Worked Solutions for *Linear Algebra Done Right*

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Chapter 1

Exercises 1A

1 Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Proof. $\forall \alpha, \beta \in \mathbb{C}$, suppose $\alpha \stackrel{\text{def}}{=} a + bi,\ \beta \stackrel{\text{def}}{=} c + di$, where $a, b, c, d \in \mathbb{R}$.
Then

$$\begin{aligned}
\alpha + \beta &= (a + bi) + (c + di) && \text{(by definition)} \\
&= (a + c) + (b + d)i && \text{(by definition of addition over } \mathbb{C}\text{)} \\
&= (c + a) + (d + b)i && \text{(holds for commutativity over } \mathbb{R}\text{)} \\
&= (c + di) + (a + bi) && \text{(by definition of addition over } \mathbb{C}\text{)} \\
&= \beta + \alpha && \text{(by definition)}
\end{aligned}$$

$\Longrightarrow$ Thus $\alpha + \beta = \beta + \alpha\ \ \forall \alpha, \beta \in \mathbb{C}$.

□

2 Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Proof. $\forall \alpha, \beta, \lambda \in \mathbb{C}$, suppose $\alpha \stackrel{\text{def}}{=} a + bi,\ \beta \stackrel{\text{def}}{=} c + di,\ \lambda \stackrel{\text{def}}{=} j + ki$, where $a, b, c, d, j, k \in \mathbb{R}$.
Then

$$\begin{aligned}
(\alpha + \beta) + \lambda &= \Big((a + bi) + (c + di)\Big) + (j + ki) && \text{(by definition)} \\
&= \Big((a + c) + (b + d)i\Big) + (j + ki) && \text{(by definition of addition over } \mathbb{C}\text{)} \\
&= \Big((a + c) + j\Big) + \Big((b + d) + k\Big)i && \text{(by definition of addition over } \mathbb{C}\text{)} \\
&= \Big(a + (c + j)\Big) + \Big(b + (d + k)\Big)i && \text{(holds for associativity over } \mathbb{R}\text{)} \\
&= (a + bi) + \Big((c + j) + (d + k)i\Big) && \text{(by definition of addition over } \mathbb{C}\text{)} \\
&= (a + bi) + \Big((c + di) + (j + ki)\Big) && \text{(by definition of addition over } \mathbb{C}\text{)} \\
&= \alpha + (\beta + \lambda) && \text{(by definition)}
\end{aligned}$$

$\Longrightarrow$ Thus $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)\ \ \forall \alpha, \beta, \lambda \in \mathbb{C}$. □

3 Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Proof. $\forall \alpha, \beta, \lambda \in \mathbb{C}$, suppose $\alpha \overset{\text{def}}{=} a + bi,\ \beta \overset{\text{def}}{=} c + di,\ \lambda \overset{\text{def}}{=} j + ki$, where $a, b, c, d, j, k \in \mathbb{R}$.
Then

$$\begin{aligned}
(\alpha\beta)\lambda &= \Big((a+bi)(c+di)\Big)(j+ki) && \text{(by definition)}\\
&= \Big((ac-bd)+(ad+bc)i\Big)(j+ki) && \text{(by definition of multiplication over } \mathbb{C})\\
&= \Big((ac-bd)j-(ad+bc)k\Big)+\Big((ac-bd)k+(ad+bc)j\Big)i && \text{(by definition of multiplication over } \mathbb{C})\\
&= (acj-bdj-adk-bck)+(ack-bdk+adj+bcj)i && \text{(holds for distributivity over } \mathbb{R})\\
&= (acj-adk-bdj-bck)+(ack+adj-bdk+bcj)i && \text{(holds for commutativity over } \mathbb{R})\\
&= \Big(a(cj-dk)-b(dj+ck)\Big)+\Big(a(ck+dj)+b(-dk+cj)\Big)i && \text{(holds for distributivity over } \mathbb{R})\\
&= (a+bi)\Big((cj-dk)+(ck+dj)i\Big) && \text{(by definition of multiplication over } \mathbb{C})\\
&= (a+bi)\Big((c+di)(j+ki)\Big) && \text{(by definition of multiplication over } \mathbb{C})\\
&= \alpha(\beta\lambda) && \text{(by definition)}
\end{aligned}$$

$\implies$ Thus $(\alpha\beta)\lambda = \alpha(\beta\lambda)\ \ \forall \alpha, \beta, \lambda \in \mathbb{C}$.

□

4 Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.

Proof. $\forall \lambda, \alpha, \beta \in \mathbb{C}$, suppose $\lambda \overset{\text{def}}{=} j + ki,\ \alpha \overset{\text{def}}{=} a + bi,\ \beta \overset{\text{def}}{=} c + di$, where $j, k, a, b, c, d \in \mathbb{R}$.
Then

$$\begin{aligned}
\lambda(\alpha+\beta) &= (j+ki)\Big((a+bi)+(c+di)\Big) && \text{(by definition)}\\
&= (j+ki)\Big((a+c)+(b+d)i\Big) && \text{(by definition of addition over } \mathbb{C})\\
&= \Big(j(a+c)-k(b+d)\Big)+\Big(j(b+d)+k(a+c)\Big)i && \text{(by definition of multiplication over } \mathbb{C})\\
&= (ja+jc-kb-kd)+(jb+jd+ka+kc)i && \text{(holds for distributivity over } \mathbb{R})\\
&= (ja-kb+jc-kd)+(jb+ka+jd+kc)i && \text{(holds for commutativity over } \mathbb{R})\\
&= (ja-kb)+(jc-kd)+(jb+ka)i+(jd+kc)i && \text{(holds for distributivity over } \mathbb{R})\\
&= \Big((ja-kb)+(jb+ka)i\Big)+\Big((jc-kd)+(jd+kc)i\Big) && \text{(holds for commutativity over } \mathbb{R})\\
&= (j+ki)(a+bi)+(j+ki)(c+di) && \text{(by definition of multiplication over } \mathbb{C})\\
&= \lambda\alpha+\lambda\beta && \text{(by definition)}
\end{aligned}$$

$\implies$ Thus $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta\ \ \forall \lambda, \alpha, \beta \in \mathbb{C}$. □

5 Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

Proof. $\forall \alpha \in \mathbb{C}$, we assume $\exists! \beta \in \mathbb{C}$ such that $\alpha + \beta = 0$. To this end, suppose $\alpha \stackrel{\text{def}}{=} a + bi, \beta \stackrel{\text{def}}{=} c + di$, where $a, b, c, d \in \mathbb{R}$. It follows that

$$\begin{aligned}
& \alpha + \beta = 0 && \text{(by assumption)} \\
\implies & (a + bi) + (c + di) = 0 && \text{(by definition)} \\
\implies & (a + c) + (b + d)i = 0 && \text{(by definition of addition over } \mathbb{C}) \\
\implies & (a + (-a)) + (b + (-b))i = 0 && \text{(holds for definition of additive inverse over } \mathbb{R}) \\
\implies & (a + bi) + \big((-a) + (-b)i\big) = 0 && \text{(by definition of addition over } \mathbb{C}) \\
\implies & (a + bi) + (-1)(a + bi) = 0 && \text{(holds for distributivity over } \mathbb{R}) \\
\implies & \alpha + (-1)(\alpha) = 0 && \text{(by definition)} \\
\implies & \alpha + -\alpha = 0 && \text{(by definition of multiplication over } \mathbb{C})
\end{aligned}$$

$\implies \forall \alpha \in \mathbb{C},\ \exists! \beta = -\alpha \in \mathbb{C}$ such that $\alpha + (-\alpha) = 0$.

$\square$

6 Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

Proof. $\forall \alpha \in \mathbb{C}$, we assume $\exists! \beta \in \mathbb{C}$ such that $\alpha\beta = 1$. To this end, suppose $\alpha \stackrel{\text{def}}{=} a + bi, \beta \stackrel{\text{def}}{=} c + di$, where $a, b, c, d \in \mathbb{R}$ and $\alpha \neq 0$. If follows that

$$\begin{aligned}
& \alpha\beta = 1 && \text{(by assumption)} \\
\implies & (a + bi)(c + di) = 1 && \text{(by definition)} \\
\implies & (ac - bd) + (ad + bc)i = 1 && \text{(by definition of multiplication over } \mathbb{C})
\end{aligned}$$

For the above equality to hold, $\Re(\alpha\beta) = 1$, and $\Im(\alpha\beta) = 0$, giving us two equations

$$ac - bd = 1 \qquad (*)$$

$$ad + bc = 0 \qquad (**)$$

Rearranging $(*)$ for c gives us

$$c = \frac{1 + bd}{a}$$

Rearranging $(**)$ for d gives us

$$d = -\frac{bc}{a}$$

Substituting $(*)$ into $(**)$ gives us

$$\begin{aligned} & d = -\frac{b\left(\frac{1+bd}{a}\right)}{a} \\ \implies & d = -\frac{b(1+bd)}{a^2} \\ \implies & da^2 = -b(1+bd) \\ \implies & da^2 = -b - b^2 d \\ \implies & da^2 + b^2 d = -b \\ \implies & d(a^2+b^2) = -b \\ \implies & d = -\frac{b}{a^2+b^2} \end{aligned} \tag{$***$}$$

Substituting $(***)$ into $(*)$ gives us

$$\begin{aligned} c &= \frac{1 + b\left(\frac{-b}{a^2+b^2}\right)}{a} \\ &= \frac{1 - \left(\frac{b^2}{a^2+b^2}\right)}{a} \\ &= \frac{\left(\frac{a^2+b^2}{a^2+b^2}\right) - \left(\frac{b^2}{a^2+b^2}\right)}{a} \\ &= \frac{\left(\frac{a^2}{a^2+b^2}\right)}{a} = \frac{a^2}{a(a^2+b^2)} \\ &= \frac{a}{a^2+b^2} \end{aligned}$$

Now, using our new definitions of c and d

$$\begin{aligned} & 1 = (ac - bd) + (ad + bc)i \\ \implies & 1 = \left(a\left(\frac{a}{a^2+b^2}\right) - b\left(\frac{-b}{a^2+b^2}\right)\right) + \left(a\left(\frac{-b}{a^2+b^2}\right) + b\left(\frac{a}{a^2+b^2}\right)\right)i && \text{(by definition)} \\ \implies & 1 = \left(\left(\frac{a^2}{a^2+b^2}\right) + \left(\frac{b^2}{a^2+b^2}\right)\right) + \left(\left(\frac{-ab}{a^2+b^2}\right) + \left(\frac{ab}{a^2+b^2}\right)\right)i && \text{(holds for multiplication over } \mathbb{R}) \\ \implies & 1 = \left(\frac{a^2+b^2}{a^2+b^2}\right) = 1 && \text{(holds for addition over } \mathbb{R}) \end{aligned}$$

We can redefine β in terms of α

$$\beta = \frac{a-bi}{a^2+b^2}$$
$$\beta \stackrel{\text{def}}{=} \frac{1}{\alpha}$$

$$\implies \forall \alpha \in \mathbb{C},\ \exists! \beta = \frac{1}{\alpha} \in \mathbb{C} \text{ such that } \alpha\left(\frac{1}{\alpha}\right) = 1.$$

$\square$

7 Show that $\dfrac{-1+\sqrt{3}i}{2}$ is a cube root of 1 (meaning that its cube equals 1).

Proof.

$$
\begin{aligned}
\left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= \left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right) && \text{(by definition)}\\
&= \left(\left(\frac{1}{4}-\frac{3}{4}\right)+\left(-\frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4}\right)i\right)\left(\frac{-1+\sqrt{3}i}{2}\right) && \text{(by definition of multiplication over } \mathbb{C})\\
&= \left(\frac{-1-\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right) && \text{(holds for addition over } \mathbb{R})\\
&= \left(\frac{1}{4}+\frac{3}{4}\right)+\left(-\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}\right)i && \text{(by definition of multiplication over } \mathbb{C})\\
&= 1 && \text{(holds for addition over } \mathbb{R})
\end{aligned}
$$

$$\implies \frac{-1+\sqrt{3}i}{2} = \sqrt[3]{1}\ .$$

□

8 Find two distinct square roots of i.

Let $\alpha \in \mathbb{C}$, where $\alpha \stackrel{\text{def}}{=} a+bi$, with $a, b \in \mathbb{R}$.

$$
\begin{aligned}
& i = \alpha^2\\
\implies & i = (a+bi)^2\\
\implies & i = a^2+2abi-b^2\\
\implies & i = \Im(a^2+2abi-b^2)\\
\implies & i = 2abi\\
\implies & 1 = 2ab\\
\implies & \frac{1}{2} = ab
\end{aligned}
$$

$$\boxed{\Re(\alpha^2) = 0 = a^2-b^2 \implies a^2 = b^2 \implies a = \pm b}$$

$$
\begin{aligned}
\implies & \frac{1}{2} = (a)^2\\
\implies & a = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}\\
\implies & \sqrt{i} = \pm\alpha = \pm\left(\frac{\sqrt{2}+\sqrt{2}i}{2}\right)
\end{aligned}
$$

$$\implies \sqrt{i} = \frac{-\sqrt{2}-\sqrt{2}i}{2},\ \frac{\sqrt{2}+\sqrt{2}i}{2}\ .$$

9 Find $x \in \mathbb{R}^4$ such that $(4,-3,1,7)+2x=(5,9,-6,8)$.

Let $x \in \mathbb{R}^4$, where $x \overset{\text{def}}{=} (x_1,x_2,x_3,x_4)$, with $x_1,x_2,x_3,x_4 \in \mathbb{R}$.

$$\begin{aligned}
(4,-3,1,7)+2x &= (5,9,-6,8)\\
(4,-3,1,7)+2(x_1,x_2,x_3,x_4) &= (5,9,-6,8)\\
2(x_1,x_2,x_3,x_4) &= (5,9,-6,8)-(4,-3,1,7)\\
2(x_1,x_2,x_3,x_4) &= (1,12,-5,1)\\
(x_1,x_2,x_3,x_4) &= \frac{1}{2}(1,12,-5,1)\\
(x_1,x_2,x_3,x_4) &= \left(\frac{1}{2},6,-\frac{5}{2},\frac{1}{2}\right)
\end{aligned}$$

$\implies x = \left(\frac{1}{2},6,-\frac{5}{2},\frac{1}{2}\right)$.

10 Explain why there does not exist $\lambda \in \mathbb{C}$ such that $\lambda(2-3i,5+4i,-6+7i)=(12-5i,7+22i,-32-9i)$.

We assume, for the sake of contradiction, that there exists $\lambda \in \mathbb{C}$ such that the equation above holds. This,

$$\begin{aligned}
\lambda(2-3i) &= 12-5i && \text{(by the definition of scalar multiplication)}\\
\lambda &= \frac{12-5i}{2-3i} && \text{(solving for } \lambda\text{)}\\
&= \frac{(12-5i)(2+3i)}{(2-3i)(2+3i)} && \text{(by the definition of division over } \mathbb{C}\text{)}\\
&= \frac{24+36i-10i-15}{4+9} && \text{(holds for distributivity over } \mathbb{C}\text{)}\\
&= \frac{9+26i}{13} && \text{(holds for commutativity over } \mathbb{C}\text{)}\\
&= \frac{9}{13}+\frac{26}{13}i. && \text{(separating } \Re(z) \text{ and } \Im(z) \text{ parts)}
\end{aligned}$$

Substituting $\lambda = \frac{9}{13}+\frac{26}{13}i$ into the second equation yields

$$\begin{aligned}
\lambda(5+4i) &= 7+22i && \text{(by the definition of scalar multiplication)}\\
\left(\frac{9}{13}+\frac{26}{13}i\right)(5+4i) &= 7+22i && \text{(substitute } \lambda\text{)}\\
&= \frac{1}{13}\Big(45+36i+130i-104\Big) && \text{(by the definition of multiplication over } \mathbb{C}\text{)}\\
&= \frac{1}{13}\Big(-59+166i\Big) && \text{(by commutativity over } \mathbb{C}\text{)}\\
&= -\frac{59}{13}+\frac{166}{13}i. && \text{(separating } \Re(z) \text{ and } \Im(z) \text{ parts)}
\end{aligned}$$

Comparing with $7+22i$, we observe a contradiction, as

$$-\frac{59}{13} \neq 7 \quad \text{and} \quad \frac{166}{13} \neq 22.$$

Therefore, no such $\lambda \in \mathbb{C}$ can exist.