SPECTRAL AND TOPOLOGICAL APPROACHES FOR HYPERCUBE POWER COLOURINGS

FINN STEINKE, RAJESH PEREIRA

Abstract. We investigate three prominent graph colouring invariants—the chromatic number χ, achromatic number ψ, and b-chromatic number b for the pth powers of n-dimensional hypercube graphs Q_n^p. While the chromatic number has been extensively studied, general bounds for these invariants on hypercube powers remain challenging to obtain. We develop novel algebro-combinatorial methods using spectral graph theory and the Bose–Mesner algebra of the Hamming association scheme to derive lower bounds for $\chi(Q_n^p)$, which extend naturally to $\psi(Q_n^p)$ and $b(Q_n^p)$. In particular, we express eigenvalues in terms of Kravchuk polynomials and present a Hoffman-type bound adapted to this setting. Complementing these spectral results, we introduce an algebro-topological framework based on simplicial complexes associated to hypercube powers, providing connectivity-based bounds based on Lovasz's original findings and exploring equivariant obstructions to colourings of Q_n^p.

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1. Introduction

Throughout, all graphs are simple, undirected, and finite. The problem of vertex colouring has remained combinatorially rich for a long period of time. In fact, many notions of graph colouring exist and we will discuss three of the most prominent graph colourings, each with an associated invariant.

1.A. **Proper and optimal colourings.** Let $\sqcup$ denote a disjoint union. Let us fix some $k \in \mathbb{N}$. A k-colouring of the graph G is a map

$$\mathcal{P} : V(G) \to [k],$$

where $[k] = \{1, 2, \ldots, k\}$. Then, a colouring is called proper if for all $\{u, v\} \in E(G)$, $\mathcal{P}(u) \neq \mathcal{P}(v)$, where $u, v \in V(G)$. Intuitively, this means that no adjacent vertices can

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receive the same colour. We will let the set of all proper colourings of G be denoted by $\mathscr{C}(G)$. We can immediately observe the following result for proper colourings.

Proposition 1.1. *Let $G = (V(G), E(G))$ be a graph. A proper k-colouring $\mathcal{P}$ induces a partition of $V(G)$ into k subsets of the form*

$$V(G) = \bigsqcup_{i \in [k]} A_i,$$

where $A_i = \mathcal{P}^{-1}(i)$. Each subset A_i is an independent set in G.

Proof. Let $\mathcal{P} : V(G) \to [k]$ be a proper colouring of G. For each $i \in [k]$, define

$$A_i = \mathcal{P}^{-1}(i) := \{v \in V(G) \mid \mathcal{P}(v) = i\}.$$

Since $\mathcal{P}$ is a colouring, every vertex $v \in V(G)$ is assigned exactly one colour in $[k]$. Therefore,

$$V(G) = \bigcup_{i \in [k]} A_i.$$

Moreover, if $i \neq j$, then $A_i \cap A_j = \emptyset$ because a vertex cannot receive two distinct colours. Hence, $\{A_i\}_{i \in [k]}$ is a partition of $V(G)$.

Next, we show that each A_i is an independent set. Suppose, for contradiction, that there exist $u, v \in A_i$ such that $\{u, v\} \in E(G)$. Then, by definition of A_i, $\mathcal{P}(u) = \mathcal{P}(v) = i$. But since $\{u, v\} \in E(G)$ and $\mathcal{P}$ is proper, we must have $\mathcal{P}(u) \neq \mathcal{P}(v)$, a contradiction. Therefore, no such edge exists, and each A_i is an independent set. $\square$

We define the chromatic number of G as

$$\chi(G) = \min\{k \mid \mathscr{C}(G) \neq \emptyset\},$$

where each colouring in $\mathscr{C}(G)$ is of the form $\mathcal{P} : V(G) \to [k]$ by definition. Any proper colouring $\mathcal{P} \in \mathscr{C}(G)$ with $\chi(G)$ colours is known as an optimal colouring of G. For a more traditional definition of optimal colouring, see [Wes01]. The chromatic number is the most well-studied graph colouring invariant, with wide-reaching applications in many areas of pure mathematics. Since χ is generally NP-hard to compute, this motivates our primary focus of the paper, namely the bounding of χ for a specific graph family we will introduce in section 1.D.

To make precise the definition of a proper k-colouring of G, we often describe it as a graph homomorphism from G to the complete graph on k vertices K_k. We recall that a graph homomorphism is a map $\phi : G \to H$ such that for all $\{u, v\} \in E(G)$, there exists $\{\phi(u), \phi(v)\} \in E(H)$. Then, a colouring $\mathcal{P}$ is precisely a graph homomorphism such that $\mathcal{P} : G \to K_k$. We summarize the relationship with the following diagram

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[dr, "\mathcal{P}"'] & H \arrow[d, "\mathcal{Q}"] \\ & K_k \end{tikzcd}$$

which commutes if and only if $\mathcal{P} = \mathcal{Q} \circ \phi$.

A rather important result with regard to the chromatic number is given in [Bro41] and is commonly known as Brooks' theorem, which we will restate as a lemma.

Lemma 1.2. *Let G be a connected graph with maximum degree $\Delta(G)$. If G is neither a complete graph nor an odd cycle, then*

$$\chi(G) \leq \Delta(G).$$

In all cases, we have the trivial bound $\chi(G) \leq \Delta(G) + 1$.

Proof. Given in [Bro41]. □

For any graph G, let us consider a subset $C \subseteq V(G)$. If for all distinct pairs $u, v \in C$ there exists $\{u, v\} \in E(G)$, then C is known as a clique of G. Equivalently, the subgraph of G induced by C is a complete graph. This leads us to another key insight for bounding χ for general graphs.

Proposition 1.3. *For any graph G, the chromatic number satisfies*

$$\chi(G) \geq \omega(G),$$

where $\omega(G)$ is the size of the largest clique in G.

Proof. Let G be a graph and let $\omega(G)$ denote the size of the largest clique in G. By definition, a clique of size $\omega(G)$ consists of $\omega(G)$ vertices such that every pair of distinct vertices is adjacent. In any proper coloring of G, adjacent vertices must receive different colors. Therefore, all vertices of this clique must be assigned distinct colors. Consequently, any proper coloring of G requires at least $\omega(G)$ colors, and hence

$$\chi(G) \geq \omega(G).$$

□

Proposition 1.4. *For any graph G, the chromatic number satisfies*

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1.$$

Proof. Follows directly from lemma 1.2. and proposition 1.3. □

Finally, χ was found to have a spectral lower bound, as given by Hoffman's theorem in [Hof03], another theorem we will rewrite as a lemma.

Lemma 1.5. *For any graph G, the chromatic number satisfies*

$$\chi(G) \geq 1 + \frac{\max_{1 \leq i \leq |V(G)|}\{\lambda_i \in \mathrm{Spec(A(G))}\}}{|\min_{1 \leq i \leq |V(G)|}\{\lambda_i \in \mathrm{Spec(A(G))}\}|},$$

where $A(G)$ is the adjacency matrix of G and where λ_i are the corresponding eigenvalues of $A(G)$.

Proof. Given in [Hof03]. □

1.B. **Complete colourings.** A complete k-colouring of G is a proper colouring $\mathcal{P}$ such that for every pair of distinct colours $i, j \in [k]$ with $i \neq j$, there exists $\{u, v\} \in E(G)$ where $\mathcal{P}(u) = i$ and $\mathcal{P}(v) = j$. Intuitively, this means that every pair of colours appears on adjacent vertices somewhere in the graph, and no colour is "isolated" from the others.

We can introduce the idea of complete k-colouring with more precision using an equivalence relation. Given a proper k-colouring $\mathcal{P} \in \mathscr{C}(G)$, we define the equivalence relation $\sim_{\mathcal{P}}$ on $V(G)$ such that

$$\sim_{\mathcal{P}} := \{(u, v) \in V(G) \times V(G) \mid \mathcal{P}(u) = \mathcal{P}(v)\},$$

where the equivalence classes of the equivalence relation are the colour classes of $\mathcal{P}$, given by $A_i = \mathcal{P}^{-1}(i)$, where $i \in [k]$. Now, we define the quotient graph $G/\sim_{\mathcal{P}}$, where

$$V(G/\sim_{\mathcal{P}}) = \{A_1, A_2, \ldots, A_k\},$$

and where $\{A_i, A_j\} \in E(G/\sim_{\mathcal{P}})$ if and only if there exist $u \in A_i$ and $v \in A_j$ such that $\{u, v\} \in E(G)$. It follows that if $\mathcal{P}$ is a complete k-colouring, then

$$G/\sim_{\mathcal{P}} \cong K_k.$$

We can relate complete k-colouring to graph homomorphisms as before.

Proposition 1.6. *Let $G = (V(G), E(G))$ be any graph and let $\mathcal{P} \in \mathscr{C}(G)$ be a proper k-colouring of G. Then, $\mathcal{P}$ is a surjective graph homomorphism such that $\mathcal{P} : G \to K_k$.*

Proof. Let $\mathcal{P} : V(G) \to [k]$, where the vertices of K_k are identified with the set $[k]$. For each $i \in [k]$, we define the colour class

$$A_i = \mathcal{P}^{-1}(i) := \{v \in V(G) \mid \mathcal{P}(v) = i\}.$$

Since $\mathcal{P}$ is a proper colouring, by definition for all $\{u, v\} \in E(G)$ we have $\mathcal{P}(u) \neq \mathcal{P}(v)$. Hence, $\{\mathcal{P}(u), \mathcal{P}(v)\}$ is an edge in K_k, because K_k is complete and contains all edges between distinct vertices. Thus, $\mathcal{P}$ preserves adjacency, and therefore it is a graph homomorphism.

Furthermore, $\mathcal{P}$ uses exactly k colours, so every $i \in [k]$ has a non-empty preimage $A_i \neq \emptyset$. This implies $\mathcal{P}$ is surjective on vertices, and is thus a surjective graph homomorphism. □

We define the achromatic number of G as

$$\psi(G) = \max\{k \mid \mathscr{C}(G) \neq \emptyset, G/{\sim_{\mathcal{P}}} \cong K_k\},$$

where any colouring $\mathcal{P} \in \mathscr{C}(G)$ and $\mathcal{P}$ defines $\sim_{\mathcal{P}}$. It is the case that finding ψ generally is NP-hard, as was the case with χ. A trivial, yet important result for the achromatic number is immediately observable.

Proposition 1.7. *For any graph G, the achromatic number satisfies*

$$\psi(G) \geq \chi(G).$$

Proof. Let G be a graph. By definition, any complete colouring $\mathcal{P}$ of G must be proper. Since $\chi(G)$ is the minimum number of colours required for a proper colouring, the result follows. □

We find a rather interesting result for the upper bound on ψ in general.

Theorem 1.8. *For any graph G, there is no upper bound on $\psi(G)$ that depends on both the spectrum of $A(G)$ and $\Delta(G)$.*

Proof. For each $k \in \mathbb{N}$, define G_k as follows: start with a star $K_{1,k}$, and replace each leaf by a path of length 2. Then G_k is a tree on $2k + 1$ vertices. We have:

$$\Delta(G_k) = 2, \quad \mathrm{Spec}(A(G_k)) \subseteq [-2, 2].$$

Let $v_1, \dots, v_k$ be the k leaves (endpoints of the paths). Assign each v_i a distinct colour, and extend this to a proper colouring in which every pair of colours appears on an edge. Then $\psi(G_k) \geq k$. Hence, $\psi(G_k) \to \infty$ while $\Delta(G_k)$ and the spectrum of $A(G_k)$ remain bounded. No function of these can bound $\psi(G)$. □

1.C. ***b*-colourings.** A b-colouring of G with k colours is a proper colouring $\mathcal{P}$ such that for any colour class A_i there exists at least one b-vertex $v_i \in A_i$ satisfying

$$N(v_i) \cap A_j \neq \emptyset$$

for all $j \neq i$, where $i, j \in [k]$ and $N(v_i) = \{u \in V(G) \mid \{v_i, u\} \in E(G)\}$. Intuitively, this means every colour class A_i contains a vertex adjacent to at least one vertex of every other colour class. We can immediately find a result through the perspective of graph homomorphisms as with proper and complete colourings.

Proposition 1.9. *A b-colouring of G with k colours exists if and only if there is a graph homomorphism $\mathcal{P} : G \to K_k$ such that for every $i \in [k]$, there exists a vertex $v \in \mathcal{P}^{-1}(i)$ satisfying*

$$\mathcal{P}(N(v)) = [k] \setminus \{i\}.$$

Proof. Let G be a graph. ($\Rightarrow$) Suppose G admits a b-colouring with k colours. We define $\mathcal{P} : G \to K_k$ by $\mathcal{P}(u) = i$ if $u \in A_i$. Properness of the colouring implies $\mathcal{P}$ is a graph homomorphism. For each i, the b-colouring condition guarantees a vertex $v_i \in A_i$ whose neighbourhood intersects every other colour class, hence $\mathcal{P}(N(v_i)) = [k] \setminus i$.

($\Leftarrow$) Conversely, if such a homomorphism exists, define the colour classes as $A_i = \mathcal{P}^{-1}(i)$. Properness follows from $\mathcal{P}$ being a homomorphism. The local surjectivity property guarantees each class contains a b-vertex. Thus, the colouring is a b-colouring. □

We also notice another result, this one about the maximum degree of any b-coloured graph.

Proposition 1.10. *Let $G = (V(G), E(G))$ be a graph. If G admits a b-colouring with k colours, then*

$$\Delta(G) \geq k - 1.$$

Proof. Let G be a graph where $v \in V(G)$ is a b-vertex in some colour class. By definition, v must have a neighbour in every other colour class, where there are $k-1$ colour classes remaining. Therefore, $\deg(v) \geq k-1$, and since $\deg(v) \leq \Delta(G)$, the conclusion follows directly. □

We define the b-chromatic number of G as

$$b(G) = \max\left\{k \,\middle|\, \mathcal{P} \in \mathscr{C}(G), \forall i \in [k], \exists v \in \mathcal{P}^{-1}(i) \text{ such that } \mathcal{P}(N(v)) = [k] \setminus \{i\}\right\},$$

where $N(v)$ is the open neighbourhood of a vertex, as defined previously. As with χ and ψ, it was shown in [IM99] that finding b for general G is NP-hard. It was also shown in the same text that there is only a superficial similarity between $b(G)$ and $\Gamma(G)$, the Grundy number, and that neither bounds or impacts the other in any meaningful way. Indeed, we can apply the same trivial result found for ψ to b.

Proposition 1.11. *For any graph G, the b-chromatic number satisfies*

$$b(G) \geq \chi(G)$$

Proof. Nearly identical to the proof of proposition 1.7. □

Moreover, similarly to χ, an upper bound in terms of $\Delta(G)$ can be generally constructed using proposition 1.10.

Corollary 1.12. *For any graph G, the b-chromatic number satisfies*

$$b(G) \leq \Delta(G) + 1.$$

Proof. Let G be a graph. Then, by definition of b, we know that $b(G) = k$ for the maximal b-colouring, and hence

$$\Delta(G) \geq b(G) - 1 \implies b(G) \geq \Delta(G) + 1.$$

□

Proposition 1.13. *For any graph G, the b-chromatic number satisfies*

$$\chi(G) \leq b(G) \leq \Delta(G) + 1.$$

Proof. Follows directly from proposition 1.11. and corollary 1.12. □

Let $\mathscr{A}(G)$ be the set of all colourings of G, proper and improper. It should be noted that the formal definition of b-colouring, as in [IM99], requires the definition of a relation R_G such that

$$R_G \coloneqq \left\{(\mathcal{P}, \mathcal{Q}) \in \mathscr{A}(G) \times \mathscr{A}(G) \,\middle|\, \mathcal{P}^{-1}(i) = \bigsqcup_{j \in S_i} \mathcal{Q}^{-1}(j), \forall i \in [k]\right\}$$

where $\mathcal{P} : V(G) \to [k]$, $\mathcal{Q} : V(G) \to [k+1]$, and where $\{S_i\}_{i=1}^{k}$ is a partition of $[k+1]$. The transitive closure of R_G, given by R_G^*, induces a strict ordering, where we let

$$b(G) := \max\left\{k \in \mathbb{N} \mid \mathcal{P} \in \mathscr{C}(G),\ \mathcal{P} \text{ is } R_G^*\text{-minimal}\right\}.$$

Any proper colouring $\mathcal{P}$ on G which is R_G^*-minimal is said to be a b-colouring of G. This was introduced as a refinement to the complete colouring, which was given a similar relational definition. We present this definition for clarity for the origin of b-colouring, but the earlier definition is far more concise.

The next section will focus on the family of graphs we will be finding χ, ψ, b bounds for.

1.D. **Hypercube graphs and pth powers.** We define the hypercube graph Q_n using a vector space, where

$$V(Q_n) := \mathbb{F}_2^n, \quad E(Q_n) := \{\{\mathbf{x}, \mathbf{y}\} \subseteq \mathbb{F}_2^n \mid d_H(\mathbf{x}, \mathbf{y}) = 1, \}$$

such that the Hamming metric d_H is equipped with a vector-based definition, analogous to the ℓ^1 norm,

$$d_H(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{n} |x_i - y_i|.$$

We observe some important result for hypercube graphs and their colourings.

Proposition 1.14. *Every hypercube graph Q_n is bipartite.*

Proof. Let Q_n be the hypercube graph of vector length $n \geq 2$. We define the sets

$$U := \left\{\mathbf{x} \in \mathbb{F}_2^n \,\middle|\, \textstyle\sum_{i=1}^{2} x_i \equiv 0 \pmod 2\right\},$$

and

$$W := \left\{\mathbf{y} \in \mathbb{F}_2^n \,\middle|\, \textstyle\sum_{i=1}^{2} y_i \equiv 1 \pmod 2\right\}.$$

If $\mathbf{x}$ and $\mathbf{y}$ are adjacent, only one coordinate will differ between them, so their coordinate sums will differ in parity. Since $U \cup W = \mathbb{F}_2^n$, $U \cap W = \emptyset$, and $\emptyset \notin \{U, W\}$, we have shown that $\{U, W\}$ forms a valid partition of $\mathbb{F}_2^n$, and hence the conclusion follows. □

This leads us directly to a corollary.

Corollary 1.15. *For all $n \geq 2$, the hypercube graph Q_n satisfies*

$$\chi(Q_n) = 2.$$

Proof. Follows directly from proposition 1.14. □

We then define the pth power of the hypercube graph Q_n^p such that

$$V(Q_n^p) := V(Q_n) = \mathbb{F}_2^n, \quad E(Q_n^p) := \{\{\mathbf{x}, \mathbf{y}\} \subseteq \mathbb{F}_2^n \mid 1 \leq d_H(\mathbf{x}, \mathbf{y}) \leq p\},$$

where $1 \leq p \leq n$. It is trivial that $Q_n^1 \cong Q_n$, and equally trivial that $Q_n^n \cong K_{2^n}$, making cases where $p = 1$ and $p = n$ generally uninteresting to study for powers. For this reason, the first nontrivial case is Q_3^2, a typical construction of which is given in 1.1.

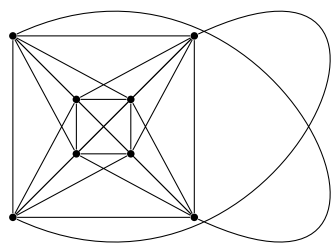

FIGURE 1.1. Typical construction of Q_3^2

We give a trivial result for $\chi(Q_n^p)$ that follows a general result for graphs.

Proposition 1.16. *Let G be a graph and $H \subseteq G$ a subgraph. Then, the chromatic numbers of both G and H satisfy*

$$\chi(H) \leq \chi(G).$$

Proof. Let G be a graph with subgraph H. Suppose there exists a proper k-colouring of G, denoted $\mathcal{P}$. We then define the restriction map

$$\mathrm{res}_H : \mathscr{C}(G) \to \mathscr{C}(H),$$

where for any colouring $\mathcal{P}$ of G, we set

$$\mathrm{res}_H(\mathcal{P}) = \mathcal{P}|_{V(H)}.$$

Then, since $E(H) \subseteq E(G)$, if $\mathcal{P}$ is proper on G, then $\mathcal{P}|_{V(H)}$ must be proper on H. Since $\mathcal{P}$ uses k colours, then $\mathcal{P}|_{V(H)}$ will use at most k colours, meaning

$$\chi(H) \leq k = \chi(G).$$

□

Extending to graph powering, we find a similar result.

Corollary 1.17. *Let G be a graph and G^p its pth power, based on the standard graph metric d_G. Then, the chromatic numbers of both G and G^p satisfy*

$$\chi(G) \leq \chi(G^p).$$

Proof. Let $G = (V, E)$ be a graph. It is enough to show that $G \subseteq G^p$, where G is a subgraph. Recall that G^p is defined on the same vertex set $V(G)$, where two vertices u, v are adjacent in G^p if and only if $d_G(u, v) \leq p$. In particular, if u and v are adjacent in G, then $d_G(u, v) = 1 \leq p$, so every edge of G is also an edge of G^p. Therefore,

$$E(G) \subseteq E(G^p),$$

meaning G is a subgraph of G^p. □

Proposition 1.18. *For all $n \geq 2$, $1 \leq p \leq n$, the pth power hypercube graph Q_n^p satisfies*

$$\chi(Q_n^p) \geq 2.$$

Proof. Follows directly from corollary 1.15. and corollary 1.17. □

[FFR17] presented some strong results for certain values of $b(Q_n^p)$, $\chi(Q_n^p)$, and $\omega(Q_n^p)$. These results highly useful for most if not all graphs Q_n^p, and were found combinatorially. We list four of the text's results below, as they will be important to compare with our results later.

Theorem 1.19. *For all $n \geq 2$,*

$$b(Q_n^{n-1}) = 2^{n-1}.$$

Proof. Given in [FFR17]. □

Theorem 1.20. *For all $n \geq 2$,*

$$b(Q_n^{\lfloor \frac{n}{2} \rfloor}) = 2^{n-1}.$$

Proof. Given in [FFR17]. □

Theorem 1.21. (i) *For all $n \geq 3$ and $1 \leq p \leq n-1$,*

$$\omega(Q_n^p) = \begin{cases} \sum_{i=0}^{\frac{p}{2}} \binom{n}{i}, & \text{if } p \text{ is even,} \\ 2\sum_{i=0}^{\frac{p-1}{2}} \binom{n-1}{i}, & \text{if } p \text{ is odd,} \end{cases}$$

(ii) *For all* $n \geq 2$ *and* $\left\lfloor \frac{n}{2} \right\rfloor < p < n-1$,

$$2^{n-1} \leq b(Q_n^p) \leq 2^{n-1} + \left\lfloor \frac{\omega(Q_n^p)}{2} \right\rfloor.$$

Proof. Given in [FFR17]. □

Theorem 1.22. *For all* $n \geq 2$ *and* $\left\lceil \frac{2(n-1)}{3} \right\rceil \leq p \leq n-1$,

$$\chi(Q_n^p) = 2^{n-1}.$$

Proof. Given in [FFR17]. □

In fact, we can state a quick result that builds on theorem 1.21.(i).

Corollary 1.23. *For all* $n \geq 3$ *and* $1 \leq p \leq n-1$,

$$\chi(Q_n^p) \geq \begin{cases} \sum_{i=0}^{\frac{p}{2}} \binom{n}{i}, & \text{if } p \text{ is even,} \\ 2\sum_{i=0}^{\frac{p-1}{2}} \binom{n-1}{i}, & \text{if } p \text{ is odd.} \end{cases}$$

Proof. Follows directly from proposition 1.3. and theorem 1.21.(i). □

The specific interest of this text is finding general bounds for $\chi(Q_n^p)$ using novel methods, expanding to discuss $\psi(Q_n^p)$ and $b(Q_n^p)$ whenever possible. The methods of interest will be presented in the following sections as they are used.

2. Algebro-combinatorial results

We introduce the Kravchuk polynomials, first presented in [Kra29] to use in this section's main result. Let $\mathcal{K}_i(x; n, q)$ denote the ith Kravchuk polynomial of x, where we define

$$\mathcal{K}_i(x; n, q) = \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} \binom{x}{j} \binom{n-x}{i-j},$$

where $n \in \mathbb{N}$, q is a prime power, and $i \in [n]$. In particular, we will make use of the case where $q = 2$, where the polynomials reduce to

$$\mathcal{K}_i(x; n, 2) =: \mathcal{K}_i(x; n) = \sum_{j=0}^{i} (-1)^j \binom{x}{j} \binom{n-x}{i-j}.$$

We define an n-class Hamming scheme as the ordered pair $(\mathbb{F}_2^n, \{R_j\}_{j=0}^n)$, where

$$R_j := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{F}_2^n \times \mathbb{F}_2^n \mid d_H(\mathbf{x}, \mathbf{y}) = j\}.$$

To do this, we define $(A_j)_{\mathbf{x},\mathbf{y}} := \begin{cases} 1, & \text{if } (\mathbf{x}, \mathbf{y}) \in R_j \\ 0, & \text{otherwise,} \end{cases}$ such that $\{A_j\}_{j=0}^n$ generate the Bose-Mesner algebra $\mathcal{A} = \operatorname{span}_{\mathbb{R}}\left\{\{A_j\}_{j=0}^n\right\} \subseteq \mathbb{R}^{n \times n}$. There are certain key facts about the Bose-Mesner algebra we discuss, taken from [BI84].

Proposition 2.1. *Let* $(X, \{R_j\}_{j=0}^n)$ *be an* n*-class association scheme. Then, the Bose-Mesner algebra* $\mathcal{A}$ *generated by the associated* $\{A_j\}_{j=0}^n$ *satisfies the following:*

(i) $\mathcal{A}$ *is a commutative, semisimple subalgebra of* $\mathbb{R}^{X \times X}$,

(ii) *The matrices* $\{A_j\}$ *are real symmetric and simultaneously diagonalizable,*

(iii) *There exists a unique basis* $\{J_0, \ldots, J_n\}$ *of* $\mathcal{A}$ *consisting of pairwise orthogonal idempotents (i.e.,* $J_i J_j = \delta_{ij} J_i$ *and* $\sum_{i=0}^n J_i = I$*, where* δ_{ij} *denotes the Kronecker delta).*

Proof. Given in [BI84]. □

We also know a key fact about the eigenvalues of each A_j.

Lemma 2.2. *Let* $(\mathbb{F}_2^n, \{R_j\}_{j=0}^n)$ *be the* n*-class Hamming scheme with corresponding Bose-Mesner algebra* $\mathcal{A} = \operatorname{span}_{\mathbb{R}}\big\{\{A_j\}_{j=0}^n\big\}$. *Then the eigenvalues of* A_j *satisfy*

$$A_j \mathbf{v}_k = \mathcal{K}_j(k;n)\mathbf{v}_k,$$

where $\mathbf{v}_k \in \operatorname{Im}(J_k)$ *with* $0 \le k \le n$, *where* J_k *is a primitive idempotent in* $\mathcal{A}$.

Proof. Given in [BI84]. □

We are now ready for our main result of the section.

Theorem 2.3. *For all* $n \ge 1$ *and* $1 \le p \le n$,

$$\chi(Q_n^p) \ge 1 + \frac{\sum_{i=1}^p \binom{n}{i}}{-\min\limits_{0\le k\le n}\left\{\sum_{i=1}^p \sum_{j=0}^i (-1)^j \binom{k}{j}\binom{n-k}{i-j}\right\}}.$$

Proof. Let $(\mathbb{F}_2^n, \{R_i\}_{i=0}^n)$ be the Hamming scheme. By proposition 2.1.(i,iii), the adjacency matrices $\{A_i\}_{i=0}^n$ generate a commutative, semisimple subalgebra of $\mathbb{R}^{2^n\times 2^n}$ with a basis of primitive orthogonal idempotents $\{J_k\}_{k=0}^n$. Since these matrices are simultaneously diagonalizable by proposition 2.1.(ii), there exists a common orthonormal basis of eigenvectors $\{\mathbf{v}_k\}_{k=0}^n$ such that each $\mathbf{v}_k \in \operatorname{Im}(J_k)$. We also know that

$$A(Q_n^p) = \sum_{i=1}^p A_i.$$

By lemma 2.2., we find

$$A(Q_n^p)\mathbf{v}_k = \left(\sum_{i=1}^p \mathcal{K}_i(k;n)\right)\mathbf{v}_k,$$

where $\lambda_k = \sum_{i=1}^p \mathcal{K}_i(k;n)$ for $0 \le k \le n$. The largest eigenvalue is given by

$$\lambda_0 = \sum_{i=1}^p \binom{n}{i},$$

and the smallest by

$$\min_{0\le k\le n} \lambda_k.$$

Since A_i are real symmetric matrices,

$$\operatorname{tr}\big(A(Q_n^p)\big) = \sum_{i=1}^p \operatorname{tr}(A_i) = 0.$$

However, we know that

$$\sum_{k=0}^n \dim\big(\operatorname{Im}(J_k)\big)\lambda_k = 0,$$

where $\lambda_0 > 0$, then $\min\limits_{0\le k\le n} \lambda_k < 0$ to satisfy the trace requirement. Finally, applying lemma 1.5. and writing out the Kravchuk polynomials explicitly, the conclusion follows. □

It can be noted that the above result can be proved without the use of the Kravchuk polynomials, instead relying on the machinery of Hadamard matrices and counting principles. We give this proof below, after a lemma.

Lemma 2.4. *Let* $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ *be the* 2×2 *Hadamard matrix. Then, for all* $n \geq 1$*, a* $2^n \times 2^n$ *Hadamard matrix satisfies*

$$H_{2^n} = H_2^{\otimes n},$$

where $\otimes$ *denotes the Kronecker product.*

Proof. We prove the lemma using weak induction. The base case, $n = 1$, trivially gives us $H_{2^1} = H_2^{\otimes 1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. We assume that the conclusion holds for some $k \in \mathbb{N}$, where $k \leq n$. Thus, we take the inductive step, where $n = k + 1$. Here, by results concerning construction of Hadamard matrices in [Syl67], we find that $H_{2^{k+1}} = \begin{bmatrix} H_{2^k} & H_{2^k} \\ H_{2^k} & -H_{2^k} \end{bmatrix}$, and by definition of the Kronecker product

$$\begin{bmatrix} H_{2^k} & H_{2^k} \\ H_{2^k} & -H_{2^k} \end{bmatrix} = \begin{bmatrix} 1 \cdot H_{2^k} & 1 \cdot H_{2^k} \\ 1 \cdot H_{2^k} & (-1) \cdot H_{2^k} \end{bmatrix} = H_2 \otimes H_2^{\otimes k} = H_2^{\otimes (k+1)},$$

and therefore the conclusion follows. $\square$

This directly leads us to the following alternative proof of theorem 2.3.

Proof. of ***Theorem 2.3.*** Let $(\mathbb{F}_2^n, \{R_i\}_{i=0}^n)$ be the Hamming scheme. We observe that

$$A_i = \sum_{S \subseteq [n], |S| = i} \bigotimes_{j=1}^{n} M_j^S, \text{ where } M_j^S = \begin{cases} J_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \text{if } j \in S, \\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \text{if } j \notin S. \end{cases}$$

The 2×2 Hadamard matrix H_2 diagonalizes both I_2 and J_1 simultaneously with eigenvalues:

$$\mathrm{Spec}(I_2) = \{1, 1\}, \quad \mathrm{Spec}(J_1) = \{1, -1\}.$$

Therefore by constructing H_{2^n} using lemma 2.4., the $2^n \times 2^n$ Hadamard matrix diagonalizes all A_i simultaneously. An eigenvector of H_{2^n} corresponds to a vector $\mathbf{k} \in \mathbb{F}_2^n$ with Hamming weight $k = |\mathbf{k}|$, where on coordinate j the eigenvalue is

$$\lambda_j = \begin{cases} 1, & \text{if } k_j = 0, \\ -1, & \text{if } k_j = 1. \end{cases}$$

For a fixed $\mathbf{k}$, the eigenvalue $\lambda_i(k)$ of A_i is then

$$\lambda_i(k) = \sum_{S \subseteq [n], |S| = i} \prod_{j \in S} (-1)^{k_j} = \sum_{S \subseteq [n], |S| = i} (-1)^{|S \cap \mathrm{supp}(\mathbf{k})|},$$

where $\mathrm{supp}(\mathbf{k}) := \{j \mid k_j = 1\}$. Counting subsets S of size i with exactly j elements in $\mathrm{supp}(\mathbf{k})$, we get

$$\lambda_i(k) = \sum_{j=0}^{i} (-1)^j \binom{k}{j} \binom{n-k}{i-j}.$$

As we know, the adjacency matrix of the graph Q_n^p satisfies

$$A(Q_n^p) = \sum_{i=1}^{p} A_i,$$

and its eigenvalues on $\mathbf{v}_k$ are

$$\lambda(k) = \sum_{i=1}^{p} \lambda_i(k) = \sum_{i=1}^{p} \sum_{j=0}^{i} (-1)^j \binom{k}{j} \binom{n-k}{i-j}.$$

The remainder of the proof follows almost exactly as before until the desired result is obtained. $\square$

We can extend this lower bound on $\chi(Q_n^p)$ to the other invariants discussed earlier, where we obtain general lower bounds for $\psi(Q_n^p)$ and $b(Q_n^p)$.

Corollary 2.5. *For all $n \geq 1$ and $1 \leq p \leq n$,*

$$\psi(Q_n^p) \geq \left\lceil 1 + \frac{\sum_{i=1}^{p} \binom{n}{i}}{-\min\limits_{0 \leq j \leq n} \left\{ \sum_{i=1}^{p} \sum_{r=0}^{i} (-1)^r \binom{j}{r} \binom{n-j}{i-r} \right\}} \right\rceil.$$

Proof. From proposition 1.7., we know that $\chi(Q_n^p) \leq \psi(Q_n^p)$, where $\chi(Q_n^p) \in \mathbb{N}$. It follows by theorem 2.3. that

$$\psi(Q_n^p) \geq \chi(Q_n^p) \geq 1 + \frac{\sum_{i=1}^{p} \binom{n}{i}}{-\min\limits_{0 \leq j \leq n} \left\{ \sum_{i=1}^{p} \sum_{r=0}^{i} (-1)^r \binom{j}{r} \binom{n-j}{i-r} \right\}},$$

and since $\chi(Q_n^p) \in \mathbb{N}$, the conclusion follows. $\square$

Corollary 2.6. *Given Q_n^p, then*

$$b(Q_n^p) \geq \left\lceil 1 + \frac{\sum_{i=1}^{p} \binom{n}{i}}{-\min\limits_{0 \leq j \leq n} \left\{ \sum_{i=1}^{p} \sum_{r=0}^{i} (-1)^r \binom{j}{r} \binom{n-j}{i-r} \right\}} \right\rceil.$$

Proof. Nearly identical to the proof of corollary 2.5. $\square$

From theorem 1.8., we also arrive at another result for the upper bound of $\psi(Q_n^p)$, which we note here.

Corollary 2.7. *There is no upper bound on $\psi(Q_n^p)$ that depends on both the spectrum of $A(Q_n^p)$ and $\Delta(Q_n^p)$.*

Proof. By Theorem 1.8., there exists a family of graphs $\{G_k\}_{k \in \mathbb{N}}$ such that each G_k has uniformly bounded spectrum and maximum degree, but $\psi(G_k) \to \infty$ as $k \to \infty$. For each fixed k, since G_k is finite, it can be embedded as a connected induced subgraph into Q_n^p for sufficiently large n and p (this follows from the fact that powers of hypercubes contain all finite trees and similar subgraphs as induced subgraphs). Thus, for these embeddings $G'_k \subseteq Q_n^p$, we have:

$$\psi(G_k) \leq \psi(Q_n^p), \quad \Delta(G_k) \leq \Delta(Q_n^p), \quad \text{and} \quad \mathrm{Spec}(A(G_k)) \subseteq [-M, M] \subseteq \mathrm{Spec}(A(Q_n^p))$$

for some uniform bound M independent of k. If there were an upper bound on $\psi(Q_n^p)$ depending only on $\mathrm{Spec}(A(Q_n^p))$ and $\Delta(Q_n^p)$, then this bound would restrict $\psi(G_k)$ for all k as well, contradicting $\psi(G_k) \to \infty$. Hence, no such upper bound exists for $\psi(Q_n^p)$. $\square$

We can make a quick note about a certain bound for $b(G)$ presented in [AK11], which states that

$$b(G) \leq \left\lfloor \frac{2|V(G)| - \Delta(G) - \delta(G) - 3}{3|V(G)| - 2\Delta(G) - \delta(G) - 4} |V(G)| \right\rfloor,$$

where $\delta(G)$ denotes the minimum degree in G. Generally, for $G \cong Q_n^p$, this bound is less tight than $\Delta(Q_n^p)+1$, but there are certain values of n, p for which the bound from [AK11] outperforms $\Delta(Q_n^p)+1$. For example, if we consider Q_3^2, where

$$b(Q_3^2) \leq \left\lfloor \frac{16-6-6-3}{24-12-6-4}(8) \right\rfloor = 4,$$

as opposed to

$$b(Q_3^2) \leq 6+1 = 7.$$

We rewrite the [AK11] bound, substituting for general Q_n^p,

$$b(Q_n^p) \leq \left\lfloor \frac{2(2^n)-2D-3}{3(2^n)-3D-4}(2^n) \right\rfloor,$$

where $D = \Delta(Q_n^p) = \delta(Q_n^p) = \sum_{i=1}^{p} \binom{n}{i}$. To find when the above bound is tighter than $D+1$, we can manipulate the inequality

$$\frac{(2^{n+1}-2D-3)(2^n)}{3(2^n)-3D-4} < D+1,$$

where since $3(2^n)-3D-4 > 0$ for $n \in \mathbb{N}$,

$$\begin{aligned}
&\implies (2^{n+1}-2D-3)(2^n) < (D+1)(3(2^n)-3D-4),\\
&\iff 2^{2n+1}-2^{n+1}D-3(2^n) < 3(2^n)D-3D^2-4D+3(2^n)-3D-4,\\
&\iff 3D^2+\big(-2^{n+1}-3(2^n)+7\big)D+\big(2^{2n+1}-6(2^n)+4\big) < 0.
\end{aligned}$$

This is only worth mentioning as a side-note, as there are very few instances we would choose to use the more cumbersome bound for $b(Q_n^p)$ over $\Delta(Q_n^p)+1$.

3. Algebro-topological results

Recalling the definition of a simplex, we define a finite abstract simplicial complex Δ on a set of vertices V and a collection of subsets of V such that $\{v\} \in \Delta$ for all $v \in V$. If a simplex $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. We define the functor

$$|\cdot| : \mathbf{sSet} \to \mathbf{CGHaus},$$

such that for any simplicial complex Δ, the associated topological space $|\Delta|$ is known as the geometric realization of Δ, a compactly generated Hausdorff space. Specifically, each simplex is given a geometric definition. We use the convex hull $\mathrm{conv}(\cdot)$ such that

$$|\Delta| := \bigcup_{\sigma \in \Delta} \mathrm{conv}(\sigma),$$

where we identify each vertex $v_i \in V$ with the standard basis vector $\mathbf{e}_i \in \mathbb{R}^{|V|}$, and

$$\mathrm{conv}(\sigma) := \left\{ \sum_{v_i \in \sigma} t_i \mathbf{e}_i \in \mathbb{R}^{|V|} \;\middle|\; t_i \geq 0,\ \sum_{v_i \in \sigma} t_i = 1 \right\}.$$

The geometric realization $|\Delta|$ is endowed with the weak topology such that any subset $U \subseteq |\Delta|$ is open if and only if $U \cap \mathrm{conv}(\sigma)$ is open in $\mathrm{conv}(\sigma)$ for each $\sigma \in \Delta$.

With these primitives, we can define specific simplicial complexes that will become valuable for our main result in this section.

Let $G = (V(G), E(G))$ be a graph where $U \subseteq V(G)$. The induced subgraph $G[U]$ is defined as

$$V(G[U]) := U, \quad E(G[U]) := \{\{u,v\} \in E(G) \mid u,v \in U\}.$$

Then, $\mathrm{Cl}(G)$ denotes the clique complex of G, where

$$\mathrm{Cl}(G) \coloneqq \{\sigma \subseteq V(G) \mid \exists r \in \mathbb{N}, G[\sigma] \cong K_r\}.$$

Another important simplicial complex for graphs is the Vietoris-Rips complex, a special case of a clique complex first used in [Vie27]. More formally, we define the Vietoris-Rips complex of a metric space (X, d) as

$$\mathrm{VR}(X; \delta) \coloneqq \{\sigma \subseteq V(X) \mid \forall u, v \in \sigma, d(u, v) \leq \delta\},$$

where $\delta \in \mathbb{N}$ and d is the metric. For our purposes, we will consider the metric space $(\mathbb{F}_2^n, d_H)$, which is easily verified to be such a space by definition. For this space, we find

$$\mathrm{VR}(\mathbb{F}_2^n; p) = \{\sigma \subseteq \mathbb{F}_2^n \mid \forall \mathbf{x}, \mathbf{y} \in \sigma, d_H(\mathbf{x}, \mathbf{y}) \leq p\}.$$

A result about these two complexes becomes clear for Q_n^p.

Proposition 3.1. *Let Q_n^p be the pth power of the n-dimensional hypercube graph. Then,*

$$|\mathrm{VR}(\mathbb{F}_2^n; p)| \cong |\mathrm{Cl}(Q_n^p)|,$$

where $\cong$ denotes homeomorphism.

Proof. Since edges of Q_n^p connect precisely those vertices at Hamming distance at most p, it follows that σ is a clique in Q_n^p if and only if every pair of vertices in σ has Hamming distance at most p. Thus, the simplices of $\mathrm{VR}(\mathbb{F}_2^n; p)$ and $\mathrm{Cl}(Q_n^p)$ coincide. The geometric realization depends only on the simplicial complex, so the conclusion follows. □

Two topological spaces (X, τ_X) and (Y, τ_Y) are homotopy equivalent if there exist continuous maps

$$f : X \to Y, \quad \text{and} \quad g : Y \to X,$$

together with homotopies

$$H : X \times [0, 1] \to X, \quad \text{and} \quad K : Y \times [0, 1] \to Y,$$

such that

$$\begin{aligned} H(x, 0) &= (g \circ f)(x),\ H(x, 1) = x,\ \text{for all } x \in X, \\ K(y, 0) &= (f \circ g)(y),\ K(y, 1) = y,\ \text{for all } y \in Y. \end{aligned}$$

In this case, we write $X \simeq Y$. For the definiton of a continuous map and other basic topological definitions, see [Mun00].

Another important idea in algebraic topology is given immediately thereafter. Let $\vee$ denote the wedge sum of topological spaces, which is defined as follows: given a family of pointed topological spaces $\{(X_i, \tau_i, x_i)\}_{i \in I}$, the wedge sum $\bigvee_{i \in I} X_i$ is the quotient space

$$\bigvee_{i \in I} X_i \coloneqq \left(\bigsqcup_{i \in I} X_i \right) \Big/ \sim,$$

where the equivalence relation $\sim$ identifies all basepoints, such that

$$x_i \sim x_j \text{ for all } i, j \in I.$$

[AA21] showed that for $p = 2$, there was a general theorem concerning homotopy type for Vietoris-Rips complexes of hypercube graphs. We restate that theorem as a lemma for our results.

Lemma 3.2. *For all $n \geq 3$,*

$$|\mathrm{VR}(\mathbb{F}_2^n; 2)| \simeq \bigvee_{\sum\limits_{0 \leq j < i < n} (2^{n-2} - 2^{i-1})} S^3,$$

where S^3 is the 3-sphere.

Proof. Given in [AA21]. □

We require some more topology in order to continue. Let (X, τ, x_0) be a pointed topological space. For each integer $n \geq 1$, the nth homotopy group $\pi_n(X, x_0)$ is the set of homotopy classes of based continuous maps

$$f : (S^n, s_0) \to (X, x_0),$$

where two maps f, g are equivalent if there exists a based homotopy

$$H : S^n \times [0,1] \to X,$$

satisfying

$$H(-,0) = f, \ H(-,1) = g, \text{ and } H(s_0, t) = x_0, \text{ for all } t \in [0,1].$$

For $n = 1$, $\pi_1(X, x_0)$ is a group with concatenation of loops as the operation; for $n \geq 2$, $\pi_n(X, x_0)$ are abelian groups under the standard operation induced by sphere concatenation. For more detail on loop and sphere concatenation, reference [Hat02].

This leads us to another topological definition. A topological space (X, τ) is called k-connected (for $k \geq -1$) if

$$\pi_0(X) = \pi_1(X) = \cdots = \pi_k(X) = 0,$$

where $\pi_0(X)$ denotes the set of path-connected components of X. We dentote the connectedness of a space with conn$(\cdot)$.

It's known that (S^3, τ_{std}) is 2-connected for any basepoint, where $\pi_1(S^3) = \pi_2(S^3) = 0$. We prove a result about k-connectedness and the wedge sum, after a known result from [Hil55].

Proposition 3.3. *For all $k < n$, where $n, k \in \mathbb{N}$,*

$$\pi_k\left(\bigvee_{i\in I} S^n\right) = 0$$

Proof. Given in [Hil55]. □

Thus, the following result about wedges of similar dimensional spheres becomes clear.

Lemma 3.4. *For $n \geq 0$, $\bigvee\limits_{i\in I} S^n$ is $(n-1)$-connected.*

Proof. Let S^n be the n-dimensional sphere. By proposition 3.3., we know it is the case that $\pi_k\left(\bigvee_{i\in I} S^n\right) = 0$ for all $k < n$, so by the definition of k-connectedness, the conclusion follows. □

We can now state our first of two main theorems of the section, which will become more useful soon.

Theorem 3.5. *For all $n \geq 3$, $|\mathrm{Cl}(Q_n^2)|$ is 2-connected.*

Proof. By Proposition 3.1., we have $|\mathrm{Cl}(Q_n^2)| \cong |\mathrm{VR}(\mathbb{F}_2^n; 2)|$. Lemma 3.2. states that this space is homotopy equivalent to a wedge sum of S^3's, i.e.,

$$|\mathrm{Cl}(Q_n^2)| \simeq \bigvee_{i\in I} S^3.$$

Finally, by Lemma 3.4., wedge sums of 3-spheres are 2-connected. Since connectedness is a homotopy invariant, $|\mathrm{Cl}(Q_n^2)|$ is 2-connected. □

Finally, we move to understand the second main theorem of the section, which will lead to a direct result about $\chi(Q_n^2)$. We introduce the notion of a $\mathbb{Z}_2$-space.

Let $\mathbb{Z}_2 \coloneqq \{0,1\} \cong (\mathbb{Z}/2\mathbb{Z},+)$ where $\mathbb{Z}_2$ denotes the cyclic group of order 2. A $\mathbb{Z}_2$-space is a topological space (X,τ) equipped with a continuous function

$$\alpha : \mathbb{Z}_2 \times X \to X,$$

where $\mathbb{Z}_2$ acts on X such that

$$\alpha(0,x) = x \text{ for all } x \in X, \quad \alpha(1,\alpha(1,x)) = x \text{ for all } x \in X.$$

That is, $1 \in \mathbb{Z}_2$ acts as an involution on X. More details on $\mathbb{Z}_2$-maps are given in [Koz08]. Since a free group action requires that no non-identity element in the group fixes a point, the definition of a free $\mathbb{Z}_2$-action follows. Finally, for any two topological spaces (X,τ_X) and (Y,τ_Y), we define a $\mathbb{Z}_2$-equivariant map to be a map $f : X \to Y$ such that $gf(x) = f(gx)$ for all $x \in X$ and $g \in \mathbb{Z}_2$.

We give a useful variant of the Borsuk-Ulam theorem as a lemma for use in our second theorem.

Lemma 3.6. *Let (X,τ_X) be a free $\mathbb{Z}_2$-space that is k-connected and let (Y,τ_Y) be a topological space with trivial $\mathbb{Z}_2$-action and $\dim(Y) = d$. Then, there is no continuous $\mathbb{Z}_2$-equivariant map*

$$f : X \to Y$$

if $d \le k+1$.

Proof. Given in [Bor33]. ☐

We are ready for our second main theorem of the section.

Theorem 3.7. *For all $p < n$,*

$$\chi(Q_n^p) \ge \operatorname{conn}\big(|\mathrm{Cl}(Q_n^p)|\big) + 3,$$

where $\operatorname{conn}(\cdot)$ denotes the connectedness of a space.

Proof. Let $p < n$, where $n,p \in \mathbb{N}$. We suppose, for the sake of contradiction, that

$$\chi(Q_n^p) \le \operatorname{conn}(|\mathrm{Cl}(Q_n^p)|) + 2.$$

Then, there exists a proper colouring

$$\mathcal{P} : V(Q_n^p) \to [k],$$

where $k = \operatorname{conn}(|\mathrm{Cl}(Q_n^p)|) + 2$. This induces a simplicial map

$$\phi : \mathrm{Cl}(Q_n^p) \to \Delta^{k-1},$$

where Δ^{k-1} is the standard $(k-1)$-simplex with vertex set $[k]$. The map ϕ sends each vertex $v \in V(Q_n^p)$ to its colour $\mathcal{P}(v)$, and each clique to the corresponding set of colours, which forms a face of Δ^{k-1} since $\mathcal{P}$ is proper. Taking the geometric realization, we obtain a continuous map

$$|\phi| : |\mathrm{Cl}(Q_n^p)| \to |\Delta^{k-1}| \cong D^{k-1},$$

where D^{k-1} is homeomorphic to the closed $(k-1)$-dimensional ball.

We consider the involutive map $\alpha : \mathbb{F}_2^n \to \mathbb{F}_2^n$ defined by

$$\alpha(\mathbf{x}) \coloneqq \mathbf{x} + \mathbf{1},$$

where $\mathbf{1} \coloneqq (1,\ldots,1) \in \mathbb{F}_2^n$. Since $p < n$, the map α is a fixed-point-free graph automorphism of Q_n^p, and thus induces a free $\mathbb{Z}_2$-action on $V(Q_n^p)$ that preserves adjacency. This action extends to a simplicial automorphism of $\mathrm{Cl}(Q_n^p)$ and hence a free $\mathbb{Z}_2$-action on the topological space $|\mathrm{Cl}(Q_n^p)|$.

Since $|\mathrm{Cl}(Q_n^p)|$ is a free $\mathbb{Z}_2$-space and $k = \mathrm{conn}(|\mathrm{Cl}(Q_n^p)|) + 2$, it follows from lemma 3.6. that there does not exist a continuous $\mathbb{Z}_2$-equivariant map

$$|\mathrm{Cl}(Q_n^p)| \to |\Delta^{k-1}|.$$

But this contradicts the existence of the map $|\phi|$, which is continuous and equivariant with respect to the trivial $\mathbb{Z}_2$-action on Δ^{k-1}. Therefore, our assumption must be false, and we conclude

$$\chi(Q_n^p) \geq \mathrm{conn}(|\mathrm{Cl}(Q_n^p)|) + 3.$$

□

We obtain a corollary, even if relatively weak.

Corollary 3.8. *For all $n \geq 3$,*

$$\chi(Q_n^2) \geq 5.$$

Proof. Follows directly from theorem 3.5. and theorem 3.7. □

The same bound can also be applied to both $\psi(Q_n^2)$ and $b(Q_n^2)$.

4. Discussion of results

When directly comparing theorem 2.3. to known lower bounds on $\chi(Q_n^p)$ such as the one found in corollary 1.23., while more computationally dense, it still represents a novel combinatorial construction that aids in bounding. The same is true for corollary 2.5., though corollary 2.6. represents an interesting break to this pattern. From theorems 1.19. and 1.20. we already have easily-computable exact values for many $n, p \in \mathbb{N}$. Additionally, while corollary 2.6. works for any general n and p, we can rely on theorem 1.21.(ii) for most values of p, given some n. The "Hoffman-like" bound in theorem 2.3. is, in fact, challenging to compute due to the additional minimization problem given in the denominator of the fractional term. For this reason, $\chi(Q_n^p)$ and $\psi(Q_n^p)$ obtain useful, but generally difficult lower bounds for all n and p. It is likely that brute force optimization could produce most lower bounds using theorem 2.3. though for high enough n and p this can obviously be challenging as well.

While corollary 3.8. is quite unhelpful for large n, the process using theorems 3.5. and 3.7. is novel and warrants further investigation. Particularly, corollary 3.8. is weaker than corollary 1.23. for almost all cases, making it combinatorially useless. Further investigation into general application of theorem 3.7. might prove useful for larger n and p, particularly in light of [AV23], which gives some homotopical and homological results and bounds for $\mathrm{VR}(Q_n; p)$ for $p \neq 2$. Additionally, trying other simplicial complexes associated with Q_n^p may prove useful in making the topological obstruction arguments such as those made in our "Lovasz-type" proof of theorem 3.7. For example, [ASS23] studies some results for the Čech complexes of some Q_n^p, which could be useful to apply to methods presented in this text.

4.A. Open problems.

We are left with some remaining open problems:

(i) *Is it possible to apply theorem 3.7. using general results for the homotopy type of $|\mathrm{Cl}(Q_n^p)|$ for general p?*
(ii) *Does the bound for $b(Q_n^p)$ from* [AK11] *have any meaningful results based off the inequality presented in section 2? Can it be used to generate better lower bounds for $b(Q_n^p)$?*
(iii) *Can the minimum eigenvalues in theorem 2.3. always be found simply in general?*
(iv) *Can these results for Q_n^p be extended to other vertex-transitive graphs with high adjacency?*

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References

[AA21] Michał Adamaszek and Henry Adams, *On vietoris–rips complexes of hypercube graphs*, arXiv preprint arXiv:2103.01040 (2021).

[AK11] M. Alkhateeb and A. Kohl, *Upper bounds on the b-chromatic number and results for restricted graph classes*, Discussiones Mathematicae. Graph Theory **31** (2011), 709–735.

[ASS23] Henry Adams, Samir Shukla, and Anurag Singh, *Čech complexes of hypercube graphs*, arXiv preprint arXiv:2212.05871 (2023).

[AV23] Henry Adams and Žiga Virk, *Lower bounds on the homology of vietoris–rips complexes of hypercube graphs*, arXiv preprint arXiv:2309.06222 (2023).

[BI84] Eiichi Bannai and Tatsurō Itō, *Algebraic combinatorics i: Association schemes*, The Benjamin/Cummings Publishing Co., Menlo Park, CA, 1984.

[Bor33] Karol Borsuk, *Drei sätze über die n-dimensionale euklidische sphäre*, Fundamenta Mathematicae **20** (1933), no. 1, 177–190.

[Bro41] R. L. Brooks, *On colouring the nodes of a network*, Proceedings of the Cambridge Philosophical Society **37** (1941), no. 2, 194–197.

[FFR17] P. Francis and S. Francis Raj, *On b-coloring of powers of hypercubes*, Discrete Applied Mathematics **225** (2017), 74–86.

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[Hil55] Peter J. Hilton, *On the homotopy groups of the union of spheres*, Journal of the London Mathematical Society **30** (1955), no. 2, 154–172.

[Hof03] Alan J. Hoffman, *On eigenvalues and colorings of graphs*, Selected Papers of Alan J. Hoffman, World Scientific, 2003, pp. 407–419.

[IM99] R. W. Irving and D. F. Manlove, *The b-chromatic number of a graph*, Discrete Applied Mathematics **91** (1999), no. 1–3, 127–141.

[Koz08] Dmitry N. Kozlov, *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics, vol. 21, Springer, 2008.

[Kra29] M. Kravchuk, *Sur une généralisation des polynômes d'hermite*, Comptes Rendus Mathématique **189** (1929), 620–622, JFM 55.0799.01.

[Mun00] James R. Munkres, *Topology*, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2000.

[Syl67] James Joseph Sylvester, *Thoughts on inverse orthogonal matrices, simultaneous sign-successions, and tessellated pavements in two or more colours, with applications to newton's rule, ornamental tile-work, and the theory of numbers*, Philosophical Magazine **34** (1867), no. 232, 461–475.

[Vie27] L. Vietoris, *Über den höheren zusammenhang kompakter räume und eine klasse von zusammenhangstreuen abbildungen*, Mathematische Annalen **97** (1927), 454–472.

[Wes01] Douglas B. West, *Introduction to graph theory*, 2nd ed., Prentice Hall of India, 2001.