Dynamic Model of Quanser Aero

This document develops a dynamic model of the Quanser Aero system using the Euler-Lagrange framework.

```
clear all;
```

Symbolic Values

We define the symbolic values need for the following development.

```
% STATES
syms theta theta_dot psi psi_dot theta_ddot psi_ddot real

% AERO PARAMETERS
syms d_x d_t l_t real % body distances
syms m_e m_rod m_pa m_mt m_tt m_tc real % body masses
syms m_body real % total body mass as a separate variable for simplicity

syms r_y h_y real % forked yoke distances
syms m_y real % forked yoke mass

syms g real % gravitational acceleration
```

Kinematics

We define the kinematics of the system.

Rotation Matrices

```
% Rotation matrix from azimuth frame to inertial frame
R_ai = [
    cos(psi), -sin(psi), 0;
    sin(psi), cos(psi),  0;
    0,        0,         1;
]
```

R_ai =

$$\begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
% Rotation matrix from body frame azimuth frame
R_ba = [
    cos(theta), 0, -sin(theta);
    0,          1,           0;
    sin(theta), 0,  cos(theta);
]
```

R_ba =

$$\begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

```
% Rotation matrix from body frame to inertial frame
R_bi = R_ai * R_ba
```

R_bi =

$$\begin{pmatrix} \cos(\psi)\cos(\theta) & -\sin(\psi) & -\cos(\psi)\sin(\theta) \\ \cos(\theta)\sin(\psi) & \cos(\psi) & -\sin(\psi)\sin(\theta) \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

Angular Velocity

The angular velocity of the body and azimuth frames with respect to the inertial frame is computed.

```
% angular velocity of azimuth frame expressed in inertial frame
omega_fyi = [
          0;
          0;
    psi_dot;
]
```

omega_fyi =

$$\begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

```
% angular velocity of body frame expressed in inertial frame
omega_bi = omega_fyi + R_ai * [
             0;
    -theta_dot;
             0;
]
```

omega_bi =

$$\begin{pmatrix} \dot{\theta}\sin(\psi) \\ -\dot{\theta}\cos(\psi) \\ \dot{\psi} \end{pmatrix}$$

Center of Mass

We first define the location of the masses in the simplified body model.

```
% location of main assembly
r_mainassembly = [
    d_t;
    0;
    0;
];

% location of tail assembly
r_tailassembly = [
    -d_t;
    0;
    0;
];

% location of main motor
r_mainmotor = [
    d_t;
    0;
    -d_x;
];

% location of tail motor
r_tailmotor = [
    -d_t;
    d_x;
    0;
];
```

We locate the center of mass of the body as it will contribute to the total energy with kinetic and potential energy.

```
m_body_expr = m_rod + 2*m_pa + 2*m_e
```

m_body_expr = $2\,m_e + 2\,m_{\mathrm{pa}} + m_{\mathrm{rod}}$

```
r_cmbodyb = ( ...
    r_mainassembly * m_pa + ...
    r_tailassembly * m_pa + ...
    r_mainmotor * m_e + ...
    r_tailmotor * m_e ...
) / m_body
```

r_cmbodyb =

$$\begin{pmatrix} 0 \\ \frac{d_x\, m_e}{m_{\mathrm{body}}} \\ -\frac{d_x\, m_e}{m_{\mathrm{body}}} \end{pmatrix}$$

```
r_cmbodyi = R_bi * r_cmbodyb
```

r_cmbodyi =

$$\begin{pmatrix} \frac{d_x\, m_e\cos(\psi)\sin(\theta)}{m_{\mathrm{body}}} - \frac{d_x\, m_e\sin(\psi)}{m_{\mathrm{body}}} \\ \frac{d_x\, m_e\cos(\psi)}{m_{\mathrm{body}}} + \frac{d_x\, m_e\sin(\psi)\sin(\theta)}{m_{\mathrm{body}}} \\ -\frac{d_x\, m_e\cos(\theta)}{m_{\mathrm{body}}} \end{pmatrix}$$

We also want to know the velocity of the center of mass.

```
v_cmbodyi = simplify(jacobian(r_cmbodyi, [theta; psi]) * [
    theta_dot; psi_dot;
])
```

v_cmbodyi =

$$\begin{pmatrix} -\frac{d_x\, m_e\, \left(\dot{\psi}\cos(\psi) - \dot{\theta}\cos(\psi)\cos(\theta) + \dot{\psi}\sin(\psi)\sin(\theta)\right)}{m_{\mathrm{body}}} \\ \frac{d_x\, m_e\, \left(\dot{\psi}\cos(\psi)\sin(\theta) - \dot{\psi}\sin(\psi) + \dot{\theta}\cos(\theta)\sin(\psi)\right)}{m_{\mathrm{body}}} \\ \frac{d_x\, m_e\, \dot{\theta}\sin(\theta)}{m_{\mathrm{body}}} \end{pmatrix}$$

And we will need its norm squared for later calculations.

```
v_norm2 = simplify(v_cmbodyi' * v_cmbodyi)
```

v_norm2 =

$$-\frac{{d_x}^2\, {m_e}^2\, \left(\dot{\psi}^2\cos(\theta)^2 - 2\,\dot{\psi}^2 + 2\,\dot{\psi}\,\dot{\theta}\cos(\theta) - \dot{\theta}^2\right)}{{m_{\mathrm{body}}}^2}$$

Kinetics

We define the kinetics of the system.

Inertia Matrices

```
% -- Inertia matrix of slender rod --
I_rodb = [
    0,                 0,                 0;
    0, 1/12*m_rod*l_t^2,                 0;
    0,                 0, 1/12*m_rod*l_t^2;
]
```

I_rodb =

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{{l_t}^2\, m_{\mathrm{rod}}}{12} & 0 \\ 0 & 0 & \frac{{l_t}^2\, m_{\mathrm{rod}}}{12} \end{pmatrix}$$

```
% -- Inertia matrices of point masses --
I_pmb = zeros(3,3);

% Contribution from main assembly
I_pmb = I_pmb + point_mass_inertia_matrix(r_mainassembly, m_pa);

% Contribution from tail assembly
I_pmb = I_pmb + point_mass_inertia_matrix(r_tailassembly, m_pa);

% Contribution from main motor
I_pmb = I_pmb + point_mass_inertia_matrix(r_mainmotor, m_e);

% Contribution from tail motor
I_pmb = I_pmb + point_mass_inertia_matrix(r_tailmotor, m_e);

% -- Total inertia matrix of body --
I_bodybfull = simplify(I_rodb + I_pmb)
```

I_bodybfull =

$$\begin{pmatrix} 2\,{d_x}^2\, m_e & d_t\, d_x\, m_e & d_t\, d_x\, m_e \\ d_t\, d_x\, m_e & \sigma_1 & 0 \\ d_t\, d_x\, m_e & 0 & \sigma_1 \end{pmatrix}$$

where

$$\sigma_1 = 2\,{d_t}^2\, m_e + {d_x}^2\, m_e + 2\,{d_t}^2\, m_{\mathrm{pa}} + \frac{{l_t}^2\, m_{\mathrm{rod}}}{12}$$

```
% -- Inertia matrix of forked yoke --
I_fyafull = [
    1/12*m_y*(3*r_y^2+h_y^2), 0, 0;
    0, 1/12*m_y*(3*r_y^2+h_y^2), 0;
    0, 0, 1/2*m_y*r_y^2;
]
```

I_fyafull =

$$\begin{pmatrix} \frac{m_y\, \left({h_y}^2 + 3\,{r_y}^2\right)}{12} & 0 & 0 \\ 0 & \frac{m_y\, \left({h_y}^2 + 3\,{r_y}^2\right)}{12} & 0 \\ 0 & 0 & \frac{m_y\, {r_y}^2}{2} \end{pmatrix}$$

Kinetic Energy

We will now find the contribution of the forked yoke to the total kinetic energy.

```
T_rotfyfull = 1/2 * omega_fyi' * R_ai * I_fyafull * R_ai' * omega_fyi
```

T_rotfyfull =
$$\frac{m_y\,\dot{\psi}^2\,r_y{}^2}{4}$$

And the contribution of the body to the total kinetic energy is

```
T_rotbfull = simplify(1/2 * omega_bi' * R_bi * I_bodybfull * R_bi' * omega_bi)
```

T_rotbfull =
$$d_x{}^2\,m_e\,\dot{\psi}^2 + d_t^2\,m_e\,\dot{\theta}^2 + \frac{d_x{}^2\,m_e\,\dot{\theta}^2}{2} + d_t^2\,m_{\mathrm{pa}}\,\dot{\theta}^2 + \frac{l_t^2\,m_{\mathrm{rod}}\,\dot{\theta}^2}{24} + d_t^2\,m_e\,\dot{\psi}^2\cos(\theta)^2 - \frac{d_x{}^2\,m_e\,\dot{\psi}^2\cos(\theta)^2}{2} + d_t^2\,m_{\mathrm{pa}}\,\dot{\psi}^2\cos(\theta)^2 + \frac{l_t^2\,m_{\mathrm{rod}}\,\dot{\psi}^2\cos(\theta)^2}{24} + \frac{d_t\,d_x\,m_e\,\dot{\psi}^2\sin(2\,\theta)}{2} - d_t\,d_x\,m_e\,\dot{\psi}\,\dot{\theta}\sin(\theta)$$

The expressions for the kinetic energies are long and complicated so we compute them the symbolic inertia matrices instead

```
syms I_bxx I_byy I_bxy real
syms I_fyxx I_fyyy I_fyzz real
I_bodyb = [
    I_bxx, I_bxy, I_bxy;
    I_bxy, I_byy,     0;
    I_bxy,     0, I_byy;
]
```

I_bodyb =
$$\begin{pmatrix} I_{\mathrm{bxx}} & I_{\mathrm{bxy}} & I_{\mathrm{bxy}} \\ I_{\mathrm{bxy}} & I_{\mathrm{byy}} & 0 \\ I_{\mathrm{bxy}} & 0 & I_{\mathrm{byy}} \end{pmatrix}$$

```
T_rotb = simplify(1/2 * omega_bi' * R_bi * I_bodyb * R_bi' * omega_bi)
```

T_rotb =
$$\frac{I_{\mathrm{bxx}}\,\dot{\psi}^2}{2} + \frac{I_{\mathrm{byy}}\,\dot{\theta}^2}{2} - \frac{I_{\mathrm{bxx}}\,\dot{\psi}^2\cos(\theta)^2}{2} + \frac{I_{\mathrm{byy}}\,\dot{\psi}^2\cos(\theta)^2}{2} + I_{\mathrm{bxy}}\,\dot{\psi}^2\cos(\theta)\sin(\theta) - I_{\mathrm{bxy}}\,\dot{\psi}\,\dot{\theta}\sin(\theta)$$

```
I_fya = [
    I_fyxx,      0,      0;
         0, I_fyyy,      0;
         0,      0, I_fyzz;
]
```

I_fya =
$$\begin{pmatrix} I_{\mathrm{fyxx}} & 0 & 0 \\ 0 & I_{\mathrm{fyyy}} & 0 \\ 0 & 0 & I_{\mathrm{fyzz}} \end{pmatrix}$$

```
T_rotfy = 1/2 * omega_fyi' * R_ai * I_fya * R_ai' * omega_fyi
```

T_rotfy =
$$\frac{I_{\mathrm{fyzz}}\,\dot{\psi}^2}{2}$$

We can also find the kinetic energy due to translation of the center of mass.

```
T_cmbody = simplify(1/2*m_body*v_norm2)
```

T_cmbody =
$$-\frac{d_x{}^2\,m_e{}^2\,\left(\dot{\psi}^2\cos(\theta)^2 - 2\,\dot{\psi}^2 + 2\,\dot{\psi}\,\dot{\theta}\cos(\theta) - \dot{\theta}^2\right)}{2\,m_{\mathrm{body}}}$$

Note that we have not expressed the inertia tensor from the center of mass, so this does not affect the kinetic energy of the body.

The total kinetic energy of the system is therefore

```
Tfull = simplify(T_rotfyfull + T_rotbfull)
```

Tfull =
$$d_x{}^2\,m_e\,\dot{\psi}^2 + d_t^2\,m_e\,\dot{\theta}^2 + \frac{d_x{}^2\,m_e\,\dot{\theta}^2}{2} + d_t^2\,m_{\mathrm{pa}}\,\dot{\theta}^2 + \frac{l_t^2\,m_{\mathrm{rod}}\,\dot{\theta}^2}{24} + \frac{m_y\,\dot{\psi}^2\,r_y{}^2}{4} + d_t^2\,m_e\,\dot{\psi}^2\cos(\theta)^2 - \frac{d_x{}^2\,m_e\,\dot{\psi}^2\cos(\theta)^2}{2} + d_t^2\,m_{\mathrm{pa}}\,\dot{\psi}^2\cos(\theta)^2 + \frac{l_t^2\,m_{\mathrm{rod}}\,\dot{\psi}^2\cos(\theta)^2}{24} + \frac{d_t\,d_x\,m_e\,\dot{\psi}^2\sin(2\,\theta)}{2} - d_t\,d_x\,m_e\,\dot{\psi}\,\dot{\theta}\sin(\theta)$$

```
T = simplify(T_rotfy + T_rotb)
```

T =
$$\frac{I_{\mathrm{bxx}}\,\dot{\psi}^2}{2} + \frac{I_{\mathrm{fyzz}}\,\dot{\psi}^2}{2} + \frac{I_{\mathrm{byy}}\,\dot{\theta}^2}{2} - \frac{I_{\mathrm{bxx}}\,\dot{\psi}^2\cos(\theta)^2}{2} + \frac{I_{\mathrm{byy}}\,\dot{\psi}^2\cos(\theta)^2}{2} + I_{\mathrm{bxy}}\,\dot{\psi}^2\cos(\theta)\sin(\theta) - I_{\mathrm{bxy}}\,\dot{\psi}\,\dot{\theta}\sin(\theta)$$

Potential Energy

The potential energy of the system depends only on the position of the body center of mass.

```
g_vec = [
    0;
    0;
    g;
];
V = m_body * g_vec' * r_cmbodyi
```

V = $-d_x\,g\,m_e\cos(\theta)$

Euler-Lagrange Equation

We first define the vector of generalized coordinates and its time derivatives.

```
q = [
    theta;
    psi;
];
q_dot = [
    theta_dot;
    psi_dot
];
q_ddot = [
    theta_ddot;
    psi_ddot;
];
```

We now define the dynamics of the system using the Euler-Lagrange equation.

```
L = simplify(T - V) % Lagrangian
```

L =

$$\frac{I_{\mathrm{bxx}}\,\dot{\psi}^2}{2}+\frac{I_{\mathrm{fyzz}}\,\dot{\psi}^2}{2}+\frac{I_{\mathrm{byy}}\,\dot{\theta}^2}{2}-\frac{I_{\mathrm{bxx}}\,\dot{\psi}^2\cos(\theta)^2}{2}+\frac{I_{\mathrm{byy}}\,\dot{\psi}^2\cos(\theta)^2}{2}+I_{\mathrm{bxy}}\,\dot{\psi}^2\cos(\theta)\sin(\theta)+d_x\,g\,m_e\cos(\theta)-I_{\mathrm{bxy}}\,\dot{\psi}\,\dot{\theta}\sin(\theta)$$

```
%L = Lfull;
```

We simplify the expression by introducing new symbolics

```
% syms k_1 k_2 k_3 k_4 k_5 k_6 k_7 real
% L = psi_dot^2 * (k_1 + k_2 * cos(theta)^2 + k_3 * cos(theta)*sin(theta)) ...
%     + k_4 * theta_dot^2 ...
%     + psi_dot * theta_dot * (k_5 * cos(theta) + k_6 * sin(theta)) ...
%     + k_7 * cos(theta)
```

We compute its partial derivatives

```
dLdq = simplify(jacobian(L, q))
```

dLdq = $\left(I_{\mathrm{bxy}}\,\dot{\psi}^2\,\left(2\cos(\theta)^2-1\right)+I_{\mathrm{bxx}}\,\dot{\psi}^2\cos(\theta)\sin(\theta)-I_{\mathrm{byy}}\,\dot{\psi}^2\cos(\theta)\sin(\theta)-I_{\mathrm{bxy}}\,\dot{\psi}\,\dot{\theta}\cos(\theta)-d_x\,g\,m_e\sin(\theta)\quad 0\right)$

```
dLdq_dot = simplify(jacobian(L, q_dot))
```

dLdq_dot = $\left(I_{\mathrm{byy}}\,\dot{\theta}-I_{\mathrm{bxy}}\,\dot{\psi}\sin(\theta)\quad I_{\mathrm{bxx}}\,\dot{\psi}+I_{\mathrm{fyzz}}\,\dot{\psi}-I_{\mathrm{bxy}}\,\dot{\theta}\sin(\theta)-I_{\mathrm{bxx}}\,\dot{\psi}\cos(\theta)^2+I_{\mathrm{byy}}\,\dot{\psi}\cos(\theta)^2+2\,I_{\mathrm{bxy}}\,\dot{\psi}\cos(\theta)\sin(\theta)\right)$

```
ddLdq_dotdt = simplify(jacobian(dLdq_dot, q_dot) * q_ddot + jacobian(dLdq_dot, q) * q_dot)
```

ddLdq_dotdt =

$$\begin{pmatrix} I_{\mathrm{byy}}\,\ddot{\theta}-I_{\mathrm{bxy}}\,\ddot{\psi}\sin(\theta)-I_{\mathrm{bxy}}\,\dot{\psi}\,\dot{\theta}\cos(\theta) \\ \frac{I_{\mathrm{bxx}}\,\ddot{\psi}}{2}+\frac{I_{\mathrm{byy}}\,\ddot{\psi}}{2}+I_{\mathrm{fyzz}}\,\ddot{\psi}-I_{\mathrm{bxy}}\,\ddot{\theta}\sin(\theta)-\frac{I_{\mathrm{bxx}}\,\ddot{\psi}\cos(2\,\theta)}{2}+\frac{I_{\mathrm{byy}}\,\ddot{\psi}\cos(2\,\theta)}{2}-I_{\mathrm{bxy}}\,\dot{\theta}^2\cos(\theta)+I_{\mathrm{bxy}}\,\ddot{\psi}\sin(2\,\theta)+2\,I_{\mathrm{bxy}}\,\dot{\psi}\,\dot{\theta}\cos(2\,\theta)+I_{\mathrm{bxx}}\,\dot{\psi}\,\dot{\theta}\sin(2\,\theta)-I_{\mathrm{byy}}\,\dot{\psi}\,\dot{\theta}\sin(2\,\theta) \end{pmatrix}$$

The equations of motion are therefore

```
syms tau1 tau2 real
tau = [tau1; tau2];
ddLdq_dotdt - dLdq' == tau
```

ans =

$$\begin{pmatrix} I_{\mathrm{byy}}\,\ddot{\theta}-I_{\mathrm{bxy}}\,\ddot{\psi}\sin(\theta)-I_{\mathrm{bxy}}\,\dot{\psi}^2\,\left(2\cos(\theta)^2-1\right)-I_{\mathrm{bxx}}\,\dot{\psi}^2\cos(\theta)\sin(\theta)+I_{\mathrm{byy}}\,\dot{\psi}^2\cos(\theta)\sin(\theta)+d_x\,g\,m_e\sin(\theta)=\tau_1 \\ \frac{I_{\mathrm{bxx}}\,\ddot{\psi}}{2}+\frac{I_{\mathrm{byy}}\,\ddot{\psi}}{2}+I_{\mathrm{fyzz}}\,\ddot{\psi}-I_{\mathrm{bxy}}\,\ddot{\theta}\sin(\theta)-\frac{I_{\mathrm{bxx}}\,\ddot{\psi}\cos(2\,\theta)}{2}+\frac{I_{\mathrm{byy}}\,\ddot{\psi}\cos(2\,\theta)}{2}-I_{\mathrm{bxy}}\,\dot{\theta}^2\cos(\theta)+I_{\mathrm{bxy}}\,\ddot{\psi}\sin(2\,\theta)+2\,I_{\mathrm{bxy}}\,\dot{\psi}\,\dot{\theta}\cos(2\,\theta)+I_{\mathrm{bxx}}\,\dot{\psi}\,\dot{\theta}\sin(2\,\theta)-I_{\mathrm{byy}}\,\dot{\psi}\,\dot{\theta}\sin(2\,\theta)=\tau_2 \end{pmatrix}$$

where τ_1, τ_2 are the generalized forces acting on the system.

Reduced Model

We know $I_{\mathrm{bxx}}, I_{\mathrm{bxy}}, I_{\mathrm{bxz}} \ll \min(I_{yy}, I_{zz})$ so we let $I_{\mathrm{bxx}} = I_{\mathrm{bxy}} = I_{\mathrm{bxz}} = 0$ and obtain a reduced model

```
subs(ddLdq_dotdt - dLdq' == tau, [I_bxx,I_bxy], [0,0])
```

ans =

$$\begin{pmatrix} I_{\mathrm{byy}}\cos(\theta)\sin(\theta)\,\dot{\psi}^2+I_{\mathrm{byy}}\,\ddot{\theta}+d_x\,g\,m_e\sin(\theta)=\tau_1 \\ \frac{I_{\mathrm{byy}}\,\ddot{\psi}}{2}+I_{\mathrm{fyzz}}\,\ddot{\psi}+\frac{I_{\mathrm{byy}}\,\ddot{\psi}\cos(2\,\theta)}{2}-I_{\mathrm{byy}}\,\dot{\psi}\,\dot{\theta}\sin(2\,\theta)=\tau_2 \end{pmatrix}$$

Generalized Forces

We now identify and model the non-conservative forces and torques (generalized forces) affecting the system.

Propeller Forces

We introduce symbolics for the longitudinal and transversal forces from the main and tail propellers.

```
syms F_mtheta F_mpsi F_ttheta F_tpsi real
```

The position of the main propeller $\vec{r}_m$ is

```
r_mb = [
    d_t;
    0;
    0;
];
r_mi = R_bi * r_mb;
```

The longitudinal force from the main propeller $\overrightarrow{F}_{m\theta}$ is

```
F_mthetab = [
    0;
    0;
    F_mtheta;
];
F_mthetai = R_bi * F_mthetab;
```

The moment about the pitch axis as a result of this force as a generalized force is therefore

```
tau_mtheta = simplify(jacobian(r_mi, theta)' * F_mthetai)
```

tau_mtheta = $F_{\mathrm{mtheta}}\, d_t$

The transversal force from the main propeller $\overrightarrow{F}_{m\psi}$ is

```
F_mpsib = [
    0;
    -F_mpsi;
    0;
];
F_mpsii = R_bi * F_mpsib;
```

So the moment about the yaw axis as a result of this force is

```
tau_mpsi = simplify(jacobian(r_mi, psi)' * F_mpsii)
```

tau_mpsi = $-F_{\mathrm{mpsi}}\, d_t \cos(\theta)$

The position of the tail propeller $\overrightarrow{r}_t$ is

```
r_tb = [
    -d_t;
    0;
    0;
];
r_ti = R_bi * r_tb;
```

The longitudinal and transverse forces $\overrightarrow{F}_{t\theta}, \overrightarrow{F}_{t\psi}$ from the tail propeller are

```
F_tthetab = [
    0;
    0;
    -F_ttheta;
];
F_tthetai = R_bi * F_tthetab;
F_tpsib = [
    0;
    -F_tpsi;
    0;
];
F_tpsii = R_bi * F_tpsib;
```

Which leads to the moment about the pitch axis

```
tau_ttheta = simplify(jacobian(r_ti, theta)' * F_tthetai)
```

tau_ttheta = $F_{\mathrm{ttheta}}\, d_t$

and about the yaw axis

```
tau_tpsi = simplify(jacobian(r_ti, psi)' * F_tpsii)
```

tau_tpsi = $F_{\mathrm{tpsi}}\, d_t \cos(\theta)$