Dynamics Effects of Added Mass

This document computes the dynamic effects of the added mass.

```
clear all;
```

Symbolic Values

We define the symbolic values need for the following development.

```
% STATES
syms theta theta_dot psi psi_dot theta_ddot psi_ddot real

% AERO PARAMETERS
syms d_m d_t mbar real
syms g real % gravitational acceleration
```

Kinematics

Rotation Matrices

```
% Rotation matrix from azimuth frame to inertial frame
R_ai = [
    cos(psi), -sin(psi), 0;
    sin(psi), cos(psi),  0;
    0,        0,         1;
]
```

R_ai =

$$\begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```

% Rotation matrix from body frame azimuth frame
R_ba = [
    cos(theta), 0, -sin(theta);
    0,          1,           0;
    sin(theta), 0,  cos(theta);
]
```

R_ba =

$$\begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

```

% Rotation matrix from body frame to inertial frame
R_bi = R_ai * R_ba
```

R_bi =

$$\begin{pmatrix} \cos(\psi)\cos(\theta) & -\sin(\psi) & -\cos(\psi)\sin(\theta) \\ \cos(\theta)\sin(\psi) & \cos(\psi) & -\sin(\psi)\sin(\theta) \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

Angular Velocity

The angular velocity of the body and azimuth frames with respect to the inertial frame is computed.

```
% angular velocity of azimuth frame expressed in inertial frame
omega_fyi = [
          0;
          0;
    psi_dot;
]
```

omega_fyi =

$$\begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

```

% angular velocity of body frame expressed in inertial frame
omega_bi = omega_fyi + R_ai * [
             0;
    -theta_dot;
             0;
]
```

omega_bi =

$$\begin{pmatrix} \dot{\theta}\sin(\psi) \\ -\dot{\theta}\cos(\psi) \\ \dot{\psi} \end{pmatrix}$$

Position of added mass

We define the location of the added mass.

```
% location of main assembly
r_mb = [
    d_t;
    0;
    -d_m;
]
```

r_mb =

$$\begin{pmatrix} d_t \\ 0 \\ -d_m \end{pmatrix}$$

```
r_mi = R_bi * r_mb
```

r_mi =

$$\begin{pmatrix} d_t\cos(\psi)\cos(\theta) + d_m\cos(\psi)\sin(\theta) \\ d_t\cos(\theta)\sin(\psi) + d_m\sin(\psi)\sin(\theta) \\ d_t\sin(\theta) - d_m\cos(\theta) \end{pmatrix}$$

Inertia Matrix

```
% -- Moment of inertia of the added mass --
I_mb = point_mass_inertia_matrix(r_mb, mbar)
```

I_mb =

$$\begin{pmatrix} {d_m}^2\,\mathrm{mbar} & 0 & d_m\,d_t\,\mathrm{mbar} \\ 0 & \mathrm{mbar}\,\left({d_m}^2 + {d_t}^2\right) & 0 \\ d_m\,d_t\,\mathrm{mbar} & 0 & {d_t}^2\,\mathrm{mbar} \end{pmatrix}$$

Kinetic Energy

The kinetic energy of the added mass is

```
T_m = simplify(1/2 * omega_bi' * R_bi * I_mb * R_bi' * omega_bi)
```

T_m =

$$\mathrm{mbar}\,\left({d_m}^2\,\dot{\psi}^2 + {d_t}^2\,\dot{\psi}^2 + 2\,{d_m}^2\,\dot{\theta}^2 + 2\,{d_t}^2\,\dot{\theta}^2 - {d_m}^2\,\dot{\psi}^2\cos(2\,\theta) + {d_t}^2\,\dot{\psi}^2\cos(2\,\theta) + 2\,d_m\,d_t\,\dot{\psi}^2\sin(2\,\theta)\right)$$

Potential Energy

The potential energy of the system depends only on the position of the body center of mass.

```
g_vec = [
    0;
    0;
    g;
];
V_m = mbar * g_vec' * r_mi
```

V_m = $-g\,\mathrm{mbar}\,\left(d_m\cos(\theta) - d_t\sin(\theta)\right)$

Euler-Lagrange Equation

We first define the vector of generalized coordinates and its time derivatives.

```
q = [
    theta;
    psi;
];
q_dot = [
    theta_dot;
    psi_dot
];
q_ddot = [
    theta_ddot;
    psi_ddot;
];
```

We define the Lagrangian for the added mass

```
L = simplify(T_m - V_m)
```

L =

$$\frac{\mathrm{mbar}\,\left(-{d_m}^2\,\dot{\psi}^2\cos(\theta)^2 + {d_m}^2\,\dot{\psi}^2 + {d_m}^2\,\dot{\theta}^2 + 2\sin(\theta)\,d_m\,d_t\,\dot{\psi}^2\cos(\theta) + 2\,g\,d_m\cos(\theta) + {d_t}^2\,\dot{\psi}^2\cos(\theta)^2 + {d_t}^2\,\dot{\theta}^2 - 2\,g\sin(\theta)\,d_t\right)}{2}$$

We compute its partial derivatives

```
dLdq = simplify(jacobian(L, q))
```

dLdq =

$$\left(-\frac{\mathrm{mbar}\,\left(2\,d_t\,g\cos(\theta) + 2\,d_m\,g\sin(\theta) - 2\,{d_m}^2\,\dot{\psi}^2\cos(\theta)\sin(\theta) + 2\,{d_t}^2\,\dot{\psi}^2\cos(\theta)\sin(\theta) - 2\,d_m\,d_t\,\dot{\psi}^2\,\left(2\cos(\theta)^2 - 1\right)\right)}{2} \quad 0\right)$$

```
dLdq_dot = simplify(jacobian(L, q_dot))
```

dLdq_dot = $\left(\mathrm{mbar}\,\dot{\theta}\,\left({d_m}^2 + {d_t}^2\right) \quad \mathrm{mbar}\,\dot{\psi}\,\left(-{d_m}^2\cos(\theta)^2 + {d_m}^2 + \sin(2\,\theta)\,d_m\,d_t + {d_t}^2\cos(\theta)^2\right)\right)$

```
ddLdq_dotdt = simplify(jacobian(dLdq_dot, q_dot) * q_ddot + jacobian(dLdq_dot, q) * q_dot)
```

ddLdq_dotdt =

$$\begin{pmatrix} \mathrm{mbar}\,\ddot{\theta}\,\left({d_m}^2 + {d_t}^2\right) \\ \mathrm{mbar}\,\ddot{\psi}\,\left(\frac{{d_t}^2\cos(2\,\theta)}{2} - \frac{{d_m}^2\cos(2\,\theta)}{2} + \frac{{d_m}^2}{2} + \frac{{d_t}^2}{2} + d_m\,d_t\sin(2\,\theta)\right) + \mathrm{mbar}\,\dot{\psi}\,\dot{\theta}\,\left(\sin(2\,\theta)\,{d_m}^2 + 2\cos(2\,\theta)\,d_m\,d_t - \sin(2\,\theta)\,{d_t}^2\right) \end{pmatrix}$$

The equations of motion are therefore

```
syms tau1 tau2 real
tau = [tau1; tau2];
ddLdq_dotdt - dLdq' == tau
```

ans =

$$\begin{pmatrix} \frac{\mathrm{mbar}\,\left(2\,d_t\,g\cos(\theta) + 2\,d_m\,g\sin(\theta) - 2\,{d_m}^2\,\dot{\psi}^2\cos(\theta)\sin(\theta) + 2\,{d_t}^2\,\dot{\psi}^2\cos(\theta)\sin(\theta) - 2\,d_m\,d_t\,\dot{\psi}^2\,\left(2\cos(\theta)^2 - 1\right)\right)}{2} + \mathrm{mbar}\,\ddot{\theta}\,\left({d_m}^2 + {d_t}^2\right) = \tau_1 \\ \mathrm{mbar}\,\ddot{\psi}\,\left(\frac{{d_t}^2\cos(2\,\theta)}{2} - \frac{{d_m}^2\cos(2\,\theta)}{2} + \frac{{d_m}^2}{2} + \frac{{d_t}^2}{2} + d_m\,d_t\sin(2\,\theta)\right) + \mathrm{mbar}\,\dot{\psi}\,\dot{\theta}\,\left(\sin(2\,\theta)\,{d_m}^2 + 2\cos(2\,\theta)\,d_m\,d_t - \sin(2\,\theta)\,{d_t}^2\right) = \tau_2 \end{pmatrix}$$

where τ_1, τ_2 are the generalized forces acting on the system.