Estimators

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Suppose we have samples (X_t, Y_t) for $t = 1...n$ drawn from the joint distribution $P(X, Y)$ and we want to estimate $\mu = P(Y = 1|X = 1)$

1 The standard estimator

$$\hat{\mu} = \frac{\sum_{t=1}^{n} Y_t \mathbb{1}\{X_t = 1\}}{\sum_{t=1}^{n} \mathbb{1}\{X_t = 1\}} \tag{1}$$

We can't actually say that this estimator is unbiased - as it can be undefined if $n = 0$

Is there an unbiased estimator for this simple problem if the $P(X = 1)$ is not known?

Let's consider variance and bounds.

2 The importance sampling estimator

If we know the probability that $X = 1$ we can instead use importance sampling.

$$\hat{\mu} = \frac{1}{n} \sum_{t=1}^{n} \frac{Y_t \mathbb{1}\{X_t = 1\}}{P(X_t = 1)} \tag{2}$$

The estimator is unbiased:

$$E[\hat{\mu}] = \frac{1}{n} \sum_{t=1}^{n} E\left[\frac{Y_t \mathbb{1}\{X_t = 1\}}{P(X_t = 1)}\right] = \frac{1}{np} \sum_{t=1}^{n} E\left[Y_t \mathbb{1}\{X_t = 1\}\right] \tag{3}$$

$$= \frac{1}{n} \sum_{t=1}^{n} \frac{(0 * P(Y_t = 0 \text{ or } X_t = 0) + 1 * P(Y_t = 1 \text{ and } X_t = 1))}{P(X_t = 1)} \tag{4}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{P(Y = 1, X = 1)}{P(X = 1)} = \frac{1}{n} \sum_{i=1}^{n} P(Y = 1|X = 1) = \frac{1}{n} \sum_{i=1}^{n} \mu = \mu \tag{5}$$

We can use Hoeffding's inequality to get a bound on how far the estimator is likely to be from the true value. Let:

$$Z_t = \frac{Y_t \mathbb{1}\{X_t = 1\}}{p} \in \{0, \frac{1}{p}\} \tag{6}$$

Hoeffding's inequality: If $X_1 ... X_n$ are independent observations such that $a_i < X_i < b_i$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ then:

$$P(\bar{X} - E[\bar{X}] \geq \epsilon) \leq \exp \frac{-2n^2\epsilon^2}{\sum_{i=1}^{n}(b_i - a_i)^2} \tag{7}$$

In our case this gives:

$$P(\frac{1}{n}\sum_{i=1}^{n} Z_t - \mu \geq \epsilon) \leq \exp\left(-2n\epsilon^2 p^2\right) \implies P(\frac{1}{n}\sum_{i=1}^{n} Z_t - \mu \geq \frac{1}{p}\sqrt{\frac{1}{2n}\log\frac{1}{\delta}}) \leq \delta \tag{8}$$

This is not so good because $\frac{1}{p}$ can be very large if p is small and its outside the log - so the bounds will grow quickly as p gets small.

We can get a tighter bound by using an Chernoff's inequality that takes account of the variance of X. Let $W_t = \mathbb{1}\{Y_t = 1, X_t = 1\} = pZ_t$, then:

$$P(\frac{1}{n}\sum_{i=1}^{n} Z_t - \mu \geq \epsilon) = P(\frac{1}{n}\sum_{i=1}^{n} pZ_t - p\mu \geq p\epsilon) = P(\frac{1}{n}\sum_{i=1}^{n} W_t - p\mu \geq p\epsilon) \tag{9}$$

Now W_t is a bernoulli random variable so:

$$V[W_t] = P(W_t = 1)P(W_t = 0) \leq P(W_t = 1) = P(Y_t = 1, X_t = 1) \leq P(X_t = 1) = p \tag{10}$$

Now do the same for the standard estimator