Intervention Bandits

Blah blah

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Abstract

An abstract.

1 Introduction

Useful references are: **?**.

2 Notation

Assume we have a known causal model with binary variables $\boldsymbol{X} = \{X_1..X_K\}$ that independently cause a target variable of interest Y. We can run sequential experiments on the system, where at each timestep t we can select a variable on which to intervene and then we observe the complete result, $(\boldsymbol{X}_t, Y_t)$ This problem can be viewed as a variant of the multi-armed bandit problem.

Let $p \in [0,1]^K$ be a fixed and known vector. In each time-step t:

1. The learner chooses an $I_t \in \{1,\ldots,K\}$ and $J_t \in \{0,1\}$.
2. Then $X_t \in \{0,1\}^K$ is sampled from a product of Bernoulli distributions, $X_{t,i} \sim \text{Bernoulli}(p_i)$
3. The learner observes $\tilde{X}_t \in \{0,1\}^K$, which is defined by

$$\tilde{X}_{t,i} = \begin{cases} X_{t,i} & \text{if } i \neq I_t \\ J_t & \text{otherwise}\,. \end{cases}$$

4. The learner receives reward $Y_t \sim \text{Bernoulli}(q(\tilde{X}))$ where $q : \{0,1\}^K \to [0,1]$ is unknown and arbitrary.

The expected reward of taking action i, j is $\mu_{i,j} = \mathbb{E}[q(X)|do(X_i = j)]$. The optimal reward and action are μ^* and (i^*, j^*) respectively, where $(i^*, j^*) = \arg\max_{i,j} \mu_{i,j}$ and $\mu^* = \mu(i^*, j^*)$. The n-step cumulative expected regret is

$$R_n = \mathbb{E}\sum_{t=1}^{n} \left(\mu^* - \mu_{I_t,J_t}\right).$$

3 Estimating $\mu_{i,j}$

The most natural way to estimate $\mu_{i,j}$ is to compute an empirical estimate based on samples when that action was taken. This approach would lead directly to the UCB algorithm with $2K$ actions and a regret bound that depended linearly on K. In this instance we can significantly outperform this approach by exploiting the known causal structure of the problem.

$$
\begin{aligned}
P(Y|do(X_i = j)) &= P(Y|X_i = j) \\
&= \sum_b P(Y|X_i = j, X_a = b)P(X_a = b|X_i = j) \\
&= \sum_b P(Y|X_i = j, X_a = b)P(X_a = b),\ \forall\, a \in \{1...K\}/i \text{ as } X_a \perp\!\!\!\perp X_i \\
&= \sum_b P(Y|X_i = j, do(X_a = b))P(X_a = b)
\end{aligned}
$$

Fix some time-step t and $i \in \{1, \dots, K\}$ and $j \in \{0, 1\}$.
Let $\hat{\mu_a}$ be an empirical estimator for $P(Y|do(X_i = j))$ obtained via marginalization over a.

$$
\hat{\mu_a} = \begin{cases} \frac{m_{a,1}}{n_{a,1}} p_a + \frac{m_{a,0}}{n_{a,0}}(1 - p_a) & \text{if } a \neq i \\ \frac{m_{i,j}}{n_{i,j}} & \text{if } a = i \end{cases}
$$

where:

$$
m_{a,b} = \sum_{s=1}^{t} \mathbb{1}\{X_i = j, I = a, J = b, Y = 1\}_s
$$

$$
n_{a,b} = \sum_{s=1}^{t} \mathbb{1}\{X_i = j, I = a, J = b\}_s
$$

This gives K estimators $\{\hat{\mu_1}...\hat{\mu_K}\}$ to be pooled into a single estimator $\hat{\mu}$.

$$
\hat{\mu} = \sum_{a=1}^{K} \eta_a \hat{\mu}_a = \eta_i \frac{m_{i,j}}{n_{i,j}} + \sum_{a \neq i} \eta_a \left[p_a \frac{m_{a,1}}{n_{a,1}} + (1 - p_a) \frac{m_{a,0}}{n_{a,0}} \right]
$$

where:

$$
\eta_a = \frac{n_a}{\sum_{a=1}^{K} n_a} \text{ and } n_{i,j} = \begin{cases} n_{i,j} & \text{if } a = i \\ \frac{1}{2} \min\left\{ \frac{n_{a,1}}{p_a}, \frac{n_{a,0}}{1-p_a} \right\} & \text{otherwise} \end{cases}
$$

If p is not known, these expression are unchanged except that p_a is replaced with $\hat{p_a}$

$$
\hat{p_a} = \frac{\sum_{s=1}^{t} \mathbb{1}\{X_a = 1, I \neq a\}_s}{\sum_{s=1}^{t} \mathbb{1}\{I \neq a\}_s}
$$

Theorem 1. *With probability at least* $1 - \delta$ *we have that:* $|\hat{\mu}_t - \mu| \leq \sqrt{\frac{\beta}{\sum_a n_a} \log \frac{1}{\delta}}$, *where* $\beta > 0$ *is some constant.*

Proof. First note that $n_{a,b}$ is a random variable that is bounded by t for all a, b. We use the short-hand $\mu_{i,j}^{a,b} = \mathbb{E}[q(X)|X_i = j, X_a = b]$. Then

$$
\mu_{i,j} = p_a \mu_{i,j}^{a,1} + (1 - p_a) \mu_{i,j}^{a,0} \,.
$$

Now we can apply Hoeffding's bound and the union bound to show that

$$
\mathrm{P}\left\{ \left| \frac{m_{a,b}}{n_{a,b}} - \mu_{i,j}^{a,b} \right| \geq \sqrt{\frac{1}{2n_{a,b}} \log \frac{4t}{\delta}} \right\} \leq \frac{\delta}{2} \,.
$$

Therefore by the union bound

$$\mathrm{P}\left\{\left|p_a\frac{m_{a,1}}{n_{a,1}} + (1-p_a)\frac{m_{a,0}}{n_{a,0}} - \mu_{i,j}\right| \geq p_a\sqrt{\frac{1}{2n_{a,1}}\log\frac{4t}{\delta}} + (1-p_a)\sqrt{\frac{1}{2n_{a,0}}\log\frac{4t}{\delta}}\right\} \leq \delta$$

Now by Jensen's inequality

$$\begin{aligned}
p_a\sqrt{\frac{1}{2n_{a,1}}\log\frac{4t}{\delta}} + (1-p_a)\sqrt{\frac{1}{2n_{a,0}}\log\frac{4t}{\delta}} &\leq \sqrt{\left(\frac{p_a}{2n_{a,1}} + \frac{1-p_a}{2n_{a,0}}\right)\log\frac{4t}{\delta}} \\
&\leq \sqrt{\max\left\{\frac{p_a}{n_{a,1}}, \frac{1-p_a}{n_{a,0}}\right\}\log\frac{4t}{\delta}} \\
&= \sqrt{\frac{1}{2n_a}\log\frac{4t}{\delta}}\,.
\end{aligned}$$

Similarly,

$$\mathrm{P}\left\{\left|\frac{m_{i,j}}{n_{i,j}} - \mu_{i,j}\right| \geq \sqrt{\frac{1}{2n_a}\log\frac{4t}{\delta}}\right\} \leq \mathrm{P}\left\{\left|\frac{m_{i,j}}{n_{i,j}} - \mu_{i,j}\right| \geq \sqrt{\frac{1}{2n_a}\log\frac{2t}{\delta}}\right\} \leq \delta\,.$$

□

4 Algorithm

Algorithm 1 UCB

```
Input: Number of variables $K$, vector $p \in [0,1]^K$, horizon $n$
for $t \in 1, \dots, n$ do
    for $i \in 1, \dots, K$ do
        for $j \in \{0,1\}$ do
            Compute $\tilde{\mu}_{i,j} = \hat{\mu}_{i,j} + \sqrt{\frac{\alpha}{\sum_a n_a}\log n}$
        end for
    end for
    Choose $I_t, J_t = \arg\max_{i,j}\tilde{\mu}_{i,j}$
end for
```

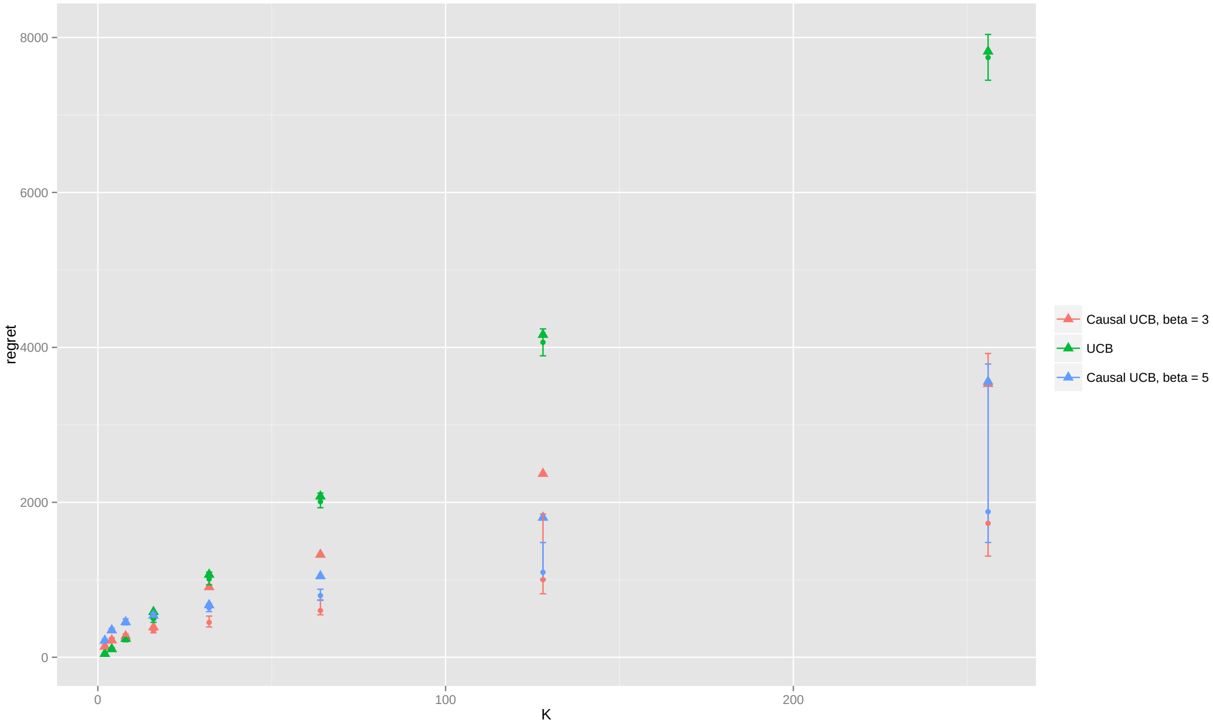
5 Theorems

6 Experiments

Simululations to compare the performance of standard UCB with our modified algorithm. For each number of arms, 100 bandits of each type were created and run upto to a horizon of 1000 timesteps. The mean regret and its standard error from these simulations is plotted in figure **??** The true data was generated from a model where:

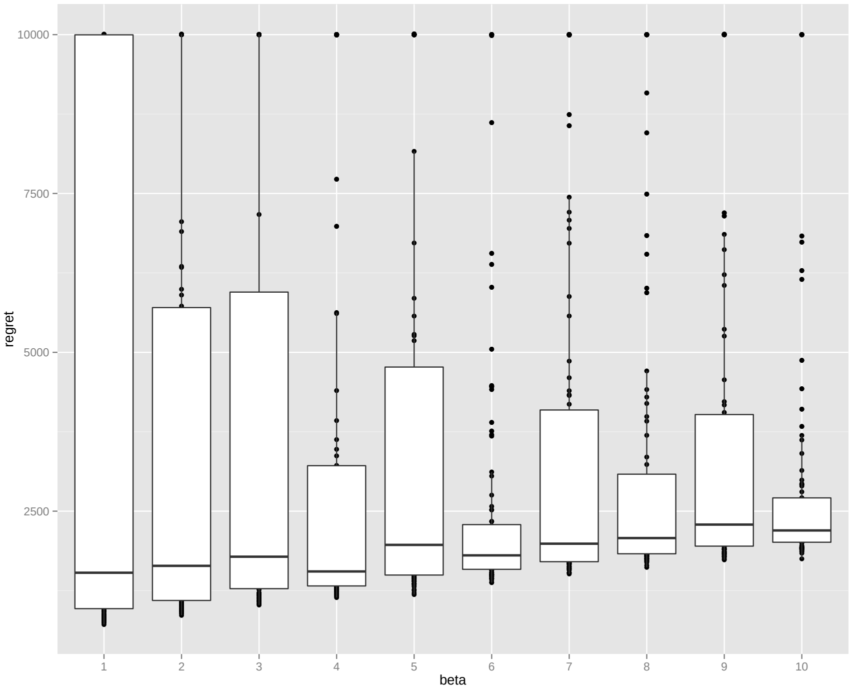
$$\begin{aligned}
p &= [0.5]^K \\
q(\boldsymbol{X}) &= \begin{cases} 0.5 & \text{if } X_1 = 0 \\ 0.6 & \text{otherwise} \end{cases}
\end{aligned}$$

Figure 1: Comparison of the performance of standard UCB versus causal UCB with $\beta = 3$ and $\beta = 5$. 100 simulations were run for each algorithm up to a horizon of 10^5 per value of K. Error bars span the 1st to 3rd quantile of the regret, round points mark the median and triangular points show the mean. For standard UCB the regret increases linearly with the number of arms K. For causal UCB the increase is sub-linear. Increasing β leads to slower convergence but lower variance.



7 Conclusion

Figure 2: The distribution of regret varies with the β parameter in the bound in the estimator. As beta increases, the mean regret increases but the variance decreases. The plot shows the results of running 100 independent bandits, with $K = 256$ and $\epsilon = 0.1$, up to a horizon $h = 10^5$ for each value of β.


regret
beta