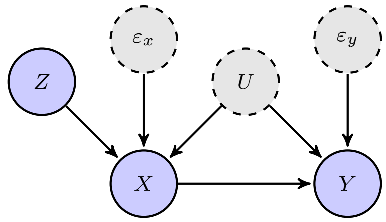


$$
\begin{aligned}
Z &\sim \mathcal{N}\left(\mu_z \ , \ v_z\right) \\
U &\sim \mathcal{N}\left(\mu_u \ , \ v_u\right) \\
\varepsilon_x &\sim \mathcal{N}\left(0 \ , \ \epsilon_x\right) \\
\varepsilon_y &\sim \mathcal{N}\left(0 \ , \ \epsilon_y\right)
\end{aligned}
$$

$$
\begin{aligned}
X &= w_{x0} + w_{xz}Z + w_{xu}U + \varepsilon_x \\
Y &= w_{y0} + w_{yu}U + w_{yx}X + \varepsilon_y
\end{aligned}
$$

As all the exogenous variables $(Z, U, \varepsilon_x, \varepsilon_y)$ are Gaussian and all the functions linear, $\mathrm{P}\left\{Z, U, X, Y\right\}$ and thus $\mathrm{P}\left\{Z, X, Y\right\}$ is a multivarite Gaussian.

$$
\mathrm{P}\left\{Z, X, Y\right\} \sim \mathcal{N}\left(\begin{bmatrix} \mu_z \\ w_{x0} + w_{xz}\mu_z + w_{xu}\mu_u \\ (w_{y0} + w_{yx}w_{x0}) + (w_{yu} + w_{yx}w_{xu})\mu_u + w_{yx}w_{xz}\mu_z \end{bmatrix} , \ \Sigma\right)
$$

where,

$$
\Sigma = \begin{bmatrix} \Sigma_{zz} & \Sigma_{zx} & \Sigma_{zy} \\ . & \Sigma_{xx} & \Sigma_{xy} \\ . & . & \Sigma_{yy} \end{bmatrix} = \begin{bmatrix} v_z & w_{xz}\Sigma_{zz} & w_{yx}\Sigma_{zx} \\ . & w_{xz}^2 v_z + w_{xu}^2 v_u + \epsilon_x & w_{xu}w_{yu}v_u + w_{yx}\Sigma_{xx} \\ . & . & w_{yx}^2\Sigma_{xx} + 2w_{yx}w_{yu}w_{xu}v_u + w_{yu}^2 v_u + \epsilon_y \end{bmatrix}
$$

We want to identify,

$$
\begin{aligned}
Y|do(X=x) &= w_{y0} + w_{yu}U + w_{yx}x + \varepsilon_y \\
\implies \mathrm{P}\left\{Y|do(X=x)\right\} &= \mathcal{N}\left(w_{y0} + w_{yu}\mu_u + w_{yx}x \ , \ w_{yu}^2 v_u + \epsilon_y\right) \\
&= \mathcal{N}\left(\mu_y + w_{yx}(x - \mu_x) \ , \ w_{yu}^2 v_u + \epsilon_y\right) \\
&= \mathcal{N}\left(\mu_y + \frac{\Sigma_{zy}}{\Sigma_{zx}}(x - \mu_x) \ , \ \Sigma_{yy} + \frac{\Sigma_{zy}}{\Sigma_{zx}}\left(\frac{\Sigma_{zy}}{\Sigma_{zx}}\Sigma_{xx} - 2\Sigma_{xy}\right)\right)
\end{aligned}
$$

The final equation is purely in terms of $x$ and properties of the joint (non interventional) distribution $\mathrm{P}\left\{X, Y, Z\right\}$ so we get unbiased point estimates for both the mean and variance of $\mathrm{P}\left\{Y|do(X=x)\right\}$ provided $\Sigma_{zx} \neq 0$, that is if $v_z \neq 0$ and $w_{xz} \neq 0$.

We can identify the difference in the average causal effect for two interventions on $X$ (assuming $v_z, w_{xz} > 0$) .

$$
\mathbb{E}\left[Y|do(X=x)\right] - \mathbb{E}\left[Y|do(X=x')\right] = w_{yx}(x - x') = \frac{\Sigma_{zy}}{\Sigma_{zx}}(x - x')
$$

We can also identify the variance of P $\{Y|do(X=x)\}$.

$$w_{yu}^2 v_u + \epsilon_y = \Sigma_{yy} + \frac{\Sigma_{zy}}{\Sigma_{zx}} \left( \frac{\Sigma_{zy}}{\Sigma_{zx}} \Sigma_{xx} - 2\Sigma_{xy} \right)$$

Conditional distributions are also gaussian.

$$\begin{aligned} P(Y|X=x) &= \mathcal{N}\left( \mu_y + \frac{\Sigma_{xy}}{\Sigma_{yy}}(x-\mu_x)\ ,\ \Sigma_{yy} - \frac{\Sigma_{xy}^2}{\Sigma_{yy}} \right) \\ &= \end{aligned}$$

Intuitively we want the link $Z \rightarrow X \rightarrow Y$ to be stronger than $Z \rightarrow X \rightarrow U \rightarrow Y$. If we can see that X is mostly determined by Z then this must be the case. Similarly it would be good if Y is strongly determined by X (then U cannot have much influence). Quantify this statement.

Start by looking at the variance of $P(X|Z)$ - relative to the mean? Also the variance of $P(Y|X)$.

Look at some extreme examples - what happens when $Z$ exactly determines $X$.

We know all the diagonal terms in the covariance matrix are positive. This will create bounds on terms that cannot be exactly identified.