About Stationary Block Bootstrap

$$\sigma_\infty^2 = \gamma(0) + 2\sum_{k=1}^{\infty} \gamma(k), \quad \gamma(k) = Cov[X_0, X_k]$$

$\{X_t\}$ is stationary with $\mu = 0$.

Block Size $L \sim Geometric(p)$

- $P(L = \ell) = p(1-p)^{\ell - 1}, \quad \ell = 1, 2 \dots$
- $E[L] = \frac{1}{p}$ = block length (b)
- Memoryless $\rightarrow P(L > k+m \mid L > k) = P(L > k+m)$

where p = probability the block ends.

Using the key identity:

$$E[\hat{\gamma}(k)] = (1-p)^k \gamma(k) + \underset{\top}{O\left(\frac{k}{N}\right)}$$

Describes the growth rate of a function.

- O indicates the upper bound
- o indicates strictly slower

$$E[\hat{\sigma}_\infty^2] = \gamma(0) + 2\sum_{k=1}^{\infty} (1-p)^k \gamma(k) + o(1)$$

$$\text{Bias} = E[\hat{\sigma}_b^2] - \sigma_\infty^2$$

$$= -\frac{1}{b} G + o\left(\frac{1}{b}\right)$$

$$G = \sum_{k=-\infty}^{\infty} \underbrace{|k|}_{\text{lag } k} \underbrace{R(k)}_{\text{autocovariance}}$$

block size

$$V[\hat{\sigma}_b^2] = \frac{b}{N} D + o\left(\frac{b}{N}\right)$$

total sample

$$D = 2g^2(0)$$

$$g(w) = \sum_{s=-\infty}^{\infty} R(s) \cos(ws)$$

Spectral Density

autocovariance

$$MSE(\hat{\theta}) = E[(\hat{\theta}-\theta)^2]$$

$$= V[\hat{\theta}] + Bias(\hat{\theta}, \theta)^2$$

$$= \frac{b}{N} D + o\left(\frac{b}{N}\right) + \left(-\frac{1}{b} G + o\left(\frac{1}{b}\right)\right)^2$$

$$= \frac{b}{N} D + o\left(\frac{b}{N}\right) + \left(\frac{G^2}{b^2} - \frac{2G}{b} o\left(\frac{1}{b}\right) + o\left(\frac{1}{b^2}\right)\right)$$

Follow Big O Rules

$$= \frac{G^2}{b^2} + D\frac{b}{N} + o(b^{-2}) + o\left(\frac{b}{N}\right)$$

Set $b^* = \left(\frac{2G^2}{D}\right)^{1/3} N^{1/3}$

$$MSE \approx \frac{3}{\sqrt[3]{4}} \cdot \frac{G^{\frac{2}{3}} D^{\frac{2}{3}}}{N^{\frac{2}{3}}}$$