Algorithmic Game Theory and Applications - Coursework 1

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Let A be the bimatrix defining the 2-player strategic game G:

$$A = \begin{bmatrix} (6,8) & (2,9) & (3,8) & (2,8) \\ (0,5) & (2,3) & (2,6) & (8,4) \\ (7,0) & (2,7) & (4,4) & (4,3) \\ (2,3) & (5,3) & (2,5) & (5,4) \end{bmatrix} \tag{1}$$

1.1 Elimination

First we can eliminate the strongly dominated pure strategies from A to solve an equivalent but easier game.

For a pure strategy $x_i \in X_i$ to be strictly dominated there must exist another strategy x'_i such that, for all $x_{-i} \in X_{-i}$:

$$U_i(x_{-i}; x'_i) > U_i(x_{-i}; x_i) \tag{2}$$

Note that this x'_i does not need to be a pure strategy. There are no pure strategies that strictly dominate any others in this game. But there are mixed strategies that dominate pure strategies.

Take x'_2 to be $(0, \frac{1}{4}, \frac{3}{4}, 0)$. This mixed strategy strictly dominates $\pi_{2,1}$ and $\pi_{2,4}$ as:

$$U_2(\pi_{1,1}; x'_2) = 8.25 \tag{3}$$
$$U_2(\pi_{1,2}; x'_2) = 5.25 \tag{4}$$
$$U_2(\pi_{1,3}; x'_2) = 4.75 \tag{5}$$
$$U_2(\pi_{1,4}; x'_2) = 4.5 \tag{6}$$

Hence we can eliminate columns 1 and 4 from A:

$$A' = \begin{bmatrix} (2,9) & (3,8) \\ (2,3) & (2,6) \\ (2,7) & (4,4) \\ (5,3) & (2,5) \end{bmatrix} \tag{7}$$

By the same method we can choose x_1' to be $(0, 0, \frac{2}{3}, \frac{1}{3})$ which strictly dominates $\pi_{1,1}$ and $\pi_{1,2}$.

Therefore we can solve the following reduced matrix to find all Nash Equilibrium for the game G:

$$A'' = \begin{bmatrix} (2,7) & (4,4) \\ (5,3) & (2,5) \end{bmatrix} \tag{8}$$

1.2 Computing NE

It is clear by inspection that there are no pure Nash Equilibria. This is because, for every profile of pure strategies, there is always a player that is better off unilaterally switching to the other available pure strategy. By the definition of a Nash Equilibrium, this rules out pure NEs for the game G. By a similar argument there are no NE where one player plays a pure strategy and the other a mixed. However Nash's theorem tells us that a NE does exist, and as it must be a mixed NE, both players must play each strategy with positive probability. This means that each pure strategy, for both players, is a best response in a NE, by the Useful Corollary to Nash's Theorem.

Let $x_1^*(1)$, the probability of player 1 playing the strategy 1 in the NE, be a and $x_1^*(2)$ be $1-a$. Let $x_2^*(1)$ be b and $x_2^*(2)$ be $1-b$.

$$\begin{align} U_1^*(x_{-1}^*; \pi_{1,1}) &= U_1^*(x_{-1}^*; \pi_{1,2}) \tag{9} \\ 2b + 4(1-b) &= 5b + 2(1-b) \tag{10} \\ 4 - 2b &= 2 + 3b \tag{11} \\ b &= \frac{2}{5} \tag{12} \end{align}$$

Hence $x_2^* = (\frac{2}{5}, \frac{3}{5})$, or $(0, \frac{2}{5}, \frac{3}{5}, 0)$ under the original game.

$$\begin{align} U_2^*(x_{-1}^*; \pi_{2,1}) &= U_2^*(x_{-1}^*; \pi_{2,2}) \tag{13} \\ 7a + 3(1-a) &= 4a + 5(1-a) \tag{14} \\ 3 + 4a &= 5 - a \tag{15} \\ a &= \frac{2}{5} \tag{16} \end{align}$$

Hence $x_1^* = x_2^*$ (under the reduced game) and the profile of the NE for game G is

$$[(0, 0, \frac{2}{5}, \frac{3}{5}), (0, \frac{2}{5}, \frac{3}{5}, 0)] \tag{17}$$

As there was no loss of generality in these assumptions, it is clear that this is the only NE that exists for this game.

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The LP corresponding to the given 2-player zero sum game can be specified by:

$$\begin{aligned}
\textbf{maximise} \quad & v \\
\textbf{subject to:} \\
v - (7x_1 + 2x_2 + 6x_3 + 5x_4 + 2x_5) &\leq 0 \\
v - (x_1 + 6x_2 + 3x_3 + 5x_4 + 8x_5) &\leq 0 \\
v - (6x_1 + 8x_2 + 8x_3 + 4x_4 + 2x_5) &\leq 0 \\
v - (4x_1 + 3x_2 + 3x_3 + 7x_4 + 8x_5) &\leq 0 \\
v - (2x_1 + 5x_2 + 4x_3 + 4x_4 + 9x_5) &\leq 0 \\
x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \\
x_i \geq 0 \quad \text{for} \quad j &= 1, ..., 5
\end{aligned}$$

This was computed using the `linprog` function in `MATLAB` in the following manner: Constraints 1,...,5 were encoded in the form $A\mathbf{x} \leq \mathbf{b}$ where:

$$\mathbf{b} = \mathbf{0} \tag{18}$$

$$A = \begin{bmatrix} 1 & -7 & -2 & -6 & -5 & -2 \\ 1 & -1 & -6 & -3 & -5 & -8 \\ 1 & -6 & -8 & -8 & -4 & -2 \\ 1 & -4 & -3 & -3 & -7 & -8 \\ 1 & -2 & -5 & -4 & -4 & -9 \end{bmatrix} \tag{19}$$

Constraint 6 was encoded in the form:

$$\text{beq} = 1 \tag{20}$$

$$Aeq = [0, 1, 1, 1, 1, 1] \tag{21}$$

And Contraint 7 was encoded as lower bounds for the variables $x_1, ..., x_5$
The program produced the following results:

$$v = 4.8333 \tag{22}$$

$$x_1 = 0 \tag{23}$$

$$x_2 = 0 \tag{24}$$

$$x_3 = 0.3333 \tag{25}$$

$$x_4 = 0.5 \tag{26}$$

$$x_5 = 0.1667 \tag{27}$$

Hence the minimax value of this game is 4.8333 and the optimal strategy for player 1 is $x_1^* = (0, 0, 0.3333, 0.5, 0.1667)$.

The variables of the dual of the linear program can be interpreted as the optimal strategy for player 2.

$$
\begin{aligned}
\textbf{minimise} \quad & u \\
\textbf{subject to:} \\
u - (7y_1 + 1y_2 + 6y_3 + 4y_4 + 2y_5) &\leq 0 \\
u - (2y_1 + 6y_2 + 8y_3 + 3y_4 + 5y_5) &\leq 0 \\
u - (6y_1 + 3y_2 + 8y_3 + 3y_4 + 4y_5) &\leq 0 \\
u - (5y_1 + 5y_2 + 4y_3 + 7y_4 + 4y_5) &\leq 0 \\
u - (2y_1 + 8y_2 + 2y_3 + 8y_4 + 9y_5) &\leq 0 \\
y_1 + y_2 + y_3 + y_4 + y_5 &= 1 \\
y_i \geq 0 \quad \text{for} \quad j &= 1, ..., 5
\end{aligned}
$$

Computing this dual LP using `MATLAB` gives the following values:

$$u = 4.8333 \tag{28}$$
$$y_1 = 0.5556 \tag{29}$$
$$y_2 = 0.2778 \tag{30}$$
$$y_3 = 0 \tag{31}$$
$$y_4 = 0 \tag{32}$$
$$y_5 = 0.1667 \tag{33}$$

Hence player 2's optimal strategy is $x_2^* = (0.5556, 0.2778, 0, 0, 0.1667)$.

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$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{34}$$

(a)

The unique NE of the Matching Pennies game is given by $x^* = [(0.5, 0.5), (0.5, 0.5)]$

(b)