ESSENTIALS OF BAYESIAN MODELING

Course overview and introduction to Bayesian statistics

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Hi everybody!

Overview

- Introduction to Bayesian inference
- Linear and generalized linear models
- Hierarchical models
- Prior selection and model parameterization
- Model selection
- Model criticism, diagnostics, and visualization
- Markov chain Monte Carlo
- Bayesian workflow

In other words, what are Bayesian models? How can we use them to do cool science things? How can we interrogate their limitations?

Website: `https://github.com/fintzij/BRB_Bayes_course`

Hi everybody!

Materials

- Lectures: Richard McElreath's Statistical Rethinking (StaRt), Winter 2019.
- Software: leaning towards raw **Stan**, but **brms**/**RStanArm** also an option.
- Additional books, papers, and case studies linked on website.

Plan

- At home: watch ~2 lecture videos (approx. 2hrs/wk, 1 hour at 2x speed).
- When we meet: summarize, supplement with examples, case studies, and additional material.

VERY IMPORTANT: I want this to be useful. Tell me if I'm going too fast/slow, can explain something more clearly, or there's something you to cover. Don't tell me if my jokes are lame.

This week - “Basics” of Bayesian inference

Lectures 1 and 2 of StaRt touched on the following topics:

- Statistical procedures for interepretting natural phenomena.
  - Models as golems.
  - *Core idea:* Count all the ways data can happen, according to assumptions. Assumptions with more ways that are consistent with data are more plausible.
- Small world (model) vs. large world (real world).
- Workflow: design model (globe) $\longrightarrow$ collect then condition on data (throw it around, compute posterior) $\longrightarrow$ evaluate (sample to summarize, simulate to critique).

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I want to talk more about

- Bayesian inference and how it inference differs from frequentist inference.
- What is a Bayesian model?

Quantifying uncertainty

Dicing with the unknown (T. O'Hagan, 2004):

- Two kinds of uncertainty: *aleatory* and *epistemic*.
  - Aleatory: due to randomness, e.g., outcomes of rolling dice.
  - Epistemic: uncertainty about things one could know, but doesn't in practice, e.g., disease risk factors for participants in a clinical trial.
  - *N.B.* Two people may have different epistemic uncertainty about the same question.
- Two definitions of probability: *frequency* and *degree of belief*.
  - Frequency of occurrence under infinite replication, describes aleatory uncertainty.
  - For Bayesians, probability represents the degree of belief about a proposition and may describe both aleatory and epistemic uncertainty.
- Implications:
  - p-values and CIs are statements about aleatory uncertainty.
  - Bayesians quantify uncertainty using probability distributions conditioned on the data.
  - "Bayesian statistics is about the statistician, for whatever reason they may have, guessing or estimating the distribution of the next outcome" (Walker, 2013).
  - Bayesians quantify uncertainty about parameters, θ, given data, y, in the *posterior*, $\pi(\theta|y)$.

Bayesian inference

Some notation:

- θ: unobserved parameter, e.g., $\theta = \Pr(Y = \text{heads})$.
- y: observed data, e.g., $y \in \{\text{heads}, \ \text{tails}\}$.
- $\widetilde{y}$: unknown but possibly observable quantities, e.g., future data.

Bayesian inference

Bayesian inference *always* starts with a model for the *joint* probability distribution of θ and y:

$$\pi(\theta, y) = f(y|\theta)\pi(\theta).$$

- $f(y|\theta)$ is the *sampling distribution* for y given θ.
- $\pi(\theta)$ is the prior distribution of θ.

Bayes rule yields the *posterior density*

$$\pi(\theta|y) = \frac{f(y,\theta)}{\pi(y)} = \frac{f(y|\theta)\pi(\theta)}{\pi(y)},$$

where $\pi(y) = \int \pi(y,\theta)\mathrm{d}\theta = \int f(y|\theta)\pi(\theta)\mathrm{d}\theta$.

Important distributions

At various points, we will be interested in the following distributions:

- *Prior distirbution*: $\pi(\theta)$.
- *Sampling distribution*: $f(y|theta)$.
- *Joint distribution*: $\pi(y,\theta) = f(y|\theta)\pi(\theta)$.
- *Marginal distribution*: $\pi(y) = \int \pi(y,\theta)\mathrm{d}\theta = \int f(y|\theta)\pi(\theta)\mathrm{d}\theta$.
- *Posterior distribution*: $\pi(\theta|y) = \frac{f(y|\theta)\pi(\theta)}{\pi(y)} \propto f(y|\theta)\pi(\theta)$.
- *Posterior predictive*: $f(\widetilde{y}|y) = \int f(y|$.

Bayesian inference

The term $\pi(y) = \int f(y|\theta)\pi(\theta)\mathrm{d}\theta$ is a normalizing constant.

- Sometimes solvable analytically, e.g., conjugate priors.
- Can be difficult to evaluate without conjugacy, especially in high dimensions.
- Markov chain Monte Carlo (next time) $\implies$ don't evaluate explicitly:

$$\pi(\theta|y) \propto f(y|\theta)\pi(\theta)$$
$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}$$

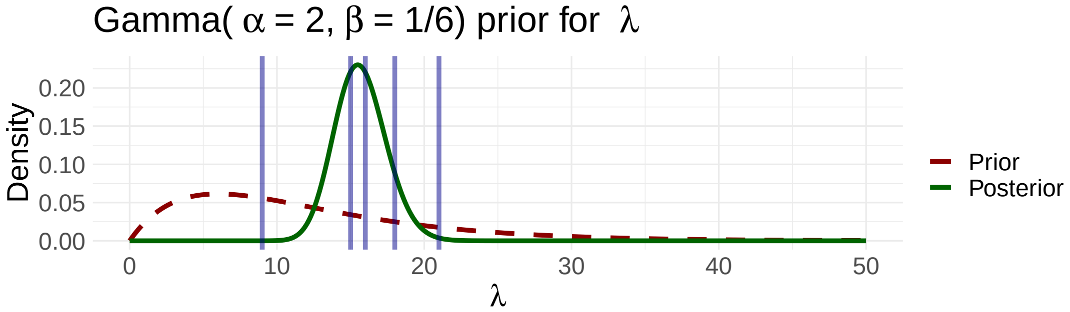
Why Bayes?

- Incorporate prior information.
- Sample to summarize: all possible inferences computable from the posterior.
- Flexibility: specify prior + likelihood, turn the crank.
- Self-consistent framework for handling missing data.
- Diagnostics.

Example

Let Y = number of earthquakes of magnitude >4.0 per year in southern California.

- Model: $Y|\lambda \sim \text{Poisson}(\lambda)$, where λ = rate of earthquake occurrence.
- Prior: $\lambda \sim \text{Gamma}(\alpha, \beta) \implies \text{E}(\lambda) = \alpha/\beta,\ \text{SD}(\lambda) = \sqrt{\alpha}/\beta$. Set α, β s.t., $\text{E}(\lambda) = 12$, and $SD(\lambda) \approx 8.5$.
- Data: 5 years, observe 21, 9, 15, 16, and 18 earthquakes w/magnitude > 4.0 in each of the years.
- Posterior: $\lambda|y\ \text{Gamma}(\alpha + \sum_{i=1}^{5} y_i, \beta + 5)$.



Next week

Skipping ahead a bit. Watch lecture 10 (SmaRt).

We'll talk about:

- Sampling from probability distributions,
- Markov chain Monte Carlo,
- Stan.

REFERENCES

T. O'Hagan "Dicing with the unknown." *Significance* 1.3 (2004): 132-133.

S.G. Walker "Bayesian inference with misspecified models." *Journal of Statistical Planning and Inference* 143.10 (2013): 1621-1633.