ESSENTIALS OF BAYESIAN MODELING

Course overview and introduction to Bayesian statistics

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Jon Fintzi

Biostatistics Research Branch

National Institute of Allergy and Infectious Diseases

National Institutes of Health

Hi everybody!

Overview

- Introduction to Bayesian inference
- Linear and generalized linear models
- Hierarchical models
- Prior selection and model parameterization
- Model selection
- Model criticism, diagnostics, and visualization
- Markov chain Monte Carlo
- Bayesian workflow

In other words, what are Bayesian models? How can we use them to do cool science things? How can we interrogate their limitations?

Website: `https://github.com/fintzij/BRB_Bayes_course`

Hi everybody!

Materials

- Lectures: Richard McElreath's Statistical Rethinking (StaRt), Winter 2019.
- Software: leaning towards raw **Stan**, but **brms**/**RStanArm** also an option.
- Additional books, papers, and case studies linked on website.

Plan

- At home: watch ~2 lecture videos (approx. 2hrs/wk, 1 hour at 2x speed).
- When we meet: summarize, supplement with examples, case studies, and additional material.

VERY IMPORTANT: I want this to be useful. Tell me if I'm going too fast/slow, can explain something more clearly, or there's something you to cover. Don't tell me if my jokes are lame.

This week - “Basics” of Bayesian inference

Lectures 1 and 2 of StaRt touched on the following topics:

- Statistical procedures for interepretting natural phenomena.
    - Models as golems.
    - *Core idea:* Count all the ways data can happen, according to assumptions. Assumptions with more ways that are consistent with data are more plausible.
- Small world (model) vs. large world (real world).
- Workflow: design model (globe) $\longrightarrow$ collect then condition on data (throw it around, compute posterior) $\longrightarrow$ evaluate (sample to summarize, simulate to critique).

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I want to talk more about

- Bayesian inference and how it inference differs from frequentist inference.
- What is a Bayesian model?

Quantifying uncertainty

Dicing with the unknown (T. O'Hagan, 2004):

- Two kinds of uncertainty: *aleatory* and *epistemic*.
  - Aleatory: due to randomness, e.g., outcomes of rolling dice.
  - Epistemic: uncertainty about things one could know, but doesn't in practice, e.g., disease risk factors for participants in a clinical trial.
  - *N.B.* Two people may have different epistemic uncertainty about the same question.
- Two definitions of probability: *frequency* and *degree of belief*.
  - Frequency of occurrence under infinite replication, describes aleatory uncertainty.
  - For Bayesians, probability represents the degree of belief about a proposition and may describe both aleatory and epistemic uncertainty.
- Implications:
  - p-values and CIs are statements about aleatory uncertainty.
  - Bayesians quantify uncertainty using probability distributions conditioned on the data.
  - "Bayesian statistics is about the statistician, for whatever reason they may have, guessing or estimating the distribution of the next outcome" (Walker, 2013).
  - Bayesians quantify uncertainty about parameters, θ, given data, y, in the *posterior*, $\pi(\theta|y)$.

Bayesian inference

Some notation:

- θ: unobserved parameter, e.g., $\theta = \Pr(Y = \text{heads})$.
- y: observed data, e.g., $y \in \{\text{heads},\ \text{tails}\}$.
- $\widetilde{y}$: unknown but possibly observable quantities, e.g., future data.

Bayesian inference

Bayesian inference *always* starts with a model for the *joint* probability distribution of θ and y:

$$\pi(\theta, y) = f(y|\theta)\pi(\theta).$$

- $f(y|\theta)$ is the *sampling distribution* for y given θ.
- $\pi(\theta)$ is the prior distribution of θ.

Bayes rule yields the *posterior density*

$$\pi(\theta|y) = \frac{f(y,\theta)}{\pi(y)} = \frac{f(y|\theta)\pi(\theta)}{\pi(y)},$$

where $\pi(y) = \int \pi(y,\theta)\mathrm{d}\theta = \int f(y|\theta)\pi(\theta)\mathrm{d}\theta$.

Important distributions

At various points, we will be interested in the following distributions:

- *Prior distirbution*: $\pi(\theta)$.
- *Sampling distribution*: $f(y|theta)$.
- *Joint distribution*: $\pi(y, \theta) = f(y|\theta)\pi(\theta)$.
- *Marginal distribution*: $\pi(y) = \int \pi(y, \theta)\mathrm{d}\theta = \int f(y|\theta)\pi(\theta)\mathrm{d}\theta$.
- *Posterior distribution*: $\pi(\theta|y) = \frac{f(y|\theta)\pi(\theta)}{\pi(y)} \propto f(y|\theta)\pi(\theta)$.
- *Posterior predictive*: $f(\widetilde{y}|y) = \int f(\widetilde{y}|\theta, y)\pi(\theta|y)\mathrm{d}\theta$.

Bayesian inference

The term $\pi(y) = \int f(y|\theta)\pi(\theta)\mathrm{d}\theta$ is a normalizing constant.

- Sometimes solvable analytically, e.g., conjugate priors.
- Can be difficult to evaluate without conjugacy, especially in high dimensions.
- Markov chain Monte Carlo (next time) $\implies$ don't evaluate explicitly:

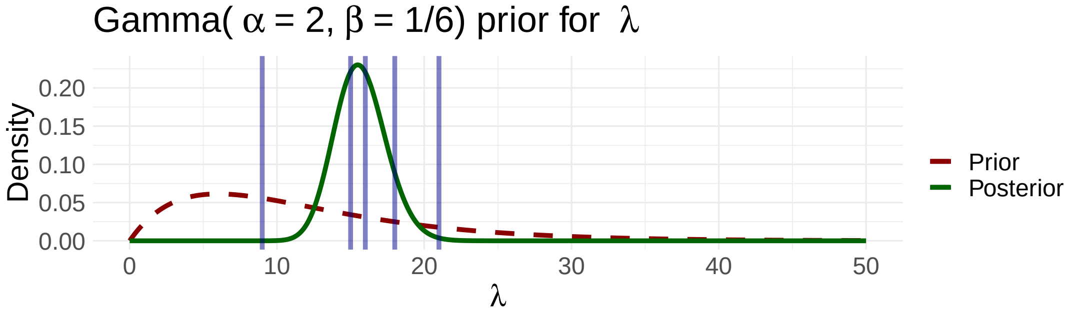
$$\pi(\theta|y) \propto f(y|\theta)\pi(\theta)$$
$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}$$

Why Bayes?

- Incorporate prior information.
- Sample to summarize: all possible inferences computable from the posterior.
- Flexibility: specify prior + likelihood, turn the crank.
- Self-consistent framework for handling missing data.
- Diagnostics.

Let Y = number of earthquakes of magnitude >4.0 per year in southern California.

- Model: $Y|\lambda \sim \text{Poisson}(\lambda)$, where $\lambda =$ rate of earthquake occurrence.
- Prior: $\lambda \sim \text{Gamma}(\alpha, \beta) \implies \text{E}(\lambda) = \alpha/\beta,\ \text{SD}(\lambda) = \sqrt{\alpha}/\beta$. Set α, β s.t., $\text{E}(\lambda) = 12$, and $SD(\lambda) \approx 8.5$.
- Data: 5 years, observe 21, 9, 15, 16, and 18 earthquakes w/magnitude > 4.0 in each of the years.
- Posterior: $\lambda|y \ \text{Gamma}(\alpha + \sum_{i=1}^{5} y_i, \beta + 5)$.



Next week

Skipping ahead a bit. Watch lecture 10 on MCMC (SmaRt).

We'll talk about:

- Sampling from probability distributions,
- Markov chain Monte Carlo,
- Stan.

REFERENCES

T. O'Hagan "Dicing with the unknown." *Significance* 1.3 (2004): 132-133.

S.G. Walker "Bayesian inference with misspecified models." *Journal of Statistical Planning and Inference* 143.10 (2013): 1621-1633.

This lecture also borrowed material from Vladimir Minin's MCMC for infectious diseases short course, and from Aki Vehtari's Bayesian data analysis course. These are both fantastic resources and you should check them out.