Brief description of MCMC for reparameterized model

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Let $\boldsymbol{\theta} = (\beta, \mu, \rho, p_{init})$ and $\boldsymbol{\eta} = f(\boldsymbol{\theta}) = (\log(R_0), \log(\mu), \text{logit}(\rho), \text{logit}(p_{init})) = (\eta_1, \eta_2, \eta_3, \eta_4)$. For each parameter in turn, we propose a new value on the estimation scale (the scale corresponding to $\boldsymbol{\eta}$) from a normal distribution centered at the current value with tuning parameter $\sigma_i, i = 1, \ldots, 4$. We then accept the proposed parameter value with probability,

$$a_{\boldsymbol{\eta}_{cur} \longrightarrow \boldsymbol{\eta}_{new}} = \min\left\{1, \frac{\pi(\boldsymbol{\eta}_{new} \mid \mathbf{X}, \mathbf{Y})}{\pi(\boldsymbol{\eta}_{cur} \mid \mathbf{X}, \mathbf{Y})} \cancel{\frac{q(\boldsymbol{\eta}_{cur} \mid \boldsymbol{\eta}_{new})}{q(\boldsymbol{\eta}_{new} \mid \boldsymbol{\eta}_{cur})}}\right\} = \left\{1, \frac{\pi(\boldsymbol{\eta}_{new} \mid \mathbf{X}, \mathbf{Y})}{\pi(\boldsymbol{\eta}_{cur} \mid \mathbf{X}, \mathbf{Y})}\right\}$$

Now,

$$\begin{aligned}
\pi(\boldsymbol{\eta}_{new} \mid \mathbf{X}, \mathbf{Y}) &\propto \pi(\mathbf{X}, \mathbf{Y} \mid \boldsymbol{\eta}_{new}) \times \pi(\boldsymbol{\eta}_{new}) \\
&= \pi(\mathbf{X}, \mathbf{Y} \mid \boldsymbol{\theta}_{new})|J| \times \pi(\boldsymbol{\eta}_{new}) \\
\pi(\boldsymbol{\eta}_{cur} \mid \mathbf{X}, \mathbf{Y}) &\propto \pi(\mathbf{X}, \mathbf{Y} \mid \boldsymbol{\theta}_{cur})|J| \times \pi(\boldsymbol{\eta}_{cur})
\end{aligned}$$

where J is the Jacobian matrix of the inverse tranformation, $\boldsymbol{\theta} = f^{-1}(\boldsymbol{\eta})$, and $\pi(\boldsymbol{\eta}) = \pi(\eta_1)\pi(\eta_2)\pi(\eta_3)\pi(\eta_4)$ is the prior probability of $\boldsymbol{\eta}$. As, $f^{-1}(\boldsymbol{\eta}) = \left(\frac{\mu}{N}e^{R_0}, e^{\mu}, \frac{\mathrm{e}^{\rho}}{1+\mathrm{e}^{\rho}}, \frac{\mathrm{e}^{p_{init}}}{1+\mathrm{e}^{p_{init}}}\right)$ the Jacobian is

$$J = \left(\frac{\partial f^{-1}(\boldsymbol{\eta}_i)}{\partial \boldsymbol{\eta}_j}\right)_{i,j=1,\ldots,4} = \begin{pmatrix} \frac{\mu}{N}\mathrm{e}^{R_0} & \frac{1}{N}\mathrm{e}^{R_0} & 0 & 0 \\ 0 & \mathrm{e}^{\mu} & 0 & 0 \\ 0 & 0 & \frac{\mathrm{e}^{\rho}}{(1+e^{\rho})^2} & 0 \\ 0 & 0 & 0 & \frac{\mathrm{e}^{p_{init}}}{(1+e^{p_{init}})^2} \end{pmatrix}$$

with determinant equal to the product of the diagonal terms.