Double fibration transforms with conjugate points

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Double fibration

Following Mazzucchelli-Salo-Tzou [2], we introduce double fibration transforms.

- Let $\mathcal{G}$ and X be oriented smooth manifolds without boundaries. $N := \dim(\mathcal{G})$ and $n := \dim(X)$. Denote by $d\mathcal{G}$ and dX the orientation forms of $\mathcal{G}$ and X respectively.
- Let Z be an oriented embedded submanifold of $\mathcal{G} \times X$, and let dZ be the orientation form.
- Assume that $N + n > \dim(Z) > N \geqq n \geqq 2$, and set $n' := \dim(Z) - N$ and $n'' := n - n'$. Then $\dim(Z) = N + n'$, $n = n' + n''$ and $n', n'' = 1, \dots, n-1$.

$$\begin{tikzcd} & Z \arrow[dl, "\pi_{\mathcal{G}}"'] \arrow[dr, "\pi_X"] & \\ \mathcal{G} & & X \end{tikzcd}$$

- We assume that Z is a double fibration, that is, the natural projections $\pi_{\mathcal{G}} : Z \to \mathcal{G}$ and $\pi_X : Z \to X$ are submersions respectively.
- Then $G_z := \pi_x \circ \pi_{\mathcal{G}}^{-1}(z)$ becomes an n'-dim submanifold of X for any $z \in \mathcal{G}$, and $H_x := \pi_{\mathcal{G}} \circ \pi_X^{-1}(x)$ forms an $(N - n'')$-dim submanifold of $\mathcal{G}$ for any $x \in X$.

Orientation forms on G_z and H_x

Fix arbitrary $(z,x) \in Z$, and let $\{v_1,\dots,v_n\}$ and $\{w_1,\dots,w_N\}$ be bases of T_xX and $T_z\mathcal{G}$ respectively such that

$$T_{(z,x)}Z = \operatorname{span}\langle v_1,\dots,v_{n'},w_1,\dots,w_N\rangle = \operatorname{span}\langle v_1,\dots,v_n,w_1,\dots,w_{N-n''}\rangle.$$

The induced orientation forms dG_z on G_z and dH_x on H_x are given by

$$\begin{aligned}
dG_z\big(d\pi_X(v_1),\dots,d\pi_X(v_{n'})\big) &:= dZ_{\pi_{\mathcal{G}}^{-1}(z)}(v_1,\dots,v_{n'}) \\
&= \frac{dZ(v_1,\dots,v_{n'},w_1,\dots,w_N)}{d\mathcal{G}\big(d\pi_{\mathcal{G}}(w_1),\dots,d\pi_{\mathcal{G}}(w_N)\big)}, \\
dH_x\big(d\pi_{\mathcal{G}}(w_1),\dots,d\pi_X(w_{N-n''})\big) &:= dZ_{\pi_X^{-1}(x)}(w_1,\dots,w_{N-n''}) \\
&= \frac{dZ(v_1,\dots,v_n,w_1,\dots,w_{N-n''})}{dX\big(d\pi_X(v_1),\dots,d\pi_X(v_n)\big)}.
\end{aligned}$$

Double fibration transform

Suppose that a weight function $\kappa(z,x) \in C^\infty(\mathcal{G}\times X)$ is nowhere vanishing. A double fibration transform $\mathcal{R}$ associated with the double fibration Z is defined by

$$\mathcal{R}f(z) := \left(\int_{G_z} \kappa(z,x) \frac{f}{|dX|^{1/2}}(x)\, dG_z(x)\right) |d\mathcal{G}(z)|^{1/2}$$

for $f \in \mathscr{D}(X, \Omega_X^{1/2})$. The adjoint $\mathcal{R}^*$ is given by

$$\mathcal{R}^* u(x) = \left(\int_{H_x} \overline{\kappa(z,x)} \frac{u}{|d\mathcal{G}|^{1/2}}(z)\, dH_x(z)\right) |dX(x)|^{1/2}$$

for $u \in \mathscr{D}(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2})$. Then we deduce that

$$\mathcal{R} : \mathscr{D}(X, \Omega_X^{1/2}) \to \mathscr{E}(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}), \quad \mathcal{R}^* : \mathscr{D}(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}) \to \mathscr{E}(X, \Omega_X^{1/2}),$$

are continuous linear mappings, and so are

$$\mathcal{R} : \mathscr{E}'(X, \Omega_X^{1/2}) \to \mathscr{D}'(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}), \quad \mathcal{R}^* : \mathscr{E}'(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}) \to \mathscr{D}'(X, \Omega_X^{1/2}).$$

More precisely $\mathcal{R}$ and $\mathcal{R}^*$ are elliptic Fourier integral operators.

Mapping properties of double fibration transforms

Theorem 1

Suppose that Z is a double fibration with $\dim(Z) = N + n'$. Then $\mathcal{R}$ and $\mathcal{R}^*$ are elliptic Fourier integral operators of order $-(N+2n'-n)/4$ with canonical relations $(N^*Z\setminus 0)'$ and $((N^*Z\setminus 0)^T)'$ respectively. More precisely

$$\mathcal{R} \in \mathcal{I}^{-(N+2n'-n)/4}(\mathcal{G}\times X, N^*Z\setminus 0; \Omega^{1/2}_{\mathcal{G}\times X}),$$
$$\mathcal{R}^* \in \mathcal{I}^{-(N+2n'-n)/4}(X\times\mathcal{G}, (N^*Z\setminus 0)^T; \Omega^{1/2}_{X\times\mathcal{G}}),$$

where

$$\begin{aligned} N^*Z\setminus 0 &= \{(z, A(z,x)\eta, x, \eta) : (z,x)\in Z, \eta\in N^*_x G_z\setminus\{0\}\} \\ &= \{(z, \zeta, x, B(z,x)\zeta) : (z,x)\in Z, \zeta\in N^*_z H_x\setminus\{0\}\}, \end{aligned}$$

$A(z,x)\in \mathrm{Hom}(N^*_x G_z, T^*_z\mathcal{G})$ *and* $B(z,x)\in\mathrm{Hom}(N^*_z H_x, T^*_x X)$ *smoothly depend on* $(z,x)\in Z$ *respectively.*

Preliminaries

For local coordinates $(z, x) = (z', z'', x', x'') \in \mathbb{R}^{N-n''} \times \mathbb{R}^{n''} \times \mathbb{R}^{n'} \times \mathbb{R}^{n''}$, There exist $\mathbb{R}^{n''}$-valued functions $\phi(z, x')$ and $b(x, z')$ such that we have locally

$$Z = \{x'' = \phi(z, x')\} = \{z'' = b(x, z')\}.$$

Lemma 2 ([2, Lemmas 2.4, 2.5 and 2.6])

$$N^*_{(z,x)}Z = \left\{\Big(-\phi_z(z, x')^T\eta'', (-\phi_{x'}(z, x')^T\eta'', \eta'')\Big) : \eta'' \in \mathbb{R}^{n''}\right\},$$

$$A(z, x)\begin{bmatrix} -\phi_{x'}(z, x')^T \\ I_{n''} \end{bmatrix}\eta'' = -\phi_z(z, x')^T\eta'', \quad \eta'' \in \mathbb{R}^{n''}.$$

Similar results hold for $b(x, z')$ and $B(z, x)$.

Variation fields and conjugate points

Fix arbitrary $(z, w) \in T\mathcal{G}$, and consider a curve in $\mathcal{G}$ of the form

$$z(s) = z + sw + \mathcal{O}(s^2) \quad \text{near} \quad s = 0.$$

Then $(G_{z(s)})$ is said to be a variation of G_z, and the variation field $J_w : G_z \to (N_x^* G_z)^*$ associated to $(G_{z(s)})$ is defined by

$$J_w(x) := A(z, x)^* w \in (N_x^* G_z)^* \simeq N_x G_z = T_x X / T_x G_z$$

for $x \in G_z$. Note that

$$A(z, x)^* \in \mathrm{Hom}\big(T_z\mathcal{G}, (N_x^* G_z)^*\big) \simeq \mathrm{Hom}\big(T_z\mathcal{G}, N_x G_z\big), \quad (z, x) \in Z,$$

For $z \in \mathcal{G}$ and $x, y \in G_z$, set

$$V_z(x, y) := \{J_w(x) : w \in T_z\mathcal{G}, J_w(y) = 0\}.$$

Note that $\dim\big(V_z(x, y)\big) \leqq n''$ holds since $\mathrm{rank}\big(A(z, x)^*\big) = n''$, and $\dim\big(V_z(x, y)\big) = \dim\big(V_z(y, x)\big)$ holds for any $z \in \mathcal{G}$ and $x, y \in G_z$.

Z-conjugate triplets

Normal operators without conjugate points

Known results on geodesic X-ray transforms with conjugate points

Normal operators with conjugate points

Outline of the proof

References

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