RIEMANNIAN SUBMERSION AND MASLOV QUANTIZATION CONDITION

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Abstract. We consider a relation of spectra of Laplacians on the total space and the base space of a Riemannian submersion.

If a Riemannian submersion commutes with the Laplacians, then the spectrum of the base space are a part of that of the total space and in this case the Riemannian submersion must be a harmonic map (and vise versa).

We are interesting what is happening when a Riemannian submersion need not be a harmonic map and will talk an aspect from a geometric quantization. Here the important role is carried out by Lagrangian submanifolds in the cotangent bundle.

So I will talk

(1) "Eigenvalue Theorem" by A. Weinstein and its generalization for a sub-Laplacian case together with a review of Maslov quantization condition,

(2) Behavior of Lagrangian submanifolds under submersion,

(3) Some example of Lagrangian submanifolds satisfying Maslov quantization condition.

This study is still on the way and this talk is a report of some results obtained with M. Tamura based on papers

(1) Riemann submersion and Maslov quantization condition (with M.Tamura, submitted) and,

(2) Lagrangian submanifolds satisfying Maslov quantization condition (in preparation)

Contents

1. Configuration space and manifold

As one of the framework in physics, we may understand a manifold as a **configuration space** of a physical system, where a Riemannian metric will be naturally installed according to the physical condition of the system.

Then intuitively, in this system a geodesic will be seen as an orbit of a free particle, which can be observed by a classical mechanical method. Especially, closed geodesics can be seen as orbits of stable free particles corresponding to eigenvalues of the Laplacian.

In fact, a result proved by J. V. Ralston in the paper

• *Approximate eigenfunctions of the Laplacian*, J. Diff. Geom., **12**(1977), 87-100,

shows that the existence of eigenfunctions corresponding to a certain kind of closed geodesics (also there are several papers by math-physicists in relation to this correspondence).

The existence of closed geodesics is a basic problem in the Riemannian geometry. If we have an one-to-one correspondence between closed geodesics and eigenstates, then the problem is solved, since the Laplacian is essentially selfadjoint and with compact resolvents. But not so simple in fact.

If there is a closed geodesics, then it means in the classical mechanics the existence of a particle and will imply the existence of quantum phenomena, but it will not hold the opposite direction, since not all the quantum phenomenon will be observed in the macro level of energy as a classical mechanical phenomena.

There are many studies. But I can cite one historical work by N. Bohr (**Bohr-Sommerfeld condition**) and was formulated later as the cohomology class of a symplectic form being integral. For example,

• D. J. Simms, *Bohr-Sommerfeld orbits and quantizable symplectic manifolds*, Proc. Cambridge Philos. Soc., **73**(1973), 489-491.

Here the existence of closed geodesices will be clear, since an electron is around the hydrogen nucleus and the problem is to calculate the energy level, on which the electron is rotating. This condition restricts the energy level to be discrete.

In this talk I will explain a famous quantization condition, called

"**Maslov quantization condition**".

This gives a specific sequence of eigenvalues.

Let X be a (compact) oriented Riemannian manifold with the Riemannian metric g_X and denote its Laplacian by Δ.

The Hamiltonian is understood as the principal symbol $\sigma_{\Delta^X} \in C^\infty(T^*(X))$ of the Laplacian, which is a smooth function defined on the whole cotangent bundle $T^*(X)$.

In some cases or it may happen often that the movement of free particles is restricted to specific directions. In such a case there exist a subbundle $\mathcal{H}$ in the tangent bundle $T(X)$, to which directions free particles can move and we call the structure "sub-Rieannian",

that is a manifold is called to be a sub-Riemannian manifold, if there is a subbundle $\mathcal{H}$ satisfying a condition, so called **bracket generating**, or **non-holonomic**,

if the totality of iterated brackets of vector fields taking values in the given subbundle coincides with the space of all vector fields.

By a basic theorem of "**Chow**",

- W. L. Chow, *Über Systeme von linearlen partiellen Differentialgleichungen erster Ordnung*, Math. Ann. **117**(1939), 98-105

under such a condition any two points can be joined by piece-wise smooth horizontal curves, that is any two states can transfer each other.

Here horizontal curves mean that the tangent vectors of the curves are all belonging to the given subbundle $\mathcal{H}$(= given direction).

So in general and in nice cases, starting from any subbundle $\mathcal{H}$ in $T(X)$ iterations of brackets of vector fields taking values in $\mathcal{H}$ will be stable after finite steps, that is, totality of iterations of such vector fields will become a foliation structure.

Then by famous theorem of "**Frobenius**", for any point there exist a maximal integral submanifold passing through the point. Hence it will be enough to consider the each leaf as a physics system, since there are no "transversal" interaction of the movement of free particles between the different leaves at this level.

However in the limiting process there might happen a complicated phenomena interacting between different leaves.

In this case of sub-Riemannian structure we also have a second order differential operator so called "**sub-Laplacian**", which will be denoted as Δ_{sub} and is proved to be "sub-elliptic" by "**Hörmander theorem**" of a priori estimate with a loss of derivative, which is also called a sub-elliptic estimate.

2. Maslov quantization condition

As I stated in the beginning, closed geodesics will be periodic orbits of free particles and intuitively it corresponds to a quantum eigenstate, which will be expressed by a function, that is a state function ψ is an eigenfunction of the Laplacian

$$\Delta\,\phi = \lambda\cdot\psi.$$

Then in the micro region, energy level is restricted, but this is usual, since the eigenvalues of operators with compact resolvents are always discrete.

Then, to observe the eigenvalues (or eigenstates) we need a (sufficient) condition which asserts the existence of a special family of eigenvalues.

A condition called "Maslov quantization condition" gives a series of eigenvalues reflecting the existence of a periodic geodesic.

For the formulation we need some notions:

(0) I assume the manifold X is compact.

(1) Let θ^X be the Liouville one-form on the cotangent, bundle $T^*(X$, which is expressed locally as $\sum \xi_i dx_i$ by local coordinates $x = (x_1, \cdots, x_n) \in X$ and the associated local coordinates $(x_1, \cdots, x_n; \xi_1, \cdots, \xi_n) \longleftrightarrow \sum \xi_i dx_i \in T^*_x(X)$ of the cotangent bundle.

The differential $d\theta^X := \omega^X$ defines an intrinsic symplectic structure on the cotangent bundle $T^*(X)$.

(2) With respect to this symplectic form $\omega^X = d\theta^X$ a submanifold $\Lambda \subset T^*(X)$ is said to be a Lagrangian submanifold, if $\dim \Lambda = \dim X$ and the symplectic form ω^X vanishes on it. Or this is the same thing to say that the tangent space $T_\lambda(\Lambda)$ at each point $\lambda \in \Lambda$ is a Lagrangian subspace in the symplectice vector space $T_\lambda(T^*(X))$.

(3) For Lagrangian submanifolds, a cohomology class is defined, called "**Keller-Maslov-Arnold-Hörmander**" characteristic class (shortly call it **Maslov class**) $\mathfrak{m}_\Lambda \in H^1(\Lambda, \mathbb{Z})$ of the Lagrangian submanifold Λ. This is seen as a homomorphism $\mathfrak{m}_\Lambda : \pi_1(X) \to \mathbb{Z}$.

This cohomology class expresses the variation of the intersection of the tangent space $T_\lambda(\Lambda)$ of Λ and the vertical tangent space $\mathrm{Ker}(d\pi^X)_\lambda$ of the natural projection map $\pi^X : T^*(X) \to X$.

(4) Also let's denote the **geodesic flow** by $\{\Phi_t^M\}_{t\in\mathbb{R}}$, $\Phi_t^X : T^*(X) \to T^*(X)$, the Hamilton flow whose Hamiltonian is the principal symbol σ_{Δ^X} of the Laplacian (or the square root of it).

The geodesics are the projection of the orbits of this flow to the base manifold. So an orbit, or a bi-characteristic curve of the Laplacian carries location and momentum of a classical particle.

We assume that there is a specific Lagrangian submanifold $\Lambda \subset T_0^*(X)$ satisfying the conditions :

Mas[1]: $\sigma_{\Delta^X}\big|_\Lambda \equiv E_\Lambda : \text{const} > 0$,

Mas[2]: For any closed curve $\{\gamma\}$ in Λ, we assume

$$\frac{1}{2\pi}\int_\gamma \theta^X - \frac{1}{4} < \mathfrak{m}_\Lambda, \gamma > \in \mathbb{Z}$$

Mas[3]: there exists a geodesic flow invariant positive measure on Λ ("measure" means a no-where vanishing highest degree differential form, in case Λ is orientable.)

Note that Λ itself is invariant under the geodesic flow action by the condition $Mas[1]$.

3. Existence of Eigenvalues of Laplacian : Theorem by Weinstein

Theorem 3.1. (*W.Weinstein, Springer LNM No. 459, 1974*) *: Existence of certain series of eigenvalues)*

Under the existence of a Lagrangian submanifold satisfying three condition above, there exists a sequences $\{\lambda_j\}_{j\geq 0}$ of eigenvalues of the Laplacian such that

$$sup_j \left| \lambda_j - E_\Lambda(dj+1) \right|^2 < +\infty,$$

that is there is a series of eigenvalues closed to the values $\left\{ E_\Lambda(dj+1) \right\}_{j\geq 0}$, where d is one of $1, 2$ or 4 and determined by the condition for $\mathfrak{m}_\Lambda$.

First, I remark a possible example of such a Lagrangian submanifold.

If the geodesic flow is "completely integrable" with the first integrals $\{\sigma_{\Delta^X} = f_0, f_1, \cdots, f_{m-1}\}$ ($m = \dim X$), that is,

(i) $\Lambda = \left\{(x,\xi) \in T_x^*(X)\backslash\{0\} \;\middle|\; \sigma_{\Delta^X} = E_\Lambda,\ f_i = C_i,\ 1 \leq i \leq m-1 \right\}$

(ii) they are Poisson commuting: $\{f_i\,,\,f_j\} \equiv 0$, moreover

(iii) on an open dense subset in $T^*(X)$ their differentials $\{df_i\}$ are linearly independent.

Then we will be able to find such a Lagrangian submanifold among the m- dimensional torus

$$\begin{aligned}&\mathbb{T}_{E,C_1,\ldots,C_{m-1}}\\ &= \{ (x,\xi) \in T_0^*(X) \mid \sigma_{\Delta^X} \equiv E,\\ &\qquad f_1 \equiv C_1, \cdots, f_{m-1} \equiv C_{m-1} \},\end{aligned}$$

which also suggest that the Lagrangian submanifold can be seen as a higher dimensional analog of a closed geodesic.

In the more concrete examples like spheres or projective spaces their geodesic flows are not only completely integrable but periodic with a common period, so that the existence of an invariant measure on Λ will be apparent, but I will explain another construction of such a measure, or half density on such cases, which may be interesting.

4. An outline of the proof of Eigenvalue Theorem

A proof of the Eigenvalue theorem consists of three arguments:

(i) Construction of a conic Lagrangian submanifold L in $T_0^*(S^1)\times T_0^*(X) \subset T_0^*(S^1\times X)$. Here "conic" means it is invariant under the natural $\mathbb{R}_+$-action (dilation) on the cotangent bundle $T_0^*(S^1 \times X)$.

(ii) Construction of an operator in the class of so called "Fourier integral operator" F

$$F : C^\infty(S^1) \to C^\infty(X)$$

which is an approximate intertwining operator between $-E_\Lambda \dfrac{d^2}{dx^2}$ on S^1 and Δ on X

(iii) There exist eigenvalues closed to the eigenvalues of $-E_\Lambda \frac{d^2}{dx^2}$.

Fourier integral operator has a form

$$\int e^{\sqrt{-1}\varphi(x,y;\eta)} a(x,y;\eta) f(y) dy d\eta$$

with a phase function φ which is defined by the conic Lagrangian submanifold micro locally and a homogeneous function of the variable η and of degree one.

The condition Mas[1] is used to construct the conic Lagrangian submanifold and the condition Mas[2] is used for the global definition of the operator together with the condition Mas[3] for the (global) construction of the amplitude function $a = a(x, y; \eta)$, by which the operator will be an isometric operator from $L_2(S^1)$ to $L_2(X)$.

4.1. An outline of the proof is as follows.

Step(1): By the condition Mas[2], the Lagrangian submanifold $\frac{d_L}{2\pi} \cdot L$ is integral, so that we can construct a conic Lagrangian submanifold $\widehat{\frac{d_L}{2\pi} \cdot L}$ in $T_0^*(X) \times T_0^*(U(1))$. This construction will be explained in the next subsection.

Step(2): The operator $A : C^\infty(U(1)) \to C^\infty(X)$, or rather its kernel distribution K_A we are going to construct, is a Lagrangian distribution with respect to the conic Lagrangian submanifold $\widehat{\frac{d_L}{2\pi} \cdot L}$, which satisfies the condition that the Lagrangian distribution

$$\mathcal{D}(K_A) := \left(\Delta^X \otimes Id + Id \otimes E_L \frac{d^2}{dt^2}\right) K_A$$

is of order 0 *"mod half density factor"*. So we take a Lagrangian distribution whose principal symbol on $\widehat{\frac{d_L}{2\pi}\cdot L}$ is $\sqrt{d\mu_L}\otimes\sqrt{d\tau}\otimes \mathbf{s}_{\mathrm{Pft}(\hat{L})}$, where $d\mu_L$ is the invariant measure assumed in Mas[3] (note that we may regard that $\hat{L}$ is more or less $L\times\mathbb{R}_+$).

Here $\mathbf{s}_{\mathrm{Pft}(\hat{L})}$ is a global section of the Maslov line bundle on the conic Lagrangian submanifold which is constructed based on a covering by phase-function-triples for the given Lagrangian submanifold Λ (see §4.2 below).

By applying the operator $\mathcal{D}=\Delta^X\otimes Id+Id\otimes E_L\frac{d^2}{dt^2}$ to such a distribution K_A, we know that the principal symbol of the distribution $\mathcal{D}(K_A)$ (as a Lagrangian distribution) vanishes, since on the conic Lagrangian submanifold $\widehat{\frac{d_L}{2\pi}\cdot L}$ the principal symbol of the operator $\mathcal{D}$ vanishes by the condition Mas[1]. Then as the $1st$ order Lagrangian distribution, the principal symbol of the distribution $\mathcal{D}(K_A)$ again vanishes, because the subprincipal symbol of the operator $\mathcal{D}$ vanishes and the principal symbol of our distribution K_A is invariant under the geodesic flow action of the Laplacian by Mas[1] (precisely to say, its lift to $T^*_0(X)\times T^*_0(U(1))$, cf. [7]). So the operator $\mathcal{D}=\Delta^X\circ A+E_LA\circ\frac{d^2}{dt^2}:C^\infty(U(1))\to C^\infty(X)$ is bounded as an operator : $L_2(U(1))\to L_2(X)$. Since the action of $\mathbb{Z}_{d_L}$ on $\widehat{\frac{d_L}{2\pi}\cdot L}$ comes from the natural action on the base space $U(1)$ and by the assumption Mas[1] and Mas[3] we may find a candidate of such an operator A always $\mathbb{Z}_{d_L}$ action *equivariant*, so that the operator A is descended to an operator acting on the space of sections $\Gamma\big(\mathcal{E}\otimes|\bigwedge|^{1/2}(U(1)/\mathbb{Z}_{d_L})\big)$ of the line bundle $\mathcal{E}\otimes|\bigwedge|^{1/2}(U(1)/\mathbb{Z}_{d_L})$ on $U(1)/\mathbb{Z}_{d_L}$, where $\mathcal{E}$ is the line bundle on $U(1)/\mathbb{Z}_{d_L}$ associated to the principal bundle $U(1)\to U(1)/\mathbb{Z}_{d_L}$ via the natural representation of the structure group $\mathbb{Z}_{d_L}$ ($\cong\{1\}$ or $\{\pm1\}$ or $\{\pm1, e^{\pm\frac{\pi}{2}\sqrt{-1}}\}$) to $U(1)$ and $|\bigwedge|^{1/2}(U(1)/\mathbb{Z}_{d_L})$ is the half density line bundle of $U(1)/\mathbb{Z}_{d_L}$ (this is the reason why the eigenvalues of the form $(d_L\cdot k+1)^2$ are appearing in this space as eigenvalues of the operator $-\frac{d^2}{dt^2}$, see Proposition A.5).

Step(3) By some non-trivial modification (based on clean intersection theorem) of A in the lower order terms, we can make the operator A maps isometrically the subspace consisting of eigenspaces of $-\frac{d^2}{dt^2}$ with the eigenvalues $(d_Lk+1)^2$ $(k=0,1,\dots)$ in $L_2(U(1))$ to $L_2(X)$ with the bounded commutator. Note that we can define an inner product on the space $\Gamma\big(\mathcal{E}\otimes|\bigwedge|^{1/2}(U(1)/\mathbb{Z}_{d_l})\big)$ in an intrinsic way. We also note that if we consider the operator A as $\Gamma\big(\mathcal{E}\otimes|\bigwedge|^{1/2}(U(1)/\mathbb{Z}_{d_L})\big)\to\Gamma\big(|\bigwedge|^{1/2}(X)\big)$, the operator

$$^tA\circ A:\Gamma\big(\mathcal{E}\otimes|\bigwedge|^{1/2}(U(1)/\mathbb{Z}_{d_L})\big)\to\Gamma\big(\mathcal{E}\otimes|\bigwedge|^{1/2}(U(1)/\mathbb{Z}_{d_L})\big)$$

is a pseudo-differential operator of order zero whose principal symbol can be identified in a natural way with the Heaviside function on each fiber of $T^*(U(1))\cong U(1)\times\mathbb{R}\backslash\{0\}$.

Then by using the spectral decomposition of the Laplacian we can prove the existence of eigenvalues of the Laplacian Δ^X described in the statement in the Theorem.

4.2. **Construction of conic Lagrangian submanifold.** Here I explain precisely the construction of a conic Lagrangian submanifold from a compact Lagrangian submanifold Λ satisfying Maslov quantization condition Mas[2].

- **Parametrization of Lagrangian submanifolds by phase functions**

Let Λ be a Lagrangian submanifold in the cotangent bundle $T^*(X)$. Then the canonical one-form θ^X restricted to Λ is a closed form and so, it defines a de Rham cohomology class $[\theta^X{}_{|\Lambda}]\in H^1_{dR}(\Lambda)$.

Let $p\in\Lambda$, then there exists a coordinate neighborhood U of $\pi^X(p)$, an open subset D in $\mathbb{R}^k$ and a smooth function $\phi\in C^\infty(U\times D)$ satisfying the following properties:

Put

$$C_\phi$$
$$= \left\{ (x,\eta) \in U \times D \;\middle|\; \frac{\partial \phi(x,\eta)}{\partial \eta_j} = \phi_{\eta_j} = 0,\ j \le \dim D \right\}$$

Then,

(1) the one-forms $\left\{ d\phi_{\eta_j} \right\}_{j=1}^{k=\dim D}$ are linearly independent on C_ϕ,

(2) the map

$$\rho_\phi : C_\phi \ni (x,\eta) \longmapsto (x;\phi_x) \leftrightarrow \sum_{j=1}^{n} \frac{\partial \phi(x,\eta)}{\partial x_j} dx_j$$
$$= \sum_{j=1}^{n} \phi_{x_j} dx_j \in \Lambda \subset T^*(X)$$

is a *diffeomorphism* between C_ϕ and $\rho_\phi(C_\phi) =: L_\phi \subset L$, when we take U (and D) small enough(the differential $d\rho_\phi$ is injective.

• **Relations of the map ρ_ϕ and other maps can be seen from the commutative diagram:**

$$\begin{tikzcd}
T^*(U \times D) \arrow[dd, shift right, "\pi^{U\times D}"'] & {\pi_U}^*(T^*(U)) \arrow[l, hook', "\chi_{\pi_U}"'] \arrow[rr, "\mathfrak{p}_{\pi_U}"] & & T^*(U) \arrow[r, hook] & T^*(X) \arrow[dd, "\pi^X"] \\
& C_\phi \arrow[ul, hook', "d\phi_{|C_\phi}"'] \arrow[dl, hook] \arrow[r, "\rho_\phi"] & L_\phi \arrow[ur, hook] \arrow[r, hook] \arrow[dr, "{\pi^X}_{|L_\phi}"'] & L \arrow[ur, hook] \arrow[dr, "{\pi^X}_{|L}"] & \\
U \times D \arrow[uu, shift right, "d\phi"'] \arrow[rrr, "\pi_U"'] & & & U \arrow[r, hook] & X
\end{tikzcd}$$

where $\pi_U : U \times D \to U$ is the projection map. The map $\mathfrak{p}_{\pi_U}$ is the natural projection map from the induced bundle ${\pi_U}^*(T^*(U))$ on $U \times D$ to the original bundle $T^*(U)$ and it is a submersion. The map $\chi_{\pi_U} := (d\pi_U)^*$ is the dual map of the differential $d\pi_U : T(U \times D) \to {\pi_U}^*(T(U))$ and it is an embedding.

Also note that

$$\pi^X \circ \mathfrak{p}_{\pi_U} = \pi_U \circ \pi^{U\times D} \circ \chi_{\pi_U}.$$

The subset C_ϕ can be characterized as

$$C_\phi = \left\{ (x,\eta) \in U \times D \mid d\phi\,(x,\eta) \in \chi_{\pi_U}({\pi_U}^*(T^*(U)) \right\}.$$

We call C_ϕ or $\rho_\phi(C_\phi) = L_\phi$ a local parametrization of L by a phase-function-triple $(U \times D, \phi, \rho_\phi)$, where we always assume that the open subset U is taken small enough for the map ρ_ϕ to be a diffeomorphism.

Put the functions $\psi_i \in C^\infty(L_{\phi_i})$ by $\psi_i := \phi_i \circ \rho_{\phi_i}^{-1}$. By the definition of the map ρ_{ϕ_i}, we have $d\psi_i = {\theta^X}_{|L}$ on L_{ϕ_i}, and

$$0 = d\psi_1 - d\psi_2 = d(\psi_1 - \psi_2).$$

Hence the difference $\psi_1 - \psi_2$ is locally constant on $L_{\phi_1} \bigcap L_{\phi_2}$.

Now let L be a Lagrangian submanifold and $\{L_{\phi_i}\}$ a covering by local parametrizations defined by the phase-function-triples Pft(L):= $\{(U_i \times D_i, \phi_i, \rho_{\phi_i})\}_{i\in S}$:

$$L = \bigcup_{i\in S} L_{\phi_i}.$$

Then,

Proposition 4.1. *The set of locally constant functions*

$$\{\ c_{ij} = \phi_j \circ \rho_{\phi_j}^{-1} - \phi_i \circ \rho_{\phi_i}^{-1} : L_{\phi_j} \cap L_{\phi_i} \to \mathbb{R}\ \}_{i\,j \in S}$$

defines an 1*-Čech cocycle with the values in* $\mathbb{R}$*, that is its cohomology class corresponds to the de Rham cohomology class of* ${\theta^X}_{|L}$ *according to the fine resolution of* $\mathbb{R}$*-constant sheaf* $\mathbb{R}_L$ *on* L *by the sheaves of differential forms on* L*.*

We call a Lagrangian submanifold $L \subset T_0^*(X)$ "**quasi-integral**" (or "**integral**)", if there exists a positive constant c_0 (integral in case $c_0 = 1$) such that the de Rham cohomology class $c_0[{\theta^X}_{|L}]$ is in $\check{H}^1(L, \mathbb{Z}_L) \subset \check{H}^1(L, \mathbb{R}_L) \cong H^1_{dR}(L)$, where the inclusion is the induced map from the natural inclusion map $\mathbb{Z}_L \subset \mathbb{R}_L$ of constant sheaves and the natural isomorphism between the de Rham cohomology group and the Čech cohomology group.

This is equivalent to assume that for any smooth closed curve $\{\gamma(t)\}$ in L, the integral

$$c_0 \int_\gamma \theta^X \ \in\ \mathbb{Z}. \tag{4.1}$$

Hence, in this case the Lagrangian submanifold

$$L_0 := c_0 \cdot L = \{(x, c_0\xi) \mid (x, \xi) \in L\}$$

is integral and the cohomology class $[{\theta^X}_{|L_0}] \in \check{H}^1(L_0, \mathbb{Z}_{L_0})$.

Remark 1. *By the induced map* $\check{H}^1(c_0 \cdot L, \mathbb{Z}_{c_0 \cdot L}) \to \check{H}^1(L, \mathbb{Z}_L)$ *from the diffeomorphism* $L \overset{\approx}{\to} c_0 \cdot L$*, the class* $[{\theta^X}_{|c_0 \cdot L}]$ *is mapped to the class* $c_0 \cdot [{\theta^X}_{|L}]$*.*

If $L \subset T_0^*(X)$ is an integral Lagrangian submanifold then for any positive integer $k \in \mathbb{N}$, $k \cdot L$ is also integral.

- **Remark on conic Lagrangian submanifolds**

The positive real numbers $\lambda \in \mathbb{R}_+$ act on $T^*(X)\backslash\{0\}$,

$$\begin{aligned}(x;\xi) &\longleftrightarrow \sum \xi_i dx_i \mapsto \lambda \sum \xi_i dx_i = \lambda \cdot \sum \xi_i dx_i \\ &\longleftrightarrow (x;\lambda\xi),\end{aligned}$$

the space $T_0^*(X)$(punctured cotangent bundle) is a cone bundle over the quotient space $T_0^*(X)/\mathbb{R}_+ := S^*(X)$ which is naturally a contact manifold. If a Lagrangian submanifold $L \subset T_0^*(X)$ (closed in $T_0^*(X)$) is invariant under the action of $\mathbb{R}_+$, we call it a **conic Lagrangian submanifold**. On such a Lagrangian submanifold, the canonical one-form θ^X vanishes and vice versa.

In this case the phase functions ϕ can be taken as defined on an open cone, and the phase function $\phi = \phi(x, \eta) \in C^\infty(U \times D)$ is homogeneous of degree 1 with respect to the variable $\eta \in D \subset \mathbb{R}^k\backslash\{0\}$.

Proposition 4.2. *Let* L *be an integral Lagrangian submanifold in* $T_0^*(X)$*.*

(1): There exists a function $\vartheta : L \to U(1)$ *(mod* $C^\infty(L)$*) such that* ϑ *is mapped to the cohomology class* $[{\theta^X}_{|L}]$*, that is* ϑ *expresses the cohomology class* $[{\theta^X}_{|L}]$ *through the connecting homomorphism* $\delta : C^\infty(L, U(1)) \to \check{H}^1(L, \mathbb{Z}_{\mathbb{L}})$ *associated with the exact sequence of sheaves on* L*:*

$$\{0\} \longrightarrow \mathbb{Z}_L \longrightarrow \mathcal{F}(L, \mathbb{R}) \overset{f \mapsto e^{2\pi\sqrt{-1}f}}{\longrightarrow} \mathcal{F}(L, U(1)) \longrightarrow \{0\},$$

where $\mathcal{F}(L, \mathbb{R})$ *is the sheaf of germs of real valued smooth functions on* L *and* $\mathcal{F}(L, U(1))$ *is a sheaf of germs of smooth functions taking values in* $U(1)$*.*

In fact, once we fix a set of a covering of L by local parametrizations $\{L_{\phi_i}\}$, by the phase-function-triples $\mathrm{Pft}(L) := \{(U_i \times D_i, \phi_i, \rho_{\phi_i})\}$, *then a function ϑ is given by* $\vartheta = e^{2\pi\sqrt{-1}\,\phi_i \circ \rho_{\phi_i}^{-1}}$ *on L_{ϕ_i}, since*

$$e^{2\pi\sqrt{-1}\left(\phi_j \circ \rho_{\phi_j}^{-1} - \phi_i \circ \rho_{\phi_i}^{-1}\right)} \equiv 1$$

on $L_{\phi_i} \bigcap L_{\phi_j}$.

(2): Let

$$\hat{L} = \left\{ (x; \tau \cdot \xi\,,\, \overline{\vartheta(x;\xi)}\,; \tau) \;\middle|\; (x;\xi) \in L, \tau > 0 \right\}.$$

Then $\hat{L}$ is a conic Lagrangian submanifold in $T_0^*(X) \times T_0^*(U(1))$.

In fact, it is covered by local parametrizations defined by the phase-function-triples

$$\mathrm{Pft}(\hat{L}) := \{\, (U_i \times \hat{D}_i, \hat{\phi}_i, \rho_{\hat{\phi}_i})\,\},$$

where we define a conic open subset $\hat{D}_i \subset \mathbb{R}^{k_i+1} \backslash \{0\}$ *by*

$$\hat{D}_i = \left\{ (v, \tau) \in \mathbb{R}^{k_i} \times \mathbb{R}_+ \;\middle|\; 1/\tau \cdot v \in D_i \right\}$$

and a phase function $\hat{\phi}_i$ by

$$\begin{aligned} C^\infty(U_i \times \mathbb{R} \times \hat{D}_i) \,&\ni\, \hat{\phi}_i(x, t, v, \tau) \\ &:= \tau \phi_i(x, 1/\tau \cdot v) + \tau\, t,\;\; (x, 1/\tau \cdot v) \in U_i \times D_i. \end{aligned}$$

The assertion with respect to the (local) parametrization is a basic fact in the theory of FIO,

- L. Hörmander, *Fourier integral operators I*, Acta Math. **127** (1971), 79–183.

The following paper is also very carefully written and is useful for our standing point of view:

- A. Yoshikawa, *On Maslov's canonical operator*, Hokkaido Math. J. **4** (1975), 8–38.

- **Essential part of the proof**

Let's consider the equations:

$$\frac{\partial \hat{\phi}_i(x, t, v, \tau)}{\partial v_j} = \frac{\partial \phi_i}{\partial \eta_j}(x, 1/\tau \cdot v) = 0$$

and

$$\begin{aligned} \frac{\partial \hat{\phi}_i(x, t, v, \tau)}{\partial \tau} &= \phi_i(x, 1/\tau \cdot v) - \sum_{j=1}^{k} \frac{v_j}{\tau} \frac{\partial \phi_i}{\partial \eta_j} + t \\ &= \phi_i(x, 1/\tau \cdot v) \,+\, t = 0. \end{aligned}$$

Then we can characterize the set

$$\begin{aligned} C_{\hat{\phi}_i} = \{\; &(x, t, v, \tau) \mid (x, 1/\tau \cdot v) \in C_{\phi_i}, \\ &t + \phi_i(x, 1/\tau \cdot v) = 0,\; \tau \in \mathbb{R}_+ \;\}. \end{aligned}$$

Note that we may assume that the range of the phase functions ϕ_i on $U_i \times D_i$ are included in a sufficiently small interval so that the values $e^{-2\pi\sqrt{-1}\phi_i}$ are included in a small arc and the maps $\rho_{\hat{\phi}_i}$ are given as

$$\rho_{\hat{\phi}_i} : C_{\hat{\phi}_i} \ni (x,t,v,\tau) \mapsto \Big(\tau \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j}(x, 1/\tau \cdot v) dx_j \, , \, \tau dt\Big) \\ \in \hat{L}_{\hat{\phi}_i} \subset T_0^*(U_i) \times T_0^*(U(1)),$$

where $\tau dt \in T^*_{e^{-2\pi\sqrt{-1}\phi_i(x,1/\tau\cdot v)}}(U(1))$.

Proposition 4.3. *Let $L \subset T_0^*(X)$ be an integral Lagrangian submanifold, then the corresponding conic Lagrangian submanifold to the integral Lagrangian submanifold $k \cdot L$ is given as*

$$\widehat{k \cdot L} = \big\{ (x\,; \tau k\xi \, , \, \overline{\vartheta^k(x\,;\xi)}, \tau) \;\big|\; \\ (x;\xi) \in L, \;\; \vartheta : L \to U(1), \;\; \tau > 0 \;\big\},$$

where $\vartheta : L \to U(1)$ is the map constructed in the preceding Proposition.

On the other hand let $L \subset T_0^*(X)$ be a quasi-Lagrangian submanifold and we assume that $k \cdot L$ is integral with a positive integer k.

Let $\vartheta : k \cdot L \to U(1)$ be the map constructed in the above Proposition and consider a manifold

$$\overline{L} = \big\{ (x;\xi \, , \, e^{2\pi\sqrt{-1}\, s}) \in L \times U(1) \;| \\ \vartheta(x\,; k \cdot \xi) = e^{2\pi\sqrt{-1}\, k\, s} \,\big\}.$$

Since the map $\vartheta(x\,;k\cdot\xi)$ is given locally by $\vartheta(x\,;k\cdot\xi) = e^{2\pi\sqrt{-1}\,k\;\phi\circ\rho_\phi{}^{-1}(x,\xi)}$ with a non-degenerate phase function ϕ of L, $d\phi \neq 0$ (because $L \subset T_0^*(X)$), the subset $\overline{L}$ is a smooth submanifold in $L \times U(1)$ and is a k-hold covering of L.

Now consider the map

$$\tilde{\rho} : \overline{L} \times \mathbb{R}_+ \longrightarrow T_0^*(X) \times T_0^*(U(1)), \\ \Big(x;\xi \, , \, e^{2\pi\sqrt{-1}\, s}; \tau\Big) \longmapsto \Big(x;\, \tau\xi \, , \, e^{-2\pi\sqrt{-1}\, s}; \tau\Big) \\ = \left(\tau \sum_{i=1}^{n} \xi_i dx_i \, , \, \tau dt \right) \in T_0^*(X) \times T_0^*(U(1)),$$

where $dt \in T^*_{e^{-2\pi\sqrt{-1}\, s}}(U(1))$.

Then

Proposition 4.4. *The map $\tilde{\rho}$ is an embedding and the image is a closed conic Lagrangian submanifold in $T_0^*(X) \times T_0^*(U(1))$.*

If $k = 1$, that is if L is integral, $\tilde{\rho}(\overline{L}) = \hat{L}$, so we also denote $\tilde{\rho}(\overline{L} \times \mathbb{R}_+)$ by $\hat{L}$.

5. Submersion and Maslov quantization condition

Let $\varphi : M \longrightarrow N$ be a Riemannian submersion ($\dim M \geq \dim N$).

When we decompose the tangent bundle $T(M)$ into $T(M) = \mathcal{V} \oplus_\perp \mathcal{H}$, where $\mathcal{V} = \mathrm{Ker}\, d\varphi$, the vertical subbundle and $\mathcal{H} = \mathcal{V}^\perp$ its orthogonal complement, we assume that the inner product in $\mathcal{H}$ has the following property that for any two points $x,\ x' \in M$ with $\varphi(x) = \varphi(x')$, $\mathcal{H}_x$ and $\mathcal{H}_{x'}$ are isometric through the differentials $\mathcal{H}_x \overset{d\varphi}{\to} T_{\varphi(x)}(N) = T_{\varphi(x')}(N) \overset{d\varphi}{\leftarrow} H_{x'}$. Hence the Riemannian metric on N coincides with the inner product in $\mathcal{H}$ through the map $d\varphi$.

We have a commutative diagram:

$$\begin{tikzcd}
T_0^*(M) \arrow[dr, "\pi^M"'] & \varphi^*(T_0^*(N)) \arrow[l, hook', "\chi"'] \arrow[r, "\mathcal{P}_\varphi"] \arrow[d, "\pi^M"] & T_0^*(N) \arrow[d, "\pi^N"] \\
& M \arrow[r, "\varphi"'] & N,
\end{tikzcd}$$

where the map $\mathcal{P}_\varphi$ is the natural projection from the induced bundle to the original (tangent)bundle. It is also a submersion. The space $\varphi^*(T_0^*(N))$ is regarded as a submanifold in $T_0^*(M)$ through the dual map $\chi = (d\varphi)^*$ of the differential $d\varphi : T(M) \to \varphi^*(T(N))$. Note that by the assumption the map $d\varphi$ is surjective. The map π^M (also π^N) is the natural projection to the base manifold.

Let Δ^M and Δ^N be the Laplacian on M and N respectively. Then,

Theorem 5.1. *B. Watson, Manifold maps commuting with the Laplacian, J. Diff. Geom. 8(1973), says that*

$$\Delta^M \circ \varphi^* = \varphi^* \circ \Delta^N, \text{ where } \varphi^* : C^\infty(N) \to C^\infty(M)$$

if and only if the fibers of the map φ are minimal, or equivalently the map φ is harmonic.

In this case of Theorem above, if $f \in C^\infty(N)$ is an eigenfunction of Δ^N then $\varphi^*(f)$ is an eigenfunction of Δ^M with the same eigenvalue.

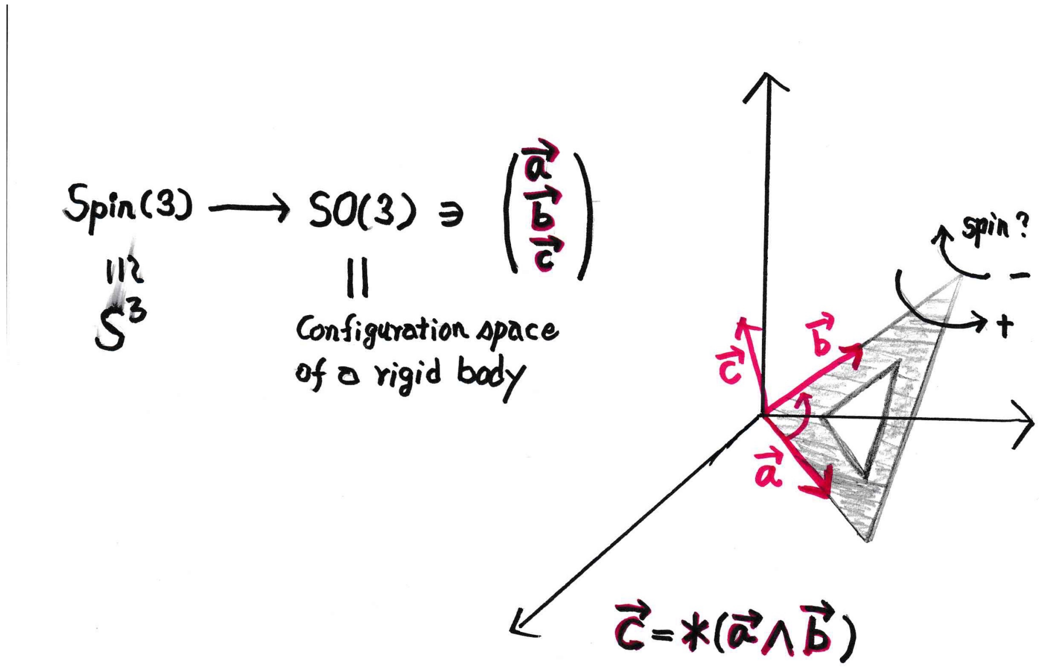
If we consider the base space of the Riemannian submersion as a configuration space of a physical system, then the fiber in the total space can be regarded as describing a structure (more fine data) of a point (= a state) together with the description by the submersion map φ.

Example 1.

$$S^3 \cong Spin(3) \xrightarrow[\text{covering}]{\text{double}} \begin{pmatrix} a \\ b \\ *(a \wedge b) \end{pmatrix} \in SO(3),$$

the matrix $\begin{pmatrix} a \\ b \\ *(a \wedge b) \end{pmatrix}$ *can be seen as a one point fixed rigid body, where $a \perp b$ and $*(a \wedge b)$ is the exterior product and back to the original space by the Hodge star operator.*

The points in S^3 distinguish "spins" of a rigid body in micro states and will not appear in the macro states.



Hence I expected the existence of eigenvalues of the Laplacian on the total space corresponding to the eigenvalues of the Laplacian on the base space without the property described in the above Proposition, that is, only under the assumption of Riemannian submersion, but based on the existence of a compact Lagrangian submanifold satisfying the conditions $Mas[1] \sim Mas[3]$.

This is one expectation and one more is the following proposition:

Proposition 5.2. *Let $\varphi : M \to N$ be a submersion and Λ be a closed Lagrangian submanifold in $T_0^*(N)$. Then the manifold $\mathcal{P}^{-1}(\Lambda)$ is also a Lagrangian submanifold in $T_0^*(M)$.*

Proof. Recall the diagram:

$$\begin{tikzcd}
T_0^*(M) \arrow[dr, "\pi^M"'] & \varphi^*(T_0^*(N)) \arrow[l, hook', "\chi"'] \arrow[r, "\mathcal{P}"] \arrow[d, "\pi^M"] & T_0^*(N) \arrow[d, "\pi^N"] \\
 & M \arrow[r, "\varphi"'] & N,
\end{tikzcd}$$

We can find a local coordinates system $(x, y) \in U \times V \cong W \subset M$, $U \times V \subset \mathbb{R}^{n+d}$, around a point $p \in W \subset M$ such that the map φ is given by the projection $(x, y) \longmapsto x$.

Then a coordinates system in $T^*(M)$ on an open subset $(\pi^M)^{-1}(W)$ is given by the correspondence

$$\begin{aligned} &U \times V \times \mathbb{R}^n \times \mathbb{R}^d \ni (x, y\,; \xi, \eta) \\ &\longleftrightarrow \sum \xi_i dx_i + \sum \eta_j dy_j \in T^*((\pi^M)^{-1}(W)) \end{aligned}$$

and a coordinates system on an open subset $(\pi^N)^{-1}(\tilde{U}) \cong U \times \mathbb{R}^n$ $(\tilde{U} \subset N)$ is given by

$$U \times \mathbb{R}^n \ni (x\,; \xi) \leftrightarrow \sum \xi_i dx_i.$$

Then the canonical two forms ω^M on $T^*(M)$ and ω^N on $T^*(N)$ are expressed as $\omega^M = \sum d\xi_i \wedge dx_i + \sum d\eta_j \wedge dy_j$ and $\omega^N = \sum d\xi_i \wedge dx_i$ by these coordinates.

Since the map $\mathcal{P}$ is a submersion, $\dim \mathcal{P}^{-1}(\Lambda) = \dim N + (\text{fiber dim of } \varphi) = \dim M$ and $\mathcal{P}^*(\omega^N) = \chi^*(\omega^M)$. In fact, locally $\mathcal{P}^{-1}(\Lambda)$ is given as

$$\mathcal{P}^{-1}(\Lambda) \cap (\pi^M)^{-1}(W) \cong \{(x, y\,; \xi, 0) \mid x \in U\}$$

and so $\sum d\xi_i \wedge dx_i + \sum d\eta_j \wedge dy_j$ on $\mathcal{P}^{-1}(\Lambda)$ coincides with $\sum d\xi_i \wedge dx_i$. □

From the arguments above,

Corollary 5.3. *more generally, if Λ is an isotropic submanifold, then $\mathcal{P}^{-1}(\Lambda)$ is also isotropic. Also if Λ is conic, then $\mathcal{P}^{-1}(\Lambda)$ is conic, and if Λ is compact, then $\mathcal{P}^{-1}(\Lambda)$ is also compact, since we assumed M is compact.*

On the other hand, let $\tilde{\Lambda}$ be a Lagrangian submanifold included in $\varphi^*(T_0^*(N))$.

Proposition 5.4. *We assume $\mathcal{P}^{-1}(\mathcal{P}(\tilde{\Lambda})) = \tilde{\Lambda}$ (as a set), then $\mathcal{P}(\tilde{\Lambda})$ is a Lagrangian submanifold.*

In particular, if the fibers of the submersion φ are connected, then the condition $\mathcal{P}^{-1}(\mathcal{P}(\tilde{\Lambda})) = \tilde{\Lambda}$ is automatically satisfied.

Proof. Let $\varphi : (x, y) \mapsto x$ be local coordinates as before. Then

$$\mathcal{P} : \varphi^*(T^*(N)\backslash\{0\}) \ni (x, y; \xi, 0) \mapsto (x, \xi) \leftrightarrow \sum \xi_i dx_i \in T_x^*(N)$$

Let $f_0, \cdots, f_{m-1}$ be local defining functions of $\tilde{\Lambda}$. Then by the assumption, the variables $\{y_j\}$ are free, so that we may assume $f_0(x, y, \xi) = f_0(x, \xi), \cdots, f_{n-1}(x, y, \xi) = f_{n-1}(x, \xi)$, which is a set of local defining functions of $\mathcal{P}(\tilde{\Lambda})$. The Lagrangian property follows in the same way as before. □

Hence we know in the both directions that a submanifold being a Lagrangian is deduced from other.

Then, next is to consider whether Λ or $\tilde{\Lambda}$ satisfies the conditions $Mas[1] \sim Mas[3]$, how is the Lagrangian submanifold $\mathcal{P}^{-1}(\Lambda)$ or $\mathcal{P}(\tilde{\Lambda})$?

So assume Λ or $\tilde{\Lambda}$ satisfies the conditions $Mas[1] \sim Mas[3]$.

Let Φ_t^M and Φ_t^N be the geodesic flow on M and N respectively. Then

Lemma 5.5. *The following diagram is commutative and the commutativity follows from only the assumption of the Riemannian submersion (= coincidence of the values σ_{Δ^M} and $\mathcal{P}^*\big(\sigma_{\Delta^N}\big)$ on $\varphi^*(T^*(N)\backslash\{0\})$): $\sigma_{\Delta^M}\big|_{\varphi^*(T^*(N)\backslash\{\})} = \mathcal{P}^*(\sigma_{\Delta^N})$.*

Note that we do not need the relation $\Delta^M \circ \varphi^* = \varphi^* \circ \Delta^N$.

$$\begin{array}{ccc}
T_0^*(M) & \xrightarrow{\Phi_t^M} & T_0^*(M) \\
\chi\big\uparrow & & \big\uparrow\chi \\
\varphi^*(T_0^*(N)) & \longrightarrow & \varphi^*(T_0^*(N)) \\
\mathcal{P}\big\downarrow & & \big\downarrow\mathcal{P} \\
T_0^*(N) & \xrightarrow{\Phi_t^N} & T_0^*(M).
\end{array}$$

This implies that not only the space $\mathcal{P}^{-1}(\Lambda)$ (or in the case we start from $\tilde{\Lambda}$ in $\varphi^*(T_0^*(N))$) but also the space $\varphi^*(T_0^*(N))$ being invariant under the action of the geodesic flow $\{\Phi_t^M\}$.

As for the condition $Mas[2]$ we assume that Λ is satisfying

$$\frac{1}{2\pi}\int_{\gamma}\theta^{N}\ -\ \frac{1}{4} < \mathcal{M}_{\Lambda}, \gamma > \in\ \mathbb{Z},$$

for any loop $\{\gamma(t)\} \subset \Lambda$. Then, by noting the facts:

$$\bullet \ \int_{\tilde\gamma}\theta^{M} = \int_{\tilde\gamma}\mathcal{P}^*(\theta^N) = \int_{\mathcal{P}(\tilde\gamma)}\theta^{N}$$

where $\{\tilde\gamma\}$ is a loop in $\mathcal{P}^{-1}(\Lambda)$, we can prove

$$\bullet \ \mathcal{M}_{\mathcal{P}^{-1}(\Lambda)} = \mathcal{P}^*\left(\mathcal{M}_{\Lambda}\right).$$

Hence

Lemma 5.6. *The condition $Mas[2]$ is satisfied by $\mathcal{P}^{-1}(\Lambda)$. Also it is OK for the case that $\mathcal{P}(\tilde\Lambda)$.*

As for the condition $Mas[3]$, there is a difference.

(1) First, let $\tilde\mu$ be an invariant measure on $\tilde\Lambda$ under the geodesic flow action of $\{\Phi^M_t\}$. Then in this case the push forward measure $\mathcal{P}_*(\tilde\mu)$ is an invariant measure on $\mathcal{P}(\tilde\Lambda)$ under the geodesic flow action of $\{\Phi^N_t\}$.

(2) Let $\mu_\Lambda \in \Gamma(\overset{max}{\bigwedge} T^*(\Lambda))$ be a volume form on Λ which is invariant under the geodesic flow $\{\Phi^N_t\}$ action.

Let dv^M and dv^N be the Riemannian volume forms on M and N respectively. According to the assumption of the Riemannian submersion and the decomposition $T(M) = \mathcal{V}\oplus_\perp\mathcal{H}$, there is a differential form $\theta_{V^*} \in \Gamma(\overset{max}{\bigwedge}\mathcal{V}^*)$ such that $dv^M = \theta_{V^*}\bigwedge\varphi^*(dv^N)$.

With these, the volume form $\tilde\mu$ on $\mathcal{P}^{-1}(\Lambda)$ is expressed as

$$\tilde\mu = w\cdot\left(\pi^M\right)^*(\theta_{V^*})\bigwedge\mathcal{P}^*(\mu_\Lambda)$$

with a positive function $w \in C^\infty(\mathcal{P}^{-1}(\Lambda))$.

We assume a measure $\tilde\mu = w\cdot\left(\pi^M\right)^*(\theta^{V^*})\bigwedge\mathcal{P}^*(\mu_\Lambda)$ is invariant under the action of $\{\Phi^M_t\}$, then since the form $\mathcal{P}^*(\mu_\Lambda)$ is invariant under the action of $\{\Phi^M_t\}$ by the assumption, the form $\left(\pi^M\right)^*(\theta_{V^*})$ is also invariant under the action of $\{\Phi^M_t\}$. The opposite is clear, so that we should assume there exist an invariant measure on $\mathcal{P}^{-1}(\Lambda)$, and it is equivalent to assume the form $\left(\pi^M\right)^*(\theta_{V^*})$ is invariant under the action of $\{\Phi^M_t\}$.

Or another type assumption on the submersion $\varphi : M \to N$ can be putted on. For example, if the geodesic flow $\{\Phi^M_t\}$ is periodic (M is called a *C_ℓ-manifold*), then we can find an invariant measure without any assumption on the submersion $\varphi : M \to N$. Maybe in this case the possible such Riemannian submersion will be highly restricted.

Remark 2. *Let $C : T^*_0(N) \to T^*_0(N)$ be a homogeneous symplectomorphism and denote by G'_C its sign changed (or twisted)graph;*

$$\begin{aligned} G'_C = &\left\{(x,\xi\,;\,y,-\eta)\ \middle|\ (x,\xi)\in T^*_x(N)\backslash\{0\}, (y,\eta) = C(x,\xi)\ \right\}\\ &\subset T^*_0(N)\times T^*_0(N) \subset T^*_0(N\times N).\end{aligned}$$

Then this is a Lagrangian submanifold in $T^*(N\times N)$. In this case the Lagrangian submanifold $\mathcal{P}^{-1}(G'_C)$ of the product map $\varphi\times\varphi : M\times M \to N\times N$ is not an interesting one, nor it is not a sign changed graph of a symplectomorphism of $T^*_0(M) \to T^*_0(M)$.

6. Sub-Riemannian case

Now we discuss shortly a sub-Riemannian situation of the submersion. There are various cases and I only restrict the discussion to a special case.

(1) The subbundle $\mathcal{H}$ is bracket generating, that is we assume

$$\Gamma(\mathcal{H}) + [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] = \Gamma(T(M)),$$

or it is the same thing that $\Gamma(\mathcal{V}) = [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})]$.

Instead of the Laplacian Δ^M, we can define the sub-Laplacian intrinsically which can be seen as a horizontal part of the Laplacian Δ^M and can be treated as before for the Riemannian submersion case.

(2) Next most simple case will be the case that there exists a subbundle $\mathcal{H}^1_x \subset \mathcal{H}$ such that if $\varphi(x) = \varphi(x')$ then

$$\mathcal{H}^1 \stackrel{d\varphi}{\to} d\varphi(\mathcal{H}^1_x) = d\varphi(\mathcal{H}^1_{x'}) \stackrel{d\varphi}{\leftarrow} \mathcal{H}^1_{x'}, \text{ and isometrically isomorphic}$$

$$\Gamma(\mathcal{V}) + [\Gamma(\mathcal{H}^1),\, \Gamma(\mathcal{H}^1)] = \Gamma(T(M)).$$

In this case we put $d\varphi(\mathcal{H}^1) = \overline{\mathcal{H}^1}$. Then

$$\Gamma(T(N)) = \Gamma(\overline{\mathcal{H}^1}) + [\Gamma(\overline{\mathcal{H}^1}),\, \Gamma(\overline{\mathcal{H}^1})],$$

that is the subbundle $\mathcal{H}^1$ is a 2 step sub-Riemannian structure, so that we can define a sub-Laplacian on N intrinsically.

If we assume there exists a compact Lagrangian submanifold in $T^*_0(N)$ satisfying the conditions Mas[1] $\sim$ Mas[3], then similar existence theorem for the eigenvalues of sub-Laplacians holds, where we replace the invariance of the geodesic flow action by the invariance of bi-characteristic flow actions. The difference comes from the existence of non-trivial characteristic variety, but this does not affect the conclusion.

7. Examples of Lagrangian submanifolds

7.1. **Closed one-form and Lagrangian submanifold.** Let X be a closed manifold and $\varphi : X \to U(1)$ a submersion to $U(1) = \{e^{\sqrt{-1}s} \mid s \in \mathbb{R}\} \cong S^1$. Then the set of local solutions $\{f_i\}$, where each real valued function f_i is defined on an open set U_i and satisfying the equation $e^{2\pi\sqrt{-1}\,f_i} = \varphi$, defines an one-Čeck cochain $\{c_{ji} = f_j - f_i\}$ of the $\mathbb{Z}$-valued constant sheaf $\mathbb{Z}_X$ on X and globally defines a closed one-form η ($:= df_j$ on U_j, which also coincides with $\varphi^*(ds)$). The cohomology class $[\eta] \in H^1_{dR}(X)$ is integral.

Conversely, let $\alpha \in H^1_{dR}(X)$ and assume

(1) α is in $H^1(X)$, that is α is an integral class, and

(2) there is a nowhere vanishing closed one-form η representating the class α.

Then, the set of local solutions $\{f_i\}$, $df_i = \eta$ on U_i where $\{U_i\}$ is an open covering of X, defines a submersion $\varphi : X \to U(1)$, $\varphi := e^{2\pi\sqrt{-1}\,f_i}$ on U_i, since by assumptions $f_j - f_i \in \mathbb{Z}$ and df_i does not vanish at any point.

Hence

Proposition 7.1. *Let X be a closed manifold. Then there is a submersion $\varphi : X \to U(1)$, if and only if there is a closed one-form η such that its cohomology class $[\eta]$ is integral and the one-form η never vanish.*

So, we assume that there is a closed one-form η satisfying the above conditions (1) and (2).

Then the image $\eta(X)$, $\eta : X \to T^*(X)$, is included in $T_0^*(X)$ and by the fact that $\eta^*(\theta^X) = \eta$, $\eta(X)$ is a Lagrangian submanifold and also coincides with the pul-back $\varphi^*(ds(S^1))$ of the Lagrangian submanifold $ds(S^1) = \{(e^{\sqrt{-1}s}, 1) \mid s \in \mathbb{R}\} \subset T^*(S^1) \cong S^1 \times \mathbb{R}$.

Moreover we see that the cohomology class $[\theta^X{}_{|\eta(X)}]$ of the restriction of the Liouville one-form to $\eta(X)$ is in $H^1(\eta(X)) \cong H^1(X) \subset H^1_{dR}(X)$.

In this case, since the tangent bundle $T(\eta(X))$ is transversal to the vertical subbundle $\mathcal{V}$ ($\mathcal{V} =$ Ker$(d\varphi)$) at all the points in $\eta(X)$, the Maslov class $\mathfrak{m}_{\eta(X)}$ is zero. So if the dimension $\dim H^1(X) = 1$, then a constant multiple $c_0\eta(X)$ satisfys the condition Mas[2].

The above case is a special case. In fact let $\varphi : M \to N$ be a submersion between closed manifolds and assume there exist a compact Lagrangian submanifold $\Lambda \subset T_0^*(N)$, then the submanifold $\varphi^*(\Lambda) \subset \varphi^*(T_0^*(N)) \overset{\chi}{\hookrightarrow} T_0^*(M)$ is a Lagrangian submanifold, where χ is the dual map of the differential $d\varphi$. If Λ satisfies the condition Mas[2], then $\varphi^*(\Lambda)$ also satisfys the condition Mas[2] (cf. [5]).

7.2. **Nilmanifolds.** In this subsection we treat a typical example satisfying the equivalent condition explained in Proposition 7.1.

Let $\mathbf{N}$ be a simply connected nilpotent Lie group having a lattice Γ. Then by Nomizu theorem (cf. [10]), the de Rham cohomology group $H^*_{dR}(\Gamma\backslash\mathbf{N})$ of the compact nilmanifold $\Gamma\backslash\mathbf{N}$ is isomorphic to the cohomology group of the corresponding Lie algebra $\mathfrak{n}$ through the induced map from the natural inclusion map of the differential subcomplex consisting of left invariant differential forms on $\mathbf{N}$ to the Γ-left action invariant differential forms, i.e., the differential complex on the nilmanifold $\Gamma\backslash\mathbf{N}$. In particular, $H^1_{dR}(\Gamma\backslash\mathbf{N}) \cong \{\eta \in \mathfrak{n}^* \mid \eta([X, Y]) = 0,\ X,\ Y \in \mathfrak{n}\}$.

So by Malcev theorem, let $\{X_i\}$ be a linear basis of the Lie algebra $\mathfrak{n}$ such that the structure constants $\{c^k_{ij}]\}$, $[X_i,\ X_j] = \sum c^k_{ij}X_k$ are all rational numbers, then $\{\exp X_i\}$ generates a lattice. Let $\{\eta_i\}$ be the dual basis of the space $\mathfrak{n}^*$ and assume $\eta_1([\mathfrak{n}, \mathfrak{n}]) = 0$. Then the space $\eta(\mathbf{N}) = \{(g, \eta) \mid g \in \mathbf{N}\} \subset \mathbf{N} \times \mathfrak{n}^* \cong T^*(\mathbf{N})$ is a Lagrangian subspace. In this case if we consider a left invariant Riemannian metric on $\mathbf{N}$, then the energy function is constant on $\eta(\mathbf{N})$ and the transformed Haar measure on $\eta(\mathbf{N})$ by the map $\eta : \mathbf{N} \to T^*(\mathbf{N})$ is invariant under the geodesic flow.

7.3. **Contact manifold and Lagrangian submanifold.** Let (M, α) be a compact contact manifold with a contact form α ($\dim M = 2n + 1$) and denote by $\Sigma_\alpha = \{\, t\alpha \mid t > 0\,\} \subset T_0^*(M)$, the cone bundle on M which is isomorphic to $M \times \mathbb{R}_+$. Then throught this isomorphism it holds

$$(\omega^M{}_{|\Sigma_\alpha})^{n+1} = (n+1)t^n \cdot dt \wedge \alpha \wedge (d\alpha)^n. \tag{7.1}$$

Hence the cone Σ_α is a symplectic manifold with the symplectic form $\omega^M{}_{|\Sigma_\alpha}$, the restriction of the natural symplectic form ω^M of $T^*(M)$ (and vise versa).

We assume that

$[\mathcal{RP}]$: *the action generated by the "Reeb vector field" $\mathcal{R}$ reduces to the $U(1)$-free action on M.*

The vector field $\mathcal{R}$ is uniquley determined by the conditions that $d\alpha(\mathcal{R}, \bullet) \equiv 0$ and $\alpha(\mathcal{R}) \equiv 1$. We may assume that the period is "2π".

Under this assumption the orbit space becomes a symplectic manifold in a natural way. In fact, let $\pi_\alpha : M \to M/U(1) =: \mathcal{O}$ be the projection map to the space of orbits, which is a $U(1)$-principal bundle and together with the Darboux's theorem for contact form, for any point $q \in \mathcal{O}$ we can find a local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$ defined on a small neighborhood $\overline{V} \ni q$ such that on which we have a local trivialization $\overline{V} \times U(1) \cong \pi_\alpha{}^{-1}(\overline{V}) := V$, $(x_1, \ldots, x_n, y_1, \ldots, y_n\,;\, e^{\sqrt{-1}s}) \in \mathbb{R}^{2n} \times U(1)$ and the contact form α is expressed as $\alpha = ds + \sum x_i dy_i$. The Reeb vector field $\mathcal{R}$ is expressed as $\partial/\partial s$ in terms of this coordinates and the projection is $\pi_\alpha(x_1, \ldots, x_n, y_1, \ldots, y_n\,;\, e^{\sqrt{-1}s}) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_n)$. The differential $d\alpha = \sum dx_i \wedge dy_i$ is invariant under the structure group action (= action generated by the Reeb vector field). Hence it defines a symplectic structure $\omega^{\mathcal{O}}$ on the orbit space $\mathcal{O}$.

Let $({x_1}', \ldots, {x_n}', {y_1}', \ldots, {y_n}')$ be another Darboux coodinates defined on $\overline{V}\,' \ni q$ and on which we have a local trivialization $\overline{V}\,' \times U(1) \cong {\pi_\alpha}^{-1}(\overline{V}\,') := V\,'$, then on $V \cap V\,'$ we have

$$\sum x_i\, dy_i - \sum {x_i}'\, {dy_i}' = ds - ds\,'.$$

Since $e^{\sqrt{-1}s} = g \cdot e^{\sqrt{-1}s\,'}$ with a transition function $g : \overline{V} \cap \overline{V}\,' \to U(1)$, also $e^{\sqrt{-1}s\,'} = h \cdot e^{\sqrt{-1}s\,''}$ on $\overline{V}\,' \cap \overline{V}\,''$ and so on, it holds that

$$e^{\sqrt{-1}s}\, e^{-\sqrt{-1}s'}\, e^{\sqrt{-1}s''} \;\equiv\; 1$$

on the intersection $\overline{V} \cap \overline{V}\,' \cap \overline{V}\,''$. This implies that the symplectic form $\frac{\omega^{\mathcal{O}}}{2\pi}$ is integral, i.e., the cohomology class $\frac{[\omega^{\mathcal{O}}]}{2\pi} \in H^2_{dR}(M)$ is in the image of the natural map $\check{H}^2(M, \mathbb{Z}) \to H^2_{dR}(M)$.

In this case, the maximal non-integrable subbundle $\mathrm{Ker}\,(\alpha) = \{Z \in T(M) \mid \alpha(Z) = 0\}$ defines a connection to the principal bundle $\pi_\alpha : M \to \mathcal{O}$ and is bracket generating so that it defines a 2-step sub-Riemannian structure on M.

Now let L be a Lagrangian submanifold in $\mathcal{O}$. Then from the expression (7.1)

Proposition 7.2. *The submanifold $\alpha({\pi_\alpha}^{-1}(L))$ is a $U(1)$-action invariant Lagrangian submanifold in Σ_α, where we regard $\alpha : M \to \Sigma_\alpha \subset T_0^*(M)$. Conversely if $\Lambda \subset M$ is $U(1)$-action invariant and $\alpha(\Lambda)$ is a Lagrangian submanifold in Σ_α, then $\pi_\alpha(\Lambda)$ is a Lagrangian submanifold.*

Let Z be a compact symplectic manifold with an integral symplectic form $\frac{1}{2\pi}\omega^Z$, that is $\frac{1}{2\pi}[\omega^Z]$ is in the image of $H^2(Z, \mathbb{Z}) \subset H^2_{dR}(Z)$, then we can construct a compact contact manifold such that the action generated by the Reeb vector field satisfies the condition $[\mathcal{RP}]$ and come back to $\mathcal{O} \cong Z$. These are explained precisely in [9].

Next we put one more strong assumption on the contact manifold M together with the assumption $[\mathcal{RP}]$:

$[\mathcal{PS}]$: *There exists a closed Riemannian manifold X with a Riemannnian metric $g(\cdot, \cdot)$ and its dual inner product Q^g on $T^*(X)$. When we realize the cotangent sphere bundle $S^*(X)$ as a submanifold $S^*(X) \cong \{(x, \xi) \in T^*(X) \mid Q^g(\xi, \xi) = 1\}$ with the contact form ${\theta^X}_{|S^*(X)}$, then we assume that there exists an isomorphism $\mathcal{C} : S^*(X) \to M$ keeping the contact structures, i.e., $\mathcal{C}^*(\alpha) = {\theta^X}_{|S^*(X)}$.*

Hence under these two assumptions $[\mathcal{RP}]$ and $[\mathcal{PS}]$ we may restate Prorposition 7.2

Proposition 7.3. *Let L be a Lagrangian submanifold in $\mathcal{O}$, then ${\pi_\alpha}^{-1}(L)$ is a Lagrangian submanifold in $T_0^*(X)$.*

These two assumptions $[\mathcal{RP}]$ and $[\mathcal{PS}]$ say that the manifold X must be a $SC_{2\pi}$-manifold and at the moment we may mention spheres and projective spaces or Zoll surface as such manifolds.

7.4. **Sphere case.** In this section we consider the sphere case from the point discussed in the last section.

We can realize the cotangent bundle of the sphere $S^n \subset \mathbb{R}^{n+1}$ as

$$T^*(S^n) = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \\ |x| = 1,\ < x, \xi >= 0\,\}$$

by identifying tangent and cotangent bundles using the standard Riemannian metric and we will denote hence force by $X^S := \{(x,\xi) \in T^*(S^n) \mid \xi \neq 0\}$, the punctured cotangent bundle $T_0^*(S^n)$.

By this realization of the cotangent bundle, the Liouville one-form $\theta^{S^n} := \theta^S$ and the symplectic form $\omega^{S^n} := \omega^S$ are expressed as

$$\theta^S = \sum \xi_i \, dx_i, \ \omega^S = \sum d\xi_i \wedge dx_i,$$

that is these can be seen as restrictions of those for $\mathbb{R}^{n+1}$.

Then by the map $\tau_S : X^S \longrightarrow \mathbb{C}^{n+1}$

$$\tau_S : (x,\xi) \longmapsto z = |\xi| x + \sqrt{-1}\xi$$

the punctured cotangent bundle $X^S = T_0^*(S^n)$ is identified with the quadrics

$$Q_2 = \left\{ z \in \mathbb{C}^{n+1} \backslash \{0\} \mid z^2 = \sum z_i^2 = 0 \right\}$$

and the symplectic form is expressed as

$$\omega^S = (\tau_S)^* \big(\sqrt{2}\sqrt{-1}\, \overline{\partial}\partial \, |z| \big), \ |z| = \sqrt{\sum |z_i|^2}, \tag{7.2}$$

which says that the space $T_0^*(S^n)$ has a Kähler manifold structure. By this realization of the space $T_0^*(S^n)$, the geodesic flow is expressed as the scalar multiplication of complex numbers of mudulus 1. Moreover let

$$\sigma = \frac{2}{|z|^2} \sum_j \overline{z}_j dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n,$$

then σ is a nowhere vanishing holomorphic n-form on $T_0^*(S^n) \overset{\tau_S}{\cong} Q_2$ and

$$\sigma \wedge \overline{\sigma} = \sqrt{-1}^{\,n} 2^{n/2+3} |z|^{n-2} \frac{(-1)^{n(n-1)/2}}{n!} (\omega^S)^n \tag{7.3}$$

through the identification by the map τ_S. These relations are found in [15].

We consider an $n+1$-dimensional submanifold Z in Q_2 defined by

$$Z = \{ e^{\sqrt{-1}\tau}(s_0, \ldots, s_p, \sqrt{-1}t_{p+1}, \ldots, \sqrt{-1}t_{n-p}) \mid s_i, \, t_j \in \mathbb{R}, \text{ and } \sum {s_i}^2 = \sum {t_j}^2 > 0 \},$$

where we assume $p \geq 2$ and $n - p \geq 3$ (hence $n \geq 5$).

Let $\mathcal{H} : \mathbb{R} \times \mathbb{R}^{p+1} \times \mathbb{R}^{n-p}$ be a map

$$\mathcal{H} : \mathbb{R} \times \mathbb{R}^{p+1} \times \mathbb{R}^{n-p} \ni (\tau, s, t) \longmapsto e^{\sqrt{-1}\tau}(s, \sqrt{-1}t) \in \mathbb{C}^{n+1},$$

then the map $\mathcal{H}$ restricted to an $n+1$-dimensional submanifold $\mathbb{R} \times \{(s,t) \in \mathbb{R}^{p+1} \times \mathbb{R}^{n-p} \mid |s| = |t| > 0\} := \mathbb{R} \times H$ is a covering map to Z and can be descended to a double covering map from $U(1) \times H$ to Z, which we can see from the expession of the manifold Z. Then

$$\begin{aligned} &{\tau_S}^{-1}(Z) \\ &= \Big\{ \Big(\frac{s_0 \cos\tau}{\sqrt{|s|^2\cos^2\tau + |t|^2 \sin^2\tau}}, \ldots, \frac{s_p \cos\tau}{\sqrt{|s|^2\cos^2\tau + |t|^2 \sin^2\tau}}, \\ &\qquad \frac{-t_{p+1} \sin\tau}{\sqrt{|s|^2\cos^2\tau + |t|^2 \sin^2\tau}}, \ldots, \frac{-t_n \sin\tau}{\sqrt{|s|^2\cos^2\tau + |t|^2 \sin^2\tau}}; \\ &\qquad s_0 \sin\tau, \ldots, s_p \sin\tau, t_{p+1}\cos\tau, \ldots, t_n \cos\tau \Big) \Big\}. \end{aligned} \tag{7.4}$$

We put $|s| = |t| = 1$ and denote $Z_1 := \mathcal{H}(U(1) \times S^p \times S^{n-p-1})$, then

Proposition 7.4. *$L_1 := {\tau_S}^{-1}(Z_1) \cong (U(1)\times S^p \times S^{n-p})/\mathbb{Z}_2$ is a geodesic flow action invariant Lagrangian submanifold in $T^*_0(S^n)$. The action by $\mathbb{Z}_2$ is given by*

$$U(1)\times S^p\times S^{n-p} \ni (e^{\sqrt{-1}\tau}, s, t) \longmapsto$$
$$(e^{\sqrt{-1}(\tau+\pi)}, -s, -t) = -(e^{\sqrt{-1}}, s, t) \in U(1)\times S^p \times S^{n-p}.$$

Proof. By definition it will be apparent of the geodesic flow invariance. Since,

$$\begin{aligned}
(7.5)\quad & {\tau_S}^{-1}(Z_1) = L_1 \\
&= \{(s_0\cos\tau,\dots,s_p\cos\tau,-t_{p+1}\sin\tau,\dots,-t_n\sin\tau; \\
&\qquad s_0\sin\tau,\dots,s_p\sin\tau,t_{p+1}\cos\tau,\dots.t_n\cos\tau) \mid \tau\in\mathbb{R}, \sum {s_i}^2 = \sum {t_{p+j}}^2 = 1\}, \\
&(\mathcal{H}\circ{\tau_S}^{-1})^*(\omega^S) = \sum d(s_i\sin\tau)\wedge d(s_i\cos\tau) - \sum d(t_{p+j}\cos\tau)\wedge d(t_{p+j}\sin\tau) \\
&= \sum s_i d\tau\wedge ds_i - \sum t_{p+j}dt_{p+j}\wedge d\tau = \frac{1}{2}\,d\tau\wedge d\left(\sum {s_i}^2 + \sum {t_{p+j}}^2\right) = 0,
\end{aligned}$$

which shows that the submanifold L_1 is a Lagrangian submanifold. □

Let $\{c^0(\tau)\}_{0\le\tau<frm-e\pi}$ be a loop,

$$(7.6)\qquad c^0(\tau) = (x^0(\tau),\xi^0(\tau)) = (\cos\tau,\underbrace{0,\dots,0}_{n-1},-\sin\tau;\sin\tau,\underbrace{0,\dots,0}_{n-1},\cos\tau)\in L_1,$$

then the loop $\{c^0(\tau)\}_{0\le\tau\le 2\pi}$ is twice of the generator of $\pi_1(L_1)\cong\mathbb{Z}$.

Proposition 7.5. *The action integral*

$$\frac{1}{2\pi}\int_{c^0(\tau)}\theta^S = -1.$$

Proof. By the explicit expression of the curve we have

$$\int_{c^0(\tau)}\theta^S = \int\sum \xi_i^0(\tau)\,dx_i^0(\tau) = \int_0^{2\pi}\sin\tau d(\cos\tau) - \cos\tau d\sin\tau = -\int_0^{2\pi} d\tau = -2\pi.$$

□

Next, we determine the Maslov class $\mathfrak{m}_{L_1}$ of L_1. For this purpose, first we determine the points $(x^0(\tau),\xi^0(\tau)) = c^0(\tau)$ at which $T_{c^0(\tau)}(L_1)\cap {\mathcal{V}^{L_1}}_{c^0(\tau)} \ne \{0\}$, where $\mathcal{V}^{L_1}$ denote the vertical subbundle of the kernel of the differential of the projection map $\pi^S : T^*(S^n)\to S^n$.

By the expression (7.5) put a map $\mathcal{F}:\mathbb{R}\times\mathbb{R}^{p+1}\times\mathbb{R}^{n-p}\longrightarrow\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ by

$$\begin{aligned}
(7.7)\quad & \mathcal{F}:\mathbb{R}\times\mathbb{R}^{p+1}\times\mathbb{R}^{n-p}\ni(\tau,s,t)\longmapsto \\
&\qquad (s_0\cos\tau,\dots,s_p\cos\tau,-t_{p+1}\sin\tau,\dots,-t_n\sin\tau; \\
&\qquad\qquad s_0\sin\tau,\dots,s_p\sin\tau,t_{p+1}\cos\tau,\dots.t_n\cos\tau),
\end{aligned}$$

then $\mathcal{F}(\{(\tau,s,t)\mid |s|=|t|=1\}) = L_1$. On the curve $\{p(\tau)\}_{0\le\tau<2\pi}$

$$p(\tau) = (\tau,1,\underbrace{0,\dots,0}_{n-1},1)\in\mathbb{R}\times\mathbb{R}^{p+1}\times\mathbb{R}^{n-p},$$

the map $\mathcal{F}$ is injective, $\mathcal{F}(p(\tau)) = c^0(\tau)$ and

$$\begin{aligned}
d\mathcal{F}_{p(\tau)}\left(\frac{\partial}{\partial\tau}\right) &= -\sum s_i\sin\tau\frac{\partial}{\partial x_i} - \sum t_{p+j}\cos\tau\frac{\partial}{\partial x_{p+j}} \\
&\quad + \sum s_i\cos\tau\frac{\partial}{\partial\xi_i} - \sum t_{p+j}\sin\tau\frac{\partial}{\partial\xi_{p+j}} \\
&= -\sin\tau\frac{\partial}{\partial x_0} - \cos\tau\frac{\partial}{\partial x_n} + \cos\tau\frac{\partial}{\partial\xi_0} - \sin\tau\frac{\partial}{\partial\xi_n}\in T_{c^0(\tau)}(L_1),
\end{aligned}$$

$$d\mathcal{F}_{p(\tau)}\left(\frac{\partial}{\partial s_i}\right) = \cos\tau\frac{\partial}{\partial x_i} + \sin\tau\frac{\partial}{\partial \xi_i},\ i = 0,\dots,p,$$
$$d\mathcal{F}_{p(\tau)}\left(\frac{\partial}{\partial t_{p+j}}\right) = -\sin\tau\frac{\partial}{\partial x_{p+j}} + \cos\tau\frac{\partial}{\partial \xi_{p+j}},\ j = 1,\dots,n-p.$$

Let α, β_i and $\delta_j \in \mathbb{R}$ with the conditions that $\sum_{i=0}^{p}\beta_i s_i = 0$ and $\sum_{j=1}^{n-p}\delta_j t_{p+j} = 0$, that is we take

$$\alpha\frac{\partial}{\partial\tau} + \sum\beta_i\frac{\partial}{\partial s_i} + \sum\delta_j\frac{\partial}{\partial t_{p+j}} \in T_{p(\tau)}(\mathbb{R}\times S^p\times S^{n-p-1}),$$

where $\beta_0 = 0 = \delta_{n-p}$, and α, β_i $(i \geq 1)$ and δ_j $(1 \leq j \leq n-p-1)$ can be taken arbitrarily. The tangent space $T_{c^0(\tau)}(L_1) = d\mathcal{F}_{p(\tau)}(T_{p(\tau)}(U(1)\times S^p\times S^{n-p-1}))$ is expressed as

$$\begin{aligned}(7.8)\quad T_{c^0(\tau)}(L_1) = \Big\{&\alpha\big(-\sin\tau\frac{\partial}{\partial x_0} - \cos\tau\frac{\partial}{\partial x_n} + \cos\tau\frac{\partial}{\partial\xi_0} - \sin\tau\frac{\partial}{\partial\xi_n}\big)\\ &+\sum_{i=1}^{p}\beta_i\big(\cos\tau\frac{\partial}{\partial x_i} + \sin\tau\frac{\partial}{\partial\xi_i}\big)\\ &+\sum_{j=1}^{n-p-1}\delta_j\big(-\sin\tau\frac{\partial}{\partial x_{p+j}} + \cos\tau\frac{\partial}{\partial\xi_{p+j}}\big)\ \Big|\ \alpha,\ \beta_i,\ \delta_j\ \in\mathbb{R}\Big\}.\end{aligned}$$

If a tangent vector

$$\alpha\frac{\partial}{\partial\tau} + \sum_{i=1}^{p}\beta_i\frac{\partial}{\partial s_i} + \sum_{j=1}^{n-p-1}\delta_j\frac{\partial}{\partial t_{p+j}} \in T_{p(\tau)}(U(1)\times S^p\times S^{n-p-1})$$

satisfies

$$\begin{aligned}&d\pi^{S^n}_{c^0(\tau)}\circ d\mathcal{F}_{p(\tau)}\left(\alpha\frac{\partial}{\partial\tau} + \sum_{i=1}^{p}\beta_i\frac{\partial}{\partial s_i} + \sum_{j=1}^{n-p-1}\delta_j\frac{\partial}{\partial t_{p+j}}\right)\\ &= -\alpha\sin\tau\frac{\partial}{\partial x_0} - \alpha\cos\tau\frac{\partial}{\partial x_n} + \sum_{i=1}^{p}\beta_i\cos\tau\frac{\partial}{\partial x_i} - \sum_{j=1}^{n-p-1}\delta_j\sin\tau\frac{\partial}{\partial x_{p+j}} = 0,\end{aligned}$$

then $\alpha = 0$ and

(1) at the points $c^0(\tau)$ for $\tau \neq \pi/2$ nor $3\pi/2$, that is $\cos\tau \neq 0$ we have

$$\beta_i\cos\tau = 0\ (i = 1,\dots,p),\ \ \delta_j\sin\tau = 0\ (j = 1,\dots,n-p-1),\ \text{ and}$$

(2) at the points $c^0(\tau)$ for $\tau \neq 0$ nor π, that is $\sin\tau \neq 0$

$$\beta_i\cos\tau = 0\ (i = 1,\dots,p),\ \ \delta_j\sin\tau = 0\ (j = 1,\dots,n-p-1).$$

Hence except four points of $c^0(\tau)$ at $\tau = 0,\ \pi/2,\ \pi,\ 3\pi/2$, the intersection $T_{c^0(\tau)}(L_1)\cap {\mathcal{V}^{L_1}}_{c^0(\tau)} = \{0\}$ and non-trivial intersections are given as

Case 1: $\tau = 0$ or $\tau = \pi$,

$$T_{c^0(\tau)}(L_1)\cap {\mathcal{V}^{L_1}}_{c^0(\tau)} = \left\{\ \sum_{1\leq j\leq n-p-1}\delta_j\frac{\partial}{\partial\xi_{p+j}}\ \right\}$$

Case 2: $\tau = \pi/2$ or $3\pi/2$,

$$T_{c^0(\tau)}(L_1)\cap {\mathcal{V}^{L_1}}_{c^0(\tau)} = \left\{\ \sum_{1\leq i\leq p}\beta_i\frac{\partial}{\partial\xi_i}\ \right\}.$$

To determine the Malsov class of the Lagrangian submanifold L_1, it is enough to calculate the Maslov indeces on the small intervals including these four points. We follow our definition of the Malsov index and α-construction explained in §2.

So, before the calculation we notice a Lemma whose proof will be apparent.

Lemma 7.6. *Let* $\mathbf{E}$ *be a symplectic vector space and* $\mathbf{F} \subset \mathbf{E}$ *a symplectic subspace. Let* λ *be a Lagrangian subspace of* $\mathbf{E}$ *and assume there is a continuous curve of Lagrangian subspaces* $\{\gamma(t)\}_{|t|\leq\epsilon\ll 1}$ *of* $\mathbf{E}$*. These satisfy the conditions* (R1), (R2) *and* (R3) such that

(R1) $\lambda_F := \lambda \cap \mathbf{F}$ *is a Lagrangian subspace of* $\mathbf{F}$,

(R2) *the curve of the intersections* $\gamma_F(t) := \gamma(t) \cap \mathbf{F}$ *is a continuous family of Lagrangian subspaces of* $\mathbf{F}$ *and,*

(R3) *the intersection* $\lambda \cap \gamma(t) \subset \mathbf{F}$ *for each* t.

Then,

$$Mas(\{\gamma(t)\}_{|t|\leq\epsilon}, \lambda) = Mas(\{\gamma_F(t)\}_{|t|\leq\epsilon}, \lambda_F). \tag{7.9}$$

Let $\{\mathbf{e}_i, \mathbf{f}_i\}_{i=1}^n$ be the standard symplectic basis of the symplectic vector space $\mathbf{E} := \mathbb{R}^{2n}$ with the symplectic form ω^{2n}, that is they satisfy the conditions

$$\omega^{2n}(\mathbf{e}_i, \mathbf{e}_j) = \omega^{2n}(\mathbf{f}_i, \mathbf{f}_j) = 0, \omega^{2n}(\mathbf{e}_i, \mathbf{f}_j) = -\omega^{2n}(\mathbf{f}_j, \mathbf{e}_i) = \delta_{i\,j}.$$

Now we show that our cases can be proved by applying the Lemma 7.6 above.

We must treat the two Cases 1 and 2 separately.

Case 1:

Let τ be around $\tau = 0$ or π, say $|\tau| \leq \epsilon \ll 1$, or $|\tau - \pi| \leq \epsilon \ll 1$. Then the tangent space $T_{c^0(\tau)}(X^S)$ at $c^0(\tau)$ are characterized as

(7.10)

$$\begin{aligned} &T_{c^0(\tau)}(X^S) \\ &= \Big\{ \sum_{i=0}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{i=0}^{n} b_i \frac{\partial}{\partial \xi_i} \;\Big|\; a_0 \cos\tau = a_n \sin\tau, (a_n + b_0)\cos\tau = (b_n - a_0)\sin\tau \Big\} \end{aligned}$$

(7.11)

$$= \Big\{ \sum_{i=1}^{n-1} a_i \frac{\partial}{\partial x_i} + a_n\big(\tan\tau \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_n} - \frac{1}{\cos^2\tau}\frac{\partial}{\partial \xi_0}\big) + \sum_{i=1}^{n-1} b_i \frac{\partial}{\partial \xi_i} + b_n\big(\tan\tau \frac{\partial}{\partial \xi_0} + \frac{\partial}{\partial \xi_n}\big)\Big\}$$

Based on these expressions we define symplectic isomorphisms $S_\tau : \mathbf{E} \to T_{c^0(\tau)}(X^S)$ by

$$\begin{cases} S_\tau : \mathbf{e}_i \longmapsto \frac{\partial}{\partial x_i}, \text{ for } i = 1, \dots, n-1, \\ S_\tau : \mathbf{e}_n \longmapsto \cos\tau\big\{\tan\tau\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_n} - \frac{1}{\cos^2\tau}\frac{\partial}{\partial \xi_0}\big\}, \\ S_\tau : \mathbf{f}_i \longmapsto \frac{\partial}{\partial \xi_i}, \text{ for } i = 1, \dots, n-1, \\ S_\tau : \mathbf{f}_n \longmapsto \cos\tau\big\{\tan\tau\frac{\partial}{\partial \xi_0} + \frac{\partial}{\partial \xi_n}\big\}. \end{cases}$$

Since $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_j}\}_{i,j=0}^n$ are symplectic basis of the space $T(T^*(\mathbb{R}^{n+1}))$, it will be easily seen that these maps are symplectic, that is it leaves the symplectic forms.

Then the symplectic subspace $\mathbf{F}$ in $\mathbf{E}$ spanned by the basis vectors $\{\mathbf{e}_i, \mathbf{f}_i\}_{i\leq n-1}$ is maped to the subspace $S_\tau(\mathbf{F}) = F_{c^0(\tau)}$, where $F_{c^0(\tau)}$ is a symplectic subspace in $T_{c^0(\tau)}(X^S)$ spanned by the basis vectors $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_i}\}_{i=1}^{n-1}$ for each $|\tau| \ll \pi/2$ or $|\tau - \pi| \ll \pi/2$.

Also by (7.11) the vertical subbundle $\mathcal{V}^{L_1}{}_{c^0(\tau)} = \mathrm{Ker}\,(d\pi^{S^n}{}_{c^0(\tau)})$, $\pi^{S^n} : T(X^S) \to S^n$, is characterized as

$$\mathcal{V}^{L_1}{}_{c^0(\tau)} = \Big\{ \sum_{i=1}^{n-1} b_i \frac{\partial}{\partial \xi_i} + b_n \big(\tan\tau \frac{\partial}{\partial \xi_0} + \frac{\partial}{\partial \xi_n} \big) \;\Big|\; b_i \in \mathbb{R}, \Big\}$$

Hence let λ_E be a Lagrangian subspace in $\mathbf{E}$ spanned by $\{\mathbf{f}_i\}_{i=1}^n$, then $\mathcal{V}^{L_1}{}_{c^0(\tau)} = S_\tau(\lambda_E)$.

For each τ, let $\gamma(\tau)$ be a subspace in $\mathbf{E}$ spanned by the vectors

$$\{\cos\tau\cdot\mathbf{e}_i + \sin\tau\cdot\mathbf{f}_i \ (i = 1, \dots, p), -\sin\tau\cdot\mathbf{e}_{p+j} + \cos\tau\cdot\mathbf{f}_{p+j} \ (j = 1, \dots, n-p-1), \mathbf{e}_n + \tan\tau\cdot\mathbf{f}_n\}$$

then $S_\tau(\gamma(\tau)) = T_{c^0(\tau)}(L_1)$.

Now we prove that these subspaces λ_E, $\mathbf{F}$ and $\{\gamma(\tau)\}_{|\tau|\leq\epsilon\ll\pi/2}$ (also $\{\gamma(\tau)\}_{|\tau-\pi|\leq\epsilon\ll\pi/2}$ satisfy the conditions (R1) $\sim$ (R3) in Lemma 7.6.

Proposition 7.7. (R1) *It will be apparent that the intersection $\mathbf{F}\cap\lambda_E =: \lambda_F$ is generated by $\{\mathbf{f}_1, \dots, \mathbf{f}_{n-1}\}$ and is a Lagrangian subspace in $\mathbf{F}$.*

(R2) $\gamma(\tau) \cap \mathbf{F} =: \gamma_F(\tau)$ *is a curve of Lagrangian subspace of* $\mathbf{F}$.

(R3) $\gamma(\tau) \cap \lambda_E \subset \mathbf{F}$.

Proof. Since the intersection $\gamma(\tau) \cap \mathbf{F}$ is spanned by the vectors

$$\{\cos\tau\cdot\mathbf{e}_i + \sin\tau\cdot\mathbf{f}_i \ (i = 1, \dots, p), \ -\sin\tau\cdot\mathbf{e}_{p+j} + \cos\tau\cdot\mathbf{f}_{p+j} \ (j = 1, \dots, n-p-1)\},$$

we know that $\gamma(\tau)\cap\mathbf{F}$ is a family of Lagrangian subspace in $\mathbf{F}$, which shows (R2) condition.

(R3) condition will be seen by the expression that

$$\gamma(\tau) \cap \lambda_E = \{0\} \text{ or a subspace spanned by } \{\mathbf{f}_{p+j}\}_{j=1}^{n-p-1} \text{ for } \tau = 0 \text{ or } \pi,$$

which is a subspace in $\mathbf{F}$. □

Then

Proposition 7.8. *The curve $\{\gamma(\tau)\}_{|\tau|\leq\epsilon}$ and a fixed Lagrangian subspace λ_E in $\mathbf{E}$ are mapped to the curves of Lagrangian subspaces $\{T_{c^0(\tau)}(L_1)\}_{|\tau|\leq\epsilon}$ and $\{\mathcal{V}^{L_1}{}_{c^0(\tau)}\}$ in $T_{c^0(\tau)}(X^S)$. Hence by α-construction*

$$Mas(\{T_{c^0(\tau)}(L_1)\}_{|\tau|\leq\epsilon}, \{\mathcal{V}^{L_1}{}_{c^0(\tau)}\}_{|\tau|\leq\epsilon}) = Mas(\{\gamma_E(\tau)\}_{|\tau|\leq\epsilon}, \lambda_E).$$

Also by applying Lemma 7.6

$$Mas(\{\gamma_E(\tau)\}_{|\tau|\leq\epsilon}, \lambda_E) = Mas(\{\gamma_F(\tau)\}_{|\tau|\leq\epsilon}, \lambda_F).$$

The explicit determination of the value $Mas(\{\gamma_F(\tau)\}_{|\tau|\leq\epsilon}, \lambda_F)$ is done as follows:

Let μ be the Lagrangian subspace of $\mathbf{F}$ spanned by the basis vectors $\{\mathbf{e}_i + \mathbf{f}_i\}_{i=1}^{n-1}$, then μ and λ_F are transversal, and also μ and $\gamma_F(\tau)$ are transversal when $|\tau| \leq \epsilon \ll \pi/2$ and the subspace $\gamma_F(\tau)$ is spanned by the basis vectors

$$\cos\tau\cdot\mathbf{e}_i + \sin\tau\cdot\mathbf{f}_i \ (i = 1, \dots, p), -\sin\tau\cdot\mathbf{e}_{p+j} + \cos\tau\cdot\mathbf{f}_{p+j} \ (j = 1, \dots, n-p-1).$$

For each τ we define a map

$$\begin{aligned}
&A_\tau : \lambda_F \to \mu, \\
&A_\tau(\mathbf{f}_i) = \frac{\cos\tau}{\sin\tau - \cos\tau}(\mathbf{e}_i + \mathbf{f}_i) \text{ for } i = 1, \dots, p, \\
&A_\tau(\mathbf{f}_{p+j}) = \frac{-\sin\tau}{\sin\tau + \cos\tau}(\mathbf{e}_{p+j} + \mathbf{f}_{p+j}) \text{ for } j = 1, \dots, n-p-1.
\end{aligned}$$

Then the space spanned by vectors $\{\mathbf{f}_i + A_\tau(\mathbf{f}_i)\}_{i=1}^{n-1}$ coincides with the subspace $\gamma_F(\tau)$ and the map A_τ can be seen as a symmetric matrix

$$A_\tau(\tau) = \begin{pmatrix} \frac{\cos\tau}{\sin\tau-\cos\tau} I_p & \mathcal{O} \\ \\ \mathcal{O} & \frac{-\sin\tau}{\sin\tau+\cos\tau} I_{n-p-1}, \end{pmatrix}$$

where I_k denotes the identity matrix of size k. Then by Lemma 7.10 below

Proposition 7.9.

$$Mas(\{\gamma_F(\tau)\}_{|\tau|\le\epsilon}, \lambda_F) = \operatorname{sign}(\dot{A}_0) \text{ on } \operatorname{Ker}(A_0) = 1+p-n,$$

and

$$Mas(\{\gamma_F(\tau)\}_{|\tau-\pi|\le\epsilon}, \lambda_F) = \operatorname{sign}(\dot{A}_\pi) \text{ on } \operatorname{Ker}(A_\pi) = 1+p-n.$$

Lemma 7.10. *Let $\{A_t\}_{|t|\ll\epsilon}$ be a continuously differentialble family of $k\times k$ symmetric matrices defined for small t such that*

$$\text{the matrix } \dot{A}_0 \text{ is non-singular on } \operatorname{Ker}(A_0). \tag{7.12}$$

Then for sufficiently small $0 < t \ll \epsilon$, "the number of the positive eigenvalues of A_t" coincides with
"the number of the positive eigenvalues of $\dot{A}_0$ on $\operatorname{Ker}(A_0)$*" + "the number of the positive eigenvalues of A_0 on the orthogonal complement of* $\operatorname{Ker}(A_0)$*" and*
"the number of the negative eigenvalues of A_t" coincides with
"the number of the negative eigenvalues of $\dot{A}_0$ on $\operatorname{Ker}(A_0)$ + *"the number of the negative eigenvalues of A_0 on the orthogonal complement of* $\operatorname{Ker}(A_0)$*".*

Also for sufficiently small $0 > t \gg -\epsilon$, it holds similar statements.

The determination of the Maslov indeces around the points $c^0(\pi/2)$ and $c^0(3\pi/2)$ can be carried out by the same way. We list the necessary data here. Assume $|\tau-\pi/2| \le \epsilon \ll \pi/2$ or $|\tau - 3\pi/2| \le \epsilon \ll \pi/2$. Then

(7.13)

$$\begin{aligned} &T_{c^0(\tau)}(X^S) \\ &= \Big\{ a_0\Big(\frac{\partial}{\partial x_0} + \cot\tau \frac{\partial}{\partial x_n} + \frac{1}{\sin^2\tau}\frac{\partial}{\partial \xi_n}\Big) + \sum_{i=1}^{n-1} a_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n-1} b_i \frac{\partial}{\partial \xi_i} + b_0\Big(\frac{\partial}{\partial \xi_0} + \cot\tau\frac{\partial}{\partial \xi_n}\Big)\Big\} \end{aligned}$$

The symplectic isomorphism $U_\tau : \mathbf{E} \to T_{c^0(\tau)}(X^S)$ is defined as

$$\begin{cases} U_\tau : \mathbf{e}_i \longmapsto \frac{\partial}{\partial x_i}, \text{ for } i = 1,\dots,n-1, \\ \\ U_\tau : \mathbf{e}_n \longmapsto \sin\tau\Big(\frac{\partial}{\partial x_0} + \cot\tau\frac{\partial}{\partial x_n} + \frac{1}{\sin^2\tau}\frac{\partial}{\partial\xi_n}\Big) \\ \\ U_\tau : \mathbf{f}_i \longmapsto \frac{\partial}{\partial \xi_i}, \text{ for } \ i = 1,\dots,n-1, \\ \\ U_\tau : \mathbf{f}_n \longmapsto \sin\tau\Big(\frac{\partial}{\partial \xi_0} + \cot\tau\frac{\partial}{\partial\xi_n}\Big). \end{cases}$$

The vertical subbundle ${\mathcal{V}^{L_1}}_{c^0(\tau)}$ is

$${\mathcal{V}^{L_1}}_{c^0(\tau)} = \Big\{ \sum_{i=1}^{n-1} b_i \frac{\partial}{\partial \xi_i} + b_0\Big(\frac{\partial}{\partial \xi_0} + \cot\tau\frac{\partial}{\partial\xi_n}\Big)\Big\} \tag{7.14}$$

The Lagrangian subspace $\mathbf{F}$ and λ_E are the same spaces with the first case. Then

$$U_\tau(\mathbf{F}) = \Big\{ \sum_{i=1}^{n-1} a_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n-1} b_i \frac{\partial}{\partial \xi_i} \Big\}.$$

For each τ, $|\tau - \pi/2| \leq \epsilon$ (or $|\tau - 3\pi/2| \leq \epsilon$), let $\varphi(\tau)$ be a family of Lagrangian subspaces of $\mathbf{E}$ spanned by the vectors

$$\{\cos\tau \cdot \mathbf{e}_i + \sin\tau \cdot \mathbf{f}_i \ (i = 1, \ldots, p), \\ - \sin\tau \cdot \mathbf{e}_{p+j} + \cos\tau \cdot \mathbf{f}_{p+j} \ (j = 1, \ldots, n-p-1), \ \mathbf{e}_n - \cot\tau \cdot \mathbf{f_n}\},$$

then $U_\tau(\varphi(\tau)) = T_{c^0(\tau)}(L_1)$.

We can take the same Lagrangian subspace μ in $\mathbf{F}$ which is transversal to $\varphi(\tau) \cap \mathbf{F}$ and when we express the space $\varphi(\tau) \cap \mathbf{F}$ as the graph of a map $B_\tau : \lambda_F \to \mu$ the operator has the same expression of A_τ, so that finnaly we have

Proposition 7.11.

$$Mas(\{T_{c^0(\tau)}(L_1)\}_{|\tau-\pi/2|\leq\epsilon}, \{\mathcal{V}^{L_1}{}_{c^0(\tau)}\}_{|\tau-\pi/2|\leq\epsilon}) = Mas(\{\varphi(\tau)\}\}_{|\tau-\pi/2|\leq\epsilon} \cap \mathbf{F}, \lambda_F) \\ = \mathrm{sign}\,(\dot{A}_{\pi/2}) \ \textit{on} \ \mathrm{Ker}\,(A_{\pi/2}) = -p,$$

and

$$Mas(\{T_{c^\tau}(L_1)\}_{|\tau-\pi/2|\leq\epsilon}, \{\mathcal{V}^{L_1}{}_{c^0(\tau)}\}_{|\tau-3\pi/2|\leq\epsilon}) = Mas(\{\varphi(\tau)\}_{|\tau-3\pi/2|\leq\epsilon} \cap \mathbf{F}, \lambda_F) \\ = \mathrm{sign}\,(\dot{A}_{3\pi/2}) \ \textit{on} \ \mathrm{Ker}\,(A_{3\pi/2}) = -p.$$

Summing up these calculation we have

Proposition 7.12.

$$\mathfrak{m}_{L_1} : \pi_1(L_1) \cong \mathbb{Z} \longrightarrow \mathbb{Z}, 1 \longmapsto (1-n).$$

Corollary 7.13. *For $n = 4k + 3$, then L_1 satisfies the condition* Mas[2].
For $n = 4k + 2$, then $1/2 \cdot L_1$ satisfies the condition Mas[2].
For $n = 4k + 1$, then $2 \cdot L_1$ satisfies the condition Mas[2].
For $n = 4k$, then $3/2 \cdot L_1$ satisfies the condition Mas[2].

It will be clear that on *const* $\cdot L_1$ the principal symbol of the Laplacian is constant $= 1$ (Condition Mas[1]).

As for the condition Mas[3], there is a way to construct a measure on any geodesic flow invariant Lagrangian submanifold in $T_0^*(S^n) \cong Q_2$ based on the Kähler structure and the properties (7.2) and (7.3).

In fact, the property (7.3) says that $|\sigma|$ is a nowhere vanishing half density on the whole space Q_2. If Λ is a $U(1)$-invariant Lagrangian subspace, then by the characterization of Q_2 and the relation (7.3) we can regard that the complexification $T^*(\Lambda) \otimes \mathbb{C}$ is isomorphic to $T^*(Q_2)_{|\Lambda}$ considered as a complex vector bundle, or it is the same thing that it is isomorphic to the restriction to Λ of the holomorphic part $T^{*\prime}(Q_2)$ of the complexification $T^*(Q_2) \otimes \mathbb{C} = T^{*\prime}(Q_2) \oplus T^{*\prime\prime}(Q_2)$, hence

$$\bigwedge^n (T^*(\Lambda) \otimes \mathbb{C}) = \Big(\bigwedge^n T^*(\Lambda) \Big) \otimes \mathbb{C} \cong \bigwedge^n T^{*\prime}(Q_2)_{|\Lambda}.$$

Then we can define a half density on Λ by restricting the half density $|\sigma|$ to Λ.

Appendix A. Maslov index and Maslov class

In this Appendix, we recall a definition of the Maslov class for two Lagrangian subbundles in a symplectic vector bundle based on the Maslov index defined for arbitrary paths (cf. [6], [16], [4], [5]) and prove some properties based on our definition of the Maslov index.

A.1. **Maslov index for path and Maslov class.** We consider $\mathbb{C}^n$ as a typical symplectic vector space with the anti-symmetric and non-degenerate bilinear form $\omega^{(n)}(z,w)$

$$\omega^{(n)}(z,w) := \mathrm{Im}\left(\sum z_i\overline{w}_i\right) = \sum x_{n+i}y_i - x_iy_{n+i},$$

where $z = (z_1,\dots,z_n) = (x_1,x_2,\dots,x_n\,;x_{n+1},\dots,x_{2n})$, $z_i = x_i + x_{n+i}\sqrt{-1}$ and $w = (w_1,\dots,w_n) = (y_1,y_2,\dots,y_n\,;y_{n+1},\dots,y_{2n})$, $w_i = y_i + y_{n+i}\sqrt{-1}$.

A real subspace λ in $\mathbb{C}^n$ is said to be an isotropic subspace, if the anti-symmetric bilinear form $\omega^{(n)}{}_{|\lambda\times\lambda} \equiv 0$. Then $\dim_{\mathbb{R}}\lambda \le n$. In case $\dim\lambda = n$, i.e., it has the maximal dimension among isotropic subspaces, then it is called a Lagrangian subspace.

For h a subspace (real vector space) in $\mathbb{C}^n$, we denote by h° the subspace defined by

$$h^\circ = \{z\in\mathbb{C}^n \mid \omega^{(n)}(z,v) = 0 \text{ for any } v\in h\}.$$

So, h is isotropic, if and only if $h\subset h^\circ$ and h is a Lagrangian subspace, if and only if $h = h^\circ$.

The subspaces

$$\lambda_{\mathrm{Re}} := \{\ (x_1,\dots,x_n\,;0,\dots,0)\ \}$$

and

$$\lambda_{\mathrm{Im}} := \{\ (0,\dots,0\,;x_{n+1},\dots,x_{2n})\ \}$$

are typical Lagrangian subspaces and $\mathbb{C}^n = \lambda_{\mathrm{Re}}\oplus\lambda_{\mathrm{Im}}$.

We denote the space of all Lagrangian subspaces in $\mathbb{C}^n$ by $\Lambda(n)$, which as is well known isomorphic to the quotient space $U(n)/O(n)$ and is called the Lagrangian-Grassmaniann and together with the projection map

$$\pi_{\mathcal{F}} : U(n)\to\Lambda(n),\ U(n)\ni U\longmapsto U(\lambda_{\mathrm{Im}})$$

it is a principal bundle with the structure group $O(n)$.

Let $\lambda\in\Lambda(n)$ and denote by $\mathcal{P}_\lambda$ the orthogonal projection operator $\mathbb{C}^n\to\lambda\subset\mathbb{C}^n$. Then the operator $\tau_\lambda := 2\mathcal{P}_\lambda - Id$ is an involution with λ as the 1-eigenspace and the orthogonal complement $\lambda^\perp$ as the -1-eigenspace. Also for $U\in U(n)$ let's denote the operator $\tau_\lambda\circ U^*\circ\tau_\lambda$ by $\theta_\lambda(U)$. In particular, if $\lambda = \lambda_{\mathrm{Re}}$ and we express the matrix $U = \left(u_{ij}\right)$ with the standard orthonormal basis $\{e_i\}$ of $\mathbb{C}^n$, then $\theta_{\lambda_{\mathrm{Re}}}(U) = \overline{U}$, that is $\overline{U} = \left(\overline{u}_{ij}\right)$.

For each $\lambda\in\Lambda(n)$, let $\mathcal{S}_\lambda : \Lambda(n)\to U(n)$ be a map, called "*Souriou map*", defined by

$$\mathcal{S}_\lambda : \Lambda(n)\ni\mu\longmapsto U\circ\theta_\lambda(U)\in U(n),$$

where $\mu = U(\lambda_{\mathrm{Im}})$. In fact this does not depend on the operator U for $\mu = U(\lambda^\perp)$, since we have an expression

$$\mathcal{S}_\lambda(\mu) = -\tau_\mu\circ\tau_\lambda.$$

Let $U_{\mathfrak{M}}$ be a subset in $U(n)$ defined by

$$U_{\mathfrak{M}} = \{\,U\in U(n)\mid U+Id \text{ is not invertible}\,\}. \tag{A.1}$$

Then we call the subset defined by

$$\mathcal{M}_\lambda := {\mathcal{S}_\lambda}^{-1}(U_{\mathfrak{M}}) = \left\{\mu\in\Lambda(n)\ \middle|\ \mu\bigcap\lambda\neq\{0\}\right\} \tag{A.2}$$

the "*Maslov cycle*" passing through a Lagrangian subspace $\lambda\in\Lambda(n)$.

Let $\gamma : [0,\,1]\to\Lambda(n)$ be a continuous curve. We define an intersection number of γ and $\mathcal{M}_\lambda$ in the following way (cf. [4]):

We can find a partition $\{0 = t_0 < t_1 < t_2 < \cdots < t_\ell = 1\}$ of the interval $[0,\,1]$ and a set of small positive numbers $\{\,0 < \varepsilon_j \ll 1\,\}_{j=0}^{\ell}$ satisfying the condition that for $j = 0,\dots,\ell-1$

$$\begin{cases} \text{the values } e^{\sqrt{-1}(\pi\,\pm\,\varepsilon_j)} \text{ are not eigenvalues of the operators} \\ \mathcal{S}_\lambda(\gamma(t)) \text{ for } t_j\le t\le t_{j+1}. \end{cases} \tag{A.3}$$

This condition means that the eigenvalues of the operators $\{\mathcal{S}_\lambda(\gamma(t))\}_{t_j \leq t \leq t_{j+1}}$ included in the arc $\{e^{\sqrt{-1}s} \mid \pi - \varepsilon_j \leq s \leq \pi + \varepsilon_j\}$ stay there when the parameter $t_j \leq t \leq t_{j+1}$. Then we define an integer $Mas(\{\gamma\}, \lambda)$, and call it "*Maslov index*" for a path $\{\gamma\}$ with respect to the Maslov cycle $\mathcal{M}_\lambda$ by

Definition A.1.

$$
\begin{aligned}
Mas(\{\gamma\}, \lambda) := \sum_{j=0}^{\ell-1} \\
& \textit{the number of the eigenvalues of the operator } \mathcal{S}_\lambda(\gamma(t_{j+1})) \\
& \qquad \textit{in the sector } \left\{e^{\sqrt{-1}s} \mid \pi \leq s \leq \pi + \varepsilon_j\right\} \\
& - \textit{the number of the eigenvalues of the operator } \mathcal{S}_\lambda(\gamma(t_j)) \\
& \qquad \textit{in the sector } \left\{e^{\sqrt{-1}s} \mid \pi \leq s \leq \pi + \varepsilon_j\right\}.
\end{aligned}
$$

Then,

[M-Ind(1)] The integer $Mas(\{\gamma\}, \lambda)$ does not depend on the partition $\{t_j\}$ of the interval $[0, 1]$ and the small positive numbers $\{\varepsilon_j\}$ satisfying the condition (A.3).

[M-ind(2)] It is a homotopy invariant for the paths with the fixed end points.

[M-ind(3)] It satisfies the additivity under catenations of paths.

Let $\Psi : E \to X$ be a symplectic vector bundle over a space X (we put the fiber dimension $= n$). The space X will have suitable properties satisfied by manifolds. We denote the anti-symmetric non-degenerate bilinear form on E by ω^E, then we can install an inner product $< \cdot, \cdot >$ on E "*compatible*" with the symplectic structure ω^E in such a sense that there exists an almost complex structure $J : E \to E$, $J^2 = -Id$, $\Psi \circ J = \Psi$ such that

$$\omega^E(u, v) =< J(u), v >, \ < J(u), J(v) >=< u, v >, \ u, \ v \in E_x.$$

We assume that there exist two Lagrangian sub-bundles F and G in E, that is their fibers at each point x are Lagrangian subspaces in E_x.

Let $\{\gamma(t)\}$ be a continuous curve, $\gamma : [0, 1] \to X$. We divide it into small segments $\left\{\{\gamma(t)\}_{t_i \leq t \leq t_{i+1}}\right\}$ in such a way that there exist a finite open covering $\{O_i\}_i$ around the curve $\{\gamma(t)\}$ and $\gamma([t_i, t_{i+1}]) \subset O_i$, such that the vector bundle E has local trivializations

$$\psi_i : O_i \times \mathbb{C}^n \cong \Psi^{-1}(O_i)$$

satisfying the property that by this trivialization for each $x \in O_i$, $(x, \lambda_{\mathrm{Im}})$ is mapped to $\psi_i(x, \lambda_{\mathrm{Im}}) = F_x = \Psi^{-1}(x) \bigcap F$. Then we can assign an integer $Mas_{(F,G)}(\{\gamma(t)\})$ for an arbitrary continuous path $\gamma : [0, 1] \to X$ as the sum

$$Mas_{(F,G)}(\{\gamma(t)\}) = \sum_i Mas(\{{\psi_i}^{-1}(G_{\gamma(t)})\}_{t_i \leq t \leq t_{i+1}}, \lambda_{\mathrm{Im}}). \tag{A.4}$$

This quantity can be defined for all paths and has the properties:

- $\mathcal{M}(0)$: The definition does not depend on the partition of the interval $[0, 1]$, nor the local trivializations of the symplectic vector bundle E satisfying the conditions above nor does not depend on the inner product installed which satisfies the "*compatibility properties*",
- $\mathcal{M}(1)$: Homotopy invariance for paths with fixed end points,
- $\mathcal{M}(2)$: Additivity under catenations.

Hence, let $\pi : \tilde{X} \to X$ be the universal covering space of X consisting of homotopy classes of paths starting from a fixed initial point $x_0 \in X$. Then we can define a function

$$Mas_{(F,G)} : \tilde{X} \longrightarrow \mathbb{Z},\ \tilde{X} \ni \{\gamma\} \longmapsto Mas_{(F,G)}(\{\gamma(t)\}). \tag{A.5}$$

Especially its restriction to the fiber $\pi^{-1}(x_0)$ defines a homomorphism:

$$Mas_{(F,G)} : \pi^{-1}(x_0) \cong \pi_1(X) \to \mathbb{Z}.$$

Consequently, we have a cohomology class $\in H^1(X, \mathbb{Z})$, which we denote by $\mathfrak{m}_{(F,G)}$ and is called the "*Maslov class*" of the pair of Lagrangian subbundles F and G. Note that $\mathfrak{m}_{(F,G)} = -\mathfrak{m}_{(G,F)}$.

Proposition A.2. *It will be apparent if the intersection $F \cap G$ on a curve $\{\gamma(t)\}$ is trivial bundle, then $Mas_{(F,G)}(\{\gamma\}) = 0$*

Definition A.3. *Let $\chi_{\pi/2}$ be the representation $\chi_{\pi/2} : \mathbb{Z} \to U(1)$, $n \mapsto e^{\pi/2\sqrt{-1}n}$ and we define an associated complex line bundle $L_{\mathfrak{m}_{(F,G)}}$ on X to the principal bundle $\pi : \tilde{X} \to X$ through the representation $\pi_1(X) \overset{Mas_{(F,G)}}{\longrightarrow} \mathbb{Z} \overset{\chi_{\pi/2}}{\longrightarrow} U(1)$. It is called "Maslov line bundle".*

Let E be symplectic a vector bundle on a space X with two Lagrangian subbundle F and G. Let $\mathfrak{f} : Y \to X$ be a continuous map, then we can define the symplectic vector bundle $\mathfrak{f}^*(E)$ on Y with two Lagrangian subbundles $\mathfrak{f}^*(F)$ and $\mathfrak{f}^*(G)$. Let $\tilde{\mathfrak{f}} : \tilde{Y} \to \tilde{X}$ be the map between their universal covering spaces $\tilde{Y}$ and $\tilde{X}$. Then

$$Mas_{(F,G)} \circ \tilde{\mathfrak{f}} = Mas_{(\mathfrak{f}^*(F), \mathfrak{f}^*(G))}. \tag{A.6}$$

Now let L be a Lagrangian submanifold in the cotangent bundle $T^*(X)$. Then the restriction of the tangent bundle $T(T^*(X))$ to L is a symplectic vector bundle together with two Lagrangian subbundles, the tangent bundle of L, $T(L)$, and the restriction of $\mathrm{Ker}\, d\pi^X$ on L, the vertical subbundle with respect to the projection map $\pi^X : T^*(X) \to X$.

Hence we have a cohomology class $\mathfrak{m}_{\left(\mathrm{Ker}\, d\pi^X, T(L)\right)}$ as a homomorphism

$$\mathfrak{m}_{\left(\mathrm{Ker}\, d\pi^X, T(L)\right)} : \pi_1(L) \to \mathbb{Z},$$

which we will denote simply by $\mathfrak{m}_L$.

Remark 3. *The definition of Maslov index for arbitrary paths given in* [16] *has a modification term at the end points and is not natural one. In* [6] *it was noticed for the first time without any modification term and in* [1] *and* [4]*, it was given based on the arguments by* [12] *including the infinite dimensional symplectic Hilbert space case.*

A.2. **Three remarks.**

Proposition A.4. *Let L be a compact Lagrangian submanifold in $T_0^*(X)$. Then for any positive real number $A > 0$ and any closed curve $\{\gamma\}$ in L,*

$$< \mathfrak{m}_L,\ \gamma > = < \mathfrak{m}_{A\cdot L},\ A \cdot \gamma > . \tag{A.7}$$

Proof. Since the Maslov index $< \mathfrak{m}_L, \gamma >$ for a path $\{\gamma\}$ is defined based on the data

$$\left\{ \dim \left(T_{\gamma(t)}(L) \bigcap \left(\mathrm{Ker}\, d\pi^X\right)_{\gamma(t)} \right) \right\}_{t \in [0,1]}$$

and it holds that

$$\dim \left(T_{\gamma(t)}(\lambda) \bigcap \left(\mathrm{Ker}\, d\pi^X\right)_{\gamma(t)} \right) = \dim \left(T_{A\cdot\gamma(t)}(A \cdot \lambda) \bigcap \left(\mathrm{Ker}\, d\pi^X\right)_{A\cdot\gamma(t)} \right)$$

for any t, since the dilation $A\cdot : T_0^*(X) \longrightarrow T_0^*(X)$, $(x;\,\xi) \longmapsto A\cdot(x;\,\xi) = (x;\, A\cdot\xi)$, $A > 0$, is a diffeomorphism. Hence (A.7) holds. ∎

Let L be a compact Lagrangian submanifold appearing in the Eigenvalue Theorem 3.1 and $\hat{L}$ the corresponding conic Lagrangian submanifold in $T_0^*(X) \times T_0^*(U(1))$. We also note the obvious free action of the group $\mathbb{Z}_k$ (the cyclic group of order k) on the space $U(1)$ is lifted to $T_0^*(U(1))$ and the lifted action leaves invariant the Lagrangian submanifold $\hat{L}$. Moreover the Maslov class $\mathfrak{m}_{\hat{L}}$ is invariant under this action. Hence

Proposition A.5. *The conic Lagrangian submanifold $\hat{L}$ is descended to the conic Lagrangian submanifold $\hat{L}/\mathbb{Z}_k$ in $T_0^*(X) \times T_0^*(U(1)/\mathbb{Z}_k)$, and the pull-back of the Maslov class $\mathfrak{m}_{\hat{L}/\mathbb{Z}_k}$ to $\hat{L}$ coincides with $\mathfrak{m}_{\hat{L}}$.*

At the end of the Appendix, we remark a construction called "α-construction" given in [8] in relation to our definition of the Maslov index for arbitrary paths.

First we recall the construction of the universal covering space $\Phi : \tilde{X} \to X$ from the very beginning. The space $\tilde{X}$ consists of homotopy classes of paths $\{\gamma\}$ starting from a fixed common point $\gamma(0) = x_0 \in X$. So, for each homotopy class $[\gamma] \in \tilde{X}$, $\Phi([\gamma]) = \gamma(1)$, the end point. Then $\Phi : \tilde{X} \to X$ is a principal bundle with the structure group $\pi_1(X) = \pi_1(X, x_0)$ (homotopy classes of loops with the base point x_0) so that there is an open covering $\{U_\ell\}$ of X and homeomorphisms $\{\phi_\ell\}$

$$\phi_\ell : U_\ell \times \pi_1(X) \overset{\sim}{\to} \Phi^{-1}(U_\ell)$$

which we define as follows:

For any point $x \in X$ we take a sufficiently small "*simply*" connected open neighborhood U_x (existence of such neighborhoods is assumed) and fix a path $\{\sigma_x\}$ connecting x_0 and x, $\sigma_x(0) = x_0$, $\sigma_x(1) = x$. Let $y \in U_x$ and we connect x and y by an arbitrary fixed path $s(U_x, y)$ in U_x. Since U_x is simply connected, the homotopy class of the path $s(U_x, y)$ is uniquely determined. Then, let $\phi_x : U_x \times \pi_1(X, x_0) \ni (y, [\gamma]) = [\gamma * \sigma_x * s(U_x, y)]$, where we mean by $[\gamma * \sigma_x * s(U_x, y)]$ the homotopy class of catenations of the loop γ and the paths σ_x and $s(U_x, y)$ with the end point y.

Let $y \in U_x \cap U_{x'}$. Then,

$$\phi_x(y, [\gamma]) = \phi_{x'}(y, [\mu]) \tag{A.8}$$

implies that there exists a unique element $[C_{x,x'}] \in \pi_1(X, x_0)$ such that the paths

$$\{C_{x,x'} * \gamma * \sigma_x * s(U_x, y)\} \text{ and } \{\mu * \sigma_{x'} * s(U_{x'}, y)\} \text{ are homotopic} \tag{A.9}$$

and the correspondence $U_x \cap U_{x'} \ni y \longmapsto C_{x,x'}(y)$ gives the transition functions of the principal bundle $\Phi : \tilde{X} \to X$. If we can connect $y_1, y_2 \in U_x \cap U_{x'}$ by a path in $U_x \cap U_{x'}$, then the loops $\{\sigma_x * s(U_x, y_1) * s(U_{x'}, y_1)^{-1} * \sigma_{x'}{}^{-1}\}$ and $\{\sigma_x * s(U_x, y_2) * s(U_{x'}, y_2)^{-1} * \sigma_{x'}{}^{-1}\}$ are homotopic. Hence we see that the correspondence $C_{x,x'} : U_x \cap U_{x'} \ni y \mapsto C_{x,x'} = C_{x,x'}(y)$ is a locally constant map on $U_x \cap U_{x'}$ taking values in $\pi_1(X) = \pi_1(X, x_0)$.

Now let $\Psi : E \to X$ be a symplectic vector bundle over a suitable space X as before with two Lagrangian subbundles F and G.

Then the integer valued locally constant functions $\{Mas_{F,G}(\{C_{x,x'}\})\}$,

$$Mas_{F,G} \circ C_{x,x'} : U_x \cap U_{x'} \longrightarrow \pi_1(X) \overset{Mas_{F,G}}{\longrightarrow} \mathbb{Z}$$

define a 1-$\check{C}$ech cocycle which cohomology class in $\check{H}^1(X, \mathbb{Z})$ corresponds to the Maslov class $\mathfrak{m}_{F,G}$ (cf. [2]).

Here we explain a realization of a set of the transition functions $\{C_{x,x'}\}$ given in terms of "*Hörmander index*".

Let λ_1 and λ_2 be two Lagrangian subspaces in $\mathbb{C}^n$ and we consider two Lagrangian subspaces μ, ν such that each of μ and ν are transversal to both of λ_1 and λ_2. Then the index called "*Hörmander index*" (cf. [8]) can be defined as

$$\sigma(\lambda_1, \lambda_2; \mu, \nu) := Mas(\{\gamma\}, \lambda_2) - Mas(\{\gamma\}, \lambda_1), \tag{A.10}$$

where $\{\gamma\}$ is a path connecting μ and ν. Then by the fact that the Maslov index for loop does not depend on the particular Maslov cycle $\mathcal{M}_\lambda$, the integer (A.10) is well-defined. In fact, for two paths $\{\gamma\}$ and $\{\gamma'\}$ connecting μ and ν we have

$$Mas([\gamma * \gamma'^{-1}]) = Mas(\{\gamma\}, \lambda_1) - Mas(\{\gamma'\}, \lambda_1) = Mas(\{\gamma\}, \lambda_2) - Mas(\{\gamma'\}, \lambda_2).$$

Let fix a point $x, x' \in X$, and take simply connected open neighborhoods $U_x \ni x$ and $U_{x'} \ni x'$ such that the principal bundle $\Phi : \tilde{X} \to X$ is trivial on each of them as before. Then for $y \in U_x \cap U_{x'}$, the difference

$$Mas_{F,G}(\{\mu * \sigma_{x'} * s(U_{x'}, y)\}) - Mas_{F,G}(\{\gamma * \sigma_x * s(U_x, y)\})$$

coincides with the Maslov index $Mas_{F,G}(\{C_{x,x'}\})$:

$$\begin{aligned} &Mas_{F,G}(\{\mu * \sigma_{x'} * s(U_{x'}, y)\}) - Mas_{F,G}(\{\gamma * \sigma_x * s(U_x, y)\}) \\ &= Mas_{F,G}(\{C_{x,x'}\}) = \mathfrak{m}_{F,G}([C_{x,x'}]), \end{aligned}$$

where we assumed (A.9).

Then for $y \in U_x \cap U_{x'}$, we can find two Lagrangian subspaces μ, ν in E_y such that each of the Lagrangian subspaces F_y and G_y is transversal to μ and ν, and moreover the Hörmander index $\sigma(F_y, G_y; \mu, \nu) = \mathfrak{m}_{F,G}([C_{x,x'}])$, since for any pair of Lagrangian subspaces F_y and G_y the values $\sigma(F_y, G_y; \mu, \nu)$ can take any integer by taking the suitable Lagrangian subspaces μ, ν in E_y transversal to F_y and G_y. Then under the local trivialization of the associated the Lagrangian-Grassmannian bundle $\Psi_\Lambda : \Lambda(E) \to X$ (the fibers $\Lambda(E)_x = {\Psi_\Lambda}^{-1}(x)$ are the Lagrangian-Grassmannian $\cong \Lambda(n)$) the transversality condition for μ and ν at y allows us to leave the value $\sigma(F_z, G_z; \mu, \nu)$ invariant around the point y and coincides with the value $\mathfrak{m}_{F,G}([C_{x,x'}])$. Hence a collection of Hörmander index $\{\sigma(F_z, G_z; \mu, \nu)\}$ is a realization of a set of the transition functions of the $\check{C}$eck cohomology class $\in \check{H}^1(X, \mathbb{Z}_X)$ corresponding to the Maslov class $\mathfrak{m}_{F,G} \in H^1(X)$.

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