Double fibration transforms with conjugate points

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Double fibration

Following Mazzucchelli-Salo-Tzou [2], we introduce double fibration transforms.

- Let $\mathcal{G}$ and X be oriented smooth manifolds without boundaries. $N := \dim(\mathcal{G})$ and $n := \dim(X)$. Denote by $d\mathcal{G}$ and dX the orientation forms of $\mathcal{G}$ and X respectively.
- Let Z be an oriented embedded submanifold of $\mathcal{G} \times X$, and let dZ be the orientation form.
- Assume that $N + n > \dim(Z) > N \geqq n \geqq 2$, and set $n' := \dim(Z) - N$ and $n'' := n - n'$. Then $\dim(Z) = N + n'$, $n = n' + n''$ and $n', n'' = 1, \ldots, n-1$.

$$\begin{tikzcd} & Z \arrow[dl, "\pi_{\mathcal{G}}"'] \arrow[dr, "\pi_X"] & \\ \mathcal{G} & & X \end{tikzcd} \qquad \begin{tikzcd} & (N^*Z \setminus 0)' \arrow[dl, "\pi_{\mathrm{L}}"'] \arrow[dr, "\pi_{\mathrm{R}}"] & \\ T^*\mathcal{G} \setminus 0 & & T^*X \setminus 0 \end{tikzcd}$$

- We assume that Z is a double fibration, that is, the natural projections $\pi_{\mathcal{G}} : Z \to \mathcal{G}$ and $\pi_X : Z \to X$ are submersions respectively.

Orientation forms on $G_z := \pi_x \circ \pi_{\mathcal{G}}^{-1}(z)$ and $H_x := \pi_{\mathcal{G}} \circ \pi_X^{-1}(x)$

- $G_z := \pi_x \circ \pi_{\mathcal{G}}^{-1}(z)$ becomes an n'-dim submanifold of X for any $z \in \mathcal{G}$, and $H_x := \pi_{\mathcal{G}} \circ \pi_X^{-1}(x)$ forms an $(N - n'')$-dim submanifold of $\mathcal{G}$ for any $x \in X$.
- Fix arbitrary $(z, x) \in Z$, and let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_N\}$ be bases of T_xX and $T_z\mathcal{G}$ respectively such that

$$T_{(z,x)}Z = \mathrm{span}\langle v_1, \ldots, v_{n'}, w_1, \ldots, w_N\rangle = \mathrm{span}\langle v_1, \ldots, v_n, w_1, \ldots, w_{N-n''}\rangle.$$

The induced orientation forms dG_z on G_z and dH_x on H_x are given by

$$\begin{aligned} dG_z\big(d\pi_X(v_1), \ldots, d\pi_X(v_{n'})\big) &: = dZ_{\pi_{\mathcal{G}}^{-1}(z)}(v_1, \ldots, v_{n'}) \\ &= \frac{dZ(v_1, \ldots, v_{n'}, w_1, \ldots, w_N)}{d\mathcal{G}\big(d\pi_{\mathcal{G}}(w_1), \ldots, d\pi_{\mathcal{G}}(w_N)\big)}, \\ dH_x\big(d\pi_{\mathcal{G}}(w_1), \ldots, d\pi_X(w_{N-n''})\big) &: = dZ_{\pi_X^{-1}(x)}(w_1, \ldots, w_{N-n''}) \\ &= \frac{dZ(v_1, \ldots, v_n, w_1, \ldots, w_{N-n''})}{dX\big(d\pi_X(v_1), \ldots, d\pi_X(v_n)\big)}. \end{aligned}$$

Double fibration transform

Suppose that a weight function $\kappa(z,x) \in C^\infty(\mathcal{G}\times X)$ is nowhere vanishing. A double fibration transform $\mathcal{R}$ associated with the double fibration Z is defined by

$$\mathcal{R}f(z) := \left(\int_{G_z} \kappa(z,x)\frac{f}{|dX|^{1/2}}(x)dG_z(x)\right)|d\mathcal{G}(z)|^{1/2}$$

for $f \in \mathscr{D}(X,\Omega_X^{1/2})$. The adjoint $\mathcal{R}^*$ is given by

$$\mathcal{R}^*u(x) = \left(\int_{H_x} \overline{\kappa(z,x)}\frac{u}{|d\mathcal{G}|^{1/2}}(z)dH_x(z)\right)|dX(x)|^{1/2}$$

for $u \in \mathscr{D}(\mathcal{G},\Omega_{\mathcal{G}}^{1/2})$. Then we deduce that

$$\mathcal{R} : \mathscr{D}(X,\Omega_X^{1/2}) \to \mathscr{E}(\mathcal{G},\Omega_{\mathcal{G}}^{1/2}), \quad \mathcal{R}^* : \mathscr{D}(\mathcal{G},\Omega_{\mathcal{G}}^{1/2}) \to \mathscr{E}(X,\Omega_X^{1/2}),$$

are continuous linear mappings, and so are

$$\mathcal{R} : \mathscr{E}'(X,\Omega_X^{1/2}) \to \mathscr{D}'(\mathcal{G},\Omega_{\mathcal{G}}^{1/2}), \quad \mathcal{R}^* : \mathscr{E}'(\mathcal{G},\Omega_{\mathcal{G}}^{1/2}) \to \mathscr{D}'(X,\Omega_X^{1/2}).$$

More precisely $\mathcal{R}$ and $\mathcal{R}^*$ are elliptic Fourier integral operators.

Mapping properties of double fibration transforms

Theorem 1 ([2, Theorem 2.2] & [Hörmander IV, Theorem25.2.2)

Suppose that Z is a double fibration with $\dim(Z) = N + n'$. Then $\mathcal{R}$ and $\mathcal{R}^*$ are elliptic Fourier integral operators of order $-(N+2n'-n)/4$ with canonical relations $(N^*Z \setminus 0)'$ and $((N^*Z \setminus 0)^T)'$ respectively. More precisely

$$\mathcal{R} \in \mathcal{I}^{-(N+2n'-n)/4}(\mathcal{G} \times X, N^*Z \setminus 0; \Omega^{1/2}_{\mathcal{G}\times X}),$$
$$\mathcal{R}^* \in \mathcal{I}^{-(N+2n'-n)/4}(X \times \mathcal{G}, (N^*Z \setminus 0)^T; \Omega^{1/2}_{X\times\mathcal{G}}),$$

where

$$\begin{aligned} N^*Z \setminus 0 &= \big\{\big(z, A(z,x)\eta, x, \eta\big) : (z,x) \in Z, \eta \in N^*_x G_z \setminus \{0\}\big\} \\ &= \big\{\big(z, \zeta, x, B(z,x)\zeta\big) : (z,x) \in Z, \zeta \in N^*_z H_x \setminus \{0\}\big\}, \end{aligned}$$

$A(z,x) \in \mathrm{Hom}(N^*_x G_z, T^*_z \mathcal{G})$ and $B(z,x) \in \mathrm{Hom}(N^*_z H_x, T^*_x X)$ smoothly depend on $(z,x) \in Z$ respectively.

Preliminaries

For local coordinates $(z, x) = (z', z'', x', x'') \in \mathbb{R}^{N-n''} \times \mathbb{R}^{n''} \times \mathbb{R}^{n'} \times \mathbb{R}^{n''}$, There exist $\mathbb{R}^{n''}$-valued functions $\phi(z, x')$ and $b(x, z')$ such that we have locally

$$Z = \{x'' = \phi(z, x')\} = \{z'' = b(x, z')\}.$$

Lemma 2 ([2, Lemmas 2.4, 2.5 and 2.6])

$$N^*_{(z,x)}Z = \left\{ \left(-\phi_z(z, x')^T \eta'', (-\phi_{x'}(z, x')^T \eta'', \eta'') \right) : \eta'' \in \mathbb{R}^{n''} \right\},$$

$$A(z, x) \begin{bmatrix} -\phi_{x'}(z, x')^T \\ I_{n''} \end{bmatrix} \eta'' = -\phi_z(z, x')^T \eta'', \quad \eta'' \in \mathbb{R}^{n''}.$$

Similar results hold for $b(x, z')$ and $B(z, x)$.

Variation fields and conjugate points

Fix arbitrary $(z, w) \in T\mathcal{G}$, and consider a curve in $\mathcal{G}$ of the form

$$z(s) = z + sw + \mathcal{O}(s^2) \quad \text{near} \quad s = 0.$$

Then $(G_{z(s)})$ is said to be a variation of G_z, and the variation field $J_w : G_z \to (N_x^* G_z)^*$ associated to $(G_{z(s)})$ is defined by

$$J_w(x) := A(z, x)^* w \simeq -\phi_z(z, x')w \in (N_x^* G_z)^* \simeq N_x G_z = T_x X / T_x G_z$$

for $x \in G_z$. Note that

$$A(z, x)^* \in \mathrm{Hom}\big(T_z\mathcal{G}, (N_x^* G_z)^*\big) \simeq \mathrm{Hom}\big(T_z\mathcal{G}, N_x G_z\big), \quad (z, x) \in Z,$$

For $z \in \mathcal{G}$ and $x, y \in G_z$, set

$$V_z(x, y) := \{J_w(x) : w \in T_z\mathcal{G}, J_w(y) = 0\}.$$

Note that $\dim\big(V_z(x, y)\big) \leqq n''$ holds since $\mathrm{rank}\big(A(z, x)^*\big) = n''$, and $\dim\big(V_z(x, y)\big) = \dim\big(V_z(y, x)\big)$ holds for any $z \in \mathcal{G}$ and $x, y \in G_z$.
cf. If $x = \exp_y(tu)$, then $J_w(x) \simeq Y(t) := tD\exp_y(tu)w$.

Z-conjugate triplets

Definition 3

Suppose that Z is a double fibration and $N \geqq 2n''$. Let $k = 1, \dots, n''$.

- Z-conjugate triplet of degree k: Let $z \in \mathcal{G}$ and let $x, y \in G_z$ with $x \neq y$. We say that $(z; x, y)$ is a Z-conjugate triplet of degree k if $\dim\big(V_z(x, y)\big) = n'' - k$.
- Regular Z-conjugate triplet of degree k: We say that a Z-conjugate triplet $(z; x, y)$ of degree k is regular if there exit a nbd U_x of x in X, a nbd U_y of y in X, and a nbd W_z of z in $\mathcal{G}$ such that any Z-conjugate triplet $(z'; x', y') \in W_z \times U_x \times U_y$ is also of degree k. The set of all the regular Z-conjugate triplets of degree k is denoted by $C_{R,k}$.
- The set of all the Z-conjugate triplets which are not regular is denoted by C_S.

Lemma 4

Suppose that Z is a double fibration and $N \geqq 2n''$. For any $k = 1, \dots, n''$, $C_{R,k}$ is an $(N + n')$-dimensional embedded submanifold of $\mathcal{G} \times X \times X$.

Normal operators without conjugate points

$$\mathcal{R}^*\mathcal{R}f(x) = \left(\iint_{H_x \times G_z} \overline{\kappa(z,x)}\kappa(z,y)\frac{f}{|dX|^{1/2}}(y)dG_z(y)dH_x(z)\right)|dX(x)|^{1/2}, \quad x \in X.$$

Set $\mathcal{C} := (N^*Z \setminus 0)'$, which is the canonical relation of $\mathcal{R}$.

Theorem 5

Suppose that Z is a double fibration. In addition, we assume $N \geqq 2n''$ and the following:

- $\pi_X : Z \to X$ *is proper, and* $\pi_X^{-1}(x)$ *is connected for any* $x \in X$.
- *There are no Z-conjugate triplets, and $D\pi_L$ is injective at all $(z, \zeta, x, \eta) \in \mathcal{C}$.*

Then $\mathcal{C}^T \circ \mathcal{C}$ is a clean intersection with excess $e = N - n$, and $\mathcal{R}^*\mathcal{R}$ is an elliptic pseudodifferential operator of order $-n'$ on X.

Proof: Some lemmas in [2] and the assumptions guarantee the Bolker condition. $\square$

Known results on geodesic X-ray transforms with conjugate points

- Let (M,g) be a compact Riemannian manifold with strictly convex boundary. Set $\gamma_w(t) := \exp_{\pi_M(w)}(tw)$ for $w \in \partial_- SM$. Consider the geodesic X-ray fransform

$$\mathcal{X} f(w) := \left(\int_0^{\tau(w)} \kappa(\gamma_w(t), \dot{\gamma}_w(t)) \frac{f}{|dM|^{1/2}} (\gamma_w(t))\, dt \right) |d\partial_- SM(w)|^{1/2}.$$

- Stefanov and Uhlmann (2012) [3]: If $v_0 = |v_0|\theta_0$ is a fold conjugate vector at p_0, and v_0 is the only singularity of $\exp_{p_0}(v)$ on γ_{θ_0} near p_0, then the localized normal operator is decomposed as

$$\mathcal{X}^* \chi \mathcal{X} = A + F \quad \text{near} \quad p_0,$$

where A is a PsDOs of order -1, and F is a *FIO* of order $-n/2$.

- Holman and Uhlmann (2018) [1]: If $C_S = \emptyset$, then

$$\mathcal{X}^* \mathcal{X} = A + \sum_{k=1}^{n-1} \sum_{\alpha=1}^{M_k} F_{k,\alpha},$$

where A is a PsDOs of order -1, and F is a *FIO* of order $-(n-k+1)/2$.

Normal operators with conjugate points

Theorem 6

Suppose that Z is a double fibration, $C_S = \emptyset$, $N \geqq (n-1) + n''$ and the following:

- *π_X is proper, and $\pi_X^{-1}(x)$ is connected for any $x \in X$.*
- *If $\pi_L^{-1}((z,\zeta)) = \{(z,\zeta,x,\eta)\}$ for $(z,\zeta,x,\eta) \in \mathcal{C}$, then $D\pi_L|_{(z,\zeta,x,\eta)}$ is injective.*

Then we have a decomposition of $\mathcal{R}^*\mathcal{R}$ of the form

$$\mathcal{R}^*\mathcal{R} = A + \sum_{k=1}^{n''} \sum_{\alpha \in \Lambda_k} F_{k,\alpha},$$

where A is an elliptic PsDO of order $-n'$ on X,
$F_{k,\alpha}$ is a FIO in $\mathcal{I}^{-(n+n'-k)/2}(X \times X, \mathcal{C}'_{A_{k,\alpha}}; \Omega^{1/2}_{X \times X})$ with some canonical relation of $\mathcal{C}_{F_{k,\alpha}}$,
associated to the decomposition of connected components $C_{R,k} = \bigcup_{\alpha \in \Lambda_k} C_{R,k,\alpha}$.

Outline of the proof

- $\mathcal{R}^*\mathcal{R}$ is given by

$$\frac{\mathcal{R}^*\mathcal{R}f}{|dX|^{1/2}}(x) = \iint_{H_x \times G_z} \overline{\kappa(z,x)}\kappa(z,y)\frac{f}{|dX|^{1/2}}(y)dG_z(y)dH_x(z).$$

- Follow the idea of Holman and Uhlmann [1]: a partition of unity of $\mathcal{G} \times X \times X$.
- Set $C_\delta := \{(z;x,x) : z \in \mathcal{G}, x \in X\}$, which is related to the elliptic term.
- $C_{R,k,\alpha}$ are disjoint since $C_S = \emptyset$, so are $C_{R,k,\alpha}$ and C_δ.
  Pick up disjoint nbds $U_{k,\alpha}$ and U_δ of $C_{R,k,\alpha}$ and C_δ respectively in $\mathcal{G} \times X \times X$.
- We can find an open set U_0 in $\mathcal{G} \times X \times X$ such that

$$U_0 \bigcup U_\delta \bigcup (\cup U_{k,\alpha}) = \mathcal{G} \times X \times X, \quad U_0 \bigcap \Big(C_\delta \bigcup (\cup C_{R,k,\alpha})\Big) = \emptyset.$$

- Pick up a partition of unity subordinated to $\{U_0, U_\delta, U_{k,\alpha}\}$, and split the Schwartz kernel of $\mathcal{R}^*\mathcal{R}$. U_0-part of $\mathcal{R}^*\mathcal{R}$ is a smoothing operator, and is absorbed in U_δ-part A.

References

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