Semiclassical distribution of resonances associated with an energy-level crossing

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joint work with

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System of 1D Schrödinger equations

$$P(h)u = Eu$$

where $P(h)$ is a 2×2 matrix-valued Schrödinger operator:

$$P(h) = \begin{pmatrix} P_1(h) & hW \\ hW^* & P_2(h) \end{pmatrix}$$

with

$$P_j(h) = h^2 D_x^2 + V_j(x), \quad (D_x = \frac{1}{i}\frac{d}{dx}),$$
$$W = W(x, hD_x) = r_0(x) + ir_1(x)hD_x$$

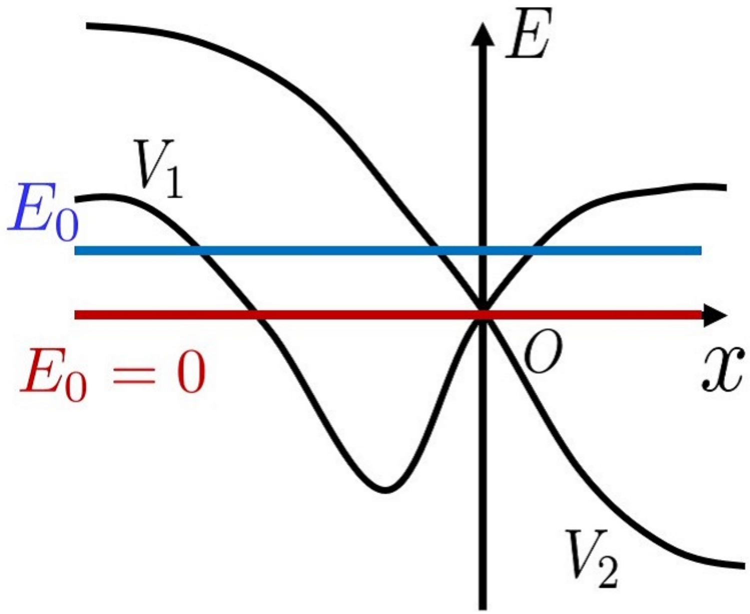
h : small positive parameter (semiclassical parameter)

Problem and Contents

We consider a one-dimensional model where $V_1(x)$ and $V_2(x)$ cross each other, and study the asymptotic distribution as $h \to 0$ of (quantum) resonances near a fixed energy $E_0 \in \mathbb{R}$.

1st part : E_0 is near crossing level
Fujiié-Martinez-W. 2016, 2017

2nd part : E_0 is above crossing level
work in progress



Potentials V_1 V_2

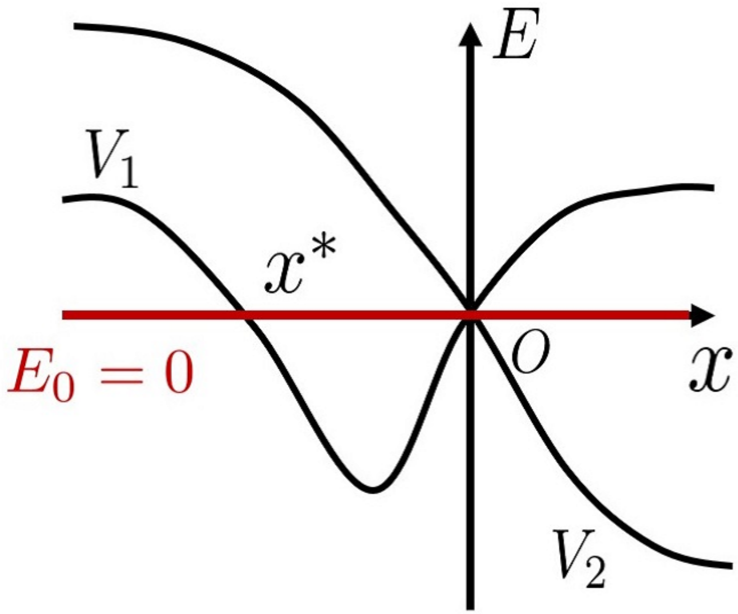
Assumptions

(A1) $\boldsymbol{V_1(x), V_2(x)}$: real-valued and real analytic on $\mathbb{R}$ and analytic in $\boldsymbol{\Gamma}$:

$$\boldsymbol{\mathcal{S} := \{x \in \mathbb{C}\,;\ |\mathrm{Im}\, x| < \exists\delta_0 \langle \mathrm{Re}\, x\rangle\}}$$

(A2) $\boldsymbol{\exists V_1^+, V_1^-, V_2^- > 0}$, $\boldsymbol{\exists V_2^+ < 0}$ s.t.

$\boldsymbol{V_j(x) \to V_j^\pm}$ as $\mathrm{Re}\, \boldsymbol{x} \to \pm\infty$ in $\boldsymbol{\mathcal{S}}$



Potentials V_1 V_2

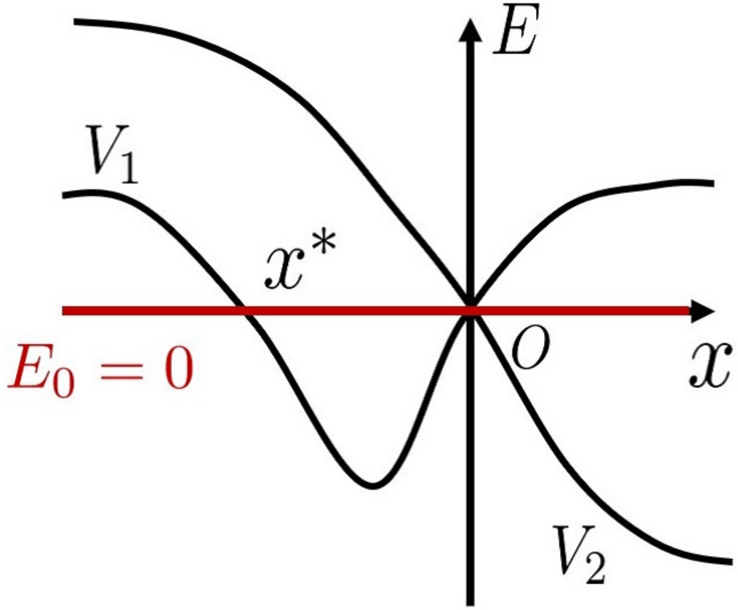
Assumptions

(A3) $\exists x^* < 0$ s.t.

- $V_1, V_2 > 0$ on $(-\infty, x^*)$
- $V_1 < 0 < V_2$ on $(x^*, 0)$
- $V_2 < 0 < V_1$ on $(0, \infty)$

Let x^* and 0 be simple zeros and for simplicity

$$V_1'(x^*) < 0, \quad V_1'(0) = 1,$$
$$V_2'(0) = -1.$$



Potentials V_1 V_2

Assumptions

(A4) $W(x, hD_x)$: 1st order differential operator s.t.

$$W(x, hD_x) = r_0(x) + ir_1(x)hD_x$$

where $r_0(x)$ and $r_1(x)$ are analytic and bound in $\mathcal{S}$.

In the 1st Part, we treat two cases assuming **(A1) - (A4)** & (**(B1)** or **(B2)**)

(B1) $r_0(x)$ is real-valued on $\mathbb{R}$.

(B2) $r_0(x) \equiv 0$ and $r_1(x)$ is real-valued on $\mathbb{R}$.

Def $\boldsymbol{E \in \mathbb{C}_-}$ is a resonance iff $\boldsymbol{\exists u(x,h) \neq 0}$ s.t.

$$\begin{cases} \boldsymbol{Pu = Eu} \\ \quad \boldsymbol{u} : \text{"out-going"} \end{cases}$$

$\boldsymbol{u}$: "out-going" $\Leftrightarrow \boldsymbol{u(xe^{i\theta}) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})}$ for small $\boldsymbol{\theta > 0}$.

Def Res$\boldsymbol{(P) := \{}$resonances of $\boldsymbol{P\} \subset \mathbb{C}_-}$.

Remark If $\boldsymbol{W \equiv 0}$, denoting a small interval near $\boldsymbol{0}$ by $\boldsymbol{I \subset \mathbb{R}}$, we know

$$\boldsymbol{\sigma(P) \cap I = \sigma_{\text{disc}}(P_1) \cup \sigma_{\text{ess}}(P_2).}$$

Perturbations (i.e. $\boldsymbol{W, W^*}$) create resonances near $\boldsymbol{\sigma_{\text{disc}}(P_1)}$.

We expect to find resonances in the complex nbd of $\boldsymbol{\sigma}_{\text{disc}}(\boldsymbol{P_1})$ when $\boldsymbol{h} \downarrow \mathbf{0}$ and we want to study in particular the imaginary part of the resonance (so-called width of resonance) .

Physical interpretation of quantum resonances

$$\text{eigenvalue} \in \mathbb{R} \quad \longleftrightarrow \quad \text{bounded state}$$

$$\text{resonance} \in \mathbb{C}_{-} \quad \longleftrightarrow \quad \text{quasi-bounded state}$$

Note that the life span of trapped quantum particle $\propto \mathbf{1}/\text{Im(resonance)}$

Moreover we also expect that

Im (resonance) $\leftrightarrow$ “Strength of trapping” of classical orbit

Box of resonances and Action from a simple well

Let $C_0 > 0$ be (large) fixed.

$$\mathcal{D}_h(C_0) := [-C_0 h^{2/3}, C_0 h^{2/3}] - i[0, C_0 h]$$

For $E \in \mathbb{C}_-$ small enough, we define

$$\mathcal{A}(E) := \int_{x_1^*(E)}^{x_1(E)} \sqrt{E - V_1(t)}\, dt$$

where $x_1(E)$ (resp. $x_1^*(E)$) is the unique root of $V_1(x) = E$ close to 0 (resp. x^*).

Let us characterize the distribution of resonances by using e.v. of P_1.

$$\lambda_k(h) := \frac{-\mathcal{A}(0) + (k+\frac{1}{2})\pi h}{\mathcal{A}'(0)h^{\frac{2}{3}}} \in \mathbb{R} \quad (k \in \mathbb{Z}).$$

Remark Bohr-Sommerfeld quantization.condition. for P_1, i.e.

$$\mathcal{A}(E) = \left(k + \frac{1}{2}\right)\pi h + \mathcal{O}(h^2) \quad (h \downarrow 0).$$

$\lambda_k(h)$ are roots of the equation $\mathcal{A}(0) + \mathcal{A}'(0)E = (k+\frac{1}{2})\pi h$ with $E = \lambda_k(h)h^{\frac{2}{3}}$.

Remark $E_k(h) = \lambda_k(h)h^{\frac{2}{3}} \in \mathcal{D}_h$ implies that k is taken large enough as $\mathcal{O}(h^{-1})$.

Theorem (Fujiié-Martinez-W. 2016)

Assume **(A1)**-**(A4)** and **(B1)**. For h small enough, one has

$$\mathrm{Res}(P(h)) \cap \mathcal{D}_h = \{E_k(h); k \in \mathbb{Z}\} \cap \mathcal{D}_h,$$

where the $E_k(h)$'s are complex numbers satisfying

$$\mathrm{Re}\, E_k = \lambda_k h^{\frac{2}{3}} - \frac{\mathcal{A}''(0)}{2\mathcal{A}'(0)} \lambda_k^2 h^{\frac{4}{3}} + \mathcal{O}(h^{\frac{5}{3}})$$

$$\mathrm{Im}\, E_k = -\frac{\pi^2 r_0(0)^2}{2^{\frac{2}{3}} \mathcal{A}'(0)} \left(\mathbf{Ai}\,(-2^{\frac{2}{3}} \lambda_k)\right)^2 h^{\frac{5}{3}} + \mathcal{O}(h^2)$$

where $\mathbf{Ai}$ stands for the Airy function

$$\mathbf{Ai}\,(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left[ix\xi + \xi^3/3\right] d\xi.$$

Theorem (Fujiié-Martinez-W. 2017)

Assume **(A1)**-**(A4)** and **(B2)**. For h small enough, one has

$$\mathrm{Res}(P(h)) \cap \mathcal{D}_h = \{E_k(h); k \in \mathbb{Z}\} \cap \mathcal{D}_h,$$

where the $E_k(h)$'s are complex numbers satisfying

$$\mathrm{Re}\, E_k = \lambda_k h^{\frac{2}{3}} - \frac{\mathcal{A}''(0)}{2\mathcal{A}'(0)} \lambda_k^2 h^{\frac{4}{3}} - \frac{\mathcal{A}^{(3)}(0)}{6\mathcal{A}'(0)} \lambda_k^3 h^{\frac{6}{3}} + \mathcal{O}(h^{\frac{7}{3}})$$

$$\mathrm{Im}\, E_k = -\frac{\pi^2 r_1(0)^2}{2^{\frac{4}{3}} \mathcal{A}'(0)} \left(\mathrm{Ai}'(-2^{\frac{2}{3}}\lambda_k)\right)^2 h^{\frac{7}{3}} + \mathcal{O}(h^{\frac{8}{3}})$$

Related works (Focus on the width of resonances and the fixed energy-level)

The imaginary part of resonances was shown to be $\sim$

Crossing case

— Near the bottom of the well V_1 (Grigis-Martinez 2014); $e^{-2S/h}$,
S: Agmon distance $\min(V_1 - E_0, V_2 - E_0)dx^2$

— Below the crossing (Ashida 2018); $e^{-2S/h}$

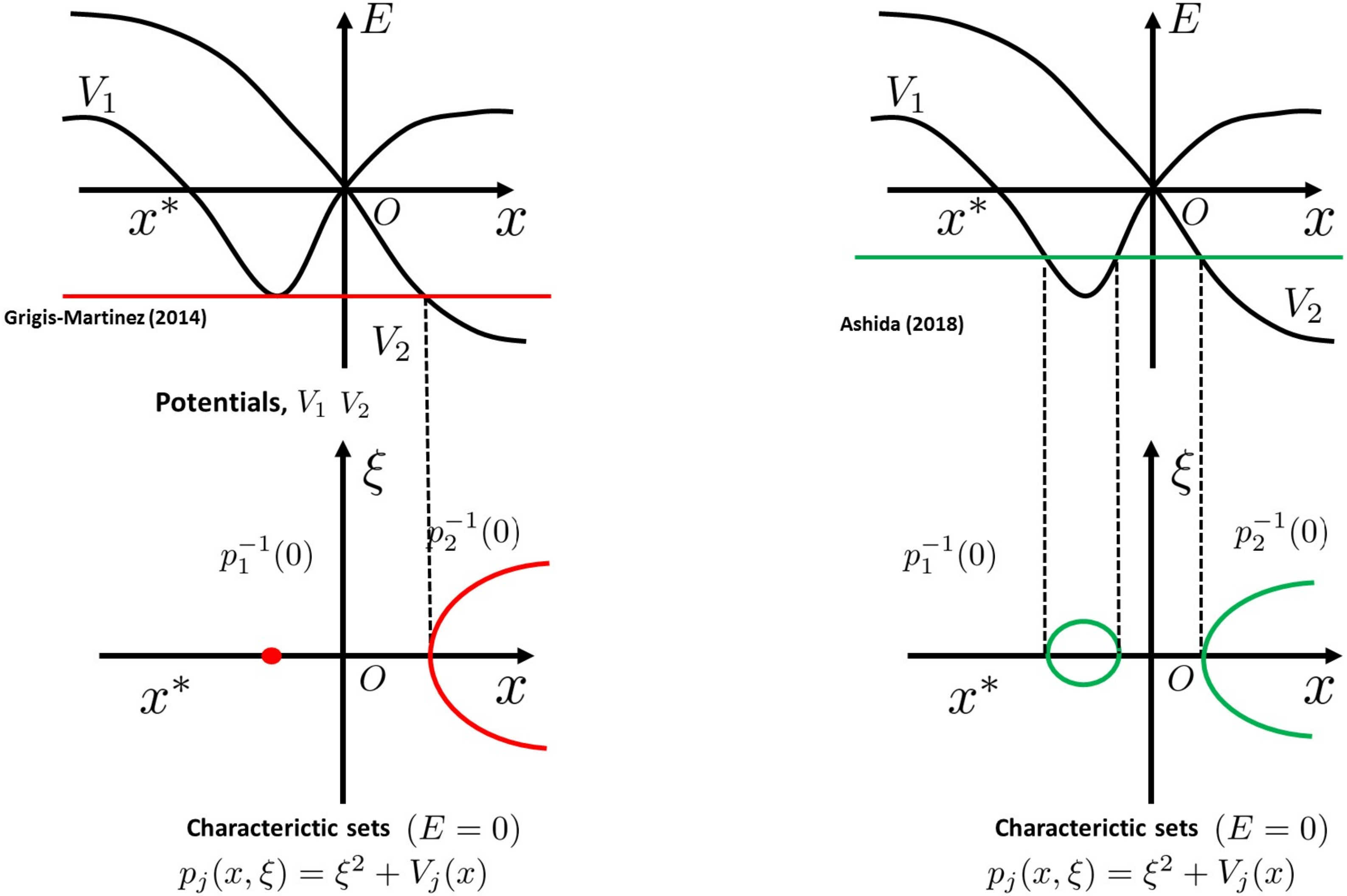
— Near the crossing

* $*$ elliptic interaction (F-M-W 2016); $h^{5/3}$
* $*$ vector field interaction (F-M-W 2017); $h^{7/3}$

— Above the crossing (work in progress); h^2

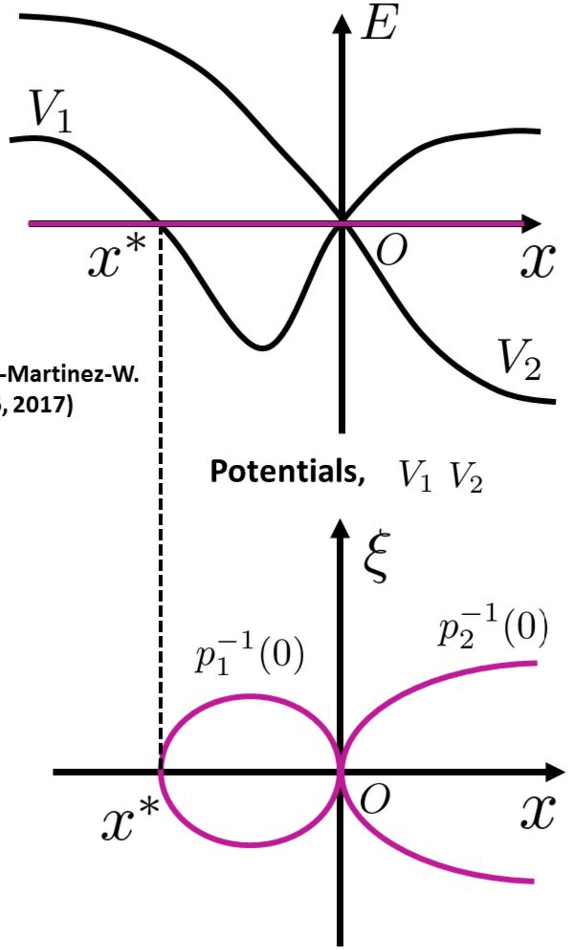
Non crossing case

— The case $V_1 = x^2, V_2 = -x - 1, E_0 = 0$
(Martinez 1994, Nakamura 1994, Baklouti 1998); $e^{-\delta/h}$



Width of resonances is exp. small $\longleftrightarrow$ well in a island

Fujiié-Martinez-W.
(2016, 2017)

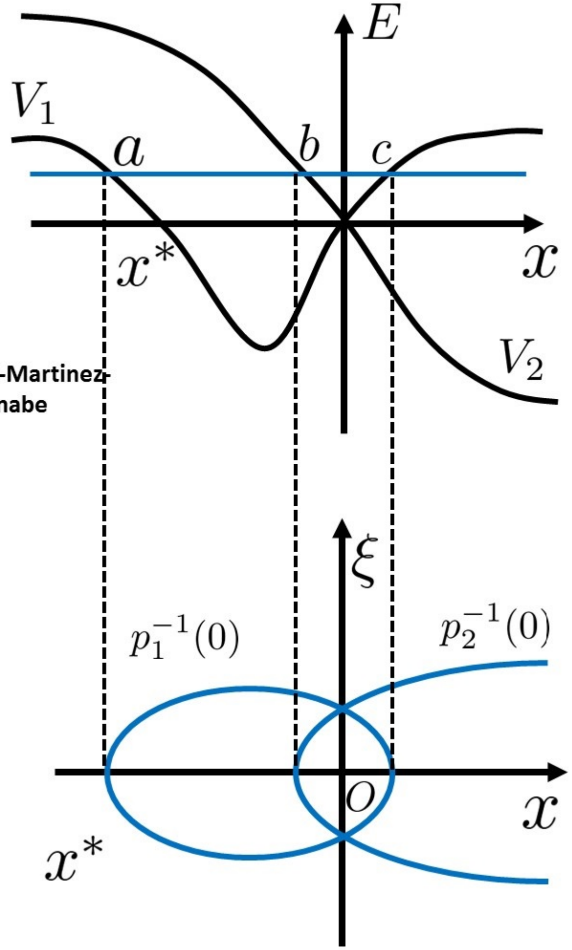


Potentials, V_1 V_2

Characterictic sets $(E = 0)$

$p_j(x, \xi) = \xi^2 + V_j(x)$

Fujiié-Martinez-Watanabe



Characterictic sets

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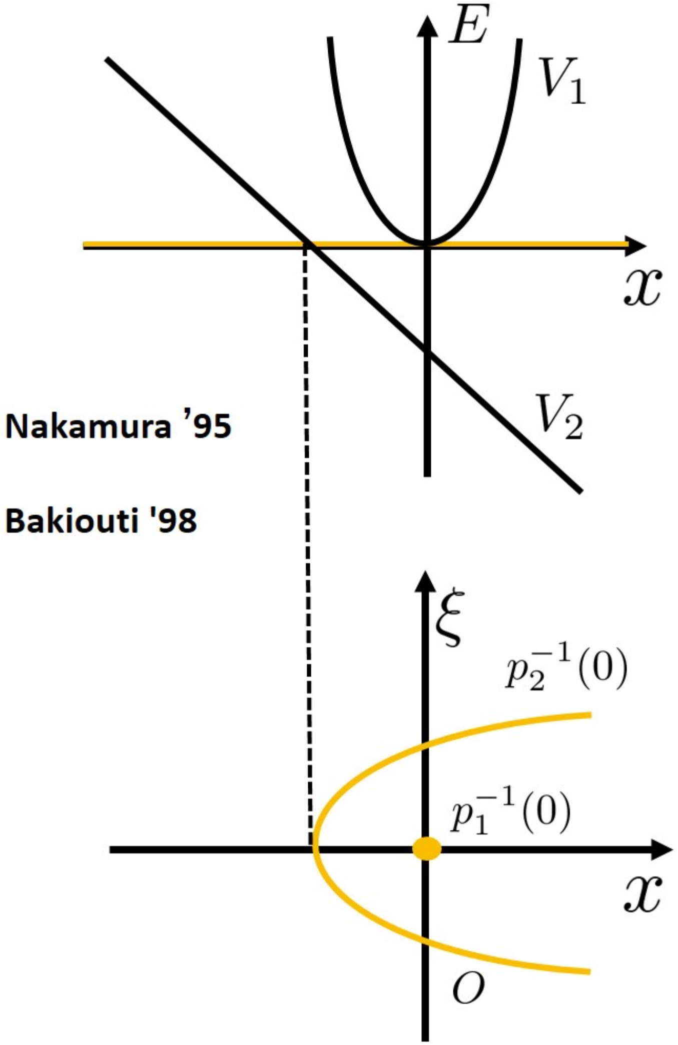
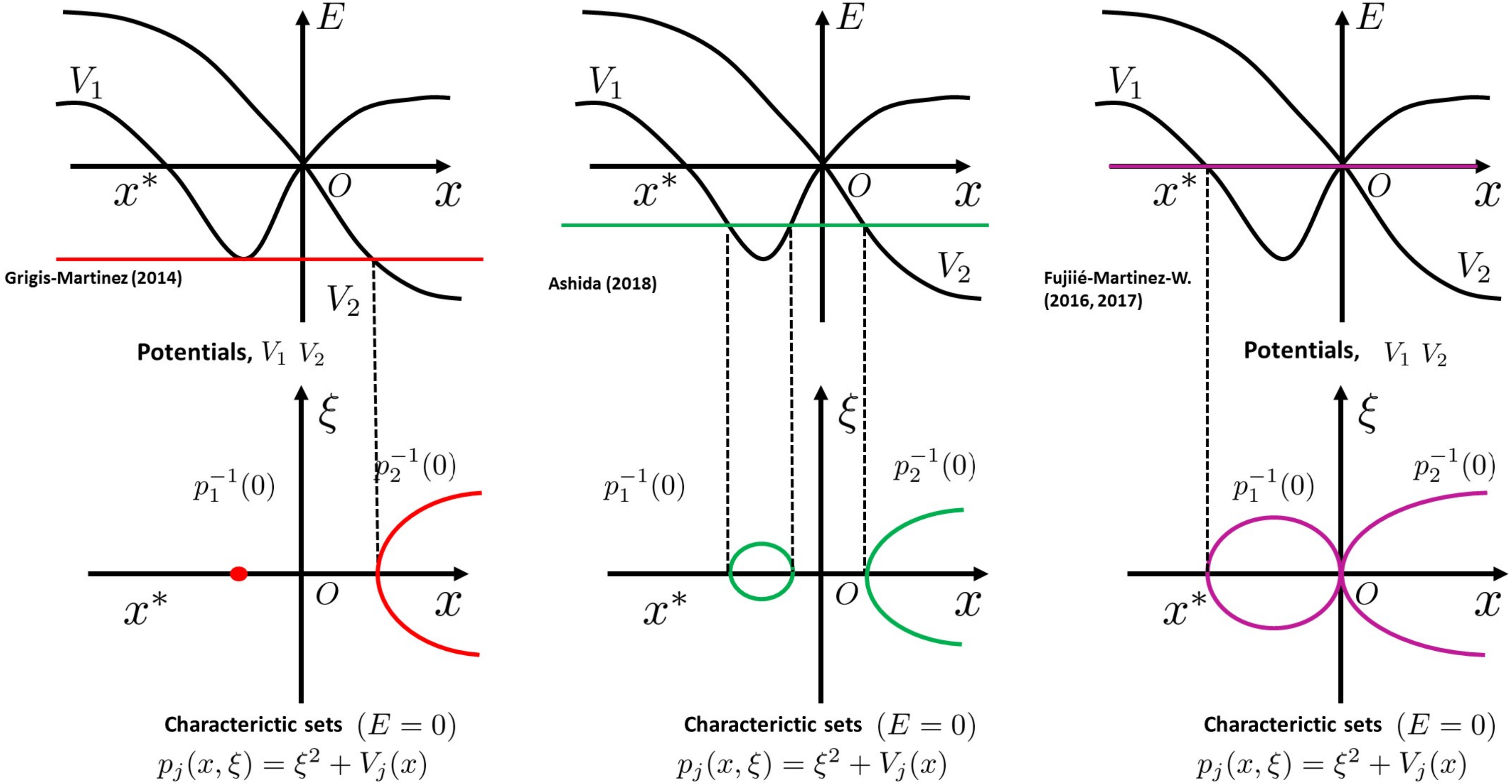


Characterictic sets $(E = 0)$

$$p_j(x,\xi) = \xi^2 + V_j(x)$$

Remark on the descriptions by Airy functions

If $\boldsymbol{\lambda_k} \to -\infty$, then $\mathbf{Ai}\,(-\mathbf{2}^{\frac{2}{3}}\boldsymbol{\lambda_k})$ & $\mathbf{Ai}\,'(-\mathbf{2}^{\frac{2}{3}}\boldsymbol{\lambda_k})$ exp. decay.



The physical background of our system comes from a molecular dynamics described by the many body Hamiltonian.

Born-Oppenheimer approximation:

$h := \sqrt{\frac{m}{M}} \to 0$, where M, m : mass of nuclear & electron.

Feshbach reduction:
Consider the Grushin problem focusing on two electron energy-levels.

Our operator $P(h)$ is the effective Hamiltonian of the Grushin problem.
Such reduction was given in Klein-Martinez-Seiler-Wang (1992).

Notice **(B1)** : coupling Schrödinger equation, **(B2)** : effective Hamiltonian

Quantization condition

$E = \rho h^{2/3} \in \mathcal{D}_h(C_0)$ is a resonance of P iff,

$$\cos \frac{\mathcal{A}(E)}{h} = h^{\frac{2}{3}} (\sin \frac{\mathcal{A}(E)}{h}) F(E,h),$$

with $\mathrm{Re}\, F(E,h) = \mathcal{O}(1)$,

$$\mathrm{Im}\, F(E,h) = 4\pi^2 r_0(0)^2 \mu_A (\mathrm{Re}\rho)^2 + \mathcal{O}(h^{\frac{1}{3}}).$$

Here $\mu_A(t) = \int_0^\infty \mathrm{Ai}\,(y-t)\mathrm{Ai}\,(-y-t)dy$.

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Here $\mu_A(t) = \int_0^\infty \mathrm{Ai}\,(y-t)\mathrm{Ai}\,(-y-t)dy$.

If moreover $r_0(x) \equiv 0$ and $r_1(x)$ is real-valued, then
$F(E,h) = h^{\frac{2}{3}} G(E,h)$ with $\mathrm{Re}\, G(E,h) = \mathcal{O}(1)$,

$$\mathrm{Im}\, G(E,h) = \pi^2 r_1(0)^2 \mu_A' (\mathrm{Re}\rho)^2 + \mathcal{O}(h^{\frac{1}{3}}).$$

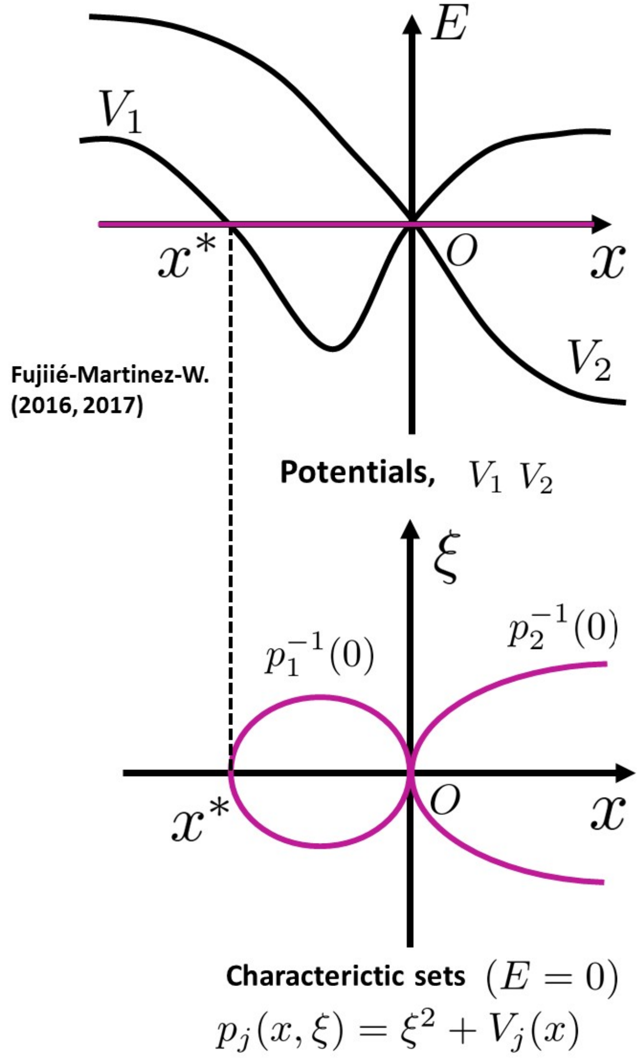
Main difficulty : Connection at the crossing

◇ (Complex) WKB approach:

The energy crossing point is the turning point of both potential $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$. Complex WKB solutions (out-going) cannot prolongate analytically upto the crossing point (turning point).

◇ Microlocal approach:

Classical orbits associated with a semiclassical principal symbol (i.e. $\boldsymbol{\xi^4 - x^2}$ near $\boldsymbol{(0,0)}$) cross tangentially at the crossing.



Idea (Fundamental solutions for a single-valued problem)
Considering the fundamental solution for each diagonal and constructing the out-going solutions by an iteration method combing both fundamental solutions, which is based on Yafaev' construction (2010,2011).

Point 1 (Derive a good estimate by treating both together)
One of the estimates of fundamental solutions is not good for working the iteration but the other is good thanks to an ellipticity.

Point 2 (Make clear where contributions in the Wronskian come)
The sequences constructed in the iteration generate vector-valued asymptotic sequences w.r.t. $\boldsymbol{h^{\frac{1}{3}}}$.

Idea (Fundamental solutions for a single-valued problem)

Considering the fundamental solution for each diagonal and constructing the out-going solutions by an iteration method combing both fundamental solutions, which is based on Yafaev' construction (2010,2011).

Fundamental solution $K_{j,L} := (P_j - E)^{-1}$ on $I_L = (-\infty, 0]$.

$$K_{j,L}[v](x) := \frac{u_{j,L}^{+}(x)}{h^2 \mathcal{W}[u_{j,L}^{+}, u_{j,L}^{-}]} \int_{-\infty}^{x} u_{j,L}^{-}(t) v(t)\, dt$$

$$+ \frac{u_{j,L}^{-}(x)}{h^2 \mathcal{W}[u_{j,L}^{+}, u_{j,L}^{-}]} \int_{x}^{0} u_{j,L}^{+}(t) v(t)\, dt.$$

$u_{j,L}^{+}$: solution of $(P_j - E)u = 0$ with exp. growth at ∞

$u_{j,L}^{-}$: solution of $(P_j - E)u = 0$ with exp. decay at $-\infty$

Point 1 (Derive a good estimate by treating both together)

One of the estimates of fundamental solutions is not good for working the iteration but the other is good thanks to an ellipticity.

Prop As h goes to 0, we have

$$\| hK_{2,L}W^* \|_{\mathcal{L}(C_b(I_L))} = \mathcal{O}(h^{\frac{1}{3}}),$$

$$\| h^2 K_{1,L} W K_{2,L} W^* \|_{\mathcal{L}(C_b(I_L))} = \mathcal{O}(h^{\frac{2}{3}}).$$

We construct out-going solution $w_{1,L}$ by an iteration with an initial condition $w^0_{1,L} = {}^t(u^-_{1,L}, 0)$. The first procedure gives $w^1_{1,L} = {}^t(u_1, u_2)$ as

$$\begin{cases} u_1 = u^-_{1,L} + h^2 K_{1,L} W K_{2,L} W^* u^-_{1,L} \\ u_2 = -hK_{2,L} W^* u^-_{1,L} \end{cases}$$

Point 2 (Make clear where contributions in the Wronskian come)

The sequences constructed in the iteration generate vector-valued asymptotic sequences w.r.t. $h^{\frac{1}{3}}$.

We get the restriction of out-going solutions to $x = 0$.

$$w_{1,S}(0) = \begin{pmatrix} u_{1,S}^{-}(0) + \mathcal{O}(h^{\frac{2}{3}}) \\ \mathcal{O}(h^{\frac{1}{3}}) \end{pmatrix} + \mathcal{O}(h)$$

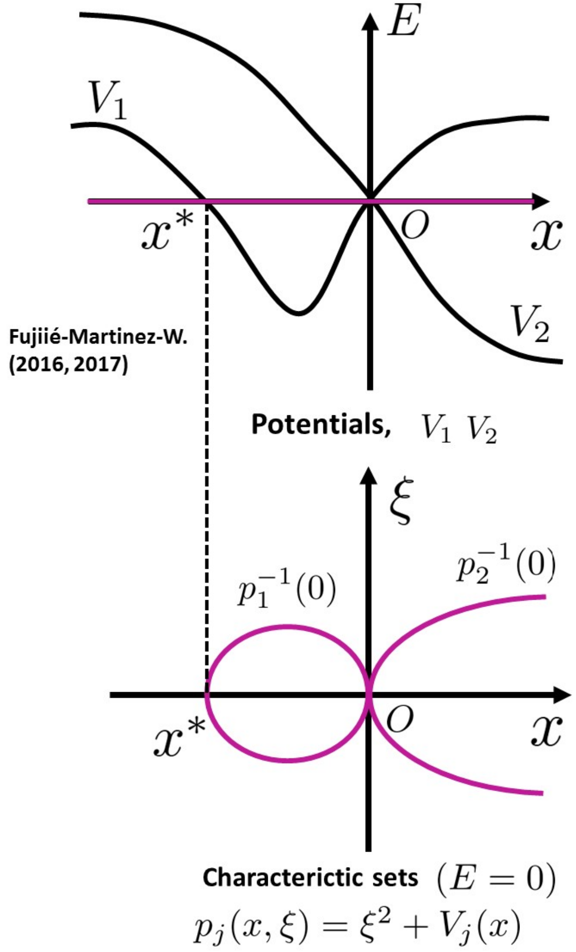
$$w_{2,S}(0) = \begin{pmatrix} \mathcal{O}(h^{\frac{1}{3}}) \\ u_{2,S}^{-}(0) + \mathcal{O}(h^{\frac{2}{3}}) \end{pmatrix} + \mathcal{O}(h)$$

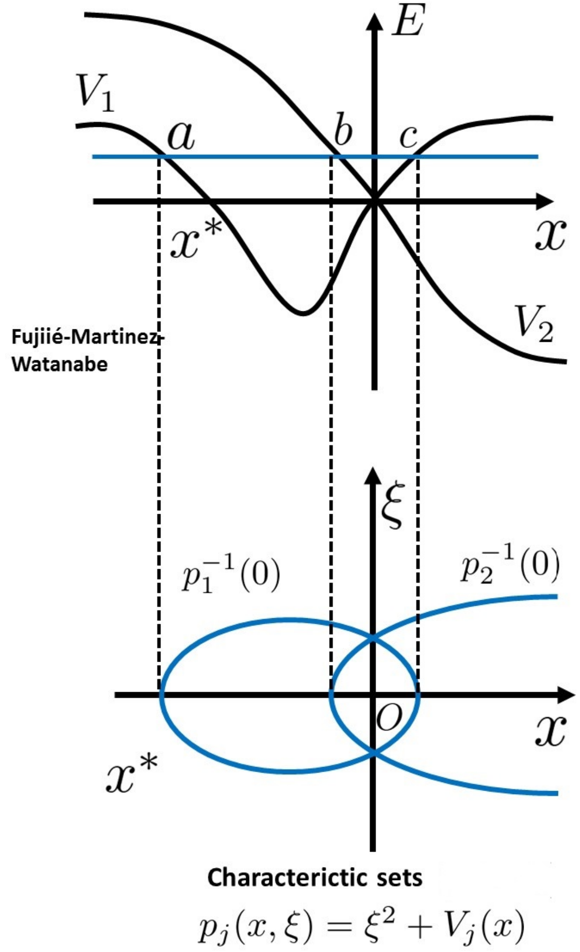
Resonances are given by quantization condition:

$$\mathcal{W}[w_{1,L}, w_{2,L}, w_{1,R}, w_{2,R}]|_{x=0} = 0$$

2nd Part: Above crossing level. From the descriptions by Airy functions

If $\boldsymbol{\lambda_k} \to \boldsymbol{\infty}$, the leading term $\mathbf{Ai}\,(-\mathbf{2}^{\frac{2}{3}}\boldsymbol{\lambda_k})$ vanishes periodically.





Assumptions

(A1) & **(A2)** are same. Let $\boldsymbol{E_0}$ be

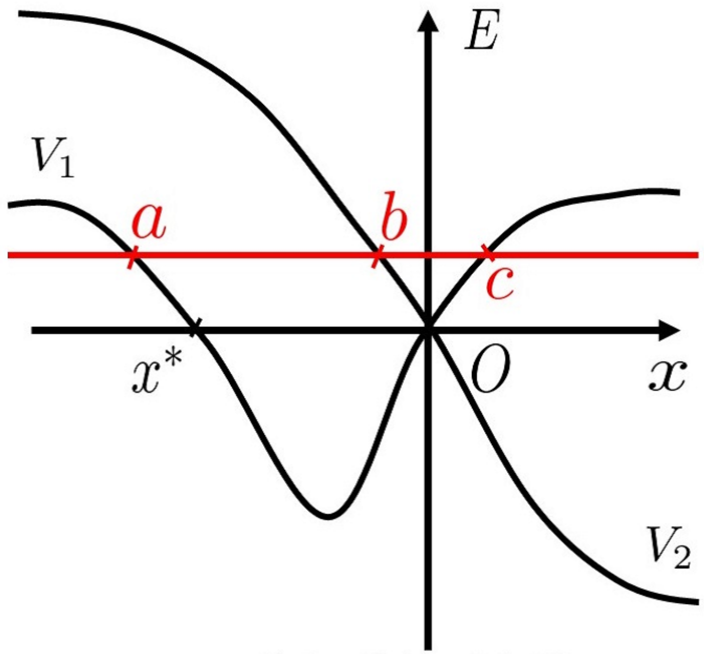
$$\boldsymbol{V_2^+ < 0 < E_0 < \min\{V_1^{\pm}, V_2^-\}}$$

(A3-1) $\boldsymbol{V_1(0) = V_2(0) = 0}$,
$\boldsymbol{V_1'(0) > 0, V_2'(0) < 0.}$

(A3-2) $\boldsymbol{V_1'(a) < 0}$, $\boldsymbol{V_1'(c) > 0}$,
$\boldsymbol{V_2'(b) < 0.}$

(A4) is the same and the ellipticity condition:

$$\boldsymbol{(r_0(0), r_1(0)) \neq (0, 0).}$$



Potentials, V_1 V_2

Compare each difficulty :

◇ Crossing point of the energy-levels:

In the 1st part, we have, near the crossing point, the ellipticity of either $P_j - E$. In the 2nd, we do not get the ellipticity near the crossing point $]b, c[$.

◇ Crossing point of the characteristics set:

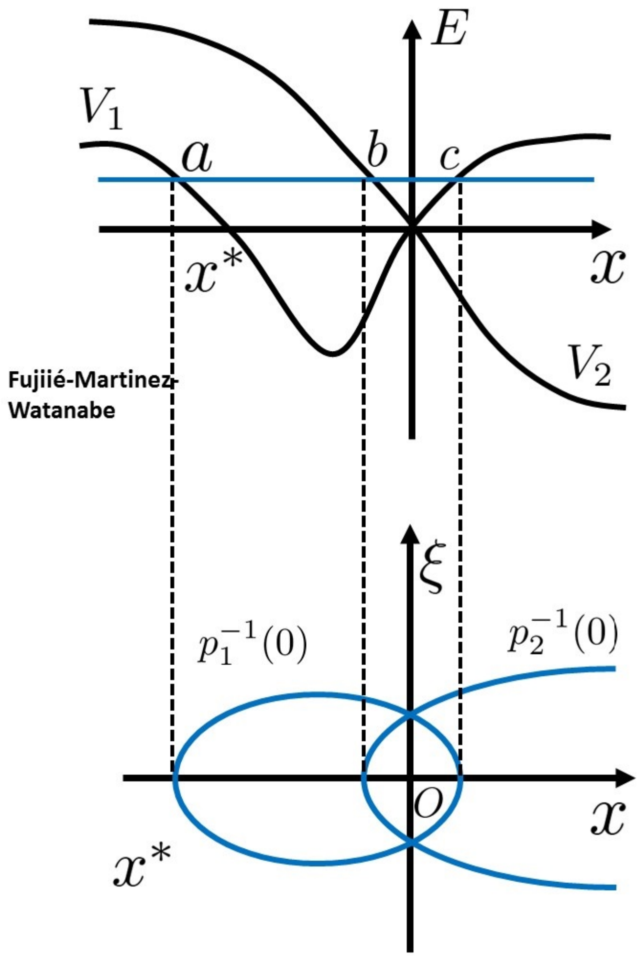
While, in the 1st part, the characteristics set on the phase space cross tangentially, i.e.

$$\xi^4 - x^2 \text{ near } (0,0)$$

those in the 2nd part cross transversally, i.e.

$$(\xi^2 + V_1(x) - E)(\xi^2 + V_2(x) - E)$$

near $(0, \pm\sqrt{E})$.



Characterictic sets
$p_j(x,\xi) = \xi^2 + V_j(x)$

Rough estimate of resonances

We can apply the same method as our previous work for this problem. However, near the crossing point, we lose the ellipticity concerning fundamental solutions.

Compute the wronskian $W(E,h)$ of out-going solutions:

$$\exists C h^{\frac{4}{3}} W(E,h) = \cos\frac{\mathcal{A}(E)}{h} + \mathcal{O}(h^{1/6}).$$

Deduce a rough estimate of resonances from Q.C. $W(E,h) = 0$:

$$E_k(h) = e_k(h) + \mathcal{O}(h^{7/6}).$$

$$\left(\Longrightarrow \ \mathrm{Im}\, E_k(h) = \mathcal{O}(h^2)\right)$$

Microlocal method (For studying more precise estimate of resonances)

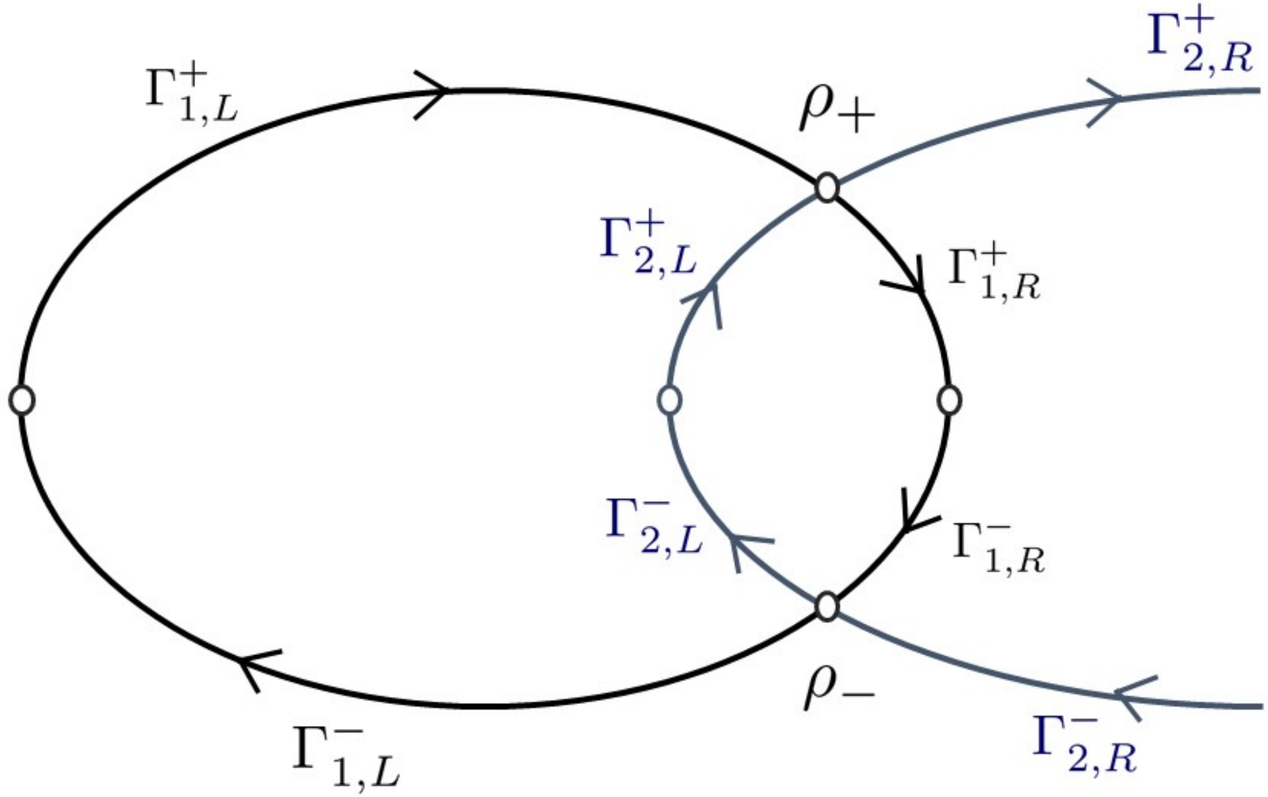
If u is a (suitably normalized) solution to $(P - E)u = 0$, u is microlocally supported on the characteristic set,

$$\Gamma_j = \{(x, \xi); |\xi|^2 + V_j(x) = E\}, \quad \Gamma = \Gamma_1 \cup \Gamma_2.$$

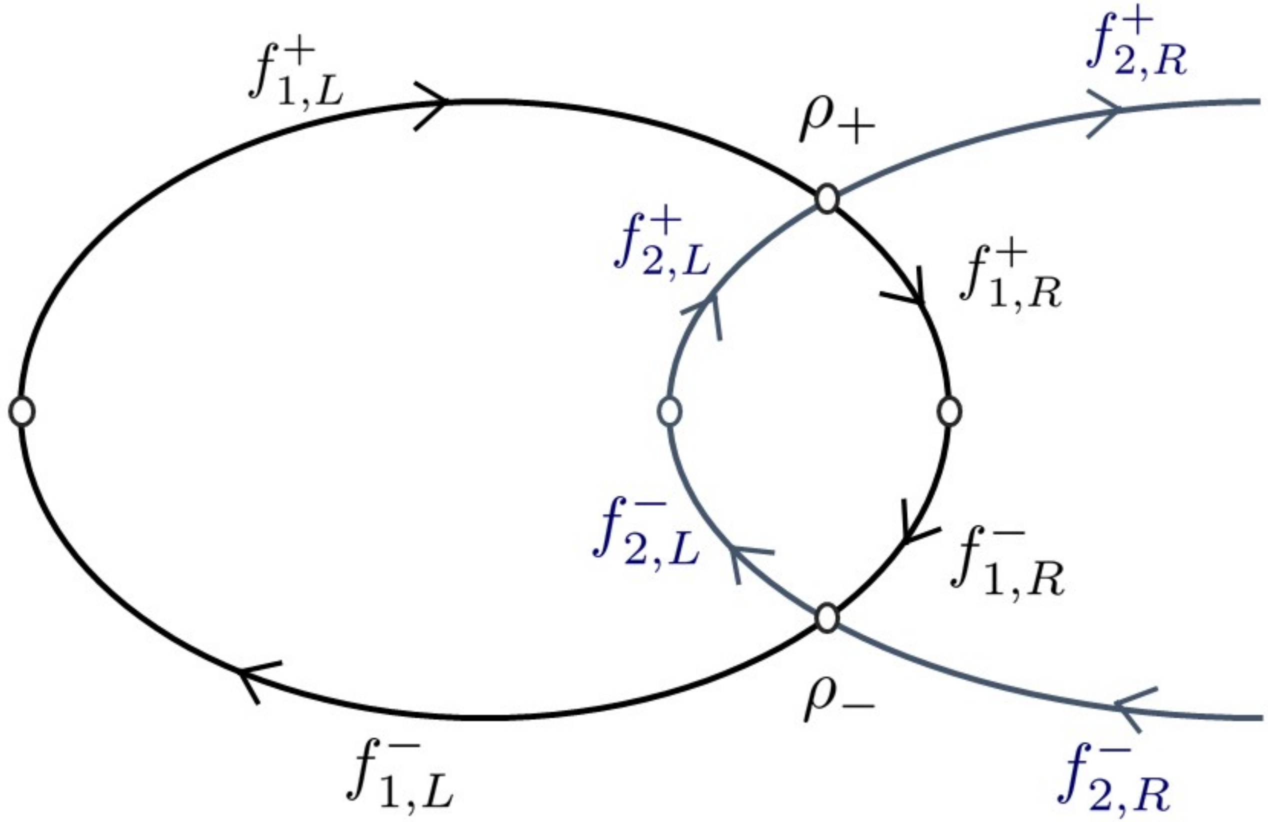
On Γ, WKB solutions $f^{\pm}_{j,S}$ are microlocally defined on each of 8 curves $\Gamma^{\pm}_{j,S}$ $(j = 1, 2, S = L, R)$, except at

- 3 turning points $(a(E), 0)$, $(b(E), 0)$ and $(c(E), 0)$
- 2 crossing points $\rho_+ = (0, \sqrt{E})$ and $\rho_- = (0, -\sqrt{E})$.

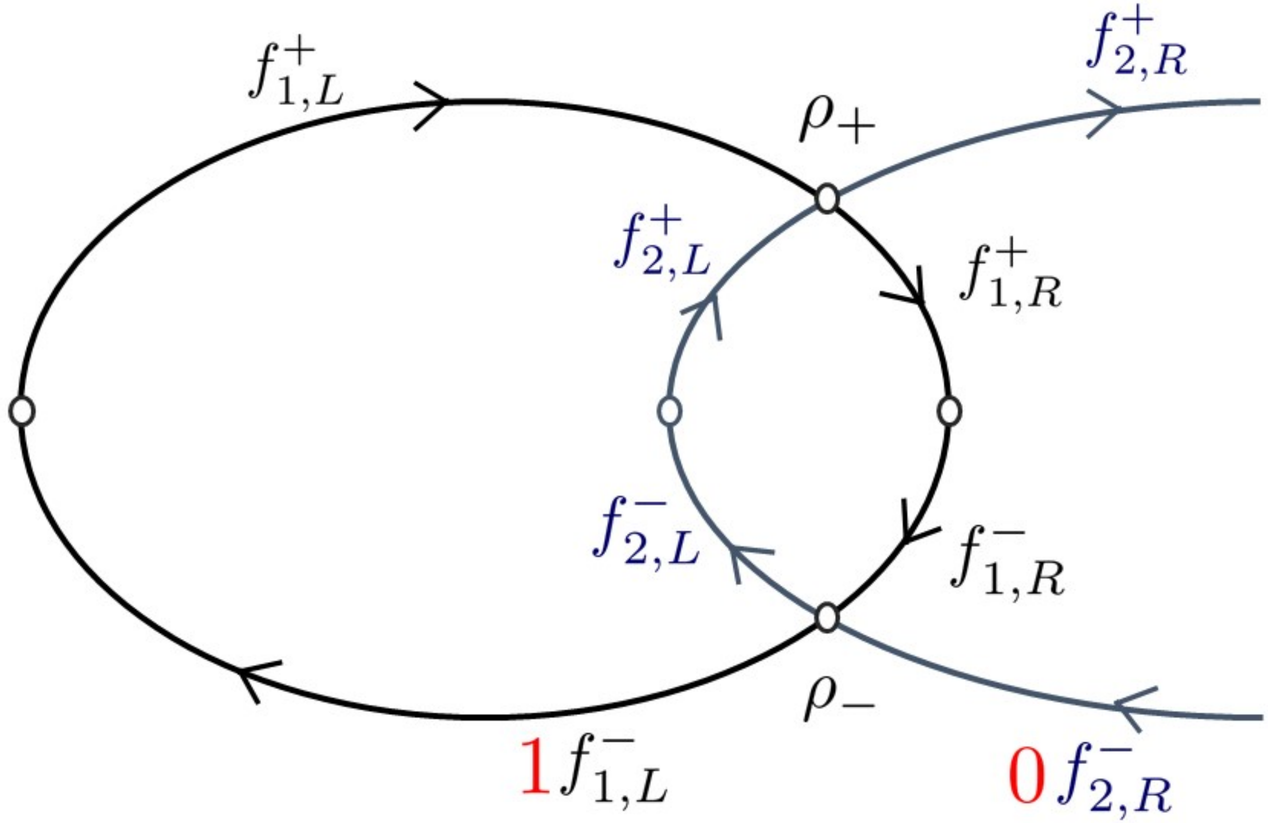
If u is a resonant state, $u \sim 0$ on the incoming trajectory $\Gamma^{-}_{2,R}$.



$\Gamma_{1,L}^{+}$
$\Gamma_{2,R}^{+}$
ρ_{+}
$\Gamma_{2,L}^{+}$
$\Gamma_{1,R}^{+}$
$\Gamma_{2,L}^{-}$
$\Gamma_{1,R}^{-}$
ρ_{-}
$\Gamma_{2,R}^{-}$
$\Gamma_{1,L}^{-}$



$f_{1,L}^{+}$
$f_{2,R}^{+}$
ρ_{+}
$f_{2,L}^{+}$
$f_{1,R}^{+}$
$f_{2,L}^{-}$
$f_{1,R}^{-}$
ρ_{-}
$f_{1,L}^{-}$
$f_{2,R}^{-}$



$f^+_{1,L}$
$f^+_{2,R}$
ρ_+
$f^+_{2,L}$
$f^+_{1,R}$
$f^-_{2,L}$
$f^-_{1,R}$
ρ_-
$1\, f^-_{1,L}$
$0\, f^-_{2,R}$

Microlocal WKB solutions (near $\rho_-(E) = (0, -\sqrt{E})$)

On each $\Gamma^{\pm}_{j,S}(E)$ ($j = 1, 2$, $S = R, L$), the space of microlocal solutions is one dimensional, and a basis is given by

$$f^{\pm}_{1,S} \sim \begin{pmatrix} a_1^{\pm} \\ h a_2^{\pm} \end{pmatrix} e^{\pm i\nu_1(x)/h} \text{ on } \Gamma^{\pm}_{1,S}(E), \quad f^{\pm}_{2,S} \sim \begin{pmatrix} h b_1^{\pm} \\ b_2^{\pm} \end{pmatrix} e^{\pm i\nu_2(x)/h} \text{ on } \Gamma^{\pm}_{2,S}(E),$$

Here, for $j = 1, 2$, $\nu_j(x) := \int_0^x \sqrt{E - V_j(t)}\, dt$,

$$a_j^{\pm}(x;h) \sim \sum_{k\geq 0} h^k a^{-}_{j,k}(x), \quad b_j^{\pm}(x;h) \sim \sum_{k\geq 0} h^k b^{-}_{j,k}(x)$$

$$a_{1,0} = \frac{1}{(E - V_1)^{1/4}} \quad ; \quad a_{2,0} = \frac{r_0 + i r_1 \sqrt{E - V_1}}{(V_1 - V_2)(E - V_1)^{1/4}}.$$

$$b_{2,0} = \frac{1}{(E - V_2)^{1/4}} \quad ; \quad b_{1,0} = \frac{r_0 - i r_1 \sqrt{E - V_2}}{(V_2 - V_1)(E - V_2)^{1/4}}.$$

Remark These microlocal solutions are not defined at turning points $V_1(x) = V_2(x) = E$ & crossing points $V_1(x) = V_2(x)$.

Although we lost the ellipticity of fundamental solutions, the characteristics cross transversally on the phase space, whose geometrical configuration is better than the precious one with tangentially crossing.

The elliptic condition of W (i.e. **(A5)**) implies $(P - E)v \sim 0 \Leftrightarrow$

$$\begin{cases} Qv_1 \sim 0, \quad Q := W(P_2 - E)W^{-1}(P_1 - E) - h^2 WW^* \\ v_2 \sim Rv_1, \quad R = -h^{-1}W^{-1}(P_1 - E) \end{cases}$$

microlocally near $\rho_-(E)$. Since $\rho_-(E)$ is a saddle point of the principal symbol of Q: $(\xi^2 + V_1(x) - E)(\xi^2 + V_2(x) - E)$, it is reduced to:

$$UF(Q,h)U^{-1} = \frac{1}{2}(yhD + hDy) \; (= (y\eta)^w),$$

with a Fourier integral operator $Uu = \int_{\mathbb{R}} e^{i\psi(x,y)/h} c(x,y;h)u(y)dy$ and an analytic symbol $F(t,h)$ (Helffer-Sjöstrand).

In our case, we have

$$\psi(x,y) = -\frac{\tau_2}{4\sqrt{E}}x^2 - \frac{\sqrt{E}}{\tau_1+\tau_2}y^2 + xy + \sqrt{E}x + \mathcal{O}(|(x,y)|^3),$$

$$F(0,h) = -\frac{i}{2}h + \mu h^2, \quad \mu = -\frac{r_0(0)^2 + r_1(0)^2 E}{2(\tau_1+\tau_2)\sqrt{E}} + \mathcal{O}(h).$$

The reduced equation for $v = Uu_1$ is

$$\frac{1}{2}(yhD + hDy)v = F(0,h)v$$

and it has a basis of solutions

$$v^{\vdash}(y) = H(y)y^{i\mu h}, \quad v^{\dashv}(y) = H(-y)|y|^{i\mu h}.$$

The problem is thus reduced to the analysis of an integral of the form

$$u_1^{\vdash}(x) := Uv^{\vdash} = \int_0^{\infty} e^{i\psi(x,y)/h} c(x,y;h) y^{i\mu h} dy.$$

The contributions coming from stationary point and the end point correspond to the asymptotics on $\Gamma_{j,S}^{-}(E)$. This gives the connection formula over $\rho_{-}(E)$.

Thank you for
your attentions

Fundamental solutions for $P_j - E$

Construct fundamental solution
$K_{j,L} := (P_j - E)^{-1}$ on $I_L = (-\infty, 0]$.

$$K_{j,L}[v](x) := \frac{u^{+}_{j,L}(x)}{h^2 \mathcal{W}[u^{+}_{j,L}, u^{-}_{j,L}]} \int_{-\infty}^{x} u^{-}_{j,L}(t) v(t)\, dt + \frac{u^{-}_{j,L}(x)}{h^2 \mathcal{W}[u^{+}_{j,L}, u^{-}_{j,L}]} \int_{x}^{0} u^{+}_{j,L}(t) v(t)\, dt.$$

$u^{+}_{j,L}$: sol of $(P_j - E)u = 0$ with exp. growth at ∞
$u^{-}_{j,L}$: sol of $(P_j - E)u = 0$ with exp. decay at $-\infty$

Fundamental solutions for $P_j - E$

Construct fundamental solution
$K_{j,R} := (P_j - E)^{-1}$ on $I_R^\theta = F_\theta([0,\infty))$.

$$K_{j,R}[v](x) := \frac{u_{j,R}^+(x)}{h^2\mathcal{W}[u_{j,R}^+, u_{j,R}^-]} \int_x^\infty u_{j,R}^-(t)v(t)\,dt + \frac{u_{j,R}^-(x)}{h^2\mathcal{W}[u_{j,R}^+, u_{j,R}^-]} \int_0^x u_{j,R}^+(t)v(t)\,dt.$$

$u_{j,R}^+$: sol. of $(P_j - E)u = 0$ with exp. growth at ∞ along I_R^θ
$u_{j,R}^-$: sol. of $(P_j - E)u = 0$ with exp. decay at ∞ along I_R^θ

Prop As h goes to 0, we have

$$\| hK_{2,L}W^* \|_{\mathcal{L}(C_b(I_L))} = \mathcal{O}(h^{\frac{1}{3}}),$$
$$\| h^2K_{1,L}WK_{2,L}W^* \|_{\mathcal{L}(C_b(I_L))} = \mathcal{O}(h^{\frac{2}{3}}).$$

We construct out-going solution $w_{1,L}$ by an iteration with an initial condition $w^0_{1,L} = {}^t(u^-_{1,L}, 0)$. The first procedure gives $w^1_{1,L} = {}^t(u_1, u_2)$ as

$$\begin{cases} u_1 = u^-_{1,L} + h^2K_{1,L}WK_{2,L}W^*u^-_{1,L} \\ u_2 = -hK_{2,L}W^*u^-_{1,L} \end{cases}$$

The next procedures succeed thanks to the above proposition.

Prop As h goes to 0, we have

$$\| hK_{1,R}W \|_{\mathcal{L}(C_b(I_R^\theta))} = \mathcal{O}(h^{\frac{1}{3}}),$$
$$\| h^2 K_{2,R}W^* K_{1,R}W \|_{\mathcal{L}(C_b(I_R^\theta))} = \mathcal{O}(h^{\frac{2}{3}}).$$

We construct out-going solution $w_{1,R}$ by an iteration with an initial condition $w_{1,R}^0 = {}^t(u_{1,R}^-, 0)$. The first procedure gives $w_{1,R}^1 = {}^t(u_1, u_2)$ as

$$\begin{cases} u_1 = u_{1,R}^- + h^2 K_{1,R} W K_{2,R} W^* u_{1,R}^- \\ u_2 = -hK_{2,R}W^* u_{1,R}^- \end{cases}$$

The next procedures succeed thanks to the above proposition.

Similarly we can construct out-going solution $w_{2,S}$ with ${}^t(u_1, u_2) = {}^t(0, u^-_{2,S})$ for $S \in \{L, R\}$.

Moreover we get the restriction of out-going solutions to $x = 0$.

$$w_{1,S}(0) = \begin{pmatrix} u^-_{1,S}(0) + \mathcal{O}(h^{\frac{2}{3}}) \\ \mathcal{O}(h^{\frac{1}{3}}) \end{pmatrix} + \mathcal{O}(h)$$

$$w_{2,S}(0) = \begin{pmatrix} \mathcal{O}(h^{\frac{1}{3}}) \\ u^-_{2,S}(0) + \mathcal{O}(h^{\frac{2}{3}}) \end{pmatrix} + \mathcal{O}(h)$$

Resonances are given by quantization condition:

$$\mathcal{W}[w_{1,L}, w_{2,L}, w_{1,R}, w_{2,R}]|_{x=0} = 0$$

Prop Case $r_0 \equiv 0$. As $h \downarrow 0$, we have

$$\| hK_{2,L}W^* \|_{\mathcal{L}(C_b(I_L))} = \mathcal{O}(h^{\frac{2}{3}}),$$
$$\| h^2 K_{1,L} W K_{2,L} W^* \|_{\mathcal{L}(C_b(I_L))} = \mathcal{O}(h).$$

And the same estimates hold on I_R^{θ}.

Similarly we get the restriction of out-going solutions at $x = 0$ for $S \in \{L, R\}$

$$w_{1,S}(0) = \begin{pmatrix} u_{1,S}^{-} + \mathcal{O}(h^{\frac{4}{3}}) \\ \mathcal{O}(h^{\frac{2}{3}}) \end{pmatrix} + \mathcal{O}(h^{\frac{5}{3}})$$

$$w_{2,S}(0) = \begin{pmatrix} \mathcal{O}(h^{\frac{2}{3}}) \\ u_{2,S}^{-} + \mathcal{O}(h^{\frac{4}{3}}) \end{pmatrix} + \mathcal{O}(h^{\frac{5}{3}})$$

Previous considerations reduce the computations of the Wronskian to studying the integral, for example,

$$K_2[W^* u_{1,L}^-](0) = \frac{u_{2,L}^+(0)}{h^2 \mathcal{W}[u_{2,L}^+, u_{2,L}^-]} \times \int_{-\infty}^{0} u_{2,L}^-(t) W^* u_{1,L}^-(t) dt$$

Essentially we must derive a decay estimate $\mathcal{O}(h^{\frac{1}{3}})$ from

$$\int_{\lambda h^{\frac{2}{3}}}^{1} e^{-ct^{\frac{3}{2}}/h} \, dt = \mathcal{O}(e^{-c\lambda^{\frac{3}{2}}}) = \mathcal{O}(h^{\frac{1}{3}})$$

by introducing a large parameter λ with $\lambda = (\frac{1}{3c}|\log h|)^{\frac{2}{3}}$

When the derivatives at the crossing are general i.e.

$$V_1'(0) = \tau_1 > 0, \quad V_2'(0) = -\tau_2 < 0,$$

the first result is modified as follows:

$$\mathrm{Im}\, E_k = -\frac{2\pi^2 r_0(0)^2}{\mathcal{A}'(0)} (\tau_1 \tau_2)^{\frac{1}{3}}$$

$$\times \left(\mu_1(\lambda_k)^2 + \mu_2(\lambda_k)^2\right) h^{\frac{5}{3}} + \mathcal{O}(h^2),$$

where μ_l is given for $l = 1, 2$ with $\tau_3 = \tau_1$ by

$$\mu_l(t) = \int_0^\infty \mathrm{Ai}\,(\tau_l^{-\frac{2}{3}}(\tau_l y - t))$$

$$\times \mathrm{Ai}\,(\tau_{l+1}^{-\frac{2}{3}}(\tau_{l+1} y + t))dy.$$

When the derivatives at the crossing are general i.e.

$$V_1'(0) = \tau_1 > 0, \quad V_2'(0) = -\tau_2 < 0,$$

the second result (case $r_0(x) \equiv 0$) is modified as follows:

$$\mathrm{Im}\, E_k = -\frac{\pi^2 r_1(0)^2}{\mathcal{A}'(0)} \frac{\tau_3^{\frac{1}{3}}}{\tau_1 + \tau_2} \times \left(\mathrm{Ai}\,'(-\tau_3^{\frac{2}{3}} \lambda_k) \right)^2 h^{\frac{7}{3}} + \mathcal{O}(h^{\frac{8}{3}}),$$

with $\tau_3^{-1} = \tau_1^{-1} + \tau_2^{-1}$.