```
→   /* eigenvalues, eigenvectors, diagonalization
       and Jordan normal form of square matrices
    - the rotation matrix of π/2 radian on the plane
    - a square matrix of order 3 with distinct eigenvalues
    - a diagonalizable square matrix of order 3
    - a Jordan normal form of a square matrix of order 3
    - the package for matrix normalization
    */

(%i1) /* the rotation matrix of π/2 radian on the plane */
      R:matrix([0,-1],[1,0]);
```

(R) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

```
(%i2) /* characteristic polynomial */
      expand(charpoly(R, z));
```

(%o2) z^2+1

```
(%i3) /* eigenvalues and multiplicity */
      eigenvalues(R);
(%o3) [[-%i,%i],[1,1]]

(%i4) /* eigenvalues, multiplicity and associated eigenvectors */
      eigenvectors(R);
(%o4) [[[-%i,%i],[1,1]],[[[1,%i]],[[1,-%i]]]]

(%i5) /* a diagonalizer */
      P0:transpose(matrix([1,%i],[1,-%i]));
```

(P0) $\begin{pmatrix} 1 & 1 \\ \%i & -\%i \end{pmatrix}$

```
(%i7) /* diagonalization */
      D0:invert(P0).R.P0;
```

(D0) $\begin{pmatrix} -\%i & 0 \\ 0 & \%i \end{pmatrix}$

```
(%i8) /* a square matrix of order 3 with distinct eigenvalues */
      A1:matrix([5,-3,6],[2,0,6],[-4,4,-1]);
```

(A1) $\begin{pmatrix} 5 & -3 & 6 \\ 2 & 0 & 6 \\ -4 & 4 & -1 \end{pmatrix}$

```
(%i9) /* characteristic polynomial */
expand(charpoly(A1, z));
```

(%o9) $-z^3+4z^2-z-6$

```
(%i10) /* eigenvalues and multiplicity */
eigenvalues(A1);
```

(%o10) [[3,-1,2],[1,1,1]]

```
(%i11) /* eigenvalues, multiplicity and eigenvectors */
eigenvectors(A1);
```

(%o11) $[[[3,-1,2],[1,1,1]],[[[1,\frac{4}{3},\frac{1}{3}]],[[1,1,-\frac{1}{2}]],[[1,1,0]]]]$

```
(%i12) /* a diagonalizer */
P1:transpose(matrix([1,4/3,1/3],[1,1,-1/2],[1,1,0]));
```

(P1) $\begin{pmatrix} 1 & 1 & 1 \\ \frac{4}{3} & 1 & 1 \\ \frac{1}{3} & -\frac{1}{2} & 0 \end{pmatrix}$

```
(%i13) /* diagonalization */
D1:invert(P1).A1.P1;
```

(D1) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

```
(%i14) /* a diagonalizable square matrix of order 3 */
A2:matrix([-3,-2,-2],[4,3,2],[8,4,5]);
```

(A2) $\begin{pmatrix} -3 & -2 & -2 \\ 4 & 3 & 2 \\ 8 & 4 & 5 \end{pmatrix}$

```
(%i15) /* characteristic polynomial */
expand(charpoly(A2, z));
```

(%o15) $-z^3+5z^2-7z+3$

```
(%i16) /* eigenvalues and multiplicity */
eigenvalues(A2);
```

(%o16) [[3,1],[1,2]]

(%i17) /* eigenvalues, multiplicity and eigenvectors */
```
eigenvectors(A2);
```
(%o17) [[[3,1],[1,2]],[[[1,-1,-2]],[[1,0,-2],[0,1,-1]]]]

(%i18) /* a diagonalizer */
```
P2:transpose(matrix([1,-1,-2],[1,0,-2],[0,1,-1]));
```
(P2) $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -2 & -2 & -1 \end{pmatrix}$

(%i19) /* diagonalization */
```
D2:invert(P2).A2.P2;
```
(D2) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(%i20) /* a Jordan normal form of a square matrix of order 3 */
```
A3:matrix([2,-1,2],[1,0,2],[-2,2,-1]);
```
(A3) $\begin{pmatrix} 2 & -1 & 2 \\ 1 & 0 & 2 \\ -2 & 2 & -1 \end{pmatrix}$

(%i21) /* characteristic polynomial */
```
expand(charpoly(A3, z));
```
(%o21) $-z^3+z^2+z-1$

(%i22) /* eigenvalues and multiplicity */
```
eigenvalues(A3);
```
(%o22) [[-1,1],[1,2]]

(%i23) /* eigenvalues, multiplicity and eigenvectors */
```
eigenvectors(A3);
```
(%o23) [[[-1,1],[1,2]],[[[1,1,-1]],[[1,1,0]]]]

→ /* A3 cannot be diagonalizable.
Find the generalized eigenspace subordinated to the eigenvalue 1. */

(%i24) /* It is the kernel of (E-A3)^2 */
```
(diagmatrix(3,1)-A3).(diagmatrix(3,1)-A3);
```
(%o24) $\begin{pmatrix} -4 & 4 & -4 \\ -4 & 4 & -4 \\ 4 & -4 & 4 \end{pmatrix}$

→ /* The generalized eigenspace is the set of all {}^t[x,y,x] satisfying x-y+z=0. */

(%i25) /* Solve x-y+z=0. */
```
linsolve([x-y+z=0], [x,y,z]);
```
(%o25) $[x=\%r2-\%r1, y=\%r2, z=\%r1]$

(%i26) /* a normalizer */
```
P3:transpose(matrix([1,1,-1],[1,1,0],[-1,0,1]));
```
(P3) $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

(%i27) /* the Jordan normal form of A3 */
```
D3:invert(P3).A3.P3;
```
(D3) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(%i28) /* Use the package for matrix normalization */
```
load("diag")$
```

(%i29) /* the Jordan blocks of the diagonalization of A1 J(3,1)+J(-1,1)+J(2,1) */
```
J1:jordan(A1);
```
(J1) [[3,1],[-1,1],[2,1]]

(%i30) /* the matrix expession of the diagonalization A1 */
```
dispJordan(J1);
```
(%o30) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

```
(%i31) /* the Jordan blocks of the diagonalizatin of A2
          J(3,1)+J(1,1)+J(1,1) */
       J2:jordan(A2);
(J2)   [[3,1],[1,1,1]]
```

```
(%i32) /* the matrix expression of the diagonalization of A2 */
       dispJordan(J2);
```

(%o32)
$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
(%i33) /* the Jordan blocks of the normalization of A3
          J(-1,1)+J(1,2) */
       J3:jordan(A3);
(J3)   [[-1,1],[1,2]]
```

```
(%i34) /* the matrix expression of the normalization of A3 */
       dispJordan(J3);
```

(%o34)
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$