Riemannian submersion and Maslov quatization condition

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Contents

Title: Riemannian submersion and Maslov quantization condition

Abstract: We consider a relation of spectra of Laplacians on the total space and the base space of a Riemannian submersion.
If a Riemannian submersion commutes with the Laplacians, then the spectrum of the base space are a part of that of the total space and in this case the Riemannian submersion must be a harmonic map (and vise versa).

We are interesting what is happening when a Riemannian submersion need not be a harmonic map and will talk an aspect from a geometric quantization.
Here the important role is carried out by Lagrangian submanifolds in the cotangent bundle.
So first I will survey on

(1) ”Eigenvalue Theorem” by A. Weinstein and its generalization for a sub-Laplacian case together with a review of Maslov quantization condition.

Then I explain

(2) Behavior of Lagrangian submanifolds under submersion,

(3) an example.

This study is still on the way and this talk is a report of some results obtained with M. Tamura:

(1) Riemann submersion and Maslov quantization condition

2) Lagrangian submanifolds satisfying Maslov quantization condition

By Kenro Furutani and Mitsuji Tamura

Configulation space and manifold

As one of the framework in physics, we may understand a manifold as a **configulation space** of a physical system, where a Riemannian metric will be naturally installed according to the physical condition of the system.

Then intuitively, in this system a geodesic will be seen as an orbit of a free particle, which can be observed by a classical mechanical method. Especially, closed geodesics can be seen as orbits of stable free particles corresponding to eigenvalues of the Laplacian. In fact, a result proved by J. V. Ralston in the paper

• *Approximate eigenfunctions of the Laplacian*, J. Diff. Geom., **12**(1977), 87-100,

shows that the existence of eigenfunctions corresponding to a certain kind of closed geodesics (also there are several papers by math-physicists in relation to this correspondence).

The existence of closed geodesics is a basic problem in the Riemannian geometry. If we have an one-to-one correspondence between closed geodesics and eigenstates, then the problem is solved, since the Laplacian is essentially selfadjoint and with compact resolvents. But not so simple in fact.

If there is a closed geodesics, then it means in the classical mechanics the existence of a particle and will imply the existence of quantum phenomena, but it will not hold the opposite direction, since not all the quantum phenomenon will be observed in the macro level of energy as a classical mechanical phenomena.

There are many studies. But I can cite one historical work by N. Bohr (**Bohr-Sommerfeld condition**) and was formulated later as the cohomology class of a symplectic form being integral. For example,

D. J. Simms,
Bohr-Sommerfeld orbits and quantizable symplectic manifolds, Proc. Cambridge Philos. Soc., **73**(1973), 489-491.

Here the existence of closed geodesices will be clear, since an electron is around the hydrogen nucleus and the problem is to calculate the energy level, on which the electron is rotating. This condition restricts the energy level to be discrete.

In this talk I will talk a condition called

"**Maslov quantization condition**".

This gives a specific sequence of eigenvalues.

Let X be a (compact) oriented Riemannian manifold with the Riemannian metric g_X and denote its Laplacian by Δ.

The Hamiltonian is understood as the principal symbol $\sigma_{\Delta^X} \in C^\infty(T^*(X))$ of the Laplacian, which is a smooth function defined on the whole cotangent bundle $T^*(X)$.

In some cases or it may happen often that the movement of free particles is restricted to specific directions. In such a case there exist a subbundle $\mathcal{H}$ in the tangent bundle $T(X)$, to which directions free particles can move and we call the structure "sub-Rieannian", that is a manifold is called to be a sub-Riemannian manifold, if there is a subbundle $\mathcal{H}$ satisfying a condition, so called **bracket generating**, or **non-holonomic**,

if the totality of iterated brackets of vector fields taking values in the given subbundle coincides with the space of all vector fields.

By a basic theorem of ”**Chow**”,

• W. L. Chow, *Über Systeme von linearlen partiellen Differentialgleichungen erster Ordnung*, Math. Ann. **117**(1939), 98-105

under such a condition any two points can be joined by piece-wise smooth horizontal curves, that is

any two states can transfer each other.

Here horizontal curves mean that the tangent vectors of the curves are all belonging to the given subbundle $\mathcal{H}$(= given direction).

So in general and in nice cases, starting from any subbundle $\mathcal{H}$ in $T(X)$ iterations of brackets of vector fields taking values in $\mathcal{H}$ will be stable after finite steps, that is, totality of iterations of such vector fields will become a foliation structure.

Then by famous theorem of "**Frobenius**", for any point there exist a maximal integral submanifold passing through the point. Hence it will be enough to consider the each leaf as a physics system, since there are no "transversal" interaction of the movement of free particles between the different leaves at this level.

However in the limiting process there might happen a complicated phenomena interacting between different leaves.

In this case of sub-Riemannian structure we also have a second order differential operator so called "**sub-Laplacian**", which will be denoted as Δ_{sub} and is proved to be "sub-elliptic" by "**Hörmander theorem**" of a priori estimate with a loss of derivative, which is also called a sub-elliptic estimate.

Maslov quantization condition

As I stated in the begining, closed geodesics will be periodic orbits of free particles and intuitively it corresponds to a quantum eigenstate, which will be expressed by a function, that is a state function ψ is an eigenfunction of the Laplacian

$$\Delta\psi = \lambda \cdot \psi.$$

Then in the micro region, energy level is restricted, but this is usual, since the eigenvalues of operators with compact resolvents are always discrete.

Then, to observe the eigenvalues (or eigenstates) we need a (sufficient) condition which asserts the existence of a special family of eigenvalues.

A condition called “Maslov quantization condition” gives a series of eigenvalues reflecting the existence of a periodic geodesic.

For the formulation we need some notions:

(0) I assume the manifold X is compact.

(1) Let θ^X be the Liouville one-form on the cotangent, bundle $T^*(X$, which is expressed locally as $\sum \xi_i dx_i$ by local coodinates $x = (x_1, \cdots, x_n) \in X$ and the associated local coodinates $(x_1, \cdots, x_n; \xi_1, \cdots, \xi_n) \longleftrightarrow \sum \xi_i dx_i \in T^*_x(X)$ of the cotangent bundle.

The differential $d\theta^X := \omega^X$ defines an intrinsic symplectic structure on the contangent bundle $T^*(X)$.

(2) With respect to this symplectic form $\omega^X = d\theta^X$ a submanifold $\Lambda \subset T^*(X)$ is said to be a Lagrangian submanifold, if $\dim \Lambda = \dim X$ and the symplectic form ω^X vanishes on it. Or this is the same thing to say that the tangent space $T_\lambda(\Lambda)$ at each point $\lambda \in \Lambda$ is a Lagrangian subspace in the symplectice vector space $T_\lambda(T^*(X))$.

(3) For Lagrangian submanifolds, a cohomology class is defined, called "**Keller-Maslov-Arnold-Hörmander**" characteristic class (shortly call it **Maslov class**) $\mathfrak{m}_\Lambda \in H^1(\Lambda, \mathbb{Z})$ of the Lagrangian submanifold Λ. This is seen as a homomorphism $\mathfrak{m}_\Lambda : \pi_1(X) \to \mathbb{Z}$.

This cohomology class expresses the variation of the intersection of the tangent space $T_\lambda(\Lambda)$ of Λ and the vertical tangent space $\mathrm{Ker}(d\pi^X)_\lambda$ of the natural projection map $\pi^X : T^*(X) \to X$.

(4) Also let's denote the **geodesic flow** by $\{\Phi_t^M\}_{t\in\mathbb{R}}$, $\Phi_t^X : T^*(X) \to T^*(X)$, the Hamilton flow whose Hamiltonian is the principal symbol σ_{Δ^X} of the Laplacian (or the squre root of it).

The geodesics are the projection of the orbits of this flow to the base manifold. So an orbit, or a bi-characteristic curve of the Laplacian carries location and momentum of a classical particle.

Assumptions

We assume a specific Lagrangian submanifold $\Lambda \subset T_0^*(X)$ satisfying the conditions :

Mas[1] $\quad \sigma_{\Delta^X}\big|_{\Lambda} \equiv E_\Lambda$: const $> \mathbf{0}$,

Mas[2] $\quad$ For any closed curve $\{\gamma\}$ in Λ, we assume

$$\frac{1}{2\pi}\int_\gamma \theta^X - \frac{1}{4} < \mathfrak{m}_\Lambda, \gamma > \in \mathbb{Z}$$

Mas[3] there exists a geodesic flow invariant positive measure on Λ ("measure" means a no-where vanishing highest degree differential form, in case Λ is orientable.)

Note that Λ itself is invariant under the geodesic flow action by the condition ***Mas*[1]**.

Existence of eigenvalues of Laplacian

Theorem

(W.Weinstein, Springer LNM No. 459, 1974) : Existence of certain series of eigenvalues)
Under the existence of a Lagrangian submanifold satisfying three condition above, there exist a sequences $\{\lambda_j\}_{j\geq 0}$ of eigenvalues of the Laplacian such that

$$\sup_j \left| \lambda_j - E_\Lambda(dj + 1) \right|^2 < +\infty,$$

that is there are series of eigenvalues closed to the values $\{ E_\Lambda(dj + 1) \}_{j\geq 0}$, where d is one of $1, 2$ or 4 and determined by the condition for $\mathfrak{m}_\Lambda$.

I remark a possible example of such a Lagrangian submanifold.

If the geodesic flow is “completely integrable” with the first integrals $\{\sigma_{\Delta^X} = f_0, f_1, \cdots, f_{m-1}\}$ $(m = \dim X)$, that is

(i) $\Lambda = \{(x,\xi) \in T_x^*(X)\backslash\{0\} \,\big|\, \sigma_{\Delta^X} = E_\Lambda,\ f_i = C_i,\ 1 \leq i \leq m-1\,\}$

(ii) they are Poisson commuting: $\{f_i\,,\ f_j\} \equiv 0$, moreover

(iii) on an open dense subset in $T^*(X)$ their differentials $\{df_i\}$ are linearly independent.

Then we will be able to find such a Laglangian submanifold among the m- dimensionl torus

$$\begin{aligned}&\mathbb{T}_{E,C_1,\dots,C_{m-1}}\\ &= \{\, (x,\xi) \in T_0^*(X) \mid \sigma_{\Delta^X} \equiv E,\\ &\qquad f_1 \equiv C_1, \cdots, f_{m-1} \equiv C_{m-1} \,\},\end{aligned}$$

which also suggest that the Lagrangian submanifold can be seen as a higher dimensional analog of a closed geodesic.

In the more concrete examples like spheres or projective spaces their geodesic flows are not ony completely integrable but periodic with a common period, so that the existence of the invariant measure on Λ will be apparent, but I will give another construction of such a measure, or half density on such cases, which may be interesting.

An out line of the proof and the meaning of the assumptions:

(i) Construction of a conic Lagrangian submanifold L in $T_0^*(S^1) \times T_0^*(X) \subset T_0^*(S^1 \times X)$. Here "conic" means it is invariant under the natural $\mathbb{R}_+$-action (dilation) on the cotangent bundle $T_0^*(S^1 \times X)$.

(ii) Construction of an operator in the class of so called "Fourier integral operator" F

$$F : C^\infty(S^1) \to C^\infty(X)$$

which is an approximate intertwining operator between $-E_\Lambda \dfrac{d^2}{dx^2}$ on S^1 and Δ on X

(iii) There exist eigenvalues closed to the eigenvalues of $-E_\Lambda \frac{d^2}{dx^2}$.

Fourier integral operator has a form

$$\int e^{\sqrt{-1}\varphi(x,y;\eta)} a(x, y; \eta) f(y) dy d\eta$$

with a phase function φ which is defined by the conic Lagrangian submanifold micro locally and a homogeneous function of the variable η and of degree one.

The condition $Mas[1]$ is used to construct the conic Lagrangian submanifold and the condition $Mas[2]$ is used for the global definition of the operator together with the conditon $Mas[3]$ for the (global) construction of the amplitude function $a = a(x, y; \eta)$, by which the operator will be an isometric operator from $L_2(S^1)$ to $L_2(X)$.

Here I explain precisely the construction of a conic Lagrangian submanifold from a compact Lgrangian submanifold Λ satisfying Maslov quantization condition Mas[2].

Parametrization of Lagrangian submanifolds by phase functions

Let Λ be a Lagrangian submanifold in the cotangent bundle $T^*(X)$. Then the canonical one-form θ^X restricted to Λ is a closed form and so, it defines a de Rham cohomology class $[\theta^X{}_{|\Lambda}] \in H^1_{dR}(\Lambda)$.

Let $p \in \Lambda$, then there exists a coordinate neighborhood U of $\pi^X(p)$, an open subset D in $\mathbb{R}^k$ and a smooth function $\phi \in C^\infty(U \times D)$ satisfying the following properties:
Put

$$C_\phi$$
$$= \left\{(x,\eta) \in U \times D \,\middle|\, \frac{\partial\phi(x,\eta)}{\partial\eta_j} = \phi_{\eta_j} = 0,\ j \leq \dim D\right\}$$

Then,

(1) the one-forms $\left\{d\phi_{\eta_j}\right\}_{j=1}^{k=\dim D}$ are linearly independent on C_ϕ,

(2) the map

$$\rho_\phi : C_\phi \ni (x,\eta) \longmapsto (x;\phi_x) \leftrightarrow \sum_{j=1}^{n} \frac{\partial\phi(x,\eta)}{\partial x_j} dx_j$$

$$= \sum_{j=1}^{n} \phi_{x_j} dx_j \in \Lambda \subset T^*(X)$$

is a *diffeomorphism* between C_ϕ and $\rho_\phi(C_\phi) =: L_\phi \subset L$, when we take U (and D) small enough(the differential $d\rho_\phi$ is injective.

Relations of the map ρ_ϕ and other maps can be seen from the commutative diagram:

$$
\begin{tikzcd}
T^*(U \times D) \arrow[dd, shift right, "\pi^{U \times D}"'] & \pi_U{}^*(T^*(U)) \arrow[l, hook', "\chi_{\pi_U}"'] \arrow[rr, "\mathfrak{p}_{\pi_U}"] & & T^*(U) \arrow[rr, hook] & & T^*(X) \arrow[dd, "\pi^X"] \\
& C_\phi \arrow[ul, "d\phi_{|C_\phi}"'] \arrow[dl, hook'] \arrow[r, "\rho_\phi"] & L_\phi \arrow[ur, hook] \arrow[rr, hook] \arrow[dr, "\pi^X{}_{|L_\phi}"'] & & L \arrow[ur, hook] \arrow[dr, "\pi^X{}_{|L}"] & \\
U \times D \arrow[uu, shift right, "d\phi"'] \arrow[rrr, "\pi_U"'] & & & U \arrow[rr, hook] & & X
\end{tikzcd}
$$

where $\pi_U : U \times D \to U$ is the projection map. The map $\mathfrak{p}_{\pi_U}$ is the natural projection map from the induced bundle $\pi_U{}^*(T^*(U))$ on $U \times D$ to the original bundle $T^*(U)$ and it is a submersion. The map $\chi_{\pi_U} := (d\pi_U)^*$ is the dual map of the differential $d\pi_U : T(U \times D) \to \pi_U{}^*(T(U))$ and it is an embedding.

Also note that

$$\pi^X \circ \mathfrak{p}_{\pi_U} = \pi_U \circ \pi^{U\times D} \circ \chi_{\pi_U}.$$

The subset C_ϕ can be characterized as

$$C_\phi = \{(x,\eta) \in U \times D \mid d\phi\,(x,\eta) \in \chi_{\pi_U}({\pi_U}^*(T^*(U))\}.$$

We call C_ϕ or $\rho_\phi(C_\phi) = L_\phi$ a local parametrization of L by a phase-function-triple $(U \times D, \phi, \rho_\phi)$, where we always assume that the open subset U is taken small enough for the map ρ_ϕ to be a diffeomorphism.

Put the functions $\psi_i \in C^\infty(L_{\phi_i})$ by $\psi_i := \phi_i \circ \rho_{\phi_i}^{-1}$. By the definition of the map ρ_{ϕ_i}, we have $d\psi_i = \theta^X{}_{|L}$ on L_{ϕ_i}, and

$$0 = d\psi_1 - d\psi_2 = d(\psi_1 - \psi_2).$$

Hence the difference $\psi_1 - \psi_2$ is locally constant on $L_{\phi_1} \bigcap L_{\phi_2}$.

Now let L be a Lagrangian submanifold and $\{L_{\phi_i}\}$ a covering by local parametrizations defined by the phase-function-triples Pft(L):= $\left\{(U_i \times D_i, \phi_i, \rho_{\phi_i})\right\}_{i \in S}$:

$$L = \bigcup_{i \in S} L_{\phi_i}.$$

Then,

Proposition

The set of locally constant functions

$$\{ c_{ij} = \phi_j \circ \rho_{\phi_j}^{-1} - \phi_i \circ \rho_{\phi_i}^{-1} : L_{\phi_j} \cap L_{\phi_i} \to \mathbb{R} \}_{i\,j \in S}$$

defines an $\mathbf{1}$-$\check{C}$ech cocycle with the values in $\mathbb{R}$, that is its cohomology class corresponds to the de Rham cohomology class of $\theta^X{}_{|L}$ according to the fine resolution of $\mathbb{R}$-constant sheaf $\mathbb{R}_L$ on L by the sheaves of differential forms on L.

We call a Lagrangian submanifold $L \subset T^*_0(X)$ "**quasi-integral** " (or "**integral**)", if there exists a positive constant c_0 (integral in case $c_0 = 1$) such that the de Rham cohomology class $c_0[\theta^X{}_{|L}]$ is in $\check{H}^1(L, \mathbb{Z}_L) \subset \check{H}^1(L, \mathbb{R}_L) \cong H^1_{dR}(L)$, where the inclusion is the induced map from the natural inclusion map $\mathbb{Z}_L \subset \mathbb{R}_L$ of constant sheaves and the natural isomorphism between the de Rham cohomology group and the $\check{C}$ech cohomology group.

This is equivalent to assume that for any smooth closed curve $\{\gamma(t)\}$ in L, the integral

$$c_0 \int_\gamma \theta^X \in \mathbb{Z}. \tag{1}$$

Hence, in this case the Lagrangian submanifold

$$L_0 := c_0 \cdot L = \{(x, c_0\xi) \mid (x,\xi) \in L\}$$

is integral and the cohomology class $[\theta^X{}_{|L_0}] \in \check{H}^1(L_0, \mathbb{Z}_{L_0})$.

Remark

By the induced map $\check{H}^1(c_0 \cdot L, \mathbb{Z}_{c_0 \cdot L}) \to \check{H}^1(L, \mathbb{Z}_L)$ from the diffeomorphism $L \xrightarrow{\approx} c_0 \cdot L$, the class $[\theta^X{}_{|c_0 \cdot L}]$ is mapped to the class $c_0 \cdot [\theta^X{}_{|L}]$.

If $L \subset T_0^*(X)$ is an integral Lagrangian submanifold then for any positive integer $k \in \mathbb{N}$, $k \cdot L$ is also integral.

Remark on conic Lagrangian submanifolds

The positive real numbers $\lambda \in \mathbb{R}_+$ act on $T^*(X)\backslash\{0\}$,

$$(x;\xi) \longleftrightarrow \sum \xi_i dx_i \mapsto \lambda \sum \xi_i dx_i = \lambda \cdot \sum \xi_i dx_i$$
$$\longleftrightarrow (x;\lambda\xi),$$

where $T^*(X)\backslash\{0\}$ means the zero section removed cotangent space and we call it the **punctured cotangent bundle** and sometimes we denote it by $T^*_0(X)$.

The space $T^*_0(X)$ is a cone bundle over the quotient space $T^*_0(X)/\mathbb{R}_+ := S^*(X)$ which is naturally a contact manififold and we call it the **cotangent sphere bundle** .

If a Lagrangian submanifold $L \subset T^*_0(X)$ (closed in $T^*_0(X)$) is invariant under the action of $\mathbb{R}_+$, we call it a **conic Lagrangian submanifold**.
On such a Lagrangian submanifold,
the canonical one-form θ^X vanishes and vise versa.

In this case the phase functions ϕ can be taken as defined on an open cone, and the phase function $\phi = \phi(x, \eta) \in C^\infty(U \times D)$ is homogeneous of degree 1 with respect to the variable $\eta \in D \subset \mathbb{R}^k \backslash \{0\}$.

Proposition

Let L be an integral Lagrangian submanifold in $T^*_0(X)$.
(1) *There exists a function $\vartheta : L \to U(1)$ (mod $C^\infty(L)$) such that ϑ is mapped to the cohomology class $[\theta^X{}_{|L}]$, that is ϑ expresses the cohomology class $[\theta^X{}_{|L}]$ through the connecting homomorphism $\delta : C^\infty(L, U(1)) \to \check{H}^1(L, \mathbb{Z}_{\mathbb{L}})$ associated with the exact sequence of sheaves on L:*

$$\{0\} \longrightarrow \mathbb{Z}_L \longrightarrow \mathcal{F}(L, \mathbb{R}) \xrightarrow{f \mapsto e^{2\pi\sqrt{-1}f}} \mathcal{F}(L, U(1)) \longrightarrow \{0\},$$

where $\mathcal{F}(L, \mathbb{R})$ is the sheaf of germs of real valued smooth functions on L and $\mathcal{F}(L, U(1))$ is a sheaf of germs of smooth functions taking values in $U(1)$.

In fact, once we fix a set of a covering of L by local parametrizations $\{L_{\phi_i}\}$, by the phase-function-triples *Pft*$(L) := \{(U_i \times D_i, \phi_i, \rho_{\phi_i})\}$, then a function ϑ is given by $\vartheta = e^{2\pi\sqrt{-1}\,\phi_i \circ \rho_{\phi_i}^{-1}}$ on L_{ϕ_i}, since

$$e^{2\pi\sqrt{-1}\left(\phi_j \circ \rho_{\phi_j}^{-1} - \phi_i \circ \rho_{\phi_i}^{-1}\right)} \equiv 1$$

on $L_{\phi_i} \bigcap L_{\phi_j}$.

(2) Let

$$\hat{L} = \left\{ (x; \tau \cdot \xi\, , \overline{\vartheta(x;\xi)}\, ; \tau) \,\middle|\, (x;\xi) \in L, \tau > 0 \right\}.$$

Then $\hat{L}$ is a conic Lagrangian submanifold in $T_0^*(X) \times T_0^*(U(1))$.

In fact, it is covered by local parametrizations defined by the phase-function-triples

$$Pft(\hat{L}) := \{\, (U_i \times \hat{D}_i, \hat{\phi}_i, \rho_{\hat{\phi}_i}) \,\},$$

where we define a conic open subset $\hat{D}_i \subset \mathbb{R}^{k_i+1} \backslash \{0\}$ by

$$\hat{D}_i = \left\{ (v, \tau) \in \mathbb{R}^{k_i} \times \mathbb{R}_+ \,\middle|\, 1/\tau \cdot v \in D_i \right\}$$

and a phase function $\hat{\phi}_i$ by

$$
\begin{aligned}
C^\infty(U_i \times \mathbb{R} \times \hat{D}_i) \ \ni \ \hat{\phi}_i(x, t, v, \tau) \\
:= \tau\phi_i(x, 1/\tau \cdot v) + \tau\, t, \ (x, 1/\tau \cdot v) \in U_i \times D_i.
\end{aligned}
$$

The assertion with respect to the (local) parametrization is a basic fact in the theory of FIO,

L. Hörmander, *Fourier integral operators I*, Acta Math. **127** (1971), 79–183.

The following paper is also very carefully written and is usueful for our standing point of view:

A. Yoshikawa, *On Maslov's canonical operator*, Hokkaido Math. J. **4** (1975), 8–38.

Essential part of the proof

Let's consider the equations:

$$\frac{\partial \hat{\phi}_i(x,t,v,\tau)}{\partial v_j} = \frac{\partial \phi_i}{\partial \eta_j}(x, 1/\tau \cdot v) = 0$$

and

$$\begin{aligned}\frac{\partial \hat{\phi}_i(x,t,v,\tau)}{\partial \tau} &= \phi_i(x, 1/\tau \cdot v) - \sum_{j=1}^{k} \frac{v_j}{\tau}\frac{\partial \phi_i}{\partial \eta_j} + t \\ &= \ \phi_i(x, 1/\tau \cdot v) \ + \ t \ = 0.\end{aligned}$$

Then we can characterize the set

$$C_{\hat{\phi}_i} = \{ (x, t, v, \tau) \mid (x, 1/\tau \cdot v) \in C_{\phi_i},$$
$$t + \phi_i(x, 1/\tau \cdot v) = 0,\ \tau \in \mathbb{R}_+ \}.$$

Note that we may assume that the range of the phase functions ϕ_i on $U_i \times D_i$ are included in a sufficiently small interval so that the values $e^{-2\pi\sqrt{-1}\phi_i}$ are included in a small arc and the maps $\rho_{\hat{\phi}_i}$ are given as

$$\rho_{\hat{\phi}_i} : C_{\hat{\phi}_i} \ni (x, t, v, \tau) \mapsto \Big(\tau \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j}(x, 1/\tau \cdot v) dx_j,\ \tau dt\Big)$$
$$\in \hat{L}_{\hat{\phi}_i} \subset T_0^*(U_i) \times T_0^*(U(1)),$$

where $\tau dt \in T^*_{e^{-2\pi\sqrt{-1}\phi_i(x, 1/\tau \cdot v)}}(U(1))$.

Proposition

Let $L \subset T_0^*(X)$ *be an integral Lagrangian submanifold, then the corresponding conic Lagrangian submanifold to the integral Lagrangian submanifold* $k \cdot L$ *is given as*

$$\widehat{k \cdot L} = \{(x\,;\tau k\xi\,, \overline{\vartheta^k(x\,;\xi)}, \tau) \;\Big|$$

$$(x;\xi) \in L,\ \vartheta : L \to U(1),\ \tau > 0\,\},$$

where $\vartheta : L \to U(1)$ *is the map constructed in the preceding Proposition.*

On the other hand let $L \subset T_0^*(X)$ be a quasi-Lagrangian submanifold and we assume that $k \cdot L$ is integral with a positive integer k.

Let $\vartheta : k \cdot L \to U(1)$ be the map constructed in the above Proposition and consider a manifold

$$\overline{L} = \{(x; \xi\,,\, e^{2\pi\sqrt{-1}\,s}) \in L \times U(1) \mid \\ \vartheta(x\,;k\cdot\xi) = e^{2\pi\sqrt{-1}\,k\,s}\}.$$

Since the map $\vartheta(x\,;k\cdot\xi)$ is given locally by $\vartheta(x\,;k\cdot\xi) = e^{2\pi\sqrt{-1}\,k\;\phi\circ\rho_\phi{}^{-1}(x,\xi)}$ with a non-degenerate phase function ϕ of L, $d\phi \neq 0$ (because $L \subset T_0^*(X)$), the subset $\overline{L}$ is a smooth submanifold in $L \times U(1)$ and is a k-hold covering of L.

Now consider the map

$$\tilde{\rho} : \overline{L} \times \mathbb{R}_+ \longrightarrow T_0^*(X) \times T_0^*(U(1)),$$

$$\left(x; \xi\,,\ e^{2\pi\sqrt{-1}\,s}; \tau\right) \longmapsto \left(x;\ \tau\xi\,,\ e^{-2\pi\sqrt{-1}\,s}; \tau\right)$$

$$= \left(\tau \sum_{i=1}^{n} \xi_i dx_i\,,\ \tau dt\right) \in T_0^*(X) \times T_0^*(U(1)),$$

where $dt \in T^*_{e^{-2\pi\sqrt{-1}\,s}}(U(1))$.

Then

Proposition

The map $\tilde{\rho}$ is an embedding and the image is a closed conic Lagrangian submanifold in $T_0^*(X) \times T_0^*(U(1))$.

If $k = 1$, that is if L is integral, $\tilde{\rho}(\overline{L}) = \hat{L}$, so we also denote $\tilde{\rho}(\overline{L} \times \mathbb{R}_+)$ by $\hat{L}$.

Submersion and Maslov quantization condition

$\varphi : M \longrightarrow N$: Riemannian submersion:

(1) φ is a surjective submersion ($\dim M \geq \dim N$,

(2) When we decompose the tangent bundle $T(M)$ into $T(M) = \mathcal{V} \oplus_{\perp} \mathcal{H}$, where $\mathcal{V} = \mathrm{Ker}\, d\varphi$, the vertical subbundle and $\mathcal{H} = \mathcal{V}^{\perp}$ its orthogonal complement, we assume that the inner product in $\mathcal{H}$ has the following property that for any two points $x,\ x' \in M$ with $\varphi(x) = \varphi(x')$, $\mathcal{H}_x$ and $\mathcal{H}_{x'}$ are isometric through the differentials $\mathcal{H}_x \overset{d\varphi}{\to} T_{\varphi(x)}(N) = T_{\varphi(x')}(N) \overset{d\varphi}{\leftarrow} H_{x'}$. Hence the Riemannian metric on N coincides with the inner product in $\mathcal{H}$ through the map $d\varphi$.

We have a commutative diagram:

$$
\begin{tikzcd}
T_0^*(M) \arrow[dr, "\pi^M"'] & \varphi^*(T_0^*(N)) \arrow[l, hook', "\chi"'] \arrow[r, "\mathcal{P}_\varphi"] \arrow[d, "\pi^M"] & T_0^*(N) \arrow[d, "\pi^N"] \\
 & M \arrow[r, "\varphi"'] & N,
\end{tikzcd}
$$

where the map $\mathcal{P}_\varphi$ is the natural projection from the induced bundle to the original (tangent)bundle. It is also a submersion. The space $\varphi^*(T_0^*(N))$ is regarded as a submanifold in $T_0^*(M)$ through the dual map $\chi = (d\varphi)^*$ of the differential $d\varphi : T(M) \to \varphi^*(T(N))$. Note that by the assumption the map $d\varphi$ is surjective. The map π^M (also π^N) is the natural projection to the base manifold.

Let Δ^M and Δ^N be the Laplacian on M and N respectively. Then,

Theorem

B. Watson, Manifold maps commuting with the Laplacian, J. Diff. Geom. 8(1973)

$$\Delta^M \circ \varphi^* = \varphi^* \circ \Delta^N, \quad \textit{where} \ \ \varphi^* : C^\infty(N) \to C^\infty(M)$$

if and only if the fibers of the map φ are minimal, or equivalently the map φ is harmonic.

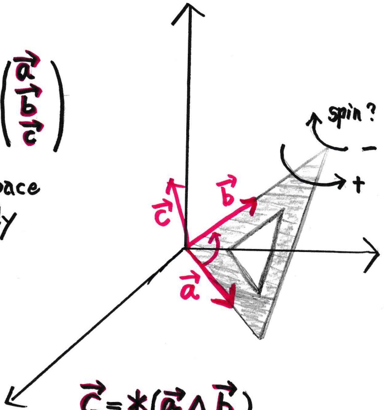
In this case of Theorem above, if $f \in C^\infty(N)$ is an eigenfunction of Δ^N then $\varphi^*(f)$ is an eigenfunction of Δ^M with the same eigenvalue.

If we consider the base space of the Riemannian submersion as a configulation space of a physical system, then the fiber in the total space can be regarded as describing a structure (more fine data) of a point (= a state) together with the description by the submersion map φ. An example:

$$S^3 \cong Spin(3) \xrightarrow[\text{covering}]{\text{double}} \begin{pmatrix} a \\ b \\ *(a \wedge b) \end{pmatrix} \in SO(3), \quad \text{the matrix}$$

$\begin{pmatrix} a \\ b \\ *(a \wedge b) \end{pmatrix}$ can be seen as a one point fixed rigid body, where $a \perp b$ and $*(a \wedge b)$ is the exterior product and back to the original space by Hodge star operator.

$$\begin{array}{ccccc} \mathrm{Spin}(3) & \longrightarrow & SO(3) & \ni & \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} \\ \cong & & \| & & \\ S^3 & & \text{Configuration space of a rigid body} & & \end{array}$$



$$\vec{c} = *(\vec{a} \wedge \vec{b})$$

The points in S^3 distinguish “spins” of a rigid body in micro states and wil not appear in the macro states.

Hence I expected the existence of eigenvalues of the Laplacian on the total space corresponding to the eigenvalues of the Laplacian on the base space without the property described in the above Proposition, that is, only under the assumption of Riemannian submersion, but based on the existence of a compact Lagrangian submanifold satisfying the conditions $Mas[1] \sim Mas[3]$.

This is one expectation and one more is the following proposition:

Proposition

Let $\varphi : M \to N$ be a submersion and Λ be a closed Lagrangian submanifold in $T_0^*(N)$. Then the manifold $\mathcal{P}^{-1}(\Lambda)$ is also a Lagrangian submanifold in $T_0^*(M)$.

Proof

Recall the diagram:

$$
\begin{tikzcd}
T_0^*(M) \arrow[dr, "\pi^M"'] & \varphi^*(T_0^*(N)) \arrow[l, hook', "\chi"'] \arrow[r, "\mathcal{P}"] \arrow[d, "\pi^M"] & T_0^*(N) \arrow[d, "\pi^N"] \\
 & M \arrow[r, "\varphi"'] & N,
\end{tikzcd}
$$

We can find a local coordinates system $(x, y) \in U \times V \cong W \subset M$, $U \times V \subset \mathbb{R}^{n+d}$, around a point $p \in W \subset M$ such that the map φ is given by the projection $(x, y) \longmapsto x$.

Then a coodinates system in $T^*(M)$ on an open subset $(\pi^M)^{-1}(W)$ is given by the correspondence

$$U \times V \times \mathbb{R}^n \times \mathbb{R}^d \ni (x, y\,; \xi, \eta)$$
$$\longleftrightarrow \sum \xi_i dx_i + \sum \eta_j dy_j \in T^*((\pi^M)^{-1}(W))$$

and a coordinates system on an open subset $(\pi^N)^{-1}(\tilde{U}) \cong U \times \mathbb{R}^n$ ($\tilde{U} \subset N$) is given by

$$U \times \mathbb{R}^n \ni (x\,; \xi) \leftrightarrow \sum \xi_i dx_i.$$

Then the canonical two forms ω^M on $T^*(M)$ and ω^N on $T^*(N)$ are expressed as
$\omega^M = \sum d\xi_i \wedge dx_i + \sum d\eta_j \wedge dy_j$ and $\omega^N = \sum d\xi_i \wedge dx_i$ by these coodinates.

Since the map $\mathcal{P}$ is a submersion,
$\mathbf{dim}\,\mathcal{P}^{-1}(\Lambda) = \mathbf{dim}\,N +$ (fiber dim of φ) $= \mathbf{dim}\,M$ and $\mathcal{P}^*(\omega^N) = \chi^*(\omega^M)$. In fact, locally $\mathcal{P}^{-1}(\Lambda)$ is given as

$$\mathcal{P}^{-1}(\Lambda) \cap (\pi^M)^{-1}(W) \cong \{(x, y\,; \xi, 0) \mid x \in U\}$$

and so $\sum d\xi_i \wedge dx_i + \sum d\eta_j \wedge dy_j$ on $\mathcal{P}^{-1}(\Lambda)$ coindides with $\sum d\xi_i \wedge dx_i$. ■

From the arguments above, more generally, if $\boldsymbol{\Lambda}$ is an isotropic submanifold, then $\mathcal{P}^{-1}(\boldsymbol{\Lambda})$ is also isotropic. Also if $\boldsymbol{\Lambda}$ is conic, then $\mathcal{P}^{-1}(\boldsymbol{\Lambda})$ is conic, and if $\boldsymbol{\Lambda}$ is compact, then $\mathcal{P}^{-1}(\boldsymbol{\Lambda})$ is also compact, since we assumed M is compact.

On the other hand, let $\tilde{\Lambda}$ be a Lagrangian submanifold included in $\varphi^*(T^*_0(N))$.

Proposition

We assume $\mathcal{P}^{-1}(\mathcal{P}(\tilde{\Lambda})) = \tilde{\Lambda}$ (as a set), then $\mathcal{P}(\tilde{\Lambda})$ is a Lagrangian submanifold.

In particular, if the fibers of the submersion φ are connected, then the condiion $\mathcal{P}^{-1}(\mathcal{P}(\tilde{\Lambda})) = \tilde{\Lambda}$ is automatically satisfied.

Proof.

Let $\varphi : (x, y) \mapsto x$ be local coordinates as before. Then

$$\mathcal{P} : \varphi^*(T^*(N)\backslash\{0\}) \ni (x, y; \xi, 0) \\ \mapsto (x, \xi) \leftrightarrow \sum \xi_i dx_i \in T^*_x(N)$$

Let $f_0, \cdots, f_{m-1}$ be local defining functions of $\tilde{\Lambda}$. Then by the assumption, the variables $\{y_j\}$ are free, so that we may assume $f_0(x, y, \xi) = f_0(x, \xi), \cdots, f_{n-1}(x, y, \xi) = f_{n-1}(x, \xi)$, which is a set of local defining functions of $\mathcal{P}(\tilde{\Lambda})$. The Lagrangian property follows in the same way as before. $\square$

So in the both directions, a submanifold being a Lagrangian is rather a mild condition.

Based on these facts it may be worthwhile to consider if Λ or $\tilde{\Lambda}$ satisfies the conditions $\boldsymbol{Mas[1] \sim Mas[3]}$, then how is the Lagrangian submanifold $\mathcal{P}^{-1}(\Lambda)$ or $\mathcal{P}(\tilde{\Lambda})$?

So assume Λ or $\tilde{\Lambda}$ satisfies the conditions $\boldsymbol{Mas[1] \sim Mas[3]}$.

Let Φ_t^M and Φ_t^N be the geodesic flow on M and N respectively. Then

Lemma The following diagram is commutative and the commutativity follows from only the assumption of the Riemannian submersion (= coincidence of the values σ_{Δ^M} and $\mathcal{P}^*(\sigma_{\Delta^N})$ on $\varphi^*(T^*(N)\backslash\{0\})$):

$$\begin{array}{ccc}
T_0^*(M) & \xrightarrow{\Phi_t^M} & T_0^*(M) \\
\chi \uparrow & & \uparrow \chi \\
\varphi^*(T_0^*(N)) & \longrightarrow & \varphi^*(T_0^*(N)) \\
\mathcal{P} \downarrow & & \downarrow \mathcal{P} \\
T_0^*(N) & \xrightarrow{\Phi_t^N} & T_0^*(M).
\end{array}$$

This implies that not only the space $\mathcal{P}^{-1}(\Lambda)$ (or in the case we start from $\tilde{\Lambda}$ in $\varphi^*(T_0^*(N))$) but also the space $\varphi^*(T_0^*(N))$ being invariant under the action of the geodesic flow $\left\{\Phi_t^M\right\}$.

As for the condition $\boldsymbol{Mas[2]}$ we assume that $\boldsymbol{\Lambda}$ is satisfying

$$\frac{1}{2\pi}\int_{\gamma}\theta^{N} - \frac{1}{4} < \mathcal{M}_{\Lambda}, \gamma > \in \mathbb{Z},$$

for any loop $\{\gamma(t)\} \subset \Lambda$,
we can prove by noting the facts:

- $$\int_{\tilde{\gamma}}\theta^{M} = \int_{\tilde{\gamma}}\mathcal{P}^{*}(\theta^{N}) = \int_{\mathcal{P}(\tilde{\gamma})}\theta^{N}$$

where $\{\tilde{\gamma}\}$ is a loop in $\mathcal{P}^{-1}(\Lambda)$

- $$\mathcal{M}_{\mathcal{P}^{-1}(\Lambda)} = \mathcal{P}^{*}(\mathcal{M}_{\Lambda}).$$

Hence

Lemma

The condition $\color{red}{\boldsymbol{Mas}[2]}$ *is satisfied by* $\mathcal{P}^{-1}(\boldsymbol{\Lambda})$. *Also it is OK for the case that* $\mathcal{P}(\tilde{\boldsymbol{\Lambda}})$.

As for the condition $\color{red}{\boldsymbol{Mas}[3]}$, there is a difference.

(1) First, let $\tilde{\mu}$ be an invariant mesure on $\tilde{\boldsymbol{\Lambda}}$ under the geodesic flow action of $\{\boldsymbol{\Phi}_t^M\}$. Then in this case the pushforward measure $\mathcal{P}_*(\tilde{\mu})$ is an invariant measure on $\mathcal{P}(\tilde{\boldsymbol{\Lambda}})$ under the geodesic flow action of $\{\boldsymbol{\Phi}_t^N\}$.

(2) Let $\mu_\Lambda \in \Gamma(\bigwedge^{max} T^*(\Lambda))$ be a volume form on Λ which is invariant under the geodesic flow $\{\Phi_t^N\}$ action.

Let dv^M and dv^N be the Riemannian volume forms on M and N respectively. According to the assumption of the Riemannian submersion and the decomposition $T(M) = \mathcal{V} \oplus_\perp \mathcal{H}$, there is a differential form $\theta_{V^*} \in \Gamma(\bigwedge^{max} \mathcal{V}^*)$ such that $dv^M = \theta_{V^*} \bigwedge \varphi^*(dv^N)$.

With these, the volume form $\tilde{\mu}$ on $\mathcal{P}^{-1}(\Lambda)$ is expressed as

$$\tilde{\mu} = w \cdot (\pi^M)^*(\theta_{V^*}) \bigwedge \mathcal{P}^*(\mu_\Lambda)$$

with a positive function $w \in C^\infty(\mathcal{P}^{-1}(\Lambda))$.

We assume a measure $\tilde{\mu} = w \cdot (\pi^M)^*(\theta^{V^*}) \wedge \mathcal{P}^*(\mu_\Lambda)$ is invariant under the action of $\{\Phi_t^M\}$, then since the form $\mathcal{P}^*(\mu_\Lambda)$ is invariant under the action of $\{\Phi_t^M\}$ by the assumption, the form $(\pi^M)^*(\theta_{V^*})$ is also invariant under the action of $\{\Phi_t^M\}$. The opposite is clear, so that we should assume there exist an invariant measure on $\mathcal{P}^{-1}(\Lambda)$, and it is equivalent to assume the form $(\pi^M)^*(\theta_{V^*})$ is invariant under the action of $\{\Phi_t^M\}$.

Or another type assumption on the submersion $\varphi : M \to N$ can be putted on. For example, if the geodesic flow $\{\Phi_t^M\}$ is periodic (M is called a C_ℓ-*manifold*), then we can find an invariant measure without any assumption on the submersion $\varphi : M \to N$. Maybe in this case the possible such Riemannian submersion will be highly restricted.

Remark

Let $C : T_0^*(N) \to T_0^*(N)$ be a homogeneous symplectomorphism and denote by G'_C its sign changed (or twisted)*graph*;

$$G'_C = \left\{(x, \xi\,;\, y, -\eta) \,\middle|\, (x, \xi) \in T_x^*(N)\backslash\{0\}, (y, \eta) = C(x, \xi)\right\}$$
$$\subset T_0^*(N) \times T_0^*(N) \subset T_0^*(N \times N).$$

Then this is a Lagrangian submanifold in $T^*(N \times N)$. In this case the Lagrangian submanifold $\mathcal{P}^{-1}(G'_C)$ of the product map $\varphi \times \varphi : M \times M \to N \times N$ is not an interesting one, nor it is not a sign changed *graph* of a symplectomorphism of $T_0^*(M) \to T_0^*(M)$.

Sub-Riemannian case

Now we discuss sub-Riemannian situation of the submersion. There are various cases and I only restrict the discussion to special cases.

(1) The subbundle $\mathcal{H}$ is bracket generating, that is we assume

$$\Gamma(\mathcal{H}) + [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] = \Gamma(T(M)),$$

or it is the same thing that $\Gamma(\mathcal{V}) = [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})]$.
Instead of the Laplacian Δ^M, we can define the sub-Laplacian intrinsically which can be seen as a horizontal part of the Laplacian Δ^M and can be treated as before for the Riemannian submersion case.

(2) Next most simple case will be the case that there exists a subbundle $\mathcal{H}^1_x \subset \mathcal{H}$ such that if $\varphi(x) = \varphi(x')$ then

$$\mathcal{H}^1 \overset{d\varphi}{\to} d\varphi(\mathcal{H}^1_x) = d\varphi(\mathcal{H}^1_{x'}) \overset{d\varphi}{\leftarrow} \mathcal{H}^1_{x'},$$

and isometrically isomorphic

$$\Gamma(\mathcal{V}) + [\Gamma(\mathcal{H}^1), \Gamma(\mathcal{H}^1)] = \Gamma(T(M)).$$

In this case we put $d\varphi(\mathcal{H}^1) = \overline{\mathcal{H}^1}$. Then

$$\Gamma(T(N)) = \Gamma(\overline{\mathcal{H}^1}) + [\Gamma(\overline{\mathcal{H}^1}), \Gamma(\overline{\mathcal{H}^1})],$$

that is the subbundle $\mathcal{H}^1$ is a 2 step sub-Riemannian structure, so that we can define a sub-Laplacian on N intrinsically.

Theorem

If there exists a compact Lagrangian submanifold in $T^*(N)$ satisfying the conditions $Mas[1] \sim Mas[3]$, then similar existence theorem for the eigenvalues of sub-Laplacians hold, where we replace the invariance of the geodesic flow action by the invariance of bi-characteristic flow actions. The difference comes from the existence of non-trivial characteristic varieties, but this does not affect the conclution.

Some example of integral Lagrangian submanifolds

[I] *A general case*

The image of an one-form $\eta : X \to T^*(X)$, $\eta(X)$, is a Lagrangian submanifold $\iff$ η is closed.

Hence if there exists such a closed one-form η that $\eta(X) \subset T_0^*(X)$, then the cohomology class $[\eta] \neq 0$, that is, η is not exact, hence $\mathbf{dim}\, H^1(X) \geq 1$.

Here is a problem.

Is it true, for any cohomology class $\alpha \in H^1_{dR}(X)$, is there a representative η by a nowhere vanishing closed one-form.

For our case (Mas[2] is to be satisfied), the Lagrangian submanifold must be at least quasi-integral, and from the relation $\eta^*(\theta^X) = \eta$ we may assume the cohomology calss $[\eta] \in H^1(M, \mathbb{Z})$.

So assume there exists a nowhere vanishing closed one-form η whose cohomology class is in $H^1(X, \mathbb{Z})$, then there exists a fucntion $\vartheta : X \to U(1)$ such that

$$\delta([\vartheta]) = [\eta],$$

where δ is the connecting homomorphism $C^\infty(X, U(1)) \xrightarrow{\delta} \check{H}(X, \mathbb{Z}) \cong H^1(X, \mathbb{Z}) \subset H^1_{dR}(X)$ and the function ϑ is a submersion.

Let $\{\phi_i\}$ be a family of local solutions $d\phi_i = \eta$, then the functions $\{\phi_i \circ \eta^{-1}\}$ are phase functions locally parametrizing the Lagrangian submanifold $L = \eta(X)$. Hence on a Riemannian manifold X if there is such an one-form η and the Lagrangian submanifold $L = \eta(X)$ satisfying another two conditions Mas[1] an Mas[3], then we can apply Eigenvalue Theorem to X.

Conversely, let $\psi : X \to U(1)$ be a submersion, then a closed one-form η corresponding $\check{C}$eck cohomology class $\delta([\psi]) \in \check{H}^1(X, \mathbb{Z})$ never vanish and define a Lagrangian submanifold $\eta(X)$ in $T_0^*(X)$.

In fact, let $\{f_i\}$ be the local solutions of the equation $e^{2\pi\sqrt{-1}f_i} = \psi$, then by the assumption the differences $f_j - f_i$ are locally constant,
hence

$$df_j = df_i,$$

which define a globally defined one-form $\eta = df_i$ and does not vanish anywhere, since the function ψ is a submersion.

Note that in this case if $\dim H^1(X) = 1$, we can always make such a Lagrangian submanifold $\eta(X)$ to be integral by applying a suitable rational number to the value $\frac{1}{2\pi}\int_\gamma \theta^X$, or to the one-form η in advance.

As for the condition Mas[2], the Maslov class $\mathfrak{m}_{\eta(X)}$ is zero, Hence if such a representative of an integral cohomology class $H^1_{dR}(X)$ only non-trivial condition is Mas[3] after changing the Riemannian metric.

[II] *sphere case*

We realize the cotangent bundle of the sphere S^n $(n \geq 3)$ as

$$T_0^*(S^n) = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \\ |x| = 1, \ < x, \xi >= 0, \xi \neq 0\}$$

By this realization, the Liouville one-form θ^{S^n} and the symplectic form ω^{S^n} on $T^*(S^n)$ are expressed as

$$\theta^{S^n} = \sum \xi_i \, dx_i, \ \omega^{S^n} = \sum d\xi_i \wedge dx_i,$$

that is these can be seen as restrictions of those on $\mathbb{R}^{n+1}$.

Then by the map $\tau_S : T_0^*(S^n) \longrightarrow \mathbb{C}^{n+1}$

$$\tau_S : (x, \xi) \longmapsto z = |\xi| x + \sqrt{-1}\xi$$

$$T_0^*(S^n) \overset{\tau_S}{\cong} Q_2 = \{z \in \mathbb{C}^{n+1} \backslash \{0\} \mid z^2 = \sum z_i^2 = 0\}$$

and the symplectic form is expressed as

$$\omega^{S^n} = 2\sqrt{-1}\,\bar{\partial}\partial\,|z|.$$

More over in this realization of the space $T_0^*(S^n)$, the geodesic flow is the scalar multiplication by modulus 1 complex numbers.

We consider an $n+1$-dimensional submanifold Z in Q_2 defined by

$$Z = \{e^{\sqrt{-1}\tau}(s_0, \ldots, s_p, \sqrt{-1}t_{p+1}, \ldots, \sqrt{-1}t_{n-p}) \mid \tau, s_i, t_j \in \mathbb{R}, \text{ and } \sum s_i^2 = \sum t_j^2 > 0\},$$

where $p \geq 2$ and $n-p \geq 3$ (hence $n \geq 5$).
Especially

$$Z_1 = \{e^{\sqrt{-1}\tau}(s_0, \ldots, s_p, \sqrt{-1}t_{p+1}, \ldots, \sqrt{-1}t_{n-p}) \mid \tau, s_i, t_j \in \mathbb{R}, \text{ and } \sum s_i^2 = \sum t_j^2 = 1 > 0\}$$
$$\cong (U(1) \times S^p \times S^{n-p-1})/\mathbb{Z}_2$$

The loop

$$c^0(\tau) = (x^0(\tau), \xi^0(\tau)) = (\cos\tau, \underbrace{0, \ldots, 0}_{n-1}, -\sin\tau;$$

$$\sin\tau, \underbrace{0, \ldots, 0}_{n-1}, \cos\tau) \in L_1 =: (\tau_S)^{-1}(Z_1),$$

is "twice" of the generator of $\pi_1(L_1) \cong \mathbb{Z}$.

Proposition

The manifold L_1 is a compact Lagrangian submanifold. The value $\sigma_{\Delta^{S^n}} \equiv 1$

$$\frac{1}{2\pi}\int_{c^0} \theta^{S^n} = -1.$$

Proposition

The Maslov class $\mathfrak{m}_{L_1}$ *is given by*

$$\begin{aligned} \mathfrak{m}_{L_1} : \pi_1(L_1) \cong \mathbb{Z} &\longrightarrow \mathbb{Z} \\ k &\longmapsto k(1-2n) \end{aligned}$$

This can be determined by
(1) Determine the points on the curve $\{c^0\}$ at which the projection map $d\pi^{S^N}$ has non-trivial kernel and we know they are four points at $\tau = 0, \pi/2, \pi, 3\pi/2$.
(2) Caluculate the Malsov indeces around the small arc around these four points.
(3) $2(1-2n)$ is the sum of these indeces.

Proposition

For $n = 4k + 3$, L_1 *satisfies the condition* Mas[2].

For $n = 4k + 2$, $1/2 \cdot L_1$ *satisfies the condition* Mas[2].

For $n = 4k + 1$, $2 \cdot L_1$ *satisfies the condition* Mas[2].

For $n = 4k$, $3/2 \cdot L_1$ *satisfies the condition* Mas[2].

As for the condition Mas[3], there is a way to construct a measure on any geodesic flow invariant Lagrangian submanifold in $T_0^*(S^n) \cong Q_2$ based on the Kähler structure and a relation: Let

$$\sigma = \frac{2}{|z|^2} \sum_j \bar{z}_j dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n,$$

then σ is a nowhere vanishing holomorphic n-form on $Q_2 \overset{\tau_S{}^{-1}}{\cong} T_0^*(S^n)$ and

$$\sigma \wedge \overline{\sigma} = \sqrt{-1}^{\,n} 2^{n/2+3} |z|^{n-2} \frac{(-1)^{n(n-1)/2}}{n!} (\omega^S)^n \qquad (2)$$

through the identification by the map τ_S.

In fact, the property (**??**) says that $|\sigma|$ is a nowhere vanishing half density on the whole space Q_2.

If Λ is a $U(1)$-invariant Lagrangian subspace, then by the characterization of Q_2 and the relation (**??**) we can regard that the complexification $T^*(\Lambda) \otimes \mathbb{C}$ is isomorphic to $T^*(Q_2)_{|\Lambda}$ which is considered as a complex vector bundle,

or it is the same thing that it is isomorphic to the restriction to Λ of the holomorphic part $T^{*\prime}(Q_2)$ of the complexification $T^*(Q_2) \otimes \mathbb{C} = T^{*\prime}(Q_2) \oplus T^{*\prime\prime}(Q_2)$, hence

$$\bigwedge^{n} (T^*(\Lambda) \otimes \mathbb{C}) = \left(\bigwedge^{n} T^*(\Lambda) \right) \otimes \mathbb{C} \cong \bigwedge^{n} T^{*\prime}(Q_2)_{|\Lambda}.$$

Then we can define a half density on Λ by restricting the half density $|\sigma|$ to Λ.


Thank you
for your
Attention!