Geodesic X-ray transform and streaking artifacts on simple surfaces or on spaces of constant curvature

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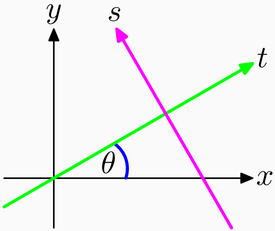
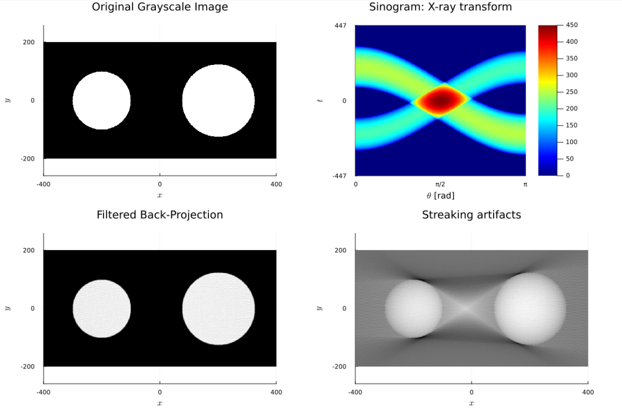
X-ray transform on the plane

- All the planar lines are parametrized by $(\theta, t) \in [0, \pi] \times \mathbb{R}$:
$$\ell = \left\{(-s \sin\theta + t\cos\theta, s\cos\theta + t\sin\theta) : s \in \mathbb{R}\right\}.$$
The X-ray transform of $f(x, y)$ on $\mathbb{R}^2$ is defined by
$$\mathcal{R}f(\theta, t) := \int_\ell f = \int_{-\infty}^{\infty} f(-s\sin\theta + t\cos\theta, s\cos\theta + t\sin\theta)ds.$$
This is considered to be the measurements of CT scanners for normal tissue. The FBP formula $f = (-\partial_x^2 - \partial_y^2)^{1/2} \circ \mathcal{R}^T \circ \mathcal{R}f$ is well-known.
- We consider a model of human body f containing a metal region D such as dental implants, stents in blood vessels, and etc. We observe that the metal streaking artifacts caused by beam hardening effect in the energy level of X-ray. The main term is the filtered back-projection of nonlinear term
$$(-\partial_x^2 - \partial_y^2)^{1/2} \circ \mathcal{R}^T\left[(\mathcal{R}1_D)^2\right],$$
This is a conormal distribution whose singular support is the streaking artifact.

Figures: metal streaking artifacts





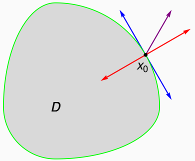
The main part of artifacts: $(-\partial_x^2 - \partial_y^2)^{1/2}\mathcal{R}^T\left[(\mathcal{R}1_D)^2\right]$.

Conormal distributions

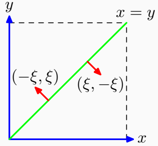
Definition 1 (Conormal distributions)

Let X be an N-dim manifold, and let Y be a closed submanifold of X. We say that $u \in \mathscr{D}'(X)$ is conormal with respect to Y of degree m if $L_1 \cdots L_\mu u \in {}^\infty H^{\mathrm{loc}}_{(-m-N/4)}(X)$ for all $\mu = 0, 1, 2, \ldots$ and all vector fields $L_1, \ldots, L_\mu$ tangential to Y. Denote by $I^m(N^*(Y))$ the set of all distributions on X conormal wrt Y of degree m. Note that $\mathrm{WF}(u) \subset N^*(Y) \setminus 0$.

- The characteristic function of a domain: $1_D \in I^{-1/2-n/4}(N^*(\partial D))$ for $D \subset \mathbb{R}^n$, which is a domain with smooth boundary.



- The Schwartz kernel of a PsDO: $\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,\xi)\, d\xi \in I^m(N^*(\Delta))$, $\Delta = \{(x,x)\}$ for $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$.



Geodesic X-ray transform 1

Suppose that (M, g) is a compact nontrapping simple Riemannian manifold with strictly convex smooth boundary. A map $\pi : S(M) \to M$ is the natural projection. Denote by $\partial_- S(M)$ the set of all unit incoming tangent vectors on the boundary ∂M:

$$\partial_- S(M) = \{w \in S(M) : \pi(w) \in \partial M, \langle \nu, w \rangle < 0\},$$

where $\nu(x)$ is the unit outer normal vector at $x \in \partial M$. Note that the nontrapping condition ensures that $\partial_- S(M)$ is identified with the manifold of all the normal geodesics on (M, g):

$$\partial_- S(M) \simeq \mathcal{G} := \{\gamma_v : \ \nabla_{\dot\gamma_v(t)} \dot\gamma_v(t) = 0, \ \dot\gamma_v(0) = v \in S(M)\}.$$

The geodesic X-ray transform of a function (more precisely a half-density) f on M is defined by

$$\mathcal{X} f(w) := \int_0^{\tau(w)} f\big(\gamma_w(s)\big) ds, \quad w \in \partial_- S(M),$$

where $\tau(w)$ is the exit time of γ_w.

Geodesic X-ray transform 2

Set $n = \dim(M)$. Then $\dim\big(S(M)\big) = 2n-1$ and $\dim\big(\partial_- S(M)\big) = 2n-2$.

Let $F : S(M) \to \partial_- S(M)$ be the submersion defined by $F\big(\dot\gamma_w(t)\big) = w$ for $w \in \partial_- S(M)$ and $t \in [0, \tau(w)]$. Then we have $\mathcal{X} = F_* \circ \pi^*$ and $\mathcal{X}^T = \pi_* \circ F^*$. See Holman-Uhlmann (2018).

Proposition 2

$\mathcal{X}$ is an elliptic Fourier integral operator, and its Schwartz kernel belongs to

$$I^{-n/4}\big(\partial_- S(M) \times M^{int}, C'_{\mathcal{X}}; \Omega^{1/2}_{\partial_- S(M)\times M^{int}}\big),$$

where $C_{\mathcal{X}}$ is the canonical relation of $\mathcal{X}$: we say that $(\xi, \eta) \in C_{\mathcal{X}}$ if $\exists v \in S(M^{int})$ such that

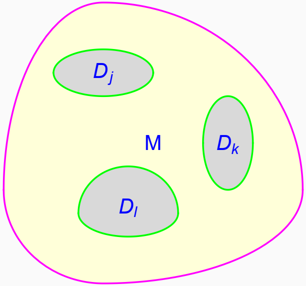
$$\xi \in T^*_{F(v)}\big(\partial_- S(M)\big) \setminus \{0\}, \quad \eta \in T^*_{\pi(v)}(M^{int}) \setminus \{0\}, \quad DF|_v^T \xi = D\pi|_v^T \eta,$$

Assumption 1

- Assume that $\dim(M) = 2$ or (M, g) is a space of constant curvature.

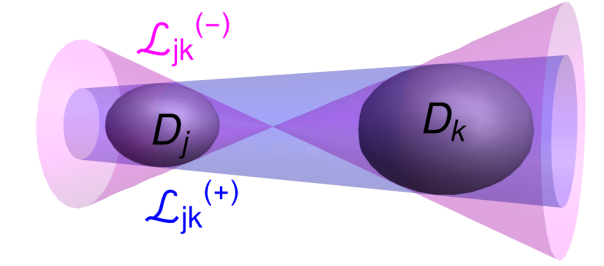
  This ensures that all the Jacobi fields are of the form scalar function $\times$ parallel transport.

- Suppose that the metal region $D \subset M^{\text{int}}$ is a disjoint union of D_j $(j = 1 \ldots, J)$ which are simply connected, strictly convex and bounded with smooth boundaries ∂D_j.



A hypersurface $\mathscr{L}$ surrounding the metal region D

- For any j and $x \in \partial D_j$, denote by $\nu_j(x)$ the unit outer normal vector at x. Consider the tangent hyperplane $\exp_x \nu_j(x)^{\perp} \cap M^{\mathrm{int}}$ at $x \in \partial D_j$.
- There are some common tangent hyperplanes of ∂D_j and ∂D_k for $j \neq k$. In this case there is common tangent geodesics in such hyperplanes. The union of all these geodesics forms a conical or cylindrical hypersurface denoted by $\mathscr{L}_{jk}^{(\pm)}$. Set $\mathscr{L} := \bigcup \left(\mathscr{L}_{jk}^{(+)} \cup \mathscr{L}_{jk}^{(-)} \right)$.



Assumption 2 (The simple model of beam hardening effect)

Let $E \geqq 0$ be a parameter describing the energy level of the X-ray beam, and let E_0 be the fixed standard level for the normal tissue. The measurement P is of the form:

$$P = -\log\left\{\int_0^\infty \rho(E)\exp(-\mathcal{X} f_E)dE,\right\},$$

where $\rho(E)$ is a probability density function on $[0, \infty)$ and is called the spectral function. Let f_{CT} be the FBP of P. We employ the simple model of the form

$$f_E(x) = f_{E_0}(x) + \alpha(E - E_0)1_D(x), \quad \rho(E) = \frac{1}{2\varepsilon}1_{[E_0-\varepsilon, E_0+\varepsilon]}(E)$$

with small parameters $\alpha > 0$ and $\varepsilon > 0$.

Main Theorem

Then the nonlinear effect f_{MA} in the CT image becomes

$$f_{\text{MA}} := f_{\text{CT}} - f_{E_0} = \sum_{k=1}^{\infty} (\alpha\varepsilon)^{2k} A_k Q \mathcal{X}^T [(\mathcal{X} 1_D)^{2k}] \mod C^\infty(M^{\text{int}}), \quad \{A_k\} \subset \mathbb{R},$$

where Q is a parametrix of $\mathcal{X}^T \circ \mathcal{X}$: $Q\mathcal{X}^T\mathcal{X} = Id$ modulo smoothing operators locally.

Our main result is as follows:

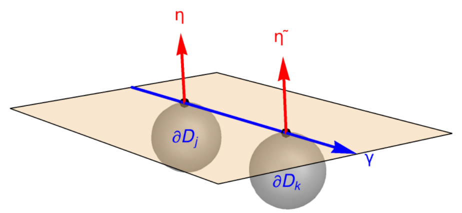
Theorem 3

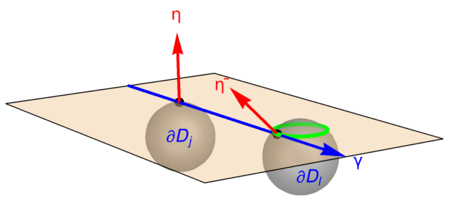
$f_{MA} \in I^{-3n/4-1/2}(N^*(\mathscr{L}))$ away from ∂D, and $\sigma_{prin}(Q\mathcal{X}^T[(\mathcal{X}1_D)^2]) \neq 0$.

- Park-Choi-Seo (2017) proved that $\text{WF}(f_{\text{MA}}) \subset N^*(\mathscr{L})$ for $M = \mathbb{R}^2$.
- Palacios-Uhlmann-Wang (2018) proved Theorem 3 for $M = \mathbb{R}^2$.
- C (2022) proved Theorem 3 for the d-plane transform on $\mathbb{R}^n$.
  We could NOT understand the meaning in many parts of this paper.

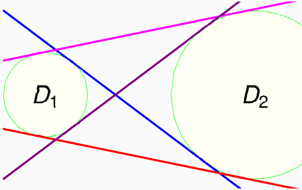
What does Theorem 3 say?

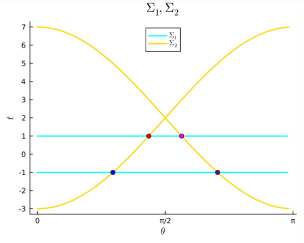
- If ∂D_j and ∂D_k have a common tangent hyperplane, then the conormal singularities propagate along the common tangent geodesic. See the left figure.
- Suppose $n \geqq 3$. If ∂D_j and ∂D_k have a common tangent geodesic, but the conormal directions at the tangent points are different, then the conormal singularities do not propagate along the common tangent geodesic. See the right figure.





- $1_{D_j} \in I^{-1/2-n/4}\big(N^*(\partial D_j) \setminus 0\big)$.
- $\mathcal{X}1_{D_j} \in I^{-(n+1)/2}\big(N^*(\Sigma_j) \setminus 0\big)$ with some hypersurface Σ_j in $\partial_- S(M)$.
- For $j \neq k$, Σ_j is transversal to Σ_k.





- Set $\Sigma_{jk} := \Sigma_j \cap \Sigma_k$ for short. For $j \neq k$,

$$\mathcal{X}1_{D_j} \cdot \mathcal{X}1_{D_k} \in \begin{cases} I^{-(n+1)/2-1}\big(N^*(\Sigma_{jk}) \setminus 0\big) & \text{at } \Sigma_{jk}, \\ I^{-(n+1)/2}\big(N^*(\Sigma_j) \setminus 0\big) + I^{-(n+1)/2}\big(N^*(\Sigma_k) \setminus 0\big) & \text{away from } \Sigma_{jk}. \end{cases}$$

Key: $C_{\mathcal{X}^T} \circ \Sigma_{jk} \setminus 0 = N^*(\mathcal{L}_{jk}) \setminus 0$

- Fix arbitrary geodesic $\gamma_w \simeq w \in \Sigma_{jk}$.
- If $\xi, \tilde{\xi} \in T_w^*(\partial_- S(M))$, $\quad w = F(v) = F(\tilde{v}), \quad \pi(v) \in \partial D_j, \quad \pi(\tilde{v}) \in \partial D_k$,

$$DF|_v^T \xi = D\pi|_v^T \eta, \quad \eta \in N_v^*(\partial D_j) \setminus \{0\}, \quad DF|_{\tilde{v}}^T \tilde{\xi} = D\pi|_{\tilde{v}}^T \tilde{\eta}, \quad \tilde{\eta} \in N_{\tilde{v}}^*(\partial D_k) \setminus \{0\},$$

then ξ and $\tilde{\xi}$ are linearly independent, and the nonlinear effect on the geodesic γ_w creates two-dimensional singularity span$\langle \xi, \tilde{\xi} \rangle$ in $T_w^*(\partial_- S(M))$ due to the simplicity condition.
- WLOG WMA η and $\tilde{\eta}$ are unit covectors.
- WLOG WMA η is the parallel transport of $\tilde{\eta}$ if $\eta \parallel \tilde{\eta}$.
- We shall show that if $\tilde{\eta}$ is the parallel transport of η, then

$$C_{\mathcal{X}}^T \circ \mathrm{span}\langle \xi, \tilde{\xi} \rangle = \bigcup_{a \in \mathbb{R}} (\text{the parallel transport of } a\eta \text{ along } \gamma_w) = \bigcup_{t \in [0, \tau(w)]} N^*_{\gamma_w(t)}(\mathcal{L}_{jk}),$$

otherwise, $C_{\mathcal{X}}^T \circ \mathrm{span}\langle \xi, \tilde{\xi} \rangle = N^*_{\pi(v)}(\partial D_j) \bigcup N^*_{\pi(\tilde{v})}(\partial D_k)$.

When $\tilde{\eta}$ is the parallel transport of η

- Set $\gamma_w(t_0) = \pi(\eta) \in \partial D_j$ and $\gamma_w(\tilde{t}_0) = \pi(\tilde{\eta}) \in \partial D_k$, and suppose $\tilde{\eta} = P(\tilde{t}_0, t_0; \gamma_w)^T \eta$, where $P(t_0, \tilde{t}_0; \gamma_w)$ is the parallel transport of $T_{\gamma_w(\tilde{t}_0)}(M)$ onto $T_{\gamma_w(t_0)}(M)$ along γ_w. Set $\eta(s) := P(s, t_0; \gamma_w)^T \eta \in T^*_{\gamma_w(s)}(M^{\mathrm{int}})$ for $s \in (0, \tau(w))$. Then $\eta(\tilde{t}_0) = \tilde{\eta}$.
- Let $k(x)$ be a sectional curvature at $x \in M$, which is a constant when $n \geqq 3$.
- Let $a(t; s), b(t; s) \in C^\infty(0, \tau(w))$ be solutions to

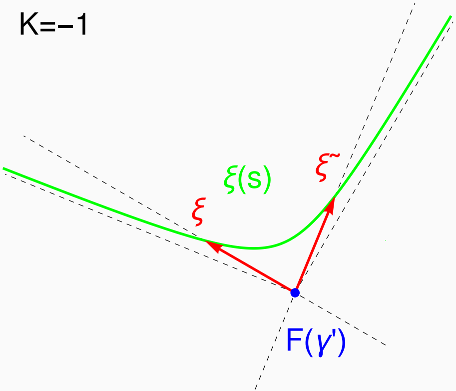
$$
\begin{aligned}
a_{tt}(t; s) + k\big(\gamma_w(t)\big)a(t; s) = 0, \quad a(s; s) = 1, \quad a_t(s; s) = 0, \\
b_{tt}(t; s) + k\big(\gamma_w(t)\big)b(t; s) = 0, \quad b(s; s) = 0, \quad b_t(s; s) = 1.
\end{aligned}
$$

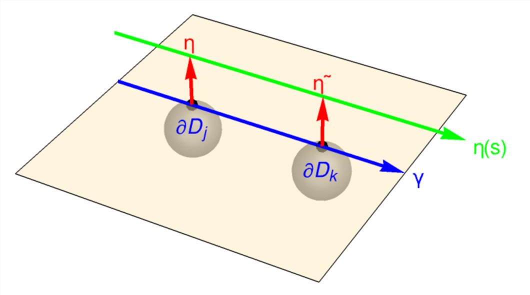
- $\Delta(t_0, \tilde{t}_0; s) := \det \begin{bmatrix} a(t_0; s) & a(\tilde{t}_0; s) \\ b(t_0; s) & b(\tilde{t}_0; s) \end{bmatrix}$ never vanish due to the simplicity. If we set

$$
\xi(s) := \frac{b(\tilde{t}_0; s)}{\Delta(t_0, \tilde{t}_0; s)}\xi - \frac{b(t_0; s)}{\Delta(t_0, \tilde{t}_0; s)}\tilde{\xi} \in \operatorname{span}\langle \xi, \tilde{\xi} \rangle, \quad s \in (0, \tau(w)),
$$

then we have $DF|^T_{\dot{\gamma}_w(s)}\xi(s) = D\pi|^T_{\dot{\gamma}_w(s)}\eta(s)$ in $T^*_{\dot{\gamma}_w(s)}\big(S(M^{\mathrm{int}})\big)$ for $s \in (0, \tau(w))$.

$\xi(s)$ in $\text{span}\langle\xi,\tilde{\xi}\rangle \subset T^*_{F(\gamma)}(\partial_- S(M))$ and $\eta(s)$ in $T^*(S(M^{\text{int}}))$ for $K=-1$





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