Microlocal analysis of d-plane transform on the Euclidean space

Hiroyuki Chihara (University of the Ryukyus)

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What this talk is about?

- We study d-plane transform

$$f(x) \mapsto \mathcal{R}_d f(\Xi) := \int_{\Xi} f, \quad \Xi \in G(d,n).$$

$\mathcal{R}_1 f$ for $n = 2$ is considered to be the measurements of CT scanners for normal tissue.

- We give a concrete expression of the canonical relation $\Lambda'_\phi \subset T^*\big(G(d,n) \times \mathbb{R}^n\big)$ of $\mathcal{R}_d$.
- We consider a model of human body f containing a metal region D such as dental implants, stents and etc. We observe that the metal streaking artifact caused by beam hardening effect of measurements, which is the filtered back-projection of nonlinear term

$$(-\Delta_x)^{d/2} \circ \mathcal{R}_d^*\big[(\mathcal{R}_d \chi_D)^2\big],$$

is a singular support of some conormal distribution.

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d-plane transform

d-plane transform

- $n = 2, 3, 4, \ldots, \; d = 1, \ldots, n-1$.
- The Grassmannian $G_{d,n}$ is the set of all d-dimensional vector subspaces of $\mathbb{R}^n$. $\dim G_{d,n} = d(n-d)$.
- The affine Grassmannian $G(d,n)$ is the set of all d-dimensional planes in $\mathbb{R}^n$, that is, $G(d,n) = \{x'' + \sigma \; : \; \sigma \in G_{d,n}, x'' \in \sigma^\perp\}$. $N(d,n) := \dim G(d,n) = (d+1)(n-d)$. We use notation $x'' + \sigma = (\sigma, x'')$.
- Denote $x = x' + x'' \in \sigma \oplus \sigma^\perp = \mathbb{R}^n$. The d-plane transform of $f(x) = f(x' + x'') = \mathcal{O}(\langle x \rangle^{-d-\varepsilon})$ is defined by

$$\mathcal{R}_d f(\sigma, x'') := \int_{x''+\sigma} f = \int_\sigma f(x' + x'')dx', \tag{1}$$

where $\langle x \rangle = \sqrt{1 + |x|^2}$ and dx' is the Lebesgue measure on σ.
- $\mathcal{R}_1 f$ is called the X-ray transform of f, and $\mathcal{R}_{n-1} f$ is called the Radon transform of f.

Filtered back-projection

The formal adjoint of $\mathcal{R}_d$ is given by

$$\begin{aligned}\mathcal{R}_d^*\varphi(x) &= \frac{1}{C(d,n)}\int_{\{\Xi\in G(d,n):x\in\Xi\}}\varphi(\Xi)d\mu(\Xi)\\ &= \frac{1}{C(d,n)}\int_{O(n)}\varphi(x+k\cdot\sigma)dk,\end{aligned}$$

where $x\in\mathbb{R}$, $\varphi\in C\big(G(d,n)\big)$, $C(d,n)=(4\pi)^{d/2}\Gamma(n/2)/\Gamma\big((n-d)/2\big)$, $C(d,n)^{-1}d\mu$ and $C(d,n)^{-1}dk$ are normalized measure, and $\sigma\in G_{d,n}$.

Proposition 1 (FBP (filtered back-projection))

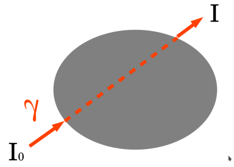
For $f(x)=\mathcal{O}(\langle x\rangle^{-d-\varepsilon})$,

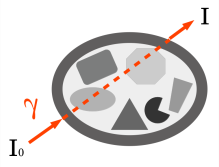
$$f=(-\Delta_x)^{d/2}\mathcal{R}_d^*\mathcal{R}_d f=\mathcal{R}_d^*(-\Delta_{x''})^{d/2}\mathcal{R}_d f, \tag{2}$$

where $-\Delta_x=-\partial_{x_1}^2-\cdots-\partial_{x_n}^2$, and $-\Delta_{x''}$ is the Laplacian on $\sigma^\perp$.

CT scanner

- $f(x, y)$ is the attenuation coefficient distribution of the section of an object.
- The X-ray beam is supposed to have no width, and traverses the object along a line γ. I_0 and I denote the intensities of the beam before and after passing through the object respectively.



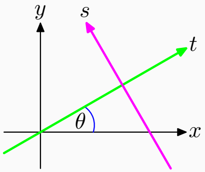


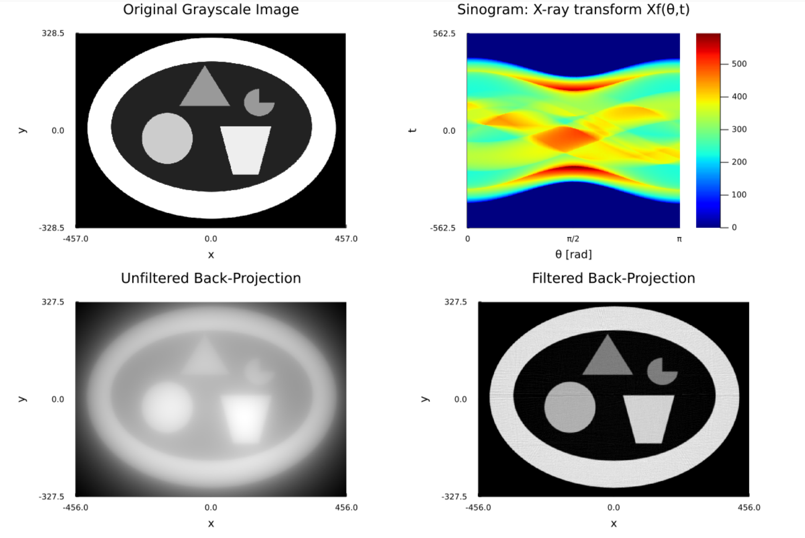
- If the object is uniform, that is, $f = a \cdot \chi_\Omega$ and the travel length in the object is ℓ, then the Beer-Lambert law obtains

$$\log\left(\frac{I_0}{I}\right) = a \cdot \ell = \int_\gamma f = \mathcal{R}_1 f(\gamma).$$

The same formula can be obtained for more general f, and we can regard $\mathcal{R}_1 f(\gamma)$ as the measurement of CT scanners. c.f. $\log(I_0/I_1) + \log(I_1/I_2) + \cdots + \log(I_p/I) = \log(I_0/I)$.

Figures: $\mathcal{R}_1$, UFBP and FBP on $\mathbb{R}^2$





Beam hardening

- There are some factors causing artifacts in CT images:
  beam width, partial volume effect, beam hardening,
  noise in measurements, numerical errors and etc.
- In the formulation of CT scanners in Page 6, the X-ray is supposed to be monochromatic with a fixed energy, say $E_0 > 0$.
- Actually, however, the X-ray beam has a wide range of energy E and the attenuation coefficient distribution f_E depends on E. This is described by the spectral function $\rho(E)$ which is a probability density function of $E \in [0, \infty)$. Click on an NIH page. The formulation of the measurements P of CT scanners becomes

$$P := \log\left(\frac{I_0}{I}\right) = -\log\left\{\int_0^\infty \rho(E)\exp(-\mathcal{R}_1 f_E)dE\right\}.$$

If f_E is independent of E, i.e., $f_E = f_{E_0}$, then $\log(I_0/I) = \mathcal{R}_1 f_{E_0}$.

Metal streaking artifacts

- Consider a simple model of the beam hardening:

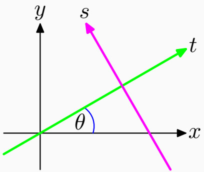
$$\rho(E) = \frac{1}{2\varepsilon}\chi_{[E_0-\varepsilon, E_0+\varepsilon]}(E),$$
$$f_E(x) = f_{E_0}(x) + \alpha(E - E_0)\chi_D(x),$$

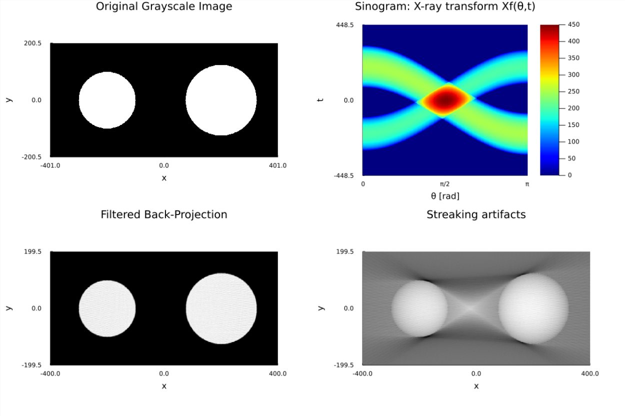
  where f_{E_0} is an attenuation coefficient distribution of human tissue, ε and α are small positive constants, and D is a metal region. Then the measurement P becomes

$$P - \mathcal{R}_1 f_{E_0} = -\log\left\{\frac{\sinh(\alpha\varepsilon\mathcal{R}_1\chi_D)}{\alpha\varepsilon\mathcal{R}_1\chi_D}\right\} = \sum_{k=1}^{\infty} A_k(\alpha\varepsilon\mathcal{R}_1\chi_D)^{2k}.$$

- Park-Choi-Seo (CPAM, 2017) proved that the metal streaking artifacts are propagation of $\mathrm{WF}(\chi_D)$.
- Palacios-Uhlmann-Wang (SIAM J. Math. Anal., 2018) proved that the streaking artifacts are conormal distributions.

Figures: metal streaking artifacts





The principal part of artifact: $\mathcal{R}_1^*(-\Delta_{x''})^{1/2}\left\{-\frac{1}{3}(\alpha\varepsilon\mathcal{R}_1\chi_D)^2\right\}$.

Conormal distributions

Conormal distributions

Definition 2 (Conormal distributions)

Let X be an N-dim manifold, and let Y be a closed submanifold of X. $u \in \mathscr{D}'(X)$ is said to be conormal with respect to Y of degree m if

$$L_1 \cdots L_M u \in {}^{\infty}H^{\mathrm{loc}}_{(-m-N/4)}(X)$$

for all $M = 0, 1, 2, \ldots$ and all vector fields $L_1, \ldots, L_M$ tangential to Y. Denote by $I^m(N^*Y)$, the set of all distributions on X conormal with respect to Y of degree m.

Note that $N^*_y Y := T^*_y X / T^*_y Y$ for any $y \in Y$. If $u \in I^m(N^*Y)$, then $\mathrm{WF}(u) \subset N^*Y \setminus 0$.

$$\|u\|_{{}^{\infty}H_{(s)}(\mathbb{R}^N)} := \sup_{j=0,1,2,\ldots} \left(\int_{A_j} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{1/2},$$

$$A_0 := \{|\xi| < 1\}, \quad A_j := \{2^{j-1} \leqq |\xi| < 2^j\}, j = 1, 2, 3, \ldots .$$

Conormal distributions and oscillatory integrals

Proposition 3 (Characterization of conormal distributions)

Let $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and let $Y = \mathbb{R}^k \times \{0\} = \{x'' = 0\}$. Then $u \in \mathscr{D}'(\mathbb{R}^N)$ belongs to $I^{m+k/2-N/4}(N^*Y)$ if and only if there exists an amplitude $a(x'', \xi') \in S^m(\mathbb{R}^{N-k} \times \mathbb{R}^k)$ such that

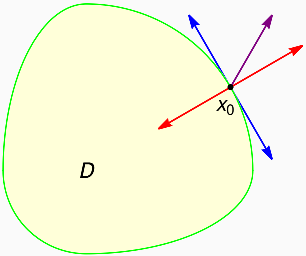
$$u(x) = \int_{\mathbb{R}^k} e^{ix' \cdot \xi'} a(x'', \xi') d\xi'.$$

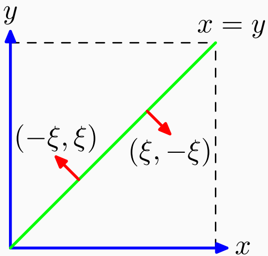
We can replace the conormal bundle N^*Y by more general Lagrangian distributions Λ. The elements of $I^m(\Lambda)$ is said to be Lagrangian distributions on X. These are characterized as oscillatory integrals with more general phase functions. The distributions kernels of Fourier integral operators are Lagrangian distributions.

Examples of conormal distributions

- $\chi_D \in I^{-1/2-n/4}(N^*\partial D)$, where D is a domain in $\mathbb{R}^n$ with smooth boundary.
- Set $\Delta = \{(x,x) : x \in \mathbb{R}^N\}$. If $a(x,\xi) \in S^m(\mathbb{R}^N \times \mathbb{R}^N)$, then

$$K(x,y) = \int_{\mathbb{R}^N} e^{i(x-y)\cdot\xi} a(x,\xi)d\xi \in I^m(N^*\Delta).$$





Canonical relation of d-plane transform

The canonical relation of the d-plane transform

Theorem 4

$\mathcal{R}_d$ is an elliptic Fourier integral operator whose distribution kernel belongs to

$$I^{-d(n-d+1)/4}\big(G(d,n)\times\mathbb{R}^n;\Lambda_\phi\big),$$

$$\begin{aligned}\Lambda'_\phi &= \Big\{\big(\sigma, y-\pi_\sigma y, y;\ \eta(y\cdot\omega_1,\dots,y\cdot\omega_d,1,1)\big):\\ &\qquad \sigma=\langle\omega_1,\dots,\omega_d\rangle\in G_{d,n},\ \omega_1,\dots,\omega_d\in\mathbb{S}^{n-1},\ y\in\mathbb{R}^n,\ \eta\in\sigma^\perp\Big\}\\ &= \Big\{\big(\sigma, x'', x''+t_1\omega_1+\cdots+t_d\omega_d;\ \xi(t_1,\dots,t_d,1,1)\big):\\ &\qquad (\sigma,x'')\in G(d,n),\ \sigma=\langle\omega_1,\dots,\omega_d\rangle\in G_{d,n},\\ &\qquad \omega_1,\dots,\omega_d\in\mathbb{S}^{n-1},\ t_1,\dots,t_d\in\mathbb{R},\ \xi\in\sigma^\perp\Big\},\end{aligned}$$

where π_σ is the orthogonal projection of $\mathbb{R}^n$ onto $\sigma\in G_{d,n}$.

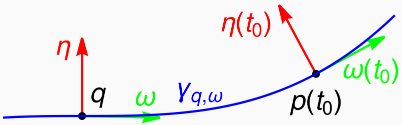
What is the meaning of Lemma 4?

Let $\mathcal{G}$ be the manifold of all the normal geodesics of a Riemannian manifold (M, g). The canonical relation of the geodesic X-ray transform on (M, g) is a conic Lagrangian submanifold of $T^*(\mathcal{G} \times M) \setminus 0$:

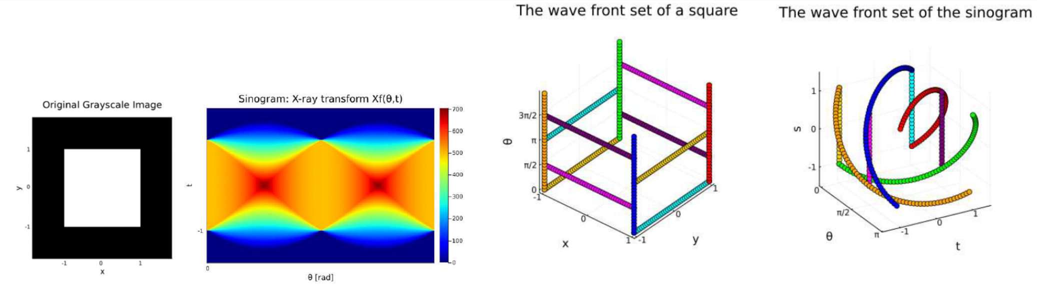
$$\Lambda' = \{(\gamma_{q,\omega}, q; \eta(t_0), -\Gamma^i_{jk}(p(t_0))\omega_j(t_0)\eta_k(t_0), \eta) : (q,\eta)\in T^*(M) \setminus 0, \omega\in S^*_q(M) \cap \eta^\perp\}$$

where $\gamma_{q,\omega} = \exp_p \cdot\omega$, $p(t) = \gamma_{q,\omega}(t)$, $\omega(t) = \dot\gamma_{q,\omega}(t)$, $\eta(t)$ is the parallel transport of η along $\gamma_{q,\omega}$ at $p(t)$, and $t_0 \in \mathbb{R}$ is some constant. Λ' says that

> *the geodesic X-ray transform maps the visible singularity η at point q to the horizontal lift of the parallel transport of η along the geodesic flow $(\gamma_{q,\omega}, \dot\gamma_{q,\omega})$.*



Figures: Canonical relation of $\mathcal{R}_1$ on $\mathbb{R}^2$



$$\Lambda' = \left\{ \left(\theta, t, \begin{bmatrix} t\cos\theta - s\sin\theta \\ t\sin\theta + s\cos\theta \end{bmatrix}; \ -\tau s \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}, \tau, \tau \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right) : t, s, \tau \in \mathbb{R}, \ \theta \in [0, \pi] \right\}$$

$$= \left\{ \left(\frac{\xi}{|\xi|}, \frac{x\cdot\xi}{|\xi|}, x; \mp|\xi| \left(x - \frac{x\cdot\xi}{|\xi|^2} \right) \xi, \pm|\xi|, \xi \right) : (x,\xi) \in T^*(\mathbb{R}^2) \setminus 0 \right\}.$$

Analysis of streaking artifacts

Assumption and notation

- **Assumption on the metal region**. The metal region $D \subset \mathbb{R}^n$ is supposed to be a disjoint union of D_j $(j = 1, \ldots, J)$ which are simply connected, strictly convex and bounded with smooth boundaries $\Sigma_j := \partial D_j$. Set $\Sigma := \partial D$.
- Denote by $\nu(y_j)$ the unit outer normal vector of Σ_j at $y_j \in \Sigma_j$. We consider the set of pairs $(y_j, y_k) \in \Sigma_j \times \Sigma_k$ such as

$$\mathcal{M}_{jk}^{(\pm)} := \{(y_j, y_k) \in \Sigma_j \times \Sigma_k \ : \ y_j + T_{y_j}\Sigma_j = y_k + T_{y_k}\Sigma_k, \ \nu(y_j) = \pm\nu(y_k)\},$$

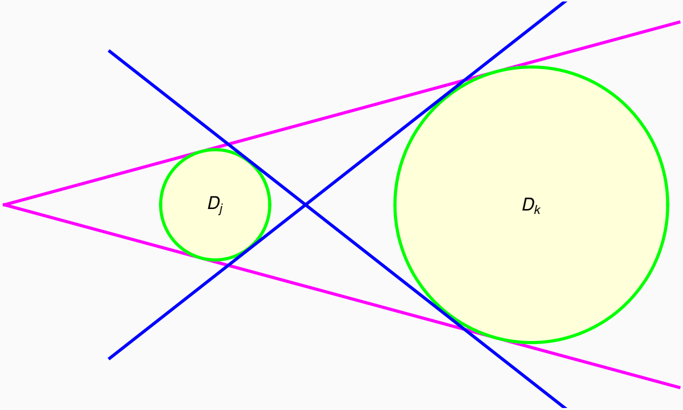
which is an $(n-2)$-dimensional submanifold of $\Sigma_j \times \Sigma_k$. We introduce the set of lines

$$\mathcal{L}_{jk}^{(\pm)} := \{y_j + t(y_k - y_j) \ : \ (y_j, y_k) \in \mathcal{M}_{jk}^{(\pm)}, \ t \in \mathbb{R}\},$$

Then $\mathcal{L}_{jk}^{(\pm)}$ becomes a cylindrical surface or a cone which is tangent to Σ_j at y_j and to Σ_k at y_k for all $(y_j, y_k) \in \mathcal{M}_{jk}^{(\pm)}$. Set $\mathcal{L}_{jk} := \mathcal{L}_{jk}^{(+)} \cup \mathcal{L}_{jk}^{(-)}$ and $\mathcal{L} := \bigcup_{j<k} \mathcal{L}_{jk}$.

Figures: D_j, D_k, Σ_j, Σ_k, $\mathcal{L}_{jk}^{(+)}$ and $\mathcal{L}_{jk}^{(-)}$

Σ_j, Σ_k, $\mathcal{L}_{jk}^{(+)}$, and $\mathcal{L}_{jk}^{(-)}$.



Main Theorem

The nonlinear part of the CT image is

$$f_{MA} := f_{CT} - f_{E_0} = \sum_{k=1}^{\infty} A_k (a\varepsilon)^{2k} \mathcal{R}_d^* (-\Delta_{x''})^{d/2} [(\mathcal{R}_d \chi_D)^{2k}]$$

Theorem 5

Away from Σ,

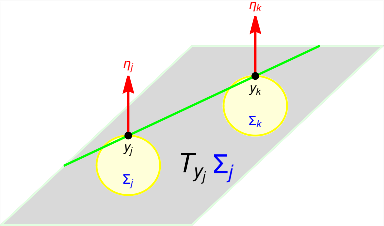
$$f_{MA} \in I^{-(d+2+n/4)+d(n-d)/2}(N^*\mathscr{L}), \quad \mathrm{WF}(f_{MA}) \subset N^*\mathcal{L}.$$

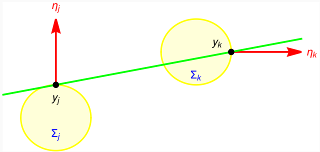
The principal symbol of the FBP of $(\mathcal{R}_d \chi_D)^2$ does not vanish.

- Note that $\chi_D \in I^{-1/2-n/4}(N^*\Sigma)$.
- For $(n, d) = (2, 1)$,
  - Park-Choi-Seo (2017) proved that $\mathrm{WF}(f_{\mathrm{MA}}) \subset N^*\mathscr{L}$.
  - Palacios-Uhlmann-Wang (2018) proved Theorem 5.

What does Theorem 5 say?

- If Σ_j and Σ_k have a common tangential hyperplane, then the common conormal singularity propagates all over the line connecting the tangential points. This is the true identity of the metal streaking artifacts.
- If Σ_j and Σ_k have a common tangential plane of codimension two, then the normal directions at the tangential points are different and no singularity propagates along the connecting line.





Clean intersection and transversal intersection

Definition 6

Let X be a smooth manifold, and let Y and Z be submanifolds of X.

- We say that Y and Z intersect transversely if $N_x^* Y \cap N_x^* Z = \{0\}$ for all $x \in Y \cap Z$. Note that this condition is equivalent to that $T_x Y \cup T_x Z = T_x X$ for all $x \in Y \cap Z$.
- We say that Y and Z intersect cleanly if $Y \cap Z$ is smooth and $T_x Y \cap T_x Z = T_x(Y \cap Z)$ for all $x \in Y \cap Z$. Moreover,

$$e := \mathrm{codim}(Y) + \mathrm{codim}(Z) - \mathrm{codim}(Y \cap Z)$$

is said to be the excess of the intersection.

Note that transverse intersection is clean intersection with no excess.

Clean intersection or not

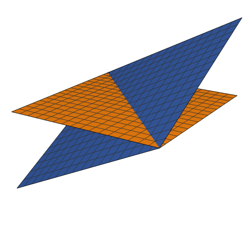
In xyz-space $\mathbb{R}^3$, set

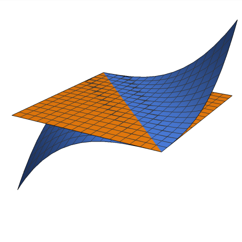
$$Y_1 = \{z = x + y\}, \quad Y_3 = \{z = (x+y)^3\}, \quad Z = \{z = 0\}.$$

Then $Y_k \cap Z = \{(x, -x, 0)\}$. For any $p \in Y_k \cap Z$ we have

$$T_p(Y_k \cap Z) = T_p Y_1 \cap T_p Z = \langle (1, -1, 0) \rangle_{\mathbb{R}}$$
$$\subsetneq T_p Y_3 \cap T_p Z = \langle (1, -1, 0), (1, 1, 0) \rangle_{\mathbb{R}} = T_p Z.$$

$Y_1 \cap Z$ is clean and $Y_3 \cap Z$ is not clean.





The canonical transform of D

We need to consider

$$(\mathcal{R}_d\chi_D)^2 = \sum_{j=1}^{J}(\mathcal{R}_d\chi_{D_j})^2 + 2\sum_{1\leqq j<k\leqq J}\mathcal{R}_d\chi_{D_j}\cdot\mathcal{R}_d\chi_{D_k},$$

$$\Lambda'_\phi\circ N^*\Sigma_j = \left\{\left(\sigma, y-\pi_\sigma y;\ \eta(\cdots,1)\right):\ (y,\eta)\in N^*\Sigma_j, \sigma\in G_{d,n}\cap\eta^\perp\right\},$$
$$S_j := \pi_{G(d,n)}(\Lambda'_\phi\circ N^*\Sigma_j) = \{(\sigma, y-\pi_\sigma y): y\in\Sigma_j, \sigma\in G_{d,n}\cap T_y\Sigma_j\}.$$

Set $S_{jk} := S_j\cap S_k \subset G(d,n)$.

Lemma 7

- codim $S_j = 1$, *and* $N^*S_j = \Lambda'_\phi\circ N^*\Sigma_j$.
- *If* $j\neq k$ *and* $S_j\cap S_k \neq \emptyset$, *then* S_j *intersects* S_k *transversally, that is,* $N^*_{(\sigma,x'')}S_j\cap N^*_{(\sigma,x'')}S_k = \{0\}$ *for any* $(\sigma,x'')\in S_j\cap S_k$.

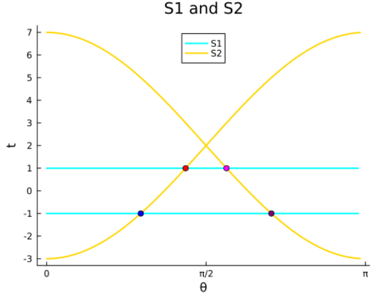
Figures: Examples of $S_j \cap S_k \neq \emptyset$

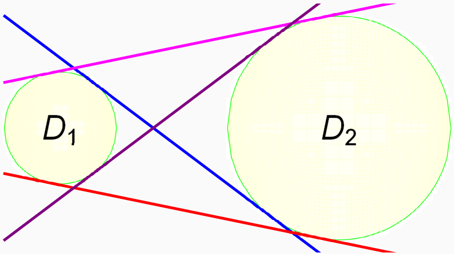
Let $n = 2$, and let

$$D_1 = \{x^2 + y^2 < 1\}, \quad D_2 = \{(x-5)^2 + y^2 < 4\}.$$

Then

$$S_1 = \{(\theta, \pm 1) : \theta \in [0, \pi]\}, \quad S_2 = \{(\theta, 2 \pm 5\cos\theta) : \theta \in [0, \pi]\}.$$





Intersection calculus of S_{jk}

Lemma 7 implies that codim $S_{jk} = 2$. If $(\sigma, x'') \in S_{jk}$, then there exist $y_j \in \Sigma_j$ and $y_k \in \Sigma_k$ such that $\sigma \subset T_{y_j}\Sigma_j \cap T_{y_k}\Sigma_k$ and $x'' = y_j - \pi_\sigma y_j = y_k - \pi_\sigma y_k$. S_{jk} is a disjoint union of

$$S_{jk}^{(1)} = \{(\sigma, x'') \in S_{jk} : N^*_{y_j}\Sigma = N^*_{y_k}\Sigma\}, \quad S_{jk}^{(2)} = \{(\sigma, x'') \in S_{jk} : N^*_{y_j}\Sigma \neq N^*_{y_k}\Sigma\}.$$

Lemma 8

We have clean intersections

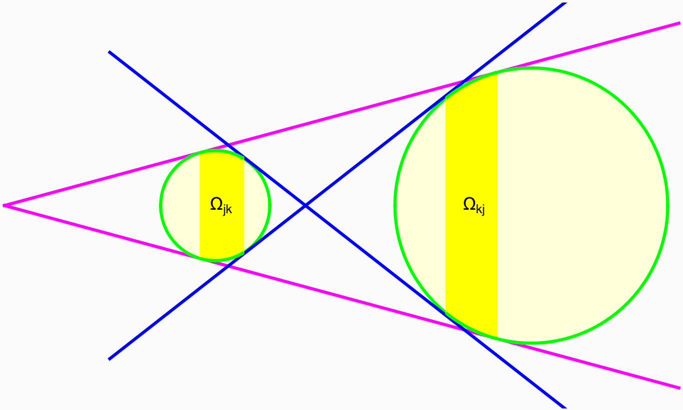
$$(\Lambda'_\phi)^* \circ N^* S_j = N^*\Sigma_j \setminus 0, \quad (\Lambda'_\phi)^* \circ N^* S_{jk}^{(1)} = N^*\mathcal{L}_{jk} \setminus 0, \quad e = d(n-d-1),$$

$$(\Lambda'_\phi)^* \circ N^* S_{jk}^{(2)} = (N^*\Omega_{jk} \setminus 0) \cup (N^*\Omega_{kj} \setminus 0), \quad e = d(n-d-2),$$

where Ω_{jk} is the connected subdomain in Σ_j enclosed by

$$B_{jk}^{(\pm)} := \{y_j \in \Sigma_j \mid \exists y_k \in \Sigma_k \;\; s.t.\; (y_j, y_k) \in \mathcal{M}_{jk}^{(\pm)}\}.$$

Figure: Ω_{jk} and Ω_{kj}



Definition 9 (Paired Lagrangian distributions)

Let $\mu, \nu \in \mathbb{R}$. Suppose that Λ_0 and Λ_1 are cleanly intersecting conic Lagrangian submanifolds of $T^*X \setminus 0$, that is,

$$T_{(x,\xi)}\Lambda_0 \cap T_{(x,\xi)}\Lambda_1 = T_{(x,\xi)}(\Lambda_0 \cap \Lambda_1), \quad \forall (x,\xi) \in \Lambda_0 \cap \Lambda_1.$$

We say that $u \in \mathscr{D}'(X)$ belongs to $I^{\mu,\nu}(\Lambda_0, \Lambda_1)$ if $\mathrm{WF}(u) \subset \Lambda_0 \cup \Lambda_1$ and away from $\Lambda_0 \cap \Lambda_1$, we have $u \in I^{\mu+\nu}(\Lambda_0 \setminus \Lambda_1)$ and $u \in I^{\mu}(\Lambda_1)$.

Products of paired Lagrangian distributions

Lemma 10 (Greenleaf-Uhlmann, 1993)

Let X be an N-dimensional manifold, and let Y and Z be transversally intersecting submanifolds of X. Set $\operatorname{codim} Y = k_1$. $\operatorname{codim} Z = l_1$ $\operatorname{codim} Y \cap Z = k_1 + k_2 = l_1 + l_2$. *Then*

$$\begin{aligned} & I^{\mu+k_1/2-N/4}(N^*Y) \cdot I^{\nu+k_1/2-N/4}(N^*Z) \\ \subset & I^{\mu+k_1/2-N/4,\nu+k_2/2}(N^*(Y \cap Z), N^*Y) \\ + & I^{\nu+l_1/2-N/4,\mu+l_2/2}(N^*(Y \cap Z), N^*Z). \end{aligned}$$

The transversality $N^*Y \cap N^*Z = \{0\}$ guarantees that the product can be well-defined since

$$\xi + \eta \neq 0 \quad \text{for} \quad \xi \in N_x^*Y, \quad \eta \in N_x^*Z, \quad x \in Y \cap Z.$$

Outline of Proof of Theorem 5 i

- Set $\mathscr{A} := \sum_{j \neq k} I^{-(d+1)/2-N(d,n)/4,-(d+1)/2}(N^*S_{jk}, N^*S_j)$.
- Note that $\chi_{D_j} \in I^{-1/2-n/4}(N^*\Sigma_j)$, $\mathcal{R}_d\chi_{D_j} \in I^{-(d+1)/2-N(d,n)/4}(N^*S_j) \subset \mathscr{A}$.
- Lemma 10 proves that $(\mathcal{R}_d\chi_D)^2 \in \mathscr{A}$.
- It follows that $\mathscr{A}$ is an algebra. In particular $P_{MA} := \sum_{k=1}^{\infty} A_k(\alpha\varepsilon)^{2k}(\mathcal{R}_d\chi_D)^{2k} \in \mathscr{A}$.
- Applying Lemmas 8, 11, and 12 to P_{MA}, we prove Theorem 5.

Lemma 11

$\mathcal{R}_d^*(-\Delta_{x''})^{d/2}$ is a FIO of order $\dfrac{d}{2}+\dfrac{N(d,n)}{4}-\dfrac{n}{4}$ with a canonical relation

$$(\Lambda'_\phi)^* := \{(x,y,\xi,\eta) : (y,x,;\eta,\xi)\in\Lambda'_\phi\}.$$

Lemma 12 (Hörmander IV, Theorem 25.2.3)

Assume that

$$A_1 \in I^{m_1}(X\times Y, C_1'),\quad A_2 \in I^{m_2}(Y\times Z, C_2')$$

are properly supported, and that $C := C_1\circ C_2$ is clean with excess e, proper and connected. Then

$$A_1\circ A_2 \in I^{m_1+m_2+e/2}(X\times Z, C').$$

$I^{\mu,\nu}(N^*S_{jk}, N^*S_j)$ is given by oscillatory integrals.

If $u \in I^{-(d+1)/2-N(d,b)/4,-(d+1)/2}(N^*S_{jk}, N^*S_j)$, we can choose local coordinates $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N(d,n)-2}$ and find $a(x, y, z, \xi, \eta)$ such that $S_j = \{x = 0\}$, $S_{jk} = \{x = y = 0\}$,

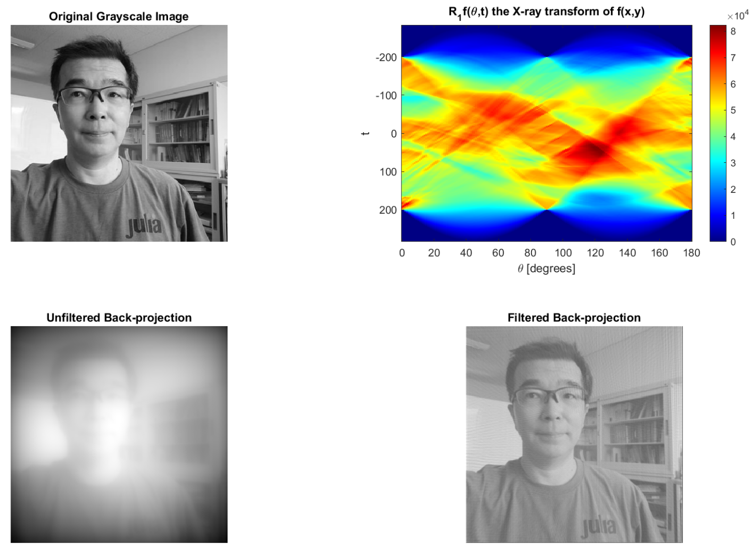
$$\partial^{\gamma}_{x,y,z}\partial^{\alpha}_{\xi}\partial^{\beta}_{\eta}a(x, y, z, \xi, \eta) = \mathcal{O}(\langle \xi; \eta\rangle^{-(d+2)/2-\alpha}\langle \eta\rangle^{-(d+2)/2-\beta}),$$

$$u(x, y, z) = \iint_{\mathbb{R}^2} e^{i(x\xi+y\eta)}a(x, y, z, \xi, \eta)d\xi d\eta$$

near $(x, y, z) = 0$. Using formulas like this, we can obtain

$$\begin{aligned}\left(I^{\mu,\nu}(N^*S_{jk}, N^*S_j)\right)^2 &\subset \mathscr{A},\\ I^{\mu,\nu}(N^*S_{jk}, N^*S_{j}) \cdot I^{\mu,\nu}(N^*S_{jk}, N^*S_{k}) &\subset \mathscr{A}.\end{aligned}$$

Thank you for your attention!



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