THE EXPLICIT FORMULA OF SOLUTIONS TO THE CAUCHY PROBLEM FOR A CERTAIN HYPERBOLIC SYSTEM OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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ABSTRACT. We present an explicit formula of solutions to the Cauchy problem for a certain 2×2 hyperbolic system of first order partial differential equations with constant coefficients. The principal part of our system is diagonalized, and we see the off-diagonal part of the lower order terms as the perturbation. The Cauchy problem is equivalent to a system of integral equations, and we obain our formula by using successive approximation. Our formula includes d'Alembert's formula of solutions to the Cauchy problem for the one-dimensional wave equation.

In this note we present the explicit formula of solutions to the Cauchy problem of the form

$$\frac{\partial u}{\partial t} + a \cdot \frac{\partial u}{\partial x} + \gamma u + \lambda v = 0, \qquad \text{in} \quad \mathbb{R} \times \mathbb{R}^d, \tag{1}$$

$$\frac{\partial v}{\partial t} + b \cdot \frac{\partial v}{\partial x} + \nu u + \mu v = 0, \qquad \text{in} \quad \mathbb{R} \times \mathbb{R}^d, \tag{2}$$

$$u(0,x) = \varphi(x), \quad \text{in} \quad \mathbb{R}^d, \tag{3}$$

$$v(0,x) = \psi(x), \quad \text{in} \quad \mathbb{R}^d, \tag{4}$$

where $u(t,x)$ and $v(t,x)$ are complex-valued unknown functions of $(t,x) = (t, x_1, \dots, x_d) \in \mathbb{R} \times \mathbb{R}^d$, $a, b \in \mathbb{R}^d$ and $\gamma, \lambda, \nu, \mu \in \mathbb{C}$ are constants, $\partial u/\partial x = (\partial u/\partial x_1, \dots, \partial u/\partial x_d)$, and $\varphi(x)$ and $\psi(x)$ are given initial data. We denote by $\mathscr{B}(\mathbb{R}^d)$ the set of all complex-valued bounded continuous functions on $\mathbb{R}^d$, whose norm is defined by $\|\varphi\| = \sup_{x \in \mathbb{R}^d} |\varphi(x)|$. We denote by $C(\mathbb{R}; \mathscr{B}(\mathbb{R}^d))$ the set of all $\mathscr{B}(\mathbb{R}^d)$-valued continuous functions on $\mathbb{R}$.

The system (1)-(2) is a diagonalized hyperbolic system, and it is easy to handle it from a point of view of mathematical analysis. However, it is not easy to give a comprehensive explicit formula of solutions to the Cauchy problem. More precisely, we would like to obtain an explicit formula with local property. For this purpose, some elementary methods do not necessarily work for (1)-(2) since the matrices of the principal part and the lower order part are not commutative in general. In other words, the characteristic roots of the full symbol of the system are complicated because of square roots of complex numbers. The systems like (1)-(2) arise in various fields of applied science, including population dynamics, transport phenomenon of several kinds of matters and etc. We believe that a comprehensive explicit formula for solutions to the Cauchy problem (1)-(2)-(3)-(4) plays an important role in numerical simulations in applied science.

Set $\tilde{u}(t,x) = e^{\gamma t} u(t,x)$ and $\tilde{v}(t,x) = e^{\mu t} v(t,x)$ for short. We see the off-diagonal part of (1)-(2) as the perturbation. By using the Fourier transform in x or the integration along the characteristics, we can obtain the system of integral equations of the form

$$\tilde{u}(t,x) = \varphi(x - ta) - \lambda \int_0^t e^{-(\mu - \gamma) t_1} \tilde{v}\big(t_1, x - (t - t_1) a\big) dt_1, \tag{5}$$

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$$\tilde{v}(t,x)=\psi(x-tb)-\lambda\int_0^t e^{-(\gamma-\mu)t_1}\tilde{u}\bigl(t_1,x-(t-t_1)b\bigr)dt_1, \tag{6}$$

which is equivalent to the Cauchy problem (1)-(2)-(3)-(4). Substitute (6) into the right hand side of (5). We have

$$\begin{aligned}\tilde{u}(t,x)&=\varphi(x-ta)-\lambda\int_0^t e^{-(\mu-\gamma)t_1}\psi\bigl(x-ta-t_1(b-a)\bigr)dt_1\\&\quad+\lambda\nu\int_0^t dt_1\int_0^{t_1}e^{(\mu-\gamma)(-t_1+t_2)}\tilde{u}\bigl(t_2,x-ta-t_1(b-a)+t_2b\bigr)dt_2.\end{aligned} \tag{7}$$

Repeat this argument. In other words, we employ successive approximation of (5)-(6) with the initial step $\bigl(\tilde{u}(t,x),\tilde{v}(t,x)\bigr)\sim\bigl(\varphi(x),\psi(x)\bigr)$. We can prove the following.

Theorem 1. *Suppose that $\varphi,\psi\in\mathscr{B}(\mathbb{R}^d)$.*

(i) *The system of integral equations* (5)-(6) *has a unique solution $(u,v)\in C\bigl(\mathbb{R};\mathscr{B}(\mathbb{R}^d)\bigr)$. Moreover, for any $T>0$, there exists a constant $C_T>0$ such that*

$$\|u(t)\|,\|v(t)\|\leqslant C_T\bigl(\|\varphi\|+\|\psi\|\bigr),\quad t\in[-T,T].$$

(ii) *$u(t,x)$ and $v(t,x)$ are given by the formulas*

$$\begin{aligned}e^{\gamma t}u(t,x)&=\varphi(x-ta)-\lambda\int_0^t e^{-(\mu-\gamma)t_1}\psi\bigl(x-ta-t_1(b-a)\bigr)dt_1\\&\quad+\sum_{k=1}^\infty\lambda^k\nu^k\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{2k-1}}\exp\left((\mu-\gamma)\sum_{j=1}^{2k}(-1)^jt_j\right)\\&\qquad\times\varphi\left(x-ta+\sum_{j=1}^{2k}(-1)^jt_j(b-a)\right)dt_{2k}\\&\quad-\sum_{k=1}^\infty\lambda^{k+1}\nu^k\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{2k}}\exp\left((\mu-\gamma)\sum_{j=1}^{2k+1}(-1)^jt_j\right)\\&\qquad\times\psi\left(x-ta+\sum_{j=1}^{2k+1}(-1)^jt_j(b-a)\right)dt_{2k+1},\end{aligned} \tag{8}$$

$$\begin{aligned}e^{\mu t}v(t,x)&=\psi(x-tb)-\nu\int_0^t e^{-(\gamma-\mu)t_1}\varphi\bigl(x-tb-t_1(a-b)\bigr)dt_1\\&\quad+\sum_{k=1}^\infty\lambda^k\nu^k\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{2k-1}}\exp\left((\gamma-\mu)\sum_{j=1}^{2k}(-1)^jt_j\right)\\&\qquad\times\psi\left(x-tb+\sum_{j=1}^{2k}(-1)^jt_j(a-b)\right)dt_{2k}\\&\quad-\sum_{k=1}^\infty\lambda^k\nu^{k+1}\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{2k}}\exp\left((\gamma-\mu)\sum_{j=1}^{2k+1}(-1)^jt_j\right)\\&\qquad\times\varphi\left(x-tb+\sum_{j=1}^{2k+1}(-1)^jt_j(a-b)\right)dt_{2k+1}.\end{aligned} \tag{9}$$

The right hand sides of (8) *and* (9) *converge uniformly on* $[-T,T]\times\mathbb{R}^d$ *for any* $T>0$.

Sketch of the Proof. Part (i) can be proved by the contraction mapping theorem. We omit the detail. We shall obtain the formula (8) and show its uniform convergence. Repeat using (7). We can obtain for any $n=1,2,3,\dots,$

$$\begin{aligned}
\tilde{u}(t,x) &= \varphi(x-ta)-\lambda\int_0^t e^{-(\mu-\gamma)t_1}\psi\big(x-ta-t_1(b-a)\big)dt_1\\
&\quad+\sum_{k=1}^n \lambda^k\nu^k\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{2k-1}}\exp\left((\mu-\gamma)\sum_{j=1}^{2k}(-1)^j t_j\right)\\
&\qquad\qquad\times\varphi\left(x-ta+\sum_{j=1}^{2k}(-1)^j t_j(b-a)\right)dt_{2k}\\
&\quad-\sum_{k=1}^n \lambda^{k+1}\nu^k\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{2k}}\exp\left((\mu-\gamma)\sum_{j=1}^{2k+1}(-1)^j t_j\right)\\
&\qquad\qquad\times\psi\left(x-ta+\sum_{j=1}^{2k+1}(-1)^j t_j(b-a)\right)dt_{2k+1}\\
&\quad+R_{n+1}(t,x),\\
R_{n+1}(t,x) &= \lambda^{n+1}\nu^{n+1}\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{2n+1}}\exp\left((\mu-\gamma)\sum_{j=1}^{2n+2}(-1)^j t_j\right)\\
&\qquad\qquad\times\tilde{u}\left(t_{2n+2},x-ta+\sum_{j=1}^{2k+1}(-1)^j t_j(b-a)+t_{2n+2}b\right)dt_{2n+2}.
\end{aligned}$$

Fix arbitrary $T>0$. Set $A=\max\{|\lambda|,|\nu|,|\gamma|+|\mu|\}$ and $N(T)=\max_{t\in[-T,T]}\|\tilde{u}(t)\|$ for short. Then, we have for $(t,x)\in[-T,T]\times\mathbb{R}^d$

$$\begin{aligned}
|R_{n+2}(t,x)| &\leqslant (Ae^{AT})^{2n+2}N(T)\left|\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{2n+1}}dt_{2n+2}\right|\\
&= (Ae^{AT})^{2n+2}N(T)\frac{|t|^{2n+2}}{(2n+2)!}\leqslant N(T)\frac{\{ATe^{AT}\}^{2n+2}}{(2n+2)!}.
\end{aligned}$$

Hence

$$0\leqslant\sup_{t\in[-T,T]}\|R_{n+1}(t)\|\leqslant N(T)\frac{\{ATe^{AT}\}^{2n+2}}{(2n+2)!}\to 0\quad(n\to\infty).$$

This completes the proof. □

Finally we remark that our formula (8) obtains d'Alembert's formula of solutions to the Cauchy problem for the one-dimensional wave equation of the form

$$u_{tt}-u_{xx}=0,\quad u(0,x)=\phi_0(x),\quad u_t(0,x)=\phi_1(x),\quad (t,x)\in\mathbb{R}\times\mathbb{R}. \tag{10}$$

If we set $v=u_t+u_x$, the Cauchy problem (10) becomes

$$\begin{aligned}
u_t+u_x-v=0,&\quad u(0,x)=\phi_0(x),\\
v_t-v_x=0,&\quad v(0,x)=\phi_0'(x)+\phi_1(x).
\end{aligned}$$

Substitute $a = 1$, $b = -1$, $\lambda = -1$, $\gamma = \nu = \mu = 0$, $\varphi = \phi_0$ and $\psi = \phi_0' + \phi_1$ into (8). We obtain

$$\begin{aligned}
u(t,x) &= \phi_0(x-t) + \int_0^t (\phi_0' + \phi_1)(x - t + 2\tau) d\tau \\
&= \phi_0(x-t) + \frac{1}{2}\int_{x-t}^{x+t} (\phi_0' + \phi_1)(y) dy \\
&= \phi_0(x-t) + \frac{\phi_0(x+t) - \phi_0(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \phi_1(y) dy \\
&= \frac{\phi_0(x+t) + \phi_0(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \phi_1(y) dy,
\end{aligned}$$

which is d'Alembert's formula for the Cauchy problem (10).