Double fibration transforms with conjugate points

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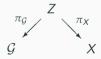
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Double fibration

Following Mazzucchelli-Salo-Tzou [2], we introduce double fibration transforms.

- Let \mathcal{G} and X be oriented smooth manifolds without boundaries. $N := \dim(\mathcal{G})$ and $n := \dim(X)$. Denote by $d\mathcal{G}$ and dX the orientation forms of \mathcal{G} and X respectively.
- Let Z be an oriented embedded submanifold of $\mathcal{G} \times X$, and let dZ be the orientation form.
- Assume that $N+n>\dim(Z)>N\geqq n\geqq 2$, and set $n':=\dim(Z)-N$ and n'':=n-n'. Then $\dim(Z)=N+n'$, n=n'+n'' and n', $n''=1,\ldots,n-1$.



- We assume that Z is a double fibration, that is, the natural projections $\pi_{\mathcal{G}}: Z \rightarrow \mathcal{G}$ and $\pi_X: Z \rightarrow X$ are submersions respectively.
- Then $G_z := \pi_x \circ \pi_{\mathcal{G}}^{-1}(z)$ becomes an n'-dim submanifold of X for any $z \in \mathcal{G}$, and $H_x := \pi_{\mathcal{G}} \circ \pi_X^{-1}(x)$ forms an (N n'')-dim submanifold of \mathcal{G} for any $x \in X$.

Orientation forms on G_z and H_x

Fix arbitrary $(z, x) \in Z$, and let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_N\}$ be bases of $T_x X$ and $T_z \mathcal{G}$ respectively such that

$$T_{(z,x)}Z=\text{span}\langle v_1,\ldots,v_{n'},w_1,\ldots,w_N\rangle=\text{span}\langle v_1,\ldots,v_n,w_1,\ldots,w_{N-n''}\rangle.$$

The induced orientation forms dG_z on G_z and dH_x on H_x are given by

$$\begin{split} dG_{z}\big(d\pi_{X}(v_{1}),\ldots,d\pi_{X}(v_{n'})\big) &:= dZ_{\pi_{\mathcal{G}}^{-1}(z)}(v_{1},\ldots,v_{n'}) \\ &= \frac{dZ(v_{1},\ldots,v_{n'},w_{1},\ldots,w_{N})}{d\mathcal{G}\big(d\pi_{\mathcal{G}}(w_{1}),\ldots,d\pi_{\mathcal{G}}(w_{N})\big)}, \\ dH_{x}\big(d\pi_{\mathcal{G}}(w_{1}),\ldots,d\pi_{X}(w_{N-n''})\big) &:= dZ_{\pi_{X}^{-1}(x)}(w_{1},\ldots,w_{N-n''}) \\ &= \frac{dZ(v_{1},\ldots,v_{n},w_{1},\ldots,w_{N-n''})}{dX\big(d\pi_{X}(v_{1}),\ldots,d\pi_{X}(v_{n})\big)}. \end{split}$$

Double fibration transform

Suppose that a weight function $\kappa(z,x) \in C^{\infty}(\mathcal{G} \times X)$ is nowhere vanishing. A double fibration transform \mathcal{R} associated with the double fibration Z is defined by

$$\mathcal{R}f(z) := \left(\int_{G_z} \kappa(z, x) \frac{f}{|dX|^{1/2}}(x) dG_z(x) \right) |d\mathcal{G}(z)|^{1/2}$$

for $f \in \mathcal{D}(X, \Omega_X^{1/2})$. The adjoint \mathcal{R}^* is given by

$$\mathcal{R}^* u(x) = \left(\int_{\mathcal{H}_x} \overline{\kappa(z, x)} \frac{u}{|d\mathcal{G}|^{1/2}}(z) d\mathcal{H}_x(z) \right) |dX(x)|^{1/2}$$

for $u \in \mathcal{D}(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2})$. Then we deduce that

$$\mathcal{R}: \mathscr{D}(X,\Omega_X^{1/2}) \to \mathscr{E}(\mathcal{G},\Omega_\mathcal{G}^{1/2}), \quad \mathcal{R}^*: \mathscr{D}(\mathcal{G},\Omega_\mathcal{G}^{1/2}) \to \mathscr{E}(X,\Omega_X^{1/2}),$$

are continuous linear mappings, and so are

$$\mathcal{R}: \mathscr{E}'(X,\Omega_X^{1/2}) \to \mathscr{D}'(\mathcal{G},\Omega_\mathcal{G}^{1/2}), \quad \mathcal{R}^*: \mathscr{E}'(\mathcal{G},\Omega_\mathcal{G}^{1/2}) \to \mathscr{D}'(X,\Omega_X^{1/2}).$$

More precisely $\mathcal R$ and $\mathcal R^*$ are elliptic Fourier integral operators.

Mapping properties of double fibration transforms

Theorem 1

Suppose that Z is a double fibration with $\dim(Z) = N + n'$. Then \mathcal{R} and \mathcal{R}^* are elliptic Fourier integral operators of order -(N+2n'-n)/4 with canonical relations $(N^*Z\setminus 0)'$ and $((N^*Z\setminus 0)^T)'$ respectively. More precisely

$$\begin{split} \mathcal{R} &\in \mathcal{I}^{-(N+2n'-n)/4} \big(\mathcal{G} \times X, N^* Z \setminus 0; \Omega_{\mathcal{G} \times X}^{1/2} \big), \\ \mathcal{R}^* &\in \mathcal{I}^{-(N+2n'-n)/4} \big(X \times \mathcal{G}, (N^* Z \setminus 0)^T; \Omega_{X \times \mathcal{G}}^{1/2} \big), \end{split}$$

where

$$N^*Z \setminus 0 = \left\{ \left(z, A(z, x)\eta, x, \eta \right) : (z, x) \in Z, \eta \in N_x^*G_z \setminus \{0\} \right\}$$
$$= \left\{ \left(z, \zeta, x, B(z, x)\zeta \right) : (z, x) \in Z, \zeta \in N_z^*H_x \setminus \{0\} \right\},$$

 $A(z,x) \in \text{Hom}(N_x^*G_z, T_z^*G)$ and $B(z,x) \in \text{Hom}(N_z^*H_x, T_x^*X)$ smoothly depend on $(z,x) \in Z$ respectively.

Preliminaries

For local coordinates $(z,x)=(z',z'',x',x'')\in\mathbb{R}^{N-n''}\times\mathbb{R}^{n''}\times\mathbb{R}^{n''}\times\mathbb{R}^{n''}$, There exist $\mathbb{R}^{n''}$ -valued functions $\phi(z,x')$ and b(x,z') such that we have locally

$$Z = \{x'' = \phi(z, x')\} = \{z'' = b(x, z')\}.$$

Lemma 2 ([2, Lemmas 2.4, 2.5 and 2.6])

$$\begin{split} N_{(z,x)}^* Z &= \Big\{ \Big(-\phi_z(z,x')^T \eta'', \big(-\phi_{x'}(z,x')^T \eta'', \eta'' \big) \Big) : \eta'' \in \mathbb{R}^{n''} \Big\}, \\ A(z,x) &\left[\begin{matrix} -\phi_{x'}(z,x')^T \\ I_{n''} \end{matrix} \right] \eta'' = -\phi_z(z,x')^T \eta'', \quad \eta'' \in \mathbb{R}^{n''}. \end{split}$$

Similar results hold for b(x, z') and B(z, x).

Variation fields and conjugate points

Fix arbitrary $(z,w)\in T\mathcal{G}$, and consider a curve in \mathcal{G} of the form

$$z(s) = z + sw + \mathcal{O}(s^2)$$
 near $s = 0$.

Then $(G_{z(s)})$ is said to be a variation of G_z , and the variation field $J_w: G_z \to (N_x^*G_z)^*$ associated to $(G_{z(s)})$ is defined by

$$J_w(x) := A(z, x)^* w \in (N_x^* G_z)^* \simeq N_x G_z = T_x X / T_x G_z$$

for $x \in G_z$. Note that

$$A(z,x)^* \in \operatorname{Hom}(T_z\mathcal{G}, (N_x^*G_z)^*) \simeq \operatorname{Hom}(T_z\mathcal{G}, N_xG_z), \quad (z,x) \in Z,$$

For $z \in \mathcal{G}$ and $x, y \in \mathcal{G}_z$, set

$$V_z(x,y) := \{J_w(x) : w \in T_z \mathcal{G}, J_w(y) = 0\}.$$

Note that $\dim(V_z(x,y)) \leq n''$ holds since $\operatorname{rank}(A(z,x)^*) = n''$, and $\dim(V_z(x,y)) = \dim(V_z(y,x))$ holds for any $z \in \mathcal{G}$ and $x,y \in \mathcal{G}_z$.

Z-conjugate triplets

Normal operators without conjugate points

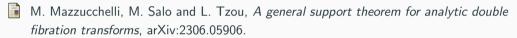
Known results on geodesic X-ray transforms with conjugate points

Normal operators with conjugate points

Outline of the proof

References





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H. Chihara, *Microlocal analysis of double fibration transforms with conjugate points*, arXiv:2412.14520.