Normal operators of double fibration transforms

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d-plane transform on $\mathbb{R}^n$

- Let $n = 2, 3, 4, \ldots,$ and let $d = 1, \ldots, n-1$.
- Grassmannian $G_{d,n} :=$ the set of all d-dimensional vector subspaces of $\mathbb{R}^n$.
- For any $\sigma \in G_{d,n}$, we have an orthogonal decomposition $x = x' + x'' \in \sigma \oplus \sigma^\perp = \mathbb{R}^n$.
- Affine Grassmannian $G(d,n) :=$ the set of all d-dimensional planes in $\mathbb{R}^n$, i.e.,

$$G(d,n) := \{(\sigma, x'') : \sigma \in G_{d,n},\ x'' \in \sigma^\perp\}.$$

- The d-plane transform of $f \in \mathscr{S}(\mathbb{R}^n)$ is defined by

$$R_d f(\sigma, x'') := \int_\sigma f(x' + x'')dx', \quad (\sigma, x'') \in G(d,n),$$

  where dx' is the Lebesgue measure on σ.
- R_1 and R_{n-1} are said to be the X-ray transsorm and the Radon transform on $\mathbb{R}^n$ respectively, and $R_1 = R_{n-1}$ when $n = 2$.

The normal operator of the d-plane transform

- The formal adjoint of R_d of a continuous function φ on $G(d,n)$ is explicitly given by

$$R_d^*\varphi(x) := \frac{1}{C(d,n)}\int_{O(n)} \varphi(x + k\cdot\sigma)dk,$$

  where $C(d,n) = (4\pi)^d\Gamma(n/2)/\Gamma((n-1)/2)$, $\Gamma(\cdot)$ is the gamma function, $O(n)$ is the orthogonal group, dk is the normalized measure which is invariant under rotations, and $\sigma \in G_{d,n}$ is arbitrary. R_d and R_d^* are elliptic Fourier integral operators.
- The normal operator is $R_d^*R_d = (-\Delta_{\mathbb{R}^n})^{-d/2}$, and the excess e for $R_d^*R_d$, which is the degeneracy of the phase function of $R_d^*R_d$, is

$$e = \dim\big(G(d,n)\big) - \dim(\mathbb{R}^n) = (d+1)(n-d) - n = d(n-d-1).$$

  This gives the inversion formula $f = (-\Delta_{\mathbb{R}^n})^{d/2}R_d^*R_d f$ for $f(x) = \mathcal{O}(\langle x\rangle^{-d-\varepsilon})$.
- We can consider the invertibility for more general operators arising in integral geometry via the normal operators.

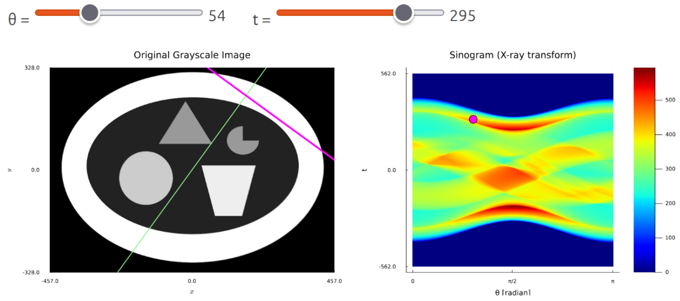
X-ray transform on $\mathbb{R}^2$

Let $n = 2$. A planar line ℓ is parametrized by $(\theta, t) \in [0, \pi] \times \mathbb{R}$ as

$$\ell = \{(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta) \ : \ s \in \mathbb{R}\}.$$

The X-ray transform of a function f of $(x, y) \in \mathbb{R}^2$ is defined by

$$R_1 f(\theta, t) := \int_\ell f = \int_{-\infty}^{\infty} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta) ds, \quad (\theta, t) \in [0, \pi] \times \mathbb{R}.$$



Conormal distributions

Conormal distributions

Definition 1 (Conormal distributions)

Let X be an N-dim manifold, and let Y be a closed submanifold of X. $u \in \mathscr{D}'(X)$ is said to be conormal with respect to Y of degree m if

$$L_1 \cdots L_M u \in {}^{\infty}H^{\mathrm{loc}}_{(-m-N/4)}(X)$$

for all $M = 0, 1, 2, \ldots$ and all vector fields $L_1, \ldots, L_M$ tangential to Y. Denote by $I^m(X, N^*Y)$, the set of all distributions on X conormal with respect to Y of degree m.

Note that $N^*_y Y := T^*_y X / T^*_y Y$ for any $y \in Y$. If $u \in I^m(X, N^*Y)$, then $\mathrm{WF}(u) \subset N^*Y \setminus 0$.

$$\|u\|_{{}^{\infty}H_{(s)}(\mathbb{R}^N)} := \sup_{j=0,1,2,\ldots} \left(\int_{A_j} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{1/2},$$

$$A_0 := \{|\xi| < 1\}, \quad A_j := \{2^{j-1} \leqq |\xi| < 2^j\}, j = 1, 2, 3, \ldots .$$

Conormal distributions and oscillatory integrals

Proposition 2 (Characterization of conormal distributions)

Let $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and let $Y = \mathbb{R}^k \times \{0\} = \{x'' = 0\}$. Then $u \in \mathscr{D}'(\mathbb{R}^N)$ belongs to $I^{m+k/2-N/4}(\mathbb{R}^n, N^*Y)$ if and only if there exists an amplitude $a(x'', \xi') \in S^m(\mathbb{R}^{N-k} \times \mathbb{R}^k)$ such that

$$u(x) = \int_{\mathbb{R}^k} e^{ix' \cdot \xi'} a(x'', \xi') d\xi'.$$

We can replace the conormal bundle N^*Y by more general Lagrangian distributions Λ. The elements of $I^m(X, \Lambda)$ is said to be Lagrangian distributions on X. These are characterized as oscillatory integrals with more general phase functions. The distributions kernels of Fourier integral operators are Lagrangian distributions. Rigorously we should use the set of distribution section of the half-density bundle $I^m(X, \Lambda; \Omega_X^{1/2})$.

Double fibration transforms

Double fibration

Following Mazzucchelli-Salo-Tzou [2], we introduce double fibration transforms.

- Let $\mathcal{G}$ and X be oriented smooth manifolds without boundaries. $N := \dim(\mathcal{G})$ and $n := \dim(X)$. Denote by $d\mathcal{G}$ and dX the orientation forms of $\mathcal{G}$ and X respectively.
- Let Z be an oriented embedded submanifold of $\mathcal{G}\times X$, and let dZ be the orientation form.
- Assume that $N + n > \dim(Z) > N \geqq n \geqq 2$, and set $n' := \dim(Z) - N$ and $n'' := n - n'$. Then $\dim(Z) = N + n'$, $n = n' + n''$ and $n', n'' = 1, \ldots, n-1$.

$$\begin{tikzcd} & Z \arrow[dl, "\pi_{\mathcal{G}}"'] \arrow[dr, "\pi_X"] & \\ \mathcal{G} & & X \end{tikzcd} \qquad \begin{tikzcd} & (N^*Z \setminus 0)' \arrow[dl, "\pi_{\mathrm{L}}"'] \arrow[dr, "\pi_{\mathrm{R}}"] & \\ T^*\mathcal{G} \setminus 0 & & T^*X \setminus 0 \end{tikzcd}$$

- We assume that Z is a double fibration, that is, the natural projections $\pi_{\mathcal{G}} : Z \to \mathcal{G}$ and $\pi_X : Z \to X$ are submersions respectively.

Orientation forms on $G_z := \pi_x \circ \pi_{\mathcal{G}}^{-1}(z)$ and $H_x := \pi_{\mathcal{G}} \circ \pi_X^{-1}(x)$

- $G_z := \pi_x \circ \pi_{\mathcal{G}}^{-1}(z)$ becomes an n'-dim submanifold of X for any $z \in \mathcal{G}$, and $H_x := \pi_{\mathcal{G}} \circ \pi_X^{-1}(x)$ forms an $(N - n'')$-dim submanifold of $\mathcal{G}$ for any $x \in X$.
- Fix arbitrary $(z, x) \in Z$, and let $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_N\}$ be bases of $T_x X$ and $T_z\mathcal{G}$ respectively such that

$$T_{(z,x)}Z = \operatorname{span}\langle v_1, \dots, v_{n'}, w_1, \dots, w_N\rangle = \operatorname{span}\langle v_1, \dots, v_n, w_1, \dots, w_{N-n''}\rangle.$$

The induced orientation forms dG_z on G_z and dH_x on H_x are given by

$$\begin{aligned} dG_z\big(d\pi_X(v_1), \dots, d\pi_X(v_{n'})\big) &:= dZ_{\pi_{\mathcal{G}}^{-1}(z)}(v_1, \dots, v_{n'}) \\ &= \frac{dZ(v_1, \dots, v_{n'}, w_1, \dots, w_N)}{d\mathcal{G}\big(d\pi_{\mathcal{G}}(w_1), \dots, d\pi_{\mathcal{G}}(w_N)\big)}, \\ dH_x\big(d\pi_{\mathcal{G}}(w_1), \dots, d\pi_X(w_{N-n''})\big) &:= dZ_{\pi_X^{-1}(x)}(w_1, \dots, w_{N-n''}) \\ &= \frac{dZ(v_1, \dots, v_n, w_1, \dots, w_{N-n''})}{dX\big(d\pi_X(v_1), \dots, d\pi_X(v_n)\big)}. \end{aligned}$$

Double fibration transform

Suppose that a weight function $\kappa(z,x) \in C^\infty(\mathcal{G}\times X)$ is nowhere vanishing. A double fibration transform $\mathcal{R}$ associated with the double fibration Z is defined by

$$\mathcal{R}f(z) := \left(\int_{G_z} \kappa(z,x)\frac{f}{|dX|^{1/2}}(x)dG_z(x)\right)|d\mathcal{G}(z)|^{1/2}$$

for $f \in \mathscr{D}(X, \Omega_X^{1/2})$. The adjoint $\mathcal{R}^*$ is given by

$$\mathcal{R}^*u(x) = \left(\int_{H_x} \overline{\kappa(z,x)}\frac{u}{|d\mathcal{G}|^{1/2}}(z)dH_x(z)\right)|dX(x)|^{1/2}$$

for $u \in \mathscr{D}(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2})$. Then we deduce that

$$\mathcal{R} : \mathscr{D}(X, \Omega_X^{1/2}) \to \mathscr{E}(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}), \quad \mathcal{R}^* : \mathscr{D}(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}) \to \mathscr{E}(X, \Omega_X^{1/2}),$$

are continuous linear mappings, and so are

$$\mathcal{R} : \mathscr{E}'(X, \Omega_X^{1/2}) \to \mathscr{D}'(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}), \quad \mathcal{R}^* : \mathscr{E}'(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}) \to \mathscr{D}'(X, \Omega_X^{1/2}).$$

More precisely $\mathcal{R}$ and $\mathcal{R}^*$ are elliptic Fourier integral operators.

Mapping properties of double fibration transforms

Theorem 3 ([2, Theorem 2.2] & [Hörmander IV, Theorem25.2.2)

Suppose that Z is a double fibration with $\dim(Z) = N + n'$. Then $\mathcal{R}$ and $\mathcal{R}^*$ are elliptic Fourier integral operators of order $-(N+2n'-n)/4$ with canonical relations $(N^*Z \setminus 0)'$ and $((N^*Z \setminus 0)^T)'$ respectively. More precisely

$$\mathcal{R} \in \mathcal{I}^{-(N+2n'-n)/4}(\mathcal{G}\times X, N^*Z \setminus 0; \Omega^{1/2}_{\mathcal{G}\times X}),$$
$$\mathcal{R}^* \in \mathcal{I}^{-(N+2n'-n)/4}(X \times \mathcal{G}, (N^*Z \setminus 0)^T; \Omega^{1/2}_{X\times\mathcal{G}}),$$

where

$$\begin{aligned} N^*Z \setminus 0 &= \big\{\big(z, A(z,x)\eta, x, \eta\big) : (z,x) \in Z, \eta \in N^*_x G_z \setminus \{0\}\big\} \\ &= \big\{\big(z, \zeta, x, B(z,x)\zeta\big) : (z,x) \in Z, \zeta \in N^*_z H_x \setminus \{0\}\big\}, \end{aligned}$$

$A(z,x) \in \mathrm{Hom}(N^*_x G_z, T^*_z\mathcal{G})$ *and* $B(z,x) \in \mathrm{Hom}(N^*_z H_x, T^*_x X)$ *smoothly depend on* $(z,x) \in Z$ *respectively.*

Examples of double fibration transforms

- *d*-plane transform on $\mathbb{R}^n$: $\mathcal{G} := G(d, m)$, $X := \mathbb{R}^n$.
- Geodesic X-ray transform: Let (M, g) be a compact and nontrapping Riemannian manifold with a strictly convex boundary. Denote by $\nu(x)$ the unit outer normal vector at $x \in \partial M$. Then

$$\mathcal{G} := \partial_- SM = \{(x, u) \in SM : x \in \partial M, \langle u, \nu(x)\rangle < 0\}, \quad X := M^{\mathrm{int}}.$$

- Null bicharacteristics: Let P be a real-principal-type pseudodifferential operator on a manifold X with the principal symbol $p_m(x, \xi)$. Let $\mathcal{G}$ be the set of all Hamilton flows for $p_m(x, \xi) = 0$ on T^*X, and consider the integration over all the $\gamma \in \mathcal{G}$.
- Light ray transform: This is a special case of the above. Let (M, g) be a Lorentzian manifold, and set P be the d'Alembertian.

Z-conjugate points

For local coordinates $(z,x) = (z', z'', x', x'') \in \mathbb{R}^{N-n''} \times \mathbb{R}^{n''} \times \mathbb{R}^{n'} \times \mathbb{R}^{n''}$, There exist $\mathbb{R}^{n''}$-valued functions $\phi(z,x')$ and $b(x,z')$ such that we have locally

$$Z = \{x'' = \phi(z,x')\} = \{z'' = b(x,z')\}.$$

Lemma 4 ([2, Lemmas 2.4, 2.5 and 2.6])

$$N^*_{(z,x)}Z = \left\{\left(-\phi_z(z,x')^T\eta'', (-\phi_{x'}(z,x')^T\eta'', \eta'')\right) : \eta'' \in \mathbb{R}^{n''}\right\},$$

$$A(z,x)\begin{bmatrix} -\phi_{x'}(z,x')^T \\ I_{n''} \end{bmatrix}\eta'' = -\phi_z(z,x')^T\eta'', \quad \eta'' \in \mathbb{R}^{n''}.$$

Similar results hold for $b(x,z')$ and $B(z,x)$.

Variation fields and conjugate points

Fix arbitrary $(z, w) \in T\mathcal{G}$, and consider a curve in $\mathcal{G}$ of the form

$$z(s) = z + sw + \mathcal{O}(s^2) \quad \text{near} \quad s = 0.$$

Then $(G_{z(s)})$ is said to be a variation of G_z, and the variation field $J_w : G_z \to (N_x^* G_z)^*$ associated to $(G_{z(s)})$ is defined by

$$J_w(x) := A(z, x)^* w \simeq -\phi_z(z, x')w \in (N_x^* G_z)^* \simeq N_x G_z = T_x X \,/\, T_x G_z$$

for $x \in G_z$. Note that $T_z H_x = \mathrm{Ker}\big(A(z, x)^*\big)$ since $\phi\big(z(s), x'\big) = x''$ for $z(s) \in H_x$, and

$$A(z, x)^* \in \mathrm{Hom}\big(T_z\mathcal{G}, (N_x^* G_z)^*\big) \simeq \mathrm{Hom}\big(T_z\mathcal{G}, N_x G_z\big), \quad (z, x) \in Z,$$

For $z \in \mathcal{G}$ and $x, y \in G_z$, set

$$V_z(x, y) := \{J_w(x) : w \in T_z\mathcal{G}, J_w(y) = 0\}.$$

Note that $\dim\big(V_z(x, y)\big) \leqq n''$ holds since $\mathrm{rank}\big(A(z, x)^*\big) = n''$, and $\dim\big(V_z(x, y)\big) = \dim\big(V_z(y, x)\big)$ holds for any $z \in \mathcal{G}$ and $x, y \in G_z$.
cf. If $x = \exp_y(tu)$, then $J_w(x) \simeq Y(t) := tD\exp_y(tu)w$.

Z-conjugate triplets

Definition 5

Suppose that Z is a double fibration and $N \geqq 2n''$. Let $k = 1, \ldots, n''$.

- Z-conjugate triplet of degree k: Let $z \in \mathcal{G}$ and let $x, y \in G_z$ with $x \neq y$. We say that $(z; x, y)$ is a Z-conjugate triplet of degree k if $\dim\bigl(V_z(x, y)\bigr) = n'' - k$.
- Regular Z-conjugate triplet of degree k: We say that a Z-conjugate triplet $(z; x, y)$ of degree k is regular if there exit a nbd U_x of x in X, a nbd U_y of y in X, and a nbd W_z of z in $\mathcal{G}$ such that any Z-conjugate triplet $(z'; x', y') \in W_z \times U_x \times U_y$ is also of degree k. The set of all the regular Z-conjugate triplets of degree k is denoted by $C_{R,k}$.
- The set of all the Z-conjugate triplets which are not regular is denoted by C_S.

How to describe Z-conjugacy 1/2

Let $(z_0; x_0, y_0)$ be a Z-conjugate triplet of degree $k = 1, \dots, n''$. Z is locally expressed by ϕ and ψ as $\{x'' = \phi(z, x')\}$ near (z_0, x_0) and $\{y'' = \psi(z, y')\}$ near (z_0, y_0) respectively, and

$$V_{z_0}(x_0, y_0) = \left\{ \phi_z(z_0, x_0')w : w \in \operatorname{Ker}\bigl(\psi_z(z_0, y_0')\bigr) \right\}.$$

Set $\phi = [\phi^{(1)}, \dots, \phi^{(n'')}]^T$ and $\psi = [\psi^{(1)}, \dots, \psi^{(n'')}]^T$. Then $\{\phi_z^{(1)}, \dots, \phi_z^{(n'')}\}$ and $\{\psi_z^{(1)}, \dots, \psi_z^{(n'')}\}$ are linearly independent near (z_0, x_0) and (z_0, y_0) respectively. Note that

$$\operatorname{Ker}\bigl(\psi_z(z_0, y_0')\bigr) = \operatorname{span}\left\langle \psi_z^{(1)}(z_0, y_0')^T, \dots, \psi_z^{(n'')}(z_0, y_0')^T \right\rangle^{\perp} \quad \text{in} \quad \mathbb{R}^N.$$

We deduce that $\dim\bigl(V_{z_0}(x_0, y_0)\bigr) = n'' - k$ is equivalent to

$$\dim\Bigl(\operatorname{span}\left\langle \phi_z^{(1)}(z_0, x_0'), \dots, \phi_z^{(n'')}(z_0, x_0')\right\rangle \cap \operatorname{span}\left\langle \psi_z^{(1)}(z_0, y_0'), \dots, \psi_z^{(n'')}(z_0, y_0')\right\rangle\Bigr) = k. \quad (1)$$

We express (1) by k equations.

How to describe Z-conjugacy 2/2

Denote by $\{\tilde{\psi}_z^{(1)}, \ldots, \tilde{\psi}_z^{(n'')}\}$ the Schmidt orthonormalization of $\{\psi_z^{(1)}, \ldots, \psi_z^{(n'')}\}$.

(1) is equivalent to the following: there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}^{n''}$, $\lambda_l = [\lambda_{l1}, \ldots, \lambda_{ln''}]$ $(l = 1, \ldots, k)$ such that $\lambda_1, \ldots, \lambda_k$ are linearly independent and if we set

$$\begin{aligned}
\phi_z^{\lambda_l}(z, x') &:= \lambda_l \phi_z(z, x') = \sum_{m=1}^{n''} \lambda_{lm} \phi_z^{(m)}(z, x'), \\
H^{\lambda_l}(z, x', y') &:= \phi_z^{\lambda_l}(z, x') \phi_z^{\lambda_l}(z, x')^T - \sum_{m=1}^{n''} \left| \phi_z^{\lambda_l}(z, x') \tilde{\psi}_z^{(m)}(z, y')^T \right|^2 \\
&= \lambda_l \phi_z(z, x') \big(I_N - \tilde{\psi}_z^T(z, y') \tilde{\psi}_z(z, y') \big) \phi_z(z, x')^T \lambda_l^T, \\
H^{\lambda}(z, x', y') &:= [H^{\lambda_1}(z, x', y'), \ldots, H^{\lambda_k}(z, x', y')]^T,
\end{aligned}$$

for $l = 1, \ldots, k$, then

$$H^{\lambda}(z_0, x_0', y_0') = 0. \tag{2}$$

An artificial condition (H)

Condition (H): Let $k = 1, \dots, n''$. Suppose that $(z_0; x_0, y_0) \in C_{R,k}$. Suppose that $H^\lambda(z, x', y')$ is the same as that of the previous paragraph and satisfies (2). Condition (H) is that $\mathrm{rank}\big(D_{z,x',y'} H^\lambda(z_0, x_0', y_0')\big) = 1$ holds for any choice of linearly independent $\lambda_1, \dots, \lambda_k \in \mathbb{R}^{n''}$.

Spirit of Condition (H):

- **rank one**: We refer the case of the geodesic X-ray transform for the dimension. In other words we fit our case to the geodesic X-ray transform.
- **for any choice of $\lambda_1, \dots, \lambda_k$**: We avoid the case that there exist two choices $\lambda_1, \dots, \lambda_k$ and $\lambda_1', \dots, \lambda_k'$ such that

$$\{H^\lambda(z, x', y')) = 1\} \bigcap \{H^{\lambda'}(z, x', y')) = 1\}$$

is transversal.

Lemma 6

Suppose that Z is a double fibration, $N \geqq 2n''$ and some condition (H). *For any $k = 1, \ldots, n''$, $C_{R,k}$ is an $(N + 2n' - 1)$-dimensional embedded submanifold of $\mathcal{G} \times X \times X$.*

Note that the connected component of $C_{R,k}$ containing $(z_0; x_0, y_0)$ is characterized by

$$F(x'', y'', z; x', y') := \begin{bmatrix} x'' - \phi(z, x') \\ y'' - \psi(z, y') \\ H^{\lambda}(z, x', y') \end{bmatrix} = 0 \quad \text{near} \quad (z_0; x_0, y_0).$$

We have

$$\operatorname{rank}\big(DF(x_0'', y_0'', z_0; x_0', y_0')\big) = \operatorname{rank} \begin{bmatrix} I_{n''} & O & * \\ O & I_{n''} & * \\ O & O & D_{z,x',y'} H^{\lambda}(z_0, x_0', y_0') \end{bmatrix} = 2n'' + 1$$

Structure of normal operators

Normal operators without Z-conjugate points

$$\mathcal{R}^*\mathcal{R}f(x) = \left(\iint_{H_x \times G_z} \overline{\kappa(z,x)}\kappa(z,y)\frac{f}{|dX|^{1/2}}(y)dG_z(y)dH_x(z)\right)|dX(x)|^{1/2}, \quad x \in X.$$

Set $\mathcal{C} := (N^*Z \setminus 0)'$, which is the canonical relation of $\mathcal{R}$.

Theorem 7

Suppose that Z is a double fibration. In addition, we assume $N \geqq 2n''$ and the following:

- *$\pi_X : Z \to X$ is proper, and $\pi_X^{-1}(x)$ is connected for any $x \in X$.*
- *There are no Z-conjugate triplets, and $D\pi_L$ is injective at all $(z, \zeta, x, \eta) \in \mathcal{C}$.*

Then $\mathcal{C}^T \circ \mathcal{C}$ is a clean intersection with excess $e = N - n$, and $\mathcal{R}^*\mathcal{R}$ is an elliptic pseudodifferential operator of order $-n'$ on X.

Proof: Some lemmas in [2] and the assumptions guarantee the Bolker condition. $\square$

Known results on geodesic X-ray transforms with conjugate points

- Let (M,g) be a compact Riemannian manifold with strictly convex boundary. Set $\gamma_w(t) := \exp_{\pi_M(w)}(tw)$ for $w \in \partial_- SM$. Consider the geodesic X-ray fransform

$$\mathcal{X}f(w) := \left(\int_0^{\tau(w)} \kappa(\gamma_w(t), \dot{\gamma}_w(t)) \frac{f}{|dM|^{1/2}}(\gamma_w(t))\, dt\right) |d\partial_- SM(w)|^{1/2}.$$

- Stefanov and Uhlmann (2012) [3]: If $v_0 = |v_0|\theta_0$ is a fold conjugate vector at p_0, and v_0 is the only singularity of $\exp_{p_0}(v)$ on γ_{θ_0} near p_0, then the localized normal operator is decomposed as

$$\mathcal{X}^*\chi\mathcal{X} = A + F \quad \text{near} \quad p_0,$$

where A is a PsDOs of order -1, and F is a *FIO* of order $-n/2$.

- Holman and Uhlmann (2018) [1]: If $C_S = \emptyset$, then

$$\mathcal{X}^*\mathcal{X} = A + \sum_{k=1}^{n-1} \sum_{\alpha=1}^{M_k} F_{k,\alpha},$$

where A is a PsDOs of order -1, and F is a *FIO* of order $-(n-k+1)/2$.

Theorem 8

Suppose that Z is a double fibration, $C_S = \emptyset$, $N \geqq 2n''$, condition (H) and the following:

- *π_X is proper, and $\pi_X^{-1}(x)$ is connected for any $x \in X$.*
- *If $\pi_L^{-1}((z,\zeta)) = \{(z,\zeta,x,\eta)\}$ for $(z,\zeta,x,\eta) \in \mathcal{C}$, then $D\pi_L|_{(z,\zeta,x,\eta)}$ is injective.*

Then we have a decomposition of $\mathcal{R}^*\mathcal{R}$ of the form

$$\mathcal{R}^*\mathcal{R} = A + \sum_{k=1}^{n''} \sum_{\alpha \in \Lambda_k} F_{k,\alpha},$$

where A is an elliptic PsDO of order $-n'$ on X,
$F_{k,\alpha}$ is a FIO in $\mathcal{I}^{-(n+1-k)/2}(X \times X, \mathcal{C}'_{A_{k,\alpha}}; \Omega^{1/2}_{X \times X})$ with some canonical relation of $\mathcal{C}_{F_{k,\alpha}}$, associated to the decomposition of connected components $C_{R,k} = \bigcup_{\alpha \in \Lambda_k} C_{R,k,\alpha}$.

Outline of the proof

- $\mathcal{R}^*\mathcal{R}$ is given by

$$\frac{\mathcal{R}^*\mathcal{R}f}{|dX|^{1/2}}(x) = \iint_{H_x \times G_z} \overline{\kappa(z,x)}\kappa(z,y)\frac{f}{|dX|^{1/2}}(y)dG_z(y)dH_x(z).$$

- Follow the idea of Holman and Uhlmann [1]: a partition of unity of $\mathcal{G} \times X \times X$.
- Set $C_\delta := \{(z; x, x) : z \in \mathcal{G}, x \in X\}$, which is related to the elliptic term.
- $C_{R,k,\alpha}$ are disjoint since $C_S = \emptyset$, so are $C_{R,k,\alpha}$ and C_δ.
  Pick up disjoint nbds $U_{k,\alpha}$ and U_δ of $C_{R,k,\alpha}$ and C_δ respectively in $\mathcal{G} \times X \times X$.
- We can find an open set U_0 in $\mathcal{G} \times X \times X$ such that

$$U_0 \bigcup U_\delta \bigcup (\cup U_{k,\alpha}) = \mathcal{G} \times X \times X, \quad U_0 \bigcap \Big(C_\delta \bigcup (\cup C_{R,k,\alpha}) \Big) = \emptyset.$$

- Pick up a partition of unity subordinated to $\{U_0, U_\delta, U_{k,\alpha}\}$, and split the Schwartz kernel of $\mathcal{R}^*\mathcal{R}$. U_0-part of $\mathcal{R}^*\mathcal{R}$ is a smoothing operator, and is absorbed in U_δ-part A.

References

- S. Holman and G. Uhlmann, *On the microlocal analysis of the geodesic X-ray transform with conjugate points*, J. Diff. Geom., **108** (2018), pp.459–494.
- M. Mazzucchelli, M. Salo and L. Tzou, *A general support theorem for analytic double fibration transforms*, arXiv:2306.05906.
- P. Stefanov and G. Uhlmann, *The geodesic X-ray transform with fold caustics*, Anal. PDE, **5** (2012), pp.219–260
- H. Chihara, *Microlocal analysis of double fibration transforms with conjugate points*, arXiv:2412.14520.