

# Geodesic X-ray transform and streaking artifacts on simple surfaces or on spaces of constant curvature

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## X-ray transform on the plane

- All the planar lines are parametrized by  $(\theta, t) \in [0, \pi] \times \mathbb{R}$  by

$$\ell = \{(-s \sin \theta + t \cos \theta, s \cos \theta + t \sin \theta) : s \in \mathbb{R}\}.$$

The X-ray transform of  $f(x, y)$  on  $\mathbb{R}^2$  is defined by

$$\mathcal{R}f(\theta, t) := \int_{\ell} f = \int_{-\infty}^{\infty} f(-s \sin \theta + t \cos \theta, s \cos \theta + t \sin \theta) ds.$$

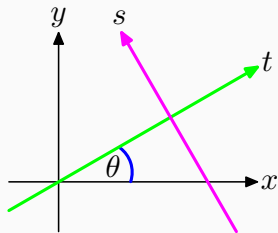
This is considered to be the measurements of CT scanners for normal tissue. The FBP formula  $f = (-\partial_x^2 - \partial_y^2)^{1/2} \circ \mathcal{R}^T \circ \mathcal{R}f$  is well-known.

- We consider a model of human body  $f$  containing a metal region  $D$  such as dental implants, stents in blood vessels, and etc. We observe that the metal streaking artifacts caused by beam hardening effect in the energy level of X-ray. The main term is the filtered back-projection of nonlinear term

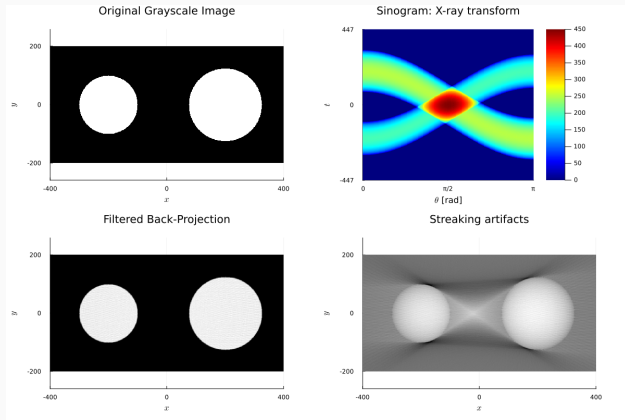
$$(-\partial_x^2 - \partial_y^2)^{1/2} \circ \mathcal{R}^T [(\mathcal{R}1_D)^2],$$

This is a conormal distribution whose singular support is the streaking artifact.

# Figures: metal streaking artifacts



The main part of artifacts:  $(-\partial_x^2 - \partial_y^2)^{1/2} \mathcal{R}^T [(\mathcal{R}1_D)^2]$ .

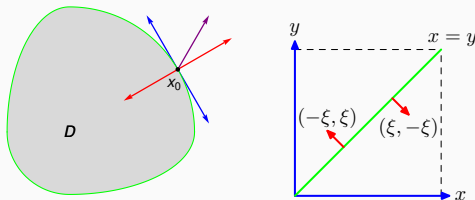


# Conormal distributions

## Definition 1 (Conormal distributions)

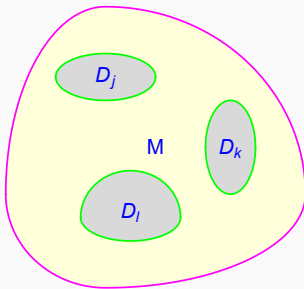
Let  $X$  be an  $N$ -dim manifold, and let  $Y$  be a closed submanifold of  $X$ . We say that  $u \in \mathcal{D}'(X)$  is conormal with respect to  $Y$  of degree  $m$  if  $L_1 \cdots L_\mu u \in {}^\infty H_{(-m-N/4)}^{\text{loc}}(X)$  for all  $\mu = 0, 1, 2, \dots$  and all vector fields  $L_1, \dots, L_\mu$  tangential to  $Y$ . Denote by  $I^m(N^*(Y))$  the set of all distributions on  $X$  conormal wrt  $Y$  of degree  $m$ . Note that  $\text{WF}(u) \subset N^*(Y) \setminus 0$ .

- If  $D \subset \mathbb{R}^n$  is a domain with smooth boundary, then  $1_D \in I^{-1/2-n/4}(N^*(\partial D))$ .
- If  $a(x, \xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ , then  $\int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} a(x, \xi) d\xi \in I^m(N^*(\Delta))$ ,  $\Delta = \{(x, x)\}$ .



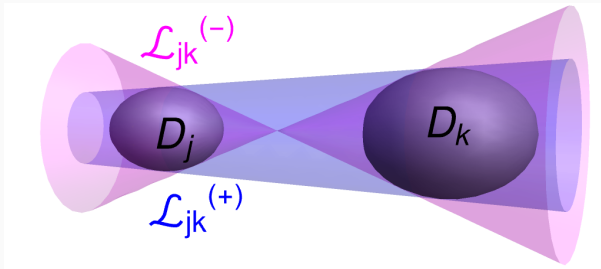
# Assumption

- Suppose that  $(M, g)$  is a compact nontrapping simple Riemannian manifold with strictly convex smooth boundary.
- In addition we assume that  $\dim(M) = 2$  or  $(M, g)$  is a space of constant curvature.
- Suppose that the metal region  $D \subset M^{\text{int}}$  is a disjoint union of  $D_j$  ( $j = 1 \dots, J$ ) which are simply connected, strictly convex and bounded with smooth boundaries  $\partial D_j$ .



## A hypersurface $\mathcal{L}$ surrounding the metal region $D$

- For any  $j$  and  $x \in \partial D_j$ , denote by  $v_j(x)$  the unit outer normal vector at  $x$ . Consider the tangent hyperplane  $\exp_x v_j(x)^\perp \cap M^{\text{int}}$  at  $x \in \partial D_j$ .
- There are some common tangent hyperplanes of  $\partial D_j$  and  $\partial D_k$  for  $j \neq k$ . In this case there is common tangent geodesics in such hyperplanes. The union of all these geodesics forms a conical or cylindrical hypersurface denoted by  $\mathcal{L}_{jk}^{(\pm)}$ . Set  $\mathcal{L} := \bigcup \left( \mathcal{L}_{jk}^{(+)} \cup \mathcal{L}_{jk}^{(-)} \right)$ .



# Main Theorem

The geodesic X-ray transform of a function  $f$  on  $M$  is defined by

$$\mathcal{X}f(\gamma_w) := \int_0^{\tau_+(w)} f(\gamma_w(s)) ds, \quad \nabla_{\dot{\gamma}_w(s)} \dot{\gamma}_w(s) = 0, \quad \dot{\gamma}_w(0) = w \in \partial_- S(M),$$

where  $\tau_+(w)$  is the exit time of  $\gamma_w$ . The nonlinear part of the CT image is supposed to be

$$f_{\text{MA}} := f_{\text{CT}} - f_{\text{normal}} = \sum_{k=1}^{\infty} A_k Q \mathcal{X}^T [(\mathcal{X}1_D)^{2k}] \mod C^\infty(M^{\text{int}}), \quad \{A_k\} \subset \mathbb{R},$$

where  $Q$  is an elliptic pseudodifferential operator of order 1 such that  $Q \mathcal{X}^T \mathcal{X} = Id$  locally.

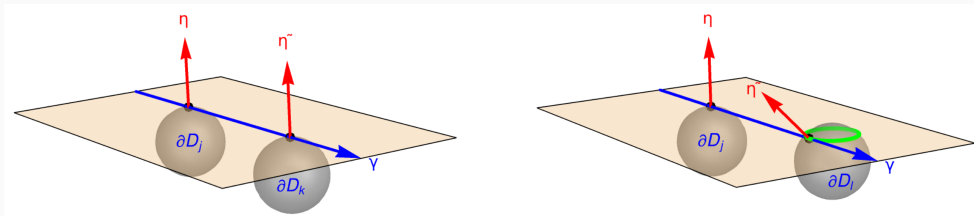
## Theorem 2

$f_{\text{MA}} \in I^{-3n/4-1/2}(N^*(\mathcal{L}))$  away from  $\partial D$ , and  $\sigma_{\text{prin}}(Q \mathcal{X}^T [(\mathcal{X}1_D)^2]) \neq 0$ .

- Park-Choi-Seo (2017) proved that  $\text{WF}(f_{\text{MA}}) \subset N^*(\mathcal{L})$  for  $M = \mathbb{R}^2$ .
- Palacios-Uhlmann-Wang (2018) proved Theorem 2 for  $M = \mathbb{R}^2$ .
- C (2022) proved Theorem 2 for the  $d$ -plane transform on  $\mathbb{R}^n$ .

## What does Theorem 2 say?

- If  $\partial D_j$  and  $\partial D_k$  have a common tangent hyperplane, then the conormal singularities propagate along the common tangent geodesic. See the left figure.
- Suppose  $n \geq 3$ . If  $\partial D_j$  and  $\partial D_k$  have a common tangent geodesic, but the conormal directions at the tangent points are different, then the conormal singularities do not propagate along the common tangent geodesic. See the right figure.





## How to prove Theorem 2

- The canonical relation  $C_{\mathcal{X}}$  of  $\mathcal{X}$ :  $(\xi, \eta) \in C_{\mathcal{X}}$  if  $\exists v \in S(M^{\text{int}})$  such that






$$\xi \in T_{F(v)}^*(\partial_- S(M)) \setminus \{0\}, \quad \eta \in T_{\pi_M(v)}^*(M^{\text{int}}) \setminus \{0\}, \quad DF|_v^T \xi = D\pi_M|_v^T \eta,$$

where  $F : S(M) \ni \dot{\gamma}_w(t) \mapsto w \in \partial_- S(M)$  for any  $t$ . See Holman-Uhlmann (2018).

- The story of the proof is basically due to Palacios-Uhlmann-Wang (2018) for  $M = \mathbb{R}^2$ :
  - $\mathcal{X}1_{D_j}$  is a conormal distribution supported on a hypersurface  $\Sigma_j$  in  $\partial_- S(M)$ .
  - $\Sigma_j \cap \Sigma_k$  is a transversal intersection, and  $\bigcup_{1 \leq j < k \leq J} C_{\mathcal{X}}^T \circ N^*(\Sigma_j \cap \Sigma_k) = N^*(\mathcal{L})$ .
- The additional strange condition ensures that

a Jacobi field along  $\gamma$  = a scalar function  $\times$  a parallel transport along  $\gamma$ .

We can obtain  $\bigcup_{1 \leq j < k \leq J} C_{\mathcal{X}}^T \circ N^*(\Sigma_j \cap \Sigma_k) = N^*(\mathcal{L})$  by using simplicity and this fact.  $\square$

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