# One-variable complex analysis and two-dimensional incompressible irrotational steady flow

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Let $\big(u(x,y),v(x,y)\big)$ be a smooth vector field on $\mathbb{R}^2$ describing the velocity field of two-dimensional steady fluid flow.

**Assumptions in Classical Mechanics**

- **Irrotational Field**: The volticity of the velocity field is supposed to be identically equal to zero, that is,

$$\operatorname{rot}(u,v,0)=(0,0,0), \quad \text{i.e.,} \quad (0,0,-u_y+v_x)=(0,0,0),$$

which means that $-u_y+v_x=0$.

- **Incompressible Flow**: The density field $\rho(x,y)$ is supposed to be constant along with the motion of the fluid flow, that is, $\rho$ is constant along any integral curve of $(u,v)$. This means that $u\rho_x+v\rho_y=0$. In view of the mass conservation law $(\rho u)_x+(\rho v)_y=0$, this is equivalent to $u_x+v_y=0$.

Under the above assumtions, there exist a harmonic funtion $\Phi$ which is the scalar potential of $(u,v)$, and its harmonic conjugate $\Psi$, that is,

$$u=\Phi_x=\Psi_y, \quad v=\Phi_y=-\Psi_x, \quad \Phi_{xx}+\Phi_{yy}=0, \quad \Psi_{xx}+\Psi_{yy}=0.$$

Indeed, if we set

$$\Phi(x,y):=\int_0^x u(t,0)dt+\int_0^y v(x,t)dt,$$
$$\Psi(x,y):=-\int_0^x v(t,0)dt+\int_0^y u(x,t)dt,$$

the irrotational condition $-u_y+v_x=0$ implies that $\Phi_x=u$ and $\Phi_y=v$, and $u_x+v_y=0$ implies that $\Psi_x=-v$ and $\Psi_y=u$. In particular, $\Phi$ and $\Psi$ satisfy the Cauchy-Riemann system of partial differential equations. So if we set

$$W(x+iy):=\Phi(x,y)+i\Psi(x,y),$$

then $W(z)$ is holomorphic in $\mathbb{C}$, and $W'(x+iy)=u(x,y)-iv(x,y)$ holds. $\Phi$, $\Psi$ and $W$ are said to be the velocity potential of $(u,v)$, the stream function of $(u,v)$, and the complex velocity potential of $(u,v)$ respectively.

Let $\gamma=\big\{\big(x(t),y(t)\big)\ \big|\ t\in[a,b]\big\}$ be the integral curve of the vector field $\big(u,v\big)$, that is, the pair of $x(t)$ and $y(t)$ satisfies the system of ordianry differential equations of the form

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}=\begin{bmatrix} u\big(x(t),y(t)\big) \\ v\big(x(t),y(t)\big) \end{bmatrix}.$$

In fluid mechanics, $\gamma$ is said to be the stream line of $(u,v)$. Since $(u,v)$ is a steady vector field, $\gamma$ coincides with the particle path of $(u,v)$ provided that $t$ is regarded as time variable. $\gamma$ also coinsides with the streak line of $(u,v)$ since the particle path is invariant for the change of the initial time. Note that the stream funtion $\Psi$ is constant along with $\gamma$. Indeed, we have

$$\frac{d}{dt}\Psi\big(x(t),y(t)\big)=\Psi_x\big(x(t),y(t)\big)x'(t)+\Psi_y\big(x(t),y(t)\big)y'(t)=-vu+uv=0.$$

This means that the countour lines of $\Psi$ are stream lines of $(u,b)$.

In what follows we show some examples of $(u, v)$.

- **Uniform Flow**. $W(z) = (a - ib)z$, $(a, b \in \mathbb{R})$, $(u, v) = (a, b)$.

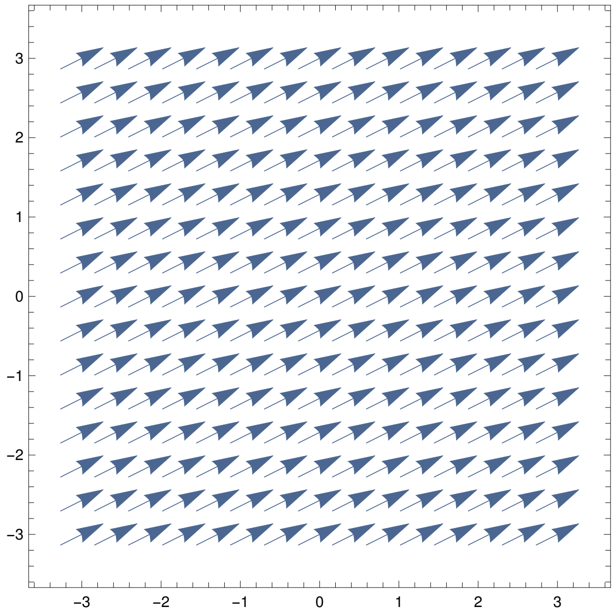


Figure of $(u, v) = (2, 1)$.

- **Quadratic Function**. $W(z) = (a - ib)z^2/2$, $(a, b \in \mathbb{R})$, $(u, v) = (ax + by, bx - ay)$.

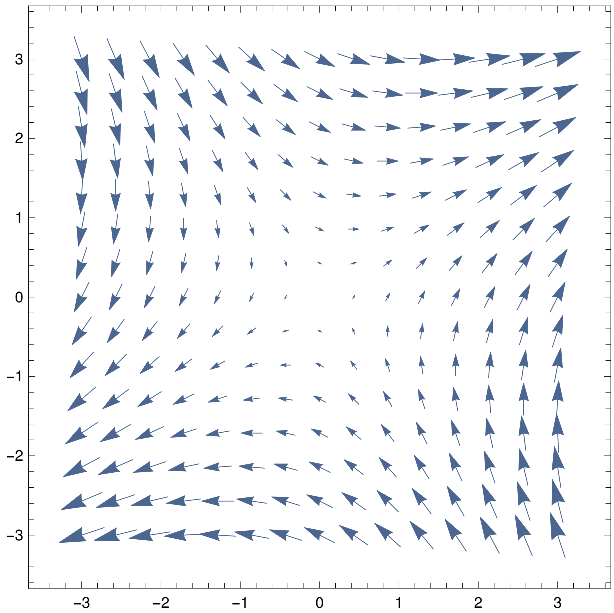


Figure of $(u, v) = (x + 2y, 2x - y)$.

In the above discussion, the complex velocity potentials are supposed to be holomorphic. Incredibly, however, Some potentials with isolated singularities which have meaning in classical mechanics.

- **Sources and Sink.** $a \log z$, $a \in \mathbb{R} \setminus \{0\}$, $\quad (u, v) = \left( \dfrac{ax}{r^2}, \dfrac{ay}{r^2} \right)$, $(x + iy = re^{i\theta})$.

  Since $\Psi = a\theta$, $\theta =$ constant, that is, all lines passing through the origin are stream lines. It is interpreted that if $a > 0$, the origin describes a source, and if $a < 0$ the origin describes a sink.

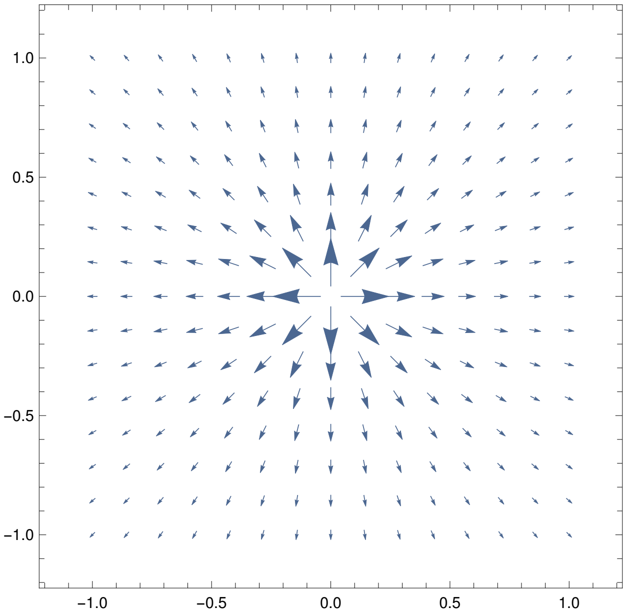
  
  

  Figure of $(u, v) = (x/r^2, y/r^2)$.

- **Vortices.** $ib \log z$, $b \in \mathbb{R} \setminus \{0\}$, $\quad (u, v) = \left( \dfrac{by}{r^2}, \dfrac{-bx}{r^2} \right)$, $(x + iy = re^{i\theta})$.

  Since $\Psi = b \log r$, $r =$ constant, that is, circles with the center at the origin are stream lines. It is interpreted that if $b > 0$ it describes a clockwise vortex, and if $b < 0$ it describes an anti-clocwise vortex.

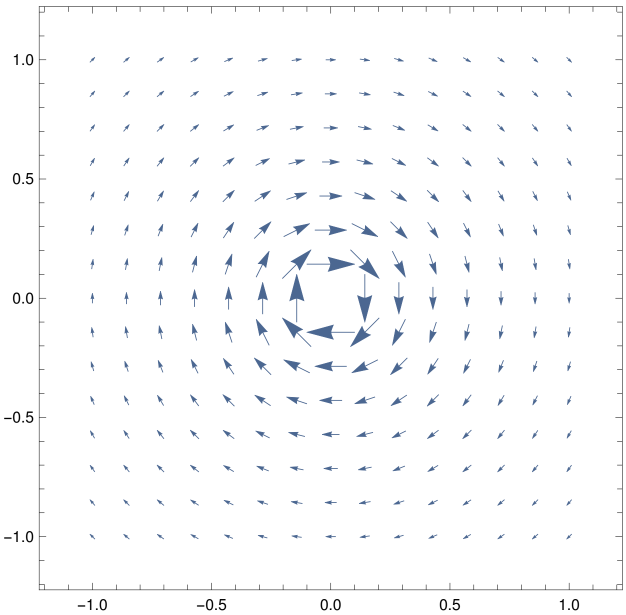
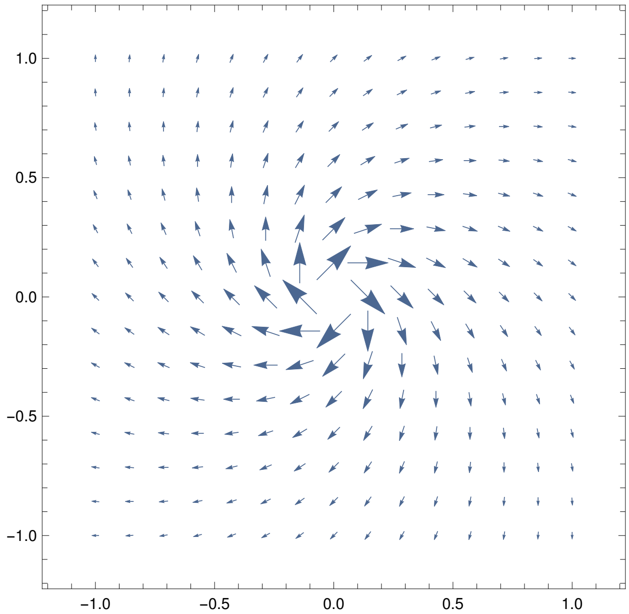
  
  

  Figure of $(u, v) = (y/r^2, -x/r^2)$.

Note that the superposition of two complex velocity potentials is a complex velocity potential.

- **Superposition of Source/Sink and Vortex**. Here we combine a source and a vortex. Figure of

$$(u,v) = \left(\frac{x}{r^2}, \frac{y}{r^2}\right) + \left(\frac{y}{r^2}, \frac{-x}{r^2}\right).$$




- **Dipole**. We combine a source and a sink satisfying some conditions. Figure of

$$(u,v) = \left(\frac{-(x-1/2)}{(x-1/2)^2+y^2}, \frac{-y}{(x-1/2)^2+y^2}\right) + \left(\frac{(x+1/2)}{(x+1/2)^2+y^2}, \frac{y}{(x+1/2)^2+y^2}\right).$$