A NOTE ON D'ALEMBERT'S FORMULA

HIROYUKI CHIHARA

ABSTRACT. We obtain d'Alembert's formula of solutions to the Cauchy problem for one-dimensional wave equation.

Consider the Cauchy problem for the one-dimensional wave equation of the form

$$u_{tt} - c^2 u_{xx} = f(x,t) \quad \text{in} \quad \mathbb{R}^2, \tag{1}$$

$$u(x,0) = \varphi(x) \quad \text{in} \quad \mathbb{R}, \tag{2}$$

$$u_t(x,0) = \psi(x) \quad \text{in} \quad \mathbb{R}, \tag{3}$$

where $u(x,t)$ is an unknown function of $(x,t) \in \mathbb{R}^2$, and $c > 0$, $f(x,t) \in C(\mathbb{R}^{\nvDash})$, $\varphi(x) \in C^2(\mathbb{R})$ and $\psi(x) \in C^1(\mathbb{R})$ are given. The following theorem is well-known.

Theorem 1. *The Cauchy problem* (1)-(2)-(3) *has a unique solution* $u(x,t) \in C^2(\mathbb{R}^2)$, *which is given by d'Alembert's formula*

$$u(x,t) = \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy \right) ds. \tag{4}$$

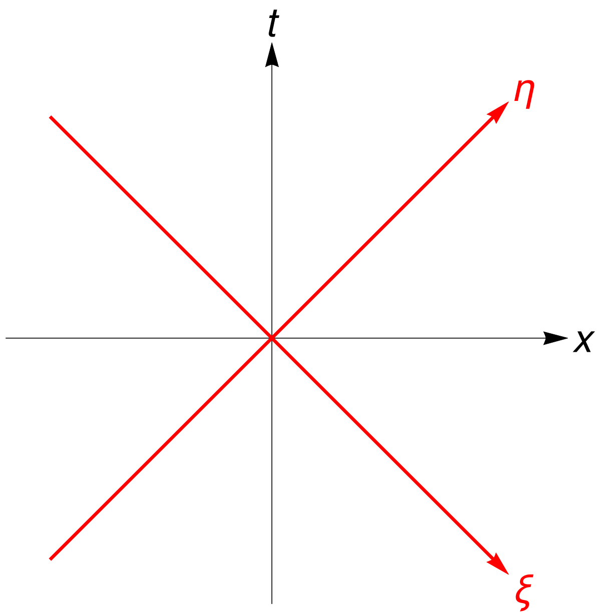
In what follows we shall obtain only d'Alembert formula (4). For this purpose we make use of a change of variables

$$(\xi, \eta) = (x - ct, x + ct),$$

that is,

$$(x,t) = \left(\frac{\xi + \eta}{2}, \frac{-\xi + \eta}{2c} \right).$$

The following figure describes the relationship between xt-plane and $\xi\eta$-plane. Note that x-axis ($t = 0$) becomes the diagonal $\{(\xi,\xi) \mid \xi \in \mathbb{R}\}$ in $\xi\eta$-plane.



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Lemma 2. *Let $u(x,t) \in C^2(\mathbb{R}^2)$ be a solution to* (1)-(2)-(3)*. If we set*

$$U(x-ct, x+ct) := u(x,t), \quad F(x-ct, x+ct) := f(x,t),$$

that is,

$$U(\xi,\eta) := u\left(\frac{\xi+\eta}{2}, \frac{-\xi+\eta}{2c}\right), \quad F(\xi,\eta) := f\left(\frac{\xi+\eta}{2}, \frac{-\xi+\eta}{2c}\right),$$

then $U(\xi,\eta) \in C^2(\mathbb{R}^2)$ solves the Cauchy problem of the form

$$U_{\xi\eta}(\xi,\eta) = -\frac{F(\xi,\eta)}{4c^2} \qquad in \quad \mathbb{R}^2, \tag{5}$$

$$U(\xi,\xi) = \varphi(\xi) \qquad in \quad \mathbb{R}, \tag{6}$$

$$U_\eta(\xi,\xi) = \frac{\varphi'(\xi)}{2} + \frac{\psi(\xi)}{2c} \qquad in \quad \mathbb{R}. \tag{7}$$

Proof. Since $u(x,t) = U(x-ct, x+ct)$, we deduce that

$$\begin{aligned}
u_t &= -cU_\xi + cU_\eta, \\
u_{tt} &= -c(-cU_\xi + cU_\eta)_\xi + c(-cU_\xi + cU_\eta)_\eta = c^2U_{\xi\xi} - 2c^2U_{\xi\eta} + c^2U_{\eta\eta}, \\
u_x &= U_\xi + U_\eta, \\
u_{xx} &= (U_\xi + U_\eta)_\xi + (U_\xi + U_\eta)_\eta = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}, \\
F &= f = u_{tt} - c^2u_{xx} = \{c^2U_{\xi\xi} - 2c^2U_{\xi\eta} + c^2U_{\eta\eta}\} - c^2\{U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\} = -4c^2U_{\xi\eta}, \\
\varphi(\xi) &= u(\xi,0) = U(\xi,\xi), \\
\varphi'(\xi) &= U_\xi(\xi,\xi) + U_\eta(\xi,\xi), \\
\psi(\xi) &= u_t(\xi,0) = -cU_\xi(\xi,\xi) + cU_\eta(\xi,\xi), \\
U_\eta(\xi,\xi) &= \frac{U_\xi(\xi,\xi) + U_\eta(\xi,\xi)}{2} + \frac{-U_\xi(\xi,\xi) + U_\eta(\xi,\xi)}{2} = \frac{\varphi'(\xi)}{2} + \frac{\psi(\xi)}{2c}.
\end{aligned}$$

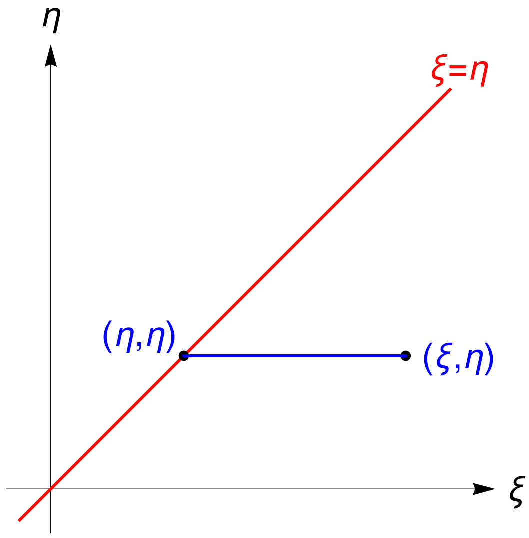
This completes the proof. □

We shall obtain d'Alembert's formula (4) by solving the Cauchy problem (5)-(6)-(7).

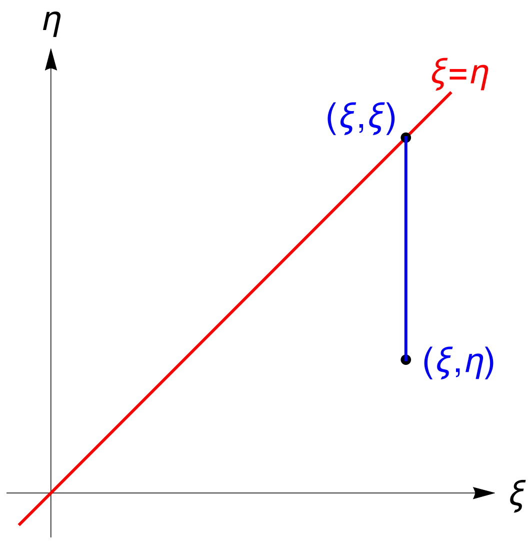
Derivation of (4). Suppose that $u(x,t) \in C^2(\mathbb{R}^2)$ is a solution to (1)-(2)-(3). Lemma 2 shows that $U \in C^2(\mathbb{R}^2)$ is a solution to the Cauchy problem (5)-(6)-(7).

Firstly we integrate the equation (5) in ξ from η to ξ. See the next figure. We have

$$\begin{aligned}
&U_\eta(\xi,\eta) \\
=&U_\eta(\eta,\eta) - \int_\eta^\xi \frac{F(z,\eta)}{4c^2}dz \\
=&\frac{\varphi'(\eta)}{2} + \frac{\psi(\eta)}{2c} + \int_\xi^\eta \frac{F(z,\eta)}{4c^2}dz.
\end{aligned} \tag{8}$$



Secondly we integrate (8) in η from ξ to η. See the next figure. We have



$$
\begin{aligned}
U(\xi,\eta) &= U(\xi,\xi) + \int_\xi^\eta \left\{ \frac{\varphi'(y)}{2} + \frac{\psi(y)}{2c} \right\} dy \\
&+ \int_\xi^\eta \left(\int_\xi^\zeta \frac{F(z,\zeta)}{4c^2} dz \right) d\zeta \\
&= \varphi(\xi) + \int_\xi^\eta \left\{ \frac{\varphi'(y)}{2} + \frac{\psi(y)}{2c} \right\} dy \\
&+ \int_\xi^\eta \left(\int_\xi^\zeta \frac{F(z,\zeta)}{4c^2} dz \right) d\zeta \\
&= \frac{\varphi(\eta) + \varphi(\xi)}{2} + \frac{1}{2c} \int_\xi^\eta \psi(y) dy \\
&+ \frac{1}{4c^2} \int_\xi^\eta \left(\int_\xi^\zeta F(z,\zeta) dz \right) d\zeta. \qquad (9)
\end{aligned}
$$

Here we change variables by $(\xi,\eta) = (x-ct, x+ct)$ and $(z,\zeta) = (y-cs, y+cs)$ in (9). We have

$$
\frac{\partial(z,\zeta)}{\partial(y,s)} = \det \begin{bmatrix} 1 & -c \\ 1 & c \end{bmatrix} = 2c.
$$

In case of $t > 0$ we have

$$
\xi = x - ct < \zeta = y + cs < \eta = x + ct, \quad \xi = x - ct < z = y - cs < \zeta = y + cs < \eta.
$$

Hence $0 \leqq s \leqq t$ and $x - c(t-s) \leqq y \leqq x + c(t-s)$. Similarly, in case of $t < 0$ we have $t \leqq s \leqq 0$ and $x + c(t-s) \leqq y \leqq x - c(t-s)$. Then we have

$$
\begin{aligned}
u(x,t) &= U(x-ct, x+ct) \\
&= \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy + \frac{1}{4c^2} \int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(y,s) \cdot (2c) dy \right) ds \\
&= \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy + \frac{1}{2c} \int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(y,s) dy \right) ds.
\end{aligned}
$$

This completes the proof. $\square$