On the complete integrability of the geodesic flow on pseudo-H-type Lie groups

Wolfram Bauer

Joint work with Daisuke Tarama, Kyoto

Leibniz Universität, Hannover

Workshop - University of the Ryukyus

15.12.2018

Outline

Outline

1. Pseudo-H-type Lie groups

Outline

1. Pseudo-H-type Lie groups

2. Complete integrability of Pseudo-H type Lie groups

Outline

1. Pseudo-H-type Lie groups
2. Complete integrability of Pseudo-H type Lie groups
3. Explicit first integrals via the isometry group

Outline

1. Pseudo-H-type Lie groups
2. Complete integrability of Pseudo-H type Lie groups
3. Explicit first integrals via the isometry group
4. First integrals on compact nilmanifolds

Outline

1. Pseudo-H-type Lie groups
2. Complete integrability of Pseudo-H type Lie groups
3. Explicit first integrals via the isometry group
4. First integrals on compact nilmanifolds
5. Complete integrability versus isospectrality

Pseudo-H-type Lie algebras

Let $r, s \in \mathbb{N}_0$ and consider $\mathbb{R}^{r,s} = \mathbb{R}^{r+s}$ with bilinear form

$$\langle x, y \rangle_{r,s} = \sum_{i=1}^{r} x_i y_i - \sum_{j=1}^{s} x_{r+j} y_{r+j}.$$

Pseudo-H-type Lie algebras

Let $r, s \in \mathbb{N}_0$ and consider $\mathbb{R}^{r,s} = \mathbb{R}^{r+s}$ with bilinear form

$$\langle x, y \rangle_{r,s} = \sum_{i=1}^{r} x_i y_i - \sum_{j=1}^{s} x_{r+j} y_{r+j}.$$

Consider $q_{r,s}(x) := \langle x, x \rangle_{r,s}$ and define

$$C\ell_{r,s} := \textit{Clifford algebra generated by } (\mathbb{R}^{r,s}, q_{r,s}).$$

Pseudo-H-type Lie algebras

Let $r, s \in \mathbb{N}_0$ and consider $\mathbb{R}^{r,s} = \mathbb{R}^{r+s}$ with bilinear form

$$\langle x, y\rangle_{r,s} = \sum_{i=1}^{r} x_i y_i - \sum_{j=1}^{s} x_{r+j} y_{r+j}.$$

Consider $q_{r,s}(x) := \langle x, x\rangle_{r,s}$ and define

$$C\ell_{r,s} := \textit{Clifford algebra generated by } (\mathbb{R}^{r,s}, q_{r,s}).$$

Clifford module

Let V be a Clifford module, i.e. V is a real vector space with Clifford module action

$$J : C\ell_{r,s} \times V \to V : J_z = J(z, \cdot) : V \to V.$$

Pseudo-H-type Lie algebras

Let $r, s \in \mathbb{N}_0$ and consider $\mathbb{R}^{r,s} = \mathbb{R}^{r+s}$ with bilinear form

$$\langle x, y \rangle_{r,s} = \sum_{i=1}^{r} x_i y_i - \sum_{j=1}^{s} x_{r+j} y_{r+j}.$$

Consider $q_{r,s}(x) := \langle x, x \rangle_{r,s}$ and define

$$C\ell_{r,s} := \textit{Clifford algebra generated by } (\mathbb{R}^{r,s}, q_{r,s}).$$

Clifford module

Let V be a Clifford module, i.e. V is a real vector space with Clifford module action

$$J : C\ell_{r,s} \times V \to V : J_z = J(z, \cdot) : V \to V.$$

This means: $J_z J_{z'} + J_{z'} J_z = -2\langle z, z' \rangle_{r,s} \mathrm{Id}$ for all $z, z' \in \mathbb{R}^{r,s}$.

Pseudo-H-type algebras

Assume: V carries a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_V$

Pseudo-H-type algebras

Assume: V carries a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_V$

Definition

We call the module $(V, \langle \cdot, \cdot \rangle_V)$ admissible if

$$\begin{aligned} \langle J_z X, J_z Y \rangle_V &= \langle z, z \rangle_{r,s} \langle X, Y \rangle_V \\ \langle J_z X, Y \rangle_V &= -\langle X, J_z Y \rangle_V \\ J_z^2 &= -\langle z, z \rangle_{r,s} \mathrm{Id}. \end{aligned}$$

Assume: V carries a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_V$

Definition

We call the module $(V, \langle \cdot, \cdot \rangle_V)$ admissible if

$$\begin{aligned}\langle J_z X, J_z Y \rangle_V &= \langle z, z \rangle_{r,s} \langle X, Y \rangle_V \\ \langle J_z X, Y \rangle_V &= -\langle X, J_z Y \rangle_V \\ J_z^2 &= -\langle z, z \rangle_{r,s} \mathrm{Id}.\end{aligned}$$

Note: the conditions are not independent.

Assume: V carries a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_V$

Definition

We call the module $(V, \langle \cdot, \cdot \rangle_V)$ admissible if

$$\begin{aligned} \langle J_z X, J_z Y \rangle_V &= \langle z, z \rangle_{r,s} \langle X, Y \rangle_V \\ \langle J_z X, Y \rangle_V &= -\langle X, J_z Y \rangle_V \\ J_z^2 &= -\langle z, z \rangle_{r,s} \mathrm{Id}. \end{aligned}$$

Note: the conditions are not independent.

Lemma

If $s > 0$, then $(V, \langle \cdot, \cdot \rangle_V)$ has positive and negative definite sub-spaces of the same dimension. In particular, $\dim V$ is even.

Define a Lie bracket $[\cdot,\cdot] : V \times V \to \mathbb{R}^{r,s}$ through the relations:

$$\langle J_z X, Y \rangle_V = \langle z, [X,Y] \rangle_{r,s}, \qquad z \in \mathbb{R}^{r,s},\ X, Y \in V.$$

Define a Lie bracket $[\cdot,\cdot] : V \times V \to \mathbb{R}^{r,s}$ through the relations:

$$\langle J_z X, Y \rangle_V = \langle z, [X,Y] \rangle_{r,s}, \qquad z \in \mathbb{R}^{r,s},\ X, Y \in V.$$

Definition

Let V be an admissible $C\ell_{r,s}$- module. With the bracket $[\cdot,\cdot]$ and the center $\mathbb{R}^{r,s}$ the sum

$$\mathcal{N}_{r,s} := V \oplus \mathbb{R}^{r,s}$$

defines a step-2 nilpotent Lie algebra called pseudo-H-type algebra.

(Pseudo)-H-type algebras

Define a Lie bracket $[\cdot,\cdot] : V \times V \to \mathbb{R}^{r,s}$ through the relations:

$$\langle J_z X, Y \rangle_V = \langle z, [X, Y] \rangle_{r,s}, \quad z \in \mathbb{R}^{r,s},\ X, Y \in V.$$

Definition

Let V be an admissible $C\ell_{r,s}$- module. With the bracket $[\cdot,\cdot]$ and the center $\mathbb{R}^{r,s}$ the sum

$$\mathcal{N}_{r,s} := V \oplus \mathbb{R}^{r,s}$$

defines a step-2 nilpotent Lie algebra called pseudo-H-type algebra.

A. Kaplan, *Fundamental solution for a class of hypo-elliptic PDE generated by composition of quadric forms*, Trans. Amer. Math. Soc. 258 (1980) no. 1. 147-153.

(Pseudo)-H-type algebras

Define a Lie bracket $[\cdot,\cdot] : V \times V \to \mathbb{R}^{r,s}$ through the relations:

$$\langle J_z X, Y \rangle_V = \langle z, [X,Y] \rangle_{r,s}, \qquad z \in \mathbb{R}^{r,s},\ X, Y \in V.$$

Definition

Let V be an admissible $C\ell_{r,s}$- module. With the bracket $[\cdot,\cdot]$ and the center $\mathbb{R}^{r,s}$ the sum

$$\mathcal{N}_{r,s} := V \oplus \mathbb{R}^{r,s}$$

defines a step-2 nilpotent Lie algebra called pseudo-H-type algebra.

A. Kaplan, *Fundamental solution for a class of hypo-elliptic PDE generated by composition of quadric forms*, Trans. Amer. Math. Soc. 258 (1980) no. 1. 147-153.

P. Ciatti, *Scalar products on Clifford modules and pseudo-H-type Lie algebras*, Ann. Mat. Pura Appl. 178 (4) (2000), 1-31.

The following is known:

The following is known:

- there is a connected, simply connected Lie group $G_{r,s}$ with Lie algebra $\mathcal{N}_{r,s}$ diffeomorphic to $\mathcal{N}_{r,s}$ (via the exp), we call it

  "Pseudo-H-type group".

  $=$ *generalization of Heisenberg or Heisenberg-type algebras.*

Some properties of pseudo-H-type Lie algebras

The following is known:

- there is a connected, simply connected Lie group $G_{r,s}$ with Lie algebra $\mathcal{N}_{r,s}$ diffeomorphic to $\mathcal{N}_{r,s}$ (via the exp), we call it

  "Pseudo-H-type group".

  $=$ *generalization of Heisenberg or Heisenberg-type algebras.*

- existence of a lattice in a general pseudo-H-type group,

Some properties of pseudo-H-type Lie algebras

The following is known:

- there is a connected, simply connected Lie group $G_{r,s}$ with Lie algebra $\mathcal{N}_{r,s}$ diffeomorphic to $\mathcal{N}_{r,s}$ (via the exp), we call it

  "Pseudo-H-type group".

  $=$ *generalization of Heisenberg or Heisenberg-type algebras.*
- existence of a lattice in a general pseudo-H-type group,

Ex.: The Heisenberg group $\mathbb{H}_{2n+1} = G_{0,1}$ of dimension $2n+1$ can be represented as a pseudo-H-type group.

The following is known:

- there is a connected, simply connected Lie group $G_{r,s}$ with Lie algebra $\mathcal{N}_{r,s}$ diffeomorphic to $\mathcal{N}_{r,s}$ (via the exp), we call it

  "Pseudo-H-type group".

  = *generalization of Heisenberg or Heisenberg-type algebras.*
- existence of a lattice in a general pseudo-H-type group,

Ex.: The Heisenberg group $\mathbb{H}_{2n+1} = G_{0,1}$ of dimension $2n+1$ can be represented as a pseudo-H-type group.

K. Furutani, I. Markina, *Complete classification of pseudo-H-type Lie algebras: I and II*, **Part I:** Geom. Dedicata, 190, 23-51, (2017).

The following is known:

- there is a connected, simply connected Lie group $G_{r,s}$ with Lie algebra $\mathcal{N}_{r,s}$ diffeomorphic to $\mathcal{N}_{r,s}$ (via the exp), we call it

  "Pseudo-H-type group".

  = *generalization of Heisenberg or Heisenberg-type algebras.*
- existence of a lattice in a general pseudo-H-type group,

Ex.: The Heisenberg group $\mathbb{H}_{2n+1} = G_{0,1}$ of dimension $2n+1$ can be represented as a pseudo-H-type group.

K. Furutani, I. Markina, *Complete classification of pseudo-H-type Lie algebras: I and II*, **Part I:** Geom. Dedicata, 190, 23-51, (2017).

K. Furutani, I. Markina, *Existence of lattice on general H-type groups*, J. Lie Theory 24, 979-1011, (2014).

Complete integrability in the sense of Liouville

Let $(G, *)$ be a Lie group with Lie algebra $\mathfrak{g}$. Fix

$$\langle \cdot, \cdot \rangle := \textit{non-degenerate scalar product on } \mathfrak{g}, \qquad (+)$$

which induces a left-invariant pseudo-Riemannian metric on G.

Complete integrability in the sense of Liouville

Let $(G, *)$ be a Lie group with Lie algebra $\mathfrak{g}$. Fix

$$\langle \cdot, \cdot \rangle := \textit{non-degenerate scalar product on } \mathfrak{g}, \qquad (+)$$

which induces a left-invariant pseudo-Riemannian metric on G.

Induced Identifications

$$T^*G \cong G \times \mathfrak{g}^* \overset{(+)}{\cong} G \times \mathfrak{g} \cong TG \qquad (\textit{Lie groups}).$$

with product pseudo-Riemannian metric on $G \times \mathfrak{g}$. Moreover,

$$T\big(T^*G\big) \cong T(TG) \cong T(G \times \mathfrak{g}) \cong \mathfrak{g} \times \mathfrak{g}.$$

Complete integrability in the sense of Liouville

Let $(G, *)$ be a Lie group with Lie algebra $\mathfrak{g}$. Fix

$$\langle \cdot, \cdot \rangle := \textit{non-degenerate scalar product on } \mathfrak{g}, \qquad (+)$$

which induces a left-invariant pseudo-Riemannian metric on G.

Induced Identifications

$$T^*G \cong G \times \mathfrak{g}^* \overset{(+)}{\cong} G \times \mathfrak{g} \cong TG \qquad (\textit{Lie groups}).$$

with product pseudo-Riemannian metric on $G \times \mathfrak{g}$. Moreover,

$$T(T^*G) \cong T(TG) \cong T(G \times \mathfrak{g}) \cong \mathfrak{g} \times \mathfrak{g}.$$

In order to simplify calculations we will:

- transfer the symplectic structure and Poisson bracket of T^*G to the Lie group $TG \cong G \times \mathfrak{g}$.

Identifications and translations:

Identifications and translations:

Let $(p, Y) \in TG \cong G \times \mathfrak{g}$ and $(U_1, V_1), (U_2, V_2) \in T_{(p,Y)}(G \times \mathfrak{g})$.

Symplectic form Ω on $T^*G \cong TG \cong G \times \mathfrak{g}$

$$\Omega_{(p,Y)}\Big((U_1, V_1), (U_2, V_2)\Big) = \langle U_1, V_2 \rangle - \langle V_1, U_2 \rangle - \langle Y, [U_1, U_2] \rangle.$$

Let $(p, Y) \in TG \cong G \times \mathfrak{g}$ and $(U_1, V_1), (U_2, V_2) \in T_{(p,Y)}(G \times \mathfrak{g})$.

Symplectic form Ω on $T^*G \cong TG \cong G \times \mathfrak{g}$

$$\Omega_{(p,Y)}\Big((U_1, V_1), (U_2, V_2)\Big) = \big\langle U_1, V_2 \big\rangle - \big\langle V_1, U_2 \big\rangle - \big\langle Y, [U_1, U_2] \big\rangle.$$

Let $f : TG \cong G \times \mathfrak{g} \to \mathbb{R}$ be smooth and $(p, Y) \in G \times \mathfrak{g}$. The Hamiltonian vector field X_f of f on TG implicitly is defined via

$$df_{(p,Y)}(U, V) = \Omega_{(p,Y)}\Big(X_f(p, Y), (U, V)\Big).$$

Identifications and translations:

Let $(p, Y) \in TG \cong G \times \mathfrak{g}$ and $(U_1, V_1), (U_2, V_2) \in T_{(p,Y)}(G \times \mathfrak{g})$.

Symplectic form Ω on $T^*G \cong TG \cong G \times \mathfrak{g}$

$$\Omega_{(p,Y)}\Big((U_1, V_1), (U_2, V_2)\Big) = \langle U_1, V_2\rangle - \langle V_1, U_2\rangle - \langle Y, [U_1, U_2]\rangle.$$

Let $f : TG \cong G \times \mathfrak{g} \to \mathbb{R}$ be smooth and $(p, Y) \in G \times \mathfrak{g}$. The Hamiltonian vector field X_f of f on TG implicitly is defined via

$$df_{(p,Y)}(U, V) = \Omega_{(p,Y)}\Big(X_f(p, Y), (U, V)\Big).$$

Hamiltonian vector field

More explicitly with $\operatorname{grad}_{(p,Y)} f = (U, V)$:

$$X_f(p, Y) = \Big(V, \operatorname{ad}^t(V)(Y) - U\Big) \in \mathfrak{g} \times \mathfrak{g} \cong T_{(p,Y)}(G \times \mathfrak{g}).$$

Identifications and translations:

Hamiltonian

The Hamiltonian for the geodesic flow w.r.t. the left-invariant pseudo-Riemannian metric is given by

$$H : T^*G \cong TG \cong G \times \mathfrak{g} \to \mathbb{R} : H(p, Y) := \frac{1}{2}\langle Y, Y \rangle.$$

Identifications and translations:

Hamiltonian

The Hamiltonian for the geodesic flow w.r.t. the left-invariant pseudo-Riemannian metric is given by

$$H : T^*G \cong TG \cong G \times \mathfrak{g} \to \mathbb{R} : H(p, Y) := \frac{1}{2}\langle Y, Y\rangle.$$

Lemma:

The Hamiltonian vector field to H at $(p, Y) \in T_pG$ has the form:

$$X_H(p, Y) = \big(Y, \operatorname{ad}^t(Y)(Y)\big) \in \mathfrak{g} \times \mathfrak{g}.$$

Hamiltonian

The Hamiltonian for the geodesic flow w.r.t. the left-invariant pseudo-Riemannian metric is given by

$$H : T^*G \cong TG \cong G \times \mathfrak{g} \to \mathbb{R} : H(p, Y) := \frac{1}{2}\langle Y, Y\rangle.$$

Lemma:

The Hamiltonian vector field to H at $(p, Y) \in T_pG$ has the form:

$$X_H(p, Y) = \big(Y, \mathrm{ad}^t(Y)(Y)\big) \in \mathfrak{g} \times \mathfrak{g}.$$

Proof: Follows from the previous formula with $U = 0$ since:

$$\mathrm{grad}_{(p,Y)} H = (0, Y).$$

□

Identifications and translations:

Take 2 smooth functions $f, g : TG \cong G \times \mathfrak{g} \to \mathbb{R}$ and $p \in G$ with

$$\text{grad } f_{(p,Y)} = (U_1, V_1) \in \mathfrak{g} \times \mathfrak{g} \cong T_{(p,Y)}(TG) \quad and \quad ,$$
$$\text{grad } g_{(p,Y)} = (U_2, V_2) \in \mathfrak{g} \times \mathfrak{g}.$$

Identifications and translations:

Take 2 smooth functions $f, g : TG \cong G \times \mathfrak{g} \to \mathbb{R}$ and $p \in G$ with

$$\text{grad } f_{(p,Y)} = (U_1, V_1) \in \mathfrak{g} \times \mathfrak{g} \cong T_{(p,Y)}(TG) \quad \textit{and} \quad ,$$
$$\text{grad } g_{(p,Y)} = (U_2, V_2) \in \mathfrak{g} \times \mathfrak{g}.$$

Poisson bracket on T^*G

The Poisson bracket $\{f, g\} = \Omega(X_f, X_g)$ of f and g is given by:

$$\{f, g\}(p, Y) = \langle V_2, U_1 \rangle - \langle V_1, U_2 \rangle + \langle Y, [V_2, V_1] \rangle.$$

Identifications and translations:

Take 2 smooth functions $f, g : TG \cong G \times \mathfrak{g} \to \mathbb{R}$ and $p \in G$ with

$$\text{grad } f_{(p,Y)} = (U_1, V_1) \in \mathfrak{g} \times \mathfrak{g} \cong T_{(p,Y)}(TG) \quad \textit{and} \quad ,$$
$$\text{grad } g_{(p,Y)} = (U_2, V_2) \in \mathfrak{g} \times \mathfrak{g}.$$

Poisson bracket on T^*G

The Poisson bracket $\{f, g\} = \Omega(X_f, X_g)$ of f and g is given by:

$$\{f, g\}(p, Y) = \langle V_2, U_1 \rangle - \langle V_1, U_2 \rangle + \langle Y, [V_2, V_1] \rangle.$$

Remark:

If G is a pseudo-H-type group we can use the relation

$$\langle J_z X, Y \rangle_V = \langle z, [X, Y] \rangle_{r,s}, \qquad z \in \mathbb{R}^{r,s},\ X, Y \in V$$

and apply the algebraic properties of the Clifford representation J_z.

First integrals and Liouville complete integrability

Definition

A function $f \in C^\infty(TG)$ is called first integral of the *geodesic flow* if f is constant along integral curves of X_H. Equivalently, if

$$\{f, H\} = 0.$$

First integrals and Liouville complete integrability

Definition

A function $f \in C^\infty(TG)$ is called first integral of the *geodesic flow* if f is constant along integral curves of X_H. Equivalently, if

$$\{f, H\} = 0.$$

"The group G has completely integrable geodesic flow in the sense of Liouville,"

if there are first integrals $f_1, \cdots, f_{\dim G}$ of the geodesic flow with:

First integrals and Liouville complete integrability

Definition

A function $f \in C^\infty(TG)$ is called first integral of the *geodesic flow* if f is constant along integral curves of X_H. Equivalently, if

$$\{f, H\} = 0.$$

"The group G has completely integrable geodesic flow in the sense of Liouville,"

if there are first integrals $f_1, \cdots, f_{\dim G}$ of the geodesic flow with:

(a) $\{f_i, f_j\} = 0$ for all $i, j = 1, \cdots, \dim G$, *(Poisson commuting)*,

First integrals and Liouville complete integrability

Definition

A function $f \in C^\infty(TG)$ is called first integral of the *geodesic flow* if f is constant along integral curves of X_H. Equivalently, if

$$\{f, H\} = 0.$$

"The group G has completely integrable geodesic flow in the sense of Liouville,"

if there are first integrals $f_1, \cdots, f_{\dim G}$ of the geodesic flow with:

(a) $\{f_i, f_j\} = 0$ for all $i, j = 1, \cdots, \dim G$, *(Poisson commuting)*,

(b) the differentials $df_1, \cdots, df_{\dim G}$ are linear independent on an open dense subset of TG.

Complete integrability $\Longrightarrow$ *Liouville-Arnold theorem.*

Some problems

Questions:

Let $G_{r,s}$ be a pseudo-H-type Lie group[a].

Some problems

Questions:

Let $G_{r,s}$ be a pseudo-H-type Lie group[a].

1. How to obtain (Poisson commuting) first integrals of the geodesic flow explicitly?

Some problems

Questions:

Let $G_{r,s}$ be a pseudo-H-type Lie group[a].

1. How to obtain (Poisson commuting) first integrals of the geodesic flow explicitly?
2. Does complete integrability of the geodesic flow on $G_{r,s}$ in the sense of Liouville hold?

Some problems

Questions:

Let $G_{r,s}$ be a pseudo-H-type Lie group[a].

1. How to obtain (Poisson commuting) first integrals of the geodesic flow explicitly?
2. Does complete integrability of the geodesic flow on $G_{r,s}$ in the sense of Liouville hold?
3. Let $L \subset G_{r,s}$ denote a lattice. How can one obtain (Poisson commuting) first integrals of the geodesic flow on $L \backslash G_{r,s}$?

Questions:

Let $G_{r,s}$ be a pseudo-H-type Lie group[a].

1. How to obtain (Poisson commuting) first integrals of the geodesic flow explicitly?
2. Does complete integrability of the geodesic flow on $G_{r,s}$ in the sense of Liouville hold?
3. Let $L \subset G_{r,s}$ denote a lattice. How can one obtain (Poisson commuting) first integrals of the geodesic flow on $L \backslash G_{r,s}$?
4. In which cases (lattice and metric) can we prove complete integrability of $L \backslash G_{r,s}$ in the sense of Liouville?

Some problems

Questions:

Let $G_{r,s}$ be a pseudo-H-type Lie group[a].

1. How to obtain (Poisson commuting) first integrals of the geodesic flow explicitly?
2. Does complete integrability of the geodesic flow on $G_{r,s}$ in the sense of Liouville hold?
3. Let $L \subset G_{r,s}$ denote a lattice. How can one obtain (Poisson commuting) first integrals of the geodesic flow on $L \backslash G_{r,s}$?
4. In which cases (lattice and metric) can we prove complete integrability of $L \backslash G_{r,s}$ in the sense of Liouville?
5. Can we even choose real analytic first integrals?

Some problems

Questions:

Let $G_{r,s}$ be a pseudo-H-type Lie group[a].

1. How to obtain (Poisson commuting) first integrals of the geodesic flow explicitly?
2. Does complete integrability of the geodesic flow on $G_{r,s}$ in the sense of Liouville hold?
3. Let $L \subset G_{r,s}$ denote a lattice. How can one obtain (Poisson commuting) first integrals of the geodesic flow on $L \backslash G_{r,s}$?
4. In which cases (lattice and metric) can we prove complete integrability of $L \backslash G_{r,s}$ in the sense of Liouville?
5. Can we even choose real analytic first integrals?
6. Is the property of "complete integrability" encoded in the spectrum of the Laplace-Beltrami operator?

[a] i.e the Lie algebra of $G_{r,s}$ is of pseudo-H type $\mathcal{N}_{r,s}$.

Some literature:

Complete integrability of the Heisenberg group G and Heisenberg manifolds $L\backslash G$ via real analytic first integrals is shown here:

Some literature:

Complete integrability of the Heisenberg group G and Heisenberg manifolds $L \backslash G$ via real analytic first integrals is shown here:

A. Koscard, G. P. Ovando, S. Reggiani, *On first integrals of the geodesic flow on Heisenberg nilmanifolds*, Diff. Geom. Appl. 49, 469-509, (2016).

Some literature:

Complete integrability of the Heisenberg group G and Heisenberg manifolds $L\backslash G$ via real analytic first integrals is shown here:

A. Koscard, G. P. Ovando, S. Reggiani, *On first integrals of the geodesic flow on Heisenberg nilmanifolds*, Diff. Geom. Appl. 49, 469-509, (2016).

If the Lie algebra is of Heisenberg-Reiter type, then the smooth complete integrability of $L\backslash G$ is shown here:

L. Butler, *Integrable geodesic flows with wild first integrals: the case of two-step nilmanifolds* Ergod. Th. & Dynam. Sys. 23, 771-797, (2003).

Some literature:

Complete integrability of the Heisenberg group G and Heisenberg manifolds $L\backslash G$ via real analytic first integrals is shown here:

A. Koscard, G. P. Ovando, S. Reggiani, *On first integrals of the geodesic flow on Heisenberg nilmanifolds*, Diff. Geom. Appl. 49, 469-509, (2016).

If the Lie algebra is of Heisenberg-Reiter type, then the smooth complete integrability of $L\backslash G$ is shown here:

L. Butler, *Integrable geodesic flows with wild first integrals: the case of two-step nilmanifolds* Ergod. Th. & Dynam. Sys. 23, 771-797, (2003).

Question 6. is addressed here:

Dorothee Schueth, *Integrability of geodesic flows and isospectrality of Riemannian manifolds*, Math. Z. 260, 595-613, (2008).

Some literature:

Complete integrability of the Heisenberg group G and Heisenberg manifolds $L\backslash G$ via real analytic first integrals is shown here:

A. Koscard, G. P. Ovando, S. Reggiani, *On first integrals of the geodesic flow on Heisenberg nilmanifolds*, Diff. Geom. Appl. 49, 469-509, (2016).

If the Lie algebra is of Heisenberg-Reiter type, then the smooth complete integrability of $L\backslash G$ is shown here:

L. Butler, *Integrable geodesic flows with wild first integrals: the case of two-step nilmanifolds* Ergod. Th. & Dynam. Sys. 23, 771-797, (2003).

Question 6. is addressed here:

Dorothee Schueth, *Integrability of geodesic flows and isospectrality of Riemannian manifolds*, Math. Z. 260, 595-613, (2008).

Related Problem:

7. Which pseudo-H-type algebras $\mathcal{N}_{r,s}$ are Heisenberg-Reiter type?

Assumptions:

Assumptions:

- $G :=$ connected, simply connected step 2 nilpotent Lie group.

Assumptions:

- $G :=$ connected, simply connected step 2 nilpotent Lie group.
- $\mathfrak{g} :=$ Lie algebra of G with $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}$= *center*.

First integrals on step-2 nilpotent groups

Assumptions:

- $G :=$ connected, simply connected step 2 nilpotent Lie group.
- $\mathfrak{g} :=$ Lie algebra of G with $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}=$ *center*.
- $\langle \cdot, \cdot \rangle=$ non-degenerate scalar product on G.

Assumptions:

- $G :=$ connected, simply connected step 2 nilpotent Lie group.
- $\mathfrak{g} :=$ Lie algebra of G with $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}=$ *center*.
- $\langle \cdot, \cdot \rangle =$ non-degenerate scalar product on G.

Known fact

The exponential $\exp : \mathfrak{g} \to G$ is a diffeomorphism. Identify $g \cong G$.

First integrals on step-2 nilpotent groups

Assumptions:

- $G :=$ connected, simply connected step 2 nilpotent Lie group.
- $\mathfrak{g} :=$ Lie algebra of G with $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}=$ *center*.
- $\langle \cdot, \cdot \rangle =$ non-degenerate scalar product on G.

Known fact

The exponential $\exp : \mathfrak{g} \to G$ is a diffeomorphism. Identify $g \cong G$.

Examples for G:

Assumptions:

- $G :=$ connected, simply connected step 2 nilpotent Lie group.
- $\mathfrak{g} :=$ Lie algebra of G with $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}$= *center*.
- $\langle \cdot, \cdot \rangle$= non-degenerate scalar product on G.

Known fact

The exponential $\exp : \mathfrak{g} \to G$ is a diffeomorphism. Identify $g \cong G$.

Examples for G:

- $G = \mathbb{H}_{2n+1}$, Heisenberg group of dimension $2n + 1$,

Assumptions:

- $G :=$ connected, simply connected step 2 nilpotent Lie group.
- $\mathfrak{g} :=$ Lie algebra of G with $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}=$ *center*.
- $\langle \cdot, \cdot \rangle =$ non-degenerate scalar product on G.

Known fact

The exponential $\exp : \mathfrak{g} \to G$ is a diffeomorphism. Identify $g \cong G$.

Examples for G:

- $G = \mathbb{H}_{2n+1}$, Heisenberg group of dimension $2n+1$,
- $G =$ free nilpotent group of dimension $\frac{n(n+1)}{2}$,

First integrals on step-2 nilpotent groups

Assumptions:

- $G :=$ connected, simply connected step 2 nilpotent Lie group.
- $\mathfrak{g} :=$ Lie algebra of G with $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z} =$ *center*.
- $\langle \cdot, \cdot \rangle =$ non-degenerate scalar product on G.

Known fact

The exponential $\exp : \mathfrak{g} \to G$ is a diffeomorphism. Identify $g \cong G$.

Examples for G:

- $G = \mathbb{H}_{2n+1}$, Heisenberg group of dimension $2n+1$,
- $G =$ free nilpotent group of dimension $\frac{n(n+1)}{2}$,
- $G =$ Pseudo-H-type group $G_{r,s}$.

Source of first integrals

Let X^* be a Killing vector field [a] on G. A first integral on TG is given by:

$$f_{X^*}(p, v) = \left\langle X^*, v \right\rangle_p \quad \textit{where} \quad v \in T_pG.$$

[a] Vector field with flow being a family of continuous isometries **or** $\mathcal{L}_{X^*} g = 0$.

Construction of first integrals

Source of first integrals

Let X^* be a Killing vector field [a] on G. A first integral on TG is given by:

$$f_{X^*}(p,v) = \langle X^*, v\rangle_p \quad where \quad v \in T_pG.$$

[a] Vector field with flow being a family of continuous isometries **or** $\mathcal{L}_{X^*} g = 0$.

Let $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ the decomposition above. Chose orthonormal bases:

- $[X_1, \cdots, X_{2n}]$ of $\mathfrak{v}$.
- $[Z_1, \cdots, Z_k]$ of $\mathfrak{z}$.

Construction of first integrals

Source of first integrals

Let X^* be a Killing vector field [a] on G. A first integral on TG is given by:

$$f_{X^*}(p,v) = \left\langle X^*, v \right\rangle_p \quad \textit{where} \quad v \in T_pG.$$

[a] Vector field with flow being a family of continuous isometries **or** $\mathcal{L}_{X^*}g = 0$.

Let $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ the decomposition above. Chose orthonormal bases:

- $[X_1, \cdots, X_{2n}]$ of $\mathfrak{v}$.
- $[Z_1, \cdots, Z_k]$ of $\mathfrak{z}$.

First integrals via left-multiplication on G: Let $f \in C^\infty(G)$:

$$[X_j^{(r)} f](p) = \frac{d}{dt}\Big|_{t=0} f\Big(\exp\big(-tX_j\big) * p\Big), \qquad (j = 1, \cdots, 2n).$$

Some Properties

- We can regard the right invariant vector fields $X_j^{(r)}$ as Killing vector fields of the action of G on itself from the left.

Some Properties

- We can regard the right invariant vector fields $X_j^{(r)}$ as Killing vector fields of the action of G on itself from the left.
- We have the relation:

$$X_j^{(r)}(p) = -X_j - \sum_{\ell=1}^{k} \frac{\langle J_{Z_\ell} X_j, W_{\mathfrak{v}} \rangle}{\langle Z_\ell, Z_\ell \rangle} Z_\ell, \qquad p = \exp(W) \in G.$$

(identifying X_j and Z_ℓ with left-invariant vector fields on G)

- We can regard the right invariant vector fields $X_j^{(r)}$ as Killing vector fields of the action of G on itself from the left.
- We have the relation:

$$X_j^{(r)}(p) = -X_j - \sum_{\ell=1}^{k} \frac{\langle J_{Z_\ell} X_j, W_{\mathfrak{v}} \rangle}{\langle Z_\ell, Z_\ell \rangle} Z_\ell, \qquad p = \exp(W) \in G.$$

(identifying X_j and Z_ℓ with left-invariant vector fields on G)

Lemma

First integrals $F_j : TG \cong G \times \mathfrak{g} \to \mathbb{R}$ for $j = 1, \cdots, 2n$ induced by the Killing vector fields $X_j^{(r)}$ with $p = \exp(W)$ are:

$$F_j(p, Y) = \Big\langle X_j^{(r)}(p), Y \Big\rangle = \Big\langle X_j, J_{Y_{\mathfrak{z}}} W_{\mathfrak{v}} - Y \Big\rangle.$$

How to produce Poisson commuting first integrals?

Observation

Let $(p(t), Y(t)) \subset TG \cong G \times \mathfrak{g}$ be a solution of the geodesic flow equation

$$\frac{d}{dt}(p, Y) = X_H(p, Y) = \big(Y, \underbrace{J_{Y_{\mathfrak{z}}} Y_{\mathfrak{v}}}_{\in \mathfrak{v}}\big).$$

In particular,

$$\frac{d}{dt} Y_{\mathfrak{v}} = J_{Y_{\mathfrak{z}}} Y_{\mathfrak{v}} \quad \text{and} \quad \frac{d}{dt} Y_{\mathfrak{z}} = 0.$$

How to produce Poisson commuting first integrals?

Observation

Let $(p(t), Y(t)) \subset TG \cong G \times \mathfrak{g}$ be a solution of the geodesic flow equation

$$\frac{d}{dt}(p, Y) = X_H(p, Y) = \big(Y, \underbrace{J_{Y_{\mathfrak{z}}} Y_{\mathfrak{v}}}_{\in \mathfrak{v}}\big).$$

In particular,

$$\frac{d}{dt} Y_{\mathfrak{v}} = J_{Y_{\mathfrak{z}}} Y_{\mathfrak{v}} \quad \text{and} \quad \frac{d}{dt} Y_{\mathfrak{z}} = 0.$$

Idea: We still have first integrals of the geod. flow if we replace in

$$F_j(p, Y) = \Big\langle X_j, J_{Y_{\mathfrak{z}}} W_{\mathfrak{v}} - Y \Big\rangle$$

X_j by arbitrary non-constant smooth functions $\alpha : \mathfrak{z} \to \mathfrak{v}$. Put

How to produce Poisson commuting first integrals?

Observation

Let $(p(t), Y(t)) \subset TG \cong G \times \mathfrak{g}$ be a solution of the geodesic flow equation

$$\frac{d}{dt}(p, Y) = X_H(p, Y) = \big(Y, \underbrace{J_{Y_\mathfrak{z}} Y_\mathfrak{v}}_{\in \mathfrak{v}}\big).$$

In particular,

$$\frac{d}{dt} Y_\mathfrak{v} = J_{Y_\mathfrak{z}} Y_\mathfrak{v} \quad \text{and} \quad \frac{d}{dt} Y_\mathfrak{z} = 0.$$

Idea: We still have first integrals of the geod. flow if we replace in

$$F_j(p, Y) = \Big\langle X_j, J_{Y_\mathfrak{z}} W_\mathfrak{v} - Y \Big\rangle$$

X_j by arbitrary non-constant smooth functions $\alpha : \mathfrak{z} \to \mathfrak{v}$. Put

$$F_\alpha(p, Y) = \big\langle \alpha(Y_\mathfrak{z}), J_{Y_\mathfrak{z}} W_\mathfrak{v} - Y \big\rangle.$$

Poisson brackets

Let $\alpha, \beta : \mathfrak{z} \to \mathfrak{v}$ be smooth functions.

Let $\alpha, \beta : \mathfrak{z} \to \mathfrak{v}$ be smooth functions.

Lemma

If $f : TG \cong G \times \mathfrak{g} \to \mathbb{R}$ is left- invariant and $(p, Y) \in G \times \mathfrak{g}$. Then

(a) $\{F_\alpha, f\} = 0$.

(b) $\{F_\alpha, F_\beta\} = \langle Y_\mathfrak{z}, [\alpha(Y_\mathfrak{z}), \beta(Y_\mathfrak{z})]\rangle$.

Poisson brackets

Let $\alpha, \beta : \mathfrak{z} \to \mathfrak{v}$ be smooth functions.

Lemma

If $f : TG \cong G \times \mathfrak{g} \to \mathbb{R}$ is left- invariant and $(p, Y) \in G \times \mathfrak{g}$. Then

(a) $\{F_\alpha, f\} = 0$.

(b) $\{F_\alpha, F_\beta\} = \langle Y_\mathfrak{z}, [\alpha(Y_\mathfrak{z}), \beta(Y_\mathfrak{z})]\rangle$.

Idea: Choose smooth functions $\alpha, \beta : \mathfrak{z} \to \mathfrak{v}$ with

$$[\alpha(Y_\mathfrak{z}), \beta(Y_\mathfrak{z})] = 0.$$

Let $\alpha, \beta : \mathfrak{z} \to \mathfrak{v}$ be smooth functions.

Lemma

If $f : TG \cong G \times \mathfrak{g} \to \mathbb{R}$ is left- invariant and $(p, Y) \in G \times \mathfrak{g}$. Then

(a) $\{F_\alpha, f\} = 0$.

(b) $\{F_\alpha, F_\beta\} = \langle Y_\mathfrak{z}, [\alpha(Y_\mathfrak{z}), \beta(Y_\mathfrak{z})]\rangle$.

Idea: Choose smooth functions $\alpha, \beta : \mathfrak{z} \to \mathfrak{v}$ with

$$[\alpha(Y_\mathfrak{z}), \beta(Y_\mathfrak{z})] = 0.$$

Then, from the Lemma we have Poisson commuting first integrals which also commute with the Hamiltonian H.

$$\{F_\alpha, F_\beta\} = 0 = \{F_\alpha, H\}.$$

Example: Let $[v_1, \cdots, v_{2n}]$ be a basis of $\mathfrak{v}$ in $\mathcal{N}_{r,0} = \mathfrak{v} \oplus \mathfrak{z}$. Put

Example: Let $[v_1, \cdots, v_{2n}]$ be a basis of $\mathfrak{v}$ in $\mathcal{N}_{r,0} = \mathfrak{v} \oplus \mathfrak{z}$. Put

$$\tilde{j}(Y_{\mathfrak{z}}) := T(Y_{\mathfrak{z}}) \circ J_{Y_{\mathfrak{z}}} \circ T^{-1}(Y_{\mathfrak{z}}) =$$

$$= \sqrt{\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}} \rangle} \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}.$$

Example: Let $[v_1, \cdots, v_{2n}]$ be a basis of $\mathfrak{v}$ in $\mathcal{N}_{r,0} = \mathfrak{v} \oplus \mathfrak{z}$. Put

$$\tilde{j}(Y_\mathfrak{z}) := T(Y_\mathfrak{z}) \circ J_{Y_\mathfrak{z}} \circ T^{-1}(Y_\mathfrak{z}) =$$

$$= \sqrt{\langle Y_\mathfrak{z}, Y_\mathfrak{z} \rangle} \begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix}.$$

Define functions $\alpha_i : \mathfrak{z} \to \mathfrak{v}$ for $i = 1, \cdots, 2n$ by:

$$\alpha_i(Y_\mathfrak{z}) := T^{-1}(Y_\mathfrak{z}) v_i.$$

Theorem, (W.-B., D. Tarama)

The geodesic flow on an H-type Lie group $G_{r,0}$ is completely integrable in the sense of Liouville.

A set of Poisson commuting functionally independent first integrals can be constructed explicitly.

Theorem, (W.-B., D. Tarama)

The geodesic flow on an H-type Lie group $G_{r,0}$ is completely integrable in the sense of Liouville.

A set of Poisson commuting functionally independent first integrals can be constructed explicitly.

Remark:

This result generalizes to the pseudo-Riemannian geodesic flow on pseudo-H-type Lie groups.

$s > 0$: The map J_z can be non-invertible on v for some $z \neq 0$.

First integrals via the isometry group

Let G be a step-two nilpotent Lie group and put:

First integrals via the isometry group

Let G be a step-two nilpotent Lie group and put:

- $I(G)=$ *isometry group of* G,

First integrals via the isometry group

Let G be a step-two nilpotent Lie group and put:

- $I(G)$= *isometry group of* G,
- K = *isotropy group of* G *with Lie algebra* $\mathfrak{k}$.

First integrals via the isometry group

Let G be a step-two nilpotent Lie group and put:

- $I(G)$= *isometry group of* G,
- $K =$ *isotropy group of* G *with Lie algebra* $\mathfrak{k}$.

The following facts are well-known:

Lemma

Let G act by left translation. Then

$$I(G) = K \ltimes G.$$

Let G be a step-two nilpotent Lie group and put:

- $I(G)=$ *isometry group of* G,
- $K=$ *isotropy group of* G *with Lie algebra* $\mathfrak{k}$.

The following facts are well-known:

Lemma

Let G act by left translation. Then

$$I(G) = K \ltimes G.$$

and

- $K = \Big\{(\Phi, T) \in O(\mathfrak{z}, \langle\cdot,\cdot\rangle_{\mathfrak{z}}) \times O(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}}) \ : \ \text{where} \quad TJ_Z T^{-1} = J_{\Phi(Z)}, \quad Z \in \mathfrak{z}\Big\}.$

First integrals via the isometry group

Let G be a step-two nilpotent Lie group and put:

- $I(G)$= *isometry group of* G,
- K = *isotropy group of* G *with Lie algebra* $\mathfrak{k}$.

The following facts are well-known:

Lemma

Let G act by left translation. Then

$$I(G) = K \ltimes G.$$

and

- $K = \Big\{(\Phi, T) \in O(\mathfrak{z}, \langle\cdot,\cdot\rangle_{\mathfrak{z}}) \times O(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}}) \ : \ \text{where} \quad TJ_Z T^{-1} = J_{\Phi(Z)}, \quad Z \in \mathfrak{z}\Big\}.$
- $\mathfrak{k} = \Big\{(A, B) \in \mathfrak{so}(\mathfrak{z}, \langle\cdot,\cdot\rangle_{\mathfrak{z}}) \times \mathfrak{so}(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}}) \ : \ \text{where} \quad BJ_Z - J_Z B = J_{AZ}, \quad Z \in \mathfrak{z}\Big\}.$

Notation:

Let G be a step-two nilpotent Lie group with Lie algebra $\mathfrak{g}$.

[1] Recall

$$K = \Big\{ (\Phi, T) \in O(\mathfrak{z}) \times O(\mathfrak{v}) \ : \ Tj(Z)T^{-1} = j\left(\Phi(Z)\right), \ Z \in \mathfrak{z} \Big\}.$$

Notation:

Let G be a step-two nilpotent Lie group with Lie algebra $\mathfrak{g}$.

- $\exp_G : \mathfrak{g} \to G$,

[1] Recall

$$K = \left\{ (\Phi, T) \in O(\mathfrak{z}) \times O(\mathfrak{v}) \ : \ Tj(Z)T^{-1} = j\left(\Phi(Z)\right), \ Z \in \mathfrak{z} \right\}.$$

Notation:

Let G be a step-two nilpotent Lie group with Lie algebra $\mathfrak{g}$.

- $\exp_G : \mathfrak{g} \to G$,
- $\exp_K : \mathfrak{k} \to K =$ isotropy group, [1]

[1] Recall

$$K = \Big\{(\Phi, T) \in O(\mathfrak{z}) \times O(\mathfrak{v}) \ : \ Tj(Z)T^{-1} = j\left(\Phi(Z)\right), \ Z \in \mathfrak{z}\Big\}.$$

Notation:

Let G be a step-two nilpotent Lie group with Lie algebra $\mathfrak{g}$.

- $\exp_G : \mathfrak{g} \to G$,
- $\exp_K : \mathfrak{k} \to K =$ isotropy group, [1]
- $\pi_{\mathfrak{z}} : K \to O(\mathfrak{z})$ and $\pi_{\mathfrak{v}} : K \to O(\mathfrak{v})$ the projections.

[1] Recall

$$K = \Big\{ (\Phi, T) \in O(\mathfrak{z}) \times O(\mathfrak{v}) \; : \; Tj(Z)T^{-1} = j\left(\Phi(Z)\right), \, Z \in \mathfrak{z} \Big\}.$$

Notation:

Let G be a step-two nilpotent Lie group with Lie algebra $\mathfrak{g}$.

- $\exp_G : \mathfrak{g} \to G$,
- $\exp_K : \mathfrak{k} \to K =$ isotropy group, [1]
- $\pi_{\mathfrak{z}} : K \to O(\mathfrak{z})$ and $\pi_{\mathfrak{v}} : K \to O(\mathfrak{v})$ the projections.

Flow of isometries

Fix $k = (A, B) \in \mathfrak{k}$ and $s \in \mathbb{R}$. Define $\Phi_{k,s} : G \to G$ by

$$\Phi_{k,s}(p) := \exp_G \Big\{ \big(\pi_{\mathfrak{z}} \circ \exp_K(sk)\big) U_{\mathfrak{z}} + \big(\pi_{\mathfrak{v}} \circ \exp_K(sk)\big) U_{\mathfrak{v}} \Big\},$$

where $p = \exp_G(U) \in G$ and $U \in \mathfrak{g}$.

[1] Recall

$$K = \Big\{ (\Phi, T) \in O(\mathfrak{z}) \times O(\mathfrak{v}) \; : \; Tj(Z)T^{-1} = j\left(\Phi(Z)\right), \; Z \in \mathfrak{z} \Big\}.$$

A flow of isometries

Proposition

For each $k = (A, B) \in \mathfrak{k}$ and $s \in \mathbb{R}$ the map $\Phi_{k,s}$ is a flow of isometries, i.e.

- $\Phi_{k,s} : G \to G$ is an isometric group homomorphism,
- $\Phi_{k,s} \circ \Phi_{k,t} = \Phi_{k,s+t}$ for alle $s, t \in \mathbb{R}$.

A flow of isometries

Proposition

For each $k = (A, B) \in \mathfrak{k}$ and $s \in \mathbb{R}$ the map $\Phi_{k,s}$ is a flow of isometries, i.e.

- $\Phi_{k,s} : G \to G$ is an isometric group homomorphism,
- $\Phi_{k,s} \circ \Phi_{k,t} = \Phi_{k,s+t}$ for alle $s, t \in \mathbb{R}$.

The Killing vector field corresponding to $k = (A, B) \in \mathfrak{k}$ is

$$X_k^*(p) = \frac{d}{ds}\Big|_{s=0} \Phi_{k,s}(p) \quad \text{where} \quad p = \exp_G(W) \in G.$$

A flow of isometries

Proposition

For each $k = (A, B) \in \mathfrak{k}$ and $s \in \mathbb{R}$ the map $\Phi_{k,s}$ is a flow of isometries, i.e.

- $\Phi_{k,s} : G \to G$ is an isometric group homomorphism,
- $\Phi_{k,s} \circ \Phi_{k,t} = \Phi_{k,s+t}$ for alle $s, t \in \mathbb{R}$.

The Killing vector field corresponding to $k = (A, B) \in \mathfrak{k}$ is

$$X_k^*(p) = \frac{d}{ds}\Big|_{s=0} \Phi_{k,s}(p) \quad \textit{where} \quad p = \exp_G(W) \in G.$$

Lemma

With the left translation L_p on G and $p = \exp_G(W) \in G$:

$$X_k^*(p) = dL_p\Big(BW_{\mathfrak{v}} - \frac{1}{2}\big[W_{\mathfrak{v}}, BW_{\mathfrak{v}}\big] + AW_{\mathfrak{z}}\Big).$$

First integrals via the isometry group

Recall: the isometry group of G has the form

$$I(G) = K \ltimes G \qquad \textit{where} \qquad K = \text{isotropy group.}$$

First integrals via the isometry group

Recall: the isometry group of G has the form

$$I(G) = K \ltimes G \qquad where \qquad K = \text{isotropy group.}$$

Lemma (first integral via the isotropy group)

To $k = (A, B) \in \mathfrak{k}$ (Lie algebra of K) we assign the Killing vector field X_k^* and the first integral:

$$f_{X_k^*} : TG \cong G \times \mathfrak{g} \to \mathbb{R} : f_{X_k^*}(p, Y) = \Big\langle Y, dL_{p^{-1}}(X_k^*(p)) \Big\rangle.$$

First integrals via the isometry group

Recall: the isometry group of G has the form

$$I(G) = K \ltimes G \qquad \textit{where} \qquad K = \text{isotropy group.}$$

Lemma (first integral via the isotropy group)

To $k = (A, B) \in \mathfrak{k}$ (Lie algebra of K) we assign the Killing vector field X_k^* and the first integral:

$$f_{X_k^*} : TG \cong G \times \mathfrak{g} \to \mathbb{R} : f_{X_k^*}(p, Y) = \Big\langle Y, dL_{p^{-1}}(X_k^*(p)) \Big\rangle.$$

It has the explicit form:

$$f_{X_k^*}(p, Y) = \Big\langle Y, BW - \frac{1}{2}\Big[W, BW\Big] + AW \Big\rangle.$$

First integrals via the isometry group

Recall: the isometry group of G has the form

$$I(G) = K \ltimes G \qquad \textit{where} \qquad K = \text{isotropy group.}$$

Lemma (first integral via the isotropy group)

To $k = (A, B) \in \mathfrak{k}$ (Lie algebra of K) we assign the Killing vector field X_k^* and the first integral:

$$f_{X_k^*} : TG \cong G \times \mathfrak{g} \to \mathbb{R} : f_{X_k^*}(p, Y) = \Big\langle Y, dL_{p^{-1}}(X_k^*(p)) \Big\rangle.$$

It has the explicit form:

$$f_{X_k^*}(p, Y) = \Big\langle Y, BW - \frac{1}{2}\Big[W, BW\Big] + AW \Big\rangle.$$

Notation: We extend A from $\mathfrak{z}$ to $\mathfrak{g}$ and B from $\mathfrak{v}$ to $\mathfrak{g}$ by zero.

Proposition

Let $k_j = (A_j, B_j) \in \mathfrak{k}$ where $j = 1, 2$ and assume that

$$g : TG \cong G \times \mathfrak{g} \to \mathbb{R}$$

is a left-invariant function with $\operatorname{grad}(p, Y) = (0, V)$.

Proposition

Let $k_j = (A_j, B_j) \in \mathfrak{k}$ where $j = 1, 2$ and assume that

$$g : TG \cong G \times \mathfrak{g} \to \mathbb{R}$$

is a left-invariant function with $\operatorname{grad}(p, Y) = (0, V)$. Then

(a) $\{f_{X^*_{k_1}}, g\} = \langle Y_{\mathfrak{v}}, BV_{\mathfrak{v}}\rangle + \langle Y_{\mathfrak{z}}, AV_{\mathfrak{z}}\rangle$,

Proposition

Let $k_j = (A_j, B_j) \in \mathfrak{k}$ where $j = 1, 2$ and assume that

$$g : TG \cong G \times \mathfrak{g} \to \mathbb{R}$$

is a left-invariant function with $\operatorname{grad}(p, Y) = (0, V)$. Then

(a) $\{f_{X^*_{k_1}}, g\} = \langle Y_{\mathfrak{v}}, BV_{\mathfrak{v}}\rangle + \langle Y_{\mathfrak{z}}, AV_{\mathfrak{z}}\rangle$,

(b) $\{f_{X^*_{k_1}}, f_{X^*_{k_2}}\} = f_{X^*_{[k_1,k_2]}}$,

Poisson brackets

Proposition

Let $k_j = (A_j, B_j) \in \mathfrak{k}$ where $j = 1, 2$ and assume that

$$g : TG \cong G \times \mathfrak{g} \to \mathbb{R}$$

is a left-invariant function with $\operatorname{grad}(p, Y) = (0, V)$. Then

(a) $\{f_{X^*_{k_1}}, g\} = \langle Y_{\mathfrak{v}}, BV_{\mathfrak{v}} \rangle + \langle Y_{\mathfrak{z}}, AV_{\mathfrak{z}} \rangle$,

(b) $\{f_{X^*_{k_1}}, f_{X^*_{k_2}}\} = f_{X^*_{[k_1,k_2]}}$,

where $[\cdot, \cdot]$ is the Lie bracket in $\mathfrak{k}$.

Proposition

Let $k_j = (A_j, B_j) \in \mathfrak{k}$ where $j = 1, 2$ and assume that

$$g : TG \cong G \times \mathfrak{g} \to \mathbb{R}$$

is a left-invariant function with $\operatorname{grad}(p, Y) = (0, V)$. Then

(a) $\{f_{X^*_{k_1}}, g\} = \langle Y_{\mathfrak{v}}, BV_{\mathfrak{v}}\rangle + \langle Y_{\mathfrak{z}}, AV_{\mathfrak{z}}\rangle$,

(b) $\{f_{X^*_{k_1}}, f_{X^*_{k_2}}\} = f_{X^*_{[k_1,k_2]}}$,

where $[\cdot,\cdot]$ is the Lie bracket in $\mathfrak{k}$. With the Hamiltonian $H = g$:

$$\{f_{X^*_k}, H\} = 0 \quad \textit{since} \quad \operatorname{grad} H(p, Y) = (0, Y).$$

A Lie algebra homomorphisms

Note: the equation:

$$\{ f_{X^*_{k_1}}, f_{X^*_{k_2}} \} = f_{X^*_{[k_1,k_2]}}$$

says that the map:

$$\Psi : \mathfrak{k} \to \Big(C^\infty(TG), \{\cdot,\cdot\} \Big) : k = (A,B) \mapsto f_{X^*_k}$$

is a Lie algebra homomorphism with range consisting of first integrals of the geodesic flow.

A Lie algebra homomorphisms

Note: the equation:

$$\left\{ f_{X^*_{k_1}}, f_{X^*_{k_2}} \right\} = f_{X^*_{[k_1, k_2]}}$$

says that the map:

$$\Psi : \mathfrak{k} \to \left(C^\infty(TG), \{\cdot, \cdot\} \right) : k = (A, B) \mapsto f_{X^*_k}$$

is a Lie algebra homomorphism with range consisting of first integrals of the geodesic flow.

Question

Is there a natural extension of Ψ to a larger Lie algebra?

A Lie algebra homomorphismus

Let $\mathrm{Der}(\mathfrak{g})$ denote the Lie algebra of derivations on $\mathfrak{g}$.

Lemma

A Lie algebra homomorphism is obtained by

$$\tau : \mathfrak{k} \to \mathrm{Der}(\mathfrak{g}) : (A, B) \mapsto \Big[\mathfrak{g} \ni U = U_{\mathfrak{z}} + U_{\mathfrak{v}} \mapsto AU_{\mathfrak{z}} + BU_{\mathfrak{v}} \in \mathfrak{g}\Big].$$

In particular, that means: for $U, W \in \mathfrak{g}$:

$$\tau(A, B)[U, W] = \Big[U, \tau(A, B)W\Big] + \Big[\tau(A, B)U, W\Big].$$

A Lie algebra homomorphismus

Let $\mathrm{Der}(\mathfrak{g})$ denote the Lie algebra of derivations on $\mathfrak{g}$.

Lemma

A Lie algebra homomorphism is obtained by

$$\tau : \mathfrak{k} \to \mathrm{Der}(\mathfrak{g}) : (A, B) \mapsto \Big[\mathfrak{g} \ni U = U_\mathfrak{z} + U_\mathfrak{v} \mapsto AU_\mathfrak{z} + BU_\mathfrak{v} \in \mathfrak{g}\Big].$$

In particular, that means: for $U, W \in \mathfrak{g}$:

$$\tau(A, B)[U, W] = \Big[U, \tau(A, B)W\Big] + \Big[\tau(A, B)U, W\Big].$$

Definition: With respect to τ we can form the semi-direct product

$$\mathfrak{k} \ltimes_\tau \mathfrak{g}$$

with Lie brackets:

$$\Big[(A, B), U\Big] = \tau(A, B)(U), \qquad (A, B) \in \mathfrak{k}, \ U \in \mathfrak{g}.$$

A Lie algebra homomorphismus

Consider the natural extension $\tilde{\Psi}$ of $\Psi : \mathfrak{k} \to C^\infty(TG)$:

$$\tilde{\Psi} : \mathfrak{k} \oplus \mathfrak{g} \to C^\infty(TG) : \Psi(k, U) := f_{X_k^*} + \sum_{i=1}^{n} a_i F_{X_i} + \sum_{\ell=1}^{j} b_\ell F_{Z_\ell},$$

where $U \in \mathfrak{g}$ has the expansion:

$$U = \sum_{i=1}^{n} a_i X_i + \sum_{\ell=1}^{j} b_\ell Z_\ell \in \mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}.$$

A Lie algebra homomorphismus

Consider the natural extension $\tilde{\Psi}$ of $\Psi : \mathfrak{k} \to C^\infty(TG)$:

$$\tilde{\Psi} : \mathfrak{k} \oplus \mathfrak{g} \to C^\infty(TG) : \Psi(k, U) := f_{X_k^*} + \sum_{i=1}^{n} a_i F_{X_i} + \sum_{\ell=1}^{j} b_\ell F_{Z_\ell},$$

where $U \in \mathfrak{g}$ has the expansion:

$$U = \sum_{i=1}^{n} a_i X_i + \sum_{\ell=1}^{j} b_\ell Z_\ell \in \mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}.$$

Theorem, (W. B., D. Tarama)

Let $G = G_{r,s}$ be a pseudo-H-type Lie group. The the map Ψ defines an injective Lie algebra homomorphism

$$\Psi : \mathfrak{k} \ltimes_\tau \mathfrak{g} \to \left(C^\infty(TG), \{\cdot, \cdot\} \right)$$

into a set of first integrals of the geodesic flow.

First integrals on compact nilmanifolds

Let $G_{r,s}$ be a pseudo-H-type group with lattice $L \subset G_{r,s}$.

Let $G_{r,s}$ be a pseudo-H-type group with lattice $L \subset G_{r,s}$.

Definition

The left-coset space $L \backslash G_{r,s}$ is called a pseudo-H-type nilmanifold.

First integrals on compact nilmanifolds

Let $G_{r,s}$ be a pseudo-H-type group with lattice $L \subset G_{r,s}$.

Definition

The left-coset space $L \backslash G_{r,s}$ is called a pseudo-H-type nilmanifold.

Problems:

First integrals on compact nilmanifolds

Let $G_{r,s}$ be a pseudo-H-type group with lattice $L \subset G_{r,s}$.

Definition

The left-coset space $L \backslash G_{r,s}$ is called a pseudo-H-type nilmanifold.

Problems:

- Complete integrability of the geodesic flow on $L \backslash G_{r,s}$.

Let $G_{r,s}$ be a pseudo-H-type group with lattice $L \subset G_{r,s}$.

Definition

The left-coset space $L \backslash G_{r,s}$ is called a pseudo-H-type nilmanifold.

Problems:

- Complete integrability of the geodesic flow on $L \backslash G_{r,s}$.
- Can we descend enough Poisson commuting first integrals from $G_{r,s}$ to $L \backslash G_{r,s}$?

Heisenberg-Reiter Lie Algebra

Definition

A step-2 nilpotent Lie algebra $\mathfrak{g}$ is called Heisenberg-Reiter Lie algebra (HR) , if it admits a decomposition

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n} \oplus \mathfrak{z},$$

such that

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}, \qquad [\mathfrak{z}, \mathfrak{g}] = 0, \qquad [\mathfrak{r}, \mathfrak{r}] = 0 \quad \text{and} \quad [\mathfrak{n}, \mathfrak{n}] = 0.$$

Definition

A step-2 nilpotent Lie algebra $\mathfrak{g}$ is called Heisenberg-Reiter Lie algebra (HR) , if it admits a decomposition

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n} \oplus \mathfrak{z},$$

such that

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}, \quad [\mathfrak{z}, \mathfrak{g}] = 0, \quad [\mathfrak{r}, \mathfrak{r}] = 0 \quad \text{and} \quad [\mathfrak{n}, \mathfrak{n}] = 0.$$

Theorem (L. Butler, 2003)

Let $\mathfrak{g}$ be a Heisenberg-Reiter Lie algebra. For each left-invariant Riemannian metric g on G and each lattice $L \subset G$ the geodesic flow of g is smoothly Liouville integrable on $T^*(L\backslash G)$.

Example

Let $\mathfrak{n} = \mathfrak{h}_3$ be the three-dimensional Heisenberg Lie algebra:

$$\mathfrak{h}_3 = \text{span}\{X\} \oplus \text{span}\{Y\} \oplus \text{span}\{Z\}, \qquad (*)$$

where $[X, Y] = Z$ and all other brackets vanish. Then $\mathfrak{h}_3$ is of Heisenberg-Reiter type.

Example

Heisenberg-Reiter pseudo-H-type algebras: Let $s > 0$, then the Lie algebra $\mathcal{N}_{0,s}$ is of Heisenberg-Reiter type.

Example

Heisenberg-Reiter pseudo-H-type algebras: Let $s > 0$, then the Lie algebra $\mathcal{N}_{0,s}$ is of Heisenberg-Reiter type.

Question:

How to descend first integrals on $G_{r,s}$ to the quotients $L\backslash G_{r,s}$?

Example

Heisenberg-Reiter pseudo-H-type algebras: Let $s > 0$, then the Lie algebra $\mathcal{N}_{0,s}$ is of Heisenberg-Reiter type.

Question:

How to descend first integrals on $G_{r,s}$ to the quotients $L\backslash G_{r,s}$?

Here we only treat $s = 0$. Consider an integral basis in $\mathcal{N}_{r,0}$

$$\mathcal{N}_{r,0} = \mathfrak{v} \oplus \mathfrak{z} = \text{span}\big\{X_1, \cdots, X_{2n}\big\} \oplus \underbrace{\text{span}\big\{Z_1, \cdots, Z_r\big\}}_{=\text{center}}.$$

Example

Heisenberg-Reiter pseudo-H-type algebras: Let $s > 0$, then the Lie algebra $\mathcal{N}_{0,s}$ is of Heisenberg-Reiter type.

Question:

How to descend first integrals on $G_{r,s}$ to the quotients $L \backslash G_{r,s}$?

Here we only treat $s = 0$. Consider an integral basis in $\mathcal{N}_{r,0}$

$$\mathcal{N}_{r,0} = \mathfrak{v} \oplus \mathfrak{z} = \mathrm{span}\{X_1, \cdots, X_{2n}\} \oplus \underbrace{\mathrm{span}\{Z_1, \cdots, Z_r\}}_{=\text{center}}.$$

Let L be the lattice in $G_{r,0}$ generated by $\exp(X_i)$ and $\exp(Z_\ell)$, i.e.

Example

Heisenberg-Reiter pseudo-H-type algebras: Let $s > 0$, then the Lie algebra $\mathcal{N}_{0,s}$ is of Heisenberg-Reiter type.

Question:

How to descend first integrals on $G_{r,s}$ to the quotients $L \backslash G_{r,s}$?

Here we only treat $s = 0$. Consider an integral basis in $\mathcal{N}_{r,0}$

$$\mathcal{N}_{r,0} = \mathfrak{v} \oplus \mathfrak{z} = \text{span}\{X_1, \cdots, X_{2n}\} \oplus \underbrace{\text{span}\{Z_1, \cdots, Z_r\}}_{=\text{center}}.$$

Let L be the lattice in $G_{r,0}$ generated by $\exp(X_i)$ and $\exp(Z_\ell)$, i.e.

$$L = \exp\left\{\sum \gamma_q X_q + \frac{1}{2}\sum \beta_\ell Z_\ell \ : \ \gamma_q, \beta_\ell \in \mathbb{Z}\right\}.$$

First integrals on compact quotients

Recall the first integrals induced by left-multiplication

$$F_{X_i}(p, Y) = \left\langle X_i, J_{Y_{\mathfrak{z}}} W_{\mathfrak{v}} - Y_{\mathfrak{v}} \right\rangle, \quad \textit{where} \quad p = \exp(W) \in G_{r,0}.$$

First integrals on compact quotients

Recall the first integrals induced by left-multiplication

$$F_{X_i}(p, Y) = \left\langle X_i, J_{Y_{\mathfrak{z}}} W_{\mathfrak{v}} - Y_{\mathfrak{v}} \right\rangle, \quad \textit{where} \quad p = \exp(W) \in G_{r,0}.$$

Lemma

Fix an element in the lattice

$$g = \exp(V) \in L \quad \textit{where} \quad V_{\mathfrak{v}} = \sum_q \gamma_q X_q \in \mathfrak{v}, \ \gamma_q \in \mathbb{Z}.$$

First integrals on compact quotients

Recall the first integrals induced by left-multiplication

$$F_{X_i}(p,Y) = \Big\langle X_i, J_{Y_\mathfrak{z}} W_\mathfrak{v} - Y_\mathfrak{v} \Big\rangle, \quad \textit{where} \quad p = \exp(W) \in G_{r,0}.$$

Lemma

Fix an element in the lattice

$$g = \exp(V) \in L \quad \textit{where} \quad V_\mathfrak{v} = \sum_q \gamma_q X_q \in \mathfrak{v}, \;\; \gamma_q \in \mathbb{Z}.$$

Then

$$\begin{pmatrix} F_{X_1} \\ \vdots \\ F_{X_{2n}} \end{pmatrix} (g * p, Y) = \begin{pmatrix} F_{X_1} \\ \vdots \\ F_{X_{2n}} \end{pmatrix} (p,Y) + M(Y_\mathfrak{z}) \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{2n} \end{pmatrix},$$

where $M(Y_\mathfrak{z}) := \big(\langle J_{Y_\mathfrak{z}} X_q, X_i \rangle\big)_{q,i=1}^{2n}$ for $Y_\mathfrak{z} \neq 0$ is invertible.

We multiply both sides of the last equation by $M(Y_{\mathfrak{z}})^{-1}$ for $Y_{\mathfrak{z}} \neq 0$:

$$\underbrace{M^{-1}(Y_{\mathfrak{z}}) \begin{pmatrix} F_{X_1} \\ \vdots \\ F_{X_{2n}} \end{pmatrix} (g * p, Y)}_{=:\widetilde{F}(g*p,Y)} = \underbrace{M(Y_{\mathfrak{z}})^{-1} \begin{pmatrix} F_{X_1} \\ \vdots \\ F_{X_{2n}} \end{pmatrix} (p, Y)}_{=:\widetilde{F}(p,Y)} + \underbrace{\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{2n} \end{pmatrix}}_{\in \mathbb{Z}^{2n}}.$$

First integrals on compact quotients

We multiply both sides of the last equation by $M(Y_\mathfrak{z})^{-1}$ for $Y_\mathfrak{z} \neq 0$:

$$\underbrace{M^{-1}(Y_\mathfrak{z}) \begin{pmatrix} F_{X_1} \\ \vdots \\ F_{X_{2n}} \end{pmatrix} (g * p, Y)}_{=:\widetilde{F}(g*p,Y)} = \underbrace{M(Y_\mathfrak{z})^{-1} \begin{pmatrix} F_{X_1} \\ \vdots \\ F_{X_{2n}} \end{pmatrix} (p, Y)}_{=:\widetilde{F}(p,Y)} + \underbrace{\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{2n} \end{pmatrix}}_{\in \mathbb{Z}^{2n}}.$$

Corollary

The $i = 1, \cdots, 2n$ functions

$$F_i^{L \backslash G_{r,0}}(p, Y) := \sin\left(2\pi \widetilde{F}(p, Y)_i\right), \quad \textit{for} \quad i = 1, \cdots, 2n$$

are invariant under the left action by L and they descend to first integrals on the compact quotient $L \backslash G_{r,0}$.

First integrals on compact quotients

We multiply both sides of the last equation by $M(Y_{\mathfrak{z}})^{-1}$ for $Y_{\mathfrak{z}} \neq 0$:

$$\underbrace{M^{-1}(Y_{\mathfrak{z}}) \begin{pmatrix} F_{X_1} \\ \vdots \\ F_{X_{2n}} \end{pmatrix} (g * p, Y)}_{=:\widetilde{F}(g*p,Y)} = \underbrace{M(Y_{\mathfrak{z}})^{-1} \begin{pmatrix} F_{X_1} \\ \vdots \\ F_{X_{2n}} \end{pmatrix} (p, Y)}_{=:\widetilde{F}(p,Y)} + \underbrace{\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{2n} \end{pmatrix}}_{\in \mathbb{Z}^{2n}}.$$

Corollary

The $i = 1, \cdots, 2n$ functions

$$F_i^{L \backslash G_{r,0}}(p, Y) := \sin\left(2\pi \widetilde{F}(p, Y)_i\right), \quad \textit{for} \quad i = 1, \cdots, 2n$$

are invariant under the left action by L and they descend to first integrals on the compact quotient $L \backslash G_{r,0}$.

If F_j Poisson commutes with F_i, then also $F_j^{L \backslash G_{r,0}}$ Poisson commutes with $F_j^{L \backslash G_{r,0}}$.

First integrals on compact quotients

Result: (W.-B. D. Tarama, 2018)

In some cases - including but exceeding the Heisenberg-Reiter type algebras among $\mathcal{N}_{r,s}$ - we can prove complete integrability of the (pseudo)-Riemannian geodesic flow on $L \backslash G_{r,s}$, where $L \subset G_{r,s}$ denotes an integral lattice.

However, we are not able to decide all possible cases.

W.-B., D. Tarama, *On the complete integrability of the geodesic flow of pseudo-H-type Lie groups*, Anal. Math. Phys. 8 (2018), no. 4, 493 - 520.

Problem

Is the complete integrability property of a closed Riemannian manifold M determined by the spectral data, i.e. by the collection of eigenvalues of the Laplace operator acting on functions?

Let $(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$ and $(\mathfrak{z}, \langle\cdot,\cdot\rangle_{\mathfrak{z}})$ be euclidean vector spaces with inner products. Consider a linear map:

$$j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v}).$$

Let $(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$ and $(\mathfrak{z}, \langle\cdot,\cdot\rangle_{\mathfrak{z}})$ be euclidean vector spaces with inner products. Consider a linear map:

$$j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v}).$$

Define a metric Lie algebra

$$\mathfrak{n}(j) := \mathfrak{v} \oplus_{\perp} \mathfrak{z}$$

Let $(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$ and $(\mathfrak{z}, \langle\cdot,\cdot\rangle_{\mathfrak{z}})$ be euclidean vector spaces with inner products. Consider a linear map:

$$j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v}).$$

Define a metric Lie algebra

$$\mathfrak{n}(j) := \mathfrak{v} \oplus_{\perp} \mathfrak{z}$$

Inner product: Define $\langle\cdot,\cdot\rangle_{\mathfrak{n}}$ by taking the inner products on $\mathfrak{v}$ and $\mathfrak{z}$ and assuming that $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal.

Let $(\mathfrak{v}, \langle\cdot,\cdot\rangle_{\mathfrak{v}})$ and $(\mathfrak{z}, \langle\cdot,\cdot\rangle_{\mathfrak{z}})$ be euclidean vector spaces with inner products. Consider a linear map:

$$j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v}).$$

Define a metric Lie algebra

$$\mathfrak{n}(j) := \mathfrak{v} \oplus_{\perp} \mathfrak{z}$$

Inner product: Define $\langle\cdot,\cdot\rangle_{\mathfrak{n}}$ by taking the inner products on $\mathfrak{v}$ and $\mathfrak{z}$ and assuming that $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal.

Lie bracket: Define $[\cdot,\cdot]^j$ by assuming that $\mathfrak{z}$ is in the center and

$$[\mathfrak{v},\mathfrak{v}]^j \subset \mathfrak{z}$$

$$\langle j(Z)X, Y\rangle_{\mathfrak{v}} = \langle Z, [X,Y]^j\rangle_{\mathfrak{z}}, \qquad X, Y \in \mathfrak{v},\ Z \in \mathfrak{z}.$$

Definition

Let $G(j)$ be the 2-step simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}(j)$ and left-invariant Riemannian metric g^j which coincides with $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$ on $\mathfrak{n}(j) = T_e G(j)$.

Consider the exponential map:

$$\exp^j : \mathfrak{n}(j) \to G(j).$$

Definition

Let $G(j)$ be the 2-step simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}(j)$ and left-invariant Riemannian metric g^j which coincides with $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$ on $\mathfrak{n}(j) = T_e G(j)$.

Consider the exponential map:

$$\exp^j : \mathfrak{n}(j) \to G(j).$$

The Baker-Campbell-Hausdorff formula implies for all $X, Y \in \mathfrak{n}(j)$:

$$\exp^j(X) * \exp^j(Y) = \exp^j\left(X + Y + \frac{1}{2}[X, Y]^j\right).$$

Definition

Let $G(j)$ be the 2-step simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}(j)$ and left-invariant Riemannian metric g^j which coincides with $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$ on $\mathfrak{n}(j) = T_e G(j)$.

Consider the exponential map:

$$\exp^j : \mathfrak{n}(j) \to G(j).$$

The Baker-Campbell-Hausdorff formula implies for all $X, Y \in \mathfrak{n}(j)$:

$$\exp^j(X) * \exp^j(Y) = \exp^j\left(X + Y + \frac{1}{2}[X, Y]^j\right).$$

Lattice: Consider a lattice $\mathfrak{l} \subset \mathfrak{n}(j)$ with the property

$$[\mathfrak{l}, \mathfrak{l}]^j \subset 2\mathfrak{l}.$$

Then

$$\Gamma = \exp^j(\mathfrak{l}) \subset G(j)$$

is a discrete subgroup.

Compact nilmanifolds

Definition

The left-coset space $\Gamma \backslash G(j)$ with the metric induced by g^j is called two-step nilmanifold. We assume that Γ is cocompact [a], i.e. $(M^j = \Gamma \backslash G(j), g^j)$ is compact.

[a] i.e. $\mathfrak{l}$ has full rank in $\mathfrak{n}(j)$

Compact nilmanifolds

Definition

The left-coset space $\Gamma\backslash G(j)$ with the metric induced by g^j is called two-step nilmanifold. We assume that Γ is cocompact [a], i.e. $(M^j = \Gamma\backslash G(j), g^j)$ is compact.

[a] i.e. $\mathfrak{l}$ has full rank in $\mathfrak{n}(j)$

Compact two-step nilmanifolds are treatable examples of closed Riemannian manifolds.

Compact nilmanifolds

Definition

The left-coset space $\Gamma\backslash G(j)$ with the metric induced by g^j is called two-step nilmanifold. We assume that Γ is cocompact [a], i.e. $(M^j = \Gamma\backslash G(j), g^j)$ is compact.

[a] i.e. $\mathfrak{l}$ has full rank in $\mathfrak{n}(j)$

Compact two-step nilmanifolds are treatable examples of closed Riemannian manifolds.

Problem:

Under which conditions are two compact two-step nilmanifolds M_1 and M_2 isospectral, i.e the spectra of M_1 and M_2 coincide?

Definition

Let $\mathfrak{z}$ and $\mathfrak{v}$ be as above:

Definition

Let $\mathfrak{z}$ and $\mathfrak{v}$ be as above:

(a) Two linear maps:

$$j, j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$$

are called isospectral, if for each $Z \in \mathfrak{z}$ the maps

$$j(Z), j'(Z) \in \mathfrak{so}(\mathfrak{v})$$

have the same eigenvalues (with multiplicities).

Definition

Let $\mathfrak{z}$ and $\mathfrak{v}$ be as above:

(a) Two linear maps:

$$j, j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$$

are called isospectral, if for each $Z \in \mathfrak{z}$ the maps

$$j(Z), j'(Z) \in \mathfrak{so}(\mathfrak{v})$$

have the same eigenvalues (with multiplicities).

(b) Two lattices [a] in a euclidean space are called isospectral, if the lengths of their elements counted with multiplicities coincide.

[a] lattice in a vector space: $\mathfrak{l} = \{a_1 b_1 + \cdots + a_m b_m \ : \ (a_1, \cdots, a_m) \in \mathbb{Z}^m\}$ and $\{b_1, \cdots, b_m\}$ a basis of $\mathfrak{l}$.

Dual lattice: Let $\mathfrak{l}_z$ be a cocompact lattice in $\mathfrak{z}$. Put:

$$\textit{dual lattice} := \mathfrak{l}_z^* = \{ Z \in \mathfrak{z} \,:\, \langle Z, \mathfrak{l}_z \rangle_{\mathfrak{z}} \subset \mathbb{Z} \} \subset \mathfrak{z}.$$

Here is a criterion for isospectrality:

Dual lattice: Let $\mathfrak{l}_z$ be a cocompact lattice in $\mathfrak{z}$. Put:

$$\textit{dual lattice} := \mathfrak{l}_z^* = \{Z \in \mathfrak{z} \, : \, \langle Z, \mathfrak{l}_z \rangle_{\mathfrak{z}} \subset \mathbb{Z}\} \subset \mathfrak{z}.$$

Here is a criterion for isospectrality:

Theorem (C. S. Gordon, D. Schueth, E. N. Wilson)

Let $j, j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ be isospectral and consider two cocompact lattices

$$\mathfrak{l}_v \subset \mathfrak{v} \qquad \textit{and} \qquad \mathfrak{l}_z \subset \mathfrak{z}.$$

Assume that:

Isospectral compact two-step nilmanifolds

Dual lattice: Let $\mathfrak{l}_z$ be a cocompact lattice in $\mathfrak{z}$. Put:

$$\textit{dual lattice} := \mathfrak{l}_z^* = \{Z \in \mathfrak{z} \; : \; \langle Z, \mathfrak{l}_z \rangle_{\mathfrak{z}} \subset \mathbb{Z}\} \subset \mathfrak{z}.$$

Here is a criterion for isospectrality:

Theorem (C. S. Gordon, D. Schueth, E. N. Wilson)

Let $j, j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ be isospectral and consider two cocompact lattices

$$\mathfrak{l}_v \subset \mathfrak{v} \qquad \textit{and} \qquad \mathfrak{l}_z \subset \mathfrak{z}.$$

Assume that:

(a) $[\mathfrak{l}_v, \mathfrak{l}_v]^j \subset 2\mathfrak{l}_z$ *and* $[\mathfrak{l}_v, \mathfrak{l}_v]^{j'} \subset 2\mathfrak{l}_z$,

Dual lattice: Let $\mathfrak{l}_z$ be a cocompact lattice in $\mathfrak{z}$. Put:

$$\textit{dual lattice} := \mathfrak{l}_z^* = \left\{ Z \in \mathfrak{z} \, : \, \langle Z, \mathfrak{l}_z \rangle_{\mathfrak{z}} \subset \mathbb{Z} \right\} \subset \mathfrak{z}.$$

Here is a criterion for isospectrality:

Theorem (C. S. Gordon, D. Schueth, E. N. Wilson)

Let $j, j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ be isospectral and consider two cocompact lattices

$$\mathfrak{l}_v \subset \mathfrak{v} \qquad \textit{and} \qquad \mathfrak{l}_z \subset \mathfrak{z}.$$

Assume that:

(a) $[\mathfrak{l}_v, \mathfrak{l}_v]^j \subset 2\mathfrak{l}_z$ *and* $[\mathfrak{l}_v, \mathfrak{l}_v]^{j'} \subset 2\mathfrak{l}_z$,

(b) *For each* $Z \in \mathfrak{l}_z^*$ *the lattices*

$$\ker\big(j(Z)\big) \cap \mathfrak{l}_v \qquad \textit{and} \qquad \ker\big(j'(Z)\big) \cap \mathfrak{l}_v$$

are isospectral in $\mathfrak{l}_v$. *(This happens e.g. if they are isometric).*

Theorem (C. S. Gordon, D. Schueth, E. N. Wilson, (continued))

Consider the corresponding lattices in $G(j)$ and $G(j')$:

$$\Gamma(j) = \exp^{j}\left(\mathfrak{l}_v + \mathfrak{l}_z\right) \quad \textit{and} \quad \Gamma(j') = \exp^{j'}\left(\mathfrak{l}_v + \mathfrak{l}_z\right).$$

Theorem (C. S. Gordon, D. Schueth, E. N. Wilson, (continued))

Consider the corresponding lattices in $G(j)$ and $G(j')$:

$$\Gamma(j) = \exp^{j}\left(\mathfrak{l}_v + \mathfrak{l}_z\right) \quad \textit{and} \quad \Gamma(j') = \exp^{j'}\left(\mathfrak{l}_v + \mathfrak{l}_z\right).$$

Then the compact Riemannian manifolds

$$\left(\Gamma(j)\backslash G(j), g^{j}\right) \quad \textit{and} \quad \left(\Gamma(j')\backslash G(j'), g^{j'}\right)$$

are isospectral for the Laplace operator on functions.

Two Examples

With respect to the bases the maps

$$j(c_iZ_i + c_jZ_j + c_kZ_k) \quad \mathit{and} \quad j'(c_iZ_i + c_jZ_j + c_kZ_k)$$

with $c_i, c_j, c_k \in \mathbb{R}$ are expressed by the skew-symmetric matrices:

$$\begin{pmatrix} 0 & 0 & 0 & -c_k & c_j \\ 0 & 0 & c_k & 0 & -c_i \\ 0 & -c_k & 0 & 0 & 0 \\ c_k & 0 & 0 & 0 & 0 \\ -c_j & c_i & 0 & 0 & 0 \end{pmatrix} \quad \mathit{and} \quad \begin{pmatrix} 0 & -c_k & 0 & 0 & 0 \\ c_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_k & c_j \\ 0 & 0 & c_k & 0 & -c_i \\ 0 & 0 & -c_j & c_i & 0 \end{pmatrix}.$$

Two Examples

With respect to the bases the maps

$$j(c_iZ_i + c_jZ_j + c_kZ_k) \quad \textit{and} \quad j'(c_iZ_i + c_jZ_j + c_kZ_k)$$

with $c_i, c_j, c_k \in \mathbb{R}$ are expressed by the skew-symmetric matrices:

$$\begin{pmatrix} 0 & 0 & 0 & -c_k & c_j \\ 0 & 0 & c_k & 0 & -c_i \\ 0 & -c_k & 0 & 0 & 0 \\ c_k & 0 & 0 & 0 & 0 \\ -c_j & c_i & 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 & -c_k & 0 & 0 & 0 \\ c_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_k & c_j \\ 0 & 0 & c_k & 0 & -c_i \\ 0 & 0 & -c_j & c_i & 0 \end{pmatrix}.$$

Let $(G(j), g^j)$ and $(G(j'), g^{j'})$ be the corresponding 2-step nilpotent Lie groups with appropriate left-invariant metrics g^j and $g^{j'}$ and standard lattices $\Gamma(j)$ and $\Gamma(j')$, respectively.

Theorem (D. Schueth)

$(\Gamma(j)\backslash G(j), g^j)$ and $(\Gamma(j')\backslash G(j'), g^{j'})$ are isospectral.

Criterion for complete integrability

:

:

Theorem (L. Butler)

Let $\mathfrak{n}$ be a Lie algebra of Heisenberg-Reiter type with corresponding simply connected Lie group G. Assume:

:

Theorem (L. Butler)

Let $\mathfrak{n}$ be a Lie algebra of Heisenberg-Reiter type with corresponding simply connected Lie group G. Assume:

(a) *There exists a discrete cocompact subgroup (lattice) Γ in G.*

Criterion for complete integrability

:

Theorem (L. Butler)

Let $\mathfrak{n}$ be a Lie algebra of Heisenberg-Reiter type with corresponding simply connected Lie group G. Assume:

(a) *There exists a discrete cocompact subgroup (lattice) Γ in G.*

(b) *There is an injective presentation, i.e. $j(Z)$ is injective on $\mathfrak{r}$ for some $Z \in \mathfrak{z}$*

$$\mathfrak{n} = \mathfrak{r} \oplus \mathfrak{m} \oplus \mathfrak{z}$$

such that $\mathfrak{r} \cup \mathfrak{m} \cup \mathfrak{z}$ contains a set of vectors which is mapped by the exponential map $\exp : \mathfrak{n} \to G$ to a generating set of Γ.

Criterion for complete integrability

:

Theorem (L. Butler)

Let $\mathfrak{n}$ be a Lie algebra of Heisenberg-Reiter type with corresponding simply connected Lie group G. Assume:

(a) *There exists a discrete cocompact subgroup (lattice) Γ in G.*

(b) *There is an injective presentation, i.e. $j(Z)$ is injective on $\mathfrak{r}$ for some $Z \in \mathfrak{z}$*

$$\mathfrak{n} = \mathfrak{r} \oplus \mathfrak{m} \oplus \mathfrak{z}$$

such that $\mathfrak{r} \cup \mathfrak{m} \cup \mathfrak{z}$ contains a set of vectors which is mapped by the exponential map $\exp : \mathfrak{n} \to G$ to a generating set of Γ.

Then, for **any** *left-invariant metric g on G, the geodesic flow of $(\Gamma \backslash G, g)$ is completely integrable in the sense of Liouville.*

Criterion for **non**-integrability

Criterion for **non**-integrability

Definition

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra. We define:

Criterion for **non**-integrability

Definition

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra. We define:

(a) With $\lambda \in \mathfrak{n}^*$ (dual space) let

$$\mathfrak{n}_\lambda := \left\{ X \in \mathfrak{n} \; : \; \lambda_{|[X,\mathfrak{n}]} \right\} = 0.$$

Criterion for **non**-integrability

Definition

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra. We define:

(a) With $\lambda \in \mathfrak{n}^*$ (dual space) let

$$\mathfrak{n}_\lambda := \left\{ X \in \mathfrak{n} \ : \ \lambda_{|[X,\mathfrak{n}]} \right\} = 0.$$

(b) An element $\lambda \in \mathfrak{n}^*$ is called regular if $\mathfrak{n}_\lambda$ has minimal dimension.

Criterion for **non**-integrability

Definition

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra. We define:

(a) With $\lambda \in \mathfrak{n}^*$ (dual space) let

$$\mathfrak{n}_\lambda := \left\{ X \in \mathfrak{n} \;:\; \lambda_{|[X,\mathfrak{n}]} \right\} = 0.$$

(b) An element $\lambda \in \mathfrak{n}^*$ is called regular if $\mathfrak{n}_\lambda$ has minimal dimension.

(c) $\mathfrak{n}$ is called non-integrable if there is a dense open subset U of $\mathfrak{n}^* \times \mathfrak{n}^*$ such that for each pair $(\lambda, \mu) \in U$ both λ and μ are regular and $[\mathfrak{n}_\lambda, \mathfrak{n}_\mu]$ has positive dimension.

Theorem (L. Butler)

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra with corresponding simply connected Lie group G. Assume that:

Criterion for **non**-integrability

Theorem (L. Butler)

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra with corresponding simply connected Lie group G. Assume that:

(a) $\mathfrak{n}$ *be non-integrable.*

Criterion for **non**-integrability

Theorem (L. Butler)

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra with corresponding simply connected Lie group G. Assume that:

(a) $\mathfrak{n}$ *be non-integrable.*

(b) *There is a discrete co-compact subgroup Γ of G.*

Criterion for **non**-integrability

Theorem (L. Butler)

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra with corresponding simply connected Lie group G. Assume that:

(a) *$\mathfrak{n}$ be non-integrable.*

(b) *There is a discrete co-compact subgroup Γ of G.*

For **any** *left-invariant metric g on G the geodesic flow of $(\Gamma \backslash G, g)$ is not completely integrable in the sense of Liouville.*

Examples: revisited

Recall $j : \mathfrak{z} \cong \mathbb{R}^3 \to \mathfrak{so}(\mathfrak{v})$ where $\mathfrak{v} \cong \mathbb{R}^5$:

$$j(c_i Z_i + c_j Z_j + c_k Z_k) \cong \begin{pmatrix} 0 & 0 & 0 & -c_k & c_j \\ 0 & 0 & c_k & 0 & -c_i \\ 0 & -c_k & 0 & 0 & 0 \\ c_k & 0 & 0 & 0 & 0 \\ -c_j & c_i & 0 & 0 & 0 \end{pmatrix}.$$

Let $(G(j), g^j)$ be the corresponding two-step nilpotent Lie group with left-invariant metric g^j.

Lemma

Let $\Gamma(j)$ be a "standard lattice" in $G(j)$. Then the two-step nilmanifold $(\Gamma(j)\backslash G(j), g^j)$ has completely integrable geodesic flow.

Examples: revisited

Recall $j : \mathfrak{z} \cong \mathbb{R}^3 \to \mathfrak{so}(\mathfrak{v})$ where $\mathfrak{v} \cong \mathbb{R}^5$:

$$j(c_i Z_i + c_j Z_j + c_k Z_k) \cong \begin{pmatrix} 0 & 0 & 0 & -c_k & c_j \\ 0 & 0 & c_k & 0 & -c_i \\ 0 & -c_k & 0 & 0 & 0 \\ c_k & 0 & 0 & 0 & 0 \\ -c_j & c_i & 0 & 0 & 0 \end{pmatrix}.$$

Let $(G(j), g^j)$ be the corresponding two-step nilpotent Lie group with left-invariant metric g^j.

Lemma

Let $\Gamma(j)$ be a "standard lattice" in $G(j)$. Then the two-step nilmanifold $(\Gamma(j)\backslash G(j), g^j)$ has completely integrable geodesic flow.

Proof: Show that $\mathfrak{n}$ is of Heisenberg-Reiter type + additional conditions.

Examples: revisited

Recall: $j' : \mathfrak{z} \cong \mathbb{R}^3 \to \mathfrak{so}(\mathfrak{v})$ where $\mathfrak{v} \cong \mathbb{R}^5$:

$$j'(c_i Z_i + c_j Z_j + c_k Z_k) \cong \begin{pmatrix} 0 & -c_k & 0 & 0 & 0 \\ c_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_k & c_j \\ 0 & 0 & c_k & 0 & -c_i \\ 0 & 0 & -c_j & c_i & 0 \end{pmatrix}.$$

Let $(G(j'), g^{j'})$ be the corresponding two-step nilpotent Lie group with left-invariant metric $g^{j'}$.

Theorem

Let $\Gamma(j')$ be a "standard lattice" in $G(j')$. Then $(\Gamma(j')\backslash G(j'), g^{j'})$ does not have completely integrable geodesic flow.

Proof: Butler's non-integrability condition.

Examples: revisited

Recall: $j' : \mathfrak{z} \cong \mathbb{R}^3 \to \mathfrak{so}(\mathfrak{v})$ where $\mathfrak{v} \cong \mathbb{R}^5$:

$$j'(c_i Z_i + c_j Z_j + c_k Z_k) \cong \begin{pmatrix} 0 & -c_k & 0 & 0 & 0 \\ c_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_k & c_j \\ 0 & 0 & c_k & 0 & -c_i \\ 0 & 0 & -c_j & c_i & 0 \end{pmatrix}.$$

Let $(G(j'), g^{j'})$ be the corresponding two-step nilpotent Lie group with left-invariant metric $g^{j'}$.

Theorem

Let $\Gamma(j')$ be a "standard lattice" in $G(j')$. Then $(\Gamma(j')\backslash G(j'), g^{j'})$ does not have completely integrable geodesic flow.

Proof: Butler's non-integrability condition.

Answer to the initial question:

Theorem (D. Schueth, 2008)

There is a pair of isospectral compact closed Riemannian manifolds: one has completely integrable geodesic flow and the other does not have completely integrable geodesic flow.

Answer to the initial question:

Theorem (D. Schueth, 2008)

There is a pair of isospectral compact closed Riemannian manifolds: one has completely integrable geodesic flow and the other does not have completely integrable geodesic flow.

or shortly:

Answer to the initial question:

Theorem (D. Schueth, 2008)

There is a pair of isospectral compact closed Riemannian manifolds: one has completely integrable geodesic flow and the other does not have completely integrable geodesic flow.

or shortly:

One cannot read complete integrability of the geodesic flow from the spectral data.

References

- W. -B., K. Furutani, M. Tamura, *First integrals of bi-characteristic curves of a sub-Laplacian and related Grushin type operators*, JNCA, 18(5), 893-917, (2017).
- W.-B., D. Tarama, *On the complete integrability of the geodesic flow of pseudo-H-type Lie groups*, Anal. Math. Phys. 8 (2018),
  no. 4, 493 - 520.
- L. Butler, *Integrable geodesic flow with wild first integrals: the case of two-step nilmanifolds*, Ergo. Th. & Dynam. Sys. (2003), 23, 771-797.
- D. Schueth, *Integrability of geodesic flows and isospectrality of Riemannian manifolds*. Math. Z. 260 (2008), no. 3, 595?613.

Thank you for your attention!