Geodesic X-ray transform and streaking artifacts on simple surfaces or on spaces of constant curvature

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X-ray transform on the plane

- All the planar lines are parametrized by $(\theta, t) \in [0, \pi] \times \mathbb{R}$ by

$$\ell = \{(-s \sin\theta + t\cos\theta, s\cos\theta + t\sin\theta) : s \in \mathbb{R}\}.$$

  The X-ray transform of $f(x, y)$ on $\mathbb{R}^2$ is defined by

$$\mathcal{R}f(\theta, t) := \int_\ell f = \int_{-\infty}^{\infty} f(-s\sin\theta + t\cos\theta, s\cos\theta + t\sin\theta)ds.$$

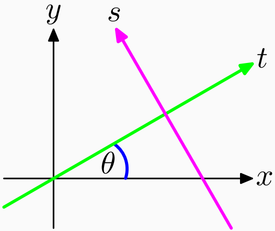
  This is considered to be the measurements of CT scanners for normal tissue. The FBP formula $f = (-\partial_x^2 - \partial_y^2)^{1/2} \circ \mathcal{R}^T \circ \mathcal{R}f$ is well-known.

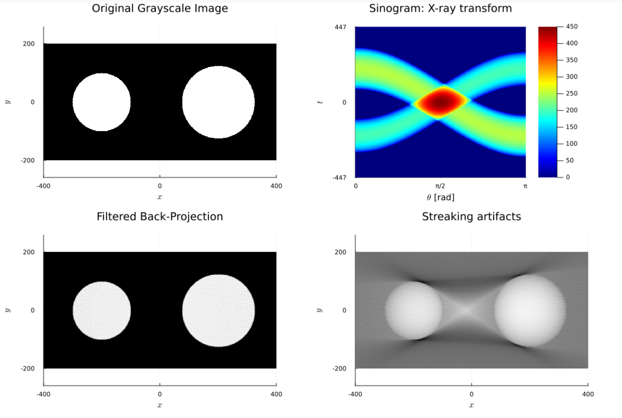
- We consider a model of human body f containing a metal region D such as dental implants, stents in blood vessels, and etc. We observe that the metal streaking artifacts caused by beam hardening effect in the energy level of X-ray. The main term is the filtered back-projection of nonlinear term

$$(-\partial_x^2 - \partial_y^2)^{1/2} \circ \mathcal{R}^T\left[(\mathcal{R}1_D)^2\right],$$

  This is a conormal distribution whose singular support is the streaking artifact.

Figures: metal streaking artifacts





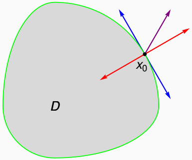
The main part of artifacts: $(-\partial_x^2 - \partial_y^2)^{1/2}\mathcal{R}^T\left[(\mathcal{R}1_D)^2\right]$.

Conormal distributions

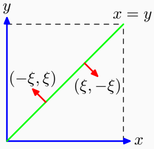
Definition 1 (Conormal distributions)

Let X be an N-dim manifold, and let Y be a closed submanifold of X. We say that $u \in \mathscr{D}'(X)$ is conormal with respect to Y of degree m if $L_1 \cdots L_\mu u \in {}^\infty H^{\mathrm{loc}}_{(-m-N/4)}(X)$ for all $\mu = 0, 1, 2, \ldots$ and all vector fields $L_1, \ldots, L_\mu$ tangential to Y. Denote by $I^m(N^*(Y))$ the set of all distributions on X conormal wrt Y of degree m. Note that $\mathrm{WF}(u) \subset N^*(Y) \setminus 0$.

- The characteristic function of a domain: $1_D \in I^{-1/2-n/4}(N^*(\partial D))$ for $D \subset \mathbb{R}^n$, which is a domain with smooth boundary.



- The Schwartz kernel of a PsDO: $\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,\xi) d\xi \in I^m(N^*(\Delta))$, $\Delta = \{(x,x)\}$ for $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$.

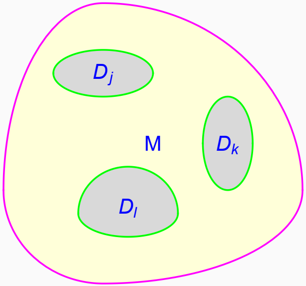


Assumption

- Suppose that (M, g) is a compact nontrapping simple Riemannian manifold with strictly convex smooth boundary.
- In addition we assume that $\dim(M) = 2$ or (M, g) is a space of constant curvature.

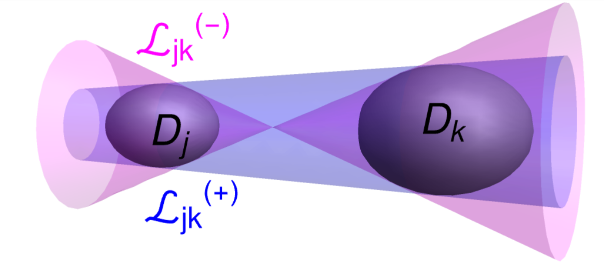
  This ensures that all the Jacobi fields are of the form scalar function $\times$ parallel transport.

- Suppose that the metal region $D \subset M^{\text{int}}$ is a disjoint union of D_j $(j = 1\ldots, J)$ which are simply connected, strictly convex and bounded with smooth boundaries ∂D_j.



A hypersurface $\mathscr{L}$ surrounding the metal region D

- For any j and $x \in \partial D_j$, denote by $\nu_j(x)$ the unit outer normal vector at x. Consider the tangent hyperplane $\exp_x \nu_j(x)^{\perp} \cap M^{\text{int}}$ at $x \in \partial D_j$.
- There are some common tangent hyperplanes of ∂D_j and ∂D_k for $j \neq k$. In this case there is common tangent geodesics in such hyperplanes. The union of all these geodesics forms a conical or cylindrical hypersurface denoted by $\mathscr{L}_{jk}^{(\pm)}$. Set $\mathscr{L} := \bigcup \left(\mathscr{L}_{jk}^{(+)} \cup \mathscr{L}_{jk}^{(-)} \right)$.



Main Theorem

The geodesic X-ray transform of a function f on M is defined by

$$\mathcal{X}f(\gamma_w) := \int_0^{\tau(w)} f(\gamma_w(s))\,ds, \quad \nabla_{\dot\gamma_w(s)}\dot\gamma_w(s) = 0, \quad \dot\gamma_w(0) = w \in \partial_- S(M),$$

where $\tau(w)$ is the exit time of γ_w. The nonlinear part of the CT image is supposed to be

$$f_{\mathrm{MA}} := f_{\mathrm{CT}} - f_{\mathrm{normal}} = \sum_{k=1}^{\infty} A_k Q\mathcal{X}^T[(\mathcal{X}1_D)^{2k}] \mod C^\infty(M^{\mathrm{int}}), \quad \{A_k\} \subset \mathbb{R},$$

where Q is a parametrix of $\mathcal{X}^T \circ \mathcal{X}$: $Q\mathcal{X}^T\mathcal{X} = Id$ modulo smoothing operators locally.

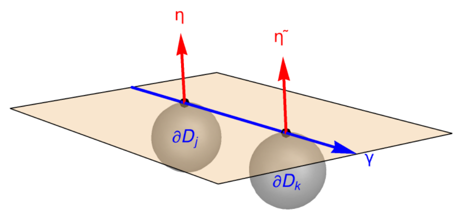
Theorem 2

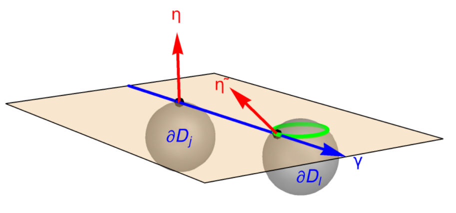
$f_{MA} \in I^{-3n/4-1/2}(N^*(\mathscr{L}))$ away from ∂D, and $\sigma_{prin}(Q\mathcal{X}^T[(\mathcal{X}1_D)^2]) \neq 0$.

- Park-Choi-Seo (2017) proved that $\mathrm{WF}(f_{\mathrm{MA}}) \subset N^*(\mathscr{L})$ for $M = \mathbb{R}^2$.
- Palacios-Uhlmann-Wang (2018) proved Theorem 2 for $M = \mathbb{R}^2$.
- C (2022) proved Theorem 2 for the d-plane transform on $\mathbb{R}^n$.

What does Theorem 2 say?

- If ∂D_j and ∂D_k have a common tangent hyperplane, then the conormal singularities propagate along the common tangent geodesic. See the left figure.
- Suppose $n \geqq 3$. If ∂D_j and ∂D_k have a common tangent geodesic, but the conormal directions at the tangent points are different, then the conormal singularities do not propagate along the common tangent geodesic. See the right figure.





How to prove Theorem 2

- The canonical relation $C_{\mathcal{X}}$ of $\mathcal{X}$: $(\xi,\eta) \in C_{\mathcal{X}}$ if $\exists v \in S(M^{\text{int}})$ such that

  $$\xi \in T^*_{F(v)}(\partial_- S(M)) \setminus \{0\}, \quad \eta \in T^*_{\pi_M(v)}(M^{\text{int}}) \setminus \{0\}, \quad DF|_v^T \xi = D\pi_M|_v^T \eta,$$

  where $F : S(M) \ni \dot\gamma_w(t) \mapsto w \in \partial_- S(M)$ for any t. See Holman-Uhlmann (2018).
- If $\xi, \tilde\xi \in T^*_w(\partial_- S(M))$, $\quad w = F(v) = F(\tilde v)$, $\quad \pi_M(v) \in \partial D_j$, $\quad \pi_M(\tilde v) \in \partial D_k$,

  $$DF|_v^T \xi = D\pi_M|_v^T \eta, \quad \eta \in N_v^*(\partial D_j) \setminus \{0\}, \quad DF|_{\tilde v}^T \tilde\xi = D\pi_M|_{\tilde v}^T \tilde\eta, \quad \tilde\eta \in N_{\tilde v}^*(\partial D_k) \setminus \{0\},$$

  then ξ and $\tilde\xi$ are linearly independent, and the interation of the nonlinear effect is $\mathrm{span}\langle \xi, \tilde\xi\rangle$ in $T^*_w(\partial_- S(M))$.
- If η is parallel to $\tilde\eta$, then $C_{\mathcal{X}}^T \circ \mathrm{span}\langle \xi, \tilde\xi\rangle$ becomes the parallel transport of $\mathbb{R}\eta$ along γ_w.

  Otherwise, $C_{\mathcal{X}}^T \circ \mathrm{span}\langle \xi, \tilde\xi\rangle = \mathbb{R}\eta \cup \mathbb{R}\tilde\eta$.

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