

Double fibration transforms with conjugate points

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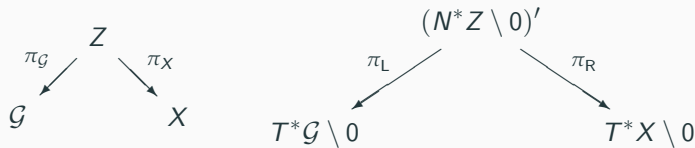
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Double fibration

Following Mazzucchelli-Salo-Tzou [2], we introduce double fibration transforms.

- Let \mathcal{G} and X be oriented smooth manifolds without boundaries. $N := \dim(\mathcal{G})$ and $n := \dim(X)$. Denote by $d\mathcal{G}$ and dX the orientation forms of \mathcal{G} and X respectively.
- Let Z be an oriented embedded submanifold of $\mathcal{G} \times X$, and let dZ be the orientation form.
- Assume that $N + n > \dim(Z) > N \geq n \geq 2$, and set $n' := \dim(Z) - N$ and $n'' := n - n'$. Then $\dim(Z) = N + n'$, $n = n' + n''$ and $n', n'' = 1, \dots, n - 1$.



- We assume that Z is a **double fibration**, that is, the natural projections $\pi_{\mathcal{G}} : Z \rightarrow \mathcal{G}$ and $\pi_X : Z \rightarrow X$ are submersions respectively.

Orientation forms on $G_z := \pi_X \circ \pi_{\mathcal{G}}^{-1}(z)$ and $H_x := \pi_{\mathcal{G}} \circ \pi_X^{-1}(x)$

- $G_z := \pi_X \circ \pi_{\mathcal{G}}^{-1}(z)$ becomes an n' -dim submanifold of X for any $z \in \mathcal{G}$, and $H_x := \pi_{\mathcal{G}} \circ \pi_X^{-1}(x)$ forms an $(N - n'')$ -dim submanifold of \mathcal{G} for any $x \in X$.
- Fix arbitrary $(z, x) \in Z$, and let $\{v_1, \dots, v_{n'}\}$ and $\{w_1, \dots, w_N\}$ be bases of $T_x X$ and $T_z \mathcal{G}$ respectively such that

$$T_{(z,x)}Z = \text{span}\langle v_1, \dots, v_{n'}, w_1, \dots, w_N \rangle = \text{span}\langle v_1, \dots, v_{n'}, w_1, \dots, w_{N-n''} \rangle.$$

The induced orientation forms dG_z on G_z and dH_x on H_x are given by

$$\begin{aligned} dG_z(d\pi_X(v_1), \dots, d\pi_X(v_{n'})) &:= dZ_{\pi_{\mathcal{G}}^{-1}(z)}(v_1, \dots, v_{n'}) \\ &= \frac{dZ(v_1, \dots, v_{n'}, w_1, \dots, w_N)}{d\mathcal{G}(d\pi_{\mathcal{G}}(w_1), \dots, d\pi_{\mathcal{G}}(w_N))}, \\ dH_x(d\pi_{\mathcal{G}}(w_1), \dots, d\pi_X(w_{N-n''})) &:= dZ_{\pi_X^{-1}(x)}(w_1, \dots, w_{N-n''}) \\ &= \frac{dZ(v_1, \dots, v_{n'}, w_1, \dots, w_{N-n''})}{dX(d\pi_X(v_1), \dots, d\pi_X(v_{n'}))}. \end{aligned}$$

Double fibration transform

Suppose that a weight function $\kappa(z, x) \in C^\infty(\mathcal{G} \times X)$ is nowhere vanishing. A double fibration transform \mathcal{R} associated with the double fibration Z is defined by

$$\mathcal{R}f(z) := \left(\int_{G_z} \kappa(z, x) \frac{f}{|dX|^{1/2}}(x) dG_z(x) \right) |d\mathcal{G}(z)|^{1/2}$$

for $f \in \mathcal{D}(X, \Omega_X^{1/2})$. The adjoint \mathcal{R}^* is given by

$$\mathcal{R}^*u(x) = \left(\int_{H_x} \overline{\kappa(z, x)} \frac{u}{|d\mathcal{G}|^{1/2}}(z) dH_x(z) \right) |dX(x)|^{1/2}$$

for $u \in \mathcal{D}(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2})$. Then we deduce that

$$\mathcal{R} : \mathcal{D}(X, \Omega_X^{1/2}) \rightarrow \mathcal{E}(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}), \quad \mathcal{R}^* : \mathcal{D}(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}) \rightarrow \mathcal{E}(X, \Omega_X^{1/2}),$$

are continuous linear mappings, and so are

$$\mathcal{R} : \mathcal{E}'(X, \Omega_X^{1/2}) \rightarrow \mathcal{D}'(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}), \quad \mathcal{R}^* : \mathcal{E}'(\mathcal{G}, \Omega_{\mathcal{G}}^{1/2}) \rightarrow \mathcal{D}'(X, \Omega_X^{1/2}).$$

More precisely \mathcal{R} and \mathcal{R}^* are elliptic Fourier integral operators.

Mapping properties of double fibration transforms

Theorem 1 ([2, Theorem 2.2] & [Hörmander IV, Theorem 25.2.2])

Suppose that Z is a double fibration with $\dim(Z) = N + n'$. Then \mathcal{R} and \mathcal{R}^ are elliptic Fourier integral operators of order $-(N + 2n' - n)/4$ with canonical relations $(N^*Z \setminus 0)'$ and $((N^*Z \setminus 0)^T)'$ respectively. More precisely*

$$\mathcal{R} \in \mathcal{I}^{-(N+2n'-n)/4}(\mathcal{G} \times X, N^*Z \setminus 0; \Omega_{\mathcal{G} \times X}^{1/2}),$$

$$\mathcal{R}^* \in \mathcal{I}^{-(N+2n'-n)/4}(X \times \mathcal{G}, (N^*Z \setminus 0)^T; \Omega_{X \times \mathcal{G}}^{1/2}),$$

where

$$\begin{aligned} N^*Z \setminus 0 &= \{ (z, A(z, x)\eta, x, \eta) : (z, x) \in Z, \eta \in N_x^*G_z \setminus \{0\} \} \\ &= \{ (z, \zeta, x, B(z, x)\zeta) : (z, x) \in Z, \zeta \in N_z^*H_x \setminus \{0\} \}, \end{aligned}$$

$A(z, x) \in \text{Hom}(N_x^*G_z, T_z^*\mathcal{G})$ and $B(z, x) \in \text{Hom}(N_z^*H_x, T_x^*X)$ smoothly depend on $(z, x) \in Z$ respectively.

For local coordinates $(z, x) = (z', z'', x', x'') \in \mathbb{R}^{N-n''} \times \mathbb{R}^{n''} \times \mathbb{R}^{n'} \times \mathbb{R}^{n''}$, There exist $\mathbb{R}^{n''}$ -valued functions $\phi(z, x')$ and $b(x, z')$ such that we have locally

$$Z = \{x'' = \phi(z, x')\} = \{z'' = b(x, z')\}.$$

Lemma 2 ([2, Lemmas 2.4, 2.5 and 2.6])

$$N_{(z,x)}^* Z = \left\{ \left(-\phi_z(z, x')^T \eta'', (-\phi_{x'}(z, x')^T \eta'', \eta'') \right) : \eta'' \in \mathbb{R}^{n''} \right\},$$

$$A(z, x) \begin{bmatrix} -\phi_{x'}(z, x')^T \\ I_{n''} \end{bmatrix} \eta'' = -\phi_z(z, x')^T \eta'', \quad \eta'' \in \mathbb{R}^{n''}.$$

Similar results hold for $b(x, z')$ and $B(z, x)$.

Variation fields and conjugate points

Fix arbitrary $(z, w) \in T\mathcal{G}$, and consider a curve in \mathcal{G} of the form

$$z(s) = z + sw + \mathcal{O}(s^2) \quad \text{near } s = 0.$$

Then $(G_{z(s)})$ is said to be a variation of G_z , and the variation field $J_w : G_z \rightarrow (N_x^* G_z)^*$ associated to $(G_{z(s)})$ is defined by

$$J_w(x) := A(z, x)^* w \simeq -\phi_z(z, x') w \in (N_x^* G_z)^* \simeq N_x G_z = T_x X / T_x G_z$$

for $x \in G_z$. Note that $T_z H_x = \text{Ker}(A(z, x)^*)$ since $\phi(z(s), x') = x''$ for $z(s) \in H_x$, and

$$A(z, x)^* \in \text{Hom}(T_z \mathcal{G}, (N_x^* G_z)^*) \simeq \text{Hom}(T_z \mathcal{G}, N_x G_z), \quad (z, x) \in Z,$$

For $z \in \mathcal{G}$ and $x, y \in G_z$, set

$$V_z(x, y) := \{J_w(x) : w \in T_z \mathcal{G}, J_w(y) = 0\}.$$

Note that $\dim(V_z(x, y)) \leq n''$ holds since $\text{rank}(A(z, x)^*) = n''$, and $\dim(V_z(x, y)) = \dim(V_z(y, x))$ holds for any $z \in \mathcal{G}$ and $x, y \in G_z$.

cf. If $x = \exp_y(tu)$, then $J_w(x) \simeq Y(t) := tD \exp_y(tu)w$.

Z-conjugate triplets

Definition 3

Suppose that Z is a double fibration and $N \geq 2n''$. Let $k = 1, \dots, n''$.

- **Z-conjugate triplet of degree k :** Let $z \in \mathcal{G}$ and let $x, y \in G_z$ with $x \neq y$. We say that $(z; x, y)$ is a Z -conjugate triplet of degree k if $\dim(V_z(x, y)) = n'' - k$.
- **Regular Z-conjugate triplet of degree k :** We say that a Z -conjugate triplet $(z; x, y)$ of degree k is regular if there exist a nbd U_x of x in X , a nbd U_y of y in X , and a nbd W_z of z in \mathcal{G} such that any Z -conjugate triplet $(z'; x', y') \in W_z \times U_x \times U_y$ is also of degree k . The set of all the regular Z -conjugate triplets of degree k is denoted by $C_{R,k}$.
- The set of all the Z -conjugate triplets which are not regular is denoted by C_S .

Lemma 4

Suppose that Z is a double fibration and $N \geq 2n''$. For any $k = 1, \dots, n''$, $C_{R,k}$ is an $(N + 2n' - 1)$ -dimensional embedded submanifold of $\mathcal{G} \times X \times X$.

Normal operators without conjugate points

$$\mathcal{R}^* \mathcal{R} f(x) = \left(\iint_{H_x \times G_z} \overline{\kappa(z, x)} \kappa(z, y) \frac{f}{|dX|^{1/2}}(y) dG_z(y) dH_x(z) \right) |dX(x)|^{1/2}, \quad x \in X.$$

Set $\mathcal{C} := (N^*Z \setminus 0)'$, which is the canonical relation of \mathcal{R} .

Theorem 5

Suppose that Z is a double fibration. In addition, we assume $N \geq 2n''$ and the following:

- $\pi_X : Z \rightarrow X$ is proper, and $\pi_X^{-1}(x)$ is connected for any $x \in X$.*
- There are no Z -conjugate triplets, and $D\pi_L$ is injective at all $(z, \zeta, x, \eta) \in \mathcal{C}$.*

Then $\mathcal{C}^T \circ \mathcal{C}$ is a clean intersection with excess $e = N - n$, and $\mathcal{R}^ \mathcal{R}$ is an elliptic pseudodifferential operator of order $-n'$ on X .*

Proof: Some lemmas in [2] and the assumptions guarantee the Bolker condition. □

Known results on geodesic X-ray transforms with conjugate points

- Let (M, g) be a compact Riemannian manifold with strictly convex boundary. Set $\gamma_w(t) := \exp_{\pi_M(w)}(tw)$ for $w \in \partial_- SM$. Consider the geodesic X-ray transform

$$\mathcal{X}f(w) := \left(\int_0^{\tau(w)} \kappa(\gamma_w(t), \dot{\gamma}_w(t)) \frac{f}{|dM|^{1/2}}(\gamma_w(t)) dt \right) |d\partial_- SM(w)|^{1/2}.$$

- Stefanov and Uhlmann (2012) [3]: If $v_0 = |v_0|\theta_0$ is a fold conjugate vector at p_0 , and v_0 is the only singularity of $\exp_{p_0}(v)$ on γ_{θ_0} near p_0 , then the localized normal operator is decomposed as

$$\mathcal{X}^* \chi \mathcal{X} = A + F \quad \text{near } p_0,$$

where A is a PsDOs of order -1 , and F is a *FIO* of order $-n/2$.

- Holman and Uhlmann (2018) [1]: If $C_S = \emptyset$, then

$$\mathcal{X}^* \mathcal{X} = A + \sum_{k=1}^{n-1} \sum_{\alpha=1}^{M_k} F_{k,\alpha},$$

where A is a PsDOs of order -1 , and F is a *FIO* of order $-(n-k+1)/2$.

Normal operators with conjugate points

Theorem 6

Suppose that Z is a double fibration, $C_S = \emptyset$, $N \geq (n-1) + n''$ and the following:

- π_X is proper, and $\pi_X^{-1}(x)$ is connected for any $x \in X$.
- If $\pi_L^{-1}((z, \zeta)) = \{(z, \zeta, x, \eta)\}$ for $(z, \zeta, x, \eta) \in \mathcal{C}$, then $D\pi_L|_{(z, \zeta, x, \eta)}$ is injective.

Then we have a decomposition of $\mathcal{R}^*\mathcal{R}$ of the form

$$\mathcal{R}^*\mathcal{R} = A + \sum_{k=1}^{n''} \sum_{\alpha \in \Lambda_k} F_{k,\alpha},$$

where A is an elliptic PsDO of order $-n'$ on X ,

$F_{k,\alpha}$ is a FIO in $\mathcal{I}^{-(n+1-k)/2}(X \times X, \mathcal{C}'_{A_{k,\alpha}}; \Omega_{X \times X}^{1/2})$ with some canonical relation of $\mathcal{C}_{F_{k,\alpha}}$, associated to the decomposition of connected components $C_{R,k} = \bigcup_{\alpha \in \Lambda_k} C_{R,k,\alpha}$.

Outline of the proof





- $\mathcal{R}^*\mathcal{R}$ is given by

$$\frac{\mathcal{R}^*\mathcal{R}f}{|dX|^{1/2}}(x) = \iint_{H_x \times G_z} \overline{\kappa(z, x)} \kappa(z, y) \frac{f}{|dX|^{1/2}}(y) dG_z(y) dH_x(z).$$

- Follow the idea of Holman and Uhlmann [1]: a partition of unity of $\mathcal{G} \times X \times X$.
- Set $C_\delta := \{(z; x, x) : z \in \mathcal{G}, x \in X\}$, which is related to the elliptic term.
- $C_{R,k,\alpha}$ are disjoint since $C_S = \emptyset$, so are $C_{R,k,\alpha}$ and C_δ .
Pick up disjoint nbds $U_{k,\alpha}$ and U_δ of $C_{R,k,\alpha}$ and C_δ respectively in $\mathcal{G} \times X \times X$.
- We can find an open set U_0 in $\mathcal{G} \times X \times X$ such that

$$U_0 \cup U_\delta \cup (\cup U_{k,\alpha}) = \mathcal{G} \times X \times X, \quad U_0 \cap \left(C_\delta \cup (\cup C_{R,k,\alpha}) \right) = \emptyset.$$

- Pick up a partition of unity subordinated to $\{U_0, U_\delta, U_{k,\alpha}\}$, and split the Schwartz kernel of $\mathcal{R}^*\mathcal{R}$. U_0 -part of $\mathcal{R}^*\mathcal{R}$ is a smoothing operator, and is absorbed in U_δ -part A .

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