Enumerative Combinatoric Algorithms

Contents

1 Enumerating vs. Counting

1.1 Example 1: Spanning Paths

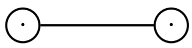
Given: n points in convex position

Question: How many straight line planar (crossing-free) spanning paths (all points are used) are there?

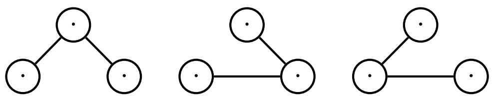
- $n = 1$

  □

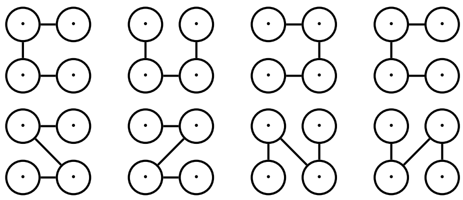
- $n = 2$

- $n = 3$: 4 paths

- $n = 4$: 8 paths

- $n = 5$:
  - more possibilites $\rightarrow$ we go towards counting rather than enumerating
  - look at possible types of paths, in CW and CCW direction for each of the five starting points. Divide by 2 for identical paths obtained with different starting points (since paths have 2 ends)

1.2 Example 2: Blokus

Definition

- consists of unit squares connected via edges (not just connected through vertices)
- n-polyomino has n unit squares
- Animal: does not contain holes

- free: rotation/mirroring is not taken into account
- fixed: rotated/mirrored polyominoes are regarded as being distinct

Question: How many n-polyominoes are there?

- $n = 1$:

  $\square$

  1

- $n = 2$:
  - $\square\square$
  - $\begin{array}{c}\square\\\square\end{array}$

  fixed: 2, free: 1

- $n = 3$:
  - $\square\ \square\ \square$
  - $\begin{array}{c}\square\\\square\\\square\end{array}$
  - $\begin{array}{cc}\square & \square\\\square & \end{array} \quad \times 4$

  fixed: 6, free: 2

- $n = 4$:
  - $\square\ \square\ \square\ \square \quad \times 2$
  - $\begin{array}{cc}\square & \\\square & \\\square & \square\end{array} \quad \times 8$
  - $\begin{array}{ccc}\square & \square & \square\\ & \square & \end{array} \quad \times 4$ (reflection has same effect as rotation)
  - $\begin{array}{cc} & \square\\\square & \square\\\square & \end{array} \quad \times 4$
  - $\begin{array}{cc}\square & \square\\\square & \square\end{array} \quad \times 1$

  fixed: 19, free: 5

Enumerate/Construct all polyominoes

- construct the $(n+1)$-ominoes from the n-ominoes. This works because we can show that from each $(n+1)$-ominoes you can remove a square and get a valid n-omino (idea: construct an arbitrary spanning tree, remove leaf)
- find and remove duplicates
- how many $(n+1)$-ominoes are generated?

maximum number: $2n+2$ or $4n-2|E|$
Why?
$\square$: 4 possibilities to add a square (at each edge)

$$\begin{array}{cccc} & & \square & \\ \square & \square & \square & \square \\ & \square & & \end{array}$$

$4n-2|E| \rightarrow 4n-2(n-1) = 2(n+2)$
How to find duplicates? compare every pair $\rightarrow O(k^2 n)$

1.2.1 Fingerprints

- maps an object (here: polyomino) to a *unique* representation, independent of translation, rotation, reflection
- sort the fingerprints (insertion sort)
- binary search

One possibility for polyominoes: use vector representations on a coordinate system

Examples:

- $\begin{array}{cccc} 3 & & & \square \\ 2 & & & \square \\ 1 & \square & \square & \square \\ & 1 & 2 & 3 \end{array}$ has vector representation $< (1,1),(2,1),(3,1),(3,2),(3,3) >$

  $\begin{array}{cccc} 3 & \square & & \\ 2 & \square & & \\ 1 & \square & \square & \square \\ & 1 & 2 & 3 \end{array}$ has vector representation $< (1,1),(2,1),(3,1),(1,2),(1,3) >$

  This is lexicographically smaller than the first one

- $\begin{array}{cccc} 3 & \square & & \\ 2 & \square & \square & \\ 1 & & \square & \square \\ & 1 & 2 & 3 \end{array}$ $\begin{array}{cccc} 3 & & \square & \square \\ 2 & \square & \square & \\ 1 & \square & & \\ & 1 & 2 & 3 \end{array}$ $< (1,1),(1,2),(2,2),(2,3),(3,3) >$

- $\begin{array}{cccc} 3 & & & \\ 2 & \square & \square & \square \\ 1 & \square & & \square \\ & 1 & 2 & 3 \end{array}$ $\quad$ $\begin{array}{cccc} 3 & \square & \square & \\ 2 & \square & & \\ 1 & \square & \square & \\ & 1 & 2 & 3 \end{array}$ $\quad < (1,1), (3,1), (1,2), (1,3), (2,3) >$

Generally:

1. generate all 8 transformations (4 rotations, 2 reflections)
2. take the lexicographically smallest one $\rightarrow O(1) \cdot O(n)$ for an n-omino

$f'(n)$ fingerprints generated in total $\rightarrow f'(n)\,(O(n) + \log(f'(n) \cdot O(n))$

$\lim\limits_{n\to\infty} \frac{f(n+1)}{f(n)} = \lambda$, $3.98 < \lambda < 4.65$

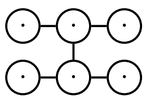
$f(n)$ is known up to $n = 56$.

1.3 Example Spanning Trees on Ladders

with n rangs

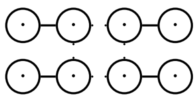
Question: Number of spanning trees on a ladder with n rangs

- $n = 1$

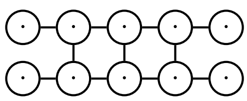
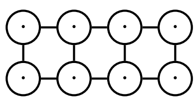
  # = 1

- $n = 2$

  # = 4

- $n = 3$

   $\Leftrightarrow$ 

  # = 15

Answer: Counting spanning trees on a ladder (with n rangs) $\Leftrightarrow$ counting spanning trees on a $2n$ grid
$n \to n+1$

1. The 2 rightmost vertices for n share a connected component $\to A(n)$
2. The 2 rightmost vertices for n are in different connected components $\to B(n)$

Consider case 1
$A(n+1) = 3A(n) + 1B(n)$
Consider case 2

$$A(1) = \frac{1}{2\sqrt{3}}\left((2+\sqrt{3})^1 - (2-\sqrt{3})^1\right) = \frac{2\sqrt{3}}{2\sqrt{3}}$$

$$A(2) = \cdots = \frac{8\sqrt{3}}{2\sqrt{3}} = 4$$

$$A(n) = 4A(n-1) - A(n-2) \tag{1}$$

$$= \frac{4}{2\sqrt{3}}\left((2+\sqrt{3})^{n-1} - (2-\sqrt{3})^{n-1}\right) - \frac{1}{2\sqrt{3}}\left((2+\sqrt{3})^{n-2} - (2-\sqrt{3})^{n-2}\right) \tag{2}$$

2 Inclusion – Exclusion Principle

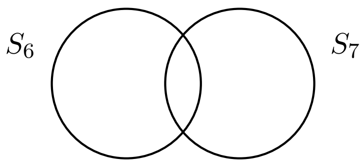
2.1 Example: Multipliers

$S = \{1, \ldots, 100\}$

Question: How many numbers in S are multipliers of 6 or 7?

$|S_6| = \lfloor \frac{100}{6} \rfloor$ = 16, $|S_6| = 14$

$|S_{6,7}| \neq |S_6| + |S_6|$:



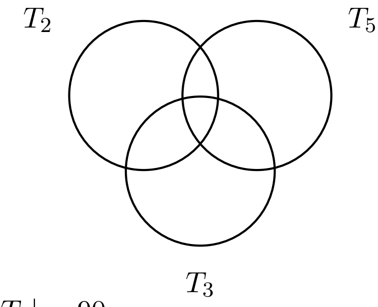
$|S_{6,7}| = |S_6| + |S_6| - |S_{6\&7}|$

2.2 Example: Relatively Prime Numbers

$S = \{1, \ldots, 180\}$

Question: How many numbers in S are relatively prime to 180?

Idea: Count all numbers *not* relatively prime to 180 $\rightarrow T$.
$|T| = ?$



$|T_2| = 90$
$|T_2| = 60$
$|T_2| = 36$

$|T| = |T_2| + |T_3| + |T_5| - (|T_2 \cap T_3| + |T_2 \cap T_5| + |T_3 \cap T_5|) + |T_2 \cap T_3 \cap T_5|$
$|T| = 90 + 60 + 36 - (30 + 18 + 12) + 6 = 132$

$|S| - |T| = 48$

In general:

$$\mathcal{A} = \{A_i\}_{i=1}^{n}, I = \{1, \dots, n\}$$

$$|_{i \in I} A_i| = \sum_{i=1}^{n} \left((-1)^{i-1} \sum_{J \subseteq I, |J| = i} |\bigcap_{j \in J} A_j| \right)$$

Proof: $x \in A_1 \cup \dots \cup A_n$, x is contained in m sets A_i.

$$|A_1 \cup \dots \cup A_n| = \sum (.) - \sum (. \cap .) + \sum (. \cap . \cap .) - \sum (. \cap . \cap . \cap .) \dots$$

1. for (single) sets *not* containing $x \rightarrow +0$
2. if in any $A_i \cap \dots \cap A_j$ at least one of the sets does *not* contain $x \rightarrow \pm 0$
3. $\rightarrow$ only consider intersections of sets where all sets contain x
   $\binom{m}{k}$ intersections of k sets, all sets containing $x \rightarrow \pm 1$

$$\Rightarrow + \binom{m}{1} - \binom{m}{2} + \dots + (-1)^{m-1} \binom{m}{m} = 1$$

$$\binom{m}{0} + \binom{m}{1} - \binom{m}{2} + \cdots + (-1)^m \binom{m}{m} = 0$$

Pascal's triangle

$$\begin{tikzcd}[column sep=small]
 & 1 \arrow[dl] \arrow[dr] & & 3 \arrow[dl] \arrow[dr] & & 3 \arrow[dl] \arrow[dr] & & 1 \arrow[dl] \arrow[dr] & \\
1 & & 4 & & 6 & & 4 & & 1
\end{tikzcd}$$

2.3 Example: Number of Words

Given: Language L with alphabet $\Sigma = \{A, B, C\}$ and every word of L consists of 10 characters.

Question: How many words of L contain each character at least once?

$$|G| = |L| - |X| = 3^{10} - (3 \cdot 2^{10} - 3 + 0) = 55980$$

$$|S_A| = |S_B| = |S_C| = 2^{10}$$
$$|S_A \cap S_B| = |S_A \cap S_C| = |S_B \cap S_C| = 1$$
$$|S_A \cap S_B \cap S_C| = 0$$
$$|X| = |S_A| + |S_B| + |S_C| - |S_A \cap S_B| - |S_A \cap S_C| - |S_B \cap S_C| + |S_A \cap S_B \cap S_C|$$

3 The Pigeon Hole Principle

If n elements are distributed to k sets, then there exists at least one set containing at least $\lceil \frac{n}{k} \rceil$ elements.

3.1 Example: People in Vienna

Claim: In Vienna there exist at least two persons with the exact same number of hairs on their head.

- human: at most $\approx 500,000$ hairs
- Vienna: $\geq 1.7 \cdot 10^6$ citizens

Idea: Take 500,000 people. If 2 of them have the same amount of hair, then we are done. If not, add one more person, whose amount of hair must then already be present.

3.2 Example: Division of odd numbers

Given: $1, 3, 7, 15, 31, \ldots ; a_i := 2^i - 1, i \geq 1$ and $q > 0$ arbitrary odd number.

Claim: At least one of the a_i's is (integer) divisible by q.

Proof: Consider all a_i, $1 \leq i \leq q$

- If one of these a_i is divisible by q, then we are done
- otherwise: $a_i = d_i \cdot q + r_i$, $0 < r_i < q$; $q \geq n > m > 0$

  $\Rightarrow (q-1)$ remainders $\overset{\Rightarrow}{\text{p.h.p.}}$ $\exists r_m, r_n$ such that $r_m = r_n$

  Then $(a_n - a_m) = (d_n - d_m)q + \underbrace{(r_n - r_m)}_{0}$

  $(2^n - 1) - (2^m - 1) = 2^n - 2^m = 2^m(\underbrace{2^{n-m} - 1}_{=a_{n-m}})$.

  2^m is even and thus not divisible by q.

In general: if $A_1, \dots A_k$ are finite (pairwise disjoint) sets *and* $|A_1 \cup \dots \cup A_k| > kr$, then $\exists i$ such that $|A_i| > r$.

Proof: Assume for the sake of contradiction that all $|A_i| \leq r$

$k \cdot r \geq \sum_{i=1}^{k} |A_i| \geq |A_1 \cup \dots \cup A_k| > k \cdot r$

3.3 Example: Lossless Data Compression

... cannot guarantee compression for all input data!

- file $\rightarrow$ string of bits
- assume that each file is transformed into a *distinct* file that is not larger
- let F be the file with the least number of bits (M) that compresses $\rightarrow N$ bits
- $N < M : 2^N + 1$ – cannot be all distinct

4 Reverse Search

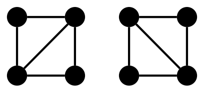
- Avis, Fukuda 1992
- enumerating data/objects/elements with almost no structure
- algorithmic
- count all objects *exactly* once, no overcounting

4.1 Example: Triangulations

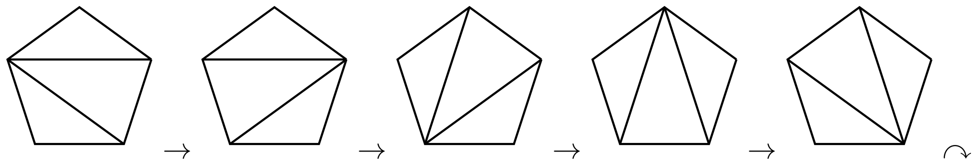
Given: set S of labelled points in convex position

Question: # of triangulations on S.

- $n = 4$



- $n = 5$



Flip on triangulations: 4 points, remove the diagonal and add the other diagonal

4.2 Basic Idea

- abstract graph $G(V, E)$
- vertices V: elements to be enumerated, $|V|$ is very large
- edges $e \in E : e = v_1, v_2; v_1, v_2 \in V$
- $\Rightarrow G$ is an undirected graph
- G must be connected
- task: enumerate all vertices of G
- idea: breadth-first or depth-first search: $O(|V| + |E|)$ which is ok, but too much space necessary

4.3 Example: Poker Chips Stacking

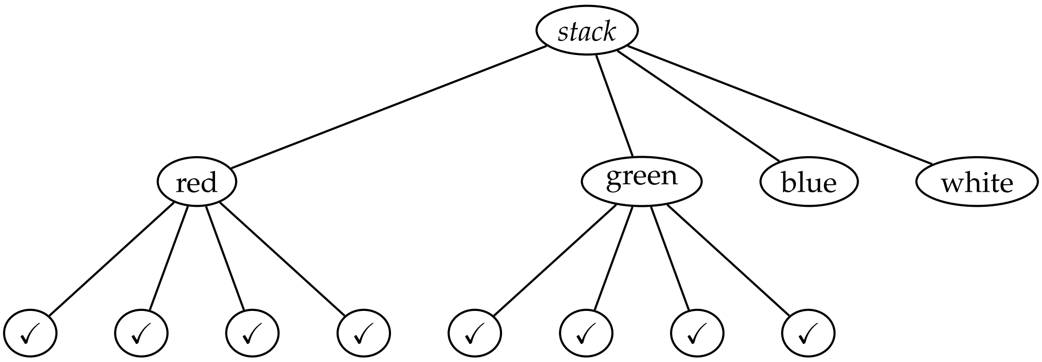
Given: four colours (red, green, blue, white) of poker chips

Task: build stacks of height $\leq$ 30 such that at most 3 chips of equal colour appear directly after another

- v_Φ: empty stack
- neighbours:

- 'successors': add a chip on top of the stack
- 'predecessors': remove the topmost chip from the stack

- depth-first search



- store current stack.
  Nodes that have already been visited can be deduced from the stack composition.

Requirements:

- unique root
- successors and predecessors in arbitrary but fixed order
  neighbour relation $\gamma(v, k) \in \Gamma(v)$ (kth neighbour)
- a unique predecessor function $f(v), f : v \to v$ in $G(V, E)$
  1. $\exists v_0 \in V : f(v_0) = v_0$: root
  2. $\forall v \in V \backslash \{v_0\} : f(v) \in \Gamma(v)$: exactly one root
  3. $\forall v \in V \backslash \{v_0\} \forall x \in \mathbb{N} : f^x(v) \neq v$: cycle-free ($f^x = f(f \ldots f(v) \ldots)$) x times
     $\to$ reaching the root in finite time $f^{|V|}(v) = v_0$

4.4 Example: Tic Tac Toe

Given:

Task: Enumerate all valid game positions (not considering symmetries)

- V: game positions
- labelled board cells
- neighbours $\gamma(v, k)$: legally add k-th mark or remove one
- predecessor function: empty board as root, otherwise remove highest legal mark
- when iterating over the possible positions, check each node's predecessor to see if it has been visited before

4.5 Example: Connect Four (4 Gewinnt)

⋮	⋮	⋮	⋮	⋮	⋮
			X		
			O		

too complex for reverse search, NP-hard

4.6 Algorithm

Complexity Analysis

4.7 Example: Triangulations