Frequency Division Multiplexing Using the Fast Fourier Transform

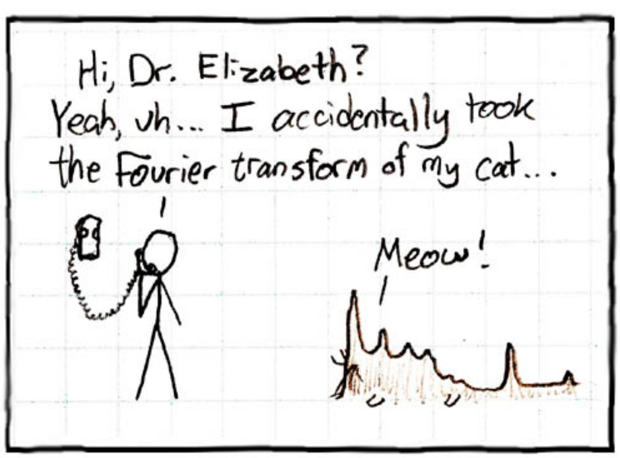
Nicholas Vanderweit
Fiona Pigott
Krishan Patel

APPM 4350 – Fourier Analysis
Harvey Segur

December 10, 2012

Abstract

The discrete Fourier transform, or DFT, is undoubtedly one of the most important and widely used operators in signal processing. Because it is able to represent many common kinds of signals without loss of information, and because there are many efficient algorithms (called fast Fourier transforms, or FFTs) to implement it on conventional hardware, the DFT has become an invaluable tool for analysing and manipulating the frequency information in digital signals. In this paper, we will investigate Frequency-Division Multiplexing, one application which focuses on sending several signals concurrently in time through a fixed-bandwidth channel by translating each of the constituent frequencies into its own allocation within the channel. In the process, we will investigate the relationship between the DFT, the DTFT (discrete-time Fourier transform), and the Fourier transform on an infinite continuous domain.



source: `http://xkcd.com/26/`

1 Mathematical Formulation of Digital Signal Processing

1.1 Nyquist-Shannon Sampling Theorem

One of the most important results in signal processing, the Nyquist-Shannon Sampling Theorem (often simply the Nyquist Sampling Theorem) helps to develop the discrete Fourier Transform. The result of this theorem is that any periodic signal can be represented by a discretely sampled set of points, with samping frequency greater than the largest frequency present in the signal.

1.1.1 Proof

The following proof of the Sampling Theorem is an adaptation of Shannon's original proof, found in *Boundary Value Problems and Partial Differential Equations* by David L Powers.

Assume the function $f(t)$ is some infinite function in the time domain, but note that this proof holds as well for a spacial $f(x)$.

A signal $f(t)$ is called *band limited* if its Fourier transform is identically zero except in a finite interval.

$$\hat{f}(\omega) \equiv 0 \quad \forall\, \omega \text{ s.t. } |\omega| > \Omega \text{ ``cutoff frequency''}$$

i.e., the signal is composed of a limited number of frequencies. Practically, most signals that are not completely random are band limited.

If $f(t)$ is band limited:

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega \; = \; \int_{-\Omega}^{\Omega} \hat{f}(\omega) e^{i\omega t}\, d\omega \tag{1}$$

Now, since $\hat{f}(\omega)$ is defined on a finite, rather than an infinite, interval, $\hat{f}(\omega)$ can be expressed as as a summation rather than an integral:

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} \hat{c}_n e^{\frac{in\pi\omega}{\Omega}}$$

Where the complex coefficients $\hat{c}_n$ are the Fourier transform of $\hat{f}(\omega)$:

$$\hat{c}_n = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \hat{f}(\bar{\omega}) e^{\frac{-in\pi\bar{\omega}}{\Omega}}\, d\bar{\omega}$$

Looking at $\hat{c}_n$, and replacing $\frac{-n\pi}{\Omega} = \bar{t}$ yield the definition of the Fourier transform of $f(\bar{t})$:

$$\hat{c}_n = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \hat{f}(\bar{\omega}) e^{i\bar{\omega}\bar{t}}\, d\bar{\omega} = \frac{1}{2\Omega} f(\bar{t}) = \frac{1}{2\Omega} f(\frac{-n\pi}{\Omega})$$

$$\therefore\; \hat{f}(\omega) = \sum_{n=-\infty}^{\infty} \hat{c}_n e^{\frac{in\pi\omega}{\Omega}} = \sum_{n=-\infty}^{\infty} \frac{1}{2\Omega} f(\frac{-n\pi}{\Omega}) e^{\frac{in\pi\omega}{\Omega}} = \frac{1}{2\Omega} \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{\Omega}) e^{-i\omega\frac{n\pi}{\Omega}} \tag{2}$$

Therefore, the Fourier transform can be completely determined by taking discrete samples of the function $f(t)$ at only times t = $\frac{n\pi}{\Omega}$. Because any function with a Fourier transform can always be recovered exactly from its Fourier transform, this implies that the band limited function $f(t)$ can be completely determined by the same discrete set of samples.

Use (**??**) and (**??**) to reconstruct $f(t)$:

$$f(t) = \int_{-\Omega}^{\Omega} \frac{1}{2\Omega} \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{\Omega}) e^{-i\omega\frac{n\pi}{\Omega}} e^{i\omega t}\, d\omega = \frac{1}{2\Omega} \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{\Omega}) \int_{-\Omega}^{\Omega} e^{-i\omega\frac{n\pi}{\Omega}} e^{i\omega t}\, d\omega \tag{3}$$

And since the Fourier transform $\hat{f}(\omega)$ is periodic with a period of 2π, the sampling frequency n is given by:

$$\frac{n\pi}{\Omega} = 2\pi \Rightarrow n = 2\Omega$$

Thus, the result of the Nyquist-Shannon sampling theorem: any bandwidth limited function with the cutoff frequency Ω in the time (or spacial) domain can be reconstructed exactly from samples taken at a rate of 2Ω.

1.1.2 Implications

The Nyquist-Shannon Sampling Theorem is important because it demonstrates how to use digital processing (which is necessarily discrete) to analyse signals.

In practice, many signals are only *very nearly* band limited, meaning that the vast majority of frequencies are under a certain Ω, but some almost random noise causes some small frequency components ω with $|\omega| > \Omega$. The signal might also be artificially band limited because the sampling frequency f_s is determined by factors other than the signal's band (such as limited storage), and so the frequencies ω in the signal are represented faithfully only for $|\omega| < \frac{f_s}{2}$. Because samples at $f_s > 2\Omega$ only correspond to unique periodic components with frequency $< \frac{f_s}{2}$, samples from components with frequency $> \frac{f_s}{2}$ are aliased to a frequency components with the same sample with frequency $< \frac{f_s}{2}$. In both cases, this aliasing results in noise and distortion of the sound; aliasing is avoided only by understanding the frequency components of the time dependent signal. When small sampling frequency cannot be avoided, the added noise from aliasing can be eliminated by artificially bandwidth limiting the signal–taking the Fourier transform and setting it equal to zero outside of some band $|\omega| > \frac{f_s}{2}$. This way, information is lost, but the remaining signal is clearer.

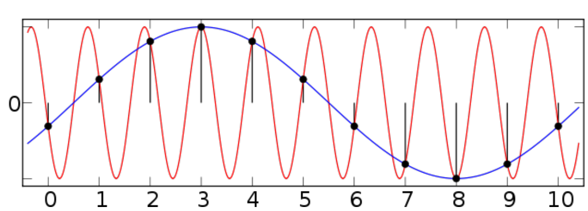


Figure 1: A graph showing the effects of aliasing. If the red sine function has a frequency $> \frac{f_s}{2}$, its samples can be mapped to a function (the blue curve) with a frequency $< \frac{f_s}{2}$. The blue sine function is *aliased* to the f_s frequency samples of the red sine function.

1.2 Discrete Fourier Transform

The Fourier transform of a continuous signal gives a continuous signal. Above, the proof of the Nyquist-Shannon Sampling Theorem shows how a Fourier transform of a discrete signal (a discrete time Fourier transform) can give an exact, continuous signal. Finally, the discrete Fourier transform takes a discrete signal and maps that to a discrete transform, which, at a high enough sampling frequency, is a set of samples from the continuous transform.

1.2.1 Formulation

The Nyquist-Shannon Sampling Theorem demonstrates that any band limited signal can be expressed as a discrete time signal. Furthermore, any practical signal with only periodic components is periodic on some interval. Call that interval T_0.

Say that for some sampling frequency f_s, $T_0 = N\frac{1}{f_s}$. Therefore, one period T_0 can be divided into N intervals with length $\frac{1}{f_s}$.

We know that the signal $f(t)$ can be represented as the infinite sum of its frequency components, with coefficients c_n.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega t}$$

$$c_n = \frac{1}{T_0}\int_0^{T_0} f(t)e^{i\omega t}dt$$

Because we defined T_0 as the period of the function, $c_n(\omega)$ should have a period of T_0, $e^{i\omega T_0} = e^{i\omega n T_0}$. Since e^{ix} has a period of 2π, ω must be equal to $\frac{2\pi n}{T_0}$.

$$\therefore c_n = \frac{1}{T_0}\int_0^{T_0} f(t)e^{i\frac{2\pi}{T_0}t}\,dt$$

Now, using the Left-Hand Riemann sum to approximate c_n with N intervals of $\frac{1}{f_s}$:

$$c_n = \frac{1}{T_0}\frac{1}{f_s}\sum_{k=0}^{N-1} f(k\frac{1}{f_s})e^{i\frac{2\pi n}{T_0}k\frac{1}{f_s}}$$

Which simplifies to:

$$c_n = \frac{1}{N}\sum_{k=0}^{N-1} f(k\frac{1}{f_s})e^{i\frac{2\pi n}{N}k}$$

Written where $f(k\frac{1}{f_s})$ = the indexed sample f_k, spaced $\frac{1}{f_s}$ apart, this Riemann sum yields the common form of the Discrete Fourier Transform (DFT):

$$c_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k\, e^{i\frac{2\pi n}{N}k}$$

And from the discussion of Nyquist sampling frequency above, we know that as long as $f_s = 2\Omega$, the signal's DFT will be non-aliased representation of its continuous Fourier transform.

This proof is founded on ideas from Li Tan's book *Digital Signal Processing* and *Applied Partial Differential Equations* by Richard Haberman.

1.2.2 Properties of the Discrete Fourier Transform

The Fourier transform lends itself well to signal processing because it provides a radically different way to view the signal while preserving all of its information. Many signals which are difficult to analyse in the time domain can be transformed into the frequency domain where they become regular. There are two important properties of the discrete Fourier transform that are integral to the digital analysis of real signals: its periodicity and its *reality condition.*

Periodicity

As to the periodicity of the function, we enforced a period of T_0 on the discrete Fourier transform in the formulation. This period results directly from the sampling frequency f_s with $T_0 = \frac{N}{f_s}$ (which is controlled during implementation). That way, when we sample a function (say, a voice) to analyse it, we know what the period of the transform and of resulting time domain signal should be, and can select only one period to analyse.

An interesting way of showing the periodic properties of the discrete transform is to recognize the discrete transform as a special kind of Dirac delta.

The Dirac delta is a *sifting function* defined as the derivative of the unit step function. By definition, a sifting function samples some other function $f(t)$ at some arbitrary point τ. The the inner product of the Dirac delta (defined below) and $f(t)$ is equal to $f(\tau)$:

$$\int_{-\infty}^{\infty} \delta(t-\tau) f(t)\, dt = f(\tau)$$

With that property, with n Dirac deltas, $f(t)$ would be sampled n times with samples spaced τ units apart:

$$\int_{-\infty}^{\infty} \delta(t-n\tau) f(t)\, dt = f(n\tau)$$

Notice that computing the discrete Fourier transform of a continuous signal is exactly that– taking $n = \frac{1}{f_s}$ Dirac deltas, computing their inner products to sample a function $f(t)$ n times, then taking the transform of those samples. For $|n| \to \infty$, the combination of all of these Dirac deltas is called an infinite Dirac comb, and the transform of those samples is the transform of the Dirac comb over some interval.

This brings us to note a fundamental property of the Dirac comb, which we will use in our analysis: the frequency of samples that make up the comb, f_s, is the inverse frequency of the transform of those samples. Another way of stating that property is that for some spacing τ between sample points in the time domain, the spacing in the frequency domain is $\frac{1}{\tau}$. A proof and examples of this can be found in the Allen and Mills, as well as the Edinburgh physics department references.

The periodicity of function and transform stated above, on which we rely to analyze signals in our implementation of Frequency Division Multiplexing, results directly from the above property of the Dirac comb.

Reality Condition

The second property that we make use of in manipulating signals, the reality condition, holds for all Fourier transforms, and states that for all Fourier transforms of all real-valued functions:

$$\hat{f}(\omega) = \overline{\hat{f}(-\omega)}$$

Which can be easily demonstrated. Recall Euler's identity: $e^{ix} = \cos(x) + i\,\sin(x)$ and make note of the fact that $\cos(x)$ is even: $\cos(x) = \cos(-x)$, and that $\sin(x)$ is odd: $\sin(-x) = -\sin(x)$. Then we invoke the Fourier transform definition:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = \int_{-\infty}^{\infty} f(t)(\cos(\omega t) - i\,\sin(\omega t))\,dt = \int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt - i\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt$$

$$\hat{f}(-\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt = \int_{-\infty}^{\infty} f(t)(\cos(\omega t) + i\,\sin(\omega t))\,dt = \int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt + i\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt$$

$$\therefore\ \hat{f}(\omega) = \overline{\hat{f}(-\omega)}$$

1.3 Fast Fourier Transform

As it is formulated, the discrete Fourier transform has a time complexity of $O(n^2)$, because each of the n output samples requires a summation over all n input samples. This is similar to the problem of sorting: a naïve solution requires making a computation using every element of the input, for each element of the input.

Like the sorting problem, the DFT has considerably more efficient "divide and conquer" algorithms like the Cooley-Tukey algorithm, which reduce the time complexity to $O(n \log n)$ by recursively breaking the DFT into subproblems. The Cooley-Tukey FFT over N elements factors N to express the DFT as a combination of several DFTs on smaller inputs. This algorithmic speedup greatly enhances the feasibility of using the DFT on large data sets.

2 Frequency Division Multiplexing

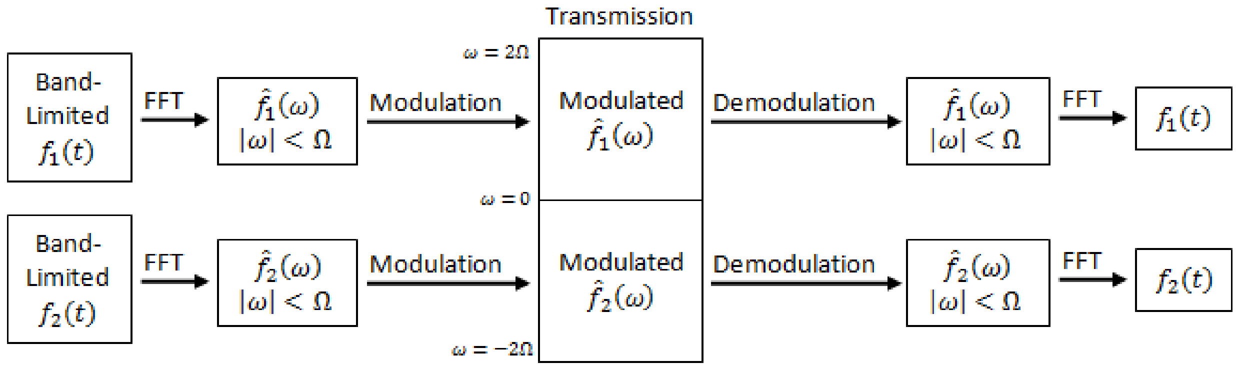


Figure 2: An illustration of the concept of FDM

2.1 Our Implementation

To simulate transmitting multiple signals over a channel of a fixed bandwidth, we created a simulation using GNU Octave (a free and open source MATLAB® clone) which reads in two single-channel WAV files sampled at some frequency f_s and performs the necessary modulation to place them in adjacent band allocations, each $f_s/2$ Hz wide.

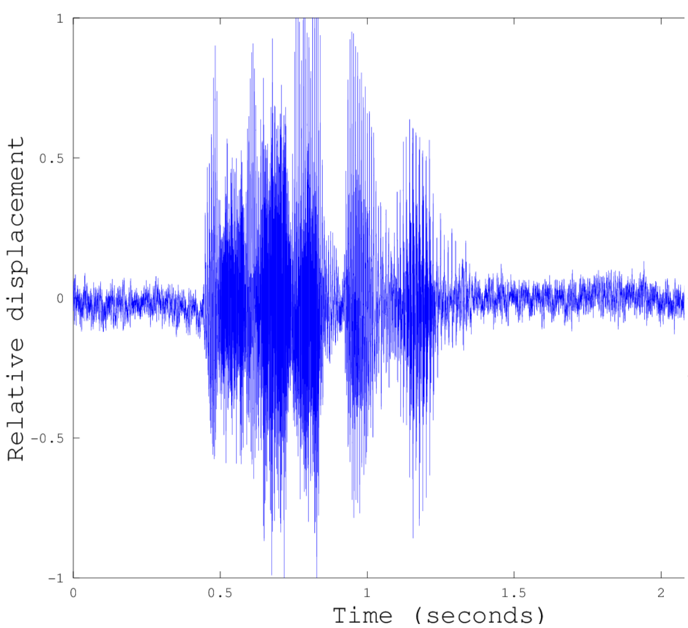


Figure 3: The first input sample in the time domain.

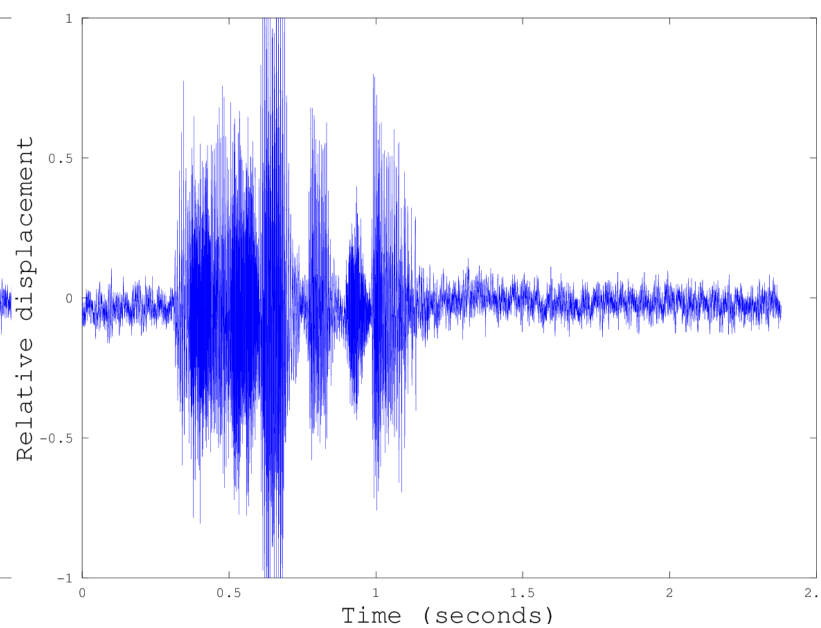


Figure 4: The second input sample in the time domain.

To accomplish this, we first take the Fourier transform of each signal. This gives us a complex-valued vector of the same length. Since the signal is real, the aforementioned reality condition holds and we can safely discard the negative frequencies without losing any information.

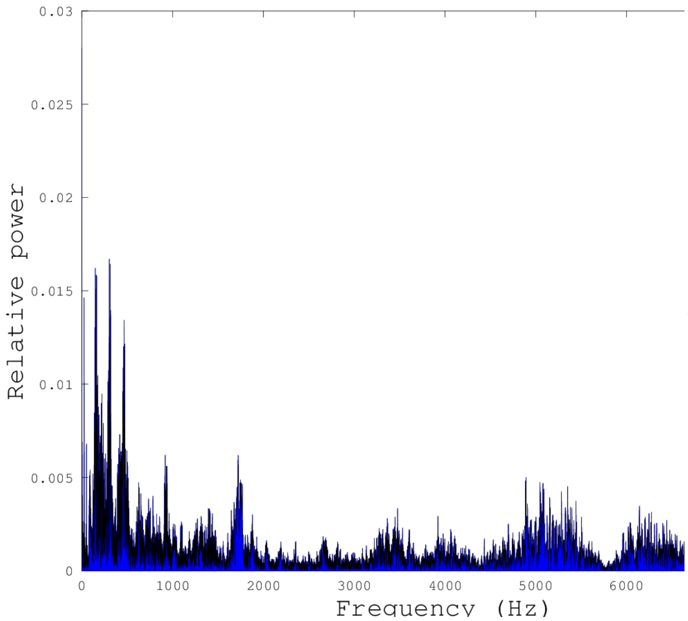


Figure 5: The first input sample in the frequency domain.

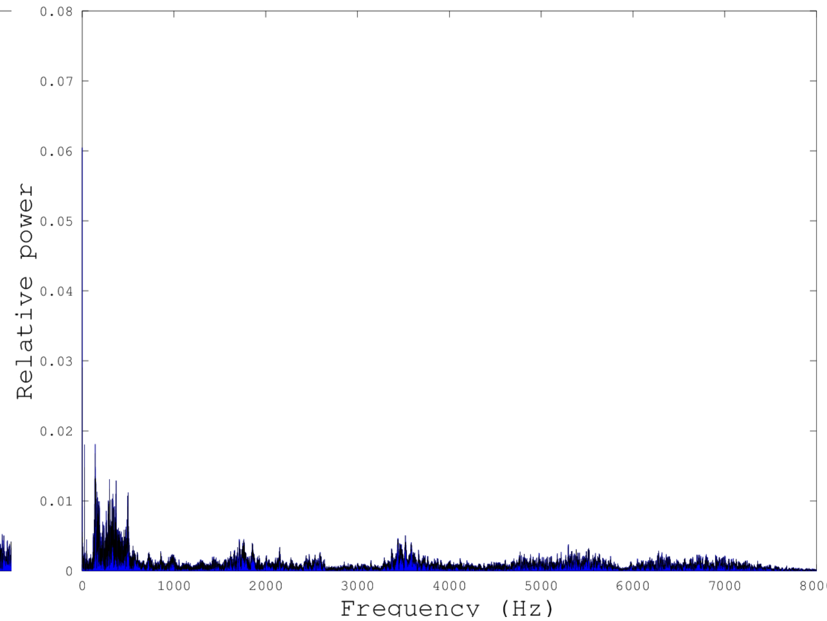


Figure 6: The second input sample in the frequency domain.

At this point, there is a simple way to combine the remaining frequencies: we simply concatenate them. By concatenating the signals, we move the translated signal into the higher portion of the frequency band. This is called modulating the signal.

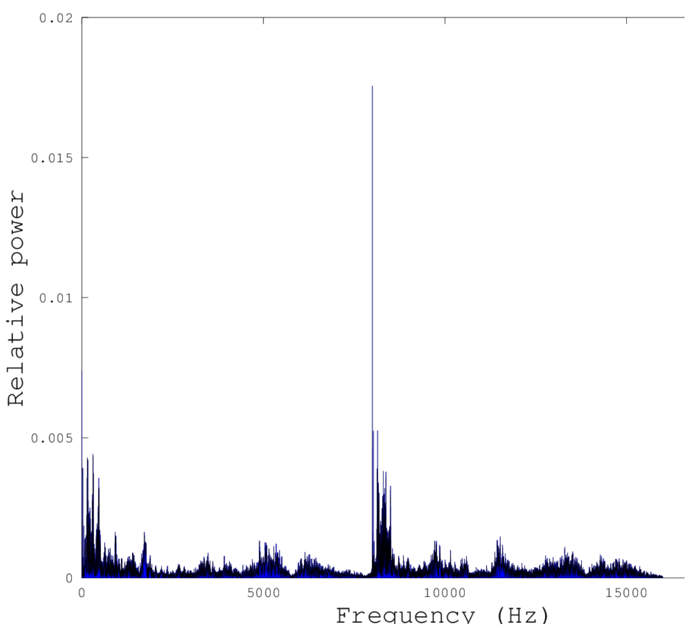


Figure 7: The modulated output in frequency

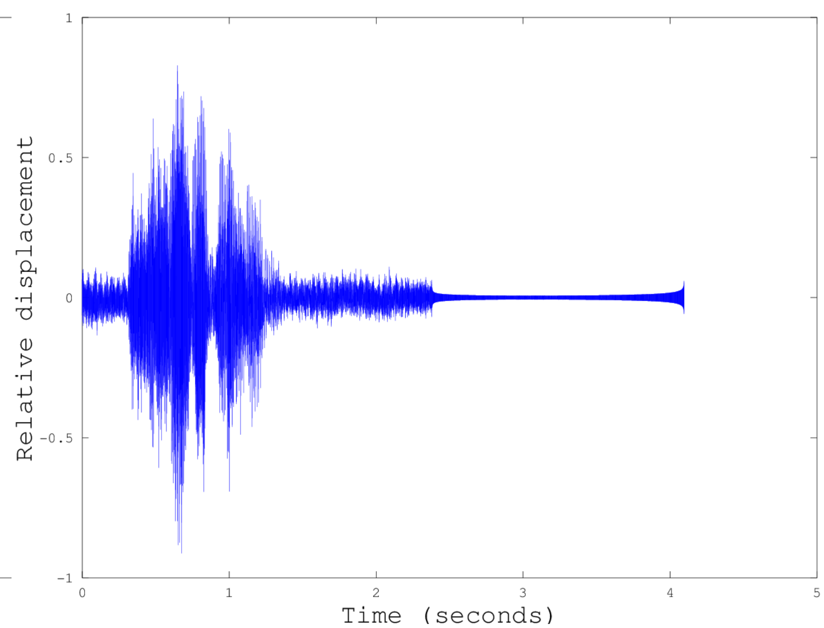


Figure 8: The modulated output in time

Finally, to satisfy the reality condition so that the inverse FFT of these new frequencies is real, we generate the negative frequencies by appending the reverse of the complex conjugate of the positive frequencies. Taking the inverse FFT, we now have a periodic signal with all of the frequency information of each input signal, and which is twice as long. Hence it must be sent at twice the input sampling frequency in order to maintain the same time scale. This is equivalent to saying that it must be sent over a channel which is wide enough to accommodate both signals, by the Nyquist theorem.

To demodulate these signals, the Octave code takes the FFT of the modulated signal. At this point, it is simply a process of stripping away the redundant frequency information and splitting the frequency vector in half to reveal the information contained in each of the individual signals. We then replicate the negative frequencies as before, and use the inverse FFT to recover the input signals.

2.2 A Broader Perspective

Frequency division multiplexing (FDM) is a method of sectioning the large frequency band of a medium into non-overlapping frequency sub-bands, each of which contains a separate signal.

One could fathom a scheme for multiplexing in which the samples in time from each signal are sent one after another. In a certain sense, this sequencing is analogous to what is done in our implementation. The difference is that, instead of sequencing the signals in *time*, we sequence them in *frequency*. That is, instead of each occupying a separate portion of the time domain, each signal takes up a distinct portion of the frequency domain. The Fourier transform gives an excellent lens to describe processes on frequencies in terms of familiar problems in time and spacial domains.

A good example of the FDM process is the telephone system. When a person talks on a telephone, the audio signal is bandwidth limited and with multiple phones there are multiple audio signals with similar bandwidths. If a telephone wire were to send only one signal in each transmission then most of the mediums bandwidth would be wasted. Instead FDM translates some of the signals (modulation) into different frequency sub-bands so that multiple signals can be sent in one transmission. When the transmission is received then the signals are inversely translated (demodulation) into their original forms and audio frequencies used by humans.

Similarly, the modern notion of a digital subscriber line uses existing telephone infrastructure to provide high-speed internet. This works by encoding digital data in higher frequency portions of the band, and using filters to keep voice and digital data separate. Radio transmissions is a usage of FDM: various agencies including the Federal Communications Commission and International Telecommunications Union set limits on bands for various channels, and operators transmit within a small portion of the frequency spectrum.

3 Extension of Principles

The same properties of real signals and Fourier analysis that make Frequency Division Multiplexing feasible lend themselves to other real-world application; indeed, the unique interplay between time/ spacial domains and the frequency domain forms the basis for the mathematics behind hundreds of software, engineering, and science applications.

The fact that most audio or visual signals are bandwidth limited (or, at least, human perception is bandwidth limited) allows for various forms of compression in which less perceptible data is filtered out.

Recent experimentation with signal tranmission, including using different wavelength lasers to send multiple signals over one fiber optic cable, makes use of fast digital modulation and demodulaton of signals using the FFT.

Analysing light signals in the frequency domain has long helped to identify chemicals in spectroscopy. Now, converting sounds and images into the frequency domain helps researchers to identify materials, chemicals, distinct voices, and detect edges.

Converting audio signals into the frequency domain and then frequency-shifing those signals gives us synthesizers and auto-tuners; selecting for and changing only certain frequencies in image manipulation software gives us color and sharpness filters.

Works Cited

"Aliasing." Wikipedia. Wikimedia Foundation, 12 Sept. 2012. Web. 09 Dec. 2012

Cooley, James W., and John W. Tukey. "An Algorithm for the Machine Calculation of Complex Fourier Series." Mathematics of Computation 19 (1965): 297-301. Mathematics of Computation. Web. 09 Dec. 2012.

Allen, Ronald L., and Duncan W. Mills. Signal Analysis: Time, Frequency, Scale, and Structure. Piscataway, NJ: IEEE, 2004. Print.

Foulkes, John D. Audio and Full Duplex Digital Data Carrier System. Teltone Corporation, assignee. Patent US4330687. 18 May 1982. Print.

Haberman, Richard. Applied Partial Differential Equations: With Fourier Series and Boundary Value Problems. Upper Saddle River, NJ: Pearson Prentice Hall, 2004. Print.

Powers, David L. Boundary Value Problems: And Partial Differential Equations. Amsterdam: Elsevier/Academic, 2010. Print.

School of Phyiscs. "The Fourier Transform (What You Need to Know)." The University of Edinburgh, 10 Sept. 2007. Web. 9 Dec. 2012. `http://www2.ph.ed.ac.uk/~wjh/teaching/Fourier/documents/booklet.pdf`

Tan, Li. Digital Signal Processing: Fundamentals and Applications. Amsterdam: Academic, 2008. Print.

Weinstein, S.; Ebert, P.; , "Data Transmission by Frequency-Division Multiplexing Using the Discrete Fourier Transform," Communication Technology, IEEE Transactions on , vol.19, no.5, pp.628-634, October 1971 `http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=1090705&isnumber=23757`