Frequency Division Multiplexing

(a fundamental application of digital signal processing)

A Project for
APPM 4350: *Fourier Series and Boundary Value Problems*

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Plan.

- Develop the mathematics behind digital signal processing
- Demonstrate Frequency Division Multiplexing
- Discuss real-world applications

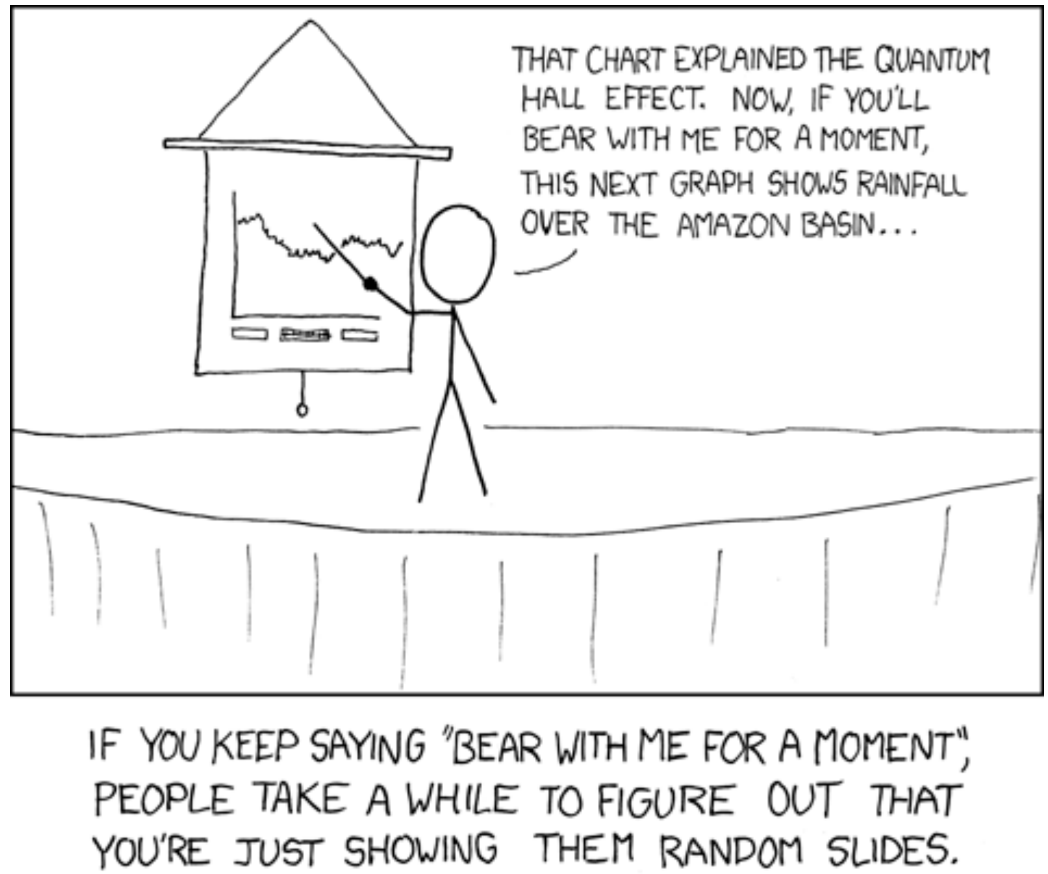

Image source: www.xkcd.com/365

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Nyquist-Shannon Sampling Theorem

Any bandwidth limited function with the cutoff frequency Ω can be reconstructed exactly from samples taken at sampling frequency $f_s > 2\,\Omega$.

- A function is bandwidth limited if:

$$\mathcal{F}(f(t)) = \hat{f}(\omega) \equiv 0 \;\; \forall\, |\omega| > \Omega$$

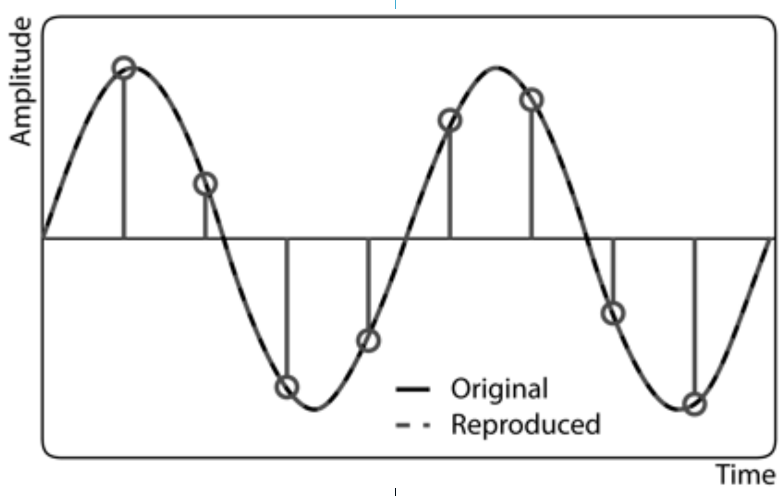


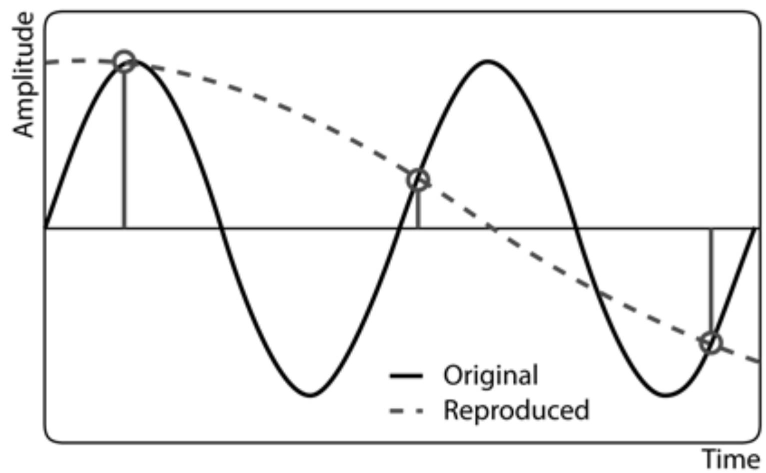
Image source: http://www.widex.pro/en/innovations/analysingsound/samplingrate/

Importance

The Nyquist Sampling Theorem demonstrates how to use discrete samples to model continuous functions with frequencies < $f_s/2$

- Most signals are nearly band limited—meaning that most of the important information is within a certain band.
  - Ex: a human voice usually only ranges between 60 and 7000 Hz, so $f_s > 1400$ Hz would accurately sample a voice

Downsampling, sampling a signal below its cutoff frequency, results in aliasing



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Discrete Fourier Transform

The DFT takes a set of discrete samples f_k of the time-domain function f(t) and produces a set of discrete samples of the Fourier transform of f(t).

For some band limited f(t) with a period of T_0 equal to some integer multiple N of the inverse sampling frequency $1/f_s$:

$$\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt = \frac{1}{T_0} \int_0^{T_0} f(t) e^{i\frac{2\pi}{T_0}t} dt$$

The Riemann sum of the Fourier transform with N intervals is:

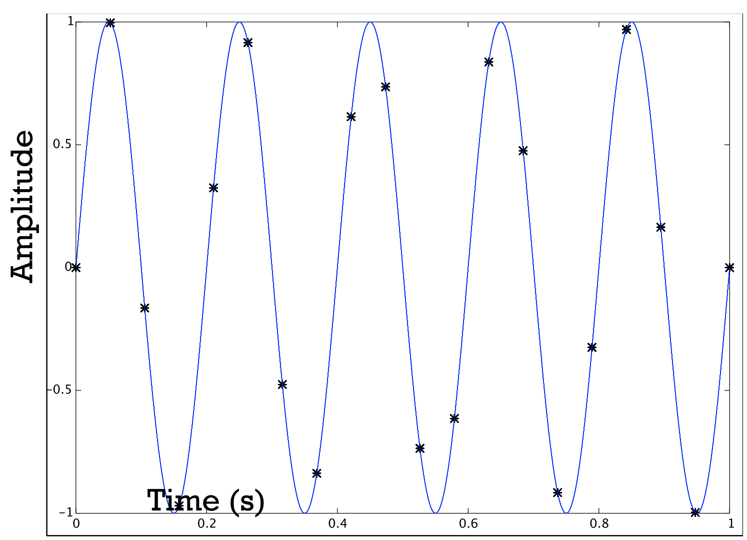
$$DFT(f(t)) = \frac{1}{N} \sum_{k=0}^{N-1} f_k \, e^{i\frac{2\pi n}{N}k}$$

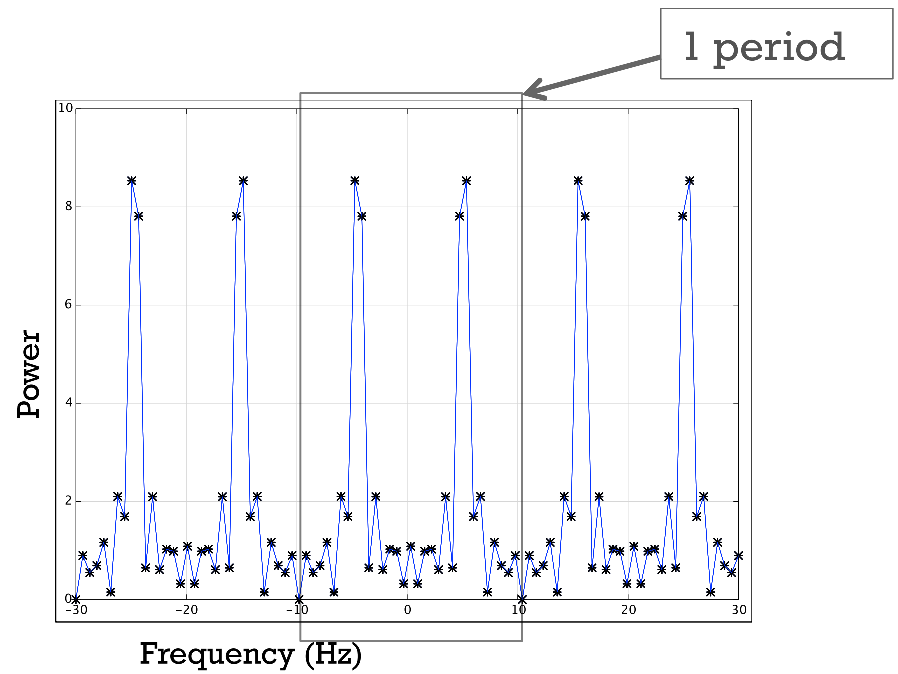
Periodicity of the DFT

In formulating the DFT, we assumed the function to be periodic on T_0, with $T_0 = N/f_s$.

$\Longrightarrow$ Then the DFT is also periodic on N samples.

$\Longrightarrow$ f_s = 20Hz, so the period of the DFT is 20Hz





Reality Condition

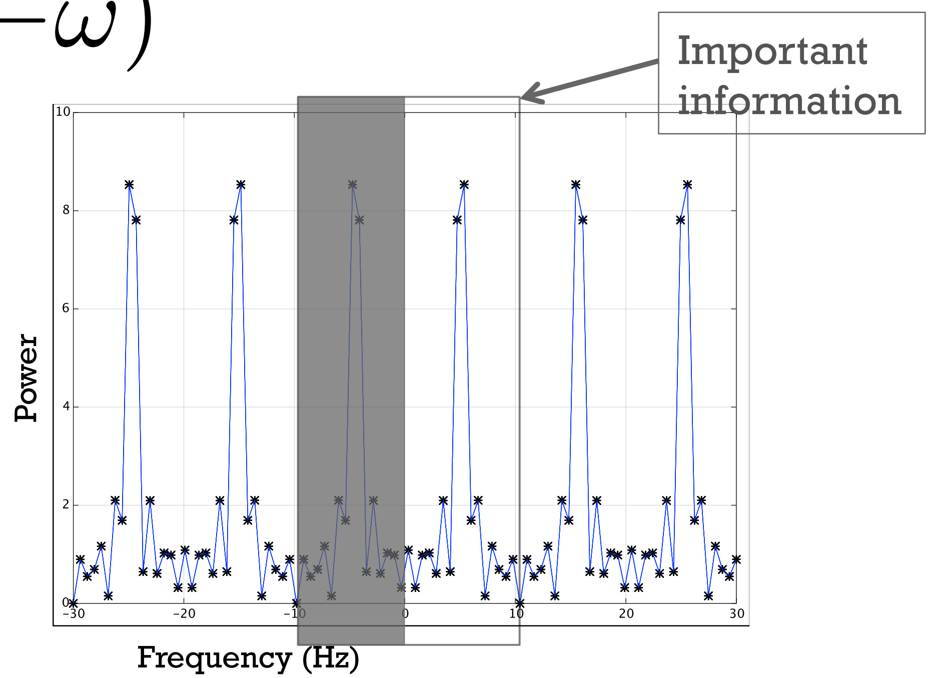
For all Fourier transforms of all real-valued functions:

$$\hat{f}(\omega) = \overline{\hat{f}(-\omega)}$$

For an even, real valued function:

$$\hat{f}(\omega) = \hat{f}(-\omega)$$

$\Longrightarrow$ *Redundant information can be recognized and thrown out*



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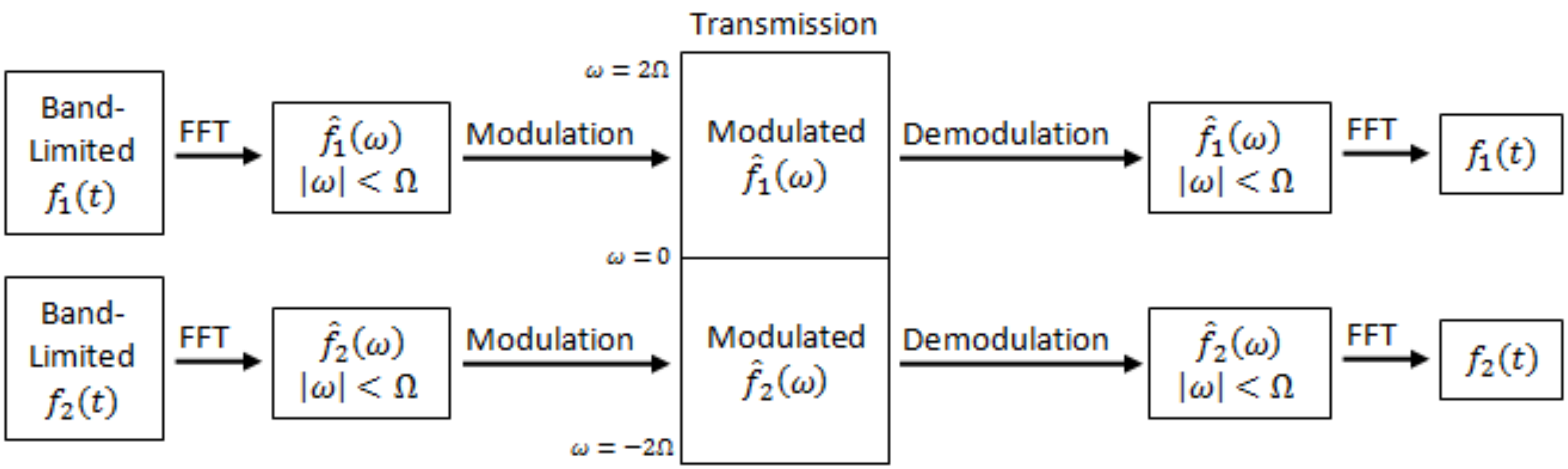
Fast Fourier Transform

A Fast Fourier Transform (FFT) is and efficient implementation of the discrete Fourier transform and its inverse.

- Computing the DFT the naïve way requires $O(n^2)$ operations. FFTs reduce the time complexity to O(n *log*(n))

- Most commonly used FFT is the Cooley-Tukey algorithm, which is a type of divide-and-conquer algorithm
  - Relies on the symmetry properties of the DFT to break the transform into smaller problems over smaller intervals
  - Base-2 case: an N element DFT is split into two N/2 element DFTs, of the even-indexed and the odd-indexed terms

Frequency Division Multiplexing (finally)



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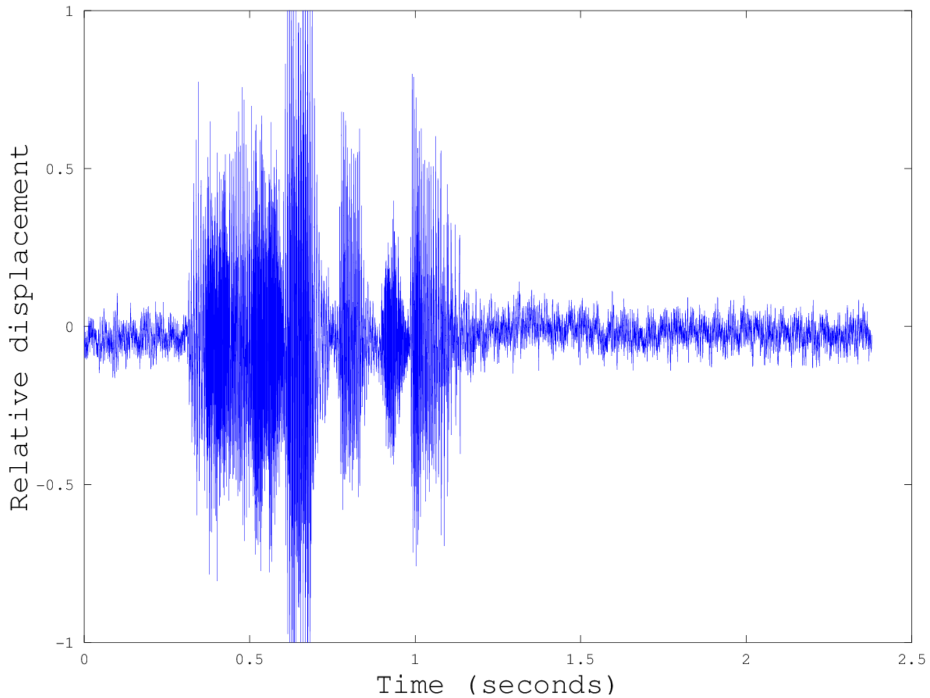
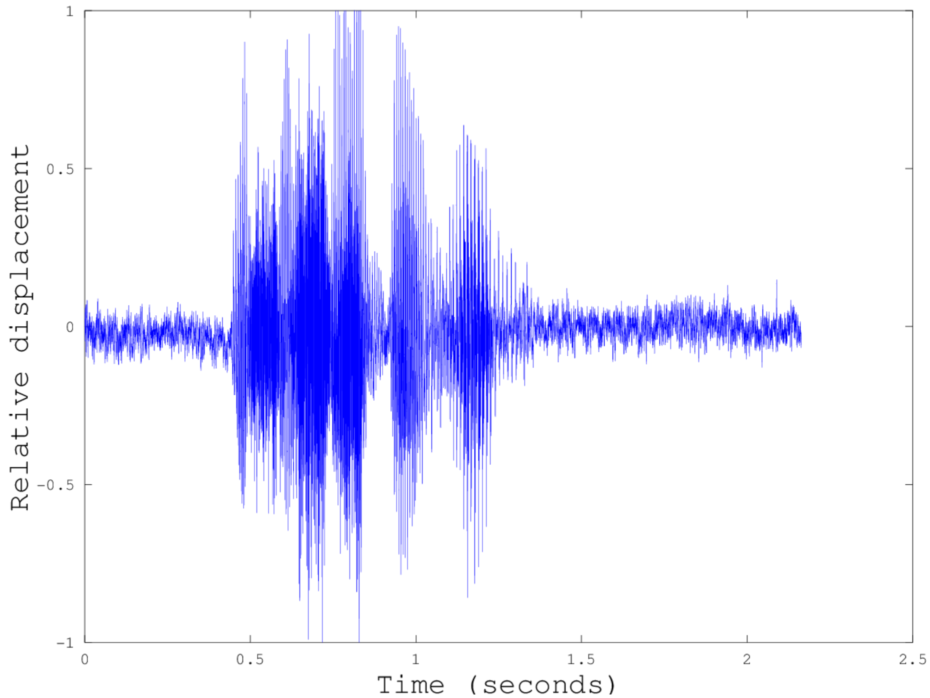
Our Implementation & Demonstration

- 2 voice samples
  - Cutoff frequency of a human voice ~7000Hz
- Sampling at f_s > 2(7000Hz)
  - f_s = 8000Hz
- Bandwidth of "transmitted" signal = 16000Hz
- GNU Octave implementation (an open-source MATLAB© alternative)

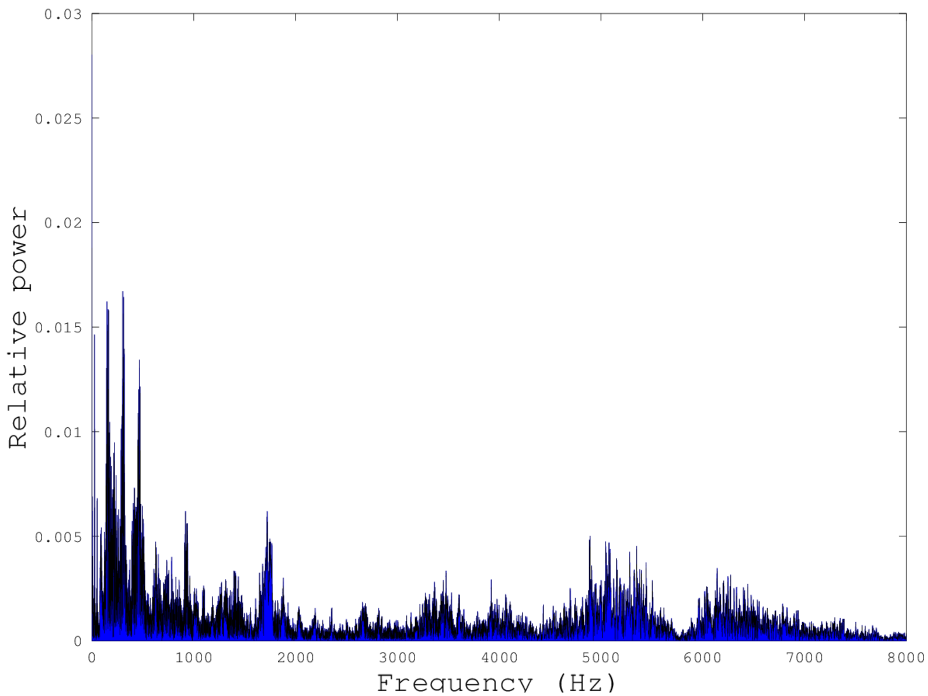
The original two sound files

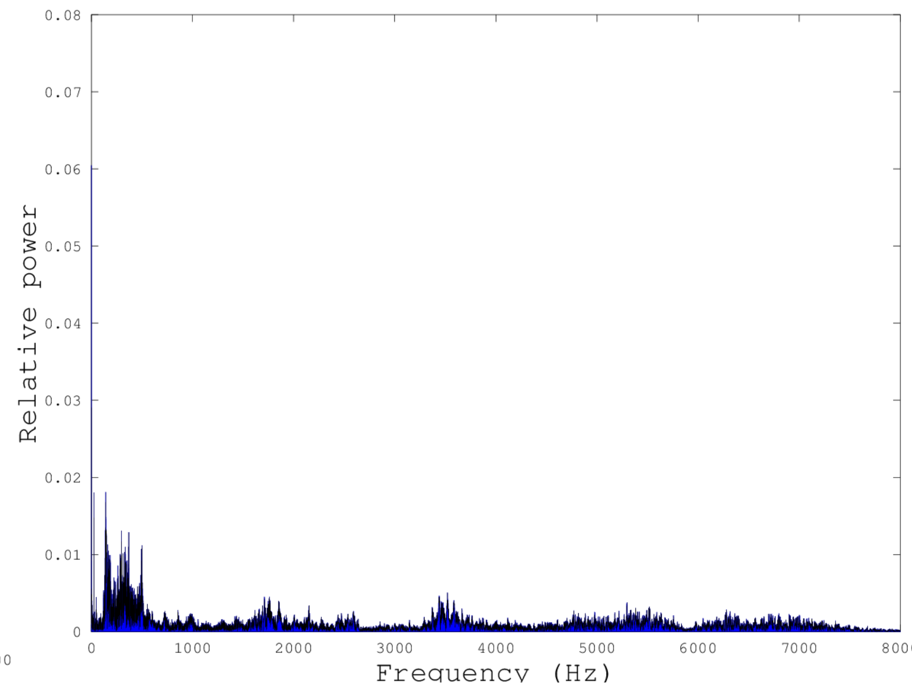




Take the Fourier Transform

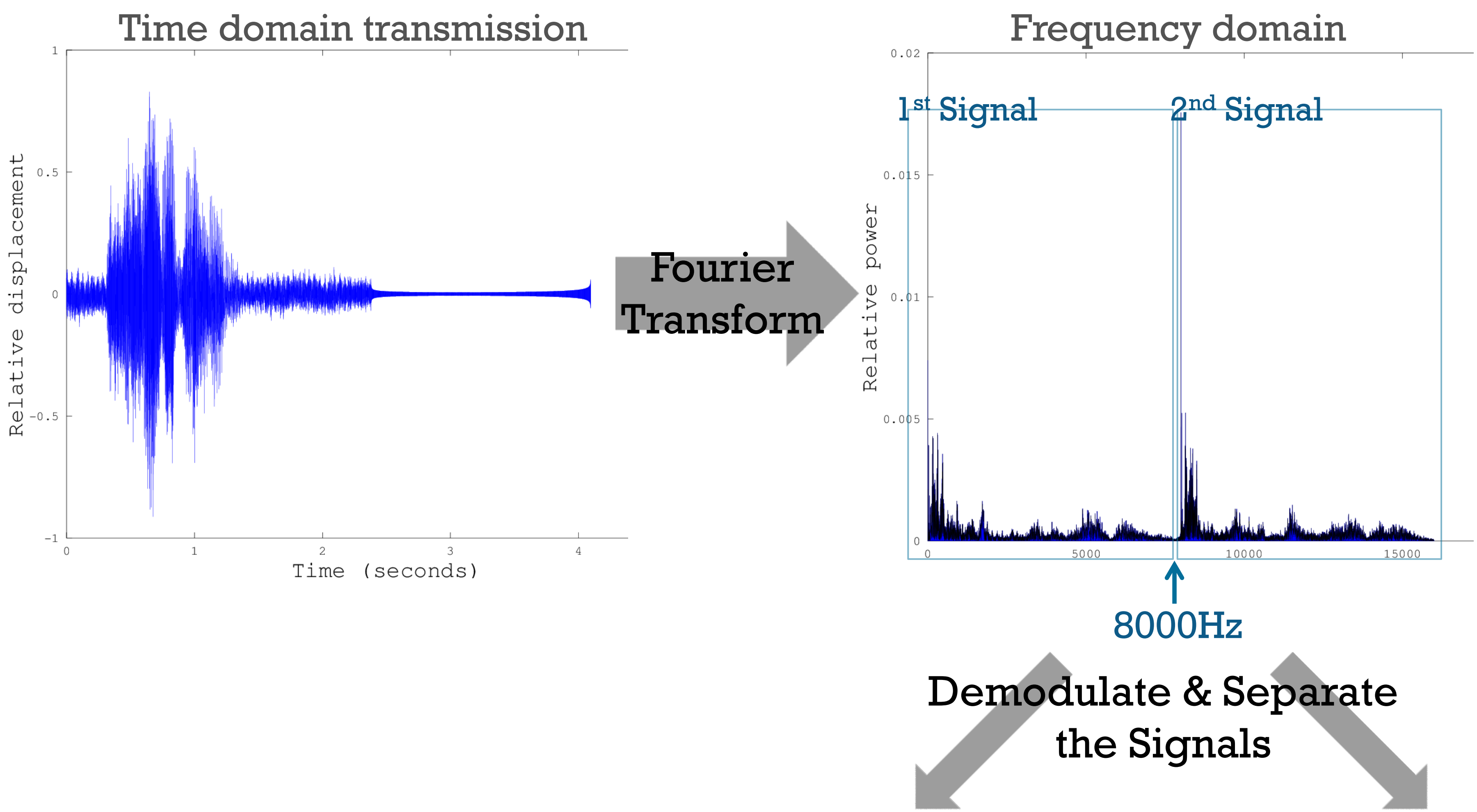
The data in the frequency domain



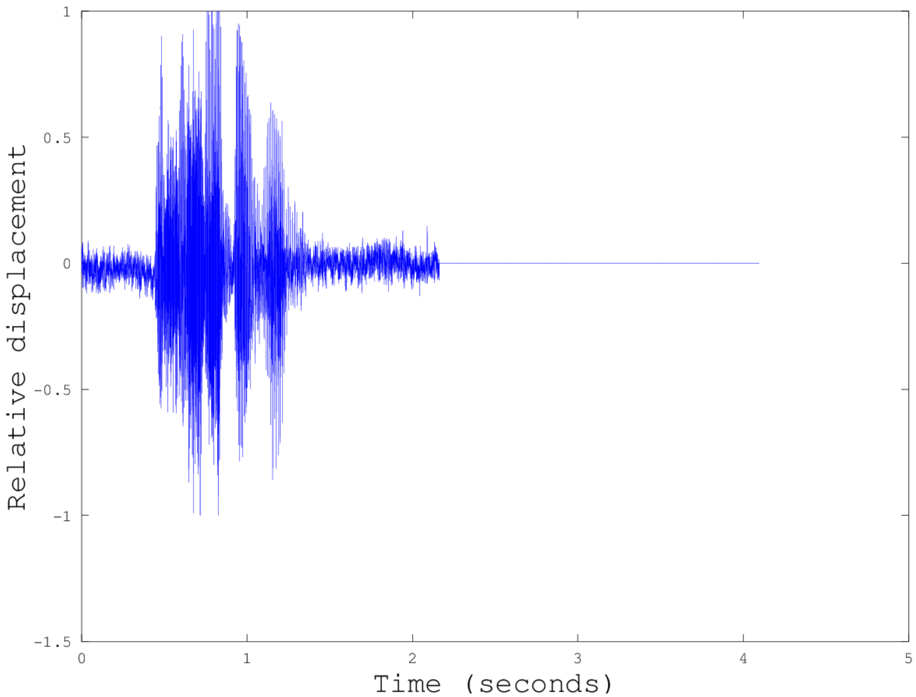


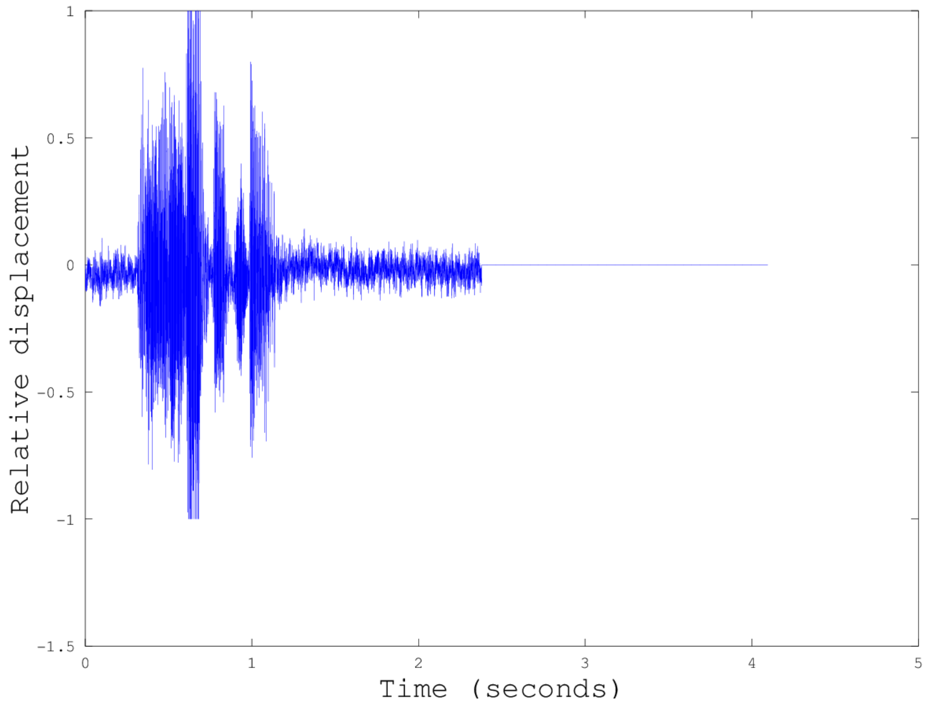
Modulate, concatenate and take the inverse transform of the signals

One time-domain signal with twice the bandwidth, ready for transmission



The received signals





Ta-da!

A Broader Perspective

- Telephone system
  - Use digital FDM to send multiple signals over the same analogue phone line
- Radio
  - Use analogue FDM to transmit multiple channels over the air at once
    - Each station uses its own frequency bandwidth & your radio demodulates on the receiving end
- Real FDM uses buffer frequency bands to prevent analogue noise from the electronics /air /phone line etc from causing the signals to mix

Extension of Principles

- Human audio-visual perception is bandwidth limited, which allows for various forms of data compression filtering out less perceptible data.
- Recent experimentation with signal transmission (including using different wavelength lasers to send multiple signals over one fiber optic cable) makes use of FDM principles.
- Converting sounds and images into the frequency domain helps re-searchers to identify materials, chemicals, distinct voices, and detect edges.
- Synthesizers and auto-tuners analyse audio signals in the frequency domain and then frequency-shift those signals.
- Image manipulation software selects for and changes only certain frequencies in an image.