Module 2.7: Predicates and Quantifiers

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Predicates

In a previous segment we have seen **two** kinds of basic statements.

- Examples of the **first** kind: $\mathit{odd}(7)$ $\mathit{odd}(8)$.

These are called **propositions** and they are either **true** or **false**.

- Examples of the **second** kind: $\mathit{odd}(p)$ $\mathit{vowel}(\ell)$.

Because p and ℓ are **variables**, these basic statements are called **predicates**.

Predicates are **undetermined**, that is, neither true nor false, because the values of the variables are not specified.

Example. The complex statement: $\mathit{integer}(x) \wedge (x > 1) \Rightarrow \neg \mathit{prime}(x^3 + 1)$ remains undetermined unless we specify a value for x.

Quantifiers

The statement

$$\text{“}integer(x) \wedge (x > 1) \;\Rightarrow\; \neg prime(x^3 + 1)\text{”}$$

is undetermined, but

“For all integers x, if $x > 1$, then $x^3 + 1$ is not prime.”

is **true** since we have proved it!

“For all x” is called a **universal quantifier**. Notation: $\forall x$.

$$\text{"}\forall x \; integer(x) \wedge (x > 1) \;\Rightarrow\; \neg prime(x^3 + 1)\text{”}$$

We also have “there exists x”, the **existential quantifier**. Notation: $\exists x$.

$$\text{"}\exists x \; integer(x) \wedge (10 < x < 20) \wedge prime(x)\text{”}$$

This is also **true**.

Notation exercises

“Val loves somebody.”

“$\exists m \; loves(\texttt{Val}, m)$”

“Val loves everybody.”

“$\forall z \; loves(\texttt{Val}, z)$.”

“Everybody loves somebody.”

“$\forall x \; \exists y \; loves(x, y)$”

”$\neg \exists x \; integer(x) \wedge even(x) \wedge odd(x)$”

“There is no integer that is both even and odd.”

The definition of “n is even” can also be written with quantifiers:

“$integer(n) \wedge \exists k \; integer(k) \wedge n = 2k$.”

ACTIVITY : Quantifier Notation

Now, try expressing the definition of "n is odd" using quantifier notation.

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Quantifier Notation (Continued)

Answer:

$integer(n) \land \exists\, k\;\; integer(k) \land n = 2k + 1.$

Notice that this is very similar to the way we wrote "n is even" using quantifier notation!

Activity : More Quantifier Notation

We often want to apply our quantifiers only to members of a particular set, and we have a way to express this succinctly.

For example, earlier we wrote the statement "There is no integer that is both even and odd," as $\neg\exists x\ \mathit{integer}(x) \wedge \mathit{even}(x) \wedge \mathit{odd}(x)$, which can be read as "There is no x such that x is an integer and x is even and x is odd."

An equivalent way to express the same statement is

$$\neg\exists x \in \mathbb{Z}\ \mathit{even}(x) \wedge \mathit{odd}(x)\,,$$

i.e., "There is no x in the integers such that x is even and x is odd."

Similarly, the statement "For all integers x, x is even or x is odd" can be written as $\forall x \in \mathbb{Z}\ \mathit{even}(x) \vee \mathit{odd}(x)$.

ACTIVITY : More Quantifier Notation (Continued)

Use quantifier notation to express the following statement:

> *For all positive integers there is a strictly bigger positive integer that is prime.*

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : More Quantifier Notation (Continued)

Answer:

$$\forall x \in \mathbb{Z}^{+}\ \exists y \in \mathbb{Z}^{+}\ (y > x) \wedge \mathit{prime}(y)$$

This can be read directly as "For all x in the positive integers, there exists a y in the positive integers such that y is greater than x and y is prime."

We will soon prove this statement, which is equivalent to the existence of infinitely many primes.

Quantifiers in English

An integer n is even if $n = 2k$ for some integer k.

"$integer(n) \wedge \exists\, k \; integer(k) \wedge n = 2k$."

"For some", just like "there exists" and "there is" indicates an existential quantifier.

"For every", or just "every", like "for all" or "for any", indicates a universal quantifier. Examples:

"For every integer there is a bigger prime integer."

Zhang's Theorem: There exists an integer N that is less than 70 million, such that for any integer x there are primes bigger than x that differ by N.

Penn Engineering

Quiz

Which of the following statements in logical notation corresponds to

"Any positive integer n that is not 1 and is not a prime has some positive integer factor that is neither 1 nor n" ?

A. $\forall n \in \mathbb{Z}^+ \; ((n \neq 1 \wedge \neg prime(n)) \Rightarrow \exists k \in \mathbb{Z}^+ \; (k \mid n \wedge k \neq 1 \wedge k \neq n))$

B. $\forall n \in \mathbb{Z}^+ \; ((n \neq 1 \wedge \neg prime(n)) \Rightarrow \forall k \in \mathbb{Z}^+ \; (k \mid n \wedge k \neq 1 \wedge k \neq n))$

C. $\exists n \in \mathbb{Z}^+ \; ((n \neq 1 \wedge \neg prime(n)) \Rightarrow \forall k \in \mathbb{Z}^+ \; (k \mid n \wedge k \neq 1 \wedge k \neq n))$

D. $\exists n \in \mathbb{Z}^+ \; ((n \neq 1 \wedge \neg prime(n)) \Rightarrow \exists k \in \mathbb{Z}^+ \; (k \mid n \wedge k \neq 1 \wedge k \neq n))$

Answer

A. $\forall n \in \mathbb{Z}^+ \ ((n \neq 1 \land \neg prime(n)) \Rightarrow \exists k \in \mathbb{Z}^+ \ (k \mid n \land k \neq 1 \land k \neq n))$
Correct. "Any" corresponds to a universal quantifier for n, and "has some" corresponds to an existential quantifier for the factor k.

B. $\forall n \in \mathbb{Z}^+ \ ((n \neq 1 \land \neg prime(n)) \Rightarrow \forall k \in \mathbb{Z}^+ \ (k \mid n \land k \neq 1 \land k \neq n))$
Incorrect. "Any" corresponds to a universal quantifier for n, and "has some" corresponds to an existential quantifier for the factor k.

C. $\exists n \in \mathbb{Z}^+ \ ((n \neq 1 \land \neg prime(n)) \Rightarrow \forall k \in \mathbb{Z}^+ \ (k \mid n \land k \neq 1 \land k \neq n))$
Incorrect. "Any" corresponds to a universal quantifier for n, and "has some" corresponds to an existential quantifier for the factor k.

D. $\exists n \in \mathbb{Z}^+ \ ((n \neq 1 \land \neg prime(n)) \Rightarrow \exists k \in \mathbb{Z}^+ \ (k \mid n \land k \neq 1 \land k \neq n))$
Incorrect. "Any" corresponds to a universal quantifier for n, and "has some" corresponds to an existential quantifier for the factor k.

Activity : Even More Quantifier Notation

Translate Zhang's Theorem into logical notation with quantifiers.

Zhang's Theorem: There exists an integer N that is less than 70 million, such that for any integer x there are two distinct primes greater than x that differ by at most N.

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Even More Quantifier Notation (Continued)

Answer:

$$\exists N \in \mathbb{Z} \Big(N < 70000000 \wedge (\forall x \in \mathbb{Z}\ \exists a, b$$
$$(a \neq b \wedge prime(a) \wedge prime(b) \wedge a > x \wedge b > x \wedge |a - b| \leq N)) \Big)$$

Read directly, this says "There exists an integer N such that N is less than 70000000 and for all integers x, there exist a and b such that a and b are not equal and a is prime and b is prime and a is greater than x and b is greater than x and the difference between a and b is at most N."