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This problem set will refer to a *standard deck of cards*. Each card in such a deck has a *rank* and a *suit*. The 13 ranks, ordered from lowest to highest, are *2*, *3*, *4*, *5*, *6*, *7*, *8*, *9*, *10*, *jack*, *queen*, *king*, and *ace*. The 4 suits are *clubs* ($\clubsuit$), *diamonds* ($\diamondsuit$), *hearts* ($\heartsuit$), and *spades* ($\spadesuit$). Cards with clubs or spades are *black*, while cards with diamonds or hearts are *red*. The deck has exactly one card for each rank–suit pair: *4 of diamonds*, *queen of spades*, etc., for a total of $13 \cdot 4 = 52$ cards.

1. [10 pts] Suppose you select a card uniformly at random from a standard deck, and then without putting it back, you select a second card uniformly at random from the remaining cards. What is the probability that both cards have rank no lower than 6, and at least one of the cards is red?

Solution.

Let Event A = "Both cards have rank ≥ 6"
Event B = "At least one of the cards is red"
We computer the cardinality of the uniform probability space Ω using the multiplication rule:
$|\Omega| = \binom{52}{2} = 1326$
We seek to compute the probability of $P_r[AB] = P_r[A] * P_r[B|A]$.
For $P_r[A]$:
Each suits has 13 ranks and 9 of them are equal or greater than 6, so total 36 cards are equal or greater than 6. We select two cards from the 36 cards, that is $\binom{36}{2} = 630$, so $P_r[A] = \binom{36}{2}/\binom{52}{2} = 630/1326$.
For $P_r[B|A]$:
We count complementarily. Event $\overline{B}|A$ = "For the two cards that are both have rank no lower than 6, none of them is red".
Out of the 36 cards that are no lower than 6, half of the cards, which is 18, are red. We select two cards from the 18 cards, so $|\overline{B}|A| = \binom{18}{2} = 153$, so $|B|A| = 630 - 153 = 477$, and

$P_r[B|A] = 477/630$.

From above we can calculate:

$P_r[AB] = P_r[A] * P_r[B|A] = 630/1326 * 477/630 = 477/1326 \approx 35.97\%$.

2. **[10 pts]** Mary and Emily have two distinguishable gardens. Initially, each of the gardens contains four roses and five daisies. Mary first picks a flower uniformly at random from the left garden and moves it to the right garden. Then, Emily picks a flower uniformly at random from the right garden.

What is the probability that Emily picks a rose?

Solution.

we have two sources of randomness: Emily picked rose or daisy, E_r or E_d, and Mary picked rose or daisy, M_r or M_d. So there are four outcomes, so let:
Event M_r = "Mary picked a rose"
Event $\overline{M_r}$ = "Mary did not picked a rose"
Event E_r = "Emily picked a rose"
We computer the cardinality of the uniform probability space Ω using the multiplication rule:
$|\Omega| = 9 * 10 = 90$
The probability we are looking for is (given by the law of total probability):
$P_r[E_r] = P_r[E_r|M_r] * P_r[M_r] + P_r[E_r|\overline{M_r}] * P_r[\overline{M_r}]$
Step 1:
There are 4 ways for Mary to pick up a rose from the left garden, and then Emily picks a flower from the right garden with no restrictions, so $P_r[M_r] = (4 * 10)/90 = 4/9$. Using P4, we know $P_r[\overline{M_r}] = 1 - 4/9 = 5/9$.
Step 2:
We know that E_r and M_r are not independent since E_r depends on M_r.
In how many ways Emily can pick a rose after Mary picked a rose? There will be 5 roses left among 10 flowers in the right garden. Hence $P_r[E_r|M_r] = 5/10 = 1/2$.
In how many ways Emily can pick a rose after Mary did not pick a rose? There will be 4 roses among 10 flowers in the right garden. Hence $P_r[E_r|\overline{M_r}] = 4/10 = 2/5$.
From above we can calculate:
$P_r[E_r] = P_r[E_r|M_r] * P_r[M_r] + P_r[E_r|\overline{M_r}] * P_r[\overline{M_r}]$
$= 1/2 * 4/9 + 2/5 * 5/9$
$= 4/9$

3. **[10 pts]** For each of three golfers g_1, g_2, g_3 the probabilities of hitting a ball on the green (the desired play!), in a bunker (sand trap), or in a water hazard are given by the following table (we make the simplifying assumption that these are the only three results of a hit):

golfer	green	bunker	water
g_1	1/2	1/3	1/6
g_2	1/3	1/6	1/2
g_3	1/6	1/2	1/3

The three golfers, playing together, hit one ball each, mutually independently.

(a) **[4 pts]** What is the probability that all three balls end up in the water?

(b) **[6 pts]** What is the probability that at least one of the balls ends up on the green?

Solution.

(a) Let Event A = "g_1's ball end up in the water".
Event B = "g_2's ball end up in the water".
Event C = "g_3's ball end up in the water".
Since the three events are mutually independent, we know that
$P_r[A \cap B \cap C] = P_r[A] * P_r[B] * P_r[C] = 1/6 * 1/2 * 1/3 = 1/36$

(b) Let Event A = "g_1's ball end up on the green" and $P_r[A] = 1/2$.
Event B = "g_2's ball end up on the green" and $P_r[B] = 1/3$.
Event C = "g_3's ball end up on the green" and $P_r[C] = 1/6$.
Event L = "At least one of the balls ends up on the green".
Then Event $\overline{L}$ = "No ball ends up on the green".
Since the three events are mutually independent, we know that
$P_r[L] = P_r[A \cup B \cup C]$
$= 1 - P_r[\overline{A \cup B \cup C}]$
$= 1 - P_r[\overline{A} \cap \overline{B} \cap \overline{C}]$
$= 1 - P_r[\overline{A}] * P_r[\overline{B}] * P_r[\overline{C}]$
$= 1 - (1 - P_r[A]) * (1 - P_r[B]) * (1 - P_r[C])$
$= 1 - (1 - 1/2) * (1 - 1/3) * (1 - 1/6)$
$= 13/18$

4. **[10 pts]** 25 participants enter a sushi-making competition! Each of them must make 4 rolls with different proteins: a salmon roll, a tuna roll, a shrimp roll, and a crab roll, for a total of 100 rolls made in the competition. All of the rolls are distinguishable, even if they have the same protein (ex. a salmon roll is distinguishable from all other salmon rolls, as well as all the other rolls with different protein). According to the competition rules, salmon and tuna rolls are *always* made with white rice, while shrimp and crab rolls are *always* made with brown rice. The judges select a sushi roll uniformly at random from the 100 rolls. Without putting it back, they select a second roll uniformly at random from the remaining 99 rolls. What is the probability that both of the rolls are made by the same participant, or both have the same protein, or one has white rice and one has brown rice?

Solution.

Let Event L1 = "Both rolls are made by the same participant"
Event L2 = "Both rolls have the same protein"
Event L3 = "One roll has white rice and one has brown rice"
Given the Inclusion-exclusion for three events, we seek to compute the probability of:
$P_r[L1 \cup L2 \cup L3] = P_r[L1]+P_r[L2]+P_r[L3]-P_r[L1 \cap L2]-P_r[L1 \cap L3]-P_r[L2 \cap L3]+P_r[L1 \cap L2 \cap L3]$
We computer the cardinality of the uniform probability space Ω using the multiplication rule:
$|\Omega| = 100 * 99 = 9900$
For L1:
We need to select one participant from 25 participants, that is $\binom{25}{1}$, and to select 2 rolls out of the 4 rolls made by this participant, that is $\binom{4}{2}$. By multiplication rule $|L1| = \binom{25}{1} * \binom{4}{2} = 150$, so $P_r[L1] = 150/9900 = 5/330$.
For L2:
We need to select one protein from 4 protein, that is $\binom{4}{1}$, and to select 2 rolls out of the 25 rolls that have the selected protein, that is $\binom{25}{2}$. By multiplication rule $|L2| = \binom{4}{1} * \binom{25}{2} = 1200$, so $P_r[L2] = 1200/9900 = 4/33$.
For L3:
We need to select one roll from 50 rolls that have white rice, that is $\binom{50}{1}$, and to select one roll from 50 rolls that have brown rice, that is $\binom{50}{1}$. By multiplication rule $|L3| = \binom{50}{1} * \binom{50}{1} = 2500$, so $P_r[L3] = 2500/9900 = 25/99$.
For $L1 \cap L2$:
Since every participant makes 4 rolls with different proteins, no participant will make two rolls that have the same protein. Therefore $L1 \cap L2 = \emptyset$, so $P_r[L1 \cap L2] = 0$.
For $L2 \cap L3$:
Since salmon and tuna rolls are always made with white rice, shrimp and crab rolls are always

made with brown rice, rolls with same protein will always have the same rice. Therefore $L1 \cap L2 = \emptyset$, so $P_r[L2 \cap L3] = 0$.

For $L1 \cap L3$:

We need to firstly select one participant from 25 participants, that is $\binom{25}{1}$, and then to select 1 roll from the 2 rolls this participant made with white rice, that is $\binom{2}{1}$, and lastly to select 1 roll from the 2 rolls this participant made with brown rice, that is $\binom{2}{1}$.By multiplication rule,$|L1 \cap L3| = \binom{25}{1} * \binom{2}{1} * \binom{2}{1} = 100$, so $P_r[L1 \cap L3] = 100/9900 = 1/99$.

For $L1 \cap L2 \cap L3$:

Since $L1 \cap L2 = \emptyset$ and $L2 \cap L3 = \emptyset$, we know $L1 \cap L2 \cap L3 = \emptyset$, so $P_r[L1 \cap L2 \cap L3] = 0$.

From above we can calculate that:

$P_r[L1 \cup L2 \cup L3] = P_r[L1] + P_r[L2] + P_r[L3] - P_r[L1 \cap L2] - P_r[L1 \cap L3] - P_r[L2 \cap L3] + P_r[L1 \cap L2 \cap L3]$

$= 5/330 + 4/33 + 25/99 - 0 - 1/99 - 0 + 0$

$= 25/66 \approx 37.88\%$

5. [10 pts] Alice and Bob each have a standard deck of cards. First, Alice selects uniformly at random a card from her deck, adds it to Bob's deck and then shuffles his deck. Second, Bob selects uniformly at random a card from his deck, adds it to Alice's deck and then shuffles her deck. Finally, Alice selects uniformly at random a card from her deck. What is the probability that this card's suit is diamond?

Solution.

YOUR SOLUTION HERE Let Event A = "Alice picked a card in diamond suit in the first round".
Event $\overline{A}$ = "Alice did not pick a card in diamond suit in the first round".
Event B = "Bob picked a card in diamond suit in the second round".
Event $\overline{B}$ = "Bob did not pick a card in diamond suit in the second round".
Event C = "Alice picked a card in diamond suit in the last round".
Then the sample space would be $\{ABC, \overline{A}BC, A\overline{B}C, \overline{AB}C, AB\overline{C}, \overline{A}B\overline{C}, A\overline{BC}, \overline{ABC}\}$ and it's not uniform.
We seek to computer the probability of:
$P_r[C] = P_r[ABC] + Pr[\overline{A}BC] + P_r[A\overline{B}C] + Pr[\overline{AB}C]$
Given the chain rule, we know that:
$P_r[ABC] = P_r[A] * P_r[B|A] * P_r[C|AB]$
$Pr[\overline{A}BC] = P_r[\overline{A}] * P_r[B|\overline{A}] * P_r[C|\overline{A}B]$
$P_r[A\overline{B}C] = P_r[A] * P_r[\overline{B}|A] * P_r[C|A\overline{B}]$
$P_r[\overline{AB}C] = P_r[\overline{A}] * P_r[\overline{B}|\overline{A}] * P_r[C|\overline{AB}]$
$P_r[A]$: There are 13 diamond suit cards in a deck of 52 cards, so $P_r[A] = 13/52 = 1/4$
$P_r[\overline{A}]$: Using P4, $P_r[\overline{A}] = 1 - P_r[A] = 1 - 1/4 = 3/4$
$P_r[B|A]$: When Alice picked a diamond suit card and put it in Bob's deck, Bob's deck will have 53 cards and 14 diamond suit cards. So $P_r[B|A] = 14/53$.
$P_r[B|\overline{A}]$: When Alice picked a non-diamond suit card and put it in Bob's deck, Bob's deck will have 53 cards and 13 diamond suit cards. So $P_r[B|\overline{A}] = 13/53$.
$P_r[\overline{B}|A]$: When Alice picked a diamond suit card and put it in Bob's deck, Bob's deck will have 53 cards and 39 non-diamond suit cards. So $P_r[P_r[\overline{B}|A] = 39/53$.
$P_r[\overline{B}|\overline{A}]$: When Alice picked a non-diamond suit card and put it in Bob's deck, Bob's deck will have 53 cards and 39 non-diamond suit cards. So $P_r[P_r[\overline{B}|\overline{A}] = 39/53$.
$P_r[ABC]$: When Alice picked a diamond suit card and put it in Bob's deck, and then Bob picked a diamond card and put it back to Alice's deck, Alice's deck will have 52 cards and 13 diamond suit cards. So $P_r[ABC] = 13/52 = 1/4$.
$P_r[C|AB]$: When Alice picked a diamond suit card and put it in Bob's deck, and then Bob

picked a diamond card and put it back to Alice's deck, Alice's deck will have 52 cards and 13 diamond suit cards. So $P_r[ABC] = 13/52 = 1/4$.

$P_r[C|\overline{A}B]$: When Alice picked a non-diamond suit card and put it in Bob's deck, and then Bob picked a diamond card and put it back to Alice's deck, Alice's deck will have 52 cards and 14 diamond suit cards. So $P_r[ABC] = 14/52$.

$P_r[C|A\overline{B}]$: When Alice picked a diamond suit card and put it in Bob's deck, and then Bob picked a non-diamond card and put it back to Alice's deck, Alice's deck will have 52 cards and 12 diamond suit cards. So $P_r[ABC] = 12/52$.

$P_r[C|\overline{AB}]$: When Alice picked a non-diamond suit card and put it in Bob's deck, and then Bob picked a non-diamond card and put it back to Alice's deck, Alice's deck will have 52 cards and 13 diamond suit cards. So $P_r[ABC] = 13/52 = 1/4$.

From above we can calculate that:

$P_r[C] = P_r[ABC] + Pr[\overline{A}BC] + P_r[A\overline{B}C] + Pr[\overline{AB}C]$

$= (1/4)*(14/53)*(1/4)+(3/4)*(13/53)*(14/52)+(1/4)*(39/53)*(12/52)+(3/4)*(39/53)*(1/4)$

$= 209/848 \approx 24.65\%$