Module 6.1: Ordinary Induction

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

What is induction good for?

For proving statements of the form:

"for all natural numbers n we have $P(n)$"

where $P(n)$ is a predicate whose truth depends on n.

In logical notation: $\forall n \in \mathbb{N} \;\; P(n)$.

Examples. Statements of this form that are of interest:

- $P(n)$ is $2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1$.
- $P(n)$ is $1 + 2 + 3 + \cdots + n = n(n+1)/2$ (for $n \geq 1$).
- $P(n)$ is $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ (for $n \geq 1$).
- $P(n)$ is "n can be written as the product of one or more (not necessarily distinct) prime numbers" (for $n \geq 2$).

Proof pattern: (ordinary) induction

Let $P(n)$ be a predicate whose truth depends on n.

Proof pattern.

(BASE CASE) Check that $P(0)$ holds true.

(INDUCTION STEP) Let k be an arbitrary natural number.
Assume $\boxed{P(k).}$ Using that derive $P(k+1)$.

Conclude $\forall\, n \in \mathbb{N}\;\; P(n)$.

The $P(k)$ inside the box in the induction step is called the **INDUCTION HYPOTHESIS (IH)**. The IH must be stated **inside the induction step** because it refers to k.

In logical notation the induction step is $\forall\, k \in \mathbb{N}\;\; P(k) \;\Rightarrow\; P(k+1)$.

Sum of a geometric progression

Problem. Prove $\forall\, n \in \mathbb{N}\;\; 2^0 + 2^1 + \cdots + 2^n \;=\; 2^{n+1} - 1.$

Answer. $P(n)$ is $\;2^0 + 2^1 + \cdots + 2^n \;=\; 2^{n+1} - 1.$

(BASE CASE) $2^0 = 1$ and $2^{0+1} - 1 = 2 - 1 = 1.$ Check.

(INDUCTION STEP) Let k be an arbitrary natural number.
Assume $2^0 + 2^1 + \cdots + 2^k \;=\; 2^{k+1} - 1.$ (This is the IH.)
(Now we **want to show** $2^0 + 2^1 + \cdots + 2^{k+1} \;=\; 2^{k+2} - 1.$)
Then:

$$
\begin{aligned}
2^0 + 2^1 + \cdots + 2^{k+1} &= (2^0 + 2^1 + \cdots + 2^k) + 2^{k+1} && \text{(Grouping)} \\
&= 2^{k+1} - 1 + 2^{k+1} && \text{(By IH)} \\
&= 2 \cdot 2^{k+1} - 1 \;=\; 2^{k+2} - 1 && \text{(Done)}
\end{aligned}
$$

Quiz

The previous identity gives the summation of the **geometric progression** with ratio 2.

Memorize the sum of a general **geometric progression** with ratio q:

$$q^0 + q^1 + q^2 + \cdots + q^n \;=\; \begin{cases} \frac{q^{n+1}-1}{q-1} & \text{if } q \neq 1, \\ n+1 & \text{if } q = 1, \end{cases}$$

Now apply this to

$$s \;=\; 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

Which of the following is true?

A. $s < 2$

B. $s = 2$

C. $s > 2$

ANSWER

Which of the following is true?

A. $s < 2$

Correct. Apply the formula with $q = \frac{1}{2}$ to derive $s = 2 - \frac{1}{2^n} < 2$.

B. $s = 2$

Incorrect. Apply the formula to derive $s = 2 - 1/2^n$.

$$s = \frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1} = \frac{\frac{1}{2^{n+1}} - 1}{-\frac{1}{2}} = -2\left(\frac{1}{2^{n+1}} - 1\right) = 2 - \frac{1}{2^n} < 2\,.$$

C. $s > 2$

Incorrect. Apply the formula to derive $s = 2 - 1/2^n$.

$$s = \frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1} = \frac{\frac{1}{2^{n+1}} - 1}{-\frac{1}{2}} = -2\left(\frac{1}{2^{n+1}} - 1\right) = 2 - \frac{1}{2^n} < 2\,.$$

When we cannot start at 0

"$1 + 2 + 3 + \cdots + n = n(n+1)/2 \quad (\text{for } n \geq 1)$"

This statement does not make sense for $n = 0$.

However, we still need a base case to use induction!

One possibility is to change the statement:

"if $n \geq 1$ then $1 + 2 + 3 + \cdots + n = n(n+1)/2$"

Then the base case $n = 0$ holds vacuously!

However, in the induction step we will need to reason separately for the case $k = 0$ and essentially prove the statement for $n = 1$.

Instead, it's much easier to adopt a **variant** of the (ordinary) induction proof pattern, as we do in the next slide.

Proof pattern variant for (ordinary) induction

Let n_0 be a natural number and let $P(n)$ be a predicate that is well defined for all natural numbers $n \geq n_0$.

Proof pattern.

(BASE CASE) Check that $P(n_0)$ holds true.

(INDUCTION STEP) Let $k \geq n_0$ be an arbitrary natural number. Assume $\boxed{P(k)}$. Using that, infer $P(k+1)$.

Conclude $\forall\, n \geq n_0 \;\; P(n)$.

Aside: From now on we agree to **abbreviate**: (BASE CASE) as (BC),(INDUCTION STEP) as (IS) and "we want to show" as WTS.

The sum $1 + 2 + 3 + \cdots + n$

Problem. Prove $\forall\, n \geq 1 \;\; 1 + 2 + 3 + \cdots + n \;=\; n(n+1)/2.$

Answer. Take $n_0 = 1$ in the proof pattern.

(BC) $1 = 1$ and $1(1+1)/2 = 2/2 = 1.$ Check.

(IS) Let $k \geq 1$ be an arbitrary natural number.

Assume (IH) $1 + 2 + \cdots + k \;=\; k(k+1)/2.$
(And **WTS** $1 + 2 + \cdots + k + (k+1) \;=\; (k+1)(k+2)/2.$) Then:

$$
\begin{aligned}
1 + 2 + \cdots + k + (k+1) &= (1 + 2 + \cdots + k) + (k+1) && \text{(Grouping)}\\
&= k(k+1)/2 + (k+1) && \text{(By IH)}\\
&= (k+1)(k/2+1) \;=\; (k+1)(k+2)/2
\end{aligned}
$$

Done.

ACTIVITY : Sum of integers

Gauss was one of the world's greatest mathematicians. Legend has it that when Gauss was very young (accounts vary between 7 and 9 years old) his teacher asked the class to add all the numbers from 1 to 100 (to keep them busy for an hour, I suppose ☺). After only a couple of moments, Gauss raised his hand. The teacher was annoyed but nonetheless had to listen to him giving . . . the correct answer!

Question: What is the correct answer to the question? With the formula you just proved you can also do it in a couple of moments!

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Sum of integers

Answer: 5050.

Did you replace $n = 100$ in $n(n+1)/2$ and therefore you calculated $(100 \cdot 101)/2 = 10100/2 = 5050$? Good!

But what did young Gauss do? Not knowing the formula, he grouped the numbers into pairs $(1, 100), (2, 99)...(50, 51)$ realizing that each of these pairs sums to the same number — 101 (or $n + 1$ in general). How many pairs are there? Exactly 50 (or, $\frac{n}{2}$). Hence, $50 \cdot 101 = 5050$. Try seeing why this works for an odd n (e.g. 101). [Hint: the middle element is $\frac{n+1}{2}$ in this case.]