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1. [10 pts] Prove by induction that for all positive integers  $n$ , we have:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

**Solution.**

(BC) ( $n=1$ ) Since  $2n - 1 = 2 * 1 - 1 = 1$  so  $1 = n^2 = 1$ . Check.

(IS) Let  $k \geq 1$  be any arbitrary positive integer.

Assume (IH)  $1 + 3 + 5 + \cdots + (2k - 1) = k^2$ .

WTS  $1 + 3 + 5 + \cdots + (2(k + 1) - 1) = (k + 1)^2$ . Then:

$$1 + 3 + 5 + \cdots + (2(k + 1) - 1)$$

$$= k^2 + 2(k + 1) - 1$$

$$= k^2 + 2k + 1$$

$$= (k + 1)^2$$

The proof is finished.

2. [10 pts] Use the recursion tree method or the telescopic method to solve the recurrence relation  $f(0) = 7$  and, for all  $n \in \mathbb{Z}^+$   $f(n) = f(n-1) - 2n$ .

**Solution.**

$$f(n) = f(n-1) - 2n$$

$$f(n-1) = f(n-2) - 2(n-1)$$

$$f(n-2) = f(n-3) - 2(n-2)$$

$$f(n-3) = f(n-4) - 2(n-3)$$

...

$$f(2) = f(1) - 2 * 2$$

$$f(1) = f(0) - 2 * 1$$

Add all the LHSs and RHSs and cancel terms that appear on both sides:

$$f(n) = f(0) - 2n - 2(n-1) - 2(n-2) - 2(n-3) - \dots - 2 * 2 - 2 * 1$$

$$= f(0) - 2 * [n + (n-1) + (n-2) + (n-3) + \dots + 2 + 1]$$

$$= f(0) - 2 * n(n+1)/2$$

$$= f(0) - n(n+1)$$

Done.

3. [10 pts] Use strong induction to prove that  $C(n) = 2^n + 3$  is a solution to the recurrence  $C(0) = 4$ ,  $C(1) = 5$ , and, for all  $n \in \mathbb{Z}^+$ ,  $n > 1$
- $$C(n) = 3 \cdot C(n-1) - 2 \cdot C(n-2) .$$

**Solution.**

$$(BC) \ (n=2) \ C(2) = 2^2 + 3 = 7$$

$$C(2) = 3 \cdot C(1) - 2 \cdot C(0) = 3 \cdot 5 - 2 \cdot 4 = 7$$

Therefore  $C(n)$  is a solution to  $3 \cdot C(n-1) - 2 \cdot C(n-2)$  when  $n = 2$ . Check.

(IS) Let  $k \geq 2$  arbitrary. Assume (IH) that for all integers  $2, 3, \dots, k$ ,  $C(k) = 2^k + 3$  is a solution to  $C(k) = 3 \cdot C(k-1) - 2 \cdot C(k-2)$ .

WTS  $C(k+1) = 2^{k+1} + 3$  is a solution to  $C(k+1) = 3 \cdot C(k) - 2 \cdot C(k-1)$ .

From IH we know  $C(k) = 2^k + 3$  is a solution to  $C(k) = 3 \cdot C(k-1) - 2 \cdot C(k-2)$  and  $C(k-1) = 2^{k-1} + 3$  is a solution to  $C(k-1) = 3 \cdot C(k-2) - 2 \cdot C(k-3)$ . Then:

$$C(k+1) = 3 \cdot C(k) - 2 \cdot C(k-1)$$

$$= 3 \cdot (2^k + 3) - 2 \cdot (2^{k-1} + 3)$$

$$= 3 \cdot 2^k + 9 - 2^k - 6$$

$$= (3 - 1) \cdot 2^k + 3$$

$$= 2^{k+1} + 3$$

Therefore we can conclude  $C(k+1) = 2^{k+1} + 3$  is a solution to  $C(k+1) = 3 \cdot C(k) - 2 \cdot C(k-1)$ .

The proof is finished.

4. [10 pts] Recall the Fibonacci sequence, where every number in the sequence is the sum of the previous two numbers (except for the first and second positions, which are 0 and 1 respectively). Let  $F_n$  represent the  $n$ th number in the Fibonacci sequence. Use strong induction to prove that for Fibonacci numbers  $F_{n+1} - F_{n-1} < 2^n$  for all positive integers  $n$ .

**Solution.**

(BC) ( $n=1$ ) Since  $F_{n+1} = F_2 = 1$ ,  $F_{n-1} = F_0 = 0$ , so  $F_{n+1} - F_{n-1} = 1$ . Since  $2^n = 2$ , therefore  $F_{n+1} - F_{n-1} < 2^n$  when  $n = 1$ . Check.

(IS) Let  $k \geq 1$  arbitrary. Assume (IH) that for all integers  $1, 2, \dots, k$ ,  $F_{k+1} - F_{k-1} < 2^k$ .

And WTS  $F_{k+2} - F_k < 2^{k+1}$ . Then:

$$\begin{aligned} F_{k+2} - F_k &= (F_{k+1} + F_k) - (F_{k-1} + F_{k-2}) \\ &= (F_{k+1} - F_{k-1}) + (F_k - F_{k-2}) \end{aligned}$$

From IH we know  $F_{k+1} - F_{k-1} < 2^k$  and  $F_k - F_{k-2} < 2^{k-1}$

Then  $(F_{k+1} - F_{k-1}) + (F_k - F_{k-2}) < 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}$

So we got  $F_{k+2} - F_k < 2^{k+1}$ .

The proof is finished.

5. [10 pts] Use ordinary induction to prove that for every positive integer  $n$ ,  $n^3 - n$  is a multiple of 6. Only proofs by induction are accepted.

**Solution.**

(BC) ( $n=1$ )  $n^3 - n = 1 - 1 = 0$ . Check.

(IS) Let  $k \geq 1$  arbitrary.

Assume (IH)  $k^3 - k$  is a multiple of 6. And WTS  $(k+1)^3 - (k+1)$  is a multiple of 6. Then

$$\begin{aligned} & (k+1)^3 - (k+1) \\ &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 + 3k^2 + 2k \\ &= k^3 - k + 3k^2 + 3k \\ &= (k^3 - k) + 3k(k+1) \end{aligned}$$

From IH we know  $(k^3 - k)$  is a multiple of 6.

Since  $k \geq 1$ , we know  $(k+1) \geq 2$  therefore  $3(k+1)$  is a multiple of 6 which means  $3k(k+1)$  is also a multiple of 6. In summary,  $(k+1)^3 - (k+1) = (k^3 - k) + 3k(k+1)$  is also a multiple of 6. The proof is finished.