Self-paced Example: The Erdős-Szekeres Theorem

Module 5

MCIT Online - CIT592 - Professor Val Tannen

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion. When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Erdős-Szekeres Theorem

In a story recounted in this module you found out more about Erdős, the most prolific mathematician of the 20th century. Furthermore, in the previous segment you learned about the Pigeonhole Principle (PHP).

In this self-paced segment, we will learn about one of Erdős' theorems that he co-authored with the mathematician George Szekeres. This theorem is called the Erdős-Szekeres Theorem and utilizes the Pigeonhole Principle. Take a moment to recall PHP.

Erdős-Szekeres Theorem *Let n be a positive integer. Every sequence of n^2+1 distinct integers must contain a monotone (increasing or decreasing) subsequence of length $n+1$.*

Proof: Before we begin the proof, we should make precise the definition of a **subsequence**.

Subsequence means that some of the elements are chosen and listed in the same order. The elements of the subsequence need not be consecutive (next to each other) in the sequence.

Suppose we have a sequence of distinct integers. (We don't need this in the proof, but w.l.o.g. you can assume that they are positive; why?)

(CONTINUED)

Erdős-Szekeres Theorem (continued)

For each integer x in the sequence let u_x be the length of a **longest increasing subsequence** that starts at x (we count x in the length) and let d_x be the length of a **longest decreasing subsequence** that starts at x (again we count x in the length).

Before we proceed further in the proof, given our definitions of u_x and d_x, we present a lemma.

Lemma: If $x \neq y$ are distinct integers in the sequence such that x occurs before y then $(u_x, d_x) \neq (u_y, d_y)$.

Proof: We have two cases to consider: when $x < y$ and when $x > y$.

Indeed, if $x < y$ then $u_x > u_y$. Can you figure out why, on your own?

Indeed, this is because, if $x < y$, then we can consider the increasing sequence that starts with x and continues with y and a longest increasing sequence that starts at y. The length of this sequence is $1 + u_y$ so we must have $u_x \geq 1 + u_y$. Hence $u_x > u_y$ and therefore $(u_x, d_x) \neq (u_y, d_y)$.

Similarly, if $x > y$ then $d_x > d_y$ and thus $(u_x, d_x) \neq (u_y, d_y)$.

Now we can return to the proof of the theorem.

(CONTINUED)

Erdős-Szekeres Theorem (continued)

We prove the theorem by contradiction. Suppose that all monotone subsequences have length at most n. Then for each x in the sequence we have $u_x, d_x \in [1..n]$

We use PHP with the pairs $(i, j) \in [1..n] \times [1..n]$ as pigeonholes, with the integers in the sequence as pigeons, and with pigeon x being placed in pigeonhole (u_x, d_x).

Since we have $n^2 + 1$ integers in the sequence but only n^2 pairs in $[1..n] \times [1..n]$, we must have $(u_x, d_x) = (u_y, d_y)$ for some distinct $x \neq y$.

This contradicts the above lemma and so we have completed our proof of the Erdős-Szekeres theorem.