Module 4.3: Functions

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Functions I

A **function** (sometimes called **mapping**) denoted $f : A \to B$, consists of

- a set A, called **domain**,
- a set B, called **codomain**, and
- a way of associating with **every** element of the domain, $x \in A$, a **unique** element of the codomain, $f(x) \in B$, write $x \mapsto f(x)$.

The **range** of a function $f : A \to B$ is:

$$\mathrm{Ran}(f) \;=\; \{\, y \mid y \in B \;\wedge\; \exists x \in A \;\; y = f(x) \,\}$$

Note that this defines a subset $\mathrm{Ran}(f) \subseteq B$.

Functions II

Examples.

$h : \mathbb{N} \to \mathbb{N}$ where $h(n) = 2n$. Ran$(h) =$ even integers ≥ 0

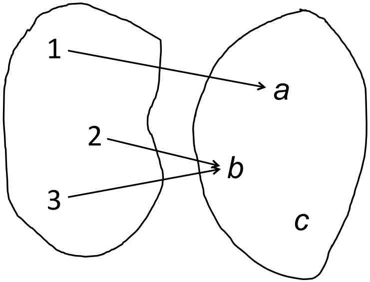
$g : [0, \infty) \to \mathbb{R}$ where $g(x) = \sqrt{x}$. Ran$(g) =$ $[0, \infty)$

$f : A \to B$ with domain $A = \{1, 2, 3\}$, codomain $B = \{a, b, c\}$

with $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto b$ or $f(1) = a$, $f(2) = b$, $f(3) = b$.

Table and diagram representations:

$x \in \{1, 2, 3\}$	$f(x) \in \{a, b, c\}$
1	a
2	b
3	b



Ran$(f) =$ $\{a, b\}$.

Quiz

Consider the function $f : [1, \infty) \to \mathbb{R}$ where $f(n) = \log_2 n$.

What is the range of this function?

A. $\mathbb{R}$

B. $\mathbb{Z}^+$

C. $[0, \infty)$

Answer

Consider the function $f : [1, \infty) \to \mathbb{R}$ where $f(n) = \log_2 n$.

What is the range of this function?

A. $\mathbb{R}$

Incorrect. A log function defined on real numbers ≥ 1 does not return negative numbers.

B. $\mathbb{Z}^+$

Incorrect. The log of a number does not have to be an integer.

C. $(0, \infty)$

Correct. The log function defined on numbers ≥ 1 returns positive real numbers (or 0).

The set of all functions

Let A, B be two sets. The set

$$\{f \mid f : A \to B\} \quad \text{is denoted by} \quad B^A$$

Proposition. If $|A| = r$ and $|B| = n$ then the number of different functions with domain A and codomain B is n^r.

Proof. Let $A = \{ a_1, \ldots, a_r \}$. We can construct a function from A to B in r steps, $i = 1, 2, \ldots, r$ as follows.

In step (i) we choose $b \in B$ to define $f(a_i) = b$, that is $a_i \mapsto b$.
This can be done in n ways.

By the multiplication rule, the number of functions is $n \cdot n \cdots n = n^r$.

Therefore $$|B^A| = |B|^{|A|}$$

Activity : Example of one-to-one correspondence

Consider a function $f : A \to B$ where A is the set of elements $\{a_1, a_2, \ldots, a_n\}$.
First notice that the number of possible functions is also the number of sequences of length n of elements in the set B.
In an activity in an earlier segment we showed that the subsets of a set A are in one-to-one correspondence with sequences of bits of size $|A|$.
Recall that we denoted the set of subsets of A by 2^A.

Question: What is the cardinality of 2^A?

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

Activity : Example of one-to-one correspondence (Continued)

Answer: $2^{|A|}$.

Now consider the particular case when B has two elements, for example $B = \{0, 1\}$. In this case, and using the formula we just learned, there are

$$|B^A| = |B|^{|A|} = 2^{|A|}$$

possible functions from A to B.

This is also the number of subsets of the set A!

Is there a connection between the subsets of A and the functions from A to $\{0, 1\}$? Yes!

We will describe a one-to-one correspondence between them.

Activity : Example of one-to-one correspondence (Continued)

Namely, to any function $f : A \to \{0,1\}$ this correspondence associates a subset S_f of A where:

$$S_f \; = \; \{x \in A \mid f(x) = 1\}$$

Conversely, to any subset $S \subseteq A$ this correspondence associates a function $f_s : A \to \{0,1\}$ defined by

$$f_s(x) \; = \; \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise} \end{cases}$$