MCIT 592 Spring 2022 **Homework 11** **Solutions**

Problem Set

1. **[10 pts]** Matrix Metro is a city with m rows and n columns of buildings, with roads connecting these houses to form a grid. Amy is visiting and wants to walk around the city. Help Amy find the length of the longest path that she can walk (i.e. she never walks to the same building twice). Provide a brief explanation as to why it is the maximum. You can assume $m, n \geq 2$.

Solution:

Matrix Metro is a grid graph of size $m \times n$. Let P be a maximum length path in our grid graph, G, that passes through k vertices. $|P| = k - 1$, as the length of a path is equal to the number of edges it passes through. Since we have mn vertices in our graph G, $|P| \leq mn - 1$. However, it is clearly possible to create a path that passes through all vertices in G (start from a corner vertex and pass through all vertices in the row before moving down to the next row and repeating). Therefore such a path has maximum length, that is $k = mn$, implying that $|P| = mn - 1$.

2. **[10 pts]** Suppose there are a series of islands connected by bridges. A d-coalition is a group of islands in which every island has exactly d bridges connected to it. Prove that there is no 7-coalition with 39 bridges.

Solution:

We create a graph with islands as vertices and bridges as edges. Assume toward a contradiction that such a graph exists and has n vertices, for some positive integer n. Then by the Handshaking Lemma $7n = 2 \cdot 39 = 78$, so $n = 78/7$, which is not an integer. This is a contradiction, so we conclude that no such graph may exist. $\square$

3. [10 pts] Suppose there exists a Facebook group with n people, where $n \geq 2$. Each of them has *p or more* connections to other members of the group. Prove that if $p > \frac{n-2}{2}$, then the group is connected (i.e. they are all linked together through some traversal of friendships).

Solution:

We can map this Facebook group as a graph G, with the friends representing n vertices. Each of these vertices has a degree of at least p (where $p > \frac{n-2}{2}$). We will prove by contradiction that G is connected.

Assume for contradiction that G is not connected hence there are at least two connected components. Let's call the first cc C_1 and the second cc C_2. C_1 consists of at least one vertex. Since this vertex is of degree at least p then there must be at least $p+1$ vertices in C_1. We can argue similarly for C_2, both C_1 and C_2 have at least $p+1$ vertices, giving a total of at least $p+1+p+1 = 2p+2$ vertices. However, we have that $p > \frac{n-2}{2}$ therefore $2p+2 > n-2+2 = n$ which contradicts the fact that the graph has n vertices (since we have more than n vertices). We have found our contradiction and this completes our proof.

4. [10 pts] Suppose there is a shipping network with n shipping locations, where $n \geq 2$. They are connected by roads, but a shipping location could be isolated (i.e. there are no roads to it). Prove that, no matter how these roads are organized, there are at least two shipping locations with the same number of roads.

Solution:

We can create a graph with shipping locations as vertices, and roads as edges. Suppose G is a graph with $n \geq 2$ vertices. Consider two cases.

- *Case 1: There is at least one isolated vertex.* Then no vertex may have degree greater than $n-2$ (since it is not adjacent to itself and not adjacent to the isolated vertex), so there are n vertices with degrees in the integer interval $[0..n-2]$, which has cardinality $n-1$. By the pigeonhole principle, two vertices must have the same degree.

- *Case 2: There is no isolated vertex.* Then the n vertices all have degrees in the integer interval $[1..n-1]$, which also has cardinality $n-1$. The pigeonhole principle applies again, and two vertices must have the same degree.

□

5. [10 pts] Consider a graph G with $2k$ vertices and k edges, where k is a positive integer. Prove that if each vertex in G has degree at least 1, then it has exactly k connected components.

Solution:

Recall that connected components, vertices, and edges can be related with the following inequality:

$$|CC| \geq |V| - |E|$$

Substituting in values given in the problem, we arrive at our first conclusion that

$$|CC| \geq k$$

For our second conclusion, observe that there are no isolated nodes. As a result, every single connected component must contain at least one edge. That is:

$$|CC| \leq |E|$$

$$|CC| \leq k$$

Combining our two conclusions, we find that $k \leq |CC| \leq k$, which is only true when $\boxed{|CC| = k}$