Module 2.1: Counting Subsets

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Counting subsets

Problem. Let A be a set with n elements, that is, $|A| = n$. How many distinct subsets of A are there?

Answer. If A is empty then it has exactly one subset, the empty set.

Suppose A has $n \geq 1$ elements, say, $A = \{x_1, x_2, \ldots, x_n\}$.

Each subset S of A can be constructed in n steps:

(1) Decide whether to include x_1 in S or not: can be done in 2 ways.

. . .

(n) Decide whether to include x_n in S or not: 2 ways.

By the multiplication rule the answer is $2 \cdot 2 \cdots 2 = 2^n$.

Cardinality of powerset

Recall the definition of the **powerset** 2^A of set A.

What is the cardinality of 2^A?

This is the same as asking how many subsets of A there are.

We have just seen that if A is empty, $|2^A| = 1$.

We have also seen that if $|A| = n \geq 1$ then $|2^A| = 2^n$.

But $2^0 = 1$.

Proposition. For *any* set A we have

$$|\, 2^A \,| \; = \; 2^{|\, A \,|}$$

Counting pets I

Problem. Animal Rescue has 5 cats and 3 dogs. How many different groups of pets can you adopt, knowing that you must adopt at least one dog and at least one cat, and you might adopt all of them?

Answer. We try to construct a group of pets as follows:

(1) Choose one of the cats: can be done in 5 ways.

(2) Choose one of the dogs: 3 ways.

(3) Choose a subset of the remaining $4 + 2 = 6$ pets: 2^6 ways.

(We have just seen that a set with n elements has 2^n subsets.)

Answer $5 \cdot 3 \cdot 2^6$? But this is wrong!

Counting pets II

Wrong! Because of *overlaps* between step 3 and each of the first two steps, some groups are counted more than once!

For example, the group {Maimu,CousCous,Archer} is counted twice.

Here's how it is counted twice:

Choose CousCous in step 1, Archer in step 2, and { Maimu } in step 3.

Choose Maimu in step 1, Archer in step 2, and { CousCous } in step 3!

We call this *overcounting*.

Counting pets III

We can still use the multiplication rule, but in a different way.

We construct a group of pets as follows:

(1) Choose a non-empty subset of cats: $2^5 - 1$ ways.

(2) Choose a non-empty subset of dogs: $2^3 - 1$ ways.

(A set with n elements has $2^n - 1$ non-empty subsets.)

By the multiplication rule, the answer is

$(2^5 - 1)(2^3 - 1) = 31 \cdot 7 = 217$ different groups of pets.

Counting pets IV

Another method is to count **complementarily**: subtract from total number of sets of pets the number of those sets of pets that you *cannot* adopt.

The total number of subsets (including the empty set) of $5 + 3 = 8$ pets: 2^8.

Which sets of pets cannot be adopted? Those of just cats or just dogs!

Total number of sets of cats: 2^5.

Total number of sets of dogs: 2^3.

Is the answer $2^8 - (2^5 + 2^3)$?

No! We have *oversubtracted*: the empty set of cats is the same as the empty set of dogs!

The answer is $2^8 - 2^5 - 2^3 \mathbf{+1} = 256 - 32 - 8 + 1 = 217$ again.