Module 7.5: Birthdays, Balls and Bins

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Lecture Notes

The birthday "paradox" I

Problem. Suppose there are k people in a room. What is the probability that at least two people in the room have the same birthday? What is the smallest value of k for which this probability is at least $1/2$?

Answer. We set up a probability space in which the outcomes are the birthdays of the k people: sequences of length k of elements from $[1..365]$. There are 365^k such sequences. Next, we make two assumptions:

- It is equally likely for any given person to be born on any of the 365 days of the year.
- We also assume that the birthdays of the different people in the room are unrelated (they are **independent**).

Based on the intuition supported by these two assumptions, we state that our probability space is **uniform**. Each outcome has probability $1/365^k$.

The birthday "paradox" II

Answer (continued). Let E be the event that **at least** two people in the room have the same birthday.

Its complement is $\overline{E} =$"all k people have distinct birthdays". The outcomes in $\overline{E}$ are the partial permutations of k out of 365.

Therefore, using **P4**:

$$\Pr[E] = 1 - \Pr[\overline{E}] = 1 - \frac{365!/(365-k)!}{365^k} = 1 - \frac{365!}{(365-k)! \cdot 365^k}$$

Using a "big integer" calculator we find that the smallest value of k for which $\Pr[E] \geq 0.5$ is . . . **23**!

Taking $k = 60$ we obtain $\Pr[E] \simeq 0.99$. Therefore, with 60 people in the room it is **almost certain** that there are two sharing the same birthday!

Balls into bins I

We have $n \geq 1$ distinguishable bins into which we throw $k \geq 0$ distinguishable balls under two assumptions:

- Each ball is equally likely to land in each of the n bins.
- The k throws are independent of each other.

As with the birthday "paradox" we assume a **uniform** probability space whose outcomes are sequences of length k of elements from $[1..n]$. There are n^k outcomes so each outcome has probability $1/n^k$.

Problem. What is the probability that ball $i \in [1..k]$ lands in bin $j \in [1..n]$?

Answer. The number of outcomes with ball i in bin j is n^{k-1}.
Therefore the probability is $n^{k-1}/n^k = 1/n$.

Balls into bins II

Problem. We throw k balls into $n \geq 2$ bins. What is the probability that Bin 1 remains empty?

Answer. Recall that in this space the outcomes are sequences of length k of elements from $[1..n]$.

Event of interest: sequences whose elements are just from $[2..n]$.

The number of such sequences is $|[2..n]|^k = (n-2+1)^k = (n-1)^k$.

Therefore the probability is

$$\frac{(n-1)^k}{n^k} = \left(1 - \frac{1}{n}\right)^k$$