Recitation Module 5

Penn Engineering

Recitation Topics

- Bijective Functions
- Principle of Inclusion Exclusion (PIE)
- Pigeonhole Principle (PHP)

Bijection

A function $f : A \rightarrow B$ is called **bijective** if it is both injective and surjective.
A bijective function is also called a **bijection** or a **one-to-one correspondence**.

The bijection rule:
If we can define a bijective function with domain A and codomain B, then $|A| = |B|$.

The **variants** of the bijection rule:
If we can define an injective function $f : A \rightarrow B$, then $|A| = |Ran(f)|$.
If $f : A \rightarrow B$ is injective, then $f' : A \rightarrow Ran(f)$ where $f'(x) = f(x)$ is bijective

Question 1

Q1. Let $m, n \geq 2$. Define:

$$f : [1..m] \times [1..n] \to [2..(m+n)] \text{ by } f(x, y) = x + y.$$

Is f an injection? Is f a surjection? Is f a bijection? Prove your answers.

Answer to Question 1

f is not injective. In order to show this, we will provide two elements $(a, b), (c, d) \in [1..m] \times [1..n]$ such that $(a, b) \neq (c, d)$ but $f(a, b) = f(c, d)$. To this end, consider $(1, 2)$ and $(2, 1)$. Clearly, $(1, 2) \neq (2, 1)$. However, since $f(1, 2) = 3 = f(2, 1)$, they both map onto the same element in the codomain, and we conclude that f cannot be injective.

f is surjective. To prove surjectivity, we must show that $\forall z \in [2..(m+n)], \exists x \in [1..m], y \in [1..n]$ such that $f(x, y) = z$. Intuitively, we want to show that everything in the codomain of f is mapped to by at least one element from the domain. Consider any z in the codomain, i.e., any $z \in [2..(m+n)]$. We consider the following cases:

Answer to Question 1 (cont.)

Case 1: $2 \leq z \leq m + 1$. In this case, let $y = 1$. Clearly, $1 \in [1..n]$. Then we need $x = z - 1$, by definition of f. Substituting into the fact that $2 \leq z \leq m + 1$, we see that $1 \leq x \leq m$, meaning x will always be an element of $[1..m]$. Thus, we know that $(x, 1) \in [1..m] \times [1..n]$, and we have found an element of the domain, namely $(x, 1)$, such that $f(x, 1) = z$.

Case 2: $m + 1 < z \leq m + n$. In this case, let $x = m$. Clearly, $m \in [1..m]$. Then we need $y = z - m$, by definition of f. As above, by substituting into the fact that $m+1 < z \leq m+n$, we see that $1 < y \leq n$, meaning y will always be an element of $[1..n]$. As above, we know that $(m, y) \in [1..m] \times [1..n]$, and we have found an element of the domain, namely (m, y), such that $f(m, y) = z$.

Answer to Question 1 (cont.)

In both cases, we have found an element of the domain (x, y) such that $f(x, y) = z$, so we know that f is surjective.

f is not bijective because a bijection must be both injective and surjective, and f is not injective.

PIE - Principle of Inclusion Exclusion

- For two disjoint sets A and B, $|A \cup B| = |A| + |B|$.
- When A and B are two sets that are **not disjoint,** then $|A| + |B|$ **overcounts.**
  - Specifically, elements that are in $A \cap B$ are counted twice.
- Subtracting $|A \cap B|$, we get the general formula:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

- For 3 disjoint sets A, B and C, we get the general formula:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

This is called **Principle of Inclusion-Exclusion (PIE).**

Question 2

Let A, B, C be nonempty finite sets such that $A \subseteq B \cup C$. Prove that

$$|A| + |A \cap B \cap C| = |A \cap B| + |A \cap C|$$

Answer to Question 2

First, we claim that $A \cup B \cup C = B \cup C$.

Consider an element $x \in LHS$. If $x \in B \cup C$, then $x \in RHS$ by definition. If $x \in A$, then since $A \subseteq B \cup C$ we know that $x \in B \cup C$ and thus, $x \in RHS$. Hence, $A \cup B \cup C \subseteq B \cup C$

Now consider an element $x \in RHS$. Since $x \in B \cup C$, then $x \in A \cup B \cup C$ by definition of set union, so we know $x \in LHS$. Hence, $B \cup C \subseteq A \cup B \cup C$

$A \cup B \cup C \subseteq B \cup C$ and $B \cup C \subseteq A \cup B \cup C \implies A \cup B \cup C = B \cup C$.

We then use PIE on $A \cup B \cup C$ to prove the problem statement.

Answer to Question 2

$$
\begin{aligned}
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
|A \cup B \cup C| - |B| - |C| + |B \cap C| &= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C| \\
|A \cup B \cup C| - |B \cup C| &= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{By PIE on } B \cup C)
\end{aligned}
$$

Using the fact that $A \cup B \cup C = B \cup C$, we see:

$$
\begin{aligned}
0 &= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C| \\
|A \cap B| + |A \cap C| &= |A| + |A \cap B \cap C|
\end{aligned}
$$

Thus, we have proved the problem statement.

This problem can also be solved by applying PIE to $A \cap B$ and $A \cap C$ directly (try it!).

Pigeonhole Principle

Theorem. If *r* objects (or pigeons) are placed into *n* categories (or holes), then there is at least one category containing **at least** $\lceil r / n \rceil$ objects.

Equivalently: If *r* objects (or pigeons) are placed into *n* categories (or holes), then for any integer k such that $r > k \cdot n$, there is at least one category containing at least k+1 objects.

In HW and assessments, state clearly what are the pigeons and what are the holes when using PHP.

Question 3

Let S be a set of 16 distinct positive integers such that $\forall x \in S,\ x < 60$.

Show that there exist distinct integers $a, b, c, d \in S$ such that $a + b = c + d$.

Answer to Question 3

Every pair of integers will have an associated sum, and there are $\binom{16}{2} = 120$ unordered pairs of distinct elements in S. Since all elements of S are between 1 and 59 inclusive, the sum of any pair of distinct elements will be between 3 and 117 inclusive, which gives 115 possibilities.

Let the unordered pairs represent the pigeons and the possible sums represent the holes. Since there are 120 pigeons and 115 holes, by PHP there exist $\lceil 120/115 \rceil = 2$ distinct pairings that map to the same sum.

However, we are not quite done yet. What if the unordered pairs overlap? If the 2 inputs that map to the same sum are $\{a, b\}$ and $\{a, c\}$ (with distinct a, b, c), then this would be invalid. This would imply, however, that $a + b = a + c \implies b = c$, which contradicts the fact that a, b, c are distinct. Thus, the two pairings that have the same sum have no overlaps.

Terms to Know

1. Surjective, injective, bijective. Know what they mean in words, mathematically, and how to prove/disprove them.
2. Principle of Inclusion-Exclusion. You do not have to expand this out for 4+ sets, but know how to use it for 2-3 sets.
3. Pigeonhole principle (PHP). Properly defining pigeons, holes, and correct PHP conclusions.
    a. Generalizing PHP (GPHP)

Before you go...

Do you have any general questions about this week's assignment? We can't go into minute details about any specific questions, but this is the time to ask any general clarifying/confusing questions!

As always, if you can't talk to us during recitation or office hours, post any questions/anything course related on Piazza or email mcitonline@seas.upenn.edu.