

**Recitation Module 10** 



#### Independent random variables

Random variables X and Y defined on the same probability space are

$$\forall x \in Val(X) \ \forall y \in Val(Y) \ (X = x) \perp (Y \models$$

independent, written  $X \perp Y$ , when  $\forall x \in Val(X) \ \forall y \in Val(Y) \ (X = x) \perp (Y = y)$   $= \bigcap_{X \in X} \bigcap_{X = x} \bigcap_{Y \in Y} \bigcup_{X = x} \bigcap_{Y \in Y} \bigcap_{X \in$ 

**Example.** Green-purple fair dice are rolled, G returns the number shown by the green die, P by the purple die. Then  $G \perp P$ , because for any  $g, p \in [1..6]$ :

$$\Pr[(G = g) \cap (P = p)] = 1/36 = (1/6)(1/6) = \Pr[G = g] \cdot \Pr[P = p].$$

When we have three or more r.v.'s, we define again pairwise and mutual independence analogously.



#### Variance

The **variance** of a random variable X is defined as

$$Var[X] = E[(X - \mu)^2]$$
 (where  $\mu = E[X]$ )

The **standard deviation** of a random variable X is

$$\sigma[X] = \sqrt{\operatorname{Var}[X]} \qquad \qquad \mathcal{E} \left( \times - y^2 \right)^2$$

Why square root? By undoing the squaring in the variance we obtain the same units of measurement as those used for the values of the random variable.

**Notation.** When the random variable is understood, its mean is often denoted by  $\mu$ , its standard deviation by  $\sigma$ , and its variance by  $\sigma^2$ .

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An alternative formula for variance

**Proposition.** Let X be an r.v. defined on  $(\Omega, Pr)$ .

$$Var[X] = E[X^2] - \mu^2$$
 (where  $\forall w \in \Omega$   $X^2(w) = (X(w))^2$ )

**Proof.** Using linearity of expectation and the fact that  $\mu$  is a constant:

$$E[(X - \mu)^{2}] = E[X^{2} - 2\mu X + \mu^{2}]$$

$$= E[X^{2}] - 2\mu E[X] + \mu^{2}$$

$$= E[X^{2}] - 2\mu^{2} + \mu^{2}$$

Linearity of variance?

**Proposition** 
$$Var[c X] = c^2 Var[X]$$

**Proof.** 
$$Var[cX] = E[(cX)^2] - (E[cX])^2 = E[c^2X^2] - (cE[X])^2$$
  
=  $c^2 E[X^2] - c^2(E[X])^2 = c^2(E[X^2] - (E[X])^2) = c^2 Var[X]$ 

In general, variance does not distribute over sums. However:

**Proposition.** if 
$$X \perp Y$$
 then  $Var[X + Y] = Var[X] + Var[Y]$ .

**Proof.** In the segment entitled "Correlated random variables" we define **product** of r.v.'s and show

- 1) Var[X + Y] = Var[X] + Var[Y] iff E[XY] = E[X]E[Y]
- 2)  $X \perp Y \Rightarrow E[XY] = E[X]E[Y]$ . The proposition follows.

When we roll two fair green-purple dice, we have S = G + P and  $G \perp P$ .

Therefore, 
$$Var[S] = Var[G] + Var[P] = (35/12) + (35/12) = 35/6$$
.

Kondom Varidey are independent

#### Binomial random variables

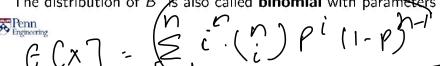
An r.v.  $B:\Omega\to\mathbb{R}$  is called **binomial** with parameters  $n\in\mathbb{N}$  and  $p\in[0,1]$ when Val(B) = [0..n] and  $\forall k \in [0..n]$   $Pr[B = k] = \binom{n}{\nu} p^k (1-p)^{n-k}$ .

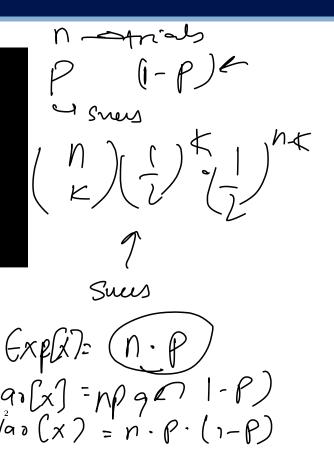
How does such an r.v. arise? For example, perform n IID Bernoulli trials with probability of success p and let B be the r.v. that returns the number of successes observed. Clearly Val(B) = [0..n]. Then:

We have seen before the probability space on which B is defined: the outcomes are the  $2^n$  sequences of length n of S's (for "success") and F's (for "failure"). An outcome with k S's has probability  $p^k(1-p)^{n-k}$ .

There are  $\binom{n}{k}$  outcomes with k S's. Therefore the probability of the event "k successes observed" is  $\binom{n}{k} p^k (1-p)^{n-k}$ .

The distribution of B is also called **binomial** with parameters n and p.





## Question 1

Linda is playing cornhole. For each throw, if the beanbag lands in the hole, then Linda gets 3 points, if the beanbag lands on the board, then Linda gets 1 point, and if the beanbag does not land on the board or in the hole, then Linda gets 0 points. Linda throws 5 beanbags (all throws are independent). What is the variance of Linda's points?

#### Question 2

Given a homework with 7 problems. For each problem, you have a 30% chance to lose 2 points for not mentioning something.

What is the expectation of total loss of points?

$$x_i = the r \cdot v$$
 denoting the interior pts lost  
in  $Q_i$ :
$$C[x_1 + x_2 + x_3 + \dots + x_7] \leftarrow Total points$$

$$= \sum_{i=1}^{7} f(x_i)$$

$$-2 \cdot \frac{1}{3} = f(x_1)^{-1} - \frac{2}{3}$$

$$= f(x_1)^{-1} - \frac{2}{3} \times 1 = -14$$

$$= -14$$

$$= -14$$

#### Question 3

You roll a die 3 times (R1, R2, R3). Let X be the random variable representing the difference of R1(R2) and let Y be the random variable representing the difference of R2-R3. Prove whether X and Y are independent or dependent.

$$P_{Y}(X=n \cap Y=y) \neq P_{Y}(X=n) \cdot P_{X}(Y=y)$$
  
 $X = R_{1} - R_{2} = 1$   
 $Y = R_{2} - R_{3} = 1$ 

$$P_{1}(X_{1}-X_{2}-1) \cap X_{2}-X_{3}=1)$$

$$(X_{1},X_{2},X_{3}) \quad X_{2}=1 \quad X_{2}=1 \quad X_{2}=1 \quad X_{3}=1$$

$$(X_{1},X_{2},X_{3}) \quad X_{2}=1 \quad X_{3}=1$$



$$6x6x6 = \begin{pmatrix} 3 & 2 \\ 4 & 32 \end{pmatrix}$$

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$$X^{2}R_{1}-R_{L}$$

$$Y=R_{2}-R_{3}$$

# Before you go...

Do you have any general questions about this week's assignment? We can't go into minute details about any specific questions, but this is the time to ask any general clarifying/confusing questions!

As always, if you can't talk to us during recitation or office hours, post any questions/anything course related on the forums or email <a href="mailto:mcitonline@seas.upenn.edu">mcitonline@seas.upenn.edu</a>. Ask questions that might be beneficial to other students on the forums, while emailing about more personal questions (regrade requests, etc).