MCIT 592 Spring 2022 **Homework 5** **Solutions**

Problem Set

1. [10 pts] Show that each of the following functions is not a bijection by giving either

- an element of the codomain that is not in the range, or
- two elements of the domain that map to the same element in the range.

Be sure to explain why each one is not a bijection!

(a) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 8x$

(b) $g : \mathbb{N} \to \mathbb{N}$ given by $g(x) = x + 7$

(c) $h : [11..16] \to [12..16]$ given by $h(x) = \begin{cases} 16 & \text{if } x = 11 \\ x & \text{otherwise} \end{cases}$

(d) $j : \mathbb{N} \to \mathbb{N}$ given by $j(x) = \begin{cases} (x+1)^2 & \text{if } x \text{ is even} \\ 2x+1 & \text{if } x \text{ is odd} \end{cases}$

(e) $k : [-7..10] \to [0..12]$ given by $k(x) = |x + 2|$

Solution:

(a) Not surjective because there is no $x \in \mathbb{N}$ such that $f(x) = 2$.

(b) Not surjective because there is no $x \in \mathbb{N}$ such that $g(x) = 4$.

(c) Not injective because $h(11) = h(16)$.

(d) Not surjective because there is no $x \in \mathbb{N}$ such that $j(x) = 0$.

(e) Not injective because $k(-4) = k(0)$.

2. [**10 pts**] Recall that a derangement is a permutation where no element ends up in its original position. In this problem we consider a different, related concept: *deranged anagrams*. We say that an anagram is deranged if no letter ends up in its original position and no letter ends up in the original position of an identical letter. For example, `ffeeco` is a deranged anagram of `coffee`, but `eefcof` is not.

There are $\frac{(2+2+1)!}{2!2!1!} = 30$ anagrams of `radar`. How many of them are deranged?

Solution:

Since neither `r` can be in its original position, there are only three possibilities for which two letters are `r` in a deranged anagram:

(i) the second and fourth letter are `r`,

(ii) the second and third letter are `r`, or

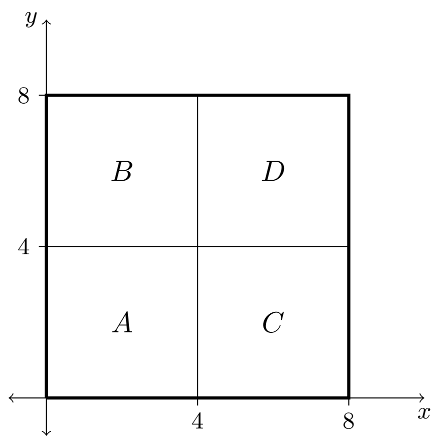
(iii) the third and fourth letter are `r`.

In the case (i), the `d` can be the first or fifth letter, so this case includes two anagrams. In case (ii), the `d` must be the fourth letter, since the fourth letter cannot be `a`, so this case includes one anagram. In case (iii), the `d` must be the second letter, since the second letter cannot be `a`, so this case includes one anagram. The cases are disjoint, so we can use the addition rule to conclude that there are $2 + 1 + 1 = \boxed{4}$ deranged anagrams. They are `drara`, `arard`, `arrda`, and `adrra`.

3. [**10 pts**] Let there be a room that is 8 feet by 8 feet. Suppose that there are 5 people who sit in this room. For simplicity, assume these people are just points. Prove that, among these people, there is some pair that is seated at most $4\sqrt{2}$ feet from each other.

Solution:

Suppose the room is represented by [0,8] $\times$ [0,8]. We use the pigeonhole principle. The "pigeons" are the five people, and the "pigeonholes" are the four smaller squares $A = [0,4] \times [0,4]$, $B = [0,4] \times (4,8]$, $C = (4,8] \times [0,4]$, and $D = (4,8] \times (4,8]$. Notice that $S = A \cup B \cup C \cup D$, and that A, B, C, and D are pairwise disjoint.



Since $5 > 4$, the pigeonhole principle tells us that at least two of the points must be in the same smaller square. Let (x_1, y_1) and (x_2, y_2) be the coordinates of two such points. The distance between them is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \leq \sqrt{4^2 + 4^2} = 4\sqrt{2}$. $\square$

4. **[10 pts]** Alex wants to renew his wardrobe by buying a new item every day. He buys three types of items; shoes, shirts, and pants, and the store never runs out of an item. Alex, being the diligent TA he is, wants to plan out his buying options for the next $n \geq 1$ days, and decides that the easiest way to do this is to create a function f such that for every day i, he will buy item $f(i)$ (where the codomain is $\{shoes, shirt, pants\}$, and he buys one item per day). Because he wants to have full outfits, Alex will only consider functions f that allow him to buy each type of item at least once. How many such functions f can Alex devise?

Solution:

We can consider this problem as finding the number of surjections from the n days to the three clothing types. We will let our domain be $A = \{1, 2, \ldots, n\}$, where each element is a day, and the codomain be $B = \{x, y, z\}$, where each letter represents a clothing item.

Instead of counting the surjections directly, we can use complementary counting. First, we will count the total number of functions $f : A \to B$, which we denote B^A. We know that $|B^A| = |B|^{|A|}$, so there are 3^n functions.

Now, we want to count the number of functions that are not surjective. Let F_i denote the set of functions $f \in B^A$ such that item i is not in the range of f. We want to find $|F_x \cup F_y \cup F_z|$, the number of functions for which at least one of x, y, or z are not mapped to. Using PIE, $|F_x \cup F_y \cup F_z| = |F_x| + |F_y| + |F_z| - |F_x \cap F_y| - |F_y \cap F_z| - |F_x \cap F_z| + |F_x \cap F_y \cap F_z|$. The number of functions in F_x is the number of functions that only map to y and z, so we can consider the functions as $\{y, z\}^A$, which has cardinality $|\{x, y\}^A| = |\{x, y\}|^{|A|} = 2^n$. This same argument can be repeated to find that $|F_x| = |F_y| = |F_z| = 2^n$.

Next, we need to find $|F_x \cap F_y|$. This is the set of functions that don't map to to x and don't map to y. The functions must all then map to z. There is only one such function, where every element in A is mapped to z. The same argument holds for $|F_y \cap F_z|$ and $|F_x \cap F_z|$, so we have that $|F_x \cap F_y| = |F_y \cap F_z| = |F_x \cap F_z| = 1$.

Next, we want to consider the functions $|F_x \cap F_y \cap F_z|$, which is the set of functions that don't map to x, y, or z. Thus, the range of such a function is $\emptyset$, which is impossible, so there are no such functions, and $|F_x \cap F_y \cap F_z| = 0$

Plugging in values into the original expression for $|F_a \cup F_b \cup F_c|$, we have

$|F_a \cup F_b \cup F_c| = |F_x| + |F_y| + |F_z| - |F_x \cap F_y| - |F_y \cap F_z| - |F_x \cap F_z| + |F_x \cap F_y \cap F_z| = 3 \cdot 2^n - 3 \cdot 1 + 0 = 3 \cdot 2^n - 3$, the number of functions $f \in B^A$ such that f is not surjective. We have 3^n functions in B^A and $3 \cdot 2^n - 3$ functions in B^A that are not surjective, so there are $3^n - (3 \cdot 2^n - 3) = \boxed{3^n - 3 \cdot 2^n + 3}$ surjective functions.

5. [10 pts] After an important and successful meeting, employees of a company shake hands with each other. There are $n \geq 2$ employees at the meeting. Any of these employees can shake hands with any number of the other employees (including zero), but two employees can only shake each others' hands once. A handshake is a mutual event between exactly two employees.

Prove or disprove the claim that there must always be at least two employees who shake the same number of hands.

Solution:

Note that each employee could have shook hands with at most $n - 1$ people (everyone else). Furthermore, note that it is impossible for there to simultaneously exist someone who shook hands with everybody else ($n - 1$ people) and another person who shook hands with 0 people, since handshakes are mutual. Therefore, the possible values of the number of handshakes people gave are either all in the range $[0..n - 2]$ or all in the range $[1..n - 1]$. In either case, there are $n - 1$ possible values for each employee's handshake total.

We allow the $n - 1$ possible values to be holes and the n employees to be the pigeons, where an employee gets assigned to the hole corresponding to the number of handshakes they gave. By the Pigeonhole Principle, there exists a hole (number of handshakes) with at least two distinct pigeons (employees). Therefore, there must be at least two employees who have shook hands with the same number of people.