Recitation Module 13

Penn Engineering

Basic terms/definitions to know!!

- Spanning Trees (two algorithms: edge pruning, and edge growing)

  A **spanning subgraph** of the graph $G = (V, E)$ is a subgraph whose vertex set is the entire set V. A **spanning tree (forest)** of G is a spanning subgraph that is a tree (a forest).

- Chromatic Number of a graph

  The smallest k such that G is k-colorable is called the **chromatic number** of G and is denoted by $\chi(G)$.

- Bipartite Graphs (i.e. 2-colorable graphs)
  - Property: A graph is bipartite iff it does not contain a cycle of odd length.

Basic terms/definitions to know!!

- Distance: the length of the shortest path between two vertices
- Cliques: complete subgraphs
- Independent Sets: subgraph where no two vertices adjacent
- Graph Complement

Let $G = (V, E)$ be a graph. The **complement** of G is the graph $\overline{G} = (V, \overline{E})$ where

$$\overline{E} = \{\{u, v\} \mid u, v \in V \wedge u \neq v \wedge \{u, v\} \notin E\}.$$

Question 1

a. How many cliques of size k does K_n have?

b. Calculate the size of a maximum clique in C_n.

Answer to Question 1

a. Since there is an edge between every two vertices in K_n, if we pick any subset of k vertices it will form a clique. Thus, there are

n choose k

cliques in K_n.

b. C_3 has a maximum clique of size 3. If $n > 3$ then no subset of 3 or more vertices is complete. Thus the maximum size of a clique is 2 (every pair of adjacent nodes.

Question 2

Prove the following statement: If a graph G is not 5-colorable (i.e. the chromatic color of G is greater than 5), then there must be at least 2 odd-length cycles in the graph that do not share any vertices.

Answer to Question 2 (Solution 1)

We will prove the contrapositive: If G does not contain at least two odd-length vertex-disjoint cycles, then the chromatic color of G is less than or equal to 5. (There are either 1 or 0 odd-length vertex-disjoint cycles in G.)

Case 1: G has no odd cycles, so G is bipartite and 2-colorable.

Case 2: G has one odd cycle. Let C be the smallest odd-length cycle in G. Let $G' = G - V(C)$ where $V(C)$ is the vertex set of C. Since all other odd-length cycles in G must have shared a vertex with C, there are no odd-length cycles in G'. Thus, G' is bipartite and 2-colorable.

Answer to Question 2 (Solution 1)

Case 2 (cont): *C* is an odd-length cycle which is 3-colorable.

Now consider vertices u,v in C and some edge (*u, v)* in *G* but not in *C*. (This might make the coloring of *C* invalid because *u* and *v* might have the same color.) We will prove that edge *(u, v)* can't exist.

Assume for contradiction that it does. Then there are two paths P_1 and P_2 from *u* to *v* along *C*, and P_1 and P_2 form *C.* Since C has an odd length, one of P_1 and P_2 must have even length and the other must have odd length. W.L.O.G., assume P_1 has even length. Then the cycle that is formed with P_1 and edge *(u, v)* is of odd length.

Answer to Question 2 (Solution 1)

However, this is a contradiction because C is the smallest odd-length cycle.

Thus, in case 2 we can color G with 5 colors.

Answer to Question 2 (Solution 2)

If G is not 5-colorable, then at least 6 colors are required to colored the graph. Let the colors used be denoted by C_1, C_2, C_3, …

Consider the subgraph induced by $C_1 \cup C_2 \cup C_3$. This subgraph must contain an odd cycle, otherwise it would 2-colorable. Similarly, there must be another odd cycle in the subgraph induced by $C_4 \cup C_5 \cup C_6$. These two odd-length cycles are vertex disjoint because each vertex can only receive one coloring.

Question 3

Suppose that $f : V \to [1..k]$ is a proper k-coloring of a graph $G = (V, E)$ for some $k \geq 1$. Prove that G has some independent set $S \subseteq V$ with $|S| \geq \frac{|V|}{k}$.

Answer to Question 3

All vertices that have the same coloring form an independent set.

For all vertices $v \in V$, we assign each v a color/number from 1 to k such that it meets the criteria of a proper k-coloring.

Let V_i be the set of all vertices which have color/number i.

Using Pigeonhole Principle:

Pigeons: vertices. there are $|V|$ pigeons

Pigeonholes: each set V_i, there are k pigeonholes

By PHP, we know that there must be at least one independent set of at least $\lceil \frac{|V|}{k|} \rceil$.

So $|S| \geq \lceil \frac{|V|}{k|} \rceil$.

(Recall: In a graph with proper coloring, all vertices that have the same color form an independent set.)

Question 4

Consider a connected graph $G = (V, E)$ and an arbitrary partition of G's vertex set V into nonempty sets S and $V \setminus S$. Prove that if there exists only one edge e between the vertices in S and the vertices in $V \setminus S$, then e must be in every spanning tree of G.

Answer to Question 4

Consider an arbitrary spanning tree of G, call it T. Since T is a tree, it is connected and acyclic. Connected means that there is a path between any pair of vertices.

Consider a vertex $x \in S$ and a vertex $y \in V \setminus S$. Because T is connected, there must be a path from x to y. Call this path p.

Since x is in S and y is in $V \setminus S$, we know that the path p must start from S and end in $V \setminus S$, so there must be at least one edge in p that has an endpoint in S and an endpoint in $V \setminus S$. By assumption, the only edge that does this is e. So e must be in p; hence; e must be in the spanning tree.

Before you go...

Do you have any general questions about this week's assignment? We can't go into minute details about any specific questions, but this is the time to ask any general clarifying/confusing questions!

As always, if you can't talk to us during recitation or office hours, post any questions/anything course related on the forums or email mcitonline@seas.upenn.edu. Ask questions that might be beneficial to other students on the forums, while emailing about more personal questions (regrade requests, etc).