Self-paced Example: Counting Pizza Pieces with Cross-points

Module 6

MCIT Online - CIT592 - Professor Val Tannen

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion. When you see the following marker:

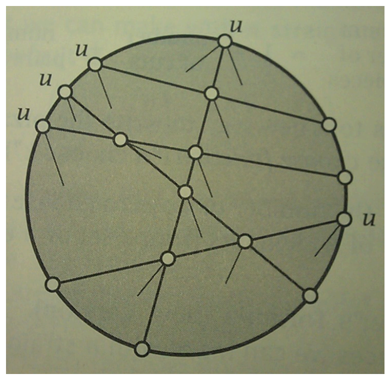
we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Counting pizza pieces with cross-points

In a previous segment, we showed you how to approach the following problem using a recurrence relation.

Problem. What is the largest number of pieces (not slices!) of pizza that can be made with n distinct straight cuts?

Answer. Now we will show you another way of counting the number of pieces.



In the picture above, observe that there are two kinds of **cross-points**: boundary cross-points (where a cut intersects the crust) and interior cross-points (where two cuts intersect).

Imagine that we rotate the pizza to make sure no cut is "horizontal."

To help with counting we think of a mapping from pieces to the cross-point at the "top" of the piece.

Note that every interior cross-point is mapped to from exactly one piece.

How many interior cross-points are there? Can you figure the answer to this question?

Counting pizza pieces with cross-points (continued)

Any two cuts give one interior cross-point. There are $\binom{n}{2}$ sets of two cuts, thus there are $\binom{n}{2}$ interior cross-points.

Every cut produces two boundary cross-points. Since we made sure no cut is horizontal, one of these two is "higher." We call these "upper" boundary cross-points and they are marked by `u` in the picture.

Note that every upper boundary cross-point is mapped to from exactly one piece except the one of the "top" of the pizza which is mapped to from two pieces.

How many upper boundary cross-points are there? Can you figure the answer to this question?

Any of the n cuts produces a distinct upper boundary cross-point. So there are n of them.

Therefore, by the addition rule, the number of interior or upper boundary cross-points

$$n + \binom{n}{2} = n + \frac{n(n-1)}{2} = \frac{2n + n^2 - n}{2} = \frac{n^2+n}{2}$$

The pieces map one-to-one to these cross-points except for the top boundary one. **Two pieces** map to that one.

So for the number of pieces we have to add 1 to the number of interior and upper boundary cross-points. We obtain the same answer as in the previous segment.

$$\frac{n^2+n}{2} + 1 = \frac{n^2+n+2}{2}$$