Module 1.1: Addition and Multiplication Rules

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Combinatorics = counting techniques

Counting techniques belong to a branch of mathematics called **Combinatorics**.

Combinatorics problems range from the super-easy to the super-complicated. Naturally, we start at the easy end :).

Problem. Grandmother's market stall has 23 apples, 18 oranges, and 31 peaches for sale. How many fruits does Granny have for sale?

Answer: Granny has for sale $23 + 18 + 31 = 72$ fruits.

We just applied here the first rule of combinatorics: how to count objects of different kinds!

The addition rule

Suppose that the objects we count can be classified into k separate kinds, $1, 2, \ldots, k$. Suppose we count

(1) n_1 objects of the first kind,

(2) n_2 objects of the second kind,

$\ldots$

(k) n_k objects of the k'th kind,

then the total number of objects counted is $n_1 + n_2 + \cdots + n_k$.

Problem. How many integers have an absolute value that is at most 10?

Answer: There are 10 integers between 1 and 10, inclusive, another 10 between -10 and -1, inclusive, and then there is 0. In total $10 + 10 + 1 = 21$.

How many types of sandwich?

Problem. A local store that serves sandwiches offers a choice of 4 kinds of bread and, for filling, it offers 2 kinds of meat or 3 kinds of cheese. How many different types of sandwich are available?

Answer: First we count, using the addition rule, the number of different kinds of filling: $2 + 3 = 5$.

Next each of the 4 kinds of bread can be combined with each of the 5 kinds of filling. This gives us $4 \cdot 5 = 20$ types of sandwich.

We have applied a different counting rule here. A sandwich is "constructed" in two *steps*:

(1) Choose one of 4 breads.

(2) Choose one of 5 fillings.

The multiplication rule

Suppose that a procedure that constructs objects of some kind can be broken down into k **steps** and

(1) the first step can be performed in n_1 ways,

(2) the second step can be performed in n_2 ways, **regardless** of how the first step was performed,

. . .

(k) and the k^{th} step can be performed in n_k ways, regardless of how **all the preceding** steps were performed,

then the entire procedure can be performed in $n_1 \cdot n_2 \cdots n_k$ different ways.

If every different way of performing the procedure constructs a different object then $n_1 \cdot n_2 \cdots n_k$ distinct objects are constructed.

How many seats?

Problem. The seats of a concert hall are to be labeled with an uppercase letter of the English alphabet and a 2-digit number between 1 and 100. What is the largest count of seats that can be labeled differently?

Answer: The English alphabet has 26 upper-case letters.

The 2-digit numbers between 1 and 100 are $10, 11, \ldots, 98, 99$. One way to count them is to subtract from 100 the count of numbers of 1 digit $(1, \ldots, 9)$ or 3 digits (100): $100 - 9 - 1 = 90$.

A seat can be labeled using the following two steps:

(1) Choose an upper-case letter. Can be done in 26 ways.

(2) Choose a number among $10, 11, \ldots, 98, 99$. Can be done in 90 ways.

By the multiplication rule a seat can be labeled in $26 \cdot 90 = 2340$ ways.

Counting club officer assignments I

Problem. Three club officers - a president, a treasurer, and a secretary - are to be chosen from among four people: Ann, Bob, Cindi, and Dan. Suppose that for various reasons, Bob cannot be the president and either Cindi or Dan must be the secretary. In how many ways can the officers be chosen?

Answer: [FIRST ATTEMPT] The three club officers could be chosen as follows:

(1) Choose the president. Can be done in 3 ways (not Bob).

(2) Choose the treasurer. In 3 ways (not the person chosen in step (1)).

(3) Choose the secretary. In 2 ways (Cindi or Dan).

Apparently (by the multiplication rule): $3 \cdot 3 \cdot 2 = 18$ officer groups.

Incorrect! Because step (3) depends on **how** steps (1) and (2) were done. What if Cindi was chosen president and Dan treasurer?!

Counting club officer assignments II

Answer: [SECOND ATTEMPT] The three club officers could be chosen as follows:

(1) Choose the secretary. Can be done in 2 ways (Cindi or Dan).

(2) Choose the president. In 2 ways (neither Bob nor the person chosen in step (1)).

(3) Choose the treasurer. In 2 ways (either of the two persons left).

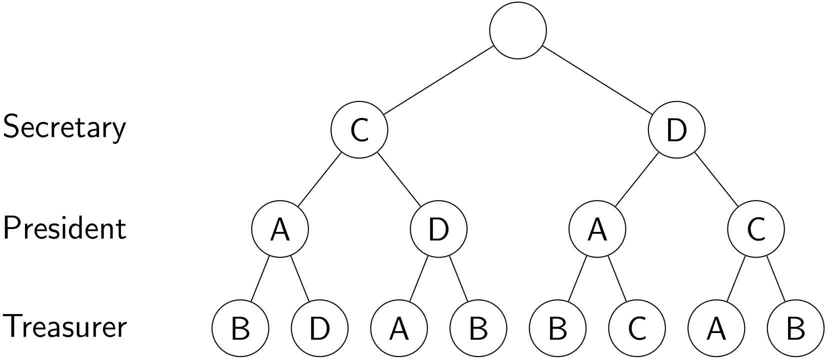
By the multiplication rule: $2 \cdot 2 \cdot 2 = 8$ officer groups.

When using the multiplication rule there may be different orders in which the steps can be performed.

Also keep in mind: it's better to make the **most restrictive** choices first and the least restrictive last.

Activity : Tree of possibilities

Possible outcomes of a sequence of choices can be drawn in a **tree**. This tree represents picking the secretary, then the president, then the treasurer:



(Remember: Bob can't be president, and Cindi or Dan must be secretary.)

Moving from top to bottom, the leftmost path corresponds to picking Cindi (C) as secretary, Ann (A) as president, and Bob (B) as treasurer.

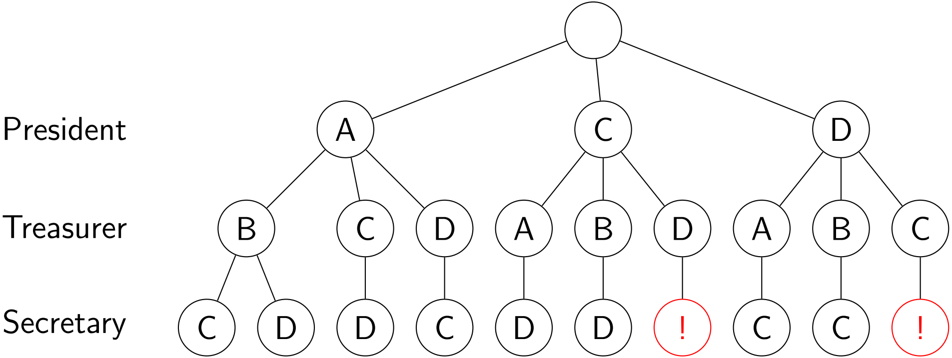
ACTIVITY : Tree of possibilities (Continued)

The tree of possibilities you just saw corresponds to the [SECOND ATTEMPT] described in the lecture.

Problem. Using a pencil and paper, draw the tree of possibilities that represents picking the president, then the treasurer, then the secretary, as we did in the lecture in the [FIRST ATTEMPT].

Activity : Tree of possibilities (Continued)

Answer.



Notice that in the paths marked with a !, there is no valid option for secretary; in both cases, Cindi and Dan already have other roles.

ACTIVITY : Tree of possibilities (Continued)

Discussion. In drawing the tree, you must have noticed that the number of options available in each step depends on the choices made in previous steps, which is why the multiplication rule does not apply here.

If we choose Ann as president and Bob as treasurer, then there are two options for secretary. But if we choose Ann as president and Cindi as treasurer, then there is only one option for secretary. And if we choose Cindi as president and Dan as treasurer, then we have reached a "dead end," and there is no valid option for secretary in this case.