Module 2.1: Counting Subsets

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Counting subsets

Problem. Let A be a set with n elements, that is, $|A| = n$. How many distinct subsets of A are there?

Answer. If A is empty then it has exactly one subset, the empty set.

Suppose A has $n \geq 1$ elements, say, $A = \{x_1, x_2, \ldots, x_n\}$.

Each subset S of A can be constructed in n steps:

(1) Decide whether to include x_1 in S or not: can be done in 2 ways.

$\ldots$

(n) Decide whether to include x_n in S or not: 2 ways.

By the multiplication rule the answer is $2 \cdot 2 \cdots 2 = 2^n$.

Cardinality of powerset

Recall the definition of the **powerset** 2^A of set A.

What is the cardinality of 2^A?

This is the same as asking how many subsets of A there are.

We have just seen that if A is empty, $|2^A| = 1$.

We have also seen that if $|A| = n \geq 1$ then $|2^A| = 2^n$.

But $2^0 = 1$.

Proposition. For *any* set A we have

$$|\,2^A\,| \;=\; 2^{|\,A\,|}$$

Counting pets I

Problem. Animal Rescue has 5 cats and 3 dogs. How many different groups of pets can you adopt, knowing that you must adopt at least one dog and at least one cat, and you might adopt all of them?

Answer. We try to construct a group of pets as follows:

(1) Choose one of the cats: can be done in 5 ways.

(2) Choose one of the dogs: 3 ways.

(3) Choose a subset of the remaining $4 + 2 = 6$ pets: 2^6 ways.

(We have just seen that a set with n elements has 2^n subsets.)

Answer $5 \cdot 3 \cdot 2^6$? But this is wrong!

Counting pets II

Wrong! Because of *overlaps* between step 3 and each of the first two steps, some groups are counted more than once!

For example, the group {Maimu,CousCous,Archer} is counted twice.

Here's how it is counted twice:

Choose CousCous in step 1, Archer in step 2, and { Maimu } in step 3.

Choose Maimu in step 1, Archer in step 2, and { CousCous } in step 3!

We call this *overcounting*.

Counting pets III

We can still use the multiplication rule, but in a different way.

We construct a group of pets as follows:

(1) Choose a non-empty subset of cats: $2^5 - 1$ ways.

(2) Choose a non-empty subset of dogs: $2^3 - 1$ ways.

(A set with n elements has $2^n - 1$ non-empty subsets.)

By the multiplication rule, the answer is

$(2^5 - 1)(2^3 - 1) = 31 \cdot 7 = 217 \quad$ different groups of pets.

Counting pets IV

Another method is to count **complementarily**: subtract from total number of sets of pets the number of those sets of pets that you *cannot* adopt.

The total number of subsets (including the empty set) of $5 + 3 = 8$ pets: 2^8.

Which sets of pets cannot be adopted? Those of just cats or just dogs!

Total number of sets of cats: 2^5.

Total number of sets of dogs: 2^3.

Is the answer $2^8 - (2^5 + 2^3)$?

No! We have *oversubtracted*: the empty set of cats is the same as the empty set of dogs!

The answer is $2^8 - 2^5 - 2^3 \, \mathbf{+1} = 256 - 32 - 8 + 1 = 217$ again.

Module 2.2: Counting Words and Strings

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Lecture Notes

Counting words

Recall that the English alphabet uses 26 letters: 5 vowels and 21 consonants.

Problem. How many words of length 3 can be formed with letters of the English alphabet?

Answer. Such a word has 3 positions. Example: $\underline{h}\ \underline{a}\ \underline{h}$.

Therefore each such word can be constructed in 3 steps:

(1) Put a letter in the first position: can be done in 26 ways.

(2) Put a letter in the second position: 26 ways.

(3) Put a letter in the third position: 26 ways.

By the *multiplication rule*, the answer is $26 \times 26 \times 26 = 26^3$.

Counting strings of bits

Recall that there are two **bits** (*bi*nary digi*ts*): 0 and 1.

Problem. How many strings of bits of length n are there?

Answer. Such a string has n positions. Example: $\underline{1}\ \underline{0}\ \cdots \underline{1}\ \underline{1}$.

Therefore each such string can be constructed in n steps:

(1) Put a bit in the first position: can be done in 2 ways.

. . .

(n) Put a bit in the n'th position: 2 ways.

Again, by the multiplication rule the answer is $2 \cdot 2 \cdots 2 = 2^n$.

Counting sequences, in general

We saw that there are 26^3 **words** of length 3 and 2^n **strings** of bits.

Similarly, let's count **words** of length 6 made only of consonants: 21^6.

Also, let's count **words** of length m made only of vowels: 5^m.

In general, using the multiplication rule, we count

n^ℓ **sequences** of length ℓ made of elements from a set of size n.

Activity : All Distinct Subsets

We have seen in the lecture segment "Counting subsets" how to apply the multiplication rule to counting the total number of distinct subsets of a set. Let us revisit that here.

Given a set $\{a_1, \ldots, a_n\}$, we construct a subset S by considering, for each element in $\{a_1, \ldots, a_n\}$, whether or not to include that element in S. This can be done in n steps by deciding, in step i, whether or not to include the element a_i in S.

ACTIVITY : All Distinct Subsets (Continued)

For example, suppose that $n = 4$ and we are constructing a subset of $\{a_1, a_2, a_3, a_4\}$. One possible way to do this is as follows.

Step 1 : Decide not to include a_1.

Step 2 : Decide to include a_2.

Step 3 : Decide to include a_3.

Step 4 : Decide not to include a_4.

The resulting subset is $S = \{a_2, a_3\}$.

Activity : All Distinct Subsets (Continued)

Constructing a subset in this way is analogous to constructing a binary string of length n. Putting a 1 in the i^{th} position corresponds to including the element a_i in S, and putting a 0 in the i^{th} position corresponds to *not* including the element a_i in S.

In this way, the subset $\{a_2, a_3\}$ of $\{a_1, a_2, a_3, a_4\}$ that we constructed above corresponds to the 4-bit binary string 0110.

This way of representating subsets gives a *one-to-one correspondence* between subsets of $\{a_1, \ldots, a_n\}$ and binary strings of length n.

ACTIVITY : All Distinct Subsets (Continued)

What does this correspondence tell us about the two problems of counting subsets of a set of cardinality n and counting binary strings of length n?

It tells us that these two counting problems are precisely equivalent!

There are exactly 2^n distinct subsets of $\{a_1, \ldots, a_n\}$, and there are exactly 2^n distinct binary strings of length n.

Question:

What subset of $\{a_1, a_2, a_3, a_4\}$ corresponds to the string 1001?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : All Distinct Subsets (Continued)

What subset of $\{a_1, a_2, a_3, a_4\}$ is represented by the string 1001?

Answer:

Since the first and fourth bits in the string are 1, and the other bits are 0, the subset is $\{a_1, a_4\}$.

Module 2.3: Permutations

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Lecture Notes

Permutations

Let A be a non-empty set with n elements, that is, $|A| = n$.
A **permutation** of A is an ordering of the elements of A in a row, i.e., a sequence of **all** the elements of A, **without repetition**.

The length of a permutation of A is n.

Example:

The set $\{x, 2, a\}$ has six permutations:

. $x2a$, $xa2$, $2xa$, $2ax$, $ax2$, $a2x$.

Sequences built from $\{x, 2, a\}$ that are **not** permutations: $aa2$ $a2$

Partial permutations

Again consider a non-empty set A with n elements. Let $1 \leq r \leq n$.

A **partial permutation of** r **out of the** n **elements of** A consists of picking r of the elements of the set and ordering them in a row, i.e., a sequence of length r, **without repetition**, whose elements are from the set A.

Example:

Here are the partial permutations of 2 out of the 3 elements of $\{x, 2, a\}$:

. $x2, \quad 2x, \quad xa, \quad ax, \quad 2a, \quad a2.$

Sequences built from $\{x, 2, a\}$ that are **not** partial permutations of 2 out of the 3 elements: $aa \quad a \quad x2a$

Examining the two definitions we have given, we see that a partial permutation of n out of the n elements of a set A is the same as a permutation of A!

Counting partial permutations

Problem. Let A be a non-empty set with n elements (i.e., $|A| = n \geq 1$) and let $1 \leq r \leq n$.
How many partial permutations of r out of the n elements of A are there?

Answer. We can construct such a partial permutation in r steps, filling its positions, numbered $1, 2, \ldots, r$, consecutively:

(1) Pick an element of A to put in position 1. Can be done in n ways.

(2) Pick one of the remaining elements to put in position 2. In $n-1$ ways.

$\ldots$

(r) Pick one of remaining $n-(r-1)$ elements to put in position r. In $n-(r-1) = n-r+1$ ways.

By the multiplication rule the answer is $n \cdot (n-1) \cdots (n-r+1)$.
This is a product of r factors.

Factorial

We computed the number of partial permutations of r out of n as:

$$n \cdot (n-1) \cdots (n-r+1)$$

Note that this number depends only on n and r, and not on the set whose elements we use (as long as there are n of them).

Now take $r = n$. This gives the number of permutations of n elements. And there is a notation for this:

$$n! \; = \; n \cdot (n-1) \cdots 2 \cdot 1$$

read "the **factorial** of n".

We will use the factorial notation to shorten expressions.

For example, the number of partial permutations of r out of n

$$n \cdot (n-1) \cdots (n-r+1) \; = \; \frac{n \cdot (n-1) \cdots (n-r+1) \cdot (n-r) \cdots 1}{(n-r) \cdots 1} \; = \; \frac{n!}{(n-r)!}$$

Quiz

Let p be the number of permutations of 6 elements, and let q be the number of partial permutations of 3 out of 6 elements. What is p/q?

A. 2

B. 3

C. 6

Answer

Let p be the number of permutations of 6 elements, and let q be the number of partial permutations of 3 out of 6 elements. What is p/q?

A. 2

   Incorrect. Since $p = 6!$ and $q = \frac{6!}{3!}$, $\frac{p}{q} \neq 2$.

B. 3

   Incorrect. Since $p = 6!$ and $q = \frac{6!}{3!}$, $\frac{p}{q} \neq 3$.

C. 6

   Correct. Since $p = 6!$ and $q = \frac{6!}{3!}$, p divided by q is $3! = 6$.

Counting words with restrictions I

Problem. Consider the set of letters $\{a, b, c, d, e, f, g, h\}$.

(a) How many possible permutations are there of these letters?

(b) How many among the permutations of these letters contain the contiguous sequence abc?

Answer. Part (a): The set has 8 elements hence there are 8! permutations.

Part (b): A permutation of $\{a, b, c, d, e, f, g, h\}$ in which a, b, c appear in consecutive positions can be constructed as follows:

(1) Pick three consecutive positions for a, b, c. Can be done in 6 ways.

(2) Pick a permutation of $\{d, e, f, g, h\}$ and place it in the remaining $8 - 3 = 5$ positions. This can be done in 5! ways.

By the multiplication rule the total number of ways is $6 \cdot 5!$.

Counting words with restrictions II

Alternative answer. Part (b):

We can construct a desired permutation differently.
Consider the set of 6 letters: $\{x, d, e, f, g, h\}$

Construct a permutation of $\{x, d, e, f, g, h\}$. For example: $edhxfg$.

Next, replace in this permutation the letter x with the string abc.
In the example: $edhabcfg$.

(Any permutation with a, b, and c in consecutive positions can be transformed into a permutation of $\{x, d, e, f, g, h\}$ by replacing the portion abc with x. Thus, counting the desired permutations is the same as counting the permutations of $\{x, d, e, f, g, h\}$.)

There are $6!$ of these. And indeed $6! = 6 \cdot 5!$.

Module 2.4: Logical Structure of Statements

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Lecture Notes

Statements

Logical terminology and notation.

Elucidate the logical **structure** of mathematical assertions (statements).

Eventually, **proof patterns.**

Examples (of statements):

13 is prime.

5 is not even.

The letter *'a'* is a vowel.

If x is prime and x is not 2 then x is odd.

A letter ℓ is either a vowel or a consonant.

Except for the first one, which is a *basic statement* (also called "atomic statement"), each of these statements contain *logical connectives*.

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Basic statements and logical connectives

In logical notation we write the basic statement "13 is prime" as $prime(13)$.

Similarly, $even(5)$, $letter('a')$, $odd(x)$, $letter(\ell)$, $vowel(\ell)$, etc.

$x = 2$ is also a basic statement.

Note that x and ℓ are **variables**. To be discussed later.

For each logical connective we give the **(fancy name)**, the corresponding "plain language name", and our corresponding mathematical notation.

- **(conjunction)** "and" $\wedge$
- **(disjunction)** "or" $\vee$
- **(implication)** "if-then" $\Rightarrow$
- **(negation)** "not" $\neg$

Logical connectives allow us to combine basic statements into more complex statements.

Statements in logical notation

13 is prime $\qquad prime(13)$

5 is not even $\qquad \neg even(5)$

The letter *'a'* is a vowel $\qquad letter('a') \wedge vowel('a')$

If x is prime and x is not 2 then x is odd

$$[prime(x) \wedge (\neg(x = 2))] \;\Rightarrow\; odd(x)$$

A letter ℓ is either a vowel or a consonant (Two equivalent translations!)

$$letter(\ell) \;\Rightarrow\; [(vowel(\ell) \wedge \neg consonant(\ell)) \;\vee\; (consonant(\ell) \wedge \neg vowel(\ell))]$$

$$letter(\ell) \;\Rightarrow\; [(vowel(\ell) \vee consonant(\ell)) \;\wedge\; \neg(vowel(\ell) \wedge consonant(\ell))]$$

Logical set-builder notation

In **set-builder notation** $A = \{ x \mid P(x) \}$ $P(x)$ is a logical statement about x.

Let's redo some of the set-builder definitions

$$C = \{ \ell \mid \mathit{letter}(\ell) \wedge \neg \mathit{vowel}(\ell) \}$$

$$\mathbb{Z}^+ = \{ x \mid x \in \mathbb{N} \wedge x \neq 0 \}$$

$$A \cup B = \{ x \mid x \in A \vee x \in B \}$$

$$A \cap B = \{ x \mid x \in A \wedge x \in B \}$$

$$A \setminus B = \{ x \mid x \in A \wedge x \notin B \}$$

Implication, conditional, and equivalence

Recall that "if P_1 then P_2" is called **implication** and is written in logical notation: $P_1 \Rightarrow P_2$

P_1 is called the **premise** of the implication and P_2 is called its **conclusion**.

Inspired by some programming languages we ask for the logical notation for the **conditional** statement "if P_1 then P_2 else P_3".

It's $(P_1 \Rightarrow P_2) \land (\neg P_1 \Rightarrow P_3)$

Another statement is the **biconditional**: "if P_1 then P_2 and if P_2 then P_1".

Logical notation: $(P_1 \Rightarrow P_2) \land (P_2 \Rightarrow P_1)$

The biconditional is commonly written as "P_1 iff P_2" where "iff" abbreviates "if and only if", and is called **equivalence**. But logically it is the same.

Activity : Set Builder Notation

The **symmetric difference** of two sets A and B is defined by
$A \triangle B \; = \; (A \setminus B) \cup (B \setminus A)$.
Give an alternative definition to $A \triangle B$ using set-builder notation.

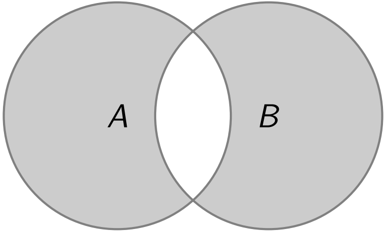
In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : Set Builder Notation (Continued)

Answer:

Using set-builder notation:
$A \triangle B = \{\ x \mid (x \in A \land x \notin B) \lor (x \notin A \land x \in B)\ \}.$

To better understand this operation, here is a diagram. Note that $A \triangle B$ contains elements that are either in A or in B but not in both.



Activity : Logical Notation

Recall a statement we proved in the first module:

> *If x is an integer such that $x > 1$, then $x^3 + 1$ is not prime.*

Write the statement above in logical notation.

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Logical Notation (Continued)

Answer:

$(int(x) \wedge greaterThanOne(x)) \Rightarrow \neg prime(x^3 + 1)$

Use the logical predicates $int(x)$, $greaterThanOne(x)$, and $prime(x)$ to represent the statements "x is an integer," "$x > 1$," and "x is prime," respectively.

Recall that the logical notation for "if x then y" is $x \Rightarrow y$, and the logical notation for "not x" is $\neg x$.

Therefore, we can write "If x is an integer such that $x > 1$ then $x^3 + 1$ is not prime" in logical notation as above.

ACTIVITY : More Logical Notation

Recall a statement we proved in the first module:

> *If p, r, s are positive integers such that $p = r \cdot s$ and p is prime, then one of r and s is 1 and the other one equals p.*

Write this statement in logical notation.

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : More Logical Notation (Continued)

Answer:

$$\big(posInt(p) \land posInt(r) \land posInt(s) \land (p = r \cdot s) \land prime(p)\big)$$
$$\Rightarrow \Big(\big((s = 1) \land (r = p)\big) \lor \big((s = p) \land (r = 1)\big)\Big)$$

Use the logical predicates $posInt(x)$ and $prime(x)$ to represent the statements "x is a positive integer" and "x is prime," respectively. We connect these predicates in the appropriate order using $\lor$, $\land$, $\Rightarrow$, and parentheses.

Module 2.5: Two Basic Proof Patterns

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Lecture Notes

Proof of an "if-then" statement

Recall the statement

> If $m + n$ is even then $m - n$ is even.

Logical structure: $\mathit{even}(m+n) \;\Rightarrow\; \mathit{even}(m-n)$.

What did we do?

We assumed the premise $\mathit{even}(m+n)$.

Then, $m + n = 2\ell$, for some integer ℓ (by definition of "even").

Then, $m = 2\ell - n$.

Then, $m - n = (2\ell - n) - n = 2\ell - 2n = 2(\ell - n)$.

Then, we satisfied the definition of "even" (by taking $k = \ell - n$).

We concluded $\mathit{even}(m-n)$.

The proof pattern for implication

You wish to prove $P_1 \Rightarrow P_2$

Proof pattern.

assert the **premise** P_1

(then derive/infer)

. . . logical/mathematical consequences . . .

(until you can)

assert the **conclusion** P_2

With all this you have proven $P_1 \Rightarrow P_2$.

A proof with cases I

Recall the statement

> If $p = r \cdot s$ and p is prime, then one of r and s equals 1 and the other one equals p.

Logical structure:

$$(p = r \cdot s) \wedge prime(p) \;\Rightarrow\; (r = 1 \wedge s = p) \vee (s = 1 \wedge r = p).$$

What did we do to prove this one? To begin with, we have an implication.

We assumed the premise $(p = r \cdot s) \wedge prime(p)$

Then, $r \mid p$

Then, since p is prime, $r = 1$ or $r = p$.

Then, we proceeded in **two cases**.

A proof with cases II

Because $r = 1$ or $r = p$ we can continue in two cases.

In the first case we assume $r = 1$.

Therefore $p = 1 \cdot s$.

And thus $s = p$.

Hence, $(r = 1 \land s = p) \lor (s = 1 \land r = p)$.

In the second case we assume $r = p$.

Therefore $p = p \cdot s$.

And thus $1 = s$.

Hence, $(r = 1 \land s = p) \lor (s = 1 \land r = p)$.

In both cases we have concluded $(r = 1 \land s = p) \lor (s = 1 \land r = p)$.

The by-cases proof pattern

Assuming $P_1 \lor P_2$ you wish to prove P_3.

Proof pattern.

assert $P_1 \lor P_2$

Case 1. assert P_1.

. . . logical/mathematical consequences . . .

assert P_3

Case 2. assert P_2.

. . . logical/mathematical consequences . . .

assert P_3

Since in both cases we obtained P_3, we have proved it assuming $P_1 \lor P_2$.

Some observations about by-cases

1. It generalizes easily to more than two cases. If we start from a disjunction of k statements, then we will have k cases.

2. The cases need not be *mutually exclusive*, as they were (almost) in our example: $(r = 1) \vee (r = p)$. We will give examples later in the course.

3. The disjunction that yields the cases need not appear as part of the assumptions in the original statement. In fact $(r = 1) \vee (r = p)$ did not. You can see a more striking example in another segment in this module.

Module 2.6: Combinations

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Lecture Notes

Combinations

Let A be a non-empty set with n elements, that is, $|A| = n$,
and let r be a natural number.

A **combination of r elements from the n elements of A** is an **unordered** selection of r of the n elements of A.

This is the same as a **subset** S of A of **size** r, that is, $S \subseteq A$ and $|S| = r$.

Example.

The combinations of 2 out of the 3 elements of $\{ x, 2, a \}$ are:

. $\{ x, 2 \}, \ \{ x, a \}, \ \{ 2, a \}$.

The combinations of 2 out of the 3 elements of $\{ o, u, i \}$ are:

. $\{ o, u \}, \ \{ o, i \}, \ \{ u, i \}$.

Clearly, the number of combinations does not depend on who the elements of A are, it only depends on how many elements there are.

The number of combinations I

Let n and r be natural numbers.

The number of combinations of r elements from the n elements of some set is denoted $\binom{n}{r}$.

Read $\binom{n}{r}$ as "n **choose** r".

We saw on the previous slide that $\binom{3}{2} = 3$.

The following can be verified easily:

$$\binom{n}{r} = \begin{cases} 0 & \text{if } r > n \\ 1 & \text{if } r = 0 \text{ or } r = n \\ n & \text{if } r = 1 \end{cases}$$

.

Quiz

Let's compare the number of combinations of 2 elements out of 4 elements with the number of (partial) permutations of 2 elements out of 4 elements. What do you think?

A. There are more combinations of 2 out of 4 than permutations of 2 out of 4.

B. There are more permutations of 2 out of 4 than combinations of 2 out of 4.

C. They are the same.

Answer

Let's compare the number of combinations of 2 elements out of 4 elements with the number of (partial) permutations of 2 elements out of 4 elements. What do you think?

A. More combinations of 2 out of 4 than permutations of 2 out of 4.
   Incorrect. For every combination of 2 elements out of 4, there are two ways of ordering these two elements.
B. More combinations of 2 out of 4 than permutations of 2 out of 4.
   Correct. For every combination of 2 elements out of 4, there are two ways of ordering these two elements.
C. They are the same.
   Incorrect. For every combination of 2 elements out of 4, there are two ways of ordering these two elements.

The number of combinations II

In a previous segment we gave a formula for the number of partial permutations of r out of n:

$$n \cdot (n-1) \cdots (n-r+1) \;=\; \tfrac{n!}{(n-r)!}$$

We give a different way to count the partial permutations. A partial permutation of r out of n can be constructed in two steps as follows:

(1) Choose r elements from the n elements. Can be done in $\binom{n}{r}$ ways.

(2) Arrange the chosen r elements in some order. In $r!$ ways.

By the multiplication rule the total number of partial permutations is $\binom{n}{r} \cdot r!$.

$$\frac{n!}{(n-r)!} = \binom{n}{r} \cdot r!$$

Hence

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Seating students

Problem. There are 15 students enrolled in a course, but exactly 12 students attend on any given day. The classroom for the course has 25 distinct (distinguishable) seats. How many different classroom seating arrangements are possible?

Answer. A classroom seating can be constructed in two steps as follows:

(1) Choose 12 students out of 15 that are enrolled.
Can be done in $\binom{15}{12}$ ways.

(2) Arrange the 12 chosen students among the 25 distinct seats available.
(A partial permutation of 12 out of 25.) In $\frac{25!}{(25-12)!} = \frac{25!}{13!}$ ways.

By the multiplication rule the total number of seating arrangements is

$$\binom{15}{12}\frac{25!}{13!} = \frac{15!}{12!\,3!} \cdot \frac{25!}{13!}$$

Committees when people are feuding I

Problem. From a group of 8 women and 6 men, how many different committees consisting of 3 women and 2 men can be formed?

What if 2 of the men are feuding and refuse to serve on the committee together?

Answer. (to the first part) Observe that a committee can be formed in two steps as follows:

(1) Choose the 3 women. Can be done in $\binom{8}{3}$ ways.

(2) Choose the 2 men. In $\binom{6}{2}$ ways.

By the multiplication rule the total number of different committees is $\binom{8}{3}\binom{6}{2}$.

Committees when people are feuding II

Answer. (to the second part) Say the feuding men are Bob and Dan.
We break the second step into **alternatives**.

(1) Choose the 3 women. Can be done in $\binom{8}{3}$ ways.

(2) Choose the 2 men:

- (2.1) Bob, but not Dan, choose 1 of the other 4. In $\binom{4}{1}$ ways.
- (2.2) Dan, but not Bob, choose 1 of the other 4. In $\binom{4}{1}$ ways.
- (2.3) No Bob, no Dan, choose 2 of the other 4. In $\binom{4}{2}$ ways.

For the alternatives we use the addition rule.
Step (2) can be done in $\binom{4}{1} + \binom{4}{1} + \binom{4}{2}$ ways.

Then, by the multiplication rule the total number of different committees is $\binom{8}{3} \cdot [\binom{4}{1} + \binom{4}{1} + \binom{4}{2}]$.

Committees when people are feuding III

Answer. (to the second part) Using **complementary counting**.

We find the number of **all** possible committees and then subtract the number of **"bad"** committees that have both Bob and Dan.

All committees (from the first part): $\binom{8}{3}\binom{6}{2}$.

The number of "bad" committees is the number of ways to choose the 3 women, $\binom{8}{3}$, since the men must be Bob and Dan.

This gives the answer $\binom{8}{3}\binom{6}{2} - \binom{8}{3}$.
Using the formula with factorials **check (!)** that

$$\binom{8}{3} \cdot \left[\binom{4}{1} + \binom{4}{1} + \binom{4}{2}\right] = \binom{8}{3}\binom{6}{2} - \binom{8}{3}$$

Module 2.7: Predicates and Quantifiers

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Lecture Notes

Predicates

In a previous segment we have seen **two** kinds of basic statements.

- Examples of the **first** kind: $odd(7) \quad odd(8)$.

These are called **propositions** and they are either **true** or **false**.

- Examples of the **second** kind: $odd(p) \quad vowel(\ell)$.

Because p and ℓ are **variables**, these basic statements are called **predicates**.

Predicates are **undetermined**, that is, neither true nor false, because the values of the variables are not specified.

Example. The complex statement: $integer(x) \wedge (x > 1) \;\Rightarrow\; \neg prime(x^3 + 1)$ remains undetermined unless we specify a value for x.

Quantifiers

The statement

$$\text{“}integer(x) \wedge (x > 1) \;\Rightarrow\; \neg prime(x^3+1)\text{”}$$

is undetermined, but

“For all integers x, if $x > 1$, then $x^3 + 1$ is not prime.”

is **true** since we have proved it!

“For all x” is called a **universal quantifier**. Notation: $\forall\, x$.

$$\text{"}\forall\, x \;\; integer(x) \wedge (x > 1) \;\Rightarrow\; \neg prime(x^3+1)\text{”}$$

We also have “there exists x”, the **existential quantifier**. Notation: $\exists\, x$.

$$\text{"}\exists\, x \;\; integer(x) \wedge (10 < x < 20) \wedge prime(x)\text{”}$$

This is also **true**.

Notation exercises

"Val loves somebody."

"$\exists m\ loves(\texttt{Val}, m)$"

"Val loves everybody."

"$\forall z\ loves(\texttt{Val}, z)$."

"Everybody loves somebody."

"$\forall x\ \exists y\ loves(x, y)$"

"$\neg \exists x\ integer(x) \wedge even(x) \wedge odd(x)$"

"There is no integer that is both even and odd."

The definition of "n is even" can also be written with quantifiers:

"$integer(n) \wedge \exists k\ integer(k) \wedge n = 2k$."

Activity : Quantifier Notation

Now, try expressing the definition of "n is odd" using quantifier notation.

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Quantifier Notation (Continued)

Answer:

$integer(n) \wedge \exists\, k \;\; integer(k) \wedge n = 2k + 1.$

Notice that this is very similar to the way we wrote "n is even" using quantifier notation!

Activity : More Quantifier Notation

We often want to apply our quantifiers only to members of a particular set, and we have a way to express this succinctly.

For example, earlier we wrote the statement "There is no integer that is both even and odd," as $\neg\exists x \; \mathit{integer}(x) \wedge \mathit{even}(x) \wedge \mathit{odd}(x)$, which can be read as "There is no x such that x is an integer and x is even and x is odd."

An equivalent way to express the same statement is

$$\neg\exists x \in \mathbb{Z} \; \mathit{even}(x) \wedge \mathit{odd}(x)\,,$$

i.e., "There is no x in the integers such that x is even and x is odd."

Similarly, the statement "For all integers x, x is even or x is odd" can be written as $\forall x \in \mathbb{Z} \; \mathit{even}(x) \vee \mathit{odd}(x)$.

Activity : More Quantifier Notation (Continued)

Use quantifier notation to express the following statement:

> *For all positive integers there is a strictly bigger positive integer that is prime.*

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : More Quantifier Notation (Continued)

Answer:

$$\forall x \in \mathbb{Z}^+ \; \exists y \in \mathbb{Z}^+ \; (y > x) \land \mathit{prime}(y)$$

This can be read directly as "For all x in the positive integers, there exists a y in the positive integers such that y is greater than x and y is prime."

We will soon prove this statement, which is equivalent to the existence of infinitely many primes.

Quantifiers in English

An integer n is even if $n = 2k$ for some integer k.

"$integer(n) \wedge \exists\, k \; integer(k) \wedge n = 2k$."

"For some", just like "there exists" and "there is" indicates an existential quantifier.

"For every", or just "every", like "for all" or "for any", indicates a universal quantifier. Examples:

"For every integer there is a bigger prime integer."

Zhang's Theorem: There exists an integer N that is less than 70 million, such that for any integer x there are primes bigger than x that differ by N.

Quiz

Which of the following statements in logical notation corresponds to "Any positive integer n that is not 1 and is not a prime has some positive integer factor that is neither 1 nor n" ?

A. $\forall n \in \mathbb{Z}^+ \ ((n \neq 1 \wedge \neg prime(n)) \Rightarrow \exists k \in \mathbb{Z}^+ \ (k \mid n \wedge k \neq 1 \wedge k \neq n))$

B. $\forall n \in \mathbb{Z}^+ \ ((n \neq 1 \wedge \neg prime(n)) \Rightarrow \forall k \in \mathbb{Z}^+ \ (k \mid n \wedge k \neq 1 \wedge k \neq n))$

C. $\exists n \in \mathbb{Z}^+ \ ((n \neq 1 \wedge \neg prime(n)) \Rightarrow \forall k \in \mathbb{Z}^+ \ (k \mid n \wedge k \neq 1 \wedge k \neq n))$

D. $\exists n \in \mathbb{Z}^+ \ ((n \neq 1 \wedge \neg prime(n)) \Rightarrow \exists k \in \mathbb{Z}^+ \ (k \mid n \wedge k \neq 1 \wedge k \neq n))$

Answer

A. $\forall n \in \mathbb{Z}^+ \ ((n \neq 1 \land \neg prime(n)) \Rightarrow \exists k \in \mathbb{Z}^+ \ (k \mid n \land k \neq 1 \land k \neq n))$
Correct. “Any” corresponds to a universal quantifier for n, and “has some” corresponds to an existential quantifier for the factor k.

B. $\forall n \in \mathbb{Z}^+ \ ((n \neq 1 \land \neg prime(n)) \Rightarrow \forall k \in \mathbb{Z}^+ \ (k \mid n \land k \neq 1 \land k \neq n))$
Incorrect. “Any” corresponds to a universal quantifier for n, and “has some” corresponds to an existential quantifier for the factor k.

C. $\exists n \in \mathbb{Z}^+ \ ((n \neq 1 \land \neg prime(n)) \Rightarrow \forall k \in \mathbb{Z}^+ \ (k \mid n \land k \neq 1 \land k \neq n))$
Incorrect. “Any” corresponds to a universal quantifier for n, and “has some” corresponds to an existential quantifier for the factor k.

D. $\exists n \in \mathbb{Z}^+ \ ((n \neq 1 \land \neg prime(n)) \Rightarrow \exists k \in \mathbb{Z}^+ \ (k \mid n \land k \neq 1 \land k \neq n))$
Incorrect. “Any” corresponds to a universal quantifier for n, and “has some” corresponds to an existential quantifier for the factor k.

Activity : Even More Quantifier Notation

Translate Zhang's Theorem into logical notation with quantifiers.

Zhang's Theorem: There exists an integer N that is less than 70 million, such that for any integer x there are two distinct primes greater than x that differ by at most N.

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : Even More Quantifier Notation (Continued)

Answer:

$$\exists N \in \mathbb{Z} \Big(N < 70000000 \wedge (\forall x \in \mathbb{Z}\ \exists a, b$$
$$(a \neq b \wedge prime(a) \wedge prime(b) \wedge a > x \wedge b > x \wedge |a - b| \leq N)) \Big)$$

Read directly, this says "There exists an integer N such that N is less than 70000000 and for all integers x, there exist a and b such that a and b are not equal and a is prime and b is prime and a is greater than x and b is greater than x and the difference between a and b is at most N."