Module 2.6: Combinations

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Combinations

Let A be a non-empty set with n elements, that is, $|A| = n$,
and let r be a natural number.

A **combination of r elements from the n elements of A** is an **unordered** selection of r of the n elements of A.

This is the same as a **subset** S of A of **size** r, that is, $S \subseteq A$ and $|S| = r$.

Example.

The combinations of 2 out of the 3 elements of $\{ x, 2, a \}$ are:

. $\{ x, 2 \}$, $\{ x, a \}$, $\{ 2, a \}$.

The combinations of 2 out of the 3 elements of $\{ o, u, i \}$ are:

. $\{ o, u \}$, $\{ o, i \}$, $\{ u, i \}$.

Clearly, the number of combinations does not depend on who the elements of A are, it only depends on how many elements there are.

The number of combinations I

Let n and r be natural numbers.

The number of combinations of r elements from the n elements of some set is denoted $\binom{n}{r}$.

Read $\binom{n}{r}$ as "n **choose** r".

We saw on the previous slide that $\binom{3}{2} = 3$.

The following can be verified easily:

$$\binom{n}{r} = \begin{cases} 0 & \text{if } r > n \\ 1 & \text{if } r = 0 \text{ or } r = n \\ n & \text{if } r = 1 \end{cases}$$

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Quiz

Let's compare the number of combinations of 2 elements out of 4 elements with the number of (partial) permutations of 2 elements out of 4 elements. What do you think?

A. There are more combinations of 2 out of 4 than permutations of 2 out of 4.
B. There are more permutations of 2 out of 4 than combinations of 2 out of 4.
C. They are the same.

Answer

Let's compare the number of combinations of 2 elements out of 4 elements with the number of (partial) permutations of 2 elements out of 4 elements. What do you think?

A. More combinations of 2 out of 4 than permutations of 2 out of 4.
   Incorrect. For every combination of 2 elements out of 4, there are two ways of ordering these two elements.
B. More combinations of 2 out of 4 than permutations of 2 out of 4.
   Correct. For every combination of 2 elements out of 4, there are two ways of ordering these two elements.
C. They are the same.
   Incorrect. For every combination of 2 elements out of 4, there are two ways of ordering these two elements.

The number of combinations II

In a previous segment we gave a formula for the number of partial permutations of r out of n:

$$n \cdot (n-1) \cdots (n-r+1) \;=\; \tfrac{n!}{(n-r)!}$$

We give a different way to count the partial permutations. A partial permutation of r out of n can be constructed in two steps as follows:

(1) Choose r elements from the n elements. Can be done in $\binom{n}{r}$ ways.
(2) Arrange the chosen r elements in some order. In $r!$ ways.

By the multiplication rule the total number of partial permutations is $\binom{n}{r} \cdot r!$.

$$\frac{n!}{(n-r)!} = \binom{n}{r} \cdot r!$$

Hence

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Seating students

Problem. There are 15 students enrolled in a course, but exactly 12 students attend on any given day. The classroom for the course has 25 distinct (distinguishable) seats. How many different classroom seating arrangements are possible?

Answer. A classroom seating can be constructed in two steps as follows:

(1) Choose 12 students out of 15 that are enrolled.
Can be done in $\binom{15}{12}$ ways.

(2) Arrange the 12 chosen students among the 25 distinct seats available.
(A partial permutation of 12 out of 25.) In $\frac{25!}{(25-12)!} = \frac{25!}{13!}$ ways.

By the multiplication rule the total number of seating arrangements is

$$\binom{15}{12}\frac{25!}{13!} = \frac{15!}{12!\,3!} \cdot \frac{25!}{13!}$$

Committees when people are feuding I

Problem. From a group of 8 women and 6 men, how many different committees consisting of 3 women and 2 men can be formed?

What if 2 of the men are feuding and refuse to serve on the committee together?

Answer. (to the first part) Observe that a committee can be formed in two steps as follows:

(1) Choose the 3 women. Can be done in $\binom{8}{3}$ ways.

(2) Choose the 2 men. In $\binom{6}{2}$ ways.

By the multiplication rule the total number of different committees is $\binom{8}{3}\binom{6}{2}$.

Committees when people are feuding II

Answer. (to the second part) Say the feuding men are Bob and Dan.
We break the second step into **alternatives**.

(1) Choose the 3 women. Can be done in $\binom{8}{3}$ ways.

(2) Choose the 2 men:

- (2.1) Bob, but not Dan, choose 1 of the other 4. In $\binom{4}{1}$ ways.
- (2.2) Dan, but not Bob, choose 1 of the other 4. In $\binom{4}{1}$ ways.
- (2.3) No Bob, no Dan, choose 2 of the other 4. In $\binom{4}{2}$ ways.

For the alternatives we use the addition rule.
Step (2) can be done in $\binom{4}{1} + \binom{4}{1} + \binom{4}{2}$ ways.

Then, by the multiplication rule the total number of different committees is $\binom{8}{3} \cdot [\binom{4}{1} + \binom{4}{1} + \binom{4}{2}]$.

Committees when people are feuding III

Answer. (to the second part) Using **complementary counting**.

We find the number of **all** possible committees and then subtract the number of **"bad"** committees that have both Bob and Dan.

All committees (from the first part): $\binom{8}{3}\binom{6}{2}$.

The number of "bad" committees is the number of ways to choose the 3 women, $\binom{8}{3}$, since the men must be Bob and Dan.

This gives the answer $\binom{8}{3}\binom{6}{2} - \binom{8}{3}$.

Using the formula with factorials **check (!)** that

$$\binom{8}{3} \cdot \left[\binom{4}{1} + \binom{4}{1} + \binom{4}{2}\right] = \binom{8}{3}\binom{6}{2} - \binom{8}{3}$$