Module 12.4: Properties of Trees

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Every tree is minimally connected

Proposition. Removing **any** edge in a tree disconnects it.

Proof. (Warm-up: This is true even if we remove the only edge incident to a leaf. It leaves a graph with 2 cc's one of which is edgeless with one vertex.)

Erase an edge, now "one more vertex than edges" fails so the graph is not a tree anymore.

But erasing an edge does not create cycles so the resulting graph is still acyclic.

The only way it can fail to be a tree is if it is disconnected.

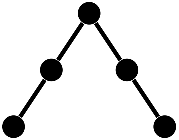
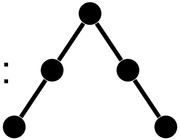
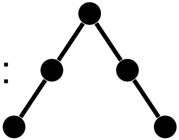
Activity: Removing edges from a tree

Consider a tree with 5 nodes. Remove 2 edges.

Question. How many cc's will the resulting graph have?

Question. Why must at least one of the resulting cc's have at least 2 nodes?

Answer. TIKZ TREE; TIKZ TWO DIFFERENT WAYS OF REMOVING 2 EDGES

T :  T_1 :  T_2 : 

Every tree is maximally acyclic

Proposition. Adding an edge between **any** two non-adjacent vertices in a tree creates a cycle.

Proof. (Warm-up: Draw a cycle graph. Remove an edge. The result is a path graph. But every path graph is a tree! Now put back the edge you removed. You have "created" a cycle!)

Now let u and v be non-adjacent vertices in $G = (V, E)$.

We add u–v and obtain $G_{uv} = (V, E \cup \{u\text{–}v\})$.

Since G is connected there is a path from u to v in G. Adding u–v to this path produces a cycle in G_{uv}.

Quiz

We add 2 edges to a tree. Clearly we can create 2 cycles. But can we create 3 cycles?

(A) Yes.

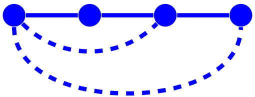
(B) No.

Answer

(A) Yes.

Correct. Consider the following example: we start with P_4 and add one edge to make C_4. We then add a diagonal to make two more cycles, as shown below:

G :



(B) No.

Incorrect. Before you see the answer, try and think the construction on your own. You may want to use pencil and paper and experiment.

Every tree is unique-path connected

Proposition. Any two distinct vertices of a tree are connected by a **unique** path.

Proof. A tree is connected so any two vertices are connected by **at least** one path. We need to prove there is only one such path.

Case 1: two adjacent vertices. u–v is a path. Suppose, toward a contradiction, that there is **another** path from u to v. This path together with u–v forms a cycle, which contradicts the acyclicity of the tree.

Case 2: two non-adjacent vertices. If $G = (V, E)$ is the tree, consider $G_{uv} = (V, E \cup \{u\text{–}v\})$ which has a cycle that includes u–v. Suppose, toward a contradiction, there were **two** distinct paths in G from u to v. Each path together with u–v creates a **distinct** cycle in G_{uv}. See next lemma that contradicts this!

At most one cycle I

Lemma. Adding an edge to an acyclic graph creates **at most** one cycle.

Proof. (Warm-up: why did we say "at most" and not "exactly one"? It is certainly "exactly one" if the graph was a tree, we proved that. But if it is a forest with at least two trees, the added edges can go between nodes in these trees!)

Let u, v be two distinct non-adjacent vertices in an acyclic graph $G = (V, E)$. We add u–v thus producing $G_{uv} = (V, E \cup \{u\text{–}v\})$.

Claim. G_{uv} has at most one cycle.

Suppose, toward a contradiction, that G_{uv} has at least two distinct cycles C_1 and C_2. Since G_{uv} was acyclic u–v must belong to both C_1 and C_2. Since C_1 and C_2 are distinct one of them must contain an edge that is not in the other one. On the next slide we will derive a contradiction from this.

At most one cycle II

Proof (continued). We had u, v distinct non-adjacent vertices in an acyclic graph $G = (V, E)$. We added u–v thus producing $G_{uv} = (V, E \cup \{u\text{–}v\})$. We claimed that G_{uv} has at most one cycle.

We assumed, toward a contradiction, that G_{uv} has two distinct cycles C_1 and C_2. Both go through u–v and one of them must contain an edge that is not in the other one. Let e be that edge.

Deleting u–v from C_1 gives us a path from u to v in G. Deleting u–v from C_2 gives us a path from v to u in G. Concatenating these two gives us a closed walk from u to u that traverses e exactly once. Next, we will see another lemma that says that such a walk must contain a cycle, which contradicts the acyclicity of G. And that will end the proof.

Cycle in a closed walk

Lemma. Any closed walk of non-zero length that traverses at least one of its edges **exactly once** contains a **cycle**.

We omit the (longish) proof but provide some visual intuition through examples.

The closed walk of length 4 u–v–w–v–u does not contain any cycles because it traverses each of its edges twice.

The closed walk of length 5 u–v–w–z–v–u traverses three of its edges, namely v–w, w–z and z–v exactly once and indeed it contains a cycle: v–w–z–v.

Unique path connectivity

Proposition. A graph such that any two distinct vertices are connected by a unique path must be a tree.

Proof. The graph is clearly connected. It remains to show that it is also acyclic.

Suppose, toward a contradiction, that the graph has a cycle. Let u and v be two distinct nodes in this cycle. The edges and the rest of the vertices of the cycle yield two **distinct** paths from u to v, contradiction.