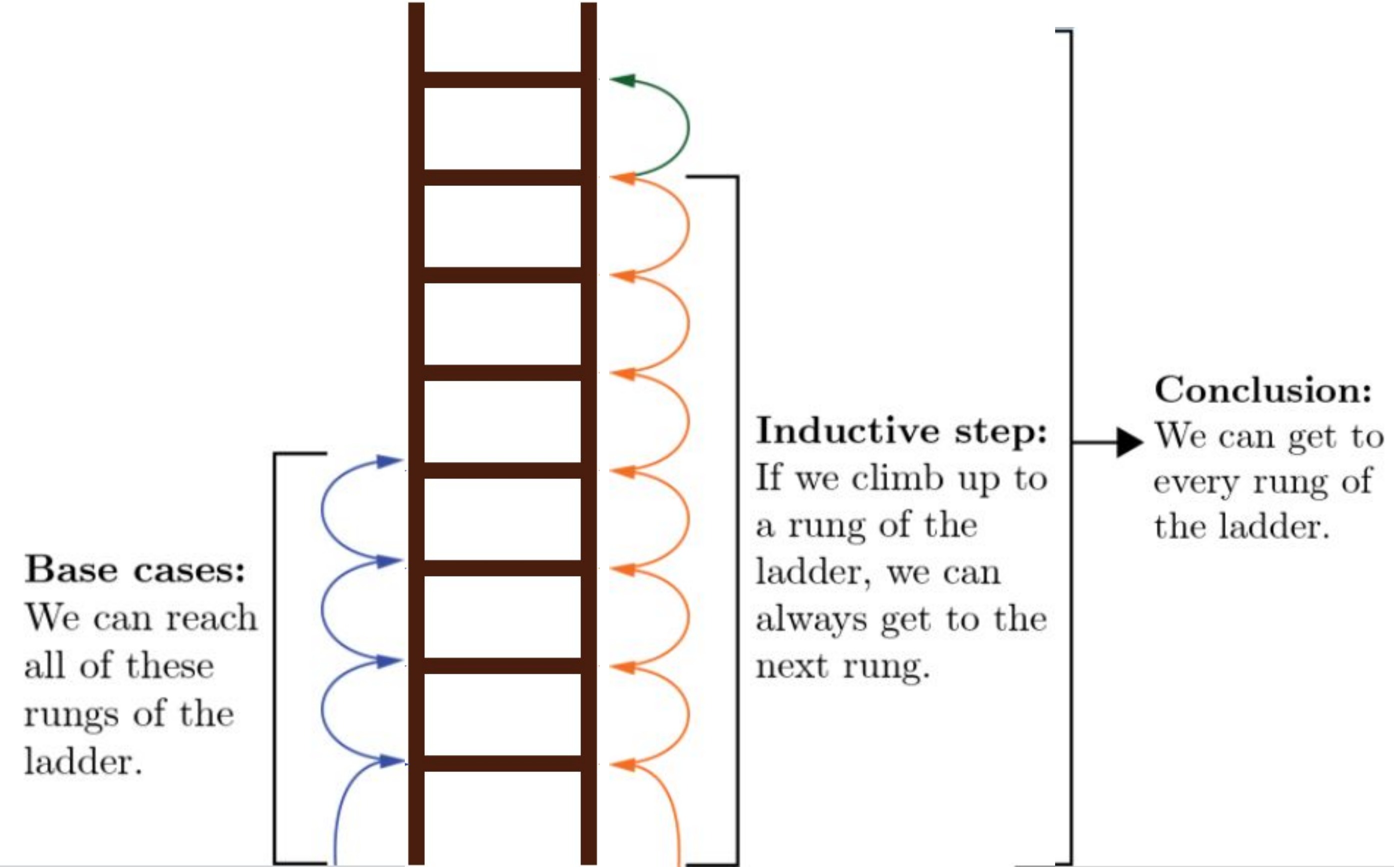
Recitation Module 6

Penn Engineering

Lecture Review

Understanding induction: The ladder analogy



What is induction good for?

(Usually) statements of the form:

“for all natural numbers n, a predicate on n, P(n), is true.”

Example: Good candidate for induction

P(n) = 1+2+3+...+n = n(n+1)/2 **(n ≥ 1)**

Example 1: Prove 1+2+...+n=n(n+1)/2 using a proof by induction.

n=1: 1=1(2)/2=1 checks. BC

Assume n=k holds: 1+2+...+k=k(k+1)/2 (Induction Hyypothesis)
Show n=k+1 holds: 1+2+...+k+(k+1)=(k+1)((k+1)+1)/2
I just substitute k and k+1 in the formula to get these lines. Notice that I write out what I

Now I start with the left side of the equation I want to show and proceed using the induct
=k(k+1)/2 + (k+1) by the Induction Hypothesis

What is the proof pattern for **ordinary** induction?

Let n_0 be a natural number and let $P(n)$ be a predicate well defined for all natural numbers $n \geq n_0$.

Proof pattern.

(BASE CASE) Check that $P(n_0)$ holds true.

(INDUCTION STEP)

Let $k \geq n_0$ be an arbitrary natural number.
Assume $\boxed{P(k)}$. Using that, infer $P(k+1)$.

Conclude $\forall\, n \geq n_0 \;\; P(n)$.

Strong induction

Let n_0 be a natural number and let $P(n)$ be a predicate that is well defined for all natural numbers $n \geq n_0$.

Proof pattern.

(BASE CASE) Derive/infer $P(n_0)$.

(INDUCTION STEP) Let $k \in \mathbb{N}$ such that $k \geq n_0$.
Assume $\boxed{P(n_0) \text{ and} \cdots \text{and } P(k)}$.
Derive/infer $P(k+1)$.

Conclude $\forall\, n \geq n_0 \;\; P(n)$.

The IH $P(n_0)$ and $\cdots$ and $P(k)$ is stronger than $P(k)$. But strong induction is mathematically equivalent to the ordinary one!

Recurrence relations

We can solve a recurrence relation (i.e., write a closed formula for the *n*th term in the sequence that doesn't depend on preceding terms) using

1. Telescopic method
2. Recursion tree method

Problems

Question 1

Prove $4^{n-1} > n^2$ for $n \geq 3$

Answer to Question 1

Base Case

n = 3: Since $4^{(3-1)} = 4^2 = 16 > 3^2$, the statement holds

Induction Hypothesis (IH)

Assume that for an arbitrary natural number $k \geq 3$, $4^{(k - 1)} > k^2$

Answer to Question 1 Continued

Induction Step
Induction Hypothesis (IH): Assume that for an arbitrary natural number k >= 3, 4^(k - 1) > k^2

Prove the statement for n = k+1 (that is, we want to show 4^(k + 1 - 1) > (k+1)^2

From the IH, we can see that 4^k / 4^1 > k^2. Therefore, we can say that 4^k > 4k^2

Lemma: 4k^2 > k^2 + 2k +1 for k>= 3:

To prove this, let us bring all terms to the LHS so that 3k^2 -2k + 1 > 0. Factoring this, we get (3k+1) (k-1) > 0. For all values of k greater than or equal to 3, we get a positive number times a positive number, which will always be greater than 0. Therefore, the inequality stated in the lemma is true for all values of k greater than or equal to 3.

Combining this with the IH, we see that 4^k > 4k^2 (from the IH) > k^2 + 2k + 1 (from lemma).

Therefore, 4^k > k^2 + 2k + 1. Therefore, 4^(k+1-1) > (k+1)^2. Thus the statement holds for n = k+1, so we have proven the original statement via induction.

Question 2

Using strong induction, prove the fundamental theorem of arithmetic:

Any integer $n \geq 2$ is either a prime or can be represented as a product of (not necessarily distinct) primes, i.e., in the form $n = p_1 p_2 ... p_r$ where each p_i is prime.

Answer to Question 2

Base Case

n = 2: Since two is prime there do not exist two integers k, m ≥ 2 s.t. km = 2 so the statement holds.

Induction Hypothesis (IH)

Assume that for all $n \in [2...k]$, n is either a prime or can be represented as a product of (not necessarily distinct) primes, i.e. in the form $n = p_1p_2...p_r$ where p_i are primes.

Answer to Question 2 Continued

Induction Step

We prove the statement for $n = k + 1$.

<u>Case 1:</u> If $k + 1$ is a prime, then the statement holds because by definition of prime numbers

if $s * z = k + 1$ then one of s and z must equal to 1 which is not a prime.

<u>Case 2:</u> If $k + 1$ is not a prime, then by definition of a composite number it must be written in the form

$k + 1 = s * z$ for some integers s,z s.t. $2 \leq s, z < k + 1$, which means that $s, z \leq k$.

By IH $s = p_1p_2...p_r$ and $z = p'_1p'_2...p'_r$ where $p_{i,}$ p_i' are primes.

Since $k + 1 = s * z = p_1p_2...p_r\ p'_1p'_2...p'_r$ the statement holds.

Thus by strong induction we have proved the fundamental theorem of arithmetic.

Question 3

Use the recursion tree method or the telescopic method to solve the recurrence relation for all positive integers

$$f(n) = f(n-1) + n(n+1), f(0) = 1$$

Answer to Question 3

Telescopic Method:

$f(n) = f(n-1) + n(n+1)$

$f(n-1) = f(n-2) + (n-1)\ n$

$f(n-2) = f(n-3) + (n-2)\ (n-1)$

…   …

$f(2) = f(1) + 2(3)$

$f(1) = f(0) + 1(2)$

$f(0) = 1$

Answer to Question 3 Continued

Looking at terms that only appear on one side of the equation, we see that

f(n) = n(n+1) + (n-1)n + (n-2)(n-1) + … + 2(3) + 1(2) + 1

f(n) = 1 + $\sum_{k=1}^{n} k(k+1)$

f(n) $= 1 + \sum_{k=1}^{n}(k^2 + k) = 1 + \sum_{k=1}^{n} k + \sum_{k=1}^{n} k^2$

f(n) = $1 + \frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Before you go...

Do you have any general questions about this week's assignment? We can't go into minute details about any specific questions, but this is the time to ask any general clarifying/confusing questions!

As always, if you can't talk to us during recitation or office hours, post any questions/anything course related on the Piazza!