OMCIT 592 Module 14 Self-Paced 01 (instructor Val Tannen)

One reference to this self-paced segment, in lecture segment 14.3.

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion.

When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Proof of topological sorting

In the lecture segment "Directed acyclic graphs (DAGs)" we stated the following:

Proposition. Every DAG has at least one topological sort.

In the lecture segment we sketched the idea of the proof as well as a **recursive** algorithm for constructing a topological sort. Here we give a detailed proof.

Proof. By induction on n, the number of vertices of the DAG.

(BC) $n = 1$. We cannot have any edges because they would form a cycle of length 1. With just one vertex v and zero edges the topological sort is the sequence v.

(IS) Let $k \geq 1$ arbitrarily. Assume (IH) that any DAG with k vertices has at least one topological sort.

Now take any DAG G with $k + 1$ vertices.

By the proposition on sources and sinks that we proved in the same lecture segment G has a source u (and a sink too, but we don't use it in this proof).

Delete u from G. This deletes also all the edges (if any) outgoing from u. There are no incoming edges to u because u is a source.

The resulting graph G_u must also be a DAG because no directed cycle is created when we just remove edges. Since G_u is a DAG with k vertices we can apply the IH and obtain a topological sort of G_u, call it σ.

Now we claim that the concatenated sequence $u\,\sigma$ is a topological sort of G.

Clearly it is a permutation of the vertices of G. Moreover, for any edge $x \to y$ of G we have two cases:

Case 1: $x \to y \in G_u$. Then x appears before y in σ, hence also in $u\,\sigma$.

Case 2: $x \to y$ is not in G_u. Then $x \equiv u$ and u occurs before y in $u\,\sigma$.