Module 4.1: Pascal's Triangle

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

The Binomial Theorem I

Binomial coefficients: $\binom{n}{r}$

Binomial Theorem. For any reals a and b and any natural number n

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$$

Proof. When $n = 0$ we get $1 = \binom{0}{0} \cdot 1 \cdot 1.$

When $n \neq 0$: $\quad (a+b)^n \quad = \quad (a+b)\cdot(a+b) \ \cdots \ (a+b)$

We obtain a sum of terms of the form $a^{n-i}b^i$ for various i between 0 and n.

The Binomial Theorem II

Binomial Theorem. For any reals a and b and any natural number n

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Proof (continued). $$(a+b)^n = (a+b) \cdot (a+b) \cdots (a+b)$$

In the resulting sum, **how many times** does each $a^{n-i}b^i$ occur?

There are n factors in the multiplication. To get $a^{n-i}b^i$ multiply a from $n-i$ of the factors and b from the other i.

That is, choose i of the n factors! This can be done in $\binom{n}{i}$ ways. So the coefficient of $a^{n-i}b^i$ is $\binom{n}{i}$.

Pascal's Triangle I

The formulas given by the Binomial Theorem for $n = 0, 1, 2, 3, 4, 5$. Keep in mind that $a^0 = b^0 = 1$.

$$(a+b)^0 = \binom{0}{0} = 1$$

$$(a+b)^1 = \binom{1}{0}a^1 + \binom{1}{1}b^1 = a + b$$

$$(a+b)^2 = \binom{2}{0}a^2 + \binom{2}{1}a^1b^1 + \binom{2}{2}b^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}b^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}b^4 = \\ = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = \binom{5}{0}a^5 + \binom{5}{1}a^4b^1 + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^5 + \binom{5}{4}a^1b^4 + \binom{5}{5}b^5 = \\ = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Pascal's Triangle II

Pascal kept just the coefficients and arranged them into a triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \cdots & & & & &
\end{array}
$$

Continuing with $n = 6, 7, \ldots$, the (infinite) result is called **Pascal's Triangle**.

Notice the pattern: $2 = 1 + 1$, $3 = 1 + 2$, $6 = 3 + 3$, $10 = 4 + 6$, etc.

We will prove this!

Module 4.2: Combinatorial Proofs

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Lecture Notes

A combinatorial identity

How many times does $a^{n-i}b^i$ occur in $(a+b)^n$? We counted $\binom{n}{i}$ factors that contributed a b. What if we counted for a ? That's $\binom{n}{n-i}$. Luckily:

Problem. Prove that

$$\binom{n}{r} = \binom{n}{n-r}$$

Answer. Consider a set A such that $|A| = n$. The left-hand side (LHS) counts subsets of size r and the right-hand side (RHS) of size $n-r$.

But the RHS also counts the number of subsets of size r by counting the ways in which elements are **not** put in the subset.

If $|S| = n-r$ then $|A \setminus S| = r$. We have a "one-to-one correspondence" between subsets of size r and of size $n-r$. This proves the identity.

Pascal's identity

Problem (Pascal's Identity). Let n and k be positive integers with $n \geq k \geq 1$. Prove

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Answer. We count the number of subsets of size k of a set with n elements in two ways. The usual way gives the LHS of Pascal's Identity.

Let $A = \{ x_1, x_2, \ldots, x_n \}$ be the set. Notice that k-element subsets of A can be classified into those that contain x_n and those that don't.

For the first kind the other $k-1$ elements come from $A \setminus \{x_n\}$, $\binom{n-1}{k-1}$ ways.

For the second kind all the k elements come from $A \setminus \{x_n\}$, in $\binom{n-1}{k}$ ways.

By the addition rule, we get the RHS of Pascal's Identity.

Activity : Pascal's Triangle

Recall the Pascal's Triangle from a previous segment.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \cdots & & & & & &
\end{array}
$$

ACTIVITY : Pascal's Triangle

We think of the rows as ordered vertically starting from row 0 at the top. We think about every row as a sequence of numbers. For instance, row 6 is $1, 6, 15, 20, 15, 6, 1$. We number the positions in each row as starting from 0.

Therefore, in row n position k, we have the value of $\binom{n}{k}$.

For example, in position 4 of row 6 we have $\binom{6}{4} = 15$.

Question: What number is in position 5 of the next row (row 7)?

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Pascal's Triangle (continued)

Answer: 21.

Why? In position 5 of row 7 we have $\binom{7}{5}$. We can use Pascal's identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

to derive the answer. Taking $n = 7$ and $k = 5$ we obtain:

$$\binom{7}{5} = \binom{6}{4} + \binom{6}{5}$$

From the sixth row in Pascal's triangle, $\binom{6}{4} = 15$ and $\binom{6}{5} = 6$. Therefore $\binom{7}{5} = 15 + 6 = 21$.

Combinatorial proofs of identities

We just proved two combinatorial identities:

$$\binom{n}{r} = \binom{n}{n-r} \qquad \text{and} \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

In both proofs the method was the same:

We posed a **counting question**.

We answered the question in **one way**, with the answer giving the LHS of the identity.

We answered the question in **another way**, with the answer giving the RHS of the identity.

Another combinatorial proof

Problem. Prove

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Answer. We pose the following counting question: how many subsets are there of a set A with n elements?

From earlier lectures, we know that the answer is 2^n. This gives us the RHS.

Another way : The powerset 2^A can be partitioned into $S_0, S_1, \ldots, S_n$, where S_i, $0 \leq i \leq n$, is the set of all subsets of A that have cardinality i.

These are pairwise disjoint so by the addition rule the answer is $\sum_{i=0}^{n} |S_i|$.

But $|S_i| = \binom{n}{i}$. This gives us the LHS.

Activity : Binomial Theorem

Recall that the Binomial Theorem (covered in a previous segment) states

$$(a+b)^n \;=\; \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

We can also derive the identity in the previous slide from the Binomial Theorem by setting $a = b = 1$. Here's how:

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} &= \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i \\
&= (1+1)^n \\
&= 2^n
\end{aligned}$$

Activity : Binomial Theorem (Continued)

Now do the following:

Question: What is your idea for proving

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0$$

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Binomial Theorem (Continued)

Answer:

One way to solve this problem is by substituting $a = 1$ and $b = -1$ in the Binomial Theorem, yielding

$$0^n = 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

Activity : Binomial Theorem (Continued)

However, a combinatorial proof will give us more insight into what the expression means. Moving some terms to the RHS, we want to prove that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$

Consider a set $X = \{x_1, x_2, x_3, \ldots, x_n\}$. We want to show that the total number of subsets of X that have **even size** equals the total number of subsets of X that have **odd size**. We will now show that both these quantities equal 2^{n-1} from which the claim follows.

ACTIVITY : Binomial Theorem (Continued)

An even-sized subset of X can be constructed as follows.

Step 1 : Decide whether x_1 belongs to the subset or not.
Step 2 : Decide whether x_2 belongs to the subset or not.
...
Step n : Decide whether x_n belongs to the subset or not.

ACTIVITY : Binomial Theorem (Continued)

In the first $n-1$ steps one can make either one of the **two choices**, in or out. But in step n only **one choice** is possible!

This is because if we have chosen an even number of elements from $X \setminus \{x_n\}$ to put in the subset then we must leave out x_n.

Otherwise, we must include x_n in the subset.

Hence using the multiplication rule the total number of even-sized subsets of X equals 2^{n-1}.

ACTIVITY : Binomial Theorem (Continued)

Another way of thinking about this is to count in two steps.

In the first step choose a subset of $\{x_1, \ldots, x_{n-1}\}$.
In the second step decide whether to add x_n to the subset chosen in the first step, making sure the result has even size (don't forget that 0 is even!).

To compute the number of odd-sized subsets we could proceed similarly.

Or, we could count complementarily: since we know that the total number of subsets of X is 2^n, the total number of odd-sized subsets of X is

$$2^n - 2^{n-1} = 2^{n-1}(2-1) = 2^{n-1}$$

Module 4.3: Functions

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Lecture Notes

Functions I

A **function** (sometimes called **mapping**) denoted $f : A \to B$, consists of

- a set A, called **domain**,
- a set B, called **codomain**, and
- a way of associating with **every** element of the domain, $x \in A$, a **unique** element of the codomain, $f(x) \in B$, write $x \mapsto f(x)$.

The **range** of a function $f : A \to B$ is:

$$\mathrm{Ran}(f) \;=\; \{\, y \mid y \in B \;\wedge\; \exists x \in A \;\; y = f(x) \,\}$$

Note that this defines a subset $\mathrm{Ran}(f) \subseteq B$.

Functions II

Examples.

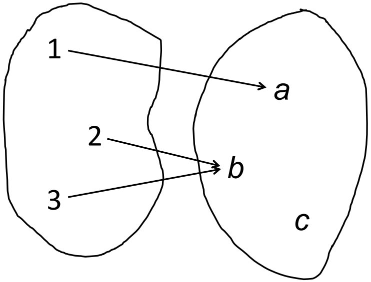
$h : \mathbb{N} \to \mathbb{N}$ where $h(n) = 2n$. Ran$(h) =$ even integers ≥ 0

$g : [0, \infty) \to \mathbb{R}$ where $g(x) = \sqrt{x}$. Ran$(g) =$ $[0, \infty)$

$f : A \to B$ with domain $A = \{1, 2, 3\}$, codomain $B = \{a, b, c\}$
with $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto b$ or $f(1) = a$, $f(2) = b$, $f(3) = b$.

Table and diagram representations:

$x \in \{1, 2, 3\}$	$f(x) \in \{a, b, c\}$
1	a
2	b
3	b



Ran$(f) =$ $\{a, b\}$.

Quiz

Consider the function $f : [1, \infty) \to \mathbb{R}$ where $f(n) = \log_2 n$.

What is the range of this function?

A. $\mathbb{R}$

B. $\mathbb{Z}^+$

C. $[0, \infty)$

Answer

Consider the function $f : [1, \infty) \to \mathbb{R}$ where $f(n) = \log_2 n$.

What is the range of this function?

A. $\mathbb{R}$

Incorrect. A log function defined on real numbers ≥ 1 does not return negative numbers.

B. $\mathbb{Z}^+$

Incorrect. The log of a number does not have to be an integer.

C. $(0, \infty)$

Correct. The log function defined on numbers ≥ 1 returns positive real numbers (or 0).

The set of all functions

Let A, B be two sets. The set

$$\{f \mid f : A \to B\} \quad \text{is denoted by} \quad B^A$$

Proposition. If $|A| = r$ and $|B| = n$ then the number of different functions with domain A and codomain B is n^r.

Proof. Let $A = \{ a_1, \ldots, a_r \}$. We can construct a function from A to B in r steps, $i = 1, 2, \ldots, r$ as follows.

In step (i) we choose $b \in B$ to define $f(a_i) = b$, that is $a_i \mapsto b$.
This can be done in n ways.

By the multiplication rule, the number of functions is $n \cdot n \cdots n = n^r$.

Therefore $$|B^A| = |B|^{|A|}$$

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ACTIVITY : Example of one-to-one correspondence

Consider a function $f : A \to B$ where A is the set of elements $\{a_1, a_2, \ldots, a_n\}$. First notice that the number of possible functions is also the number of sequences of length n of elements in the set B.

In an activity in an earlier segment we showed that the subsets of a set A are in one-to-one correspondence with sequences of bits of size $|A|$.

Recall that we denoted the set of subsets of A by 2^A.

Question: What is the cardinality of 2^A?

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

Activity : Example of one-to-one correspondence (Continued)

Answer: $2^{|A|}$.

Now consider the particular case when B has two elements, for example $B = \{0, 1\}$. In this case, and using the formula we just learned, there are

$$|B^A| = |B|^{|A|} = 2^{|A|}$$

possible functions from A to B.

This is also the number of subsets of the set A!

Is there a connection between the subsets of A and the functions from A to $\{0, 1\}$? Yes!

We will describe a one-to-one correspondence between them.

ACTIVITY : Example of one-to-one correspondence (Continued)

Namely, to any function $f : A \to \{0, 1\}$ this correspondence associates a subset S_f of A where:

$$S_f \;=\; \{x \in A \mid f(x) = 1\}$$

Conversely, to any subset $S \subseteq A$ this correspondence associates a function $f_s : A \to \{0, 1\}$ defined by

$$f_s(x) \;=\; \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise} \end{cases}$$

Module 4.4: Integer Intervals

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Lecture Notes

Integer intervals

An **integer interval** $[m..n]$ (where $m \leq n$) is the set of **all** integers that lay between m and n, inclusive. In set-builder notation:

$$[m..n] = \{k \in \mathbb{Z} \mid m \leq k \leq n\}$$

Example. $[6..10] = \{6, 7, 8, 9, 10\}$, a set of 5 elements.

Memorize: $\quad |\,[m..n]\,| = n - m + 1.$

Problem. How many two-digit numbers are there between 1 and 100?

Answer. $|\,[10..99]\,| = 99 - 10 + 1 = 90$

Quiz I

How many elements does the following set have

$$([10..30] \cup [20..40]) \setminus ([10..30] \cap [20..40])$$

A. 20

B. 42

C. 21

Answer

What is the cardinality of the following set?

$$([10..30] \cup [20..40]) \setminus ([10..30] \cap [20..40])$$

A. 20

Correct. The same as the union of $[10..19]$ and $[31..40]$, which are disjoint and contain 10 elements each.

B. 42

Incorrect. Did you forget to remove the elements in $[10..30] \cap [20..40]$?

C. 31

Incorrect. Remember to count the elements in $[10..30] \cup [20..40]$ only once.

Quiz II

What is the value of the following sum of cardinalities?

$$\sum_{i=0}^{10} |[5i \mathinner{..} (5i + 3)]|$$

A. 40

B. 44

C. 45

Answer

What is the value of the following sum of cardinalities?

$$\sum_{i=0}^{10} |[5i \mathrel{..} (5i+3)]|$$

A. 40

Incorrect. i takes 11 values, rather than 10.

B. 44

Correct. i takes 11 values, and for every value of i, we have 4 unique elements.

C. 45

Incorrect.

Functions and integer intervals I

Examples.

$$f : [0..10] \to [0..20]$$
where $f(x) \;=\; x + 10.$

$f(5) \;= 15 \qquad 5 \mapsto 15$

$f(0) \;= 10 \qquad 0 \mapsto 10$

$f(10) \;= 20 \qquad 10 \mapsto 20$

$\mathrm{Ran}(f) = \;\; [10..20].$

.

$$g : [-20..10] \to [0..20]$$
where $g(y) \;=\; \mathrm{abs}(y).$

$g(5) \;= 5 \qquad 5 \mapsto 5$

$g(-5) \;= 5 \qquad -5 \mapsto 5$

$g(-20) \;= 20 \qquad -20 \mapsto$
20

$g(10) \;= 10 \qquad 10 \mapsto 10$

$\mathrm{Ran}(g) = \;\; [0..20].$

Functions and integer intervals II

Another example.

$h : [0..n] \to [0..n]$
where $h(z) \;=\; n - z.$

$h(1) \;= n - 1 \qquad 1 \mapsto \; n - 1$

$h(0) \;= n \qquad 0 \mapsto n$

$f(n) \;= 0 \qquad n \mapsto 0$

$f(n-1) \;= 1 \qquad n - 1 \mapsto 1$

$\mathrm{Ran}(h) = \;\; [0..n].$

Functions and integer intervals III

Yet another example.

$$t : [0..2n] \to [0..n] \quad \text{where } t(w) = \begin{cases} \frac{w}{2} & \text{if } w \text{ is even} \\ \frac{w-1}{2} & \text{if } w \text{ is odd} \end{cases}$$

$t(0) = 0 \qquad 0 \mapsto 0$.

$t(1) = 0 \qquad 1 \mapsto 0$

$t(2n) = n \qquad 2n \mapsto n$

$t(2n-1) = n-1 \qquad 2n-1 \mapsto n-1$

$t(2n-2) = n-1 \qquad 2n-2 \mapsto n-1$

$\mathrm{Ran}(t) = [0..n]$.

ACTIVITY : Functions as elements

Let $A = B = C = [1..n]$.

This activity concerns functions with domain B^A and codomain 2^C. These are *functions of functions*, in the sense that they map elements of B^A (which are themselves functions from A to B) to elements of 2^C (which are subsets $S \subseteq C$).

Question: How many functions with domain B^A and codomain 2^C are there?

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Functions as elements (Continued)

Answer: Observe that $|A| = |B| = |C| = n$. Hence, $|B^A| = |B|^{|A|} = n^n$ and $|2^C| = 2^{|C|} = 2^n$. Therefore, the number of functions with domain B^A and codomain 2^C is

$$|2^C|^{|B^A|} = (2^n)^{(n^n)} = 2^{n \cdot n^n} = 2^{n^{n+1}}.$$

ACTIVITY : Functions as elements (Continued)

What do functions from B^A to 2^C look like? We give two examples.

Example 1. A function $g : [1..n]^{[1..n]} \to 2^{[1..n]}$ that maps any function $f : [1..n] \to [1..n]$ to the subset of $[1..n]$ given by $\{x \mid f(x) = 1\}$.

Example 2. A function $h : [1..n]^{[1..n]} \to 2^{[1..n]}$ that maps any function $f : [1..n] \to [1..n]$ to the set $\mathrm{Ran}(f) \subseteq [1..n]$.

Question: If $f : [1..n] \to [1..n]$ is the identity function that maps each element of $[1..n]$ to itself, then what are $g(f)$ and $h(f)$?

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Functions as elements (Continued)

Answer:

The only element of $[1..n]$ that f maps to 1 is 1, so $g(f) = \{1\}$.

The range of f is $[1..n]$, so $h(f) = [1..n]$.

Module 4.5: Surjections and Injections

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Lecture Notes

Surjective functions I

A function $f : A \to B$ is called **surjective** if $\mathrm{Ran}(f) = B$, or equivalently:

$$\text{for every } y \in B \text{ there exists } x \in A \text{ such that } y = f(x).$$

A surjective function is also called a **surjection**.

Examples.

$g : [-20..10] \to [0..20]$
where $g(y) = \mathrm{abs}(y)$. $\quad \mathrm{Ran}(g) = [0..20]$

$h : [0..n] \to [0..n]$
where $h(z) = n - z$. $\quad \mathrm{Ran}(h) = [0..n]$

Surjective functions II

$$t : [0..2n] \to [0..n] \quad \text{where } t(w) = \begin{cases} \frac{w}{2} & \text{if } w \text{ is even} \\ \frac{w-1}{2} & \text{if } w \text{ is odd} \end{cases}$$

Problem. Prove that t is surjective.

Answer. By the definition, we need to show that:

For every $y \in [0..n]$ there exists $x \in [0..2n]$ such that $y = t(x)$.

Indeed, let y be an arbitrary element of $[0..n]$.

Take $x = 2y$. x is even, therefore

$$t(x) = \frac{x}{2} = \frac{2y}{2} = y$$

Injective functions I

A function $f : A \to B$ is called **injective** if it maps distinct elements to distinct elements, that is,

$$\text{for every } \ x_1 \neq x_2 \ \text{ in the domain we have } \ f(x_1) \neq f(x_2),$$

or, equivalently, (by contrapositive)

$$\forall\, x_1, x_2 \in A \quad f(x_1) = f(x_2) \ \Rightarrow \ x_1 = x_2$$

An injective function is also called an **injection**.

Examples.

$h : \mathbb{N} \to \mathbb{N}$ where $h(n) = 2n$.

It's injective because if $n_1 \neq n_2$ then $2n_1 \neq 2n_2$.

$f : [0..10] \to [0..20]$ where $f(x) = x + 10$.

It's injective because if $x_1 \neq x_2$ then $x_1 + 10 \neq x_2 + 10$.

Injective functions II

$g : [0, \infty) \to \mathbb{R}$ where $g(x) = 2\sqrt{x} - 3$.

Problem. Prove that g is injective.

Answer. By definition, we need to show that:

$$\forall\, x_1, x_2 \in [0, \infty) \quad 2\sqrt{x_1} - 3 = 2\sqrt{x_2} - 3 \;\Rightarrow\; x_1 = x_2$$

Assume $2\sqrt{x_1} - 3 = 2\sqrt{x_2} - 3$. Then

$$\begin{aligned} 2\sqrt{x_1} &= 2\sqrt{x_2} && \text{(Adding 3)} \\ \sqrt{x_1} &= \sqrt{x_2} && \text{(Dividing by 2)} \\ x_1 &= x_2 && \text{(Squaring)} \end{aligned}$$

Done.

Quiz I

For $k \in \mathbb{Z}^+$ let $P = \{1, 2, \ldots, k\} \times \{1, 2, \ldots, k\}$,
define $g : P \to \{1, 2, \ldots, k\}$ by $g(x, y) = x$.

Is this function

A. Injective?

B. Surjective?

C. Both?

D. Neither?

Answer

For $k \in \mathbb{Z}^+$ $P = \{1, 2, \ldots, k\} \times \{1, 2, \ldots, k\}$, define $g : P \to \{1, 2, \ldots, k\}$ by $g(x, y) = x$. Is this function

A. Injective?

Incorrect. The function is not injective, because $g(1, 1) = g(1, 2) = 1$.

B. Surjective?

Correct. Every element in the codomain is mapped onto by some element in the domain: $x = g(x, 1)$.

C. Both?

Incorrect. The function is not injective, see the first answer.

D. Neither?

Incorrect. The function is surjective, see the second answer.

Quiz II

$f : \mathbb{R} \to \mathbb{R}$ defined as:

$$f(x) = \begin{cases} x^3 & x \geq 1 \\ x^3 + 1 & x < 1 \end{cases}$$

Is this function

A. Injective?

B. Surjective?

C. Both?

D. Neither?

Answer:

Define a function $f : \mathbb{R} \to \mathbb{R}$ by:

$$f(x) = \begin{cases} x^3 & x \geq 1 \\ x^3 + 1 & x < 1 \end{cases}$$

Is this function

A. Injective?

   Incorrect. $f(x)$ is not injective because $f(x) = 1$ for both $x = 0$ and $x = 1$.

B. Surjective?

   Correct. The function is surjective.

C. Both?

   Incorrect. $f(x)$ is not injective because $f(x) = 1$ for both $x = 0$ and $x = 1$.

D. Neither?

   Incorrect. The function is surjective.

Injections, surjection, and counting

Let A and B be two sets.

The **injection rule**: if we can define an injective function with domain A and codomain B then $|A| \leq |B|$.

The **surjection rule**: if we can define a surjective function with domain A and codomain B then $|A| \geq |B|$.

The **surjection rule (variant)**: if we can define a function $f : A \to B$ then $|A| \geq |\mathrm{Ran}(f)|$.

(If $f : A \to B$ then $f' : A \to \mathrm{Ran}(f)$ where $f'(x) = f(x)$ is surjective.)

If we can define a function with domain A and codomain B that is **both** a surjection and an injection then $|A| = |B|$.

In the next segment we describe this as the "bijection rule" and we discuss more about it.

Penn Engineering

ACTIVITY : Surjection in Counting

In this activity, we will use the surjection rule to explain why there are at least as many permutations of r out of n as there are combinations of r out of n.

Let A be a set with n elements, $|A| = n$. Let P_r be the set of partial permutations of length r made out of elements of A. Let C_r be the set of combinations of size r made out of elements of A (subsets of A of size r).

Now, we try defining a function which maps elements from P_r to C_r.

ACTIVITY : Surjection in Counting

We define a function $f : P_r \to C_r$ that associates to each permutation the set of elements occurring in the permutation. A permutation $\sigma \in P_r$ has no repeated elements, so the set of elements that occur in σ is of size r.

Now we prove that f is a surjection. For every $S \in C_r$, order the elements arbitrarily. This produces a permutation of length r, call it σ and $f(\sigma) = S$.

By the surjection rule:

$$|P_r| \geq |C_r|.$$

Of course, this can be also checked algebraically since

$$\frac{n!}{(n-r)!} \geq \binom{n}{r}$$

Self-paced Example: Algebraic Verification of Pascal's Identity

Module 4

MCIT Online - CIT592 - Professor Val Tannen

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion. When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Pascal's identity algebraic verification

In this module you learned about **Pascal's Identity**, which states that:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

where n and k are positive integers with $n \geq k \geq 1$.

Recall also that Pascal's identity is suggested by a pattern that can be noticed in Pascal's Triangle:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \cdots & & & & & &
\end{array}$$

(Notice in Pascal's Triangle that every (inner) number is the sum of the two numbers above it.)

When we stated it we showed a combinatorial proof for Pascal's Identity. In this segment, we will walk you through another way to prove Pascal's Identity.

Don't worry, this is a much more straightforward proof! ☺

Problem. Once again, recall **Pascal's Identity**:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

where n and k are positive integers with $n \geq k \geq 1$. Prove this combinatorial identity by algebraic verification.

(CONTINUED)

Pascal's identity algebraic verification (continued)

Answer. We are going to consider two cases, when $k > n$, and when $k \leq n$.

First consider the case where $k > n$. In this case, proving that the identity holds is trivial since

$$\binom{n}{k} = 0 = \binom{n-1}{k-1} = \binom{n-1}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Now we are going to consider the case where $k \leq n$.

We start by expanding the right-hand side (RHS) of the identity[1] using the formula we had derived for combinations:

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\ &= (n-1)!(\frac{1}{(k-1)!(n-k)!} + \frac{1}{k!(n-k-1)!})\end{aligned}$$

Observe that

$$\frac{1}{(k-1)!} = \frac{k}{k(k-1)!} = \frac{k}{k!}$$

Similarly,

$$\frac{1}{(n-k-1)!} = \frac{n-k}{(n-k)(n-k-1)!} = \frac{n-k}{(n-k)!}$$

It follows that:

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= (n-1)!(\frac{k}{k(k-1)!(n-k)!} + \frac{n-k}{k!(n-k)(n-k-1)!}) \\ &= (n-1)!(\frac{k}{k!(n-k)!} + \frac{n-k}{k!(n-k)!})\end{aligned}$$

[1] *Note: think about why we choose to expand the RHS; it is more informative in some way... This type of thinking will allow you to tackle future problems!*

Pascal's identity algebraic verification (continued)

With some more simple algebraic manipulations we obtain in the end the left-hand side (LHS):

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= (n-1)!\frac{k+n-k}{k!(n-k)!} \\ &= \frac{n(n-1)!}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k)!} = \binom{n}{k}\end{aligned}$$

We saw how we can provide an algebraic proof, where we earlier provided a combinatorial proof. This pattern is true for many combinatorial identities, namely there is a way to prove all of them through combinatorics (what we did in lecture), and a way to prove it through algebraic manipulations (what we just did).

If you are still skeptical (...and even if you are not) you should try and follow a similar approach and solve the other combinatorial identities that we saw through algebraic verification. You will see that there is always a way!

Self-paced Example: Counting Two Gender Committees

Module 4

MCIT Online - CIT592 - Professor Val Tannen

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Counting committees of r out of n and m

You have learned earlier about a technique called **combinatorial proof**. Specifically, in the lecture segment "Combinatorial proofs" you saw an example of this technique in a proof of the following identity:

$$\binom{n}{r} = \binom{n}{n-r}$$

and another one in a proof of Pascal's Identity.

Take a moment to recall this technique: To prove an identity we pose a counting question. We then answer the question in two ways: one answer corresponds to LHS and the other corresponds to the RHS of the identity.

We will use this same technique to solve the following problem.

Problem. Give a combinatorial proof to show that

$$\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k} = \binom{n+m}{r}$$

(CONTINUED)

Counting committees of r out of n and m (continued)

Answer. We answer this problem in three steps.

Step 1: We pose the following counting question.

> Consider a group of n women and m men. How many ways are there to form a committee of r people from this group?

Step 2 (RHS): the total number of people is $m+n$. A committee of r people is a subset of size r of the set of people, therefore there $\binom{n+m}{r}$ distinct such committees. This gives us the RHS.

Step 3 (LHS): The set S of all possible committees of r people can be partitioned into subsets $S_0, S_1, S_2, ..., S_r$, where S_k is the set of committees in which there are exactly k women and the rest, $r-k$ people, are men.

However, what is $|S_k|$?

We can construct a committee in S_k in two steps. Think about how you would do it.

Counting committees of r out of n and m (continued)

Indeed in the first of the two steps we choose k out of the n women. This can be done in $\binom{n}{k}$ ways.

In the second of the two steps we choose $r-k$ out of the m men. This can be done in $\binom{m}{r-k}$ ways.

By the multiplication rule

$$|S_k| = \binom{n}{k}\binom{m}{r-k}$$

Since the sets $S_0, S_1, S_2, ..., S_r$ are pairwise disjoint we can apply the addition rule:

$$|S| \;=\; |S_0| + |S_1| + \cdots + |S_r| \;=\; \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$$

which gives us the LHS.

The technique used to solve this problem, combinatorial proofs, is very useful in proving many identities. You can revisit some or all of the identities we proved using combinatorial proofs in the lecture segment "Combinatorial proofs."