Module 3.1: Stars and Bars

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Stars and bars I

Problem. How many different ways are there to buy a dozen (12) donuts when 5 given glazes (chocolate, dulce de leche, etc.) are available? Assume

(1) that there is an unlimited supply of donuts of each glaze.
(2) that donuts of the same glaze are indistinguishable.

Answer. Consider 12 donuts **before glazing** when they cannot be distinguished. Represent them by 12 stars in a row.

Now place 4 bars between some of the stars. For example:

$$***|**|*|****|**$$

This separates the unglazed donuts into 5 contiguous parts.

Glaze the donuts in the first part with chocolate `(C)`, the second part with dulce de leche `(D)` and three other flavors, `(E)`, `(F)`, `(G)`, as follows:

```
***|**|*|****|**
CCC DD E FFFF GG
```

Stars and bars II

Answer. (continued)

Note that you may not buy all the glazes. This is represented with bars at the beginning or end, or adjacent bars:

`***||******|***|`	`||*********|***|`	`*|*|||**********`
`CCC||EEEEEE|FFF|`	`||EEEEEEEEE|FFF|`	`C|D|||GGGGGGGGGG`

Note also that the ordering of `C,D,E,F,G` does not matter, only **how many** stars are in each part. Thus, to avoid overcounting, we **fix an ordering** of `C,D,E,F,G` and then we count.

The 12 stars and the 4 bars form a sequence with $12 + 4 = 16$ positions. Out of these, we choose 4 positions where we put the bars.

The answer is $\binom{16}{4}$.

Stars and bars

The donuts and glazes problem is an example of a general class of counting problems that can all be solved with **stars and bars**. These include:

- Counting the number of ways of putting n indistinguishable marbles in r distinguishable urns.
- Counting the number of ways of distributing n indistinguishable coins to r distinguishable children (see next).
- Counting the number of nonnegative solutions to the Diophantine equation $x_1 + x_2 + \cdots + x_r = n$ (see another segment in this module).
- Counting the number of bags/multisets of size n made from elements of a set of size r (see lecture segment "Counting anagrams" in this module).

In all these cases, stars and bars applies and the answer is :

$$\binom{n+r-1}{r-1}.$$

Penn Engineering

Coins to children I

Problem. In how many ways can we distribute 11 indistinguishable coins to 3 distinguishable children?

What if each child must receive at least one coin?

Answer. (to the first part) We notice the analogy: the coins correspond to the unglazed donuts and the glazes to the children!

Using stars and bars with children `A,B,C`:

```
***|******|**
AAA|BBBBBB|CC
```

The answer is $\binom{11+2}{2} = \binom{13}{2}$

Coins to children II

Answer. (to the second part)

We need to modify our solution to the first part, because now we need every child to have at least one coin. This is solved with a simple trick: we begin by giving each child a coin!

The remaining distributions of $11 - 3 = 8$ coins are counted by stars and bars.

The answer is $\binom{8+2}{2} = \binom{10}{2}$.

Activity : Counting Marbles

The goal of this activity is to count the number of ways in which n indistinguishable marbles can be put in r distinguishable urns such that each urn contains at least 2 marbles.

Before we get to the answer, let us first consider several questions:

Question: What are the minimum number of marbles that are needed?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : Counting Marbles (Continued)

Answer: $\boxed{2r}$.

We have r urns, each of which needs at least 2 marbles.

Now consider:

Question: Draw a stars and bars diagram for $r = 4$ and $n = 15$.

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : Counting Marbles (Continued)

Answer: $* \mid * \mid ** \mid ***$

Any diagram with $15 - 2 \cdot 4 = 7$ stars and 3 bars is a correct answer for this question. We remove $2r$ marbles from the stars and bars diagram because they must belong to their respective urns.

Now we go back to the question asked in the beginning: count the number of ways in which n indistinguishable marbles can be put in r distinguishable urns such that each urn contains at least 2 marbles.

Question: What do you think could be the answer?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : Counting Marbles (Continued)

Answer:

$$\binom{n-2r+r-1}{r-1} = \binom{n-r-1}{r-1}$$

We apply the stars and bars formula to $n-2r$ stars and $r-1$ bars, removing the ones that must belong to their respective urns from consideration.