MIDTERM ONE

MODULE 1

Multiplication Rule	If there are a ways of doing something and b ways of doing another thing, then there are a x b ways of performing both (when the **number of ways to do each step is independent from another step**)
Addition Rule	If we have A ways of doing something and B ways of doing another thing and we **can not do both at the same time**, then there are A + B ways to choose one of the actions
Rule of Thumbs	P1: Even plus even is even P2: Even plus odd is odd P3: Odd plus even is odd P4: Odd plus odd is even M1: Even minus even is even M2: Even minus odd is odd M3: Odd minus even is odd M4: Odd minus odd is even
Prime Number	Integer with exactly two positive factors (1 and itself) - where $p \geq 2$
Twin Primes	Primes which are 2 apart (e.g. 3 & 5, 5 & 7, 11 & 13)
Set Notation	**Set Theory Notation.** $\{,\}$ We use these ***braces*** to enclose the elements of a set. So $\{1,2,3\}$ is the set containing 1, 2, and 3. $:$ $\{x : x > 2\}$ is the set of all x ***such that*** x is greater than 2. $\in$ $2 \in \{1,2,3\}$ asserts that 2 is ***an element of*** the set $\{1,2,3\}$. $\notin$ $4 \notin \{1,2,3\}$ because 4 ***is not an element of*** the set $\{1,2,3\}$. $\subseteq$ $A \subseteq B$ asserts that ***A is a subset of B***: every element of A is also an element of B. $\subset$ $A \subset B$ asserts that ***A is a proper subset of B***: every element of A is also an element of B, but $A \neq B$. $\cap$ $A \cap B$ is the ***intersection of A and B***: the set containing all elements which are elements of both A and B. $\cup$ $A \cup B$ is the ***union of A and B***: is the set containing all elements which are elements of A or B or both. $\times$ $A \times B$ is the ***Cartesian product of A and B***: the set of all ordered pairs (a,b) with $a \in A$ and $b \in B$. $\setminus$ $A \setminus B$ is ***A set-minus B***: the set containing all elements of A which are not elements of B. $\overline{A}$ The ***complement of A*** is the set of everything which is not an element of A. $\lvert A\rvert$ The ***cardinality (or size) of A*** is the number of elements in A. **Special sets.** $\emptyset$ The ***empty set*** is the set which contains no elements. $\mathcal{U}$ The ***universe set*** is the set of all elements. $\mathbb{N}$ The set of natural numbers. That is, $\mathbb{N} = \{0,1,2,3\ldots\}$. $\mathbb{Z}$ The set of integers. That is, $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,3,\ldots\}$. $\mathbb{Q}$ The set of rational numbers. $\mathbb{R}$ The set of real numbers. $\mathcal{P}(A)$ The ***power set*** of any set A is the set of all subsets of A.
Set Builder	The following are equivalent: $B = \{x \in X \mid P'(x)\}$ $B = \{x \mid x \in X \text{ and } P'(x)\}$
Symmetric Difference	The **symmetric difference** of two sets A and B is defined by $A \triangle B = (A \setminus B) \cup (B \setminus A)$.
Disjoint	Two sets A and B are said to be **disjoint** when they have no elements in common. Equivalently, A and B are disjoint when $A \cap B = \emptyset$.

Pairwise Disjoint	Three sets, A_1, A_2, A_3, are **pairwise disjoint** when A_1 and A_2 are disjoint, A_1 and A_3 are disjoint, and A_2 and A_3 are disjoint. This generalizes: three or more sets $A_1, A_2, \ldots, A_n$ $(n \geq 3)$ are **pairwise disjoint** when A_i and A_j are disjoint for all $i, j \in \{1, 2, \ldots, n\}$ such that $i \neq j$. **Example.** $\{b, c, d\}$, $\{f, g, h\}$, V, $\{m, n, p\}$, and $\{x, y, z\}$ are pairwise disjoint.
Set Difference	The **difference** of two sets A and B is the set whose elements are elements of A but **not** elements of B. Notation: $A \setminus B$. Using set-builder notation: $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$ **Examples.** $\{1,2,3\} \setminus \{2,3,4\} = \{1\}$ $\mathbb{N} \setminus \{0\} = \mathbb{Z}^+$ $\mathbb{N} \setminus \mathbb{Z}^+ = \{0\}$ $\mathbb{Z}^+ \setminus \mathbb{N} = \emptyset$ More generally, if $A \subseteq B$ then $A \setminus B = \emptyset$.
Cardinality	$\lvert \{3,5,7,9\} \rvert = 4.$ $\qquad \lvert \emptyset \rvert = 0.$ $\lvert \{x \in \mathbb{N} \mid 3 \leq x < 9\} \rvert = 6.$ $\qquad \lvert \{\emptyset, \{\emptyset\}\} \rvert = 2.$ If A and B are disjoint then $\lvert A \cup B \rvert = \lvert A \rvert + \lvert B \rvert$.
Empty Power Set	$2^{\{\emptyset\}} = \{ \emptyset , \{\emptyset\} \}$
Sequence	A **sequence** is an ordered collection of elements, with possible repetitions. Alternative terminology **list, array, string, tuple, word**. However, mathematically, these are all the same as sequences. A sequence has **positions**, 1,2,3, etc. and **length**. **Examples:** Consider the set $\{x, 2, a\}$. The sequences of length 2 whose elements are from this set: . $xx, x2, xa, 22, 2x, 2a, aa, a2, ax$. A string of digits of length 6: 737334 (has 7 in position 3) A word made of letters from the English alphabet: `floccinaucinihilipilification`.
Cartesian Product	The **cartesian product** (or **cross product**) of two sets A and B is the set whose elements are pairs whose first component is an element of A and whose second component is an element of B. Notation: $A \times B$. Using set-builder notation: $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ **Examples:** Let's enumerate the set $\{p, q\} \times \{2, 3\}$: . $\{(p, 2), (p, 3), (q, 2), (q, 3)\}$.

MODULE 2

Cardinality of Power Set	$\vert 2^A \vert = 2^{\vert A \vert}$ There are exactly 2^n distinct subsets of $\{a_1, \ldots, a_n\}$, and there are exactly 2^n distinct binary strings of length n.
Partial Permutation	For example, the number of partial permutations of r out of n $n \cdot (n-1) \cdots (n-r+1) = \frac{n \cdot (n-1) \cdots (n-r+1) \cdot (n-r) \cdots 1}{(n-r) \cdots 1} = \frac{n!}{(n-r)!}$
Permutation with Restrictions	Replace the desired result of the restriction (e.g. “hip”) with a single element (e.g. x) and proceed normally
Logic Notation	“and” $\wedge$ “or” $\vee$ “if-then” $\Rightarrow$ “not” $\neg$
Implication	Recall that “if P_1 then P_2” is called **implication** and is written in logical notation: $P_1 \Rightarrow P_2$ P_1 is called the **premise** of the implication and P_2 is called its **conclusion**.
Conditional	Inspired by some programming languages we ask for the logical notation for the **conditional** statement “if P_1 then P_2 else P_3”. It's $(P_1 \Rightarrow P_2) \wedge (\neg P_1 \Rightarrow P_3)$
Biconditional	Another statement is the **biconditional**: “if P_1 then P_2 and if P_2 then P_1”. Logical notation: $(P_1 \Rightarrow P_2) \wedge (P_2 \Rightarrow P_1)$ The biconditional is commonly written as “P_1 iff P_2” where “iff” abbreviates “if and only if”, and is called **equivalence**. But logically it is the same.
Proof with Implication	You wish to prove $P_1 \Rightarrow P_2$ **Proof pattern.** assert the **premise** P_1 (then derive/infer) . . . logical/mathematical consequences . . . (until you can) assert the **conclusion** P_2 With all this you have proven $P_1 \Rightarrow P_2$.
Proof by Cases	Assuming $P_1 \vee P_2$ you wish to prove P_3. **Proof pattern.** assert $P_1 \vee P_2$ **Case 1.** assert P_1. . . . logical/mathematical consequences . . . assert P_3 **Case 2.** assert P_2. . . . logical/mathematical consequences . . . assert P_3 Since in both cases we obtained P_3, we have proved it assuming $P_1 \vee P_2$.

Combinations	$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$
Propositions & Predicates	• Examples of the **first** kind: $odd(7)$ $odd(8)$. These are called **propositions** and they are either **true** or **false**. • Examples of the **second** kind: $odd(p)$ $vowel(\ell)$. Because p and ℓ are **variables**, these basic statements are called **predicates**. Predicates are **undetermined**, that is, neither true nor false, because the values of the variables are not specified. **Example.** The complex statement: $integer(x) \wedge (x > 1) \Rightarrow \neg prime(x^3 + 1)$ remains undetermined unless we specify a value for x.

MODULE 3

Stars and Bars	`***||******|***|` `||*********|***|` `*|*|||**********`
	`CCC||EEEEEE|FFF|` `||EEEEEEEEE|FFF|` `C|D|||GGGGGGGGGG`
	Distributing indistinguishable objects to distinguishable recipients, where the answer is:
	$$\binom{n+r-1}{r-1}.$$

Negation of disjunction / conjunction	Memorize: "the negation of disjunction is conjunction" and "the negation of conjunction is disjunction" In logical notation: $\neg(P_1 \vee P_2)$ is $(\neg P_1) \wedge (\neg P_2)$ and $\neg(P_1 \wedge P_2)$ is $(\neg P_1) \vee (\neg P_2)$ These are known as **De Morgan's Laws**. **Examples:** "not($x < 0$ or $x > 0$)" is "$x \geq 0$ and $x \leq 0$". "not($x \in A \cup B$)" is "$x \notin A$ and $x \notin B$". "not($x \in A \cap B$)" is "$x \notin A$ or $x \notin B$".
Negation of Quantifiers	Memorize: "the negation of universal is existential" and "the negation of existential is universal" In logical notation: $\neg(\forall x\ P(x))$ is $\exists x\ \neg P(x)$ and $\neg(\exists x\ P(x))$ is $\forall x\ \neg P(x)$ **Examples:** "not($\forall x\ \exists y\ loves(x, y)$)" is "$\exists x\ \forall y\ \neg loves(x, y)$"
Negation of Implication	Memorize: the negation of "if premise then conclusion" is "premise **and** the negation of the conclusion" In logical notation: $\neg(P_1 \Rightarrow P_2)$ is $P_1 \wedge \neg P_2$

Converse & Contrapositive

Memorize:
the **converse** of “if P_1 then P_2” is “if P_2 then P_1”, and

the **contrapositive** of “if P_1 then P_2” is “if (not P_2) then (not P_1)”.

In logical notation:
the **converse** of $P_1 \Rightarrow P_2$ is $P_2 \Rightarrow P_1$, and
the **contrapositive** of $P_1 \Rightarrow P_2$ is $\neg P_2 \Rightarrow \neg P_1$.

The contrapositive is **logically equivalent** to the original implication. This leads to:
Proof pattern: instead of the implication, prove its contrapositive,

The converse is **not** logically equivalent to the original implication (in general).

Truth Tables

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Vacuously True Statements

If premise is always false, the statement is vacuously true

Logical Equivalence

Two boolean expressions are **logically equivalent** when they yield the same truth value for the same truth assignments to their variables. Notation: $\equiv$

Here are some logically equivalent boolean expressions:

$$
\begin{aligned}
p \Rightarrow q &\equiv \neg q \Rightarrow \neg p \quad \text{(Contrapositive)} \\
p \vee q \Rightarrow r &\equiv (p \Rightarrow r) \wedge (q \Rightarrow r) \quad \text{(By-cases)} \\
p \Rightarrow q &\equiv \neg p \vee q \quad \text{(Law of Implication)} \\
\neg(p \Rightarrow q) &\equiv p \wedge \neg q \quad \text{(Disproving implication)} \\
\neg(p \vee q) &\equiv \neg p \wedge \neg q \quad \text{(De Morgan's Law I)} \\
\neg(p \wedge q) &\equiv \neg p \vee \neg q \quad \text{(De Morgan's Law II)} \\
\neg\neg p &\equiv p \quad \text{(Law of Double Negation)}
\end{aligned}
$$

Proof by Contradiction

Proof pattern. To prove “P” we can instead prove “if (not P) then C”.

This proof pattern is justified by the following logical equivalence:

. $p \equiv \neg p \Rightarrow \mathrm{F}$.

Proof pattern (variant). To prove “if P then Q” we can instead prove “if P and (not Q) then C”.

Counting Anagrams

$$\frac{(n_1 + n_2 + \ldots + n_k)!}{n_1! \cdot n_2! \cdots n_k!}$$

MODULE 4

Binomial Theorem

Binomial coefficients: $\binom{n}{r}$

Binomial Theorem. For any reals a and b and any natural number n

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$$

$$(a+b)^0 = \tbinom{0}{0} = 1$$

$$(a+b)^1 = \tbinom{1}{0}a^1 + \tbinom{1}{1}b^1 = a+b$$

$$(a+b)^2 = \tbinom{2}{0}a^2 + \tbinom{2}{1}a^1b^1 + \tbinom{2}{2}b^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = \tbinom{3}{0}a^3 + \tbinom{3}{1}a^2b^1 + \tbinom{3}{2}a^1b^2 + \tbinom{3}{3}b^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\begin{aligned}(a+b)^4 &= \tbinom{4}{0}a^4 + \tbinom{4}{1}a^3b^1 + \tbinom{4}{2}a^2b^2 + \tbinom{4}{3}a^1b^3 + \tbinom{4}{4}b^4 = \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

$$\begin{aligned}(a+b)^5 &= \tbinom{5}{0}a^5 + \tbinom{5}{1}a^4b^1 + \tbinom{5}{2}a^3b^2 + \tbinom{5}{3}a^2b^5 + \tbinom{5}{4}a^1b^4 + \tbinom{5}{5}b^5 = \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \cdots & & & & &
\end{array}$$

Combinatorial Proofs

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\sum_{i=0}^{n}\binom{n}{i} = 2^n$$

Function Range

The **range** of a function $f : A \to B$ is:

$$\mathrm{Ran}(f) = \{\ y \mid y \in B \ \wedge\ \exists x \in A\ \ y = f(x)\ \}$$

Counting Functions

Let A, B be two sets. The set

$$\{f \mid f : A \to B\} \quad \text{is denoted by} \quad B^A$$

Proposition. If $|A| = r$ and $|B| = n$ then the number of different functions with domain A and codomain B is n^r.

$$|B^A| = |B|^{|A|}$$

Integer Intervals	An **integer interval** $[m..n]$ (where $m \leq n$) is the set of **all** integers that lay between m and n, inclusive. In set-builder notation: $[m..n] \;=\; \{k \in \mathbb{Z} \mid m \leq k \leq n\}$ Memorize: $\mid [m..n] \mid \;=\; n - m + 1.$ $h : [0..n] \to [0..n]$ where $h(z) \;=\; n - z.$ $h(1) \;= n - 1 \quad 1 \mapsto \; n - 1$ $h(0) \;= n \quad 0 \mapsto n$ $f(n) \;= 0 \quad n \mapsto 0$ $f(n-1) \;= 1 \quad n - 1 \mapsto 1$ $\text{Ran}(h) = \;\; [0..n].$
Surjective	A function $f : A \to B$ is called **surjective** if $\text{Ran}(f) = B$, or equivalently: for every $y \in B$ there exists $x \in A$ such that $y = f(x)$. A surjective function is also called a **surjection**. **Examples.** $g : [-20..10] \to [0..20]$ where $g(y) \;=\; \text{abs}(y)$. $\quad$ $\text{Ran}(g) = [0..20]$ $h : [0..n] \to [0..n]$ where $h(z) \;=\; n - z.$ $\quad$ $\text{Ran}(h) = [0..n]$
Injective	A function $f : A \to B$ is called **injective** if it maps distinct elements to distinct elements, that is, for every $x_1 \neq x_2$ in the domain we have $f(x_1) \neq f(x_2)$, or, equivalently, (by contrapositive) $\forall\, x_1, x_2 \in A \quad f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2$ An injective function is also called an **injection**. **Examples.** $h : \mathbb{N} \to \mathbb{N}$ where $h(n) \;=\; 2n.$ It's injective because if $n_1 \neq n_2$ then $2n_1 \neq 2n_2$. $f : [0..10] \to [0..20]$ where $f(x) \;=\; x + 10.$ It's injective because if $x_1 \neq x_2$ then $x_1 + 10 \neq x_2 + 10$.
Surjective & Injective Rules	Let A and B be two sets. The **injection rule**: if we can define an injective function with domain A and codomain B then $\lvert A\rvert \leq \lvert B\rvert$. The **surjection rule**: if we can define a surjective function with domain A and codomain B then $\lvert A\rvert \geq \lvert B\rvert$. The **surjection rule (variant)**: if we can define a function $f : A \to B$ then $\lvert A\rvert \geq \lvert\text{Ran}(f)\rvert$. (If $f : A \to B$ then $f' : A \to \text{Ran}(f)$ where $f'(x) = f(x)$ is surjective.) If we can define a function with domain A and codomain B that is **both** a surjection and an injection then $\lvert A\rvert \;=\; \lvert B\rvert$.

MODULE 5

Bijective	A function $f : A \to B$ is called **bijective** if it is both injective and surjective. A bijective function is also called a **bijection** or a **one-to-one correspondence**. **Example.** $h : [0..n] \to [0..n]$ where $h(z) = n - z$. h is injective because $n - z_1 = n - z_2 \Rightarrow z_1 = z_1$ h is surjective because for $y \in [0..n]$ we can take $z = n - y$ and check $n - (n - y) = y$. The **bijection rule**: if we can define a bijective function with domain A and codomain B then $\|A\| = \|B\|$. The **bijection rule (variant)**: if we can define an **injective** function $f : A \to B$ then $\|A\| = \|\mathrm{Ran}(f)\|$. (If $f : A \to B$ is injective then $f' : A \to \mathrm{Ran}(f)$ where $f'(x) = f(x)$ is bijective.) **Example.** $f : [m..n] \to \mathbb{Z}$ where $f(z) = z + p$. f is injective and $\mathrm{Ran}(f) = [(m+p)..(n+p)]$. By the variant of the bijection rule $\| [m..n] \| = \| [(m+p)..(n+p)] \|$.
Principle of Inclusion Exclusion	Subtracting those, we get $\| A \cup B \| = \|A\| + \|B\| - \| A \cap B \|$. This is called the **Principle of Inclusion-Exclusion (PIE)** for two sets. The **Principle of Inclusion-Exclusion (PIE)** for **three** sets: $\| A \cup B \cup C \| = \|A\| + \|B\| + \|C\| - \| A \cap B \| - \| B \cap C \| - \| A \cap C \| + \| A \cap B \cap C \|$
Counting by Divisibility	**Problem.** How many integers in $[1..150]$ are divisible by 3, or by 5, or by 7? **Answer.** Let's first ask a simpler question. Given integers $1 < k < n$, how many multiples of k are in $[1..n]$? Let m be the largest multiple of k that is smaller than (or equal to) n. Then there are m/k multiples of k in $[1..n]$. Why? Because n lies between m (multiple of k) and $m + k$ (next multiple of k). For example, there are $150/3 = 50$ multiples of 3 and $150/5 = 30$ multiples of 5 in $[1..150]$. As for the multiples of 7, note that 147 is a multiple of 7. Since $147/7 = 21$, there are 21 multiples of 5 in $[1..150]$.
Derangements	An arrangement where some item does not occur in its original position
PHP	**PHP:** Let $f : A \to B$ be a function. If $\|A\| > \|B\|$ then there exist at least two elements $x_1, x_2 \in A$ such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. This is saying that if $\|A\| > \|B\|$ then f is not injective! From this formulation we see that the PHP is the **contrapositive** of the injection rule!

GPHP	**GPHP:** r objects are placed into n boxes. For any integer k such that $r > k\,n$ there is at least one box containing at least $k+1$ objects. Equivalently: Let $f : A \to B$ and $k \in \mathbb{Z}^+$. If $\lvert A\rvert > k\,\lvert B\rvert$ then there exist at least $k+1$ pairwise distinct elements of A that f maps to the same element of B.
Friends & Strangers	Among 6 friends - there are 3 that are pairwise FB friends or pairwise FB strangers

MODULE 6

Ordinary Induction	Let $P(n)$ be a predicate whose truth depends on n. **Proof pattern.** **(BASE CASE)** Check that $P(0)$ holds true. **(INDUCTION STEP)** Let k be an arbitrary natural number. Assume $\boxed{P(k).}$ Using that derive $P(k+1)$. Conclude $\forall\, n \in \mathbb{N}\;\; P(n)$. The $P(k)$ inside the box in the induction step is called the **INDUCTION HYPOTHESIS (IH)**. The IH must be stated **inside the induction step** because it refers to k. In logical notation the induction step is $\forall\, k \in \mathbb{N}\;\; P(k) \;\Rightarrow\; P(k+1)$.
Sum of Integers	$\dfrac{n(n+1)}{2}$
Strong Induction	Let n_0 be a natural number and let $P(n)$ be a predicate that is well defined for all natural numbers $n \geq n_0$. **Proof pattern.** **(BASE CASE)** Derive/infer $P(n_0)$. **(INDUCTION STEP)** Let $k \in \mathbb{N}$ such that $k \geq n_0$. Assume $\boxed{P(n_0) \text{ and} \cdots \text{and } P(k)}$. Derive/infer $P(k+1)$. Conclude $\forall\, n \geq n_0\;\; P(n)$. The IH $P(n_0)$ and $\cdots$ and $P(k)$ is stronger than $P(k)$. But strong induction is mathematically equivalent to the ordinary one!
Recursion Tree	**Answer (continued).** We will separate the addition terms in $C(0) \;=\; 1 \qquad C(n) \;=\; C(n-1) \;+\; n$ We draw a "tree of additions": $\begin{array}{ccccccccccc} n & & n-1 & & \cdots & & 2 & & 1 & & 1 \\ \vert & & \vert & & & & \vert & & \vert & & \vert \\ C(n) & — & C(n-1) & —— & \cdots & —— & C(2) & —— & C(1) & —— & C(0) \end{array}$ Therefore $C(n) \;=\; (n \;+\; (n-1) \;+\; \cdots \;+\; 2 \;+\; 1) \;+\; 1$ Using the formula $C(n) \;=\; n(n+1)/2 \;+\; 1 \;=\; (n^2+n+2)/2$

Telescopic Method	**Answer (continued).** We write the recurrence relation for $n, \ldots, 1$: $$\begin{aligned} C(n) &= C(n-1) + n \\ C(n-1) &= C(n-2) + n - 1 \\ C(n-2) &= C(n-3) + n - 2 \\ &\cdots \\ C(2) &= C(1) + 2 \\ C(1) &= C(0) + 1 \end{aligned}$$ Add all the LHSs and RHSs and cancel terms that appear on both sides: $C(n) \ = \ C(0) + 1 + 2 + \cdots + n \ = \ 1 + n(n+1)/2 \ = \ (n^2 + n + 2)/2$ (The method is called "telescopic" because the n equalities "collapse" into just one.)
Fibonacci Numbers	$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \qquad \text{for } n \geq 2 \end{aligned}$$ Hence the sequence of **Fibonacci numbers**: . 0 1 1 2 3 5 8 13 21 34 $\cdots$