Module 12.2: Cycles

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Closed walks and cycles

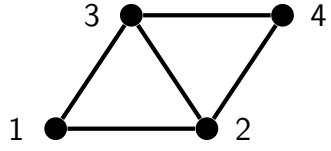
A **closed walk** is a walk in which the first and the last vertex are the same.

A **cycle** is a closed walk **of length at least 3** in which all nodes are pairwise distinct, except for the last and the first.

The **length** of the cycle is the length of the closed walk.

Examples of closed walks (cw's) and cycles:

G :



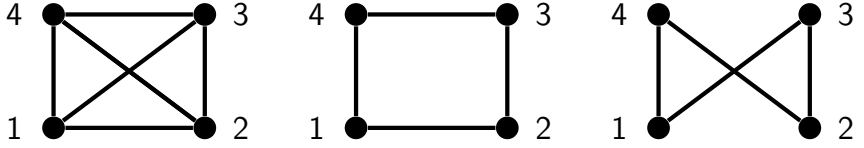
- 1–2–3–4–2–1 cw, **not** cycle
- 2–3–4–2 cycle, length 3
- 2 cw, **not** cycle
- 2–4–2 cw, **not** cycle
- 1–2–4–3–1 cycle, length 4

Counting cycles

When we count **cycles of length** n in a graph G, we count in fact the subgraphs of G that **are** cycle graphs on n vertices.

Problem. Consider the complete graph on nodes $\{1, 2, 3, 4\}$. Find two **different** cycle subgraphs that have the **same** set of nodes.

Answer.



The cycles 1–2–3–4–1 and 1–3–2–4–1 have the same set of nodes but different sets of edges. They correspond to the two distinct cycle subgraphs shown.

Counting cycles in K_4

Problem. How many cycles are there in K_4?

Answer. The subgraph induced by any three vertices is a cycle subgraph.

Therefore, the number of cycles of length 3 is $\binom{4}{3} = 4$.

We count the cycles of length 4 from the perspective of one node, say node 1.

Any cycle going through 1 uses 2 of the 3 edges incident to node 1. Once we choose these 2 edges the rest of the cycle of length 4 is determined. Indeed, there is only one more node to go through and this can be done in only one way.

Therefore, the number of cycles of length 4 is $\binom{3}{2} = 3$.

And the total number of cycles in K_4 is $4 + 3 = 7$.

Quiz I

How many cycles of length 4 are there in K_5?

(A) 15

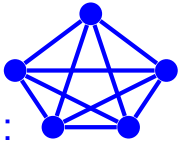
(B) 12

Answer

(A) 15

Correct. Construct a cycle subgraph of length 4 in two steps:
Step 1: Choose 4 out of 5 vertices in $\binom{5}{4} = 5$ ways
Step 2: Construct a cycle of length 4 on the chosen 4 vertices. Recall that earlier in this video we showed that this can be done in 3 ways.
In total there are $5 \cdot 3 = 15$ cycles of length 4. You can make sure that this counting is correct by checking K_5 manually

K_5 :



(B) 12

Incorrect.

Quiz II

What is the total number of cycles in K_5?

(A) 35

(B) 36

(C) 37

Answer

(A) 35

Incorrect.

(B) 36

Incorrect.

(C) 37

Correct. We have $\binom{5}{3} = 10$ cycles of length 3 and from the previous quiz we have 15 cycles of length 4. For length 5 begin similarly to length 4 in K_4: from the perspective of node 1. Choose 2 of the 4 edges incident to node 1 in $\binom{4}{2} = 6$ ways. Next, let u and v be the other endpoints of the chosen 2 edges. To complete the cycle we need a path of length 3 from u to v that does not go through 1. There are two ways to order the two intermediate nodes on this path. In total there are $6 \cdot 2 = 12$ cycles of length 5. Finally, $10 + 15 + 12 = 37$.

More Information

You can make sure that this counting is correct by checking K_5 manually

K_5 :