OMCIT 592 Module 08 Self-Paced 01 (instructor Val Tannen)

Reference to this self-paced segment in seg.08.02

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion.

When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Proofs of independence properties

Throughout this segment, let $(\Omega, \Pr)$ be an arbitrary probability space, and let E, A, B, F be arbitrary events in this space.

Recall the following from the lecture segment "Probability properties":

Property P0. $\Pr[E] \ \geq \ 0$

Property P1. $\Pr[\Omega] \ = \ 1$

Property P2. If A, B are disjoint then $\Pr[A \cup B] \ = \ \Pr[A] \ + \ \Pr[B]$

Property P3. If $A \subseteq B$ then $\Pr[A] \ \leq \ \Pr[B]$

Property P4. $\Pr[\overline{E}] \ = \ 1 \ - \ \Pr[E]$

Property P5. $\Pr[\emptyset] \ = \ 0$

Property P6. $\Pr[A \cup B] \ = \ \Pr[A] \ + \ \Pr[B] \ - \ \Pr[A \cap B]$

Property P7. $\Pr[A \cup B] \ \leq \ \Pr[A] \ + \ \Pr[B]$

The proofs of these properties were given in the segment "Proofs of probability properties".

In the lecture segment "Independence" we stated four properties of **event independence**:

Property Ind (i). If $\Pr[A] = 0$ then $A \perp B$ for any B.

Property Ind (ii). $\Omega \perp E$ for any E.

Property Ind (iii). If $A \perp B$ then $\Pr[A \cup B] \ = \ 1 - (1 - \Pr[A])(1 - \Pr[B])$.

Property Ind (iv). $A \perp B$ iff $\overline{A} \perp B$ iff $A \perp \overline{B}$ iff $\overline{A} \perp \overline{B}$

Properties Ind (i) and Ind (iii) were proved in lecture.

Here we prove properties Ind (ii) and Ind (iv).

Proof of Ind (ii). First, note that $\Omega \cap E = E$. Using also (P1) we obtain

$$\begin{aligned}
\Pr[\Omega \cap E] &= \Pr[E] \\
&= \Pr[E] \cdot 1 \\
&= \Pr[E] \cdot \Pr[\Omega]
\end{aligned}$$

It folows that $\Omega \perp E$.

Proofs of independence properties (continued)

Proof of Ind (iv).

Lemma. For any two events E, F if $E \perp F$ then $\overline{E} \perp F$.

Proof (of Lemma). Observe that $F = \Omega \cap F$ and that $E \cup \overline{E} = \Omega$. Using these, and the fact that set intersection distributes over set union (check it with an Euler-Venn diagram) we obtain

$$F \;=\; (E \cup \overline{E}) \cap F \;=\; (E \cap F) \cup (\overline{E} \cap F)$$

Notice that $E \cap F$ and $\overline{E} \cap F$ are disjoint so we can apply (P2):

$$\Pr[F] \;=\; \Pr[(E \cap F) \cup (\overline{E} \cap F)] \;=\; \Pr[E \cap F] + \Pr[\overline{E} \cap F]$$

The assumption $E \perp F$ means that $\Pr[E \cap F] = \Pr[E] \cdot \Pr[F]$. Plugging this into the previous equality we obtain

$$\Pr[F] \;=\; \Pr[E] \cdot \Pr[F] + \Pr[\overline{E} \cap F]$$

Hence

$$\Pr[\overline{E} \cap F] \;=\; \Pr[F] - \Pr[E] \cdot \Pr[F] \;=\; \Pr[F](1 - \Pr[E]) \;=\; \Pr[F] \cdot \Pr[\overline{E}])$$

Therefore $\overline{E} \perp F$.

Now we apply the lemma to prove that the four independence assertions in Ind (iv) are equivalent. Because implication is transitive it will suffice to show that

$$A \perp B \;\;\Rightarrow\;\; \overline{A} \perp B \;\;\Rightarrow\;\; \overline{A} \perp \overline{B} \;\;\Rightarrow\;\; A \perp \overline{B} \;\;\Rightarrow\;\; A \perp B$$

We will also need to use the symmetry of independence, that is, $G \perp H$ iff $H \perp G$, and the fact that for any event G we have $\overline{\overline{G}} = G$.

To show $A \perp B \;\Rightarrow\; \overline{A} \perp B$ we take $E = A$ and $F = B$ in the lemma.

To show $\overline{A} \perp B \;\Rightarrow\; \overline{A} \perp \overline{B}$ we take $E = B$ and $F = \overline{A}$ in the lemma.

To show $\overline{A} \perp \overline{B} \;\Rightarrow\; A \perp \overline{B}$ we take $E = \overline{A}$ and $F = \overline{B}$ in the lemma.

To show $A \perp \overline{B} \;\Rightarrow\; A \perp B$ we take $E = \overline{B}$ and $F = A$ in the lemma.