Module 1.6: Set Operations and Cardinality

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Union

The **union** of two sets A and B is the set whose elements are elements of A or elements of B (including those who are elements of both).

Notation: $A \cup B$.

Using set-builder notation: $A \cup B \;=\; \{x \mid x \in A \text{ or } x \in B\}$

Examples.

Recall that V is the set of vowels and C the set of consonants.

Then $V \cup C$ is the whole alphabet!

$\mathbb{Z}^+ \cup \{0\} \;=\; \mathbb{N}$

If $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$ then $A \cup B \subseteq \mathbb{N}$.

Intersection

The **intersection** of two sets A and B is the set whose elements are elements of both A and B.

Notation: $A \cap B$.

Using set-builder notation: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Examples.

$V \cap C = \emptyset$.

$\mathbb{Z}^+ \cap \mathbb{N} = \mathbb{Z}^+$

More generally, if $A \subseteq B$ then $A \cap B = A$.

Union/intersection of more than two sets

The **union** and the **intersection** of the sets $A_1, A_2, \ldots, A_n$ are defined by

$$A_1 \cup A_2 \cup \cdots \cup A_n \; = \; \{x \mid x \in A_1 \text{ or } x \in A_2 \text{ or } \ldots \text{ or } x \in A_n\}$$

respectively,

$$A_1 \cap A_2 \cap \cdots \cap A_n \; = \; \{x \mid x \in A_1 \text{ and } x \in A_2 \text{ and } \ldots \text{ and } x \in A_n\}$$

For $n = 2$ we get exactly the definition for union/intersection of two sets that we had before.

How about for $n = 1$? We get $A_1 \; = \; \{x \mid x \in A_1\}$.

True but not interesting!

Disjoint and pairwise disjoint sets

Two sets A and B are said to be **disjoint** when they have no elements in common. Equivalently, A and B are disjoint when $A \cap B = \emptyset$.

Example.

V and C are disjoint. $\{o, u, a, i, s\}$ and C are **not** disjoint.

Three sets, $A_1, A_2, A_3,$ are **pairwise disjoint** when A_1 and A_2 are disjoint, A_1 and A_3 are disjoint, and A_2 and A_3 are disjoint.

This generalizes: three or more sets $A_1, A_2, \ldots, A_n$ $(n \geq 3)$ are **pairwise disjoint** when A_i and A_j are disjoint for all $i, j \in \{1, 2, \ldots, n\}$ such that $i \neq j$.

Example.

$\{b, c, d\}$, $\{f, g, h\}$, V, $\{m, n, p\}$, and $\{x, y, z\}$ are pairwise disjoint.

Quiz

If three sets X, Y and Z have an empty intersection ($X \cap Y \cap Z = \emptyset$), then they must be pairwise disjoint.

(A) True

(B) False

Answer

If three sets X, Y and Z have an empty intersection ($X \cap Y \cap Z = \emptyset$), then they must be pairwise disjoint.

(A) True

Incorrect. See B for a counterexample.

(B) False

Correct. Consider the sets $X = \{1, 2\}$, $Y = \{2, 3\}$, $Z = \{1, 3\}$. They have an empty intersection, but they are not pairwise disjoint.

More Information

Consider again the sets $\{1, 2\}, \{2, 3\}, \{1, 3\}$. They have an empty intersection, but they are not pairwise disjoint, in fact the intersection of any two of them is non-empty. On the other hand, if some number of sets are pairwise disjoint, then their intersection is empty.

This distinction leads to possible confusion and this is why we avoid defining "disjoint" for more than three sets. "Pairwise disjoint" is more useful anyway.

Difference

The **difference** of two sets A and B is the set whose elements are elements of A but **not** elements of B. Notation: $A \setminus B$.

Using set-builder notation: $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$

Examples.

$$\{1, 2, 3\} \setminus \{2, 3, 4\} = \{1\}$$

$$\mathbb{N} \setminus \{0\} = \mathbb{Z}^+$$

$$\mathbb{N} \setminus \mathbb{Z}^+ = \{0\}$$

$$\mathbb{Z}^+ \setminus \mathbb{N} = \emptyset$$

More generally, if $A \subseteq B$ then $A \setminus B = \emptyset$.

Cardinality

The **cardinality** of a finite set A is the number of elements of A.
Notation: $|A|$.

Examples.

$$|\{3,5,7,9\}| = 4. \qquad |\emptyset| = 0.$$

$$|\{x \in \mathbb{N} \mid 3 \leq x < 9\}| = 6. \qquad |\{\emptyset, \{\emptyset\}\}| = 2.$$

If A and B are disjoint then $|A \cup B| = |A| + |B|$.

An **aside**: We have only defined cardinality for finite sets. However, the concept can be defined for any number of sets. I will not say much more about this but here are some intriguing facts about cardinality of infinite sets:

$$|\mathbb{N}| = |\mathbb{Z}^+| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}| = |\mathbb{C}|$$