Recitation Module 4

Penn Engineering

Lecture Review

Binomial Theorem

Binomial coefficients: $\binom{n}{r}$

Binomial Theorem. For any reals a and b and any natural number n

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Combinatorial Proofs

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Functions

A **function** (sometimes called **mapping**) denoted $f : A \to B$, consists of

- a set A, called **domain**,
- a set B, called **codomain**, and
- a way of associating with **every** element of the domain, $x \in A$, a **unique** element of the codomain, $f(x) \in B$, write $x \mapsto f(x)$.

The **range** of a function $f : A \to B$ is:

$$\mathrm{Ran}(f) \;=\; \{\, y \mid y \in B \;\wedge\; \exists x \in A \;\; y = f(x) \,\}$$

Note that this defines a subset $\mathrm{Ran}(f) \subseteq B$.

Counting Functions

Let A, B be two sets. The set

$$\{f \mid f : A \to B\} \quad \text{is denoted by} \quad B^A$$

Proposition. If $|A| = r$ and $|B| = n$ then the number of different functions with domain A and codomain B is n^r.

Integer Intervals

An **integer interval** $[m..n]$ (where $m \leq n$) is the set of **all** integers that lay between m and n, inclusive. In set-builder notation:

$$[m..n] \;=\; \{k \in \mathbb{Z} \mid m \leq k \leq n\}$$

Surjection and Injection

A function $f : A \to B$ is called **surjective** if $\mathrm{Ran}(f) = B$, or equivalently:

for every $y \in B$ there exists $x \in A$ such that $y = f(x)$.

A surjective function is also called a **surjection**.

A function $f : A \to B$ is called **injective** if it maps distinct elements to distinct elements, that is,

for every $x_1 \neq x_2$ in the domain we have $f(x_1) \neq f(x_2)$,

or, equivalently, (by contrapositive)

$$\forall\, x_1, x_2 \in A \quad f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2$$

An injective function is also called an **injection**.

Injection Rule and Surjection Rule

Let A and B be two sets.

The **injection rule**: if we can define an injective function with domain A and codomain B then $|A| \leq |B|$.

The **surjection rule**: if we can define a surjective function with domain A and codomain B then $|A| \geq |B|$.

Practice Questions

Question 1

Prove whether the following function is injective, surjective, or bijective (or none of the above):

$f : \mathbb{R} \to \mathbb{R}$ defined as:

$$f(x) = \begin{cases} x^3 & x \geq 1 \\ x^3 + 1 & x < 1 \end{cases}$$

Answer to Question 1

$$f(x) = \begin{cases} x^3 & x \geq 1 \\ x^3 + 1 & x < 1 \end{cases}$$

Let's start with surjection.

Case 1: $\quad y \geq 1$

We observe that $y^{1/3} \geq 1$.

Let $x = y^{1/3}$. Then $f(x) = x^3 = (y^{1/3})^3 = y$.

So for any $y \geq 1$, we've shown there exists some x such that $f(x) = y$.

Case 2: $\quad y < 1$

We observe that $y – 1 < 0$ and that $(y – 1)^{1/3} < 0 < 1$.

Let $x = (y – 1)^{1/3}$. Then $f(x) = x^3 + 1 = ((y – 1)^{1/3})^3 + 1 = y$.

So for any $y < 1$, we've shown there exists some x such that $f(x) = y$.

So we have shown that, for any y, there exists some x such that $f(x) = y$. Thus, by definition, f is a surjection.

Answer to Question 1

$$f(x) = \begin{cases} x^3 & x \geq 1 \\ x^3 + 1 & x < 1 \end{cases}$$

Let's now proceed to injection.

Are there two distinct elements of the domain, a and b, such that $a \neq b$ and $f(a) = f(b)$? If yes, then the function is *not* injective.

Consider $a = 0$ and $b = 1$.

If $a = 0$, $f(a) = a^3 + 1 = 1$.

If $b = 1$, $f(b) = b^3 = 1$.

Since two distinct elements of the domain map to the same element in the domain, f is not injective. By definition, this also means f cannot be a bijection, since it is not both surjective and injective.

Question 2

Give a combinatorial proof of the following:

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$$

Answer to Question 2

Give a combinatorial proof of the following:

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$$

Consider the following (more concrete) example: We have a graduating class of *n* students and we are deciding which *k* of them will receive an award. Suppose that 2 of these *n* students are economics majors, Julie and John.

Answer to Question 2

The most straightforward approach is to simply choose *k* of the *n* students to give awards to – this is the LHS.

Or, we could look at things from the perspective of the economics students.

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$$

Case 1: None of the 2 economics students win the award, so we still need to give *k* awards to *n* – 2 students.

Case 2: 1 of the 2 economics students wins the award, so we still need to give *k* – *1* awards to *n* – 2 students.*

Case 3: Both of the economics students win the award, so we need to give *k* – *2* awards to *n* – 2 students.

***The reason we multiply by 2 is because there are two scenarios in which 1 of the 2 economics students wins an award: Julie wins an award and John doesn't, or John wins an award and Julie doesn't. We need to count both possibilities.**

Answer to Question 2

We can put everything together on the RHS using the addition rule.

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$$

Question 3

The janitor needed to distribute soap bars and toilet paper to customers of the hotel. He starts his shift with 10 bars of soap and 10 rolls of toilet paper. After the 6th room, he wonders to himself:

How many ways could he have distributed the toilet paper rolls and soap bars to the 6 different rooms?

He cannot tell the difference between any two toilet paper rolls and between any two soap bars. However, he can easily tell the difference between toilet paper and soap bars.

Answer to Question 3

Since we're allocating objects to distinguishable categories (in this case, bars of soap and toilet paper to rooms), this is a **stars and bars problem**!

To distribute 10 soap boars to 6 rooms, we have n = 10 and r = 6, so the number of possible arrangements is $\binom{10+6-1}{6-1} = \binom{15}{5}$.

The math is the same for distributing 10 rolls of toilet paper to the 6 rooms.

Since the processes of distributing toilet paper and soap bars are separate from each other, we can use the multiplication rule to arrivate at $\binom{15}{5}\binom{15}{5}$.

Question 4

Consider the intervals A = [3..7], B = [4..10], C = [1..5], D = [1..3].

How many different functions are possible if…

(a) The domain is A and codomain is B
(b) The domain is A and codomain is D; functions are surjective
(c) The domain is A and codomain is B; functions are injective
(d) The domain is A and codomain is C; functions are bijective

Answer to Question 4

Consider the intervals A = [3..7], B = [4..10], C = [1..5], D = [1..3].

(a) The domain is A and codomain is B

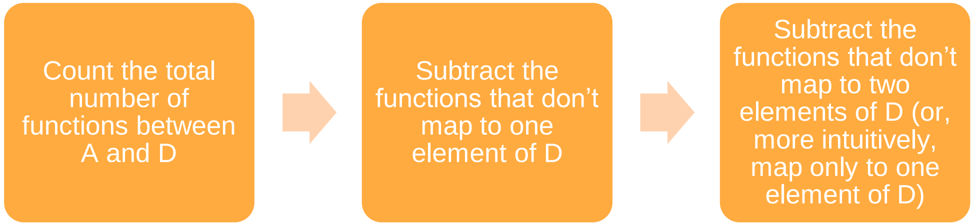
We know $|A| = 5$ and $|B| = 7$. The number of possible functions from a domain A to codomain B is $|B|^{|A|} = 7^5$.

Answer to Question 4

Consider the intervals A = [3..7], B = [4..10], C = [1..5], D = [1..3].

(b) The domain is A and codomain is D; functions are surjective

This is a little tricky, but a good application of complementary counting. We can do the following:



First, count the total number of possible functions from A to D. Since |A| = 5 and |D| = 3, we know this number is 3^5.

Answer to Question 4

Consider the intervals A = [3..7], B = [4..10], C = [1..5], D = [1..3].

(b) The domain is A and codomain is D; functions are surjective

Second, subtract from 3^5 the total number of functions where one element of D is not mapped to (which would violate the requirement that the counted functions be surjective).

This is like picking 1 of the 3 elements from D to ignore and then counting functions between A and the reduced set of D, with only 2 elements. For example, we count functions that don't map to 1 in D, and only map to 2 and 3 in D.

Answer to Question 4

Consider the intervals A = [3..7], B = [4..10], C = [1..5], D = [1..3].

(b) The domain is A and codomain is D; functions are surjective

But we need to be careful, since counting functions between A and the reduced set D will include two functions that map to only one of the 2 remaining elements, and we are counting these functions in the third step (in which we count functions that only map to one element of D). So, using our example of counting functions that don't map to 1 in D, if we count all functions between A and the reduced set of elements in D {2, 3}, we'll be including two functions that *only* map to 2 or 3.

So we subtract two to get the following: $\binom{3}{1}(2^5 - 2)$.

Answer to Question 4

Consider the intervals A = [3..7], B = [4..10], C = [1..5], D = [1..3].

(b) The domain is A and codomain is D; functions are surjective

Third and finally, we count the number of functions from A to D that map only to one element of D. This is like picking which element in D gets mapped to by all elements of A, and we have 3 options (as |D| = 3).

So there are 3 such functions.

Answer to Question 4

Consider the intervals A = [3..7], B = [4..10], C = [1..5], D = [1..3].

Putting this all together, we have the following equation resulting from our complementary counting:

$3^5 - \binom{3}{1}(2^5 - 2) - 3$

= 150 surjective functions from A to D

Answer to Question 4

Consider the intervals A = [3..7], B = [4..10], C = [1..5], D = [1..3].

(c) The domain is A and codomain is B; functions are injective

We know |A| = 5 and |B| = 7. As a first step, we need to chose *which |A| = 5* elements of B get mapped to, since no element of A can map to multiple elements in B. We have $\binom{7}{2}$ choices.

Next, we need to assign these elements of B to elements of A, so that elements of A map to elements of B. This is akin to generating permutations of the elements of B. There are 5! different arrangements.

Answer to Question 4

Consider the intervals A = [3..7], B = [4..10], C = [1..5], D = [1..3].

(c) The domain is A and codomain is B; functions are injective

Putting everything together by the multiplication rule, our answer is $\binom{7}{2}$ 5!.

Answer to Question 4

Consider the intervals A = [3..7], B = [4..10], C = [1..5], D = [1..3].

(d) The domain is A and codomain is C; functions are bijective

By definition of a bijection, there must be a one-to-one mapping between elements of A and C. So all we need to do is generate some arrangement of the elements of C and assign the arrangement to A.

We know |C| = 5, and there are 5! ways to order the elements of C. Thus there are 5! bijective functions between A and C.

Question 5 (time permitting)

For any natural number $n \in \mathbb{N}$ and for any function $f : [1..n] \to [1..n]$ prove that f is injective iff it is surjective.

Answer to Question 5

First note that this is an iff statement. That is, we have to prove the statement in both directions. Also note that the initial proofs ignore the case where $n = 0$.

(f is injective if it is surjective) Assume for contradiction that f is not injective. That is, there exist two elements $x \neq y$ such that $f(x) = f(y)$. However, this leads to a contradiction since this function could not be surjective. That is, the size of the domain and codomain are both n. If two elements in the domain map to the same element in the codomain, the remaining $n - 2$ elements can map to at most $n - 2$ elements in the codomain. Thus at most $n - 1$ of the n elements could be mapped to, concluding that f is not surjective - a contradiction. Note that this argument mirrors the contrapositive of the surjection rule.

Answer to Question 5

(f is surjective if it is injective) Again assume for contradiction that f is not surjective. That means there exists an element y in the codomain such that for all x in the domain, $y \neq f(x)$. However, this also leads to a contradiction as the function could not be injective. Without element y, the size of the codomain is $n-1$ but the size of the domain is n. Thus, it's not possible for every element in the domain to map to a unique element.

Note that this argument mirrors the contrapositive of the injection rule.

Now for the case where $n = 0$. The previous arguments do not apply to this case, since f is not a valid function. That is, [1.0] is an invalid set, making the domain and codomain of f undefined. Since f is not a valid function, it cannot be injective or surjective, making the statement f is injective iff it is surjective vacuously true.