Module 5.1: Bijections

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Lecture Notes

Bijective functions I

A function $f : A \to B$ is called **bijective** if it is both injective and surjective. A bijective function is also called a **bijection** or a **one-to-one correspondence**.

Example. $h : [0..n] \to [0..n]$ where $h(z) = n - z$.

h is injective because $n - z_1 = n - z_2 \Rightarrow z_1 = z_1$

h is surjective because for $y \in [0..n]$ we can take $z = n - y$ and check $n - (n - y) = y$.

The **bijection rule**: if we can define a bijective function with domain A and codomain B then $|A| = |B|$.

Recall from an earlier segment the one-to-one correspondence between subsets of a set with n elements and strings of bits of length n.

Bijective functions II

The **bijection rule (variant)**: if we can define an **injective** function $f : A \to B$ then $|A| = |\mathrm{Ran}(f)|$.

(If $f : A \to B$ is injective then $f' : A \to \mathrm{Ran}(f)$ where $f'(x) = f(x)$ is bijective.)

Example. $f : [m..n] \to \mathbb{Z}$ where $f(z) = z + p$.

f is injective and $\mathrm{Ran}(f) = [(m+p)..(n+p)]$.

By the variant of the bijection rule $|\,[m..n]\,| = |\,[(m+p)..(n+p)]\,|$.

A bijection for counting pets I

Problem. Animal Rescue has 5 cats and 3 dogs. How many different groups of these pets that include at least one cat and at least one dog can you adopt?

Answer. (again) Let C be the set of the 5 cats and D the set of the 3 dogs. Let also, $P = C \cup D$ be the set of all pets.

A group of adopted pets is a subset $S \subseteq P$, that is, $S \in 2^P$.

Because pets are either cats or dogs (that is, C and D are **disjoint**) any $S \subseteq P$ is $S = A \cup B$ where $A \subseteq C$ and $B \subseteq D$.

Specifically, A is the set of those pets in S which are cats and B is the set of those pets in S which are dogs.

In set notation $A = S \cap C$ and $B = S \cap D$.

That is, A and B are completely determined by S.

A bijection for counting pets II

Answer. (continued) The discussion on the previous slide was in fact about the properties of

$$f : 2^C \times 2^D \rightarrow 2^P \qquad \text{where } f(A, B) = A \cup B$$

Indeed, f is **surjective** because any $S \in 2^P$ can be written

. $S = A \cup B$ where $A \in 2^C$ and $B \in 2^D$.

And f is **injective** because any $S = f(A, B)$ uniquely determines (A, B) since

. $A = S \cap C$ and $B = S \cap D$.

Therefore f is a **bijection**.

However, what about the condition "at least one cat and at least one dog"?

A bijection for counting pets III

Answer. (continued) Notation for the set of all **non-empty** subsets:

$$\text{nonempty}(2^X) \;=\; \{T \subseteq X \mid T \neq \emptyset\}$$

Define f' on pairs of non-empty subsets:

$$f' : \text{nonempty}(2^C) \times \text{nonempty}(2^D) \;\to\; 2^P \quad \text{where} \quad f'(A, B) \;=\; A \cup B$$

Because $f'(A, B) = f(A, B)$ and f is injective, f' is also **injective**.

Therefore, the **variant of the bijection rule** applies:

$$|\,\text{nonempty}(2^C) \times \text{nonempty}(2^D)\,| \;=\; |\,\text{Ran}(f')\,|$$

We already seen that $|\,\text{nonempty}(2^X)\,| \;=\; 2^{|X|} - 1$

The sets of pets in $\text{Ran}(f')$ are exactly the ones that have at least one cat and at least one dog! All this was implicit when we solved the problem the first time. The bijection rule is often used implicitly, as it was there.