OMCIT 592 Spring 2022 Exam 2

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This assignment is due at the beginning of the the first section on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - **no collaboration is allowed**.

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1. **[10 pts]** Alice conducts in private the following random experiment. First she flips a fair coin. If the coin comes up *heads* she (independently) rolls a fair die *once*. If the coin comes up *tails* she rolls a fair die *twice*, independently, and independently of the coin flip. Alice does not tell Bob what the coin showed but she tells Bob a number that she obtains as follows:

   - If she rolled the die *once* she just tells Bob what number the die showed.
   - If she rolled the die *twice* she adds the two numbers shown and tells Bob that sum.

   (a) **[5 pts]** Calculate the probability that Alice tells the number 6 to Bob.

   (b) **[5 pts]** Suppose Alice tells Bob the number 6. What seems more likely to Bob: that the coin came up heads or that it came up tails?

   Remember to describe the probability space and to justify your answers.

   Solution.

   (a) Let H mean the coin comes up heads and T mean the coin comes up tails. Let X be the random variable that returns the number that Alice tells Bob. I would discuss three probability space:
   First probability space - When Alice flips a fair coin:
   There are two outcomes when flipping a coin - heads and tails, so $|\Omega_1| = 2$. Since this is a fair coin, the two outcomes has the same probability, $P_r[H] = P_r[T] = 1/2$. This probability space is uniform.
   Second probability space - Alice rolls a fair die once:
   There are 6 outcomes when rolling a dice, 1,2,3,4,5,6, so $|\Omega_2| = 6$. Since this is fair die, all outcomes has the same probability, 1/6. $P_{r1}[X = 1] = \cdots = P_{r1}[X = 6] = 1/6$. This probability space is uniform.
   Third probability space - Alice rolls a fair die twice and sum the two rolls up:

Since the two rolls are independent, there will be $6*6 = 36$ outcomes, so $|\Omega_3| = 36$. Since this is fair die, all outcomes has the same probability, 1/36. The sum of the two rolls are [2,12], where one outcome will sum to $2(P_{r2}[X = 2] = 1/36)$, 2 outcomes will sum to 3 ($P_{r2}[X = 3] = 2/36$), ..., 5 outcomes will sum to 6 ($P_{r2}[X = 6] = 5/36$),. . . .This probability space is not uniform.
In order for Alice to tells the number 6 to Bob (X=6), there are two situations:
First: The coin comes up heads and Alice rolls a 6.
From the probability space description we know that the probability of the coin comes up heads is 1/2, and the probability of Alice rolls a 6 is 1/6. Since the coin flip and dice rolling are in dependent, we use multiplication rule to calculate that $P_r[(X = 6)|H] = P_r[H \cap (X = 6)] = P_r[H] * P_{r1}[X = 6] = 1/2 * 1/6 = 1/12$.
Second: The coin comes up tails and Alice rolls a 6 as the sum of the two dice rolls.
In order to let the sum of the two dice rolls be 6, there are 5 outcomes: $\{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$ where the first number in every pair is the first dice roll, and the second number in every pair is the second dice roll. From the description of the third probability space we know that the probability of Alice rolls a 6 as the sum of two rolls are 5/36. Since the coin flip and dice rollings are in dependent, we use multiplication rule to calculate that $P_r[(X = 6)|T] = P_{r2}[T \cap (X = 6)] = P_r[T] * P_{r2}[X = 6] = 1/2 * 5/36 = 5/72$.
Therefore the probability that Alice tells the number 6 to Bob is $P_r[X = 6] = P_r[(X = 6)|H] + P_r[(X = 6)|T] = 1/12 + 5/72 = 11/72$

(b) Since $P_r[(X = 6)|H] = 1/12$ which is bigger than $P_r[(X = 6)|T] = 5/72$, it's more likely that the coin came up heads to Bob.

2. **[10 pts]** We generate uniformly at random a sequence of 10 (ten) decimal digits, that is, elements of `[0..9]`. Digits can repeat, for example, `2675673377` or `0884480491`. What is the probability of each of the following events?

   (a) **[5 pts]** The first three digits of the sequence are `801` *or* `555` *or* `090`.

   (b) **[5 pts]** The string begins *or* ends with `999`.

   Remember to describe the probability space and to justify your answers.

   Solution.

   (a) The probability space would have all the sequences of 10 digits where each position can be any number in [0..9], thus $|\Omega| = 10^{10}$. Since we generate it uniformly at random, the probability space is uniform.

Let A be event "The first three digits of the sequence are 801".

Let B be event "The first three digits of the sequence are 555".

Let C be event "The first three digits of the sequence are 090".

Since $|A| = |B| = |C| = 10^7$, $P_r[A] = P_r[B] = P_r[C] = 10^7/10^{10} = 1/1000$

Since an event can not have 801, 555 and 090 as the first three digits at the same time, we know that event A, B and C are disjoint, which means $A \cap B \cap C = \emptyset$. By probability properties we know that $P_r[A \cap B \cap C] = 0$.

Similarly since an event can not have 801 and 555 as the first three digits at the same time, we know that event A and B are disjoint and $P_r[A \cap B] = 0$.

Similarly we can know that $P_r[A \cap C] = P_r[B \cap C] = 0$.

The probability we are trying to computer is $P_r[A \cup B \cup C]$. By probability properties we know that:

$P_r[A \cup B \cup C] = P_r[A] + P_r[B] + P_r[C] - P_r[A \cap B] - P_r[A \cap C] - P_r[B \cap C] + P_r[A \cap B \cap C]$

$=1/1000 + 1/1000 + 1/1000 - 0 - 0 - 0 + 0 = 3/1000$

(b) The probability space would have all the sequences of 10 digits where every position can be any number in [0..9], thus $|\Omega| = 10^{10}$. Since we generate it uniformly at random, the probability space is uniform.

Let A be event "The string begins with 999".

Let B be event "The string ends with 999".

Since $|A| = |B| = 10^7$, $P_r[A] = P_r[B] = 10^7/10^{10} = 1/1000$

Since a string can both begin and end with 999, $|A \cap B| = 10^4$, so $P_r[A \cap B] = 10^4/10^{10} = 1/10^6$

The probability we are trying to computer is $P_r[A \cup B]$. By probability properties we know that:

$P_r[A \cup B] = P_r[A] + P_r[B] - P_r[A \cap B]$

$=1/1000 + 1/1000 - 1/10^6$

$=1999/10^6$

3. **[10 pts]** Consider a random permutation of the decimal digits, that is, of the numbers $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and the following two events:
E is "the first two digits are 0 and 1 in that order".
F is "the last two digits are 8 and 9 in that order".
Are E and F independent?

We know this probability space from lecture, no need to describe it. However, justify your answer!

Solution.

By definition $E \perp F$ iff $P_r[E] * P_r[F] = P_r[E \cap F]$.

From the probability space we know $|\Omega| = 10!$ and it is uniform.

Since $|E| = |F| = 8!$ therefore $P_r[E] = P_r[F] = 8!/10!$.

$E \cap F$ represents the event that "the first two digits are 0 and 1 in that order and the last two digits are 8 and 9 in that order", so $|E \cap F| = 6!$ and $P_r[E \cap F] = 6!/10!$

Since $P_r[E] * P_r[F] = 8!/10! * 8!/10! = 1/8100$ and apparent it's different from $P_r[E \cap F]$, by definition E and F are not independent.

4. **[15 pts]** Let X and Y be two independent Bernoulli random variables, both with parameter 1/2, defined on the same probability space. Consider the random variables $Z = (1-X)(1-Y)$ and $W = 1 - XY$.

   (a) **[5 pts]** Among Z and W, which one has the larger expectation? Justify your answer.

   (b) **[5 pts]** Are Z and W independent? Justify your answer.

   (c) **[5 pts]** Compute the variance of W.

   No need to describe the probability space. However, justify your answer!

   Solution.

   (a) by the given definition of X, Y, Z, W, we know that

   X=1, Y=1 $\implies$ Z=0, W=0

   X=1, Y=0 $\implies$ Z=0, W=1

   X=0, Y=1 $\implies$ Z=0, W=1

   X=0, Y=0 $\implies$ Z=1, W=1

   From above we can tell that the probability space for Z is not uniform, and $Val(Z) = \{0, 1\}$, $P_r[Z = 1] = 1/4$ and $P_r[Z = 0] = 3/4$.

   The probability space for W is not uniform, and $Val(W) = \{0, 1\}$, $P_r[W = 1] = 3/4$ and $P_r[W = 0] = 1/4$.

   Both Z and W are Bernoulli random variables, and parameter of Z $p_Z = 1/4$, parameter of W $p_W = 3/4$.

   Since Z and W are both Bernoulli r.v., so we know that:

   $E[Z] = p_Z = 1/4$

   $E[W] = p_W = 3/4$

   Therefore W has the larger expectation.

   (b) By definition, in order to achieve $Z \perp W$, $\forall a \in Z$ and $\forall b \in W$, $P_r[Z = a] * P_r[W = b] =$

$P_r[(Z = a) \cap (W = b)]$
From above we know that:
$P_r[Z = 0] = 3/4$
$P_r[Z = 1] = 1/4$
$P_r[W = 0] = 1/4$
$P_r[W = 1] = 3/4$
$P_r[(Z = 0) \cap (W = 0)] = 1/4$
$P_r[(Z = 0) \cap (W = 1)] = 1/2$
$P_r[(Z = 1) \cap (W = 1)] = 1/4$
$P_r[(Z = 1) \cap (W = 0)] = 0$
Thus we know:
$P_r[Z = 0] * P_r[W = 0] = 3/4 * 1/4 = 3/16 \neq P_r[(Z = 0) \cap (W = 0)]$
Similarly we can show:
$P_r[Z = 0] * P_r[W = 1] = 3/4 * 3/4 = 9/16 \neq P_r[(Z = 0) \cap (W = 1)]$ etc
Therefore we found a counterexample that shows $\exists a \in Z$ and $\exists b \in W$, $P_r[Z = a] * P_r[W = b] \neq P_r[(Z = a) \cap (W = b)]$
Thus we disapproved $Z \perp W$, so Z and W are not independent.

(c) Since W is a Bernoulli r.v., we can calculate:
$Var[W] = p_W(1 - p_W)$
$=3/4 * 1/4 = 3/16$

5. **[15 pts]** You offer Alice and Bob a game of chance. The game works as follows. You give each of them a *biased* coin. Let A be Alice's coin and B be Bob's coin. Let's denote by $\Pr[A = H] = p_A$ and $\Pr[B = H] = p_B$. Each of them flips their coin independently. There is also a bank that you can assume will never run out of money.

- If $A = H$ (A shows heads) and $B = T$ then Alice *takes* \$1 from the bank and Bob *gives* \$1 to the bank. You also *give* \$1 to the bank.
- If $A = T$ and $B = H$ then Alice *gives* \$1 to the bank and Bob *takes* \$1 from the bank. You also *give* \$1 to the bank.
- If $A = H$ and $B = H$ then Alice and Bob do nothing but you *take* \$2 from the bank.
- If $A = T$ and $B = T$ then everybody does nothing.

Assume that Alice and Bob don't care if you make money but they care very much that the game treats the two of them equally, that is, that the two of them have the same expected gains/losses. As long as you respect their desire, you are free to set p_A and p_B as you wish.

(a) **[5 pts]** Prove that for Alice and Bob to have the same expected gains/losses you must bias the two coins in exactly the same way, that is $p_A = p_B$.

(b) **[10 pts]** Assume (just for this part) that you set $p_A = p_B = 3/4$. Calculate *your* expected gains after 100 repetitions of the game. Justify your answer.

Remember to describe the probability space.

Solution.

(a) Probability Space A is for Alice to flip the biased coin. There are two outcomes, heads(H) and tails (T). Since this is a biased coin, this probability space is not uniform, with $P_r[A = H] = p_A$ and $P_r[A = T] = 1 - p_A$ (unless $p_A = 1/2$ then the probability space is uniform).

Probability Space B is for Bob to flip the biased coin. There are two outcomes, heads(H) and tails (T). Since this is a biased coin, this probability space is not uniform, with $P_r[B = H] = p_B$ and $P_r[B = T] = 1 - p_B$(unless $p_B = 1/2$ then the probability space is uniform).

Since Alice and Bob flips their coin independently, by definition we know that:

$P_r[(A = H) \cap (B = H)] = P_r[A = H] * P_r[B = H] = p_A * p_B$

$P_r[(A = H) \cap (B = T)] = P_r[A = H] * P_r[B = T] = p_A * (1 - p_B)$

$P_r[(A = T) \cap (B = H)] = P_r[A = T] * P_r[B = H] = (1 - p_A) * p_B$

$P_r[(A = T) \cap (B = T)] = P_r[A = T] * P_r[B = T] = (1 - p_A) * (1 - p_B)$

By definition of expectation and what's given in the question, we can compute $E[A]$ and $E[B]$ as below:

$E[A] = P_r[(A = H) \cap (B = T)] * 1 + P_r[(A = T) \cap (B = H)] * (-1) + P_r[(A = H) \cap (B = H)] * 0 + P_r[(A = T) \cap (B = T)] * 0$

$= p_A - p_B$

$E[B] = P_r[(B = H) \cap (A = T)] * 1 + P_r[(B = T) \cap (A = H)] * (-1) + P_r[(A = H) \cap (B = H)] * 0 + P_r[(A = T) \cap (B = T)] * 0$

$= p_B - p_A$

In order to let $E[A] = E[B]$, we need:

$p_A - p_B = p_B - p_A$

Which can be express as: $2p_A = 2p_B$

Which is $p_A = p_B$

(b) The probability Space for me has three outcomes, do nothing(denote by 0), give \$1 to bank(denote by -1), and take \$2 out of the bank(denote by 2). Let I be the random variable that returns the amount of money I take (or gives if negative) on every game.

From what's given from the question we know that:

$P_r[I = 0] = P_r[(A = T) \cap (B = T)] = (1 - p_A) * (1 - p_B) = 1/16$

$P_r[I = -1] = P_r[(A = H) \cap (B = T)] + P_r[(A = T) \cap (B = H)] = p_A * (1 - p_B) + (1 - p_A) * p_B = p_A + p_B - 2p_A p_B = 6/16$

$P_r[I = 2] = P_r[(A = H) \cap (B = H)] = P_r[A = H] * P_r[B = H] = p_A * p_B = 9/16$

So the probability space is not uniform.

By definition of expectation, and the given $p_A = p_B = 3/4$:

$E[I] = 0 * P_r[I = 0] + (-1) * P_r[I = -1] + 2 * P_r[I = 2]$

$= 3/4$

So my expectation of every game is 3/4.

By the linearity of expectation:

$E[100I] = 100 * E[I] = 300/4$