MCIT 592 Spring 2022 **Homework 4** **Solutions**

Problem Set

1. **[10 pts]** We can use the binomial theorem to expand $(2x+7)^{12}$ and $(2+7x)^{12}$ as sums of terms (monomials). Indicate all the terms (monomials) that are the same in both sums and explain why you know this, without fully expanding out all of the terms.

Solution:

According to the binomial theorem, each term for these two expressions takes the following form:

$$\binom{12}{k}(2x)^k(7)^{12-k} = \binom{12}{j}(2)^j(7x)^{12-j}$$

Note that the number of x's must equal for the terms to be the same, meaning we can state that the following must be true: $j = 12 - k$. Substituting in this value:

$$\binom{12}{k}(2x)^k(7)^{12-k} = \binom{12}{k}(2)^{12-k}(7x)^k$$

Now we can focus on the coefficients, setting them equal to each other. That is, $\binom{12}{k}(2)^k(7)^{12-k} = \binom{12}{k}(2)^{12-k}(7)^k \rightarrow (2)^k(7)^{12-k} = (2)^{12-k}(7)^k$. Since 2 and 7 are prime numbers, the only way for these two expressions to equal each other is if $\boxed{k=6}$. Expressing the actual term:

$$\boxed{\binom{12}{6}(2x)^6(7)^6} = \boxed{6957291264x^6}$$

2. **[10 pts]** For each of the following functions, prove that the function is neither injective nor surjective. Then, show how you could restrict the domain and codomain — without changing the mapping rule — to make the function both injective and surjective. (Restricting the domain (codomain) means replacing it with a subset that must be clearly defined using our notation, including, if needed, set-builder notation.) Your restricted domain and codomain should be *as large as you can think of* (you get more points for larger domains and codomains).

F or example, the function $f : [-3..3] \to [-9..9]$ given by $f(x) = x^2$ is neither injective (since, e.g., $f(-3) = f(3)$) nor surjective (since, e.g., 2 is not a square), but we can restrict the domain and codomain to define the function $f_1 : [0..3] \to \{0, 1, 4, 9\}$ with the same mapping rule, that is, $f_1(x) = x^2$, which is both injective and surjective. The codomain is as large as possible because adding every other number in $[-9..9]$ that is not a square would break surjectivity and the domain is as large as possible because adding any negative number breaks injectivity.

(a) $g : \{\mathtt{v}, \mathtt{x}, \mathtt{y}, \mathtt{z}\} \to \{\mathtt{a}, \mathtt{e}, \mathtt{i}, \mathtt{o}, \mathtt{u}\}$ given by:

x	$g(x)$
v	i
x	o
y	e
z	i

(b) $h : 2^{[1..n]} \to [0..2n]$ given by $h(S) = |S|$ where $n \geq 2$.

Solution:

(a) This function is not injective because $g(\mathtt{v}) = g(\mathtt{z})$. It is not surjective because there is no element of the domain that maps to a.

The function $\hat{g} : \{\mathtt{v}, \mathtt{x}, \mathtt{y}\} \to \{\mathtt{e}, \mathtt{i}, \mathtt{o}\}$ with the same mapping rule is both injective and surjective; no two elements of $\{\mathtt{v}, \mathtt{x}, \mathtt{y}\}$ map to the same element of $\{\mathtt{e}, \mathtt{i}, \mathtt{o}\}$, and the codomain is equal to the range.

Another valid function is $\hat{g} : \{\mathtt{x}, \mathtt{y}, \mathtt{z}\} \to \{\mathtt{e}, \mathtt{i}, \mathtt{o}\}$ with the same mapping rule, for the same reasons stated above.

(b) This function is not injective since $h(\{1\}) = h(\{2\})$, that is both sets have a size of 1. The function is not surjective since no element in the domain maps to $2n$, as the maximum sized subset is n.

The function $\hat{h} : \{[1..k] \mid \forall k \in [1..n]\} \cup \emptyset \to [0..n]$ with the same mapping rule is both injective and surjective. Note that the elements in the domain are the sets:

$\{\}, \{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, n\}$. The function is injective since no two elements in the domain are the same size and the function is surjective since the domain consists of sets with sizes from 0 to n.

3. [10 pts] Answer the following questions about a scheduling system for assigning TAs to office hours. Within the system, every single TA and every single office hour time is assigned an integer ID. These IDs start with 1 and increment by 1. That is, if there is a TA with ID equal to 6, there must be TAs with IDs equal to 1, 2, 3, 4 and 5. The same restriction applies to the office hour IDs. Note that any office hour not assigned a TA will be covered by Professor Tannen.

(a) Let p, q, r, s be integers with $p \leq q \leq r \leq s$. Consider the TAs with IDs ranging from p to r inclusive, and consider the office hour slots with IDs ranging from q to s, inclusive. How many distinct functions for assigning TAs to office hours are there? (The TAs are the domain and the office hours are the codomain)

(b) Let n be a positive integer. Suppose there are n TAs and $2n$ office hour slots. How many distinct functions for assigning TAs to office hours are there, such that every TA is assigned an office hour with an ID that is either strictly less than their ID or greater than or equal to two times their ID? (ex. if $n = 10$, TA 4 can be assigned office hour x, where $x < 4$ or $x \geq 8$)

Solution:

(a) Each integer in $[p..r]$ can be mapped to any integer in $[q..s]$, which can be done in $s - q + 1$ ways. Since there are $r - p + 1$ integers in $[p..r]$, the multiplication rule tells us that there are $(s - q + 1)^{r-p+1}$ such functions.

(b) For each element in the domain, calculate the number of elements in the codomain to which it can map. For any element x, the range of numbers it **can't** map to is x to $2x - 1$, since it has to be either less than x or greater than or equal to $2x$. The number of integers in this range is: $(2x - 1) - x + 1 = x$. Thus, the number of elements that an element x in the domain can map to is: $2n - x$.

For each element in the domain, independently map it to an element in the codomain. By the multiplication rule, there are: $\boxed{(2n-1)(2n-2)(2n-3)...(2n-n)}$ possible functions. This expression can also be stated as:

$$\boxed{\frac{(2n-1)!}{(n-1)!}}$$

4. [10 pts] Recall that a *combinatorial proof* for an identity proceeds as follows:

1. State a counting question.
2. Answer the question in two ways:
   (i) one answer must correspond to the left-hand side (LHS) of the identity
   (ii) the other answer must correspond to the right-hand side (RHS).
3. Conclude that the LHS is equal to the RHS.

With that in mind, give a combinatorial proof of the identity

$$\binom{2n}{n}\binom{n}{2} = \binom{2n}{2}\binom{2n-2}{n-2}$$

where $n \geq 2$.

Solution:

We pose the following counting question:

> How many ways can we form a team of n people and appoint 2 of these people co-captains from a group of $2n$ distinct people, where $n \geq 2$?

(LHS): One procedure is to first count the number of people who will be on the team. More specifically, choose n out of the $2n$ people to be on the team, which is $\binom{2n}{n}$. Now from these n people, choose two of them to be co-captains. This can be done in $\binom{n}{2}$ ways. Finally, use the multiplication rule on these two independent steps to conclude that there are $\binom{2n}{n}\binom{n}{2}$ ways to count these teams, exactly the LHS.

(RHS): Another procedure is to first choose the two co-captains then form the rest of the team. First, choose 2 of the $2n$ people to be captains. This can be done in $\binom{2n}{2}$ ways. Then from the remaining $2n-2$ people, pick $n-2$ more people to be part of the team. This can be done in $\binom{2n-2}{n-2}$ ways. Finally use the multiplication rule on these two independent steps to conclude that there are $\binom{2n}{2}\binom{2n-2}{n-2}$, exactly the RHS.

Both sides of the expression answer the same question, so they must be equal.

5. **[10 pts]** For each of the following, prove that $|A| \leq |B|$ by defining an injective function $f : A \to B$ and then using the injection rule.

(a) A is any set and $B = 2^A$.

(b) A is the set of all prime numbers and B is the real interval $[0, 1]$. (Even though these sets have infinite cardinalities, the injection rule still applies!)

Solution:

(a) First we consider the case $A \neq \emptyset$. Define a function $f : A \to 2^A$ by $f(x) = \{x\}$, i.e., f maps single elements of A to singleton subsets of A. Since $x \neq y$ implies $\{x\} \neq \{y\}$, this function is an injection. Thus, by the injection rule, $|A| \leq |2^A|$.

Next we consider the case $A = \emptyset$. Note that $2^\emptyset$ has exactly one element, namely $\emptyset$. So $|\emptyset| = 0 \leq 1 = |2^\emptyset|$. $\square$

(b) Let P be the set of all prime numbers. Define a function $f : P \to [0, 1]$ by $f(p) = 1/p$. Clearly if $p \neq q$ then $f(p) \neq f(q)$, so the function is injective. Thus, by the injection rule, $|P| \leq |[0, 1]|$. $\square$