Module 6.5: Fibonacci Numbers

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

The Fibonacci numbers

Problem. A farmer raises rabbits. Each rabbit pair gives birth to another rabbit pair when it turns one month old, and thereafter to one rabbit pair each month. Rabbits never die. How many rabbit pairs will the farmer have at the end of the nth month if he starts with one newborn rabbit pair in the first month?

Answer. Denoting the number of rabbits in month n by F_n, we have the following **recurrence relation**:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \qquad \text{for } n \geq 2 \end{aligned}$$

Hence the sequence of **Fibonacci numbers**:

. $0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad \cdots$

Solving the Fibonacci recurrence I

Proposition. $$F_n = (\varphi^n - \psi^n)/\sqrt{5}$$
where $\varphi > \psi$ are the two roots of the equation $x^2 - x - 1 = 0$.

$\varphi = (1+\sqrt{5})/2 \simeq 1.618$ is known as the **Golden Ratio** (see optional segment "Daisies, sunflowers, and shells") and $\psi = 1 - \varphi = -1/\varphi = (1-\sqrt{5})/2$.

Proof. Can $F_n = F_{n-1} + F_{n-2}$ have a solution $F_n = r^n$ where $r \neq 0$?

Note that $r^n = r^{n-1} + r^{n-2} \Rightarrow r^2 = r + 1$ hence $r^2 - r - 1 = 0$.

Therefore, both $F_n = \varphi^n$ and $F_n = \psi^n$ are solutions of $F_n = F_{n-1} + F_{n-2}$.
But what about $F_0 = 0$ and $F_1 = 1$?

Luckily (because $F_n = F_{n-1} + F_{n-2}$ is **homogeneous**), any **linear combination** of its solutions is also a solution.

ACTIVITY : Homogeneous recurrence, linear combination

By calling the Fibonacci recurrence **homogeneous** we point out that all its terms have the same form: $F_{n \pm i}$.

In contrast, we saw in the lecture segment "Pizza cutting recurrence" a non-homogeneous recurrence: $C_n = C_{n-1} + n$.

A **linear combination** of functions $f(n)$ and $g(n)$ is a function of the form $h(n) = \alpha f(n) + \beta g(n)$ where α and β are two real constants.

ACTIVITY : Homogeneous recurrence, linear combination (Continued)

Consider the homogeneous recurrence $A_{n+1} = 2A_{n-1} + 3A_{n-3}$. Suppose $f(n)$ and $g(n)$ are two solutions, let $\alpha, \beta \in \mathbb{R}$, and let $h(n) = \alpha f(n) + \beta g(n)$. Here is how we show that $h(n)$ is also a solution of the recurrence:

$$
\begin{aligned}
f(n+1) &= 2f(n-1) + 3f(n-3) && \\
\alpha f(n+1) &= 2\alpha f(n-1) + 3\alpha f(n-3) && \text{multiply by } \alpha \\
g(n+1) &= 2g(n-1) + 3g(n-3) && \\
\beta g(n+1) &= 2\beta g(n-1) + 3\beta g(n-3) && \text{multiply by } \beta \\
h(n+1) &= 2h(n-1) + 3h(n-3) && \text{add sides}
\end{aligned}
$$

Solving the Fibonacci recurrence II

Proof (continued). We have established that for any $\lambda_1, \lambda_2 \in \mathbb{R}$ the sequence defined by $F_n = \lambda_1 \varphi^n + \lambda_2 \psi^n$ satisfies $F_n = F_{n-1} + F_{n-2}$.

Now we determine λ_1, λ_2 to make sure that $F_0 = 0$ and $F_1 = 1$.

$$\begin{aligned} \lambda_1 + \lambda_2 &= 0 \\ \lambda_1 \varphi + \lambda_2 \psi &= 1 \end{aligned}$$

Solving, we get $\lambda_2 = -\lambda_1$

.$\lambda_1 = 1/(\varphi - \psi) = 1/\sqrt{5}$

.$\lambda_2 = 1/(\psi - \varphi) = -1/\sqrt{5}$

We conclude that $F_n = (\varphi^n - \psi^n)/\sqrt{5}.$

Done.

Strong induction (Fibonacci variant) I

Problem. A car needs 1 unit of length to park while a truck needs 2 units of length. Assume that cars are indistinguishable and so are trucks. How many distinct car/truck parking patterns are possible along an n unit long sidewalk?

Answer. We write the parking patterns as a string of `C`'s and `T`'s. Here are two distinct ways in which 3 cars and 2 trucks can be parked along a sidewalk that is 7 units long: `CTCCT` and `TCTCC`.

length	patterns	#
1	C	1
2	CC T	2
3	CT CCC TC	3
4	CCT TT CTC CCCC TCC	5
5	CTT CCCT TCT CCTC TTC CTCC CCCCC TCCC	8

Strong induction (Fibonacci variant) II

Answer (continued). We prove by induction that the number of distinct parking patterns along a sidewalk of length $n \geq 1$ is F_{n+1}. It's a special strong induction with an IH only for k and $k-1$, and hence two base cases.

(BC 1) $(n = 1)$ Only 1 pattern, `C`. $F_2 = 1$. Check.

(BC 2) $(n = 2)$ 2 patterns, `CC` and `T`. $F_3 = 2$. Check.

(IS) Let $k \geq 1$. Assume (IH) that the number of patterns for length $k-1$ is F_k and that the number of patterns for length k is F_{k+1}.

Now consider a pattern p for length $k+1$. Depending on whether this pattern ends with a car or a truck, we have two cases.

Strong induction (Fibonacci variant) III

Answer (continued).

Case 1. The last vehicle in p is a car, that is, $p = r\mathtt{C}$.

Then r has length $k + 1 - 1 = k$.

By IH there are F_{k+1} distinct r's.

Therefore in this case we have F_{k+1} distinct patterns.

Case 2. The last vehicle in p is a truck, that is, $p = s\mathtt{T}$.

Then s has length $k + 1 - 2 = k - 1$.

By IH there are F_k distinct s's therefore F_k distinct p's in this case.

By the addition rule, there are $F_{k+1} + F_k = F_{k+2}$ distinct patterns. Done.

ACTIVITY : Proof by induction

In this activity we will give a proof by ordinary induction of **Cassini's identity**:

$$F_{n-1} \cdot F_{n+1} \; - \; F_n^2 \; = \; (-1)^n \qquad\qquad (n \geq 1)$$

First we want to determine the relevant base case for this proof. Since we want to prove this identity for all $n \geq 1$, then we need to set the base case as $n = 1$.

(BC) $F_0 \cdot F_2 \; - \; F_1^2 \; = \; 0 \cdot 1 - 1^2 = -1 = (-1)^1$, so the identity holds for $n = 1$.

(IS) Let $k \geq 1$ be an arbitrary positive integer. Assume the inductive hypothesis...

Question: What should the inductive hypothesis be?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

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ACTIVITY : Proof by induction (Continued)

Answer: $F_{k-1} \cdot F_{k+1} \ - \ F_k^2 \ = \ (-1)^k$.

Now consider $k+1$. Assuming the inductive hypothesis (IH), we want to show that $F_k \cdot F_{k+2} \ - \ F_{k+1}^2 \ = \ (-1)^{k+1}$, which we derive as follows.

Activity : Proof by induction (Continued)

$$
\begin{aligned}
F_k \cdot F_{k+2} - F_{k+1}^2 &= F_k \cdot (F_{k+1} + F_k) - F_{k+1}^2 \\
&= F_k^2 + F_k \cdot F_{k+1} - F_{k+1}^2 \\
&= F_k^2 + F_{k+1}(F_k - F_{k+1}) \\
&= F_k^2 - F_{k+1}(F_{k+1} - F_k) \\
&= F_k^2 - F_{k+1} \cdot F_{k-1} \\
&= (-1) \cdot (F_{k+1} \cdot F_{k-1} - F_k^2) \\
\text{Here is where we use the IH} \rightarrow \quad &= (-1) \cdot (-1)^k \\
&= (-1)^{k+1}.
\end{aligned}
$$