Recitation Module 8

Penn Engineering

Probability properties

Property P0: $\Pr[E] \geq 0$

Property P1: $\Pr[\Omega] = 1$

Property P2: If A, B are disjoint then $\Pr[A \cup B] = \Pr[A] + \Pr[B]$

Property P3: If $A \subseteq B$ then $\Pr[A] \leq \Pr[B]$

Property P4: $\Pr[\overline{E}] = 1 - \Pr[E]$

Property P5: $\Pr[\varnothing] = 0$

Property P6: $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$

Property P7: $\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$

Properties of event independence

Property Ind (i). If $\Pr[A] = 0$ then $A \perp B$ for any B.

Property Ind (ii): $\Omega \perp E$ for any E.

Property Ind (iii). If $A \perp B$ then $\Pr[A \cup B] = 1 - (1 - \Pr[A])(1 - \Pr[B])$.

Property Ind (iv). $A \perp B$ iff $\overline{A} \perp B$ iff $A \perp \overline{B}$ iff $\overline{A} \perp \overline{B}$

Pairwise and Mutual independence

Events $A1, \ldots, An$ are called **pairwise independent** when any 2 of these events are independent of each other
i.e. $Ai \perp Aj$ for all $1 < i < j < n$

Events $A1, \ldots, An$ are called **mutually independent** when for any subset of these events we have:
$\Pr[A1 \cap \cdots \cap Ak] = \Pr[A1] \cdots \Pr[Ak]$

Mutual independence $\Rightarrow$ *pairwise independence*
BUT pairwise independence does not imply mutual independence

Conditional Probabilities and Chain Rule

Conditional Probability:

$$\Pr[E|U] = \frac{\sum_{w \in E \wedge w \in U} \Pr[w]}{\sum_{w \in U} \Pr[w]} = \frac{\Pr[E \cap U]}{\Pr[U]} \qquad (\text{provided} \quad \Pr[U] \neq 0)$$

In plain English, Pr[*E*|*U]* is the probability of event *E* happening, given that event *U* has already happened.

Chain Rule - (illustrate with tree diagrams)

For any events $A_1, \ldots, A_n$ in the same probability space we have

$$\Pr[A_1 \cap A_2 \cap A_3 \cdots \cap A_n] =$$
$$= \Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2] \cdots \Pr[A_n|A_1 \cap \cdots \cap A_{n-1}]$$

Conditional probabilities rules

Bayes Rule:

$$\Pr[A \mid B] = \frac{\Pr[A]\ \Pr[B \mid A]}{\Pr[B]}$$

Rule of Total Probability:

Proposition (The Rule of Total Probability). Let $A_1, \ldots, A_n$ be **two or more** events, each of **of non-zero probability** in the same probability space such that:

- $A_1, \ldots, A_n$ are **pairwise disjoint**, and
- $A_1 \cup \cdots \cup A_n = \Omega$

(we say that $A_1, \ldots, A_n$ form a **partition** of Ω). Then, for any event E

$$\Pr[E] = \sum_{i=1}^{n} \Pr[E \mid A_i]\ \Pr[A_i]$$

"Partition" -> Collection of sets that is pairwise disjoint and their union = whole sample space

Question 1

We draw two cards from a deck of shuffled cards without replacement. Find the probability of getting the second card a queen using the total probability theorem.

Answer to Question 1

Let A be the event that the second card is a queen, and let B be the event that the first card is not a queen. We seek to compute the probability (given by the law of total probability)

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

We compute the cardinality of the probability space Ω using the multiplication rule:

$$|\Omega| = 52 * 51$$

In how many ways can we satisfy B: 52 - 4 = 48 ways to pick the first card s.t it is not a Queen, and there are 51 ways to pick the second card with no restriction.

Hence by the multiplication rule P(B) = 48 * 51 / (52*51) = 48/52

Answer to Question 1 (Continued III)

Thus $P(B^c) = 1 - P(B) = 1 - 48/52 = 4/52$

Now we compute $P(A|B)$; we know that A and B are not independent since A depends on B.

In how many ways can we pick a queen after we have not picked a queen? There are 4 queens left, and 51 cards in the deck. Hence, $P(A|B) = 4/51$

Now we compute $P(A|B^c)$;

In how many ways can we pick a queen after we have already picked a queen? There are 3 queens left, and 51 cards in the deck. Hence, $P(A|B^c) = 3/51$

Thus, $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = (48/52)*(4/51) + (4/52)*(3/51) = 204/(51*52) =$ **1/13**

Question 2

If three indistinguishable dice are thrown what is the probability that one shows a 6 given that no two show the same face?

Answer to Question 2

Let A be the event that one of the three die shows 6, and B the event that no two die have the same value.

We seek the probability $P(A|B) = P(A \cap B) / P(B)$ (by Bayes Rule)

We compute the cardinality of the probability space $|\Omega| = 6 * 6 * 6 = 216$

In how many ways can we throw the dice that it satisfies B? $6 * 5 * 4 = 120$.
Hence, $P(B) = 120/216 = 60/108 = 30/54 = 15/27 =$ **5 / 9**

Now we seek to compute $P(A \cap B)$. In how many ways can we throw the dice s.t. it satisfies $A \cap B$? $(3C1)*5 * 4 = 60$. Hence, $P(A \cap B) = 60 / 216 =$ **5/18**

Thus $P(A|B) = (5/18) / (5 / 9) = 9 / 18 =$ **1 / 2**

Question 3

A drawer contains 10 distinct pairs of gloves. Eight gloves are selected at random. What is the probability that there is at least one pair among the eight selected?

Answer to Question 3

We compute the cardinality of the probability space: $|\Omega| = (2*10)C8 = 20C8$

We count *complementarily*. Let A be the event that there is at least one pair of gloves in the 4 chosen.

Hence, A^c is the event that there is **no** pair of gloves in the 8 gloves chosen. In how many ways can we pick the gloves s.t. it satisfies A^c? There are 10 pairs of gloves. Only one glove from each pair will be chosen. We choose which pairs will contribute one glove by 10C8. Then for each of these pairs there are two options hence 2^8 ways. Hence $|A^c| = 10C8 * 2^8$

Hence $P(A^c) = |A^c| / |\Omega| = (10C8 * 2^8) / 20C8$

Thus, $P(A) =$ **$1 - (10C8 * 2^8) / 20C8 \sim 0.908$**

Before you go...

Do you have any general questions about this week's assignment? We can't go into minute details about any specific questions, but this is the time to ask any general clarifying/confusing questions!

As always, if you can't talk to us during recitation or office hours, post any questions/anything course related on the forums or email mcitonline@seas.upenn.edu. Ask questions that might be beneficial to other students on the forums, while emailing about more personal questions (regrade requests, etc).