Recitation Module 1

Penn Engineering

Basic terms/definitions to know!!

- The addition rule (and disjoint)
- The multiplication rule (independence)
- Restrictions and orderings when using multiplication rule
- Odd and evens
- Prime numbers
- Subset vs. proper subset
- Set Builder: $A = \{x \mid P(x)\}$
- Intersection vs. Union
- Powerset

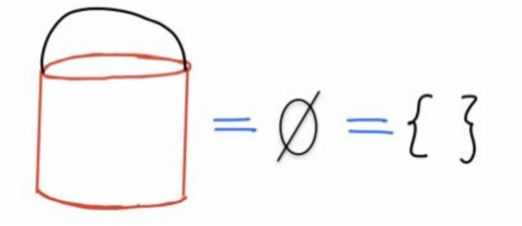
Note that this is a dense module with many important concepts. This is by no means an exhaustive summary.

Tips on proofs

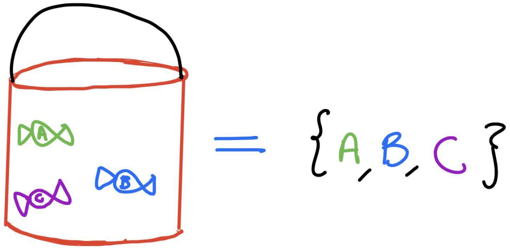
- Make sure you are understanding what you are trying to prove
- Know definitions
- Try some inputs to verify the proof (make sure it makes sense to you, and try to find a counterexample if it is a prove/disprove question)
- Try to identify the proof pattern needed
  - direct proof: start from what is known and apply operations that lead you to what you want to prove! Make sure you consider all possible cases.
- Reach out in OH, Piazza, and Recitations!

Clarifications on Sets

- Think of sets as … buckets
- An empty set is like an empty bucket - its cardinality is 0 b/c there are 0 elements in it.

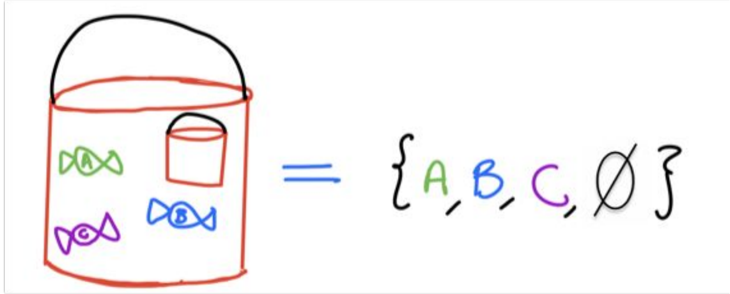


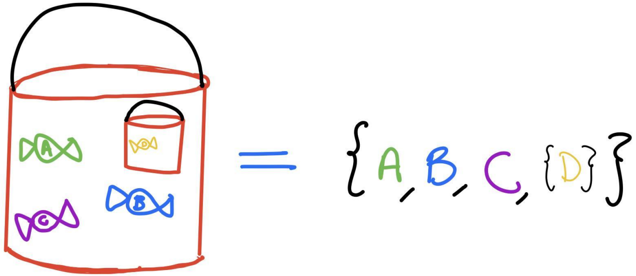
- A non-empty set is like a non-empty bucket - and its cardinality is equal to the number of elements in it.



Clarifications on Sets (continued)

- Sets as elements in sets





Question 1

Prove the following: If m and n have different parities, then mn is even.

Answer to Question 1

WLOG (without loss of generality), assume *m* is odd and *n* is even.

$$
\begin{aligned}
m &= 2k+1,\ \ k \in \mathbb{Z} \\
n &= 2p,\ p \in \mathbb{Z} \\
mn &= (2k+1) \times (2p) \\
&= 4kp + 2p \\
&= 2(2kp + p)
\end{aligned}
$$

2kp + *p* is an integer, therefore *mn* is even by the definition of even. The proof is exactly the same as if *m* were even and *n* were odd, so we omit it.

Question 2

We roll two distinguishable dice: one green die, and one purple die.

Let set $A = \{\ (i, j) \mid \text{green die shows i and purple die shows j and i + j is odd}\}$

1. How many elements does *A* have? How many subsets?

Now, let *B* be the set of numbers *i* shown by the green die such that the tuple (*i*, *j*) is in *A*.

2. Define *B* using set notation.

3. How many elements does *B* have?

$A = \{ (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 2), (3, 4), (3, 6)$

$(4, 1), (4, 3), (4, 5), (5, 2), (5, 4), (5, 6), (6, 1), (6, 3), (6, 5) \}$

Answer to Question 2

1. A consists of the following 18 elements:

$$A = \{ (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 2), (3, 4), (3, 6)$$

$$(4, 1), (4, 3), (4, 5), (5, 2), (5, 4), (5, 6), (6, 1), (6, 3), (6, 5) \}$$

Another way to count the number of elements in A is to realize that it consists of twice the number of combinations of numbers between 1 and 6 whose sum is odd (since the dice are distinguishable).

Recall that the sum of two numbers is odd if one number is odd and the other is even. There are 3 odd and 3 even numbers between 1 and 6. Thus, there are 3 x 3 = 9 pairs that have an odd sum. Thus, A consists of 9 x 2 = 18 elements.

It follows that 2^A consists of $2^{|A|} = 2^{18} = 262{,}144$ elements

Answer to Question 2 (continued)

2. Using set builder notation we define *B* to be:

$$B = \{\ i \mid \text{there exists j such that (i, j)} \in A\}$$

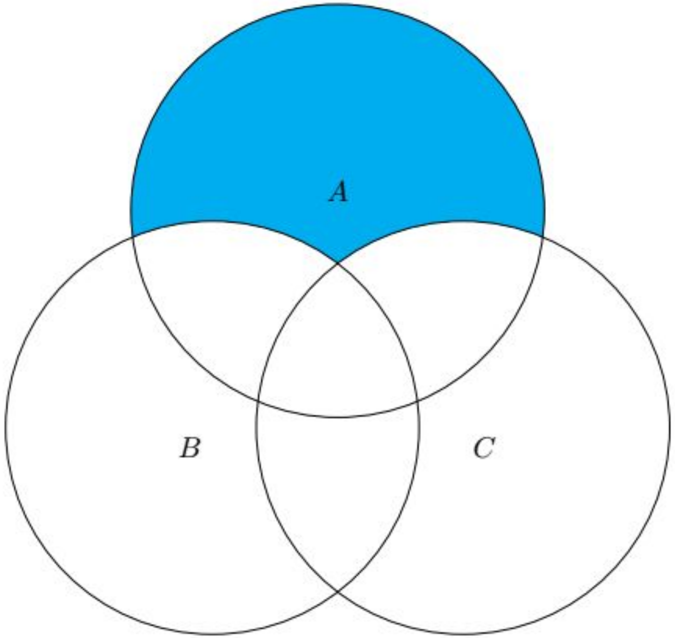
3. Recall that there are no duplicate elements in a set. Therefore, |*B*| = 6 since for every number *i* shown by the green die we can find a number *j* shown by the purple die such that *i* + *j* is odd.

Question 3

Use an Euler Venn diagram to describe which portion of the diagram corresponds to:

$$(A \setminus B) \cap (A \setminus C)$$

Answer to Question 3



Question 4

Prove that, for any integer x, $x^2 + x$ is even.

Answer to Question 4

Using algebra, we can see that $x^2 + x = x(x + 1)$. Now x is an integer, which means it must be either even or odd.

Consider the case in which x is even. In this case, by definition, $x = 2k$ for some integer k. Therefore $x(x + 1) = 2k(2k + 1) = 2(2k^2 + k)$. Since k is an integer, $2k^2 + k$ is an integer. Thus $x^2 + x = 2n$ for some integer n, namely $n = 2k^2 + k$, which means $x^2 + x$ is even.

Now consider the case in which x is odd. In this case, by definition, $x = 2k + 1$ for some integer k. Therefore $x(x + 1) = (2k + 1)(2k + 2) = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$. Since k is an integer, $2k^2 + 3k + 1$ is an integer. Thus $x^2 + x = 2n$ for some integer n, namely $n = 2k^2 + 3k + 1$, which means $x^2 + x$ is even.

We have shown that, in both possible cases, $x^2 + x$ is even. Thus we have proven that $x^2 + x$ is even.