Module 1.1: Addition and Multiplication Rules

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Combinatorics = counting techniques

Counting techniques belong to a branch of mathematics called **Combinatorics**.

Combinatorics problems range from the super-easy to the super-complicated. Naturally, we start at the easy end :).

Problem. Grandmother's market stall has 23 apples, 18 oranges, and 31 peaches for sale. How many fruits does Granny have for sale?

Answer: Granny has for sale $23 + 18 + 31 = 72$ fruits.

We just applied here the first rule of combinatorics: how to count objects of different kinds!

The addition rule

Suppose that the objects we count can be classified into k separate kinds, $1, 2, \ldots, k$. Suppose we count

(1) n_1 objects of the first kind,

(2) n_2 objects of the second kind,

$\ldots$

(k) n_k objects of the k'th kind,

then the total number of objects counted is $n_1 + n_2 + \cdots + n_k$.

Problem. How many integers have an absolute value that is at most 10?

Answer: There are 10 integers between 1 and 10, inclusive, another 10 between -10 and -1, inclusive, and then there is 0. In total $10 + 10 + 1 = 21$.

How many types of sandwich?

Problem. A local store that serves sandwiches offers a choice of 4 kinds of bread and, for filling, it offers 2 kinds of meat or 3 kinds of cheese. How many different types of sandwich are available?

Answer: First we count, using the addition rule, the number of different kinds of filling: $2 + 3 = 5$.

Next each of the 4 kinds of bread can be combined with each of the 5 kinds of filling. This gives us $4 \cdot 5 = 20$ types of sandwich.

We have applied a different counting rule here. A sandwich is "constructed" in two *steps*:

(1) Choose one of 4 breads.

(2) Choose one of 5 fillings.

The multiplication rule

Suppose that a procedure that constructs objects of some kind can be broken down into k **steps** and

(1) the first step can be performed in n_1 ways,

(2) the second step can be performed in n_2 ways, **regardless** of how the first step was performed,

. . .

(k) and the k^{th} step can be performed in n_k ways, regardless of how **all the preceding** steps were performed,

then the entire procedure can be performed in $n_1 \cdot n_2 \cdots n_k$ different ways.

If every different way of performing the procedure constructs a different object then $n_1 \cdot n_2 \cdots n_k$ distinct objects are constructed.

How many seats?

Problem. The seats of a concert hall are to be labeled with an uppercase letter of the English alphabet and a 2-digit number between 1 and 100. What is the largest count of seats that can be labeled differently?

Answer: The English alphabet has 26 upper-case letters.

The 2-digit numbers between 1 and 100 are $10, 11, \ldots, 98, 99$. One way to count them is to subtract from 100 the count of numbers of 1 digit $(1, \ldots, 9)$ or 3 digits (100): $100 - 9 - 1 = 90$.

A seat can be labeled using the following two steps:

(1) Choose an upper-case letter. Can be done in 26 ways.

(2) Choose a number among $10, 11, \ldots, 98, 99$. Can be done in 90 ways.

By the multiplication rule a seat can be labeled in $26 \cdot 90 = 2340$ ways.

Counting club officer assignments I

Problem. Three club officers - a president, a treasurer, and a secretary - are to be chosen from among four people: Ann, Bob, Cindi, and Dan. Suppose that for various reasons, Bob cannot be the president and either Cindi or Dan must be the secretary. In how many ways can the officers be chosen?

Answer: [FIRST ATTEMPT] The three club officers could be chosen as follows:

(1) Choose the president. Can be done in 3 ways (not Bob).

(2) Choose the treasurer. In 3 ways (not the person chosen in step (1)).

(3) Choose the secretary. In 2 ways (Cindi or Dan).

Apparently (by the multiplication rule): $3 \cdot 3 \cdot 2 = 18$ officer groups.

Incorrect! Because step (3) depends on **how** steps (1) and (2) were done. What if Cindi was chosen president and Dan treasurer?!

Counting club officer assignments II

Answer: [SECOND ATTEMPT] The three club officers could be chosen as follows:

(1) Choose the secretary. Can be done in 2 ways (Cindi or Dan).

(2) Choose the president. In 2 ways (neither Bob nor the person chosen in step (1)).

(3) Choose the treasurer. In 2 ways (either of the two persons left).

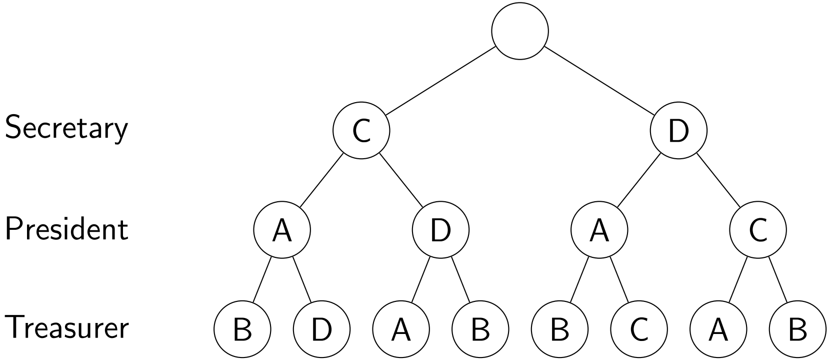
By the multiplication rule: $2 \cdot 2 \cdot 2 = 8$ officer groups.

When using the multiplication rule there may be different orders in which the steps can be performed.

Also keep in mind: it's better to make the **most restrictive** choices first and the least restrictive last.

Activity : Tree of possibilities

Possible outcomes of a sequence of choices can be drawn in a **tree**. This tree represents picking the secretary, then the president, then the treasurer:



(Remember: Bob can't be president, and Cindi or Dan must be secretary.)

Moving from top to bottom, the leftmost path corresponds to picking Cindi (C) as secretary, Ann (A) as president, and Bob (B) as treasurer.

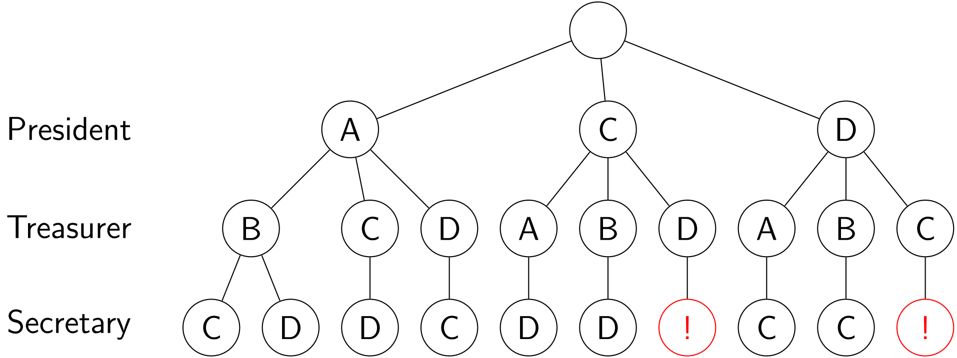
Activity : Tree of possibilities (Continued)

The tree of possibilities you just saw corresponds to the [Second Attempt] described in the lecture.

Problem. Using a pencil and paper, draw the tree of possibilities that represents picking the president, then the treasurer, then the secretary, as we did in the lecture in the [First Attempt].

Activity : Tree of possibilities (Continued)

Answer.



Notice that in the paths marked with a !, there is no valid option for secretary; in both cases, Cindi and Dan already have other roles.

ACTIVITY : Tree of possibilities (Continued)

Discussion. In drawing the tree, you must have noticed that the number of options available in each step depends on the choices made in previous steps, which is why the multiplication rule does not apply here.

If we choose Ann as president and Bob as treasurer, then there are two options for secretary. But if we choose Ann as president and Cindi as treasurer, then there is only one option for secretary. And if we choose Cindi as president and Dan as treasurer, then we have reached a "dead end," and there is no valid option for secretary in this case.

Module 1.2: Odd and Even

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Lecture Notes

Even and odd I

Let's ignore the negative integers.

Here are the **even** numbers: $0, 2, 4, 6, 8, 10, \ldots$

And here are the **odd** ones: $1, 3, 5, 7, 9, 11, \ldots$

Problem. Prove that if the sum of two integers is even then so is their difference.

Since we know even and odd numbers "by example" we could verify this on some values.

$6 + 4 = 10 \qquad 6 - 4 = 2 \qquad\qquad 7 + 3 = 10 \qquad 7 - 3 = 4$

But this is **not** a proof!

We need a mathematical proof for the **general** assertion in the problem.

Even and odd II

We realize that we do not even have mathematical **definitions** for "even" or for "odd"!

Here are the definitions:

An integer n is **even** when $n = 2k$ for *some* integer k.

An integer n is **odd** when $n = 2k + 1$ for *some* integer k.

Examples.

6 is even because $6 = 2 \cdot 3$ and we can take $k = 3$.

9 is odd because $9 = 2 \cdot 4 + 1$ and we can take $k = 4$.

Even and odd III

Problem. Prove that if the sum of two integers is even then so is their difference.

Answer. Reformulate the statement as follows:

> Let m and n be any two integers.
> If $m + n$ is even then $m - n$ is even.

By definition of "even", we have $m + n = 2\ell$, for some integer ℓ.

Then $m = 2\ell - n$.

Now $m - n$ can be written as follows:

. $m - n \;=\; (2\ell - n) - n \;=\; 2\ell - 2n \;=\; 2(\ell - n)$

Since $\ell - n$ is an integer, we take $k = \ell - n$ to satisfy the def. of "even".

We conclude that $m - n$ is even.

Lessons from this “even and odd” story

1. We cannot hope to **prove** a mathematical assertion rigorously unless all the concepts in the statement have an unambiguous mathematical definition.

2. “Verifying” an assertion on a few **examples** is **not** a proof. In fact as we shall see very soon, it can be misleading.

3. What about “rules of thumb” such as “even plus even is even” and “odd minus even is odd”, etc. But how do we know that these are true? Prove them! However, we cannot prove everything! In assignments, we will make clear what you can assume and what you still need to prove.

Activity: Proof using rules of thumb

Assume the following rules of thumb:

P1: Even plus even is even

P2: Even plus odd is odd

P3: Odd plus even is odd

P4: Odd plus odd is even

M1: Even minus even is even

M2: Even minus odd is odd

M3: Odd minus even is odd

M4: Odd minus odd is even

Which of the rules above would you use to prove for any even integers m, n that if $m + n$ is even then $m - n$ is even?

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

Activity: Proof using rules of thumb

Answer.

If m is even and n is even we will use P1 and M1 to prove that if $m + n$ is even then $m - n$ is even.

$m + n$ is even (by P1)
$m - n$ is even (by M1)

ACTIVITY: Proof using rules of thumb

Below we show the full proof of the statement "for any two integers m, n if $m+n$ is even then $m-n$ is even. Consider all four possible cases:

Case 1: m is even and n is even
$m+n$ is even (by P1)
$m-n$ is even (by M1)

Case 2: m is even and n is odd
$m+n$ is odd (by P2)
$m-n$ is odd (by M2)

Case 3: m is odd and n is even
$m+n$ is even (by P3)
$m-n$ is even (by M3)

Case 4: m is odd and n is odd
$m+n$ is odd (by P4)
$m-n$ is odd (by M4)

In all cases we showed that if $m+n$ is even then $m-n$ is even. Thus the statement must be true, since we covered all possible scenarios.

Module 1.3: Divisibility and Primes

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Lecture Notes

Divisibility

The positive integer d is a **divisor** (or **factor**) of an integer n when $n = d \cdot k$ for some integer k.

If d is a divisor of n we say that d **divides** n and we write $d \mid n$.

We also say that n is **divisible** by d or n is a **multiple** of d.

Therefore, "even" is the same as "has 2 as a factor", or as "is divisible by 2".

Examples: 37 is a divisor of 111 because $111 = 37 \cdot 3$.
The positive factors of 24 are 1,2,3,4,6,8,12,24.

We will be interested only in factors/divisors that are *positive*.

Quiz I

What is the only integer divisible by 0?

(A) 1

(B) 0

(C) All integers are divisible by 0.

Answer

What is the only integer divisible by 0?

(A) 1

Incorrect. Since $0 \cdot k \neq 1$ for all integers k, 1 is not divisible by 0.

(B) 0

Correct. Since $0 = 0 \cdot k$ for whatever integer k we want. No other integer is divisible by 0.

(C) All integers are divisible by 0.

Incorrect. For any integer $\ell \neq 0$ we have $0 \cdot k \neq \ell$ for all integers k. Thus, $\ell \neq 0$ is not divisible by 0.

More Information

Note, however, that 0 is divisible by *any* integer! Indeed $0 = d \cdot 0$ for all d.

Quiz II

Suppose that an integer n is divisible by -2. Then

(A) n is even.

(B) n is odd.

(C) n could be either odd or even.

Answer

Suppose that an integer n is divisible by -2. Then

(A) n is even.
Correct. Since n is divisible by -2 we have $n = 2 \cdot (-k)$ for some integer k.

(B) n is odd.
Incorrect. Since n is divisible by -2 it should also be divisible by 2.

(C) n could be either odd or even.
Incorrect. n cannot be odd (see B).

More Information

Since n is divisible by -2 we have $n = (-2) \cdot k$ for some integer k. Then

$$n = (-2) \cdot k = (-1) \cdot 2 \cdot k = 2 \cdot (-1) \cdot k = 2 \cdot (-k)$$

so $n = 2 \cdot \ell$ for some integer ℓ (specifically $\ell = -k$).

Primes

An integer p is **prime** when p has exactly two (positive) factors: 1 and itself, and moreover $p \geq 2$.

Thus, no negative integers, neither 0 nor 1 are primes.

Here are the first few primes (up to 70):

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67 ...

Primes which are 2 apart are called **twin primes**:

$$(3, 5),\ (5, 7),\ (11, 13),\ (17, 19),\ (29, 31),\ (41, 43),\ (59, 61), \ldots$$

Quiz III

How many primes are there between 80 and 100? (You can look this up online.)

(A) 3

(B) 4

(C) 5

Answer

How many primes are there between 80 and 100? (You can look this up online.)

(A) 3

Correct. The only primes between 80 and 100 are 83, 89, and 97.

(B) 4

Incorrect.

(C) 5

Incorrect.

More Information

Only three prime numbers between 80 and 100! Isn't that interesting?

Of course, looking online for a list of all the prime numbers up to at least 100 was not the pedagogical purpose of this quiz. We wanted you to become curious about how to find these primes!

We have an optional segment about the Sieve of Eratosthenes, which is an algorithm for finding all prime numbers up to any given limit. The optional segment features animations that explain this ancient algorithm.

Fermat's "primes"

Consider the sequence of numbers:

$$f_n \;=\; 2^{2^n} + 1 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

$f_0 = 3, f_1 = 5, f_2 = 17$ are primes. Check that that $f_3 = 257$ is also a prime. Fermat checked that $f_4 = 65537$ is a prime too!

He seems to have conjectured that *all the f_n's are prime*. However...

Euler showed that $f_5 = 4294967297 = 641 \cdot 6700417$, hence *not a prime*.

So much for "proof by example"!

To date, no f_n with $n > 4$ was shown to be prime and many larger and larger f_n's were shown *not* prime!

Module 1.4: Two proofs and primes

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Lecture Notes

A proof about primes

Proposition. If p, r, s are positive integers such that $p = r \cdot s$ and p is prime then one of r and s is 1 and the other one equals p.

Proof. Assume that $p = r \cdot s$ and p is prime.

Then r is a factor of p $\quad (r \mid p)$.

Since p is prime, $r = 1$ or $r = p$.

In the first case $r = 1$ therefore $p = 1 \cdot s$ and thus $s = p$.

In the second case $r = p$ therefore $p = p \cdot s$.

Dividing both sides of $p = p \cdot s$ by p we get $1 = s$.

Done.

Another proof about primes

Proposition. For all integers x, if $x > 1$, then $x^3 + 1$ is *not* prime.

Proof. Let x be any integer such that $x > 1$ and let's denote $x^3 + 1$ by n.

We are going to show that n has a factor that is neither 1 nor equal to n and therefore n cannot be a prime.

First observe that $x^3 + 1 = (x+1)(x^2 - x + 1)$. (Multiply and check!)

Let's also denote $x + 1$ by r and $x^2 - x + 1$ by s.
Note that both r and s are factors of n, since $n = r \cdot s = s \cdot r$.

Now, because $x > 1$ we have $r = x + 1 > 1$.

Another proof about primes (continued)

We just derived $r > 1$.

Now, multiply both sides of $r > 1$ with s. We get $r \cdot s > s$.

However $r \cdot s = n$. Therefore $n > s$. underline

We can also show $s > 1$ underline by the following reasoning:

$$\begin{aligned} x &> 1 && \text{(Recall assumption)} \\ x^2 &> x && \text{(Multiplying both sides by x.)} \\ x^2 - x &> 0 && \text{(Subtracting x from both sides.)} \\ x^2 - x + 1 &> 1 && \text{(Adding 1 to both sides.)} \end{aligned}$$

To summarize, we have shown $1 < s < n$.underline Therefore n has a factor, namely s, that is neither 1 nor equal to n. Done.

Module 1.5: Subsets and Set-builder Notation

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Lecture Notes

Sets and their elements

A **set** is an unordered collection of distinct **elements**.

Define by enumeration: $V = \{a, e, i, o, u\}$

The elements of V are the vowels of the English alphabet.

Same set : $V = \{u, a, i, e, o\}$ (order does not matter).

Not a set: $\{o, e, a, e\}$ (elements must be distinct).

Notation for "element of" (membership in a set): $\in$.

Letter e is an element of V: $e \in V$.

Letter z is not an element of V: $z \notin V$.

Subset and proper (strict) subset

We say that the set A is a **subset** of the B
and we write $A \subseteq B$
when every element of A is also an element of B

Example: a subset of $V = \{a, e, i, o, u\}$? $\{o, u, i\} \subseteq V$.

Example: not a subset? $\{o, u, a, i, s\} \not\subseteq V$.

A set is its own subset: $V \subseteq V$.

Proper (strict) subset: a subset that is not itself.

Notation for proper subset: $\{o, u, i\} \subsetneq V$.

Empty set

The **empty set** has no elements.

Notation for the empty set: $\emptyset$

The empty set is a subset of *any* set: $\emptyset \subseteq A$.

The empty set is a proper subset of any *non-empty* set! $\emptyset \subsetneq V$

Standard sets of numbers

The **integers** are a proper subset of the **real numbers**: $\mathbb{Z} \subsetneq \mathbb{R}$.

Reminder: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Others: the **rational numbers** $\mathbb{Q}$; the **complex numbers** $\mathbb{C}$.

These sets are **infinite**. We will work mostly with **finite** sets.

We also use:

The **positive integers**: $\mathbb{Z}^+ = \{1, 2, \ldots\}$.

The **natural numbers**: $\mathbb{N} = \{0, 1, 2, \ldots\}$.

0 is a natural number, but it is not a positive integer (in this course!).

Set-builder notation

$$A = \{x \mid P(x)\}$$

This defines A as the set consisting of those elements x that have the property $P(x)$.

Often used in the form: $B = \{x \in X \mid P'(x)\}.$

This is the same as $B = \{x \mid x \in X \text{ and } P'(x)\}.$

Defines B as the subset of X consisting of those elements x that have the property $P'(x)$.

The set-builder notation is also called **set comprehension** notation.

Set-builder examples

The consonants

$$C = \{\ell \mid \ell \text{ is a Latin alphabet letter but not a vowel}\}$$

The positive integers can be defined, for example, in at least two ways:

$$\mathbb{Z}^+ = \{x \in \mathbb{Z} \mid x \geq 1\} \qquad \mathbb{Z}^+ = \{x \mid x \in \mathbb{N} \text{ but } x \neq 0\}$$

The rational numbers can be defined as

$$\mathbb{Q} = \{r \mid \text{there exists } x \in \mathbb{Z} \text{ and there exists } y \in \mathbb{Z}^+ \text{ such that } r = \tfrac{x}{y}\}$$

Module 1.6: Set Operations and Cardinality

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Lecture Notes

Union

The **union** of two sets A and B is the set whose elements are elements of A or elements of B (including those who are elements of both).

Notation: $A \cup B$.

Using set-builder notation: $A \cup B \;=\; \{x \mid x \in A \text{ or } x \in B\}$

Examples.

Recall that V is the set of vowels and C the set of consonants.

Then $V \cup C$ is the whole alphabet!

$\mathbb{Z}^+ \cup \{0\} \;=\; \mathbb{N}$

If $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$ then $A \cup B \subseteq \mathbb{N}$.

Intersection

The **intersection** of two sets A and B is the set whose elements are elements of both A and B.

Notation: $A \cap B$.

Using set-builder notation: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Examples.

$V \cap C = \emptyset$.

$\mathbb{Z}^+ \cap \mathbb{N} = \mathbb{Z}^+$

More generally, if $A \subseteq B$ then $A \cap B = A$.

Union/intersection of more than two sets

The **union** and the **intersection** of the sets $A_1, A_2, \ldots, A_n$ are defined by

$$A_1 \cup A_2 \cup \cdots \cup A_n \;=\; \{x \mid x \in A_1 \text{ or } x \in A_2 \text{ or } \ldots \text{ or } x \in A_n\}$$

respectively,

$$A_1 \cap A_2 \cap \cdots \cap A_n \;=\; \{x \mid x \in A_1 \text{ and } x \in A_2 \text{ and } \ldots \text{ and } x \in A_n\}$$

For $n = 2$ we get exactly the definition for union/intersection of two sets that we had before.

How about for $n = 1$? We get $A_1 \;=\; \{x \mid x \in A_1\}$.

True but not interesting!

Disjoint and pairwise disjoint sets

Two sets A and B are said to be **disjoint** when they have no elements in common. Equivalently, A and B are disjoint when $A \cap B = \emptyset$.

Example.

V and C are disjoint. $\{o, u, a, i, s\}$ and C are **not** disjoint.

Three sets, A_1, A_2, A_3, are **pairwise disjoint** when A_1 and A_2 are disjoint, A_1 and A_3 are disjoint, and A_2 and A_3 are disjoint.

This generalizes: three or more sets $A_1, A_2, \ldots, A_n$ $(n \geq 3)$ are **pairwise disjoint** when A_i and A_j are disjoint for all $i, j \in \{1, 2, \ldots, n\}$ such that $i \neq j$.

Example.

$\{b, c, d\}$, $\{f, g, h\}$, V, $\{m, n, p\}$, and $\{x, y, z\}$ are pairwise disjoint.

Quiz

If three sets X, Y and Z have an empty intersection ($X \cap Y \cap Z = \emptyset$), then they must be pairwise disjoint.

(A) True

(B) False

Answer

If three sets X, Y and Z have an empty intersection ($X \cap Y \cap Z = \emptyset$), then they must be pairwise disjoint.

(A) True

Incorrect. See B for a counterexample.

(B) False

Correct. Consider the sets $X = \{1, 2\}$, $Y = \{2, 3\}$, $Z = \{1, 3\}$. They have an empty intersection, but they are not pairwise disjoint.

More Information

Consider again the sets $\{1, 2\}, \{2, 3\}, \{1, 3\}$. They have an empty intersection, but they are not pairwise disjoint, in fact the intersection of any two of them is non-empty. On the other hand, if some number of sets are pairwise disjoint, then their intersection is empty.

This distinction leads to possible confusion and this is why we avoid defining "disjoint" for more than three sets. "Pairwise disjoint" is more useful anyway.

Difference

The **difference** of two sets A and B is the set whose elements are elements of A but **not** elements of B. Notation: $A \setminus B$.

Using set-builder notation: $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$

Examples.

$$\{1, 2, 3\} \setminus \{2, 3, 4\} = \{1\}$$

$$\mathbb{N} \setminus \{0\} = \mathbb{Z}^+$$

$$\mathbb{N} \setminus \mathbb{Z}^+ = \{0\}$$

$$\mathbb{Z}^+ \setminus \mathbb{N} = \emptyset$$

More generally, if $A \subseteq B$ then $A \setminus B = \emptyset$.

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Cardinality

The **cardinality** of a finite set A is the number of elements of A.
Notation: $|A|$.

Examples.

$$|\{3,5,7,9\}| = 4. \qquad |\emptyset| = 0.$$

$$|\{x \in \mathbb{N} \mid 3 \leq x < 9\}| = 6. \qquad |\{\emptyset, \{\emptyset\}\}| = 2.$$

If A and B are disjoint then $|A \cup B| = |A| + |B|$.

An **aside**: We have only defined cardinality for finite sets. However, the concept can be defined for any number of sets. I will not say much more about this but here are some intriguing facts about cardinality of infinite sets:

$$|\mathbb{N}| = |\mathbb{Z}^+| = |\mathbb{Z}| = |\mathbb{Q}| \ < \ |\mathbb{R}| = |\mathbb{C}|$$

Module 1.7: Powerset and Cartesian Product

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Lecture Notes

Powerset

The **powerset** of a set A is the set whose elements are all the subsets of A.
Notation: 2^A.

Using set-builder notation: $2^A = \{X \mid X \subseteq A\}$.

Examples:

$$2^{\{1,2,3\}} = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,3\},\{1,2,3\},\emptyset\}.$$

Is $\{1,3\}$ in $2^{\{1,2\}} \cup 2^{\{2,3\}}$? No.

What is the powerset of the set whose only element is the empty set?
$2^{\{\emptyset\}} = \{\ \emptyset\ ,\ \{\emptyset\}\ \}$.

Quiz

Which of the following is **not** an element of the powerset of $\{0, \emptyset, \{0\}\}$?

(A) 0

(B) $\{0\}$

(C) $\emptyset$

(D) $\{\emptyset, \{0\}\}$

Answer

Which of the following is not an element of the powerset of $\{0, \emptyset, \{0\}\}$?

(A) 0

Correct. Any element of a powerset must be a set.

(B) $\{0\}$

Incorrect. Since 0 is an element of the set, $\{0\}$ **is** an element of the powerset.

(C) $\emptyset$

Incorrect. The empty set is an element of any powerset.

(D) $\{\emptyset, \{0\}\}$

Incorrect. This is a subset of the given set hence an element of the powerset.

More Information

The power set is the set of all subsets, so the power set of $\{0, \emptyset, \{0\}\}$ is

$\{\emptyset, \{0\}, \{\emptyset\}, \{\{0\}\}, \{0, \{0\}\}, \{\emptyset, \{0\}\}, \{0, \emptyset\}, \{0, \emptyset, \{0\}\}\}$

Sequences

A **sequence** is an ordered collection of elements, with possible repetitions.

Alternative terminology **list, array, string, tuple, word**.

However, mathematically, these are all the same as sequences.

A sequence has **positions**, 1,2,3, etc. and **length**.

Examples:

Consider the set $\{x, 2, a\}$. The sequences of length 2 whose elements are from this set:

. $xx, x2, xa, 22, 2x, 2a, aa, a2, ax$.

A string of digits of length 6: 737334 (has 7 in position 3)

A word made of letters from the English alphabet:

`floccinaucinihilipilification`.

Tuples, triples, pairs

Sometimes we write sequences as

$(2, a, 2, x)$ instead of $2a2x$

and call them **tuples**, or more specifically, n-**tuples** where n is the length.

$(2, a, 2, x)$ is a 4-tuple.

Triples are the same as 3-tuples.

Pairs are the same as 2-tuples.

In a pair (a, b) we call a the **first component** and b the **second component**.

Cartesian (cross) product

The **cartesian product** (or **cross product**) of two sets A and B is the set whose elements are pairs whose first component is an element of A and whose second component is an element of B.

Notation: $A \times B$.

Using set-builder notation: $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

Examples:

Let's enumerate the set $\{p, q\} \times \{2, 3\}$:

. $\{(p, 2), (p, 3), (q, 2), (q, 3)\}$.

(e, f) is an element of $V \times C$.

$(2, 2)$ is an element of $\mathbb{Z}^+ \times \mathbb{N}$.

If $A \subseteq B$ then $A \times B \subseteq B \times B$.

ACTIVITY : Subsets

Name all the subsets of $\{1, 2, 3\}$ containing 2 but not 3.

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Subsets (Continued)

Answer.

The subsets of $\{1, 2, 3\}$ containing 2 but not 3 are $\{2\}$ and $\{1, 2\}$.

Observe that all the subsets containing 2 are $\{2\}$, $\{1, 2\}$, $\{1, 2, 3\}$, and $\{2, 3\}$. From these subsets, we do not consider the subsets that contain 3, and thus we get $\{2\}$ and $\{1, 2\}$.

ACTIVITY : Cartesian Product

Problem. Consider two sets $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Name all the elements of $A \times B$ (the cartesian product of A and B) whose second component is b.

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Activity : Cartesian Product (Continued)

Answer.

The elements of $A \times B$ whose second component is b are:

$$(1, b), (2, b), (3, b).$$

The cartesian product of A and B contains all ordered pairs where the first element is from set A and the second element is from set B, i.e.,

$$A \times B = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b), (1, c), (2, c), (3, c)\}.$$

From these elements the ones whose second component is b are:

$$(1, b), (2, b), (3, b).$$