Module 9.2: Expectation

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Expectation (expected value)

Average, or **mean**, value returned by a random variable. Such an average should take into account that some values may "weigh" more than others. The **weights** are given by the probability distribution!

Notation $\mathsf{E}[X]$. Two candidates, but they give the same answer:

Proposition. For a random variable X defined on $(\Omega, \Pr)$ we have

$$\mathsf{E}[X] \;=\; \sum_{x \in \mathsf{Val}(X)} x \cdot \Pr[X = x] \;=\; \sum_{w \in \Omega} X(w) \cdot \Pr[w]$$

The first expression corresponds directly to a **weighted average** of the values taken by random variable. Recall that the weights $\Pr[X = x]$ sum up to 1.

The second expression takes the average of values by outcomes they map from. Multiple outcomes may be mapped to the same value taken by the r.v.

Expected number shown by a die

Problem. Compute $\mathrm{E}[D]$ where D is the number shown by a fair die.

Answer.

$$1 \cdot (1/6) + 2 \cdot (1/6) + 3 \cdot (1/6) + 4 \cdot (1/6) + 5 \cdot (1/6) + 6 \cdot (1/6) \;=$$
$$= \;(1+2+3+4+5+6)/6 \;=\; 3.5$$

Since all weights are the same, this is the usual average. More generally:

Proposition. The expectation of a uniform r.v. that takes values $v_1, \ldots, v_n$ is

$$\frac{v_1 \; + \; \cdots \; + \; v_n}{n}$$

Expectation of the Bernoulli r.v.

Problem. Compute the expectation of the Bernoulli r.v. X with parameter p.

Answer. Recall that $\mathsf{Val}(X) = \{0, 1\}$

and the distribution is $\Pr[X = 1] = p$ and $\Pr[X = 0] = 1 - p$.

Then $\mathsf{E}[X] = 1 \cdot \Pr[X = 1] + 0 \cdot \Pr[X = 0] = 1 \cdot p + 0 \cdot (1 - p) = p$.

Expectation of a constant r.v.

Let $c \in \mathbb{R}$ and let $(\Omega, \Pr)$ be a probability space.

Consider the r.v. $C : \Omega \to \mathbb{R}$ such that for all outcomes $w \in \Omega$ we have $C(w) = c$.

The distribution is trivial $\Pr[C = c] = \sum_{w \in \Omega} \Pr[w] = 1$.

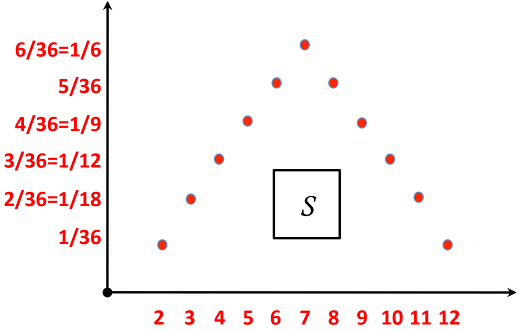
Proposition. $\mathrm{E}[C] = c$ (Of course!)

Proof.

$$\mathrm{E}[C] = \sum_{w \in \Omega} C(w) \cdot \Pr[w] = \sum_{w \in \Omega} c \cdot \Pr[w] = c \cdot \sum_{w \in \Omega} \Pr[w] = c.$$

ACTIVITY : Expected sum of two fair dice

In this activity we will compute the expectation of the random variable S that returns the sum of two fair dice. Recall the distibution of S



Question. List all the outcomes that correspond to $S = 6$. How many are there? What does this imply for $\Pr[S = 6]$?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : Expected sum of two fair dice (continued)

Answer. The outcomes are $(1,5)$, $(2,4)$, $(3,3)$ $(4,2)$, $(5,1)$. There are 5 of them. Since the probability space is uniform with 36 outcomes it follows that $\Pr[S=6] = 5/36$, as the figure indicated.

Now, we compute

$$\begin{aligned} \mathrm{E}[S] &= 2\cdot(1/36) + 3\cdot(2/36) + 4\cdot(3/36) + 5\cdot(4/36) + 6\cdot(5/36) \\ &+ 7\cdot(6/36) + 8\cdot(5/36) + 9\cdot(4/36) + 10\cdot(3/36) + 11\cdot(2/36) \\ &+ 12\cdot(1/36) = \frac{252}{36} = 7 \end{aligned}$$

Recall that earlier in this segment we computed the expectation of the value a single die shows as 3.5. Now observe that $\mathrm{E}[S] = 7 = 3.5 + 3.5$ so it's the sum of the expected values of each die! Is this a coincidence? No, as we will see when we learn about the **linearity of expectation**.

Expected sum of three dice

Problem. Compute $\mathrm{E}[T]$ where T is the sum of the numbers shown by **three** fair dice rolled together.

Answer. Clearly $\mathrm{Val}(T) = [3..18]$ so T takes $18 - 3 + 1 = 16$ values. For values $T = 3, 4, 5, 6, 7, 8$ we can use stars and bars. Let's find $\Pr[T = 9]$.

$9 = 1 + 2 + 6 = 1 + 3 + 5 = 1 + 4 + 4 = 2 + 2 + 5 = 2 + 3 + 4 = 3 + 3 + 3$

$3 + 3 + 3$ is one outcome, probability $1/216$. $1 + 2 + 6$, $1 + 3 + 5$, and $2 + 3 + 4$ each correspond to $3!$=6 outcomes so each has probability $6/216$. $1 + 4 + 4$ and $2 + 2 + 5$ each correspond to $3!/2!1!$=3 outcomes so each has probability $3/216$.

$\Pr[T = 9]$ is therefore $(1/216)$+$3(6/216)$+$2(3/216)$=$25/216$.

This is too hard! I give up! Is there an easier way?