OMCIT 592 Module 11 Self-Paced 01 (instructor Val Tannen)

No reference to this self-paced segment in the lecture segments.

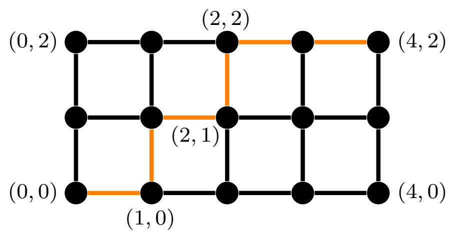
This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion.

When you see the following marker:

We suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Shortest paths in a grid

In the lecture segment "Special graphs" we introduced **grid graphs** intuitively, as follows. Let m, n be positive integers. An $m \times n$ grid graph has m rows of n vertices and each vertex is linked by an edge to the vertices "closest" to it. We supported this intuitive description visually with an example of a 3×5 grid graph (ignore for the moment the labels of some of the nodes):



Note also (drawn in orange) a path of **minimum length** from the "lower left corner" to the "upper right corner". It has length 6.

In this segment we will give a precise definition of grid graphs and prove the following:

"Proposition". In an $m \times n$ grid graph every path of minimum length from the "lower left corner" to the "upper right corner" has length $m + n - 2$ and traverses edges only "upwards" or "rightwards".

Our first goal is to state the "proposition" above in a precise manner (removing the quotes, as it were).

Shortest paths in a grid (continued)

Let m, n be positive integers. The $m \times n$ graph that we will work with makes precise "m rows of n vertices" by taking as vertices $V = [0..(n-1)] \times [0..(m-1)]$. These correspond to points of natural number coordinates in the usual 2-dimensional plane. The "lower left corner" is $(0,0)$ and the "upper right corner" is $(n-1, m-1)$. All four "corners" and three other vertices are illustrated on the figure above.

We define the set of **horizontal** edges, Hor, the set of **vertical** edges, Ver, and the set of all edges of the grid graph as

$$
\begin{aligned}
\mathrm{Hor} &= \{\{(i,j),(k,j)\} \mid i,k \in [0..(n-1)] \wedge j \in [0..(m-1)] \wedge (k = i+1)\} \\
\mathrm{Ver} &= \{\{(i,j),(i,\ell)\} \mid i \in [0..(n-1)] \wedge j,\ell \in [0..(m-1)] \wedge (\ell = j+1)\} \\
E &= \mathrm{Hor} \cup \mathrm{Ver}
\end{aligned}
$$

Now we can describe precisely the path drawn in orange in the figure above:

$$(0,0)\text{-}(1,0)\text{-}(1,1)\text{-}(2,1)\text{-}(2,2)\text{-}(3,2)\text{-}(4,2)$$

We can now also state in a precise manner the proposition of interest.

Proposition. Let m, n be positive integers. In an $m \times n$ grid graph every path of minimum length from $(0,0)$ to $(n-1, m-1)$ has the form $u_0, \ldots, u_p$ where

- The path has length $m + n - 2$, that is, $p = m + n - 2$.
- When $p \geq 1$, for every $q = 0, \ldots, p-1$ the edge $u_q\text{-}u_{q+1}$ of the path is
  - either $u_q \equiv (i,j)\text{-}(i,j+1) \equiv u_{q+1}$ ("upwards")
  - or $u_q \equiv (i,j)\text{-}(i+1,j) \equiv u_{q+1}$ ("rightwards"),

Shortest paths in a grid (continued)

Proof. The proof is by ordinary induction on $p = m + n - 2$. This may seem a bit strange (!) at first. However, consider that a grid can "incrementally" grow in two directions: upwards or rightwards. Induction on something like $m + n$ will allow to consider the two directions by cases. The presence of -2 is simply to make sure the base case $\ell = 0$ corresponds to $m = n = 1$.

(BC) $p = 0$, hence $m + n = 2$, therefore $m = n = 1$. This is is the one-node grid. There is only one path from $(0, 0)$ to $(0, 0)$, the path of length 0. But $0 = p$ and there are no edges. Check.

(IS) Let $k \in \mathbb{N}$ arbitrary. Assume (IH) that in any $m \times n$ grid graph such that $m+n-2 = k$ every path of minimum length from $(0, 0)$ to $(n - 1, m - 1)$ has the properties stated in the proposition (taking p to be k).

Now consider an $m \times n$ grid graph G such that $m + n - 2 = k + 1$. Consider also a path P of minimum length in G from $(0, 0)$ to $(n - 1, m - 1)$. Such a path must reach $(n - 1, m - 1)$ going through $(n - 1, m - 2)$ or going through $(n - 2, m - 1)$. (In the figure above, where $m = 3$ and $n = 5$ the path drawn in orange goes through $(3, 2)$ and note that $3 = 5 - 2$ and $2 = 3 - 1$.)

Therefore we have two cases.

Case 1: The path P of minimum length in G is $(0, 0)$-$\cdots$-$(n-1, m-2)$-$(n-1, m-1)$. Let's denote P' the part $(0, 0)$-$\cdots$-$(n - 1, m - 2)$ of P (remove the last edge).

Consider the subgraph G' of G obtained by removing the nodes $(0, m - 1), \ldots, (n - 1, m - 1)$ (the nodes in the "uppermost" row). Observe that G' is an $(m - 1) \times n$ grid graph.

Since $(m - 1) + n - 2 = (m + n - 2) - 1 = (k + 1) - 1 = k$ it must be the case that the IH applies to G'. But P' is a path in G' from $(0, 0)$ to $(n - 1, m - 2)$

Now we claim that among such paths P' is of minimum length. Indeed, if there was a strictly shorter one we would add the edge $(n - 1, m - 2)$-$(n - 1, m - 1)$ at the end and contradict the fact that P is of minimum length!

By IH, P' has length $(m - 1) + n - 2$ hence P has length $(m - 1) + n - 2 + 1 = m + n - 2$.

Again by IH each edge of P' is either "upwards" or "rightwards". The last edge of P is "rightwards". This concludes the induction step in Case 1.

Case 2, in which P goes through $(n - 2, m - 1)$ is similar. (It's a good exercise to write it up without looking at the proof above!)