OMCIT 592 Module 13 Self-Paced 02 (instructor Val Tannen)

One reference to this self-paced segment in the lecture segment "No odd cycles".

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion.

When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Proof of 2-colorability

In the lecture segment "No odd cycles" stated the following chracterization of bipartite graphs:

Proposition. A graph is bipartite iff it does not contain a cycle of odd length (an odd cycle).

We proved one of the directions in lecture, showing that a bipartite graph cannot have odd cycles. For the other direction we observed in lecture that w.l.o.g. we can assume that the graph is **connected**. Indeed, since no edges go between connected components, a graph has a proper coloring iff each of its connected components have a proper coloring.

And, we defined in lecture for any connected graph the concept of distance in a graph as follows.

Let $G = (V, E)$ be a connected graph. The **distance** between two vertices $u, v \in V$, notation $d(u, v)$, is the length of a shortest path from u to v (the existence of such a path is guaranteed by the Well-Ordering Principle).

Further, in lecture, using the concept of distance we embarked on showing that a connected graph $G = (V, E)$ without odd cycles is bipartite. Specifically, we continued as follows.
Fix an arbitrary vertex w_o. Now color the vertices of G:

- w is colored **red** when $d(w_0, w)$ is **even** (in particular, w_0 is colored red).
- w is colored **blue** when $d(w_0, w)$ is **odd**.

So, this is a 2-coloring, but is it **proper**? And we promised a proof of propriety in this segement. Here it goes.

We need another useful observation:

Lemma (Locality of shortest paths). Consider a shortest path p from u to v and let x and y be two vertices in this path. Then the portion x–$\cdots$–y of p is a shortest path from x to y, so $d(x, y) \leq d(u, v)$.

Proof (of Lemma). Indeed, if there exists a strictly shorter path from x to y then we can replace the portion x–$\cdots$–y of p with that shorter path and get a strictly shorter path from u to v.

Proof of 2-colorability (continued)

Back to the proof of the remaining direction of the proposition. We have a connected graph $G = (V, E)$ without odd cycles and the only thing left to show is that the 2-coloring defined using the even/odd parity of distance is proper.

Suppose, toward a contradiction, that we have an edge u–$v \in E$ such that both u and v are colored blue. (Later we consider the case when they are both colored red.) Our hope is to show that this implies that G has an odd cycle, hence we have a contradiction.

Note that w_0, u, v are pairwise distinct. Let p_1 be a shortest path w_0–$\cdots$–u and p_2 be a shortest path w_0–$\cdots$–v. Since u, v are colored blue the lengths of both p_1 and p_2 are odd.

It is is true that from p_1, u–v and (reversing) p_2 we get a closed walk w_0–$\cdots$–u–v–$\cdots$–w_0 of length $d(w_0, u) + 1 + d(w_0, v)$, which is odd. But this walk is, in general, not a cycle because p_1 and p_2 may have intermediate vertices in common and having a closed walk of odd length does not, by itself, imply that we have an odd cycle.

Let S be the set of vertices that p_1 and p_2 have in common. Suppose $u \in S$. Then, by the lemma above (locality of shortest paths) we have $d(w_0, v) = d(w_0, u) + 1$, but both $d(w_0, v)$ and $d(w_0, u)$ are odd, contradiction. Hence $u \notin S$. Similarly $v \notin S$.

S is not empty because $w_0 \in S$. Let $w \in S$ be the vertex "closest" to u, i.e., all the nodes in the portion w–$\cdots$–u of p_1 are not in S. It follows that all the nodes in the portion w–$\cdots$–v of p_2 are also not in S. Thus the closed walk w–$\cdots$–u–v–$\cdots$–w forms a cycle, C. We shall prove that C has odd length.

By the lemma above (locality of shortest paths) the portion w_0–$\cdots$–w of p_1 has length $d(w_0, w)$. Similarly, the portion w_0–$\cdots$–w of p_2 has length $d(w_0, w)$. Hence the length of C is $d(w_0, u) - d(w_0, w) + 1 + d(w_0, v) - d(w_0, w) = d(w_0, u) + 1 + d(w_0, v) - 2d(w_0, w)$ which is odd, since $d(w_0, u)$ and $d(w_0, v)$ are both odd.

So we proved that u and v cannot both be colored blue.

Now suppose, again toward a contradiction, that we have an edge u–$v \in E$ such that u and v are both colored red. Then $u \neq w_0$, otherwise $d(w_0, v) = 1$ and v should be blue. Similarly $v \neq w_0$.

Now that we have established that w_0, u, v are pairwise distinct the proof proceeds as in the earlier case, again resulting in a cycle of length $d(w_0, u) + 1 + d(w_0, v) - 2d(w_0, w)$ which is again odd, even though now $d(w_0, u)$ and $d(w_0, v)$ are both even.