CIT 592 Spring 2022 Homework 5

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1. [10 pts] Show that each of the following functions is not a bijection by giving either

- an element of the codomain that is not in the range, or
- two elements of the domain that map to the same element in the range.

Be sure to explain why each one is not a bijection!

(a) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 8x$

(b) $g : \mathbb{N} \to \mathbb{N}$ given by $g(x) = x + 7$

(c) $h : [11..16] \to [12..16]$ given by $h(x) = \begin{cases} 16 & \text{if } x = 11 \\ x & \text{otherwise} \end{cases}$

(d) $j : \mathbb{N} \to \mathbb{N}$ given by $j(x) = \begin{cases} (x+1)^2 & \text{if } x \text{ is even} \\ 2x+1 & \text{if } x \text{ is odd} \end{cases}$

(e) $k : [-7..10] \to [0..12]$ given by $k(x) = |x + 2|$

Solution.

(a) Since the domain is $\mathbb{Z}$, it means $x \in \{..., -2, -1, 0, 1, 2, ...\}$, so $Ran(f) = \{y|8|y\}$, which means $y \in \{..., -16, -8, 0, 8, 16, ...\}$. Therefore we can find an element of the codomain, such as 2, that is not in $Ran(f)$. The proof of function f is not a bijection is finished.

(b) Since the domain is $\mathbb{N}$, it means $x \in \{0, 1, 2, ...\}$, so $Ran(g) = \{y|y \geq 7\}$. Therefore we can find an element of the codomain, such as 3, that is not in $Ran(g)$. The proof of function g is not a bijection is finished.

(c) By the mapping rule of function h, we can find $h(11) = h(16) = 16$. The proof of function h is not a bijection is finished.

(d) Since the domain is $\mathbb{N}$, it means $x \in \{0, 1, 2, ...\}$. We can calculate that $j(0) = 1$, $j(1) = 3$, $j(2) = 9$, and go on. Therefore we can find an element of the codomain, 2, that is not in $Ran(g)$. The proof of function g is not a bijection is finished.

(e) By the mapping rule of function h, we can find $k(-4) = k(0) = 2$. The proof of function k is not a bijection is finished.

2. [**10 pts**] Recall that a derangement is a permutation where no element ends up in its original position. In this problem we consider a different, related concept: *deranged anagrams*. We say that an anagram is deranged if no letter ends up in its original position and no letter ends up in the original position of an identical letter. For example, `ffeeco` is a deranged anagram of `coffee`, but `eefcof` is not.

There are $\frac{(2+2+1)!}{2!2!1!} = 30$ anagrams of `radar`. How many of them are deranged?

Solution.

To achieve derangement, we know the letter d cannot be on position 3:
Case 1: d is on position 1 - then the position 2 and 4 have to be letter r, which would leave position 3 and 5 to the letter a. So we get only one derangement, drara.
Case 2: d is on position 2 - then position 1 and 5 have to be letter a, which would leave position 3 and 4 to the letter r. So we get only one derangement, adrra.
Case 3: d is on position 4 - then position 1 and 5 have to be letter a, which would leave position 2 and 3 to the letter r. So we get only one derangement, arrda.
Case 4: d is on position 5 - then position 2 and 4 have to be letter r, which would leave position 1 and 3 to the letter a. So we get only one derangement, arard.

Summary: From above we have exhausted all cases and get 4 derangements.

3. [**10 pts**] Let there be a room that is 8 feet by 8 feet. Suppose that there are 5 people who sit in this room. For simplicity, assume these people are just points. Prove that, among these people, there is some pair that is seated at most $4\sqrt{2}$ feet from each other.

Solution.

Step 1: Since the room is 8x8, we can divide the room to 4 smaller 4x4 rooms. Since there are 5 people (pigeons) and only 4 smaller rooms (pigeonholes), $5 > 4 * 1$, thus we know there must be 2 people in the same smaller room.
Step 2: The size of smaller room is 4x4, so the longest distance between two people happen when they sit diagonally across the room. Based on Pythagorean theorem we can calculate the distance between the diagonal corners of a 4x4 room is $4\sqrt{2}$.
Step 3: Since there must be 2 people in the same smaller 4x4 room and the longest distance between them is $4\sqrt{2}$, we can conclude that among the 5 people, there exists some pair that is seated at most $4\sqrt{2}$ feet from each other.

4. **[10 pts]** Alex wants to renew his wardrobe by buying a new item every day. He buys three types of items; shoes, shirts, and pants, and the store never runs out of an item. Alex, being the diligent TA he is, wants to plan out his buying options for the next $n \geq 1$ days, and decides that the easiest way to do this is to create a function f such that for every day i, he will buy item $f(i)$ (where the codomain is $\{shoes, shirt, pants\}$, and he buys one item per day). Because he wants to have full outfits, Alex will only consider functions f that allow him to buy each type of item at least once. How many such functions f can Alex devise?

Solution.
Method 1: Do not use PIE - treat it as a counting question.
Step 1: Without any restrictions there would be 3^n functions in total to buy items.
Step 2: If Alex doesn't buy any shoes, there would be 2^n functions in total, including one function for all pants and one function for all shirts.
Step 3: Similar to step 2, there would be 2^n functions when Alex doesn't buy shirts (including one function for all pants and one function for all shoes), and 2^n functions when Alex doesn't buy pants, including one function for all shirts and one function for all shoes.
Step 4: From above we can see the functions for only one type item were counted twice, so we need to add back 3.
Step 5: The functions f that buy each type of items at least one would be $3^n - 3 * 2^n + 3$.

Method 2: Use PIE
Step 1: Let A be the set of functions that doesn't allow Alex to buy pants, B be the set of functions that doesn't allow Alex to buy shoes, C be the set of functions that doesn't allow Alex to buy shirts. Then $|A \cup B \cup C|$ would cover all the functions other than the functions that allow Alex buys at least one of each type of items.
Step 2: Based on the multiplication rule we know that $|A| = |B| = |C| = 2^n$.
Step 3: Since we definite A as the set of functions that doesn't buy pants and B as the set of functions that doesn't buy shoes, $A \cap B$ would mean the function that only buys shirts, so $|A \cap B| = 1$. Similarly, we can conclude $|B \cap C| = |A \cap C| = 1$. Since the three sets, A,B,C are disjoint, $|A \cap B \cap C| = 0$ (Alex cannot ONLY buy pants/shoes/shirts at the same time).
Step 4: Based on PIE we know that
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$
$= 2^n + 2^n + 2^n - 1 - 1 - 1 + 0 = 3 * 2^n - 3$
Step 5: Without any restrictions there would be 3^n functions in total to buy items.
Step 6: The function f that allows Alex to buy each type of items at least one would be $3^n - (3 * 2^n - 3) = 3^n - 3 * 2^n + 3$.

5. [**10 pts**] After an important and successful meeting, employees of a company shake hands with each other. There are $n \geq 2$ employees at the meeting. Any of these employees can shake hands with any number of the other employees (including zero), but two employees can only shake each others' hands once. A handshake is a mutual event between exactly two employees.

Prove or disprove the claim that there must always be at least two employees who shake the same number of hands.

Solution.

Step 1: For any random employee i, let a_i be how many handshakes that employee i would have, then $a_i \in [0..(n-1)]$.
Step 2: If an employee decides not to shake hand with anyone, then employee i would at most shake hands with (n-2) people so $a_i \in [0..(n-2)]$.
Step 3: If an employee decides to shake hand with everyone, then employee i would at least shake hands with one people so $a_i \in [1..(n-1)]$..
Step 4: From the two edge cases from Step 2 and Step 3, we can tell at most $|a_i| = (n-2)-0+1 = (n-1)-1+1 = n-1$
Step 5: There are n employees (pigeons) but at most $n-1$ handshake situations (pigeonholes), so based on PHP, we can conclude at least two employees will have the same number of handshakes.