Self-paced Example: Inverse Functions

Module 5

MCIT Online - CIT592 - Professor Val Tannen

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion. When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Inverse functions

Given $f : A \to B$, an **inverse** of f is a function $g : B \to A$ such that

$$\forall x \in A \quad g(f(x)) = x \qquad \text{and} \qquad \forall y \in B \quad f(g(y)) = y$$

The definition above implies that an inverse of f, if it exists, is completely determined by f. Therefore we will talk about **the** inverse of a function.

Examples:

- The inverse of $\quad \mathsf{squ} : [0, \infty) \to [0, \infty) \quad \mathsf{squ}(x) = x^2 \quad$ is the function $\mathsf{sqrt} : [0, \infty) \to [0, \infty) \quad \mathsf{sqrt}(x) = \sqrt{x}$.

- The inverse of $\quad \mathsf{exp} : \mathbb{R} \to (0, \infty) \quad \mathsf{exp}(x) = 2^x \quad$ is the function $\mathsf{loga} : (0, \infty) \to \mathbb{R} \quad \mathsf{loga}(x) = \log_2 x$.

- The inverse of $f : \{1, 2, 3\} \to \{a, b, c\}$ given by the table

$x \in \{1, 2, 3\}$	$f(x) \in \{a, b, c\}$
1	c
2	a
3	b

is the function $g : \{a, b, c\} \to \{1, 2, 3\}$ given by the table

$y \in \{a, b, c\}$	$g(y) \in \{1, 2, 3\}$
a	2
b	3
c	1

Bijections and inverse functions

Proposition. A function has an inverse iff it is a bijection. The inverse of a bijection is also a bijection.

Proof. We have to prove an "iff". This means proving two implications.

Claim. If $f : A \to B$ has an inverse, $g : B \to A$, then f is a bijection.

To prove that f is bijection we have to prove that it is both an injection and a surjection.

1. f is injective.

   Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$. We are going to show that $x_1 = x_2$ thus verifying the contrapositive of the definition of injectivity.

   Using the definition of inverse, we have $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$. Done.

   

2. f is surjective.

   Let $y \in B$. We want to show that there exists $x \in A$ such that $f(x) = y$. For that, we can take $x = g(y)$. Indeed $f(g(y)) = y$ using the definition of inverse.

   

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Bijections and inverse functions (continued)

Claim. If $f : A \to B$ is a bijection, then it has an inverse, $g : B \to A$.

To define g observe that for any $y \in B$ there exists, because f is surjective, an $x \in A$ such that $f(x) = y$.
Moreover, that x is the only element of A that f maps to y, because f is injective.
Now we define $g(y)$ to be that x.

Since $f(x) = y$ we have $g(f(x)) = g(y) = x$. And since $g(y) = x$ we have $f(g(y)) = f(x) = y$. So f and g are inverses.

There is one more part to the proposition, namely to show that the inverse is also a bijection. But notice that the definition of inverses is **symmetric**. Therefore the argument made in the first Claim applies to the inverse!

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Functions and sequences

Let $n \in \mathbb{Z}^+$ and consider the set $F = \{0,1\}^{[1..n]}$ the elements of F are functions with domain $[1..n]$ and codomain $\{0,1\}$.

Consider also the set S of sequences of bits (elements of $\{0,1\}$) of length n. Notice that the positions in such a sequence are exactly the numbers in $[1..n]$.

We are going to show that the sets F and S are in one-to-one correspondence, that is, there is a bijection with domain F and codomain S.

And we will show this by defining a pair of inverse function.

Define $\varphi : F \to S$ as follows. For any function $f \in F$ define $\varphi(f)$ as the sequence of bits of length n that in position k has the bit $f(k)$, for all $k \in [1..n]$.

Now define $\psi : S \to F$ as follows. For any sequence of bits of length n, $s \in S$ define $\psi(s)$ as the function $f : [1..n] \to \{0,1\}$ that maps $k \in [1..n]$ to the bit in position k in s.

The hard work is done. Convince yourselves (intuition suffices) that φ and ψ are inverse to each other, that is,

$$\varphi(\psi(s)) = s \qquad\qquad \psi(\varphi(f)) = f$$

Many mathematicians do not distinguish between sequences and functions, even preferring to **define** a sequence as a special kind of function, making the one-to-one correspondence that we have shown **implicit**.

However the formalities involved in working with functions can obscure the intuition. It's better to think of sequences as their own kind of object studied in Discrete Mathematics.