Module 3.2: Negating Statements

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Disprove

Problem. Prove or disprove: 0 is both a natural number and a positive integer.

To **disprove** a statement P means to prove its **negation**: not P.

This problem is simple, but when we have complex statements we need **rules** for negating them.

In addition we will see that manipulating negation is very effective in some proof techniques.

Negation of disjunction/conjunction

Memorize: "the negation of disjunction is conjunction"
and "the negation of conjunction is disjunction"

In logical notation: $\neg(P_1 \vee P_2)$ is $(\neg P_1) \wedge (\neg P_2)$
and $\neg(P_1 \wedge P_2)$ is $(\neg P_1) \vee (\neg P_2)$

These are known as **De Morgan's Laws**.

Examples:

"not($x < 0$ or $x > 0$)" is "$x \geq 0$ and $x \leq 0$".

"not($x \in A \cup B$)" is "$x \notin A$ and $x \notin B$".

"not($x \in A \cap B$)" is "$x \notin A$ or $x \notin B$".

Negation of quantifiers

Memorize: "the negation of universal is existential"
and "the negation of existential is universal"

In logical notation: $\neg(\forall x\ P(x))$ is $\exists x\ \neg P(x)$
and $\neg(\exists x\ P(x))$ is $\forall x\ \neg P(x)$

Examples:

"not$(\forall x\ \exists y\ loves(x, y))$" is "$\exists x\ \forall y\ \neg loves(x, y)$"

"not(everybody loves somebody)" is "there is somebody who loves nobody".

"not$(\exists x\ \ x \in A \cap B)$" is "$\forall x\ \ x \notin A$ or $x \notin B$".

"not $(\exists k\ \ n = 2k)$" is "$\forall k\ \ n \neq 2k$"

(For simplicity we omitted the conditions that n and k are integers.)

Shouldn't the negation of "n is even" be the statement "n is odd"?

With some arithmetic help, yes! (See next activity.)

ACTIVITY : Negation of Even

Recall that you can always write any number n as $n = 2q + r$ where q is called quotient and r is called remainder.

Question:

What possible remainders are there when we divide a positive integer n by 2?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : Negation of Even (Continued)

Answer: 0 and 1.

There are only two possibilities for n when we divide n by 2. Notice that $r = 0$ corresponds to our definition of even and $r = 1$ corresponds to our definition of odd.

Activity : Negation of Even (Continued)

Using this fact, we now prove that the negation of "n is even" is "n is odd." In other words, n is odd if and only if n is not even.

If n is not even, then by the definition of even, there is no integer k such that $n = 2k$. In particular, we cannot have $n = 2q$, so the remainder of n divided by 2 cannot be 0. The only other possibility is that the remainder is 1, meaning that $n = 2q + 1$, so n is odd.

On the other hand, if n is odd, then by the definition of odd there is some integer ℓ such that $n = 2\ell + 1$, so the remainder of n divided by 2 must be 1. This implies that there is no integer k such that $n = 2k$, so n is not even.

We will give proofs that use negation in future segments, and from now on we will assume that the negation of "n is even" is "n is odd" and the negation of "n is odd" is "n is even."

Negation of implication

Memorize: the negation of "if premise then conclusion" is
"premise **and** the negation of the conclusion"

In logical notation: $\neg(P_1 \Rightarrow P_2)$ is $P_1 \wedge \neg P_2$

Examples:

"not(if p is prime and p is even then $p = 2$)" is
"p is prime and p is even and $p \neq 2$"

"not(for all x if x is odd then x is prime)" is
"there is an x that is odd and not prime"

Counterexamples

Problem. Fermat: prove or disprove that for every natural number n, the number $2^{2^n} + 1$ is prime.

Answer. Euler disproved this, showing that $2^{2^5} + 1$ is not prime.

Terminology: we say that Euler showed that $n = 5$ is a **counterexample** for the universally quantified statement:

$$\forall n \;\; natural(n) \;\Rightarrow\; prime(2^{2^n} + 1)$$

A counterexample to this statement is a proof of its negation, which (by the rules we saw) is an existentially quantified statement:

$$\exists n \;\; natural(n) \;\land\; \neg prime(2^{2^n} + 1)$$

Quiz

According to our rules, the **negation** of "every prime number is odd" is

A. 2 is a prime number but is also an even number.
B. There exists a prime number that is not odd.
C. There exists a prime number that is not even.

Answer

According to our rules, the **negation** of “every prime number is odd” is

A. 2 is a prime number but is also an even number.
   Incorrect. This is true, but it is not the negation of the statement above.

B. There exists a prime number that is not odd.
   Correct. The negation of a universal quantifier is an existential qualifier.

C. There exists a prime number that is not even.
   Incorrect. This is true, but it is not the negation of the statement above.

More Information

The given statement “every prime number is odd” is false. Its negation is therefore true. Interestingly, all the answers are true statements, but only (B) is the negation of the given statement.

We can think of (A) as proving the negation statement, rather than the negation itself. Therefore (A) is a counterexample.