Module 8.1: Inclusion-exclusion for Probability

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

At least one die shows a six I

We computed **for two dice** the probability of the event "at least one of the dice shows a 6" using **P6** (inclusion-exclusion).

Problem. We roll three fair dice. Compute the probability of the event "at least one of the dice shows a 6"

Answer. The probability space has $6 \cdot 6 \cdot 6 = 216$ outcomes and is uniform .

Let the dice be d_1, d_2, d_3 and define $D_i =$ "d_i shows a 6", $i = 1, 2, 3$.

The event of interest is $D_1 \cup D_2 \cup D_3$.

D_1, D_2, D_3 are **not** pairwise disjoint so **P2gen** does not apply.

Need **P6** but for 3 events!

Inclusion-exclusion for three events

Proposition. For any events A, B, C in the same probability space

$$\begin{aligned}\Pr[A \cup B \cup C] \;=\;& \Pr[A] + \Pr[B] + \Pr[C] \\ & - \Pr[A \cap B] - \Pr[B \cap C] - \Pr[C \cap A] \\ & + \Pr[A \cap B \cap C]\end{aligned}$$

The proof is in the segment entitled "Inclusion-exclusion for three events".

Answer (continued).

We will apply this to $\Pr[D_1 \cup D_2 \cup D_3]$.

At least one die shows a six II

Answer (continued). We calculate the probabilities in the formula.

$$\Pr[D_1] = \Pr[D_2] = \Pr[D_3] \ = \frac{36}{216} = 1/6$$

$$\Pr[D_1 \cap D_2] = \Pr[D_2 \cap D_3] = \Pr[D_3 \cap D_1] \ = \frac{6}{216} = 1/36$$

$$\Pr[D_1 \cap D_2 \cap D_3] \ = 1/216$$

Now we apply the formula.

$$\Pr[D_1 \cup D_2 \cup D_3] \ = \ 1/6 + 1/6 + 1/6 - 1/36 - 1/36 - 1/36 + 1/216 = 91/216$$

Note that for two dice we had $11/36 \simeq 0.31$. For three dice we have, of course, a bigger probability $91/216 \simeq 0.42$.

At least one die shows a six III

Problem. We roll **ten** fair dice. Compute the probability of the event "at least one of the dice shows a 6".

Answer. The probability space has 6^{10} outcomes and is uniform.

To proceed like we did for three dice we would need an inclusion-exclusion formula for ten events! Such a formula exists but it is completely unwieldy.

Are we stuck? We forgot one advantage: the ten dice roll **independently**!

Using independence we can find the answer: $1 - (5/6)^{10}$! But **how**? We will come back to this problem as we study independence.

A car and two goats

On a game show there are three doors. There is a car behind one of the doors and goats (!) behind the others.

The contestant chooses a door. Then the host opens a **different** door behind which there is **always a goat**. The contestant is then given a choice whether to **switch** to the **third** door or not.

The contestant wins as a prize whatever is behind the door she or he chooses. Is it to the contestant's **benefit** to switch doors?

The surprising and counterintuitive answer is **"yes"!**

This problem has a fascinating history (see the segment entitled "The Monty Hall problem").

The argument for "no" relies on assuming that the contestant's decision is **independent** of the choice of door opened first.

Module 8.2: Independence

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LECTURE NOTES

Independent events

Let $(\Omega, \Pr)$ be a probability space. Two events $A, B \subseteq \Omega$ are **independent**, write $A \perp B$, when $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$.

Note that $A \perp B$ iff $B \perp A$ (independence is **symmetric**).

Independence can be **checked** when we have an alternative way of computing $\Pr[A \cap B]$. This does not happen often.

Still, we can do such checking, for example, in the space of rolls of a fair die, or in the space of random permutations, see next.

Much more often, independence is **assumed** based on our intuition about the problem. Then, it allows us to calculate probabilities by multiplication. We will give such examples also.

ACTIVITY : Checking independence

We roll a fair die twice. As we did before, we assume that corresponding probability space is uniform with 36 outcomes.

Consider the event $A =$ "the **first** roll shows a number divisible by 2" and the event $B =$ "the **second** roll shows a number divisible by 3".

Question. Do you think that events A and B are independent?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : Checking independence (continued)

Answer. Yes.

In fact, even before we defined formally independence, the assumption that probability spaces such as the one we use here are uniform relies on our intuition that the die rolls do not interfere with or influence each other, an intuitive manifestation of independence.

But does the formal definition fit this intuition?

We are about to check this.

ACTIVITY : Checking independence (continued)

Counting the outcomes in A we have 3 possibilities for the first roll and 6 for the second, thus $3 \cdot 6 = 18$ by the multiplication rule. Therefore $\Pr[B] = 18/36 = 1/2$.

Counting the outcomes in B we have 6 possibilities for the first roll and 2 for the second, thus $6 \cdot 2 = 12$ by the multiplication rule. Therefore $\Pr[B] = 12/36 = 1/3$.

Counting the outcomes in $A \cap B$ we have 3 possibilities for the first roll and 2 for the second, thus $3 \cdot 2 = 6$ by the multiplication rule. Therefore $\Pr[A \cap B] = 6/36 = 1/6$.

Now $\Pr[A \cap B] = 1/6 = (1/2)(1/3) = \Pr[A] \cdot \Pr[B]$ therefore $A \perp B$.

Independence and random permutations

Problem. Assume that all permutations of a, b, c are equally likely. Are the events $E =$ "a occurs in position 1" and $F =$ "b occurs in position 2" independent?

Answer. In a previous segment we calculated that the probability of a given object occurring in a given position in a random permutation is $1/n$.

Here $n = 3$ therefore $\Pr[E] = \Pr[F] = 1/3$.

However, $E \cap F$ consists of a single outcome, abc, because fixing the positions of a and of b also determines the position of c.

Therefore $\Pr[E \cap F] = 1/3! = 1/6$ but $\Pr[E] \cdot \Pr[F] = 1/9$

E and F are **not** independent.

Properties of independence I

Consider an arbitrary probability space $(\Omega, \Pr)$ and arbitrary events E, A, B in this space.

Property Ind (i). If $\Pr[A] = 0$ then $A \perp B$ for any B.

In particular, $\emptyset \perp E$ for any E.

Proof. $A \cap B \subseteq A$ so by **P3** (monotonicity) $\Pr[A \cap B] \leq \Pr[A] = 0$. If $\Pr[A] = 0$ then $\Pr[A \cap B] = 0$. $A \perp B$ follows.

Property Ind (ii). $\Omega \perp E$ for any E.

The proof is in the segment entitled "Proofs of independence properties".

Properties of independence II

Consider an arbitrary probability space (Ω, Pr) and arbitrary events E, A, B in this space.

Property Ind (iii). If $A \perp B$

then $\mathrm{Pr}[A \cup B] = 1 - (1 - \mathrm{Pr}[A])(1 - \mathrm{Pr}[B])$.

Proof. By **P6** (inclusion-exclusion) and using independence, the LHS becomes

. $\mathrm{Pr}[A \cup B] = \mathrm{Pr}[A] + \mathrm{Pr}[B] - \mathrm{Pr}[A \cap B] = \mathrm{Pr}[A] + \mathrm{Pr}[B] - \mathrm{Pr}[A] \cdot \mathrm{Pr}[B]$

By algebraic manipulation this equals the RHS.

Property Ind (iv). $A \perp B$ iff $\overline{A} \perp B$ iff $A \perp \overline{B}$ iff $\overline{A} \perp \overline{B}$

The proof is in the segment entitled "Proofs of independence properties".

Independent vs. disjoint

Don't confuse "independent" with "disjoint"! In fact, disjoint events are typically **not** independent of each other!

Proposition. Let A, B be disjoint events in $(\Omega, \Pr)$. If $A \perp B$ then at least one of A, B has probability 0.

Proof. If A, B are both disjoint and independent then by **Ind (iii)** and by **P2** (addition rule) we have:

. $1 - (1 - \Pr[A])(1 - \Pr[B]) = \Pr[A \cup B] = \Pr[A] + \Pr[B]$

. $\Pr[A] + \Pr[B] - \Pr[A] \cdot \Pr[B] = \Pr[A] + \Pr[B]$

. $\Pr[A] \cdot \Pr[B] = 0.$

Corollary. If $E \perp \overline{E}$ then $\Pr[E]$ is 0 or 1.

Quiz

We know from **Ind (i)** and **Ind (ii)** that $\emptyset \perp \emptyset$ and $\Omega \perp \Omega$. Are $\emptyset$ and Ω the **only** events E such that $E \perp E$?

(A) True

(B) False

Answer

We know from **Ind (i)** and **Ind (ii)** that $\emptyset \perp \emptyset$ and $\Omega \perp \Omega$. Are $\emptyset$ and Ω the **only** events E such that $E \perp E$?

(A) True

Incorrect. Any other event of probability 0 or 1 would have this property.

(B) False

Correct. We can construct a space with an non-empty event of probability 0.

More information

Let $p = \Pr[E]$. Since $E \cap E = E$ it follows that $E \perp E$ iff $p = p^2$. This last holds iff $p = 0$ or $p = 1$.

Now consider the probability space $(\Omega, \Pr)$ where $\Omega = \{w_1, w_2\}$ and also $\Pr[w_1] = 0$ as well as $\Pr[w_2] = 1$.

In this space, taking $E = \{w_1\}$ we have $\Pr[E] = 0$ therefore $E \perp E$.

Module 8.3: Pairwise and Mutual Independence

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Lecture Notes

Two independent Bernoulli trials

Problem. We perform **independently** two Bernoulli trials, with **identical** probabilities of success p and failure $q = 1 - p$. Compute the probability of the event "one success and one failure are observed, in any order".

Answer. Let $\Omega = \{\mathtt{SS}, \mathtt{SF}, \mathtt{FS}, \mathtt{FF}\}$ where $\mathtt{S}$ stands for success and $\mathtt{F}$ for failure in a Bernoulli trial.

Since the trials are declared **independent** we define $(\Omega, \Pr)$ by **multiplying** probabilities of success/failure:

. $\Pr[\mathtt{SS}] = p \cdot p = p^2 \quad \Pr[\mathtt{SF}] = \Pr[\mathtt{FS}] = p \cdot q = pq \quad \Pr[\mathtt{FF}] = q \cdot q = q^2$

The probability of the event of interest is $\Pr[\mathtt{SF}] + \Pr[\mathtt{FS}] = pq + pq = 2pq$

Recall the biased coin problem solved with urns in a previous segment. There, $p = 1/3$ so $2pq = 2(1/3)(1 - 1/3) = 2(1/3)(2/3) = 4/9$. Same answer.

Independence for three events?

Proposition. There exist three events, A, B, C, in some space, such that $A \perp B,\ B \perp C,\ C \perp A$ but $\Pr[A \cap B \cap C] \neq \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$.

Proposition. There exist three events, A, B, C, in some space, such that $\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$ but $A \not\perp B,\ B \not\perp C,\ C \not\perp A$

The proofs are shown in the segment entitled "Counterexamples for independence of three events".

We are therefore justified in introducing two kinds of independence for three or more events, see next.

Pairwise and mutual independence

Events $A_1, \ldots, A_n$ are called **pairwise independent** when for any $1 \le i < j \le n$ we have $A_i \perp A_j$.

Events $A_1, \ldots, A_n$ are called **mutually independent** when for any $\{i_1, \ldots, i_k\} \subseteq [1..n]$ we have

$$\Pr[A_{i_1} \cap \cdots \cap A_{i_k}] = \Pr[A_{i_1}] \cdots \Pr[A_{i_k}]$$

Mutual independence implies pairwise independence but the converse is not true, as we saw in the first proposition on the previous slide.

Multiple IID Bernoulli trials

Problem. We perform n **mutually independent** and **identically** distributed Bernoulli trials (they all have probability of success p and failure $q = 1 - p$). Compute the probability of the event "at least one success and at least one failure are observed".

Answer. The outcomes are the 2^n sequences of $\mathtt{S}$'s and $\mathtt{F}$'s.

Since the trials are **mutually independent** we **multiply** the probabilities: $\Pr[s] = p^k q^{n-k}$ where k is the number of $\mathtt{S}$'s in s. Not uniform!

Let E be the event of interest.

$\overline{E}$ has just two outcomes: the sequence of all $\mathtt{S}$'s and that of all $\mathtt{F}$'s.

Hence, $\Pr[\overline{E}] = p^n + q^n$.

Thus, by **P4**, $\Pr[E] = 1 - p^n - q^n$.

Multiple balls into three bins

Problem. k balls are thrown **mutually independently** into **three** bins, A, B, C. Each ball is **equally likely** to fall in A, B or C. Show that the resulting probability space is uniform.

Answer. The outcomes are the 3^k sequences of length k built with the letters A, B, C. By mutual independence we **multiply** the probabilities, much as we did for multiple IID Bernoulli trials. For an outcome s:

. $$\Pr[s] = (1/3)^a(1/3)^b(1/3)^c = (1/3)^{a+b+c}$$

where a is the number of A's, b is the number of B's, and c is the number of C's in s. But $a + b + c = k$. Each outcome has probability $(1/3)^k$.

Note that the assumption that each ball is equally likely to fall into one or another of the bins is crucial for uniformity.

Unions of mutually independent events

Proposition. **(generalizes Ind (iii))** Let $A_1, \ldots, A_n$ be mutually independent events in the same probability space. Then we have

$$\Pr[A_1 \cup \cdots \cup A_n] \;=\; 1 \;-\; (1 - \Pr[A_1]) \cdots (1 - \Pr[A_n])$$

Proof. We could use induction. We present a more interesting method, for the case $n = 3$. We use **P4** and a De Morgan law for sets:

$\overline{A \cup B \cup C} \;=\; \overline{A} \cap \overline{B} \cap \overline{C}$ together with a generalization of **Ind (iv)**:

Lemma. A, B, C are mutually independent iff $\overline{A}, \overline{B}, \overline{C}$ are mutually independent.

In the following activity we will guide you through this alternative proof. It will be followed by a quiz that applies this proposition.

Activity : An alternative proof

First we show why it's a De Morgan law. For each outcome w

$$
\begin{aligned}
w \in \overline{A \cup B \cup C} &\Leftrightarrow \neg(w \in A \cup B \cup C) \\
&\Leftrightarrow \neg(w \in A \vee w \in B \vee w \in C) \\
&\Leftrightarrow w \notin A \wedge w \notin B \wedge w \notin C \\
&\Leftrightarrow w \in \overline{A} \wedge w \in \overline{B} \wedge w \in \overline{C} \\
&\Leftrightarrow w \in \overline{A} \cap \overline{B} \cap \overline{C}
\end{aligned}
$$

ACTIVITY : An alternative proof (continued)

Now we get back to the proof of the proposition. Using the DeMorgan law for sets as well as property **P4**:

$$\Pr[A \cup B \cup C] \;=\; 1 \;-\; \Pr[\overline{A \cup B \cup C}] \;=\; 1 \;-\; \Pr[\overline{A} \cap \overline{B} \cap \overline{C}]$$

By the Lemma, $\overline{A}, \overline{B}, \overline{C}$ are also mutually independent. Therefore:

$$\Pr[\overline{A} \cap \overline{B} \cap \overline{C}] \;=\; \Pr[\overline{A}] \cdot \Pr[\overline{B}] \cdot \Pr[\overline{C}] \;=\; (1 - \Pr[A])(1 - \Pr[B]) \cdot (1 - \Pr[C])$$

Putting these together we are done.

Quiz

A fair die is rolled 10 times such that the rolls are mutually independent. What is the probability that at least one roll shows a six?

(A) $1 - (5/6)^{10}$

(B) $1/6^{10}$

Answer

(A) $1 - (5/6)^{10}$

Correct. This follows from the proposition we have just proved.

(B) $1/6^{10}$

Incorrect. This is the probability that **all** rolls show a six.

More information I

Recall the proposition we have just proved:

Proposition. (generalizes Ind (iii)) Let $A_1, \ldots, A_n$ be mutually independent events in the same probability space. Then we have

$$\Pr[A_1 \cup \cdots \cup A_n] \;=\; 1 \;-\; (1 - \Pr[A_1]) \cdots (1 - \Pr[A_n])$$

We apply this to 10 mutually independent rolls of a fair die. Event $A_i,\ i = 1, \ldots, 10$ corresponds to a six showing in roll i.

Our event of interest is $A_1 \cup \cdots \cup A_{10}$.

As we computed before $\Pr[A_i] = 1/6$ hence $1 - \Pr[A_i] = 5/6$.

Now the proposition gives us $\Pr[A_1 \cup \cdots \cup A_{10}] \;=\; 1 \;-\; (5/6)^{10}$

More information II

This also justifies the answer we gave in the lecture segment "Inclusion-exclusion for probability" to the problem: We roll 10 fair dice. What is the probability that at least one die shows a six?

That's because rolling 10 fair dice independently and rolling a fair die 10 times independently both give raise to the same probability space.

Module 8.4: Sketching the Monty Hall Problem

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Lecture Notes

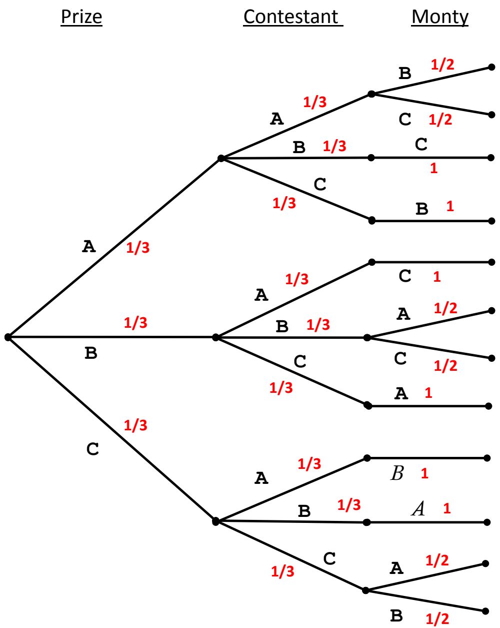
A car and two goats, again!

The Monty Hall Problem. On a game show (hosted by Monty Hall) there are three doors. There is a car behind one of the doors and goats behind the others. The contestant (Ann) chooses a door. Then Monty opens a **different** door behind which there is **always a goat**. Ann can choose whether to **switch** to the **third** door. Is it to Ann's benefit to switch doors?

Answer. We only sketch the answer. For full details see the segment entitled "The Monty Hall problem". Let the doors be `A`,`B` and `C`. The probability space that Ann analyzes has outcomes of the form $D_C D_A D_M$ where D_C is the door that hides the car, D_A is the door Ann chose, and D_M is the door opened by Monty. Because Monty never opens D_C only the following 12 outcomes are possible:

`AAB, AAC, ABC, ACB, BAC, BBA, BBC, BCA, CAB, CBA, CCA, CCB`

Tree of all possibilities



Outcome	Probability
AAB	1/18
AAC	1/18
ABC	1/9
ACB	1/9
BAC	1/9
BBA	1/18
BBC	1/18
BCA	1/9
CAB	1/9
CBA	1/9
CCA	1/18
CCB	1/18

Assumptions

(a) The car is placed by game staff behind one of the three doors with **equal likelihood**.

(b) Ann does not know where the car is, she chooses her door **independently** of that.

Moreover, we assume that she chooses one of the three doors with **equal likelihood**.

(c) Monty must never reveal the car, so his action does **depend** on where the car was placed and on which door Ann opens.

Moreover, if it turns out that Monty has a choice among two doors, we assume that he opens one of them with **equal likelihood**.

Should the contestant switch doors?

Problem (continued). What is the probability that Ann wins the car if she switches doors? What is the probability that Ann wins the car if she stays with the door she chose first?

Answer (continued). The outcomes contain all the information. For example, in outcome `ABC` Ann wins the car if she switches doors, while in outcome `AAB` she wins the car if she stays with her first choice.

The event of interest is $E =$ "win if switch." From the tree:

$$E = \{\mathtt{ABC}, \mathtt{ACB}, \mathtt{BAC}, \mathtt{BCA}, \mathtt{CAB}, \mathtt{CBA}\}$$

$$\Pr[E] \; = \; 1/9 + 1/9 + 1/9 + 1/9 + 1/9 + 1/9 \; = \; 6 \cdot (1/9) \; = \; 2/3$$

On the other hand $\Pr[\overline{E}] \; = \; 1/3$. Therefore Ann improves her chances of winning the car (from $1/3$ to $2/3$) if she switches doors.

Module 8.5: Conditional Probability

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Lecture Notes

Conditional probability

Given a probability space (Ω, P) we are interested in an event E in a context in which we already know (for sure!) that another event, U has happened (or is happening).

We write this as $E \mid U$ and we read it as E **conditioned** on U.

The **conditional probability** that we assign to this situation is denoted and defined by

$$\Pr[E|U] = \frac{\sum_{w \in E \wedge w \in U} \Pr[w]}{\sum_{w \in U} \Pr[w]} = \frac{\Pr[E \cap U]}{\Pr[U]} \qquad \text{(provided } \Pr[U] \neq 0)$$

When $\Pr[U] = 0$ the conditional probability $\Pr[E|U]$ is **undefined**.

One heads or two? (I)

Problem. Alice flips a fair coin twice, independently. Bob did not see the flips but Alice tells him (variant 1) that the **first flip** was heads and asks him the probability that **both** flips were heads. Bob correctly answers $1/2$ (why?). But what if Alice tells him (variant 2) that **at least one of the flips** was heads and asks him the probability that both were heads?

Answer. In variant 1 Bob can simply reason that two heads show iff the second flip is heads. By independence, this happens with probability $1/2$.

For variant 2, Bob might be tempted to answer $1/2$ as well, as the probability of the **other** flip also being heads. This would be OK if Bob knew that the other flip is, say, the second one, as we saw in variant 1. But Bob does not know that.

One heads or two? (II)

Answer (continued). In effect, in both variants Bob is asked to compute a **conditional probability**.

Let E be the event "both flips are heads", F be the event "first flip is heads", and let G be the event "at least one of the flips is heads", that is, $\{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}\}$.

In variant 1 Bob is asked to calculate

$$\Pr[E|F] = \Pr[E \cap F]/\Pr[F] = (1/4)/(1/2) = 1/2$$

Bob's reasoning led to the correct result.

In variant 2 Bob is asked to calculate

$$\Pr[E|G] = \Pr[E \cap G]/\Pr[G] = (1/4)/(3/4) = 1/3$$

This is **different** from the $1/2$ that Bob might impetuously answer.

Testing for a rare disease I

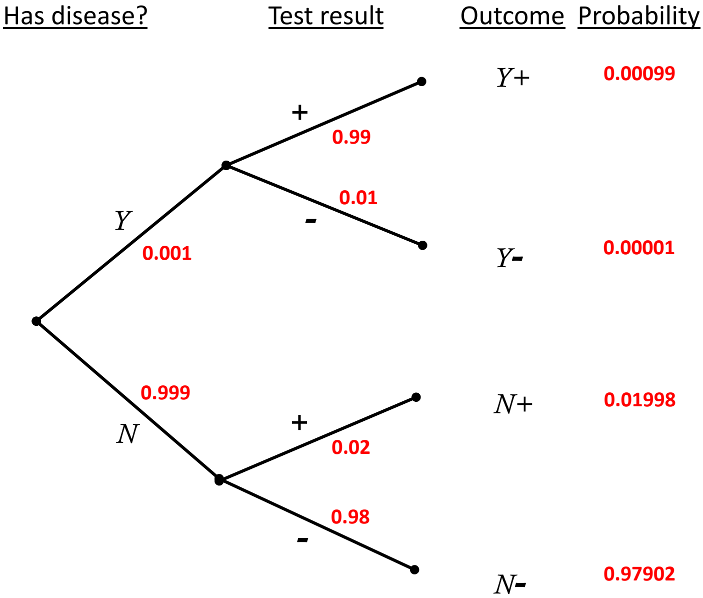
Problem. A test for a rare disease that affects 0.1% of the population is 99% effective on people with the disease (i.e., it gives a **false negative** with probability 0.01). On people who do not suffer from the disease the test gives a **false positive** with probability 0.02. What is the probability that someone who tests positive does, in fact, have the disease?

Answer. Given a person, V, we have two sources of randomness: V has the disease, Y(es) or N(o), and V tests positive $+$ or negative $-$.

Thus we have four outcomes, which we denote $Y+, Y-, N+, N-$ and correspondingly we have four events: $Y = \{Y+, Y-\}$, $T_+ = \{Y+, N+\}$, $N = \{N+, N-\}$, $T_- = \{Y-, N-\}$. Clearly, they are **not** independent!

We use the false negative and false positive statistics as estimates for conditional probabilities: $\Pr[T_- | Y] = 0.01$, $\Pr[T_+ | N] = 0.02$.

Testing for a rare disease II



Testing for a rare disease III

Answer (continued). Let's first show that multiplying along the branches is justified by the conditional probability definition:

$\Pr[T_-|Y] = \Pr[T_- \cap Y]/\Pr[Y]$ therefore $\Pr[T_- \cap Y] = \Pr[Y] \cdot \Pr[T_-|Y]$

So $\Pr[Y-] \;=\Pr[T_- \cap Y] = \Pr[Y] \cdot \Pr[T_-|Y] = 0.001 \cdot 0.01 = 0.00001$

Back to the problem. It asks for the probability that V actually has the disease, knowing that V tested positive. A conditional probability!

$$\begin{aligned}\Pr[Y|T_+] \;&=\; \Pr[Y \cap T_+]/\Pr[T_+] \;=\; \Pr[Y+]/(\Pr[Y+] + \Pr[N+]) \\ &=\; (0.00099)/(0.00099 + 0.01998) \;=\; 0.0472\end{aligned}$$

How long will your battery last?

Problem. Consumers have reported that 80% of new car batteries still work after 10,000 miles and that 40% of them still work after 20,000 miles. If your car battery still works after 10,000 miles, what is the probability that it will last another 10,000 miles?

Answer. Consider the events $L_1 =$ "battery still works after 10,000 mi" and $L_2 =$ "battery still works after 20,000 mi".

Based on the consumer reports we estimate $\Pr[L_1] = 0.8$ and $\Pr[L_2] = 0.4$.

The problem asks for the conditional probability $\Pr[L_2 | L_1]$.

Note that $L_2 \subseteq L_1$ therefore $L_2 \cap L_1 = L_2$.

So $\Pr[L_2 | L_1] = \Pr[L_2 \cap L_1] / \Pr[L_1] = \Pr[L_2] / \Pr[L_1] = 0.4/0.8 = 0.5$.

ACTIVITY : A tree for car batteries

In this activity we will set up a tree of all possibilities for a problem we just solved with another method that did not require us to describe the underlying probability space. We repeat the problem here.

Problem. Consumers have reported that 80% of new car batteries still work after 10,000 miles and that 40% of them still work after 20,000 miles. If your car battery still works after 10,000 miles, what is the probability that it will last another 10,000 miles?

Question. How many outcomes do you think we will use in the probability space?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : A tree for car batteries (continued)

Answer. There are **three** outcomes, corresponding to the following:

(w_1) The battery still works after 20,000mi.

(w_2) The battery did not last 20,000mi but it still worked after 10,000mi.

(w_3) The battery did not last 10,000mi.

This is because we identify the following two sources of randomness. The first is whether the battery lasts 10,000mi or not. The second only affects the battery life after it has lasted 10,000mi and it is whether it will last another 10,000mi or not.

ACTIVITY : A tree for car batteries (continued)

Now consider the events $L_1 =$ "battery still works after 10,000mi" and $L_2 =$ "battery still works after 20,000mi".

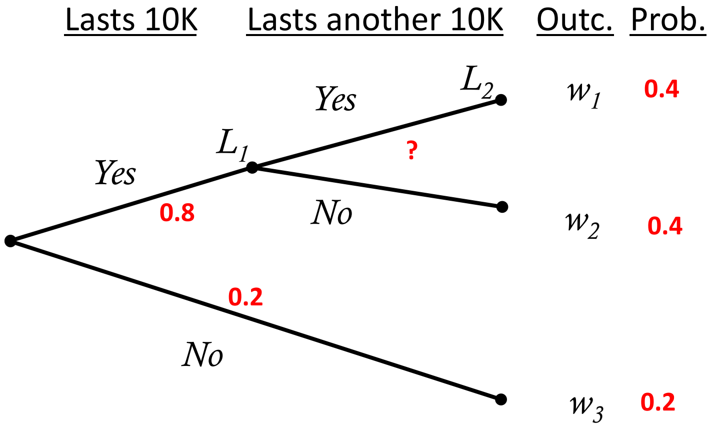
In terms of the outcomes we have identified we have $L_1 = \{w_1, w_2\}$ and $L_2 = \{w_1\}$.

The problem tells us that $\Pr[L_1] = 0.8$ and $\Pr[L_2] = 0.4$.

Therefore we know that $\Pr[w_1] = 0.4$. We can then calculate from $\Pr[L_1] = \Pr[w_1] + \Pr[w_2]$ that $\Pr[w_2] = 0.8 - 0.4 = 0.4$. Also, $\Pr[w_3] = 1 - (\Pr[w_1] + \Pr[w_2]) = 1 - 0.8 = 0.2$.

This leads to the following tree of all possibilities

Activity : A tree for car batteries (continued)



Note that the problem does not give probabilities for the branches that split out of L_1. In fact, it **asks** for one of them (where we put a question mark).

ACTIVITY : A tree for car batteries (continued)

However, recalling how we computed outcome probabilities in the tree of all possibilities for the Monty Hall problem we can recover the answer.

Let p be the probability that labels the Yes branch out of L_1 (the one where we put a question mark in the tree).

Since the outcome probabilities are computed by multiplying along branches we must have $(0.8) \cdot p = 0.4$. Hence $p = (0.4)/(0.8) = 0.5$.

The problem asked: If your car battery still works after 10,000 miles, what is the probability that it will last another 10,000 miles?

That's exactly p and the answer is 0.5 just as in the first solution.

Module 8.6: The Chain Rule

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Independence and conditional probability

Intuitively, $\Pr[E|U]$ should be the same as $\Pr[E]$ when E does not really depend on U. Indeed:

Proposition. For any two events A, B in the same probability space the following two statements are **equivalent**:

(i) $A \perp B$ (ii) $\Pr[B] = 0$ or ($\Pr[B] \neq 0$ and $\Pr[A|B] = \Pr[A]$)

Proof. To prove the logical equivalence of (i) and (ii) we have to show that: (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Assume $A \perp B$. When $\Pr[B] \neq 0$ we have

$$\Pr[A|B] = \Pr[A \cap B]/\Pr[B] = (\Pr[A] \cdot \Pr[B])/\Pr[B] = \Pr[A]$$

(ii) $\Rightarrow$ (i): If $\Pr[B] = 0$ then by **Ind (i)** $A \perp B$. If $\Pr[B] \neq 0$ then $\Pr[A|B] = \Pr[A]$ becomes $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$ hence $A \perp B$.

The chain rule

We regard the equality $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B|A]$ as **generalizing** the equality $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$ that defines independence. For more than two events we have a further generalization:

Proposition (The chain rule). For any events A, B, C in the same probability space we have

$$\Pr[A \cap B \cap C] \;=\; \Pr[A] \cdot \Pr[B|A] \cdot \Pr[C|A \cap B]$$

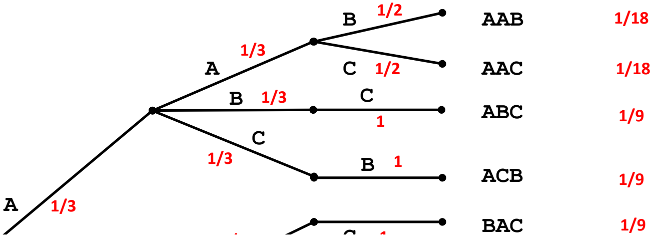
For any events $A_1, \ldots, A_n$ in the same probability space we have

$$\begin{aligned} \Pr[A_1 \cap A_2 \cap A_3 \cdots \cap A_n] \;&= \\ &= \; \Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2] \cdots \Pr[A_n|A_1 \cap \cdots \cap A_{n-1}] \end{aligned}$$

The proof is given in the segment entitled "Conditional probability rules."

Tree diagrams and the chain rule I

Here is the upper part of the Monty Hall "tree of all possibilities":



Why is $\Pr[\texttt{AAB}] = 1/18$?

Define $E =$ "car is behind door $\texttt{A}$", $F =$ "Ann chooses door $\texttt{A}$",
$G =$ "Monty opens door $\texttt{B}$".

Then $E \cap F \cap G = \{\texttt{AAB}\}$.

The chain rule gives: $\Pr[E \cap F \cap G] \; = \; \Pr[E] \cdot \Pr[F | E] \cdot \Pr[G | E \cap F]$

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Tree diagrams and the chain rule II

The chain rule gives: $\Pr[E \cap F \cap G] = \Pr[E] \cdot \Pr[F|E] \cdot \Pr[G|E \cap F]$

We have $\Pr[E] = 1/3$

We have $\Pr[F|E] = \Pr[F] = 1/3$

And we have $\Pr[G|F \cap E] = 1/2$

Therefore $\Pr[\texttt{AAB}] = (1/3)(1/3)(1/2) = 1/18.$

In general, the branches are labeled with conditional probabilities and along each branch the chain rule computes the probability of the outcome.