Module 8.3: Pairwise and Mutual Independence

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Two independent Bernoulli trials

Problem. We perform **independently** two Bernoulli trials, with **identical** probabilities of success p and failure $q = 1 - p$. Compute the probability of the event "one success and one failure are observed, in any order".

Answer. Let $\Omega = \{\mathtt{SS}, \mathtt{SF}, \mathtt{FS}, \mathtt{FF}\}$ where $\mathtt{S}$ stands for success and $\mathtt{F}$ for failure in a Bernoulli trial.

Since the trials are declared **independent** we define $(\Omega, \Pr)$ by **multiplying** probabilities of success/failure:

. $\Pr[\mathtt{SS}] = p \cdot p = p^2 \quad \Pr[\mathtt{SF}] = \Pr[\mathtt{FS}] = p \cdot q = pq \quad \Pr[\mathtt{FF}] = q \cdot q = q^2$

The probability of the event of interest is $\Pr[\mathtt{SF}] + \Pr[\mathtt{FS}] = pq + pq = 2pq$

Recall the biased coin problem solved with urns in a previous segment. There, $p = 1/3$ so $2pq = 2(1/3)(1 - 1/3) = 2(1/3)(2/3) = 4/9$. Same answer.

Independence for three events?

Proposition. There exist three events, A, B, C, in some space, such that $A \perp B,\ B \perp C,\ C \perp A$ but $\Pr[A \cap B \cap C] \neq \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$.

Proposition. There exist three events, A, B, C, in some space, such that $\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$ but $A \not\perp B,\ B \not\perp C,\ C \not\perp A$

The proofs are shown in the segment entitled "Counterexamples for independence of three events".

We are therefore justified in introducing two kinds of independence for three or more events, see next.

Pairwise and mutual independence

Events $A_1, \ldots, A_n$ are called **pairwise independent** when for any $1 \leq i < j \leq n$ we have $A_i \perp A_j$.

Events $A_1, \ldots, A_n$ are called **mutually independent** when for any $\{i_1, \ldots, i_k\} \subseteq [1..n]$ we have

$$\Pr[A_{i_1} \cap \cdots \cap A_{i_k}] = \Pr[A_{i_1}] \cdots \Pr[A_{i_k}]$$

Mutual independence implies pairwise independence but the converse is not true, as we saw in the first proposition on the previous slide.

Multiple IID Bernoulli trials

Problem. We perform n **mutually independent** and **identically** distributed Bernoulli trials (they all have probability of success p and failure $q = 1 - p$). Compute the probability of the event "at least one success and at least one failure are observed".

Answer. The outcomes are the 2^n sequences of $\mathtt{S}$'s and $\mathtt{F}$'s.

Since the trials are **mutually independent** we **multiply** the probabilities: $\Pr[s] = p^k q^{n-k}$ where k is the number of $\mathtt{S}$'s in s. Not uniform!

Let E be the event of interest.

$\overline{E}$ has just two outcomes: the sequence of all $\mathtt{S}$'s and that of all $\mathtt{F}$'s.

Hence, $\Pr[\overline{E}] = p^n + q^n$.

Thus, by **P4**, $\Pr[E] = 1 - p^n - q^n$.

Multiple balls into three bins

Problem. k balls are thrown **mutually independently** into **three** bins, A, B, C. Each ball is **equally likely** to fall in A, B or C. Show that the resulting probability space is uniform.

Answer. The outcomes are the 3^k sequences of length k built with the letters A, B, C. By mutual independence we **multiply** the probabilities, much as we did for multiple IID Bernoulli trials. For an outcome s:

. $$\Pr[s] = (1/3)^a(1/3)^b(1/3)^c = (1/3)^{a+b+c}$$

where a is the number of A's, b is the number of B's, and c is the number of C's in s. But $a+b+c=k$. Each outcome has probability $(1/3)^k$.

Note that the assumption that each ball is equally likely to fall into one or another of the bins is crucial for uniformity.

Unions of mutually independent events

Proposition. **(generalizes Ind (iii))** Let $A_1, \ldots, A_n$ be mutually independent events in the same probability space. Then we have

$$\Pr[A_1 \cup \cdots \cup A_n] \; = \; 1 \; - \; (1 - \Pr[A_1]) \cdots (1 - \Pr[A_n])$$

Proof. We could use induction. We present a more interesting method, for the case $n = 3$. We use **P4** and a De Morgan law for sets:

$\overline{A \cup B \cup C} \; = \; \overline{A} \cap \overline{B} \cap \overline{C}$ together with a generalization of **Ind (iv)**:

Lemma. A, B, C are mutually independent iff $\overline{A}, \overline{B}, \overline{C}$ are mutually independent.

In the following activity we will guide you through this alternative proof. It will be followed by a quiz that applies this proposition.

Activity : An alternative proof

First we show why it's a De Morgan law. For each outcome w

$$\begin{aligned} w \in \overline{A \cup B \cup C} &\Leftrightarrow \neg(w \in A \cup B \cup C) \\ &\Leftrightarrow \neg(w \in A \vee w \in B \vee w \in C) \\ &\Leftrightarrow w \notin A \wedge w \notin B \wedge w \notin C \\ &\Leftrightarrow w \in \overline{A} \wedge w \in \overline{B} \wedge w \in \overline{C} \\ &\Leftrightarrow w \in \overline{A} \cap \overline{B} \cap \overline{C} \end{aligned}$$

$\textsc{Activity}$: An alternative proof (continued)

Now we get back to the proof of the proposition. Using the DeMorgan law for sets as well as property **P4**:

$$\Pr[A \cup B \cup C] = 1 - \Pr[\overline{A \cup B \cup C}] = 1 - \Pr[\overline{A} \cap \overline{B} \cap \overline{C}]$$

By the Lemma, $\overline{A}, \overline{B}, \overline{C}$ are also mutually independent. Therefore:

$$\Pr[\overline{A} \cap \overline{B} \cap \overline{C}] = \Pr[\overline{A}] \cdot \Pr[\overline{B}] \cdot \Pr[\overline{C}] = (1 - \Pr[A])(1 - \Pr[B]) \cdot (1 - \Pr[C])$$

Putting these together we are done.

Quiz

A fair die is rolled 10 times such that the rolls are mutually independent. What is the probability that at least one roll shows a six?

(A) $1 - (5/6)^{10}$

(B) $1/6^{10}$

Answer

(A) $1 - (5/6)^{10}$

Correct. This follows from the proposition we have just proved.

(B) $1/6^{10}$

Incorrect. This is the probability that **all** rolls show a six.

More information I

Recall the proposition we have just proved:

Proposition. (generalizes Ind (iii)) Let $A_1, \ldots, A_n$ be mutually independent events in the same probability space. Then we have

$$\Pr[A_1 \cup \cdots \cup A_n] \;=\; 1 \;-\; (1 - \Pr[A_1]) \cdots (1 - \Pr[A_n])$$

We apply this to 10 mutually independent rolls of a fair die. Event A_i, $i = 1, \ldots, 10$ corresponds to a six showing in roll i.

Our event of interest is $A_1 \cup \cdots \cup A_{10}$.

As we computed before $\Pr[A_i] = 1/6$ hence $1 - \Pr[A_i] = 5/6$.

Now the proposition gives us $\Pr[A_1 \cup \cdots \cup A_{10}] \;=\; 1 \;-\; (5/6)^{10}$

More information II

This also justifies the answer we gave in the lecture segment "Inclusion-exclusion for probability" to the problem: We roll 10 fair dice. What is the probability that at least one die shows a six?

That's because rolling 10 fair dice independently and rolling a fair die 10 times independently both give raise to the same probability space.