Module 6.2: Induction Examples

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

We prove an inequality by induction I

Problem. Prove $\forall\, n \geq 5 \;\; n^2 < 2^n$.

Answer. First of all, where did 5 come from?
We just tried the first natural numbers one after the other (see the adjacent table). We see that $n^2 < 2^n$ moves between true and false until $n = 5$ and from there on it seems to settle to true. So we **guessed** 5 as a start. And we hope the proof will validate this guess!

n	n^2	2^n	$n^2 < 2^n$
0	0	1	T
1	1	2	T
2	4	4	F
3	9	8	F
4	16	16	F
5	25	32	T
6	36	64	T
7	49	128	T
8	64	256	T

We prove an inequality by induction II

Answer (continued). We use the variant pattern with $n_0 = 5$.

(BC) $(n = 5)$ $5^2 = 25 < 32 = 2^5$. Check.

(IS) Let $k \geq 5$.

Assume (IH) $k^2 < 2^k$. (And WTS $(k+1)^2 < 2^{k+1}$.)

Let's collect what we know:

$$\begin{array}{rcll} k^2 & < & 2^k & \qquad\qquad (\text{IH}) \\ (k+1)^2 & = & k^2 + 2k + 1 & \\ 2^{k+1} & = & 2 \cdot 2^k \ = \ 2^k + 2^k & \end{array}$$

From these we observe that it would suffice to also show $2k + 1 \leq 2^k$. (Why?)

We prove an inequality by induction III

Answer (continued). We will prove a lemma and that will do it.

Lemma. For all $m \in \mathbb{N}$, $m \geq 5$ we have $2m + 1 \leq 2^m$.

Proof of lemma. By ... induction!

(BC) $(m = 5)$ $2 \cdot 5 + 1 = 11 \leq 32 = 2^5$. Check.

(IS) Let $\ell \geq 5$.

Assume (IH) $2\ell + 1 \leq 2^\ell$. (And WTS $2(\ell + 1) + 1 \leq 2^{\ell+1}$.)

Using the IH we derive the following:

$$2(\ell + 1) + 1 \;=\; (2\ell + 1) + 2 \;\leq\; 2^\ell + 2 \;\leq\; 2^\ell + 2^\ell \;=\; 2^{\ell+1}$$

Done.

Induction can be tricky I

Induction can trick us into proving ... false facts!

False proposition! All the sheep in (my grandfather) Abe's flock were the same color!

Wrong proof! By (careless) induction on the size of the flock.

(BC) (Flock of size 1) One sheep, one color. Check.

(IS) Assume (IH) that in any flock of size k the sheep are all the same color.

WTS the same for any flock of size $k+1$. Let F be such a flock.

Tricky induction II

- See drawing of F in the corresponding video lecture segment.

- See also drawing of $F \setminus \texttt{Dolly}$.

- See also drawing of $F \setminus \texttt{Polly}$.

The proof is written out on the next slide.

Tricky induction III

Wrong proof (continued)!

Let F be a flock of size $k+1$.

Take one sheep, `Dolly`, out. The remaining flock $F \setminus \{\texttt{Dolly}\}$ has size k so by IH they are all the same color.

Now put `Dolly` back in and take another sheep, `Polly`, out.

The flock $F \setminus \{\texttt{Polly}\}$ has size k so by IH all its sheep are the same color.

But $\texttt{Dolly} \in F \setminus \{\texttt{Polly}\}$ so `Dolly` must be the same color as the rest!

The proof **must** be wrong! But where is the error?!?

Quiz

Do you think the error in the preceding proof is

A. In the base case?

B. In the induction hypothesis?

C. In the induction step?

Quiz

Do you think the error in the preceding proof is

A. In the base case?

Incorrect. See third answer then continue the video.

B. In the induction hypothesis?

Incorrect. See third answer then continue the video.

C. In the induction step?

Correct. To see **where** continue the video.

Indeed induction can be tricky I

Wrong proof (explained)!

(IS) First of all we forgot to state "let $k \geq 1$" before we stated the (IH): "in any flock of size k the sheep are all the same color".

Then we considered a flock F of size $k+1$.

Since $k \geq 1$ the flock F has size at least 2 so we are sure to find both a `Dolly` and a `Polly` in it.

The flocks $F \setminus \{\texttt{Dolly}\}$ and $F \setminus \{\texttt{Polly}\}$ have indeed size k so the IH applies to them.

Indeed induction can be tricky II

Wrong proof (explained)!

However, we need to be able to assert that `Dolly` is the same color as the rest of the sheep! In particular that `Dolly` and `Polly` are the same color!

For that there **must exist** at least **one more** sheep in $F \setminus \{\texttt{Polly}\}$ in addition to `Dolly`!

See drawing in the corresponding video lecture segment.

And this is true when $k + 1 \geq 3$ but false when $k + 1 = 2$!

The IS is true for $k \geq 2$ when, in fact, we need it to be true for $k \geq 1$.