Module 1.2: Odd and Even

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Even and odd I

Let's ignore the negative integers.

Here are the **even** numbers: $0, 2, 4, 6, 8, 10, \ldots$

And here are the **odd** ones: $1, 3, 5, 7, 9, 11, \ldots$

Problem. Prove that if the sum of two integers is even then so is their difference.

Since we know even and odd numbers "by example" we could verify this on some values.

$6 + 4 = 10 \qquad 6 - 4 = 2 \qquad\qquad 7 + 3 = 10 \qquad 7 - 3 = 4$

But this is **not** a proof!

We need a mathematical proof for the **general** assertion in the problem.

Even and odd II

We realize that we do not even have mathematical **definitions** for "even" or for "odd"!

Here are the definitions:

An integer n is **even** when $n = 2k$ for *some* integer k.

An integer n is **odd** when $n = 2k + 1$ for *some* integer k.

Examples.

6 is even because $6 = 2 \cdot 3$ and we can take $k = 3$.

9 is odd because $9 = 2 \cdot 4 + 1$ and we can take $k = 4$.

Even and odd III

Problem. Prove that if the sum of two integers is even then so is their difference.

Answer. Reformulate the statement as follows:

> Let m and n be any two integers.
> If $m + n$ is even then $m - n$ is even.

By definition of "even", we have $m + n = 2\ell$, for some integer ℓ.

Then $m = 2\ell - n$.

Now $m - n$ can be written as follows:

. $m - n \; = \; (2\ell - n) - n \; = \; 2\ell - 2n \; = \; 2(\ell - n)$

Since $\ell - n$ is an integer, we take $k = \ell - n$ to satisfy the def. of "even".

We conclude that $m - n$ is even.

Lessons from this "even and odd" story

1. We cannot hope to **prove** a mathematical assertion rigorously unless all the concepts in the statement have an unambiguous mathematical definition.

2. "Verifying" an assertion on a few **examples** is **not** a proof. In fact as we shall see very soon, it can be misleading.

3. What about "rules of thumb" such as "even plus even is even" and "odd minus even is odd", etc. But how do we know that these are true? Prove them! However, we cannot prove everything! In assignments, we will make clear what you can assume and what you still need to prove.

ACTIVITY: Proof using rules of thumb

Assume the following rules of thumb:

- P1: Even plus even is even
- P2: Even plus odd is odd
- P3: Odd plus even is odd
- P4: Odd plus odd is even
- M1: Even minus even is even
- M2: Even minus odd is odd
- M3: Odd minus even is odd
- M4: Odd minus odd is even

Which of the rules above would you use to prove for any even integers m, n that if $m + n$ is even then $m - n$ is even?

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

ACTIVITY: Proof using rules of thumb

Answer.

If m is even and n is even we will use P1 and M1 to prove that if $m + n$ is even then $m - n$ is even.

$m + n$ is even (by P1)
$m - n$ is even (by M1)

ACTIVITY: Proof using rules of thumb

Below we show the full proof of the statement "for any two integers m, n if $m + n$ is even then $m - n$ is even. Consider all four possible cases:

Case 1: m is even and n is even
$m + n$ is even (by P1)
$m - n$ is even (by M1)

Case 2: m is even and n is odd
$m + n$ is odd (by P2)
$m - n$ is odd (by M2)

Case 3: m is odd and n is even
$m + n$ is even (by P3)
$m - n$ is even (by M3)

Case 4: m is odd and n is odd
$m + n$ is odd (by P4)
$m - n$ is odd (by M4)

In all cases we showed that if $m + n$ is even then $m - n$ is even. Thus the statement must be true, since we covered all possible scenarios.