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1. [10 pts] Tautologies are logical expressions that are always true. Decide if the following proposition forms are tautologies using a truth table. Make sure your truth table shows **all** intermediate logical expressions — for example, in showing the truth table for $(p \vee \neg q) \wedge p$, your table should contain separate columns for p, q, $\neg q$, $p \vee \neg q$, as well as the final expression. You should also clearly state your final answer to the question.

(a) $[(\neg p \implies q) \implies (\neg p \wedge q)] \wedge (p \vee q)$

(b) $[p \wedge (q \implies r)] \implies (q \implies r)$

Solution.

(a) Below is the truth table for proposition $[(\neg p \implies q) \implies (\neg p \wedge q)] \wedge (p \vee q)$

p	q	$\neg p$	$\neg p \implies q$	$\neg p \wedge q$	$p \vee q$	$[(\neg p \implies q) \implies (\neg p \wedge q)]$	$[(\neg p \implies q) \implies (\neg p \wedge q)] \wedge$
T	T	F	T	F	T	F	F
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	F	T	F	F	F	T	F

From above we can conclude that Proposition $[(\neg p \implies q) \implies (\neg p \wedge q)] \wedge (p \vee q)$ is not a Tautologies.

(b) Below is the truth table for proposition $[p \wedge (q \implies r)] \implies (q \implies r)$

p	q	r	$q \implies r$	p $\wedge (q \implies r)$	$[p \wedge (q \implies r)] \implies (q \implies r)$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

From above we can conclude that Proposition $[p \wedge (q \implies r)] \implies (q \implies r)$ is a Tautologies.

2. [10 pts] Michael is a manager for his company, which covers multiple regions of the country.

(a) In region 1, there are 5 unique office buildings to which he would like to assign 10 indistinguishable senior workers and 40 indistinguishable entry-level workers. How many ways are there to assign workers to office buildings, such that each office has at least 1 senior worker and 4 entry-level workers?

(b) After some time, region 1 is idle (no jobs) and is looking to steal employees from the 7 other distinguishable regions. Suppose that all 7 other regions are overworked, each having a large number of indistinguishable senior employees. How many ways can region 1 take exactly 23 senior workers from the other regions, such that there are at least 3 regions from which 6 or more senior employees are taken?

Solution.

(a) This is a Stars and Bars question where we have 5 unique office buildings (r=5), and 10 indistinguishable senior workers ($n_1 = 10$), and 40 indistinguishable entry level workers ($n_2 = 40$).

Step 1: We assign at least 1 senior worker to each of the 5 unique office buildings, and there will be $\binom{n_1-r+r-1}{r-1} = \binom{9}{4}$
Step 2: We assign at least 4 senior worker to each of the 5 unique office buildings, and there will be $\binom{n_2-r*4+r-1}{r-1} = \binom{24}{4}$
Step 3: The number of ways to assign at least 1 senior worker and 4 entry level workers would be $\binom{9}{4} * \binom{24}{4}$

(b) This is a Stars and Bars question where we have 7 unique office buildings (r=7), and 23 indistinguishable senior workers ($n = 23$).
Step 1: We assign at least 6 senior workers to any combination of 3 office buildings, and there will be $\binom{n-6*3+r-1}{r-1} = \binom{11}{6}$
Step 2: We calculate how many ways we can pick up 3 office buildings from the 7 office buidlings, and there will be $\binom{7}{3}$ ways.
Step 3: If Region 1 is taking 6 or more senior workers from one individual region, Region 1 can at most take 6 or more senior workers from 3 regions since there are totally 23 senior works taken. Since the question is asking Region one to take 6 or more senior workers from at least 3 regions, we can know that Region 1 is taking 6 or more senior workers from exact 3 regions. Therefore the number of ways to assigned at lease 6 senior workers to is $\binom{11}{6} * \binom{7}{3}$

3. **[10 pts]** Three integers are *consecutive* if they immediately follow each other in enumerating the integers. For example, $-13, -12, -11$; or $5, 6, 7$; or $-2, -1, 0$. Prove that if a, b, c are consecutive integers then $a + b + c$ is divisible by 3 but $a^2 + b^2 + c^2$ is *not* divisible by 3.

Solution.

Step 1 we try to prove if a,b,c are consecutive integers then a+b+c is divisible by 3.
Since a,b,c are consecutive integers, by definition of consecutive we can write a,b,c as n, n+1, n+2 where n is an integer.
$a + b + c = n + (n + 1) + (n + 2)$
= 3n + 3
= 3(n+1)
Since n is an integer, n+1 would also be an integer. Since a+b+c divided by 3 results in n+1, we can conclude a+b+c is divisible by 3.

Step 2 we try to prove if a,b,c are consecutive integers then $a^2 + b^2 + c^2$ is *not* divisible by 3.
Since a,b,c are consecutive integers, by definition of consecutive we can write a,b,c as n, n+1, n+2 where n is an integer.
$a^2 + b^2 + c^2 = n^2 + (n + 1)^2 + (n + 2)^2$
$=n^2 + n^2 + 2n + 1 + n^2 + 4n + 4$
$=3n^2 + 6n + 5$
$=3(n^2 + 2n + 1) + 2$
Since n is an integer, $(n^2 + 2n + 1)$ would also be an integer. Therefore we can conclude that $a^2 + b^2 + c^2$ divided by 3 would have a remainder of 2, which means $a^2 + b^2 + c^2$ is not divisible by 3.

From above two steps we can conclude that if a, b, c are consecutive integers then $a + b + c$ is divisible by 3 but $a^2 + b^2 + c^2$ is *not* divisible by 3.

4. [**10 pts**] How many anagrams of `raspberries` are there that have at least two consecutive `r`'s?

Solution.

The word "raspberries" has 11 letters To make sure there are at least two consecutive r's, we treat the two r's as one letter, then we will have 10 letters. We start with permutations of 10 letters, and that's 10!s.

Then we consider:

(1)Permutation of the 1 r, that's 1! times.

(2)Permutation of the 1 a, that's 1! times.

(3)Permutation of the 2 s's, that's 2! times.

(4)Permutation of the 1 p, that's 1! times.

(5)Permutation of the 1 b, that's 1! times.

(6)Permutation of the 2 e, that's 2! times.

(7)Permutation of the 1 rr, that's 1! times.

(8)Permutation of the 1 i, that's 1! times.

So the same anagram is counted 1!*1!*2!*1!*1!*2!*1!*1! times.

Therefore the number of anagrams is $\frac{10!}{1! * 1! * 2! * 1! * 1! * 2! * 1! * 1!}$, which is $\frac{10!}{4}$.

5. **[10 pts]** Consider the following statement.

There exist integers a and c such that for all integers x if $x \geq a$ then $x^2 < c \cdot x$.

Disprove this statement. (Hint: first write the negation of this statement then prove this negation.)

Solution.

The negation of this statement is as below,
For any integers a and c there exists an integer x that $(x \geq a) \wedge (x^2 \geq c \cdot x)$
Step 1 we prove there exists an integer x that for any integer a $x \geq a$.
Case 1: We consider an arbitrary integer a, no matter x and c are positive or negative, there would exist an integer x = a + k where k is any positive integer such as when k = 1. Thus we conclude there exists an integer x, $x > a$.
Case 2: We consider an arbitrary integer a, no matter x and c are positive or negative, there would exist an integer x = a + 0. Thus we conclude there exists an integer x, x = a.
From above we can conclude there exists an integer $x \geq a$.

Step 2 we prove there exists an integer x that for any integer c that $x^2 \geq c \cdot x$.
We consider an arbitrary integer c. From Step 1 we know there would exist an integer that $x \geq c$.
Case 1: Let integer x = c, no matter x and c are positive or negative, we multiply both sides of the equation by x to reach to $x^2 = c \cdot x$.
Case 2: Let integer $c \geq 0$. For any integer $x \geq c$, such as x = c + 1, we multiply both sides of the equation by x would reach to $x^2 > c \cdot x$. Thus such integer x exists.
Case 3: Let integer $c < 0$. For any integer x that $\mid x \mid \geq \mid c \mid$, such as x = -c + 1, we can still multiply both sides of the equation by x to reach to $x^2 > c \cdot x$. Thus such integer x exists.
From above we can conclude that there exists an integer $x^2 \geq c \cdot x$.

Step 3 we prove there exists an integer x that for any integers a and c $(x \geq a) \wedge (x^2 \geq c \cdot x)$
Case 1: Let $a \leq 0 \leq c$
From Step 1 and Step 2 above, we can find integer x that $x = c+1$ to allow $(x \geq a) \wedge (x^2 \geq c \cdot x)$
Case 2: Let $a \leq c \leq 0$
From Step 1 and Step 2 above, we can find integer x that $x = -c+1$ to allow $(x \geq a) \wedge (x^2 \geq c \cdot x)$
Case 3: Let $0 \leq a \leq c$
From Step 1 and Step 2 above, we can find integer x that $x = c+1$ to allow $(x \geq a) \wedge (x^2 \geq c \cdot x)$
Case 4: Let $0 \leq c \leq a$

From Step 1 and Step 2 above, we can find integer x that $x = a+1$ to allow $(x \geq a) \wedge (x^2 \geq c \cdot x)$

Case 5: Let $c \leq 0 \leq a$

From Step 1 and Step 2 above, we can find integer x that $x = -c+a+1$ to allow $(x \geq a) \wedge (x^2 \geq c \cdot x)$ Case 6: Let $c \leq a \leq 0$

From Step 1 and Step 2 above, we can find integer x that $x = -c+1$ to allow $(x \geq a) \wedge (x^2 \geq c \cdot x)$

The proof of the negation of the original statement is finished, thus the disprove of the original statement is finished.