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1. [10 pts] Over the years, Francesca, who loves fashion, has become an adamant lover of shoes. As such, she has many types in her collection, including: (1) sneakers, (2) flip flops, (3) sandals, (4) slippers, and (5) combat boots. Francesca realizes that she has too many shoes, so she decides to have a garage sale. $n \geq 3$ TAs, including Sasha, Kevin, and Arnav, show up to the sale, and since they're Francesca's friends, Francesca kindly offers to discount all of her shoes. However, in order to be eligible for the discount, Francesca mandates that each customer can only buy at most one pair of each type (ex. someone can buy a pair of sneakers and a pair of flip flops, but not two pairs of flip flops). Sasha must buy 1 pair of each of the 5 types, Kevin must also buy 1 pair of each of the 5 types, and Arnav must buy 1 pair of exactly 2 of the 5 types. All other customers can buy 0 or 1 pair of all of the 5 types. Assume that Francesca has enough shoes for each of the n TA's to purchase 1 pair of each type. How many distinct ways can these n customers buy Francesca's shoes and be eligible for the discount? Treat a pair of shoes as a single item.

Solution.

Sasha and Kevin: Since they both must buy 1 pair of each of the 5 types, they both have only 1 way to get eligible for the discount.
Arnav: Since Arnav must buy 1 pair of 2 of the 5 types, there are 10 ways for him to get eligible for the discount - $\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}$.
All other TA: Every other TA has 2 ways (buy 0 or buy 1) to be eligible for the discount for each type of the 5 types of shoes, so for each other TA, they have 2^5 ways. Since there are (n-3) other TAs, there are totally $(2^5)^{n-3}$ ways for all the other TAs.
Since each person's way is independent from the others, therefore the total ways = 1 * 1 * 10 * $(2^5)^{n-3}$

2. [**10 pts**] Let $A = \{2, 3\}$, $B = \{3, 4, 5\}$, and $C = \{3, 5, 7, 9\}$ and let P be the set whose elements are all the proper subsets of $(A \cup B) \setminus C$. List all the elements of 2^P. Show your work.

Solution.

$A \cup B = \{2,3,4,5\}$
$(A \cup B) \setminus C = \{2,4\}$
$P = \{\emptyset, \{2\}, \{4\}\}$
$2^P = \{\emptyset, \{\emptyset\}, \{\{2\}\}, \{\{4\}\}, \{\emptyset, \{2\}\}, \{\emptyset, \{4\}\}, \{\{2\}, \{4\}\}, \{\emptyset, \{2\}, \{4\}\}\}$

3. [**10 pts**] $n \geq 2$ distinguishable Hogwarts students participate in Professor Snape's experiment. Each student is given potion A, or philter B, or neither, or both. We know that Harry and Hermione are the only students among the n that are given both A and B. In how many distinct ways could Snape have distributed his experimental liquids?

Solution.

For Harry : only 1 way
For Hermione: only 1 way
For all other students: 4 ways
Since every student is independent from the others, the total distinct ways = 1*1*4^{n-2}

4. [10 pts] Prove that each of the following integers is composite (not prime).

(a) $3^{222}+1$

(b) $2^{35}(2^{33}-1)+1$

Solution.

(a) $3^{222}+1$
$= (3^{74})^3+1$
$= (3^{74}+1)((3^{74})^2$ - 3^{74} $+1)$

From above we can tell that both (3^{74}+1) and (($3^{74})^2$ - 3^{74} +1) are factors of $3^{222}+1$. Since both (3^{74}+1) and (($3^{74})^2$ - 3^{74} +1) are integers that are bigger than 1 but smaller than $3^{222}+1$, we can conclude that $3^{222}+1$ has factors other than 1 and itself. Therefore by definition $3^{222}+1$ is a composite.

(b) Let's set a = 2^{33}
$2^{35}(2^{33}-1)+1$
$= 2^2$*a(a-1)+1
= 4a(a-1)+1
= $4a^2$ - 4a +1
$= (2a-1)^2$

From above we can tell that 2a-1 is a factor of $2^{35}(2^{33}-1)+1$.
Since a = 2^{33}, 2a-1 is a integer that's bigger than 1 but smaller than $2^{35}(2^{33}-1)+1$, thus $2^{35}(2^{33}-1)+1$ has at least a factor other than 1 and itself. Therefore by definition $2^{35}(2^{33}-1)+1$ is a composite.

5. **[10 pts]** z is said to be a *Broadway integer* (this is not a standard term, it was made up for this problem) when $z = 4k + 2$ for some integer k. Prove that the difference of the squares of two Broadway integers is always divisible by 16.

Solution.

Let us assume two integers, a and b.

m = 4a + 2

n = 4b + 2

l = m^2 - n^2

= $(4a + 2)^2$ - $(4b + 2)^2$

= 16a^2 + 16a + 4 - (16b^2 + 16b + 4)

= 16a^2 + 16a + 4 - 16b^2 - 16b - 4

= 16(a^2 + a - b^2 - b)

Since both a and b are integers, a^2 + a - b^2 - b is also integers. Therefore by definition, l, the difference of the squares of two Broadway integers is divisible by 16.