Module 5.4: The Pigeonhole Principle

MCIT Online - CIT592 - Professor Val Tannen

LECTURE NOTES

The Pigeonhole Principle (PHP)

Get a pair of socks

Problem. A drawer in a dark room contains red, green, blue, and orange socks. How many socks must you take from the drawer to be sure that you have at least one matching pair? (We assume that any two socks of the same color form a matching pair.)

Answer. We apply PHP as follows:

Pigeons: single socks.

Pigeonholes: the four colors (red, green, blue, and orange).

If we pick five or more socks we are guaranteed to have a matching pair.

With similar reasoning: if you have three gloves then at least two of them are for the left hand or at least two of them are for the right hand.

PHP and the injection rule

PHP: Let $f : A \to B$ be a function. If $|A| > |B|$ then there exist at least two elements $x_1, x_2 \in A$ such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

This is saying that if $|A| > |B|$ then f is not injective!

From this formulation we see that the PHP is the **contrapositive** of the injection rule!

Coloring the points of the plane

Problem. Suppose that each point in the plane is colored either red or blue. Show that there always exist two points of the same color that are exactly one unit apart.

Answer. Consider an equilateral triangle ΔABC with the length of each side being one unit. Like all points in the plane, A, B, C are colored red or blue.

By PHP, two of the three points A, B, C must have the same color. By construction, these two are one unit apart.

(Although the set of points in the plane is infinite, we apply PHP to finite sets: the pigeons are A, B, C and the pigeonholes are red and blue.)

The Generalized PHP (GPHP)

In a big enough city there exist two people with exactly the same number of hairs on their head. Apparently the number of hairs on a person's head is at most 200,000. So this requires a city with 200,000 non-bald inhabitants. What about a city of 1M or 10M? There we can say more!

GPHP: r objects are placed into n boxes. For any integer k such that $r > k\,n$ there is at least one box containing at least $k+1$ objects.

Equivalently:

Let $f : A \to B$ and $k \in \mathbb{Z}^+$. If $|A| > k\,|B|$ then there exist at least $k+1$ pairwise distinct elements of A that f maps to the same element of B.

New York City (pop. 8.5M) may have as many as 42 people with the same number of hairs on their head. Because $8,500,000 > 42 \cdot 200,000$.

Quiz I

What is the fewest number of people we can ask to be sure that at least 10 of them are born in the same month?

A. 108

B. 109

C. 110

Answer

What is the fewest number of people we can ask to be sure that at least 10 of them are born in the same month?

A. 108

Incorrect. It could be that out of these 108, 9 are born in each month. $108 = 9 \cdot 12$.

B. 109

Correct. By GPHP, since $109 > 9 \cdot 12$.

C. 110

Incorrect. 110 is **enough** but not the fewest; see second answer.

Quiz II

How about if we want at least 10 of them to have been born in December?

A. 120

B. It is impossible.

C. 1440

Answer

How about if we want at least 10 of them to have been born in December?

A. 120

   Incorrect. See second answer!

B. It is impossible.

   Correct. For any number N we may have a set of N people all born outside of December. In fact they may all be born on January 1st.

C. 1440

   Incorrect. See second answer!