MCIT 592 Online - Spring 2022 **(instructor Val Tannen)**

Additional Problems (Packet 1)

1. Let $n, k \in \mathbb{Z}^+$ such that $k \leq n-1$. Which one is bigger $\binom{n}{k-1}$ or $\binom{n-1}{k}$?

2. A *study quad* consists of 4 students that participate in *study budds*. A study budd is a partnership of two students. Moreover, in a study quad, each student participates in exactly two study budds. Prove that, given 4 students, they can form exactly 3 distinct study quads.

3. Let x, y, z, a, b, c be six integers such that $x = a + y$ and $y = b + z$. Prove that if the average of a, b and c is not 0 then $z \neq c + x$.

4. Let P and Q be the following mathematical statements:

   - P: there are infinitely many pairs of prime numbers that differ by exactly 2 (this is known as the Twin Primes Conjecture).
   - Q: every even integer strictly greater than 2 can be written as the sum of two primes (this is known as Goldbach's Conjecture).

   As of this writing (early 2021) we do not know if either one of these conjectures is true or false.

   (a) Write P and Q as formal logical statements using logical connectives, quantifiers, and the predicates $prime(z)$ and $odd(z)$.

   (b) Show that regardless of whether either of the conjectures is true or false the statement $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$ is always true.

5. Prove that if P is a group of 4 people that can be split $P = R \cup B$ such that R and B are disjoint, and such that each person in R does not have FB friends in R and each person in B does not have FB friends in B then there exist 3 people in P that are not pairwise friends (that is, at least 2 of the 3 are not FB friends).

6. Prove that there exists a subset a $X \subseteq [0..2]^{[0..2]}$ such that $|X| = 9$ and for any two $f, g \in X$ we have $f(1) = g(1)$.

7. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a *very brief* explanation of your answer.

   (a) $\binom{100}{51}$ is strictly bigger than $\binom{100}{49}$.

   (b) In Pascal's Triangle, there exist four binomial coefficients c_1, c_2, c_3, c_4 such that $c_1 = c_2 + c_3 + c_4$.

(c) The boolean expressions $\neg[(p \Rightarrow q) \vee q]$ and $p \wedge \neg q$ are logically equivalent.

(d) Let A be a finite set. All functions $f : A \to A$ are bijections.

(e) Let A, B be finite set. Then, there are exactly as many subsets of $A \times B$ as there are functions with domain A and codomain 2^B.

8. Count the number of distinct sequences of bits (0's, 1s) of length 101 such that:

   - there are 3 more 1's than 0's in the sequence; **and...**
   - **...also** the middle bit is a 1.

9. Prove that for any $x, y, z \in \mathbb{Z}$ such that $x + 2y = z$, if $z - x$ is *not* divisible by 4 then $x + y + z$ is odd.

10. Let A, B be any sets such that $A \cap \{1, 2\} = B \cap \{1, 2\}$. Prove that the sets $(A \setminus B) \cup (B \setminus A)$ and $\{1, 2\}$ are disjoint.

11. Let $n \in \mathbb{N}$ and $n \geq 3$. Give a combinatorial proof (no other kinds of proofs will be accepted) for the following identity
$$\binom{n+2}{3} = \binom{n}{1} + 2\binom{n}{2} + \binom{n}{3}$$

12. Let X be a nonempty finite set. Consider the set $W = \{(A, B) \mid A, B \in 2^X \wedge A \subseteq B\}$.
Prove that W has exactly as many elements as there are functions with domain X and codomain $\{1, 2, 3\}$.

13.

14. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a *very brief* explanation of your answer.

(a) Assume that B is a set with 7 elements and that A is a set with 15 elements. Then, for any function $f : A \to B$ there exist at least 3 distinct elements of A that are mapped by f to the same element of B. TRUE or FALSE?

(b) There are exactly three surjective functions with domain $\{1, 2\}$ and codomain $\{a, b\}$. TRUE or FALSE?

(c) Exactly two of the following three boolean expressions: $p \Rightarrow q$, $p \wedge \neg q$, and $\neg p \vee q$ are logically equivalent. TRUE or FALSE?

(d) Let A be a finite set. For any function $f : A \to A$ we have $|\mathrm{Ran}(f)| = |A|$. TRUE or FALSE?

(e) There exist two distinct functions with domain and codomain $\{a, b\}$ that are their own inverses. TRUE or FALSE?

(f) For any two finite sets A, B, $|2^{A \times B}| > |2^A \times 2^B|$. TRUE or FALSE?

(g) Recall that for any $n = 0, 1, 2, 3, \ldots$ row n of the Pascal Triangle contains the binomial coefficients of the form $\binom{n}{k}$ for $k = 0, 1, \ldots, n$. $\binom{7}{4}$ can be expressed as a sum of binomial coefficients from row 5. TRUE or FALSE?

15. Assume the following formula for the sum of the squares of the first n positive integers: $1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = n(n+1)(2n+1)/6$.

    Using *only* the formula given above, derive the following formula for the sum of the squares of first m **odd** positive integers. Show your work.

$$1^2 + 3^2 + 5^2 + \cdots + (2m-3)^2 + (2m-1)^2 \;=\; \frac{m(4m^2-1)}{3}$$

16. Give a boolean expression e with three variables p, q, r such that e has the following properties:

    - $e = T$ when $p = q = T$ and $r = F$, AND
    - $e = F$ when $p = F$ and $q = r = T$.

    Also, construct a truth table for e. Make sure to include all intermediate propositions as a separate column. (Yes, there are many possible answers.)

17. In how many different ways can we arrange *all* the letters from the English alphabet (26 characters) in a sequence such that:

    - each letter occurs exactly once, AND
    - the 5 vowels (a,e,i,o,u) occur in 5 consecutive positions.

18. Give a combinatorial proofs (no other kind of proofs will be accepted) for the following identities

    (a)
$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k} \qquad \text{(where } k \le r \le n\text{)}$$

    (b)
$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1} \qquad \text{(where } k \le n\text{)}$$

19. Prove that for any integers $a, b \in \mathbb{Z}$ we have $a^2 - 4b \neq 2$.

20. Consider n (distinguishable) bins labeled $B_1, \ldots, B_n$ and r indistinguishable (identical) marbles. We wish to put the r marbles into the n bins in such a way that each bin will contain at least **one** marble and at least **three** of the bins will contain **two or more** marbles. Assume $r \geq n+3$. In how many different ways can this be done?

21. For each statement below, decide whether it is TRUE or FALSE. In each case attach a *very brief* explanation of your answer.

    (a) $\binom{100}{51}$ is strictly bigger than $\binom{100}{49}$. TRUE or FALSE?

    (b) The word QWERTY has 6! anagrams. TRUE or FALSE? (Recall that a word is a valid anagram of itself.)

    (c) The contrapositive of $p \Rightarrow q$ is logically equivalent to $p \wedge \neg q$. TRUE or FALSE?

    (d) For any $2 \leq k < n$, if A has n elements then the number of subsets of A of k elements is $\frac{n!}{(n-k)!}$. TRUE or FALSE?

(e) If the set A has n elements then there are $n!$ injective functions with domain A and codomain A. TRUE or FALSE?

(f) There is no set X such that $2^X = \emptyset$. TRUE or FALSE?

22. The Taney Dragons are going to the Little League World Series! In appreciation, each of the 12 distinct team members (players) can pick 2 hats from a supply of red (Philly Phillies), blue (Boston Red Sox), and green (Ploiesti Frackers) hats. For each color, the supply is unlimited. For each of the three questions below (see also next page), give the answer and an explanation of how you derived it. No proofs required.

    In how many different ways can the hat picking be done if:

    (a) There is no ordering among the two hats that each player picks, and both hats can even be of the same color.

    (b) The ordering matters and the two hats have a different color: let's say each player picks a hat to wear in the morning and then a hat (of a different color) to wear in the afternoon.

    (c) What is the count for part (22a) above, if you also know that at least one of the hats that Dragon's pitcher Mo'ne Davis picks is *red*.

23. Recall from homework that the boolean expression e_2 is a **logical consequence** of the boolean expression e_1 if every truth assignment to the variables that makes e_1 true also makes e_2 true.

    Let x, y be arbitrary boolean variables. Prove, using truth tables, that $x \to y$ is a logical consequence of $\neg x \wedge y$.

24. In the following, just give the examples, you do not have to prove that they work.

    (a) For arbitrary $n \geq 1$, give an example of a set Y and a function $f : [1..n] \to Y$ such that f is injective but *not* surjective.

    (b) For arbitrary $n \geq 2$, give an example of a set X and a function $g : X \to [1..n]$ that is *not* injective and moreover $|\,\mathrm{Ran}(g)| = n - 1$.

25. Punch happily tells Judy that he proved two new theorems and he shares his proofs with her.

    (a) *Punch's First Theorem*: If n is odd then $n^2 - 1$ is a multiple of 4.

        Punch's Proof: "We prove the contrapositive instead. Suppose n is even, then n^2 is even, then $n^2 - 1$ is odd so it cannot be a multiple of 4. Done." Upon reading these, Judy remarks that while the theorem is true, the proof is not proving the theorem, but another statement, which is not the contrapositive of of the theorem.

        i. What is the contrapositive of the theorem and what statement is Punch actually proving?

        ii. Give a correct proof of Punch's First Theorem.

    (b) *Punch's Second Theorem*: For any finite sets A, B, if $|A|$ and $|B|$ are even then $|A \setminus B|$ is even.
        Punch's Proof: "The difference of two even numbers is an even number. Done."

        i. Now, Judy remarks that this other theorem is not even true. Give a counterexample that supports Judy's contention.

ii. Judy also remarks that Punch's "proof" relies on a false statement about set cardinalities. (Since the theorem is not true, there had to be a bug in the proof!) What is that false statement?

26. How many sequences of bits (0's, 1's) are there that each sequence has all of the following properties:

- Their length is either 3 or 5 or 7.
- Their middle bit is a 1.
- The number of 0's they have equals the number of 1's they have minus one.

27. How many sequences of bits (0's, 1's) of length 100 can we make such that:

- the number of 0's in the sequence is equal to the number of 1's in the sequence; and
- the sequence begins with a 1 and ends with a 1.

28. A cookie shop has k different flavors of cookies. Arnav wishes to purchase cookies for his recitation, and he has enough money to buy up to 250 cookies. Assuming that he does not have to spend all of the money that he has, in how many ways can he purchase cookies? (For full credit, your solution should be in closed form, so no open summations!)

29. Give a combinatorial proof (no other kind of proofs will be accepted) for the following identity

$$\frac{n!}{(n-r)!} = \frac{(n-1)!}{(n-r-1)!} + r \cdot \frac{(n-1)!}{(n-r)!} \qquad \text{(where } 1 \leq r \leq n\text{)}$$

30. We distribute indistinguishable ungraded papers to $n \geq 3$ distinguishable TAs in the following way. First we select two "lucky" TAs to have the designation of Head TA. Next we distribute the papers in such a way that each TA gets at least 1 paper to grade, and both of the Head TAs get at least 2 papers. What is the minimum number of papers needed to make this work? Now, assume that we have r papers, where r is large enough to make this kind of distribution work, in how many different ways can the papers be distributed?

31. Give a combinatorial proof for the following identity:

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

32. For each statement below, decide whether it is true or false. In each case attach a *very brief* explanation of your answer.

(a) Consider the proposition P where:

$$P: \ \exists N > 0, \ \forall n \geq N, \ \ 100n \leq n^2/100$$

True or false: $\neg P$ is the following proposition:

$$\forall N > 0, \ \forall n \geq N, \ \ 100n > n^2/100$$

(b) Let X be a finite nonempty set. The number of functions with domain X and codomain $\{0,1\}$ is $2^{|X|}$, true or false?

33. For sets A, B, C, and D, suppose that $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$ then $x \in B$.

34. You are choosing a sequence of five characters for a license plate. Your choices for characters are any letter in PERM and any digit in 1223. Your five-character sequence can contain any of these characters at most the number of times they appear in either PERM or 1223. If there are no other restrictions, how many such sequences are possible?

35. Prove that if for some integer a, $a \geq 3$, then $a^2 > 2a + 1$.

36. Give a combinatorial proof of the following identity for $N, a, b \in \mathbb{N}$:

$$\binom{N}{a}\binom{N}{b} = \sum_{i=0}^{\min(a,b)} \binom{N}{i}\binom{N-i}{a-i}\binom{N-a}{b-i}$$

37. There are 100 guests at a fundraising party, excluding the host. As part of a "fun" party game, the host pairs up the dinner guests into 50 pairs that the host calls "fundraising pairs". In the game, the individual with the smaller net worth in each pair declares the amount of money that they wish to donate, which the individual with the higher net worth must match in double. For example, if the individual with the smaller net worth in one pair donates \$100 dollars, the individual with the larger net worth must donate \$200 dollars.

    The host says that the aim of the game is to raise a total of 9 million dollars between all of the individuals. Given this set up, how many ways can the game unfold? Assume that the net worth of each of the individuals is unique, that all donations are in whole dollars, and that all of them can donate up to 9 million dollars each.

38. Let x, y be arbitrary boolean variables. Give a truth table for the expression $x \Rightarrow (\neg y \Rightarrow F)$. Then, find, *two* other, distinct, boolean expressions that are logically equivalent to the previous expression.

39. Consider sequences of bits such that m of the bits are 0, where $m \geq 2$, n of the bits are 1, where $n \geq 1$, and start with a 1 and end with two 0's. How many such sequences are there?

40. For each statement below, decide whether it is true or false. In each case attach a *very brief* explanation of your answer.

    (a) Let A, B be finite sets with $|A| = 2$ and $|B| = 3$. There are more functions $A \to B$ than functions $B \to A$, true or false?

    (b) Let X, Y be nonempty finite sets such that $|Y| = 1$ and such that there exists an injection $f : X \to Y$. Then $|X| = 1$, true or false?

    (c) There are as many sequences of bits of length 100 that start with a 0 as sequences of bits of length 100 that end with a 1, true or false?

    (d) Let S be the set of the first 100 natural numbers: $S = 0, 1, \ldots, 99$. There are as many subsets of S of size 40 that contain the number 40 as subsets of S of size 40 that do not contain the number 40, true or false?

41. In the remote town of Plictisitor a local ordinance prevents inhabitants from having first names, they can only have last names. These last names must start with an upper case letter followed by one to three lower case letters followed by a number between 1 and 22 (to accomodate families, you see). The lower case letters must be distinct among themselves but they can be the same letter as the upper case at the beginning of the names. Moreover, no two inhabitants can have the same name. The alphabet used in Plictisitor has 31 letters, with lower and upper case for each of them.

    What is the maximum population of Plictisitor? (Just give it as an arithmetical expression since you cannot use a calculator during the exam.)