MCIT 592 Spring 2022 **Homework 7** **Solutions**

PROBLEM SET

1. **[10 pts]** Suppose Jeff has 5 cars; a Ferrari, an Audi, a Bugatti, a Bentley, and a Lamborghini. He wants to organize his garage consisting of these 5 cars. He does this by selecting a uniformly random permutation of the 5 cars. What is the probability that the first car is the Audi, and the last car is the Bentley?

Solution:

Define a sample space of garages made up of a random permutation of the cars $\{F, A, Bu, Be, L\}$ (essentially using what the problem gives us). Since the cars are uniformly distributed, the sample space is uniform and the probability that a permutation has the Audi as its first car and the Bentley as its last letter can be found by counting possibilities.

There are 5! possible outcomes total. If the first car is A and the last car is Be, then there are 3! ways to arrange the middle three cars, so there are 3! such outcomes. Thus, the probability of this event is $3!/5! = \boxed{1/20}$.

2. **[10 pts]** Among $2 \leq k \leq 7$ randomly selected people, what is the probability that at least two of them were born on the same day of the week?

Solution:

Define the sample space as a k element ordered tuple, where each element is a day, corresponding to the birthday of a different person. This space is uniform as each person independently has an equal likelihood of being born on any day. There are 7^k possible outcomes in the sample space.

Use complementary counting to approach this problem. The number of outcomes in which no two people were born on the same day of the week is the number of partial permutations of length k of 7 items, which is $7!/(7-k)!$. A partial permutation works well as it guarantees that no two elements are the same day, meaning no two people have the same birthday. So the probability that no two people were born on the same day of the week is $\frac{7!/(7-k)!}{7^k}$, and the probability that at least two of them were born on the same day is

$$\boxed{1 - \frac{7!/(7-k)!}{7^k}}.$$

3. **[10 pts]** Suppose you roll three fair six-sided dice and add up the numbers you get. What is the probability that the sum is at most 5?

Solution:

Define the sample space Ω as an ordered tuple of three numbers. The first number contains the value of the first dice, the second contains the value of the second dice, and the third the third dice. Notice that this sample space is uniform since each dice is uniformly distributed and the rolls are independent. Thus, the probability that the sum is at most 5 can be found by finding the number of outcomes with a sum less than or equal to 5 and dividing by total number of outcomes.

Since there are 6 sides to a dice, each number of the tuple has 6 possibilities. By the multiplication rule, there are $6^3 = 216$ possible outcomes. For the sum to be less than 5, it can equal 3, 4, or 5. It is impossible to get a sum less than 3 as the minimum value of each dice is 1. To count the number of possibilities:

- There is 1 outcome with sum 3: $(1, 1, 1)$
- There are 3 outcomes with sum 4: $(1, 1, 2), (1, 2, 1), (2, 1, 1)$
- There are 6 outcomes with sum 5: $(1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1)$

Thus, by the addition rule (P2), the probability of getting sum at most 5 is

$$\frac{1}{216} + \frac{3}{216} + \frac{6}{216} = \frac{10}{216} = \boxed{\frac{5}{108}}.$$

4. [10 pts]

Sasha and Professor Tannen are planning the end of the semester dinner party for the CIT Staff. Answer the following questions. **You don't need to show your work.**

Questions (a)-(d) are based on the following scenario: Sasha and Professor Tannen decide to order 4 dishes for the team of TAs. For any meal, Laila feels like eating 29% of the time, Alex feels like eating 68% of the time, Francesca 54% of the time, and Rohan feels like eating 0% of the time (he's always eats before the dinner party!). All four dishes are independent of each other.

Questions (e)-(i) are based on the following scenario: after creating the planning and successfully running the dinner party, Sasha examines the dishes herself. She visits 5 dishes independently and gets one spoonful of food from each, which is equally likely to be either warm or cold.

(a) What is the probability that exactly 3 TAs eat at the dinner party?

(b) What is the probability that exactly 0 TAs eat at the dinner party?

(c) What is the probability that at least 1 TAs eat at the dinner party?

(d) What is the probability that exactly 1 TAs eat at the dinner party?

(e) What is the probability that every spoonful Sasha gets is of the same temperature type (warm or cold)?

(f) What is the probability that the first 3 spoonfuls of food she gets are warm and the other 2 are cold?

(g) What is the probability that she gets exactly 3 warm spoonfuls of food?

(h) What is the probability that the first 2 spoonfuls of food she gets are cold?

(i) What is the probability that she gets at least 1 cold spoonful of food?

Solution:

For (a) through (d), we define our sample space as the power set of the set of symbols $\{\ell, a, f, r\}$, where ℓ represents Laila, a represents Alex, f represents Francesca, and r represents Rohan. In each element of the power set, if a symbol is in the set, then the TA it represents ate at the dinner party.

We now define our events as follows:

Let A be the event that Laila decides to eat at the dinner party.
Let B be the event that Alex decides to eat at the dinner party.
Let C be the event that Francesca decides to eat at the dinner party.
Let D be the event that Rohan decides to eat at the dinner party.

As given, we have

$$\Pr[A] = 0.29 \qquad \Pr[B] = 0.68$$
$$\Pr[C] = 0.54 \qquad \Pr[D] = 0$$

For (e) through (i), we will represent the outcomes of the dinner party as $\Omega = \{t, p\}^5$ where each element is a 5-tuple $(c_1, c_2, c_3, c_4, c_5)$, where the c_i in the 5-tuple represents whether the ith spoonful taken by Sasha was warm (t) or cold (p). E.g. $(t, p, p, t, t) \in \Omega$. Note that each outcome is equally likely, so we have a uniform probability space. Furthermore, note that $|\Omega| = 2^5 = 32$.

(a) Let E be the event that exactly 3 TAs eat at the dinner party. Note that if two events A and B are independent, then A and $\overline{B}$ is also independent (see the lemma in the solution to problem 5).

$$\begin{aligned}
\Pr[E] =& \Pr[\overline{A} \cap B \cap C \cap D] + \Pr[A \cap \overline{B} \cap C \cap D] \\
&+ \Pr[A \cap B \cap \overline{C} \cap D] + \Pr[A \cap B \cap C \cap \overline{D}] \\
=& \Pr[\overline{A}] \cdot \Pr[B] \cdot \Pr[C] \cdot \Pr[D] + \Pr[A] \cdot \Pr[\overline{B}] \cdot \Pr[C] \cdot \Pr[D] \\
&+ \Pr[A] \cdot \Pr[B] \cdot \Pr[\overline{C}] \cdot \Pr[D] + \Pr[A] \cdot \Pr[B] \cdot \Pr[C] \cdot \Pr[\overline{D}] \\
&\text{(by independence)} \\
=& (1 - \Pr[A]) \cdot \Pr[B] \cdot \Pr[C] \cdot \Pr[D] + \Pr[A] \cdot (1 - \Pr[B]) \cdot \Pr[C] \cdot \Pr[D] \\
&+ \Pr[A] \cdot \Pr[B] \cdot (1 - \Pr[C]) \cdot \Pr[D] + \Pr[A] \cdot \Pr[B] \cdot \Pr[C] \cdot (1 - \Pr[D]) \\
=& (1 - 0.29)(0.68)(0.54)(0) + (0.29)(1 - 0.68)(0.54)(0) \\
&+ (0.29)(0.68)(1 - 0.54)(0) + (0.29)(0.68)(0.54)(1 - 0) \\
=& (0.29)(0.68)(0.54) \qquad \text{(from above)} \\
=& \boxed{0.106488}
\end{aligned}$$

(b) Let E be the event that exactly 0 TAs eat at the dinner party. Note that E happens when the complements of A, B, C, and D occur.

$$\begin{aligned}
\Pr[\text{E}] =& \Pr[\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}] \\
=& \Pr[\overline{A}] \cdot \Pr[\overline{B}] \cdot \Pr[\overline{C}] \cdot \Pr[\overline{D}] \qquad \text{(by independence)} \\
=& (1 - \Pr[A]) \cdot (1 - \Pr[B]) \cdot (1 - \Pr[C]) \cdot (1 - \Pr[D]) \\
=& (1 - 0.29)\,(1 - 0.68)\,(1 - 0.54)\,(1 - 0) \\
=& (0.71)(0.32)(0.46)(1) \\
=& \boxed{0.104512}
\end{aligned}$$

(c) Let E be the event that at least 1 TAs eat at the dinner party. E happens when at least one of A, B, C, and D occur. Therefore E is equivalent to $A \cup B \cup C \cup D$.

$$\begin{aligned}
\Pr[E] &= \Pr[A \cup B \cup C \cup D] \\
&= 1 - \Pr[\overline{(A \cup B \cup C \cup D)}] \\
&= 1 - \Pr[\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}] && \text{(DeMorgan's)} \\
&= 1 - 0.104512 && \text{(from (b))} \\
&= \boxed{0.895488}
\end{aligned}$$

(d) Let E be the event that exactly 1 TAs eat at the dinner party.

$$\begin{aligned}
\Pr[E] =& \Pr[A \cap \overline{B} \cap \overline{C} \cap \overline{D}] + \Pr[\overline{A} \cap B \cap \overline{C} \cap \overline{D}] \\
&+ \Pr[\overline{A} \cap \overline{B} \cap C \cap \overline{D}] + \Pr[\overline{A} \cap \overline{B} \cap \overline{C} \cap D] \\
=& \Pr[A] \cdot \Pr[\overline{B}] \cdot \Pr[\overline{C}] \cdot \Pr[\overline{D}] + \Pr[\overline{A}] \cdot \Pr[B] \cdot \Pr[\overline{C}] \cdot \Pr[\overline{D}] \\
&+ \Pr[\overline{A}] \cdot \Pr[\overline{B}] \cdot \Pr[C] \cdot \Pr[\overline{D}] + \Pr[\overline{A}] \cdot \Pr[\overline{B}] \cdot \Pr[\overline{C}] \cdot \Pr[D] \\
& \text{(by independence)} \\
=& (0.29)(1-0.68)(1-0.54)(1-0) + (1-0.29)(0.68)(1-0.54)(1-0) \\
&+ (1-0.29)(1-0.68)(0.54)(1-0) + (1-0.29)(1-0.68)(1-0.54)(0) \\
=& \boxed{0.387464}
\end{aligned}$$

(e) Let H be the event that all of the spoonfuls of food are the same temperature type. Note that this occurs exactly when Sasha gets all warm or all cold spoonfuls of food. Therefore, $H = \{(t,t,t,t,t),(p,p,p,p,p)\}$, so $|H| = 2$. Since the probability space is uniform, $\Pr[H] = \frac{|H|}{|\Omega|}$. Hence, $\Pr[H] = \frac{2}{32} = \boxed{\dfrac{1}{16}}$.

(f) Let J be the event that the first 3 spoonfuls are warm and the last 2 are cold. Note that this describes precisely one outcome, so $J = \{(p,p,p,t,t)\}$, and $|J| = 1$. Since the probability space is uniform, $\Pr[J] = \frac{|J|}{|\Omega|}$. Hence, $\Pr[J] = \boxed{\dfrac{1}{32}}$.

(g) Let K be the event that exactly 3 of the spoonfuls are warm. $|K| = \binom{5}{3}$, since that is the number of ways to select 3 of the spoonfuls to be warm. Note that the other two must be cold spoonfuls.

Since the probability space is uniform, $\Pr[K] = \frac{|K|}{|\Omega|}$. Hence, $\Pr[K] = \frac{\binom{5}{3}}{32} = \boxed{\frac{5}{16}}$.

(h) Let L be the event that the first 2 spoonfuls are cold. We note that the probability space is uniform, therefore $\Pr[L] = \frac{|L|}{|\Omega|}$. Observe that there are $2^3 = 8$ ways to select temperature types for the remaining 3 spoonfuls of food, given that the first 2 are cold spoonfuls. Therefore, $|L| = 8$. Hence, $\Pr[L] = \frac{8}{32} = \boxed{\frac{1}{4}}$.

(i) Let M be the event that there is at least one cold spoonful of food. Note that $\overline{M}$ is the event that there are no cold. Observe that $\overline{M} = \{(p, p, p, p, p)\}$, so $|\overline{M}| = 1$. Hence, $\Pr[M] = 1 - \Pr[\overline{M}] = 1 - \frac{|\overline{M}|}{\Omega} = 1 - \frac{1}{32} = \boxed{\frac{31}{32}}$.

5. [10 pts] It is spring break, and Alice wants to go on vacation for two weeks. She wants to go to either Cancun, the Bahamas, or Hawaii. She only has money to go to one place for the first week, and another for the second week, for a total of two locations. She puts a tickets labeled "Cancun", b tickets labeled "Bahamas", and c tickets labeled "Hawaii" into a hat. She will pick a ticket from the hat uniformly at random to decide where to travel the first week, then return that ticket to the hat. She'll then pick another ticket from the hat uniformly at random and go to that location the second week. What is the probability that she goes to two different locations? Be sure to define the sample space and any relevant events. Assume that $a + b + c \geq 2$.

Solution:

Let S be the set of all locations in the hat. Our sample space is the set of all ordered pairs of locations: $\Omega = S \times S$. Observe that the probability space is uniform, since Alice picks each ticket uniformly at random, so each pair of locations is also equally likely. Thus, for any event E, we have $\Pr[E] = \frac{|E|}{|\Omega|} = \frac{|E|}{n^2}$, where $n = a + b + c$. We define the following events:

> D: event that Alice picks two tickets with different locations.
> A_j: event that Alice chooses Cancun for the j^{th} week, $j \in \{1, 2\}$.
> B_j: event that Alice chooses the Bahamas for the j^{th} week, $j \in \{1, 2\}$.
> C_j: event that Alice chooses Hawaii for the j^{th} week, $j \in \{1, 2\}$.

We seek to calculate $\Pr[D]$. Since Alice must pick two different places on each week for D to occur, we see that we can express D as:

$$D = (A_1 \cap B_2) \cup (A_1 \cap C_2) \cup (B_1 \cap A_2) \cup (B_1 \cap C_2) \cup (C_1 \cap A_2) \cup (C_1 \cap B_2)$$

Furthermore, each pair of elements in this union are disjoint, since Alice cannot choose 2 different places for the same week simultaneously. Applying the Addition Rule for probabilities, we see that:

$$\begin{aligned}\Pr[D] &= \Pr\Big[(A_1 \cap B_2) \cup (A_1 \cap C_2) \cup (B_1 \cap A_2) \cup (B_1 \cap C_2) \cup (C_1 \cap A_2) \cup (C_1 \cap B_2)\Big] \\ &= \Pr[A_1 \cap B_2] + \Pr[A_1 \cap C_2] + \Pr[B_1 \cap A_2] + \Pr[B_1 \cap C_2] + \Pr[C_1 \cap A_2] + \Pr[C_1 \cap B_2]\end{aligned}$$

Additionally, since Alice is picking uniformly at random from the hat each time, we see that Alice's second location choice is independent of her first location choice. Using the fact that, if X and Y are two independent events, then $\Pr[X \cap Y] = \Pr[X]\Pr[Y]$, we have:

$$= \Pr[A_1]\Pr[B_2] + \Pr[A_1]\Pr[C_2] + \Pr[B_1]\Pr[A_2] + \Pr[B_1]\Pr[C_2] + \Pr[C_1]\Pr[A_2] + \Pr[C_1]\Pr[B_2]$$

We now calculate each of the probabilities above. We do so for $\Pr[A_1]$; the rest follow similarly. Since the sample space is uniform, we can simply compute $|A_1|$, which is all elements of $S \times S$ with Cancun as the first element. We see that we then have a choices for the first week and n choices for the second week, so we find that $\Pr[A_1] = \frac{|A_1|}{n^2} = \frac{an}{n^2} = \frac{a}{n}$. Plugging this in yields:

$$\begin{aligned}
\Pr[D] &= \left(\frac{a}{n}\right)\left(\frac{b}{n}\right) + \left(\frac{a}{n}\right)\left(\frac{c}{n}\right) + \left(\frac{b}{n}\right)\left(\frac{a}{n}\right) + \left(\frac{b}{n}\right)\left(\frac{c}{n}\right) + \left(\frac{c}{n}\right)\left(\frac{a}{n}\right) + \left(\frac{c}{n}\right)\left(\frac{b}{n}\right) \\
&= \frac{ab + ac + ba + bc + ca + cb}{n^2} \\
&= \boxed{\frac{2(ab + ac + bc)}{(a + b + c)^2}}
\end{aligned}$$

Alternate Solution:

We can also calculate $\Pr[D]$ by calculating its complement $\Pr[\overline{D}]$. $\overline{D}$ is the event that Alice selects the same location for both weeks:

$$\overline{D} = (A_1 \cap A_2) \cup (C_1 \cap C_2) \cup (B_1 \cap B_2)$$

Similar to above, the three intersections that make up $\overline{D}$ are pairwise disjoint. Hence, by the Sum Rule for probabilities, $\Pr[\overline{D}] = \Pr[A_1 \cap A_2] + \Pr[C_1 \cap C_2] + \Pr[B_1 \cap B_2]$.

Again, each intersection is the intersection of two independent events as mentioned above. So,

$$\begin{aligned}
\Pr[\overline{D}] &= \Pr[A_1 \cap A_2] + \Pr[C_1 \cap C_2] + \Pr[B_1 \cap B_2] \\
&= \Pr[A_1] \cdot \Pr[A_2] + \Pr[C_1] \cdot \Pr[C_2] + \Pr[B_1] \cdot \Pr[B_2] \\
&= \left(\frac{a}{n}\right)\left(\frac{a}{n}\right) + \left(\frac{c}{n}\right)\left(\frac{c}{n}\right) + \left(\frac{b}{n}\right)\left(\frac{b}{n}\right) \\
&= \frac{a^2 + b^2 + c^2}{n^2}
\end{aligned}$$

Now we can calculate $\Pr[D]$ using $\Pr[\overline{D}]$:

$$\begin{aligned}
\Pr[D] &= 1 - \Pr[\overline{D}] \\
&= 1 - \frac{a^2 + c^2 + b^2}{n^2} \\
&= \frac{(a + c + b)^2 - a^2 - c^2 - b^2}{n^2} \\
&= \frac{a^2 + c^2 + b^2 + 2ac + 2ab + 2cb - a^2 - c^2 - b^2}{n^2} \\
&= \boxed{\frac{2(ac + ab + cb)}{(a + c + b)^2}}
\end{aligned}$$

Which is the same as above!