OMCIT 592 Module 12 Self-Paced 01 (instructor Val Tannen)

No reference to this self-paced segment in the lecture segments.

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion.

When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

A property of cut edges

Let $G = (V, E)$ be a graph. An edge of G is a **cut edge** if by removing it we obtain a graph with strictly **more** connected components (cc's) than G.

Proposition. Removing a cut edge increases the number of connected components by exactly 1.

Proof. Let $e \equiv u\text{–}v$ be a cut edge in $G = (V, E)$. Denote by $G_e = (V, E \setminus \{e\})$ the graph obtained from G by removing e.

By the definition of cut edge G_e has strictly more cc's than G.

Because we have a walk from u to v (the edge!) u and v must belong to the same connected component of G, which we denote by D.

D is the only connected component of G affected by the removal of e. We want to show that D splits into exactly two cc's in G_e. We will prove this by contradiction.

Suppose, toward a contradiction, that in G_e the component D splits into three or more distinct components. Let D_1, D_2, D_3 be three of these distinct components. Every cc is non-empty so we can consider three distinct vertices $w_1 \in D_1, w_2 \in D_2, w_3 \in D_3$.

In G there existed walks, hence **paths**, $w_1\text{–}\cdots\text{–}w_2$, $w_2\text{–}\cdots\text{–}w_3$, $w_3\text{–}\cdots\text{–}w_1$ but in G_e these paths cannot exist. Thus, e appears in all three of these paths.

We will show that this situation implies that there is a walk in G_e between at least two of the three vertices w_1, w_2, w_3. This will contradict the fact that in G_e these are distinct connected components.

Consider the paths $w_1\text{–}\cdots\text{–}w_2$, $w_2\text{–}\cdots\text{–}w_3$, $w_3\text{–}\cdots\text{–}w_1$ as sequences of vertices. All three paths traverse e in some direction but there are only two traversal directions for e so by the Pigeonhole Principle two of these three paths must traverse e in the same direction.

W.l.o.g. we can assume that these two paths are $p_1 \equiv w_1\text{–}\cdots\text{–}u\text{–}v\text{–}\cdots\text{–}w_2$ and $p_2 \equiv w_2\text{–}\cdots\text{–}u\text{–}v\text{–}\cdots\text{–}w_3$. From p_1 and p_2 we construct in G a walk $w_1\text{–}\cdots\text{–}u\text{–}\cdots\text{–}w_2$ that does not contain e and is therefore also a walk in G_e. Hence D_1 and D_2 cannot be distinct. Contradiction.