# Module 3.6: Counting Anagrams

## MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

# Counting anagrams I

An **anagram** is a reordering of the letters of a word.

**Example.** Anagrams (that make some sense in English) for `Mathematics`: `ThematicSam`, `MismatchTea`, `ActHammiest`.

**Problem.** How many anagrams does `mississippi` have (including itself)?

**Answer.** Notice that what matters is which letters occur, and how many times each. We use the following representation:

$$. \quad \{1 \cdot \mathtt{m},\, 4 \cdot \mathtt{i},\, 4 \cdot \mathtt{s},\, 2 \cdot \mathtt{p}\}$$

to capture all the information we need for the count.

`mississippi` and its anagrams are sequences of length 11 but it is pretty clear that the answer is not 11!, the number of permutations. Why?

Because we can swap around the different occurrences of the **same** letter and the sequence stays the **same**!

# Counting anagrams II

**Problem.** How many anagrams does `mississippi` have (including itself)?

**Answer (continued).** Notice that we can construct an anagram as follows:

(1) Choose 1 out of 11 positions to put `m`. In $\binom{11}{1}$ ways.

(2) Choose 4 out of $11 - 1 = 10$ remaining positions to put `i`'s. In $\binom{10}{4}$ ways.

(3) Choose 4 out of $10 - 4 = 6$ remaining positions to put `s`'s. In $\binom{6}{4}$ ways.

(4) Choose 2 out of $6 - 4 = 2$ remaining positions to put `p`'s. In $\binom{2}{2}$ ways.

By the multiplication rule the answer is

$$\binom{11}{1} \cdot \binom{10}{4} \cdot \binom{6}{4} \cdot \binom{2}{2} = \frac{11!}{1! \cdot 10!} \cdot \frac{10!}{4! \cdot 6!} \cdot \frac{6!}{4! \cdot 2!} \cdot \frac{2!}{2! \cdot 0!} = \frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!}$$

# Counting anagrams III

**Problem.** How many anagrams does `mississippi` have (including itself)?

**Alternative answer .** We start with permutations of 11 letters. That's 11! and it would be correct if all 11 letters were distinct. As we saw, however, for `mississippi` this is **overcounting**. But by **how much**?

When we count all permutations the **same** anagram is counted as many times as there are

(1) Permutations of the 1 `m`. That's 1! times.
(2) Permutations of the 4 `i`'s. That's 4! times.
(3) Permutations of the 4 `s`'s. That's 4! times.
(4) Permutations of the 2 `p`'s. That's 2! times.

So the same anagram is counted $1! \cdot 4! \cdot 4! \cdot 2!$ times.

Therefore the number of anagrams is $\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!}$

# Anagrams are permutations of bags

A **multiset** or **bag** is an unordered collection in which repetitions are allowed.

The letters in `mississippi` form a bag:

. $\{ 1 \cdot \mathtt{m}, 4 \cdot \mathtt{i}, 4 \cdot \mathtt{s}, 2 \cdot \mathtt{p} \}$ same as $\{ \mathtt{m,i,i,i,i,s,s,s,s,p,p} \}$

The **size** of a bag is the total number of elements, repetitions included. The bag $\{ \mathtt{m,i,i,i,i,s,s,s,s,p,p} \}$ has size 11.

A **permutation** of a bag lists all its elements, repetitions included, in some order. Thus, a permutation of $\{ \mathtt{m,i,i,i,i,s,s,s,s,p,p} \}$ is exactly an anagram of `mississippi`.

Consider the bag $\{ n_1 \cdot a_1, \ n_2 \cdot a_2, \ldots, \ n_k \cdot a_k \}$

The number of permutations of this bag is

$$\frac{(n_1 + n_2 + \ldots + n_k)!}{n_1! \cdot n_2! \cdots n_k!}$$

## Activity : Example of distinguishable v. indistinguishable

Consider $n$ distinguishable books (different titles) and $k$ bins labeled $B_1, B_2, \ldots, B_k$. How many ways are there to distribute the books among the bins so that bin $B_i$ receives $n_i$ books and $n_1 + n_2 + \ldots + n_k = n$?

# ACTIVITY : Example of distinguishable v. indistinguishable (continued)

Let the $n$ distinguishable books be arranged in a sequence: $b_1, \ldots, b_n$. (In a moment we shall see that it does not matter in what order we arrange them, as long as the same ordering is used during the whole counting procedure.)

For each distribution of the books to the bins, modify the sequence above by replacing each book with a piece of paper on which is written the label of the bin the book goes into.

# ACTIVITY : Example of distinguishable v. indistinguishable (continued)

Now it might help to think of the resulting sequences as words over the alphabet $B_1, \ldots, B_k$. The distributions correspond one-to-one to anagrams of length $n$ using $n_i$ copies of letter $B_i$, etc.! And now you see that it does not matter in what order we arranged the books originally.

As we learned in this segment, the answer is the number of permutations of the bag $\{n_1 \cdot B_1, \ldots, n_k \cdot B_k\}$, that is:

$$\binom{n}{n_1 \; n_2 \; \cdots \; n_k} = \frac{n!}{n_1! \, n_2! \cdots n_k!}$$

## ACTIVITY : Example of distinguishable v. indistinguishable (continued)

Now suppose that the bins are still labeled but the books are all copies of "Moby Dick". In other words, the bins are distinguishable but the books are indistinguishable.

**Question:**

Now, how many ways are there to distribute the books among the bins?

*In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!*

# ACTIVITY : Example of distinguishable v. indistinguishable (continued)

**Answer:**

1 way!

In this case we put any $n_i$ copies of Moby in $B_i$ for $i = 1 \ldots, k$ and this can be done in exactly 1 way!

## ACTIVITY : Example of distinguishable v. indistinguishable (continued)

Finally suppose that both the bins and books are indistinguishable.

**Question:**

Now, how many ways are there to distribute the books among the bins?

*In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!*

## ACTIVITY : Example of distinguishable v. indistinguishable (continued)

**Answer:** 1 way!

We can proceed in two steps.

In step 1 we label the bins with the labels $1, \ldots, k$. Since the bins were indistinguishable to begin with this can be done in exactly 1 way. (That is, any way will work!)

In step 2 we distribute the books among the bins just as we did in the previous question. As we saw there is only 1 way of doing this.

By the multiplication rule, there is exactly $1 \cdot 1 = 1$ way here too.