MCIT 592 Spring 2022 **Homework 2** **Solutions**

1. [10 pts] Let p, q, and r be the following propositions.

p: Comedies are the best type of movie.

q: Everyone loves comedies.

r: Amy will only watch comedies.

Express the following propositions using p, q, r and logical operators.

For this question specifically, a line or two explaining your answer may help, but don't worry too much about providing justification.

(a) Everyone loves comedies and Amy will only watch comedies.

(b) If comedies are not the best type of movie or not everyone loves comedies, then Amy will not watch only comedies.

(c) It is necessary for everyone to love comedies for comedies to be the best type of movie. It is also necessary for everyone to love comedies for Amy to only watch comedies.

(d) Amy will only watch comedies if and only if comedies are the best type of movie.

(e) Amy will not only watch comedies and comedies are the best type of movie.

(f) Comedies being the best type of movie is sufficient for everyone to love comedies and Amy to only watch comedies.

Solution:

(a) $q \wedge r$

(b) $(\neg p \vee \neg q) \Rightarrow \neg r$ or $\neg(p \wedge q) \Rightarrow \neg r$

(c) $(p \vee r) \Rightarrow q$ or $(p \Rightarrow q) \wedge (r \Rightarrow q)$

(d) $r \Leftrightarrow p$

(e) $\neg r \wedge p$

(f) $p \Rightarrow (q \wedge r)$

2. [10 pts] Let A, B, C be three finite sets.

(a) Prove that if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.

(b) Prove that if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

Solution:

(a) Assume $A \subseteq C$ and $B \subseteq C$.
To prove $A \cup B \subseteq C$ let x be an arbitrary element of $A \cup B$.

Case 1 $x \in A$
Then, by the first assumption, $x \in C$.

Case 2 $x \in B$
Then, by the second assumption, $x \in C$.

In both cases, $x \in C$. Therefore $A \cup B \subseteq C$.

(b) Assume $A \subseteq B$ and $A \subseteq C$.
To prove $A \subseteq B \cap C$ let x be an arbitrary element of A.

By the first assumption ($A \subseteq B$), $x \in B$ and by the second, $x \in C$.

Since $x \in B$ and $x \in C$, $x \in B \cap C$.

Since x is an arbitrary element of A, $A \subseteq B \cap C$.

3. [10 pts] Thirteen distinguishable families want to participate in the TV game show "Family Rivalry". Each family consists of a mommy, a daddy, and a child. The game show host must select 20 *people* for the show. The host also wants to see the full family dynamic at play, and thus maximizes the number of times all three members of a family are selected. How many different selections can the host make for this game show?

Solution:

Since we want to maximize the number of times all three members of a family are selected, we want to maximize the number of full families we include. For a 20 person game show, we can fit 6 families of 3. (7 families would equal $7 \times 3 = 21 > 20$ people.) Thus, break this problem into two independent steps.

First, select the full families to include. Since the order in which the families are chosen does not matter, we can calculate this value as $\binom{13}{6}$. Note that within each family, the order in which each family member is chosen does not matter either.

The second step is to choose the remaining two members who will appear on the game show. Since there are 7 families remaining after the first step, there are 21 people left who can appear on the game show. Also observe at this point that it does not matter if the remaining two members come from the same family, as that does not impact the requirement of maximizing the number of full families included. As with before, the order in which these two members are chosen does not matter, so it can be calculated as $\binom{21}{2}$.

Finally, use multiplication rule on the two steps independent:

$$\boxed{\binom{13}{6} \cdot \binom{21}{2}}$$

4. **[10 pts]** Instructor Tiny and the 14 TAs of an online course are having lunch at a round table (they are all vaccinated and boosted). Two seatings are the same when everybody has the same left neighbor and the same right neighbor. How many seatings are possible such that two of the TAs, Biggie and Tupac, do *not* sit next to each other?

Solution:

As seen by Tiny, every seating corresponds to a unique permutation of the 14 TAs in front of him so in general there are 14! distinct seatings. Counting complementarily, we will subtract from this the number of seatings in which Biggie and Tupac are next to each other. To count these, again as seen by Tiny, we count separately:

(a) When Biggie sits to the left of Tupac. Considering Biggie and Tupac fused as one (large) TA, there are 13! such seatings.

(b) When Tupac sits to the left of Biggie. Again, there are 13! such seatings.

By the addition rule there are $2 \cdot 13!$ seatings in which Biggie and Tupac sit next to each other.

Therefore there are $\boxed{14! - 2 \cdot 13!} = 12 \cdot 13!$ seatings in which Biggie and Tupac do not sit next to each other.

5. **[10 pts]** A group of children consists of m families, each family consisting of n siblings. The children decide to arrange themselves in a formation of n rows, each row being a sequence of m children from *different* families. In how many different ways can such a formation be arranged?

Solution:

- Arrange row 1
  - Choose a child from the 1st family (n ways).

    $\ldots$
  - Choose a child from the m'th family (n ways).

  - Order the m children chosen ($m!$ ways).

  Therefore row 1 can be arranged in $n \cdots n \cdot m! = n^m m!$ ways.

- For row 2 each family has only $n-1$ sigblings left to choose for. Similarly to row 1, we can construct row 2 in $(n-1)^m m!$ ways.

  ...

- Similarly, row k can be arranged in $(n-k+1)^m m!$ ways. ...

- Finally row n can be arranged in $m!$ ways because there only one sibling from each family left.

The choices were independent in the sense needed for the multiplication rule so the total number of ways to arrange the formation is

$$(n^m m!)((n-1)^m m!) \cdots (2^m m!)(m!) \;=\; (n\cdot(n-1)\cdots 2\cdot 1)^m (m!)^n \;=\; \boxed{(n!)^m (m!)^n}$$