Module 8.2: Independence

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Independent events

Let (Ω, Pr) be a probability space. Two events $A, B \subseteq \Omega$ are **independent**, write $A \perp B$, when $\mathrm{Pr}[A \cap B] = \mathrm{Pr}[A] \cdot \mathrm{Pr}[B]$.

Note that $A \perp B$ iff $B \perp A$ (independence is **symmetric**).

Independence can be **checked** when we have an alternative way of computing $\mathrm{Pr}[A \cap B]$. This does not happen often.

Still, we can do such checking, for example, in the space of rolls of a fair die, or in the space of random permutations, see next.

Much more often, independence is **assumed** based on our intuition about the problem. Then, it allows us to calculate probabilities by multiplication. We will give such examples also.

ACTIVITY : Checking independence

We roll a fair die twice. As we did before, we assume that corresponding probability space is uniform with 36 outcomes.

Consider the event $A =$ "the **first** roll shows a number divisible by 2" and the event $B =$ "the **second** roll shows a number divisible by 3".

Question. Do you think that events A and B are independent?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : Checking independence (continued)

Answer. Yes.

In fact, even before we defined formally independence, the assumption that probability spaces such as the one we use here are uniform relies on our intuition that the die rolls do not interfere with or influence each other, an intuitive manifestation of independence.

But does the formal definition fit this intuition?

We are about to check this.

Activity : Checking independence (continued)

Counting the outcomes in A we have 3 possibilities for the first roll and 6 for the second, thus $3 \cdot 6 = 18$ by the multiplication rule. Therefore $\Pr[B] = 18/36 = 1/2$.

Counting the outcomes in B we have 6 possibilities for the first roll and 2 for the second, thus $6 \cdot 2 = 12$ by the multiplication rule. Therefore $\Pr[B] = 12/36 = 1/3$.

Counting the outcomes in $A \cap B$ we have 3 possibilities for the first roll and 2 for the second, thus $3 \cdot 2 = 6$ by the multiplication rule. Therefore $\Pr[A \cap B] = 6/36 = 1/6$.

Now $\Pr[A \cap B] = 1/6 = (1/2)(1/3) = \Pr[A] \cdot \Pr[B]$ therefore $A \perp B$.

Independence and random permutations

Problem. Assume that all permutations of a, b, c are equally likely. Are the events $E =$ "a occurs in position 1" and $F =$ "b occurs in position 2" independent?

Answer. In a previous segment we calculated that the probability of a given object occurring in a given position in a random permutation is $1/n$.

Here $n = 3$ therefore $\Pr[E] = \Pr[F] = 1/3$.

However, $E \cap F$ consists of a single outcome, abc, because fixing the positions of a and of b also determines the position of c.

Therefore $\Pr[E \cap F] = 1/3! = 1/6$ but $\Pr[E] \cdot \Pr[F] = 1/9$

E and F are **not** independent.

Properties of independence I

Consider an arbitrary probability space (Ω, Pr) and arbitrary events E, A, B in this space.

Property Ind (i). If $\mathrm{Pr}[A] = 0$ then $A \perp B$ for any B.
In particular, $\emptyset \perp E$ for any E.

Proof. $A \cap B \subseteq A$ so by **P3** (monotonicity) $\mathrm{Pr}[A \cap B] \leq \mathrm{Pr}[A] = 0$.
If $\mathrm{Pr}[A] = 0$ then $\mathrm{Pr}[A \cap B] = 0$. $A \perp B$ follows.

Property Ind (ii). $\Omega \perp E$ for any E.

The proof is in the segment entitled "Proofs of independence properties".

Properties of independence II

Consider an arbitrary probability space $(\Omega, \Pr)$ and arbitrary events E, A, B in this space.

Property Ind (iii). If $A \perp B$

then $\Pr[A \cup B] = 1 - (1 - \Pr[A])(1 - \Pr[B])$.

Proof. By **P6** (inclusion-exclusion) and using independence, the LHS becomes

$$. \Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B] = \Pr[A] + \Pr[B] - \Pr[A] \cdot \Pr[B]$$

By algebraic manipulation this equals the RHS.

Property Ind (iv). $A \perp B$ iff $\overline{A} \perp B$ iff $A \perp \overline{B}$ iff $\overline{A} \perp \overline{B}$

The proof is in the segment entitled "Proofs of independence properties".

Independent vs. disjoint

Don't confuse "independent" with "disjoint"! In fact, disjoint events are typically **not** independent of each other!

Proposition. Let A, B be disjoint events in $(\Omega, \Pr)$. If $A \perp B$ then at least one of A, B has probability 0.

Proof. If A, B are both disjoint and independent then by **Ind (iii)** and by **P2** (addition rule) we have:

. $$1 - (1 - \Pr[A])(1 - \Pr[B]) = \Pr[A \cup B] = \Pr[A] + \Pr[B]$$

. $$\Pr[A] + \Pr[B] - \Pr[A] \cdot \Pr[B] = \Pr[A] + \Pr[B]$$

. $$\Pr[A] \cdot \Pr[B] = 0.$$

Corollary. If $E \perp \overline{E}$ then $\Pr[E]$ is 0 or 1.

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Quiz

We know from **Ind (i)** and **Ind (ii)** that $\emptyset \perp \emptyset$ and $\Omega \perp \Omega$. Are $\emptyset$ and Ω the **only** events E such that $E \perp E$?

(A) True

(B) False

Answer

We know from **Ind (i)** and **Ind (ii)** that $\emptyset \perp \emptyset$ and $\Omega \perp \Omega$. Are $\emptyset$ and Ω the **only** events E such that $E \perp E$?

(A) True

Incorrect. Any other event of probability 0 or 1 would have this property.

(B) False

Correct. We can construct a space with an non-empty event of probability 0.

More information

Let $p = \Pr[E]$. Since $E \cap E = E$ it follows that $E \perp E$ iff $p = p^2$. This last holds iff $p = 0$ or $p = 1$.

Now consider the probability space $(\Omega, \Pr)$ where $\Omega = \{w_1, w_2\}$ and also $\Pr[w_1] = 0$ as well as $\Pr[w_2] = 1$.

In this space, taking $E = \{w_1\}$ we have $\Pr[E] = 0$ therefore $E \perp E$.