Module 8.6: The Chain Rule

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Independence and conditional probability

Intuitively, $\Pr[E|U]$ should be the same as $\Pr[E]$ when E does not really depend on U. Indeed:

Proposition. For any two events A, B in the same probability space the following two statements are **equivalent**:

$$\text{(i)}\ \ A \perp B \qquad \text{(ii)}\ \ \Pr[B] = 0 \text{ or } (\ \Pr[B] \neq 0 \text{ and } \Pr[A|B] = \Pr[A]\ )$$

Proof. To prove the logical equivalence of (i) and (ii) we have to show that: (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Assume $A \perp B$. When $\Pr[B] \neq 0$ we have

$$\Pr[A|B] \ = \Pr[A \cap B]/\Pr[B] \ = \ (\Pr[A] \cdot \Pr[B])/\Pr[B] \ = \ \Pr[A]$$

(ii) $\Rightarrow$ (i): If $\Pr[B] = 0$ then by **Ind (i)** $A \perp B$. If $\Pr[B] \neq 0$ then $\Pr[A|B] = \Pr[A]$ becomes $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$ hence $A \perp B$.

The chain rule

We regard the equality $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B|A]$ as **generalizing** the equality $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$ that defines independence. For more than two events we have a further generalization:

Proposition (The chain rule). For any events A, B, C in the same probability space we have

$$\Pr[A \cap B \cap C] \;=\; \Pr[A] \cdot \Pr[B|A] \cdot \Pr[C|A \cap B]$$

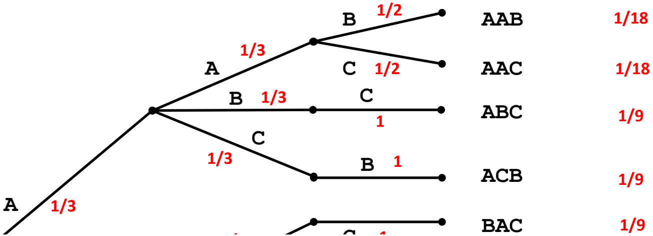
For any events $A_1, \ldots, A_n$ in the same probability space we have

$$\Pr[A_1 \cap A_2 \cap A_3 \cdots \cap A_n] \;=$$
$$= \; \Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2] \cdots \Pr[A_n|A_1 \cap \cdots \cap A_{n-1}]$$

The proof is given in the segment entitled "Conditional probability rules."

Tree diagrams and the chain rule I

Here is the upper part of the Monty Hall "tree of all possibilities":



Why is $\Pr[\texttt{AAB}] = 1/18$?

Define $E =$ "car is behind door $\texttt{A}$", $F =$ "Ann chooses door $\texttt{A}$",
$G =$ "Monty opens door $\texttt{B}$".

Then $E \cap F \cap G = \{\texttt{AAB}\}$.

The chain rule gives: $\Pr[E \cap F \cap G] \;=\; \Pr[E] \cdot \Pr[F|E] \cdot \Pr[G|E \cap F]$

Tree diagrams and the chain rule II

The chain rule gives: $\Pr[E \cap F \cap G] = \Pr[E] \cdot \Pr[F|E] \cdot \Pr[G|E \cap F]$

We have $\Pr[E] = 1/3$

We have $\Pr[F|E] = \Pr[F] = 1/3$

And we have $\Pr[G|F \cap E] = 1/2$

Therefore $\Pr[\texttt{AAB}] = (1/3)(1/3)(1/2) = 1/18.$

In general, the branches are labeled with conditional probabilities and along each branch the chain rule computes the probability of the outcome.