Questions

This assignment is due in about one week from when the assignment opens. The exact deadline and full instructions for submission are provided in Coursera. To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - **no collaboration is allowed**.

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1. **[10 pts]** Over the years, Francesca, who loves fashion, has become an adamant lover of shoes. As such, she has many types in her collection, including: (1) sneakers, (2) flip flops, (3) sandals, (4) slippers, and (5) combat boots. Francesca realizes that she has too many shoes, so she decides to have a garage sale. $n \geq 3$ TAs, including Sasha, Kevin, and Arnav, show up to the sale, and since they're Francesca's friends, Francesca kindly offers to discount all of her shoes. However, in order to be eligible for the discount, Francesca mandates that each customer can only buy at most one pair of each type (ex. someone can buy a pair of sneakers and a pair of flip flops, but not two pairs of flip flops). Sasha must buy 1 pair of each of the 5 types, Kevin must also buy 1 pair of each of the 5 types, and Arnav must buy 1 pair of exactly 2 of the 5 types. All other customers can buy 0 or 1 pair of all of the 5 types. Assume that Francesca has enough shoes for each of the n TA's to purchase 1 pair of each type. How many distinct ways can these n customers buy Francesca's shoes and be eligible for the discount? Treat a pair of shoes as a single item.

2. **[10 pts]** Let $A = \{2, 3\}$, $B = \{3, 4, 5\}$, and $C = \{3, 5, 7, 9\}$ and let P be the set whose elements are all the proper subsets of $(A \cup B) \setminus C$. List all the elements of 2^P. Show your work.

3. **[10 pts]** $n \geq 2$ distinguishable Hogwarts students participate in Professor Snape's experiment. Each student is given potion A, or philter B, or neither, or both. We know that Harry and Hermione are the only students among the n that are given both A and B. In how many distinct ways could Snape have distributed his experimental liquids?

4. **[10 pts]** Prove that each of the following integers is composite (not prime).

(a) $3^{222} + 1$

(b) $2^{35}(2^{33} - 1) + 1$

5. **[10 pts]** z is said to be a *Broadway integer* (this is not a standard term, it was made up for this problem) when $z = 4k + 2$ for some integer k. Prove that the difference of the squares of two Broadway integers is always divisible by 16.