CIT 592 Spring 2022 Homework 9

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1. [10 pts] The 10 decimal digits, $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are arranged in a uniformly random permutation. We denote by a the integer formed in base 10 by the first five positions in this permutation and by b the integer formed in base 10 by the last five positions in this permutation (either a or b may begin with 0 which in such a case is ignored). For example, if the random permutation is 8621705394 then $a = 86217$ and $b = 5394$. Consider the probability space whose outcomes are these random permutations and a random variable X defined on this probability space such $X = 1$ when the product ab is even and $X = 0$ when that product is odd. Calculate $\mathrm{E}[X]$.

Solution.

Rules of thumb:
We know that for an integer, like a and b, the parity is the same as the parity of the last digit.
Also we know that:
The product of two odd integers is odd.
The product of an even integer and an odd integer is even.
The product of two even integers is even.
From the rules of thumb we know that in order for $X = 0$, both a and b need to be odd, which means position 5 and position 10 should have an odd number. For $X = 1$, both a and b need to be even, or one of them is even, which means for position 5 and position 10, at least one of them should be even .
Step 1:
we computer the number of permutation of the 10 decimal digits.We computer the cardinality of the uniform probability space Ω using the multiplication rule:
$|\Omega| = 10!$.
Step 2:

Calculate $P_r[X = 0]$.

Among 5 odd numbers - we have 5 ways to pick for the 5th position, and 4 ways to pick for the 10th position. We have 8! ways for the other 8 positions. By multiplication rule the number of permutations to make $X = 0$ is $5 * 4 * 8! = 20 * 8!$. Therefore $P_r[X = 0] = 20 * 8!/10! = 2/9$

Step 3:

Calculate $P_r[X = 1]$.

Using P4, $P_r[X = 1] = 1 - P_r[X = 0] = 7/9$

Step 4:

Calculate $E[X]$.

By the definition of Expectation, we know that:

$E[X] = 1 * P_r[X = 1] + 0 * P_r[X = 0] = 7/9$

2. **[10 pts]** The digits 1, 4, and 7 are randomly arranged to form a one digit number and a two-digit number. Each digit can only be used once; for example, if the one-digit number is 7, then the two-digit number is either 14 or 41. What is the expected value of the product of the two numbers?

Solution.

The possible arrangement are as below:
1 and 47
1 and 74
4 and 17
4 and 71
7 and 14
7 and 41
We computer the cardinality of the uniform probability space Ω using the multiplication rule:
$|\Omega| = 6$.
Let Event T = "the two digit number in the defined arrangement", then
$Val[T] = \{47, 74, 17, 71, 14, 41\}$, and
$P_r[T = 47] = P_r[T = 74] = P_r[T = 17] = P_r[T = 71] = P_r[T = 14] = P_r[T = 41] = 1/6$
So by the definition of expectation:
$E[T] = 47 * P_r[T = 47] + 74 * P_r[T = 74] + 17 * P_r[T = 17] + 71 * P_r[T = 71] + 14 * P_r[T = 14] + 41 * P_r[T = 41]$
$= 44$

3. **[10 pts]** You have a standard deck of cards (see the preamble to hw08). You divide this deck in half by selecting uniformly at random 26 of the deck's 52 cards and placing them in your left hand. You hold the remaining 26 cards in your right hand. Prove that the expected number of aces in your right hand is 2.

Solution.

We want E[X] where X is the random variable that returns the number of aces in the right hand.
We can express X as $X = I_1 + I_2 + I_3 + I_4$
Where I_i is the indicator r.v. of the event $L_i =$ "ace i ends up in the right hand".
Since for each ace there are only two outcomes - (1) it's in the right hand, and (2) it's in the left hand, so $P_r[L_i] = 1/2$.
Therefore $E[I_i] = P_r[L_i] = 1/2$.
Using linearity of expectation we calculate $E[X] = 1/2 + 1/2 + 1/2 + 1/2 = 2$.

4. **[10 pts]** Jay has \$500 in the bank when he decides to try a savings experiment. On each day $i \in [1..30]$, Jay flips a fair coin. If it comes up heads, he deposits i dollars into the bank; if it comes up tails, he withdraws \$10. How many dollars should he expect to have in the bank after 30 days?

Solution.

We want $E[X]$ where X is the random variable that returns the amount of money that changed in Jay's bank account, either deposit or withdraw.

We know that $X = X_1 + X_2 + X_3 + \cdots + X_{30}$. Using linearity of expectation:

$E[X] = E[X_1 + X_2 + X_3 + \cdots + X_{30}]$

$=E[X_1] + E[X_2] + E[X_3] + \cdots + E[X_{30}]$

For each day, there are only two outcomes - Jay throws a head (Event H) or tail (Event T), and since it's a fair coin, $P_r[H] = P_r[T] = 1/2$.

So for X_k, we can express $X_k = c_{H_k} * P_r[H] + c_T * P_r[T]$. Since $P_r[H] = P_r[T] = 1/2$, $c_{H_k} = k$, c_T is a constant, which is -10(it means to withdraw money), we can express $E[X_k] = k * 1/2 + (-10) * 1/2 = k/2 - 5$.

Thus we can calculate:

$E[X] = E[X_1] + E[X_2] + E[X_3] + \cdots + E[X_{30}]$

$=(1/2 - 5) + (2/2 - 5) + (3/2 - 5) + \cdots + (30/2 - 5)$

$=(1 + 2 + 3 + \cdots + 30)/2 - 5 * 30$

$=82.5$

5. [10 pts] Let A be a set of $n \geq 2$ distinct numbers and let $a_1 a_2 \cdots a_n$ be a permutation of A. For $i = 2, 3, \ldots, n$ we say that position i in the permutation is a *step* if $a_{i-1} < a_i$. We also go ahead and just consider position 1 a step. What is the expected number of steps in a random permutation of A?

Solution.

Denote C be the r.v. that returns the number of steps in a permutation of A. Then we are trying to compute $E[C]$.
Since we consider position 1 a step, we know $C = I_2 + I_3 + \cdots + I_n$
where, for $k = 2, 3, \ldots, n$, I_k is the indicator r.v. of the event $A_k =$ "position k in the permutation is a step", which means $a_{k-1} < a_k$.
Recall that $E[I_k] = P_r[I_k = 1] = P_r[A_k]$
Since there are only two outcomes, a_k is a step or is not a step, $P_r[A_k] = 1/2$.
By linearity of expectation
$E[C] = E[I_2] + E[I_3] + \cdots + E[I_n]$
$=1/2 + 1/2 + \cdots + 1/2 = (n-1) * 1/2 = (n-1)/2$