Module 4.4: Integer Intervals

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Integer intervals

An **integer interval** $[m..n]$ (where $m \leq n$) is the set of **all** integers that lay between m and n, inclusive. In set-builder notation:

$$[m..n] = \{k \in \mathbb{Z} \mid m \leq k \leq n\}$$

Example. $[6..10] = \{6, 7, 8, 9, 10\}$, a set of 5 elements.

Memorize: $\quad |\,[m..n]\,| = n - m + 1.$

Problem. How many two-digit numbers are there between 1 and 100?

Answer. $|\,[10..99]\,| = 99 - 10 + 1 = 90$

Quiz I

How many elements does the following set have

$$([10..30] \cup [20..40]) \setminus ([10..30] \cap [20..40])$$

A. 20
B. 42
C. 21

Answer

What is the cardinality of the following set?

$$([10..30] \cup [20..40]) \setminus ([10..30] \cap [20..40])$$

A. 20

Correct. The same as the union of $[10..19]$ and $[31..40]$, which are disjoint and contain 10 elements each.

B. 42

Incorrect. Did you forget to remove the elements in $[10..30] \cap [20..40]$?

C. 31

Incorrect. Remember to count the elements in $[10..30] \cup [20..40]$ only once.

Quiz II

What is the value of the following sum of cardinalities?

$$\sum_{i=0}^{10} |[5i \,..\, (5i+3)]|$$

A. 40

B. 44

C. 45

Answer

What is the value of the following sum of cardinalities?

$$\sum_{i=0}^{10} |[5i \,..\, (5i+3)]|$$

A. 40

Incorrect. i takes 11 values, rather than 10.

B. 44

Correct. i takes 11 values, and for every value of i, we have 4 unique elements.

C. 45

Incorrect.

Functions and integer intervals I

Examples.

$$f : [0..10] \to [0..20]$$

where $f(x) = x + 10.$

$f(5) = 15 \qquad 5 \mapsto 15$

$f(0) = 10 \qquad 0 \mapsto 10$

$f(10) = 20 \qquad 10 \mapsto 20$

$\text{Ran}(f) = [10..20].$

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$$g : [-20..10] \to [0..20]$$

where $g(y) = \text{abs}(y).$

$g(5) = 5 \qquad 5 \mapsto 5$

$g(-5) = 5 \qquad -5 \mapsto 5$

$g(-20) = 20 \qquad -20 \mapsto$
20

$g(10) = 10 \qquad 10 \mapsto 10$

$\text{Ran}(g) = [0..20].$

Functions and integer intervals II

Another example.

$$h : [0..n] \to [0..n]$$
where $h(z) \; = \; n - z$.

$h(1) \; = n - 1 \qquad 1 \mapsto \; n - 1$

$h(0) \; = n \qquad 0 \mapsto n$

$f(n) \; = 0 \qquad n \mapsto 0$

$f(n-1) \; = 1 \qquad n - 1 \mapsto 1$

$\mathrm{Ran}(h) = \; [0..n].$

Functions and integer intervals III

Yet another example.

$$t : [0..2n] \to [0..n] \quad \text{where } t(w) = \begin{cases} \frac{w}{2} & \text{if } w \text{ is even} \\ \frac{w-1}{2} & \text{if } w \text{ is odd} \end{cases}$$

$$t(0) = 0 \qquad 0 \mapsto 0\,.$$

$$t(1) = 0 \qquad 1 \mapsto 0$$

$$t(2n) = n \qquad 2n \mapsto n$$

$$t(2n-1) = n-1 \qquad 2n-1 \mapsto n-1$$

$$t(2n-2) = n-1 \qquad 2n-2 \mapsto n-1$$

$$\mathrm{Ran}(t) = [0..n].$$

ACTIVITY : Functions as elements

Let $A = B = C = [1..n]$.

This activity concerns functions with domain B^A and codomain 2^C. These are *functions of functions*, in the sense that they map elements of B^A (which are themselves functions from A to B) to elements of 2^C (which are subsets $S \subseteq C$).

Question: How many functions with domain B^A and codomain 2^C are there?

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

Activity : Functions as elements (Continued)

Answer: Observe that $|A| = |B| = |C| = n$. Hence, $|B^A| = |B|^{|A|} = n^n$ and $|2^C| = 2^{|C|} = 2^n$. Therefore, the number of functions with domain B^A and codomain 2^C is

$$|2^C|^{|B^A|} = (2^n)^{(n^n)} = 2^{n \cdot n^n} = 2^{n^{n+1}}\ .$$

ACTIVITY : Functions as elements (Continued)

What do functions from B^A to 2^C look like? We give two examples.

Example 1. A function $g : [1..n]^{[1..n]} \to 2^{[1..n]}$ that maps any function $f : [1..n] \to [1..n]$ to the subset of $[1..n]$ given by $\{x \mid f(x) = 1\}$.

Example 2. A function $h : [1..n]^{[1..n]} \to 2^{[1..n]}$ that maps any function $f : [1..n] \to [1..n]$ to the set $\mathrm{Ran}(f) \subseteq [1..n]$.

Question: If $f : [1..n] \to [1..n]$ is the identity function that maps each element of $[1..n]$ to itself, then what are $g(f)$ and $h(f)$?

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Functions as elements (Continued)

Answer:

The only element of $[1..n]$ that f maps to 1 is 1, so $g(f) = \{1\}$.

The range of f is $[1..n]$, so $h(f) = [1..n]$.