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1. [10 pts] Let X and Y be independent random variables such that $\text{Var}[X] = 5.3$ and $\text{Var}[Y] = 8.9$. What is the standard deviation of $3X + 2Y$?

Solution.

By definition we know that $\sigma[3X + 2Y] = \sqrt{\text{Var}[3X + 2Y]}$.

Since X and Y are independent random variables,

$$\text{Var}[3X + 2Y] = \text{Var}[3X] + \text{Var}[2Y]$$

$$= 3^2 * \text{Var}[X] + 2^2 * \text{Var}[Y]$$

$$= 9 * 5.3 + 4 * 8.9 = 83.3$$

$$\text{Therefore } \sigma[3X + 2Y] = \sqrt{\text{Var}[3X + 2Y]} = \sqrt{83.3} \approx 9.13$$

2. [10 pts] Suppose James has a garden. Let X be the random variable representing the heights of the flowers in the garden, and let Y be the random variable representing the number of petals the flowers have. Suppose that X and Y are non-negative and independent. Help James prove that $X^2 \perp Y^2$.

Solution.

Solution 1:

Since X is the random variable representing the heights of the flowers, Y is the random variable representing the number of petals the flower have, we know that $x \geq 0$ and $y \geq 0$.

Let $X: \Omega_1 \rightarrow R$ and $Y: \Omega_2 \rightarrow R$

Then $X^2: \Omega_1 \rightarrow R$ and $Y^2: \Omega_2 \rightarrow R$, and:

$$\forall \omega_1 \in \Omega_1 \quad X^2(\omega_1) = (X(\omega_1))^2 = x^2$$

$$\forall \omega_2 \in \Omega_2 \quad Y^2(\omega_2) = (Y(\omega_2))^2 = y^2$$

Since $x \geq 0$ and $y \geq 0$, x^2 and y^2 are only decided by x and y , which means the function $Val(X)$ to $Val(X^2)$ have one to one correspondence and function $Val(Y)$ to $Val(Y^2)$ have one to one correspondence too.

Thus X and X^2 have the same distribution, and Y and Y^2 have the same distribution.

$$\text{So } P_r[X(\omega_1)] = P_r[X^2(\omega_1)] \implies P_r[X = x] = P_r[X^2 = x^2]$$

$$P_r[Y(\omega_2)] = P_r[Y^2(\omega_2)] \implies P_r[Y = y] = P_r[Y^2 = y^2]$$

By definition X and Y are independent then for $\{\omega_1 \in \Omega_1 | X(\omega_1) = x\}$, $\{\omega_2 \in \Omega_2 | Y(\omega_2) = y\}$

$$P_r[(X = x) \cap (Y = y)] = P_r[X = x] * P_r[Y = y]$$

$$P_r[(X^2 = x^2) \cap (Y^2 = y^2)] = P_r[(X = x) \cap (Y = y)] = P_r[X = x] * P_r[Y = y] = P_r[X^2 = x^2] * P_r[Y^2 = y^2]$$

$$\text{In short we noted that } P_r[(X^2 = x^2) \cap (Y^2 = y^2)] = P_r[X^2 = x^2] * P_r[Y^2 = y^2]$$

By definition we know that $X^2 \perp Y^2$.

Solution 2:

By definition X and Y are independent when

$$\forall x \in Val(X) \quad \forall y \in Val(Y) \quad (X = x) \perp (Y = y)$$

Since X and Y are non-negative, we know that $\forall x \in Val(X) x \geq 0$ and $\forall y \in Val(Y) y \geq 0$. Since X and Y are independent:

$$P_r[X = x] = P_r[X = x | Y = y] = P_r[X = x | Y = y^2]$$

$$\text{Let } k = x^2, \text{ then } P_r[X = k] = P_r[X = k | Y = y^2]$$

$$\text{Which can be expressed as } P_r[X = x^2] = P_r[X = x^2 | Y = y^2]$$

By definition we know that $X^2 \perp Y^2$.

3. [10 pts] Suppose you roll $n \geq 1$ fair dice. Let X be the random variable for the sum of their values, and let Y be the random variable for the number of times an odd number comes up. Prove or disprove: X and Y are independent.

Solution.

We guess X and Y are not independent. So we try to disprove it.

We know that:

X = the sum of the value of n fair dice

Y = the number of dice that show an odd number

By definition X and Y are independent when

$$\forall x \in Val(X) \forall y \in Val(Y) (X = x) \perp (Y = y)$$

Then the negation of it is that $\exists x \in Val(X), \exists y \in Val(Y), (X = x)$ and $(Y = y)$ are not independent.

Let $n = 4$, then $Val(X) = [4..24]$ and $Val[Y] = [0..4]$

Let $x = 5$ and $y = 2$. Then we can calculate:

$$P_r[X = 5] = 4/6^4$$

$$P_r[Y = 2] = 6/2^4$$

$P_r[X = 5 \cap Y = 2] = 0$ since in order to get $X = 5$ we would need 3 dice show 1 and one shows 2, so when $X = 5$, Y would always be 3.

From above we can tell $P_r[X = 5 \cap Y = 2] \neq P_r[X = 5] * P_r[Y = 2]$, thus $P_r[X = 5]$ and $P_r[Y = 2]$ not independent. So we proved $\exists x \in Val(X), \exists y \in Val(Y), (X = x)$ and $(Y = y)$ are not independent, which is the negation of $X \perp Y$.

Therefore we disproved that random variables X and Y are independent.

4. [10 pts] Suppose that you generate a 12-character password by selecting each character independently and uniformly at random from $\{\mathbf{a}, \mathbf{b}, \dots, \mathbf{z}\} \cup \{\mathbf{A}, \mathbf{B}, \dots, \mathbf{Z}\} \cup \{0, 1, \dots, 9\}$.
- (a) What is the probability that exactly 6 of the characters are digits?
 - (b) What is the expected number of digits in a password?
 - (c) What is the variance of the number of digits in a password?

Solution.

- (a) For each character in the password, the probability that it is a digit is:

$$p = 10/62 = 5/31$$

Let D be the binomial r.v with parameters $Val(D) = [0..12]$ and $p = 5/31$.

$$\text{Thus } Pr[I_k = 0] = 1 - p = 52/62 = 26/31$$

Here $n = 12$, $k = 6$, so we compute the probability that exactly 6 of the characters are digits is:

$$Pr(D = 6) = \binom{n}{k} p^k (1 - p)^{n-k} = \binom{12}{6} \left(\frac{5}{31}\right)^6 \left(\frac{26}{31}\right)^6$$

- (b) Let D_k be the event that the k th character in the password is a digit, then we have

$$Pr[D_k] = p = 5/31.$$

Let I_k be the indicator r.v. of the event D_k .

$$E[I_k] = Pr[I_k = 1] = Pr[D_k] = p = 5/31$$

$$B = D_1 + D_2 + \dots + D_{12}$$

By linearity of expectation, we know that:

$$E[D] = E[I_1] + E[I_2] + \dots + E[I_{12}] = np = 12 * 5/31 = 60/31 \approx 1.9$$

- (c) Since the characters in the password can be repeated, Event D_1, D_2, \dots, D_{12} are mutually independent, thus the indicator r.v I_k are also mutually independent.

Therefore:

$$Var[D] = Var[I_1] + Var[I_2] + \dots + Var[I_{12}]$$

$$= p(1 - p) + p(1 - p) + \dots + p(1 - p)$$

$$= 12 * p(1 - p)$$

$$= 12 * 5/31 * 26/31 \approx 1.62$$

5. [10 pts] Suppose Jay shoots a basketball. Let X be the Bernoulli random variable that returns 1 if he makes the shot, and 0 if he misses. Let Y be the Bernoulli random variable (independent of X) that returns 1 if he hits the backboard, and 0 if he does not hit it. X and Y both have parameter $1/2$. Let Z be the random variable that returns the remainder of the division of $X + Y$ by 2.
- Prove that Z is also a Bernoulli random variable, also with parameter $1/2$.
 - Prove that X, Y, Z are pairwise independent but not mutually independent.
 - By computing $\text{Var}[X + Y + Z]$ according to the alternative formula for variance and using the variance of Bernoulli r.v.'s, verify that $\text{Var}[X + Y + Z] = \text{Var}[X] + \text{Var}[Y] + \text{Var}[Z]$ (observe that this also follows from the proposition on slide 5 of the lecture segment entitled "Binomial distribution").

Solution.

- We know that $\text{Val}(X) = \text{Val}(Y) = 0, 1$

Since Z is the random variable that returns the remainder of the division of $(X + Y)/2$, we know that:

When $X = 0$ and $Y = 0$, $Z = 0$

When $X = 1$ and $Y = 0$, $Z = 1$

When $X = 1$ and $Y = 1$, $Z = 0$

When $X = 0$ and $Y = 1$, $Z = 1$

From above we can tell that $\text{Val}(Z) = 0, 1$ and $P_r[Z = 0] = P_r[Z = 1] = 1/2$

By the definition we proved that Z is also a Bernoulli random variable, also with parameter $p = 1/2$.

- It's given that X and Y are independent. Now we need to prove (1) X and Z are independent, and (2) Y and Z are independent.

Since X , Y and Z are all Bernoulli random variables with parameter $1/2$, we know that $P_r[X = 0] = P_r[X = 1] = P_r[Y = 0] = P_r[Y = 1] = P_r[Z = 0] = P_r[Z = 1] = 1/2$

We also know that the possible outcomes are as below:

When $X = 0$ and $Y = 0$, $Z = 0$

When $X = 1$ and $Y = 0$, $Z = 1$

When $X = 1$ and $Y = 1$, $Z = 0$

When $X = 0$ and $Y = 1$, $Z = 1$

(1) $P_r[X = 0 \cap Z = 0] = 1/4$ since there is only one outcome "when $X = 0$ and $Y = 0$, $Z = 0$ " out of the 4 outcomes.

So $P_r[X = 0] * P_r[Z = 0] = 1/2 * 1/2 = 1/4 = P_r[X = 0 \cap Z = 0]$.

Similarly we can prove that:

$$P_r[X = 0] * P_r[Z = 1] = 1/2 * 1/2 = 1/4 = P_r[X = 0 \cap Z = 1].$$

$$P_r[X = 1] * P_r[Z = 0] = 1/2 * 1/2 = 1/4 = P_r[X = 1 \cap Z = 0].$$

$$P_r[X = 1] * P_r[Z = 1] = 1/2 * 1/2 = 1/4 = P_r[X = 1 \cap Z = 1].$$

Above discussion would cover all the possible outcomes, so we can conclude that for

$$\forall a \in Val(X) \forall c \in Val(Z), P_r[(X = a) \cap (Z = c)] = P_r[X = a] * P_r[Z = c].$$

By definition, X and Z are independent.

(2) $P_r[Y = 0 \cap Z = 0] = 1/4$ since there is only one outcome "when $X = 0$ and $Y = 0$, $Z = 0$ " out of the 4 outcomes.

$$\text{So } P_r[Y = 0] * P_r[Z = 0] = 1/2 * 1/2 = 1/4 = P_r[Y = 0 \cap Z = 0].$$

Similarly we can prove that:

$$P_r[Y = 0] * P_r[Z = 1] = 1/2 * 1/2 = 1/4 = P_r[Y = 0 \cap Z = 1].$$

$$P_r[Y = 1] * P_r[Z = 0] = 1/2 * 1/2 = 1/4 = P_r[Y = 1 \cap Z = 0].$$

$$P_r[Y = 1] * P_r[Z = 1] = 1/2 * 1/2 = 1/4 = P_r[Y = 1 \cap Z = 1].$$

Above discussion would cover all the possible outcomes, so we can conclude that for

$$\forall b \in Val(Y) \forall c \in Val(Z), P_r[(Y = b) \cap (Z = c)] = P_r[Y = b] * P_r[Z = c].$$

By definition, Y and Z are independent.

Lastly, we know that:

In order to let X,Y,Z be mutually independent, for $\forall a \in Val(X) \forall b \in Val(Y) \forall c \in Val(Z)$,

$$P_r[(X = a) \cap (Y = b) \cap (Z = c)] = P_r[X = a] * P_r[Y = b] * P_r[Z = c]$$

Therefore to prove X,Y,Z are not mutually independent, we can prove it by finding a counterexample to show that:

$$\exists a \in Val(X) \exists b \in Val(Y) \exists c \in Val(Z) P_r[(X = a) \cap (Y = b) \cap (Z = c)] \neq P_r[X = a] * P_r[Y = b] * P_r[Z = c].$$

We know that $(X = 1) \cap (Y = 1) \cap (Z = 1) = \emptyset$ and $P_r[X = 1] * P_r[Y = 1] * P_r[Z = 1] = 1/2 * 1/2 * 1/2 = 1/8$,

$$\text{so } P_r[(X = 1 \cap Y = 1 \cap Z = 1)] \neq P_r[X = 1] * P_r[Y = 1] * P_r[Z = 1].$$

Further $P_r[(X = 0 \cap Y = 0 \cap Z = 0)] = 1/4$, and $P_r[X = 0] * P_r[Y = 0] * P_r[Z = 0] = 1/2 * 1/2 * 1/2 = 1/8$

$$\text{So } P_r[X = 0 \cap Y = 0 \cap Z = 0] \neq P_r[X = 0] * P_r[Y = 0] * P_r[Z = 0].$$

By the definition, we can conclude that X, Y and Z are not mutually independent.

(c) Step 1:

We first calculate the expectation of X, Y and Z. Since they are all Bernoulli r.v with parameter p, we know that :

$$E[X] = E[Y] = E[Z] = p = 1/2$$

Step 2:

Since X,Y and Z are pairwise independent, we know that:

$$E[XY] = E[X] * E[Y], E[XZ] = E[X] * E[Z], E[YZ] = E[Y] * E[Z]$$

Step 3:

By variance of Bernoulli r.v., we know that When X is a Bernoulli r.v. with parameter p, then X^2 is also Bernoulli with the same distribution and $X = X^2$. Thus $E[X^2] = E[X]$. Similarly we know $E[Y^2] = E[Y]$, and $E[Z^2] = E[Z]$.

Step 4:

By Variance for Binomial II, we know that:

$$Var[X] = Var[Y] = Var[Z] = np(1-p) = 1 * 1/2 * (1 - 1/2) = 1/4$$

$$\text{Thus } Var[X] + Var[Y] + Var[Z] = 1/4 + 1/4 + 1/4 = 3/4$$

Step 5:

By the alternative formula for variance, we know that:

$$\begin{aligned} Var[X + Y + Z] &= E[(X + Y + Z)^2] - (E[X + Y + Z])^2 \\ &= E[(X + Y + Z) * (X + Y + Z)] - (E[X] + E[Y] + E[Z])^2 \\ &= E[X^2 + Y^2 + Z^2 + 2XY + 2XZ + 2YZ] - (E[X] + E[Y] + E[Z])^2 \\ &= E[X^2] + E[Y^2] + E[Z^2] + E[2XY] + E[2XZ] + E[2YZ] - (E[X] + E[Y] + E[Z])^2 \\ &= E[X] + E[Y] + E[Z] + 2E[XY] + 2E[XZ] + 2E[YZ] - (E[X] + E[Y] + E[Z])^2 \\ &= 1/2 + 1/2 + 1/2 + 2 * 1/2 * 1/2 + 2 * 1/2 * 1/2 + 2 * 1/2 * 1/2 - (1/2 + 1/2 + 1/2)^2 \\ &= 3/4 \end{aligned}$$

Step 6:

From above we can tell that

$$Var[X + Y + Z] = Var[X] = Var[Y] = Var[Z] = 3/4.$$

Verification is completed.