$\boxed{\textbf{MCIT 592 Online - Spring 2022} \text{ (instructor Val Tannen)}}$

Additional Problems (Packet 2)

With Solutions

1. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a *very brief* explanation of your answer.

   (a) If X_1 and X_2 are Bernoulli random variables with $\Pr[X_1 = 1] = 1/2$ and $\Pr[X_2 = 1] = 1/3$ then $\mathrm{E}[X_1 - X_2] = 0$.

   (b) For any two events A, B in the same probability space $(\Omega, \Pr)$ such that $\Pr[B] \neq 0$ we have $\Pr[A \cup B \mid B] = 1$.

   ANSWER

   (a) $\boxed{FALSE.}$ By linearity of expectation and by the formula for expectation of Bernoulli random variables:

   $$\mathrm{E}[X_1 - X_2] = \mathrm{E}[X_1] - \mathrm{E}[X_2] = \Pr[X_1 = 1] - \Pr[X_2 = 1] = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \neq 0$$

   (b) $\boxed{TRUE.}$ Since $B \subseteq A \cup B$ we have $(A \cup B) \cap B = B$. Therefore

   $$\Pr[A \cup B \mid B] = \frac{\Pr[(A \cup B) \cap B]}{\Pr[B]} = \frac{\Pr[B]}{\Pr[B]} = 1$$

2. My 6th grade teacher of Russian was unable to pay attention to what we were answering and it appeared to us that he was assigning grades completely randomly. Let's assume that his grading rubric consisted of tossing a fair coin six times, counting the number k of heads and assigning the grade $4 + k$ (our grades were in the 1-10 range).

   (a) What was the probability that I would get a 10?

   (b) What was the probability of the following event: "my grade was divisible by 4 or (non-exclusive or!) it was bigger than or equal to Lady Gaga's shoe size (a 6)"?

   ANSWER

   We work with a uniform probability space Ω with 2^6 outcomes. Each outcome is a sequence of length 6 of H's and T's and each outcome has probability $1/2^6$.

(a) To get a 10 we must have $k = 6$ therefore the event E of interest consists of the one outcome with exactly 6 heads.

$$\Pr[E] \;=\; \frac{|E|}{|\Omega|} = \frac{1}{2^6}$$

(b) The grade can be (4 or 8) or (6 or 7 or 8 or 9 or 10) therefore $k = 0$ or $k \geq 2$. The event G of interest consists of sequences with no heads or with two or more heads. Its complement, $\overline{G}$ consists of sequences with exactly one head. The one head can be in any of the 6 flips so there are 6 such sequences. Therefore

$$\Pr[G] \;=\; 1 - \Pr[\overline{G}] \;=\; 1 - \frac{|\overline{G}|}{|\Omega|} = 1 - \frac{6}{2^6}$$

3. (20pts)

A fair coin is flipped *twice*. Let $(\Omega, \Pr)$ be the resulting probability space. Let X_H be random variable defined on Ω that returns the number of heads observed and X_T similarly the number of tails observed.

(a) Describe the probability space $(\Omega, \Pr)$. That is, list the outcomes and their probabilities.

(b) Show that the random variable Z defined by $\quad \forall w \in \Omega \quad Z(w) \;=\; X_H(w) \cdot X_T(w)$ is a Bernoulli random variable and find its probability of success.

(c) Show that $\mathrm{E}[Z] \neq \mathrm{E}[X_H]\mathrm{E}[X_T]$.

Answer

(a) Our sample space is all possible outcomes of two coin tosses. Therefore,

$$\Omega = \{HH, HT, TH, TT\}$$

Since we are flipping a fair coin, the probability space is uniform, so each outcome has a probability of $\frac{1}{4}$. Alternatively, since each coin flip has $\Pr[H] = \Pr[T] = \frac{1}{2}$ and the coin flips are independent, the probability of an outcome in our sample space is simply $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

(b) Observe that X_H and X_T are defined as follows:

$$X_H(w) = \begin{cases} 0 & w = TT \\ 1 & w \in \{HT, TH\} \\ 2 & w = HH \end{cases} \qquad\qquad X_T(w) = \begin{cases} 0 & w = TT \\ 1 & w \in \{HT, TH\} \\ 2 & w = TT \end{cases}$$

Now in order to find the distribution of $Z(w) = X_H(w) \cdot X_T(w)$, we plug in each $w \in w$:

$$Z(HH) = X_H(HH) \cdot X_T(HH) = 2 \cdot 0 = 0$$
$$Z(HT) = X_H(HT) \cdot X_T(HT) = 1 \cdot 1 = 1$$
$$Z(TH) = X_H(TH) \cdot X_T(TH) = 1 \cdot 1 = 1$$
$$Z(TT) = X_H(TT) \cdot X_T(TT) = 0 \cdot 2 = 0$$

Therefore, we have:

$$Z(w) = \begin{cases} 0 & w \in \{HH, TT\} \\ 1 & w \in \{HT, TH\} \end{cases}$$

Since Z only takes on the values 1 and 0, we can define success as the outcomes in $\{HT, TH\}$, and failure as the outcomes in $\{HH, TT\}$. This means that Z is a Bernoulli random variable with a success probability p equal to $\Pr[Z = 1]$:

$$p = \Pr[Z = 1] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Therefore, Z is a Bernoulli random variable with probability success of $p = \dfrac{1}{2}$.

(c) We use our answer to part b to calculate the expectation of Z, X_H, and X_T.

$$\begin{aligned}
\mathrm{E}[Z] &= 0 \cdot \Pr[Z = 0] + 1 \cdot \Pr[Z = 1] = 0 + 1 \cdot \frac{1}{2} = \frac{1}{2} \\
\mathrm{E}[X_H] &= 0 \cdot \Pr[X_H = 0] + 1 \cdot \Pr[X_H = 1] + 2 \cdot \Pr[X_H = 2] = 0 + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1 \\
\mathrm{E}[X_T] &= 0 \cdot \Pr[X_T = 0] + 1 \cdot \Pr[X_T = 1] + 2 \cdot \Pr[X_T = 2] = 0 + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1
\end{aligned}$$

Therefore, we can conclude that

$$\mathrm{E}[Z] = \frac{1}{2} \neq 1 \cdot 1 = \mathrm{E}[X_H] \cdot \mathrm{E}[X_T]$$

4. Alice has a strange coin that shows the number 3 on one side and the number 5 on the other. Still, the coin is fair. Bob has strange die that shows the numbers 5,6,7,8,9,10 on its six faces. Still, the die is fair. Alice flips the coin and, independently, Bob rolls the die. What is the probability that the number on the die is divisible by the number on the coin?

Answer

Let c be the number shown by the coin and d be the number shown by the die. Let E denote the event that $c \mid d$.

Solution 1 The sample space, Ω, consists of all possible ordered pairs (c, d), where c denotes the result of the coin flip and d denotes the result of the die roll. Because the coin and die are fair, and the flip and roll are independent, we have a uniform probability space with $|\Omega| = 2 \times 6 = 12$ outcomes. E consists of 4 of these outcomes:

$$E = \{(c = 3, d = 6), (c = 3, d = 9), (c = 5, d = 5), (c = 5, d = 10)\}.$$

Therefore, we see that

$$\Pr[E] = \frac{|E|}{|\Omega|} = \frac{4}{12} = 1/3.$$

Solution 2 (This solution gets the right result but relies, as is commonly the case, on more complex assumptions.) Since the die is fair, we have $\Pr[d = 6 \text{ or } d = 9] = (1/6) + (1/6) = 2/6 = 1/3$.

Similarly, $\Pr[d = 5 \text{ or } d = 10] = 1/3$. Now, since the coin flip and the die roll are independent, and since the coin is fair

$$\Pr[(d = 6 \text{ or } d = 9) \text{ and } c = 3] = \Pr[d = 6 \text{ or } d = 9]\Pr[c = 3] = (1/3)(1/2) = 1/6.$$

Similarly,

$$\Pr[(d = 5 \text{ or } d = 10) \text{ and } c = 5] = \Pr[d = 5 \text{ or } d = 10]\Pr[c = 5] = (1/3)(1/2) = 1/6.$$

These events are disjoint, so the probability we seek is $(1/6) + (1/6) = 1/3$.

5. (25pts) Alice has a *fair* coin that shows the number 2 on one side and the number 3 on the other. Bob has a *fair* tetrahedral die (a tetradie) that shows the numbers 1,2,3 and 4 on its four faces. They play the following game:

- Alice flips the coin showing the number a and, independently, Bob rolls the tetradie showing the number b
- If $a > b$ then Alice wins and Bob pays Alice $a - b$ dollars. If $a = b$ then it's a tie and no money changes hands. If $b > a$ then Bob wins and Alice pays Bob $b - a$ dollars.

(a) Draw the tree of possibilities for a single game.

(b) Compute the probability that Alice wins a single game.

(c) Suppose that Alice and Bob play the game 3 times in a row, independently. Assume that Alice starts with 10 dollars. Let Z be the random variable that returns the amount of dollars that Alice has after these 3 games. Compute $\mathrm{E}[Z]$.

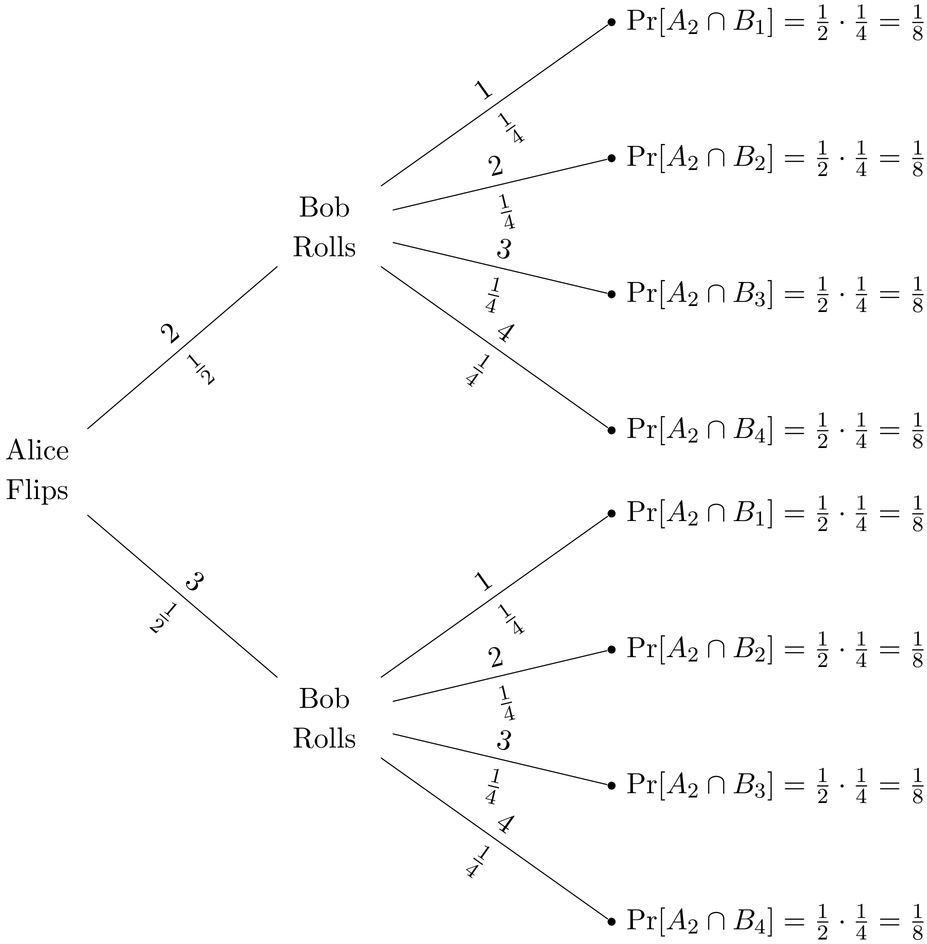
Answer

(a) We first note that our sample space consists of ordered pairs, where the first element corresponds to the result of Alice's coin flip and the second corresponds to the result of Bob's die roll. More formally, we see that:

$$\Omega = \{(a, b) \mid a \in \{2, 3\}, b \in [1..4]\}$$

We define the event of Alice flipping 2 and 3 to be A_2, A_3, respectively, and the event of Bob rolling 1, 2, 3, and 4 to be B_1, B_2, B_3, B_4, respectively.

The tree of possibilities is thus:



(b) Let the event Alice wins be W_A. We seek $\Pr[W_A]$. As seen in the above tree, Alice wins when she flips a 3 and Bob rolls either a 1 or 2, and when she flips a 2 and Bob rolls a 1. These are disjoint events. We find

$$\begin{aligned}
\Pr[W_A] &= \Pr\left[(A_3 \cap B_2) \cup (A_3 \cap B_1) \cup (A_2 \cap B_1)\right] \\
&= \Pr[A_3 \cap B_2] + \Pr[A_3 \cap B_1] + \Pr[A_2 \cap B_1] \\
&= \Pr[A_3] \cdot \Pr[B_2] + \Pr[A_3] \cdot \Pr[B_1] + \Pr[A_2] \cdot \Pr[B_1] \\
&= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} \\
&= \boxed{\frac{3}{8}}
\end{aligned}$$

(c) Let Y_1, Y_2, Y_3 be the amount Alice gains in games 1, 2, and 3 respectively. Since each game has the same rules, $\mathrm{E}[Y_1] = \mathrm{E}[Y_2] = \mathrm{E}[Y_3]$. Further, $Z = 10 + Y_1 + Y_2 + Y_3$. By the Linearity of Expectation,

$$\mathrm{E}[Z] = \mathrm{E}[10] + \mathrm{E}[Y_1] + \mathrm{E}[Y_2] + \mathrm{E}[Y_3] = 10 + 3\mathrm{E}[Y_1]$$

Now we simply apply the definition of expectation:

$$
\begin{aligned}
\mathrm{E}[Y_1] &= \sum_{w\in\Omega} \Pr[w] Y_1(w) \\
&= \Pr[A_2 \cap B_1](2-1) + \Pr[A_2 \cap B_2](2-2) + \Pr[A_2 \cap B_3](2-3) + \Pr[A_2 \cap B_4](2-4) \\
&\quad + \Pr[A_3 \cap B_1](3-1) + \Pr[A_3 \cap B_2](3-2) + \Pr[A_3 \cap B_3](3-3) + \Pr[A_3 \cap B_4](3-4) \\
&= \frac{1}{8}(1) + \frac{1}{8}(0) + \frac{1}{8}(-1) + \frac{1}{8}(-2) + \frac{1}{8}(2) + \frac{1}{8}(1) + \frac{1}{8}(0) + \frac{1}{8}(-1) \\
&= \frac{1}{8}(1+0-1-2+2+1+0-1) \\
&= 0
\end{aligned}
$$

Alternatively, we could have noticed that $Y_1 = a - b$, and applied linearity again to have

$$
\begin{aligned}
\mathrm{E}[Y_1] &= \mathrm{E}[a] - \mathrm{E}[b] \\
&= \sum_{w\in\Omega} \Pr[w]a(w) - \sum_{w\in\Omega} \Pr[w]b(w) \\
&= \left(\frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 3\right) - \left(\frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 3 + \frac{1}{4}\cdot 4\right) \\
&= \frac{5}{2} - \frac{5}{2} \\
&= 0
\end{aligned}
$$

Plugging in to our original equation,

$$\mathrm{E}[Z] = 10 + 3\mathrm{E}[Y_1] = 10 + 3 \cdot 0 = \boxed{10}$$

6. For each statement below, decide whether it is TRUE or FALSE In each case attach a *very brief* explanation of your answer.

   (a) Let $(\Omega, \Pr)$ be a probability space with *three* outcomes. Let E, F be two nonempty events in this space such that $\Pr[E \cup F] = \Pr[E] + \Pr[F]$. Then $E \cap F = \emptyset$.

   (b) Let A, B, C be three events of non-zero probability in a probability space (Ω, P). If $A \cap B = B \cap C$, $A \perp B$, and $B \perp C$ then $\Pr[A] = \Pr[C]$.

   (c) If a probability space has an *event* of probability 2/3 then it must have some *outcome* of probability at most 1/3, TRUE or FALSE?

   (d) Let A, B be events in a probability space such that $\Pr[A] = 0$ and $\Pr[B] \neq 0$. Then, $\Pr[A \mid B] = 0$, true or false?

   (e) For *any* probability space (Ω, P) and *any* event $A \subseteq \Omega$ such that $\Pr[A] \neq 0$ we have $\Pr[\Omega \mid A] = \Pr[A \mid \Omega]$, true or false?

   (f) Let E, F be two events in a finite probability space. If $|E| = |F|$ then $\Pr[E] = \Pr[F]$, true or false?

   (g) If E, F are two events in a finite probability space such that $\Pr[E \cap F] > 0$ then E and F can be disjoint, true or false?

(h) Let A, B be events in a finite probability space such that $\Pr[A] = 1/4$ and $\Pr[A \cup B] = 1/2$. Then, $1/4 \leq \Pr[B] \leq 1/2$, true or false?

(i) For any three events E, F, G in the same probability space. if $E \perp F$ and $F \perp G$ then $E \perp G$.

Answer

(a) FALSE.

Take $\Omega = \{w_1, w_2, w_3\}$ with $\Pr[w_1] = \Pr[w_2] = 1/2$ and $\Pr[w_3] = 0$, and with $E = \{w_1, w_3\}$ and $F = \{w_2, w_3\}$. We have $\Pr[E \cup F] = \Pr[\Omega] = 1 = 1/2 + 1/2 = (1/2+0) + (1/2+0) = \Pr[E] + \Pr[F]$. But $E \cap F = \{w_3\} \neq \emptyset$.

(b) TRUE.

We first observe that, as A and B are independent, we have that

$$\Pr[A] \cdot \Pr[B] = \Pr[A \cap B]$$

However, we also have that $A \cap B = B \cap C$, meaning $\Pr[A \cap B] = \Pr[B \cap C]$. Furthermore, since B and C are independent, we have that:

$$\Pr[B] \cdot \Pr[C] = \Pr[B \cap C]$$

Combining these facts, we have that:

$$\Pr[A] \cdot \Pr[B] = \Pr[B] \cdot \Pr[C]$$

(c) TRUE.

Let A be the event such that $\Pr[A] = \frac{2}{3}$. Then, we know that the complement of A, $\overline{A}$, has probability

$$\Pr[\overline{A}] = \frac{1}{3}$$

Since $\Pr[\overline{A}] \neq 0$, $\overline{A} \neq \emptyset$, meaning $\overline{A}$ contains at least one outcome, call it w. Then, we know that:

$$\Pr[w] \leq \Pr[\overline{A}] = \frac{1}{3}$$

Thus, w is an outcome with probability at most $\frac{1}{3}$.

(d) TRUE.

Since $A \cap B \subset A$ it follows by monotonicity of probability that $0 \leq \Pr[A \cap B] \leq \Pr[A] = 0$ so $\Pr[A \cap B] = 0$.

Therefore $\Pr[A \mid B] = \Pr[A \cap B] / \Pr[B] = 0$.

(e) FALSE.

We proceed with a disproof by counterexample.

Let $(\Omega, \Pr)$ be the probability space of one flip of a fair coin. Further, let A be the event that the coin shows heads.

$$
\begin{aligned}
\Pr[\Omega \mid A] &= \frac{\Pr[\Omega \cap A]}{\Pr[A]} \\
&= \frac{\Pr[A]}{\Pr[A]} && \text{(Because } A \subseteq \Omega\text{)} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\Pr[A \mid \Omega] &= \frac{\Pr[A \cap \Omega]}{\Pr[\Omega]} \\
&= \frac{\Pr[A]}{\Pr[\Omega]} \\
&= \frac{1}{2} \neq 1
\end{aligned}
$$

(f) FALSE.

We can find a counterexample by defining a non-uniform probability space and letting E and F be sets of single outcomes with different probabilities. For example, consider the roll of two indistinguishable dice. Let E be the event that the roll results in two 6's and let F be the event that the roll results in one 5 and one 6. $|E| = |F| = 1$, but $\Pr[E] = \frac{1}{36}$ and $\Pr[F] = \frac{1}{18}$.

(g) FALSE.

If $\Pr[E \cap F] > 0$, there exists some outcome $\omega \in E \cap F$ that occurs with a positive probability, so $E \cap F \neq \emptyset$.

(h) TRUE.

By the Principle of Inclusion-Exclusion,

$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$$

Substituting, we have

$$\frac{1}{2} = \frac{1}{4} + \Pr[B] - \Pr[A \cap B]$$

Hence $\Pr[B] = \frac{1}{4} + \Pr[A \cap B]$. Since probabilities must be non-negative, $\Pr[A \cap B] \geq 0$; thus

$$\Pr[B] \geq \frac{1}{4}$$

Further, since $B \subseteq A \cup B$, we know that

$$\Pr[B] \leq \Pr[A \cup B] = \frac{1}{2}$$

Thus, $\frac{1}{4} \leq \Pr[B] \leq \frac{1}{2}$.

(i) FALSE. Take E, F such that $E \perp F$, E such that $\Pr[E] = 1/2$, and $G = E$. Obviously, since $E \perp F$ and $G = E$, then $G \perp F$. However, let's look at if $E \perp G$. We can say that this is not the case ($E \not\perp G$) because $\Pr[E \cap G] = \Pr[E \cap E] = \Pr[E] = 1/2$ while $\Pr[E] \cdot \Pr[G] = (1/2)(1/2) = 1/4$.

7. Let A, B, C be three events in the same probability space such that $A \subseteq B$, $A \subseteq C$, $B \perp C$, and $\Pr[A] = 1$. Prove that $\Pr[A \cap B \cap C] = \Pr[A]\Pr[B]\Pr[C]$.

   Answer

   Since $A \subseteq B$ and $A \subseteq C$, we know that $A \subseteq B \cap C$ (one way to reason about this is to observe that $A \cap C \subseteq B \cap C$, and plug in $A = A \cap C$). Therefore, $A \cap B \cap C = A$.

   Moreover, by monotonicity of probability, $A \subseteq B$ implies $\Pr[A] \leq \Pr[B]$. Since $1 = \Pr[A] \leq \Pr[B] \leq 1$ we have $1 \leq \Pr[B] \leq 1$, which means that we must have $\Pr[B] = 1$.

   Similarly, we can show that $\Pr[C] = 1$.

   Therefore, $\Pr[A \cap B \cap C] = \Pr[A] = 1 = 1 \cdot 1 \cdot 1 = \Pr[A]\Pr[B]\Pr[C]$.

8. Let E, F be two events in a finite probability space such that $\Pr[E \cap F] > 0$. Prove that $\Pr[E \setminus F] + \Pr[F \setminus E] < \Pr[E \cup F]$.

   Answer

   By Euler-Venn diagram, $(E \setminus F) \cup (E \cap F) \cup (F \setminus E) = E \cup F$. By the Sum Rule, since the LHS sets are pairwise disjoint:

   $$\Pr[E \setminus F] + \Pr[E \cap F] + \Pr[F \setminus E] = \Pr[E \cup F]$$

   Therefore $\Pr[E \setminus F] + \Pr[F \setminus E] = \Pr[E \cup F] - \Pr[E \cap F] < \Pr[E \cup F]$.

9. Alice has an urn with three marbles labeled 1, 2, and 3. Each of the marbles is equally likely to be extracted. Bob has a separate, similar urn. They play the following game of chance:

   (1) Alice extracts a marble from her urn and obtains $a \in \{1, 2, 3\}$.

   (2) Independently, Bob extracts a marble from his urn and obtains $b \in \{1, 2, 3\}$.

   (3) If $a > b$ then Alice wins. If $b > a$ then Bob wins. If $a = b$ they flip a fair coin and if the coin shows heads, Alice wins. If the coin shows tails, Bob wins.

   In the calculations below, do not spend time on the arithmetic. It's OK to leave your results as products and fractions.

   (a) Draw the "tree of possibilities" diagram for this game, with all the outcomes and their probabilities.

   (b) Compute the probability that the game was decided by a coin flip.

   (c) Compute the conditional probability that Alice wins, knowing that Bob extracted the marble labeled 2.

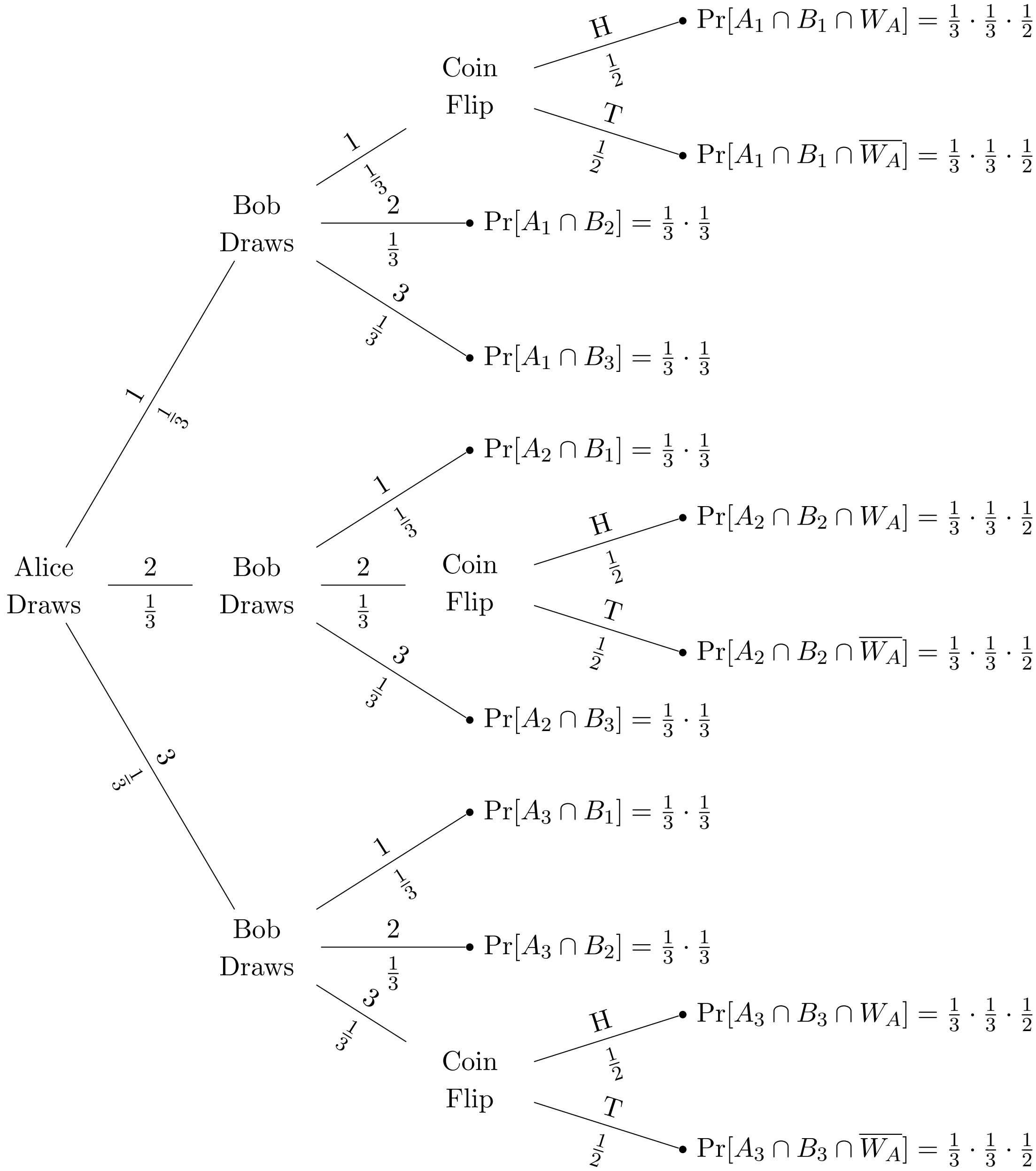
   (d) Alice and Bob put bets on the game. If Alice wins without a coin flip Bob pays her 2$. If Alice wins with a coin flip then Bob pays her 1$. If Bob wins then Alice pays him 1.5$.

   What is Alice's expected monetary win/loss (wins are positive, losses are negative) after n such games?

   Answer

   Let A_i be the event that Alice removes the marble with value i and B_j be the event that Bob removes the marble with value j. Let C be the event that the coin is flipped. Finally, let W_A be the event that Alice wins.

(a) The tree of possibilities:



(b) We want to find $\Pr[C]$. Note that we flip a coin exactly when $a = b$, i.e. we are interested in the event $\bigcup_{i=1}^{3} A_i \cap B_i$. Since each of these are disjoint, we apply the Sum Rule:

$$\Pr\left[\bigcup_{i=1}^{3} A_i \cap B_i\right] = \Pr[A_1 \cap B_1] + \Pr[A_2 \cap B_2] + \Pr[A_3 \cap B_3]$$

Since the marbles are drawn independently, $A_i \perp B_j$ for all i, j:

$$= \Pr[A_1]\Pr[B_1] + \Pr[A_2]\Pr[B_2] + \Pr[A_3]\Pr[B_3]$$

Since we know each marble is equally likely to be extracted:

$$
\begin{aligned}
&= \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} \\
&= \boxed{\frac{1}{3}}
\end{aligned}
$$

(c) We want to find $\Pr[W_A \mid B_2]$.

$$
\begin{aligned}
\Pr[W_A \mid B_2] &= \frac{\Pr[W_A \cap B_2]}{\Pr[B_2]} \\
&= \frac{\Pr[W_A \cap B_2 \cap C] + \Pr[W_A \cap B_2 \cap \overline{C}]}{\Pr[B_2]} \\
&= \frac{\Pr[W_A \mid C \cap B_2] \Pr[C \mid B_2] \Pr[B_2] + \Pr[W_A \mid \overline{C} \cap B_2] \Pr[\overline{C} \mid B_2] \Pr[B_2]}{\Pr[B_2]}
\end{aligned}
$$

Note that $\Pr[W_A \mid C \cap B_i] = \frac{1}{2}$, since we are flipping a coin at this point. Additionally, $\Pr[C \mid B_i] = \frac{1}{3}$, since we flip the coin exactly when $a = b$. It follows that $\Pr[\overline{C} \mid B_i] = 1 - \Pr[C \mid B_i] = 1 - \frac{1}{3} = \frac{2}{3}$. Finally, $\Pr[W_A \mid \overline{C} \cap B_2] = \frac{1}{2}$, since Alice is equally likely to have drawn a 1 or a 3.

$$
\begin{aligned}
&= \frac{\frac{1}{2} \times \frac{1}{3} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} \times \frac{1}{3}}{\frac{1}{3}} \\
&= \boxed{\frac{1}{2}}
\end{aligned}
$$

(d) Let G be the random variable denoting Alice's monetary gain after a single game. We seek $\mathrm{E}[G]$. We express G as a piecewise function:

$$
G = \begin{cases} 2 & W_A \cap \overline{C} \\ 1 & W_A \cap C \\ -1.5 & \overline{W_A} \end{cases}
$$

Then we can find $\mathrm{E}[G]$ thus:

$$
\begin{aligned}
\mathrm{E}[G] &= G(W_A \cap \overline{C}) \cdot \Pr[W_A \cap \overline{C}] + G(W_A \cap C) \cdot \Pr[W_A \cap C] + G(\overline{W_A}) \cdot \Pr[\overline{W_A}] \\
&= (2) \cdot \Pr[W_A \cap \overline{C}] + (1) \cdot \Pr[W_A \cap C] + (-1.5) \cdot \Pr[\overline{W_A}]
\end{aligned}
$$

We proceed by finding the relevant probabilities. We have $W_A \cap C$ when the coin is flipped and comes up with a head. Then:

$$
\begin{aligned}
\Pr[W_A \cap C] &= \Pr[W_A | C] \Pr[C] \\
&= \frac{1}{2} \cdot \frac{1}{3} && \text{(from part (b))} \\
&= \frac{1}{6}
\end{aligned}
$$

We also know the event $W_A \cap \overline{C}$ occurs exactly when Alice draws a value strictly greater than Bob's. That is:

$$\begin{aligned}
\Pr[W_A \cap \overline{C}] &= \Pr[A_3 \cap B_2] + \Pr[A_3 \cap B_1] + \Pr[A_2 \cap B_1] \\
&= \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} && \text{(because } A_i \perp B_j \text{ from part (b))} \\
&= \frac{1}{3}
\end{aligned}$$

By the Law of Total Probabilities:

$$\Pr[\overline{W_A}] = 1 - \Pr[W_A \cap C] - \Pr[W_A \cap \overline{C}] = 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}$$

We now plug these values in:

$$\begin{aligned}
\mathrm{E}[G] &= (2) \cdot \Pr[W_A \cap \overline{C}] + (1) \cdot \Pr[W_A \cap C] + (-1.5) \cdot \Pr[\overline{W_A}] \\
&= 2 \cdot \frac{1}{3} + 1 \cdot \frac{1}{6} - 1.5 \cdot \frac{1}{2} \\
&= \frac{1}{12}
\end{aligned}$$

Now, if we play n games, we can apply linearity of expectation to see that Alice's expected winnings are $\boxed{\frac{n}{12}}$.

10. A fair coin is flipped $2n$ times ($n \geq 1$), independently. Let X_H the random variable that returns the number of heads that occurred and X_T the random variable that returns the number of tails that occurred. Compute $P(X_H > X_T)$.

    Answer

    We first observe that we are working in a uniform probability space of size 2^{2n}, since any series of tosses is equally likely. Additionally, note that there is a bijection between outcomes where $X_H > X_T$ and $X_T > X_H$, by inverting the results of every flip in the sequence. In other words, $\Pr[X_H > X_T] = \Pr[X_T > X_H]$.

    But we know that our sample space can be partitioned into the events $[X_H > X_T]$, $[X_T > X_H]$, and $[X_H = X_T]$. Thus,

$$1 = \Pr[X_H > X_T] + \Pr[X_T > X_H] + \Pr[X_H = X_T] = 2\Pr[X_H > X_T] + \Pr[X_H = X_T]$$

    We can count $\binom{2n}{n}$ outcomes where $X_H = X_T$ (choose n spots of the $2n$ for the heads to occur). Since the probability space is uniform, $\Pr[X_H = X_T] = \dfrac{|[X_H = X_T]|}{|\Omega|} = \dfrac{\binom{2n}{n}}{2^{2n}}$.

    Plugging this into our equation from above gives us:

$$\Pr[X_H > X_T] = \frac{1 - \frac{\binom{2n}{n}}{2^{2n}}}{2} = \boxed{\frac{2^{2n} - \binom{2n}{n}}{2^{2n+1}}}$$

11. Let A, B be events in the same probability space and let I_A, I_B be their indicator random variables. If $E(I_A + I_B) = 1$ then $P(A) = P(\overline{B})$, true or false?

ANSWER

TRUE. By LOE and the definition of a random variable, we have $\mathrm{E}[I_A + I_B] = \mathrm{E}[I_A] + \mathrm{E}[I_B] = \Pr[A] + \Pr[B] = 1$. So $\Pr[A] = 1 - \Pr[B] = \Pr[\overline{B}]$

12. Let (Ω, P) be a probability space and let X be a random variable defined on Ω such that $\mathrm{Val}(X) = \{a, b\}$ where $a < b$. We also denote $\mu = E(X)$.

(a) Express $P(X \le (a+b)/2)$ in terms of a, b and μ.

(b) Let $a = -1$ and $b = 1$. Show that if $E(X) = 0$ then there exists an event $A \subseteq \Omega$ such that $P(A) = 1/2$.

ANSWER

(a) Since $a < b$ we have $a < \frac{a+b}{2} < b$. Then, $\left\{\omega \mid X(\omega) \le \frac{a+b}{2}\right\} = \{\omega \mid X(\omega) = a\}$. Therefore $\Pr[X \le \frac{a+b}{2}] = \Pr[X = a]$ so it suffices to express $\Pr[X = a]$ in terms of μ, a, b.

$$\mu = \mathrm{E}[X] = a \Pr[X = a] + b \Pr[X = b]$$

But we also have $\Pr[X = a] + \Pr[X = b] = 1$:

$$\begin{aligned} &= a \Pr[X = a] + b(1 - \Pr[X = a]) \\ \mu &= (a - b)\Pr[X = a] + b \end{aligned}$$

Hence

$$\Pr[X = a] = \frac{\mu - b}{a - b} = \frac{b - \mu}{b - a}$$

.We conclude that

$$\Pr[X \le \frac{a+b}{2}] = \frac{b - \mu}{b - a}$$

(b) Plugging $a = -1, b = 1, \mu = 0$ in to the result of part (a), we get

$$\Pr[X \le \frac{a+b}{2}] = \frac{1 - 0}{1 - (-1)} = \frac{1}{2}$$

Therefore we can take $A = \left\{w \mid X(w) \le \frac{-1+1}{2}\right\} = \{w \mid X(w) \le 0\}$

13. Weird Al (WAl) is playing with his coins. The game uses two *fair coins* and one *urn*. The result of the game is one of H (heads) or T (tails) and is determined as follows:

- WAl places both coins in the urn.
- WAl reaches inside the urn and (a) with probability 2/3 WAl grabs one of the coins and tosses it, OR (b) with probability 1/3 WAl grabs both coins, then tosses them separately in some order (doesn't matter which order).

- If WAl has tossed just one coin then whatever that coin shows is the result of the game. If WAl has tossed both coins then applying the weird $\otimes$ operation to what the two coins show is the result of the game, where $T \otimes T = T$, $T \otimes H = H$, $H \otimes T = H$, and $H \otimes H = T$.

(a) Draw the "tree of possibilities" diagram for WAl's game.

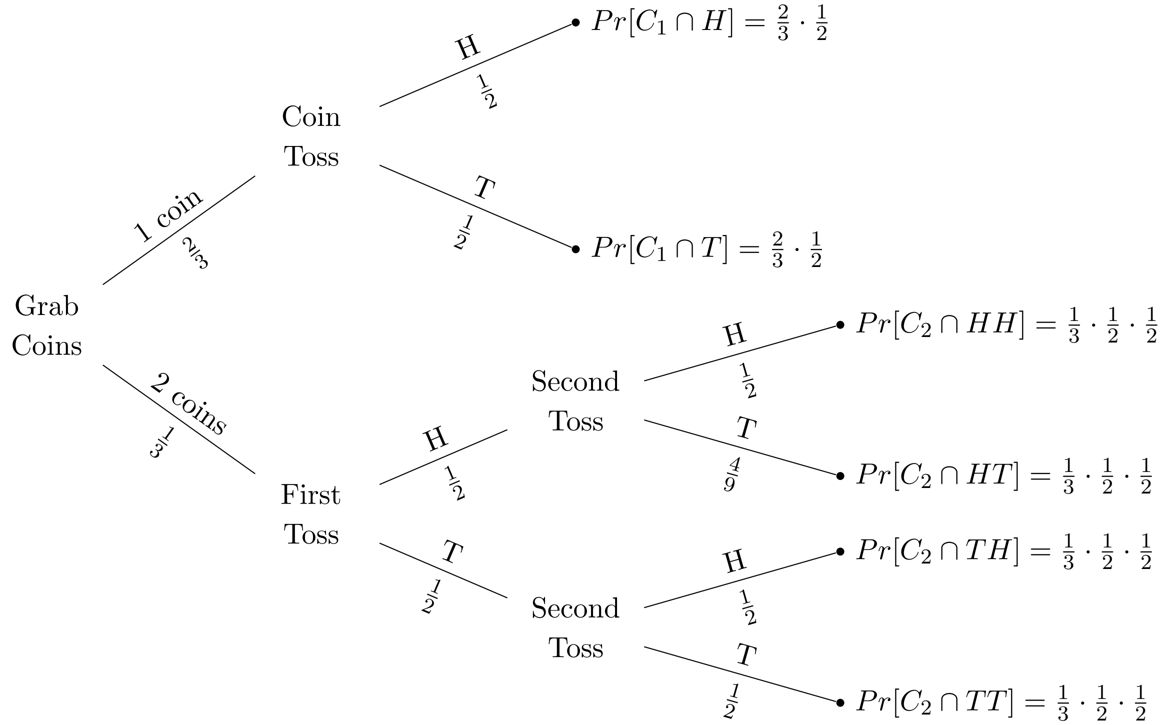
(b) Calculate the probability that the result of of the game is H.

(c) What simpler game could Weird Al play that would give him exactly the same odds?

ANSWER

We define the following events: C_1 indicating the event WAl flips one coin, H_1 indicating the event that flip is a head, C_2 indicating the event he flips 2 coins, HH, HT, TH, TT indicating the sequence of flips he gets.

(a) The tree of possibilities:



(b) The outcome of the game is H if 1) WAl flips one coin (event C_1) and it lands on H (event H_1) or 2) WAl flips two coins (event C_2) and gets one H, one T (event HT). If WAl flips two coins, he will get one H, one T with probability $\frac{1}{2}$, since both HT and TH will yield a head, and both occur with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

$$\begin{aligned}
Pr[H] &= Pr[C_1 \cap H_1] + Pr[C_2 \cap HT] \\
&= Pr[C_1]Pr[H_1|C_1] + Pr[C_2]Pr[HT|C_2] \\
&= \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

(c) Since the result of WAl's game is H with probability $\frac{1}{2}$ and T otherwise, this is equivalent to simply flipping a fair coin once.

14. Let S be the probability space $(\Omega, \Pr)$ with $\Omega = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = ax + b\}$ such that $a, b \in \mathbb{N}$, $1 \leq a \leq 10$, and $1 \leq b \leq 10$, and Pr is the uniform probability distribution on Ω. For each natural number k, Let X_k be the random variable that takes on the value of $f(k)$.

    (a) What is $\mathrm{E}[X_5]$?

    (b) What is $\Pr[X_5 > 1]$?

    (c) What is $\mathrm{E}[X_k]$ in terms of k?

    (d) If you remember derivatives, consider the probability space $S' = (\frac{d\Omega}{dx}, \Pr)$, where $\frac{d\Omega}{dx} = \{\frac{df}{dx} \mid f \in \Omega\}$. Let Y be the random variable that takes on $f(5)$. What is $\mathrm{E}[[]Y]$? (If you don't remember derivatives, no problem ☺)

    Answer

    (a) We apply the definition of expectation:

    $$\mathrm{E}[X_5] = \sum_{\omega \in \Omega} X_5(\omega) \Pr[\omega]$$

    Since the probability distribution is uniform, we can count the outcomes and divide by $|\Omega|$:

    $$\begin{aligned}
    &= \sum_{i=1}^{10} \sum_{j=1}^{10} \frac{5i + j}{100} \\
    &= \sum_{i=1}^{10} \frac{50i + 55}{100} \\
    &= \sum_{i=1}^{10} \frac{i}{2} + \sum_{i=1}^{10} \frac{55}{100} \\
    &= \frac{1}{2} \left(\sum_{i=1}^{10} i \right) + \frac{11}{2} \\
    &= \boxed{33}
    \end{aligned}$$

    (b) Since the minimum value of X_5 is $1 \cdot 5 + 1 = 6$, $\Pr[X_5 > 1] = 1$.

(c) We again proceed by using the definition of expectation:

$$\begin{aligned}
\mathrm{E}[X_k] &= \sum_{\omega\in\Omega} X_k(\omega)\Pr[\omega] \\
&= \sum_{i=1}^{10}\sum_{j=1}^{10} \frac{ki+j}{100} \\
&= \sum_{i=1}^{10} \frac{10ki+55}{100} \\
&= \frac{k}{10}\left(\sum_{i=1}^{10} i\right) + \frac{11}{2} \\
&= \frac{11}{2}(k+1)
\end{aligned}$$

(d) Since this new probability consists of constant functions $f(x) = c$ for $1 \le c \le 10$, and each has the same probability, the average value of $f(x)$ for any x must be 5.5.

15. You have a standard deck of 52 cards, from which you draw 13 cards, without replacement.

(a) Given that you drew the 4 of spades, what is the probability that all the other cards that you drew are aces, twos, threes, or fours?

(b) Define S to be the number of spades you draw. What is $E[S]$?

(c) For what $s \in \mathrm{Val}(S)$ do we have the maximum value of $\Pr[S = s]$?

(d) Suppose that the number of spades you have in your hand is equal to the number you found in part (c). What is the probability that the sum of their numerical values (letting J = 11, Q = 12, K = 13, A = 1) is odd?

Answer

(a) You know that one of the cards you drew is the 4 of spades. You know that 12 cards remain in your hand, and there are exactly 51 cards you could have chosen from, of which 15 are aces, twos, threes, and fours. Any combination of 12 cards is equally likely; that is, our probability distribution is uniform. Thus, defining A as the event that you draw only aces, twos, threes and fours, given that one of your cards is the 4 of spades:

$$\Pr[A] = \frac{\binom{15}{12}}{\binom{51}{12}} = \frac{15!39!}{3!51!}$$

(b) Define indicator random variables S_i for $i = 1, 2, \ldots, 13$ with $S_i = 1$ if the i^{th} spade is drawn and $S_i = 0$ otherwise. We know that

$$\begin{aligned}
\mathrm{E}[S] &= \mathrm{E}[S_1 + S_2 + \cdots + S_{13}] \\
&= \sum_{i=1}^{13} \mathrm{E}[S_i] \\
&= \sum_{i=1}^{13} \Pr[S_i = 1]
\end{aligned}$$

In the uniform probability space whose outcomes are all possible sets of 13 cards, there are $\binom{52}{13}$ total outcomes, and there are $\binom{51}{12}$ outcomes in which the i^{th} spade is drawn (1 way to include the i^{th} spade, $\binom{51}{12}$ ways to choose the remaining 12 cards). Therefore,

$$\Pr[S_i = 1] = \frac{\binom{51}{12}}{\binom{52}{13}} = \frac{51!13!39!}{12!39!52!} = \frac{13}{52} = \frac{1}{4}$$

and we get

$$\mathrm{E}[S] = \sum_{i=1}^{13} \frac{1}{4} = \frac{13}{4}.$$

Alternatively, you could note that, letting H, C, and D be the number of hearts, clubs, and diamonds respectively,

$$\mathrm{E}[S] = \mathrm{E}[H] = \mathrm{E}[C] = \mathrm{E}[D]$$

and, since $S + H + C + D = 13$,

$$\mathrm{E}[S] + \mathrm{E}[H] + \mathrm{E}[C] + \mathrm{E}[D] = \mathrm{E}[S + H + C + D] = \mathrm{E}[13] = 13$$

$$4 \cdot \mathrm{E}[S] = 13$$
$$\mathrm{E}[S] = \frac{13}{4}$$

(c) For $s = 0, 1, \dots, 13$, we can express the probability of choosing s spades by determining how many ways there are to choose 13 cards at random (our outcome space) and how many ways there are to choose s spades and $13 - s$ other cards at random (our desired event). Note that our probability distribution is uniform.

$$\Pr[S = s] = \frac{\binom{13}{s}\binom{39}{13-s}}{\binom{52}{13}}$$

Since all of these probabilities have the same denominator - the same outcome space - we only compare the numerator to determine the maximum $\Pr[S = s]$ value.

$$\binom{13}{s}\binom{39}{13-s} = \frac{13!39!}{s!(13-s)!(13-s)!(26+s)!}$$

To find where the maximum probability lies, we find the ratio

$$\frac{\Pr[S = s]}{\Pr[S = s+1]} = \frac{(s+1)!(12-s)!(12-s)!(27+s)!}{s!(13-s)!(13-s)!(26+s)!} = \frac{(s+1)(27+s)}{(13-s)(13-s)}$$

By calculating the values for which this ratio is > 1 or < 1, we can determine where the maximum lies:

$$\Pr[S = s] > \Pr[S = s+1]$$
$$\frac{(s+1)(27+s)}{(13-s)(13-s)} = \frac{\Pr[S = s]}{\Pr[S = s+1]} > 1$$
$$(s+1)(27+s) > (13-s)(13-s)$$
$$s^2 + 28s + 27 > s^2 - 26s + 169$$

$$54s > 142$$
$$s > \frac{142}{54} \approx 2.63$$

Clearly, we can simply reverse the sign in all of the above inequalities to have that $\Pr[S = s] < \Pr[S = s+1]$ if $s < \frac{142}{54} \approx 2.63$. Thus, the value of $\Pr[S = s]$ is increasing from 0 to 2 and decreasing after 3. Either s=2 or s=3 yields the greatest $\Pr[S = s]$ value. Plugging in, we find

$$\frac{\Pr[S = 2]}{\Pr[S = 3]} = \frac{3 \cdot 29}{11 \cdot 11} = \frac{87}{121} < 1$$
$$\Pr[S = 2] < \Pr[S = 3]$$

$s = 3$ yields the greatest $\Pr[S = s]$ value; it is most likely that we draw 3 spades. In fact, plugging in our values, we will find

$$\Pr[S = 3] \approx 0.286$$

(d) Since you have three cards, the only ways to have an odd sum are with 3 odd cards (call this event O_1) or 1 odd and 2 even cards (call this event O_2). Our outcome space Ω is the possible ways of choosing 3 spades out of the 13 available. Note again that our probability distribution is uniform, so we can simply sum over the two cases and divide by total number of possible outcomes. There are 7 odd spades and 6 even spades. We have:

$$\frac{|O_1| + |O_2|}{|\Omega|} = \frac{\binom{7}{3} + \binom{7}{1}\binom{6}{2}}{\binom{13}{3}} = \frac{140}{286} \approx .49$$

16. Consider X and Y, two independent Bernoulli random variables defined on the same probability space. We are given $\Pr[X = 1] = 1/3$ and $\Pr[Y = 1] = 1/4$. Compute $\mathrm{E}[(X+Y)^2]$.

Answer

Method 1: compute the distribution of $(X + Y)^2$

We have $\mathrm{Val}(X + Y) = \{0, 1, 2\}$. We compute the distribution of $X + Y$ using independence:

$$\begin{aligned}
\Pr[X+Y=0] &= \Pr[X=0 \text{ and } Y=0] = \Pr[X=0]\cdot\Pr[Y=0] = \frac{2}{3}\cdot\frac{3}{4} = \frac{1}{2} \\
\Pr[X+Y=1] &= \Pr[(X=0 \text{ and } Y=1) \text{ or } (X=1 \text{ and } Y=0)] \\
&= \Pr[X=0]\cdot\Pr[Y=1] + \Pr[X=1]\cdot\Pr[Y=0] \\
&= \frac{2}{3}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{3}{4} = \frac{5}{12} \\
\Pr[X+Y=2] &= \Pr[X=1 \text{ and } Y=1] = \Pr[X=1]\cdot\Pr[Y=1] = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}
\end{aligned}$$

Note that $X + Y$ takes only non-negative values. Based on this, $\mathrm{Val}((X + Y)^2) = \{0, 1, 2\}$ with the similar distribution:

$$\Pr[(X+Y)^2 = 0] = \frac{1}{2} \qquad \Pr[(X+Y)^2 = 1] = \frac{5}{12} \qquad \Pr[(X+Y)^2 = 4] = \frac{1}{12}$$

Finally, by the formula for expectation

$$\mathrm{E}[(X+Y)^2] = 0\cdot\frac{1}{2} + 1\cdot\frac{5}{12} + 4\cdot\frac{1}{12} = \boxed{\frac{3}{4}}$$

Method 2: uses a result that is not in the lecture notes (optional)

The product of random variables is defined in the manner sums are defined. It can be shown that if X and Y are independent then $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$.

Using this, also linearity of expectation, and that for Bernoulli r.v.'s we have $X^2 = X$ as well as $\mathrm{E}[X] = \Pr[X = 1]$ we get

$$\begin{aligned} \mathrm{E}[(X+Y)^2] &= \mathrm{E}[X^2 + 2XY + Y^2] \;=\; \mathrm{E}[X^2] + 2\mathrm{E}[XY] + \mathrm{E}[Y^2] \\ &= \mathrm{E}[X] + 2\mathrm{E}[X]\mathrm{E}[Y] + \mathrm{E}[Y] \;=\; \frac{1}{3} + 2 \cdot \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{4} \;=\; \boxed{\frac{3}{4}} \end{aligned}$$

17. Let $(\Omega, \Pr)$ be a probability space with 2 or more outcomes and $X : \Omega \to \mathbb{R}$ a random variable such that $\mathrm{Val}(X) = \{-1, 1\}$. Show that if $\mathrm{E}[X] = 0$ then $\Pr[X = 1] = 1/2$.

    ANSWER

    $\mathrm{E}[X] = (-1) \cdot \Pr[X = -1] + (1) \cdot \Pr[X = 1] = \Pr[X = 1] - \Pr[X = -1] = \Pr[X = 1] - (1 - \Pr[X = 1]) = 2 \cdot \Pr[X = 1] - 1$.

    When $\mathrm{E}[X] = 0$ we compute $\Pr[X = 1] = 1/2$.

18. Sophie is playing the following game:

    - First she chooses with equal probability one of the numbers $0, 1, \ldots, 4$, call it a.
    - Then she chooses with equal probability one of the *remaining* numbers, call it b.
    - Then she adds $a + b = c$ and c is the result of her game.

    (a) What possible results can Sophie's game have?

    (b) Draw the "tree of possibilities" diagram for Sophie's game.

    (c) We denote with $R(c)$ the event that the result of the game is c. What is the probability of $R(2)$?

    (d) What is the probability of $R(2) \cup R(3)$?

    (e) What is the conditional probability $\Pr[R(2) \mid R(2) \cup R(3)]$?

    (f) Find two events E, F in the probability space of Sophie's game such that neither is empty, neither equals the whole sample space and $E \perp F$.

    ANSWER

    (a) Any natural number between 1 and 7 can be expressed as the sum of two *distinct* numbers between 0 and 4:

    $$1 = 0 + 1, \quad 2 = 0 + 2, \quad 3 = 0 + 3, \quad 4 = 0 + 4, \quad 5 = 1 + 4, \quad 6 = 2 + 4, \quad 7 = 3 + 4$$

    but 8 (or bigger numbers) cannot. So the possible results are 1,2,3,4,5,6,7.

    (b) See Figure 1.

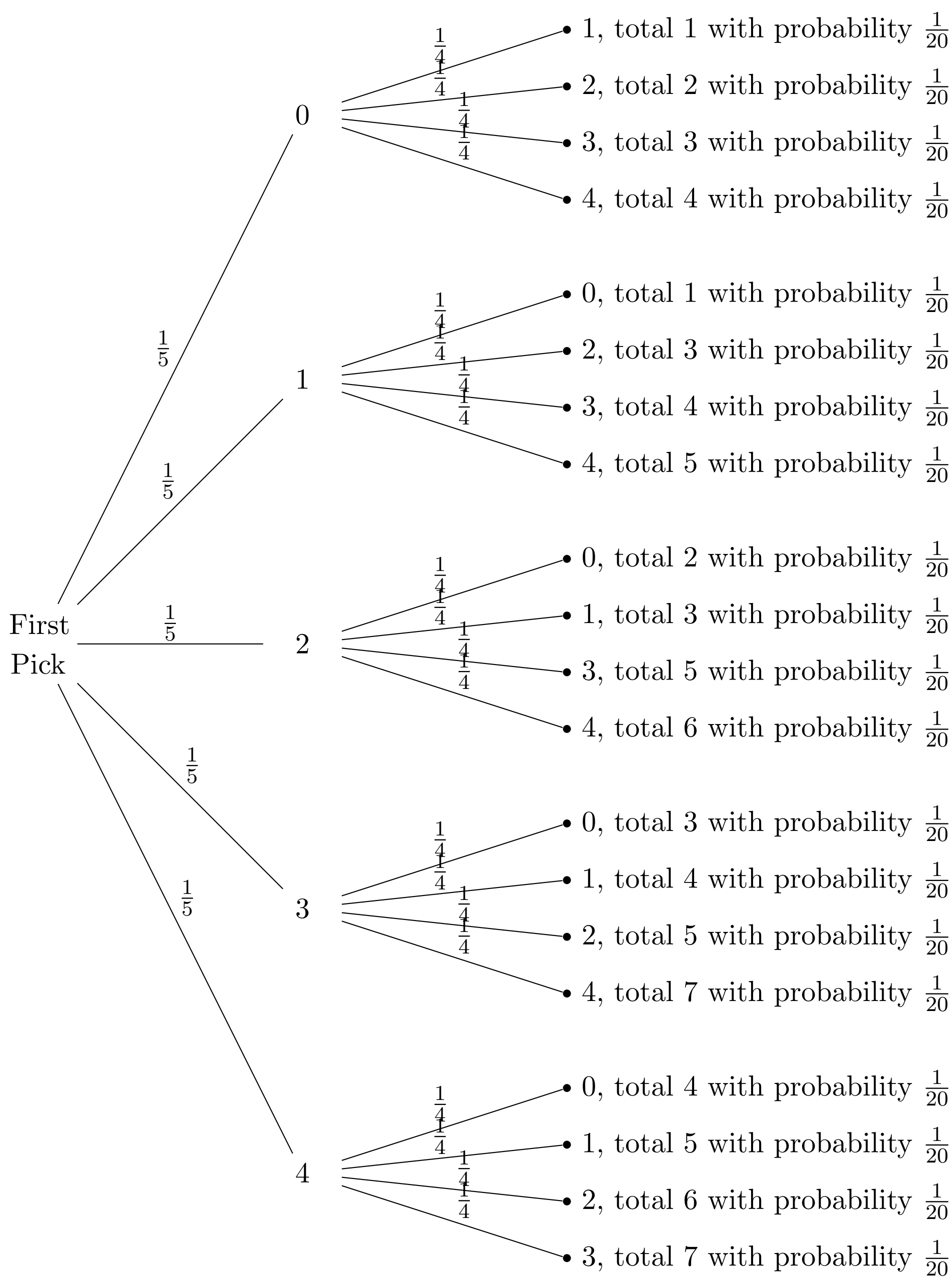


Figure 1: The “tree of possibilities” for Sophie’s Game.

(c) Here are all of the probabilities of events $R(c)$ by adding up probabilities of outcomes in the tree of possibilities:
$\Pr[R(1)] = \Pr[R(2)] = 2/20$
$\Pr[R(3)] = \Pr[R(4)] = \Pr[R(5)] = 4/20$
$\Pr[R(6)] = \Pr[R(7)] = 2/20$
The probability of $R(2)$ is $\boxed{2/20}$.

(d) Note that $R(2)$ and $R(3)$ are disjoint. Then $\Pr[R(2) \cup R(3)] = 2/20 + 4/20 = \boxed{6/20}$

(e) $\Pr[R(2) \mid R(2) \cup R(3)] \ = \ \Pr[R(2) \cap (R(2) \cup R(3))] \ / \ \Pr[R(2) \cup R(3)] \ =$
$\Pr[R(2)] \ / \ \Pr[R(2) \cup R(3)] \ = \ (2/20)/(6/20) = \boxed{1/3}$

(f) By trial and error we decide to take $E = R(1) \cup R(2)$ and $F = R(2) \cup R(3) \cup R(4)$.
$\Pr[E] = 2/20 + 2/20 = 4/20 \qquad \Pr[F] = 2/20 + 4/20 + 4/20 = 10/20$
$\Pr[E \cap F] = \Pr[R(2)] = 2/20 = (4/20) \cdot (10/20)$
Hence $E \perp F$.

19. (25pts) Alice and Bob are playing a game of chance. They roll a fair die. If the die shows an even number then Alice pays Bob \$1. If the die shows an odd number then Bob pays Alice \$1.

(a) What is the probability that Alice wins \$1?

(b) They play the game 2 times in a row, independently. What is the probability that Bob wins \$2?

(c) Alice and Bob start with 100 dollars each and play the game 100 times in a row, independently. Let X be the random variable representing how much money Alice has after all this. Express X as a sum of 101 random variables.

(d) Compute $\mathrm{E}[X]$.

ANSWER

(a) The probability that the die shows an odd number (1,3, or 5) is $(1/6)+(1/6)+(1/6) = 3/6 = 1/2$. Hence the probability that Alice wins \$1 is $1/2$.

(b) The probability that the die shows an even number is also $1/2$. For Bob to win \$2 the die has to roll an even number in both games. By independence, the probability of this happening is $(1/2) \cdot (1/2) = 1/4$.

(c) Let X_i be random variable that returns the gain/loss for Alice in game i, for $i = 1, \ldots, 100$. We have $\mathrm{Val}(X_i) = \{1, -1\}$ and by the above, $\Pr[X_i = 1] = 1/2$ and $\Pr[X_i = -1) = 1/2$.
We also consider the initial amount of money with which Alice started. This is also a random variable (albeit a trivial one!) which returns 100 with probability 1. Then

$$X \ = \ 100 \ + \ X_1 \ + \ \cdots \ + \ X_{100}$$

(d) $\mathrm{E}[X_i] = (1)(1/2) + (-1)(1/2) = 0$. Now, by linearity of expectation

$$\mathrm{E}[X] \ = \ \mathrm{E}[100] \ + \ \mathrm{E}[X_1] \ + \ \cdots \ + \ \mathrm{E}[X_{100}] = 100 + 0 + \cdots 0 = 100$$

20. A **biased coin** shows heads with probability 1/3 and tails with probability 2/3. The coin is flipped $n \geq 3$ times, independently.

    Compute the expected number of occurrences of **consecutive** heads, tails, tails.

    ANSWER

    Let X be the r.v. that returns the number of occurrences of consecutive heads, tails, tails in n flips.

    For each $i = 1, \ldots, n-2$, let X_i be the indicator r.v. associated withe the event "heads in flip i and tails in flips $i+1$ and $i+2$". By independence the probability of this event is $(1/3)\cdot(2/3)\cdot(2/3) = 4/27$. Therefore $\mathrm{E}[X_i] = 4/27$.

    Now $X = \sum_{i=1}^{n-2} X_i$ therefore, by linearity of expectation

$$\mathrm{E}[X] \;=\; \sum_{i=1}^{n-2} \mathrm{E}[X_i] \;=\; \boxed{\frac{4(n-2)}{27}}$$

21. Alice receives 20 candies from her granny.

    - On day 1, for each candy that she has, Alice is equally likely to *keep* it or to give it to Bob.
    - On day 2, for each candy that he has (all given previously by Alice), Bob is equally likely to *eat* it or to give it back to Alice. (Bob is weird.)
    - On day 3, for each candy that she has (either kept in day 1 or received from Bob in day 2), Alice is equally likely again to *eat* it or to throw it in the trash. (Less appetizing by now.)

    (a) Given a particular candy, c, what is the probability that Alice eats c?

    (b) Calculate the expected number of candies that Alice eats.

    ANSWER

    (a) We draw a tree-of-possibilities diagram for the candy c.

```
A ----1/2----- B ----1/2---- A----1/2---- (1/8) A eats c
 \              \             \
  \              \             \
   1/2           1/2           1/2
    \              \             \
     \              \             \
   A keeps c      (1/4)          (1/8) c in trash
       |  \       B eats c
       |   \
      1/2  1/2
       |     \
       |      \
    (1/4)    (1/4)
   A eats c  c in trash
```

    The probability that Alice eats c is $(1/4) + (1/8) = 3/8$.

(b) For each candy $c = 1, \ldots, 20$ let X_c be the indicator variable of the event "Alice eats c". From part (a), the probability of this event is $3/8$ therefore $\mathrm{E}[X_c] = 3/8$.

The random variable that gives the number of candies that Alice eats can be expressed as the sum $X_1 + \cdots + X_{20}$. The expected number of candies that Alice eats is $\mathrm{E}[X_1 + \cdots + X_{20}]$. By linearity of expectation this equals $20 \cdot (3/8) = 15/2 = \boxed{7.5}$