Recitation Module 10

Penn Engineering

Concept Review

Independent random variables

Random variables X and Y defined on the same probability space are **independent**, written $X \perp Y$, when

$$\forall x \in \text{Val}(X) \;\; \forall y \in \text{Val}(Y) \quad (X = x) \perp (Y = y)$$

$$\Pr[X = x \cap Y = y] = \Pr[X = x] \cdot \Pr[Y = y]$$

Example. Green-purple fair dice are rolled, G returns the number shown by the green die, P by the purple die. Then $G \perp P$, because for any $g, p \in [1..6]$:

$$\Pr[(G = g) \cap (P = p)] = 1/36 = (1/6)(1/6) = \Pr[G = g] \cdot \Pr[P = p].$$

When we have three or more r.v.'s, we define again **pairwise** and **mutual independence** analogously.

Concept Review

Variance

$u = E[x]$

The **variance** of a random variable X is defined as

$$\text{Var}[X] = \text{E}[(X - \mu)^2] \quad (\text{where } \mu = \text{E}[X])$$

The **standard deviation** of a random variable X is

$$\sigma[X] = \sqrt{\text{Var}[X]}$$

$E((x - u)^2)^2$

Why square root? By undoing the squaring in the variance we obtain the same units of measurement as those used for the values of the random variable.

Notation. When the random variable is understood , its mean is often denoted by μ, its standard deviation by σ, and its variance by σ^2.

An alternative formula for variance

Proposition. Let X be an r.v. defined on (Ω, Pr).

$$\text{Var}[X] = \text{E}[X^2] - \mu^2 \qquad (\text{where } \forall w \in \Omega \;\; X^2(w) = (X(w))^2 \;)$$

Proof. Using linearity of expectation and the fact that μ is a constant:

$$\begin{aligned} \text{E}[(X - \mu)^2] &= \text{E}[X^2 - 2\mu X + \mu^2] \\ &= \text{E}[X^2] - 2\mu\text{E}[X] + \mu^2 \\ &= \text{E}[X^2] - 2\mu^2 + \mu^2 = \underline{\text{E}[X^2] - \mu^2} \end{aligned}$$

Concept Review

Linearity of variance?

Proposition. $\mathrm{Var}[c\,X] \;=\; c^2\,\mathrm{Var}[X]$

Random Variables are independent

Proof. $\mathrm{Var}[c\,X] \;=\; \mathrm{E}[(cX)^2] - (\mathrm{E}[cX])^2 \;=\; \mathrm{E}[c^2\,X^2] - (c\,\mathrm{E}[X])^2$

$= \; c^2\,\mathrm{E}[X^2] - c^2(\mathrm{E}[X])^2 \;=\; c^2(\mathrm{E}[X^2] - (\mathrm{E}[X])^2) \;=\; c^2\,\mathrm{Var}[X]$

In general, variance does not distribute over sums. However:

Proposition. if $X \perp Y$ then $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.

Proof. In the segment entitled "Correlated random variables" we define **product** of r.v.'s and show

1) $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ iff $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$
2) $X \perp Y \;\Rightarrow\; \mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$. The proposition follows.

When we roll two fair green-purple dice, we have $S = G + P$ and $G \perp P$.

Therefore, $\mathrm{Var}[S] = Var[G] + \mathrm{Var}[P] = (35/12) + (35/12) = 35/6$.

Concept Review

Binomial random variables

An r.v. $B : \Omega \to \mathbb{R}$ is called **binomial** with parameters $n \in \mathbb{N}$ and $p \in [0, 1]$ when $\text{Val}(B) = [0..n]$ and $\forall k \in [0..n]$ $\Pr[B = k] = \binom{n}{k} p^k (1-p)^{n-k}$.

How does such an r.v. arise? For example, perform n IID Bernoulli trials with probability of success p and let B be the r.v. that returns the number of successes observed. Clearly $\text{Val}(B) = [0..n]$. Then:

We have seen before the probability space on which B is defined: the outcomes are the 2^n sequences of length n of S's (for "success") and F's (for "failure"). An outcome with k S's has probability $p^k(1-p)^{n-k}$.

There are $\binom{n}{k}$ outcomes with k S's. Therefore the probability of the event "k successes observed" is $\binom{n}{k} p^k (1-p)^{n-k}$.

The distribution of B is also called **binomial** with parameters n and p.

$n \to$ trials

p $(1-p)$

succ

$$\binom{n}{k}\left(\frac{1}{2}\right)^k \cdot \left(\frac{1}{2}\right)^{n-k}$$

Succ

$\text{Exp}[X] = n \cdot p$

$\text{Var}[X] = npq \; (1-p)$

$\text{Var}(X) = n \cdot p \cdot (1-p)$

$$E[X] = \sum_{i=1}^{n} i \cdot \binom{n}{i} p^i (1-p)^{n-i}$$

Question 1

Linda is playing cornhole. For each throw, if the beanbag lands in the hole, then Linda gets 3 points, if the beanbag lands on the board, then Linda gets 1 point, and if the beanbag does not land on the board or in the hole, then Linda gets 0 points. Linda throws 5 beanbags (all throws are independent). What is the variance of Linda's points? $(1/3)$

X_i = pts achieved on the i^{th} throw

$$\mathrm{Var}\left[\sum_{i=1}^{5} X_i\right] = \sum_{i=1}^{5} \mathrm{Var}[X_i]$$

Answer to Question 1

$$\mathrm{Var}(x_i) = E[x_i^2] - E(x_i)^2$$

μ

$$E(x_i) = \frac{1}{3} \times 3 + \frac{1}{3} \times 1 + \frac{1}{3} \times 0$$

$E[x_i^2]$

$$= 1 + \frac{1}{3} = 4/3$$

$$E(x_i^2) \quad \frac{1}{3} \times 3^2 + \frac{1}{3} \times 1^2 = 3 + \frac{1}{3} = 10/3$$

$$\mathrm{Var}(x_i) = 2 \; ; \Rightarrow 10$$

Question 2

Given a homework with 7 problems. For each problem, you have a 30% chance to lose 2 points for not mentioning something.
What is the expectation of total loss of points?

X_i = the r.v denoting the number of pts lost in Q_i

$E[X_1 + X_2 + X_3 \ldots + X_7]$ ← Total points

$= \sum_{i=1}^{7} E[X_i]$

?

Answer to Question 2

$$-2 \cdot \frac{1}{3} = E[x_i] = -2/3$$

$$\sum_{i=1}^{7} E[x_i] \quad -\frac{2}{3} \times 7 = -\frac{14}{3}$$

Question 3

You roll a die 3 times (R1, R2, R3). Let X be the random variable representing the difference of R1-R2 and let Y be the random variable representing the difference of R2-R3. Prove whether X and Y are independent or dependent.

$$\Pr[X = x \cap Y = y] \neq \Pr[X = x] \cdot \Pr[Y = y]$$

$$X = R_1 - R_2 = 1$$

$$Y = R_2 - R_3 = 1$$

Answer to Question 3

$X_1 - X_2 = 1$

$Pr[(X_1 - X_2) = 1] = \frac{5}{36}$

$$\left[\begin{array}{c} (6, 5) \\ (5, 4) \\ (4, 3) \\ (3, 2) \\ (2, 1) \end{array}\right]$$

$Pr[(X_2 - X_3) = 1] = 5/36$

Answer to Question 3

$P_r[X_1 - X_2 = 1 \cap X_2 - X_3 = 1]$

(x_1, x_2, x_3)

$x_2 = 1$ $\times$

$x_1 = 3,\quad x_2 = 2,\quad x_1 = 1$

$4,\quad 3,\quad x_1 = 2$

Answer to Question 3

$$\frac{4}{6}$$

Answer to Question 3

$$6 \times 6 \times 6 = \begin{bmatrix} (3\ 2\ 1) \\ (4\ 3\ 2) \\ (5\ 4\ 3) \\ (6, 5, 4) \end{bmatrix} 4$$

$$\Pr[\qquad \frac{4}{6^3} \neq \frac{5}{36} \times \frac{5}{36}$$

Answer to Question 3

$$X = R_1 - R_2$$

$$Y = R_2 - R_3$$

Before you go...

Do you have any general questions about this week's assignment? We can't go into minute details about any specific questions, but this is the time to ask any general clarifying/confusing questions!

As always, if you can't talk to us during recitation or office hours, post any questions/anything course related on the forums or email mcitonline@seas.upenn.edu. Ask questions that might be beneficial to other students on the forums, while emailing about more personal questions (regrade requests, etc).