Recitation Module 11

Penn Engineering

Tool for Graphs in LaTeX

For the next assignments you will be required to draw your graphs in LaTeX. This is usually achieved by a graph library called **tikz**. However, we understand that this is a complex library to learn, which is why we introduce you to

https://www.mathcha.io/editor   http://www.madebyevan.com/fsm/

You can design your graphs with a very user friendly interface and then export them on LaTeX source code form.

Tips:

- Make sure you include all packages required in your tex file preamble with

\usepackage{packagename}

Graphs that are not written in LaTeX (either using this tool, or tikz directly) will result in points deducted.

Basic terms/definitions to know!!

- Graph terminology and notation
- Handshaking lemma: $\sum_{v \in V} \deg(v) = 2|E|$
- Types of graphs: edgeless, complete, path, cycle, grid
- Walk vs path

  A **walk** is a non-empty sequence of vertices consecutively linked by edges: $u_0, u_1, \ldots, u_k$ such that $u_0 - u_1 - \cdots - u_k$. This walk is **from** u_0 **to** u_k (the **endpoints** of the walk) and u_0 and u_k are **connected** by this walk. The **length** of this walk is the number k of edges (**not** $k+1$!).

  A **path** is a walk in which all the vertices are **distinct**.

- Connected components

  A **connected component** of a graph $G = (V, E)$ is a set of vertices $C \subseteq V$ such that:

  - any two vertices in C are **connected**, and
  - there is **no strictly bigger** set of vertices $C \subsetneq C' \subseteq V$ such that any two vertices in C' are connected.

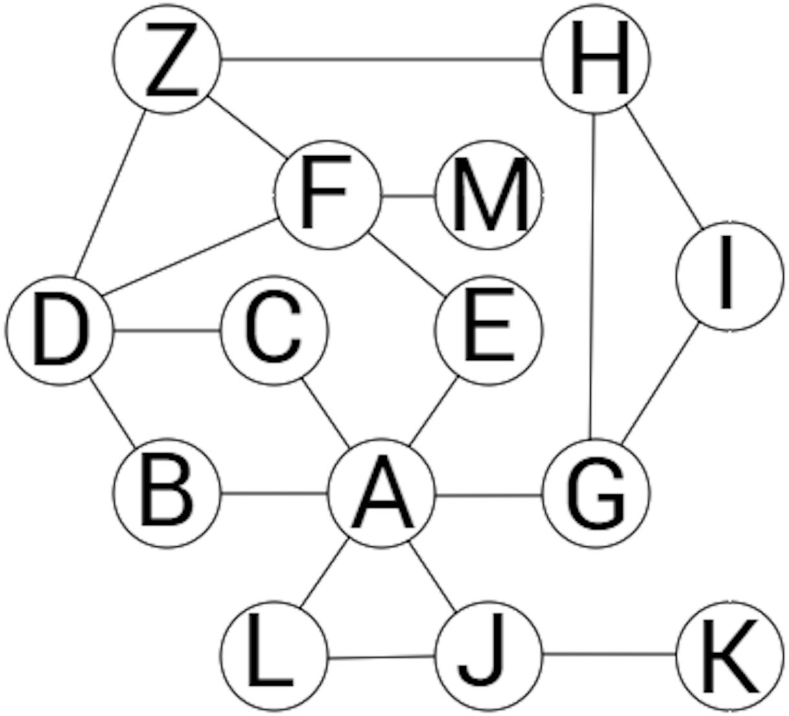
  We say that C is a **maximally connected** set of vertices of G.

- Relation properties: reflexivity, transitivity, symmetry

Note that this is a dense module with many important concepts. This is by no means an exhaustive summary.

Question 1.1

Verify that the handshaking lemma holds for the following graph.

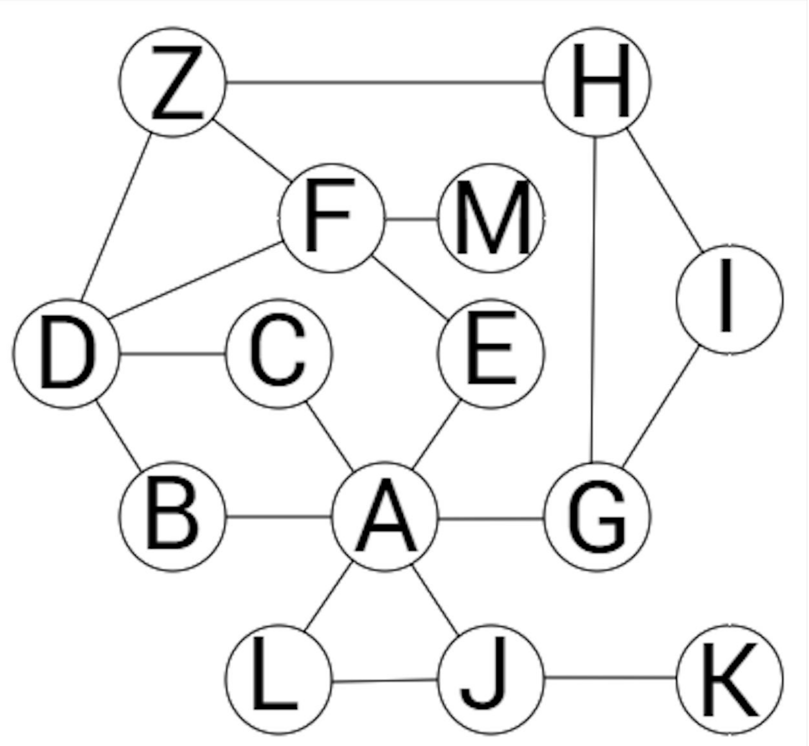


Answer to Question 1.1

The handshaking lemma states that the total degree is twice the number of edges. Let's count the total degree T:

T = deg(Z) + deg(H) + deg(F) + deg(M)
+ deg(I) + deg(D) + deg(C) +
deg(A) + deg(E) + deg(B) + deg(G)
+ deg(L) + deg(J) + deg(K)
= 3 + 3 + 4 + 1 + 2 + 4 + 2 + 6 + 2 + 2
+ 3 + 2 + 3 + 1 = **38**

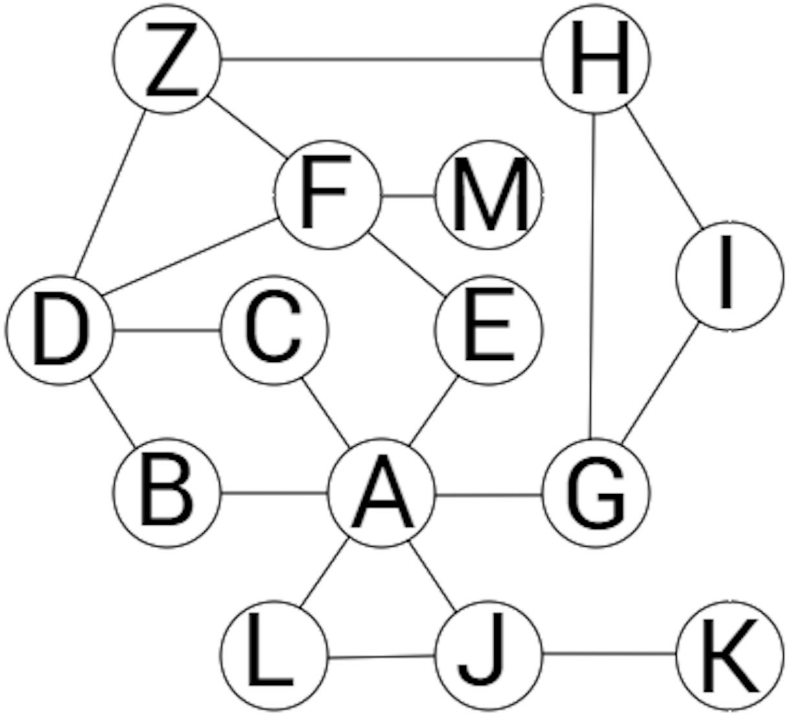
We now count the number of edges: **19**
Indeed, **38 = 2 * 19.**



Question 1.2

Find the complete subgraphs of the graph with more than 2 vertices.

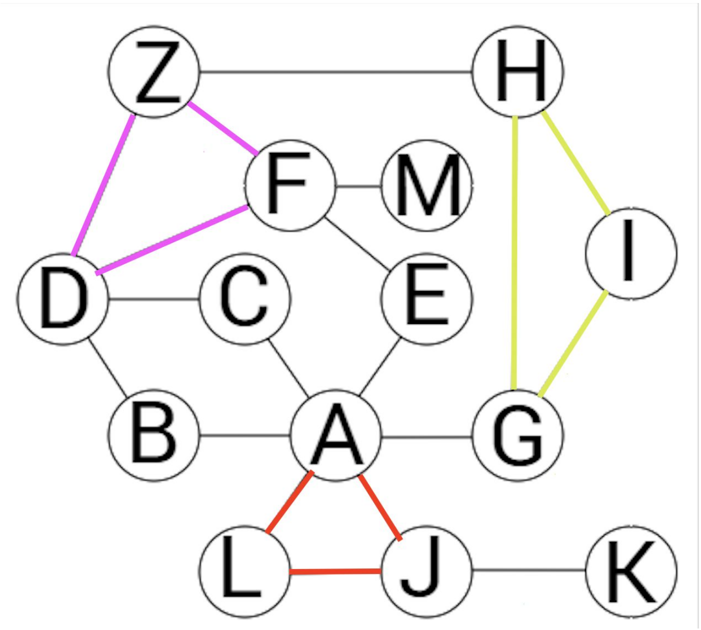
We are currently relying on intuition by what is meant by a subgraph but it will formally defined in the next module.



Answer to Question 1.2

We have highlighted with distinct colors the **3** complete subgraphs with more than two vertices.

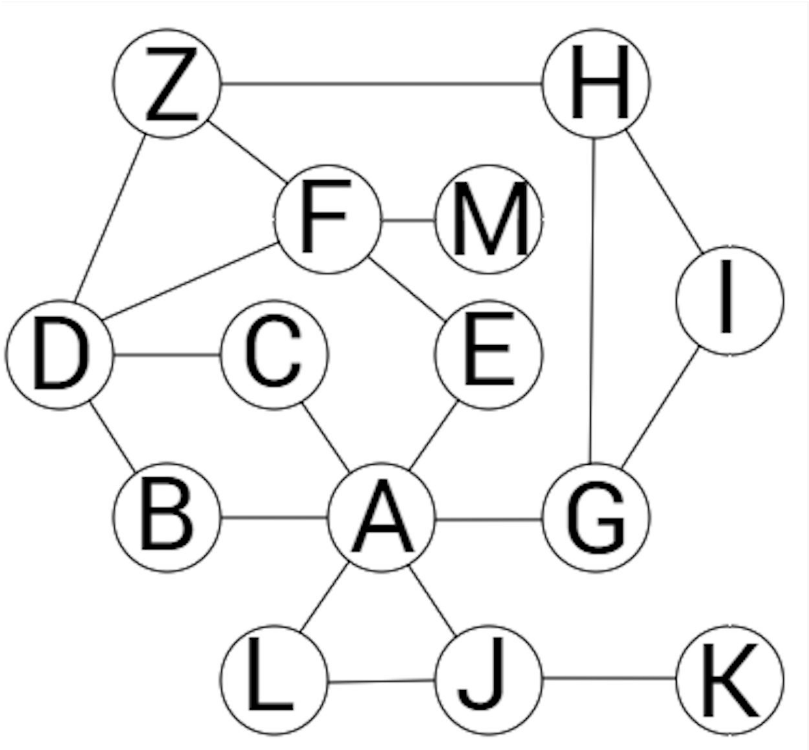
Note: every vertex is technically a complete subgraph with 1 vertex and no edges, and every two adjacent vertices are a complete subgraph with 2 vertices.



Question 1.3

How many cycle subgraphs with strictly less than 5 vertices are there?

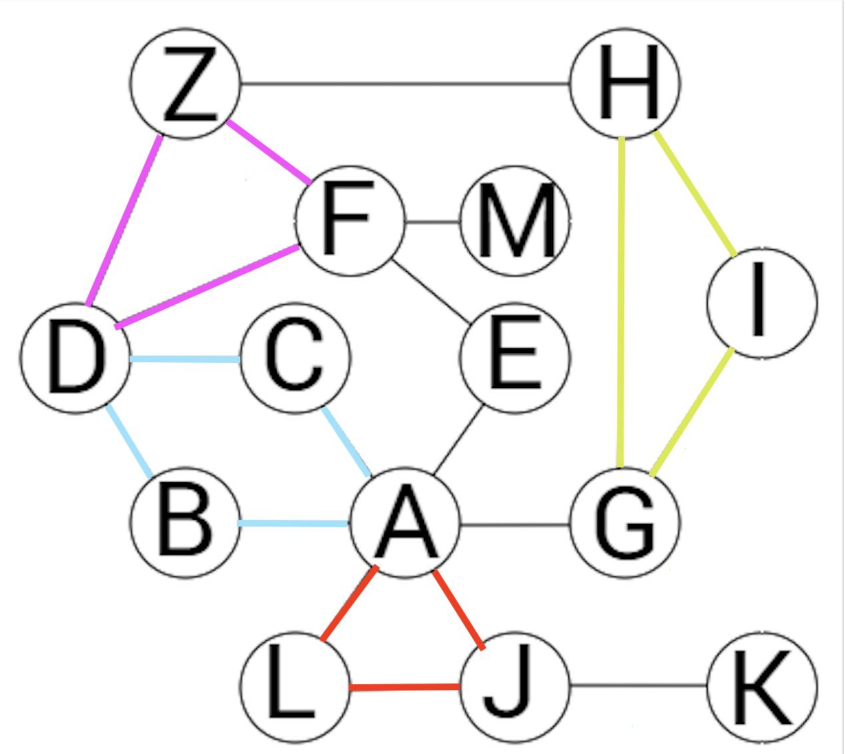
We are currently relying on intuition by what is meant by a subgraph but it will formally defined in the next module.



Answer to Question 1.3

3. We have highlighted with distinct colors the **4** cycle subgraphs of size strictly smaller than 5.

Notice how K_3 is "the same as" C_3. We will learn in a later module that K_3, C_3 are called *isomorphic!*

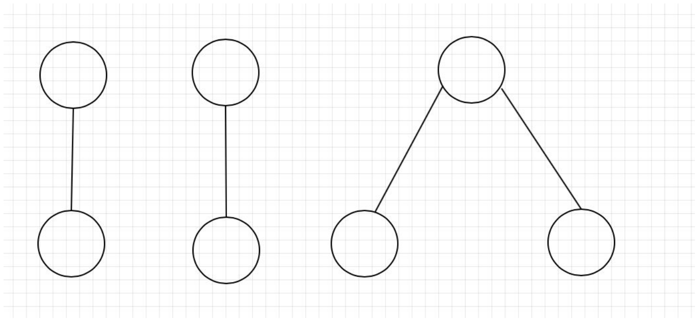


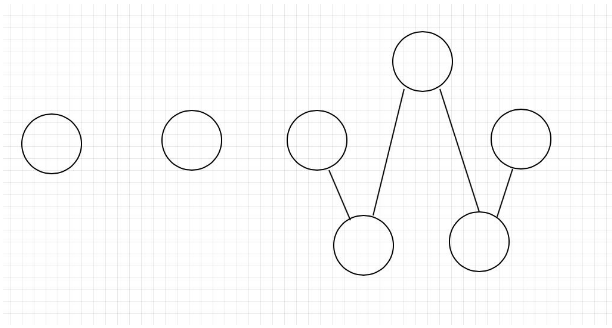
Question 2

What are the minimum and the maximum number of edges that a graph with 7 nodes and 3 connected components can have?

Answer to Question 2

Recall the fact that |E| >= |V| - |CC|. Thus the minimum number of edges the graph can have is 7 - 3 = **4**. See the examples below:





Answer to Question 2 (Continued I)

- For the maximum constraint it is a bit harder.
- Let’s formalize what we are looking for:
- We seek numbers n_1, n_2, n_3 representing the number of nodes in each component, such that
  - $n_1 + n_2 + n_3 = 7$ (there must be 7 nodes total), and
  - the sum of all the maximum numbers of edges in the 3 components is maximum.
- Recall that there are (n choose 2) possible edges between n nodes (this is just the number of ways we can choose two vertices and place an edge between them).

Answer to Question 2 (Continued II)

- Thus, we want to maximize $\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2}$ subject to the constraint $n_1 + n_2 + n_3 = 7$
- Our possible options are *(note: order doesn't matter for the total number of edges):*

(1) $n_1 = 1, n_2 = 2, n_3 = 4$ where $\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} = 0 + 1 + 6 = 7$

(2) $n_1 = 2, n_2 = 2, n_3 = 3$ where $\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} = 1 + 1 + 3 = 5$

(3) $n_1 = 1, n_2 = 1, n_3 = 5$ where $\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} = 0 + 0 + 10 = 10$

(4) $n_1 = 1, n_2 = 3, n_3 = 3$ where $\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} = 0 + 3 + 3 = 6$

Hence the maximum number of edges is **10**.

Question 3

Let S be the set of integers strictly greater than 1.

Define the relation ~ on S by:

$m \sim n$ if $\gcd(m, n) > 1$, for m, n in S.

Determine whether ~ is reflexive, symmetric and transitive.

Answer to Question 3

Reflexivity: Let m > 1. We have: gcd(m,m) = m > 1, so m ~ m. Thus, the relation is reflexive.

Symmetry: Let m, n > 1 such that m ~ n. This means gcd(m, n) > 1. We have that gcd(n, m) = gcd(m, n) > 1, so n ~ m follows. Thus, the relation is symmetric.

Transitivity: We show it is not transitive by counterexample:

gcd(25, 15) = 5 => 25 ~ 15

gcd(15, 21) = 3 => 15 ~ 21

BUT gcd(25, 21) = 1 => not(25 ~ 21).

Fun activity: Olympus meets graphs! (optional)

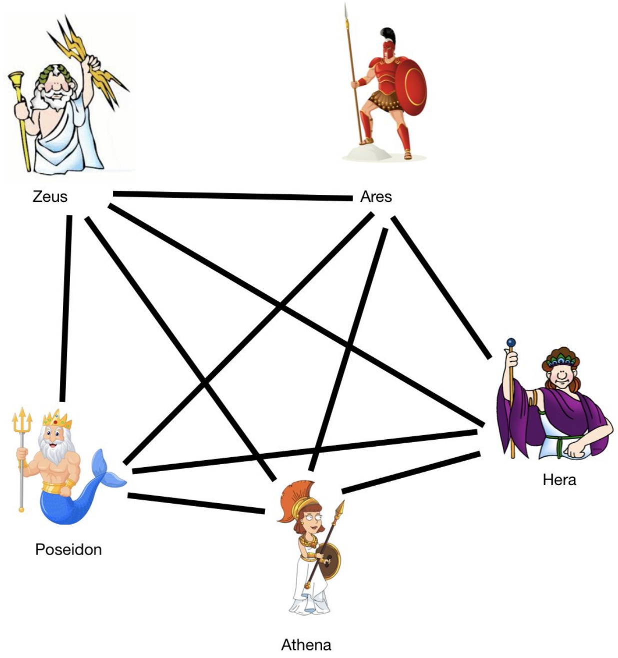
Ancient Greek mythology is all about the feuds and friendships between the gods. Their strong temperament made the friendship/alliance relations change all the time. In this question we will model the alliances between the ancient greek gods in order to satisfy certain path properties. We model the alliances by representing the gods as nodes in an undirected graph, s.t. there is an edge between them only if they are friends/allies.
Let the 5 gods of interest be: Zeus, Hera, Poseidon, Ares, and Athena (arguably the most feisty ones). Model their alliances such that the resulting graph is:

1. A Complete Graph
2. A Cycle Graph
3. A path graph.

Try out other types of graphs and verify the properties we have learned in this module! By studying the different types of graphs visually it will be easier to absorb the material. And fun images are easier to remember!!

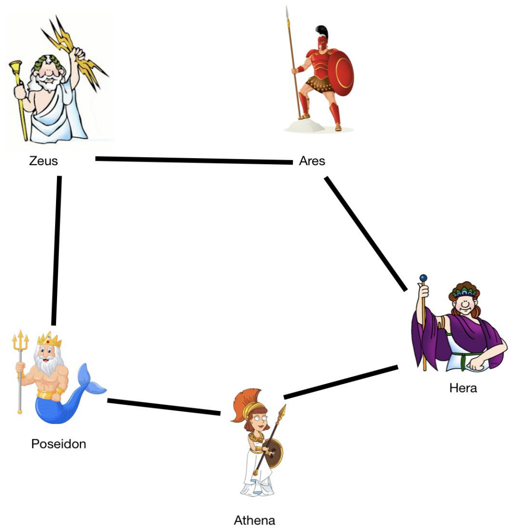
Answer to Activity

1. **Complete graph:** everyone is friends with everyone (what a nice day for Olympus…)



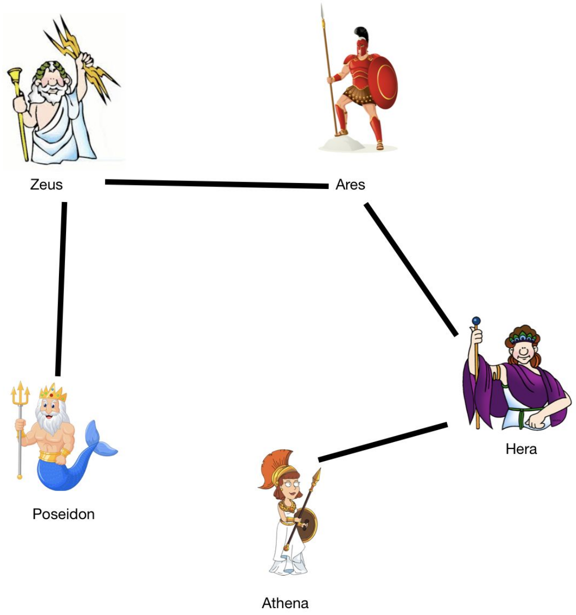
Answer to Activity

2. **Cycle graph:** Zeus has a good relations with his son (Ares) and his brother (Poseidon), but not a good day for him and his wife (Hera) and his daughter (Athena). There also seems to be a feud between Athena and Ares (another pair of siblings), and surprisingly Athena is having a good rapport with her step mom (Hera). Even more surprisingly, Poseidon and Athena seem to have made up after fighting over the capital of Greece, Athens (we all know who won that one).



Answer to Activity

3. **Path graph:** As before, but this time Poseidon and Athena went back to old habits..



Before you go...

Do you have any general questions about this week's assignment? We can't go into minute details about any specific questions, but this is the time to ask any general clarifying/confusing questions!

As always, if you can't talk to us during recitation or office hours, post any questions/anything course related on the forums or email mcitonline@seas.upenn.edu. Ask questions that might be beneficial to other students on the forums, while emailing about more personal questions (regrade requests, etc).