MCIT 592 Spring 2022 **Homework 6** # Solutions

Problem Set

1. [10 pts] Prove by induction that for all positive integers n, we have:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

Solution:

We denote by $P(n)$ be the statement $1 + 3 + 5 + \cdots + (2n-1) = n^2$ and we prove by ordinary induction $\quad \forall \in \mathbb{Z}^+ \; P(n)$.

(BC) $n = 1$. The sum is 1 but $1 = 1^2$, so $P(1)$ holds.

(IS) Let k be an arbitrary positive integer. Assume $P(k)$, this is our IH. We now show that this implies $P(k+1)$.

In $P(k+1)$ the sum is: $1 + 3 + 5 + \cdots + (2k-1) + (2(k+1)-1)$

Apply the IH to the sum of the first k terms:

$$\begin{aligned}
1 + 3 + 5 + \cdots + (2k-1) + (2(k+1)-1) &= k^2 + (2(k+1)-1) \\
&= k^2 + 2k + 2 - 1 \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

We have shown that $P(k)$ implies $P(k+1)$.

We conclude that $P(n)$ holds for all positive integers n.

2. [10 pts] Use the recursion tree method or the telescopic method to solve the recurrence relation $f(0) = 7$ and, for all $n \in \mathbb{Z}^+ \;\; f(n) = f(n-1) - 2n$.

Solution:

We use the telescopic method.

$$
\begin{aligned}
f(n) &= f(n-1) - 2n \\
f(n-1) &= f(n-2) - 2(n-1) \\
&\vdots \\
f(2) &= f(1) - 2(2) \\
f(1) &= f(0) - 2(1) \\
f(0) &= 7\,.
\end{aligned}
$$

We add all LHS and all RHS and remove the terms that appear on both sides. Looking at the terms that appear on only one side of these equations, we can see that

$$
\begin{aligned}
f(n) &= -2n - 2(n-1) - 2(n-2) - \cdots - 2(2) - 2(1) + 7 \\
&= -2(n + (n-1) + (n-2) + \cdots + 2 + 1) + 7 \\
&= -2(\frac{n(n+1)}{2}) + 7 \\
&\qquad \text{(using sum of first n positive integers formula)} \\
&= -n(n+1) + 7
\end{aligned}
$$

Therefore our closed formula for f is $\boxed{f(n) = -n(n+1) + 7}$.

3. [10 pts] Use strong induction to prove that $C(n) = 2^n + 3$ is a solution to the recurrence $C(0) = 4$, $C(1) = 5$, and, for all $n \in \mathbb{Z}^+$, $n > 1$
$C(n) = 3 \cdot C(n-1) - 2 \cdot C(n-2)$.

Solution:

Let $P(n)$ be "$C(n) = 2^n + 3$."

(BC) $C(0) = 4 = 2^0 + 3$ and $C(1) = 5 = 2^1 + 3$, so $P(0)$ and $P(1)$ hold.

(IS) Let k be an arbitrary positive integer. Assume $P(j)$ holds for all $0 \le j \le k$.

This is our IH. We want to show that this implies $P(k+1)$.

The recurrence tells us that:

$$C(k+1) = 3C(k) - 2C(k-1)$$

Applying the IH:

$$\begin{aligned} C(k+1) &= 3C(k) - 2C(k-1) \\ &= 3(2^k + 3) - 2(2^{k-1} + 3) \\ &= 3 \cdot 2^k + 9 - 2^k - 6 \\ &= 2 \cdot 2^k + 3 \\ &= 2^{k+1} + 3 \end{aligned}$$

Thus, we have shown that $P(j)$ holding for all $0 \leq j \leq k$ implies $P(k+1)$. By strong induction, we conclude that $P(n)$ holds for all $n \in \mathbb{N}$, i.e., $C(n) = 2^n + 3$ for all $n \in \mathbb{N}$. $\square$

4. [10 pts] Recall the Fibonacci sequence, where every number in the sequence is the sum of the previous two numbers (except for the first and second positions, which are 0 and 1 respectively). Let F_n represent the *nth* number in the Fibonacci sequence. Use strong induction to prove that for Fibonacci numbers $F_{n+1} - F_{n-1} < 2^n$ for all positive integers n.

Solution:

This problem works for both defs of the Fibonacci sequences:

lecture notes	$F_0 = 0 \quad F_1 = 1 \quad F_2 = 1 \quad F_3 = 2$ etc.
Francesca on Piazza	$F_0 = 1 \quad F_1 = 1 \quad F_2 = 2 \quad F_3 = 3$ etc.

SOLUTION ONE

Let $P(n)$ be $F_{n+1} - F_{n-1} < 2^n$.

(BC) $n = 1$. $F_2 - F_0 = 2 - 1 = 1 < 2 = 2^1$. $P(1)$ holds. (Same with the other def: $F_2 - F_0 = 1 - 0 = 1$)

$n = 2$. $F_3 - F_1 = 3 - 1 = 2 < 4 = 2^2$. $P(2)$ holds. (Same with the other def: $F_3 - F_1 = 2 - 1 = 1$)

Let $k \geq 2$ be an arbitrary integer. Assume assume that $P(k)$ and $P(k-1)$ hold; this is our (strong) IH.

We want to show that this implies $P(k + 1)$. That is:

$$\begin{aligned}
F_{k+2} - F_k &= F_{k+1} + F_k - (F_{k-1} + F_{k-2}) \\
&= F_{k+1} - F_{k-1} + F_k - F_{k-2} \\
&< 2^k + 2^{k-1} && \text{(By IH)} \\
&< 2^k + 2^k \\
&= 2^{k+1}
\end{aligned}$$

Thus, $P(k)$ and $P(k - 1)$ together imply $P(k + 1)$. By strong induction, we conclude that $P(n)$ holds for all positive integers n, i.e., that $F_{n+1} - F_{n-1} < 2^n$ for all positive integers n $\square$

Solution Two

First notice that $F_{n+1} - F_{n-1} = F_n$. Now prove by induction that $F_n < 2^n$

Let $P(n)$ be "$F_n < 2^n$."

(BC) $F_1 = 1 < 2 = 2^1$ and $F_2 = 2 < 4 = 2^2$, so $P(1)$ and $P(2)$ hold. (Same with the other def: $F_2 = 1 < 4$.)

(IS) Let $k \geq 2$ be an arbitrary integer. Assume assume that $P(k)$ and $P(k - 1)$ hold; this is our (strong) IH.

Then

$$F_{k+1} = F_k + F_{k-1} < 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1} .$$

Thus, $P(k - 1)$ and $P(k)$ together imply $P(k + 1)$.

By strong induction, we conclude that $P(n)$ holds for all positive integers n, i.e., that $F_n < 2^n$ for all positive integers n. It then follows that $F_{n+1} - F_{n-1} < 2^n$ for all positive integers n $\square$

5. **[10 pts]** Use ordinary induction to prove that for every positive integer n, $n^3 - n$ is a multiple of 6. Only proofs by induction are accepted.

Solution:

Let $P(n)$ be "$n^3 - n$ is a multiple of 6."

(BC) $1^3 - 1 = 0 = 0 \cdot 6$, so $P(1)$ holds.

(IS) Let k be an arbitrary positive integer. Assume $P(k)$, this is our IH. Then

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1) = (k^3 - k) + 3(k^2 + k)$$

Our IH tells us that $(k^3 - k)$ is a multiple of 6.

We will also use the following:

Lemma For any integer m, $m^2 + m$ is even.

Proof of Lemma For $m \in \mathbb{N}$ this can be easily shown by induction on m and this suffices for what follows (the rubric will reflect this alternative proof).

However, here is a different proof which works for arbitrary $m \in \mathbb{Z}$. Notice that that $m^2 + m = m(m+1)$ so this is the product of two *consecutive* integers. But for any two consecutive integers one of them is even and the other is odd. And even times odd is even so the product of two consecutive integers is always even. **End proof of lemma**

Back to our induction step. We have just shown in the Lemma that $k^2 + k$ is even. Therefore $3(k^2 + k)$ is a multiple of 6. It follows that

$(k^3 - k) + 3(k^2 + k)$ is a multiple of 6, and $P(k+1)$ holds. We have shown that $P(k)$ implies $P(k+1)$.

By ordinary induction, we conclude that $P(n)$ holds for all $n \in \mathbb{Z}^+$, i.e., for every $n \in \mathbb{Z}^+$, $n^3 - n$ is a multiple of 6. $\square$