Module 10.3: Linearity of Variance?

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Lecture Notes

Variance of Bernoulli r.v.'s

Problem. Let X be a Bernoulli r.v. with parameter $\Pr[X=1]=p$. Calculate $\mathrm{Var}[X]$.

Answer. We have already calculated the expectation of a Bernoulli r.v. as $\mu = \mathrm{E}[X] = p$.

To compute the variance we will need the expectation of X^2. For any outcome w we have:

. $X^2(w) = 1$ iff $X(w)=1$ and $X^2(w)=0$ iff $X(w)=0$.

Therefore, X^2 is also Bernoulli, with the same distribution. In fact, X^2 equals X!

We conclude that $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mu^2 = p - p^2 = p(1-p)$.

Quiz

Let X be a Bernoulli r.v. with parameter $1/3$ and let $Y = 1 - X$. Then

(A) X and Y have the **same** expectation and the **same** variance.

(B) X and Y have the **same** expectation and **different** variances.

(C) X and Y have **different** expectations and the **same** variance.

Answer

(A) X and Y have the **same** expectation and the **same** variance.
Incorrect. Observe that Y is also Bernoulli but with parameter $1 - (1/3) = 2/3$. Hence $\mathrm{E}[X] = 1/3$ but $\mathrm{E}[Y] = 2/3$.

(B) X and Y have the **same** expectation and **different** variances.
Incorrect. Expectations are different, see (A).

(C) X and Y have **different** expectations and the **same** variance.
Correct. Observe that Y is also Bernoulli but with parameter $1 - (1/3) = 2/3$. Therefore
$\mathrm{Var}[X] = (1/3)(1 - (1/3)) = (2/3)(1 - (2/3)) = \mathrm{Var}[Y]$

Variance of the sum of two fair dice

Problem. Recall S the r.v. that returns the sum of numbers shown by two fair dice rolled together. Calculate $\mathrm{Var}[S]$.

Answer. In an earlier segment we have calculated $\mathrm{E}[S] = 7$. It remains to calculate $\mathrm{E}[S^2]$.

Like S, S^2 takes 11 values. Namely, the squares of the 11 values of S with the same probabilities. We show only some of the 11 terms in the variance sum:

$$\mathrm{E}[S^2] \;=\; 4 \cdot (1/36) + \cdots + 25 \cdot (4/36) + \cdots + 49 \cdot (6/36) + \cdots + 144 \cdot (1/36)$$

Can we avoid this calculation...?

From previous segments, we know that $S = G + P$ and we calculated $Var[G] = \mathrm{Var}[P] = \mathrm{Var}[D] = 35/12$. Is there **linearity of variance**?

Linearity of variance?

Proposition. $\mathrm{Var}[c\,X] \;=\; c^2\,\mathrm{Var}[X]$

Proof. $\mathrm{Var}[c\,X] \;=\; \mathrm{E}[(cX)^2] - (\mathrm{E}[cX])^2 \;=\; \mathrm{E}[c^2\,X^2] - (c\,\mathrm{E}[X])^2$

$$= \; c^2\,\mathrm{E}[X^2] - c^2(\mathrm{E}[X])^2 \;=\; c^2(\mathrm{E}[X^2] - (\mathrm{E}[X])^2) \;=\; c^2\,\mathrm{Var}[X]$$

In general, variance does not distribute over sums. However:

Proposition. if $X \perp Y$ then $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.

Proof. In the segment entitled "Correlated random variables" we define **product** of r.v.'s and show

1) $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ iff $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$

2) $X \perp Y \;\Rightarrow\; \mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$. The proposition follows.

When we roll two fair green-purple dice, we have $S = G + P$ and $G \perp P$.

Therefore, $\mathrm{Var}[S] = Var[G] + \mathrm{Var}[P] = (35/12) + (35/12) = 35/6$.