MCIT 592 Spring 2022 **Homework 10** # Solutions

Problem Set

1. [10 pts] Let X and Y be independent random variables such that $\text{Var}[X] = 5.3$ and $\text{Var}[Y] = 8.9$. What is the standard deviation of $3X + 2Y$?

Solution:

In general $\text{Var}[c \cdot Z] = c^2\text{Var}[Z]$, so we have $\text{Var}[3X] = 9\text{Var}[X] = 47.7$ and $\text{Var}[2Y] = 4\cdot\text{Var}[Y] = 35.6$. Since X and Y are independent, $3X$ and $2Y$ are also independent, so $\text{Var}[3X + 2Y] = \text{Var}[3X] + \text{Var}[2Y] = 83.3$. Finally, the standard deviation of $3X + 2Y$ is the square root of the variance, or $\boxed{\sqrt{83.3}}$.

2. [10 pts] Suppose James has a garden. Let X be the random variable representing the heights of the flowers in the garden, and let Y be the random variable representing the number of petals the flowers have. Suppose that X and Y are non-negative and independent. Help James prove that $X^2 \perp Y^2$.

Solution:

We want to show, assuming $X \perp Y$, that for all $a \in \text{Val}(X^2)$ and $b \in \text{Val}(Y^2)$,

$$\Pr[(X^2 = a) \cap (Y^2 = b)] = \Pr[X^2 = a] \cdot \Pr[Y^2 = b]\,.$$

Notice that since $a \in \text{Val}(X^2)$ and X is non-negative, we must have $a = x^2$ for a unique $x \in \text{Val}(X)$, and similarly $b = y^2$ for a unique $y \in \text{Val}(Y)$. So we can restate the desired condition as, for all $x \in \text{Val}(X)$ and $y \in \text{Val}(Y)$,

$$\Pr[(X^2 = x^2) \cap (Y^2 = y^2)] = \Pr[X^2 = x^2] \cdot \Pr[Y^2 = y^2]\,.$$

We now show that this holds.

$$\begin{aligned}\Pr[(X^2 = x^2) \cap (Y^2 = y^2)] &= \Pr[(X = x) \cap (Y = y)] \\ &= \Pr[X = x] \cdot \Pr[Y = y] \\ &\qquad \text{(by our assumption that } X \perp Y) \\ &= \Pr[X^2 = x^2] \cdot \Pr[Y^2 = y^2]\,.\end{aligned}$$

□

3. **[10 pts]** Suppose you roll $n \geq 1$ fair dice. Let X be the random variable for the sum of their values, and let Y be the random variable for the number of times an odd number comes up. Prove or disprove: X and Y are independent.

Solution:

There are 2 interpretations on this problem:
Proof 1: we disprove "$\forall n \geq 1$, X and Y are independent"
$\forall n \geq 1, X \perp Y$ means that for all possible n, $x \in Val(X)$ and $y \in Val(Y)$,

$$\Pr[X = x \text{ and } Y = y] = \Pr[X = x] \cdot \Pr[Y = y]\,.$$

But $n = 2$, $x = 2$ and $y = 0$ is a counterexample to that condition, because the only way X can be 2 is if every die came up as a 1, which is odd, so there cannot be 0 odd numbers.

$$\begin{aligned}\Pr[X = 2 \text{ and } Y = 0] &= 0 \\ &\neq \left(\frac{1}{12}\right)^2 \\ &= \left(\frac{1}{6}\right)^2 \cdot \left(\frac{1}{2}\right)^2 \\ &= \Pr[X = 2] \cdot \Pr[Y = 0]\,.\end{aligned}$$

Proof 2: $\forall n \geq 1$, *we disprove "X and Y are independent"*

$X \perp Y$ means that for all $x \in Val(X)$ and $y \in Val(Y)$,

$$\Pr[X = x \text{ and } Y = y] = \Pr[X = x] \cdot \Pr[Y = y]\,.$$

But $x = n$ and $y = 0$ are a counterexample to that condition, because the only way X can be n is if every die came up as a 1, which is odd, so there cannot be 0 odd numbers.

$$\begin{aligned}\Pr[X = n \text{ and } Y = 0] &= 0 \\ &\neq \left(\frac{1}{12}\right)^n \\ &= \left(\frac{1}{6}\right)^n \cdot \left(\frac{1}{2}\right)^n \\ &= \Pr[X = n] \cdot \Pr[Y = 0]\,.\end{aligned}$$

4. [10 pts] Suppose that you generate a 12-character password by selecting each character independently and uniformly at random from $\{\texttt{a}, \texttt{b}, \ldots, \texttt{z}\} \cup \{\texttt{A}, \texttt{B}, \ldots, \texttt{Z}\} \cup \{\texttt{0}, \texttt{1}, \ldots, \texttt{9}\}$.

(a) 4 What is the probability that exactly 6 of the characters are digits?

(b) 4 What is the expected number of digits in a password?

(c) 2 What is the variance of the number of digits in a password?

Solution:

(a) Note that there are 26 lowercase letters, 26 uppercase letters, and 10 digits. We can now define our sample space Ω as the set of all possible passwords we can create out of these three sets. Since there are 62 total characters and 12 locations, $|\Omega| = 62^{12}$.

Let E be the event we have exactly 6 digits in our password. Note that what we want is $|E|/|\Omega|$. We count $|E|$ as follows:

Step 1. Choose 6 of the 12 locations as places we will place our digits ($\binom{12}{6}$ ways)

Step 2. Assign digits to each of the chosen locations (10^6 ways)

Step 3. Assign lowercase or uppercase letters to the remaining locations (52^6 ways)

Hence, by the multiplication rule, $|E| = \binom{12}{6} \times 10^6 \times 52^6$. Plugging into our expression, the probability we want is

$$\frac{|E|}{|\Omega|} = \frac{\binom{12}{6} \times 10^6 \times 52^6}{62^{12}}.$$

(b) Let X be a random variable denoting the number of digits in a password. Let S_i be the event that the ith character location contains a digit. Let X_i be an indicator random variable that is 1 if its corresponding S_i occurs and 0 otherwise.

Note that $X = X_1 + X_2 + \cdots + X_{12}$. By linearity of expectation, $\mathrm{E}[X] = \sum_{i=1}^{12} \mathrm{E}[X_i]$. We have $\mathrm{E}[X_i] = 10/62 = 5/31$ since there are 10 digits and 62 total characters, and we choose our characters uniformly at random. Hence,

$$\mathrm{E}[X] = \sum_{i=1}^{12} \frac{5}{31} = \frac{60}{31}.$$

(c) Notice that our random variable X as defined in part (b) is a Binomial random variable as it is composed of the independent Bernoulli random variables X_i. In other words, our X is done over $n = 12$ trials and the ith Bernoulli trial succeeds when a digit is chosen for the ith position with success probability $p = 5/31$. Hence,

$$\mathrm{Var}[X] = np(1-p) = 12 \times \frac{5}{31} \times \frac{26}{31}.$$

5. [10 pts] Suppose Jay shoots a basketball. Let X be the Bernoulli random variable that returns 1 if he makes the shot, and 0 if he misses. Let Y be the Bernoulli random variable (independent of X) that returns 1 if he hits the backboard, and 0 if he does not hit it. X and Y both have parameter 1/2. Let Z be the random variable that returns the remainder of the division of $X+Y$ by 2.

(a) 2 Prove that Z is also a Bernoulli random variable, also with parameter 1/2.

(b) 4 Prove that X, Y, Z are pairwise independent but not mutually independent.

(c) 4 By computing $\text{Var}[X+Y+Z]$ according to the alternative formula for variance and using the variance of Bernoulli r.v.'s, verify that $\text{Var}[X+Y+Z] = \text{Var}[X] + \text{Var}[Y] + \text{Var}[Z]$ (observe that this also follows from the proposition on slide 5 of the lecture segment entitled "Binomial distribution").

Solution:

(a) The values taken by Z are shown in the following table (that also includes the values taken by $X+Y+Z$ in anticipation of the last part of this problem):

X	Y	$X+Y$	Z	$X+Y+Z$
0	0	0	0	0
0	1	1	1	2
1	0	1	1	2
1	1	2	0	2

Therefore, $\text{Val}(Z) = \{0, 1\}$. Moreover, using $X \perp Y$,

$\Pr[Z = 1] = \Pr[(X = 0 \;\cap\; Y = 1) \cup (X = 1 \;\cap\; Y = 0)] = 1/2 \cdot 1/2 + 1/2 \cdot 1/2 = 1/4 + 1/4 = 1/2$. It follows that Z is Bernoulli with parameter 1/2.

(b) We already know that $X \perp Y$. We verify $X \perp Z$ as follows:

$$\begin{aligned}
\Pr[X = 1 \ \cap \ Z = 1] &= \Pr[(X = 1) \cap ((X = 0 \ \cap \ Y = 1) \cup (X = 1 \ \cap \ Y = 0))] \\
&= \Pr[X = 1 \ \cap \ Y = 0] \\
&= \Pr[X = 1] \cdot \Pr[Y = 0] = 1/2 \cdot 1/2 = \Pr[X = 1] \cdot \Pr[Z = 1]
\end{aligned}$$

We would also need to verify that, for example, $\Pr[X = 0 \ \cap \ X = 1] = \Pr[X = 0] \cdot \Pr[Z = 1]$ and for the other intersections. However, for Bernoulli variables, these follow from the one we just proved in view of property Ind (iv) from the lecture segment entitled "Independence".

With the same observation, here is the verification of $Y \perp Z$:

$$\begin{aligned}
\Pr[Y = 1 \ \cap \ Z = 1] &= \Pr[(Y = 1) \cap ((X = 0 \ \cap \ Y = 1) \cup (X = 1 \ \cap \ Y = 0))] \\
&= \Pr[X = 0 \ \cap \ Y = 1] \\
&= \Pr[X = 0] \cdot \Pr[Y = 1] = 1/2 \cdot 1/2 = \Pr[Y = 1] \cdot \Pr[Z = 1]
\end{aligned}$$

Therefore X, Y, Z are pairwise independent. To show that they are not mutually independent observe that $\Pr[X = 1 \ \cap \ Y = 1 \ \cap \ Z = 1] = \Pr[\emptyset] = 0$ but $\Pr[X = 1] \cdot \Pr[Y = 1] \cdot \Pr[Z = 1] = 1/2 \cdot 1/2 \cdot 1/2 = 1/8$.

(c) From the table above $\text{Val}(X + Y + Z) = \{0, 2\}$. Each row in the table corresponds to a probability of $1/2 \cdot 1/2 = 1/4$ therefore the distribution of $X + Y + Z$ is 0 with probability 1/4 and 2 with probability 3/4.

It also follows that $(X + Y + Z)^2$ is $0^2 = 0$ with probability 1/4 and $2^2 = 4$ with probability 3/4. We now use the alternative formula for variance:

$$\begin{aligned}
\mathrm{Var}[X+Y+Z] &= \mathrm{E}[(X+Y+Z)^2] - \mathrm{E}[X+Y+Z]^2 \\
&= \left(\frac{1}{4}\cdot 0 + \frac{3}{4}\cdot 4\right) - \left(\frac{1}{4}\cdot 0 + \frac{3}{4}\cdot 2\right)^2 \\
&= 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}\,.
\end{aligned}$$

Using the formula for variance of a Bernoulli variable with a given parameter we have $\mathrm{Var}[X] + \mathrm{Var}[Y] + \mathrm{Var}[Z] = 3(1/2 - (1/2)^2) = 3/4$. This verifies the proposition that says that for pairwise independent r.v.'s variance distributes over sums.