Recitation Module 2

Penn Engineering

Concept Review

A powerset of A 2^A $|2^A| = 2^{|A|}$

- Cardinality of Power Set
- Complementary Counting

5 coin flips $\underset{\text{total}}{2^5} - \underset{\text{all tails}}{1} = 32 - 1 = 31$

- Number of Binary Strings

r out of n $\frac{n!}{(n-r)!}$

- Partial Permutations $\{0, a, x\}$ $0a,\ 0x,\ a0,\ x0,\ ax,\ xa$
- Logical Connectives and $\wedge$, or $\vee$, if-then $\Rightarrow$, not $\neg$
- Implication, conditional, and equivalence

if P_1 then P_2

P_1: premise, P_2: conclusion

if P_1 then P_2, else P_3

$(P_1 \Rightarrow P_2) \wedge (\neg P_1 \Rightarrow P_3)$

biconditional

P_1 iff P_2

$P_1 \Leftrightarrow P_2$

- Combinations

r out of n $\frac{n!}{r!\,(n-r)!}$

- Quantifiers

universal $\forall x$: for all x

existential $\exists x$: there exists an x

Question 1

Consider the following set of Elves: Haldir, Arwen, Legolas, Galadriel, Tauriel, Luthien, Elrond

1. How many groups of these Elves contain Arwen but do not contain Tauriel?
2. How many groups of these Elves contain Haldir and Legolas OR contain Elrond?
3. How many different permutations can we construct out of the Elves whose names contain at least one letter “a”?

Question 1

Elves = {Haldir, Arwen, Legolas, Galadriel, Tauriel, Luthien, Elrond}

1. How many groups of these Elves contain Arwen but do not contain Tauriel?

$B = \{H, L_e, G, L_u, E\}$

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$

$2^{|B|} = 2^5 = \boxed{32}$

Question 1

Elves = {Haldir, Arwen, Legolas, Galadriel, Tauriel, Luthien, Elrond}

2. How many groups of these Elves contain Haldir and Legolas OR contain Elrond?

complementary counting

I) groups that contain H but not Le and E → $2^4 = 16$

II) groups that contain Le but not H and E → $2^4 = 16$

III) groups w/o H, Le, and E → $2^4 = 16$

$16 \times 3 = 48$

$2^7 = 128$

$128 - 48 = \boxed{80}$

Question 1

Elves = {Haldir, Arwen, Legolas, Galadriel, Tauriel, Luthien, Elrond}

3. How many different permutations can we construct out of the Elves whose names contain at least one letter “a”?

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$

Question 2

1. Write the following statement using logical notation:

   "If n and m are both odd, then nm is odd"

2. Then, prove the statement.

3. Finally, use proof by cases to prove that

   "If n and m are not both odd, then nm is even"

Question 2

1. Write the following statement using logical notation:

"If n and m are both odd, then nm is odd"

$(odd(n) \wedge odd(m)) \Rightarrow odd(nm)$

2. Then, prove the statement.

$n = 2k+1$
$m = 2g+1$

$$nm = (2k+1)(2g+1) = 4gk + 2k + 2g + 1$$
$$= 2(2gk + k + g) + 1$$
$$= 2j + 1$$

Question 2

3. Finally, use proof by cases to prove that

"If n and m are not both odd, then nm is even"

Case 1: n is even, m is odd

$n = 2k$

$m = 2g+1$

$nm = (2k)(2g+1) = 4gk + 2k$

$= 2(2gk+k)$

Case 2: n is odd, m is even

$n = 2k+1$

$m = 2g$

$nm = (2k+1)(2g) = 4gk + 2g =$

$2(2gk+g)$

Case 3: n is even, m is even

$nm = (2g)(2k) = 4gk$

$= 2(2gk)$

Question 3

In how many ways can we arrange the digits 1-9 in a permutation so that either 5 appears in the middle or 9 appears at the end?

$8!+8!-7! = 2\cdot 8! - 7!$

$7!(8+8-1)$

$= 75{,}600$

_ _ _ _ 5 _ _ _ _ $= 8!$

_ _ _ _ _ _ _ _ 9 $= 8!$

_ _ _ _ 5 _ _ _ 9 $= 7!$

Question 4

Count the number of sequences of bits of length 8 in which every 0 is followed immediately by a 1. (*Note:* this means the sequence cannot end with a 0!)

$\underbrace{01}$

no zeros → 1 way   _ _ _ _ _ _ _ _ 1

1 zero → 7 ways

two $\underbrace{01}$ blocks, four $\underbrace{1}$ blocks → 2 zeros $6c2 =$ 15 ways   _ _ _ _ _ _

three $\underbrace{01}$ blocks, two $\underbrace{1}$ blocks   3 zeros $5c3 =$ 10 ways

4 zeros → + 1 way

34 ways total

Question 5 (time permitting)

7. [6 pts] EXTRA CREDIT CHALLENGE PROBLEM

To cheer up your friend who is safely social distancing at home, you decide to put together and send them a box of chocolates! Let j, k, l, m be natural numbers such that $1 \leq k \leq j$ and $1 \leq 2l \leq m$. You have a total of j dark chocolates and m milk chocolates for you to choose from. All of the chocolates (even the same type) have different fillings and are therefore all distinguishable. However, you remember that l of the milk chocolates have nuts inside of them and that your friend isn't the biggest fan of nuts with milk chocolate. Thus, you decide to include **at most** one of these l milk chocolates with nuts (yes, you can include none). How many different chocolate boxes can you form consisting of exactly k dark chocolates and l milk chocolates? For full credit, your answer must be in closed form.

dark chocolate

$$\underbrace{\binom{j}{k}\binom{m-\ell}{\ell}}_{\text{no milk choc w/ nuts}}$$

$$\underbrace{\binom{j}{k}\binom{m-\ell}{\ell-1}\binom{\ell}{1}}_{\text{one milk choc w/ nuts}}$$

$$\binom{j}{k}\binom{m-\ell}{\ell}+\binom{j}{k}\binom{m-\ell}{\ell-1}\ell$$

$$\binom{j}{k}\left(\binom{m-\ell}{\ell}+\ell\binom{m-\ell}{\ell-1}\right)$$