Module 3.3: Converse and Contrapositive

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Lecture Notes

Converse and contrapositive

Memorize:

the **converse** of "if P_1 then P_2" is "if P_2 then P_1", and

the **contrapositive** of "if P_1 then P_2" is "if (not P_2) then (not P_1)".

In logical notation:

the **converse** of $P_1 \Rightarrow P_2$ is $P_2 \Rightarrow P_1$, and

the **contrapositive** of $P_1 \Rightarrow P_2$ is $\neg P_2 \Rightarrow \neg P_1$.

The contrapositive is **logically equivalent** to the original implication. This leads to:

Proof pattern: instead of the implication, prove its contrapositive,

The converse is **not** logically equivalent to the original implication (in general).

A false converse I

Problem. Consider the statement “for all integers z, if z is divisible by 4 then it is even”. Prove it. State the converse (of the implication under the quantifier) and disprove it.

Answer. (first part)

Assume that z is divisible by 4.

Then $z = 4k$ for some integer k.

Then $z = 2\ell$ where $\ell = 2k$.

ℓ is an integer, hence z is even.

A false converse II

Answer. (second part)

The converse is "for all integers z, if z is even then it is divisible by 4".

To disprove this, we prove the negation:

"there exists an even integer z that is not divisible by 4".

Of course there is such an integer! Take $z = 6$.

Often, we avoid stating the negation explicitly and we just say that $z = 6$ is a **counterexample**, as discussed in a prior slide.

In this problem we found an implication statement that is not logically equivalent to its converse since one is true and the other one is false.

A proof by contrapositive I

Problem. Consider the statement "for any integers x, y, if xy is even then at least one of x, y must be even". Write the contrapositive (of the implication under the quantifiers) and prove it.

Answer. Let's rewrite the statement :

> "for any integers x, y, if xy is even then x is even or y is even"

Using De Morgan's Laws and recalling that we agreed that the negation of even is odd we write the contrapositive :

> "for any integers x, y, if x is odd and y is odd then xy is odd"

A proof by contrapositive II

Answer. (continued) We wish to show that:

"for any integers x, y, if x is odd and y is odd then xy is odd"

Assume that x and y are both odd.

Then $x = 2k + 1$ for some integer k.
And $y = 2\ell + 1$ for some integer ℓ.

Then

$$. \quad xy = (2k+1)(2\ell+1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1$$

$2k\ell + k + \ell$ is an integer, hence xy is odd.

This was an example where the (logically equivalent) contrapositive was much easier to prove.