Module 9.5: More Expectations

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Expectation: success followed by failure I

Problem. Consider again n IID Bernoulli trials, each with probability of success p. Let C be the random variable that returns the number of successes which are followed immediately (in the next trial) by failure. Compute the expectation of C.

Answer. Using **indicator** r.v.'s and **linearity of expectation**.

Notice that $C = I_1 + \cdots + I_{n-1}$

where, for $k = 1, \ldots, n-1$, I_k is the indicator r.v. of the event $A_k =$ "success in trial k and failure in trial $k+1$"

Recall that $\mathrm{E}[I_k] = \Pr[I_k = 1] = \Pr[A_k]$.

Expectation: success followed by failure II

Answer (continued).

Since trial k is independent of trial $k+1$, $\Pr[A_k] \;=\; p(1-p)$.

Hence, $\quad \mathrm{E}[C] = p(1-p) + \cdots + p(1-p) = (n-1)p(1-p)$.

Notice that for $p \neq 0$ or 1 the events $A_1, \ldots, A_{n-1}$ are not mutually independent. They are not even pairwise independent, because $A_k \not\perp A_{k+1}$!
Indeed, $\Pr[A_k] \cdot \Pr[A_{k+1}] = p^2(1-p)^2$.
However, $\Pr[A_k \cap A_{k+1}] = \Pr[\emptyset] = 0$.

Penn Engineering

The hats and gangsters problem I

Problem. n hat-wearing gangsters leave their distinguishable hats with a restaurant cloakroom attendant. After the meal, the attendant gives them back their hats **randomly**. How many of the gangsters get back their own hat, "on average"?

Answer. We assume that the returned hats form a **random permutation** of n elements. We studied them in a previous segment.

By definition, these form a uniform probability space: each outcome has probability $1/n!$.

We also saw that the probability that a given element occurs in a given position of a random permutation is $1/n$.

The hats and gangsters problem II

Answer (continued). Let X be the random variable that returns the number of gangsters that get back their own hat. We wish to compute $\mathrm{E}[X]$.

Figuring out the distribution X is complicated. Therefore, computing $\mathrm{E}[X]$ directly from the definition of expectation is also complicated. **Linearity of expectation** gives us, again, an easy solution.

Clearly $X = X_1 + \cdots + X_n$

where X_k is the indicator r.v. of the event E_k = "Gangster k gets back his own hat".

Recall again that $\mathrm{E}[X_k] = \Pr[X_k = 1] = \Pr[E_k]$

As I reminded you, we earlier calculated $\Pr[E_k] = 1/n$.

Therefore, by linearity of expectation $\mathrm{E}[X] = n \cdot (1/n) = 1$

On average, only **one** gangster gets back his own hat!

“Sorting” by random permutation

The hats and gangsters problem has a computer science equivalent!

Suppose I want to sort a set of n distinct real numbers.

I try to sort them by producing a **random permutation** of the elements of the set.

My brilliant intuition tells me that this cannot be **too** bad, can it? I should get, on average, maybe half of the numbers in their place? Am I right?

I am wrong, as we saw. On average, only **one element** ends up in the right place. Did you guess that it would be that bad?