Module 11.1: Graphs, Handshaking

MCIT Online - CIT592 - Professor Val Tannen

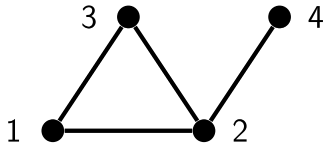
Lecture Notes

Graph terminology and notation I

An **undirected graph** is a pair $G = (V, E)$ where V is a finite **non-empty** set of **vertices** or **nodes** and $E \subseteq 2^V$ is a finite (possibly empty) set of **edges** consisting only of subsets of cardinality 2. We shall omit "undirected" and just say "graph".

Example:

G :



$$G = (\{1,2,3,4\} , \{ \{1,2\} , \{1,3\} , \{2,3\} , \{2,4\} \})$$

We use the notation u–v (equivalently, v–u) for both the edge $\{u, v\}$ and to say that u and v are **linked** by an edge.

For the example, $G = (\{1,2,3,4\} , \{ 1–2 , 1–3 , 2–3 , 2–4 \})$

Graph terminology and notation II

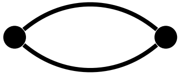
We will say that the edge u–v is **incident** to either of its **endpoints** u and v.

Two vertices such that u–v are called **adjacent** (or **neighbors**).

Note that our definition of an edge as a set of nodes of cardinality 2 precludes "loops" or "parallel edges". The kind of undirected graphs we work with are often called **simple graphs**.

.

Not
allowed:



Vertex degree and handshaking

The **degree** of a vertex, $\deg(u)$, is the number of neighbors of u (the number of vertices adjacent to u).

A vertex of degree 0 is called **isolated**.

Proposition (The Handshaking Lemma). The sum of the degrees of all nodes in a graph is **twice** the number of edges.

.

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof. Since each edge is incident to exactly two vertices, each edge contributes two to the sum of degrees of the vertices.

For the name "Handshaking Lemma" imagine that the graph represents handshakes between some of the people at a meeting.

Number of vertices of odd degree

Problem. Prove that in any graph there are an even number of vertices of odd degree.

Answer. Consider an arbitrary graph $G = (V, E)$. Let $V_e \subseteq V$ and $V_o \subseteq V$ be the set of vertices with even degree, respectively odd degree.

Then, $\sum_{v \in V} \deg(v) = \sum_{v \in V_e} \deg(v) + \sum_{v \in V_o} \deg(v)$.

The first term on the RHS is even since it is a sum of even numbers.

From the Handshaking Lemma it follows that the LHS is even.

Thus, the second term on the RHS must be even.

Since each vertex in V_o has odd degree, for the sum of the degrees of vertices in V_o to be even, $|V_o|$ must be even.

Quiz

In the graph $G = (V, E)$ all vertices have degree 1. Can G have

(A) 999 vertices?

(B) 1000 vertices?

(C) As many vertices as we wish?

Answer

(A) 999 vertices.
Incorrect. Recall the Handshake Lemma and apply it to this problem in order to find the correct answer.

(B) 1000 vertices.
Correct. Since every vertex has degree 1 the sum of all the degrees must equal the number of vertices, $|V|$. By the Handshake Lemma this sum also equals $2|E|$. Therefore, in such a graph $|V| = 2|E|$ so the graph must have an even number of vertices.

(C) As many vertices as we wish.
Incorrect. Recall the Handshake Lemma and apply it to this problem in order to restrict the number of vertices.

More information

We can reach the same solution with an alternative explanation. Recall that we proved that the number of vertices of odd degree must be even. But in G *all* vertices have odd degree (1 is odd). Thus G must have an even number of vertices. Therefore, the only solution that satisfies this condition is (B) 1000 vertices.

Quiz

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In a graph $G = (V, E)$ all vertices have even degree. Then G can have as many vertices as we wish, true or false?

(A) True.

(B) False.

Answer

(A) True.

Correct. The reason is that for any n, in the graph with n vertices and **no** edges is such that all vertices have even degree, since 0 is an even number!

(B) False.

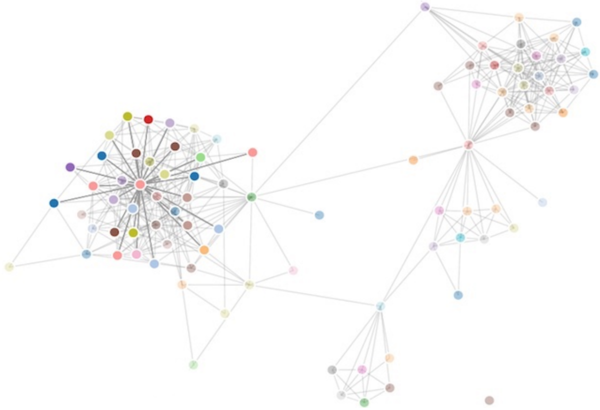
Incorrect. We can construct a graph with as many vertices as we want were all vertices have an even degree; try to think the construction that makes this possible.

More information

We will soon see that a graph with **no** edges is called **edgeless**.

Activity

In this activity you will verify the handshaking lemma for a real life graph - what we call a graph found in the "wild". Below you can see a small piece of the LinkedIn network.



Below you can see a sub-graph of the LinkedIn network above.

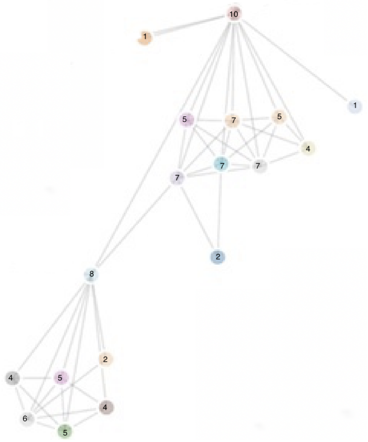
Question. For this activity: first find the sum of the degrees of all vertices in the sub-graph above and verify the handshaking lemma. Feel free to do the same for the larger graph as well (the reason we do not ask you to, is because the process of counting can be a bit tedious).

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Answer. First we count the sum of degrees of all vertices in the sub-graph:

$$10 + 1 + 1 + 4 + 5 + 5 + 7 + 7 + 7 + 7 + 2 + 8 + 2 + 4 + 4 + 5 + 5 + 6 = 90$$

As can be seen from the figure below, that shows the degree of each vertex by the number on it.



Activity

Answer (continued). Second we count the number of edges to be 45. Therefore, the handhsaking lemma holds since $90 = 2 \cdot 45$.

Module 11.2: Special Graphs

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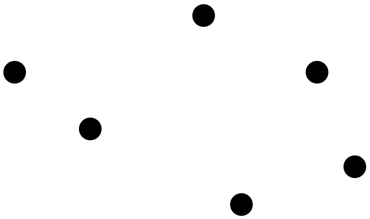
LECTURE NOTES

Edgeless graphs

An **edgeless** graph $G = (V, E)$ has some vertices (recall that it must have at least one so $|V| \geq 1$) but **no edges**.

Therefore, $E = \emptyset$ hence $|E| = 0$.

Example:

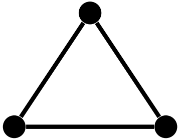


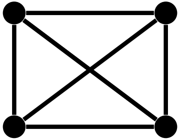
Complete graphs

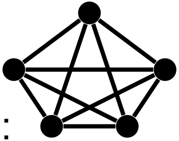
A **complete** graph on $n \geq 1$ vertices, notation K_n, has edges between **any two vertices**.

Therefore, $K_n = (V, E)$ where $|V| = n$ and $E = \{ \{u, v\} \mid u, v \in V \}$.

Examples:

K_3 : 

K_4 : 

K_5 : 

How many edges are there in K_n? The number of unordered pairs: $\binom{n}{2}$

Path graphs

A **path** graph on $n \geq 1$ vertices, notation P_n, has n vertices and $n-1$ edges arranged "in a row".

Examples:

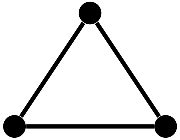
P_1 :  P_2 :  P_4 : 

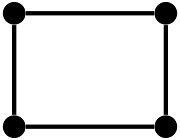
Note that for $n \geq 3$ we have two vertices of degree 1 in P_n and the rest have degree 2.

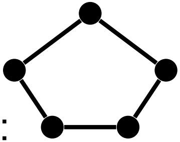
Cycle graphs

A **cycle** graph on $n \geq 1$ vertices, notation C_n, has n vertices and n edges arranged "in a circle".

Examples:

C_3 : 

C_4 : 

C_5 : 

Note that neither C_1 nor C_2 are defined.

Note also that in cycle graphs all vertices have degree 2.

Quiz

We saw that in a cycle graph all vertices have degree 2. Is the converse true? That is, if all vertices in a graph G have degree 2 must G be a cycle graph?

(A) Yes

(B) No

Answer

(A) Yes

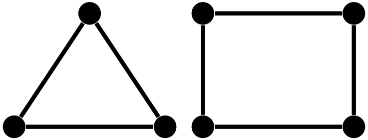
Incorrect. See More Information for a counterexample.

(B) No

Correct. There may be more than one cycle, see More information.

More information

In the counterexample below all nodes have degree 2 but the graph is not a cycle. (Although it is “made” of two cycles.)

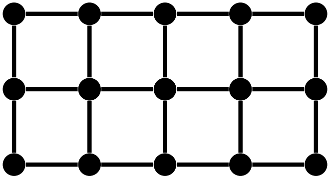


Grid graphs

Let $m, n \geq 1$. An $m \times n$ **grid** graph has m rows of n vertices. Each vertex is linked by an edge to the vertices "closest" to it.

$1 \times n$ grid is the same as path graph P_n. 2×2 grid is the same as cycle graph C_4.

Here is an example of 3×5 grid graph:



Note that an $m \times n$ grid graph has mn vertices.

When $m, n \geq 3$, we have 4 vertices with degree 2 and the other vertices have degree 3 or 4.

ACTIVITY : Counting edges in grid graphs

In this activity we will count the number of edges in an $m \times n$ grid graph in two different ways. (In both cases the answer will be $2mn - m - n$.)

First approach. Appealing to your visual intuition and referring to the figure in the previous slide, let's call the edges in the rows **horizontal** edges, and let's also call the edges between nodes in different rows **vertical** edges.

Question. How many horizontal edges and how many vertical edges are there in total?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Counting edges in grid graphs (continued)

Answer (first part). There are n vertices in each row so there are $n-1$ edges. Since there are m rows we have in total $m(n-1)$ horizontal edges.

Just like we think of the grid graph as consisting of horizontal rows we can also think of it as consisting of vertical columns. The vertical edges are grouped by column.

Answer (second part). There are m nodes in each column therefore $m-1$ edges. And there are n columns so there are $n(m-1)$ vertical edges in total.

In an $m \times n$ grid graph we have in total

$$m(n-1) + n(m-1) \; = \; 2mn - m - n$$

edges. Now for the second approach.

Activity : Counting edges in grid graphs (continued)

Second approach. We are going to use the Handshaking Lemma. Thus we have to count the sum the degrees of all the nodes. Let's assume $m, n \geq 2$ and we will consider the other cases separately.

There are 4 vertices of degree 2, the "corners" of the grid. So their contribution toward the sum of degrees is $4 \cdot 2$.

There are $m - 2$ vertices of degree 3 on each of the 2 vertical sides. Indeed there are a total of m vertices on each of those sides but two of these m are "corners" and have degree 2. Their contribution toward the sum of degrees is $2(m-2) \cdot 3$. Similarly there are $n - 2$ vertices of degree 3 on each of the horizontal sides. Their contribution toward the sum of degrees is $2(n-2) \cdot 3$.

Question. How many vertices of degree 4 are there?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Counting edges in grid graphs (continued)

Answer. The vertices of degree 4 form an $(m-2)\times(n-2)$ grid so there are $(m-2)(n-2)$ of them.

Their contribution to the sum of of degrees is $(m-2)(n-2)\cdot 4$.

Now we calculate the sum of degrees:

$$4\cdot 2+2(m-2)\cdot 3+2(n-2)\cdot 3+(m-2)(n-2)\cdot 4 \ =$$

$$8+6m-12+6n-12+4mn-8m-8n+16 \ = \ 4mn-2m-2n$$

By the Handshaking Lemma we divide by 2 and obtain the number of edges; $2mn-m-n$.

It is easy to check that the formula also applies for $m=1$ and/or $n=1$.

Module 11.3: Walks and Paths

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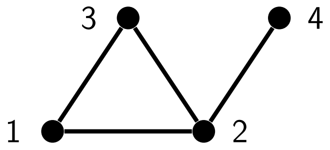
Lecture Notes

Walks

A **walk** is a non-empty sequence of vertices consecutively linked by edges: $u_0, u_1, \ldots, u_k$ such that $u_0–u_1–\cdots–u_k$. This walk is **from** u_0 **to** u_k (the **endpoints** of the walk) and u_0 and u_k are **connected** by this walk. The **length** of this walk is the number k of edges (**not** $k+1$!).

Example: (recall the graph)

G :



- 1–2–4 length 2
- 1–2–3–1–2–4 length 5
- 3–1 length 1
- 2 length 0

If u is a vertex, then u is also, by definition, a walk, whose length is 0. (Note that we cannot have edges linking a node with itself, but we have walks of length 0 !)

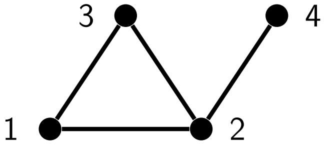
Paths

A **path** is a walk in which all the vertices are **distinct**.

Walks of length 0 are paths. Walks of length 1 are already paths because we cannot have loops. u–v–u is a walk of length 2 but not a path.

Example: (from previous slide)

G :

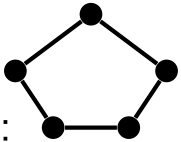


- 1–2–4 It's a path.
- 1–2–3–1–2–4 Not a path!
- 3–1 A path of length 1.
- 2 A path of length 0.

In G (above) there are paths of lengths 0, 1, 2, and 3. Can you find at least one path of each these lengths?

Quiz I

Recall the cycle graph C_n (assume $n \geq 3$). Below you can see C_5 to help you find the correct answer through the use of a concrete example.

C_5 : 

The maximum length of a path in C_n is:

(A) $n - 1$.

(B) n.

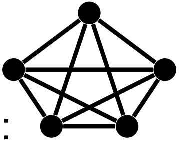
Answer

(A) $n - 1$.
Correct. Recall that that a path is a walk in which all the vertices are distinct. Therefore, the longest path will include n vertices connected by $n - 1$ edges. Therefore, the length of the longest path is $n - 1$.

(B) n.
Incorrect. Recall that a path is a walk in which all the vertices are distinct. How many edges are required to for a path with n vertices (since this would be the longest path in C_n)?

Quiz II

Recall the complete graph K_n. Below you can see K_5 to help you find the correct answer through the use of a concrete example.

K_5 : 

The maximum length of a path in K_n is:

(A) $n - 1$.

(B) n.

Answer

(A) $n - 1$.

Correct. Recall that that a path is a walk in which all the vertices are distinct. Therefore, the longest path will include n vertices connected by $n - 1$ edges. Therefore, the length of the longest path is $n - 1$.

(B) n.

Incorrect. Recall that a path is a walk in which all the vertices are distinct. How many edges are required to for a path with n vertices (since this would be the longest path in K_n)?

Quiz III

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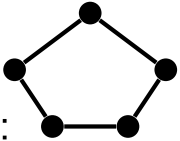
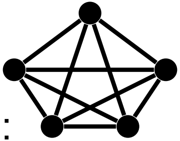
Let's look again at C_n and K_n (assume $n \geq 3$)

C_5 : 

K_5 : 

We have seen in previous two quizzes that the maximum length of a path in both is $n-1$. Clearly there are more paths of length $n-1$ in K_n than in C_n. We will see in a future segment how to count paths precisely. However, what would be your guess,

(A) There are 100 times more paths of length 99 in K_{100} than in in C_{100}.
(B) There are 2^{100} times more paths of length 99 in K_{100} than in in C_{100}.
(C) Much more than that :)

Answer

(A) There are 100 times more paths of length 99 in K_{100} than in in C_{100}.

(B) There are 2^{100} times more paths of length 99 in K_{100} than in in C_{100}.

(C) Much more than that :)

Where there is a walk there is a path I

Proposition. If $u_0 – u_1 – \cdots – u_{n-1} – u_n$ is a **walk** of length $n \geq 3$ such that $u_0 \neq u_n$, then there exist vertices $v_1, \ldots, v_m$ such that $u_0 – v_1 – \cdots – v_m – u_n$ is a **path** of length at most n. (Where there is a walk, there is a path that is not longer!).

Proof. Consider the walk $u_0 – u_1 – \cdots – u_{n-1} – u_n$ of length $n \geq 3$ and such that $u_0 \neq u_n$. Define A to be a set of natural numbers that are lengths of some walk from u_0 to u_n. We have $n \in A$ so A is non-empty.

The Well-Ordering Principle. Every non-empty set of natural numbers has a least element.

So A has a least element, call it p, with a walk W of length p from u_0 to u_n. p cannot be 0 because $u_0 \neq u_n$. If $p = 1$ then the walk W is a path of length 1 and we are done. The remaining case is $p \geq 2$ (see next slide).

Where there is a walk there is a path II

Proof (continued). We are left with the case when we have a walk $W = u_0 – v_1 – \cdots – v_m – u_n$ whose length $m + 1 \geq 2$ is 2 is the **shortest possible** length of any walk from u_0 to u_n.

(Now, if W would be a path we would be done!)

Claim. W is a path.

Suppose, toward a contradiction, that two of the nodes of W coincide.

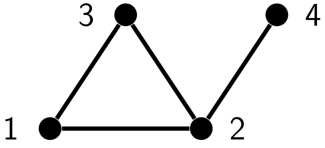
This can happen in three different ways: $u_0 \equiv v_i$ for some $1 \leq i \leq m$, or $v_i \equiv v_j$ for some $1 \leq i < j \leq m$, or $u_n \equiv v_i$ for some $1 \leq i \leq m$.

In each case we **delete** the edges between the repeated nodes leaving a **strictly shorter** walk between u_0 and u_n. This contradicts the property of W.

A shorter walk

Example:

G :



Consider the walk 1–2–3–1–2–4.

The vertex 2 is repeated. We **delete** the edges between the two occurrences of 2.

We get a **shorter** walk (actually a path) 1–2–4.

This is like "cutting off" the cycle 2–3–1–2 and replacing it with 2, a walk of length 0.

A stronger version

Proposition. If $u_0 - u_1 - \cdots - u_{n-1} - u_n$ is a walk of length $n \geq 3$ such that $u_0 \neq u_n$, then there exist vertices $v_1, \ldots, v_m$ such that $u_0 - v_1 - \cdots - v_m - u_n$ is a path **whose sequence of nodes and edges is a subsequence of the sequence of nodes and edges** of $u_0 - u_1 - \cdots - u_{n-1} - u_n$.

Here, "subsequence" preserves order, but it does not necessarily consist of consecutive elements, i.e., $e_1 e_3 e_4$ is a subsequence of $e_1 e_2 e_3 e_4$.

This is useful in some circumstances but we left the proof (by strong induction on the length of the walk) for an optional segment in this module.

Module 11.4: Connected Components (cc's)

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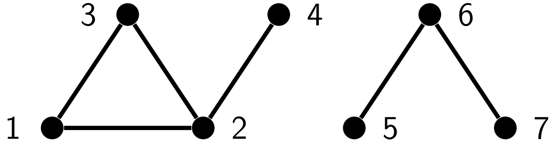
Lecture Notes

Connectivity

Two vertices u and v of a graph $G = (V, E)$ are said to be **connected**, **notation** $u{-}\cdots{-}v$, if there exists some walk (and therefore also some path!) with **endpoints** u and v. We call ${-}\cdots{-}$ the **connectivity relation**.

Example:

G :



In this graph we have, for example, $1{-}\cdots{-}4$, $3{-}\cdots{-}2$, and $5{-}\cdots{-}7$, however, we do **not** have $4{-}\cdots{-}6$ **nor** $3{-}\cdots{-}5$.

Reflexivity, symmetry, transitivity

- For any u we have $u\!-\!\cdots\!-\!u$ so connectivity is **reflexive**.

  Indeed, u is connected to u by the walk/path of length 0.

- For any u and v, if $u\!-\!\cdots\!-\!v$ then $v\!-\!\cdots\!-\!u$ so connectivity is **symmetric**.

  Indeed, if $u\!-\!w_1\!-\!\cdots\!-\!w_n\!-\!v$ is a walk, then its **reversal** $v\!-\!w_n\!-\!\cdots\!-\!w_1\!-\!u$ is also a walk.

- For any u, v, and w, if $u\!-\!\cdots\!-\!v$ and $v\!-\!\cdots\!-\!w$ then $u\!-\!\cdots\!-\!w$ so connectivity is **transitive**.

  Indeed, if $u\!-\!y_1\!-\!\cdots\!-\!y_m\!-\!v$ and $v\!-\!z_1\!-\!\cdots\!-\!z_n\!-\!w$ are walks, then their **concatenation** $u\!-\!y_1\!-\!\cdots\!-\!y_m\!-\!v\!-\!z_1\!-\!\cdots\!-\!z_n\!-\!w$ is also a walk.

Note that the reversal of a path is also a path, but the concatenation of two paths need not be a path.

Quiz

Recall the definition of event independence. As we pointed out $A \perp B$ is **symmetric**. But is it **transitive**?

(A) Yes.

(B) No.

Answer

(A) Yes.
Incorrect. Consider just two independent events $A \perp B$ such $\Pr[A] \neq 0$ and $\Pr[A] \neq 1$. Since independence is symmetric it follows that also $B \perp A$ and by transitivity $A \perp A$ which is impossible.

(B) No.
Correct. Consider just two independent events $A \perp B$ such $\Pr[A] \neq 0$ and $\Pr[A] \neq 1$. Since independence is symmetric it follows that also $B \perp A$ and by transitivity $A \perp A$ which is impossible.

More information

Let two independent events A, B, such that $\Pr[A] \neq 0$ and $\Pr[A] \neq 1$. We have that $A \perp B \Leftrightarrow B \perp A$ since independence is symmetric. Assume for the sake of contradiction that event independence is also transitive, it follows that $A \perp A$. By definition of event independence we have that $\Pr[A \cap A] = \Pr[A] \cdot \Pr[A] = \Pr[A]$. However, this is a contradiction since $\Pr[A] \neq 0$ and $\Pr[A] \neq 1$. Therefore, event independence is not transitive.

Connected components

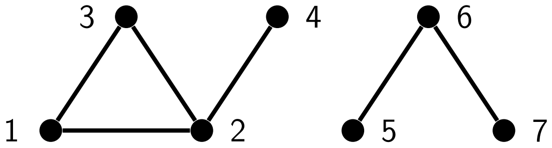
A **connected component** of a graph $G = (V, E)$ is a set of vertices $C \subseteq V$ such that:

- any two vertices in C are **connected**, and
- there is **no strictly bigger** set of vertices $C \subsetneq C' \subseteq V$ such that any two vertices in C' are connected.

We say that C is a **maximally connected** set of vertices of G.

Example:

G :



In this graph we have two connected components: $\{1, 2, 3, 4\}$ and $\{5, 6, 7\}$.

Partition

Clearly, every vertex is in **some** connected component (at worst, it may be just by itself).

Proposition. Any two distinct connected components are **disjoint**.

Proof. Suppose, toward a contradiction that two **distinct** connected components, C_1 and C_2, are not disjoint, namely there is some vertex $w \in C_1 \cap C_2$.

Any vertex $u \in C_1$ is connected to w. Similarly, any vertex $v \in C_2$ is connected to w. By symmetry and transitivity $u - \cdots - v$. But C_1 and C_2 are **maximally** connected. Hence, $u \in C_2$ and $v \in C_1$. We conclude $C_1 = C_2$. Contradiction.

We say that the connected components form a **partition** of the vertices.

Connected graphs

A graph in which any two vertices are connected is called **connected**, otherwise it is called **disconnected**.

A connected graph has exactly one connected component. A disconnected graph has two or more connected components.

Complete graphs, K_n, path graphs, P_n, and cycle graphs, C_n, are all connected.

In edgeless graphs, each vertex forms a separate connected component. Thus, edgeless graphs with two or more vertices are disconnected, while 1-vertex graphs are connected.

The $m \times n$ grid graphs are connected.

Is the Facebook graph connected?

The answer is **yes, essentially**. A study from 2011 found that 99.91% of Facebook users are in one gigantic connected component.

The length of the shortest path between two vertices is called **distance** in graph theory. In social networks we define the "degree of separation" as the number of intermediate nodes on the shortest path. In 2011 the average degree of separation among the 721M users of Facebook was about 3.74. The average number of friends (what we call **degree**) was about 190.

Another study found in 2016 that the average degree of separation had shrunk to about 3.57 while the total number of users had grown to 1.59B. The average degree has grown significantly: as of 2018, it is over 330.

arxiv.org/abs/1111.4503 research.fb.com/three-and-a-half-degrees-of-separation

Module 11.5: How many cc's?

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

How many connected components?

Consider a graph $G = (E, V)$ and denote by CC the set of connected components of G. Shorthand: cc for "connected component."

Proposition. $\max(\, |V| - |E|\, ,\, 1\,) \;\leq\; |CC| \;\leq\; |V|$.

Proof. Every cc contains at least one vertex so $|CC|$ attains maximum for the edgeless graph where $|CC| = |V|$.

$|V| - |E|$ is possibly 0 or even negative. That's why we have max with 1. We already know that any graph has at least one cc, so we remove the max.

Fix an $n \geq 1$ and consider for the rest of this proof only graphs with n vertices. We will prove by induction the following:

Claim. For every natural number m, for any graph $G = (E, V)$ with n vertices and m edges, the number of cc's of G is $n - m$ or more.

Proof of claim I

(BC) $m = 0$. Edgeless graph and $|CC| = n \geq n - 0$. Check.

(IS) Let k be an arbitrary natural number. Assume (IH) that any graph with n vertices and k edges has at least $n - k$ connected components.

We want to prove that any graph, G, with n vertices and $k + 1$ edges has at least $n - (k + 1) = n - k - 1$ cc's.

Note that G has at least one edge. Consider the graph G' that has the same nodes as G and is obtained by removing an edge (does not matter which one), say $\{u, v\}$, from G.

Proof of claim II

We removed edge $\{u, v\}$ from G resulting in G' which has n vertices and k edges. By IH, G' has at least $n - k$ cc's.
Now add $\{u, v\}$ back to G' to obtain the graph G. We have two cases:

Case 1: u and v belong to the same cc of G'. Then, adding $\{u, v\}$ to G' is not changing any cc's of G'. Hence, the number of cc's of G is the same as the number of cc's of G' which is at least $n - k > n - k - 1$.

Case 2: u and v belong to different cc's of G'. Then, the two cc's containing u and v become one cc in G. All other cc's in G' remain unchanged. Thus, G has one less cc than G'. Hence, G has at least $n - k - 1$ cc's.

How many edges?

Proposition. In any graph $G = (V, E)$ we have $|E| \ \geq \ |V| \ - \ |CC|$.

Proof. Equivalent to the inequality we proved in the previous proposition: $|CC| \ \geq \ |V| \ - \ |E|$.

Proposition. $|E| \ \leq \ \binom{|V|}{2}$.

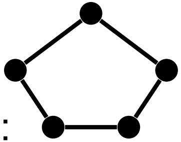
Proof. The maximum number of edges is attained for the complete graph.

Corollary Every connected graph with n vertices has $n - 1$ or more edges.

Proof. In a connected graph $|CC| = 1$ therefore $|E| \geq |V| - |CC| = n - 1$.

Quiz I

Recall that when "removing an edge" we keep the endpoints so we do not remove any vertices. Consider C_5 below as a concrete example to answer the following question.



What is the minimum number of edges that we need to remove to transform the cycle graph C_n $(n \geq 3)$ into a disconnected graph:

(A) It is enough to remove any edge.

(B) It is enough to remove any two edges.

(C) You must remove all n edges

Answer

(A) It is enough to remove any edge.
Incorrect. If we remove an edge from C_n we still have a connected graph, and in particular P_n.

(B) It is enough to remove any two edges.
Correct. If we remove any two edges from C_n we get a disconnected graph composed by two paths. If the edges removed are adjacent we get a path of length 0 (a single vertex) and a path of length $n - 2$.

(C) You must remove all n edges.
Incorrect. If we remove all n edges we do get a disconnected graph, but we can achieve this by removing a smaller number of edges.

Quiz II

WE ARE NOT MAKING THIS QUIZ INTO AN IN-VIDEO ITEM. WE MIGHT PUT SOMETHING SIMILAR IN THE Module 11 quiz prep

Recall that when "removing an edge" we keep the endpoints so we do not remove any vertices. Consider P_4 below as a concrete example to answer the following question.

P_4 : 

What is the minimum number of edges that we need to remove to transform the path graph P_n $(n \geq 1)$ into a disconnected graph:

(A) It is enough to remove any edge.
(B) It is enough to choose a specific edge and remove it.
(C) It is enough to remove any edge unless $n = 1$.
(D) It is enough to choose a specific edge and remove it, unless $n = 1$.

Answer

(A) It is enough to remove any edge.
Incorrect. If $n = 1$ there are no edges to remove and as such it cannot become a disconnected graph.

(B) It is enough to choose a specific edge and remove it.
Incorrect. If $n = 1$ there are no edges to remove and as such it cannot become a disconnected graph.

(C) It is enough to remove any edge unless $n = 1$.
Correct. Removing any edge will yield two disconnected path graphs, unless $n = 1$ in which case there are no edges to remove and as such it cannot become a disconnected graph.

(D) It is enough to choose a specific edge and remove it, unless $n = 1$.
Incorrect. Removing any edge will yield two disconnected path graphs, unless $n = 1$ in which case there are no edges to remove and as such it cannot become a disconnected graph.

Quiz III

A graph has n vertices and n edges. Then, the graph **must** be connected, true or false?

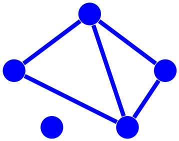
(A) True.

(B) False.

Answer

A graph has n vertices and n edges. Then, the graph **must** be connected, true or false?

(A) True.
Incorrect. Consider the counter-example below:



The graph above has n vertices and n edges but it is disconnected.

(B) False.
Correct.