# Module 12.3: Forests, Trees, Leaves

## MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

# Connected components as graphs

Let $G = (V, E)$ be a graph. We defined **connected components** (cc's) as subsets $C \subseteq V$ and observed that they form a **partition** of $V$.

Now, we regard a cc $C$ as a graph, namely the subgraph of $G$ **induced** by the set of vertices in $C$. The cc's also partitions the set of edges $E$:

**Proposition.** Every edge $\{u, v\} \in E$ belongs to **exactly one** of the subgraphs induced by the cc's of $G$.

**Proof.** Indeed, both $u$ and $v$ must be in the same cc.

**Proposition.** To count the number of paths of length $k$ in $G$ we add up the number of paths of length $k$ in each of its connected components. Similar for cycles.

# Acyclic graphs and trees

A graph in which there are no cycles is called **acyclic**. The cc's of an acyclic graph are also acyclic.

A graph that is both connected and acyclic is called a **tree**.

Consequently, an acyclic graph is also called a **forest** since all its cc's are trees!

**Proposition.** Let $G_1$ and $G_2$ be two **isomorphic** graphs, $G_1 \simeq G_2$. Then:

- $G_1$ is acyclic iff $G_2$ is acyclic.
- $G_1$ is connected iff $G_2$ is connected.
- $G_1$ is a tree iff $G_2$ is a tree.

(A formal proof would be tedious and these facts are intuitively obvious as we think of isomorphism as "copy".)

# How many edges in a tree/forest?

**Proposition.** A tree has one more vertex than edges. That is, if $G = (V, E)$ is a tree then $|E| = |V| - 1$.

**Corollary.** If $G = (V, E)$ is a forest then $|E| = |V| - |CC|$ where $CC$ is the set of cc's of $G$.

**Proof.** Let's say that $G$ has two cc's, $(V_1, E_1)$ and $(V_2, E_2)$. The cc's partition both vertices and edges: $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.

. $|V| = |V_1| + |V_2| \qquad |E| = |E_1| + |E_2|$

The two cc's must be trees so by the proposition:

. $|E_1| = |V_1| - 1 \qquad |E_2| = |V_2| - 1$

Put together the equalities: $|E| = |V| - 2 = |V| - |CC|$ since here we have just two cc's. The general case is similar.

# Vertices and edges in a tree

**Proposition.** Any tree with $n$ vertices has $n - 1$ edges.

**Proof.** By induction on the number $n$ of vertices.

**(BC)** $n = 1$: An edgeless graph with one vertex is a tree. (It is the only edgeless tree!) It has 0 edges. Check.

**(IS)** Let $k$ be an arbitrary natural number $\geq 1$. Assume (IH) that every tree with $k$ vertices has $k - 1$ edges.

Now we want to prove that any tree, $G$, with $k + 1$ vertices has $k$ edges.

To apply the IH we would like to delete a vertex from $G$. But which one? And how do we insure that after the deletion we still have a tree?

# Trees have leaves

In a graph a **leaf** is a node of degree 1.

**Lemma.** Every tree with edges has at least one leaf (actually, at least two!).

**Proof.** Consider the set $L \subseteq \mathbb{N}$ of lengths of paths (not walks!) in $G$. Since paths cannot have length more than the total number of edges, $L$ is finite. The **Well-Ordering Principle** implies that $L$ has a greatest element.

Therefore, there exists a path $W$ of maximum length.

**Claim.** The endpoints $u$ and $v$ of $W$ are leaves.

Indeed, suppose, toward a contradiction, that $u$ is not a leaf. Then it has degree $\geq 2$. Thus $W$ can be extended by one more edge. The other end of that edge is not in $W$ since there are no cycles. Thus $W$ does not have maximum length. Contradiction.

# Remove a leaf

We considered an arbitrary $k \geq 1$. We assumed (IH) that every tree with $k$ vertices has $k - 1$ edges. Then we took an arbitrary tree, $G$, with $k + 1$ vertices and wanted to show it has $k$ edges.

Since $k + 1 \geq 2$, $G$ has edges. By the lemma it has at least one leaf. Let $G'$ be the graph obtained by removing this leaf. Nicely:

**Lemma.** Removing a leaf from a tree with edges leaves again a tree.

**Proof.** The resulting graph is still acyclic. It is also still connected because the only paths affected have the removed leaf as an endpoint.

# Vertices and edges in a tree (finale)

We considered an arbitrary $k \geq 1$. We assumed (IH) that every tree with $k$ vertices has $k-1$ edges. Then we took an arbitrary tree, $G$, with $k+1$ vertices and wanted to show it has $k$ edges. We removed a leaf from $G$ obtaining $G'$ and showed that $G'$ is still a tree.

$G'$ has $k+1-1=k$ nodes, so we can apply the IH.
By IH, $G'$ has $k-1$ edges.

When we removed the leaf from $G$ we removed exactly one edge, because the degree of the leaf is 1.

Hence $G$ has one more edge that $G'$, i.e., it has $k-1+1=k$ edges. Done.

# Quiz

I have a tree and in my tree all nodes have the same degree! Then my tree has:

(A) Two edges.

(B) Strictly more than two edges.

(C) Strictly less than two edges.

# Answer

(A) Two edges.
Incorrect. Any tree with two edges will have a node with degree 1 (i.e. a leaf) and a node with degree equal to 2.

(B) Strictly more than two edges.
Incorrect. Any tree with more than two edges will have a node with degree equal to 1 (i.e. a leaf) and a node of degree at least 2.

(C) Strictly less than two edges.
Correct. The tree consists of two nodes connected by one edge, as shown below:

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