Module 1.3: Divisibility and Primes

MCIT Online - CIT592 - Professor Val Tannen

LECTURE NOTES

Divisibility

The positive integer d is a **divisor** (or **factor**) of an integer n when $n = d \cdot k$ for some integer k.

If d is a divisor of n we say that d **divides** n and we write $d \mid n$.

We also say that n is **divisible** by d or n is a **multiple** of d.

Therefore, "even" is the same as "has 2 as a factor", or as "is divisible by 2".

Examples: 37 is a divisor of 111 because $111 = 37 \cdot 3$.
The positive factors of 24 are 1,2,3,4,6,8,12,24.

We will be interested only in factors/divisors that are *positive*.

Quiz I

What is the only integer divisible by 0?

(A) 1

(B) 0

(C) All integers are divisible by 0.

Answer

What is the only integer divisible by 0?

(A) 1

Incorrect. Since $0 \cdot k \neq 1$ for all integers k, 1 is not divisible by 0.

(B) 0

Correct. Since $0 = 0 \cdot k$ for whatever integer k we want. No other integer is divisible by 0.

(C) All integers are divisible by 0.

Incorrect. For any integer $\ell \neq 0$ we have $0 \cdot k \neq \ell$ for all integers k. Thus, $\ell \neq 0$ is not divisible by 0.

More Information

Note, however, that 0 is divisible by *any* integer! Indeed $0 = d \cdot 0$ for all d.

Quiz II

Suppose that an integer n is divisible by -2. Then

(A) n is even.

(B) n is odd.

(C) n could be either odd or even.

Answer

Suppose that an integer n is divisible by -2. Then

(A) n is even.
Correct. Since n is divisible by -2 we have $n = 2 \cdot (-k)$ for some integer k.

(B) n is odd.
Incorrect. Since n is divisible by -2 it should also be divisible by 2.

(C) n could be either odd or even.
Incorrect. n cannot be odd (see B).

More Information

Since n is divisible by -2 we have $n = (-2) \cdot k$ for some integer k. Then

$$n = (-2) \cdot k = (-1) \cdot 2 \cdot k = 2 \cdot (-1) \cdot k = 2 \cdot (-k)$$

so $n = 2 \cdot \ell$ for some integer ℓ (specifically $\ell = -k$).

Primes

An integer p is **prime** when p has exactly two (positive) factors: 1 and itself, and moreover $p \geq 2$.

Thus, no negative integers, neither 0 nor 1 are primes.

Here are the first few primes (up to 70):

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67 ...

Primes which are 2 apart are called **twin primes**:

$$(3, 5),\ (5, 7),\ (11, 13),\ (17, 19),\ (29, 31),\ (41, 43),\ (59, 61), \ldots$$

Quiz III

How many primes are there between 80 and 100? (You can look this up online.)

(A) 3

(B) 4

(C) 5

Answer

How many primes are there between 80 and 100? (You can look this up online.)

(A) 3

Correct. The only primes between 80 and 100 are 83, 89, and 97.

(B) 4

Incorrect.

(C) 5

Incorrect.

More Information

Only three prime numbers between 80 and 100! Isn't that interesting?

Of course, looking online for a list of all the prime numbers up to at least 100 was not the pedagogical purpose of this quiz. We wanted you to become curious about how to find these primes!

We have an optional segment about the Sieve of Eratosthenes, which is an algorithm for finding all prime numbers up to any given limit. The optional segment features animations that explain this ancient algorithm.

Fermat's "primes"

Consider the sequence of numbers:

$$f_n \;=\; 2^{2^n}+1 \quad \text{for} \quad n=0,1,2,\ldots$$

$f_0=3, f_1=5, f_2=17$ are primes. Check that that $f_3=257$ is also a prime.

Fermat checked that $f_4=65537$ is a prime too!

He seems to have conjectured that *all the f_n's are prime*. However...

Euler showed that $f_5=4294967297=641 \cdot 6700417$, hence *not a prime*.

So much for "proof by example"!

To date, no f_n with $n>4$ was shown to be prime and many larger and larger f_n's were shown *not* prime!