MCIT 592 Spring 2022 **Homework 13** **Solutions**

Problem Set

1. [10 pts] There exists a group of users on LinkedIn with $n \geq 3$ people. This group has a clique of size $n-2$, but does *not* have a clique of size $n-1$. Prove that this group has two distinct independent sets of size 2.

Solution:

Since there is a clique of size $n-2$, we know that $n-2$ vertices must be fully connected. Consider the one of the remaining two vertices. Call this vertex a. We know that there must be at least one vertex in the connected clique of $n-2$ vertices to which this vertex is not connected. Otherwise, there would be a clique of size $n-1$ (the clique of $n-2$ plus a). Call this other vertex b. Thus, an independent set of size 2 can be formed from a and b.

The same logic can be applied to the other one of the remaining two vertices (call it d) to form a second independent set of size 2. These two sets are distinct because one includes a while the other includes d.

2. [15 pts] Show that there is a graph G with exactly 5 nodes where both G and its complement, $\overline{G}$, have chromatic number ≥ 3.

Solution:

Take G to be the cycle graph $(\{1,2,3,4,5\}, \{1–2, 2–3, 3–4, 4–5, 5–1\}$. Its complement is the graph

$$\overline{G} = (\{1,2,3,4,5\}, \{1–3, 3–5, 5–2, 2–4, 4–1\})\,.$$

which is also a cycle graph with 5 nodes. We have proved in lecture that cycle graphs of odd length have chromatic number 3.

3. **[10 pts]** Suppose that G is a connected graph. It contains *exactly one* spanning tree. Prove that G itself is a tree.

Solution:

We will prove the contrapositive, namely that if a connected graph G is not a tree then it has at least two distinct spanning trees.

Assume that $G = (V, E)$ is connected but not a tree. To remove worries about corner cases, note that G has 2 or more edges and 3 or more nodes (connected graphs with fewer edges and nodes are trees).

We know from lecture that G has at least one spanning tree, call it T. Since G is not a tree there must exist one edge in G that is not in T. Call it $e = u$-v. In T there is a path P from u to v. This path has at least 2 edges (no parallel edges). Let e' be one of the edges of P.

Consider the subgraph T' obtained from T by deleting e' and adding e. Clearly, T' is distinct from T and it has all the vertices of G so it is a spanning subgraph.

We now argue that T' is connected. Since T is connected, there exists a path in T between every pair of vertices. Some of these paths may go through e' and therefore will not exist in T'. However, these paths can be rerouted to use e by going along the remaining pieces of P. Thus, there still exist paths in T' connecting all the vertices.

Next we argue that T' is a tree. Indeed, the connected graph T' has as many nodes and as many edges as T who is a tree. By HW12 Q6, T' is a tree also.

We have shown that G has two distinct spanning trees.

4. **[10 pts]** Let G be a connected graph with at least one cycle. Prove the following statement:

We can remove some edges from G such that the resulting subgraph is *bipartite and connected.*

Solution:

As proven in lecture, every connected graph has a spanning tree. Consider a spanning tree and remove all edges of G that are not in said spanning tree. (Essentially, run the edge-pruning algorithm.)

By definition of a tree, this spanning tree is still connected. As also proven in lecture, every tree is bipartite so the resulting spanning tree is also bipartite. Thus, we have proven that it is always possible to remove edges from G (through the edge-pruning algorithm) to result in a graph that is connected and bipartite - a spanning tree.

5. [**10 pts**] Suppose G be a connected graph with $n \geq 3$ vertices such that $\chi(G) = 3$. Consider a proper 3-coloring of G with colors, purple, yellow, and orange. Prove that there exists a orange node that has both a purple neighbor and a yellow neighbor.

Solution:

Since G is connected and has strictly more than one node, there are no isolated nodes.

In the proper coloring of G considered there must exist some orange nodes; otherwise it would be a 2-coloring which would contradict the fact that the chromatic number, $\chi(G)$, is 3.

Moreover, among the orange nodes there must exist some that have 2 or more neighbors. Indeed, if each orange node has exactly one neighbor (that is, all orange nodes are leaves) then we can change its orange color to purple or yellow as long as it is different from the color of its neighbor. This would result in a proper 2-coloring which again would contradict that fact that $\chi(G) = 3$.

Now, suppose, toward a contradiction, that all the orange nodes that have

2 or more neighbors have either only yellow neighbors or only purple neighbors. Then, we can again change their orange color to purple or yellow as long as it is different from the color of their neighbors. This would again result in a proper 2-coloring which again would contradict that fact that $\chi(G) = 3$.

We conclude that there exist orange nodes with 2 or more neighbors such that one (or more) of their neighbors is yellow and another (or more) is purple.