Module 4.5: Surjections and Injections

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Lecture Notes

Surjective functions I

A function $f : A \to B$ is called **surjective** if $\mathrm{Ran}(f) = B$, or equivalently:

$$\text{for every } y \in B \text{ there exists } x \in A \text{ such that } y = f(x).$$

A surjective function is also called a **surjection**.

Examples.

$g : [-20..10] \to [0..20]$
where $g(y) = \mathrm{abs}(y)$. $\qquad \mathrm{Ran}(g) = [0..20]$

$h : [0..n] \to [0..n]$
where $h(z) = n - z$. $\qquad \mathrm{Ran}(h) = [0..n]$

Surjective functions II

$$t : [0..2n] \to [0..n] \quad \text{where } t(w) = \begin{cases} \frac{w}{2} & \text{if } w \text{ is even} \\ \frac{w-1}{2} & \text{if } w \text{ is odd} \end{cases}$$

Problem. Prove that t is surjective.

Answer. By the definition, we need to show that:

For every $y \in [0..n]$ there exists $x \in [0..2n]$ such that $y = t(x)$.

Indeed, let y be an arbitrary element of $[0..n]$.

Take $x = 2y$. x is even, therefore

$$t(x) = \frac{x}{2} = \frac{2y}{2} = y$$

Injective functions I

A function $f : A \to B$ is called **injective** if it maps distinct elements to distinct elements, that is,

$$\text{for every } \; x_1 \neq x_2 \; \text{ in the domain we have } \; f(x_1) \neq f(x_2),$$

or, equivalently, (by contrapositive)

$$\forall\, x_1, x_2 \in A \quad f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2$$

An injective function is also called an **injection**.

Examples.

$h : \mathbb{N} \to \mathbb{N}$ where $h(n) = 2n$.

It's injective because if $n_1 \neq n_2$ then $2n_1 \neq 2n_2$.

$f : [0..10] \to [0..20]$ where $f(x) = x + 10$.

It's injective because if $x_1 \neq x_2$ then $x_1 + 10 \neq x_2 + 10$.

Injective functions II

$g : [0, \infty) \to \mathbb{R}$ where $g(x) = 2\sqrt{x} - 3.$

Problem. Prove that g is injective.

Answer. By definition, we need to show that:

$$\forall\, x_1, x_2 \in [0, \infty) \quad 2\sqrt{x_1} - 3 = 2\sqrt{x_2} - 3 \;\Rightarrow\; x_1 = x_2$$

Assume $2\sqrt{x_1} - 3 = 2\sqrt{x_2} - 3.$ Then

$$\begin{aligned} 2\sqrt{x_1} &= 2\sqrt{x_2} && \text{(Adding 3)} \\ \sqrt{x_1} &= \sqrt{x_2} && \text{(Dividing by 2)} \\ x_1 &= x_2 && \text{(Squaring)} \end{aligned}$$

Done.

Quiz I

For $k \in \mathbb{Z}^+$ let $P = \{1, 2, \ldots, k\} \times \{1, 2, \ldots, k\}$,
define $g : P \to \{1, 2, \ldots, k\}$ by $g(x, y) = x$.

Is this function

A. Injective?

B. Surjective?

C. Both?

D. Neither?

Answer

For $k \in \mathbb{Z}^+$ $P = \{1, 2, \ldots, k\} \times \{1, 2, \ldots, k\}$, define $g : P \to \{1, 2, \ldots, k\}$ by $g(x, y) = x$. Is this function

A. Injective?

Incorrect. The function is not injective, because $g(1, 1) = g(1, 2) = 1$.

B. Surjective?

Correct. Every element in the codomain is mapped onto by some element in the domain: $x = g(x, 1)$.

C. Both?

Incorrect. The function is not injective, see the first answer.

D. Neither?

Incorrect. The function is surjective, see the second answer.

Quiz II

$f : \mathbb{R} \to \mathbb{R}$ defined as:

$$f(x) = \begin{cases} x^3 & x \geq 1 \\ x^3 + 1 & x < 1 \end{cases}$$

Is this function

A. Injective?

B. Surjective?

C. Both?

D. Neither?

ANSWER:

Define a function $f : \mathbb{R} \to \mathbb{R}$ by:

$$f(x) = \begin{cases} x^3 & x \geq 1 \\ x^3 + 1 & x < 1 \end{cases}$$

Is this function

A. Injective?

   Incorrect. $f(x)$ is not injective because $f(x) = 1$ for both $x = 0$ and $x = 1$.

B. Surjective?

   Correct. The function is surjective.

C. Both?

   Incorrect. $f(x)$ is not injective because $f(x) = 1$ for both $x = 0$ and $x = 1$.

D. Neither?

   Incorrect. The function is surjective.

Injections, surjection, and counting

Let A and B be two sets.

The **injection rule**: if we can define an injective function with domain A and codomain B then $|A| \leq |B|$.

The **surjection rule**: if we can define a surjective function with domain A and codomain B then $|A| \geq |B|$.

The **surjection rule (variant)**: if we can define a function $f : A \to B$ then $|A| \geq |\mathrm{Ran}(f)|$.

(If $f : A \to B$ then $f' : A \to \mathrm{Ran}(f)$ where $f'(x) = f(x)$ is surjective.)

If we can define a function with domain A and codomain B that is **both** a surjection and an injection then $|A| = |B|$.

In the next segment we describe this as the "bijection rule" and we discuss more about it.

ACTIVITY : Surjection in Counting

In this activity, we will use the surjection rule to explain why there are at least as many permutations of r out of n as there are combinations of r out of n.

Let A be a set with n elements, $|A| = n$. Let P_r be the set of partial permutations of length r made out of elements of A. Let C_r be the set of combinations of size r made out of elements of A (subsets of A of size r).

Now, we try defining a function which maps elements from P_r to C_r.

ACTIVITY : Surjection in Counting

We define a function $f : P_r \to C_r$ that associates to each permutation the set of elements occurring in the permutation. A permutation $\sigma \in P_r$ has no repeated elements, so the set of elements that occur in σ is of size r.

Now we prove that f is a surjection. For every $S \in C_r$, order the elements arbitrarily. This produces a permutation of length r, call it σ and $f(\sigma) = S$.

By the surjection rule:

$$|P_r| \ \geq \ |C_r|.$$

Of course, this can be also checked algebraically since

$$\frac{n!}{(n-r)!} \ \geq \ \binom{n}{r}$$