Module 6.4: Pizza Cutting Recurrence

MCIT Online - CIT592 - Professor Val Tannen

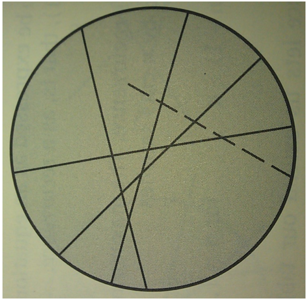
LECTURE NOTES

The pizza-cutting problem I

Problem. What is the largest number of pieces (not slices!) of pizza that can be made with n distinct straight cuts?

Answer. Since we only want the maximum **number** of pieces, it does not matter where the edge of the pizza is. (In fact this problem is known as Steiner's **Plane** Cutting Problem.)

The following picture shows some cuts that maximize the number of pieces.



The pizza-cutting problem II

Answer (continued). Some experimentation and some reasoning (see the optional segment entitled "Pizza cutting") leads us to the following **maximizing conditions**:

(1) Every cut must cross every other cut.

(2) No three cuts cross each other at the same point.

Let $C(n)$ be the number of pieces produced by a set of n cuts satisfying (1) and (2). Clearly $C(0) = 1$ (the whole pizza!).

When we add the nth cut we add a new piece for intersecting each of the existing $n-1$ cuts, plus one more for intersecting the edge! Therefore:

$$C(0) = 1 \qquad\qquad C(n) = C(n-1) + n$$

This is a **recurrence relation**. Recurrences are extremely useful in the analysis of the running time of algorithms.

Solving the recurrence relation

Answer (continued). Solving the recurrence relation that we obtained

$$C(0) = 1 \qquad\qquad C(n) = C(n-1) + n$$

Method 1. Guess the answer and prove by induction that your guess was correct. Here the answer is $(n^2 + n + 2)/2$.

Method 2. Analyze the "recursion tree" constructed from the recurrence.

Method 3. Use a "telescopic" trick that repeats the recurrence and simplifies terms.

We illustrate methods 2 and 3 in what follows.

The recursion tree method

Answer (continued). We will separate the addition terms in

$$C(0) \;=\; 1 \qquad\qquad C(n) \;=\; C(n-1) \;+\; n$$

We draw a "tree of additions":

$$\begin{array}{ccccccccccc}
n & & n-1 & & \cdots & & 2 & & 1 & & 1 \\
| & & | & & & & | & & | & & | \\
C(n) & — & C(n-1) & —— & \cdots & —— & C(2) & —— & C(1) & —— & C(0)
\end{array}$$

Therefore $C(n) \;=\; (n \;+\; (n-1) \;+\; \cdots \;+\; 2 \;+\; 1) \;+\; 1$

Using the formula $\quad C(n) \;=\; n(n+1)/2 \;+\; 1 \;=\; (n^2+n+2)/2$

The "telescopic" method

Answer (continued). We write the recurrence relation for $n, \ldots, 1$:

$$\begin{aligned}
C(n) &= C(n-1) + n \\
C(n-1) &= C(n-2) + n - 1 \\
C(n-2) &= C(n-3) + n - 2 \\
&\cdots \\
C(2) &= C(1) + 2 \\
C(1) &= C(0) + 1
\end{aligned}$$

Add all the LHSs and RHSs and cancel terms that appear on both sides:

$$C(n) = C(0) + 1 + 2 + \cdots + n = 1 + n(n+1)/2 = (n^2 + n + 2)/2$$

(The method is called "telescopic" because the n equalities "collapse" into just one.)