Self-paced Example: Counting Two Gender Committees

Module 4

MCIT Online - CIT592 - Professor Val Tannen

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion. When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Counting committees of r out of n and m

You have learned earlier about a technique called **combinatorial proof**. Specifically, in the lecture segment "Combinatorial proofs" you saw an example of this technique in a proof of the following identity:

$$\binom{n}{r} = \binom{n}{n-r}$$

and another one in a proof of Pascal's Identity.

Take a moment to recall this technique: To prove an identity we pose a counting question. We then answer the question in two ways: one answer corresponds to LHS and the other corresponds to the RHS of the identity.

We will use this same technique to solve the following problem.

Problem. Give a combinatorial proof to show that

$$\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k} = \binom{n+m}{r}$$

(CONTINUED)

Counting committees of r out of n and m (continued)

Answer. We answer this problem in three steps.

Step 1: We pose the following counting question.

> Consider a group of n women and m men. How many ways are there to form a committee of r people from this group?

Step 2 (RHS): the total number of people is $m + n$. A committee of r people is a subset of size r of the set of people, therefore there $\binom{n+m}{r}$ distinct such committees. This gives us the RHS.

Step 3 (LHS): The set S of all possible committees of r people can be partitioned into subsets $S_0, S_1, S_2, ..., S_r$, where S_k is the set of committees in which there are exactly k women and the rest, $r - k$ people, are men.

However, what is $|S_k|$?

We can construct a committee in S_k in two steps. Think about how you would do it.

Counting committees of r out of n and m (continued)

Indeed in the first of the two steps we choose k out of the n women. This can be done in $\binom{n}{k}$ ways.

In the second of the two steps we choose $r-k$ out of the m men. This can be done in $\binom{m}{r-k}$ ways.

By the multiplication rule

$$|S_k| = \binom{n}{k}\binom{m}{r-k}$$

Since the sets $S_0, S_1, S_2, ..., S_r$ are pairwise disjoint we can apply the addition rule:

$$|S| \;=\; |S_0| + |S_1| + \cdots + |S_r| \;=\; \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$$

which gives us the LHS.

The technique used to solve this problem, combinatorial proofs, is very useful in proving many identities. You can revisit some or all of the identities we proved using combinatorial proofs in the lecture segment "Combinatorial proofs."