CIT 592 Spring 2022 Homework 12

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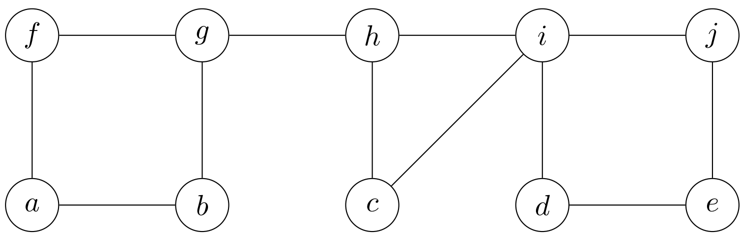
1. [10 pts]



In the graph above, how many connected components are in the subgraph induced by each of the following subsets of the vertices?

(a) $\{a, b, c, d, e\}$

(b) $\{f, g, h, i, j\}$

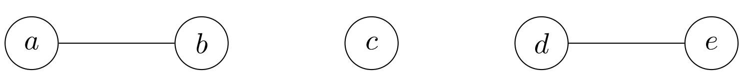
(c) $\{a, b, e, h, i\}$

(d) $\{c, f, g, i, j\}$

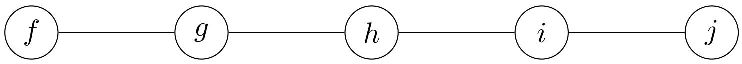
(e) $\{a, c, d, g, j\}$

Solution.

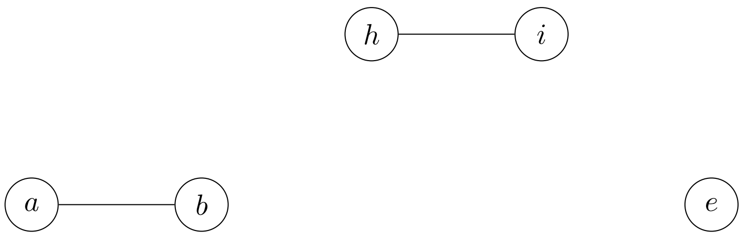
(a) The subgraph induced by subsets$\{a, b, c, d, e\}$ is like below. Therefore by the definition of connected component we can tell there are 3 connected components.



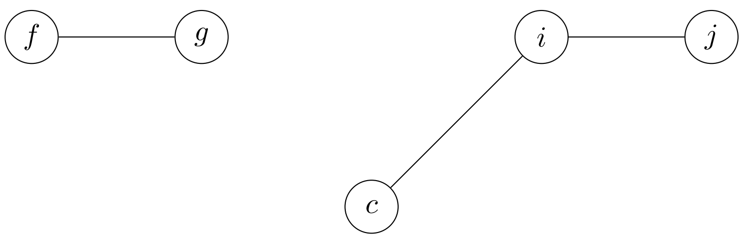
(b) The subgraph induced by subsets$\{f, g, h, i, j\}$ is like below. Therefore by the definition of connected component we can tell there are 1 connected components.



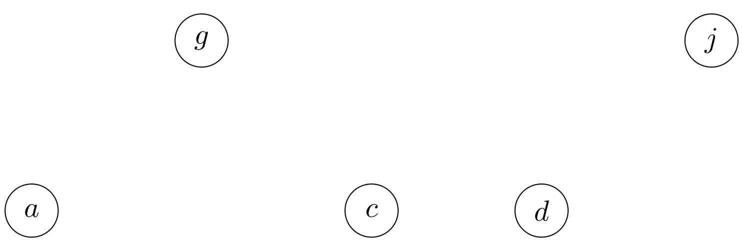
(c) The subgraph induced by subsets$\{a, b, e, h, i\}$ is like below. Therefore by the definition of connected component we can tell there are 3 connected components.



(d) The subgraph induced by subsets$\{c, f, g, i, j\}$ is like below. Therefore by the definition of connected component we can tell there are 2 connected components.



(e) The subgraph induced by subsets$\{a, c, d, g, j\}$ is like below. Therefore by the definition of connected component we can tell there are 5 connected components.



2. [**10 pts**]

Suppose Kruskal's Kingdom consists of $n \geq 3$ farmhouses, which are connected in a cyclical manner. That is, there is a road between farmhouse 1 and 2, between farmhouse 2 and 3, and so on until we connect farmhouse n back to farmhouse 1. In the center of these is the king's castle, which has a road to every single farmhouse. Besides these, there are no other roads in the kingdom.

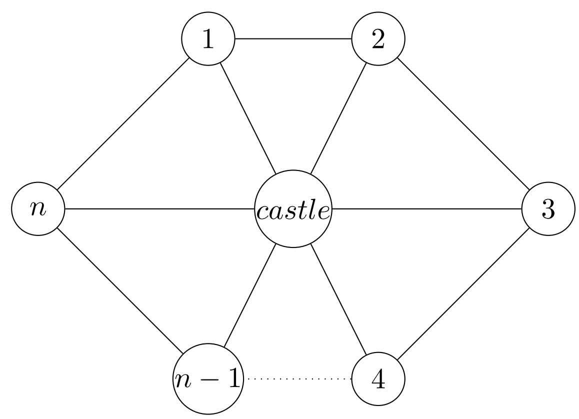
(a) Find the number of paths of length 2 in the kingdom in terms of n. Justify your answer.

(b) Find the number of cycles of length 3 in the kingdom in terms of n. Justify your answer.

(c) Find the number of cycles in the kingdom in terms of n. Justify your answer.

Solution.

(a) The graph of the Kingdom is shown as below based on the given condition:



The number of paths of lengths 2 is the subgraphs that are path graphs on 3 vertices.
Situation 1: Both edges are edges between farmhouses.
In order to have a path of length 2 and both edges are edges between farmhouses, we need 3 farmhouses that are adjacent such as farmhouse 1,2,3, or 2,3,4 etc, where every farmhouse can be the starting point of the path. Therefore since there are n farmhouses, there are n paths of 2.

Situation 2: One edge is between 2 farmhouses and the other edge is between farmhouse and the castle.
For every farmhouse, there are 2 roads that connect other farmhouses. Once we chose one of the 2 roads and go to the farmhouse the road connects to, there are only one way to finish the path, which is to go to the castle. Such as, from 2 we can choose to go to 1 or

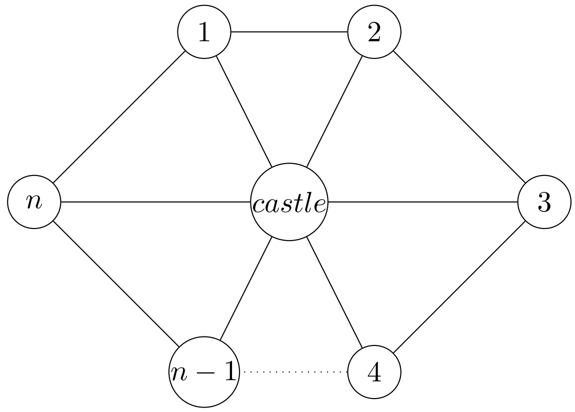
3. Once we go to 1, we can only go to the castle, the path will be 2-1-castle. Once we go to 3, we also can only go to the castle, the path will be 2-3-castle.
So for every farmhouse there are 2 ways, so for n farmhouses there are $2n$ ways.

Situation 3: Both edges are between farmhouse and the castle.
Since all farmhouses have a road that connects to the castole, we just need to choose 2 farmhouses from the n farmhouses. So there are $\binom{n}{2}$ ways.

Combine all the above situations we have $2n + n + \binom{n}{2} = 3n + n(n-1)/2$ ways.

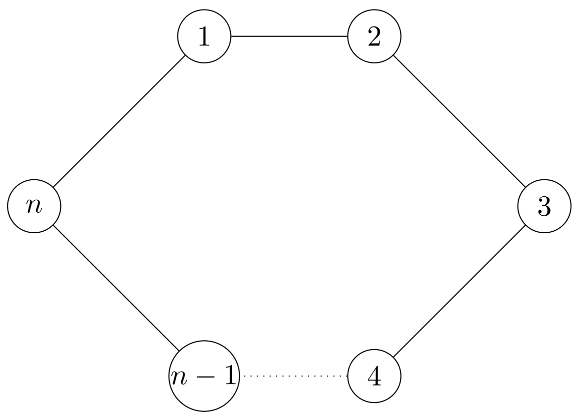
(b) Counting cycles of length 3 in a graph means counting the subgraphs of the graph that are cycle graphs on 3 vertices.
We know that any farmhouse has 3 degrees, which means each farmhouse is connected to the two adjacent farmhouses and the castle in the middle. So any C_3 will have 2 farmhouses and castle as its vertices. Since any farmhouse is only connected with the 2 farmhouses that are adjacent to it, the 2 farmhouses in any C_3 would need to be adjacent. So essentially we are counting how many distinct pairs of adjacent farmhouses.
Starting from every farmhouse we can find two pairs of adjacent houses, so for n farmhouses we will have total $2n$. However since the cycle "castle - 1 - 2 - castle" and the cycle "castle - 2 - 1 - castle" have the same nods and edges, they are essentially the same cycle, so we double counted every cycle.
Therefore the number of C_3 is $2n/2 = n$.

(c) Counting cycles of length n in a graph means counting the subgraphs of the graph that are cycle graphs on n vertices.
Since the graph of the Kingdom is shown as below based on the given condition:



Situation 1: The cycle doesn't have the castle as one of the vertices.

There is only one cycle that fits in this situation as below.



Situation 2: We count the number of C_3.

When $n = 3$, the vertices can be 2 farmhouses and the castle, or 3 farmhouses. We have discussed the situation where all vertices are farmhouse under situation 1. So we only discuss when one of the vertices is the castle.

We know that any farmhouse has 3 degrees, which means each farmhouse is connected to the two adjacent farmhouses and the castle in the middle. So any C_3 will have 2 farmhouses and castle as its vertices. Since any farmhouse is only connected with the 2 farmhouses that are adjacent to it, the 2 farmhouses in any C_3 would need to be adjacent. So essentially we are counting how many distinct pairs of adjacent farmhouses.

Starting from every farmhouse we can find two pairs of adjacent houses, so for n farmhouses we will have total $2n$. However since the cycle "castle - 1 - 2 - castle" and the cycle "castle - 2 - 1 - castle" have the same nods and edges, they are essentially the same cycle, so we double counted every cycle.

Therefore the number of C_3 is $2n/2 = n$.

Situation 3: We count the number of C_4.

Any C_4 will have 3 farmhouses and castle as its vertices.

Since any farmhouse is only connected with the 2 farmhouses that are adjacent to it, the 3 farmhouses in any C_4 would need to be adjacent. So essentially we are counting how many distinct set of 3 adjacent farmhouses.

Starting from every farmhouse we can find two pairs of 3 adjacent houses, so for n farmhouses we will have total $2n$. However since the cycle "castle - 1 - 2 - 3 - castle" and the cycle "castle - 3 - 2 - 1 - castle" have the same nods and edges, they are essentially the same cycle, so we double counted every cycle.

Therefore the number of C_4 is $2n/2 = n$.

Similarly we will have n cycles for C_5, C_6, $\dots$, $C_n + 1$.

All above mentioned situation combined we will have $1+n+n+...+n = 1+n(n+1-2) = 1+n(n-1)$ cycles.

3. **[10 pts]** Suppose we have a neighborhood of n houses. For any two houses we pick, there is a road between them.

(a) The landlord wants to cut maintenance costs by removing some of the roads. Let k be the minimum number of roads he can remove such that the neighborhood is still connected (every house can be walked to from every other house) and there are no cycles. Determine the value of k as an expression in terms of n. Then indicate how to remove the minimum number of roads from the neighborhood such that the requirements are satisfied.

(b) Suppose now instead that the landlord wants to remove *houses*. Let ℓ be the minimum number of houses that must be removed such that the neighborhood is still connected and has no cycles. Removing a house also removes the roads it is connected to. Determine the value of ℓ as an expression in terms of n. Then indicate how to remove the minimum number of houses from the neighborhood such that the requirements are satisfied.

Solution.

(a) By the description of the neighborhood, any two houses have a road between them, we know that the neighborhood is a complete graph $G = (V, E)$ where the houses are vertices and edges are roads. So we know $|V| = n$.
Every vertex has $(n-1)$ degrees so G with n vertices has $n(n-1)$ degrees in total. Since $2|E| =$ sum of degrees, we know that $|E| = n(n-1)/2$, which means there are $n(n-1)/2$ road currently in the neighborhood.
After remove k roads, there will be $n(n-1)/2 - k$ roads left, so the new graph $G^{'}$ $=(V, E^{'})$ where $|V| = n$ and $|E^{'}| = n(n-1)/2 - k$.
Since the new graph $G^{'}$ is still connected and there are no cycles, by definition it is a tree, so we know $|E^{'}| = |V| - 1$, which is $n(n-1)/2 - k = n - 1$.
So we got $k = (n-1)(n-2)/2$.
How to remove:
first, the landlord should remove all the roads except the those connect each house to its two adjacent houses. This will leave a cycle made of houses.
second, the landlord should select two adjacent houses and remove the road between them.

(b) By the description of the neighborhood, any two houses have a road between them, we know that the neighborhood is a complete graph $G = (V, E)$ where the houses are vertices and edges are roads. So we know $|V| = n$.
When one house is removed and all the roads it is connected to are also removed, the neighborhood is left with $n-1$ houses and any two houses is still connected since any road

that is not connected to the house that got removed is not impacted. Therefore as long as there are more than 3 houses left, the new neighborhood after removing one house is still a complete graph with a cycle.

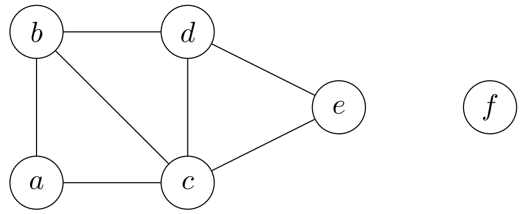
Similarly, the new neighborhood after removing 2 houses, 3 houses, $\dots$ are still a complete graph with a cycle, until there are only 2 houses left, since a cycle would need at least 3 houses.

Therefore we know the landlord at least need to remove $n - 2$ houses, so $l = n - 2$.

How to remove:

The landlord should pick two adjacent houses and keep the road between them, then all the other houses and roads.

4. [10 pts] We define a graph's *degree sequence* as a list of the degrees of all the vertices in the graph, **in increasing order of degree**. For example, the graph



has degree sequence $(0, 2, 2, 3, 3, 4)$ because there is one node with degree 0 (f), two nodes with degree 2 (a and e), two nodes with degree 3 (b and d), and one node with degree 4 (c).

For each of the following, either list the set of edges of a **tree** with vertex set $\{a, b, c, d, e, f\}$ that has the stated degree sequence, or show that no such tree exists.

(a) $(1, 1, 1, 3, 3, 3)$

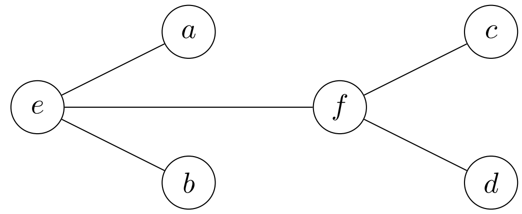
(b) $(1, 1, 1, 1, 3, 3)$

(c) $(1, 1, 1, 1, 3, 4)$

Solution.

(a) Since there are 6 nods, we know that $|V| = 6$.
Since it is a tree, $|E| = |V| - 1 = 5$.
We know that sum of degrees of a graph $= 2|E| = 2 * 5 = 10$
Since sum of degrees of vertex set $(1, 1, 1, 3, 3, 3) = 1 + 1 + 1 + 3 + 3 + 3 = 12 \neq 10$, no such tree exists.

(b) Since there are 6 nods, we know that $|V| = 6$.
Since it is a tree, $|E| = |V| - 1 = 5$.
We know that sum of degrees of a graph $= 2|E| = 2 * 5 = 10$
Since sum of degrees of vertex set $(1, 1, 1, 1, 3, 3) = 1 + 1 + 1 + 1 + 3 + 3 = 10$, such tree exists.
By definition vertex set $(1, 1, 1, 1, 3, 3)$ means there are 4 vertices have 1 degree and 2 vertices have 2 degrees and the graph has 5 edges as mentioned above.
Therefore the set of edges are:$\{a - e, b - e, f - e, c - f, d - f\}$ as shown below.



(c) Since there are 6 nods, we know that $|V| = 6$.

Since it is a tree, $|E| = |V| - 1 = 5$.

We know that sum of degrees of a graph $= 2|E| = 2 * 5 = 10$

Since sum of degrees of vertex set $(1, 1, 1, 1, 3, 4) = 1 + 1 + 1 + 1 + 3 + 4 = 11 \neq 10$, no such tree exists.

5. [**10 pts**] Suppose there exists a connected group of Facebook friends. That is, there is a way to reach every person from every other person through some traversal of friendships. Additionally, there is no cycle of friendships (ex. we would *not* have the following case: 1 is friends with 2, who is friends with 3, who is friends with 1). Suppose that some person u in this group has at least d friends. Prove that there exists at least d people in this group with exactly 1 friend. *Hint:* Think about what specific type of graph this is based on the definition!

Solution.

By the description of the friend group, we can tell the group is a tree since it's connected and acyclic. The statement we are trying to prove, "there exists at least d people in this group with exactly 1 friend" means the tress has at least d leaves.

Since some person u in this group has at least d friends, we know that there are at least d vertices connected with u directly, let's call those vertices v_1, v_2, ..., v_d, v_{d+1}, ... Let's assume there are d vertices connected to u for now.

Suppose, toward a contradiction that there are less than d leaves, let's say, there are $d-1$ leaves. By PHP, the d vertices (Pigeons) need to be connected to $d-1$ leaves (Pigeon holes). Since $d-1 < d$, we know that there are at least 2 out of the d vertices will be connected to the same leaf, w.

Let's call the 2 vertices that are connected to the leaf w v_1 and v_2. Since v_1 and v_2 connected to the same leaf w and u, we can find a walk $u - v_1 - \cdots - w - \cdots - v_2 - u$, which by definition is a cycle.

However by the given description of this group, the group is a tree and it has no cycle. Contradiction.

Proof of there exists at least d people in this group with exactly 1 friend is completed.