Module 12.1: Subgraphs and Counting Paths

MCIT Online - CIT592 - Professor Val Tannen

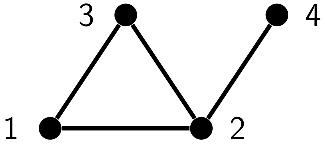
Lecture Notes

Graph isomorphism

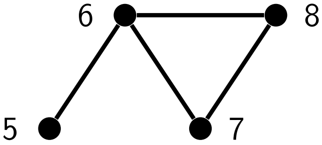
Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic**, notation $G_1 \simeq G_2$, when there is a **bijection** $\beta : V_1 \to V_2$ such that for any $u_1, v_1 \in V_1$ we have $u_1 – v_1 \in E_1$ **iff** $\beta(u_1) – \beta(v_1) \in E_2$.

Example:

G_1 :



G_2 :



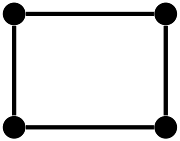
$G_1 \simeq G_2$ by the bijection $4 \mapsto 5,\ 2 \mapsto 6,\ 1 \mapsto 7,\ 3 \mapsto 8.$

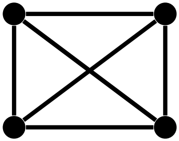
$1 \mapsto 8,\ 3 \mapsto 7$ also works! Note that the bijection must preserve node degree.

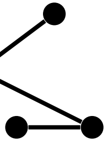
More graph isomorphism examples

Proposition. Any two complete graphs are isomorphic iff they have the same number of vertices. The same holds for path, cycle, and edgeless graphs. Any two $m \times n$ grids are isomorphic, as well as isomorphic to any $n \times m$ grids!

(Proving isomorphism formally is tedious. Visual intuition is much better! Below is just a reminder of what some of these graphs look like.)

C_4 : 

K_4 : 

P_4 : 

ACTIVITY : Special graphs revisited

We can now give a precise definition: a **path graph on n vertices** is a graph isomorphic to P_n. Hence, a **path graph of length ℓ** is a graph isomorphic to $P_{\ell+1}$.

But who is P_n? Recall that we did not name its vertices when we drew it. We didn't because it does not matter, as you see! You can choose any names you want, for instance: $P_3 = \{\{1,2,3\},\{1\text{–}2, 2\text{–}3\}\}$. Regardless of the vertex names chosen, the class of path graphs is the same. In many problems the names of the vertices do not matter, and can talk about **the** graph P_n, as we already did.

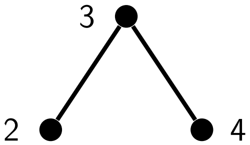
The same definition and terminology will be used for, and the same discussion applies to **cycle**, **complete**, and **edgeless graphs on n vertices**, and **grid graphs on $m \times n$ vertices**..

Subgraphs

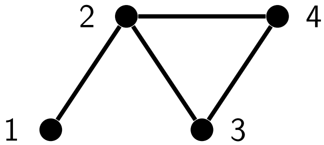
A graph $G_1 = (V_1, E_1)$ is a **subgraph** of the graph $G_2 = (V_2, E_2)$ when $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. (Beware: not all pairs of such subsets form graphs!)

Example:

G_1 :



G_2 :



Here, G_1 is a subgraph of G_2. Another subgraph of G_2 has vertices $2, 3, 4$ and edges 2–3, 3–4, 4–2.

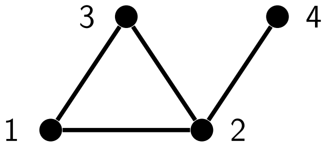
Yet another subgraph of G_2 has vertices $1, 2, 3, 4$ and edges 2–1, 2–3, 2–4.

Induced subgraphs

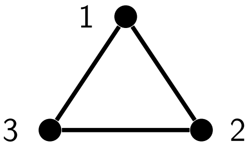
If $G = (V, E)$ is a graph and $V' \subseteq V$ is a set consisting of some of G's nodes, the subgraph of G **induced** by V' is the graph $G' = (V', E')$ where E' consists of all the edges of G whose endpoints are both in V'.

Example:

G :



G' :



G' is the subgraph of G **induced** by the subset of vertices $\{1, 2, 3\}$.

The subgraph of G induced by $\{2, 3, 4\}$ is path graph of length 2.

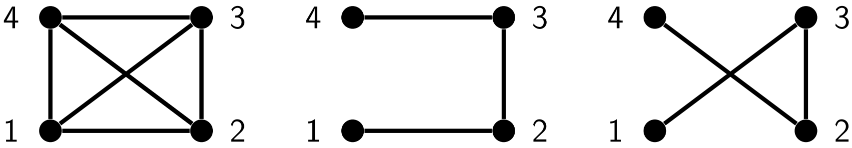
The subgraph of G induced by $\{1, 3, 4\}$ is disconnected.

Counting paths

When we count **paths of length** n in a graph G, we count in fact the subgraphs of G that **are** path graphs on $n+1$ vertices.

Problem. Consider the complete graph on nodes $\{1, 2, 3, 4\}$. Find two **different** path subgraphs that have the **same** set of nodes.

Answer.

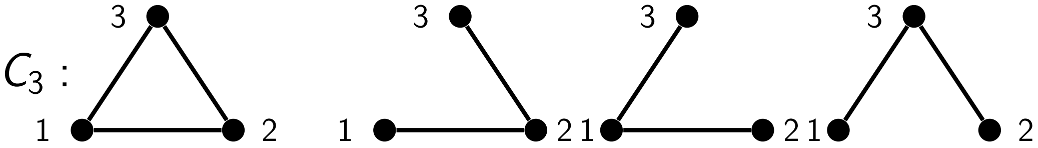


The paths 1–2–3–4 and 1–3–2–4 have the same set of nodes but different sets of edges. They correspond to the two distinct path subgraphs shown.

Counting paths in cycles I

Problem. How many paths of length 2 are there in C_3?

Answer.



Any path of length 2 has two edges. In C_2 any two edges form a path of length 2 since they must have an endpoint in common. There are $\binom{3}{2} = 3$ combinations of two edges therefore 3 path subgraphs (see figure).

Counting paths in cycles II

Problem. How many paths of length 2 are there in C_n $(n \geq 3)$?

Answer. For $n \geq 4$ there are combinations of two edges that do not form a path subgraph. We must count in a different way.

Observe that we can define (in any graph) a function from path subgraphs of length 2 to vertices by associating to each such path subgraph its middle vertex. In general, this function is neither injective nor surjective but in C_n it is both, hence it's a bijection.

By the bijection rule there are as many path subgraphs of length 2 as there are vertices. This gives us the answer: n. This answer is valid for $n = 3$ and it's the same as on the previous slide.

Quiz

How many paths (of any length) are there in P_4?

(A) 6

(B) 15

(C) 10

Answer

(A) 6

Incorrect. Have you counted the paths of length 0?

(B) 15

Incorrect. Recall that P_4 has length 3.

(C) 10

Correct. There are 4 paths of length 0, 3 paths of length 1, 2 paths of length 2, and 1 path of length 3. In total there are $4 + 3 + 2 + 1 = 10$ paths.