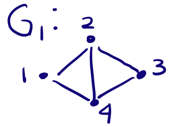
Recitation Module 12

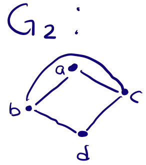
Penn Engineering

Basic terms/definitions to know!!

- Isomorphic graphs

  Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic**, notation $G_1 \simeq G_2$, when there is a **bijection** $\beta : V_1 \to V_2$ such that for any $u_1, v_1 \in V_1$ we have $u_1{-}v_1 \in E_1$ **iff** $\beta(u_1){-}\beta(v_1) \in E_2$.





- Subgraphs

  A graph $G_1 = (V_1, E_1)$ is a **subgraph** of the graph $G_2 = (V_2, E_2)$ when $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. (Beware: not all pairs of such subsets form graphs!)

- Closed walks and cycles

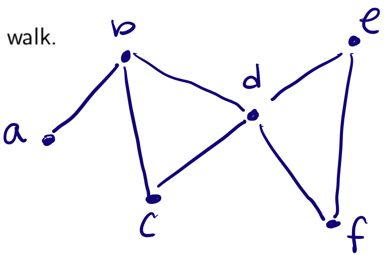
  Cycle length with k nodes is k.

  A **closed walk** is a walk in which the first and the last vertex are the same.

  A **cycle** is a closed walk **of length at least 3** in which all nodes are pairwise distinct, except for the last and the first.

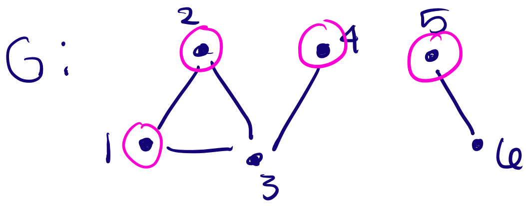
  The **length** of the cycle is the length of the closed walk.

- Counting paths and cycles of a graph (for example Q3)



Induced Subgraphs (examples)

"Regular" Subgraph

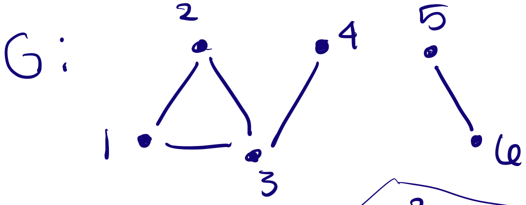


$$(\underbrace{\{1,2,4,5\}}_{V_1}, \underbrace{\{(1,2)\}}_{E_1})$$

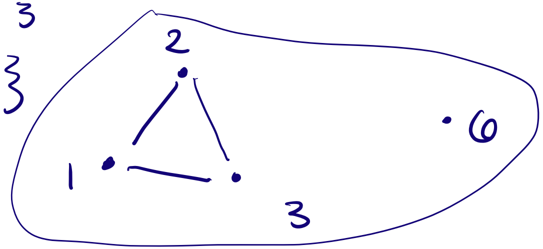
$$(\{1,2,4,5\}, \emptyset)$$

Induced Subgraph

Take some subset of nodes and include as many edges as possible.



$\{1,2,3,6\}$



Basic terms/definitions to know!!

- Acyclic graphs, trees, and forests

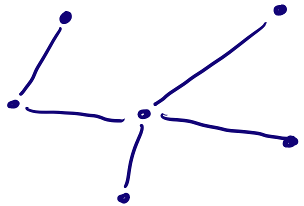
A graph in which there are no cycles is called **acyclic**. The cc's of an acyclic graph are also acyclic.

A graph that is both connected and acyclic is called a **tree**.

Consequently, an acyclic graph is also called a **forest** since all its cc's are trees!

- Properties of trees $|E| = |V| - 1$
  - every tree is minimally connected:
    i.e. removing any edge in a tree disconnects it
  - every tree is maximally acyclic
  - adding an edge between **any** two non adjacent vertices in a tree creates a cycle
  - adding an edge to an acyclic graph creates at most one cycle
  - ~~adding an edge between any two non-adjacent vertices in a tree creates a cycle~~
  - Any two distinct vertices of a tree are connected by a **unique** path

T:



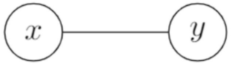
- Cut edges

Let $G = (V, E)$ be a graph. An edge of G is a **cut edge** if by removing it we obtain a graph with strictly **more** connected components (cc's) than G.

Note that this is a dense module with many important concepts. This is by no means an exhaustive summary.

0 / 1 po

1. The graph



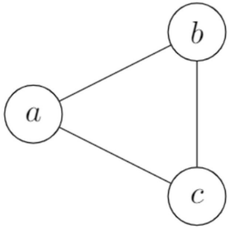
1) $\bullet\, x$ ✓

2) $\bullet\, y$ ✓

3) $\overset{x}{\bullet}\ \ \overset{y}{\bullet}$ ✗

4) $\overset{x}{\bullet}\!-\!\!\!-\!\overset{y}{\bullet}$ ✓

Has four subgraphs $(\{x\}, \emptyset)$, $(\{y\}, \emptyset)$, $(\{x, y\}, \emptyset)$, and $(\{x, y\}, \{x - y\})$. How many different subgraphs does the graph below have?



1) singletons : $(\{a\}, \emptyset)$, ... x3

2) 2 nodes : $\{a, b\}$ w/ or w/o edge

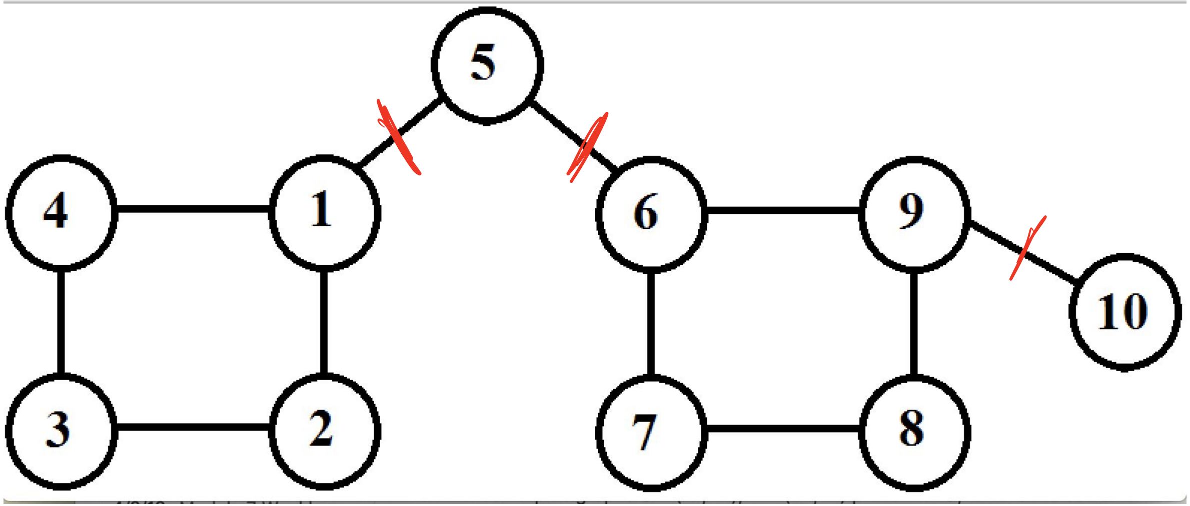
$\{a, c\}$

$\{b, c\}$

3) 3 nodes : $\{a, b, c\}$ w/ any combo of edges

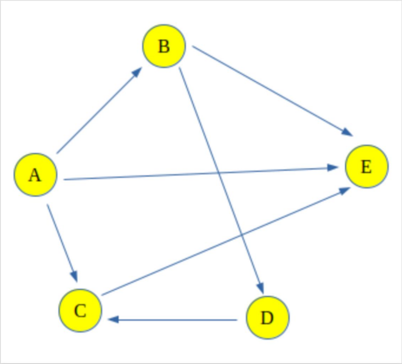
Question 1

Identify the cut edges in the following graph:



Answer to Question 1

For the following Graph:



Input: Count paths between A and E

Output : Total paths between A and E are 4

Explanation: The 4 paths between A and E are:

A -> E

A -> B -> E

A -> C -> E

A -> B -> D -> C -> E

2. Let T be a tree with at least 3 vertices. Assume that every vertex in T has either degree 3 or is a leaf. Let L be the set of leaves of T and let R be the set of vertices in T that have degree 3. Show that

$$\boxed{|R| = |L| - 2}$$

WTS

ANSWER

The total number of vertices is the number of the leaves ($|L|$) plus the number of the vertices in T with degree three ($|R|$). We thus see that:

$$\boxed{|V| = |L| + |R|}$$

Tree Property: $|E| = |V| - 1$

$\sum_{v \in V} \deg(v) = 2|E|$, by Handshaking Lemma

$$\sum_{v \in V} \deg(v) = \boxed{1 \times |L| + 3 \times |R|} = 2|E|$$

$$= 2(|V| - 1) = 2|V| - 2$$

$$= \boxed{2|L| + 2|R| - 2}$$

$$\Longrightarrow |R| = |L| - 2 \quad \blacksquare$$

and there is no directed path u->>v. Therefore, the graph is not strongly connected fo
r all {u,v} since there isn't strong connectivity between v and u. Contradiction.

Q4

15 Points

No justification is required for this question --- only your final answer will be graded. Your answers should be in closed form for full credit.

(a) $[\mathbf{5\ pt}]$ Let $n \geq 2$ be a positive integer. Count the number of *induced* subgraphs of C_{2n} (the cycle graph on $2n$ vertices) that have n vertices and are *edgeless*.

(b) $[\mathbf{5\ pt}]$ How many paths of length 3 are there in K_5?

QUESTION 4

(no title) 5 /

	+ 0 pts	incorrect/blank
	+ 5 pts	a (2)
	+ 5 pts	b (60)
✔	+ 5 pts	c (2^{n-1})

QUESTION 5

(no title) 10 /

QUESTION 6

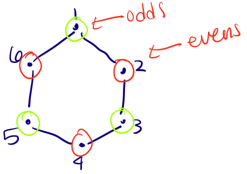
(no title) 10 /

QUESTION 7

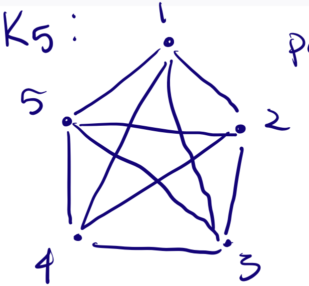
(no title) 10 /

(a) $\cancel{\binom{2n}{n}}$ $\boxed{2}$

C_6 :



(b) K_5 :



path of length 3 $\Rightarrow$ 4 nodes

choose 1st $\rightarrow$ 5

2nd $\rightarrow$ 4 $\Rightarrow$ 120 $\xRightarrow{\div 2}$

3rd $\rightarrow$ 3

4th $\rightarrow$ 2

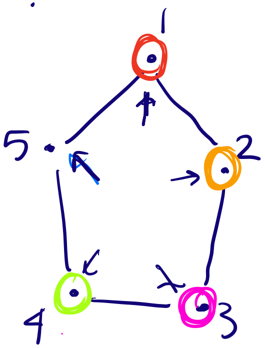
$\boxed{60}$

Question 3

How many paths of length at least 1 are in C_{100}?

A Toy Example:

C_5:



Consider C_{100}:

1) 100 nodes → how many ways to choose start and end nodes? $\binom{100}{2}$

2) How many path per pair? 2 "shorter" and "longer" path

$$\boxed{\binom{100}{2} \times 2}$$

Before you go...

Do you have any general questions about this week's assignment? We can't go into minute details about any specific questions, but this is the time to ask any general clarifying/confusing questions!

As always, if you can't talk to us during recitation or office hours, post any questions/anything course related on the forums or email mcitonline@seas.upenn.edu. Ask questions that might be beneficial to other students on the forums, while emailing about more personal questions (regrade requests, etc).

Recitation Module 11

Penn Engineering

Tool for Graphs in LaTeX

For the next assignments you will be required to draw your graphs in LaTeX. This is usually achieved by a graph library called **tikz**. However, we understand that this is a complex library to learn, which is why we introduce you to

https://www.mathcha.io/editor http://www.madebyevan.com/fsm/

You can design your graphs with a very user friendly interface and then export them on LaTeX source code form.

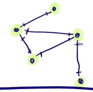
Tips:

- Make sure you include all packages required in your tex file preamble with

\usepackage{packagename}

Graphs that are not written in LaTeX (either using this tool, or tikz directly) will result in points deducted.

Basic terms/definitions to know!!



3+1+3+2+1
= 10

Edgeless:
Complete (K_n):
K_5:

Path:
Cycle: (C_n)
C_4:
Grid: $A_{m,n}$
$A_{2,3}$:

G_2:

- Graph terminology and notation
- Handshaking lemma: $\boxed{\sum_{v \in V} \deg(v) = 2|E|}$
- Types of graphs: edgeless, complete, path, cycle, grid
- Walk vs path

A walk (path) with k nodes has k-1 edges.

A **walk** is a non-empty sequence of vertices consecutively linked by edges: $u_0, u_1, \ldots, u_k$ such that $u_0 - u_1 - \cdots - u_k$. This walk is **from** u_0 **to** u_k (the **endpoints** of the walk) and u_0 and u_k are **connected** by this walk. The **length** of this walk is the number k of edges (**not** $k+1$!).

A **path** is a walk in which all the vertices are **distinct**.

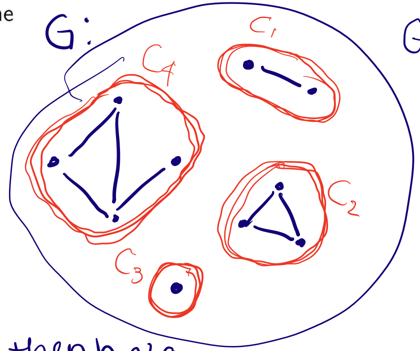
- Connected components

A **connected component** of a graph $G = (V, E)$ is a set of vertices $C \subseteq V$ such that:

- any two vertices in C are **connected**, and
- there is **no strictly bigger** set of vertices $C \subsetneq C' \subseteq V$ such that any two vertices in C' are connected.

We say that C is a **maximally connected** set of vertices of G.

G:



G_3:

- Relation properties: reflexivity, transitivity, symmetry

Relation: $\sim$ → Reflexivity means $x \sim x$; symmetry means if $a \sim b$ then $b \sim a$

Note that this is a dense module with many important concepts. This is by no means an exhaustive summary.

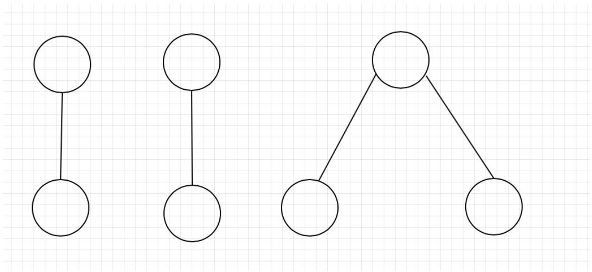
Transitivity means if $a \sim b$ and $b \sim c$, then $a \sim c$ || Equality = || Inequality <

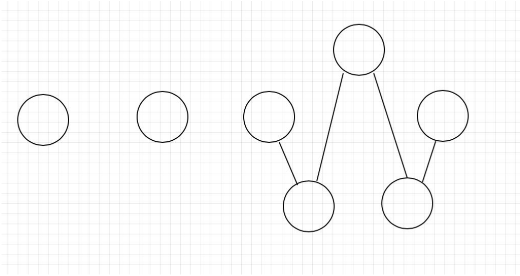
Question 2

What are the minimum and the maximum number of edges that a graph with 7 nodes and 3 connected components can have?

Answer to Question 2

Recall the fact that $\boxed{|E| >= |V| - |CC|.}$ Thus the minimum number of edges the graph can have is 7 - 3 = **4**. See the examples below:





Answer to Question 2 (Continued I)

- For the maximum constraint it is a bit harder.
- Let's formalize what we are looking for:
- We seek numbers n_1, n_2, n_3 representing the number of nodes in each component, such that
  - $n_1 + n_2 + n_3 = 7$ (there must be 7 nodes total), and
  - the sum of all the maximum numbers of edges in the 3 components is maximum.

  the 3 CC's are complete

- Recall that there are (n choose 2) possible edges between n nodes (this is just the number of ways we can choose two vertices and place an edge between them).

Answer to Question 2 (Continued II)

- Thus, we want to maximize $\boxed{\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2}}$ subject to the constraint $n_1 + n_2 + n_3 = 7$

- Our possible options are *(note: order doesn't matter for the total number of edges):*

(1) $\boxed{n_1 = 1, n_2 = 2, n_3 = 4}$ where $\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} = 0 + 1 + 6 = 7$

(2) $\boxed{n_1 = 2, n_2 = 2, n_3 = 3}$ where $\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} = 1 + 1 + 3 = 5$

(3) $\boxed{n_1 = 1, n_2 = 1, n_3 = 5}$ where $\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} = 0 + 0 + 10 = 10$

(4) $\boxed{n_1 = 1, n_2 = 3, n_3 = 3}$ where $\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} = 0 + 3 + 3 = 6$

Hence the maximum number of edges is **10**.

Question 3

Let S be the set of integers strictly greater than 1.

Define the relation ~ on S by:

$m \sim n$ if $\gcd(m, n) > 1$, for m, n in S.

Determine whether ~ is reflexive, symmetric and transitive.

Answer to Question 3

Reflexivity: Let m > 1. We have: gcd(m,m) = m > 1, so m ~ m. Thus, the relation is reflexive.

Symmetry: Let m, n > 1 such that m ~ n. This means gcd(m, n) > 1. We have that gcd(n, m) = gcd(m, n) > 1, so n ~ m follows. Thus, the relation is symmetric.

if $m \sim n$, then $n \overset{?}{\sim} m$

Transitivity: We show it is not transitive by counterexample:

gcd(25, 15) = 5 => 25 ~ 15
gcd(15, 21) = 3 => 15 ~ 21
BUT gcd(25, 21) = 1 => not(25 ~ 21).