Module 2.4: Logical Structure of Statements

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Statements

Logical terminology and notation.

Elucidate the logical **structure** of mathematical assertions (statements).

Eventually, **proof patterns.**

Examples (of statements):

13 is prime.

5 is not even.

The letter *'a'* is a vowel.

If x is prime and x is not 2 then x is odd.

A letter ℓ is either a vowel or a consonant.

Except for the first one, which is a *basic statement* (also called "atomic statement"), each of these statements contain *logical connectives*.

Basic statements and logical connectives

In logical notation we write the basic statement "13 is prime" as $\mathit{prime}(13)$.

Similarly, $\mathit{even}(5)$, $\mathit{letter}('a')$, $\mathit{odd}(x)$, $\mathit{letter}(\ell)$, $\mathit{vowel}(\ell)$, etc.

$x = 2$ is also a basic statement.

Note that x and ℓ are **variables**. To be discussed later.

For each logical connective we give the **(fancy name)**, the corresponding "plain language name", and our corresponding mathematical notation.

- **(conjunction)** "and" $\wedge$
- **(disjunction)** "or" $\vee$
- **(implication)** "if-then" $\Rightarrow$
- **(negation)** "not" $\neg$

Logical connectives allow us to combine basic statements into more complex statements.

Statements in logical notation

13 is prime $\quad prime(13)$

5 is not even $\quad \neg even(5)$

The letter *'a'* is a vowel $\quad letter('a') \wedge vowel('a')$

If x is prime and x is not 2 then x is odd

$$[prime(x) \wedge (\neg(x = 2))] \;\Rightarrow\; odd(x)$$

A letter ℓ is either a vowel or a consonant (Two equivalent translations!)

$$letter(\ell) \;\Rightarrow\; [(vowel(\ell) \wedge \neg consonant(\ell)) \vee (consonant(\ell) \wedge \neg vowel(\ell))]$$

$$letter(\ell) \;\Rightarrow\; [(vowel(\ell) \vee consonant(\ell)) \wedge \neg(vowel(\ell) \wedge consonant(\ell))]$$

Logical set-builder notation

In **set-builder notation** $A = \{ x \mid P(x) \}$ $P(x)$ is a logical statement about x.

Let's redo some of the set-builder definitions

$$C = \{ \ell \mid \mathit{letter}(\ell) \wedge \neg \mathit{vowel}(\ell) \}$$

$$\mathbb{Z}^+ = \{ x \mid x \in \mathbb{N} \wedge x \neq 0 \}$$

$$A \cup B = \{ x \mid x \in A \vee x \in B \}$$

$$A \cap B = \{ x \mid x \in A \wedge x \in B \}$$

$$A \setminus B = \{ x \mid x \in A \wedge x \notin B \}$$

Implication, conditional, and equivalence

Recall that "if P_1 then P_2" is called **implication** and is written in logical notation: $P_1 \Rightarrow P_2$

P_1 is called the **premise** of the implication and P_2 is called its **conclusion**.

Inspired by some programming languages we ask for the logical notation for the **conditional** statement "if P_1 then P_2 else P_3".

It's $(P_1 \Rightarrow P_2) \wedge (\neg P_1 \Rightarrow P_3)$

Another statement is the **biconditional**: "if P_1 then P_2 and if P_2 then P_1".

Logical notation: $(P_1 \Rightarrow P_2) \wedge (P_2 \Rightarrow P_1)$

The biconditional is commonly written as "P_1 iff P_2" where "iff" abbreviates "if and only if", and is called **equivalence**. But logically it is the same.

ACTIVITY : Set Builder Notation

The **symmetric difference** of two sets A and B is defined by
$A \triangle B \ = \ (A \setminus B) \cup (B \setminus A)$.
Give an alternative definition to $A \triangle B$ using set-builder notation.

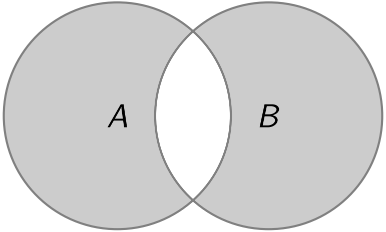
In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : Set Builder Notation (Continued)

Answer:

Using set-builder notation:
$A \triangle B = \{\ x \mid (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\ \}$.

To better understand this operation, here is a diagram. Note that $A \triangle B$ contains elements that are either in A or in B but not in both.



ACTIVITY : Logical Notation

Recall a statement we proved in the first module:

> *If x is an integer such that $x > 1$, then $x^3 + 1$ is not prime.*

Write the statement above in logical notation.

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Logical Notation (Continued)

Answer:

$(int(x) \wedge greaterThanOne(x)) \Rightarrow \neg prime(x^3 + 1)$

Use the logical predicates $int(x)$, $greaterThanOne(x)$, and $prime(x)$ to represent the statements "x is an integer," "$x > 1$," and "x is prime," respectively.

Recall that the logical notation for "if x then y" is $x \Rightarrow y$, and the logical notation for "not x" is $\neg x$.

Therefore, we can write "If x is an integer such that $x > 1$ then $x^3 + 1$ is not prime" in logical notation as above.

Activity : More Logical Notation

Recall a statement we proved in the first module:

> *If p, r, s are positive integers such that $p = r \cdot s$ and p is prime, then one of r and s is 1 and the other one equals p.*

Write this statement in logical notation.

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : More Logical Notation (Continued)

Answer:

$$\big(\mathit{posInt}(p) \land \mathit{posInt}(r) \land \mathit{posInt}(s) \land (p = r \cdot s) \land \mathit{prime}(p)\big)$$
$$\Rightarrow \Big(\big((s = 1) \land (r = p)\big) \lor \big((s = p) \land (r = 1)\big)\Big)$$

Use the logical predicates $\mathit{posInt}(x)$ and $\mathit{prime}(x)$ to represent the statements "x is a positive integer" and "x is prime," respectively. We connect these predicates in the appropriate order using $\lor$, $\land$, $\Rightarrow$, and parentheses.