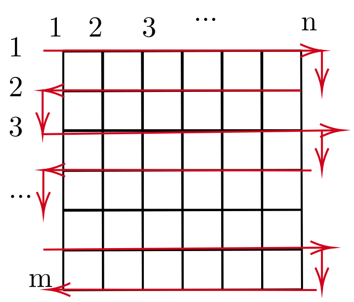
YOUR NAME HERE
YOUR PENN EMAIL HERE

1. **[10 pts]** Matrix Metro is a city with m rows and n columns of buildings, with roads connecting these houses to form a grid. Amy is visiting and wants to walk around the city. Help Amy find the length of the longest path that she can walk (i.e. she never walks to the same building twice). Provide a brief explanation as to why it is the maximum. You can assume $m, n \geq 2$.

Solution.



As shown above, the city can be seen as a grid with m rows and n columns, every intersection is a building. Suppose Amy starts at the left upper corner (1,1), where the first number means which row and the second number means which column. She should follow the red arrows to find the longest path that she can walk. Then she should walk pass all the n buildings in the first row, then take a right at (1,n) and walk to the next building (2,n), and then take another right. Then she should walk along all the n buildings in the second row, and take a left turn at (2,1) and walk to the first building in the next row at (3,1) and then take another left to walk through all the n buildings in row 3 etc etc, till she walks pass all the n buildings in the last row, row m, and reaches to the last building.

This is the longest path that she can walk is the longest path will make sure she covers as many buildings as she can without coming back to the same building. The above described path is the longest is because it lets Amy to walk along every building in every row and column.

From the graph we know that every row has n buildings so there will be $(n-1)$ edges, since

Amy walks pass every row, she will walk $m(n-1)$. She also walks one edge between every row. Since there are m rows, she walks total $(m-1)$. Therefore the total length of the path is:
$m(n-1)+(m-1)=mn-1$

2. **[10 pts]** Suppose there are a series of islands connected by bridges. A d-coalition is a group of islands in which every island has exactly d bridges connected to it. Prove that there is no 7-coalition with 39 bridges.

Solution.

From the question we know that 7-coalition with 39 bridges means that every island has exactly 7 bridges connect to it, which can be considered as a graph $G = (V, E)$, in which islands are nodes and bridges are edges.
Therefore $|V|$ is the number of islands and $|E|$ is the amount of bridges.
Assume there is no 7-coalition with 39 bridges. Then $|E| = 39$, and the degrees of every vertex is 7. Let $|V| = n$, then the sum of all the degrees of G is $7n$.
From the Handshaking Lemma we know that the sum of the degrees of all vertices in a graph is twice the number of edges. Therefore $7n = 2|E| = 2 * 39 = 78$, so $n = 78/7$.
Since n is the number of islands in the 7-coalition, we know that n needs to be a positive integer. However from above we can tell $n = 78/7$ is not an positive integer. So 7-coalition cannot have 39 bridges.
Proof is completed.

3. **[10 pts]** Suppose there exists a Facebook group with n people, where $n \geq 2$. Each of them has *p or more* connections to other members of the group. Prove that if $p > \frac{n-2}{2}$, then the group is connected (i.e. they are all linked together through some traversal of friendships).

Solution.

The group is connected means there is only one connected component.

In order to prove the original statement, we can disprove its negation. The negation of the statement would be, if $p > \frac{n-2}{2}$, then the group is not connected.

If we see the Facebook group as a graph $G = (V, E)$, in which members are nodes and friendships are edges, and member's connection is their degrees. So we know $|V| = n \geq 2$, and $deg(v) > \frac{n-2}{2}$.

Since the group is not connected, let us assume there are two separate groups(connected components), $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.

Since G_1 and G_2 are G's connected component, we know that $|V_1| + |V_2| \leq |V| = n$

For any v_1 in $|V_1|$, any v_2 in $|V_2|$,

$deg(v_1)$ at least can be 0 when G_1 is edgeless, and at most can have $(|V_1| - 1)$ when v_1 is connected with all the other nodes in G_1, therefore $0 \leq deg(v_1) \leq (|V_1| - 1)$.

Similarly, $0 \leq deg(v_2) \leq (|V_2| - 1)$.

So we know that:

$0 \leq deg(V_1) + deg(V_2) \leq (|V_1| - 1) + (|V_2| - 1) = |V_1| + |V_2| - 2$

Since $|V_1| + |V_2| \leq |V| = n$,

$0 \leq deg(V_1) + deg(V_2) \leq (|V_1| - 1) + (|V_2| - 1) = |V_1| + |V_2| - 2 \leq (n - 2)$

Which can be expressed as $0 \leq deg(V_1) + deg(V_2) \leq n - 2$

When any vertex has degree $p > \frac{n-2}{2}$, $deg(V_1) + deg(V_2) > \frac{n-2}{2} + \frac{n-2}{2} = (n - 2)$

Which can be expressed as $deg(V_1) + deg(V_2) > n - 2$, which is contradicted to $0 \leq deg(V_1) + deg(V_2) \leq n - 2$. Thus the negation is disproved.

By disproving the negation, we proved the original statement "if $p > \frac{n-2}{2}$, then the group is connected".

4. **[10 pts]** Suppose there is a shipping network with n shipping locations, where $n \geq 2$. They are connected by roads, but a shipping location could be isolated (i.e. there are no roads to it). Prove that, no matter how these roads are organized, there are at least two shipping locations with the same number of roads.

Solution.

We can see the shipping network as a graph $G = (V, E)$, in which shipping locations are nodes and roads are edges. So we know $|V| = n$.

Let us assume every shipping location has distinct amount of roads connected to it, which is also the degree of a shipping location, so let's call it $deg(v_i)$. Then $deg(v_i)$ can at least have 0 and at most n-1.

However, just like the handshake question, if any of the n shipping locations has 0 road, then for the other shipping locations, they at most have n-2 roads, so $deg(v_i) \in [0..(n-2)]$.

If any of the n shipping locations has n-1 roads, then for the other shipping locations, they at least have 1 road, so $deg(v_i) \in [1..(n-1)]$.

From two situations discussed above we can tell at most $|deg(v_i)| = (n-2) - 0 + 1 = (n-1) - 1 + 1 = n-1$

There are n shipping locations (pigeons) but at most $n-1$ choices of roads (pigeonholes), so based on PHP, we can conclude at least two shipping locations will have the same number of roads.

5. **[10 pts]** Consider a graph G with $2k$ vertices and k edges, where k is a positive integer. Prove that if each vertex in G has degree at least 1, then it has exactly k connected components.

Solution.

Step 1: Disprove G can not have more than k connected components.
Let m be a positive integer and $k < m$. Assume G has m connected components, $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_m = (V_m, E_m)$, and $|V_1| + |V_2| + \cdots + |V_m| = 2k$.
Since each vertex in G has degree at least 1, every CC in G at least has two connected vertices, therefore $|V_i| \geq 2$ and $|E_i| \geq 1$.
Since there are m connected components, $|V_1| + |V_2| + \cdots + |V_m| = 2k \geq 2 + 2 + \cdots + 2 = 2m$, so simply we got:
$2k \geq 2m$, so $k \geq m$ which is contradicted with our assumption $k < m$.
Therefore G cannot have more than k connected components.

Step 2: Disprove G can not have less than k connected components.
Let n be a positive integer and $n < k$. Assume G has n connected components, $G_1' = (V_1', E_1'), G_2' = (V_2', E_2'), \ldots, G_n = (V_n, E_n)$, and $|V_1'| + |V_2'| + \cdots + |V_n| = 2k$.
We know that in any graph $G = (V, E)$, $|E| \geq |V| - |CC|$.
So $|CC| \geq |V| - |E|$.
so when G has n connect components, $n \geq |V| - |E| = 2k - k = k$ which is contradicted with our assumption $n < k$.
Therefore G cannot have less than k connected components.

Step 3: G can have k connected components.
If G has k connect components, when each connected component will have 2 vertices and one edge, the graph has 2k vertices and k edges.
Therefore G can have k connected components.

From above two steps we proved that if each vertex in G has degree at least 1, the graph cannot have more or less than k connected components but can have k connected component.
Therefore if each vertex in G has degree at least 1, then it has exactly k connected components.