OMCIT 592 Module 08 Self-Paced 04 (instructor Val Tannen)

Reference to this self-paced segment in seg.08.06

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion.

When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Conditional probability rules

In the lecture segment "The chain rule" we stated the general property that justifies the calculation of probabilities in the tree of all possibilities by multiplying a long a branch:

Proposition (The Chain Rule). For any **two or more** events $A_1, \ldots, A_n$ in the same probability space we have

$$\Pr[A_1 \cap A_2 \cap A_3 \cdots \cap A_n] \;=\; \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \;\cdots\; \Pr[A_n \mid A_1 \cap \cdots \cap A_{n-1}]$$

Proof. By induction on n.

(BC) The case $n = 2$ follows directly from the definition of conditional probability:

$$\Pr[A_2 \mid A_1] \;=\; \frac{\Pr[A_1 \cap A_2]}{\Pr[A_1]} \qquad \text{therefore} \qquad \Pr[A_1 \cap A_2] \;=\; \Pr[A_1] \cdot \Pr[A_2 \mid A_1]$$

(IS) Let $k \geq 2$ be an arbitrary natural number. Assume (IH) that

$$\Pr[A_1 \cap \cdots \cap A_k] \;=\; \Pr[A_1] \;\cdots\; \Pr[A_k \mid A_1 \cap \cdots \cap A_{k-1}]$$

We want to show that

$$\Pr[A_1 \cap \cdots \cap A_k \cap A_{k+1}] \;=\; \Pr[A_1] \;\cdots\; \Pr[A_k \mid A_1 \cap \cdots \cap A_{k-1}] \cdot \Pr[A_{k+1} \mid A_1 \cap \cdots \cap A_k]$$

Let $B = A_1 \cap \cdots \cap A_k$. As above, by the definition of conditional probability we have

$$\Pr[B \cap A_{k+1}] \;=\; \Pr[B] \cdot \Pr[A_{k+1} \mid B]$$

Using this, as well as the IH we obtain

$$\begin{aligned}
\Pr[A_1 \cap \cdots \cap A_k \cap A_{k+1}] &= \Pr[B \cap A_{k+1}] \\
&= \Pr[B] \cdot \Pr[A_{k+1} \mid B] \\
&= \Pr[A_1] \;\cdots\; \Pr[A_k \mid A_1 \cap \cdots \cap A_{k-1}] \cdot \Pr[A_{k+1} \mid A_1 \cap \cdots \cap A_k]
\end{aligned}$$

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Conditional probability rules (continued)

Problem (Rare Disease Test (again)). A test for a disease that affects 0.1% of the population is 99% effective on people with the disease (i.e., it gives a **false negative** with probability 0.01). On people who do not suffer from the disease the test gives a **false positive** with probability 0.02. What is the probability that someone who tests positive in fact has the disease?

Answer. In the lecture segment "Conditional probability" we gave a detailed analysis of this problem using a tree of possibilities. Here, we give a different, faster, solution that illustrates two other useful properties of probability: Bayes' Rule and the Total Probability Rule.

Again let V be a random person who takes the test. Let Y be the event that V has the disease and let $+$ be the event that V tested positive. We wish to compute the conditional probability $\Pr[Y \mid +]$.

The problem gives us $\Pr[Y] = 0.001$.

We denote by N the complement of Y, which is the event that V does not have the disease. $\Pr[N] = 0.999$. The statement of the problem gives us the conditional probabilities for testing positive given that someone has, or that someone does not have the disease:

$$\Pr[+ \mid Y] = 0.99 \qquad \Pr[+ \mid N] = 0.02$$

Note that we know $\Pr[+ \mid Y]$ but we need $\Pr[Y \mid +]$. There is a property of probability that allows us to express one in terms of the other:

Proposition (Bayes' Rule).

$$\Pr[A \mid B] \;=\; \frac{\Pr[A]\;\Pr[B \mid A]}{\Pr[B]}$$

There is not much to prove, as this follows directly from the definitions of conditional probabilities $\Pr[A \mid B]$ and $\Pr[B \mid A]$.

Using Bayes' Rule we get $\Pr[Y \mid +] = \Pr[Y]\,\Pr[+ \mid Y]/\Pr[+]$ so we still need to compute $\Pr[+]$ from what we know (note that we haven't used $\Pr[+ \mid N]$ yet).

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Conditional probability rules (continued)

Again there is a useful property of probability that we can use:

Proposition. Let E and A be two events in the same probability space such that $\Pr[A] \neq 0$. Then

$$\Pr[E] \;=\; \Pr[E \mid A]\Pr[A] + \Pr[E \mid \overline{A}]\Pr[\overline{A}]$$

This is a particular case of a more general property called the Rule of Total Probability that we state without proof below. For our rare disease problem we need only this particular case so let's prove it.

Proof. Note that $E \cap A$ and $E \cap \overline{A}$ are disjoint. Note also (for example by drawing an Euler-Venn diagram) that $(E \cap A) \cup (E \cap \overline{A}) = E$. Therefore we can apply the addition rule (P2):

$$\Pr[E] \;=\; \Pr[E \cap A] + \Pr[E \cap \overline{A}]$$

By the definition of conditional probability

$$\Pr[E \cap A] = \Pr[E \mid A]\Pr[A] \qquad \text{and} \qquad \Pr[E \cap \overline{A}] = \Pr[E \mid \overline{A}]\Pr[\overline{A}]$$

and the proposition follows.

Applying this to our problem $\Pr[+] = \Pr[+ \mid Y]\Pr[Y] + \Pr[+ \mid N]\Pr[N] = 0.99 \cdot 0.001 + 0.02 \cdot 0.999 = 0.00099 + 0.01998 = 0.02097$. Then $\Pr[Y \mid +] = \Pr[Y]\Pr[+ \mid Y]/\Pr[+] = 0.00099/0.02097 \simeq 0.0472$, the same answer we obtained in our first solution that used a tree of all possibilities (in the lecture segment "Conditional probability").

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Conditional probability rules (continued)

As promised we state without proof (the proof is by induction, not hard, but somewhat tedious):

Proposition (The Rule of Total Probability). Let $A_1, \ldots, A_n$ be **two or more** events, each of **of non-zero probability** in the same probability space such that:

- $A_1, \ldots, A_n$ are **pairwise disjoint**, and
- $A_1 \cup \cdots \cup A_n = \Omega$

(we say that $A_1, \ldots, A_n$ form a **partition** of Ω). Then, for any event E

$$\Pr[E] \;=\; \sum_{i=1}^{n} \Pr[E \mid A_i]\,\Pr[A_i]$$