Fan Zhang
fanzp@seas.upenn.edu

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1. [10 pts] Prove by induction that for all positive integers n, we have:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Solution.

(BC) (n=1) Since $2n - 1 = 2 * 1 - 1 = 1$ so $1 = n^2 = 1$. Check.
(IS) Let $k \geq 1$ be any arbitrary positive integer.
Assume (IH) $1 + 3 + 5 + \cdots + (2k - 1) = k^2$.
WTS $1 + 3 + 5 + \cdots + (2(k + 1) - 1) = (k + 1)^2$. Then:
$1 + 3 + 5 + \cdots + (2(k + 1) - 1)$
$= k^2 + 2(k + 1) - 1$
$= k^2 + 2k + 1$
$= (k + 1)^2$
The proof is finished.

2. **[10 pts]** Use the recursion tree method or the telescopic method to solve the recurrence relation $f(0) = 7$ and, for all $n \in \mathbb{Z}^+$ $f(n) = f(n-1) - 2n$.

Solution.

$f(n) = f(n-1) - 2n$

$f(n-1) = f(n-2) - 2(n-1)$

$f(n-2) = f(n-3) - 2(n-2)$

$f(n-3) = f(n-4) - 2(n-3)$

$\ldots$

$f(2) = f(1) - 2 * 2$

$f(1) = f(0) - 2 * 1$

Add all the LHSs and RHSs and cancel terms that appear on both sides:

$f(n) = f(0) - 2n - 2(n-1) - 2(n-2) - 2(n-3) - \cdots - 2 - 1$

$= f(0) - 2 * [n + (n-1) + (n-2) + (n-3) + \cdots + 2 + 1]$

$= f(0) - 2 * n(n+1)/2$

$= f(0) - n(n+1)$

Done.

3. [10 pts] Use strong induction to prove that $C(n) = 2^n + 3$ is a solution to the recurrence $C(0) = 4$, $C(1) = 5$, and, for all $n \in \mathbb{Z}^+$, $n > 1$
$C(n) = 3 \cdot C(n-1) - 2 \cdot C(n-2)$.

Solution.

(BC) (n=2) $C(2) = 2^2 + 3 = 7$
$C(2) = 3 \cdot C(1) - 2 \cdot C(0) = 3 * 5 - 2 * 4 = 7$
Therefore $C(n)$ is a solution to $3 \cdot C(n-1) - 2 \cdot C(n-2)$ when $n = 2$. Check.
(IS) Let $k \geq 2$ arbitrary. Assume (IH) that for all integers $2, 3, ..., k$, $C(k) = 2^k + 3$ is a solution to $C(k) = 3 \cdot C(k-1) - 2 \cdot C(k-2)$.
WTS $C(k+1) = 2^{k+1} + 3$ is a solution to $C(k+1) = 3 \cdot C(k) - 2 \cdot C(k-1)$.
From IH we know $C(k) = 2^k + 3$ is a solution to $C(k) = 3 \cdot C(k-1) - 2 \cdot C(k-2)$ and $C(k-1) = 2^{k-1} + 3$ is a solution to $C(k-1) = 3 \cdot C(k-2) - 2 \cdot C(k-3)$. Then:
$C(k+1) = 3 \cdot C(k) - 2 \cdot C(k-1)$
$= 3 \cdot (2^k + 3) - 2 \cdot (2^{k-1} + 3)$
$= 3 \cdot 2^k + 9 - 2^k - 6$
$= (3-1) \cdot 2^k + 3$
$= 2^{k+1} + 3$
Therefore we can conclude $C(k+1) = 2^{k+1} + 3$ is a solution to $C(k+1) = 3 \cdot C(k) - 2 \cdot C(k-1)$.
The proof is finished.

4. [10 pts] Recall the Fibonacci sequence, where every number in the sequence is the sum of the previous two numbers (except for the first and second positions, which are 0 and 1 respectively). Let F_n represent the nth number in the Fibonacci sequence. Use strong induction to prove that for Fibonacci numbers $F_{n+1} - F_{n-1} < 2^n$ for all positive integers n.

Solution.

(BC) (n=1) Since $F_{n+1} = F_2 = 1$, $F_{n-1} = F_0 = 0$, so $F_{n+1} - F_{n-1} = 1$. Since $2^n = 2$, therefore $F_{n+1} - F_{n-1} < 2^n$ when $n = 1$. Check.
(IS) Let $k \geq 1$ arbitrary. Assume (IH) that for all integers $1, 2, ..., k$, $F_{k+1} - F_{k-1} < 2^k$.
And WTS $F_{k+2} - F_k < 2^{k+1}$. Then:
$F_{k+2} - F_k$
$= (F_{k+1} + F_k) - (F_{k-1} + F_{k-2})$
$= (F_{k+1} - F_{k-1}) + (F_k - F_{k-2})$
From IH we know $F_{k+1} - F_{k-1} < 2^k$ and $F_k - F_{k-2} < 2^{k-1}$
Then $(F_{k+1} - F_{k-1}) + (F_k - F_{k-2}) < 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}$
So we got $F_{k+2} - F_k < 2^{k+1}$.
The proof is finished.

5. **[10 pts]** Use ordinary induction to prove that for every positive integer n, $n^3 - n$ is a multiple of 6. Only proofs by induction are accepted.

Solution.

(BC) (n=1) $n^3 - n = 1 - 1 = 0$. Check.

(IS) Let $k \geq 1$ arbitrary.

Assume (IH) $k^3 - k$ is a multiple of 6. And WTS $(k+1)^3 - (k+1)$ is a multiple of 6. Then

$(k+1)^3 - (k+1)$

$=k^3 + 3k^2 + 3k + 1 - k - 1$

$=k^3 + 3k^2 + 2k$

$=k^3 - k + 3k^2 + 3k$

$=(k^3 - k) + 3k(k+1)$

From IH we know $(k^3 - k)$ is a multiple of 6.

Since $k \geq 1$, we know $(k+1) \geq 2$ therefore $3(k+1)$ is a multiple of 6 which means $3n(k+1)$ is also a multiple of 6. In summary, $(k+1)^3 - (k+1) = (k^3 - k) + 3k(k+1)$ is also a multiple of 6. The proof is finished.