Module 7.1: Probability Space and Events

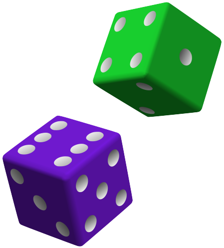
MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Rolling dice

A die has six faces, each marked with dots representing the numbers 1, 2, 3, 4, 5, or 6. Thus, when we roll a die we get (on top) one of six possible **outcomes**: the numbers 1 through 6.

With a **fair** die each of the six faces is **equally likely** to show up on top. Thus, when we roll a fair die the **chance** or **probability** of getting, for example, a 4, is $1/6$.



Distinguishable dice

Problem. We roll a pair of distinguishable fair dice (one die is green, the other is purple) together. What are the chances that we get a double (1-1, 2-2, . . . , 6-6)?

Answer. A roll of the dice shows 6 possible green sides and 6 purple ones.

By the multiplication rule, that's $6 \cdot 6 = 36$ possible outcomes.

6 of these outcomes are doubles.

The chances of getting a double are therefore $6/36 = 1/6$.

Indistinguishable dice

Problem. We roll a pair of indistinguishable fair dice (both are beige) together. What are the chances that we get a double (1-1, 2-2, ..., 6-6)?

Answer. There are two kinds of outcomes: (1) dice show same number, and (2) dice show different numbers.

There are 6 outcomes of the first kind (the doubles) and $\binom{6}{2} = 15$ outcomes of the second kind for a total of $6 + 15 = 21$.

The chances of getting a double seem to be $6/21 = 2/7$.

But $2/7 \simeq 0.28$ is significantly bigger than $1/6 \simeq 0.17$. Backgammon (tavla) players love doubles. Must they always play with indistinguishable dice?

Distinguishable vs. indistinguishable

No, it does not matter whether the pair consists of distinguishable dice or not.

Explanation. The key is to consider **how likely** each outcome is.

For the green-purple dice, since the dice are fair, and since we roll them in the same way, each of the 36 outcomes is **equally likely**.

However, for the beige dice some of the outcomes are more likely than others. For example rolling a 5 and a 6 cannot be distinguished from rolling a 6 and a 5. Therefore this is twice as likely than rolling a 6 and a 6.

This discussion motivates the definition of **probability space**.

Probability space

A **probability space** (Ω, Pr) consists of

- a finite non-empty set Ω of **outcomes** and
- a **probability distribution** function $\Pr : \Omega \to [0, 1]$ that associates with each outcome $w \in \Omega$ its **probability** $\Pr[w]$ which is a real number between 0 and 1 (inclusive), such that

$$\sum_{w \in \Omega} \Pr[w] = 1$$

What we cover in this course is **finite** probability theory. It suffices for much of computer science and it frees us from the need to use calculus.

Quiz

A **fair** coin is equally likely to show heads (abbreviated `H`) as it is to show tails (abbreviated `T`) when it is flipped (or tossed).

We flip a fair coin three times in a row. How many outcomes are there in the resulting probability space?

A. 3
B. 6
C. 8

Answer

A. 3

   Incorrect. There are 2 possibilities in each flip.

B. 6

   Incorrect. You need to multiply the 2 possibilities in each flip.

C. 8

   Correct. There are 2 possibilities in each flip and by the multiplication rule $2 \cdot 2 \cdot 2 = 8$.

More Information

Here are the 8 outcomes (`THH` means "tails in the first round followed by heads in rounds two and three):

`HHH` `HHT` `HTH` `HTT` `THH` `THT` `TTH` `TTT`

Examples of probability space I

When we roll the green-purple dice the outcomes are
$\Omega = \{(1,1),(1,2),\ldots,(6,5),(6,6)\}$ with $|\Omega| = 36$.

Following the "equally likely" intuition, the probability distribution is
$\Pr[(1,1)] = \Pr[(1,2)] = \cdots = \Pr[(6,5)] = \Pr[(6,6)] = 1/36$.

When we flip a fair coin three times the outcomes are
$\{\mathtt{HHH}, \mathtt{HHT}, \ldots, \mathtt{TTH}, \mathtt{TTT}\}$ with $|\Omega| = 8$.

Following again the "equally likely" intuition
$\Pr[\mathtt{HHH}] = \Pr[\mathtt{HHT}] = \cdots = \Pr[\mathtt{TTH}] = \Pr[\mathtt{TTT}] = 1/8$.

We flip a fair coin and then, without regard to what the coin showed, we roll a fair die. In this case the outcomes are:
$\Omega = \{(\mathtt{H},1),\ldots,(\mathtt{H},6),(\mathtt{T},1),\ldots,(\mathtt{T},6)\}$ with $|\Omega| = 2 \cdot 6 = 12$.

Again our intuition says the outcomes are "equally likely" so each of them will have probability $1/12$.

Another example of probability space

When we roll the indistinguishable (beige) dice the outcomes are
$\Omega = \{1-1, \ldots, 6-6\} \cup \{\{1,2\}, \{1,3\}, \ldots, \{6,4\}, \{6,5\}\}$
And $|\Omega| = 6 + 15 = 21$.

Outcomes $1-1, \ldots, 6-6$ are equally likely, probability p.

Outcomes $\{1,2\}, \{1,3\}, \ldots, \{6,4\}, \{6,5\}$ are also equally likely.
Probability q.

An outcome of the second kind is twice as likely as one of the first kind.

That is $q = 2p$.

But all probabilities in the space must add up to 1!

Hence $6p + 15q = 1$.

Solving, we get $p = 1/36$ and $q = 1/18$.

Events

Let (Ω, Pr) be a probability space. An **event** in this space is a subset $E \subseteq \Omega$. We extend the probability function from outcomes to events as follows

$$\mathrm{Pr}[E] \;=\; \sum_{w \in E} \mathrm{Pr}[w]$$

Note that

- $\mathrm{Pr}[E] \in [0, 1]$
- $\mathrm{Pr}[\emptyset] \;=\; 0$
- $\mathrm{Pr}[\{w\}] \;=\; \mathrm{Pr}[w]$

Now we can calculate the probability of getting a double with beige dice:

$$\mathrm{Pr}[\{1-1, \ldots, 6-6\}] = \mathrm{Pr}[\{1-1\}] + \cdots + \mathrm{Pr}[\{6-6\}] = 1/36 + \cdots + 1/36 =$$
$$= 6/36 = 1/6$$

Distinguishable or not (again)

Explanation (again). By setting up the correct probability space for the indistinguishable beige dice we have corrected the conclusion we drew earlier: that beige dice are better at getting doubles!

In fact one can formally relate the probability space of the beige dice to that of the green-purple dice. Then we can prove that for all events (not just the doubles) in the beige space the probability is the same as in the green-purple space.

Informally, we can treat the beige dice as if they are distinguishable!

Here is an intuitive argument for that: instead of rolling the beige dice simultaneously, roll them sequentially.

Event probability exercise

Problem. We roll a pair of fair dice. Compute the probability of the event "at least one of the dice shows a 6". Solve the problem in both the green-purple space and the beige space.

Answer. First in the green-purple space. The outcomes in the event of interest are of two kinds. Green 6 with any purple: six outcomes. And purple 6 with any green: six outcomes.

However, we double counted green 6 purple 6! So only 11 outcomes and the answer is $1/36 + \cdots + 1/36 = 11/36$

Now in the beige space the outcomes in the event of interest are $\{1,6\}, \ldots, \{5,6\}, 6-6$. The answer is

$$(1/18 + \cdots + 1/18) + 1/36 = 5/18 + 1/36 = 11/36$$