MCIT 592 Spring 2022 **Homework 9** **Solutions**

PROBLEM SET

1. [10 pts] The 10 decimal digits, $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are arranged in a uniformly random permutation. We denote by a the integer formed in base 10 by the first five positions in this permutation and by b the integer formed in base 10 by the last five positions in this permutation (either a or b may begin with 0 which in such a case is ignored). For example, if the random permutation is 8621705394 then $a = 86217$ and $b = 5394$. Consider the probability space whose outcomes are these random permutations and a random variable X defined on this probability space such $X = 1$ when the product ab is even and $X = 0$ when that product is odd. Calculate $\mathrm{E}[X]$.

Solution:

Since it takes values 0 and 1, X is a Bernoulli random variable. As with all such random variables, by definition of expectation,

$$\mathrm{E}[X] = 0 \cdot \Pr[X = 0] + 1 \cdot \Pr[X = 1] = \Pr[X = 1].$$

Define $(\Omega, \Pr)$ to be the uniform probability space of random permutations of the 10 decimal digits. We have $\Pr[X = 1] = |X = 1|/|\Omega|$ with $|\Omega| = 10!$, the total number of permutations.

To find $|X = 1|$, we count the permutations $d_0d_1d_2d_3d_4d_5d_6d_7d_8d_9$ for which ab is even. Note that the parity of ab is determined by d_4 and d_9: by the multiplication rules of thumb, ab is odd if both d_4 and d_9 are odd, and even otherwise. A permutation where both d_4 and d_9 are odd can be formed by choosing d_4 in 5 ways, choosing d_9 in 4 ways (since we have already chosen one of the odd digits), and permuting the remaining digits in $8!$ ways. Therefore, there are $5 \cdot 4 \cdot 8!$ such permutations and by complementary counting, $|X = 1| = 10! - 5 \cdot 4 \cdot 8!$. Then

$$\mathrm{E}[X] = \frac{10! - 5 \cdot 4 \cdot 8!}{10!} = 1 - \frac{5 \cdot 4}{10 \cdot 9} = \frac{7}{9}.$$

2. **[10 pts]** The digits 1, 4, and 7 are randomly arranged to form a one digit number and a two-digit number. Each digit can only be used once; for example, if the one-digit number is 7, then the two-digit number is either 14 or 41. What is the expected value of the product of the two numbers?

Solution:

Let the sample space be a random permutation of $\{1, 4, 7\}$, where the first number is the single digit number and the following numbers are the two digit number. Thus, there are six outcomes of equal probability.

The expected value is the value of each case times the probability that it occurs. That is:

$$\frac{1}{6}(1 \cdot 47) + \frac{1}{6}(1 \cdot 74) + \frac{1}{6}(4 \cdot 17) + \frac{1}{6}(4 \cdot 71) + \frac{1}{6}(7 \cdot 14) + \frac{1}{6}(7 \cdot 41)$$
$$= \frac{47 + 74 + 68 + 284 + 98 + 287}{6} = \frac{858}{6} = \boxed{\frac{429}{3}}.$$

3. **[10 pts]** You have a standard deck of cards (see the preamble to hw08). You divide this deck in half by selecting uniformly at random 26 of the deck's 52 cards and placing them in your left hand. You hold the remaining 26 cards in your right hand. Prove that the expected number of aces in your right hand is 2.

Solution:

Define the sample space Ω as a sets of 26 elements, representing the cards chosen to be in the left hand. Note that the remaining 26 elements not in each outcome are the cards in the right hand. Ω is uniform sample space since the cards are divided uniformly at random between the two hands.

Define random variable X_1 such that it equals 1 if the ace of diamonds is in the right hand and 0 otherwise. Define X_2, X_3, and X_4 the same way for the ace of hearts, spades, and clubs. Because these are Bernoulli random variables, for $i \in [1..4]$, $E[X_i] = \Pr[X_i = 1]$.

Define random variable X as the number of aces in your right hand. Note that $X = X_1 + X_2 + X_3 + X_4$. We are interested in the value $E[X] = E[X_1 + X_2 + X_3 + X_4]$ and by the linearity of expectations,

$$E[X] = E[X_1] + E[X_2] + E[X_3] + E[X_4] = 4E[X_1] = 4\Pr[X_1 = 1]$$

We turn our attention to $Pr[X_1 = 1]$. Because Ω is a uniform distribution, $\Pr[X_1 = 1] = \frac{|X_1=1|}{|\Omega|}$. $|\Omega| = \binom{52}{26}$, since each outcome in the sample space can be counted by choosing 26 cards to be in the right hand. $|X_1 = 1|$ can be counted by first adding the ace of diamonds to the right hand. Then, choose the remaining cards: $\binom{51}{25}$. So, $\Pr[X_1 = 1] = \binom{51}{25}/\binom{52}{26} = \frac{1}{2}$.

So $\boxed{E[X] = 4(1/2) = 2}$.

Note that a symmetry argument can be made for $\Pr[X_1 = 1] = 1/2$. That is, the two hands are symmetrical. For every outcome in the sample space L, where L is the set of cards in the left hand, it can be mapped one to one to an outcome L' containing all the elements not in L. (These two outcomes are just the cards switching between the right and left hands!) Thus, $\Pr[X_1 = 1] = \Pr[X_1 = 0]$. And since these two events are complements of each other, it must be the case that $\Pr[X_1 = 1] = \Pr[X_1 = 0] = 1/2$.

4. [10 pts]

Jay has \$500 in the bank when he decides to try a savings experiment. On each day $i \in [1..30]$, Jay flips a fair coin. If it comes up heads, he deposits i dollars into the bank; if it comes up tails, he withdraws \$10. How many dollars should he expect to have in the bank after 30 days?

Solution:

Let X_i be the random variable for the amount Jay deposits on day i (this will negative if he withdraws). Then the amount in the bank after 30 days is $500 + X = 500 + \sum_{i=1}^{30} X_i$. By the definition of expectation, $\mathrm{E}[X_i] = \frac{i-10}{2}$. By linearity of expectation,

$$\begin{aligned}
\mathrm{E}[500 + X] &= \mathrm{E}\left[500 + \sum_{i=1}^{30} X_i\right] \\
&= 500 + \sum_{i=1}^{30} \mathrm{E}[X_i] \\
&= 500 + \sum_{i=1}^{30} \frac{i-10}{2} \\
&= 350 + \frac{1}{2}\sum_{i=1}^{30} i \\
&= 350 + \frac{1}{2}\frac{30^2 + 30}{2} \\
&= \boxed{582.50}
\end{aligned}$$

5. [10 pts] Let A be a set of $n \geq 2$ distinct numbers and let $a_1 a_2 \cdots a_n$ be a permutation of A. For $i = 2, 3, \ldots, n$ we say that position i in the permutation is a *step* if $a_{i-1} < a_i$. We also go ahead and just consider position 1 a step. What is the expected number of steps in a random permutation of A?

Solution:

Define the set of outcomes Ω as the set of permutations of the elements of A. As we have defined it in lecture this probability space is uniform and $|\Omega| = n!$.

For $i \in [1..n]$, define the random variable X_i to equal 1 if position i is a step and 0 otherwise. Since X_i is a Bernoulli random variable as it only takes on values of 0 and 1, $E[X_i] = \Pr[X_i = 1]$. Define random variable $X = \sum_{i=1}^{n} X_i$. The value we're interested in is $E[X]$ and according to the linearity of expectation, $E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \Pr[X_i = 1]$.

Since position 1 is defined as a step in the problem, $\Pr[X_1 = 1] = 1$. As for $i \in [2..n]$, we calculate the number of outcomes in the event $X_i = 1$ by the multiplication rule. First we select the numbers for positions i and $i - 1$. Do so by first choosing a set of 2 numbers from A ($\binom{n}{2}$ ways) and then placing the greater one in position i and the smaller one in position $i - 1$ (1 way). Then, the remaining $n - 2$ positions can be filled in $(n - 2)!$ ways with the remaining elements of A.

$$\Pr[X_i = 1] = \frac{\binom{n}{2}(n-2)!}{n!} = \frac{n(n-1)/2}{n(n-1)} = \frac{1}{2}$$

Therefore the expected number of steps is

$$E[X] = \sum_{i=1}^{n} \Pr[X_i = 1] = 1 + (n-1)\frac{1}{2} = \boxed{\frac{n+1}{2}}$$