MCIT 592 Online - Spring 2022 **(instructor Val Tannen)**

Additional Problems (Packet 3)

1. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a *very brief* explanation of your answer.

   (a) There exists an undirected graph G with 4 vertices such that both G and its complement, $\overline{G}$, are connected.

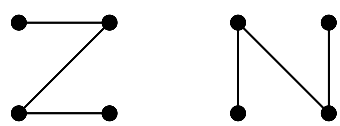
   (b) Let $G = (V, E)$ be an undirected graph in which every node has degree 3. Then $|E|/3 = |V|/2$.

   (c) For any tree if we add one edge between any two existing vertices the resulting graph is *not* bipartite.

   (d) For any positive integer $n \geq 2$, if the complete bipartite graph $K_{2,n}$ has a cycle of length $n + 2$ then n must be even.

   Answer

   (a) $\boxed{TRUE.}$ Here is one example

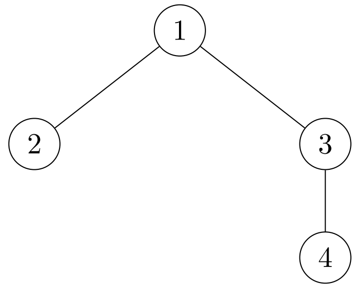
   (b) $\boxed{TRUE.}$ By the Handshaking Lemma, we have

$$
\begin{aligned}
2|E| &= \sum_{v \in V} \deg v \\
&= \sum_{v \in V} 3 && \text{(because } \deg v = 3 \text{ for all } v\text{)} \\
&= 3|V|.
\end{aligned}
$$

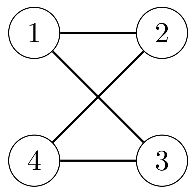
   Dividing through by 6, we have

$$\frac{|E|}{3} = \frac{|V|}{2}.$$

(c) $\boxed{FALSE.}$ Here is our original tree



We can draw an edge between nodes 2 and 4 to get the complete bipartite graph $K_{2,2}$



(d) $\boxed{TRUE.}$ From lecture we know that a bipartite graph has no odd length cycles. Then, we know $n+2$ is even, i.e. $n+2 = 2k$ for some positive integer k. Solving for n, we see that

$$n = 2(k-1).$$

Since $k - 1 \in \mathbb{Z}$, we know n must be even.

2. Let T be a tree with at least 3 vertices. Assume that every vertex in T has either degree 3 or is a leaf. Let L be the set of leaves of T and let R be the set of vertices in T that have degree 3. Show that

$$|R| \;=\; |L| - 2$$

Answer

The total number of vertices is the number of the leaves ($|L|$) plus the number of the vertices in T with degree three ($|R|$). We thus see that:

$$|V| = |L| + |R|$$

We can find the total degree of the tree as follows:

$$\begin{aligned}\sum_{v\in V} \deg(v) &= \sum_{v\in L} \deg(v) + \sum_{v\in R} \deg(v)\\ &= \sum_{v\in L} 1 + \sum_{v\in R} 3\\ &= |L| + 3|R|\end{aligned}$$

However, we also know by the Handshaking Lemma that:

$$\begin{aligned}\sum_{v\in V} \deg(v) &= 2|E|\\ &= 2\left(|V| - 1\right) && (|E| = |V| - 1, \text{ as } T \text{ is a tree})\end{aligned}$$

Combining these two facts, we conclude that:

$$\begin{aligned}
|L| + 3|R| &= 2(|V| - 1) \\
|L| + 3|R| &= 2(|L| + |R| - 1) && \text{(plugging in } |V| = |L| + |R|\text{)} \\
|L| + 3|R| &= 2|L| + 2|R| - 2 \\
|R| &= |L| - 2
\end{aligned}$$

3. Prove **by induction** that any tree with at least 3 vertices must have at least one vertex of degree ≥ 2. (Only proofs by induction will receive credit.)

   Answer

   (BASE CASE): $n = 3$. We note that there is only one tree with 3 vertices, P_3. We see that P_3 has an internal vertex with degree 2.

   (INDUCTION STEP): Let $k \in \mathbb{Z}^+$, $k \geq 3$. Assume **(IH)** that any tree with k vertices must have at least one vertex of degree ≥ 2. We now wish to show that any tree with $k+1$ vertices must have at least one vertex of degree ≥ 2.

   Let T be a tree with $k+1$ vertices. Since we know that $k \geq 3$, we have that T has a nonzero number of edges. The lemma on slide 6 of module 12.3 gives us that T has at least one leaf, call it v. Remove v from T to form a new graph T'. By the lemma on slide 7 of module 12.3, T' is a tree with k vertices. By the Induction Hypothesis, we know T' has at least one vertex u with $\deg(u) \geq 2$.

   Now consider adding v back to T' to form T. Adding back a vertex cannot decrease u's degree in T. Therefore, $\deg(u) \geq 2$ still holds and T has at least one vertex with degree ≥ 2 (namely u). Thus, we have shown our claim is true when $n = k+1$, concluding our Induction Step and completing our proof.

4. Let $n \geq 3$ be a positive integer. Consider $K_{3,n}$, the complete bipartite graph with 3 red nodes and n blue nodes.

   (a) Consider a cycle of length 6 in $K_{3,n}$. How many blue nodes must such a cycle have? Briefly explain your answer.

   Answer

   Since $K_{3,n}$ is bipartite, every edge in $K_{3,n}$ must have different-colored endpoints. In other words, the vertices along any walk, including the cycle, must alternate colors. Since there are 6 total vertices in a cycle of length 6, exactly half of the vertices must be blue, for a total of $\boxed{3}$ blue vertices.

   (b) Count the number of paths of length 3 in $K_{3,n}$.

   Answer

   Note that paths of length 3 contain 4 vertices. Observe that the two endpoints must be different colors. Thus, we can count by starting at the red end, and building up the path. We apply the Multiplication Rule:

   Step 1: Choose a red endpoint. (3 ways)
   Step 2: Choose a blue vertex. (n ways)
   Step 3: Choose a different red vertex. (2 ways)
   Step 4: Choose a different blue vertex. ($n-1$ ways)

Observe that each of these steps is valid, since the edges will always exist between selected vertices in $K_{3,n}$. Additionally, we are not missing any paths, since any path of length 3 must be of the form *red-blue-red-blue* (direction does not matter, since we are working in an undirected graph, so it would be the same if you started by choosing a blue vertex instead of a red one). By Multiplication Rule, there are $\boxed{6n(n-1)}$ such paths.

5. Let $G = (V, E)$ be an undirected graph with $|V| = n \geq 3$ vertices and satisfying the following property. For any $u, v, w \in V$, *distinct* vertices, at least one of these three is adjacent to the other two, that is,

$$v\text{–}u\text{–}w \qquad \text{OR} \qquad u\text{–}v\text{–}w \qquad \text{OR} \qquad u\text{–}w\text{–}v$$

(Note that it is also possible that the 3 vertices are pairwise adjacent.) Prove that G has at least $\frac{n^2}{2} - n$ edges.

Answer

We first show that every vertex has degree at least $n-2$. Assume otherwise, that is, there exists a vertex $v \in V$ with at most $n-3$ neighbors. Consider v and two of its non-neighbors. Since v is not adjacent to either of these two vertices, the given condition cannot be satisfied for this group of 3.

From this, we use the Handshake Lemma to show that:

$$\begin{aligned}
|E| &= \frac{1}{2}\sum_{v \in V} \deg(v) \\
&\geq \frac{1}{2}\sum_{v \in V}(n-2) \\
&= \frac{1}{2}n(n-2) \\
&= \frac{n^2}{2} - n
\end{aligned}$$

Alternate Solution:

Let $P(n)$ be:

> Any graph $G = (V, E)$ with $|V| = n$ vertices where, for any three distinct vertices $u, v, w \in V$, at least one of u, v, w is adjacent to the other two, then G has at least $\dfrac{n^2}{2} - n$ edges.

We proceed with a proof by **strong** induction on n (see note below why strong induction is needed).

(BASE CASE): $n = 3, 4$.

For $n = 3$, we observe that picking all three vertices means that at least two edges have to be present. $2 \geq \frac{3^2}{2} - 3 = \frac{3}{2}$, so the condition holds.

For $n = 4$, let the four vertices be a, b, c, d. We then wish to show that any graph satisfying the condition contains at least $\frac{4^2}{2} - 4 = 4$ edges.

Consider the triple $\{a, b, c\}$. By the condition in the problem, at least two edges must occur between these three vertices. First suppose all three edges occur, meaning a, b, c form a triangle. Then consider the triple $\{a, b, d\}$; d must have at least one edge to a or b, so there are at least 4 edges in total.

In the case where there are exactly two edges, suppose WLOG these edges are $\{a, b\}$ and $\{a, c\}$. Then consider the triple $\{b, c, d\}$. By the condition in the problem, at least two other edges exist amongst these three vertices, thus showing us that there are at least 4 edges in total.

(INDUCTION STEP): Let $k \in \mathbb{Z}^+$, $k \geq 4$. Assume **(IH)** that $P(j)$ holds, for all integers $3 \leq j \leq k$ We want to show that $P(k+1)$ holds.

Consider a graph $G = (V, E)$ with $k+1$ vertices where, for any three distinct vertices $u, v, w \in V$, at least one of u, v, w is adjacent to the other two. We consider the following cases:

Case 1: For any pair of distinct vertices $s, t \in V$, the edge $\{s, t\}$ is in G.

In this case, we see that $G \simeq K_{k+1}$. Thus, we know that

$$|E| = \binom{n}{2} = \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2} > \frac{n^2}{2} - n$$

as desired.

Case 2: There exists a pair of distinct vertices $s, t \in V$ such that the edge $\{s, t\}$ is not in G.

Let G' be the graph formed by removing s and t from G. Note that G' is a graph on $k-1$ vertices where between any three distinct vertices u, v, w, at least one of u, v, w is adjacent to the other two. Thus, we can apply our strong IH to determine that G' has at least $\frac{(k-1)^2}{2} - (k-1)$ edges.

Now consider vertices s and t. In order for our condition to be satisfied for G, we know that every other vertex $w \in V \setminus \{s, t\}$ must have edges to both s and t; indeed, if some vertex $x \in V \setminus \{s, t\}$ did not have edges to both s and t, then the triple $\{s, t, x\}$ would violate the condition of the problem. Noting there are at least $k-1$ other vertices in G and that every edge in G' also exists in G, we see:

$$\begin{aligned}
|E| &\geq \frac{(k-1)^2}{2} - (k-1) + 2(k-1) \\
&= \frac{k^2}{2} - k + \frac{1}{2} - k + 1 + 2k - 2 \\
&= \frac{k^2}{2} - \frac{1}{2} \\
&= \frac{k^2 + 2k + 1}{2} - \frac{2k + 2}{2} \\
&= \frac{(k+1)^2}{2} - (k+1)
\end{aligned}$$

as desired.

Thus, we have shown our claim is true when $n = k+1$ in both cases, concluding our Induction Step and completing our proof.

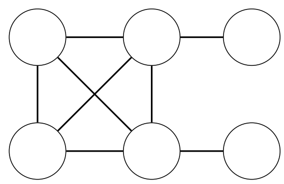
Note: In order to show the above claim, we needed a strong Induction Hypothesis; in other words, we showed that $\forall k \in \mathbb{Z}^+, k \geq 4, ((P(k-1) \land P(k)) \implies P(k+1))$. One may wonder if we could have performed the same proof using ordinary induction instead. Say we removed a single vertex, say v, from G to get a graph G' on k vertices, which by the IH has at least $\frac{k^2}{2} - k$ edges. In order to complete the proof, we would need to be able to show that v has at least $k - \frac{1}{2}$ edges, since

$$\frac{k^2}{2} - k + \left(k - \frac{1}{2}\right) = \frac{k^2}{2} - \frac{1}{2} = \frac{(k+1)^2}{2} - (k+1)$$

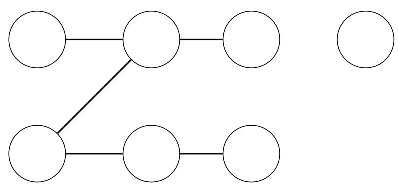
similar to above. However, this would mean that we would need to show that v has exactly k edges, since the degree of any vertex must be an integer. However, this is not necessarily always the case – there are examples of graphs satisfying the property in our claim where no vertex has degree $n-1$. A simple example is C_4. Thus, we know that we cannot actually complete this proof, and must use strong induction instead.

6. In this problem the graphs are undirected.

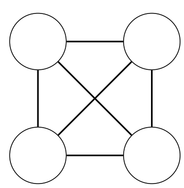
   (a) Draw a connected graph with 6 nodes and exactly 2 cut edges.



   (b) Draw an acyclic graph with 5 edges and 7 nodes.



   (c) Draw a connected graph in which every node has degree 3.



7. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a *very brief* explanation of your answer.

   (a) There exists a tree in which every node is a leaf.

   (b) The complete bipartite graph $K_{5,5}$ has a cycle of length 5.

   (c) There exists an undirected graph with 23 vertices such that 11 of them have degree 11 and 12 of them have degree 12.

   (d) There exists a connected undirected graph with 100 vertices and 50 edges.

   (e) A graph has an edge that is not a cut edge. Then it must have three edges that are not cut edges.

Answer

(a) $\boxed{TRUE.}$ An example of this is a graph G with only two vertices connected by an edge. G is connected and acyclic, so it is a tree. In addition, the two vertices each have degree 1, so each node is a leaf.

(b) $\boxed{FALSE.}$ As proved in lecture, a graph is bipartite iff it has only even length cycles (regardless of whether or not the graph is complete). Hence, if $K_{5,5}$ has an odd cycle, it cannot be bipartite.

(c) $\boxed{FALSE.}$ We know by the Handshaking Lemma that there must be an even number of vertices with odd degree. Since 11 vertices have degree 11, this would violate the aforementioned property, so it would not be a valid graph.

(d) $\boxed{FALSE.}$ We know that a tree is a minimally connected graph with n vertices and $n-1$ edges. This would mean that any connected graph with n vertices will have at least $n-1$ edges. $100-1 = 99 > 50$. Thus, a graph with 100 vertices and 50 edges would not be connected. Alternatively, we also know a graph has at least $|V|-|E|$ connected components. This graph has at least $100-50 = 50 > 1$ CCs which means it is not connected.

(e) $\boxed{TRUE.}$ If a graph has a non-cut edge, that would mean it is not minimally connected, so it must have a cycle. The smallest cycle we can create in a graph is a cycle of length 3. Each of the 3 edges in this cycle are not cut edges since removing any of them will still leave the graph connected.

8. Let G be a bipartite graph in which every connected component is a cycle.

   (a) Draw the smallest such G. (Just the drawing, no need for explanation)

   Answer

   Answer $G = (\{r_1, r_2, b_1, b_2\}, \{r_1\text{-}b_1, b_1\text{-}r_2, r_2\text{-}b_2, b_2\text{-}r_1\})$

   (b) Prove that, not just in the smallest, but in *any* such G the number of red nodes is equal to the number of blue nodes.

   Answer

   Answer Recall from lecture that a graph G is bipartite if and only if it does not contain any odd length cycles. Then, every connected component in the graph G in this problem must be an even length cycle. If we show that an arbitrary connected component in G has an equal number of blue nodes and red nodes, we can take the sum over all components and get our desired result.

   Consider an arbitrary connected component in G, call it C. We know that C is an even length cycle; suppose that C has $2n$ edges, for some $n \geq 2 \in \mathbb{Z}$. Since C is a cycle with $2n$ edges, we know that C must have $2n$ vertices. Call these vertices $v_1, v_2, ..., v_{2n}$, where v_i is adjacent to v_{i+1} $\forall i \leq 2n-1$, and v_{2n} is adjacent to v_1. Without loss of generality, suppose that v_1 is red. Then, since an edge exists between v_1 and v_2, v_2 must be blue. Continuing, we require that all odd indexed vertices be red and all even indexed vertices be blue for G to be bipartite. Note that there are n odd indexed vertices and n even vertices, so the number of blue vertices in G is n and the number of red vertices in G is also n. Then, there is an equal number of blue and red vertices in any given connected component of G. Taking the sum over all connected components in G proves our claim.

9. Recall that $K_{m,n},\ m, n \geq 1$ is the complete bipartite undirected graph with m vertices colored (say) red, and n vertices colored (say) blue and with edges between any red node and any blue node.

(a) Draw $K_{3,2}$. Name the red vertices r_1, r_2, r_3 and the blue vertices b_1, b_2.

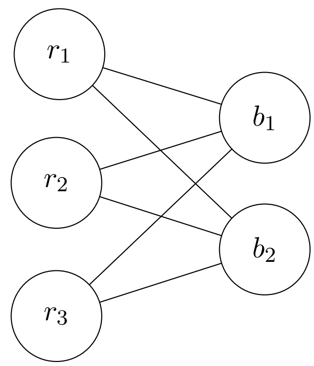
(b) Draw a spanning tree of $K_{3,2}$ using the same vertex names as in part 9a.

(c) How many edges do we have to delete from $K_{m,n}$ so we are left with a spanning tree? Give the answer in terms of m and n and a short explanation of how you obtained it.

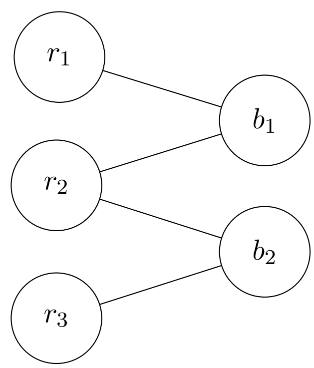
(d) Assume that $m, n \geq 2$. What is the biggest length that a cycle can have in $K_{m,n}$? Give the answer in terms of m and n and a short explanation of how you obtained it.

ANSWER

(a)



(b)



(c) There are mn edges in $K_{m,n}$ since there is exactly one edge for each pairing of a red with a blue vertex, and there are mn such pairs. For any tree, the number of edges is one less than the number of vertices, so since any spanning tree of $K_{m,n}$ has $m+n$ vertices, it must have $m+n-1$ edges. Thus, we must delete $mn-(m+n-1) = (m-1)(n-1)$ edges. Any cycle in $K_{m,n}$ must alternate between red and blue vertices.

(d) Since we cannot repeat vertices (except the first and last), the length cannot be more than double the number of red vertices or more than double the number of blue vertices, so we cannot have a length that is more than $\min(2m, 2n) = 2\min(m, n)$.

To show that this length is indeed attainable, we consider the cycle r_1-b_1-r_2-b_2-$\cdots$-r_k-b_k-r_1 where $k = \min(m, n)$.

10. There exists a tree with exactly 3 leaves, in which the length of a longest path is 1000, true or false?

ANSWER

ANSWER $\boxed{TRUE.}$ Consider the distinct vertices

$$a, b, v_1, v_2, \ldots, v_{999}, v_{1000}$$

and the edges

$$a\text{–}v_1, b\text{–}v_1, v_1\text{–}v_2, \ldots, v_{999}\text{–}v_{1000}$$

This a tree with leaves a, b, v_{1000}, and there are two longest paths, both of length 1000.

11. Consider a connected graph $G = (V, E)$ such that $|E| = |V|$. Prove that G contains *exactly* one cycle.

    ANSWER

    As the graph G is connected, we know it has at least one spanning tree T that we can arbitrarily select. From the definitions of a tree proved in lecture, we know T to have $n - 1$ edges from the original graph, to be connected, and to be acyclic. From this, we know there is only one edge e that isn't included in the spanning tree, but is in the original graph. Thus, if there was a cycle to exist in the original graph, it would have to include the edge e, as there are no cycles in the spanning tree by definition.

    Looking at the two vertices adjacent on e, v_a and v_b, we know from the connectivity of the spanning tree that there is a path connecting the two vertices. Combining this path with the edge e, we can see that a cycle forms as there is a path from a vertex to itself. Furthermore, we know that there can be no more cycles that form, as otherwise, there would have had to have been a different path connecting v_a and v_b (as the new cycle would have to be distinct), which would contradict our definition of a tree as being acyclic (as two unique paths connecting two vertices implies a cycle to already exist without e).

    As such we have proven the desired claim.

12. Consider an *undirected* graph with 3 or more vertices and with *exactly* 3 connected components. In order to make this into a connected graph we must add at least 2 edges, true or false?

    ANSWER

    $\boxed{TRUE.}$ In order to make the graph connected, we need to find a way to create paths between the vertices in each of the three components. Call the connected components A, B, and C. We can add an edge between any vertex in A and any vertex in B. This create paths between all of the vertices in A and B via the edge we just added. However, the graph is not connected yet because the vertices in A and B cannot reach the vertices in C.

    We can add another edge between any vertex in C and any vertex in B (or A) to create paths between the two connected components. Now, we have connected all three connected components with only two edges. Note that you can add more edges to the graph, but this would not affect the connectedness of the graph once we add the two edges as described above.

13. Define the complement of a string of bits w of length $n \geq 1$ to be the string obtained by replacing all the 0's in w with 1 and all the 1's with 0's. Clearly, the complement of the complement of w is w, Now construct an undirected graph whose nodes are all the strings of bits of length n and such that

there is an edge u-v exactly when v is the complement of u (equivalently, when u is the complement of v). Prove that the resulting graph is bipartite.

Answer

Let $G = (V, E)$ be the graph that was constructed given the constrains above. In order to show that it is bipartite, we will show a partition of V into two sets, X and Y, such that for any two vertices in X or Y, there is no edge between them. We can do this as follows: for each string w, place it in X iff its complement is not in X, and place it in Y otherwise. We will now prove that all the edges in G are in the form (x, y) such that $x \in X$ and $y \in Y$. Assume toward contradiction that there exists an edge (u, v) such that both $u, v \in X$ or both $u, v \in Y$.

Case 1: $u, v \in X$

By our construction of X, no two strings in X are complements of each other. This is because we place a string w in X iff its complement is not in X. Therefore, if an edge exists such that both $u, v \in X$, this means that there is an edge between two strings that are not complements of each other, a contradiction.

Case 2: $u, v \in Y$

We will also prove that no two strings in Y are complements of each other. Consider, during our construction, an instance where we are adding a string w such that the complement of w (denoted $\overline{w}$) is already contained in Y. Since $\overline{w} \in Y$, $\overline{w} \notin X$, as X and Y are disjoint by our construction. Therefore, w must be added to X and not Y, due to the condition that a string is added to X iff its complement is not in X. Using this fact, if an edge existed such that both $u, v \in Y$, then there would be an edge between two strings that were not complements of each other, a contradiction.

Therefore, since we have shown a contradiction in both cases, we know that there cannot exist an edge (u, v) such that both $u, v \in X$ or both $u, v \in Y$. Therefore, all edges are between X and Y, and thus the graph is bipartite.

14. Let X, Y, Z be three finite nonempty sets such that $X \cap Y = \emptyset$, $Z \cap Y = \emptyset$, $X \cap Z = \emptyset$ and denote $|X| = m$, $|Y| = n$, $|Z| = p$. Assume that $m < n < p$. Let also $f : X \to Y$ and $g : Y \to Z$ be two functions. Consider the undirected graph $G = (V, E)$ where $V = X \cup Y \cup Z$ and

$$E = \{ \ \{x, f(x)\} \mid x \in X\} \cup \{ \ \{y, g(y)\} \mid y \in Y\}$$

(a) What is $|V|$ and what is $|E|$ (in terms of m, n, p)?.
(b) What is the maximum number of nodes of degree 0 that G can have (in terms of m, n, p)?.
(c) What is the minimum number of nodes of degree 0 that G can have (in terms of m, n, p)?.
(d) What is the maximum length that a path in G can have?
(e) Prove that G is acyclic.

Answer

(a) Since X, Y, Z are disjoint, We know

$$|V| = |X \cup Y \cup Z| = |X| + |Y| + |Z| = m + n + p.$$

Since f, g are functions there is exactly one edge of the form $\{x, f(x)\}$ for each $x \in X$ and exactly one set of the form $\{y, g(y)\}$ for each $y \in Y$. Therefore $|E| = m + n$.

(b) Since every node in X and Y is the endpoint of an edge only nodes from Z can have degree 0. They correspond to elements *outside* of the range of g To maximize their number we must *minimize* the size of $g(Y)$. This happens when g maps all the elements of Y to one element of Z. Hence, the answer is $p-1$.

(c) Reason like in part (b) then *maximize* the size of $g(Y)$. This happens when g is injective and the size of $g(Y)$ is n. The answer is $p-n$. (Note that we need $n \leq p$ or else we cannot define g to be an injection.)

(d) First we observe that X is nonempty so either $m=1$ or $m \geq 2$. We now consider the maximum length path in each case.

<u>CASE 1</u> $m=1$:
Let $X=\{a\}$. Since $n>m$ we have $n \geq 2$ so let $b_1, b_2 \in Y$ with $b_1 \neq b_2$. Define $f(a)=b_1$ and $g(b_1)=g(b_2)$. Then we have a path of length 3 in G:

$$a - f(a) = b_1 - g(b_1) = g(b_2) - b_2.$$

We now prove that there is no path of length 4. Since $a \notin Y \cup Z$, there is exactly one edge incident to the a. In any path of length 4, therefore, there must be 3 edges between Y-nodes and Z-nodes. Then two of these edges would be incident to the same Y-node. Since g is a function, the edges would be incident to the same Z-node. We cannot repeat edges, however, so no such path exists. Hence, when $m=1$, the maximum path length is 3.

<u>CASE 2</u> $m \geq 2$:
Let $a_1, a_2 \in X$ with $a_1 \neq a_2$. Since also $n \geq 2$ let $b_1, b_2 \in Y$ with $b_1 \neq b_2$. Define $f(a_1)=b_1$ and $f(a_2)=b_2$ and $g(b_1)=g(b_2)$. Then we have a path of length 4 in G:

$$a_1 - f(a_1) = b_1 - g(b_1) = g(b_2) - b_2 = f(a_2) - a_2.$$

We now prove that no path of length 5 can exist. Every edge either connects an X-node and Y-node or a Y-node and Z-node. By PHP, any path P with 5 edges must have at least three 3 edges between X-nodes and Y-nodes or between Y-nodes and Z-nodes. In either case, at least 2 of the 3 edges would have to share an X-node (in the first case) or a Y-node (in the second case). Since f and g are functions, these two edges would also be incident to the same Y node (in the first case) or Z node (in the second case). This means the two edges are equal, which is a contradiction. Hence, when $m \geq 2$ the maximum path length is 4.

(e) Assume, for the sake of contradiction, that a cycle C exists. Since f maps each X-node to one Y-node, the degree of each X-node is 1. Any vertex in a cycle must have at least two incident edges. Thus, cycle C cannot contain an X-node.

Now, assume there is a Y-node y in C. From above, we know that C contains no X-nodes, so we know y is incident to some Z-node z in C. Vertex y is only incident to one Z-node because g is a function. Since C does contain any X-nodes, there is no edge that can return to y.

Thus, cycle C must only contain Z-nodes. This, however, is impossible because there is no edge between two Z-nodes. Hence, no cycle exists, and G must be acyclic.

15. Prove that a connected undirected graph in which every node has exactly degree 2 is a cycle. (**Hint:** use induction on the number of vertices.)

    ANSWER

    A graph in which every node has degree 2 is called 2-regular (see the next problem). Let $G = (V, E)$ be such a graph. By the handshake lemma $2|E| = 2|V|$ therefore $|E| = |V|$. We will make use of this observation later.

    (Since $|E| = |V|$ you might be tempted to invoke Problem 11 on the mock exam above. But that only gives us that the graph contains exactly one cycle, not that *it is* a cycle!)

    So we need to show that any connected 2-regular graph is a cycle. We shall do so by induction on the number n of vertices.

    For $n = 1$ and $n = 2$ the statement holds vacuously because graphs with one or two nodes cannot be 2-regular.

    (BASE CASE): $n = 3$. Since it also has 3 edges the only possibility is K_3, a triangle, hence a cycle.

    (INDUCTION STEP): Let $k \geq 3$ be an arbitrary natural number. Assume **(IH)** that any 2-regular graph with k nodes is a cycle. Now let $G = (V, E)$ be a connected 2-regular graph with $k + 1$ nodes. We want to show that G is a cycle.

    Pick a node $u \in V$. We have exactly two nodes adjacent to u in G call them v and w. Since $|V| \geq 4$ we must have yet another node in G, call it z.

    (CLAIM) v–$w \notin E$
    (PROOF OF CLAIM) By contradiction. If v–$w \in E$ then u, v, w are all adjacent to each other and not adjacent to any other node in V (because the graph is 2-regular). Therefore there cannot exist a walk from z to, say, v and thus G is not connected, contradiction.

    Now we construct from G a new graph G' by deleting the node u, the edges u–v and u–w, and then adding an edge v–w. Since v and w still have degree 2 in G', G' must be 2-regular. Moreover, G' is connected. Indeed, any walk in G that passed through u must have used the edges v–u and u–w. These edges, as well as u are gone, but we can "repair" the path in G' by using the edge v–w.

    It follows that G' is a connected 2-regular graph with k nodes. By IH it is a cycle. Now we go back to G from the cycle G' by deleting the edge v–w and adding instead the vertex u and edges v–u and u–w. Clearly this constructs a cycle.

16. An r-regular graph is a graph in which the degree of each vertex is exactly r. Show that any 3-regular graph must have an even number of vertices and a number of edges divisible by 3.

    ANSWER

    Let $G = (V, E)$ be a 3-regular graph. As we have shown in lecture using the handshake lemma, every graph has an even number of vertices of odd degree. Since, every vertex in G has degree 3 by the definition of 3-regular, every vertex in G has odd degree. With this we can say that the graph must have an even number of vertices because there are only odd degree vertices.

    Also, by the handshake lemma we know that $2|E| = \sum_{v \in V} deg(v)$. Every, vertex in G has degree 3, substituting into the above formula, we get that $2|E| = 3|V|$. We know that $|E|$ is the number of edges in G, and since 2 is not divisible by 3, we can say that the number of edges must be divisible by 3.

Now, we have show that the for any 3-regular graph G there are an even number of vertices, and a number of edges that is divisible by 3, and we are done.

17. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a *very brief* explanation of your answer.

    (a) The complement of the path graph P_4 is isomorphic to P_4.

    Answer

    TRUE. If you draw out P_4 and it's complement, you can make a bijection of vertices and edges to satisfy the definition of isomorphism.

    (b) Let G be a DAG with $n \geq 2$ vertices and let the sequence σ be a topological sort of G. If u appears before v in σ then there exists a directed path from u to v in G.

    Answer

    FALSE. A simple counter example is a DAG where there are no edges. Any sorting order of the vertices is thus a valid topological sort, but there does not exist a directed path from any vertex to another vertex.

    (c) A strongly connected digraph with at least two nodes can have neither sources nor sinks.

    Answer

    TRUE. In a strongly connected digraph, there exists a directed path from every vertex to every other vertex. In a strongly connected digraph with at least two nodes, take any two nodes. If there is a strong directed path from the first node to the second and vice versa, the first and second node but have at least outdegree one and indegree one, which mean they can neither be sources nor sinks. This holds for any two vertices and thus all vertices.

18. For the three parts below, use the following definition: for any digraph $G = (V, E)$ *without self-loops* and *without cycles of length 2* define an undirected graph $G^u = (V, E^u)$ that has the same vertices as G and moreover in G^u we have an edge v–w whenever we have the edge $v{\to}w$ or the edge $w{\to}v$ in G.

    (a) If G is strongly connected then G^u is connected. Prove or disprove.

    Answer

    TRUE. Let v, w be two vertices in G^u (hence in G). Since G is strongly connected, there exists a directed walk $v{\to}\cdots{\to}w$ in G. Erasing the direction of the edges in this walk gives a walk $v\text{–}\cdots\text{–}w$ in G^u.

    (b) If G is a DAG then G^u is acyclic. Prove or disprove.

    Answer

    FALSE. Counterexample: $G = (\{a, b, c\}, \{a{\to}b, a{\to}c, c{\to}b\})$ is a DAG but G^u is a cycle of length 3.

    (c) Prove that if G is a DAG in which every sink is reachable from every source then G^u is connected.

    Answer

Assume G is such a DAG and let's prove that G^u is connected.

Let v, w be two distinct vertices. We want to show that there is a walk $v\text{-}\cdots\text{-}w$ in G^u.

CLAIM In G there exists a source s such that v is reachable from s.

PROOF OF CLAIM Consider the set S of all *paths* in G from some vertex to v. There is at least one such path, the path of length 0. By the Well-Ordering Principle at least one of these paths has maximum length among the paths in S, we call it a *maximal* path.

If this path has length 0 then v itself must be a source, otherwise we can extend the length of the path by including one of v's predecessors. So v is reachable from a source (itself).

Suppose the maximal path is not of length 0, let it be $s \to \cdots \to v$. Then s must be a source. Indeed, if s has a predecessor p then either p is among the vertices of the maximal path (and then G has a cycle, contradiction) or p is not among the path's vertices and then the path is not maximal (also a contradiction) because we can extend it with the edge $p \to s$. So v is reachable from the source s. This ends the proof of the claim.

Similarly, we prove the claim

CLAIM In G there exists a sink t such that t is reachable from w.

So we have a source s and a sink t such that $s \twoheadrightarrow v$ and $w \twoheadrightarrow t$. In addition we know that in G every sink is reachable from every source therefore $s \twoheadrightarrow t$. Now we erase direction on the walks that give $s \twoheadrightarrow v$, $w \twoheadrightarrow t$ and $s \twoheadrightarrow t$. This gives a walk $v\text{-}\cdots\text{-}s\text{-}\cdots\text{-}t\text{-}\cdots w$ in G^u. Done.

19. Recall the complete undirected graph on n vertices, K_n. Prove that for any $n \geq 4$ it is possible to assign direction to each of the edges of K_n such that the resulting digraph has exactly $n-2$ strongly connected components.

ANSWER

SOLUTION ONE We proceed by induction on n.

(BASE CASE:) $n = 4$. If our graph is

$$\big(\{a, b, c, 1\}, \{(a, b), (b, c), (c, a), (a, 1), (b, 1), (c, 1)\}\big),$$

we can see that the $n-2 = 2$ strongly connected components are $\{a, b, c\}$ and $\{1\}$ since the subgraph induced on $\{a, b, c\}$ is a cycle and since a, b, and c are not reachable from 1.

(INDUCTION STEP:) Let $k \in \mathbb{Z}^+$, $k \geq 4$, and assume that there is a digraph $G = (V, E)$ with exactly $k - 2$ strongly connected components that is the result of assigning direction to each of the edges of K_k **(IH)**. Now consider the graph G' formed by adding a vertex v to G and directing edges to v from every other vertex. Since no other vertex is reachable from v, it forms its own strongly connected component, so G' has $(k + 1) - 2$ strongly connected components, and the induction step holds.

SOLUTION TWO Let's name the vertices $1, 2, \ldots, n-3, a, b, c$ (check that there are indeed n vertices). We have three types of edges in K_n: edges between two numbered vertices, edges between two vertices labeled with letters, and edges between one vertex labeled with a number and one with a letter. For the edges with numbers, we orient the edge so that it points from the lower number to the higher number. For the edges with letters, we use the edges (a, b), (b, c), and (c, a). For the edges between letters and numbers, we orient the edge so that it points from the letter to the number.

We claim that each vertex $1, 2, \ldots, n-3$ forms a distinct strongly connected component and that $\{a, b, c\}$ forms another. To prove this, we show that no two vertices from different strongly connected components are mutually reachable from each other.

Since every edge that begins in a numbered vertex ends in a numbered vertex, we cannot have a path from a numbered vertex to a non-numbered vertex, so the lettered vertices are not reachable from the numbered vertices and thus cannot be in the same strongly connected component. In addition, since the only edge from any numbered vertex is to a higher-numbered vertex, we cannot have any paths from a higher-numbered vertex to a lower-numbered vertex, and thus the lower-numbered vertices are not reachable from the higher-numbered vertices; therefore no two numbered vertices are in the same strongly connected component, and since they are all separate from the letters, they each constitute their own strongly connected component. Finally, since a, b, and c are part of the cycle $a \to b \to c \to a$, they are in the same strongly connected component.

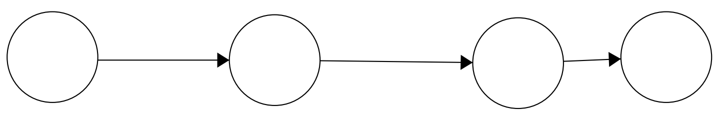
We have shown that the strongly connected components are $\{1\}, \{2\}, \ldots, \{n-3\}, \{a, b, c\}$. Thus there are exactly $n-2$, and we are done.

20. Let's call a *slug* a DAG $G = (V, E)$ with at least 4 vertices, ($|V| \geq 4$), and such that G has exactly one source r and exactly one sink s.

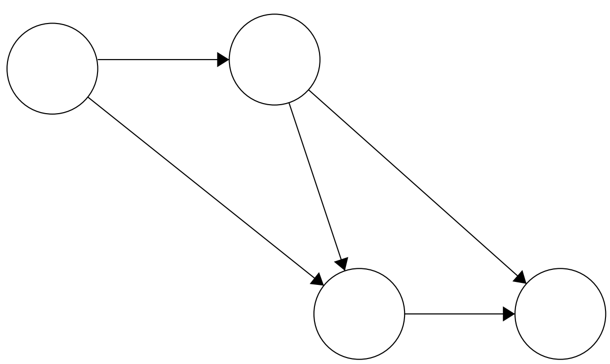
    (a) Draw two different slugs, both with 4 vertices, one of them with 3 edges and the other one with 5 edges.

    Answer

    3 edges:

    5 edges:

    (b) Prove that in any slug, for every node u that is not r or s, there exists a directed path from r to s that passes through u.

    Answer

    First, we must note that there is no possibility of an isolated vertex. If there were an isolated vertex, it would either mean there exists more than one sink or that there is a cycle elsewhere in the graph. Both of these contradict the definition of a slug given in the problem to be a DAG and exactly one source and one sink.

Second, let us show that there is a path from u to s. We know that in a DAG, for every vertex v, there exists a sink t such that there is a path from v to t. This can be done by "following the arrows" from v until you reach a sink. Because a slug by definition has exactly one sink s, we know there must be a path from u to s.

Third, let us show that there is a path from r to u. We know that u is not a source, so it has indegree of at least 1. Similar to the paragraph above, we can "follow the arrows" backwards from u until we reach a node that has indegree 0, at which point we have reached the only sink in this graph, r.

We know there is a path from r to u and a path from u to s. We can now conclude that there exists a path from r to s that passes through u by connecting these two paths.