CIT 592 Spring 2022 Homework 13

Fan Zhang
fanzp@seas.upenn.edu

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1. **[10 pts]** There exists a group of users on LinkedIn with $n \geq 3$ people. This group has a clique of size $n-2$, but does *not* have a clique of size $n-1$. Prove that this group has two distinct independent sets of size 2.

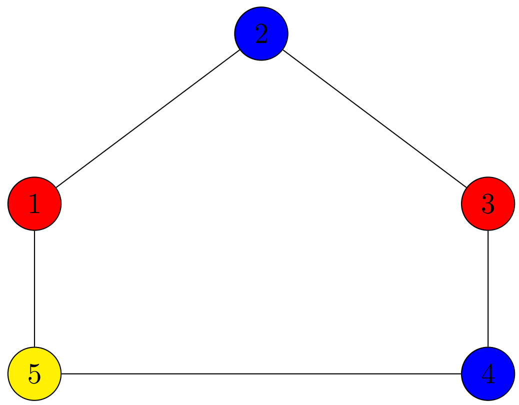
Solution.

We can see the group of users as a graph G with n vertices and the clique of size $n-2$ can be called $G^{'}$.
By definition of clique, we know that $G^{'}$ is a complete subgraph of G with $n-2$ vertices. So we know that there are 2 other vertices in G that are not in $G^{'}$, let's call them u and v.
Since G does not have a clique of size $n-1$, there exists at least a vertex w_1 of $G^{'}$ that is not connected to u. The reason is that if every vertex of $G^{'}$ is connected to u, then $G^{'}$ and u will become a clique of size n-1, which is contradicted to what's given in the question. Therefore u and w_1 is an independent sets of size 2.
Similarly, there exists at least a vertex of w_2 of $G^{'}$ that is not connected to v. Therefore v and w_2 is an independent sets of size 2.
From above we proved that this group has at least two distinct independent sets of size 2.

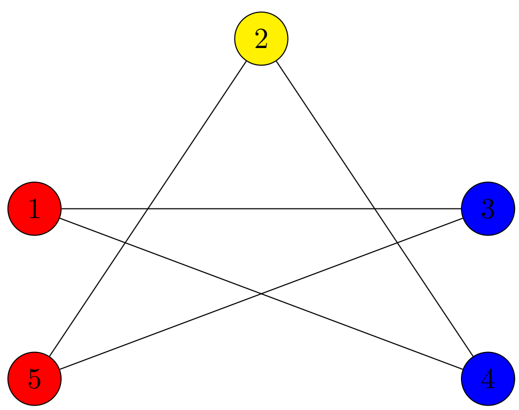
2. [**15 pts**] Show that there is a graph G with exactly 5 nodes where both G and its complement, $\overline{G}$, have chromatic number ≥ 3.

Solution.

Therefore G can be a cycle graph as C_5 as below.
From the lecture slides $\chi(C_n) = 3$ when n is odd. Since 5 is odd and G is a cycle graph, we know that the chromatic number of C_5, which is G, is 3.



By definition $\overline{G}$ can be shown as below, and it also has one cycle $1-3-5-2-4-1$ with all 5 nods in it.
From the lecture slides $\chi(C_n) = 3$ when n is odd. Since 5 is odd and $\overline{G}$ is a cycle graph, we know that the chromatic number of C_5, which is $\overline{G}$, is 3.

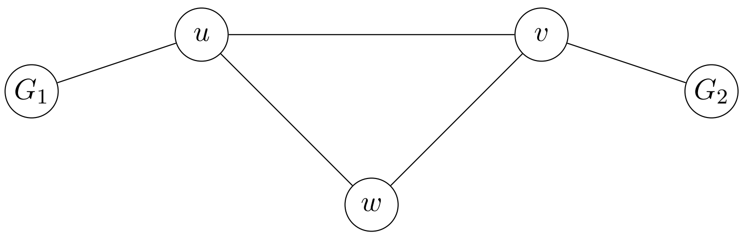


3. **[10 pts]** Suppose that G is a connected graph. It contains *exactly one* spanning tree. Prove that G itself is a tree.

Solution.

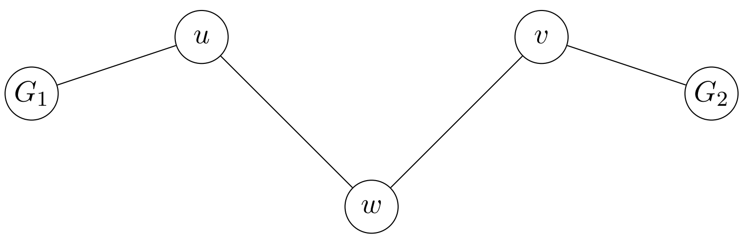
Suppose, toward a contradiction that G is not a tree so it must have at least one cycle. Let's assume G has one cycle u-w-v-u.

Since u-w-v-u is the only cycle let's assume vertex u is connected to G_1, and vertex u is connected to G_2, where G_1 and G_2 both trees. G can be shown as below:

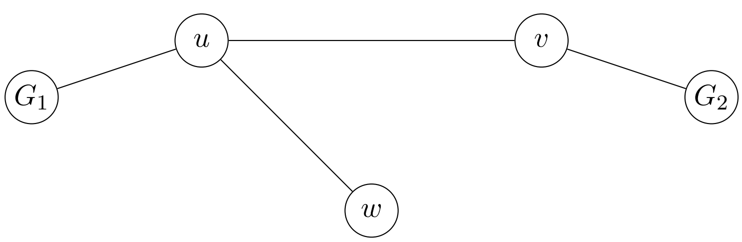


Then by the definition of spanning tree, we will have spanning trees as below:

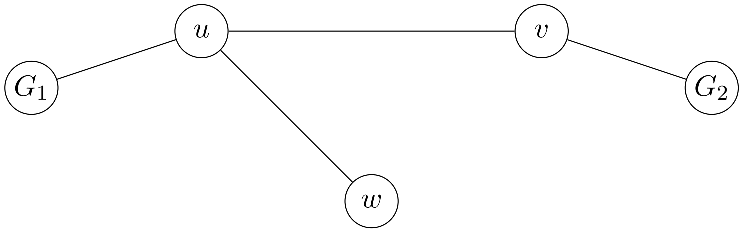
Spanning tree NO. 1:



Spanning tree NO. 2:



Spanning tree NO. 3:

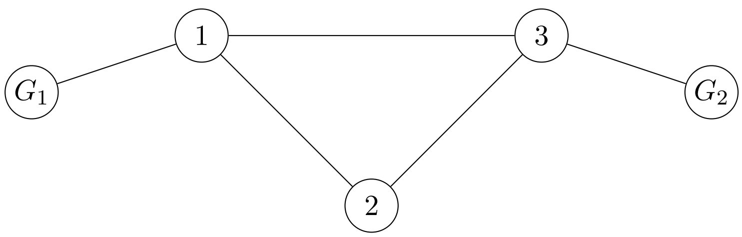


From above we can see that G would have more than 1 spanning tree which is a contradiction. So we proved that if G contains exactly one spanning tree, G is a tree.

4. **[10 pts]** Let G be a connected graph with at least one cycle. Prove the following statement:
We can remove some edges from G such that the resulting subgraph is bipartite and connected.

Solution.

When G has one cycle as below. Since 1-2-3-1 is the only cycle let's assume vertex 1 is connected to G_1, and vertex 3 is connected to G_2, where G_1 and G_2 both trees.
We can remove edge 1-2 or edge 2-3 to break the only cycle in G so that the resulting subgraph G' is a tree.
Since every tree is connected and bipartite, we proved that we can remove some edges from G such that the resulting subgraph is bipartite and connected.



Similarly, when G has 2, 3, $\ldots$, n cycle, we can always remove the edges from those cycles to break the cycles to let the resulting subgraph be a tree. Since every tree is connected and bipartite, we proved that we can remove some edges from G such that the resulting subgraph is bipartite and connected.

5. [10 pts] Suppose G be a connected graph with $n \geq 3$ vertices such that $\chi(G) = 3$. Consider a proper 3-coloring of G with colors, purple, yellow, and orange. Prove that there exists a orange node that has both a purple neighbor and a yellow neighbor.

Solution.

Since $\chi(G) = 3$ we know that G at least has three colors. Since every tree is bipartitie, we know G is not a tree. Since G would be bipartite iff it does not contain a cycle of odd length, we know G has at least one cycle of odd length.
Let's assume C_k is a cycle of odd length in G. The nodes on C_k is called $v_1, v_2, \ldots, v_k$, and the edges of C_k are $v_1 - v_2, v_2 - v_3, \ldots, v_{k-1} - v_k, v_k - v_1$, where k is an odd number since a cycle graph C_k has k vertices and k edges. Since $n \geq 3$, and C_k is a cycle of odd length in G, we know $k \geq 3$. From the lecture slides we know that $\chi(C_k) = 3$ when k is odd.
Since G is a proper 3-coloring, we know that every edge has two different colors.
Suppose, toward a contradiction that all orange nodes have either only purple neighbor or only yellow neighbor.
Situation 1: all orange nodes only have purple neighbors.
Let's say v_1 is orange, v_2 has to be purple. Then v_3 can be either yellow or orange.
When v_3 is yellow, v_4 has to be purple since orange can't have yellow neighbors.
When v_3 is orange, v_4 has to be purple since orange can't have yellow neighbors.
So no matter v_3 is yellow or orange, v_4 has to be purple. Therefore we can tell all the nodes in even position such as $v_2, v_4, \ldots, v_{k-1}$ are purple.
For the nodes in odd position, we already know v_1 is orange, and other odd positions such as as $v_3, v_5, \ldots, v_{k-2}$ can be orange or yellow and at least of them should be yellow in order to make $\chi(G) = 3$.
For v_k, since v_1 is orange, so v_k can't be orange. Since $v_k - v_1$ is proper coloring, v_k has to be yellow, therefore v_1 has to have one purple neighbor v_2 and a yellow neighbor v_k. Contradiction. So orange nodes can not only have purple neighbors.

Situation 2: all orange nodes only have yellow neighbors.
Let's say v_1 is orange, v_2 has to be yellow. Then v_3 can be either purple or orange.
When v_3 is purple, v_4 has to be yellow since orange can't have purple neighbors.
When v_3 is orange, v_4 has to be yellow since orange can't have purple neighbors.
So no matter v_3 is purple or orange, v_4 has to be yellow. Therefore we can tell all the nodes in even position such as $v_2, v_4, \ldots, v_{k-1}$ are yellow.
For the nodes in odd position, we already know v_1 is orange, and other odd positions such as as $v_3, v_5, \ldots, v_{k-2}$ can be orange or purple and at least of them should be purple in order to make

$\chi(G) = 3$.

For v_k, since v_1 is orange, so v_k can't be orange. Since $v_k - v_1$ is proper coloring, v_k has to be purple, therefore v_1 has to have one yellow neighbor v_2 and a purple neighbor v_k. Contradiction.

So orange nodes can not only have yellow neighbors.

Therefore we disproved the contradiction that all orange nodes have either only purple neighbor or only yellow neighbor.

So we proved there exists an orange node that has both a purple neighbor and a yellow neighbor.