Self-paced Example: Floor, Ceiling, and Pigeons

Module 5

MCIT Online - CIT592 - Professor Val Tannen

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion. When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Floor, ceiling, and pigeons

First, let us introduce some notation that will prove useful later.

Ceiling and Floor Notation

Let $x \in \mathbb{R}$. The **ceiling** of x, denoted as $\lceil x \rceil$ is the **smallest** integer z such that $z \geq x$. Note that:

$$x \leq \lceil x \rceil < x + 1$$

The **floor** of x, denoted $\lfloor x \rfloor$ is the **largest** integer z such that $z \leq x$. Note that:

$$x - 1 \ < \ \lfloor x \rfloor \ \leq \ x$$

Examples:

- If $z \in \mathbb{Z}$ then $\lceil z \rceil = \lfloor z \rfloor = z$.
- $\lceil 1/2 \rceil = 1 \qquad \lfloor 1/2 \rfloor = 0$
- $\lceil \pi \rceil = 4 \qquad \lfloor \pi \rfloor = \lceil e \rceil = 3 \qquad \lfloor e \rfloor = 2$

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Floor, ceiling, and pigeons (continued)

We can now provide an alternative formulation for the Generalized Pigeonhole Principle that we learned in the lecture segment "The pigeonhole principle."

Theorem. If r objects are placed into n boxes then there is at least one box containing at least $\lceil \frac{r}{n} \rceil$ objects.

Proof. We prove the contrapositive.

That is, we will show that if each box contains at most $\lceil \frac{r}{n} \rceil - 1$ objects, then the total number of objects is not equal to r.

Assume that each box contains at most $\lceil \frac{r}{n} \rceil - 1$ objects. Then the total number of objects is at most

$$n \left(\left\lceil \frac{r}{n} \right\rceil - 1 \right) < n \left(\frac{r}{n} + 1 - 1 \right) = r$$

Thus we have shown that the total number of objects is strictly less than r.

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Floor, ceiling, and pigeons (continued)

Let's try a harder problem involving PHP.

Problem. A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but, in order not to tire himself, he decides not to play more than 12 games during any calendar week. Show that there exists consecutive days during which the chess master will have played exactly 21 games.

Answer. Let a_i, $1 \leq i \leq 77$, be the total number of games that the chess master has played during the first i days. Note that the sequence of numbers $a_1, a_2, \ldots, a_{77}$ is a strictly increasing sequence.

We have $\quad 1 \leq a_1 < a_2 < \ldots < a_{77} \leq 11 \times 12 = 132.$

Now consider the sequence $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$. We have

$$22 \leq a_1 + 21 < a_2 + 21 < \ldots < a_{77} + 21 \leq 153$$

Clearly, this sequence is also a strictly increasing sequence.

The numbers $a_1, a_2, \ldots, a_{77}, a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ (154 in all) belong to the set $\{1, 2, \ldots, 153\}$. By the pigeonhole principle there must be two numbers out of the 154 numbers that must be the same.

Since no two numbers in $a_1, a_2, \ldots, a_{77}$ are equal and no two numbers in $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ are equal there must exist i and j such that $a_i = a_j + 21$. Hence during the days $j + 1, j + 2, \ldots, i$, exactly 21 games must have been played.