Questions

This assignment is due in about one week from when the assignment opens. The exact deadline and full instructions for submission are provided in Coursera. To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - **no collaboration is allowed**.

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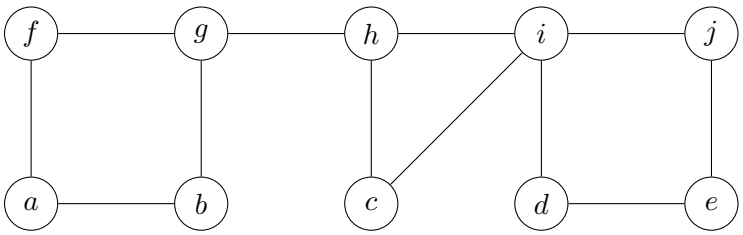
1. [10 pts]



In the graph above, how many connected components are in the subgraph induced by each of the following subsets of the vertices?

(a) $\{a, b, c, d, e\}$

(b) $\{f, g, h, i, j\}$

(c) $\{a, b, e, h, i\}$

(d) $\{c, f, g, i, j\}$

(e) $\{a, c, d, g, j\}$

2. [10 pts] Suppose Kruskal's Kingdom consists of $n \geq 3$ farmhouses, which are connected in a cyclical manner. That is, there is a road between farmhouse 1 and 2, between farmhouse 2 and 3, and so on until we connect farmhouse n back to farmhouse 1. In the center of these is the king's castle, which has a road to every single farmhouse. Besides these, there are no other roads in the kingdom.

(a) Find the number of paths of length 2 in the kingdom in terms of n. Justify your answer.

(b) Find the number of cycles of length 3 in the kingdom in terms of n. Justify your answer.

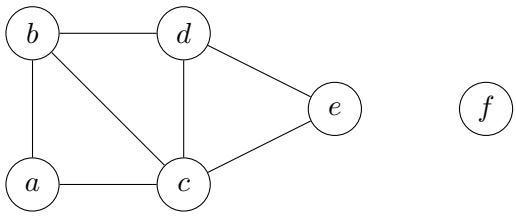
(c) Find the number of cycles in the kingdom in terms of n. Justify your answer.

3. **[10 pts]** Suppose we have a neighborhood of n houses. For any two houses we pick, there is a road between them.

(a) The landlord wants to cut maintenance costs by removing some of the roads. Let k be the minimum number of roads he can remove such that the neighborhood is still connected (every house can be walked to from every other house) and there are no cycles. Determine the value of k as an expression in terms of n. Then indicate how to remove the minimum number of roads from the neighborhood such that the requirements are satisfied.

(b) Suppose now instead that the landlord wants to remove *houses*. Let ℓ be the minimum number of houses that must be removed such that the neighborhood is still connected and has no cycles. Removing a house also removes the roads it is connected to. Determine the value of ℓ as an expression in terms of n. Then indicate how to remove the minimum number of houses from the neighborhood such that the requirements are satisfied.

4. **[10 pts]** We define a graph's *degree sequence* as a list of the degrees of all the vertices in the graph, **in increasing order of degree**. For example, the graph



has degree sequence $(0, 2, 2, 3, 3, 4)$ because there is one node with degree 0 (f), two nodes with degree 2 (a and e), two nodes with degree 3 (b and d), and one node with degree 4 (c).

For each of the following, either list the set of edges of a **tree** with vertex set $\{a, b, c, d, e, f\}$ that has the stated degree sequence, or show that no such tree exists.

(a) $(1, 1, 1, 3, 3, 3)$

(b) $(1, 1, 1, 1, 3, 3)$

(c) $(1, 1, 1, 1, 3, 4)$

5. **[10 pts]** Suppose there exists a connected group of Facebook friends. That is, there is a way to reach every person from every other person through some traversal of friendships. Additionally, there is no cycle of friendships (ex. we would *not* have the following case: 1 is friends with 2, who is friends with 3, who is friends with 1). Suppose that some person u in this group has at least d friends. Prove that there exists at least d people in this group with exactly 1 friend. *Hint:* Think about what specific type of graph this is based on the definition!