Module 1.7: Powerset and Cartesian Product

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Powerset

The **powerset** of a set A is the set whose elements are all the subsets of A.
Notation: 2^A.

Using set-builder notation: $2^A \; = \; \{X \mid X \subseteq A\}$.

Examples:

$$2^{\{1,2,3\}} \; = \; \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}, \emptyset\}.$$

Is $\{1,3\}$ in $2^{\{1,2\}} \cup 2^{\{2,3\}}$? No.

What is the powerset of the set whose only element is the empty set?
$2^{\{\emptyset\}} \; = \; \{\; \emptyset \;,\; \{\emptyset\} \;\}$.

Quiz

Which of the following is **not** an element of the powerset of $\{0, \emptyset, \{0\}\}$?

(A) 0

(B) $\{0\}$

(C) $\emptyset$

(D) $\{\emptyset, \{0\}\}$

Answer

Which of the following is not an element of the powerset of $\{0, \emptyset, \{0\}\}$?

(A) 0

Correct. Any element of a powerset must be a set.

(B) $\{0\}$

Incorrect. Since 0 is an element of the set, $\{0\}$ **is** an element of the powerset.

(C) $\emptyset$

Incorrect. The empty set is an element of any powerset.

(D) $\{\emptyset, \{0\}\}$

Incorrect. This is a subset of the given set hence an element of the powerset.

More Information

The power set is the set of all subsets, so the power set of $\{0, \emptyset, \{0\}\}$ is

$\{\emptyset, \{0\}, \{\emptyset\}, \{\{0\}\}, \{0, \{0\}\}, \{\emptyset, \{0\}\}, \{0, \emptyset\}, \{0, \emptyset, \{0\}\}\}$

Sequences

A **sequence** is an ordered collection of elements, with possible repetitions.

Alternative terminology **list, array, string, tuple, word**.

However, mathematically, these are all the same as sequences.

A sequence has **positions**, 1,2,3, etc. and **length**.

Examples:

Consider the set $\{x, 2, a\}$. The sequences of length 2 whose elements are from this set:

. $xx, x2, xa, 22, 2x, 2a, aa, a2, ax$.

A string of digits of length 6: 737334 (has 7 in position 3)

A word made of letters from the English alphabet:

`floccinaucinihilipilification`.

Tuples, triples, pairs

Sometimes we write sequences as

$(2, a, 2, x)$ instead of $2a2x$

and call them **tuples**, or more specifically, n-**tuples** where n is the length.

$(2, a, 2, x)$ is a 4-tuple.

Triples are the same as 3-tuples.

Pairs are the same as 2-tuples.

In a pair (a, b) we call a the **first component** and b the **second component**.

Cartesian (cross) product

The **cartesian product** (or **cross product**) of two sets A and B is the set whose elements are pairs whose first component is an element of A and whose second component is an element of B.

Notation: $A \times B$.

Using set-builder notation: $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

Examples:

Let's enumerate the set $\{p, q\} \times \{2, 3\}$:

. $\{(p, 2), (p, 3), (q, 2), (q, 3)\}$.

(e, f) is an element of $V \times C$.

$(2, 2)$ is an element of $\mathbb{Z}^+ \times \mathbb{N}$.

If $A \subseteq B$ then $A \times B \subseteq B \times B$.

Penn Engineering

ACTIVITY : Subsets

Name all the subsets of $\{1, 2, 3\}$ containing 2 but not 3.

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : Subsets (Continued)

Answer.

The subsets of $\{1, 2, 3\}$ containing 2 but not 3 are $\{2\}$ and $\{1, 2\}$.

Observe that all the subsets containing 2 are $\{2\}$, $\{1, 2\}$, $\{1, 2, 3\}$, and $\{2, 3\}$. From these subsets, we do not consider the subsets that contain 3, and thus we get $\{2\}$ and $\{1, 2\}$.

Activity : Cartesian Product

Problem. Consider two sets $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Name all the elements of $A \times B$ (the cartesian product of A and B) whose second component is b.

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : Cartesian Product (Continued)

Answer.

The elements of $A \times B$ whose second component is b are:

$$(1, b), (2, b), (3, b).$$

The cartesian product of A and B contains all ordered pairs where the first element is from set A and the second element is from set B, i.e.,

$$A \times B = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b), (1, c), (2, c), (3, c)\}.$$

From these elements the ones whose second component is b are:

$$(1, b), (2, b), (3, b).$$