Module 5.2: Counting Injections

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Lecture Notes

Counting injections I

We already counted the number of **arbitrary** functions: $|B^A| = |B|^{|A|}$.

Problem. Let A be a set with r elements and B be a set with n elements. How many injective functions with domain A and codomain B can be defined?

Answer. By the injection rule, there is no injective function when $r > n$.

Assume $r \leq n$. W.l.o.g., let $A = \{ a_1, \ldots, a_r \}$.
Why w.l.o.g? Because the **number** of functions should not depend on what the elements of A are, just on **how many** there are.

We construct a function $f: A \to B$ in r steps where in step (i) we map a_i to an element that we pick in B, making sure f is injective.

Counting injections II

Answer (continued). We assumed $r \leq n$ and $A = \{ a_1, \ldots, a_r \}$.

We construct an injection $f : A \to B$ in r steps as follows:

(1) Pick an element of B to map a_1 to. Can be done in n ways.

(2) Pick one of the remaining elements to map a_2 to. In $n - 1$ ways.

. . .

(r) Pick one of the remaining $n - (r - 1)$20mm elements to map a_r to. In $n - (r - 1) = n - r + 1$ ways.

This is the same as counting partial permutations of r out of n!

The number of injections is therefore $\frac{n!}{(n-r)!}$.

Counting bijections

Problem. Let A be a set with r elements and B be a set with n elements. How many bijective functions with domain A and codomain B can be defined?

Answer. By the bijection rule, to have any bijective function $f : A \to B$ we must have $r = n$.

Then we can count bijections in the same way we counted injections, except that r is replaced by n.

The number of bijections is the same as the number of permutations of n elements, namely $n!$.

ACTIVITY : Bijections, injections and surjections

Let's assume that A and B have the same nonzero cardinality, n.

How many bijections are there? On the previous slide we showed there are $n!$ bijections.

Similarly, how many injections are there? There are $\frac{n!}{(n-n)!} = n!$ injections, according to how we counted them on a previous slide.

Therefore, there are as many bijections as injections: $n!$.

Question: Does this give a proof of the following?

Proposition If the domain and codomain have the same number of elements then every injection is also a surjection.

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Bijections, injections and surjections (continued)

Answer: Yes!

Let I, S and J be the set of injections, surjections, and bijections, respectively, from A to B.

Then the proposition follows from $J = I \cap S$ that we knew by definition and $|I| = |J|$ that we just observed.

Here are the details:

Since $J \subseteq I$ and $|I| = |J|$, we must have $I = J$.

Thus, every injection from A to B is a bijection, and therefore is also a surjection.

Counting surjections?

First of all, by the surjection rule, to have any surjective functions of domain A and codomain B it must be that $|A| \geq |B|$.

W.l.o.g., assume $A = \{ a_1, \ldots, a_r \}$. We only consider the particular case when B has 2 elements and we have $r \geq 2$. Again w.l.o.g., assume $B = \{0, 1\}$.

We count **complementarily**: we subtract from the total number of functions the number of those functions which are **not surjections**.

If a function $f : A \to B$ is not a surjection there must be some element of B that is not in $\mathrm{Ran}(f)$. Define

$$\begin{aligned} F_0 &= \{f : A \to \{0, 1\} \mid 0 \notin \mathrm{Ran}(f)\} \\ F_1 &= \{f : A \to \{0, 1\} \mid 1 \notin \mathrm{Ran}(f)\} \end{aligned}$$

Now, $F_0 \cup F_1$ is the set of functions $f : A \to \{0, 1\}$ that are not surjections.

How many are there? We need $|\, F_0 \cup F_1 \,|$.

Still counting surjections?

$$F_0 \cup F_1 \quad \text{where} \quad \begin{aligned} F_0 &= \{f : A \to \{0,1\} \mid 0 \notin \mathrm{Ran}(f)\} \\ F_1 &= \{f : A \to \{0,1\} \mid 1 \notin \mathrm{Ran}(f)\} \end{aligned}$$

Lemma. The sets of functions F_0 and F_1 are **disjoint**.

Proof of Lemma. Suppose (toward a contradiction) that there is some $f \in F_0 \cap F_1$. Then neither 0 nor 1 are in $\mathrm{Ran}(f)$. Therefore $\mathrm{Ran}(f) = \emptyset$, which is impossible.

By the Lemma and by the addition rule, $|\, F_0 \cup F_1 \,| = |F_0| + |F_1|$.
There is exactly one function in F_0, the one that maps all a_i's to 1. Similarly for F_1. Therefore $|\, F_0 \cup F_1 \,| = 2$.

And the number of surjections is $2^r - 2$.