OMCIT 592 Module 08 Self-Paced 02 (instructor Val Tannen)

Reference to this self-paced segment in seg.08.03

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion.

When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Counterexamples for independence of three events

In this segment we prove two propositions that were used in the lecture segment "Pairwise and mutual independence" to analyze the subtleties involved in generalizing the concept of independence beyond two events.

Proposition. There exist three events, A, B, C, in some space, such that $A \perp B$, $B \perp C$, $C \perp A$ but $\Pr[A \cap B \cap C] \neq \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$.

Proof. We consider the probability space $(\Omega, \Pr)$ associated with tossing a fair coin twice, independently. Let A be the event that a head is obtained on the first toss, B be the event that a head is obtained on the second toss, and C be the event that either two heads or two tails are obtained.

We are going to show that $A \perp B$, $B \perp C$, $C \perp A$ but $\Pr[A \cap B \cap C] \neq \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$.

Note that $\Omega = \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}$ and that, as we argued in previous examples, the probability space is uniform. We have $A = \{\mathtt{HH}, \mathtt{HT}\}$, $B = \{\mathtt{HH}, \mathtt{TH}\}$, $C = \{\mathtt{HH}, \mathtt{TT}\}$, $A \cap B = \{\mathtt{HH}\}$, $A \cap C = \{\mathtt{HH}\}$, $B \cap C = \{\mathtt{HH}\}$, $A \cap B \cap C = \{\mathtt{HH}\}$. Since the tosses are independent, the probabilities of the relevant events are as follows, and they prove the proposition.

$$
\begin{array}{rclcrclcl}
\Pr[A] & = & 1/2 & \qquad & \Pr[A \cap B] & = & 1/4 & = & \Pr[A] \cdot \Pr[B] \\
\Pr[B] & = & 1/2 & \qquad & \Pr[A \cap C] & = & 1/4 & = & \Pr[A] \cdot \Pr[C] \\
\Pr[C] & = & 1/2 & \qquad & \Pr[B \cap C] & = & 1/4 & = & \Pr[B] \cdot \Pr[C]
\end{array}
$$

$$\Pr[A \cap B \cap C] = 1/4 \neq 1/8 = \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$$

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Counterexamples for independence of three events

Proposition. There exist three events, A, B, C, in some space, such that $\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$ but $A \not\perp B$, $B \not\perp C$, $C \not\perp A$.

Proof. We consider the probability space $(\Omega, \Pr)$ associated with rolling a fair die twice, independently. Consider the following events:
A: the first roll results in a 1, 2, or a 3,
B: the first roll results in a 3, 4, or a 5,
C: the sum of the two rolls is a 9.

We will show that $\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$ but $A \not\perp B$, $B \not\perp C$, and $C \not\perp A$.

Again we have a uniform probability space in which each outcome has probability 1/36. The relevant events and their probabilities are as follows and they prove the proposition

$$\begin{aligned} A &= \{(i,j) \,|\, 1 \le i \le 3 \text{ and } 1 \le j \le 6\} & A \cap B &= \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\} \\ B &= \{(i,j) \,|\, 3 \le i \le 5 \text{ and } 1 \le j \le 6\} & B \cap C &= \{(3,6),(4,5),(5,4)\} \\ C &= \{(3,6),(6,3),(4,5),(5,4)\} & C \cap A &= \{(3,6)\} \;=\; A \cap B \cap C \end{aligned}$$

$$\begin{aligned} \Pr[A] &= 1/2 & \Pr[A \cap B] &= 1/6 \;\neq\; \Pr[A] \cdot \Pr[B] \\ \Pr[B] &= 1/2 & \Pr[B \cap C] &= 1/12 \;\neq\; \Pr[B] \cdot \Pr[C] \\ \Pr[C] &= 1/9 & \Pr[C \cap A] &= 1/36 \;\neq\; \Pr[C] \cdot \Pr[A] \end{aligned}$$

$$\Pr[A \cap B \cap C] \;=\; 1/36 \;=\; \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$$