Module 4.1: Pascal's Triangle

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Lecture Notes

The Binomial Theorem I

Binomial coefficients: $\binom{n}{r}$

Binomial Theorem. For any reals a and b and any natural number n

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$$

Proof. When $n = 0$ we get $1 = \binom{0}{0} \cdot 1 \cdot 1.$

When $n \neq 0$: $(a+b)^n = (a+b)\cdot(a+b) \cdots (a+b)$

We obtain a sum of terms of the form $a^{n-i}b^i$ for various i between 0 and n.

The Binomial Theorem II

Binomial Theorem. For any reals a and b and any natural number n

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Proof (continued). $$(a+b)^n = (a+b)\cdot(a+b) \cdots (a+b)$$

In the resulting sum, **how many times** does each $a^{n-i}b^i$ occur?

There are n factors in the multiplication. To get $a^{n-i}b^i$ multiply a from $n-i$ of the factors and b from the other i.

That is, choose i of the n factors! This can be done in $\binom{n}{i}$ ways. So the coefficient of $a^{n-i}b^i$ is $\binom{n}{i}$.

Pascal's Triangle I

The formulas given by the Binomial Theorem for $n = 0, 1, 2, 3, 4, 5$. Keep in mind that $a^0 = b^0 = 1$.

$$(a+b)^0 = \binom{0}{0} = 1$$

$$(a+b)^1 = \binom{1}{0} a^1 + \binom{1}{1} b^1 = a + b$$

$$(a+b)^2 = \binom{2}{0} a^2 + \binom{2}{1} a^1 b^1 + \binom{2}{2} b^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = \binom{3}{0} a^3 + \binom{3}{1} a^2 b^1 + \binom{3}{2} a^1 b^2 + \binom{3}{3} b^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$

$$\begin{aligned}(a+b)^4 &= \binom{4}{0} a^4 + \binom{4}{1} a^3 b^1 + \binom{4}{2} a^2 b^2 + \binom{4}{3} a^1 b^3 + \binom{4}{4} b^4 = \\ &= a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4\end{aligned}$$

$$\begin{aligned}(a+b)^5 &= \binom{5}{0} a^5 + \binom{5}{1} a^4 b^1 + \binom{5}{2} a^3 b^2 + \binom{5}{3} a^2 b^5 + \binom{5}{4} a^1 b^4 + \binom{5}{5} b^5 = \\ &= a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5\end{aligned}$$

Pascal's Triangle II

Pascal kept just the coefficients and arranged them into a triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \cdots & & & & &
\end{array}
$$

Continuing with $n = 6, 7, \ldots$, the (infinite) result is called **Pascal's Triangle**.

Notice the pattern: $2 = 1 + 1$, $3 = 1 + 2$, $6 = 3 + 3$, $10 = 4 + 6$, etc.

We will prove this!