**MCIT 592 Spring 2022** **Homework 14** **Solutions**

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Problem Set

**1.** [**10 pts**] Suppose $G$ is a digraph with $n \geq 1$ vertices, where $G$ is acyclic.

(a) How many strongly connected components does $G$ have? Justify your answer.

(b) Suppose we add a directed edge from each sink of $G$ to each source of $G$. Let this resulting digraph be named $G'$. How many strongly connected components does $G'$ have? Justify your answer.

**Solution:**

(a) There are $n$ strongly connected components, each corresponding to one of the $n$ vertices. Because $G$ is a DAG, two vertices can never belong to the same strongly connected component. Assume for contradiction that vertices $a$ and $b$ are in the same strongly connected component. That means there is a path from $a$ to $b$ and also from $b$ back to $a$. However, the two paths form a closed walk hence there exists a cycle, contradicting the fact that $G$ is a DAG. Thus, all $n$ vertices must form their own strongly connected component.

(b) There is exactly 1 strongly connected component. We will argue that for any vertex in $G$, there is a path from that vertex to some sink and also there is a path from some source to that vertex. Once we have proven these two facts, we can connect any vertex $a$ to any other vertex $b$ in the following manner: First take the path from $a$ to some sink. Since every sink is connected to every source, go from that sink to a source that can reach $b$. Finally take the path from the source to $b$.

To argue that there is a path from any vertex $v$ to some sink, consider the subset $S$ of natural numbers that are length of directed paths that start in $v$. We have $0 \in S$ corresponding to the length of the path from $v$ to $v$. Moreover, since paths cannot repeat vertices there are only finitely many paths in $G$ therefore $S$ is finite. By the Well-Ordering Principle $S$ has a maximum element $k$. Thus we have a path $p$ that starts in $v$ and whose length is maximum which means that $p$ cannot be extended with another edge. (By the way, such a path is called *maximal*.) It follows that the node at the end of $p$ is a sink.

Similarly we consider the set of lengths of all the paths that end in $v$ and conclude that there is a path from some source to $v$.

**2.** **[10 pts]** As seen in lecture, a rooted tree can be seen as a digraph. More specifically, it is a DAG with the root as the unique source.

Consider the rooted tree $(T, r)$ where $T = (V, E)$ is the undirected tree with nodes $V = \{r, x, y, z\}$, and $r$ is the root. We are given *all* the topological sorts of this DAG:

$$r\ z\ x\ y \qquad\qquad r\ y\ z\ x \qquad\qquad r\ z\ y\ x$$

List the edges in the tree. Justify your answer. You can list the edges as directed or undirected.

**Solution:**

Notice that $z$ follows immediately after $r$ in one of the toposorts (actually in two of them). This must mean that $z$ is a child of $r$. If this were not true, there would be an arrow "pointing backwards" from $x$ or $y$ back to $z$.

Similarly, $y$ must be a child of $r$.

Since the above contains *all* toposorts and $x$ does not follow immediately after $r$ in any of them, $x$ cannot be a child of $r$. That is, $x$ must be preceded by its parent. Since there is one toposort in which $x$ precedes $y$ but $z$ always precedes $x$, it must be the case that $x$ is a child of $z$.

We conclude that $E = \{r{-}z, r{-}y, c{-}x\}$. It's also OK to list the edges as directed: $E = \{r \to z,\ r \to y,\ z \to x\}$

**3.** [**10 pts**] We define an *orientation* of an undirected graph $G = (V, E)$ to be a directed graph $G' = (V, E')$ that has the same set of vertices $V$ and whose set of directed edges $E'$ is obtained by giving each of the edges in $G$ a direction. That is, for each edge $u$-$v$ in $E$ we can put in $E'$ either the directed edge $u{\to}v$ or the directed edge $v{\to}u$, but *not both.*

Let $G = (V, E)$ be an undirected graph with $n \geq 2$ nodes and let $a, b$ be any two nodes in $V$. Prove that $G$ has some orientation that is a DAG in which $c$ is a source and $d$ is a sink.

**Solution:**

We know that a digraph is a DAG if and only if it can be topologically sorted. Consider $G = (V, E)$ and form a permutation $\sigma$ of the vertices in $V$ such that the first element is $c$ and the last element is $d$. Note that any permutation, as long as $c$ and $d$ are in these set locations, works! For each edge $u$-$v \in E$ if $u$ occurs before $v$ in $\sigma$ we put $u{\to}v$ in $E'$, otherwise $v$ occurs before $u$ and we put $v{\to}u$ in $E'$. Clearly the digraph $G'$ is an orientation of the undirected graph $G$. Moreover, by the way we constructed the edges of $G'$, $\sigma$ is a topological sort of $G'$. Therefore $G'$ is a DAG. Finally, we know that $c$ must be a source since it is the first vertex in $\sigma$ so no edges can point towards it. By the same reasoning, $d$ must be a sink since it is the last vertex in $\sigma$ so no edges can point away from it.

**4.** [**10 pts**] Let $G$ be a digraph with $n \geq 2$ vertices. The graph is strongly connected, and *every* node has indegree 1. Prove that $G$ is the directed cycle with $n$ vertices.

**Solution:**

Let $u, v$ be two distinct vertices of $G$. Since $G$ is strongly connected, there exist two paths $u \to w_1 \to \cdots \to w_m \to v$ and $v \to z_1 \to \cdots \to z_n \to u$. Suppose, toward a contradiction, that there exists some $i$ and $j$ such that $w_i = z_j$. Since the indegrees are all 1 it follows that $w_{i-1} = z_{j-1}$. We can similarly backtrack until some $w$ or some $z$ must equal $u$ or $v$ or $u = v$, none of which is possible. Therefore $u \to w_1 \to \cdots \to w_m \to v \to z_1 \to \cdots \to z_n \to u$ is a directed cycle. We will show that that's the whole $G$.

Indeed, assume for contradiction that $G$ had another vertex $x$, distinct from the ones in the directed cycle above. Since $G$ is strongly connected, we must have a path $x \to y_1 \to \cdots \to y_p \to u$. Again, because the indegrees are 1 we must have $y_p = z_n$ then $y_{p-1} = z_{n-1}$ and so on, until $x$ equals one of the vertices in the directed cycle, contradicting our assumption.

**5. [10 pts]** Let $T = (V, E)$ be a tree, and let $r, r' \in V$ be two nodes in the tree. Prove that the height of the tree rooted at vertex $r$, $(T, r)$, is at most twice the height of the tree rooted at vertex $r'$, $(T, r')$.
*Hint: consider using the triangle inequality*

**Solution:**

Let $h$ be the height of the rooted tree $(T, r)$, which is the maximum distance from $r$ to any other node in the tree, and let $h'$ be the height of $(T, r')$, the maximum distance from $r'$ to any other node in the tree. We want to show that $h \leq 2h'$.

Suppose that $\ell$ is a node at the maximum distance from $r$, so that the distance between $r$ and $\ell$ is $h$. We use the following notation $d(r, \ell) = h$. Now, since we know that $h'$ is the height of rooted tree $(T, r')$, we can say that $d(r', \ell) \leq h'$ and $d(r', r) \leq h'$. By the triangle inequality, $d(r, l) \leq d(r', r) + d(r', l) \leq 2h'$. So we have $h \leq 2h'$, which was the desired conclusion.