Self-paced Example: Derangements

Module 5

MCIT Online - CIT592 - Professor Val Tannen

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion. When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Derangements

Recall the derangements (gangsters and hats) problem from the lecture segment "Inclusion-exclusion for cardinality":

Problem. n hat-wearing gangsters (this problem is from the 1930s) leave their distinguishable hats with a restaurant cloakroom attendant. After the meal, the attendant gives them back their hats in such a way that none of the gangsters gets their own hat.

The returned hats form what is called a "derangement" or a "deranged permutation." How many **derangements** are possible?

In the lecture segment where we introduced the problem we did give the answer, but we did not go through the full justification. Here we will work with you through the answer in some detail.

Answer. As was mentioned in the lecture segment already, we start by denoting the gangsters as $G_1, G_2, \ldots, G_n$ and their respective hats as $h_1, h_2, \ldots, h_n$ where G_i's hat is h_i.

Using the above notation, a derangement is a permutation of the set $H = \{h_1, \ldots, h_n\}$ in which h_i does **not** occur in position i for any $i = 1, \ldots, n$.

For example, when $n = 3$ we have only 2 derangements:

$h_2 h_3 h_1$, and $h_3 h_1 h_2$

As you must recall, the lecture segment also included an activity in which you computed the total number of derangements for $n = 4$ obtaining the answer 9.

Here we will show how to count the derangements for n elements, that is, we want to compute the number of permutations of $H = \{h_1, \ldots, h_n\}$ in which h_i does not occur in position i for any $i \in [1 \ldots n]$.

As mentioned in the segment where we introduced the problem, the idea is to count **complementarily**.

Counting derangements the rest of the argument (continued)

We define B_i to be the set of permutations in which h_i **does** occur in position i. It follows that $B_1 \cup \ldots \cup B_n$ is the set of permutations that are **not** derangements.

As we know, the total number of permutations is $n!$. It follows that the total number of derangements is:

$$n! - |B_1 \cup \ldots \cup B_n|$$

Recall that earlier in the module we used the Principle of Inclusion-Exclusion (PIE) to compute the cardinality of the union of two sets, and then you used it to compute the cardinality of the union of three sets. For four sets we may still have the patience to write down the inclusion-exclusion rule (do it!), but for five sets it takes utmost dedication. It is easier to figure out a way to write it in general:

$$|\bigcup_{i=1}^{n} A_i| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subseteq [1 \ldots n] \\ |J = j|}} |\bigcap_{j \in J} A_j|$$

We accept this without proof.

By applying the general formula for PIE we have that:

$$|B_1 \cup \ldots \cup B_n| = |\bigcup_{i=1}^{n} B_i| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subseteq [1 \ldots n] \\ |J = k|}} |\bigcap_{j \in J} B_j|$$

Therefore, the number of derangements is:

$$\begin{aligned} n! - |B_1 \cup \ldots \cup B_n| &= n! - |\bigcup_{i=1}^{n} B_i| \\ &= n! - \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subseteq [1 \ldots n] \\ |J = k|}} |\bigcap_{j \in J} B_j| \end{aligned}$$

Counting derangements the rest of the argument (continued)

Consider any set $J \subseteq [1 \ldots n]$ such that $|J| = k$ and observe that

$$|\bigcap_{j \in J} B_j| = (n-k)!$$

This is because we already know which elements go in the k positions from J and we are left with placing elements in the other $n-k$ positions.

Since there are $\binom{n}{k}$ such sets J, it follows that:

$$\sum_{\substack{J \subseteq [1 \ldots n] \\ |J|=k}} |\bigcap_{j \in J} B_j| = \binom{n}{k}(n-k)!$$

Therefore, the number of derangements is:

$$\begin{aligned}
n! - |B_1 \cup \ldots \cup B_n| &= n! - \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subseteq [1 \ldots n] \\ |J=j|}} |\bigcap_{j \in J} B_j| \\
&= n! - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k}(n-k)! \\
&= n! - \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!(n-k)!}(n-k)! \\
&= n! - \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!}
\end{aligned}$$

Counting derangements the rest of the argument (continued)

By factoring out $n!$ and some algebra we get that the number of derangements is:

$$\begin{aligned} n! - n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} &= n!(1 - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}) \\ &= n!(\frac{(-1)^0}{0!} + \sum_{k=1}^{n} \frac{(-1)^k}{k!}) \\ &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \end{aligned}$$

This formula does not apply only to hats and gangsters, of course!

In general, the number of derangements of a set of n elements is:

$$n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$