Recitation Module 3

Penn Engineering

Topics for Today!

- Stars and bars method
- Growing toolbox:
  - Contrapositive
  - Contradiction
- Truth tables
- Counting anagrams

Stars and Bars

- Count the # of ways to put *n* indistinguishable elements into *r* distinct categories
- *r - 1* bars that divide the *r* categories
- *n + r - 1* slots, choose either *n* slots to be stars or *r -1* slots to be bars
- Formula:

$$\binom{n+r-1}{r-1}.$$

Logical statements

Statement: If it rains, then the ground is wet.

Negation: It rains and the ground is not wet.

Contrapositive: If the ground is not wet, then it did not rain.

What is negation?

Statement	Negation
"A or B"	"not A and not B"
"A and B"	"not A or not B"
"if A, then B"	"A and not B"
"For all x, A(x)"	"There exist x such that not A(x)"
"There exists x such that A(x)"	"For every x, not A(x)"

In logical notation: $\neg(P_1 \vee P_2)$ is $(\neg P_1) \wedge (\neg P_2)$
and $\neg(P_1 \wedge P_2)$ is $(\neg P_1) \vee (\neg P_2)$

These are known as **De Morgan's Laws**.

In logical notation: $\neg(\forall x\ P(x))$ is $\exists x\ \neg P(x)$
and $\neg(\exists x\ P(x))$ is $\forall x\ \neg P(x)$

Contrapositive and Contradiction

Original Statement: If *p,* then *q.*

Contrapositive: If not *q*, then not *p.*

- logically equivalent to the original statement

Contradiction: Assume *p* and not *q*.

Anagrams

- Anagrams are permutations of bags

Consider the bag $\{ n_1 \cdot a_1, \; n_2 \cdot a_2, \ldots, \; n_k \cdot a_k \}$

The number of permutations of this bag is

$$\frac{(n_1 + n_2 + \ldots + n_k)!}{n_1! \cdot n_2! \cdots n_k!}$$

e.g. Consider the word *anagram*: 3 x a's and 4 unique other letters

$$\frac{(3+4)!}{3!\,1!\,1!\,1!\,1!} = \frac{7!}{3!}$$

Question 1

How many integer solutions are there to the equation

$$x + y + z = 8$$

1. If x, y, z are all positive?
2. If x, y, z are all nonnegative?
3. If x, y, z are all AT MOST 5?

Question 1

How many integer solutions are there to the equation

$$x + y + z = 8$$

1. If x, y, z are all positive?
2. If x, y, z are all nonnegative?
3. If x, y, z are all AT MOST 5?

Number of stars will vary but we will always have 2 bars

1. x, y, z all positive ($x + y + z = 8$)

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- Distribute one star to each variable
- Left with 5 stars ($8 - 3*1 = 5$) and 2 bars (since there are 3 variables)
- Thus the answer is (7 choose 2) = 21

2. x, y, z nonnegative $(x + y + z = 8)$

2. x, y, z nonnegative ($x + y + z = 8$)

- In this case, x, y, z can’t be less than 0, but can be 0
- No need to distribute one star to each variable
- Thus we have 8 stars and 2 bars and the answer is (10 choose 2) = 45

3. x, y, z AT MOST 5

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- Assume that x, y, z all equal 5
- 7 “negative numbers to distribute” to get from 3*5 = 15 down to 8
- Thus we have 7 “anti-stars” and 2 bars and the answer is (9 choose 2) = 36

Question 2

For all integers x, there exists a positive integer y such that x + y = 4.

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}^{+} \text{ such that } x + y = 4$$

State the negation and disprove the original statement with a counterexample.

Answer to Question 2

1. Negation: $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}^+ \text{ such that } x + y \neq 4$
2. Counterexample: Consider x = 4.

   General rule for negation: negate all quantifiers and finally negate conclusion.

Question 3

Prove the following statement using contradiction.

"If mn is even and m is odd, then n is even."

Answer to Question 3

Assume (towards a contradiction) that mn is even and m is odd and n is odd. By definition, if m is odd and n is odd, then $m = 2k + 1$ and $n = 2j + 1$ for some integers k and j. It follows algebraically that
$mn = (2k + 1)(2j + 1) = (4kj + 2k + 2j + 1) = 2(2kj + k + j) + 1$

Now k and j are both integers, which means $(2kj + k + j)$ must be an integer. Therefore $mn = 2i + 1$ for some integer i, namely $i = 2kj + k + j$, and thus, by definition, mn is odd. But this contradicts our assumption that mn is even, since mn cannot be both even and odd. Therefore it cannot be the case that mn is even and m is odd and n is odd, so it follows that, if mn is even and m is odd, then n is even, as we wanted to show.

Question 4

Show that

$$(p \implies q) \land (\neg p \implies r) \text{ and } (p \implies q) \lor (\neg p \implies r)$$

Do not have the same truth table.

4. Show these logical statements do not have the same truth table.

$(p \implies q) \wedge (\neg p \implies r)$ and $(p \implies q) \vee (\neg p \implies r)$

p	q	r	$\neg p$	$p \implies q$	$\neg p \implies r$	$(p \implies q) \wedge (\neg p \implies r)$	$(p \implies q) \vee (\neg p \implies r)$

P	Q	$P \implies Q$
T	F	F
T	T	T
F	T	T
F	F	T

Not Equal!

Question 5

How many anagrams of “level” are there

a) In total?
b) Beginning with e?
c) With two l’s adjacent to each other?

5 a) How many anagrams of “level” are there in total?

Answer to Question 5

Part 1, solution 1:

Choose 2 of 5 positions to put l, choose 2 of remaining 3 positions to put e, choose 1 of remaining 1 positions to put v:

$$\binom{5}{2}\binom{3}{2}\binom{1}{1} = \frac{5!}{2!\cdot\cancel{3!}}\cdot\frac{\cancel{3!}}{2!\cdot\cancel{1!}}\cdot\frac{\cancel{1!}}{1!\cdot 0!} = \frac{5!}{2!\cdot 2!\cdot 1!} = 30$$

Part 1, solution 2:

There are 5! different permutations of $l_1e_1v_1e_2l_2$, so we divide by 2! to remove ordering of l's, divide by 2! to remove ordering of e's, and divide by 1! to remove ordering of v's, giving us the same answer.

5 b) How many anagrams of “level” are there beginning with e?

5 c) How many anagrams of “level” are there with 2 l’s adjacent to each other?

Answer to Question 5 (cont.)

Part 2:
Put e at the beginning, then find number of anagrams of lvel, which is 4! / (2! * 1! * 1!)

Part 3:
Treat ll as a single letter, which we'll represent as X. The answer is the number of anagrams of Xeve, which is 4! / (2! * 1! * 1!)