Fan Zhang
fanzp@seas.upenn.edu

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1. [10 pts] Suppose G is a digraph with $n \geq 1$ vertices, where G is acyclic.

(a) How many strongly connected components does G have? Justify your answer.

(b) Suppose we add a directed edge from each sink of G to each source of G. Let this resulting digraph be named G'. How many strongly connected components does G' have? Justify your answer.

Solution.

(a) Every vertex in G is a strongly connected component, and there do not exist two distinct vertices in G are strongly connected, so there are n strongly connected components in G.
Suppose, toward a contradiction that there exist two distinct vertices in G, u and v, are strongly connected. By definition we know that u is reachable from v and v is reachable from u, then we would get a cycle that has u and v in it. However G is acyclic. Contradiction.
Thus we proved there do not exist two distinct vertices in G are strongly connected. Since every vertex in G is a strongly connected components, G has n strongly connected components.

(b) G' has 1 strongly connected component.
Let's assume G' has m sources, $u_1, u_2, \ldots, u_m$ and n sinks, $v_1, v_2, \ldots, v_n$.
Lemma: For any node, k, there should be a path connect k to at least one sink, v_k.
Now we try to prove the lemma:
By the Well-Ordering Principle, there is a direct path of maximum length p. By definition, the direct path of p cannot go further with another edge, which means the end point of

the direct path of p is a sink, v_k.

Also for any node, k, it at least has one source u_k, which is easy to understand because any nodes before k needs to start somewhere and we can track back to one of the sources, u_k.

For any two nodes a and b in G', a can connect to at least one sink, v_a and one source u_a, and b can connect to at least one sink, v_b and one source u_b.

Since each sink has a directed edge to each source, we know there must exist a walk $a \to v_a \to u_b \to b \to v_b \to u_a \to a$.

Therefore G' has 1 strongly connected component.

2. **[10 pts]** As seen in lecture, a rooted tree can be seen as a digraph. More specifically, it is a DAG with the root as the unique source.

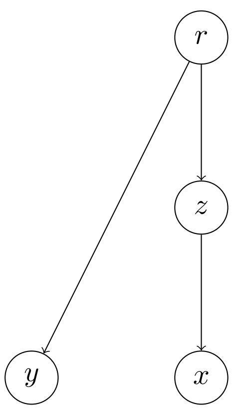
Consider the rooted tree (T, r) where $T = (V, E)$ is the undirected tree with nodes $V = \{r, x, y, z\}$, and r is the root. We are given *all* the topological sorts of this DAG:

$$r\ z\ x\ y \qquad\qquad r\ y\ z\ x \qquad\qquad r\ z\ y\ x$$

List the edges in the tree. Justify your answer. You can list the edges as directed or undirected.

Solution.

We observed that in all the topological sorts of this DAG, z is always in front of x, so z must be parent of x.
Since z can be in front of y or after y, z is not parent of y and z is not child of y.
Since r is the only source, z and y must both be child of r.
Therefore the DAG might look something like below:



Therefore the edges in the tree are $\{r - z, z - x, r - y\}$ or$\{r \to z, z \to x, r \to y\}$.

3. [**10 pts**] We define an *orientation* of an undirected graph $G = (V, E)$ to be a directed graph $G' = (V, E')$ that has the same set of vertices V and whose set of directed edges E' is obtained by giving each of the edges in G a direction. That is, for each edge u–v in E we can put in E' either the directed edge $u \to v$ or the directed edge $v \to u$, but *not both*.

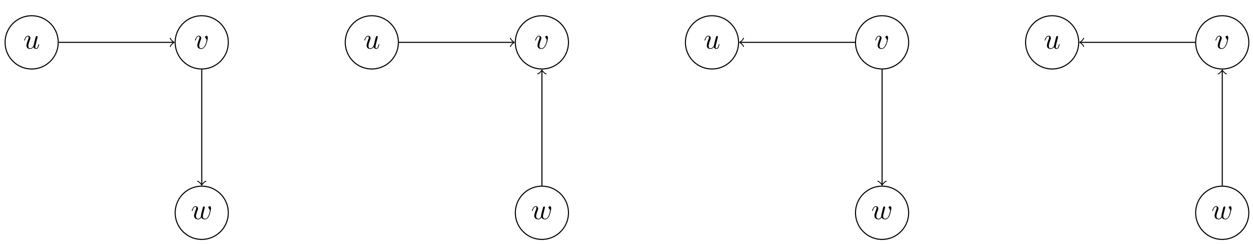
Let $G = (V, E)$ be an undirected graph with $n \geq 2$ nodes and let c, d be any two nodes in V. Prove that G has some orientation that is a DAG in which c is a source and d is a sink.

Solution.

Situation 1: G only has edgeless nodes.
Then any two nodes are a DAG, and they are both a source and a sink, since isolated vertices are both sources and sinks.
Then we proved G has some orientation that is a DAG in which c is a source and d is a sink.

Situation 2: G only has one edge, and $n = 2$, u and v.
Since for each edge it can only have one direction, the edge is either $u \to v$ or $v \to u$. In both cases there is one source and one sink and G' is a DAG. The prove is completed.

Situation 3:
When G has 2 edges and 3 vertices, u and v and w. 2 edges can not make up to a cycle so G' itself must be a DAG. Since for each edge it can only have one direction, the edges will be like below. In all cases there is at there is at least one source and one sink. The prove is completed.



Situation 4:
When G has more than 2 edges, and $n \geq 3$. Let's say it has n vertices, $1, 2, \ldots, c, d, n$.
Situation 4.1:
For any node c and d, if c and d are isolated, per Situation 1 we know that c and d are both sources and sinks since isolated nodes are both sources and sinks. Proof is completed.
Situation 4.2:
If only one of c and d is isolated, let's say only c is isolated, then c is both a source and a

sink. For d, d is in some connected component of G. We can add directions so that d only has indegrees or outdegrees, and every node in the connected component is a strongly connected component which can be treated as a reduced graph. Since reduced graph has no directed cycles, and isolated nodes have no directed cycle, we know it's a DAG.

So we prove that c and d can be seen as a DAG in which c can be seen as a source and d is a sink.

Situation 4.3:

G is a connected.

We can start with any node c, and have a direct path from c to any node d. Then c will be a source and d will be a sink.

Meanwhile for other nodes, we can add directions to the edges of those nodes so that every node is a strongly connected component.

Since every node in G' will be a strongly connected component, we know there is no directed cycle. Therefore G' is a DAG. Proof is completed.

In summary of all situations above, we proved G has some orientation that is a DAG in which c is a source and d is a sink.

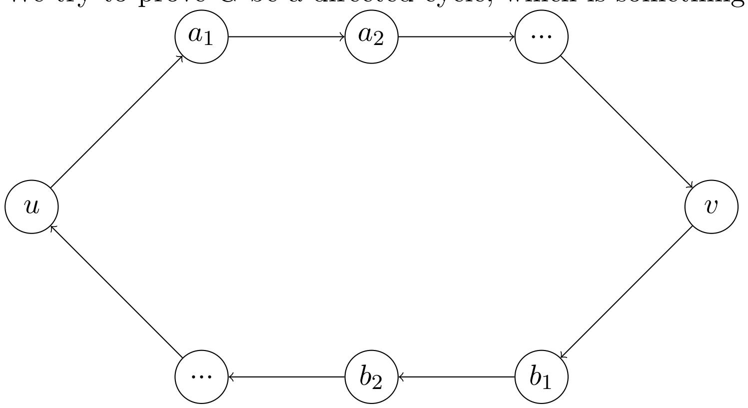
4. **[10 pts]** Let G be a digraph with $n \geq 2$ vertices. The graph is strongly connected, and *every* node has indegree 1. Prove that G is the directed cycle with n vertices.

Solution.

Since G is strongly connected with $n \geq 2$ vertices, and every node has indegree 1, we know G will be a connected digraph without any edgeless vertex.

Since G is strongly connected, G can not be acyclic, therefore G will not a tree.

We try to prove G be a directed cycle, which is something like below:



When $n = 2$, let's say G has 2 nodes, u and v. Since every node has indegree 1, G has two edges, $u \to v$, and $v \to u$. G is a directed cycle with 2 vertices. Proof completed.

When $n > 2$, as shown above, assume we have two distinct nodes in G named u and v.

Since G is strongly connected, we know that u is reachable from v and v is reachable from u, so we know there are two paths $u \to a_1 \to a_2 \to \cdots \to v$ and $v \to b_1 \to b_2 \to \cdots \to u$.

Suppose, toward a contradiction that G is not a directed cycle. Then out of $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ there exists $a_m = b_n$ where m and n are arbitrary.

Since every node has indegree 1, then $a_{m-1} = b_{n-1}$, $a_{m-2} = b_{n-2}$, etc.

Eventually we can trace back to node u and v, and conclude u=v, which is contradict to the assumption that u and v and two distinct nodes. Therefore we know G should be $u \to a_1 \to a_2 \to \cdots \to v \to b_1 \to b_2 \to \cdots \to u$. Thus we proved G is the directed cycle with n vertices.

5. **[10 pts]** Let $T = (V, E)$ be a tree, and let $r, r' \in V$ be two nodes in the tree. Prove that the height of the tree rooted at vertex r, (T, r), is at most twice the height of the tree rooted at vertex r', (T, r').

Hint: consider using the triangle inequality

Solution.

The ratio of the height of the tree rooted at vertex r, (T, r) to the height of the tree rooted at vertex r', (T, r') is the largest when the height of the tree rooted at vertex r, (T, r) is the largest and the height of the tree rooted at vertex r', (T, r') is the smallest.

Situation 1: $n = 1$:

Since $n = 1$, T only has one node, we cannot find $r, r' \in V$ be two nodes in the tree and the height of the tree is always 0. Check.

Situation 2: $n = 2$:

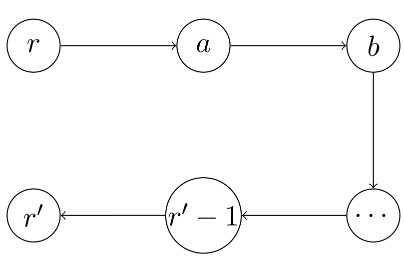
Since $n = 2$, $r, r' \in V$ are the only two nodes in the tree. Therefore the height of the tree rooted at vertex r, (T, r), is the same with the height of the tree rooted at vertex r', (T, r'). The ratio of the height of the tree rooted at vertex r, (T, r) to the height of the tree rooted at vertex r', (T, r') is 1. $1 < 2$, proof is completed.

Situation 3: $n = 3$:

Since $n = 3$, the largest height of the tree is 2 and the smallest height of the tree is 1. Therefore the height of the tree rooted at vertex r, (T, r), is at most twice the height of the tree rooted at vertex r', (T, r'). Proof is completed.

Situation 4: $n > 3$:

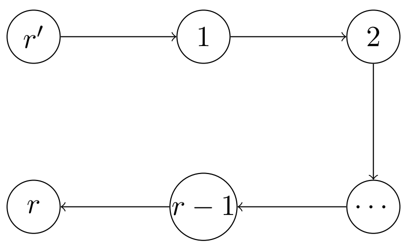
To simplify the question, let's assume T is a tree similarly as below, and r is the first node (the root):



Assume there are n elements in V, so T will have n nodes. The height of the tree rooted at r is the largest when r is the first node, and the height $h_r = n - 1$.

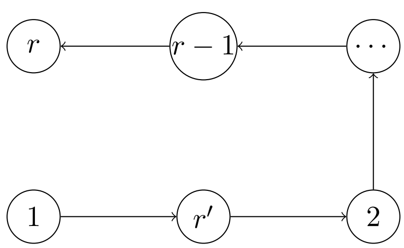
Now we can discuss the height of the tree rooted at r'.

When r' is the first node, $h_{r'} = n - 1$ such as below.



When r' moves down the tree and has 1 node before it, the tree has two leaves, node 1 and node n. The distance from r' to the node r is $n-2$ and the distance from r' to the another leaf, node 1 is 1.

Since $n > 3$, we know $n - 2 > 1$, by definition $h_{r'}$ is the distance from r' to r, so $h_{r'} = n - 2$ such as below.



Similarly when r' keeps move down the tree and there are 2 nodes before r', he distance from r' to the node r is $n - 3$ and the distance from r' to the another leaf, node 1 is 2; when there are 3 nodes before r', the distance from r' to the node r is $n-4$ and the distance from r' to the another leaf, node 1 is 3 etc.

As we move r' down along the tree, the height of the tree would depend on which is the longer distance, the distance between r' and node r or the distance between r' and node 1.

Say r' is in position k of the tree, which means there are k nodes above r'. Then the distance between r' and node 1 would be k, and the distance between r' and r would be $n - 1 - k$.

Therefore we can calculate that the smallest height of tree rooted in r' is when the distance between r' and node r equals the distance between r' and node 1, which is $k = n - 1 - k$.

So we know the smallest height of the tree rooted at vertex r', is $h_{r'} = k = (n - 1)/2$.

Now we know the largest height of the tree rooted at r h_r is $n-1$ and the smallest height of the tree rooted at r' $h_{r'}$is $k = (n-1)/2$.

Therefore the largest ratio of the height of the tree rooted at vertex r, (T, r) to the height of the tree rooted at vertex r', (T, r') is $h_r/h_{r'} = (n-1)/(n-1)/2 = 2$.

Proof is completed.