Module 10.1: Independent Random Variables

MCIT Online - CIT592 - Professor Val Tannen

LECTURE NOTES

Independent random variables

Random variables X and Y defined on the same probability space are **independent**, written $X \perp Y$, when

$$\forall x \in \mathrm{Val}(X) \ \ \forall y \in \mathrm{Val}(Y) \quad (X = x) \perp (Y = y)$$

Example. Green-purple fair dice are rolled, G returns the number shown by the green die, P by the purple die. Then $G \perp P$, because for any $g, p \in [1..6]$:

$$\Pr[(G = g) \cap (P = p)] = 1/36 = (1/6)(1/6) = \Pr[G = g] \cdot \Pr[P = p].$$

When we have three or more r.v.'s, we define again **pairwise** and **mutual independence** analogously.

Independence of indicators and events

Proposition. Let A, B be two events in the probability space $(\Omega, \Pr)$. Then $I_A \perp I_B$ iff $A \perp B$

Proof. First we show $I_A \perp I_B \Rightarrow A \perp B$.

Assume $I_A \perp I_B$. By definition, $(I_A = 1) \perp (I_B = 1)$. Hence $A \perp B$.

Next, we show $A \perp B \Rightarrow I_A \perp I_B$.

If $A \perp B$ then by **Ind (iv)** we have $\overline{A} \perp \overline{B}$, $\overline{A} \perp B$, and $A \perp \overline{B}$. These cover all the cases needed for the definition of $I_A \perp I_B$.

This proposition can be extended straightforwardly to **mutual independence** of an arbitrary number of r.v.'s (we omit the statement and proof).

When we considered multiple IID Bernoulli trials "performed independently" we assumed that the corresponding Bernoulli random variables are **mutually independent**.

Quiz

A fair coin is tossed twice. Let X_H be random a variable that returns the number of heads observed. Clearly $\mathrm{Val}(X_H) = \{0, 1, 2\}$. Which one is the **bigger** probability?

(A) $\Pr[X_H = 0]$

(B) $\Pr[X_H = 1]$

(C) $\Pr[X_H = 2]$

Answer

(A) $\Pr[X_H = 0]$
Incorrect. There is only one outcome in which there are no heads.

(B) $\Pr[X_H = 1]$
Correct. There are **two** outcomes in which there is exactly one head.

(C) $\Pr[X_H = 2]$
Incorrect. There is only one outcome in which there are two heads.

More information

The probability space is uniform with 4 outcomes: $\{\mathtt{HH},\ \mathtt{HT},\ \mathtt{TH},\ \mathtt{TT}\}$.

The events for the distribution and their probabilities are

$(X_H = 0) = \{\mathtt{TT}\}$ with $\Pr[X_H = 0] = 1/4$

$(X_H = 1) = \{\mathtt{HT},\ \mathtt{TH}\}$ with $\Pr[X_H = 1] = 1/2$

$(X_H = 2) = \{\mathtt{HH}\}$ with $\Pr[X_H = 2] = 1/4$

Two r.v.'s that are not independent

Problem. A fair coin is flipped twice. Let X_H and X_T be random variables that return, respectively, the number of heads and tails observed. Are X_H and X_T independent?

Answer. Intuitively, the r.v.'s are **not** independent. For example, $X_H = 1$ **forces** $X_T = 1$. Let's verify this in detail.

The probability space on which the two r.v.'s are defined is uniform and has the set of outcomes $\{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}$ each with probability $1/4$.

Then, $\Pr[X_T = 1 \,|\, X_H = 1] \;=\; \Pr[(X_T = 1) \cap (X_H = 1)] \,/\, \Pr[X_H = 1]$.

Note that $(X_T = 1) \cap (X_H = 1) = \{\mathtt{HT}, \mathtt{TH}\} = (X_H = 1) = (X_T = 1)$.

Thus, $\Pr[X_T = 1 \,|\, X_H = 1] \;=\; (1/2)/(1/2) = 1$. Also, $\Pr[X_T = 1] = 1/2$.

It follows that $(X_T = 1) \not\perp (X_H = 1)$ and therefore $X_H \not\perp X_T$.

ACTIVITY : Constant r.v.'s are independent of any r.v.

In this activity we prove the following

Proposition, For any probability space $(\Omega, \Pr)$, any random variable $X : \Omega \to \mathbb{R}$, and any $c \in \mathbb{R}$, the constant r.v. $C : \Omega \to \mathbb{R}$ defined by $\forall w \in \Omega \quad C(w) = c$ is independent of X, $C \perp X$.

Proof. Since $\mathrm{Val}(C) = \{c\}$ all we have to prove is

$$\forall x \in \mathrm{Val}(X) \quad (X = x) \perp (C = c)$$

This follows from the observation that $(C = c) = \Omega$ together with property **Ind (ii)** in lecture segment "Independence" which states that Ω is independent of any event.