OMCIT 592 Spring 2022 — Timed Exam 3

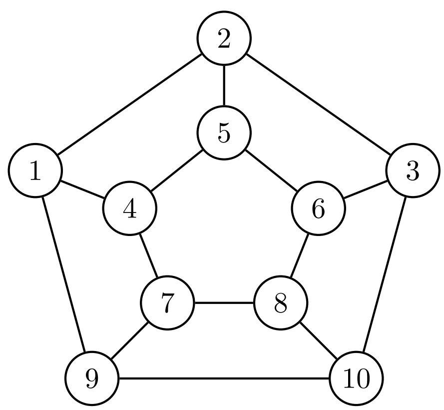
Fan Zhang fanzp@seas.upenn.edu April 30, 2022

To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - **no collaboration is allowed**.

Six (6) Problems for a Total of 60 Points

Below, *graph* always refers to an undirected graph; directed graphs are called digraphs.

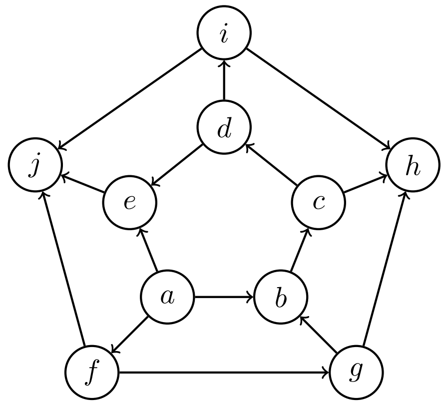
1. [10pts] By determining which vertices should be red, green, and blue, give a proper 3-coloring of the graph below. Your answer should consist of three lists: a list of the red vertices, a list of the green vertices, and a list of the blue vertices. You are not required to draw anything in your answer.



Solution.

Red vertices: 2,4,6,9
Blue vertices: 1,5,7,10
Green vertices: 3,8

2. [10pts] Topologically sort the vertices of the digraph below. Your answer should be a sequence (list) of vertices that forms a topological sort. You are not required to draw anything in your answer.



Solution.

Source: a

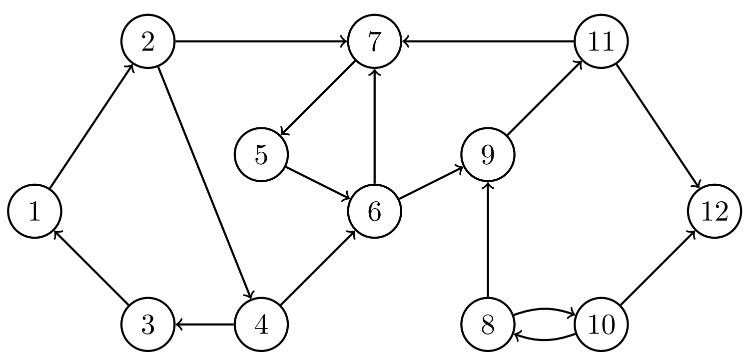
Sink: h,j

Sequence: $\{a \to f \to g \to b \to c \to d \to e \to j \to i \to h\}$

or

$\{a \to f \to g \to b \to c \to d \to i \to h \to e \to j\}$

3. [10pts] What are the strongly connected components of the digraph below? Your answer should consist of a list of strongly connected components where each component is represented as a set of vertices. You are not required to draw anything in your answer.



Solution.

SCC with one vertex: Every single vertex is a strongly connected component, so $\{1\}$,$\{2\}$,$\{3\}$,$\{4\}$,$\{5\}$,$\{6\}$,$\{7\}$, $\{8\}$,$\{9\}$,$\{10\}$,$\{11\}$,$\{12\}$.

SCC with 2 vertices: $\{8, 10\}$
SCC with 3 vertices: $\{5, 6, 7\}$
SCC with 4 vertices: $\{1, 2, 4, 3\}$
SCC with 5 vertices: $\{6, 9, 11, 7, 5\}$

4. [10pts] Let $G = (V, E)$ be a connected graph in which every vertex is a leaf. Prove that G has exactly one edge. You are not required to draw anything in your proof.

   Solution.

   Assume G has n vertices. Since every vertex is a leaf, we know that every vertex has degree 1, thus sum of degree of G is n.
   From the handshake lemma we know that $|E|$ = sum of degree / 2, so $|E| = n/2$.
   Since G is a connected graph and every vertex is a leaf, G is a tree, thus $|E| = |V| - |CC| = n - 1$.
   Thus $|E| = n/2 = n - 1$. We can calculate that $n = 2$ and $|E| = 1$.
   Proof is completed.

5. [15pts] Let $G = (V, E)$ be a digraph in which every vertex is a source, or a sink, or both a sink and a source.

   (a) [7pts] Prove that G has neither self-loops nor anti-parallel edges.

   (b) [8pts] Let $G^u = (V, E^u)$ be the undirected graph obtained by erasing the direction on the edges of G. Prove that G^u has chromatic number 1 or 2.

   You are not required to draw anything in your proofs.

   Solution.

   (a) [7pts] Since every vertex is a source, or a sink, or both a sink and a source, we know that the vertices in G can have only in degrees, or only out degrees or no degrees.
   By definition self-loops has one in degree and one out degree, which is contradicted to

the definition of vertices in G.

By definition anti-parallel edges $u \to v$ and $v \to u$, for both u and v, $in(u) = in(v) = out(u) = out(v) = 1$, so u and v have both in degree and out degree 1. So it's contradicted to the definition of vertices in G.

Thus self-loops and anti-parallel edges are both contradicted to the definition of vertices in G. Thus G has neither self-loops nor anti-parallel edges.

Proof is completed.

(b) [8pts] Situation 1: When G only has edgeless vertices.

When all vertices are isolated vertices, they are both a sink and source so they are qualified for the definition in G.

Since there are no edges, $G^u = (V, E^u)$ is also edgeless. Therefore we can color all the vertices with one color to proper color G^u. Thus the chromatic number of G^u is 1.

Situation 2: When G has edges.

Any arbitrary edge in G, $u \to v$, u is a source and v is a sink. It means there is no node before u and there is no node after v. So the path between any source and any sink in G, the length of the path is 1. Thus we can tell there is no cycle in G, so there is no cycle in G^u.

After erasing the direction on the arbitrary edge, it becomes $u - v$. Since G^u does not have any cycle, the connected components in G^u also do not have cycle, so the connected components in G^u are trees, and G^u is a tree or a forest.

From the slides in 592 course we know tree is bipartite, thus the chromatic number of connected component in G^u is 2 thus the chromatic number of G^u is 2.

In summary we proved that G^u has chromatic number 1 or 2.

6. [5pts] Let $G = (V, E)$ be a graph such that

- $|V| \geq 3$,
- G has exactly 2 leaves, and
- in G all the non-leaf vertices have degree 3 or more.

Prove that G has at least one cycle. You are not required to draw anything in your proof.

Solution.

Assume $|V| = n$, $n \geq 3$. Then we know:

sum of degree $= 2|E| \geq 3(n-2)+2$

So $|E| \geq (3n-4)/2$

Suppose, toward contradiction G does not have any cycle. Then G is a tree or a forest.

We know for a forest $|E| = |V| - |CC|$, and $|E|$ is the largest when $|CC| = 1$.

So $|E| \geq |V| - 1 = n - 1$.

When $|E| = (3n-4)/2$ and $|E| = n-1$, we can calculate n=2 which is contradicted to the given assumption that $|V| \geq 3$.

When $(3n-4)/2 < n-1$, $n < 2$ which is contradicted to the given assumption that $|V| \geq 3$.

Proof is completed.