Module 3.4: Truth Tables

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

From statements to boolean expressions

How are proof patterns **justified**? We will explain them with the help of **boolean expressions**.

The proof patterns were described **abstractly** using symbols: P_1, P_2, P_3 .

To justify the proof patterns **all that matters** is whether the symbols P_1, P_2, P_3 represent true or false statements.

So we replace them with **boolean variables** that can take one of two **truth values**: T (for true!), and F (for false!). We use $p, q, r, s, p_1, p_2, \ldots$.

Boolean expressions are obtained from boolean variables using logical connectives: $\vee, \wedge, \neg, \Rightarrow$.

Examples: $(p \wedge \neg q) \Rightarrow r \qquad p_1 \wedge (p_2 \Rightarrow s)$

Truth assignments and truth tables

A **truth assignment** gives each boolean variable a value, T or F.

Given a truth assignment to its variables, a boolean expression also yields T or F.

We use **truth tables** to specify this. For each logical connective:

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Memorize these tables!

Quiz

Given the boolean expression $(p \wedge \neg p) \vee (\neg q)$ and a truth assignment $p = \mathtt{T}$ and $q = \mathtt{F}$, what is the resulting truth value?

A. `T`.

B. `F`.

Answer

Given the boolean expression $(p \wedge \neg p) \vee (\neg q)$ and a truth assignment $p = \mathtt{T}$ and $q = \mathtt{F}$, what is the resulting truth value?

A. T.

   Correct. Please refer to the truth table in the next slide.

B. F.

   Incorrect. Please refer to the truth table in the next slide.

More Information

This is the truth table for the boolean expression. As you can see, the quiz corresponds to the second line in this truth table.

p	q	$\neg p$	$\neg q$	$(p \wedge \neg p)$	$(p \wedge \neg p) \vee (\neg q)$
T	T	F	F	F	F
T	F	F	T	F	T
F	T	T	F	F	F
F	F	T	T	F	T

False implies anything!

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

If the premise is false then the implication is true, regardless of the conclusion.

Or, "false implies anything".

For example, consider

"if $2 + 2 = 5$ then there exist infinitely many twin primes".

This implication is true...

...although we don't know (as of the time of this writing) whether there are infinitely many twin primes (this is the **Twin Prime Conjecture**).

The implication will remain true even after the conjecture is settled!

Holds vacuously

Some people say that such implications, in which the premise is always false, "hold vacuously".

This is inspired by a particular case with implication under a universal quantifier. For example:

"for any natural number n such that $kn = k^2 + 1$ for some integer $k > 1$, there exist twin primes bigger than n"

But the set of natural numbers n such that $kn = k^2 + 1$ for some integer $k > 1$ is empty!

$$. \quad \{\, n \in \mathbb{N} \mid \exists k \in \mathbb{Z} \;\; k > 1 \wedge kn = k^2 + 1 \} \;=\; \emptyset$$

ACTIVITY : Vacuously true statement

In this activity, we prove: "there are no natural numbers n such that $kn = k^2 + 1$ for some integer $k > 1$."

First consider the statement $kn = k^2 + 1$. We can rewrite this statement as $n = \frac{k^2+1}{k}$. For n to be a natural number we must have $k \mid (k^2 + 1)$.

Since $k \mid k^2$, it follows that the remainder of the division of for $k^2 + 1$ by k is 1. This problem then reduces to finding integers k for which $k \mid 1$.

Question: For what integers k do we have $k \mid 1$?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

Activity : Vacuously true statement (Continued)

Answer: The only integers that divide 1 are -1 and 1.

We can now complete the proof.

There are **no** integers $k > 1$, such that $k \mid (k^2 + 1)$.

Therefore, there are **no** natural numbers n such that $kn = k^2 + 1$ for some integer $k > 1$.

We justify "by contrapositive"

Proof pattern. Instead of "if P_1 then P_2" you can prove "if (not P_2) then (not P_1)".

The contrapositive transformation on boolean expressions goes

from $p \Rightarrow q$2cm to $\neg q \Rightarrow \neg p$.

The following truth table justifies this transformation:

p	q	$p \Rightarrow q$	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

We justify "by cases"

Proof pattern. To prove "if P_1 or P_2 then P_3" you prove "if P_1 then P_3 and if P_2 then P_3".

The by-cases transformation on boolean expressions goes

from $p \vee q \Rightarrow r$ to $(p \Rightarrow r) \wedge (q \Rightarrow r)$.

The following justifies this (only 4 of 8 rows are shown):

p	q	r	$p \vee q$	$p \vee q \Rightarrow r$	$p \Rightarrow r$	$q \Rightarrow r$	$(p \Rightarrow r) \wedge (q \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F

ACTIVITY : "By-cases" Justification

Now we try finishing the truth table for the "by-cases" justification by providing the final 4 rows of the truth table.

Recall that we want to convert from $p \vee q \Rightarrow r$ to $(p \Rightarrow r) \wedge (q \Rightarrow r)$.

Question:

Assuming $p = \mathtt{F}$, $q = \mathtt{T}$ and $r = \mathtt{T}$, what are the truth values of $p \vee q \Rightarrow r$ and $(p \Rightarrow r) \wedge (q \Rightarrow r)$? Are they the same?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : "By-cases" Justification

Answer: Look at the first row below. Both truth values are T.

p	q	r	$p \vee q$	$p \vee q \Rightarrow r$	$p \Rightarrow r$	$q \Rightarrow r$	$(p \Rightarrow r) \wedge (q \Rightarrow r)$
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Logical equivalence

Two boolean expressions are **logically equivalent** when they yield the same truth value for the same truth assignments to their variables. Notation: $\equiv$

Here are some logically equivalent boolean expressions:

$$
\begin{array}{rcll}
p \Rightarrow q & \equiv & \neg q \Rightarrow \neg p & \text{(Contrapositive)} \\
p \vee q \Rightarrow r & \equiv & (p \Rightarrow r) \wedge (q \Rightarrow r) & \text{(By-cases)} \\
p \Rightarrow q & \equiv & \neg p \vee q & \text{(Law of Implication)} \\
\neg(p \Rightarrow q) & \equiv & p \wedge \neg q & \text{(Disproving implication)} \\
\neg(p \vee q) & \equiv & \neg p \wedge \neg q & \text{(De Morgan's Law I)} \\
\neg(p \wedge q) & \equiv & \neg p \vee \neg q & \text{(De Morgan's Law II)} \\
\neg\neg p & \equiv & p & \text{(Law of Double Negation)}
\end{array}
$$

We verified the first two with the truth tables in the previous slides. More is discussed in the activities that follow.

Activity : More truth tables

The logical equivalence $p \wedge \neg p \equiv F$ is also known as **Law of Contradiction**. The truth table to verify this logical equivalence is shown below.

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

ACTIVITY : More truth tables (Continued)

Similarly, $p \lor \neg p \equiv T$ is the **Law of the Excluded Middle**.

It is also known as **Tertium Non Datur** (Latin for "a third (possibility) is not given").

The truth table to verify this logical equivalence is shown below.

p	$\neg p$	$p \lor \neg p$
T	F	T
F	T	T