Recitation Module 9

Penn Engineering

Concept Review

- Random Variables *(are functions)*
- A random variable X on (Ω,Pr) is a function $X: \Omega \rightarrow \mathbf{R}$

- Uniform r.v. and distribution:
  - The uniform random variable, $U: [1 \ldots n] \rightarrow \mathbf{R}$, is associated with the uniform probability space (all n outcomes have a 1/n probability of occurring)

- Bernoulli r.v. and distribution:
  - The Bernoulli random variable, $B: \{S,F\} \rightarrow \{0,1\}$, is associated with the Bernoulli probability space (probability of success is p)

Concept Review

- Expectation (Expected Value)

$$\mathsf{E}[X] \;=\; \sum_{x \in \mathsf{Val}(x)} x \cdot \Pr[X = x] \;=\; \sum_{w \in \Omega} X(w) \cdot \Pr[w]$$

- Linearity of Expectation

$$\mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i].$$

- Indicator r.v.

Let A be an event in a probability space $(\Omega, \Pr)$. The **indicator** random variable of the event A, notation I_A, is defined by

$$I_A(w) \;=\; \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A \end{cases}$$

Question 0

You roll a fair die twice, independently. Consider the random variable Z that takes the values obtained by subtracting the second roll number from the first roll number.

a. What is Val(Z)?
b. What is Pr[Z=1]?
c. What is E[Z]?

Answer to Question 0

a. The largest value of Z is 6-1 = 5 and the smallest value is 1-6 = -5. We now show that Z can take all the in between values in at least one way:
4 = 6-2, 3 = 6-3, 2 = 6-4, 1 = 6-5, 0 = 1-1, -1 = 5-6, -2 = 4-6, -3 = 3-6, -4 = 2-6
Thus, $Val(Z) = [-5..5]$

b. We first define the probability space to be the 36 different outcomes of rolling a die twice. This is a random, uniform event, so each of these outcomes is equally probable. In how many ways can $Z=1$? (6,5), (5,4), (4,3), (3,2), (2,1) → 5 ways! Hence, $Pr[Z = 1] = 5/36$.

Answer to Question 0

c. In order to find E[Z] we need to find the probability of every value in Val(Z), but as we saw in (b) this can be tedious!!! We thus employ linearity of expectation.

Let X be the r.v. representing the value of the first roll, and Y be the r.v. representing the value of the second roll.

By linearity of expectation, we have that $E[Z] = E[X - Y] = E[X] - E[Y]$

Since we know (segment 9.2) that the expectation of the number shown by a die is 3.5:
$E[Z] = E[X] - E[Y] = 3.5 - 3.5 = 0$

Question 1

A box contains 10 tickets, 4 of the tickets carry a prize of $6 and the others carry a prize of $3.

a. If one ticket is drawn, what is the expected value of the prize?

b. If two tickets are drawn at the same time, what is the expected value of the prize?

Answer to Question 1a

Let X be the r.v. representing the value of the prize. For one ticket, X = \$6 or \$3. In other words, Val(X) = {3, 6}. We then need to find Pr[X = 3] and Pr[X = 6] in order to calculate E[X] = 3*Pr[X = 3] + 6*Pr[X=6]

1. Identify the sample space Ω to be all the possible ways to pick a ticket: since there are 10 tickets and we pick one, |Ω| = 10. Since this is a random, uniform event, the probability of each outcome is 1/10.
2. Compute |X=6|. We have that 4 of the tickets carry a \$6 prize. Thus, Pr[X=6] = |X=6| / |Ω| = 4/10 = 0.4.
3. Similarly, we compute the number of ways for the event X=3. We know that 6 tickets carry a \$3. Thus, Pr[X=3] = 6/10 = 0.6
4. It follows that E[X] = 6*Pr[X=6] + 3*Pr[X=3] = 6*0.4 + 3*.6 = 4.2

Answer to Question 1b

Let X be the r.v. representing the value of the prize. In this case, we pick two tickets, each of which can have either a \$3 or \$6 prize. It follows that X can take on the following values:

- $3 + 3 = 6$
- $6 + 3 = 3 + 6 = 9$
- $6 + 6 = 12$

(i.e. $Val(X) = \{6, 9, 12\}$)
Hence, we apply the expectation formula: $E[X] = 6*Pr[X = 6] + 9*Pr[X = 9] + 12*Pr[X = 12]$

To calculate the probabilities, we first determine the sample space Ω as the set of all possible pairs of tickets. This time, since we are randomly choosing 2 out of 10 possible tickets, there are 10C2 = 45 possible outcomes. This is a uniform sample space because each ticket is equally likely to be chosen.

1. Compute Pr[X = 6]: There are 6 \$3-tickets, meaning there are 6C2 = 15 ways to pick two \$3 tickets.
Hence, $Pr[X = 6] = 15/45$
2. Compute Pr[X = 12]: There are 4 \$6-tickets, meaning there are 4C2 = 6 ways to pick two \$6 tickets.
Hence, $Pr[X = 12] = 6/45$
3. Compute Pr[X = 9]: There are 4 \$6-tickets and 6 \$3-tickets. Therefore, there are 6*4 = 24 ways of choosing one of each.
Hence, $Pr[X = 9] = 24/45$

Thus, $E[X] = 6*(15/45) + 9*(24/45) + 12*(6/45) = 8.4$

Question 2

Use an indicator random variable to compute the following:

Roll a die 100 times. What is the expected number of 6s (i.e. the number of times you roll a 6)?

Answer to Question 2

Let I_k be the indicator variable where $I_k = 1$ if we roll a six on the kth roll, and $I_k = 0$ if we did not roll a six on the kth roll.

We have that $Pr[I_k = 1] = 1/6$ and $Pr[I_k = 0] = 5/6$ for every roll. Therefore, for any value of k, the expectation of I_k will be:

$E[I_k] = 1*(1/6) + 0*(5/6) = 1/6$

Let X be the r.v. representing the number of 6s we get in 100 trials. By the linearity of expectation:

$$E[\sum_{k=0}^{100} I_k] = \sum_{k=0}^{100} E[I_k] = 100 * \frac{1}{6} = \frac{100}{6} = 16.67$$

Therefore, the expected number of 6s in 100 rolls of a die will be 16.67.

Question 3

A restaurant has the following schedule of daily demand for Danish pecan.

# of Danish pecan demanded	0	1	2	3	4	5	6	7	8	9
Probability	0.02	0.07	0.09	0.12	0.20	0.20	0.18	0.10	0.01	0.01

Find the expected number of Danish pecan demanded per day.

Answer to Question 3

This is a non-uniform probability space since the outcomes have different probabilities. (Note that the probabilities still sum up to 1).

Let X be the r.v. representing the number of Danish pecan dishes ordered in a day.

Thus, using the definition of expectation we get

$$E[X] = \Sigma_{i \in [0...9]} Pr[X = i] * i = 0.02 * 0 + 0.07 * 1 + 0.09 * 2 + 0.12 * 3 + 0.20 * 4 + 0.18 * 6 +$$

$$0.10 * 7 + 0.01 * 8 + 0.01 * 9 = \mathbf{3.36}$$

Before you go...

Do you have any general questions about this week's assignment? We can't go into minute details about any specific questions, but this is the time to ask any general clarifying/confusing questions!

As always, if you can't talk to us during recitation or office hours, post any questions/anything course related on Piazza. As always, please request regrades via Gradescope, and email mcitonline@seas.upenn.edu to request an extension