Module 2.3: Permutations

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Lecture Notes

Permutations

Let A be a non-empty set with n elements, that is, $|A| = n$.
A **permutation** of A is an ordering of the elements of A in a row, i.e., a sequence of **all** the elements of A, **without repetition**.

The length of a permutation of A is n.

Example:

The set $\{x, 2, a\}$ has six permutations:

. $x2a$, $xa2$, $2xa$, $2ax$, $ax2$, $a2x$.

Sequences built from $\{x, 2, a\}$ that are **not** permutations: $aa2$ $a2$

Partial permutations

Again consider a non-empty set A with n elements. Let $1 \leq r \leq n$.

A **partial permutation of r out of the n elements of A** consists of picking r of the elements of the set and ordering them in a row, i.e., a sequence of length r, **without repetition**, whose elements are from the set A.

Example:

Here are the partial permutations of 2 out of the 3 elements of $\{x, 2, a\}$:

. $x2, \quad 2x, \quad xa, \quad ax, \quad 2a, \quad a2.$

Sequences built from $\{x, 2, a\}$ that are **not** partial permutations of 2 out of the 3 elements: $aa \quad a \quad x2a$

Examining the two definitions we have given, we see that a partial permutation of n out of the n elements of a set A is the same as a permutation of A!

Counting partial permutations

Problem. Let A be a non-empty set with n elements (i.e., $|A| = n \geq 1$) and let $1 \leq r \leq n$.
How many partial permutations of r out of the n elements of A are there?

Answer. We can construct such a partial permutation in r steps, filling its positions, numbered $1, 2, \ldots, r$, consecutively:

(1) Pick an element of A to put in position 1. Can be done in n ways.

(2) Pick one of the remaining elements to put in position 2. In $n-1$ ways.

$\ldots$

(r) Pick one of remaining $n-(r-1)$ elements to put in position r. In $n-(r-1) = n-r+1$ ways.

By the multiplication rule the answer is $n \cdot (n-1) \cdots (n-r+1)$.
This is a product of r factors.

Factorial

We computed the number of partial permutations of r out of n as:

$$n \cdot (n-1) \cdots (n-r+1)$$

Note that this number depends only on n and r, and not on the set whose elements we use (as long as there are n of them).

Now take $r = n$. This gives the number of permutations of n elements. And there is a notation for this:

$$n! \;=\; n \cdot (n-1) \cdots 2 \cdot 1$$

read "the **factorial** of n".

We will use the factorial notation to shorten expressions.

For example, the number of partial permutations of r out of n

$$n \cdot (n-1) \cdots (n-r+1) \;=\; \frac{n \cdot (n-1) \cdots (n-r+1) \cdot (n-r) \cdots 1}{(n-r) \cdots 1} \;=\; \frac{n!}{(n-r)!}$$

Quiz

Let p be the number of permutations of 6 elements, and let q be the number of partial permutations of 3 out of 6 elements. What is p/q?

A. 2

B. 3

C. 6

Answer

Let p be the number of permutations of 6 elements, and let q be the number of partial permutations of 3 out of 6 elements. What is p/q?

A. 2

Incorrect. Since $p = 6!$ and $q = \frac{6!}{3!}$, $\frac{p}{q} \neq 2$.

B. 3

Incorrect. Since $p = 6!$ and $q = \frac{6!}{3!}$, $\frac{p}{q} \neq 3$.

C. 6

Correct. Since $p = 6!$ and $q = \frac{6!}{3!}$, p divided by q is $3! = 6$.

Counting words with restrictions I

Problem. Consider the set of letters $\{a, b, c, d, e, f, g, h\}$.

(a) How many possible permutations are there of these letters?

(b) How many among the permutations of these letters contain the contiguous sequence abc?

Answer. Part (a): The set has 8 elements hence there are $8!$ permutations.

Part (b): A permutation of $\{a, b, c, d, e, f, g, h\}$ in which a, b, c appear in consecutive positions can be constructed as follows:

(1) Pick three consecutive positions for a, b, c. Can be done in 6 ways.

(2) Pick a permutation of $\{d, e, f, g, h\}$ and place it in the remaining $8 - 3 = 5$ positions. This can be done in $5!$ ways.

By the multiplication rule the total number of ways is $6 \cdot 5!$.

Counting words with restrictions II

Alternative answer. Part (b):

We can construct a desired permutation differently.
Consider the set of 6 letters: $\{x, d, e, f, g, h\}$

Construct a permutation of $\{x, d, e, f, g, h\}$. For example: *edhxfg*.

Next, replace in this permutation the letter x with the string *abc*.
In the example: *edhabcfg*.

(Any permutation with a, b, and c in consecutive positions can be transformed into a permutation of $\{x, d, e, f, g, h\}$ by replacing the portion *abc* with x. Thus, counting the desired permutations is the same as counting the permutations of $\{x, d, e, f, g, h\}$.)

There are $6!$ of these. And indeed $6! = 6 \cdot 5!$.