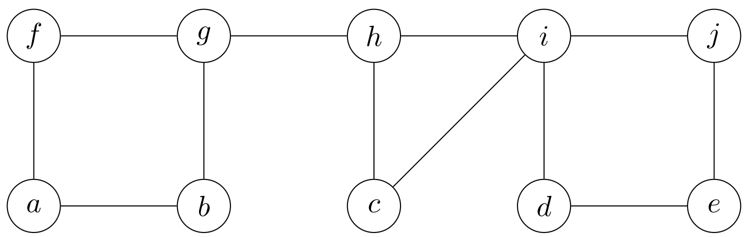
MCIT 592 Spring 2022 **Homework 12** **Solutions**

Problem Set

1. [10 pts]



In the graph above, how many connected components are in the subgraph induced by each of the following subsets of the vertices?

(a) $\{a, b, c, d, e\}$

(b) $\{f, g, h, i, j\}$

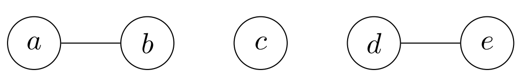
(c) $\{a, b, e, h, i\}$

(d) $\{c, f, g, i, j\}$

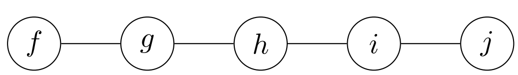
(e) $\{a, c, d, g, j\}$

Solution:

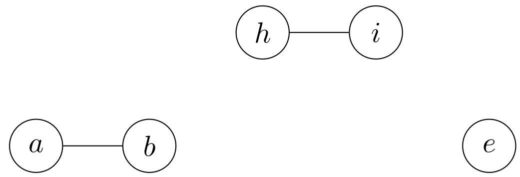
(a) 3



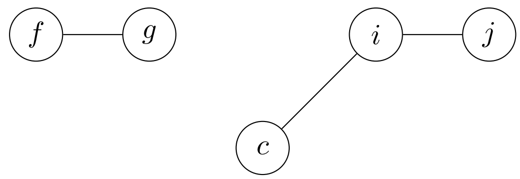
(b) 1



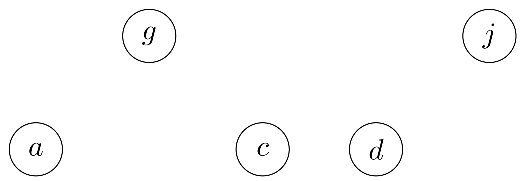
(c) 3



(d) 2



(e) 5



2. [10 pts] Suppose Kruskal's Kingdom consists of $n \geq 3$ farmhouses, which are connected in a cyclical manner. That is, there is a road between farmhouse 1 and 2, between farmhouse 2 and 3, and so on until we connect farmhouse n back to farmhouse 1. In the center of these is the king's castle, which has a road to every single farmhouse. Besides these, there are no other roads in the kingdom.

(a) Find the number of paths of length 2 in the kingdom in terms of n. Justify your answer.

(b) Find the number of cycles of length 3 in the kingdom in terms of n. Justify your answer.

(c) Find the number of cycles in the kingdom in terms of n. Justify your answer.

Solution:

We map the kingdom as a graph, with the farmhouses and castle representing vertices, and the roads representing edges. Lets let c be the vertex that represents the castle.

(a) Consider two types of paths: those which don't include c and those that do. First count the number of paths without c. Under that condition, every vertex in A is the midpoint of a single path of length 2. That is, the number of paths in this scenario is equal to the number of vertices, yielding n such paths. For paths with c, c can either by a midpoint or an endpoint. To count number of paths with c as a midpoint, choose 2 vertices to connect to c. There are $\binom{n}{2}$ paths. Finally to find paths with c as an endpoint, first pick an edge in c (n choices), then connect c to one of the endpoints of that edge (2 choices), leading to $2n$ paths.

So in total, there are $n + \binom{n}{2} + 2n = \boxed{3n + \binom{n}{2}}$ paths.

(b) The number of cycles with length 3 actually depends on the value of n. If $n > 3$, the number of cycles can only be formed by selected an edge in A and connecting both endpoints to c. It's impossible to form a cycle of length 3 using only vertices in A because A itself is a bigger cycle. There are n such cycles since there are n edges.

Now for the corner case of $n = 3$. If $n = 3$, the vertices in A actually form a cycle of length 3 as well! Taking into account the cycles including c as mentioned above, the number of cycles in this case is 4.

Thus the number of cycles of length 3 would actually be a piece-wise function. $\boxed{\text{If } n = 3\text{, it's 4. Else, it's } n}$.

(c) First, there is the cycle induced by the vertices in A, so there's 1 cycle without c. Now, consider cycles which incorporate c. These cycles can be formed by connecting c to two different vertices in A and then connecting these 2 vertices through a path. Note that there are always 2 paths to take (you can think of it starting at one vertex a and going clockwise to b or starting at b and going clockwise to a). Thus, the number of cycles including c is $\binom{n}{2} \times 2$.

All together, the number of cycles is $\boxed{1 + 2\binom{n}{2}} = \boxed{1 + n(n-1)}$.

An alternative approach can be used to count the number of cycles that incorporate c. First for the cycle induced by A, impose an "ordering" to it. That is, call looping through the cycle in one direction going clockwise, and looping through in the other direction going counterclockwise. To count cycles, first pick a starting vertex in A (n choices). Then, pick an end vertex in A ($n - 1$ choices). A cycle can now be formed by taking the first vertex, traversing the cycle clockwise until reaching the end vertex, connecting to c and finally the first vertex. This yields the equivalent answer $\boxed{1 + n(n-1)}$.

3. [10 pts] Suppose we have a neighborhood of n houses. For any two houses we pick, there is a road between them.

(a) The landlord wants to cut maintenance costs by removing some of the roads. Let k be the minimum number of roads he can remove such that the neighborhood is still connected (every house can be walked to from every other house) and there are no cycles. Determine the value of k as an expression in terms of n. Then indicate how to remove the minimum number of roads from the neighborhood such that the requirements are satisfied.

(b) Suppose now instead that the landlord wants to remove *houses*. Let ℓ be the minimum number of houses that must be removed such that the neighborhood is still connected and has no cycles. Removing a house also removes the roads it is connected to. Determine the value of ℓ as an expression in terms of n. Then indicate how to remove the minimum number of houses from the neighborhood such that the requirements are satisfied.

Solution:

We map houses as vertices, and roads as edges. In both problems, we are looking to create a tree (connected and acyclic).

(a) Since we are removing only edges the resulting tree must have n nodes. A tree with n nodes has $n-1$ edges. K_n has $\binom{n}{2}$ edges. Therefore the minimum number of edges that must be removed is

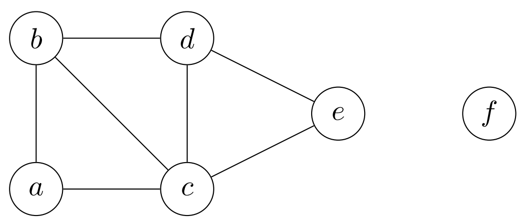
$$k \;=\; \binom{n}{2} - (n-1) \;=\; \frac{n(n-1)}{2} - (n-1) \;=\; \frac{(n-1)(n-2)}{2}.$$

The procedure for removing the edges is the following. Choose a vertex in K_n, call it v. Let W be the set of vertices in K_n that are *not* v. Notice that $|W| = n-1$. Remove all the edges in K_n whose endpoints are both in W. Keep all the edges that link v to a vertex in W. The resulting graph (sometimes called a "star graph with center v") is a tree with n vertices and $n-1$ edges. Because any two vertices in W are adjacent, the number of removed edges is $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$ and therefore equals k, the minimum number of edges that we determined above.

The procedure above works for K_n. For a connected graph in general, we can also remove edges to obtain a tree, as we shall see when we study spanning trees.

(b) In K_n each node is adjacent to all the other $n-1$ nodes and thus has $n-1$ incident edges. When we remove a node (and its incident edges), we are left with a graph in which any two nodes are still adjacent; that is, we are left with K_{n-1}. So we can choose any node and remove and continue, obtaining the sequence $K_n, K_{n-1}, K_{n-2}, \ldots, K_3$, all of which are connected but have cycles. Finally, we remove one more node to obtain K_2, which is a tree. Therefore, $\ell = n - 3 + 1 = n - 2$. The procedure is very simple. Choose any node, remove it, and repeat until 2 nodes are left. Therefore the removal step must be repeated $n - 2$ times which is the minimum number of nodes ℓ that we determined earlier. THIS SOLUTION IS INCOMPLETE; WE MUST ALSO ARGUE WHY THIS NUMBER IS MINIMUM FOR THAT WE NEED A LEMMA: K_n is acyclic iff n=2

4. [10 pts] We define a graph's *degree sequence* as a list of the degrees of all the vertices in the graph, **in increasing order of degree**. For example, the graph



has degree sequence $(0, 2, 2, 3, 3, 4)$ because there is one node with degree 0 (f), two nodes with degree 2 (a and e), two nodes with degree 3 (b and d), and one node with degree 4 (c).

For each of the following, either list the set of edges of a **tree** with vertex set $\{a, b, c, d, e, f\}$ that has the stated degree sequence, or show that no such tree exists.

(a) $(1, 1, 1, 3, 3, 3)$

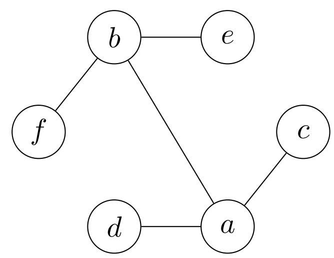
(b) $(1, 1, 1, 1, 3, 3)$

(c) $(1, 1, 1, 1, 3, 4)$

Solution:

(a) No such tree exists. By the handshaking lemma, a graph with degree sequence $(1,1,1,3,3,3)$ would have $(1,1,1,3,3,3)/2 = 6$ edges. But we saw in lecture that every tree with n vertices has $n-1$ edges, so a tree with vertex set $\{a,b,c,d,e,f\}$ must have 5 edges.

(b) $\{a–b, a–c, a–d, b–e, b–f\}$



(c) No such graph exists. By the handshaking lemma, the sum of all degrees must be even. $1+1+1+1+3+4 = 11$, which is odd.

5. [10 pts] Suppose there exists a connected group of Facebook friends. That is, there is a way to reach every person from every other person through some traversal of friendships. Additionally, there is no cycle of friendships (ex. we would *not* have the following case: 1 is friends with 2, who is friends with 3, who is friends with 1). Suppose that some person u in this group has at least d friends. Prove that there exists at least d people in this group with exactly 1 friend. *Hint:* Think about what specific type of graph this is based on the definition!

Solution:

By definition, this graph is a tree, which we can name T. Following the hint, we consider the subgraph of T induced by the set of vertices $V \setminus \{u\}$. Call this subgraph G.

Consider two distinct neighbors of u in T. The unique path that connects these two neighbors in T goes through u therefore in G this path does not exist. Thus, these two neighbors belong to different connected components (cc). It follows that each of the d neighbors of u in T must belong in G to a different cc. Since T is acyclic, G is also acyclic therefore each of these cc's is a tree. There are two possibilities for each cc of G. If a cc is a single node, then this single node is a leaf adjacent to u in T. If the cc has at least two nodes, then it has at least two leaves. At most one of the leaves will be u's neighbor and therefore not a leaf in T, but any other leaf in this cc is still a leaf in T. In any case, each of the d components contains at least one leaf of T and hence T must have at least d leaves. $\square$