Module 6.1: Ordinary Induction

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

What is induction good for?

For proving statements of the form:

"for all natural numbers n we have $P(n)$"

where $P(n)$ is a predicate whose truth depends on n.

In logical notation: $\forall n \in \mathbb{N} \;\; P(n)$.

Examples. Statements of this form that are of interest:

- $P(n)$ is $2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1$.
- $P(n)$ is $1 + 2 + 3 + \cdots + n = n(n+1)/2$ (for $n \geq 1$).
- $P(n)$ is $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ (for $n \geq 1$).
- $P(n)$ is "n can be written as the product of one or more (not necessarily distinct) prime numbers" (for $n \geq 2$).

Proof pattern: (ordinary) induction

Let $P(n)$ be a predicate whose truth depends on n.

Proof pattern.

(BASE CASE) Check that $P(0)$ holds true.

(INDUCTION STEP) Let k be an arbitrary natural number.
Assume $\boxed{P(k).}$ Using that derive $P(k+1)$.

Conclude $\forall\, n \in \mathbb{N}\ \ P(n)$.

The $P(k)$ inside the box in the induction step is called the **INDUCTION HYPOTHESIS (IH)**. The IH must be stated **inside the induction step** because it refers to k.

In logical notation the induction step is $\forall\, k \in \mathbb{N}\ \ P(k) \Rightarrow P(k+1)$.

Sum of a geometric progression

Problem. Prove $\forall\, n \in \mathbb{N}\ \ 2^0 + 2^1 + \cdots + 2^n \ = \ 2^{n+1} - 1.$

Answer. $P(n)$ is $\ 2^0 + 2^1 + \cdots + 2^n \ = \ 2^{n+1} - 1.$

(BASE CASE) $2^0 = 1$ and $2^{0+1} - 1 = 2 - 1 = 1.$ Check.

(INDUCTION STEP) Let k be an arbitrary natural number.
Assume $2^0 + 2^1 + \cdots + 2^k \ = \ 2^{k+1} - 1.$ (This is the IH.)
(Now we **want to show** $2^0 + 2^1 + \cdots + 2^{k+1} \ = \ 2^{k+2} - 1.$)
Then:

$$\begin{aligned}
2^0 + 2^1 + \cdots + 2^{k+1} \ &= \ (2^0 + 2^1 + \cdots + 2^k) + 2^{k+1} && \text{(Grouping)} \\
&= \ 2^{k+1} - 1 + 2^{k+1} && \text{(By IH)} \\
&= \ 2 \cdot 2^{k+1} - 1 \ = \ 2^{k+2} - 1 && \text{(Done)}
\end{aligned}$$

Quiz

The previous identity gives the summation of the **geometric progression** with ratio 2.

Memorize the sum of a general **geometric progression** with ratio q:

$$q^0 + q^1 + q^2 + \cdots + q^n \;=\; \begin{cases} \frac{q^{n+1}-1}{q-1} & \text{if } q \neq 1, \\ n+1 & \text{if } q = 1, \end{cases}$$

Now apply this to

$$s \;=\; 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

Which of the following is true?

A. $s < 2$

B. $s = 2$

C. $s > 2$

Answer

Which of the following is true?

A. $s < 2$

Correct. Apply the formula with $q = \frac{1}{2}$ to derive $s = 2 - \frac{1}{2^n} < 2$.

B. $s = 2$

Incorrect. Apply the formula to derive $s = 2 - 1/2^n$.

$$s = \frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1} = \frac{\frac{1}{2^{n+1}} - 1}{-\frac{1}{2}} = -2\left(\frac{1}{2^{n+1}} - 1\right) = 2 - \frac{1}{2^n} < 2\,.$$

C. $s > 2$

Incorrect. Apply the formula to derive $s = 2 - 1/2^n$.

$$s = \frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1} = \frac{\frac{1}{2^{n+1}} - 1}{-\frac{1}{2}} = -2\left(\frac{1}{2^{n+1}} - 1\right) = 2 - \frac{1}{2^n} < 2\,.$$

When we cannot start at 0

"$1 + 2 + 3 + \cdots + n = n(n+1)/2 \quad (\text{for } n \geq 1)$"

This statement does not make sense for $n = 0$.

However, we still need a base case to use induction!

One possibility is to change the statement:

"if $n \geq 1$ then $1 + 2 + 3 + \cdots + n = n(n+1)/2$"

Then the base case $n = 0$ holds vacuously!

However, in the induction step we will need to reason separately for the case $k = 0$ and essentially prove the statement for $n = 1$.

Instead, it's much easier to adopt a **variant** of the (ordinary) induction proof pattern, as we do in the next slide.

Proof pattern variant for (ordinary) induction

Let n_0 be a natural number and let $P(n)$ be a predicate that is well defined for all natural numbers $n \geq n_0$.

Proof pattern.

(BASE CASE) Check that $P(n_0)$ holds true.

(INDUCTION STEP) Let $k \geq n_0$ be an arbitrary natural number. Assume $\boxed{P(k)}$. Using that, infer $P(k+1)$.

Conclude $\forall\, n \geq n_0 \;\; P(n)$.

Aside: From now on we agree to **abbreviate**: (BASE CASE) as (BC),(INDUCTION STEP) as (IS) and "we want to show" as WTS.

The sum $1 + 2 + 3 + \cdots + n$

Problem. Prove $\forall\, n \geq 1 \;\; 1 + 2 + 3 + \cdots + n \;=\; n(n+1)/2.$

Answer. Take $n_0 = 1$ in the proof pattern.

(BC) $1 = 1$ and $1(1+1)/2 = 2/2 = 1.$ Check.

(IS) Let $k \geq 1$ be an arbitrary natural number.

Assume (IH) $1 + 2 + \cdots + k \;=\; k(k+1)/2.$
(And **WTS** $1 + 2 + \cdots + k + (k+1) \;=\; (k+1)(k+2)/2.$) Then:

$$
\begin{aligned}
1 + 2 + \cdots + k + (k+1) &= (1 + 2 + \cdots + k) + (k+1) && \text{(Grouping)} \\
&= k(k+1)/2 + (k+1) && \text{(By IH)} \\
&= (k+1)(k/2+1) \;=\; (k+1)(k+2)/2
\end{aligned}
$$

Done.

ACTIVITY : Sum of integers

Gauss was one of the world's greatest mathematicians. Legend has it that when Gauss was very young (accounts vary between 7 and 9 years old) his teacher asked the class to add all the numbers from 1 to 100 (to keep them busy for an hour, I suppose ☺). After only a couple of moments, Gauss raised his hand. The teacher was annoyed but nonetheless had to listen to him giving . . . the correct answer!

Question: What is the correct answer to the question? With the formula you just proved you can also do it in a couple of moments!

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Sum of integers

Answer: 5050.

Did you replace $n = 100$ in $n(n+1)/2$ and therefore you calculated $(100 \cdot 101)/2 = 10100/2 = 5050$? Good!

But what did young Gauss do? Not knowing the formula, he grouped the numbers into pairs $(1, 100), (2, 99) ... (50, 51)$ realizing that each of these pairs sums to the same number — 101 (or $n+1$ in general). How many pairs are there? Exactly 50 (or, $\frac{n}{2}$). Hence, $50 \cdot 101 = 5050$. Try seeing why this works for an odd n (e.g. 101). [Hint: the middle element is $\frac{n+1}{2}$ in this case.]

Module 6.2: Induction Examples

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Lecture Notes

We prove an inequality by induction I

Problem. Prove $\forall\, n \geq 5 \;\; n^2 < 2^n$.

Answer. First of all, where did 5 come from?

We just tried the first natural numbers one after the other (see the adjacent table). We see that $n^2 < 2^n$ moves between true and false until $n = 5$ and from there on it seems to settle to true. So we **guessed** 5 as a start. And we hope the proof will validate this guess!

n	n^2	2^n	$n^2 < 2^n$
0	0	1	T
1	1	2	T
2	4	4	F
3	9	8	F
4	16	16	F
5	25	32	T
6	36	64	T
7	49	128	T
8	64	256	T

We prove an inequality by induction II

Answer (continued). We use the variant pattern with $n_0 = 5$.

(BC) $(n = 5)$ $5^2 = 25 < 32 = 2^5$. Check.

(IS) Let $k \geq 5$.

Assume (IH) $k^2 < 2^k$. (And WTS $(k+1)^2 < 2^{k+1}$.)

Let's collect what we know:

$$\begin{aligned} k^2 &< 2^k && \text{(IH)} \\ (k+1)^2 &= k^2 + 2k + 1 \\ 2^{k+1} &= 2 \cdot 2^k = 2^k + 2^k \end{aligned}$$

From these we observe that it would suffice to also show $2k + 1 \leq 2^k$. (Why?)

We prove an inequality by induction III

Answer (continued). We will prove a lemma and that will do it.

Lemma. For all $m \in \mathbb{N}$, $m \geq 5$ we have $2m + 1 \leq 2^m$.

Proof of lemma. By ... induction!

(BC) $(m = 5)$ $2 \cdot 5 + 1 = 11 \leq 32 = 2^5$. Check.

(IS) Let $\ell \geq 5$.

Assume (IH) $2\ell + 1 \leq 2^\ell$. (And WTS $2(\ell + 1) + 1 \leq 2^{\ell+1}$.)

Using the IH we derive the following:

$$2(\ell + 1) + 1 \;=\; (2\ell + 1) + 2 \;\leq\; 2^\ell + 2 \;\leq\; 2^\ell + 2^\ell \;=\; 2^{\ell+1}$$

Done.

Induction can be tricky I

Induction can trick us into proving ... false facts!

False proposition! All the sheep in (my grandfather) Abe's flock were the same color!

Wrong proof! By (careless) induction on the size of the flock.

(BC) (Flock of size 1) One sheep, one color. Check.

(IS) Assume (IH) that in any flock of size k the sheep are all the same color.

WTS the same for any flock of size $k+1$. Let F be such a flock.

Tricky induction II

- See drawing of F in the corresponding video lecture segment.

- See also drawing of $F \setminus \texttt{Dolly}$.

- See also drawing of $F \setminus \texttt{Polly}$.

The proof is written out on the next slide.

Tricky induction III

Wrong proof (continued)!

Let F be a flock of size $k+1$.

Take one sheep, `Dolly`, out. The remaining flock $F \setminus \{\texttt{Dolly}\}$ has size k so by IH they are all the same color.

Now put `Dolly` back in and take another sheep, `Polly`, out.

The flock $F \setminus \{\texttt{Polly}\}$ has size k so by IH all its sheep are the same color.

But $\texttt{Dolly} \in F \setminus \{\texttt{Polly}\}$ so `Dolly` must be the same color as the rest!

The proof **must** be wrong! But where is the error?!?

Quiz

Do you think the error in the preceding proof is

A. In the base case?

B. In the induction hypothesis?

C. In the induction step?

Quiz

Do you think the error in the preceding proof is

A. In the base case?

Incorrect. See third answer then continue the video.

B. In the induction hypothesis?

Incorrect. See third answer then continue the video.

C. In the induction step?

Correct. To see **where** continue the video.

Indeed induction can be tricky I

Wrong proof (explained)!

(IS) First of all we forgot to state "let $k \geq 1$" before we stated the (IH): "in any flock of size k the sheep are all the same color".

Then we considered a flock F of size $k+1$.

Since $k \geq 1$ the flock F has size at least 2 so we are sure to find both a `Dolly` and a `Polly` in it.

The flocks $F \setminus \{\texttt{Dolly}\}$ and $F \setminus \{\texttt{Polly}\}$ have indeed size k so the IH applies to them.

Indeed induction can be tricky II

Wrong proof (explained)!

However, we need to be able to assert that `Dolly` is the same color as the rest of the sheep! In particular that `Dolly` and `Polly` are the same color!

For that there **must exist** at least **one more** sheep in $F \setminus \{\texttt{Polly}\}$ in addition to `Dolly`!

See drawing in the corresponding video lecture segment.

And this is true when $k+1 \geq 3$ but false when $k+1=2$!

The IS is true for $k \geq 2$ when, in fact, we need it to be true for $k \geq 1$.

Module 6.3: Strong Induction

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Lecture Notes

We get stuck with ordinary induction I

Problem. Prove that any integer $n \geq 2$ can be written as the product of one or more (not necessarily distinct) prime numbers.

Answer (first attempt). We proceed by ordinary induction.

(BC) $(n_0 = 2)$ 2 is already prime. Check.

(IS) Let $k \geq 2$ arbitrary. Assume (IH) that k can be written as the product of one or more (not necessarily distinct) prime numbers.

Now consider $k+1$. If $k+1$ is prime we are done.

If $k+1$ is not prime then it has a factor r such that $1 < r < k+1$. Moreover, $k+1 = r \cdot s$ such that $1 < s < k+1$ too.

So now we would like to use the IH on r and s.

We get stuck with ordinary induction II

Answer (first attempt, continued).

We have established that $k+1 = r \cdot s$ such that $1 < r, s < k+1$, that is, $2 \leq r, s \leq k$.

Let's show that $r \neq k$. Suppose (toward a contradiction) that $r = k$.

Then $k+1 = ks$ so $1 = k(s-1)$.

Therefore $k = 1$ which contradicts "let $k \geq 2$"

Similarly $s \neq k$.

So we cannot apply the IH to r or s.

On the other hand, since $2 \leq r, s < k$ the induction "process" must have gone through them already! We need a **stronger** induction hypothesis!

Proof pattern for strong induction

Let n_0 be a natural number and let $P(n)$ be a predicate that is well defined for all natural numbers $n \geq n_0$.

Proof pattern.

(BASE CASE) Derive/infer $P(n_0)$.

(INDUCTION STEP) Let $k \in \mathbb{N}$ such that $k \geq n_0$.
Assume $\boxed{P(n_0) \text{ and} \cdots \text{and } P(k)}$.
Derive/infer $P(k+1)$.

Conclude $\forall\, n \geq n_0 \;\; P(n)$.

The IH $P(n_0)$ and $\cdots$ and $P(k)$ is stronger than $P(k)$. But strong induction is mathematically equivalent to the ordinary one!

We succeed with strong induction

Problem. Prove that any integer $n \geq 2$ can be written as the product of one or more (not necessarily distinct) prime numbers.

Answer (second attempt). We proceed by strong induction. The base case is the same.

(IS) Let $k \geq 2$ arbitrary. Assume (IH) that all integers $2, 3, \ldots, k$ can be written as the product of one or more (not necessarily distinct) prime numbers.

Again, if $k+1$ is prime we are done and if $k+1$ is not prime then, as before, $k+1 = r \cdot s$ where $2 \leq r, s < k$.

Now we **can** use the IH on r and s! We have $r = p_1 \cdots p_u$ and $s = q_1 \cdots q_v$. Hence $k+1 = rs = p_1 \cdots p_u \cdot q_1 \cdots q_v$ with all p's and q's prime. Done.

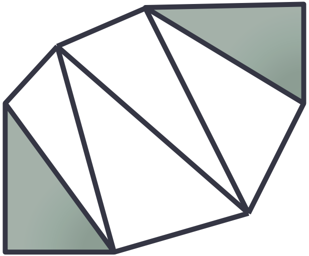
Polygon triangulation I

Problem. Prove that if a polygon with four or more sides is **triangulated** then at least two of the triangles thus formed are **exterior**.

Triangulating a polygon means drawing non-intersecting diagonals until all resulting regions are triangles.

Exterior triangles share **two** of their sides with the polygon.

Example:



Polygon triangulation II

Problem. Prove that if a polygon with four or more sides is **triangulated** then at least two of the triangles thus formed are **exterior**.

Answer. We proceed by strong induction on the number n of vertices of the polygon.

(BC) $(n_0 = 4)$ To triangulate a quadrilateral we draw one diagonal. Both resulting triangles are exterior.

.See drawing in the corresponding video lecture segment.

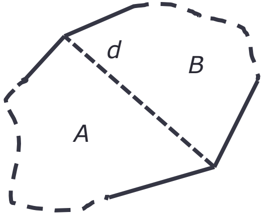
(IS) Let $k \geq 4$. Assume (IH) that for any triangulated polygon with a number of sides between 4 and k at least two of the formed triangles are exterior.

Polygon triangulation III

Answer (continued).

Assume (IH) that for any triangulated polygon with a number of sides between 4 and k at least two of the formed triangles are exterior.

Let P be triangulated with $k+1$ sides. Let d be one of the diagonals (used in the triangulation) which divides P into A and B:



Crucial observation: both A and B have at most k sides!

Polygon triangulation IV

Claim. The triangulation of A has at least one triangle that is exterior for the triangulation of P.

Proof of claim. If A is itself a triangle we are done.

Otherwise, A has between 4 and k sides and the IH applies, so the triangulation of A has at least two triangles which are exterior for A.

At most one of these two triangles has d as a side. Therefore, the other one must be exterior for P as well.

Now we can finish our main proof. The lemma applies to B as well so, in total, we have at least two exterior triangles for P.

Module 6.4: Pizza Cutting Recurrence

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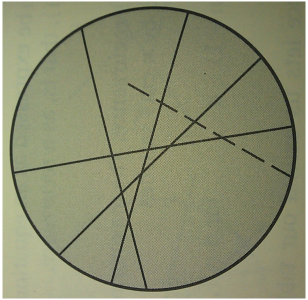
Lecture Notes

The pizza-cutting problem I

Problem. What is the largest number of pieces (not slices!) of pizza that can be made with n distinct straight cuts?

Answer. Since we only want the maximum **number** of pieces, it does not matter where the edge of the pizza is. (In fact this problem is known as Steiner's **Plane** Cutting Problem.)

The following picture shows some cuts that maximize the number of pieces.



The pizza-cutting problem II

Answer (continued). Some experimentation and some reasoning (see the optional segment entitled "Pizza cutting") leads us to the following **maximizing conditions**:

(1) Every cut must cross every other cut.

(2) No three cuts cross each other at the same point.

Let $C(n)$ be the number of pieces produced by a set of n cuts satisfying (1) and (2). Clearly $C(0) = 1$ (the whole pizza!).

When we add the nth cut we add a new piece for intersecting each of the existing $n-1$ cuts, plus one more for intersecting the edge! Therefore:

$$C(0) = 1 \qquad\qquad C(n) = C(n-1) + n$$

This is a **recurrence relation**. Recurrences are extremely useful in the analysis of the running time of algorithms.

Solving the recurrence relation

Answer (continued). Solving the recurrence relation that we obtained

$$C(0) = 1 \qquad\qquad C(n) = C(n-1) + n$$

Method 1. Guess the answer and prove by induction that your guess was correct. Here the answer is $(n^2 + n + 2)/2$.

Method 2. Analyze the "recursion tree" constructed from the recurrence.

Method 3. Use a "telescopic" trick that repeats the recurrence and simplifies terms.

We illustrate methods 2 and 3 in what follows.

The recursion tree method

Answer (continued). We will separate the addition terms in

$$C(0) \;=\; 1 \qquad\qquad C(n) \;=\; C(n-1) \;+\; n$$

We draw a "tree of additions":

$$\begin{array}{ccccccccccc}
n & & n-1 & & \cdots & & 2 & & 1 & & 1 \\
| & & | & & & & | & & | & & | \\
C(n) & — & C(n-1) & —— & \cdots & —— & C(2) & —— & C(1) & —— & C(0)
\end{array}$$

Therefore $C(n) \;=\; (n \;+\; (n-1) \;+\; \cdots \;+\; 2 \;+\; 1) \;+\; 1$

Using the formula $\;C(n) \;=\; n(n+1)/2 \;+\; 1 \;=\; (n^2+n+2)/2$

The "telescopic" method

Answer (continued). We write the recurrence relation for $n, \ldots, 1$:

$$\begin{aligned}
C(n) &= C(n-1) + n \\
C(n-1) &= C(n-2) + n - 1 \\
C(n-2) &= C(n-3) + n - 2 \\
&\cdots \\
C(2) &= C(1) + 2 \\
C(1) &= C(0) + 1
\end{aligned}$$

Add all the LHSs and RHSs and cancel terms that appear on both sides:

$$C(n) = C(0) + 1 + 2 + \cdots + n = 1 + n(n+1)/2 = (n^2 + n + 2)/2$$

(The method is called "telescopic" because the n equalities "collapse" into just one.)

Module 6.5: Fibonacci Numbers

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LECTURE NOTES

The Fibonacci numbers

Problem. A farmer raises rabbits. Each rabbit pair gives birth to another rabbit pair when it turns one month old, and thereafter to one rabbit pair each month. Rabbits never die. How many rabbit pairs will the farmer have at the end of the nth month if he starts with one newborn rabbit pair in the first month?

Answer. Denoting the number of rabbits in month n by F_n, we have the following **recurrence relation**:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \qquad \text{for } n \geq 2 \end{aligned}$$

Hence the sequence of **Fibonacci numbers**:

. $0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad \cdots$

Solving the Fibonacci recurrence I

Proposition. $$F_n = (\varphi^n - \psi^n)/\sqrt{5}$$
where $\varphi > \psi$ are the two roots of the equation $x^2 - x - 1 = 0$.

$\varphi = (1+\sqrt{5})/2 \simeq 1.618$ is known as the **Golden Ratio** (see optional segment "Daisies, sunflowers, and shells") and $\psi = 1 - \varphi = -1/\varphi = (1-\sqrt{5})/2$.

Proof. Can $F_n = F_{n-1} + F_{n-2}$ have a solution $F_n = r^n$ where $r \neq 0$?

Note that $r^n = r^{n-1} + r^{n-2} \quad \Rightarrow \quad r^2 = r + 1$ hence $r^2 - r - 1 = 0$.

Therefore, both $F_n = \varphi^n$ and $F_n = \psi^n$ are solutions of $F_n = F_{n-1} + F_{n-2}$.
But what about $F_0 = 0$ and $F_1 = 1$?

Luckily (because $F_n = F_{n-1} + F_{n-2}$ is **homogeneous**), any **linear combination** of its solutions is also a solution.

Activity : Homogeneous recurrence, linear combination

By calling the Fibonacci recurrence **homogeneous** we point out that all its terms have the same form: $F_{n\pm i}$.

In contrast, we saw in the lecture segment "Pizza cutting recurrence" a non-homogeneous recurrence: $C_n = C_{n-1} + n$.

A **linear combination** of functions $f(n)$ and $g(n)$ is a function of the form $h(n) = \alpha f(n) + \beta g(n)$ where α and β are two real constants.

Activity : Homogeneous recurrence, linear combination (Continued)

Consider the homogeneous recurrence $A_{n+1} = 2A_{n-1} + 3A_{n-3}$. Suppose $f(n)$ and $g(n)$ are two solutions, let $\alpha, \beta \in \mathbb{R}$, and let $h(n) = \alpha f(n) + \beta g(n)$. Here is how we show that $h(n)$ is also a solution of the recurrence:

$$
\begin{aligned}
f(n+1) &= 2f(n-1) + 3f(n-3) && \\
\alpha f(n+1) &= 2\alpha f(n-1) + 3\alpha f(n-3) && \text{multiply by } \alpha \\
g(n+1) &= 2g(n-1) + 3g(n-3) && \\
\beta g(n+1) &= 2\beta g(n-1) + 3\beta g(n-3) && \text{multiply by } \beta \\
h(n+1) &= 2h(n-1) + 3h(n-3) && \text{add sides}
\end{aligned}
$$

Solving the Fibonacci recurrence II

Proof (continued). We have established that for any $\lambda_1, \lambda_2 \in \mathbb{R}$ the sequence defined by $F_n = \lambda_1 \varphi^n + \lambda_2 \psi^n$ satisfies $F_n = F_{n-1} + F_{n-2}$.

Now we determine λ_1, λ_2 to make sure that $F_0 = 0$ and $F_1 = 1$.

$$\begin{aligned} \lambda_1 + \lambda_2 &= 0 \\ \lambda_1 \varphi + \lambda_2 \psi &= 1 \end{aligned}$$

Solving, we get $\lambda_2 = -\lambda_1$

.$\lambda_1 = 1/(\varphi - \psi) = 1/\sqrt{5}$

.$\lambda_2 = 1/(\psi - \varphi) = -1/\sqrt{5}$

We conclude that $F_n = (\varphi^n - \psi^n)/\sqrt{5}$.

Done.

Strong induction (Fibonacci variant) I

Problem. A car needs 1 unit of length to park while a truck needs 2 units of length. Assume that cars are indistinguishable and so are trucks. How many distinct car/truck parking patterns are possible along an n unit long sidewalk?

Answer. We write the parking patterns as a string of `C`'s and `T`'s. Here are two distinct ways in which 3 cars and 2 trucks can be parked along a sidewalk that is 7 units long: `CTCCT` and `TCTCC`.

length	patterns	#
1	`C`	1
2	`CC` `T`	2
3	`CT` `CCC` `TC`	3
4	`CCT` `TT` `CTC` `CCCC` `TCC`	5
5	`CTT` `CCCT` `TCT` `CCTC` `TTC` `CTCC` `CCCCC` `TCCC`	8

Strong induction (Fibonacci variant) II

Answer (continued). We prove by induction that the number of distinct parking patterns along a sidewalk of length $n \geq 1$ is F_{n+1}. It's a special strong induction with an IH only for k and $k-1$, and hence two base cases.

(BC 1) $(n = 1)$ Only 1 pattern, `C`. $F_2 = 1$. Check.

(BC 2) $(n = 2)$ 2 patterns, `CC` and `T`. $F_3 = 2$. Check.

(IS) Let $k \geq 1$. Assume (IH) that the number of patterns for length $k-1$ is F_k and that the number of patterns for length k is F_{k+1}.

Now consider a pattern p for length $k+1$. Depending on whether this pattern ends with a car or a truck, we have two cases.

Strong induction (Fibonacci variant) III

Answer (continued).

Case 1. The last vehicle in p is a car, that is, $p = r\mathtt{C}$.

Then r has length $k + 1 - 1 = k$.

By IH there are F_{k+1} distinct r's.

Therefore in this case we have F_{k+1} distinct patterns.

Case 2. The last vehicle in p is a truck, that is, $p = s\mathtt{T}$.

Then s has length $k + 1 - 2 = k - 1$.

By IH there are F_k distinct s's therefore F_k distinct p's in this case.

By the addition rule, there are $F_{k+1} + F_k = F_{k+2}$ distinct patterns. Done.

ACTIVITY : Proof by induction

In this activity we will give a proof by ordinary induction of **Cassini's identity**:

$$F_{n-1} \cdot F_{n+1} \; - \; F_n^2 \; = \; (-1)^n \qquad\qquad (n \geq 1)$$

First we want to determine the relevant base case for this proof. Since we want to prove this identity for all $n \geq 1$, then we need to set the base case as $n = 1$.

(BC) $F_0 \cdot F_2 \; - \; F_1^2 \; = \; 0 \cdot 1 - 1^2 = -1 = (-1)^1$, so the identity holds for $n = 1$.

(IS) Let $k \geq 1$ be an arbitrary positive integer. Assume the inductive hypothesis. . .

Question: What should the inductive hypothesis be?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Proof by induction (Continued)

Answer: $F_{k-1} \cdot F_{k+1} \; - \; F_k^2 \; = \; (-1)^k$.

Now consider $k+1$. Assuming the inductive hypothesis (IH), we want to show that $F_k \cdot F_{k+2} \; - \; F_{k+1}^2 \; = \; (-1)^{k+1}$, which we derive as follows.

Activity : Proof by induction (Continued)

$$
\begin{aligned}
F_k \cdot F_{k+2} - F_{k+1}^2 &= F_k \cdot (F_{k+1} + F_k) - F_{k+1}^2 \\
&= F_k^2 + F_k \cdot F_{k+1} - F_{k+1}^2 \\
&= F_k^2 + F_{k+1}(F_k - F_{k+1}) \\
&= F_k^2 - F_{k+1}(F_{k+1} - F_k) \\
&= F_k^2 - F_{k+1} \cdot F_{k-1} \\
&= (-1) \cdot (F_{k+1} \cdot F_{k-1} - F_k^2) \\
\text{Here is where we use the IH} \rightarrow \quad &= (-1) \cdot (-1)^k \\
&= (-1)^{k+1} .
\end{aligned}
$$