Module 5.3: Inclusion-exclusion for Cardinality

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Cardinality of union of two sets

When A, B are two **disjoint** sets we have $|\, A \cup B \,| \; = \; |A| + |B|$.
But what can we say when the sets are **not** disjoint?

$|A| + |B|$ **overcounts**. It counts twice the elements in **both** A and B.

Subtracting those, we get $\quad |\, A \cup B \,| \; = \; |A| + |B| - |\, A \cap B \,|$.

This is called the **Principle of Inclusion-Exclusion (PIE)** for two sets.

Inclusion because we include the count of the elements of A and of B.
Exclusion because we exclude the count of the elements common to both A and B.

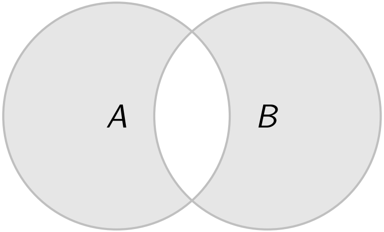
Activity : Principle of inclusion-exclusion

In this activity, we will prove the principle of inclusion-exclusion,

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

in two ways.

First, consider an Euler-Venn diagram in which $B \setminus A$ and $A \setminus B$ and $A \cap B$ appear.



Activity : Principle of inclusion-exclusion (continued)

Observe that $A \setminus B$, $A \cap B$, and $B \setminus A$ are pairwise disjoint and that the following three equations hold.

$$\begin{aligned} A \cup B &= (A \setminus B) \cup (A \cap B) \cup (B \setminus A) \\ A &= (A \setminus B) \cup (A \cap B) \\ B &= (B \setminus A) \cup (A \cap B) . \end{aligned}$$

We can apply the addition rule to each equation to see that

$$\begin{aligned} |A \cup B| &= |A \setminus B| + |A \cap B| + |B \setminus A| \\ |A| &= |A \setminus B| + |A \cap B| \\ |B| &= |B \setminus A| + |A \cap B| . \end{aligned}$$

ACTIVITY : Principle of inclusion-exclusion (continued)

From these last three equations, we can derive the Principle of Inclusion-Exclusion:

$$\begin{aligned}|A \cup B| &= |A \setminus B| + |A \cap B| + |B \setminus A| \\ &= |A| - |A \cap B| + |A \cap B| + |B| - |A \cap B| \\ &= |A| + |B| - |A \cap B| \,.\end{aligned}$$

This completes the proof.

Activity : Principle of inclusion-exclusion (continued)

An alternative approach is applying the addition rule four times to see that

$$\begin{aligned}
|A \cup B| &= |A| + |B \setminus A| \\
|A \cup B| &= |B| + |A \setminus B| \\
|A| &= |A \setminus B| + |A \cap B| \\
|B| &= |B \setminus A| + |A \cap B|
\end{aligned}$$

The sum of the first two equations is

$$2|A \cup B| = |A| + |B| + |B \setminus A| + |A \setminus B| .$$

Substituting in the last two equations into this and canceling like terms gives

$$2|A \cup B| = 2|A| + 2|B| - 2|A \cap B| .$$

Dividing both sides by 2 yields the PIE.

Cardinality of union of three sets

The **Principle of Inclusion-Exclusion (PIE)** for **three** sets:

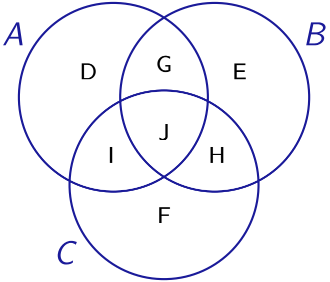
$$\begin{aligned} |\, A \cup B \cup C \,| \;&=\; |A| \;+\; |B| \;+\; |C| \\ &\quad -\; |\, A \cap B \,| \;-\; |\, B \cap C \,| \;-\; |\, A \cap C \,| \\ &\quad +\; |\, A \cap B \cap C \,| \end{aligned}$$

We can justify this similarly. An element in $(A \cap B) \setminus (A \cap B \cap C)$ is added to the count **twice** in $|A| + |B| + |C|$. This justifies subtracting $-|A \cap B|$.

An element in $A \cap B \cap C$ is added to the count **three times** in $|A| + |B| + |C|$ and then subtracted **three times** in $-|A \cap B| - |B \cap C| - |A \cap C|$. This justifies adding $+|A \cap B \cap C|$ at the end.

Activity : Understanding PIE

Consider the following Euler-Venn diagram of sets A, B, and C, with regions labeled D through J.



ACTIVITY : Understanding PIE (Continued)

Question: Identify the three regions whose elements are counted exactly once in $|A| + |B| + |C|$.

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Understanding PIE (Continued)

Answer: The regions are D, E, and F.

The three regions whose elements are counted twice in $|A| + |B| + |C|$ and once in $|A \cap B| + |B \cap C| + |A \cap C|$ are $\boxed{\text{G, H, and I}}$.

The one region whose elements are counted three times in $|A| + |B| + |C|$ and three times in $|A \cap B| + |B \cap C| + |A \cap C|$ is $\boxed{\text{J}}$.

Counting by divisibility criteria I

Problem. How many integers in $[1..150]$ are divisible by 3, or by 5, or by 7?

Answer. Let's first ask a simpler question. Given integers $1 < k < n$, how many multiples of k are in $[1..n]$?

Let m be the largest multiple of k that is smaller than (or equal to) n.

Then there are m/k multiples of k in $[1..n]$. Why?

Because n lies between m (multiple of k) and $m + k$ (next multiple of k).

For example, there are $150/3 = 50$ multiples of 3 and $150/5 = 30$ multiples of 5 in $[1..150]$.

As for the multiples of 7, note that 147 is a multiple of 7. Since $147/7 = 21$, there are 21 multiples of 5 in $[1..150]$.

Quiz I

We divide 150 by 35. The (integer) quotient and the remainder are:

A. 3 and 45.
B. 10 and 4.
C. 4 and 10.

Answer

We divide 150 by 35. The (integer) quotient and the remainder are:

A. 3 and 45.
   Incorrect.

B. 10 and 4.
   Incorrect.

C. 4 and 10.
   Correct.

Counting by divisibility criteria II

Answer (continued). We introduce some notation:

$$A = \{n \mid n \in [1..150] \text{ and } 3 \mid n\}$$
$$B = \{n \mid n \in [1..150] \text{ and } 5 \mid n\}$$
$$C = \{n \mid n \in [1..150] \text{ and } 7 \mid n\}$$

The problem asks for $|A \cup B \cup C|$. Sets **overlap**: use PIE.

We saw on the previous slide that $|A| = 50, |B| = 30$ and $|C| = 21$.

Note that $3, 5, 7$ are primes. Therefore

$A \cap B$ consists of the multiples of $3 \cdot 5 = 15$,

$|A \cap C|$ consists of the multiples of $3 \cdot 7 = 21$,

$|B \cap C|$ consists of the multiples of $5 \cdot 7 = 35$,

$|A \cap B \cap C|$ consists of the multiples of $3 \cdot 5 \cdot 7 = 105$.

Counting by divisibility criteria III

Answer (continued).

Similarly to how we computed $|A|, |B|$ and $|C|$ we obtain:

$|A \cap B| = 150/15 = 10,$

$|A \cap C| = 147/21 = 7,$

$|B \cap C| = 140/35 = 4,$ and

$|A \cap B \cap C| = 105/105 = 1.$

By PIE we have

$$|A \cup B \cup C| = 50 + 30 + 21 - 10 - 7 - 4 + 1 = 81$$

Derangements I

Problem. n hat-wearing gangsters leave their distinguishable hats with a restaurant cloakroom attendant. After the meal, the attendant gives them back their hats in a such a way that none of the gangsters gets their own hat.

The returned hats form what is called a "derangement" or a "deranged permutation". How many derangements are possible?

Answer. Let's say the gangsters are $G_1, G_2, \ldots, G_n$ and their respective hats are $h_1, h_2, \ldots, h_n$ (G_i's hat is h_i).

A derangement is a permutation of the set $H = \{h_1, \ldots, h_n\}$ in which h_i does **not** occur in position i for any $i = 1, \ldots, n$.

For example, when $n = 3$ we have only 2 derangements:

. $h_2 h_3 h_1$ $h_3 h_1 h_2$

Quiz II

How many derangements of 4 elements are there?

A. 6

B. 8

C. 9

Answer

How many derangements of 4 elements are there?

A. 6

Incorrect. Please refer to the next slide for more information.

B. 8

Incorrect. Please refer to the next slide for more information.

C. 9

Correct. Please refer to the next slide for more information.

More Information

There are three cases for derangement of 4 elements.

Case 1: a_2 is in position 1.

Case 1.1: a_1 is in position 2. Then the rest of the elements must form a derangement of a set with two elements, and there is exactly one of those.

Case 1.2: a_1 is not in position 2. Then we can replace a_1 with a_2 and erase the first element of the sequence. The result is a derangement of a set with 3 elements and we counted those, there are two of them.

For Case 1, there are $\boxed{1 + 2 = 3}$ possible derangements.

Case 2: a_3 is in position 1. This case is symmetric to Case 1. Therefore, there are three derangements in this case.

Cases 3: a_4 is in position 1.This case is symmetric to Case 1. Therefore, there are three derangements in this case.

In total, there are $\boxed{3+3+3=9}$ derangements.

Derangements II

Problem. Count the number of derangements of n elements.

Answer (continued). The idea is to count **complementarily**.

Define B_i to be the set of permutations in which h_i **does** occur in position i.

Then the set of permutations that are **not** derangements is $B_1 \cup \cdots \cup B_n$.

The total number of permutations is $n!$.

Hence the number of derangements is $n! \; - \; |B_1 \cup \cdots \cup B_n|$.

This ends up as

$$n! \; \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$$

$B_1, \ldots, B_n$ clearly overlap: need a general PIE!

Read the continuation in a segment entitled "Derangements".