CIT 592 Spring 2022 Homework 4

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1. [10 pts] We can use the binomial theorem to expand $(2x+7)^{12}$ and $(2+7x)^{12}$ as sums of terms (monomials). Indicate all the terms (monomials) that are the same in both sums and explain why you know this, without fully expanding out all of the terms.

Solution.

From Binomial Theorem, we know that any term in the sum of $(2x+7)^{12}$ can be expressed as $\binom{12}{a} * (2x)^{12-a} * 7^a$, where $0 \le a \le 12$.
From Binomial Theorem, we know that any term in the sum of $(2+7x)^{12}$ can be expressed as $\binom{12}{b} * 2^{12-b} * (7x)^b$, where $0 \le b \le 12$.
The common terms in the sums of $(2x+7)^{12}$ and $(2+7x)^{12}$ can be expressed as
$\binom{12}{a} * (2x)^{12-a} * 7^a = \binom{12}{b} * 2^{12-b} * (7x)^b$
Which can be expressed as:
$\binom{12}{a} * 2^{12-a} * x^{12-a} * 7^a = \binom{12}{b} * 2^{12-b} * x^b * 7^b$
Therefore:
$\binom{12}{a} = \binom{12}{b}$, which means $a = b$ or $a = 12 - b$
$2^{12-a} = 2^{12-b}$ which means $a = b$
$x^{12-a} = x^b$, which means $12 - a = b$
$7^a = 7^b$, which means $a = b$
In summary we can say a and b needs to satisfy: $a = b$ and $a + b = 12$
So we can calculate that $a = b = 6$.
Therefore their common term would be $\binom{12}{6} * 2^6 * x^6 * 7^6$

2. [10 pts] For each of the following functions, prove that the function is neither injective nor surjective. Then, show how you could restrict the domain and codomain — without changing the mapping rule — to make the function both injective and surjective. (Restricting the domain (codomain) means replacing it with a subset that must be clearly defined using our notation, including, if needed, set-builder notation.) Your restricted domain and codomain should be *as large as you can think of* (you get more points for larger domains and codomains).

For example, the function $f : [-3..3] \to [-9..9]$ given by $f(x) = x^2$ is neither injective (since, e.g., $f(-3) = f(3)$) nor surjective (since, e.g., 2 is not a square), but we can restrict the domain and codomain to define the function $f_1 : [0..3] \to \{0, 1, 4, 9\}$ with the same mapping rule, that is, $f_1(x) = x^2$, which is both injective and surjective. The codomain is as large as possible because adding every other number in $[-9..9]$ that is not a square would break surjectivity and the domain is as large as possible because adding any negative number breaks injectivity.

(a) $g : \{\mathtt{v}, \mathtt{x}, \mathtt{y}, \mathtt{z}\} \to \{\mathtt{a}, \mathtt{e}, \mathtt{i}, \mathtt{o}, \mathtt{u}\}$ given by:

x	$g(x)$
v	i
x	o
y	e
z	i

(b) $h : 2^{[1..n]} \to [0..2n]$ given by $h(S) = |S|$ where $n \geq 2$.

Solution.

(a) Section 1: Proof the function g is neither injective or surjective.

Step 1: Since g(v) = g(z) = i, by definition the function $g : \{\mathtt{v}, \mathtt{x}, \mathtt{y}, \mathtt{z}\} \to \{\mathtt{a}, \mathtt{e}, \mathtt{i}, \mathtt{o}, \mathtt{u}\}$ is not injective.

Step 2: Since a and u are in codomain but can't find any element x in domain that maps to a or u, by definition the function $g : \{\mathtt{v}, \mathtt{x}, \mathtt{y}, \mathtt{z}\} \to \{\mathtt{a}, \mathtt{e}, \mathtt{i}, \mathtt{o}, \mathtt{u}\}$ is not surjective.

Step 3: From above, we can proof the function $g : \{\mathtt{v}, \mathtt{x}, \mathtt{y}, \mathtt{z}\} \to \{\mathtt{a}, \mathtt{e}, \mathtt{i}, \mathtt{o}, \mathtt{u}\}$ is neither injective nor surjective.

Section 2: Make the function g both injective and surjective.

Step 1: Since no element in the domain maps to a or u, I would restrict the codomain to $\{\mathtt{e}, \mathtt{i}, \mathtt{o}\}$ so that every element in the codomain can find an element in the domain that maps to it, which, by definition, makes the function surjective.

Step 2: Since g(v) = g(z) = i, I would restrict the domain to $\{\mathtt{v}, \mathtt{x}, \mathtt{y}\}$ or $\{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$ to make every element in the domain maps to only one element in the codomain, which, by definition, makes the function injective.

Step 3: In summary the function would be restricted to $g : \{\mathtt{v}, \mathtt{x}, \mathtt{y}\} \to \{\mathtt{e}, \mathtt{i}, \mathtt{o}\}$ or $g : \{\mathtt{x}, \mathtt{y}, \mathtt{z}\} \to \{\mathtt{e}, \mathtt{i}, \mathtt{o}\}$ to be both injective and surjective.

(b) Section 1: Proof the function h is neither injective or surjective.

Step 1: When $n = 2$, $[1..n] = [1..2]$, so the domain is 2^A where $A = \{1, 2\}$, which means the domain is $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Since $h(\{1\}) = h(\{2\}) = 1$, by definition the function $h : 2^{[1..n]} \to [0..2n]$ is not injective.

Step 2: The subset with the most elements (largest cardinality) would be the set itself, which is the set $\{1, ..., n\}$, and its cardinality $h(\{1, ..., n\}) = n$. The subset with the least elements (smallest cardinality) would be the empty set, and its cardinality $h(\{\emptyset\}) = 0$. Therefore $Ran(h) = [0..n]$. Since the codomain is $[0..2n]$, the codomain is larger than $Ran(h)$, and no elements in domain maps to $[n + 1..2n]$. Therefore by definition the function $h : 2^{[1..n]} \to [0..2n]$ is not surjective.

Step 3: From above, we can proof the function $h : 2^{[1..n]} \to [0..2n]$ is neither injective nor surjective.

Section 2: Make the function h both injective and surjective.

Step 1: Since the powerset of $[1..n]$ has multiple subsets that have the same cardinality, I would restrict the domain to $2^{[1..n]} \cap \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, ..., n\}\}$.

The intersection of $2^{[1..n]}$ and $\{\emptyset, \{1\}, \{1, 2\}, \{1, 2, ..., n\}\}$ is a subset of is $2^{[1..n]}$ and essentially the same with $\{\emptyset, \{1\}, \{1, 2\}, \{1, 2, ..., n\}\}$ so no two subsets share the same cardinality, which means for each element x in domain maps only one elements $h(S)$ in the codomain. By definition the function h would be injective.

Step 2: Since the cardinalities of elements in $\{\emptyset, \{1\}, \{1, 2\}, \{1, 2, ..., n\}\}$ are $0, 1, 2, ..., n$, it means $Ran(h) = [0..n]$. Therefore I would revise the codomain to $[0..n]$ to make every element in the domain can find an element in the doamin that maps to it, which, by defintion, makes the function surjective.

Step 3: In summary the function would be restricted to $h : 2^{[1..n]} \cap \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, ..., n\}\} \to [0..n]$ to be both injective and surjective.

3. [**10 pts**] Answer the following questions about a scheduling system for assigning TAs to office hours. Within the system, every single TA and every single office hour time is assigned an integer ID. These IDs start with 1 and increment by 1. That is, if there is a TA with ID equal to 6, there must be TAs with IDs equal to 1, 2, 3, 4 and 5. The same restriction applies to the office hour IDs. Note that any office hour not assigned a TA will be covered by Professor Tannen.

(a) Let p, q, r, s be integers with $p \leq q \leq r \leq s$. Consider the TAs with IDs ranging from p to r inclusive, and consider the office hour slots with IDs ranging from q to s, inclusive. How many distinct functions for assigning TAs to office hours are there? (The TAs are the domain and the office hours are the codomain)

(b) Let n be a positive integer. Suppose there are n TAs and $2n$ office hour slots. How many distinct functions for assigning TAs to office hours are there, such that every TA is assigned an office hour with an ID that is either strictly less than their ID or greater than or equal to two times their ID? (ex. if $n = 10$, TA 4 can be assigned office hour x, where $x < 4$ or $x \geq 8$)

Solution.

(a) The number of TAs $= [p..r] = r - p + 1$
The number of OHs $= [q..s] = s - q + 1$
Let A is the set of TAs, then $A = \{TA_1, TA_2, ..., TA_{r-p+1}\}$ and $|A| = r - p + 1$.
Let B is the set of OHs, then $|B| = s - q + 1$.
We can construct a function from A to B in $r - p + 1$ steps, $i = 1, 2, ..., r - p + 1$.
In step (i) we choose $OH \in B$ to define $f(TA_i) = OH$, that is $TA_i \mapsto OH$. This can be done in $s - q + 1$ ways.
By the multiplication rule, the number of function is $(s-q+1)*(s-q+1)*...*(s-q+1) = (s - q + 1)^{r-p+1}$.
Therefore there are $(s - q + 1)^{r-p+1}$ distinct functions for assigning TAs to office hours.

(b) Let A is the set of TAs, then $A = \{TA_1, TA_2, ..., TA_n\}$
For any TA_i, his/her office hour x options is either $x < i$, where the amount of x is $i - 1$ or $2i \leq x \leq 2n$, where the amount of x is $2n - (2i - 1)$.
Let B is the set of the OH for TA_i, then $|B| = (i - 1) + 2n - (2i - 1) = 2n - i$
So we can say for any TA_i, there have $2n - i$ ways to assign he/she to an OH.Therefore for n TAs, there are $(2n - 1) * (2n - 2) * (2n - 3) * ... * (2n - n)$ distinct functions.

4. [10 pts] Recall that a *combinatorial proof* for an identity proceeds as follows:

1. State a counting question.
2. Answer the question in two ways:
   (i) one answer must correspond to the left-hand side (LHS) of the identity
   (ii) the other answer must correspond to the right-hand side (RHS).
3. Conclude that the LHS is equal to the RHS.

With that in mind, give a combinatorial proof of the identity

$$\binom{2n}{n}\binom{n}{2} = \binom{2n}{2}\binom{2n-2}{n-2}$$

where $n \geq 2$.

Solution.

Step 1: We pose the following counting question.
Consider a group of 2n people. How many ways are there to form a club of total n people from the group with 2 of the n people are leaders?
Step 1(LHS): First we want to select n people out of the group of 2n to form the club - a club of n people is a subset of size 2n, therefore there are $\binom{2n}{n}$ distinct such clubs. Then we select 2 people out of the n club members as leaders - a set of 2 leaders is a subset of club with n people, therefore there are $\binom{n}{2}$ ways to select the leaders. This gives us $\binom{2n}{n}\binom{n}{2}$ ways, which is the LHS.
Step 2 (RHS): First we select 2 people as leaders from a group of 2n people - a set of 2 leaders is a subset of group with 2n people, therefore there are $\binom{2n}{2}$ ways to select the leaders. Then we want to select n-2 people out of the 2n-2 people left in group to form the rest of the club - a club of n-2 people is a subset of size 2n-2, therefore there are $\binom{2n-2}{n-2}$ distinct such clubs.This gives us $\binom{2n}{2}\binom{2n-2}{n-2}$ ways, which is the RHS.
Step 3: Therefore we conclude that the LHS is equal to the RHS. The proof is finished.

5. [10 pts] For each of the following, prove that $|A| \leq |B|$ by defining an injective function $f : A \to B$ and then using the injection rule.

(a) A is any set and $B = 2^A$.

(b) A is the set of all prime numbers and B is the real interval $[0, 1]$. (Even though these sets have infinite cardinalities, the injection rule still applies!)

Solution.

(a) Step 1: We define a function $f : A \to 2^A$ where $f(x) = 2^A$.
Step 2: By definition, 2^A is the powerset of A whose elements are all the subsets of A. Since different element will have different powersets, and the same elements will have the same powersets, we can conclude that for every $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$. Thus function $f : A \to B$ is injective.
Step 3: By the injection rule, $|A| \leq |Ran(f)|$. Since $Ran(f) = B$, we can also conclude $|A| \leq |B|$.

(b) Step 1: We define a function $f : A \to 1/A$ where $f(x) = 1/A$.
Step 2: Since A is the set of all prime numbers, we know $A > 1$, thus $0 < 1/A < 1$. Since different elements x_1 and x_2 in A will result in different $1/x_1$ and $1/x_2$, and the same elements A will result the same, we can conclude that for every $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.Thus function $f : A \to B$ is injective.
Step 3: By the injection rule, $|A| \leq |Ran(f)|$. Since $Ran(f)$ and B both have infinity cardinalities, $|Ran(f)| = |B|$, we can also conclude $|A| \leq |B|$.