Module 8.1: Inclusion-exclusion for Probability

MCIT Online - CIT592 - Professor Val Tannen

LECTURE NOTES

At least one die shows a six I

We computed **for two dice** the probability of the event "at least one of the dice shows a 6" using **P6** (inclusion-exclusion).

Problem. We roll three fair dice. Compute the probability of the event "at least one of the dice shows a 6"

Answer. The probability space has $6 \cdot 6 \cdot 6 = 216$ outcomes and is uniform .

Let the dice be d_1, d_2, d_3 and define $D_i =$ "d_i shows a 6", $i = 1, 2, 3$.

The event of interest is $D_1 \cup D_2 \cup D_3$.

D_1, D_2, D_3 are **not** pairwise disjoint so **P2gen** does not apply.

Need **P6** but for 3 events!

Inclusion-exclusion for three events

Proposition. For any events A, B, C in the same probability space

$$\begin{aligned}\Pr[A \cup B \cup C] \;=\;& \Pr[A] + \Pr[B] + \Pr[C] \\ & - \Pr[A \cap B] - \Pr[B \cap C] - \Pr[C \cap A] \\ & + \Pr[A \cap B \cap C]\end{aligned}$$

The proof is in the segment entitled "Inclusion-exclusion for three events".

Answer (continued).

We will apply this to $\Pr[D_1 \cup D_2 \cup D_3]$.

At least one die shows a six II

Answer (continued). We calculate the probabilities in the formula.

$$\Pr[D_1] = \Pr[D_2] = \Pr[D_3] \ = \tfrac{36}{216} = 1/6$$

$$\Pr[D_1 \cap D_2] = \Pr[D_2 \cap D_3] = \Pr[D_3 \cap D_1] \ = \tfrac{6}{216} = 1/36$$

$$\Pr[D_1 \cap D_2 \cap D_3] \ = 1/216$$

Now we apply the formula.

$$\Pr[D_1 \cup D_2 \cup D_3] \ = \ 1/6 + 1/6 + 1/6 - 1/36 - 1/36 - 1/36 + 1/216 = 91/216$$

Note that for two dice we had $11/36 \simeq 0.31$. For three dice we have, of course, a bigger probability $91/216 \simeq 0.42$.

At least one die shows a six III

Problem. We roll **ten** fair dice. Compute the probability of the event "at least one of the dice shows a 6".

Answer. The probability space has 6^{10} outcomes and is uniform.

To proceed like we did for three dice we would need an inclusion-exclusion formula for ten events! Such a formula exists but it is completely unwieldy.

Are we stuck? We forgot one advantage: the ten dice roll **independently**!

Using independence we can find the answer: $1 - (5/6)^{10}$! But **how**? We will come back to this problem as we study independence.

A car and two goats

On a game show there are three doors. There is a car behind one of the doors and goats (!) behind the others.

The contestant chooses a door. Then the host opens a **different** door behind which there is **always a goat**. The contestant is then given a choice whether to **switch** to the **third** door or not.

The contestant wins as a prize whatever is behind the door she or he chooses. Is it to the contestant's **benefit** to switch doors?

The surprising and counterintuitive answer is **"yes"!**

This problem has a fascinating history (see the segment entitled "The Monty Hall problem").

The argument for "no" relies on assuming that the contestant's decision is **independent** of the choice of door opened first.