MCIT 592 Spring 2022 **Homework 8** **Solutions**

This problem set will refer to a *standard deck of cards*. Each card in such a deck has a *rank* and a *suit*. The 13 ranks, ordered from lowest to highest, are *2*, *3*, *4*, *5*, *6*, *7*, *8*, *9*, *10*, *jack*, *queen*, *king*, and *ace*. The 4 suits are *clubs* (♣), *diamonds* (♢), *hearts* (♡), and *spades* (♠). Cards with clubs or spades are *black*, while cards with diamonds or hearts are *red*. The deck has exactly one card for each rank–suit pair: *4 of diamonds*, *queen of spades*, etc., for a total of $13 \cdot 4 = 52$ cards.

Problem Set

1. [10 pts] Suppose you select a card uniformly at random from a standard deck, and then without putting it back, you select a second card uniformly at random from the remaining cards. What is the probability that both cards have rank no lower than 6, and at least one of the cards is red?

Solution:

There are $52 \cdot 51$ possible outcomes. These outcomes are our sample space Ω. Note that Ω has a uniform probability distribution, which will allow us to find the probability by simply dividing event space size by sample space size. Let E be the event where both cards have rank no lower than 6 and at least one of the cards is red. There are 9 ranks (6, 7, 8, 9, 10, *jack*, *queen*, *king*, *ace*) that are not lower than 6 - note that we defined at the beginning of the assignment that *ace* is the highest rank. The number of outcomes in our sample space satisfying the condition that both cards have rank no lower than 6 is $36 \cdot 35$, since there 4 suits of each rank. This number is overcounting since it includes the outcomes in which both of the cards are black. Account for this overcounting by subtracting out those cases: $36 \cdot 35 - 18 \cdot 17$. (There are two black suits per rank.) Thus, the final probability is:

$$Pr[E] = \boxed{\frac{36 \cdot 35 - 18 \cdot 17}{52 \cdot 51}} = \frac{954}{2652} = \frac{159}{442}$$

2. [10 pts] Mary and Emily have two distinguishable gardens. Initially, each of the gardens contains four roses and five daisies. Mary first picks a flower uniformly at random from the left garden and moves it to the right garden. Then, Emily picks a flower uniformly at random from the right garden.

What is the probability that Emily picks a rose?

Solution:

The sample space here is the possibilities of what Mary and Emily pick. $\Omega = \{RR, RD, DR, DD\}$ where R represents "picks a rose" and D represents "picks a daisy." Let R_1 = "Mary picks a rose," D_1 = "Mary picks a daisy," R_2 = "Emily picks a rose," and D_2 = "Emily picks a daisy." Then we want to find $\Pr[R_2]$.

R_2 consists of two outcomes: Mary picks a rose and Emily picks a rose, or Mary picks a daisy and Emily picks a rose. More formally, $R_2 = (R_2 \cap R_1) \cup (R_2 \cap D_1)$. And since $R_2 \cap R_1$ and $R_2 \cap D_1$ are disjoint, the addition rule applies. This gives us

$$\begin{aligned}\Pr[R_2] &= \Pr[R_2 \cap R_1] + \Pr[R_2 \cap D_1] \\ &= \Pr[R_2 \mid R_1] \cdot \Pr[R_1] + \Pr[R_2 \mid D_1] \cdot \Pr[D_1]\,.\end{aligned}$$

Now, four out of the nine flowers initially in the left pond are roses, so $\Pr[R_1] = 4/9$ and $\Pr[D_1] = 5/9$. If Mary moves a rose to the right garden, then five out of the ten flowers there when Emily makes her choice are roses. If she instead moves a daisy, then four of the ten flowers are roses. Thus, $\Pr[R_2 \mid R_1] = 5/10 = 1/2$ and $\Pr[R_2 \mid D_1] = 4/10 = 2/5$. We have

$$\Pr[R_2] = \frac{1}{2} \cdot \frac{4}{9} + \frac{2}{5} \cdot \frac{5}{9} = \frac{2}{9} + \frac{2}{9} = \boxed{\frac{4}{9}}.$$

3. **[10 pts]** For each of three golfers g_1, g_2, g_3 the probabilities of hitting a ball on the green (the desired play!), in a bunker (sand trap), or in a water hazard are given by the following table (we make the simplifying assumption that these are the only three results of a hit):

golfer	green	bunker	water
g_1	1/2	1/3	1/6
g_2	1/3	1/6	1/2
g_3	1/6	1/2	1/3

The three golfers, playing together, hit one ball each, mutually independently.

(a) **[4 pts]** What is the probability that all three balls end up in the water?

(b) **[6 pts]** What is the probability that at least one of the balls ends up on the green?

Solution:

(a) Let $W_i,\ i = 1,2,3$ be the event that golfer $g_i,\ i = 1,2,3$ hits their ball in the water. The event that all three balls end up in the water is $W_1 \cap W_2 \cap W_3$. By mutual independence

$$\Pr[W_1 \cap W_2 \cap W_3] \;=\; \Pr[W_1] \cdot \Pr[W_2] \cdot \Pr[W_3] \;=\; \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{3} \;=\; \boxed{\frac{1}{36}}$$

(b) Let $G_i,\ i = 1,2,3$ be the event that golfer $g_i,\ i = 1,2,3$ hits their ball on the green. The event that at least one of the balls ends up on the green is $G_1 \cup G_2 \cup G_3$. By using probability properties, De Morgan's laws, mutual independence and properties of independence we obtain:

$$\begin{aligned}
\Pr[G_1 \cup G_2 \cup G_3] &= 1 - \Pr[\overline{G_1 \cup G_2 \cup G_3}] \\
&= 1 - \Pr[\overline{G}_1 \cap \overline{G}_2 \cap \overline{G}_3] \\
&= 1 - \Pr[\overline{G}_1] \cdot \Pr[\overline{G}_2] \cdot \Pr[\overline{G}_3] \\
&= 1 - (1 - \Pr[G_1]) \cdot (1 - \Pr[G_2]) \cdot (1 - \Pr[G_3]) \\
&= 1 - (1 - \frac{1}{2}) \cdot (1 - \frac{1}{3}) \cdot (1 - \frac{1}{6}) = \boxed{\frac{13}{18}}
\end{aligned}$$

Instead of the derivation shown above we could have directly used the proposition on unions of mutually independent events that we have stated in the lecture segment entitled "Pairwise and mutual independence". it

4. [10 pts] 25 participants enter a sushi-making competition! Each of them must make 4 rolls with different proteins: a salmon roll, a tuna roll, a shrimp roll, and a crab roll, for a total of 100 total rolls made in the competition. All of the rolls are distinguishable, even if they have the same protein (ex. a salmon roll is distinguishable from all other salmon rolls, as well as all the other rolls with different protein). According to the competition rules, salmon and tuna rolls are *always* made with white rice, while shrimp and crab rolls are *always* made with brown rice. The judges select a sushi roll uniformly at random from the 100 rolls. Without putting it back, they select a second roll uniformly at random from the remaining 99 rolls. What is the probability that both of the rolls are made by the same participant, or both have the same protein, or one has white rice and one has brown rice?

Solution:

There are $100 \cdot 99$ possible outcomes. These outcomes are our sample space Ω. Note that Ω has a uniform probability distribution which will allow us to find probability by simply dividing event space size by sample space size. Let R be the event that both rolls are made by the same participant, S be the event that both rolls have the same protein, and C be the event that one has white rice and one has brown rice. We are interested in $\Pr[R \cup S \cup C]$, which we will calculate using the principle of inclusion-exclusion for three events.

There are $100 \cdot 3$ ways to choose the rolls such that both are made by the same participant: 100 ways to choose the first roll and then $4 - 1 = 3$ ways to choose a second roll from the same participant from the remaining rolls. Thus, by the multiplication rule, $|R| = 100 \cdot 3$. Similarly, $|S| = 100 \cdot 24$ (there are 100 ways to choose the first rolls and 24 ways to choose a second roll with the same protein). There are 100 ways to choose the first roll, and then 50 ways to choose a second roll with a different type of rice from the remaining rolls, so $|C| = 100 \cdot 50$. We have $\Pr[R] = |R|/|\Omega| = 100 \cdot 3/(100 \cdot 99) = 3/99$, $\Pr[S] = 100 \cdot 24/(100 \cdot 99) = 24/99$, and $\Pr[C] = 100 \cdot 50/(100 \cdot 99) = 50/99$.

Once the first roll has been chosen, there are no remaining rolls from the same participant with the same protein, so $|R \cap S| = 0$. There is also no way for the two rolls to have the same protein but different types of rice, so $|S \cap C| = 0$. Once the first roll has been chosen, there are 2 ways to choose a second roll from the same participant but with a different type of rice, so $|R \cap C| = 100 \cdot 2$. Finally, $|R \cap S \cap C| = 0$ because $R \cap S \cap C \subseteq R \cap S = \emptyset$. We have $\Pr[R \cap C] = |R \cap C|/|\Omega| = 100 \cdot 2/(100 \cdot 99) = 2/99$ and $\Pr[R \cap S] = \Pr[S \cap C] = \Pr[R \cap S \cap C] = 0$.

We can now apply the principle of inclusion-exclusion for three events:

$$\begin{aligned}
\Pr[R \cup S \cup C] &= \Pr[R] + \Pr[S] + \Pr[C] - \Pr[R \cap S] - \Pr[S \cap C] - \Pr[R \cap C] + \Pr[R \cap S \cap C] \\
&= \frac{3}{99} + \frac{24}{99} + \frac{50}{99} - 0 - 0 - \frac{2}{99} + 0 \\
&= \boxed{\frac{75}{99}}.
\end{aligned}$$

5. [10 pts] Alice and Bob each have a standard deck of cards. First, Alice selects uniformly at random a card from her deck, adds it to Bob's deck and then shuffles his deck. Second, Bob selects uniformly at random a card from his deck, adds it to Alice's deck and then shuffles her deck. Finally, Alice selects uniformly at random a card from her deck. What is the probability that this card's suit is diamond?

Solution:

To simplify the vocabulary in this problem, let's categorize the cards in the deck into two groups: diamond (D) or not a diamond (N). Represent the sample space with a string of letters representing whether the drawn cards were diamonds or not diamonds. The possible outcomes are $\{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$.

Let E be the event where the final card Alice draws is a diamond. Approach this problem with cases, summing up the probabilities of each of the favorable outcomes which make up E: $DDD, DND, NDD,$ and NND. For each of these cases, apply the chain rule. The full explanation will be detailed for one of these cases, but then the same logic will be carried over to the other 3.

DDD: For this outcome, there are 3 events that have to happen. First Alice has to pick a diamond (D_1), then Bob also has to pick a diamond (D_2) and finally Alice has to pick a diamond (D_3). Since these events all impact each other, the chain rule must be applied. That is:

$$\Pr[DDD] = \Pr[D_1] \cdot \Pr[D_2 \mid D_1] \cdot \Pr[D_3 \mid D_2 \cap D_1]$$

$\Pr[D_1] = \frac{13}{52}$, because 13 of the cards are diamonds. $\Pr[D_2 \mid D_1] = \frac{14}{53}$ though, since an extra diamond has been added to Bob's deck. Finally, $\Pr[D_3 \mid D_2 \cap D_1] = \frac{13}{52}$, since a diamond was added to Alice's deck, but she also removed a diamond as the first step. Thus,

$$\Pr[DDD] = \frac{13}{52} \cdot \frac{14}{53} \cdot \frac{13}{52}$$

Now using similar logic for the rest of the cases:

$$\Pr[DND] = \frac{13}{52} \cdot \frac{39}{53} \cdot \frac{12}{52}$$

$$\Pr[NDD] = \frac{39}{52} \cdot \frac{13}{53} \cdot \frac{14}{52}$$

$$\Pr[NND] = \frac{39}{52} \cdot \frac{40}{53} \cdot \frac{13}{52}$$

Finally applying addition rule to sum up all the cases,

$$\Pr[E] = \frac{13}{52} \cdot \frac{14}{53} \cdot \frac{13}{52} + \frac{13}{52} \cdot \frac{39}{53} \cdot \frac{12}{52} + \frac{39}{52} \cdot \frac{13}{53} \cdot \frac{14}{52} + \frac{39}{52} \cdot \frac{40}{53} \cdot \frac{13}{52} = \boxed{\frac{1}{4}}$$