Module 7.3: Biased Coins and Bernoulli Trials

MCIT Online - CIT592 - Professor Val Tannen

LECTURE NOTES

Biased coins, Bernoulli trials

A **fair** coin corresponds to a uniform probability space with two outcomes, heads ($\mathtt{H}$) and tails ($\mathtt{T}$). Each of the outcomes has probability $1/2$.

A **biased** coin corresponds (in general) to a **non-uniform** space with the same outcomes, $\mathtt{H}$ and $\mathtt{T}$, that is parameterized by $\Pr[\mathtt{H}] = p \in [0, 1]$. It follows that $\Pr[\mathtt{T}] = 1 - p$.

The biased coin with parameter p is commonly invoked under a different name. A **Bernoulli trial** corresponds to a probability space with two outcomes, conventionally called "success" and "failure" that is parameterized by a probability of success which is p. The probability of failure is then $1 - p$.

Therefore flipping a biased coin is a Bernoulli trial in which heads is conventionally designated as success.

Biased coin flipped twice I

Problem. A biased coin has a probability $1/3$ of showing heads. We flip this coin twice. What is the probability that we obtain one tails and one heads (in either order)?

Answer. There are four outcomes $\mathtt{HH}$, $\mathtt{HT}$, $\mathtt{TH}$, $\mathtt{TT}$. However, they are **not** equally likely. So, what is the probability distribution?

Consider an urn holding three identical marbles. On one of them we write $\mathtt{H}$ and on the others $\mathtt{T}_1$ and $\mathtt{T}_2$. Assuming that each marble is equally likely to be extracted, sampling one marble corresponds to flipping our biased coin.

Now consider **two** such urns, U and U'. Sampling a marble from U then one from U' corresponds to flipping our biased coin **twice**.

Biased coin flipped twice II

Answer (continued). We now have $3 \cdot 3 = 9$ outcomes:

$$\mathtt{HH},\ \mathtt{HT}_1,\ \mathtt{HT}_2,\ \mathtt{T}_1\mathtt{H}, \ldots\ \mathtt{T}_1\mathtt{T}_2,\ \mathtt{T}_2\mathtt{H}, \ldots$$

Extractions from each urn happen in the same way so these 9 outcomes are **equally likely**. We have a **uniform space**.

The event of interest (a heads and a tails, in some order) is $\{\mathtt{HT}_1, \mathtt{HT}_2, \mathtt{T}_1\mathtt{H}, \mathtt{T}_2\mathtt{H}\}$.

That's 4 outcomes out of a total of 9, so its probability is $4/9$.

Random permutations

Distinct objects $a_1, \ldots, a_n$. A **random permutation** of $a_1, \ldots, a_n$ is an element of the **uniform** probability space whose outcomes are all the permutations . Each outcome has probability $1/n!$.

Problem. Let $i, j \in [1..n]$ (not necessarily distinct). Calculate the probability that a_i occurs in position j in a random permutation.

Answer. Let E_{ij} be the event consisting of all outcomes in which a_i occurs in position j. The probability we want is $|E_{ij}|/n!$.

A permutation in E_{ij} can be constructed in two steps: (1) put a_i in position j (1 way) then (2) put any permutation of $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ in the rest of the positions ($(n-1)!$ ways).

By the multiplication rule $|E_{ij}| = 1 \cdot (n-1)! = (n-1)!$.

The answer is $(n-1)!/n! \;=\; 1/n$.