Module 9.1: Random Variables

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Random variables

A **random variable** on (Ω, Pr) is a function $X : \Omega \to \mathbb{R}$.

Denote $\mathsf{Val}(X) = \{x \in \mathbb{R} \mid \exists w \in \Omega \;\; X(w) = x\}$.
(The set of values **taken** by X.)

Like Ω, $\mathsf{Val}(X)$ is also a finite set.

Denote with x the real values that X takes and with $X = x$ the **event** $\{w \in \Omega \mid X(w) = x\}$. Its probability $\mathrm{Pr}[X = x]$ is of particular interest.

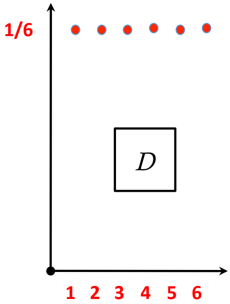
The **distribution** of the random variable X is the function

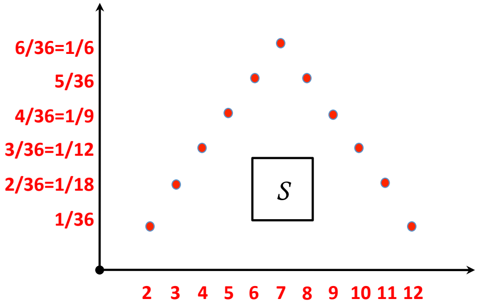
$$f : \mathsf{Val}(X) \to [0, 1] \quad \text{where} \quad f(x) = \mathrm{Pr}[X = x].$$

Examples of random variables I

Problem. We roll a fair die. What is the distribution of the random variable D that returns the number shown by the die? We roll two fair dice. What is the distribution of the random variable S which returns the sum of the numbers the two dice show?

Answer. Note that $\mathrm{Val}(D) = [1..6]$ and $\mathrm{Val}(S) = [2..12]$. Here are the graphs of the distribution functions:

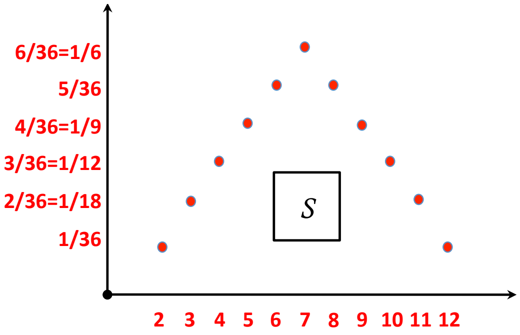




Examples of random variables II

Problem. Let S be the r.v. (short for random variable) which returns the sum of the numbers two fair dice show when rolled. Calculate $\Pr[5 \leq S \leq 9]$.

Answer. We use the distribution of S:



$.\Pr[5 \leq S \leq 9] =$
$.\Pr[S = 5] + \Pr[S = 6] +$
$.\Pr[S = 7] + \Pr[S = 8] +$
$.\Pr[S = 9] =$
$.(1/9) + (5/36) + (1/6) +$
$.(5/36) + (1/9) = 2/3$

Activity : Three-dice slot machine

A casino has a slot machine that shows three fair dice rolled independently. The player wins if the sum of the three values the dice show is in $[5..8]$ or in $[13..17]$. Note that

$$|[5..8]| + |[13..17]| = (8 - 5 + 1) + (17 - 13 + 1) = 4 + 5 = 9$$

out of a total of

$$|[3..18]| = 18 - 3 + 1 = 16$$

thus a player might think this is a good bet since more than half of the possible values the sum takes are winning values. But casinos always win in the long run!

There may be 9 winning values for the sum out of 16 but the values are **not** distributed uniformly! Let's compute the probability that the sum gives a winning value.

ACTIVITY : Three-dice slot machine (continued)

The probability space is uniform with $6 \cdot 6 \cdot 6 = 216$ outcomes.

Therefore, to compute the probabilities of the distribution it suffices to count the number of outcomes corresponding to each value taken by the sum of three dice.

Question. List all the possible ways in which a sum of 5 can be obtained. Count the number of ways. What is the connection with "stars and bars"?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Three-dice slot machine (continued)

Answer. This is the same as the number of ways of distributing 5 indistinguishable coins to 3 children in such a way that each child gets at least a coin. We did essentially this in the lecture segment "Stars and bars" except that we distributed 11 coins and here we have 5. The number of ways (here) is:

$$\binom{(5-3)+(3-1)}{3-1} = \binom{4}{2} = 6$$

And here are the six ways of getting a sum of 5:

$$3+1+1 = 1+3+1 = 1+1+3 = 2+2+1 = 2+1+2 = 1+2+2 \ (=5)$$

Activity : Three-dice slot machine (continued)

By thoroughly analyzing sums of three numbers, each from [1..6], we fill in the following table. In the first column is the value of the sum and in the second is the number of outcomes that gives the sum that value (which we can then divide by 216 to obtain the corresponding probability).

sum	# outc.
3	1
4	3
5	6
6	10
7	15
8	21

sum	# outc.
9	25
10	27
11	27
12	25

sum	# outc.
13	21
14	15
15	10
16	6
17	3
18	1

ACTIVITY : Three-dice slot machine (continued)

Let U be the random variable that returns the sum of the values shown by three fair dice rolled independently. Using the table we calculate:

$$\Pr[5 \le U \le 8] = \frac{6}{216} + \frac{10}{216} + \frac{15}{216} + \frac{21}{216} = \frac{52}{216}$$
$$\Pr[13 \le U \le 17] = \frac{21}{216} + \frac{15}{216} + \frac{10}{216} + \frac{6}{216} + \frac{3}{216} = \frac{55}{216}$$

Thus the probability of getting a winning value is:

$$\frac{52}{216} + \frac{55}{216} = \frac{107}{216} < \frac{1}{2}$$

The casino always wins (in the long run).

Probabilities add up to 1

For the distribution of S the sum of **all** the $12 - 2 + 1 = 11$ probabilities is 1. This is true for any random variable X:

Proposition. $\sum_{x \in \mathsf{Val}(X)} \Pr[X = x] \; = \; 1.$

Proof. Recall that $(X = x) \; = \; \{w \in \Omega \mid X(w) = x\}$.

The events $(X = x)$ for $x \in \mathsf{Val}(X)$ are pairwise disjoint.

Also, $\bigcup_{x \in \mathsf{Val}(X)} (X = x) \; = \; \Omega$.

By **P2** (the addition rule) and by **P1**

we have $\sum_{x \in \mathsf{Val}(X)} \Pr[X = x] \; = \; \Pr[\Omega] \; = \; 1.$

Uniform r.v. and distribution

Let $v_1, \ldots, v_n$ be n distinct values in $\mathbb{R}$.

Given $(\Omega, \Pr)$, an r.v. $U : \Omega \to \mathbb{R}$ is **uniform** with these values

when $\mathsf{Val}(U) = \{v_1, \ldots, v_n\}$

and $\Pr[U = v_i] = 1/n$ for $i = 1, \ldots, n$.

The corresponding distribution

. $f : \{v_1, \ldots, v_n\} \to [0, 1] \qquad f(v_i) = 1/n$ for $i = 1, \ldots, n$

is also called **uniform**.

The r.v. D associated with a fair die is uniform

with $n = 6$ and $v_i = i$ for $i = 1, \ldots, 6$.

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Bernoulli r.v. and distribution

Recall **Bernoulli trials**. Similarly, we can define:

Given $(\Omega, \Pr)$, an r.v. $X : \Omega \to \mathbb{R}$

with $\mathsf{Val}(X) = \{0, 1\}$

and $\Pr[X = 1] = p$

is called a **Bernoulli random variable** with parameter p.

A Bernoulli r.v. X defines implicitly a Bernoulli trial where "success" is $X = 1$ and "failure" is $X = 0$.

The corresponding distribution

$f : \{0, 1\} \to [0, 1]$ where $f(1) = p$ and $f(0) = 1 - p$

is called a **Bernoulli distribution** with parameter p.