MCIT 592 Spring 2022 **Homework 1** **Solutions**

Problem Set

Note. Whenever you use the Multiplication Rule (MR) or the Addition Rule in your solutions you have to state this explicitly: e.g., "by the multiplication rule or by MR". You must also persuade yourselves and explicitly state that the number of ways in which each step in the MR is performed does not depend on how the previous steps were performed.

1. [10 pts] Over the years, Francesca, who loves fashion, has become an adamant lover of shoes. As such, she has many types in her collection, including: (1) sneakers, (2) flip flops, (3) sandals, (4) slippers, and (5) combat boots. Francesca realizes that she has too many shoes, so she decides to have a garage sale. $n \geq 3$ TAs, including Sasha, Kevin, and Arnav, show up to the sale, and since they're Francesca's friends, Francesca kindly offers to discount all of her shoes. However, in order to be eligible for the discount, Francesca mandates that each customer can only buy at most one pair of each type (ex. someone can buy a pair of sneakers and a pair of flip flops, but not two pairs of flip flops). Sasha must buy 1 pair of each of the 5 types, Kevin must also buy 1 pair of each of the 5 types, and Arnav must buy 1 pair of exactly 2 of the 5 types. All other customers can buy 0 or 1 pair of all of the 5 types. Assume that Francesca has enough shoes for each of the n TA's to purchase 1 pair of each type. How many distinct ways can these n customers buy Francesca's shoes and be eligible for the discount? Treat a pair of shoes as a single item.

Solution:

We begin by counting separately the number of ways in which Arnav can buy shoes from Francesca. There are 10 ways to choose two of the five shoe types: 1 and 2, 1 and 3, 1 and 4, 1 and 5, 2 and 3, 2 and 4, 2 and 5, 3 and 4, 3 and 5, and finally 4 and 5.

We also count separately the number of ways in which a customer *other than Sasha, Kevin, and Arnav* can buy shoes. The shoes they could have bought form a subset of $\{1, 2, 3, 4, 5\}$, including the empty subset. There are 2^5 such subsets.

Using these we count the number of ways to buy shoes as follows:

Step 1: Select shoes for Sasha. (1 way)

Step 2: Select shoes for Kevin. (1 way)

Step 3: Select shoes for Arnav. (10 ways, see above)

Step 4: Select shoes for customer $k = 4$. ($2^5 = 32$ ways, see above)

$\vdots$

Step n: Select shoes for customer $k = n$. ($2^5 = 32$ ways)

Since the choices made in each step do not depend on how the choices were made in previous steps, we can use the Multiplication Rule. There are a total of $n - 3$ customers such that $4 \leq k \leq n$, so this gives us

$$\boxed{10 \cdot 32^{n-3} = 10 \cdot 2^{5(n-3)}}$$

different ways to buy shoes.

2. [**10 pts**] Let $A = \{2, 3\}$, $B = \{3, 4, 5\}$, and $C = \{3, 5, 7, 9\}$ and let P be the set whose elements are all the proper subsets of $(A \cup B) \setminus C$. List all the elements of 2^P. Show your work.

Solution:

First construct $A \cup B = \{2, 3, 4, 5\}$
Next construct $(A \cup B) \setminus C = \{2, 4\}$

Next construct P, whose elements are all the proper subsets of $(A \cup B) \setminus C$.
Thus, $P = \{\emptyset, \{2\}, \{4\}\}$.

Finally construct 2^P. Recall that 2^P is the powerset of set P and is defined as the set whose elements are all of the subsets of P.
Thus, $2^P = \big\{\emptyset, \{\emptyset\}, \{\{2\}\}, \{\{4\}\}, \{\emptyset, \{2\}\}, \{\emptyset, \{4\}\}, \{\{2\}, \{4\}\}, \{\emptyset, \{2\}, \{4\}\}\big\}$

3. [10 pts]

$n \geq 2$ distinguishable Hogwarts students participate in Professor Snape's experiment. Each student is given potion A, or philter B, or neither, or both. We know that Harry and Hermione are the only students among the n that are given both A and B. In how many distinct ways could Snape have distributed his experimental liquids?

Solution:

Since Harry and Hermione are the only students given both, the remaining $n-2$ students have potion A, philter B, or neither. These options are assigned to each student independently, so we can use the Multiplication Rule here. Consider the following steps:

Step 1: Assign student 1 to potion A, philter B, or neither. (3 ways)
Step 2: Assign student 2 to potion A, philter B, or neither. (3 ways)

$\vdots$

Step $n-2$*:* Assign student $n-2$ to potion A, philter B, or neither. (3 ways)

Step $n-1$: Assign Harry to both. (1 way)

Step n: Assign Hermione to both. (1 way)

By the Multiplication Rule, there are $\boxed{3^{n-2}}$ ways for Snape to distribute his experimental liquids.

4. [10 pts] Prove that each of the following integers is composite (not prime).

(a) $3^{222}+1$

(b) $2^{35}(2^{33}-1)+1$

Hint: Think abstractly about the numbers. For example, we know from algebra that $x^2-1=(x-1)(x+1)$. When we replace x with 3^{33} we get $(3^{33})^2-1=(3^{33}-1)(3^{33}+1)$. Therefore $n=(3^{33})^2-1=3^{66}-1$ has factors that neither 1 nor n. It follows that $n=3^{66}-1$ is composite.

Solution:

Recall from lecture that x^3+1 is not prime for any integer $x>1$. Note that $3^{222}+1=(3^{74})^3+1$, and that $3^{74}>1$. Therefore $3^{222}+1$ is composite.

For the other one $2^{35}(2^{33}-1)+1=2^{68}-2^{35}+1=(2^{34}-1)^2$ so it has a divisor $2^{34}-1>1$ so it's composite.

5. [10 pts] z is said to be a *Broadway integer* (this is not a standard term, it was made up for this problem) when $z=4k+2$ for some integer k. Prove that the difference of the squares of two Broadway integers is always divisible by 16.

Solution:

We can name z_1 and z_2 to be two Broadway integers, such that $z_1=4i+2$ and $z_2=4j+2$ for some integers i and j. We can say that the difference of their squares, without loss of generality, is equal to $z_1^2 - z_2^2 = (4i+2)^2 - (4j+2)^2$. Expanding the squares goes as follows:

$$(4i+2)^2 - (4j+2)^2$$

$$= 16i^2 + 16i + 4 - 16j^2 - 16j - 4$$

$$= 16i^2 + 16i - 16j^2 - 16j$$

$$= 16(i^2 - j^2 + i - j)$$

Because i and j are both integers, we know that $i^2 = i \cdot i$ and $j^2 = j \cdot j$ will also be integers. Therefore, the quantity $i^2 - j^2 + i - j$ must also be an integer, which we can arbitrarily name to be k. We thus have that the quantity is equal to $16k$. Because k is an integer, $16k$ is always divisible by 16. Therefore, the difference of the squares of two Broadway numbers is always divisible by 16.