MCIT 592 Online - Spring 2022 **(instructor Val Tannen)**

Additional Problems (Packet 2)

1. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a *very brief* explanation of your answer.

   (a) If X_1 and X_2 are Bernoulli random variables with $Pr[X_1 = 1] = 1/2$ and $Pr[X_2 = 1] = 1/3$ then $E[X_1 - X_2] = 0$.

   (b) For any two events A, B in the same probability space (Ω, Pr) such that $Pr[B] \neq 0$ we have $Pr[A \cup B \mid B] = 1$.

2. My 6th grade teacher of Russian was unable to pay attention to what we were answering and it appeared to us that he was assigning grades completely randomly. Let's assume that his grading rubric consisted of tossing a fair coin six times, counting the number k of heads and assigning the grade $4 + k$ (our grades were in the 1-10 range).

   (a) What was the probability that I would get a 10?

   (b) What was the probability of the following event: "my grade was divisible by 4 or (non-exclusive or!) it was bigger than or equal to Lady Gaga's shoe size (a 6)"?

3. A fair coin is flipped *twice*. Let (Ω, Pr) be the resulting probability space. Let X_H be random variable defined on Ω that returns the number of heads observed and X_T similarly the number of tails observed.

   (a) Describe the probability space (Ω, Pr). That is, list the outcomes and their probabilities.

   (b) Show that the random variable Z defined by $\quad \forall w \in \Omega \quad Z(w) = X_H(w) \cdot X_T(w)$ is a Bernoulli random variable and find its probability of success.

   (c) Show that $E[Z] \neq E[X_H]E[X_T]$.

4. Alice has a strange coin that shows the number 3 on one side and the number 5 on the other. Still, the coin is fair. Bob has strange die that shows the numbers 5,6,7,8,9,10 on its six faces. Still, the die is fair. Alice flips the coin and, independently, Bob rolls the die. What is the probability that the number on the die is divisible by the number on the coin?

5. Alice has a *fair* coin that shows the number 2 on one side and the number 3 on the other. Bob has a *fair* tetrahedral die (a tetradie) that shows the numbers 1,2,3 and 4 on its four faces. They play the following game:

- Alice flips the coin showing the number a and, independently, Bob rolls the tetradie showing the number b
- If $a > b$ then Alice wins and Bob pays Alice $a - b$ dollars. If $a = b$ then it's a tie and no money changes hands. If $b > a$ then Bob wins and Alice pays Bob $b - a$ dollars.

(a) Draw the tree of possibilities for a single game.

(b) Compute the probability that Alice wins a single game.

(c) Suppose that Alice and Bob play the game 3 times in a row, independently. Assume that Alice starts with 10 dollars. Let Z be the random variable that returns the amount of dollars that Alice has after these 3 games. Compute $E[Z]$.

6. For each statement below, decide whether it is TRUE or FALSE In each case attach a *very brief* explanation of your answer.

   (a) Let (Ω, Pr) be a probability space with *three* outcomes. Let E, F be two nonempty events in this space such that $Pr[E \cup F] = Pr[E] + Pr[F]$. Then $E \cap F = \emptyset$.

   (b) Let A, B, C be three events of non-zero probability in a probability space (Ω, Pr). If $A \cap B = B \cap C$, $A \perp B$, and $B \perp C$ then $Pr[A] = Pr[C]$.

   (c) If a probability space has an *event* of probability $2/3$ then it must have some *outcome* of probability at most $1/3$.

   (d) Let A, B be events in a probability space such that $Pr[A] = 0$ and $Pr[B] \neq 0$. Then, $Pr[A \mid B] = 0$.

   (e) For *any* probability space (Ω, Pr) and *any* event $A \subseteq \Omega$ such that $Pr[A] \neq 0$ we have $Pr[\Omega \mid A] = P[rA \mid \Omega]$.

   (f) Let E, F be two events in a finite probability space. If $|E| = |F|$ then $Pr[E] = Pr[F]$, true or false?

   (g) If E, F are two events in a finite probability space such that $Pr[E \cap F] > 0$ then E and F can be disjoint.

   (h) Let A, B be events in a finite probability space such that $Pr[A] = 1/4$ and $Pr[A \cup B] = 1/2$. Then, $1/4 \leq Pr[B] \leq 1/2$.

   (i) For any three events E, F, G in the same probability space. if $E \perp F$ and $F \perp G$ then $E \perp G$.

7. Let A, B, C be three events in the same probability space such that $A \subseteq B$, $A \subseteq C$, $B \perp C$, and $Pr[A] = 1$. Prove that $Pr[A \cap B \cap C] = Pr[A]\, Pr[B]\, Pr[C]$.

8. Let E, F be two events in a finite probability space such that $Pr[E \cap F] > 0$. Prove that $Pr[E \setminus F] + Pr[F \setminus E] < Pr[E \cup F]$.

9. Alice has an urn with three marbles labeled 1, 2, and 3. Each of the marbles is equally likely to be extracted. Bob has a separate, similar urn. They play the following game of chance:

   (1) Alice extracts a marble from her urn and obtains $a \in \{1, 2, 3\}$.

   (2) Independently, Bob extracts a marble from his urn and obtains $b \in \{1, 2, 3\}$.

(3) If $a > b$ then Alice wins. If $b > a$ then Bob wins. If $a = b$ they flip a fair coin and if the coin shows heads, Alice wins. If the coin shows tails, Bob wins.

In the calculations below, do not spend time on the arithmetic. It's OK to leave your results as products and fractions.

(a) Draw the "tree of possibilities" diagram for this game, with all the outcomes and their probabilities.

(b) Compute the probability that the game was decided by a coin flip.

(c) Compute the conditional probability that Alice wins, knowing that Bob extracted the marble labeled 2.

(d) Alice and Bob put bets on the game. If Alice wins without a coin flip Bob pays her 2\$. If Alice wins with a coin flip then Bob pays her 1\$. If Bob wins then Alice pays him 1.5\$.

What is Alice's expected monetary win/loss (wins are positive, losses are negative) after n such games?

10. A fair coin is flipped $2n$ times ($n \geq 1$), independently. Let X_H the random variable that returns the number of heads that occurred and X_T the random variable that returns the number of tails that occurred. Compute $P(X_H > X_T)$.

11. Let A, B be events in the same probability space and let I_A, I_B be their indicator random variables. If $E(I_A + I_B) = 1$ then $P(A) = P(\bar{B})$, true or false?

12. Let (Ω, P) be a probability space and let X be a random variable defined on Ω such that $\mathrm{Val}(X) = \{a, b\}$ where $a < b$. We also denote $\mu = E(X)$.

(a) Express $P(X \leq (a+b)/2)$ in terms of a, b and μ.

(b) Let $a = -1$ and $b = 1$. Show that if $E(X) = 0$ then there exists an event $A \subseteq \Omega$ such that $P(A) = 1/2$.

13. Weird Al (WAl) is playing with his coins. The game uses two *fair coins* and one *urn*. The result of the game is one of H (heads) or T (tails) and is determined as follows:

- WAl places both coins in the urn.
- WAl reaches inside the urn and (a) with probability 2/3 WAl grabs one of the coins and tosses it, OR (b) with probability 1/3 WAl grabs both coins, then tosses them separately in some order (doesn't matter which order).
- If WAl has tossed just one coin then whatever that coin shows is the result of the game. If WAl has tossed both coins then applying the weird $\otimes$ operation to what the two coins show is the result of the game, where $T \otimes T = T$, $T \otimes H = H$, $H \otimes T = H$, and $H \otimes H = T$.

(a) Draw the "tree of possibilities" diagram for WAl's game.

(b) Calculate the probability that the result of of the game is H.

(c) What simpler game could Weird Al play that would give him exactly the same odds?

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14. Let S be the probability space (Ω, Pr) with $\Omega = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = ax + b\}$ such that $a, b \in \mathbb{N}$, $1 \le a \le 10$, and $1 \le b \le 10$, and Pr is the uniform probability distribution on Ω. For each natural number k, Let X_k be the random variable that takes on the value of $f(k)$.

    (a) What is $E[X_5]$?

    (b) What is $Pr[X_5 > 1]$?

    (c) What is $E[X_k]$ in terms of k?

    (d) If you remember derivatives, consider the probability space $S' = (\frac{d\Omega}{dx}, Pr)$, where $\frac{d\Omega}{dx} = \{\frac{df}{dx} \mid f \in \Omega\}$. Let Y be the random variable that takes on $f(5)$. What is $E[Y]$? (If you don't remember derivatives, no problem ☺)

15. You have a standard deck of 52 cards, from which you draw 13 cards, without replacement.

    (a) Given that you drew the 4 of spades, what is the probability that all the other cards that you drew are aces, twos, threes, or fours?

    (b) Define S to be the number of spades you draw. What is $E[S]$?

    (c) For what $s \in \text{Val}(S)$ do we have the maximum value of $\Pr[S = s]$?

    (d) Suppose that the number of spades you have in your hand is equal to the number you found in part (c). What is the probability that the sum of their numerical values (letting J = 11, Q = 12, K = 13, A = 1) is odd?

16. Consider X and Y, two independent Bernoulli random variables defined on the same probability space. We are given $Pr[X = 1] = 1/3$ and $Pr[Y = 1] = 1/4$. Compute $E[(X + Y)^2]$.

17. Let (Ω, Pr) be a probability space with 2 or more outcomes and $X : \Omega \to \mathbb{R}$ a random variable such that $\text{Val}(X) = \{-1, 1\}$. Show that if $E[X] = 0$ then $Pr[X = 1] = 1/2$.

18. Sophie is playing the following game:

    - First she chooses with equal probability one of the numbers $0, 1, \ldots, 4$, call it a.
    - Then she chooses with equal probability one of the *remaining* numbers, call it b.
    - Then she adds $a + b = c$ and c is the result of her game.

    (a) What possible results can Sophie's game have?

    (b) Draw the "tree of possibilities" diagram for Sophie's game.

    (c) We denote with $R(c)$ the event that the result of the game is c. What is the probability of $R(2)$?

    (d) What is the probability of $R(2) \cup R(3)$?

    (e) What is the conditional probability $\Pr[R(2) \mid R(2) \cup R(3)]$?

    (f) Find two events E, F in the probability space of Sophie's game such that neither is empty, neither equals the whole sample space and $E \perp F$.

19. Alice and Bob are playing a game of chance. They roll a fair die. If the die shows an even number then Alice pays Bob \$1. If the die shows an odd number then Bob pays Alice \$1.

(a) What is the probability that Alice wins \$1?

(b) They play the game 2 times in a row, independently. What is the probability that Bob wins \$2?

(c) Alice and Bob start with 100 dollars each and play the game 100 times in a row, independently. Let X be the random variable representing how much money Alice has after all this. Express X as a sum of 101 random variables.

(d) Compute $E[X]$.

20. A **biased coin** shows heads with probability 1/3 and tails with probability 2/3. The coin is flipped $n \geq 3$ times, independently.

    Compute the expected number of occurrences of **consecutive** heads, tails, tails.

21. Alice receives 20 candies from her granny.

    - On day 1, for each candy that she has, Alice is equally likely to *keep* it or to give it to Bob.
    - On day 2, for each candy that he has (all given previously by Alice), Bob is equally likely to *eat* it or to give it back to Alice. (Bob is weird.)
    - On day 3, for each candy that she has (either kept in day 1 or received from Bob in day 2), Alice is equally likely again to *eat* it or to throw it in the trash. (Less appetizing by now.)

(a) Given a particular candy, c, what is the probability that Alice eats c?

(b) Calculate the expected number of candies that Alice eats.