Recitation Module 5

Penn Engineering

Lecture Review

Recitation Topics

- Bijective Functions
- Principle of Inclusion Exclusion (PIE)
- Pigeonhole Principle (PHP)

Bijection

A function $f : A \to B$ is called **bijective** if it is both injective and surjective.
A bijective function is also called a **bijection** or a **one-to-one correspondence**.

The bijection rule:
if we can define a bijective function with domain A and codomain B **then** $|A| = |B|$.

PIE - Principle of Inclusion Exclusion

For two disjoint set A, B $|A \cup B| = |A| + |B|$

When A, B are two sets that are **not disjoint** $|A| + |B|$ **overcounts.**

Subtracting those, we get $|A \cup B| = |A| + |B| - |A \cap B|$

This is called **Principle of Inclusion-Exclusion(PIE)**

Pigeonhole Principle

Theorem (PHP). If *r* objects are placed into *n* boxes, with $r > n$, then there is at least one box containing **at least** 2 objects.

Theorem (GPHP). If *r* objects are placed into *n* boxes then there is at least one box containing **at least** $\lceil r/n \rceil$ objects. (How to write in latex: `$\lceil\frac{r}{n}\rceil$`)

In HW and assessments, state clearly what are the pigeons and what are the holes when using PHP or GPHP.

Practice Questions

Question 1

Determine whether the following functions are bijections:

(a) $a : \mathbb{Z}^+ \to \mathbb{Z}^+$ given by $a(x) = x^2$

(b) $b : \mathbb{Z} \to \mathbb{Z}^+$ given by $b(x) = \begin{cases} -2x & \text{if } x < 0 \\ 2x + 1 & \text{otherwise} \end{cases}$

(c) $c : (\mathbb{R} \setminus \{0\}) \to \{y \mid y \in \mathbb{R} \wedge y > 0\}$ given by $c(x) = \frac{1}{|x|}$

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Answer to Question 1

A. The range only includes perfect squares **- for example, no element maps to 3**. The function is **not a surjection** and therefore is not a bijection.

B. The **negative integers map to the even positive integers**, and the **natural numbers map to the odd positive integers**. For any positive integer, there **is exactly one integer that maps to it**, and therefore the function is a bijection.

C. For **any positive real number y**, there are **two elements in the domain** that **map to y**, namely **1/y** and **-1/y**. The function is **not an injection** and therefore is not a bijection.

Question 2

Let n be a positive integer. How many *injective* functions with domain [1..n] and codomain [1..(n+1)] are there?

Answer to Question 2

Remember: injective means every element in the domain maps to a *unique* element in the codomain. In other words, no two elements in the domain map to the same element in the codomain.

Notation: Let X be the domain:

$$\forall a, b \in X, f(a) = f(b) \Rightarrow a = b$$

Contrapositive: $\forall a, b \in X, a \neq b \Rightarrow f(a) \neq f(b)$

Make sure to know and understand injective vs surjective vs bijective.

Answer to Question 2 continued

Step 1: Map the first element in the domain to an element in the codomain ($n+1$ choices).

Step 2: Map the second element in the domain to an element in the codomain (n choices, cannot map to whatever the first element mapped to.

Step 3: Map the third element in the domain to an element in the codomain ($n-1$ choices, cannot map to either of what the first two mapped to.)

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By the multiplication rule, there are $(n+1)!$ ways to do this.

Why is this the same as ordering (n + 1) things in a line?

Question 3

Count the number of permutations of the set $\{a,b,c,d\}$ in which:

- a does NOT occur in the FIRST position OR

- b DOES occur in the SECOND position OR

- either c or d occurs in the THIRD position

Answer to Question 3

Let A, B, C be the following events:
A = the event in which a does NOT occur in the first position.
B = the event in which b DOES occur in the second position.
C = the event in which c or d occurs in the third position.

We are trying to calculate: $|A \cup B \cup C|$.

By the principle of inclusion-exclusion, we have:
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Answer to Question 3 Continued

Let's look at these individually:

$|A|$:
Step 1: Pick an element for the first spot (3 choices, everything *except* a).
Step 2: Pick an element for the second spot (3 choices).
Step 3: Pick an element for the third spot (2 choices).
Step 4: Pick an element for the last spot (1 choice).
By the multiplication rule, $|A| = 3 * 3 * 2 * 1 = 18$

$|B|$:
Step 1: Pick the element that goes in the second spot (1 choice, must be b).
Step 2: Pick an element for the first spot (3 choices).
Step 3: Pick an element for the third spot (2 choices).
Step 4: Pick an element for the fourth spot (1 choice).
By the multiplication rule, $|B| = 1 * 3 * 2 * 1 = 6$

$|C|$:
Step 1: Pick an element to go in the third position (2 choices, either c or d).
Step 2: Pick an element for the first spot (3 choices).
Step 3: Pick an element for the second spot (2 choices).
Step 4: Pick an element for the fourth spot (1 choice).
By the multiplication rule, $|C| = 2 * 3 * 2 * 1 = 12$

Answer to Question 3 Continued

$|A \cap B|$:
Step 1: Pick an element for the second position (1 way, must choose b).
Step 2: Pick an element for the first position (2 choices, cannot be a).
Step 3: Pick an element for the third position (2 choices).
Step 4: Pick an element for the fourth position (1 choice).
By the multiplication rule, $|A \cap B| = 1 * 2 * 2 * 1 = 4$

$|A \cap C|$:
Step 1: Pick an element for the third spot (2 choices, either c or d).
Step 2: Pick an element for the first spot (2 choices, cannot be a).
Step 3: Pick an element for the second spot (2 choices).
Step 4: Pick an element for the fourth spot (1 choice).
By the multiplication rule, $|A \cap C| = 2 * 2 * 2 * 1 = 8$

$|B \cap C|$:
Step 1: Pick an element for the second spot (1 choice, must be b).
Step 2: Pick an element for the third spot (2 choices, must be either c or d).
Step 3: Pick an element for the first spot (2 choices).
Step 4: Pick an element for the fourth spot (1 choice).
By the multiplication rule, $|B \cap C| = 1 * 2 * 2 * 1 = 4$

Answer to Question 3 Continued

$|A \cap B \cap C|$:
Step 1: Pick an element for the second spot (1 choice, must be b).
Step 2: Pick an element for the third spot (2 choices, must be either c or d)
Step 3: Pick an element for the first spot (1 choice, cannot be a).
Step 4: Pick an element for the fourth spot (1 choice).
By the multiplication rule: $|A \cap B \cap C| = 1 * 2 * 1 * 1 = 2$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 18 + 6 + 12 - 4 - 8 - 4 + 2 = 22$$

Question 4

- Let *n* be a natural number. Using the Pigeonhole Principle, prove that there exist positive integers *a* and *b* such that $n^a - n^b$ is divisible by 10.

Answer to Question 4

First, let's try to find an example to make sense of things:
Let $n = 2, a = 5, b = 1$
$2^5 - 2^1 = 32 - 2 = 30$, which is divisible by 10 because $30 = 10 * 3$.

Lemma: If $a, b \in \mathbb{Z}$ end in the same digit, their difference is divisible by 10. For example, 20 and 30, 25 and 35, 101 and 10001.

Simple proof/explanation for lemma: In subtraction, you start from right to left and any number minus itself is 0. Any number that ends in 0 is divisible by 10 because there must exist some number by which you can multiply (the last digit will multiple with the 0 in the 10, giving a last digit of 0). Note: You do not need a rigorous proof for this lemma because it uses fundamental operations of subtraction. This is different from odd/even because we define what it means to be odd even. In this case, you cannot "prove" that a number minus itself is 0, it is just a given fact of numbers!

Answer to Question 4 Continued

Now, back to the question: Let X be a set of 11 integers where each element is a positive integer.

Let Y be some set of 11 integers (can be any number > 10) where each element is n to the power of (some element from X).

For example, if $n = 2$ and $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, $Y = \{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048\}$

Let the pigeons be the 11 numbers in Y. Let the holes be the last digit of a number (0 through 9, 10 digits possible). By the pigeonhole principle, there is at least 1 digit (0 - 9) with at least $\lceil 11/10 \rceil = 2$ two numbers (elements in Y) that end with the same digit. (Be careful of pigeonhole principle conclusion wording!!!!)

We can conclude that there are at least two elements in Y that end in the same digit. By the lemma briefly proved above, we know that the difference between these two elements is divisible by 10. Thus we have proved the question.

Question 5

A company has 45 employees around the world, all of whom work remotely and set their own schedules. Each of the employees works exactly 40 hours every week. Prove that, in any given week, there is some point in time at which at least 11 of the employees are working.

Answer to Question 5

- The total number of hour worked by all employees per week is $45 * 40 = 1800$. Assume (towards a contradiction) that there are never more than 10 employees working at the same time. Since there are $24 * 7 = 168$ hours in a week, there cannot be more than 1680 total hours worked. But we know there were 1800 total hours worked, so our assumption must be false, and there must be some point during the week at which more than 10 employees (in other words, at least 11) were working.

Terms/definitions to know

1. Surjective, injective, bijective. Know what they mean in words, mathematically, and how to prove/disprove them.
2. Principle of Inclusion-Exclusion. You do not have to expand this out for 4+ sets, but know how to use it for 2-3 sets.
3. Pigeonhole principle (PHP). Properly defining pigeons, holes, and correct PHP conclusions.
    a. Generalizing PHP (GPHP)