CIT 592 Spring 2022 Homework 7

YOUR NAME HERE
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1. **[10 pts]** Suppose Jeff has 5 cars; a Ferrari, an Audi, a Bugatti, a Bentley, and a Lamborghini. He wants to organize his garage consisting of these 5 cars. He does this by selecting a uniformly random permutation of the 5 cars. What is the probability that the first car is the Audi, and the last car is the Bentley?

Solution.

Step 1: Put Audi on the first position and Bentley on the last position.
Step 2: The number of ways to put the other three cars in the three positions in the middle is $3!$.
Step 3: The number of permutation of 5 is $5!$. Thus the probability we want is $3!/5! = 1/20$.

2. **[10 pts]** Among $2 \leq k \leq 7$ randomly selected people, what is the probability that at least two of them were born on the same day of the week?

Solution.

We set up a probability space in which the outcomes are the birthdays of the k people: sequence of length k of elements from [2..7]. There are 7^k such sequences. Next, we make two assumptions:
1. It is equally likely for any given person to be born on any of the 7 days of the week.
2. We assume that the birthdays of the different people selected are independent.
So each outcome has probability $1/7^k$.
Let E be the event that at least two people selected have the same birthday. Its complement is $\overline{E}$ = "all k people have distinct birthdays". The outcomes in $\overline{E}$ are the partial permutation of k out of 7.
Therefore, using P4:

$$\Pr[E] = 1 - \Pr[\overline{E}] = 1 - \frac{7!}{(7-k)! \cdot 7^k}$$

3. [**10 pts**] Suppose you roll three fair six-sided dice and add up the numbers you get. What is the probability that the sum is at most 5?

Solution.

Situation 1: Since there are 3 dice, the numbers added up is at least 3, where all three dice roll to 1. The probability is $1/6^3$.

Situation 2: When 2 dice roll to 1 and the 3rd rolls to 2, the numbers add up to 4. The probability is $\binom{3}{1}/6^3 = 3/6^3$.

Situation 3: Since $5 = 1 + 1 + 3 = 1 + 2 + 2$, the numbers add up to 5 when 2 dice roll to 1 and the 3rd rolls to 3, or 2 dice roll to 2 and the 3rd rolls to 1.The probability is $\binom{3}{1}/6^3 * 2 = 6/6^3$.

Therefore the probability that the sum is at most 5 is $1/6^3 + 3/6^3 + 6/6^3 = 10/6^3$.

4. [**10 pts**] Sasha and Professor Tannen are planning the end of the semester dinner party for the CIT Staff. Answer the following questions. **You don't need to show your work.**

Questions (a)-(d) are based on the following scenario: Sasha and Professor Tannen decide to order 4 dishes for the team of TAs. For any meal, Laila feels like eating 29% of the time, Alex feels like eating 68% of the time, Francesca 54% of the time, and Rohan feels like eating 0% of the time (he's always eats before the dinner party!). All four dishes are independent of each other.

Questions (e)-(i) are based on the following scenario: after creating the planning and successfully running the dinner party, Sasha examines the dishes herself. She visits 5 dishes independently and gets one spoonful of food from each, which is equally likely to be either warm or cold.

(a) What is the probability that exactly 3 TAs eat at the dinner party?

(b) What is the probability that exactly 0 TAs eat at the dinner party?

(c) What is the probability that at least 1 TAs eat at the dinner party?

(d) What is the probability that exactly 1 TAs eat at the dinner party?

(e) What is the probability that every spoonful Belinda gets is of the same temperature type (warm or cold)?

(f) What is the probability that the first 3 spoonfuls of food she gets are warm and the other 2 are cold?

(g) What is the probability that she gets exactly 3 warm spoonfuls of food?

(h) What is the probability that the first 2 spoonfuls of food she gets are cold?

(i) What is the probability that she gets at least 1 cold spoonful of food?

Solution.

(a) Since Rohan eats 0% of the time, it means he does not eat. To have exactly 3 TAs eat, all the other TAs need to eat. Therefore the probability is $29\% * 68\% * 54\% = 10.6\%$.

(b) Since Rohan eats 0% of the time, it means he does not eat. To have exactly 0 TAs eat, none of the other TAs eats. Therefore using P4, the probability is $(1 - 29\%) * (1 - 68\%) * (1 - 54\%) = 10.5\%$.

(c) The probability of at least 1 TA eats is the compliment of the probability of 0 TA eats. Therefore using P4, the probability of at least 1 TA eats is $1 - 10.5\% = 89.5\%$.

(d) The probability of only Laila eats is $29\% * (1 - 68\%) * (1 - 54\%) = 4.3\%$.
Similarly, the probability of only Alex and Francesca eats are $(1-29\%)*68\%*(1-54\%) = 22.2\%$ and $(1 - 29\%) * (1 - 68\%) * 54\% = 12.3\%$.
Therefore the probability of 1 TA eats is $29\% * (1 - 68\%) * (1 - 54\%) + (1 - 29\%) * 68\% * (1 - 54\%) + (1 - 29\%) * (1 - 68\%) * 54\% = 38.7\%$.

(e) By multiplication rule there would be total 2^5 situations and every spoonful can be either cold or warm, thus the probability of every spoonful is of the same temperature type is $2/2^5 = 1/2^4$.

(f) By multiplication rule there would be total 2^5 situations, thus the first 3 spoonfuls of food she gets are warm and the other 2 are cold is $1/2^5$.

(g) There are $\binom{5}{3} = 10$ ways to select 3 spoonfuls of warm food and By multiplication rule there would be total 2^5 situations. Therefore the probability that she gets exactly 3 warm spoonfuls of food is $10/2^5 = 5/2^4$.

(h) When the first 2 spoonfuls are cold, there are 2^3 situations for the following 3 spoonfuls which can be either cold or warm independently. By multiplication rule there would be total 2^5 situations. Therefore the probability that the first 2 spoonfuls of food she gets are cold is $2^3/2^5 = 1/4$.

(i) Let E be the event that she gets at least one spoonful of cold. Its complement is $\overline{E} =$ "she gets 0 spoonful of cold". The outcome in $\overline{E}$ is 1. By multiplication rule there would be total 2^5 situations.
Therefore, using P4:

$$\Pr[E] = 1 - \Pr[\overline{E}] = 1 - \frac{1}{2^5}$$

5. [10 pts] It is spring break, and Alice wants to go on vacation for two weeks. She wants to go to either Cancun, the Bahamas, or Hawaii. She only has money to go to one place for the first week, and another for the second week, for a total of two locations (note that she can also pick the same place twice, going there for the entire two weeks). She puts a tickets labeled "Cancun", b tickets labeled "Bahamas", and c tickets labeled "Hawaii" into a hat. She will pick a ticket from the hat uniformly at random to decide where to travel the first week, then return that ticket to the hat. She'll then pick another ticket from the hat uniformly at random and go to that location the second week. What is the probability that she goes to two different locations? Be sure to define the sample space and any relevant events. Assume that $a + b + c \geq 2$.

Solution.

The probability to pick Cancun for the first week is $\frac{a}{a+b+c}$. Similarly, the probably to pick Bahamas and Hawaii for the first week is $\frac{b}{a+b+c}$ and $\frac{c}{a+b+c}$ respectively.
Similarly, the probability to pick Cancun, Bahamas and Hawaii for the second week is $\frac{a}{a+b+c}$, $\frac{b}{a+b+c}$ and $\frac{c}{a+b+c}$ respectively.
Therefore the probability to pick Cancun for both weeks is $\frac{a^2}{(a+b+c)^2}$. Similarly the probability to pick Bahamas and Hawaii for both weeks is $\frac{b^2}{(a+b+c)^2}$ and $\frac{c^2}{(a+b+c)^2}$.
Let E be the event that Alice goes to two different locations. Its complement is $\overline{E} =$ "Alice goes to the same location for both weeks". The outcomes in $\overline{E}$ is $\frac{a^2}{(a+b+c)^2} + \frac{b^2}{(a+b+c)^2} + \frac{c^2}{(a+b+c)^2}$.
Therefore, using P4, $\Pr[E] = 1 - \Pr[\overline{E}] = 1 - (\frac{a^2}{(a+b+c)^2} + \frac{b^2}{(a+b+c)^2} + \frac{c^2}{(a+b+c)^2})$.