OMCIT 592 Module 13 Self-Paced 01 (instructor Val Tannen)

No reference to this self-paced segment in the lecture segments.

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion.

When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Colorability and maximum degree

Proposition. Every graph G is $\Delta(G) + 1$-colorable, where $\Delta(G)$ is the maximum degree of a node in G. Moreover, there exist graphs G whose chromatic number, $\chi(G)$, is $\Delta(G) + 1$.

Proof (part 1). By induction on the number n of vertices of G.

(BC) $(n = 1)$ G has just one, isolated, vertex, so $\Delta(G) = 0$. Clearly G is 1(=0+1)-colorable. Check.

(IS) Let k be an arbitrary natural number ≥ 1. Assume (IH) that any graph G with k vertices is $\Delta(G) + 1$-colorable.

For any graph G' with $k + 1$ vertices pick any vertex u of G'. Let N be the set of neighbors of u in G'.

Now remove u as well all the edges incident to u and let G'' be the resulting graph. Since removing edges cannot increase any node's degree it must be the case that $\Delta(G'') \leq \Delta(G')$.

Now, G'' has k vertices and thus, by IH, it is $\Delta(G'') + 1$-colorable. Since $\Delta(G'') \leq \Delta(G')$ it must be the case that G'' is also $\Delta(G') + 1$-colorable. So we have a proper $\Delta(G') + 1$-colorable of G'', call it f.

In G', $\deg(u) \leq \Delta(G')$ so u has at most $\Delta(G')$ neighbors, i.e., $|N| \leq \Delta(G')$.

Thus at most $\Delta(G')$ of the colors are used by f to color the vertices in N. This leaves available at least one color in the coloring f that is not used on the vertices in N. We put u back into G'' (the removed edges too) and we extend f to G' using the available color to color u.

This gives a proper $\Delta(G') + 1$-coloring of G', and this finishes the induction step.

Proof (part 2). That graph is K_n, the complete graph on n vertices. In K_n every vertex has degree $n - 1$ so $\Delta(K_n) = n - 1$. And we have seen in the lecture segment "Graph coloring" that $\chi(K_n) = n$. Hence $\chi(K_n) = \Delta(K_n) + 1$.