CIT 592 Spring 2022 Homework 10

Fan Zhang
fanzp@seas.upenn.edu

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1. [10 pts] Let X and Y be independent random variables such that $\mathrm{Var}[X] = 5.3$ and $\mathrm{Var}[Y] = 8.9$. What is the standard deviation of $3X + 2Y$?

Solution.

By definition we know that $\sigma[3X + 2Y] = \sqrt{Var[3X + 2Y]}$.
Since X and Y are independent random variables,
$Var[3X + 2Y] = Var[3X] + Var[2Y]$
$= 3^2 * Var[X] + 2^2 * Var[Y]$
$= 9 * 5.3 + 4 * 8.9 = 83.3$
Therefore $\sigma[3X + 2Y] = \sqrt{Var[3X + 2Y]} = \sqrt{83.3} \approx 9.13$

2. **[10 pts]** Suppose James has a garden. Let X be the random variable representing the heights of the flowers in the garden, and let Y be the random variable representing the number of petals the flowers have. Suppose that X and Y are non-negative and independent. Help James prove that $X^2 \perp Y^2$.

Solution.

Solution 1:

Since X is the random variable representing the heights of the flowers, Y is the random variable representing the number of petals the flower have, we know that $x \geq 0$ and $y \geq 0$.
Let X: $\Omega_1 -> R$ and Y: $\Omega_2 -> R$
Then X^2: $\Omega_1 -> R$ and Y^2: $\Omega_2 -> R$, and:
$\forall \omega_1 \in \Omega_1 \; X^2(\omega_1) = (X(\omega_1))^2 = x^2$
$\forall \omega_2 \in \Omega_2 \; Y^2(\omega_2) = (Y(\omega_2))^2 = y^2$
Since $x \geq 0$ and $y \geq 0$, x^2 and y^2 are only decided by x and y, which means the function $Val(X)$ to $Val(X^2)$ have one to one correspondence and function $Val(Y)$ to $Val(Y^2)$ have one to one correspondence too.
Thus X and X^2 have the same distribution, and Y and Y^2 have the same distribution.
So $P_r[X(\omega_1)] = P_r[X^2(\omega_1)] \implies P_r[X = x] = P_r[X^2 = x^2]$
$P_r[Y(\omega_2)] = P_r[Y^2(\omega_2)] \implies P_r[Y = y] = P_r[Y^2 = y^2]$
By definition X and Y are independent then for $\{\omega_1 \in \Omega_1 | X(\omega_1) = x\}$, $\{\omega_2 \in \Omega_2 | Y(\omega_2) = y\}$
$P_r[(X = x) \cap (Y = y)] = P_r[X = x] * P_r[Y = y]$
$P_r[(X^2 = x^2) \cap (Y^2 = y^2)] = P_r[(X = x) \cap (Y = y)] = P_r[X = x] * P_r[Y = y] = P_r[X^2 = x^2] * P_r[Y^2 = y^2]$
In short we noted that $P_r[(X^2 = x^2) \cap (Y^2 = y^2)] = P_r[X^2 = x^2] * P_r[Y^2 = y^2]$
By definition we know that $X^2 \perp Y^2$.

Solution 2:

By definition X and Y are independent when
$\forall x \in Val(X) \; \forall y \in Val(Y) \; (X = x) \perp (Y = y)$
Since X and Y are non-negative, we know that $\forall x \in Val(X) x \geq 0$ and $\forall y \in Val(Y) y \geq 0$. Since X and Y are independent:
$P_r[X = x] = P_r[X = x | Y = y] = P_r[X = x | Y = y^2]$
Let $k = x^2$, then $P_r[X = k] = P_r[X = k | Y = y^2]$
Which can be expressed as $P_r[X = x^2] = P_r[X = x^2 | Y = y^2]$
By definition we know that $X^2 \perp Y^2$.

3. **[10 pts]** Suppose you roll $n \geq 1$ fair dice. Let X be the random variable for the sum of their values, and let Y be the random variable for the number of times an odd number comes up. Prove or disprove: X and Y are independent.

Solution.

We guess X and Y are not independent. So we try to disprove it.

We know that:

X = the sum of the value of n fair dice

Y = the number of dice that show an odd number

By definition X and Y are independent when

$\forall x \in Val(X)\ \forall y \in Val(Y)\ (X = x) \perp (Y = y)$

Then the negation of it is that $\exists x \in Val(X), \exists y \in Val(Y)$, $(X = x)$ and $(Y = y)$ are not independent.

Let $n = 4$, then $Val(X) = [4..24]$ and $Val[Y] = [0..4]$

Let $x = 5$ and $y = 2$. Then we can calculate:

$P_r[X = 5] = 4/6^4$

$P_r[Y = 2] = 6/2^4$

$P_r[X = 5 \cap Y = 2] = 0$ since in order to get $X = 5$ we would need 3 dice show 1 and one shows 2, so when $X = 5$, Y would always be 3.

From above we can tell $P_r[X = 5 \cap Y = 2] \neq P_r[X = 5] * P_r[Y = 2]$, thus $P_r[X = 5]$ and $P_r[Y = 2]$ not independent. So we proved $\exists x \in Val(X), \exists y \in Val(Y)$, $(X = x)$ and $(Y = y)$ are not independent, which is the negation of $X \perp Y$.

Therefore we disproved that random variables X and Y are independent.

4. [10 pts] Suppose that you generate a 12-character password by selecting each character independently and uniformly at random from $\{\texttt{a}, \texttt{b}, \ldots, \texttt{z}\} \cup \{\texttt{A}, \texttt{B}, \ldots, \texttt{Z}\} \cup \{\texttt{0}, \texttt{1}, \ldots, \texttt{9}\}$.

(a) What is the probability that exactly 6 of the characters are digits?

(b) What is the expected number of digits in a password?

(c) What is the variance of the number of digits in a password?

Solution.

(a) For each character in the password, the probability that it is a digit is:
$p = 10/62 = 5/31$
Let D be the binomial r.v with parameters $Val(D) = [0..12]$ and $p = 5/31$.
Thus $P_r[I_k = 0] = 1 - p = 52/62 = 26/31$
Here $n = 12$, $k = 6$, so we compute the probability that exactly 6 of the characters are digits is:
$P_r(D = 6) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{12}{6} (\frac{5}{31})^6 (\frac{26}{31})^6$

(b) Let D_k be the event that the kth character in the password is a digit, then we have $P_r[D_k] = p = 5/31$.
Let I_k be the indicator r.v. of the event D_k.
$E[I_k] = Pr[I_k = 1] = Pr[D_k] = p = 5/31$
$B = D_1 + D_2 + \cdots + D_{12}$
BY linearity of expectation, we know that:
$E[D] = E[I_1] + E[I_2] + \cdots + E[I_{12}] = np = 12 * 5/31 = 60/31 \approx 1.9$

(c) Since the characters in the password can be repeated, Event D_1, D_2,..., D_{12} are mutually independent, thus the indivator r.v I_k are also mutually independent.
Therefore:
$Var[D] = Var[I_1] + Var[I_2] + \cdots + Var[I_{12}]$
$= p(1-p) + p(1-p) + \cdots + p(1-p)$
$= 12 * p(1-p)$
$= 12 * 5/31 * 26/31 \approx 1.2$

5. [10 pts] Suppose Jay shoots a basketball. Let X be the Bernoulli random variable that returns 1 if he makes the shot, and 0 if he misses. Let Y be the Bernoulli random variable (independent of X) that returns 1 if he hits the backboard, and 0 if he does not hit it. X and Y both have parameter 1/2. Let Z be the random variable that returns the remainder of the division of $X + Y$ by 2.

(a) Prove that Z is also a Bernoulli random variable, also with parameter 1/2.

(b) Prove that X, Y, Z are pairwise independent but not mutually independent.

(c) By computing $\text{Var}[X + Y + Z]$ according to the alternative formula for variance and using the variance of Bernoulli r.v.'s, verify that $\text{Var}[X + Y + Z] = \text{Var}[X] + \text{Var}[Y] + \text{Var}[Z]$ (observe that this also follows from the proposition on slide 5 of the lecture segment entitled "Binomial distribution").

Solution.

(a) We know that $Val(X) = Val(Y) = 0, 1$
Since Z is the random variable that returns the remainder of the division of $(X+Y)/2$, we know that:
When $X = 0$ and $Y = 0$, $Z = 0$
When $X = 1$ and $Y = 0$, $Z = 1$
When $X = 1$ and $Y = 1$, $Z = 0$
When $X = 0$ and $Y = 1$, $Z = 1$
From above we can tell that $Val(Z) = 0, 1$ and $P_r[Z = 0] = P_r[Z = 1] = 1/2$
By the definition we proved that Z is also a Bernoulli random variable, also with parameter $p = 1/2$.

(b) It's given that X and Y are independent. Now we need to prove (1) X and Z are independent, and (2) Y and Z are independent.
Since X, Y and Z are all Bernoulli random variables with parameter 1/2, we know that $P_r[X = 0] = P_r[X = 1] = P_r[Y = 0] = P_r[Y = 1] = P_r[Z = 0] = P_r[Z = 1] = 1/2$
We also know that the possible outcomes are as below:
When $X = 0$ and $Y = 0$, $Z = 0$
When $X = 1$ and $Y = 0$, $Z = 1$
When $X = 1$ and $Y = 1$, $Z = 0$
When $X = 0$ and $Y = 1$, $Z = 1$
(1)$P_r[X = 0 \cap Z = 0] = 1/4$ since there is only one outcome "when $X = 0$ and $Y = 0$, $Z = 0$" out of the 4 outcomes.
So $P_r[X = 0] * P_r[Z = 0] = 1/2 * 1/2 = 1/4 = P_r[X = 0 \cap Z = 0]$.

Similarly we can prove that:

$P_r[X = 0] * P_r[Z = 1] = 1/2 * 1/2 = 1/4 = P_r[X = 0 \cap Z = 1]$.

$P_r[X = 1] * P_r[Z = 0] = 1/2 * 1/2 = 1/4 = P_r[X = 1 \cap Z = 0]$.

$P_r[X = 1] * P_r[Z = 1] = 1/2 * 1/2 = 1/4 = P_r[X = 1 \cap Z = 1]$.

Above discussion would cover all the possible outcomes, so we can conclude that for $\forall a \in Val(X)\ \forall c \in Val(Z)$, $P_r[(X = a) \cap (Z = c)] = P_r[X = a] * P_r[Z = c]$.

By definition, X and Z are independent.

(2)$P_r[Y = 0 \cap Z = 0] = 1/4$ since there is only one outcome "when $X = 0$ and $Y = 0$, $Z = 0$" out of the 4 outcomes.

So $P_r[Y = 0] * P_r[Z = 0] = 1/2 * 1/2 = 1/4 = P_r[Y = 0 \cap Z = 0]$.

Similarly we can prove that:

$P_r[Y = 0] * P_r[Z = 1] = 1/2 * 1/2 = 1/4 = P_r[Y = 0 \cap Z = 1]$.

$P_r[Y = 1] * P_r[Z = 0] = 1/2 * 1/2 = 1/4 = P_r[Y = 1 \cap Z = 0]$.

$P_r[Y = 1] * P_r[Z = 1] = 1/2 * 1/2 = 1/4 = P_r[Y = 1 \cap Z = 1]$.

Above discussion would cover all the possible outcomes, so we can conclude that for $\forall b \in Val(Y)\ \forall c \in Val(Z)$, $P_r[(Y = b) \cap (Z = c)] = P_r[Y = b] * P_r[Z = c]$.

By definition, Y and Z are independent.

Lastly, we know that:

In order to let X,Y,Z be mutually independent, for $\forall a \in Val(X)\ \forall b \in Val(Y)\ \forall c \in Val(Z)$, $P_r[(X = a) \cap (Y = b) \cap (Z = c)] = P_r[X = a] * P_r[Y = b] * P_r[Z = c]$

Therefore to prove X,Y,Z are not mutually independent, we can prove it by finding a counterexample to show that:

$\exists a \in Val(X)\ \exists b \in Val(Y)\ \exists c \in Val(Z) P_r[(X = a) \cap (Y = b) \cap (Z = c)] \neq P_r[X = a] * P_r[Y = b] * P_r[Z = c]$.

We know that $(X = 1) \cap (Y = 1) \cap (Z = 1) = \emptyset$ and $P_r[X = 1] * P_r[Y = 1] * P_r[Z = 1] = 1/2 * 1/2 * 1/2 = 1/8$,

so $P_r[(X = 1 \cap Y = 1 \cap Z = 1)] \neq P_r[X = 1] * P_r[Y = 1] * P_r[Z = 1]$.

Further $P_r[(X = 0 \cap Y = 0 \cap Z = 0)] = 1/4$, and $P_r[X = 0] * P_r[Y = 0] * P_r[Z = 0] = 1/2 * 1/2 * 1/2 = 1/8$

So $P_r[X = 0 \cap Y = 0 \cap Z = 0] \neq P_r[X = 0] * P_r[Y = 0] * P_r[Z = 0]$.

By the definition, we can conclude that X, Y and Z are not mutually independent.

(c) Step 1:

We first calculate the expectation of X, Y and Z. Since they are all Bernoulli r.v with parameter p, we know that :

$E[X] = E[Y] = E[Z] = p = 1/2$

Step 2:

Since X,Y and Z are pairwise independent, we know that:

$E[XY] = E[X] * E[Y]$, $E[XZ] = E[X] * E[Z]$,$E[YZ] = E[Y] * E[Z]$

Step 3:

By variance of Bernoulli r.v., we know that When X is a Bernoulli r.v. with parameter p, then X^2 is also Bernoulli with the same distribution and $X = X^2$. Thus $E[X^2] = E[X]$. Similarly we know $E[Y^2] = E[Y]$, and $E[Z^2] = E[Z]$.

Step 4:

By Variance for Binomial II, we know that:

$Var[X] = Var[Y] = Var[Z] = np(1-p) = 1 * 1/2 * (1 - 1/2) = 1/4$

Thus $Var[X] + Var[Y] + Var[Z] = 1/4 + 1/4 + 1/4 = 3/4$

Step 5:

By the alternative formula for variance, we know that:

$Var[X+Y+Z] = E[(X+Y+Z)^2] - (E[X+Y+Z])^2$

$=E[(X+Y+Z)*(X+Y+Z)] - (E[X]+E[Y]+E[Z])^2$

$=E[X^2+Y^2+Z^2+2XY+2XZ+2XZ] - (E[X]+E[Y]+E[Z])^2$

$=E[X^2]+E[Y^2]+E[Z^2]+E[2XY]+E[2XZ]+E[2XZ] - (E[X]+E[Y]+E[Z])^2$

$=E[X]+E[Y]+E[Z]+2E[XY]+2E[XZ]+2E[XZ] - (E[X]+E[Y]+E[Z])^2$

$=1/2+1/2+1/2+2*1/2*1/2+2*1/2*1/2+2*1/2*1/2-(1/2+1/2+1/2)^2$

$=3/4$

Step 6:

From above we can tell that

$Var[X+Y+Z] = Var[X] = Var[Y] = Var[Z] = 3/4$.

Verification is completed.