Self-paced Example: Algebraic Verification of Pascal's Identity

Module 4

MCIT Online - CIT592 - Professor Val Tannen

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion. When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Pascal's identity algebraic verification

In this module you learned about **Pascal's Identity**, which states that:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

where n and k are positive integers with $n \geq k \geq 1$.

Recall also that Pascal's identity is suggested by a pattern that can be noticed in Pascal's Triangle:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \cdots & & & & & &
\end{array}$$

(Notice in Pascal's Triangle that every (inner) number is the sum of the two numbers above it.)

When we stated it we showed a combinatorial proof for Pascal's Identity. In this segment, we will walk you through another way to prove Pascal's Identity.

Don't worry, this is a much more straightforward proof! ☺

Problem. Once again, recall **Pascal's Identity**:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

where n and k are positive integers with $n \geq k \geq 1$. Prove this combinatorial identity by algebraic verification.

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Pascal's identity algebraic verification (continued)

Answer. We are going to consider two cases, when $k > n$, and when $k \leq n$.

First consider the case where $k > n$. In this case, proving that the identity holds is trivial since

$$\binom{n}{k} = 0 = \binom{n-1}{k-1} = \binom{n-1}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Now we are going to consider the case where $k \leq n$.

We start by expanding the right-hand side (RHS) of the identity[1] using the formula we had derived for combinations:

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\ &= (n-1)!(\frac{1}{(k-1)!(n-k)!} + \frac{1}{k!(n-k-1)!})\end{aligned}$$

Observe that

$$\frac{1}{(k-1)!} = \frac{k}{k(k-1)!} = \frac{k}{k!}$$

Similarly,

$$\frac{1}{(n-k-1)!} = \frac{n-k}{(n-k)(n-k-1)!} = \frac{n-k}{(n-k)!}$$

It follows that:

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= (n-1)!(\frac{k}{k(k-1)!(n-k)!} + \frac{n-k}{k!(n-k)(n-k-1)!}) \\ &= (n-1)!(\frac{k}{k!(n-k)!} + \frac{n-k}{k!(n-k)!})\end{aligned}$$

[1] *Note: think about why we choose to expand the RHS; it is more informative in some way... This type of thinking will allow you to tackle future problems!*

Pascal's identity algebraic verification (continued)

With some more simple algebraic manipulations we obtain in the end the left-hand side (LHS):

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= (n-1)!\frac{k+n-k}{k!(n-k)!} \\ &= \frac{n(n-1)!}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k)!} = \binom{n}{k}\end{aligned}$$

We saw how we can provide an algebraic proof, where we earlier provided a combinatorial proof. This pattern is true for many combinatorial identities, namely there is a way to prove all of them through combinatorics (what we did in lecture), and a way to prove it through algebraic manipulations (what we just did).

If you are still skeptical (...and even if you are not) you should try and follow a similar approach and solve the other combinatorial identities that we saw through algebraic verification. You will see that there is always a way!