Module 7.4: Probability Properties

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Probability properties I

In this segment we give the statements of the properties, some intuition that justifies them, and some examples of putting the properties to use. Rigorous proofs of the properties appear in the segment entitled "Proofs of probability properties".

Consider an arbitrary probability space (Ω, Pr) and arbitrary events E, A, B in this space.

Property P0. $\mathrm{Pr}[E] \geq 0$

Since it's the sum of non-negative numbers.

Property P1. $\mathrm{Pr}[\Omega] = 1$

Since it adds up the probabilities of all the outcomes in the space.

Probability properties II

Property P2. If A, B are disjoint then $\Pr[A \cup B] = \Pr[A] + \Pr[B]$

This is called the **addition rule** and is analogous to the addition rule in counting applied to set cardinality: $A \cap B = \emptyset \;\Rightarrow\; |A \cup B| = |A| + |B|$.

Property P3. If $A \subseteq B$ then $\Pr[A] \leq \Pr[B]$

This is called **monotonicity** and it has an analogous property of set cardinality: $A \subseteq B \;\Rightarrow\; |A| \leq |B|$.

If $E \subseteq \Omega$ is an event then the **complement** of E is the event $\overline{E} = \Omega \setminus E$.

Property P4. $\Pr[\overline{E}] = 1 - \Pr[E]$

Some applications

Problem. We roll a pair of fair dice. Compute the probability of the event "none of the dice shows a 6".

Answer. In a previous segment we computed the probability of the event $G =$ "at least one of the dice shows a 6" to be $11/36$. The event here is $\overline{G}$, the **complement** of G. By **P4** the answer here is:

$$\Pr[\overline{G}] = 1 - 11/36 = (36 - 11)/36 = 25/36.$$

Problem. We roll a pair of fair dice. Compute the probability of the event "the numbers add up to 2, 3, 4, 6, 8, 10 or 12".

Answer. The event of interest is $C \cup D$ where $C =$ "the numbers add up to an even number" and $D =$" the numbers add up to 3" are **disjoint**. By **P2**:

$$\Pr[C \cup D] = \Pr[C] + \Pr[D] = 1/2 + (1/36 + 1/36) = 5/9.$$

More probability properties

Property P5. $\Pr[\emptyset] = 0$

By the definition of event probability, this is a sum with **no terms**! A common convention is that such a sum is 0. However, see the next activity.

Property P6. $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$

This is called (of course!) **inclusion-exclusion** for two events and is analogous to PIE for two sets.

Property P7. $\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$

This is called the **union bound**, and it plays a major role in the analysis of **probabilistic algorithms**. It's quite clear that it follows immediately from **P6**.

ACTIVITY : Two proofs of property P5

In this activity we will present two different ways of deriving:

Property P5. $\Pr[\emptyset] = 0$

from some of the other probability properties.

First proof. Recall

Property P1. $\Pr[\Omega] = 1$

and

Property P4. $\Pr[\overline{E}] = 1 - \Pr[E]$

Question. How should we choose E in P4 to help (together with P1) prove P5?

In the video, there is a box here for learners to put in an answer to the question above. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Two proofs of property P5 (continued)

Answer. Choose $E = \Omega$.

Now replacing $E = \Omega$ in P4 and observing that $\overline{\Omega} = \emptyset$ we obtain

$$\Pr[\emptyset] \;=\; 1 \;-\; \Pr[\Omega]$$

Using P1 we obtain $\Pr[\emptyset] = 1 - 1 = 0$.

Second proof. In this proof we use

Property P2. If A, B are disjoint then $\Pr[A \cup B] \;=\; \Pr[A] \;+\; \Pr[B]$

ACTIVITY : Two proofs of property P5 (continued)

We wish to take $A = B = \emptyset$ in P2. We can do this if A and B are disjoint. And they are: $\emptyset \cap \emptyset = \emptyset$! This can feel surprising. Any non-empty set is **not** disjoint from itself. But the empty set is disjoint from itself!

Taking $A = B = \emptyset$ in P2 we obtain $\Pr[\emptyset \cup \emptyset] = \Pr[\emptyset] + \Pr[\emptyset]$.

Since $\emptyset \cup \emptyset = \emptyset$ we get $\Pr[\emptyset] = \Pr[\emptyset] + \Pr[\emptyset]$.

It follows that $\Pr[\emptyset] = 0$.

An application and generalizations

Problem. We roll a pair of fair dice. Compute the probability of the event "at least one of the dice shows a 6" using **P6** (inclusion-exclusion).

Answer. In the green-purple dice space define $C =$ "green die rolls 6", $D =$ "purple die rolls 6". By **P6**:

$$\Pr[C \cup D] = \Pr[C] + \Pr[D] - \Pr[C \cap D] = 1/6 + 1/6 - 1/36 = 11/36$$

Finally, we note that **P2** and **P7** generalize:

Property P2gen. If $A_1, \ldots, A_n$ are pairwise disjoint then

$$\Pr[A_1 \cup \cdots \cup A_n] = \Pr[A_1] + \cdots + \Pr[A_n]$$

Property P7gen. $\Pr[A_1 \cup \cdots \cup A_n] \leq \Pr[A_1] + \cdots + \Pr[A_n]$