Module 7.2: Uniform Spaces

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Uniform probability spaces

A probability space $(\Omega, \Pr)$ is called **uniform** if all the outcomes have the **same** probability.

Denote $n = |\Omega|$. Since the probabilities are equal and sum up to 1:

. $\Pr[w] = 1/n$ for each outcome $w \in \Omega$.

Proposition. In a uniform probability space $\Pr[E] = m/n$ where $m = |E|$ and $n = |\Omega|$.

Proof.

$$\Pr[E] = \sum_{w \in E} \Pr[w] = \sum_{w \in E} \frac{1}{n} = m \cdot \frac{1}{n} = \frac{m}{n}$$

Dice add up to an even number

Problem. Compute the probability of the event "when we roll a pair of fair dice the numbers add up to an even number".

Answer. W.l.o.g. we work in the green-purple dice probability space. This space is uniform with 36 outcomes.

Let g be the number shown by the green die and p the one shown by the purple one.
Each outcome corresponds to a pair (g, p) where $g, p \in \{1, \dots, 6\}$

$g + p$ is even iff both g and p are even or both are odd.

The event of interest contains exactly **half of the outcomes** because for each die there are as many even faces as there are odd ones.

The answer is $18/36 = 1/2$.

Three dice show the same number

Problem. We roll three fair dice. What is the probability that all three show the same number?

Answer. Each outcome corresponds to a triple $(d_1, d_2, d_3) \in \{1, \ldots, 6\}^3$. By the multiplication rule there are $6 \cdot 6 \cdot 6 = 216$ outcomes.

Since the dice are fair and rolled in the same way, each of the outcomes (d_1, d_2, d_3) is **equally likely**.

Therefore the space is uniform: each outcome has probability $1/216$.

The event of interest consists of outcomes
. $(1, 1, 1), \ldots, (6, 6, 6)$. That's six outcomes.

Hence the answer is $6/216 \;=\; 1/36$.

Quiz

We flip a fair coin 3 times. The probability that we get one heads and two tails (in some order) is

(A) $3/8$

(B) $1/3$

(C) $1/8$

Answer

The correct answer is (A) $3/8$. This is a uniform probability space with 8 outcomes `HHH` `HHT` `HTH` `HTT` `THH` `THT` `TTH` `TTT`

Of these, the outcomes in the event of interest are `HTT` `THT` `TTH` so 3 outcomes. Hence the probability is $3/8$.

Activity

We flip a fair coin n times. Compute the probability that we get at least one heads.

Describe an outcome as a sequence of length n of $\mathtt{H}$ and $\mathtt{T}$.

ASK HOW MANY OUTCOMES

By the multiplication rule there are $2 \cdot 2 \cdots 2 = 2^n$ outcomes.

This is a uniform probability space so we just need to count how many outcomes have at least one heads?

It's easier to count complementarily!

Only one outcome has no heads at all: $\mathtt{TT} \cdots \mathtt{T}$.

$$\text{Answer:} \qquad \frac{2^n - 1}{2^n}$$