

Additional Problems (Packet 1)

1. Let $n, k \in \mathbb{Z}^+$ such that $k \leq n - 1$. Which one is bigger $\binom{n}{k-1}$ or $\binom{n-1}{k}$?

ANSWER

Not that $k \geq 1$ and $n \geq 2$ and that $n \geq k + 1$ We need to compare (and guess!)

$$a = \frac{n!}{(n-k+1)!(k-1)!} \quad \text{and} \quad b = \frac{(n-1)!}{(n-k-1)!k!}$$

Intuition suggests that this depends on how close n is to k . The closest is when $n = k + 1$. In this case $b = 1$ and $a = k(k+1)/2 \geq 1 = b$.

However, if we take $k = 1$ then $a = 1$ and $b = n - 1 \geq 1 = a$.

Therefore, in general, both $a \geq b$ and $a \leq b$ are possible.

The same conclusion can be reached by staring at Pascal's triangle and finding in it various instances of these two binomial coefficients.

2. A *study quad* consists of 4 students that participate in *study budds*. A study budd is a partnership of two students. Moreover, in a study quad, each student participates in exactly two study budds. Prove that, given 4 students, they can form exactly 3 distinct study quads.

ANSWER

We prove that students (WLOG) A,B,C,D can form exactly 3 study quads. A must have two study partners, one in each of the two study budds that A participates in. The choice of two study partners can be done in $\binom{3}{2} = 3$ ways.

We argue that once the two study partners of A have been chosen the entire study quad is uniquely determined. Indeed, let the two study partners of A be (WLOG) B and C.

We show by contradiction that B and C cannot be partners in a study budd. Indeed, if that were so then A,B,C participate in two study budds each and there is nobody left to partner with D.

Therefore B and C are not partners and for each them there is only D left to partner with. Thus we have uniquely determined the study quad: $\{\{A, B\}, \{A, C\}, \{B, D\}, \{C, D\}\}$.

3. Let x, y, z, a, b, c be six integers such that $x = a + y$ and $y = b + z$. Prove that if the average of a, b and c is not 0 then $z \neq c + x$.

ANSWER

We prove the contrapositive, namely, if $z = c + x$ then $(a + b + c)/3 = 0$.

Assume $z = c + x$, that is $c = z - x$. From the other assumptions, $b = y - z$ and $a = x - y$. Therefore $(a + b + c)/3 = (x - y + y - z + z - x)/3 = 0/3 = 0$.

4. Let P and Q be the following mathematical statements:

- P : there are infinitely many pairs of prime numbers that differ by exactly 2 (this is known as the Twin Primes Conjecture).
- Q : every even integer strictly greater than 2 can be written as the sum of two primes (this is known as Goldbach's Conjecture).

As of this writing (early 2021) we do not know if either one of these conjectures is true or false.

- Write P and Q as formal logical statements using logical connectives, quantifiers, and the predicates $prime(z)$ and $odd(z)$.
- Show that regardless of whether either of the conjectures is true or false the statement $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$ is always true.

ANSWER

- $P : \forall x \in \mathbb{Z} \exists y, z \in \mathbb{Z} (y \geq x) \wedge prime(y) \wedge prime(z) \wedge (z = y + 2)$
 $Q : \forall x \in \mathbb{Z} \neg odd(x) \wedge (x > 2) \Rightarrow \exists y, z \in \mathbb{Z} prime(y) \wedge prime(z) \wedge (x = y + z)$
- Showing this is the same as showing that the boolean expression $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$ where p, q are boolean variables is true for any truth assignment to p and q (we say that such an expression is a *tautology*). Indeed, this is the case as the following truth table shows:

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \Rightarrow p$	$((p \Rightarrow q) \Rightarrow p) \Rightarrow p$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

- Prove that if P is a group of 4 people that can be split $P = R \cup B$ such that R and B are disjoint, and such that each person in R does not have FB friends in R and each person in B does not have FB friends in B then there exist 3 people in P that are not pairwise friends (that is, at least 2 of the 3 are not FB friends).

ANSWER

Case 1: There are 0 or 1 people in R (hence B has 4 or 3 people). In that case there are at least one group of 3 people, all in B . No 2 of these 3 are FB friends and we are done. The case in which there are 0 or 1 people in B is similar (symmetric).

Case 2: There are 2 people in each of R and B . Then our group of 3 consists of 2 from R and 1 from B (or the other way around). The 2 from R are guaranteed not to be FB friends and we are done in this case too.

6. Prove that there exists a subset $X \subseteq [0..2]^{[0..2]}$ such that $|X| = 9$ and for any two $f, g \in X$ we have $f(1) = g(1)$.

ANSWER

We apply the Generalized Pigeonhole Principle (GPHP). The pigeons are the elements of $[0..2]^{[0..2]}$, that is, all the functions with domain $[0..2]$ and codomain $[0..2]$. There are $3^3 = 27$ pigeons. The pigeonholes are the elements of codomain $[0..2]$, that is, 1, 2 and 3. Pigeon $f \in [0..2]^{[0..2]}$ flies into pigeonhole $f(1) \in [0..2]$.

By GPHP at least one of the pigeonholes will have at least $\lceil 27/3 \rceil = 9$ pigeons. WLOG we refer to such a pigeonhole (with 9 or more pigeons). If f, g are any two pigeons that fly into H then $f(1) = g(1)$. Therefore we can take X to be any set of 9 distinct pigeons/functions from H .

Alternate Solution:

We construct explicitly the set of functions X . Each of the functions $f \in X$ has domain $[0..2]$, codomain $[0..2]$, and is constructed in steps as follows:

Step 1 Define $f(1) = 1$. (1 way)

Step 2 Define $f(0)$ as 0, 1, or 2. (3 ways)

Step 3 Define $f(2)$ as 0, 1, or 2. (3 ways)

By the multiplication rule there are $1 \times 3 \times 3 = 9$ distinct such functions f . We take X to be the set of these 9 functions. By construction any two functions f, g in X are defined such that $f(1) = 1 = g(1)$.

7. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a *very brief* explanation of your answer.

- (a) $\binom{100}{51}$ is strictly bigger than $\binom{100}{49}$.
- (b) In Pascal's Triangle, there exist four binomial coefficients c_1, c_2, c_3, c_4 such that $c_1 = c_2 + c_3 + c_4$.
- (c) The boolean expressions $\neg[(p \Rightarrow q) \vee q]$ and $p \wedge \neg q$ are logically equivalent.
- (d) Let A be a finite set. All functions $f : A \rightarrow A$ are bijections.
- (e) Let A, B be finite set. Then, there are exactly as many subsets of $A \times B$ as there are functions with domain A and codomain 2^B .

ANSWER

- (a) *FALSE* In fact, we see that they are equal. This stems from the identity

$$\binom{n}{k} = \binom{n}{n-k}$$

presented in lecture.

- (b) *TRUE* Consider the following values for c_1, c_2, c_3 , and c_4 :

$$c_1 = \binom{3}{1} \qquad c_2 = \binom{160}{0} \qquad c_3 = \binom{160}{160} \qquad c_4 = \binom{0}{0}$$

We thus see that:

$$\begin{aligned} c_1 &= c_2 + c_3 + c_4 \\ \binom{3}{1} &= \binom{160}{0} + \binom{160}{160} + \binom{0}{0} \\ 3 &= 1 + 1 + 1 \end{aligned}$$

which we know to be true.

Alternate Solution:

We also could have solved this problem by applying Pascal's Identity twice, for example:

$$\begin{aligned} \binom{n}{k} &= \binom{n-1}{k-1} + \binom{n-1}{k} \\ &= \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k} \end{aligned}$$

By choosing appropriate values for n and k (i.e., $n \geq k \geq 1$, we know the above equality holds. Concretely, take $n = 5$ and $k = 3$. We thus have:

$$\begin{aligned} \binom{5}{3} &= \binom{4}{2} + \binom{3}{2} + \binom{3}{3} \\ 10 &= 6 + 3 + 1 \end{aligned}$$

which we know to be true.

- (c) TRUE We demonstrate this by showing they have equivalent truth tables, as follows:

p	q	$p \implies q$	$(p \implies q) \vee q$	$\neg[(p \implies q) \vee q]$	$\neg q$	$p \wedge \neg q$
T	T	T	T	F	F	F
T	F	F	F	T	T	T
F	T	T	T	F	F	F
F	F	T	T	F	T	F

- (d) FALSE For a counterexample, consider the set $A = \{1, 2\}$ and the function $f : A \rightarrow A$ given by $f(1) = f(2) = 1$. The function is not injective, since $f(1) = 1 = f(2)$. Therefore it cannot be a bijection. Alternatively, we can argue that f is not surjective because there is no $a \in A$ with $f(a) = 2$, and hence, it is not bijective.
- (e) TRUE Let $|A| = n$ and $|B| = m$. From class, we know that:

$$|A \times B| = |A| \cdot |B| = nm$$

Furthermore, for a given set S , we also know that $|2^S| = 2^{|S|}$. Thus, we conclude that the number of subsets of $A \times B$ is given by:

$$|2^{A \times B}| = 2^{|A \times B|} = 2^{nm}$$

From class, we have also know that the number of functions with domain X and codomain Y is given by $|Y|^{|X|}$. In this problem, we know that $X = A$ and $Y = 2^B$, thus the number of such functions is:

$$|2^B|^{|A|} = (2^m)^n = 2^{nm}$$

Therefore, we have shown that there are exactly as many subsets of $A \times B$ as there are functions with domain A and codomain 2^B .

8. Count the number of distinct sequences of bits (0's, 1s) of length 101 such that:

- there are 3 more 1's than 0's in the sequence; **and...**
- ... **also** the middle bit is a 1.

ANSWER

Note that given the first constraint, there must be exactly 52 1's and 49 0's in the sequence. We find this by letting x be the total number of 0's in the sequence, and we see that

$$\begin{aligned}x + (x + 3) &= 101 \\2x &= 98 \\x &= 49\end{aligned}$$

To count the total number of sequences, we use multiplication rule.

Step 1: place a 1 bit in the middle position.

Step 2: place the remaining 1's.

Step 3: place the remaining 0's.

There's only one way of placing a 1 bit in the middle position. For Step 2, we are left with 51 1's to distribute among $101 - 1 = 100$ spots (since the middle spot is now occupied). This can be done in $\binom{100}{51}$ ways. Finally, note that once the 1's are in their positions, there's only one way to place the 0's. Specifically, the 0's *must* go into the empty positions. Thus our final answer is, by the Multiplication Rule:

$$1 \cdot \binom{100}{51} \cdot 1 = \boxed{\binom{100}{51}}$$

9. Prove that for any $x, y, z \in \mathbb{Z}$ such that $x + 2y = z$, if $z - x$ is *not* divisible by 4 then $x + y + z$ is odd.

ANSWER

Proof by contradiction. Let $x, y, z \in \mathbb{Z}$ such that $x + 2y = z$ and $z - x$ is *not* divisible by 4. Assume toward a contradiction that $x + y + z$ is *not* odd.

Hence $x + y + z$ is even, and therefore $x + y + z = 2k$, for some $k \in \mathbb{Z}$. We also have that $z = x + 2y$, so we may write:

$$\begin{aligned}x + y + (x + 2y) &= 2k \\2x + 3y &= 2k \\3y &= 2(k - x)\end{aligned}$$

Therefore $3y$ must be even. It follows that y must also be even. Indeed, if y were odd then $3y$ would also be odd. So $y = 2\ell$ for some $\ell \in \mathbb{Z}$.

But then we have

$$\begin{aligned} z - x &= (x + 2y) - x \\ &= 2y \\ &= 4\ell \end{aligned}$$

Since ℓ is an integer, this means that $4 \mid z - x$, and we have reached a *contradiction*.

Alternate Solution: (direct proof).

We rewrite the given equation as

$$2y = z - x$$

Since $y \in \mathbb{Z}$, this tells us that $z - x$ is even. Additionally, since $z - x$ is not divisible by 4, we know that $\frac{z - x}{2}$ must be odd – if it were even, $z - x$ would be divisible by 4.

We write

$$z - x = 2(2k + 1), k \in \mathbb{Z}$$

But then $2y = 2(2k + 1)$, so $y = 2k + 1$ is odd.

Finally, we know that $2x$ is even, since $x \in \mathbb{Z}$. Putting this together gives us:

$$\begin{aligned} x + y + z &= (z - x) + 2x + y \\ &= 2(2k + 1) + 2x + 2k + 1 \\ &= 2(3k + x + 1) + 1 \end{aligned}$$

As the sum of products of integers, $3k + x + 1$ must be an integer, so $x + y + z$ is odd.

10. Let A, B be any sets such that $A \cap \{1, 2\} = B \cap \{1, 2\}$. Prove that the sets $(A \setminus B) \cup (B \setminus A)$ and $\{1, 2\}$ are disjoint.

ANSWER

If we can show that no element in $((A \setminus B) \cup (B \setminus A))$ can be in $\{1, 2\}$, then clearly the intersection of these 2 sets is empty and we are done.

We consider an arbitrary element $x \in ((A \setminus B) \cup (B \setminus A))$ and show that $x \notin \{1, 2\}$. There are 2 cases:

Case 1: $x \in A \setminus B$. Then $x \in A \wedge x \notin B$. Assume for the sake of contradiction that $x \in \{1, 2\}$. Then $x \in A \cap \{1, 2\}$ as it is in both sets and $x \notin B \cap \{1, 2\}$ as it is not in B . Therefore, $A \cap \{1, 2\} \neq B \cap \{1, 2\}$, a contradiction. So, $x \notin \{1, 2\}$.

Case 2: This analogous to Case 1, by symmetry. (Switch A and B in all of the above statements).

In both cases $x \notin \{1, 2\}$.

Since no element $x \in ((A \setminus B) \cup (B \setminus A))$ can be in $\{1, 2\}$, these 2 sets are disjoint.

Alternate Solution:

We prove the *contrapositive*. Assume that the sets $(A \setminus B) \cup (B \setminus A)$ and $\{1, 2\}$ are *not* disjoint, i.e.,

$$\text{there exists some } x \in ((A \setminus B) \cup (B \setminus A)) \cap \{1, 2\}$$

Therefore

$$[(x \in A \setminus B) \vee (x \in B \setminus A)] \wedge (x = 1) \vee (x = 2)]$$

We have 4 cases.

Case 1: $(x \in A \setminus B) \wedge (x = 1)$ i.e., $1 \in A \setminus B$. Then $1 \in A$ and $1 \notin B$. Since $1 \in A$ we have $1 \in A \cap \{1, 2\}$. Since $1 \notin B$ we have $1 \notin B \cap \{1, 2\}$. Therefore, $A \cap \{1, 2\} \neq B \cap \{1, 2\}$.

The other three cases:

Case 2: $(x \in A \setminus B) \wedge (x = 2)$.

Case 2: $(x \in B \setminus A) \wedge (x = 1)$.

Case 2: $(x \in B \setminus A) \wedge (x = 2)$.

are analogous by symmetry. (Switch A and B and/or 1 and 2.)

In all four cases we have shown $A \cap \{1, 2\} \neq B \cap \{1, 2\}$ and this proves the contrapositive.

11. Let $n \in \mathbb{N}$ and $n \geq 3$. Give a combinatorial proof (no other kinds of proofs will be accepted) for the following identity

$$\binom{n+2}{3} = \binom{n}{1} + 2\binom{n}{2} + \binom{n}{3}$$

ANSWER

We ask the following question:

Since the weather is starting to get warmer Ali, Serena, and Constance decide to go get some ice cream. The ice cream shop, Flora's Creamery, has just 2 kinds of soft serve ice cream, and n kinds of hard ice cream. In how many ways can Ali, Serena, and Constance order 3 different kinds of ice cream if among the 3 of them it does not matter who gets which kind?

We see that the LHS counts this question. There are $n + 2$ different kinds of ice cream, since there are 2 kinds of soft serve and n kinds of hard ice cream. Since we want to choose 3 of these kinds in any order, there are $\binom{n+2}{3}$ ways to do this.

We now count the RHS. We can do this by casing on the number of kinds of soft ice cream that we choose. Since there are only 2 kinds of soft serve ice cream, we can choose 0, 1, or 2 kinds of soft ice cream.

Case 1: We choose 0 kinds of soft ice cream. In this case we are choosing 3 kinds of hard ice cream. This can be done in $\binom{n}{3}$ ways since the flavors of hard ice cream are distinguishable.

Case 2: We choose 1 kind of soft ice cream. In this case we are choosing 1 kind of soft serve and 2 kinds of hard ice cream. This can be done in $2\binom{n}{2}$ ways. We can choose 2 kinds of hard ice cream in $\binom{n}{2}$ ways, and then there are 2 ways to choose a kind of soft serve. Combining these, we have $2\binom{n}{2}$ ways.

Case 3: We choose 2 kinds of soft serve ice cream. In this case we are choosing 2 kinds of soft serve and 1 kind of hard ice cream. There is only 1 way to choose 2 kinds of soft serve (since there are only 2 kinds). There are n ways to choose a kind of hard ice cream. Combining these, we have $n = \binom{n}{1}$ ways.

Combining all these cases together, we see that, by the Sum Rule, there are

$$\binom{n}{1} + 2\binom{n}{2} + \binom{n}{3}$$

which is exactly the RHS and we are done.

12. Let X be a nonempty finite set. Consider the set $W = \{(A, B) \mid A, B \in 2^X \wedge A \subseteq B\}$. Prove that W has exactly as many elements as there are functions with domain X and codomain $\{1, 2, 3\}$.

ANSWER

We find the sizes of these two sets: First, we consider W . We are counting ordered pairs (A, B) such that $A \subseteq B \subseteq X$. For this constraint to hold, each element $x \in X$ must have either $x \notin B$, $x \in B \setminus A$ or $x \in A$. (You may recall this from an earlier homework.) We can construct two such sets A, B in $|X|$ steps; in each step, one element $x \in X$ is assigned one of the 3 states $x \notin B$, $x \in B \setminus A$ or $x \in A$. Thus, this whole process can be done in $3^{|X|}$ ways.

For the size of the other set we note that the size of the codomain is 3 therefore (as shown in lecture) there are $3^{|X|}$ functions with domain X and codomain $\{1, 2, 3\}$. Thus, W and the set of functions f described in the problem have equal size.

NOTE This also follows from the Bijection Rule. Indeed, in each (A, B) of W , each $x \in X$ must be mapped to one of 3 states "not in B ", "in $B \setminus A$ " and "in A ". This establishes a one-to-one correspondence between the pairs (A, B) in W and the functions with domain X and a codomain formed by these three states. As this codomain is of the same size as the set $\{1, 2, 3\}$, then there are also an equal number of functions with domain X and codomain $\{1, 2, 3\}$.

13. ANSWER

Setting this problem was a mistake. It leads to very complicated counting. Please ignore it.

14. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a *very brief* explanation of your answer.
- (a) Assume that B is a set with 7 elements and that A is a set with 15 elements. Then, for any function $f : A \rightarrow B$ there exist at least 3 distinct elements of A that are mapped by f to the same element of B . TRUE or FALSE?
 - (b) There are exactly three surjective functions with domain $\{1, 2\}$ and codomain $\{a, b\}$. TRUE or FALSE?
 - (c) Exactly two of the following three boolean expressions: $p \Rightarrow q$, $p \wedge \neg q$, and $\neg p \vee q$ are logically equivalent. TRUE or FALSE?
 - (d) Let A be a finite set. For any function $f : A \rightarrow A$ we have $|\text{Ran}(f)| = |A|$. TRUE or FALSE?
 - (e) There exist two distinct functions with domain and codomain $\{a, b\}$ that are their own inverses. TRUE or FALSE?
 - (f) For any two finite sets A, B , $|2^{A \times B}| > |2^A \times 2^B|$. TRUE or FALSE?

- (g) Recall that for any $n = 0, 1, 2, 3, \dots$ row n of the Pascal Triangle contains the binomial coefficients of the form $\binom{n}{k}$ for $k = 0, 1, \dots, n$. $\binom{7}{4}$ can be expressed as a sum of binomial coefficients from row 5. TRUE or FALSE?

ANSWER

- (a) TRUE.

Since $15 > 2 \cdot 7$, by the GPHP there must exist some $b \in B$ such that at least 3 distinct elements of A map to b under f .

- (b) FALSE.

There are exactly two. One that maps 1 to a and 2 to b and one that maps 1 to b and 2 to a .

- (c) TRUE.

The first and the third are logically equivalent. The second is logically equivalent to the negation of the first. We see this with a quick table:

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$p \wedge \neg q$	$\neg p \vee q$
T	T	F	F	T	F	T
T	F	F	T	F	T	F
F	T	T	F	T	F	T
F	F	T	T	T	F	T

- (d) FALSE.

Counterexample: Let $A = \{0, 1\}$, and $f(0) = f(1) = 0$.

Then $|A| = 2$ but $|\text{Ran}(f)| = 1$ since $\text{Ran}(f) = \{0\}$.

- (e) TRUE.

These functions are f defined by $f(a) = a$ and $f(b) = b$, and g defined by $g(a) = b$ and $g(b) = a$.

- (f) FALSE.

Take $A = \{a\}$ and $B = \{b\}$. Then $A \times B = \{(a, b)\}$.

So $|A| = |B| = |A \times B| = 1$.

$|2^{A \times B}| = 2^{|A \times B|} = 2$.

$|2^A \times 2^B| = |2^A| \times |2^B| = 2^{|A|} \times 2^{|B|} = 2 \times 2 = 4$.

$2 \not\asymp 4$.

- (g) TRUE.

By Pascal's Identity

$$\binom{7}{4} = \binom{6}{3} + \binom{6}{4} = \left(\binom{5}{2} + \binom{5}{3} \right) + \left(\binom{5}{3} + \binom{5}{4} \right)$$

15. Assume the following formula for the sum of the squares of the first n positive integers: $1^2 + 2^2 + \dots + (n-1)^2 + n^2 = n(n+1)(2n+1)/6$.

Using *only* the formula given above, derive the following formula for the sum of the squares of first m **odd** positive integers. Show your work.

$$1^2 + 3^2 + 5^2 + \dots + (2m-3)^2 + (2m-1)^2 = \frac{m(4m^2-1)}{3}$$

ANSWER

- (a) Let $D(m) = \sum_{k=1}^m (2k-1)^2$. To this we add $E(m) = \sum_{k=1}^m (2k)^2$ and we obtain $T(m) = \sum_{i=1}^{2m} i^2$. By the formula, $T(m) = 2m(2m+1)(4m+1)/6$. Meanwhile, $E(m) = \sum_{k=1}^m (2k)^2 = 2^2 \sum_{k=1}^m k^2 = 2^2 \cdot m(m+1)(2m+1)/6$ (using again the formula). Hence

$$\begin{aligned} D(m) &= T(m) - E(m) = \frac{2m(2m+1)(4m+1)}{6} - \frac{4m(m+1)(2m+1)}{6} \\ &= \frac{2m(2m+1)(4m+1-2(m+1))}{6} = \frac{2m(2m+1)(2m-1)}{6} = \frac{m(4m^2-1)}{3} \end{aligned}$$

16. Give a boolean expression e with three variables p, q, r such that e has the following properties:

- $e = T$ when $p = q = T$ and $r = F$, AND
- $e = F$ when $p = F$ and $q = r = T$.

Also, construct a truth table for e . Make sure to include all intermediate propositions as a separate column. (Yes, there are many possible answers.)

ANSWER

Here's how to solve such a problem systematically. Note that $p \wedge q \wedge \neg r$ is T exactly when $p = q = T$ and $r = F$. Similarly $\neg p \wedge q \wedge r$ is T exactly when $p = F$ and $q = r = T$, therefore $\neg(\neg p \wedge q \wedge r)$ is F exactly when $p = F$ and $q = r = T$. If there are more such constraints, you find more such conjunctions or negated conjunctions.

Then can take $e = (p \wedge q \wedge \neg r) \vee \neg(\neg p \wedge q \wedge r)$.

(Actually, because we only have two constraints, it suffices to take $e = p \wedge q \wedge \neg r$, for example.)

We skip the truth table in this solution set.

17. In how many different ways can we arrange *all* the letters from the English alphabet (26 characters) in a sequence such that:

- each letter occurs exactly once, AND
- the 5 vowels (a,e,i,o,u) occur in 5 consecutive positions.

ANSWER

We can solve this using the multiplication rule.

Step 1: Arrange the consonants. (21! ways)

Step 2: Choose a position for the vowels. (22 ways)

Step 3: Arrange the vowels. (5! ways)

Thus, our final answer is $\boxed{21! \times 22 \times 5! = 22! \times 5!}$

Alternate Solution:

Since the vowels appear in consecutive positions think of them as a block. There are 21 consonants plus the vowel block, a total of 22 elements, and in Step 1 they are arranged in a sequence. This can

be done in $22!$ ways. However, we must still arrange the vowels within the block, and again we can arrange them in $5!$ ways.

Answer: $22! \times 5!$ (the same)

18. Give a combinatorial proofs (no other kind of proofs will be accepted) for the following identities

(a)

$$\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k} \quad (\text{where } k \leq r \leq n)$$

(b)

$$\sum_{i=k}^n \binom{i}{k} = \binom{n+1}{k+1} \quad (\text{where } k \leq n)$$

ANSWER

(a) Our question is as follows:

Given n pieces of (distinguishable) sushi, how many ways can we choose r pieces to eat if we also want to put wasabi on k of them?

LHS: We first choose the r pieces of sushi to eat $\binom{n}{r}$, and then we choose k of them out of r to put wasabi on $\binom{r}{k}$.

RHS: We first choose the k pieces of sushi to both eat and put wasabi on. This can be done in $\binom{n}{k}$ ways. Then we choose the $r - k$ ones without wasabi to eat from the remaining $n - k$, which can be done in $\binom{n-k}{r-k}$ ways.

(b) Our question is as follows:

Artemis is writing a new song. He wants to include exactly $k+1$ (distinguishable) chords at some point in his song out of a possible $n+1$ chords (labeling them $\{1, 2, \dots, n+1\}$). How many collections of $k+1$ can he make?

LHS: We split it up into cases. The first is that we include chord 1. There are $\binom{n}{k}$ ways to make such a collection. The next is we don't include chord 1 but include chord 2. There are $\binom{n-1}{k}$ such collections. The next is we don't include chords 1 and 2 but include chord 3. This can be done $\binom{n-2}{k}$ ways. We can do this up until chord $n - k$ because then we must include the next k chords. We can express this as the sum $\sum_{i=k}^n \binom{i}{k}$

RHS: We are asked to pick $k+1$ chords from a total of $n+1$ distinguishable chords. There are $\binom{n+1}{k+1}$ ways to do this.

19. Prove that for any integers $a, b \in \mathbb{Z}$ we have $a^2 - 4b \neq 2$.

ANSWER

Assume, toward a contradiction that $a^2 - 4b = 2$. Then $a^2 = 4b + 2 = 2(2b + 1)$ and therefore a^2 is even. We have shown in lecture that in that case a is even.

Let $a = 2k$ for some integer k .

$$(2k)^2 = 4b + 2$$

$$4k^2 = 4b + 2$$

$$2k^2 = 2b + 1$$

$$2k^2 - 2b = 1$$

$$2(k^2 - b) = 1$$

Now, because $k^2 - b$ is an integer, it must be the case that $2(k^2 - b)$ is even. Therefore 1 is even. Contradiction.

20. Consider n (distinguishable) bins labeled B_1, \dots, B_n and r indistinguishable (identical) marbles. We wish to put the r marbles into the n bins in such a way that each bin will contain at least **one** marble and at least **three** of the bins will contain **two or more** marbles. Assume $r \geq n + 3$. In how many different ways can this be done?

ANSWER

First we compute the number of ways to put r marbles into the n bins such that each bin will contain at least one marble. We have done this before with stars and bars and the answer is

$$\binom{n + (r - n) - 1}{r - n} = \binom{r - 1}{r - n}$$

But not all of these ways ensure that at least three of the bins will contain two or more marbles. To correct for this over counting, we will count these bad cases and subtract from the number above.

If it is not the case that at least three of the bins will contain two or more marbles then only exactly zero or exactly one or exactly two bins contain two or more marbles. Note that these are disjoint cases, we do not have to worry about counting a bad case twice.

Case 1: Exactly zero bins with two or more marbles.

This is impossible. We have $r \geq n + 3$ marbles, and if we distribute one to each, we will be left with $r - n \geq 3$ marbles. As no more bins have more marbles, we have not distributed all the marbles.

Case 2: Exactly one bins with two or more marbles.

This means that all $r - n$ additional marbles go into one bin. There are n ways to choose that one bin, so this can be done in n ways.

Case 3: Exactly two bins with two or more marbles.

This means means that all $r - n$ additional marbles go into two bins. We can do this in three steps. In Step 1 we choose two of the bins, in $\binom{n}{2}$ ways. In Step 2 we put one more marble in each of the two bins (recall that each bin already had a marble). This guarantees that both bins will have two or more marbles. This is done in one way and we are left with $r - n - 2$ marbles. In Step 3 we put the remaining $r - n - 2$ marbles in the two (distinguishable) bins we chose, any way we want. So this can be done in $(r - n - 2) + 1 = r - n - 1$ ways (think $r - n - 2$ stars and 1 bar). By multiplication rule the count in the “exactly two” case is:

$$\binom{n}{2} (r - n - 1)$$

Putting this all together, the total number of ways is:

$$\binom{r-1}{r-n} - n - \binom{n}{2} (r-n-1)$$

21. For each statement below, decide whether it is TRUE or FALSE. In each case attach a *very brief* explanation of your answer.

- (a) $\binom{100}{51}$ is strictly bigger than $\binom{100}{49}$. TRUE or FALSE?
- (b) The word QWERTY has 6! anagrams. TRUE or FALSE? (Recall that a word is a valid anagram of itself.)
- (c) The contrapositive of $p \Rightarrow q$ is logically equivalent to $p \wedge \neg q$. TRUE or FALSE?
- (d) For any $2 \leq k < n$, if A has n elements then the number of subsets of A of k elements is $\frac{n!}{(n-k)!}$. TRUE or FALSE?
- (e) If the set A has n elements then there are $n!$ injective functions with domain A and codomain A . TRUE or FALSE?
- (f) There is no set X such that $2^X = \emptyset$. TRUE or FALSE?

ANSWER

- (a) FALSE. They are equal. Choosing 51 people to be on a committee from 100 people can be done in the same number of ways as choosing 49 people to not be on the committee.
Alternatively, we see that $\frac{100!}{51!(100-51)!} = \frac{100!}{49!(100-49)!}$.
- (b) TRUE. There are no multiple occurrences of the same letter so the bag is the same as the set and they have the same number of permutations. Alternatively, you could state

$$\frac{6!}{1!1!1!1!1!1!} = 6!$$

- (c) FALSE. The contrapositive is $\neg q \Rightarrow \neg p$ and it is not logically equivalent to $p \wedge \neg q$ because, for example, for the truth assignment $p = T, q = F$ we have $\neg q \Rightarrow \neg p = F$ but $p \wedge \neg q = T$. *Alternative explanation:* We know from class that $p \Rightarrow q$ is logically equivalent to its contrapositive. However we also know that $p \wedge \neg q$ is logically equivalent to the *negation* of $p \Rightarrow q$. A proposition cannot be logically to its negation.
- (d) FALSE. $(n)_k$ is the number of permutations (not subsets) of k elements out of n . *Alternative explanation:* The number of subsets of size k of a set of n elements is different, namely $\binom{n}{k}$.
- (e) TRUE. The injective functions from A to A correspond one-to-one to the permutations of the elements of A and there are $n!$ such permutations.
(Not required but FYI: The one-to-one correspondence can be described as follows. Let $A = \{a_1, \dots, a_n\}$ and let $f : A \rightarrow A$. When f is injective, the sequence $f(a_1), \dots, f(a_n)$ consists of n distinct elements of A and is therefore a permutation of A . Conversely, given a permutation b_1, \dots, b_n of the elements of A then the function defined by $f(a_i) = b_i$, $i = 1, \dots, n$ must be injective because the b_i 's are distinct.)
- (f) TRUE. For any X , the set 2^X contains at least one element, namely \emptyset (the empty subset of X).

22. The Taney Dragons are going to the Little League World Series! In appreciation, each of the 12 distinct team members (players) can pick 2 hats from a supply of red (Philly Phillies), blue (Boston Red Sox), and green (Ploiesti Frackers) hats. For each color, the supply is unlimited. For each of the three questions below (see also next page), give the answer and an explanation of how you derived it. No proofs required.

In how many different ways can the hat picking be done if:

- (a) There is no ordering among the two hats that each player picks, and both hats can even be of the same color.
- (b) The ordering matters and the two hats have a different color: let's say each player picks a hat to wear in the morning and then a hat (of a different color) to wear in the afternoon.
- (c) What is the count for part (22a) above, if you also know that at least one of the hats that Dragon's pitcher Mo'ne Davis picks is *red*.

ANSWER

- (a) Each player can pick an unordered set of two hats of different color, and there are 3 such options, or two hats of the same color, and there are 3 such options. By the sum (addition) rule each player has 6 options. Because the supplies are unlimited, a hat picking is just like sequence of length 12 of options. There are 6^{12} such sequences (equivalently 6^{12} functions from the set of 12 players to the set of 6 options). Hence 6^{12} ways.
- (b) Each player has 3 options for her/his morning hat and once he/she picks that, is left with 2 options for for her/his afternoon hat. Therefore, by the multiplication rule, each player has $3 \cdot 2 = 6$ options. Again we count the number of sequences of options (or functions from the set of players to the set of options) and we get 6^{12} ways.
- (c) We are back to the 6 options described in the solution to part (22a) above. But now these 6 options are only available to Mo'ne's 11 teammates so they can pick hats in 6^{11} ways. Mo'ne herself must pick a red Philly hat and for her other hat she can pick any of the colors so she has 3 options. Thus she only has 3 options.

By the multiplication rule the total number of ways for this part is $3 \cdot 6^{11}$.

23. Recall from homework that the boolean expression e_2 is a **logical consequence** of the boolean expression e_1 if every truth assignment to the variables that makes e_1 true also makes e_2 true.

Let x, y be arbitrary boolean variables. Prove, using truth tables, that $x \rightarrow y$ is a logical consequence of $\neg x \wedge y$.

ANSWER

We compute the truth table:

x	y	$\neg x$	$\neg x \wedge y$	$x \rightarrow y$
T	T	F	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	F	T

To prove if this is a logical consequence, we need check to that every assignment of x, y that makes $\neg x \wedge y$ true also makes $x \rightarrow y$ true. Indeed, there is only one assignment $x = F, y = T$ that makes the first expression true, and the same assignment makes the second expression true.

24. In the following, just give the examples, you do not have to prove that they work.

- (a) For arbitrary $n \geq 1$, give an example of a set Y and a function $f : [1..n] \rightarrow Y$ such that f is injective but *not* surjective.
- (b) For arbitrary $n \geq 2$, give an example of a set X and a function $g : X \rightarrow [1..n]$ that is *not* injective and moreover $|\text{Ran}(g)| = n - 1$.

ANSWER

- (a) Define $Y = [1..n + 1]$ and for each $1 \leq k \leq n$ define $f(k) = k$.
(*Not required but FYI:* The function is clearly injective since every value 1 to n maps to itself, and it is not surjective because there is no $k \in [1..n]$ such that $f(k) = n + 1$.)
- (b) ANSWER Define $X = [1..n]$ and for each $1 \leq k \leq n$ define

$$g(k) = \begin{cases} k & \text{if } 1 \leq k \leq n - 1 \\ n - 1 & \text{if } k = n \end{cases}$$

(*Not required but FYI:* The function is not injective because two different values map to $n - 1$, specifically $g(n - 1) = n - 1 = g(n)$. Moreover $\text{Ran}(g) = \{1, \dots, n - 1\}$ therefore $|\text{Ran}(g)| = n - 1$.)

25. Punch happily tells Judy that he proved two new theorems and he shares his proofs with her.

- (a) *Punch's First Theorem:* If n is odd then $n^2 - 1$ is a multiple of 4.
Punch's Proof: "We prove the contrapositive instead. Suppose n is even, then n^2 is even, then $n^2 - 1$ is odd so it cannot be a multiple of 4. Done." Upon reading these, Judy remarks that while the theorem is true, the proof is not proving the theorem, but another statement, which is not the contrapositive of the theorem.
 - i. What is the contrapositive of the theorem and what statement is Punch actually proving?
 - ii. Give a correct proof of Punch's First Theorem.
- (b) *Punch's Second Theorem:* For any finite sets A, B , if $|A|$ and $|B|$ are even then $|A \setminus B|$ is even.
Punch's Proof: "The difference of two even numbers is an even number. Done."
 - i. Now, Judy remarks that this other theorem is not even true. Give a counterexample that supports Judy's contention.
 - ii. Judy also remarks that Punch's "proof" relies on a false statement about set cardinalities. (Since the theorem is not true, there had to be a bug in the proof!) What is that false statement?

ANSWER

- (a) i. The contrapositive is:
 “If $n^2 - 1$ is not a multiple of 4 then n is even (or you can say “is not odd”).”
 What Punch actually proved is:
 “If n is even then $n^2 - 1$ is not a multiple of 4.”
 (*Not required but FYI:* This is the converse of the contrapositive and in general it’s not logically equivalent to the theorem.)
- ii. Direct proof: if n is odd then $n = 2k + 1$ for some integer k . Thus,

$$\begin{aligned} n^2 - 1 &= (2k + 1)^2 - 1 \\ &= 4k^2 + 4k + 1 - 1 \\ &= 4(k^2 + k) \end{aligned}$$

which is clearly divisible by 4.

(*Not required but FYI:* Suppose that we attempt to prove the contrapositive. This leads to the following strange but interesting proof:

Suppose $n^2 - 1$ is not a multiple of 4 and W.T.S. n is even.

Case 1: $n^2 - 1$ is odd. In this case n^2 is even and therefore n is even.

Case 2: $n^2 - 1$ is even but still not a multiple of 4. In this case n^2 is odd so $n = 2k + 1$ for some integer k and then $n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4(k^2 + k)$ hence a multiple of 4. But our statement says that $n^2 - 1$ couldn’t have been a multiple of 4, and thus this case is impossible. Therefore since we showed in this case the premise is *false*, and false implies anything, the statement still holds true!

n is even in both cases. Done.

The strange thing is that in the process of doing a proof by contrapositive you essentially discover the direct proof too, see inside Case 2.)

- (b) i. The counterexample is $A = \{1, 2\}$, $B = \{2, 3\}$ therefore $A \setminus B = \{1\}$.
 ii. Punch assumes that $|A \setminus B| = |A| - |B|$. Note that it is actually $|A \setminus B| = |A| - |A \cap B|$.

26. How many sequences of bits (0’s, 1’s) are there that each sequence has all of the following properties:

- Their length is either 3 or 5 or 7.
- Their middle bit is a 1.
- The number of 0’s they have equals the number of 1’s they have minus one.

ANSWER

We partition the possible sequences into three cases by their lengths.

Case 1: Sequences with 3 bits Since the middle bit must be a 1 and the bit must contain a total of two 1’s and one 0, there are only 2 ways to arrange the bits: 011 and 110.

Case 2: Sequences with 5 bits Since the middle bit must be a 1, we only need to consider the possible arrangements of the remaining 1’s and 0’s. To do this, we choose the 2 of the 4 non-middle positions to place the 1’s and then place the 0’s in the remaining spots. This can be done in $\binom{4}{2}$ ways.

Case 3: Sequences with 7 bits Again, the middle bit must be a 1, so we only need to consider the possible arrangements of the remaining 1’s and 0’s. To do this, we choose 3 of the 6 non-middle

positions to place the 1's and then place the 0's in the remaining spots. This can be done in $\binom{6}{3}$ ways.

Since the cases are disjoint, we can apply the Sum Rule to get

$$2 + \binom{4}{2} + \binom{6}{3} = 2 + \frac{4!}{2!2!} + \frac{6!}{3!3!} = 28$$

27. How many sequences of bits (0's, 1's) of length 100 can we make such that:

- the number of 0's in the sequence is equal to the number of 1's in the sequence; and
- the sequence begins with a 1 and ends with a 1.

ANSWER

Since there are an equal number of 0's and 1's, we know that there are 50 0's and 50 1's. The sequence begins and ends with 1's, so we only need to count the number of ways to arrange the middle 98 bits, which are 50 0's and 48 1's. This can be done by choosing 50 of 98 positions to place the 0's and then filling the remaining spots with 1's. Alternatively, we can choose 48 of 98 positions to place the 1's and fill the remaining spots with 0's, which will give us the same answer. Thus, the total number of distinct bit sequences is

$$\binom{98}{50} = \binom{98}{48}$$

28. A cookie shop has k different flavors of cookies. Arnav wishes to purchase cookies for his recitation, and he has enough money to buy up to 250 cookies. Assuming that he does not have to spend all of the money that he has, in how many ways can he purchase cookies? (For full credit, your solution should be in closed form, so no open summations!)

ANSWER

The problem essentially asks for the number of ways to distribute 250 cookies into distinguishable categories.

Each of these cookies will be either one of the k flavors or "not purchased," so we can imagine $k + 1$ bins to place the cookies in. Since any two unpurchased cookies or cookies of the same flavor are indistinguishable, we can use the stars and bars method, where the cookies are the stars and the $(k + 1) - 1 = k$ dividers between the $k + 1$ categories are the bars.

For example, if Arnav purchases 1 cookie of each of n varieties and then $250 - n$ potential cookies are left unpurchased (because he's feeling stingy), then we would represent this with one star followed by one bar k times, and then $250 - k$ stars representing unpurchased cookies.

Thus, using the formula for the stars and bars method, the answer is $\binom{250+k}{k}$.

Alternative (incomplete) solution: You could choose to sum over all possibilities. For each i between 0 and 250 equal to number of cookies purchased, we want to count the number of ways we can split i up into the k varieties. This can still be done with stars and bars, but we now have k bins, one for each variety. Summing, we have

$$\sum_{i=0}^{250} \binom{i+k-1}{k-1} = \sum_{j=k-1}^{250+k-1} \binom{j}{k-1}$$

It's OK to leave your answer like this.

29. Give a combinatorial proof (no other kind of proofs will be accepted) for the following identity

$$\frac{n!}{(n-r)!} = \frac{(n-1)!}{(n-r-1)!} + r \cdot \frac{(n-1)!}{(n-r)!} \quad (\text{where } 1 \leq r \leq n)$$

ANSWER

We use this counting question: There are n TAs but only r (distinguishable) chairs at grading. How many ways can we place the n TAs into the r chairs?

The LHS is a fairly direct way to solve this problem - for the first chair, there are n TAs we can place in it. For the second chair, there are $n-1$ TAs we can place in it. This will evaluate to the falling factorial (i.e. the permutation) $(n)_r$ on the LHS.

Now we try to solve this problem in such a way that yields the expression on the RHS. Since we add two expressions, it seems logical that we might have two cases. In case 1, Hanzhao, as the TA who always arrives first, decides to be self-sacrificing and chooses to stand so the other TAs can sit. In case 2, he decides to be selfish and takes a seat.

Case 1: Since we know Hanzhao is not going to sit, there are only $n-1$ TAs who are going to sit, and there are r chairs. We apply the definition of a permutation to get $(n-1)_r$.

Case 2:

Step 1: Hanzhao chooses a seat for himself. (r ways)

Step 2: We place $r-1$ of the remaining $n-1$ TAs in the remaining $r-1$ chairs. Applying again the definition of a permutation, there are $(n-1)_{r-1}$ ways to do this.

We apply the multiplication rule to get $r(n-1)_{r-1}$ in this case.

We add the two cases to get $(n-1)_r + r(n-1)_{r-1}$, which is exactly the expression on the RHS. Thus, as we have solved the same problem in two valid ways to get the expressions on the LHS and RHS respectively, we have a combinatorial proof.

30. We distribute indistinguishable ungraded papers to $n \geq 3$ distinguishable TAs in the following way. First we select two "lucky" TAs to have the designation of Head TA. Next we distribute the papers in such a way that each TA gets at least 1 paper to grade, and both of the Head TAs get at least 2 papers. What is the minimum number of papers needed to make this work? Now, assume that we have r papers, where r is large enough to make this kind of distribution work, in how many different ways can the papers be distributed?

ANSWER

We need at least $n-2$ papers for each of the $n-2$ "unlucky", non-Head TAs to have 1 paper. Therefore, the minimum, and then 4 for the 2 Head TAs to each have 2, so the number of papers we need is $n-2+4 = n+2$.

Before we proceed with counting the number of ways in which the papers can be distributed we observe that the problem can be interpreted in two ways:

- (a) (A) The designation of head TA for two of the TAs is part of the configurations that need to be counted.
- (b) (B) The verbiage about head TA is just a way to ensure that *at least 2 of the TAs get at least 2 papers each*.

If we choose the interpretation (B) then the problem can be rewritten as follows. In how many ways can r papers be distributed to n TAs such that each TA gets at least 1 paper to grade and, moreover, at least 2 of the TAs get at least 2 papers each to grade. This formulation is a slightly simpler version of Problem 20 above (examine its solution).

Here we count under interpretation (A), perhaps the more reasonable interpretation of what this problem asks.

For this interpretation a distribution can be constructed as follows (assuming we have $r \geq n + 2$ papers):

Step 1: Choose two (out of n) TAs to be Head TAs.

Step 2: Give 2 papers each to Head TAs and 1 paper each to the remaining $n - 2$ TAs.

Step 3: Distribute the remaining $r - n - 2$ papers to all the n TAs.

This process works because, after we designate the head TAs and give everyone their minimum number of papers, the number of ways to distribute the rest of the papers is the same as the number of total distributions.

Step 1 can be done in $\binom{n}{2}$ ways, as we are simply choosing 2 TAs from n without order. Step 2 can be done in 1 way. To do Step 3, we apply stars and bars. There are $r - n - 2$ papers, which, as the items being distributed, represent the stars. Then there are n TAs - n “categories” into which we sort the items - so we have $n - 1$ bars. Thus we need to choose $n - 1$ bars from the $r - n - 2 + (n - 1) = r - 3$ stars and bars. So we have that Step 3 can be done in $\binom{r-3}{n-1}$ ways. By the multiplication rule the answer is

$$\binom{n}{2} \binom{r-3}{n-1}$$

31. Give a combinatorial proof for the following identity:

$$\sum_{k=0}^r \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

ANSWER

We pose the following counting question.

Sriram bought m (distinguishable) math textbooks and n (distinguishable) CS textbooks as summer reading, but Aryan forgot to buy books to read, so Sriram agrees to give him r of his textbooks. How many different combinations of r math and CS textbooks can Sriram give Aryan?

We need to solve this problem in 2 ways which yield the RHS and LHS respectively.

First, we look at the RHS. Since we are choosing r books from $m + n$ total distinguishable books, we simply apply the definition of a combination to have $\binom{m+n}{r}$.

We now look at the LHS. Since we see a summation, we consider breaking this expression into cases. Of the r textbooks, Sriram could choose 0 to be math textbooks, 1 to be a math textbook, or 2, 3, etc. (the remainder being made up of CS textbooks). We let this value be k . That is, k is the number of math textbooks and $r - k$ is the number of CS textbooks Sriram chooses. So we solve this problem in 2 steps:

Step 1: Sriram chooses k of the m math textbooks.

Step 2: Sriram chooses $r - k$ of the n CS textbooks.

We again apply the definition of the combination to find that there are $\binom{m}{k}$ ways to perform Step 1 and $\binom{n}{r-k}$ ways to perform Step 2. With the multiplication rule, we have $\binom{m}{k}\binom{n}{r-k}$. Then we sum over all possible values of k , which range over all integer values from 0 to r (Sriram could pick any value from 0 to r of the r textbooks to be math books):

$$\sum_{k=0}^r \binom{m}{k} \binom{n}{r-k}$$

This is exactly the expression on the RHS. Thus, as we have solved the same problem in two valid ways to get the expressions on the LHS and RHS respectively, we have a combinatorial proof.

32. For each statement below, decide whether it is true or false. In each case attach a *very brief* explanation of your answer.

- (a) Consider the proposition P where:

$$P: \exists N > 0, \forall n \geq N, 100n \leq n^2/100$$

True or false: $\neg P$ is the following proposition:

$$\forall N > 0, \forall n \geq N, 100n > n^2/100$$

- (b) Let X be a finite nonempty set. The number of functions with domain X and codomain $\{0, 1\}$ is $2^{|X|}$, true or false?

ANSWER

- (a) FALSE. If we negate the expression and move the negation as far right as possible, we derive:

$$\neg[\exists N > 0, \forall n \geq N, 100n \leq n^2/100] \tag{1}$$

$$\forall N > 0, \neg[\forall n \geq N, 100n \leq n^2/100] \tag{2}$$

$$\forall N > 0, \exists n \geq N, \neg[100n \leq n^2/100] \tag{3}$$

$$\forall N > 0, \exists n \geq N, 100n > n^2/100 \tag{4}$$

which is a different statement.

- (b) TRUE. We have seen in class that, for a domain of size a and a codomain of size b , there are b^a functions. We also accept a solution by counting. For every element of X , we need to choose exactly one element of $\{0, 1\}$ to map it to. Thus, there are $|X|$ decisions each with two choices, and so the number of possibilities is $2 \cdot 2 \cdots 2 = 2^{|X|}$.

33. For sets A, B, C , and D , suppose that $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$ then $x \in B$.

ANSWER

We will prove the claim by proving the contrapositive. Suppose that $A \setminus B \subseteq C \cap D$ and $x \in A$ but $x \notin B$. Since $x \notin B$ and $x \in A$, it must be that $x \in A \setminus B$ and hence $x \in C \cap D$. Thus $x \in D$.

34. You are choosing a sequence of five characters for a license plate. Your choices for characters are any letter in PERM and any digit in 1223. Your five-character sequence can contain any of these characters at most the number of times they appear in either PERM or 1223. If there are no other restrictions, how many such sequences are possible?

ANSWER

We want to count the number of distinct ways to order 5 characters from “PERM” and “1223”.

“2” is the only character that appears more than once in “PERM” and “1223”. Thus, we case upon the number of “2”s.

Case 1: Zero “2”s

In this case, all 5 letters come from $\{P, E, R, M\}$ or $\{1, 3\}$. Since there are 6 options, there are $\binom{6}{5}$ ways to select the letters for the string. There are $5!$ ways to order the letters once they are selected. By Multiplication Rule, Case 1 has $\binom{6}{5} \cdot 5!$ solutions.

Case 2: Exactly one “2”

In this case, 4 of the letters come from $\{P, E, R, M\}$ or $\{1, 3\}$, and 1 of the letters is fixed to be “2”. Since there are 6 options for the unknown characters, there are $\binom{6}{4}$ ways to select the letters for the string. There are $5!$ ways to order the letters once they are selected since there are still no repetitions. By Multiplication Rule, Case 2 has $\binom{6}{4} \cdot 5!$ solutions.

Case 3: Two “2”s

In this case, 3 of the 5 letters come from $\{P, E, R, M\}$ or $\{1, 3\}$, and 2 of the letters are fixed to be “2”. Since there are 6 options for the unknown characters, there are $\binom{6}{3}$ ways to select the letters for the string. A simple linear ordering of $5!$ counts each distinct solution exactly twice since the positions of the “2”s could be reversed, so there are $\frac{5!}{2}$ ways to order the 5 characters. By Multiplication Rule, Case 3 has $\binom{6}{3} \cdot \frac{5!}{2}$ solutions.

By Sum Rule, the number of solutions is:

$$\binom{6}{5} \cdot 5! + \binom{6}{4} \cdot 5! + \binom{6}{3} \cdot \frac{5!}{2} = \boxed{3720}$$

35. Prove that if for some integer a , $a \geq 3$, then $a^2 > 2a + 1$.

ANSWER

Let $a \in \mathbb{Z}$ s.t. $a \geq 3$.

We observe that $3a > 2a + 1$ since $a > 1$. So if we can show that $a^2 \geq 3a$, then we’re done.

Note that $a^2 = a \times a$, and $3a = 3 \times a$. Since we know that $a \geq 3$, we can conclude $a \times a \geq 3 \times a$. Hence our proof is complete.

36. Give a combinatorial proof of the following identity for $N, a, b \in \mathbb{N}$:

$$\binom{N}{a} \binom{N}{b} = \sum_{i=0}^{\min(a,b)} \binom{N}{i} \binom{N-i}{a-i} \binom{N-a}{b-i}$$

ANSWER

Consider the situation: there are N people, and Oprah gives a of them a boat, and b of them a house (some may receive both). We pose the question: how many ways can Oprah do this? We can choose the people to give a boat to in $\binom{N}{a}$ ways, and we can choose the people to give a house to in $\binom{N}{b}$ ways. Thus there are $\binom{N}{a} \times \binom{N}{b}$ total ways to distribute the boats and houses, which is exactly the left side of the equation.

The RHS starts with iterating on the number of people who are going to get both boats and houses (i iterates from 0 to $\min(a, b)$). We then choose the i people who receive both boats and houses. Next out of the remaining $N - i$ we choose $a - i$ who receive only boats. Note, that so far we have chosen $i + a - i = a$ people. Finally, out of the remaining $N - a$ people we choose $b - i$ who receive only houses.

37. There are 100 guests at a fundraising party, excluding the host. As part of a “fun” party game, the host pairs up the dinner guests into 50 pairs that the host calls “fundraising pairs”. In the game, the individual with the smaller net worth in each pair declares the amount of money that they wish to donate, which the individual with the higher net worth must match in double. For example, if the individual with the smaller net worth in one pair donates \$100 dollars, the individual with the larger net worth must donate \$200 dollars.

The host says that the aim of the game is to raise a total of 9 million dollars between all of the individuals. Given this set up, how many ways can the game unfold? Assume that the net worth of each of the individuals is unique, that all donations are in whole dollars, and that all of them can donate up to 9 million dollars each.

ANSWER

First, let us calculate how many ways there are to pair up the individuals. We can do this by arranging all 100 individuals in line and interpreting the $2k + 1$ th and $2k + 2$ th individuals as paired up, where $0 \leq k \leq 49$, and then removing the ordering from these pairs. There are $100!$ ways of arranging the individuals in a line, which we divide by $50!$ to remove the ordering of the pairs. We also need to divide by 2^{50} , since there is no ordering within the pairs either.

Second, we seek the number of possible ways that the individuals can donate money. Since the individual with the higher net worth must donate twice what the individual with the lower net worth donates, each pairs donation must be a multiple of 3. We can therefore find the number of arrangements using the stars and bars method seen in lecture. Each star is a donation of \$3 dollars, and there are 3 million stars. There are 49 bars since there are 50 pairs. Therefore, the number of arrangements is $\binom{3000000+49}{49}$.

Finally, we seek to the number of ways that the two individuals within the pair can donate the money. There is simply 1 way, since the ratio of donation between the two is fixed.

Therefore, the total number of ways that the game can unfold is $\frac{100!}{50! \cdot 2^{50}} \times \binom{3000049}{49}$.

38. Let x, y be arbitrary boolean variables. Give a truth table for the expression $x \Rightarrow (\neg y \Rightarrow F)$. Then, find, *two* other, distinct, boolean expressions that are logically equivalent to the previous expression.

ANSWER

The two boolean expressions are $x \rightarrow y$ and $\neg x \vee y$. This is justified by the truth table:

x	y	$x \Rightarrow (\neg y \Rightarrow F)$	$x \Rightarrow y$	$\neg x \vee y$
T	T	T	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Additional explanations, not required for your solution: The statement $\neg y \rightarrow F$ is logically equivalent to y , so $x \rightarrow (\neg y \rightarrow F)$ is logically equivalent to $x \rightarrow y$. The only time $\neg x \vee y$ will be false is when $x = T$ and $y = F$, which is consistent with the truth values for the other two expressions.

Thus $x \rightarrow (\neg y \rightarrow F)$ is logically equivalent to $x \rightarrow y$ and $\neg x \vee y$.

(These are not the only possibilities. In fact there are infinitely many solutions!)

39. Consider sequences of bits such that m of the bits are 0, where $m \geq 2$, n of the bits are 1, where $n \geq 1$, and start with a 1 and end with two 0's. How many such sequences are there?

ANSWER

Since the sequence of bits must start with a one and end with 2 zeroes, we can place those three elements in their respective positions. We now need to deal with ordering the remaining $m - 2$ zeroes and $n - 1$ ones. There are $m - 2 + n - 1 = m + n - 3$ "spots" in which we need to organize the remaining zeroes and ones. We note that we can choose $m - 2$ of them to be zeroes and leave the other $n - 1$ to be ones, which yields the answer:

$$\binom{m+n-3}{m-2}$$

40. For each statement below, decide whether it is true or false. In each case attach a *very brief* explanation of your answer.
- (a) Let A, B be finite sets with $|A| = 2$ and $|B| = 3$. There are more functions $A \rightarrow B$ than functions $B \rightarrow A$, true or false?
 - (b) Let X, Y be nonempty finite sets such that $|Y| = 1$ and such that there exists an injection $f : X \rightarrow Y$. Then $|X| = 1$, true or false?
 - (c) There are as many sequences of bits of length 100 that start with a 0 as sequences of bits of length 100 that end with a 1, true or false?
 - (d) Let S be the set of the first 100 natural numbers: $S = 0, 1, \dots, 99$. There are as many subsets of S of size 40 that contain the number 40 as subsets of S of size 40 that do not contain the number 40, true or false?

ANSWER

(a) TRUE.

The number of functions $f : X \rightarrow Y$ is $|Y|^{|X|}$, since for each element $x \in X$ we can have $|Y|$ possible values for $f(x)$. Each such assignment results in a function, since we map every element of X to exactly one element of Y . There are $|B|^{|A|} = 3^2 = 9$ functions $A \rightarrow B$ and $|A|^{|B|} = 2^3 = 8$ functions $B \rightarrow A$, there are more functions $A \rightarrow B$.

(b) TRUE.

X is nonempty so we must have $|X| \geq 1$. Suppose towards contradiction that $|X| \geq 2$. Then there exist $x_1 \neq x_2 \in X$. Since Y has exactly one element, $y \in Y$ we must have $f(x_1) = y = f(x_2)$. This contradicts the fact that f is injective, as we have two different elements of X with the same image in Y . Thus, the only possibility left is $|X| = 1$.

(c) TRUE.

If we fix any specific bit in a 100-bit sequence, we can ignore it and treat the others as a 99-bit sequence, which has 2^{99} possible configurations. We find that this is the case because each of these 99 positions has two choices, independent of all other positions, and applying the multiplication rule. Here, we can fix the first to be a 0 and consider only the following 99, or fix the last bit to be a 1 and consider only the first 99, both of which lead to 2^{99} possible sequences.

(d) FALSE.

There are $\binom{99}{39}$ sets that do contain the number 40 and $\binom{99}{40}$ that don't contain the number 40. By simplifying these values, we see that

$$\frac{99!}{60!39!} = \frac{40}{60} \times \frac{99!}{59!40!} < \frac{99!}{59!40!}$$

So the two quantities are not the same; there are more such subsets of S that do not contain the number 40 than do.

41. In the remote town of Plectisitor a local ordinance prevents inhabitants from having first names, they can only have last names. These last names must start with an upper case letter followed by one to three lower case letters followed by a number between 1 and 22 (to accomodate families, you see). The lower case letters must be distinct among themselves but they can be the same letter as the upper case at the beginning of the names. Moreover, no two inhabitants can have the same name. The alphabet used in Plectisitor has 31 letters, with lower and upper case for each of them.

What is the maximum population of Plectisitor? (Just give it as an arithmetical expression since you cannot use a calculator during the exam.)

ANSWER

All we know about Plectisitor is how many different names are allowed in this weird town. So we will assume that the maximum population of Plectisitor is the same as the total number of distinct names which we can create given the constraints mentioned in the question.

Let A be the set containing all the names which have only one lower case letter, B be the set containing all names which have two lower case letters and C be the set whose members are name that have three lower case letters. It is important to note that three sets are pairwise disjoint. Therefore the total number of distinct possible names is just the sum of the cardinalities of the three sets.

We compute the number of ways of forming names that belong to set A :

Choose the first upper case letter. There are 31 ways of doing this (As there are 31 letters in Plictisitor's alphabet). Choose the lower case letter. There are again 31 ways of doing this. Finally, choose the number. There are 22 ways of doing this. Therefore the $|A|$ is $= 31 \times 31 \times 22$.

Now we compute the number of ways of forming names that belong to B :

There are 31 ways of choosing the first uppercase letter. There are 31×30 ways of choosing the two lower case letters (As they have to be distinct). Finally, there are 22 ways of choosing the number. Therefore, $|B| = 31 \times 31 \times 30 \times 22$.

Finally, we compute the number of ways of forming names that belong to C :

There are, once again, 31 ways of choosing the first uppercase letter. There $31 \times 30 \times 29$ of choosing the three lower case letters and finally 22 ways of choosing the number. therefore, the size of C is $31 \times 31 \times 30 \times 29 \times 22$.

Overall, the maximum possible population of Plictisitor is equal to the number of possible unique names:

$$|A| + |B| + |C| = 31 \times 31 \times 22 + 31 \times 31 \times 30 \times 22 + 31 \times 31 \times 30 \times 29 \times 22$$

You can leave it like this. Or, if you suffer from MOCD like me you can write $31 \times 31 \times 22(1 + 30 + 30 \times 29) = 31 \times 31 \times 22(1 + 30 \times 30) = 31^2 \times 22 \times 901$. (Even my MOCD has limits!:) Anyway, that's at least 16 million so Plictisitor has, unfortunately, lots of room to grow.