Module 5.5: Friends and Strangers

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

Sums of consecutive subsequences I

Problem. Given any sequence of n integers (not necessarily distinct), show that we can always pick some of them (a **subsequence**) which appear in **consecutive** positions in the sequence and whose **sum** is a **multiple** of n.

Answer. Let's first look at some examples.

Sequence: $3\ 7\ 5\ 5$
Consecutive subsequences whose sum is divisible by 4:
$7\ 5 \qquad 3\ 7\ 5\ 5$

Sequence: $7\ (-4)\ 7$
Consecutive subsequences whose sum is divisible by 3:
$7\ (-4) \qquad (-4)\ 7$

Sums of consecutive subsequences II

Answer (continued). Let $x_1 \; x_2 \; \cdots \; x_n$ be the sequence of n integers.
Consider the following n sums.

$$s_1 = x_1$$

$$s_2 = x_1 + x_2$$

$$\cdots$$

$$s_n = x_1 + x_2 + \cdots + x_n$$

If any of $s_1, s_2, \ldots, s_n$ is divisible by n, then we are done.

On the next slide we consider the other case, namely when each of $s_1, s_2, \ldots, s_n$ is not divisible by n, that is, $n \nmid s_1, \ldots, n \nmid s_n$.

Sums of consecutive subsequences III

Answer (continued). In the case where $n \nmid s_1, \ldots, n \nmid s_n$.

Let r_i be the remainder of the integer division of s_i by n for $i = 1, \ldots, n$. Each $r_i \neq 0$.

Note that there are $n - 1$ different possible non-zero remainders: $1, 2, \ldots, n - 1$.

We apply PHP with $r_1, \ldots, r_n$ as pigeons and $1, 2, \ldots n - 1$ as pigeonholes. Hence there exist distinct p and q such that $r_p = r_q$.

By integer division, for some integers k and ℓ we have

$$s_p = kn + r_p \quad \text{and} \quad s_q = \ell n + r_q$$

W.l.o.g. assume $p < q$. Subtracting both sides, since $r_p = r_q$ we get

$$s_q - s_p \;=\; x_{p+1} + \cdots + x_q \;=\; (\ell - k)n$$

We conclude that $x_{p+1} + \cdots + x_q$ is divisible by n.

The theorem of friends and strangers I

Theorem. In any group of 6 Facebook (FB) users, there are 3 that are pairwise FB **friends** or there are 3 that are pairwise FB **strangers** (that is, **not** FB friends).

(This theorem is a particular case of a famous result of Ramsey that created an entirely new branch of Combinatorics called Ramsey Theory. We will mention this again later.)

Proof. Let A, B, C, D, E, F be a group of six FB users.

Each of B, C, D, E, F is either friends with A or not.

We apply PHP placing B, C, D, E, F into the two categories "friend of A" / "not friend of A".

Since $5 > 2 \cdot 2$ at least 3 of B, C, D, E, F belong to the same category. W.l.o.g., let these 3 be B, C, D. Now we have two cases.

The theorem of friends and strangers II

Proof (continued).

Case 1: B, C, D are in the "friend of A" category. We continue with two subcases.

> **Subcase 1.1:** B, C, D are pairwise strangers. Done.
>
> **Subcase 1.2:** B, C, D are **not** pairwise strangers. Then at least two of them, say w.l.o.g. B and C, are friends. Therefore A, B, C are pairwise friends. Done again.

The theorem of friends and strangers III

Proof (continued).

Case 2: B, C, D are in the "not friend of A" category. Again we have two subcases.

Subcase 2.1: B, C, D are pairwise friends. Done, yet again.

Subcase 2.2: B, C, D are **not** pairwise friends. Then at least two of them, say w.l.o.g. B and C, are strangers. Therefore A, B, C are pairwise strangers. Finally, done!

Does it feel like we did some redundant work? Indeed Case 1 and Case 2 use exactly the same reasoning, except that "friend" and "stranger" are swapped! Mathematicians would skip Case 2 entirely, saying that it proceeds **analogously** or **similarly**. We shall do the same in the future!