Module 4.2: Combinatorial Proofs

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

A combinatorial identity

How many times does $a^{n-i}b^i$ occur in $(a+b)^n$? We counted $\binom{n}{i}$ factors that contributed a b. What if we counted for a ? That's $\binom{n}{n-i}$. Luckily:

Problem. Prove that

$$\binom{n}{r} = \binom{n}{n-r}$$

Answer. Consider a set A such that $|A| = n$. The left-hand side (LHS) counts subsets of size r and the right-hand side (RHS) of size $n-r$.

But the RHS also counts the number of subsets of size r by counting the ways in which elements are **not** put in the subset.

If $|S| = n - r$ then $|A \setminus S| = r$. We have a "one-to-one correspondence" between subsets of size r and of size $n-r$. This proves the identity.

Pascal's identity

Problem (Pascal's Identity). Let n and k be positive integers with $n \geq k \geq 1$. Prove

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Answer. We count the number of subsets of size k of a set with n elements in two ways. The usual way gives the LHS of Pascal's Identity.

Let $A = \{ x_1, x_2, \ldots, x_n \}$ be the set. Notice that k-element subsets of A can be classified into those that contain x_n and those that don't.

For the first kind the other $k-1$ elements come from $A \setminus \{x_n\}$, $\binom{n-1}{k-1}$ ways.

For the second kind all the k elements come from $A \setminus \{x_n\}$, in $\binom{n-1}{k}$ ways.

By the addition rule, we get the RHS of Pascal's Identity.

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ACTIVITY : Pascal's Triangle

Recall the Pascal's Triangle from a previous segment.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \cdots & & & & & &
\end{array}
$$

ACTIVITY : Pascal's Triangle

We think of the rows as ordered vertically starting from row 0 at the top. We think about every row as a sequence of numbers. For instance, row 6 is $1, 6, 15, 20, 15, 6, 1$. We number the positions in each row as starting from 0.

Therefore, in row n position k, we have the value of $\binom{n}{k}$.

For example, in position 4 of row 6 we have $\binom{6}{4} = 15$.

Question: What number is in position 5 of the next row (row 7)?

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

ACTIVITY : Pascal's Triangle (continued)

Answer: 21.

Why? In position 5 of row 7 we have $\binom{7}{5}$. We can use Pascal's identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

to derive the answer. Taking $n = 7$ and $k = 5$ we obtain:

$$\binom{7}{5} = \binom{6}{4} + \binom{6}{5}$$

From the sixth row in Pascal's triangle, $\binom{6}{4} = 15$ and $\binom{6}{5} = 6$. Therefore $\binom{7}{5} = 15 + 6 = 21$.

Combinatorial proofs of identities

We just proved two combinatorial identities:

$$\binom{n}{r} = \binom{n}{n-r} \qquad \text{and} \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

In both proofs the method was the same:

We posed a **counting question**.

We answered the question in **one way**, with the answer giving the LHS of the identity.

We answered the question in **another way**, with the answer giving the RHS of the identity.

Another combinatorial proof

Problem. Prove

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Answer. We pose the following counting question: how many subsets are there of a set A with n elements?

From earlier lectures, we know that the answer is 2^n. This gives us the RHS.

Another way : The powerset 2^A can be partitioned into $S_0, S_1, \ldots, S_n$, where S_i, $0 \leq i \leq n$, is the set of all subsets of A that have cardinality i.

These are pairwise disjoint so by the addition rule the answer is $\sum_{i=0}^{n} |S_i|$.

But $|S_i| = \binom{n}{i}$. This gives us the LHS.

ACTIVITY : Binomial Theorem

Recall that the Binomial Theorem (covered in a previous segment) states

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

We can also derive the identity in the previous slide from the Binomial Theorem by setting $a = b = 1$. Here's how:

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} &= \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i \\ &= (1+1)^n \\ &= 2^n \end{aligned}$$

Activity : Binomial Theorem (Continued)

Now do the following:

Question: What is your idea for proving

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0$$

In the video, there is a box here for learners to put in an answer. As you read these notes, try it yourself using pen and paper!

Activity : Binomial Theorem (Continued)

Answer:

One way to solve this problem is by substituting $a = 1$ and $b = -1$ in the Binomial Theorem, yielding

$$0^n = 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

ACTIVITY : Binomial Theorem (Continued)

However, a combinatorial proof will give us more insight into what the expression means. Moving some terms to the RHS, we want to prove that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$

Consider a set $X = \{x_1, x_2, x_3, \ldots, x_n\}$. We want to show that the total number of subsets of X that have **even size** equals the total number of subsets of X that have **odd size**. We will now show that both these quantities equal 2^{n-1} from which the claim follows.

ACTIVITY : Binomial Theorem (Continued)

An even-sized subset of X can be constructed as follows.

Step 1 : Decide whether x_1 belongs to the subset or not.
Step 2 : Decide whether x_2 belongs to the subset or not.
...
Step n : Decide whether x_n belongs to the subset or not.

ACTIVITY : Binomial Theorem (Continued)

In the first $n - 1$ steps one can make either one of the **two choices**, in or out. But in step n only **one choice** is possible!

This is because if we have chosen an even number of elements from $X \setminus \{x_n\}$ to put in the subset then we must leave out x_n.

Otherwise, we must include x_n in the subset.

Hence using the multiplication rule the total number of even-sized subsets of X equals 2^{n-1}.

ACTIVITY : Binomial Theorem (Continued)

Another way of thinking about this is to count in two steps.

In the first step choose a subset of $\{x_1, \ldots, x_{n-1}\}$.
In the second step decide whether to add x_n to the subset chosen in the first step, making sure the result has even size (don't forget that 0 is even!).

To compute the number of odd-sized subsets we could proceed similarly.

Or, we could count complementarily: since we know that the total number of subsets of X is 2^n, the total number of odd-sized subsets of X is

$$2^n - 2^{n-1} = 2^{n-1}(2-1) = 2^{n-1}$$