Questions

This assignment is due in about one week from when the assignment opens. The exact deadline and full instructions for submission are provided in Coursera. To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - **no collaboration is allowed**.

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1. **[10 pts]** We can use the binomial theorem to expand $(2x+7)^{12}$ and $(2+7x)^{12}$ as sums of terms (monomials). Indicate all the terms (monomials) that are the same in both sums and explain why you know this, without fully expanding out all of the terms.

2. **[10 pts]** For each of the following functions, prove that the function is neither injective nor surjective. Then, show how you could restrict the domain and codomain — without changing the mapping rule — to make the function both injective and surjective. (Restricting the domain (codomain) means replacing it with a subset that must be clearly defined using our notation, including, if needed, set-builder notation.) Your restricted domain and codomain should be *as large as you can think of* (you get more points for larger domains and codomains).

   For example, the function $f : [-3..3] \to [-9..9]$ given by $f(x) = x^2$ is neither injective (since, e.g., $f(-3) = f(3)$) nor surjective (since, e.g., 2 is not a square), but we can restrict the domain and codomain to define the function $f_1 : [0..3] \to \{0, 1, 4, 9\}$ with the same mapping rule, that is, $f_1(x) = x^2$, which is both injective and surjective. The codomain is as large as possible because adding every other number in $[-9..9]$ that is not a square would break surjectivity and the domain is as large as possible because adding any negative number breaks injectivity.

   (a) $g : \{\mathtt{v}, \mathtt{x}, \mathtt{y}, \mathtt{z}\} \to \{\mathtt{a}, \mathtt{e}, \mathtt{i}, \mathtt{o}, \mathtt{u}\}$ given by:

x	$g(x)$
v	i
x	o
y	e
z	i

(b) $h : 2^{[1..n]} \to [0..2n]$ given by $h(S) = |S|$ where $n \geq 2$.

3. [10 pts] Answer the following questions about a scheduling system for assigning TAs to office hours. Within the system, every single TA and every single office hour time is assigned an integer ID. These IDs start with 1 and increment by 1. That is, if there is a TA with ID equal to 6, there must be TAs with IDs equal to 1, 2, 3, 4 and 5. The same restriction applies to the office hour IDs. Note that any office hour not assigned a TA will be covered by Professor Tannen.

(a) Let p, q, r, s be integers with $p \leq q \leq r \leq s$. Consider the TAs with IDs ranging from p to r inclusive, and consider the office hour slots with IDs ranging from q to s, inclusive. How many distinct functions for assigning TAs to office hours are there? (The TAs are the domain and the office hours are the codomain)

(b) Let n be a positive integer. Suppose there are n TAs and $2n$ office hour slots. How many distinct functions for assigning TAs to office hours are there, such that every TA is assigned an office hour with an ID that is either strictly less than their ID or greater than or equal to two times their ID? (ex. if $n = 10$, TA 4 can be assigned office hour x, where $x < 4$ or $x \geq 8$)

4. [10 pts] Recall that a *combinatorial proof* for an identity proceeds as follows:

1. State a counting question.
2. Answer the question in two ways:
   (i) one answer must correspond to the left-hand side (LHS) of the identity
   (ii) the other answer must correspond to the right-hand side (RHS).
3. Conclude that the LHS is equal to the RHS.

With that in mind, give a combinatorial proof of the identity

$$\binom{2n}{n}\binom{n}{2} = \binom{2n}{2}\binom{2n-2}{n-2}$$

where $n \geq 2$.

5. [10 pts] For each of the following, prove that $|A| \leq |B|$ by defining an injective function $f : A \to B$ and then using the injection rule.

(a) A is any set and $B = 2^A$.

(b) A is the set of all prime numbers and B is the real interval $[0, 1]$. (Even though these sets have infinite cardinalities, the injection rule still applies!)