Module 6.3: Strong Induction

MCIT Online - CIT592 - Professor Val Tannen

Lecture Notes

We get stuck with ordinary induction I

Problem. Prove that any integer $n \geq 2$ can be written as the product of one or more (not necessarily distinct) prime numbers.

Answer (first attempt). We proceed by ordinary induction.

(BC) $(n_0 = 2)$ 2 is already prime. Check.

(IS) Let $k \geq 2$ arbitrary. Assume (IH) that k can be written as the product of one or more (not necessarily distinct) prime numbers.

Now consider $k+1$. If $k+1$ is prime we are done.

If $k+1$ is not prime then it has a factor r such that $1 < r < k+1$.
Moreover, $k+1 = r \cdot s$ such that $1 < s < k+1$ too.

So now we would like to use the IH on r and s.

We get stuck with ordinary induction II

Answer (first attempt, continued).

We have established that $k + 1 = r \cdot s$ such that $1 < r, s < k + 1$, that is, $2 \leq r, s \leq k$.

Let's show that $r \neq k$. Suppose (toward a contradiction) that $r = k$.

Then $k + 1 = ks$ so $1 = k(s - 1)$.

Therefore $k = 1$ which contradicts "let $k \geq 2$"

Similarly $s \neq k$.

So we cannot apply the IH to r or s.

On the other hand, since $2 \leq r, s < k$ the induction "process" must have gone through them already! We need a **stronger** induction hypothesis!

Proof pattern for strong induction

Let n_0 be a natural number and let $P(n)$ be a predicate that is well defined for all natural numbers $n \geq n_0$.

Proof pattern.

(BASE CASE) Derive/infer $P(n_0)$.

(INDUCTION STEP) Let $k \in \mathbb{N}$ such that $k \geq n_0$.
Assume $\boxed{P(n_0) \text{ and} \cdots \text{and } P(k)}$.
Derive/infer $P(k+1)$.

Conclude $\forall\, n \geq n_0 \;\; P(n)$.

The IH $P(n_0)$ and $\cdots$ and $P(k)$ is stronger than $P(k)$. But strong induction is mathematically equivalent to the ordinary one!

We succeed with strong induction

Problem. Prove that any integer $n \geq 2$ can be written as the product of one or more (not necessarily distinct) prime numbers.

Answer (second attempt). We proceed by strong induction. The base case is the same.

(IS) Let $k \geq 2$ arbitrary. Assume (IH) that all integers $2, 3, \ldots, k$ can be written as the product of one or more (not necessarily distinct) prime numbers.

Again, if $k+1$ is prime we are done and if $k+1$ is not prime then, as before, $k+1 = r \cdot s$ where $2 \leq r, s < k$.

Now we **can** use the IH on r and s! We have $r = p_1 \cdots p_u$ and $s = q_1 \cdots q_v$. Hence $k+1 = rs = p_1 \cdots p_u \cdot q_1 \cdots q_v$ with all p's and q's prime. Done.

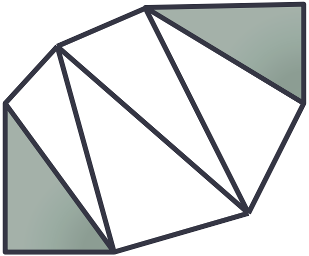
Polygon triangulation I

Problem. Prove that if a polygon with four or more sides is **triangulated** then at least two of the triangles thus formed are **exterior**.

Triangulating a polygon means drawing non-intersecting diagonals until all resulting regions are triangles.

Exterior triangles share **two** of their sides with the polygon.

Example:



Polygon triangulation II

Problem. Prove that if a polygon with four or more sides is **triangulated** then at least two of the triangles thus formed are **exterior**.

Answer. We proceed by strong induction on the number n of vertices of the polygon.

(BC) $(n_0 = 4)$ To triangulate a quadrilateral we draw one diagonal. Both resulting triangles are exterior.

.See drawing in the corresponding video lecture segment.

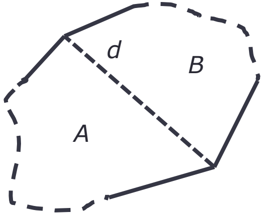
(IS) Let $k \geq 4$. Assume (IH) that for any triangulated polygon with a number of sides between 4 and k at least two of the formed triangles are exterior.

Polygon triangulation III

Answer (continued).

Assume (IH) that for any triangulated polygon with a number of sides between 4 and k at least two of the formed triangles are exterior.

Let P be triangulated with $k+1$ sides. Let d be one of the diagonals (used in the triangulation) which divides P into A and B:



Crucial observation: both A and B have at most k sides!

Polygon triangulation IV

Claim. The triangulation of A has at least one triangle that is exterior for the triangulation of P.

Proof of claim. If A is itself a triangle we are done.

Otherwise, A has between 4 and k sides and the IH applies, so the triangulation of A has at least two triangles which are exterior for A.

At most one of these two triangles has d as a side. Therefore, the other one must be exterior for P as well.

Now we can finish our main proof. The lemma applies to B as well so, in total, we have at least two exterior triangles for P.