Recitation Module 2

Penn Engineering

Concept Review

- Cardinality of Power Set
- Complementary Counting
- Number of Binary Strings
- Partial Permutations
- Logical Connectives
- Implication, conditional, and equivalence
- Combinations
- Quantifiers

Question 1

Consider the following set of Elves: Haldir, Arwen, Legolas, Galadriel, Tauriel, Luthien, Elrond

1. How many groups of these Elves contain Arwen but do not contain Tauriel?
2. How many groups of these Elves contain Haldir and Legolas OR contain Elrond?
3. How many different permutations can we construct out of the Elves whose names contain at least one letter “a”?

Answer to Question 1

1. How many groups of these Elves contain Arwen but do not contain Tauriel?

We are counting the number of sets of Elves that include Arwen but do not contain Tauriel.

This is equivalent to counting the number of sets that can be made using only Haldir, Legolas, Galadriel, Luthien, Elrond, since Arwen will be in all of these sets. In other words, this is equivalent to finding the ***cardinality of the powerset*** of {Haldir, Legolas, Galadriel, Luthien, and Elrond}.

We can make $2^5 = 32$ possible sets with those 5 Elves, so there are 32 possible sets containing Arwen.

Answer to Question 1

2. How many groups of these Elves contain Haldir and Legolas OR contain Elrond?

We are going to solve this problem by **counting complementarily** (i.e. we will subtract from the total number of the sets that can be made by all 7 elves those that do not satisfy the problem conditions).

The total number of sets using these 7 elves is $2^7 = 128$.

Now, we need to count the number of sets that:

I. Contain Haldir but do not contain Legolas, AND do not contain Elrond
II. Contain Legolas but do not contain Haldir, AND do not contain Elrond
III. Do not contain any of Haldir, Legolas, or Elrond

Answer to Question 1

I. There are $2^4 = 16$ sets that contain Haldir but do not contain Legolas, AND do not contain Elrond.

II. There are $2^4 = 16$ sets that contain Legolas but do not contain Haldir, AND do not contain Elrond.

III. There are $2^4 = 16$ sets that do not contain any of Haldir, Legolas, or Elrond.

In total there are $16 + 16 + 16 = 48$ sets that do not satisfy the conditions of the question. Thus, the answer is $128 – 48 =$ **80** sets.

Answer to Question 1

3. How many different permutations can we construct out of the Elves whose names contain at least one letter “a”?

Elves = {Haldir, Arwen, Legolas, Galadriel, Tauriel, Luthien, Elrond}

Five out of the seven Elves contain at least one letter “a” in their name: Haldir, Arwen, Legolas, Galadriel, Tauriel

It follows that there are 5! = 5 • 4 • 3 • 2 • 1 = 120 ways we can order those 5 Elves.

Question 2

1. Write the following statement using logical notation:

   “If n and m are both odd, then nm is odd”

2. Then, prove the statement.

3. Finally, use proof by cases to prove that

   “If n and m are not both odd, then nm is even”

Answer to Question 2

"If n and m are both odd, then nm is odd"

Let the predicate *odd(x)* represent the statement "x is odd" and the predicate *even(x)* the statement "x is even."

1. Then we write the statement in logical notation as:

$$(odd(m) \wedge odd(n)) \implies odd(mn)$$

2. We can prove the statement using the "if…then" proof pattern

Answer to Question 2

“If… then” proof pattern

It follows:

$$mn = (2k+1)(2l+1) = 4kl + 2l + 2k + 1 = 2(2kl + l + k) + 1$$

Since $2kl + l + k$ is an integer, mn is odd.

Answer to Question 2

“By cases” proof pattern

Case 1: (m, n are both even)
We can write m, n for two integers k, l as:

$$m = 2k \text{ and } n = 2l$$

It follows:

$$mn = 2k2l = 2(2kl)$$

Since $2kl$ is an integer, mn is even.

Case 2: (m is odd and n is even)
We can write m, n for two integers k, l as:

$$m = 2k + 1 \text{ and } n = 2l$$

It follows:

$$mn = (2k + 1)2l = 2(2kl + l)$$

Since $2kl + 1$ is an integer, mn is even.

Case 3: (m is even and n is odd)
We can write m, n for two integers k, l as:

$$m = 2k \text{ and } n = 2l + 1$$

It follows:

$$mn = 2k(2l + 1) = 2(2kl + k)$$

Since $2kl + k$ is an integer, mn is even.

Question 3

In how many ways can we arrange the digits 1-9 in a permutation so that either 5 appears in the middle or 9 appears at the end?

Answer to Question 3

In how many ways can we arrange the digits 1-9 in a permutation so that either 5 appears in the middle or 9 appears at the end?

Case 1: _ _ _ _ **5** _ _ _ _ → 5 is stationary, and other 8 digits permute (8!)

Case 2: _ _ _ _ _ _ _ _ **9** → 9 is stationary, and other 8 digits permute (8!)

Case 3: _ _ _ _ **5** _ _ _ **9** → 5 and 9 are stationary, and other 7 digits permute (7!)

We "overcounted" the number of times 5 appears in the middle AND 9 appears at the end, so we must subtract one overcounting out. For example, the sequence (1, 2, 3, 4, 5, 6, 7, 8, 9) is counted in the first 8! ways **and** the second 8! ways.

The solution is 8! + 8! – 7! = 2(8!) – 7! = 75,600.

Question 4

Count the number of sequences of bits of length 8 in which every 0 is followed immediately by a 1.

(*Note:* this means the sequence cannot end with a 0!)

Answer to Question 4

Count the number of sequences of bits of length 8 in which every 0 is followed immediately by a 1.

(*Note:* this means the sequence cannot end with a 0!)

_ _ _ _ _ _ _ **<u>1</u>**

How many zeroes can we have? **0, 1, 2, 3, or 4!**
We proceed by cases.

Answer to Question 4

- How many ways to order **zero** 0s?
  - All of the numbers in the sequence will be “1” → **1** way
- How many ways to order **one** 0?
  - I can put the “01” block (and fill in the remaining position with “1”s) in every position except for the last one → **7** ways
- How many ways to order **two** 0s?
  - I need to arrange two “01” blocks amongst four “1”s, which is like (6 choose 2) → **15** ways
- How many ways to order **three** 0s?
  - I need to arrange three “01” blocks amongst two “1”s, which is like (5 choose 3) → **10** ways
- How many ways to order **four** 0s?
  - Only one way: “01010101” → **1** way
- There are **1 + 7 + 15 + 10 + 1 = 34** ways, in total, to create sequences of bits of length 8 in which every 0 is followed immediately by a 1

Question 5 (time permitting)

7. [6 pts] EXTRA CREDIT CHALLENGE PROBLEM

To cheer up your friend who is safely social distancing at home, you decide to put together and send them a box of chocolates! Let j, k, l, m be natural numbers such that $1 \leq k \leq j$ and $1 \leq 2l \leq m$. You have a total of j dark chocolates and m milk chocolates for you to choose from. All of the chocolates (even the same type) have different fillings and are therefore all distinguishable. However, you remember that l of the milk chocolates have nuts inside of them and that your friend isn't the biggest fan of nuts with milk chocolate. Thus, you decide to include **at most** one of these l milk chocolates with nuts (yes, you can include none). How many different chocolate boxes can you form consisting of exactly k dark chocolates and l milk chocolates? For full credit, your answer must be in closed form.

Answer to Question 5

There are two steps, which we can combine using the multiplication rule (as the selection of dark chocolates has no effect on selection of milk chocolates, and vice versa).

1. **Pick dark chocolates.** Choose *k* dark chocolates out of *j* total → $\binom{j}{k}$ ways
2. **Pick milk chocolates.** We have *m* total milk chocolates, out of which *l* have nuts, so *m* – *l* milk chocolates have no nuts.
   1. **Pick 0 milk chocolates with nuts.** So we pick *l* milk chocolates out of *m* – *l* options → $\binom{m-l}{l}$ ways
   2. **Pick exactly 1 milk chocolate with nuts.** We must choose 1 of *l* milk chocolates to include → $\binom{l}{1} = l$ ways. Next, we must choose *l* – *1* milk chocolates with no nuts out of *m* – *l* options → $\binom{m-l}{l-1}$ ways.

Answer to Question 5

Combining the steps together, we get:

$$\boxed{\binom{j}{k}\left(\binom{m-l}{l} + (l)\binom{m-l}{l-1}\right)}$$

Picking dark choco- lates

Picking milk chocolates (NO NUTS)

Picking milk chocolates (exactly 1 with nuts)