OMCIT 592 Module 09 Self-Paced 01 (instructor Val Tannen)

Reference to this self-paced segment in seg.09.02

This is a segment that contains material meant to be learned *at your own pace*. We are trying to assist you in this endeavor by organizing the material in a manner similar to the way it is outlined in the recorded segments, however with one additional suggestion.

When you see the following marker:

we suggest that you stop and make sure you thoroughly understood the material presented so far before you proceed further.

Proof of equivalence of the two definitions of expectation

In the lecture segment “Expectation” you learned about the two equivalent definitions of expectation. Here we will prove the equivalence of the two definitions of expectation.

Proposition. For a random variable X defined on (Ω, Pr) we have

$$\sum_{x \in Val(X)} x \cdot \mathrm{Pr}[X = x] \;=\; \sum_{w \in \Omega} X(w) \cdot \mathrm{Pr}[w]$$

Proof:

$$\begin{aligned}
\sum_{x \in Val(X)} x \cdot \mathrm{Pr}[X = x] &= \sum_{x \in Val(X)} x \cdot \left(\sum_{w \in [X = x]} \mathrm{Pr}[w] \right) \\
&= \sum_{x \in Val(X)} \; \sum_{w \in [X = x]} x \cdot \mathrm{Pr}[w] \\
&= \sum_{x \in Val(X)} \; \sum_{w \in [X = x]} X(w) \cdot \mathrm{Pr}[w] \\
&= \sum_{w \in \Omega} X(w) \cdot \mathrm{Pr}[w]
\end{aligned}$$

where

- in the third equality step we have used $w \in [X = w]$ iff $X(w) = x$;
- in the fourth equality step we have use that the events $[X = x]$ for all $x \in Val(X)$ are pairwise disjoint and their union equals Ω.

The **expectation** (mean) of a random variable, notation $\mathrm{E}[X]$, is defined by one of the two sums shown equal in the previous proposition.

Remarkably, the first sum in the proposition can be computed without knowledge of the probability space on which X is defined: we just use the values that X returns and the distribution of X. Consequently, when we talk about the mean or expectation of a distribution, this is understood to be the expectation of a random variable with that given distribution.