

Module 4.1: Pascal's Triangle

MCIT Online - CIT592 - Professor Val Tannen

LECTURE NOTES

The Binomial Theorem I

Binomial coefficients: $\binom{n}{r}$

Binomial Theorem. For any reals a and b and any natural number n

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n = \sum_{i=0}^n \binom{n}{i}a^{n-i}b^i$$

Proof. When $n = 0$ we get $1 = \binom{0}{0} \cdot 1 \cdot 1$.

When $n \neq 0$: $(a+b)^n = (a+b) \cdot (a+b) \cdots (a+b)$

We obtain a sum of terms of the form $a^{n-i}b^i$ for various i between 0 and n .

The Binomial Theorem II

Binomial Theorem. For any reals a and b and any natural number n

$$(a + b)^n = \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i$$

Proof (continued). $(a + b)^n = (a + b) \cdot (a + b) \cdots (a + b)$

In the resulting sum, **how many times** does each $a^{n-i} b^i$ occur?

There are n factors in the multiplication. To get $a^{n-i} b^i$ multiply a from $n - i$ of the factors and b from the other i .

That is, choose i of the n factors! This can be done in $\binom{n}{i}$ ways. So the coefficient of $a^{n-i} b^i$ is $\binom{n}{i}$.

Pascal's Triangle I

The formulas given by the Binomial Theorem for $n = 0, 1, 2, 3, 4, 5$. Keep in mind that $a^0 = b^0 = 1$.

$$(a + b)^0 = \binom{0}{0} = 1$$

$$(a + b)^1 = \binom{1}{0}a^1 + \binom{1}{1}b^1 = a + b$$

$$(a + b)^2 = \binom{2}{0}a^2 + \binom{2}{1}a^1b^1 + \binom{2}{2}b^2 = a^2 + 2ab + b^2$$

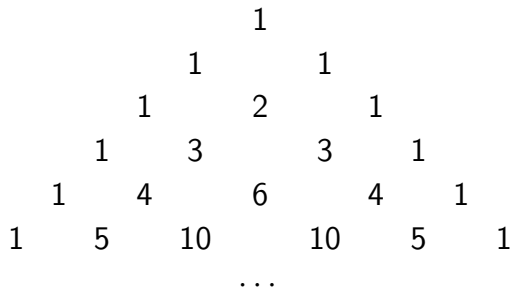
$$(a + b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}b^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\begin{aligned}(a + b)^4 &= \binom{4}{0}a^4 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}b^4 = \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

$$\begin{aligned}(a + b)^5 &= \binom{5}{0}a^5 + \binom{5}{1}a^4b^1 + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}a^1b^4 + \binom{5}{5}b^5 = \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

Pascal's Triangle II

Pascal kept just the coefficients and arranged them into a triangle:



Continuing with $n = 6, 7, \dots$, the (infinite) result is called **Pascal's Triangle**.

Notice the pattern: $2 = 1 + 1$, $3 = 1 + 2$, $6 = 3 + 3$, $10 = 4 + 6$, etc.

We will prove this!