MCIT 592 Spring 2022 **Homework 3** # Solutions

Problem Set

1. [10 pts]

Tautologies are logical expressions that are always true. Decide if the following proposition forms are tautologies using a truth table. Make sure your truth table shows **all** intermediate logical expressions — for example, in showing the truth table for $(p \vee \neg q) \wedge p$, your table should contain separate columns for p, q, $\neg q$, $p \vee \neg q$, as well as the final expression. You should also clearly state your final answer to the question.

(a) $[(\neg p \implies q) \implies (\neg p \wedge q)] \wedge (p \vee q)$

(b) $[p \wedge (q \implies r)] \implies (q \implies r)$

Solution:

(a) The expression is not a tautology.

p	q	$\neg p$	$\neg p \implies q$	$\neg p \wedge q$	$p \vee q$	$(\neg p \implies q) \implies (\neg p \wedge q)$	$[(\neg p \implies q) \implies (\neg p \wedge q)] \wedge (p \vee q)$
T	T	F	T	F	T	F	F
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	F	T	F	F	F	T	F

(b) The expression is a tautology.

p	q	r	$q \implies r$	$p \wedge (q \implies r)$	$[p \wedge (q \implies r)] \implies (q \implies r)$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

2. [10 pts]

Michael is a manager for his company, which covers multiple regions of the country.

(a) In region 1, there are 5 unique office buildings to which he would like to assign 10 indistinguishable senior workers and 40 indistinguishable entry-level workers. How many ways are there to assign workers to office buildings, such that each office has at least 1 senior worker and 4 entry-level workers?

(b) After some time, region 1 is idle (no jobs) and is looking to steal employees from the 7 other distinguishable regions. Suppose that all 7 other regions are overworked, each having a large number of indistinguishable senior employees. How many ways can region 1 take exactly 23 senior workers from the other regions, such that there are at least 3 regions from which 6 or more senior employees are taken?

Solution:

(a) Since senior employees are indistinguishable from each other, and likewise entry-level workers are indistinguishable among themselves, we can first assign 1 senior and 4 entry-level workers to each office building. We can then assign the remaining workers however we like.

Step 1: Assign 1 senior and 4 entry-level workers to each office. (1 way)

Step 2: Assign the $10 - 5 = 5$ senior workers left. ($\binom{5+5-1}{4}$ ways)

Step 3: Assign the $40 - (4 \times 5) = 20$ entry-level workers left. ($\binom{5+20-1}{4}$ ways)

Where from stars and bars there are $\binom{5+5-1}{4}$ ways to distribute 5 indistinguishable workers among 5 distinguishable offices, and $\binom{5+20-1}{4}$ is the number of ways to distribute 20 indistinguishable workers among 5 distinguishable offices. In total, there are

$$1 \times \binom{5+5-1}{5} \times \binom{5+20-1}{4} = \boxed{\binom{9}{4}\binom{24}{4}} = \boxed{\binom{9}{5}\binom{24}{20}} = \boxed{126 \cdot 10626} = \boxed{1338876}$$

ways to distribute these workers.

(b) Observe that exactly 3 regions can give 6 or more senior employees because otherwise cluster 1 ends up with > 23 senior workers. So, we can first pick 3 regions to first give exactly 6 employees. Then for the remaining $23 - (6 \times 3) = 5$ workers we can pick however we like.

Step 1: Choose 3 regions to take exactly 6 employees. ($\binom{7}{3}$ ways)

Step 2: Take the remaining 5 workers from any region. ($\binom{5+7-1}{5}$ ways)

By the Multiplication Rule, there are

$$\binom{7}{3}\binom{5+7-1}{5} = \boxed{\binom{7}{3}\binom{11}{5}} = \boxed{\binom{7}{3}\binom{11}{6}} = \boxed{16170}$$

ways for region 1 to steal employees.

3. [10 pts]

Three integers are *consecutive* if they immediately follow each other in enumerating the integers. For example, $-13, -12, -11$; or $5, 6, 7$; or $-2, -1, 0$. Prove that if a, b, c are consecutive integers then $a + b + c$ is divisible by 3 but $a^2 + b^2 + c^2$ is *not* divisible by 3.

Solution:

If a, b, c are consecutive then $a = b - 1$ and $c = b + 1$, hence $a + b + c = b - 1 + b + b + 1 = 3b$ which is divisible by 3.

Now for the second statement,

$$a^2+b^2+c^2 = (b-1)^2+b^2+(b+1)^2 = b^2-2b+1+b^2+b^2+2b+1 = 3b^2+2$$

Now suppose toward a contradiction that this is divisible by 3. So there exists an integer k such that $3b^2+2 = 3k$. Therefore $2 = 3(k-b^2)$. Since $k-b^2$ is an integer, it implies that 2 is divisible by 3 which is impossible.

4. [10 pts] How many anagrams of `raspberries` are there that have at least two consecutive `r`'s?

Solution:

We can construct such an anagram by first choosing an anagram of the ten-character string `raspbeies`, then deciding where in the resulting string to put `rr`. There are $9!/(2!2!)$ anagrams of `raspbeies`, since there are 2 `s`'s and 2 `e`'s. For each of these, we could put `pp` in one of 10 places: before the first letter, before the second letter, etc., or after the last letter. Putting `rr` immediately before the other `r` will result in the same string as putting `rr` immediately after the other `r`, so there are 9 distinct ways to insert `rr`. In other words, we remove the potential spot for rr

positioned directly after r in the sequence. Thus, by the multiplication rule, there are $\boxed{9!/(2!2!) \cdot 9}$ such anagrams.

5. [10 pts] Consider the following statement.

> There exist integers a and c such that for all integers x if $x \geq a$ then $x^2 < c \cdot x$.

Disprove this statement. (Hint: first write the negation of this statement then prove this negation.) new

Solution:

The negation is "For all integers a and c there exists an integer x such that $x \geq a$ and $x^2 \geq c \cdot x$."

Let a and c be any integers. Next we can continue along several alternate variants of the solution.

We can let (variant 1) $x = \max\{|a|, |c|\}$ (i.e., let x be the absolute value of a or the absolute value of c, whichever is greater), or we can let (variant 2) $x = |a| + |c|$, or we can let (variant 3) $x = \max\{a, c, 0\}$.

In all variants x is an integer, $x \geq a$, (both of which we need to have) but also, $x \geq c$ and $x \geq 0$. We show that these last two inequalities imply $x^2 \geq c \cdot x$. Indeed, because $x \geq 0$ we can multiply both sides of $x \geq c$ by x and we obtain $x^2 = x \cdot x \geq c \cdot x$.

Thus, in all three variants, the chosen x proves the negation and is a counterexample to the statement. □