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Semistable models of hyperelliptic curves over residue characteristic 2

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Preface

Let (K, v) be a complete discrete-valued field. Given a smooth geometrically connected K-curve[1] $X \to \operatorname{Spec}(K)$, a *normal model* of X is an extension of X to a proper, flat, normal scheme $\mathcal{X} \to \operatorname{Spec}(\mathcal{O}_K)$ defined over the ring of integers $\mathcal{O}_K$. The spectrum of $\mathcal{O}_K$ consists only of a generic point, η, and a closed point, s; a model of X can thus be thought of as a family of two curves: the generic fiber $\mathcal{X}_\eta$, defined over K, and the special fiber $\mathcal{X}_s$, defined over the residue field k. While $\mathcal{X}_\eta$ is canonically identified with X and is thus a smooth curve, nothing prevents the special fiber $\mathcal{X}_s$ from possibly being singular, reducible, and even non-reduced.

In the realm of complex geometry, an example is given by the case in which $\mathcal{O}_K$ is the stalk of the germs of holomorphic functions at a point P of a Rieamnn surface: X is then to be viewed as a family of Riemann surfaces defined over an infinitesimal punctured neighbourhood of P, and building a model $\mathcal{X}$ means continuing its definition also at the point P; in this scenario, the fields $K \cong \mathbb{C}((t))$ and $k \cong \mathbb{C}$ both have characteristic zero. The context we will mainly deal with in this thesis, however, is that of number theory, in which K is a local field, and the ring of integers $\mathcal{O}_K$ consequently has mixed characteristic.

It is desirable that the model of a curve preserves as many properties of the curve as possible; since X is assumed to be smooth, it is thus interesting to ask whether it admits a smooth model (a model is smooth when not only its generic fiber, but also the special one is a smooth curve). When this happens, we say that X has *good reduction*; moreover, we will say that X has *potential good reduction* if it acquires good reduction over a large enough extension of K. Since most smooth curves fail to have potential good reduction, if we want to recover some general existence result smoothness has to be replaced with some weaker requirement on the model. The suitable notion turns out to be the one of *semistability*: a model of X is said to be *semistable* if its special fiber is a geometrically reduced curve whose singularities are all nodes (i.e. ordinary double points); if X admits such a model, it is said to have *semistable reduction*. Semistable models, when they exist, are infinitely many; however, provided that the genus of X is at least 2, there is a minimum one among them, which is named the *stable model* and is the only one exhibiting an ample canonical bundle.

An important theorem ensures that all smooth curves have potential semistable reduction. Its first proof was obtained by Deligne and Mumford in 1969 (see [DM69, Corollary 2.7, p. 92]) by applying to the Jacobian of X a result by

[1]By *curve* we mean a proper scheme of pure dimension 1 over the spectrum of a field.

Grothendieck on semiabelian reduction of abelian varieties ([SGA7.I, Exposé I, Theorème 6.1, p. 21]). The proof we will present in this thesis (following [Liu] e [Des81]) is, instead, due to Artin and Winters (see [AW71]) and, although it does not require the cited Grothendieck's result, is equally based on the relation between the singularities of a curve and the structure of its Jacobian.

These proves of the semistable reduction theorem are, however, non-constructive, and the actual determination of semistable models of a given smooth curve remains a non-trivial task. A possible strategy (already explored in other works, e.g. [LL99]) is based on the following remark. Let $Y \to X$ be a Galois covering of smooth geometrically connected K-curves: it is possible to prove that, if Y has semistable reduction, X also does, and the stable model $\mathcal{Y}_{\mathrm{st}}$ of Y (assuming that the genus of Y is at least 2) can be obtained by normalizing in the function field $K(Y)$ of Y a suitable semistable model of X, which we will name the *ur-stable* model and denote by $\mathcal{X}_{\mathrm{urst}}$. Hence, assuming that a description of the family $\mathbf{Semist}_X$ of all semistable models of X is available (notice that X has lower genus, and is thus easier to study than Y), to build $\mathcal{Y}_{\mathrm{st}}$ it is enough to scan through all of them, computing the normalization in $K(Y)$ of each one and studying its stability, until $\mathcal{X}_{\mathrm{urst}}$ is detected.

From now on, we will work under the hypothesis that the base X of the Galois covering is a projective line; what motivates this choice is the availability, in this case, of a simple description of $\mathbf{Semist}_X$: semistable models of the line, in fact, are all obtained by composing (Definition 6.2.3) smooth models – and smooth models of the line, in turn, are all isomorphic to $\mathbb{P}^1_{\mathcal{O}_K}$ as $\mathcal{O}_K$-schemes, and they differ from each other only for the choice of the K-isomorphism $(\mathbb{P}^1_{\mathcal{O}_K})_\eta \xrightarrow{\sim} X$ that provides them with the structure of models of X. All that is left to do now is to identify the set $\mathbf{Urst}_X^Y \subseteq \mathbf{Sm}_X$ of the smooth models of the line that compose $\mathcal{X}_{\mathrm{urst}}$: at the end of Chapter 6 we will prove a criterion to determine whether a smooth model $\mathcal{X}$ of the line belongs to $\mathbf{Urst}_X^Y$ depending on the singularities of the special fiber $\mathcal{Y}_s$ of its normalization $\mathcal{Y}$ in $K(Y)$ (Theorem 6.4.6); we will also show that, even if $\mathcal{X}$ does not contribute to the ur-stable model, the singularities of $\mathcal{Y}_s$ still give precious indication on the position of the elements of $\mathbf{Urst}_X^Y$ relative to $\mathcal{X}$ (Theorem 6.4.7).

After deducing these results, we will apply them to the particular case in which $Y \to X = \mathbb{P}^1_K$ is a tame hyperelliptic curve with rational branch locus. We preliminarily observe that, if $\mathcal{R} = \{A_1, \dots A_{2g+2}\} \subseteq X(K)$ denotes the branch locus and x is the standard coordinate on the line X, then the equation of Y can be written in the form $y^2 = c\prod_i(x - x(A_i))$: the datum of the A_i's, together with the class of the coefficient c in $K^\times/(K^\times)^2$, determines the curve Y and the covering map $Y \to X$ completely. Once a smooth model $\mathcal{X}$ of the line is fixed, we can study the reduction of the branch points: although they are all distinct in the generic fiber $\mathcal{X}_\eta = X$, the A_i's can clearly cluster together in $\mathcal{X}_s$, and the speed at which two of them collapse is measured by the valuation $v(A_i - A_j)$ of their difference. The valuations $v(A_i - A_j)$ determine the so-called *cluster picture* of $\mathcal{R}$ with respect to $\mathcal{X}$ (Definitions 6.3.5 and 6.3.7), which can be graphically represented by a diagram like the following one.

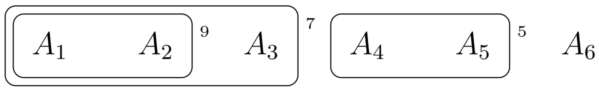

Figure 0.1.: A possible cluster picture of a genus-2 hyperelliptic curve $Y \to X$ with respect to some smooth model $\mathcal{X}$ of the line X. Looking at the diagram, we can deduce, for example, that $v(A_1 - A_6) = 0$, $v(A_1 - A_3) = 7$, and $v(A_1 - A_2) = 7 + 9 = 16$. There are three maximal clusters, namely $\{A_1, A_2, A_3\}$, $\{A_4, A_5\}$ and $\{A_6\}$, corresponding to the three points of the special fiber $\mathcal{X}_s$ to which the A_i's reduce.

When all the $2g+2$ branch points reduce to the same point of $\mathcal{X}_s$, $\mathcal{X}$ is said to be *crushed* (Definition 6.3.17). A non-crushed model is completely determined, among all smooth models of the line, by its associated cluster picture (Proposition 6.3.18); crushed models, instead, do not enjoy this property, and, more in general, they cannot be parameterized by means of a discrete invariant. Nonetheless, each crushed smooth model $\mathcal{X}$ is *anchored* to some uniquely determined non-crushed smooth model $\mathcal{X}^{\mathrm{nc}}$ (Definition 6.3.20).

If the residue characteristic of (K, v) is not 2, then it is easy to compute the model $\mathcal{Y}$ of the hyperelliptic curve that corresponds to a given smooth model $\mathcal{X}$ of the line, and the structure of the special fiber $\mathcal{Y}_s$ only depends on the cluster picture and on the parity of the valuation of the coefficient c appearing in the equation of Y (see Section 7.2). In particular, we have that $\mathbf{Urst}_X^Y$ consists precisely of the *very unwound* smooth models of the line (Definition 6.3.23), i.e. those whose associated cluster picture has more than three maximal clusters, or exactly three maximal clusters, as far as no more than one of them is a singleton (see Theorems 7.2.3 and 7.2.4 for details). In particular, $\mathbf{Urst}_X^Y$ does not contain crushed models.

The residue characteristic 2 case is far more complicated. First, computing the normalization $\mathcal{Y}$ in $K(Y)$ of a given smooth model $\mathcal{X}$ of the line is harder: the technique we propose will be presented at the beginning of Section 7.3. In particular, the coefficient c and the cluster picture do not codify enough information to predict the structure of $\mathcal{Y}_s$. Moreover, the possible inseparability and the presence of wild ramification phoenomena make the behaviour of the quadratic covering $\mathcal{Y}_s \to \mathcal{X}_s$ more diversified, and harder to dominate: a relevant consequence is the fact that $\mathbf{Urst}_X^Y$ may also contain crushed models.

To determine $\mathbf{Urst}_X^Y$ and consequently $\mathcal{Y}_{\mathrm{st}}$, our strategy will consist in scanning through all non-crushed smooth models $\mathcal{X}$ of the line, computing the normalization $\mathcal{Y}$ in $K(Y)$ of each of them: the study of the singularities of $\mathcal{Y}_s$ will then allow us to decide which models $\mathcal{X}$ belong to $\mathbf{Urst}_X^Y$, and which are anchors of crushed smooth models of the line falling within $\mathbf{Urst}_X^Y$. It is reasonable to expect that this approach can be turned into a full algorithm capable of computing the stable model of any hyperelliptic curve over residue characteristic 2: our attempts in this direction are detailed in Subsection 7.3.4. As an application, we will discuss the case of genus-2 hyperelliptic curves.

The use of cluster pictures to study semistable models of hyperelliptic curves over residue characteristic different from 2 has been introduced in [Dok+18], where this theory is developed without any assumption on the rationality of the branch locus. The attempt to generalize that approach to the case of residue characteristic 2 is, to the best of our knowledge, original, and has originated from the construction proposed in [Yel19] for semistable models of elliptic curves over residue characteristic 2.

Chapters 1 to 3 collect some necessary preliminary material: the main topics discussed include blowups, birational morphisms, the Formal Functions Theorem, Zariski's Main Theorem, Picard's functor, the study of singularities of curves and their relation to the Jacobian. Chapter 4 discusses the general theory of models of curves, whereas Chapter 5 focuses on smooth and semistable models. Chapter 6 introduces the classification of the models of the line and the tool of cluster pictures; then, it inquires how the models of a Galois covering of the line can be built out of models of the line itself. Finally, Chapter 7 applies all this machinery to the determination of semistable models of hyperelliptic curves, first over residue characteristic different from 2, then over residue characteristic 2.

Acknowledgement The research presented in this thesis was developed in collaboration with dr. Jeffrey Yelton, who proposed me the problem: the constructions for genus-1 and genus-2 curves presented in Section 7.4 are an adaptation of the computations done together during the last months. I would also like to thank my supervisor, prof. Fabrizio Andreatta, for the valuable clarifications he gave me on many theoretical topics.

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Contents

1. Divisors and line bundles

Given X a scheme, a closed subscheme $Z \hookrightarrow X$ is said to be *regular of codimension* r if it can be locally described as the zero-locus of a regular system of r functions. Regular closed subschemes of codimension 1 are also named *effective Cartier divisors* and they play a prominent role in the study of the geometry of a scheme. If $Z \hookrightarrow X$ is any closed subscheme, there exists a universal way of transforming X that turns Z into an effective Cartier divisor: this operation goes under the name of *blowup* and will later turn out to be a main tool to manipulate arithmetic surfaces.

The study of blowups will be one of the main aims of this chapter, which will also introduce a number of fundamental notions related to divisors, line bundles (i.e. invertible sheaves) and morphisms from a scheme to the projective space.

1.1. Closed subschemes

Closed subschemes of a given scheme X correspond biiectively to quasi-coherent sheaves of ideals: given a closed subscheme $j : Z \hookrightarrow X$ of a scheme X, we can associate to it the quasicoherent sheaf of ideals $\mathcal{Z} := \ker(\mathcal{O}_X \to j_*\mathcal{O}_Z)$; conversely, if $\mathcal{Z}$ is any given quasicoherent sheaf of ideals on a scheme X, its zero locus $Z := V(\mathcal{Z}) \hookrightarrow X$ defines a closed subscheme of X. The closed subschemes of a scheme X form a complete lattice $\mathrm{ClSub}(X)$; every morphism $f : X' \to X$ induces a monotonic Galois connection between $\mathrm{ClSub}(X')$ and $\mathrm{ClSub}(X)$, which we will now describe.

- If we take a closed subscheme $Z \in \mathrm{ClSub}(X)$, the pullback $j' : Z' \to X'$ of the the closed immersion $Z \hookrightarrow X$ along f is still a closed immersion; the assignment $Z \mapsto f^{-1}(Z) := Z'$ provides the right adjoint $f^{-1} : \mathrm{ClSub}(X) \to \mathrm{ClSub}(X')$ of the Galois connection. We will equivalently name $f^{-1}(Z)$ the *inverse image*, the *pullback*, or the *total transform* of Z in X'.

  If $\mathcal{Z}$ is the ideal sheaf defining Z in X, the ideal sheaf $\mathcal{Z}'$ defining $f^{-1}(Z)$ in X' is the extension $\mathcal{Z}\mathcal{O}_{X'}$ of $\mathcal{Z}$ in $\mathcal{O}_{X'}$; more formally, $\mathcal{Z}'$ is the image in $\mathcal{O}_{X'}$ of the morphism $f^*\mathcal{Z} \to \mathcal{O}_{X'}$ that is obtained by applying the pullback of quasi-coherent sheaves functor f^* to the inclusion $\mathcal{Z} \hookrightarrow \mathcal{O}_X$.

- Conversely, if we start with a closed subscheme $Z' \in \mathrm{ClSub}(X')$, we can compute the scheme-theoretic image $f(Z')$ of the composition $Z' \hookrightarrow X' \to$

X, i.e. the minimum closed subscheme of X through which $Z' \to X$ factors. The assignment $Z' \mapsto f(Z')$ we have defined provides the left adjoint f : $\mathrm{ClSub}(X') \to \mathrm{ClSub}(X)$ of the Galois connection.

If f is quasi-compact and $\mathcal{Z}' \subseteq \mathcal{O}_{X'}$ is the sheaf of ideals defining Z', then the ideal sheaf defining $f(Z')$ is the contraction of $\mathcal{Z}'$ in $\mathcal{O}_X$, i.e. the kernel of $\mathcal{O}_X \to f_*(\mathcal{O}_{X'}/\mathcal{Z}')$, or equivalently the kernel of $\mathcal{O}_X \to f_*(\mathcal{O}_{X'})/f_*(\mathcal{Z}')$. If f is not quasi-compact, this kernel may not be quasicoherent, and the sheaf of ideals defining $f(Z')$ will be the largest quasicoherent sheaf contained inside it (see for example [Stacks, `01R5`] for details).

Actually, for any given scheme X, $\mathrm{ClSub}(X)$ is not only a complete lattice, but also a commutative monoid: the monoid operation corresponds to the product of quasi-coherent sheaves of ideals. In particular, if $Z \in \mathrm{ClSub}(X)$ is a closed subscheme defined by a quasicoherent sheaf of ideals $\mathcal{Z}$, it is meaningful to consider its *thickenings* $Z, 2Z, 3Z, \ldots$ defined by the ideal sheaves $\mathcal{Z}$, $\mathcal{Z}^2$, $\mathcal{Z}^3$, and so on. If $X' \to X$ is a morphism of schemes, the inverse image $f^{-1} : \mathrm{ClSub}(X) \to \mathrm{ClSub}(X')$ is clearly a morphism of monoids.

1.1.1. Locally principal closed subschemes

A closed subscheme Z of a scheme X is called *locally principal* if X admit an open covering $X = \cup_\alpha U_\alpha$ such that $Z \cap U_\alpha$ can be described as the zero locus $V(\varphi_\alpha)$ of a single function $\varphi_\alpha \in \mathcal{O}_X(U_\alpha)$. We will say that $\varphi_\alpha = 0$ is a *local equation* for X on U_α. Since two equations $\varphi = 0$ and $\psi = 0$ describe the same closed closed subscheme if and only if they differ by a never-vanishing function, we can also say that a locally principal closed subscheme of X is a global section of the quotient sheaf $\mathcal{O}_X/\mathcal{O}_X^\times$ (which is a sheaf of commutative monoids on X). Locally principal closed subschemes of X form a submonoid of $\mathrm{ClSub}(X)$; moreover, if $f : X' \to X$ is any morphism of schemes, the inverse image of a locally principal closed subscheme is still locally principal.

1.1.2. Effective Cartier divisors

If the local equations f_α defining a locally principal closed subscheme $Z \subseteq X$ are all regular (meaning that they are not zero divisors, i.e. that the homothety $\cdot f_\alpha : \mathcal{O}_{U_\alpha} \to \mathcal{O}_{U_\alpha}$ is injective), then Z is named an *effective Cartier divisor*. If $\mathcal{O}_X^{\mathrm{reg}} \subseteq \mathcal{O}_X$ denotes the system of all regular sections of $\mathcal{O}_X$, an effective Cartier divisor is a global section of the quotient sheaf $\mathcal{O}_X^{\mathrm{reg}}/\mathcal{O}_X^\times$. Effective Cartier divisors form a submonoid of the locally principal closed subschemes on X, and they can be characterized as those locally principal closed subschemes that do not contain any weakly associated point of X (see [Stacks, 056K] for a definition). The complement of an effective Cartier divisor is a scheme-theoretically dense open subset of X.

Effective Cartier divisors, in some favorable situations, are nothing but closed subschemes of pure codimension 1:

Proposition 1.1.1. Let X be a locally Noetherian scheme, and $Z \subseteq X$ a closed subscheme. If Z is an effective Cartier divisor, then it has pure codimension 1 in X. If X is regular, the converse is also true.

Proof. If C is a Cartier divisor on X, it clearly avoids the generic points of X; hence, for each irreducible component Z_i of Z, we have that $\mathrm{cod}_X(Z_i) \geq 1$. The converse bound $\mathrm{cod}_X(Z_i) \leq 1$ is a corollary of Krull's principal ideal theorem.

If X is now supposed to be regular we have that, for all $x \in X$, $\mathcal{O}_{X,x}$ is an UFD, and hence all its primes of height 1 are principal. This easily implies that closed subschemes of pure codimension 1 are locally principal, and hence they are effective Cartier divisors (fore a more general statement, see [Stacks, 0BXH]). $\square$

The sheaf of ideals defining an effective Cartier divisor Z is generally denoted by $\mathcal{O}_X(-Z)$, and it is invertible. Actually, the invertibility of the defining sheaf of ideals characterizes effective Cartier divisors among all closed subschemes. The dual of $\mathcal{O}_X(-Z)$ will be denoted by $\mathcal{O}_X(Z)$; since $\mathcal{O}_X(-Z)$ is a subsheaf of $\mathcal{O}_X$, $\mathcal{O}_X(Z)$ comes with a distinguished injection $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(Z)$, i.e. with a distinguished regular section $f \in \mathcal{O}_X(Z)$, and Z can be recovered as $V(f)$. Conversely, the zero-locus $V(f)$ of a regular section $f \in H^0(\mathcal{L})$ of any given invertible sheaf $\mathcal{L}$ on X is an effective Cartier divisor on X.

To summarize, an effective Cartier divisor is defined as a closed subscheme Z locally described by a single regular equation, and can be equivalently thought as (1) a global section of $\mathcal{O}_X^{\mathrm{reg}}/\mathcal{O}_X^{\times}$, (2) an invertible sheaf of ideals $\mathcal{O}_X(-Z)$ in $\mathcal{O}_X$, and (3) an invertible sheaf $\mathcal{O}_X(Z)$ together with a distinguished regular section.

The assignment $Z \mapsto \mathcal{O}_X(Z)$ defines a morphism from the monoid of effective Cartier divisors to the Picard group of X (i.e., the group of isomorphism classes of invertible sheaves on X). The kernel of this morphism consists of those effective Cartier divisors which can be described by a unique global regular equation, and are consequently named *principal* effective Cartier divisors. If Z_1 and Z_2 are two effective Cartier divisors such that $\mathcal{O}_X(Z_1) \cong \mathcal{O}_X(Z_2)$, we say that they are *linearly equivalent.*

1.1.3. Pullback of Cartier divisors

We have remarked that the pullback of a locally principal closed subscheme $Z \subseteq X$ along a morphism of schemes $X' \to X$ is still a locally principal closed subscheme. Anyway, if Z is an effective Cartier divisor on X, Z' may fail in general to be an effective Cartier divisor, and this problem arises precisely whenever Z' contains some associated component of X', i.e. when Z meets the image in X of an associated point of X':

Proposition 1.1.2. Let $f : X' \to X$ a morphism of schemes. Then, (a) the effective Cartier divisors Z on X that pullback to effective Cartier divisors on X' are precisely those avoiding $f(\mathrm{WAss}(\mathcal{O}_{X'}))$, where $\mathrm{WAss}(\mathcal{O}_X)$ is the set of weakly associated points of X, and (b) if Z is as such, $\mathcal{O}_{X'}(f^{-1}Z) \cong f^*\mathcal{O}_X(Z)$.

Proof. The discussion above is enough to prove (a), and (b) is immediate. See also [Stacks, 0200], [Liu, Lemma 7.1.29]. □

Proposition 1.1.3. If $f : X' \to X$ is a flat morphism of schemes, all effective Cartier divisors on X pull back to effective Cartier divisors on X'.

Proof. Flatness clearly implies that $f^*(\mathcal{O}_X^{\mathrm{reg}}) \subseteq \mathcal{O}_{X'}^{\mathrm{reg}}$, and from this the proposition easily follows. See [Stacks, 0200] for a reference. □

1.1.4. Blowup

Given $Z \subseteq X$ a closed subscheme, we may ask whether there is a universal way of replacing X by another scheme $f : X' \to X$ so $Z' := f^{-1}Z$ becomes an effective Cartier divisor. If such an X' exists, it is named the *blowup of X along Z*, and the Cartier divisor Z' is named the *exceptional divisor* of the blowup.

Remark 1.1.4. From a categorical point of view, we are working in the category $\mathcal{C}$ where objects are couples (X, Z) consisting of a scheme together with closed subschemes and morphisms $(X_1, Z_1) \to (X_2, Z_2)$ are morphisms of schemes $X_1 \to X_2$ such that $f^{-1}Z_2 = Z_1$; we are considering the full subcategory $\mathcal{C}'$ consisting of couples (X, Z) such that Z is an effective Cartier divisor in X: the blowup of an object $(X, Z) \in \mathrm{Ob}(\mathcal{C})$ is just its reflection into $\mathcal{C}'$; provided that the blowup exists for all objects of $\mathcal{C}$, blowing up gives a left adjoint to the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$.

We will soon prove that the blowup always exists. But before this, we can easily derive many of its most interesting aspects directly from the universal property defining it: their proofs are immediate, and will thus be omitted. Our discussion mainly follows [Vakil, Chapter 22].

1. The formation of the blowup always commutes with flat base-change. This means that, if $X' \to X$ is the blowup of X along some closed subscheme $Z_X \subseteq X$, $Y \to X$ is a flat morphism of schemes and $Z_Y \subseteq Y$ is the pullback of Z_X to Y, then $Y' \to Y$ (where $Y' := Y \times_X X'$) is the blowup of Y along Z_Y.
$$\begin{tikzcd} (Y', Z'_Y) \arrow[r] \arrow[d, "\text{blowup}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (X', Z'_X) \arrow[d, "\text{blowup}"] \\ (Y, Z_Y) \arrow[r, "\text{flat}"] & (X, Z_X) \end{tikzcd} \tag{†}$$
In particular, this implies that the proof of existence and the construction itself of the blowup of a scheme (X, Z_X) can be carried out affine-locally.

2. The blowup of a scheme X along the total subscheme $Z = X$ is empty. The blowup of a scheme X along the empty subscheme $Z = \emptyset$ is X itself. If $f : (X', Z') \to (X, Z)$ is any blowup, then $f^{-1}(X \smallsetminus Z) = X' \smallsetminus Z'$ and f is an isomorphism over $X \smallsetminus Z$.

3. The blowup of a reduced or irreducible scheme is still reduced or irreducible, respectively: this follows immediately from the point above, if we recall that the complement of an effective Cartier divisor is scheme-theoretically dense.

Remark 1.1.5. Blowing up does not in general commute with arbitrary base-change: if in the setting of diagram (†) we relax the flatness hypothesis on $Y \to X$, we can no longer be guaranteed that $Y' \to Y$ is the blowup of Y along Z_Y. Yet, $Y' \to Y$ still has some important property, and it consequently deserves a name: we will refer to it as the *total transform* of Y through the blowup of X along Z_X. In particular, the total transform $Y' \to Y$ pulls Z_Y back to a closed subscheme Z'_Y that, even though it may not be an effective Cartier divisor, is certainly at least locally principal; moreover, it is easy to see that $Y' \to Y$ still satisfies the mapping property of the blowup, meaning that any morphism of schemes $T \to Y$ that pulls Z_Y back to an effective Cartier divisor must factor uniquely through Y'. As a consequence, the blowup of Y along Z_Y and the blowup of the total transform Y' along Z'_Y must coincide (meaning that, if one of them exists, the other exists, and they are the same): the following diagram summarizes the situation.

$$\begin{tikzcd}
Z''_Y \arrow[r] \arrow[d, hook, "\text{Cartier}"] \arrow[dr, phantom, "\lrcorner", very near start] & Z'_Y \arrow[r] \arrow[d, hook, "\text{loc.princ.}"] \arrow[dr, phantom, "\lrcorner", very near start] & Z_Y \arrow[d, hook] \\
(Y'', Z_{Y''}) \arrow[r, "\text{blowup}"] \arrow[rr, bend right, "\text{blowup}"'] & (Y', Z'_Y) \arrow[r, "\text{total transform of } Y \text{ along}", "\text{the blowup } X' \to X"'] & (Y, Z_Y)
\end{tikzcd}$$

Our strategy will now be to effectively build the blowup in some particular situations, and then invoke all the considerations above to extend our existence result to every scheme. The first situation we will consider is the blowup of a closed subscheme $Z \subseteq X$ which is already locally principal. In this case, turning it into a Cartier divisor in a universal way simply means removing from X all those associated components that Z contains (see figure 1.1):

Proposition 1.1.6. Suppose $Z \subseteq X$ is a locally principal closed subscheme of a scheme X, and let us denote by X' the scheme-theoretic closure of $X \smallsetminus Z$ in X. Then, the closed immersion $X' \hookrightarrow X$ is the blowup of X along Z.

Proof. Let A be a ring and $a \in A$ an element. Let I be the kernel of the localization $A \to A[a^{-1}]$, i.e. the ideal consisting of all those elements $b \in A$ such that $ba^k = 0$ for some $k \geq 1$. It is clear that the image $\overline{a}$ of a in A/I is a regular element, and it is not difficult to show that any ring homomorphism $A \to B$ turning a into a regular element must contain I in its kernel. This is enough to prove that $\mathrm{Spec}(A/I)$ is the blowup of $V(a) \subseteq \mathrm{Spec}(A)$, and $\mathrm{Spec}(A/I)$ is clearly nothing but the scheme-theoretic closure of $\mathrm{Spec}(A[a^{-1}]) = \mathrm{Spec}(A) \smallsetminus V(a)$ in $\mathrm{Spec}(A)$. The argument we have presented is clearly enough to prove the proposition. $\square$

Proposition 1.1.7. The blowup of the affine space $X := \mathbb{Z}[x_1, \ldots, x_n]$ at the origin is the closed subscheme X' of $\mathrm{Spec}(\mathbb{Z}[x_1, \ldots, x_n]) \times \mathrm{Proj}(\mathbb{Z}[t_1, \ldots, t_n])$ defined by the equations $t_i x_j = t_j x_i$.

Proof. The origin of X, and hence its pullback in X', are defined by the n equations $x_1 = 0, \ldots, x_n = 0$. If we restrict ourselves to the affine chart $t_i \neq 0$ of X', then, for all $j \neq i$, x_j can be written as $(t_j/t_i)x_i$, and hence the n equations $x_1 = 0, \ldots, x_n = 0$ reduced to the single, regular equation $x_i = 0$. This is enough to ensure that $X' \to X$ pulls back the origin of X to an effective Cartier divisor. Let now suppose that $W \to X$ is another scheme on which the equations $x_1 = \ldots = x_n = 0$

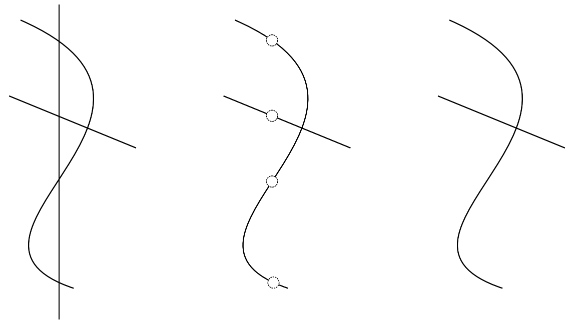


Figure 1.1.: The process of blowing up X (first figure) along the vertical line Z. The second figure displays $X \smallsetminus Z$, and the third one the blowup X'.

define an effective Cartier divisor Z_W: $x_1, \dots x_n$ will be sections of the line bundle $\mathcal{O}_W(Z_W)$ and will induce a morphism $W \to \mathrm{Proj}(\mathbb{Z}[t_1, \dots, t_n])$ that pulls back $\mathcal{O}(1)$ to $\mathcal{O}_W(D)$ and each of the sections t_i of $\mathcal{O}(1)$ to the section x_i of $\mathcal{O}_W(Z_W)$. From this, it is immediate to build a factorization of $W \to X$ through X'. To prove that this factorization in unique, it is enough to observe that, since $X' \to X$ is an isomorphism away from the origin of X, two factorizations $W \rightrightarrows X'$ would be forced to agree on the complement of the effective Cartier divisor D, which is a scheme-theoretically dense open subscheme of W. But X' is separated, and hence they will be necessarily equal everywhere. □

Remark 1.1.8. In the proposition above, the hypothesis that the number n of coordinates is finite is inessential.

Now, suppose we have a scheme X together with a closed subscheme $Z \subseteq X$ defined by a globally generated ideal sheaf $\mathcal{Z}$ (a condition which is always satisfied if X is affine). Making a choice of global sections $s_1, \dots, s_n \in H^0(\mathcal{Z})$ generating $\mathcal{Z}$ (we are supposing that they are finitely many just to keep notation simpler) induces a morphism $s : X \to \mathbb{A}^n_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[x_1, \dots, x_n])$ such that $s^* x_i = f_i$, so the pullback of the origin of $\mathbb{A}^n_{\mathbb{Z}}$ is precisely the closed subscheme Z. Hence, thanks to Remark 1.1.5, the blowup of X along Z can be computed in two stages (see Figure 1.2):

A) First, we compute the total transform X'_1 of X along the blowup of the affine space $\mathbb{A}^n_{\mathbb{Z}}$ at the origin: as a result, we will get the closed subscheme of $\mathbb{P}^{n-1}_X$ defined by the equations $s_i t_j = s_j t_i$ (if $t_1, \dots, t_n$ denote the coordinates of $\mathbb{P}^{n-1}_X$). This step will turn $Z \subseteq X$ into a locally principal closed subscheme $Z'_1 \subseteq X'_1$.

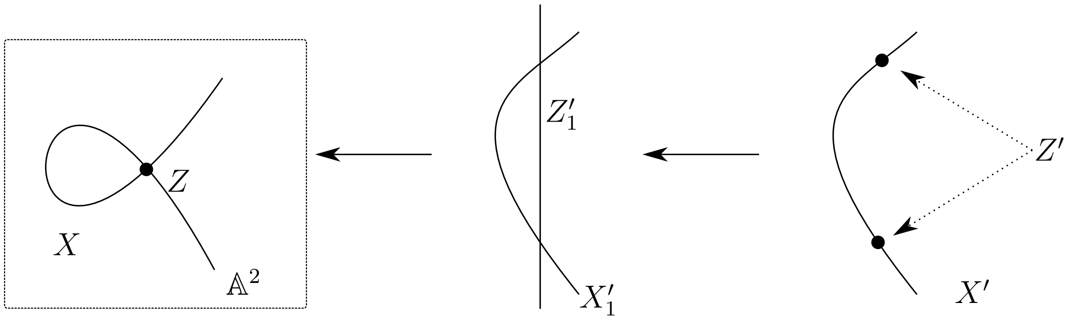


Figure 1.2.: The two-stage process of blowing up the cuspidal cubic $y^2 = x^3 + x^2$ at the cusp Z; if x and y are the chosen generators for $\mathcal{Z}$, the morphism s is just the standard immersion of the curve in the plane.

B) If $Z'_1 \subseteq X'_1$ does not contain any associated component of X'_1, we are done: $X' := X'_1$ provides the desired blowup of X. Otherwise, we compute the scheme-theoretic closure X' of $X'_1 \smallsetminus Z'_1$ in X'_1: X' will be the blowup of X along Z that we were looking for.

If $Z \subseteq X$ is an arbitrary closed subscheme of a scheme X, we can find an open cover $X = \cup_i X_i$ such that Z is defined by a globally generated sheaf over each of the X_i: we can form the blowup of each of the X_i, and then glue them together to obtain the blowup of X. We have consequently proved that the blowup always exists and given a fairly explicit construction, which clearly shows that:

Proposition 1.1.9. If $Z \subseteq X$ is a closed subscheme defined by a finite-type sheaf of ideals, the blowup $X' \to X$ of X along Z is proper.

There are some occasions in which, if the generators of the defining sheaf of ideals are chosen properly, step A is actually sufficient to produce the blowup of X. For example, suppose that $Z \subseteq X$ is a regular closed subscheme of a regular scheme. To describe the blowup, it is clearly enough to study the case in which $X = \operatorname{Spec}(A)$ is the spectrum of a local ring $(A, \mathfrak{m})$; let us choose a minimal system $s_1, \ldots, s_r$ of generators of $\mathcal{Z}$, where $\mathcal{Z} \subseteq \mathfrak{m}$ is the ideal defining Z, and r is the codimension of $Z \subseteq X$. It is not difficult to verify that $X'_1 := \operatorname{Proj}\left(A[t_1, \ldots, t_n]/(s_i t_j - s_j t_i)\right)$ is a regular scheme (see the proof of [Liu, Theorem 8.1.19] for details). In particular, the fact that X'_1 is integral (together with the observation that Z'_1 cannot coincide with the whole X'_1) implies that step B does nothing, so the blowup is $X' := X'_1$. In particular,

Proposition 1.1.10. The blowup X' of a regular scheme X along a regular closed subscheme Z is regular, and can be described as above. In particular, the exceptional divisor Z' is isomorphic, as a Z-scheme, to $\mathbb{P}_Z^{r-1}$, being r the codimension of Z in X (supposing X connected).

1.1.5. Cycles

If X is a locally Noetherian scheme, a *prime cycle* on X is simply a point $x \in X$. Points of a scheme correspond to its integral closed subschemes: the term "prime cycle" will be employed indifferently to mean a point $x \in X$ or the corresponding integral closed subscheme $\overline{\{x\}} \subseteq X$. A *cycle* on X is a locally finite formal combination with integer coefficients of prime cycles. A cycle has dimension (resp. codimension) d if all the prime cycles composing it have dimension (resp. codimension) d. Cycle dimensions and codimensions provide two alternative gradings $Z_\bullet(X)$ and $Z^\bullet(X)$ of the abelian group $Z(X)$ of all cycles. A cycle Z is said to be *effective* (and we write $Z \geq 0$) if all the coefficients of the linear combination the defines it are ≥ 0: effective cycles form a submonoid $Z_+(X) \subseteq Z(X)$. Codimension 1 cycles are also named *Weyl divisors*, and the group they form is also denoted by $\mathrm{Div}(X) := Z^1(X)$.

If C is a closed subscheme of a locally Noetherian scheme X, we can associated to C the cycle $[C] := \sum \mathrm{length}(\mathcal{O}_{C,\xi_C})\overline{\{\xi_C\}}$, where the sum ranges on all the generic points ξ_C of C, and the natural number $\mathrm{length}(\mathcal{O}_{C,\xi_C})$ is named the *multiplicity* of $\overline{\{\xi_C\}}$ in C.

Proposition 1.1.11. If X is a locally Noetherian normal scheme, the operator $[\cdot]$ establishes an isomorphism between the monoid of closed subschemes of X having pure codimension 1 and the monoid $\mathrm{Div}^+(X)$ of effective Weyl divisors on X.

Proof. Normality ensures that the local rings of X at its codimension 1 points are discrete valuation rings. From this, the proposition easily follows. □

As a consequence of Propositions 1.1.1 and 1.1.11, we have that, on locally Noetherian normal schemes, effective Cartier divisors form a submonoid of effective Weyl divisors:

$$\{\text{eff. Cartier div.}\} \hookrightarrow \{\text{closed sub. of pure cod. } 1\} \xrightarrow{\sim} \{\text{eff. Weyl divisors}\}.$$

On a regular scheme, we also know that all closed subschemes of pure codimension 1 are effective Cartier divisors (see Proposition 1.1.1), and hence effective

Weyl and effective Cartier divisors will coincide:

$$\{\text{eff. Cartier div.}\} = \{\text{closed sub. of pure cod. } 1\} \xrightarrow{\sim} \{\text{eff. Weyl divisors}\}.$$

We have discussed how an effective Cartier divisor Z on a scheme X gives rise to an invertible sheaf $\mathcal{O}_X(Z) \in \operatorname{Pic}(X)$ (Subsection 1.1.2); if X is a regular scheme, we can extend by linearity this correspondence and turn it into a morphism of abelian groups $\operatorname{Div}(X) \to \operatorname{Pic}(X)$. It is possible to prove that

Proposition 1.1.12. If X is a regular scheme, every invertible sheaf on X arises from a (possibly non-effective) divisor; in other words, the canonical morphism $\operatorname{Div}(X) \to \operatorname{Pic}(X)$ is surjective.

Proof. This is a classical result; it can be seen, for example, as a consequence of [Liu, Corollary 7.1.19]. □

1.1.6. Pushforward of cycles

If $f : X \to S$ is a proper morphism of locally Noetherian schemes, we can define a graded pushforward operator $Z_\bullet(X) \to Z_\bullet(S)$ as follows. Take $\overline{\{x\}} \subseteq X$ a d-dimensional prime cycle, and let $s \in S$ denote the image of x through f. There are two cases:

A) if $\overline{\{x\}}$ has still dimension d, then $\overline{\{x\}} \to \overline{\{s\}}$ is a finite surjective morphism of schemes, inducing a finite extension of fields $k(x) \subseteq k(s)$: we define the pushforward of the cycle $\overline{\{x\}}$ as $f_*(\overline{\{x\}}) := [k(x) : k(s)]\overline{\{s\}}$.

B) if $\overline{\{s\}}$ has dimension lower than d, then we set $f_*(\overline{\{x\}}) = 0$, instead.

Extending this assignement by $\mathbb{Z}$-linearity clearly defines a pushforward operator $f_* : Z_\bullet(X) \to Z_\bullet(S)$. A particularly simple and important case is the one of zero-cycles: if we take a 0-cycle $Z := \sum_i n_i x_i$ on X (where $n_i \in \mathbb{Z}$ and x_i are closed points of X), and x_i lies over $s_i \in S$, then $f_*(Z) = \sum_i n_i[k(x_i) : k(s_i)]s_i$. If S is the spectrum of a field, so all closed points of X lie over the unique point $\star \in \operatorname{Spec}(k)$, then $f_*(Z)$ is just an integer, which is named the *degree* of Z.

1.1.7. Existence results for divisors

Given a locally Noetherian scheme, it can generally be an important but not always easy task to prove existence of divisors satisfying constraints such that having some prescribed support, passing through a point, avoiding a point, etc. Later on, we will present multiple results of this kind for curves and surfaces; in this subsection, we prove an easy proposition, valid in general for any locally Noetherian scheme, that will turn out later to be useful.

Proposition 1.1.13. Given x a non-associated point of codimension ≥ 2 of a locally Noetherian scheme X, then (a) there exists a neighborhood U of x and an effective Cartier divisor D on U passing through x and avoiding any given finite set of prime Weyl divisors passing through x; and (b) there exists an infinite number of prime Weyl divisors passing through x.

Proof. Let us prove (a). Let $D_1, \ldots, D_n$ denote the Weyl divisors we want to avoid and let us denote by $\mathfrak{p}_i$ the corresponding primes in the local ring $\mathcal{O}_{X,x}$. Since x has codimension ≥ 2, none of them will coincide with $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$. Let us also denote by $\mathfrak{q}_i$ the associated primes of $\mathcal{O}_{X,x}$; since x is not an associated point, none of them will coincide with $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$.

By prime avoidance, $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ cannot be the union of a finite number of smaller primes. Consequently, we can find an element $f_x \in \mathfrak{m}_x$ outside the union of all the $\mathfrak{p}_i$ and $\mathfrak{q}_i$. Our $f_x \in \mathfrak{m}_x$ will clearly extend to some regular f defined over a neighborhood of $V \ni x$: the effective Cartier divisor $V(f)$ satisfies all the requirements that point (a) of our proposition imposes.

Let us now take any irreducible component D of $V(f)$ passing through x: it will be a prime Weyl divisor different from all the D_i's. This is enough to ensure that an infinite number of Weyl divisors passing through x exist; hence, we have also proved (b). $\square$

1.2. Morphisms to the projective space

It is a well-known fact that a linear system $V \subseteq |D|$ on a complete variety X over a field k induces a rational map $X \dashrightarrow \mathbb{P}(V)^\vee$ (where $\mathbb{P}(V)^\vee$ is the dual projectivization of the vector space V), associating to every point $x \in X$ outside the base locus the hyperplane of V consisting of all divisors passing through x; if V is base-point free, this rational map is actually a morphism from X to a projective variety. In this section, we will seek present a generalization of this kind of result, adopting the language of line bundles and Proj constructions. To begin, we will describe morphisms having a Proj as a target, and then we will see how a line bundle can induce such a morphism.

1.2.1. The homogeneous spectrum of an algebra

Given S a scheme and $\mathcal{A}_\bullet$ a quasi-coherent $\mathcal{O}_S$-algebra, it is possible to build the relative homogeneous spectrum of $\mathcal{A}_\bullet$, which we will denote by $p : \mathrm{Proj}_S(\mathcal{A}_\bullet) \to S$. Suppose now we are given another S-scheme $f : X \to S$, which we will suppose quasi-compact and quasi-separated over S, together with an invertible sheaf $\mathcal{L}$. We can build, out of $\mathcal{L}$, the $\mathcal{O}_X$-algebra $\mathcal{L}_\bullet := \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L})$ of tensor powers of $\mathcal{L}$ (i.e. $\mathcal{L}_n := \mathcal{L}^n$). We will now show that:

Proposition 1.2.1. Any given morphism of $\mathcal{O}_S$-graded algebras $\varphi : \mathcal{A}_\bullet \to f_*(\mathcal{L}_\bullet)$ induces an S-morphism from an open subscheme $U \subseteq X$ to $\mathrm{Proj}_S(\mathcal{A}_\bullet)$.

Proof. Suppose, for simplicity, that $S = \mathrm{Spec}(R)$ is affine. Our starting point is the graded R-algebra morphism $\varphi : \mathcal{A}_\bullet \to H^0(\mathcal{L}_\bullet)$: if we take an homogeneous element $\beta \in \mathcal{A}_n$ $(n \geq 1)$, its image $\varphi(\beta)$ will be a global section of $\mathcal{L}^n$, and we can consider the following.

1. The affine open subset $D_+(\beta) \subseteq \mathrm{Proj}_S(\mathcal{A}_\bullet)$, which is simply the specturm of the homogeneous localization of A at β: $D_+(\beta) = \mathrm{Spec}((A_\bullet)_{(\beta)})$.

2. The complement of the zero-locus of $\varphi(\beta)$ on X, which is an open subset $X_{\varphi(\beta)} \subseteq X$. The global functions on $X_{\varphi(\beta)}$ can be canonically identified with $H^0(\mathcal{L}_\bullet)_{(\varphi(\beta))}$: in fact, there is an obvious canonical map $H^0(\mathcal{L}_\bullet)_{(\varphi(\beta))} \to H^0(X_{\varphi(\beta)})$, that can be showed to be an isomorphism thanks to the quasi-compactness and quasi-separatedness of X (see [Stacks, `01PW`] for details).

By localizing $\varphi : A_\bullet \to H^0(\mathcal{L}_\bullet)$, we obtain an R-algebra homomorphism $\varphi_{(\beta)} : (A_\bullet)_{(\beta)} \to H^0(\mathcal{L}_\bullet)_{(\beta)}$, which corresponds to some morphism of S-schemes $f_\beta : X_{\varphi(\beta)} \to D_+(\beta)$. It is clear that, as β varies, the targets $D_+(\beta)$ will cover the whole $\mathrm{Proj}_S(\mathcal{A}_\bullet)$, and the morphisms f_β will glue together giving an S-scheme homomorphism $f : X \supseteq U \to \mathrm{Proj}_S(\mathcal{A}_\bullet)$, where $U = \cup_\beta X_{\varphi(\beta)}$, such that, for every β, the restriction of f to $f^{-1}(D_+(\beta)) \to D_+(\beta)$ coincides with f_β.

Should S not be affine, we can carry out the construction described above affine-locally on the basis, and glue the resulting morphisms of schemes: we will still end up with the desired S-morphism $X \supseteq U \to \mathrm{Proj}_S(\mathcal{A}_\bullet)$ induced by φ. □

1.2.2. Line bundles and morphisms to the projective space

Given $f : X \to S$ a quasi-compact, quasi-separated S-scheme, and $\mathcal{L}$ an invertible sheaf on X, the construction illustrated in the proof of Proposition 1.2.1 above gives a canonical morphism of S-schemes $X \supseteq U \to \mathrm{Proj}_S(f_*\mathcal{L}_\bullet)$, corresponding to the identity $\mathrm{id} : f_*\mathcal{L}_\bullet \to f_*\mathcal{L}_\bullet$: we will denote it as $\psi_{\mathcal{L}}$.

Suppose now just for simplicity that S is affine, and let us use the letter β to denote a global section of some a tensor power $\mathcal{L}^n$ of $\mathcal{L}$ ($n \geq 1$). The morphism $\psi_{\mathcal{L}}$ can be described as follows:

- The domain of definition of $\psi_{\mathcal{L}}$ is $U = \cup_\beta X_\beta$.
- $\psi_{\mathcal{L}}^{-1}(D_+(\beta)) = X_\beta$, and the restriction of $\psi_{\mathcal{L}}$ to X_β is nothing but the canonical morphism mapping X_β to its affinization $\mathrm{Spec}(H^0(X_\beta)) = D_+(\beta) \subseteq \mathrm{Proj}_S(f_*\mathcal{L}_\bullet)$; in particular, $\psi_{\mathcal{L}}$ is $\mathcal{O}$-connected (see Definition 2.2.8) and consequently has dense image.
- $\psi_{\mathcal{L}}$ separates two points $x_1, x_2 \in X$ whenever there exists some β which is able to distinguish between them, i.e. some β such that $\beta(x_1) \neq 0$ and $\beta(x_2) = 0$ or vice versa.

A further remarkable property of the construction of $\psi_{\mathcal{L}}$ is that it is invariant under the replacement of $\mathcal{L}$ with some tensor power $\mathcal{L}^N$ (for any $N \geq 1$), in the sense that:

Proposition 1.2.2. If $\mathcal{L}' := \mathcal{L}^N$, then the domains $U \subseteq X$ and $U' \subseteq X$ of $\psi_{\mathcal{L}}$ and $\psi_{\mathcal{L}'}$ coincide, and exists a canonical isomorphism $h : \mathrm{Proj}_S(f_*\mathcal{L}_\bullet) \to \mathrm{Proj}_S(f_*\mathcal{L}'_\bullet)$ such that $h \circ \psi_{\mathcal{L}} = \psi_{\mathcal{L}'}$.

Proof. It is a well-known fact that, given a scheme S and a quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{S}_\bullet$, if we define $\mathcal{S}_\bullet^{(N)}$ by $(\mathcal{S}^{(N)})_n := \mathcal{S}_{Nn}$, then $\mathrm{Proj}_S(\mathcal{S}_\bullet)$ and $\mathrm{Proj}_S(\mathcal{S}_\bullet^{(N)})$ are canonically isomorphic as S-schemes: from this, and from the explicit form of the isomorphism in question, it is not difficult to prove the proposition. As a reference, see, for example, the material in [Stacks, `01NS`]. □

Two crucial properties of line bundles, i.e. semiampleness and ampleness, can be easily defined by means of the morphism $\psi_{\mathcal{L}}$ that we have introduced:

Definition 1.2.3. Let $X \to S$ be a quasi-compact, quasi-separated morphism of schemes. We say that a line bundle $\mathcal{L}$ on X is semiample (relative to f) if $\psi_{\mathcal{L}} : X \supseteq U \to \mathrm{Proj}_S(f_*\mathcal{L}_\bullet)$ is everywhere defined on X (i.e. $U = X$). We say that $\mathcal{L}$ is ample (relative to f) if, moreover, $\psi_{\mathcal{L}}$ is an open immersion ([Stacks, `01VJ`]).

Example 1.2.4. Let S be a locally Noetherian scheme and $\mathcal{A}_\bullet$ be a quasi-coherent, finite type graded $\mathcal{O}_S$-algebra. Suppose that $\mathcal{A}_\bullet$ is generated over $\mathcal{O}_S$ by its elements of degree 1, meaning that $\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{A}_1) \to \mathcal{A}_\bullet$ is an epimorphism of sheaves on S. Then, it is not difficult to verify that $\mathcal{O}(1)$ is an S-ample sheaf on $X := \mathrm{Proj}_S(\mathcal{A}_\bullet)$ – which is, moreover, a proper S-scheme by [Stacks, `0800`].

Remark 1.2.5. If S is quasi-compact, and $f : X \to S$ is a quasi-compact, quasi-separated S-scheme, that an invertible sheaf $\mathcal{L}$ on X is f-semiample simply means that some tensor power of it is f-globally generated, i.e. that the counit $f^* f_*(\mathcal{L}^n) \to \mathcal{L}^n$ is an epimorphism for $n \gg 0$.

Proposition 1.2.6. If S is locally Noetherian, $f : X \to S$ is proper and $\mathcal{L}$ is f-semiample, then the quasi-coherent $\mathcal{O}_S$-algebra $f_*\mathcal{L}_\bullet$ is of finite type, and $\mathrm{Proj}_S(f_*\mathcal{L}_\bullet)$ is consequently a proper S-scheme.

Proof. We lose no generality in supposing $\mathcal{L}$ globally generated (relative to f). Let us denote by $\mathcal{S}_\bullet := \mathrm{Sym}_{\mathcal{O}_S}(f_*\mathcal{L})$ the symmetric $\mathcal{O}_S$-algebra on $f_*\mathcal{L}$: as $f_*\mathcal{L}$ is a coherent $\mathcal{O}_S$-module, $\mathcal{S}$ is a quasi-coherent $\mathcal{O}_S$-algebra of finite type. As $\mathcal{L}$ is f-globally generated, we have that $f^*(\mathcal{S}_\bullet) \to \mathcal{L}_\bullet$ is an epimorphism of quasi-coherent $\mathcal{O}_X$-algebras; in particular, $\mathcal{L}_\bullet$ is a finite type, quasi-coherent $f^*(\mathcal{S}_\bullet)$-module. Hence, we may apply [EGAIII.1, Proposition 3.3.1, p. 118] and get that $f_*(\mathcal{L}_\bullet)$ is a finite-type graded module on $\mathcal{S}_\bullet$; from this, it follows easily that $f_*(\mathcal{L}_\bullet)$ will also be of finite type as a quasi-coherent $\mathcal{O}_S$-algebra. As a consequence, $\mathrm{Proj}_S(f_*\mathcal{L}_\bullet)$ is certainly a proper S-scheme (by [Stacks, 0800]). □

As a consequence of proposition above, if we suppose that S is locally Noetherian and X is a proper S-scheme, we have that a line bundle $\mathcal{L}$ on X is ample if and only if $\psi_\mathcal{L}$ is everywhere defined on X and it is an isomorphism. Hence, we can say that

Definition/Proposition 1.2.7. Given a morphism of schemes $f : X \to S$, with S Noetherian, the following are equivalent: (a) $X \to S$ is proper, and there exists an S-ample invertible sheaf on X; (b) X is of the form $\mathrm{Proj}_S(\mathcal{A}_\bullet)$ for some quasi-coherent, finite-type graded $\mathcal{O}_S$-algebra $\mathcal{A}_\bullet$. If the equivalent conditions (a) and (b) hold, we say that X is a *projective* S-scheme.

Proof. Suppose (a) holds; then, we can pick an ample invertible sheaf $\mathcal{L}$ on $\mathcal{X}$, and $\psi_\mathcal{L}$ will provide an isomorphism between $\mathcal{X}$ and $\mathrm{Proj}_S(\mathcal{A}_\bullet)$, where $\mathcal{A}_\bullet := f_*\mathcal{L}_\bullet$ is a quasi-coherent, finite-type graded $\mathcal{O}_S$-algebra (Proposition 1.2.6).

Suppose (b) holds. First, we observe that replacing $\mathcal{A}_\bullet$ with $\mathcal{A}_\bullet^{(N)}$ for some $N > 0$, where $\mathcal{A}_\bullet^{(N)}$ is defined as $\mathcal{A}_n^{(N)} := \mathcal{A}_{Nn}$, does not alter the homogeneous spectrum: $\mathrm{Proj}_S(\mathcal{A}_\bullet) = \mathrm{Proj}_S(\mathcal{A}_\bullet^{(N)})$. Exploiting this fact and the quasi-compactness of S, we easily obtain that no generality is lost if we suppose $\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{A}_1) \to \mathcal{A}_\bullet$ surjective. Hence, we are reduce to the setting of Example 1.2.4, and we obtain that X is a proper scheme bearing an S-ample invertible sheaf. □

Another important result on ampleness of invertible sheaves on proper schemes is the following one.

Proposition 1.2.8. If $f : X \to S$ is a proper morphism of locally Noetherian schemes, and $\mathcal{L}$ is an invertible sheaf on X, then the following are equivalent:

(a) $\mathcal{L}_{|X_s}$ is relatively ample on the fiber $X_s \to \mathrm{Spec}(k(s))$;

(b) there exists an open neighborhood $s \in V \subseteq X$ such that $\mathcal{L}_{|f^{-1}(V)}$ is relatively ample on $f^{-1}(V) \to V$.

Proof. See [EGAIII.1, Théorème 4.7.1, p. 145]. □

1.3. The canonical bundle

If $f : X \to S$ is a smooth proper morphism of locally Noetherian schemes of relative dimension r, then the sheaf $\Omega_{X/S} \in \mathrm{Coh}(X)$ of relative differentials is locally free of rank r, and its top exterior power $\det(\Omega_{X/S}) = \wedge^r \Omega_{X/S}$ is an invertible sheaf, which is named the *canonical bundle* of X over S and can have a crucial role in the study of the geometry of X. The canonical bundle will also be a major ingredient of our discussion on arithmetic surfaces; in particular, we will study its behaviour with respect to the intersection form and derive the extremely useful *adjunction formula*. However, since most of the surfaces we will work with will not be smooth, a more general definition of the canonical bundle is necessary to obtain such a result.

If the proper morphism $f : X \to S$ is not smooth, a canonical bundle cannot in general be defined; anyway, a more sophisticated object, known as the *relative dualizing complex* $\omega^\bullet_{X/S}$, does always exists, and will be an object of the derived category $D^+(\mathrm{QCoh}(X))$. Under some favorable conditions, $\omega^\bullet_{X/S}$ is actually a quasi-coherent sheaf on X, that we will name the *canonical sheaf* of X and denote by $\omega_{X/S}$; under further hypotheses, we can guarantee that $\omega_{X/S}$ is invertible and consequently name it the *canonical bundle* of X relative to S.

The theory underlying these constructions, known as *Grothendieck duality*, requires an extensive use of advanced homological techniques, and it would be too ambitious presenting it here. We will anyway spend some words to illustrate, without proofs and without even recalling all necessary definitions, some of the results that are necessary to introduce and manipulate dualizing complexes, so we will later be able to present a proof of the adjunction formula for arithmetic surfaces.

If $f : X \to S$ is a proper morphism of locally Noetherian schemes, then the total derived direct image functor $Rf_* : D^+(\mathrm{QCoh}(X)) \to D^+(\mathrm{QCoh}(S))$ admits a right adjoint, which we will denote by $f^!$.

Example 1.3.1. If X is a locally Noetherian scheme and $i : Z \hookrightarrow X$ is a regular closed immersion of codimension d, let us denote by $\mathcal{Z}$ the ideal sheaf defining Z

in X: by definition of regular immersion, $\mathcal{Z}_{|Z}$ is a locally free sheaf of rank r on Z. Then, it is possible to prove that, for any $\mathcal{F}^\bullet \in D^+(\mathrm{QCoh}(X))$, we have that $i^!(\mathcal{F}^\bullet) = Li^*\mathcal{F}^\bullet[-d] \otimes_{\mathcal{O}_Z} \det(\mathcal{Z})^\vee_{|Z}$. In particular, if $\mathcal{L}$ is an invertible sheaf, then $i^!(\mathcal{L}) = (\mathcal{L} \otimes_{\mathcal{O}_X} \det(\mathcal{Z})^\vee)_{|Z}[-d]$. See [Stacks, `0BR0`] and [Stacks, `0B6U`].

The complex $\omega^\bullet_{X/S} := f^!\mathcal{O}_S \in D^+(\mathrm{QCoh}(X))$ is named the *relative dualizing complex* for the morphism $X \to S$. If there exists a coherent sheaf $\omega_{X/S}$ such that $\omega^\bullet_{X/S} \cong \omega_{X/S}[n]$ for some $n \in \mathbb{Z}$, then we say that X *admits a canonical sheaf* over S, and $\omega_{X/S}$, which can be recovered as $\omega_{X/S} = H^{-n}(\omega^\bullet_{X/S})$ and is consequently uniquely determined up to isomorphism, is named the *canonical sheaf* of X over S. Two important cases in which the canonical sheaf exists and has some good properties are the following ones.

- If $i : Z \to X$ is a regular closed immersion of codimension d of locally Noetherian schemes, then it admits an invertible canonical sheaf, which is given by $\omega_{Z/X} = \det(\mathcal{Z}_{|Z})^\vee$, and $\omega^\bullet_{Z/X} = \omega_{Z/X}[-d]$. Moreover, $\omega_{Z/X}$ represents the functor $\mathrm{QCoh}(Z)^{\mathrm{op}} \to \mathrm{Set}$, $\mathcal{F} \mapsto \mathrm{Ext}^d(i_*\mathcal{F}, \mathcal{O}_X)$.

- If $f : X \to S$ is a flat Cohen-Macaulay morphism of relative dimension d, then it admits a canonical sheaf $\omega_{X/S}$, and $\omega^\bullet_{X/S} = \omega_{X/S}[d]$. Moreover, $\omega_{X/S}$ represents the functor $\mathrm{QCoh}(X)^{\mathrm{op}} \to \mathrm{Set}$, $\mathcal{F} \mapsto \mathrm{Hom}(R^d f_*\mathcal{F}, \mathcal{O}_S)$. If f is further supposed to be Gorenstein (e.g., if X is regular), then $\omega_{X/S}$ is invertible.

2. Birationality

Two schemes are said to be *birational* if they are generically isomorphic. Birationality is an equivalence relation, and within each of its equivalence classes there is a natural way to define a preorder relation, which goes under the name of *dominance*. Birationality will be the exclusive subject of Section 2.1; the rest of this chapter (Sections 2.2 and 2.3), instead, discusses a number of important results and techniques which are not directly related to this notion: most of them, however, turn out to have fruitful applications in the realm of birational geometry.

2.1. Birational morphisms and birational equivalences

2.1.1. The scheme of generic points

Given X a scheme, we will denote by $X^0 \to X$ the *scheme of generic points* of X, i.e.

$$X^0 := \coprod_{\eta \text{ generic point of } X} \operatorname{Spec}(\mathcal{O}_{X,\eta}).$$

The morphism $X^0 \hookrightarrow X$ is quasi-compact if and only if the number of irreducible components of X is locally finite (i.e. each point of X has a neighborhood containing only finitely many generic points of X). Under this condition, which we will always assume to be true, it is not difficult to see that $i : X^0 \to X$ is affine, it is the inverse limit of the inclusions of all dense open subschemes of X, and it is a topological immersion (i.e., the topology on X^0 is the subspace topology).

In the particular case in which X is locally Noetherian without embedded points, $i : X^0 \to X$ has a particularly simple description: for every affine open subscheme $\operatorname{Spec}(A_i)$ of X, it is given by the localization $\operatorname{Spec}(Q(A_i)) \to \operatorname{Spec}(A_i)$, where $Q(A_i)$ is the total ring of fractions of A_i.

2.1.2. Birational morphisms

A morphism of schemes $f : X \to Y$ is said to be *birational* if X and Y locally have only a finite number of irreducible components and f sends the generic points of X to the generic points of Y inducing an isomorphism $X^0 \xrightarrow{\sim} Y^0$ (see [Stacks, `01RO`]).

Example 2.1.1. Suppose that X has locally a finite number of irreducible components and $Z \subseteq X$ is a closed subscheme not containing any irreducible component of X. Then, the blowup $X' \to X$ of X along Z is a birational morphism.

It is not difficult to deduce from the definition that a birational morphism is always an isomorphism above Y^0:

Proposition 2.1.2. If $f : X \to Y$ is a birational morphism, f is an isomorphism above $\mathrm{Spec}(\mathcal{O}_{Y,\eta})$ for every $\eta \in Y^0$.

Proof. Let us consider the base change $f' : X' \to Y'$ of f, where $Y' := \mathrm{Spec}(\mathcal{O}_{Y,\eta})$: it will still obviously be a birational morphism, and this implies that X' has a unique generic point $\xi \in X'$, and that the composition $\mathrm{Spec}(\mathcal{O}_{X,\xi}) \to X' \to \mathrm{Spec}(\mathcal{O}_{Y,\eta})$ is an isomorphism. In particular, this implies that $\mathrm{Spec}(\mathcal{O}_{X,\xi}) \to X'$ is a split monomorphism, and hence, for every point $x \in X'$, the localization $\mathrm{Spec}(\mathcal{O}_{X,\xi}) \to \mathrm{Spec}(\mathcal{O}_{X,x})$ will also be a split monomorphism of affine schemes, and consequently a closed immersion. But this immediately forces $x = \xi$; hence, the only point of X' is ξ, and f' is consequently an isomorphism. □

A stronger conclusion holds under the additional hypothesis that f is of finite presentation:

Proposition 2.1.3. Let $f : X \to Y$ be a birational morphism of finite presentation. Then, f is an isomorphism over some dense open subscheme $V \subseteq Y$, and the maximal V satisfying such property can be characterized as $V = \{y \in Y : f \text{ is an isomorphism over } \mathrm{Spec}(\mathcal{O}_{Y,y})\}$.

Proof. This can be deduced from the proposition above by means of a standard "spreading out" argument, that we defer to the appendix (Corollary B.0.4). □

Convention 2.1.4. We will denote by $\mathrm{E}(f)$ the complement of the maximal $V \subseteq Y$ over which the birational morphism of finite presentation $f : X \to Y$ is an isomorphism; in other words, $\mathrm{E}(f)$ is the closed subscheme of Y over which $f : X \to Y$ is not an isomorphism.

2.1.3. Birational equivalences and their graphs

Given S a scheme, and X_1, X_2 two S-schemes, we will say they are *birational* if the number of their irreducible components is locally finite and X_1^0 and X_2^0 are S-isomorphic. For the rest of our discussion, we will fix a distinguished isomorphism $X_1^0 \cong X_2^0$, which we will name a *birational equivalence*, and we will denote X_1^0 and X_2^0, which are now identified, using the symbol X^0.

Definition 2.1.5. The *graph* Γ of the birational equivalence between X_1 and X_2 is the scheme-theoretic image of the canonical morphism $j : X^0 \to X_1 \times_S X_2$.

Since $X^0 \hookrightarrow X_i$ are both quasi-compact, $j : X^0 \to X_1 \times_S X_2$ is easily seen to also be quasi-compact; hence, $\Gamma \subseteq X_1 \times_S X_2$ coincides, set-theoretically, with the Zariski-closure of the set-theoretic image of j (see [Stacks, `01R8`]). The canonical morphism $X^0 \to \Gamma$ will also be quasi-compact, and it is not difficult to see that, by construction, it provides precisely the scheme of generic points of Γ: each of the canonical projections $p_i : \Gamma \to X_i$ is consequently a birational morphism. In this way, a birational equivalence between two schemes can be seen as a couple of birational morphisms coming from a common dominating scheme, and this simple observation will allow us to essentially reduce the study of birational equivalences to that of birational morphisms.

2.1.4. Extending birational equivalences

Let S be a scheme, and suppose that X_1 and X_2 are two S-schemes having a locally finite number of irreducible components. It is clear that any densely defined birational S-morphism $f : X_1 \supseteq V_1 \to X_2$ induces a birational equivalence between X_1 and X_2 over S: we will say that f *extends* (or *represents*) the birational equivalence in question, and that the birational equivalence *arises from* f. In general, it is not guaranteed that a given birational equivalence $X_1^0 \cong X_2^0$ will admit such an extension, nor that two extensions agree on their common locus of definition. To get results of this kind, we have to add some hypotheses.

Existence of extensions Let us first discuss some existence result: let us suppose that S is locally Noetherian, and that X_1 and X_2 are of finite type and birational over S. Under these hypotheses, the two projections $p_i : \Gamma \to X_i$ of the graph Γ of the birational equivalence to X_1 and X_2 also become of finite presentation, and we can exploit Proposition 2.1.3 to find two dense open subsets $U_i \subseteq X_i$ over which p_i are isomorphisms – and we will preferably choose U_i maximal with respect to this property, i.e. $U_i := X_i \smallsetminus \mathrm{E}(p_i)$. In particular,

Proposition 2.1.6. In the setting above (X_1 and X_2 of finite type over a locally Noetherian basis S), a birational equivalence $X_1^0 \cong X_2^0$ induces:

a) two densely-defined morphisms $p_2 p_1^{-1} : X_1 \supseteq U_1 \to X_2$ and $p_1 p_2^{-1} : X_2 \supseteq U_2 \to X_1$ extending the birational equivalence;
b) an isomorphism between the two (smaller) dense open subschemes $U_1' \subseteq X_1$ and $U_2' \subseteq X_2$, where $U_i' := p_i(p_1^{-1}(U_1) \cap p_2^{-1}(U_2))$.

Uniqueness of extensions Let us now prove that, under some reasonable hypotheses, two extensions of a given birational equivalence always agree.

Proposition 2.1.7. If X_1 has no embedded points and X_2 is a separated S-scheme, then, given any extension $f : X_1 \supseteq V_1 \to X_2$ of the birational equivalence, we have that p_1 is an isomorphism above V_1, and that f coincides with $p_2 p_1^{-1}$ on V_1, where $p_i : \Gamma \to X_i$ are the projections of the graph Γ of the birational equivalence to X_1 and X_2.

Proof. Since X_2 is separated, $V_1 \to V_1 \times_S X_2$ is a closed immersion: hence, if we denote by Γ_f the scheme-theoretic image of $V_1 \to X_1 \times_S X_2$, we have that the projection $X_1 \times_S X_2 \to X_1$ restricts to an isomorphism $\Gamma_f \cap (V_1 \times X_2) \xrightarrow{\sim} V_1$. Since X_1 has no embedded points, X^0 is scheme-theoretically dense in X_1 (and hence in V_1); in particular, the scheme theoretic images of $X^0 \to X_1 \times_S X_2$ and $V_1 \to X_1 \times_S X_2$ coincide; in other words, $\Gamma = \Gamma_f$ as closed subschemes of $X_1 \times_S X_2$.

These two observations together clearly imply that $p_1 : \Gamma \to X_1$ is an isomorphism above $V_1 \subseteq X_1$, and that f agrees with $p_2 p_1^{-1}$. □

2.1.5. Birational classes of schemes

In the following sections and chapters, will often speak about birational schemes using the language of categories:

Definition 2.1.8. Given a scheme S, a *birational class of S-schemes* will be a category $\mathcal{A}$ such that:

- the objects of $\mathcal{A}$ are S-schemes having locally finitely many irreducible components, and all sharing the scheme of generic points X^0;
- given two objects $X_1, X_2 \in \mathrm{Ob}(\mathcal{A})$, a morphism $f \in \mathrm{Hom}_{\mathcal{A}}(X_1, X_2)$ is a birational morphism $f : X_1 \to X_2$ of S-schemes extending the birational equivalence between X_1 and X_2 (i.e., inducing the identity on X^0).

Let $\mathcal{A}$ be a birational class of S-schemes. The uniqueness result we have discussed in the previous subsection implies that:

Proposition 2.1.9. If all objects of $\mathcal{A}$ are separated S-schemes without embedded points, then $\mathcal{A}$ is a preorder, in the sense that its Hom-sets all consist of at most one element.

In the hypotheses of the proposition above, the name we give to the preorder relation that the category $\mathcal{A}$ incarnates is *dominance*: given $X_1, X_2 \in \mathcal{A}$, that X_1 dominates X_2 means that the S-birational equivalence $X_1^0 = X_2^0$ extends (uniquely) to a birational S-morphism $X_1 \to X_2$. In particular, X_1 and X_2 mutually dominate each other if and only if their birational equivalence extends (uniquely) to an isomorphism of S-schemes $X_1 \cong X_2$.

2.2. Formal methods

Given a morphism of locally Noetherian schemes $f : X \to S$, and a point $s \in S$, we are interested in studying f near s. We can get near to s in many senses: we can, for example, restrict f to (1) a Zariski neighborhood of s, (2) the spectrum of the local ring $\mathcal{O}_{S,s}$, (3) the spectrum of the complete local ring $\widehat{\mathcal{O}}_{S,s}$, (4) the spectrum of the residue field $k(s)$. Steps (1), (2) and (3) consist of subsequent flat base changes, which do not significantly alter the S-scheme X; in particular, cohomological computations always commute with flat base changes. Step (4), instead, involves a non-flat base-change, and hence there is no easy way to transfer information (e.g. regularity properties or the cohomology of a coherent sheaf) between X and a fiber.

To make the jump between (3) and (4) more gradual, one can introduce the so-called *thickened fibers*, which are defined as $X_{ns} := X \times \operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^n)$: for $n = 1$, we recover the usual fiber $X_{1s} = X_s$ and, as n grows, X_{ns} gives increasingly better approximations of $X \times_S \operatorname{Spec}(\widehat{\mathcal{O}}_{S,s})$. In this section, we discuss how to exploit this approximation process to relate properties of $X \times_S \operatorname{Spec}(\widehat{\mathcal{O}}_{S,s})$ to properties of the fiber X_s and of the thickened fibers X_{ns}. We will first introduce two main theoretical results (namely, the theorems on formal functions and algebrization of sheaves by Grothendieck), and then see some important applications to the study of birational morphisms (namely, Zariski's Main Theorem).

2.2.1. Formal function theorem

Let $f : X \to S$ a proper scheme over the spectrum S of a complete Noetherian local ring $(A, \mathfrak{m})$. Let $P_k = \operatorname{Spec}(A/\mathfrak{m}^k)$ be the k-th thickened closed point of A, and $X_k \to P_k$ the k-th thickened special fiber of X. Given $\mathcal{F}$ a coherent sheaf on X, we clearly have canonical morphisms of finite A-modules $H^i(\mathcal{F}) \to H^i(\mathcal{F}_{|X_k})$: the *theorem on formal functions* ensures that they become isomorphisms in the limit $k \to \infty$.

Theorem 2.2.1. The canonical A-module homomorphism $H^i(\mathcal{F}) \to \varprojlim_k H^i(\mathcal{F}_{|X_k})$ is an isomorphism.

Proof. We will begin by writing down the cokernel sequence for each of the maps $r_{i,k} : H^i(\mathcal{F}) \to H^i(\mathcal{F}_k)$, where $\mathcal{F}_k := \mathcal{F}_{|X_k}$:

$$0 \to H^i(\mathcal{F})/\ker(r_{i,k}) \to H^i(\mathcal{F}_k) \to H^i(\mathcal{F}_k)/r_{i,k}(H^i(\mathcal{F})) \to 0$$

As the inverse system on the left has surjective transition functions, it satisfies the Mittag-Leffler condition; hence, passing to the inverse limits preserves exactness (see [Stacks, 0598]):

$$0 \to \varprojlim_k H^i(\mathcal{F})/\ker(r_{i,k}) \to \varprojlim_k H^i(\mathcal{F}_k) \to \varprojlim_k H^i(\mathcal{F}_k)/r_{i,k}(H^i(\mathcal{F})) \to 0$$

We will now prove that,

(1) as k grows, $\ker(r_{i,k}) \subseteq H^i(\mathcal{F})$ tends to 0 in the $\mathfrak{m}$-adic topology; hence, the left term $\varprojlim_k H^i(\mathcal{F})/\ker(r_{i,k})$ is simply $H^i(\mathcal{F})$;

(2) the transition map from the k-th module to the $(k+r)$-th module of the projective system $H^i(\mathcal{F}_k)/r_{i,k}(H^i(\mathcal{F}))$ is zero for $k, r \gg 0$, so the right term $\varprojlim_k H^i(\mathcal{F}_k)/r_{i,k}(H^i(\mathcal{F}))$ is 0.

It is clear that (1) and (2) are enough to prove our theorem. To prove these two results, we will exploit the short exact sequence:

$$0 \to \mathfrak{m}^k\mathcal{F} \to \mathcal{F} \to \mathcal{F}_k \to 0.$$

If we consider the corresponding long exact cohomological sequence, we can write that:

$$M_k := \ker(r_{i,k}) = \operatorname{im}(H^i(\mathfrak{m}^k\mathcal{F}) \to H^i(\mathcal{F})),$$

and that:

$$\begin{aligned} N_k &:= H^i(\mathcal{F}_k)/r_{i,k}(H^i(\mathcal{F})) = \\ &= \ker(H^{i+1}(\mathfrak{m}^k\mathcal{F}) \to H^{i+1}(\mathcal{F})) = \operatorname{im}(H^i(\mathcal{F}_k) \to H^{i+1}(\mathfrak{m}^k\mathcal{F})). \end{aligned}$$

Proof of (1) We want to show that the images M_k of $H^i(\mathfrak{m}^k\mathcal{F})$ inside $H^i(\mathcal{F})$ tend to zero in the $\mathfrak{m}$-adic topology. For this to be true, it is enough to prove that $\oplus_k H^i(\mathfrak{m}^k\mathcal{F})$ is finite as a graded module over $R_\bullet := \oplus_k \mathfrak{m}^k$, i.e. that the cohomology groups of $\oplus_k \mathfrak{m}^k\mathcal{F}$ are finite $R_\bullet$-modules. But $\oplus_k \mathfrak{m}^k\mathcal{F}$ may be thought as the pullback of the coherent sheaf $\mathcal{F}$ to $X \times_S \operatorname{Spec}(R_\bullet)$, and we can now simply invoke the fact that the cohomology groups of any coherent sheaf on the proper $R_\bullet$-scheme $X \times_S \operatorname{Spec}(R_\bullet)$ are finite $R_\bullet$-modules.

Proof of (2) We have to show that the transition maps between the N_k's are eventually zero. First, we notice that, since N_k is an image of $H^i(\mathcal{F}_k)$, and the latter is an $A/\mathfrak{m}^k$-module, we have that $\mathfrak{m}^k N_k = 0$. Secondly, $\oplus_k N_k$ is a finitely-generated $R_\bullet$-graded module, because it is a submodule of $\oplus_k H^{i+1}(\mathfrak{m}^k \mathcal{F})$, whose finiteness has been already proved. These two observations together imply that $\oplus_k N_k$ is annihilated by the ideal $\oplus_{k \geq N} \mathfrak{m}^k \subseteq R_\bullet$ for $N \gg 0$.

Consider now the composition $N_k \otimes \mathfrak{m}^r \to N_{k+r} \to N_k$: since $\oplus_k N_k$ is a finitely-generated $R_\bullet$-graded module, the first map is surjective for all r if $k \gg 0$; but, if $r \gg 0$, $\mathfrak{m}^r$ annihilates N_k for all k, and hence the composition is zero. We deduce from this that the second map, i.e. $N_k \to N_{k+r}$, is zero for $k, r \gg 0$. □

2.2.2. Grothendieck's Existence Theorem

Let X be a proper scheme over the spectrum S of a Noetherian complete local ring $(A, \mathfrak{m})$, and let $f : X \to S$ denote the structure morphism. Given a coherent sheaf $\mathcal{F}$ on X, one can consider its restrictions $\mathcal{F}_{|X_n} = \mathcal{F} \otimes \mathcal{O}_X / \mathfrak{m}^n \mathcal{O}_X$ to the thickened closed fibers X_n of X: this gives rise to a functor $J : \operatorname{Coh}(X) \to \operatorname{Coh}(X, \mathfrak{m})$, where $\operatorname{Coh}(X, \mathfrak{m})$ is the category whose objects are families of coherent sheaves $\{\mathcal{F}_n \in \operatorname{Coh}(X_n)\}_n$ endowed with a system of isomorphisms $\mathcal{F}_{n+1|X_n} \cong \mathcal{F}_n$. It is possible to prove that

Theorem 2.2.2 (Grothendieck's Existence)**.** If X is a proper scheme over the spectrum S of a Noetherian complete local ring $(A, \mathfrak{m})$, then $J : \operatorname{Coh}(X) \to \operatorname{Coh}(X, \mathfrak{m})$ is an equivalence of categories.

We will not discuss here the essential surjectivity of J, but we will present the proof that J is fully faithful as an application of the formal function theorem. As a first step, we will describe, for an ideal I of a Noetherian ring R, the structure of the category $\operatorname{Coh}(R, I)$ whose objects are families of finite modules $\{N_n \in \operatorname{Mod}^{\text{fg}}_{R/I^n}\}_n$ endowed with a system of isomorphisms $N_{n+1}/I^n N_{n+1} \cong N_n$.

Lemma 2.2.3. $\operatorname{Coh}(R, I)$ is equivalent to the category $\operatorname{Mod}^{\text{fg}}_{\widehat{R}}$ of finite $\widehat{R}$-modules, where $\widehat{R}$ denotes the completion of R with respect to the ideal I.

Proof. The equivalence is obtained by associating to every module $M \in \operatorname{Mod}(\widehat{R})$ the inverse system $(M/I^n M)_n$, and to every inverse system $(N_n) \in \operatorname{Coh}(R, I)$ the inverse limit $\varprojlim_n N_n$. The only non-trivial point we have to verify is that, for any inverse system $(N_n) \in \mathrm{Coh}(R, I)$, the projective limit $N := \varprojlim_n N_n$ is a finitely generated as an $\widehat{R}$-module, and that the canonical morphisms $N_n \to N/I^n N$ are all isomorphisms. For a verification of these two statements, see [Stacks, 087W]. $\square$

Proposition 2.2.4. If X is a proper scheme over the spectrum S of a Noetherian complete local ring $(A, \mathfrak{m})$, then $J : \mathrm{Coh}(X) \to \mathrm{Coh}(X, \mathfrak{m})$ is fully faithful.

Proof. Given any two coherent sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$, then it is easy to notice that the assignment $U \mapsto \mathrm{Hom}_{\mathrm{Coh}(U)}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$ defines a coherent sheaf, that we will denote by $[\mathcal{F}, \mathcal{G}]$; moreover, the formation of this internal Hom can be seen to commute with the $\mathfrak{m}$-adic completion of coherent sheaves, so $\widehat{[\mathcal{F}, \mathcal{G}]} = [\widehat{\mathcal{F}}, \widehat{\mathcal{G}}]$.

For any two $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$, also the homomorphisms from $J\mathcal{F}$ to $J\mathcal{G}$ can be naturally given the nature of a sheaf on X: we will denote by $\langle F, G \rangle$ the sheaf (of sets) defined as $U \mapsto \mathrm{Hom}_{\mathrm{Coh}(U,\mathfrak{m})}(J\mathcal{F}_{|U}, J\mathcal{G}_{|U})$.

By globalizing the result we have proved in the lemma above, we immediately get an isomorphism of sheaves $\langle \mathcal{F}, \mathcal{G} \rangle \cong [\widehat{\mathcal{F}}, \widehat{\mathcal{G}}]$; at the level of global sections, we get that:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Coh}(X)}(\mathcal{F}, \mathcal{G}) = H^0([\mathcal{F}, \mathcal{G}]) = H^0(\widehat{[\mathcal{F}, \mathcal{G}]}) = H^0([\widehat{\mathcal{F}}, \widehat{\mathcal{G}}]) = \\ = H^0(\langle \mathcal{F}, \mathcal{G} \rangle) = \mathrm{Hom}_{\mathrm{Coh}(X,\mathfrak{m})}(J\mathcal{F}, J\mathcal{G}) \end{aligned}$$

where the equality $H^0([\mathcal{F}, \mathcal{G}]) = H^0(\widehat{[\mathcal{F}, \mathcal{G}]})$ immediately follows from the theorem on formal functions (Theorem 2.2.1). $\square$

2.2.3. Formal neighborhoods and the Picard group

As an immediate consequence of Grothendieck's Existence Theorem, we obtain that, for every scheme X proper over the spectrum S of a complete Noetherian local ring A, we have that

Corollary 2.2.5. $\mathrm{Pic}(X) = \varprojlim_n \mathrm{Pic}(X_n)$, where X_n denotes the n-th thickened closed fiber of $X \to S$.

We will now try to study more in detail the transition maps $\mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$ of the inverse system on the right, following mainly [Des81, proof of corollaire 3.3]. Let us start from the short exact sequence:

$$0 \to \mathcal{J}_n \to \mathcal{O}_{X_{n+1}}^\times \to \mathcal{O}_{X_n}^\times \to 0$$

where $\mathcal{J}_n := \mathfrak{m}^n\mathcal{O}_X/\mathfrak{m}^{n+1}\mathcal{O}_X$, and $\mathcal{J}_n \to \mathcal{O}_{X_{n+1}}^\times$ is the exponential map $x \mapsto 1+x$. The corresponding long exact sequence in cohomology gives:

$$\ldots \to H^0(\mathcal{O}_{X_n})^\times \to H^1(\mathcal{J}_n) \to \mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n) \to H^2(\mathcal{J}_n) \to \ldots \qquad (\dagger)$$

We remark that, by how $\mathcal{J}_n$ is defined, we have that $\mathfrak{m}\mathcal{J}_n = 0$; hence, the cohomology groups $H^p(\mathcal{J}_n)$ are k-vector spaces, where k is the residue field of A; in particular, for any integer ℓ coprime to the residue characteristic p, they are uniquely ℓ-divisible abelian groups. On the other hand, $H^0(\mathcal{O}_{X_n})$ is a finite algebra over the local Artinian ring $A/\mathfrak{m}^n$; in particular, $H^0(\mathcal{O}_{X_n})$ is itself Artinian. If we suppose that k is algebraically closed, we can be sure that $H^0(\mathcal{O}_{X_n})^\times$ is ℓ-divisible, for every ℓ prime to p: the easy argument is reported in the lemma below.

Lemma 2.2.6. Let A be an Artinian ring whose residue fields are algebraically closed. Then, $A^\times$ is an ℓ-divisible abelian group, for every ℓ prime to all residue characteristics.

Proof. We can clearly suppose A local; let $\mathfrak{m}$ be its maximal ideal and p its residue characteristic. We will show inductively that, for all n, $(A/\mathfrak{m}^n)^\times$ is ℓ-divisible for any ℓ prime to p. Since $\mathfrak{m}$ is nihilpotent, this is clearly enough to prove the lemma.

The base case $n = 1$ is immediately deduced from the fact that k is algebraically closed, and hence $k^\times$ is certainly ℓ-divisible. For every $n \geq 1$, we have the exact sequence

$$0 \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to (A/\mathfrak{m}^{n+1})^\times \to (A/\mathfrak{m}^n)^\times \to 0,$$

where $\mathfrak{m}^n/\mathfrak{m}^{n+1} \to (A/\mathfrak{m}^{n+1})^\times$ is the exponential map $x \mapsto 1 + x$. Since $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a k-vector space, it is uniquely ℓ-divisbile; hence, if $(A/\mathfrak{m}^n)^\times$ is ℓ-divisible, also $(A/\mathfrak{m}^{n+1})^\times$ will inherit the same property. □

We have now collected enough information on the objects appearing in (†) to state the following proposition:

Proposition 2.2.7. If the complete local ring A has algebraically closed residue field, then, for every ℓ prime to the residue characteristic, the canonical group homomorphism $\mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$ has the following properties: its kernel is uniquely ℓ-divisible; the multiplication by ℓ is injective on its cokernel; it consequently restricts to an isomorphism $\mathrm{Pic}(X_{n+1})[\ell] \to \mathrm{Pic}(X_n)[\ell]$ at the level of the ℓ-torsion subgroups of the two Picard groups.

In particular, for any such ℓ, the restriction morphisms $\mathrm{Pic}(X) \to \mathrm{Pic}(X_n)$ have uniquely ℓ-divisible kernels, and induce isomorphisms $\mathrm{Pic}(X)[\ell] \to \mathrm{Pic}(X_n)[\ell]$ at the level of their ℓ-torsion subgroups.

2.2.4. Zariski's Main Theorem

There are many different statements that go under the name of Zariski's Main Theorem. The version we will need is generally referred to as *Zariski's connectedness principle*, and will be crucial to describe proper birational morphisms having a normal target. Our theorem will actually apply to the wider class of proper $\mathcal{O}$-connected morphisms, where the latter condition is defined as follows:

Definition 2.2.8. A morphism of schemes $f : X \to S$ is said to be $\mathcal{O}$-connected whenever $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism.

The property of being $\mathcal{O}$-connected is clearly preserved under flat base change. Isomorphism of schemes are trivially $\mathcal{O}$-connected; conversely, an affine morphism of schemes can be $\mathcal{O}$-connected only if it is an isomorphism. Proper birational morphisms with normal target are effectively $\mathcal{O}$-connected:

Lemma 2.2.9. Let $f : X' \to X$ a proper birational morphism between integral schemes. If X is normal, f is $\mathcal{O}$-connected.

Proof. Since f is proper and birational, for every affine $U \subseteq X$ we have that the canonical morphism $\mathcal{O}_X(U) \to \mathcal{O}_{X'}(f^{-1}(U))$ is an integral extension of subdomains of the common function field $K = K(X) = K(X')$. But $\mathcal{O}_X(U)$ is normal, hence the extension is trivial. $\square$

We are now ready to state and prove Zariski's Connectedness Principle:

Theorem 2.2.10. Let $f : X \to S$ be a proper $\mathcal{O}$-connected morphism of locally Noetherian schemes. Then, the fibers of f are all connected.

Proof. This is a corollary of the formal function theorem: here is the proof in full detail. Suppose by contradiction that the fiber over some point $s \in S$ has two connected components. Replacing S with $\operatorname{Spec}(\widehat{\mathcal{O}}_{S,s})$, we may assume S is the spectrum of a complete local ring $(A, \mathfrak{m})$, with closed point s (in fact, the base change in question preserves properness and $\mathcal{O}$-connectedness, and leaves the fiber over s unchanged). Let us name $X_k \to S_k$ the k-th thickened fiber of f over the closed point $s \in S$. We know, by hypothesis, that the X_k's, which are all homeomorphic topological spaces, are disconnected. Let us fix two disconnecting open subsets U_1 and U_2; this topological decomposition corresponds to a uniquely determined couple of idempotent regular functions $e_1^{(k)}, e_2^{(k)} \in H^0(\mathcal{O}_{X_k})$ on each thickened fiber X_k. Since the topological decomposition is the same for all thickened fibers, the canonical maps $H^0(\mathcal{O}_{X_{k+1}}) \to H^0(\mathcal{O}_{X_k})$ send $e_1^{(k+1)} \mapsto e_1^{(k)}$

and $e_2^{(k+1)} \mapsto e_2^{(k)}$. Hence, $e_1 := \lim_k e_1^{(k)}$ and $e_2 := \lim_k e_2^{(k)}$ are a well-defined non-trivial pair of orthogonal idempotent elements in $\varprojlim_k H^0(\mathcal{O}_{X_k})$.

Now, by the formal function theorem, $\varprojlim_k H^0(\mathcal{O}_{X_k}) = H^0(\mathcal{O}_X)$; on the other hand, by $\mathcal{O}$-connectedness, we know that $H^0(\mathcal{O}_X) = A$: we have thus reached a contradiction, because local rings obviously always have a connected spectrum, and hence they cannot contain a non-trivial pair of orthogonal idemponent elements. □

We have seen that a proper $\mathcal{O}$-connected morphism $X \to S$ has connected fibers; in particular, a fiber X_s is either a connected variety whose irreducible components have all dimension ≥ 1, or it consists of a single point. In the latter case, we can prove that f is an isomorphism above s:

Proposition 2.2.11. Let $f : X \to S$ a proper, $\mathcal{O}$-connected morphism of locally Noetherian schemes. Let $V \subseteq S$ be the set of point $s \in S$ such that the fiber X_s consists of a single point. Then, V is open and f is an isomorphism above V.

Proof. Take $s \in V$. We want to restrict our study to an infinitesimal neighborhood of s; hence, we will introduce $S' := \mathrm{Spec}(\mathcal{O}_{S,s})$, $X' := X \times_S S'$, and we will focus on the restricted morphism $f : X' \to S'$, which will still be proper and $\mathcal{O}$-connected.

Now, $X' \to S'$ is a closed morphism, s is the unique closed point of S' and x is the unique point of X' lying over s: we can consequently be sure that x is the unique closed point of X'. Since X' is a quasi-compact scheme having a unique closed point, it is affine, but an affine $\mathcal{O}$-connected morphism is an isomorphism; hence, $X' \to S'$ is an isomorphism.

We have thus proved that $X \to S$ is an isomorphism over an "infinitesimal neighborhood" of s; to complete the proof of the proposition, we have to extend the result to a whole Zariski neighborhood of s. The spreading-out argument which is needed to conclude is deferred to the appendix (Corollary B.0.4). □

We will generically use the name *Zairki's Main Theorem* to mean the combined provisions of this proposition and of Zariski's Connectedness Principle (Theorem 2.2.10).

Given a morphism of schemes $f : X \to S$, one can factor it as a composition $X \to \mathrm{Spec}_S(f_*\mathcal{O}_X) \to S$ of an $\mathcal{O}$-connected morphism followed by an affine morphism: this factorization is named *Stein factorization*, and it is universal among all the factorizations $X \to X' \to S$ with $X' \to S$ affine (this is because $\mathrm{Spec}_S(f_*\mathcal{O}_X)$ is, by definition, the reflection of X into the subcategory of affine S-schemes). If $f : X \to S$ is assumed to be a proper morphism of locally Noetherian schemes, then its Stein factorization consists of a proper $\mathcal{O}$-connected morphism (to which the results we have proved in this subsection apply) followed by a finite morphism. As an example of how Stein factorization can be applied in combination with Zariski's Main Theorem, we present the following corollary:

Proposition 2.2.12. A quasi-finite, proper morphism between locally Noetherian schemes is finite.

Proof. Let $f : X \to S$ any such morphism, and let $X \to \operatorname{Spec}_S(f_*\mathcal{O}_X) \to S$ be its Stein factorization. The morphism $X \to \operatorname{Spec}_S(f_*\mathcal{O}_X)$ is proper, quasi-finite and $\mathcal{O}$-connected; hence, thanks to Zariski's Main Theorem (Theorem 2.2.10 and Proposition 2.2.11), it is an isomorphism. Consequently, $f : X \to S$ is affine, and we are done, since finite is equivalent to proper and affine. $\square$

2.3. Valuations on schemes

If X is an integral scheme, then all its local rings $\mathcal{O}_{X,x}$ are subrings of its function field $K(X)$. Moreover, a dominant morphism $f : X' \to X$ of integral schemes induces a function field extension $K(X) \subseteq K(X')$, so the local rings of X and X' at their points can all be seen as subrings of $K(X')$. If $x \in X$, $x' \in X'$ and $f(x') = x$, then we have a particular relation between $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X',x'}$, namely, $\mathcal{O}_{X',x'} \supseteq \mathcal{O}_{X,x}$ and $\mathfrak{m}_{x'} \cap \mathcal{O}_{X,x} = \mathfrak{m}_x$: we will say that $(\mathcal{O}_{X',x'}, \mathfrak{m}_{x'})$ *dominates* $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ (observe that this notion of dominance, although it is somehow related to the one defined in Subsection 2.1.5, is not equivalent to it).

We will start this section with a brief abstract study of the local subrings of a field with respect to the dominance order relation, and we will in particular characterize valuation rings as the maximal ones. We will then discuss how valuation rings can give an interesting perspective on the study of birational equivalences and birational morphisms of integral proper schemes, especially when they are normal.

2.3.1. Valuations and valuation rings

Let K be a field. Given two local subrings $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ of K, we say that B *dominates* A whenever $B \supseteq A$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$. This is equivalent to say that there is a morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ that makes the triangle

$$\begin{tikzcd} & \operatorname{Spec}(K) \arrow[dl, hook] \arrow[dr, hook] & \\ \operatorname{Spec}(B) \arrow[rr] & & \operatorname{Spec}(A) \end{tikzcd}$$

commute and sends the closed point of $\mathrm{Spec}(B)$ to the closed point of $\mathrm{Spec}(A)$. Dominance defines a partial order among local subrings of K, and it is easy to verify that the hypotheses of Zorn's lemma are verified; hence, every local subring of K is dominated by some maximal local subring of K. These maximal elements turn out to be nothing but the valuation rings of K:

Proposition 2.3.1. Given a field K and a local subring $(A, \mathfrak{m}_A)$, A is a valuation ring of K if and only if it is maximal among the local subrings of K (with respect to the dominance order).

Proof. Suppose $(A, \mathfrak{m}_A)$ is a valuation ring, and let $(B, \mathfrak{m}_B)$ any local subring of K dominating A. Suppose, by contradiction, that there exists $b \in B \setminus A$. Then, since A is a valuation ring, $b^{-1} \in \mathfrak{m}_A$, but this implies $b^{-1} \in \mathfrak{m}_B$, which is clearly incompatible with the hypothesis $b \in B$.

Conversely, suppose that $(A, \mathfrak{m}_A)$ is maximal. It is easy to show that A is integrally closed: if $A \subseteq A' \subseteq K$ is the normalization of A in K, and we fix any closed point $\mathfrak{p} \in \mathrm{Max}(A')$, we have that $\mathfrak{p} \cap A = \mathfrak{m}_A$ (since integral morphisms are closed), and hence $(A'_{\mathfrak{p}}, \mathfrak{p}A'_{\mathfrak{p}})$ dominates $(A, \mathfrak{m}_A)$. By the maximality of $(A, \mathfrak{m}_A)$, we can conclude that $A'_{\mathfrak{p}} = A$; consequenly, since $A \subseteq A' \subseteq A'_{\mathfrak{p}}$, we also have that $A = A'$.

Now, we are ready to prove that A is a valuation ring. We will take $\alpha \in K \setminus A$ and try to prove that $\alpha^{-1} \in A$. Consider the extension $A \subseteq A[\alpha] \subseteq K$. If $\mathfrak{p}$ is any point of $\mathrm{Spec}(A[\alpha])$ lying above $\mathfrak{m}_A$, we have, by the maximality of $(A, \mathfrak{m}_A)$, that $A[\alpha]_{\mathfrak{p}} = A$; since $A \subseteq A[\alpha] \subseteq A[\alpha]_{\mathfrak{p}}$, this means that $A = A[\alpha]$, which contradicts $\alpha \in K \setminus A$; hence, no such $\mathfrak{p}$ can exist, meaning that the fiber of $\mathrm{Spec}(A[\alpha])$ over $\mathfrak{m}_A$ is empty: in other words, $1 \in \mathfrak{m}_A A[\alpha]$. This last statement can be rephrased as the existence of a polynomial $P(X) = a_0X^d + a_1X^{d-1} + \ldots + a_d \in \mathfrak{m}_A[X]$ such that $P(\alpha) = 1$. If we introduce $Q(X) := (a_d - 1)X^d + a_{d-1}X^{d-1} + \ldots + a_0$, we can rewrite $P(\alpha) = 1$ as $Q(\alpha^{-1}) = 0$. But $Q(X) \in A[X]$ has leading coefficient $a_d - 1 \in A^*$, hence, it gives an integral equation for α^{-1} over A. Since we have already proved that A is integrally closed in K, we can conclude that $\alpha^{-1} \in A$, as we wanted. $\square$

2.3.2. Valuations for a scheme

Given S an integral scheme, ν a valuation of its function field $K(S)$, and $s \in S$ a point, we have that both $\mathcal{O}_\nu$ and $\mathcal{O}_{S,s}$ are local subrings of $K(S)$. We say that ν *admits s as a center* if $(\mathcal{O}_\nu, \mathfrak{m}_\nu)$ dominates $(\mathcal{O}_{S,s}, \mathfrak{m}_s)$. A valuation ν of $K(S)$ with a center induces a birational morphism $\mathrm{Spec}(\mathcal{O}_\nu) \to S$; conversely, a birational morphism from the spectrum of a valuation ring to S identifies a point $x \in X$ and a valuation of S centered at x:

$$\left\{ \begin{array}{c} (\nu, s)\colon\ s \in S \text{ and } \nu \text{ is a} \\ \text{valuation of } K(S) \\ \text{admitting } s \text{ as a center} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} (\nu, h)\colon\ \nu \text{ is a valuation of } K(S) \text{ and} \\ h : \operatorname{Spec}(\mathcal{O}_\nu) \to S \text{ is a birational morphism} \\ \text{restricting to the identity on } K(S) \end{array} \right\}$$

We will now reproduce the same discussion in the relative case: let $f : X \to S$ be a dominant morphism of integral schemes. Every valuation of $K(X)$ restricts to some valuation of $K(S)$; an *S-valuation of $K(X)$* will be, by definition, a valuation ν of $K(X)$ endowed with a distinguished center $s \in S$ for $\nu_{|K(S)}$; in other words, an S-valuation ν comes with a canonical morphism $\operatorname{Spec}(\mathcal{O}_\nu) \to S$ that restricts to the function field extension $K(S) \subseteq K(X)$ at the level of generic points. We say that an S-valuation ν (centered at a given $s \in S$) *admits a center* at some point $x \in X$ if $(\mathcal{O}_\nu, \mathfrak{m}_\nu)$ dominates $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ and $f(x) = s$.

The centers in X for an S-valuation ν of $K(X)$ correspond to the dashed arrows solving the following lifting problem:

$$\begin{tikzcd} \operatorname{Spec}(K(X)) \arrow[r, hook] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec}(\mathcal{O}_\nu) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

In other words:

$$\left\{ \begin{array}{c} (\nu, x)\colon\ x \in X \text{ and } \nu \text{ is an} \\ S\text{-valuation of } K(S) \\ \text{admitting } x \text{ as a center} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} (\nu, h)\colon\ \nu \text{ is a valuation of } K(S) \text{ and} \\ h : \operatorname{Spec}(\mathcal{O}_\nu) \to X \text{ is a bir. } S\text{-morphism} \\ \text{restricting to the identity on } K(X) \end{array} \right\}$$

Given an S-valuation of $K(X)$, it is natural to ask whether it admits a center in X, and whether this center is unique. It turns out that the relevant conditions are, respectively, separatedness and universal closedness of the morphism $X \to S$:

Proposition 2.3.2. Let $X \to S$ a dominant morphism of integral schemes, and ν an S-valuation of $K(X)$. If f is universally closed, ν admits at least one center in X. If f is separated, ν admits at most one center in X.

The result is actually a corollary of the more general valutative criteria for the separatedness and universal closedness, that we will present and prove in the following subsection.

2.3.3. Valutative criteria

Let X be any scheme, and R a valuation ring. Giving an R-point of X (i.e. a morphism $\mathrm{Spec}(R) \to X$) means essentially taking a specialization of points $x \rightsquigarrow x'$ in X (i.e., two points $x, x' \in X$ such that $x' \in \overline{\{x\}}$), together with a distinguished local extension $\mathcal{O}_{X,x'}/(\mathfrak{m}_x \cap \mathcal{O}_{X,x'}) \hookrightarrow R$. Conversely, given a specialization of points $x \rightsquigarrow x'$, one can consider the valuation rings dominating $\mathcal{O}_{X,x'}/\mathfrak{m}_x\mathcal{O}_{X,x'}$ and build R-points corresponding to that specialization. We will interchangeably use the expressions *R-point* and *R-specialization* to mean a morphism $\mathrm{Spec}(R) \to X$; as we have just seen, an R-specialization identifies a *topological specialization* $x \rightsquigarrow x'$ in X, but also carries some additional algebraic data.

Given a morphism of schemes $f : X \to S$, we want to address the lifing problems for R-specializations, i.e. : given an R-specialization $s \rightsquigarrow s'$ on S and a K-lifting x of the point s, can this be extended to an R-lifting $x \rightsquigarrow x'$ of the specialization $s \rightsquigarrow s'$? More formally, we are willing to study the dashed arrows solving the problem represented by a square of the form:

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd} \qquad (\diamond)$$

Definition 2.3.3. We say that a morphism of schemes $f : X \to S$ satisfies the existence (resp. the uniqueness) part of the valutative criterion if, for any field K and any valuation ring R with fraction field K, every lifting problem of the form $(\diamond)$ always has at least (resp. at most) one solution.

Both properties are clearly stable under arbitrary base change.

Valutative criterion for universal closedness

The following proposition shows that the possibility of lifting of R-specializations is equivalent to the one of lifting topological specializations universally.

Proposition 2.3.4. A morphism of schemes $f : X \to S$ satisfies the existence part of the valutative criterion if and only if (topological) specializations lift along any base change $f' : X' \to S'$ of f (in the sense that every such f' sends specialization-closed subsets of X' to specialization-closed subsets of S').

Proof. Suppose that f satisfies the existence part of the valutative criterion: we want to prove that specializations lift along f (since the property of satisfying the existence part of the valutative criterion is stable under base-change, this is clearly sufficient to prove the "only if" part of the proposition). Suppose thus that we are given $s \rightsquigarrow s'$ a specialization in S, and x a lifting of s. In this setting, $\mathcal{O}_{S,s'}/(\mathfrak{m}_s \cap \mathcal{O}_{S,s'})$ is a local subring of $k(s) \subseteq k(x)$: to conclude the proof, it is now enough to take any valuation ring R of $k(x)$ dominating $\mathcal{O}_{S,s'}/(\mathfrak{m}_s \cap \mathcal{O}_{S,s'})$ (so, in particular, $s \rightsquigarrow s'$ acquires the structure of an R-specialization), and then exploit the existence part of the valutative criterion to find an R-specialization $x \rightsquigarrow x'$ above $s \rightsquigarrow s'$ (so, in particular, $x \rightsquigarrow x'$ will be a lifting of $s \rightsquigarrow s'$ as topological specializations).

Let us now prove the converse implication. Suppose now that specializations lift along any base-change of f. Then, given an R-specialization $s \rightsquigarrow s'$ in S, and x a lifting of the K-point s (where $K := \mathrm{Frac}(R)$), we can perform a base-change and suppose, without losing generality, that $S = \mathrm{Spec}(R)$, so s coincides with the generic point and s' with the closed point of R. By hypothesis, x admits a specialization x' over s', which will clearly induce a dominance relation $(R, \mathfrak{m}_R) \subseteq (\mathcal{O}_{X,x'}/\mathfrak{m}_x\mathcal{O}_{X,x'}, \mathfrak{m}_{x'})$ of local subrings of $k(x') \subseteq K$. Since R is a valuation ring of K (and hence, a maximal local subring of K), this implies $\mathcal{O}_{X,x'}/\mathfrak{m}_x\mathcal{O}_{X,x'} = R$, so $x \rightsquigarrow x'$ has the structure of an R-specialization, and it clearly lifts the R-specialization $s \rightsquigarrow s'$. $\square$

It is now clearly interesting to further study along which morphisms $f : X \to S$ do topological specializations lift. A closed morphism will obviously have this property; moreover, it turns out that the converse also holds supposing f is quasi-compact:

Proposition 2.3.5. Let $f : X \to S$ be a quasi-compact morphism of schemes such that specializations lift along f. Then, f is closed.

Proof. Take $Z \subseteq X$ a closed subset: we want to prove that its image is closed. We may suppose, without loss of generality, that X is reduced and $Z = X$. Moreover, since X is now reduced, f factors through the closed reduced subscheme $\overline{f(X)} \subseteq S$, and we lose no generality in supposing $S = \overline{f(X)}$.

We are now left with a quasi-compact dominant morphism of reduced schemes $X \to S$ along which specializations lift, and we simply have to prove that it is surjective. The crucial observation is that, since f is quasi-compact and dominant, $f(X)$ contains all the generic points of S (see [Stacks, `01RL`]): hence, the image $f(X)$, which is closed under specialization by hypothesis, must coincide with the whole S. $\square$

As a consequence of what we have proved,

Corollary 2.3.6. A quasi-compact morphism of schemes $f : X \to S$ is universally closed if and only if it satisfies the existence part of the valutative criterion.

Valutative criterion for separatedness

It is easy to prove that

Proposition 2.3.7. If $f : X \to S$ is a separated morphism of schemes, then it satisfies the existence part of the valutative criterion.

Proof. Let R be a valuation ring and K its field of fraction. Suppose we are given a square:

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed, shift left] \arrow[ur, dashed, shift right] & S \end{tikzcd}$$

and two dashed solutions g_1 and g_2. Since X is a separated S-scheme, the equalizer of the two S-morphisms g_1 and g_2 is a closed subscheme $V(I) \hookrightarrow \mathrm{Spec}(R)$ (see [Stacks, `01KM`]). But we know that $\mathrm{Spec}(K) \hookrightarrow \mathrm{Spec}(R)$ equalizes g_1 and g_2; hence, $V(I)$ contains the generic point of R and must consequently coincide with the whole $\mathrm{Spec}(R)$. Hence, $g_1 = g_2$. □

A partial converse of this result holds, and its proof relies on the application of the valutative criterion for universal closedness to the diagonal morphism:

Proposition 2.3.8. Let X a quasi-separated S-scheme satisfying the uniqueness part of the valutative criterion. Then, X is separated over S.

Proof. Since $\Delta_{X/S}$ is quasi-separated by hypothesis, to show that $\Delta_{X/S}$ is closed (and hence $X \to S$ separated) it is enough to prove that $\Delta_{X/S}$ satisfies the existence part of the valutative criterion. Let us thus take any valuation ring R (with fraction field K) and consider a lifting problem for the diagonal $\Delta_{X/S} : X \hookrightarrow X \times_S X$:

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d, hook] & X \arrow[d, hook, "\Delta_{X/S}"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & X \times_S X \end{tikzcd}$$

It is clear that the existence of a dashed solution is equivalent to the statement that the two components f_1 and f_2 of $\mathrm{Spec}(R) \to X \times_S X$ are equal. But f_1 and

f_2 clearly solve, by construction, the lifting problem

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

and hence we can be sure that $f_1 = f_2$, because, by hypothesis, X satisfies the uniqueness part of the valutative criterion. □

2.3.4. Valuations and birationality

Let us now fix an integral, Noetherian basis S. If we take X a proper, dominant, integral S-scheme, every S-valuation of $K(X)$ has one and only one center in X (Proposition 2.3.2); in other words, we may think of a point of X as an equivalence class of S-valuations (the equivalence relation being "having the same center in X"). If $x \in X$ is a regular point of codimension 1, the situation is particularly simple, since $\mathcal{O}_{X,x}$ is itself a valuation ring, and hence the set of all S-valuations centered at x reduces to a singleton $\{\nu_x\}$:

Definition 2.3.9. The S-valuations centered at regular points of codimension 1 of X are named *valuations of the first kind*, and they bijectively correspond to codimension 1 regular points of X.

If we take two proper, dominant, integral S-schemes X_1 and X_2 that are S-birationally equivalent, then the local rings of X_1 and of X_2 at their points are all subrings of the same function field $K := K(X_1) = K(X_2)$, and we may imagine the S-valuations of K aggregating together in two different ways to give the points of X and the points of X'; more formally, X and X' define two different equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$ on the set of all S-valuations of K, and we can try to compare them.

Proposition 2.3.10. If X_1 dominates X_2, then $\mathcal{R}_1$ is finer than $\mathcal{R}_2$. If X_1 is normal, the converse is also true.

Proof. Suppose X_1 dominates X_2, and name $f : X_1 \to X_2$ the birational morphism. If $x \in X_1$ is any point, then there is clearly a dominance relation $\mathcal{O}_{X_2,f(x)} \subseteq \mathcal{O}_{X_1,x}$. Hence, all S-valuations centered at $x \in X_1$ are also centered at $f(x) \in X_2$; consequently, $\mathcal{R}_1$ is finer then $\mathcal{R}_2$. Suppose now that X_1 is normal, and that $\mathcal{R}_1$ is finer than $\mathcal{R}_2$. Then, any given point $x \in X_1$ determines a unique point $x' \in X_2$, which is the common center in X_2 of all S-valuations centered at $x \in X_1$. If Γ

is the graph of the birational equivalence between X_1 and X_2, then it is clear that only finitely many points of Γ can simultaneously lie over x and x'. Let now $z \in \Gamma$ be a point lying over x: as $\mathcal{O}_{\Gamma,z}$ dominates $\mathcal{O}_{X_1,x}$, all S-valuations centered at z will have center at x, and hence at x'; consequently, z must lie over x'. In other words, we have a set-theoretical inclusion $\Gamma_x \subseteq \Gamma_{x'}$, whence we can conclude that Γ_x only consists of finitely many points. By Zariski's Main Theorem (Theorem 2.2.10 and Proposition 2.2.11), $\Gamma \to X_1$ is consequently an isomorphism over $x \in X_1$; since x is arbitrary, we conclude that $\Gamma = X_1$: in other words, X_1 dominates X_2. □

Suppose we are in the case in which a birational morphism $f : X_1 \to X_2$ exists, so the relation $\mathcal{R}_1$ is finer than $\mathcal{R}_2$: each equivalence class of $\mathcal{R}_2$ (which will correspond to a point $x_2 \in X_2$) splits into multiple equivalence classes of $\mathcal{R}_1$ (corresponding to all points $x_1 \in X_1$ lying over x_2). Let ν be a valuation, and let us denote by x_1 and x_2 its centers in X_1 and X_2 respectively, so we have a dominance relation $\mathcal{O}_{X_2,x_2} \subseteq \mathcal{O}_{X_1,x_1}$. If ν is of the first kind in X_2, then x_2 is a regular point of codimension 1 of X_2 and $\mathcal{O}_{X_2,x_2} = \mathcal{O}_\nu$; by the maximality of local rings, this implies that also $\mathcal{O}_{X_1,x_1} = \mathcal{O}_\nu$, so ν is a valuation of the first kind in X_1 too; moreover, its center $x_1 \in X_1$ is evidently the unique point of X_1 lying over $x_2 \in X_2$, and hence (essentially by Proposition 2.2.11) $f : X_1 \to X_2$ is an isomorphism above x_2:

Proposition 2.3.11. If S is a Noetherian integral basis, and $f : X_1 \to X_2$ is a birational morphism of integral proper dominant S-schemes, then every S-valuation ν of $K = K(X_1) = K(X_2)$ that is of the first kind on X_2 is also of the first kind on X_1; furthermore, f is an isomorphism over the center $x_2 \in X_2$ of ν in X_2 (meaning that $x_2 \in X_2 \smallsetminus E(f)$, see Convention 2.1.4).

An obvious but relevant corollary is the following one:

Corollary 2.3.12. Let $f : X' \to X$ be a proper birational morphism of integral Noetherian schemes, and suppose that X is normal. Then, the locus $E(f) \subseteq X$ above which f fails to be an isomorphism (see Convention 2.1.4) has codimension ≥ 2.

3. Curves

With the term *curve* we will always refer to a proper scheme of pure dimension 1 over a field k. In particular, we assume no regularity conditions: our curves may have singularities, they may have multiple irreducible components, and they may even be non-reduced: the main objective of this chapter will actually be the development of tools for measuring and studying singularities of curves.

As a fist step (Sections 3.1 and 3.2), we will recall a number of useful facts on divisors and line bundles on curves. Then, we will derive some remarkable results on the classification of singularities from a more general discussion on the structure of birational morphisms of curves (Sections 3.3 and 3.4). Finally, after a brief general discussion on the Picard functor, we will introduce the Jacobian of a curve and see how it codifies information on its singularities (Sections 3.5 and 3.6). The final section (Section 3.7) collects some results on genus zero curves that will be useful later.

3.1. Genera of curves

Given $X \to \mathrm{Spec}(k)$ a curve, its only non-trivial cohomology spaces are $H^0(X) := H^0(X, \mathcal{O}_X)$ and $H^1(X) := H^1(X, \mathcal{O}_X)$: thanks to properness, they will both have finite dimension over k, i.e. $h^p_k(X) := \dim_k(H^p(X)) < \infty$. Moreover, as the computation of cohomology commutes with flat base-change, we clearly have that $h^p_{k'}(X_{k'}) = h^p_k(X)$ for every field extension $k \subseteq k'$.

The *Euler-Poincaré characteristic* $\chi(X)$ of a curve $X \to \mathrm{Spec}(k)$ is defined as the Euler-Poincaré characteristic over k of its structure sheaf, i.e. $\chi(X) := \chi(\mathcal{O}_X) = \sum_p (-1)^p h^p(X) = h^0(X) - h^1(X)$. If X is a geometrically integral curve, then $H^0(X)$ simply coincides with the base field k, and hence $h^0(X) = 1$: in this case, we will define the *genus* of X to be $g(X) := h^1(X)$, and the Euler-Poincaré characteristic can consequently be written as $\chi(X) = 1 - g(X)$. An extremely relevant property of the Euler-Poincaré characteristic is its invariance within a family of curves:

Proposition 3.1.1. If $\mathcal{X} \to S$ is a family of curves (i.e. a proper flat morphism whose fibers have pure dimension 1) over a locally Noetherian basis S, then the function associating to each point $s \in S$ the genus of the Euler-Poincaré characteristic of $\mathcal{X}_s \to \mathrm{Spec}(k(s))$ is locally constant.

Proof. This is a consequence of the more general result [Vakil, Theorem 24.7.1]. □

3.2. Divisors and line bundles on curves

3.2.1. Existence results for Cartier divisors

Effective Cartier divisors on curves have a rather simple form, as their support is just a finite collection of closed points. If $x \in X$ is a closed point on a curve (or, more generally, of a Noetherian scheme of pure dimension 1), then it is perfectly clear that

Proposition 3.2.1. (a) x, as a reduced closed subscheme of X, is an effective Cartier divisor if and only if it is a regular point of X, and (b) there exists an effective Cartier divisor whose support is $\{x\}$ if and only if x is not an associated point of X.

Let now X be a Noetherian scheme of pure dimension 1, and $X' \subseteq X$ a closed subscheme of pure dimension 1 (for example, X' may be an irreducible component of X, it may coincide with X_{red}, ...). We ask ourselves whether, given D' an effective Cartier divisor of X', is it true that D' admits a *strict Cartier lifting* to X, i.e. an effective Cartier divisor D on X such that $\mathrm{Supp}(D) = \mathrm{Supp}(D')$ and $D \cap X' = D'$. An answer to this question is given by the following proposition:

Proposition 3.2.2. If D' does not contain any embedded point of X and the support of X' contains every irreducible component of X intersecting D', then a strict Cartier lifting D exists.

Proof. We lose no generality in supposing that D' is supported on a single closed point $x \in X'$. Let $f' = 0$ be an equation for $D' \subseteq X'$ on some neighborhood $x \in U' \subseteq X'$: as D' is an effective Cartier divisor on X', $f' \in \mathcal{O}_{X'}(U')$ will be a regular function. Let now U be an affine neighborhood of x in X such that $U \cap X' \subseteq U'$, and let $f \in \mathcal{O}_X(U)$ be any lifting of f'. Thanks to our hypotheses, we can shrink U so (1) it does not contain any embedded point of X, and (2) all generic points of U belong to X'. If these two conditions are satisfied, the fact that $f' \in \mathcal{O}_{X'}(U')$ is regular implies that $f \in \mathcal{O}_X(U)$ is also regular; up to further restricting U, we may moreover suppose that the only point at which f vanishes is x. Now, it is enough to set D equal to the Cartier divisor defined by the equation $f = 0$ on U, and by the trivial equation $1 = 0$ on $X \smallsetminus \{x\}$. □

Corollary 3.2.3. If X is a Noetherian scheme of pure dimension 1 and D' an effective Cartier divisor on X_{red} avoiding the embedded points of X, then D' can be lifted to an effective Cartier divisor on X.

3.2.2. Degree of divisors

If X is a curve, a Weyl divisor $Z \in \mathrm{Div}(X)$ will be in particular a 0-cycle, and its push-forward to $\mathrm{Spec}(k)$ will be nothing but a multiple $n[*]$ of the unique point of $* \in \mathrm{Spec}(k)$, where n is named the *degree* of the Weyl divisor Z (see Subsection 1.1.6). The degree defines a group homomorphism $\deg_X : \mathrm{Div}(X) \to \mathbb{Z}$.

A notion of degree is also well-defined for line bundles on a curve: given $\mathcal{L} \in \mathrm{Pic}(X)$, we can set $\deg(\mathcal{L}) := \chi(\mathcal{L}) - \chi(\mathcal{O}_X)$. While it is not immediate, it is possible to show that:

Proposition 3.2.4. The function $\deg_X : \mathrm{Pic}(X) \to \mathbb{Z}$ is a group homomorphism. Moreover, if $X_1, \dots, X_k$ are the irreducible components of X (to be thought as reduced closed subschemes of X), then, for every $\mathcal{L} \in \mathrm{Pic}(X)$, $\deg_X(\mathcal{L}) = \sum_i m_i \deg_{X_i}(\mathcal{L}_{|X_i})$, where m_i is the multiplicity of X_i in X.

Proof. See [Stacks, `0AYQ`]. □

Definition 3.2.5. The degrees $\deg_{X_i}(\mathcal{L}_{|X_i})$ mentioned in the proposition above constitute the so-called *multi-degree* of $\mathcal{L}$ on X.

The line bundles of multi-degree 0 form a subgroup of $\mathrm{Pic}(X)$, that we will denote by $\mathrm{Pic}^0(X)$. The formation of $\mathrm{Pic}^0(X)$ commutes with all extensions of the base field k in the following sense: if $k \subseteq k'$ is any extension, then the pullback of line bundles defines an injection $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(X_{k'})$ ([Stacks, `0CC5`]), and we have $\mathrm{Pic}^0(X) = \mathrm{Pic}(X) \cap \mathrm{Pic}^0(X_{k'})$.

Given an effective Cartier divisor D on X, one can compute the degree of its associated Weyl divisor $[D]$, and the degree of its associated line bundle $\mathcal{O}_X(D)$: it is not difficult to realize that they always coincide.

Proposition 3.2.6. For every effective Cartier divisor D on X, $\deg([D]) = \deg(\mathcal{O}_X(D))$.

Proof. We have an exact sequence of coherent sheaves $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0$; therefore, $\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_D(D))$. But $\mathcal{O}_D(D) \cong \mathcal{O}_D$ is a skyscraper sheaf on X, hence $\chi(\mathcal{O}_D(D)) = h^0(\mathcal{O}_D)$, and it is immediate to deduce from the definition of the degree of a Weyl divisor that $h^0(\mathcal{O}_D) = \deg([D])$. The proposition now easily follows. □

3.2.3. Ampleness

We want to inquire when a line bundle $\mathcal{L}$ on a curve $X \to \mathrm{Spec}(k)$ satisfies the following conditions: 1) some tensor power of $\mathcal{L}$ has a regular section, 2) $\mathcal{L}$ is semiample, 3) $\mathcal{L}$ is ample (definitions in Subsection 1.2.2). There is an obvious chain of implications $3 \implies 2 \implies 1$; the propositions that follow will try to depict further how the three conditions are related to each other.

Proposition 3.2.7. If some tensor power of $\mathcal{L}$ has a regular section, $\mathcal{L}$ is semiample. In other words, conditions 1 and 2 are equivalent.

Proof. Up to replacing $\mathcal{L}$ with some tensor power of it, we may suppose $\mathcal{L} = \mathcal{O}_X(D)$ for some effective Cartier divisor D. Consider the exact sequence $0 \to \mathcal{L}^{n-1} \to \mathcal{L}^n \to \mathcal{L}^n_{|D} \to 0$. Since $\mathcal{L}^n_{|D}$ is skyscraper, taking cohomology one gets, for every $n \geq 1$, a surjective morphism $\pi_n : H^1(\mathcal{L}^{n-1}) \twoheadrightarrow H^1(\mathcal{L}^n)$ of finite-dimensional k-vector spaces: an obvious dimensional argument ensures that π_n must eventually be an isomorphism; hence, $H^0(\mathcal{L}^n) \to H^0(\mathcal{L}^n_{|D})$ is eventually surjective. We will now show how this implies that $\mathcal{L}^n$ is eventually globally generated. Take any point $x \in X$: we have to exhibit a global section of $\mathcal{L}^n$ not vanishing at x. It is clear that there is really something to check only if $x \in \operatorname{Supp}(D)$. But $\mathcal{L}^n_{|D}$ is trivial on D (as all line bundles on a zero-dimensional variety), and hence it will obviously have a global section φ on D not vanishing at $x \in D$; if n is large enough, we can rely on the surjectivity of $H^0(\mathcal{L}^n) \to H^0(\mathcal{L}^n_{|D})$ to lift φ to a global section of $\mathcal{L}^n$ on X. $\square$

Proposition 3.2.8. $\mathcal{L}$ is ample (i.e., 3 holds) if and only if $\mathcal{L}$ is semiample (i.e., 1/2 holds), and $\mathcal{L}_{|X_i}$ is non-trivial for all i (where the X_i's are the irreducible components of X, endowed with the reduced closed subscheme structure).

Proof. Let us prove the only non-trivial implication (i.e., the "if" part). Since $\mathcal{L}$ is semiample, it determines a surjective, $\mathcal{O}$-connected morphism $\psi_{\mathcal{L}} : X \to X'$, where $X' := \operatorname{Proj}_k(H^0(\mathcal{L}_\bullet))$ is a proper scheme over k, of dimension ≤ 1 (see Section 1.2 for notation and construction). Suppose, by contradiction, that $\psi_{\mathcal{L}}(X_i)$ is reduced to a point: this implies that the global sections of $\mathcal{L}$ on X must either be everywhere-vanishing or nowhere-vanishing on X_i; as $\mathcal{L}$ is semiample, at least one section $\varphi \in H^0(X, \mathcal{L})$ will vanish at no point of X_i, but this contradicts our hypothesis that $\mathcal{L}_{|X_i}$ is not trivial. Hence, $\psi_{\mathcal{L}}(X_i)$ is not reduced to a point for any i, and this implies that $\psi_{\mathcal{L}}$ is quasi-finite, and hence finite thanks to Zariski's Main Theorem (Proposition 2.2.12). But a finite $\mathcal{O}$-connected morphism is an isomorphism; hence, $\mathcal{L}$ is ample. $\square$

Now, let us try to see how to characterize semiample (condition 1 or 2) and ample (condition 3) line bundles on a curve X. It is clear that, if $\deg(\mathcal{L}_{|X_i}) < 0$ for some irreducible component X_i of X, or if $\deg(\mathcal{L}_{|X_i}) = 0$ but $\mathcal{L}_{|X_i}$ is non-trivial, $\mathcal{L}$

cannot be semiample. It is equally clear that ampleness forces $\deg(\mathcal{L}_{|X_i}) > 0$ for all i. We want to achieve some results in the converse direction.

Proposition 3.2.9. If X is an integral curve over k, a line bundle $\mathcal{L}$ is semiample if and only if $\deg(\mathcal{L}) > 0$ or $\mathcal{L}$ is trivial; it is ample if and only if $\deg(\mathcal{L}) > 0$.

Proof. The only point we really have to prove is that a line bundle of positive degree on an integral curve is semiample. Suppose thus that $\deg(\mathcal{L}) > 0$. We remark that $\dim_k H^0(\mathcal{L}) \geq \chi(\mathcal{L}) = \deg(\mathcal{L}) + \chi(\mathcal{O}_X)$. As a consequence, up to replacing $\mathcal{L}$ with some tensor power of it, we may suppose that $H^0(\mathcal{L}) \neq 0$. Hence, $\mathcal{L}$ has a non-zero section, which is certainly regular since X is integral. □

Proposition 3.2.10. A line bundle $\mathcal{L}$ on a curve X is ample if and only if $\deg(\mathcal{L}_{|X_i}) > 0$ for all i (X_i here denote the irreducible components of X, to be thought as reduced closed subschemes).

Proof. This can be deduced from proposition above with the help of [Stacks, `0B5V`] or [Liu, Corollary 5.3.8]. □

Corollary 3.2.11. All curves are projective.

Proof. With the help of Proposition 3.2.1, it is not difficult to exhibit an effective Cartier divisor D on X whose restriction to each of the X_i's is not trivial. It follows from Proposition 3.2.10 that such D has to be ample. □

3.3. Birational morphisms of curves

For a curve, regularity and normality are two equivalent conditions, and two normal curves are birational if and only if they are isomorphic (see Corollary 2.3.12). Anyway, if we allow curves to be non-normal, the structure of their birational classes is no longer trivial.

We will start by observing that a birational morphism of curves $f : X' \to X$ is always finite (because it is quasi-finite and proper, and hence Proposition 2.2.12 applies). Moreover, to any such morphism we can associate an exact sequence $0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \to \Delta_f \to 0$, where Δ_f is a skyscraper sheaf on X. Since f is affine, $\chi(\mathcal{O}_{X'}) = \chi(f_*\mathcal{O}_{X'})$; hence, the exact sequence gives us $\chi(\mathcal{O}_{X'}) = \chi(\mathcal{O}_X) + h^0(\Delta_f)$. In particular, the Euler-Poincaré characteristic decreases along birational morphisms.

If X is a reduced curve, a crucial example of a birational morphism to X is given by the normalization $\widetilde{X} \to X$. Moreover, since birational morphisms of proper curves are finite, all proper curves within the same birational class have the same normalization.

These considerations ensure that:

Proposition 3.3.1. If $\mathcal{A}$ is a birational class of reduced curves, then we have the following:

(a) $\mathcal{A}$ has a maximum element, which coincides with the only normal curve within that birational class, and will be the common normalization of all curves of $\mathcal{A}$;
(b) $\mathcal{A}$ satisfies the ascending chain condition.

Proof. (a) follows from what we have observed about the normalization of reduced curves; (b) follows from (a) together with the fact that χ decreases along birational morphisms. □

An important source of birational morphisms is given by blowups: if X is a curve, $x \in X$ a closed point, and X' the blowup of X at x, then $X' \to X$ is a birational morphism. We know that $X' \to X$ is an isomorphism if and only if x is already a Cartier divisor in X, but this happens if and only if X is regular at x. Since birational classes of reduced curves satisfy the ascending chain condition, we can be sure that:

Proposition 3.3.2. The normalization of a curve can be computed by repeatedly blowing up its singular locus, a finite number of times.

If blowing up resolves the singularities of a curve, we will now explore the opposite direction, and try to add singularities to a curve, while remaining within its birational class. In particular, given a curve X' and a finite number of rational points $x'_1, \ldots, x'_n$ on it, we want to glue them together; in other words, we want to compute the pushout X of the diagram

$$\begin{tikzcd} \coprod_i \mathrm{Spec}(k) \arrow[r] \arrow[d] & X' \arrow[d, dashed, "f"] \\ \mathrm{Spec}(k) \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner"] & X \end{tikzcd} \tag{†}$$

Colimits do not always exist in the category of schemes, so some verification is needed to ensure that the pushout exists: we defer a careful discussion of this issue to Appendix A. We remark that, as X' is projective (Corollary 3.2.11), there certainly exists an open affine subscheme of X' containing all the x'_i's, and this guarantees that the hypotheses of Corollary A.4.9 are satisfied. As a consequence, the results collected in the appendix are enough to ensure that (a) X is a well-defined curve over k; (b) topologically, X is just the quotient of X' under the identification of the points $x'_i \in X'$ to a single points $x \in X$; (c) $f : X' \to X$ is a finite, scheme-theoretically surjective morphism, and it is an isomorphism over $X \smallsetminus \{x\}$; (d) the structure sheaf $\mathcal{O}_X$ is the subsheaf of $f_*\mathcal{O}_{X'}$ consisting of those functions on X' that agree at the points x'_i. This last statement can be rephrased saying that the pushout diagram defining X induces a pullback diagram

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, hook] \arrow[d, two heads] & f_*\mathcal{O}_{X'} \arrow[d, two heads] \\
k \arrow[r, hook] & \prod_i k
\end{tikzcd}$$

of coherent sheaves on X (where k means the structure sheaf of the point x, and $\prod_i k$ the structure sheaf of the points x'_i). In particular, the cokernel Δ_f of $\mathcal{O}_X \hookrightarrow f_*\mathcal{O}_{X'}$ is a skyscraper sheaf concentrated at x, and $h^0(\Delta_f) = \chi(X') - \chi(X) = n-1$.

We want now to study more in detail what happens over the point $x \in X$ to which the x'_i glue. By the universal property of the pushout, we have morphisms $\prod_i \operatorname{Spec}(k) \to X'_x \to \operatorname{Spec}(k)$, so the fiber X'_x must actually coincide with $\prod_i \operatorname{Spec}(k)$. Let us now look at the complete local ring at x: since the formation of the pushout (†) commutes with flat base-change, we have a pushout diagram

$$\begin{tikzcd}
\prod_i \operatorname{Spec}(k) \arrow[r] \arrow[d] & \prod_i \operatorname{Spec}(\widehat{\mathcal{O}}_{X',x_i}) \arrow[d, "f"] \arrow[dl, phantom, "\ulcorner"] \\
\operatorname{Spec}(k) \arrow[r] & \operatorname{Spec}(\widehat{\mathcal{O}}_{X,x})
\end{tikzcd}$$

that allows us to compute $\widehat{\mathcal{O}}_{X,x}$ as the fibered product of the rings $\widehat{\mathcal{O}}_{X',x'_i}$ over their common residue field k.

Example 3.3.3. If X' is regular at the points x'_i, and t_i are local coordinates at those points, then $\widehat{\mathcal{O}}_{X,x} \cong k[\![t_1, \ldots, t_n]\!]/(t_i t_j : i \neq j)$: in other words, the n branches of X at x meet as the axes of an n-dimensional affine space over k.

Gluing morphisms have a fundamental role among all birational morphisms of curves, thanks to the following factorization result:

Proposition 3.3.4. If $X' \to X$ is a birational morphism of curves over a field k, and all the points of X' lying over $E(f) \subseteq X$ (see Convention 2.1.4) are k-rational, then $X' \to X$ can be factored as the composition of a gluing morphism $X' \to Y$, followed by a universal homeomorphism $Y \to X$ (both of which are birational morphisms of curves over k). Moreover, this decomposition is unique.

Proof. Let x_i be the points of $E(f)$, and x'_{ij} the points of X' lying above x_i: our hypotheses ensure that all the x'_{ij}, and consequently all the x_i, are rational over k. We will set Y to be the scheme obtained from X' by gluing together, for each i, the points $\{x_{ij}\}_j$ to a single point $y_i \in Y$. By the universal property of the pushout, $X' \to X$ factors uniquely through Y. The morphism $Y \to X$ is clearly an isomorphism away from the points x_i, and each of the (possibly non-reduced) fibers Y_{x_i} consists of a single rational point: this is enough to guarantee that $Y \to X$ is a universal homeomorphism. The uniqueness of the decomposition is easily proved. $\square$

3.4. Classification of singularities

Let X be a reduced curve over a field k. Its normalization $\widetilde{X} \to X$ is a finite birational morphism of curves. The formation of the normalization $\widetilde{X}$:

- commutes with any separable extension $k' \subseteq k$, in the sense that, for any such extension, $\big(\widetilde{X}\big)_{\overline{k}}$ is the normalization of $X_{k'}$;
- commutes with any localization of X; in particular, for every point $x \in X$, $\widetilde{X} \times_X \operatorname{Spec}(\mathcal{O}_{X,x})$ provides the normalization of $\mathcal{O}_{X,x}$ in $\operatorname{Frac}(\mathcal{O}_{X,x}) = K(X)$;
- commutes with taking the completed local ring at a closed point $x \in X$, in the sense that $\widetilde{X} \times_X \operatorname{Spec}(\widehat{\mathcal{O}}_{X,x})$ provides the integral closure of $\widehat{\mathcal{O}}_{X,x}$ in its total ring of fractions (see [Stacks, `0C3V`]); the same is true if, instead of completing, we take the Henselianization or the strict Henselianization of $\mathcal{O}_{X,x}$.

Suppose now that X is a reduced curve over an algebraically closed field k. To the normalization morphism $f : \widetilde{X} \to X$ we can associate an exact sequence $0 \to \mathcal{O}_X \to f_*\mathcal{O}_{\widetilde{X}} \to \mathcal{S} \to 0$, where $\mathcal{S} := \Delta_f$ is a skyscraper sheaf, whose support consists precisely of the singular points of X: we have that $\chi(\widetilde{X}) = \chi(X) + h^0(\mathcal{S})$, where $h^0(\mathcal{S}) = \sum_{x \in \operatorname{Sing}(X)} h^0(\mathcal{S}_x)$.

Definition 3.4.1. If k is algebraically closed, X is a reduced curve over k and $f : \widetilde{X} \to X$ is the normalization morphism, the integer $h^0(\mathcal{S}_x) = \dim_k(\mathcal{S}_x) = \operatorname{length}_{\mathcal{O}_{X,x}}(\mathcal{S}_x)$ (where $\mathcal{S}$ has been defined above) is named the *δ-invariant* of the singular point x and is denoted by $\delta(x)$, whereas the number of points of $\widetilde{X}$ above x (i.e., the set-theoretic cardinality of the fiber $\widetilde{X}_x$) is named the *number of branches* at the singular point x, and is denoted by $m(x)$. If $m(x) = 1$, we say that x is a *unibranch singularity* of the curve X.

The results we have listed at the beginning of this section on the formation of $\widetilde{X}$ ensure that the two invariants $\delta(x)$ and $m(x)$ can be computed from the local ring at the point $\mathcal{O}_{X,x}$, or even just from its completion, Henselianization or strict Henselianization. In particular, if $\mathcal{O}_{X,x} \to \widetilde{\mathcal{O}_{X,x}}$ is the integral closure of $\mathcal{O}_{X,x}$ in its fraction field $\mathrm{Frac}(\mathcal{O}_{X,x}) = K(X)$, then $\delta(x)$ is just $\dim_k(\widetilde{\mathcal{O}_{X,x}}/\mathcal{O}_{X,x})$, and $m(x)$ is the number of points of $\widetilde{\mathcal{O}_{X,x}}$ above x, and the same result will continue to hold if we replace $\mathcal{O}_{X,x}$ with its completion, Henselianization or strict Henselianization, and $\widetilde{\mathcal{O}_{X,x}}$ with their integral closure in the respective total rings of fractions.

Since $f : \widetilde{X} \to X$ is a birational morphism of curves over an algebraically closed field, it factors uniquely as a composition $\widetilde{X} \to X^{\mathrm{ord}} \to X$ of two birational morphisms of curves, with $f_t : \widetilde{X} \to X^{\mathrm{ord}}$ a gluing morphism and $f_u : X^{\mathrm{ord}} \to X$ a universal homeomorphism (see Proposition 3.3.4). If we name $\mathcal{S}^u := \Delta_{f_u}$, and $\mathcal{S}^t := f_{u*}\Delta_{f_t}$ (see the beginning of Section 3.3 for the notation), we clearly have that, for every closed point $x \in X$, $h^0(\mathcal{S}_x) = h^0(\mathcal{S}^u_x) + h^0(\mathcal{S}^t_x)$; moreover, we have, by definition, $\delta(x) = h^0(\mathcal{S}_x)$, and from our discussion on gluing morphisms (Section 3.3) it follows that $h^0(\mathcal{S}^t_x) = m(x) - 1$. In particular, we have that $\delta(x) \geq m(x) - 1$; when equality holds, x is said to be an *ordinary multiple point*:

Definition/Proposition 3.4.2. If $x \in X$ is a singular point on a reduced curve X over an algebraically closed field k, x is named an *ordinary multiple point* (or an *ordinary singularity*) is any of the following, equivalent conditions are satisfied:

1. $\delta(x) = m(x) - 1$;
2. $X^{\mathrm{ord}} \to X$ is an isomorphism above x;
3. $\widehat{\mathcal{O}}_{X,x}$ is isomorphic, as a k-algebra, to $k[[t_1, \ldots, t_n]]/(t_it_j : i \neq j)$ for some n (and it turns out that $n = m(x)$).

Proof. In light of the discussion above, conditions 1 and 2 are clearly equivalent. That 2 implies 3 is essentially the computation we have carried out in Example 3.3.3, when speaking about local rings of curves obtained as a result of a gluing processes. Finally, the integral closure of $A := k[[t_1, \ldots, t_n]]/(t_it_j : i \neq j)$ in its total ring of fractions is given by $\widetilde{A} = k[[t_1]] \times \ldots \times k[[t_n]]$: in particular, $\widetilde{A}$ has n points over the maximal ideal $\mathfrak{m}_A = (t_1, \ldots, t_n)$ and $\dim_k(\widetilde{A}/A) = n - 1$. This computation is enough to show that 3 implies 1 and that $n = m(x)$. □

Definition 3.4.3. Let X be a reduced curve over an algebraically closed field k. An ordinary multiple point $x \in X$ such that $\delta(x) = 1$, or equivalently $m(x) = 2$, is named an *ordinary double point* or, more simply, a *node* of the curve X.

If X is a geometrically reduced curve over an arbitrary field k, and $x \in X$ is a point, we will write $\delta(x)$ and $m(x)$ to mean the value of those invariants at any point of $X_{\overline{k}}$ lying above x. So, we have that $x \in X$ is a *smooth point* if and only if $\delta(x) = 0$: smooth points are always regular, but the converse is not necessarily true. If $x \in X$ is a non-smooth point of X, then we will name it an ordinary multiple point, a node, etc., whenever any (and hence each) of the points of $X_{\overline{k}}$ lying above it are ordinary multiple points, nodes, etc.

Definition 3.4.4. A curve X over a field k is said to be a *curve with ordinary singularities* if it is geometrically reduced, and all its points are either smooth or ordinary multiple points. If, additionally, all ordinary multiple points of X are nodes, then we will say that X is a *semistable curve.*

We have already discussed how, for a curve X over a field k, the formation of the normalization $\widetilde{X} \to X$ commutes with separable extensions of the field k. If we consider instead an arbitrary extension $k \subseteq k'$, we have a canonical morphism $w : \widetilde{\left(X_{k'}\right)} \to \left(\widetilde{X}\right)_{k'}$ which is not, in general, an isomorphism, but it is certainly always at least a universal homeomorphism (see [Stacks, 0C3N]). Anyway, we have that:

Proposition 3.4.5. If X is a curve with ordinary singularities over a field k, then the formation of $\widetilde{X}$ commutes with arbitrary extensions $k \subseteq k'$ of the base field k. In particular, a closed point $x \in X$ is regular if and only if it is smooth.

Proof. It is clearly not restrictive to suppose $k' = \overline{k}$. We have morphisms $\widetilde{\left(X_{\overline{k}}\right)} \to \left(\widetilde{X}\right)_{\overline{k}} \to X_{\overline{k}}$, and we have recalled above that $\widetilde{\left(X_{\overline{k}}\right)} \to \left(\widetilde{X}\right)_{\overline{k}}$ must be a universal homeomorphism. But since X is a curve with ordinary singularities, we also know that the normalization $\widetilde{\left(X_{\overline{k}}\right)} \to X_{\overline{k}}$ of $X_{\overline{k}}$ is a gluing morphism of reduced curves. From these two considerations it is immediate to conclude that $\widetilde{\left(X_{\overline{k}}\right)} \to \left(\widetilde{X}\right)_{\overline{k}} \to X_{\overline{k}}$ must be an isomorphism. $\square$

Proposition 3.4.6. Let $x \in X$ be an ordinary multiple point of a geometrically reduced curve X over a field k, and $y \in \widetilde{X}$ a point over x. Then, we have that (a) $k(x) \subseteq k(y)$ is separable and (b) the inseparable degree of $k \subseteq k(x)$ divides $\delta(x)$.

Proof. We lose no generality in supposing that x is the only singular point of X.

To prove (a), observe that, since $\widetilde{(X_{\overline{k}})} \to X_{\overline{k}}$ is a gluing morphism, its fibers are reduced. But, thanks to the proposition above, $\widetilde{(X_{\overline{k}})} \to X_{\overline{k}}$ is nothing but the base change of $\widetilde{X} \to X$ to $\overline{k}$; hence, $\widetilde{X} \to X$ has geometrically reduced fibers, whence the separability of $k(x) \subseteq k(y)$.

To prove (b), suppose, without loss of generality, that k is separably closed. Let us denote by $f : \widetilde{X} \to X$ the normalization of X and let Δ_f be the cokernel of $\mathcal{O}_X \hookrightarrow f_*\mathcal{O}_{\widetilde{X}}$. Since the formation of $\widetilde{X}$ commutes with arbitrary field extensions, and, being k separably closed, there exists only one point $\overline{x} \in X_{\overline{k}}$ above $x \in X$, we can be sure that $\delta(x) = h^0(\Delta_{f,x})$. But $h^0(\Delta_{f,x}) = [k(x) : k] \operatorname{length}_{\mathcal{O}_{X,x}}(\Delta_{f,x})$ is certainly a multiple of $[k(x) : k]$, whence (b) follows. □

Corollary 3.4.7. *If $x \in X$ is a node on a geometrically reduced curve X over a field k, then $k \subseteq k(x)$ is separable, and $k(x) \subseteq k(y)$ is also separable for every $y \in \widetilde{X}$ over x.*

Let X be a geometrically reduced curve over a field k, and $x \in X$ an ordinary multiple point. We will say that x is *split* if all points of $\widetilde{X}$ lying over x are rational (and x will be rational too). It is clear that, if $x \in X$ is a split ordinary multiple point of a geometrically reduced curve X, then $\widehat{\mathcal{O}}_{X,x} \cong k[[t_1, \dots, t_n]]/(t_it_j : i \neq j)$, where $n := m(x)$, and the fiber $\widetilde{X}_x$ is simply $\sqcup_{i=1}^n \operatorname{Spec}(k)$.

A curve with ordinary singularities is said to be *split* (or, equivalently, to have *split ordinary singularities*) if all its singular points are split. Every curve X with ordinary singularities becomes split after some large enough finite extension $k \subseteq k'$ of the base field. If all singular points of X are separable over k, then, in view of Proposition 3.4.6, the extension $k \subseteq k'$ can be chosen to be separable. In particular, in light of Corollary 3.4.7, $k \subseteq k'$ can always be chosen to be separable if X is a semistable curve.

3.5. The Picard functor

There are many abelian groups that can be naturally associated to a scheme X: we can compute, for example, the additive group of global function $\mathcal{O}_X(X)$, the multiplicative group of invertible global functions $\mathcal{O}_X(X)^\times$, the Picard group $\operatorname{Pic}(X)$. These three assignments all have, actually, a functorial nature, in the sense that they give rise to presheaves on the site (Sch/X) of X-schemes: we have

- the "additive group" functor $\mathbb{G}_{a,X} : (\mathrm{Sch}/X)^{\mathrm{op}} \to \mathrm{Ab}$, $X' \mapsto \mathcal{O}_{X'}(X')$;

- the “multiplicative group” functor $\mathbb{G}_{m,X} : (\mathrm{Sch}\,/X)^{\mathrm{op}} \to \mathrm{Ab}$, $X' \mapsto \mathcal{O}_{X'}(X')^\times$;
- the absolute Picard functor $\mathrm{Pic}_X : (\mathrm{Sch}\,/X)^{\mathrm{op}} \to \mathrm{Ab}$, $X' \mapsto \mathrm{Pic}(X')$.

It is immediate to realize that $\mathbb{G}_{a,X}$ and $\mathbb{G}_{m,X}$ are both representable; in other words, they define two group schemes over X, which, by abuse of notation, we will still denote by $\mathbb{G}_{a,X}$ and $\mathbb{G}_{m,X}$; as an X-scheme, $\mathbb{G}_{a,X}$ is isomorphic $\mathbb{A}^1_X$ and $\mathbb{G}_{m,X}$ is isomorphic to $\mathbb{A}^1_X \smallsetminus \{0\}$.

The absolute Picard functor Pic_X, instead, is far for being representable. To see this, it is enough to observe that Pic_X is not even a sheaf with respect to the Zariski topology; actually, since all line bundles are locally trivial, its Zariski-sheafification is the zero functor. A possible way of circumventing this problem is replacing the Picard functor with an adequate relative counterpart having better chances of being representable. This section will present, often without poofs, some fundamental results on the construction and representability of such a functor, mostly following [BLR90, Chapter 8].

Definition 3.5.1. If $f : X \to S$ is a quasi-compact and quasi-separated morphism of schemes, and τ is a Grothendieck topology on $(\mathrm{Sch}\,/S)$, we define the τ-relative Picard functor of X/S as $\mathrm{Pic}^\tau_{X/S} := (R^1 f^\tau_*)\mathbb{G}_{m,X}$, where f^τ_* is the pushforward functor $\mathrm{Sh}(\mathrm{Sch}\,/X)_\tau \to \mathrm{Sh}(\mathrm{Sch}\,/S)_\tau$ induced by f.

Let us now name $\mathcal{T}$ the set $\mathcal{T} = \{\text{Zariski}, \text{étale}, \text{fppf}\}$ of increasingly finer, subcanonical Grothendieck topologies of schemes. Our definition can be restated by saying that the functor $\mathrm{Pic}^\tau_{X/S} : (\mathrm{Sch}\,/S)_\tau \to \mathrm{Ab}$ is the τ-sheafification of the assignment $S' \mapsto H^1_\tau(X', \mathcal{O}_{X'})$, where $X' := X \times_S S'$. But, regardless of the chosen topology $\tau \in \mathcal{T}$, $H^1_\tau(X', \mathcal{O}_{X'})$ is nothing but the Picard group of X': this result is generally known as *Hilbert Theorem 90* (see, for example, [Stacks, `03P7`]) and justifies why the functors $\mathrm{Pic}^\tau_{X/S}$ we have defined can really be thought as a relativized and sheafified variant of the absolute Picard functor. The actual values of $\mathrm{Pic}^\tau_{X/S}$ can now be computed with the aid of the Leray spectral sequence:

$$0 \to H^1_\tau(S', f_*\mathbb{G}_{m,X}) \to \mathrm{Pic}(X') \to \mathrm{Pic}^\tau_{X/S}(S') \to H^2_\tau(S', f_*\mathbb{G}_{m,X}) \to H^2_\tau(X, \mathbb{G}_{m,X})$$

Remark 3.5.2. If $S' = \mathrm{Spec}(k')$ is the spectrum of a field, then all sheaves of abelian groups on S'_{Zar} clearly have trivial cohomology, and hence the sequence above gives an isomorphism $\mathrm{Pic}^{\mathrm{Zar}}_{X/S}(k') \cong \mathrm{Pic}(X_{k'})$. If k' is further assumed to be separably closed, an analogous result holds for $S'_{\text{ét}}$, and hence we have $\mathrm{Pic}^{\text{ét}}_{X/S}(k') \cong \mathrm{Pic}(X_{k'})$.

Remark 3.5.3. If $f' : X' \to S'$ is $\mathcal{O}$-connected (Definition 2.2.8), it can be shown, as an application of Hilbert Theorem 90, that $H^1_\tau(S', f_*\mathbb{G}_{m,X}) \cong \operatorname{Pic}(S')$, regardless of the chosen topology $\tau \in \mathcal{T}$.

Remark 3.5.4. If $f' : X' \to S'$ is proper, then, even if we cannot be sure that $H^1_\tau(S', f_*\mathbb{G}_{m,X}) \cong \operatorname{Pic}(S')$, it is still possible to show that $H^1_\tau(S', f_*\mathbb{G}_{m,X})$ is independent of $\tau \in \mathcal{T}$ (in fact, one can reduce to the situation of the previous remark by means of Stein factorization). In particular, if $f' : X' \to S'$ is proper and $S' = \operatorname{Spec}(k')$ is the spectrum of a field, then $H^1_\tau(S', f_*\mathbb{G}_{m,X}) = H^1_{\mathrm{Zar}}(S', f_*\mathbb{G}_{m,X}) = 0 \ \forall \tau \in \mathcal{T}$, and hence $\operatorname{Pic}(X_{k'}) \to \operatorname{Pic}^\tau_{X/S}(k')$ is injective for any topology $\tau \in \mathcal{T}$.

If, for some $\tau \in \mathcal{T}$, $\operatorname{Pic}^\tau_{X/S}$ happens to be representable, we will denote its representative by $\operatorname{Pic}_{X/S} \in (\mathrm{Sch}\,/S)$ and we will name it the *Picard scheme* of X over S. By abuse of notation, we will write $\operatorname{Pic}_{X/S}$ also for the represented functor. It is clear that:

- the representative will not depend on τ, in the sense that, if $\tau, \tau' \in \mathcal{T}$ are two topolgies such that $\operatorname{Pic}^\tau_{X/S}$ and $\operatorname{Pic}^{\tau'}_{X/S}$ are both representable, then they coincide as functors on $(\mathrm{Sch}\,/S)$, and they consequently have the same representative $\operatorname{Pic}_{X/S}$;

- if $\operatorname{Pic}^\tau_{X/S}$ is representable for some $\tau \in \mathcal{T}$, then $\operatorname{Pic}^{\tau'}_{X/S}$ is also representable for every topology $\tau' \in \mathcal{T}$ finer than τ.

3.6. Jacobians

In the case of curves over a field, the representability problem for the relative Picard functor has a positive answer:

Fact 3.6.1. If X is a curve over a field k, then the relative Picard functor $\operatorname{Pic}^{\text{ét}}_{X/k}$ is represented by a smooth commutative locally algebraic k-group (i.e., a commutative k-group scheme that is smooth and locally of finite type over k), which we will denote by $\operatorname{Pic}_{X/k}$. Moreover, if X is $\mathcal{O}$-connected over k and has a k-rational point, then $\operatorname{Pic}^{\mathrm{Zar}}_{X/k}$ is already representable.

The identity component $\operatorname{Pic}^0_{X/k}$ of the Picard group scheme is named the *Jacobian* of X, and it is a smooth connected commutative algebraic k-group (i.e., a connected commutative k-group scheme that is smooth and of finite type over k). If the curve X is smooth over k, then its Jacobian $\operatorname{Pic}^0_{X/k}$ is an abelian variety over k (i.e., a connected commutative k-group scheme that is proper over k).

Proof. We have collected here various results of [BLR90, Chapters 8 and 9]. □

Before going deeper into the study of Jacobians of curves, we need to recall some facts about algebraic groups; in particular, we will need the following structure theorem:

Fact 3.6.2. If G is a smooth connected commutative algebraic group over an algebraically closed field k, then there exists a maximum abelian (=proper over k) quotient A of G, and the kernel of $G \to A$ admits a unique decomposition as a product $\mathbb{G}_{m,k}^t \times U$, where U is a smooth connected affine commutative algebraic group admitting a composition series whose factors are all isomorphic to $\mathbb{G}_{a,k}$. The integer t, and the number u of factors isomorphic to $\mathbb{G}_{a,k}$ appearing in the composition series of U are uniquely determined by G.

Proof. This is a classical result on the structure of commutative algebraic groups, which can be found, for example, in [Mil17]. □

In particular, the rank (i.e., the Krull dimension) of G can be computed as the sum $a(G) + t(G) + u(G)$, where $a(G)$ denotes the rank of its maximum abelian quotient A and is named the *abelian rank* of G, while $t(G)$ and $u(G)$ are the two integers defined above, which we will respectively name the *toric* and *unipotent* rank of G. If G is a smooth connected commutative algebraic group over a possibly non-algebraically closed field k, we can still define its abelian, toric and unipotent ranks by simply letting $a(G) := a(G_{\overline{k}})$, $t(G) := t(G_{\overline{k}})$ and $u(G) := u(G_{\overline{k}})$. It is clear that G is an abelian variety if and only if $t(G) = u(G) = 0$; similarly, we will say that G is *unipotent* if $a(G) = t(G) = 0$, and we will say that it is *toric* (or, equivalently, that it is a *torus*) if $a(G) = u(G) = 0$. An important property of a, t and u is their additivity along exact sequences:

Fact 3.6.3. Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of smooth connected commutative algebraic groups over a field k. Then, $a(G) = a(G') + a(G'')$, $t(G) = t(G') + t(G'')$, and $u(G) = u(G') + u(G'')$.

We are now ready to move to the study of the Picard scheme and the Jacobian of a curve $X \to \mathrm{Spec}(k)$: we will follow [BLR90, Chapter 9]. First, we want a reasonably concrete description of the points of $\mathrm{Pic}_{X/k}$. From the results we have collected in the previous section, it is clear that, if k is separably closed, or if k is arbitrary but X is connected and has a rational point, then $\mathrm{Pic}_{X/k}(k) \cong \mathrm{Pic}(X)$. In the general case of a curve X over an arbitrary field k, we only have an injection $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}_{X/k}(k)$: the k-rational points $\xi \in \mathrm{Pic}_{X/S}(k)$ correspond to line bundles that may not be defined on X itself, but only on $X_{k'}$ for some finite separable extension $k' \supseteq k$, which we will name a "splitting field" for ξ: see, for example, [Lie17, Section 3.1, p. 169]. If we restrict our attention to the Jacobian, i.e. to the identity component of $\mathrm{Pic}_{X/k}$, we get that:

Fact 3.6.4. Let X be a curve over a field k. The group $\mathrm{Pic}^0_{X/k}(k)$ of k-rational points of the Jacobian of X consists precisely of those elements $\xi \in \mathrm{Pic}_{X/k}(k)$ which have multi-degree 0 (where ξ is said to have multi-degree 0 if, for some and hence all splitting fields $k' \supseteq k$ of ξ, ξ defines a line bundle of multi-degree 0 on $X_{k'}$).

Proof. See [BLR90, Corollary 9.3.13, p. 251]. □

The tangent space of the Jacobian of a curve has a particularly convenient description:

Proposition 3.6.1. There is a canonical isomorphism of k-groups between the tangent space of $\mathrm{Pic}_{X/k}$ at the identity and $H^1(X)$.

Proof. The argument we propose is taken from [Rom13, Lemma 3.3.1, p.160]. Let $k[\varepsilon]$ be the k-algebra of dual numbers (i.e., the free k-algebra on the letter ε, modulo the relation $\varepsilon^2 = 0$). If we now consider $X[\varepsilon] := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\varepsilon])$, which we should think of as a thickened version of X, we have a split exact sequence of sheaves of abelian groups on $(\mathrm{Sch}\,/X)_{\mathrm{Zar}}$

$$0 \to \mathbb{G}_{a,X} \to \mathbb{G}_{m,X[\varepsilon]} \to \mathbb{G}_{m,X} \to 0$$

where

- the first morphism $\mathbb{G}_{a,X} \to \mathbb{G}_{m,X[\varepsilon]}$ comes from viewing $X[\varepsilon]$ as an X-scheme, and is given by the exponential $\varphi \mapsto 1 + \varepsilon h^*(\varphi)$, where h is the structure morphism $X[\varepsilon] \to X$;
- the second morphism $\mathbb{G}_{m,X[\varepsilon]} \to \mathbb{G}_{m,X}$ is given by pulling back functions along the closed immersion $X \hookrightarrow X[\varepsilon]$ that identifies X with the closed subscheme of $X[\varepsilon]$ defined by the equation $\varepsilon = 0$;
- the morphism $\mathbb{G}_{m,X[\varepsilon]} \to \mathbb{G}_{m,X}$ we have just described is split because the closed immersion $X \hookrightarrow X[\varepsilon]$ splits the structure map $X[\varepsilon] \to X$, and hence a splitting of $\mathbb{G}_{m,X[\varepsilon]} \to \mathbb{G}_{m,X}$ is provided by the pullback of invertible functions $\mathbb{G}_{m,X} \to \mathbb{G}_{m,X[\varepsilon]}$ along the structure map $h : X[\varepsilon] \to X$.

If f denotes the structure morphism of X over k and we apply $R^1 f_*^{\text{ét}}$ to the split exact sequence above, we get a split exact sequence of sheaves of abelian groups on $(\mathrm{Sch}/k)_{\text{ét}}$

$$0 \to R^1 f_*(\mathbb{G}_{a,X}) \to \mathrm{Pic}_{X[\varepsilon]/k} \to \mathrm{Pic}_{X/k} \to 0 \tag{\dagger}$$

Now, the tangent space of a k-group scheme G at the identity can actually be defined as the kernel of the k-group homomorphism $G[\varepsilon] \to G$: hence, the exact sequence (†) identifies the tangent space of $\mathrm{Pic}_{X/k}$ at the identity with $R^1 f_*(\mathbb{G}_{a,X})$, which is nothing but the cohomology space $H^1(X)$ viewed as a k-group scheme. □

Remark 3.6.2. As the Jacobian of a k-curve X is a smooth algebraic k-group, its rank equals the dimension over k of its tangent space at the identity. Hence, as a consequence of the proposition above, we have that the rank of the Jacobian equals $h^1(X)$, so we may, in particular, write $h^1(X) = a(X) + t(X) + u(X)$, where $a(X)$, $t(X)$ and $u(X)$ denote the abelian, toric and unipotent ranks of the Jacobian of X, which will also be refereed to as the abelian, toric, unipotent rank of the curve X.

Remark 3.6.3. If X is a smooth curve, we know that its Jacobian is an abelian variety, so $h^1(X) = a(X)$, while $t(X) = u(X) = 0$. We will soon see how the toric and unipotent contributions to $h^1(X)$ are related to the presence of singularities on the curve.

We will now try to study how the Jacobian varies within a birational class of curves. We will begin by stating a general result, which we will then specialize in multiple directions.

Proposition 3.6.4. Let $\pi : X' \to X$ be a birational morphism of curves over a field k, and let $f : X \to \mathrm{Spec}(k)$ and $f' : X' \to \mathrm{Spec}(k)$ denote their structure morphisms. Suppose that X has no embedded points. Then, the natural morphism $\mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X'/k}$ is an epimorphism of sheaves on $(\mathrm{Sch}/k)_{\text{ét}}$, whose kernel $\mathcal{K}$ fits into the following exact sequence of sheaves on $(\mathrm{Sch}/k)_{\text{ét}}$:

$$0 \to f_*\mathbb{G}_{m,X} \to f'_*\mathbb{G}_{m,X'} \to f_*(\pi_*\mathbb{G}_{m,X'}/\mathbb{G}_{m,X}) \to \mathcal{K} \to 0 \tag{\dagger}$$

Proof. Since X has no embedded points and π is a birational morphism, π is scheme-theoretically surjective. Hence, also any flat base-change of π will remain scheme-theoretically surjective, and we consequently have a monomorphism $\mathbb{G}_{m,X} \hookrightarrow \pi_*\mathbb{G}_{m,X'}$ of sheaves on $(\mathrm{Flat\,Sch}/X)_{\text{ét}}$, given by pulling back invertible functions from X to X': let us name its cokernel $\mathcal{C}$. The exact sequence $0 \to \mathbb{G}_{m,X} \to \pi_*\mathbb{G}_{m,X'} \to \mathcal{C} \to 0$ of sheaves on $(\mathrm{Flat\,Sch}/X)_{\text{ét}}$ can be pushed forward along the structure map $f : X \to \mathrm{Spec}(k)$ and gives rise to a long exact sequence of sheaves on $(\mathrm{Sch}\,/k)_{\text{ét}}$:

$$0 \longrightarrow f_*\mathbb{G}_{m,X} \longrightarrow f_*\pi_*\mathbb{G}_{m,X'} \longrightarrow f_*\mathcal{C} \longrightarrow$$
$$\longrightarrow (R^1f_*)\mathbb{G}_{m,X} \longrightarrow (R^1f_*)\pi_*\mathbb{G}_{m,X'} \longrightarrow (R^1f_*)\mathcal{C} \longrightarrow \dots$$

Since π is an isomorphism over a dense open subscheme of X, $\mathcal{C}_{|X_{\text{ét}}}$ is only supported on a closed subscheme $Z \subseteq X$ consisting of a finite number of points, and, if we perform any base-change by replacing X by some $Y \to X$, $\mathcal{C}_{|Y_{\text{ét}}}$ will be supported on the preimage of Z in Y. This is enough to conclude that $(R^1f_*)\mathcal{C} = 0$: the complete argument is presented in the Lemma 3.6.5 below.

Now, $(R^1f_*)\mathbb{G}_{m,X}$ is, by definition, $\mathrm{Pic}_{X/k}$; moreover, since π is finite and $\pi_* : \mathrm{Sh}(\mathrm{Sch}\,/X')_{\text{ét}} \to \mathrm{Sh}(\mathrm{Sch}\,/X)_{\text{ét}}$ is consequently exact, we can identify $(R^1f_*)\pi_*\mathbb{G}_{m,X'}$ with $\mathrm{Pic}_{X'/k}$. The proposition now easily follows. □

Lemma 3.6.5. Let S be a scheme, $f : X \to S$ an S-scheme, and $\mathcal{F} \in \mathrm{Sh}(\mathrm{Sch}\,/X)_{\text{ét}}$ a sheaf of abelian groups. If, for every scheme $S' \in (\mathrm{Sch}\,/S)_{\text{ét}}$, the restriction $\mathcal{F}_{|X'_{\text{ét}}}$ (where $X' := X \times_S S'$) is supported on some closed subcheme $Z' \subseteq X'$ which is finite over S', then $(R^pf_*)\mathcal{F} = 0$ as a sheaf on $\mathrm{Sh}(\mathrm{Sch}\,/S)_{\text{ét}}$, for all $p > 1$.

Proof. The computation of higher direct images commutes with switching from the big to the small sites. In other words, if $S' \in (\mathrm{Sch}\,/S)_{\text{ét}}$ and $X' := X \times_S S'$, we have that $(R^pf_{*,\text{big}}\mathcal{F})_{|\,\mathrm{Sh}(S'_{\text{ét}})} = R^pf_{*,\text{small}}(\mathcal{F}_{|\,\mathrm{Sh}(X'_{\text{ét}})})$. Hence, it is clear that we can reduce ourselves to prove that, given an S-scheme X and a sheaf of abelian groups $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ supported on a closed subscheme $Z \subseteq X$ that is finite over S, $R^pf_*\mathcal{F} = 0$ for all $p > 0$. The proof of this consists in two steps: first, we observe that $\mathcal{F}$ is the pushforward of a sheaf of abelian groups on $Z_{\text{ét}}$ (see [Stacks, `04CA`]), then we conclude by using the fact that the pushforward of sheaves for the étale topology along finite morphisms is exact (see [Stacks, `03QP`]). □

We have seen how any birational morphism of curves over an algebraically closed field factors as a gluing morphism followed by a universal homeomorphism of curves (see Proposition 3.3.4): we will now discuss the contribution of each of them to the Jacobian.

Proposition 3.6.6. If we add to the hypotheses of Proposition 3.6.4 the assumption that k is algebraically closed and π is a gluing morphism of reduced curves, then the kernel $\mathcal{K}$ of $\mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X'/k}$ is a torus of rank equal to $(n'-n)-(r'-r)$, where n and n' denote the number of points of X and X' involved in the gluing process, whereas r and r' denote the number of connected components of X and X'.

Proof. Let $x_i \in X$ be the target points, and, for each i, let us denote by $x'_{i,j}$ the points of X' that glue to x_i. We then have a pushout square of k-schemes:

$$\begin{tikzcd} Z' \arrow[r, hook, "j'"] \arrow[d, "\gamma"] & X' \arrow[d, "\pi"] \\ Z \arrow[r, hook, "j"] \arrow[ur, phantom, "\ulcorner", very near end] & X \end{tikzcd}$$

where $\pi : X' \to X$ is a finite, birational morphism of reduced proper curves over k, $Z' := \coprod_{i,j} \operatorname{Spec}(k)$ is the reduced closed subscheme of X' formed by the points that we are gluing together, and $Z := \coprod_i \operatorname{Spec}(k)$ is the reduced closed subscheme consisting of the points of X to which the they are glued.

We will now try to determine the sheaf $\mathcal{C} := \mathbb{G}_{m,X}/\pi_*\mathbb{G}_{m,X'}$ appearing in the sequence (†) of Proposition 3.6.4. The pushout square we have just written down corresponds to a pullback square in $\mathrm{Sh}(\mathrm{Flat}\,\mathrm{Sch}\,/X)_{\text{ét}}$:

$$\begin{tikzcd} \mathbb{G}_{m,X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \pi_*\mathbb{G}_{m,X'} \arrow[d] \\ j_*\mathbb{G}_{m,Z} \arrow[r] & \pi_* j'_*\mathbb{G}_{m,Z'} \end{tikzcd} \tag{$\diamond$}$$

Notice that, since X' and Z' map surjectively to X and Z respectively, the two horizontal arrows are monomorphisms. Since $Z \to X$ is a closed immersion, the left vertical arrow is an epimorphism. Since $Z' \to X'$ is also a closed immersion, we have that $\mathbb{G}_{m,X'} \to j'_*\mathbb{G}_{m,Z'}$ is an epimorphism in $(\mathrm{Sch}\,/X')_{\text{ét}}$; moreover, since π is finite, $\pi_* : \mathrm{Sh}(\mathrm{Sch}\,/X')_{\text{ét}} \to \mathrm{Sh}(\mathrm{Sch}\,/X)_{\text{ét}}$ is exact, and this allows us to conclude that also the right vertical arrow in the diagram ($\diamond$) is an epimorphism. These observations are enough to ensure that the two horizontal arrows have isomorphic cokernels; as a consequence, $\mathcal{C} = (j_*\gamma_*\mathbb{G}_{m,Z'})/j_*\mathbb{G}_{m,Z}$. Since j is a finite morphism, $j_* : \mathrm{Sh}(\mathrm{Sch}\,/X)_{\text{ét}} \to \mathrm{Sh}(\mathrm{Sch}\,/Z)_{\text{ét}}$ is exact, and hence we can also reformulate our result as $\mathcal{C} = j_*(\gamma_*\mathbb{G}_{m,Z'}/\mathbb{G}_{m,Z})$.

Denoting as $g : Z \to \operatorname{Spec}(k)$ and $g' : Z' \to \operatorname{Spec}(k)$ the structure maps of Z and Z' respectively, we can now rewrite the exact sequence (†) of Proposition 3.6.4 as

$$0 \to f_*\mathbb{G}_{m,X} \to f'_*\mathbb{G}_{m,X'} \to g'_*\mathbb{G}_{m,Z'}/g_*\mathbb{G}_{m,Z} \to \mathcal{K} \to 0,$$

It is clear that $f_*\mathbb{G}_{m,X} = \mathbb{G}^r_{m,k}$, $f_*\mathbb{G}_{m,X'} = \mathbb{G}^{r'}_{m,k}$, $g_*\mathbb{G}_{m,Z} = \mathbb{G}^n_{m,k}$ and $g'_*\mathbb{G}_{m,Z'} = \mathbb{G}^{n'}_{m,k}$, where the integers n, n', r, r' are those we have introduced in the statement of the proposition. Hence, our exact sequence is simply:

$$0 \to \mathbb{G}^r_{m,k} \to \mathbb{G}^{r'}_{m,k} \to \mathbb{G}^{n'}_{m,k}/\mathbb{G}^n_{m,k} \to \mathcal{K} \to 0$$

and we may also rewrite it in the "two-dimensional" form:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}^r_{m,k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{G}^{r'}_{m,k} \arrow[r] \arrow[d] & \mathbb{G}^{r'}_{m,k}/\mathbb{G}^r_{m,k} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbb{G}^n_{m,k} \arrow[r] & \mathbb{G}^{n'}_{m,k} \arrow[r] & \mathbb{G}^{n'}_{m,k}/\mathbb{G}^n_{m,k} \arrow[r] \arrow[d, two heads] & 0 \\
 & & & \mathcal{K} &
\end{tikzcd}$$

where the cartesian square on the left is simply obtained by applying f_* to the diagram ($\diamond$). It is now perfectly clear that $\mathcal{K}$ is a torus of the desired rank. $\square$

Proposition 3.6.7. If we add to the hypotheses of proposition 3.6.4 the assumption that k is algebraically closed, X and X' are reduced, and π is injective, then the kernel $\mathcal{K}$ of $\mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X'/k}$ is a smooth connected unipotent algebraic k-group. It is trivial if and only if $X' \to X$ is an isomorphism.

Proof. Let us remark that, in the case we are studying, the map $f_*\mathbb{G}_{m,X} \to f'_*\mathbb{G}_{m,X'}$ appearing in the sequence (†) of Proposition 3.6.4 is clearly an isomorphism. Hence, $\mathcal{K} := \ker(\mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X'/k})$ is simply $f_*\mathcal{C}$, where $\mathcal{C} := (\pi_*\mathbb{G}_{m,X'})/\mathbb{G}_{m,X}$.

Let us suppose, just for simplicity, that f fails to be an isomorphism over a single point, and let us denote by $j : Z \hookrightarrow X$ and $j' : Z \hookrightarrow X'$ the closed immersions of that single point $Z = \mathrm{Spec}(k)$ inside X and X'.

We have to determine $\mathcal{C}$. Since π is finite, $\pi_* : \mathrm{Sh}(\mathrm{Sch}\,/X')_{\text{ét}} \to \mathrm{Sh}(\mathrm{Sch}\,/X)_{\text{ét}}$ is exact, and we can consequently write down the following diagram with exact rows in $\mathrm{Sh}(\mathrm{Flat\,Sch}\,/X)_{\text{ét}}$:

$$\begin{tikzcd}
0 \arrow[r] & 1+\mathcal{I} \arrow[r] \arrow[d, hook] & \mathbb{G}_{m,X} \arrow[r] \arrow[d, hook] & j_*\mathbb{G}_{m,Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & 1+\pi_*\mathcal{I}' \arrow[r] & \pi_*\mathbb{G}_{m,X'} \arrow[r] & \pi_*j'_*\mathbb{G}_{m,Z} \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{I}$ and $\mathcal{I}'$ are the ideal sheaves describing Z in X and X', respectively. Using the snake lemma, it is immediate to show that $\mathcal{C} \cong (1+\pi_*\mathcal{I}')/(1+\mathcal{I})$.

Our aim will now be studying $\mathcal{C} \cong (1+\pi_*\mathcal{I}')/(1+\mathcal{I})$ as a sheaf on $(\mathrm{Flat\,Sch}\,/X)_{\text{ét}}$. First, we analyze the inclusion $\mathcal{I} \subseteq \pi_*\mathcal{I}'$ of $\mathcal{O}_X$-submodules of $\pi_*\mathcal{O}_{X'}$. Thanks to our hypotheses on $f : X' \to X$, we can show that $\pi_*(\mathcal{I}')^n \subseteq \mathcal{I}$ for large enough n (we defer the argument to the Lemma 3.6.8 below). Let N the least value of n for which this is happens: then, we can construct a filtration $\mathcal{J}_n$ of quasi-coherent $\mathcal{O}_X$-submodules of $\pi_*\mathcal{O}_{X'}$ defined as

$$\mathcal{J}_1 = \pi_*\mathcal{I}', \qquad \mathcal{J}_2 = \pi_*(\mathcal{I}')^2 + \mathcal{I}, \quad \ldots, \quad \mathcal{J}_{n-1} = \pi_*(\mathcal{I}')^{n-1} + \mathcal{I}, \qquad \mathcal{J}_N = \mathcal{I}.$$

Since, by construction, $\mathcal{J}_n^2 \subseteq \mathcal{J}_{n+1}$, we get an isomorphism of sheaves $\mathcal{J}_n/\mathcal{J}_{n+1} \cong (1+\mathcal{J}_n)/(1+\mathcal{J}_{n+1})$ given by the exponential map $a \mapsto 1 + a$; as a consequence, the sheaf of ableian groups $\mathcal{C} \in (\mathrm{Flat\,Sch}\,/X)_{\text{ét}}$ can be obtained, by subsequent extensions, out of the coherent sheaves $\mathcal{J}_n/\mathcal{J}_{n+1}$. As these coherent sheaves have a finite support, and hence no higher étale cohomology over $\mathrm{Spec}(k)$, we conclude that the k-group scheme $\mathcal{K} \cong f_*\mathcal{C} \in \mathrm{Sh}(\mathrm{Sch}\,/k)_{\text{ét}}$ can be obtained, by subsequent extensions, from the vector bundles $f_*((1+\mathcal{J}_n)/(1+\mathcal{J}_{n+1})) \cong \mathbb{G}_{a,k}^{d_n}$ whose ranks are $d_n := h^0(\mathcal{J}_n/\mathcal{J}_{n+1})$. In particular, $\mathcal{K}$ is a smooth, unipotent and connected commutative algebraic group over k, whose rank can only be zero if $\mathcal{I} = \pi_*\mathcal{I}'$ (and this, in turn, can only happen if $X' \to X$ is an isomorphism). □

Lemma 3.6.8. Let $(A, \mathfrak{m}_A) \subseteq (B, \mathfrak{m}_B)$ be a finite local extension of Noetherian local rings. Then, there exists an integer $N > 0$ such that $\mathfrak{m}_B^N \subseteq \mathfrak{m}_A$.

Proof. Observe that $B/\mathfrak{m}_A$ is a finite-length A-module. Hence, there exists an $N > 0$ such that the sequence $(\mathfrak{m}_B^n + \mathfrak{m}_A)/\mathfrak{m}_A$ of A-submodules of $B/\mathfrak{m}_A$ stabilizes for $n \geq N$. As a consequence,

$$\frac{\mathfrak{m}_B^N + \mathfrak{m}_A}{\mathfrak{m}_A} = \bigcap_{n \geq 0} \frac{\mathfrak{m}_B^n + \mathfrak{m}_A}{\mathfrak{m}_A} = \bigcap_{j \geq 0} \frac{\mathfrak{m}_A^j B + \mathfrak{m}_A}{\mathfrak{m}_A} = 0$$

where the last equality comes from applying Krull's intersection theorem ([Stacks, 00IP]) to the A-module $B/\mathfrak{m}_A$. Hence, $\mathfrak{m}_B^N \subseteq \mathfrak{m}_A$. □

The two results we have presented (Propositions 3.6.6 and 3.6.7) are enough to study the Jacobian of a geometrically reduced curve. To conclude our survey, the only point which is missing is comparing the Jacobian of a possibly non-reduced curve X with the Jacobian of X_{red}. It is clear that the inclusion $X_{\text{red}} \hookrightarrow X$ can be factored as a sequence of closed immersions $X_{\text{red}} = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_r = X$ such that X_{i-1} is defined inside X_i by an ideal sheaf $\mathcal{J}_i$ such that $\mathcal{J}_i^2 = 0$. To each of these closed immersions, we may apply the following proposition:

Proposition 3.6.9. Let $j : X' \hookrightarrow X$ a closed immersions of curves over k, and suppose that X' is defined in X by an ideal sheaf $\mathcal{J}$ such that $\mathcal{J}^2 = 0$. Then,

$\mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X'/k}$ is an epimorphism of sheaves on $(\mathrm{Sch}/k)_{\text{ét}}$, and its kernel $\mathcal{K}$ is a smooth connected unipotent algebraic k-group.

Proof. We have an exact sequence $0 \to \mathcal{J} \to \mathbb{G}_{m,X} \to j_*\mathbb{G}_{m,X'} \to 0$ of sheaves on $(\mathrm{Sch}/X)_{\mathrm{Zar}}$, where $\mathcal{J} \to \mathbb{G}_{m,X}$ is given by the exponential map $a \mapsto 1 + a$. If $f : X \to \mathrm{Spec}(k)$ is the structure map of X, we can apply f_* to the sequence and get the long sequence $0 \to f_*\mathcal{J} \to f_*\mathbb{G}_{m,X} \to f_*\mathbb{G}_{m,X'} \to R^1 f_*\mathcal{J} \to \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X'/k} \to 0$ of sheaves on $(\mathrm{Sch}/k)_{\text{ét}}$ (while writing this sequence, we have identified $(R^1 f_*) j_*\mathbb{G}_{m,X'}$ with $\mathrm{Pic}_{X'/k}$ by exploiting finiteness of j, which makes the étale pushforward j_* exact). But $\mathcal{J}$ is a quasi-coherent sheaf, and hence $R^1 f_*\mathcal{J} \cong \mathbb{G}_{a,k}^n$ where $n := h^0(\mathcal{J})$. The kernel $\mathcal{K}$ of $\mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X'/k}$ is now a quotient of $R^1 f_*\mathcal{J}$, and hence will be certainly a smooth connected unipotent group. □

Remark 3.6.10. We remark that, in the proposition above, the unipotent kernel of $\mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X'/k}$ may well be trivial even if the closed immersion $X' \hookrightarrow X$ is not.

We are now ready to take a curve $X \to \mathrm{Spec}(k)$ over an algebraically closed field k and apply all the results we have listed above to the normalization morphism $\widetilde{X} \to X$. We have already observed that $\widetilde{X} \to X$ admits a factorization $\widetilde{X} \to X^{\mathrm{ord}} \to X_{\mathrm{red}} \to X$ (see Section 3.4) and each of these morphisms induces a surjective morphism of the Picard schemes whose kernel is always a smooth and connected algebraic k-group and has been carefully described in the propositions we have just proved. Hence, $\mathrm{Pic}^0_{X/k}$ can be obtained by extending the abelian variety $\mathrm{Pic}^0_{\widetilde{X}/k}$ with:

- a smooth connected toric commutative algebraic k-group introduced by $\widetilde{X} \to X^{\mathrm{ord}}$, whose rank is equal to

$$\left[\sum_{x \in \mathrm{Sing}(X)} (m(x) - 1)\right] + [\#\text{ conn. comp. of } X] - [\#\text{ irred. comp. of } X];$$

- a smooth connected unipotent commutative algebraic k-group introduced by $X^{\mathrm{ord}} \to X_{\mathrm{red}}$, which is trivial if and only if $X^{\mathrm{ord}} = X_{\mathrm{red}}$;

- a smooth connected unipotent commutative algebraic k-group introduced by $X \to X_{\mathrm{red}}$, which may however be trivial even if X is not reduced.

For X a curve over an algebraically closed field k, we consequently have that $a(X)$ is the genus of $\widetilde{X}$; $t(X)$ depends on the topology of X and, in particular, it is determined by how the irreducible components of X intersect each other; and $u(X)$ detects the presence of non-ordinary singularities in X_{red}, and may also be influenced by X being non-reduced.

As a corollary of this whole discussion, we can state a crucial criterion for X being a curve with ordinary multiple singularities:

Proposition 3.6.11. Let X be a curve over a field k. Then, X is a curve with ordinary singularities if and only if X is geometrically reduced and $u(X) = 0$.

3.7. Curves with trivial Jacobian

In this section, we will study curves X over a field k *with trivial Jacobian*, i.e., curves such that, equivalently, $H^1(X, \mathcal{O}_X) = 0$, $\mathrm{Pic}^0_{X/k}$ is trivial, or $\mathrm{Pic}^0(X_{\overline{k}})$ is trivial. We will denote by X_i the irreducible components of X viewed as reduced closed subschemes of X.

If X has trivial Jacobian, we have in particular that $\mathrm{Pic}^0(X) = 0$, and hence a line bundle on X is completely determined by the degree of its restrictions to the X_i's. In other words, the multi-degree of a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ encodes all information about it. The two propositions that follow provide, in particular, a characterization of globally generated line bundles in terms of their multi-degree.

Proposition 3.7.1. If X is a curve with trivial Jacobian over an *infinite* field k, and $\mathcal{L}$ is a line bundle on X such that $\deg(\mathcal{L}_{|X_i}) \geq 0$ for all i, then $\mathcal{L}$ has a regular section.

Proof. Let us first prove the result in the case in which the curve X is integral: it follows from the definition of degree of a line bundle that $\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \deg(\mathcal{L})$; hence, since $h^1(\mathcal{O}_X) = 0$ and $\deg \mathcal{L} \geq 0$, we get that $h^0(\mathcal{L}) \geq h^0(\mathcal{O}_X) \geq 1$. Hence, $\mathcal{L}$ must admit a non-trivial section, which, since X is integral, will be regular.

Let us now turn our attention to the general case. Let us name U_i the open subset of X consisting of all points of X_i, except those that are embedded points of X and those that also belong to other irreducible components of X. Since the field k is infinite and, by the previous part of the proof, $\mathcal{L}_{|X_i}$ is globally generated, we can certainly find a global section of $\mathcal{L}_{|X_i}$ whose zero-locus avoids any given finite set of points: in particular, we can find a regular section $s_i \in H^0(X_i, \mathcal{L}_{|X_i})$ whose zero-locus $D'_i := V(s_i)$ is contained in U_i. Now, we can clearly lift each of the D'_i's to an effective Cartier divisor D_i on X such that $\mathrm{Supp}(D_i) = \mathrm{Supp}(D'_i)$ and $D_i \cap X_i = D'_i$ (see Proposition 3.2.2). If we now define $D := \sum_i D_i$, by construction $D_{|X_i} = D'_i$, and hence $\deg(D_{|X_i}) = \deg(\mathcal{L}_{|X_i})$ for all i. But since $\mathrm{Pic}^0(X) = 0$, this is enough to ensure that $\mathcal{L} \cong \mathcal{O}_X(D)$. $\square$

Proposition 3.7.2. If X is a curve with trivial Jacobian, and $\mathcal{L}$ is a line bundle on X admitting a regular section, then $\mathcal{L}$ is globally generated and $H^1(\mathcal{L}) = 0$.

Proof. Since $\mathcal{L}$ admits a regular section, it is the line bundle associated to some effective Cartier divisor D on X. Let us consider the exact sequence $0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{L}_{|D} \to 0$: by looking at the corresponding exact sequence is cohomology, it is immediate to see how $H^1(X) = 0$ implies that $H^1(\mathcal{L}) = 0$ and that $H^0(\mathcal{L}) \to H^0(\mathcal{L}_{|D})$ is surjective. The fact that $H^0(\mathcal{L}) \to H^0(\mathcal{L}_{|D})$ is surjective, in turn, is clearly enough to ensure that $\mathcal{L}$ is globally generated. □

Finally, we observe that integral curves with trivial Jacobian often have a particularly simple description.

Proposition 3.7.3. Let X be an integral curve with trivial Jacobian over a field k, and let $k' := H^0(X)$.

- if X is Gorenstein, then it is a conic over k';
- if X has an invertible sheaf of degree 1 over k', then it is isomorphic to $\mathbb{P}^1_{k'}$;
- if k is algebraically closed, then $X \cong \mathbb{P}^1_{k'}$.

Proof. See [Stacks, `0C6L`]. □

4. Arithmetic surfaces

An *arithmetic surface* $\mathcal{X}$ is a relative curve defined over a one-dimensional regular connected base S. Its generic fiber is a curve X over the fraction field of S, and the arithmetic surface itself can be seen as a *model* of X over S, i.e. a way of extending X to a family of curves indexed by the points of S.

This chapter is an introduction to the theory of arithmetic surfaces. Section 4.1 deals with the definition; then, Sections 4.2 to 4.5 describe some fundamental aspects of their geometry. From Section 4.6 on, arithmetic surfaces are no longer treated as isolated entities, but start being viewed as representatives of their birational equivalence class: the main tools to operate on a surface by means of birational transformations are discussed in Sections 4.7 to 4.9. Then, Sections 4.10 and 4.11 will focus on some particularly important elements of the birational class of a surface, namely the minimal regular model and the canonical model. Section 4.12 discusses how, given a finite morphism of curves, the models of the first one can be used to construct models of the second one, and vice versa. Finally, Section 4.13 gives some results on the Picard group of an arithmetic surface and of its fibers that will be necessary to prove the semistable reduction theorem in the next chapter.

4.1. The definition

To define what an arithmetic surface is, we should first fix what we will name a *Dedekind base*, i.e. a regular integral base scheme S of dimension 1. Examples of commonly used Dedekind bases are $\mathrm{Spec}(\mathbb{Z})$ and spectra of discrete valuation rings. We will reserve the letter η for the generic point of the Dedekind base S, while we will write s to mean any of its closed points. Moreover, K will be the function field of S, while $k(s)$ will mean the residue field of S at its closed point s.

Once a Dedekind base S is fixed, we define an *arithmetic surface* to be a family of curves over S:

Definition 4.1.1. An *arithmetic surface* over S (or, briefly, an S-*surface*) is an integral, proper and faithfully flat S-scheme $\mathcal{X}$, whose fibers have pure dimension 1.

We will start by presenting some results that make it clearer what the multiple requirements listed in this definition actually mean and imply. Let us first consider the flatness condition that, over a Dedekind base, is a particularly easy notion to understand:

Proposition 4.1.2. Let $f : \mathcal{X} \to S$ a morphism of locally Noetherian schemes, where S is Dedekind. Then, f is flat if and only if it sends each associated point of $\mathcal{X}$ to the generic point of S.

Proof. Any flat morphism of locally Noetherian schemes sends associated points to associated points, whence the "only if" part of the statement follows. The converse depends on the fact that any torsion-free module over a DVR is flat. □

Corollary 4.1.3. Suppose S is a Dedekind base and $\mathcal{X}$ a proper, integral scheme over S. Then, $f : \mathcal{X} \to S$ is flat if and only if it is surjective.

Now that the flatness requirement has been discussed, let us consider the dimensional constraint that Definition 4.1.1 enforces:

Proposition 4.1.4. Let S be a universally catenary Noetherian integral scheme, and $\mathcal{X} \to S$ a flat, finite-type, integral S-scheme. Then, (a) the fibers of f are all pure of the same dimension d, and (b) $\dim(\mathcal{X}) = \dim(S) + d$.

Proof. This is an easy consequence of standard results in dimension theory for schemes. Let Γ be an irreducible component of some fiber of $\mathcal{X}$, and let x_Γ denote its generic point. Then, the dimension formula ([Liu, Theorem 8.2.5]) ensures that $\dim(\mathcal{X}_\eta) - \dim(\Gamma) = \mathrm{cod}_{\mathcal{X}}(x_\Gamma) - \mathrm{cod}_S(f(x_\Gamma))$. At the same time, flatness ensures (via [Liu, Theorem 4.3.12]) that $\mathrm{cod}_{\mathcal{X}}(x_\Gamma) = \mathrm{cod}_S(f(x_\Gamma))$. Hence, the fibers of $f : \mathcal{X} \to S$ have all pure dimension $d := \dim(\mathcal{X}_\eta)$, so we have proved (a). Now, (b) can be easily deduced by observing that, if a morphism of Noetherian schemes $f : \mathcal{X} \to S$ has all fibers of the same pure dimension d, then $\dim(\mathcal{X}) = \dim(S) + d$: inequality $\geq$ is obvious, while $\leq$ follows from [Liu, Theorem 4.3.12]. □

Corollary 4.1.5. Let $\mathcal{X} \to S$ a flat, finite-type, integral S-scheme, where S is a Dedekind base. Then, the following are equivalent:
(a) $\mathcal{X}$ has dimension 2;
(b) some fiber of $\mathcal{X} \to S$ has an irreducible component of dimension 1;
(c) all fibers of $\mathcal{X} \to S$ have pure dimension 1.

An important consequence of our discussion on the definition of arithmetic surface is the following one:

Proposition 4.1.6. Let $f : \mathcal{X}' \to \mathcal{X}$ a finite surjective morphism of S-schemes, and suppose $\mathcal{X}'$ and $\mathcal{X}$ are integral. Then, $\mathcal{X}'$ is a surface on S if and only if $\mathcal{X}$ is a surface on S.

Proof. As $\mathcal{X}' \to \mathcal{X}$ is finite surjective, we have that (1) $\mathcal{X}$ is proper over S if and only if $\mathcal{X}'$ is; (2) $\mathcal{X}$ is surjective on S if and only if $\mathcal{X}'$; and (3) $\mathcal{X}'$ has the same dimension of $\mathcal{X}$. Now, the proposition easily follows from Corollaries 4.1.3 and 4.1.5. □

4.2. Generic fiber and special fibers

Let us fix a surface $f : \mathcal{X} \to S$ (Definition 4.1.1). This section presents some results concerning the generic fiber $\mathcal{X}_\eta$, which is a curve over the function field K of S, and its special fibers $\mathcal{X}_s$, which will be curves over $k(s)$; in particular, we will describe how the properties of $\mathcal{X}_\eta$ and $\mathcal{X}_s$ are linked together, and how they are related to those of the S-scheme $\mathcal{X}$.

A crucial invariant that $\mathcal{X}_\eta$ and $\mathcal{X}_s$ share is the Euler-Poincaré characteristic: indeed, we have already observed in the chapter about curves (Proposition 3.1.1) that $\chi(\mathcal{X}_\eta) = \chi(\mathcal{X}_s)$. The common Euler-Poincaré characteristic of the fibers of $\mathcal{X} \to S$ will be denoted by $\chi(\mathcal{X}/S)$.

Under a cohomological point of view, we observe that, since the computation of cohomology commutes with flat base-changes, $H^0(\mathcal{X}_\eta)$ and $H^1(\mathcal{X}_\eta)$ are just the fibers at η of $f_*\mathcal{O}_\mathcal{X}$ and $R^1 f_*\mathcal{O}_\mathcal{X}$:

$$H^0(\mathcal{X}_\eta) = (f_*\mathcal{O}_\mathcal{X})_\eta, \qquad H^1(\mathcal{X}_\eta) = (R^1 f_*\mathcal{O}_\mathcal{X})_\eta.$$

We cannot expect an identical result for a special fiber $\mathcal{X}_s$: we still clearly have morphisms $(f_*\mathcal{O}_\mathcal{X})_s \to H^0(\mathcal{X}_s)$ and $(R^1 f_*\mathcal{O}_\mathcal{X})_s \to H^1(\mathcal{X}_s)$ of $k(s)$-vector spaces, but, a priori, they may fail to be isomorphisms because the base-change $S \to \operatorname{Spec}(k(s))$ is not flat. Despite this, $(R^1 f_*\mathcal{O}_\mathcal{X})_s \to H^1(\mathcal{X}_s)$ actually proves to be always an isomorphism:

Proposition 4.2.1. $H^1(\mathcal{X}_s) = (R^1 f_*\mathcal{O}_\mathcal{X})_s$.

Proof. It is not restrictive to suppose S local, so we have an exact sequence $0 \to \mathcal{O}_\mathcal{X} \xrightarrow{\cdot\pi} \mathcal{O}_\mathcal{X} \to \mathcal{O}_{\mathcal{X}_s} \to 0$, π being the uniformizer of S. By taking the corresponding long exact sequence in cohomology, we get $R^1 f_*\mathcal{O}_\mathcal{X} \xrightarrow{\cdot\pi} R^1 f_*\mathcal{O}_\mathcal{X} \to R^1 f_*\mathcal{O}_{\mathcal{X}_s} \to R^2 f_*\mathcal{O}_\mathcal{X} = 0$, but $R^1 f_*\mathcal{O}_{\mathcal{X}_s}$ is nothing but $H^1(\mathcal{X}_s)$, and hence that sequence tells us that $H^1(\mathcal{X}_s) = (R^1 f_*\mathcal{O}_\mathcal{X}) \otimes_{\mathcal{O}_S} \mathcal{O}_S/\pi\mathcal{O}_S$, which is what we wanted. □

Since $\mathcal{X}$ is an integral scheme, its generic fiber $\mathcal{X}_\eta$ will certainly be an integral curve. Most often, we will working under the stronger hypothesis that $\mathcal{X}_\eta$ is also *geometrically connected* or even *geometrically integral*:

Proposition 4.2.2. Let $f : \mathcal{X} \to S$ be a surface. If the generic fiber $\mathcal{X}_\eta$ is geometrically connected, the special fibers $\mathcal{X}_s$ will also be. If the generic fiber $\mathcal{X}_\eta$ is geometrically integral, then $f : \mathcal{X} \to S$ is $\mathcal{O}$-connected (Definition 2.2.8).

Proof. See [Liu, Corollary 8.3.6]. □

It is important to remark that regularity properties fail, in general, to pass from the generic fiber (or from the surface) to its the special fibers: even if we suppose $\mathcal{X}_\eta$ to be a smooth connected K-curve, and $\mathcal{X}$ to be a regular scheme, $\mathcal{X}_s$ may still be singular, it may have multiple irreducible components, and may even be non-reduced.

Finally, a very useful tool to study the geometry of surfaces is given by the canonical sheaf, that we have introduced and discussed in Section 1.3. For the canonical sheaf $\omega_{\mathcal{X}/S}$ to exist, it is necessary to assume that the surface $\mathcal{X}$ is Cohen-Macaulay (e.g., if $\mathcal{X}$ is normal); under this assumption, moreover, the fibers of $\mathcal{X}$ also admit canonical sheaves $\omega_{\mathcal{X}_\eta/K}$ and $\omega_{\mathcal{X}_s/k(s)}$, which will be nothing but restrictions of $\omega_{\mathcal{X}/S}$: this provides a further, useful link between the generic fiber of $\mathcal{X} \to S$ and the special ones. We will use the canonical sheaf especially to treat the geometry of *regular* arithmetic surfaces, on which $\omega_{\mathcal{X}/S}$ is an *invertible* sheaf.

4.3. Points and divisors

Let $f : \mathcal{X} \to S$ be a surface. Given a point $x \in \mathcal{X}$, we can compute the dimension $\dim(x)$ and the codimension $\mathrm{cod}(x)$ of the closed subscheme $\overline{\{x\}} \subseteq \mathcal{X}$. Under this perspective, there are four kinds of points on $\mathcal{X}$, listed below.

- The unique generic point ξ of $\mathcal{X}$, which also coincides with the only generic point of $\mathcal{X}_\eta$, has $\mathrm{cod}(\xi) = 0$ and $\dim(\xi) = 2$.

- The closed points x_H of the generic fiber have $\mathrm{cod}(x_H) = 1$ and $\dim(x_H) = 1$: in particular, they are prime Weyl divisors, and they are referred to as the *horizontal prime Weyl divisors on* $\mathcal{X}$. They are infinitely many.

- The generic points x_V of any special fiber $\mathcal{X}_s$ have still $\dim(x_V) = 1$ and $\mathrm{cod}(x_V) = 1$; they are prime Weyl divisors and they are referred to as the *vertical prime Weyl divisors on* $\mathcal{X}$ (or the *vertical components* of $\mathcal{X}$). It is clear that only finitely many such points exist for each special fiber.

- Finally, the closed points x of a special fiber $\mathcal{X}_s$ have $\dim(x) = 0$ and $\mathrm{cod}(x) = 2$. Every special fiber has infinitely many closed points.

Weyl divisors, i.e. $\mathbb{Z}$-linear combinations of codimension 1 points, are said to be vertical or horizontal if all their prime components are. We will denote by $\mathrm{Div}_{\mathrm{h}}(\mathcal{X})$, $\mathrm{Div}_{\mathrm{v}}(\mathcal{X})$ and $\mathrm{Div}_s(X)$ the groups of horizontal Weyl divisors on $\mathcal{X}$, vertical Weyl divisors on $\mathcal{X}$, and vertical Weyl divisors on $\mathcal{X}$ supported on a specific special fiber $\mathcal{X}_s$: we clearly have that

$$\mathrm{Div}(\mathcal{X}) = \mathrm{Div}_{\mathrm{v}}(\mathcal{X}) \oplus \mathrm{Div}_{\mathrm{h}}(\mathcal{X}), \quad \mathrm{Div}_{\mathrm{v}}(\mathcal{X}) = \oplus_s \mathrm{Div}_s(\mathcal{X}), \quad \mathrm{Div}_{\mathrm{h}}(\mathcal{X}) \cong \mathrm{Div}(\mathcal{X}_\eta)$$

A closed subscheme C of $\mathcal{X}$ will be said to be *vertical* or *horizontal* if its associated cycle $[C]$ (see Subsection 1.1.5) is a vertical or horizontal effective Weyl divisor.

A vertical prime Weyl divisor $V = \overline{\{x_V\}}$ lies over some closed point $s \in S$, and will be a proper curve over $k(s)$. On the contrary, a horizontal prime Weyl divisor $H = \overline{\{x_H\}}$, even if it is still a dimension one closed subscheme of $\mathcal{X}$, cannot be viewed as curve over a field; instead, it provides a finite cover of the base S:

Proposition 4.3.1. Let $H \subseteq \mathcal{X}$ be a horizontal closed subscheme of $\mathcal{X}$. Then, $H \to S$ is a finite faithfully flat morphism of schemes.

Proof. As each irreducible component of H is a 1-dimensional scheme dominating S, we have that $H \to S$ is quasi-finite (clear) and surjective (by Corollary 4.1.3). To conclude the proof, it is now enough to recall that quasi-finite, proper morphisms are finite (Proposition 2.2.12). □

We conclude this section with a useful result that we will use to ensure the existence of smooth points of the special fibers of a regular surface:

Proposition 4.3.2. If $\mathcal{X} \to S$ is a regular surface, x_H a K-rational point point of $\mathcal{X}_\eta$ and $H := \overline{\{x_H\}}$ the corresponding prime horizontal divisor, then H_s is a smooth point of $\mathcal{X}_s$.

Proof. By the valutative criterion for properness, the K-point x_H extends uniquely to an $\mathcal{O}_{S,s}$-point $h : \mathrm{Spec}(\mathcal{O}_{S,s}) \to \mathcal{X}$, which, in turn, restricts to a morphism $h_s : \mathrm{Spec}(k(s)) \to \mathcal{X}_s$: it is immediate to verify that H_s will be precisely the reduced closed subscheme of $\mathcal{X}_s$ coinciding with the $k(s)$-point h_s. Thanks to the regularity of $\mathcal{X}$, H is an effective Cartier divisor on $\mathcal{X}$ (Proposition 1.1.1), and hence H_s is an effective Cartier divisor on $\mathcal{X}_s$. So, H_s is a regular, $k(s)$-rational point of $\mathcal{X}_s$ (Proposition 3.2.1), and hence $\mathcal{X}_s$ is smooth at H_s. □

4.4. Existence results for divisors

Given x a closed point of a surface $\mathcal{X}$, the total number of prime Weyl divisor passing through it is necessarily infinite (see proposition 1.1.13). Since only finitely many of them can be vertical, we have the following

Proposition 4.4.1. Given x a closed point of a surface $\mathcal{X}$, there is an infinite number of horizontal prime Weyl divisors passing through it (i.e., an infinite number of closed point of the generic fiber that specialize to it).

To construct effective *Cartier* divisor on $\mathcal{X}$, a particularly convenient setting is the one in which the base scheme S is Henselian:

Proposition 4.4.2. If S is the spectrum of a Henselian discrete valuation ring, every effective Cartier divisor $\overline{D}$ of the closed fiber $\mathcal{X}_s$ admits a lifting to a (horizontal) effective Cartier divisor D on $\mathcal{X}$ (i.e., there exists a horizontal effective Cartier divisor D on $\mathcal{X}$ such that $D \cap \mathcal{X}_s = \overline{D}$).

Proof. It is clearly enough to prove the result for a Cartier divisor $\overline{D}$ on $\mathcal{X}_s$ whose support consists of a single closed point P. Let $\overline{h} \in \mathcal{O}_{\mathcal{X}_s,P}$ an equation for $\overline{D}$ near P, and let $h \in \mathcal{O}(U)$ denote any lifting of $\overline{h}$ to a non-zero function defined on an open neighborhood $P \in U \subseteq \mathcal{X}$. If U is chosen small enough, we can clearly suppose that $V(h) \subseteq U$ is connected, and that its closure $\overline{V(h)}$ in $\mathcal{X}$ is purely horizontal. Since $\overline{V(h)}$ is finite over S (Proposition 4.3.1) and connected, Henselianity implies that P is its unique closed point; this means, in particular, that $\overline{V(h)}$ must be entirely contained in U and hence that $V(h) = \overline{V(h)}$ defines a horizontal Cartier divisor D on the whole $\mathcal{X}$ that lifts $\overline{D}$. □

Corollary 4.4.3. Suppose that $\mathcal{X} \to S$ is a surface over the spectrum S of a Henselian discrete valuation ring (with closed points s), and that $x \in \mathcal{X}$ is a non-associated closed point of $\mathcal{X}_s$. Then, there exists an effective horizontal Cartier divisor D such that x is the unique closed point of the special fiber $\mathcal{X}_s$ in the support of D.

We conclude by stating a slightly more general version of Proposition 4.4.2, which is proved exactly in the same way and will turn out to be useful later.

Proposition 4.4.4. Suppose that $\mathcal{X} \to S$ is a surface over the spectrum S of an Henselian Noetherian local ring, let V be any vertical closed subscheme of $\mathcal{X}$, and $\overline{D}$ a Cartier divisor of V. Let us name Γ_i the vertical components of $\mathcal{X}$ that form the support of V, and Γ'_j all the remaining vertical components of $\mathcal{X}$. If $\overline{D}$ avoids all the points where V intersects the curves Γ'_j, then $\overline{D}$ can be lifted to an effective horizontal Cartier divisor on $\mathcal{X}$.

4.5. Intersection theory

Intersection theory is a valuable tool for studying the reciprocal position of divisors on a surface $\mathcal{X} \to S$: its main idea is to attach, to a couple of divisors, an integer representing the way the intersect. We will actually define intersection numbers only when one of the two divisors considered is vertical, and only under the assumption the surface $\mathcal{X}$ is *regular*. Before proceeding further, we recall that, on a regular scheme, the notions of effective Cartier and effective Weyl divisor coincide (see Subsection 1.1.5); hence, we can simply refer to them as *effective divisors*, without further qualifications. We will say *divisor* to mean a (possibly non-effective) Weyl divisor.

Let thus now D and V be two divisors on a regular surface $\mathcal{X}$, and suppose V is vertical (lying over some closed point $s \in S$): we want to define their intersection number, which we will denote by $D \cdot V$. Actually, $D \cdot V$ will not be exactly an integer, but rather a 0-cycle on the base S, id est a (finite) collection of integers attached to the closed points of the base S.

A first definition If the divisors D and V are *effective* and intersect *properly*, meaning that they share no irreducible component, then their scheme-theoretic intersection $D \cap V := V(\mathcal{I}_D + \mathcal{I}_V)$ is a 0-dimensional closed subscheme of $\mathcal{X}$. We can now pass to the associated 0-cycle $[D \cap V]$, and push it down to S (Subsection 1.1.6): the resulting 0-cycle on S will be, by definition, the intersection number $D \cdot V$.

An equivalent reformulation is the following one: to compute $D \cdot V$, we first compute the pullback $D_{|V}$ of the Cartier divisor D to V; now, $D_{|V}$ is a Cartier divisor on the 1-dimensional scheme V, and hence it is a 0-dimensional scheme: we define $D \cdot V$ as its associated 0-cycle, pushed down to S. The same could clearly be done by switching the roles of V and D; hence, to summarize:

$$D \cdot V := f_*([D \cap V]) = f_*([D|_V]) = f_*([V|_D])$$

From this definition of $D \cdot V$, it is immediate to see the bilinearity and symmetry of the intersection pairing; however, it only works whenever D and V are effective divisors that do not share any irreducible component; in particular, it is not suitable to define the *self-intersection number* $V^2 := V \cdot V$ of a vertical divisor, which will play a prominent role in the next sections.

A second definition To circumvent the problem, we will adopt an alternative perspective: suppose V is a prime vertical divisor, and D any effective divisor. While D does not, in general, restrict to a Cartier divisor on V, its associated invertible sheaf $\mathcal{O}_{\mathcal{X}}(D)$ can certainly be restricted to an invertible sheaf $\mathcal{O}_{\mathcal{X}}(D)_{|V}$ on V. As V is a curve over the field $k(s)$, this restricted sheaf has a well defined *degree* (see Subsection 3.2.2), and we can set:

$$D \cdot V := \deg_{k(s)}(\mathcal{O}_{\mathcal{X}}(D)_{|V})[s]$$

The intersection number $D \cdot V$ defined in this way is linear in D; extending by linearity in both entries, we obtain a well-defined intersection number for every pair of divisors, provided one of them is vertical. This new definition treats D and V asymmetrically, but has two remarkable advantages: it works without the hypothesis that D and V meet properly, and clearly shows that $D \cdot V$ only depends on the linear equivalence class of D.

Properties of the intersection form The two definitions we have proposed of $D \cdot V$ prove to agree (whenever they are both applicable): this is an easy consequence of the results on divisors on curves that we have presented in Subsection 3.2.2). We have thus introduced a well-defined intersection form $\mathrm{Div}(\mathcal{X}) \times \mathrm{Div}_{\mathrm{v}}(\mathcal{X}) \to \mathbb{Z}, (D, V) \mapsto D \cdot V$, which has the following properties:

- it is bilinear (as definition 2 clearly shows);
- $D \cdot V$ only depends on the linear equivalence class $\mathcal{O}_{\mathcal{X}}(D) \in \mathrm{Pic}(\mathcal{X})$ of D (it follows from definition 2); in other words, the intersection form descends to a pairing $\mathrm{Pic}(\mathcal{X}) \times \mathrm{Div}_{\mathrm{v}}(\mathcal{X}) \to \mathbb{Z}$;
- it is symmetric, meaning that, if $V_1, V_2 \in \mathrm{Div}_{\mathrm{vert}}(\mathcal{X})$, then $V_1 \cdot V_2 = V_2 \cdot V_1$ (as it is easy to see from definition 1);
- if $D, V \geq 0$, then $D \cdot V \geq 0$, and $D \cdot V = 0$ if and only if $D \cap V = \emptyset$ (as definition 1 clearly shows).

4.5.1. Intersection form on the special fibers

Let us name $\Gamma_1, \ldots, \Gamma_r$ the irreducible components of a special fiber $\mathcal{X}_s$, and d_i their multiplicities: we are now interested in determining how the Γ_i's intersect each other, i.e. what the intersection pairing $\mathrm{Div}_s(\mathcal{X}) \times \mathrm{Div}_s(\mathcal{X}) \to \mathbb{Z}$ looks like, where $\mathrm{Div}_s(\mathcal{X}) = \oplus_i \mathbb{Z}\Gamma_i$ is the group of (vertical) divisors supported on the special fiber $\mathcal{X}_s$. First, we observe that, clearly, $\Gamma_i \cdot \Gamma_j \geq 0$ whenever $i \neq j$. Moreover, the whole special fiber $\mathcal{X}_s = \sum_i d_i Z_i$ is a principal divisor over some open neighborhood $s \in V \subseteq S$, and hence it is numerically trivial, in the sense that $V \cdot \mathcal{X}_s = 0$ for every vertical divisor V.

From these considerations, it is immediate to deduce that the intersection paring on $\mathrm{Div}_s(\mathcal{X})_{\mathbb{R}} \cong \mathbb{R}^r$ is *negative semi-definite*, with radical spanned by the connected components of $\mathcal{X}_s$: this is a consequence of the linear algebra lemma that follows.

Lemma 4.5.1. Let $V = \langle v_1, \ldots v_r \rangle$ an r-dimensional vector space over $\mathbb{R}$, endowed with a bilinear symmetric form $(-,-)$. Suppose that $w := v_1 + \ldots + v_r$ is in the radical, and that $(v_i, v_j) \geq 0 \ \forall i \neq j$. Then, $(-,-)$ is negative semidefinite, and a vector $v \in V$ is isotropic if and only if its components along v_i and v_j are equal for each pair v_i, v_j of basis vectors such that $(v_i, v_j) > 0$.

Proof. Since w lies in the radical, we can be sure that $\sum_i (v_i, v_j) = 0$ for any given j. Hence, given any vector $v = \sum_i a_i v_i$, we can rewrite its norm $v^2 := (v, v)$ as

$$\begin{aligned} 2v^2 &= 2\sum_{ij} a_i a_j (v_i, v_j) \\ &= 2\sum_{ij} a_i a_j (v_i, v_j) - \sum_j a_j^2 \sum_i (v_i, v_j) - \sum_i a_i^2 \sum_j (v_i, v_j) \\ &= -\sum_{ij} (a_i - a_j)^2 (v_i, v_j), \end{aligned}$$

so we clearly have $v^2 \leq 0$. Furthermore, $v^2 = 0$ if and only if $a_i = a_j$ whenever $(v_i, v_j) > 0$. $\square$

In particular, if $\mathcal{X}$ is a surface having geometrically connected generic fiber (and hence geometrically connected special fibers, see Proposition 4.2.2), then the numerically trivial vertical divisors are precisely the rational multiples of $\mathcal{X}_s$.

Remark 4.5.2. The fact that $\mathcal{X}_s$ is numerically trivial allows us to give an interpretation of the self-intersection number of a vertical component. In fact, if $\Gamma_1, \ldots, \Gamma_n$ are the vertical components of a special fiber $\mathcal{X}_s$, and $m_1, \ldots, m_n$ are their multiplicities, then $\Gamma_i \cdot \mathcal{X}_s = 0$ may be rewritten as

$$\Gamma_i^2 = -\sum_{j \neq i} \frac{m_j}{m_i} \Gamma_i \cdot \Gamma_j$$

and this shows that $-\Gamma_i^2$ can be thought as the number of times Γ_i intersects the remaining components of $\mathcal{X}_s$, if one takes carefully into account multiplicities.

4.5.2. Adjunction formula

A regular surface $\mathcal{X}$ always admits a invertible canonical sheaf $\omega_{\mathcal{X}/S}$: it is hence natural to ask how it intersects the components of a special fiber $\mathcal{X}_s$. The main result, in this sense, is known as *adjunction formula* (Proposition 4.5.3): we will now sketch its derivation, relying on the definitions and techniques we have introduced in Section 1.3.

If V is an effective divisor on $\mathcal{X}$ contained in some special fiber $\mathcal{X}_s$, we can draw the following commutative diagram:

$$\begin{tikzcd} V \arrow[r, hook, "i"] \arrow[d, "g"] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec}(k(s)) \arrow[r, hook, "j"] & S \end{tikzcd}$$

where i and j are regular closed immersions of codimension 1, and f and g are flat Gorenstein morphisms of relative dimension 1. We can now compute the relative dualizing complex $\omega^\bullet_{V/S}$, equivalently, as $i^! f^! \mathcal{O}_S$ or as $g^! j^! \mathcal{O}_S$; in light of what we have observed in Section 1.3, we have that:

- $f^!(\mathcal{O}_S)$ is just $\omega_{\mathcal{X}/S}[1]$; moreover, since $\omega_{\mathcal{X}/S}$ is invertible and i is a regular closed immersion, we have that $i^! f^! \mathcal{O}_S = i^!(\omega_{\mathcal{X}/S})[1] = (\omega_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{X}}(V))_{|V}[0]$;
- Since, up to replacing S with some open neighborhood of s, $\operatorname{Spec}(k(s))$ is a principal effective Cartier divisor in S, we have that $j^!(\mathcal{O}_S)$ is simply $\mathcal{O}_{k(s)}[-1]$. Hence, $g^! j^! \mathcal{O}_S = g^! \mathcal{O}_{k(s)}[-1] = \omega_{V/k(s)}[0]$.

Comparing these two results, we get that $\omega_{V/k(s)}$ and $(\omega_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{X}}(V))_{|V}$ are isomorphic invertible sheaves on the curve V. Their degrees over $k(s)$ can be respectively written as $\deg(\omega_{V/k(s)}) = -2\chi(\mathcal{O}_V)$ and $\deg((\omega_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{X}}(V))_{|V}) = \omega_{\mathcal{X}/S} \cdot V + V^2$: hence, we obtain that

Proposition 4.5.3 (Adjunction formula)**.** If $\mathcal{X} \to S$ is a regular surface, and V a vertical divisor such that $V \leq \mathcal{X}_s$ for some special fiber $\mathcal{X}_s$, then:

$$\omega_{\mathcal{X}/S} \cdot V = -2\chi(V) - V^2.$$

As an immediate application of the adjunction formula, we can characterize the vertical components of $\mathcal{X}$ that have non-positive intersection with the canonical sheaf. Let thus Γ denote a vertical component of the regular surface $\mathcal{X}$ and let k' be the function field of Γ, which will be a finite extension of $k := k(s)$, where $s \in S$ is the closed point over which Γ lies.

- Suppose $\omega_{\mathcal{X}/S} \cdot \Gamma < 0$. Then, there can only be two possibilities.
  - ⋆ $\Gamma^2 = 0$ and $\chi(\Gamma) > 0$: in this case, Γ must exhaust a whole connected component of $\mathcal{X}_s$, and the fibers of $\mathcal{X}$ consequently have positive Euler-Poincaré characteristic: $\chi(\mathcal{X}/S) > 0$.
  - ⋆ $\Gamma^2 = -[k' : k]$ and $\chi(\Gamma) > 0$: in this case, we will name Γ a (-1)-*curve*.
- Suppose $\omega_{\mathcal{X}/S} \cdot \Gamma = 0$. Then, there are only two possibilities:
  - ⋆ $\Gamma^2 = 0$ and $\chi(\Gamma) = 0$. In this case, Γ must exhaust a whole connected component of $\mathcal{X}_s$, and the fibers of the surface consequently have Euler-Poincaré characteristic $\chi(\mathcal{X}/S) = 0$.
  - ⋆ $\Gamma^2 = -2[k' : k]$, and $\chi(\Gamma) > 0$: in this case, Γ is named a (-2)-*curve*.

Let us systematize these results.

Definition 4.5.4. Given $d \geq 1$, a $(-d)$-*curve* on a regular surface $\mathcal{X}$ is a vertical component Γ satisfying $\chi(\Gamma) > 0$ and $\Gamma^2 = -d[k' : k]$, where k' denotes the function field of Γ.

Remark 4.5.5. It is possible to prove that, if Γ is a (-1)-curve, then it must be isomorphic to $\mathbb{P}^1_{k'}$. If Γ is a (-2)-curve, instead, the best we can say is that it is a conic over k', not necessarily isomorphic to $\mathbb{P}^1_{k'}$. If k is algebraically closed, then all $(-d)$-curves are always $\cong \mathbb{P}^1_k$. These results all follow from the characterization of curves with trivial Jacobian that we have presented in Proposition 3.7.3.

Proposition 4.5.6. Let $\mathcal{X}$ be a regular surface. Then,

- all (-1)-curves of $\mathcal{X}$ have negative intersection with the canonical sheaf $\omega_{\mathcal{X}/S}$; moreover, if $\chi(\omega_{\mathcal{X}/S}) \leq 0$, this property characterizes (-1)-curves among all vertical components;
- all (-2)-curves of $\mathcal{X}$ have zero intersection with the canonical sheaf $\omega_{\mathcal{X}/S}$; moreover, if $\chi(\omega_{\mathcal{X}/S}) < 0$, this property characterizes (-2)-curves among all vertical components.

4.5.3. Projection formula

To conclude our survey of intersection theory, we have to say something about how intersection form changes along morphisms of surfaces. The main result in this direction is known as *projection formula*:

Proposition 4.5.7 (Projection Formula)**.** Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a surjective morphism of regular surfaces over the same base S. Then, for every two divisors $D_1 \in \mathrm{Div}(\mathcal{X}_1), D_2 \in \mathrm{Div}(\mathcal{X}_2)$, supposing that at least one of them is vertical, we have:

$$D_1 \cdot f^{-1} D_2 = f_* D_1 \cdot D_2$$

where f_* denotes the pushforward of cycles along f (as defined in Subsection 1.1.6) and f^{-1} the pullback of effective Cartier divisors along f (as defined in Section 1.1 and discussed in Subsection 1.1.3).

Proof. See [Liu, Theorem 9.2.12]. □

4.6. Models

Given $\mathcal{X} \to S$ a surface, we name a *model* of $\mathcal{X}$ any element of the birational class of S-surfaces to which $\mathcal{X}$ belongs. In other words, a model for $\mathcal{X}$ is any surface $\mathcal{Y}$ endowed with a distinguished S-birational equivalence between $\mathcal{Y}$ and $\mathcal{X}$. The models of $\mathcal{X}$, together with the birational morphisms between them, form a category $\mathcal{M}_S(\mathcal{X})$, which is actually a preordered ordered set: the preorder relation is named *dominance* (this is just a particular instance of the general discussion developed in Subsection 2.1.5).

If $X \to K$ is an integral curve (where K denotes the function field of S), then a *model* of X over S is an S-surface $\mathcal{X}$, together with a distinguished birational equivalence over S between X and $\mathcal{X}$; again, the model of a curve form a category $\mathcal{M}_S(X)$ which is a preordered set.

The difference between the notions of model of an integral curve and model of a surface is mostly terminological: given a surface $\mathcal{X}$ over S, the category $\mathcal{M}_S(\mathcal{X})$ of the models of $\mathcal{X}$ and the category $\mathcal{M}_S(X)$ of the models of its generic fiber $X := \mathcal{X}_\eta$ are identical. Conversely, given an integral K-curve X, it is often not difficult to find a surface $\mathcal{X}$ over S birational to X (it is always possible if S is affine), and hence the category $\mathcal{M}_S(X)$ of the models of the curve X coincides with $\mathcal{M}_S(\mathcal{X})$.

As a first simple property of the category $\mathcal{M}_S(\mathcal{X})$ of the models of a surface $\mathcal{X}$, we show that the supremum of two models always exists:

Proposition 4.6.1. Given $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{M}_S(\mathcal{X})$ two models of a same surface $\mathcal{X} \to S$, the graph Γ of the birational equivalence between them (see Definition 2.1.5) is still an S-surface, and it is the supremum of $\mathcal{X}_1$ and $\mathcal{X}_2$ in $\mathcal{M}(\mathcal{X})$.

Proof. It is clear that Γ is integral, and that it is a proper, dominant S-scheme. Moreover, the dimension of the generic fiber Γ_η is clearly 1, as $K(\Gamma) = K(\mathcal{X}_1) = K(\mathcal{X}_2)$ has transcendence degree 1 over K. This is enough to ensure that $\Gamma \to S$ is a surface (see discussion in Section 4.1). Proving that it is the sup of $\mathcal{X}_1$ and $\mathcal{X}_2$ is now straightforward. □

Most of the time, we will be actually concerned not with the class of all models $\mathcal{M}_S(\mathcal{X})$ of a surface $\mathcal{X}$, but instead with some full subcategories $\mathcal{M}_S^{\mathcal{P}}(\mathcal{X}) \subseteq \mathcal{M}_S(\mathcal{X})$ consisting of models having some particular property: the most important cases are those in which $\mathcal{P}$ is the property of being normal or regular. Within a single class models $\mathcal{M}_S^{\mathcal{P}}(\mathcal{X})$, we will often look for the simplest ones, i.e. those satisfying some minimality condition with respect to the dominance order:

Definition 4.6.2. Given an S-surface $\mathcal{X}$, we say that a model $\mathcal{Y} \in \mathcal{M}_S^{\mathcal{P}}(\mathcal{X})$ is $\mathcal{P}$-*relatively minimal* if it is a minimal element of the preordered set $\mathcal{M}_S^{\mathcal{P}}(\mathcal{X})$, i.e. if no other model in $\mathcal{M}_S^{\mathcal{P}}(\mathcal{X})$ is strictly dominated by $\mathcal{Y}$. It is said to be $\mathcal{P}$-*minimal* if it is a minimum element of the preordered set $\mathcal{M}_S^{\mathcal{P}}(\mathcal{X})$, i.e. if all models in $\mathcal{M}_S^{\mathcal{P}}(\mathcal{X})$ dominate it. A $\mathcal{P}$-surface $\mathcal{X} \to S$ is said to be *relatively minimal* (resp. *minimal*) if it is a relatively minimal (resp. minimal) $\mathcal{P}$-model of itself.

4.6.1. Normal models

Normal models will have a central role in our discussion, because they are both easy to construct and well-behaved. First, we will give a useful, sufficient condition for normality.

Proposition 4.6.3. Let $\mathcal{X} \to S$ be a surface. If its generic fiber $\mathcal{X}_\eta$ is a normal curve, and all its special fibers $\mathcal{X}_s$ are reduced, then $\mathcal{X}$ is normal.

Proof. This is an immediate consequence of Serre's criterion for normality ([Stacks, `031S`]). □

The following propositions discuss the construction of normal models:

Proposition 4.6.4. If the Dedekind base S is Nagata (e.g., if K has characteristic 0, or if S is the spectrum of a complete Noetherian local ring), and $\mathcal{X}$ is any surface on S, then the normalization $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ is a normal model of $\mathcal{X}$; more precisely, it is the minimal normal model of $\mathcal{X}$ dominating $\mathcal{X}$.

Proof. Since S is Nagata, $\widetilde{\mathcal{X}} \to \mathcal{X}$ is finite; hence, Proposition 4.1.6 ensures that it is still a surface. The rest of the proposition is obvious. □

Remark 4.6.5. If we want to reinterpret the proposition above under a more categorical point of view, we can state that, if S is Nagata, $\mathcal{M}_{\mathrm{norm}}(\mathcal{X}) \subseteq \mathcal{M}(\mathcal{X})$ is a reflective subcategory, the reflector being given by normalization. As a consequence, in particular, the supremum of two normal models exists in $\mathcal{M}_{\mathrm{norm}}(\mathcal{X})$, and is nothing but the reflection in $\mathcal{M}_{\mathrm{norm}}(\mathcal{X})$ of their sup in $\mathcal{M}(\mathcal{X})$, i.e. the normalization of the graph of their birational equivalence.

Remark 4.6.6. As two birational normal curves are isomorphic, all normal models of a surface share the same generic fiber.

Birational morphisms of normal surfaces have a particularly simple structure, thanks to Zariski's Main Theorem:

Proposition 4.6.7. Let $f : \mathcal{X}' \to \mathcal{X}$ be a birational morphism of surfaces over S, and suppose that $\mathcal{X}$ is normal. Then, f will be an isomorphism over a dense open subset of $\mathcal{X}$, whose complement $\mathrm{E}(f) \subseteq \mathcal{X}$ (see Convention 2.1.4 for notation) only contains of a finite number of closed point of $\mathcal{X}$, the fibers above whom are connected vertical closed subschemes of $\mathcal{X}'$.

Proof. That f is an isomorphism over a dense open subset of $\mathcal{X}$ was already proved in Proposition 2.1.3; moreover, the normality of $\mathcal{X}$ ensures that $\mathrm{E}(f)$ has codimension ≥ 2 (see Corollary 2.3.12): this implies that $\mathrm{E}(f)$ only consists of a finite number of closed points of $\mathcal{X}$. If we take any $x \in \mathrm{E}(f)$ an name s the closed point of S over which x lies, Zariski's Main Theorem (Theorem 2.2.10 and Proposition 2.2.11) ensures that $\mathcal{X}'_x$ is connected and cannot contain any isolated point: it must consequently be pure of dimension 1, and will hence be a vertical closed subscheme of $\mathcal{X}'$. □

Another important consequence of normality is the fact that each vertical component Γ of a normal surface $\mathcal{X}$ corresponds to an S-valuation ν_Γ of the first kind of the function field $K(\mathcal{X})$ (see Definition 2.3.9): we will use the notation $\mathrm{Vert}(\mathcal{X})$ to mean set $\{\nu_\Gamma : \Gamma \text{ vertical component of } \mathcal{X}\}$ collecting all these S-valuations. Since birational equivalences leave the function field unvaried, given any two birational normal surfaces $\mathcal{X}$ and $\mathcal{X}'$, the sets $\mathrm{Vert}(\mathcal{X})$ and $\mathrm{Vert}(\mathcal{X}')$ can be compared, and it is clear that, if $\mathcal{X}'$ dominates $\mathcal{X}$, then $\mathrm{Vert}(\mathcal{X}') \supseteq \mathrm{Vert}(\mathcal{X})$ (see Proposition 2.3.11). A remarkable consequence of the structure result on birational morphisms of normal surfaces stated above (Proposition 4.6.7) is that the converse is also true, and hence that $\mathrm{Vert}(\cdot)$ characterizes a normal model completely.

Proposition 4.6.8. Let $\mathcal{X}$ and $\mathcal{X}'$ be two normal models of a same surface. Then, $\mathcal{X}'$ dominates $\mathcal{X}$ if and only if $\mathrm{Vert}(\mathcal{X}') \supseteq \mathrm{Vert}(\mathcal{X})$.

Proof. We still only need to prove the "if" implication. Take the graph Γ of the birational equivalence between $\mathcal{X}$ and $\mathcal{X}'$, i.e. their sup in $\mathcal{M}(\mathcal{X})$: we have two birational morphisms $\pi : \Gamma \to \mathcal{X}$ and $\pi' : \Gamma \to \mathcal{X}'$. Suppose, by contradiction, that $\mathcal{X}'$ does not dominate $\mathcal{X}$, or, in other words, that $\mathrm{E}(\pi') \neq \emptyset$. Since $\mathcal{X}'$ is normal, $\mathrm{E}(\pi') \subseteq \mathcal{X}'$ will consist of a finite number of closed points: let $x' \in \mathrm{E}(\pi')$ be one of them. We know that the fiber $\Gamma_{x'}$ is connected of pure dimension 1 (Proposition 4.6.7); hence, the image $\pi(\Gamma_{x'})$ is a connected closed subset of $\mathcal{X}$ that cannot be reduced to a single point. Consequently, $\pi(\Gamma_{x'})$ will contain some point $x \in \mathcal{X}$ of codimension 1, corresponding to some valuation of the first kind ν on $\mathcal{X}$. The valuation ν will remain of the first kind also in Γ, with center at some point of $\Gamma_{x'}$ (see Proposition 2.3.11). Then, the center of ν in $\mathcal{X}'$ must be the closed point x', and we have proved that $\nu \in \mathrm{Vert}(\mathcal{X}) \smallsetminus \mathrm{Vert}(\mathcal{X}')$, which contradicts our hypothesis $\mathrm{Vert}(\mathcal{X}') \supseteq \mathrm{Vert}(\mathcal{X})$. □

Remark 4.6.9. Under a categorical point of view, $\mathrm{Vert}(\cdot)$ can be seen as a functor from a birational class of normal surfaces $\mathcal{M}_{\mathrm{norm}}(\mathcal{X})$ to the power set of all possible valuations on $K(\mathcal{X})$: the proposition we have just proved says that this functor is fully faithful.

Definition 4.6.10. If $\Gamma \in \mathrm{Vert}(\mathcal{X})$ is a vertical component of a normal surface $\mathcal{X}$, and $\mathcal{X}'$ is a normal model dominating $\mathcal{X}$, so that $\mathrm{Vert}(\mathcal{X}) \subseteq \mathrm{Vert}(\mathcal{X}')$, then we say *the strict transform of* Γ *in* $\mathcal{X}'$ to mean Γ, thought as an element of $\mathrm{Vert}(\mathcal{X}')$, i.e. as a vertical component of $\mathcal{X}'$. Extending by linearity, we may similarly define the *strict transform* in $\mathcal{X}'$ of any vertical Weyl divisor on $\mathcal{X}$.

4.6.2. Vertical components of normal models

Take now $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{M}_{\mathrm{norm}}(\mathcal{X})$ two birational normal surfaces, and let $\Gamma \in \mathrm{Vert}(\mathcal{X}_1) \cap \mathrm{Vert}(\mathcal{X}_2)$ a common vertical component (lying over some $s \in S$): we will write Γ_1 and Γ_2 to mean the curve Γ viewed as a vertical component of $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively. We are now going to inquire what properties Γ_1 and Γ_2 have in common; in other words, we are interested in those properties that Γ exhibits unaltered inside *any* normal model of $\mathcal{X}$ in which it appears. First, we observe that

Proposition 4.6.11. Γ_1 and Γ_2 are canonically birational $k(s)$-curves; moreover, if $\mathcal{X}_1$ dominates $\mathcal{X}_2$, then Γ_1 dominates Γ_2.

Proof. Γ_1 and Γ_2 correspond, by hypothesis, to the same valuation ν of $K(\mathcal{X})$; hence, they share the same function field $\mathrm{Frac}(\mathcal{O}_\nu)$, and are consequently birational. If $\mathcal{X}_1$ dominates $\mathcal{X}_2$, then the birational morphism $\mathcal{X}_1 \to \mathcal{X}_2$ must clearly restrict to a morphism $\Gamma_1 \to \Gamma_2$, inducing the identity on the common function field $\mathrm{Frac}(\mathcal{O}_\nu)$ of the two curves: in other words, Γ_1 dominates Γ_2. □

Corollary 4.6.12. Γ_1 and Γ_2 have the same normalization, and, in particular, the same abelian rank.

Another important invariant is given by multiplicity:

Proposition 4.6.13. Γ has the same multiplicity in $(\mathcal{X}_1)_s$ and $(\mathcal{X}_2)_s$.

Proof. If π denotes a local parameter of S at s, and ν the valuation of $K(\mathcal{X})$ corresponding to Γ, then we have that the multiplicity of Γ in $(\mathcal{X}_i)_s$ can be computed as $\nu(\pi)$: hence, it is the same for $i = 1, 2$. □

4.6.3. Regular models

Existence of regular models is much harder to guarantee: this is known as the problem of *desingularization*.

Definition 4.6.14. If $\mathcal{X}$ is a surface with normal generic fiber, then a *desingularization* of $\mathcal{X}$ is a regular model $\mathcal{X}'$ of $\mathcal{X}$ that dominates $\mathcal{X}$, and such that the birational morphism $\mathcal{X}' \to \mathcal{X}$ is an isomorphism over the regular points of $\mathcal{X}$.

In many important cases, desingularizing a surface is actually possible:

Fact 4.6.1. The following hold.

(a) If $\mathcal{X}$ is a surface with smooth generic fiber, or if $\mathcal{X}$ is any surface, but the base scheme S is *excellent*, then $\mathcal{X}$ admits a regular model.
(b) If a surface $\mathcal{X}$ has normal generic fiber and admits a regular model, then it also admits a desingularization.

Proof. See, for example [Liu, Subsection 8.3.4], and especially [Liu, Theorem 8.3.50]. □

It can be interesting to study more in detail how desingularization reshapes the special fibers of a normal surface:

Definition 4.6.15. Let $\mathcal{X}$ be a normal surface, and $\Gamma \leq \mathcal{X}_s$ a vertical component of multiplicity 1. Let $P_1, \ldots, P_k$ be the singular points of $\mathcal{X}_s$ that lie on Γ. The *entanglement number* of Γ in $\mathcal{X}$ is the total number of points of $\widetilde{\Gamma}$ lying over $P_1, \ldots, P_k$, where $\widetilde{\Gamma}$ denotes the normalization of Γ.

Proposition 4.6.16. Let $\mathcal{X}$ be a normal surface and $\Gamma \in \operatorname{Vert}(\mathcal{X})$ a component of multiplicity 1. If $\mathcal{X}$ admits a desingularization $\mathcal{X}'$, and Γ' denotes the strict transform of Γ in $\mathcal{X}'$ (Definition 4.6.10), then the entanglement number of Γ' in $\mathcal{X}'$ is equal to the one of Γ in $\mathcal{X}$.

Proof. The birational morphism of surfaces $\mathcal{X}' \to \mathcal{X}$ restricts to a birational morphism of curves $\Gamma' \to \Gamma$. Let $P_1, \dots, P_k$ the singular points of $\mathcal{X}_s$ lying on Γ. If we take a closed point $x \in \Gamma \setminus \{P_1, \dots, P_k\}$, then x is a regular point of $\mathcal{X}_s$ and hence a regular point of $\mathcal{X}$; consequently, $\mathcal{X}' \to \mathcal{X}$ is an isomorphism above x: this means, in particular, that the unique preimage x' of x in $\mathcal{X}'$ will be a regular point of $\mathcal{X}'_s$. If, instead, we take $x \in \{P_1, \dots, P_k\}$, then $\mathcal{X}' \to \mathcal{X}$ may or may not be an isomorphism above x: in the first case, x has a unique preimage $x' \in \mathcal{X}'$, lying on Γ', that must also be a singular point of $\mathcal{X}'_s$; in the second case, the fiber $\mathcal{X}'_x$ consists of a certain number of vertical curves $V_1, \dots, V_r$ (Proposition 4.6.7), and each point $x' \in \Gamma'$ lying over x will also be a point of one of the $V_1, \dots, V_r$: in other words, multiple components of $\mathcal{X}'_s$ cross at x', which will hence be, in particular, a singular point of $\mathcal{X}'_s$.

We have showed that the singular points of $\mathcal{X}'_s$ belonging to Γ' are precisely those lying over the singular points of $\mathcal{X}_s$ that belong to Γ. From this, the proposition immediately follows (recalling that Γ and Γ' have the same normalization). $\square$

The regular models of a surface $\mathcal{X}$ form a subcategory $\mathcal{M}_{\mathrm{reg}}(\mathcal{X}) \subseteq \mathcal{M}_{\mathrm{norm}}(\mathcal{X})$ of normal models, whose study will be the main aim of the next sections. We will show, in particular, that, under reasonable hypotheses, $\mathcal{M}_{\mathrm{reg}}(\mathcal{X})$ is a *reflective* subcategory of $\mathcal{M}_{\mathrm{norm}}(\mathcal{X})$, and that it has a minimum element, which will be named the *minimal regular model* of $\mathcal{X}$.

4.7. Blowing up

Now that we have introduced the categories of models, normal models and regular models of a surface $\mathcal{X}$, we will try to populate them by finding new surfaces dominating or dominated by $\mathcal{X}$. If we work within the realm of normal surfaces, this essentially means studying the image of the fully faithful functor $\operatorname{Vert}(\cdot)$ that we have introduced in Subsection 4.6.1, i.e. determining how to make new vertical components appear and how to contract existing vertical components.

This section will start by discussing blowups at closed points, which are the main technique to find new surfaces dominating a given surface:

Proposition 4.7.1. If $\mathcal{X} \to S$ is a surface, $x \in \mathcal{X}$ is a closed point, and $\mathcal{X}' \to \mathcal{X}$ is the blowup of $\mathcal{X}$ at x, then $\mathcal{X}'$ is still a surface on S, and $\mathcal{X}' \to \mathcal{X}$ is a birational morphism of surfaces.

Proof. It is easy to see, from the properties of blowups (Subsection 1.1.4), that $\mathcal{X}'$ is an integral scheme, surjective and proper over S; moreover, $\mathcal{X}' \to \mathcal{X}$ is clearly an isomorphism at the level of generic fibers. This is enough to guarantee that $\mathcal{X}'$ is still an S-surface (see the results in Section 4.1). Moreover, it is perfectly clear that the blowup morphism $\mathcal{X}' \to \mathcal{X}$ is birational. $\square$

Blowing up a normal surface at a closed point may not produce a normal surface as a result; regularity, instead, is always preserved (see Proposition 1.1.10): the blowup of a *regular* surface $\mathcal{X}$ at a closed point $x \in \mathcal{X}$ is still a regular surface $\mathcal{X}'$, which is moreover particularly simple to describe; indeed, denoting as $k' := k(x)$ the residue filed of $\mathcal{X}$ at x, and writing E for the fiber $\mathcal{X}'_x$, we have the following.

- $E \cong \mathbb{P}^1_{k'}$ and $E^2 = -[k' : k]$; in other words, E is a (-1)-curve (Definition 4.5.4).
- $\mathrm{Vert}(\mathcal{X}') = \mathrm{Vert}(\mathcal{X}) \sqcup \{E\}$.
- For every effective divisor D on $\mathcal{X}$, its total transform in $\mathcal{X}'$ (as defined in Section 1.1 and Subsection 1.1.3) is $f^{-1}(D) = D' + \mu E$, where D' denotes the strict transform (Definition 4.6.10) of D in $\mathcal{X}'$ and $\mu \in \mathbb{N}$ can be written (thanks to the projection formula, Proposition 4.5.7) as $\mu = (D' \cdot E)/(-E^2)$.
- If D_1, D_2 are two vertical divisors on $\mathcal{X}$, and D_1', D_2' denote their strict transforms in $\mathcal{X}'$, we have (still thanks to the projection formula) that $D_1 \cdot D_2 = D_1' \cdot D_2' + (E \cdot D_1')(E \cdot D_2')/(-E^2)$. In particular, blowup makes the self-intersection of vertical components become more negative, in the sense that, if C is a vertical component of $\mathcal{X}$ and C' is its strict transform in $\mathcal{X}'$, then $(C')^2 = C^2 - (E \cdot C')/(-E^2) \leq C^2$.

Blowups at closed points actually play a fundamental role in the realm of regular surfaces, since their composition allows us to recover any given birational morphism:

Proposition 4.7.2. Let $f : \mathcal{X}' \to \mathcal{X}$ be any birational morphism of regular surfaces, $\mathcal{E}(f) := \mathrm{Vert}(\mathcal{X}') \smallsetminus \mathrm{Vert}(\mathcal{X}) = \{E_1, \ldots, E_n\}$ the set of vertical components contracted by f, and $x_1, \ldots, x_n$ the (not necessarily distinct) points of $\mathcal{X}$ to which the E_i's contract. Then, f can be factored as a sequence of n blowups at closed points; moreover, the center of the first blowup may be chosen to be any of the points x_i.

Proof. Let x be one of the points x_i, and let $\mathcal{X}_1$ be the blowup of $\mathcal{X}$ at x: $\mathcal{X}_1$ dominates $\mathcal{X}$, and $\mathrm{Vert}(\mathcal{X}_1) \smallsetminus \mathrm{Vert}(\mathcal{X})$ only consists of the exceptional divisor of the blowup. Since $\mathcal{X}'$ is regular, $\mathcal{X}'_x$ will be a Cartier divisor on $\mathcal{X}'$ (Proposition 1.1.1); hence, by the universal property of blowups, the morphism $\mathcal{X}' \to \mathcal{X}$ factors as

$$\mathcal{X}' \longrightarrow \mathcal{X}_1 \xrightarrow{\text{blowup}} \mathcal{X}.$$

Now, if we apply the same argument to $\mathcal{X}' \to \mathcal{X}_1$, we obtain a further factorization

$$\mathcal{X}' \longrightarrow \mathcal{X}_2 \xrightarrow{\text{blowup}} \mathcal{X}_1 \xrightarrow{\text{blowup}} \mathcal{X};$$

and, proceeding recursively in this way, we will clearly be able to eventually write $\mathcal{X}' \to \mathcal{X}$ as the composition of n subsequent blowups at closed points. □

An important corollary of this structure result is the following fact.

Corollary 4.7.3. If $\mathcal{X}' \to \mathcal{X}$ is a nontrivial birational morphism of regular surfaces, $\mathcal{X}'$ contains at least one (-1)-curve $E \in \mathrm{Vert}(\mathcal{X}') \smallsetminus \mathrm{Vert}(\mathcal{X})$: in other words, a relatively minimal regular surface (Definition 4.6.2) contains no (-1)-curves.

Moreover, if we take into account the fact that only finitely many closed fibers of a regular surface can contain (-1)-curves (see [Liu, Lemma 9.3.17]), we have the following

Corollary 4.7.4. A normal surface $\mathcal{X} \to S$ can only dominate finitely many of its regular models. In particular, $\mathcal{M}_{\mathrm{reg}}(\mathcal{X})$ is an Artinian preordered set (in the sense that it satisfies the descending chain condition).

4.8. Blowing down

We will now start considering the problem of blowing down a surface $\mathcal{X}$, i.e. of building from $\mathcal{X}$ a new surface $\mathcal{X}'$ within the same birational class and dominated by $\mathcal{X}$. In the context of normal surfaces, this corresponds to the problem of contracting vertical components: given a normal surface $\mathcal{X}$ and a finite set $\mathcal{E} \subseteq \mathrm{Vert}(\mathcal{X})$ of vertical components, we seek a normal model $\mathcal{X}_{\mathcal{E}}$ of $\mathcal{X}$ such that $\mathrm{Vert}(\mathcal{X}_{\mathcal{E}}) = \mathrm{Vert}(\mathcal{X}) \smallsetminus \mathcal{E}$. The main technique to blow down a surface will be constructing an appropriate semiample divisor (Definition 1.2.3) on it.

Proposition 4.8.1. Suppose $\mathcal{L}$ is a semiample invertible sheaf on a surface $\mathcal{X} \to S$ such that $\mathcal{L}_{|\mathcal{X}_\eta}$ is not trivial. Let us consider the morphism $\psi_{\mathcal{L}} : \mathcal{X} \to \mathcal{X}'$ induced by $\mathcal{L}$, where $\mathcal{X}' := \mathrm{Proj}_S(f_*\mathcal{L}_\bullet)$: see Section 1.2 for notation. We then have that $\mathcal{X}'$ is a projective S-surface, and $\psi_\mathcal{L} : \mathcal{X} \to \mathcal{X}'$ is a birational morphism. Moreover, if $\mathcal{X}$ is normal, $\mathcal{X}'$ will also be normal, and $\psi_\mathcal{L} : \mathcal{X} \to \mathcal{X}'$ contracts precisely those vertical components of $\mathcal{X}$ to which the restriction of $\mathcal{L}$ is trivial:

$$\mathcal{E}(\psi_\mathcal{L}) := \mathrm{Vert}(\mathcal{X}) \smallsetminus \mathrm{Vert}(\mathcal{X}') = \{\Gamma \in \mathrm{Vert}(\mathcal{X}) : \mathcal{L}_{|\Gamma} \cong \mathcal{O}_\Gamma\}.$$

Proof. The proof of this result follows [Rom13, Theorem 2.4.5, p. 156], and is essentially an application of the study of the properties of $\psi_\mathcal{L}$ that we have presented in Section 1.2.

First, we observe that, since $\mathcal{X}$ is proper over S, $\mathcal{X}'$ will be a projective (and hence, in particular, proper) S-scheme (Definition/Proposition 1.2.7). Since $\mathcal{X} \to \mathcal{X}'$ is $\mathcal{O}$-connected (Definition 2.2.8), $\mathcal{X}'$ inherits from $\mathcal{X}$ the property of being integral; still thanks to $\mathcal{O}$-connectedness, $\mathcal{X}'$ also inherits normality, if $\mathcal{X}$ is normal. The restriction $\mathcal{L}_{|\mathcal{X}_\eta}$ of $\mathcal{L}$ to the generic fiber of $\mathcal{X}$ is a semiample, non-trivial invertible sheaf on an integral curve; hence, it is ample (by Proposition 3.2.8), and hence $\psi_\mathcal{L}$ induces an isomorphism $\mathcal{X}_\eta \to \mathcal{X}'_\eta$ at the level of generic fibers. All these observations are easily seen to be enough to ensure that $\mathcal{X}'$ is still a surface over S, that $\psi_\mathcal{L}$ is a birational morphism, and that $\mathcal{X}'$ is normal whenever $\mathcal{X}$ is.

Now, all that is left is to determine, in the case $\mathcal{X}$ normal, which vertical components $\psi_\mathcal{L}$ contracts. For each $\Gamma \in \mathrm{Vert}(\mathcal{X})$, we have two mutually exclusive possibilities: either $\Gamma \in \mathcal{E}(\psi_\mathcal{L})$, and hence $\psi_\mathcal{L}(\Gamma)$ consists of a single point, or $\Gamma \notin \mathcal{E}(\psi_\mathcal{L})$, and then $\psi_\mathcal{L}(\Gamma)$ is a 1-dimensional closed subscheme of $\mathcal{X}'$. Suppose $\mathcal{L}_{|\Gamma}$ is trivial: then, the global sections of $\mathcal{L}_{|\Gamma}$ and of its tensor powers are all constant; as a consequence, all points of Γ have the same image through $\psi_\mathcal{L}$, so $\psi_\mathcal{L}(\Gamma)$ is a single point. Conversely, suppose that $\mathcal{L}_{|\Gamma}$ is not trivial and, without loss of generality, assume that S is affine and $\mathcal{L}$ globally generated. Since $\mathcal{L}$ is globally generated and $\mathcal{L}_{|\Gamma}$ is not trivial, it is possible to find a global section of $\mathcal{L}$ on $\mathcal{X}$ that does not vanish neither at all points of Γ, neither at no point of Γ: hence, $\psi_\mathcal{L}(\Gamma)$ is not reduced to a single point. □

The proposition we have just proved inspires the following definition:

Definition 4.8.2. Given a normal surface $\mathcal{X}$ and finite subset $\mathcal{E} \subseteq \mathrm{Vert}(\mathcal{X})$, a semiample invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ such that (1) $\mathcal{L}_{|\mathcal{X}_\eta}$ is not trivial, (2) $\mathcal{L}_{|\Gamma}$ is trivial for $\Gamma \in \mathcal{E}$ and (3) $\mathcal{L}_{|\Gamma}$ is not trivial for $\Gamma \in \mathrm{Vert}(\mathcal{X}) \smallsetminus \mathcal{E}$ is named a *contraction line bundle* for $\mathcal{E}$.

And, in light of this definition, we can restate Proposition 4.8.1 by saying that:

Theorem 4.8.3. If a normal surface $\mathcal{X}$ admits a contraction line bundle $\mathcal{L}$ for a set of vertical components $\mathcal{E} \subseteq \mathrm{Vert}(\mathcal{X})$, then the contraction $\mathcal{X} \to \mathcal{X}_{\mathcal{E}}$ exists, and is provided by the morphism $\psi_{\mathcal{L}}$ induced by $\mathcal{L}$. Moreover, the normal surface $\mathcal{X}_{\mathcal{E}}$ is projective over S.

Now it is clearly important to seek ways of building semiample invertible sheaves on a surface. We will start by stating a useful and simple semiampleness criterion:

Proposition 4.8.4. Horizontal effective Cartier divisors on a surface $f : \mathcal{X} \to S$ are all semiample.

Proof. The proof presented here follows [Liu, Lemma 8.3.32, p.359]. If $D = 0$, there is nothing to prove; hence, we may suppose that D is non-zero. Since $D \to S$ is an affine morphism (Proposition 4.3.1), it is trivially true that $\mathcal{O}_D(D)$ is globally generated over S. In order to prove that D is semiample, it is consequently enough to show that $f_*\mathcal{O}_{\mathcal{X}}(nD) \to f_*\mathcal{O}_D(nD)$ is surjective for $n \gg 0$. To do this, we will consider the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}((n-1)D) \to \mathcal{O}_{\mathcal{X}}(nD) \to \mathcal{O}_D(nD) \to 0$$

of coherent sheaves on $\mathcal{X}$, which induces a long exact sequence of coherent $\mathcal{O}_S$-modules:

$$\begin{aligned} 0 \longrightarrow f_*\mathcal{O}_{\mathcal{X}}((n-1)D) \longrightarrow f_*\mathcal{O}_{\mathcal{X}}(nD) \longrightarrow f_*(\mathcal{O}_D(nD)) \longrightarrow \\ \longrightarrow R^1f_*\mathcal{O}_{\mathcal{X}}((n-1)D) \longrightarrow R^1f_*\mathcal{O}_{\mathcal{X}}(nD) \longrightarrow R^1f_*(\mathcal{O}_D(nD)) \longrightarrow \dots \end{aligned}$$

First, since $D \to S$ is affine, the last term, i.e. $R^1f_*(\mathcal{O}_D(nD))$, is 0 for all n.

Since $nD|_{\mathcal{X}_\eta}$ is a non-zero effective Cartier divisor on an integral curve, it is ample (see Proposition 3.2.10), and its cohomology in positive degree consequently vanishes for n large enough (see [Stacks, `0B5U`]): from this, we deduce that the two terms $R^1f_*\mathcal{O}_{\mathcal{X}}((n-1)D)$ and $R^1f_*\mathcal{O}_{\mathcal{X}}(nD)$ of our sequence are eventually torsion coherent sheaves on S; since S is a Dedekind scheme, this implies that they have finite length.

To summarize, we have that the map $R^1f_*\mathcal{O}_{\mathcal{X}}((n-1)D) \to R^1f_*\mathcal{O}_{\mathcal{X}}(nD)$ is eventually a surjection of finite length coherent sheaves on S; hence, the (finite) length of $R^1f_*\mathcal{O}_{\mathcal{X}}(nD)$ becomes stable for $n \gg 0$, and, as a consequence, $R^1f_*\mathcal{O}_{\mathcal{X}}((n-1)D) \to R^1f_*\mathcal{O}_{\mathcal{X}}(nD)$ is eventually an isomorphism. From this, we can clearly conclude that $f_*\mathcal{O}_{\mathcal{X}}(nD) \to f_*(\mathcal{O}_D(nD))$ becomes surjective for large enough n, which was what we wanted. □

We can now combine this semiampleness criterion with the existence results on Cartier divisors presented in Section 4.4 to prove the following existence result:

Proposition 4.8.5. If $\mathcal{X}$ is a normal surface on the spectrum S of a Henselian discrete valuation ring, then a contraction line bundle (Definition 4.8.2) exists for any proper subset $\mathcal{E} \subsetneq \mathrm{Vert}(\mathcal{X})$ of vertical components of $\mathcal{X}$.

Proof. Let $\mathcal{E} = \{\Gamma_1, \dots, \Gamma_m\}$ the set of all vertical curves we want to contract, and $\mathrm{Vert}(\mathcal{X}) = \{\Gamma_1, \dots, \Gamma_m, \Gamma'_1, \dots, \Gamma'_n\}$ the set of all vertical curves (i.e. the irreducible components of $\mathcal{X}_s$). We have seen that the contraction can be obtained as the morphism $\psi_{\mathcal{L}}$ associated to a globally generated invertible sheaf $\mathcal{L}$ such that the restrictions $\mathcal{L}_{|\Gamma_i}$ are trivial while the restrictions $\mathcal{L}_{|\Gamma'_i}$ are not.

Let us pick closed points $x_1, \dots, x_n$ on $\mathcal{X}$ such that x_i belongs to Γ'_i and to no other irreducible component of the special fiber: it is immediate to construct a Cartier divisor $\overline{D}$ on $\mathcal{X}_s$ supported precisely at those points (see Proposition 3.2.1). As we showed in Proposition 4.4.2, Henselianity allows us to lift $\overline{D}$ to an effective Cartier horizontal divisor D on $\mathcal{X}$ that will satisfy $\mathrm{Supp}(D \cap \mathcal{X}_s) = \{x_1, \dots, x_n\}$. It is now straightforward to verify that $\mathcal{O}_{\mathcal{X}}(D)$ is a contraction line bundle for $\mathcal{E}$. □

Corollary 4.8.6. Every normal surface over the spectrum of an Henselian discrete valuation ring is projective.

Proof. This is obtained by applying Theorem 4.8.3 and Proposition 4.8.5 with $\mathcal{E} = \emptyset$. □

4.9. Singularities produced by contractions

Given a set of vertical components $\mathcal{E} \subseteq \mathrm{Vert}(\mathcal{X})$ of a *regular* surface $\mathcal{X} \to S$, we are clearly not only interested in determining whether a contraction $\mathcal{X}_{\mathcal{E}} \in \mathcal{M}_{\mathrm{norm}}(\mathcal{X})$ exists, but also in establishing whether or not $\mathcal{X}_{\mathcal{E}}$ is still a regular surface.

To obtain positive answers to the first question, we have already clarified that our strategy is proving the existence of a *contraction line bundle* for $\mathcal{E}$ (Definition 4.8.2). If S is the spectrum of an Henselian discrete valuation ring, this is an easy task for every normal surface $\mathcal{X}$ (Proposition 4.8.5); for regular surfaces, some interesting results are also available over a general Dedekind base.

Theorem 4.9.1. If $\mathcal{X} \to S$ is a regular surface, and $\mathcal{E} \subsetneq \mathrm{Vert}(\mathcal{X})$ is a finite set of vertical components of $\mathcal{X}$ only consisting of (-1) and (-2)-curves (Definition 4.5.4) that does not contain a whole connected component of a special fiber, then a contraction line bundle for $\mathcal{E}$ exists.

Proof. This follows from [Liu, Theorem 9.4.2 and Lemma 9.4.5]. □

Corollary 4.9.2. Regular S-surfaces are projective.

Proof. Apply Theorems 4.8.3 and 4.9.1 with $\mathcal{E} = \emptyset$. □

This section is devoted to analyzing the second question, i.e. whether or not a contraction $\psi : \mathcal{X} \to \mathcal{X}_{\mathcal{E}}$ of a regular surface $\mathcal{X}$ produces singularities. We will suppose, just for simplicity, that all curves of $\mathcal{E} = \{E_1, \ldots, E_n\}$ contract to the same closed point $P \in \mathcal{X}_{\mathcal{E}}$, which will lie over some $s \in S$. We will denote by $E \subseteq \mathcal{X}$ the connected reduced divisor $E = E_1 + \ldots + E_n$ and we will let $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ be the sheaf of ideals defining it.

As a first step, we introduce the notion of *fundamental divisor*:

Definition 4.9.3. A *fundamental divisor* Z for $\mathcal{E}$ is a divisor $E \leq Z \leq \mathcal{X}_s$, such that $\mathrm{Supp}(Z) = \mathrm{Supp}(E)$, $Z \cdot E_i \leq 0$ for all i and $H^0(Z, \mathcal{O}_Z)$ is a field.

It is easy to verify that a fundamental divisor always exists, as we can build it by means of the following simple iterative method. We start with $Z_0 = E$; then, to obtain Z_{j+1} from Z_j, we proceed as follows: if there exists an $E_i \in \mathcal{E}$ such that $Z_j \cdot E_i > 0$, we set $Z_{j+1} = Z_j + E_i$; otherwise, we stop.

Proposition 4.9.4. The sequence Z_j defined above stops after a finite number of steps, providing a fundamental divisor for $\mathcal{E}$.

Proof. First, we observe that all the Z_j's forming the sequence are $\leq \mathcal{X}_s$: indeed, suppose that $Z_j \leq \mathcal{X}_s$, and that Z_j contains E_i a number of times which is already equal to the multiplicity d_i of E_i in $\mathcal{X}_s$. Then, $Z_j \cdot E_i = -(\mathcal{X}_s - Z_j)E_i \leq 0$; hence, by construction, Z_{j+1} will not contain an additional copy of E_i. This is enough to prove that $Z_j \leq \mathcal{X}_s$ for all j, and that Z_j consequently stops after a finite number of iterations. If Z_∞ is the last term of the sequence, to be sure that it is a fundamental divisor we still only have to prove that $H^0(Z_\infty)$ is a field. To do this, let us further analyze a single iterative step $Z_{j+1} = Z_j + E_i$: we have an exact sequence of sheaves $0 \to \mathcal{O}_{E_i}(-Z_j) \to \mathcal{O}_{Z_{j+1}} \to \mathcal{O}_{Z_j} \to 0$. Since $Z_j \cdot E_i > 0$, $\mathcal{O}_{E_i}(-Z_j)$ has negative degree, and hence $H^0(\mathcal{O}_{E_i}(-Z_j)) = 0$. As a result, the canonical morphism $H^0(Z_{j+1}) \to H^0(Z_j)$ is an injection; hence, $H^0(Z_\infty) \hookrightarrow H^0(Z_0)$ will also be injection: now, since $Z_0 = E$ is a reduced connected divisor, $H^0(Z_0) = H^0(E)$ is a field, so $H^0(Z_\infty)$ is certainly a domain. But, as $Z_\infty \leq X_s$, $H^0(Z_\infty)$ is also clearly a finite algebra over $k := k(s)$. As a result, it is a field, as we wanted. □

It turns out that the crucial hypothesis we need to extract information about the regularity of $\mathcal{X}_{\mathcal{E}}$ at P is that the fundamental divisor Z has trivial Jacobian: we will now present two crucial lemmas relying on that hypothesis (Lemmas 4.9.5 and 4.9.6), after which we will be able to provide a criterion for the regularity of $\mathcal{X}_{\mathcal{E}}$ (Theorem 4.9.8).

Lemma 4.9.5. If $H^1(Z, \mathcal{O}_Z) = 0$, then, for every $h \geq 0$, $\mathcal{O}_{\mathcal{X}}(-hZ)$ is ψ-globally generated, $H^1(Z, \mathcal{O}_Z(-hZ)) = 0$ and $R^1\psi_*(\mathcal{O}_{\mathcal{X}}(-hZ)) = 0$.

Proof. The definition of fundamental divisor ensures that the restrictions of $\mathcal{O}_{\mathcal{X}}(-hZ)$ to all curves $E_i \in \mathcal{E}$ have degrees ≥ 0. As Z is a curve with trivial Jacobian, we can be guaranteed that $\mathcal{O}_Z(-hZ)$ is a globally generated invertible sheaf on Z such that $H^1(\mathcal{O}_Z(-hZ)) = 0$: when the residue field $k := k(s)$ is infinite, this is just Proposition 3.7.2; when k is a finite field, a subtler argument is necessary, for which we refer to [Liu, Lemma 9.4.11].

Now, thanks to the formal function theorem (Theorem 2.2.1), to prove that $R^1f_*(\mathcal{O}_{\mathcal{X}}(-hZ)) = 0$ one may proceed by showing that $H^1(\mathcal{O}_{rZ}(-hZ)) = 0$ for all $r \geq 1$, and then taking the limit as $r \to \infty$: the family $\{rZ : r \geq 1\}$ is indeed cofinal in the collection of all closed subschemes of $\mathcal{X}$ supported on the fiber of $\mathcal{X}$ above P. We have already proved $H^1(\mathcal{O}_{rZ}(-hZ)) = 0$ result in the case $r = 1$ for all $h \geq 0$; the case $r > 1$ can easily be deduced reasoning by induction, exploiting the exact sequence

$$0 \to \mathcal{O}_Z(-(h+r)Z) \to \mathcal{O}_{(r+1)Z}(-hZ) \to \mathcal{O}_{rZ}(-hZ) \to 0.$$

To conclude the proof of the lemma, we still only have to show that $\mathcal{O}_{\mathcal{X}}(-hZ)$ is ψ-globally generated for all $h \geq 0$: but this is easy, because we have already showed that $-hZ$ is globally generated as a divisor on Z, and we can be sure that $\psi_*\mathcal{O}_{\mathcal{X}}(-hZ) \to \psi_*\mathcal{O}_Z(-hZ)$ is surjective as $R^1\psi_*\mathcal{O}_{\mathcal{X}}(-(h+1)Z) = 0$. □

For each $h \geq 0$, we have a closed subscheme hZ or $\mathcal{X}$, defined by the sheaf of ideals $\mathcal{O}_{\mathcal{X}}(-hZ) \hookrightarrow \mathcal{O}_{\mathcal{X}}$. If we take the scheme-theoretic image of hZ in $\mathcal{X}_{\mathcal{E}}$ (see Section 1.1), we obtain a closed subscheme $f(hZ)$ of $\mathcal{X}_{\mathcal{E}}$, whose defining sheaf of ideals, since ψ is $\mathcal{O}$-connected (Lemma 2.2.9), is just $f_*\mathcal{O}_{\mathcal{X}}(-hZ) \hookrightarrow f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{\mathcal{E}}}$.

Lemma 4.9.6. If $H^1(Z, \mathcal{O}_Z) = 0$ and P is the point of $\mathcal{X}_{\mathcal{E}}$ to which E contracts, considered as a reduced closed subscheme of $\mathcal{X}_{\mathcal{E}}$, then the scheme-theoretic image of hZ in $\mathcal{X}_{\mathcal{E}}$ is hP; moreover $f_*\mathcal{O}_{hZ} \cong \mathcal{O}_{hP}$ as $\mathcal{O}_{\mathcal{X}_{\mathcal{E}}}$-algebras.

Proof. For brevity, we will name $\mathcal{Z}_h := f_*(\mathcal{O}_{\mathcal{X}}(-hZ))$ the sheaf of ideals defining $f(hZ)$ in $\mathcal{X}_{\mathcal{E}}$, we will write $\mathcal{P}_1$ for the sheaf of ideals defining P, and $\mathcal{P}_h :=$

$\mathcal{P}_1^h$ for the one defining hP. First, we notice that, since $R^1\psi_*(\mathcal{O}_{\mathcal{X}}(-hZ)) = 0$ (Lemma 4.9.5), we have a canonical isomorphism $f_*\mathcal{O}_{hZ} \cong \mathcal{O}_{f(hZ)}$ of $\mathcal{O}_{\mathcal{X}_\mathcal{E}}$-algebras.

The closed subscheme $f(Z)$ is only supported at P. Moreover, a consequence of the definition of fundamental divisor, $(\mathcal{O}_{f(Z)})_P = (f_*\mathcal{O}_Z)_P$ is a field, and $f(Z)$ is consequently a closed reduced subscheme of $\mathcal{X}_\mathcal{E}$: in other words, $f(Z) = P$, which means that the lemma holds for $h = 1$.

To prove the lemma for $h \geq 1$, it is now enough to verify that $\mathcal{Z}_h = \mathcal{Z}_1^h$. The inclusion $\mathcal{Z}_1^h \subseteq \mathcal{Z}_h$ is trivial. The other inclusion follows from the fact that $\mathcal{Z}_1$ is f-globally generated and $R^1f_*\mathcal{O}_{\mathcal{X}} = 0$, as the following cohomological lemma ensures. □

Lemma 4.9.7. Let $f : X \to Y$ a proper morphism of locally Noetherian schemes whose fibers have Krull dimension ≤ 1, and such that $R^1f_*\mathcal{O}_X = 0$. Then, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two f-globally generated quasi-coherent sheaves on X, the canonical morphism $(f_*\mathcal{F}_1) \otimes_{\mathcal{O}_Y} (f_*\mathcal{F}_2) \to f_*(\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2)$ is surjective.

Proof. See, for example, the point (d) of the proof of [Liu, Lemma 9.3.6]. □

We are now ready to state and proof our regularity criterion.

Theorem 4.9.8 (Castelnuovo's Criterion)**.** Suppose that $H^1(Z, \mathcal{O}_Z) = 0$, let $k' := H^0(Z, \mathcal{O}_Z)$ be the function field of the fundamental divisor Z and set $d := \deg_{k'} \mathcal{O}_Z(-Z)$, so $Z^2 = -d[k' : k(s)]$. Then, the residue field of $\mathcal{X}_\mathcal{E}$ at P is k', and the tangent space of $\mathcal{X}_\mathcal{E}$ at P has dimension $d+1$ over k'. In particular, $\mathcal{X}_\mathcal{E}$ is regular if and only if $d = 1$.

Proof. Lemma 4.9.6 ensures that $\mathcal{O}_P = f_*\mathcal{O}_Z$, which simply means that the residue field at P is nothing but the function field k' of Z.

The tangent space at P is, by definition, $\ker(\mathcal{O}_{2P} \to \mathcal{O}_P)$, which, thanks to Lemma 4.9.6, is just $\psi_* \ker(\mathcal{O}_{2Z} \to \mathcal{O}_Z) = H^0(\mathcal{O}_Z(-Z))$. But, by definition of degree of an invertible sheaf on a curve (Subsection 3.2.2), we have that the dimension over k' of $H^0(\mathcal{O}_Z(-Z))$ can be written as

$$h^0_{k'}(\mathcal{O}_Z(-Z)) = d + h^0_{k'}(\mathcal{O}_Z) - h^1_{k'}(\mathcal{O}_Z) + h^1_{k'}(\mathcal{O}_Z(-Z)).$$

Now, $h^1_{k'}(\mathcal{O}_Z) = 0$ by hypothesis, $h^1_{k'}(\mathcal{O}_Z(-Z)) = 0$ by Lemma 4.9.5, and, as we have already remarked, $h^0_{k'}(\mathcal{O}_Z) = 1$ by Lemma 4.9.6: from this, the theorem clearly follows. □

4.10. The minimal regular model

We will now apply the results we have obtained to the study of the contraction of a single vertical component E on a regular surface $\mathcal{X}$. First, observe that, if the contraction $\mathcal{X}_{\mathcal{E}}$ exists (with $\mathcal{E} = \{E\}$) and is a regular surface, then $\mathcal{X} \to \mathcal{X}_{\mathcal{E}}$ must coincide with the blowup of $\mathcal{X}_{\mathcal{E}}$ at the closed point x to which E contracts (Proposition 4.7.2), and E is consequently a (-1)-curve (Definition 4.5.4) on $\mathcal{X}$. Castelnuovo's criterion (Theorem 4.9.8) allows us to also prove the converse of this result.

Theorem 4.10.1. If E is a (-1)-curve on a regular surface $\mathcal{X}$, then it can be contracted preserving the regularity of $\mathcal{X}$.

Proof. Let $\mathcal{E} := \{E\}$. By Theorem 4.9.1, contraction $\mathcal{X}_{\mathcal{E}}$ exists as a normal, but a priori not necessarily regular surface. It is clear that $Z = E$ is a fundamental divisor for $\mathcal{E}$ (Definition 4.9.3), and Castelnuovo's criterion (Theorem 4.9.8) ensures that $\mathcal{X}_{\mathcal{E}}$ is regular. $\square$

This result on contractibility of (-1)-curves has many relevant consequences, which we will now discuss. First, it provides a criterion for relative minimality of regular surfaces:

Proposition 4.10.2. A regular surface $\mathcal{X}$ is relatively minimal (Definition 4.6.2) if and only if it contains no (-1)-curves.

Proof. That a relatively minimal regular surface cannot contain (-1)-curves was already proved as a consequence of the structure of birational morphism of regular surfaces (see Corollary 4.7.3). The converse implication, instead, is new, and it is an immediate consequence of the fact that a (-1)-curve can be contracted preserving regularity (Theorem 4.10.1). $\square$

To continue, we will now need a brief digression on how stable the property of being a (-1)-curve is when a surface undergoes birational transformations. Let us in particular suppose that a regular surface $\mathcal{X}$ contains two distinct (-1)-curves $E_1, E_2 \in \mathrm{Vert}(\mathcal{X})$ and let us consider the contraction $\mathcal{X} \to \mathcal{X}_1$ of E_1: we want to determine whether E_2 is still a (-1)-curve in $\mathcal{X}_1$ or not. There are only two cases:

- If E_1 and E_2 are disjoint in $\mathcal{X}$, the contraction of E_1 will not affect E_2 at all, that will consequently remain a (-1)-curve inside $\mathcal{X}_1$.

- If E_1 and E_2 are irreducible components of a same special fiber $\mathcal{X}_s$ and they intersect, the contraction of E_1 will make the self-intersection number of E_2 become greater; hence, $(E_2)^2$, which is equal to $-[K(E_2) : k(s)]$ in $\mathcal{X}$, becomes 0 in $\mathcal{X}_1$. This implies that E_2 will exhaust a whole connected component of $(\mathcal{X}_2)_s$, from which it follows easily that E_1 and E_2 form an entire connected component of $\mathcal{X}_s$. It is now easy to see, via the adjunction formula (Proposition 4.5.3), that $\chi(\mathcal{X}/S) > 0$: in other words, the case we are discussing can only occur for surfaces whose fibers have positive Euler-Poincaré characteristic.

These observations allow to prove the following, crucial existence of results for minimal regular models.

Proposition 4.10.3. If $\mathcal{X}$ is a regular surface such that $\chi(\mathcal{X}/S) \leq 0$, then it admits a minimal regular model (Definition 4.6.2).

Proof. Since $\mathcal{M}_{\mathrm{reg}}(\mathcal{X})$ has already been proved to be Artinian (see Corollary 4.7.4), it is sufficient to show that some dominance relation always exists between any two given relatively minimal regular models of $\mathcal{X}$ (Definition 4.6.2). Let thus $\mathcal{X}_1, \mathcal{X}_2$ be two such models and suppose, by contradiction, that no dominance relation exists between them. Let $\mathcal{Y}$ be a regular model dominating them both: its existence is guaranteed by Fact 4.6.1, and, thanks to the Artinianity of $\mathcal{M}_{\mathrm{reg}}(\mathcal{X})$, we may clearly choose it so it is relatively minimal among all regular models of $\mathcal{X}$ dominating both $\mathcal{X}_1$ and $\mathcal{X}_2$. As no dominance relation exists between $\mathcal{X}_1$ and $\mathcal{X}_2$, $\mathcal{Y}$ must dominate $\mathcal{X}_1$ and $\mathcal{X}_2$ strictly, and hence, in particular, it must contain a (-1)-curve E. It is clear that E cannot appear as a vertical component of $\mathcal{X}_1$ nor in $\mathcal{X}_2$, because otherwise the discussion above shows that it would be still a (-1)-curve in $\mathcal{X}_i$, contradicting the relative minimality of $\mathcal{X}_i$. We thus have to conclude that E is reduced to a point both in $\mathcal{X}_1$ and in $\mathcal{X}_2$, but then, if we contract E on $\mathcal{Y}$, we obtain a smaller regular surface $\mathcal{Y}'$ that dominates both $\mathcal{X}_1$ and $\mathcal{X}_2$, and this contradicts the minimality assumption we have formulated on $\mathcal{Y}$. □

Contracting (-1)-curves on a regular surfaces can be seen as a way of making the canonical divisor numerically effective, by getting rid of all those vertical components that intersect it negatively.

Proposition 4.10.4. A regular surface $\mathcal{X}$ such that $\chi(\mathcal{X}/S) \leq 0$ is minimal if and only if its canonical line bundle $\omega_{\mathcal{X}/S}$ is numerically effective, i.e. $\omega_{\mathcal{X}/S} \cdot \Gamma \geq 0$ for all $\Gamma \in \mathrm{Vert}(\mathcal{X})$.

Proof. This follows from the fact that, on a regular surface $\mathcal{X}$ whose fibers have non-positive Euler-Poincaré characteristic, (-1)-curves can be characterized as those vertical components that have negative intersection with the canonical line bundle (see Proposition 4.5.6). □

This characterization of minimal regular surfaces is useful to show that the formation of the minimal regular model commutes with étale base change:

Proposition 4.10.5. Let $\mathcal{X} \to S$ be a surface such that $\chi(\mathcal{X}/S) \leq 0$, and let $\mathcal{X}'$ be its base change along some flat morphism of Dedekind bases $S' \to S$. Then:

1. if $S' \to S$ is surjective, and $\mathcal{X}'$ is a minimal regular surface, then $\mathcal{X}$ is also a minimal regular surface;
2. if $\mathcal{X}$ is a minimal regular surface, and $\mathcal{X}'$ happens to be regular, then $\mathcal{X}'$ is a minimal regular surface;
3. if $\mathcal{X}$ is a minimal regular surface, and $S' \to S$ is a localization, an étale base change, or if $S = \mathrm{Spec}(R)$ is the spectrum of a discrete valuation ring and S' is its Henselianization, strict Henselianization, or completion, then $\mathcal{X}'$ is also a minimal regular surface.

Proof. Suppose that both $\mathcal{X}$ and $\mathcal{X}'$ are regular; then, since the computation of the canonical line bundles commutes with any base change, we have that $\omega_{\mathcal{X}'/S'} = f^*\omega_{\mathcal{X}/S}$, where f denotes the morphism $\mathcal{X}' \to \mathcal{X}$. Let us fix a point $s' \in S'$ mapping to $s \in S$: the special fiber $\mathcal{X}'_{s'}$ is nothing but the base-change of $\mathcal{X}_s$ along the field extension $k(s) \subseteq k(s')$. From these observations, it is not difficult to see that the two conditions "$\omega_{\mathcal{X}/S} \cdot \Gamma \geq 0$ for all $\Gamma \in \mathrm{Vert}_s(\mathcal{X})$" and "$\omega_{\mathcal{X}'/S'} \cdot \Gamma' \geq 0$ for all $\Gamma' \in \mathrm{Vert}_{s'}(\mathcal{X}')$" are equivalent. We conclude that, if $\mathcal{X}$ and $\mathcal{X}'$ are both regular, the minimality of $\mathcal{X}'$ implies that of $\mathcal{X}$, and the converse is also true if $S' \to S$ is surjective.

From this, (2) follows immediately. To prove (1), it is now enough to recall that regularity descends along any flat morphism. And finally (3) is just a particular case of (2), since all the base-changes $S' \to S$ listed there preserve regularity. □

Let us finally show how Castelnuovo's criterion allows us to prove the existence of minimal desingularizations of surfaces.

Definition/Proposition 4.10.6. Let $\mathcal{X}$ be a normal surface admitting a regular model. Then, there exists a minimal regular model $\mathcal{Y}$ of $\mathcal{X}$ dominating $\mathcal{X}$ (meaning that $\mathcal{Y}$ is dominated by any other regular model of $\mathcal{X}$ dominating $\mathcal{X}$: see Definition 4.6.2). Moreover, $\mathcal{Y}$ is a desingularization of $\mathcal{X}$ (Definition 4.6.14), it is named the *minimal desingularization* of $\mathcal{X}$, and can be characterized as a regular model $\mathcal{Y}$ of $\mathcal{X}$ dominating $\mathcal{X}$ and such that $\mathrm{Vert}(\mathcal{Y}) \setminus \mathrm{Vert}(\mathcal{X})$ does not contain any (-1)-curve of $\mathcal{Y}$.

Proof. The result can be proved by means of a technique similar to the one we have adopted to prove the existence of minimal regular models (Proposition 4.10.3): see [Liu, Proposition 9.3.32]. $\square$

Remark 4.10.7. In categorical terms, what Definition/Proposition 4.10.6 ensures is that, if $\mathcal{X}$ is a normal surface such that $\mathcal{M}_{\mathrm{reg}}(\mathcal{X}) \neq \emptyset$, then $\mathcal{M}_{\mathrm{reg}}(\mathcal{X}) \subseteq \mathcal{M}_{\mathrm{norm}}(\mathcal{X})$ is a reflective subcategory.

4.11. The canonical model

Let us take a regular surface $\mathcal{X}$ such that $\chi(\mathcal{X}/S) < 0$: in light of Proposition 4.10.3, $\mathcal{X}$ will, in particular, admit a minimal regular model $\mathcal{X}_{\min}$. Let us denote by $\mathcal{E} \subsetneq \mathrm{Vert}(\mathcal{X}_{\min})$ the set of all (-2)-curves on $\mathcal{X}_{\min}$: then, we have that

Proposition 4.11.1. A contraction line bundle for $\mathcal{E}$ (Definition 4.8.2) exists.

Proof. This result is a consequence of Theorem 4.9.1: to apply it, we have to verify that $\mathcal{E}$ is finite and does not contain entire connected components of special fibers of $\mathcal{X}$. But, as $\chi(\mathcal{X}_{\min}/S) < 0$, we can exploit adjunction formula to characterize (-2)-curves on $\mathcal{X}_{\min}$ as those vertical components having zero intersection with the canonical bundle $\omega_{\min}$, and be sure that they never exhaust an entire connected component of a special fiber (see the results in Subsection 4.5.2). All that is left to show is the fact that the (-2)-curves are finitely many, but this is easily proved: since $\chi(\mathcal{X}_{\min}/S) < 0$, in fact, the restriction of $\omega_{\min}$ to the generic fiber has positive degree and is consequently ample; hence, $\omega_{\min}$ has ample restriction to all but finitely many special fibers (Proposition 1.2.8), in which all (-2)-curves must consequently be confined. $\square$

We are thus now ready to define the canonical model:

Definition 4.11.2. Given a regular surface $\mathcal{X}$ such that $\chi(\mathcal{X}/S) < 0$, the normal projective S-surface $(\mathcal{X}_{\min})_{\mathcal{E}}$ that results from the contraction of the set $\mathcal{E}$ of all (-2)-curves of the minimal regular model $\mathcal{X}_{\min}$ is named the *canonical model* of $\mathcal{X}$, and is denoted by $\mathcal{X}_{\mathrm{can}}$.

This section is devoted to the study of the properties of $\mathcal{X}_{\mathrm{can}}$. First, observe that $\mathcal{X}_{\mathrm{can}}$ bears a number of distinguished points $P_1, \ldots, P_r$, namely the ones to which the (-2)-curves of $\mathcal{X}_{\min}$ contract. If Z_i is the fundamental divisor (Definition 4.9.3) of those (-2)-curves contracting at P_i, it is clear that $\omega_{\min} \cdot Z_i = 0$, and hence, by the adjunction formula (Proposition 4.5.3), we have that $\chi(Z_i) > 0$ (i.e. $H^1(Z_i) = 0$) and $Z_i^2 = -2[k(Z_i) : k(s)]$. Hence, Theorem 4.9.8 applies and ensures that the tangent space of $\mathcal{X}_{\mathrm{can}}$ at each of the P_i's has dimension 3.

Proposition 4.11.3. The singular locus of $\mathcal{X}_{\mathrm{can}}$ consists of the closed points $\mathrm{E}(f) = \{P_1, \ldots, P_r\} \subseteq \mathcal{X}_{\mathrm{can}}$ over which $f : \mathcal{X}_{\min} \to \mathcal{X}_{\mathrm{can}}$ is not an isomorphism. Moreover, for each of these points, $\dim_{k(P_i)}(\mathfrak{m}_{P_i}/\mathfrak{m}_{P_i}^2) = 3$.

A Noetherian local ring $(A, \mathfrak{m}, k)$ such that $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq \dim(A) + 1$ is Gorenstein (actually, it is a local complete intersection ring); hence, we have that

Corollary 4.11.4. The canonical model $\mathcal{X}_{\mathrm{can}} \to S$ is Gorenstein, and hence admits an invertible canonical sheaf.

We will now discuss the relation between the invertible canonical sheaves $\omega_{\min}$ and ω_{can} of the minimal regular model and the canonical model. If we write f for the morphism $\mathcal{X}_{\min} \to \mathcal{X}_{\mathrm{can}}$, we have that

Proposition 4.11.5. $f^*\omega_{\mathrm{can}} \cong \omega_{\min}$ and $\omega_{\mathrm{can}} \cong f_*\omega_{\min}$

Proof. If we adopt the notation $\omega^\bullet$ for the relative dualizing complexes of the two surfaces over S, we clearly have $\omega^\bullet_{\min} \cong f^!(\omega^\bullet_{\mathrm{can}})$; but, for each of the two surfaces, $\omega^\bullet \cong \omega[1]$ where ω is the canonical sheaf, and hence $\omega_{\min} \cong f^!(\omega_{\mathrm{can}})$. Let $V \subseteq \mathcal{X}_{\mathrm{can}}$ be the complement of $E(f) \subseteq \mathcal{X}_{\mathrm{can}}$; then, $f : f^{-1}(V) \to V$ is an isomorphism, and hence, over V, $f^!$ simply coincides the pullback f^*, so $\omega_{\min|U} \cong (f^*\omega_{\mathrm{can}})_{|U}$, if $U := f^{-1}(V) = \mathcal{X}_{\min} \smallsetminus \cup_i E_i$, where the E_i's are the (-2)-curves of $\mathcal{X}_{\min}$. Moreover, $f^*\omega_{\mathrm{can}}$ is clearly trivial on a neighborhood of each connected component of $\cup_i E_i$, and hence, in particular, $f^*\omega_{\mathrm{can}} \cdot E_i = 0$ for all i; but from the definition of (-2)-curve we also have that $\omega_{\min} \cdot E_i = 0$ for all i. This is enough to conclude that $f^*\omega_{\mathrm{can}} \cong \omega_{\min}$, as the lemma below shows.

The second identity, i.e. $f_*(\omega_{\min}) \cong \omega_{\mathrm{can}}$, can easily be derived from the first one by using the projection formula for invertible sheaves ([Stacks, `01E8`]), thanks to the $\mathcal{O}$-connectedness of f (Definition 2.2.8, Lemma 2.2.9): $\omega_{\mathrm{can}} \cong \omega_{\mathrm{can}} \otimes \mathcal{O}_{\mathcal{X}_{\mathrm{can}}} \cong \omega_{\mathrm{can}} \otimes f_*\mathcal{O}_{\mathcal{X}_{\min}} \cong f_*(f^*(\omega_{\mathrm{can}}) \otimes \mathcal{O}_{\mathcal{X}_{\min}}) \cong f_*(\omega_{\min})$. □

Lemma 4.11.6. Let $\{\Gamma_1, \ldots, \Gamma_r\} \subsetneq \mathrm{Vert}(\mathcal{X})$ be vertical prime divisors on a regular surface $\mathcal{X}$, such that the intersection form on $\oplus_i \mathbb{Z}\Gamma_i$ is negative definite (this condition simply means that no connected component of a special fiber is entirely contained in $\Gamma := \cup_i \Gamma_i$). Then, a line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ is completely determined by its restriction $\mathcal{L}_{|\mathcal{X} \smallsetminus \Gamma} \in \mathrm{Pic}(\mathcal{X} \smallsetminus \Gamma)$ together with the intersection numbers $\mathcal{L} \cdot \Gamma_i$.

Proof. Since $\mathcal{X}$ is regular, we can write an exact sequence of abelian groups $\oplus_i \mathbb{Z}\Gamma_i \to \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X} \smallsetminus \Gamma) \to 0$; in particular, exactness in the middle ensures that, if two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathcal{X}$ have isomorphic restrictions to $\mathcal{X} \smallsetminus \Gamma$, then their difference $\mathcal{L}_1 \otimes \mathcal{L}_2^\vee$ is isomorphic to $\mathcal{O}_{\mathcal{X}}(D)$ for some $D \in \oplus_i \mathbb{Z}\Gamma_i$. If we further suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ have the same intersection number with each of the Γ_i, we clearly get that $D \cdot \Gamma_i = 0$ for all i; since the intersection form on $\oplus_i \mathbb{Z}\Gamma_i$ has been supposed to be non-degenerate, we conclude that $D = 0$, and hence $\mathcal{L}_1 \cong \mathcal{L}_2$. □

We are now ready to prove the most prominent property of the canonical model.

Proposition 4.11.7. The canonical sheaf of the canonical model is ample.

Proof. Let Γ be a vertical component of $\mathcal{X}_{\mathrm{can}}$ (lying over some $s \in S$), and Γ' denote its strict transform in $\mathcal{X}_{\min}$ along $f : \mathcal{X}_{\min} \to \mathcal{X}_{\mathrm{can}}$. It is clear that, as $f^*\omega_{\mathrm{can}} = \omega_{\min}$, we have that $\deg_{k(s)}(\omega_{\mathrm{can}|\Gamma}) = \deg_{k(s)}(\omega_{\min|\Gamma'}) = \omega_{\min} \cdot \Gamma'$ But, as $\mathcal{X}_{\min}$ has no (-1)-curves, and Γ' is clearly not a (-2)-curve, we can be sure that $\omega_{\min} \cdot \Gamma' > 0$. We have thus proved that ω_{can} has a positive degree restriction to every vertical component of $\mathcal{X}_{\mathrm{can}}$; in light of Proposition 3.2.10, ω_{can} has thus ample restriction to all special fibers of $\mathcal{X}_{\mathrm{can}}$: but this is enough to ensure that it is an S-ample invertible sheaf on $\mathcal{X}_{\mathrm{can}}$ (Proposition 1.2.8). □

In the previous section, given any regular surface $\mathcal{X}$ such that $\chi(\mathcal{X}/S) \leq 0$, we have characterized its minimal regular model as the unique regular model bearing a *numerically effective* canonical bundle (Proposition 4.10.4). But the proposition we have just proved ensures that, if $\chi(\mathcal{X}/S) < 0$, the canonical sheaf $\omega_{\min}$ of $\mathcal{X}_{\min}$ is the pullback of the S-ample invertible sheaf $\omega_{\mathrm{can}} \in \mathrm{Pic}(\mathcal{X}_{\mathrm{can}})$, and hence is *semiample*, which is clearly a stronger property than being numerically effective. We know that semiample line bundles induce contractions (Section 4.8), and it is immediate to see that the contraction induced by the semiample line bundle $\omega_{\min}$ is precisely the one producing the canonical model $\mathcal{X}_{\mathrm{can}}$.

Proposition 4.11.8. Given $\mathcal{X}_{\min}$ a minimal regular surface such that $\chi(\mathcal{X}_{\min}/S) < 0$, the canonical sheaf $\omega_{\min}$ on $\mathcal{X}_{\min}$ is semiample, and it is a contraction line bundle (Definition 4.8.2) for $\mathcal{X}_{\min} \to \mathcal{X}_{\mathrm{can}}$.

As the minimal regular model, the canonical model is also uniquely determined up to (unique) isomorphisms of models. Moreover, its formation commutes with étale base-change.

Proposition 4.11.9. Let $\mathcal{X} \to S$ be a canonical surface, and let $\mathcal{X}'$ be its base change along some flat morphism of Dedekind bases $S' \to S$. If $S' \to S$ is a localization, or an étale base change, or if $S = \mathrm{Spec}(R)$ is the spectrum of a discrete valuation ring and S' is its Henselianization, strict Henselianization, or completion, then $\mathcal{X}'$ is also a canonical surface.

Proof. This follows easily from the analogue result we have proved for minimal regular models (Proposition 4.10.5). □

4.12. Normalizing and taking quotients

Suppose we are given a Dedekind base S with function field K and a finite extension $F \subseteq L$ of fields having transcendence degree 1 over K. In this section, we will seek a relation between the birational class of surfaces having function fields F (which we will name, for brevity, F-surfaces) and that of surfaces having function field L (which we will refer to as L-surfaces).

If we start with a normal surface $\mathcal{X}$ with function field F, the normalization $\mathcal{Y}$ of $\mathcal{X}$ in L produces an integral S-scheme having function field L.

Proposition 4.12.1. If S is Nagata or $F \subseteq L$ is separable, then $\mathcal{Y}$ is still an S-surface, and the normalization morphism $\mathcal{Y} \to \mathcal{X}$ is a finite surjective morphism of normal surfaces. Moreover, $\mathcal{Y} \to \mathcal{X}$ is flat over the regular points of $\mathcal{X}$, and, if $F \subseteq L$ is separable and $B_i \subseteq \mathcal{X}$ is an irreducible component of the branch locus of $\mathcal{Y} \to \mathcal{X}$, then either B_i is a prime Weyl divisor on $\mathcal{X}$, or it consists of a singular (closed) point of $\mathcal{X}$.

Proof. If S is Nagata or $F \subseteq L$ is separable, then the normalization morphism $\mathcal{Y} \to \mathcal{X}$ is certainly finite, and from this we can deduce that $\mathcal{Y}$ is an S-surface (see Proposition 4.1.6). The flatness of $\mathcal{Y} \to \mathcal{X}$ over the regular points of $\mathcal{X}$ follows from [Stacks, `00R4`], while the purity result on the branch locus is [Stacks, `0BMB`]. □

In fact, $\mathcal{Y}$ is not just an S-scheme: since the normalization is unique up to unique isomorphism, $\mathcal{Y}$ is naturally endowed with an action of the group $\mathrm{Aut}_F(L)$, in the sense that every automorphism of the function field $L = K(\mathcal{Y})$ over $F = K(\mathcal{X})$ extends uniquely to an automorphism of $\mathcal{Y}$ over $\mathcal{X}$. We will now rewrite this observation in more formal terms.

Definition 4.12.2. Let S a Dedekind domain with function field K, $\mathcal{X}$ a surface, and $G \leq \mathrm{Aut}_K(K(\mathcal{X}))$ a finite group of automorphisms of the function field of $\mathcal{X}$.

We say that G *acts admissibly on* $\mathcal{X}$ if the action of G on $K(\mathcal{X})/K$ extends (and hence, extends uniquely) to an admissible action on $\mathcal{X}/S$ (Definition A.4.5).

Proposition 4.12.3. Supposing that $F \subseteq L$ is separable or that S is Nagata, normalization in L takes F-normal surfaces to L-normal surfaces admissibly acted on by $\mathrm{Aut}_F(L)$.

Let us now address the converse problem, which is more difficult: if we start with an L-surface $\mathcal{Y}$, there is in general no way to descend it to an F-surface. But if $\mathcal{Y}$ is admissibly acted on by $G := \mathrm{Aut}(L/F)$, and the extension $F \subseteq L$ is Galois, we can compute the quotient of $\mathcal{Y}$ under the action on G, and get a finite morphism $\mathcal{Y} \to \mathcal{Y}/G$ to an integral scheme $\mathcal{Y}/G$ whose function field is precisely $L^G = F$ (all these results are guaranteed by the general discussion on quotients of scheme that we have deferred to appendix A). The finiteness of $\mathcal{Y} \to \mathcal{Y}/G$, thanks to Proposition 4.1.6, ensures that $\mathcal{Y}/G$ is still a surface; moreover, if $\mathcal{Y}$ is normal, its normality will be inherited by the quotient $\mathcal{Y}/G$ (appendix A.5).

It is not difficult to see that the two operations we have defined are converse one of the other one, so:

Proposition 4.12.4. Let S be a Dedekind scheme, and $F \subseteq L$ a Galois extension of fields of transcendence degree 1 over K. There is an equivalence of categories between (a) the birational class of normal surfaces on S having function field F and (b) the birational class of normal surfaces on S having function field L and acted on by $\mathrm{Gal}(L/F)$. The equivalence is given by normalizing (a $\to$ b) and taking quotients (b $\to$ a).

In light of this result, it is clearly interesting to determine whether a given surface $\mathcal{X} \to S$ is acted on or not by a given finite group of automorphisms $G \leq \mathrm{Aut}_K(K(\mathcal{X}))$. It is a crucial fact that a positive answer can be given for those models that can be uniquely identified among those of some birational class; in particular:

Proposition 4.12.5. If $\mathcal{X}$ is a minimal regular surface or a canonical surface, then it is admissibly acted on by any finite group of automorphism $G \leq \mathrm{Aut}_K(K(\mathcal{X}))$ of its function field.

Proof. Suppose that $\mathcal{X}$ is a minimal regular surface. We recall that a model of $\mathcal{X}$ is a pair $(\mathcal{X}', h)$ where $\mathcal{X}'$ is an S-surface and h is a K-isomorphism $h : K(\mathcal{X}) \xrightarrow{\sim} K(\mathcal{X}')$. If we pick an automorphism $\psi \in G \leq \mathrm{Aut}_K(K(X))$, we can consider the two models $(\mathcal{X}'_1, h_1) = (\mathcal{X}, \mathrm{id})$ and $(\mathcal{X}'_2, h_1) = (\mathcal{X}, \psi)$. As $\mathcal{X}$ is a minimal regular surface, both $(\mathcal{X}'_1, h_1) = (\mathcal{X}, \mathrm{id})$ and $(\mathcal{X}'_2, h_2) = (\mathcal{X}, \mathrm{id})$ are minimum elements of

$\mathcal{M}_{\mathrm{reg}}(\mathcal{X})$, hence, they must necessarily be isomorphic as models of $\mathcal{X}$: but this precisely means that there exists an automorphism of the S-surface $\mathcal{X}$ extending ψ. We have thus proved that the action of G is well-defined: we still, however, have to verify its admissibility (Definition A.4.5), which can easily be deduced from the fact that regular surfaces are always projective (Corollary 4.9.2).

The canonical model of a surface is also unique up to (unique) isomorphisms, and projective: an identical argument hence applies. □

Corollary 4.12.6. Let $Y \to X$ be a Galois covering of integral K-curves, and fix a Dedekind base S with function field K. Then, the minimal regular model and the canonical model of Y over S (supposing they exist) can be obtained by normalizing in $K(Y)$ some suitable normal S-models of X.

4.13. Picard groups of regular surfaces

Given $\mathcal{X} \to S$ a regular surface, we can study its Picard group $\mathrm{Pic}(\mathcal{X})$, as well as the Picard groups of the fibers $\mathrm{Pic}(\mathcal{X}_\eta)$ and $\mathrm{Pic}(\mathcal{X}_s)$ and their Jacobians $\mathrm{Pic}_{\mathcal{X}_\eta/K}$ and $\mathrm{Pic}_{\mathcal{X}_s/k(s)}$.

First, we will present some results on the group homomorphisms $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_\eta)$ and $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_s)$ induced by the restriction of invertible sheaves, following [Des81]. Then, we will study the Jacobian of the special fibers of a surface $\mathcal{X}$, showing how their abelian, toric, and unipotent ranks vary along birational transformations of surfaces, taken from [Liu, Subsection 10.3.3]. Finally, we will present a result relating the toric rank of the special fibers with the intersection form, following [Liu, Subsection 10.4.2]. Most of the material in this section will prove its importance only in the next chapter, when the results presented here will allow us to conclude that smooth curves always have potential semistable reduction.

4.13.1. Restricting to the special fiber

Suppose $\mathcal{X} \to S$ is a regular surface over the spectrum S of a discrete valuation ring $(A, \mathfrak{m}, k)$: we are interested in studying the restriction of line bundles to the special fiber. First, we observe that, if A is Henselian, effective Cartier divisors on $\mathcal{X}_s$ can be lifted to effective horizontal Cartier divisors on $\mathcal{X}$ (Proposition 4.4.2), and from this it is easy to verify that $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_s)$ is surjective. The preimage of $\mathrm{Pic}^0(\mathcal{X}_s)$ through this morphism clearly corresponds to the subgroup $\mathrm{Pic}^0(\mathcal{X}) \leq \mathrm{Pic}(\mathcal{X})$ of numerically trivial line bundles on $\mathcal{X}$. To summarize, if A is Henselian, we have sujections

$$\mathrm{Pic}(\mathcal{X}) \twoheadrightarrow \mathrm{Pic}(\mathcal{X}_s) \qquad \mathrm{Pic}^0(\mathcal{X}) \twoheadrightarrow \mathrm{Pic}^0(\mathcal{X}_s).$$

If we make the stronger hypothesis that A is complete with algebraically closed residue field, and we fix ℓ prime to the residue characteristic p of A, then we have also seen, as a consequence of Grothendieck's Existence Theorem, that the kernel of $\mathrm{Pic}(\mathcal{X}) \twoheadrightarrow \mathrm{Pic}(\mathcal{X}_s)$ is uniquely ℓ-divisible and $\mathrm{Pic}(\mathcal{X})[\ell] \to \mathrm{Pic}(\mathcal{X}_s)[\ell]$ is an isomorphism (Proposition 2.2.7).

The same result can also clearly be restated in terms of the Pic^0 groups: $\mathrm{Pic}^0(\mathcal{X}) \twoheadrightarrow \mathrm{Pic}^0(\mathcal{X}_s)$ has a uniquely ℓ-divisible kernel, and $\mathrm{Pic}^0(\mathcal{X})[\ell] \to \mathrm{Pic}^0(\mathcal{X}_s)[\ell]$ is an isomorphism. Moreover, the group $\mathrm{Pic}^0(\mathcal{X}_s)$ is itself ℓ-divisible because it coincides with the group of k-rational points of the Jacobian of $\mathcal{X}_s$, which is a smooth connected algebraic k-group. We deduce from these facts that $\mathrm{Pic}^0(\mathcal{X})$ is an ℓ-divisible group.

4.13.2. Restricting to the generic fiber

Let $\mathcal{X} \to S$ be again a regular surface over the spectrum S of a discrete valuation ring $(A, \mathfrak{m}, k)$, and let us now consider the restriction of invertible sheaves on $\mathcal{X}$ to the generic fiber $\mathcal{X}_\eta$. We clearly have an exact sequence

$$\mathrm{Div}_{\mathrm{v}}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_\eta) \to 0 \tag{$\dagger$}$$

that, if we are only interested in numerically trivial line bundles on $\mathcal{X}$, restricts to an exact sequence:

$$\mathbb{Z}(d^{-1}\mathcal{X}_s) \to \mathrm{Pic}^0(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_\eta)$$

where d denotes the greatest common divisor of the multiplicities of the vertical components of $\mathcal{X}$. Since $\mathcal{X}_s$ is a principal divisor, the kernel of $\mathrm{Pic}^0(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_\eta)$ is a cyclic group, whose order is divided by d; on the other hand, the cokernel of $\mathrm{Pic}^0(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_\eta)$ turns out to be strictly related to the intersection matrix of vertical curves of $\mathcal{X}$:

Proposition 4.13.1. The cokernel of $\mathrm{Pic}^0(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_\eta)$ injects into the cokernel $I(\mathcal{X})$ of the canonical group homomorphism $\mathrm{Div}_{\mathrm{v}}(\mathcal{X}) \to \mathrm{Div}_{\mathrm{v}}(\mathcal{X})^\vee$ induced by the intersection form. If A is complete with algebraically closed residue field, the injection is actually an isomorphism.

Proof. If we take the group homomorphism $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Div}_{\mathrm{v}}(\mathcal{X})^\vee$ induced by the intersection form, and we take the quotient of both sides modulo vertical divisors, we get a homomorphism $\operatorname{Pic}(\mathcal{X}_\eta) \to I(\mathcal{X})$:

$$\begin{array}{ccccccc}
\operatorname{Div}_{\mathrm{v}}(\mathcal{X}) & \longrightarrow & \operatorname{Pic}(\mathcal{X}) & \longrightarrow & \operatorname{Pic}(\mathcal{X}_\eta) & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
\operatorname{Div}_{\mathrm{v}}(\mathcal{X}) & \longrightarrow & \operatorname{Div}_{\mathrm{v}}(\mathcal{X})^\vee & \longrightarrow & I(\mathcal{X}) & \longrightarrow & 0
\end{array}$$

and from this diagram, since the kernel of $\operatorname{Pic}(\mathcal{X}) \to \operatorname{Div}_{\mathrm{v}}(\mathcal{X})^\vee$ is $\operatorname{Pic}^0(\mathcal{X})$ by definition, we get, by diagram chasing, an exact sequence:

$$\operatorname{Pic}^0(\mathcal{X}) \to \operatorname{Pic}(\mathcal{X}_\eta) \to I(\mathcal{X})$$

and the cokernel of $\operatorname{Pic}^0(\mathcal{X}) \to \operatorname{Pic}(\mathcal{X}_\eta)$ consequently injects into $I(\mathcal{X})$.

Suppose now that A is complete with algebraically closed residue field. Then, every irreducible component Γ_i of the special fiber $\mathcal{X}_s$ has a smooth point P_i not belonging to any other vertical component; by Henselianity (see Proposition 4.4.4), P_i can be lifted to a horizontal effective Cartier divisor H_i on $\mathcal{X}$, which will satisfy $H_i \cdot \Gamma_j = \delta_{ij}$. This is enough to ensure that $\operatorname{Pic}(\mathcal{X}) \to \operatorname{Div}_{\mathrm{v}}(\mathcal{X})^\vee$, and hence $\operatorname{Pic}(\mathcal{X}_\eta) \to I(\mathcal{X})$, are surjective. □

To summarize, under the assumption that A is complete with algebraically closed residue field, we have an exact sequence:

$$\mathbb{Z}(d^{-1}\mathcal{X}_s) \to \operatorname{Pic}^0(\mathcal{X}) \to \operatorname{Pic}(\mathcal{X}_\eta) \to I(\mathcal{X}) \to 0$$

Let us now fix ℓ prime to both the residue characteristic p and to d. As $d^{-1}\mathcal{X}_s \in \operatorname{Pic}^0(\mathcal{X})$ is a torsion element, whose order is divisible by d, and ℓ has been chosen coprime to d, we have that $\operatorname{Pic}^0(\mathcal{X})[\ell] \to \operatorname{Pic}^0(\mathcal{X}_\eta)[\ell]$ is injective. Moreover, we have already remarked (Subsection 4.13.1) that, as ℓ has been chosen prime to p, $\operatorname{Pic}^0(\mathcal{X})$ is an ℓ-divisible group. These two observations are enough to deduce, from the sequence above, the following exact sequence of ℓ-torsion groups:

$$0 \to \operatorname{Pic}^0(\mathcal{X})[\ell] \to \operatorname{Pic}^0(\mathcal{X}_\eta)[\ell] \to I(\mathcal{X})[\ell] \to 0$$

And, finally, if we recall that the restriction of invertible sheaves to the special fiber $\operatorname{Pic}^0(\mathcal{X}) \to \operatorname{Pic}^0(\mathcal{X}_s)$ induces an isomorphism at the level of ℓ-torsion elements (Subsection 4.13.1), we may also rewrite the sequence as:

$$0 \to \operatorname{Pic}^0(\mathcal{X}_s)[\ell] \to \operatorname{Pic}^0(\mathcal{X}_\eta)[\ell] \to I(\mathcal{X})[\ell] \to 0$$

4.13.3. Abelian, toric and unipotent rank of special fibers

Given $\mathcal{X}$ a surface over a Dedekind base S, we may fix one of its special fibers $\mathcal{X}_s$ and write $h^1(\mathcal{X}_s)$ as the sum $a(\mathcal{X}_s) + t(\mathcal{X}_s) + u(\mathcal{X}_s)$ of its abelian, toric, and unipotent ranks (see Remark 3.6.2). Since the formation of the Jacobian commutes with arbitrary base changes, it is clear that, if we consider an arbitrary extension $S' \to S$ of Dedekind schemes, and consider $s' \in S'$ a closed point of S' mapping to $s \in S$, the abelian, toric and unipotent ranks of $\mathcal{X}'_{s'}$ will be equal to those of $\mathcal{X}_s$. It is, instead, a non-trivial problem determining how u, t and a vary within a birational class of normal surfaces.

Concerning $a(\mathcal{X}_s)$, it is enough to recall that the abelian rank of a curve C over a field k only depends on its normalization, so we have that $a(\mathcal{X}_s) = \sum_{\Gamma \in \mathrm{Vert}(\mathcal{X})} a(\Gamma) = \sum_{\Gamma \in \mathrm{Vert}(\mathcal{X})} a(\widetilde{\Gamma})$, and hence

Proposition 4.13.2. If $f : \mathcal{X} \to \mathcal{Y}$ is a birational morphism of normal surfaces contracting a set of vertical curves $\mathcal{E} = \{E_1, \dots, E_k\} \subseteq \mathrm{Vert}(\mathcal{X})$ to a single point $y \in \mathcal{Y}$ (lying over some $s \in S$), we have that $a(\mathcal{X}_s) = a(\mathcal{Y}_s) + \sum_i a(E_i)$.

Let us now turn our attention to toric ranks.

Proposition 4.13.3. If $f : \mathcal{X} \to \mathcal{Y}$ is a birational morphism of normal surfaces contracting a set of vertical curves $\mathcal{E} = \{E_1, \dots, E_k\} \subseteq \mathrm{Vert}(\mathcal{X})$ to a single point $y \in \mathcal{Y}$ (lying over some $s \in S$), and we let $E := (\mathcal{X}_y)_{\mathrm{red}}$, we have that $t(\mathcal{X}_s) = t(\mathcal{Y}_s) + t(E)$.

Proof. We will set, for brevity, $Z_{\mathcal{X}} := (\mathcal{X}_s)_{\mathrm{red}}$ and $Z_{\mathcal{Y}} := (\mathcal{Y}_s)_{\mathrm{red}}$. Each irreducible component of $Z_{\mathcal{X}}$ is alternatively part of E or of the strict transform $Z'_{\mathcal{Y}}$ of $Z_{\mathcal{Y}}$ in $\mathcal{X}$, so that we have a birational morphism $g_1 : Z'_{\mathcal{Y}} \sqcup E \to Z_{\mathcal{X}}$ of reduced curves over $k(s)$; moreover, we clearly also have a birational morphism of reduced $k(s)$-curves $g_2 : Z'_{\mathcal{Y}} \to Z_{\mathcal{Y}}$. If we denote by $x_1, \dots, x_r$ the points of $Z'_{\mathcal{Y}}$ lying above y, we have that:

1. $Z'_{\mathcal{Y}}$ intersects E precisely at the points $x_1, \dots, x_r$; hence, the preimage of $x_1, \dots, x_r \in Z_{\mathcal{X}}$ through g_1 consists of $2r$ points in total. Moreover, we have that g_1 is an isomorphism over $Z_{\mathcal{X}} \setminus \{x_1, \dots, x_r\}$, and the domain of g_1 has one connected component more than its codomain. As a result, $t(Z_{\mathcal{X}}) = t(Z'_{\mathcal{Y}}) + t(E) + r - 1$;

2. If we now look at g_2, we see that it is an isomorphism over $Z_{\mathcal{Y}} \setminus \{y\}$, its domain and codomain have the same number of connected components, and the preimage of y consists of r points, hence $t(Z_{\mathcal{Y}}) = t(Z'_{\mathcal{Y}}) + r - 1$.

We can thus conclude that $t(Z_{\mathcal{X}}) = t(Z_{\mathcal{Y}}) + t(E)$, and, since we clearly have that $t(Z_{\mathcal{X}}) = t(\mathcal{X}_s)$ and $t(Z_{\mathcal{Y}}) = t(\mathcal{Y}_s)$, this is what we wanted. $\square$

Now that we have seen how $a(\mathcal{X}_s)$ and $t(\mathcal{X}_s)$ vary through birational transformations of normal surfaces, studying the variation of $u(\mathcal{X}_s)$ is equivalent to studying that of $h^1(\mathcal{X}_s) = a(\mathcal{X}_s) + t(\mathcal{X}_s) + u(\mathcal{X}_s)$.

Proposition 4.13.4. Let $f : \mathcal{X} \to \mathcal{Y}$ a birational morphism of normal surfaces. If $R^1 f_* \mathcal{O}_{\mathcal{X}} = 0$, then $h^1(\mathcal{X}_s) = h^1(\mathcal{Y}_s)$ for each closed point $s \in S$.

Proof. Let us denote by $p : \mathcal{X} \to S$ and $q : \mathcal{Y} \to S$ the structure morphisms. Since $f : \mathcal{X} \to \mathcal{Y}$ is a birational morphism of normal schemes, it is $\mathcal{O}$-connected (Definition 2.2.8, Lemma 2.2.9); consequently, $R^1 p_* \mathcal{O}_{\mathcal{X}}$ and $R^1 q_* \mathcal{O}_{\mathcal{Y}} = (R^1 q_*) f_* \mathcal{O}_{\mathcal{X}}$ fit into the Leray spectral exact sequence $0 \to R^1 q_* \mathcal{O}_{\mathcal{Y}} \to R^1 p_* \mathcal{O}_{\mathcal{X}} \to q_*(R^1 f_* \mathcal{O}_{\mathcal{X}}) = 0$. Hence $R^1 p_* \mathcal{O}_{\mathcal{X}} \cong R^1 q_* \mathcal{O}_{\mathcal{Y}}$, and if we take the fibers of these two sheaves of $\mathcal{O}_S$-modules at a closed point $s \in S$, we get, thanks to Proposition 4.2.1, an isomorphism $H^1(\mathcal{X}_s) \cong H^1(\mathcal{Y}_s)$. $\square$

Let us now suppose that $\mathcal{X}$ is a regular surface, and $f : \mathcal{X} \to \mathcal{Y} := \mathcal{X}_{\mathcal{E}}$ a morphism to a normal surface that only contracts some (-1)-curves and (-2)-curves of $\mathcal{X}$. Then, Propositions 4.13.2 and 4.13.3 apply and give that $a(\mathcal{X}_s) = a(\mathcal{Y}_s)$ and $t(\mathcal{X}_s) = t(\mathcal{Y}_s)$. Moreover, since, in the case we are considering, the fundamental divisor Z of the contraction f has $H^1(Z, \mathcal{O}_Z) = 0$, we have that the hypothesis $R^1 f_* \mathcal{O}_{\mathcal{X}} = 0$ of Proposition 4.13.4 holds (see Lemma 4.9.5), and we can thus conclude that also $u(\mathcal{X}_s) = u(\mathcal{Y}_s)$. In particular,

Proposition 4.13.5. If $\mathcal{X}$ and $\mathcal{X}'$ are two birational regular surfaces, or if $\mathcal{X}$ is a regular surface such that $\chi(\mathcal{X}/S) < 0$ and $\mathcal{X}'$ its canonical model, then, for any closed $s \in S$, $\mathcal{X}_s$ and $\mathcal{X}'_s$ have equal abelian, toric and unipotent ranks.

4.13.4. The cokernel of the intersection form

We are now willing to study how the cokernel $I(\mathcal{X})$ of the intersection form $\mathrm{Div}_s(\mathcal{X}) \to \mathrm{Div}_s(\mathcal{X})^\vee$ varies within a birational class of regular surfaces. Suppose that the base S is the spectrum of a discrete valuation ring $(A, \mathfrak{m})$, with algebraically closed residue field k. As usual, we may reduce ourselves to consider the case of the blowup $\mathcal{X}' \to \mathcal{X}$ of a regular surface $\mathcal{X}$ at a closed point x (Proposition 4.7.2). To fix some notation, let us suppose that $\mathrm{Vert}(\mathcal{X}) = \{\Gamma_1, \dots, \Gamma_n\}$, and $\mathrm{Vert}(\mathcal{X}') = \{\Gamma'_0, \Gamma'_1, \dots, \Gamma'_n\}$, where $\Gamma'_0 = E$ is the exceptional divisor of the blowup, while $\Gamma'_1, \dots, \Gamma'_n$ are the strict transforms of the Γ_i's. Apart from the standard one $\mathcal{B}_1 = \{\Gamma'_0, \ldots, \Gamma'_n\}$, $\mathrm{Div}_s(\mathcal{X}')$ also clearly admits a $\mathbb{Z}$-basis $\mathcal{B}_2$ consisting of the exceptional divisor $E = \Gamma'_0$ along with all total transforms $f^{-1}\Gamma_i$ of the Γ_i's, where $f^{-1}\Gamma_i = \Gamma'_i + \mu_i E$, $\mu_i := \Gamma'_i \cdot E$. Thanks to the projection formula (Proposition 4.5.7), if we represent the intersection form with respect to the pair of bases $(\mathcal{B}_1, \mathcal{B}_2)$, we get a matrix:

$$\begin{pmatrix} -1 & 0 \ldots\ldots 0 \\ \mu_1 & \\ \vdots & (\Gamma_i \cdot \Gamma_j) \\ \mu_n & \end{pmatrix}$$

It is now immediate to deduce that this connection between the intersection forms on $\mathcal{X}$ and $\mathcal{X}'$ defines an isomorphism $I(\mathcal{X}) \cong I(\mathcal{X}')$. Hence:

Proposition 4.13.6. For any two birational regular surfaces $\mathcal{X}$ and $\mathcal{X}'$ over the spectrum S of a discrete valuation ring with algebraically closed residue field, we have that $I(\mathcal{X}) \cong I(\mathcal{X}')$.

A crucial property of the abelian group $I(\mathcal{X})$ is its relation with the toric rank of the special fiber $\mathcal{X}_s$.

Proposition 4.13.7. Let $\mathcal{X}$ be a regular surface over the spectrum S of a discrete valuation ring with algebraically closed residue field, and suppose that $\mathcal{X}_\eta$ is a smooth geometrically connected curve. Assume that $\mathcal{X}_\eta$ has genus $g := \dim_K H^1(\mathcal{X}_\eta) > 0$ or, in other words, that $\chi(\mathcal{X}/S) \leq 0$. Then, there exists an integer $c > 0$, only depending on g, such that, for any prime ℓ not dividing c, $\dim_{\mathbb{F}_\ell}(I(\mathcal{X})[\ell]) \geq t(\mathcal{X}_s)$.

Proof. We will sketch the proof following [Liu, Corollary 10.4.20]. By subsequently blowing up $\mathcal{X}$ at the singular points of $\mathcal{X}_s$, it is possible to find a regular surface $\mathcal{X}'$ birational to $\mathcal{X}$ such that $\mathcal{X}'_s$ has normal crossings. But Propositions 4.13.3 and 4.13.6 guarantee that $t(\mathcal{X}'_s) = t(\mathcal{X}_s)$ and $I(\mathcal{X}) \cong I(\mathcal{X}')$, hence, it is enough to prove our proposition for $\mathcal{X}'$: in other words, we lose no generality in supposing that $\mathcal{X}_s$ has normal crossings.

The rank of $I(\mathcal{X})[\ell]$ obviously only depends on the intersection matrix of $\mathcal{X}$. But the fact that $\mathcal{X}$ is a regular surface whose generic fiber is geometrically connected and whose special fiber has normal crossings implies that $t(\mathcal{X}_s)$ can also be computed from the intersection matrix alone: $t(\mathcal{X}_s) = \frac{1}{2}(\sum_{i \neq j} \Gamma_i \cdot \Gamma_j) - n + 1$, where n is the number of vertical components of $\mathcal{X}$. Now, the proposition follows from a combinatorial result that can be proved within the theory of numerical types of surfaces: the details can be found, for example, in [Des81]. $\square$

5. Semistable and smooth models

In the last chapter, we have discussed the general theory of arithmetic surfaces over a Dedekind base S; this one, instead, will only focus on *smooth* and *semistable* surfaces. We will give their definition and discuss their main properties in Sections 5.1 to 5.4. Section 5.5 proves that a smooth curve always admits a semistable model if we allow for ramified extensions of the base S: this is a crucial result that goes under the name *semistable reduction theorem*. Section 5.6 starts dealing with the problem of concretely constructing semistable models, which will be the main topic of the next chapters. Finally, Section 5.7 considers a setting in which a Galois covering $Y \to X$ of curves is given, and inquires whether and how semistability transfers from models of Y to models of X and vice versa.

5.1. Introduction

A surface $\mathcal{X}$ over a Dedekind base S is said to be *smooth* if $\mathcal{X} \to S$ is smooth as a morphism of schemes. It is clear that:

Proposition 5.1.1. If a surface $\mathcal{X}$ with $\chi(\mathcal{X}/S) \leq 0$ admits a smooth model $\mathcal{X}_{\mathrm{sm}}$, it is unique and it is the minimal regular model of $\mathcal{X}$; if $\chi(\mathcal{X}/S) < 0$, $\mathcal{X}_{\mathrm{sm}}$ is also the canonical model of $\mathcal{X}$.

However, a surface with smooth generic fiber cannot be guaranteed, in general, to admit a smooth model. To recover some general existence results, the requirements on the special fibers have to be weakened:

Definition 5.1.2. A surface $\mathcal{X} \to S$ is said to be *semistable* if its generic fiber $\mathcal{X}_\eta$ is smooth, and its special fibers are all semistable curves (Definition 3.4.4).

The notion of a semistable model is clearly a generalization of that of a smooth model. Throughout this chapter, we will fist present the most relevant properties of semistable models and we will show how it can be proved that, after a possibly ramified base-change, a semistable model can always be constructed.

It is customary to speak about the existence of smooth of semistable models in terms of reduction types of curves:

Definition 5.1.3. Let $X \to K$ be a smooth geometrically connected curve and suppose we have fixed a discrete valuation ring R with fraction field K. X is said to have *good reduction* if it admits a smooth model over R; it is said to have *semistable reduction* if it admits a semistable model over R; it is said to have *potential good* or *potential semistable* reduction if it admits a smooth model or a semistable model after some generically finite local extension $R' \supseteq R$ of discrete valuation rings.

5.2. First properties and criteria

Smoothness and semistability conditions are prescriptions on the geometric fibers of surfaces; as such, they are preserved and reflected by arbitrary base-changes:

Proposition 5.2.1. Let $\mathcal{X}$ be a surface over a Dedekind base S, $S' \to S$ any flat morphism of Dedekind schemes, and $\mathcal{X}'$ the base-change of $\mathcal{X}$ to S'. If $\mathcal{X}$ is smooth or semistable, $\mathcal{X}'$ inherits those properties. If $S' \to S$ is surjective, then the converse is also true.

From our discussion on the relation between the singularities of a curve and the structure of its Jacobian (Section 3.6, and in particular Proposition 3.6.11), we see that the semistability condition is tightly related to the vanishing of the unipotent rank of the special fibers:

Proposition 5.2.2. A surface with smooth generic fiber $\mathcal{X}$ is semistable if and only if its special fibers are geometrically reduced, have unipotent rank zero, and every geometric special fiber has at most two branches (Definition 3.4.1) at each of its points.

It is also clear that, if $\mathcal{X}$ is assumed to be regular, the tangent space of a geometric special fiber at any of its points must have dimension ≤ 2, and hence the condition that Proposition 5.2.2 imposes on the number of branches is always satisfied:

Proposition 5.2.3. A regular surface with smooth generic fiber $\mathcal{X}$ is semistable if and only if its special fibers are geometrically reduced and have unipotent rank zero.

Finally, checking semistability is even simpler for minimal regular surfaces.

Proposition 5.2.4. Let S a Dedekind base such that, for each closed point $s \in S$, the residue field $k(s)$ is algebraically closed. Let $\mathcal{X} \to S$ be a minimal regular surface such that $\chi(\mathcal{X}/S) < 0$ and $\mathcal{X}_\eta$ is smooth and geometrically connected. Then, $\mathcal{X}$ is semistable if and only if its special fibers have unipotent rank zero.

Proof. In light of Proposition 5.2.3, it is enough to verify that $\mathcal{X}_s$ is reduced for each s. The two vertical divisors $\mathcal{X}_{s,\mathrm{red}}$ and $\mathcal{X}_s$ have the same support, hence they share the same abelian and toric rank; since our hypothesis ensures $u(\mathcal{X}_s) = 0$, they also clearly have the same unipotent rank, equal to 0. Hence, $h^1(\mathcal{X}_s) = h^1(\mathcal{X}_{s,\mathrm{red}})$. Now, adjunction formula (Proposition 4.5.3) ensures that:

$$\omega \cdot \mathcal{X}_s = -2\chi(\mathcal{X}_s);$$
$$\omega \cdot \mathcal{X}_{s,\mathrm{red}} = -2\chi(\mathcal{X}_{s,\mathrm{red}}) - \mathcal{X}_{s,\mathrm{red}}^2.$$

Subtracting the two equations and recalling that $h^1(\mathcal{X}_s) = h^1(\mathcal{X}_{s,\mathrm{red}})$, we obtain:

$$\omega \cdot (\mathcal{X}_s - \mathcal{X}_{s,\mathrm{red}}) = 2[h^0(\mathcal{X}_{s,\mathrm{red}}) - h^0(\mathcal{X}_s)] + \mathcal{X}_{s,\mathrm{red}}^2$$

Being $\mathcal{X}$ minimal regular, ω is numerically effective (Proposition 4.10.4), and hence the left hand side of the equation is ≥ 0. Moreover, it is clear that the summands on the right hand side, i.e. $2[h^0(\mathcal{X}_{s,\mathrm{red}}) - h^0(\mathcal{X}_s)]$ and $\mathcal{X}_{s,\mathrm{red}}^2$, are both ≤ 0. Hence, we can deduce that:

$$\text{(a)} \quad \omega \cdot (\mathcal{X}_s - \mathcal{X}_{s,\mathrm{red}}) = 0, \qquad \text{(b)} \quad h^0(\mathcal{X}_{s,\mathrm{red}}) = h^0(\mathcal{X}_s), \qquad \text{(c)} \quad \mathcal{X}_{s,\mathrm{red}}^2 = 0.$$

From (c) it clearly follows that $\mathcal{X}_s = d\mathcal{X}_{s,\mathrm{red}}$ for some $d \geq 1$. If we had $d > 1$, then (a) would imply $\omega \cdot \mathcal{X}_s = 0$, and hence, via the adjunction formula, $\chi(\mathcal{X}_s) = 0$, which goes against our hypothesis $\chi(\mathcal{X}/S) < 0$. Hence, $d = 1$, i.e. $\mathcal{X}_s = \mathcal{X}_{s,\mathrm{red}}$. □

Remark 5.2.5. The proposition above still holds if, instead of $\chi(\mathcal{X}/S) < 0$, we assume that $\chi(\mathcal{X}/S) \leq 0$ and $\mathcal{X}_\eta$ has a K-rational point. In this case, in fact, the same argument presented above still ensures that $\mathcal{X}_s = d\mathcal{X}_{s,\mathrm{red}}$ for some $d \geq 1$, and $\mathcal{X}_\eta(K) \neq \emptyset$ ensures that $\mathcal{X}_s$ has non-empty smooth locus (Proposition 4.3.2), so d must be 1.

5.3. Local structure of semistable surfaces

By definition, semistable models have smooth generic fiber, and reduced special fibers: this is clearly enough to guarantee that they are always normal surfaces (Proposition 4.6.3). If $\mathcal{X}_s$ is a special fiber of a semistable surface $\mathcal{X}$, the points of $\mathcal{X}_s$ are of two kinds: smooth points and nodes. Since the generic fiber is smooth, and smoothness is an open condition, the nodes are confined in a finite number of special fibers; they are consequently finitely many and, after a finite suitable étale base-change $S' \to S$, they all become split (see the results in Section 3.4).

The completed local ring $\widehat{\mathcal{O}}_{\mathcal{X},x}$ of a surface $\mathcal{X}$ at a closed rational point x can easily be described whenever x is smooth or a split node:

Proposition 5.3.1. Let $x \in \mathcal{X}$ a closed rational point of a surface $\mathcal{X}$, lying over some $s \in S$. If x is a smooth point, $\widehat{\mathcal{O}}_{\mathcal{X},x} \cong \widehat{\mathcal{O}}_{S,s}[[a]]$; if x is a split node, $\widehat{\mathcal{O}}_{\mathcal{X},x} \cong \widehat{\mathcal{O}}_{S,s}[[a,b]]/(ab - u\pi^k)$ for some unit $u \in \widehat{\mathcal{O}}_{S,s}^{\times}$ and some $k \geq 1$, where π denotes some local parameter of S at s.

Proof. We will keep the uniformizer π of $\mathcal{O}_{S,s}$ fixed, and we will set $k := k(s) = k(x)$.

Suppose x is smooth, and let us pick any $a \in \mathcal{O}_{\mathcal{X},x}$ whose reduction modulo π is a local parameter for $\mathcal{X}_s$, so that $\widehat{\mathcal{O}}_{\mathcal{X}_s,x} = k[[\overline{a}]]$ and $\mathfrak{m}_x \subseteq \mathcal{O}_{\mathcal{X},x}$ is generated by a and π. We clearly have a canonical morphism $\psi : \widehat{\mathcal{O}}_{S,s}[[a]] \to \widehat{\mathcal{O}}_{\mathcal{X},x}$, which is certainly surjective since the residue extension $k(s) \subseteq k(x)$ is trivial, and reduces to an isomorphism $k[[\overline{a}]] \cong \widehat{\mathcal{O}}_{\mathcal{X}_s,x}$ modulo π. Let N denote the kernel of ψ: since $\widehat{\mathcal{O}}_{\mathcal{X},x}$ is flat over $\widehat{\mathcal{O}}_{S,s}$, the exact sequence $0 \to K \to \widehat{\mathcal{O}}_{S,s}[[a]] \to \widehat{\mathcal{O}}_{\mathcal{X},x} \to 0$ remains exact modulo π: if we combine this observation with the fact that ψ is an isomorphism modulo π, we can conclude that $\pi N = N$; hence, $N = 0$ by the Nakayama lemma, and this concludes the proof of the fact that $\widehat{\mathcal{O}}_{S,s}[[a]] \to \widehat{\mathcal{O}}_{\mathcal{X},x}$ is an isomorphism.

Let us now suppose that $x \in \mathcal{X}$ is a split node: we will proceed in a similar way. The theory we have developed about singularities of curves (Section 3.4) ensures that that $\widehat{\mathcal{O}}_{\mathcal{X}_s,x}$ has the form $k[[\overline{a}_1, \overline{b}_1]]/(\overline{a}_1\overline{b}_1)$ for two elements $a_1, b_1 \in \widehat{\mathcal{O}}_{\mathcal{X},x}$ that provide local parameters for the two branches of $\mathcal{X}_s$ intersecting at x; $\mathfrak{m}_x \subseteq \mathcal{O}_{\mathcal{X},x}$ will have a_1, b_1, π as generators.

We will now show how a_1 and b_1 can be replaced with a new couple of elements $a, b \in \mathfrak{m}_x\widehat{\mathcal{O}}_{\mathcal{X},x}$, congruent to a_1 and b_1 respectively modulo π, whose product ab lies in the subring $\widehat{\mathcal{O}}_{S,s} \subseteq \widehat{\mathcal{O}}_{\mathcal{X},x}$, i.e. $ab = u\pi^k$ for some unit $u \in \widehat{\mathcal{O}}_{S,s}$ and some $k \geq 1$. The idea is to build, out of the initial parameters a_1, and b_1, two convergent sequences a_n and b_n in $\widehat{\mathcal{O}}_{\mathcal{X},x}$ such that a_n and b_n retain the same class modulo π of a_1 and b_1, and $a_nb_n \in \widehat{\mathcal{O}}_{S,s} + \varepsilon_n$ for some increasingly small correction $\varepsilon_n \to 0$.

Here are the details of the inductive procedure: for $n = 1$, we trivially have that $a_1b_1 \in \widehat{\mathcal{O}}_{S,s} + \varepsilon_1$ for a correction $\varepsilon_1 \in \pi\widehat{\mathcal{O}}_{\mathcal{X},x}$. Suppose we have already built a_n and b_n, and that $a_nb_n \in \widehat{\mathcal{O}}_{S,s} + \varepsilon_n$ for some $\varepsilon_n \in \pi^n\widehat{\mathcal{O}}_{\mathcal{X},x}$. Since $\mathfrak{m}_x = (a_n, b_n, \pi)$ and the residue field extension $k(s) \subseteq k(x)$ is trivial, we may write $\varepsilon_n = \zeta\pi^n + \alpha\pi^n a_n + \beta\pi^n b_n + \gamma\pi^{n+1}$, for some $\zeta \in \widehat{\mathcal{O}}_{S,s}$ and $\alpha, \beta, \gamma \in \widehat{\mathcal{O}}_{\mathcal{X},x}$. Now it is immediate to see that $(a_n - \beta\pi^n)(b_n - \alpha\pi^n) \in \widehat{\mathcal{O}}_{S,s} + \alpha\beta\pi^{2n} + \gamma\pi^{n+1}$, but clearly $\varepsilon_{n+1} := \alpha\beta\pi^{2n} + \gamma\pi^{n+1} \in \pi^{n+1}\widehat{\mathcal{O}}_{\mathcal{X},x}$, so we can perform our induction step setting $a_{n+1} := a_n - \beta\pi^n$ and $b_{n+1} := b_n - \alpha\pi^n$.

Now, we clearly have a canonical homomorphism $\psi : \widehat{\mathcal{O}}_{S,s}[[a,b]]/(ab - u\pi^k) \to \widehat{\mathcal{O}}_{\mathcal{X},x}$, which is surjective since the extension $k(s) \subseteq k(x)$ is trivial and becomes an isomorphism modulo π. The proof of the injectivity of ψ is identical to the one we have presented for smooth points. $\square$

We will now take a ring A of the form $A := R[a,b]/(ab - u\pi^k)$, where R is a discrete valuation ring with uniformizer π, $u \in R^\times$ is a unit, and $k \geq 1$ is an integer that we will refer to as the *thickness*. We are interested in studying the properties of $X = \mathrm{Spec}(A)$.

- First, we observe that X is an integral scheme, and it is a smooth (and hence regular) R-scheme away from the closed point x that corresponds to the ideal (a, b, π). The point x is never smooth; if $k = 1$, x is a regular point of X, while, if $k > 1$, it will not even be regular.

- Suppose now $k > 1$, and let us to compute the blowup Y of X at x via the techniques we have introduced in Subsection 1.1.4: we have first to consider the closed subscheme Y' of $\mathrm{Spec}(R[a,b]) \times \mathrm{Proj}(R[x_0, x_1, x_2])$ defined by the equations $ab = u\pi^k, \pi x_1 = a x_0, \pi x_2 = b x_0, a x_2 = b x_1$, and Y will be the unique irreducible component of Y' on which a, b and π are not identically zero. Let $\mathrm{Spec}(B_i)$ and $\mathrm{Spec}(B'_i)$, where $i = 0, 1, 2$, denote the three affine charts covering Y and Y' respectively, and let $x_{ij} = x_i/x_j$ be the affine coordinates.

  Let us start with $i = 0$: we have that $B'_0 \cong R[x_{10}, x_{20}]/(\pi^2 x_{10} x_{20} - u\pi^k)$; B_0 will be $B'_0/\mathfrak{p}_0$ where $\mathfrak{p}_0$ is the unique minimal prime of B'_0 not containing π, hence $B_0 \cong R[x_{10}, x_{20}]/(x_{10} x_{20} - u\pi^{k-2})$.

  Let us now study the affinization $i = 1$: we have $B'_1 \cong R[a, x_{01}, x_{21}]/(a^2 x_{21} - u a^k x_{01}^k, a x_{01} - \pi)$; B_1 will be defined as $B'_1/\mathfrak{p}_1$ where $\mathfrak{p}_1$ is the unique minimal prime of B'_1 not containing a, hence $B_1 \cong R[a, x_{01}, x_{21}]/(x_{21} - u a^{k-2} x_{01}^k, a x_{01} - \pi) \cong R[a, x_{01}]/(a x_{01} - \pi)$. The discussion of the affine chart $i = 2$ is completely analogous to the one we have presented for $i = 1$, thanks to the symmetric role that a and b play in A.

  We are ready to draw a picture of the special fiber of Y: if we write a_1 and b_1 for the two coordinates x_{10} and x_{20}, we can say that the blowup replaces the node at which the lines $\overline{a}$ and $\overline{b}$ of X_s meet with a pair of lines $\overline{a}_1$ and $\overline{b}_1$, such that $\overline{a}_1$ meets $\overline{a}$ at a node of thickness 1, $\overline{b}_1$ meets $\overline{b}$ at a node of thickness 1, and $\overline{a}_1$ and $\overline{b}_1$ meet each other at a node of thickness $k-2$ (in the case $k = 2$, the two lines $\overline{a}_1$ and $\overline{b}_1$ coincide): see figure 5.1.

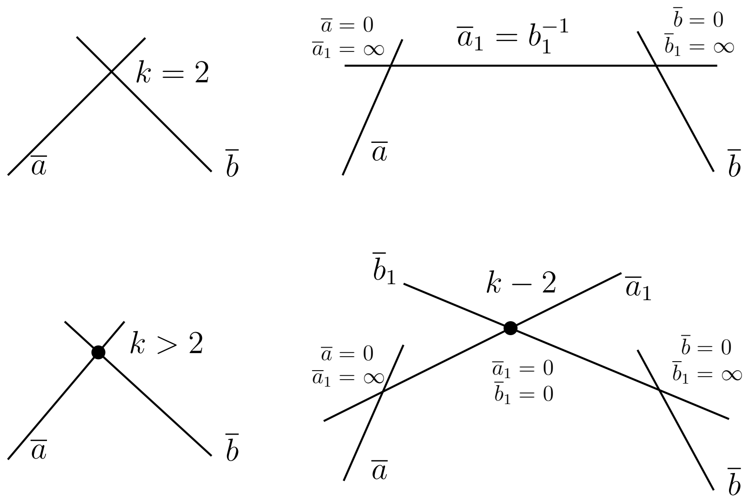


Figure 5.1.: On the left, the special fiber of $X = \operatorname{Spec}(R[a,b]/(ab - u\pi^k)))$ is displayed; the figure on the right, instead, describes the special fiber of its blowup Y in the $k = 2$ case and in the $k > 2$ case.

The description of the blowup presented above makes it clear that $\lfloor k/2 \rfloor$ repeated blowups of the singular locus turn X into a regular scheme Y, and the node of X_s is replaced with a chain of $k-1$ lines in Y_s. This also ensures that the thickness k is an intrinsic property of A (independent of the equations used to describe it), so that the notion of *thickness* of a node is unambiguously defined:

Definition 5.3.2. Let $\mathcal{X} \to S$ be a surface, and $x \in \mathcal{X}_s$ a split node of a special fiber $\mathcal{X}_s$. Then, the *thickness* of $x \in \mathcal{X}$ is defined as the unique integer $k \geq 1$ such that $\widehat{\mathcal{O}}_{\mathcal{X},x} \cong \widehat{\mathcal{O}}_{S,s}[[a,b]]/(ab - u\pi^k)$ for some unit $u \in \widehat{\mathcal{O}}_{S,s}^{\times}$.

5.4. Blowing up and blowing down

We are now ready to draw some consequences on the desingularization of semistable models:

Proposition 5.4.1. Let $\mathcal{X}$ be a semistable surface, whose nodes, which we will denote by $x_1, \ldots, x_n$, are all split with thicknesses $k_1, \ldots, k_n$ (Definition 5.3.2). Then, the minimal desingularization $\mathcal{X}'$ of $\mathcal{X}$ (Definition/Proposition 4.10.6) can be obtained by repeated blowups of the singular locus, it is still semistable, and $\operatorname{Vert}(\mathcal{X}') \smallsetminus \operatorname{Vert}(\mathcal{X})$ consists of a chain of $k_i - 1$ (-2)-curves on $\mathcal{X}'$ for each of the nodes x_i.

We will now to derive some result in the converse direction, showing that semistability is preserved under the contraction of (-1) and (-2)-curves. We preliminarily observe that

Proposition 5.4.2. Let $\mathcal{X}$ be a regular semistable surface whose nodes are all split. Then,

(a) the points at which two distinct vertical component $\Gamma, \Gamma' \in \mathrm{Vert}_s(\mathcal{X})$ meet are all $k(s)$-rational, and their number equals $\Gamma \cdot \Gamma'$;

(b) the self intersection number $-\Gamma^2$ of a vertical component $\Gamma \leq \mathcal{X}_s$ equals the total number of points at which Γ meets the other vertical components of $\mathcal{X}_s$.

Proof. A point $P \in \mathcal{X}_s$ can either be a smooth point, a split node at which a component meets itself, or a split node at which two distinct vertical components of $\mathcal{X}_s$ (which will then be smooth at P) meet.

If Γ, Γ' are two distinct components of $\mathcal{X}_s$, the 0-cycle $[\Gamma \cap \Gamma']$ consists precisely of the nodes at which Γ and Γ' meet, which are all $k(s)$-rational points, each counted with multiplicity one; $\Gamma'_i \cdot \Gamma$ will hence be simply the number of those nodes, and this proves (a). Observe now that, since $\mathcal{X}_s$ is reduced, $-\Gamma^2 = \sum_{\Gamma' \in \mathrm{Vert}_s(\mathcal{X}) \smallsetminus \Gamma} \Gamma \cdot \Gamma'$ (see Remark 4.5.2): from this, it is immediate to also deduce (b). □

Corollary 5.4.3. Let $\Gamma \leq \mathcal{X}_s$ be a vertical component of a regular semistable surface $\mathcal{X}$ whose nodes are all split;. Then, Γ is a $(-d)$-curve (for some $d \geq 1$) if and only if $\Gamma \cong \mathbb{P}^1_k$ and Γ meets the rest of $\mathcal{X}_s$ at d ($k(s)$-rational) points.

Now, we can easily state and prove the promised results.

Proposition 5.4.4. Let $\mathcal{X}$ a regular semistable surface, and $\mathcal{E} \subseteq \mathrm{Vert}(\mathcal{X})$ any finite set of (-1)-curves on $\mathcal{X}$. Then, the contraction $\mathcal{X}_{\mathcal{E}}$ is still semistable.

Proof. We know that contracting (-1)-curves on a regular surface does not make the unipotent rank of special fibers vary (see Subsection 4.13.3), so $u((\mathcal{X}_{\mathcal{E}})_s) = 0$ for all $s \in S$. Since $\mathcal{X}_s$ is geometrically reduced, it is perfectly clear that $(\mathcal{X}_{\mathcal{E}})_s$ will also be geometrically reduced. But, since $\mathcal{X}_{\mathcal{E}}$ is regular, this is enough to ensure that it is semistable (Proposition 5.2.3). □

Proposition 5.4.5. Let $\mathcal{X}$ a regular semistable surface, and $\mathcal{E} \subseteq \mathrm{Vert}(\mathcal{X})$ any finite set of (-2)-curves on $\mathcal{X}$. Then, the contraction $\mathcal{X}_{\mathcal{E}}$ is still semistable.

Proof. The same proof we have presented for the previous proposition guarantees that the fibers $(\mathcal{X}_{\mathcal{E}})_s$ are geometrically reduced with unipotent rank 0. But since $\mathcal{X}_{\mathcal{E}}$, in the case we are discussing, cannot be guaranteed to be regular, we will also have to verify that the number of branches at the points where the curves of $\mathcal{E}$ contract never exceeds 2 (Proposition 5.2.2). But this is clear from the explicit description we have given of (-2)-curves on a semistable regular surface (Corollary 5.4.3). $\square$

Remark 5.4.6. For the three propositions above to hold, the regularity assumption on $\mathcal{X}$ is only necessary because we have rigorously defined what a (-1) or a (-2)-curve is only on regular surfaces. If we define a component $\Gamma \in \mathrm{Vert}(\mathcal{X})$ of a normal (but not necessarily regular) surface $\mathcal{X}$ to be a (-1) or a (-2)-curve whenever $\Gamma \in \mathrm{Vert}(\mathcal{X}')$ is, where $\mathcal{X}'$ denotes the minimal desingularization of $\mathcal{X}$, then it is easy to see that Corollary 5.4.3 and Propositions 5.4.4 and 5.4.5 above hold also without assuming regularity.

To summarize, if Γ is a (-1)-curve on a semistable surface, it is anchored to the rest of the special fiber $\mathcal{X}_s$ at a single point x, and contracting Γ turns x into a smooth point of $\mathcal{X}$. If Γ is a (-2)-curve on a semistable surface, it connects two nodes, that will merge into a single one when the (-2)-curve gets contracted: the thickness of this new node is moreover easily seen to be equal to the sum of those of its parents.

We conclude with an immediate, important consequence of the results we have presented so far about the manipulation of semistable models:

Proposition 5.4.7. For surface $\mathcal{X} \to S$ with smooth generic fiber, the following are equivalent:
(a) $\mathcal{X}$ has a semistable model;
(b) $\mathcal{X}$ has a regular semistable model;
(c) if $\chi(\mathcal{X}/S) \leq 0$, the minimal regular model of $\mathcal{X}$ is semistable;
(d) if $\chi(\mathcal{X}/S) < 0$, the canonical model of $\mathcal{X}$ is semistable.

We have seen how two semistable models of a same surface can only differ from each other because of the presence or absence of some (-1)-curves and (-2)-curves. If we assume $\chi(\mathcal{X}/S) < 0$, the canonical model of $\mathcal{X}$, being devoid of (-1) and (-2)-curves, is clearly the minimum among all semistable models of $\mathcal{X}$: because of this peculiar role, it deserves a particular name.

Definition/Proposition 5.4.8. A surface $\mathcal{X} \to S$ with smooth geometrically connected generic fiber such that $\chi(\mathcal{X}/S) < 0$ is said to be *stable* if $\mathcal{X}$ is semistable, and one of the following equivalent conditions hold:
(a) $\mathcal{X}$ is canonical;
(b) each geometric special fiber $\mathcal{X}_{\bar{s}}$ of $\mathcal{X}$ does not contain lines meeting the other irreducible components of $\mathcal{X}_{\bar{s}}$ at fewer than three points.

Proof. The equivalence of the two conditions is now an immediate consequence of the description of $(-d)$-curves we have given for semistable surfaces in Corollary 5.4.3. □

It is clear that the formation of the stable model commutes with arbitrary base-changes:

Proposition 5.4.9. Let $\mathcal{X} \to S$ be a surface with smooth geometrically connected generic fiber such that $\chi(\mathcal{X}/S) < 0$, $S' \to S$ any flat morphism of Dedekind schemes, and $\mathcal{X}'$ the base-change of $\mathcal{X}$ to S'. If $\mathcal{X}$ is stable, then $\mathcal{X}'$ will also be. If $S' \to S$ is surjective, the converse is also true.

Proof. Since the stability of $\mathcal{X} \to S$ can be rephrased as a condition on its geometric special fibers, the proposition easily follows from arguing as in Proposition 5.2.1. □

The formation of the minimal regular model does not commute with ramified extensions of the base. However, as far as we work with semistable surfaces, we have that:

Proposition 5.4.10. If a surface $\mathcal{X}$ is semistable and minimal regular (with $\chi(\mathcal{X}/S) \geq 0$), $S' \to S$ is any flat morphism of Dedekind schemes, and $\mathcal{X}' := \mathcal{X} \times_S S'$, then the minimal desingularization $\mathcal{X}''$ of $\mathcal{X}'$ is semistable and minimal regular.

Proof. The surface $\mathcal{X}$ is semistable and does not contain (-1)-curves: it is clear that also $\mathcal{X}'$ will then be semistable (Proposition 5.2.1) and its special fibers will also not contain (-1)-curves (this can be easily deduced with the help of the characterization of (-1)-curves of semistable surfaces that we have presented in Corollary 5.4.3 and Remark 5.4.6). The desingularization process $\mathcal{X}'' \to \mathcal{X}'$ preserves semistability and only adds some (-2)-curves to the surface; as a consequence, also $\mathcal{X}''$ will be semistable and will not contain (-1)-lines, whence the proposition clearly follows. □

5.5. Existence of semistable models

We will now show that:

Theorem 5.5.1. Let $\mathcal{X} \to S$ be a regular surface whose generic fiber is smooth, geometrically connected and has genus $g(\mathcal{X}_\eta) \geq 1$ (i.e., $\chi(\mathcal{X}/S) \leq 0$). Then, after possibility replacing S with a finite flat generically étale extension $S' \to S$, $\mathcal{X}$ admits a semistable model.

Proof. We will only draw here a sketch of the proof, following [Liu, Section 10.4]. First, it is possible to show that we can reduce ourselves to the case in which S is the spectrum $S = \operatorname{Spec}(R)$ of a complete discrete valuation ring $(R, \mathfrak{m}_R)$ with algebraically closed residue field. We will denote by K the fraction field of R, and we will let ℓ be some fixed prime.

Up to replacing R with some finite generically étale extension (and $\mathcal{X}$ with its desingularization over that finite extension), we may assume that X has a K-rational point. As a consequence, $\mathcal{X}_s$ has non-empty smooth locus (Proposition 4.3.2) – in particular, it will have a component of multiplicity one. Is is easy to see that $h^0(\mathcal{X}_\eta)$ and $h^0(\mathcal{X}_s)$ must be both equal to 1; moreover, since $\mathcal{X}_\eta$ and $\mathcal{X}_s$ always have equal Euler-Poincaré characteristic, we also deduce from this that $h^1(\mathcal{X}_\eta) = h^1(\mathcal{X}_s)$.

If we choose ℓ prime to the residue characteristic, then we have an exact sequence

$$0 \to \operatorname{Pic}^0(\mathcal{X}_s)[\ell] \to \operatorname{Pic}^0(\mathcal{X}_\eta)[\ell] \to I(\mathcal{X})[\ell] \to 0$$

of $\mathbb{F}_\ell$-vector spaces (we have obtained it at the end of Subsection 4.13.2): we will now try to estimate the dimension of each term.

- Since $\mathcal{X}_\eta$ is smooth, $\operatorname{Pic}^0_{\mathcal{X}_\eta/K}$ is an abelian variety over K of rank $h^1(\mathcal{X}_\eta)$. Up to replacing R with some finite generically étale extension (which can be done without losing generality), we may suppose that all ℓ-torsion points of the Jacobian $\operatorname{Pic}_{\mathcal{X}_\eta/K}$ are K-rational. Hence, it follows from the theory of abelian varieties that

  $$\dim_{\mathbb{F}_\ell}(\operatorname{Pic}^0_{\mathcal{X}_\eta/K}(K)[\ell]) = 2g(\mathcal{X}_\eta). \tag{$\diamond$}$$

  Since $\mathcal{X}_\eta$ satisfies $h^0(\mathcal{X}_\eta) = 1$ and admits a K-rational point, $\operatorname{Pic}^0_{\mathcal{X}_\eta/K}(K)$ can be identified with $\operatorname{Pic}^0(\mathcal{X}_\eta)$ (see the results in Section 3.6); moreover, we have already observed that $h^1(\mathcal{X}_s) = h^1(\mathcal{X}_\eta)$: we can exploit these remarks to rewrite ($\diamond$) as

  $$\dim_{\mathbb{F}_\ell} \operatorname{Pic}^0(\mathcal{X}_\eta)[\ell] = 2[a(\mathcal{X}_s) + t(\mathcal{X}_s) + u(\mathcal{X}_s)].$$

- We have already remarked that

  $$\dim_{\mathbb{F}_\ell} I(\mathcal{X})[\ell] \leq t(\mathcal{X}_s)$$

  if ℓ is above some threshold that solely depends on the genus of $\mathcal{X}_\eta$ (see Proposition 4.13.7).

- Finally, as k is algebraically closed, we can be sure that $\mathrm{Pic}^0(\mathcal{X}_s)[\ell] = \mathrm{Pic}_{\mathcal{X}_s/k}(k)[\ell]$, and it follows from the theory of algebraic groups that
$$\dim_{\mathbb{F}_\ell}(\mathrm{Pic}^0(\mathcal{X}_s)[\ell]) \leq t(\mathcal{X}_s) + 2a(\mathcal{X}_s).$$

Putting together these computations, we immediately get that $u(\mathcal{X}_s) = 0$. But if we suppose that $\mathcal{X}$ is a minimal regular surface (and this can be assumed without loss of generality), the vanishing of the unipotent rank of $\mathcal{X}_s$ is enough to conclude that $\mathcal{X}$ is semistable (Proposition 5.2.4 and Remark 5.2.5). □

5.6. Forming semistable models

For this section, let R be a complete discrete valuation ring with algebraically closed residue field k, and let X be a smooth geometrically connected curve over $K = \mathrm{Frac}(R)$, of genus $g \geq 1$. We have devoted the last section to proving that, after possibly replacing R with a finite extension, X has semistable reduction; in particular, its minimal regular model and (if $g \geq 2$) its canonical model will be semistable.

However, the explicit determination of a semistable model can be a hard task. A possibly convenient approach is constructing a semistable model as the supremum in $\mathcal{M}_{\mathrm{norm}}(X)$ of a number of normal, simpler models of X: we will make an extensive use of this technique in the last chapters. Under this point of view, it is clearly useful to establish a criterion to determine whether a given normal model $\mathcal{X}$ of X is dominated or not by some semistable model of X, and to determine, in particular, how $\mathcal{X}$ is positioned with respect to the minimal regular (semistable) model and to the canonical (stable) model of X.

We start with the following, obvious remark:

Proposition 5.6.1. Let $\mathcal{X}$ be a normal model of X. If $\mathcal{X}_s$ is not reduced, then $\mathcal{X}$ cannot be dominated by any semistable model of X.

Proof. The multiplicity of a vertical component Γ of $\mathcal{X}_s$ does not change if we replace $\mathcal{X}$ with any other normal surface birational to it. Hence, if $\mathcal{X}_s$ is not reduced, and $\mathcal{X}'$ is any normal model of X dominating $\mathcal{X}$, $\mathcal{X}'_s$ cannot be reduced, and in particular $\mathcal{X}'$ cannot be semistable. □

The main technical result we will need is the following lemma, which studies the (-1) and (-2)-curves of the desingularization of a normal model.

Lemma 5.6.2. Let $\mathcal{X}$ be a normal model of X and let $\mathcal{X}'$ denote its minimal desingularization. Let us also fix a vertical component Γ of $\mathcal{X}$ having multiplicity 1, and let us denote its strict transform in $\mathcal{X}'$ by Γ', which will also have multiplicity 1 (Proposition 4.6.13). Let us write a to mean the common abelian rank of Γ and Γ' (see Corollary 4.6.12), and n_e for their common entanglement number (see Definition 4.6.15 and Proposition 4.6.16). We have that

(a) if Γ' is a (-1)-curve, then $a = 0$ and $n_e = 1$;
(b) if Γ' is a (-2)-curve, then $a = 0$ and $n_e \in \{1, 2\}$;
(c) assuming $\mathcal{X}'$ is semistable, Γ' is a (-1) curve if and only if $a = 0$ and $n_e = 1$;
(d) assuming $\mathcal{X}'$ is semistable and $g \geq 2$, Γ' is a (-2) curve if and only if $a = 0$ and $n_e = 2$.

Proof. Suppose that Γ' is a $(-d)$-curve (with $d = 1$ or $d = 2$). In this case, it is a line (i.e., $\Gamma' \cong \mathbb{P}^1_k$: see Remark 4.5.5), and, if we recall that its multiplicity is 1, we have (via Remark 4.5.2) that it must intersect the rest of $\mathcal{X}_s$ at exactly one point (if $d = 1$) or at no more than two points (if $d = 2$). From this, (a) and (b) follow immediately.

Now, assume that $\mathcal{X}'$ is semistable, that $a = 0$, and that $n_e \in \{1, 2\}$. Each node of Γ' gives a $+2$ contribution to n_e; hence, there are only two possibilities.

1. Γ' is a smooth curve; in this case we have that
   - its genus equals its abelian rank, and is consequently 0, and
   - since $\mathcal{X}_s$ is semistable, the self-intersection number $-(\Gamma')^2$ equals the entanglement number n_e.

   As a consequence, Γ' is a $(-d)$-curve, where $d = n_e$.

2. Γ' has precisely one node: but this can only occur if $n_e = 2$ and no other singular point of $(\mathcal{X}')_s$ lies on Γ'. In particular, Γ' must be a genus-1 curve exhausting the whole special fiber of the semistable model $\mathcal{X}'$, and this clearly implies that the genus of X is $g = 1$. $\square$

We are now ready to present the fundamental results of this section.

Theorem 5.6.3. Let $\mathcal{X}$ be a normal model of X. Then,

(a) if $\mathcal{X}_s$ is reduced, and all its vertical components have either abelian rank > 0, or abelian rank 0 but entanglement number > 1, then $\mathcal{X}$ is dominated by the minimal regular model $\mathcal{X}_{\text{min}}$;
(b) (assuming $g \geq 2$) if $\mathcal{X}_s$ is reduced, and all vertical components of X have either abelian rank > 0, or abelian rank 0 but entanglement number > 2, then $\mathcal{X}$ is dominated by the canonical model $\mathcal{X}_{\text{can}}$ of X;

(c) if X has semistable reduction, the two converse implications also hold.

Proof. Saying that $\mathcal{X}$ is dominated by the minimal regular model $\mathcal{X}_{\mathrm{min}}$ is equivalent to assuming that $\mathcal{X}_{\mathrm{min}}$ coincides with the minimal desingularization $\mathcal{X}'$ of $\mathcal{X}$; this is in turn equivalent to stating that no vertical component of $\mathcal{X}$ becomes a (-1)-curve in $\mathcal{X}'$. Hence, (a) is an immediate consequence of Lemma 5.6.2 (a).

Assume now that the hypotheses of implication (b) hold. Then, in particular, $\mathcal{X}$ satisfies the hypotheses of (a), and hence its minimal desingularization $\mathcal{X}'$ is the minimal regular model of X. Let C be a (-2)-curve of $\mathcal{X}'$; then, since we are assuming that all genus-0 components of $\mathcal{X}_s$ have entanglement number > 2, Lemma 5.6.2 (b) implies that C cannot be the strict transform of a vertical component of $\mathcal{X}$: in other words, $C \in \mathrm{Vert}(\mathcal{X}') \smallsetminus \mathrm{Vert}(\mathcal{X})$. As a consequence, the canonical model of $\mathcal{X}$, which is obtained by contracting all (-2)-curves of $\mathcal{X}'$, still dominates $\mathcal{X}$. This concludes the proof of (b).

Suppose now that X has semistable reduction, and that $\mathcal{X}$ is dominated by $\mathcal{X}_{\mathrm{min}}$. We have already observed that $\mathcal{X}_s$ must be reduced (Proposition 5.6.1). Moreover, $\mathcal{X}_{\mathrm{min}}$ is the minimal desingularization of $\mathcal{X}$, and hence no vertical component of $\mathcal{X}$ becomes a (-1)-curve in its minimal desingularization: Lemma 5.6.2 (c) thus ensures that no vertical component of $\mathcal{X}$ can have abelian rank 0 and entanglement number 1. If we further assume $g \geq 2$, Lemma 5.6.2 (d) ensures that all vertical components of $\mathcal{X}$ having abelian rank 0 and entanglement number 2 become (-2)-curves in $\mathcal{X}_{\mathrm{min}}$; hence, if any such component exists in $\mathrm{Vert}(\mathcal{X})$, $\mathcal{X}$ will not be dominated by $\mathcal{X}_{\mathrm{can}}$. These remarks are clearly enough to prove (c). □

Theorem 5.6.4. Let $\mathcal{X}$ be a normal model of X, and assume that X has semistable reduction. Then,

(a) if $\mathcal{X}$ is dominated by the minimal regular model, then $\mathcal{X}_{\mathrm{min}} \to \mathcal{X}$ fails to be an isomorphism precisely over the points at which the special fiber $\mathcal{X}_s$ exhibits nodes of thickness > 1, or non-nodal singularities;

(b) (assuming $g \geq 2$) if $\mathcal{X}$ is dominated by the stable model, then $\mathcal{X}_{\mathrm{st}} \to \mathcal{X}$ fails to be an isomorphism precisely over the points at which the special fiber $\mathcal{X}_s$ exhibits non-nodal singularities.

Proof. Let us prove (a). If $Q \in \mathcal{X}_s$ is a node of thickness > 1, $\mathcal{X}$ is not regular at Q; then, $\mathcal{X}_{\mathrm{min}} \to \mathcal{X}$ cannot be an isomorphism above Q because $\mathcal{X}_{\mathrm{min}}$ is a regular surface. If Q is a non-nodal singularity, then $\mathcal{X}_{\mathrm{min}} \to \mathcal{X}$ cannot be an isomorphism above Q because $\mathcal{X}_{\mathrm{min}}$ is a semistable surface. Finally, if Q is neither a node of thickness > 1 nor a non-nodal singularity of $\mathcal{X}_s$, then it is certainly a regular point of $\mathcal{X}$; since $\mathcal{X}_{\mathrm{min}} \to \mathcal{X}$ is the minimal desingularization of $\mathcal{X}$, it must then be an isomorphism above Q.

Let us now prove (b). If $Q \in \mathcal{X}_s$ is a non-nodal singularity, then $\mathcal{X}_{\mathrm{st}} \to \mathcal{X}$ cannot be an isomorphism above Q, because $\mathcal{X}_{\mathrm{st}}$ is a semistable surface. If $Q \in \mathcal{X}_s$ is a node of thickness 1 or a smooth point of $\mathcal{X}_s$, then let us consider the chain of birational morphisms $\mathcal{X}_{\min} \to \mathcal{X}_{\mathrm{st}} \to \mathcal{X}$: the morphism $\mathcal{X}_{\min}$ is an isomorphism above Q because of point (a), and hence, a fortiori, $\mathcal{X}_{\mathrm{st}} \to \mathcal{X}$ will also be. We still only have to discuss what happens if Q is a node of thickness > 1. In that case, the description of the desingularization of thick nodes that we have presented earlier in this chapter easily implies that the components of $\mathcal{X}_{\min}$ contracting to $Q \in \mathcal{X}_s$ are all (-2)-curves; hence, $\mathcal{X}_{\min} \to \mathcal{X}_{\mathrm{st}}$ already contracts them all, and $\mathcal{X}_{\mathrm{st}} \to \mathcal{X}$ is consequently an isomorphism over Q. $\square$

5.7. Behaviour with respect to quotients

Suppose now we have a surface $\mathcal{X} \to S$, and a finite group G acting faithfully on $\mathcal{X}$ (i.e., $G \leq \mathrm{Aut}_S(\mathcal{X})$). We will suppose that the action is admissible (Definition A.4.5), so to be able to compute the quotient scheme $\mathcal{Y} := \mathcal{X}/G$: admissibility is always guaranteed whenever $\mathcal{X}$ is projective, and this is always the case when $\mathcal{X}$ is regular, canonical or semistable, or when the base S is the spectrum of a Henselian discrete valuation ring.

From our discussion on the properties of quotients of schemes in appendix A, it is straightforward to verify that $\mathcal{Y}$ is integral and that $\mathcal{X} \to \mathcal{Y}$ is a finite surjective morphism; in particular, $\mathcal{Y}$ is still a surface over S (via Proposition 4.1.6); moreover, if $\mathcal{X}$ is normal then $\mathcal{Y}$ will also be normal. The main result of this section is that not only normality, but also semistability and smoothness always pass from $\mathcal{X}$ to its quotient $\mathcal{Y}$.

Both semistability and smoothness of a surface can be tested by looking at completed local rings at closed point, and completed local rings in the quotient surface $\mathcal{Y}$ can be obtained as the invariant subrings of those of the original surface $\mathcal{X}$: more precisely, if $x \in \mathcal{X}$ is a closed point, H is its stabilizer under the action of G and $y \in \mathcal{Y}$ is its image in the quotient, we have that $\widehat{\mathcal{O}}_{\mathcal{Y},y} = (\widehat{\mathcal{O}}_{\mathcal{X},x})^H \subseteq \widehat{\mathcal{O}}_{\mathcal{X},x}$.

We are now essentially reduced to the problem of considering rings of the form $A = R[[a]]$ and $A = R[[a,b]]/(ab - u\pi^k)$, where R is a complete discrete valuation ring, and determining their invariant subring under a certain finite group of R-algebras automorphisms $H \leq \mathrm{Aut}_R(A)$. In what follows, k, K and π denote respectively the residue field, the fraction field and the uniformizer of R; n will denote the order of H, $K(A)$ the field of fractions of A, $K(A)^H$ the invariant subfield of $K(A)$, which clearly coincides with the fraction field of A^H. We observe that $K(A)^H \subseteq K(A)$ is a Galois extension of degree n.

Let us start with the case $A = R[[a]]$. The element $a_1 = \prod_{h \in H} a^h$, i.e. the norm of a, is obviously invariant, so, if we let $A_1 := R[[a_1]]$ be the algebra of power series in the variable a_1, the canonical morphism $\psi : A_1 \to A$ factors through $A^H \subseteq A$.

If we now study ψ modulo π, we get a morphism $\overline{\psi} : k[[\overline{a}_1]] \to k[[\overline{a}]]$. Since the automorphisms of A must leave the maximal ideal $(\pi, a) \subseteq A$ invariant, the action modulo π of each $h \in H$ will simply consist in multiplying $\overline{a}$ by a unit of $k[[\overline{a}]]$; hence, $\overline{a}_1 = \overline{\eta}\, \overline{a}^n$, where $\overline{\eta} \in k[[\overline{a}]]^\times$. In particular, $\overline{\psi}$ is a finite and free extension of domains of degree n.

Since both A and A_1 are π-complete, the finiteness of $\overline{\psi}$ implies that of ψ. Since both A and A_1 are domains of dimension 2 and ψ is finite, it must also be injective, so we have a chain of finite extensions $A_1 \subseteq A^H \subseteq A$. Let us now consider the corresponding extensions of function fields $K(A_1) \subseteq K(A)^H \subseteq K(A)$: it is clear that the degree of $K(A_1) \subseteq K(A)$ cannot be less than $\deg(\overline{\psi}) = n$; on the other hand, we know that $K(A)^H \subseteq K(A)$ has precisely degree n. As a consequence, $K(A_1) = K(A)^H$; hence, $A_1 \subseteq A^H$ is a finite birational extension of normal domains: it will necessarily be trivial.

Let us now consider the ring $A = R[[a, b]]/(ab - u\pi^k)$; its special fiber has two minimal primes (π, a) and (π, b), which correspond to the two branches meeting at the origin: the action of a finite group $H \leq \mathrm{Aut}_R(A)$ may alternatively permute or not permute them; we will now suppose to be in the case in which they are left invariant.

If we name $a_1 := \prod_{h \in H} a^h$ and $b_1 := \prod_{h \in H} b^h$ the norms of a and b, they will clearly be invariant and satisfy the equation $a_1 b_1 = u^n \pi^{nk}$. Hence, if we define $A_1 := R[[a_1, b_1]]/(a_1 b_1 - u^n \pi^{nk})$, we have a canonical morphism $\psi : A_1 \to A$ that factors through $A^H \subseteq A$.

As before, we will first study ψ modulo π, i.e. $\overline{\psi} : k[[\overline{a}_1, \overline{b}_1]]/(\overline{a}_1 \overline{b}_1) \to k[[\overline{a}, \overline{b}]]/(\overline{a}\overline{b})$. Since H leaves (π, a) and (π, b) invariant, the action of $h \in H$ will simply consist in multiplying $\overline{a}$ and $\overline{b}$ by a unit; hence, $\overline{a}_1 = \overline{\eta}_a \overline{a}^n$ and $\overline{b}_1 = \overline{\eta}_b \overline{b}^n$, where $\overline{\eta}_a, \overline{\eta}_b \in (k[[\overline{a}, \overline{b}]]/(\overline{a}\overline{b}))^\times$.

From this it is perfectly clear that $\overline{\psi}$ is finite; we are now interested in estimating its degree at the generic points of $A_1/\pi A_1$. Let us now consider $\mathfrak{p}_a := (\pi, a)$, $\mathfrak{p}_b := (\pi, a)$, $\mathfrak{p}_{a1} := (\pi, a_1)$, $\mathfrak{p}_{b1} := (\pi, b_1)$ the minimal primes of the special fiber of A and A_1 respectively. Their residue fields are $k(\mathfrak{p}_a) = k((\overline{b}))$, $k(\mathfrak{p}_b) = k((\overline{a}))$, $k(\mathfrak{p}_{a_1}) = k((\overline{b}_1))$ and $k(\mathfrak{p}_{b_1}) = k((\overline{a}_1))$. It is not difficult to see that the fiber of $\overline{\psi}$ over $\mathfrak{p}_{a_1}$ is given by the extension $k((\overline{b}_1)) \subseteq k((\overline{b}))$, whose degree is n (in other words, $\mathfrak{p}_a$ is the unique point over $\mathfrak{p}_{a1}$, there is no ramification and the intertia index is n). The same is clearly also true for $\mathfrak{p}_{b_1}$ and $\mathfrak{p}_b$. Hence, the degree of $\overline{\psi}$ at both generic points of $A_1/\pi A_1$ is n.

As in the case we have presented before, the results we have derived on $\overline{\psi}$ are enough to ensure that $\psi : A_1 \to A$ is injective, and that $A_1 = A^H$.

Suppose now that $A = R[[a,b]]/(ab - u\pi^k)$ again, but that H is now generated by an involution σ that swaps (π, a) and (π, b). Let $a_1 := a + \sigma(a)$ denote the trace of a, which is clearly an invariant element of A, and let $A_1 := R[[a_1]]$: the canonical morphism $\psi : A_1 \to A$ factors through A^H.

If we now study ψ modulo π, we get $\overline{\psi} : k[[\overline{a}_1]] \to k[[\overline{a}, \overline{b}]]/(\overline{a}\overline{b})$, and we have that $\sigma(\overline{a})$ will equal $\overline{b}$ up to a unit $\overline{\eta} \in (k[[\overline{a}, \overline{b}]]/(\overline{a}\overline{b}))^\times$; hence $\overline{a}_1 = \overline{a} + \overline{\eta}\overline{b}$. From this it is immediate to see that $\overline{\psi}$ is a finite free rank 2 extension of domains.

As we did in the previous cases, it is now easy to conclude that $\psi : A_1 \to A$ must be an injection, and $A_1 = A^H$.

Let us now finally suppose again $A = R[[a,b]]/(ab - u\pi^k)$, and that $H \leq \mathrm{Aut}_R(A)$ is now any finite group of automorphism swapping (π, a) and (π, b). Then, the stabilizer of (π, a) under the action of H equals the one of (π, b), and is a normal subgroup $H' \trianglelefteq H$ of index 2. The idea is now clearly to compute A^H in two stages: first, $A^{H'}$ is computed and gives $A^{H'} = R[[a_1, b_1]]/(a_1 b_1 - u^n \pi^{nk})$ (where a_1 and b_1 are the norms of a_1); now, H/H' is a group of order 2 that acts on $A^{H'}$ swapping (π, a_1) and (π, b_1), so $A^H = (A^{H'})^{H/H'} = R[[a_2]]$ (where a_2 is the trace of a_1).

These computations ensure that:

Proposition 5.7.1. If G is a finite group acting on a semistable surface $\mathcal{X} \to S$, then $\mathcal{X}/G$ is a semistable surface. More precisely, on each special fiber, the quotient map $f : \mathcal{X} \to \mathcal{X}/G$ takes smooth points to smooth points, and nodes to smooth points or nodes, depending on whether the action of G swaps the two branches passing through the node or not. Moreover, if $x \in \mathcal{X}_s$ is a split node that remains a node in $\mathcal{X}/G$, then the thickness of $f(x)$ equals the one of x times the order of the stabilizer of x.

Corollary 5.7.2. If $Y \to X$ is a Galois covering of smooth geometrically connected curves over some complete discrete valued field K, with $g(Y) \geq 1$, and if Y has semistable (resp. good) reduction, then some semistable (resp. smooth) model of Y can be computed as the normalization of a suitable semistable (resp. smooth) model of X.

Proof. If we assume Y has semistable reduction and $\chi(Y) \leq 0$, Y will admit a minimal regular model $\mathcal{Y}_{\min}$ which will also be semistable. But, thanks to its minimality, $\mathcal{Y}_{\min}$ is admissibly acted by $\mathrm{Gal}_{K(X)}(K(Y))$ (Definition 4.12.2 and Proposition 4.12.5), and hence, thanks to the results of Section 4.12, it can be seen as the normalization in $K(Y)$ of the model $\mathcal{Y}_{\min}/G$ of X. But Proposition 5.7.1 ensures that $\mathcal{Y}_{\min}/G$ is semistable. If Y is further assumed to have good reduction, then $\mathcal{Y}_{\min}$ will be the unique smooth model of Y, and hence, still thanks to Proposition 5.7.1, $\mathcal{Y}_{\min}/G$ is also smooth. □

This corollary ensures that a semistable model of the curve Y, if it exists, can be certainly be discovered by scrolling through all semistable models of X and normalizing them in the extension. When X is simple enough, for example when X is a line, this is actually a feasible task, and Corollary 5.7.2 will be in particular our reference point for determining semistable models of hyperelliptic curves in the next chapters.

6. Semistable models of hyperelliptic curves: part I

Now that semistable models of a curve have been introduced and their existence established, the aim of this chapter and of the next one is the construction of semistable models of a given tame hyperelliptic curve, both over residue characteristic $\neq 2$ and over residue characteristic 2.

This chapter is actually devoted to develop a number of techniques and notions that are not specific to hyperelliptic curves, but will all be heavily used in the next one to treat the hyperelliptic case. Nonetheless, the first section of the present chapter anticipates some introductory notions on hyperelliptic curves, so as to motivate how the tools that we will introduce can be useful for the study of their models.

6.1. Introduction

Let K be a field, and let $X := \mathbb{P}^1_K$ be the line over K.

Definition 6.1.1. A hyperelliptic curve over K is a smooth, geometrically connected K-curve Y admitting a generically étale (branched) covering morphism $h : Y \to X$ of degree 2 to the line, which will be named a *hyperelliptic map.*

If $\mathrm{char}(K) \neq 2$ (i.e., if we consider *tame* hyperelliptic curves), the Riemann-Hurwitz formula ensures that the ramification locus of $Y_{\overline{K}} \to X_{\overline{K}}$ consists of $2g+2$ points of $Y_{\overline{K}}$, lying over $2g+2$ distinct branch points of $X_{\overline{K}}$, where $g \geq 0$ is the genus of Y. Over the affine chart $x \neq 0$, the equation of Y can be written in the form $y^2 = af(x)$, where $a \in K^\times$ and $f(x) \in K[x]$ is the monic polynomial of degree $2g+1$ or $2g+2$ whose roots correspond to the branch points of h (excepting the possible branch point at $x = \infty$). The branch locus determines a hyperelliptic curve almost completely:

Proposition 6.1.2. Given a field K such that $\mathrm{char}(K) \neq 2$, and letting X be the line $\mathbb{P}^1_K$, the following data are equivalent:

(a) a hyperelliptic curve Y of genus g having rational ramification locus, endowed with a distinguished hyperelliptic map $h : Y \to X$;

(b) a set of $2g+2$ distinct K-rational points of the line X, together with an element $a \in K^\times/(K^\times)^2$.

This result will allow us to treat a hyperelliptic curve essentially as a marked line, and is peculiar of the hyperelliptic case: if we were to deal with a tame Galois covering of the line of degree > 2, the same branch locus would in general be shared by multiple branched coverings corresponding to various possible monodromy actions.

Our aim will be constructing semistable models of a given tame hyperelliptic curve $h : Y \to X$ by means of the technique we have presented at the end of last chapter, i.e. by normalizing some carefully chosen semistable models of the line X in the quadratic extension $K(X) \subseteq K(Y)$. To do this, we will first need some preliminaries on the classification of semistable models of the line (Section 6.2) and of a marked line (Section 6.3). We will then explain how to identify, among all semistable models of the line, those that can give rise to semistable models of a given Galois covering of the line (Section 6.4): the next chapter will then specialize this technique to the hyperelliptic case.

6.1.1. Notation and conventions

For this whole chapter and the next one, we will work using a complete discrete valuation ring R as a base, with algebraically closed residue field k. The fraction field of R will be denoted by K, and we will fix a uniformizer π for R. The generic and closed point of $\mathrm{Spec}(R)$ will be denoted respectively as η and s. We will use R' to mean some finite extension of R, and we will write K' for the fraction field of R'. We will denote by X the projective line $\mathbb{P}^1_K$, while Y will mean a smooth geometrically connected curve endowed with a Galois generically étale (branched) covering morphism $Y \to X$.

6.2. Models of the line

Let $X = \mathbb{P}^1_K$ be the line over K. Any fixed automorphism $\psi \in \mathrm{Aut}_K(\mathbb{P}^1_K)$ gives rise to a smooth model $\mathcal{X}^\psi$ of X, which is described as follows: as an R-scheme, $\mathcal{X}^\psi$ is just $\mathbb{P}^1_R$, and the distinguished isomorphism $(\mathcal{X}^\psi)_\eta \to X$ that gives $\mathcal{X}^\psi$ the structure of a model of X is provided by ψ. It is clear that two automorphisms $\psi_1, \psi_2 \in \mathrm{Aut}_K(\mathbb{P}^1_K)$ give rise to isomorphic models if and only if they differ by an automorphism of $\mathbb{P}^1_R$; more formally, we have an injective map:

$$\mathrm{Aut}_K(\mathbb{P}^1_K)/\,\mathrm{Aut}_R(\mathbb{P}^1_R) \hookrightarrow \left\{ \begin{array}{c} \text{Smooth models of the line} \\ \text{(modulo isomorphism)} \end{array} \right\}, \qquad \psi \mapsto \mathcal{X}^\psi$$

where $\mathrm{Aut}_K(\mathbb{P}^1_K)/\mathrm{Aut}_R(\mathbb{P}^1_R)$ denotes the quotient of the group $\mathrm{Aut}_K(\mathbb{P}^1_K)$ over its normal subgroup $\mathrm{Aut}_R(\mathbb{P}^1_R)$.

Proposition 6.2.1. The correspondence described above is actually a biiection.

Proof. The only point that still needs verification is surjectivity, which can be proved following [Liu, Excercise 8.3.5]. □

Proposition 6.2.2. Every normal model $\mathcal{X}$ of $X = \mathbb{P}^1_K$ whose special fiber is integral is smooth.

Proof. Since the special and the generic fiber of $\mathcal{X}$ must have the same Euler-Poincaré characteristic, the integral curve $\mathcal{X}_s$ has genus 0 and is consequently isomorphic to $\mathbb{P}^1_k$ (recall that the residue field k has been assumed algebraically closed). In particular, $\mathcal{X} \to \mathrm{Spec}(R)$ is smooth. □

Given a number of distinct smooth models $\mathcal{X}_1, \ldots, \mathcal{X}_n$ of X, we can compute the minimum normal model dominating them all, which we will denote by $\mathcal{X}_{1\ldots n}$.

Definition 6.2.3. We will name $\mathcal{X}_{1\ldots n}$ the *composite* of the smooth models $\mathcal{X}_1, \ldots, \mathcal{X}_n$, and we will say that $\mathcal{X}_1, \ldots, \mathcal{X}_n$ are the *smooth components* of $\mathcal{X}_{1\ldots n}$. We will say *a composite model of the line* to mean any model of $X = \mathbb{P}^1_K$ that can be obtained as the composite of a finite number of smooth models of X.

It is immediate to see that, if $\mathrm{Vert}(\mathcal{X}_i) = \{L_i\}$, then $\mathrm{Vert}(\mathcal{X}_{1\ldots n}) = \{L_1, \ldots, L_n\}$. The smooth models $\mathcal{X}_1, \ldots, \mathcal{X}_n$ consequently correspond biiectively to the vertical components of $\mathcal{X}_{1\ldots n}$ and can be recovered as the only smooth models of X dominated by $\mathcal{X}_{1\ldots n}$. Hence,

Proposition 6.2.4. There is a bijective correspondence

$$\mathrm{P}^{\mathrm{fin}*}(\mathrm{Aut}_K(\mathbb{P}^1_K)/\mathrm{Aut}_R(\mathbb{P}^1_R)) \leftrightarrow \left\{ \begin{array}{c} \text{Composite models of the} \\ \text{line (modulo isomorphism)} \end{array} \right\}$$

where $P^{\mathrm{fin}*}(\cdot)$ denotes the finite nonempty subsets. Singletons (on the left) correspond to smooth models (on the right).

For each i, $L_i \in \mathrm{Vert}(\mathcal{X}_i)$ has multiplicity 1 and is isomorphic to $\mathbb{P}^1_k$, and the same will also consequently be true for its strict transform $L_i \in \mathrm{Vert}(\mathcal{X}_{1,\ldots,n})$. In particular, composite models of the line have reduced special fibers, and all their vertical components are lines. It is immediate to see that the converse is also true:

Proposition 6.2.5. A normal model $\mathcal{X}$ of the line X is composite if and only if it has reduced special fiber.

Proof. We only have to prove the "if" part of the statement. If we write $\mathcal{X}_i$ for the normal model of X we obtain from $\mathcal{X}$ by contracting $\mathrm{Vert}(\mathcal{X}) \smallsetminus \{L_i\}$, then $\mathcal{X}_i$ is necessarily a smooth model of X (by Proposition 6.2.2). Now, if $\mathcal{X}_{1,\dots,n}$ denotes the composite of all the $\mathcal{X}_i's$, we have that $\mathcal{X}$ and $\mathcal{X}_{1,\dots,n}$ are normal models of the line X having the same set of vertical components: hence, they are isomorphic, and $\mathcal{X}$ is consequently a composite model of the line. □

In particular, we have that

Corollary 6.2.6. All semistable models of the line are composite.

The converse of this corollary is not true; however, there is an easy way of characterizing semistable models among the composite ones:

Proposition 6.2.7. A composite model $\mathcal{X}$ of the line X is semistable if and only if no more than two vertical components pass through each closed point of $\mathcal{X}_s$. In particular, all composite regular models of the line are semistable.

Proof. If $\mathcal{X}$ is a composite model of X, it has reduced special fiber (Proposition 6.2.5). Moreover, as X is a line, $\chi(\mathcal{X}/S) > 0$, and, since $\mathcal{X}_s$ is a connected reduced curve over an algebraically closed field, this clearly means that $h^1(\mathcal{X}_s) = 0$; in particular, $u(\mathcal{X}_s) = 0$. Hence we may invoke Proposition 5.2.2 and be sure that $\mathcal{X}$ is semistable if and only if the number of branches of $\mathcal{X}_s$ at each closed point $P \in \mathcal{X}_s$ is ≤ 2. But since all vertical components of $\mathcal{X}_s$ are lines, and hence, in particular, smooth curves, the number of branches at P is nothing but the number of vertical components to which P belongs, whence the proposition follows. □

The hypotheses of Proposition 6.2.7 are trivially satisfied if $\mathcal{X} = \mathcal{X}_{12}$ is the composite of two smooth models $\mathcal{X}_1$ and $\mathcal{X}_2$ of the line. The special fiber $(\mathcal{X}_{12})_s$, in this case, is a semistable curve with a unique node P: let us denote by P_1 and P_2 the images of P in $(\mathcal{X}_1)_s$ and $(\mathcal{X}_2)_s$.

Definition 6.2.8. Given two smooth models $\mathcal{X}_1$ and $\mathcal{X}_2$ of the line, and letting $\mathcal{X}_{12}$, P, P_1 and P_2 be as above, we say that P_1 is *the point of* $(\mathcal{X}_1)_s$ *at which* $\mathcal{X}_2$ *meets* $\mathcal{X}_1$. The thickness of the node $P \in (\mathcal{X}_{12})_s$ is named the *distance* between the smooth models $\mathcal{X}_1$ and $\mathcal{X}_2$.

Definition 6.2.9. Two smooth models $\mathcal{X}_1$ and $\mathcal{X}_2$ of the line are said to be *adjacent* if their distance is equal to 1.

Given $\mathcal{X}$ a smooth model of the line, and $Q \in \mathcal{X}_s$ a closed point, it is immediate to verify that there exists a unique smooth model of the line $\mathcal{X}_Q$ adjacent to $\mathcal{X}$ (Definition 6.2.9) and meeting $\mathcal{X}$ at Q (Definition 6.2.8): $\mathcal{X}_Q$ is obtained by blowing up $\mathcal{X}$ at Q and then blowing down the original vertical component of $\mathcal{X}$.

Definition 6.2.10. The model $\mathcal{X}_Q$ defined above is named the *elementary alteration* of $\mathcal{X}$ at $Q \in \mathcal{X}_s$.

Remark 6.2.11. Given two smooth models of the line $\mathcal{X}^a$ and $\mathcal{X}^b$, there exists a uniquely determined chain $\mathcal{X}_0, \dots, \mathcal{X}_n$ of smooth models of the line such that $\mathcal{X}_0 = \mathcal{X}^a$, $\mathcal{X}_n = \mathcal{X}^b$ and any two subsequent terms $\mathcal{X}_i$ and $\mathcal{X}_{i+1}$ are adjacent. In other words, adjacency turns the set of all smooth models of the line into a tree, and the *distance* of Definition 6.2.8 is nothing but the distance on the tree.

Remark 6.2.12. Giving a composite model $\mathcal{X}$ of the line means selecting a finite set $V = \{\ell_1, \dots, \ell_n\}$ of vertices in the tree $\mathcal{T}$ of all smooth models of the line, and it is easy to realize that

(a) $\mathcal{X}$ is regular if and only if V is the set of vertices of a subtree of $\mathcal{T}$;
(b) more generally, the vertices of the subtree $\langle V \rangle$ generated by V correspond to the smooth components of the minimal desingularization of $\mathcal{X}$ (which will also be a composite model of X);
(c) the singular points $P \in \mathcal{X}_s$ correspond to the maximal subtrees of $\langle V \rangle$ whose leaves fall in V and whose internal vertices fall outside V: the leaves correspond to the vertical components of $\mathcal{X}$ meeting at the point P.

We end this section by observing that the composition of smooth models of the line clearly commutes with extensions of the base.

Proposition 6.2.13. Let $\mathcal{X}_1, \dots \mathcal{X}_n$ be smooth models of the line over R and $\mathcal{X}_{1,\dots,n}$ be their composite. Then, given any finite extension $R \subseteq R'$, $(\mathcal{X}_{1,\dots,n})_{R'}$ is the composite of the smooth models $(\mathcal{X}_1)_{R'}, \dots, (\mathcal{X}_n)_{R'}$.

6.3. Cluster data

Suppose we are now given two K-rational points A and B on the line $X = \mathbb{P}^1_K$, and let us fix a smooth model $\mathcal{X}$ of X: then, thanks to the valutative criterion for properness, A and B, thought of as points of $\mathcal{X}_\eta$, will extend uniquely to a pair of R-points of $\mathcal{X}$, which we will also name A and B. There will certainly exist some affine chart of $\mathcal{X}$ containing them both; once such an affine chart has been fixed, A and B get identified with two elements of R, and we can in particular compute the valuation $v(A - B)$. It is however not difficult to see that $v(A - B)$ does not depend on the chosen affine chart, and it can be characterized as the number of times that $\mathcal{X}$ has to be blown up at a closed point to separate $\overline{A} \in \mathcal{X}_s$ and $\overline{B} \in \mathcal{X}_s$: in other words, $v(A - B)$ is the minimum distance from $\mathcal{X}$, in the sense of Definition 6.2.8, of a smooth model of X in which A and B have distinct reductions.

Example 6.3.1. Let us fix the smooth model $\mathcal{X} = \mathcal{X}^\psi$ of the line X corresponding to $\psi = \mathrm{id}$ (see Section 6.2 for notation); in other words, we have $\mathcal{X} = \mathbb{P}^1_R$, and $X = \mathcal{X}_\eta = \mathbb{P}^1_K$. We will denote by x the standard coordinate on the line $\mathbb{P}^1_R$.

Let us consider the two points $A_1, B_1 \in X(K)$ given by $x(A_1) = \pi^{-1}$ and $x(B_1) = \pi^2(1+\pi)$. As A_1 reduces to $\overline{x} = \infty$, to compute $v(A_1 - B_1)$ we have to pick on $\mathcal{X}$ a coordinate different from x. For example, $x' := 1/(x-1)$ is a good choice, because the two R-points $A_1, B_1 : R \to \mathcal{X}$ both factor through the open subset $\{x' \neq \infty\} \subseteq \mathbb{P}^1_R$. An easy computation shows that that $x'(A_1) - x'(B_1)$ is a unit of R; hence, we have that $v(A_1 - B_1) = 0$ by definition. We may have also deduced the same result without introducing x', by simply observing that, since $\overline{x}(A_1) = \infty$ and $\overline{x}(B_1) = 0$, A_1 and B_1 reduce to distinct points of the special fiber $\mathcal{X}_s$.

Let us now consider $A_2, B_2 \in X(K)$ given by $x(A_2) = \pi$ and $x(B_2) = \pi(1+\pi)$: the two corresponding R-points of $\mathcal{X}$ are both contained in the standard chart $\{x \neq \infty\} \subseteq \mathcal{X}$, and we can thus say that $v(A_2 - B_2) = v(x(A_2) - x(B_2)) = 2$. In the special fiber of $\mathcal{X}$, both A_2 and B_2 reduce to the same point $\overline{x} = 0$; if we blow $\mathcal{X}$ up once at $\overline{x} = 0$, a new coordinate $x' = x/\pi$ will be introduced, but A and B will keep having the same reduction; however, if we perform a second blowup (centered at the common reduction $\overline{x}' = 1$ of A and B), a further coordinate $x'' = (x'-1)/\pi$ will arise, and we finally obtain that $\overline{x}''(A_2) = 0$ and $\overline{x}''(B_2) = 1$ are distinct.

Remark 6.3.2. It is immediate to observe that, given a smooth model $\mathcal{X}$ of $X = \mathbb{P}^1_K$, the assignment $(A, B) \mapsto e^{-v(A-B)}$ defines a non-Archimedean metric on the set $X(K)$ of K-rational points of the line: we will denote it as $\delta_{\mathcal{X}}$.

Let us now fix a smooth model $\mathcal{X}$ of the line $X = \mathbb{P}^1_K$, and a set of rational points $\mathcal{R} = \{A_1, \ldots, A_n\} \subseteq X(K)$ of X.

Definition 6.3.3. Let $\mathfrak{s} \subseteq \mathcal{R}$ be a nonempty subset of $\mathcal{R}$ cut out by a closed disk for the metric $\delta_{\mathcal{X}}$. We can compute the minimum radius of a closed disk cutting out $\mathfrak{s} \subseteq \mathcal{R}$ as $e^{-d_{\mathfrak{s}}}$, where $d_{\mathfrak{s}} := \min_{A,B\in\mathfrak{s}} v(A-B)$. If $d_{\mathfrak{s}} > 0$, then $\mathfrak{s}$ is named a *cluster* of $\mathcal{R}$, and $d_{\mathfrak{s}} \in \mathbb{N}^+ \cup \{\infty\}$ is said to be the *depth* of the cluster.

Definition 6.3.4. Given a cluster $\mathfrak{s}$ of $\mathcal{R}$, either $\mathfrak{s}$ is maximal among the clusters of $\mathcal{R}$, or there exists a minimum cluster strictly larger than $\mathfrak{s}$ that contains it, which we will name the *parent* of $\mathfrak{s}$ and we will denote by $p(\mathfrak{s})$. The difference $d_{\mathfrak{s}} - d_{p(\mathfrak{s})} \in \mathbb{N}^+ \cup \{\infty\}$ is named the *relative depth* of $\mathfrak{s}$, and is denoted by $\delta_{\mathfrak{s}}$. If $\mathfrak{s}$ is a maximal cluster, we will conventionally define its relative depth as equal to the absolute one: $\delta_{\mathfrak{s}} := d_{\mathfrak{s}} \in \mathbb{N}^+ \cup \{\infty\}$.

Definition 6.3.5. Let $\mathrm{P}^*(\mathcal{R})$ denote the set of nonempty subsets of $\mathcal{R}$. The function $\mathrm{P}^*(\mathcal{R}) \to \mathbb{N} \cup \{\infty\}$ that maps a nonempty subset $\mathfrak{s} \subseteq \mathcal{R}$ to 0 if $\mathfrak{s}$ is not a cluster, and to the relative depth $\delta_{\mathfrak{s}}$ if $\mathfrak{s}$ is a cluster, is named the *$\mathcal{R}$-cluster picture* associated to the smooth model $\mathcal{X}$ of X.

Remark 6.3.6. Once a smooth model of the line $\mathcal{X}$ and a finite set of points $\mathcal{R} \subseteq X(K)$ are fixed, the following data are clearly equivalent:
(a) the $\mathcal{R}$-cluster picture associated to $\mathcal{X}$;
(b) the set of clusters of $\mathcal{R}$ and the indication of their depths;
(c) the valuations $v(A_i - A_j)_{i \neq j}$.
In fact, it is clear that the cluster picture is simply a way of codifying what are the clusters and what are their depths. Moreover, the valuations $v(A_i - A_j)_{i \neq j}$ clearly determine the cluster picture, and conversely $v(A_i - A_j)$ can be recovered as the depth of the smallest cluster containing both A_i and A_j.

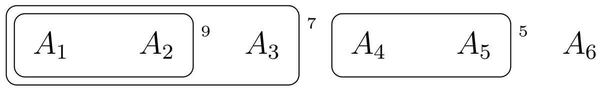
We could have also proceeded by introducing an abstract, combinatorial notion of what a cluster picture is, and then saying that every smooth model of the line X defines some distinguished, concrete cluster picture:

Definition 6.3.7. An *$\mathcal{R}$-cluster picture* is any function $\Sigma : \mathrm{P}^*(\mathcal{R}) \to \mathbb{N} \cup \{\infty\}$ satisfying the following properties:
(a) if $E_1, E_2 \in \mathrm{P}^*(\mathcal{R})$ are such that $\Sigma(E_1) > 0$ and $\Sigma(E_2) > 0$, then $E_1 \subseteq E_2$, $E_2 \subseteq E_1$, or $E_1 \cap E_2 = \emptyset$;
(b) $\Sigma(E) = \infty$ if and only if E is a singleton.
Given Σ an $\mathcal{R}$-cluster picture, a *Σ-cluster* is any $\mathfrak{s} \subseteq \mathcal{R}$ such that $\Sigma(\mathfrak{s}) > 0$; the value $\Sigma(\mathfrak{s})$ is named the *relative depth* of $\mathfrak{s}$ and is denoted by $\delta_{\mathfrak{s}}$. Clearly, we can easily define, in this abstract setting, the notions of parenthood of clusters, absolute depth, etc.

Definition 6.3.8. The set of all $\mathcal{R}$-cluster pictures is denoted by $\mathrm{CP}_{\mathcal{R}}$. We will denote by $\mathrm{ccp}_{\mathcal{R}}$ the map taking every smooth model $\mathcal{X}$ of X to its associated cluster picture $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}) \in \mathrm{CP}_{\mathcal{R}}$. The image of $\mathrm{ccp}_{\mathcal{R}}$ will be denoted by $\mathrm{CCP}_{\mathcal{R}} \subseteq \mathrm{CP}_{\mathcal{R}}$, and its elements will be named *the concrete $\mathcal{R}$-cluster pictures.*

Convention 6.3.9. If $\mathcal{P}$ is some property of $\mathcal{R}$-cluster pictures, and $\mathcal{X}$ is a smooth model of X, we will say that $\mathcal{X}$ satisfies $\mathcal{P}$ if the associated cluster picture $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ does. If $\mathcal{X} = \mathcal{X}_{1\ldots n}$ is the composite of multiple smooth models $\mathcal{X}_1, \ldots, \mathcal{X}_n$ of the line (Definition 6.2.3), we will say that $\mathcal{X}$ satisfies $\mathcal{P}$ if the cluster picture $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}_i)$ does for all i.

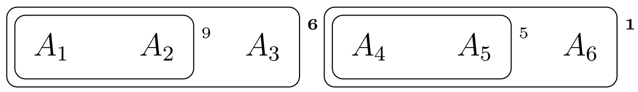
This is an example of how, for a set $\mathcal{R}$ of 6 distinct rational points of $X = \mathbb{P}^1_K$, the cluster picture associated to some smooth model $\mathcal{X}$ of X could look:

$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$:


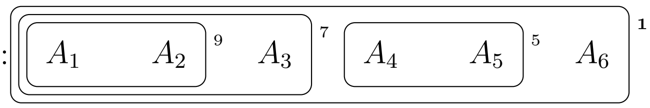
Ovals correspond to clusters (we are not drawing the singleton clusters, of depth ∞, around the single A_i's); the index reported on the top right of each oval is the relative depth of the cluster. In our example, the points of $\mathcal{R}$ reduce to three distinct points $\overline{A}_{123}$, $\overline{A}_{45}$ and $\overline{A}_6$ in $\mathcal{X}_s$. In general, we have that:

Remark 6.3.10. Given a smooth model $\mathcal{X}$ of the line X, the points of the special fiber $\mathcal{X}_s$ to which the $A_i \in \mathcal{R}$ reduce are in one-to-one correspondence with the maximal clusters of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$. More in general, if $\mathcal{X}_{1\ldots n}$ is the composite of n smooth models $\mathcal{X}_1, \ldots, \mathcal{X}_n$ of the line X, then two points $A_i, A_j \in \mathcal{R}$ reduce to the same point of $(\mathcal{X}_{1\ldots n})_s$ if and only if, for each i, they fall within the same maximal cluster of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}_i)$.

The cluster picture does effectively depend on the chosen model. Suppose, for example, that we switch from $\mathcal{X}$ to its elementary alteration $\mathcal{X}'$ at $\overline{A}_{123} \in \mathcal{X}_s$ (Definition 6.2.10): the cluster picture of $\mathcal{R}$ will change to

$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}')$:


If we had taken $\mathcal{X}'$ to be the unit transformation of $\mathcal{X}$ at some point $Q \in \mathcal{X}_s$ distinct from $\overline{A}_{123}$, $\overline{A}_{45}$ and $\overline{A}_6$, then cluster picture associated to $\mathcal{X}'$ would instead have become:

$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}')$:


To abstractly codify these phenomena, we introduce the following definition.

Definition 6.3.11. Given a cluster picture $\Sigma : \mathrm{P}^*(\mathcal{R}) \to \mathbb{N} \cup \{\infty\}$, and given $\mathfrak{s}$ a maximal cluster or $\mathfrak{s} = \emptyset$, the *elementary alteration* of Σ at $\mathfrak{s}$ is the function $\Sigma' : \mathrm{P}^*(\mathcal{R}) \to \mathbb{N} \cup \{\infty\}$ defined as:

$$\Sigma'(E) = \begin{cases} \Sigma(E) - 1 & \text{if } E = \mathfrak{s}, \\ \Sigma(E) + 1 & \text{if } E = \mathcal{R} \smallsetminus \mathfrak{s}, \\ \Sigma(E) & \text{otherwise.} \end{cases}$$

We will say that $\mathfrak{s}$ is the *center* of the elementary alteration that turns Σ into Σ'.

It is immediate to verify that Σ' is still a valid cluster picture in the sense of Definition 6.3.7. It is also clear that, if Σ' is the elementary alteration of Σ at $\mathfrak{s}$, then Σ is the elementary alteration of Σ' at $\mathcal{R} \smallsetminus \mathfrak{s}$. Two abstract cluster pictures that can be obtained one from the other one by means of an elementary alteration are said to be *adjacent*; if we can move from the one to the other one by means of finitely many elementary alterations they are said to be *equivalent*.

It is simply a matter of combinatorics to prove that adjacency turns the set of all $\mathcal{R}$-cluster pictures $\mathrm{CP}_{\mathcal{R}}$ into an acyclic undirected graph, or, in other words, it turns each equivalence class of $\mathrm{CP}_{\mathcal{R}}$ into a tree. This means that, given two equivalent cluster pictures $\Sigma^a, \Sigma^b \in \mathrm{CP}_{\mathcal{R}}$, there is a unique chain of cluster pictures $\Sigma_0, \ldots, \Sigma_d \in \mathrm{CP}_{\mathcal{R}}$ such that $\Sigma_0 = \Sigma^a$, $\Sigma_d = \Sigma^b$ and Σ_{i+1} is adjacent to Σ_i for all i. The integer d is named the *distance* of Σ^a and Σ^b. If $\mathfrak{s} \subseteq \mathcal{R}$ is the center of the elementary alteration that turns Σ_0 into Σ_1, we say that Σ^b *meets* Σ^a *at* $\mathfrak{s}$.

Before moving further, let us remark that a cluster $\mathfrak{s} \subseteq \mathcal{R}$ of a cluster picture Σ may well stop being a cluster if we replace Σ with an equivalent cluster picture Σ'. However, whenever an elementary alteration dissolves a cluster, its complement is always guaranteed to either remain or become a cluster. This elementary observation inspires the following definition.

Definition 6.3.12. Given a non-empty proper subset $\emptyset \subsetneq \mathfrak{s} \subsetneq \mathcal{R}$, and Σ a cluster picture, we say that the partition $\mathcal{R} = \mathfrak{s} \sqcup (\mathcal{R} \smallsetminus \mathfrak{s})$ is a *cluster cut* for Σ if at lest one among $\mathfrak{s}$ and its complement $\mathcal{R} \smallsetminus \mathfrak{s}$ is a cluster for Σ. The sum $\delta_{\mathfrak{s}} + \delta_{\mathcal{R} \smallsetminus \mathfrak{s}}$ is named the *depth* of the cluster cut, and is denoted by $\delta_{\mathfrak{s} \sqcup (\mathcal{R} \smallsetminus \mathfrak{s})}$.

Proposition 6.3.13. Two equivalent cluster pictures always have the same set of cluster cut. Moreover, a cluster cut always has the same depth in two equivalent cluster pictures.

Proof. It is immediate to verify that elementary alterations of cluster pictures leave the set of cluster cuts and their depths unvaried. $\square$

Proposition 6.3.14. Let $[\Sigma]$ be an equivalence class of cluster pictures, and $\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{s}'$ a cluster cut of depth $t := \delta_{\mathfrak{s} \sqcup \mathfrak{s}'}$. Then, there exist precisely $t + 1$ cluster pictures $\Sigma_0, \ldots, \Sigma_t$ equivalent to Σ for which $\mathfrak{s}$ or $\mathfrak{s}'$ are maximal clusters; moreover, the Σ_i's form a sequence of adjacent cluster pictures, and the depths of $\mathfrak{s}$ and $\mathfrak{s}'$ in Σ_i are i and $t - i$ respectively.

Proof. This is a merely combinatorial result. $\square$

We would now like to compare cluster picture adjacency (Definition 6.3.11) to model adjacency (Definition 6.2.9).

Proposition 6.3.15. The function $\mathrm{ccp}_{\mathcal{R}}$ taking each smooth model of the line X to its associated cluster picture preserves adjacency, in the sense that, if $\mathcal{X}$ is a smooth model of the line X and $\mathcal{X}'$ is its elementary alteration at a point $Q \in \mathcal{X}_s$, then,

(a) if Q coincides with the point $\bar{\mathfrak{s}} \in \mathcal{X}_s$ at which a maximal cluster $\mathfrak{s}$ of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ reduces, then $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}')$ is the elementary alteration of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ at $\mathfrak{s}$;

(b) otherwise, $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}')$ is the elementary alteration of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ at $\emptyset$.

Moreover, the image of $\mathrm{ccp}_{\mathcal{R}}$ (i.e. the set of concrete cluster pictures $\mathrm{CCP}_{\mathcal{R}} \subseteq \mathrm{CP}_{\mathcal{R}}$) coincides with an equivalence class of $\mathrm{CP}_{\mathcal{R}}$.

Proof. This is an immediate generalization of the examples presented above. □

Corollary 6.3.16. If $\mathcal{X}$ is a smooth model of the line, and $\mathcal{X}'$ a second smooth model meeting $\mathcal{X}$ at a point $Q \in \mathcal{X}_s$, then:

(a) if Q coincides with the point $\bar{\mathfrak{s}} \in \mathcal{X}_s$ at which a maximal cluster $\mathfrak{s}$ of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ reduces, then $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}')$ meets $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ at $\mathfrak{s}$;

(b) otherwise, $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}')$ meets $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ at $\emptyset$.

We now want to study the map $\mathrm{ccp}_{\mathcal{R}}$ further; in particular, we want to inquire how many smooth models $\mathcal{X}$ of the line X realize a given concrete cluster picture $\Sigma \in \mathrm{CCP}_{\mathcal{R}}$.

Definition 6.3.17. A cluster picture $\Sigma \in \mathrm{CP}_{\mathcal{R}}$ is said to be *crushed* if $\Sigma(\mathcal{R}) > 0$, i.e. if $\mathcal{R}$ is a cluster; it is said to be *non-crushed* otherwise.

Proposition 6.3.18. Let $\Sigma \in \mathrm{CCP}_{\mathcal{R}}$ be a concrete cluster picture, and assume $|\mathcal{R}| \geq 3$. If Σ is non-crushed, then the smooth model $\mathcal{X}$ of the line X realizing Σ is unique.

Proof. Suppose, by contradiction, that $\mathcal{X}_1$ and $\mathcal{X}_2$ are two distinct smooth models whose associated cluster picture is Σ, and let $\mathcal{X}_{12}$ be their composite; we will denote by L_1 and L_2 the two lines forming its special fiber $(\mathcal{X}_{12})_s$, which will meet at a single node P. The points of $(\mathcal{X}_{12})_s$ to which the points of $\mathcal{R}$ reduce correspond biiectively to the maximal clusters of Σ (see Remark 6.3.10).

Suppose that the reductions in $(\mathcal{X}_{12})_s$ of two maximal clusters $\mathfrak{s}$ and $\mathfrak{s}'$ both lie on L_1. Then, contracting L_1 will make $\bar{\mathfrak{s}}$ and $\overline{\mathfrak{s}'}$ collapse; in other words, $\mathfrak{s}$ and $\mathfrak{s}'$ reduce to the same point of $(\mathcal{X}_2)_s$, which is only possible if $\mathfrak{s} = \mathfrak{s}'$.

Suppose now that the reduction in $(\mathcal{X}_{12})_s$ of a maximal cluster $\mathfrak{s}$ lies on $L_1 \setminus L_2$. Then, it is clear that the depth of $\mathfrak{s}$ in $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}_2)$ should be equal to its depth in

$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}_1)$ plus the thickness of the node P. Since $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}_1) = \mathrm{ccp}_{\mathcal{R}}(\mathcal{X}_2) = \Sigma$ by assumption, this can only happen if the depth of $\mathfrak{s}$ in Σ is ∞, which means that $\mathfrak{s}$ is a singleton.

In light of the arguments presented in the last two paragraphs, we are only left with the two following possible cases.

1. Σ has a unique maximal cluster $\mathfrak{s}$ reducing at $P \in (\mathcal{X}_{12})_s$.
2. The maximal clusters of Σ all reduce to points of $(\mathcal{X}_{12})_s \smallsetminus \{P\}$; for this to be possible, they must all be singletons, and they cannot be more than 2.

But the first possibility is ruled out by the non-crushed hypothesis, and the second one by the assumption $|\mathcal{R}| \geq 3$. $\square$

In light of the proposition above and of Convention 6.3.9, we can say that there is a biiective correspondence (actually, thanks to Proposition 6.3.15, an isomorphism of trees) between non-crushed concrete cluster pictures and non-crushed smooth models of the line X, and this correspondence makes the family of non-crushed models something easy to describe.

Crushed models are harder to parameterize: their associated cluster pictures do not identify them unambiguously, and it is actually not difficult to realize that there are infinitely many smooth crushed models that realize any given concrete crushed cluster picture. However, there is an obvious way to associate, to any given smooth crushed model of the line, a uniquely determined non-crushed one:

Definition 6.3.19. Given a crushed cluster picture $\Sigma \in \mathrm{CP}_{\mathcal{R}}$, its *non-crushed anchor* is the non-crushed equivalent cluster picture $\Sigma' \in \mathrm{CP}_{\mathcal{R}}$ that we obtain by removing the outer cluster and leaving all the remaining clusters and their relative depths unvaried. In other words, Σ' is obtained by means of $d_{\mathcal{R}}$ elementary alterations of Σ at $\mathcal{R}$, and can be characterized, among all non-crushed cluster pictures equivalent to Σ, as the one that is closest to Σ.

Definition 6.3.20. Let $\mathcal{X}$ be a crushed model. If $\Sigma := \mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ is its associated (crushed) cluster picture and Σ' is the non-crushed anchor of Σ in the sense of the definition above, then the unique smooth model $\mathcal{X}'$ of the line such that $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}') = \Sigma'$ is named the *non-crushed anchor* (or simply the *anchor*) of $\mathcal{X}$, and we say that $\mathcal{X}$ is *anchored to* $\mathcal{X}'$. The point $P \in \mathcal{X}'_s$ where $\mathcal{X}$ meets $\mathcal{X}'$ is also named the *anchor of* $\mathcal{X}$, and we say that $\mathcal{X}$ is *anchored at* P.

Remark 6.3.21. Let $\mathcal{X}'$ be the non-crushed anchor of a crushed model $\mathcal{X}$, and let $P \in \mathcal{X}'_s$ the point at which $\mathcal{X}$ is anchored. Then, it is clear that no point of $\mathcal{R}$ can reduce to $P \in \mathcal{X}'_s$.

A simple yet important refinement of the dichotomy between crushed and non-crushed models is presented in the following definitions.

Definition 6.3.22. A cluster picture $\Sigma \in \mathrm{CP}_{\mathcal{R}}$ is said to be *crushed* if it contains a cluster of size $n := |\mathcal{R}|$ (i.e., if it has a unique maximal cluster: this is just a reformulation of Definition 6.3.17). It is said to be *semi-crushed* if it contains a cluster of size $n-1$ (i.e., it has two maximal clusters, one of which is a singleton). It is said to be *unwound* if it is neither crushed nor semi-crushed (in other words: a cluster picture is *unwound* if it has more than two maximal clusters, or exactly two maximal clusters, neither of which is a singleton).

Definition 6.3.23. A cluster picture is said to be *very unwound* if it has more than three maximal clusters, or it has exactly three maximal clusters, but no more than one of them is a singleton.

Proposition 6.3.24. Let $\Sigma \in \mathrm{CP}_{\mathcal{R}}$ be a cluster picture. Then,

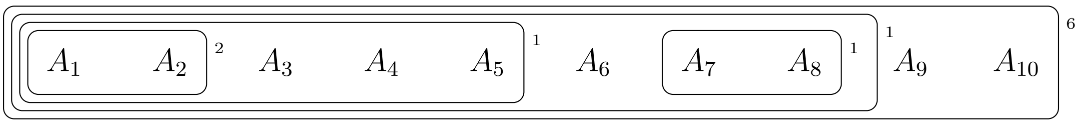
(a) an unwound cluster picture equivalent to Σ exists if and only if $|\mathcal{R}| \geq 3$;
(b) a very unwound cluster picture equivalent to Σ exists if $|\mathcal{R}| \geq 5$;
(c) assuming $|\mathcal{R}| \geq 3$, the unwound cluster pictures equivalent to Σ are the vertices of a finite subtree of $\mathrm{CP}_{\mathcal{R}}$;
(d) assuming $|\mathcal{R}| \geq 3$, there exists a unique unwound cluster picture $\widehat{\Sigma}$ equivalent to Σ whose distance from Σ is minimal (we remark that this result on unwound models is analogous to the one we have stated in Definition 6.3.19 for non-crushed models);
(e) the unwound cluster pictures equivalent to Σ (and different from Σ) meet Σ precisely at its maximal, non-singleton clusters;
(f) assuming $|\mathcal{R}| \geq 3$, for every given point $A_i \in \mathcal{R}$ there exists exactly one unwound cluster picture Σ_i equivalent to Σ for which the singleton $\{A_i\}$ is a maximal cluster.

Proof. This is a purely combinatorial result. □

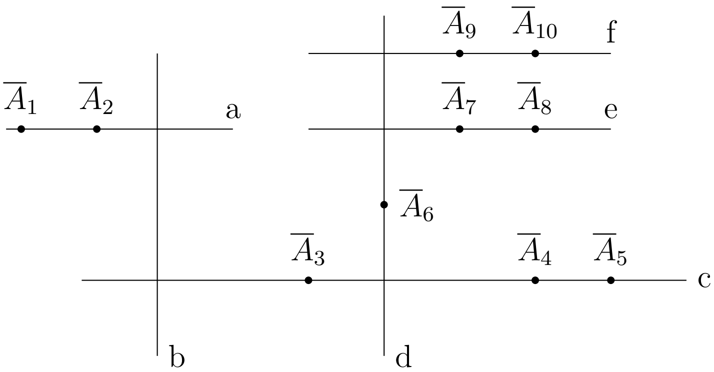
In light of proposition above, we are allowed to give the following definition:

Definition 6.3.25. Suppose that $|\mathcal{R}|$ is at least 3 (resp. 5), so that an unwound (resp. very unwound) smooth model of X certainly exists. We will denote by $\mathcal{X}_{\mathrm{unw}}$ (resp. $\mathcal{X}_{\mathrm{vunw}}$) the composite of all unwound (resp. very unwound) smooth models of X.

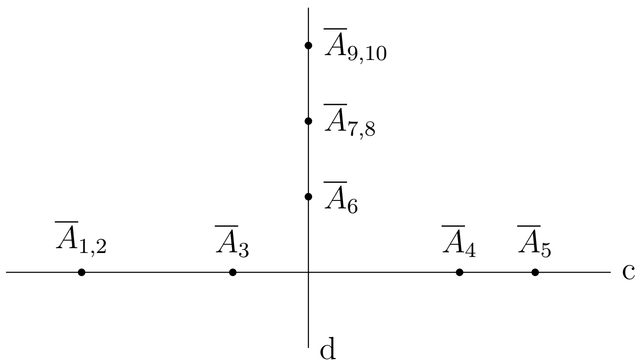
Example 6.3.26. Suppose that, for some set $\mathcal{R} \subseteq X(K)$ of 10 rational points of the line X, and for some smooth model $\mathcal{X}$ of X, we have that $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ is the following cluster picture:



This is the special fiber of $\mathcal{X}_{\mathrm{unw}}$:



The six lines a, b, c, d, e, f correspond to the six unwound smooth models $\mathcal{X}_a, \ldots, \mathcal{X}_e$ of X; only $\mathcal{X}_c$ and $\mathcal{X}_d$ are very unwound, so the special fiber of $\mathcal{X}_{\mathrm{vunw}}$ looks like this:



As the example clearly shows,

Proposition 6.3.27. Assuming $|\mathcal{R}| \geq 3$, the following hold:

(a) $\mathcal{X}_{\mathrm{unw}}$ is a regular and semistable model of the line X;
(b) none of the points of $\mathcal{R}$ reduces to a node of $(\mathcal{X}_{\mathrm{unw}})_s$;
(c) let L be the vertical line of $\mathcal{X}_{\mathrm{unw}}$ corresponding to an unwound smooth model $\mathcal{X}$: then, (i) the number of non-singleton maximal clusters of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ equals the self-intersection number $-L^2$ and (ii) the number of singleton maximal

clusters of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ coincides with the number of points of $\mathcal{R}$ whose reduction lies on L.

(d) the reductions of the points of $\mathcal{R}$ are all distinct in $(\mathcal{X}_{\mathrm{unw}})_s$, and $\mathcal{X}_{\mathrm{unw}}$ is the minimal regular composite model of the line satisfying this property (where minimality is to be interpreted in the sense of Definition 4.6.2);

Proof. Unwound concrete cluster pictures are the vertices of a subtree of CCP_Σ (Proposition 6.3.24), whence the regularity of $\mathcal{X}_{\mathrm{unw}}$ follows (Remark 6.2.12); moreover since it is a composite regular model of the line, $\mathcal{X}_{\mathrm{unw}}$ is certainly semistable (Proposition 6.2.7): this proves (a).

Point (b) can be easily be proved by means of an elementary argument, or it can be seen as a consequence of regularity (Proposition 4.3.2).

Let us now fix $\mathcal{X}$ an unwound smooth model of the line and let L be the corresponding vertical line of $\mathcal{X}_{\mathrm{unw}}$: $-L^2$ will be equal to the number of points where L meets the other unwound smooth models; in light of Proposition 6.3.24 (e) and Corollary 6.3.16, it is then equal to the number of non-singleton maximal clusters of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$. The points of $\mathcal{R}$ whose reductions lie on $L \subseteq (\mathcal{X}_{\mathrm{unw}})_s$ will clearly be the same ones that reduce to the points of $\mathcal{X}_s$ at which $\mathcal{X}$ meets no other unwound smooth model; hence, they correspond to the singleton maximal clusters of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$. This completes the proof of (c).

Let us now move to the proof of (d). Recall that, by Proposition 6.3.24 (f), every point $A_i \in \mathcal{R}$ appears as a singleton maximal cluster for some unwound smooth model $\mathcal{X}$: this is enough to ensure that no two points of $\mathcal{R}$ can collapse in the special fiber of $\mathcal{X}_{\mathrm{unw}}$ (see Remark 6.3.10). Now, suppose that $\mathcal{X}'$ is any regular composite model of the line such that the points of $\mathcal{R}$ have distinct reductions in $\mathcal{X}'_s$: we want to verify that $\mathcal{X}'$ dominates $\mathcal{X}_{\mathrm{unw}}$. Any given point $A_i \in \mathcal{R}$ will reduce to a smooth point $\overline{A}_i \in (\mathcal{X}_{1\ldots r})_s$ (see Proposition 4.3.2), which will lie on the vertical line corresponding to some smooth component $\mathcal{X}_i$ of $\mathcal{X}'$: it is perfectly clear that $\{A_i\}$ must be a singleton maximal cluster for $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}_i)$. By means of purely combinatorial arguments, one can deduce from this that the subtree of $\mathrm{CCP}_{\mathcal{R}}$ generated by the vertices $\{\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}_i)\}_i$ covers the whole set of unwound concrete cluster pictures. We conclude that $\mathcal{X}'$ dominates $\mathcal{X}_{\mathrm{unw}}$. □

Proposition 6.3.28. Assuming $|\mathcal{R}| \geq 5$, we have that:

(a) $\mathcal{X}_{\mathrm{vunw}}$ is a semistable model of X;

(b) none of the points of $\mathcal{R}$ reduces to a node of $(\mathcal{X}_{\mathrm{vunw}})_s$;

(c) no more than two points of $\mathcal{R}$ reduce to the same point of $(\mathcal{X}_{\mathrm{vunw}})_s$.

Moreover, $\mathcal{X}_{\mathrm{vunw}}$ can be characterized as the minimum (Definition 4.6.2) among all semistable model of the line satisfying these properties.

Proof. All these results be obtained from the properties of $\mathcal{X}_{\text{unw}}$ (see Proposition 6.3.27) by carefully considering the effects of the contraction morphism $\mathcal{X}_{\text{unw}} \to \mathcal{X}_{\text{vunw}}$. □

Let us now study what happens if we perform a (totally ramified) extension $R \subseteq R'$ of degree e: it is clear that a smooth model $\mathcal{X}$ of the line X over R is unwound (or very unwound) if and only if $\mathcal{X}_{R'}$ is an unwound (resp. very unwound) model of the line over R'. This follows from the obvious observation that:

Proposition 6.3.29. $\Sigma \in \mathrm{CP}_{\mathcal{R}}$ is an unwound (or very unwound) cluster picture if and only if $e\Sigma$ is.

Anyway, there are unwound smooth models over R' that cannot be realized over R. As a consequence, the composite $\mathcal{X}'_{\text{unw}}$ of all unwound smooth models over R' is not just $(\mathcal{X}_{\text{unw}})_{R'}$; however, it is easy to see that it can be obtained as the minimal desingularization of $(\mathcal{X}_{\text{unw}})_{R'}$, i.e. by replacing each node of $(\mathcal{X}_{\text{unw}})_{R'}$ with a chain of $e-1$ lines. This is the corresponding result in terms of cluster pictures:

Proposition 6.3.30. If $\Sigma \in \mathrm{CP}_{\mathcal{R}}$ is a cluster picture, and Σ' is an unwound cluster picture equivalent to $e\Sigma$, then there are only two (mutually exclusive) cases:

(a) Σ'/e is a cluster picture (i.e., the depths of all clusters of Σ' are divisible by e); in this case, Σ'/e is clearly also unwound, and it is equivalent to Σ;

(b) Σ' has exactly two maximal clusters $\mathfrak{s}_1$ and $\mathfrak{s}_2$, neither of which is a singleton, whose depths d_1 and d_2 are not divisible by e (but their sum is). If Σ'_i is the cluster picture we obtain from Σ' by means of d_i unit transforms at $\mathfrak{s}_i$, we have that Σ'_1/e and Σ'_2/e are both unwound cluster pictures equivalent to Σ.

Proof. This is a purely combinatorial result. □

In particular, we clearly have that the unwound smooth models of the line that sprout only after the extension $R \subseteq R'$ are *not* very unwound, and we can consequently be sure that:

Proposition 6.3.31. Assume $|\mathcal{R}| \geq 5$. Then all very unwound models of the line X over R' come from very unwound models of X defined over R; in particular, $\mathcal{X}'_{\text{vunw}} = (\mathcal{X}_{\text{vunw}})_{R'}$.

6.4. Models of Galois coverings of the line

Let $Y \to X$ be a Galois (branched) covering of smooth geometrically connected curves, where $X = \mathbb{P}^1_K$ is the line, and Y has genus $g \geq 1$. We have already discussed (Section 4.12) how a model $\mathcal{Y}$ of Y can be constructed by normalizing a normal model $\mathcal{X}$ of X in $K(Y)$, and how $\mathcal{X}$ can be recovered from $\mathcal{Y}$ by taking the quotient under the action of the Galois group $G := \mathrm{Gal}_{K(X)}(K(Y))$.

Definition 6.4.1. Let $\mathcal{P}$ be a property of models. A normal model $\mathcal{X}$ of the line X is said to be *ur-$\mathcal{P}$* for the Galois covering $Y \to X$ if its normalization $\mathcal{Y}$ in $K(Y)$ satisfies $\mathcal{P}$.

In particular, Corollary 4.12.6 ensures that minimal regular model and (if $g \geq 2$) the canonical model of Y bear an admissible action by the Galois group G: taking their quotients, we obtain the *ur-minimal model of* X and the *ur-canonical model of* X.

In Section 5.7, we have discussed how smoothness and semistability properties behave with respect to taking quotients, and the observations collected there, in particular, ensure that

(a) Y has good (resp. semistable) reduction if and only if an ur-smooth (resp. ur-semistable) model of X exists;
(b) the ur-smooth model, if it exists, is unique, smooth, and coincides with the ur-minimal model; if $g \geq 2$, it also coincides with the ur-canonical model;
(c) ur-semistable models are semistable;
(d) if $g \geq 2$ and Y has semistable reduction, then the ur-stable model of X exists, is unique, and coincides with the ur-canonical model.

The normalization of models of X in $K(Y)$ does not in general commute with extensions $R \subseteq R'$ of the base. However, we have that:

Proposition 6.4.2. If $\mathcal{X}$ is a model of the line whose normalization $\mathcal{Y}$ in $K(Y)$ has reduced special fiber, then $\mathcal{Y}_{R'}$ is the normalization of $\mathcal{X}_{R'}$ in $K(Y_{K'})$ for every base extension $R \subseteq R'$.

Proof. As $\mathcal{Y}$ has reduced special fiber and smooth generic fiber, $\mathcal{Y}_{R'}$ will also have reduced special fiber and smooth generic fiber, therefore it will be a normal surface. From this observation, the proposition immediately follows. □

Ur-smoothness, ur-stability and ur-semistability are preserved under any extension $R \subseteq R'$:

Proposition 6.4.3. Let $\mathcal{X}$ be an ur-smooth, ur-semistable or ur-stable model of X for the Galois covering $Y \to X$. Then, for any given base extension $R \subseteq R'$, $\mathcal{X}_{R'}$ is an ur-smooth (resp., ur-semistable, ur-stable) model of $X_{K'}$ with respect to the Galois covering $Y_{K'} \to X_{K'}$.

Proof. This easily follows from Proposition 6.4.2, recalling that smoothness, semistability and stability of a surface are preserved under arbitrary extensions of the base (Propositions 5.2.1 and 5.4.9). □

The formation of the ur-minimal model, on the other hand, does not commute with extensions of R. However, we can still say that:

Proposition 6.4.4. If Y has semistable reduction over R, $R \subseteq R'$ is any extension, and $\mathcal{X}_{\text{urmin}}$, $\mathcal{X}'_{\text{urmin}}$ are the ur-minimal models of the line over R and R' respectively, then $\mathcal{X}'_{\text{urmin}}$ dominates $(\mathcal{X}_{\text{urmin}})_{R'}$.

Proof. If $\mathcal{Y}_{\min}$ is the minimal regular model of Y over R, then $(\mathcal{Y}_{\min})_{R'}$ is still semistable, and the minimal regular model of Y over R' can be obtained as its minimal desingularization (Proposition 5.4.10). In particular, the normal R'-model $\mathcal{Y}'_{\min}$ dominates the normal R'-model $(\mathcal{Y}_{\min})_{R'}$, and hence an analogous dominance relation will exist between their quotients $\mathcal{X}'_{\text{urmin}}$ and $(\mathcal{X}_{\text{urmin}})_{R'}$. □

Our efforts will now essentially be focused on the construction of the ur-minimal, the ur-canonical, the ur-smooth, an ur-semistable, the ur-stable model of X (supposing they exist), in order to obtain a minimal regular, canonical, smooth, semistable or stable model of Y by normalizing them in $K(Y)$. We have seen that, to identify a model of the line X, it is important to study the smooth models it dominates (Section 6.2); if Y has semistable reduction, in particular, all "ur-interesting" models of the line X listed above are semistable, and hence they can be completely recovered from the knowledge of the smooth models they dominate (in other words, they are composite: see Definition 6.2.3 and Corollary 6.2.6).

If Y does not have semistable reduction, it will still gain semistable reduction after a suitable finite extension $R \subseteq R'$ (Theorem 5.5.1); supposing that the genus g of Y is ≥ 2, we consequently have a uniquely determined ur-stable model $\mathcal{X}'_{\text{urst}}$ of X over R', whose formation commutes with any further extension $R' \subseteq R''$ (Proposition 6.4.3). There are two reasons why Y may fail to have semistable reduction over R: either the smooth components of $\mathcal{X}'_{\text{urst}}$ cannot be realized over R, or, even if they descend to R, their normalization in $K(Y)$ does not have a reduced special fiber:

Proposition 6.4.5. Suppose that Y has genus $g \geq 2$, let $R \subseteq R'$ be an extension of R over which Y has semistable reduction, and let $\mathcal{X}'_{\text{urst}}$ be the ur-stable model of X over R'. Then, Y has semistable reduction over R if and only if the following conditions both hold:

(a) all smooth components $\mathcal{X}'_i$ of $\mathcal{X}'_{\text{urst}}$ can be realized over R, meaning that, for each i, $\mathcal{X}_i = (\mathcal{X}_i)_{R'}$ for some smooth model $\mathcal{X}_i$ of X (which will clearly be uniquely determined up to isomorphism);

(b) the normalization $\mathcal{Y}_i$ of each of the $\mathcal{X}_i$'s in $K(Y)$ has reduced special fiber.

Proof. The "if" part of the proposition is the only one deserving a proof, so, assume that (a) and (b) hold. If we name $\mathcal{X}$ the composite of the $\mathcal{X}_i$'s, we clearly have that $\mathcal{X}_{R'} \cong \mathcal{X}'_{\mathrm{urst}}$ as models of X over R' (Proposition 6.2.13) and that the normalization $\mathcal{Y}$ of $\mathcal{X}$ in $K(Y)$ has reduced special fiber. From this it follows that $\mathcal{Y}_{R'}$ is the normalization of $\mathcal{X}_{R'} \cong \mathcal{X}'_{\mathrm{urst}}$ in $K(Y_{K'})$ (Proposition 6.4.2), and hence that it is the stable model of Y. But this is enough to ensure that $\mathcal{Y}$ is stable (Proposition 5.4.9), and $\mathcal{X}$ is consequently ur-stable by definition. □

Let us now consider a smooth model $\mathcal{X}$ of the line X, and let $\mathcal{Y}$ denote its normalization in $K(Y)$: we want to determine whether $\mathcal{X}$ contributes or not to some "ur-interesting" model of X. The special fiber of $\mathcal{Y}$ consists of a number of components $\mathrm{Vert}(\mathcal{Y}) = \{C_1, \dots, C_t\}$ which are transitively permuted by $G := \mathrm{Gal}_{K(X)}(K(Y))$. In particular, they will all be isomorphic k-curves, and they will all have the same abelian rank, toric rank, unipotent rank, number of singularities, entanglement number, etc. We will write C_i to mean any of those vertical components.

Theorem 6.4.6. Suppose Y has semistable reduction. Let $\mathcal{X}$ be a smooth model of the line X and C_i any vertical component of its normalization $\mathcal{Y}$ in $K(Y)$. Then,

(a) $\mathcal{X}$ is a smooth component of the ur-minimal model if and only if C_i has either multiplicity 1 and abelian rank > 0, or multiplicity 1, abelian rank 0 and entanglement number > 1 (see Definition 4.6.15 for the definition of entanglement number);

(b) (assuming $g \geq 2$) $\mathcal{X}$ is a smooth component of the ur-stable model if and only if C_i has either multiplicity 1 and abelian rank > 0, or multiplicity 1, abelian rank 0 and entanglement number > 2.

Proof. This immediately follows from Theorem 5.6.3. □

Suppose we are in the case in which Y has semistable reduction and C_i has multiplicity 1: by studying the singularities of $\mathcal{Y}_s$ it is not difficult to extract precious information about where the smooth components of the ur-minimal or the ur-stable model are located with respect to $\mathcal{X}$, even when $\mathcal{X}$ is not itself part of the ur-minimal or of the ur-stable model:

Theorem 6.4.7. Suppose that Y has semistable reduction, that $\mathcal{X}$ is a smooth model of the line, and that the vertical components C_i's of the normalization $\mathcal{Y}$ of $\mathcal{X}$ in $K(Y)$ have multiplicity 1 (i.e., $\mathcal{Y}_s$ is reduced). Then,

(a) $\mathcal{X}$ meets (Definition 6.2.8) the components of the ur-minimal model (distinct from $\mathcal{X}$) precisely at the points of $\mathcal{X}_s$ over which $\mathcal{Y}_s$ has non-nodal singularities, or nodes of thickness > 1;
(b) supposing $g \geq 2$, $\mathcal{X}$ meets the smooth components of the ur-stable model (different from $\mathcal{X}$) precisely at the points of $\mathcal{X}_s$ over which $\mathcal{Y}_s$ has non-nodal singularities.

Proof. Let $\mathcal{Y}'$ be the minimal desingularization of $\mathcal{Y}$, and C_i' the strict transform in $\mathcal{Y}'$ of $C_i \in \mathrm{Vert}(\mathcal{Y})$ (Definition 4.6.10). Let us denote by n_e the common entanglement number of C_i and C_i' (Definition 4.6.15 and Proposition 4.6.16), and by a their common abelian rank (Corollary 4.6.12). There are two possibilities.

i) $\mathcal{X}$ is a component of the ur-minimal model. In this case, the theorem simply rephrases Theorem 5.6.4;

ii) $\mathcal{X}$ is not a component of the ur-minimal model, which is clearly equivalent to saying that C_i' is a (-1)-line of $\mathcal{Y}'$: the previous theorem ensures that this case occurs precisely when $a = 0$ and $n_e = 1$. Since C_i has entanglement number 1, $\mathcal{Y}_s$ can only be singular over a unique point $P \in \mathcal{X}_s$. Now, it is clear that C_i', which also has entanglement number 1, will meet the rest of $\mathcal{Y}_s'$ at a unique point lying over P. As $\mathcal{Y}'$ dominates $\mathcal{Y}_{\min}$, this implies that $\mathcal{X}$ meets all smooth components of the ur-minimal model of X at P (and hence also all smooth components of the ur-stable model, if $g \geq 2$). It is also not difficult to realize that, since $g \geq 1$, the singularities that $\mathcal{Y}_s$ exhibits over P cannot be nodes. $\square$

Let us suppose that $\mathcal{X}$ is a smooth component of an ur-semistable model $\mathcal{X}'$: we want to compare the special fibers of $\mathcal{Y}$ and $\mathcal{Y}'$. Let us thus name $P_1, \ldots, P_t \in \mathcal{X}_s$ the points at which $\mathcal{X}$ meets the other smooth components of $\mathcal{X}'$. If C_i is a vertical component of $\mathcal{Y}$, then its strict transform C_i' in $\mathcal{Y}'$ can be obtained simply by desingularizing C_i over the points P_j's:

Proposition 6.4.8. The curve C_i' is smooth over the points P_j's, and the birational morphism of reduced curves $b : C_i' \to C_i$ that we obtain by restricting $\mathcal{Y}' \to \mathcal{Y}$ is an isomorphism away from the points P_j's.

Proof. The vertical fiber of $\mathcal{Y}'$ consists of the strict transforms C_i''s of the C_i's, together with some newly added vertical components that did not appear in $\mathcal{Y}$. By hypothesis, P_j is a point at which $\mathcal{X}$ meets some other smooth component of $\mathcal{X}'$; as a consequence, $C_i' \in \mathrm{Vert}(\mathcal{X}_s')$ will intersect some of the new vertical components of $\mathcal{Y}'$ at each of the points of C_i' lying over P_j. Recalling that $\mathcal{Y}_s'$ is semistable, this means that C_i' must be smooth above each of the P_j's. Away from the P_j's, instead, $\mathcal{Y}' \to \mathcal{Y}$, and hence also $C_i' \to C_i$, are clearly isomorphisms. $\square$

7. Semistable models of hyperelliptic curves: part II

We will retain, for this chapter, the notation and conventions adopted for part I (see Subsection 6.1.1): now, however, we suppose that Y is not just any geometrically connected smooth K-curve, but a tame hyperelliptic curve with K-rational ramification locus, endowed with a distinguished hyperelliptic map $h : Y \to X = \mathbb{P}^1_K$. We will now apply the results of part I to determine whether Y has semistable reduction and to explicitly build a semistable model of Y. To construct such a model, the crucial point will be seeking a way of computing the normalization in $K(Y)$ of any given smooth model $\mathcal{X}$ of the line, so that Theorems 6.4.6 and 6.4.7 can then be exploited to locate the smooth components of an ur-semistable model of the line.

The main results in residue characteristic $\neq 2$ are those collected in Theorems 7.2.3 and 7.2.4. Once we have proved them, we will move to the case of residue characteristic 2, which is significantly harder and will remain our topic for the rest of the chapter.

7.1. The equation of a hyperelliptic curve

Let us fix a smooth model $\mathcal{X}$ of the line X. As an R-scheme, $\mathcal{X}$ is isomorphic to $\mathbb{P}^1_R$: we will choose a distinguished isomorphism $\mathcal{X} \cong \mathbb{P}^1_R$, so that we have a distinguished coordinate x on $\mathcal{X}$. Let now $\mathcal{R} \subseteq X(K)$ denote the set consisting of the $2g+2$ branch points of the hyperelliptic map $Y \to X$, let $\mathfrak{s}_1, \ldots, \mathfrak{s}_r$ be the maximal clusters of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$, and let $\bar{\mathfrak{s}}_1, \ldots, \bar{\mathfrak{s}}_r$ be the points of $\mathcal{X}_s$ to which they reduce. The hyperelliptic curve Y will be described by an equation of the form $y^2 = f(x)$, where $f \in K[x]$ is some (not necessarily monic) splitting polynomial of degree $2g+1$ or $2g+2$, whose roots correspond to the branch points $A_1, \ldots, A_{2g+2}$ (omitting the one that possibly coincides with $x = \infty$). We may now partition the roots of f in two families $\mathcal{R} = \mathcal{R}_0 \sqcup \mathcal{R}_\infty$, where $\mathcal{R}_0$ collects all the A_i's reducing to $\overline{x} = 0$, $\mathcal{R}_\infty$ those reducing to $\overline{x} = \infty$, and each root whose reduction is neither $\overline{x} = 0$ nor $\overline{x} = \infty$ can be indifferently assigned to $\mathcal{R}_0$ or $\mathcal{R}_\infty$. Correspondingly, f can be factored as $f(x) = cf_0(x)f_\infty(x)$, with

$$f_0(x) = \prod_{A \in \mathcal{R}_0} (x - x_A), \qquad f_\infty(x) = \prod_{A \in \mathcal{R}_\infty} (1 - x_A^{-1}x), \qquad c \in K^\times,$$

where x_A denotes the x-coordinate of $A \in \mathcal{R}$. The polynomials f_0 and f_∞ have both integral coefficients; moreover, f_0 is monic, while f_∞ has constant term equal to 1.

Definition 7.1.1. Given $\mathcal{X}$ a smooth model of the line, and x a coordinate on $\mathcal{X}$, a 0∞-*equation* for the hyperelliptic curve is an equation of the form $y^2 = cf_0(x)f_\infty(x)$, with $c \in K^\times$, $f_0(x) \in R[x]$ monic and $f_\infty(x) \in R[x]$ having constant term equal to 1. If $c = 1$, we will name it a *normalized* 0∞-*equation*; if $f_0 = 1$ or $f_\infty = 1$, it will be said to be, respectively, an ∞-*equation* or a 0-*equation*.

Remark 7.1.2. If the smooth model $\mathcal{X}$ of the line and the coordinates x and y are fixed, the possible 0∞-equations we can write for Y correspond biiectively to the possible ways of partitioning $\mathcal{R}$ in two classes $\mathcal{R}_0 \sqcup \mathcal{R}_\infty$ such that the points of $\mathcal{R}_0$ reduce away from $\overline{x} = \infty$ and the points of $\mathcal{R}_\infty$ reduce away from $\overline{x} = 0$.

A 0∞-equation $y^2 = f(x) = cf_0(x)f_\infty(x)$ actually only describes the hyperelliptic curve Y over the affine chart $x \neq \infty$. The equation over the chart $x \neq 0$ can be written as $(y/x^{g+1})^2 = \check{f}(1/x)$, where

$$\begin{aligned}
\check{f}\left(\frac{1}{x}\right) &= \frac{f(x)}{x^{2g+2}} &&= c\check{f}_0\left(\frac{1}{x}\right)\check{f}_\infty\left(\frac{1}{x}\right); \\
\check{f}_0\left(\frac{1}{x}\right) &= \frac{f_\infty(x)}{x^{|\mathcal{R}_\infty|}} &&= \prod_{A\in\mathcal{R}_\infty}\left(\frac{1}{x} - x_A^{-1}\right); \\
\check{f}_\infty\left(\frac{1}{x}\right) &= \frac{f_0(x)}{x^{|\mathcal{R}_0|}} &&= \prod_{A\in\mathcal{R}_0}\left(1 - x_A\frac{1}{x}\right).
\end{aligned}$$

Let us now turn our attention to the leading coefficient c. If we perform the change of variable $y \mapsto \eta y$ for some $\eta \in K^\times$, the equation $y^2 = cf_0(x)f_\infty(x)$ becomes $y^2 = \eta^2 cf_0(x)f_\infty(x)$: it is clear, in particular, that the actual value of c depends on the choice of the coordinate y. However,

- the class of c in $K^\times/(K^\times)^2$ is uniquely determined for a given smooth model $\mathcal{X}$ of the line X, once a specific coordinate x has been fixed on it and the partition $\mathcal{R} = \mathcal{R}_0 \sqcup \mathcal{R}_\infty$ has also been chosen; and

- the parity of the valuation $v(c)$ only depends on the isomorphism class of the model $\mathcal{X}$.

Definition 7.1.3. Given a smooth model $\mathcal{X}$ of the line, we will denote by $v_{Y/\mathcal{X}} \in \mathbb{Z}/2\mathbb{Z}$ the parity of $v(c)$, where c is the coefficient appearing in any 0∞-equation $y^2 = cf_0(x)f_\infty(x)$ of Y (where x is some coordinate on $\mathcal{X}$).

Furthermore, we observe the following.

- If the residue characteristic $\mathrm{char}(k)$ is not 2, then $K^\times/(K^\times)^2$ is cyclic of order 2, generated by the uniformizer π. In this case, for every given choice of the model $\mathcal{X}$, of the coordinate x and of the partition $\mathcal{R} = \mathcal{R}_0 \sqcup \mathcal{R}_\infty$, we may select y so that the 0∞-equation of Y becomes $y^2 = cf_0(x)f_\infty(x)$ with either $c = 1$ (if $v_{Y/\mathcal{X}}$ is even) or $c = \pi$ (if $v_{Y/\mathcal{X}}$ is odd). If we also allow ourselves to replace R with its unique quadratic extension $R' := R[\pi^{1/2}]$, then y can clearly be always chosen so that $c = 1$.

- If $\mathrm{char}(k) = 2$, then the structure of $K^\times/(K^\times)^2$ is more complicated, and the parity of $v(c)$ alone will not determine the class of c modulo squares. If we allow ourselves to change y, however, we can still clearly obtain that $v(c) \in \{0, 1\}$, and if we also allow ourselves to replace R with a suitable quadratic extension, y can be chosen so that $c = 1$.

It can also be interesting to determine how $v_{Y/\mathcal{X}}$ varies when the smooth model of the line $\mathcal{X}$ is changed. Suppose that $\mathcal{X}$ and $\mathcal{X}'$ are two adjacent smooth models of the line, meeting each other at $P \in \mathcal{X}_s$ and $P' \in \mathcal{X}'_s$ (Definitions 6.2.8 and 6.2.9); let $\mathfrak{s}$ be the center of the elementary alteration that turns $\mathrm{ccp}_\mathcal{R}(\mathcal{X})$ into $\mathrm{ccp}_\mathcal{R}(\mathcal{X}')$, and $\mathfrak{s}' = \mathcal{R} \smallsetminus \mathfrak{s}$ the center of the elementary alteration turning $\mathrm{ccp}_\mathcal{R}(\mathcal{X}')$ into $\mathrm{ccp}_\mathcal{R}(\mathcal{X})$: we clearly have that $|\mathfrak{s}|$ is the number of points of $\mathcal{R}$ reducing to $P \in \mathcal{X}_s$, $|\mathfrak{s}'|$ is the number of points of $\mathcal{R}$ reducing to $P' \in \mathcal{X}'_s$, and $|\mathfrak{s}| \equiv |\mathfrak{s}'| \bmod 2$. It is not difficult to prove that

Proposition 7.1.4. We have that $v_{Y/\mathcal{X}'} \equiv v_{Y/\mathcal{X}} + |\mathfrak{s}|$ in $\mathbb{Z}/2\mathbb{Z}$.

Proof. Let us choose a coordinate x on $\mathcal{X}$ such that $\overline{x}(P) = 0$. We can partition $\mathcal{R}$ as $\mathcal{R} = \mathcal{R}_0 \sqcup \mathcal{R}_\infty$, where $\mathcal{R}_0 := \mathfrak{s} = \{C_1, \ldots, C_h\}$ will be the set of points of $\mathcal{R}$ reducing to $P \in \mathcal{X}_s$, and $\mathcal{R}_\infty := \mathcal{R} \smallsetminus \mathfrak{s} = \{D_1, \ldots, D_k\}$ will collect all the remaining points of $\mathcal{R}$. Let us write down a 0∞-equation $y^2 = f(x) = cf_0(x)f_\infty(x)$ for Y, with $f_0(x) = \prod_i (x - x_{C_i})$ and $f_\infty(x) = \prod_j (1 - x_{D_j}^{-1} x)$. The elementary alteration that allows us to move from $\mathcal{X}$ to $\mathcal{X}'$ corresponds to the coordinate change $x' := \pi^{-1}x$. If we now multiply both sides of the equation $y^2 = f(x)$ by π^{-h}, we obtain

$$\pi^{-h} y^2 = c \prod_i \left(\frac{x}{\pi} - \frac{x_{C_i}}{\pi} \right) \prod_j \left(1 - \frac{\pi}{x_{D_j}} \frac{x}{\pi} \right)$$

which, introducing $y' := \pi^{-\lfloor h/2 \rfloor}$, becomes

$$(y')^2 = c\pi^e \prod_i \left(x' - x'_{C_i} \right) \prod_j \left(1 - \frac{1}{x'_{D_j}} x' \right)$$

where $e = 0$ or $e = 1$ depending on whether $h = |\mathcal{R}_0| = |\mathfrak{s}|$ is even or odd. As all the C_i's reduce to $\overline{x} = 0$, we have that $x_{C_i} \in \pi R$ for all i, and hence $x'_{C_i} \in R$ for all i. It is also clear that, for all j, since $1/x_{D_j} \in R$, then a fortiori $1/x'_{D_j} = \pi/x_{D_j} \in R$. Hence, what we have obtained is a 0∞-equation $(y')^2 = f'(x') = c' f'_0(x') f'_\infty(x')$ relative to the model $\mathcal{X}'$, where $c' = c\pi^e$ has a valuation $v(c')$ whose parity is equal or opposite to the one of $v(c)$ depending on whether $h := |\mathfrak{s}|$ is even or odd. $\square$

7.2. Residue characteristic not 2

We will retain the setting and notations of Section 7.1, and assume that the residue characteristic of R is $\neq 2$. In this case, the normalization $\mathcal{Y}$ of the smooth model of the line $\mathcal{X}$ in $K(Y)$ is easy to compute.

Proposition 7.2.1. Let $y^2 = f(x) = cf_0(x)f_\infty(x)$, with $v(c) \in \{0, 1\}$, be a 0∞-equation describing Y over $x \neq \infty$, and let $(y/x^{g+1})^2 = \check{f}(1/x) = c\check{f}_0(1/x)\check{f}_\infty(1/x)$ be the corresponding description of Y over $x \neq 0$. Then, the normalization $\mathcal{Y}$ of $\mathcal{X}$ in $K(Y)$ is the gluing of the two affine R-schemes

$$\operatorname{Spec}\left(\frac{R[x,y]}{y^2 - f(x)}\right) \text{ and } \operatorname{Spec}\left(\frac{R[1/x, y/x^{g+1}]}{(y/x^{g+1})^2 - \check{f}(1/x)}\right),$$

which provide the description of $\mathcal{Y}$ over $x \neq \infty$ and over $x \neq 0$ respectively.

Proof. It is clearly enough to show the result for the chart $x \neq \infty$: what we have to prove is that the integral closure of $R[x]$ in $K(x)[y]/(y^2 - f(x))$ is the subring $D := R[x,y]/(y^2 - f(x)) \subset K(x)[y]/(y^2 - f(x))$. Since $f \in R[x]$, it is clear that y, and hence all elements of D, are integral over $R[x]$: we still only have to show that D is normal. Since D is described by a single, non-trivial equation in the affine space $\mathbb{A}^2_R$, it is a complete intersection; hence, it is Cohen-Macaulay, and its normality is thus equivalent to regularity in codimension 1. Since D_η is certainly regular, because it is just an affine chart of the smooth hyperelliptic curve Y, the only points at which regularity must actually be checked are the codimension-1 points of D lying over s, i.e. the generic points of the special fiber D_s. Two cases should now be distinguished.

- If $v(c) = 0$, then $\overline{f} \in k[x] \smallsetminus \{0\}$, and, since $\operatorname{char}(k) \neq 2$, it is immediate to realize that $D_s = k[x,y]/(y^2 - \overline{f}(x))$ is reduced. In this case, for any generic point $\mathfrak{p}_i$ of D_s, the maximal ideal of $D_{\mathfrak{p}_i}$ can clearly be generated by π alone, which implies that $D_{\mathfrak{p}_i}$ is regular.

- If $v(c) = 1$, we have that $c = u\pi$ for some $u \in R^\times$, and $\overline{f} = 0 \in k[x]$: the equation of D_s is $y^2 = 0$, so the unique generic point of D_s is $\mathfrak{p} = (\pi, y) \in \operatorname{Spec}(D)$. The equation $y^2 = f(x) = u\pi f_0(x) f_\infty(x)$ may be rewritten as $\pi = y^2/(uf_0(x)f_\infty(x))$; since the integral polynomial $uf_0(x)f_\infty(x)$ is not zero mod π and is thus a unit in $D_\mathfrak{p}$, we conclude that y alone suffices to generate $\mathfrak{p}D_\mathfrak{p}$, and $D_\mathfrak{p}$ is consequently regular. $\square$

As a consequence of the proposition above, $\mathcal{Y}_s$ is described over k by the affine equations $y^2 = \overline{f}(x)$ over $\overline{x} \neq \infty$ and $(y/x^{g+1})^2 = \overline{\check{f}}(1/x)$ over $\overline{x} \neq 0$, and it is thus clear that

Proposition 7.2.2. The special fiber of $\mathcal{Y}$ can be described as follows.

O) If $v_{Y/\mathcal{X}}$ is odd, $\mathcal{Y}_s$ is the double line $y^2 = 0$;

E) If $v_{Y/\mathcal{X}}$ is even, given any 0∞-equation $y^2 = f(x) = cf_0(x)f_\infty(x)$ of Y having $v(c) = 0$, the equation of $\mathcal{Y}_s$ is $y^2 = \overline{f}(x)$, where $\overline{f}(x) = \overline{c}\overline{f}_0(x)\overline{f}_\infty(x) \in k[x]$ is not the zero polynomial. In this case, $\mathcal{Y}_s$ is a reduced curve, providing a degree-2 generically étale covering of the line $\mathcal{X}_s$ branched over the points $\overline{\mathfrak{s}}_i \in \mathcal{X}_s$ to which the maximal clusters $\mathfrak{s}_i$ of $\operatorname{ccp}_\mathcal{R}(\mathcal{X})$ reduce. Two further cases should be distinguished.

   E1) If all the maximal clusters $\mathfrak{s}_i$'s of $\mathcal{R}$ are even, then $\overline{f}(x) \in k[x]$ is a square: hence, $\mathcal{Y}_s$ consists of the two distinct lines $y = \pm\overline{f}(x)^{1/2}$, each providing a trivial covering of $\mathcal{X}_s$, which intersect each other over all the $\overline{\mathfrak{s}}_i$'s.

   E2) If the maximal clusters $\mathfrak{s}_i$'s are not all even, then $\overline{f}(x) \in k[x]$ is not a square, and $\mathcal{Y}_s$ is consequently an integral curve, described by the equation $y^2 = \overline{f}(x)$. If we decompose $\overline{f}(x)$ into the product of a square $\overline{g}(x)^2$ and a square-free part $\overline{s}(x)$, we have that the affine curve $z^2 = \overline{s}(x)$, where $z := y/\overline{g}(x)$, provides the normalization $\widetilde{\mathcal{Y}_s}$ of $\mathcal{Y}_s$: $\widetilde{\mathcal{Y}_s}$ is a hyperelliptic curve, endowed with a covering map $\widetilde{\mathcal{Y}_s} \to \mathcal{X}_s$ that ramifies precisely over the odd maximal clusters $\overline{\mathfrak{s}}_i$.

Case O is somehow inessential, as a quadratic extension of R makes $v_{Y/\mathcal{X}}$ even and makes us fall within case E. In the case E, the branch points of the ramified, generically étale covering map $\mathcal{Y}_s \to \mathcal{X}_s$ coincide with the specializations of the branch points $A_1, \ldots, A_{2g+2}$ of $Y \to X$: this is a manifestation of the purity result we have presented in Proposition 4.12.1. The only possible singularities of $\mathcal{Y}_s$ are the ramification points lying over the $\mathfrak{s}_i$'s, and a direct singularity estimate, based on the equations of $\mathcal{Y}_s$ we have given in Proposition 7.2.2, immediately shows that

- $\mathcal{Y}_s$ is singular over $\bar{\mathfrak{s}}_i$ if and only if $|\mathfrak{s}_i| > 1$;
- the δ-invariant of $\mathcal{Y}_s$ at the ramification point lying over $\bar{\mathfrak{s}}_i$ is precisely $\lfloor |\mathfrak{s}_i|/2 \rfloor$, and the number of branches of $\mathcal{Y}_s$ at that point (Definition 3.4.1) is equal to either 2 or 1 depending on whether $|\mathfrak{s}_i|$ is even or odd.

If we write $\mathfrak{s}$ for a maximal cluster, $\mathfrak{e}$ for an even maximal cluster, and $\mathfrak{o}$ for an odd maximal cluster, we have that:

$$a(\mathcal{Y}_s) = \frac{1}{2}\left(\sum_{\mathfrak{o}} 1\right) - 1$$

(to which 1 should be added if all maximal clusters are even)

$$t(\mathcal{Y}_s) = \sum_{\mathfrak{e}} 1$$

(to which 1 should be subtracted if all maximal clusters are even)

$$u(\mathcal{Y}_s) = \frac{1}{2}\left[\sum_{\mathfrak{e}}(|\mathfrak{e}| - 2) + \sum_{\mathfrak{o}}(|\mathfrak{o}| - 1)\right]$$

If we are in case E2, then the entanglement number (Definition 4.6.15) of the unique component of $\mathcal{Y}_s$ is $\sum_{\mathfrak{e}} 2 + \sum_{\mathfrak{o},|\mathfrak{o}|>1} 1$. If we are instead in case E1, each of the two lines of which $\mathcal{Y}_s$ consists clearly has entanglement number $\sum_{\mathfrak{e}} 1$. As a consequence, we can say that:

Theorem 7.2.3. Suppose that the hyperelliptic curve Y has semistable reduction.

(a) The smooth model of the line $\mathcal{X}$ contributes to the ur-minimal model if and only if $v_{Y/\mathcal{X}}$ is even (i.e., we are in case E), and $\mathcal{X}$ is $\mathcal{R}$-unwound (Definition 6.3.22);

(b) (if $g \geq 2$) $\mathcal{X}$ contributes to the ur-stable model if and only if $v_{Y/\mathcal{X}}$ is even (i.e., we are in case E) and $\mathcal{X}$ is $\mathcal{R}$-very unwound (Definition 6.3.23).

Proof. This follows directly from the computation of abelian ranks and entanglement numbers we have presented above, by applying the criterion that we have codified in Theorem 6.4.6. $\square$

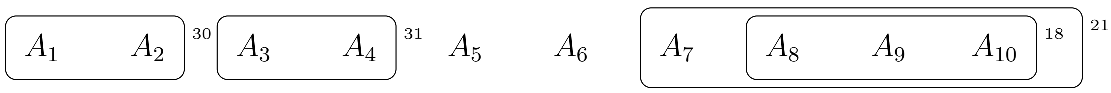
This result not only allows us to identify the ur-minimal and ur-stable model of a hyperelliptic curve Y, but also provides a criterion for semistable reduction:

Theorem 7.2.4. Suppose that the hyperelliptic curve Y has genus $g \geq 2$. Then Y has semistable reduction if and only if $v_{Y/\mathcal{X}}$ is even for every very unwound smooth model $\mathcal{X}$ of X.

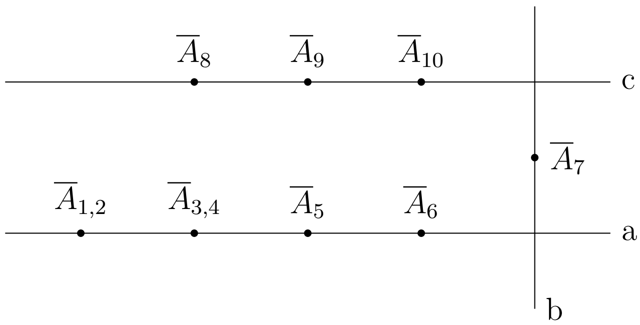
Proof. This result is essentially an application of the criterion for semistable reduction we have presented in Proposition 6.4.5. Let $R \subseteq R'$ be an extension over which Y has semistable reduction (it certainly exists thanks to Theorem 5.5.1): the theorem we have just proved ensures that the ur-stable model over R' is the composite $\mathcal{X}'_{\mathrm{vunw}}$ of all very unwound smooth models of the line over R'. But we have observed in Proposition 6.3.31 that all such models descend to R, hence condition (a) of Proposition 6.4.5 is always satisfied. On the other hand, condition (b) of Proposition 6.4.5 is clearly equivalent to requiring that $v_{Y/\mathcal{X}}$ is even for all very unwound smooth models of X over R. From these two observations the theorem immediately follows. □

7.2.1. An example

Suppose that a hyperelliptic curve $Y \to X$, with respect to some smooth model $\mathcal{X}^a$ of the line X, exhibits the following cluster picture



and let us assume that $v_{Y/\mathcal{X}^a}$ is even. It is immediate to observe that $\mathcal{X}^a$ is very unwound, and that there are three very unwound smooth models of the line in total, whose composite $\mathcal{X}_{\mathrm{vunw}}$ has the following special fiber



Let us write P_{ab} (resp. P_{bc}) for the node of $\mathcal{X}_{\mathrm{vunw}}$ at which the lines a and b (resp. b and c) meet: P_{ab} has thickness 21 and corresponds to the cluster cut $\mathcal{R} = \{A_1, \dots, A_6\} \sqcup \{A_7, \dots, A_{10}\}$ of size 4/6, while P_{bc} has thickness 18 and corresponds to the cluster cut $\mathcal{R} = \{A_1, \dots, A_7\} \sqcup \{A_8, \dots, A_{10}\}$ of size 3/7.

By assumption, $v_{Y/\mathcal{X}^a} \equiv 0 \bmod 2$; by applying Proposition 7.1.4, we can compute

$$v_{Y/\mathcal{X}^b} \equiv v_{Y/\mathcal{X}^a} + 4 \cdot 21 \equiv 0;$$
$$v_{Y/\mathcal{X}^c} \equiv v_{Y/\mathcal{X}^b} + 3 \cdot 18 \equiv 0.$$

We can thus invoke Theorem 7.2.4 and conclude that Y has semistable reduction over R; moreover, by Theorem 7.2.3 we know that

- the ur-stable model is $\mathcal{X}_{\mathrm{vunw}}$;
- the ur-minimal model is obtained by enriching $\mathcal{X}_{\mathrm{vunw}}$ with those unwound, not-very-unwound smooth models $\mathcal{X}$ of the line such that $v_{Y/\mathcal{X}}$ is even; in light of Proposition 7.1.4, this means that $\mathcal{X}_{\mathrm{urmin}}$ can be obtained form $\mathcal{X}_{\mathrm{vunw}}$ in the following way.
  - The thickness-21 node $P_{ab} \in (\mathcal{X}_{\mathrm{vunw}})_s$, which corresponds to an even cluster cut, must be replaced by a chain of $21-1=20$ lines $L_1^{ab}, \ldots, L_{20}^{ab}$ hold together by thickness-1 nodes. Each L_i^{ab} corresponds to an unwound smooth model of the line having precisely two even maximal clusters.
  - The thickness-18 node $P_{bc} \in (\mathcal{X}_{\mathrm{vunw}})_s$, which corresponds to an odd cluster cut, must be replaced by a chain of $18/2-1=8$ lines $L_1^{bc}, \ldots, L_8^{bc}$ hold together by thickness-2 nodes. Each L_i^{bc} corresponds to an unwound smooth model of the line having precisely two odd maximal clusters.
  - The point $\overline{A}_{1,2} \in (\mathcal{X}_{\mathrm{vunw}})_s$ must be replaced by a sequence of 30 lines $L_1^{1,2}, \ldots, L_{30}^{1,2}$ hold together by thickness-1 nodes. Each $L_i^{1,2}$ corresponds to an unwound smooth model of the line having two even maximal clusters (of sizes 8 and 2), except the last one, $L_{30}^{1,2}$, whose maximal clusters have sizes 8-1-1.
  - The point $\overline{A}_{3,4} \in (\mathcal{X}_{\mathrm{vunw}})_s$ must be replaced by a sequence of 31 lines $L_1^{3,4}, \ldots, L_{31}^{3,4}$ hold together by thickness-1 nodes. Each $L_i^{3,4}$ corresponds to an unwound smooth model of the line having two even maximal clusters (of sizes 8 and 2), except the last one, $L_{31}^{3,4}$, whose maximal clusters have sizes 8-1-1.

Let $\mathcal{Y}^a$, $\mathcal{Y}^b$ and $\mathcal{Y}^c$ be the normalizations of $\mathcal{X}^a$, $\mathcal{X}^b$ and $\mathcal{X}^c$ in $K(Y)$: the computations we have presented in this section allow us to explicitly describe their special fibers.

- $(\mathcal{Y}^a)_s$ is an integral curve C_a of abelian rank 0, branched over $\overline{A}_{1,2}$, $\overline{A}_{3,4}$, $\overline{A}_5$, $\overline{A}_6$, $\overline{A}_{7,8,9,10}$, whose singularities are a node over $\overline{A}_{1,2}$, a node over $\overline{A}_{3,4}$, and a non-ordinary singularity, at which $(\mathcal{Y}^a)_s$ has two branches, over $\overline{A}_{7,8,9,10}$. Thanks to Proposition 6.4.8, we can thus be sure that the strict transform C_a' of C_a in $\mathcal{Y}_{\mathrm{st}}$ is an integral curve of abelian rank 0, branched over $\overline{A}_{1,2}$,

$\overline{A}_{3,4}$, $\overline{A}_5$ and $\overline{A}_6$, whose singularities are a node $Q_{1,2}$ over $\overline{A}_{1,2}$ and a node $Q_{3,4}$ over $\overline{A}_{3,4}$.

- $(\mathcal{Y}^b)_s$ is an integral curve C_b of abelian rank 0, branched over $\overline{A}_{1,2,3,4,5,6}$, $\overline{A}_7$ and $\overline{A}_{8,9,10}$, whose singularities are a non-ordinary singularity, at which $(\mathcal{Y}^b)_s$ has two branches, over $\overline{A}_{1,2,3,4,5,6}$, and a unibranch (non-ordinary) singularity over $\overline{A}_{8,9,10}$. Thanks to Proposition 6.4.8, we can thus be sure that the strict transform C'_b of C_b in $\mathcal{Y}_{\mathrm{st}}$ is a line branched over $\overline{A}_7$ and P_{cb}.

- $(\mathcal{Y}^c)_s$ is an integral curve C_c of abelian rank 1, branched over $\overline{A}_{1,2,3,4,5,6,7}$, $\overline{A}_8$, $\overline{A}_9$ and $\overline{A}_{10}$, whose unique singular point is a unibranch (non-ordinary) singularity over $\overline{A}_{1,2,3,4,5,6,7}$. Thanks to Proposition 6.4.8, we can thus be sure that the strict transform C'_c of C_c in $\mathcal{Y}_{\mathrm{st}}$ is an elliptic curve branched over $\overline{A}_8$, $\overline{A}_9$, $\overline{A}_{10}$ and P_{cb}.

The special fiber $(\mathcal{Y}_{\mathrm{st}})_s$ has thus three irreducible components: C'_a, which is a curve of abelian rank 0 bearing two nodes $Q_{1,2}$ and $Q_{3,4}$, C'_b, which is a line, and C'_c, which is an elliptic curve. Since C'_a and C'_b are not branched over P_{ab}, they will meet at two distinct nodes $Q_{ab}^{(1)}$ and $Q_{ab}^{(2)}$ lying over P_{ab}; since C'_b and C'_c are branched over P_{bc}, they will meet at a unique node Q_{bc} lying over P_{bc}. The ultimate reason of such a difference between P_{ab} and P_{bc} is the fact that P_{ab} corresponds to an even cluster cut, while P_{bc} to an odd one.

We have already discussed what additional smooth components appear when we move from $\mathcal{X}_{\mathrm{vunw}} = \mathcal{X}_{\mathrm{urst}}$ to the ur-minimal model $\mathcal{X}_{\mathrm{urmin}}$: it is thus now easy to describe how the special fiber $\mathcal{Y}_{\mathrm{min}}$ can be obtained from that of $\mathcal{Y}_{\mathrm{st}}$; the knowledge of $(\mathcal{Y}_{\mathrm{min}})_s$ can then be exploited to extract information on the thicknesses of the nodes of $\mathcal{Y}_{\mathrm{st}}$.

- Over each of the 20 lines L_i^{ab} of $(\mathcal{X}_{\mathrm{urmin}})_s$, two (-2)-lines $\ell_i^{ab(1)}$ and $\ell_i^{ab(2)}$ of $(\mathcal{Y}_{\mathrm{min}})_s$ appear: $\ell_1^{ab(1)}, \ldots, \ell_{20}^{ab(1)}$ replace $Q_{ab}^{(1)} \in (\mathcal{Y}_{\mathrm{st}})_s$, while $\ell_1^{ab(2)}, \ldots, \ell_{20}^{ab(2)}$ replace $Q_{ab}^{(2)} \in (\mathcal{Y}_{\mathrm{st}})_s$; in particular, the nodes $Q_{ab}^{(1)}$ and $Q_{ab}^{(2)}$ of $\mathcal{Y}_{\mathrm{st}}$ have thickness 21.

- Over each of the 8 lines L_i^{bc} of $(\mathcal{X}_{\mathrm{urmin}})_s$, a unique (-2)-line ℓ_i^{bc} shows up in $(\mathcal{Y}_{\mathrm{min}})_s$: all together, they form a chain that replaces Q_{bc}, whose thickness in $\mathcal{Y}_{\mathrm{st}}$ is consequently 9.

- Over each of the 30 lines $L_i^{1,2}$, two (-2)-lines of $(\mathcal{X}_{\mathrm{urmin}})_s$ appear: the only exception is represented by the last one, i.e. $L_{30}^{1,2}$, over which a unique (-2)-line of $(\mathcal{X}_{\mathrm{urmin}})_s$ appears. As a consequence, a chain of 59 (-2)-lines of $(\mathcal{Y}_{\mathrm{min}})_s$ replaces $Q_{1,2}$, whose thickness in $\mathcal{Y}_{\mathrm{st}}$ is consequently 60.

- Over each of the 31 lines $L_i^{3,4}$, two (-2)-lines of $(\mathcal{X}_{\mathrm{urmin}})_s$ appear: the only exception is represented by the last one, i.e. $L_{31}^{1,2}$, over which a unique (-2)-line of $(\mathcal{X}_{\mathrm{urmin}})_s$ appears. As a consequence, a chain of 61 (-2)-lines of $(\mathcal{Y}_{\min})_s$ replaces $Q_{3,4}$, whose thickness in $\mathcal{Y}_{\mathrm{st}}$ is consequently 62.

That the thicknesses of $Q_{ab}^{(1)}$ and $Q_{ab}^{(2)}$ equal the one of P_{ab} is a consequence of the fact $\mathcal{Y}_{\mathrm{st}} \to \mathcal{X}_{\mathrm{urst}}$ is étale over P_{ab}. That the unique node Q_{bc} lying over P_{bc} must have half the thickness of P_{bc} could be directly deduced from Proposition 5.7.1.

7.3. Residue characteristic 2

Let us retain the notation of Section 7.1, but let us now suppose that the residue characteristic of R is 2. As before, we are interested in studying the normalization $\mathcal{Y}$ in the quadratic extension $K(X) \subseteq K(Y)$ of a given smooth model $\mathcal{X}$ of the line. We will work over the affine chart $\mathcal{X}_{(0)} \subseteq \mathcal{X}$ given by $x \neq \infty$, where x is the distinguished coordinate we have fixed on $\mathcal{X}$. Let us also fix a 0∞-equation $y^2 = f(x) = cf_0(x)f_\infty(x) \in R[x]$, with $v(c) \in \{0,1\}$, describing the corresponding affine chart $Y_{(0)}$ of the hyperelliptic curve Y in the plane $\mathbb{A}^2_K$. To determine the normalization $\mathcal{Y}_{(0)}$ of $\mathcal{X}_{(0)}$ in $K(Y)$, the idea is the following one.

1. We perform a suitable a change in the coordinate y, after which the equation of Y in $\mathbb{A}^2_K$ will assume the form $y^2 + H(x)y = G(x)$. We will check that the chosen change of coordinate preserves the integrality of the equation, i.e. $H(x), G(x) \in R[x]$.

2. We consider the R-scheme W defined by the same equation $y^2 + H(x)y = G(x)$ in $\mathbb{A}^2_R$: such a W clearly comes with a finite flat degree-2 map $W \to \mathcal{X}_{(0)}$ that recovers $Y_{(0)} \to X_{(0)}$ at the level of generic fibers.

3. We check that W is actually a normal scheme. In this regard, we remark that W is always certainly a Cohen-Macaulay scheme, since it is a complete intersection in $\mathbb{A}^2_R$. The generic fiber of W, moreover, is nothing but $Y_{(0)}$, and is consequently normal. Verifying that W is normal is hence equivalent to check that it is regular at the generic points of its special fiber W_s. As a consequence, in particular, the normality of W is always guaranteed whenever W_s happens to be reduced.

4. If the normality check gives an affirmative outcome, we can clearly be sure that W actually coincides with the normalization $\mathcal{Y}_{(0)}$ we are looking for.

To determine the right transformation that has to be performed, we will first decompose $f(x)$ as $q^2(x) + \rho(x)$, where $q(x), \rho(x) \in R[x]$: in other words, we are approximating $f(x) \in R[x]$ with a square $q^2(x) \in R[x]$, and $\rho(x)$ can be viewed as an error term.

Definition 7.3.1. Given a polynomial $F(x) \in K[x]$, its valuation $v(F)$ is, by definition, the minimum of the valuations of its coefficients.

Definition 7.3.2. Let $\mathcal{X}$ be a smooth model of the line, and x a coordinate on it. Given $y^2 = f(x) = cf_0(x)f_\infty(x)$ a 0∞-equation of Y such that $v(c) \in \{0, 1\}$, a decomposition $f(x) = q^2(x) + \rho(x)$ with $q(x), \rho(x) \in R[x]$ is named a *$q\rho$-decomposition* of f. Given such a decomposition, the valuation $v(\rho)$ will be denoted by t. A $q\rho$-decomposition is said to be *optimal* if no $q\rho$-decomposition of f over R can be found exhibiting a higher value of t.

Lemma 7.3.3. If $f(x) = q^2(x) + \rho(x)$ is a $q\rho$-decomposition, then we have that $t := v(\rho) \in \{0, 1\}$ or $v(q) = 0$.

Proof. Suppose, by contradiction, that $v(\rho) \geq 2$ and $v(q) > 0$. Then, $v(q) \geq 1$, and hence $v(q^2) \geq 2$. We conclude that $v(f) = v(q^2 + \rho) \geq 2$, but this clearly contradicts the fact that $v(f) = v(c) \in \{0, 1\}$. □

Lemma 7.3.4. Let $f(x) = q^2(x) + \rho(x)$ be a $q\rho$-decomposition with $t := v(\rho) < 2v(2)$. Then,

(a) if t is odd, the decomposition is always optimal;

(b) if t is even, the decomposition is optimal if and only if $\overline{r} \in k[x]$ is not a square, where $r(x) := \rho(x)/\pi^t$.

Proof. Suppose that $f(x) = q^2(x) + \rho(x)$ is not optimal: another decomposition $f(x) = \widetilde{q}^2(x) + \widetilde{\rho}(x)$ with $\widetilde{t} := v(\widetilde{\rho}) > t$ can thus be found. Let us now consider $q + \widetilde{q}$ and $q - \widetilde{q}$: their product has valuation $v(q^2 - \widetilde{q}^2) = v(\widetilde{\rho} - \rho) = t$; their difference has valuation $v(2\widetilde{q}^2) \geq v(2) > t/2$. From this, it is immediate to deduce that they must both have valuation equal to $t/2$. This is clearly a contradiction if t is odd, whence (a) follows. Let us now focus on the case t even. We may clearly write:

$$\rho = \widetilde{\rho} + \widetilde{q}^2 - q^2 = \widetilde{\rho} + (q + \widetilde{q})^2 - 2q(q + \widetilde{q})$$

but, looking at the valuations, we see that

- $v(\widetilde{\rho}) > t$,
- $v((q + \widetilde{q})^2) = 2v(q + \widetilde{q}) = t$, and
- $v(2q(q + \widetilde{q})) \geq v(2) + v(q + \widetilde{q}) > t$.

Hence, we can be sure that $r := \rho/\pi^t$ reduces to $\overline{r} = \overline{\pi^{-t/2}(q+\widetilde{q})}^2$ and is consequently a square in $k[x]$.

We have thus proved the "if" part of statement (b): let us now consider the converse implication. Suppose that t is even, and $\overline{r}(x) \in k[x]$ is a square. This means that we may write $r(x) = q_1^2(x) + \rho_1(x)$, where $q_1(x) \in R[x], \rho_1(x) \in \pi R[x]$. Hence, we can rewrite $f(x) = \left[q(x) + \pi^{t/2}q_1(x)\right]^2 + \left[\pi^t\rho_1(x) - 2\pi^{t/2}q(x)q_1(x)\right]$: we have found a new decomposition $f(x) = \widetilde{q}^2(x) + \widetilde{\rho}(x)$ with $\widetilde{t} := v(\widetilde{\rho}) > t$. Hence, the original decomposition $f(x) = q^2(x) + \rho(x)$ was not optimal. □

We are now ready to compute $\mathcal{Y}$.

- Suppose that a $q\rho$-decomposition of $f(x)$ exists with $t \geq 2v(2)$. In this case, the change of variable $y \mapsto [y - q(x)]/2$ leads to an (integral) equation $y^2 + q(x)y = r(x)$, where $r(x) := \rho(x)/4 \in R[x]$. In the reduction, this becomes $y^2 + \overline{q}(x)y = 0$ if $t > 2v(2)$, and $y^2 + \overline{q}(x)y = \overline{r}(x)$, with $\overline{r}(x)$ not identically zero, if t is precisely $2v(2)$. By Lemma 7.3.3, $\overline{q}(x) \in k[x]$ is not the zero polynomial, so W_s, in this case, is certainly reduced, and hence W is normal.

- Suppose, instead, that an optimal $q\rho$-decomposition $f(x) = q^2(x) + \rho(x)$ exists with $t < 2v(2)$. In this case, we will perform the change of variable $y \mapsto (y - q(x))/\pi^{\lfloor t/2 \rfloor}$, and the result is the following one.
  - If t is odd, then we obtain the integral equation $y^2 + (2/\pi^{(t-1)/2})q(x)y = \rho(x)/\pi^{t-1}$, which becomes $y^2 = 0$ in the reduction. The only generic point of W_s corresponds to the prime ideal (π, y), and a direct computation shows that, locally near that point, π can be written as a function of y. This ensures that W is normal.
  - If t is even, then the equation we obtain is also still integral, and has the form $y^2 + (2/\pi^{t/2})q(x)y = r(x)$, where $r(x) := \rho(x)/\pi^t \in R[x]$. As a consequence, W_s is described by $y^2 = \overline{r}(x)$, where $\overline{r}(x) \in k[x]$ is not a square (Lemma 7.3.4): in particular, W_s is reduced, and W is hence normal.

As a consequence,

Theorem 7.3.5. One and only one of the three following cases occurs.

H) "High case": a $q\rho$-decomposition $f(x) = q^2(x) + \rho(x)$ exists with $t := v(\rho) \geq 2v(2)$ (we will name it an *H-decomposition*); $\mathcal{Y}_s$ is a reduced curve providing a generically étale quadratic covering of $\mathcal{X}_s$, branched precisely over the points $\overline{\mathfrak{s}}_i \in \mathcal{X}_s$ to which the maximal clusters $\mathfrak{s}_i$ of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ reduce.

LO) "Low odd case": a (necessarily optimal) $q\rho$-decomposition $f(x) = q^2(x) + \rho(x)$ exists with $t < 2v(2)$ and odd (we will name it a *LO-decomposition*); $\mathcal{Y}_s$ is a double line;

LE) "Low even case": an optimal $q\rho$-decomposition $f(x) = q^2(x) + \rho(x)$ exists such that t is $< 2v(2)$ and even (we will name it a *LE-decomposition*); $\mathcal{Y}_s$ is an integral curve, inseparable over $\mathcal{X}_s$.

Case	**Change of var.**	**Equation of $\mathcal{Y}$**	**Equation of $\mathcal{Y}_s$**
H	$y \mapsto \dfrac{y - q(x)}{2}$	$y^2 + q(x)y = r(x)$, where $r(x) := \rho(x)/4$	$y^2 + \overline{q}(x)y = \overline{r}(x)$
LO	$y \mapsto \dfrac{y - q(x)}{\pi^{(t-1)/2}}$	$y^2 + \dfrac{2}{\pi^{(t-1)/2}}q(x)y = \dfrac{\rho(x)}{\pi^{t-1}}$, where $t := v(\rho)$	$y^2 = 0$
LE	$y \mapsto \dfrac{y - q(x)}{\pi^{t/2}}$	$y^2 + \dfrac{2}{\pi^{t/2}}q(x)y = r(x)$, where $r(x) := \rho(x)/\pi^t$, $t := v(\rho)$	$y^2 = \overline{r}(x)$

We will write case L to mean "case LE or case LO". We will now try to better describe $\mathcal{Y}$ in each of the three cases, and we will discuss some necessary and some sufficient conditions for a smooth model of the line to fall within each of the three cases.

7.3.1. Totally odd decompositions and case LO

To state Theorem 7.3.5 conveniently, we had to introduce a number of special $q\rho$-decompositions:

Definition 7.3.6. A $q\rho$-decomposition which is either an H-decomposition, a LO-decomposition or a LE-decomposition will be said to be a *good* $q\rho$-decomposition.

A good $q\rho$-decomposition falling in case H or in case LE will clearly still be good, and will fall within the same case, also after an arbitrary extension of R. It is equally clear that this cannot be true for LO-decompositions; actually, it is possible to show that case LO always dissolves after a suitable extension of R. To see this, let us first introduce the following definition.

Definition 7.3.7. A $q\rho$-decomposition $f = q^2 + \rho$ is said to be *totally odd* if the polynomial ρ only consists of odd degree terms.

Totally odd decompositions are particularly convenient, because of the following, obvious fact.

Proposition 7.3.8. After possibly replacing R with a quadratic extension, a totally odd $q\rho$-decomposition either falls within case H or case LE.

Proof. Since ρ is a non-zero polynomial whose terms all have odd degree, $\overline{\rho/\pi^t}$, where $t := v(\rho)$, cannot be a square of $k[x]$. □

Let us now show that totally odd decompositions actually exist.

Proposition 7.3.9. There always exists a finite extension $R \subseteq R'$ over which f admits a totally odd $q\rho$-decomposition.

Proof. Given a polynomial $F(x) \in R[x]$, we will denote by $F_e(x)$ and $F_o(x)$ the sums of its even and odd degree monomials respectively, so that $F(x) = F_e(x) + F_o(x)$; in particular, we will write $f(x) = f_e(x) + f_o(x)$ and denote by $2m$ the degree of f_e. As all terms of f_e have even degree, we may write $f_e(x) = g(x^2)$ for some uniquely determined $g(x) \in R[x]$ of degree m. Let $\alpha_1, \ldots, \alpha_m$ be the roots of g, and let us denote by a its leading coefficient. Let us also choose a square root $\sqrt{a}$ of a, and a square root $\sqrt{\alpha_i}$ of each root of g, and let us define

$$\begin{aligned} g_+(x) :=& \sqrt{a}\prod_i (x + \sqrt{\alpha_i}) = c_0x^m + c_1x^{m-1} + \ldots + c_m, \\ g_-(x) :=& \sqrt{a}\prod_i (x - \sqrt{\alpha_i}) = c_0x^m - c_1x^{m-1} + \ldots + (-1)^m c_m. \end{aligned}$$

It is clear that $f_e(x) = g_+(x)g_-(x)$; moreover, from the fact that f_e is integral, it follows that g_+ and g_- also are (their Newton polygons, in fact, coincide with that of f_e scaled by $1/2$): in other words, $v(c_i) \geq 0$ for all i. Exploiting this factorization of f_e, we may write its k-th order coefficient as

$$\sum_{i+j=2m-k} (-1)^i c_i c_j. \tag{†}$$

If we now choose a square root $\sqrt{-1}$ of -1, and we define

$$c_i' := \begin{cases} c_i & \text{if } i \text{ is even,} \\ \sqrt{-1}\cdot c_i & \text{if } i \text{ is odd,} \end{cases}$$

we may rewrite the expression (†), for all even values of k, in the more symmetric form

$$\sum_{i+j=2m-k} c'_i c'_j.$$

If we now set $q(x) := c'_0 x^m + \ldots + c'_m$, it is clear that q has integral coefficients, and that the even degree terms of q^2 reproduce f_e. Hence, $\rho(x) := f(x) - q^2(x)$ only consists of odd degree terms, and $f(x) = q^2(x) + \rho(x)$ is a totally odd $q\rho$-decomposition for f. □

Corollary 7.3.10. Suppose that $\mathcal{X}$ falls within case LO. Then, after replacing R with some finite extension, $\mathcal{X}$ will either fall within case LE or case H.

We conclude our discussion of case LO by remarking that

Proposition 7.3.11. If $\mathcal{X}$ is a smooth model of the line such that $v_{Y/\mathcal{X}}$ is odd, then $\mathcal{X}$ falls within case LO.

Proof. If $v_{Y/\mathcal{X}}$ is odd, we can take a 0∞-equation $y^2 = f(x) = cf_0(x)f_\infty(x)$ for Y where $v(c) = 1$. If we set $q(x) := 0$ and $\rho(x) := cf_0(x)f_\infty(x)$, we have that $f(x) = q^2(x) + \rho(x)$ is clearly a LO-decomposition ($t := v(\rho)$ equals 1), whence the proposition follows. □

7.3.2. Case LE

Let us now turn our attention to case LE. We recall that, in this case, $\mathcal{Y}_s$ is an integral curve endowed with a (purely) inseparable degree-2 covering map $\mathcal{Y}_s \to \mathcal{X}_s$. The normalization $\widetilde{\mathcal{Y}_s}$ of $\mathcal{Y}_s$ is thus a line, and the degree-2 covering map $\widetilde{\mathcal{Y}_s} \to \mathcal{X}_s$ is the Frobenious covering. In particular, $\mathcal{Y}_s$ has abelian and toric rank 0, and hence unipotent rank $u(\mathcal{Y}_s) = g$. Given a LE-decomposition $f(x) = q^2(x) + \rho(x)$, we know that $\mathcal{Y}_s$ is described by the equation $y^2 = \overline{r}(x)$ where $r(x) := \rho(x)/\pi^t$ and $t := v(\rho)$. The singularities $\mathcal{Y}_s$ correspond to the zeros of $\overline{r}'(x)$, and they are all unibranch (Definition 3.4.1).

Let us now see some sufficient conditions for the smooth model $\mathcal{X}$ of the line to fall within case LE.

Proposition 7.3.12. Suppose that $\mathcal{X}$ is a smooth model of the line such that $v_{Y/\mathcal{X}}$ is even and whose associated cluster picture $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ has at least one odd maximal cluster. Then, we are in case LE, and the singular points of $\mathcal{Y}_s$ correspond to the zeros of $\overline{f}'$. In particular, for all i, $\mathcal{Y}_s$ is singular over the point $\bar{\mathfrak{s}}_i \in \mathcal{X}_s$ to which the maximal cluster $\mathfrak{s}_i$ reduces if and only if $|\mathfrak{s}_i| > 1$.

Proof. Asking that $v_{Y/\mathcal{X}}$ is even and that $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ has an odd maximal cluster is clearly equivalent to assuming that $\overline{f} \in k[x]$ is not a square. The hypotheses of the proposition hence ensure that the $q\rho$-decomposition $f(x) = q^2(x) + \rho(x)$ having $q(x) := 0$ and $\rho(x) := f(x)$ is a LE-decomposition, for which we clearly have that $t := v(\rho) = 0$. In the LE case, the singularities of $\mathcal{Y}_s$ correspond to the zeros of $\overline{r}'(x)$, but, in our specific setting, $r(x) := \pi^{-t}\rho(x) = \rho(x) = f(x)$, and $\overline{\mathfrak{s}}_i$ is a zero of $\overline{f}'(x)$ if and only if $\overline{\mathfrak{s}}_i$ is a multiple root of $\overline{f}$, which in turn happens if and only if $|\mathfrak{s}_i| > 1$. $\square$

Proposition 7.3.13. Suppose that $\mathcal{X}$ is a semi-crushed smooth model of the line such that $v_{Y/\mathcal{X}}$ is even. Then, we are in case LE, and $\mathcal{Y}_s$ has exactly one singularity, which lies over the point $\overline{\mathfrak{s}} \in \mathcal{X}_s$ at which the unique non-singleton maximal cluster $\mathfrak{s}$ of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ reduces.

Proof. If the smooth model of the line $\mathcal{X}$ is semi-crushed, then $\overline{f}$ has a simple root at $\overline{A}_1$, where $\{A_1\}$ is the singleton maximal cluster, and a root of multiplicity $2g+1$ at $\overline{\mathfrak{s}}$, where $\mathfrak{s} = \mathcal{R} \setminus \{A_1\}$ is the non-singleton maximal cluster. A straightforward computation shows that $\overline{f}'$ will consequently have only one root, which will coincide with $\overline{\mathfrak{s}}$ and will have multiplicity $2g$. It is now clear that this proposition is just a particular case of the last one (Proposition 7.3.12). $\square$

7.3.3. Case H

Suppose now that f has a $q\rho$-decomposition $f(x) = q^2(x) + \rho(x)$ with $t := v(\rho) \geq 2v(2)$ (in other words, suppose we are in H). We know that $\mathcal{Y}_s$ is described by the equation $y^2 + \overline{q}(x)y = \overline{r}(x)$, where $r(x) := \rho(x)/4$, and $\mathcal{Y}_s \to \mathcal{X}_s$ is a generically étale degree-2 covering morphism, branched over the $\overline{\mathfrak{s}}_i$'s: we will denote by $Q_i \in \mathcal{Y}_s$ the ramification point lying over $\overline{\mathfrak{s}}_i$. It is clear that $\mathcal{Y}_s$ can only be possibly singular at the Q_i's, and the Jacobian criterion shows that Q_i is actually a singular point if and only if $\overline{r}'(\overline{\mathfrak{s}}_i)^2 = \overline{q}'(\overline{\mathfrak{s}}_i)^2 r(\overline{\mathfrak{s}}_i)$. For each i, there are clearly three mutually exclusive possibilities:

(a) $Q_i \in \mathcal{Y}_s$ is a smooth point of $\mathcal{Y}_s$; in this case, $\widetilde{\mathcal{Y}_s} \to \mathcal{X}_s$ branches over $\overline{\mathfrak{s}}_i$;

(b) $Q_i \in \mathcal{Y}_s$ is a singular point at which the number of branches of $\mathcal{Y}_s$ is $m(Q_i) = 1$; in this case, $\widetilde{\mathcal{Y}_s} \to \mathcal{X}_s$ branches over $\overline{\mathfrak{s}}_i$;

(c) $Q_i \in \mathcal{Y}_s$ is a singular point at which the number of branches of $\mathcal{Y}_s$ is $m(Q_i) = 2$; in this case, $\widetilde{\mathcal{Y}_s} \to \mathcal{X}_s$ does not branch over $\overline{\mathfrak{s}}_i$.

If case (c) occurs for all the maximal clusters $\mathfrak{s}_i$'s, then $\widetilde{\mathcal{Y}_s} \to \mathcal{X}_s$ is étale; since a line does not admit non-trivial étale coverings, $\widetilde{\mathcal{Y}_s}$ must be a pair of lines, each providing a trivial covering of $\mathcal{X}_s$. From this, it is immediate to deduce that $\mathcal{Y}_s$ also consists of two lines, each providing a trivial covering of $\mathcal{X}_s$, intersecting over the $\bar{\mathfrak{s}}_i$'s. If at least one maximal cluster of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ falls within case (a) or (b), instead, $\widetilde{\mathcal{Y}_s}$ is a (wild) hyperelliptic curve, and hence, in particular, $\mathcal{Y}_s$ is integral.

Remark 7.3.14. Suppose that $\mathcal{Y}_s$ is singular at the point $Q_i \in \mathcal{Y}_s$ lying over $\bar{\mathfrak{s}}_i \in \mathcal{X}_s$. Then, Q_i is a node if and only if $\bar{\mathfrak{s}}_i$ is a single root of $\overline{q}(x)$, and this in turn happens if and only if $|\mathfrak{s}_i| = 2$.

Remark 7.3.15. If the strict inequality $t := v(\rho) > 2v(2)$ holds (we will refer to this scenario as *case H+*), then $\overline{r}$ is the zero polynomial, and hence $\mathcal{Y}_s$ consists of a pair of lines $y = 0$ and $y = \overline{q}(x)$ each providing a trivial covering of $\mathcal{X}_s = \mathbb{P}^1_k$, meeting over the points $\bar{\mathfrak{s}}_i \in \mathcal{X}_s$ to which the maximal clusters $\mathfrak{s}_i$ of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ reduce. In this case, it is also not difficult to also estimate the δ-invariant of $\mathcal{Y}_s$ at the singular point over $\bar{\mathfrak{s}}_i$, which is equal to $|\mathfrak{s}_i|/2$.

Now that we have discussed what $\mathcal{Y}_s$ looks like in case H, we ask ourselves under what circumstances case H actually occurs.

Remark 7.3.16. Case H can only occur if all maximal clusters of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ are even and $v_{Y/\mathcal{X}}$ is also even: in fact, if $v_{Y/\mathcal{X}}$ is odd then Proposition 7.3.11 ensures that $\mathcal{X}$ falls within case LO, and, if $v_{Y/\mathcal{X}}$ is even but an odd maximal cluster of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ exists, then $\mathcal{X}$ falls within case LE by Proposition 7.3.12.

Proposition 7.3.17. Suppose that all maximal clusters $\mathfrak{s}_i$ of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ are even and have depths $\geq 2v(2)$. Then, after possibly replacing R with a suitable quadratic extension, $\mathcal{X}$ falls within case H; moreover, the special fiber $\mathcal{Y}_s$ is singular precisely over the points $\bar{\mathfrak{s}}_i \in \mathcal{X}_s$ corresponding to the maximal clusters $\mathfrak{s}_i$ having cardinality $|\mathfrak{s}_i| > 2$ or depth $d_{\mathfrak{s}_i} > 2v(2)$. Moreover, if all maximal clusters have depth $> 2v(2)$, then we are in case H+.

Proof. Fix a coordinate x on $\mathcal{X}$ such that no point of $\mathcal{R}$ reduces to $\overline{x} = \infty$. Let $\mathfrak{s}_i$ denote a maximal cluster of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ and let $A_{i,j}$ be its points, whose coordinates will be written as:

$$x(A_{i,1}) = -a_i, \quad x(A_{i,2}) = -a_i - \Delta_{i,2}, \quad \ldots, \quad x(A_{i,j}) = -a_i - \Delta_{i,j}, \quad \ldots.$$

By convention, we will set $\Delta_{i,1} := 0$, and we will write Δ_i to mean $\Delta_{i,2}$ if $|\mathfrak{s}_i| = 2$. Up to a quadratic extension of R, we may write the equation of Y over $x \neq \infty$ in the form $y^2 = f(x)$, with

$$f(x) = \prod_{i,j}(x + a_i + \Delta_{i,j}).$$

Thanks to our hypothesis on cluster depths, $v(\Delta_{i,j}) \geq 2v(2)$ for all i and j. Since all maximal clusters have even cardinality, we have that the polynomial $\prod_{i,j}(x+a_i) = \prod_i(x+a_i)^{|\mathfrak{s}_i|} \in R[x]$ is a square: we will denote it as $q^2(x)$. If we approximate $f(x)$ with $q^2(x)$, the remainder is

$$\rho(x) := f(x) - q^2(x) = \sum_i \left[\left(\sum_j \Delta_{i,j} \right) (x+a_i)^{|\mathfrak{s}_i|-1} \prod_{l \neq i}(x+a_l)^{|\mathfrak{s}_l|} \right] + O(4v(2))$$

where $O(4v(2))$ denotes a polynomial whose coefficients all have valuations $\geq 4v(2)$.

It is clear that $t := v(\rho) \geq 2v(2)$, and that the strict inequality $t > 2v(2)$ holds if $d_{\mathfrak{s}_i} > 2v(2)$ for all i: in other words, we have achieved an H-decomposition of $f(x)$, which is moreover an H+-decomposition if all depths $d_{\mathfrak{s}_i}$ are $> 2v(2)$. If we let $r(x) := \rho(x)/4$, we have that, for each i,

$$\overline{r}(\overline{\mathfrak{s}}_i) = 0, \qquad \overline{r}'(\overline{\mathfrak{s}}_i) = \begin{cases} \overline{\Delta_i/4} \prod_{\ell \neq i} \overline{a_l - a_i}^{|\mathfrak{s}_l|} & \text{if } |\mathfrak{s}_i| = 2, \\ 0 & \text{if } |\mathfrak{s}_i| > 2, \end{cases}$$

and in particular $\mathcal{Y}_s$, which is described by the equation $y^2 + \overline{q}(x)y = \overline{r}(x)$, is singular over $\overline{\mathfrak{s}}_i$ if and only if $|\mathfrak{s}_i| > 2$, or $|\mathfrak{s}_i| = 2$ and $d_{\mathfrak{s}_i}$ (which is equal to $v(\Delta_i)$) is $> 2v(2)$. □

Remark 7.3.18. Under the hypotheses of the previous proposition, $\mathcal{Y}$ is a semistable model if and only if $|\mathfrak{s}_i| = 2$ for all i, and it is a smooth model if and only if $|\mathfrak{s}_i| = 2$ and $d_{\mathfrak{s}_i} = 2v(2)$ for all i.

7.3.4. Our strategy

Thanks to the fundamental results we have proved in Theorems 6.4.6 and 6.4.7, the description we have provided of the special fiber $\mathcal{Y}_s$ in the previous subsections allows us to draw many relevant conclusions on the semistable models of Y. In particular, we have that

Proposition 7.3.19. Suppose that Y has semistable reduction, and that $\mathcal{X}$ is a smooth component of the ur-minimal model. Then, $v_{Y/\mathcal{X}}$ is even, and $\mathcal{X}$ is either:

(a) an unwound smooth model of the line (Definition 6.3.22), or
(b) a crushed smooth model of the line (Definition 6.3.17) that is anchored (Definition 6.3.20) to an unwound smooth model of the line that, possibly after extending R, falls within case LE.

Proof. If $v_{Y/\mathcal{X}}$ were odd, then $\mathcal{X}$ would fall within case LO (Proposition 7.3.11), $\mathcal{Y}_s$ would be non-reduced and hence $\mathcal{X}$ could not contribute to the ur-minimal model (via Theorem 6.4.6). We have thus proved that $v_{Y/\mathcal{X}}$ is even.

We observe that, by combining Proposition 7.3.13 and Theorem 6.4.6, it is easy to see that $\mathcal{X}$ cannot be semi-crushed. Hence, $\mathcal{X}$ is either unwound or crushed.

Suppose, from now on, that $\mathcal{X}$ is crushed, let $\mathcal{X}^{\mathrm{nc}}$ denote its non-crushed anchor, and let $P \in (\mathcal{X}^{\mathrm{nc}})_s$ denote the point at which $\mathcal{X}$ is anchored (Definition 6.3.20). We have that

(i) since $v_{Y/\mathcal{X}}$ is even, $v_{Y/\mathcal{X}^{\mathrm{nc}}}$ will also be even (via Proposition 7.1.4);
(ii) no branch point $A_i \in \mathcal{R}$ can reduce to $P \in (\mathcal{X}^{\mathrm{nc}})_s$ (see Remark 6.3.21); and
(iii) if $(\mathcal{Y}^{\mathrm{nc}})_s$ is reduced, then it is singular over $P \in (\mathcal{X}^{\mathrm{nc}})_s$ (Theorem 6.4.7).

If $\mathcal{X}^{\mathrm{nc}}$ fell within case H, $(\mathcal{Y}^{\mathrm{nc}})_s$ would be a reduced curve whose singularities are confined over the reduction in $(\mathcal{X}^{\mathrm{nc}})_s$ of the points of $\mathcal{R}$, and this would contradict (ii)+(iii); we conclude that $\mathcal{X}^{\mathrm{nc}}$ must fall within case L. For a similar reason, we can be sure that $\mathcal{X}^{\mathrm{nc}}$ cannot be semi-crushed (see Proposition 7.3.13), and it will consequently be unwound.

Now, since $\mathcal{X}$ is a smooth component of the ur-minimal model over R and Y has semistable reduction over R, $\mathcal{X}_{R'}$ will be a smooth component of the ur-minimal model over R' for any extension $R \subseteq R'$ (Proposition 6.4.4). As a consequence, the conclusions we have drawn on $\mathcal{X}^{\mathrm{nc}}$ over R will also be true for $(\mathcal{X}^{\mathrm{nc}})_{R'} = (\mathcal{X}_{R'})^{\mathrm{nc}}$ over R', for any extension $R \subseteq R'$. In particular, $\mathcal{X}^{\mathrm{nc}}$ falls within case L also after replacing R with an arbitrary finite extension; in light of Corollary 7.3.10, this implies that $\mathcal{X}^{\mathrm{nc}}$ falls within case LE over large enough extensions of R. □

Proposition 7.3.20. Suppose that Y has semistable reduction, and that $\mathcal{X}$ is a crushed smooth component of the ur-minimal model. Then, the distance of $\mathcal{X}$ from its anchor $\mathcal{X}^{\mathrm{nc}}$ is $\leq 2v(2)$.

Proof. If the maximal cluster $\mathfrak{s} = \mathcal{R}$ of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$ had depth $> 2v(2)$, then, after possibly taking a quadratic extension of R, $\mathcal{X}$ would fall in case H+ and $\mathcal{Y}_s$ would consist of a pair of lines meeting at a unique singular point (Proposition 7.3.17). But then Theorem 6.4.6 would prevent $\mathcal{X}$ from being a component of the ur-minimal model. □

Proposition 7.3.21. Suppose that Y has semistable reduction, and that $\mathcal{X}$ is a crushed smooth model of the line, falling within case H. Then, the following are equivalent:

(a) $\mathcal{X}$ is a component of the ur-minimal model;
(b) (assuming $g \geq 2$) $\mathcal{X}$ is a component of the ur-stable model;
(c) $\mathcal{Y}_s$ is an integral curve of abelian rank > 0.

Proof. It is obvious that (b) implies (a); moreover, it is an immediate consequence of Theorem 6.4.6 that (c) implies (a) and (b).

All that we have to show is thus that (a) implies (c): the proof will be a simple application of the criterion provided by Theorem 6.4.6, together with the explicit description we have given of $\mathcal{Y}$ while discussing case H.

Let $\mathfrak{s}$ denote the unique maximal cluster of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$; $\mathcal{Y}_s$ is only possibly singular at the unique point $Q \in \mathcal{Y}_s$ lying over $\bar{\mathfrak{s}}$. The number of branches $m(Q)$ of $\mathcal{Y}_s$ at Q (Definition 3.4.1) can either be equal to 1 or 2. If it were equal to 2, $\widetilde{\mathcal{Y}_s} \to \mathcal{X}_s$ would be étale and $\mathcal{Y}_s$ would consist of two lines meeting at Q: in light of Theorem 6.4.6, this clearly contradicts the hypothesis that $\mathcal{X}$ contributes to the ur-minimal model. Hence, $m(Q) = 1$ and $\widetilde{\mathcal{Y}_s} \to \mathcal{X}_s$ is a hyperelliptic curve. If $\widetilde{\mathcal{Y}_s}$ had genus 0, again Theorem 6.4.6 would prevent $\mathcal{X}$ from being part of the ur-minimal model; hence, the abelian rank of $\mathcal{Y}_s$ is > 0. □

Proposition 7.3.22. Suppose that Y has semistable reduction, and that its genus is ≥ 2. Suppose that $\mathcal{X}$ is a crushed smooth component of the ur-stable model falling within case LE. Then, in the tree of all smooth models of the line, $\mathcal{X}$ lies between two other crushed smooth components $\mathcal{X}_1$ and $\mathcal{X}_2$ of the ur-stable model.

Proof. Since $\mathcal{X}$ falls within case LE, the special fiber $\mathcal{Y}_s$ of its normalization $\mathcal{Y}$ in $K(Y)$ is an integral curve of abelian rank 0 curve whose singularities are all unibranch, providing a purely inseparable covering of $\mathcal{X}_s$. In light of Theorem 6.4.6, as $\mathcal{X}$ is assumed to be a component of the ur-stable model, $\mathcal{Y}_s$ must have at least three singular points. Hence, if $\mathfrak{s}$ is the unique maximal cluster of $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X})$, there exists at lest two points $P_1, P_2 \in \mathcal{X}_s$ different from $\bar{\mathfrak{s}}$ over which $\mathcal{Y}_s$ is singular. Now, Theorem 6.4.7 ensures that there exists two smooth components $\mathcal{X}_1$ and $\mathcal{X}_2$ of the ur-stable model meeting $\mathcal{X}$ at P_1 and P_2 respectively: it is clear that they are also necessarily crushed. □

Suppose now that the hyperelliptic curve Y has genus $g \geq 2$; let $\mathcal{X}'_{\mathrm{urst}}$ be its ur-stable model over some extension $R \subseteq R'$ after which Y acquires semistable reduction (the extension $R \subseteq R'$, a priori, is unknown). This is our strategy for determining $\mathcal{X}'_{\mathrm{urst}}$.

1. We scan through all unwound smooth models $\mathcal{X}$ of the line defined over finite extensions $R \subseteq R_{\mathcal{X}}$.

2. For each such model $\mathcal{X}$, first we move to an extension $R_{\mathcal{X}} \subseteq R'_{\mathcal{X}}$ that makes $\mathcal{X}$ fall within case LE or case H (Corollary 7.3.10); then, we compute the normalization $\mathcal{Y}$ of $\mathcal{X}$ in $K(Y)$, whose special fiber $\mathcal{Y}_s$ will be reduced. Proposition 6.4.2 ensures that such a normalization does not depend on the chosen extension $R_{\mathcal{X}} \subseteq R'_{\mathcal{X}}$.

3. Studying the structure of $\mathcal{Y}_s$ and its singularities, we can determine whether $\mathcal{X}$ contributes to $\mathcal{X}'_{\mathrm{urst}}$ (see Theorem 6.4.6), we can locate the (remaining) smooth components of $\mathcal{X}'_{\mathrm{urst}}$ with respect to $\mathcal{X}$ (Theorem 6.4.7), and we can describe the strict transform of $\mathcal{Y}_s$ in the special fiber of $\mathcal{Y}_{\mathrm{st}}$ (Proposition 6.4.8).

4. Whenever $\mathcal{X}$ falls within case LE, and $\mathcal{Y}$ happens to be singular above a point $P \in \mathcal{X}_s$ to which no point of $\mathcal{R}$ reduces, then P is the anchor of one or more crushed smooth models of the line contributing to the ur-stable model (Theorem 6.4.7). We thus have to scan through all crushed smooth models of the line (defined over extensions of R) anchored at P and study their normalization in $K(Y)$, in order to single out those contributing to the ur-stable model (via Theorem 6.4.6) and to study the vertical components of $\mathcal{Y}_{\mathrm{st}}$ they give rise to (via Proposition 6.4.8).

5. We can now be sure that all smooth components of $\mathcal{X}'_{\mathrm{urst}}$ have been found (Proposition 7.3.19).

To turn this procedure into a full algorithm capable of computing $\mathcal{Y}_{\mathrm{st}}$, we have to face two main difficulties. First, concerning point 1, we observe that, although there are only finitely many unwound smooth models of the line defined over R, those defined over arbitrary extensions of R are infinitely many. Secondly, a similar problem arises from point 4, since the crushed smooth models anchored at a given point of a given unwound smooth model of the line are also infinitely many. We will devote the remaining part of this subsection to sketching possible ways of overcoming these difficulties.

Scanning through unwound models Let $\mathcal{U}$ denote the collection of all unwound smooth models of the line defined over extensions of R. Although $\mathcal{U}$ is infinite, its elements can still be arranged in a finite number of families, one for each non-trivial cluster cut of $\mathcal{R}$. More precisely, we can associate to each non-trivial cluster cut $\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{s}'$ of $\mathcal{R}$ the collection $\mathcal{U}_{\mathfrak{s}\sqcup\mathfrak{s}'}$ of the unwound smooth models of the line in which $\mathfrak{s}$ or $\mathfrak{s}'$ play the role of maximal clusters: the total depth $d := (d_{\mathfrak{s}} + d_{\mathfrak{s}'})/v(2) > 0$ is constant within the family, while the individual depths $d_{\mathfrak{s}}/v(2) \in [0, d]$ and $d_{\mathfrak{s}'}/v(2) \in [0, d]$ can be used to parameterize $\mathcal{U}_{\mathfrak{s}\sqcup\mathfrak{s}'}$.

It is actually possible to determine a parametric totally odd $q\rho$-equation for Y valid for all models of a given family $\mathcal{U}_{\mathfrak{s}\sqcup\mathfrak{s}'}$, so that their normalization in $K(Y)$ also becomes reasonably simple to describe in a parametric way: we will now try to give an idea of how this can be done. Fix two branch points $P \in \mathfrak{s}$ and $P' \in \mathfrak{s}'$ and consider the smooth unwound model $\mathcal{X} \in \mathcal{U}_{\mathfrak{s}\sqcup\mathfrak{s}'}$ of the line such that $d_{\mathfrak{s}'} = 0$. If we choose the coordinate x of $\mathcal{X}$ properly, we may clearly obtain that $x(P) = 0$ and $x(P') = \infty$; in particular, a normalized 0∞-equation for Y can be written in the form $y^2 = f(x) = \sum_{i=1}^{2g+1} f_i x^i = f_0(x) f_\infty(x)$, where

$$f_0(x) = \prod_{A \in \mathfrak{s}} (x - x_A), \qquad f_\infty(x) = \prod_{A \in \mathfrak{s}'} (1 - x/x_A).$$

We will consider, for f, the totally odd $q\rho$-decomposition $f = q^2 + \rho$ we have built in Proposition 7.3.9.

To move from $\mathcal{X}$ to the other models $\mathcal{X}'$ of the family $\mathcal{U}_{\mathfrak{s}\sqcup\mathfrak{s}'}$, it is now clearly enough to perform a change of variable $x' := x/\omega$, where $\omega \in \overline{K}$ has valuation $v(\omega)/v(2) \in [0, d]$. It is not difficult to verify that, up to suitably changing y, we can write a normalized 0∞-equation $y^2 = f'(x') = f_0'(x') f_\infty'(x')$ for Y by setting

$$f'(x') := \omega^{-|\mathfrak{s}|} f(\omega x'), \quad f_0'(x') := \omega^{-|\mathfrak{s}|} f_0(\omega x'), \quad f_\infty'(x') := f_\infty(\omega x'),$$

and we can obtain a totally odd $q\rho$-decomposition $f' = (q')^2 + \rho'$ by setting

$$\begin{aligned} q'(x') &:= \omega^{-|\mathfrak{s}|/2} q(\omega x'), \\ \rho'(x') &:= \omega^{-|\mathfrak{s}|} \rho(\omega x'). \end{aligned}$$

In particular, the valuations of the coefficients of q' and ρ' can be written down as functions of $v(\omega)/v(2) \in [0, d]$ as follows:

$$\begin{aligned} v(q_i') &= v(q_i) + \left(i - \frac{|\mathfrak{s}|}{2}\right) v(\omega), \\ v(\rho_i') &= v(\rho_i) + (i - |\mathfrak{s}|)\, v(\omega). \end{aligned}$$

The valuations $v(q') := \min_i v(q_i')$ and $v(\rho') := \min_i v(\rho_i)$ will also be functions of $v(\omega)$. The models of $\mathcal{U}_{\mathfrak{s}\sqcup\mathfrak{s}'}$ we would like to single out are those that contribute to the ur-stable model, and those that are anchors of crushed components of the ur-stable model. Let us remark the following.

- If $v(\omega)/v(2) \in (0, d)$, $v(\rho) < 2v(2)$, and only one of the ρ_i's has valuation equal $v(\rho)$, then $\mathcal{X}'$ falls within case LE, and the special fiber of $\mathcal{Y}'$ is only possibly singular over $\overline{x}' = 0$ and $\overline{x}' = \infty$. This means that $\mathcal{X}'$ is neither part of the ur-stable model, nor the anchor of a crushed smooth component of the ur-stable model (Theorem 6.4.7).
- If $v(\omega)/v(2) \in (0, d)$ and $v(\rho) > 2v(2)$, then $\mathcal{X}'$ falls within case H+, and the special fiber of $\mathcal{Y}'$ consists of two lines that can only possibly intersect each other over $\overline{x}' = 0$ and $\overline{x}' = \infty$. In particular, $\mathcal{X}'$ is neither part of the ur-stable model, nor the anchor of a crushed smooth component of the ur-stable model (Theorem 6.4.7).

It is easy to see that the two cases discussed above capture together all but finitely many models of $\mathcal{U}_{\mathfrak{s} \sqcup \mathfrak{s}'}$; in other words, we are reduced to deal with finitely many, well-identifiable possibly interesting values for $v(\omega)/v(2)$.

Finding crushed components of $\mathcal{X}_{\mathbf{urst}}$ Let us now consider the following setting: we have an unwound smooth model of the line $\mathcal{X}$, falling within case LE, such that $\mathcal{Y}_s$ is singular over a point $P \in \mathcal{X}_s$ to which no branch point reduces. In light of Theorem 6.4.7, this means that $\mathcal{X}$ is the anchor of one or more crushed smooth components of the ur-stable model, which we want to determine.

If x denotes some coordinate on $\mathcal{X}$, the generic crushed smooth model anchored at P can be obtained from $\mathcal{X}$ by means of a change of coordinate $x_{\alpha,\beta} := \beta^{-1}(x - \alpha)$, where the coefficients $\alpha, \beta \in \overline{K}$ satisfy $\overline{\alpha} = P$ and $v(\beta) > 0$. We will denote as $\mathcal{X}^{\alpha,\beta}$ the crushed smooth model of the line corresponding to the coordinate $x_{\alpha,\beta}$; we clearly have that all points of $\mathcal{R}$ reduce to $\overline{x}_{\alpha,\beta} = \infty$ in $(\mathcal{X}^{\alpha,\beta})_s$.

For every possible choice of α, let us introduce, on $\mathcal{X}$, the translated coordinate $x_\alpha := x_{\alpha,1} = x - \alpha$: none of the branch points reduces to $\overline{x}_\alpha = 0$, and hence we may write down a normalized ∞-equation $y^2 = f_\alpha(x_\alpha)$ for Y and compute a totally odd $q\rho$-decomposition $f_\alpha(x_\alpha) = q_\alpha^2(x_\alpha) + \rho_\alpha(x_\alpha)$. Now, if we switch to the coordinate $x_{\alpha,\beta}$, the hyperelliptic curve Y is represented by the normalized ∞-equation $y^2 = f_{\alpha,\beta}(x_{\alpha,\beta})$ with $f_{\alpha,\beta}(x_{\alpha,\beta}) := f_\alpha(\beta x_{\alpha,\beta})$, and a totally odd $q\rho$-decomposition $f_{\alpha,\beta} = q_{\alpha,\beta}^2 + \rho_{\alpha,\beta}$ is obtained by setting $q_{\alpha,\beta}(x_{\alpha,\beta}) := q_\alpha(\beta x_{\alpha,\beta})$ and $\rho_{\alpha,\beta}(x_{\alpha,\beta}) := \rho_\alpha(\beta x_{\alpha,\beta})$. We remark that $v(\rho_{\alpha,\beta})$ is a continuous, strictly increasing and unbounded function of $v(\beta)$; in particular, there exists a threshold $B(\alpha) > 0$ such that $\mathcal{X}^{\alpha,\beta}$ stays in case LE for $v(\beta) < B(\alpha)$ and falls in case H for $v(\beta) \geq B(\alpha)$. By Proposition 7.3.20, we know that $B(\alpha) \leq 2v(2)$.

As the dependence of $q_{\alpha,\beta}$ and $\rho_{\alpha,\beta}$ on β is quite simple, we have that, for every fixed α, the crushed models corresponding to all possible choices of β can easily be studied together. All we have to do now is to find a convenient way of choosing α.

We observe that, if for some choice α we have $\rho'_\alpha(0) = 0$, then we also have $\rho'_{\alpha,\beta}(0) = 0$ for all β. As far as $v(\beta) < B(\alpha)$, $\rho'_\alpha(0) = 0$ implies that $(\mathcal{Y}^{\alpha,\beta})_s$ has a unibranch singularity at $\overline{x}_{\alpha,\beta} = 0$, which means that some crushed component of $\mathcal{X}_{\mathrm{urst}}$ must meet $\mathcal{X}^{\alpha,\beta}$ at $\overline{x}_{\alpha,\beta} = 0$ (this is a consequence of Theorem 6.4.7). In particular, there exists a crushed component $\mathcal{X}'_\alpha$ of $\mathcal{X}_{\mathrm{urst}}$ that, for all β with $v(\beta) < B(\alpha)$, meets $\mathcal{X}^{\alpha,\beta}$ at $\overline{x}_{\alpha,\beta} = 0$. For $v(\beta) = B(\alpha)$, $\mathcal{X}^{\alpha,\beta}$ falls instead within case H, and hence it can only meet $\mathcal{X}'_\alpha$ at $\overline{x}_{\alpha,\beta} = \infty$: this is easily seen to be a contradiction, from which we conclude that $\mathcal{X}^{\alpha,\beta}$, when $v(\beta) = B(\alpha)$, cannot meet $\mathcal{X}'_\alpha$, and hence it must coincide with it.

To summarize, for every chosen α for which $\rho'_\alpha(0) = 0$, we have found a uniquely determined crushed component $\mathcal{X}'_\alpha = \mathcal{X}^{\alpha,B(\alpha)}$ of $\mathcal{X}_{\mathrm{urst}}$ falling within case H: this approach to find crushed H-components of $\mathcal{X}_{\mathrm{urst}}$ is the one we will concretely use in the examples we have collected in the next section. Although we will not prove it, it is actually reasonable to expect that all crushed H-components of $\mathcal{X}_{\mathrm{urst}}$ can always be found in this way. Once the crushed H-components have all been determined, those falling within case LE are easier to detect thanks to Proposition 7.3.22.

7.4. Some computations over residue characteristic 2

7.4.1. Elliptic curves

If $Y \to X$ is an elliptic curve, the unwound concrete cluster pictures of the four branch points $\mathcal{R} \subseteq X(K)$ have all clearly the form:

$$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}) : \ \boxed{A_1 \qquad B_1}^{\,m_1} \qquad\qquad \boxed{A_2 \qquad B_2}^{\,m_2}$$

where the depths $m_1, m_2 \geq 0$ of $\mathfrak{s}_1 = \{A_1, B_1\}$ and $\mathfrak{s}_2 = \{A_2, B_2\}$ depend on the chosen unwound smooth model $\mathcal{X}$ of the line, but their sum $m := m_1 + m_2$ remains constant (and is hence an intrinsic property of the elliptic curve $Y \to X$).

Deep clusters

If we suppose that $m \geq 4v(2)$, $\mathcal{X}$ can be chosen so that m_1 and m_2 are both $\geq 2v(2)$. For any such $\mathcal{X}$, after possibly a replacing R with a suitable quadratic extension, we have that (via Proposition 7.3.17 and Remark 7.3.18)

- $\mathcal{X}$ falls within case H;
- $\mathcal{Y}$ is a semistable model;
- if the coordinate x on $\mathcal{X}$ is chosen so that no point of $\mathcal{R}$ reduces to $\overline{x} = \infty$, and we name $-(a_1), -(a_1+\delta_1), -(a_2), -(a_2+\delta_2)$ the x-coordinates of the four points of $\mathcal{R}$, $\mathcal{Y}_s$ is described by an equation of the form
$$y^2 + (x+\overline{a}_1)(x+\overline{a}_2)y = \left[\overline{\delta_1/4}(x+\overline{a}_2) + \overline{\delta_2/4}(x+\overline{a}_1)\right](x+\overline{a}_1)(x+\overline{a}_2);$$
- if m_1 and m_2 are both $> 2v(2)$, then $\mathcal{X}$ falls within case H+, and $\mathcal{Y}_s$ consists of a pair of lines meeting over $\bar{\mathfrak{s}}_1$ and $\bar{\mathfrak{s}}_2$;
- if $m_1 = 2v(2)$ but $m_2 > 2v(2)$, then $\mathcal{Y}_s$ is integral and has a unique node lying over $\bar{\mathfrak{s}}_2$; its normalization $\widetilde{\mathcal{Y}_s}$ is a line providing a double generically étale covering of $\mathcal{X}_s$;
- if $m_1 = 2v(2)$ and $m_2 = 2v(2)$, then $\mathcal{Y}_s$ is an elliptic curve.

We conclude, in particular, that, if $m = 4v(2)$, then $\mathcal{X}$ has potential good reduction, while, if $m > 4v(2)$, it doesn't.

Shallow clusters

Let us now address the case $0 \leq m < 4v(2)$: we will essentially follow the computations of [Yel19, §2]. After possibly changing R to a quadratic extension, we can suppose that m is even, and we can choose $\mathcal{X}$ to be the unwound smooth model whose associated concrete cluster picture is

$$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}) : \boxed{A_1 \qquad B_1}^{m/2} \qquad\qquad \boxed{A_2 \qquad B_2}^{m/2}$$

If we choose the coordinate x on $\mathcal{X}$ properly, we may clearly assume that

$$\begin{array}{lll} x(A_1) = 0 & x(A_2) = \infty & \text{where } \lambda, \mu \in K, \text{ and} \\ x(B_1) = -\lambda & x(B_2) = -1/\mu & v(\lambda) = v(\mu) = m/2. \end{array}$$

After possibly changing R to a further quadratic extension, we may write a normalized 0∞-equation for Y as

$$y^2 = f(x) \qquad f(x) := x(x+\lambda)(1+\mu x) = \mu x^3 + (1+\lambda\mu)x^2 + \lambda x$$

If we let $q(x) := \left(\sqrt{1+\lambda\mu}\right) x$ and $\rho(x) := x(\mu x^2 + \lambda)$, we obtain a totally odd $q\rho$-decomposition $f(x) = q^2(x) + \rho(x)$ with $v(\rho) = m/2$; supposing $m/2$ even (which is certainly true after a further quadratic extension of R), we are thus in case LE. As a consequence, after performing a coordinate change $y \mapsto (y - q(x))/\pi^{m/4}$, we can describe $\mathcal{Y}$ by means of an equation of the form

$$y^2 + 2\pi^{-m/2} q(x) = r(x), \qquad \text{where } r(x) := \pi^{-m/2}\rho(x) = \pi^{-m/2} x(\mu x^2 + \lambda),$$

which reduces to

$$y^2 = \overline{r}(x), \qquad \text{where } \overline{r}(x) = x\left[(\overline{\pi^{-m/2}\mu})x^2 + (\overline{\pi^{-m/2}\lambda})\right].$$

We have that $\mathcal{Y}_s \to \mathcal{X}_s$ is an inseparable quadratic covering map and $\mathcal{Y}_s$ has a unique (unibranch) singular point, lying over the unique zero $P \in \mathcal{X}_s$ of $\overline{r}'$, whose coordinate is

$$\overline{x}(P) = \left(\overline{\lambda/\mu}\right)^{1/2}.$$

It is immediate to see that P does not coincide with any of the points $\overline{A}_1, \overline{A}_2, \overline{B}_1, \overline{B}_2 \in \mathcal{X}_s$. Via Theorem 6.4.7, we can thus conclude that (provided that we change R with some extension over which Y has semistable reduction) any smooth model of the line contributing to the ur-minimal model of X is a crushed model anchored at $P \in \mathcal{X}_s$; in other words, it will be obtained from $\mathcal{X}$ by means of a change of variable $x_1 = \beta^{-1}(x - \alpha)$ for some coefficients α, β such that α reduces to $\overline{x}(P)$ (in particular, $v(\alpha) = 0$), and $0 < v(\beta) \leq 2v(2)$ (see Proposition 7.3.20). If we name $\mathcal{X}^1$ the smooth model of the line corresponding to the coordinate x_1, the cluster picture associated to it will have the following shape:

$$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}^1) : \boxed{\boxed{A_1 \qquad B_1}^{m/2} \qquad\qquad \boxed{A_2 \qquad B_2}^{m/2}}^{v(\beta)}$$

The branch points will now have coordinates

$$x_1(A_1) = -\beta^{-1}\alpha \quad x_1(B_1) = -\beta^{-1}(\alpha+\lambda) \quad x_1(A_2) = -\beta^{-1}(\alpha+1/\mu) \quad x_1(B_2) = \infty,$$

so, after suitably replacing y, we may write a normalized ∞-equation of the elliptic curve in the form

$$y^2 = f_1(x_1), \qquad f_1(x_1) = \left(1 + \frac{\beta}{\alpha}x_1\right)\left(1 + \frac{\beta}{\alpha+\lambda}x_1\right)\left(1 + \frac{\beta}{\alpha+1/\mu}x_1\right).$$

If we now denote by $P_i(x)$ the elementary symmetric degree-i polynomial in three variables evaluated at $(x, x+\lambda, x+1/\mu)$, and we let $p_i := P_i(\alpha)$, we can rewrite f_1 in the form

$$f_1(x_1) = 1 + \beta\frac{p_2}{p_3}x_1 + \beta^2\frac{p_1}{p_3}x_1^2 + \beta^3\frac{1}{p_3}x_1^3,$$

where the p_i's can be explicitly computed as

$$\begin{aligned}
&p_1 = 3\alpha + \lambda + 1/\mu && v(p_1) \geq -m/2 \text{ (equality holds if } m > 0),\\
&p_2 = 3\alpha^2 + 2(\lambda + 1/\mu)\alpha + \lambda/\mu && v(p_2) \geq \min(-m/2 + v(2), 0),\\
&p_3 = \alpha^3 + (\lambda + 1/\mu)\alpha^2 + (\lambda/\mu)\alpha && v(p_3) = -m/2.
\end{aligned}$$

If we approximate $f_1(x_1)$ with the square of the polynomial

$$q_1(x_1) := 1 + \beta a x_1,$$

where the choice of the parameter a is deferred, the error $\rho_1(x_1) := f_1(x_1) - q_1^2(x_1)$ equals

$$\rho_1(x_1) = \beta\left(\frac{p_2}{p_3} - 2a\right)x_1 + \beta^2\left(\frac{p_1}{p_3} - a^2\right)x_1^2 + \beta^3\left(\frac{1}{p_3}\right)x_1^3.$$

We want $f_1(x_1) = q_1^2(x_1) + \rho_1(x_1)$ to be a totally odd $q\rho$-decomposition having $\rho_1'(0) = 0$: in other words, we want that $2ap_3 = p_2$ and $a^2p_3 = p_1$, which is equivalent to ask that

$$4p_1p_3 = p_2^2, \qquad a = \sqrt{p_1/p_3} = p_2/(2p_3). \tag{†}$$

The first equation can be interpreted as a condition on α, in the sense that it is satisfied if α is chosen to be a root of the polynomial $F(x) := P_2^2(x) - 4P_1(x)P_3(x)$. An explicit computation shows that

$$F(x) = -3x^4 - 4(\lambda + 1/\mu)x^3 - 6(\lambda/\mu)x^2 + (\lambda/\mu)^2$$

whence it is clear that $F(x)$ has integral coefficients, and that its roots all reduce to $P \in \mathcal{X}_s$: it is thus legitimate to choose α to be a root of $F(x)$.

Now that α is fixed, we can choose a as prescribed by the second equation of (†): since $v(p_1) \geq -m/2$ and $v(p_3) = -m/2$, we have that $v(a) \geq 0$ (equality holds if $m > 0$), and hence $q_1(x_1)$ certainly has integral coefficients. To summarize, we have found a $q\rho$-decomposition $f_1(x_1) = q_1^2(x_1) + \rho_1(x_1)$ with

$$q_1(x_1) = (1 + \beta a x_1) \in R[x], \qquad \rho_1(x_1) = \frac{\beta^3}{p_3} x_1^3 \in R[x].$$

We will now choose β to have valuation $v(\beta) = (4 - m)/6$, so that $v(\rho) = 2v(2)$. For this choice of β, $\mathcal{X}^1$ falls within case H, and hence the equation of $\mathcal{Y}^1$ is obtained by performing the change of variable $y \mapsto (y - q(x_1))/2$, which gives

$$y^2 + (1 + \beta a x_1)\, y = \frac{\beta^3}{4p_3} x_1^3, \qquad v(a) \geq 0, \qquad v\left(\frac{\beta^3}{4p_3}\right) = 0.$$

In the reduction, we obtain the elliptic curve $y^2 + y = \overline{\beta^3/(4p_3)} x_1^3$; in particular, $\mathcal{Y}^1$ is a smooth model of Y, and Y consequently has potential good reduction. The generically étale covering map $\mathcal{Y}_s \to \mathcal{X}_s$ only branches over $\overline{x}_1 = \infty$, i.e. over the common reduction in $(\mathcal{X}^1)_s$ of A_1, A_2, B_1, B_2.

7.4.2. Genus 2

Let us now address the case in which the hyperelliptic curve $Y \to X$ has genus 2. The special fiber of the stable model $\mathcal{Y}_{\mathrm{st}}$ must be a semistable genus-2 curve whose genus-0 components intersect the rest of the fiber at no less than 3 points. It is easy to see that the number of possibilities is quite limited: more precisely, $(\mathcal{Y}_{\mathrm{st}})_s$ can only be (see [Liu, Example 10.3.6]):

(a) a hyperelliptic curve of genus 2;
(b) an integral curve of abelian rank 1, with one node;
(c) an integral curve of abelian rank 0, with two nodes;
(d) a pair of genus-1 curves meeting at single node, where each of the genus-1 components can either be
  (i) an elliptic curve, or
  (ii) an integral curve of abelian rank 0 with a node;
(e) a pair of lines intersecting each other at three nodes.

The ur-stable model $\mathcal{X}_{\mathrm{urst}}$, in particular, consists of either one or two smooth components. Moreover, we have that

Proposition 7.4.1. If Y has genus 2, and $\mathcal{X}$ is a smooth model of the line falling within case LE, then $\mathcal{X}$ cannot contribute to the ur-stable model.

Proof. Let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ in $K(Y)$: we know that the special fiber $\mathcal{Y}_s$ is an integral curve of abelian rank 0 whose singularities are all unibranch and which is described by an equation of the form $y^2 = \overline{r}(x)$, where $\overline{r}$ has degree at most 6; its derivative $\overline{r}'$ will hence be the square of a polynomial of degree at most 2. As a consequence, $\mathcal{Y}_s$ cannot have more than two singular points, and Theorem 6.4.6 thus ensures that $\mathcal{X}$ cannot contribute to $\mathcal{X}_{\mathrm{urst}}$. $\square$

The components of $\mathcal{X}_{\mathrm{urst}}$ must therefore be searched for among those smooth models of X that fall (after possibly extending the base ring R) within case H. In particular, if $\mathcal{X}$ is a smooth component of $\mathcal{X}_{\mathrm{urst}}$, and $\mathcal{Y}$ denotes the normalization of $\mathcal{X}$ in $K(Y)$, we have, combining the description of case H we have provided in Subsection 7.3.3 with Theorems 6.4.6 and 6.4.7 and Proposition 6.4.8, that those listed below are the only possible cases.

- $\mathcal{X}$ is crushed (i.e. it has a unique maximal cluster $\mathfrak{s} = \mathcal{R}$), $\mathcal{Y}_s$ is an integral curve, and one of the the two following sub-cases occurs.
  - $\mathcal{Y}_s$ is a hyperelliptic curve of genus 2; in this case, $\mathcal{Y} = \mathcal{Y}_{\mathrm{st}}$ is a smooth model.
  - $\mathcal{Y}_s$ is singular over $\overline{\mathfrak{s}}$: in this case, $\mathcal{Y}$ is strictly dominated by $\mathcal{Y}_{\mathrm{st}}$, and the strict transform of $\mathcal{Y}_s$ in $(\mathcal{Y}_{\mathrm{st}})_s$ is an elliptic curve.
- $\mathcal{X}$ has two maximal clusters (one of cardinality 2, $\mathfrak{s}_1$, the other of cardinality 4, $\mathfrak{s}_2$); in this case, $\mathcal{Y}_s$ is an integral curve, and one of the four following sub-cases occurs.
  - $\mathcal{Y}_s$ is a hyperelliptic curve of genus 2, so $\mathcal{Y} = \mathcal{Y}_{\mathrm{st}}$ is a smooth model.
  - $\mathcal{Y}_s$ is smooth over $\overline{\mathfrak{s}}_2$ but not over $\overline{\mathfrak{s}}_1$: in this case, $\mathcal{Y} = \mathcal{Y}_{\mathrm{st}}$, and $\mathcal{Y}_s$ is an integral curve of abelian rank 1 with a node.
  - $\mathcal{Y}_s$ is smooth over $\overline{\mathfrak{s}}_1$ but not over $\overline{\mathfrak{s}}_2$: in this case, $\mathcal{Y}$ is strictly dominated by $\mathcal{Y}_{\mathrm{st}}$, and the strict transform of $\mathcal{Y}_s$ in $(\mathcal{Y}_{\mathrm{st}})_s$ is an elliptic curve.
  - $\mathcal{Y}_s$ is singular over $\overline{\mathfrak{s}}_1$ and $\overline{\mathfrak{s}}_2$: in this case, $\mathcal{Y}$ is strictly dominated by $\mathcal{Y}_{\mathrm{st}}$, and the strict transform of $\mathcal{Y}_s$ in $(\mathcal{Y}_{\mathrm{st}})_s$ is an integral curve of abelian rank 0 with a node.
- $\mathcal{X}$ has three maximal clusters $\overline{\mathfrak{s}}_1$, $\overline{\mathfrak{s}}_2$ and $\overline{\mathfrak{s}}_3$ (all of cardinality 2); in this case, $\mathcal{Y} = \mathcal{Y}_{\mathrm{st}}$, and, up to resorting the three maximal clusters, one of the four following sub-cases occurs.
  - $\mathcal{Y}_s$ is a hyperelliptic curve of genus 2 and $\mathcal{Y} = \mathcal{Y}_{\mathrm{st}}$ is a smooth model.
  - $\mathcal{Y}_s$ is smooth over $\overline{\mathfrak{s}}_1$ and $\overline{\mathfrak{s}}_2$, but not over $\overline{\mathfrak{s}}_3$: in this case, $\mathcal{Y}_s$ is an integral curve of abelian rank 1 with a node on it.

- $\mathcal{Y}_s$ is smooth over $\bar{\mathfrak{s}}_1$, but not over $\bar{\mathfrak{s}}_2$ and $\bar{\mathfrak{s}}_3$: in this case, $\mathcal{Y}_s$ is an integral curve of abelian rank 0 with two nodes on it.
- $\mathcal{Y}_s$ is singular over $\bar{\mathfrak{s}}_1$, $\bar{\mathfrak{s}}_2$ and $\bar{\mathfrak{s}}_3$: in this case, $\mathcal{Y}_s$ consists of two lines meeting each other at three nodes.

We will now try to understand what the stable model $\mathcal{Y}_{\mathrm{st}}$ of the hyperelliptic curve looks like depending on the number and sizes of the cluster cuts of $\mathcal{R}$ (Definition 6.3.12). The only possibilities are those listed in the following table.

Name	Sizes of non-trivial cluster cuts	Subsection
No cluster		Subsection 7.4.3
Three of a kind	3/3	Subsection 7.4.4
One pair	2/4	Subsection 7.4.5
One pair and one triple	2/4 and 3/3	Subsection 7.4.6
Two pairs	2/4 and 2/4	Subsection 7.4.7
Two pairs and one triple	2/4, 2/4 and 3/3	Subsection 7.4.7
Three pairs	2/4, 2/4 and 2/4	Subsection 7.4.8

7.4.3. Genus 2, no cluster

In the no cluster case, the unique unwound smooth model $\mathcal{X}$ of X is the one corresponding to the trivial cluster picture

$$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}): \quad A_0 \qquad A_1 \qquad A_2 \qquad A_3 \qquad A_4 \qquad A_5$$

We may clearly choose the coordinate x of $\mathcal{X}$ in a way that

$$x(A_0) = \infty \qquad x(A_i) = -\lambda_i \text{ for } i = 1, \dots, 5$$

where λ_i are some elements of R. Up to possibly a quadratic extension of R, we may write a normalized 0-equation of Y in the form

$$y^2 = f(x) \qquad f(x) := (x+\lambda_1)\cdots(x+\lambda_5) = x^5 + \Lambda_1 x^4 + \ldots + \Lambda_5$$

where Λ_i is the elementary symmetric polynomial of degree i in 5 variables, evaluated at $(\lambda_1, \dots, \lambda_5)$. We are clearly under the hypotheses of Proposition 7.3.12: hence, f has a LE-decomposition of the form $f(x) = q(x)^2 + \rho(x)$ with $q(x) := 0$ and $\rho(x) := f(x)$; the special fiber $\mathcal{Y}_s$ of the normalization $\mathcal{Y}$ of $\mathcal{X}$ in $K(Y)$ is described by an equation of the form $y^2 = \overline{f}(x)$, and its singular points coincide with the zeros of $\overline{f}'$; since the cluster picture is trivial and $\overline{f}$ is consequently square-free, none of them can coincide with any of the $\overline{A}_i \in \mathcal{X}_s$. We conclude from this that all smooth components of the ur-stable model are crushed (Theorem 6.4.7), and that in particular $(\mathcal{Y}_{\mathrm{st}})_s$ is consequently either a hyperelliptic genus-2 curve, or a pair of elliptic curves crossing at a single node (see the discussion in Subsection 7.4.2).

To seek the (crushed) smooth components of the ur-stable model, we will now have to study the smooth models $\mathcal{X}^1$ that can be obtained from $\mathcal{X}$ by means of a change of variable $x_1 = \beta^{-1}(x - \alpha)$, where $\overline{\alpha} \in \mathcal{X}_s$ is a root of $\overline{f}'$, and $0 < v(\beta) \leq 2v(2)$. Up to suitably replacing y, we can write a normalized ∞-equation for Y in the form

$$y^2 = f_1(x_1) \qquad f_1(x_1) = \left(1 + \frac{\beta}{\alpha + \lambda_1} x_1\right) \cdots \left(1 + \frac{\beta}{\alpha + \lambda_5} x_1\right).$$

If we now denote by $P_i(x)$ the elementary symmetric polynomial of degree i in 5 variables evaluated at $(x + \lambda_1, \ldots, x + \lambda_5)$, and we let $p_i := P_i(\alpha)$, we may rewrite f_1 as

$$f_1(x_1) = 1 + \beta \frac{p_4}{p_5} x_1 + \beta^2 \frac{p_3}{p_5} x_1^2 + \beta^3 \frac{p_2}{p_5} x_1^3 + \beta^4 \frac{p_1}{p_5} x_1^4 + \beta^5 \frac{1}{p_5} x_1^5.$$

Since $\overline{\alpha}$ does not coincide with any of the $\overline{A}_i$'s, we have that $v(p_5) = 0$. Let us now look for a $q\rho$-decomposition of f_1: if we introduce $q_1(x_1) := 1 + c_1\beta x_1 + c_2\beta^2 x_1^2$, where the constants $c_1, c_2 \in K$ are still to be determined, we can decompose $f_1(x_1)$ as $q_1^2(x_1) + \rho_1(x_1)$, where

$$\begin{aligned}\rho_1(x_1) = &\beta\left(\frac{p_4}{p_5} - 2c_1\right) x_1 + \beta^2\left(\frac{p_3}{p_5} - c_1^2 - 2c_2\right) x_1^2 + \beta^3\left(\frac{p_2}{p_5} - 2c_1c_2\right) x_1^3 + \\ &\beta^4\left(\frac{p_1}{p_5} - c_2^2\right) x_1^4 + \beta^5\left(\frac{1}{p_5}\right) x_1^5.\end{aligned}$$

We want our $q\rho$-decomposition to be totally odd and to satisfy $\rho_1'(0) = 0$: hence, we will force the coefficients of x_1, x_1^2 and x_1^4 to be equal to zero. Some brief computations show that this is equivalent to asking that

$$p_4^4 - 8p_3p_5p_4^2 + 16p_3^2p_5^2 - 64p_1p_5^3 = 0, \qquad c_1 = p_4/(2p_5), \qquad c_2 = \sqrt{p_1/p_5}. \qquad (\dagger)$$

We will interpret the first of the three equations as a condition on α; in other words, we will choose α to be a root of the degree-16 polynomial

$$F := P_4^4 - 8P_3P_5P_4^2 + 16P_3^2P_5^2 - 64P_1P_5^3$$

whose coefficients are reported in table 7.1. Notice that $F(x) \in R[x]$ reduces to

$\overline{P}_4(x)^4$, and $P_4(x)$ is nothing but $f'(x)$: hence, our choice of α actually fulfills the requirement $\overline{f}'(\overline{\alpha}) = 0$ we had formulated at the beginning; looking at the form of F, it is actually immediate to realize that the stronger result $v(p_4) \geq v(2)$ holds.

Now that α has been chosen, we may use the two remaining equations of (†) to fix c_1 and c_2. We remark that, since $v(p_5) = 0$, $v(p_4) \geq v(2)$ and $v(p_1) \geq 0$, we can easily guarantee that $q_1(x_1) \in R[x_1]$. To summarize, we have obtained, for every chosen $\beta \in R$, a $q\rho$-decomposition $f_1(x_1) = q_1^2(x_1) + \rho_1(x_1)$ of f_1, where

$$\begin{aligned} q_1(x_1) =& 1 + \beta \frac{p_4}{2p_5} x_1 + \beta^2 \frac{\sqrt{p_1}}{\sqrt{p_5}} x_1^2, \\ \rho_1(x_1) =& \beta^3 R_3 x_1^3 + \beta^5 R_5 x_1^5, \qquad R_3 := \frac{p_2\sqrt{p_5} - p_4\sqrt{p_1}}{\sqrt{p_5}^3}, \qquad R_5 := \frac{1}{p_5}, \end{aligned}$$

with $v(p_1) \geq 0$, $v(p_2) \geq 0$, $v(p_4) \geq v(2)$, $v(p_5) = 0$. If we let κ denote the valuation of R_3, we clearly have that $\kappa \geq 0$, and the two coefficients of ρ_1 have valuations $\kappa + 3v(\beta)$ and $5v(\beta)$ respectively.

Remark 7.4.2. Observe that the construction of F commutes with translations and unit scalings of the coordinate x, in the sense that if $x_a := u(x + \delta)$ for some $u \in R^\times$ and $\delta \in R$, then $F_a(x_a) = F(u(x + \delta))$.

Remark 7.4.3. By looking at the expression of R_3, we easily deduce that $\kappa < 1$ if and only if $v(p_2) < 1$, and that, whenever the two equivalent conditions hold, we have $\kappa = v(p_2)$. But an explicit computation also shows that $p_2 = 10\alpha^2 + 4\Lambda_1\alpha + \Lambda_2$, which allows us to further extend our chain of equalities: we can be sure that $\kappa = v(p_2) = v(\Lambda_2)$, provided that one of them is $< v(2)$.

We should now distinguish two cases depending on the value of κ.

High κ case Suppose $\kappa \geq \frac{4}{5}v(2)$. Then, if we choose β as any element of valuation $v(\beta) = \frac{2}{5}v(2)$, we obtain that $v(\beta^5 R_5) = 2v(2)$, while $v(\beta^3 R_3) \geq 2v(2)$ (the strict inequality holds if and only if $\kappa > \frac{4}{5}v(2)$). Hence, $f_1 = q_1^2 + \rho_1$ is an H-decomposition, and after a change of variable $y \mapsto (y - q_1(x_1))/2$ we may write the equation of $(\mathcal{Y}^1)_s$ as

$$y^2 + y = \left(\overline{\beta^3 R_3/4}\right) x_1^3 + \left(\overline{\beta^5 R_5/4}\right) x_1^5,$$

where $\overline{\beta^5 R_5/4} \neq 0$, whence it readily follows that $\mathcal{Y}_s$ is a hyperelliptic curve of genus 2: we have found that Y has potential good reduction, and $\mathcal{Y}_{\mathrm{st}} = \mathcal{Y}^1$.

x^0	$(+1\Lambda_4^4 - 2^3\Lambda_3\Lambda_4^2\Lambda_5 + 2^4\Lambda_3^2\Lambda_5^2 - 2^6\Lambda_1\Lambda_5^3)$
x^1	$(-2^3\cdot 3\Lambda_2\Lambda_4^2\Lambda_5 + 2^5\cdot 3\Lambda_2\Lambda_3\Lambda_5^2 - 2^6\cdot 3\Lambda_1\Lambda_4\Lambda_5^2 - 2^6\cdot 5\Lambda_5^3)$
x^2	$(-2^2\cdot 3\Lambda_2\Lambda_4^3 + 2^4\cdot 3\Lambda_2\Lambda_3\Lambda_4\Lambda_5 - 2^4\cdot 15\Lambda_1\Lambda_4^2\Lambda_5 + 2^4\cdot 9\Lambda_2^2\Lambda_5^2 - 2^6\cdot 15\Lambda_4\Lambda_5^2)$
x^3	$(-2^3\Lambda_2\Lambda_3\Lambda_4^2 - 2^5\cdot 3\Lambda_1\Lambda_4^3 + 2^5\Lambda_2\Lambda_3^2\Lambda_5 + 2^4\cdot 9\Lambda_2^2\Lambda_4\Lambda_5 - 2^8\Lambda_1\Lambda_3\Lambda_4\Lambda_5 + 2^7\cdot 3\Lambda_1\Lambda_2\Lambda_5^2 - 2^4\cdot 65\Lambda_4^2\Lambda_5 - 2^7\cdot 5\Lambda_3\Lambda_5^2)$
x^4	$(+2^1\cdot 15\Lambda_2^2\Lambda_4^2 - 2^3\cdot 27\Lambda_1\Lambda_3\Lambda_4^2 + 2^3\cdot 15\Lambda_2^2\Lambda_3\Lambda_5 - 2^5\cdot 3\Lambda_1\Lambda_3^2\Lambda_5 + 2^5\cdot 9\Lambda_1\Lambda_2\Lambda_4\Lambda_5 + 2^7\cdot 3\Lambda_1^2\Lambda_5^2 - 2^2\cdot 95\Lambda_4^3 - 2^4\cdot 105\Lambda_3\Lambda_4\Lambda_5)$
x^5	$(+2^4\cdot 3\Lambda_2^2\Lambda_3\Lambda_4 - 2^6\cdot 3\Lambda_1\Lambda_3^2\Lambda_4 - 2^3\cdot 3\Lambda_1\Lambda_2\Lambda_4^2 + 2^3\cdot 9\Lambda_2^3\Lambda_5 + 2^6\cdot 3\Lambda_1\Lambda_2\Lambda_3\Lambda_5 + 2^7\cdot 3\Lambda_1^2\Lambda_4\Lambda_5 - 2^4\cdot 63\Lambda_3\Lambda_4^2 - 2^8\cdot 3\Lambda_3^2\Lambda_5 - 2^4\cdot 45\Lambda_2\Lambda_4\Lambda_5 + 2^8\cdot 3\Lambda_1\Lambda_5^2)$
x^6	$(+2^4\Lambda_2^2\Lambda_3^2 - 2^6\Lambda_1\Lambda_3^3 + 2^2\cdot 9\Lambda_2^3\Lambda_4 - 2^4\cdot 7\Lambda_1\Lambda_2\Lambda_3\Lambda_4 + 2^4\cdot 3\Lambda_1^2\Lambda_4^2 + 2^4\cdot 15\Lambda_1\Lambda_2^2\Lambda_5 + 2^8\Lambda_1^2\Lambda_3\Lambda_5 - 2^6\cdot 15\Lambda_3^2\Lambda_4 - 2^2\cdot 161\Lambda_2\Lambda_4^2 - 2^4\cdot 53\Lambda_2\Lambda_3\Lambda_5 + 2^5\cdot 13\Lambda_1\Lambda_4\Lambda_5 + 2^7\cdot 5\Lambda_5^2)$
x^7	$(+2^3\cdot 3\Lambda_2^3\Lambda_3 - 2^5\cdot 3\Lambda_1\Lambda_2\Lambda_3^2 + 2^4\cdot 3\Lambda_1\Lambda_2^2\Lambda_4 + 2^7\cdot 3\Lambda_1^2\Lambda_2\Lambda_5 - 2^6\cdot 5\Lambda_3^3 - 2^4\cdot 87\Lambda_2\Lambda_3\Lambda_4 - 2^4\cdot 15\Lambda_1\Lambda_4^2 - 2^6\cdot 3\Lambda_2^2\Lambda_5 + 2^5\cdot 15\Lambda_4\Lambda_5)$
x^8	$(+9\Lambda_2^4 - 2^3\cdot 3\Lambda_1\Lambda_2^2\Lambda_3 - 2^4\cdot 3\Lambda_1^2\Lambda_3^2 + 2^5\cdot 3\Lambda_1^2\Lambda_2\Lambda_4 + 2^6\cdot 3\Lambda_1^3\Lambda_5 - 2^8\cdot 3\Lambda_2\Lambda_3^2 - 2^2\cdot 129\Lambda_2^2\Lambda_4 - 2^4\cdot 51\Lambda_1\Lambda_3\Lambda_4 + 2^5\cdot 9\Lambda_1\Lambda_2\Lambda_5 - 2^1\cdot 45\Lambda_4^2 + 2^3\cdot 15\Lambda_3\Lambda_5)$
x^9	$(+2^3\Lambda_1\Lambda_2^3 - 2^5\Lambda_1^2\Lambda_2\Lambda_3 + 2^6\Lambda_1^3\Lambda_4 - 2^7\cdot 5\Lambda_2^2\Lambda_3 - 2^6\cdot 9\Lambda_1\Lambda_3^2 - 2^4\cdot 37\Lambda_1\Lambda_2\Lambda_4 + 2^6\cdot 7\Lambda_1^2\Lambda_5 - 2^5\cdot 15\Lambda_3\Lambda_4 + 2^3\cdot 25\Lambda_2\Lambda_5)$
x^{10}	$(-2^2\cdot 47\Lambda_2^3 - 2^4\cdot 63\Lambda_1\Lambda_2\Lambda_3 - 2^5\cdot 3\Lambda_1^2\Lambda_4 - 2^7\cdot 3\Lambda_3^2 - 2^2\cdot 105\Lambda_2\Lambda_4 + 2^4\cdot 33\Lambda_1\Lambda_5)$
x^{11}	$(-2^5\cdot 15\Lambda_1\Lambda_2^2 - 2^7\cdot 3\Lambda_1^2\Lambda_3 - 2^3\cdot 93\Lambda_2\Lambda_3 - 2^6\cdot 3\Lambda_1\Lambda_4 + 2^4\cdot 15\Lambda_5)$
x^{12}	$(-2^5\cdot 13\Lambda_1^2\Lambda_2 - 2^1\cdot 193\Lambda_2^2 - 2^3\cdot 75\Lambda_1\Lambda_3 - 2^2\cdot 15\Lambda_4)$
x^{13}	$(-2^7\Lambda_1^3 - 2^3\cdot 87\Lambda_1\Lambda_2 - 2^4\cdot 15\Lambda_3)$
x^{14}	$(-2^4\cdot 21\Lambda_1^2 - 2^2\cdot 75\Lambda_2)$
x^{15}	$(-2^4\cdot 19\Lambda_1)$
x^{16}	(-95)

Table 7.1.: The coefficients of F.

Low κ case Suppose $\kappa < \frac{4}{5}v(2)$. In this case, if we choose β to be any element of valuation $v(\beta) = \kappa/2$, we have that $v(\beta^3 R_3) = v(\beta^5 R_5) = 5\kappa/2 < 2v(2)$; in particular, after possibly a quadratic extension of R, $f_1(x_1) = q_1^2(x_1) + \rho_1(x_1)$ is a LE-decomposition. After performing the change of variable $y \mapsto (y - q_1(x_1))/\pi^{5\kappa/4}$, we may thus write the equation of $\mathcal{Y}_s^1$ as

$$y^2 = \left(\overline{\pi^{-5\kappa/2}\beta^3 R_3}\right) x_1^3 + \left(\overline{\pi^{-5\kappa/2}\beta^5 R_5}\right) x_1^5$$

where both coefficients are $\neq 0$ in k. It is immediate to see that $\mathcal{Y}_s$ has two distinct singular points, one at $\overline{x}_1 = 0$ and the other at $\overline{x}_1 = \overline{R_3/(\beta^2 R_5)}^{1/2}$: this implies, via Theorem 6.4.7, that $\mathcal{X}_{\mathrm{urst}}$ has two crushed components, which meet $\mathcal{X}^1$ at $\overline{x}_1 = 0$ and $\overline{x}_1 = \overline{R_3/(\beta^2 R_5)}^{1/2}$ respectively. In particular, the special fiber of $\mathcal{Y}_{\mathrm{st}}$ consists of two elliptic curves meeting at a node.

One crushed component of the ur-stable model is the smooth model of the line $\mathcal{X}^1$ we obtain from $\mathcal{X}$ by taking β with $v(\beta) = (2v(2) - \kappa)/3$. Such a choice of $v(\beta)$, in fact, makes $v(\beta^3 R_3) = 2v(2)$ and $v(\beta^5 R_5) > 2v(2)$: $f_1 = q_1^2 + \rho_1$ thus becomes an H-decomposition for f_1, and after the change of variable $y \mapsto (y - q_1(x_1))/2$ we obtain that $(\mathcal{Y}^1)_s$ has equation

$$y^2 + y = \left(\overline{\beta^3 R_3/4}\right) x_1^3$$

where $\overline{\beta^3 R_3/4} \neq 0$. The special fiber $(\mathcal{Y}^1)_s$ is a curve of abelian rank 1 having a unique singular point, which is unibranch and lies over $\overline{x}_1 = \infty$; in particular $\mathcal{X}^1$ certainly contributes to the ur-stable model (Theorem 6.4.6). The problem of determining the remaining smooth component of $\mathcal{X}_{\mathrm{urst}}$, however, remains open: the answer is hidden under the following proposition.

Proposition 7.4.4. If $\kappa < \frac{4}{5}v(2)$, there exist two roots of the polynomial $F(x)$ (i.e., two admissible choices for α) whose difference has valuation $< (2v(2) - \kappa)/3$.

Proof. Recall from Remark 7.4.3 that the hypothesis $\kappa < \frac{4}{5}v(2)$ may be equivalently restated as $v(\Lambda_2) < \frac{4}{5}v(2)$. Let α_1 be a root of $F(x)$. If we perform a coordinate change $x_a := x - \alpha_1$, the coefficient Λ_2 gets replaced by $10\alpha_1^2 - 4\Lambda_1\alpha_1 + \Lambda_2$, which also clearly has a valuation $< \frac{4}{5}v(2)$. Hence, switching from x to x_a preserves the validity of the hypothesis; moreover, the formation of F also commutes with the change of variable (Remark 7.4.2): we conclude that we lose no generality in supposing $\alpha_1 = 0$, and $x_a = x$.

The constant term of F is now zero; in other words,

$$\Lambda_4^4 - 2^3\Lambda_3\Lambda_4^2\Lambda_5 + 2^4\Lambda_3^2\Lambda_5^2 - 2^6\Lambda_1\Lambda_5^3 = 0.$$

We deduce, in particular, that $v(\Lambda_4) \geq v(2)$, and that $v(\Lambda_4^2 + 2^2\Lambda_3\Lambda_5) \geq \frac{5}{2}v(2)$.

If we now look at the x^8 coefficient of F, we obtain that it is equal to

$$+9\Lambda_2^4 + 2^3\Lambda_1\Lambda_2^2\Lambda_3 + 2^2\Lambda_2^2\Lambda_4 + 2(\Lambda_4^2 + 2^2\Lambda_3\Lambda_5) + O(4v(2)).$$

We have that

$$\begin{aligned} v(9\Lambda_2^4) &= 4\kappa, \\ v(2^3\Lambda_1\Lambda_2^2\Lambda_3) &\geq 2\kappa + 3v(2) > 4\kappa, \\ v(2^2\Lambda_2^2\Lambda_4) &\geq 2\kappa + 3v(2) > 4\kappa, \\ v(2(\Lambda_4^2 + 2^2\Lambda_3\Lambda_5)) &\geq \frac{7}{2}v(2) > 4\kappa, \\ v(O(4v(2))) &\geq 4v(2) > 4\kappa. \end{aligned}$$

We can thus conclude that the coefficient of x^8 in $F(x)$ has valuation 4κ. Recalling that the leading coefficient of F is unit, we can deduce from this that F, other than $\alpha_1 = 0$, certainly has another root α_2 of valuation $\leq \kappa/2$. But since we are assuming $\kappa < \frac{4}{5}v(2)$, we have that $\kappa/2 < (2v(2) - \kappa)/3$. □

Now, if we let α_1, α_2 be two roots of F such that $v(\alpha_1 - \alpha_2) < (2v(2) - \kappa)/3$, we clearly have that the two changes of variable $x_1 = \beta^{-1}(x - \alpha_1)$ and $x_1 = \beta^{-1}(x - \alpha_2)$, where $v(\beta) = (2v(2) - \kappa)/3$, determine two non-isomorphic models of the line X. As a consequence, setting $v(\beta) = (2v(2) - \kappa)/3$, we have not found just one, but both smooth components of the ur-stable model. The distance between them will be

$$2\left(\frac{2v(2) - \kappa}{3} - \frac{\kappa}{2}\right) = \frac{4v(2) - 5\kappa}{3}$$

and the thickness of the node of $\mathcal{Y}_{\mathrm{st}}$ will be half of this distance (Proposition 5.7.1).

7.4.4. Genus 2, one triple

Let us now assume that $Y \to X$ is a genus-2 hyperelliptic curve, and that $\mathcal{R}$ has a unique non-trivial cluster cut, of size 3/3. The cluster picture associated to an unwound smooth model $\mathcal{X}$ of the line will have the form

$$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}) : \boxed{G_1 \quad G_2 \quad G_3}^{\,m_1} \qquad \boxed{H_1 \quad H_2 \quad H_3}^{\,m_2}$$

where $m_1, m_2 \geq 0$ can vary depending on the chosen $\mathcal{X}$, but their sum $m := m_1 + m_2 > 0$ remains constant. Let $\mathcal{X}^a$ be the unwound smooth model we obtain setting $m_1 = 0$, and $\mathcal{X}^b$ the one we obtain for $m_2 = 0$:

$$\begin{array}{llll} \mathrm{ccp}_{\mathcal{R}}(\mathcal{X}^a): & G_1 \quad G_2 \quad G_3 & & \boxed{H_1 \quad H_2 \quad H_3}^{\,m} \\ \mathrm{ccp}_{\mathcal{R}}(\mathcal{X}^b): & \boxed{G_1 \quad G_2 \quad G_3}^{\,m} & & H_1 \quad H_2 \quad H_3 \end{array}$$

If x_a is some coordinate on $\mathcal{X}^a$, and we take an equation $y^2 = f_a(x_a)$ describing Y, a direct computation immediately shows that the roots of f_a' reduce to two distinct points of $(\mathcal{X}^a)_s$: one is the point $\bar{\mathfrak{s}}_H \in (\mathcal{X}^a)_s$ to which H_1, H_2 and H_3 collapse; the second one, which we will denote by $P_a \in (\mathcal{X}^a)_s$, does not coincide, instead, with the reduction of any of the six points of $\mathcal{R}$. Since $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}^a)$ has an odd maximal cluster, the normalization $\mathcal{Y}^a$ of $\mathcal{X}^a$ in $K(Y)$ is readily described (see Proposition 7.3.12): $(\mathcal{Y}^a)_s$ is an inseparable covering of the line $(\mathcal{X}^a)_s$ bearing two unibranch singularities over $\bar{\mathfrak{s}}_H \in (\mathcal{X}^a)_s$ and $P_a \in (\mathcal{X}^a)_s$; in particular, P_a is the anchor of some crushed smooth component $\mathcal{X}^{a1}$ of the ur-stable model (Theorem 6.4.7). To determine $\mathcal{X}^{a1}$, it is possible to proceed exactly as in the no cluster case, by letting G_1 play the role of A_0 (in other words, we choose x_a so that $x_a(G_1) = \infty$); moreover, it is immediate to realize that the presence of the cluster around the H_i's forces $\kappa = v(\Lambda_2) = 0$. We thus have that $\mathcal{X}^{a1}$ can be obtained from $\mathcal{X}^a$ by means of the change of coordinate $x_{a1} := {\beta_a}^{-1}(x_a - \alpha_a)$, where $v(\beta_a) = \frac{2}{3}v(2)$ and α_a is any of the roots of the polynomial $F_a(x_a)$ (the degree-16 polynomial we have defined while treating the no collapse case) that reduces to $P_a \in (\mathcal{X}^a)_s$. The special fiber $(\mathcal{Y}^{a1})_s$ is a curve of abelian rank 1, singular over $\overline{x}_{a1} = \infty$.

If we perform the same operations starting from $\mathcal{X}^b$, we obtain another crushed smooth component $\mathcal{X}^{b1}$ of the ur-stable model, anchored at a point $P_b \in (\mathcal{X}^b)_s$. Hence, $\mathcal{X}_{\mathrm{urst}}$ consists of two crushed smooth models of the line, whose distance is $m + \frac{4}{3}v(2)$ (m is the distance between their non-crushed anchors, and $\frac{2}{3}v(2)$ is the distance of each of them from its anchor). The stable model $\mathcal{Y}_{\mathrm{st}}$ of Y has a special fiber consisting of two elliptic curves meeting at a unique node, whose thickness equals $\frac{m}{2} + \frac{2}{3}v(2)$.

7.4.5. Genus 2, one pair

In the one pair case, the cluster picture of $\mathcal{R}$ with respect to any unwound smooth model $\mathcal{X}$ of X has the following form

$$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}) : \boxed{A_0 \qquad A_1}^{\,m_1} \qquad\qquad \boxed{B_0 \qquad B_1 \qquad B_2 \qquad B_3}^{\,m_2}$$

with $m_1, m_2 \geq 0$. The sum $m := m_1 + m_2 > 0$ is an invariant only depending on the hyperelliptic curve $Y \to X$, and the unwound smooth models of the line correspond to the possible ways of decomposing m as a sum $m_1 + m_2$. By properly choosing the coordinate x on the model $\mathcal{X}$, we may obtain that

$$x(A_0) = 0 \quad x(A_1) = -\lambda \quad x(B_0) = \infty \quad x(B_i) = -1/\mu_i \text{ for } i = 1, 2, 3,$$

where $v(\lambda) = m_1$, $v(\mu_i) = m_2$. After possibly changing R to a quadratic extension, we may consequently write a normalized 0∞-equation of the hyperelliptic curve in the form

$$y^2 = f(x) \qquad f(x) = x(x + \lambda)(1 + \mu_1 x)(1 + \mu_2 x)(1 + \mu_3 x);$$

setting $M_1 := \mu_1 + \mu_2 + \mu_3$, $M_2 := \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3$, $M_3 := \mu_1\mu_2\mu_3$, we have that

$$f(x) = x(x + \lambda)(1 + M_1 x + M_2 x^2 + M_3 x^3).$$

Proposition 7.4.5. The polynomial f admits a $q\rho$-decomposition $f(x) = q^2(x) + \rho(x)$ where

$$q(x) := x(1 + \sqrt{M_2}x) \qquad \rho(x) := \lambda x + \left(M_1 - 2\sqrt{M_2}\right) x^3 + M_3 x^5 + \text{h.o.t.}$$

and h.o.t. denotes some polynomial whose valuation is strictly greater than the one of ρ.

Proof. If we compute $f(x) - q^2(x)$, we immediately realize that it consists of the three summands displayed in the expression for ρ written above, plus $\varepsilon := \lambda(M_1 x + M_2 x^2 + M_3 x^3)$. Now,

- if $m_2 > 0$, then M_1, M_2, M_3 have all valuations > 0 and thus $v(\varepsilon) > m_1$; the linear coefficient of ρ has thus valuation $v(\lambda) = m_1$, and we can be sure that $v(\varepsilon) > v(\rho)$;
- if $m_2 = 0$, then $v(M_3) = 0$, while $v(\varepsilon) \geq m_1 > 0$; the coefficient of x^5 in ρ has thus valuation 0, and we can be sure that $v(\varepsilon) > v(\rho)$.

□

Remark 7.4.6. Proposition 7.4.5 still holds when $m_2 = 0$, if $v(M_1 - 2\sqrt{M_2})$ or $v(M_3)$ happens to be $< m_1$.

If we now let $t := v(\rho)$ be the minimum among the valuations of the coefficients of ρ, we have that

$$t = \min\{v(\lambda), v(M_1 - 2\sqrt{M_2}), v(M_3)\}$$

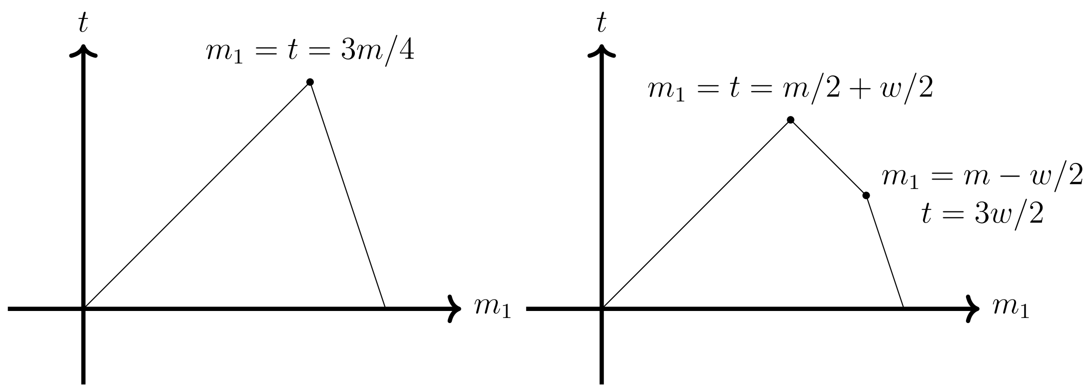
and it is clear that $r(x) := \pi^{-t}\rho(x)$ does not reduce to a square. Assuming t even (which is certainly true up to possibly changing R to a quadratic extension), we thus have that $f(x) = q^2(x) + \rho(x)$ is a LE-decomposition if $t < 2v(2)$, and an H-decomposition if $t \geq 2v(2)$.

Observe that the changes of variable $x \mapsto \pi x$ and $x \mapsto \pi^{-1}x$ allow us to switch model and tune the depth of the two clusters: the description we have given can be repeated for each of the smooth unwound models of the line we obtain letting m_1 vary between 0 and m. The valuations of the coefficients of ρ will clearly be functions of m_1:

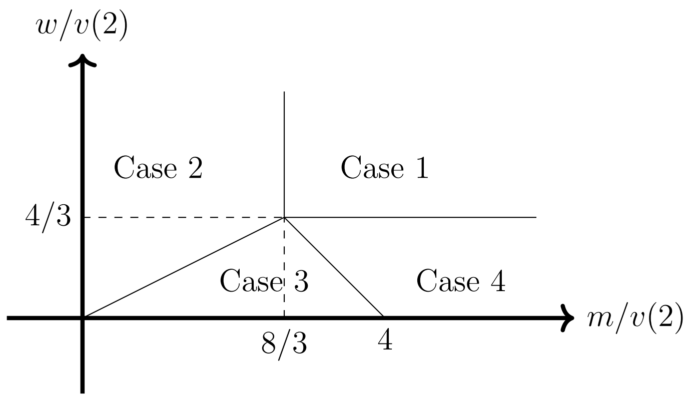
$$v(\lambda) = m_1 \qquad v(M_1 - 2\sqrt{M_2}) = w + m - m_1 \qquad v(M_3) = 3(m - m_1)$$

where $w \geq 0$ is a constant.

If we plot $t(m_1)$, which is the minimum of these three valuations, we obtain two possible shapes, depending on whether $w > m/2$ (on the left) or $w < m/2$ (on the right):



We will distinguish the following cases, depending on the values of w and m:



Case 1 Suppose that $m \geq \frac{8}{3}v(2)$ and $w \geq \frac{4}{3}v(2)$. In this case, setting

$$m_1 = m - \frac{2}{3}v(2) \qquad m_2 = \frac{2}{3}v(2),$$

we obtain that $t = v(M_3) = 2v(2)$. This implies that we are in case H, and, after the change of variable $y \mapsto (y - q(x))/2$, the special fiber $\mathcal{Y}_s$ is described by the equation

$$y^2 + xy = \left(\overline{M_3/4}\right) x^5 + \left(\overline{(M_1 - 2\sqrt{M_2})/4}\right) x^3 + \left(\overline{\lambda/4}\right) x$$

where $\overline{M_3/4} \neq 0$.

- If $\overline{\lambda/4} = 0$ (i.e. $m > \frac{8}{3}v(2)$), this is clearly a semistable curve of abelian rank 1, having a unique node at $x = 0$;
- if $\overline{\lambda/4} \neq 0$ (i.e. $m = \frac{8}{3}v(2)$), it is a smooth hyperelliptic curve of genus 2.

In both cases, $\mathcal{X}$ is the ur-stable model of X.

Case 2 Suppose that $m < \frac{8}{3}v(2)$ and $w \geq m/2$. In this case, for

$$m_1 = \frac{3}{4}m \qquad m_2 = \frac{1}{4}m,$$

we obtain that $t = v(\lambda) = v(M_3) < 2v(2)$; hence, we are in case LE, and the special fiber $\mathcal{Y}_s$ is an inseparable covering of $\mathcal{X}_s$, which, after the change of variable $y \mapsto (y - q(x))/\pi^{t/2}$, can be described by the equation

$$y^2 = \left(\overline{M_3/\pi^t}\right) x^5 + \left(\overline{(M_1 - 2\sqrt{M_2})/\pi^t}\right) x^3 + \left(\overline{\lambda/\pi^t}\right) x.$$

where $\overline{M_3/\pi^t} \neq 0$ and $\overline{\lambda/\pi^t} \neq 0$.

- If $w > m/2$, then $\overline{(M_1 - 2\sqrt{M_2})/\pi^t} = 0$, and $\mathcal{Y}_s$ has a unique singular point at $\overline{x} = \overline{\lambda/M_3}^{1/4}$, which coincides with neither of the points $\overline{x} = 0$ and $\overline{x} = \infty$ to which the two clusters reduce. This implies (Theorem 6.4.7) that the each component of the ur-stable model is crushed, and shall be found by means of a change of variable $x_i = \beta_i^{-1}(x - \alpha_i)$, for some α_i, β_i such that $\overline{\alpha}_i = \overline{\lambda/M_3}^{1/4}$, $0 < v(\beta_i) \leq 2v(2)$; its associated (crushed) cluster picture will be of the form

  $$\boxed{\boxed{A_0 \quad A_1}^{3m/4} \qquad \boxed{B_0 \quad B_1 \quad B_2 \quad B_3}^{m/4}}^{v(\beta_i)}$$

  Depending on whether $\mathcal{X}_{\mathrm{urst}}$ has one or two components, we obtain that $(\mathcal{Y}_{\mathrm{st}})_s$ either consists of a hyperelliptic genus-2 curve, or of a pair of elliptic curves crossing at a unique node.

- If $w = m/2$, $\mathcal{Y}_s$ is singular over two distinct points $P_1, P_2 \in \mathcal{X}_s$, neither of which coincides with $\overline{x} = 0$ or $\overline{x} = \infty$; the ur-stable model, in this case, certainly has two crushed smooth components (Theorem 6.4.7), which will be obtained by two suitable variable changes $x_i = \beta_i^{-1}(x - \alpha_i)$, where $\overline{\alpha_i} = P_i$, $0 < v(\beta_i) \leq 2v(2)$. As a consequence, $(\mathcal{Y}_{\mathrm{st}})_s$ consists of a pair of elliptic curves crossing at a unique node.

To actually determine the values of α and β that allow us to move from $\mathcal{X}$ to the crushed smooth component(s) $\mathcal{X}^1$ of the ur-stable model anchored to $\mathcal{X}$, we may adopt a similar technique to the one we have presented while discussing the no cluster case, by letting $0, \lambda, 1/\mu_1, 1/\mu_2, 1/\mu_3$ play the role of $\lambda_1, \ldots, \lambda_5$. Let us retain the notation adopted in the no cluster case. By means of a direct computation, it is possible to verify that the degree-16 polynomial $F := P_4^4 - 8P_3P_5P_4^2 + 16P_3^2P_5^2 - 64P_1P_5^3$ we had introduced has still integral coefficients, and that its leading and constant term are units; if we choose α to be a root of F, we consequently have that $\overline{\alpha}$ reduces neither to $\overline{x} = 0$, nor to $\overline{x} = \infty$.

If we set $x_1 := \beta^{-1}(x - \alpha)$ and we suitably replace y, we may write a normalized ∞-equation for Y in the form $y^2 = f_1(x_1)$, where $f_1(x_1) = q_1^2(x_1) + \rho_1(x_1)$ with

$$\begin{aligned} q_1(x_1) &= 1 + \beta\frac{p_4}{2p_5}x_1 + \beta^2\frac{\sqrt{p_1}}{\sqrt{p_5}}x_1^2, \\ \rho_1(x_1) &= \beta^3 R_3 x_1^3 + \beta^5 R_5 x_1^5, \qquad R_3 := \frac{p_2\sqrt{p_5} - p_4\sqrt{p_1}}{\sqrt{p_5}^3}, \qquad R_5 := \frac{1}{p_5}. \end{aligned}$$

Now, it is possible to show that $v(R_5) = 3m/4$, $v(R_3) \geq 3m/4$, and that the polynomial $q_1(x_1)$ has integral coefficients and reduces to 1 for every chosen β such that $v(\beta) > 0$. Two cases should now be distinguished.

- If $v(R_3) \geq \frac{4}{5}\left(v(2) + \frac{9m}{16}\right)$, then, choosing $\beta = \frac{2}{5}\left(v(2) - \frac{3}{8}m\right)$, we obtain that $v(\beta^3 R_3) \geq 2v(2)$ and $v(\beta^5 R_5) = 2v(2)$, and, reasoning as we did in the no cluster case, $\mathcal{Y}^1$ is readily proved to be a smooth model of Y.

- If, instead, $v(R_3) < \frac{4}{5}\left(v(2) + \frac{9m}{16}\right)$, then, choosing $\beta = \frac{1}{2}\left(v(R_3) - \frac{3}{4}m\right)$, we obtain that $v(\beta^3 R_3) = v(\beta^5 R_5) < 2v(2)$. As a consequence, reasoning as we did in the no cluster case, we can prove that $(\mathcal{Y}^1)_s$ is an inseparable covering of the line $(\mathcal{X}^1)_s$ bearing two distinct, unibranch singular points. In particular, we can be sure that $(\mathcal{Y}_{\mathrm{st}})_s$ consists of two elliptic curves crossing at a node (see Theorem 6.4.7 and the discussion in Subsection 7.4.2). One of the two crushed components of $\mathcal{X}_{\mathrm{urst}}$ is the model $\mathcal{X}^1$ we obtain by picking β of valuation $v(\beta) = \frac{2v(2)-v(R_3)}{3}$. It is reasonable to conjecture that the remaining one will also be obtained in this way, by choosing a different root α of F, as we did in the no cluster case.

Case 3 Suppose that $w < m/2$ and $w + m < 4v(2)$. In this case, two relevant values of m_1 have to be considered.

- Let us name $\mathcal{X}^a$ the smooth unwound model of the line we obtain for

$$m_1 = \frac{m}{2} + \frac{w}{2} \qquad m_2 = \frac{m}{2} - \frac{w}{2},$$

  and let x_a be its coordinate. We have $t = v(\lambda) = v(M_1 - 2\sqrt{M_2}) < 2v(2)$; in particular we are in case LE and $(\mathcal{Y}^a)_s$ is an inseparable covering of $(\mathcal{X}^a)_s$, which, after the usual change of variable $y \mapsto (y - q(x))/\pi^{t/2}$, is described by the equation

$$y^2 = \left(\overline{(M_1 - 2\sqrt{M_2})/\pi^t}\right) x_a^3 + \left(\overline{\lambda/\pi^t}\right) x_a$$

  in which both coefficients are $\neq 0$. The special fiber $(\mathcal{Y}^a)_s$ has two singular points, one at $\overline{x}_a = \infty$ and the other at $\overline{x}_a = \overline{\lambda/(M_1 - 2\sqrt{M_2})}^{-1/2}$. The latter coincides neither with $\overline{x}_a = \infty$ nor with $\overline{x}_a = 0$.

- Let us name $\mathcal{X}^b$ the smooth unwound model of the line we obtain for

$$m_1 = m - \frac{w}{2} \quad m_2 = \frac{w}{2},$$

and let x_b denote its coordinate. We have $t = 3w/2 = v(M_1 - 2\sqrt{M_2}) = v(M_3) < 2v(2)$; in particular we are in case LE and $(\mathcal{Y}^b)_s$ is an inseparable covering of $(\mathcal{X}^b)_s$ that, after the usual change of variable $y \mapsto (y - q(x))/\pi^{t/2}$, is described by the equation

$$y^2 = \left(\overline{M_3/\pi^t}\right) x_b^5 + \left(\overline{(M_1 - 2\sqrt{M_2})/\pi^t}\right) x_b^3;$$

in which both coefficients are $\neq 0$. The special fiber $(\mathcal{Y}^b)_s$ has two singular points, one at $\overline{x}_b = 0$ and the other at $\overline{x}_b = \overline{(M_1 - 2\sqrt{M_2})/M_3}^{1/2}$. The latter does not coincide with any of the points to which the elements of $\mathcal{R}$ reduce (even in the $w = 0$ case).

Both $(\mathcal{X}^a)_s$ and $(\mathcal{X}^b)_s$ have a singularity at a point to which none of the six roots of $\mathcal{R}$ reduce: this implies that $\mathcal{X}^a$ and $\mathcal{X}^b$ are the anchors of two crushed component $\mathcal{X}^{a1}$ and $\mathcal{X}^{b1}$ of $\mathcal{X}_{\mathrm{urst}}$ (Theorem 6.4.7); in particular $\mathcal{Y}_{\mathrm{st}}$ will consist of a pair of elliptic curves crossing at a single node (see discussion in Subsection 7.4.2). To find $\mathcal{X}^{a1}$, one has to perform a suitable change of variable $x_{a1} = \beta_a^{-1}(x_a - \alpha_a)$ such that $0 < v(\beta_a) \leq 2v(2)$ and $\alpha_{a,1}$ reduces to the singular point $\overline{x}_a = \overline{\lambda/(M_1 - 2\sqrt{M_2})}^{1/2}$; the associated cluster picture of $\mathcal{X}^{a1}$ will have the form:

$$\mathrm{ccp}(\mathcal{X}^{a1}) : \boxed{\boxed{A_0 \quad A_1}_{m/2 + w/2} \quad \boxed{B_0 \quad B_1 \quad B_2 \quad B_3}_{3m/2 - w/2}}_{v(\beta_a)}$$

Similarly, determining $\mathcal{X}^{b1}$ means finding a suitable change of variable $x_{b1} = \beta_b^{-1}(x_b - \alpha_b)$, where $0 < v(\beta_b) \leq 2v(2)$ and α_b reduces to $\overline{x}_b = \overline{(M_1 - 2\sqrt{M_2})/M_3}^{1/2}$:

$$\mathrm{ccp}(\mathcal{X}^{b1}) : \boxed{\boxed{A_0 \quad A_1}_{m - w/2} \quad \boxed{B_0 \quad B_1 \quad B_2 \quad B_3}_{w/2}}_{v(\beta_b)}$$

Case 4 Suppose that $w < \frac{4}{3}v(2)$ and $w + m \geq 4v(2)$. In this case, the two interesting values of m_1 are $m_1^{(a)} := w + m - 2v(2)$ and $m_1^{(b)} := m - w/2$:

- For $m_1 = m_1^{(a)}$, we have $t = v(M_1 - 2\sqrt{M_2}) = 2v(2)$: we are thus in case H, and after the usual change $y \mapsto (y - q(x))/2$ we can write for the special fiber of $\mathcal{Y}^a$ the equation

$$y^2 + x_a y = \left(\overline{(M_1 - 2\sqrt{M_2})/4}\right) x_a^3 + \left(\overline{\lambda/4}\right) x_a,$$

where $\overline{(M_1 - 2\sqrt{M_2})/4} \neq 0$.

- If $w + m > 4v(2)$, $\overline{\lambda/4} = 0$, and $(\mathcal{Y}^a)_s$ is an integral curve of abelian rank 0, having a unibranch singularity at $\overline{x}_a = \infty$ and a node at $\overline{x}_a = 0$.
- If $w + m = 4v(2)$, $(\mathcal{Y}^a)_s$ is an integral curve of abelian rank 1, having a unibranch singularity at $\overline{x}_a = \infty$.

- The description of $\mathcal{X}^b$ is perfectly analogous to the one given in case 3.

As a consequence, via Theorems 6.4.6 and 6.4.7, $\mathcal{X}_{\text{urst}}$ consist of two smooth components, one non-crushed ($\mathcal{X}^a$) and the other crushed ($\mathcal{X}^{b1}$). The special fiber of $\mathcal{Y}_{\text{st}}$ consists two genus 1 curves crossing at a node: one of them is an elliptic curve, the other is either also an elliptic curve ($w + m = 4v(2)$) or a nodal curve of abelian rank 0 ($w + m > 4v(2)$).

7.4.6. Genus 2, one pair and one triple

Suppose now that we are given a genus-2 hyperelliptic curve $Y \to X$ such that $\mathcal{R} = \{S_0, S_1, R_0, T_1, T_2, T_3\}$ has two non-trivial cluster cuts: $\mathcal{R} = \{S_0, S_1, R_0\} \sqcup \{T_1, T_2, T_3\}$ (of size 3/3) and $\mathcal{R} = \{S_0, S_1\} \sqcup \{R_0, T_1, T_2, T_3\}$ (of size 2/4). There is a unique unwound smooth model $\mathcal{X}^0$ of the line X whose associated cluster picture has the form

$$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}^0) : \quad \boxed{S_0 \qquad S_1}^{\,m} \quad R_0 \quad \boxed{T_1 \qquad T_2 \qquad T_3}^{\,e}$$

Via Proposition 7.3.12, it is immediate to see that $(\mathcal{Y}^0)_s$ is an inseparable covering of $(\mathcal{X}^0)_s$ bearing two singular points, lying over $\overline{S} := \overline{S}_0 = \overline{S}_1 \in (\mathcal{X}^0)_s$ and $\overline{T} := \overline{T}_0 = \overline{T}_1 = \overline{T}_2 \in (\mathcal{X}^0)_s$; hence, thanks to Theorem 6.4.7, the ur-stable model of X consists of two smooth components $\mathcal{X}^S$ and $\mathcal{X}^T$, meeting $\mathcal{X}^0$ at $\overline{S}$ and $\overline{T}$ respectively.

Computing $\mathcal{X}^S$ To determine $\mathcal{X}^S$, we will focus on the cluster cut $\mathcal{R} = \{S_0, S_1\} \sqcup \{R_0, T_1, T_2, T_3\}$, and treat it as in the single pair case: R_0 will play the role of B_0, T_i the role of B_i for $i = 1, 2, 3$, and S_i the role of A_i for $i = 1, 2$. The presence of the 3/3 cluster cut implies that the constant w is equal to 0. If $m < 4v(2)$, then $\mathcal{X}^S$ the model $\mathcal{X}^{a1}$ built in case 3: it is a crushed model whose associated cluster picture is

$$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}^S) : \quad \boxed{\boxed{S_0 \qquad S_1}^{\,\frac{m}{2}} \ \boxed{R_0 \qquad \boxed{T_1 \qquad T_2 \qquad T_3}^{\,e}}^{\,\frac{m}{2}}}^{\,v(\beta_S)}$$

where $0 < v(\beta_S) \leq 2v(2)$, and the special fiber of $\mathcal{Y}^S$ is a curve of abelian rank 1 with a unibranch singularity on it. If $m \geq 4v(2)$, then $\mathcal{X}^S$ coincides with the model $\mathcal{X}^a$ built in case 4: it is thus the unwound smooth model of the line corresponding to the cluster picture

$$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}^S): \quad \boxed{S_0 \quad S_1}^{\,m-2v(2)} \qquad \boxed{R_0 \quad \boxed{T_1 \quad T_2 \quad T_3}^{\,e}}^{\,2v(2)}$$

and $(\mathcal{Y}^S)_s$ is an integral curve of abelian rank 0, having a node and a unibranch singularity on it (if $m > 4v(2)$) or an integral curve of abelian rank 1 bearing a unibranch singularity (if $m = 4v(2)$).

Computing $\mathcal{X}^T$ To determine $\mathcal{X}^T$, we can focus on the cluster cut $\mathcal{R} = \{S_0, S_1, R_0\} \sqcup \{T_1, T_2, T_3\}$ only. It is easy to see that $\mathcal{X}^T$ can be computed by the same procedure we have presented discussing the one-triple case; in particular, it will be a crushed model whose associated cluster picture is

$$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}^T): \quad \boxed{\boxed{\boxed{S_0 \quad S_1}^{\,m} \quad R_0}^{\,e} \quad T_1 \quad T_2 \quad T_3}^{\,2v(2)/3}$$

and $(\mathcal{Y}^T)_s$ will be a curve of abelian rank 1 bearing a unibranch singularity.

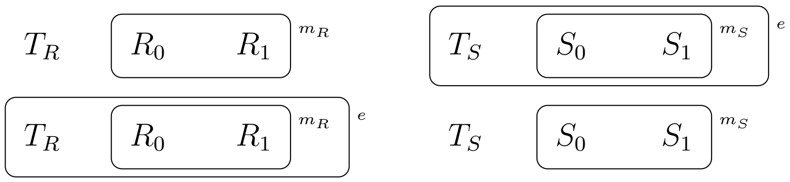
The stable model As a consequence, the special fiber of $\mathcal{Y}_{\mathrm{st}}$ consists of two genus 1 curves meeting at a node; if $m \leq 4v(2)$, they are both elliptic curves, whereas, if $m > 4v(2)$, one of them is an elliptic curve, and the other one a nodal curve of abelian rank 0.

7.4.7. Genus 2, two pairs

Consider now a genus-2 hyperelliptic curve $Y \to X$ such that $\mathcal{R} = \{R_0, R_1, S_0, S_1, T_0, T_1\}$ has two 2/4 cluster cuts, $\mathcal{R} = \{R_0, R_1\} \sqcup \{S_0, S_1, T_R, T_S\}$ and $\mathcal{R} = \{S_0, S_1\} \sqcup \{R_0, R_1, T_R, T_S\}$. We will also admit the possibility that a further 3/3 cluster cut $\mathcal{R} = \{R_0, R_1, T_R\} \sqcup \{S_0, S_1, T_S\}$ exists.

Two uniquely determined unwound smooth models $\mathcal{X}^{0R}$ and $\mathcal{X}^{0S}$ of X exist corresponding to the following cluster pictures:

$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}^{0R})$: , $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}^{0S})$:

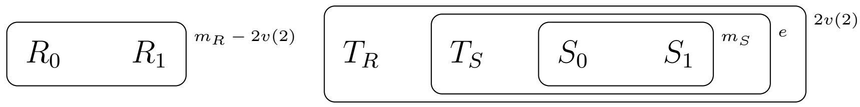


Clearly, $\mathcal{X}^{0R}$ and $\mathcal{X}^{0S}$ coincide if $e = 0$ (i.e., if the 3/3 cluster cut is not present). It is clear that (see Proposition 7.3.12) $(\mathcal{Y}^{0R})_s$ is an inseparable covering of $(\mathcal{X}^{0R})_s$ bearing two unibranch singularities over $\overline{R} := \overline{R}_0 = \overline{R}_1 \in (\mathcal{X}^{0R})_s$ and $\overline{S} := \overline{S}_0 = \overline{S}_1 \in (\mathcal{X}^{0R})_s$, whence we immediately deduce (Theorem 6.4.7) that $\mathcal{X}_{\mathrm{urst}}$ has two components, which we will name $\mathcal{X}^R$ and $\mathcal{X}^S$ and will meet $\mathcal{X}^{0R}$ at $\overline{R} \in (\mathcal{X}^{0R})_s$ and $\overline{S} \in (\mathcal{X}^{0R})_s$ respectively. Something identical can clearly be said working with $\mathcal{X}^{0S}$ instead of $\mathcal{X}^{0R}$.

To determine $\mathcal{X}^R$, we may reason as in the single pair case, letting R_0, R_1 play the role of A_0 and A_1, T_R that of B_0, and finally S_0, S_1 and T_S those of B_1, B_2 and B_3. It is immediate to realize that $w = 0$.

- If $m_R \geq 4v(2)$, $\mathcal{X}^R$ is the model $\mathcal{X}^a$ built in case 4; in other words, it is the unwound model associated to the cluster picture

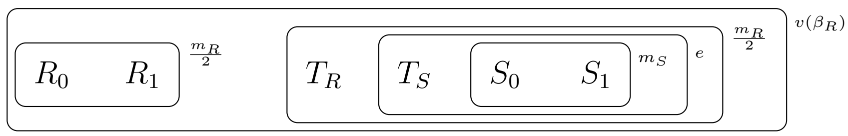
  $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}^R)$:

  The special fiber $(\mathcal{Y}^R)_s$ will either be an integral abelian rank 1 curve with a unibranch singularity ($m_R = 2v(2)$), or an abelian rank 0 curve with a node and a unibranch singularity ($m_R > 2v(2)$).

- If $m_R < 4v(2)$, $\mathcal{X}^R$ is the model $\mathcal{X}^{a1}$ built in case 3; it is thus crushed with cluster picture

  $\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}^R)$:

  where $0 < v(\beta_R) \leq 2v(2)$. The special fiber $(\mathcal{Y}^R)_s$ will be an integral abelian rank 1 curve with a unibranch singularity.

The computation of $\mathcal{X}^S$ is completely analogous. As a result, we have that the special fiber of the stable model $\mathcal{Y}_{\mathrm{st}}$ consists of two genus 1 curves crossing each other at a unique node. The one corresponding to $\mathcal{X}^R$ is an elliptic curve if $m_R \leq 2v(2)$, and it is a nodal curve of abelian rank 0 if $m_R > 2v(2)$; something analogous holds for the one corresponding to $\mathcal{X}^S$.

7.4.8. Genus 2, three pairs

Suppose now that $Y \to X$ is a hyperelliptic curve of genus 2 such that $\mathcal{R} = \{A_0, A_1, B_0, B_1, C_0, C_1\}$ has three 2/4 cluster cuts: $\mathcal{R} = \{A_0, A_1\} \sqcup \{B_0, B_1, C_0, C_1\}$, $\mathcal{R} = \{B_0, B_1\} \sqcup \{A_0, A_1, C_0, C_1\}$ and $\mathcal{R} = \{C_0, C_1\} \sqcup \{A_0, A_1, B_0, B_1\}$. Then, there exists a unique unwound smooth model $\mathcal{X}^0$ of X whose associated cluster picture has the form

$$\mathrm{ccp}_{\mathcal{R}}(\mathcal{X}^0): \quad \boxed{A_0 \qquad A_1}^{\,m_1} \ \boxed{B_0 \qquad B_1}^{\,m_2} \ \boxed{C_0 \qquad C_1}^{\,m_3}$$

We know that, if $m_i \geq 2v(2)$ for all i, then $\mathcal{X}^0$ falls within case H and is the ur-stable model (see Proposition 7.3.17 and Remark 7.3.18). Depending on how many equalities $m_i = 2v(2)$ hold, $(\mathcal{Y}_{\mathrm{st}})_s$ is either a hyperelliptic curve of genus 2, a curve of abelian rank 1 with a node, a curve of abelian rank 0 with two nodes, or a pair of lines meeting each other at three nodes.

We omit the determination of the ur-stable model in the other cases: it can be attempted by computations similar to those we have carried out in the previous subsections.

A. Colimits of schemes

Although the category of schemes is not even finitely cocomplete, under some hypotheses certain colimits exist. We will mainly be interested in two particular cases: computing the quotient of a scheme under a finite group action, and gluing schemes. Our aim will be, in each of these two cases, deriving some existence results and discussing what properties the colimit inherits.

We remark that, unlike the category of schemes, those of affine schemes ($\mathrm{Ring}^{\mathrm{op}}$), ringed topological spaces (RS) and locally ringed topological spaces (LRS) are cocomplete: we will thus start by discussing how colimits are can be described in each of these three categories, and how colimits in these categories are related to colimits of schemes.

The discussion on quotients of schemes essentially follows [SGA1, Exposé V], while the results on gluing schemes are mostly taken from [Stacks].

A.1. Affine schemes

Studying colimits of affine schemes is equivalent to studying limits in Ring. To do this, the key remark is that the forgetful functor Ring $\to$ Set is monadic, and hence it creates limits in Ring: in other words, the limit of a diagram in Ring always exists, and it is just the Set-theoretic limit endowed with the obvious canonical ring structure.

It is important to observe that the formation of colimits of affine schemes commutes with all flat base-changes: in other words, if X_i is a finite diagram of affine schemes, $X_i \to X$ is its colimit and $X' \to X$ is any flat morphism of affine schemes, then $X_i \times_X X' \to X'$ is the colimit of the diagram $X_i \times_X X'$: this is simply a manifestation of the exactness of flat base-changes.

A.2. Ringed topological spaces

Let us first show that RS is cocomplete. Suppose thus we are given a diagram D (of shape I) of ringed topological spaces, whose vertices and arrows will be respectively denoted by $\{X_i : i \in \mathrm{Ob}(I)\}$ and $\{f_\alpha : X_{i(\alpha)} \to X_{j(\alpha)} : \alpha \in \mathrm{Arr}(I)\}$; our aim is constructing its colimit $X \in \mathrm{RS}$. The forgetful functor RS $\to$ Top is cocontinuous (because it admits a right adjoint, which is the functor endowing every topological space with the locally constant sheaf of rings $\underline{\mathbb{Z}}$); hence, X, as a topological space, must coincide with the colimit in Top of the X_i's. We should now endow X with a structure sheaf. Observe that, on the topological space X, we do have some sheaves of rings, namely the pushforwards $f_{i*}(\mathcal{O}_{X_i})$ of the structure sheaves of the X_i's (where f_i denotes the continuous map $X_i \to X$). It is also clear that they actually form a diagram of sheaves of rings on X whose shape is I^{op}: we will define the structure sheaf $\mathcal{O}_X$ as the limit of this diagram; such a limit does effectively exist, and it can be computed pointwise (i.e., over any open subset of X) as a limit in Ring. We omit the straightforward verification of the fact that our X is actually the colimit of the X_i's in RS.

The construction also clearly shows that the computation of the colimits in RS is local in the following sense: given $X = \cup_j V_j$ an open covering, a cocone $f_i : X_i \to X$ over the diagram X_i is a colimit if and only if, for every j, the cocone $\{f_i^{-1}(V_j) \to V_j\}_i$ is a colimit for the diagram $\{f_i^{-1}(V_j)\}_i$.

A.3. Locally ringed topological spaces

We will now prove the cocompleteness of LRS. First, recall that the forgetful functor LRS $\to$ RS is left adjoint and conservative; consequently, if colimits exist in LRS, they have to coincide with those computed in RS. In other words, to show that colimits do exist in LRS, it is enough to prove that, given a diagram X_i of locally ringed topological spaces, its colimit X in RS is still a locally ringed topological space (i.e., $\mathcal{O}_{X,x}$ must be a local ring for every point $x \in X$), and that the canonical morphisms of ringed topological spaces $X_i \to X$ are local (in the sense that they induces local morphisms at the level of local rings). It is clearly enough to carry out this verification in the case of coproducts and coequalizers; the case of coproducts is trivial, while the verification for coequalizer requires some work:

Lemma A.3.1. If we are given two morphisms $f, g : X \rightrightarrows Y$ of locally ringed topological spaces, and we denote by $q : Y \to Z$ their coequalizer in RS, we have that

(a) Z is also a locally ringed topological space, and
(b) q is a morphism of locally ringed topological spaces.

Proof. See [DG71, I.§1.1]. □

A.4. Schemes

Schemes are not a cocomplete category, and the full embedding Sch $\to$ LRS is not cocontinuous: this means that colimits may not exist in Sch, or may exist but differ from those computed in LRS. In some favorable cases, anyway, the colimit in LRS of a diagram of schemes X_i turns out to be a scheme, and in this case it will also obviously serve as the colimit of the diagram X_i in Sch: this subsection aims to present some sufficient conditions for this to happen.

Let us first address the affine case. We have seen that a diagram of affine schemes certainly has a colimit in Ring$^{\mathrm{op}}$ and a colimit in LRS: in some favorable cases, they coincide and they consequently provide the colimit in Sch.

Lemma A.4.1. Suppose we are given a finite diagram of affine schemes X_i, and let X be its colimit in the category Ring$^{\mathrm{op}}$ of affine schemes. If X also happens to be the colimit of the diagram X_i in Top, then it serves as the colimit in LRS as well (and hence, a fortiori, in Sch).

Proof. Let us denote by $f_i : X_i \to X$ the canonical morphisms. To show that X is the colimit in LRS, we have only to prove that the cone $\mathcal{O}_X \to f_{i*}\mathcal{O}_{X_i}$ is a limit of the diagram $f_{i*}\mathcal{O}_{X_i}$ of sheaves of rings on X; for this to be true, it is enough to prove that (†) $\mathcal{O}_X(U)$ is the limit of the diagram of rings $(f_{i*}\mathcal{O}_{X_i})(U)$ for every affine open subscheme $U \subseteq X$.

We know that X is the colimit in Ring$^{\mathrm{op}}$, and this is equivalent to the statement (†) with $U = X$. But since the formation of colimits in Ring$^{\mathrm{op}}$ commutes with localization, (†) will hold equally well for every affine open subscheme $U \subseteq X$. $\square$

Before continuing our discussion, we will present two remarkable cases in which the lemma applies.

Proposition A.4.2. Let Y be an affine scheme, G a finite group acting on it, and X the quotient of Y under G in Ring$^{\mathrm{op}}$. Then,

(a) X is the quotient of Y under G in LRS and hence, a fortiori, in Sch;

(b) $f : Y \to X$ is an integral, scheme-theoretically surjective morphism of affine schemes.

Proof. Let us start by proving (b): if $Y = \mathrm{Spec}(A)$, then its quotient X as an affine scheme is just $\mathrm{Spec}(A^G)$, where $A^G \subseteq A$ is the subring of G-invariant elements. Since $A \subseteq A^G$ is injective, $f : Y \to X$ is scheme-theoretically surjective; it is also integral because any $a \in A$ is a root of the G-invariant monic polynomial $F_a(x) := \prod_g (x - a^g)$.

Let us now turn our attention to (a). Being integral, $f : Y \to X$ is certainly closed: to show that Y is the topological quotient of X under G, it is thus enough to show that two primes of A have the same contraction in A^G if and only if they are G-conjugated. It is obvious that f identifies G-conjugated primes. Conversely, if two primes $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$ are identified by f (i.e. $\mathfrak{p} \cap A^G = \mathfrak{q} \cap A^G$), then every element of $\mathfrak{p}$, once multiplied by its conjugates, falls into $\mathfrak{q}$; in particular, $\mathfrak{p} \subseteq \bigcup_{g \in G} \mathfrak{q}^g$, whence, by prime avoidance, it follows that $\mathfrak{p} \subseteq \mathfrak{q}^g$ for some $g \in G$. But, since f is integral, its fibers are zero-dimensional, and the inclusion $\mathfrak{p} \subseteq \mathfrak{q}^g$ must then actually be an equality. □

Proposition A.4.3. Let $Z_1 \hookrightarrow Y_1$ be a closed immersion of affine schemes, and $Z_1 \to Y_2$ any morphism of affine schemes. Let us denote by $X := Y_2 \cup_{Z_1} Y_2$ the pushout of the diagram $Y_1 \hookleftarrow Z_1 \to Y_2$ in $\mathrm{Ring}^{\mathrm{op}}$. Then,

(a) X is also the pushout in LRS, and hence, a fortiori, in Sch;
(b) the canonical morphism $Y_2 \to X$ is a closed immersion of affine schemes;
(c) if $Z_1 \to Y_2$ is integral, then its pushout $Y_1 \to X$ is also integral, and $Y_1 \sqcup Y_2 \to Z$ is an integral, scheme-theoretically surjective morphism of schemes.

Proof. Statement (b) immediately follows from the fact that surjective morphisms are pullback-stable in Ring. Statement (c) is a straightforward verification (see [Stacks, `0E1S`]). Finally, statement (a) will be just an application of Lemma A.4.1, once we have showed that X is the topological pushout of $Y_1 \hookleftarrow Z_1 \to Y_2$: a proof of this fact can be found in [Stacks, `0B7J`]. □

We will now see how these results in the affine case can help to determine colimits of schemes. First, let us introduce some ad-hoc terminology.

We will use the term "a *class of diagrams*" to mean the collection of all diagrams in Sch having some given shape, and whose arrows are required to belong to some families of morphisms of schemes that are stable under flat base-change. For example, we will say "the class of pushout diagrams $Y_1 \leftarrow Z_1 \to Y_2$ such that $Z_1 \leftarrow Y_1$ is a closed immersion", or, given a finite group G, "the class of all G-schemes" (which is nothing but the collection of all diagram of schemes of shape G).

We will say that a class of diagrams is *good* if, whenever a diagram of the class consisting of affine schemes is given, its colimits in $\mathrm{Ring}^{\mathrm{op}}$ and in LRS coincide. For example, both of the classes that the last paragraph mentions have been proved to be good (Propositions A.4.2 and A.4.3). A diagram of schemes will be said to be *good* whenever it belongs to some good class of diagrams.

Proposition A.4.4. Let X_i be a good diagram of schemes. If its colimit X in LRS admits an open covering $X = \cup_j U_j$ such that the restriction of X_i over U_j is affine for all i and j, then

(a) X is a scheme, and hence it serves as the colimit of the diagram X_i in Sch;
(b) the canonical morphisms $X_i \to X$ are affine morphisms of schemes;
(c) the formation of the colimit commutes with flat base-change, in the sense that, for every flat morphism of schemes $X' \to X$, X' is the colimit, in LRS and hence in Sch, of the base-changed diagram $X'_i := X_i \times_X X'$.

Proof. Since X is the colimit of the diagram $\{X_i\}_i$ in LRS, then, for every j, U_j is certainly the colimit the colimit of the restricted diagram $\{X_{i|U_j}\}_i$ in LRS. But, since $\{X_i\}_i$, and hence also $\{X_{i|U_j}\}_i$, is a good diagram of schemes, $\{X_{i|U_j}\}_i$ has the same colimit in $\mathrm{Ring}^{\mathrm{op}}$ and LRS: hence, U_j is an affine scheme for all j, and from this (a) and (b) clearly follow.

To prove (c), fix any affine open subscheme $U' \subseteq X'$ factoring through some affine open subscheme $U \subseteq X$. The cocone $X'_{i|U'} \to U'$ is the pullback of $X_{i|U} \to U$ along the flat morphism $U' \to U$; since $X_{i|U} \to U$ is a colimit in the category of affine schemes, where the formation of colimits commutes with flat base-change, we have that $X'_{i|U'} \to U'$ is a colimit in $\mathrm{Ring}^{\mathrm{op}}$. Moreover, as X_i has been assumed to be a good diagram of schemes, $X'_{i|U'}$ will also be and hence $X'_{i|U'} \to U'$ is its colimit also in LRS. Varying U', it is possible to cover the whole X', and hence we can conclude that $X'_i \to X'$ is the colimit of the diagram X'_i in LRS. $\square$

Let us specialize the result to the case of finite group actions on schemes. We have already seen that the action of finite groups on schemes always gives good diagrams (see Proposition A.4.2).

Definition A.4.5. We will say that the action of a finite group G on a scheme Y is *admissible* whenever the hypothesis of Proposition A.4.4 is satisfied, i.e. if Y can be covered by G-invariant affine open subschemes (or, equivalently, if every orbit lies in some affine open subscheme).

With this terminology, we can rephrase Proposition A.4.4 as follows.

Corollary A.4.6. If Y is a scheme endowed with an admissible action by a finite group G, then its quotient $X := Y/G$ in LRS is a scheme, the projection to the quotient $f : Y \to X$ is an affine morphism of schemes, and, for every flat morphism of schemes $X' \to X$, the base-changed morphism $f' : Y' \to X'$ provides the quotient of $Y' := Y \times_X X'$ under the action of G.

Remark A.4.7. Under the hypotheses of the corollary above, because of how quotients of affine schemes behave (see Proposition A.4.2), we know that the quotient morphism $f : Y \to X$ is not only affine, but also integral and scheme-theoretically surjective.

Let us now turn to the case of pushouts. We have already seen that a pushout diagram $Y_1 \leftarrow Z \to Y_2$ is good whenever we suppose $Z_1 \to Y_1$ to be a closed immersion (Proposition A.4.3). But some additional hypotheses on $Z_1 \to Y_2$ are necessary if we want Proposition A.4.4 to apply.

Definition A.4.8. A diagram of schemes $Y_1 \hookleftarrow Z_1 \xrightarrow{g} Y_2$ is an *admissible gluing diagram* if $Z_1 \hookrightarrow Y_1$ is a closed immersion and $g : Z_1 \to Y_2$ is an integral morphism of schemes such that each fiber of g is contained in some affine open subscheme of Y_1.

Corollary A.4.9. If $Y_1 \hookleftarrow Z_1 \xrightarrow{g} Y_2$ is an admissible gluing diagram of schemes, then its colimit $X := Y_1 \cup_{Z_1} Y_2$ in LRS is still a scheme, the canonical morphisms $a_1 : Y_1 \to X$ and $a_2 : Y_2 \to X$ are affine morphism of schemes, and, after any flat base-change $X' \to X$, we still have that $X' = Y_1' \cup_{Z_1'} Y_2'$.

Proof. We have to prove that X can be covered by open subsets U_j's such that $Y_{1|U_j}$ and $Y_{2|U_j}$ are affine for all j: this can be obtained as an immediate consequence of [Stacks, `0ECJ`]. Now, our result is a corollary of Proposition A.4.4. □

Remark A.4.10. Under the hypotheses of the corollary above, our discussion on pushouts of affine schemes (see Proposition A.4.3) actually ensures that $a_1 : Y_1 \to X$ is integral, $a_2 : Y_2 \to X$ is a closed immersion, and $Y_1 \sqcup Y_2 \to X$ an integral, scheme-theoretically surjective morphism of schemes.

A.5. Properties of the colimit

In the previous subsection, we have showed, under some particular hypotheses (see Corollaries A.4.6 and A.4.9), how to compute the quotient of a scheme Y under the action of a finite group, or how to glue two schemes Y_1 and Y_2 along a closed subscheme $Z_1 \subseteq Y_1$. We want now to discuss what properties of the scheme Y (resp. of Y_1) can be transferred to the colimit X. Let us first recall some basic results on the descent of properties of schemes along surjective morphisms.

Lemma A.5.1. If S is a scheme, and $f : Y \to X$ is a surjective morphism of S-schemes, we have that

(a) the properties of being closed, universally closed or quasi-compact over S, as well as the property of being an irreducible scheme, always descend along f;
(b) if f is scheme-theoretically surjective, the property of being a reduced scheme descends;
(c) if f is universally closed, the properties of being quasi-separated or separated over S descend (see [Stacks, `09MQ`]);
(d) if f is integral, the property of being affine over S descends (see [Stacks, `05YU`]);
(e) if f is integral, and S locally Noetherian, the property of being locally of finite type over S descends (this is the Artin-Tate lemma, see [Stacks, `00IS`]).

Let us first address the case of quotients. Let thus S be a scheme, Y an S-scheme endowed with an admissible S-action by a finite group G, and let $X := Y/G$ denote its quotient. We know that $f : Y \to X$ is an integral, scheme-theoretically surjective morphism of schemes (Remark A.4.7). If we apply Lemma A.5.1 we can thus obtain the following.
(a) If Y is irreducible, reduced or integral, X will also be.
(b) Assume that S is locally Noetherian. Then, if Y is locally of finite type over S, X will also be; moreover, in this case, f will be finite (recall that finite is equivalent to locally of finite type and integral).
(c) Assume that S is locally Noetherian. Then, if Y is proper over S, X is also a proper S-scheme.

Normality also descends when taking quotients, meaning that, if Y is a normal scheme, X is also. This is a consequence of the fact that, if A is a normal domain endowed with the action of a finite group, the subring of G-invariant elements $A^G \subseteq A$ (whose fraction field is simply $\mathrm{Frac}(A)^G$) is also readily seen to be normal.

Let us now see how the fact that a quotient map $Y \to X := Y/G$ is étale is strictly related to the action of G being free.

Proposition A.5.2. Let S be a scheme, Y an S-scheme, and G a finite group acting admissibly on Y by S-automorphisms. Assume that the quotient map $f : Y \to X := Y/G$ is finite, and let us fix a point $x \in X$. Then, he action of G on the points of the geometric fiber of Y over x is always transitive; if it is free, then f is étale over x.

Proof. Let $X' := \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,x})$ be the strict Henselianization of the local ring at x. If we let $f' : Y' \to X'$ the base-change of f, we have that the closed points y'_i of Y' are the same as those of the geometric fiber of Y over x and, by Henselianity, Y' is a finite disjoint union $Y' = \sqcup_i Y'_i$ of local schemes, one for each of the y'_i's; moreover, since the base-change $X' \to X$ is flat, we have that $X' = Y'/G$. From this, it is immediate to deduce that G must act transitively on the closed points y_i''s and hence on the Y_i''s; moreover, if $G_{y_i'}$ denotes the stabilizer of y_i', we clearly have that $X' = Y'/G = Y_i'/G_{y_i'}$; in particular, if G acts freely on the y_i''s, $Y_i' \to X'$ is an isomorphism for all i, which is equivalent to say that f is étale over x. □

Proposition A.5.3. If Y is a finite étale scheme over a base S, and G is any finite group of S-automorphism of Y acting admissibly on Y, both the quotient map $f : Y \to Y/G$ and the structure map $Y/G \to S$ will also be finite étale.

Proof. Since being finite étale is fpqc-local on the basis, and the formation of quotients commutes with flat base change, we may reduce ourselves to the case in which S is the spectrum of a strictly Henselian local ring (as we did in the proof of the proposition before). Since Y is finite étale over the spectrum S of a strictly Henselian local ring, it will be just the disjoint union of a finite number of copies of S, and the action of G only amounts to permuting them. It is thus evident that Y/G will also be a finite disjoint union of copies of S, and hence a finite étale S-scheme, and it is also completely clear that $f : Y \to Y/G$ will be finite étale. □

Let us now consider the case of gluing morphisms. Let thus S be a scheme, $Y_1 \hookleftarrow Z_1 \xrightarrow{g} Y_2$ an admissible gluing diagram of S-schemes, and X its colimit: we know that the canonical morphism $a_1 : Y_1 \to X$ is integral (Remark A.4.10). Let us further assume that g is scheme-theoretically surjective: we then have that a_1 is scheme-theoretically surjective too (this is an easy consequence of the fact that, in Ring, injective morphisms are preserved by pullbacks). The following may be thus deduced from Lemma A.5.1.

(a) If Y_1 is irreducible, reduced or integral, X will also be.
(b) Assume that S is locally Noetherian. Then, if Y_1 is locally of finite type over S, X will also be; moreover, in this case, f will be finite (recall that finite is equivalent to locally of finite type and integral).
(c) Assume that S is locally Noetherian. Then, if Y_1 is proper over S, X is also a proper S-scheme.

B. Morphisms between schemes of finite presentation

Given a scheme S and a point $s \in S$, $\operatorname{Spec}(\mathcal{O}_{S,s})$ can be thought of as the inverse limit of all the open subschemes of S containing s. All the results in this appendix deal with the same scenario: we have an S-scheme X, we suppose that some property or some construction is available over $\operatorname{Spec}(\mathcal{O}_{S,s})$, and we try to extend it to some Zariski neighborhood $V \ni s$.

Proposition B.0.1. Let X be a finite-type S-scheme, and suppose that $X_{|\operatorname{Spec}(\mathcal{O}_{S,s})} = \emptyset$. Then, there is an open neighborhood $s \in V \subseteq S$ such that $X_{|V} = \emptyset$.

Proof. We may suppose, without loss of generality, that $S = \operatorname{Spec}(R)$ is affine. Let $\mathfrak{p}$ be the prime of R corresponding to the point s. As X is quasi-compact (over S), it can be covered by a finite number of affine open subschemes $X_i = \operatorname{Spec}(A_i)$. For each i, let us choose a finite set of generators $\{T_{ij}\}$ for the R-algebra A_i. Since $X_{i|\operatorname{Spec}(\mathcal{O}_{S,s})} = \emptyset$, we have that $A_i \otimes_R R_{\mathfrak{p}} = 0$, and hence we can find elements $r_{ij} \in R \smallsetminus \mathfrak{p}$ such that $r_{ij}T_{ij} = 0$ in A_i. If $r_i := \prod_j r_{ij}$, it is clear that $A_i[r_i^{-1}] = 0$; in other words, X_i is empty over the affine open neighborhood $V_i := D(r_i)$ of $s \in S$. It is now clear that, if we let $V := \cap_i V_i$, we have $X_{|V} = \emptyset$. □

We will now consider the problem of extending an S-morphism of schemes which is initially only defined over $\operatorname{Spec}(\mathcal{O}_{S,s})$ to a morphism defined over an open neighborhood $V \ni s$. The two following propositions establish uniqueness and existence results for the problem.

Proposition B.0.2. Let X, Y be two finite type S-schemes and let $f_1, f_2 : X \rightrightarrows Y$ be two S-morphisms. If f_1 and f_2 coincide over $\operatorname{Spec}(\mathcal{O}_{S,s})$, then they coincide over some open neighborhood $V \ni s$.

Proof. We can suppose, without any loss of generality, that $S = \operatorname{Spec}(R)$ is affine; let $\mathfrak{p}$ be the prime ideal corresponding to the point s.

For every point $x \in X_{|\operatorname{Spec}(\mathcal{O}_{S,s})}$, since $y := f_1(x) = f_2(x)$ by hypothesis, we can find (by continuity) two affine open neighborhoods $x \in X' \subseteq X$ and $y \in Y' \subseteq Y$ such that $f_1(X') \subseteq Y'$ and $f_2(X') \subseteq Y'$. Let us denote by $\varphi_i : A \to B$ the ring map induced by f_i, where $X' = \operatorname{Spec}(B)$ and $Y' = \operatorname{Spec}(A)$. Let $T_1, \dots, T_n$ be a system of generators of the R-algebra A. We know that, for each generator T_j, $\varphi_1(T_j) = \varphi_2(T_j)$ in $B \otimes_R R_{\mathfrak{p}}$, i.e. $\varphi_1(T_j)r_j = \varphi_2(T_j)r_j$ for some $r_j \in R \smallsetminus \mathfrak{p}$. If we let

$r := \prod_j r_j$, we consequently get that $\varphi_1, \varphi_2 : A[r^{-1}] \rightrightarrows B[r^{-1}]$ coincide; in other words, $f_1, f_2 : X' \rightrightarrows Y'$ coincide over the open neighborhood $V' = D(r) \ni s$.

If we now let the point $x \in X_{|\operatorname{Spec}(\mathcal{O}_{S,s})}$ vary, we get a collection X^α of open subschemes of X covering the whole $X_{|\operatorname{Spec}(\mathcal{O}_{S,s})}$ and such that $f_1 = f_2$ on ${X^\alpha}_{|V^\alpha}$ for some open neighborhood $s \in V^\alpha \subseteq S$. By the proposition before (Proposition B.0.1), the X^α's will cover $X_{|W}$ for some affine open neighborhood $s \in W \subseteq S$; moreover, since X is a quasi-compact S-scheme, a finite subfamily $X^{\alpha_1}, \dots, X^{\alpha_k}$ will be enough to cover $X_{|W}$. If we define $V := W \cap V^{\alpha_1} \cap \dots \cap V^{\alpha_k}$, we get that f_1 and f_2 coincide on $X_{|V}$. □

Proposition B.0.3. Let X, Y be two finitely-presented S-schemes. Then, any S-morphism $f : X_{|\operatorname{Spec}(\mathcal{O}_{S,s})} \to Y_{|\operatorname{Spec}(\mathcal{O}_{S,s})}$ admits an extension to an S-morphism $\widetilde{f} : X_{|V} \to Y_{|V}$ for some open neighborhood $s \in V \subseteq S$.

Proof. We may suppose, without any loss of generality, that $S = \operatorname{Spec}(R)$ is affine; let $\mathfrak{p}$ be the prime of R corresponding to $s \in S$. Let $Y' = \operatorname{Spec}(A)$ be an affine open subscheme of Y, and let $X' = \operatorname{Spec}(B)$ an affine open subscheme of X such that $f(X'_{|\operatorname{Spec}(\mathcal{O}_{S,s})}) \subseteq Y'_{|\operatorname{Spec}(\mathcal{O}_{S,s})}$. Our f induces a ring homomorphism $\varphi : A \otimes_R R_\mathfrak{p} \to B \otimes_R R_\mathfrak{p}$; since A is a finitely-presented R-algebra, it is not difficult to construct an extension $\widetilde{\varphi}' : A[r^{-1}] \to B[r^{-1}]$ of φ, for some $r \in R \setminus \mathfrak{p}$, which corresponds to an S-morphism $\widetilde{f}' : X'_{|V'} \to Y$ extending f, where $V' := D(r) \ni s$.

We can now repeat this construction letting X' and Y' vary among all admissible choices, and this will produce a collection X^α, Y^α, V^α of affine open subschemes of X, Y and S respectively, and a collection of S-morphisms $\widetilde{f}^\alpha : X^\alpha_{|V^\alpha} \to Y$ extending f. It is clear that the X^α's cover the whole $X_{|\operatorname{Spec}(\mathcal{O}_{S,s})}$; by Proposition B.0.1, they will also cover $X_{|W}$ for some affine open neighborhood $s \in W \subseteq S$. Since X is quasi-compact (over S), a finite number $X^{\alpha_1}, \dots, X^{\alpha_k}$ of the X^α's will suffice to cover the whole $X_{|W}$. Let us denote by V the intersection $V := W \cap V^{\alpha_1} \cap \dots \cap V^{\alpha_k}$. Since X is quasi-separated over S, $X^{\alpha_i} \cap X^{\alpha_j}$ is quasi-compact over S, and we can apply the proposition before, which ensures that, up to shrinking V, the two extensions $\widetilde{f}^{\alpha_i} : X^{\alpha_i}_{|V} \to Y$ and $f^{\alpha_j} : X^{\alpha_j}_{|V} \to Y$ will coincide on $X^{\alpha_i}_{|V} \cap X^{\alpha_j}_{|V}$. Hence, we can glue all the $\widetilde{f}^{\alpha_i}$'s together and get the desired extension $\widetilde{f} : X_{|V} \to Y_{|V}$. □

As a useful corollary, we can "spread out" the property of being an isomorphism:

Corollary B.0.4. Let us consider $f : X \to Y$ a morphism of finitely-presented S-schemes, and let $s \in S$ be a point. If f is an isomorphism over $\operatorname{Spec}(\mathcal{O}_{S,s})$, then it is an isomorphism also over some open neighborhood $s \in V \subseteq S$.

Proof. Let $g : X \times_S \operatorname{Spec}(\mathcal{O}_{S,s}) \to Y \times_S \operatorname{Spec}(\mathcal{O}_{S,s})$ denote the base-change of f, which is, by hypothesis, an isomorphism. By Proposition B.0.3, up to shrinking S

to some neighborhood of $s \in S$, we can build an extension $f' : Y \to X$ of g^{-1}. By construction $f \circ f'$ and $f' \circ f$ give the identity over $\mathrm{Spec}(\mathcal{O}_{S,s})$; by Proposition B.0.2, the same will be true over some affine open neighborhood $s \in V \subseteq S$. Hence, f is an isomorphism (with inverse f') over V. $\square$

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