Proof for HW7Q6

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1 Proof

We want to find the expected value of the random variable (r.v.) $\boldsymbol{e}$ which is defined by:

$$\boldsymbol{e} = \min(\boldsymbol{e}_1, \boldsymbol{e}_2) \tag{1}$$

Where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are uniformly distributed r.v.s in the interval $[0, 1]$. We now define a CDF for $\boldsymbol{e}$ such that $\Pr(E \leq x)$, which is the probability that the r.v. E is less than a value x. We wish to arrive at this form because we can differentiate this expression to get the PDF, and then calculate the expected value (which we wish to find) by integrating the product of the PDF and the free variable x w.r.t. x.

We cannot easily create an expression by using the direct form of the CDF, as the way to express this would be equivalent to saying 'the probability the r.v. E is less than or equal to x is equivalent to the probability that either E_1 or E_2 are less than or equal to x'. Intuitively, we wish to get a similar statement, but in an "AND" form, which means we simply take the product.

It it as this point we turn to the CCDF, which gives us $\Pr(E > x)$. We can now get a much easier statement, which is simply 'the probability the r.v. E is greater than x is equivalent to the probability that E_1 is greater than x AND E_2 is greater than x'. This statement makes sense, since we are taking the minimum of the uniform r.v.s, therefore if we want to guarantee that E is greater than a certain value x, both of these r.v.s must be greater than that value x, hence the AND operation. This allows us to write the following equation:

$$\Pr(E > x) = \Pr(E_1 > x) \times \Pr(E_2 > x) \tag{2}$$

Since the two probability expressions on the right hand side are simply the CCDFs of the uniform distribution, we can express these both as $(1 - x)$. This means we can jump into an expression of the CDF for $\boldsymbol{e}$:

$$\Pr(E \leq x) = 1 - (1 - x)^2 \tag{3}$$

$$\Pr(E \leq x) = 2x - x^2 \tag{4}$$

Giving us a PDF of:

$$p_E(x) = 2 - 2x \tag{5}$$

Finally, we can calculate the expected value as follows:

$$\exp[E] = \int_0^1 x(2 - 2x)dx \tag{6}$$

Note the limits, as the PDF is only relevant between 0 and 1, as the value will be 0 in all other intervals

This gives us the final result of 1/3.