THE ROOT FUNCTOR

FRANCESCA PRATALI

ABSTRACT. In this paper, we show that any ∞-operad is equivalent to the localization of a discrete Σ-free operad; this result extend Joyal's delocalization theorem for categories to the operadic setting. Along the way, we pursue a systematic study of operadic localization in the dendroidal context and its compatibility with un/straightening equivalences, deducing another description of algebras over ∞-operads.

CONTENTS

INTRODUCTION AND MAIN RESULTS

Introduction. Operads were introduced by May [May06] and Boardman-Vogt [BV73] in order to describe the up-to-homotopy algebraic structures on iterated loop spaces. In modern homotopy theory, one works with topological spaces up to weak homotopy equivalence, and for this purpose the language of ∞-*operads* offers a tractable way to articulate homotopy-coherence. The theory of ∞-operads has led to advancements in mathematical questions that predate its formulation ([GW99], [Wei99], [BdBW13], [KK24], [Lur17]).

Intuitively, a (colored) ∞-operad $\mathcal{P}$ is given by a set of objects and, for any choice of objects $x_1, \ldots, x_n, y$, a space of operations $\mathcal{P}(x_1, \ldots, x_n; y)$, together with operadic partial composition laws

$$\circ_{x_i} : P(x_1, \ldots, x_k; y) \times P(z_1, \ldots, z_m; x_i) \longrightarrow P(x_1, \ldots, x_{i-1}, z_1, \ldots, z_m, x_{i+1}, \ldots, x_k; y)$$

equivariant with respect to some specified action of the symmetric groups given by 'permuting the variables'.

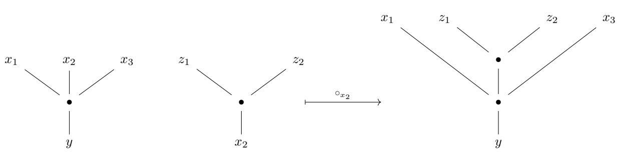

FIGURE 1. A graphical representation of the operadic composition $\circ_{x_2}$.

A $\mathcal{P}$-algebra is a family of spaces $F = \{F(x)\}_{x \in \mathsf{Ob}(P)}$ on which the operad acts via homotopy coherent maps

$$P(x_1, \ldots, x_n; y) \longrightarrow \mathsf{Map}(F(x_1) \times \cdots \times F(x_n); F(y)).$$

Key words and phrases. Dendroidal sets, ∞-operads, localization, covariant model structure.

We say that $\mathcal{P}$ is *discrete* if it is 0-truncated, that is, if its spaces of operations are sets.

Given an ∞-operad $\mathcal{P}$ and a class of unary morphisms $\mathcal{W}$, the *localization* of $\mathcal{P}$ at $\mathcal{W}$ is the ∞-operad $\mathcal{L}_{\mathcal{W}}\mathcal{P}$ obtained by freely inverting the arrows in $\mathcal{W}$. Expressing an ∞-operad $\mathcal{Q}$ as a localization $\mathcal{L}_{\mathcal{W}}\mathcal{P}$ can allow to infer properties of $\mathcal{Q}$ from those of $\mathcal{P}$, notably about the ∞-categories of algebras over the operads.

Localization of ∞-operads is related to localization of ∞-categories, which generalizes (there is a fully faithful inclusion $\mathsf{Cat}_\infty \hookrightarrow \mathsf{Op}_\infty$); also, it plays an important role in the theory of $\mathbb{E}_n$-algebras and factorization homology of manifolds.

Let us recall that the *little n-disks operad* $\mathbb{E}_n$ parametrizes the theory of homotopy associative and 'commutative up to homotopies of dimension n' algebras. The operad $\mathbb{E}_n$ is a single object ∞-operad whose space of k-ary operations is homotopy equivalent to the space of configurations of k points in $\mathbb{R}^n$. In [Lur17, Theorem 5.4.5.9], Lurie shows that the data of the homotopy coherences in $\mathbb{E}_n$ can be resolved by a discrete operad Disk_n equipped with a map $\mathcal{N}\mathsf{Disk}_n^{\otimes} \to \mathbb{E}_n$, in the sense that this map induces an equivalence between the ∞-category of $\mathbb{E}_n$-algebras and that of *locally constant* Disk_n-algebras. An analogous result is true for operads obtained by replacing $\mathbb{R}^n$ with a smooth (stratified) manifold M ([AFT17]). In both cases, expressing $\mathbb{E}_n$, resp. $\mathbb{E}_M$, as the localization of a discrete operad plays a fundamental role in recognizing certain colimit expressions and proving the existence of operadic Kan extensions.[1]

In the context of ∞-categories, this phenomenon is systematic: *Joyal's delocalization theorem*, formulated in [Joy07, §13.6] and proven by Stevenson in [Ste17], states that every ∞-category is weakly equivalent to the localization of a discrete one. It is hence natural to ask the following

Question. *Can we express* any *∞-operad as the localization of a* discrete *operad?*

In this paper we give a positive answer to this (Theorem 3.14), and provide a model for such discrete operad. We do this in the formalism of dendroidal sets, based on a category of trees which extends the simplex category ([MW07]). With out construction, we recover Joyal's delocalization theorem from Theorem 3.14 by specializing to simplicial sets, which sit inside dendroidal sets fully faithfully.

Let us explain in more details in the

Main results. Let us start with Joyal's delocalization theorem. For any simplicial set X, one may form its category of elements Δ/X, whose objects are string of composable morphisms in X, i.e. a map $\alpha\colon \Delta^n \to X$. We denote such an object by $([n],\alpha)$ and represent it as $\alpha(0) \to \cdots \to \alpha(n)$. There is a morphism from the nerve of this category back to X,

$$\mathcal{N}(\Delta/X) \longrightarrow X \quad ([n],\alpha) \mapsto \alpha(n)\ ;$$

which sends a string to its last vertex. The existence of such a functor is already proven in [Wal85, §1.6]. The delocalization theorem states that this map is in fact a weak categorical equivalence upon localizing $\mathcal{N}(\Delta/X)$ at the preimage of the identities. One consequence is that every simplicial set has the weak homotopy type of the nerve of a category[2]; it can be used to show that Barwick-Kan? Model for Rezk-nerve ?? tutte le cose dette con Victor e Kensuke

For the operadic generalization, we pass to the category of *dendroidal sets*, which we denote by **dSets**. These are presheaves on a category $\mathbf{\Omega}$ whose objects are trees and with fully faithful inclusions $\Delta \subset \mathbf{\Omega} \subset \mathsf{Op}$. Canonically, one may form the dendroidal nerve functor

$$\mathcal{N}_d\colon \mathsf{Op} \to \mathbf{dSets}$$

[1]Lurie uses the notion of weak approximations [Lur17, §2.3.3]; this notion is closely related to that of localization, notably by the characterization of algebras, but an explicit comparison has not been given yet.

[2]Although things were proven in the reverse order.

from the category of discrete operads to dendroidal sets. There is a model structure on dendroidal sets where fibrant objects are *quasioperads* ([CM11]), and it extends Joyal's model structure on simplicial sets for quasicategories.

In this paper, we construct a natural transformation $\mathbf{\Omega}/-\colon \mathbf{dSets} \to \mathsf{Op}$ fitting into the commutative diagram

$$\begin{tikzcd}
\mathbf{dSets} \arrow[r, "\mathbf{\Omega}/"] & \mathsf{Op} \arrow[r, "\mathcal{N}_d"] & \mathbf{dSets} \\
\mathbf{sSets} \arrow[u, hook] \arrow[r, "\Delta/"] & \mathsf{Cat} \arrow[u, hook] \arrow[r, "\mathcal{N}"] & \mathbf{sSets} \arrow[u, hook]
\end{tikzcd}$$

The operad $\mathbf{\Omega}/X$ is defined in Theorem 3.6. An object of $\mathbf{\Omega}/X$ is given by (T, α), with T a tree in $\mathbf{\Omega}$ and $\alpha\colon T \to X$ a map of dendroidal sets, that is, a labelling of T by objects (the edges) and operations (the vertices) of X, together with all the higher coherences.

A k-ary operation $(T_1, \alpha_1), \dots (T_n, \alpha_n) \longrightarrow (S, \beta)$ is given by a *wide* and *independent* map of trees $T_1 \sqcup \dots T_n \to S$ over X.

The *root* of T is a canonical edge, and evaluating an object (T, α) at the root yields an assignemt $\mathsf{Ob}(\mathbf{\Omega}/X) \to X_0$. In some sense, $\mathbf{\Omega}/X$ can be characterised as the maximal suboperad of the symmetric monoidal category $\mathbf{\Omega}^{\sqcup}/X$ for which the assingment $(*)$ extends to a functor of dendroidal set

$$\mathfrak{r}_X\colon \mathcal{N}_d(\mathbf{\Omega}/X) \longrightarrow X$$

cocontinuously in X as a natural transformation of functors. We call this map the *root functor* (Theorem 3.12). When X is a simplicial set, the operad $\mathbf{\Omega}/X$ is just the category of elements of X, and the root functor is the last vertex map $\mathcal{N}(\Delta/X) \to X$. Our main result is the following.

Theorem (3.14)**.** *Let X be a normal*[3] *dendroidal set, and let $\mathcal{R}$ be the set of morphisms of $\mathbf{\Omega}/X$ sent to identities by $\mathfrak{r}_X$. The root functor $\mathfrak{r}_X$ induces an operadic weak equivalence of dendroidal sets*

$$\overline{\mathfrak{r}_X}\colon \mathcal{N}_d(\mathbf{\Omega}/X)[\mathcal{R}^{-1}] \xrightarrow{\sim} X$$

between the localization of $\mathcal{N}_d(\mathbf{\Omega}/X)$ at $\mathcal{R}$ and X.

Let us refer to Theorem 3.14 as the *operadic delocalization theorem*.

When X is a simplicial set, we recover our starting motivating theorem.

Corollary (Joyal, Stevenson)**.** *For any simplicial set M, the last vertex functor $\mathcal{N}(\Delta/M) \to M$ induces a weak categorical equivalence*

$$\mathcal{N}(\Delta/X)[\mathcal{R}^{-1}] \longrightarrow X$$

between the localization of $\mathcal{N}(\Delta/X)$ at last vertex preserving morphisms and X.

In particular, the last vertex functor is a weak homotopy equivalence of simplicial sets.

From Theorem 3.14 we deduce that in the stable model structure on dendroidal sets ([BN14]), any ∞-operad X is weakly equivalent to the infinite loop space of the discrete operad $\mathbf{\Omega}/X$ (Corollary 3.15)[4]. When specialized to an ∞-category, it gives another proof of the well-known fact that every ∞- category has the weak homotopy type of the nerve of a category (used for instance in [Wal85]).

In Section 4, we study the ∞-category of algebras over an ∞-operad in light of the delocalization result given by the root functor, i.e. Theorem 3.14. The operadic un/straightening equivalences establish an equivalence of ∞-categories between algebras over an ∞-operad and operadic left fibrations over the ∞-operad.([Heu11], [BdBM20], [Ram22], [Pra25b], [Ker23]). In this final

[3] It means cofibrant in the operadic model structure. It corresponds to an operad being Σ-free.

[4] The stable model structure is a left Bousfield localization of the operadic model structure. It is Quillen equivalent to that of group-like $\mathbb{E}_\infty$-spaces ([BdBM20]), or equivalently that of infinite loop spaces. Under the equivalence $\mathbf{dSets}/\eta \simeq \mathbf{sSets}$, one recovers Kan-Quillen model structure.

section, we keep using the dendroidal model and study compatibility of dendroidal localization with the covariant model structure for left fibrations.

For a dendroidal set X and a choice of morphisms $S \subseteq X_1$, we define a model structure on $\mathbf{dSets}/X$ as a certain left Bousfield localization of the covariant model structure. The fibrant objects are those dendroidal left fibrations $E \to X$ for which a morphism $f: x \to y$ in S induces a weak homotopy equivalence of Kan complexes $f_!: E_x \to E_y$ between the fibres. We denote this model structure by $\mathbf{dSets}^S/X$. Our main result is the following

Proposition (Theorem 4.4)**.** *For any dendroidal set X and set of 1-morphisms $S \subseteq X_{C_1}$, the localization map $X \to X[S^{-1}]$ induces a Quillen equivalence*

$$\lambda_! : \mathbf{dSets}^S/X \rightleftarrows \mathbf{dSets}/X[S^{-1}] : \lambda^*$$

between the left Bousfield localization of the covariant model structure on $\mathbf{dSets}/X$ at $S_{/X}$ and the covariant model structure on $\mathbf{dSets}/X[S^{-1}]$.

With this result in hand, one can study the model structure (or in fact, semi model structure) on the category $\mathsf{Alg}_{\mathbf{\Omega}/X}(\mathbf{sSets})$ of simplicial algebras over the discrete operad $\mathbf{\Omega}/X$ induced by the localization of the covariant model structure under un/straightening equivalences. We talk about the semimodel structure for *$\mathcal{R}$-locally constant* $\mathbf{\Omega}/X$ algebras (ref) and denote it by $\mathsf{Alg}^{\mathcal{R}}_{\mathbf{\Omega}/X}(\mathbf{sSets})$. Say what a l.c. algebra is? We hence obtain the following localization result at the level of algebras.

Theorem (4.11)**.** *Let X be a normal dendroidal ∞-operad. Let $\mathsf{Alg}_X(\mathbf{sSets})$ be the category of algebras over X with the projective model structure.* [5] *There is a zig-zag of Quillen equivalences of semimodel categories*

$$\mathsf{Alg}_X(\mathbf{sSets}) \xleftarrow{\sim} \bullet \xrightarrow{\sim} \mathsf{Alg}^{\mathcal{R}_X}_{\mathbf{\Omega}/X}(\mathbf{sSets}).$$

In other words, X-algebras are equivalent to $\mathcal{R}_X$-locally constant $\mathbf{\Omega}/X$-algebras.

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Relation with other works. The real core of the root functor lies in its combinatorial definitions and in the constructions themselves, as there is no 'canonical' way of extending simplicial results (that is, results for categories) to dendroidal results (that is, results for operads), and in fact sometimes there may not be at all. Once the good construction is figured it out, the formal steps of the proof of Theorem 3.14 follow the ones used by Stevenson in the proof of Joyal delocalization theorem in [Ste17]. A new contracting homotopy for trees has to be built to prove Theorem 3.14 (the $\mathfrak{l}_T$ in the proof), and extra care in extending arguments from simplicial to dendroidal sets is needed, a some instances are the hypothesis of X normal.

Here comment on symmetric monoidal ∞-categories, the remark about the envelope and Kensuke's work.

Outline.

(1) In Section 1 we recall the necessary preliminary notions and set up notation.
(2) In Section 2, we set up the theory of dendroidal localization. Our definition extends Lurie's simplicial localization for quasicategories in [Lur17] to quasioperads.
(3) Section 3 is the heart of the article: here we define the operad of elements (Theorem 3.6), construct the root functor (Theorem 3.12) and prove the delocalization theorem (Theorem 3.14).
(4) In Section 4, we study interaction of dendroidal localization with operadic un/straightening equivalences, and then specify to the case of the operad of elements to deduce local un/straightening equivalences.

[5] The ∞-category of algebras over a dendroidal ∞-operad X is modeled by the projective model structure on $\mathsf{Alg}_{W_!(X)}(\mathbf{sSets})$, where $W_!(X)$ is the simplicial operad given by the Boardman-Vogt resolution of X([CM13]).

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1. Dendroidal recollections

For this section, complete proofs and definitions can be found in [HM22].

1.1. **Discrete operads.** From now on, we will call *operad* a discrete, strict operad P. We write Op for the category of operads and morphisms between them. A morphism of operads $f\colon P \to Q$ is the datum of a map of sets $\mathsf{Ob}(P) \to \mathsf{Ob}(Q)$ together with maps $P(c_1,\dots,c_n;d) \to Q(f(c_1),\dots,f(c_n);f(d))$ for any choice of objects $c_1,\dots,c_n,d$, compatible with the operadic composition and the symmetric group action.

Remark 1.1. The set $\mathsf{Ob}(P)$ has the structure of a poset, where $c \leqslant d$ if and only if P has an operation with target d and set of inputs containing c. Observe that a morphism of operads $f\colon P \to Q$ induces a morphism of posets $f\colon \mathsf{Ob}(P) \to \mathsf{Ob}(Q)$.

There is a fully faithful functor $j_!\colon \mathsf{Cat} \to \mathsf{Op}$, where $j_!$ acts by 'extension by zero', in the sense that

$$j_!C(c_1,\dots,c_n;d) = \begin{cases} \varnothing & \text{if } n \neq 1 \\ C(c_1;d) & \text{if } n = 1. \end{cases} \tag{1.1}$$

Its right adjoint $j^*\colon \mathsf{Op} \to \mathsf{Cat}$ is called the *underlying category* functor.

Definition 1.2. An operad P is Σ*-free* if, for any natural number n and any choice of objects $c_1,\dots,c_n,c$, the symmetric group on n-elements Σ_n acts freely on the set $\bigcup_{\sigma\in\Sigma_n} P(c_{\sigma(1)},\dots,c_{\sigma(n)};c)$.

Given any (small) symmetric monoidal category $(\mathbb{V},\otimes)$, one may form the operad $\mathbb{V}^{\otimes}$, with objects the objects of $\mathbb{V}$ and operations $\mathbb{V}^{\otimes}(x_1,\dots,x_n;y) = \mathbb{V}(x_1\otimes\cdots\otimes x_n;y)$, where composition is that of multivariable functions. A P-algebra in $\mathbb{V}$ is a morphism of operads $F\colon P \to \mathbb{V}^{\otimes}$. In this paper, we use $\mathbb{V} = \mathrm{sSets}$ and the subcategory $\mathbb{V} = \mathrm{Sets}$, symmetric monoidal with the cartesian product.

Denote by $\mathsf{Alg}_P(\mathbb{V})$ the category of P-algebras in $\mathbb{V}$ and morphisms between them. Observe that for a category C, we have a natural identification $\mathsf{Alg}_{j_!C}(\mathbb{V}) = \mathsf{Fun}(C,\mathbb{V})$.

1.2. **The dendroidal category.** Let Δ be the category of finite linear orders $[n] = \{0 < 1 < \cdots < n\}$, $n \geqslant 0$, and morphisms the maps of posets. Simplicial sets sSets are the category of presheaves over Δ. There is a fully faithful functor $\Delta \to \mathsf{Cat}$, which by restricted Yoneda embedding induces the nerve functor $\mathcal{N}\colon \mathsf{Cat} \to \mathrm{sSets}$.

The dendroidal category $\mathbf{\Omega}$ has objects non planar finite rooted trees.

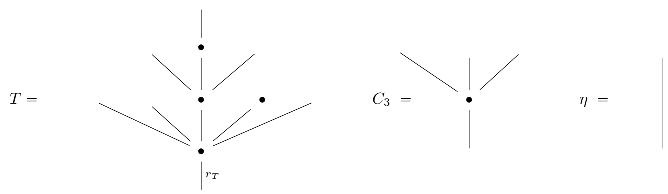
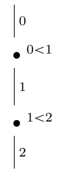


FIGURE 2. Some typical trees in $\mathbf{\Omega}$.

Any such tree T yields an operad $\Omega(T)$[6], so morphisms are defined by declaring $\Omega(-)\colon \mathbf{\Omega} \to \mathsf{Op}$ a fully faithful embedding[7].

There is a fully faithful embedding $\Delta \to \mathbf{\Omega}$, which looks like this:

$$\Delta \ni \qquad [2] = \{0 < 1 < 2\} \qquad \xmapsto{i}$$



$$\in \mathbf{\Omega}$$

FIGURE 3. The ordinal $[2]$ is realized as the linear tree with 3 edges and 2 unary vertices.

It is customary, and we adopt this convention here as well, to denote by η the image of the $[0]$ under the above inclusion. It is the unique tree with no vertex.

Dendroidal sets $\mathbf{dSets}$ are the category of presheaves on $\mathbf{\Omega}$. The restricted Yoneda embedding yields the fully faithful functor $\mathcal{N}_d\colon \mathsf{Op} \to \mathbf{dSets}$, called the *dendroidal nerve functor*. Explicitly, we have $\mathcal{N}_d(P) := \mathsf{Op}(\Omega(-), P)$.

The diagram on the left commutes, making the one the right commute as well.

$$\begin{tikzcd} \mathsf{Cat} \arrow[d] & \Delta \arrow[l] \arrow[d] \\ \mathsf{Op} & \mathbf{\Omega} \arrow[l] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathsf{Cat} \arrow[r, "\mathcal{N}"] \arrow[d] & \mathbf{sSets} \arrow[d, "i_!"] \\ \mathsf{Op} \arrow[r, "\mathcal{N}_d"'] & \mathbf{dSets} \end{tikzcd},$$

where $i_!$ is the left Kan extension of $i\colon \Delta \to \mathbf{\Omega}$ along the Yoneda embedding.

There is a natural isomorphism $\Delta \simeq \mathbf{\Omega}/\eta$, which yields the isomorphism $\mathbf{dSets}/\eta \simeq \mathbf{sSets}$, under which the inclusion $i_!$ is the forgetful functor $\mathbf{dSets}/\eta \to \mathbf{dSets}$. More generally, for any simplicial set M, one has $\mathbf{dSets}/i_!M \simeq \mathbf{sSets}/M$.

1.2.1. *Dendroidal notions.* Let T be a tree in $\mathbf{\Omega}$. We denote by r_T the root of T. Every vertex has a single output edge and n input edges for some $n \geqslant 0$ (the *arity* of v). The root is the unique edge without an input vertex. The *leaves* of T are those edges which have no output vertex. If a vertex has no output edge, we call it a *stump*. A vertex is *external* if its input edges are contained in the leaves of T or if it is a stump. We call the essentially unique tree with one vertex and n leaves the n-*corolla* and denote it by C_n.

Do I use all these notions in the following?

We say that S is a subtree of T if it can be obtained from T by successively pruning away external vertices and the outer edges attached to them from T.

Any tree is obtained by *grafting* of smaller trees. Formally, the grafting of a tree R onto a leaf ℓ of another tree S is obtained as the following pushout in $\mathbf{\Omega}$

$$\begin{tikzcd} \eta \arrow[r, "\ell"] \arrow[d, "r_R"'] & S \arrow[d] \\ R \arrow[r] & S \cup_\ell R \end{tikzcd}$$

On the set of edges $E(T)$ of a tree T, the poset structure of Remark 1.1 translates as $e \leqslant f$ if and only if the (unique) path from e to the root of T contains f. Hence the root r_T is

[6]The set of objects of $\Omega(T)$ is the set of edges of T, and for any choice of edges $e_1, \dots, e_n, e$, one has $\Omega(T)(e_1, \dots, e_n; e) = \{*\}$ if and only if there exists a subtree of T with leaves $\{e_1, \dots, e_n\}$ and root e (necessarily unique), and $\Omega(T)(e_1, \dots, e_n; e) = \varnothing$ otherwise. The operadic composition corresponds to *grafting* of subtrees, which means successive identifications of the root of a subtree with a leaf of another.

[7]That is, $\mathbf{\Omega}(S,T) = \mathsf{Op}(\Omega(S), \Omega(T))$.

the unique maximal element, while the minimal elements are the leaves of T and the input edges of stumps. In particular, any map of trees $f\colon S \to T$ induces a map of posets $f\colon E(S) \to E(T)$ between the sets of edges of S and T.

For any tree T, we denote by $\Omega[T]$ the corresponding representable dendroidal set. Observe that there is a natural isomorphism of dendroidal sets

$$\Omega[T] \simeq \mathcal{N}_d(\Omega(T)).$$

1.3. **Tensor product of dendroidal sets.** The *Boardman-Vogt tensor product* ([BV73]) of two operads P, Q is the operad $P \otimes Q$ characterized by the property that there are equivalences of categories

$$\mathsf{Alg}_Q(\mathsf{Alg}_P(\mathbf{sSets})) \simeq \mathsf{Alg}_{P\otimes Q}(\mathbf{sSets}) \simeq \mathsf{Alg}_P(\mathsf{Alg}_Q(\mathbf{sSets})),$$

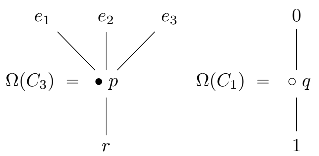
where the category of algebras over an operad is monoidal with the objectwise cartesian product. The operad $P \otimes Q$ is a quotient of $P \times Q$ under some relations, notably the *interchange relation*:

$$(p \otimes y) \circ (c_1 \otimes q, \ldots, c_n \otimes q) = \sigma^*_{n,m}(d \otimes q) \circ (p \otimes d_1, \ldots, p \otimes d_m),$$

where $\circ$ denotes the total operadic composition and the permutation $\sigma_{n,m}$ is the unique element of Σ_{nm} making sense of the above formula.

Let us depict this in an example. We depict tensoring by the 1-corolla $C_1 \simeq [1]$, as it will also be the only case we use the Boardman-Vogt tensor product.

Example 1.3. *Consider the operads*



The Boardman-Vogt interchange relation for $\Omega(C_3) \otimes \Omega(C_1)$ yields the identification:

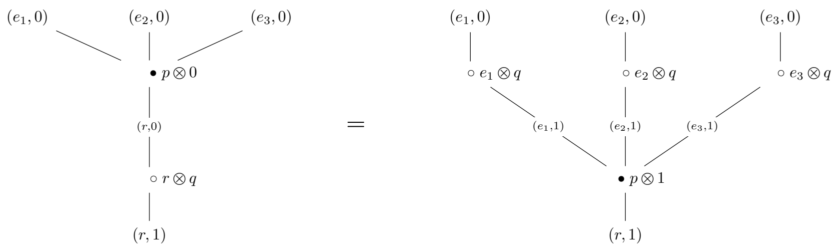


FIGURE 4. The Boardman-Vogt interchange relation for $\Omega(C_3) \otimes \Omega(C_1)$.

The bilinear functor on $\mathbf{\Omega}$ extends to a bifunctor of dendroidal sets

$$\otimes\colon \mathbf{dSets} \times \mathbf{dSets} \longrightarrow \mathbf{dSets}$$

such that on trees S, T one has

$$\Omega[T] \otimes \Omega[S] = \mathcal{N}_d(\Omega(T) \otimes \Omega(S))$$

and it is cocontinuous in each variable.

There are natural isomorphisms $X \otimes Y \simeq Y \otimes X$, but this tensor product is not associative up to coherent isomorphisms, hence it does not make $\mathbf{dSets}$ symmetric monoidal. It is however associative up to coherent homotopies [HM22, §4.4], and this makes the ∞-category of dendroidal ∞-operads (see Section 1.4) a symmetric monoidal ∞-categories ([HM24]). In particular, it is the good homotopical notion of product to consider on $\mathbf{dSets}$. Observe that it restricts to the cartesian product on simplicial sets.

There is an inner-hom object bifunctor

$$\mathrm{Hom}\colon \mathbf{dSets} \times \mathbf{dSets} \longrightarrow \mathbf{dSets},$$

right adjoint to $\otimes$ in each variable. We denote by $\hom(-,-)$ the underlying simplicial set, that is

$$\hom(X,Y) = i^* \,\mathrm{Hom}(X,Y), \quad \hom(X,Y)_n \simeq \mathbf{dSets}(i_!(\Delta^n)\otimes X, Y).$$

1.4. **Dendroidal homotopy theory.** The homotopy theory of dendroidal ∞-operads was defined in [MW07] and [CM11] by extending Joyal's theory of ∞-categories as quasicategories. We will not need explicit definition of the following fundamental notions, which we will treat syntetically:

- Analogously to the description of the morphisms in Δ as generated by faces and degeneracies under the simplicial identities, morphisms in $\mathbf{\Omega}$ are also generated ([HM22, §3.3.4]) by dendroidal faces and degeneracies, together with isomorphisms of trees, which satisfy the *dendroidal identities*.
- Some of the notions of the homotopy theory of simplicial sets can be formulated in the context of dendroidal sets. In particular, one can talk about the *boundary* of a tree and inner and external horns of a tree. Inner horns are relative to inner edges, while external horns divide into to classes, relative respectively to leaf vertices or root vertices [8].

Definition 1.4. A dendroidal set X is a *dendroidal ∞-operad* if it has the right lifting property against inner horn inclusions of trees.

A dendroidal set X is *normal* if, for any tree T, the action of $\mathsf{Aut}(T)$ on the set X_T is free.

Observe that, if $X \simeq \mathcal{N}_d P$, then X is normal if and only if P is Σ-free.

Theorem 1.5 ([CM11])**.** *There exists a model structure on the category* $\mathbf{dSets}$ *of dendroidal sets, called the* operadic model structure*, with the following properties:*

(1) The cofibrations are the normal monomorphisms.
(2) The fibrant objects are the dendroidal ∞-operads.
(3) A map between normal dendroidal sets is a weak equivalence if and only if for every dendroidal ∞-operad X, the map

$$\hom(B,X) \longrightarrow \hom(A,X)$$

is a categorical equivalence of ∞-categories.

Moreover, this model structure is left proper and cofibrantly generated.

The homotoy theory of dendroidal left fibrations, which will appear in Section 4, was first defined in [Heu11] and generalizes that for left fibration of simplicial sets. It models ∞-operads cofibred in groupoids.

Definition 1.6. A morphism of dendroidal sets $p\colon Y \to X$ is a *dendroidal left fibration* if it has the right lifting properties against inner and leaf horn inclusions of trees.

Notation 1.7. For a morphism of dendroidal sets $p\colon Y \to X$ and an object x of X, we denote by Y_x the fibre over x, that is,

$$Y_x \simeq p^{-1}(x).$$

If p is a dendroidal left fibration, Y_x is a Kan complex ([HM22, Remark 9.60]).

Theorem 1.8 ([Heu11])**.** *Let X be a dendroidal set. The category* $\mathbf{dSets}/X$ *carries a left proper cofibrantly generated model structure, called the* covariant model structure*, with the following properties:*

[8]They extend, respectively, left and right fibrations of simplicial sets. Contrarily to these latter, they are not dual to each other as $\mathbf{\Omega}$ does not admit a self duality.

(1) The cofibrations are the normal monomorphisms over X.
(2) The fibrant objects are the dendroidal left fibrations over X.
(3) A map $(A,u) \to (B,v)$ between normal objects over X is a weak equivalence if and only if for any dendroidal left fibration (Y,p), the map

$$\hom_X(B,Y) \longrightarrow \hom_X(A,Y)$$

is a weak homotopy equivalence of Kan complexes, where

$$\hom_X(A,Y) = \hom(A,Y) \times_{\hom(A,X)} \{u\}.$$

(4) A map $(Y,p) \to (Z,q)$ between dendroidal left fibrations is a weak equivalence if and only if, for any object x of X, the induced map $Y_x \to Z_x$ between the fibres over x is a weak homotopy equivalence of Kan complexes.

Moreover, if $f\colon X \to Y$ is a map of dendroidal sets, the adjunction

$$f_! \colon \mathbf{dSets}/X \rightleftarrows \mathbf{dSets}/Y : f^*$$

is a Quillen adjunction with respect to the covariant model structure, which is a Quillen equivalence when f is an operadic weak equivalence.

Slicing over the point, that is under the equivalence sSets $\simeq \mathbf{dSets}/\eta$, the operadic model structure yields the Joyal model structure, and the covariant model structure yields the homonimous one for left fibrations over a simplicial set.

Coming back to the discussion after Example 1.3, let us observe that $\hom(-,-)$ homotopy-enriches **dSets** with the operadic model structure in simplicial sets with the Joyal model structure ([HM22, Proposition 9.28]). When localizing for the covariant model structure on $\mathbf{dSets}/X$, we find a simplicial enrichment with respect to the Kan-Quillen model structure ([HM22, Proposition 9.66]). In particular, $\hom(-,-)$, resp. $\hom_X(-,-)$, gives a model for the mapping space when computed on bifibrant objects, see next Section 1.5.

1.5. **Last conventions.** When in presence of the canonical fully faithful functors $i\colon \Delta \to \mathbf{\Omega}$, $i_!\colon$ sSets $\to$ **dSets**, $\mathbf{\Omega} \to \mathsf{Op} \to$ **dSets**. In particular, given a tree T, we still write T for the operad $\Omega(T)$ and for the dendroidal set $\Omega[T]$. Observe that there are isomorphisms of dendroidal sets $\eta \simeq \Delta^0$ and $C_1 \simeq \Delta^1$[9]. We will freely use any of these functors explicitly whenever it is convenient or helps to emphasize a point.

Given a model category M, we write $\mathsf{Map}_M(-,-)$, or just $\mathsf{Map}(-,-)$ when clear from the context for its mapping space. In particular, when $M =$ **dSets** with the operadic model structure, A is a normal dendroidal set and X a dendroidal ∞-operad, one may choose

Don't really know what to do with this one...

$$\mathsf{Map}(A,X) = \hom(A,X)^{\simeq}$$

the maximal sub Kan complex of the simplicial mapping object $\hom(A,X)$. When we take $\mathbf{dSets}/Y$ with the covariant model structure, we can choose

$$\mathsf{Map}((A,\alpha),(X,\beta)) = \hom_Y(A,X) := \hom(A,X) \times_{\hom(A,Y)} \{\alpha\}$$

whenever (A,α) is a normal dendroidal set over Y and (X,β) a dendroidal left fibration over Y.

2. Localization of dendroidal ∞-operads

We now define derived localization of dendroidal sets, in a way that extends that of quasi-categories in the sense of Joyal and Lurie ([Lur17]). We then construct a model for normal dendroidal sets in Theorem 2.4. For compatibility of localization of dendroidal sets with the covariant model structure we direct the reader to Section 4.1.

[9]The analogy stops here: for $n \neq 1$, there is not even a map $C_n \to \Delta^n$.

2.1. **The definition of localization.** Let us start by some preliminary

Definition 2.1. A *normalization* of a dendroidal set X is a trivial fibration $X' \xrightarrow{\sim} X$ in the operadic model structure, with X' normal.

Given a morphism of dendroidal sets $f\colon X \to Y$, a *normalization* of f is a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\wr"'] & Y' \arrow[d, "\wr"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where both vertical arrows are normalizations.

Normalizations exist and are unique up to operadic weak equivalence. As explained in [HM22, Remark 9.21], an explicit construction of a normalizarion with contractible fibres is given by the projection $X \times W^*\mathcal{E} \to X$, where $\mathcal{E}$ is the simplicial Barrat-Eccles operad and W^* is the operadic homotopy coherent nerve functor, right adjoint to $W_!$.

Definition 2.2. Let $\lambda\colon X \to Y$ be a morphism between normal dendroidal sets and let $S \subseteq X_{C_1}$ be a subset of 1-morphisms. The map λ is a *localization of X at the set of morphisms S* if, for any dendroidal ∞-operad Z, the morphism between the simplicial hom objects

$$\mathrm{hom}(Y, Z) \longrightarrow \mathrm{hom}(X, Z)$$

is fully faithful, with essential image given by those maps $X \to Z$ sending S to equivalences.

If $\lambda\colon X \to Y$ is a morphism between non-necessarily normal dendroidal sets, we say that λ is a localization if any, or equivalently one, normalization λ' of λ is.

Localization is unique up to operadic weak equivalence, and we denote by $X[S^{-1}]$ 'the' localization.

Observe that, if we denote by $\mathrm{hom}_S(X, Z)$ the full sub simplicial set of $\mathrm{hom}(X, Z)$ spanned by those maps $X \to Z$ sending S to equivalences in Z, the universal property of the localization allows to identify $\mathrm{hom}(X[S^{-1}], Z)$ with $\mathrm{hom}_S(X, Z)$.

Remark 2.3. It is immediate to see that the definition applied to a morphism of dendroidal sets recovers the localization of quasi-categories in the sense of [Lur17, Definition 1.3.4.1].

We can construct an explicit model for the localization of a normal dendroidal set.

Proposition 2.4. *Denote by J be nerve of the connected groupoid on two objects. Given a normal dendroidal set X and a subset $S \subseteq X_{C_1}$ of 1-morphisms in X, the localization of X at S is realized by the map $\lambda\colon X \to \mathcal{L}(X, S)$ defined by the pushout diagram*

$$\begin{tikzcd} \coprod_{s\in S} C_1 \arrow[r] \arrow[d] & X \arrow[d, "\lambda"] \\ \coprod_{s\in S} J \arrow[r] & \mathcal{L}(X, S) . \end{tikzcd} \tag{2.1}$$

Proof. Observe that, by left properness of the operadic model structure, the pushout in Equation (2.1) is a homotopy pushout. Let Z be a dendroidal ∞-operad and consider the map

$$\lambda^*\colon \mathrm{hom}(\mathcal{L}(X, S), Z) \longrightarrow \mathrm{hom}(X, Z).$$

The essential image of λ^* consists of all functors sending S to equivalences, so we just need to show λ is also fully faithful. To this end, it is sufficient to show that the diagram

$$\begin{tikzcd}
\hom(\mathcal{L}(X,S)\otimes C_1, Z) \arrow[r] \arrow[d] & \hom(X\otimes C_1, Z) \arrow[d] \\
\hom(\mathcal{L}(X,S)\otimes \partial C_1, Z) \arrow[r] & \hom(X\otimes \partial C_1, Z)
\end{tikzcd}$$

is a homotopy pullback for the Joyal model structure, which happens if the diagram

$$\begin{tikzcd}
X\otimes \partial C_1 \arrow[r] \arrow[d] & X\otimes C_1 \arrow[d] \\
\mathcal{L}(X,S)\otimes \partial C_1 \arrow[r] & \mathcal{L}(X,S)\otimes C_1
\end{tikzcd} \tag{2.2}$$

is a homotopy pushout for the operadic model structure. Since the diagram in (2.1) is a transfinite composition of homotopy pushouts along the morphisms $\{C_1 \xrightarrow{s} J\}_{s\in S}$, it is sufficient to show that it is an homotopy pushout just in the case when $S = \{s\}$, where it appears as the front face of the commutative cube

$$\begin{tikzcd}
C_1\otimes\partial C_1 \arrow[rr] \arrow[dr, "s\otimes\mathrm{id}"] \arrow[dd] & & C_1\otimes C_1 \arrow[dr, "s\otimes\mathrm{id}"] \arrow[dd] & \\
& X\otimes\partial C_1 \arrow[rr] \arrow[dd] & & X\otimes C_1 \arrow[dd] \\
J\otimes\partial C_1 \arrow[rr] \arrow[dr] & & J\otimes C_1 \arrow[dr] & \\
& \mathcal{L}(X,S)\otimes\partial C_1 \arrow[rr] & & \mathcal{L}(X,S)\otimes C_1
\end{tikzcd}$$

All but the front face of the cube are homotopy pushouts, hence we conclude that the front face is one as well. This concludes the proof. □

The above construction essentially means that to localize a dendroidal set at a set of 1-morphisms, we can first localize its underlying simplicial set and then glue it to the original dendroidal set X, as we explain in the next

Remark 2.5. Let X be a normal dendroidal set, and write $M := i^*X$ for its underlying simplicial set. Of course, we have $S \subseteq X_{C_1} = M_1$, so we can localize M at S. Let $\overline{\lambda}\colon M \to \mathcal{L}(M,S)$ be the localization map. The localization of X at S can be realized as the following homotopy pushout

$$\begin{tikzcd}
M = i^*X \arrow[r, "\epsilon_X"] \arrow[d, "\overline{\lambda}"] & X \arrow[d, "\lambda"] \\
\mathcal{L}(M,S) \arrow[r] & \mathcal{L}(X,S)
\end{tikzcd}$$

where the top horizontal map is the counit of the adjunction $(i_!, i^*)$.

Given two endofunctors of X which preserve S, there is an easy way to check when they are homotopy equivalent as endofunctors of the localization $X[S^{-1}]$. Recall the following

Definition 2.6. Let $f, g\colon X \to Y$ be two maps of dendroidal sets. An *homotopy* between f and g is a morphism $h\colon X\otimes C_1 \to Y$ in **dSets** with the property that $h_0 = f$ and $h_1 = g$, where h_i is the restriction of h along the leaf, resp. root, inclusions $\eta \xrightarrow{\{i\}} C_1$, $i = 0$, resp. $i = 1$.

For $x \in X_\eta$, the arrow $h_x\colon f(x) \to g(x)$ in Y_{C_1} is called a *component* of h.

Lemma 2.7. *Let X be normal, $S \subseteq X_{C_1}$ a set of 1-morphisms. Let $f, g\colon X \to X$ be two maps sending S to itself. If f and g are homotopic via a homotopy whose components are in S, then f and g are homotopy equivalent as maps $f, g\colon X[S^{-1}] \to X[S^{-1}]$.*

Proof. Let $h\colon X \otimes C_1 \to X$ be the homotopy between f and g whose components lie in S. The transpose of $h^*\colon \hom(X,Z) \to \hom(X \otimes C_1, Z) \simeq \hom(C_1, \hom(X,Z))$ induces a morphism of simplicial sets $\hom_S(X,Z) \times \Delta^1 \to \hom(X,Z)$. By the hypotheses on f and g and fullness of $\hom_S(X,Z)$ in $\hom(X,Z)$, there is an induced homotopy $\mathfrak{h}\colon \hom_S(X,Z) \times \Delta^1 \to \hom_S(X,Z)$. To prove that $\mathfrak{h}$ is a natural equivalence, it suffices to see that, for any object ϕ of $\hom_S(X,Z)$, the arrow $\mathfrak{h}_\phi\colon f^*(\phi) \to g^*(\phi)$ is an equivalence in $\hom_S(X,Z)$, that is in $\hom(X,Z)$. So we only need to check that for any $x \in X_\eta$, the 1-morphism $(\mathfrak{h}_\phi)_x\colon \phi(f(x)) \to \phi(g(x))$ is an equivalence in Z. As $(\mathfrak{h}_\phi)_x = \phi(h_x)$, and since by hypothesis $\phi(h_x)$ is a weak equivalence as h_x is an arrow in S, we have that $\mathfrak{h}$ establishes an equivalence between f^* and g^*. This concludes the proof. □

The homotopy theory of algebras over a dendroidal ∞-operad X is governed by the covariant model structure on the over-category $\mathbf{dSets}/X$, so it is a natural question to investigate compatibility of the localization of dendroidal ∞-operads with the covariant model structure for dendroidal left fibrations. We study this in Section 4.1 (Theorem 4.4).

3. The delocalization theorem

We start Section 3.1 by introducing the formalism of forests and wide independent maps between them. We define the operad of elements and the root functor of a dendroidal set in Section 3.3, and prove the localization result, namely Theorem 3.14, in Section 3.4.

3.1. **The category of forests.** A *forest* is a finite disjoint union of trees. We denote by Φ be the *category of forests* obtained from $\mathbf{\Omega}$ by formally adjoining finite coproducts. By extending the inclusion $\mathbf{\Omega} \hookrightarrow \mathsf{Op}$ to forests under finite-coproduct, one has $\Omega\colon \Phi \hookrightarrow \mathsf{Op}$, where, given a forest $F = \coprod_{i=1}^n T_i$, the operad $\Omega(F)$ is defined as $\Omega(F) := \coprod_{i=1}^n \Omega(T_i)$. For any such F, every tree T_i is called a *constituent* of F, and $\mathbf{\Omega}$ embeds into Φ by seeing a tree to a forest with one constituent. Let ϕ be the subcategory of Φ whose objects are non-empty forests and where morphisms are the *independent* maps of forests, where

Definition 3.1. A morphism of forests $f\colon F \longrightarrow G$ is called *independent* if F and G are non-empty and, writing $F = \coprod_{i=1}^n T_i$ and $G = \coprod_{j=1}^m S_j$, f is specified by a map of sets $\alpha\colon \{1,\dots,n\} \to \{1,\dots,m\}$ and morphisms of trees $f_i\colon T_i \to S_{\alpha(i)}$, for $i = 1,\dots,n$, such that, whenever $i \neq i'$ and $\alpha(i) = \alpha(i') = j$, for all edges $e \in E(T_i)$, $e' \in E(T_{i'})$ the edges $f_i(e)$ and $f_{i'}(e')$ are incomparable in the poset $E(S_j)$.

By the independency condition, it suffices to check that the edges $f(r_{T_i})$ and $f(r_{T_{i'}})$ are independent.

The restriction of the disjoint union to ϕ endows it with a non-unital symmetric monoidal structure; we write $\oplus := \coprod_{|\phi}$ and call it direct sum. In the category ϕ, every morphism can be written as a direct sum of maps whose target has only one constituent. Observe also that ϕ is no longer the coproduct in ϕ, as a candidate for the codiagonal $F \oplus F \to F$ would not satisfy the independence property.

Remark 3.2. The two categories of forests presented here are the ones appearing in various comparisons of the dendroidal and Lurie's model for ∞-operads: the category Φ is the one considered in [HM24], while the category ϕ is the one appearing in [HHM16].

3.2. **Wide maps of forests.** These were first introduced in [HM22].

Definition 3.3. A map of forests $f\colon F \to G$ is called *wide* if, for any constituent tree S of the forest G, any maximal monotonic path in the poset $E(S)$, that is from a minimal element to the root r_S, contains one element of the form $f(r_T)$ for some T constituent tree of F.

Observe that if f is also independent, if there exists such a constituent T, then it is unique.

When the target of a map of forest has only one constituent, we can reformulate the wideness condition in the following useful way.

Lemma 3.4. *Let $T_1, \ldots, T_n, S$ be trees and $f \colon T_1 \oplus \cdots \oplus T_n \to S$ a map of forests. If f is independent, then f is wide if and only if $S(f(r_{T_1}), \ldots, f(r_{T_n}); r_S) \neq \varnothing$.*

Proof. It suffices to observe that, given edges $e_1, \ldots, e_n$ of S, the only obstruction to the existence of a subtree with leaves $e_1, \ldots, e_n$ and root r_S is the existence of a maximal monotonic path $\mathfrak{p} = \{l < a_1 < \cdots < a_m < r_S\}$ in the poset of edges of S such that $\{e_1, \ldots, e_n\} \cap \mathfrak{p} = \varnothing$. $\square$

Every wide morphism is the direct sum of morphisms of the above type, and with Theorem 3.4 it is immediate to see that wide maps are closed under composition. We introduce some classes of wide maps in the following

Example 3.5.

(1) *All maps of linear trees $[n] \to [m]$ are wide and independent.*
(2) *All maps of trees $T \to S$ are independent maps of forests.*
(3) *A morphism of trees $f \colon T \to S$ is* root preserving *if $f(r_T) = r_S$. Any root preserving map is wide, and the converse is true whenever the root vertex of S is not unary.*
(4) *Given a forest F, one can construct the tree $\overline{F}$ as follows. Write $F = \oplus_{i=1}^n T_i$ and let $\overline{F}$ be the the tree obtained as the grafting of the trees $(T_1, \ldots, T_n)$ on the leaves $\{l_1, \ldots, l_n\}$ of an n-corolla C_n,*

$$\overline{F} = C_n \circ (T_1, \ldots, T_n).$$

The forest root face *of $\overline{F}$ is the natural inclusion*

$$F = T_1 \oplus \cdots \oplus T_n \longrightarrow C_n \circ (T_1, \ldots, T_n) = \overline{F}.$$

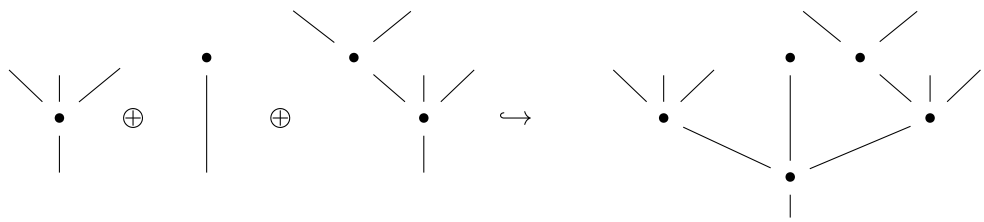


FIGURE 5. A forest root face.

In fact, it is easy to see that wide independent maps of forests are generated, under composition and direct sum, by forest root face inclusions and root preserving morphisms.

3.3. **The operad of elements and the root functor.** We want to functorially associate to every dendroidal set X a discrete operad $\mathbf{\Omega}/X$, with the requirement that, when X is a simplicial set, $\mathbf{\Omega}/X$ is a category and coincides with the category of elements of X. Also, the nerve of the constructed operad will need to map back to X.

Recall that, for every dendroidal set X, the overcategory $\mathbf{dSets}_{/X}$ is a symmetric monoidal category with the disjoint union $\sqcup$. Let $(\mathbf{dSets}_{/X})^{\sqcup}$ be the operad associated to it. A first, naive way would be to define $\mathbf{\Omega}/X$ as the suboperad of $(\mathbf{dSets}_{/X})^{\sqcup}$ spanned by the representable objects. However, it turns out this is not the good definition (see later Remark 3.11), so we need to give a more elaborate one.

Definition 3.6. Let X be a dendroidal set. Its *operad of elements* $\mathbf{\Omega}/X$ is the sub operad of $(\mathbf{dSets}_{/X})^{\sqcup}$ specified by the following data:

- the set of objects of $\mathbf{\Omega}/X$ is the set of elements of X as a presheaf, that is,

$$\mathsf{Ob}(\mathbf{\Omega}/X) = \{(T, \alpha) \mid T \in \mathbf{\Omega} \text{ and } \alpha \colon T \to X\};$$

- for objects $(S_1, \alpha_1), \dots, (S_n, \alpha_n)$, (R, β), the set of operations
$$\mathbf{\Omega}/X((S_1, \alpha_1), \dots, (S_n, \alpha_n); (R, \beta))$$
is given by the wide and independent maps of forests $f\colon S_1 \oplus \dots \oplus S_n \to R$ such that
$$\beta \circ f = (\alpha_1, \dots, \alpha_n)\colon S_1 \bigsqcup \dots \bigsqcup S_n \to X$$
as maps of dendroidal sets.

To be explicit, the partial composition of f as above with $g \in (\mathbf{\Omega}/X)((Q_1, \beta_1), \dots, (Q_m, \beta_m); (S_i, \alpha_i))$ along (S_i, α_i) is given by the wide map of forests
$$(S_1 \oplus \dots \oplus S_{i-1}) \oplus (\oplus_{j=1}^m Q_j) \oplus S_{i+1} \dots \oplus S_n \xrightarrow{(\mathrm{id}^{\oplus i-1}, g, \mathrm{id}^{\oplus n-i})} \oplus_{h=1}^n S_h \xrightarrow{f} R.$$

Remark 3.7. By the independency condition on forest morphisms, the operad of elements $\mathbf{\Omega}/X$ is Σ-free.

Example 3.8.

- *For a simplicial set M, the operad of elements $\mathbf{\Omega}/M$ coincides with the category of elements of M, usually denoted by Δ/M. Its objects are the pairs $([n], f)$, with $n \geqslant 0$ and $f\colon \Delta^n \to M$; a morphism $F\colon ([n], f) \to ([m], g)$ is just a map of representable simplicial sets $F\colon \Delta^n \to \Delta^m$ over M.*
- *If $X \simeq N_d P$ for an operad P, an object of $\mathbf{\Omega}/X$ can be written as a pair (T, α), with α in $\mathrm{Hom}_{\mathsf{Op}}(T, P)$. Similarly, if $X \simeq W^*\mathbb{P}$, where W^* is the homotopy coherent nerve of a simplicial operad $\mathbb{P}$, then for an object (S, β) of $\mathbf{\Omega}/X$ β is an element in $\mathrm{Hom}_{\mathsf{sOp}}(W(S), \mathbb{P})$, where $W(S)$ is the simplicial operad given by the Boardman-Vogt resolution of $\Omega(S)$.*

The construction $X \mapsto \mathbf{\Omega}/X$ is functorial in X via postcomposition, and we get an endofunctor
$$N_d(\mathbf{\Omega}/-)\colon \mathbf{dSets} \to \mathbf{dSets}.$$
Let us postpone to later the following very useful

Proposition 3.9. *The functor $N_d(\mathbf{\Omega}/-)$ is cocontinuous. Moreover, it preserves normal monomorphisms of dendroidal sets.*

In other words, it means that $N_d(\mathbf{\Omega}/-)$ is equivalent to the left Kan extension of its restriction to representables, and moreover in an homotopy-coherent way

Now, let T be a tree and (S, α) an object of $\mathbf{\Omega}/T$. Evaluation of α at the root of S yields an assignment
$$\mathsf{Ob}(\mathbf{\Omega}/T) \ni (S, \alpha) \mapsto \alpha(r_S) \in \mathsf{Ob}(T) = E(T). \tag{3.1}$$

Proposition 3.10. *The assignment in Equation (3.1) extends to a map of operads*
$$\mathfrak{r}_T\colon \mathbf{\Omega}/T \longrightarrow T$$
that we call the root functor *for T.*

Proof. As the set of operations of T is either empty or a singleton, we just need to check that, if there exists a wide independent map of forests $f\colon (S_1, \alpha_1), \dots, (S_n, \alpha_n) \to (R, \beta)$, then $T(\alpha_1(r_{S_1}), \dots, \alpha_n(r_{S_n}); \beta(r_R)) \neq \varnothing$. By Lemma 3.4, $R(f(r_{S_1}), \dots, f(r_{S_n}); r_R) \neq \varnothing$, and since β is a map of operads and $\beta \circ f = (\alpha_1, \dots, \alpha_n)$ we have the thesis, as wanted. □

Remark 3.11. The assignment in Equation 3.1 makes sense even for the *full* suboperad $\mathbf{\Omega}/X^{\sqcup} \subseteq (\mathbf{dSets}/X)^{\sqcup}$, where one considers all maps of forests $T_1 \oplus \dots \oplus T_n \to S$. However, Proposition 3.10 does *not* hold for $\mathbf{\Omega}/X^{\sqcup}$. An easy way to see this is taking $X = C_n$ any n-corolla; the candidate image for the binary operation $(\mathrm{id}, \mathrm{id})\colon (C_3, \mathrm{id}), (C_3, \mathrm{id}) \to (C_3, \mathrm{id})$ has to be a subtree of C_n with leaves (r_{C_n}, r_{C_n}) and root r_{C_n}. Such subtree does not exist.

Proposition 3.10 characterises $\mathbf{\Omega}/X$ as the maximal sub operad of $\mathbf{\Omega}/X^{\sqcup}$ such that Proposition 3.10 holds.

Because of cocontinuity of $\mathcal{N}_d(\mathbf{\Omega}/-)$ (Theorem 3.9), we can extend the root functor to every dendroidal set.

Definition 3.12. Let X be a dendroidal set. The *root functor* of X is the morphism of dendroidal sets

$$\mathfrak{r}_X : \mathcal{N}_d(\mathbf{\Omega}/X) \longrightarrow X$$

defined as the colimit

$$\mathfrak{r}_X := \operatorname*{colim}_{T \to X} \mathfrak{r}_T : \mathcal{N}_d(\mathbf{\Omega}/T) \longrightarrow T.$$

In particular, given an object (S, α) of $\mathcal{N}_d(\mathbf{\Omega}/X)$, image via the root functor consists in the evaluation of αX at the root of S,

$$\mathfrak{r}_X(S, \alpha) = \alpha(r_S) \in X_\eta.$$

Observe that, for any tree T and morphism $\alpha\colon T \to X$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{\Omega}/T \arrow[r, "\mathbf{\Omega}/\alpha"] \arrow[d, "\mathfrak{r}_T"'] & \mathbf{\Omega}/X \arrow[d, "\mathfrak{r}_X"] \\
T \arrow[r, "\alpha"] & X
\end{tikzcd}$$

Remark 3.13. If M is a simplicial set, the root functor coincides with the *last vertex functor*

$$\mathfrak{r}_M\colon \Delta/M \to M, \quad \mathfrak{r}_M([n], f) = f(n).$$

We conclude this section with the promised proof of cocontinuity of the nerve operad of elements construction.

of Proposition 3.9. We denote by θ the restriction of $\mathcal{N}_d(\mathbf{\Omega}/-)$ to the representables, that is $\theta = \mathcal{N}_d(\mathbf{\Omega}/-)_{|\mathbf{\Omega}}$, and we write $\widehat{\theta}$ for its left Kan extension along the Yoneda embedding.

Given a dendroidal set X and a tree T, the set $\widehat{\theta}(X)_T$ may be described as

$$\widehat{\theta}(X)_T = \{((Q, x), (T, u)) \mid Q \in \mathbf{\Omega}, x\colon Q \to X, u\colon T \to \theta(Q)\}/\sim,$$

where, for any $\alpha\colon S \to Q$, one identifies

$$((Q, x), (R, \alpha_* u)) \sim ((S, x\alpha), (R, u)),$$

where $\alpha_* = \mathcal{N}_d(\mathbf{\Omega}/\alpha)$. Functoriality in T is given by letting faces and degeneracies act on the second component, while functoriality in X is obtained via the first component.

We construct a natural equivalence

$$\psi_X : \widehat{\theta}(X) \to \mathcal{N}_d(\mathbf{\Omega}/X)$$

by defining its components $(\psi_X)_T$ by induction on the number of vertices of the tree T. When $T \simeq \eta$, we set

$$\psi_X((Q, x), (\eta, u)) := x_* \circ u\colon \eta \to \mathcal{N}_d(\mathbf{\Omega}/X).$$

The assignment respects the equivalence relation, as (with the same notations as above) one has

$$\psi_X((Q, x), (\eta, \alpha_* u)) = x_*(\alpha_* u) = (x\alpha)_* u = \psi_X((S, x\alpha), (\eta, u)),$$

and it is straightforward to see that it induces a bijection.

Similarly, given a n-corolla C_n, $n \geqslant 0$, and an element $((S, x), (C_n, u))$, we set

$$\psi_X((S, x), (C_n, u)) := x_* u\colon C_n \to \mathcal{N}_d(\mathbf{\Omega}/X),$$

and it is the same calculation which shows that it descends to the quotient and s a bijection.

For the inductive step, let us consider a tree T with at least two vertices and decompose it as the grafting $T = R \cup_a S$, with $R, S \neq \eta$. For any dendroidal set Y, there is a natural isomorphism

$$\mathcal{N}_d(\mathbf{\Omega}/Y)_T \simeq \mathcal{N}_d(\mathbf{\Omega}/Y)_R \underset{\mathcal{N}_d(\mathbf{\Omega}/Y)_\eta}{\times} \mathcal{N}_d(\mathbf{\Omega}/Y)_S.$$

The isomorphism is compatible with the equivalence relation defining $\widehat{\theta}(X)$, which means that $\widehat{\theta}(X)$ satisfies the same strict Segal condition, that is, there is a natural isomorphism

$$\widehat{\theta}(X)_T \simeq \widehat{\theta}(X)_R \underset{\widehat{\theta}(X)_\eta}{\times} \widehat{\theta}(X)_S.$$

One defines the map $(\psi_X)_T$ as

$$(\psi_X)_T := (\psi_X)_R \underset{(\psi_X)_\eta}{\times} (\psi_X)_S.$$

It respects the equivalence relation and is a bijection, so we only need to check that ψ_X is well defined. This follows from the fact that any tree T decomposes as the grafting of corollas, and the decomposition is unique up to isomorphism and operadic associativity relations, and the Segal isomorphism is compatible with these latter, which shows that the definition of $(\psi_X)_T$ does not depend on the decomposition of T, as wanted. $\square$

3.4. **The delocalization result.** In Theorem 3.5 we introduced root preserving morphisms of trees, which are in particular wide independent maps of forests. Given a dendroidal set X, we say that a morphism $f\colon (S,\alpha) \to (R,\beta)$ is *root preserving* if $f\colon S \to R$ is a root preserving map of trees. Denote by $\mathcal{R}_X$ the set of root preserving morphisms of $\mathbf{\Omega}/X$.

If $f\colon (S,\alpha) \to (R,\beta)$ is root preserving, then $\mathfrak{r}(f) = \mathrm{id}_{f(r_S)}$, so the root functor factors via the localization of $\mathcal{N}_d(\mathbf{\Omega}/X)$ at $\mathcal{R}_X$, as

$$\begin{tikzcd}
\mathcal{N}_d(\mathbf{\Omega}/X) \arrow[rr, "\mathfrak{r}_X"] \arrow[dr, "\lambda"'] & & X \\
& \mathcal{N}_d(\mathbf{\Omega}/X)[\mathcal{R}_X^{-1}] \, . \arrow[ur, "\overline{\mathfrak{r}_X}"'] &
\end{tikzcd}$$

Theorem 3.14. *For any normal dendroidal set X, the root functor induces an operadic weak equivalence of dendroidal sets*

$$\overline{\mathfrak{r}_X}\colon \mathcal{N}_d(\mathbf{\Omega}/X)[\mathcal{R}_X^{-1}] \xrightarrow{\sim} X.$$

Proof. Let us use Theorem 2.4 and choose $\mathcal{L}(\mathcal{N}_d(\mathbf{\Omega}/X), \mathcal{R}_X)$ for the model of the localization of $\mathcal{N}_d(\mathbf{\Omega}/X)$ at $\mathcal{R}_X$. By Theorem 3.9, the functor $\mathcal{L}(-,\mathcal{R}_-)\colon \mathbf{dSets} \to \mathbf{dSets}$ is cocontinuous and preserves normal monomorphisms, so we proceed by skeletal filtration, reducing the proof to the case of $X \simeq T$ a tree, that is a representable dendroidal set. The root functor is the nerve of the map of operads $\mathfrak{r}\colon \mathbf{\Omega}/T \to T$, and we construct a section $\mathfrak{l}_T$ of $\mathfrak{r}_T$,

$$\mathfrak{l}_T\colon T \to \mathbf{\Omega}/T$$

and a homotopy

$$h\colon \mathcal{N}_d(\mathbf{\Omega}/T) \otimes C_1 \to \mathcal{N}_d(\mathbf{\Omega}/T)$$

between $\mathfrak{l}_T \circ \mathfrak{r}_T$ and $\mathrm{id}_{\mathbf{\Omega}/T}$, such that its components are root preserving morphisms. After Theorem 2.7, this implies that h becomes an equivalence between $\mathfrak{l}_T \circ \mathfrak{r}_T$ and id_T after localizing at $\mathcal{R}_T$, which means that $\mathfrak{l}_T$ and $\mathfrak{r}_T$ are homotopy inverses of the other once localized.

To construct h, consider an edge e of T, and let $T_e^\uparrow$ be the biggest subtree of T having e as root. We denote by $\iota_e\colon T_e^\uparrow \hookrightarrow T$ the associated subtree inclusion. Observe that, if $e \leqslant f$, then T_e is a subtree of T_f.

We define the morphism $\mathfrak{l}_T$ on an edge e of T as

$$\mathfrak{l}_T(e) := (T_e^\uparrow, \iota_e) \in \mathsf{Ob}(\mathbf{\Omega}/T).$$

Given edges $e_1, \ldots, e_n, e$ such that $T(e_1, \ldots, e_n; e) = \{*\}$, the map

$$\mathfrak{l}_T\colon \{*\} \longrightarrow \mathbf{\Omega}/T((T_{e_1}^\uparrow, \iota_{e_1}), \ldots, (T_{e_n}^\uparrow, \iota_{e_n}); (T_e^\uparrow, \iota_e))$$

selects the operation of $\mathbf{\Omega}/T$ given by the forest inclusion

$$\oplus_{i=1}^{n} T_{e_i}^{\uparrow} \hookrightarrow T_e^{\uparrow},$$

defined by the composition of the forest root face $\oplus_{i=1}^{n} T_{e_i}^{\uparrow} \to \overline{\oplus_{i=1}^{n} T_{e_i}^{\uparrow}}$ and the inner face $\partial\colon \overline{\oplus_{i=1}^{n} T_{e_i}^{\uparrow}} \to T_e^{\uparrow}$ sending each $T_{e_i}^{\uparrow}$ to itself and the new root vertex to the subtree $T_e^{\underline{e}}$.
As the operadic composition for T is the grafting of subtrees, we can check that it is a well defined map of operads $\mathfrak{l}_T\colon T \to \mathbf{\Omega}/T$. One checks that $\mathfrak{r}_T \circ \mathfrak{l}_T = \mathrm{id}_T$; for the other composition $\mathfrak{l}_T \circ \mathfrak{r}_T$, consider an object (S, α) in $\mathbf{\Omega}/T$. We have that

$$\mathfrak{l}_T \circ \mathfrak{r}_T(S, \alpha) = \mathfrak{l}_T(\alpha(r_S))) = (T_{\alpha(r_S)}^{\uparrow}, \iota_{\alpha(r_S)}) .$$

Since the image of α is contained in the subtree $T_{\alpha(r_S)}^{\uparrow}$, we can always write α as a composition

$$\begin{tikzcd} S \arrow[rr, "\alpha"] \arrow[dr, "h_{(S,\alpha)}"'] & & T \\ & T_{\alpha(r_S)}^{\uparrow} \arrow[ur, "\iota_{\alpha(r_S)}"'] & \end{tikzcd}$$

for an unique root preserving morphism $h_{(S,\alpha)}$. In particular, $h_{(S,\alpha)}$ is a morphism in $\mathbf{\Omega}/T$ and we obtain the collection of morphisms

$$h := \{h_{(S,\alpha)}\colon (S, \alpha) \to (T_{\alpha(r_S)}, \iota_{\alpha(r_S)})\}_{(S,\alpha)\in \mathsf{Ob}(\mathbf{\Omega}/T)}.$$

Let us show that h defines an homotopy between $\mathrm{id}_{\mathbf{\Omega}/T}$ and $\mathfrak{l}_T \circ \mathfrak{r}_T$. For this purpose, we need to check the Boardman-Vogt interchange relation, so consider objects $(S_1, \beta_1), \dots, (S_n, \beta_n), (R, \gamma)$ and an operation $f \in \mathbf{\Omega}/T((S_1, \beta_1), \dots, (S_n, \beta_n); (R, \gamma))$. Since $\gamma \circ f = (\beta_1, \dots, \beta_n)$, we have that

$$\gamma \circ f = (\iota_{\mathfrak{r}_T(\beta_1)}, \dots, \iota_{\mathfrak{r}_T(\beta_n)}) \circ (h_{(S_1,\beta_1)}, \dots, h_{(S_n,\beta_n)}),$$

which is precisely the wanted relation. Consider the functor of dendroidal set given by the dendroidal nerve of h; precomposing $\mathcal{N}_d(h)$ with the natural map $\mathcal{N}_d(\mathbf{\Omega}/T) \otimes C_1 \to \mathcal{N}_d(\mathbf{\Omega}/T \otimes C_1)$, we obtain an homotopy between between the identity of $\mathbf{\Omega}/T$ and the composition $\mathfrak{l}_T \circ \mathfrak{r}_T$. As the components of this homotopy are root preserving morphisms, it becomes an equivalence after localizing at $\mathcal{R}_T$, and this concludes the argument. □

The operadic model structure on dSets admits a left Bousfield localization whose homotopy category is equivalent to that of group-like $\mathbb{E}_\infty$-algebras, in turn equivalent to infinite loop spaces. It is called the *stable model structure*, and was first introduced by Bašić-Nikolaus in [BN14]. It has the property that the induced model structure on the overcategory sSets $\simeq$ dSets$/\eta$ coincides with the Kan-Quillen model structure. In particular, one localizes also by the arrow $C_1 \to J$, which becomes an equivalence, so from Theorem 3.14 we deduce the following

Corollary 3.15. *For any normal dendroidal set X, the root functor $\mathfrak{r}_X\colon \mathcal{N}(\mathbf{\Omega}/X) \to X$ is a weak equivalence in the stable model structure for* dSets.

Specializing the above results to simplicial sets, we hence obtain the following well-known results.

Corollary 3.16 (Joyal [Joy07], Stevenson [Ste17])**.** *For any simplicial set M, the last vertex functor induces a categorical weak equivalence*

$$\Delta/M[\mathcal{R}_M^{-1}] \xrightarrow{\sim} M,$$

where $\mathcal{R}_M$ is the class of morphisms $f\colon ([n], \alpha) \to ([m], \beta)$ such that $f(n) = m$.
In particular, the last vertex functor $\Delta/M \to M$ is a weak homotopy equivalence.

4. Localization and un/straightening equivalences

In this section, we describe the homotopy theory of algebras over a dendroidal ∞-operad in terms of locally constant algebras over its operad of elements. To this end, we study the compatibility of dendroidal localization with the covariant model structure in Section 4.1, and exploit the operadic weak equivalence given by the root functor (Theorem 3.14), combined with two operadic un/straightening equivalences: one for Σ-free discrete operads, constructed in [Pra25a], and another for dendroidal ∞-operads, proven by Heuts in [Heu11], which we recall in due course.

4.1. **Localization and the covariant model structure.** For a dendroidal set X and any subset of morphisms $S \subseteq X_{C_1}$, the localization map $\lambda\colon X \to X[S^{-1}]$ induces an adjunction of over-categories

$$\lambda_!\colon \mathbf{dSets}/X \rightleftarrows \mathbf{dSets}/X[S^{-1}] : \lambda^*,$$

which is a Quillen adjunction with respect to the covariant model structure for dendroidal left fibrations ([Heu11, Proposition 2.4]). It is natural to expect dendroidal left fibrations over the localization to be weakly equivalent to those left fibrations over X for which all the maps of fibres $f_!\colon Y_a \to Y_b$ induced by the morphisms $f\colon a \to b$ in S are weak homotopy equivalences of spaces. This is indeed the case, as we show in Theorem 4.4. Let us introduce some constructions first.

Construction 4.1. Let X be a dendroidal set and S a subset of X_{C_1}. For any $f\colon a \to b$ in S, one can construct the morphism

$$r_f\colon (\eta, \{b\}) \longrightarrow (C_1, f)$$

in $\mathbf{dSets}/X$ defines as follows:

$$\begin{tikzcd} \eta \arrow[rr, hook, "r"] \arrow[dr, "b"'] & & C_1 \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

where the arrow $r\colon \eta \to C_1$ is the inclusion of the edge η into the root of C_1, or more familiary the map $\{1\}\colon [0] \to [1]$.

We write $S_{/X}$ for the set of morphisms of the form r_f, for f ranging in S.

Given a model category $\mathcal{M}$ and some set $\mathcal{S}$ of morphisms in it, we can talk about $\mathcal{S}$*-local* objects in $\mathcal{M}$: there are the fibrant objects M for which, for any morphism $s\colon A \to B$ in $\mathcal{S}$, the morphism of mapping spaces

$$s_*\colon \mathsf{Map}_{\mathcal{M}}(B, M) \longrightarrow \mathsf{Map}_{\mathcal{M}}(A, M)$$

is a weak homotopy equivalence of spaces. For $\mathcal{M}$ given by the covariant model structure for dendroidal left fibrations over a dendroidal set X, we have an explicit way of computing mapping spaces between fibrant-cofibrant objects:

Remark 4.2. If (A, u) is a normal dendroidal set over X and (E, p) a dendroidal left fibration over X, we have an equivalence of spaces

$$\mathsf{Map}_{\mathbf{dSets}/X}((A, u), (E, p)) \simeq \hom_X(A, E) = \hom(A, E) \times_{\hom(A,X)} \{u\}.$$

As maps in $S_{/X}$ have cofibrant domains and codomains, we can rephrase $S_{/X}$-locality as follows.

Definition 4.3. A (E, p) dendroidal left fibration over X is $S_{/X}$-local if, for any f in S, the morphism

$$(r_f)_*\colon \hom(C_1, E) \times_{\hom(C_1,X)} \{f\} \longrightarrow \hom(\eta, E) \times_{\hom(\eta,X)} \{b\} \simeq E_b = p^{-1}(b),$$

induced by precomposition with the root inclusion $\eta \hookrightarrow C_1$, is a weak homotopy equivalence of Kan complexes.

We write $\mathbf{dSets}^S/X$ for the left Bousfield localization of the covariant model structure on $\mathbf{dSets}/X$ at the set of arrows $S_{/X}$, which, recall, is left proper and cofibrantly generated. In this model structure, the fibrant objects are the $S_{/X}$-local dendroidal left fibrations, while the cofibrations are the normal monomorphisms over X. In particular, a morphism between $S_{/X}$-local left fibrations $\varphi\colon (E,p) \to (E',p')$ is a weak equivalence in the localization if and only if it is a covariant weak equivalence, hence a fibrewise weak homotopy equivalence. We call the model structure

$$\mathbf{dSets}^S/X$$

the *$S_{/X}$-local covariant model structure*. It is precisely this localization that encodes the compatibility of localization with the covariant model structure:

Proposition 4.4. *Let X be a normal dendroidal set and $S \subseteq X_{C_1}$ a subset of morphisms. The localization map $\lambda\colon X \to X[S^{-1}]$ induces a Quillen adjunction*

$$\lambda_! : \mathbf{dSets}^S/X \rightleftarrows \mathbf{dSets}/X[S^{-1}] : \lambda^*$$

between the $S_{/X}$-local covariant model structure on $\mathbf{dSets}/X$ and the covariant model structure on $\mathbf{dSets}/X[S^{-1}]$. Moreover, the adjunction is a Quillen equivalence.

Proof. We know that the adjunction

$$\lambda_! : \mathbf{dSets}/X \rightleftarrows \mathbf{dSets}/X[S^{-1}] : \lambda^*$$

is a Quillen adjunction between covariant model structures ([HM22, Proposition 9.62]). To prove that it is still a Quillen adjunction after localizing on the left, it suffices to show that, given any dendroidal left fibration (E,p) over X, the element (E,p) is $S_{/X}$-local if and only if there exists a dendroidal left fibration (Y,p') over $X[S^{-1}]$ and a covariant weak equivalence $(E,p) \xrightarrow{\sim} \lambda^*(Y,p)$ in $\mathbf{dSets}/X$.

Fix any such dendroidal left fibration (E,p). Write $S \times C_1$, resp. $S \times J$, for the simplicial set given by the union $S \times C_1 = \bigsqcup_{s\in S} C_1$, resp. $S \times J = \bigsqcup_{s\in S} J$, where J is the nerve of the connected groupoid on two objects.

The pullback

$$E' := (S \times C_1) \times_X E \longrightarrow S \times C_1$$

is a left fibration of simplicial sets, and after [Lur09, Proposition 2.1.3.1] it is a Kan fibration. As such, it can be factored as the composition $E' \to E'' \to S \times C_1$, where $E' \to E''$ is a trivial Kan fibration and $E'' \to S \times C_1$ is a minimal Kan fibration. As explained in [GZ67, §5.4], there is an isomorphism $E'' \simeq S \times C_1 \times M$ for some minimal Kan complex M, and the map $E'' \to S \times C_1$ is the projection. In particular, the map $S \times J \times M \to S \times J$ is a minimal Kan fibration, fitting into the pullback diagram

$$\begin{tikzcd} E'' \arrow[r, "\phi"] \arrow[d] & S \times J \times M \arrow[d] \\ S \times C_1 \arrow[r] & S \times J \end{tikzcd}$$

An argument due to Joyal (see [KL21, Lemma 2.2.5]) shows that we can find a trivial Kan fibration $Z \to S \times J \times M$ and an isomorphism $E' \simeq \phi^*(Z)$ over E''. Thus we obtain a commutative diagram

$$\begin{tikzcd} E \arrow[d] & E' \arrow[l] \arrow[r] \arrow[d] & Z \arrow[d] \\ X & S \times C_1 \arrow[l] \arrow[r] & S \times J \end{tikzcd}$$

where the right hand square is a pullback and the map $Z \to S \times J$ is a Kan fibration. Define

$$Y := E \bigcup_{E'} Z$$

there is a canonical map $p' \colon Y \to X[S^{-1}]$, and its pullback along the surjection $(S \times J) \sqcup X \to X[S^{-1}]$ consists in the dendroidal left fibration $Z \sqcup E \to (S \times J) \sqcup X$. In particular, p' is a dendroidal left fibration as well. Moreover, for any object x of X, there is a weak homotopy equivalence of fibres $(E,p)_x \to \lambda^*(Y,p')_x$, hence a covariant weak equivalence $(E,p) \to \lambda^*(Y,p')$ over X, as wanted. This concludes the proof that there is an induced Quillen adjunction. It is straightforward to check that $\mathbb{R}\lambda^*$ is fully faithful, hence we conclude that the induced adjunction is a Quillen equivalence, as wanted. □

Remark 4.5. The above proof essentially extends to the dendroidal context Stevenson's proof of [Ste17, Proposition 5.11] for simplicial sets.

4.2. **Locally constant algebras.** Whenever we are given an operad P with a choice of weak equivalences S, one can study the homotopy theory of *S-locally constant P-algebras*, where

Definition 4.6. A simplicial P-algebra A is *S-locally constant* if A sends S to equivalences, that is for any $f \colon a \to b$ in S, the map $A(f) \colon A(a) \to A(b)$ is a weak homotopy equivalence of simplicial sets.

As a model category structure on a category $\mathcal{M}$ is determined, if it exists, by the cofibrations and the fibrant objects, we can safely give the following

Definition 4.7. Let P be a Σ-free operad and S a subset of morphisms of P. The *projective model structure for S-locally constant algebras* is the model structure on simplicial P-algebras whose fibrant objects are S-locally constant and projectively fibrant P-algebras, and where the cofibrations are the projective cofibrations. We denote it by $\mathsf{Alg}^S_{\mathbf{P}}(\mathbf{sSets})$.

Let us consider the operadic un/straightening equivalence of [Pra25a, Theorem 5.9], which consists, for any Σ-free operad P, in a Quillen equivalence

$$\rho^P_! \colon \mathbf{dSets}/\mathcal{N}_d(P) \rightleftarrows \mathsf{Alg}_P(\mathbf{sSets}) : \rho^*_P \tag{4.1}$$

between the covariant model structure on the left and the projective model structure on the right. The left adjoint $\rho^P_!$ is defined as follows.

Definition 4.8. The functor $\rho^P_! \colon \mathbf{dSets}/\mathcal{N}_d P \to \mathsf{Alg}_P(\mathbf{sSets})$ is the essentially unique cocontinuous functor characterized by the fact that, for any tree T and morphism $\alpha \colon T \to \mathcal{N}_d(P)$, the P-algebra $\rho^P_!(T,\alpha)$ is defined as

$$\mathsf{Ob}(P) \ni c \mapsto \rho^P_!(T,\alpha)(c) \simeq \mathsf{Env}(T) \underset{\mathsf{Env}(P)}{\times} \mathsf{Env}(P)_{/c},$$

where Env denotes the symmetric monoidal envelope [Lur17, §2.2.4].

We can now prove the following

Proposition 4.9. *Let P be a discrete Σ-free operad. The operadic un/straightening adjunction $(\rho^P_!, \rho^*_P)$ induces a Quillen equivalence*

$$\rho^P_! \colon \mathbf{dSets}^S/\mathcal{N}_d(P) \rightleftarrows \mathsf{Alg}^S_P(\mathbf{sSets}) : \rho^*_P$$

between the covariant model structure for S-local left fibrations and the projective model structure for S-locally constant P-algebras.

Proof. In general, given a Quillen adjunction of model categories $F \colon \mathcal{M} \rightleftarrows \mathcal{N} : G$ and a choice of (cofibrant) arrows $\mathcal{S}$ in $\mathcal{M}$, one can consider the left Bousfield localizations $\mathcal{L}_{\mathcal{S}}\mathcal{M}$ on $\mathcal{M}$ and the transfered localization on $\mathcal{N}$, resp. $\mathcal{L}_{\mathbb{L}F(\mathcal{S})}\mathcal{N}$, of $\mathcal{M}$ at $\mathcal{S}$, resp. of $\mathcal{N}$ at $\mathbb{L}F(\mathcal{S})$. It is a standard result of model categories ([Hir03]) that, if (F,G) is a Quillen equivalence, then there is an induced Quillen equivalence $F \colon \mathcal{L}_{\mathcal{S}}\mathcal{M} \rightleftarrows \mathcal{L}_{\mathbb{L}F(\mathcal{S})}\mathcal{N} : G$, where $\mathcal{L}_{\mathbb{L}F(\mathcal{S})}\mathcal{N}$ is the left Bousfield localization of $\mathcal{N}$ at $\mathbb{L}F(\mathcal{S})$. This implies that we only need to characterize the fibrant objects of $\mathcal{L}_{\rho^P_!(S_{/\mathcal{N}_d(P)})}\mathsf{Alg}_P(\mathbf{sSets})$ as the locally constant fibrant P-algebras.

An object A in $\mathcal{L}_{\rho_!^P(\mathcal{S})}(\mathsf{Alg}_P(\mathbf{sSets}))$ is fibrant if and only if it is projectively fibrant and $\rho_!^P(\mathcal{S}_{/\mathcal{N}_d P})$-local. As the pair $(\rho_!^P, \rho_P^*)$ is a Quillen adjunction, this is equivalent to asking $\rho_P^*(A)$ to be $S_{/\mathcal{N}_d P}$-local, that is, that

$$\mathsf{Map}_{\mathbf{dSets}/\mathcal{N}_d(P)}((C_1, f), \rho_P^*(A)) \longrightarrow \mathsf{Map}_{\mathbf{dSets}/\mathcal{N}_d(P)}((\eta, \{b\}), \rho_P^*(A))$$

is a weak homotopy equivalence. As A is projectively fibrant, $\rho_P^*(A)$ is a dendroidal left fibration, so in particular local with respect to the map $(\eta, \{a\}) \to (C_1, f)$ induced by the leaf inclusion $\ell\colon \eta \hookrightarrow C_1$, so the above is an equivalence if and only if the map of spaces

$$\mathsf{Map}_{\mathbf{dSets}/\mathcal{N}_d(P)}((\eta, \{a\}), \rho_P^*(A)) \longrightarrow \mathsf{Map}_{\mathbf{dSets}/\mathcal{N}_d(P)}((\eta, \{b\}), \rho_P^*(A))$$

is a weak homotopy equivalence. After Theorem 4.2, there is an equivalence of spaces

$$\mathsf{Map}_{\mathbf{dSets}/\mathcal{N}_d(P)}((\eta, \{x\}) \simeq \rho_P^*(A)_x.$$

As the right adjoint ρ_P^* enjoys the property that for any P-algebra A and any object x of P, there is an equivalence

$$\rho_P^*(A)_c \simeq A(c)$$

between the fibre of $\rho_P^*(A)$ over c and the value of A on c ([Pra25a, Lemma 2.9]), we conclude that $\rho_P^*(A)$ is $S_{/\mathcal{N}_d P}$-local if and only if the map

$$A(f)\colon A(a) \longrightarrow A(b)$$

is a weak homotopy equivalence of spaces, which means that A is S-locally constant. This concludes the proof. □

Remark 4.10. The above result does not really depend on the definition of $\rho_!^P$, but only on the fact that $\rho_!^P$ preserves and detects fibrant S-locally constant algebras. This follows from the equivalence between the fiber of the unstraightening of an algebra and evaluation, $\rho_!^P(F)_a \simeq F(a)$.

4.3. **Algebras over dendroidal ∞-operads.** We can now state our main results for this section.

Corollary 4.11. *Let X be a normal dendroidal ∞-operad.*

(1) The operadic un/straightening equivalence of [Pra25a] *for the operad of dendrices $\mathbf{\Omega}/X$ induces a Quillen equivalence*

$$\rho_!^{\mathbf{\Omega}/X}\colon \mathbf{dSets}^{\mathcal{R}_X}/\mathcal{N}_d(\mathbf{\Omega}/X) \rightleftarrows \mathsf{Alg}^{\mathcal{R}_X}_{\mathbf{\Omega}/X}(\mathbf{sSets}) : \rho^*_{\mathbf{\Omega}/X}$$

between the covariant model structure for $\mathcal{R}_X$-local dendroidal left fibrations over $\mathcal{N}_d\mathbf{\Omega}/X$ and the projective semimodel structure for $\mathcal{R}_X$-locally constant $\mathbf{\Omega}/X$-algebras.

(2) There is a zig-zag of Quillen equivalences

$$\mathsf{Alg}_{W_!(X)}(\mathbf{sSets}) \xleftarrow{\sim} \bullet \xrightarrow{\sim} \mathsf{Alg}^{\mathcal{R}_X}_{\mathbf{\Omega}/X}(\mathbf{sSets})$$

between the projective model structures for simplicial $W_!(X)$-algebras and for $\mathcal{R}_X$-locally constant algebras on $\mathbf{\Omega}/X$.

Proof. The first point is just Theorem 4.9 instantiated with $P = \mathbf{\Omega}/X$, which we know to be Σ-free, so let us focus on point (2). After [Heu11, Proposition 2.4], if $f\colon X \to Y$ is an operadic weak equivalence between normal dendroidal sets, the induced adjunction between covariant model structures

$$f_!\colon \mathbf{dSets}/X \rightleftarrows \mathbf{dSets}/Y : f^*$$

is a Quillen equivalence. Together with Theorem 4.9, we obtain that the root functor yields a Quillen equivalence

$$(\imath_X)_!\colon \mathbf{dSets}^{\mathcal{R}_X}/\mathbf{\Omega}/X \rightleftarrows \mathbf{dSets}/X : \imath_X^*$$

between the localization of the covariant model structure for S-local dendroidal left fibrations over $\mathbf{\Omega}/X$ and the covariant model structure on $\mathbf{dSets}/X$. Following [Heu11, Theorem 2.7], this latter is Quillen equivalent to the projective model structure on $\mathsf{Alg}_{W_!(X)}(\mathbf{sSets})$, and combining this with point (1) we get the wanted zig-zag of Quillen equivalences

$$\mathsf{Alg}_{W_!(X)}(\mathbf{sSets}) \xleftarrow{\sim} \bullet \xrightarrow{\sim} \mathsf{Alg}^{\mathcal{R}_X}_{\mathbf{\Omega}/X}(\mathbf{sSets}).$$

□

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F.P.: Institut Galilée, Université Sorbonne Paris Nord, 99 avenue Jean-Baptiste Clément, 93430 Villetaneuse, France

Email address: `pratali@math.univ-paris13.fr`