Homework 1

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Exercise 1

Consider the ATLAS robot from Boston Dynamics in the pictures above. On the left, ATLAS is standing. On the right, ATLAS is performing a backflip. If ATLAS actuators can produce unbounded torques, establish whether each of the following statements is true or not, and briefly justify your answer.

- *a. While standing, ATLAS is fully actuated.*
- *b. While doing backflip, ATLAS is fully actuated.*

If we consider the definition of ACTUATED systems provided in the slides, we can analyze the two statements as follows:

1. **The first statement is TRUE.**

When ATLAS is standing, it is in contact with the ground, meaning it can exert reaction forces and torques through its legs. This allows it to generate any desired acceleration instantaneously in all its degrees of freedom, making it a FULLY ACTUATED system. Since its actuators can apply unbounded torques, ATLAS has complete control over its motion in this scenario.

2. **The second statement is FALSE.**

During a backflip, ATLAS is airborne for part of the motion, meaning it no longer has ground contact. In this phase, it cannot apply external forces to modify its position freely in all directions. For example, it cannot instantaneously accelerate left or right at will, as its motion is governed by the conservation of linear and angular momentum. Since it cannot achieve arbitrary accelerations in all degrees of freedom at any instant, ATLAS is UNDERACTUATED during the backflip.

Exercise 2

Consider the spatial mechanism on the above-left (an experimental surgical manipulator developed at the National University of Singapore, with three identical parts, each with a prismatic joint, P, and two universal joints, U) and the spatial mechanism on the above-right *(made by 6 identical bars with all spherical joints). Determine the number of the degrees of freedom for each mechanism, comparing the result with your intuition about the possible motions of these mechanisms. For each mechanism, write its configuration space topology.*

1. **Mechanism in the above-left**

$m = 6$ since it's a spacial mechanism
$N = 3 + 3 + 1 + 1$ if we consider the 6 long parts, the ground and the last common part
$J = 3 + 3 + 3$ if we consider the 3 prismatic joints and the 6 universal ones
$f = 3 + 6 + 6$ considering that each prismatic joint has 1 DoFs and each universal joint has 2 DoFs
Applying Grübler's formula:

$$DoFs = 6(8 - 1 - 9) + 15 = 3$$

Intuitively, it may not seem that this mechanism has exactly three degrees of freedom. However, analyzing its structure, we observe that:

- The three prismatic joints are independent and allow translations.
- Each set of three universal joints allows rotation around two distinct axes.

Thus, the mechanism permits three independent translations and four constrained rotations. For this reason we can say that the mechanism in figure has **7 DoFs**, confirming that the Grübler's formula only gives a lower bound in case of not independent joints. This leads to the following configuration space topology:

$$I^3 \times T^4 \text{ } (I^7 \text{ if we also consider the joint limits })$$

2. **Mechanism in the above-right**

$m = 6$ since it's a spacial mechanism
$N = 6 + 1 + 1$ if we consider the 6 bars, the ground and the above common part
$J = 6 + 6$ if we consider the two set of 6 spherical joints
$f = 3 \times 12$ considering that each spherical joint has 3 DoFs
By applying the Grubler's formula we get:

$$DoFs = 6(8 - 1 - 12) + 36 = 6$$

The mechanism in figure can rotate around the z-axis (even if it can rotate by a small angle) and can move along a spherical shell. Thanks to the presence of 12 spherical joints, each vertical link can rotate around its z-axis independently by all the other links. Therefore, in conclusion, the system has **9 DoFs**. Even in this case, the result confirms how Grübler's formula represents a lower bound for the degrees of freedom in mechanisms with interdependent constraints.

Configuration space topology:

$$T^6 \times S^2 \times S^1$$

Exercise 3

State whether the following sentences regarding underactuation or fully actuation are true or false. Briefly justify your answers.

- *a. A car with inputs the steering angle and the throttle is underactuated.*
- *b. The KUKA youBot system on the slides is fully actuated.*
- *c. The hexarotor system with co-planar propellers is fully actuated.*
- *d. The KUKA iiwa 7-DOF robot is redundant and it cannot be underactuated because we know that all redundant systems are not.*

1. **The first statement is TRUE.**
A car with only steering (which affects the yaw rate but does not provide direct lateral acceleration) and throttle (which controls forward acceleration) as inputs is underactuated because it cannot move sideways instantaneously. Instead, to achieve lateral displacement, it must perform a sequence of maneuvers, such as turning and moving forward or backward. This indirect control over lateral movement confirms that the system is underactuated.

2. **The second statement is TRUE.** The KUKA YouBot with 4 Mecanum wheels is fully actuated. The Mecanum wheels allow the robot to move in any direction instantaneously, providing independent control over all degrees of freedom: forward/backward, sideways, and rotation (yaw). This means the robot can generate arbitrary accelerations in all directions, making it fully actuated.

3. **The third statement is FALSE.**
In general, drones with any number of co-planar propellers are underactuated. This is because, in such drones, it is impossible to move parallel to the ground without affecting the drone's orientation. In the case of a hexarotor with co-planar propellers, the system cannot independently control all six degrees of freedom, as the available actuators cannot fully decouple translation and rotation. Consequently, the hexarotor is unable to generate arbitrary accelerations in all directions simultaneously.

4. **The fourth statement is FALSE.**
The 7-DoFs KUKA iiwa robot is redundant because it has one more degree of freedom than the 6 needed to position and orient the end-effector in space; but redundancy and underactuation are separate concepts. In fact, redundancy is related to the kinematics, while actuation is related to the dynamics of the system. A redundant system like the KUKA iiwa does not imply it is underactuated. It simply means it has more control capabilities than needed for a particular task.

Exercise 4

State whether each of following distributions are involutive or not and briefly justify your answer. If possible, find the annihilator for each distribution.

a. $\Delta_1 = \left\{ \begin{bmatrix} -3x_2 \\ 1 \\ -1 \end{bmatrix} \right\}, U \in \mathbb{R}^3$

b. $\Delta_2 = \left\{ \begin{bmatrix} -1 \\ 0 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_2 \\ -\alpha \\ x_1 \end{bmatrix} \right\}, U \in \mathbb{R}^3$ with α the last digit of your matriculation number

c. $\Delta_3 = \left\{ \begin{bmatrix} 2x_3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2x_2 \\ x_1 \\ -1 \end{bmatrix} \right\}, U \in \mathbb{R}^3$

4a

$$\Delta_1 = \left\{ \begin{bmatrix} -3x_2 \\ 1 \\ -1 \end{bmatrix} \right\}, U \in \mathbb{R}^3$$

By definition, any 1-dimensional distribution is involutive since $[f(x), f(x)] = 0$, and the zero vector belongs to any distribution by default.
In this case $d = 1$ and $n = 3$. The dimension of the distribution is equal to 1, therefore we can compute the dimension of the annihilator:

$$dim(\Delta(x)) = 1 \implies dim(\Delta^{\perp}(x)) = n - 1 = 2$$

The annihilator is identified by a set of co-vectors such that $\omega * F(x) = 0$, so we get:

$$\begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \end{pmatrix} * \begin{pmatrix} -3x_2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies -3x_2\omega_1 + \omega_2 - \omega_3 = 0$$

So, in the end, we get:

$$\omega_3 = \omega_2 - 3x_2\omega_1$$

so we have two free parameters. By choosing $\omega_1 = 1$ and $\omega_2 = 0$ and then $\omega_1 = 0$ and $\omega_2 = 1$ we get the two co-vectors:

$$\omega_1^* = \begin{bmatrix} 1 & 0 & -3x_2 \end{bmatrix}$$
$$\omega_2^* = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

Therefore:

$$\Delta^{\perp}(x) = \text{span}\left\{ \begin{bmatrix} 1 & 0 & -3x_2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \right\}$$

4b

Matr. P38000265, so:

$$\Delta_2 = \left\{ \begin{bmatrix} -1 \\ 0 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_2 \\ -5 \\ x_1 \end{bmatrix} \right\}$$

Here $d = 2$ and $n = 3$.

$$F = \begin{bmatrix} -1 & x_2 \\ 0 & -5 \\ x_3 & x_1 \end{bmatrix} \implies rank(F) = 2 \ \forall x \in U \in \mathbb{R}^3$$

Therefore the distribution is non-singular $\forall x \in U$ and $dim(\Delta(x) = 2$.

Now we need to check if the distribution is involutive by computing the Lie Brackets:

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ -5 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 - x_1 \end{bmatrix}$$

By computing the determinant of F', it's possible to see that the involutiveness depends on some conditions:

$$F' = \begin{bmatrix} -1 & x_2 & 0 \\ 0 & -5 & 0 \\ x_3 & x_1 & -1 - x_1 \end{bmatrix} \implies det(F') = -5(1 + x_1)$$

$$rank(F') = \begin{cases} 3 \text{ if } x_1 \neq -1 \implies \Delta(x) \text{ is non involutive} \\ 2 \text{ if } x_1 = -1 \implies \Delta(x) \text{ is involutive} \end{cases}$$

Here, the annihilator has dimension 1 since:

$$dim(\Delta(x)) = 2 \implies dim(\Delta^{\perp}(x)) = n - 2 = 1$$

The annihilator is identified by a set of co-vectors such that $\omega * F(x) = 0$, so we get:

$$\begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \end{pmatrix} * \begin{pmatrix} -1 & x_2 \\ 0 & -5 \\ x_3 & x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{cases} -\omega_1 + x_3\omega_3 = 0 \\ x_2\omega_1 - 5\omega_2 + x_1\omega_3 = 0 \end{cases}$$

So, in the end, we get:

$$\begin{cases} \omega_1 = x_3\omega_3 \\ \omega_2 = \frac{1}{5}[x_2x_3 + x_1]\omega_1 \end{cases}$$

so we have one parameter which is ω_3. So, by choosing $\omega_3 = 1$ we get:

$$\omega^* = [x_3, \ \frac{1}{5}(x_2x_3 + x_1), \ 1]$$

and:

$$\Delta^{\perp}(x) = \text{span}\left\{ \begin{bmatrix} x_3 & \frac{1}{5}(x_2x_3 + x_1) & 1 \end{bmatrix} \right\}$$

4c

$$\Delta_1 = \left\{ \begin{bmatrix} 2x_3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2x_2 \\ x_1 \\ -1 \end{bmatrix} \right\}, \; U \in \mathbb{R}^3$$

Here $d = 2$ and $n = 3$.

$$F = \begin{bmatrix} 2x_3 & -2x_2 \\ 1 & x_1 \\ 0 & -1 \end{bmatrix} \implies rank(F) = 2 \; \forall x \in U \in \mathbb{R}^3$$

Now we need to check if the distribution is involutive by computing the Lie Brackets:

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2x_3 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2x_2 \\ x_1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2x_3 \\ 0 \end{bmatrix}$$

By computing the determinant of F', it's possible to see check the involutiveness:

$$F' = \begin{bmatrix} 2x_3 & -2x_2 & -4 \\ 1 & x_1 & 2x_3 \\ 0 & -1 & 0 \end{bmatrix} \implies det(F') = 4(1 + x_3^2)$$

$$4(1 + x_3^2) \neq 0 \; \forall x_3 \in U \implies rank(F) = 3 \; \forall x \in U$$

Therefore, the distribution is non-involutive.

The dimension of the distribution is equal to 2, therefore we can compute the dimension of the annihilator:

$$dim(\Delta(x)) = 2 \implies dim(\Delta^{\perp}(x)) = n - 1 = 1$$

The annihilator is identified by a set of co-vectors such that $\omega * F(x) = 0$, so we get:

$$\begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \end{pmatrix} * \begin{pmatrix} 2x_3 & -2x_2 \\ 1 & x_1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{cases} 2x_3\omega_1 + \omega_2 = 0 \\ -2x_2\omega_1 + x_1\omega_2 - \omega_3 = 0 \end{cases}$$

So, in the end, we get:

$$\begin{cases} \omega_2 = -2x_3\omega_1 \\ \omega_3 = -2(x_2 + x_3x_1)\omega_1 \end{cases}$$

By choosing $\omega_1 = 1$ we get the co-vector:

$$\omega_1^* = \begin{bmatrix} 1 & -2x_3 & -2(x_2 + x_3x_1) \end{bmatrix}$$

Therefore:

$$\Delta^{\perp}(x) = \text{span}\left\{ \begin{bmatrix} 1 & -2x_3 & -2(x_2 + x_3x_1) \end{bmatrix} \right\}$$

Exercise 5

Using the accessibility rank condition, show that a set of Pfaffian constraints that does not depend on the generalized coordinates, $A\dot{q} = 0$*, with* A *constant, is always integrable (completely holonomic system).*

Let's consider $A \in \mathcal{R}^{kxn}$ and $q, \dot{q} \in \mathcal{R}^n$, where $A(q)$ is the Pfaffian matrix. $A\dot{q} = 0$ tells me which are the kinematics constraints my system is subject to. Now, let's consider the orthogonal distribution to A, which is given by :

$$\Delta = span\{g_j(q) \in \mathcal{R}^n : Ag_j(q) = 0 \ j = 1...m\}$$

If I have k constraints, I will find $m = n - k$ vectors which are independent one to the other. Since the Pfaffian matrix is constant, the vectors $g_j(q)$ will not depend on the generalized coordinates. Therefore, a generic vector belonging to the above distribution can be written like a linear combination of the vector fields spanning the distribution:

$$\dot{q} = \sum_{j=1}^{m} g_j u_j = Gu, \quad G = [g_1 \ \ldots \ g_m].$$

Now, for DRIFTLESS systems, controllability is easy to check thanks to the ACCESSIBILITY DISTRIBUTION made of all the g_j vectors and their relative Lie Brackets. Since g_j vectors do not depend on the generalized coordinates, the Lie Brackets will all be null, so the dimension of the accessibility distribution (which is the distribution generated by the vector fields $f\ _1\ g_2$ ... g_m and all the Lie Brackets that can be generated by these vector fields) will be equal to the dimension of the G matrix. Therefore:

$$dim(\Delta_A) = m < n$$

Then, according to the accessibility rank condition, is possible to conclude that $\dot{q} = Gu$ is NOT CONTROLLABLE. So kinematics distribution has only integrable constraints, and the system is completely holonomic.

Therefore, we conclude that if the Pfaffian constraints $A\dot{q} = 0$ do not depend on the generalized coordinates (i.e., A is constant), the system is always integrable and completely holonomic.

Exercise 6

Consider the Raibert's hooper robot of the picture above. It has the following kinematic constraint in the Pfaffian form, with I the moment of inertia of the body and m the leg mass concentrated at the foot. Compute a kinematic model of such a robot and show whether this system is holonomic or not. [Hint: Use Matlab symbolic toolbox and the null command to ease your work. Do not care about the physical meaning of the kinematic inputs.]

The MATLAB script performs the following steps:

1. **Define symbolic variables:** The generalized coordinates are θ, ψ, and I, while l, m, and d are system parameters.

2. **Construct the Pfaffian constraint matrix:** The kinematic constraint is expressed in Pfaffian form $A\dot{q} = 0$, where the matrix A is defined as:

$$A = \begin{bmatrix} I + m(l+d)^2 & m(l+d)^2 & 0 \end{bmatrix}$$

3. **Compute the kinematic distribution:** The script determines the null space of matrix A using the `null(A)` function, which provides a basis for the allowable velocity space G:

$$G = \text{null}(A)$$

   The columns of G represent the basis vectors of the kinematic distribution.

4. **Compute the Lie bracket:** If the kinematic distribution has more than one basis vector, the script calculates the Lie bracket of the two basis vectors g_1 and g_2. The Lie bracket is computed as:

$$[g_1, g_2] = \frac{\partial g_1}{\partial q} g_2 - \frac{\partial g_2}{\partial q} g_1$$

   using the `jacobian` function in MATLAB.

5. **Construct the accessibility matrix:** The accessibility matrix Δ_A is formed by combining the basis vectors of G with the computed Lie bracket:

$$\Delta_A = \begin{bmatrix} G(:,1) & G(:,2) & \text{Lie bracket} \end{bmatrix}$$

6. **Determine system holonomy:** The rank of Δ_A is computed using `rank(Delta_A)`. If the rank is equal to the number of generalized coordinates (i.e., 3 in this case), the system is nonholonomic. Otherwise, the constraints are integrable, and the system is holonomic.

In conclusion, the rank of Δ_A is 3, which matches the dimension of the configuration space. This confirms that the system is **nonholonomic**, meaning that the kinematic constraints are non-integrable.

```
clc; clear; close all;
syms I m l d theta psi real
A = [I + m*(l + d)^2, m*(l + d)^2, 0];
G = null(A);
disp('Kinematics distribution:')
disp(G)
if length(G) > 1
    Lie_bracket = jacobian(G(:,1), [theta, psi, I]) * G(:,2) - jacobian(G(:,2), [theta, psi, I]) * G(:,1);
    disp('Lie bracket:')
    disp(Lie_bracket)
end
Delta_A = [G(:,1), G(:,2), Lie_bracket];
disp('Accessibility matrix:')
disp(Delta_A)
disp('Accessibility matrix rank:')
disp(rank(Delta_A))
```

```
Kinematics distribution:
```

$$\begin{pmatrix} -\frac{m\,(d+l)^2}{m\,d^2+2\,m\,d\,l+m\,l^2+\mathrm{I}} & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
Lie bracket:
```

$$\begin{pmatrix} \frac{m\,(d+l)^2}{(m\,d^2+2\,m\,d\,l+m\,l^2+\mathrm{I})^2} \\ 0 \\ 0 \end{pmatrix}$$

```
Accessibility matrix:
```

$$\begin{pmatrix} -\frac{m\,(d+l)^2}{m\,d^2+2\,m\,d\,l+m\,l^2+\mathrm{I}} & 0 & \frac{m\,(d+l)^2}{(m\,d^2+2\,m\,d\,l+m\,l^2+\mathrm{I})^2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

```
Accessibility matrix rank:
    3
```

Figure 1: MATLAB script: Kinematics distribution, Lie bracket, Accessibility matrix, and its rank.