UNIVERSITÀ BOCCONI | BAI

30540 Lecture Notes
Linear Programming

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Linear Programming (3 hrs - Tue, Apr 16, 2024)

1.1 Introductory Examples

1.1.1 A production problem. Each week, a food industry produces 2 types of flour:

- x_1 is the quantity (in a given weight unit) of type 1 flour produced per week
- x_2 is the quantity (in a given weight unit) of type 2 flour produced per week

x_1 and x_2 can be any non-negative, possibly decimal, number since they can sell all the flour that is produced. The profit for each x_1 is 3 and that for each x_2 is 5.

Flour is manufactured in three plants which have different capacities per week.

- Plant 1 has 3 hrs available
- Plant 2 has 21 hrs available
- Plant 3 has 25 hrs available

- The production of each x_1 uses 6 hours of Plant 2, and 3 hours of Plant 3
- The production of each x_2 uses 1 hour of Plant 1, 2 hours of Plant 2, and 7 hours of Plant 3

The goal of the company is to maximize $z = 3x_1 + 5x_2$. Summarizing:

$$\max z = 3x_1 + 5x_2,$$

$$x_1, x_2 \in \mathbb{R}$$

subject to

$$\begin{aligned}
&x_2 \leq 3 && \text{(P1: used hrs less than avail. hrs)}\\
&6x_1 + 2x_2 \leq 21 && \text{(P2: used hrs less than avail. hrs)}\\
&3x_1 + 7x_2 \leq 25 && \text{(P3: used hrs less than avail. hrs)}\\
&x_1 \geq 0 &&\\
&x_2 \geq 0 &&
\end{aligned}$$

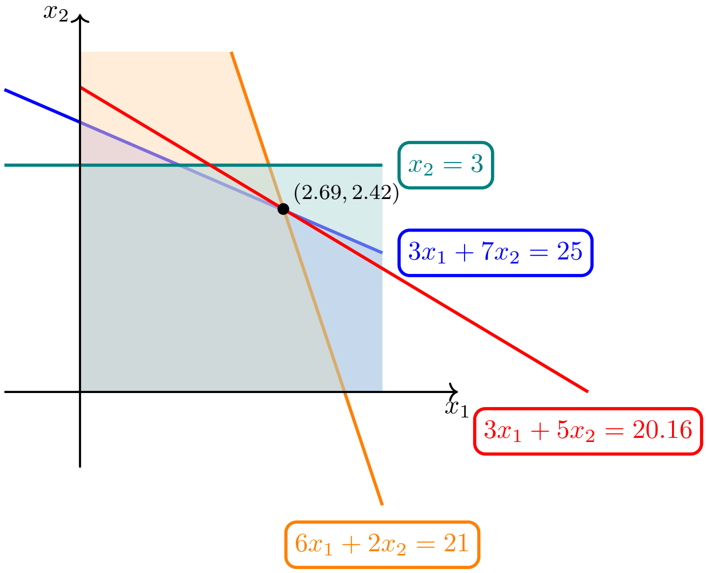

Figure 1.1. Feasible region and optimal value.

The optimal solution is $x_1^* = 97/36 \approx 2.69, x_2^* = 29/12 \approx 2.42$. At (x_1*, x_2^*) we have:

$$3 \cdot \frac{97}{36} + 5 \cdot \frac{29}{12} \approx 20.16$$

subject to

$$97/36 \leq 3 \text{ (P1: hrs available: NOT BINDING)}$$
$$21 \leq 21 \text{ (P2: full capacity: BINDING)}$$
$$25 \leq 25 \text{ (P3: full capacity: BINDING)}$$

1.1.2 A mix problem. For a healthy life, every day we must get a minimum quantity of certain substances (vitamins, fats, fiber, etc.) contained in some ingredients (flour, sugar, milk, etc.) that contain the substances in various proportions.

To simplify things, consider just two ingredients and two substances. One unit of ingredient I_1 contains 7 units of substance A and 2 units of substance B. One unit of ingredient I_2 contains 2 units of substance A and 12 units of substance B. Every day, at least 28 units of A and 24 of B are required.

One unit of ingredient I_1 costs 5 and one of I_2 costs 10. We want to get at least the minimum of both substances paying the minimum cost.

- x_1 is the quantity of ingredient 1
- x_2 is the quantity of ingredient 2

Summarizing:

$$\min 5x_1 + 10x_2$$

subject to

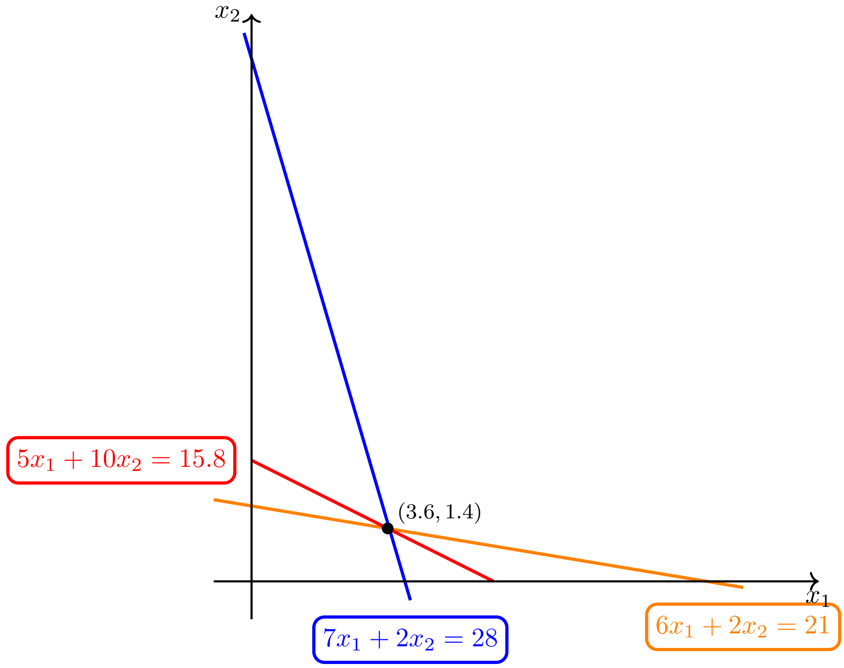

Figure 1.2. Feasible region and optimal value.

$$7x_1 + 2x_2 \geq 28 \text{ (min req. for subst. A)}$$
$$2x_1 + 12x_2 \geq 24 \text{ (min req. for subst. B)}$$

$$x_1 \geq 0$$
$$x_2 \geq 0$$

and the minimum is 15.8 attained at $(3.6, 1.4)$.

1.1.3 The production problem, revisited. Suppose that, in the first production problem, x_1 and x_2 are quantities of barrels of flour. A barrel is the common measure for the flour market (a barrel equals 196 pounds). In this case, x_1 and x_2 can't take decimal values and their domain is instead restricted to the non-negative integeres. The problem then becomes

$$\max_{x_1, x_2 \in \mathbb{N}} 3x_1 + 5x_2,$$
subject to

$$x_2 \leq 3 \quad \text{(P1: used hrs less than avail. hrs)}$$
$$6x_1 + 2x_2 \leq 21 \quad \text{(P2: used hrs less than avail. hrs)}$$
$$3x_1 + 7x_2 \leq 25 \quad \text{(P3: used hrs less than avail. hrs)}$$

$$x_1 \geq 0$$
$$x_2 \geq 0$$

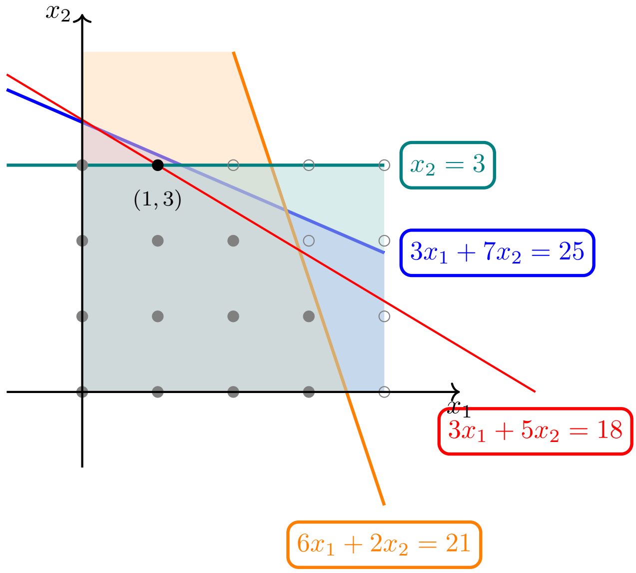

Figure 1.3. Feasible region and optimal value in Integer Problem.

and the analytical geometry technique does not work anymore. In this simple case we can actually examine all the feasible solutions, the points with integer coordinates in the feasible set found for the first case. The optimal solution is found to be $(1, 3)$ with a value of 18 for the profit. What is remarkable is that the integer optimal solution is "far" from the non-integer optimal solution $(2.69, 2.42)$: approzimating the non-integer solution gives $(3, 2)$, which is non-feasible. Truncating the non-integer solution gives a feasible solution which is not optimal.

1.2 Formalization

The problems in the previous lecture share some features.

1.2.1 Names. In the models, a choice of the value for the variables represents a decision about an activity to be done. For historical reasons, this is called "programming", in the sense of planning, and these problems are called **programming problems**. Accordingly, the variables are called **decision variables**. Sometimes, the problem itself is called "program".

1.2.2 Variables. The modeling dictates the type of variables to use in the problem. When, in the units considered in the problem, decimal numbers are legal values, the model uses real numbers. Otherwise, integers or even binary numbers are used.

1.2.3 Constraints. In general, variables cannot take any value in their domain.

Very often, variables are restricted in sign and can only take non-negative values. In addition there are further requirements on the values taken by the variables, written in terms of equations and inequalities. All these equations and inequalities are collectively called **constraints**. The set of all values the decision variables can take that satisfy all the constraints is called the **feasible set**. In other words, the feasible set is the solution set to the system of the constraints.

All inequalities are in the weak ($\leq$ or $\geq$) form. If the constraints are continuous (which is almost always the case), the **feasible set is a closed region** of $\mathbb{R}^n$.

The presence of constraints justifies the term **constrained programming problems**. However, the true distinction between the two categorizes lies in the fact that the domain of the variables is closed (constrained) rather than open (unconstrained). Essentially, in the second case necessary first order and second order conditions apply (like $f' = 0$) while in the first they do not.

1.2.4 Objective. The solution to every problem is "optimal", in the sense that it is the minimum value or the maximum value of a given function. The function to be maximized or minimized is called **objective function** and the process of finding the extreme value is called "optimization". Therefore, problems like these are collectively called constrained optimization problems.

1.2.5 Linearity. What has been said above is a very general framework. Here, we deal with a subclass of problems, namely those where all the functions in the model are linear in the decision variables.

The problems with only real variables are called **linear programming**, LP, problems. The problems where variables are integers are called **integer programming**. IP, problems.

The two types of problem are related because, when the formulation of the problem is the same, the feasible set for the IP problem is a subset of that for the LP problem. Given an IP problem, the corresponding LP problem where the constraint about the integer variables is removed, is called "relaxation". The algorithms for IP problems are not the same as those for LP problems and are beyond the scope of these notes.

1.3 Standard form of an LP problem.

The two (non-integer) LP problems in the previous section have these forms:

$$\begin{aligned} &\max \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \\ &A\mathbf{x} \leq \mathbf{b} \\ &\mathbf{x} \geq \mathbf{0} \end{aligned} \qquad \text{or} \qquad \begin{aligned} &\min \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \\ &A\mathbf{x} \geq \mathbf{b} \\ &\mathbf{x} \geq \mathbf{0} \end{aligned}$$

However, it can be proved that both forms, together with other forms, can be manipulated to get to the following standard form:

$$\begin{aligned} &\min / \max \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \\ &A\mathbf{x} = \mathbf{b} \\ &\mathbf{x} \geq \mathbf{0} \end{aligned}$$

The transformation into the standard form uses the following remarks:

- every inequality can be transformed into an equation by adding a "slack" or "surplus" variable

- if a variable x_i is unrestricted in sign then we can add two non-negative variables x_i' and x_i'' and set $x_i = x_i' - x_i''$.

Definition 1.1. Pick a subset of the inequalities. If there is a unique point that satisfies them with equality, and this point happens to be feasible, then it is a vertex. $\Diamond$

Remark 1.2. Each vertex is specified by a set of n (in)equalities.

Definition 1.3. Two vertices are neighbors if they have $n-1$ defining (in)equalities in common. $\Diamond$

Under sufficiently general hypotheses, one can prove the following theorem.

Theorem 1.4. *Given an LP problem, if there exists an optimal solution, there also exists an optimal solution which is at a vertex of the feasible set.*

Then, given the LP problem

$$\begin{aligned} \min / \max \mathbf{c}^T\mathbf{x} \\ \text{subject to} \\ A\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \\ A \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m, \mathbf{x} \in \mathbb{R}^n, \end{aligned}$$

we have m equalities from the matrix A, and n equalities from the non-negativity constraints. Overall, we have $m+n$ equalities. $A\mathbf{x} = \mathbf{b}$ has a solution only if $\operatorname{rank}(A) = n$.

A brute-force approach would be the following.

(1) pick all possible subsets of n linearly independent constraints out of the $m+n$ constraints
(2) Solve (in worst case) $\binom{m+n}{n}$ systems of equations of the type $A^*\mathbf{x} = \mathbf{b}^*$ where $A^*, \mathbf{b}^*$ are the restrictions of A and $\mathbf{b}$ to the subset of n constraints. This can be done, for example, by Gaussian elimination
(3) Check feasibility of all solutions, evaluate the objective function at each solution, and pick the best.

This algorithm is correct but inefficient because

$$\binom{m+n}{n} \leq \sum_{k=0}^{m+n} \binom{m+n}{k} = 2^{m+n},$$

that is, with respect to n,

$$\binom{m+n}{n} = O\left(2^n\right)$$

Example 1.5. The problem is

$$\begin{aligned} \max\, z = 2x_1 + 4x_2 \\ \text{subject to} \end{aligned}$$

$$\begin{aligned} x_1 &\le 50 \\ x_2 &\le 80 \\ x_1 + x_2 &\le 100 \end{aligned}$$

$$\begin{aligned} x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

and there five constraints. Turning all of them into equations we get

$$\begin{aligned} x_1 &= 50 \\ x_2 &= 80 \\ x_1 + x_2 &= 100 \\ x_1 &= 0 \\ x_2 &= 0. \end{aligned}$$

To find all vertices we need to pick two equations out of the five and solve the simultaneously. There are $\binom{5}{2} = 10$ ways to pick two elements out of a set of five and these get

$$\begin{cases} x_1 = 50 \\ x_2 = 80 \end{cases} \begin{cases} x_1 = 50 \\ x_1 + x_2 = 100 \end{cases} \begin{cases} x_1 = 50 \\ x_1 = 0 \end{cases} \begin{cases} x_1 = 50 \\ x_2 = 0. \end{cases}$$

$$\begin{cases} x_2 = 80 \\ x_1 + x_2 = 100 \end{cases} \begin{cases} x_2 = 80 \\ x_1 = 0 \end{cases} \begin{cases} x_2 = 80 \\ x_2 = 0. \end{cases} \begin{cases} x_1 + x_2 = 100 \\ x_1 = 0 \end{cases}$$

$$\begin{cases} x_1 + x_2 = 100 \\ x_2 = 0. \end{cases} \begin{cases} x_1 = 0 \\ x_2 = 0. \end{cases}$$

whose solutions are, respectively,

$$\begin{cases} x_1 = 50 \\ x_2 = 80 \end{cases} \text{not feasible} \begin{cases} x_1 = 50 \\ x_2 = 50 \end{cases} z = 300 \begin{cases} x_1 = 50 \\ x_1 = 0 \end{cases} \text{no solution} \begin{cases} x_1 = 50 \\ x_2 = 0. \end{cases} z = 100$$

$$\begin{cases} x_2 = 80 \\ x_1 = 20 \end{cases} z = 360 \begin{cases} x_2 = 80 \\ x_1 = 0 \end{cases} z = 320 \begin{cases} x_2 = 80 \\ x_2 = 0. \end{cases} \text{no solution} \begin{cases} x_2 = 100 \\ x_1 = 0 \end{cases} \text{not feasible}$$

$$\begin{cases} x_1 = 100 \\ x_2 = 0 \end{cases} \text{not feasible} \begin{cases} x_1 = 0 \\ x_2 = 0. \end{cases} z = 0$$

△

1.4 Exercises

Exercise 1.4.1 (DPV 7.2)**.** Duckwheat is produced in Kansas and Mexico and consumed in New York and California. Kansas produces 15 shnupells of duckwheat and Mexico 8. Meanwhile, New York consumes 10 shnupells and California 13. The transportation costs per shnupell are \$4 from Mexico to New York, \$1 from Mexico to California, \$2 from Kansas to New York, and \$3 from Kansas to California.

Write a linear program that decides the amounts of duckwheat (in shnupells and fractions of a shnupell) to be transported from each producer to each consumer, so as to minimize the overall transportation cost.

Exercise 1.4.2 (DPV 7.4)**.** Moe is deciding how much Regular Duff beer and how much Duff Strong beer to order each week. Regular Duff costs Moe \$1 per pint and he sells it at \$2 per pint; Duff Strong costs Moe \$1.50 per pint and he sells it at \$3 per pint. However, as part of a complicated marketing scam, the Duff company will only sell a pint of Duff Strong for each two pints or more of Regular Duff that Moe buys.

Furthermore, due to past events that are better left untold, Duff will not sell Moe more than 3,000 pints per week. Moe knows that he can sell however much beer he has. Formulate a linear program for deciding how much Regular Duff and how much Duff Strong to buy, so as to maximize Moe's profit. Solve the program geometrically.

Exercise 1.4.3 (DPV 7.7)**.** Find necessary and sufficient conditions on the reals a and b under which the linear program

$$\begin{aligned} &\max x + y \\ &\text{subject to} \\ &ax + by \leq 1 \\ &\mathbf{x} \geq \mathbf{0}. \end{aligned}$$

(1) Is infeasible.
(2) Is unbounded.
(3) Has a finite and unique optimal solution.

Exercise 1.4.4 (DPV 7.8)**.** You are given the following points in the plane:

$$(1,3), (2,5), (3,7), (5,11), (7,14), (8,15), (10,19).$$

You want to find a line $y = ax + b$ that approximately passes through these points (no line is a perfect fit). Write a linear program (you don't need to solve it) to find the line that minimizes the maximum absolute error,

$$\max_{1 \leq i \leq 7} |ax_i + b - y_i|.$$

Hint: if $z \geq |w|$ then $z \geq w$ and $z \geq -w$.

2

The Simplex algorithm (2 hrs - Thu, Apr 18, 2024)

2.1 The simplex algorithm

The simplex algorithm provides a method to reduce the number of vertices to visit in order to find the optimum. On each iteration, simplex has two tasks:

(1) Check whether the current vertex is optimal (and if so, halt).
(2) Determine where to move next.

As we will see, both tasks are easy if the vertex happens to be at the origin. And if the vertex is elsewhere, we will transform the coordinate system to move it to the origin!

2.1.1 A worked example. A wooden toy factory produces cars and trains. The demand of cars is 50 units per month, while that of trains in 80 units. Overall, the factory cannot produce more than 100 items per month. Every car is sold for 2 and every train for 4 money units. Find the production plan that maximizes the revenues.

The problem is:

$$\begin{gathered}\max\ 2x_1 + 4x_2 \\ \text{subject to} \\ x_1 \leq 50 \\ x_2 \leq 80 \\ x_1 + x_2 \leq 100 \\ x_1 \geq 0 \\ x_2 \geq 0\end{gathered}$$

and graphically we find the optimal solution at $x_1 = 20$, $x_2 = 80$ with an optimal value for the objective of 360.

The first step transforms the original problem into a more complex one, adding new variables. z, an unconstrained variable, is added to the objective function which is transformed into

$$z = 2x_1 + 4x_2$$

and each $\leq$ constraint is transformed into an equation by means of an additional "slack" variable:

$$\begin{aligned} x_1 \quad &+s_1 &&= 50 \\ +x_2 \quad &+ s_2 &&= 80 \\ x_1 + x_2 \quad &+s_3 &&= 100 \end{aligned}$$

with

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

We transformed a system of linear inequalities into a system of equations. A solution to this system is, obviously, $x_1 = x_2 = 0$ and $s_1 = 50, s_2 = 80, s_3 = 100$, that is $(0, 0, 50, 80, 100)$. *Variables set at 0 are called non-basic, the others are called basic.* Geometrically, this represents the origin. Plugging this solution into the objective equation gives $z = 0$.

Is this optimal? Likely, it is not, because in the objective the non-basic variables have a positive coefficient. It is possible that increasing one of them (they can only take positive values), will increase the value of z. We need to verify this is indeed possible.

Say that we increase x_2 to $0 + \delta$ since its coefficient is larger than that of x_1. We have

$$\begin{aligned} &z = 2 \cdot 0 + 4\delta \\ &0 \quad +s_1 \quad = 50 \\ &\delta \quad + s_2 \quad = 80 \\ &0 + \delta \quad +s_3 = 100 \end{aligned}$$

From the second equation we get $s_2 = 80 - \delta$ and from the third we get $s_3 = 100 - \delta$. The solution is then $(0, \delta, 50, 80 - \delta, 100 - \delta)$. Since all variables must be non-negative δ must be such that $80 - \delta \geq 0$ and $100 - \delta \geq 0$. This implies $\delta \leq 80$ so 80 is the maximum increase for x_2. If we set $x_2 = 80$ we have $x_1 = s_2 = 0$ and $x_2 = 80, s_1 = 50, s_3 = 20$. In practice, what we have done is moving s_2 from basic to non-basic, and x_2 from non-basic to basic. Algebraically, we have used $x_2 = 80 - s_2$. Plugging this into the original problem gets

$$\begin{aligned} &z = 2x_1 + 4(80 - s_2) \\ &x_1 \quad +s_1 \quad = 50 \\ &+x_2 \quad + s_2 \quad = 80 \\ &x_1 + 80 - s_2 \quad +s_3 = 100 \end{aligned}$$

that is

$$\begin{aligned} &z = 320 + 2x_1 - 4s_2 \\ &x_1 \quad +s_1 \quad = 50 \\ &+x_2 \quad + s_2 \quad = 80 \\ &x_1 \quad - s_2 \quad +s_3 = 20 \end{aligned}$$

Note that in going from one problem to the next we preserved three facts:

(1) each constraint contains a variable that appears only there and has coefficient 1 (they were s_1, s_2, s_3 and now are s_1, x_2, s_3)

(2) the RHS of the constraints are non-negative

(3) the candidate basic variables do not appear in the objective.

This means the change of variable is actually a change of the origin of the reference system. We are still at $(0,0)$, but in the new reference where the axes are x_1 and s_2. The complete solution is $(0,80,50,0,20)$.

If we had chosen x_1 that would have moved us to another vertex of the polytope and would have followed another path.

We are again with an LP problem as we were at the beginning but now in the objective function one non-basic variable has a negative coefficient. Since any non-basic variable has a zero value and can change only to positive values, any change in that variable would decrease the value of z and therefore changing that variable would not result in an improvement.

We then try changing x_1 from 0 to δ:

$$z = 320 + 2\delta - 4s_2$$

$$\begin{array}{lllll} \delta & & +s_1 & & = 50 \\ & +x_2 & & + s_2 & = 80 \\ \delta & & & - s_2 \quad +s_3 & = 20 \end{array}$$

whence we get $s_1 = 50 - \delta$ and $s_3 = 20 - \delta$. Both must be non-negative so the maximum increase is $\delta = 20$ and $x_1 = 20 + s_2 - s_3$. Substituting back into the problem gives

$$z = 360 + 2s_2 - 2s_3 - 4s_2 = 360 - 2s_2 - 2s_3$$

$$\begin{array}{lllll} & & +s_1 + s_2 & -s_3 & = 30 \\ & +x_2 & \quad\;\; + s_2 & & = 80 \\ x_1 & & \quad\;\; - s_2 & +s_3 & = 20 \end{array}$$

The solution at the origin of the last problem is $(20,80,30,0,0)$. Is this optimal? It is, because now all the non-basic variables have negative coefficients and therefore increasing them would decrease the objective, something we don't want.

2.1.2 Some implementation issues. While running, the simplex algorithm might encounter some issues.

- It may be that there is no bound for δ so that the objective can be brought to $+\infty$. Then the algorithm stops returning "unbounded solution".

- If may be that an initial solution is not immediately available. A common method is to create an artificial problem which includes the constraints and the objective of the given problem. The artificial problem has an easy to find starting vertex which might not be feasible for the original problem. But with a careful choice of the coefficient of the artificial variables in the artificial objective the simplex will "move" from the solution of the artificial problem to the optimal solution of the original problem.

For example,

$$\begin{aligned} \min 4x_1 + x_2 \\ \text{subject to} \\ 3x_1 + x_2 = 3 \\ 4x_1 + 3x_2 \geq 6 \\ x_1 + 2x_2 \leq 3 \\ x_1, x_2 \geq 0 \end{aligned}$$

Here, $x_1 = x_2 = 0$ is not feasible and we need a feasible solution to start the simplex. We could transform the problem into standard form but that wouldn't change the issue. We then add an artificial variable a_1 to get $3x_1 + x_2 + a_1 = 3$ with the constraint $a_1 \geq 0$. The problem has now changed and its feasible solutions are not necessarily feasible for the original problem. They would be if $a_1 = 0$. This is obtained by adding a_1 to the objective multiplied by M, where M is a very large number: $\min 4x_1 + x_2 + Ma_1$. The initial solution to this problem is $x_1 = x_2 = 0$, $a_1 = 3$. If we run the simplex for this problem it will end up to with $a_1 = 0$ (to minimize the objective) and as a byproduct will return a feasible solution to the original problem.

By the way, the same idea is used to deal with negative right-hand sides: for example, the inequality $4x_1 + 3x_2 \geq 6$ is transformed into $4x_1 + 3x_2 - s_1 = 6$, $s_1 \geq 0$. s_1 is called a "surplus" variable. The issue with surplus variables is that they canì't be basic variables because they would get a negative value (in this case, $s_1 = -6$). To fix this, another artificial variable is added, $a_2 \geq 0$, to get $4x_1 + 3x_2 - s_1 + a_2 = 6$. After these operations, the problem is

$$\begin{aligned} \min 4x_1 + x_2 + Ma_1 + Ma_2 \\ \text{subject to} \\ 3x_1 + x_2 + a_1 = 3 \\ 4x_1 + 3x_2 - s_1 + a_2 = 6 \\ x_1 + 2x_2 + s_2 = 3 \\ x_1, x_2, s_1, s_2, a_1, a_2 \geq 0 \end{aligned}$$

but it needs a final adjustment. In fact, in the standard simplex algorithm, the starting point has no basic variables in the objective. We then need to remove a_1 and a_2 from the objective. This is done using the first and second equations:

$$\begin{aligned} 4x_1 + x_2 + Ma_1 + Ma_2 &= 4x_1 + x_2 + \\ &+ M(3 - 3x_1 - x_2) + M(6 - 4x_1 - 3x_2 + s_1) \\ &= (4 - 7M)x_1 + (1 - 4M)x_2 + Ms_1 + 9M \end{aligned}$$

and the problem finally becomes

$$\min(4 - 7M)x_1 + (1 - 4M)x_2 + Ms_1 + 9M$$

subject to

$$\begin{array}{ccccccccccccc} 3x_1 & + & x_2 & & & + & a_1 & & & & & = & 3 \\ 4x_1 & + & 3x_2 & - & s_1 & & & + & a_2 & & & = & 6 \\ x_1 & + & 2x_2 & & & & & & & + & s_2 & = & 3 \end{array}$$

$$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$$

Now a feasible solution is $x_1 = x_2 = s_1 = 0$ and $a_1 = 3$, $a_2 = 6$, $s_2 = 3$. From this, the simplex algorithm would find the optimal solution, $x_1 = 0.6, x_2 = 1.2$.

- There can be cases (although rare) in which a vertex of a polytope is the endpoint of more than n edges, say $m > n$ edges. This is the degenerate case. In this case the algorithm might fall into a cycle, substituting one of the n constraints with one of the remaining $m - n > 0$. And back again, ... This is might be solved by rules for choosing variables to move from basic to non-basic and vice versa.

2.1.3 Time of execution. The simplex algorithm shows that a linear program can always be solved in finite time, and in fact in time that is at most exponential in the number of variables. This is because each iteration takes polynomial time and moves to a new vertex, and if there are m inequalities and n variables there can be at most $\binom{m}{n}$ vertices. We previously showed that this is $O(2^{m+n})$.

Actually, in the 1970s it has been shown that for all the known variants of the simplex method (which differ in the way they choose the vertex to move to, when there is more than possible choice) there are examples of linear programs on which the algorithm takes exponential time.

Later, it has been proved that these "worst cases" are actually sufficiently rare so that, on average, the simplex algorithm is usually very fast, even on linear programs with tens or hundreds of thousands of variables and constraints.

3

Duality (2 hrs - Fri, Apr 19, 2024)

3.1 A numerical example

We can approach the problem of the optimality of the solution of the original problem from another point of view. The problem is:

$$\begin{aligned} \max 2x_1 + 4x_2 \\ \text{subject to} \end{aligned}$$

$$\begin{aligned} x_1 &\leq 50 \\ x_2 &\leq 80 \\ x_1 + x_2 &\leq 100 \end{aligned}$$

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

and graphically we found the optimal solution at $x_1 = 20$, $x_2 = 80$ with an optimal value for the objective of 360. From $x_1 \leq 50$ and $x_2 \leq 80$, multiplying the first by 2 and the second by 4 we get $2x_1 \leq 100$ and $4x_2 \leq 320$. Adding the two together we get

$$2x_1 + 4x_2 \leq 420$$

which gives an upper bound, even if not very tight. However, there is a better choice of the factors that shows the solution is indeed optimal. If we multiply the second and the third inequality by 2 and add them together we get

$$2x_1 + 4x_2 \leq 360$$

and this shows that $(20, 80)$ is indeed optimal. Is there a way to get the best coefficients?

Say we assign every constraint a non-negative multiplicative coefficient, y_1, y_2 and y_3. We have

$$y_1 x_1 \le 50 y_1$$
$$y_2 x_2 \le 80 y_2$$
$$y_3(x_1 + x_2) \le 100 y_3$$

(we choose non-negative numbers because we need to keep the sign of the inequalities). Adding the three we get

$$x_1(y_1 + y_3) + x_2(y_2 + y_3) \le 50y_1 + 80y_2 + 100y_3.$$

We want to build the best upper bound to $2x_1+4x_2$ so any expression like $c_1x_1+c_2x_2$ where $c_1 \ge 2$ and $c_2 \ge 4$ on the left-hand side would provide an upper bound, because all variables are non-negative, and we would like the upper bound, the right-hand side, to be as small as possible. Then, we want to solve the following problem:

$$\min 50y_1 + 80y_2 + 100y_3$$
$$\text{subject to}$$
$$y_1 + y_3 \ge 2$$
$$y_2 + y_3 \ge 4$$
$$y_1 \ge 0$$
$$y_2 \ge 0$$
$$y_3 \ge 0$$

The link between the last problem and the first one is much clearer if we write them in matrix form. The original problem is

$$\max \mathbf{c}^T \mathbf{x}$$
$$\text{subject to}$$
$$A\mathbf{x} \le \mathbf{b}$$
$$\mathbf{x} \ge \mathbf{0}$$

where

$$\mathbf{c} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \mathbf{b} = \begin{bmatrix} 50 \\ 80 \\ 100 \end{bmatrix}.$$

and, with the same objects, the second one is

$$\min \mathbf{y}^T \mathbf{b}$$
$$\text{subject to}$$
$$\mathbf{y}^T A \ge \mathbf{c}^T$$
$$\mathbf{y} \ge \mathbf{0}$$

which is another LP problem. This is written in terms of rows, instead of columns, so it is an LP problem in the $\mathbf{y}$'s, which are rows and can be identified as vectors in the dual space of the $\mathbf{x}$'s. This is the reason why the second problem is called the dual of the first, which, in turn, is called the primal problem.

The optimal solution to the dual is $(0, 2, 2)$, and the corresponding objective is 360.

3.2 Exercises

Exercise 3.2.1 (DPV 7.11)**.** Write the dual to the following linear program.

$$\max x + y$$
$$\text{subject to}$$
$$2x + y \leq 3$$
$$x + 3y \leq 5$$
$$x, y \geq 0.$$

Find the optimal solutions to both primal and dual LPs.

Exercise 3.2.2. Write the dual to the following linear program.

$$\max -100y_1 + 480y_2 + 800y_3$$
$$\text{subject to}$$
$$-y_1 + 4y_2 + 2y_3 \leq 2$$
$$-4y_1 + 20y_2 + 40y_3 \leq 11$$
$$y_i \geq 0\, i = 1, 2, 3$$

Find the optimal solutions to the dual.

4

More on Duality (3 hrs - Tue, Apr 23, 2024)

4.1 Duality theory

Given an LP problem (primal)

$$\begin{aligned} \max \mathbf{c}^T\mathbf{x} \\ \text{subject to} \\ A\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

where $\mathbf{x}, \mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$ and A is an $m \times n$ matrix, its dual is

$$\begin{aligned} \min \mathbf{y}^T\mathbf{b} \\ \text{subject to} \\ \mathbf{y}^T A \geq \mathbf{c}^T \\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

where $\mathbf{y} \in \mathbb{R}^m$. Note that each decision variable in the primal problem corresponds to a constraint in the dual problem, and each constraint in the primal problem corresponds to a variable in the dual problem.

In case we have equations instead of inequalities, like $\mathbf{a}^T\mathbf{x} = \mathbf{b}$, then we can transform this into two inequalities:

$$\begin{aligned} \mathbf{a}^T\mathbf{x} \geq b \\ \mathbf{a}^T\mathbf{x} \leq b \end{aligned}$$

which in turn transforms into

$$\begin{aligned} \mathbf{a}^T\mathbf{x} \geq b \\ -\mathbf{a}^T\mathbf{x} \geq -b. \end{aligned}$$

We multiply each of the two $\geq$ constraint by two non-negative variables, say y_1 and y_2. Then we have

$$y_1 \mathbf{a}^T \mathbf{x} \geq y_1 b$$
$$-y_2 \mathbf{a}^T \mathbf{x} \geq -y_2 b.$$

and adding the two together we get

$$(y_1 - y_2)\mathbf{a}^T \mathbf{x} \geq (y_1 - y_2) b$$

which is an inequality with the $\geq$ sign multiplied by an unsigned variable $y_1 - y_2$.

Theorem 4.1 (Weak duality)**.** *For any feasible solutions* $\mathbf{x}$ *and* $\mathbf{y}$ *to primal and dual linear programs, we have* $\mathbf{c}^T \mathbf{x} \leq \mathbf{y}^T \mathbf{b}$.

Proof. If $\mathbf{y}$ is a feasible solution of the dual, then $\mathbf{y}^T A \geq \mathbf{c}^T$. Because $\mathbf{x} \geq \mathbf{0}$ we can right-multiply the previous inequality by $\mathbf{x}$ to get

$$\mathbf{y}^T A \mathbf{x} \geq \mathbf{c}^T \mathbf{x}.$$

If $\mathbf{x}$ is a feasible solution of the primal, then $A\mathbf{x} \leq \mathbf{b}$. Similarly, because $\mathbf{y} \geq \mathbf{0}$, we can left-multiply the last inequality by $\mathbf{y}^T$ to get

$$\mathbf{y}^T A \mathbf{x} \leq \mathbf{y}^T \mathbf{b}.$$

Combining the two inequalities we get

$$\mathbf{c}^T \mathbf{x} \leq \mathbf{y}^T A \mathbf{x} \leq \mathbf{y}^T \mathbf{b}.$$

□

Theorem 4.2 (Certificate of Optimality)**.** *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are feasible solutions of the primal and dual and* $\mathbf{c}^T \mathbf{x} = \mathbf{y}^T \mathbf{b}$*, then* $\mathbf{x}$ *and* $\mathbf{y}$ *must be optimal solutions to the primal and dual.*

There is another interesting consequence of weak duality that relates infiniteness of optimal values in the primal/dual with feasibility of the dual/primal. Let $\mathbf{y}$ be a feasible solution of the dual. By weak duality, we have $\mathbf{c}^T \mathbf{x} \leq \mathbf{y}^T \mathbf{b}$ for all feasible $\mathbf{x}$. If the optimal value in the primal is ∞, then $\infty \leq \mathbf{y}^T \mathbf{b}$. This is not possible, so the dual cannot have a feasible solution.

Theorem 4.3. *If the optimal value in the primal is* ∞*, then the dual must be infeasible. If the optimal value of the dual is* $-\infty$*, then the primal must be infeasible.*

Note that the converse does not hold. So, for example, if the dual is infeasible, then the primal might either be unbounded or infeasible as well.

Theorem 4.1 leaves the door open to the possibility that the optimal solution to the dual is strictly greater than the optimal solution to the primal. Actually, that fact that the optimal value of the objective is equal in both the primal and the dual is due to the next result whose proof is beyond the scope of this course.

Theorem 4.4 (Strong Duality)**.** *The dual has an optimal solution if and only if the primal does. If* $\mathbf{x}^*$ *and* $\mathbf{y}^*$ *are optimal solutions to the primal and dual, then* $\mathbf{c}^T \mathbf{x}^* = (\mathbf{y}^*)^T \mathbf{b}$.

4.2 Sensitivity analysis

Suppose that the resource quantities $\mathbf{b}$ change by a small amount $\Delta \in \mathbb{R}^m$. Then, the primal and dual become

$$\begin{aligned} \max \mathbf{c}^T\mathbf{x} \\ \text{subject to} \\ A\mathbf{x} \leq \mathbf{b} + \Delta \\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

and

$$\begin{aligned} \min \mathbf{y}^T(\mathbf{b} + \Delta) \\ \text{subject to} \\ \mathbf{y}^T A \geq \mathbf{c}^T \\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

Let the optimal value of the problem in both the primal and the dual be $z(\Delta)$, so that $z(0)$ is the optimal value of the original problem. Suppose that the optimal solution to the dual is unique and that, for sufficiently small Δ, the optimal solution to the dual does not change. In this case the optimal value changes by $z(\Delta) - z(0) = \mathbf{y}^T\Delta$. By strong duality, the optimal solution to the primal changes by the same amount, $\mathbf{y}^T\Delta$. To summarize, a small change in the level of the resources in the primal induces a change in the optimal value which is "scaled" by the dual solution. Every dual optimal variable scales the corresponding variation in the objective of the primal.

In the example problem, suppose that the production constraint, buying a new machine, could increase the weekly production by a certain amount δ. Are we going to spend on this to increase our revenues?

The above reasoning tells us that the change from 100 to $100+\delta$ in the working hours constraint would get a variation in revenues of 2δ. Now the question is: what is the cost of the machine? If it's less than 2δ, then the operation is profitable, otherwise, it is not. This is why the dual optimal solutions are sometimes called "shadow prices": the LP problem implies that every "additional working hour" has a price of 2. This might or might not be the real price of it, but it is a price one must consider when making choices.

The same idea does not apply to the first constraint, that about cars. This is because increasing the demand for cars does not change the optimal solution. In fact, at the optimal solution they already have an excess in the demand so there is no point in further increasing it. Consequently, the shadow price associated with cars is 0.

This property is general, as the following theorem guarantees.

Theorem 4.5 (Complementary Slackness)**.** *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be feasible solutions to symmetric form primal and dual linear programs. Then,* $\mathbf{x}$ *and* $\mathbf{y}$ *are optimal solutions to the primal and dual if and only if* $\mathbf{y}^T(\mathbf{b} - A\mathbf{x}) = 0$ *and* $(\mathbf{y}^T A - \mathbf{c}^T)\mathbf{x} = 0$.

Proof. Feasibility implies that $\mathbf{y}^T(\mathbf{b} - A\mathbf{x}) \geq 0$ and $(\mathbf{y}^T A - \mathbf{c}^T)\mathbf{x} \geq 0$.

$\Rightarrow$ Suppose that $\mathbf{x}$ and $\mathbf{y}$ are optimal. Adding the two inequalities gets

$$\mathbf{y}^T(\mathbf{b} - A\mathbf{x}) + (\mathbf{y}^T A - \mathbf{c}^T)\mathbf{x} = \mathbf{y}^T\mathbf{b} - \mathbf{y}^T A\mathbf{x} + \mathbf{y}^T A\mathbf{x} - \mathbf{c}^T\mathbf{x} = \mathbf{y}^T\mathbf{b} - \mathbf{c}^T\mathbf{x} \geq 0.$$

However, $\mathbf{x}$ and $\mathbf{y}$ are also supposed to be optimal, not just feasible. Then, by strong duality (Th. 4.4),

$$\mathbf{y}^T\mathbf{b} = \mathbf{c}^T\mathbf{x} \quad \Rightarrow \quad \mathbf{y}^T\mathbf{b} - \mathbf{c}^T\mathbf{x} = 0.$$

Going back to $\mathbf{y}^T(\mathbf{b} - A\mathbf{x})$ and $(\mathbf{y}^T A - \mathbf{c}^T)\mathbf{x}$, their sum is 0. Since $\mathbf{y}^T(\mathbf{b} - A\mathbf{x}) \geq 0$ and $(\mathbf{y}^T A - \mathbf{c}^T)\mathbf{x} \geq 0$, they must be both equal to 0.

$\Leftarrow$ Suppose that $\mathbf{y}^T(\mathbf{b} - A\mathbf{x}) = 0$ and $(\mathbf{y}^T A - \mathbf{c}^T)\mathbf{x} = 0$. Then

$$\mathbf{y}^T\mathbf{b} - \mathbf{c}^T\mathbf{x} = 0 \quad \Rightarrow \quad \mathbf{y}^T\mathbf{b} = \mathbf{c}^T\mathbf{x}$$

and so they are optimal by theorem 4.2 $\square$

4.3 Exercises

Exercise 4.3.1 (DPV 7.12)**.** For the linear program

$$\begin{aligned} \max x_1 - 2x_3 \\ \text{subject to} \\ x_1 - x_2 \leq 1 \\ 2x_2 - x_3 \leq 1 \\ \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

prove that the solution $(x_1, x_2, x_3) = (3/2, 1/2, 0)$ is optimal.

Exercise 4.3.2. Consider the following problem:

$$\begin{aligned} \max x_1 + 2x_2 + 4x_3 \\ \text{subject to} \\ x_1 + 3x_2 \leq 8 \\ 2x_2 + x_3 \leq 7 \\ 3x_1 + x_3 \leq 6 \\ \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Using complementary slackness equations (for both the primal and the dual), find whether

$$\mathbf{x}^* = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \mathbf{x}^{**} = \begin{bmatrix} 0 \\ 0.5 \\ 6 \end{bmatrix}$$

are optimal or not.

Exercise 4.3.3. Consider the linear program

$$\begin{aligned} \max x_1 - 3x_2 + 3x_3 \\ \text{subject to} \\ 2x_1 - x_2 + x_3 \leq 4 \\ -4x_1 + 3x_2 \leq 2 \\ 3x_1 - 2x_2 - x_3 \leq 5 \\ x_i \geq 0, i = 1, 2, 3 \end{aligned}$$

(1) Write the dual program

(2) Is the primal solution $x^* = (0, 0, 4)$ optimal? Explain your answer.

5
Answers

1.4.1 The decision variables are the quantities transported from each production site to each consuming site: $x_{KN}, x_{KC}, x_{MN}, x_{MC}$. The problem is

$$\begin{aligned} \min 2x_{KN} + 3x_{KC} + 4x_{MN} + x_{MC} \\ \text{subject to} \\ x_{KN} + x_{MN} \geq 10 \\ x_{KC} + x_{MC} \geq 13 \\ x_{KN} + x_{KC} \leq 15 \\ x_{MN} + x_{MC} \leq 8 \\ \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

and its optimal solution is $x_{KN} = 10, x_{KC} = 5, x_{MN} = 0, x_{MC} = 8$.

1.4.2 The program is

$$\begin{aligned} \max R + 1.5S \\ \text{subject to} \\ R + S \leq 3000 \\ R \geq 2S \\ R, S \geq 0 \end{aligned}$$

and the feasible set is the triangle with vertices $O = (0, 0)$, $A = (2000, 1000)$ and $B = (3000, 0)$. The optimal value is 3500 at A.

1.4.3 Since $a \times 0 + b \times 0 = 0 \leq 1$ the feasible set always contains $(0, 0)$ and the program is never infeasible, for any a and b.

If $a = b = 0$ all (x, y) in the first quadrant are feasible and the program is unbounded.

If $b = 0$ and $a > 0$ then the feasible set is the vertical stripe $\{(x, y) : 0 \leq x \leq 1/a, y \geq 0\}$ and the program is unbounded. If $b = 0$ and $a < 0$ then the feasible set is the first quadrant, and the program is still unbounded.

Similarly, for $a = 0$ and $b \lesseqgtr 0$ the program is unbounded.

If $a, b > 0$, then the feasible set is a triangle with vertices $(0,0)$, $A = (1/a, 0)$ and $B = (0, 1/b)$. If $a \neq b$, then the program has a finite and unique solution on A or B. If $a = b$ the program has infinitely many finite solutions: all the points on the segment AB.

If $a < 0, b > 0$, any point $(t, 0)$ where $t > 0$ is a feasible solution and the objective is clearly unbounded. Similarly for $a > 0, b < 0$ with the points $(0, s)$, $s > 0$. If both $a < 0$ and $b < 0$ the feasible set is the first quadrant and the program is unbounded.

1.4.4 Using the hint and setting z the quantity to minimize we have

$$z \geq ax_i + b - y_i \quad \forall i$$
$$z \geq -ax_i - b + y_i \quad \forall i.$$

Because one of the two right-hand sides is positive, z is always greater than the maximum of all absolute value of errors. Minimizing it we find the actual max.

The solution is

$$y = \frac{12}{7}x + \frac{13}{7}.$$

3.2.1 The optimal solution to the primal is $(4/5, 7/5)$ with objective equal to $11/5$. The dual program is

$$\begin{aligned} \min 3z + 5w \\ \text{subject to} \\ 2z + w \geq 1 \\ z + 3w \geq 1 \\ z, w \geq 0. \end{aligned}$$

and its optimal solution is $(2/5, 1/5)$ with optimal value equal to $11/5$, obviously because of the Strong Duality Theorem.

3.2.2 The dual program is

$$\begin{aligned} \min 2x_1 + 11x_2 \\ \text{subject to} \\ x_1 + 4x_2 \leq 100 \\ 4x_1 + 20x_2 \geq 480 \\ 2x_1 + 40x_2 \geq 800 \\ x_1, x_2 \geq 0. \end{aligned}$$

and its optimal solution is $(20, 20)$ with optimal value equal to 260.

4.3.1 To prove the statement, we write the dual:

$$\begin{aligned} \min y_1 + y_2 \\ \text{subject to} \\ y_1 \geq 1 \\ -y_1 + 2y_2 \geq 0 \\ -y_2 \geq -2. \\ \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

The feasible set is a triangle whose vertices are $A = (1, 1/2)$, $B = (1, 2)$ and $C = (4, 2)$. The objective is minimum at A and equals $3/2$. Since this is the dual program, by the duality theorem the primal is optimal and its optimal value is 3/2. The value of the primal objective at $(x_1, x_2, x_3) = (3/2, 1/2, 0)$ is 3/2 and is therefore optimal.

4.3.2 The complementary slackness equations for both the primal and the dual are

$$\begin{aligned} y_1(x_1 + 3x_2 - 8) &= 0 \\ y_2(2x_2 + x_3 - 7) &= 0 \\ y_3(3x_1 + x_3 - 6) &= 0 \\ x_1(y_1 + 3y_3 - 1) &= 0 \\ x_2(3y_1 + 2y_2 - 2) &= 0 \\ x_3(y_2 + y_3 - 4) &= 0. \end{aligned}$$

At $\mathbf{x}^*$, the first equation implies $y_1 = 0$. Substituting this into the fourth, fifth and sixth equations gives $y_3 = 1/3$, $y_2 = 1$ and $y_2 + y_3 = 4$, which is not feasible. Then, by the complementary slackness theorem, $\mathbf{x}^*$ is not an optimal solution.

At $\mathbf{x}^{**}$, the first equation still implies $y_1 = 0$. The fourth equation is automatically satisfied and the fifth and sixth equations give $y_2 = 1$ and $y_3 = 3$. Thus,

$$\mathbf{y}^{**} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

is a feasible solution to the dual and, again by the complementary slackness theorem, $\mathbf{x}^{**}$ is an optimal solution.

The optimal value of the objective in both the primal and the dual is 25.

4.3.3 The dual program is

$$\begin{aligned} \min 4y_1 + 2y_2 + 5y_3 \\ \text{subject to} \\ 2y_1 - 4y_2 + 3y_3 \geq 1 \\ -y_1 + 3y_2 - 2y_3 \geq -3 \\ y_1 - y_3 \geq 3 \\ y_i \geq 0, i = 1, 2, 3 \end{aligned}$$

The primal solution $(0, 0, 4)$ has an objective value of 12. In addition, it has both the second and the third constraints not binding. By complementary slackness, this implies $y_2 = 0$ and $y_3 = 0$. Solving the dual problem with these additional constraints gives

$$2y_1 \geq 1, -y_1 \geq -3, y_1 \geq 3$$

whose only solution is $y_1 = 3$. Thus, the dual solution corresponding to the primal $(0, 0, 4)$ is $(3, 0, 0)$ and at $(3, 0, 0)$ the dual objective is 12. By the strong duality theorem, this is indeed the optimal solution to both problems.