BMS1901 Calculus for Life Sciences

Week 12

Vectors

Vectors

Vectors

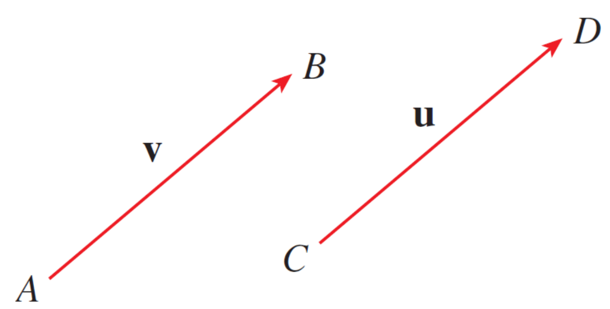
- **Vector:** indicate a quantity (e.g. displacement, velocity or force) that has both *magnitude* and *direction*
    - an arrow or a directed line segment
    - length of the arrow: magnitude of the vector
    - arrow points in the direction of the vector
    - **v** or $(\vec{v})$

Vectors

- a particle moves along a line segment from point *A* to point *B*

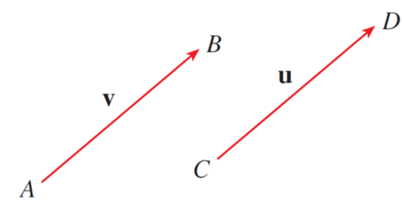
- **displacement vector v: initial point** *A* (the tail) and **terminal point** *B* (the tip)

$$\mathbf{v} = \overrightarrow{AB}$$



Equivalent vectors

Vectors



Equivalent vectors

$$\mathbf{u} = \overrightarrow{CD}$$

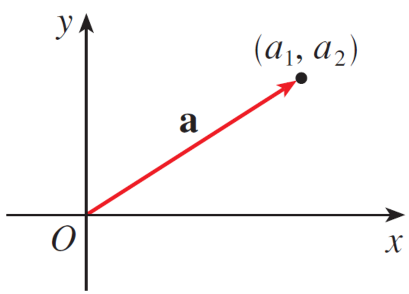
•same length
•same direction as **v** (different position)

•**u** and **v** are **equal** → **u** = **v**
•**zero vector**: **0**
- length 0
- only vector with no specific direction

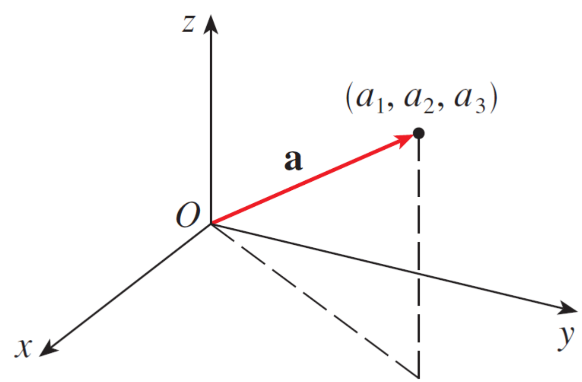
Components

Components

- Coordinate system: treat vectors algebraically
- place the initial point of a vector **a** at the origin of a rectangular coordinate system
  - terminal point of **a** has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3)



$\mathbf{a} = [a_1, a_2]$



$\mathbf{a} = [a_1, a_2, a_3]$

Components

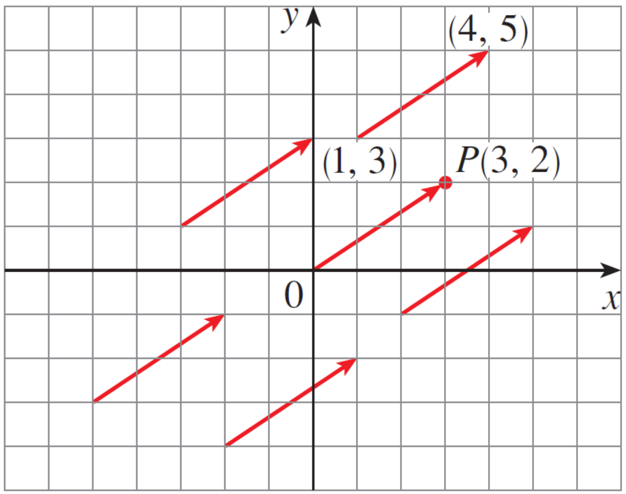
- Coordinates: **components** of **a**

$\mathbf{a} = [a_1, a_2]$ or $\mathbf{a} = [a_1, a_2, a_3]$

- Notation: $[a_1, a_2]$ for the ordered pair that refers to a vector
  - Not to confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane

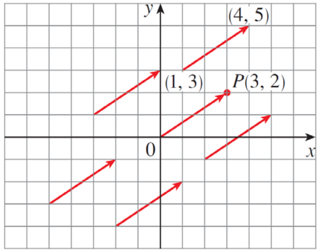
Components

- vectors are all equivalent to the vector $\overrightarrow{OP} = [3, 2]$ whose (terminal point is *P*(3, 2))



Representations of the vector **a** = [3, 2]

Components



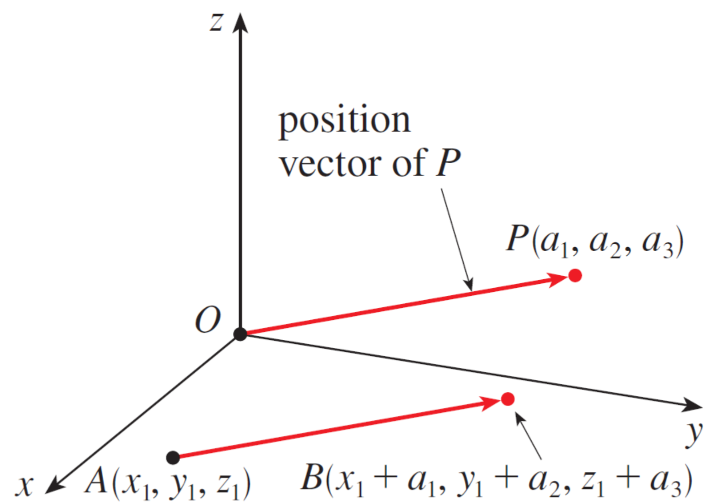
- in common: terminal point is reached from the initial point by a displacement of three units to the right and two upward

- **representations** of the algebraic vector **a** = [3, 2]

- $\overrightarrow{OP}$: **position vector** of the point *P*
    - from the origin to the point *P*(3, 2)

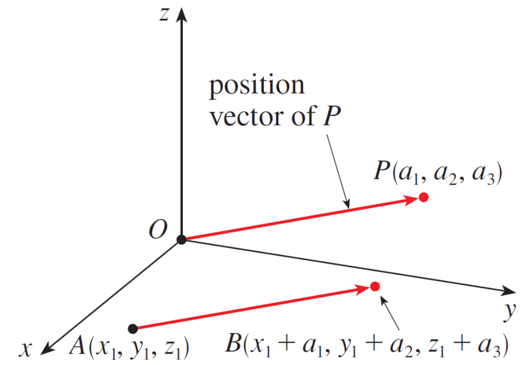
Components

- **a** = $\overrightarrow{OP}$ = [a_1, a_2, a_3] : **position vector** of the point P(a_1, a_2, a_3)



Representations of **a** = [a_1, a_2, a_3]

Components



- $\overrightarrow{AB}$ of **a** :
  initial point is $A(x_1, y_1, z_1)$
  terminal point is $B(x_2, y_2, z_2)$
  - $x_2 = x_1 + a_1$, $y_2 = y_1 + a_2$, $z_2 = z_1 + a_3$
  - $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, $a_3 = z_2 - z_1$

(1) Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$$

Components

- **magnitude** or **length** of a vector **a** : length of any of its representations ($|\mathbf{a}|$ or $\|\mathbf{a}\|$)

- distance formula : compute its length

The length of the two-dimensional vector $\mathbf{a} = [a_1, a_2]$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = [a_1, a_2, a_3]$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Components

If $\mathbf{a} = [a_1, a_2]$ and $\mathbf{b} = [b_1, b_2]$, then

$$\mathbf{a} + \mathbf{b} = [a_1 + b_1, a_2 + b_2] \qquad \mathbf{a} - \mathbf{b} = [a_1 - b_1, a_2 - b_2]$$

$$c\mathbf{a} = [ca_1, ca_2]$$

Similarly, for three-dimensional vectors,

$$[a_1, a_2, a_3] + [b_1, b_2, b_3] = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

$$[a_1, a_2, a_3] - [b_1, b_2, b_3] = [a_1 - b_1, a_2 - b_2, a_3 - b_3]$$

$$c[a_1, a_2, a_3] = [ca_1, ca_2, ca_3]$$

- **unit vector** : vector whose length is 1
- $\mathbf{a} \neq \mathbf{0}$ → the unit vector that has the same direction as **a:**

(2)
$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Example 5

If **a** = [4, 0, 3] and **b** = [–2, 1, 5], find | **a** | and the vectors **a** + **b**, **a** – **b**, 3**b**, and 2**a** + 5**b**.

Solution:

The length of the two-dimensional vector $\mathbf{a} = [a_1, a_2]$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = [a_1, a_2, a_3]$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$|\mathbf{a}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$$

$$\mathbf{a} + \mathbf{b} = [4, 0, 3] + [-2, 1, 5]$$

$$= [4 + (-2), 0 + 1, 3 + 5] \quad = [2, 1, 8]$$

$$\mathbf{a} - \mathbf{b} = [4, 0, 3] - [-2, 1, 5]$$

$$= [4 - (-2), 0 - 1, 3 - 5] \quad = [6, -1, -2]$$

Example 5 – *Solution*

$$3\mathbf{b} = 3[-2, 1, 5] = [3(-2), 3(1), 3(5)]$$

$$= [-6, 3, 15]$$

$$2\mathbf{a} + 5\mathbf{b} = 2[4, 0, 3] + 5[-2, 1, 5]$$

$$= [8, 0, 6] + [-10, 5, 25]$$

$$= [-2, 5, 31]$$

Components

- length of a vector from V_n is calculated by using the distance formula:

Properties of Vectors If **a**, **b**, and **c** are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	**2.** $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$	**4.** $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$	**6.** $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$	**8.** $1\mathbf{a} = \mathbf{a}$

The Dot Product

The Dot Product

(1) Definition If $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- dot product of **a** and **b**: multiply corresponding components and add

- result is not a vector
  - real number / a scalar
  - dot product = **scalar product**

The Dot Product

- Dot product of two-dimensional vectors:

$$[a_1, a_2] \cdot [b_1, b_2] = a_1 b_1 + a_2 b_2$$

- *n*-dimensional vectors:

$$[a_1, \ldots, a_n] \cdot [b_1, \ldots, b_n] = a_1 b_1 + \cdots + a_n b_n$$

Example 1

$$[2, 4] \cdot [3, -1]$$

$$[-1, 7, 4] \cdot \left[6, 2, -\tfrac{1}{2}\right]$$

$$[1, 2, -3] \cdot [0, 2, -1]$$

Example 1

$$
\begin{aligned}
[2, 4] \cdot [3, -1] &= 2(3) + 4(-1) \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
[-1, 7, 4] \cdot \left[6, 2, -\tfrac{1}{2}\right] &= (-1)(6) + 7(2) + 4\left(-\tfrac{1}{2}\right) \\
&= 6
\end{aligned}
$$

$$
\begin{aligned}
[1, 2, -3] \cdot [0, 2, -1] &= 1(0) + 2(2) + (-3)(-1) \\
&= 7
\end{aligned}
$$

The Dot Product

(2) Properties of the Dot Product If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

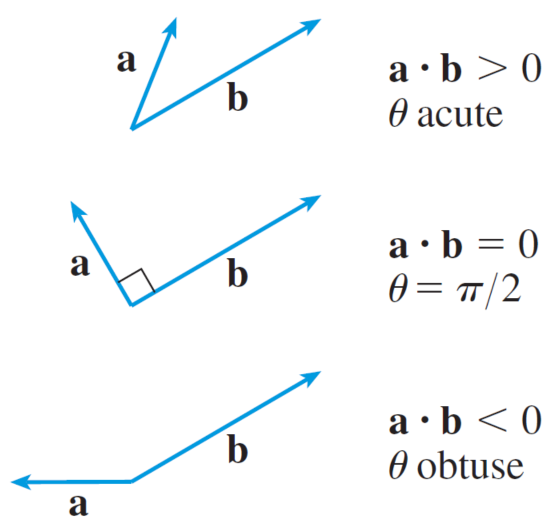
(4) An Alternative Formula for the Dot Product

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

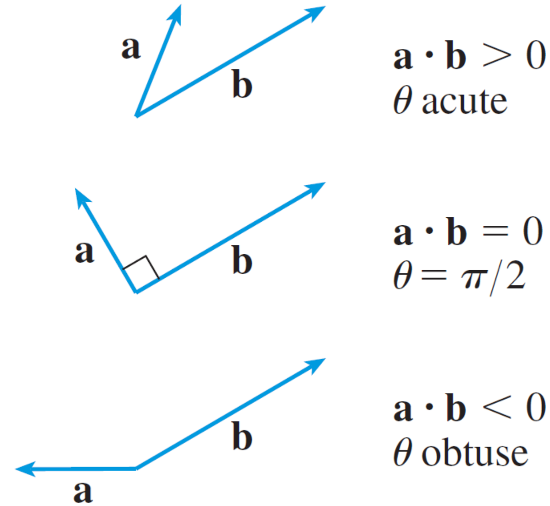
where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ $(0 \leqslant \theta \leqslant \pi)$. ($\theta$ is the smaller angle between the two vectors when drawn from the same initial point.)

The Dot Product

- dot product of two nonzero vectors is zero
    - ○ cos $\theta = 0$
- ○ $\theta = \pi/2$ and the two vectors are perpendicular (**orthogonal**)



The Dot Product



- $0 \leq \theta < \pi/2 \rightarrow \cos\theta > 0$
- $\pi/2 < \theta \leq \pi \rightarrow \cos\theta < 0$
- $\theta < \pi/2 \rightarrow$ **a** · **b** is positive
  - negative for $\theta > \pi/2$

- **a** · **b** : measures the extent to which **a** and **b** point in the same direction

- **a** and **b** point in the same general direction : dot product **a** · **b** is positive
  - 0 if they are perpendicular
  - negative if they point in generally opposite directions

The Dot Product

$$\boxed{\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta}$$

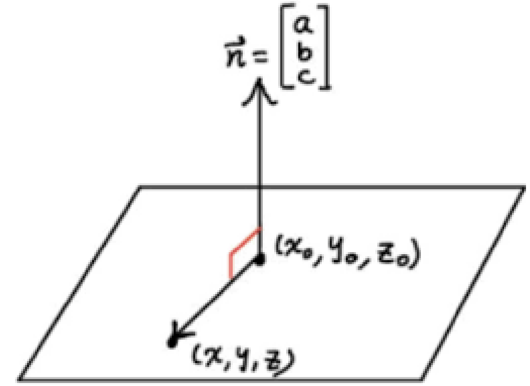
- <u>Extreme case</u>:
  - **a** and **b** point in exactly the same direction ($\theta = 0$)
  - $\cos\theta = 1$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$$

  - **a** and **b** point in exactly opposite directions ($\theta = \pi$)
  - $\cos\theta = -1$

$$\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|.$$

The Dot Product



- A plane :
  - a point $P_0(x_0, y_0, z_0)$ in the plane
  - a vector **n** = [a, b, c] that is orthogonal to the plane
  - **normal vector :** orthogonal vector **n**

(5) An equation of the plane that passes through the point $P_0(x_0, y_0, z_0)$ and is perpendicular to the vector $[a, b, c]$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Example 5

(5) An equation of the plane that passes through the point $P_0(x_0, y_0, z_0)$ and is perpendicular to the vector $[a, b, c]$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Find an equation of the plane through the point (2, 4, –1) with normal vector **n** = [2, 3, 4].

Solution:

•$a = 2$, $b = 3$, $c = 4$, $x_0 = 2$, $y_0 = 4$, and $z_0 = -1$ → Equation 5

• equation of the plane:

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

or

$$2x + 3y + 4z = 12$$