Tutorial 5 (Chapter 5)

1. Choose a number X at random from the set of numbers $\{1, 2, 3, 4, 5\}$. Now choose a second number Y at random from the subset no larger than X. Find the joint mass function of X and Y.

   Solution

   $$P(Y = j|X = i) = \begin{cases} 1/i & i \geq j \\ 0 & i < j \end{cases}$$

   Thus we have

   $$P(X = i, Y = j) = P(Y = j|X = i)P(X = i) = \begin{cases} \frac{1}{5i} & i \geq j \\ 0 & i < j \end{cases}$$

2. A television store owner figures that 45 percent of the customers entering this store will purchase an ordinary television set, 15 percent will purchase a plasma television set, and 40 percent will just be browsing. If 5 customers enter his store on a given day, what is the probability that he will sell 2 ordinary sets and 1 plasma set on the day?

   Solution

   X_1: Number of persons who buy ordinary TV among the 5 customers
   X_2: Number of persons who buy plasma TV among the 5 customers
   X_3: Number of persons who just browse among the 5 customers

   $(X_1, X_2, X_3) \sim multinomial(5, 0.45, 0.15, 0.4)$
   Thus $P(X_1 = 2, X_2 = 1, X_3 = 2) = \frac{5!}{2!1!2!} 0.45^2 \times 0.15 \times 0.4^2 = 0.146$.

3. A segment AC has length $2l$. B is the midpoint of AC. Pick a point D randomly on the segment AB. Pick a point E randomly on the segment BC. What is the probability that AD,DE and EC can form a triangle?

   Solution

   Denote the length of AD and EC by x and y respectively. The sample space is $0 \leq x \leq l, 0 \leq y \leq l$. Then the length of DE is $2l - x - y$. A triangle can be formed iff AD+DE>EC, AD+EC>DE and DE+EC>AD, i.e. $x + y > l, x < l, y < l$. If you draw the area defined by these three equations on the $x - y$ plane, you will find this is half of the sample space, so the probability is $1/2$.

4. Two random variables X and Y are identically distributed, with density

   $$f(x) = \begin{cases} \frac{3}{8}x^2 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

   Suppose the events $P(X \leq C)$ and $P(Y \leq C)$ are independent, and $P(X \leq C, Y \leq C) = 1/4$. Find C.

   Solution By independence, $P(X \leq C)P(Y \leq C) = P(X \leq C, Y \leq C) = 1/4$, so $P(X \leq C) = 1/2$. Since $P(X \leq C) = \int_0^C \frac{3}{8}x^2\,dx = \frac{1}{8}C^3$, we get $C = 4^{1/3}$.

5. Suppose two continuous random variables X and Y have joint CDF F, and marginal CDF F_X and F_Y respectively, Find the following probabilities in terms of the CDF's.

   (a) $P(X > a, Y < b)$

   (b) $P(X > a, Y > b)$

   (c) $P(X < a \text{ or } Y < b)$

   (d) $P(X < a \text{ or } Y > b)$

Solution

There might be more than one way to represent each probability, but all should be equivalent.

(a) $F_Y(b) - F(a,b)$

(b) $1 - F_X(a) - F_Y(b) + F(a,b)$

(c) $F_X(a) + F_Y(b) - F(a,b)$

(d) $1 + F(a,b) - F_Y(b)$

6. Suppose the joint PMF of (X,Y) is $p(0,0) = p_1, p(0,1) = p_2, p(1,0) = p_3, p(1,1) = p_4$ (thus $\sum p_i = 1$). Find the joint CDF.

Solution

$$F(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ p_1 & 0 \le x < 1, 0 \le y < 1 \\ p_1 + p_2 & 0 \le x < 1, y \ge 1 \\ p_1 + p_3 & x \ge 1, 0 \le y < 1 \\ 1 & x \ge 1, y \ge 1 \end{cases}$$

7. Suppose the joint probability mass function for (X,Y) is

	-1	1	2
-1	$1/4$	$1/10$	$3/10$
2	$3/20$	$3/20$	$1/20$

Find the pmf of the following r.v: (a)$X+Y$ (b)$\max\{X,Y\}$ (c)$\sin\frac{\pi(XY)}{2}$

Solution

$X+Y$	-2	0	1	3	4
P	$1/4$	$1/10$	$9/20$	$3/20$	$1/20$

$\max\{X,Y\}$	-1	1	2
P	$1/4$	$1/10$	$13/20$

$\sin\frac{\pi(XY)}{2}$	-1	0	1
P	$1/10$	$13/20$	$1/4$

8. The joint probability mass function of X and Y is given by $p(1,1) = 1/8, p(1,2) = 1/4, p(2,1) = 1/8, p(2,2) = 1/2$.

(a) Compute the conditional mass function of X given $Y = i, i = 1,2$.

(b) Are X and Y independent?

(c) Compute $P\{XY \le 3\}$ and $P\{X+Y > 2\}$.

Solution

(a) $P(X=1|Y=1) = 1/2, P(X=2|Y=1) = 1/2, P(X=1|Y=2) = 1/3, P(X=2|Y=1) = 2/3$

(b) No.

(c) $P(XY \le 3) = p(1,1) + p(2,1) + p(1,2) = 1/2, P(X+Y > 2) = p(1,2) + p(2,1) + p(2,2) = 7/8$

9. Joint density for (X, Y) is given by

$$f(x,y) = \begin{cases} 1 & |y| < x, 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the conditional densities $f_{X|Y}$ and $f_{Y|X}$.

Solution

When $0 < x < 1$, the marginal density of X is $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-x}^{x} 1dy = 2x$. The marginal density of X is 0 when $x \geq 1$ or $x \leq 0$.

When $0 < y < 1$, $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_y^1 dx = 1-y$. When $-1 < y < 0$, $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-y}^{1} dx = 1+y$. When $y \leq -1$ or $y \geq 1$, $f_Y(y) = 0$.

Thus when $0 < x < 1$, $f_{Y|X}(y|x) = \begin{cases} 1/(2x) & -x < y < x \\ 0 & \text{o.w.} \end{cases}$

when $-1 < y < 1$, $f_{X|Y}(x|y) = \begin{cases} 1/(1-|y|) & |y| < x < 1 \\ 0 & \text{o.w.} \end{cases}$

10. Suppose the variance of X exists and is nonzero. Let $Y = kX + a, k > 0$, find $\rho(X, Y)$.

Solution Suppose $Var(X) = \sigma^2$, then $Cov(Y, X) = kCov(X, X) + Cov(X, a) = k\sigma^2$ and $Var(Y) = k^2\sigma^2$. Thus $\rho(X, Y) = \frac{k\sigma^2}{\sqrt{\sigma^2}\sqrt{k^2\sigma^2}} = \frac{k}{|k|} = 1$.

11. For n random variables $X_1, \ldots, X_n$, the covariance matrix is defined as the $n \times n$ symmetric matrix

$$\begin{bmatrix} Var(X_1) & Cov(X_1, X_2) & Cov(X_1, X_3) & \cdots & Cov(X_1, X_n) \\ Cov(X_2, X_1) & Var(X_2) & Cov(X_2, X_3) & \cdots & Cov(X_2, X_n) \\ \vdots & & \vdots & & \vdots \\ Cov(X_n, X_1) & Cov(X_n, X_2) & \cdots & & Var(X_n) \end{bmatrix}$$

Suppose X and Y are i.i.d. $Pois(\lambda)$. Find the covariance matrix of $(2X + Y, 2X - Y)$.

Solution $Var(2X + Y) = Var(2X - Y) = 4Var(X) + Var(Y) = 5\lambda$. $Cov(2X + Y, 2X - Y) = 4Var(X) + Cov(Y, 2X) - Cov(2X, Y) - Var(Y) = 4Var(X) - Var(Y) = 3\lambda$. So the covariance matrix is $\begin{bmatrix} 5\lambda & 3\lambda \\ 3\lambda & 5\lambda \end{bmatrix}$

12. Suppose X and Y are identically distributed, but not necessarily independent, show that $X + Y$ and $X - Y$ are uncorrelated.

Solution

$Cov(X + Y, X - Y) = Var(X) + Cov(Y, X) - Cov(X, Y) - Var(Y) = Var(X) - Var(Y) = 0$