1.2 Set Theory

- Informally, a set is a collection of "objects", which may include numbers, points, or even set itself. $x \in S$ means *x* is an element of set *S*.
- Set is denotes by braces e.g. $S = \{1, 2, 3, 4, 5\}$.
- The order of the elements does not matter. Repeated elements are ignored. E.g. {1, 2, 3}, {3, 2, 1}, {1, 2, 3, 2, 1} are all the same.
- Denoted by description: $\mathbb{P} = \{x: x \text{ is a prime number}\}$
- Commonly used set: $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Empty Set: {} or $\emptyset$

Examples:

- If $A = \{1, 2, 3\}$ and $B = \{x : x \in \mathbb{N} \text{ and } x^2 < 10\}$, then $A = B$.
- $\{x: x \text{ is a real number and } x^2 = -1\} = \{\}$
- {1} is a set, different from 1 (a number).
- $\{(x, y): x^2 + y^2 = 1\}$ is a set of points, representing the unit circle.

Subset, power set

- *A* is a subset of *B* means every element of *A* is also an element of *B*. It is denoted by $A \subseteq B$. Two sets are equal if they have the same elements i.e.
$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A$$
- *A* is a proper subset of *B* ($A \subset B$) if $A \subseteq B$ and $A \neq B$.
- The number of elements, or cardinality, of a set *S* is denoted by |*S*|.
- Set may contain other sets - If *A* is a set, the *set of all subsets of A* is the power set of *A*, denoted by P(*A*).

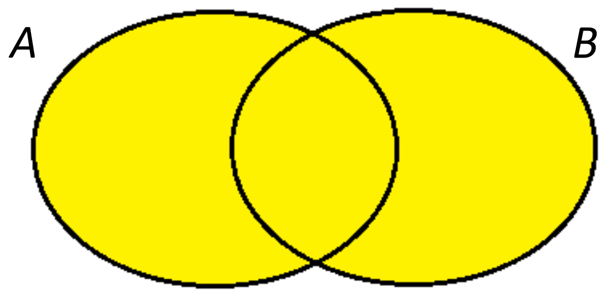
Examples:

- For any set *S*, $\emptyset \subseteq S, S \subseteq S$ but $S \not\subset S$.
- For set of numbers, $\mathbb{P} \subseteq \mathbb{N} \subseteq \mathbb{Z}$.
- If *S* = {1, 1, 2, 2}, then |*S*| = 2. P(*S*) = {{}, {1}, {2}, {1, 2}} and |P(*S*)| = 4. If *T* = {1}, $T \subseteq S$ and $T \in P(S)$, but $T \notin S$ and $T \not\subseteq P(S)$.

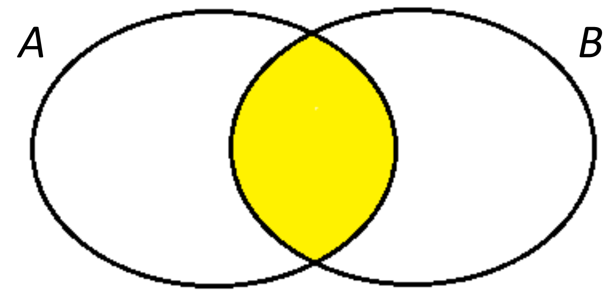
Union and Intersection

- Informally, union is "or", intersection is "and".
- The union of sets *A* and *B* is defined as $A \cup B = \{x : x \in A$ or $x \in B\}$
- The intersection of sets *A* and *B* is defined as $A \cap B = \{x : x \in A$ and $x \in B\}$

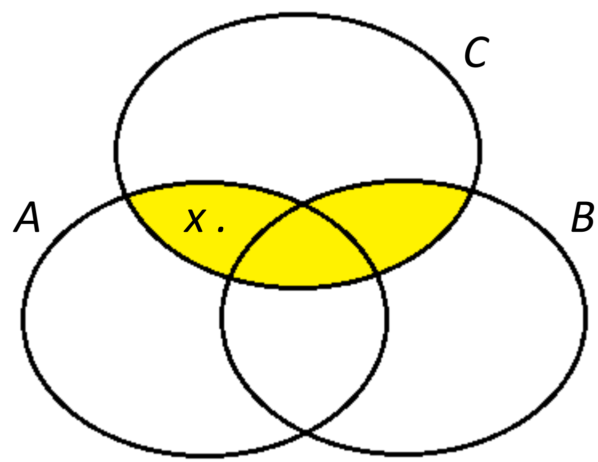
They are best illustrated in Venn Diagrams:



$A \cup B$



$A \cap B$



$x \in (A \cup B) \cap C$

Russell's Paradox

Suppose R is a set of all sets that are not members of itself.

- If R is not a member of itself, by definition, it is contained in R.
- If R contains itself, it contradicts its own definition.

$$\text{Let } R = \{x \mid x \notin x\}, \text{ then } R \in R \iff R \notin R$$

This contradiction is the Russell's Paradox and it shows that sets cannot be arbitrarily defined.

A popular version is the barber paradox. There is a barber who shave all those, and only those, who do not shave themselves. Does the barber shave himself?

$$S = \{x \in T : \text{some property of } x\}$$

In order to avoid this paradox, we only construct sets from existing set and specification.

"You shave 'all the men who don't shave themselves'? What do you mean? Who the fuck is shaving you then?"

From: The Math kid

Russell's Paradox

Everything on this page
and
everything on next page

are false

Russell's Paradox

You love Mathematics!