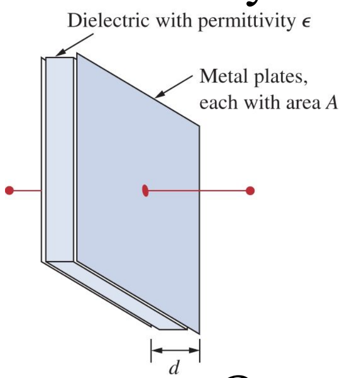
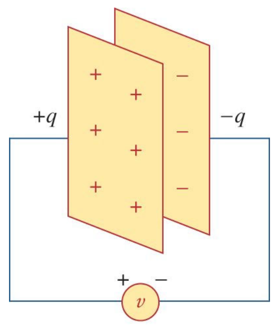
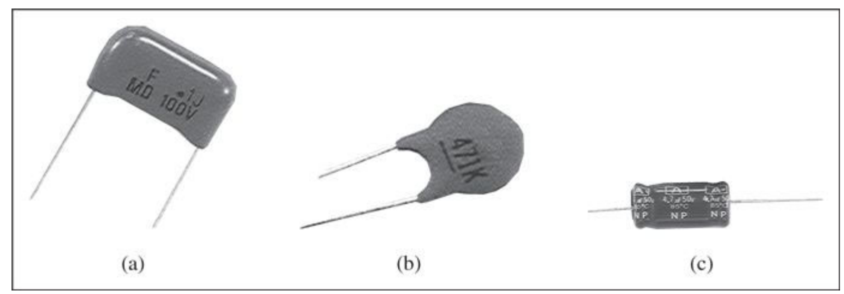
Capacitor and Capacitance

Capacitor

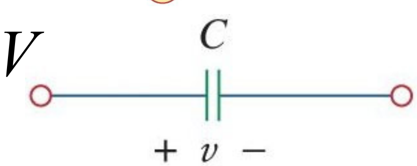
- Simplest way to form a capacitor is to sandwich an insulator (technical term is dielectric) between a pair of parallel conducting plates
- Hence the symbol for a capacitor is two parallel lines separated by a gap







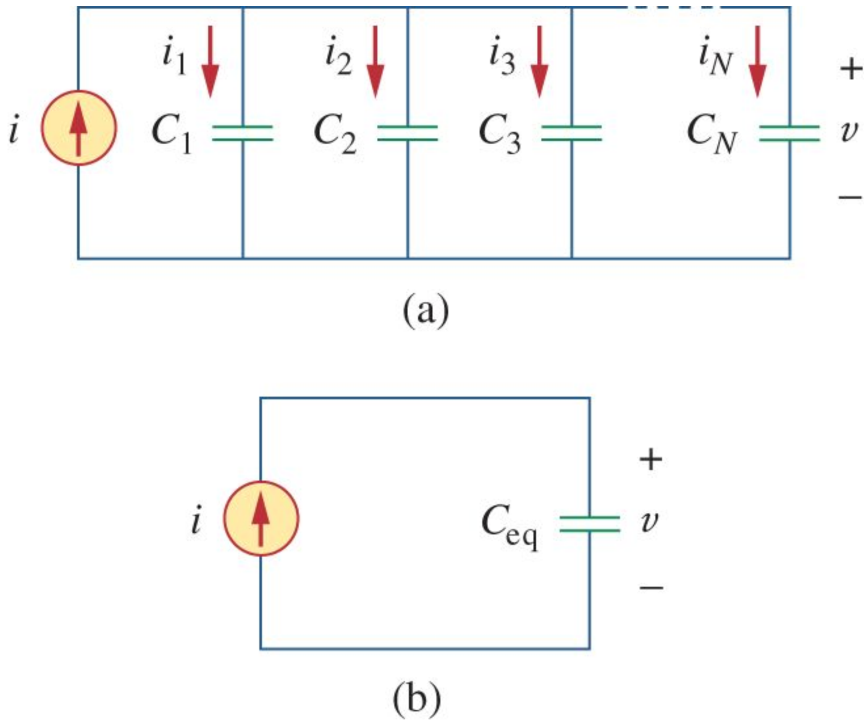
$$Q = CV$$



Capacitance

- Commonly symbolized by the letter C with unit of Farads (F)
- Relates the amount of charge stored for a given voltage applied
- No energy dissipated unlike resistors
- Energy is stored (in keeping the plates apart)

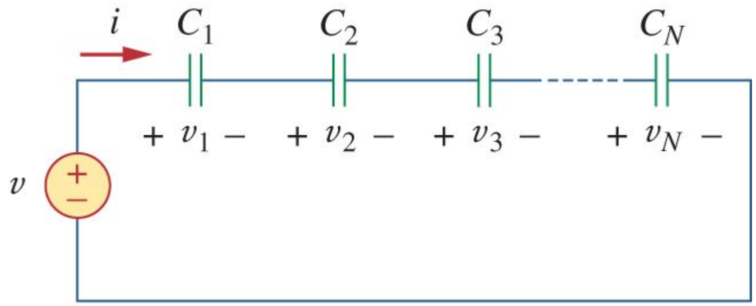
Capacitors in parallel



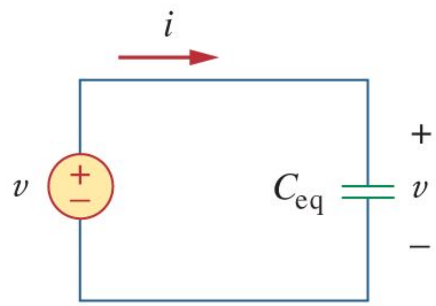
$$C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N$$

What is the value of Q_{eq}?

Capacitors in series



(a)

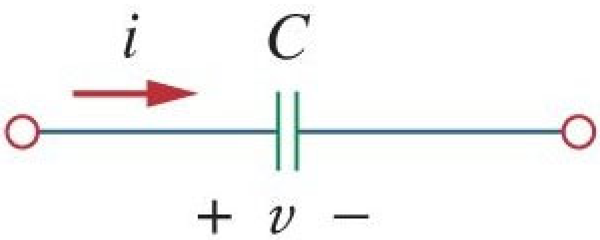


(b)

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N}$$

I-V relation in a capacitor

If the voltage across the capacitor is time-varying, then the charge stored on the capacitor must also be time-varying:



$$Q(t) = CV(t)$$ Differentiating with respect to time (t):

$$i(t) = C\frac{dV(t)}{dt}$$

Current depends on the **rate of change** of voltage

Case study to consider:

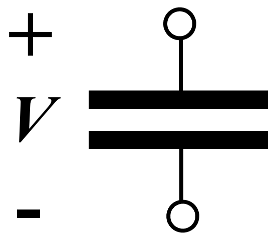
Given that C = 1 nF

If V = 1V → Q = ______

If we reverse V, such that now V = -1V → Q = _______

If the above change was made gradually over 1 ms, what would be the resulting current?

DC and AC response in Capacitor



$$\boldsymbol{Q(t) = CV(t)} \xrightarrow[\text{with respect to time}]{\text{Differentiate}} \boxed{i(t) = C\frac{dV(t)}{dt}}$$

If voltage is constant with time: **dV/dt = 0 → I = 0**

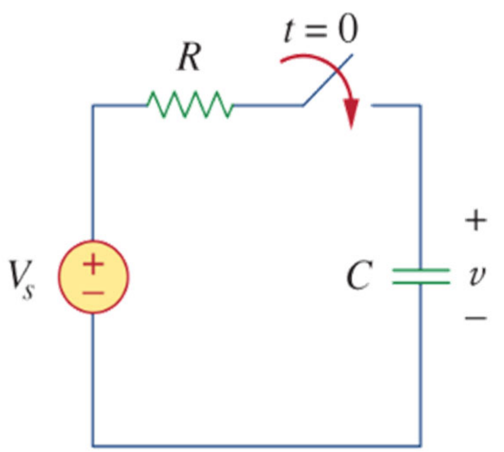
No voltage change → No current

Case study to consider: Given that C = 1 nF If V(t) = 1 sin (10t) → Q(t) = ________________ → i(t) = ________________

If voltage changes with time: **dV/dt ≠ 0 → I ≠ 0**

Voltage change → Charge changes → Current

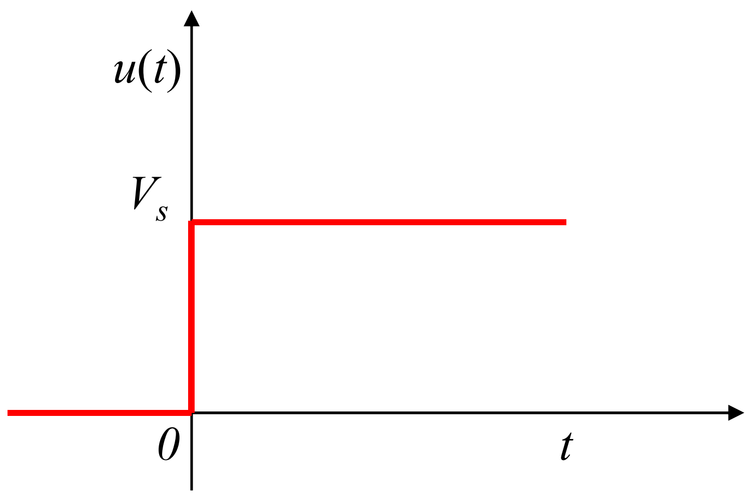
First-order Transient Circuit (RC)



An *RC* circuit with a step input voltage.

$$\xrightarrow[\text{for } t \geq 0]{\text{KVL}}$$

$$\boxed{RC\frac{dV(t)}{dt} + V(t) = Vs}$$



Input step voltage signal

First-order Transient Circuit (RC)

$$RC\frac{dV(t)}{dt} + V(t) = Vs$$

Consider the standard first-order linear equation

$$\frac{dV(t)}{dt} + PV(t) = Q$$

Integrating factor method

$$P = \frac{1}{RC}, Q = \frac{V_s}{RC}$$

also

$$\mu = e^{\int_{0-}^{t} P dt}$$

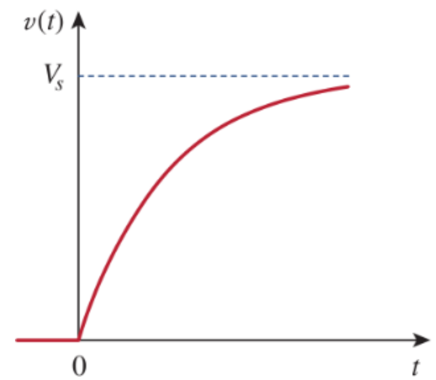
$$= e^{\frac{1}{RC}t}$$

$$V(t) = \frac{1}{\mu}\int_{0-}^{t} \mu Q dt$$

$$= e^{-\frac{t}{RC}} \frac{V_s}{RC}\int_{0-}^{t} e^{\frac{1}{RC}t} dt$$

$$= V_s\, e^{-\frac{t}{RC}} \left[\, e^{\frac{t}{RC}}\Big|_0^t \,\right]$$

$$= \boxed{V_s \left[\, 1 - e^{-\frac{t}{RC}} \,\right] (t \geq 0)}$$



Step response of an *RC* circuit

Summary: Capacitor response to DC & AC

1) If applied voltage is DC

- Insulating dielectric blocks the current from flowing through
- Plates will charge up
- At DC, capacitor blocks current from flowing through

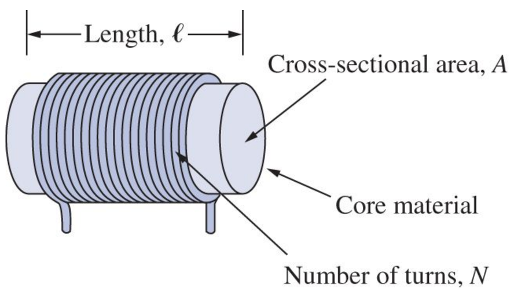
2) If applied voltage is AC (e.g. sinewave)

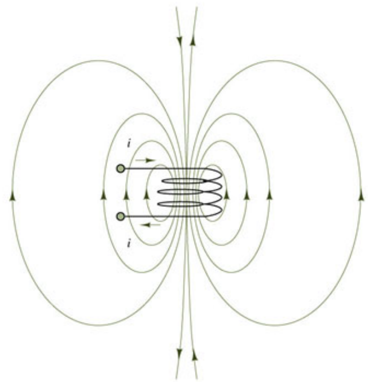
- Charge on the plates also likewise varies in time with the AC
- Capacitor therefore does not act as an open circuit in the presence of an AC
- In AC, capacitor allows current to pass through

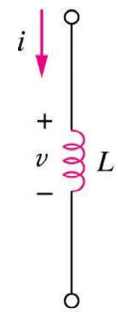
Inductors and Inductance

Inductor

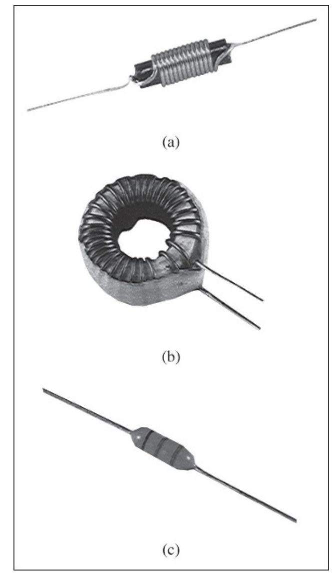
- Simplest way to form an inductor is to winding a coil around a core that concentrates magnetic field lines (flux)
- Hence symbol of an inductor is coil between two terminals







$$\varphi = Li$$

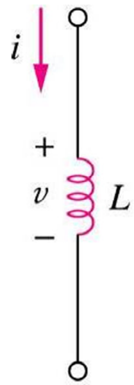


Inductance

- Inductance is commonly symbolized by letter L with unit of Henrys (H)
- Passing a current through an inductor produces a magnetic flux (φ) that is related to the inductance (L)
- No energy dissipated unlike resistors (note that wires are assumed to have no resistance by definition)

I-V relation in an inductor

If the current through an inductor is time-varying, then the generated voltage must also be time-varying:



Flux: $\varphi = Li$

Faraday's Law: $v(t) = \dfrac{d\emptyset(t)}{dt}$

Differentiating with respect to time (t): $v(t) = L\dfrac{di(t)}{dt}$

Voltage depends on the **rate of change** of current

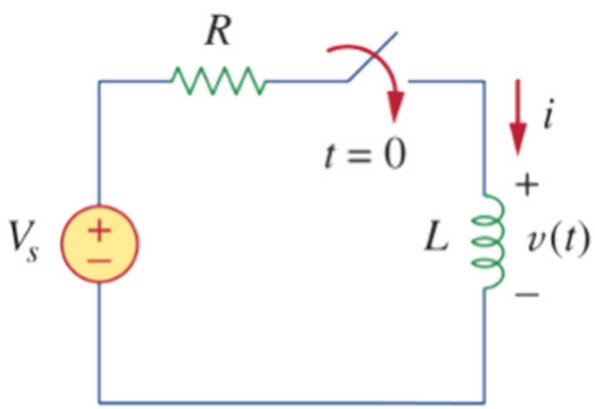
If current is constant with time → di/dt = 0 → V = 0 (No voltage)

No current change → No voltage difference

If current changes with time → dV/dt ≠ 0 → I ≠ 0 (There is voltage)

Current change → Voltage difference

First-order Transient Circuit (*RL*)

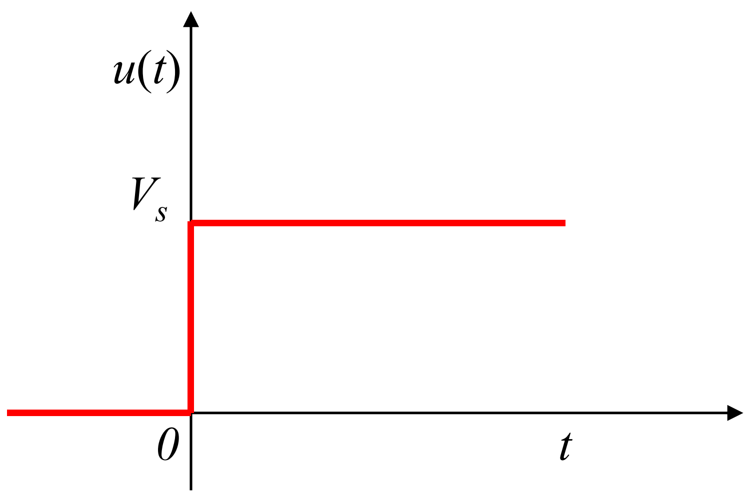


An *RL* circuit with a step input voltage.

KVL

for $t \geq 0$

$$\boxed{Ri(t) + L\frac{di(t)}{dt} = Vs}$$



Input step voltage signal

First-order Transient Circuit (*RL*)

$$Ri(t) + L\frac{di(t)}{dt} = Vs$$

Consider the standard first-order linear equation

$$\frac{di(t)}{dt} + Pi(t) = Q$$

Integrating factor method:

$$P = \frac{R}{L}, Q = \frac{V_s}{L}$$

also

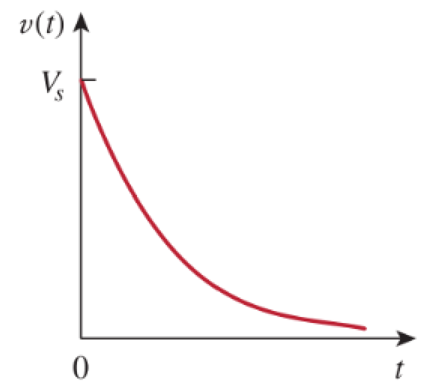
$$\mu = e^{\int_{0-}^{t} P dt}$$

$$= e^{\frac{R}{L}t}$$

$$i(t) = \frac{1}{\mu}\int_{0-}^{t} \mu Q dt$$

$$= e^{-\frac{R}{L}t}\frac{V_s}{L}\int_{0-}^{t} e^{\frac{R}{L}t} dt$$

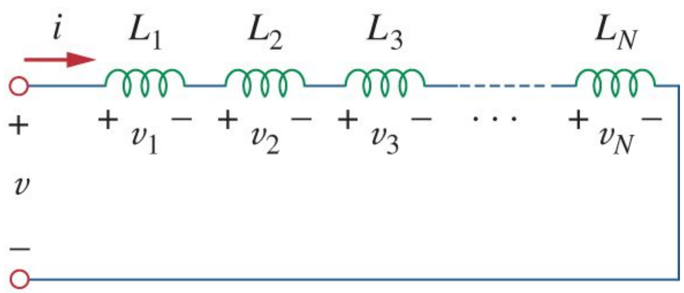
$$= \frac{V_s}{R}[1 - e^{-\frac{R}{L}t}], t \geq 0$$



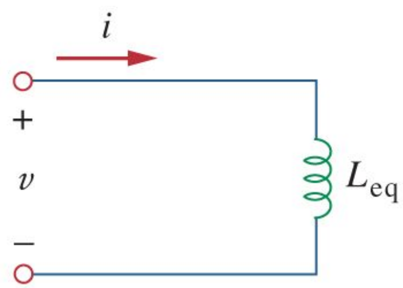
Step response of an *RL* circuit

$$V(t) = V_s - Ri(t) = \boxed{V_s\, e^{-\frac{R}{L}t}, (t \geq 0)}$$

Inductors in series



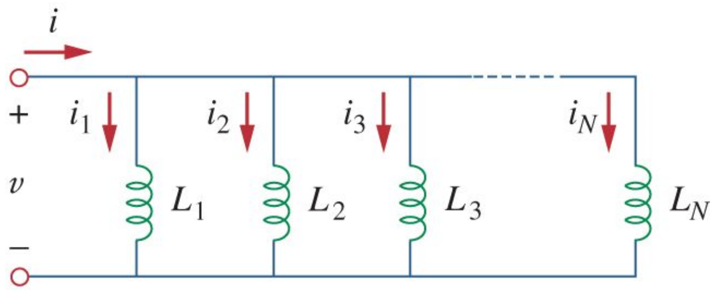
(a)



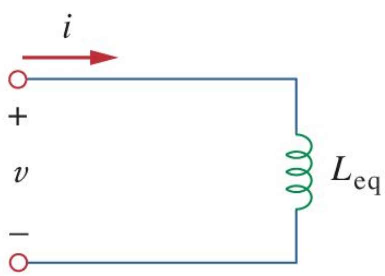
(b)

$$L_{eq} = L_1 + L_2 + L_3 + \cdots + L_N$$

Inductors in parallel



(a)



(b)

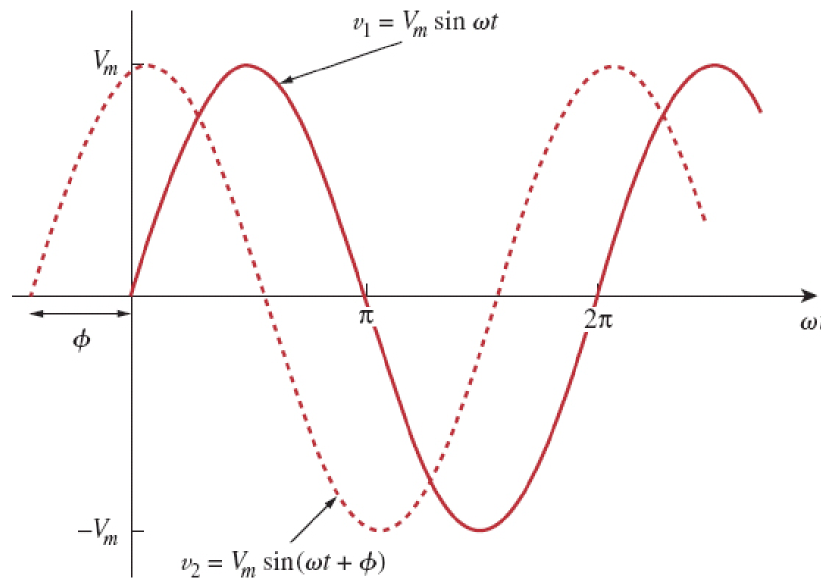
$$\frac{1}{\mathrm{L_{eq}}}\int v(t)dt = \frac{1}{\mathrm{L_1}}\int v(t)dt + \frac{1}{\mathrm{L_2}}\int v(t)dt + \frac{1}{\mathrm{L_3}}\int v(t)dt + \cdots + \frac{1}{\mathrm{L_N}}\int v(t)dt$$

$$\frac{1}{\mathrm{L_{eq}}} = \frac{1}{\mathrm{L_1}} + \frac{1}{\mathrm{L_2}} + \frac{1}{\mathrm{L_3}} + \cdots + \frac{1}{\mathrm{L_N}}$$

Sinusoids

Sinusoids can be written as sine or cosine functions

$$x(t) = A\sin(\omega t + \emptyset)$$



A = amplitude

ω = radian frequency = $2\pi f$ (rad/s)

f = natural frequency (cycles/s or Hz)

$\emptyset$ = phase (with reference to a cycle: 2π)

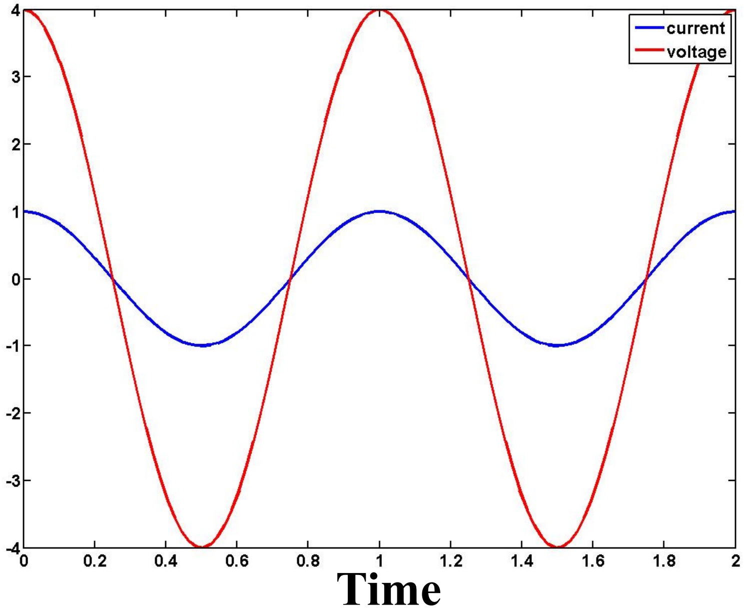
We compare the two waves by:

1) The ratio of their amplitudes

2) Phase difference

We can immediately see that an AC signal has as lot more key features than a DC signal. For a DC signal, the magnitude alone is a sufficient quantitative description. But in the case of sinusoidal AC signals, concepts like frequency and phase also need to be considered.

I-V relationship in a resistor

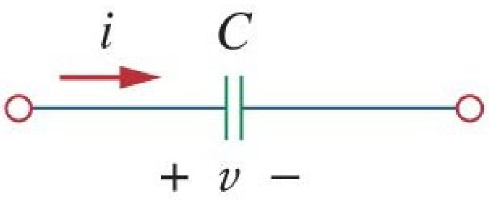


This plot shows the relationship between the current and voltage in a 4Ω resistor when the signal is an AC sinusoid.

It can be seen that there is no phase difference between the voltage and current.

This is because R is simply given by the ratio of voltage to current and there is therefore no phase shift between the two, but simply a scaling in the amplitude.

I-V relationship in a capacitor

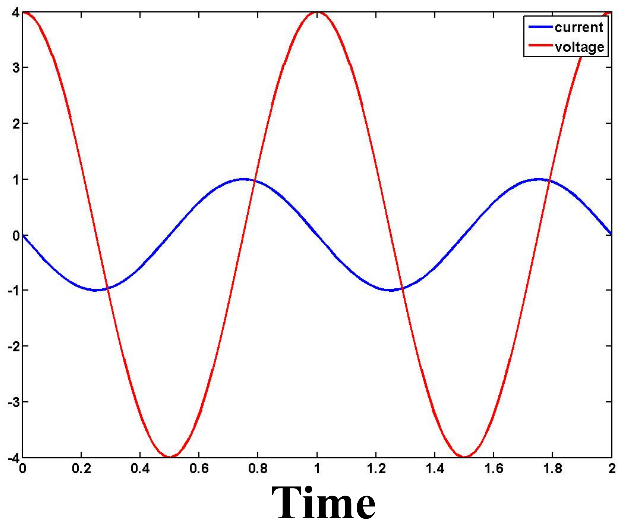


If the voltage across a capacitor is: $V(t) = V_m \cos(\omega t)$

Then the current through it will be:

$I(t) = C(dV/dt) = -\omega C V_m \sin(\omega t) = \omega C V_m \cos(\omega t + 90°)$

Shift forward by 90°



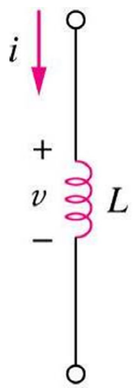
Therefore we say that I <u>leads</u> V by 90°

(or $\pi/2$ in radians)

$$\frac{V_m}{I_m} = \frac{1}{\omega C} \qquad \angle V - \angle I = -\frac{\pi}{2}$$

1) Current goes through a phase shift <u>**relative**</u> to the voltage (in a resistor, there is no phase shift between the current and voltage)
2) Ratio of voltage to current depends on the <u>**capacitance**</u> **AND** <u>**frequency**</u> of the sinusoid (in a resistor, the ratio between voltage and current is simply R and independent of frequency)

I-V relationship in an inductor

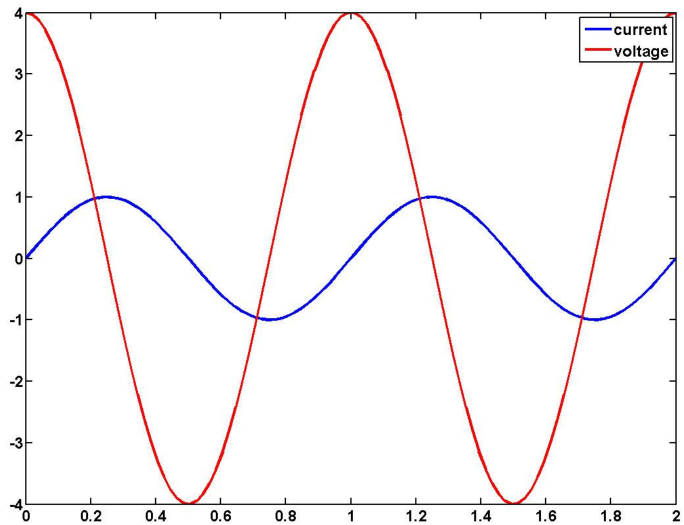


If the current through an inductor is: $i(t) = I_m cos(\omega t)$

Then the current through it will be:

$v(t) = L(di/dt) = -\omega L I_m \sin(\omega t) = \omega L I_m \cos(\omega t + 90°)$

Shift forward by 90°



Time

Therefore we say that I <u>lags</u> V by 90° (or $\pi/2$ in radians)

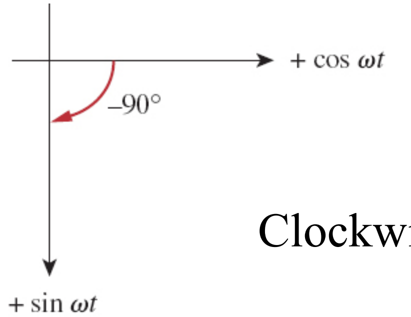
$$\frac{V_m}{I_m} = \omega L \qquad \angle V - \angle I = +\frac{\pi}{2}$$

1) Current goes through a phase shift **<u>relative</u>** to the voltage
2) Ratio of voltage to current depends on **<u>inductance</u>** AND **<u>frequency</u>** of the sinusoid

Graphical methods

Calculate the phase angle between:

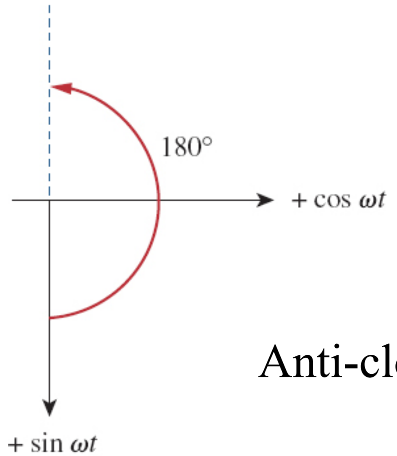
$v_1 = -10\cos(\omega t + 50°)$ and $v_2 = 12\sin(\omega t - 10°)$



(a)

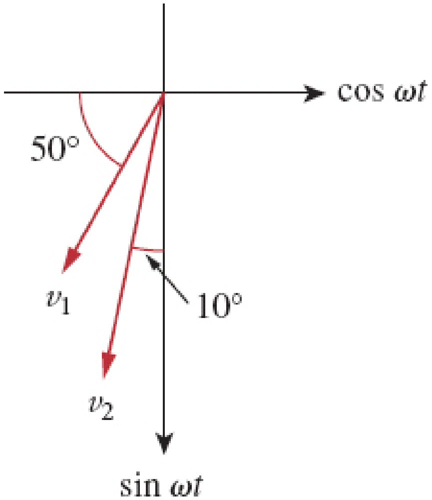
Clockwise: Phase goes backward (-ve)

$\angle v_1 - \angle v_2 = -30°$



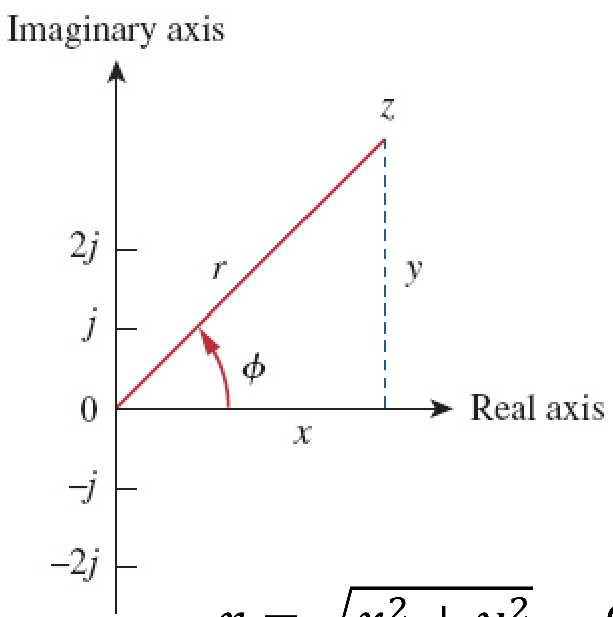
(b)

Anti-clockwise: Phase goes forward (+ve)



Phasors

- A phasor is a complex number that represents the amplitude and phase of a sinusoid
- Phasors are more convenient to work with than sine and cosine functions and are a powerful tool for analyzing circuits



A complex number z can be expressed in rectangular form as:

$z = x + jy$ Definition: $j = \sqrt{-1}$

The complex number z can be expressed in polar form or exponential form as:

$z = r\angle\emptyset = re^{j\emptyset}$

$r = \sqrt{x^2 + y^2}$ $\emptyset = \tan^{-1}\frac{y}{x}$

$x = r\cos\emptyset$ $y = r\sin\emptyset$

Euler's identity
$e^{\pm j\emptyset} = \cos\emptyset + j\sin\emptyset$

Complex number arithmetic

The following mathematical operations are important.

Addition

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication

$$z_1 z_2 = r_1 r_2 \angle(\emptyset_1 + \emptyset_2)$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\emptyset_1 - \emptyset_2)$$

Reciprocal

$$\frac{1}{z} = \frac{1}{r}\angle(-\emptyset)$$

Square Root

$$\sqrt{z} = \sqrt{r}\angle(\emptyset/2)$$

Complex Conjugate

$$z* = x - jy = r\angle - \emptyset = re^{-j\emptyset}$$

Phasors

A phasor represents a sinusoid as the real component of a vector in the complex plane representation based on Euler's identity:

$$e^{\pm j\emptyset} = \cos\emptyset \pm j\sin\emptyset$$

$$\cos\emptyset = \mathrm{Re}\left(e^{j\emptyset}\right) \qquad \sin\emptyset = Im\left(e^{j\emptyset}\right)$$

Given a sinusoid: $v(t) = V_m\cos(\omega t + \emptyset)$

$$v(t) = Re\left(V_m e^{j(\omega t+\emptyset)}\right) = Re\left(V_m e^{j\omega t} e^{j\emptyset}\right)$$

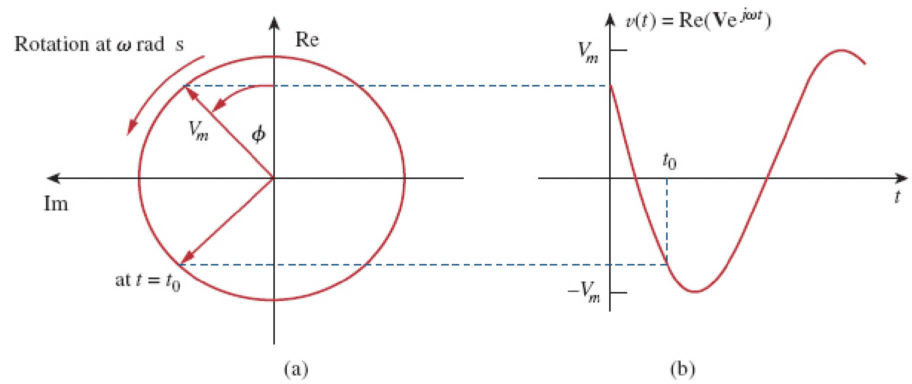
$$v(t) = Re\left(V_m e^{j(\omega t+\emptyset)}\right) = Re\left(\boldsymbol{V} e^{j\omega t}\right)$$

, where $\boldsymbol{V} = V_m e^{j\emptyset} = V_m\angle\emptyset$

$$\boxed{v(t) = V_m\cos(\omega t + \emptyset) \Leftrightarrow \boldsymbol{V} = V_m\angle\emptyset}$$

(Time-domain representation) (Phasor-domain representation)

The **length** of the vector is the **amplitude** of the sinusoid.
The **angle** φ of the vector with respect to the positive real axis is the **phase**.



Sinusoid-phase transformation

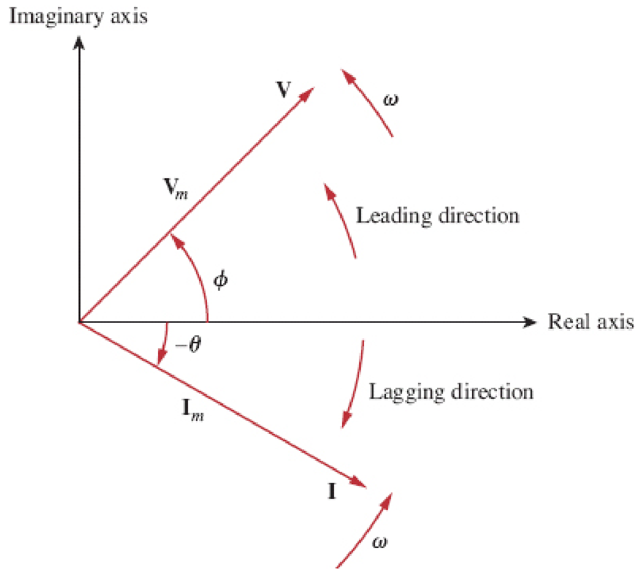
TABLE 9.1

Sinusoid-phasor transformation.

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \underline{/\phi}$
$V_m \sin(\omega t + \phi)$	$V_m \underline{/\phi - 90^\circ}$
$I_m \cos(\omega t + \theta)$	$I_m \underline{/\theta}$
$I_m \sin(\omega t + \theta)$	$I_m \underline{/\theta - 90^\circ}$

Applying a *derivative* to a phasor yields:

$$\frac{dv}{dt} \Rightarrow j\omega \boldsymbol{V}$$

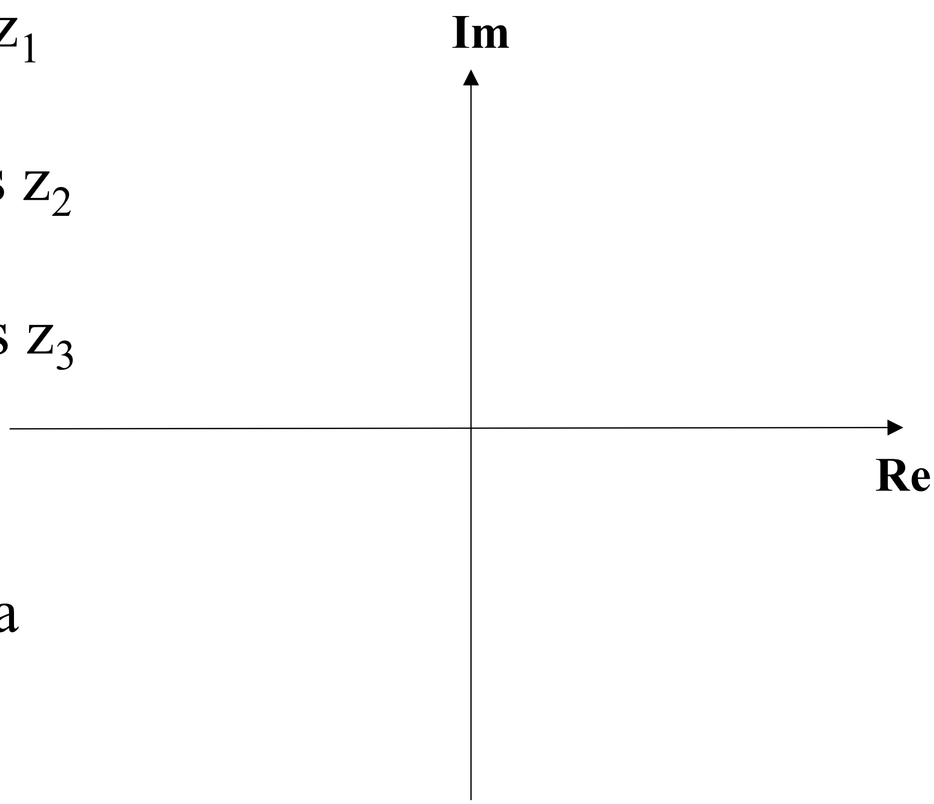


Differences between v(t) and **V** to keep in mind:

1) v(t) is the time domain form, while **V** is the frequency domain form
2) v(t) is therefore time-dependent, while **V** is not
3) v(t) is always real, while **V** is general complex

The meaning of “j”

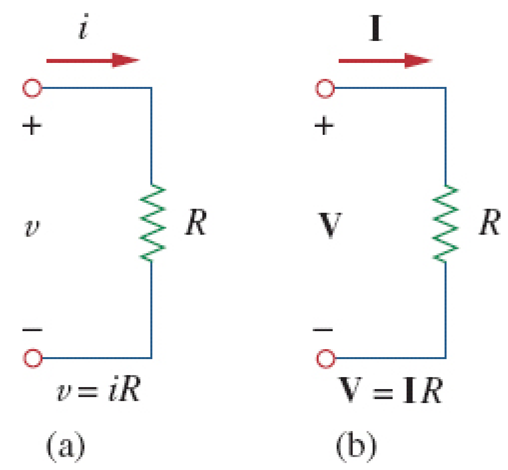
1) Draw $z = 3$ on the Im-Re graph
2) Multiply z by j, name this as z_1 and draw it on the graph
3) Multiply z by $-j$, name this as z_2 and draw it on the graph
4) Multiply z by -1, name this as z_3 and draw it on the graph

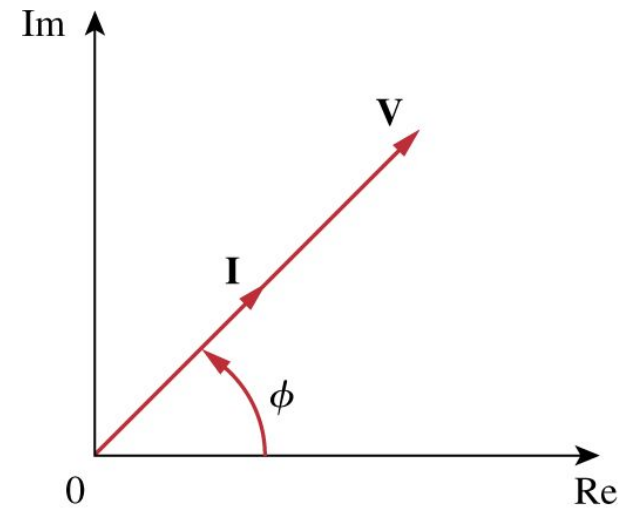


What is the effect of multiplying a number by j, -j, or -1?

Phasor Relationships for Resistors

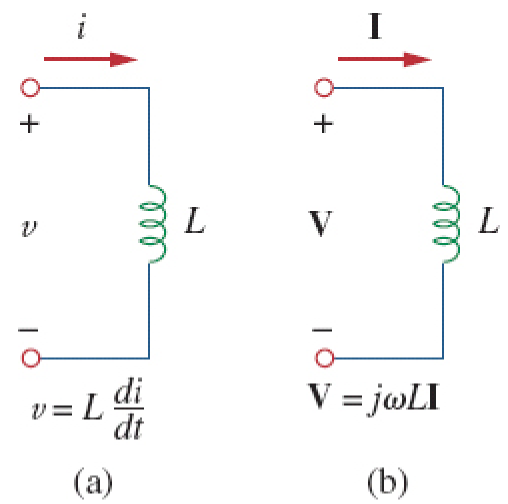
1) Each circuit element has a relationship between its current and voltage.
2) These can be mapped into phasor relationships very simply for resistors capacitors and inductor.
3) For the resistor, the voltage and current are related via Ohm's law.
4) As such, the voltage and current are in phase with each other.

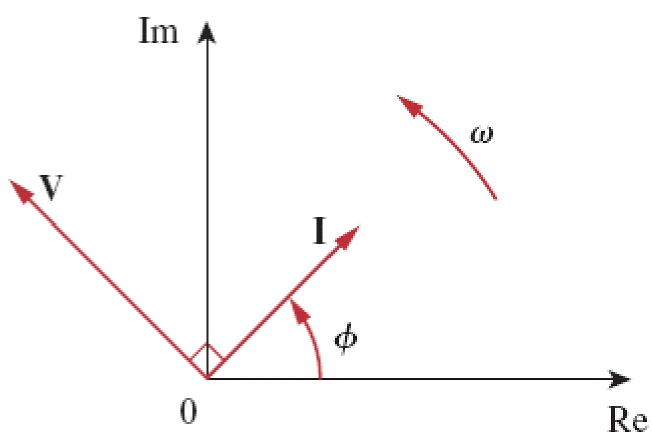




Phasor Relationships for Inductors

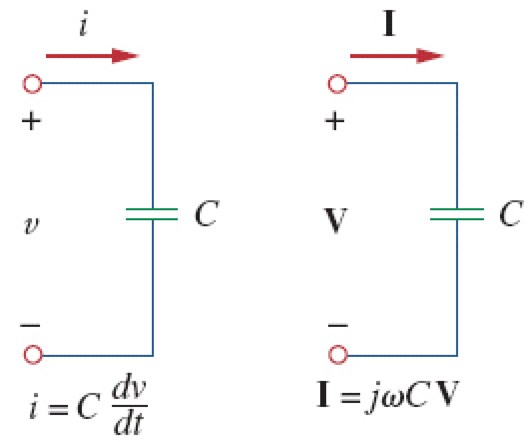
1) Inductors on the other hand have a phase shift between the voltage and current.
2) In this case, the voltage leads the current by 90°.
3) Or one says the current lags the voltage, which is the standard convention.
4) This is represented on the phasor diagram by a positive phase angle between the voltage and current.



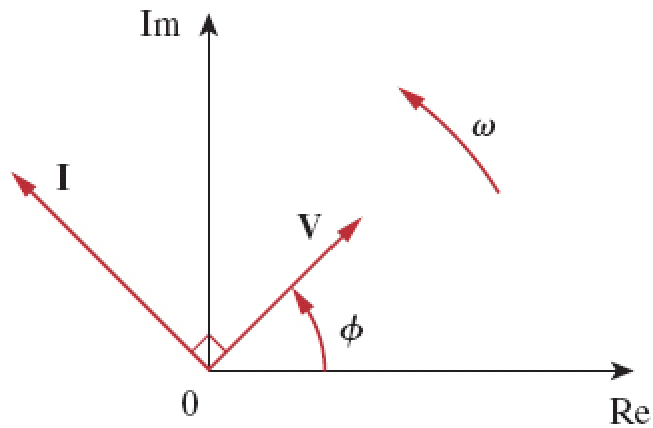


Phasor Relationships for Capacitors

1) Capacitors have the opposite phase relationship as compared to inductors.
2) In their case, the current leads the voltage.
3) In a phasor diagram, this corresponds to a negative phase angle between the voltage and current.



$i = C\frac{dv}{dt}$ (a) $\mathbf{I} = j\omega C\mathbf{V}$ (b)



Voltage current relationships

TABLE 9.2

Summary of voltage-current relationships.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

Impedance and Admittance

We can now expand Ohm's law to capacitors and inductors. This would have been otherwise tricky as the ratios of voltage and current always changing when working in the time-domain form. In frequency domain it is straightforward.

The impedance of a circuit element is the ratio of the phasor voltage to the phasor current, commonly represented by Z.

$Z = \frac{V}{I}$ or $V = ZI$ Admittance is simply the inverse of impedance: Y = 1/Z

Impedance and Admittance

- In the frequency domain, the values obtained for impedance are only valid at that frequency.
- Changing to a new frequency will require recalculating the values.
- The impedance of capacitors and inductors are shown here:

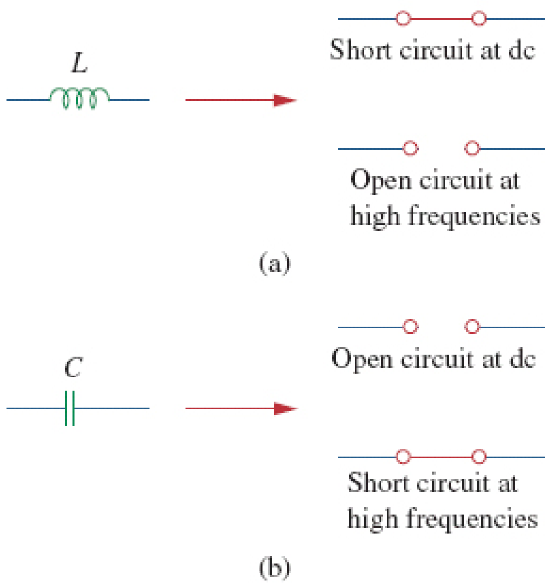


TABLE 9.3

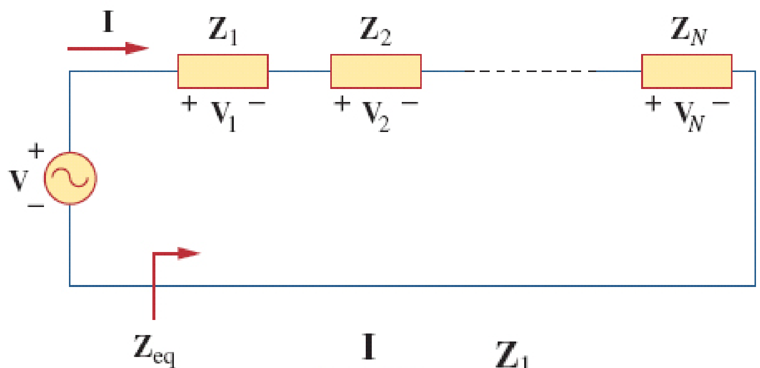
Impedances and admittances of passive elements.

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \dfrac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \dfrac{1}{j\omega L}$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Applying Kirchhoff's in frequency domain

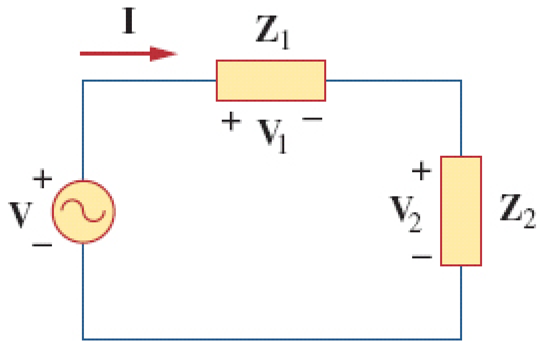
Phasors allow Kirchhoff's laws to be applicable. Therefore a circuit transformed to the frequency domain can be evaluated by the same methods developed for KVL and KCL.

Once in the frequency domain, the impedance elements can be combined using the rules for resistors.



Series combinations will result in a sum of the impedance elements:

$$Z_{eq} = Z_1 + Z_2 + Z_3 + \cdots + Z_N$$



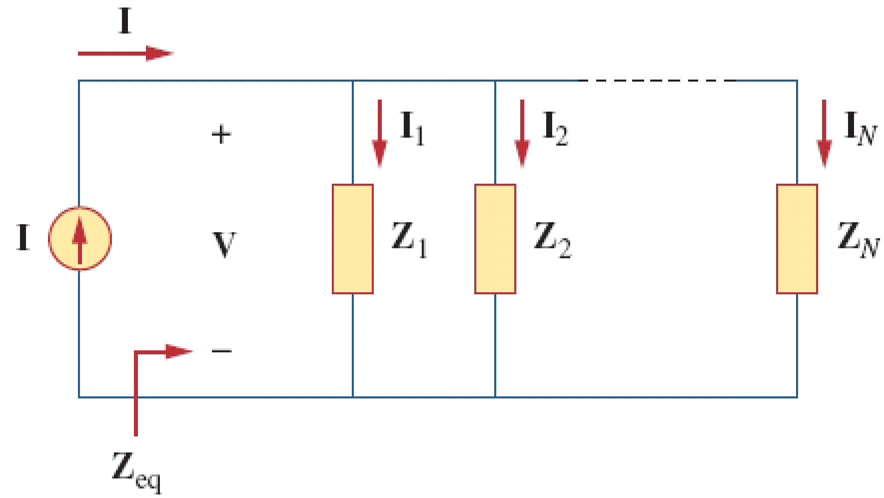
Elements in series can act like a voltage divider:

$$V_1 = \frac{Z_1}{Z_1 + Z_2} V \qquad V_2 = \frac{Z_2}{Z_1 + Z_2} V$$

Parallel combination

Elements combined in parallel will combine in the same fashion as resistors in parallel:

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots + \frac{1}{Z_N}$$



Expressed as admittance, they are again a sum:

$$Y_{eq} = Y_1 + Y_2 + Y_3 + \cdots + Y_N$$

Impedance: Example 1

Problem 4.61

Work out the impedance (Z) seen across the terminals

When (i) $\omega = 4$ rad/s, (ii) $\omega = 8$ rad/s

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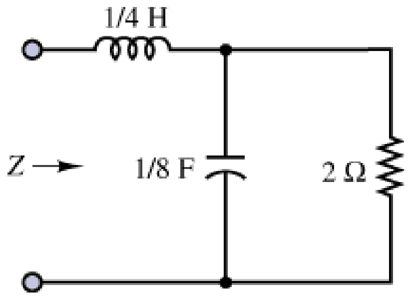

(i) At 4 rad/s:

Impedance of L:

$X_L = j\omega L = j(4)(1/4) = j$

Impedance of C:

$X_C = 1/j\omega C = 1/[j(4)(1/8)] = -j2$

Impedance: Example 1 (i) 4 rad/s

When $\omega = 4$ rad/s: $X_L = j\omega L = j$; $X_C = 1/j\omega C = -j2$

Impedance of C and parallel with R: $$Z_{RC} = \frac{X_C R}{R + X_C} = \frac{(-j2)(2)}{2 - j2} = 1 - j$$

Z_{RC} is in series with impedance of L: $$Z = Z_{RC} + X_L = 1 - j + j = 1\Omega$$

Impedance: Example 1 (ii) 8 rad/s

When ω = 8 rad/s:

$X_L = j\omega L = j(8)(1/4) = j2$;

$X_C = 1/j\omega C = -j/[(8)(1/8)] = -j$

Impedance of C and parallel with R: $Z_{RC} = \dfrac{X_C R}{R + X_C} = \dfrac{(-j)(2)}{2 - j}$

$$= \frac{-j2(2+j)}{(2-j)(2+j)} = \frac{2}{5}(1 - j2)$$

Z_{RC} is in series with impedance of L: $Z = Z_{RC} + X_L$

$$= \tfrac{2}{5}(1 - j2) + j2$$

$$= 0.4 + j1.2$$

Impedance depends on the frequency

Impedance: Example 2

Example 4.14 of Rizzoni

Find the equivalent impedance (Z_{EQ}) and admittance (Y_{EQ}) seen across the terminals

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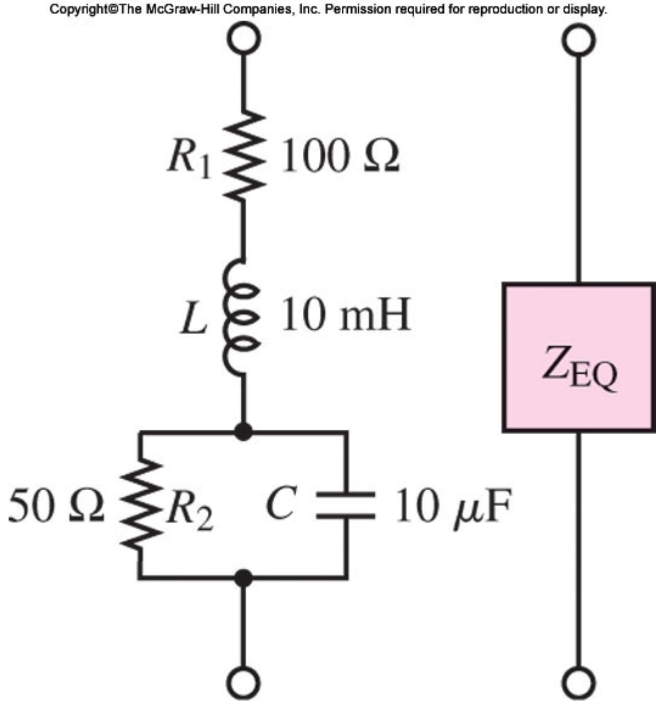

Given: $\omega = 10^4$ rad/s

Impedance of L:

$$X_L = j\omega L = j(10^4)(10^{-2}) = j100$$

Impedance of C:

$$X_C = 1/j\omega C = 1/[j(10^4)(10^{-5})] = -j10$$

Impedance: Example 2 Solution

Impedance of C parallel R_2:

$$Z_{RC} = \frac{X_C R}{R + X_C}$$

$$= \frac{(-j10)(50)}{50 - j10}$$

$$= \frac{500}{10 + j50}$$

$$= \frac{50}{1 + j5}$$

Impedance of L series R_1:

$$Z_{RL} = X_L + R$$

$$= 100 + j100$$

$$= 100(1 + j)$$

$$Z_{EQ} = Z_{RC} + Z_{RL}$$

$$= \tfrac{50}{1+j5} + 100(1 + j)$$

$$= 101.92 + j90.385$$

$$= 136.2\angle 41.57^0\,\Omega$$

$$\mathrm{Y_{EQ}} = 1/\mathrm{Z_{EQ}} = (1/136.2)\angle -41.6^0\,S$$

$$= 7.342\angle -41.6^0\,mS$$

Admittance

Admittance is simply the **inverse** of the **impedance: Y = 1/Z**

Admittance likewise comprises both real and imaginary parts

Y = G + jB

Real part is called the **AC conductance**

Imaginary part is called the **susceptance**

$$Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{1}{R + jX}\left(\frac{R - jX}{R - jX}\right)$$

$$= \frac{R - jX}{R^2 + X^2}$$

$$G = \frac{R}{R^2 + X^2}$$

$$B = -\frac{X}{R^2 + X^2}$$

Example on Admittance

$$Z_{ab} = R_1 + j\omega L$$

$$Y_{ab} = 1/Z_{ab} = 1/(R_1 + j\omega L)$$

$$Y_{ab} = \frac{R_1}{R_1^2 + (\omega L)^2} - \frac{j\omega L}{R_1^2 + (\omega L)^2}$$

$$Y_{ab} = 1/Z_{ab} = 1/R_2 + j\omega C$$

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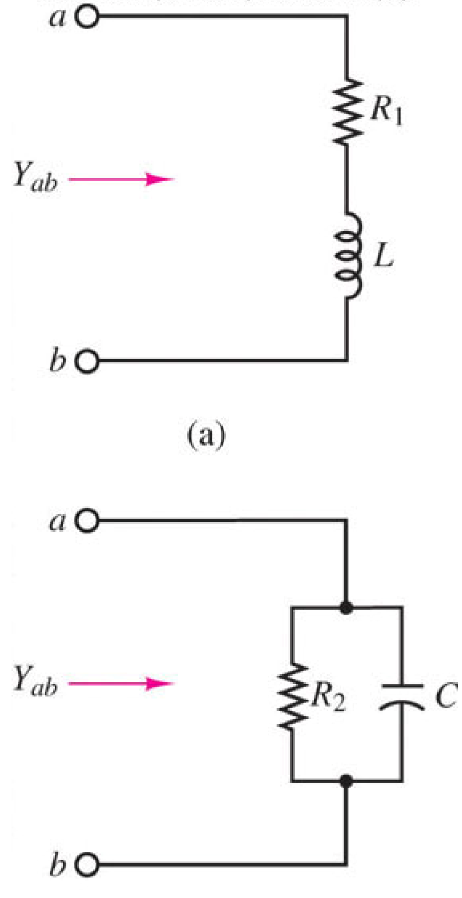

(a)

(b)

Summary

1) Capacitors: Passes current at high frequency but blocks at DC
2) Inductor: Acts as short circuit at DC but blocks current at high frequency
3) Impedance introduced as a more general form to describe V/I (magnitude & phase)
4) Impedance depends on frequency
5) Represent these as phasors so we can add up impedances just like resistances in DC