$(x+i)(x-i)$
$\|$
$x^2+1=0$

$\lambda_1, \cdots, \lambda_n \in \underline{\underline{\mathbb{C}}}$

$\underline{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0} = 0$

$\|$

$a_n(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_m)\cancel{(x^2+b_1x+c_1)\cdots(x^2+b_l x+c_l)}$

$(x-\lambda_{m+1})\cdots(x-\lambda_n)$

MA1301 Calculus and Basic Linear Algebra II
Chapter 2 Complex Numbers

As early as 250 A.D., Greek algebraist, Diophantus, attempts to solve quadratic equations of the form $ax^2+bx+c=0$. He accepts only positive rational roots and ignores all others. He rejects; equations such as $x^2+a^2=0$ as insolvable. In the sixteenth- and seventeenth-century new numbers are obtained by extending the arithmetic operation of square root to whatever numbers appeared in solving quadratic equations by the usual method of completing the square. For instance, Cartan (1545) obtains the roots of the equation $x(10-x)=40 \Leftrightarrow (x-5)^2=-15$ as $5+\sqrt{-15}$ and $5-\sqrt{-15}$. However he regarded them as useless. By the eighteenth century, through the work of Euler, de Moivre, d'Alembert and Cauchy, complex numbers were finally accepted by mathematicians. In here you will study the arithmetic operations on complex numbers, geometrical representation and use of de Moivre's theorem to simplify complex expressions. Finally you will look at the solutions of the equation $z^n+z_0=0$.

1 Basic idea of complex numbers (p.52 – p. 53)

(a) $\sqrt{-1}$ is defined as a number whose square is -1 and it is denoted by i and is called the (purely) imaginary unit. If b is real then bi is said to be a (purely) imaginary number.

(b) A complex number is the sum of a real number and a (purely) imaginary number such that for real $a, b, c, d,$ $a+bi=c+di$ iff $a=c$ and $b=d$.

Thus a complex number can be uniquely written as $z=a+bi$ where a, b are real and in this representation, a is termed the real part of z, and b is termed the (purely) imaginary part of z.
We use Re(z) to denote the real part of z and Im(z) to denote the imaginary part of z. For example, $\text{Re}(3-4i)=3$ and $\text{Im}(3-4i)=-4$.

(c) A complex number cannot be said to be larger or smaller than the other.

(d) The complex conjugate of $a+bi$ is $\overline{a+bi}=a-bi$.

2 Operations with complex numbers (p.53 – p.54)

We operate with complex numbers in exactly the same way as we operate with real numbers; for example:

(a) $(a+bi)+(c+di)=(a+c)+(b+d)i$
e.g. $(3+4i)+(2-5i)=(3+2)+[4+(-5)]i=5-i$
Note: $\text{Re}((3+4i)+(2-5i))=3+2=5$ and $\text{Im}((3+4i)+(2-5i))=4-5=-1$

(b) $(a+bi)-(c+di)=(a-c)+(b-d)i$
e.g. $(3+4i)-(2-5i)=(3-2)+[4-(-5)]i=1+9i$

(c) $(a+bi)(c+di)=(ac-bd)+(bc+ad)i$, since $i^2=-1$

$\|$ $\|$

$ac+adi+bci+bd\boxed{i^2}=-1$

$e+fi$

e.g. $(3+4i)(2-5i)=3(2)-3(5i)+(4i)(2)-(4i)(5i)=6-15i+8i-20i^2=6-15i+8i+20=26-7i$

In particular, the product of $a+bi$ and its conjugate $a-bi$ is a^2+b^2.

e.g. $(3+4i)(3-4i)=3(3)-3(4i)+(4i)(3)-(4i)(4i)=9-12i+12i-16i^2=9+16=25$

(d) $\dfrac{a+bi}{c+di}=\dfrac{(a+bi)(c-di)}{(c+di)(c-di)}=\dfrac{(a+bi)(c-di)}{c^2+d^2}=\dfrac{(ac+bd)+(bc-ad)i}{c^2+d^2}$, provided $c+di\neq 0$.

e.g. $\dfrac{1+4i}{3+2i}=\dfrac{1+4i}{3+2i}\cdot\dfrac{3-2i}{3-2i}=\dfrac{(1+4i)(3-2i)}{3^2+2^2}=\dfrac{3-2i+12i+8}{13}=\dfrac{11+10i}{13}=\dfrac{11}{13}+\dfrac{10}{13}i$

Hence, the sum, difference, product and quotient of two complex numbers is a complex number; the product of two conjugate complex numbers is real and so is their sum.

In particular, we have $\quad z+\bar{z}=2\operatorname{Re}(z), \quad z-\bar{z}=2\operatorname{Im}(z)\mathrm{i}, \quad z\bar{z}=|z|^2,$

$(x-z)(x-\bar{z})=x^2-2\operatorname{Re}(z)x+|z|^2$ for any x.

3 Geometrical representation of complex numbers (p.54)

The real numbers can be represented by a straight line in the sense that it is possible to set up a one-to-one correspondence between the set of all real numbers and the set of all points on a straight line.

To represent the complex numbers, a straight line is not sufficient and it is necessary to make use of a plane.

First, as shown in Figure 1, two perpendicular straight lines in the plane are drawn, one horizontal (called the real axis) and the other vertical (called the imaginary axis), intersecting at a point usually denoted by O and called the origin. Then for each axis, a one-to-one correspondence with the real numbers is set up in the 'usual' way so that the origin will correspond to the number zero in each case.

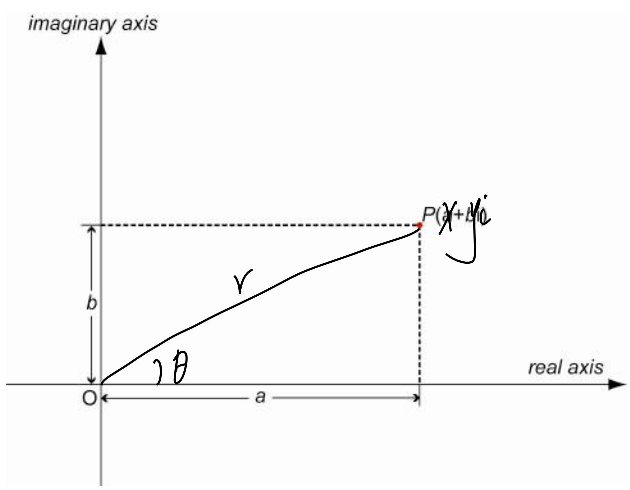


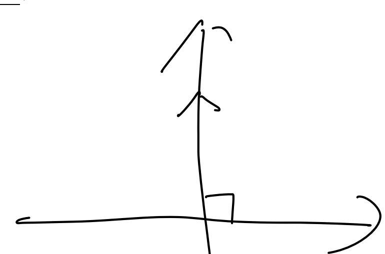
Figure 1

The plane used to represent complex numbers is called the complex plane, and the diagram used to represent complex numbers in the complex plane is called the Argand diagram.

$k\in\mathbb{Z}$.

2

$\theta+2k\pi$



$\ldots \frac{\pi}{2}-2\pi, \frac{\pi}{2}, \frac{\pi}{2}+2\pi \ldots$

4 Modulus and argument of a complex number (p.55)

If the length of OP is r and $\angle xOP = \theta$, $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$ (figure 2), r is called the modulus of z and written $|z|$; θ is called the argument or amplitude of z and written $\arg z$ or $\operatorname{amp} z$. We will measure θ in radians unless otherwise stated. Since r is by definition positive, and $\cos\theta : \sin\theta : 1 = x : y : \sqrt{x^2 + y^2}$, for a given value of $z(= x + yi)$ there is a unique value of θ in the range $-\pi < \theta \le \pi$. This is known as the principal value of $\arg z$, other values being given by the formula $\theta + 2k\pi$ where k is any integer, not zero. In subsequent work, $\arg z$ will denote the principal value unless otherwise stated.

Cartesian form, Polar form and Euler form of Complex Numbers

Since $x = r\cos\theta$ and $y = r\sin\theta$, z may be written in:

- the cartesian form $x + yi$
- the polar form $r(\cos\theta + i\sin\theta)$ which is often abbreviated as $r\angle\theta$.
- the Euler form $re^{i\theta}$, where $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ is called the Euler relation.

$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$ $x \in \mathbb{R}^1$.

$\Downarrow$

$e^{i\theta} = \sum_{n=0}^{+\infty} \frac{(i\theta)^n}{n!}$

real part imaginary part

$\cos\theta + \sin\theta\, i$

The expression $\cos\theta + i\sin\theta$ is sometimes denoted by $\operatorname{cis}\theta$
and $\cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$ by $\operatorname{cis}(-\theta)$.

Alternatively, since $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 1$, we may denote $\cos\theta - i\sin\theta$ by $(\operatorname{cis}\theta)^{-1}$.

If two complex numbers z_1 and z_2 are equal, then

$$|z_1| = |z_2| \text{ and } \arg z_1 = \arg z_2.$$

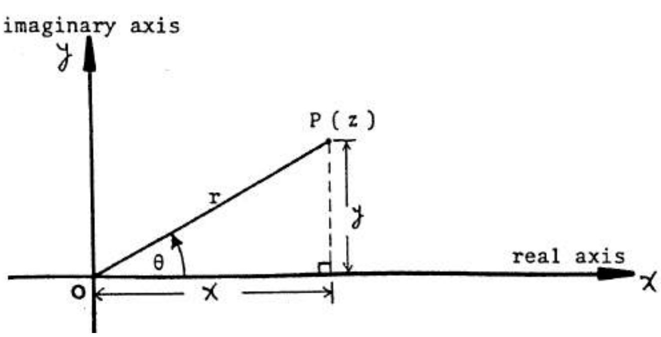


Figure 2

Example 1

Represent the following complex numbers in the Argand diagram and express them in polar form:
(i) $3-3i$ (ii) -4 (iii) $2i$ (iv) $-3+4i$.

Solution:

$3\sqrt{2}\left(\frac{3-3i}{3\sqrt{2}}\right)$

$\shortparallel$

$3\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right) = 3\sqrt{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)$

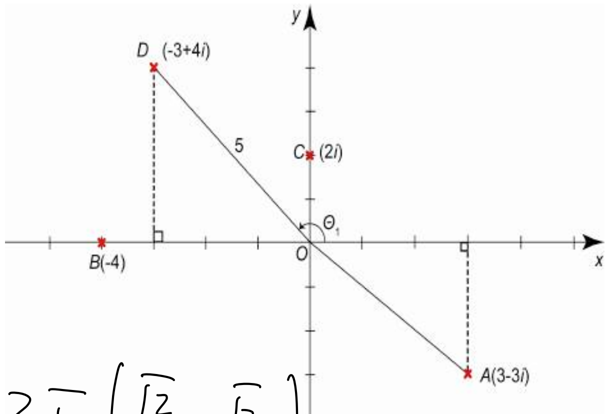


Figure 3

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(i) $3-3i = 3\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$,

(ii) $-4 = 4(\cos\pi + i\sin\pi)$,

(iii) $2i = 2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$,

(iv) Let $\theta_1 = \pi + \tan^{-1}\left(\frac{4}{-3}\right)$ ~~≈ 2.214 rad (or 126.87°)~~,

then $-3+4i = 5(\cos\theta_1 + i\sin\theta_1)$, where $\theta_1 = 2.214$ rad

$\tan\theta = \frac{4}{-3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{-3}\right)$

$\Downarrow$

$-\frac{\pi}{2} < \theta < 0$

W

Example 2

Express $\frac{2+3i}{3+4i}$ in cartesian form $a+bi$.

Solution:

$$\frac{2+3i}{3+4i} = \frac{(2+3i)(3-4i)}{(3+4i)(3-4i)} = \frac{6-8i+9i-12i^2}{9-16i^2} = \frac{18}{25}+\frac{1}{25}i, \text{ since } i^2 = -1.$$

Remark:

$$\mathrm{Re}\left(\frac{2+3i}{3+4i}\right) = \frac{18}{25} \text{ and } \mathrm{Im}\left(\frac{2+3i}{3+4i}\right) = \frac{1}{25}.$$

W

Example 3

Express $\dfrac{1+\sqrt{3}i}{1-\sqrt{3}i}$ in polar form.

Solution:

Method I

$2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=\underline{2e^{i\frac{\pi}{3}}}$

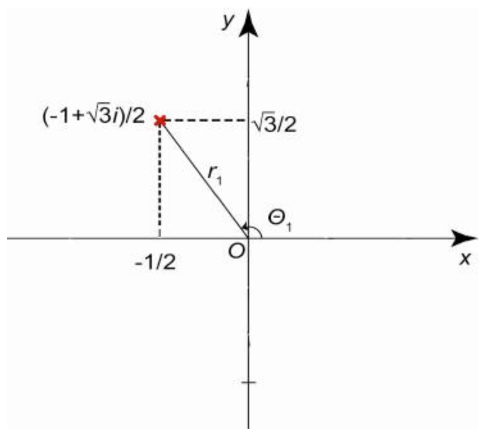
$$\frac{1+\sqrt{3}i}{1-\sqrt{3}i}=\frac{2e^{i\left(\frac{\pi}{3}\right)}}{2e^{i\left(-\frac{\pi}{3}\right)}}=e^{i\left(\frac{\pi}{3}\right)-i\left(-\frac{\pi}{3}\right)}=e^{i\left(\frac{2\pi}{3}\right)}=1\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right).$$

$2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=2\left(\cos\left(-\frac{\pi}{3}\right)+\sin\left(-\frac{\pi}{3}\right)i\right)=2e^{-i\frac{\pi}{3}}.$

Method II

$\cos\frac{2\pi}{3}$ $\quad\sin\frac{2\pi}{3}.$

The expression equals $\dfrac{\left(1+\sqrt{3}i\right)^2}{1-3i^2}=\dfrac{-2+2\sqrt{3}i}{4}=\boxed{-\dfrac{1}{2}}+\boxed{\dfrac{\sqrt{3}}{2}}i.$



$$r_1=\sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=1,\ \theta_1=\pi+\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)=\frac{2\pi}{3}.$$

Hence in polar form, the expression equals $1\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right).$

W

Example 4

Express $\sqrt{3+4i}$ the form $a+bi$.

where a and b are real numbers!!!

Solution:

If $\sqrt{3+4i}=\underline{\underline{a+bi}}$, then $3+4i=\left(a+bi\right)^2=a^2-b^2+2abi$.

Equating the real and imaginary parts, we have $\begin{cases}a^2-b^2=3\\ ab=2\end{cases}$.

$\neq 0$

$$\begin{cases} a^2-b^2=3 \\ ab=2 \end{cases} \Leftrightarrow \begin{cases} a^2-b^2=3 \\ b=\dfrac{2}{a} \end{cases} \Leftrightarrow \begin{cases} a^2-\dfrac{4}{a^2}=3 \\ b=\dfrac{2}{a} \end{cases} \Leftrightarrow \begin{cases} a^4-3a^2-4=0 \\ b=\dfrac{2}{a} \end{cases} \Leftrightarrow \begin{cases} \left(a^2-4\right)\boxed{\left(a^2+1\right)}=0 \\ b=\dfrac{2}{a} \end{cases}$$

$$\underset{\substack{a \text{ is real, therefore,} \\ a^2+1\neq 0}}{\Leftrightarrow} \begin{cases} a^2-4=0 \\ b=\dfrac{2}{a} \end{cases} \Leftrightarrow \begin{cases} a=2 \\ b=1 \end{cases} \text{ or } \begin{cases} a=-2 \\ b=-1 \end{cases}$$

The solution of these simultaneous algebraical equations is $a=\pm 2$, $b=\pm 1$ and both solutions lead to the result $\boxed{\pm(2+i) \text{ for } \sqrt{3+4i}\,.}$

W

5. Vectorial representation of a complex number

Since a complex number z is determined by its modulus r and its argument θ we may represent z by a vector of length r drawn in a direction which makes an angle θ with the positive direction of the real axis. For example, the complex number $x+yi$ which in figure 4 is represented by the point $P(x,y)$ may also be represented by the vector $\overrightarrow{OP}$ or by any equivalent vector $\overrightarrow{AB}$, that is by any line equal to, parallel to and drawn in the same sense as $\overrightarrow{OP}$.

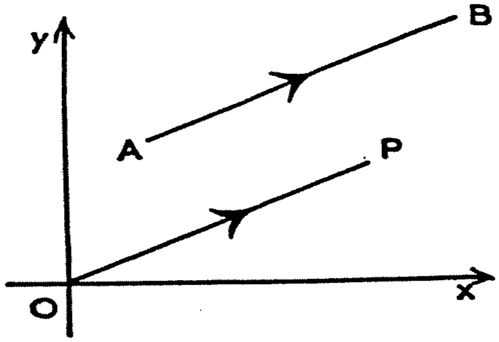


Figure 4

A real number x is represented by a vector of length $|x|$ drawn along or parallel to the real axis in the positive or negative direction according as z is positive or negative.

In the same way the purely imaginary number yi is represented by a vector of length $|y|$ drawn along or parallel to the imaginary axis in the positive or negative direction according as y is positive or negative.

We will find it convenient to use both the point and vector methods of representing a complex number.

The length of a vector $\overrightarrow{AB}$ will be denoted by AB.

6. Geometrical representation of addition or subtraction of two complex numbers

Let A and B represent the complex numbers $z_1(=x_1+y_1i)$ and $z_2(=x_2+y_2i)$ respectively in an Argand diagram (fig. 5). Complete parallelograms $OACB$ and $ODAB$.

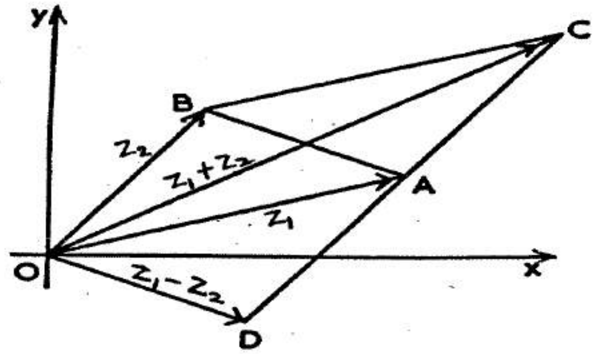
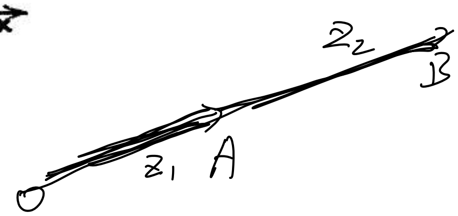


Figure 5



Then since the mid-point of AB has affix $\frac{1}{2}[(x_1+x_2)+(y_1+y_2)i]$, C has affix $(x_1+x_2)+(y_1+y_2)i$, that is, z_1+z_2. In addition, D has affix z_1-z_2.

In the vector representation z_1, z_2, z_1+z_2 are represented by $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ respectively. We thus see that this representation is in conformity with the usual parallelogram law of vector addition: $\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OC}$.

z_1-z_2 is represented by $\overrightarrow{OD}$ and also by BA. In triangle OAC, $OC < OA + AC$ $\therefore$ $|z_1+z_2|<|z_1|+|z_2|$.

This result is true for any two complex numbers except when O, A and C lie in order on a straight line. In this case $|z_1+z_2|=|z_1|+|z_2|$. Hence $|z_1+z_2|\le|z_1|+|z_2|$.

$r_1(\cos\theta_1+i\sin\theta_1)$ $\quad r_2(\cos\theta_2+i\sin\theta_2)$. we have "=" iff z_1 and z_2 share the same direction.

7. Multiplication and division of complex numbers (p.55 – p.57)

Let $z_1=\boxed{r_1\angle\theta_1}$ and $z_2=\boxed{r_2\angle\theta_2}$. Then

$$z_1z_2=r_1r_2(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)=r_1r_2[(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)]$$

$$=r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)] \quad (7.1)$$

$= r_1r_2\angle\theta_1+\theta_2$.

From this result we see that

$$|z_1z_2|=|z_1||z_2| \qquad \text{and} \qquad \arg z_1z_2=\arg z_1+\arg z_2.$$

(The latter result is not necessarily true of the principal values since the right-hand side may exceed π.)

$$\boxed{\frac{1}{\cos\theta_2+i\sin\theta_2}\overset{\checkmark}{=}\cos\theta_2-i\sin\theta_2}$$

$$\frac{\cos\theta_2 - i\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2} = \frac{\cos\theta_2 - i\sin\theta_2}{1}$$

Again,

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2) = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right] \quad (7.2)$$

Hence,

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \qquad \text{and} \qquad \arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 .$$

(The latter result is not necessarily true of the principal values.)

i.e. For $z_1 = r_1\angle\theta_1$ and $z_2 = r_2\angle\theta_2$, we have by (7.1) $\quad z_1 z_2 = r_1 r_2 \angle(\theta_1+\theta_2)$

and by (7.2) $\quad \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\angle(\theta_1-\theta_2)$.

Remark: By expressing the two complex numbers z_1 and z_2 as $r_1 e^{i\theta_1}$ and $r_2 e^{i\theta_2}$ (the Euler form), it is easy to obtain

$$\boxed{z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1+\theta_2)}} \quad \text{and} \quad \boxed{\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}}$$

<u>Illustration</u>

Question: It is given that $\omega_1 = 5e^{i\frac{\pi}{4}}$, $\omega_2 = e^{i\frac{\pi}{3}}$, $\omega_3 = 2e^{i\frac{\pi}{3}}$, $\omega_4 = e^{-i\frac{\pi}{18}}$ and $\omega_5 = 2e^{-i\frac{\pi}{9}}$, Simplify the following complex numbers and express each of them in Euler form. (Also, try locating z_1 and the result on the Argand diagram for (i) to (viii).)

(i) $\omega_1\omega_2$ (ii) $\omega_1\omega_3$ (iii) $\dfrac{\omega_1}{\omega_2}$ (iv) $\dfrac{\omega_1}{\omega_3}$ (v) $\omega_1\omega_4$ (vi) $\omega_1\omega_5$

(vii) $\dfrac{\omega_1}{\omega_4}$ (viii) $\dfrac{\omega_1}{\omega_5}$ (ix) $4e^{i\frac{2\pi}{3}} + 2e^{i\frac{\pi}{3}}$

<u>Solutions</u>

(i) $\omega_1\omega_2 = \left(5e^{i\frac{\pi}{4}}\right)\left(e^{i\frac{\pi}{3}}\right) = 5e^{i\left(\frac{\pi}{4}+\frac{\pi}{3}\right)} = 5e^{i\frac{7\pi}{12}}$

(ii)

$\omega_1\omega_3 = \left(5e^{i\frac{\pi}{4}}\right)\left(2e^{i\frac{\pi}{3}}\right) = 5(2)e^{i\left(\frac{\pi}{4}+\frac{\pi}{3}\right)} = 10e^{i\frac{7\pi}{12}}$

(iii) $\dfrac{\omega_1}{\omega_2} = \dfrac{5e^{i\frac{\pi}{4}}}{e^{i\frac{\pi}{3}}} = 5e^{i\left(\frac{\pi}{4}-\frac{\pi}{3}\right)} = 5e^{-i\frac{\pi}{12}}$

(iv) $\dfrac{\omega_1}{\omega_3} = \dfrac{5e^{i\frac{\pi}{4}}}{2e^{i\frac{\pi}{3}}} = \dfrac{5}{2}e^{i\left(\frac{\pi}{4}-\frac{\pi}{3}\right)} = 2.5e^{-i\frac{\pi}{12}}$

(v) $\omega_1\omega_4 = \left(5e^{i\frac{\pi}{4}}\right)\left(e^{-i\frac{\pi}{18}}\right) = 5e^{i\left(\frac{\pi}{4}+\left(-\frac{\pi}{18}\right)\right)} = 5e^{i\frac{7\pi}{36}}$

(vi) $\omega_1\omega_5 = \left(5e^{i\frac{\pi}{4}}\right)\left(2e^{-i\frac{\pi}{18}}\right) = 5(2)e^{i\left(\frac{\pi}{4}+\left(-\frac{\pi}{18}\right)\right)} = 10e^{i\frac{5\pi}{36}}$

(vii) $\dfrac{\omega_1}{\omega_4} = \dfrac{5e^{i\frac{\pi}{4}}}{e^{-i\frac{\pi}{18}}} = 5e^{i\left(\frac{\pi}{4}-\left(-\frac{\pi}{18}\right)\right)} = 5e^{i\frac{11\pi}{36}}$ (viii) $\dfrac{\omega_1}{\omega_5} = \dfrac{5e^{i\frac{\pi}{4}}}{2e^{-i\frac{\pi}{9}}} = \dfrac{5}{2}e^{i\left(\frac{\pi}{4}-\left(-\frac{\pi}{9}\right)\right)} = 2.5e^{i\frac{13\pi}{36}}$

(ix) $4e^{i\frac{2\pi}{3}} + 2e^{i\frac{\pi}{3}} = 4(\cos 120^\circ + i\sin 120^\circ) + 2(\cos 60^\circ + i\sin 60^\circ)$

$= (-2 + 2\sqrt{3}i) + 1 + \sqrt{3}i = -1 + 3\sqrt{3}i$

where modulus $= r = \sqrt{(-1)^2 + \left(3\sqrt{3}\right)^2} = \sqrt{28} = 2\sqrt{7}$

and argument $= 100.9^\circ = 1.76$ rad

$\therefore \quad 4e^{i\frac{2\pi}{3}} + 2e^{i\frac{\pi}{3}} = 2\sqrt{7}e^{i1.76}$

i
$=$ ← Euler formula.
$\boxed{e^{i\frac{\pi}{2}}}$
$z_1 (z_2)$

From (i)(iii) of the illustration, we can see that the effect of multiplying a complex number z_1 by a complex number with *unit modulus* and argument ϕ is to rotate the vector which represents z_1 counter-clockwise through an angle ϕ.

When $\phi = \dfrac{\pi}{2}$, $z_2 = i$ and so the vector which represents iz_1 is obtained by rotating counter-clockwise through $\dfrac{\pi}{2}$ the vector which represents z_1.

More generally, if z_1 is represented by the vector $\overrightarrow{OP}$ and $z_2 = r_2\angle\theta_2$, the product z_1z_2 is represented by a vector $\overrightarrow{OQ}$ with that $\angle POQ = \theta_2$ and length of $OQ = r_2 \cdot$length of OP.

Example 5

In the Argand diagram, PQR is an equilateral triangle of which the circumcentre is at the origin. If P represents the number $2 + i$, find the numbers represented by Q and R.

Solution:

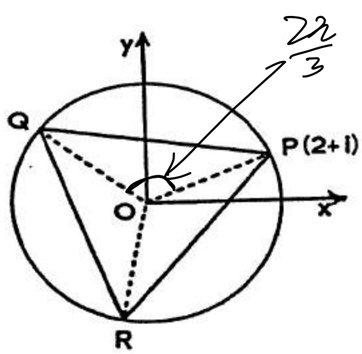


Figure 6

$P(2+i)$

$(2+i)\boxed{e^{i\frac{2\pi}{3}}}$

$=$

$(2+i)\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$=$

$\left(2\cos\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right) + \left(2\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3}\right)i.$

In Figure 6 $\angle POQ = \angle QOR = \angle ROP = \dfrac{2}{3}\pi$ and $OP = OQ = OR$, Realizing that OQ can be obtained by rotating OP in the anticlockwise direction by $\frac{2}{3}\pi$ rad, whereas OR can be obtained by rotating OP in the clockwise direction by $\frac{2}{3}\pi$ rad,

$\therefore \overrightarrow{OQ} = cis\left(\frac{2}{3}\pi\right)\overrightarrow{OP}$ and $\overrightarrow{OR} = cis\left(-\frac{2}{3}\pi\right)\overrightarrow{OP}$,

that is, $\overrightarrow{OQ} \underset{cis\left(\frac{2}{3}\pi\right)=\frac{-1+\sqrt{3}i}{2}}{=} \left(\frac{-1+\sqrt{3}i}{2}\right)(2+i) = -\left(1+\frac{1}{2}\sqrt{3}\right)+\left(\sqrt{3}-\frac{1}{2}\right)i$.

Similarly, $\overrightarrow{OR} \underset{cis\left(-\frac{2}{3}\pi\right)=\frac{-1-\sqrt{3}i}{2}}{=} \left(\frac{-1-\sqrt{3}i}{2}\right)(2+i) = -\left(1-\frac{1}{2}\sqrt{3}\right)-\left(\sqrt{3}+\frac{1}{2}\right)i$.

W

8. De Moivre's theorem (p.57)

When n is an nonnegative integer, De Moivre's theorem states that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

[Proof of De Moivre's theorem for integer n]
The following is a sketch of proof of De Moivre's theorem for any integer n by the method of Mathematical Induction.

Define $(\cos\theta + i\sin\theta)^0 = 1$, observe that $\cos(0\times\theta) + i\sin(0\times\theta) = \cos 0 + i\sin 0 = 1$. Thus, for $n = 0$, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ is valid.

Define $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$, observe that $\cos(1\times\theta) + i\sin(1\times\theta) = \cos\theta + i\sin\theta$. Thus, for $n = 1$, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ is valid.

Assume $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where $n \geq 1$.

$(\cos\theta + i\sin\theta)^{n+1} = (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta) = (\cos n\theta + i\sin n\theta)(\cos\theta + i\sin\theta)$

$= \cos n\theta\cos\theta - \sin n\theta\sin\theta + i(\sin n\theta\cos\theta + \cos n\theta\sin\theta) = \cos(n+1)\theta + i\sin(n+1)\theta$

Therefore, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \Rightarrow (\cos\theta + i\sin\theta)^{n+1} = \cos(n+1)\theta + i\sin(n+1)\theta$.

We have established by the method of induction that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ whenever n is a nonnegative integer.

When n is a negative integer, put $n = -m$ so that m is a positive integer. Then

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}.$$

But,

$$\frac{1}{\cos m\theta + i\sin m\theta} = \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)} = \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos(-m\theta) + i\sin(-m\theta) = \cos n\theta + i\sin n\theta$$

Thus, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ shows that De Moivre's theorem remains valid when n is a negative integer.

<u>Illustration</u>

Question: Simplify each of the following expressions.

(i) $\dfrac{(1-i)^8}{(-1+i)^5}$ (ii) $\dfrac{(2\text{cis}30^\circ)^7}{(4\text{cis}65^\circ)^3}$ (iii) $\dfrac{5e^{i\frac{\pi}{2}}\left(2\text{e}^{i\frac{\pi}{4}}\right)^6}{\left(2\text{e}^{i\frac{\pi}{5}}\right)^5}$

<u>Solutions</u>

(i) $$\frac{(1-i)^8}{(-1+i)^5} = \frac{\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^8}{\left(\sqrt{2}e^{i\frac{3\pi}{4}}\right)^5} = \left(\sqrt{2}\right)^{8-5} e^{i\left[8\left(-\frac{\pi}{4}\right)-5\left(\frac{3\pi}{4}\right)\right]} = 2\sqrt{2}e^{-i\frac{23\pi}{4}} = 2\sqrt{2}e^{i\frac{\pi}{4}}$$

$$\left(= 2\sqrt{2}(\cos 45^\circ + i\sin 45^\circ) = 2\sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 2 + 2i\right)$$

(ii) $$\frac{(2\text{cis}30^\circ)^7}{(4\text{cis}65^\circ)^3} = \frac{2^7}{4^3}\text{cis}\left[7(30^\circ) - 3(65^\circ)\right] = 2\text{cis}(15^\circ)$$

(iii) $$\frac{5e^{i\frac{\pi}{2}}\left(2\text{e}^{i\frac{\pi}{4}}\right)^6}{\left(2\text{e}^{i\frac{\pi}{5}}\right)^5} = \frac{5\cdot 2^6}{2^5}e^{i\left[\frac{\pi}{2}+6\left(\frac{\pi}{4}\right)-5\left(\frac{\pi}{5}\right)\right]} = 10e^{i\pi} = -10$$

Example 6

If n is an integer and $z = \cos\theta + i\sin\theta$, show that $\begin{cases} 2\cos n\theta = z^n + \dfrac{1}{z^n} \\ 2i\sin n\theta = z^n - \dfrac{1}{z^n} \end{cases}$ …(*).

Use these results to establish the formula $8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$.

Solution:

Using De Moivre's theorem we have $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Also, $z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$ and the results follow by addition and subtraction. Taking $n = 1$ in (*),

$$(2\cos\theta)^4 = \left(z + \frac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4} = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6 = 2\cos 4\theta + 8\cos 2\theta + 6,$$

where we have used (*) with $n = 4$ and $n = 2$ in the last step. The required result then follows on division by 2.

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9. Roots and fractional powers of a complex number (p.58)

Let $r(\cos\theta + i\sin\theta)$ be a complex number in polar form. For any positive integer n,

$$\boxed{[r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right),\quad k = 0, 1, 2, \ldots, n-1}$$

Illustration:

$$[8(\cos 30^\circ + i\sin 30^\circ)]^{\frac{1}{3}} = 2\left(\cos\frac{30^\circ + 360^\circ k}{3} + i\sin\frac{30^\circ + 360^\circ k}{3}\right) \qquad k = 0, 1, 2$$

i.e. the three complex numbers resulted are:

$z_0 = 2(\cos 10^\circ + i\sin 10^\circ)$ (when $k = 0$)

$z_1 = 2(\cos 130^\circ + i\sin 130^\circ)$ (when $k = 1$)

$z_2 = 2(\cos 250^\circ + i\sin 250^\circ) = 2(\cos(-110^\circ) + i\sin(-110^\circ))$(when $k = 2$)

Check: $z_0^{\,3} = 2^3(\cos(3\cdot 10^\circ) + i\sin(3\cdot 10^\circ)) = 8(\cos 30^\circ + i\sin 30^\circ)$

$z_1^{\,3} = 2^3(\cos(3\cdot 130^\circ) + i\sin(3\cdot 130^\circ)) = 8(\cos 390^\circ + i\sin 390^\circ) = 8(\cos 30^\circ + i\sin 30^\circ)$

$z_2^{\,3} = 2^3(\cos(3\cdot 250^\circ) + i\sin(3\cdot 250^\circ)) = 8(\cos 750^\circ + i\sin 750^\circ) = 8(\cos 30^\circ + i\sin 30^\circ)$

This it very useful for obtaining the complex roots of a complex number. That is, this formula can be used to solve the equation

$$\boxed{z^n = r(\cos\theta + i\sin\theta).}$$

It is expected that n roots (real and complex in total) will result and the value of each of the roots, z_k, can be obtained by:

$$\boxed{z_k = \left[r(\cos\theta + i\sin\theta)\right]^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right),\ k = 0, 1, 2, \ldots, n-1.}$$

Example 7

Let $\cos\theta + i\sin\theta$ be a complex number in polar form. Consider the equation $w^n = \cos\theta + i\sin\theta$. Find all complex numbers w such that $w^n = \cos\theta + i\sin\theta$.
(w is called a root of the equation $w^n = \cos\theta + i\sin\theta$.)

Solution:

Suppose $w = \rho(\cos\phi + i\sin\phi)$, then $w^n = \rho^n(\cos\phi + i\sin\phi)^n = \rho^n(\cos n\phi + i\sin n\phi) = \cos\theta + i\sin\theta$

$\Rightarrow \rho^n = 1,\ n\phi = \theta + 2k\pi$, where k is an integer.

$\rho > 0 \Rightarrow \rho = 1$ and $\phi = \dfrac{\theta + 2k\pi}{n} = \dfrac{\theta}{n} + k\dfrac{2\pi}{n}$.

Therefore, $w = \cos\left(\dfrac{\theta}{n} + k\dfrac{2\pi}{n}\right) + i\sin\left(\dfrac{\theta}{n} + k\dfrac{2\pi}{n}\right)$, where k is an integer.

Since $\sin x$, $\cos x$ are periodic functions with period 2π, $\cos\left(\dfrac{\theta}{n} + k\dfrac{2\pi}{n}\right) + i\sin\left(\dfrac{\theta}{n} + k\dfrac{2\pi}{n}\right)$, $k = \ldots, -1, 0, 1, \ldots$ are not all distinct. Taking, in succession, the values $k = 0, 1, 2, \ldots, n-1$ we find that $\cos\dfrac{\theta + 2k\pi}{n} + i\sin\dfrac{\theta + 2k\pi}{n}$ has exactly n distinct values.

Hence there are n distinct n^{th} roots of $\cos\theta + i\sin\theta$ given by the formula :

$$w_k = \cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n} \quad , k = 0, 1, \ldots, n-1.$$

W

Consider complex number $\cos\theta + i\sin\theta$, let $n = \dfrac{1}{q}$, where q is a positive integer, define $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$ as the collection of all roots w of the equation $w^q = \cos\theta + i\sin\theta$, that is, the collection of all q^{th} roots of $\cos\theta + i\sin\theta$.

Then, $(\cos\theta + i\sin\theta)^{\frac{1}{q}} = w = \cos\left(\dfrac{\theta}{q} + k\dfrac{2\pi}{q}\right) + i\sin\left(\dfrac{\theta}{q} + k\dfrac{2\pi}{q}\right)$, $k = 0, 1, \ldots, q-1$.

Next suppose that n is a fraction; put $n = \dfrac{p}{q}$ where p and q are integers, $q > 0$ and $\gcd(|p|, q) = 1$.

Define $(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \left[(\cos\theta + i\sin\theta)^p\right]^{\frac{1}{q}}$. Observe that $(\cos\theta + i\sin\theta)^p = \cos p\theta + i\sin p\theta$.

$(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ denotes the collection all q-th roots of $(\cos\theta + i\sin\theta)^p = \cos p\theta + i\sin p\theta$.

Therefore, $\left(\cos\theta + i\sin\theta\right)^{\frac{p}{q}} = \cos\left(\frac{p\theta}{q} + \frac{2k\pi}{q}\right) + i\sin\left(\frac{p\theta}{q} + \frac{2k\pi}{q}\right),\ k = 0, 1, \ldots, q-1$.

$$\left(\cos\theta + i\sin\theta\right)^{\frac{p}{q}} = \cos\left(\frac{p\theta}{q} + \frac{2k\pi}{q}\right) + i\sin\left(\frac{p\theta}{q} + \frac{2k\pi}{q}\right),\ k = 0, 1, \ldots, q-1$$

is <u>De Moivre's theorem for fractional indices</u>.

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Remark:
When n is a positive integer, the n^{th} roots of a complex number are by definition the values of w which satisfy the equation $w^n = z$. If $w = \rho\left(\cos\phi + i\sin\phi\right)$ and $z = r\left(\cos\theta + i\sin\theta\right)$, $\rho^n\left(\cos n\phi + i\sin n\phi\right) = r\left(\cos\theta + i\sin\theta\right)$, whence $\rho^n = r, n\phi = \theta + 2k\pi$, where k is an integer or zero. Now by definition ρ and r are positive so that $\rho = \sqrt[n]{r}$, the unique positive n^{th} root of r, also, $\phi = \dfrac{\theta + 2k\pi}{n}$, where k is an integer or zero; and taking, in succession, the values $k = 0, 1, 2, \ldots, n-1$ we find that $\cos\dfrac{\theta + 2k\pi}{n} + i\sin\dfrac{\theta + 2k\pi}{n}$ has n, and only n distinct values. Hence there are n distinct n^{th} roots of z given by the formula $w_k = \sqrt[n]{r}\left(\cos\dfrac{\theta + 2k\pi}{n} + i\sin\dfrac{\theta + 2k\pi}{n}\right),\ k = 0, 1, \ldots, (n-1)$.

In the case where n is a rational number, $n = \dfrac{p}{q}$, say, where p and q are integers, $q > 0$ and $\gcd\left(|p|, q\right) = 1$, the values of z^n are the values of w which satisfy the equation $w^q = z^p$. Hence if $z = r\left(\cos\theta + i\sin\theta\right)$ there are q values of $z^{\frac{p}{q}}$ given by the formula $w_m = \sqrt[q]{r^p}\left(\cos\dfrac{p\theta + 2m\pi}{q} + i\sin\dfrac{p\theta + 2m\pi}{q}\right),\ m = 0, 1, \ldots, q-1$, where $\sqrt[q]{r^p}$ is the unique positive q^{th} root of r^p.

<u>Example 8</u>
Solve the equation $z^3 + 8 = 0$.
<u>Solution:</u>
(Since it is a polynomial equation of degree 3, we expect to have 3 roots (real and complex in total).)
Method 1:
$z^3 + 8 = 0$
$z^3 = -8 = 8\left(\cos\pi + i\sin\pi\right)$
$z_k = 8^{\frac{1}{3}}\left(\cos\dfrac{\pi + 2k\pi}{3} + i\sin\dfrac{\pi + 2k\pi}{3}\right),\ k = 0, 1, 2$

$\therefore$ The three roots are: $z_0 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + \sqrt{3}i$

$$z_1 = 2\left(\cos\frac{\pi + 2\pi}{3} + i\sin\frac{\pi + 2\pi}{3}\right) = 2(\cos\pi + i\sin\pi) = -2$$

$$z_2 = 2\left(\cos\frac{\pi + 4\pi}{3} + i\sin\frac{\pi + 4\pi}{3}\right) = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = 1 - \sqrt{3}i$$

Method 2:

$z^3 + 8 = 0$

$(z + 2)(z^2 - 2z + 4) = 0$

$\therefore$ $z + 2 = 0$ or $z^2 - 2z + 4 = 0$

$z = -2$ or $z = \dfrac{2 \pm \sqrt{-12}}{2} = 1 \pm \sqrt{3}i$ W

Example 9

Find the roots of the equation $z^5 + 1 = 0$ in the form $\cos\theta + i\sin\theta$ and hence show that $z^4 - z^3 + z^2 - z + 1 = \left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$.

Solution:

As $z^5 = -1 = \text{cis}\pi = \text{cis}(\pi + 2k\pi)$, where k is zero or any integer, the roots are

$z_k = \text{cis}\left(\frac{\pi + 2k\pi}{5}\right)$, $k = 0,1,2,3,4$.

That is, $z_0 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$,

$z_1 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$, $\quad z_2 = \cos\pi + i\sin\pi(= -1)$,

$$z_3 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5} = \cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right) = \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5} = \overline{z_1},$$

$$z_4 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5} = \cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right) = \cos\frac{\pi}{5} - i\sin\frac{\pi}{5} = \overline{z_0},$$

We find $(z - z_0)(z - z_4) = z^2 - (z_0 + z_4)z + z_0z_4 = z^2 - (z_0 + \overline{z_0})z + z_0\overline{z_0} = z^2 - 2z\cos\frac{\pi}{5} + 1$, similarly, $(z - z_1)(z - z_3) = z^2 - 2z\cos\frac{3\pi}{5} + 1$. Then

$$(z + 1)(z^4 - z^3 + z^2 - z + 1) = z^5 + 1 = (z - z_0)(z - z_4)(z - z_1)(z - z_3)(z - z_2)$$

$$= (z - z_2)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right) \underset{\substack{z_2 = -1 \\ z+1 \neq 0}}{\Rightarrow} z^4 - z^3 + z^2 - z + 1$$

$$= \left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$$

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Example 10

If $\omega\ (\neq 1)$ is a complex cube root of unity, find the value of $(1+\omega)(2+\omega)(1+2\omega)$.

Solution:

First, observe that $\omega^3 = 1 \Leftrightarrow \omega^3 - 1 = 0 \Leftrightarrow (\omega-1)(\omega^2+\omega+1) = 0 \underset{\omega\neq 1}{\Rightarrow} \omega^2+\omega+1 = 0$.

Hence

$$(1+\omega)(2+\omega)(1+2\omega) = (2+3\omega+\omega^2)(1+2\omega)$$

$$= \left[(\omega^2+\omega+1)+2\omega+1\right](1+2\omega) = (1+2\omega)^2 = 1+4\omega+4\omega^2$$

$$= 4(\omega^2+\omega+1)-3 = -3.$$

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10. Euler relation (p.55-p.58)

When $x = i\theta$, where θ is real, we have from $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ that

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \cdots \underset{\substack{i^2=-1\\ i^3=-i\\ i^4=1}}{=} 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right).$$

Let $\mathrm{Re}(e^{i\theta})$ and $\mathrm{Im}(e^{i\theta})$ denote the real and the imaginary parts of $e^{i\theta}$ respectively. From the above series expansion of $e^{i\theta}$, we obtain

$$\mathrm{Re}(e^{i\theta}) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \quad \text{and} \quad \mathrm{Im}(e^{i\theta}) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots .$$

Recall $1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$ and $\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$ are just the Maclaurin series of $\cos\theta$ and $\sin\theta$ respectively, i.e., $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$ and $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$. Thus,

$$e^{i\theta} = \mathrm{Re}(e^{i\theta}) + i\mathrm{Im}(e^{i\theta}) = \cos\theta + i\sin\theta.$$

We call $e^{i\theta} = \cos\theta + i\sin\theta$ *Euler's relation*.

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