BMS1901 Calculus for Life Sciences

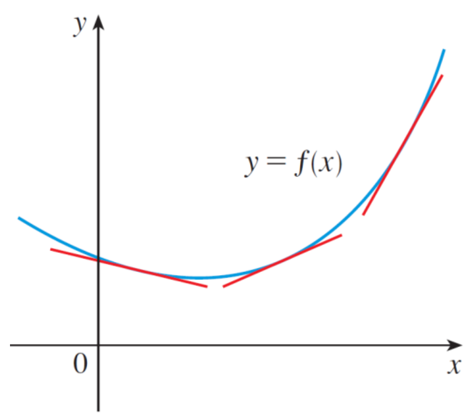
Week 6

Understand L'hospital's rule
Application of derivatives in population growth
Perform optimization

Concavity

Concavity

- how sign of $f''(x)$ affects the appearance of the graph of f?
- $f'' = (f')'$
- $f''(x)$ is positive → f' is an increasing function
- slopes of the tangent lines of the curve $y = f(x)$ increase from left to right

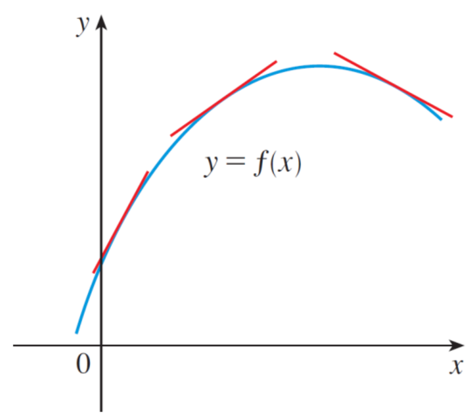


Since $f''(x) > 0$, the slopes increase and f is concave upward

Concavity

- slope of this curve becomes progressively larger as *x* increases
- curve bends upward
- **concave upward**

$f''(x)$ is negative → f' is decreasing:



Since $f''(x) < 0$, the slopes decrease and f is concave downward

Concavity

- slopes of *f* decrease from left to right
- curve bends downward
- **concave downward**

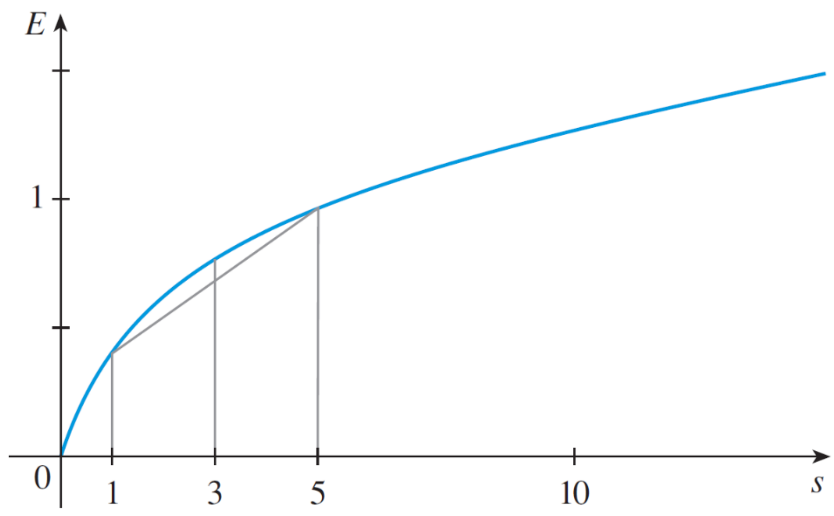
Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Example – *Risk aversion in junco foraging*

Different junco habitats yield different amounts of seeds, and individuals can choose which habitat to feed in. The amount of energy reward *E* obtained from feeding in different habitats increases with the seed abundance *s* in the habitat but it does so at a decelerating rate.



Example – *Risk aversion in junco foraging*

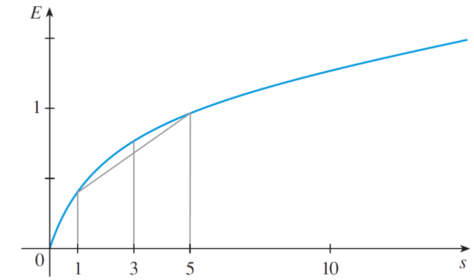
Suppose a bird can choose to feed exclusively in a habitat with $s = 3$ seeds per unit area or it can divide its time equally between two habitats with 1 and 5 seeds per unit area, respectively.

For both choices the bird experiences an average of 3 seeds per unit area. Which choice provides the greatest energy reward?

Solution:

The function $E(s)$ graphed in the figure gives the energy reward as a function of seed density.

Example – *Solution*



- graph of E : concave downward (lies below its tangent lines and above its secant lines)

- secant line: from (1, E(1)) to (5, E(5)) (lie *below* the curve)
- height of the secant line when s = 3 is the average of the heights when s = 1 and s = 5:

$$\frac{E(1) + E(5)}{2} < E(3)$$

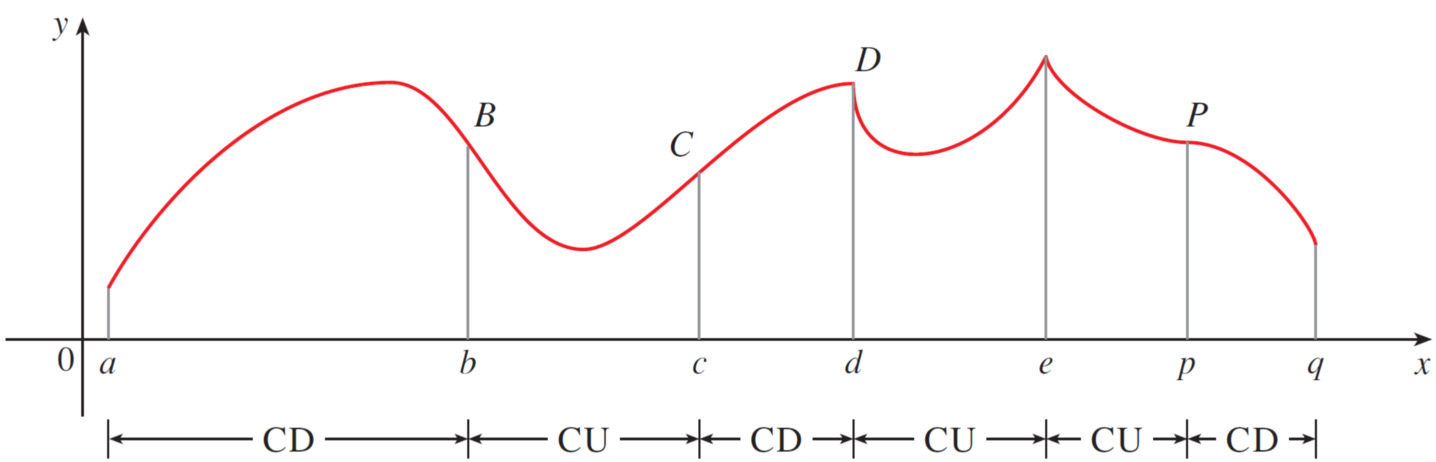
- The junco gets more energy reward in a habitat with 3 seeds per unit area than it does by splitting its time between habitats with 1 and 5 seeds per unit area

Concavity

Graph of a function that is:

•concave upward (CU) on the intervals: (b, c), (d, e), and (e, p)

•concave downward (CD) on the intervals: (a, b), (c, d), and (p, q)



Concavity

Previous figure:

•curve changes its direction of concavity when $x = b$, c, d, and p

•Points on the curve (B, C, D, and P): *inflection points*

inflection point:

•**i**f f is continuous there

•curve changes from concave upward to concave downward at P, or

•from concave downward to concave upward at P

Concavity

- Consequence of the Concavity test:

The Second Derivative Test Suppose f'' is continuous near c.

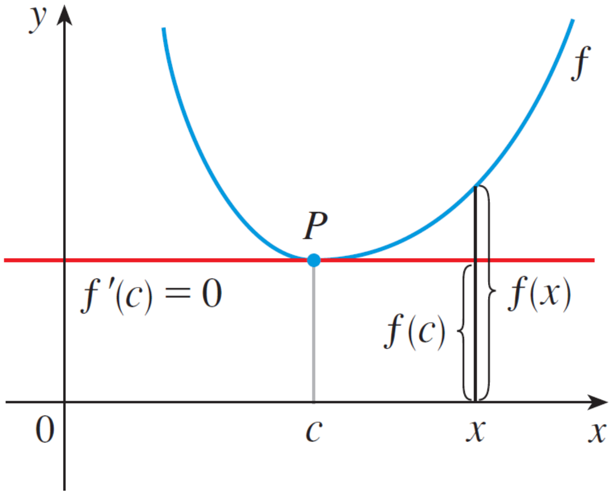
(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

E.g.: $f''(x) > 0$ near c → part (a) is true
→ f is concave upward near c.

Concavity

- graph of *f* lies *above* its horizontal tangent at *c*
- *f* has a local minimum at *c*



$f''(x) > 0$, *f* is concave upward

Example for concavity

Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Solution:

If $f(x) = x^4 - 4x^3$, then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

•critical numbers:

> **(5) Definition** A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

we set $f'(x) = 0 \rightarrow x = 0$ and $x = 3$

Example – *Solution*

> **The Second Derivative Test** Suppose f'' is continuous near c.
> (a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
> (b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

<u>Second Derivative Test</u>

• evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad\qquad f''(3) = 36 > 0$$

• $f'(3) = 0$ and $f''(3) > 0$ → $f(3) = -27$ is a local minimum

• $f''(0) = 0$ and the Second Derivative Test → no information about the critical number 0

• $f'(x) < 0$ for $x < 0$ and $0 < x < 3$ → f does not have a local maximum or minimum at 0 (First Derivative Test)

> **The First Derivative Test** Suppose that c is a critical number of a continuous function f.
> (a) If f' changes from positive to negative at c, then f has a local maximum at c.
> (b) If f' changes from negative to positive at c, then f has a local minimum at c.
> (c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

Example – *Solution*

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

- $f''(x) = 0$ when $x = 0$ or 2

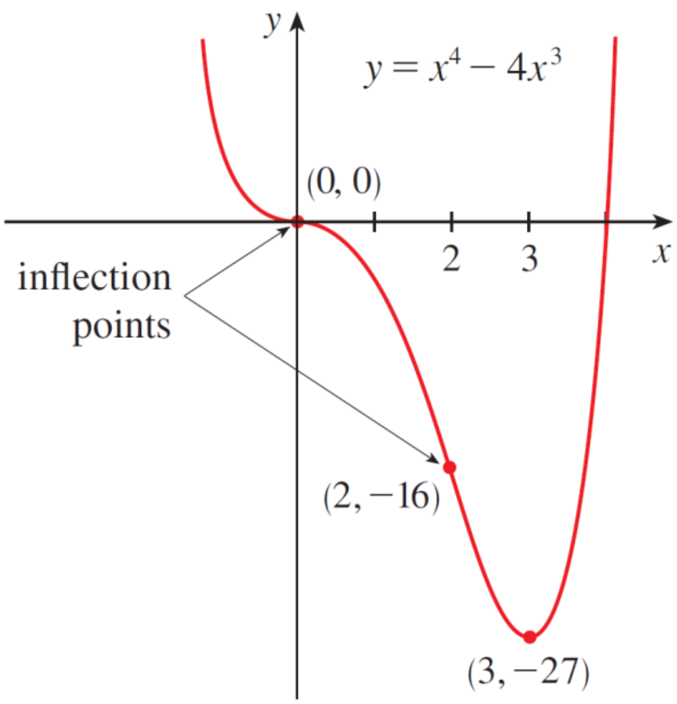
→ divide the real line into intervals with these numbers as endpoints

Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	$+$	upward
$(0, 2)$	$-$	downward
$(2, \infty)$	$+$	upward

- point (0, 0): inflection point (curve changes from concave upward to concave downward)

Example – *Solution*

- (2, –16): inflection point (curve changes from concave downward to concave upward)



Concavity

Second Derivative Test is inconclusive when $f''(c) = 0$

•at such a point there might be a maximum, a minimum, or neither

•Test fails when $f''(c)$ does not exist

- →First Derivative Test
- even when both tests apply, the **First Derivative Test is easier to use**

L'Hospital's Rule:
Comparing Rates of Growth

Indeterminate Quotients

$$F(x) = \frac{\ln(x)}{(x-1)}$$

F is not defined when $x = 1$

Q: How F behaves near 1?

$$\lim_{x \to 1} \frac{\ln x}{x-1}$$

✗ apply law of limits (∵ limit of quotient = quotient of limits)

∵ ~~limit~~ limit of denominator $= 0$

$$\Rightarrow \frac{0}{0}$$

Indeterminate Quotients

$$\lim_{x\to 1} \frac{x^2-x}{x^2-1} = \lim_{x\to 1} \frac{x(x-1)}{(x+1)(x-1)} = \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

- both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$
- Limit may or may not exist
- **indeterminate form of type** $\frac{0}{0}$

- *l'Hospital's Rule*: evaluation of indeterminate forms

Indeterminate Quotients

$$\lim_{x\to\infty}\frac{x^2-1}{2x^2+1} = \lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{2+\frac{1}{x^2}} = \frac{1-0}{2+0} = \frac{1}{2}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

- both $f(x) \to \infty$ (or $-\infty$)
- $g(x) \to \infty$ (or $-\infty$)

  →limit may or may not exist
- **indeterminate form of type** ∞/∞

Indeterminate Quotients

- L'Hospital's Rule →this type of indeterminate form

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Indeterminate Quotients

- L'Hospital's Rule: limit of a quotient of functions = limit of the quotient of their derivatives (given conditions are satisfied)
  *verify the conditions, i.e. limits of *f* and g

- valid for one-sided limits and limits at infinity or negative infinity
  - "$x \to a$" : $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$

Indeterminate Quotients

- special case : $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$
- easy to see why l'Hospital's Rule is true
- using the alternative form of the definition of a derivative:

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\displaystyle\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} = \lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

Indeterminate Quotients

- Suggest visually why l'Hospital's Rule might be true
- two differentiable functions *f* and *g* (approaches 0 as $x \to a$)
- Zoom in toward the point (*a*, 0) → linear

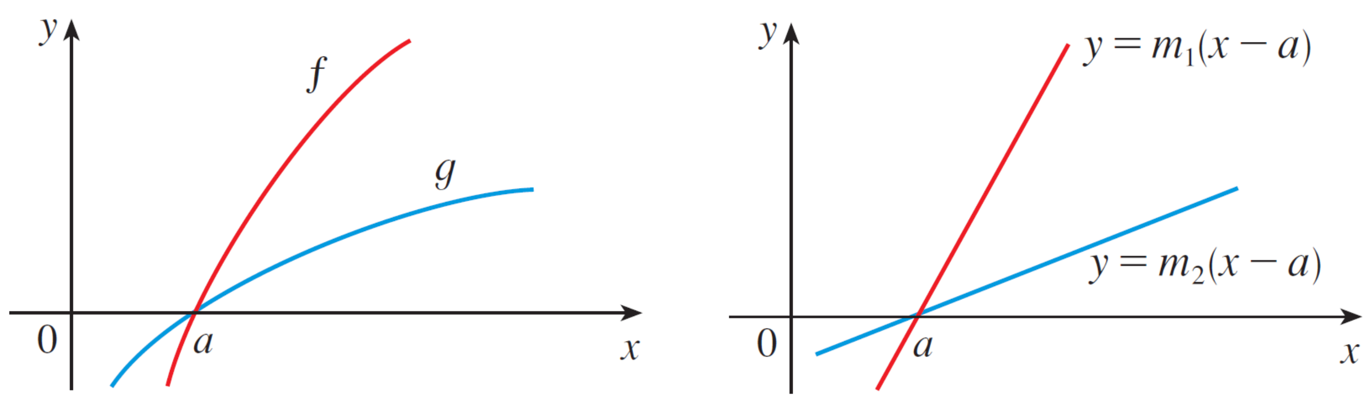


Figure 1

Indeterminate Quotients

- Linear functions: their ratio would be

$$\frac{m_1(x-a)}{m_2(x-a)} = \frac{m_1}{m_2}$$

- ratio of their derivatives:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example 1

Find $\lim_{x \to 1} \frac{\ln x}{x - 1}$.

Solution:

$$\lim_{x \to 1} \ln x = \ln 1 = 0 \qquad \text{and} \qquad \lim_{x \to 1} (x - 1) = 0$$

•l'Hospital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\dfrac{d}{dx}(\ln x)}{\dfrac{d}{dx}(x - 1)}$$

$$= \lim_{x \to 1} \frac{1/x}{1}$$

$$= \lim_{x \to 1} \frac{1}{x} = 1$$

Example 1 – *Solution*

•l’Hospital’s Rule: differentiate the numerator and denominator *separately*

- do ***not*** use the Quotient Rule

Which Functions Grow Fastest?

Which Functions Grow Fastest?

- L'Hospital's Rule: compare the rates of growth of functions

f(*x*) and *g*(*x*): become large as *x* becomes large

$$\lim_{x \to \infty} f(x) = \infty \qquad \text{and} \qquad \lim_{x \to \infty} g(x) = \infty$$

- *f*(*x*) approaches infinity **more quickly** than *g*(*x*) if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$$

- *f*(*x*) approaches infinity **more slowly** than *g*(*x*) if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

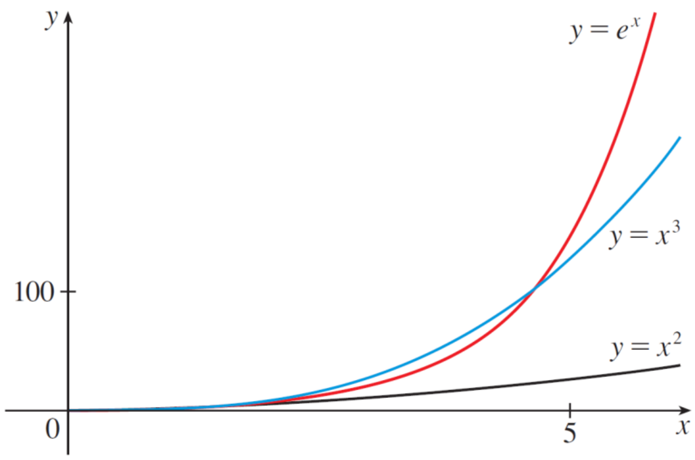
Which Functions Grow Fastest?

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \infty$$

- exponential function $y = e^x$ grows more quickly than $y = x^2$
- $y = e^x$ grows more quickly than power functions $y = x^n$

- $y = x^3$ exceeds $y = e^x$ initially

- after x = 4.5: exponential function overtakes the other functions



Example 6

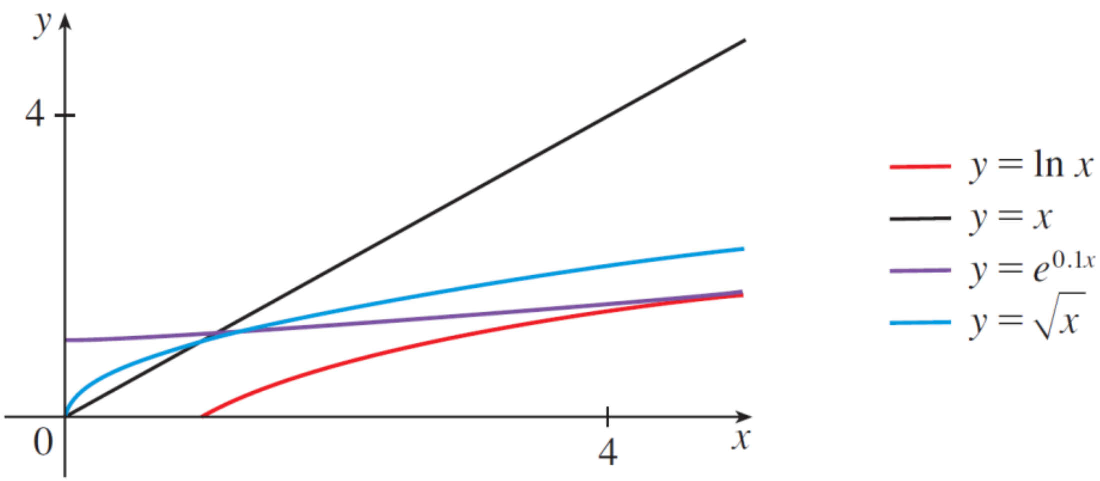
Rank the following functions in order of how quickly they approach infinity as $x \to \infty$:

$$y = \ln x \qquad y = x \qquad y = e^{0.1x} \qquad y = \sqrt{x}$$

Solution:

Ranking by plotting the four functions:

•misleading picture: it looks as if $y = x$ is the winner.



Example 6 – *Solution*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

- l'Hospital's Rule: not true

$$\lim_{x \to \infty} \frac{x}{e^{0.1x}} = \lim_{x \to \infty} \frac{1}{0.1e^{0.1x}} = 0$$

→ $y = x$ grows more slowly than $y = e^{0.1x}$

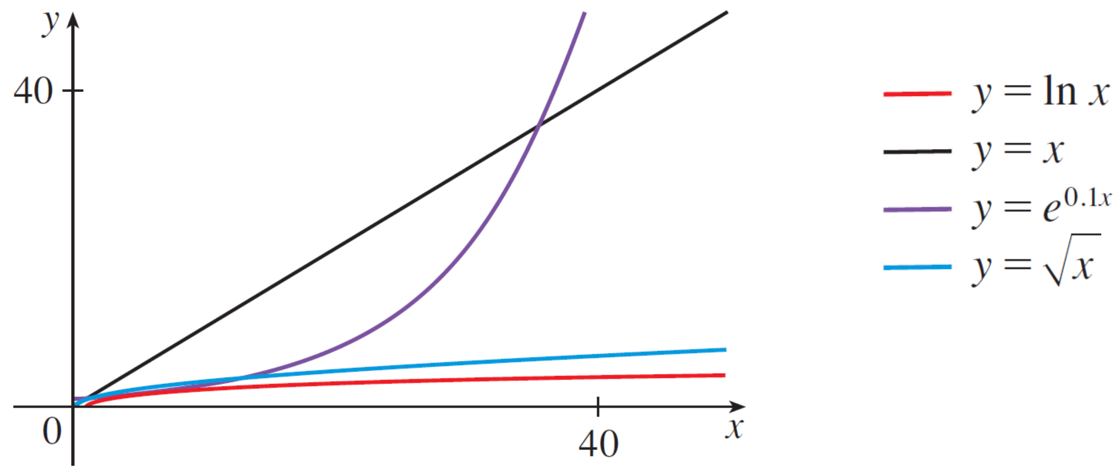
- $y = \ln x$ grows more slowly than $y = \sqrt{x}$
- $y = \sqrt{x}$ grows more slowly than $y = x$:

$$\frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} \to 0 \qquad \text{as } x \to \infty$$

Example 6 – *Solution*

- Ranking (fastest to slowest):

$$y = e^{0.1x} \qquad y = x \qquad y = \sqrt{x} \qquad y = \ln x$$



Indeterminate Products

Indeterminate Products

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$),

- $\lim_{x \to a} f(x)\, g(x)$ =?
- struggle between *f* and *g*
  - *f* wins → limit = 0
  - *g* wins→ limit = ∞ (or $-\infty$)
  - finite nonzero number
- **indeterminate form of type 0 · ∞**

Indeterminate Products

- product *fg* → quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

→indeterminate form of type $\frac{0}{0}$ or ∞/∞
→l'Hospital's Rule

Example 8

Evaluate $\lim_{x \to 0^+} x \ln x$

Use the knowledge of this limit, together with information from derivatives, to sketch the curve *y* = *x* ln *x*.

Solution:

•given limit is indeterminate:

- Because as $x \to 0^+$, the first factor (*x*) approaches 0
- second factor (ln *x*) approaches – ∞

x = 1/(1/*x*): $1/x \to \infty$

•l'Hospital's Rule gives

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

Example 8 – *Solution*

- If $f(x) = x \ln x$: $f'(x) = x \cdot \dfrac{1}{x} + \ln x = 1 + \ln x$

- $f'(x) = 0$ when $\ln x = -1$
  - $x = e^{-1}$
- $f'(x) > 0$ when $x > e^{-1}$
- $f'(x) < 0$ when $x < e^{-1}$
  - f is increasing on $(1/e, \infty)$ and decreasing on $(0, 1/e)$

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

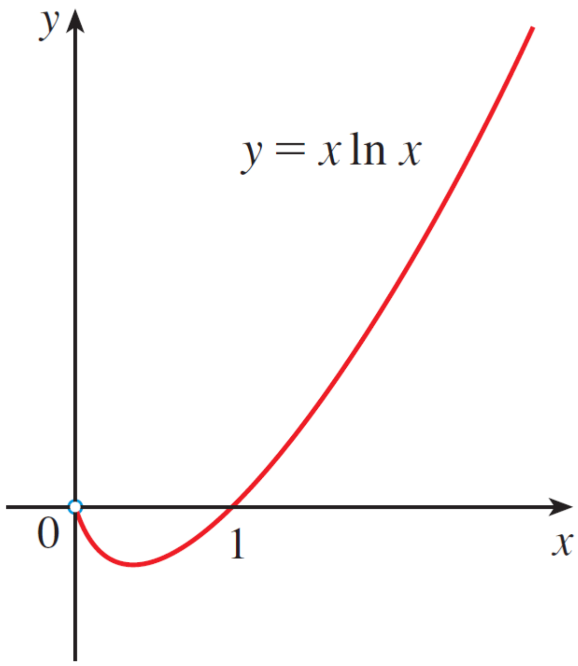
(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

<u>First Derivative Test</u>

- $f(1/e) = -1/e$ is a local (and absolute) minimum
- $f''(x) = 1/x > 0$
  - f is concave upward on $(0, \infty)$

Example 8 – *Solution*

$$\lim_{x \to 0^+} f(x) = 0$$



Indeterminate Differences

Indeterminate Differences

$\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$:

$$\lim_{x \to a} [f(x) - g(x)]$$

- **indeterminate form of type** $\infty - \infty$

- Find the limit: difference → quotient
    - common denominator / rationalization / factoring out a common factor
      → have an indeterminate form of type $\frac{0}{0}$ or ∞/∞

Example 10

Compute $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$

$\tan x = \dfrac{\sin x}{\cos x}$

$\sec x = \dfrac{1}{\cos x}$

Solution:

•sec $x \to \infty$ and tan $x \to \infty$; limit is indeterminate

<u>common denominator:</u>

$$\lim_{x \to (\pi/2)^-} (\sec x - \tan x) = \lim_{x \to (\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

• l'Hospital's Rule: $1 - \sin x \to 0$

$\cos x \to 0$ as $x \to (\pi/2)^-$

$$= \lim_{x \to (\pi/2)^-} \frac{1 - \sin x}{\cos x} = \lim_{x \to (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

Optimization Problems

Optimization Problems

- Challenge: convert the word problem into a mathematical optimization problem
  - by setting up the function that is to be maximized or minimized

Steps in Solving Optimization Problems

1. Understand the Problem

- What is the unknown?
- What are the given quantities?
- What are the given conditions?

Optimization Problems

2. Draw a Diagram
- identify the given and required quantities

3. Introduce Notation
- select symbols (*a*, *b*, *c*, . . . , *x*, *y*) for unknown quantities
- label the diagram with these symbols
- use initials as suggestive symbols
  - E.g. A for area, *h* for height, *t* for time

4. Express Q in terms of some of the other symbols

Optimization Problems

5. Use the given information to find relationships (in the form of equations) among these variables
    - use these equations to eliminate all but one of the variables in the expression for Q
        - Q will be expressed as a function of *one* variable x, e.g. $Q = f(x)$
    - Write the domain of this function

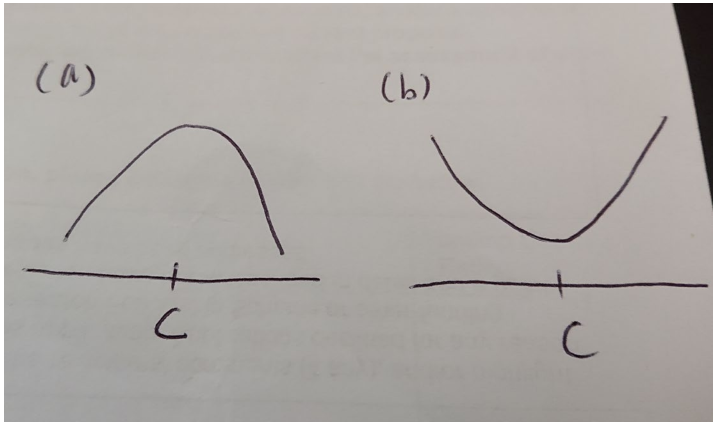
6. Find the *absolute* maximum or minimum value of f
    - Closed Interval Method

Optimization Problems

First Derivative Test for Absolute Extreme Values Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

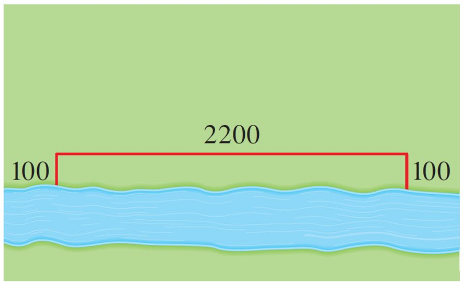


Example 1

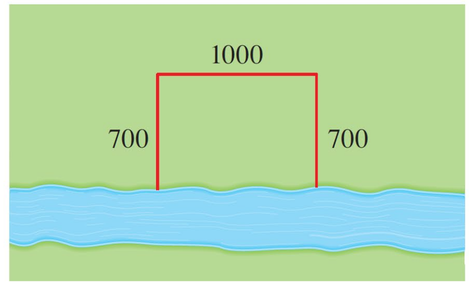
A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a river with a straight bank. He needs no fence along the river.

What are the dimensions of the field that has the largest area?

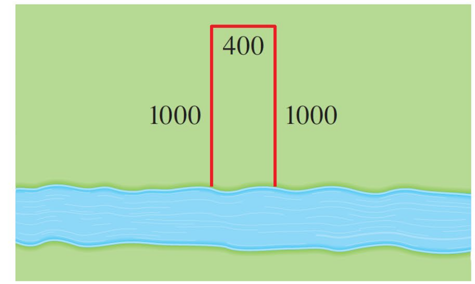
Example 1 – *Solution*



Area = $100 \cdot 2200 = 220{,}000$ ft^2



Area = $700 \cdot 1000 = 700{,}000$ ft^2

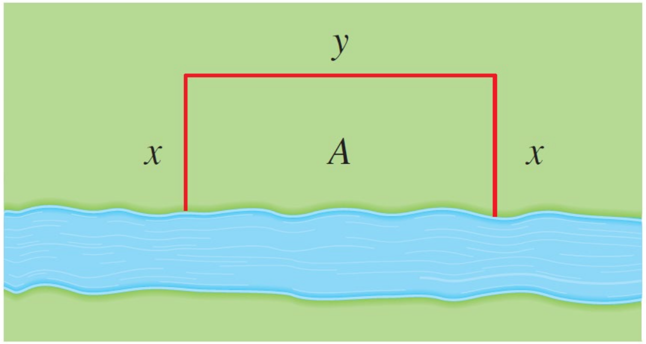


Area = $1000 \cdot 400 = 400{,}000$ ft^2

- shallow, wide fields or deep, narrow fields → relatively small areas
- intermediate configuration that produces the largest area

Example 1 – *Solution*

- General case: maximize the area A of the rectangle



- Let x and y : depth and width of the rectangle (in feet) express A in terms of x and y:

$$A = xy$$

Example 1 – *Solution*

- express A as a function of just one variable
  - eliminate y by expressing it in terms of x

- Given information:

$$2x + y = 2400$$

$y = 2400 - 2x$

$A = x(2400 - 2x) = 2400x - 2x^2$

- $x \geq 0$ and $x \leq 1200$ (otherwise $A < 0$)

Example 1 – *Solution*

- Function to maximize:

$$A(x) = 2400x - 2x^2 \qquad 0 \le x \le 1200$$

- Derivative: $A'(x) = 2400 - 4x$
- critical numbers:

$$2400 - 4x = 0$$

→ gives $x = 600$

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

- maximum value of A must occur either at this critical number or at an endpoint of the interval

Example 1 – *Solution*

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

- $A(0) = 0$
- $A(600) = 720{,}000$
- $A(1200) = 0$

→Closed Interval Method gives the maximum value as $A(600) = 720{,}000$

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

- $A''(x) = -4 < 0$ for all x:
  - A is always concave downward
  - local maximum at $x = 600$ must be an absolute maximum

- rectangular field should be 600 ft deep and 1200 ft wide