BMS 1901 Calculus for Life Sciences

Week 4

Maximum and Minimum Values
Understand Mean Value Theorem

Maximum and Minimum Values

Absolute and Local Extreme Values

Absolute and Local Extreme Values

- Highest point of the function f : (3, 5)

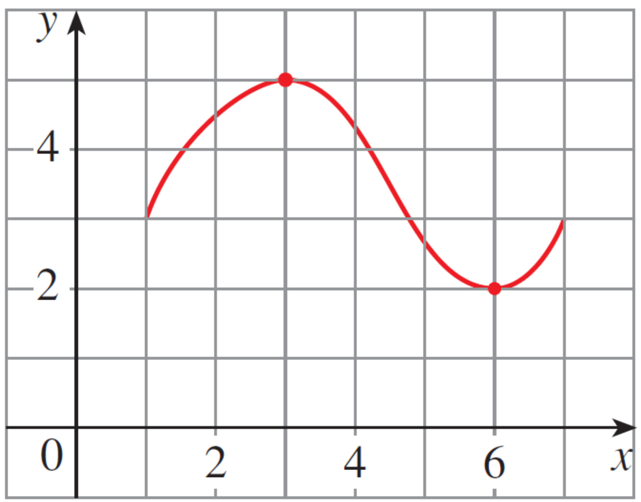


Figure 1

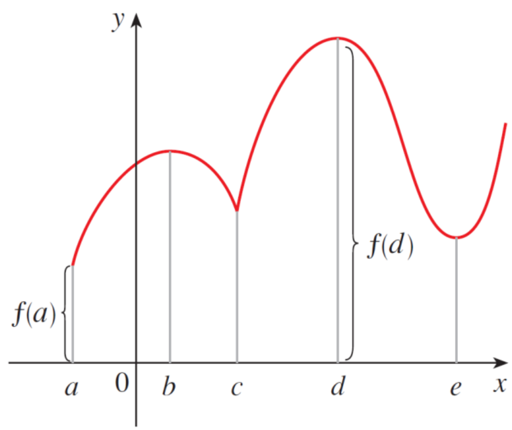
Absolute and Local Extreme Values

(1) Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

Absolute and Local Extreme Values

- **Global** maximum or minimum
- **Extreme values** of *f*



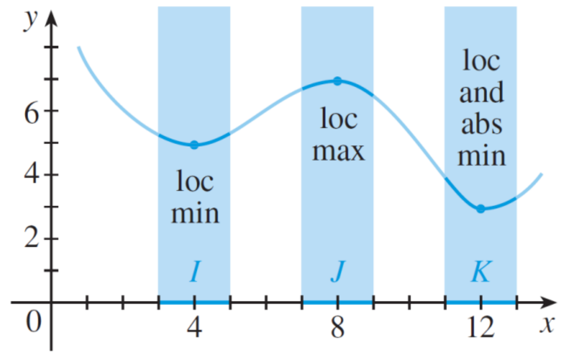
Abs min $f(a)$, abs max $f(d)$
loc min f(a), $f(c)$, $f(e)$, loc max $f(b)$, $f(d)$

Absolute and Local Extreme Values

(2) Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

Absolute and Local Extreme Values

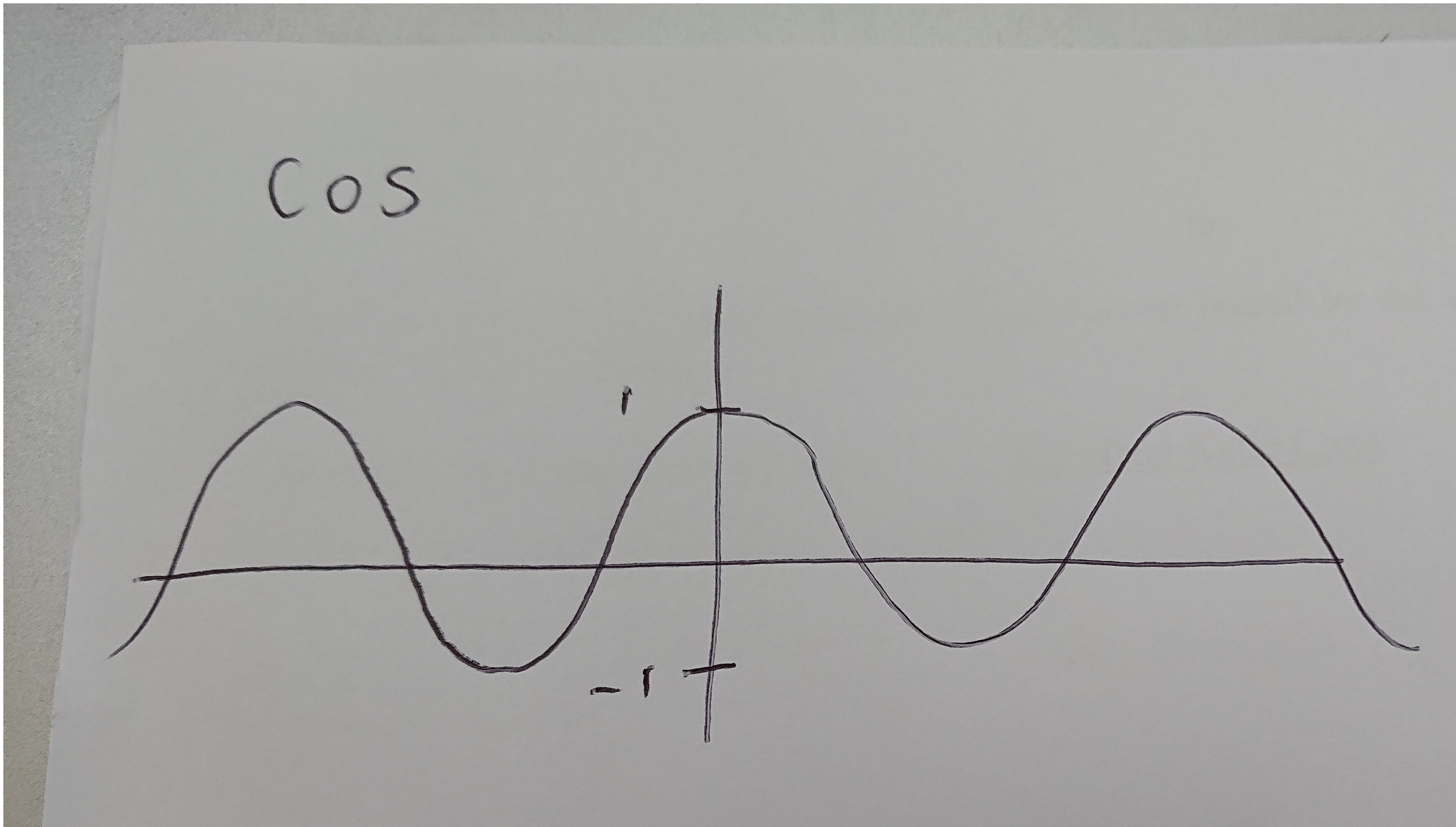


- $f(4) = 5$: local minimum
  - not the absolute minimum
  - $f(x)$ takes smaller values when x is near
- $f(12) = 3$ is both a local minimum and the absolute minimum
- $f(8) = 7$ is a local maximum
  - not the absolute maximum

Example 1

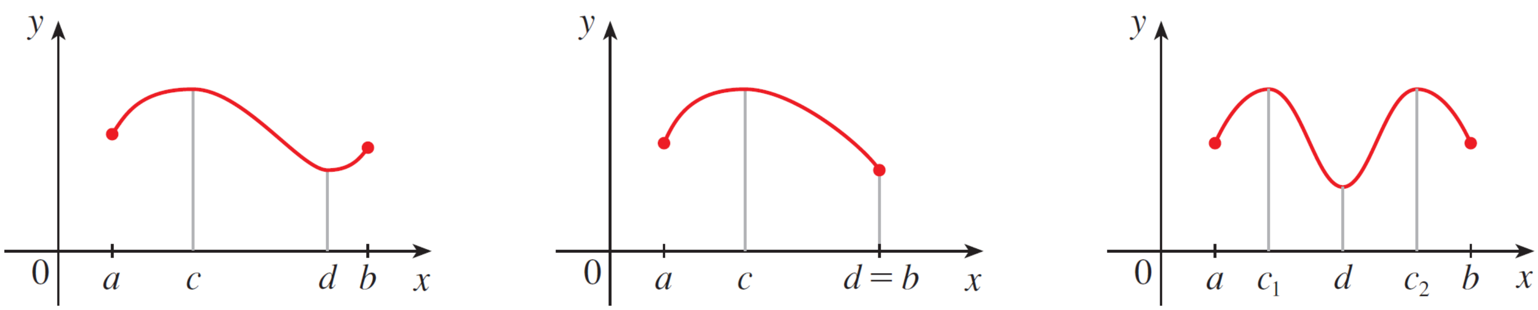
- $f(x) = \cos x$
  - takes on its (local and absolute) maximum value of 1 infinitely many times
  - $\cos 2n\pi = 1$ for any integer n and $-1 \leq \cos x \leq 1$ for all x

- $\cos(2n + 1)\pi = -1$ is its minimum value, where n is any integer


Cos
1
-1

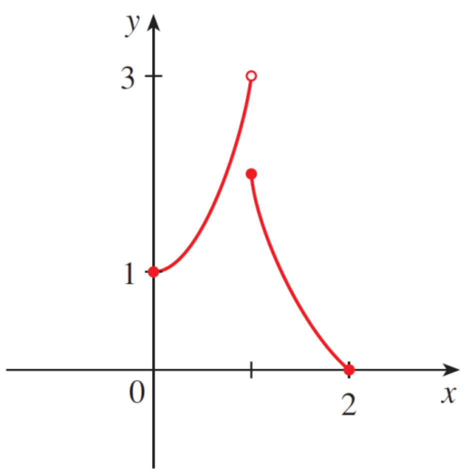
Absolute and Local Extreme Values

(3) The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

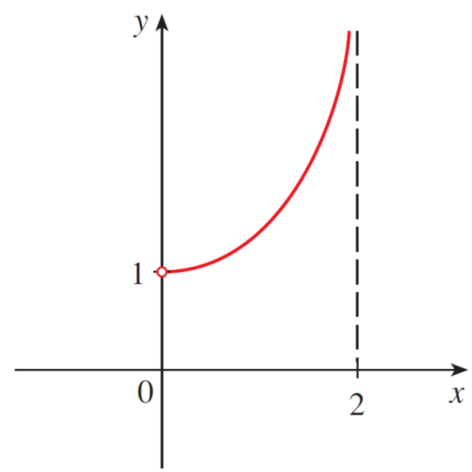


Absolute and Local Extreme Values

- a function need not possess extreme values
  - if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem



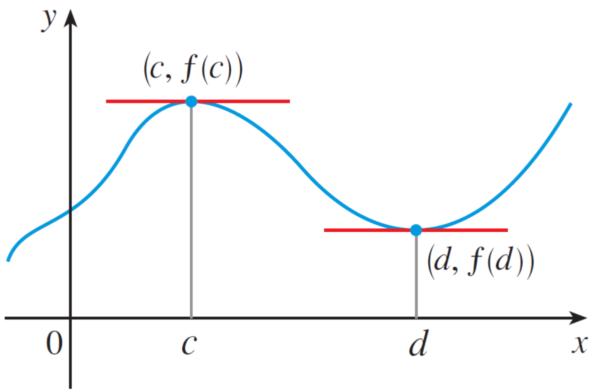
This function has a minimum value $f(2) = 0$, but no maximum value.



This continuous function g has no maximum or minimum.

Fermat’s Theorem

Fermat's Theorem



•function *f:*

- a local maximum at *c*
- a local minimum at *d*

Fermat's Theorem

- Derivative: slope of the tangent line
- $f'(c) = 0$ and $f'(d) = 0$

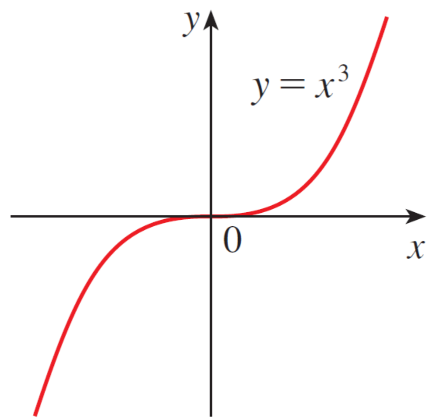
(4) Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Fermat's Theorem

$f(x) = x^3$

- $f'(x) = 3x^2$

  $f'(0) = 0$
- BUT, f has no maximum or minimum at 0



If $f(x) = x^3$, then $f'(0) = 0$ but f has no maximum or minimum.

- $f'(0) = 0$: curve $y = x^3$ has a horizontal tangent at (0, 0)

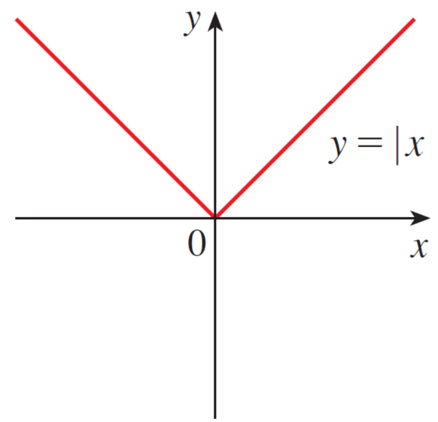
Fermat's Theorem

- No maximum nor minimum at (0, 0)
- curve crosses its horizontal tangent there
- when $f'(c) = 0$: f doesn't necessarily have a maximum or minimum at c

Fermat's Theorem

$f(x) = |x|$

- (local and absolute) minimum value at 0
- Minimum value cannot be found by setting $f'(x) = 0$
  - $f'(0)$ does not exist



If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

Fermat's Theorem

- *start* looking for extreme values of f at the numbers c
  - $f'(c) = 0$ or $f'(c)$ does not exist

(5) Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 5

Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

Solution:

The Product Rule gives

$$f'(x) = x^{3/5}(-1) + (4 - x)\left(\tfrac{3}{5}x^{-2/5}\right)$$

$$= -x^{3/5} + \frac{3(4 - x)}{5x^{2/5}}$$

$$= \frac{-5x + 3(4 - x)}{5x^{2/5}}$$

$$= \frac{12 - 8x}{5x^{2/5}}$$

Example 5

- $f(x) = 4x^{3/5} - x^{8/5}$

- $f'(x) = 0$ if $12 - 8x = 0$
  - $x = \frac{3}{2}$, and $f'(x)$ does not exist when $x = 0$

- Critical numbers are $\frac{3}{2}$ and 0

Fermat’s Theorem

- Rephrased Fermat’s Theorem:

(6) If f has a local maximum or minimum at c, then c is a critical number of f.

The Closed Interval Method

The Closed Interval Method

- find an absolute maximum or minimum of a continuous function on a closed interval
- local [in which case it occurs at a critical number by (6)] or it occurs at an endpoint of the interval

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 6 – *The Allee effect*

One of the models for the growth rate of a population of size N at time t reflects the fact that some populations decline to extinction unless they stay above a critical value. A particular case of this model is expressed by the growth rate

$$f(N) = N(N - 3)(8 - N)$$

where N is measured in hundreds of individuals. [Notice that $f(N)$ is negative when $0 < N < 3$.] Find the absolute maximum and minimum values of the growth rate function

$$f(N) = N(N - 3)(8 - N) \qquad 0 \leqslant N \leqslant 9$$

Example 6

- f is continuous on the interval [0, 9]
- Closed Interval Method:

$$f(N) = N(N - 3)(8 - N) = -N^3 + 11N^2 - 24N$$

- $f'(N)$ exists for all N, the only critical numbers of f occur when $f'(N) = 0$, i.e., $N = \frac{4}{3}$ or $N = 6.$
- each of these critical numbers lies in the interval (0, 9)

$$f'(N) = -3N^2 + 22N - 24 = -(3N - 4)(N - 6)$$

- Critical numbers and relevant values of f are:

$$f\left(\tfrac{4}{3}\right) = -\tfrac{400}{27} \qquad\qquad f(6) = 36$$

$$f'(N) = -3N^2 + 22N - 24$$

$$N = \frac{-22 \pm \sqrt{22^2 - 4(-3)(-24)}}{2(-3)}$$

$$= \frac{-22 \pm \sqrt{484 - 288}}{-6}$$

$$= \frac{-22 \pm 14}{-6}$$

$$= \frac{-22 + 14}{-6} \text{ or } \frac{-22 - 14}{-6}$$

$$= \frac{4}{3} \text{ or } 6$$

Example 6

- The values of f at the endpoints of the interval are

$$f(0) = 0 \qquad\qquad f(9) = -54$$

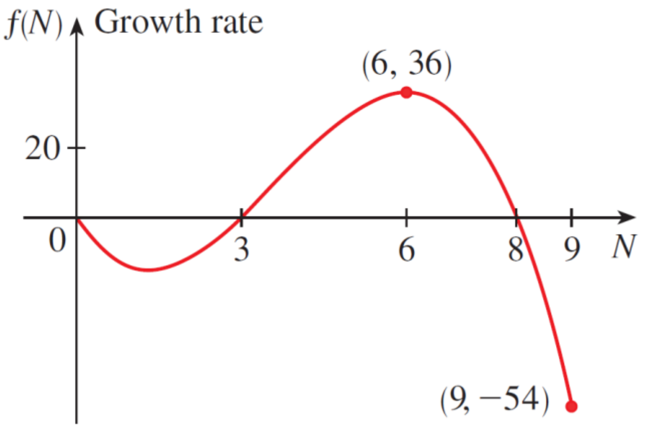
- absolute maximum value is $f(6) = 36$ and the absolute minimum value is $f(9) = -54$

- population increases fastest when $N = 6$ (the population is 600) and the absolute maximum value is: $f(6) = 36$
    - The maximum rate of increase is 3600 individuals per year

Example 6 – *Solution*

- The population decreases most rapidly on the given interval when N=9
- Absolute minimum value: f(9)=-54

*Absolute minimum occurs at an endpoint
*Absolute maximum occurs at a critical number



How Derivatives Affect the Shape of a Graph

The Mean Value Theorem

The Mean Value Theorem

The Mean Value Theorem If f is a differentiable function on the interval $[a, b]$, then there exists a number c between a and b such that

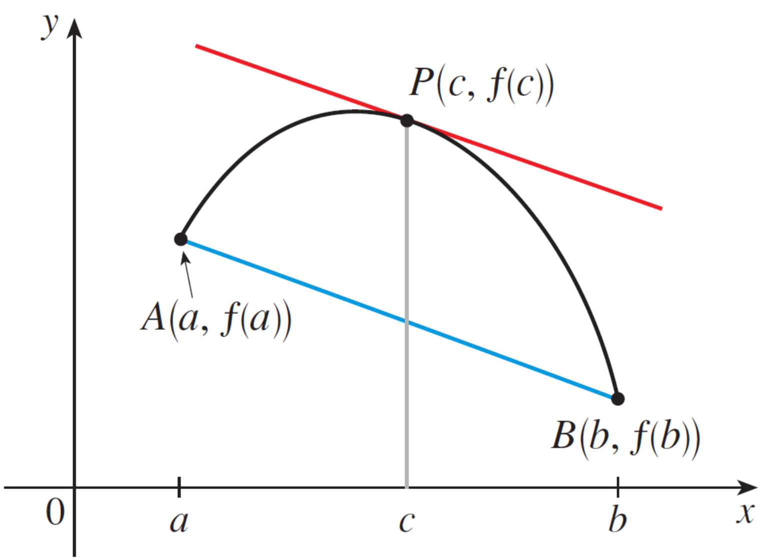
(1)
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

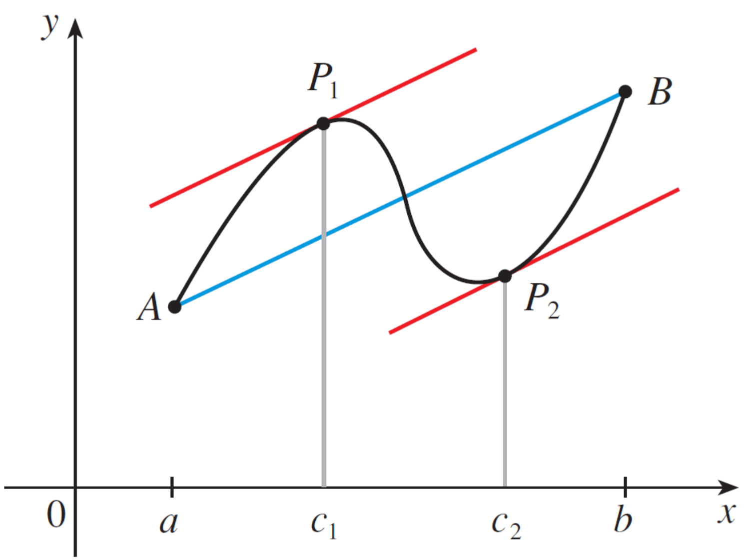
or, equivalently,

(2)
$$f(b) - f(a) = f'(c)(b - a)$$

The Mean Value Theorem

- *Points:* $A(a, f(a))$ and $B(b, f(b))$ on graphs of two differentiable functions





The Mean Value Theorem

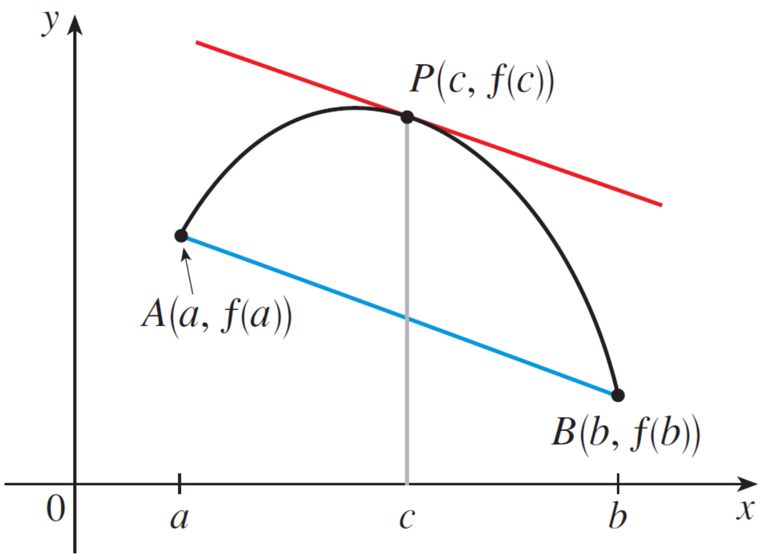
The slope of the secant line *AB* is

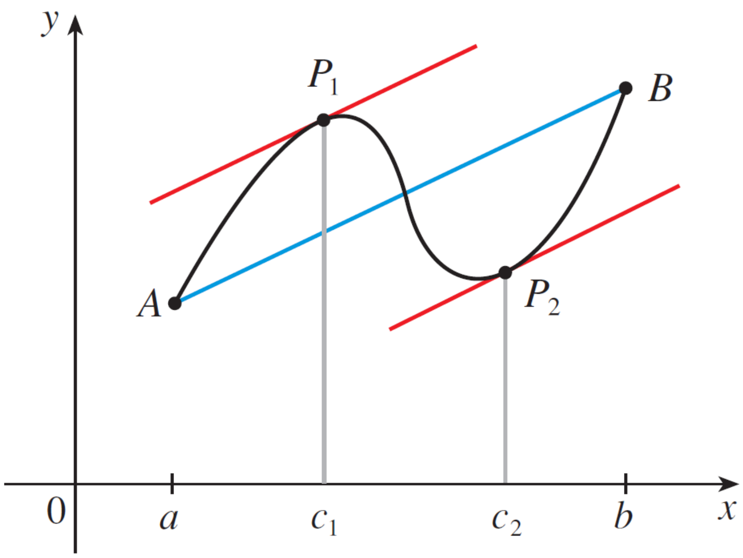
$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

- $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$
- Mean Value Theorem: there is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is = the slope of the secant line *AB*

The Mean Value Theorem

- point P where the tangent line is parallel to the secant line AB
- one such point P in left figure
- two such points P_1 and P_2 in right figure





Example for MVT

- Moving object: $s = f(t)$
- average velocity between $t = a$ and $t = b$ is

$$\frac{f(b) - f(a)}{b - a}$$

- velocity at $t = c$ is $f'(c)$

- Mean Value Theorem: at some time $t = c$ between a and b, the instantaneous velocity $f'(c)$ = average velocity

- E.g. if a car travelled 180 km in 2 hours
  - speedometer must have read 90 km/h at least once

Increasing and Decreasing Functions

Increasing and Decreasing Functions

- I/D test:

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Let x_1 and x_2 = any two numbers in the interval with $x_1 < x_2$.

To prove (a): $* f(x_1) < f(x_2)$

$f'(x) > 0 \Rightarrow f$ is differentiable on $[x_1, x_2]$.

By MVT: $\exists$ c between x_1 and x_2 such that:

$$\boxed{LHS}\ \underbrace{f(x_2) - f(x_1)} = \underbrace{f'(c)(x_2 - x_1)}\ \boxed{RHS}$$

$f'(c) > 0$ by assumption & $x_2 - x_1 > 0$ ($\because x_1 < x_2$)

RHS $> 0 \Rightarrow f(x_2) - f(x_1) > 0$ / (or) $f(x_1) < f(x_2)$

This shows that f is increasing.

Example for increasing and decreasing functions

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution:

First we calculate the derivative of f :

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1)$$

- I/D Test: need to know where $f'(x) > 0$ and where $f'(x) < 0$
  - depends on the signs of the three factors of $f'(x)$: $12x$, $x - 2$, and $x + 1$

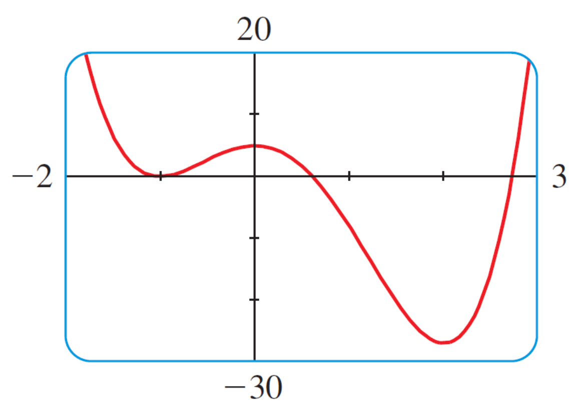
Example

- divide the real line into intervals whose endpoints are the critical numbers: –1, 0, and 2 and arrange our work in a chart
    - +: given expression is positive
    - -: it is negative
    - last column of the chart: conclusion based on the I/D Test

- *E.g.* $f'(x) < 0$ for $0 < x < 2$
    - f is decreasing on(0, 2)

Example

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$



Increasing and Decreasing Functions

- f has a local maximum or minimum at c
  - c must be a critical number of f (Fermat's Theorem)
  - not every critical number gives rise to a maximum or a minimum

- need a test that will tell us whether or not f has a local maximum or minimum at a critical number

- Previous figure: $f(0) = 5$ is a local maximum value of f
  - f increases on $(–1, 0)$
  - decreases on $(0, 2)$
  - $f'(x) > 0$ for $–1 < x < 0$ and $f'(x) < 0$ for $0 < x < 2$

Increasing and Decreasing Functions

- sign of $f'(x)$ changes from positive to negative at 0

The First Derivative Test Suppose that c is a critical number of a continuous function f.

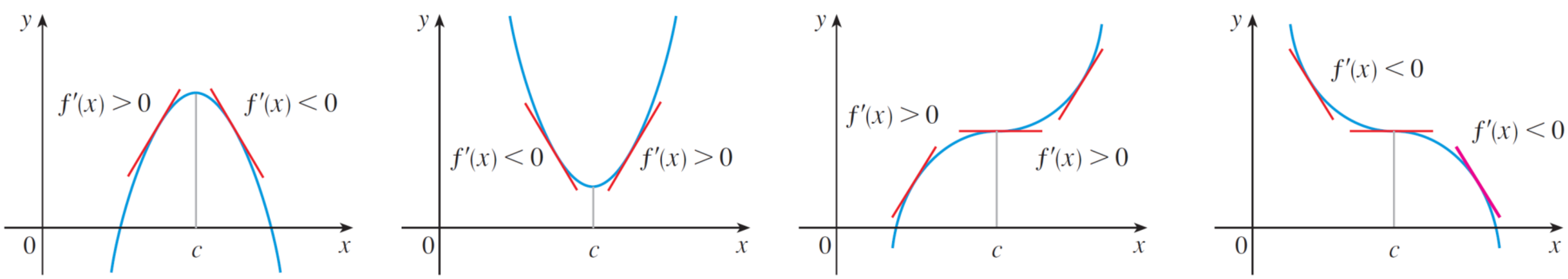
(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

Increasing and Decreasing Functions

- First Derivative Test is a consequence of the I/D Test
- (a): e.g. because the sign of $f'(x)$ changes from positive to negative at c, f is increasing to the left of c and decreasing to the right of c
    - f has a local maximum at c



(a) Local maximum (b) Local minimum (c) No maximum or minimum (d) No maximum or minimum

Example

Interval	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

Find the local minimum and maximum values of the function *f* in $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Solution:

- $f'(x)$ changes from negative to positive at –1
  - $f(-1) = 0$ is a local minimum value by the First Derivative Test
- f' changes from negative to positive at 2
  - $f(2) = -27$ is also a local minimum value
- $f'(x)$ changes from positive to negative at 0
  - $f(0) = 5$ is a local maximum value