Chapter 5

Physics of Multiple Objects

Chapter 5 Part 0

Momentum $\vec{p} = m\vec{v}$

Why use the quantity of momentum

- Like the quantity of energy, it can lead to some conservation law.
- No need to involve acceleration
- No need to consider details of forces. This is especially good in the abrupt case such as collision
- Unlike energy, it is a vector, carrying information of direction (extra).
- It can handle the case of variable mass like rocket

Impulse*

- When a force ***F acts abruptly*** for a short time Δt on an object (*e.g.*, during collision), we define the ***impulse, J***, of the encounter as

$$\vec{J} = \vec{F}\Delta t$$

- Newton's 2nd law:

$$\vec{F} = \mathrm{m}\vec{a} = m\,\frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta(m\vec{v})}{\Delta t}$$

- Or Impulse: $\vec{J} = \vec{F}\Delta t = \Delta\boxed{(m\vec{v})}$

A new physical quantity: $\Delta\vec{p} = \mathrm{m}\Delta\vec{v}$

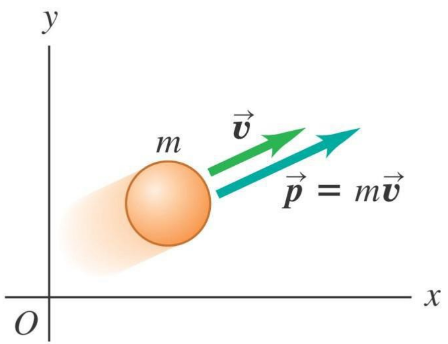
Linear momentum

- Momentum of a body is the product of its mass and its velocity:

$$\vec{p} = m\vec{v}$$

- It is a vector and has the unit of $kg{\cdot}m/s$

$$K.E. = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$



Momentum $\vec{p}$ is a vector quantity; a particle's momentum has the same direction as its velocity $\vec{v}$.

- The total linear momentum of a system of two objects is the vector sum of the pair's individual values

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{p}_1 + \vec{p}_2$$

Momentum Conservation during a collision

$$\vec{J}_1 = \vec{F}_1 \Delta t = \Delta \vec{p}_1 \qquad \vec{J}_2 = \vec{F}_2 \Delta t = \Delta \vec{p}_2$$

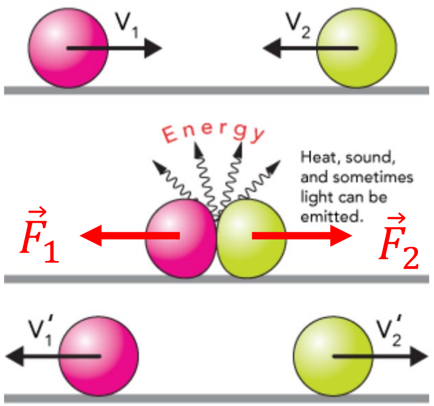
- During the collision of 2 bodies: the impulse experienced by one is equal in magnitude and opposite in direction to the impulse experienced by the other (Newton's 3rd Law):

$$\vec{F}_1 = -\vec{F}_2 \rightarrow \vec{J}_1 = -\vec{J}_2$$

$$\Delta\vec{p}_1 = -\Delta\vec{p}_2 \quad \rightarrow \quad \Delta\vec{p}_1 + \Delta\vec{p}_2 = 0$$

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{p}_1 + \vec{p}_2$$

$$\Delta\vec{P} = 0 \Rightarrow P_i = P_f$$



- Conservation of linear momentum:

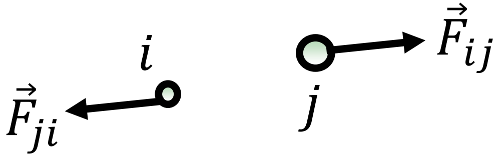
$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1' + m_2\vec{v}_2'$$

Momentum Conservation in general

For a system of multiple objects, if there is no net external forces, or short interaction time, total momentum is conserved.

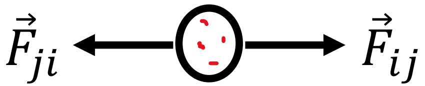
- For systems of more than two particles, the internal forces are always come in pair of action and reaction.

$$\vec{F}_{ij} = -\vec{F}_{ji} \rightarrow \vec{J}_{ij} = -\vec{J}_{ji}$$



- When consider the whole system as one, those can cancel each other

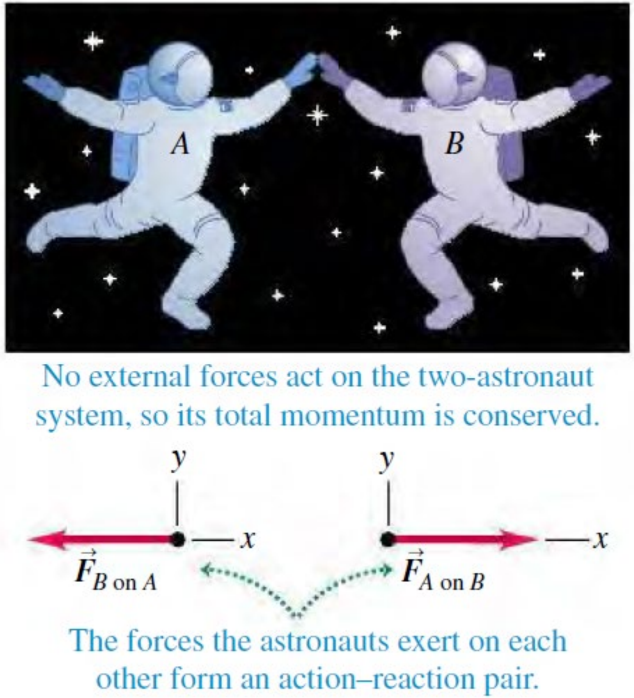
$$\Delta\vec{p}_i + \Delta\vec{p}_j = \vec{J}_{ij} + \vec{J}_{ji} \quad \rightarrow \quad \Delta\vec{p}_i + \Delta\vec{p}_j = 0$$



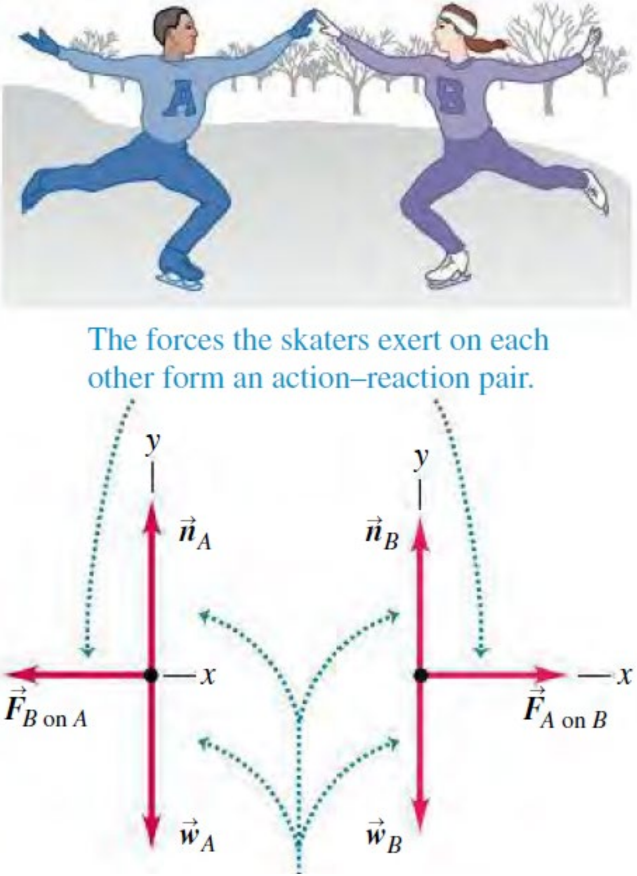
- Momentum along a certain direction is conserved when there are no external forces acting in that direction ➔ $P_{x\text{-}i} = P_{x\text{-}f}$

Examples

8.9 Two astronauts push each other as they float freely in the zero-gravity environment of space.



8.10 Two ice skaters push each other as they skate on a frictionless, horizontal surface. (Compare to Fig. 8.9.)

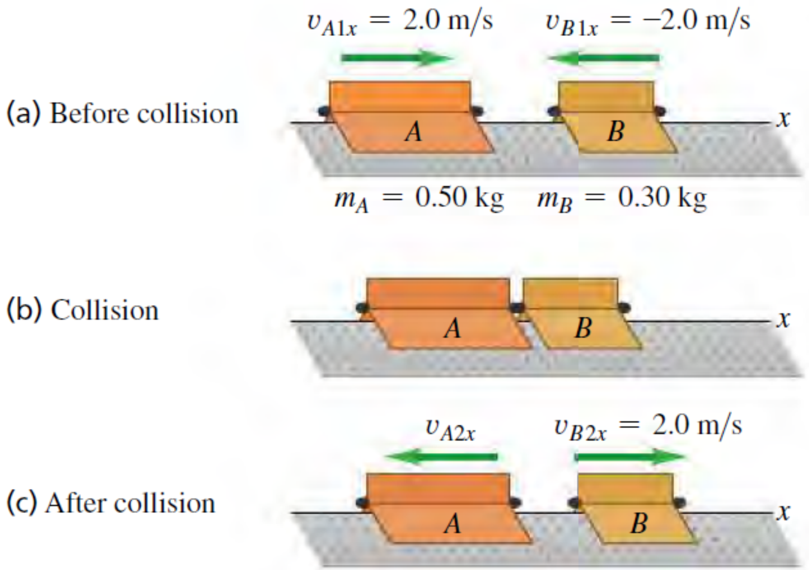


EXAMPLE 8.5 COLLISION ALONG A STRAIGHT LINE

LUTION

Two gliders with different masses move toward each other on a frictionless air track (**Fig. 8.13a**). After they collide (Fig. 8.13b), glider B has a final velocity of $+2.0$ m/s (Fig. 8.13c). What is the final velocity of glider A? How do the changes in momentum and in velocity compare?

8.13 Two gliders colliding on an air track.



SOLUTION

IDENTIFY and SET UP: As for the skaters in Fig. 8.10, the total vertical force on each glider is zero, and the net force on each individual glider is the horizontal force exerted on it by the other glider. The net external force on the *system* of two gliders is zero, so their total momentum is conserved. We take the positive x-axis to be to the right. We are given the masses and initial velocities of both gliders and the final velocity of glider B. Our target variables are v_{A2x} (the final x-component of velocity of glider A), and the changes in momentum and in velocity of the two gliders (the value *after* the collision minus the value *before* the collision).

Continued

EXECUTE: The x-component of total momentum before the collision is

$$\begin{aligned}P_x &= m_A v_{A1x} + m_B v_{B1x} \\ &= (0.50\text{ kg})(2.0\text{ m/s}) + (0.30\text{ kg})(-2.0\text{ m/s}) \\ &= 0.40\text{ kg}\cdot\text{m/s}\end{aligned}$$

This is positive (to the right in Fig. 8.13) because A has a greater magnitude of momentum than B. The x-component of total momentum has the same value after the collision, so

$$P_x = m_A v_{A2x} + m_B v_{B2x}$$

We solve for v_{A2x}:

$$\begin{aligned}v_{A2x} &= \frac{P_x - m_B v_{B2x}}{m_A} = \frac{0.40\text{ kg}\cdot\text{m/s} - (0.30\text{ kg})(2.0\text{ m/s})}{0.50\text{ kg}} \\ &= -0.40\text{ m/s}\end{aligned}$$

The changes in the x-momenta are

$$\begin{aligned}m_A v_{A2x} - m_A v_{A1x} &= (0.50\text{ kg})(-0.40\text{ m/s}) \\ &\quad - (0.50\text{ kg})(2.0\text{ m/s}) \\ &= -1.2\text{ kg}\cdot\text{m/s}\end{aligned}$$

$$\begin{aligned}m_B v_{B2x} - m_B v_{B1x} &= (0.30\text{ kg})(2.0\text{ m/s}) \\ &\quad - (0.30\text{ kg})(-2.0\text{ m/s}) \\ &= +1.2\text{ kg}\cdot\text{m/s}\end{aligned}$$

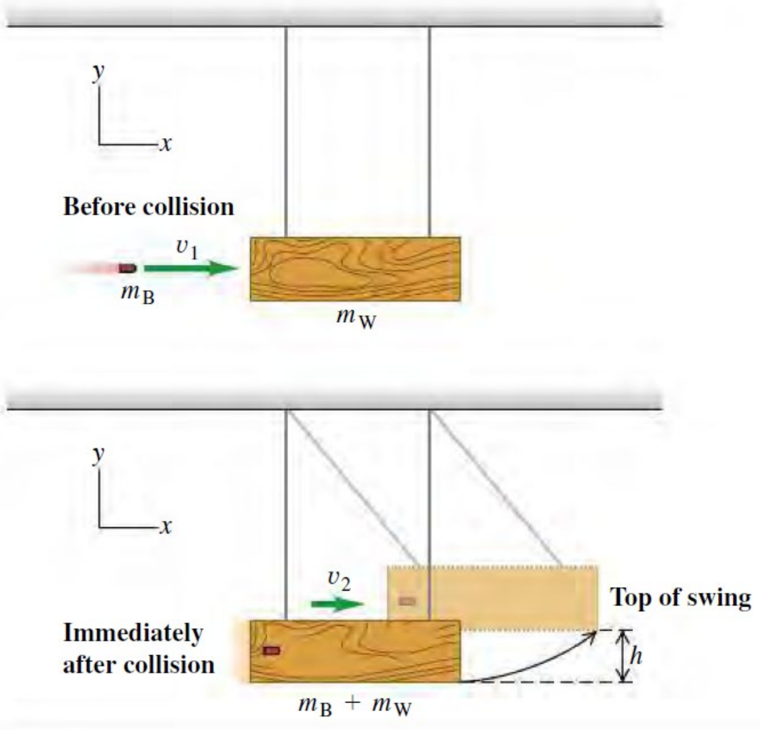
The changes in x-velocities are

$$v_{A2x} - v_{A1x} = (-0.40\text{ m/s}) - 2.0\text{ m/s} = -2.4\text{ m/s}$$
$$v_{B2x} - v_{B1x} = 2.0\text{ m/s} - (-2.0\text{ m/s}) = +4.0\text{ m/s}$$

EVALUATE: The gliders were subjected to equal and opposite interaction forces for the same time during their collision. By the impulse–momentum theorem, they experienced equal and opposite impulses and therefore equal and opposite changes in momentum. But by Newton's second law, the less massive glider (B) had a greater magnitude of acceleration and hence a greater velocity change.

EXAMPLE 8.8 THE BALLISTIC PENDULUM

Figure 8.19 shows a ballistic pendulum, a simple system for measuring the speed of a bullet. A bullet of mass m_B makes a completely inelastic collision with a block of wood of mass m_W, which is suspended like a pendulum. After the impact, the block swings up to a maximum height h. In terms of h, m_B, and m_W, what is the initial speed v_1 of the bullet?



No external force in horizontal direction!

SOLUTION

IDENTIFY: We'll analyze this event in two stages: (1) the bullet embeds itself in the block, and (2) the block swings upward. The first stage happens so quickly that the block does not move appreciably. The supporting strings remain nearly vertical, so negligible external horizontal force acts on the bullet–block system, and the horizontal component of momentum is conserved. Mechanical energy is *not* conserved during this stage, however, because a nonconservative force does work (the force of friction between bullet and block).

In the second stage, the block and bullet move together. The only forces acting on this system are gravity (a conservative force) and the string tensions (which do no work). Thus, as the block swings, *mechanical energy* is conserved. Momentum is *not* conserved during this stage, however, because there is a net external force (the forces of gravity and string tension don't cancel when the strings are inclined).

SET UP: We take the positive x-axis to the right and the positive y-axis upward. Our target variable is v_1. Another unknown quantity is the speed v_2 of the system just after the collision. We'll use momentum conservation in the first stage to relate v_1 to v_2, and we'll use energy conservation in the second stage to relate v_2 to h.

EXECUTE: In the first stage, all velocities are in the $+x$-direction. Momentum conservation gives

$$m_B v_1 = (m_B + m_W) v_2$$

$$v_1 = \frac{m_B + m_W}{m_B} v_2$$

At the beginning of the second stage, the system has kinetic energy $K = \frac{1}{2}(m_B + m_W)v_2^2$. The system swings up and comes to rest for an instant at a height h, where its kinetic energy is zero and the potential energy is $(m_B + m_W)gh$; it then swings back down. Energy conservation gives

$$\frac{1}{2}(m_B + m_W)v_2^2 = (m_B + m_W)gh$$

$$v_2 = \sqrt{2gh}$$

We substitute this expression for v_2 into the momentum equation:

$$v_1 = \frac{m_B + m_W}{m_B}\sqrt{2gh}$$

EVALUATE: Let's plug in the realistic numbers $m_B = 5.00\text{ g} = 0.00500\text{ kg}$, $m_W = 2.00\text{ kg}$, and $h = 3.00\text{ cm} = 0.0300\text{ m}$:

$$v_1 = \frac{0.00500\text{ kg} + 2.00\text{ kg}}{0.00500\text{ kg}}\sqrt{2(9.80\text{ m/s}^2)(0.0300\text{ m})}$$

$$= 307\text{ m/s}$$

$$v_2 = \sqrt{2gh} = \sqrt{2(9.80\text{ m/s}^2)(0.0300\text{ m})} = 0.767\text{ m/s}$$

Chapter 5 Part A

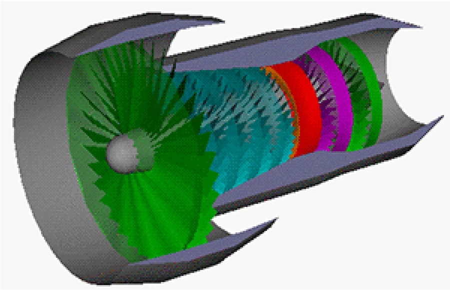
Rotation of Rigid Bodies: part 1

What we learn in Part A

- Angular coordinate, rigid body, rotation axis
- angular velocity and acceleration
- Analyze rotation with constant angular acceleration
- Relate angular quantities to the linear velocity and linear acceleration of a point on a body
- Understand moment of inertia and its relation to rotational kinetic energy

Rotating rigid bodies around us

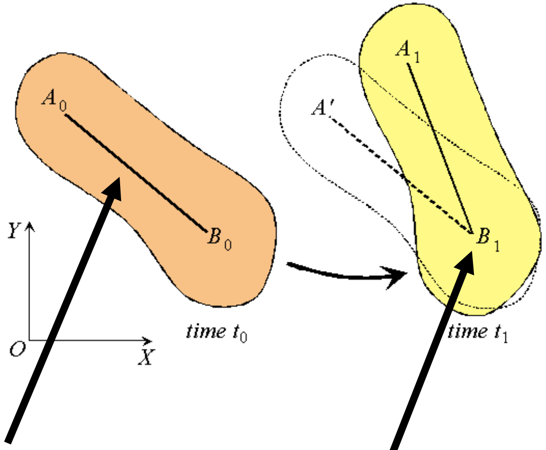
- A wind turbine, rotating blades in an engine, a ceiling fan, and a Ferris wheel all are rotating rigid objects (objects have non-zero dimensions and consist many parts of fixed structure).



Real world rotation & assumption used here

Real world rotation is complicated because:

- The object rotating can change shape
- axis of rotation changes direction.



For this introduction course, we only consider;

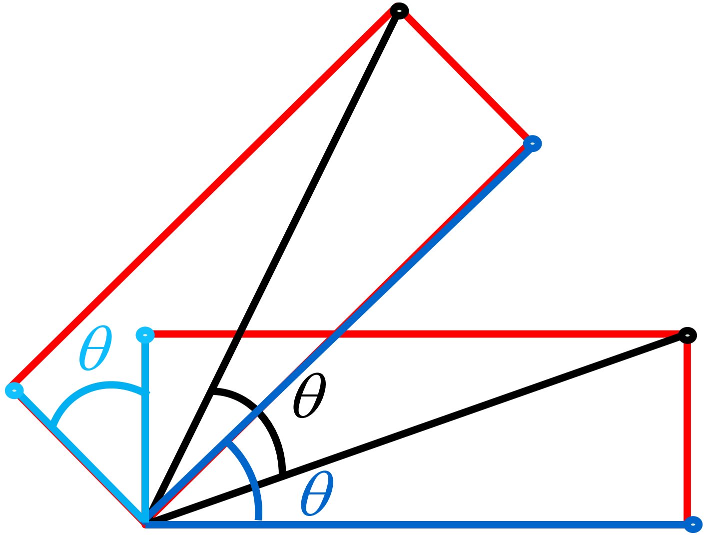
- Rigid body: object does not change its shape (distance between two fixed points is a constant, during the motion)

- Fixed axis of rotation: the axis of rotation is fixed in direction and position. the particles of the object rotate in circles; these circles are parallel

A common quantity in Rotating rigid bodies

- When a rigid body rotates, all points on the body rotate the same angle θ

- Rotation angle θ can uniquely determine the position of an rigid body.

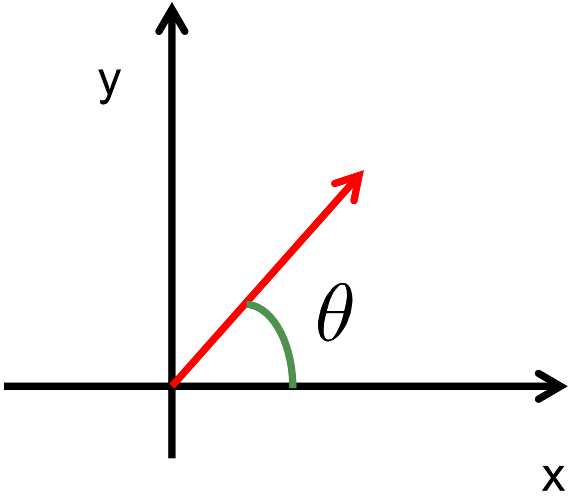


Angular coordinate: Coordinate for rotation

- Consider speedometer needle rotates about a *fixed axis*

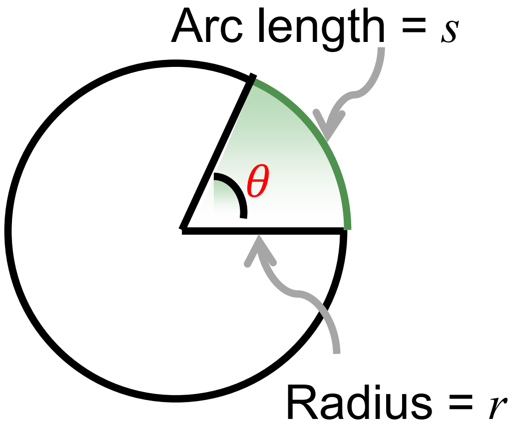
- The angle θ that the needle makes with the $+x$-axis is a coordinate for rotation.

- It tells the position of the needle (like x tells the position on the x-axis, (x,y) tells position in a 2-D plane)



Units

- An angle expressed in radians is $\theta = s/r$, as shown in the figure.
- Another unit is degree represented by $^{\circ}$
- One complete revolution is $360^{\circ} = 2\pi$ radians. $1^{\circ} = \frac{2\pi}{360}\ radian$
- In physics we usually use radian as unit of angle, as radian is easily related to the length s: $s = \theta r$ (θ in radian)



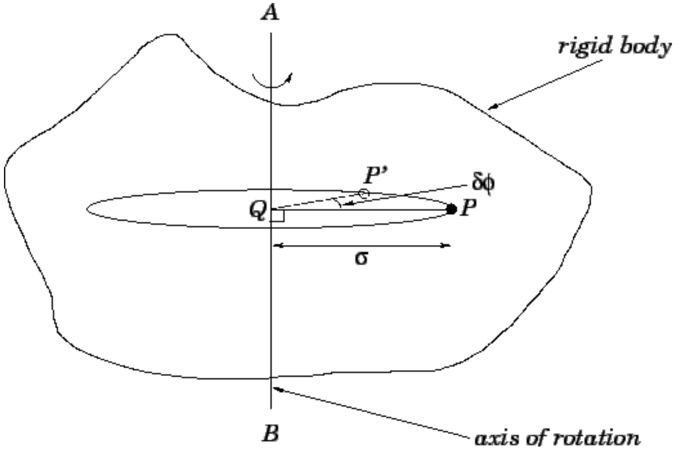
Definition of Axis of Rotation

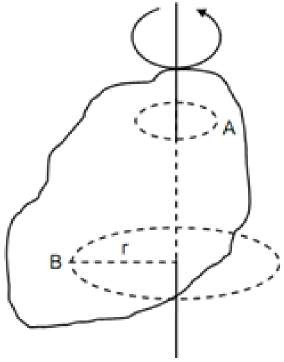
- When a rigid body rotates, the particles in the rigid body move in circles

- **All the circles of motion of the particles have their own centers**

- **These centers form a line, which is the axis of rotation**

- In this course, the axis of rotation is fixed in direction, not changing, but may move (translation only) (in T03)





The centers of the two circles A and B are on the axis of rotation They may have different radius, but same angular velocity and angular acceleration

Rotational quantities in rigid body

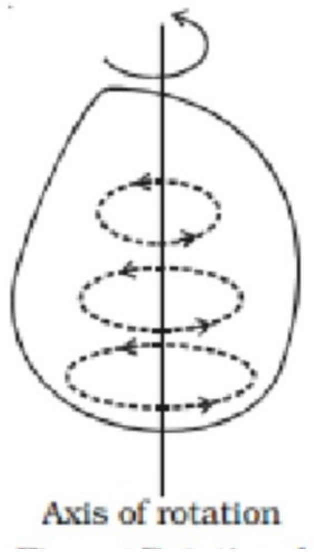
- When a rigid body rotates, all the particles in the rigid body are in circular motions **with the same angle of rotation, but with different radius.**

- As a result, they have the **same angular velocity, same angular acceleration**

To describe the rotation we only need

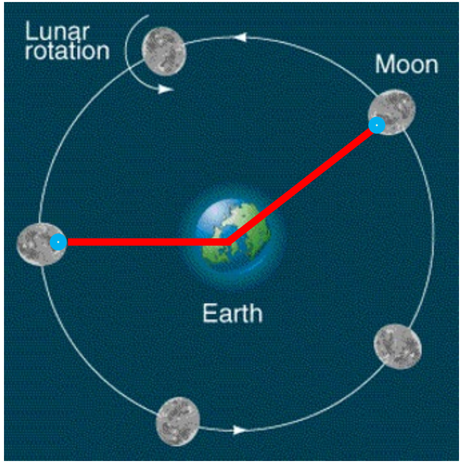
- **one rotation coordinate, one angular velocity, one angular acceleration, as all particles in the rigid body have the same values for these quantities**

- **But different particles may have different translational velocity, accelerations in a rotating object**

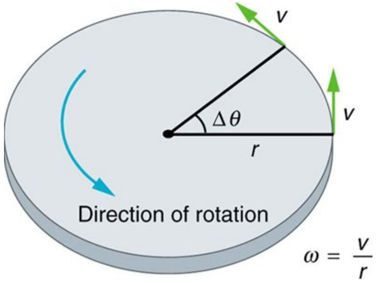


Outside axis of rotation

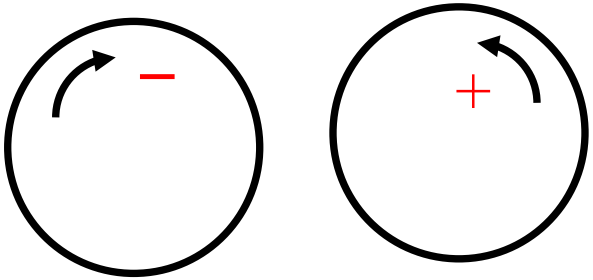
- Axis of rotation can be outside the object.
- The axis of rotation of the Moon is in the center of Earth.
- As Moon circles around the Earth, it rotates in sync with its orbit. So, the same side always faces to the Earth. We on earth can never see the other side of the Moon.



Angular displacement and velocity



- The *angular displacement of a rotating object=the change of the angular coordinate* $\Delta\theta$, $\Delta\theta = \theta_2 - \theta_1$.
- The *average angular velocity* of the object is $\omega_{av\text{-}z} = \Delta\theta/\Delta t$. It is the rate of change of the angular coordinate.
- The subscript *z* means that z –axis is axis of rotation
- The *instantaneous angular velocity is* $\omega_z = d\theta/dt$.
- A counterclockwise rotation is positive; a clockwise rotation is negative. Just like positive velocity and negative velocity, the sign depends on direction

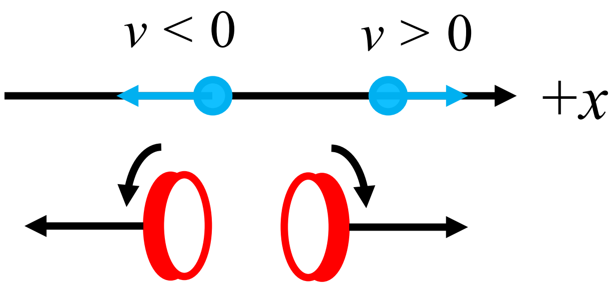


Clock wise

Counter clock wise

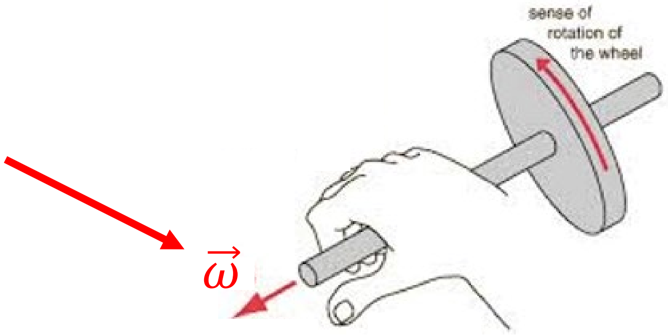
→ Similar to 1-dim motion along *x*-axis, with θ as the equivalent *x*-coordinate

Angular velocity is a vector

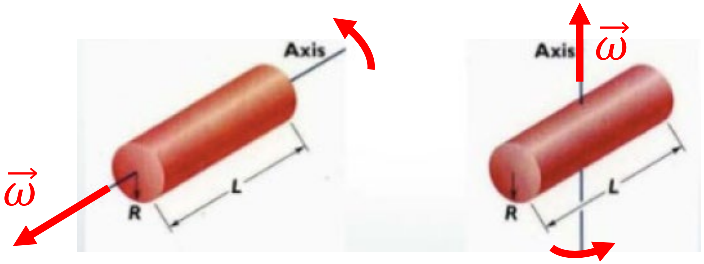


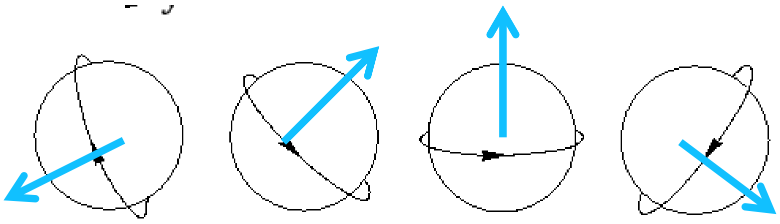
- Just like using *x*-axis to denote linear velocity direction, we use the direction of the axis of rotation to denote the direction of angular velocity → angular velocity is a vector

- Direction is given by the right-hand rule shown in Figure



- Different direction of angular velocity means different axis of rotation motion

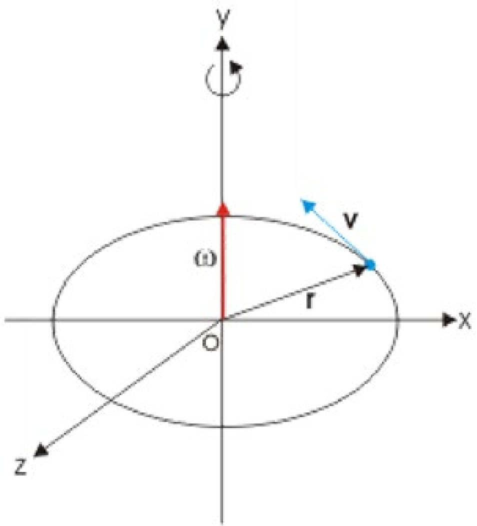




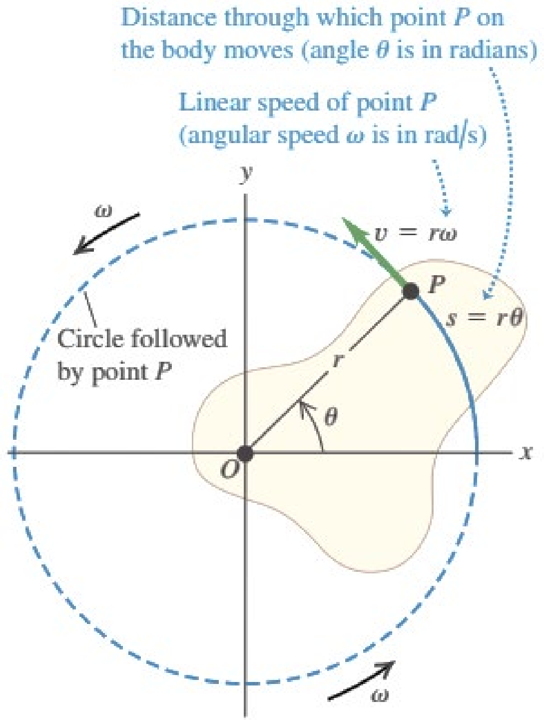
One fixed axis of rotation in this course

- The one-dimensional motion analogy only applies when the direction of the rotational axis is fixed.
- In this course, we consider only one rotational axis fixed in direction. The direction of the angular velocity is along the axis of rotation. Rotation is uniquely described by one angle θ.
- In 2nd or 3rd year of physics, you may need to consider angular velocity as a 2 or 3-dim vector, which can change direction. The orientation of a rigid body needs three parameters to be uniquely determined.

Relation between angular velocity, position vector and velocity



- Angular velocity is perpendicular to the plane of the circle of motion
- It is perpendicular to velocity $\vec{v}$, as $\vec{v}$ is tangential to the circle
- When $\vec{r}$ is in the plane of the circle, $s = r\theta \quad \rightarrow v = r\omega$

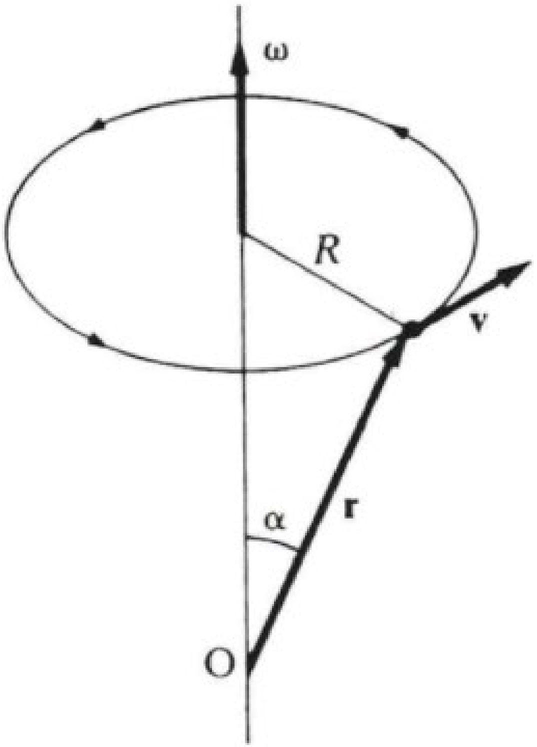


Relation between angular velocity, position vector and velocity: 3-dim case

- $\vec{\omega}$ may not be perpendicular to $\vec{r}$ as $\vec{r}$'s direction depends on the origin

- When $\vec{r}$ is not in the plane of the circle, $v = R\omega = r\omega \sin\alpha = |\vec{\omega} \times \vec{r}|$

- We can write $\vec{v} = \vec{\omega} \times \vec{r}$
- Velocity = cross product of angular velocity and position vector



Angular acceleration

- The *average angular acceleration* is $\alpha_{av\text{-}z} = \Delta\omega_z/\Delta t$. It is the rate of change of the angular velocity.

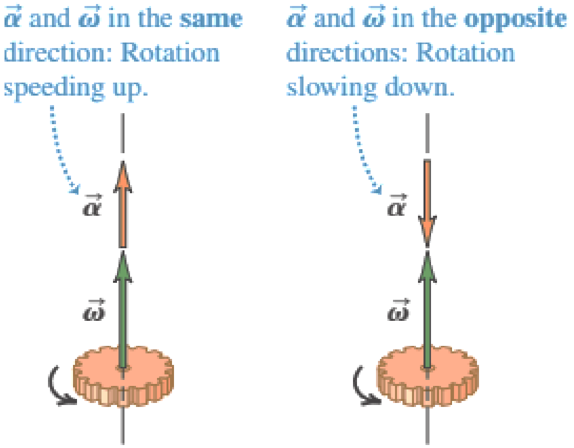
- The *instantaneous angular acceleration is* $\alpha_z = d\omega_z/dt = d^2\theta/dt^2$.

- $d^2\theta/dt^2 = \frac{d}{dt}\frac{d\theta}{dt}$ is the second derivative, which is the derivative of the derivative of θ

- Angular velocity is a vector, so is angular acceleration, as they are related.
For fixed axis of rotation, angular acceleration is also along the axis of rotation

Since all points in a rigid body have same angular displacement, angular velocity, angular acceleration,
Rotation of the rigid body is described by one angular displacement, one angular velocity and one angular acceleration

9.7 When the rotation axis is fixed, the angular acceleration and angular velocity vectors both lie along that axis.



Rotation with constant angular acceleration

- The rotational formulas have the same form as the straight-line motion formulas, as shown in The table below. They can be proved with the same angular velocity vs time graph.

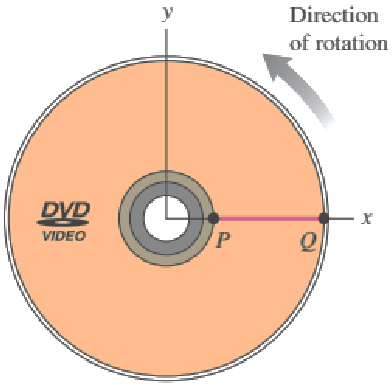
Straight-Line Motion with Constant Linear Acceleration	Fixed-Axis Rotation with Constant Angular Acceleration
$a_x = \text{constant}$	$\alpha_z = \text{constant}$
$v_x = v_{0x} + a_x t$	$\omega_z = \omega_{0z} + \alpha_z t$
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_x + v_{0x})t$	$\theta - \theta_0 = \frac{1}{2}(\omega_z + \omega_{0z})t$

Position ➔ angle, velocity ➔ angular velocity, acceleration ➔ angular acceleration.

Example: **Rotation with constant angular acceleration**

You have just finished watching a movie on DVD and the disc is slowing to a stop. The angular velocity of the disc at $t = 0$ is 27.5 rad/s and its angular acceleration is a constant -10.0 rad/s^2. A line PQ on the surface of the disc lies along the $+x$-axis at $t = 0$ (Fig. 9.8). (a) What is the disc's angular velocity at $t = 0.300$ s? (b) What angle does the line PQ make with the $+x$-axis at this time?

9.8 A line PQ on a rotating DVD at $t = 0$.



EXECUTE: (a) From Eq. (9.7), at $t = 0.300$ s we have

$$\boxed{\omega_z = \omega_{0z} + \alpha_z t} = 27.5 \text{ rad/s} + (-10.0 \text{ rad/s}^2)(0.300 \text{ s})$$
$$= 24.5 \text{ rad/s}$$

(b) From Eq. (9.11),

$$\boxed{\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2}$$

$$= 0 + (27.5\ \mathrm{rad/s})(0.300\ \mathrm{s}) + \frac{1}{2}(-10.0\ \mathrm{rad/s^2})(0.300\ \mathrm{s})^2$$

$$= 7.80\ \mathrm{rad} = 7.80\ \mathrm{rad}\left(\frac{1\ \mathrm{rev}}{2\pi\ \mathrm{rad}}\right) = 1.24\ \mathrm{rev}$$

The DVD has turned through one complete revolution plus an additional 0.24 revolution—that is, through an additional angle of $(0.24\ \mathrm{rev})(360°/\mathrm{rev}) = 87°$. Hence the line PQ is at an angle of 87° with the $+x$-axis.

Tangential and radial acceleration

When a particle is performing a circular motion, its motion is two dimension

Its position and velocity are described by two dimension vectors

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

$$\vec{v}(t) = v_x(t)\hat{i} + v_y(t)\hat{j}$$

Its acceleration is also a two-dimensional vector, we can resolve it into two components: the tangential and the radial component

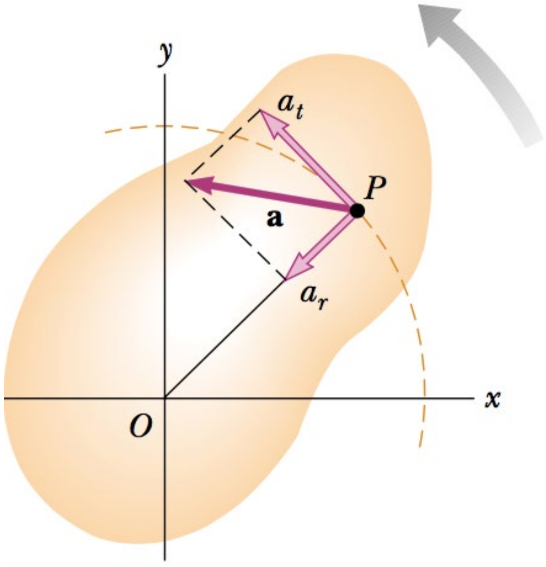
Radial and tangential acceleration

- For rotation around a fixed axis, all points are undergo a circular motion → centripetal acceleration

- This is the radial acceleration perpendicular to the tangent and points towards the center (center lies on the axis of rotation)

- Radial acceleration is not zero as long as angular velocity is not zero, even if it is a constant

- Tangential acceleration is related to the change of the length of the velocity vector (speed), which is tangential to the circle as the motion is circular.



- When the magnitude of velocity (speed) is constant, tangential acceleration is zero. It is not zero only when there is angular acceleration

Relating translational and rotational quantities

- For a point at distance r from the axis of rotation:

  its tangential linear speed is $v = r\omega$

  its centripetal (radial) acceleration is $a_{rad} = v^2/r = r\omega^2$

- Its tangential acceleration:

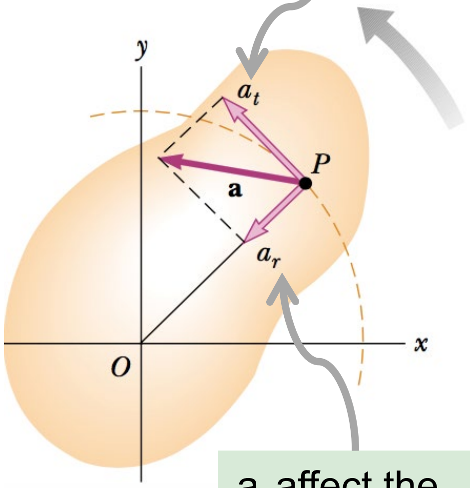
$$a_t = \frac{\Delta v}{\Delta t} = r\frac{\Delta \omega}{\Delta t} = r\alpha$$

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

$$\boxed{a_t = r\alpha}$$

$$\boxed{a_r = \frac{v^2}{r} = r\omega^2}$$

a_t affect the magnitude of velocity



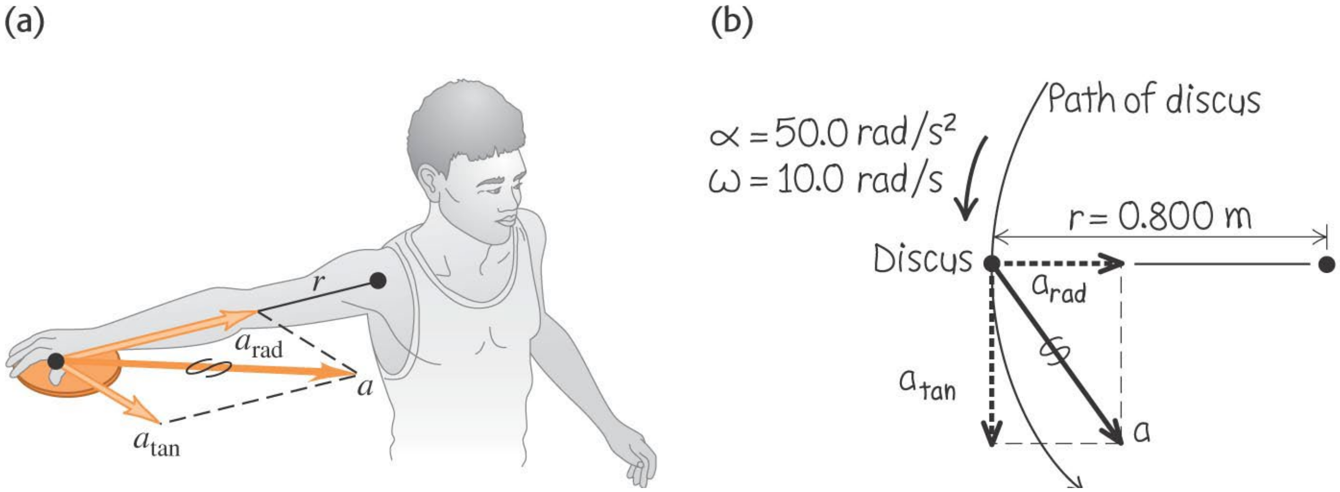
a_r affect the direction

- This example shows you how to use the concept of angular acceleration and relate the tangential acceleration to angular acceleration, relating linear and rotational kinematic quantities.

It also show how to calculate the radial acceleration from angular speed.

Example 9.4 Throwing a discus

An athlete whirls a discus in a circle of radius 80.0 cm. At a certain instant, the athlete is rotating at 10.0 rad/s and the angular speed is increasing at 50.0 rad/s^2. At this instant, find the tangential and centripetal components of the acceleration of the discus and the magnitude of the acceleration.



SOLUTION

IDENTIFY and SET UP: We treat the discus as a particle traveling in a circular path (Fig. 9.12a), so we can use the ideas developed in this section. We are given $r = 0.800$ m, $\omega = 10.0$ rad/s, and $\alpha = 50.0$ rad/s^2 (Fig. 9.12b). We'll use Eqs. (9.14) and (9.15), respectively, to find the acceleration components $a_{\tan}$ and a_{rad}; we'll then find the magnitude a using the Pythagorean theorem.

EXECUTE: From Eqs. (9.14) and (9.15),

$$\boxed{\begin{aligned} a_{\mathrm{tan}} &= r\alpha = (0.800\ \mathrm{m})(50.0\ \mathrm{rad/s^2}) = 40.0\ \mathrm{m/s^2} \\ a_{\mathrm{rad}} &= \omega^2 r = (10.0\ \mathrm{rad/s})^2(0.800\ \mathrm{m}) = 80.0\ \mathrm{m/s^2} \end{aligned}}$$

Then

$$a = \sqrt{a_{\mathrm{tan}}{}^2 + a_{\mathrm{rad}}{}^2} = 89.4\ \mathrm{m/s^2}$$

EVALUATE: Note that we dropped the unit “radian” from our results for a_{tan}, a_{rad}, and a. We can do this because “radian” is a dimensionless quantity. Can you show that if the angular speed doubles to $20.0\ \mathrm{rad/s}$ while α remains the same, the acceleration magnitude a increases to $322\ \mathrm{m/s^2}$?

Correspondence between translational and rotational quantities

- We have angular velocity equivalent to velocity
- Angular acceleration equivalent to acceleration

$$\vec{r} \Leftrightarrow \theta$$

$$\vec{v} \Leftrightarrow \omega$$

$$\vec{a}_t \Leftrightarrow \alpha$$

- Any quantity equivalent to a mass in rotational motion??
- Answer: moment of inertia *I*

$$m \Leftrightarrow I$$

- For this correspondence, let us consider the expression of kinetic energy of rotation next

Kinetic Energy of Rigid Body in Rotation

Kinetic energy in terms of Angular velocity

In rotation motion, the motion of all the particles in a rigid body can be described using the angular quantities: angles, angular velocity, angular acceleration.

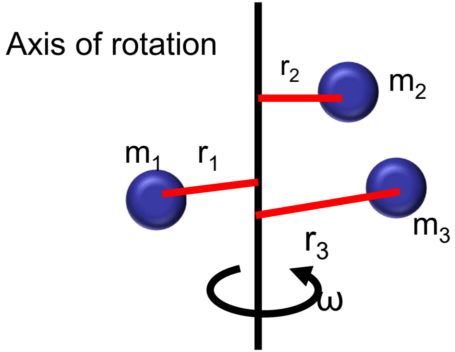
We want to express kinetic energy, Newton's law in terms of the angular quantities.

This is very convenient, different particles of the object has different translational velocities, but the angular velocity is the same for everyone in a rigid body.

Rotational kinetic energy (Moment of Inertia)

- In analogy with translational kinetic energy formula: $KE = (1/2)mv^2$, in terms of angular velocity ω, the rotational kinetic energy of a rigid body can be written as

$$K.E. = \frac{1}{2} I\omega^2.$$



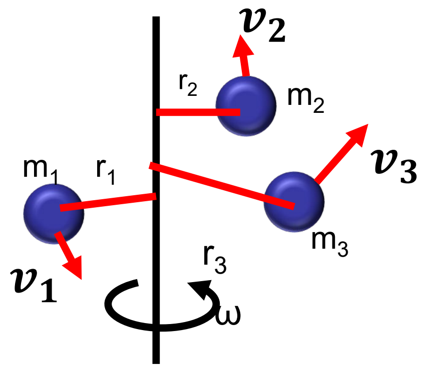
- I is the moment of inertia of a set of particles :

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots = \Sigma m_i r_i^2$$

- r_i is the perpendicular distance from the axis of rotation of the object i. (radius of the circle of motion of object i)

- Derivation in next slide

Rotational Kinetic Energy of a rigid body



Total rotational kinetic energy (RE) of rotating rigid body

= sum of the KE of all particles of the rigid body

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \ldots + \frac{1}{2}m_nv_n^2 = \sum_{i=1}^{n}\frac{1}{2}m_iv_i^2$$

r_i is the perpendicular distance of the particle from the axis of rotation, ω is the angular velocity, and $\boldsymbol{v_i = r_i\omega}$

$\boldsymbol{v_i = r_i\omega}$

v_i for different particles can be different, ω is the same for all particles

$$\rightarrow \boxed{\text{Total RE}} = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\,\omega^2 + \ldots + \frac{1}{2}m_nr_n^2\omega^2$$

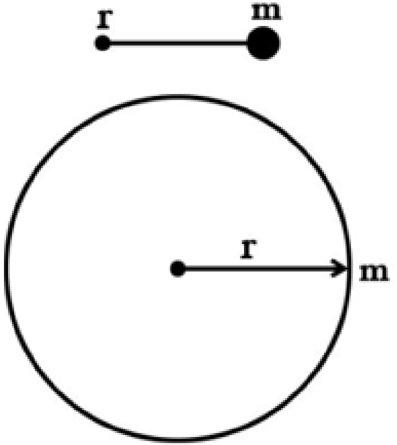
$$= \sum_{i=1}^{n}\frac{1}{2}m_ir_i^2\omega^2 = \boxed{\frac{1}{2}I\omega^2} \quad with \quad I = \sum_{i=1}^{n}m_ir_i^2$$

$$\text{KE} = \frac{1}{2}\boxed{m}\,\boxed{v^2}$$

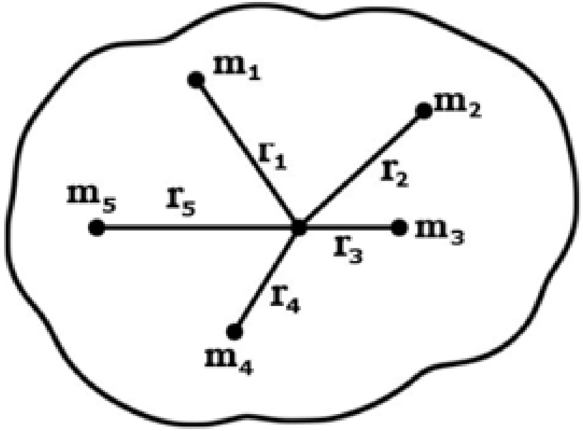
$$\text{RE} = \frac{1}{2}\boxed{I}\,\boxed{\omega^2}$$

Calculation of moment of inertia

One particle



$$I = mr^2$$



$$I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2 + m_5 r_5^2$$

Five particles

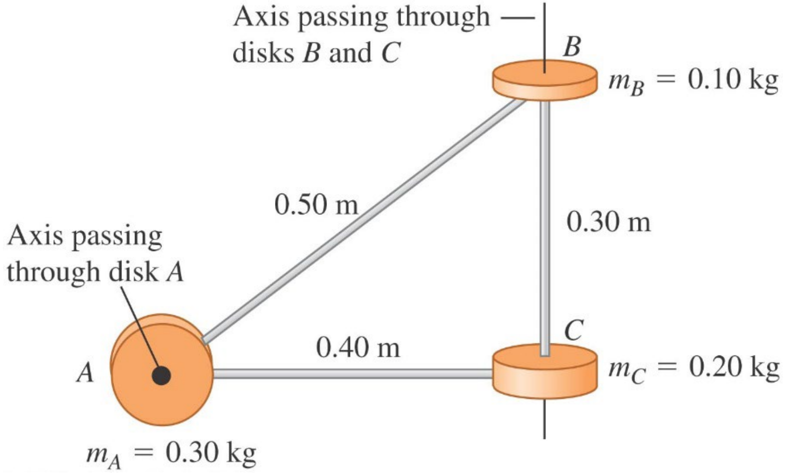
Next is an example, showing how to calculate the moment of inertia of a system of particle.

Moment of inertia depends on the axis of rotation

Different axis gives different moment of inertia

Example 9.6 Moments of inertia for different rotation axes

A machine part (Fig. 9.15) consists of three disks linked by lightweight struts. (a) What is this body's moment of inertia about an axis through the center of disk A, perpendicular to the plane of the diagram? (b) What is its moment of inertia about an axis through the centers of disks B and C? (c) What is the body's kinetic energy if it rotates about the axis through A with angular speed $\omega = 4.0\ \mathrm{rad/s}$?



Treat discs as point masses

SOLUTION

IDENTIFY and SET UP: We'll consider the disks as massive particles located at the centers of the disks, and consider the struts as massless. In parts (a) and (b), we'll use Eq. (9.16) to find the moments of inertia. Given the moment of inertia about axis A, we'll use Eq. (9.17) in part (c) to find the rotational kinetic energy.

EXECUTE: (a) The particle at point A lies *on* the axis through A, so its distance r from the axis is zero and it contributes nothing to the moment of inertia. Hence only B and C contribute, and Eq. (9.16) gives

$$I_A = \sum m_i r_i^2 = (0.10\ \text{kg})(0.50\ \text{m})^2 + (0.20\ \text{kg})(0.40\ \text{m})^2$$
$$= 0.057\ \text{kg} \cdot \text{m}^2$$

(b) The particles at B and C both lie on axis BC, so neither particle contributes to the moment of inertia. Hence only A contributes:

$$I_{BC} = \sum m_i r_i^2 = (0.30\ \text{kg})(0.40\ \text{m})^2 = 0.048\ \text{kg} \cdot \text{m}^2$$

(c) From Eq. (9.17),

$$K_A = \tfrac{1}{2} I_A \omega^2 = \tfrac{1}{2}(0.057\ \text{kg} \cdot \text{m}^2)(4.0\ \text{rad/s})^2 = 0.46\ \text{J}$$

EVALUATE: The moment of inertia about axis A is greater than that about axis BC. Hence of the two axes it's easier to make the machine part rotate about axis BC.

CAUTION **Moment of inertia depends on the choice of axis** The results of parts (a) and (b) of Example 9.6 show that the moment of inertia of a body depends on the location and orientation of the axis. It's not enough to just say, "The moment of inertia of this body is $0.048 \text{ kg} \cdot \text{m}^2$." We have to be specific and say, "The moment of inertia of this body *about the axis through B and C* is $0.048 \text{ kg} \cdot \text{m}^2$."

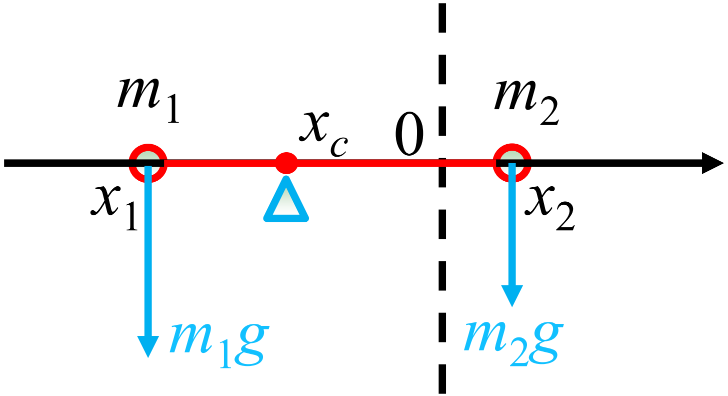
Center of mass

- Consider a rigid body of two point masses, as shown.

We can treat them as one-dim.

- Center of mass (CM)of the rigid body is

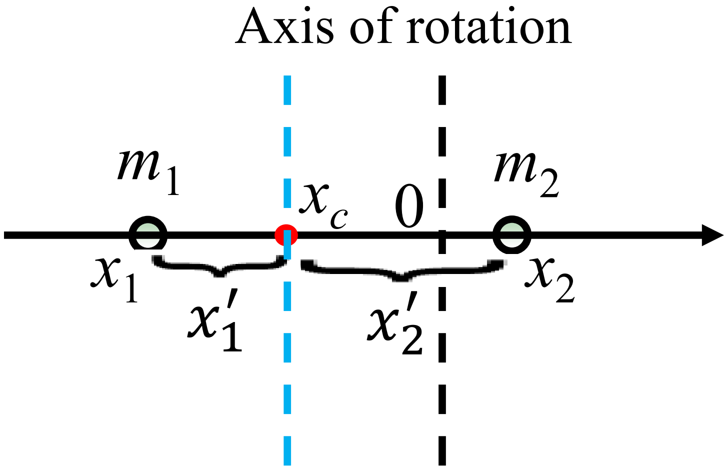
$$x_c = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_1}$$



- Gravity is balanced at this point (center of gravity)!

Center of mass and Parallel axis theorem for moment of inertia

$$x_c = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_1}$$



Conversion to CM: $x_1' = x_1 - x_c$

$x_2' = x_2 - x_c$

- Moment of inertial in CM:

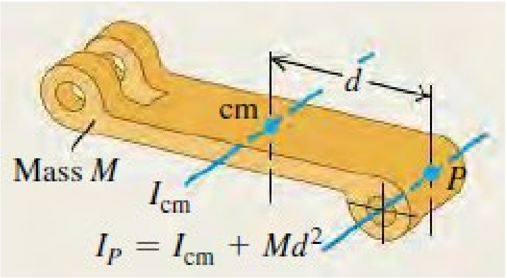
$$I_{cm} = m_1 x_1'^2 + m_2 x_2'^2 = m_1 (x_1 - x_c)^2 + m_2 (x_2 - x_c)^2$$

$$= m_1 x_1^2 + m_2 x_2^2 - 2(m_1 x_1 + m_2 x_2) x_c + (m_1 + m_2) x_c^2 = I - (m_1 + m_2) x_c^2$$

$$\rightarrow \; I = I_{cm} + (m_1 + m_2) x_c^2$$

- In general:

$I = I_{cm} + Md^2$ ← Parallel axis theorem



Moments of inertia of some common bodies

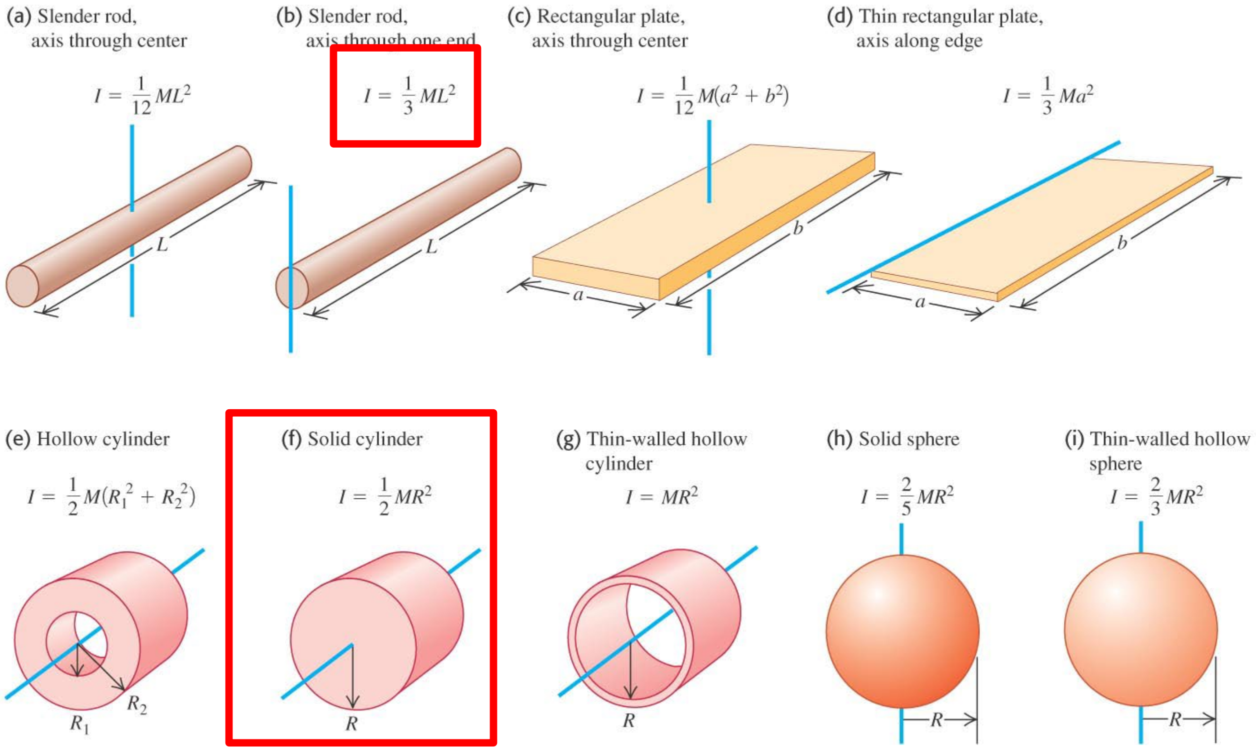
- Table 9.2 gives the moments of inertia of various bodies

•Parallel axis theorem:

$$I_{side} = I_{cm} + M\left(\frac{L}{2}\right)^2$$

$$= \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2$$

$$= \frac{1}{3}ML^2$$



Disc is same

You don’t need to learn how to calculate moment of inertia of objects with complicated shapes as it involves multi-variable integration which is beyond this course

You need to understand the concepts and know how to calculate the moment of inertia of a system of particles.

Moment of inertia is usually given in a Problem.

Work done and rotational kinetic energy

This next example show you how to use principle of conservation of work and energy to calculate the angular speed of a rotating object.

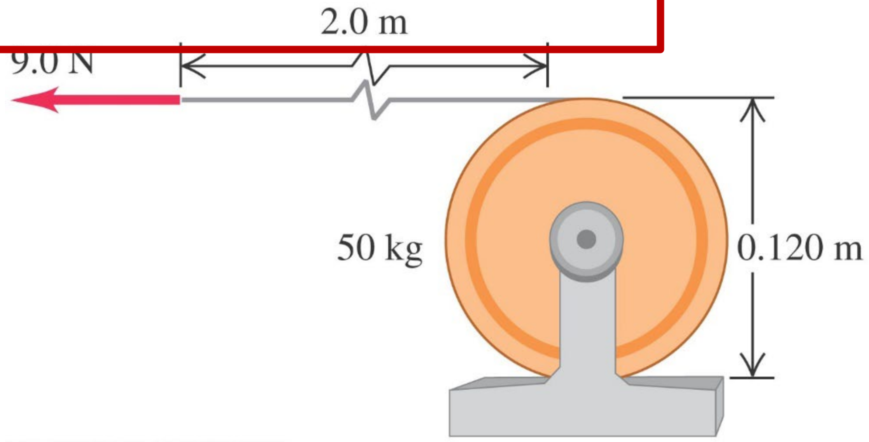
Rotational kinetic energy is the sum of the kinetic energy of the particles in the rigid body, so work done can change the rotational kinetic energy.

Work done=change in kinetic energy (work energy theorem in chapter 4) also applies to rotation motion

An unwinding cable

Example 9.7 An unwinding cable I

We wrap a light, nonstretching cable around a solid cylinder of mass 50 kg and diameter 0.120 m, which rotates in frictionless bearings about a stationary horizontal axis (Fig. 9.16). We pull the free end of the cable with a constant 9.0-N force for a distance of 2.0 m; it turns the cylinder as it unwinds without slipping. The cylinder is initially at rest. Find its final angular speed and the final speed of the cable.



SOLUTION

IDENTIFY: We'll solve this problem using energy methods. We'll assume that the cable is massless, so only the cylinder has kinetic energy. There are no changes in gravitational potential energy. There is friction between the cable and the cylinder, but because the cable doesn't slip, there is no motion of the cable relative to the cylinder and no mechanical energy is lost in frictional work. Because the cable is massless, the force that the cable exerts on the cylinder rim is equal to the applied force F.

SET UP: Point 1 is when the cable begins to move. The cylinder starts at rest, so $K_1 = 0$. Point 2 is when the cable has moved a distance $s = 2.0$ m and the cylinder has kinetic energy $K_2 = \frac{1}{2}I\omega^2$. One of our target variables is ω; the other is the speed of the cable at point 2, which is equal to the tangential speed v of the cylinder at that point. We'll use Eq. (9.13) to find v from ω.

EXECUTE: The work done on the cylinder is $W_{\text{other}} = Fs = (9.0\text{ N})(2.0\text{ m}) = 18\text{ J}$. From Table 9.2 the moment of inertia is

$$I = \tfrac{1}{2}mR^2 = \tfrac{1}{2}(50\text{ kg})(0.060\text{ m})^2 = 0.090\text{ kg}\cdot\text{m}^2$$

(The radius R is half the diameter.) From Eq. (7.14), $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$, so

$$0 + 0 + W_{\text{other}} = \tfrac{1}{2}I\omega^2 + 0$$

$$\omega = \sqrt{\frac{2W_{\text{other}}}{I}} = \sqrt{\frac{2(18\text{ J})}{0.090\text{ kg}\cdot\text{m}^2}} = 20\text{ rad/s}$$

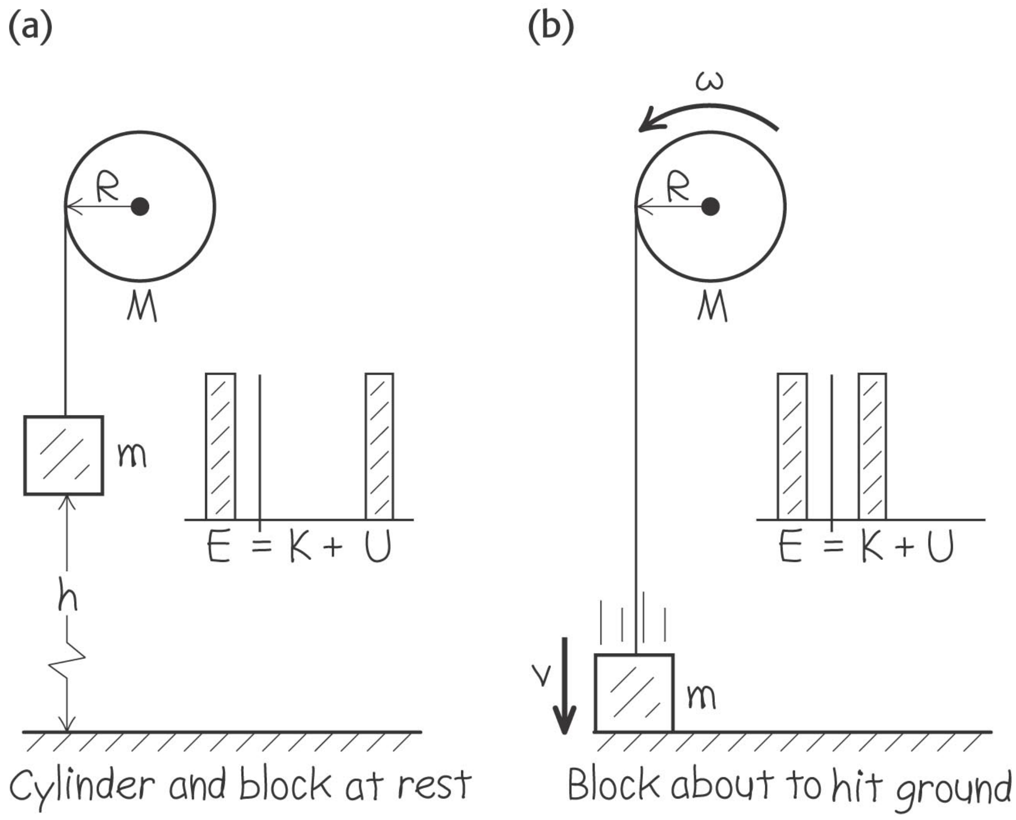
From Eq. (9.13), the final tangential speed of the cylinder, and hence the final speed of the cable, is

$$v = R\omega = (0.060\text{ m})(20\text{ rad/s}) = 1.2\text{ m/s}$$

EVALUATE: If the cable mass is not negligible, some of the 18 J of work would go into the kinetic energy of the cable. Then the cylinder would have less kinetic energy and a lower angular speed than we calculated here.

More on an unwinding cable

- Follow Example 9.8 using Figure 9.17 below.



Example 9.8 An unwinding cable II

We wrap a light, nonstretching cable around a solid cylinder with mass M and radius R. The cylinder rotates with negligible friction about a stationary horizontal axis. We tie the free end of the cable to a block of mass m and release the block from rest at a distance h above the floor. As the block falls, the cable unwinds without stretching or slipping. Find expressions for the speed of the falling block and the angular speed of the cylinder as the block strikes the floor.

SOLUTION

IDENTIFY: As in Example 9.7, the cable doesn't slip and so friction does no work. We assume that the cable is massless, so that the forces it exerts on the cylinder and the block have equal magnitudes. At its upper end the force and displacement are in the same direction, and at its lower end they are in opposite directions, so the cable does no *net* work and $W_{other} = 0$. Only gravity does work, and mechanical energy is conserved.

SET UP: Figure 9.17a shows the situation before the block begins to fall (point 1). The initial kinetic energy is $K_1 = 0$. We take the gravitational potential energy to be zero when the block is at floor level (point 2), so $U_1 = mgh$ and $U_2 = 0$. (We ignore the gravitational potential energy for the rotating cylinder, since its height doesn't change.) Just before the block hits the floor (Fig. 9.17b), both the block and the cylinder have kinetic energy, so

$$K_2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

The moment of inertia of the cylinder is $I = \frac{1}{2}MR^2$. Also, $v = R\omega$ since the speed of the falling block must be equal to the tangential speed at the outer surface of the cylinder.

EXECUTE: We use our expressions for K_1, U_1, K_2, and U_2 and the relationship $\omega = v/R$ in Eq. (7.4), $K_1 + U_1 = K_2 + U_2$, and solve for v:

$$0 + mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 + 0 = \frac{1}{2}\left(m + \frac{1}{2}M\right)v^2$$

$$v = \sqrt{\frac{2gh}{1 + M/2m}}$$

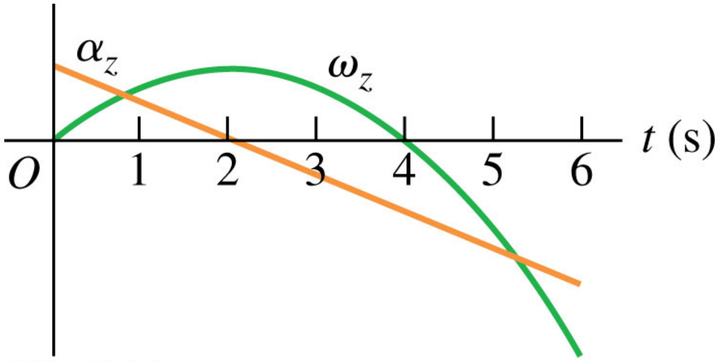
The final angular speed of the cylinder is $\omega = v/R$.

EVALUATE: When M is much larger than m, v is very small; when M is much smaller than m, v is nearly equal to $\sqrt{2gh}$, the speed of a body that falls freely from height h. Both of these results are as we would expect.

Think Questions

TQ5A.1

The graph shows the angular velocity and angular acceleration versus time for a rotating body. At which of the following times is the rotation speeding up at the greatest rate?



A. $t = 1$ s

B. $t = 2$ s

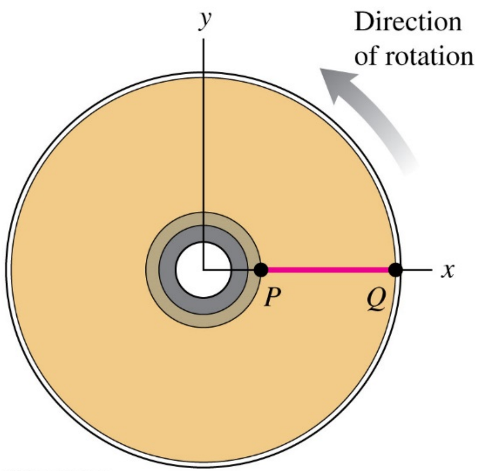
C. $t = 3$ s

D. $t = 4$ s

E. $t = 5$ s

TQ 5A.2

A DVD is rotating with an ever-increasing speed. How do the centripetal acceleration a_{rad} and tangential acceleration a_{tan} compare at points P and Q?



A. P and Q have the same a_{rad} and a_{tan}.

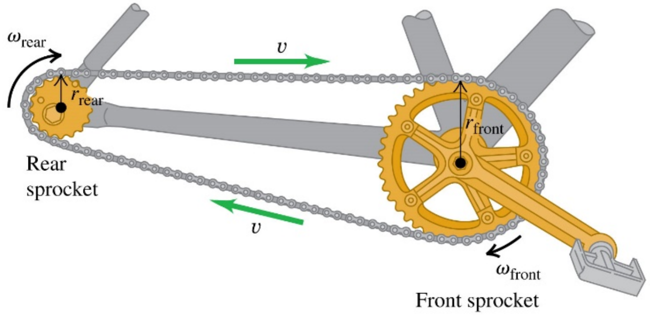
B. Q has a greater a_{rad} and a greater a_{tan} than P.

C. Q has a smaller a_{rad} and a greater a_{tan} than P.

D. P and Q have the same a_{rad}, but Q has a greater a_{tan} than P.

Tq 5A.3

Compared to a gear tooth on the rear sprocket (on the left, of small radius) of a bicycle, a gear tooth on the *front* sprocket (on the right, of large radius) has



A. a faster linear speed and a faster angular speed.

B. the same linear speed and a faster angular speed.

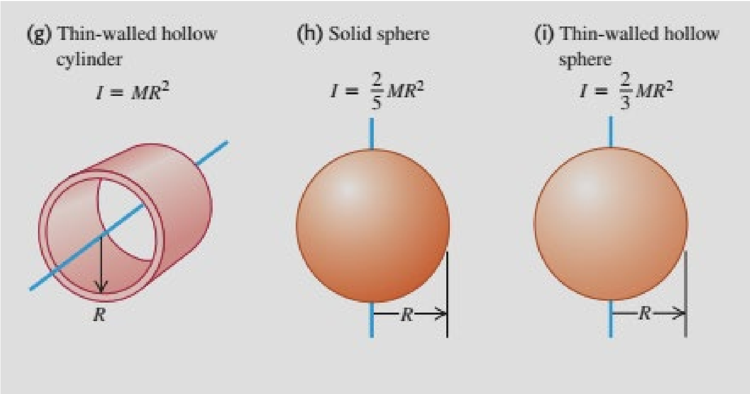
C. a slower linear speed and the same angular speed.

D. the same linear speed and a slower angular speed.

E. none of the above

TQ5A.4

The three objects shown here all have the same mass M and radius R. Each object is rotating about its axis of symmetry (shown in blue). All three objects have the *same* rotational kinetic energy. Which one is rotating *fastest*?



A. thin-walled hollow cylinder

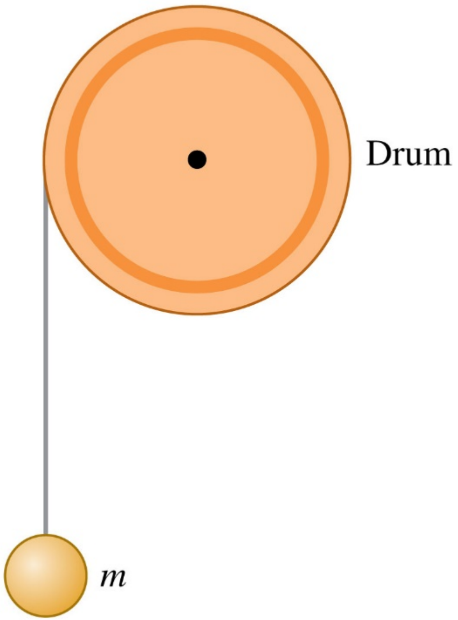
B. solid sphere

C. thin-walled hollow sphere

D. two or more of these are tied for fastest

TQ5A.5

A thin, very light wire is wrapped around a drum that is free to rotate. The free end of the wire is attached to a ball of mass m. The drum has the same mass m. Its radius is R and its moment of inertia is $I = (1/2)mR^2$. As the ball falls, the drum spins.



At an instant that the ball has translational kinetic energy K, the drum has rotational kinetic energy

A. K.

B. $2K$.

C. $K/2$.

D. none of these