Chapter 3. Discrete Random Variables

3.1 Introduction and Discrete Random Variables

If two fair dice were tossed, a probability was assigned to each of the 36 possible pairs:

P((1,1))=1/36, P((2,4))=1/36, P((1,3))=1/36 … .

If we are interested in the sum of two dice, then it makes sense to replace the 36-member sample space of (*x*,*y*) pairs with 11-member set of all possible two-dice sums, $S' = \{x + y : (x, y) \in S\} = \{2, 3, \ldots, 12\}$

$$\underbrace{X=3}_{} \Leftrightarrow \underbrace{\{(1,2),(2,1)\}}_{}$$

Definition of random variable:

A *random variable (r.v.)* is a real-valued function whose domain is the sample space S. Usually denoted by uppercase letters, often X, Y or Z.

A random variable that can take on a finite or at most countably infinite number of values (e.g. integers) is said to be *discrete*;

A random variable that can take uncountably infinite number of values (e.g. interval), either bounded or unbounded, is said to be *continuous*.

For a *discrete* random variable *X*, its *probability mass function* (pmf) gives the probability that *X* equals *a*:

$$p(a) = P\{X = a\} = P(\{s \in S : X(s) = a\})$$

Suppose X must assume one of the values $x_1, x_2, \ldots,$ then

$$p(x_i) \geq 0, \text{ and } p(x) = 0 \text{ for all other values of } x.$$

$$\sum_i p(x_i) = 1$$

$p(2) = \frac{1}{36}$

$p(3) = \frac{2}{36}$

Example : Independent trials, consisting of the flipping of a coin having probability *p* of coming up heads, are continually performed until either a head occurs or a total of *n* flips is made. Let *X* be the random variable that denotes the number of times the coin is flipped. The probability mass function for *X* is

$$P\{X = 1\} = P\{H\} = p$$
$$P\{X = 2\} = P\{(T, H)\} = (1-p)p$$
$$P\{X = 3\} = P\{(T, T, H)\} = (1-p)^2 p$$
$$P\{X = n-1\} = P\{(\underbrace{T, T, \ldots, T}_{n-2}, H)\} = (1-p)^{n-2} p$$

$(1-p)^n$ $(1-p)^{n-1} \cdot p$

$$P\{X = n\} = P\{(\underbrace{T, T, \ldots, T}_{n-1}, T), (\underbrace{T, T, \ldots, T}_{n-1}, H)\} = (1-p)^{n-1}$$

$$P(\cup_{i=1}^{n}\{X = i\}) = \sum_{i=1}^{n} P\{X = i\} = \sum_{i=1}^{n-1} p(1-p)^{i-1} + (1-p)^{n-1} = 1$$

Example : Three balls are are chosen from an urn containing 3 white, 3 red, and 5 black balls. Suppose we win $1 for each white ball selected and lose one for each red selected. What is the probability that we win some money?

3 black white red black

Solution: Let X denote our total winnings, taking values $0, \pm 1, \pm 2, \pm 3$

$$P\{X=0\} = \frac{\binom{5}{3} + \binom{3}{1}\binom{3}{1}\binom{5}{1}}{\binom{11}{3}} = 1/3$$

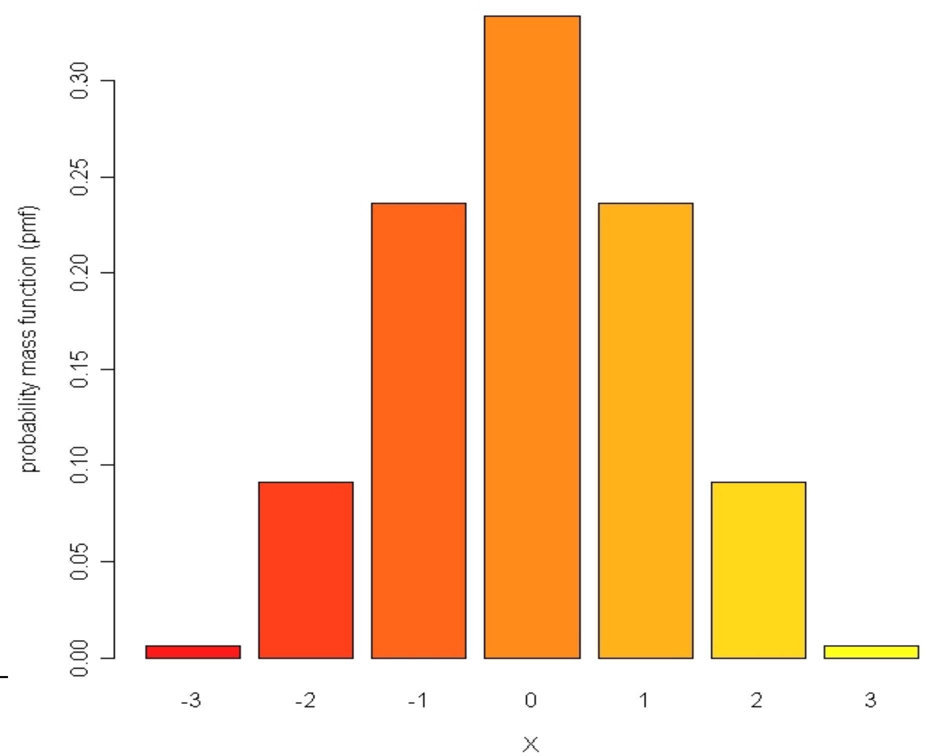
$$P\{X=1\} = P\{X=-1\} = \frac{\binom{3}{1}\binom{5}{2} + \binom{3}{2}\binom{3}{1}}{\binom{11}{3}} = 39/165$$

$$P\{X=2\} = P\{X=-2\} = \frac{\binom{3}{2}\binom{5}{1}}{\binom{11}{3}} = 15/165$$

$$P\{X=3\} = P\{X=-3\} = \frac{\binom{3}{3}}{\binom{11}{3}} = 1/165$$

$$P\{X \in \{1,2,3\}\} = \frac{39+15+1}{165} = \frac{1}{3}$$

It is often instructive to present the probability mass function in a graphical format plotting pmf against the random variable.



X take value 0, ±1, ±2, ±3

$p(X \le -2) = p(-3) + p(-2)$

Definition of Cumulative Distribution Function (CDF)

For a random variable *X*, the function *F* defined by

$\boxed{F(a) = P(X \le a)} = \sum_{\text{all } x \le a} p(x)$, is called the cumulative distribution function(cdf). The pmf can be determined uniquely from the cdf and vice versa.

$p(X=-2) = P(X \le -2) - P(X \le -3)$

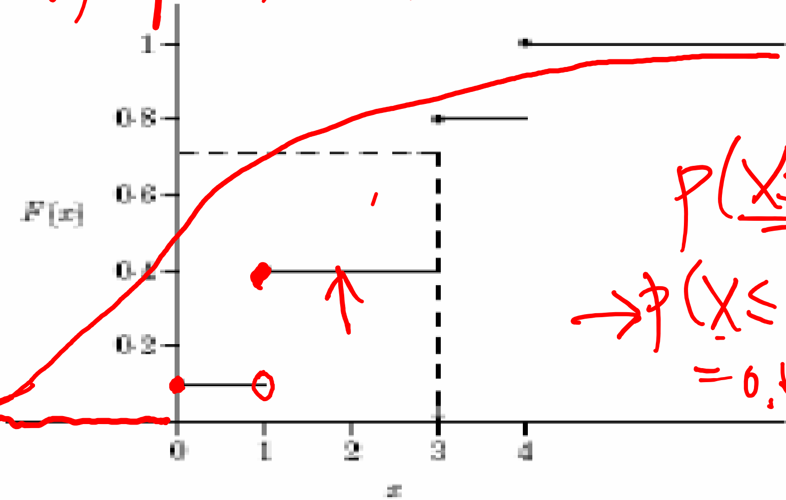
CDF

Example. Suppose that *X* takes the values 0, 1, 3 and 4 with respective probabilities 0.1, 0.3, 0.4 and 0.2. Sketch the distribution function.

$p(X \le 0) = p(X=0) = 0.1,$

$p(X \le a)$

$p(X \le -0.1) = 0$



$p(X \le 0.5)$

$\rightarrow p(X \le 1) = p(X=0) + p(X=1)$
$= 0.4$

$p(a < X \le b) = p(X \le b) - p(X \le a)$

Properties of the Cumulative Distribution Function

All probability questions about *X* can be answered in terms of the cdf *F*. For example,

$$P(a < X \le b) = F(b) - F(a) \text{ for all } a < b$$

$P(X \le 0.1) \le P(X \le 0.2)$

Properties of CDF:

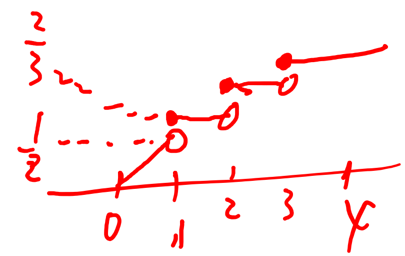
1. *F* is nondecreasing, that is, if $a < b$, then $F(a) \le F(b)$.
2. $\lim_{b \to \infty} F(b) = 1, \lim_{b \to -\infty} F(b) = 0$
3. *F* is right continuous. That is, for any *b* and any decreasing sequence $\{b_n\}$ converging to *b*, $\lim_{n \to \infty} F(b_n) = F(b)$

Remark: Any function that satisfies the above 3 properties is a CDF of some r.v. (not necessarily discrete)

$P(X \le 3) = F(3) = 1$ $\quad P(X \le 2.9) = \frac{11}{12}$

$P(X \le 2.99) = \frac{11}{12}$

Example: The distribution function (cdf) of the random variable X is given by



$$F\left(\tfrac{1}{2}\right) = \tfrac{1}{4} \Longleftarrow F(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \le x < 1 \\ 2/3 & 1 \le x < 2 \\ 11/12 & 2 \le x < 3 \\ 1 & 3 \le x \end{cases}$$

$P(X \le 1) - P(X < 1) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

Compute

$(a) P(X < 3)$; $(b) P(X = 1)$; $(c) P(X > 1/2)$; $(d) P(2 < X \le 4)$; $(e) P(2 \le X < 4)$

$1 - P(X \le \frac{1}{2})$

$F(4) - F(2)$

$P(X<4) - P(X<2)$

$= 1 - \frac{11}{12}$

Solution: (a) 11/12; (b) 1/6; (c) 3/4; (d) 1/12; (e) 1/3

1 2 3

$P(X \le a) = F(a)$ $\quad \boxed{P(X<a) = \lim_{a_n \nearrow a} F(a_n) = F(a-)}$

Expected Value

The *expected value*, *or* *expectation*, *or (informally)* *mean,* refers to the "average" value of a random variable. For a discrete random variable, the expected value of a random variable *X* is defined as

$$E[X] = \sum_{x:p(x)>0} x p(x) \qquad \sum p(x) = 1$$

Example: Find E[X] where X is the outcome when we roll a fair die.

Solution: Since $p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = 1/6$

$$E[X] = 1(\tfrac{1}{6}) + 2(\tfrac{1}{6}) + 3(\tfrac{1}{6}) + 4(\tfrac{1}{6}) + 5(\tfrac{1}{6}) + 6(\tfrac{1}{6}) = 7/2$$

Example: We say that I is an *indicator variable* for the event A if

$$I = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A^c \text{ occurs} \end{cases}$$

$p(X=1) = P(A)$

Since $p(1) = P(A), p(0) = 1 - P(A),\quad E[I] = \underline{0 \cdot p(0)} + 1 \cdot p(1) = P(A)$

That is, Expectation of an indicator variable for the event is equal to the probability that the event occurs

Expectation of Function of a Random Variable

Example: Let *X* denote a random variable that takes on the values *-1, 0, 1* with respective probabilities

$$P(X=-1)=0.2, P(X=0)=0.5, P(X=1)=0.3$$

Find the expectation of X^2

Solution: Let $Y=X^2$ and the pmf of Y is given by

$$P(Y=1)=P(X=-1)+P(X=1)=0.5$$
$$P(Y=0)=P(X=0)=0.5$$

Hence

$$E[X^2]=E[Y]=1(.5)+0(.5)=0.5$$

Proposition: If *X* is a discrete random variable that takes on one of the values $x_i,\ i \geq 1$, with respective probabilities $p(x_i)$, then for any real valued function *g*,

$$E[g(X)] = \sum_i g(x_i)p(x_i)$$

Applying the proposition to Example,

$$E[X^2] = (-1)^2(.2) + 0^2(.5) + 1^2(.3)$$

Corollary : If *a* and *b* are constants, then

E[aX+b]=aE[X]+b

X 3

2X 6

Proof:

$$
\begin{aligned}
E[aX+b] &= \sum_x (ax+b)p(x) \\
&= a\sum_x xp(x) + b\sum_x p(x) \\
&= aE[X]+b
\end{aligned}
$$

Note: more generally

$$E[aX+bY] = aE[X]+bE[Y] \text{ for any two r.v.'s } X \text{ and } Y.$$

EXY = EXEY

Variance

Another useful summary of a random variable's probability mass function is its "spread" or "variation", e.g., important in finance where investors want investments not only with good expected returns but also not to be too risky.

Example : The following two random variables have expected value 0 but very different spreads:

Y=0 with probability 1

Z=-100 with probability .5, *Z=100* with probability .5

Definition: A commonly used measure of spread is the *variance* of a random variable, defined as,

$Var(X) = \underline{E[(X-\mu)^2]}$, where $\mu = E(X)$

$V(X)$

$(X-\mu)^2 = 0 \;,\; X=\mu$

(μ is often used to denote mean, and σ^2 is often used to denote variance)

An alternative expression for *Var(X):*

$$
\begin{aligned}
Var(X) &= E[(X-\mu)^2] \\
&= E[X^2 - 2\mu X + \mu^2] \\
&= E[X^2] - 2\mu E[X] + \mu^2 \\
&= E[X^2] - \mu^2
\end{aligned}
$$

That is, $Var(X) = E[X^2] - (E[X])^2$

Example (continued). Compute the variances of *Y, Z*.

Since $E(Y) = E(Z) = 0 \quad Var(Y) = E(Y^2) = 0$

$$Var(Z) = E(Z^2) = (-100)^2(.5) + (100)^2(.5) = 10,000$$

Example : Compute *Var(X)* if *X* represents the outcome when a fair die is rolled.

Solution: it was shown previously that $E[X] = 7/2$

Also,

$$E[X^2] = 1^2(\tfrac{1}{6}) + 2^2(\tfrac{1}{6}) + 3^2(\tfrac{1}{6}) + 4^2(\tfrac{1}{6}) + 5^2(\tfrac{1}{6}) + 6^2(\tfrac{1}{6}) = \frac{91}{6}$$

Hence

$E[(X-\mu)^2]$

$$Var(X) = \frac{91}{6} - (\tfrac{7}{2})^2 = \frac{35}{12}$$

$/ = (1-\frac{7}{2})^2 \cdot \frac{1}{6} + (2-\frac{7}{2})^2 \frac{1}{6} + (3-\frac{7}{2})^2 \frac{1}{6}$

$+ \cdot =$

$Var(aX+bY) = a^2 Var(X) + b^2 Var(Y)$ ✗

Remarks on Variance:

(1) A useful identity is that for any constants *a* and *b*,

$$Var(aX+b) = a^2 Var(X)$$

$E(aX+b)$

Proof:
$$\begin{aligned} Var(aX+b) &= E[(aX+b-(a\mu-b))^2] \\ &= E[a^2(X-\mu)^2] \\ &= a^2 E[(X-\mu)^2] = a^2 Var(X) \end{aligned}$$

(2) The square root of *Var(X)* is called the standard deviation of X and usually denoted by *sd(X)* or *std(X)*

(3) The variance of a constant is 0

(4) *Var*(.) is not a linear function, you can see this from (1). Also, $Var(X+Y) \neq Var(X) + Var(Y)$

Example: The manager of a stockroom in a factory knows from his study of records that the daily demand for a certain tool has the following probability distribution:

Demand	0	1	2
Probability	0.1	0.5	0.4

Letting *X* denote the daily demand, find *E(X)* and *Var(X)*

Solution:

$$E(X) = 0 \cdot 0.1 + 1 \cdot 0.5 + 2 \cdot 0.4 = 1.3$$
$$Var(X) = E(X - 1.3)^2 = (0-1.3)^2 \cdot 0.1 + (1-1.3)^2 \cdot 0.5 + (2-1.3)^2 \cdot 0.4 = 0.41$$

or

$$E(X^2) = 0^2 \cdot 0.1 + 1^2 \cdot 0.5 + 2^2 \cdot 0.4 = 2.1$$
$$Var(X) = E(X^2) - E(X)^2 = 2.1 - 1.3^2 = 0.41$$

Example: The manager of a stockroom in a factory knows from his study of records that the daily demand for a certain tool has the following probability distribution:

Demand	0	1	2
Probability	0.1	0.5	0.4

Suppose it costs \$10 each time the tool is used, find the mean and variance of the daily cost.

Solution: $E(10X) = 10E(X) = 13$

$$Var(10X) = 100Var(X) = 41$$

Definition: *n*-th moment of a random variable

The ***n*-th moment** of a random variable *X* is defined by the expectation $E[X^n]$, where *n* is a positive integer. In particular, the expectation *E[X]* of *X* is the first moment of *X*, and $E[X^2]$ is the second moment of *X*.

The ***n*-th central moment (moment about the mean)** of a random variable X is defined by the expectation

$$E[(X - E[X])^n]$$

$EX, \; EX^2, \; \underbrace{E(X-\mu)^2}$ $\quad \mu = EX$

3.2 Special discrete distributions

Bernoulli/Binomial

Suppose that a trial (experiment) whose outcome can be classified as either a *success* or a *failure*. Let the random variable *X* equal 1 when the outcome is a success and *X* equal 0 when the outcome is a failure. Then the pmf of *X is determined by one parameter p* is

$$\begin{aligned} p(1) &= P(X=1) = p \\ p(0) &= P(X=0) = 1-p \end{aligned}$$

A random variable X that has a pmf as above is denoted as $\boxed{X \sim Bernoulli(p)}$ (or $Ber(p)$) . And p is the parameter of the distribution.

$$E(X) = p,\ Var(X) = E(X^2) - (EX)^2 = p - p^2 = p(1-p)$$

$EX = 0 \cdot p(0) + 1 \cdot p(1)$

p

$X = 0$ or 1

$X^2 = X$

Binomial Random Variables:

Suppose that *n* independent trials are performed, each of which results in a success with probability *p* and a failure with probability *1-p*. If *X* represents the number of successes that occur in *n* trials, then *X* is said to be a binomial random variable with parameters *(n, p),* denoted as

$$X \sim Binomial(n,p) \text{ or } Bin(n,p)$$

$n=5,$

$P(X=3)=\binom{5}{3}p^3(1-p)^2$

Thus, a Ber(p) is same as Bin(1,p).

$SSFFS \leftarrow p^3(1-p)^2$

$\binom{5}{3}$

The pmf of a binomial random variable having parameters (n, p) is given by

$$P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$$

$, \quad i = 0, 1, 2, \cdots, n$

You can check the probabilities sum to 1 using binomial theorem:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

$$\sum_{i=0}^{n} p(i) = \sum_{i} \binom{n}{i} p^i (1-p)^{n-i} = [p + (1-p)]^n = 1$$

Example . Five fair coins are flipped. If the outcomes are assumed independent, find the probability mass function of the number of heads obtained.

Solution: *(n=5, p=1/2)*

$$P(X=0)=\binom{5}{0}(\tfrac{1}{2})^0(\tfrac{1}{2})^5=\frac{1}{32} \qquad P(X=1)=\binom{5}{1}(\tfrac{1}{2})^1(\tfrac{1}{2})^4=\frac{5}{32}$$

$$P(X=2)=\binom{5}{2}(\tfrac{1}{2})^2(\tfrac{1}{2})^3=\frac{10}{32} \qquad P(X=3)=\binom{5}{3}(\tfrac{1}{2})^3(\tfrac{1}{2})^2=\frac{10}{32}$$

$$P(X=4)=\binom{5}{4}(\tfrac{1}{2})^4(\tfrac{1}{2})^1=\frac{5}{32} \qquad P(X=5)=\binom{5}{5}(\tfrac{1}{2})^5 1(\tfrac{1}{2})^0=\frac{1}{32}$$

Example . Suppose that a particular trait (such as eye color or left handedness) of a person is classified on the basis of one pair of genes and suppose that *d* represents a dominant gene and *r* a recessive gene. Thus a person with *dd* genes is pure dominance, one with *rr* is pure recessive, and one with *rd* is hybrid. The pure dominance and the hybrid are alike in appearance. Children receive 1 gene from each parent. If, with respect to a particular trait, 2 hybrid parents have a total of 4 children, what is the probability that exactly 3 of the 4 children have the outward appearance of the dominant gene?

$n = 4$ $\quad P(\# \text{succ} = 3)$

Solution:

Success: dd or dr

X: # chil with appear

$father:$ $\quad mother:$

dr $\quad dr$

dd or dr

$X \sim Bin(4, 3/4)$

$P(X=3)$

So the answer is $Bin(3|4, 3/4) = \binom{4}{3}(\frac{3}{4})^3(\frac{1}{4})^1 = \frac{27}{64}$

genetypes for child: $\underbrace{dd, dr, rd}_{p = \frac{3}{4}}$, rr

Properties of binomial random variables:

Computation of mean and variance of binomial RV

If *X* is a binomial random variable with parameters *(n,p)*, then

$$E[X] = np$$
$$Var(X) = np(1-p)$$

$$EX = \sum_{i=0}^{n} i \cdot p(i) = \sum_{i=1}^{n} i \cdot \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} \cancel{i} \cdot \frac{n!}{\cancel{i!}\,(n-i)!} p^i (1-p)^{n-i}$$

$(i-1)!$

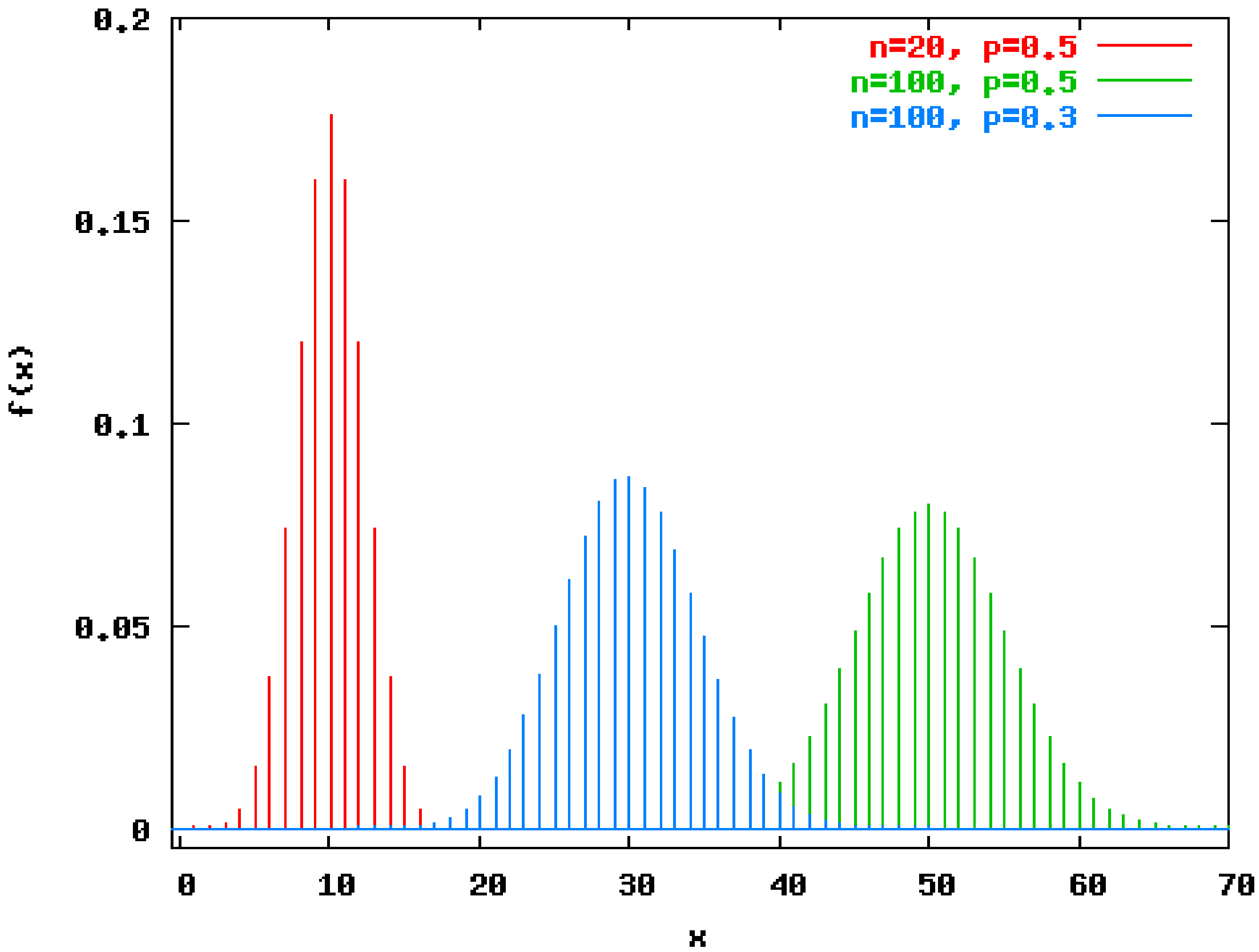
$$\boxed{\frac{(n-1)!}{(i-1)!\,(n-i)!} = \binom{n-1}{i-1}}$$

$$\boxed{j = i-1}$$

$$n-i = n-1-j$$

$$= np \sum_{i=1}^{n} \underbrace{\binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}}$$

$$= np \sum_{j=0}^{n-1} \underbrace{\binom{n-1}{j} p^j (1-p)^{n-1-j}}_{1} = np$$


n=20, p=0.5
n=100, p=0.5
n=100, p=0.3
0.2
0.15
0.1
0.05
0
f(x)
0
10
20
30
40
50
60
70
x

$P(\lambda)$

Poisson Random Variables

$Po(\lambda)$

Definition: A random variable X taking on one of the values 0, 1, 2, ... is said to be a *Poisson* random variable with parameter $\lambda > 0$, if

$pois(\lambda)$

$$p(i) = p\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!},\ i = 0, 1, 2, \ldots$$

$poisson(\lambda)$

One can check that

$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$

$$\sum_{i=0}^{\infty} p(i) = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1$$

Expected Value and Variance of Poisson Random Variables

The mean and the variance of a Poisson r.v. are both equal to λ

$$EX = \sum_{i=1}^{\infty} i \cdot e^{-\lambda}\frac{\lambda^i}{i!} = \sum_{i=1}^{\infty} e^{-\lambda}\frac{\lambda^i}{(i-1)!} = e^{-\lambda}\sum_{i=1}^{\infty}\frac{\lambda^i}{(i-1)!}$$

$$= \lambda e^{-\lambda}\underbrace{\sum_{i=1}^{\infty}\frac{\lambda^{i-1}}{(i-1)!}}_{1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\cdots}$$

$$= \lambda$$

$j = i-1$

$$\boxed{\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}} = e^{\lambda}$$

Poisson random variables for number of events occurring in a time period

One use of the Poisson is to model the number of "events" occurring in a certain period of time, e.g.,

The number of claims to an insurer in a year.

number of times the price exceeds a certain value.

Geometric Random Variable

Suppose that independent trials are performed, each having a probability *p*, 0< *p* <1, of being a success. Let *X* be the number of trials required until the first success. The probability mass function of *X* is

$$P\{X = n\} = (1-p)^{n-1}p, n = 1, 2, \ldots$$

$$\underbrace{F\ F\ \cdots\ F}_{n-1}\ \underset{\uparrow\, n}{S}$$

$$p \underbrace{\sum_{n=1}^{\infty} (1-p)^{n-1}}_{\frac{1}{1-(1-p)} = \frac{1}{p}} \cdot \cancel{p} = 1$$

$$\sum_{n=1}^{\infty} \underbrace{(1-p)}_{q}^{n-1} = \frac{1}{1-q}$$

Geometric Random Variable

Definition: A random variable that has the above pmf is called a *geometric* random variable with parameter *p* (*Geom(p) or G(p)*).

$$E[X] = \underline{\underline{1/p}}, Var(X) = \frac{1-p}{p^2}$$

$$EX = \boxed{\sum_{i=1}^{\infty} i \cdot \underbrace{(1-p)}_{q}^{i-1}} p = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

$$\sum_{i=1}^{\infty} i q^{i-1}$$

(Let *q=1-p*, then $E[X] = \sum_{n=1}^{\infty} nq^{n-1}p$, do you know how to compute the sum?)

$$(q^i)' = i q^{i-1}$$

$$\sum_{i=1}^{\infty} q^i = \frac{q}{1-q}$$

$$\sum_{i=1}^{\infty} q^{i-1} = \frac{1}{1-q}$$

$$\underline{\sum_{i=1}^{\infty} i q^{i-1}} = \frac{(1-q) - q\cdot(-1)}{(1-q)^2} = \frac{1}{(1-q)^2} = \underline{\frac{1}{p^2}}$$

$X=n:\ \underbrace{O\ S\ O \cdot\ S - O\ S}_{n\ \text{trial}}$ $r=3,$ $\binom{n-1}{r-1} p^r (1-p)^{n-r}$

Negative Binomial Distribution

Suppose that independent trials, each having a probability p, $0 < p < 1$, of being a success, are performed until r successes occur. Let X be the random variable that denotes the number of trials required. The probability mass function of X is

$$P\{X = n\} = \binom{n-1}{r-1} p^r (1-p)^{n-r}, \quad n = r, r+1, \cdots$$

Definition: A random variable whose pmf is given by the above is called a negative binomial random variable with parameters (r, p).

$$\underbrace{FF\cdot FS}_{G}\ \underbrace{F\cdots S}_{G}\ \underbrace{S}_{G} \qquad r=3$$

Remarks:

(1) geometric random variable is a negative binomial random variable with parameters (1, *p*) .

(2) Negative binomial r.v. is a sum of *r* independent geometric r.v.'s with parameter *p*

(3) The expected value and variance of a negative binomial random variable are (no need to memorize):

$$\boxed{E[X] = r/p, Var(X) = \frac{r(1-p)}{p^2}}$$

$Var(X) = r \cdot Var(X_1)$

$EX = EX_1 + \cdots + EX_r$

$= r \cdot \frac{1}{p}$

$X = X_1 + \cdots + X_r$

NB G G

Hypergeometric Random Variables

Suppose that a sample of size n is to be chosen randomly (without replacement) from an urn containing *N* balls, of which *K* are white and *N*−K are black. Let *X* denote the number of white balls selected, then

$k \le n$

$k \le K$

$$P\{X = k\} = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}, \quad k = 0, 1, 2, \ldots, n$$

$\ge$

$\min(n, K)$

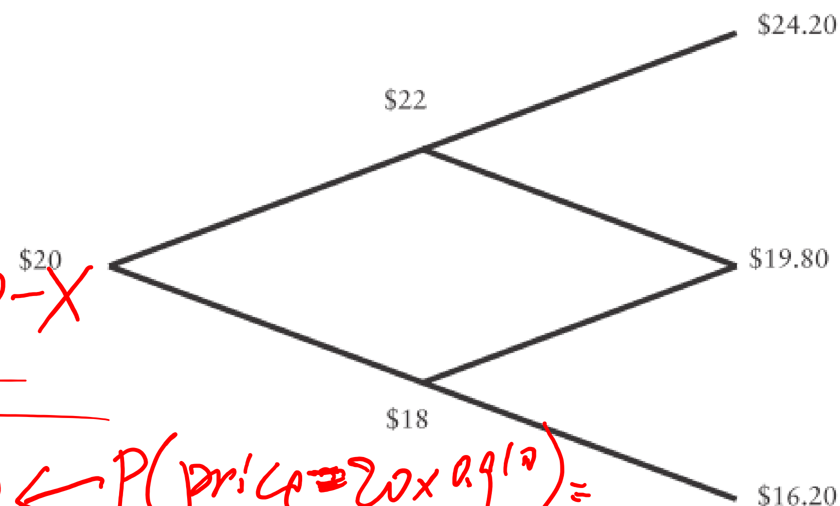
Definition: A random variable *X* whose pmf is given by the above is said to be a *hypergeometric* random variable with parameters (*n*,*N*,*K*). E(X)=nK/N

Binomial stock price model

Suppose currently the stock price is 20 and in each period in the future it can go up or down by 10% (with probability p and 1-p respectively). What is the price after 10 periods in terms of X, the number of periods where it goes up? $n=10,\ p$

X: number of days ↗

10 − X: number of days ↘



price: $\underline{20 \times 1.1^{X} \times 0.9^{10-X}}$

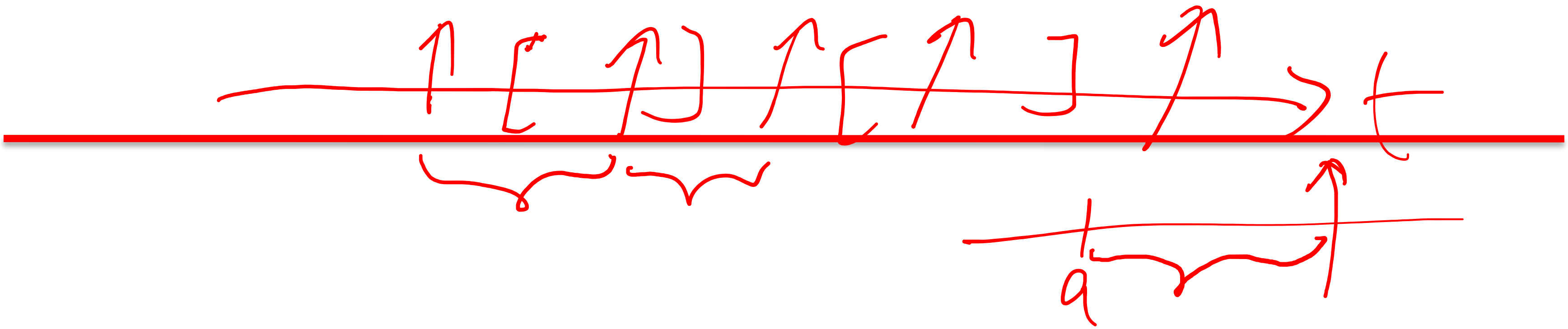
$\underbrace{X \sim Bin(10, p)}$

$P(X=0) \leftarrow P(\text{price} = 20 \times 0.9^{10}) =$

$P(X=1)$

$P(X=2)$

$\vdots$



We call some event process a *Poisson Process (with parameter λ)* if the number of events occurring in any interval of length *t* is a Poisson random variable with parameter λt.

X: # of events in the $[a, a+t]$

$X \sim Pois(\lambda t)$

① X: # claim within 1 week ~ Pois(2) $EX =$

2 Pois(4)

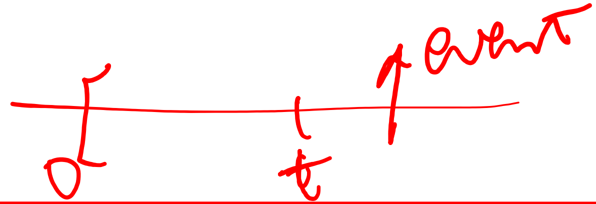
Example : Suppose that number of claims to an insurer is modeled as a Poisson process with $\lambda = 2$, with 1 week as the unit of time

$X \sim \text{Pois}(\lambda)$, $EX = \lambda$

(a) Find the probability that at least 3 claims are received during the next 2 weeks.

X: # claim 2 weeks ~ Pois(4)

$$P(X \geq 3) = \sum_{i=3}^{\infty} e^{-4} \frac{4^i}{i!} = 1 - \sum_{i=0}^{2} e^{-4} \frac{4^i}{i!}$$



Example : Suppose that number of claims to an insurer is modeled as a Poisson process with $\lambda = 2$, with 1 week as the unit of time

(b) Find the probability distribution of the time, starting from now, until the next claim

T

want: $P(T \leq t)$, $\forall t \geq 0$

$P(T > t) = P$(I need to wait more than time t to see a claim)

$= P$(within interval $[0, t]$ I see no claims)

$= P(X = 0) = e^{-\lambda t}$

X: # claims $[0,t]$ $\sim \text{Pois}(2t)$

$\text{Pois}(2t)$

$P(T \leq t) = 1 - e^{-\lambda t}$, $t \geq 0$

$$P(2\text{ cold} \mid \text{ben}) = P(\text{Pois}(3)=2) = e^{-3}\frac{3^2}{2!}$$

Example: The number of times that a person contracts a cold in a given year is *Pois(5)*. Suppose a new drug can reduce it to *Pois(3)* for 75% of the population, and no appreciable effect for the rest 25%. If a person tries the drug and has 2 colds in that year, how likely is it that the drug is beneficial for him/her.

$$P(\text{ben} \mid 2\text{ colds}) = \frac{P(2\text{ colds} \mid \text{ben})\,\overbrace{P(\text{ben})}^{75\%}}{P(2\text{ colds} \mid \text{ben})P(\text{ben}) + P(2\text{ cold} \mid \text{not ben})\,\underbrace{P(\text{not ben})}_{0.25}}$$

$$= \frac{e^{-3}\frac{3^2}{2!} \times 0.75}{e^{-3}\frac{3^2}{2!} \times 0.75 + 0.25 \times e^{-5}\frac{5^2}{2!}}$$

F F S F - S - S

$\underbrace{\quad}_{r-1\ S,\ \le m-1\ F}$ rth S

If m-1 F, rth S. occur at posit r+m-1

If k F, rth S, at r+k

Example : If independent trials, each resulting in a success with probability *p*, are performed, what is the probability of *r* successes occurring before m failures?

Solution: *r* successes occur before m failures if and only if *r*th success occurs before or at the (*r+m-1)*-th trial. And (obviously) *r*th success can not occur before the *r*th trial.

$P(\text{rth S before m F}) = P(\text{rth S at } r \text{ or } r+1, \ldots, r+m-1)$

$= P(X = r \text{ or } r+1, \ldots, r+m-1)$

so the probability is:

$= \sum_{n=r}^{r+m-1} \binom{n-1}{r-1} p^r (1-p)^{n-r}$

X: pos of rth succ $\sim NB$

$P(r \text{ suc before } m \text{ failure})$

$= P(\text{in the first } r+m-1 \text{ trials has } \geq r\ S)$

X: # S in $r+m-1$ trials

$= P(X \geq r)$

$$= \sum_{n=r}^{r+m-1} \binom{r+m-1}{n} p^n (1-p)^{r+m-1-n}$$

$r-1\ S$

$\leq m-1\ F$

rth suc

S

$\leq r+m-1$

$(r+m)$th trial

$< r\ S$ X

$\geq r$ suc $\leq m-1\ F$

$r=3 \quad m=2$

FSFSS X

FSSS ✓

Example : An urn contains N white and M black balls. Balls are randomly selected, one at a time, until a black one is obtained. If we assume that each selected ball is replaced before the next one is drawn, what is the probability that

a) exactly n draws are needed;

b) at least k draws are needed

Each step prob of seeing

$B: \frac{M}{N+M}$

X: # draws until B: $\mathrm{Geom}\left(\frac{M}{N+M}\right)$

a). $P(X=n) = \left(\frac{N}{N+M}\right)^{n-1} \frac{M}{N+M}$

$n=1,2,\cdots$

b) $P(X \geq k) = \sum_{n=k}^{\infty} \left(\frac{N}{N+M}\right)^{n-1} \frac{M}{N+M} = ?$

$= P(\text{first } k-1 \text{ all } W) = \left(\frac{N}{N+M}\right)^{k-1}$

Application of hypergeometric distribution (optional)

US dollar is appreciating against a certain foreign currency and Federal Reserve might be inclined to intervene by purchasing the foreign currency. During these N (3072) days, K (1546) is the number of days with a dollar depreciation, N-K (1508) is the number of days with a dollar appreciation. The number of days the FR interneves is n (138). Let r.v. X (51) be the number of successes within N days.

m=3 r=2

FFSS... SS...

FSFS FSFFS~ X

n=t X

When I see 2 S, How many F did I see?

2nd S

FFSFSFSSSFF

2S ref 4F