Chapter 6. Limit theorems

This (very short) section has a very different flavor than previous ones. We will talk about

(1) (weak) laws of large numbers (the average of a sequence of random variables converges to the expected average)

(2) central limit theorems (the sum of a large number of random variables has a probability distribution that is approximately normal)

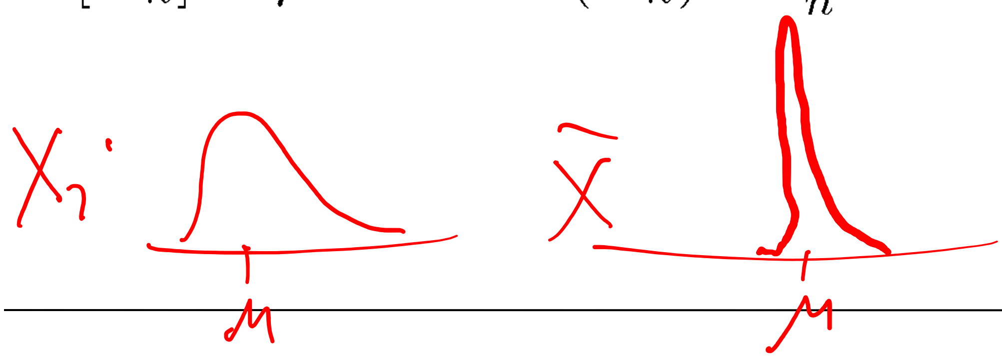
Theorem: The weak law of large numbers

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables (a sample), each having finite mean $E[X_i] = \mu$. Denote $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

Then, for any $\epsilon > 0$, $\boxed{\lim_{n\to\infty} P\{|\bar{X}_n - \mu| \geq \epsilon\} = 0}$ $\bar{X}$, $\bar{X}_n$

$\lim_{n\to\infty} P(|\bar{X} - \mu| < \epsilon) = 1$

We don't give a proof but the result can be roughly seen from $E[\bar{X}_n] = \mu$ and $Var(\bar{X}_n) = \frac{\sigma^2}{n}$



Definition. We say a sequence of random variables X_n converge to X in probability (or, weakly) if $P(|X_n - X| \geq \epsilon) \to 0, \forall \epsilon > 0$

The Central Limit Theorem (CLT)

CLT states that the sum of a large number of independent random variables has a distribution that is approximately normal

(1) provide a simple method for computing approximate probabilities for sums of independent random variables

(2) explain the fact that empirical frequencies of so many natural population exhibit bell-shaped (that is, normal) curves

$X_1+\cdots+X_n \approx N(n\mu, n\sigma^2)$, or $\bar{X} \approx N(\mu, \frac{\sigma^2}{n})$

Theorem (the Central Limit Theorem)

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables each having mean μ and variance σ^2. Then the distribution of

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$$

$E\sum X_i = n\mu$

$Var(\sum X_i) = n\sigma^2$

$E\bar{X} = \mu$, $Var(\bar{X}) = \frac{\sigma^2}{n}$

tends to the standard normal as n goes to infinity (we also say the sample mean (or sum, or Y) is asymptotically normal). $N(0,1)$

More precisely, for any $a < b$,

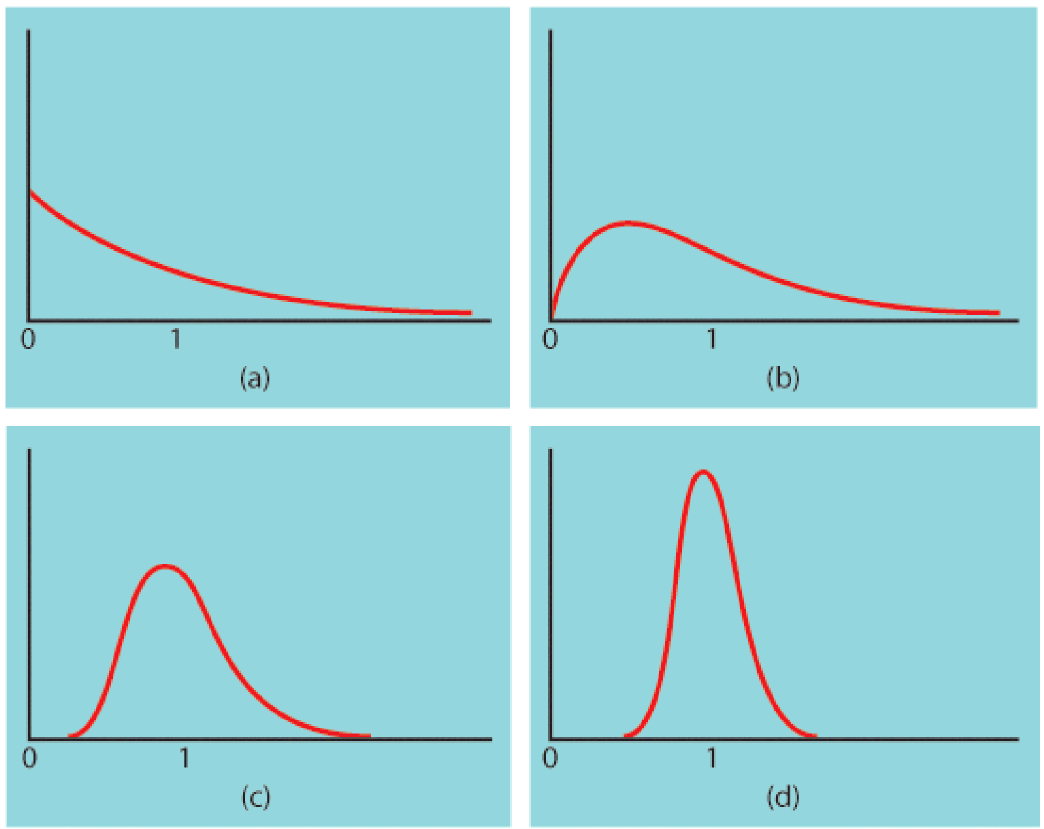
$P(a \le Z \le b)$

$$P(a \le Y_n \le b) \to \Phi(b) - \Phi(a) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, \text{ as } n \to \infty$$

Central Limit Theorem: When randomly sampling from **any** population with mean μ and standard deviation σ, **when *n* is large enough,** the sampling distribution of $\bar{x}$ is approximately normal ~ $\boldsymbol{N(\mu, \sigma^2/n)}$.

Population with strongly skewed distribution

Sampling distribution of $\bar{x}$ for n= 2 observations



Sampling distribution of $\bar{x}$ for n= 10 observations

Sampling distribution of $\bar{x}$ for n = 25 observations

Definition Suppose $X_1, X_2, \ldots, X$ are r.v. and X is continuous r.v. We say X_n converges to X in distribution if

$$P(a \leq X_n \leq b) \to P(a \leq X \leq b)$$

Compare WLLN and CLT for sample mean with EX=0.

$$\frac{X_1+...+X_n}{n} \to 0 \qquad \bar{X} \to 0 \qquad \bar{X}$$

$$\underbrace{\frac{X_1+...+X_n}{\sqrt{n}}}_{\sqrt{n}\cdot\bar{X}} \to N(0,\sigma^2)$$

We say the asymptotic distribution of $X_1 + \cdots + X_n$ or $\bar{X}$ is normal.

Example. The service times for customers coming through a checkout counter in a retail store are independent random variables with mean 1.5 minutes and variance 1. Approximate the probability that 100 customers can be served in less than 2 hours of total service time.

$X_1, \cdots X_{100}$: service times for 100 customers

$X_1 + \cdots + X_{100}$ has mean 150

Var is 100

$$P(X_1 + \cdots + X_{100} < 120)$$

$$= P\left(\frac{\sum X_i - 150}{10} < \frac{120 - 150}{10}\right)$$

$$= P(\underbrace{Z}_{N(0,1)} < -3) = \Phi(-3) = 0.0013$$

Example. The number of students who enroll in a psychology course is a Poisson random variable with mean 100. What is the probability that there are at least 120 students? $X \sim \text{Pois}(100)$

$X = X_1 + \cdots + X_{100}$, $X_1 \sim \text{Pois}(1)$

$X = X_1 + \cdots + X_{200}$, $X_1 \sim \text{Pois}(0.5)$

$$P(X \geq 120)$$

$$= P\left(\frac{X - \cancel{EX}\ 100}{\cancel{\sqrt{Var(X)}}\ 10} \geq \frac{120-100}{10}\right)$$

$$= 1 - \Phi(2)$$

Final definition: If $\sqrt{n}(Y_n - \mu) \rightarrow N(0, \sigma^2)$ in distribution, we say the asymptotic variance of Y_n is σ^2/n (we can write $aVar(Y_n) = \sigma^2/n$).

$$Y_n - \mu \approx N\left(0, \frac{\sigma^2}{n}\right)$$