MA1300 **Self Practice** # 2, Solutions

1. (P70, #15) Evaluate the limit $\lim\limits_{t\to-3}\dfrac{t^2-9}{2t^2+7t+3}$

Solution.

$$\lim_{t\to-3}\frac{t^2-9}{2t^2+7t+3}=\lim_{t\to-3}\frac{(t+3)(t-3)}{(2t+1)(t+3)}=\lim_{t\to-3}\frac{t-3}{2t+1}=\frac{6}{5}.$$

2. (P70, #19) Evaluate the limit $\lim\limits_{x\to-2}\dfrac{x+2}{x^3+8}$

Solution.

$$\lim_{t\to-2}\frac{x+2}{x^3+8}=\lim_{t\to-2}\frac{x+2}{(x+2)(x^2-2x+4)}=\lim_{t\to-2}\frac{1}{x^2-2x+4}=\frac{1}{12}.$$

3. (P70, #26) Evaluate the limit $\lim\limits_{t\to0}\left(\dfrac{1}{t}-\dfrac{1}{t^2+t}\right)$

Solution.

$$\lim_{t\to0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=\lim_{t\to0}\frac{t+1-1}{t(t+1)}=\lim_{t\to0}\frac{1}{t+1}=1.$$

4. (P70, #36) Use the Squeeze Theorem to show that

$$\lim_{x\to0}\sqrt{x^3+x^2}\sin\frac{\pi}{x}=0.$$

Solution. Since $-\frac{1}{\sqrt{2}}|x|\le-\sqrt{x^3+x^2}\le\sqrt{x^3+x^2}\sin\frac{\pi}{x}\le\sqrt{x^3+x^2}\le\sqrt{\frac{3}{2}}|x|$ when $x\in(-\frac{1}{2},0)\cup(0,\frac{1}{2})$, and by Example 7 in page 82 of the textbook, $\lim\limits_{x\to0}|x|=0$, so

$$\lim_{x\to0}-\frac{1}{\sqrt{2}}|x|=0=\lim_{x\to0}\sqrt{\frac{3}{2}}|x|,$$

then the Squeeze Theorem thus implies

$$\lim_{x\to0}\sqrt{x^3+x^2}\sin\frac{\pi}{x}=0.$$

5. (P70, #38) If $2x\le g(x)\le x^4-x^2+2$ for all x, evaluate $\lim\limits_{x\to1}g(x)$.

Solution. Since $\lim\limits_{x\to1}2x=2=\lim\limits_{x\to1}(x^4-x^2+2)$, the Squeeze Theorem implies $\lim\limits_{x\to1}g(x)=2$.

6. (P70, #40) Prove that $\lim\limits_{x\to0^+}\sqrt{x}[1+\sin^2(2\pi/x)]=0$.

Solution. Since for $x\in(0,\infty)$, $0\le\sqrt{x}[1+\sin^2(2\pi/x)]\le2\sqrt{x}$, and $\lim\limits_{x\to0^+}0=\lim\limits_{x\to0^+}2\sqrt{x}=0$, the Squeeze Theorem implies $\lim\limits_{x\to0^+}\sqrt{x}[1+\sin^2(2\pi/x)]=0$.

For Questions 7 $\sim$ 9, find the limit, if it exists. If the limit does not exist, explain why.

7. (P70, #41) $\lim\limits_{x\to3}(2x+|x-3|)$

Solution. When $x>3$, $2x+|x-3|=3x-3$, we have $\lim\limits_{x\to3^+}2x+|x-3|=6$. When $x<3$, $2x+|x-3|=x+3$, so $\lim\limits_{x\to3^-}2x+|x-3|=6$. Therefore $\lim\limits_{x\to3}2x+|x-3|=6$.

8. (P70, #42) $\lim_{x \to -6} \dfrac{2x+12}{|x+6|}$

Solution. When $x > -6$, $\dfrac{2x+12}{|x+6|} = \dfrac{2x+12}{x+6} = 2$, so $\lim_{x \to -6^+} \dfrac{2x+12}{|x+6|} = 2$. When $x < -6$, $\dfrac{2x+12}{|x+6|} = \dfrac{2x+12}{-x-6} = -2$, so $\lim_{x \to -6^-} \dfrac{2x+12}{|x+6|} = -2$. Therefore the limit $\lim_{x \to -6} \dfrac{2x+12}{|x+6|}$ does not exist.

9. (P70, #46) $\lim_{x \to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|} \right)$

Solution. When $x < 0$, $\left(\dfrac{1}{x} - \dfrac{1}{|x|} \right) = \dfrac{2}{x}$, so $\lim_{x \to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|} \right) = -\infty$.

10. (P70, #47) The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

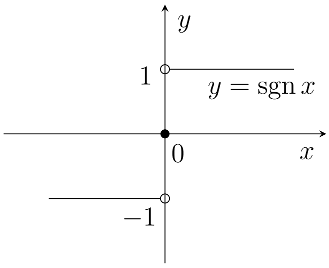
(a) Sketch the graph of this function.

(b) Find each of the following limits or explain why it does not exist.

(i) $\lim_{x \to 0^+} \operatorname{sgn} x$ (ii) $\lim_{x \to 0^-} \operatorname{sgn} x$ (iii) $\lim_{x \to 0} \operatorname{sgn} x$ (iv) $\lim_{x \to 0} |\operatorname{sgn} x|$

Solution.

(a)



(b) Since for $x > 0$, $\operatorname{sgn} x = 1$, we have $\lim_{x \to 0^+} \operatorname{sgn} x = \lim_{x \to 0^+} 1 = 1$. Similarly, $\lim_{x \to 0^-} \operatorname{sgn} x = -1$. So $\lim_{x \to 0} \operatorname{sgn} x$ does not exist. Since for $x \neq 0$, $|\operatorname{sgn} x| = 1$, so $\lim_{x \to 0} |\operatorname{sgn} x| = \lim_{x \to 0} 1 = 1$.

11. (P71, #54) In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

Solution. When $v < c$ and v is arbitrarily close to c, $c^2 - v^2$ is arbitrarily close to 0, so is $\sqrt{c^2 - v^2}$. Therefore

$$\lim_{v \to c^-} L = \frac{L_0}{c} \lim_{v \to c^-} \sqrt{c^2 - v^2} = 0.$$

The limit shows that an object with length L_0 at rest will have its length approaching 0 as its speed approaches that of light. For the left-hand limit, physically, it is because no object can move faster than light, mathematically, it is because otherwise the defining formula $L_0\sqrt{1-\frac{v^2}{c^2}}$ will be meaningless.

12. (P71, #59) If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x) = 0$.

Solution. Since $0 \le f(x) \le x^2$ and $\lim_{x\to 0} 0 = \lim_{x\to 0} x^2 = 0$, the Squeeze Theorem implies that $\lim_{x\to 0} f(x) = 0$.

13. (P71, #62) Evaluate $\lim_{x\to 2} \dfrac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$.

Solution.

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} &= \lim_{x\to 2} \frac{(\sqrt{6-x}^2-4)(\sqrt{3-x}+1)}{(\sqrt{3-x}^2-1)(\sqrt{6-x}+2)} \\
&= \lim_{x\to 2} \frac{(2-x)(\sqrt{3-x}+1)}{(2-x)(\sqrt{6-x}+2)} = \lim_{x\to 2} \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{1}{2}.
\end{aligned}$$

14. (P81, #13)

(a) Find a number δ such that if $|x-2| \le \delta$, then $|4x-8| < \varepsilon$, where $\varepsilon = 0.1$.

(b) Repeat part (a) with $\varepsilon = 0.01$.

Solution.

(a) Let $\delta = \frac{1}{80}$, then once $|x-2| \le \delta$, we have $|4x-8| = 4|x-2| \le \frac{4}{80} < 0.1 = \varepsilon$.

(b) Let $\delta = \frac{1}{800}$, then once $|x-2| \le \delta$, we have $|4x-8| = 4|x-2| \le \frac{4}{800} < 0.01 = \varepsilon$.

15. (P81, #39) If the function f is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x)$ does not exist.

Solution. Suppose $\lim_{x\to 0} f(x) = \Delta$, Let $\varepsilon = 1/4$, then for any $\delta > 0$, there exists a rational number $0 < x_\delta < \delta$, and an irrational number $0 < x'_\delta < \delta$, so $f(x_\delta) = 0$, $f(x'_\delta) = 1$, and $\varepsilon < 1 \le |f(x_\delta)-\Delta| + |f(x'_\delta)-\Delta|$. Therefore $\lim_{x\to 0} f(x)$ does not exist.

16. (P81, #41) How close to -3 do we have to take x so that

$$\frac{1}{(x+3)^4} > 10,000$$

Solution. Let $|x-(-3)| < 0.1$ to give $(x+3)^4 < \frac{1}{10^4}$, which implies $\frac{1}{(x+3)^4} > 10,000$.

17. (P81, #42, Optional) Prove that $\lim_{x \to -3} \dfrac{1}{(x+3)^4} = \infty$.

Solution. For any $M > 0$, let $\delta = \sqrt[4]{\frac{1}{2M}}$, so whenever $|x - (-3)| \leq \delta$, we have $\frac{1}{(x+3)^4} \geq 2M > M$. So by definition, $\lim_{x \to -3} \dfrac{1}{(x+3)^4} = \infty$.