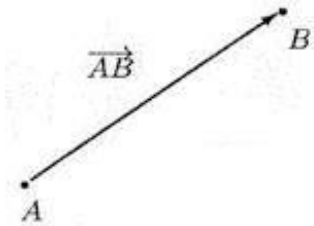
MA1301 Calculus and Basic Linear Algebra II
Chapter 1 Vector Algebra

1 Review of Basic Ideas (p.1 – p.8)

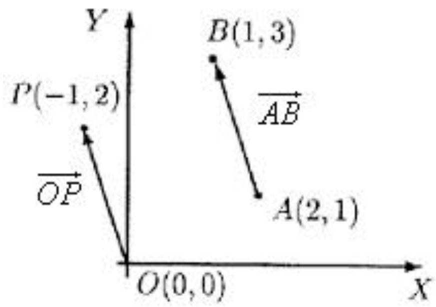
In engineering and science, physical quantities which are completely specified by their magnitude (size) are known as scalars. Examples are: mass, temperature, volume, resistance, charge, voltage, current, etc. Other quantities possess both magnitude and direction are known as vectors. Examples of vector quantities are: velocity, acceleration, force, electric field, magnetic field etc and will be denoted by $\vec{v},\ \vec{a},\ \vec{F},\ \vec{E},\ \vec{B}$, etc. Vectors may be represented geometrically by directed line segments. If A and B are two geometrical points in R^3, the directed line segment from A to B is called the vector from A to B and is denoted $\overrightarrow{AB}$. As the name implies, this is a vector quantity with direction from A to B and magnitude the distance between A and B. The vector $\overrightarrow{AB}$ is represented by an arrow from A to B as shown in the following figure.



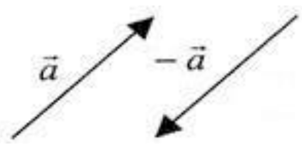
The point A is called the initial point of the vector $\overrightarrow{AB}$, and B is called the terminal point of $\overrightarrow{AB}$. The magnitude of the vector $\overrightarrow{AB}$ is called its length and is denoted $\left|\overrightarrow{AB}\right|$.

norm

1. Two vectors $\vec{a}$ and $\vec{b}$ are equal if they have the same magnitude and direction. We write $\vec{a} = \vec{b}$. Two vectors can be the same even though the initial points and terminal points are different. For example $\overrightarrow{AB} = \overrightarrow{OP}$ in R^2 because they have the same length and the same direction (they both proceed one unit to the left and two units up). Thus the same vector can be translated from one position to another; what is important is that the length and direction remain the same, and not where the initial points and terminal points are located. For this reason, we shall often denote vectors as $\vec{a},\ \vec{b},\ \vec{c}$, etc. which make no reference to the initial points and terminal points.

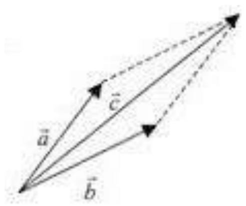


2. A vector having the same magnitude as $\vec{a}$ but the opposite direction is denoted by $-\vec{a}$.

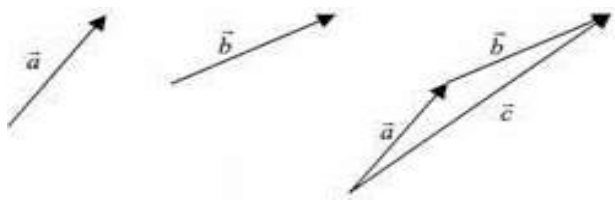


3. Geometrically the <u>sum</u> of two vectors is given by the parallelogram law of vector addition or tip-to-tail method of adding vectors.

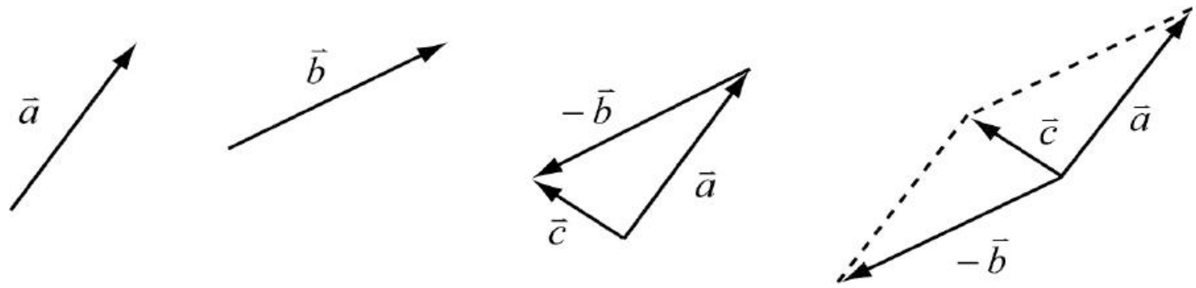
Parallelogram law of vector addition: In the parallelogram determined by two vectors $\vec{a}, \vec{b}$, the vector $\vec{a}+\vec{b}$ is the diagonal with the same initial point as $\vec{a}, \vec{b}$.



Tip-to-tail method of adding vectors: Given two vectors $\vec{a}, \vec{b}$, place the tail of $\vec{b}$ at the tip of $\vec{a}$. Then $\vec{a}+\vec{b}=\vec{c}$ is the vector from the tail of $\vec{a}$ to the tip of $\vec{b}$.

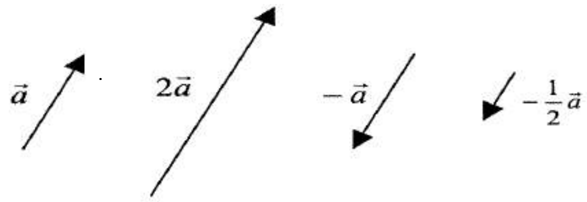


4. The <u>difference</u> of two vectors $\vec{a}$ and $\vec{b}$ represented by $\vec{c}=\vec{a}-\vec{b}$ is defined as $\vec{c}=\vec{a}+\left(-\vec{b}\right)$.



5. If $\vec{a}=\vec{b}$ then $\vec{a}-\vec{b}$ is the <u>zero vector</u> denoted by $\vec{0}$. This has magnitude 0 but no direction.

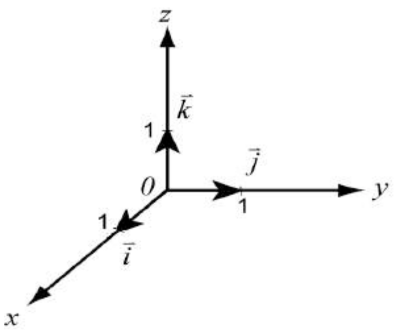
6. Multiplication of $\vec{a}$ by a scalar, *m*, produces a vector $m\vec{a}$ with magnitude *m* times that of $\vec{a}$ and direction the same as or opposite to that of $\vec{a}$ according to whether *m* is positive or negative respectively. If $m=0$ then $m\vec{a}=\vec{0}$.

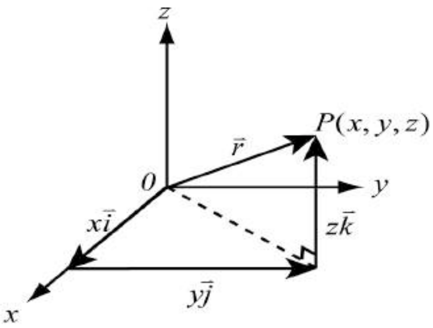


7. <u>Unit vectors</u> are vectors with magnitude 1. If $\vec{a}$ is any vector then we usually denote its magnitude / norm by $|\vec{a}|$. A unit vector with the same direction as $\vec{a}\left(\neq\vec{0}\right)$ will be $\frac{\vec{a}}{|\vec{a}|}$.

2 <u>Components of a Vector</u> (p.1 – p.8)

In a rectangular coordinate system in R^3 (the Euclidean space of dimension 3), <u>orthogonal</u> <u>(perpendicular)</u> <u>unit vectors</u> in the directions of the positive *x*, *y* and *z* axes are denoted by $\vec{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\vec{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ respectively.





The vector from the origin $O(0,0,0)$ to a point P (denoted as $\overrightarrow{OP}$) is known as the <u>position</u> <u>vector</u> of P.

If P has Cartesian coordinates (x, y, z), then the position vector of P, say $\vec{r}$ may be written as

$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = x\vec{i} + y\vec{j} + z\vec{k}$, where x, y, z are known as the <u>components</u> or <u>coordinates</u> of $\vec{r}$ with respect to the vectors $\vec{i}, \vec{j}$ and $\vec{k}$.

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, the vector from P to Q, denoted as $\overrightarrow{PQ}$ will be $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$, where the components of $\overrightarrow{PQ}$ are $x_2 - x_1, y_2 - y_1$ and $z_2 - z_1$.

Note that the ordered triple of components of a vector is unique with respect to a given coordinate system.

If $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, in terms of components we have:

<u>Equality</u> :

$\vec{a} = \vec{b}$ if and only if $a_1 = b_1, a_2 = b_2, a_3 = b_3$.

<u>Addition</u>:

$$\vec{a} + \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} = (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j} + (a_3 + b_3)\vec{k}.$$

<u>Scalar Multiplication</u>:

$$\vec{ma} = m\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ma_1 \\ ma_2 \\ ma_3 \end{pmatrix} = ma_1\vec{i} + ma_2\vec{j} + ma_3\vec{k}\ .$$

<u>Zero Vector</u> :

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

<u>Magnitude</u> :

$$\left|\vec{a}\right| = \left|a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right| = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad \text{(By Pythagoras' theorem)}$$

<u>A Unit Vector in the same direction as $\vec{a}$:</u>

$$\frac{\vec{a}}{\left|\vec{a}\right|} = \frac{1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \frac{a_1\vec{i} + a_2\vec{j} + a_3\vec{k}}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

<u>Example 1</u>

Find the vector $\overrightarrow{PQ}$ with initial point $P(3,-2,1)$ and terminal point $Q(1,2,-4)$, and a unit vector in the same direction as $\overrightarrow{PQ}$.

<u>Solution</u>:

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-3 \\ 2-(-2) \\ -4-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix} = -2\vec{i} + 4\vec{j} - 5\vec{k}\ .$$

The magnitude of $\overrightarrow{PQ}$ is $\left|\overrightarrow{PQ}\right| = \sqrt{(-2)^2 + 4^2 + (-5)^2} = \sqrt{45}$.

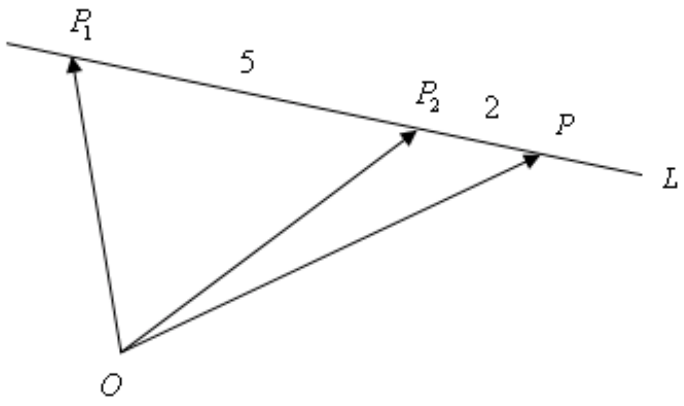
A unit vector in the same direction as $\overrightarrow{PQ}$ is given by $\vec{u} = \frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = \frac{1}{\sqrt{45}}\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix} = \frac{-2\sqrt{5}}{15}\vec{i} + \frac{4\sqrt{5}}{15}\vec{j} - \frac{\sqrt{5}}{3}\vec{k}$.

□

<u>Example 2</u>

Let L be the straight line through $P_1(3,2,3)$ and $P_2(0,2,7)$. Find a point P on L that is not on the line segment P_1P_2 and is 2 units from P_2.

Solution:



$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \begin{pmatrix} 0 \\ 2 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}, \quad \left|\overrightarrow{P_1P_2}\right| = \sqrt{(-3)^2 + 0^2 + 4^2} = 5.$$

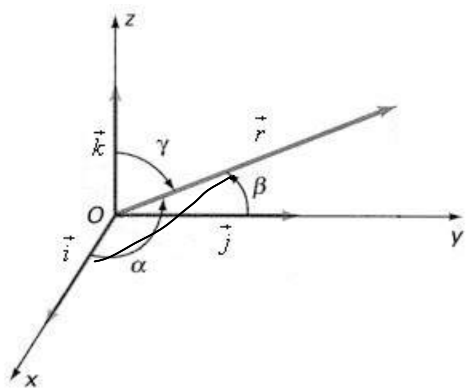
$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = \overrightarrow{OP_1} + \frac{5+2}{5}\overrightarrow{P_1P_2} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} + \frac{7}{5}\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 - \frac{21}{5} \\ 2+0 \\ 3 + \frac{28}{5} \end{pmatrix} = \begin{pmatrix} -\frac{6}{5} \\ 2 \\ \frac{43}{5} \end{pmatrix}.$$

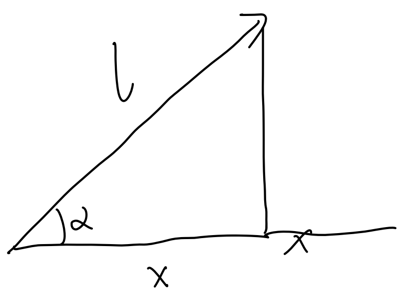
$\therefore$ The coordinates of point P is $\left(-\frac{6}{5}, 2, \frac{43}{5}\right)$.

□

3 Direction Cosines

If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, the direction of $\vec{r}$ may be specified by the cosines of the angles made by $\vec{r}$ with the 3 coordinate axes.





$$l = \cos\alpha = \frac{x}{\left|\vec{r}\right|},\ m = \cos\beta = \frac{y}{\left|\vec{r}\right|},\ n = \cos\gamma = \frac{z}{\left|\vec{r}\right|}$$

l, m, and n are known as the <u>direction cosines</u> of $\vec{r}$ and $l\vec{i} + m\vec{j} + n\vec{k}$ is a unit vector in the same direction as $\vec{r}$.

<u>Example 3</u>

Let $\vec{r}$ be the vector $\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} = 3\vec{i} + 2\vec{j} + 6\vec{k}$. Find the direction cosines of $\vec{r}$ and the angles made by $\vec{r}$ with the coordinate axes.

<u>Solution</u>:

$\left|\vec{r}\right| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$, $\quad l = \frac{3}{7},\ m = \frac{2}{7},\ n = \frac{6}{7}$ and

$\alpha = \cos^{-1}\left(\frac{3}{7}\right) \approx 64.6^{\circ},\ \beta = \cos^{-1}\left(\frac{2}{7}\right) \approx 73.4^{\circ},\ \gamma = \cos^{-1}\left(\frac{6}{7}\right) = 31^{\circ}.$

□

If $\vec{a}, \vec{b}, \vec{c} \in R^3$ are vectors and m, n are scalars (real numbers), then we have:

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$	Commutative law of vector addition
2. $\vec{a} + \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) + \vec{c}$	Associative law of vector addition
3. $\vec{a} + \vec{0} = \vec{a}$	Existence of as an additive vector identity
4. $\vec{a} + \left(-\vec{a}\right) = \vec{0}$	Existence of additive inverses
5. $m\left(\vec{a} + \vec{b}\right) = m\vec{a} + m\vec{b}$	Scalar distribution over vector addition
6. $\left(m + n\right)\vec{a} = m\vec{a} + n\vec{a}$	Vector distribution over scalar addition
7. $\left(mn\right)\vec{a} = m\left(n\vec{a}\right) = n\left(m\vec{a}\right)$	Associative law for scalar multiplication
8. $1\vec{a} = \vec{a}$	Multiplicative scalar identity

Notice that the above are also valid for the $n-$component vectors $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in R^n,\ n \geq 1$.

4 Scalar Product of two vectors (p.10 – p.12)

inner product

Let $\vec{a}$ and $\vec{b}$ be two vectors, their scalar product (or dot product), written as $\vec{a}\cdot\vec{b}$, is defined by

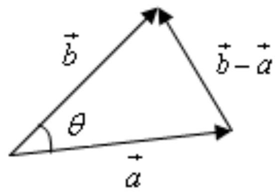
$$\vec{a}\cdot\vec{b}=\begin{cases}\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta & \text{if } \vec{a}\neq\vec{0},\ \vec{b}\neq\vec{0},\ \text{where } \theta \text{ is the angle between } \vec{a} \text{ and } \vec{b}\\ 0 & \text{otherwise}\end{cases}$$

$\begin{cases}\text{Case 1} & |\vec{a}|\,|\vec{b}| = 0\\ \text{Case 2} & \cos\theta = 0 \iff \theta = \frac{\pi}{2}.\end{cases}$

Example 4

Given two non-zero vectors $\vec{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\vec{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, show that $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$.

$\vec{b}-\vec{a}=\begin{pmatrix}b_1-a_1\\b_2-a_2\\b_3-a_3\end{pmatrix}$



Proof: According to cosine law, we have $\left|\vec{b}-\vec{a}\right|^2=\left|\vec{a}\right|^2+\left|\vec{b}\right|^2-2\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$.

$$\left|\vec{b}-\vec{a}\right|^2=(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2,\quad \left|\vec{a}\right|^2=a_1^2+a_2^2+a_3^2,\quad \left|\vec{b}\right|^2=b_1^2+b_2^2+b_3^2$$

Then we have

$$\left|\vec{b}-\vec{a}\right|^2=\left|\vec{a}\right|^2+\left|\vec{b}\right|^2-2\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$$

$$\Rightarrow (b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2=a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^2-2\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$$

$$\Rightarrow 2a_1b_1+2a_2b_2+2a_3b_3=2\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta\cdots\cdots(*) \Rightarrow$$

$\vec{a}\cdot\vec{b}=|a|\,|b|\cos\theta=a_1b_1+a_2b_2+a_3b_3$

$$\therefore\ \vec{a}\cdot\vec{b}=\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta, \text{ by definition}$$
$$=a_1b_1+a_2b_2+a_3b_3, \text{ from } (*).$$

□

Accordingly, we can define the dot product of two n-component vectors $\vec{a}=\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}$, $\vec{b}=\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix}\in R^n$ as

$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+\cdots+a_nb_n.$$

4.1 Properties of the dot product $\vec{a}\cdot\vec{b}$ where $\vec{a},\ \vec{b}\in R^n$ (p.13 – p.18)

$$(|a|+|b|)^2 \overset{?}{\geq} |a+b|^2 = (a+b)\cdot(a+b)$$

$$|a|^2+2|a||b|+|b|^2 \qquad a\cdot a + 2a\cdot b + b\cdot b = ||a||^2 + 2a\cdot b + ||b||^2$$

(i) $\vec{a}\cdot\vec{b}$ is a scalar

(ii) $\vec{a}\cdot\vec{b}$ is zero if $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$ or $\vec{a}$ and $\vec{b}$ are perpendicular (orthogonal)

(iii) $\left|\vec{a}\right| = \sqrt{\vec{a}\cdot\vec{a}}$

(iv) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (symmetry)

(v) $\left(k\vec{a}+l\vec{b}\right)\cdot\vec{c} = k\vec{a}\cdot\vec{c}+l\vec{b}\cdot\vec{c}$ for all $\vec{a},\ \vec{b},\ \vec{c}\in R^n$ and $k,\ l\in R$ (Linearity)

(vi) $\vec{a}\cdot\vec{a}\geq 0$ and $\vec{a}\cdot\vec{a}=0$ iff $\vec{a}=\vec{0}$ (Positive definiteness)

(vii) $\left|\vec{a}\cdot\vec{b}\right| \leq \left|\vec{a}\right|\left|\vec{b}\right|$ (Schwartz inequality)

$$|\vec{a}\cdot\vec{b}| = \left||a||b|\cos\theta\right| = |a||b||\cos\theta| \leq |a||b|$$

(viii) $\left|\vec{a}+\vec{b}\right| \leq \left|\vec{a}\right|+\left|\vec{b}\right|$ (Triangle inequality)

(ix) $\left|\vec{a}+\vec{b}\right|^2 + \left|\vec{a}-\vec{b}\right|^2 = 2\left(\left|\vec{a}\right|^2+\left|\vec{b}\right|^2\right)$

$$(a+b)(a+b)+(a-b)(a-b) = ||a||^2+||b||^2+\cancel{2a\cdot b}+||a||^2+||b||^2-\cancel{2a\cdot b} = 2(||a||^2+||b||^2)$$

We observe that $\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1$ and $\vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=0$.

<u>Example 5</u>

Let $\vec{a}=5\vec{i}+4\vec{j}+2\vec{k}$ and $\vec{b}=4\vec{i}-5\vec{j}+3\vec{k}$, find $\vec{a}\cdot\vec{b}$ and the angle between the vectors.

<u>Solution</u>:

$\vec{a}\cdot\vec{b} = (5\times 4)+\left[4\times(-5)\right]+(2\times 3) = 6$.

$\left|\vec{a}\right| = \sqrt{45},\ \left|\vec{b}\right| = \sqrt{50}$, hence $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|} = \dfrac{6}{\sqrt{45}\times\sqrt{50}} = \dfrac{2}{5\sqrt{10}}$.

$\therefore$ The angle between $\vec{a}$ and $\vec{b}$ is given by $\theta = \cos^{-1}\left(\dfrac{2}{5\sqrt{10}}\right) = 82.7^{\circ}$

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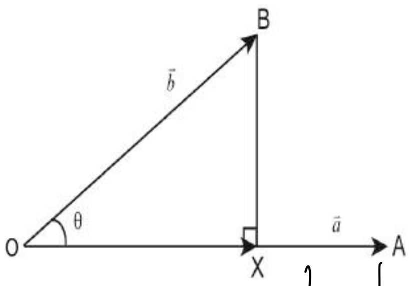
<u>Example 6</u>

Given two vectors $\vec{a}=\overrightarrow{OA},\ \vec{b}=\overrightarrow{OB}$, $\text{pro}_{\vec{a}}\vec{b}$ denotes the <u>projection vector</u> of $\vec{b}$ on $\vec{a}$ find $\text{pro}_{\vec{a}}\vec{b}$ and the <u>coefficient</u> of $\text{pro}_{\vec{a}}\vec{b}$ on $\vec{a}$.

<u>Solution</u>:



$\vec{a}\cdot\vec{b}=\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$, by definition $\Rightarrow \dfrac{\vec{a}\cdot\vec{b}}{\left|\vec{a}\right|}=\underline{\left|\vec{b}\right|\cos\theta}=OX$ the length of $\text{pro}_{\vec{a}}\vec{b}$

is a scalar

The <u>projection vector</u> $\text{pro}_{\vec{a}}\vec{b}$ of $\vec{b}$ on $\vec{a}$ is given by $\overrightarrow{OX}=(OX)\dfrac{\vec{a}}{\left|\vec{a}\right|}=\dfrac{\vec{a}\cdot\vec{b}}{\left|\vec{a}\right|}\dfrac{\vec{a}}{\left|\vec{a}\right|}=\dfrac{\vec{a}\cdot\vec{b}}{\left|\vec{a}\right|^2}\vec{a}=\boxed{\dfrac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}}\vec{a}$.

The <u>coefficient</u> of $\text{pro}_{\vec{a}}\vec{b}$ on $\vec{a}$ is:

$OX=\left|\vec{b}\right|\cos\theta=\dfrac{1}{\left|\vec{a}\right|}\vec{a}\cdot\vec{b}=\vec{b}\cdot$ (a unit vector in the same direction as $\vec{a}$)

□

Notice that the <u>coefficient</u> of $\text{pro}_{\vec{a}}\vec{b}$ on $\vec{a}$ can be negative if the angle θ between $\vec{a},\vec{b}$ is an obtuse angle.

5 <u>Vector Product of two vectors $\vec{a},\vec{b}\in R^3$</u> (p.21)

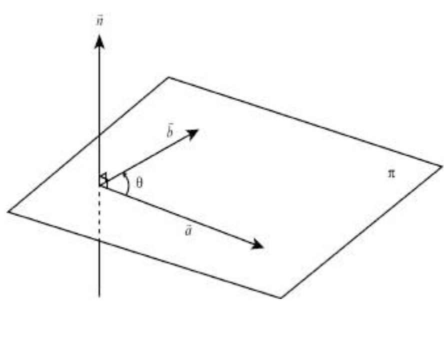
Cross Product

Let $\vec{a},\ \vec{b}\in R^3$, the <u>vector product</u> (or <u>cross product</u>) of $\vec{a}$ and $\vec{b}$, written $\vec{a}\times\vec{b}$, is defined as:

$$\vec{a}\times\vec{b}=\begin{cases}\left(\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta\right)\vec{n} & \text{if } \vec{a}\neq\vec{0},\ \vec{b}\neq\vec{0}\\ \vec{0} & \text{otherwise}\end{cases}$$

$\neq \vec{b}\times\vec{a}$

, where $\vec{n}$ is a unit vector perpendicular to the plane π containing $\vec{a}$ and $\vec{b}$ such that $\vec{a},\vec{b},\vec{n}$ form a right-handed triple and θ is the angle between $\vec{a}$ and $\vec{b}$, which is measured from $\vec{a}$ to $\vec{b}$.



Observe that $\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=\vec{0}$ and $\vec{i}\times\vec{j}=\vec{k},\ \vec{j}\times\vec{k}=\vec{i},\ \vec{k}\times\vec{i}=\vec{j}$.

not for $v=\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

general n.

Notice that cross product is defined only for $\boxed{\text{3-component vectors.}}$

5.1 Properties of the cross product $\vec{a}\times\vec{b}$, where $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k},\ \vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}\in R^3$ (p.21 – p.25)

(i) $\vec{a}\times\vec{b}$ is a vector.

(ii) $\vec{a}\times\vec{b}$ is a zero vector iff $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$ or $\vec{a}$ and $\vec{b}$ are parallel.

(iii) $\vec{a}\times\vec{a}=\vec{0}$

(vi) $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$

(v) $\left|\vec{a}\times\vec{b}\right|=\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta$ = area of the parallelogram with adjacent sides $\vec{a},\ \vec{b}$

(vi)

$$\vec{a}\times\vec{b}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\vec{i}-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\vec{j}+\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\vec{k}=\left(a_2b_3-a_3b_2\right)\vec{i}+\left(a_3b_1-a_1b_3\right)\vec{j}+\left(a_1b_2-a_2b_1\right)\vec{k}$$

Remarks:

(a) A 3×3 matrix is a rectangular array of numbers having 3 rows and 3 columns and is written as

$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, the symbol a_{21} for instance denotes the element of A which is in the second

row column

row and first column. Similarly, $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ is a 2×2 matrix.

(b) $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ has a number associated with it called $\boxed{\text{the } \textit{determinant} \text{ of } A}$, denoted as $|A|$.

$= \det(A)$

The determinant of 2×2 matrix: $\det\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=b_{11}\cdot b_{22}-b_{12}b_{21}$

1×1 matrix : $C = (C_{11})$ Definition

$\det C = C_{11}$

$|A|$ is defined as $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \underset{\text{expanded by the first row}}{=} a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$

The determinant of a 2×2 matrix $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ is defined as $|B| = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}b_{22} - b_{12}b_{21}$.

Thus, $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

(vii) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

$\mathbb{R}$

(viii) $\boxed{m}(\vec{a} \times \vec{b}) = (m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b})$

(ix) $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$ and $\vec{i} \times \vec{j} = \vec{k},\ \vec{j} \times \vec{k} = \vec{i},\ \vec{k} \times \vec{i} = \vec{j}$.

Example 7 $\begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 5 \\ -5 \\ 3 \end{pmatrix}$

If $\vec{a} = 5\vec{i} + 4\vec{j} + 2\vec{k}$ and $\vec{b} = 5\vec{i} - 5\vec{j} + 3\vec{k}$, find $\vec{a} \times \vec{b}$ and a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

Solution: $12 - (2\times(-5)) = 12 - (-10) = 22$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 4 & 2 \\ 5 & -5 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ -5 & 3 \end{vmatrix}\vec{i} - \begin{vmatrix} 5 & 2 \\ 5 & 3 \end{vmatrix}\vec{j} + \begin{vmatrix} 5 & 4 \\ 5 & -5 \end{vmatrix}\vec{k} = 22\vec{i} - 5\vec{j} - 45\vec{k}.$$

Or

$$\vec{a} \times \vec{b} = (5\vec{i} + 4\vec{j} + 2\vec{k}) \times (5\vec{i} - 5\vec{j} + 3\vec{k}) = 25\vec{i} \times \vec{i} - 25\vec{i} \times \vec{j} + 15\vec{i} \times \vec{k} + 20\vec{j} \times \vec{i} - 20\vec{j} \times \vec{j} + 12\vec{j} \times \vec{k}$$

$$+10\vec{k} \times \vec{i} - 10\vec{k} \times \vec{j} + 6\vec{k} \times \vec{k} = -25\vec{k} - 15\vec{j} - 20\vec{k} + 12\vec{i} + 10\vec{j} + 10\vec{i} = 22\vec{i} - 5\vec{j} - 45\vec{k}$$

A unit vector is $\pm\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \pm\dfrac{22\vec{i} - 5\vec{j} - 45\vec{k}}{\sqrt{2534}}$.

□

6 Triple scalar product $\vec{a} \cdot (\vec{b} \times \vec{c})$, where $\vec{a}, \vec{b}, \vec{c} \in R^3$ (p.25 – p.26)

Consider the triple scalar product $\vec{a} \cdot (\vec{b} \times \vec{c})$, which can also be written as $\vec{a} \cdot \vec{b} \times \vec{c}$ without confusion arising. We observe that $\vec{a} \cdot \vec{b} \times \vec{c}$ is a scalar.

Properties:

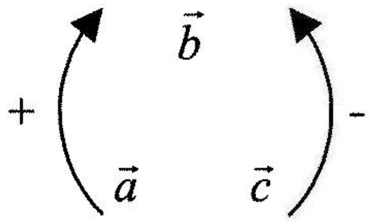
(i)

verify by yourself.

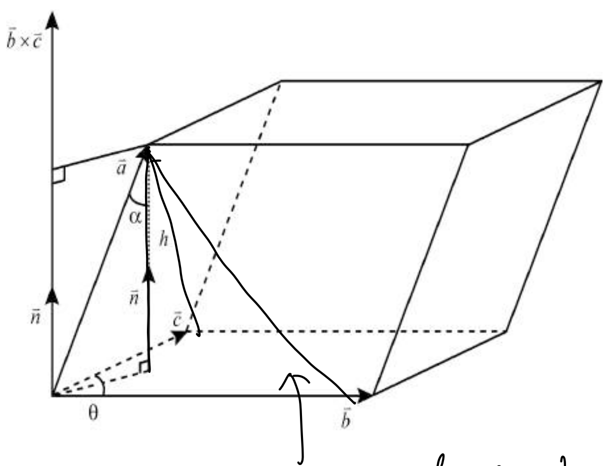
$$\vec{a}\cdot\vec{b}\times\vec{c}=\left(a_1\vec{i}+a_2\vec{j}+a_3\vec{k}\right)\cdot\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2$$

(ii) $\vec{a}\cdot\vec{b}\times\vec{c}=\vec{c}\cdot\vec{a}\times\vec{b}=\vec{b}\cdot\vec{c}\times\vec{a}=-\vec{a}\cdot\vec{c}\times\vec{b}=-\vec{c}\cdot\vec{b}\times\vec{a}=-\vec{b}\cdot\vec{a}\times\vec{c}$

– properties of determinant and the triple scalar product.



(iii) Geometrically the absolute value of $\vec{a}\cdot\vec{b}\times\vec{c}$ equals the volume of the parallelepiped with $\vec{a}$, $\vec{b}$ and $\vec{c}$ as its adjacent edges.



The area of the base: $|b|\cdot|c|\sin\theta$

Proof:

Observe that $\vec{b}\times\vec{c}=\left(|\vec{b}||\vec{c}|\sin\theta\right)\vec{n}$, where $\vec{n}$ is a unit vector in the same direction as $\vec{b}\times\vec{c}$ such that $\vec{b},\vec{c},\vec{n}$ form a right-hand triple.

Area of the base of the parallelepiped $=\left|\vec{b}\times\vec{c}\right|=|\vec{b}||\vec{c}|\sin\theta\ (>0)$.

Perpendicular height of the parallelepiped $= h = |\vec{a}|\cos\alpha = |\vec{a}\cdot\vec{n}|$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

$\therefore$ Volume of the parallelepiped = Base area of the parallelepiped $\times$ its height

$= \left(|\vec{b}||\vec{c}|\sin\theta\right)|\vec{a}\cdot\vec{n}| = \left|\vec{a}\cdot\left(|\vec{b}||\vec{c}|\sin\theta\right)\vec{n}\right| = |\vec{a}\cdot\vec{b}\times\vec{c}|$.

(iv) Three vectors are collinear or coplanar iff their triple scalar product is zero.

$= d_1\vec{i} + d_2\vec{j} + d_3\vec{k}$

7 Triple vector product $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$, where $\vec{a},\vec{b},\vec{c}\in R^3$

Consider the triple vector product $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$. We observe that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$ is a vector.

Properties:

(i) Note that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)\neq\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ in general, for example $\vec{i}\times\left(\vec{j}\times\vec{j}\right)=\vec{0}$ whereas $\left(\vec{i}\times\vec{j}\right)\times\vec{j}=-\vec{i}$.

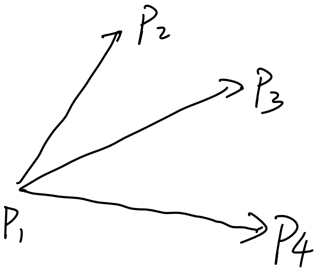
(ii) $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$ (prove by expanding both sides in components – straightforward but tedious).

Some Vector Identities:

(a) $\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)=\left(\vec{a}\cdot\vec{c}\right)\left(\vec{b}\cdot\vec{d}\right)-\left(\vec{a}\cdot\vec{d}\right)\left(\vec{b}\cdot\vec{c}\right)$

(b) $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\left(\vec{a}\cdot\vec{b}\times\vec{d}\right)\vec{c}-\left(\vec{a}\cdot\vec{b}\times\vec{c}\right)\vec{d}$

(c) $\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{b}\times\vec{c}\right)\times\left(\vec{c}\times\vec{a}\right)=\left(\vec{a}\cdot\vec{b}\times\vec{c}\right)^2$



Example 8

Let the points $P_1(1,3,-1), P_2(2,1,4), P_3(1,3,7), P_4(5,0,2)$ be given.

(a) Do these points lie in a plane, why? No!!!

(b) Find the volume of the tetrahedron with $P_1(1,3,-1), P_2(2,1,4), P_3(1,3,7), P_4(5,0,2)$ as its vertices.

Solution:

(a)

$P_1(1,3,-1), P_2(2,1,4), P_3(1,3,7), P_4(5,0,2)$

$$\Rightarrow \overrightarrow{P_1P_2}=\begin{pmatrix}2\\1\\4\end{pmatrix}-\begin{pmatrix}1\\3\\-1\end{pmatrix}=\begin{pmatrix}1\\-2\\5\end{pmatrix}, \overrightarrow{P_1P_3}=\begin{pmatrix}1\\3\\7\end{pmatrix}-\begin{pmatrix}1\\3\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\8\end{pmatrix}, \overrightarrow{P_1P_4}=\begin{pmatrix}5\\0\\2\end{pmatrix}-\begin{pmatrix}1\\3\\-1\end{pmatrix}=\begin{pmatrix}4\\-3\\3\end{pmatrix}.$$

$24 - 64 \quad = 1\cdot\left(0-(-24)\right)+2\left(0-32\right)+5\left(0-0\right)$

$\| \qquad\qquad \|$

-40

$\#0$ $\quad 1\cdot\begin{vmatrix}0&8\\-3&3\end{vmatrix}-(-2)\begin{vmatrix}0&8\\4&3\end{vmatrix}+5\begin{vmatrix}\ &\ \\4&-3\end{vmatrix}$

$//$

As $\overrightarrow{P_1P_2}\cdot\overrightarrow{P_1P_3}\times\overrightarrow{P_1P_4}=\begin{vmatrix}1&-2&5\\0&0&8\\4&-3&3\end{vmatrix}=-\begin{vmatrix}1&-2&5\\4&-3&3\\0&0&8\end{vmatrix}\overset{r_2-4r_1}{=}-\begin{vmatrix}1&-2&5\\0&5&-17\\0&0&8\end{vmatrix}=-40\neq 0$,

$P_1(1,3,-1)$, $P_2(2,1,4)$, $P_3(1,3,7)$, $P_4(5,0,2)$ don't lie in a plane.

(b)

Volume of the tetrahedron with P_1, P_2, P_3, P_4 as its vertices $=\dfrac{1}{6}\left|\overrightarrow{P_1P_2}\cdot\overrightarrow{P_1P_3}\times\overrightarrow{P_1P_4}\right|=\dfrac{1}{6}\left|-40\right|=\dfrac{20}{3}$ cu.units.

□

8 Linear Dependence and Independence (p.45 – p.47)

If $\vec{a_1},\vec{a_2},\cdots,\vec{a_k}$ $\in\mathbb{R}^n$ are any k n-component vectors, then an expression of the form $\sum_{i=1}^{k} m_i\vec{a_i}$ where m_1, m_2, $\cdots$, m_k are any k scalars is called a linear combination of $\vec{a_1},\vec{a_2},\cdots,\vec{a_k}$.

$\vec{a_1},\vec{a_2},\cdots,\vec{a_k}$ are linearly dependent if at least one of the vectors can be represented as a linear combination of the others, that is, $\vec{a_i}=\sum_{\substack{j=1\\ j\neq i}}^{k} m_j\vec{a_j}$ for some $1\le i\le k$. Otherwise $\vec{a_1},\vec{a_2},\cdots,\vec{a_k}$ are linearly independent.

$\vec{i}=(1,0,0)$, $\vec{j}=(0,1,0)$, $\vec{v}=(2,1,0)$.

$\vec{v}=2\vec{i}+1\vec{j}$

Example 9

Show that the vectors $3\vec{i}+5\vec{j}-2\vec{k}$, $4\vec{j}+2\vec{k}$ and $\vec{i}+\vec{j}-\vec{k}$ are linearly dependent.

Proof:

Observe that $3\vec{i}+5\vec{j}-2\vec{k}=\frac{1}{2}\left(4\vec{j}+2\vec{k}\right)+3\left(\vec{i}+\vec{j}-\vec{k}\right)$. Hence the vector $3\vec{i}+5\vec{j}-2\vec{k}$ is a linear combination of $4\vec{j}+2\vec{k}$ and $\vec{i}+\vec{j}-\vec{k}$, that is, $3\vec{i}+5\vec{j}-2\vec{k}$ lies in the plane expanded by the vectors $4\vec{j}+2\vec{k}$ and $\vec{i}+\vec{j}-\vec{k}$. Therefore, $3\vec{i}+5\vec{j}-2\vec{k}$, $4\vec{j}+2\vec{k}$ and $\vec{i}+\vec{j}-\vec{k}$ are linearly dependent.

W

However, we observe the vectors $\vec{i},\vec{j}$ and $\vec{k}$ are linearly independent.

$\vec{i}=(1,0,0)$, $\vec{j}=(0,1,0)$, $\vec{k}=(0,0,1)$.

$\vec{i},\vec{j},\vec{k}$ are linearly indep

$m_1\vec{i}+m_2\vec{j}+m_3\vec{k}=0 \Rightarrow m_1=m_2=m_3=0$

(m_1, m_2, m_3)

R^n

An equivalent definition is: A set of k n-components vectors $\vec{a_1},\vec{a_2},\cdots,\vec{a_k}$ are linearly independent *iff*

$m_1=m_2=m_3=\cdots=m_k=0$

$\sum_{i=1}^{k} m_i\vec{a_i}=\vec{0}$ implies $\cancel{m_1=m_2=\cdots=m_k=0}$, that is, the vector equation $\sum_{i=1}^{k} m_i\vec{a_i}=\vec{0}$ has the trivial solution $m_1=m_2=\cdots=m_k=0$ only.

$m_1\vec{a_1}+\cdots+\overset{\neq 0}{m_i}\vec{a_i}+\cdots+m_k\vec{a_k}=0$

$\vec{v_1}, \vec{v_2}$

N.B. If two vectors in R^3 are linearly dependent they must be collinear. If three vectors in R^3 are linearly dependent they must either be collinear or coplanar. Hence three vectors are linearly independent iff their triple scalar product is not zero.

Four or more vectors in R^3 will always be linearly dependent.

$\overset{\neq 0}{m_i}\vec{a_i} = -m_1\vec{a_1}-\cdots-m_{i-1}\vec{a_{i-1}}-m_{i+1}\vec{a_{i+1}}-\cdots-m_k\vec{a_k}$

$\Downarrow$

$\vec{a_i} = -\frac{m_1}{m_i}\vec{a_1}-\cdots-\frac{m_{i-1}}{m_i}\vec{a_{i-1}}-\frac{m_{i+1}}{m_i}\vec{a_{i+1}}-\cdots-\frac{m_k}{m_i}\vec{a_k}$

Example 10

If $\vec{a}=3\vec{i}+5\vec{j}-2\vec{k}$, $\vec{b}=4\vec{j}+2\vec{k}$ and $\vec{c}=\vec{i}+\vec{j}-\vec{k}$, show that $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent.

Proof:

Method I

$3\begin{vmatrix}4&2\\1&-1\end{vmatrix}-5\begin{vmatrix}0&2\\1&-1\end{vmatrix}+(-2)\begin{vmatrix}0&4\\1&1\end{vmatrix}$

$3(-4-2)-5(0-2)-2(0-4)$

As $\vec{a}\cdot\vec{b}\times\vec{c}=\begin{vmatrix}3&5&-2\\0&4&2\\1&1&-1\end{vmatrix}=0$, the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear or coplanar.

$-18+10+8=0$

Hence $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly dependent.

Method II

Consider the vector equation

$(3,5,-2)$ $(0,4,2)$ $(1,1,-1)$

$$m_1\vec{a}+m_2\vec{b}+m_3\vec{c}=\vec{0}.$$

We have the system of linear equations

$$\begin{cases} 3m_1 + m_3 = 0\text{----}(1)\\ 5m_1+4m_2+m_3=0\text{----}(2)\\ -2m_1+2m_2-m_3=0\text{----}(3)\end{cases}$$

$\doteq$ $\overset{\times}{=}$

$$\begin{cases} 3m_1 + m_3 = 0\text{----}(4)=(1)\\ 4m_2-\frac{2}{3}m_3=0\text{----}(5)=(2)-\frac{5}{3}(1)\\ 2m_2-\frac{1}{3}m_3=0\text{----}(6)=(3)-\left(-\frac{2}{3}\right)(1)\end{cases}$$

$$\begin{cases} 3m_1 \qquad + \quad m_3 = 0\text{----}(7)=(4) \\ \qquad 4m_2 - \dfrac{2}{3}m_3 = 0\text{----}(8)=(5) \\ \qquad\qquad 0 = 0\text{----}(9)=(6) - \dfrac{2}{4}(5) \end{cases}$$

Putting $m_3 = t \in R,$ we have

$$3m_1 = -t$$

$$4m_2 = \frac{2}{3}t$$

$\therefore$ The system has non-trivial solutions.

$\therefore$ The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly dependent.

□