GE2256: Game Theory Applications to Business
Lecture 7

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Continuous Strategies

- So far we have discussed one-shot simultaneous-move games with discrete strategies/actions for each player.
- Today we will look at games with continuous strategies/actions.
- Our objective will be again the same: to find the Nash equilibrium of the game.
- We will apply basic concepts from calculus (specifically the idea of partial differentiation) in order to analyze these games.
- We will restrict our attention to payoff functions that are quadratic because then one can solve these games without using calculus as well.

Continuous Strategies

- The assumption of quadratic payoff function means we can solve the games with continuous strategies using the method of calculus (partial derivatives) or via non-calculus method.
- We will discuss both approaches and you are free to use whichever method you feel is more comfortable.
- This means if you do not know calculus, you are not disadvantaged in any way for this course. Good news!!

Continuous Strategies

- Usually, there will be three steps in order to find the Nash equilibrium strategies in games with continuous strategies.
- **Step 1**: Write down the **payoff functions** for each player as a function of his strategy and other players' strategies/actions.
- **Step 2**: Write down the **best-response function** for each player. This step requires maximization of the payoff function.
- **Step 3**: As Nash equilibrium involves the play of mutual best responses, one needs to **simultaneously solve the best-response functions** of the players (obtained from step 2).

Finding a Value to Maximize a Function: CALCULUS method

- We will first discuss two methods (one calculus based and the other non-calculus based) for choosing a variable X to obtain the maximum value of a variable that is a function of it, say $Y = F(X)$.
- We will further restrict ourselves to quadratic functions only: $Y = F(X) = A + BX - CX^2$ with $B > 0$ and $C > 0$.
- Using differentiation (CALCULUS), we immediately know that:

$$\frac{dY}{dX} = F'(X) = \frac{d}{dX}(A) + \frac{d}{dX}(BX) - \frac{d}{dX}(CX^2)$$

using the fact that, if $Y(X) = G(X) + H(X)$, then:

$$\frac{dY}{dX} = \frac{dG}{dX} + \frac{dH}{dX}$$

Finding a Value to Maximize a Function: CALCULUS method

- Thus, we get:

$$\frac{dY}{dX} = 0 + B\frac{d}{dX}(X) - C\frac{d}{dX}(X^2)$$

using the fact that the differentiation of a constant is zero and $\frac{d}{dX}(kF(X)) = k\frac{d}{dX}(F(X))$ where k is a constant.

- Hence, we have:

$$\frac{dY}{dX} = B - 2CX$$

using the fact that $\frac{d}{dX}(X) = 1$ and $\frac{d}{dX}(X^a) = aX^{a-1}$ where a is a constant.

Finding a Value to Maximize a Function: CALCULUS method

- For the function $F(X)$ to attain maximum, we must have:

$$\frac{dY}{dX} = 0$$

$$\implies B - 2CX = 0 \implies X = \frac{B}{2C}$$

- Also, observe that the second order condition for a maxima is satisfied:

$$\frac{d^2Y}{dX^2} = \frac{d}{dX}\left(\frac{dY}{dX}\right) = -2C < 0$$

Finding a Value to Maximize a Function: NON-CALCULUS method

- Instead of calculus (differentiation), we can use the following method of finding the X at which $Y = F(X) = A + BX - CX^2$ is maximized.
- Observe that $Y = F(X) = A + BX - CX^2$ can be written as:

$$A + \frac{B^2}{4C} - \frac{B^2}{4C} + BX - CX^2$$

$$\implies A + \frac{B^2}{4C} - C(\frac{B^2}{4C^2} - \frac{BX}{C} + X^2)$$

$$\implies A + \frac{B^2}{4C} - C(\frac{B}{2C} - X)^2$$

Finding a Value to Maximize a Function: NON-CALCULUS method

$$Y = A + \frac{B^2}{4C} - C\left(\frac{B}{2C} - X\right)^2$$

- In the final form of the expression, X appears only in the last term, where a square involving it is being subtracted (remember $C > 0$).
- The whole expression is maximized when this subtracted term is made as small as possible, that is, zero. This happens when:

$$X = \frac{B}{2C}$$

- This is the same answer that we get using calculus.
- You can use any of the above methods (whichever you find easier!).

Solving simultaneous equations

- Suppose you have the following two equations in x and y:

$$ax + by = c$$

$$dx + ey = f$$

  where a, b, c, d, e, f are constants or parameters.

- We have to find the values of variables x and y by solving the above two linear equations simultaneously.
- You can use the **method of substitution** or **Cramer's rule** or use any other method that you know to solve simultaneous equations.

Solving simultaneous equations: substitution method

$$ax + by = c$$
$$dx + ey = f$$

- Write y in terms of x from the second equation:

$$ey = f - dx \implies y = \frac{f}{e} - \frac{d}{e}x$$

- Substitute the value of y in first equation to get:

$$ax + b(\frac{f}{e} - \frac{d}{e}x) = c$$
$$\implies (a - \frac{bd}{e})x = c - \frac{bf}{e} \implies (ae - bd)x = ce - bf$$
$$x = \frac{ce - bf}{ae - bd}$$

Solving simultaneous equations: substitution method

$$y = \frac{f}{e} - \frac{d}{e}x$$

$$\implies y = \frac{f}{e} - \frac{d}{e}(\frac{ce - bf}{ae - bd})$$

$$\implies y = \frac{1}{e}(f - \frac{cde - bdf}{ae - bd})$$

$$\implies y = \frac{1}{e}(\frac{aef - bdf - cde + bdf}{ae - bd})$$

$$\implies y = \frac{af - cd}{ae - bd}$$

Solving simultaneous equations: Cramer's rule

- Write the two equations in compact matrix form:

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$$

- We have the following determinants:

$$D_x = \begin{vmatrix} c & b \\ f & e \end{vmatrix} = ce - bf$$

$$D_y = \begin{vmatrix} a & c \\ d & f \end{vmatrix} = af - cd$$

$$D = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd$$

$$x = \frac{D_x}{D} = \frac{ce - bf}{ae - bd} \qquad y = \frac{D_y}{D} = \frac{af - cd}{ae - bd}$$

Illustration using an Effort Game

- Two individuals are playing an effort game as follows:
- If both individuals devote more effort then they are both better off.
- Each player's set of actions is the set of effort levels (non-negative numbers).
- Player i's preferences are represented by the payoff function $a_i(c + a_j - a_i)$, for $i = 1, 2$.
- a_i is i's effort level and a_j is j's effort level. c is a constant. Also, $a_i, a_j \geqslant 0$.
- What is the Nash equilibrium in this effort game?

Effort Game

We first need to write down the payoff function of each individual. For individual 1, we have:

$$u_1 = a_1(c + a_2 - a_1) = (c + a_2)a_1 - a_1^2$$

(CALCULUS method) Taking a_2 as given, the first order condition for individual 1 is:

$$\frac{\partial u_1}{\partial a_1} = 0 \implies c + a_2 - 2a_1 = 0.$$

Writing a_1 in terms of a_2, we get the individual 1's best response function:

$$a_1 = \frac{1}{2}(c + a_2)$$

Effort Game

- Individual 1's best response function is:

$$a_1 = \frac{1}{2}(c + a_2)$$

- This means that if individual 2 chooses an effort of a_2, individual 1's best response is to choose $\frac{1}{2}(c + a_2)$ (the choice of effort which maximizes individual 1's payoffs).
- For example, if $a_2 = 0$, individual 1's best response is to choose $\frac{1}{2}(c + 0) = \frac{c}{2}$. If $a_2 = c$, then individual 1's best response is $\frac{1}{2}(c + c) = c$.
- We can obtain the same answer for the best response function through non-calculus method of maximizing a quadratic function instead (see next slide).

Effort Game

- The payoff function of individual 1 is:

$$u_1 = a_1(c + a_2 - a_1) = (c + a_2)a_1 - a_1^2$$

  (NON-CALCULUS method) Writing in terms of our notations, we have: $B = (c + a_2)$, $C = 1$ and $X = a_1$.

- Then,

$$X = \frac{B}{2C} \implies a_1 = \frac{c + a_2}{2}$$

  which is the best response function for individual 1.

Effort Game

- Payoff function for individual 2 is given by:

$$u_2 = a_2(c + a_1 - a_2) = (c + a_1)a_2 - a_2^2$$

(CALCULUS method) Taking a_1 as given, the first order condition for individual 2 is:

$$\frac{\partial u_2}{\partial a_2} = 0 \implies c + a_1 - 2a_2 = 0.$$

Writing a_2 in terms of a_1, we get the individual 2's best response function:

$$a_2 = \frac{1}{2}(c + a_1)$$

Effort Game

- Individual 2's best response function is:

$$a_2 = \frac{1}{2}(c + a_1)$$

- This means that if individual 1 chooses an effort of a_1, individual 2's best response is to choose $\frac{1}{2}(c + a_1)$ (the choice of effort which maximizes individual 2's payoffs).

Effort Game

- The payoff function of individual 2 is:

$$u_2 = a_2(c + a_1 - a_2) = (c + a_1)a_2 - a_2^2$$

  (NON-CALCULUS method) Writing in terms of our notations, we have: $B = (c + a_1)$, $C = 1$ and $X = a_2$.

- Then,

$$X = \frac{B}{2C} \implies a_2 = \frac{c + a_1}{2}$$

  which is the best response function for individual 2.

Effort Game

To find the Nash equilibrium levels of a_1 and a_2, we need to find the mutual best-response, that is, solve simultaneously the two best response functions (only at that intersection point, my action is a best response to your action and your action is a best response to my action):

$$a_1 = \frac{1}{2}(c + a_2)$$

and:

$$a_2 = \frac{1}{2}(c + a_1)$$

Effort Game

- Substituting a_2 from the second equation to the first, we have:

$$a_1 = \frac{1}{2}\{c + \frac{1}{2}(c + a_1)\}$$

$$\implies (1 - \frac{1}{4})a_1 = (\frac{1}{2} + \frac{1}{4})c \implies a_1 = c$$

- Solving, we get the following unique Nash equilibrium:

$$(a_1, a_2) = (c, c)$$

- Alternatively, you can use the Cramer's rule to solve the equations simultaneously (see next two slides).

Effort game: Cramer's rule

- The two equations are:

$$2a_1 - a_2 = c$$

$$-a_1 + 2a_2 = c$$

- You can solve the equations simultaneously using the Cramer's rule as well. Write the two equations in compact matrix form:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} c \\ c \end{bmatrix}$$

Effort game: Cramer's rule

- We have the following determinants:

$$D_{a_1} = \begin{vmatrix} c & -1 \\ c & 2 \end{vmatrix} = 2c + c = 3c$$

$$D_{a_2} = \begin{vmatrix} 2 & c \\ -1 & c \end{vmatrix} = 2c + c = 3c$$

$$D = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$a_1 = \frac{D_{a_1}}{D} = \frac{3c}{3} = c \qquad a_2 = \frac{D_{a_2}}{D} = \frac{3c}{3} = c$$

Cournot Model

The Cournot model explains how firms behave if they simultaneously choose how much they produce.

In the basic Cournot model, we assume that:

- There are two firms and no others can enter the market.
- The firms have identical costs.
- The firms sell identical products.
- The firms set their quantities simultaneously.

Cournot Model

- The players of the game are the two firms.
- Each firm chooses an output level in $[0, \infty)$. This $[0, \infty)$ is the strategy space of each firm and a typical strategy is the quantity/output produced by a firm.
- The profit function for each firm is it's payoff function. Profit of a firm depends on the quantity chosen by itself and the quantity chosen by the another firm.

Cournot Model

- Let us consider the simultaneous quantity setting game between two firms, Firm 1 (F1) and Firm 2 (F2).
- The market demand function is given by:

$$P(Q) = a - Q,$$

  for $Q \leqslant a$; where P is the market price, and Q is total output produced in the market.
- Assume each firm has a constant marginal cost of c per unit of output.

Cournot Model

- Each of the players, F1 and F2, choose a non-negative quantity, q_1 and q_2.
- The market price depends on the actions of both players:

$$P(q_1, q_2) = a - q_1 - q_2.$$

- The two players' payoffs are given by

$$u_1(q_1, q_2) = q_1(a - q_1 - q_2) - cq_1;$$

$$u_2(q_1, q_2) = q_2(a - q_1 - q_2) - cq_2.$$

Best-Response Functions

The first step is to find the best response functions for F1 and F2. The best response function for a firm shows the choice of quantity which provides it with the highest profit, given the quantity of the other firm.

- (CALCULUS METHOD) This is found out by differentiating the profit function u_i w.r.t. q_i, taking the other firm's quantity as given (as a constant) and then setting this $\frac{\partial u_i}{\partial q_i}$ equal to zero (the first order condition, FOC).
- Then rewrite q_i in terms of the other firm's quantity from this FOC.

Best Response

F1's profit is given by

$$u_1(q_1, q_2) = q_1(a - q_1 - q_2) - cq_1;$$

Taking q_2 as a constant, differentiate u_1 above w.r.t. q_1 to get the F1's FOC:

$$a - 2q_1 - q_2 - c = 0$$

$$q_1 = \frac{1}{2}(a - c - q_2) = \mathscr{B}_1(q_2).$$

Similarly, you can obtain the F2's best-response function to be:

$$q_2 = \frac{1}{2}(a - c - q_1) = \mathscr{B}_2(q_1).$$

Best responses via non-calculus method

In this slide, we use the non-calculus method to obtain the best response function for firm 1.

For F1, profits are:

$$u_1(q_1, q_2) = q_1(a - q_1 - q_2) - cq_1 = 0 + (a - c - q_2)q_1 - q_1^2$$

In terms of our general quadratic function ($Y = A + BX - CX^2$), we have:

$$A = 0, B = (a - c - q_2), C = 1$$

Thus,

$$q_1 = \frac{B}{2C} = \frac{a - c - q_2}{2}$$

Best responses via non-calculus method

In this slide, we use the non-calculus method to obtain the best response function for firm 2.

For F2, profits are:

$$u_2(q_1, q_2) = q_2(a - q_1 - q_2) - cq_2 = 0 + (a - c - q_1)q_2 - q_2^2$$

In terms of our general quadratic function ($Y = A + BX - CX^2$), we have:

$$A = 0, B = (a - c - q_1), C = 1$$

Thus,

$$q_2 = \frac{B}{2C} = \frac{a - c - q_1}{2}$$

Nash Equilibrium

At the Nash equilibrium, each firm does best given what the other firm is doing. That is, given that the other firm is producing at the equilibrium, the best that the firm can do is to produce it's equilibrium quantity. There is no unilateral profitable deviation from choosing any other quantity.

In other words, the quantities corresponding to the mutual best response constitute the Nash equilibrium of this simultaneous quantity setting game.

So, to calculate the equilibrium quantities, one needs to solve the system of best response functions.

Nash Equilibrium

- The Nash equilibrium is obtained by solving the system of equations simultaneously:

$$q_1 = \frac{1}{2}(a - c - q_2)$$
$$q_2 = \frac{1}{2}(a - c - q_1)$$

- Substitute q_2 from the second equation to the first to get:

$$q_1 = \frac{1}{2}[a - c - (\frac{1}{2}(a - c - q_1))] = \frac{1}{4}(a - c) + \frac{1}{4}q_1$$

- This gives the following equilibrium quantities:

$$q_1^{NE} = q_2^{NE} = \frac{1}{3}(a - c)$$

- Alternatively, you can use the Cramer's rule to solve the equations simultaneously (see next two slides).

Cournot Model: Cramer's rule

- The two equations are:

$$2q_1 + q_2 = a - c$$

$$q_1 + 2q_2 = a - c$$

- You can solve the equations simultaneously using the Cramer's rule as well. Write the two equations in compact matrix form:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} a - c \\ a - c \end{bmatrix}$$

Cournot Model: Cramer's rule

- We have the following determinants:

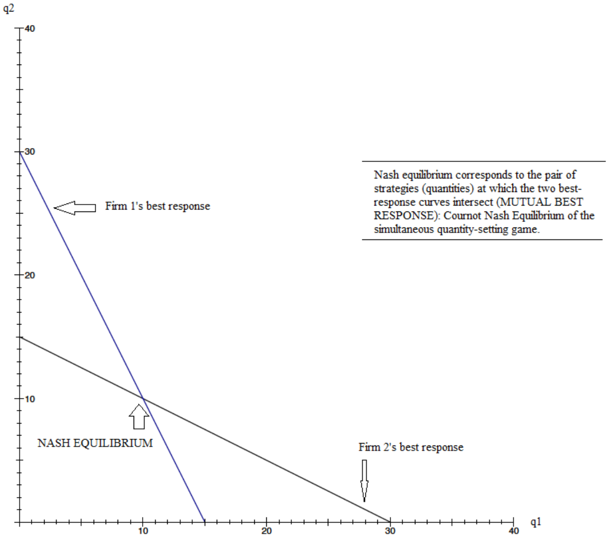
$$D_{q_1} = \begin{vmatrix} a-c & 1 \\ a-c & 2 \end{vmatrix} = 2(a-c) - (a-c) = a-c$$

$$D_{q_2} = \begin{vmatrix} 2 & a-c \\ 1 & a-c \end{vmatrix} = 2(a-c) - (a-c) = a-c$$

$$D = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$q_1 = \frac{D_{q_1}}{D} = \frac{a-c}{3} \qquad q_2 = \frac{D_{q_2}}{D} = \frac{a-c}{3}$$

Cournot-Nash Equilibrium



Bertrand Game in Differentiated Products

- Assume that there are two firms, 1 and 2, producing different brands of soft drink. Firm 1 has a demand function, $q_1 = 100 - 2p_1 + p_2$ and constant marginal cost of $c_1 = 10$. Firm 2 has a demand function, $q_2 = 100 - 2p_2 + p_1$ and constant marginal cost of $c_2 = 15$.
- What is the Nash equilibrium when firms set prices simultaneously (that is, when the model is Bertrand model)? What are the corresponding quantities and equilibrium profits? Which firm earns more in equilibrium and why?

Exercise: Nash Equilibrium with Differentiated Products in Bertrand model

- Firm 1's profit function is:

$$u_1 = (p_1 - 10)(100 - 2p_1 + p_2)$$

And the first order condition is:

$$120 - 4p_1 + p_2 = 0.$$

Firm 1's best response function is:

$$p_1 = 30 + \frac{p_2}{4}$$

Exercise: Nash Equilibrium with Differentiated Products in Bertrand model

- Alternatively, use the non-calculus method: Firm 1's profit function is

$$u_1 = (p_1 - 10)(100 - 2p_1 + p_2) = 100p_1 - 2p_1^2 + p_2p_1 - 1000 + 20p_1 - 10p_2$$

$$\Longrightarrow u_1 = (-1000 - 10p_2) + (120 + p_2)p_1 - 2p_1^2$$

$$A = -1000 - 10p_2, B = 120 + p_2, C = 2$$

Thus, we have:

$$p_1 = \frac{B}{2C} = \frac{120 + p_2}{4} = 30 + \frac{p_2}{4}$$

Exercise: Nash Equilibrium with Differentiated Products in Bertrand model

Firm 2's profit function is:

$$u_2 = (p_2 - 15)(100 - 2p_2 + p_1)$$

And the first order condition is:

$$130 - 4p_2 + p_1 = 0.$$

Firm 2's best response function is:

$$p_2 = \frac{65}{2} + \frac{p_1}{4}$$

Exercise: Nash Equilibrium with Differentiated Products in Bertrand model

- Alternatively, use the non-calculus method: Firm 2's profit function is

$$u_2 = (p_2 - 15)(100 - 2p_2 + p_1) = 100p_2 - 2p_2^2 + p_2p_1 - 1500 + 30p_2 - 15p_1$$

$$\implies u_2 = (-1500 - 15p_1) + (130 + p_1)p_2 - 2p_2^2$$

$$A = -1500 - 15p_1, B = 130 + p_1, C = 2$$

Thus, we have:

$$p_2 = \frac{B}{2C} = \frac{130 + p_1}{4} = \frac{65}{2} + \frac{p_1}{4}$$

Differentiated Bertrand Model: Cramer's rule

- The two equations are:

$$4p_1 - p_2 = 120$$

$$-p_1 + 4p_2 = 130$$

- You can solve the equations simultaneously using the Cramer's rule. Write the two equations in compact matrix form:

$$\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 120 \\ 130 \end{bmatrix}$$

Differentiated Bertrand Model: Cramer's rule

- We have the following determinants:

$$D_{p_1} = \begin{vmatrix} 120 & -1 \\ 130 & 4 \end{vmatrix} = 480 + 130 = 610$$

$$D_{p_2} = \begin{vmatrix} 4 & 120 \\ -1 & 130 \end{vmatrix} = 520 + 120 = 640$$

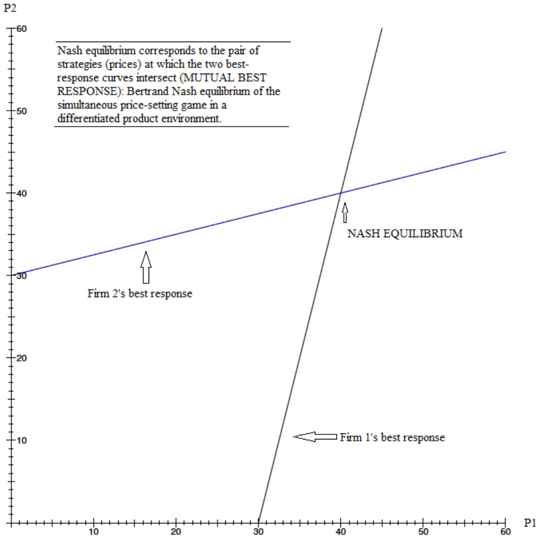
$$D = \begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} = 16 - 1 = 15$$

$$p_1 = \frac{D_{p_1}}{D} = \frac{610}{15} = 40.7 \qquad p_2 = \frac{D_{p_2}}{D} = \frac{640}{15} = 42.7$$

Exercise: Nash Equilibrium with Differentiated Products in Bertrand model

- Solving the two equations (or best response functions) simultaneously, we get the Nash equilibrium:
- $p_1^{NE} = 40.7, \quad p_2^{NE} = 42.7$
- Both prices are much higher than the corresponding marginal costs.
- Using the demand equations, the equilibrium quantities are: $q_1 = 61.3$ and $q_2 = 55.3$. The equilibrium profits are $u_1 = 1880.9$ and $u_2 = 1530.9$.
- Firm 1 has a lower price, higher output and higher profits than firm 2. Since firms have identical demand situations this arises due to the fact that firm 1 has lower costs than firm 2.

Nash equilibrium: Differentiated Bertrand



Stackelberg Model

- Stackelberg model is an example of a sequential game with continuous actions.
- It is a **sequential quantity-setting/quantity choice** model among firms.
- We will explain this model using two players/firms: this means we are in a oligopoly market structure with two firms (or duopoly).
- The two firms produce an identical good and compete in quantities.
- You might want to contrast this model to Cournot duopoly model which is the simultaneous move game of quantity competition.

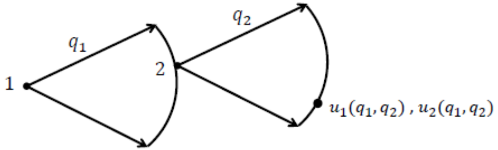
Stackelberg Model

The sequence of moves is as follows:

- In the first period, firm 1 chooses it's quantity to sell in the market, q_1.
- Firm 2 observes q_1 and then chooses the quantity to produce, q_2.

Firm 1 is referred to as the **leader** and firm 2 as the **follower**.

The extensive form representation of the game is as follows:



Stackelberg Duopoly

- Specifically, suppose the following market demand curve (or, market price is given by the linear equation):

$$p = 130 - q_1 - q_2$$

- Assume that the marginal cost of producing each unit of output is 10 for both firms.
- This means that profits are given by:

$$u_1(q_1, q_2) = pq_1 - 10q_1 = (130 - q_1 - q_2)q_1 - 10q_1 = (120 - q_1 - q_2)q_1$$

$$u_2(q_1, q_2) = pq_2 - 10q_2 = (130 - q_1 - q_2)q_2 - 10q_2 = (120 - q_1 - q_2)q_2$$

Strategies

- Firm 1 moves first without observing the choice of firm 2. Therefore, the strategy of firm 1 is simply q_1, the quantity it chooses to produce.
- Firm 2 observes q_1 before making its choice. Since a strategy for firm 2 must be a complete contingent plan of actions, it must provide a full description of what to do for each value of q_1 that could be chosen by firm 1.
- This means a strategy for firm 2 must be a function of q_1: write it as $s_2(q_1)$.
- Therefore, a strategy profile in this game is of the form: $(q_1, s_2(q_1))$.

Solving the Model: Backward Induction

- In order to solve for the SPE of the game, one can apply backward induction.
- We first need to look at firm 2's optimal decision in the second stage of the game.
- In the second stage of the game, firm 2 has observed q_1. Given this production level, firm 2 will then choose the production level that will maximize its profits. That is, firm 2 will choose the value of q_2 that maximizes:

$$u_2(q_1, q_2) = (120 - q_1 - q_2)q_2$$

- This is done using the first order condition by partial differentiation of $u_2(.)$ with respect to q_2, with q_1 as given. Alternatively, you can use the non-calculus method.

Solving the Model: Backward Induction

- FOC:

$$\frac{\partial u_2(q_1, q_2)}{\partial q_2} = 120 - q_1 - 2q_2 = 0$$

- Therefore, the best response of firm 2 in the second stage (after it observes q_1) is:

$$BR_2(q_1) = \frac{120 - q_1}{2}$$

- Since SPE requires optimal behavior in all stages of the game, in any SPE, firm 2 must follow the strategy:

$$s_2(q_1) = \frac{120 - q_1}{2}$$

Solving the Model: Backward Induction

- Now let us go back to stage 1 of the game: in stage 1, firm 1 looks ahead and realizes that if firm 2 is rational, then firm 2's strategy must be

$$s_2(q_1) = \frac{120 - q_1}{2}$$

- Using this knowledge, firm 1 can write his payoffs as

$$u_1(q_1, s_2(q_1)) = (120 - q_1 - \frac{120 - q_1}{2})q_1$$

That is,

$$u_1(q_1, s_2(q_1)) = (\frac{120 - q_1}{2})q_1 = 60q_1 - \frac{q_1^2}{2}$$

Solving the Model: Backward Induction

- Firm 1 chooses q_1 to maximize his payoffs. FOC:

$$60 - q_1 = 0$$

- That means $q_1^* = 60$ and $q_2^* = s_2(60) = \frac{120-60}{2} = 30$.

SPE

- The game has a unique SPE profile, given by:

$$(q_1^*, s_2^*(q_1)) = (60, \frac{12 - q_1}{2})$$

- The outcome of the game for the SPE profile is:

$$(q_1, q_2) = (60, 30)$$

The corresponding payoffs are:

$$(u_1, u_2) = (1800, 900)$$

SPE

- Note that there is a first mover advantage in this game: even though both firms are otherwise identical, except for the move, the firm which moves first obtains a higher payoff than the firm moving second.
- This could be contrasted with the simultaneous move version of the above game: the Cournot duopoly game (discussed earlier).
- In a Cournot duopoly competition with the same demand curve and same marginal costs, Nash equilibrium actions are $q_1 = q_2 = 40$ and equilibrium payoffs are $(1600, 1600)$. (*You can check this yourself*)

References

Although the lecture slide is self-contained, you might want to refer to the following sections of chapter 5: SIMULTANEOUS-MOVE GAMES of the textbook (GAMES OF STRATEGY by Dixit, Skeath, Reiley):

1: Pure strategies that are continuous variables

Appendix: Finding a Value to Maximize a Function