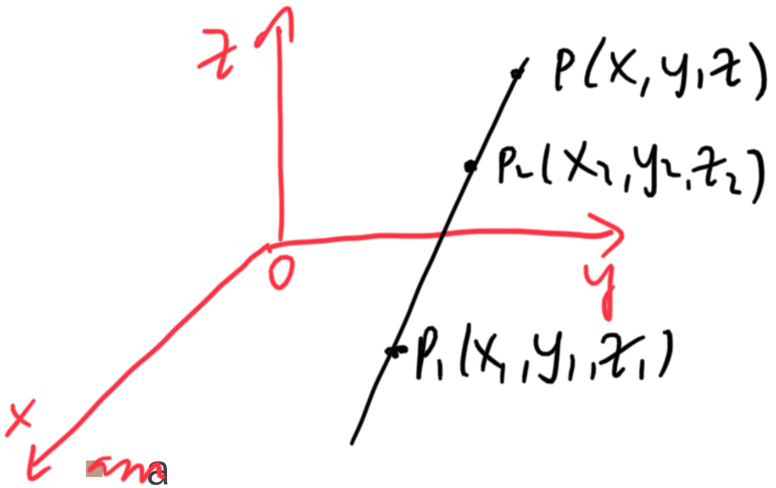
EQUATION OF A LINE PASSING THROUGH TWO GIVEN POINTS



By the theorem in pg 17

$$l = \frac{x_2 - x_1}{d} \quad m = \frac{y_2 - y_1}{d} \quad n = \frac{z_2 - z_1}{d} \quad \ldots (1)$$

let P be any point on the line $PP_1 = r$

$$\frac{x - x_1}{r}, \quad \frac{y - y_1}{r}, \quad \frac{z - z_1}{r} \quad \ldots (2)$$

let (1) = (2)

$$\boxed{\frac{x - x_1}{r} = \frac{x_2 - x_1}{d}}, \quad \frac{y - y_1}{r} = \frac{y_2 - y_1}{d}, \quad \frac{z - z_1}{r} = \frac{z_2 - z_1}{d}$$

$$x - x_1 = \frac{r}{d}(x_2 - x_1) \quad \vdots \quad \vdots$$

let $\lambda = r/d$ then...

$$x - x_1 = \lambda(x_2 - x_1), \quad y - y_1 = \lambda(y_2 - y_1), \quad z - z_1 = \lambda(z_2 - z_1)$$

$$\lambda = \frac{x - x_1}{x_2 - x_1} \qquad \lambda = \frac{y - y_1}{y_2 - y_1} \qquad \lambda = \frac{z - z_1}{z_2 - z_1}$$

$$\therefore \boxed{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}} \quad \ldots \text{(A)}$$

when we have 2 points

also note ... (1) $\boxed{\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = d}$

$$\boxed{\begin{cases} x = x_1 + ld \\ y = y_1 + md \\ z = z_1 + nd \end{cases}} \quad \ldots \text{(B)}$$

Direction ratio (l, m, n)

EXAMPLE

Ex. Obtain the equation of the line passing through $(1, -1, 3)$ and $(2, 1, 5)$.

$x_1 \; y_1 \; z_1$ $\qquad$ $x_2 \; y_2 \; z_2$

Ans.

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-1}{2-1} = \frac{y+1}{1+1} = \frac{z-3}{5-3}$$

$$\frac{x-1}{1} = \frac{y+1}{2} = \frac{z-3}{2}$$

$$\frac{2(x-1)}{\cancel{2}} = \frac{y+1}{\cancel{2}} = \frac{z-3}{\cancel{2}}$$

$$2(x-1) = y+1 = z-3$$

Alternative

$(1, -1, 3)$ $\quad$ $(2, 1, 5)$

$x_2 \; y_2 \; z_2$ $\qquad$ $x_1 \; y_1 \; z_1$

$$\frac{x-2}{1-2} = \frac{y-1}{-1-1} = \frac{z-5}{3-5}$$

$$\frac{x-2}{-1} = \frac{y-1}{-2} = \frac{z-5}{-2}$$

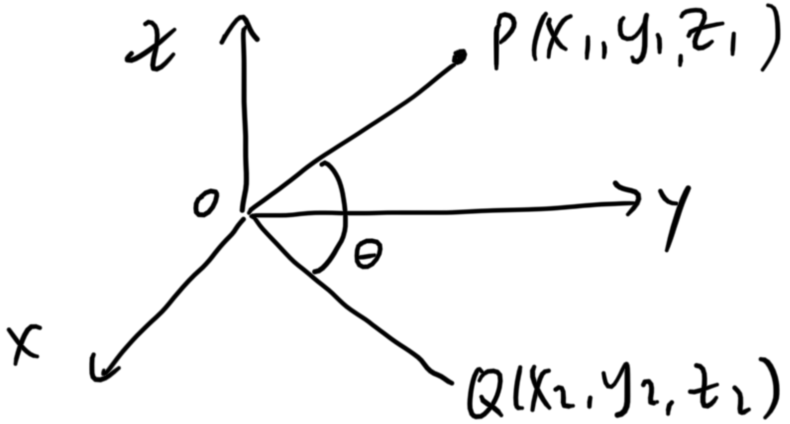
$$\frac{2(x-2)}{\cancel{-2}} = \frac{y-1}{\cancel{-2}} = \frac{z-5}{\cancel{-2}}$$

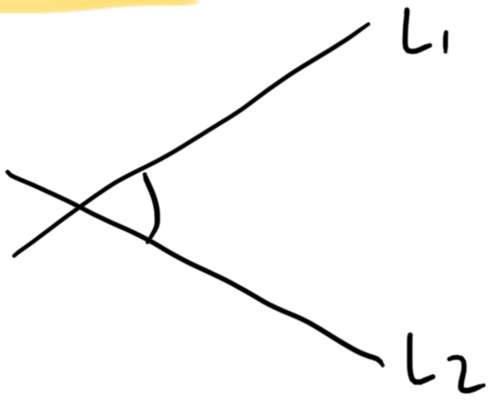
$$2(x-2) = y-1 = z-5$$

ANGLE BETWEEN TWO LINES

Theorem: If L_1 and L_2 are two lines having direction cosine l_1, m_1, n_1 and l_2, m_2, n_2 then the angle θ between them is given by

$$\boxed{\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2}$$





EXAMPLE

Find the angle between the lines having direction cosines

$$\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}} \quad \text{and} \quad \frac{1}{\sqrt{7}}, \frac{2}{\sqrt{7}}, -\frac{\sqrt{2}}{\sqrt{7}}$$

Ans.

$$\cos\theta = \frac{2\times 1}{\sqrt{14}\sqrt{7}} + \frac{3}{\sqrt{14}}\times\frac{2}{\sqrt{7}} + \frac{1}{\sqrt{14}}\times\frac{-\sqrt{2}}{\sqrt{7}} = \frac{4\sqrt{2}-1}{7}$$

$$\theta = \cos^{-1}\left(\frac{4\sqrt{2}-1}{7}\right)$$

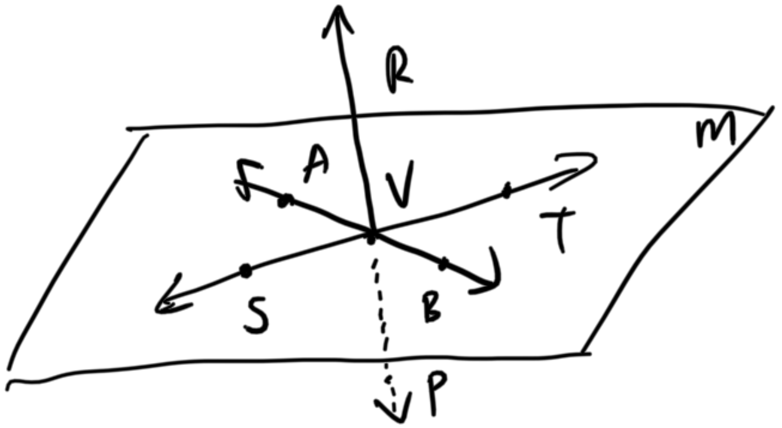
THE PLANE

The Plane in the space is detemined by a point and a vector that is perpendicular to plane.

① let $P(x_0, y_0, z_0)$ be a given point and $\vec{n}$ is the orthogonal vector

② let $P(x, y, z)$ be any point in space and $\vec{r}, \vec{r_0}$ is the position vector of point P and P_0 respectively.

PRELIMINARY CONCEPT



A, B, S, T and V all coplanar points

$\overleftrightarrow{AB}$ and $\overleftrightarrow{ST}$ are coplanar lines

$\overline{AB}$ and $\overline{ST}$ are coplanar segments

A, B, S, T and R are non-coplanar points

$\overleftrightarrow{AB}$, $\overleftrightarrow{ST}$ and $\overleftrightarrow{RP}$ are non-coplanar lines

$\overline{AB}$, $\overline{ST}$ and $\overline{RP}$ are non-coplanar segments

DEFINITION : The point of intersection of a line and a plane is called the foot of the line.

From diagram: V is the foot of $\overleftrightarrow{RP}$ in the plane m.

PLANES IN TERMS OF VECTOR

$$\boxed{n \cdot r = n\, r_0}$$

let $n = \langle a, b, c \rangle$

$r = \langle x, y, z \rangle$

$r_0 = \langle x_0, y_0, z_0 \rangle$

vector equation

$$\langle a, b, c \rangle \cdot \langle x, y, z \rangle = \langle a, b, c \rangle \cdot \langle x_0, y_0, z_0 \rangle$$

Scalar equation of plane

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$ax+by+cz+d=0$$

where $d=-(ax_0+by_0+cz_0)$

EXAMPLE

Find the equation of the plane through $(1,-1,1)$ and with normal vector $\vec{i}+\vec{j}-\vec{k}$

Ans: $\underset{x_0}{(1},\underset{y_0}{-1},\underset{z_0}{1)}$ $a=1$, $b=1$ $c=-1$

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$1(x-1)+1(y+1)-1(z-1)=0$$

$$x-1+y+1-z+1=0$$

$$x+y-z+1=0$$

THE PLANE WITH SYMMETRIC EQUATIONS

2D geometry $\Rightarrow$ $ax+by+c=0$ straight line

3D geometry $\Rightarrow$ $Ax+By+Cz+D=0$ plane. .... (1)

$P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$$
\begin{array}{rl}
 & Ax_1+By_1+Cz_1+D=0 \\
(-) & Ax_2+By_2+Cz_2+D=0 \\
\hline
\end{array}
$$

$$A(x_2-x_1)+B(y_2-y_1)+C(z_2-z_1)=0$$

Recall pg 36. $\dfrac{x-x_1}{x_2-x_1}=\dfrac{y-y_1}{y_2-y_1}=\dfrac{z-z_1}{z_2-z_1}$

$$\therefore \quad \left.\begin{aligned} x &= x_1 + (x_2 - x_1)t \\ y &= y_1 + (y_2 - y_1)t \\ z &= z_1 + (z_2 - z_1)t \end{aligned}\right\} \text{ sub into } \ldots ①$$

$$A[x_1 + (x_2 - x_1)t] + B[y_1 + (y_2 - y_1)t] + C[z_1 + (z_2 - z_1)t] + D$$

$$Ax_1 + By_1 + Cz_1 + D + t[A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1)] = 0$$

Two properties   2 planes

let
$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

then

① $A_1A_2 + B_1B_2 + C_1C_2 = 0$

perpendicular

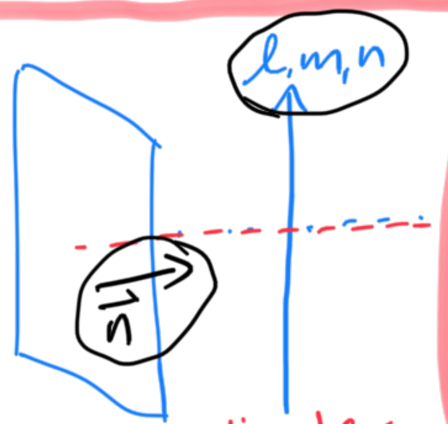
② Q4 $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2} = c$

parallel

(Find 2 parallel planes)

A plane $\boxed{Ax + By + Cz + D = 0}$

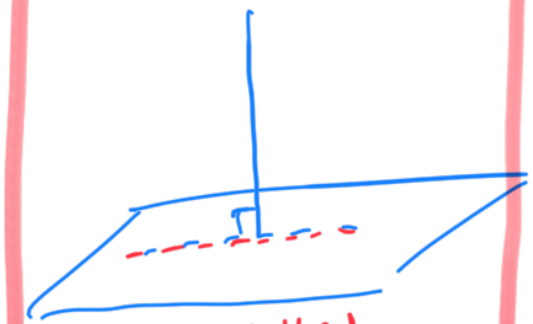
and a straight line $\boxed{\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}}$



perpendicular

$Al + Bm + Cn = 0$

parallel



parallel

$\dfrac{A}{l} = \dfrac{B}{m} = \dfrac{C}{n}$

perpendicular

Q4

EXAMPLE

Show that the plane $2x-3y+z-2=0$ is parallel to the line $\frac{x-2}{1}=\frac{y+2}{1}=\frac{z+1}{1}$

Ans: parallel $Al+Bm+Cn=0$

$A=2 \quad B=-3 \quad C=1$

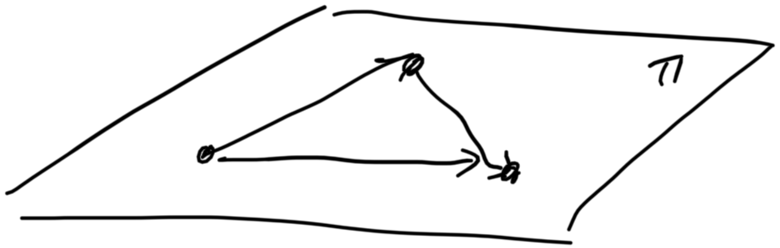
$l=1 \quad m=1 \quad n=1$

$2(1)+(-3)(1)+(1)(1)=2+-3+1=0$ ✓

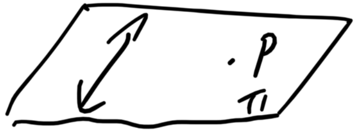
4 WAYS TO DETERMINE A PLANE

~~TWO~~ POSTULATE~~S~~ CONCERNING L~~INES AN~~D PLANES

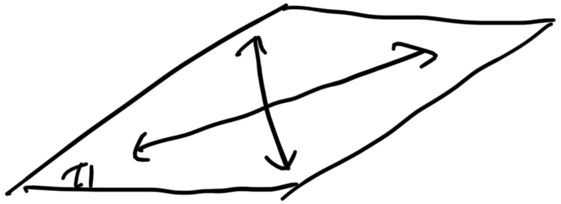
Three noncollinear points determine a plane.



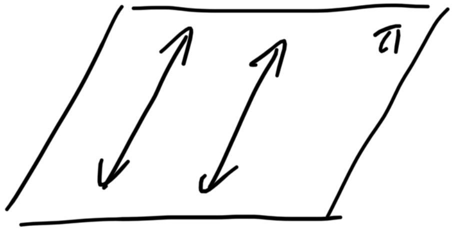
THEOREMS



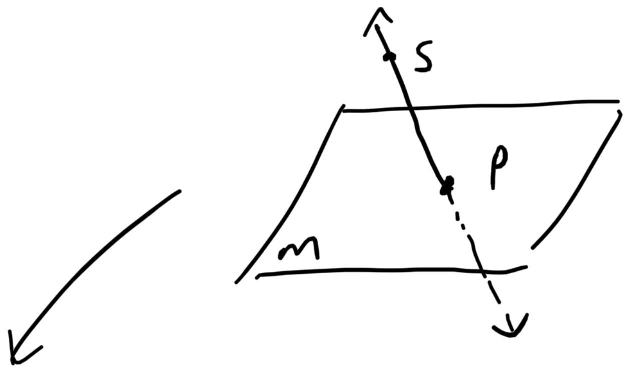
- A line and a point not on the line determine a plane
- Two intersecting lines determine a plane



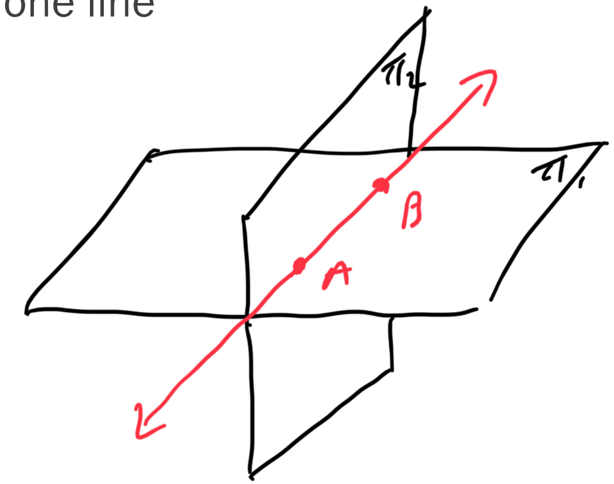
- Two parallel lines determine a plane



TWO POSTULATES CONCERNING LINES AND PLANES

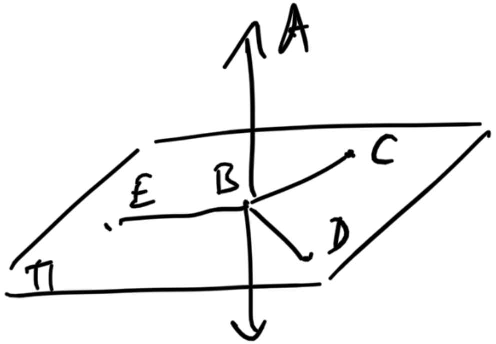


- If a line intersects a plane and not containing it, then the intersection is exactly one point
- If two planes intersect, their intersection is exactly one line



PERPENDICULARITY OF A LINE AND A PLANE : A LINE PERPENDICULAR TO A PLANE

A line is $\perp$ to a plane if it is perpendicular to every one of the lines in the plane that pass through its foot.



$\overleftrightarrow{AB} \perp \overleftrightarrow{BD}$

$\overleftrightarrow{AB} \perp \pi$

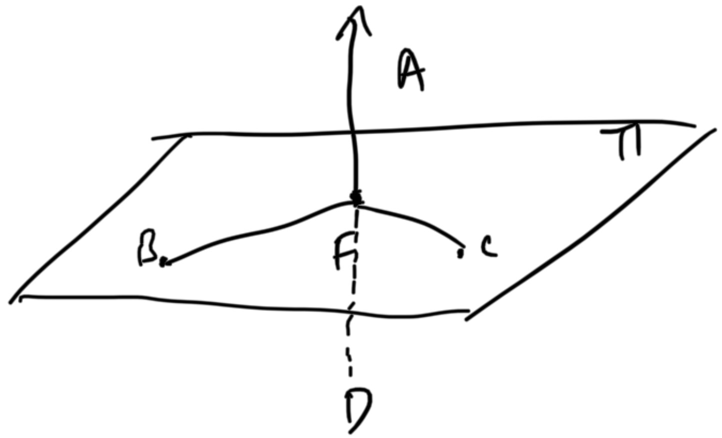
$\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$

$\overleftrightarrow{AB} \perp \overleftrightarrow{BD}$

$\overleftrightarrow{AB} \perp \overleftrightarrow{BE}$

BASIC THEOREM OF A LINE PERPENDICULAR TO A PLANE

If a line is $\perp$ to 2 distinct lines that lie in a plane and that pass through its foot, then it is $\perp$ to that plane.



$\overleftrightarrow{BF}$ and $\overleftrightarrow{CF}$ the lie in plane π

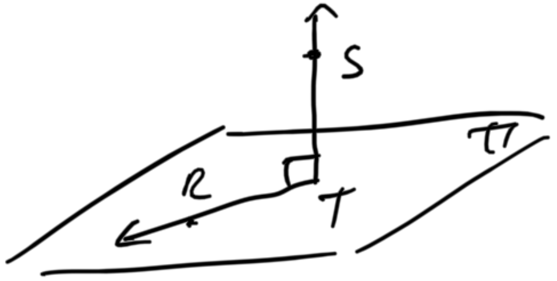
$\overleftrightarrow{AF} \perp \overleftrightarrow{FB}$

$\overleftrightarrow{AF} \perp \overleftrightarrow{FC}$

$\overleftrightarrow{DF} \perp \pi$

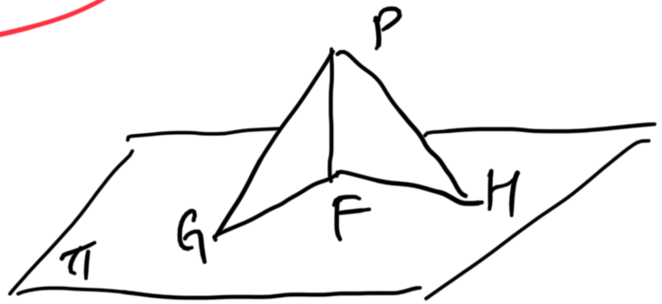
EXAMPLE

If $\angle STR$ is a right angle, we can conclude that $\overleftrightarrow{ST} \perp \pi$?



Ans To be ⊥ to plane π, $\overleftrightarrow{ST}$ must be ⊥ to at least 2 lines that lie in π passing thru T, foot of $\overleftrightarrow{ST}$

NOT EXAMINABLE



EXAMPLE

Given $\overline{PF} \perp \pi$

$\overline{PG} \cong \overline{PH}$

Proof $\angle G \cong \angle H$

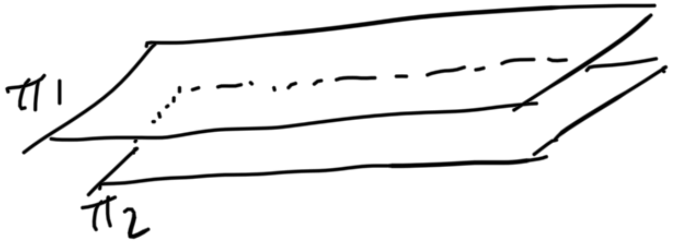
Ans

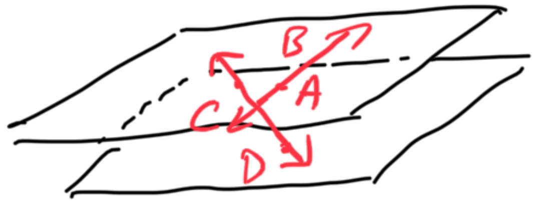
1. $\overline{PF} \perp \pi$	Given
2. $\overline{PF} \perp \overline{FG}$	By theorem on pg 56.
3. $\angle PFG$ is right $\angle PFH$ is right	$\perp$ lines that form the right
4. $\overline{PG} \cong \overline{PH}$	Given
5. $\overline{PF} \cong \overline{PF}$	Reflexive property
$\therefore \triangle PEG \cong \triangle PFH$	similarity

PARALLEL PLANES

Def A line and a plane are parallel if they don't intersect

Def Two planes are parallel if they do not intersect



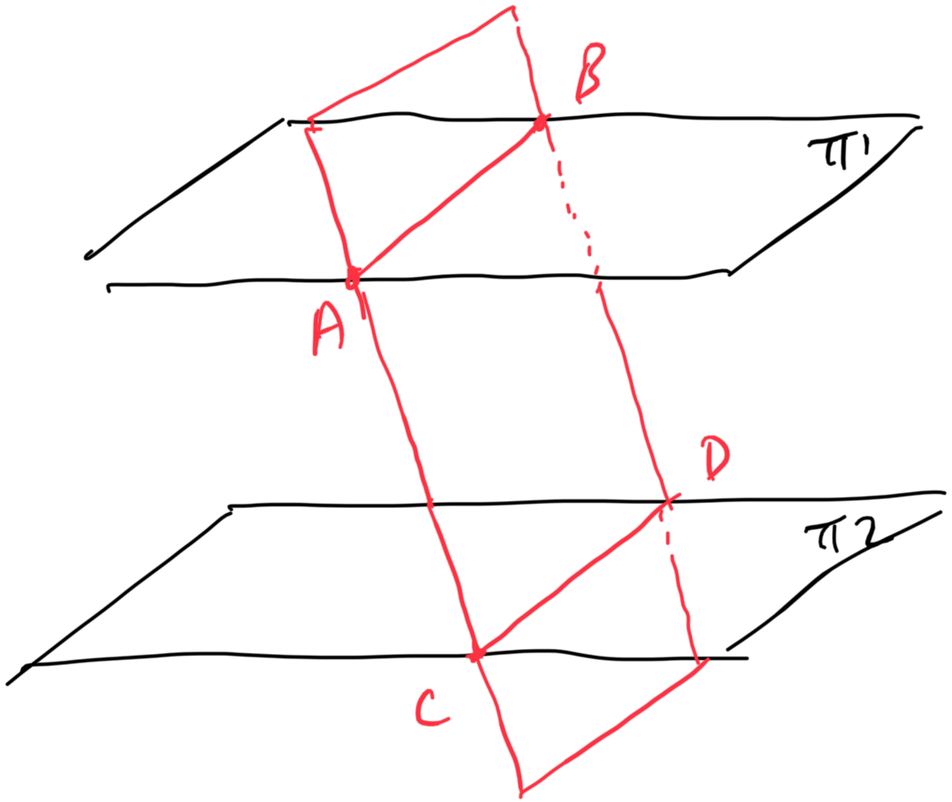


① These 2 lines are on parallel planes

② A, B, C, D don't determine the plane ∴ lines skew

THEOREM

If a plane intersects two parallel planes, the lines of intersection are parallel

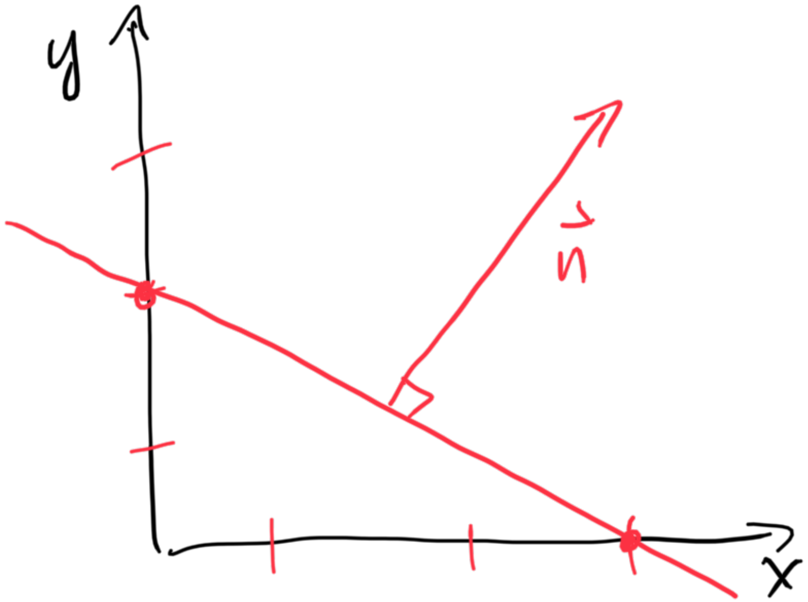


$\therefore \overline{AB} \parallel \overline{CD}$

PROPERTIES RELATING PARALLEL LINES AND PLANES

1. If 2 planes are $\perp$ to the same line, they are $\parallel$ to each other
2. If a line is $\perp$ to one of two $\parallel$ planes, it's $\perp$ to the other plane as well
3. If 2 planes are $\parallel$ to the same plane, they are $\parallel$ to each other
4. If 2 lines are $\perp$ to the same plane, they are $\parallel$ to each other
5. If a plane is $\perp$ to one of the two $\parallel$, it is $\perp$ to the other line as well

LINE IN PLANE- IMPLICIT EQUATIONS



Represent a line by giving an equation:

Dot product with normal vector is fixed.

$\{(x,y) \mid 2x+3y=6\} =$

$\{(x,y) \mid (2,3)\cdot(x,y)=6\}$

$\{(x,y) \mid \vec{n}\cdot(x,y)=\vec{n}\cdot\vec{r_0}\}$ ← General form

LINE IN PLANE- PARAMETRIC EQUATIONS

Represent a line by listing the points as a parameter varies

$\{ 3t, 2-2t \mid t \in \mathbb{R} \} =$

$\{ (0,2) + t(3,-2) \mid t \in \mathbb{R} \}$

$\{ \vec{r_0} + t\vec{v} \mid t \in \mathbb{R} \}$ General form.

EXAMPLE

Find the equation for the plane containing $P=(1,2,3)$, $Q=(-2,4,1)$ and $R=(0,6,-2)$

we need a point on a plane and the normal vector

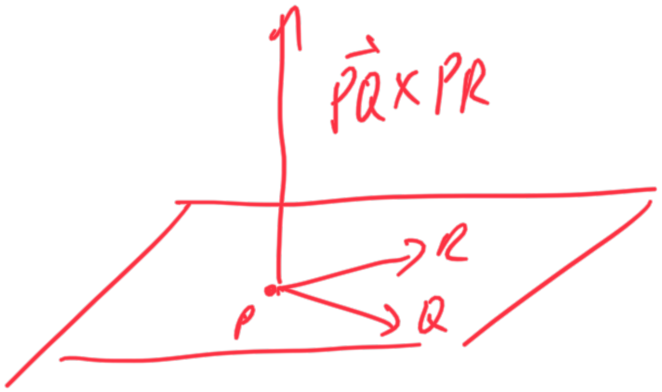
let's choose $P=(1,2,3)$

$\vec{PQ}$ and $\vec{PR}$

$\vec{PQ} = \langle -3, 2, -2 \rangle$

$\vec{PR} = \langle -1, 4, -5 \rangle$

$\vec{n} = \vec{PR} \times \vec{PR}$



$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 2 & -2 \\ -1 & 4 & -5 \end{vmatrix}$$

$$\vec{n} = -2\vec{i} - 13\vec{j} - 10\vec{k} = -\langle 2, 13, 10 \rangle$$

$$\therefore \quad 2(x-1) + 13(y-2) + 10(z-3) = 0$$

$$2x + 13y + 10z - 58 = 0$$