MA1300 Solutions to Self Practice # 13

1. (P724, #25, 30) Determine whether the sequence converges or diverges. If it converges, find the limit.

$$\text{(a).}\ a_n = \frac{3+5n^2}{n+n^2},$$
$$\text{(b).}\ a_n = \sqrt{\frac{n+1}{9n+1}}.$$

Solution:

(a). We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{3}{n^2}+5}{\frac{1}{n}+1} = 5,$$

so $\{a_n\}$ is convergent.

(b). We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{1+\frac{1}{n}}{9+\frac{1}{n}}} = \sqrt{\lim_{n\to\infty} \frac{1+\frac{1}{n}}{9+\frac{1}{n}}} = \frac{1}{3},$$

so $\{a_n\}$ is convergent.

2. (P725, #80) A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$.

(a) By mathematical induction, show that $\{a_n\}$ is increasing and bounded above by 3. Apply the Monotonic Sequence Theorem to show that $\lim_{n\to\infty} a_n$ exists.

(b) Find $\lim_{n\to\infty} a_n$.

Proof. (a) We prove the statement by induction.

Since $a_1 = \sqrt{2}$ and $a_2 = \sqrt{2+\sqrt{2}}$, we see that $a_1 < a_2$ and $a_1 \le 3$, hence the statement is true for $n = 1$.

Assume that the statement is true for k. That is, $a_k < a_{k+1}$ and $a_k \le 3$. Then we have $2+a_k < 2+a_{k+1}$ and hence

$$a_{k+1} = \sqrt{2+a_k} < \sqrt{2+a_{k+1}} = a_{k+2}.$$

Also, $a_{k+1} = \sqrt{2+a_k} \le \sqrt{2+3} = \sqrt{5} \le 3$. So the statement is true for $n = k+1$. Therefore, by mathematical induction, we know that $a_n < a_{n+1}$ and $a_n \le 3$ for every $n \in \mathbb{N}$.

By the above statement, we know from the Monotonic Sequence Theorem that $\lim_{n\to\infty} a_n = L \ge 0$ exists.

(b) Taking limits on both sides of $a_{n+1} = \sqrt{2+a_n}$, we have $L = \sqrt{2+L}$. So $L = 2$.

3. (P725, #82) Show that the sequence defined by

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{3-a_n}$$

satisfies $0 < a_n \le 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

Proof. We prove the statement by induction.

Since $a_1 = 2$ and $a_2 = \frac{1}{3-a_1} = 1$, we see that $a_1 > a_2$ and $0 < a_1 \le 2$, hence the statement is true for $n = 1$.

Assume that the statement is true for k. That is, $a_k > a_{k+1}$ and $0 < a_k \le 2$. Then we have $1 \le 3 - a_k < 3 - a_{k+1}$ and hence

$$a_{k+1} = \frac{1}{3-a_k} > \frac{1}{3-a_{k+1}} = a_{k+2}.$$

Also, $a_{k+1} = \frac{1}{3-a_k} \le 1 \in (0, 2]$. So the statement is true for $n = k+1$. Therefore, by mathematical induction, we know that $a_n > a_{n+1}$ and $0 < a_n \le 2$ for every $n \in \mathbb{N}$.

By the above statement, we know from the Monotonic Sequence Theorem that $\lim_{n\to\infty} a_n = L \ge 0$ exists.

Taking limits on both sides of $a_{n+1} = \frac{1}{3-a_n}$, we have $L = \frac{1}{3-L}$. So $L = \frac{3-\sqrt{5}}{2}$.

4. (P735, #23, 30, 40) Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\text{(a).} \quad \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n},$$

$$\text{(b).} \quad \sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2},$$

$$\text{(c).} \quad \sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right).$$

Solution. (a) The n-th partial sum is

$$s_n = \sum_{k=1}^{n} \frac{(-3)^{k-1}}{4^k} = \frac{1}{4}\frac{1-(-3/4)^n}{1-(-3/4)} = \frac{1}{7} - \frac{1}{7}(-3/4)^n \to \frac{1}{7}, \quad \text{as} \quad n \to \infty.$$

Hence the series is convergent and $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{7}$.

(b) The general term satisfies $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1+\frac{2}{k}}{(1+\frac{3}{k})^2} = 1 \ne 0$, therefore the series is divergent.

(c) The n-th partial sum is

$$s_n = \sum_{k=1}^{n} \frac{3}{5^k} + \sum_{k=1}^{n} \frac{2}{k} = \frac{3}{5}\frac{1-5^{-n}}{1-\frac{1}{5}} + 2\sum_{k=1}^{n}\frac{1}{k}.$$

Since $\lim_{n\to\infty} \frac{3}{5}\frac{1-5^{-n}}{1-\frac{1}{5}} = \frac{3}{4}$ but $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = \infty$, we know that $\lim_{n\to\infty} s_n = \infty$. So the series is divergent.

5. (P736, #64) We have seen that the harmonic series is a divergent series whose terms approach 0. Show that

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$$

is another series with this property.

Proof. The n-th partial sum is

$$s_n = \sum_{k=1}^{n} \ln\frac{k+1}{k} = \sum_{k=1}^{n} (\ln(k+1) - \ln k) = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \ldots + (\ln(n+1) - \ln n) = \ln(n+1) \to \infty$$

as $n \to \infty$. So the series is divergent. But $\lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right) = 0$.