I pledge that the answers in this examination are my own and that I will not seek or obtain an unfair advantage in producing these answers. Specifically,

- ❖ I will not plagiarize (copy without citation) from any source;
- ❖ I will not communicate or attempt to communicate with any other person during the examination; neither will I give or attempt to give assistance to another student taking the examination; and
- ❖ I will use only approved devices (e.g., calculators) and/or approved device models.
- ❖ I understand that any act of academic dishonesty can lead to disciplinary action.”

Please **reaffirm the honesty pledge by writing** “I pledge to follow the Rules on Academic Honesty and understand that violations may lead to severe penalties” onto the first examination answer sheet.

CITY UNIVERSITY OF HONG KONG

Course code & title	:	MA2506/2510 Probability and Statistics
Session	:	Semester B, 2021-2022
Time Allowed	:	Three hours

This paper has **four** pages. (including this page)

Instructions to candidates:

1. Answer **all** questions.
2. Start each main question on a new page.
3. Show all steps.

This is a closed-book examination. Materials, aids & instruments which students are permitted to use during examination:

Approved calculators

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1. (15 marks) Pick k numbers from $\{1, 2, \ldots, N\}$ one by one, with replacement (which means the same number can appear more than once). Find the probabilities for the following events:

   (a) k distinct numbers are selected.

   (b) The number 1 appears exactly m times ($m \leq k$).

   (c) k distinct numbers are selected and they are strictly increasing.

2. (10 marks) An urn initially contains 5 white and 7 black balls. Each time a ball is selected its color is noted and it is replaced in the urn along with 2 other balls of the same color.

   (a) Compute the probability that the first 2 balls selected are white and the next two black.

   (b) Compute the probability that, of the first 4 balls selected, exactly 2 are black.

3. (15 marks)

   (a) Suppose two events A and B are independent and mutually exclusive. Prove $\min\{P(A), P(B)\} = 0$

   (b) Suppose $P(A) = 1$ for an event A. Prove A is independent of any event B.

   (c) Suppose the cumulative distribution function (CDF) of a random variable X is given by

$$F(x) = \begin{cases} 1 - (1+x)e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

   Find the probability density function (PDF) of X.

4. (10 marks) Assume that Y_1, Y_2, and Y_3 are random variables, with $E(Y_1) = 2, E(Y_2) = -1, E(Y_3) = 4, Var(Y_1) = 4, Var(Y_2) = 6, Var(Y_3) = 8, Cov(Y_1, Y_2) = 1, Cov(Y_1, Y_3) = -1, Cov(Y_2, Y_3) = 0$. Find $E(3Y_1 + 4Y_2 - 6Y_3)$ and $Var(3Y_1 + 4Y_2 - 6Y_3)$.

5. (15 marks) A coin is flipped independently for 100 times, and let X be the number of heads.

   (a) Find the estimate of $P(head)$ by using method of moment.

   (b) Find the estimate of $P(head)$ by using maximum likelihood estimation.

   (c) If we observe $X = 55$, construct a 95% confidence interval for $P(head)$.

6. (20 marks) A research article reports a sample of time to repair (min) a rail break in the high rail on a curved track of a certain railway line. It is believed that the data follows a normal distribution with known standard deviation 160. The sample size is 16, and the sample mean is 240.

   (a) Is there compelling evidence for concluding that true average repair time exceeds 200 min? Conduct a hypothesis test using a significance level 0.05 by calculating the rejection region.

   (b) State the p-value for the test above.

   (c) Suppose the true mean is 300, what is the type II error probability of the test in (a); that is, $\beta(300)$.

   For (b) and (c), you can state your answers in terms of Φ, the cumulative distribution function of the standard normal distribution.

7. (15 marks) A sample of 25 pieces of laminate used in the manufacture of circuit boards was selected and the amount of warpage (mm) under particular conditions was determined for each piece, resulting in a sample mean warpage of 1.94 and a sample standard deviation of 0.3.

   (a) Suppose the amount of warpage follows a normal distribution. Construct a 95% confidence interval for the true mean warpage.

   (b) Suppose a one-sided confidence bound is in need. What is the lower confidence bound of the 95% one-sided confidence interval.

   (c) Suppose that we do not know the distribution of the amount of warpage. Instead, a random sample of 64 pieces of laminate was selected with the same sample mean and sample standard deviation. Construct a 95% confidence interval for the true mean warpage.

   For (a) and (b) the final answers can be stated using the notation $t_{n,\alpha}$ to denote the critical value of the t-distribution with n degrees of freedom and tail probability α.

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