WEEK 5/6

CHAPTER TWO

- Lines and Planes in Space

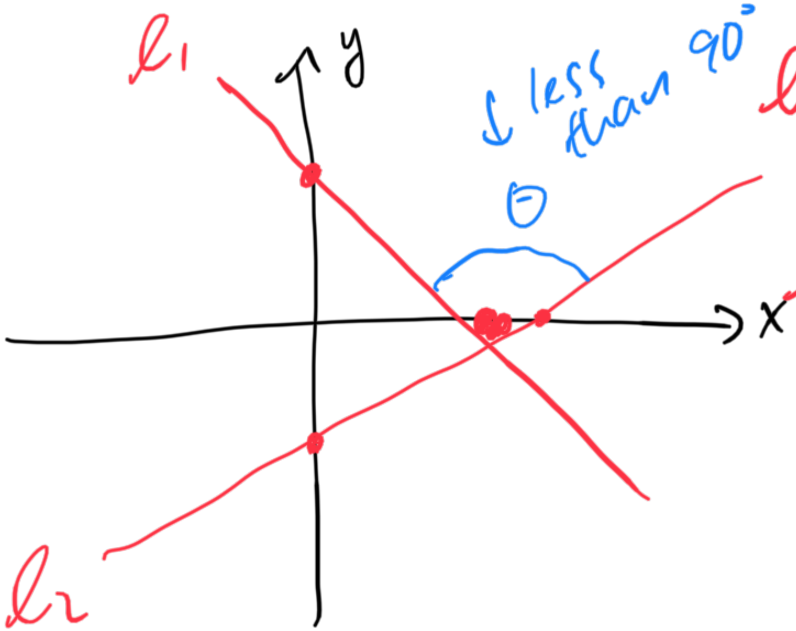
OUTLINE

- Recap with Point of Intersection of Two Lines, Angle between Two Lines, Distance of a Point from a Line
- 3D Coordinate System
- The Straight Line in 3D Space
- The Plane

find the measure of the angle between 2 lines $l_1: 2x+y=5$

$l_2: 3x-2y=8$

CLARIFICATION FROM SLIDES 35 AND 37 (WEEK 4)



l_1 $2x+y=5$

l_2 $3x-2y=8$

xint $y=0$ $x=5/2=2.5$

yint $x=0$ $y=5$

xint $y=0$ $x=8/3 = 2.6...$

yint $x=0$ $y=-4$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(1)(2)+(-2)(3)}{\sqrt{1+4}\sqrt{4+9}} = \frac{-4}{\sqrt{5}\sqrt{13}} = -\frac{4}{\sqrt{65}}$$

$\cancel{\theta = 119.7^\circ}$

$\boxed{\theta = 60.3^\circ}$

POINT OF INTERSECTION OF TWO LINES

let $y = m_1 x + c_1$ $\quad$ $y = m_2 x + c_2$ ← not parallel

let (a, b) ← point of intersection
$\quad\ \ x \ \ y$

$$\begin{aligned}
b &= m_1 a + c_1 \quad \cdots \text{①} \\
(-) \quad b &= m_2 a + c_2 \quad \cdots \text{②} \\
\hline
0 &= (m_1 - m_2) a + c_1 - c_2
\end{aligned}$$

$$c_2 - c_1 = (m_1 - m_2) a$$

$$\boxed{\frac{c_2 - c_1}{m_1 - m_2} = a}$$ sub info ... ①

$$\boxed{b = m_1 \left(\frac{c_2 - c_1}{m_1 - m_2} \right) + c_1}$$

$$\frac{m_1 (c_2 - c_1) + c_1 (m_1 - m_2)}{(m_1 - m_2)} = \frac{m_1 c_2 - m_2 c_1}{m_1 - m_2}$$

$$m_1 c_2 - \cancel{m_1 c_1} + \cancel{c_1 m_1} - c_1 m_2$$

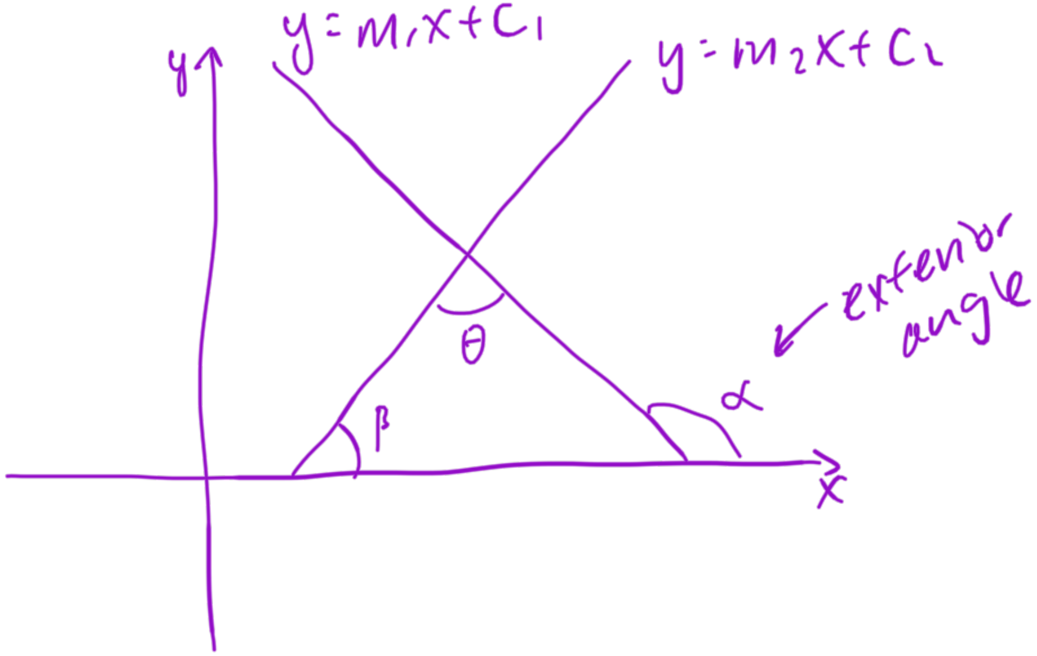
$$\therefore \left(\frac{c_2 - c_1}{m_1 - m_2}, \frac{m_1 c_2 - m_2 c_1}{m_1 - m_2} \right)$$
$\qquad\qquad\qquad\quad a \qquad\qquad\qquad\qquad b$

state condition.

$$\boxed{m_1 \neq m_2}$$

If equal then parallel

ANGLE BETWEEN TWO LINES



$\alpha = \theta + \beta$

$\theta = \alpha - \beta$

$\tan\theta = \tan(\alpha - \beta)$ by trig properties

$$\tan\theta = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$\tan\alpha = m_1$

$\tan\beta = m_2$

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\boxed{\tan\alpha = \frac{\sin\alpha}{\cos\alpha}}$$

Q: what happens if

$\theta = \pi/2 \therefore \tan\theta =$ undefined

$1 + m_1 m_2 = 0$ or $m_1 m_2 = -1$

two lines are $\perp$

the slopes of products -1

REMARK

$$\begin{cases} 3x+4y=1 \\ 4y=-3x+1 \\ y=-\frac{3}{4}x+\frac{1}{4} \\ m_1=-\frac{3}{4} \end{cases}$$

$4x-3y=-7$

$y=\frac{4x}{3}+\frac{7}{3}$

$m_2=\frac{4}{3}$

$m_1 m_2=-1$

Qt: why do we not "prove" for parallel.

Ans: only difference is $y=m_1x+\boxed{c_1}$ vs $y=m_2x+\boxed{c_2}$

$m_1=m_2$

EXAMPLE

Find eqn of the line which passes thru (1,0) and is parallel to $3x+5y=4$

Ans: $3x+5y=c$

put in $3(1)+5(0)=c$

$3+0=c$

$\therefore\ 3x+5y=3$ or $3x+5y-3=0$

DISTANCE OF A POINT FROM A LINE

$P(x_1, y_1)$, $\perp$

$y - mx - c = 0$

$\boxed{y = mx + c}$ ... ①

slope $= m$

slope

$my = -x + k$

$y = -\frac{x}{m} + k$

slope $: -\frac{1}{m}$

perpendicular of $y = mx + c$

$\rightarrow x + my = k$ since (x_1, y_1)

$x_1 + my_1 = k$ ... ②

Now suppose $Q(x_0, y_0) \leftarrow$ Intersection of ① and ②

$d^2 = (x_1 - x_0)^2 + (y - y_0)^2 \leftarrow$

①' ... $y_0 - mx_0 = c$ and $my_0 + x_0 = k$ ... ②'

② - ②' $\rightarrow$ $m(y_1 - y_0) + (x_1 - x_0) = 0 \rightarrow (x_1 - x_0) = -m(y_1 - y_0)$

$d^2 = m^2 (y_1 - y_0)^2 + (y_1 - y_0)^2 = (m^2 + 1)(y_1 - y_0)^2$ ... ③

$my_0 + x_0 = k + \quad y_0 - mx_0 = c$

$$(m^2+1)y_0 = c + km$$

$$y_0 = \frac{c+km}{m^2+1}$$

$$d^2 = (m^2+1)\left[y_1 - \frac{c+km}{m^2+1}\right]^2$$

$$d^2 = (m^2+1)\left[\frac{y_1(m^2+1) - c - km}{m^2+1}\right]^2$$

$$d^2 = (m^2+1)\left[\frac{y_1 m^2 + y_1 - c - (x_1 + my_1)\cdot m}{(m^2+1)^2}\right]^2$$

$$d^2 = \frac{(y_1 - mx_1 - c)^2}{m^2+1} = \boxed{d = \frac{|y_1 - mx_1 - c|}{\sqrt{1+m^2}}}$$

EXAMPLE

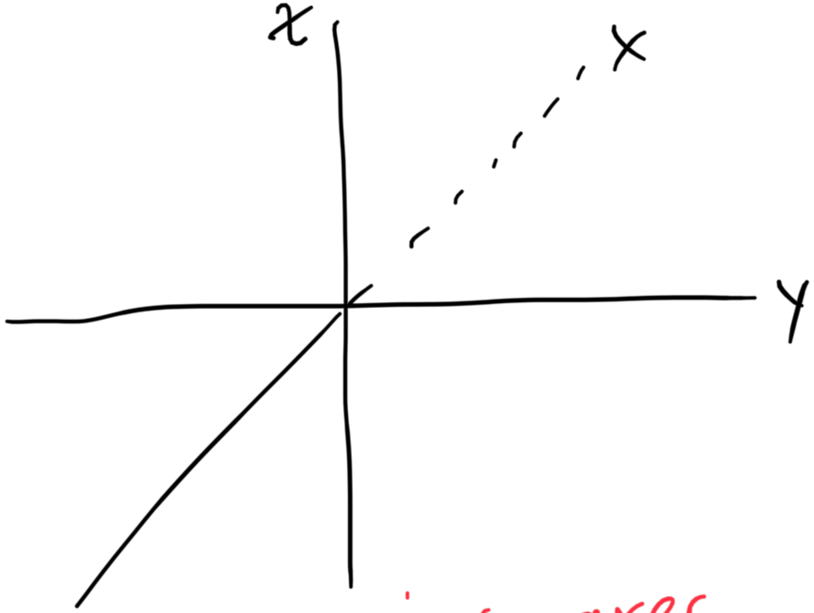
Find the length of the $\perp$ from $P(2,-3)$ to the line $6x+8y-2=0$

$$d = \frac{|6\times 2 + 8\times(-3) - 2|}{\sqrt{6^2+8^2}}$$

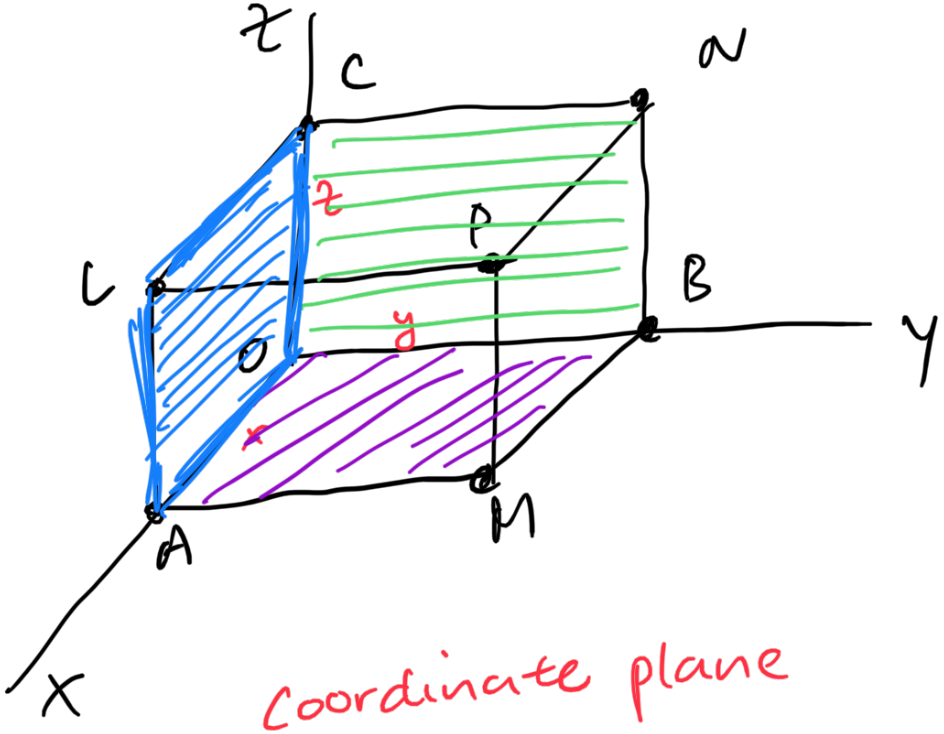
$$\boxed{d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}}$$

$$d = \frac{|-14|}{\sqrt{100}} = \frac{14}{10}$$

3D COORDINATE SYSTEM



coordinate axes



coordinate plane

xy plane

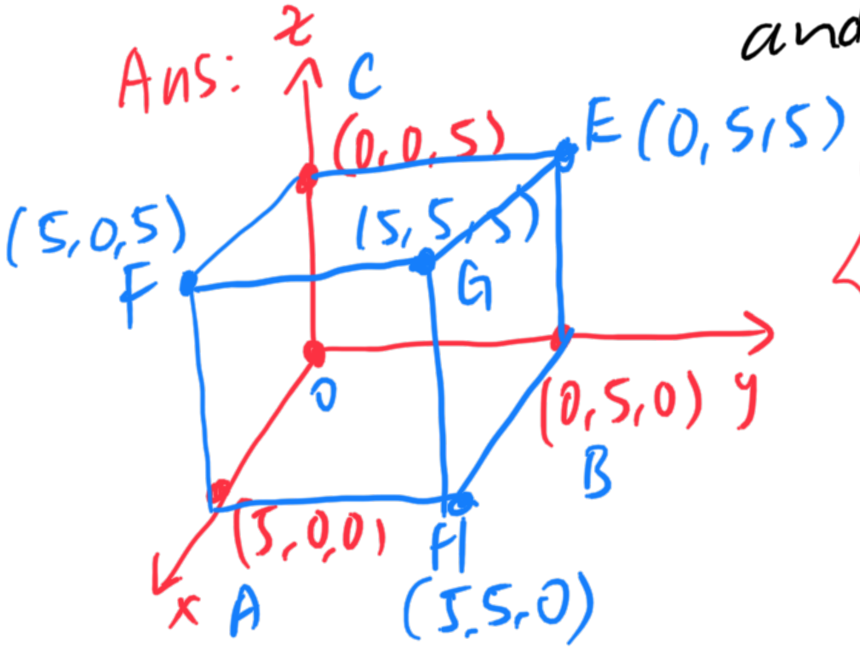
xz plane

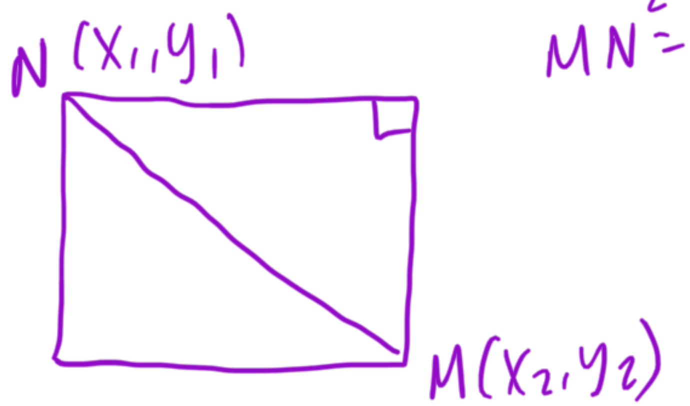
yz plane

EXAMPLE

A cube of side 5 units has one vertex at (0, 0, 0) and the 3 edges from this vertex are along the coordinates axes in the positive directions. Find the coordinates of the other vertices and the centre of the cube.

Ans:





$MN^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ } distance

⋮

Midpoint

DISTANCE BETWEEN 2 POINTS IN 3D

Ex. $P_1(2, 3, 5)$ and $P_2(4, 3, 1)$

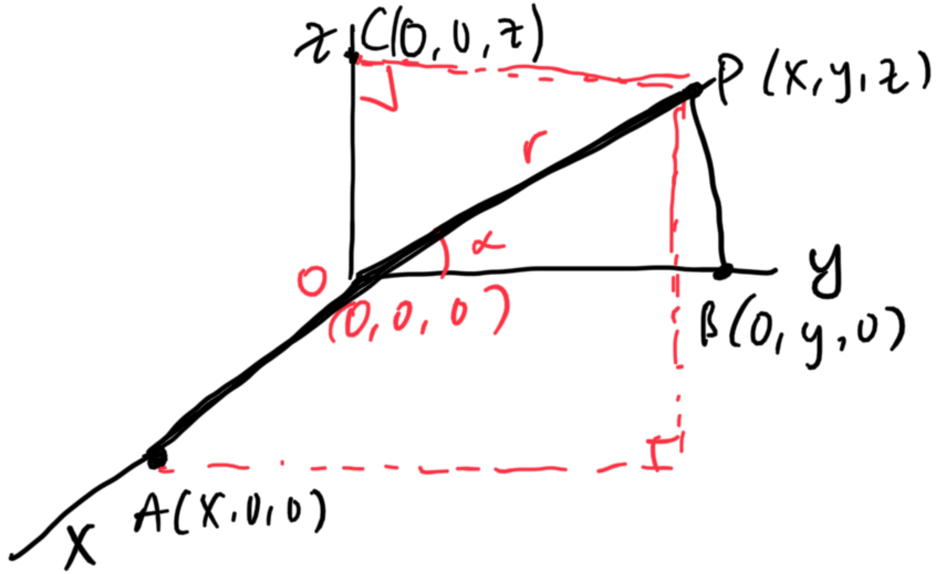
x_1, y_1, z_1 $\quad$ x_2, y_2, z_2

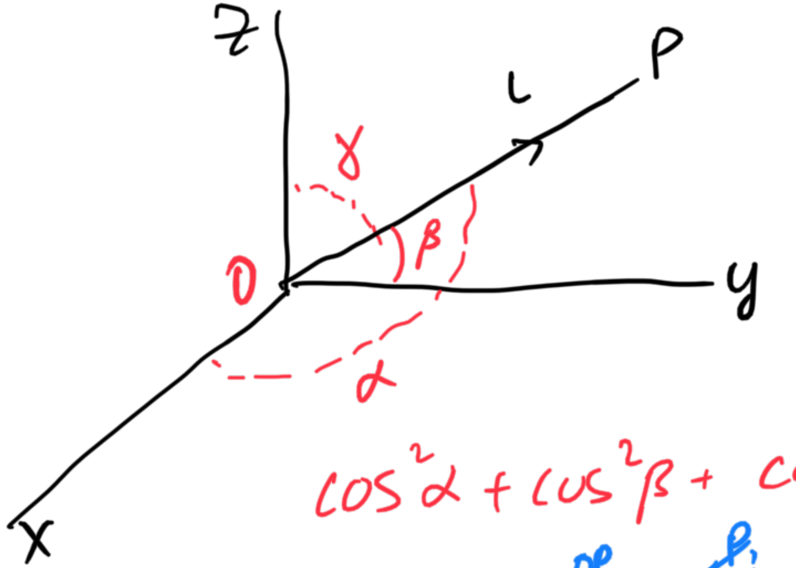
Ans: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ ☆

$d = \sqrt{(4-2)^2 + (3-3)^2 + (1-5)^2}$

$d = \sqrt{4 + 16} = \sqrt{20}$

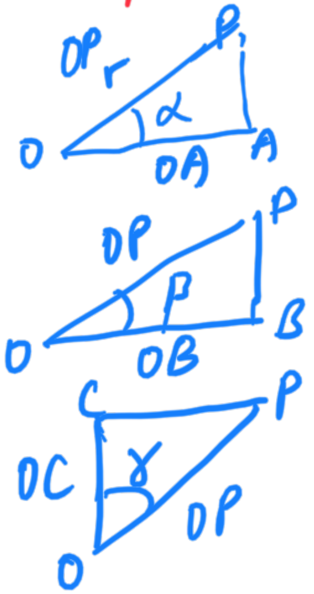
THE STRAIGHT LINE IN 3D SPACE





$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\begin{cases} \cos\alpha = \dfrac{OA}{OP} = \dfrac{x}{r} \\ \cos\beta = \dfrac{OB}{OP} = \dfrac{y}{r} \\ \cos\gamma = \dfrac{OC}{OP} = \dfrac{z}{r} \end{cases}$$



let P(x,y,z) be any pt on L

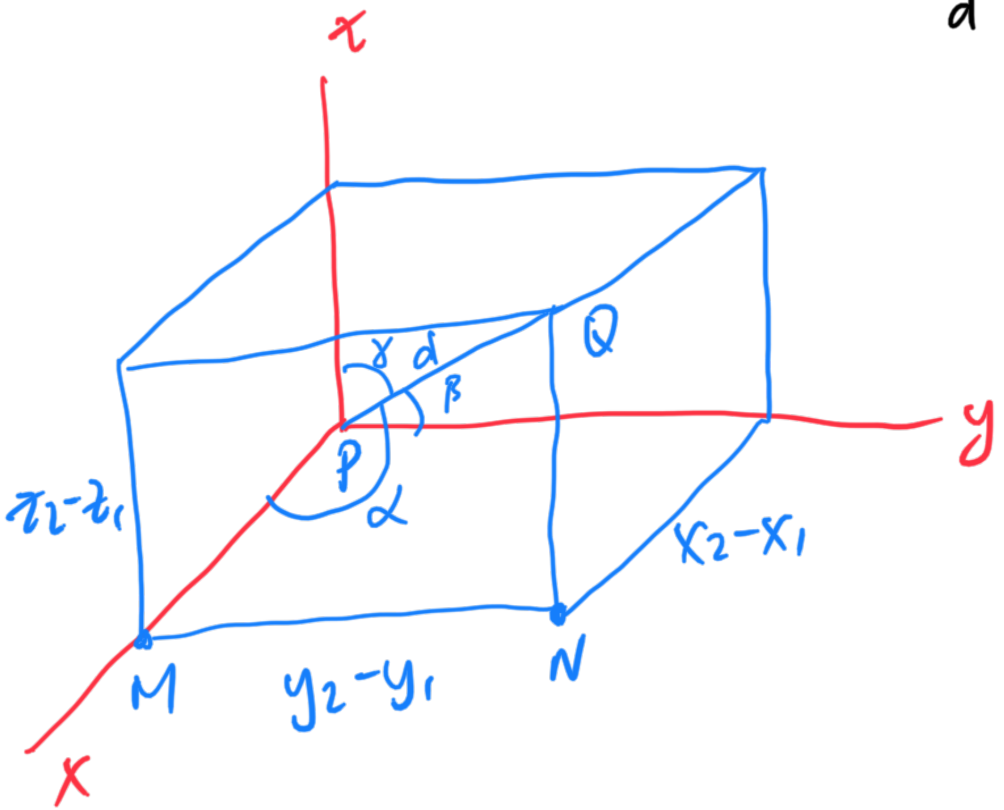
$$r^2 = OP^2 = (x-0)^2 + (y-0)^2 + (z-0)^2$$

$$r^2 = x^2 + y^2 + z^2$$

THEOREM

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ they are on line L then its direction cosines are given by. Note d distance between PQ

$$l = \frac{x_2 - x_1}{d}, \quad m = \frac{y_2 - y_1}{d}, \quad n = \frac{z_2 - z_1}{d}$$



$$\cos\alpha = \frac{x_2 - x_1}{d}$$

$$\cos\beta = \frac{y_2 - y_1}{d}$$

$$\cos\gamma = \frac{z_2 - z_1}{d}$$

Note if we remove all the "d",

$$\left\{\begin{array}{l} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{array}\right.$$

direction ratio of the line

EXAMPLE

Find the direction cosines of the line joining points

$P(1,4,3)$ and $Q(2,3,-1)$

$x_1\ y_1\ z_1$ $\quad$ $x_2\ y_2\ z_2$

Ans:

P17 $\rightarrow$ $\dfrac{2-1}{d}, \dfrac{3-4}{d}, \dfrac{-1-3}{d}$

P14 $\rightarrow$ $d = \sqrt{(2-1)^2+(3-4)^2+(-1-3)^2} = \sqrt{18} = 3\sqrt{2}$

$\left(\dfrac{1}{3\sqrt{2}}, -\dfrac{1}{3\sqrt{2}}, \dfrac{-4}{3\sqrt{2}}\right)$

$\cos\alpha \quad \cos\beta \quad \cos\gamma$

DEFINITION

A line in the space is determined by a point and a direction.

Consider a line ℓ and a point $P(x_0, y_0, z_0)$ on ℓ.

Direction of this line is determined by a vector $\vec{v}$ that is parallel to line ℓ.

let $P(x, y, z)$ be any pt on the line

let $\vec{r_0}$ be the position vector of point P_0

$\vec{r}$ be the position vector of point P.

VECTOR EQUATION OF A LINE

$$\vec{r} = \vec{r}_0 + \vec{v}t$$

$$\vec{v} = \langle a, b, c \rangle$$

$$\vec{r} = \langle x_0, y_0, z_0 \rangle$$

t - scalar

PARAMETRIC EQUATIONS OF A LINE

$$\langle x, y, z \rangle = \langle x_0 + at, \; y_0 + bt, \; z_0 + ct \rangle$$

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

SYMMETRIC EQUATION OF A LINE

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$

Explanation

$$\frac{x-x_0}{a} = \frac{at}{a}$$

$$\boxed{\frac{x-x_0}{a} = t}$$ .... everything is $t = \ldots$

EXAMPLE

Find the vector equation and Parametric eqn of the line through $(1,2,3)$ and parallel to vector $3\vec{i}+2\vec{j}-\vec{k}$

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \end{array}$$

Ans: Step1 Locate the point $r_0 = \langle 1, 2, 3 \rangle$

Step2 locate the direction $\vec{v} = \langle 3\vec{i}+2\vec{j}-\vec{k} \rangle$

$\vec{r} = i + 2\vec{j} + 3\vec{k} + t(3\vec{i}+2\vec{j}-\vec{k})$

$\vec{r} = \vec{i} + 2\vec{j} + 3\vec{k} + 3t\vec{i} + 2t\vec{j} - t\vec{k}$

$\vec{r} = (1+3t)\vec{i} + (2+2t)\vec{j} + (3-t)\vec{k}$

$\langle x, y, z \rangle$

parametric eqn

$x = 1+3t$

$y = 2+2t$

$z = 3-t$

EXAMPLE

Find the symmetric equation for line through point (1, -5, 6) and is parallel to vector $\langle -1, 2, -3 \rangle$

pg 26.

note. p28

$x = x_0 + at$

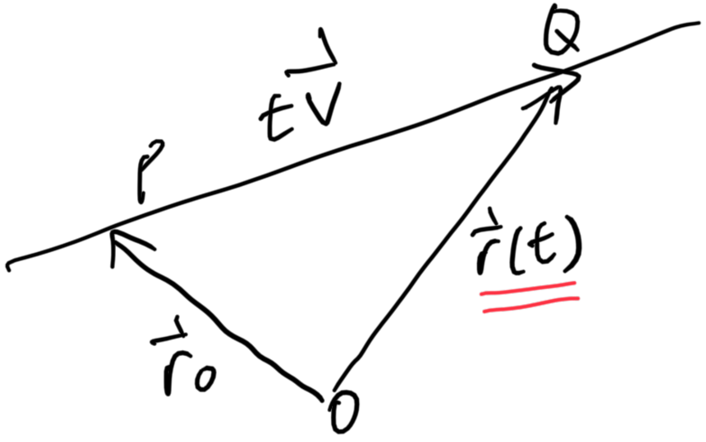
$x - x_0 = at$

$\frac{x - x_0}{a} = t$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\frac{x - 1}{-1} = \frac{y + 5}{2} = \frac{z - 6}{-3}$$

BRING IT ALL TOGETHER



vector equ

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

point

direction

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

$$\vec{r}(t) = \langle x_0 + at,\ y_0 + bt,\ z_0 + ct \rangle$$

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}\right\} \text{parametric.}$$

EXAMPLE

Find a vector equation for the line through $P = (-1, 2, 3)$ and $Q = (2, -2, 5)$

Ans $P: (-1, 2, 3)$

$\overrightarrow{PQ} = \langle 2-(-1), -2-2, 5-3 \rangle = \underbrace{\langle 3, -4, 2 \rangle}_{3\vec{i} - 4\vec{j} + 2\vec{k}}$ ← direction

$\vec{r} = \langle -1, 2, 3 \rangle + t\langle 3, -4, 2 \rangle$

$\vec{r} = \langle -1+3t, 2-4t, 3+2t \rangle$

EXAMPLE

Find a vector eqn for the line through $(5,-6,7)$ that is parallel to the line with parametric equations $x=1+t \quad y=2 \quad z=3+2t$

Ans. $P(5,-6,7)$

Direction $= \langle 1,0,2\rangle$

$\vec{r}(t) = \langle 5,-6,7\rangle + t\langle 1,0,2\rangle$

$\vec{r}(t) = \langle 5+t, -6, 7+2t\rangle$

EXAMPLE

Find the point of intersection of the lines from

$r_1(t) = \langle -1+3t, 2-4t, 3+2t \rangle$

$r_2(t) = \langle 5+t, -6, 7+2t \rangle$

Ans

$\boxed{-1+3t = 5+s} \leftarrow -1+3(2) = 5+s$ $\quad \boxed{s=0}$

$\boxed{2-4t = -6} \leftarrow -4t = -6-2$

$-4t = -8$

$3+2t = 7+2s$ $\quad \boxed{t=2}$

$r_1(2) = \langle -1+3(2), 2-4(2), 3+2(2) \rangle = \langle 5, -6, 7 \rangle$ ✓

$r_2(0) = \langle 5+0, -6, 7+2(0) \rangle = \langle 5, -6, 7 \rangle$ ✓

EXAMPLE

Show that

$$r_1(t) = \langle 1+t, -3-t, 5+2t \rangle$$

$$r_2(s) = \langle 4-s, -3+s, 6+2s \rangle$$

these are skewed (not intersecting or parallel)

Ans: step 1 Find direction of lines

step 2 equate components

step 3 see if they can intersect