1. (a) Let $y = 1 + \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = -\frac{2}{x^3} \Rightarrow dx = -\frac{x^3}{2}dy$.

Then the integral becomes

$$\int \frac{e^{1+\frac{1}{x^2}}}{x^3}dx = \int \frac{e^{1+\frac{1}{x^2}}}{x^3}\left(-\frac{x^3}{2}dy\right) = -\frac{1}{2}\int e^y dy = -\frac{1}{2}e^y + C = -\frac{1}{2}e^{1+\frac{1}{x^2}} + C.$$

(b) Let $y = 1 + x^4 \Rightarrow \frac{dy}{dx} = 4x^3 \Rightarrow dx = \frac{1}{4x^3}dy$.

Then the integral becomes

$$\int x^{11}\sqrt{1+x^4}\,dx = \int x^{11}\sqrt{1+x^4}\left(\frac{1}{4x^3}dy\right) = \frac{1}{4}\int x^8\sqrt{1+x^4}\,dy$$

$$= \frac{1}{4}\int (y-1)^2\sqrt{y}\,dy = \frac{1}{4}\int \left(y^{\frac{5}{2}} - 2y^{\frac{3}{2}} + y^{\frac{1}{2}}\right)dy$$

$$= \frac{1}{4}\left(\frac{y^{\frac{7}{2}}}{\frac{7}{2}} - 2\frac{y^{\frac{5}{2}}}{\frac{5}{2}} + \frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{14}(1+x^4)^{\frac{7}{2}} - \frac{1}{5}(1+x^4)^{\frac{5}{2}} + \frac{1}{6}(1+x^4)^{\frac{3}{2}} + C.$$

(c) Let $y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x \Rightarrow dx = -\frac{1}{\sin x}dy$

Together with the fact that $\sin 2x = 2\sin x\cos x$, the integral becomes $\int \sin 2x\sqrt{\cos x}\,dx = \int 2\sin x\cos x\sqrt{\cos x}\left(-\frac{1}{\sin x}dy\right)$

$$= -2\int y\sqrt{y}\,dy = -2\int y^{\frac{3}{2}}dy = -2\frac{y^{\frac{5}{2}}}{\frac{5}{2}} + C = -\frac{4}{5}\cos^{\frac{5}{2}}x + C.$$

(d) Let $y = x^2 - 1 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x}dy$.

When $x = 1$, $y = 1^2 - 1 = 0$. When $x = 2$, $y = 2^2 - 1 = 3$.

Then the integral becomes

$$\int_1^2 xe^{x^2-1}dx = \int_0^3 xe^{x^2-1}\left(\frac{1}{2x}dy\right) = \frac{1}{2}\int_0^3 e^y dy = \frac{1}{2}e^y\Big|_0^3 = \frac{e^3}{2} - \frac{1}{2}.$$

(e) Let $y=\ln x \Rightarrow \frac{dy}{dx}=\frac{1}{x} \Rightarrow dx = x\,dy$.

When $x=5$, $y=\ln 5$. When $x=1$, $y=\ln 1 = 0$.

Then the integral becomes

$$\int_1^5 \frac{\sin^2(\ln x)}{x}\,dx = \int_0^{\ln 5} \frac{\sin^2(\ln x)}{x}(x\,dy) = \int_0^{\ln 5} \sin^2 y\,dy$$

$$= \int_0^{\ln 5} -\frac{1}{2}(\cos(y+y)-\cos(y-y))\,dy = -\frac{1}{2}\int_0^{\ln 5}(\cos 2y - 1)\,dy$$

$$= -\frac{1}{2}\left(\frac{1}{2}\sin 2y - y\right)\Big|_0^{\ln 5} = -\frac{1}{4}\sin(2\ln 5) + \frac{1}{2}\ln 5.$$

(f) Let $y=\cos x \Rightarrow \frac{dy}{dx} = -\sin x \Rightarrow dx = -\frac{1}{\sin x}\,dy$.

The integral then becomes

$$\int \sin^7 x\,dx = \int \sin^7 x\left(-\frac{1}{\sin x}\,dy\right) = -\int \sin^6 x\,dy = -\int (1-\cos^2 x)^3\,dy$$

$$= -\int (1-y^2)^3\,dy = -\int (1-3y^2+3y^4-y^6)\,dy$$

$$= -y + y^3 - \frac{3}{5}y^5 + \frac{y^7}{7} + C = -\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{\cos^7 x}{7} + C.$$

(g) Let $x = \frac{3}{4}\sin\theta \Rightarrow \frac{dx}{d\theta} = \frac{3}{4}\cos\theta \Rightarrow dx = \frac{3}{4}\cos\theta\,d\theta$.

Then the integral becomes

$$\int \sqrt{9-16x^2}\,dx = \int \sqrt{9-16\left(\frac{3}{4}\sin\theta\right)^2}\left(\frac{3}{4}\cos\theta\,d\theta\right) = \frac{9}{4}\int \cos^2\theta\,d\theta$$

$$= \frac{9}{4}\int \frac{1}{2}(\cos(\theta+\theta)+\cos(\theta-\theta))\,d\theta = \frac{9}{8}\int \cos 2\theta + \frac{9}{8}\int 1\,d\theta$$

$$= \frac{9}{16}\sin 2\theta + \frac{9}{8}\theta + C = \frac{9}{16}\sin\left(2\sin^{-1}\frac{4}{3}x\right) + \frac{9}{8}\sin^{-1}\frac{4}{3}x + C.$$

(h) Let $x = \sin\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta \Rightarrow dx = \cos\theta\,d\theta$.

The integral then becomes

$$\int \frac{1}{x^2\sqrt{1-x^2}}\,dx = \int \frac{1}{\sin^2\theta\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \int \frac{1}{\sin^2\theta}\,d\theta = \int \csc^2\theta\,d\theta$$

$$= -\cot\theta + C = -\frac{\sqrt{1-x^2}}{x} + C.$$

2. (a) Take $u=x$ and $dv=e^{-3x}dx \Rightarrow v=\int e^{-3x}dx=-\frac{1}{3}e^{-3x}$.

Using integration by parts, we get

$$\int xe^{-3x}dx=-\frac{1}{3}xe^{-3x}-\int -\frac{1}{3}e^{-3x}dx=-\frac{1}{3}xe^{-3x}+\frac{1}{3}\int e^{-3x}dx$$

$$=-\frac{1}{3}xe^{-3x}-\frac{1}{9}e^{-3x}+C.$$

(b) Take $u=\ln x$ and $dv=\sqrt{x}\,dx \Rightarrow v=\int\sqrt{x}\,dx=\int x^{\frac{1}{2}}dx=\frac{2}{3}x^{\frac{3}{2}}$

Using integration by parts, we get

$$\int_1^e \sqrt{x}\ln x\,dx=\int_1^e \ln x\sqrt{x}\,dx=\frac{2}{3}x^{\frac{3}{2}}\ln x\Big|_1^e-\int_1^e\frac{2}{3}x^{\frac{3}{2}}d(\ln x)$$

$$=\frac{2}{3}e^{\frac{3}{2}}\ln e-\frac{2}{3}x^{\frac{3}{2}}\ln 1-\frac{2}{3}\int_1^e x^{\frac{3}{2}}\left(\frac{1}{x}dx\right)$$

$$=\frac{2}{3}e^{\frac{3}{2}}-\frac{2}{3}\int_1^e x^{\frac{1}{2}}dx=\frac{2}{3}e^{\frac{3}{2}}-\frac{2}{3}\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_1^e$$

$$=\frac{2}{3}e^{\frac{3}{2}}-\frac{4}{9}e^{\frac{3}{2}}+\frac{4}{9}=\frac{2}{9}e^{\frac{3}{2}}+\frac{4}{9}.$$

(c) Take $u=x^2$ and $dv=\sin x\,dx \Rightarrow v=\int\sin x\,dx=-\cos x$

Using integration by parts, we get

$$\int x^2\sin x\,dx=-x^2\cos x-\int -\cos x\,d(x^2)=-x^2\cos x+2\int x\cos x\,dx.$$

To compute the second integral, we take $u=x$ and $dv=\cos x\,dx \Rightarrow v=\int\cos x\,dx=\sin x$. Using integration by parts, we have

$$\int x\cos x\,dx=x\sin x-\int\sin x\,dx=x\sin x+\cos x+C.$$

Therefore, we conclude that

$$\int x^2\sin x\,dx=-x^2\cos x+2x\sin x+2\cos x+C.$$

(d) We first rewrite the integral by using product-to-sum formula:

$$\int x\sin^2 x\,dx = \int x\left[-\tfrac{1}{2}(\cos(x+x)-\cos(x-x))\right]dx$$

$$= -\frac{1}{2}\int \underbrace{x}_{u}\,\underbrace{\cos 2x\,dx}_{dv} + \frac{1}{2}\int x\,dx$$

$$= -\frac{1}{2}\left(\frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx\right) + \frac{1}{2}\left(\frac{x^2}{2}\right)$$

$$= -\frac{1}{4}x\sin 2x + \frac{1}{4}\left(-\frac{1}{2}\cos 2x\right) + \frac{x^2}{4} + C$$

$$= -\frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + \frac{x^2}{4} + C.$$

(e) Take $u=(\ln x)^2$ and $dv=\frac{1}{x^2}dx \Rightarrow v=\int\frac{1}{x^2}dx = -x^{-1} = -\frac{1}{x}$.

Using integration by parts, we have

$$\int_1^e\left(\frac{\ln x}{x}\right)^2 dx = \int_1^e(\ln x)^2\left(\frac{1}{x^2}dx\right) = -\frac{(\ln x)^2}{x}\Big|_1^e - \int_1^e -\frac{1}{x}\,d(\ln x)^2$$

$$= -\frac{(\ln e)^2}{e} + \frac{(\ln 1)^2}{1} + 2\int_1^e\frac{\ln x}{x^2}dx = -\frac{1}{e} + 2\int_1^e \underbrace{(\ln x)}_{u}\underbrace{\left(\frac{1}{x^2}dx\right)}_{dv}.$$

$$= -\frac{1}{e} + 2\left(-\frac{\ln x}{x}\Big|_1^e - \int_1^e -\frac{1}{x}\,d(\ln x)\right)$$

$$= -\frac{1}{e} + 2\left(-\frac{\ln e}{e} + \frac{\ln 1}{1} + \int_1^e\frac{1}{x^2}dx\right) = -\frac{1}{e} - \frac{2}{e} + 2\left(-\frac{1}{x}\right)\Big|_1^e = 2 - \frac{5}{e}.$$

(f) Take $u=\cos^2 x$, $dv=\cos x\,dx \Rightarrow v=\int\cos x\,dx = \sin x$.

Using integration by parts, we have

$$\int\cos^3 x\,dx = \int\cos^2 x(\cos x\,dx) = \cos^2 x\sin x - \int(\sin x)\,d(\cos^2 x)$$

$$= \cos^2 x\sin x - \int\sin x(-2\cos x\sin x)\,dx$$

$$= \cos^2 x\sin x + 2\int\sin^2 x\cos x\,dx$$

$$= \cos^2 x\sin x + 2\int(1-\cos^2 x)\cos x\,dx$$

$$= \cos^2 x\sin x + 2\int\cos x\,dx - 2\int\cos^3 x\,dx$$

$$= \cos^2 x\sin x + 2\sin x - 2\int\cos^3 x\,dx$$

$$\Rightarrow 3\int \cos^3 x\,dx = \cos^2 x \sin x + 2\sin x$$

$$\Rightarrow \int \cos^3 x\,dx = \frac{1}{3}(\cos^2 x \sin x + 2\sin x) + C.$$

(g) We take $u = \sin 3x$ and $dv = e^x dx \Rightarrow v = \int e^x dx = e^x$.

Using integration by parts, we get

$$\int e^x \sin 3x\,dx = e^x \sin 3x - \int e^x d(\sin 3x) = e^x \sin 3x - 3\int e^x \cos x\,dx$$

$$= e^x \sin 3x - 3(e^x \cos 3x - \int e^x d(\cos 3x))$$

$$= e^x \sin 3x - 3e^x \cos 3x - 9\int e^x \sin 3x\,dx.$$

$$\Rightarrow 10\int e^x \sin 3x\,dx = e^x \sin 3x - 3e^x \cos 3x$$

$$\Rightarrow \int e^x \sin 3x\,dx = \frac{1}{10} e^x \sin 3x - \frac{3}{10} e^x \cos 3x + C.$$

(h) Take $u = \tan^{-1} x$ and $dv = dx \Rightarrow v = \int dx = x$

Using integration by parts, we have

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \int x\,d(\tan^{-1} x) = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$$

$$\overset{y=1+x^2}{=} x\tan^{-1} x - \int \frac{x}{1+x^2}\left(\frac{1}{2x}dy\right) = x\tan^{-1} x - \frac{1}{2}\int \frac{1}{y}\,dy$$

$$= x\tan^{-1} x - \frac{1}{2}\ln|y| + C = x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + C.$$

3. (a) Let $y = 2e^x + 1 \Rightarrow \frac{dy}{dx} = 2e^x \Rightarrow dx = \frac{1}{2e^x}dy$

The integral then becomes

$$\int e^{2x} \sin(2e^x+1)\,dx = \int e^{2x} \sin(2e^x+1)\left(\frac{1}{2e^x}\right)dy$$

$$= \frac{1}{2}\int e^x \sin(2e^x+1)\,dy = \frac{1}{2}\int \frac{y-1}{2}\sin y\,dy = \frac{1}{4}\int y\sin y\,dy - \frac{1}{4}\int \sin y\,dy$$

$$= \frac{1}{4}\int \underbrace{y}_{u}\, \underbrace{\sin y\,dy}_{dv} + \frac{1}{4}\cos y$$

$$= \frac{1}{4}\left(-y\cos y - \int(-\cos y)\,dy\right) + \frac{1}{4}\cos y$$

$= -\frac{1}{4}y\cos y + \frac{1}{4}\sin y + \frac{1}{4}\cos y + C$

$= -\frac{1}{2}e^x\cos(2e^x+1) + \frac{1}{4}\sin(2e^x+1) + C$

(b) Let $y = 2\sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x}} \Rightarrow dx = \sqrt{x}\,dy$

When $x=1$, $y=2$. When $x=0$, $y=0$.

$\int_0^1 \sin(2\sqrt{x})\,dx = \int_0^2 \sin(2\sqrt{x})(\sqrt{x}\,dy) = \int_0^2 \frac{y}{2}\sin y\,dy = \frac{1}{2}\int_0^2 \underbrace{y}_{u}\underbrace{\sin y\,dy}_{dv}$

$= \frac{1}{2}\left(-y\cos y\Big|_0^2 - \int_0^2(-\cos y)\,dy\right) = -\cos 2 + \frac{1}{2}\sin y\Big|_0^2$

$= -\cos 2 + \frac{1}{2}\sin 2.$

(c) Let $y = 1 + x^{\frac{1}{3}} \Rightarrow \frac{dy}{dx} = \frac{1}{3x^{\frac{2}{3}}} \Rightarrow dx = 3x^{\frac{2}{3}}\,dy.$

When $x=1$, $y=2$. When $x=0$, $y=1$.

$\int_0^1 \ln(1+x^{\frac{1}{3}})\,dx = \int_1^2 3x^{\frac{2}{3}}\ln(1+x^{\frac{1}{3}})\,dy = 3\int_1^2 (y-1)^2\ln y\,dy.$

Using integration by parts with $u(y) = \ln y$ and

$dv = (y-1)^2\,dy \Rightarrow v = \int (y-1)^2\,dy = \frac{1}{3}(y-1)^3$. We get

$\int_1^2 (y-1)^2\ln y\,dy = \frac{1}{3}(y-1)^3\ln y\Big|_1^2 - \int_1^2 \frac{1}{3}(y-1)^3\,d(\ln y)$

$= \frac{1}{3}\ln 2 - \frac{1}{3}\int_1^2 \frac{y^3-3y^2+3y-1}{y}\,dy$

$= \frac{1}{3}\ln 2 - \frac{1}{3}\int_1^2 (y^2-3y+3-\frac{1}{y})\,dy$

$= \frac{1}{3}\ln 2 - \frac{1}{3}\left(\frac{y^3}{3} - \frac{3y^2}{2} + 3y - \ln y\right)\Big|_1^2 = \frac{2}{3}\ln 2 - \frac{5}{18}$

$\int_0^1 \ln(1+x^{\frac{1}{3}})\,dx = 2\ln 2 - \frac{5}{6}$

(d) Let $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow dx = x\,dy$

Then the integral becomes

$\int \cos(\ln x)\,dx = \int x\cos(\ln x)\,dy = \int e^y\cos y\,dy.$

Using integration by parts twice, we get

$\int e^y\cos y\,dy = \frac{1}{2}(e^y\cos y + e^y\sin y) + C.$

Therefore, we conclude that

$$\int \cos(\ln x)\,dx = \frac{1}{2}(e^y\cos y + e^y \sin y) + C$$
$$= \frac{1}{2}(e^{\ln x}\cos(\ln x) + e^{\ln x}\sin(\ln x)) + C$$
$$= \frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + C.$$

(e) Let $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow dx = \frac{1}{\cos x}\,dy$

Together with the fact that $\sin 2x = 2\sin x\cos x$, the integral can be written as

$$\int \sin 2x \ln(\sin x)\,dx = 2\int \sin x\cos x\ln(\sin x)\left(\frac{1}{\cos x}dy\right) = 2\int y\ln y\,dy$$

Using integration by parts with $u = \ln y$ and $dv = y\,dy$

$\Rightarrow v = \int y\,dy = \frac{y^2}{2}$, the latter integral can be computed as

$$\int y\ln y\,dy = \frac{y^2}{2}\ln y - \int \frac{y^2}{2}\,d(\ln y) = \frac{y^2}{2}\ln y - \int \frac{y^2}{2}\left(\frac{1}{y}dy\right)$$
$$= \frac{y^2}{2}\ln y - \frac{1}{2}\int y\,dy = \frac{y^2}{2}\ln|y| - \frac{y^2}{4} + C.$$

Therefore, we conclude that

$$\int \sin 2x \ln(\sin x)\,dx = 2\left(\frac{y^2}{2}\ln y - \frac{y^2}{4}\right) + C = \sin^2 x\ln|\sin x| - \frac{\sin^2 x}{2} + C.$$

(f) Let $y = x+3 \Rightarrow \frac{dy}{dx} = 1 \Rightarrow dx = dy$.

The integral becomes

$$\int (x+1)\ln(x+3)\,dx = \int (y-2)\ln y\,dy.$$

Using integration by parts with $u(y) = \ln y$ and $dv = (y-2)\,dy$

$\Rightarrow v = \int (y-2)\,dy = \frac{1}{2}(y-2)^2$. The latter integral can be computed as

$$\int (y-2)\ln y\,dy = \frac{1}{2}(y-2)^2\ln y - \int \frac{1}{2}(y-2)^2\,d(\ln y)$$
$$= \frac{1}{2}(y-2)^2\ln y - \frac{1}{2}\int \frac{y^2-4y+4}{y}\,dy$$

$$= \frac{1}{2}(y-2)^2 \ln y - \frac{1}{2}\int (y - 4 + \frac{4}{y})\, dy$$

$$= \frac{1}{2}(y-2)^2 \ln y - \frac{1}{4}y^2 + 2y - 2\ln y + C.$$

Then we conclude that

$$\int (x+1)\ln(x+3)\, dx = \frac{1}{2}(y-2)^2 \ln y - \frac{1}{4}y^2 + 2y - 2\ln y + C$$

$$= \frac{1}{2}(x+1)^2 \ln|x+3| - \frac{1}{4}(x+3)^2 + 2(x+3) - 2\ln|x+3| + C.$$

(g) Let $x = 2\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta \Rightarrow dx = 2\cos\theta\, d\theta$

Then the integral becomes

$$\int x^2\sqrt{4-x^2}\, dx = \int 4\sin^2\theta\sqrt{4-4\sin^2\theta}\,(2\cos\theta\, d\theta) = 16\int \sin^2\theta\cos^2\theta\, d\theta$$

$$= 16\int (\sin\theta\cos\theta)^2 d\theta = 16\int (\tfrac{1}{2}\sin 2\theta)^2\, d\theta = 4\int \sin^2 2\theta\, d\theta$$

$$= 4\int -\tfrac{1}{2}[\cos(2\theta+2\theta) - \cos(2\theta-2\theta)]\, d\theta$$

$$= -2\int \cos 4\theta + 2\int 1\, d\theta = -\tfrac{1}{2}\sin 4\theta + 2\theta + C$$

$$= -\tfrac{1}{2}\sin(4\sin^{-1}\tfrac{x}{2}) + 2\sin^{-1}\tfrac{x}{2} + C.$$

(h) Let $y = 4 + x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x}dy$.

Then the integral becomes

$$\int x^3\sin(4+x^2)\, dx = \int x^3\sin(4+x^2)\left(\frac{1}{2x}dy\right) = \frac{1}{2}\int (y-4)\sin y\, dy.$$

Using integration by parts with $u = y-4$ and $dv = \sin y\, dy$
$\Rightarrow v = \int \sin y\, dy = -\cos y$. The later integral can be computed as

$$\int (y-4)\sin y\, dy = -(y-4)\cos y - \int(-\cos y)\, d(y-4)$$

$$= (4-y)\cos y + \int \cos y\, dy = (4-y)\cos y + \sin y + C.$$

Hence, we can conclude that

$$\int x^3\sin(4+x^2)\, dx = \frac{1}{2}[(4-y)\cos y + \sin y] + C$$

$$= -\frac{x^2}{2}\cos(4+x^2) + \frac{1}{2}\sin(4+x^2) + C.$$

4. When $-1 \leq x \leq 1$, we have $1 \leq \sqrt{1+x^2} \leq \sqrt{2}$.

Therefore, $\int_{-1}^{1} 1\, dx \leq \int_{-1}^{1} \sqrt{1+x^2}\, dx \leq \int_{-1}^{1} \sqrt{2}\, dx$.

$$2 \leq \int_{-1}^{1} \sqrt{1+x^2}\, dx \leq 2\sqrt{2}.$$

5. The first objective is to eliminate the term "x" in the integrand. This can be done by using integration by parts.

Using integration by parts with $u = x$ and $dv = f'(x)\,dx \Rightarrow$ $v = \int f'(x)\,dx = f(x)$, we have

$$\int_a^b x f'(x)\,dx = x f(x)\Big|_a^b - \int_a^b f(x)\,dx = bf(b) - af(a) - \int_a^b f(x)\,dx$$
$$= b - a.$$

6. We first simplify the first integral $\int_0^a x^3 f(x^2)\,dx$ on the left hand side.

Let $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x}\,dy$.

When $x = a$, $y = a^2$. When $x = 0$, $y = 0$.

Then the integral $\int_0^a x^3 f(x^2)\,dx$ can be rewritten as

$$\int_0^a x^3 f(x^2)\,dx = \int_0^{a^2} x^3 f(x^2)\left(\frac{1}{2x}\,dy\right) = \frac{1}{2}\int_0^{a^2} y f(y)\,dy.$$

Since the definite integrals $\int_0^{a^2} y f(y)\,dy$ and $\int_0^{a^2} x f(x)\,dx$ give the same values. Therefore we conclude that

$$\int_0^a x^3 f(x^2)\,dx - \frac{1}{2}\int_0^{a^2} x f(x)\,dx = \frac{1}{2}\int_0^{a^2} y f(y)\,dy - \frac{1}{2}\int_0^{a^2} x f(x)\,dx = 0.$$

7. (a) We use integration by parts with $u = \cos^{n-1}x$ and $dv = \cos x\,dx$ $\Rightarrow v = \int \cos x\,dx = \sin x$ and obtain

$$I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n x\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cos^{n-1}x}_{u}\,\underbrace{(\cos x\,dx)}_{dv}$$

$$= \cos^{n-1}x \sin x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\, d(\cos^{n-1}x)$$

$$= 0 - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \left((n-1)\cos^{n-2} x \left(-\sin x \, dx \right) \right)$$

$$= (n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^{n-2} x \, dx$$

$$= (n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos^2 x) \cos^{n-2} x \, dx$$

$$= (n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-2} x \, dx - (n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n x \, dx$$

$$= (n-1) I_{n-2} - (n-1) I_n$$

Summing up, we have $I_n = (n-1) I_{n-2} - (n-1) I_n$

$\Rightarrow n I_n = (n-1) I_{n-2} \Rightarrow I_n = \frac{n-1}{n} I_{n-2}$.

(b) Using the reduction formula obtained in (a), we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, dx = I_5 = \frac{4}{5} I_3 = \frac{4}{5}\left(\frac{2}{3} I_1\right) = \frac{8}{15} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$$

$$= \frac{8}{15} \sin x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{16}{15}.$$