Chapter 3

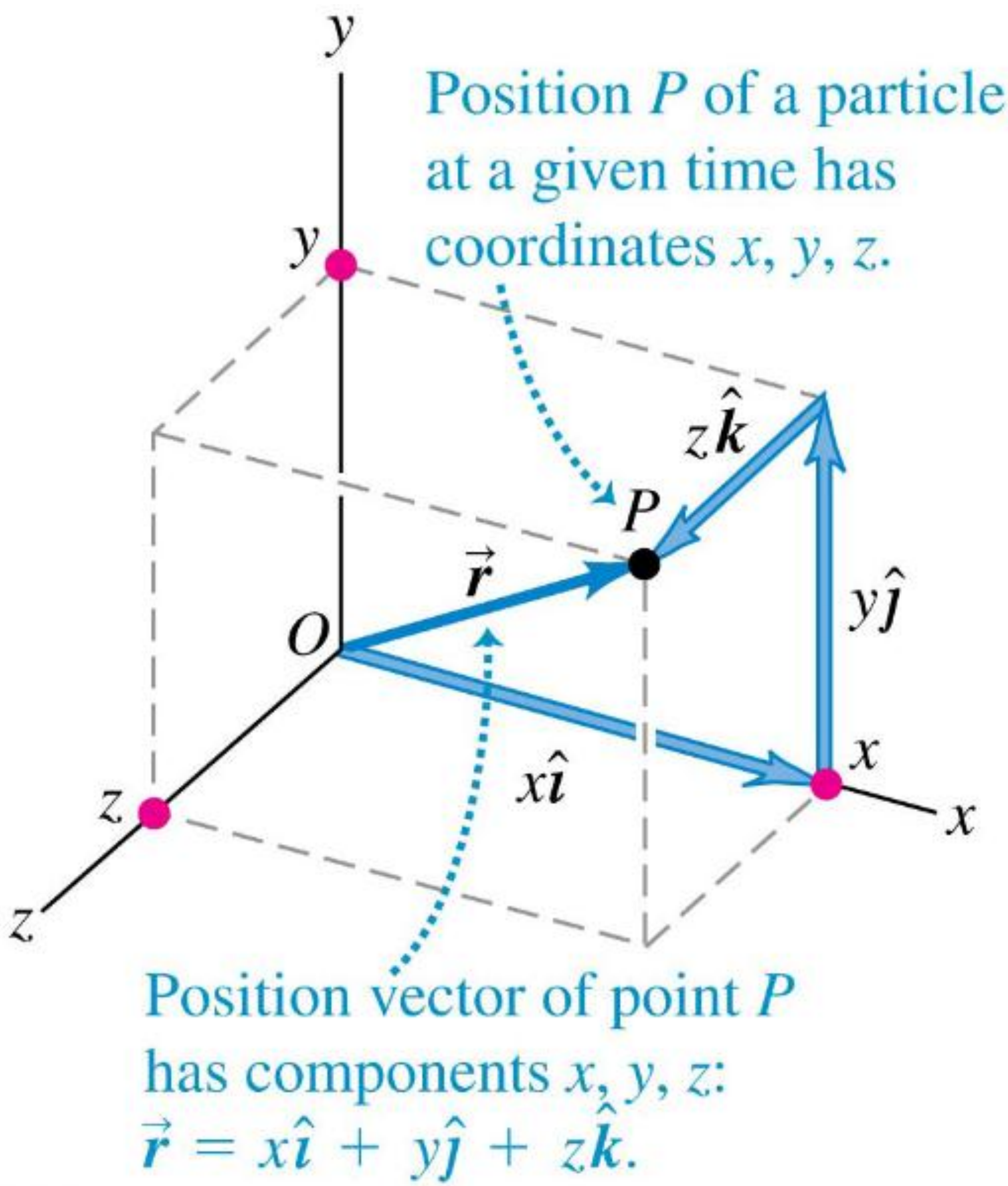
Motion in Two or Three Dimensions

Introduction

- What determines where a batted baseball lands?
- How do you describe the motion of a roller coaster car along a curved track or the flight of a circling hawk?
- Which hits the ground first, a baseball that you simply drop or one that you throw horizontally?
- We need to extend our description of motion to two and three dimensions.

Position vector

- The position vector from the origin to point P has components x, y, and z.



Velocity

- We define the **average velocity** as the displacement divided by the time interval:

Change in the particle's position vector

Average velocity vector of a particle during time interval from t_1 to t_2

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$$

Final position minus initial position

Time interval

Final time minus initial time

- **Instantaneous velocity** (a.k.a. "velocity") is the instantaneous rate of change of position with time:

The **instantaneous velocity vector** of a particle ...

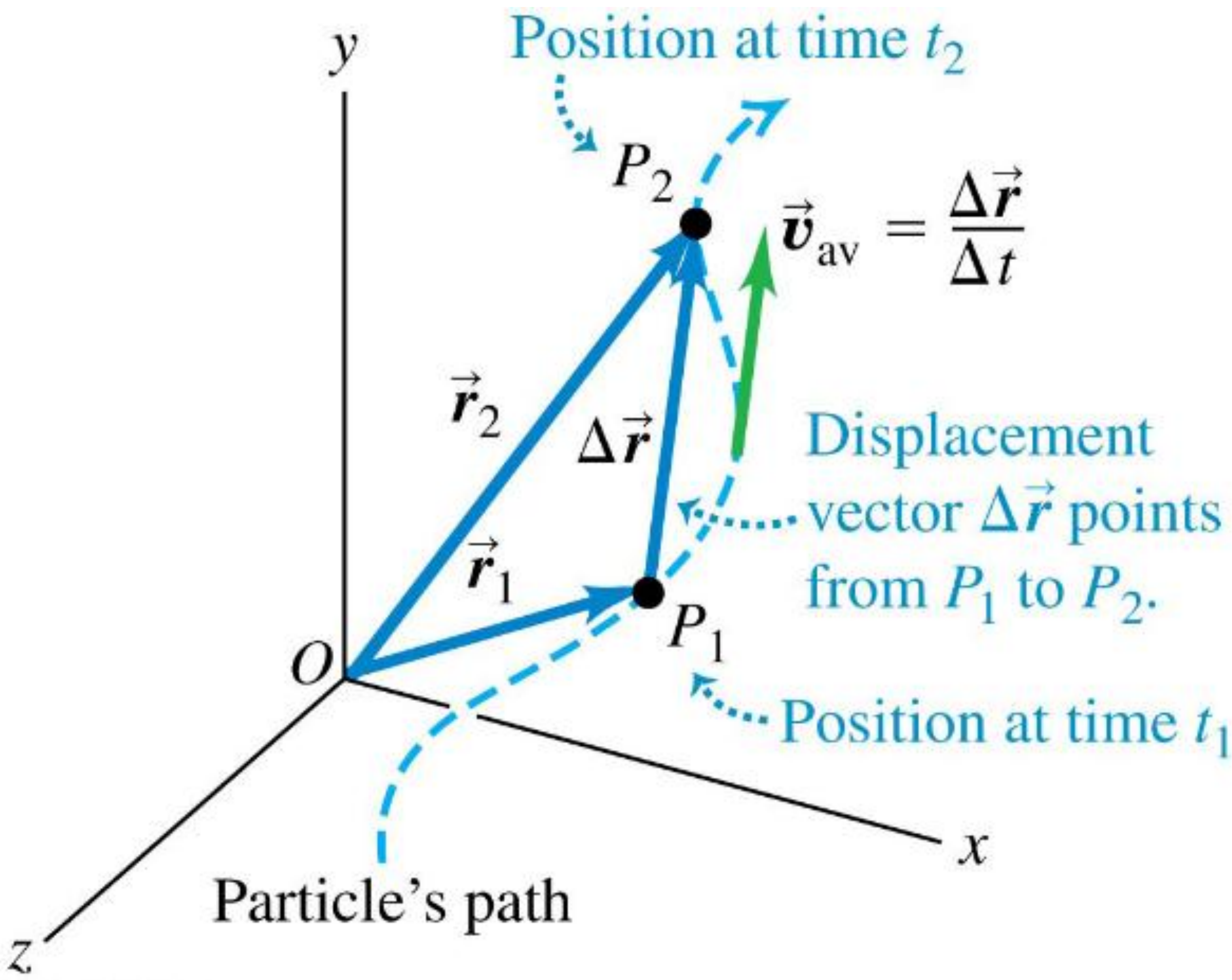
$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

... equals the limit of its average velocity vector as the time interval approaches zero ...

... and equals the instantaneous rate of change of its position vector.

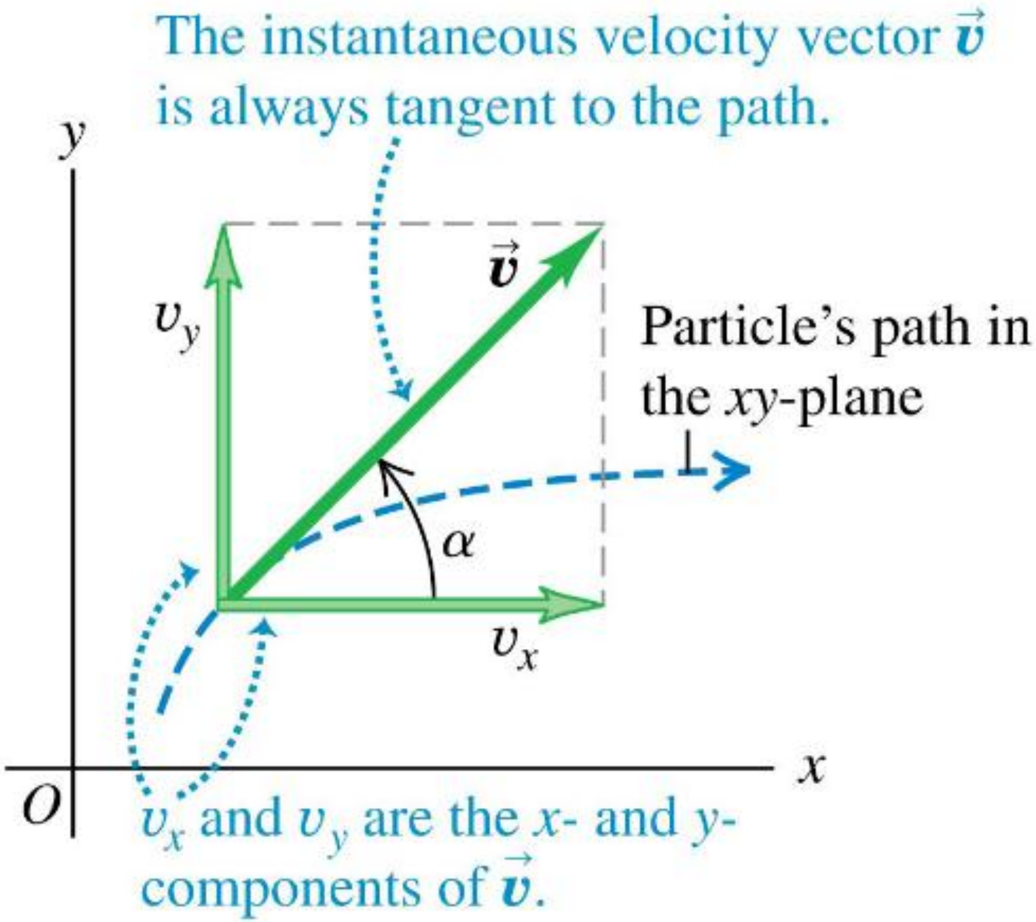
Average velocity

- The **average velocity** between two points is the displacement divided by the time interval between the two points, and it has the same direction as the displacement.



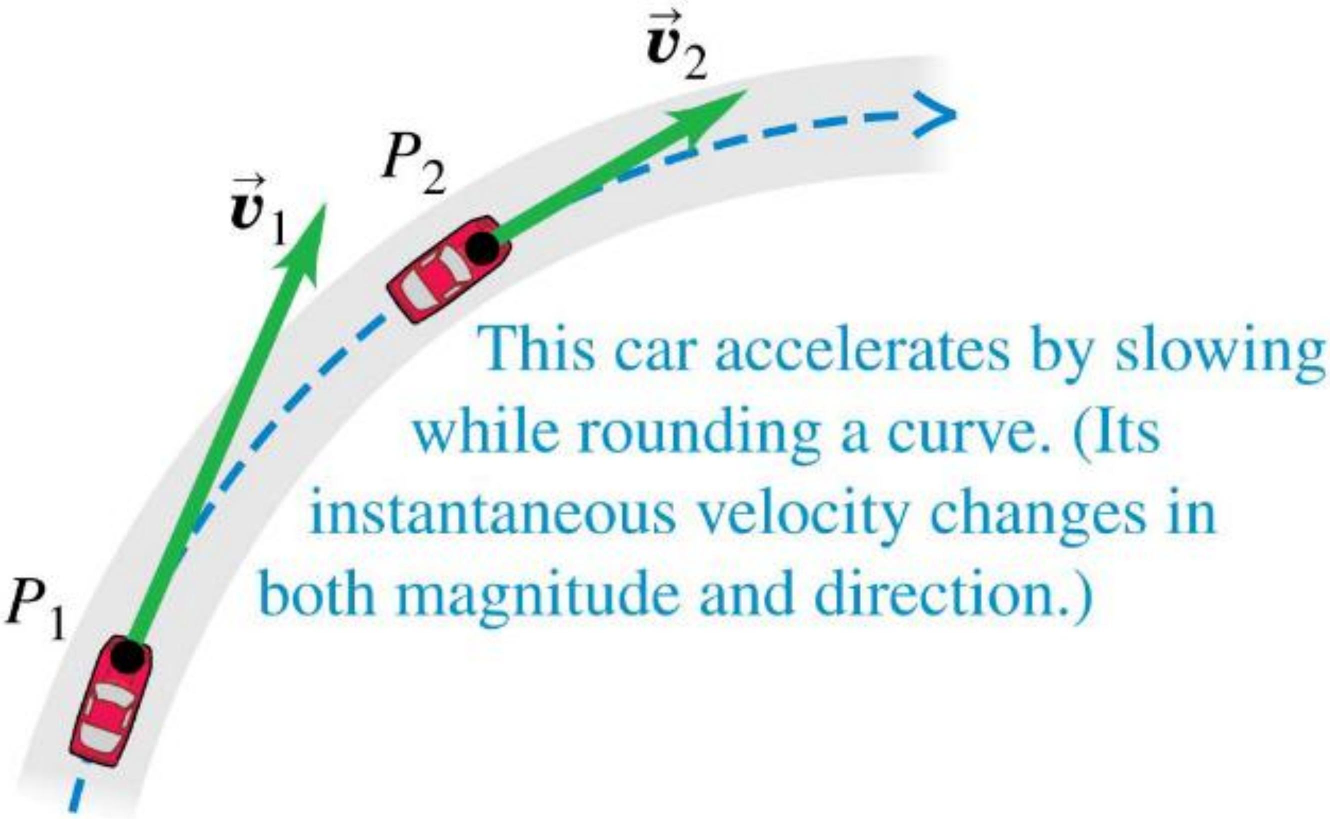
Instantaneous velocity

- The **instantaneous velocity** is the instantaneous rate of change of position vector with respect to time.
- The components of the instantaneous velocity are $v_x = dx/dt$, $v_y = dy/dt$, and $v_z = dz/dt$.
- The instantaneous velocity of a particle is always tangent to its path.



Acceleration

- Acceleration describes how the velocity changes.



Acceleration

- We define the **average acceleration** as the change in velocity divided by the time interval:

Change in the particle's velocity

Average acceleration vector of a particle during time interval from t_1 to t_2

$$\vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

Final velocity minus initial velocity

Time interval

Final time minus initial time

- **Instantaneous acceleration** (a.k.a. "acceleration") is the instantaneous rate of change of velocity with time:

The **instantaneous acceleration vector** of a particle ...

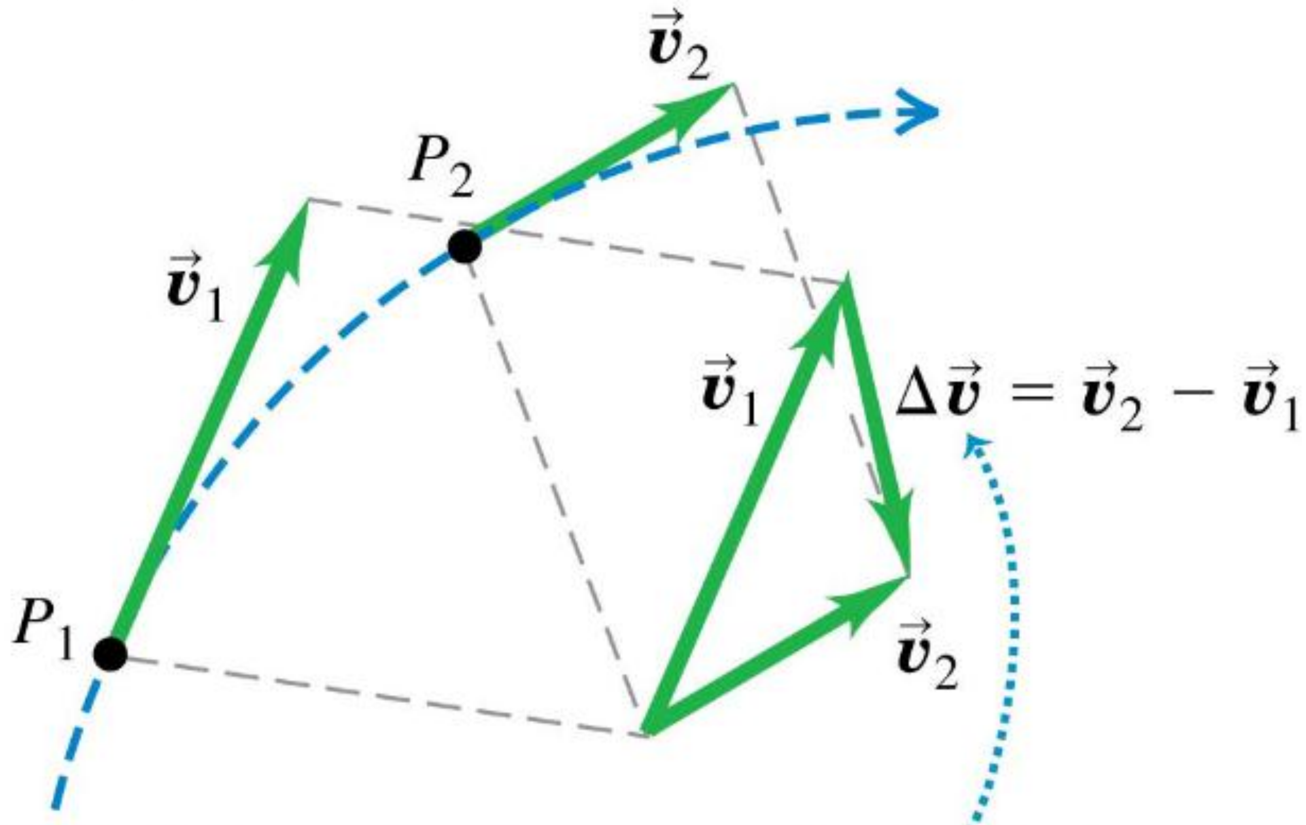
$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

... equals the limit of its average acceleration vector as the time interval approaches zero ...

... and equals the instantaneous rate of change of its velocity vector.

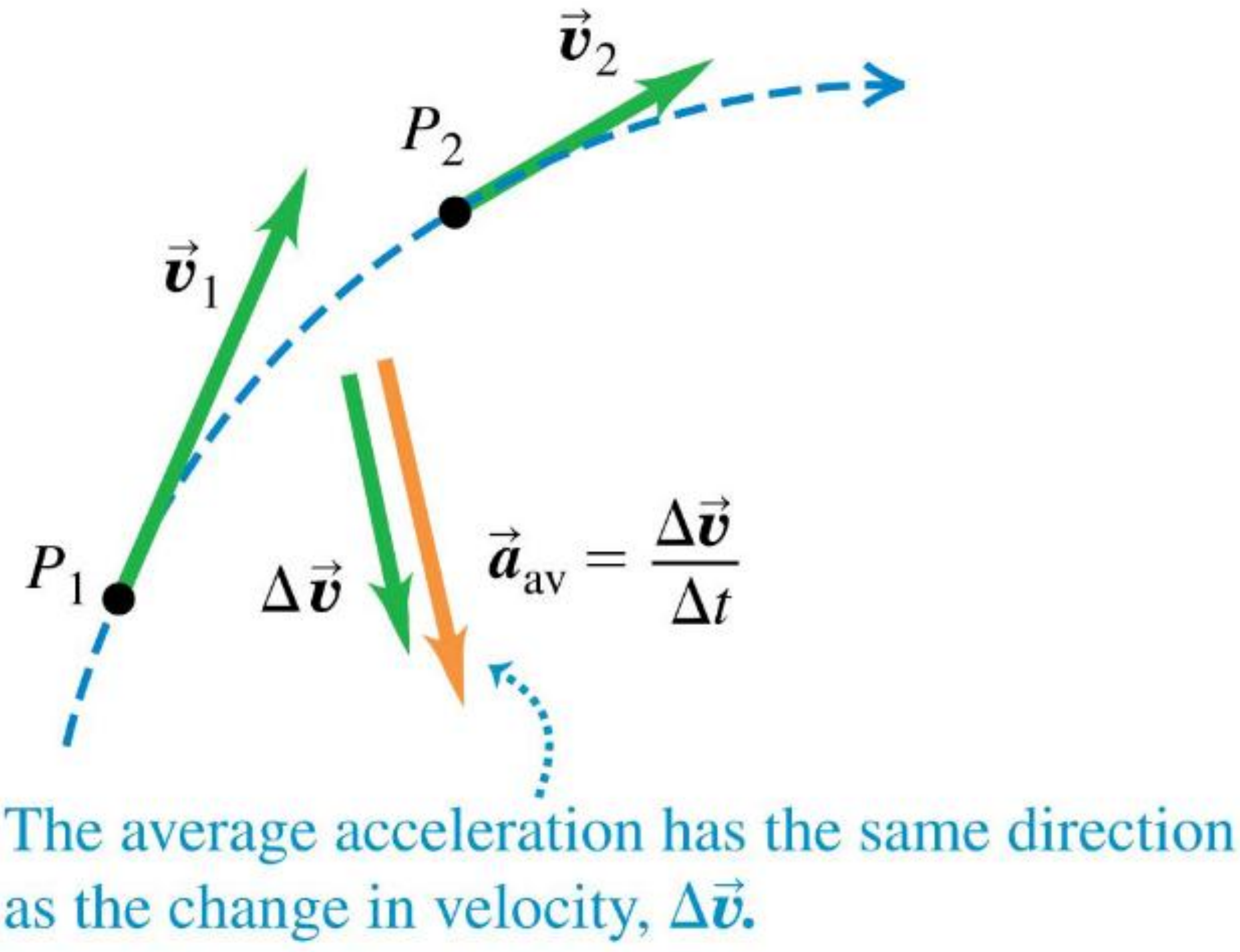
Average acceleration

- The change in velocity between two points is determined by vector subtraction.



To find the car's average acceleration between P_1 and P_2, we first find the change in velocity $\Delta\vec{v}$ by subtracting $\vec{v}_1$ from $\vec{v}_2$. (Notice that $\vec{v}_1 + \Delta\vec{v} = \vec{v}_2$.)

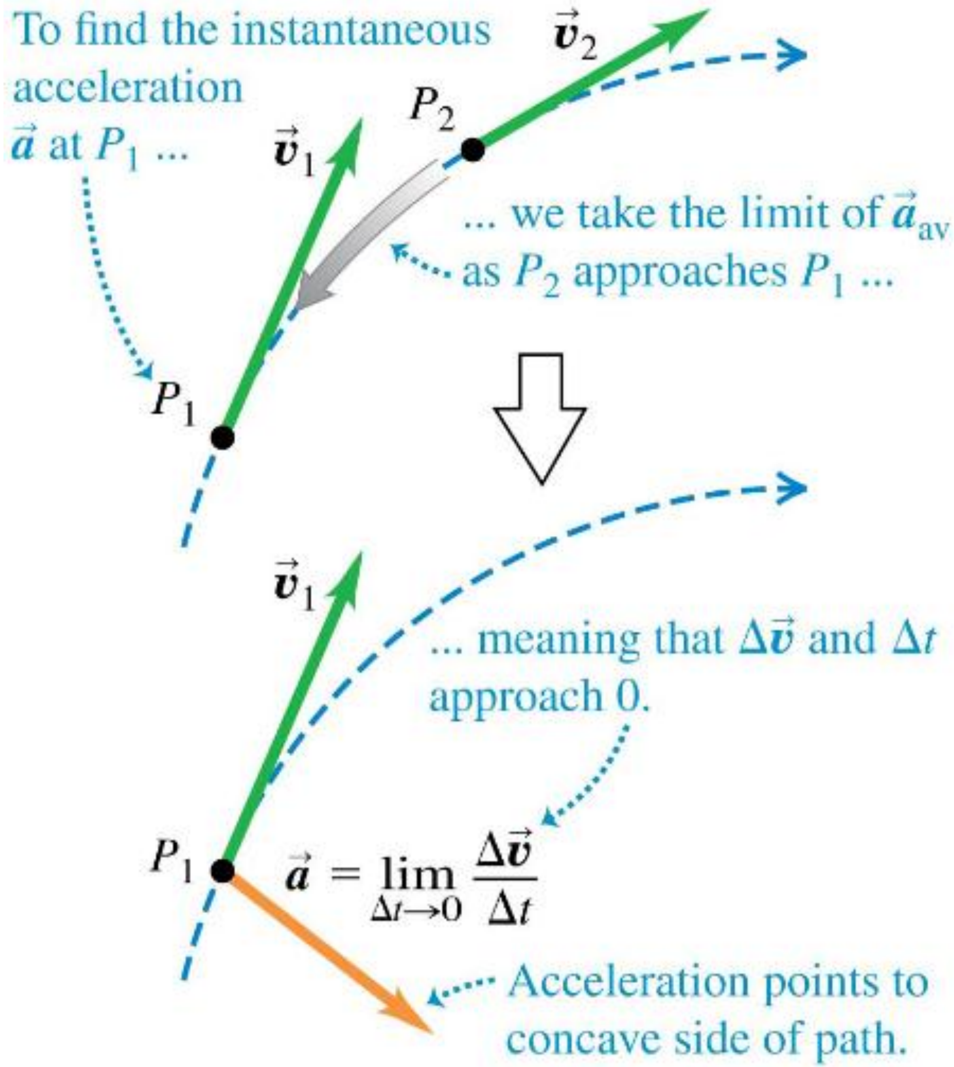
Average acceleration



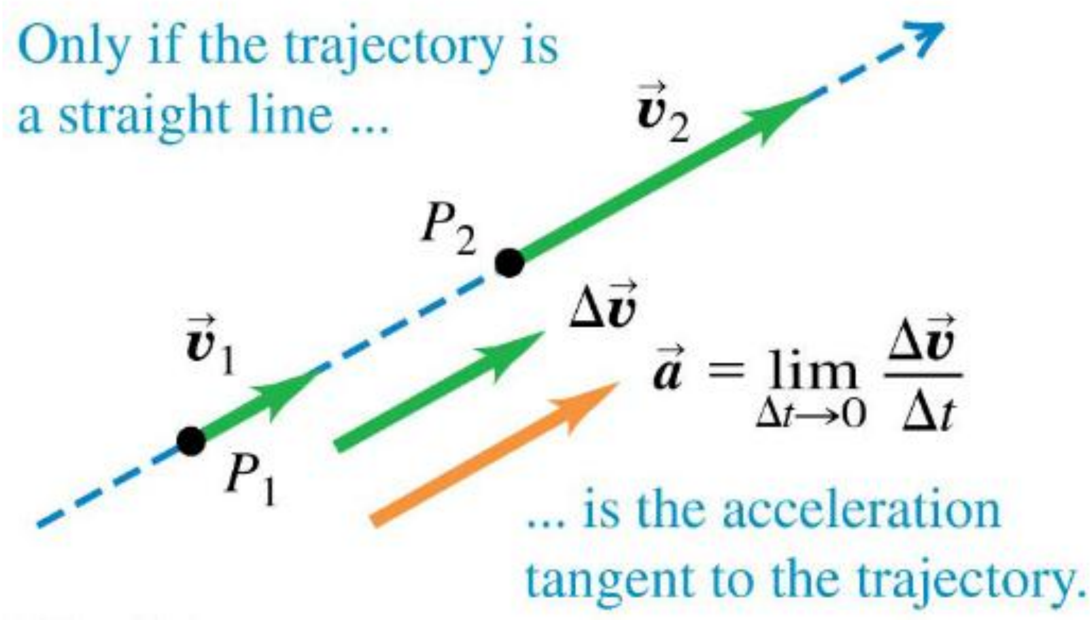
Instantaneous acceleration

- The velocity vector is always tangent to the particle's path, but the instantaneous acceleration vector does *not* have to be tangent to the path.

- If the path is curved, the acceleration points toward the concave side of the path.



Instantaneous acceleration



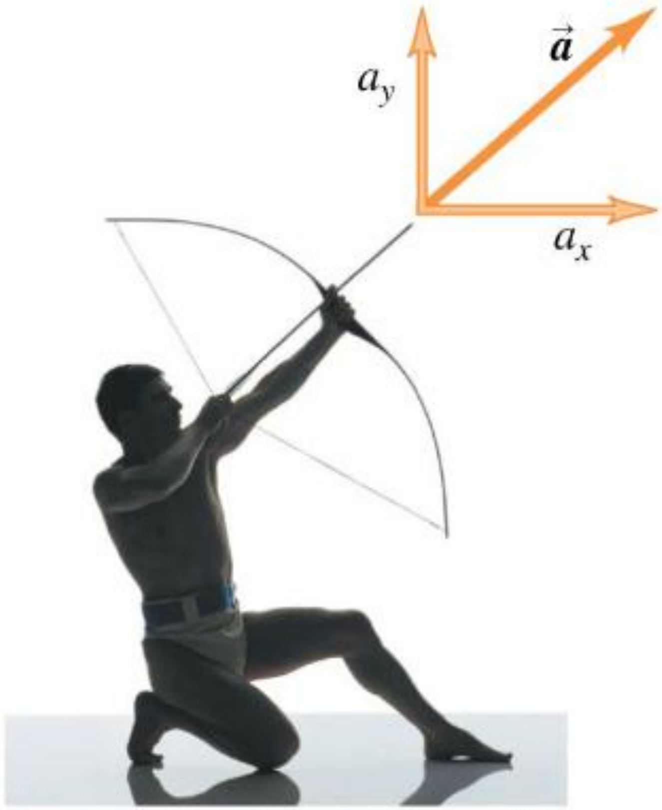
Components of acceleration

Each **component of** a particle's **instantaneous acceleration vector** ...

$$a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt} \qquad a_z = \frac{dv_z}{dt}$$

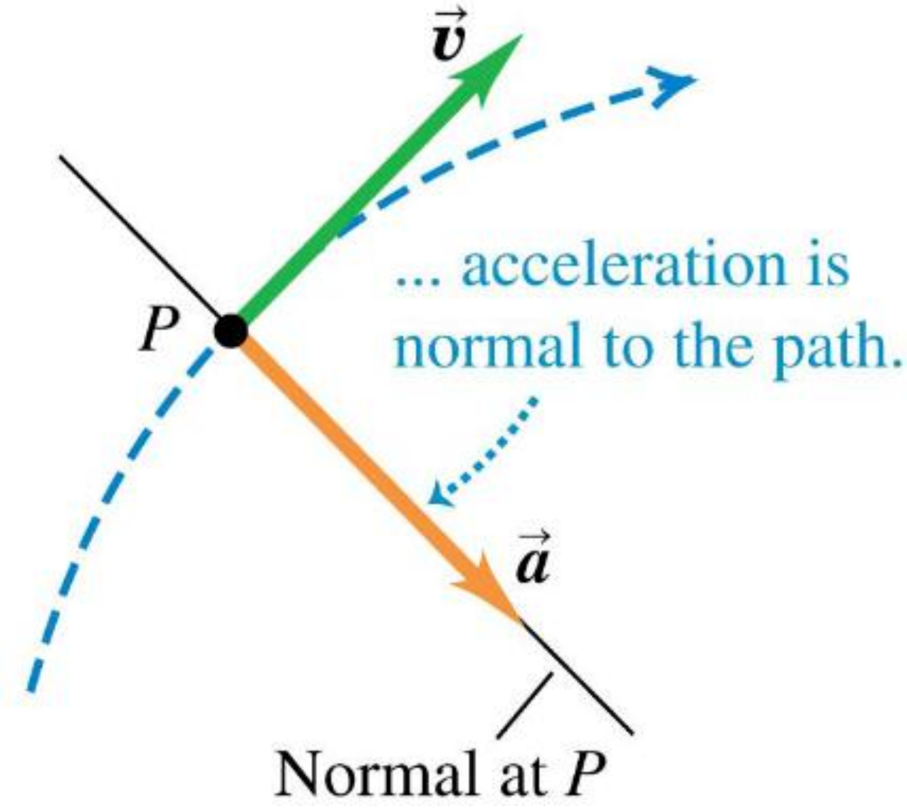
... equals the instantaneous rate of change of its corresponding velocity component.

- Shooting an arrow is an example of an acceleration vector that has both *x*- and *y*-components.



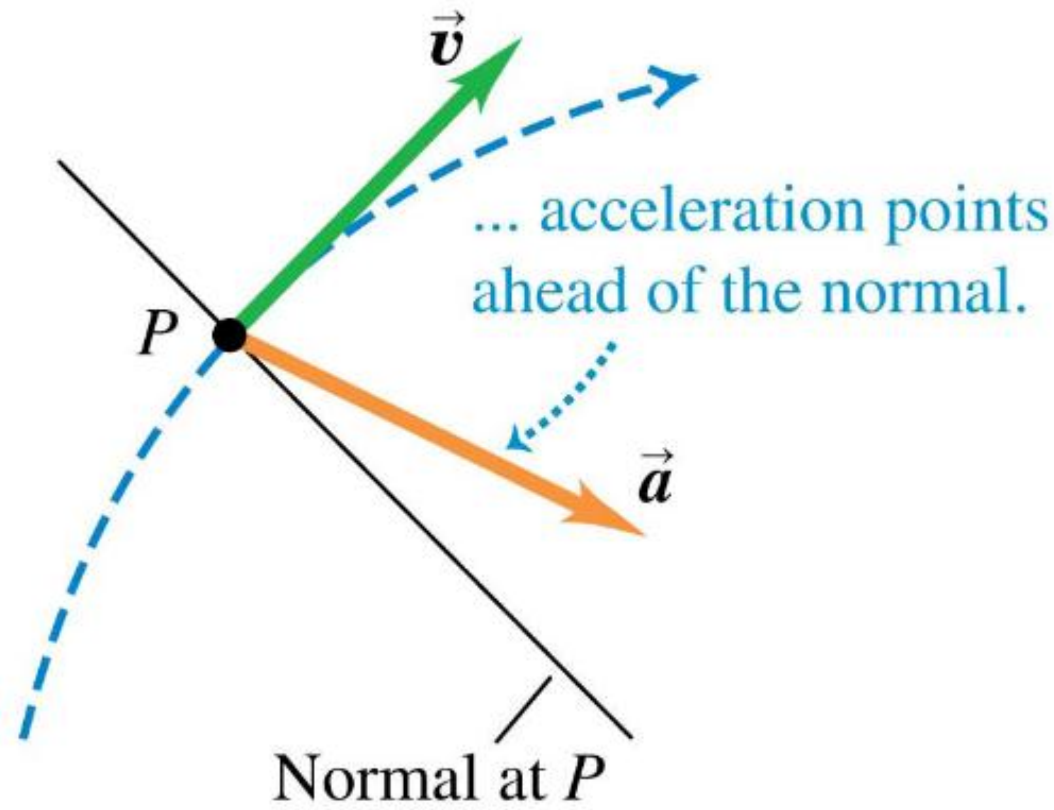
Parallel and perpendicular components of acceleration

- Velocity and acceleration vectors for a particle moving through a point P on a curved path with *constant speed*



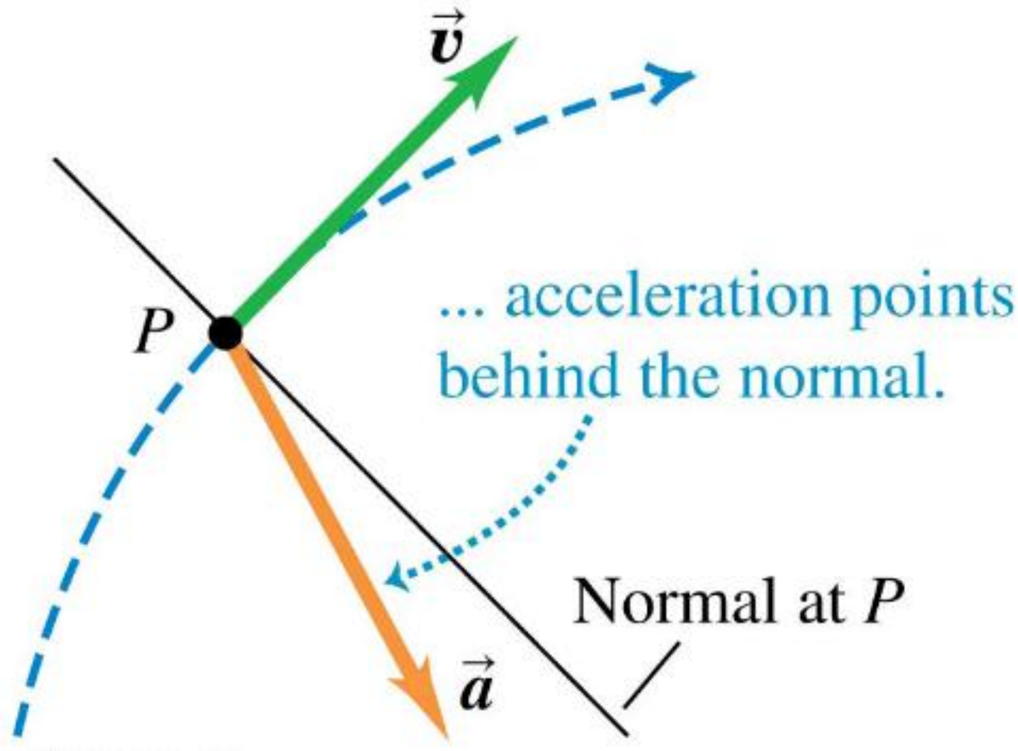
Parallel and perpendicular components of acceleration

- Velocity and acceleration vectors for a particle moving through a point *P* on a curved path with *increasing speed*



Parallel and perpendicular components of acceleration

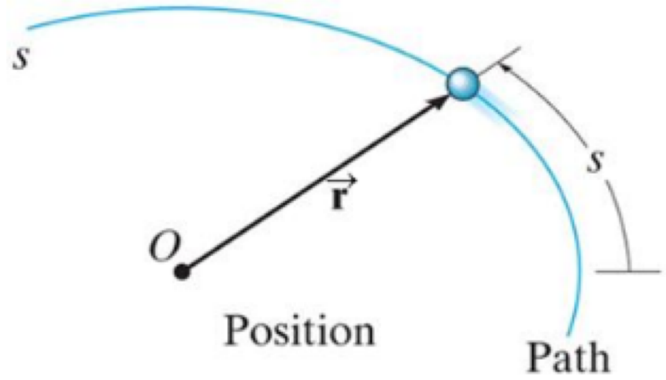
- Velocity and acceleration vectors for a particle moving through a point P on a curved path with *decreasing speed*



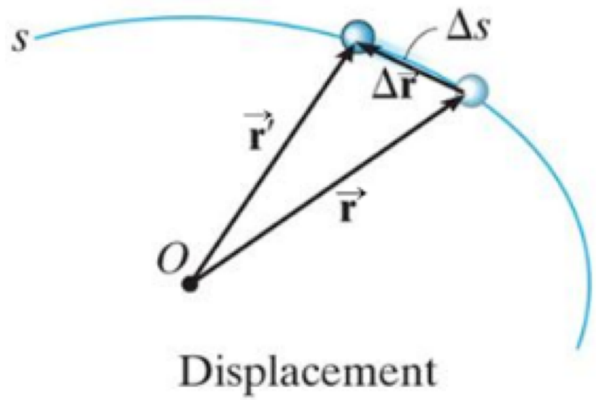
Curvilinear motion: general formalism

A particle moving along a curved path undergoes curvilinear motion. Since the motion is often three-dimensional, vectors are used to describe the motion.

A particle moves along a curve defined by the path function, s.



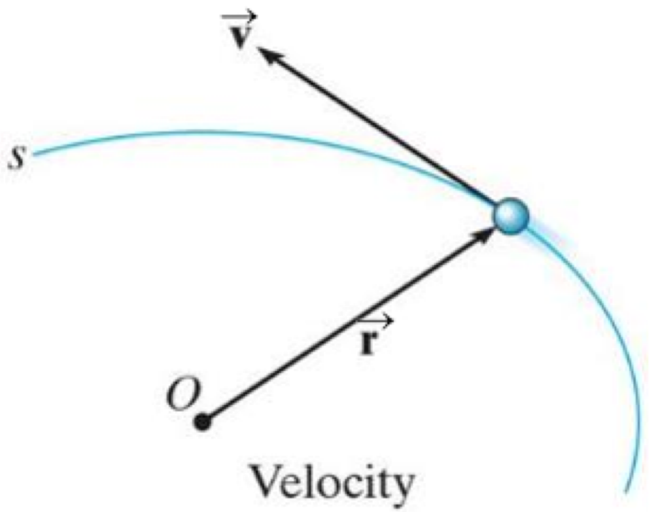
The position of the particle at any instant is designated by the vector $\vec{r} = r(t)$. Both the magnitude and direction of $\vec{r}$ may vary with time.



If the particle moves a distance Δs along the curve during time interval Δt, the displacement is determined by vector subtraction: $\Delta \vec{r} = \vec{r}\,' - \vec{r}$

Curvilinear motion: general formalism

Velocity represents the rate of change in the position of a particle.



The average velocity of the particle during the time increment Δt is $\vec{v}_{avg} = \Delta \vec{r} / \Delta t$.

The instantaneous velocity is the time-derivative of position $\vec{v} = d \vec{r} / dt$.

The velocity vector, ***v***, is always tangent to the path of motion.

The magnitude of $\vec{v}$ is called the speed. Since the arc length Δs approaches the magnitude of $\Delta \vec{r}$ as t→0, the speed can be obtained by differentiating the path function (v = ds/dt). Note that this is not a vector!

Curvilinear motion: general formalism

Acceleration represents the rate of change in the velocity of a particle.

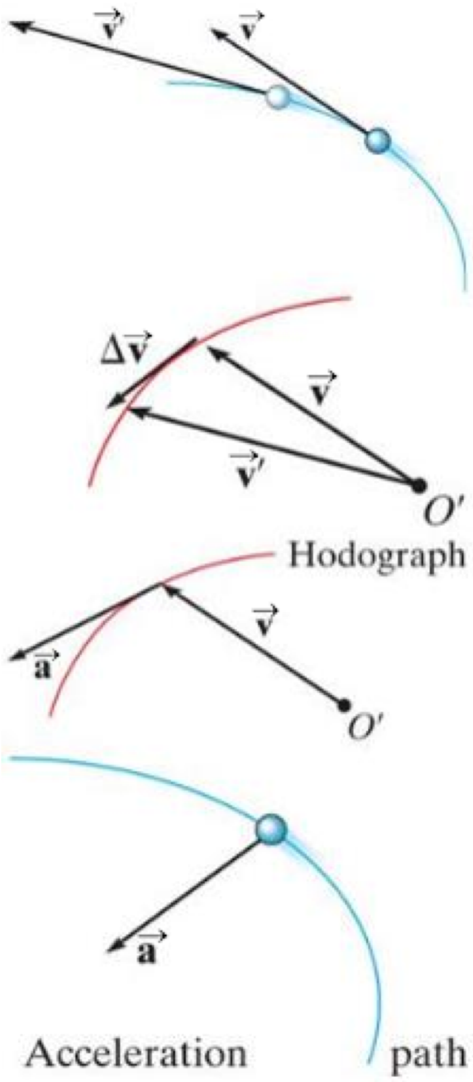
If a particle's velocity changes from $\boldsymbol{v}$ to $\boldsymbol{v'}$ over a time increment Δt, the average acceleration during that increment is:

$$\vec{a}_{avg} = \Delta\, \vec{v} /\Delta t = (\vec{v} - \vec{v}\,')/\Delta t$$

The instantaneous acceleration is the time-derivative of velocity:

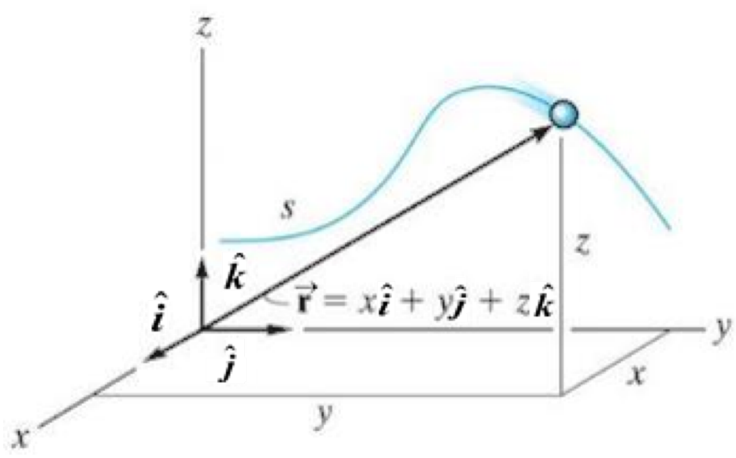
$$\vec{a} = d\, \vec{v} /dt = d^2\, \vec{r} /dt^2$$

A plot of the locus of points defined by the arrowhead of the velocity vector is called a hodograph. The acceleration vector is tangent to the hodograph, but not, in general, tangent to the path function.



Curvilinear motion: rectangular components

It is often convenient to describe the motion of a particle in terms of its x, y, z or rectangular components, relative to a fixed frame of reference.



Position

The position of the particle can be defined at any instant by the position vector

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}\ .$$

The x, y, z components may all be functions of time, i.e.,
$x = x(t)$, $y = y(t)$, and $z = z(t)$.

The magnitude of the position vector is: $r = (x^2 + y^2 + z^2)^{1/2}$

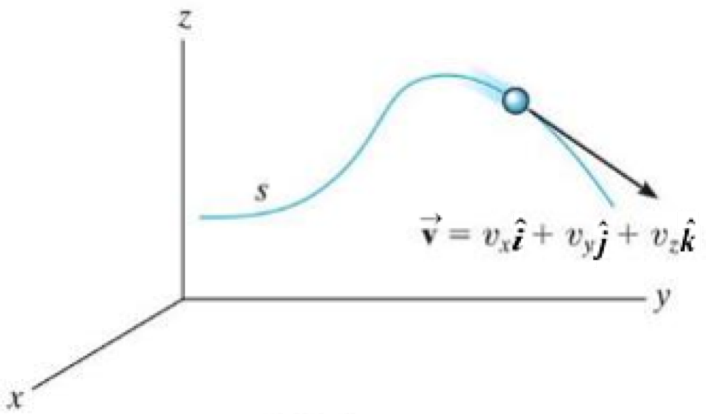
The direction of $\boldsymbol{r}$ is defined by the unit vector: $\vec{u}_r = (1/r)\vec{r}$

Rectangular components: velocity

The velocity vector is the time derivative of the position vector:

$$\vec{v} = d\,\vec{r}\,/dt = d(x\,\hat{\imath}\,)/dt + d(y\hat{\jmath}\,)/dt + d(z\hat{k}\,)/dt$$

Since the unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are constant in magnitude and direction, this equation reduces to $\vec{v} = v_x\,\hat{\imath} + v_y\,\hat{\jmath} + v_z\,\hat{k}$
where $v_x = \dot{x} = dx/dt$, $v_y = \dot{y} = dy/dt$, $v_z = \dot{z} = dz/dt$



Velocity

The magnitude of the velocity vector is

$$v = [(v_x)^2 + (v_y)^2 + (v_z)^2]^{1/2}$$

The direction of $\vec{v}$ is tangent to the path of motion.

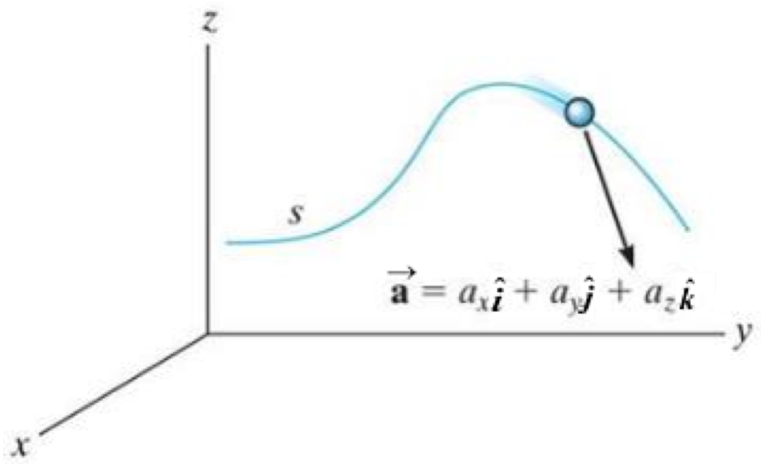
Rectangular components: acceleration

The acceleration vector is the time derivative of the velocity vector (second derivative of the position vector):

$$\vec{a} = d\,\vec{v}\,/dt = d^2\,\vec{r}\,/dt^2 = a_x\,\hat{\imath} + a_y\,\hat{\jmath} + a_z\,\hat{k}$$

where $a_x = \ddot{x} = \dot{v}_x = dv_x/dt$, $a_y = \ddot{y} = \dot{v}_y = dv_y/dt$, $a_z = \ddot{z} = \dot{v}_z = dv_z/dt$

The magnitude of the acceleration vector is $a = [(a_x)^2 + (a_y)^2 + (a_z)^2\,]^{0.5}$



Acceleration

The direction of $\vec{a}$ is usually not tangent to the path of the particle.

Quiz

1. In curvilinear motion, the direction of the instantaneous velocity is always

   A) tangent to the hodograph.
   B) perpendicular to the hodograph.
   ✔ C) tangent to the path.
   D) perpendicular to the path.

2. In curvilinear motion, the direction of the instantaneous acceleration is always

   ✔ A) tangent to the hodograph.
   B) perpendicular to the hodograph.
   C) tangent to the path.
   D) perpendicular to the path.

Example 1

Given: The box slides down the slope described by the equation y = $(0.05x^2)$ m, where x is in meters.
$v_x = -3$ m/s, $a_x = -1.5$ m/s^2 at x = 5 m.

Find: The y components of the velocity and the acceleration of the box at x = 5 m.

Plan: Note that the particle's velocity can be related by taking the first time derivative of the path's equation. And the acceleration can be related by taking the second time derivative of the path's equation.

Take a derivative of the position to find the component of the velocity and the acceleration.

Example 1

Solution:

Find the y-component of velocity by taking a time derivative of the position $y = (0.05x^2)$

$$\Rightarrow \quad \dot{y} = 2\,(0.05)\,x\,\dot{x} = 0.1\,x\,\dot{x}$$

Find the acceleration component by taking a time derivative of the velocity y

$$\Rightarrow \quad \ddot{y} = 0.1\,\dot{x}\,\dot{x} + 0.1\,x\,\ddot{x}$$

Substituting the x-component of the acceleration, velocity at x=5 into y and $\ddot{y}$.

Example 1

Since $\dot{x} = v_x = -3$ m/s, $\ddot{x} = a_x = -1.5$ m/s^2 at $x = 5$ m

$\Rightarrow \dot{y} = 0.1\, x\, \dot{x} = 0.1\,(5)\,(-3) = -1.5$ m/s

$$\Rightarrow \quad \ddot{y} = 0.1\,\dot{x}\,\dot{x} + 0.1\,x\,\ddot{x}$$
$$= 0.1\,(-3)^2 + 0.1\,(5)\,(-1.5)$$
$$= 0.9 - 0.75$$
$$= 0.15 \text{ m/s}^2$$

At $x = 5$ m

$v_y = -1.5$ m/s $= 1.5$ m/s $\downarrow$

$a_y = 0.15$ m/s^2 $\uparrow$

Quiz

1. If the position of a particle is defined by $\vec{r} = [(1.5t^2 + 1)\,\hat{\imath} + (4t - 1)\,\hat{\jmath}\,]$ (m), its speed at t = 1 s is

   A) 2 m/s B) 3 m/s

   C) 5 m/s D) 7 m/s

2. The path of a particle is defined by $y = 0.5x^2$. If the component of its velocity along the x-axis at x = 2 m is $v_x = 1$ m/s, its velocity component along the y-axis at this position is ____.

   A) 0.25 m/s B) 0.5 m/s

   C) 1 m/s D) 2 m/s

Example 2

Given: The particle travels along the path $y = 0.5\,x^2$.
When $t = 0$, $x = y = z = 0$.

Find: The particle's distance and the magnitude of its acceleration when $t = 1$ s, if $v_x = (5\,t)$ m/s, where t is in seconds.

Plan:

1) Determine x and a_x by integrating and differentiating v_x, respectively, using the initial conditions.
2) Find the y-component of velocity & acceleration by taking a time derivative of the path.
3) Determine the magnitude of the acceleration & position.

Example 2

Solution:

1) x-components:

Velocity known as: $v_x = \dot{x} = (5\ t)$ m/s $\Rightarrow$ <u>5 m/s at t=1s</u>

Position: $\int v_x dt = \int_0^t (5t)\, dt \Rightarrow x = 2.5\ t^2 \Rightarrow$ <u>2.5 m at t=1s</u>

Acceleration: $a_x = \ddot{x} = d/dt\ (5\ t) \Rightarrow$ <u>5 m/s² at t=1s</u>

2) y-components:

Position known as : $y = 0.5\ x^2 \Rightarrow$ <u>3.125 m at t=1s</u>

Velocity: $\dot{y} = 0.5\ (2)\ x\ \dot{x} = x\ \dot{x} \Rightarrow$ <u>12.5 m/s at t=1s</u>

Acceleration: $a_x = \ddot{y} = \dot{x}\ \dot{x} + x\ \ddot{x} \Rightarrow$ <u>37.5 m/s² at t=1s</u>

Example 2

3) The position vector and the acceleration vector are

Position vector: $\vec{r} = [\, x\,\hat{\imath} + y\,\hat{\jmath}\,]$ m

where $x = 2.5$ m, $y = 3.125$ m

Magnitude: $r = (2.5^2 + 3.125^2)^{1/2} = 4.00$ m

Acceleration vector: $\vec{a} = [\, a_x\,\hat{\imath} + a_y\,\hat{\jmath}\,]$ m/s^2

where $a_x = 5$ m/s^2, $a_y = 37.5$ m/s^2

Magnitude: $a = (5^2 + 37.5^2)^{1/2} = 37.8$ m/s^2

Quiz

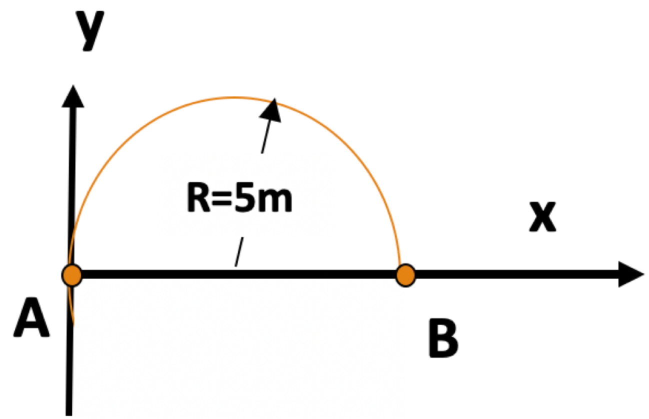
1. If a particle has moved from A to B along the circular path in 4s, what is the average velocity of the particle?

✔ A) 2.5 $\hat{j}$ m/s

B) 2.5 $\hat{\imath}$ +1.25 $\hat{j}$ m/s

C) 1.25 π $\hat{\imath}$ m/s

D) 1.25 π $\hat{j}$ m/s



2. The position of a particle is given as $\vec{r} = (4t^2\,\hat{\imath} - 2x\,\hat{j})$ m. Determine the particle's acceleration.

A) (4 $\hat{\imath}$ +8 $\hat{j}$) m/s^2

✔ B) (8 $\hat{\imath}$ -16 $\hat{j}$) m/s^2

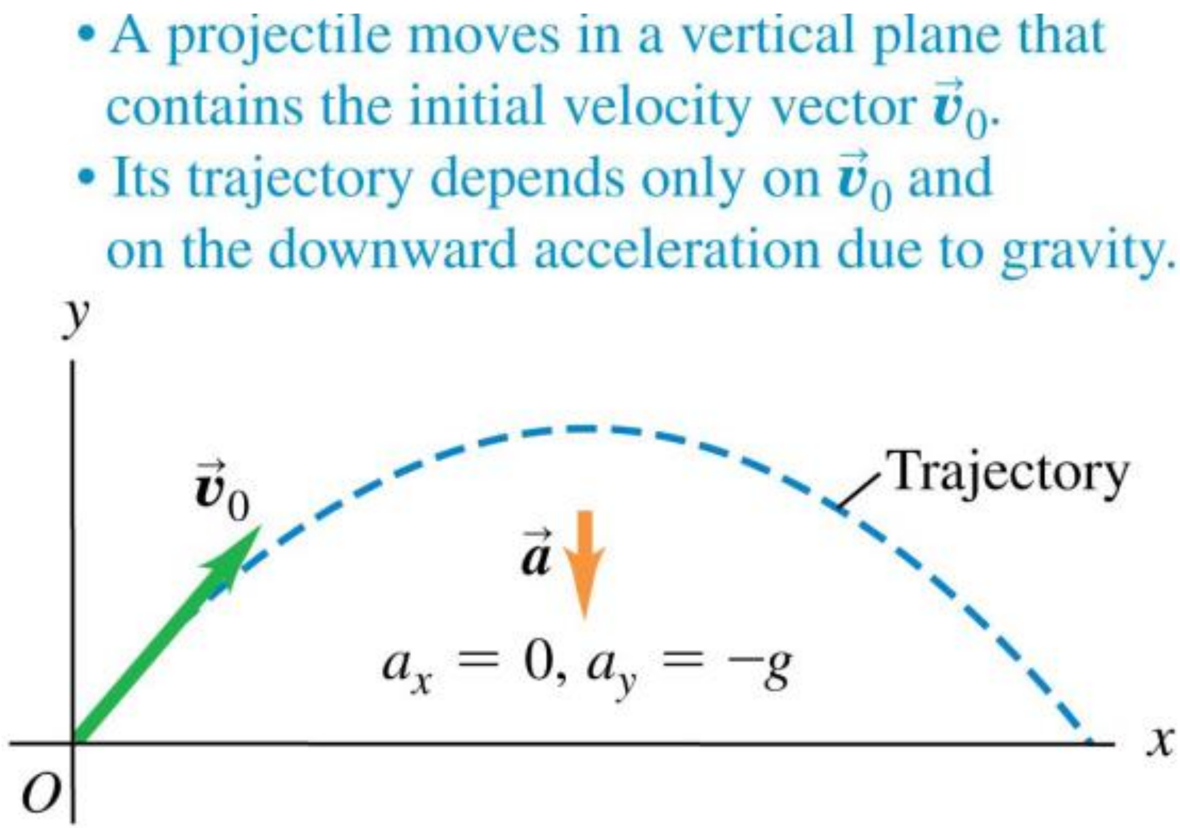
C) (8 $\hat{\imath}$) m/s^2

D) (8 $\hat{j}$) m/s^2

Projectile motion

- A **projectile** is any body given an initial velocity that then follows a path determined by the effects of gravity and air resistance.

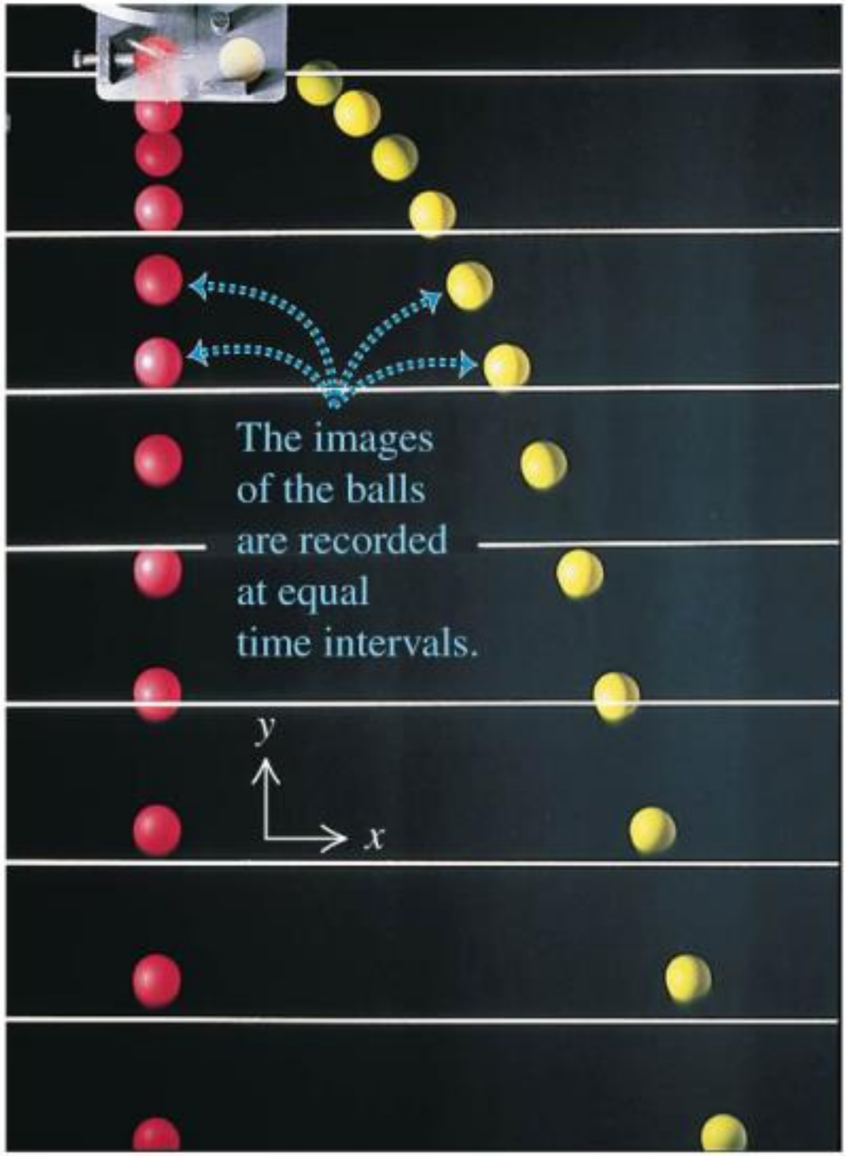
- Begin by neglecting resistance and the curvature and rotation of the earth.



The *x*- and *y*-motion are separable

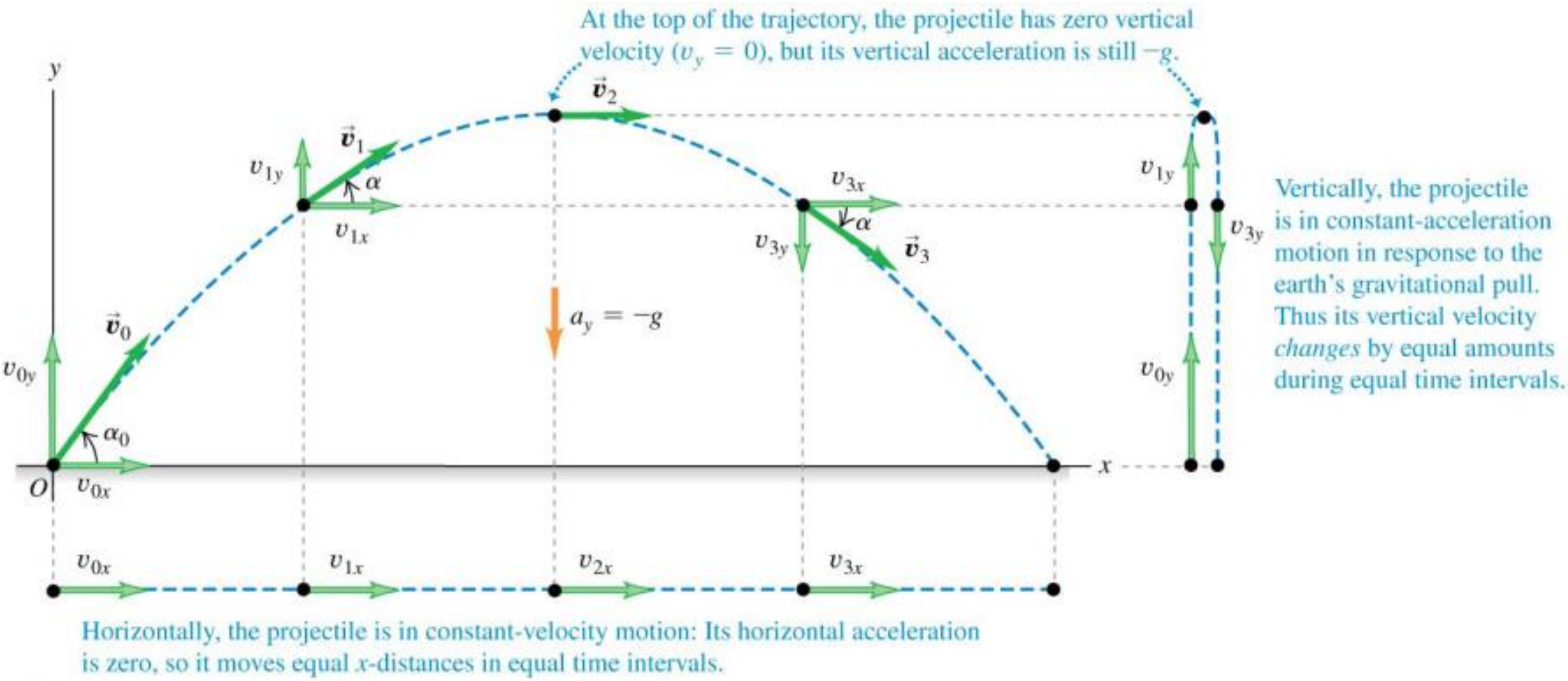
- The red ball is dropped at the same time that the yellow ball is fired horizontally.
- The strobe marks equal time intervals.
- We can analyze projectile motion as horizontal motion with constant velocity and vertical motion with constant acceleration:

$$a_x = 0 \text{ and } a_y = -g.$$



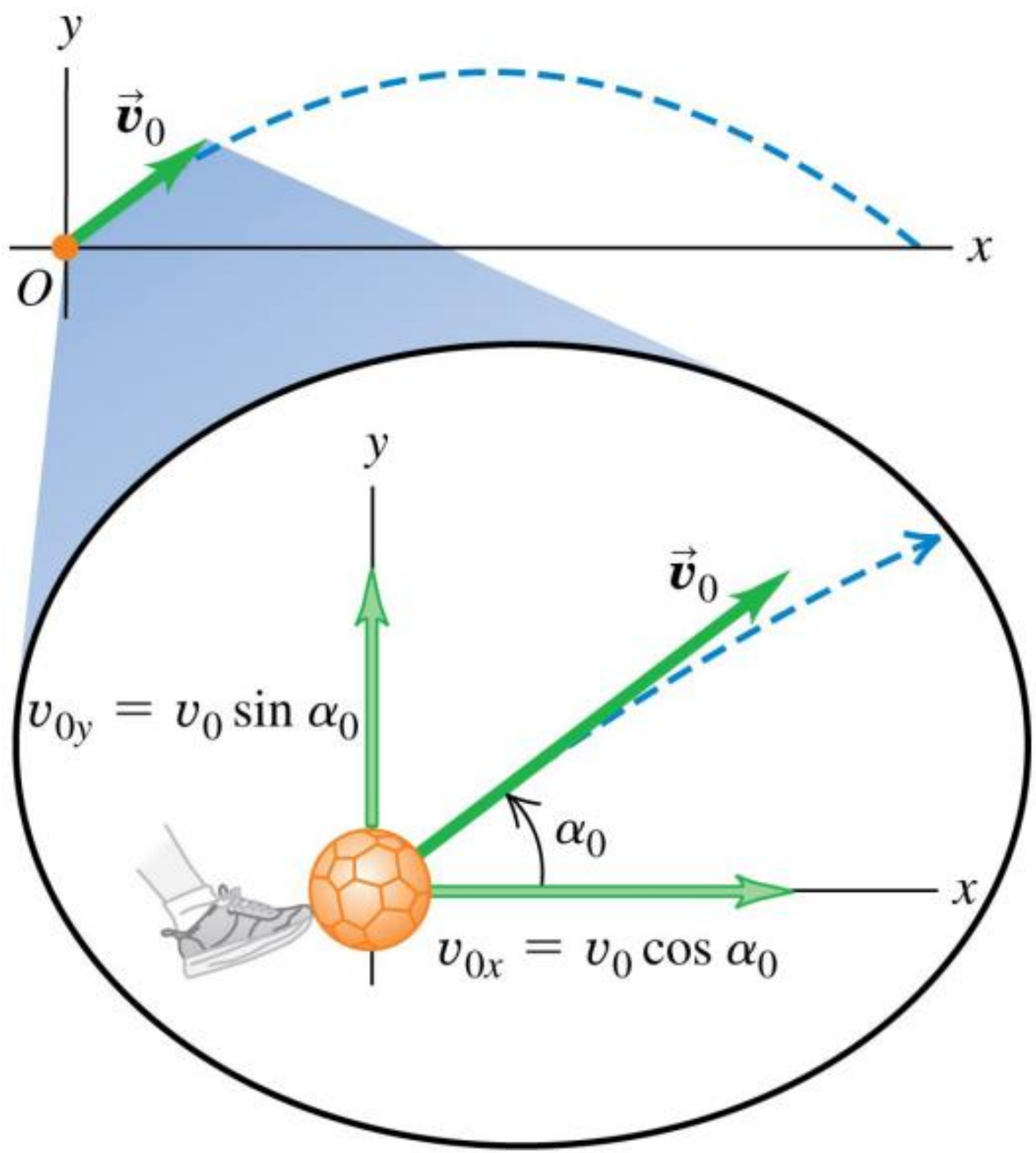
Projectile motion

- If air resistance is negligible, the trajectory of a projectile is a combination of horizontal motion with constant velocity and vertical motion with constant acceleration.

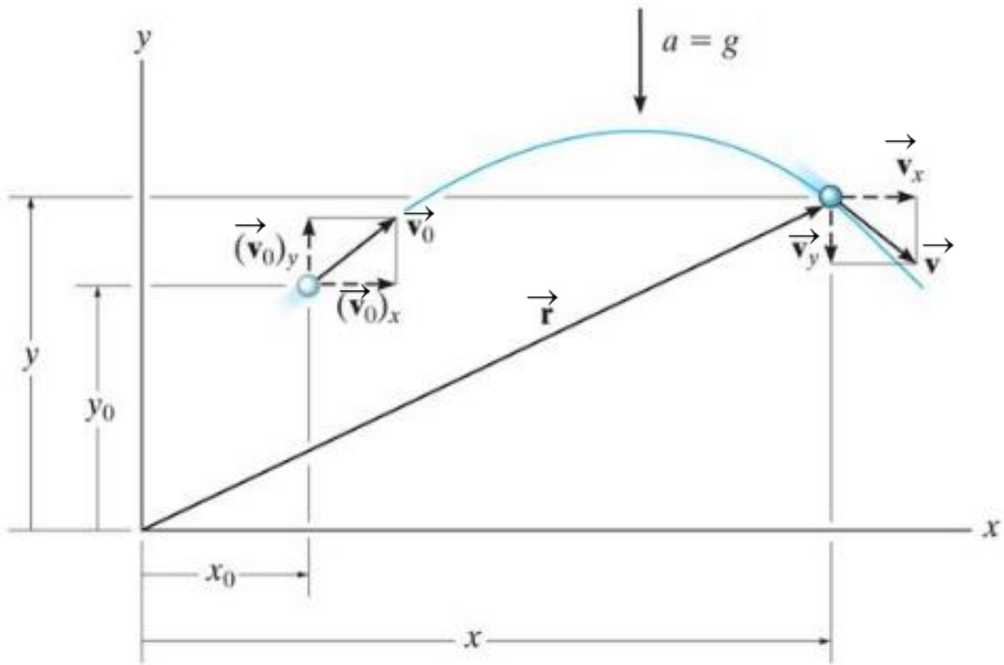


Projectile motion – Initial velocity

- The initial velocity components of a projectile (such as a kicked soccer ball) are related to the initial speed and initial angle.



Horizontal motion



Since $a_x = 0$, the velocity in the horizontal direction remains constant ($v_x = v_{ox}$) and the position in the x direction can be determined by:

$x = x_o + (v_{ox})\ t$

Why is a_x equal to zero (what assumption must be made if the movement is through the air)?

Vertical motion

Since the positive y-axis is directed upward, $a_y = -g$.
Application of the constant acceleration equations yields:

$$v_y = v_{oy} - g\,t$$

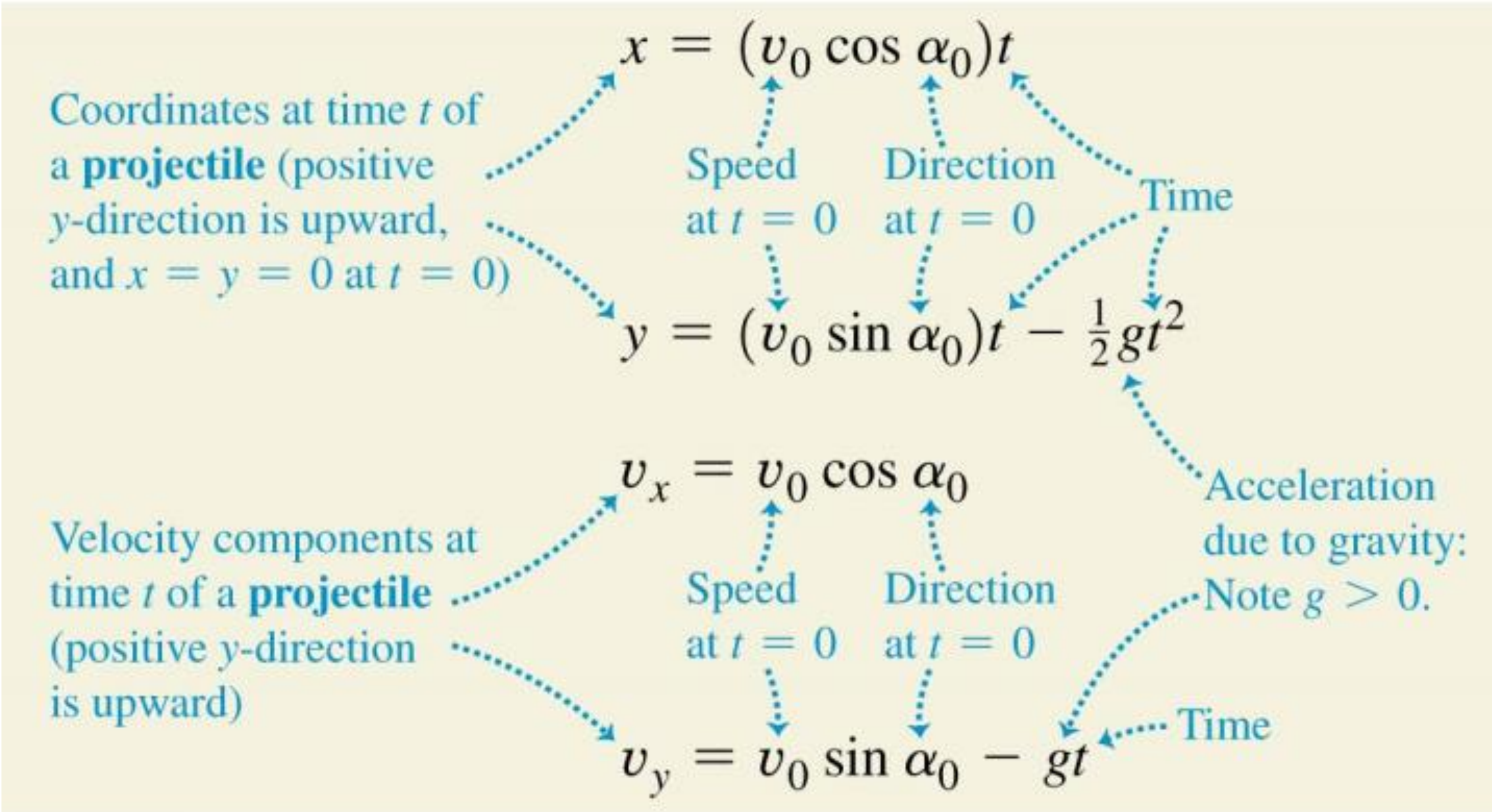
$$y = y_o + (v_{oy})\,t - \tfrac{1}{2}\,g\,t^2$$

$$v_y^{\,2} = v_{oy}^{\,2} - 2\,g\,(y - y_o)$$

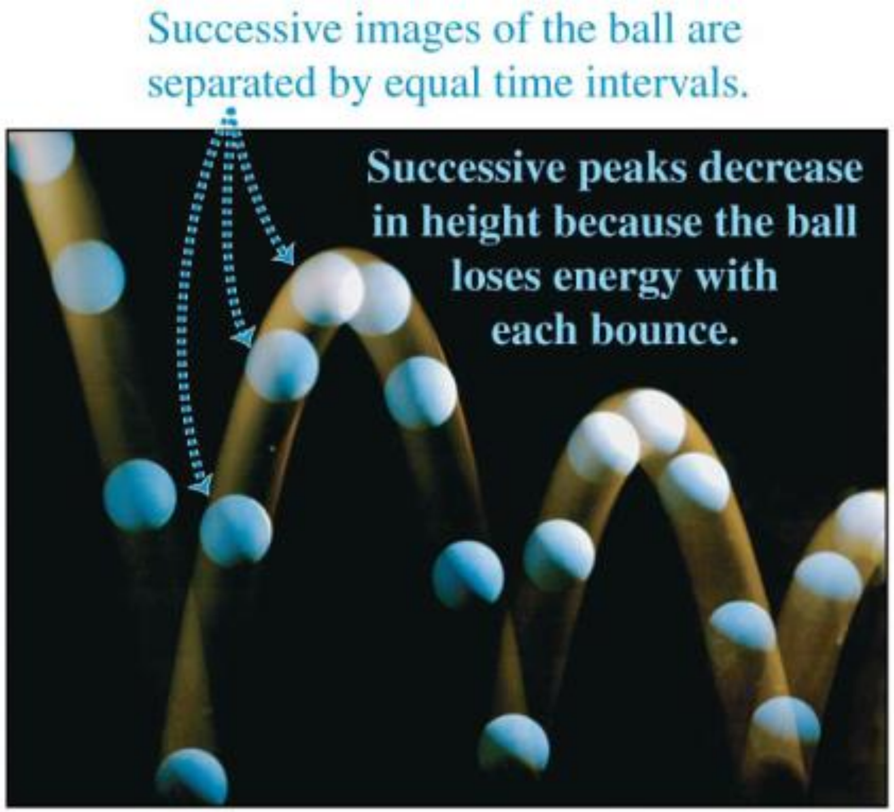
For any given problem, only two of these three equations can be used. Why?

The equations for projectile motion

- If we set $x_0 = y_0 = 0$, the equations describing projectile motion are shown below:



Parabolic trajectories of a bouncing ball



Quiz

1. The downward acceleration of an object in free-flight motion is

   A) zero. B) increasing with time.

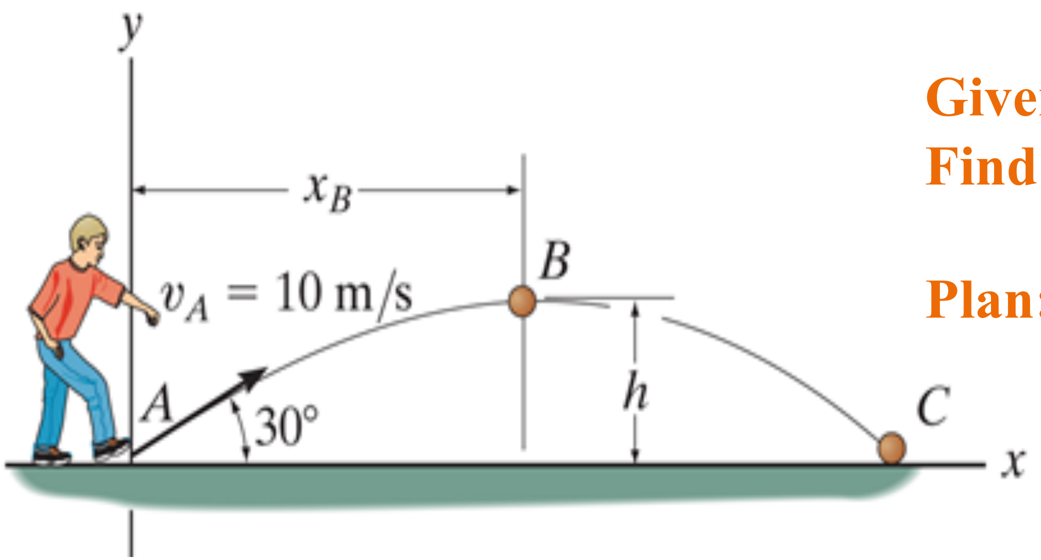
   ✔C) 9.81 m/s^2. D) decreasing with time.

2. The horizontal component of velocity remains __________ during a free-flight motion.

   A) zero ✔B) constant

   C) at 9.81 m/s^2 D) decreasing with time

Example 1



Given: v_A and θ

Find: Horizontal distance it travels and v_C.

Plan:

Apply the kinematic relations in x- and y-directions.

Solution: Using $v_{Ax} = 10 \cos 30^\circ$ and $v_{Ay} = 10 \sin 30^\circ$

We can write $v_x = 10 \cos 30^\circ$

$$v_y = 10 \sin 30^\circ - (9.81)\, t$$

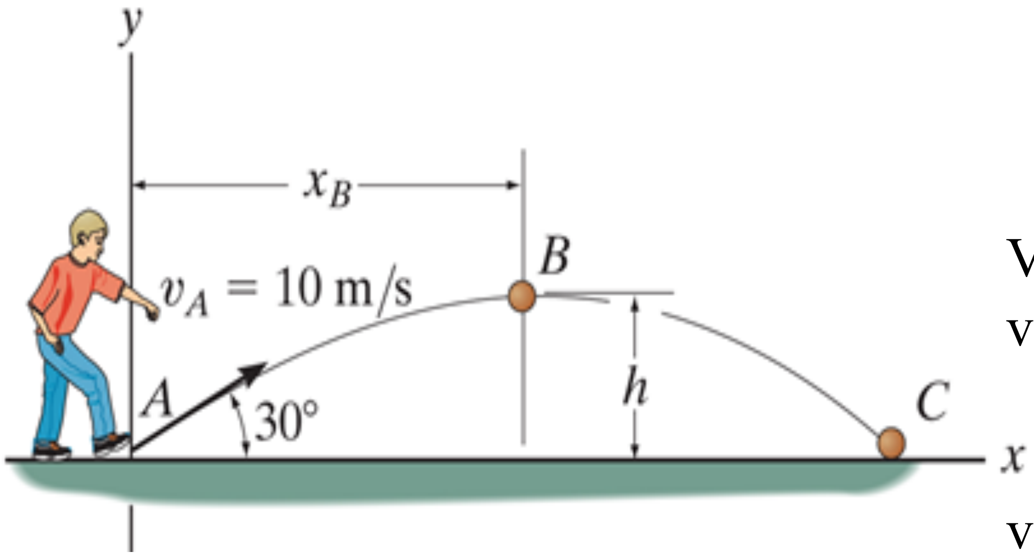
$$x = (10 \cos 30^\circ)\, t$$

$$y = (10 \sin 30^\circ)\, t - ½\, (9.81)\, t^2$$

Since y = 0 at C

$$0 = (10 \sin 30^\circ)\, t - ½\, (9.81)\, t^2 \Rightarrow t = 0,\ 1.019 \text{ s}$$

Example 1



Velocity components at C are;

$v_{Cx} = 10 \cos 30°$

$= 8.66$ m/s $\rightarrow$

$v_{Cy} = 10 \sin 30° - (9.81)(1.019)$

$= -5$ m/s $= 5$ m/s $\downarrow$

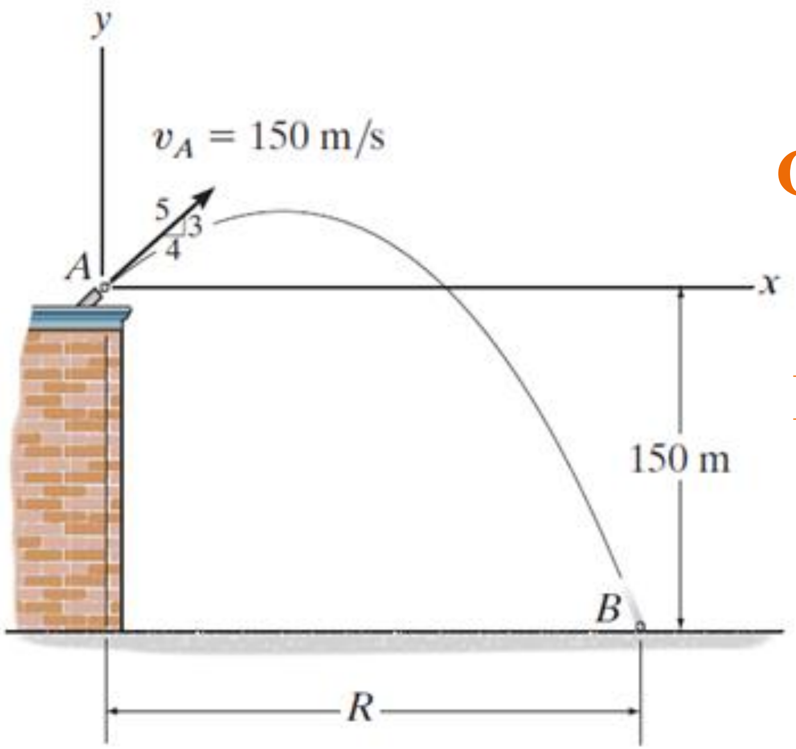
$$v_C = \sqrt{8.66^2 + (-5)^2} = 10 \text{ m/s}$$

Horizontal distance the ball travels is;

$x = (10 \cos 30°)\, t$

$x = (10 \cos 30°)\, 1.019 = 8.83$ m

Example 2



Given: Projectile is fired with v_A=150 m/s at point A.

Find: The horizontal distance it travels (R) and the time in the air.

Plan: Establish a fixed x, y coordinate system (in this solution, the origin of the coordinate system is placed at A). Apply the kinematic relations in x- and y-directions.

Example 2

Solution:

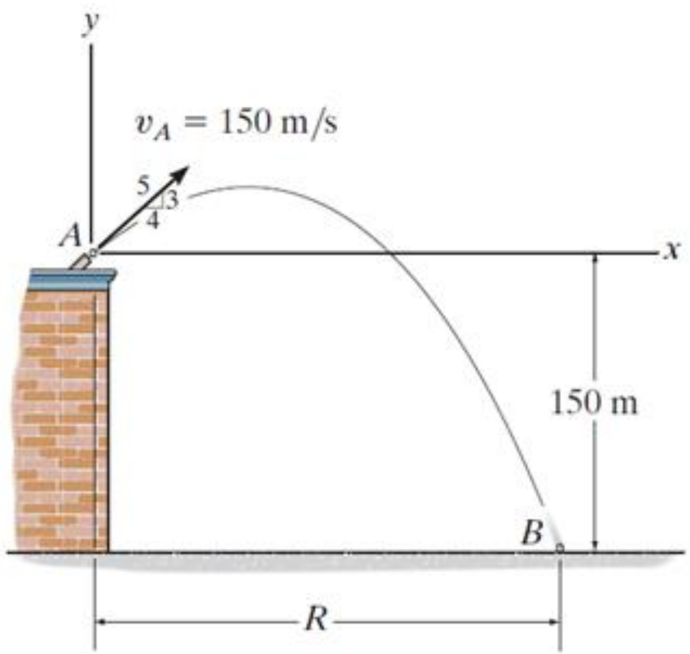
1) Place the coordinate system at point A.

Then, write the equation for horizontal motion.

$+\rightarrow \quad x_B = x_A + v_{Ax}\, t_{AB}$

where $x_B = R$, $x_A = 0$, $v_{Ax} = 150\ (4/5)$ m/s

Range, R, will be $R = 120\ t_{AB}$



2) Now write a vertical motion equation. Use the distance equation.

$+\uparrow \quad y_B = y_A + v_{Ay}\, t_{AB} - 0.5\, g\, t_{AB}^2$

where $y_B = -150$, $y_A = 0$, and $v_{Ay} = 150(3/5)$ m/s

We get the following equation: $-150 = 90\, t_{AB} + 0.5\,(-9.81)\, t_{AB}^2$

Solving for t_{AB} first, $t_{AB} = 19.89$ s.

Then, $R = 120\, t_{AB} = 120\ (19.89) = 2387$ m

Quiz

The time of flight of a projectile, fired over level ground, with initial velocity V_o at angle θ, is equal to?

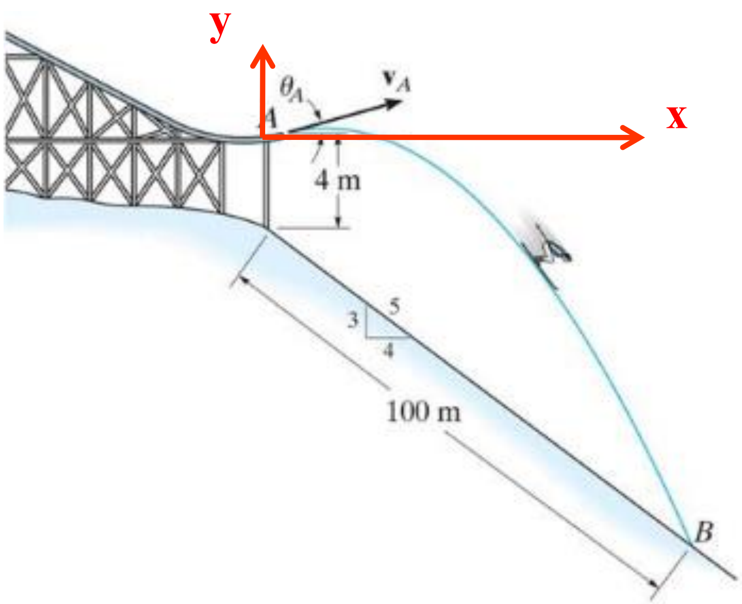
A) $(v_o \sin\theta)/g$

✔ B) $(2v_o \sin\theta)/g$

C) $(v_o \cos\theta)/g$

D) $(2v_o \cos\theta)/g$

Example 3 (T)



Given: A skier leaves the ski jump ramp at $\theta_A = 25°$ and hits the slope at B.

Find: The skier's initial speed v_A.

Plan: Establish a fixed x,y coordinate system (in this solution, the origin of the coordinate system is placed at A). Apply the kinematic relations in x and y-directions.

Example 3 (T)

Solution:

Motion in x-direction:

Using $x_B = x_A + v_{ox}(t_{AB})$ => $(4/5)100 = 0 + v_A\,(\cos 25°)\, t_{AB}$

$$t_{AB} = \frac{80}{v_A\,(\cos 25°)} = \frac{88.27}{v_A}$$

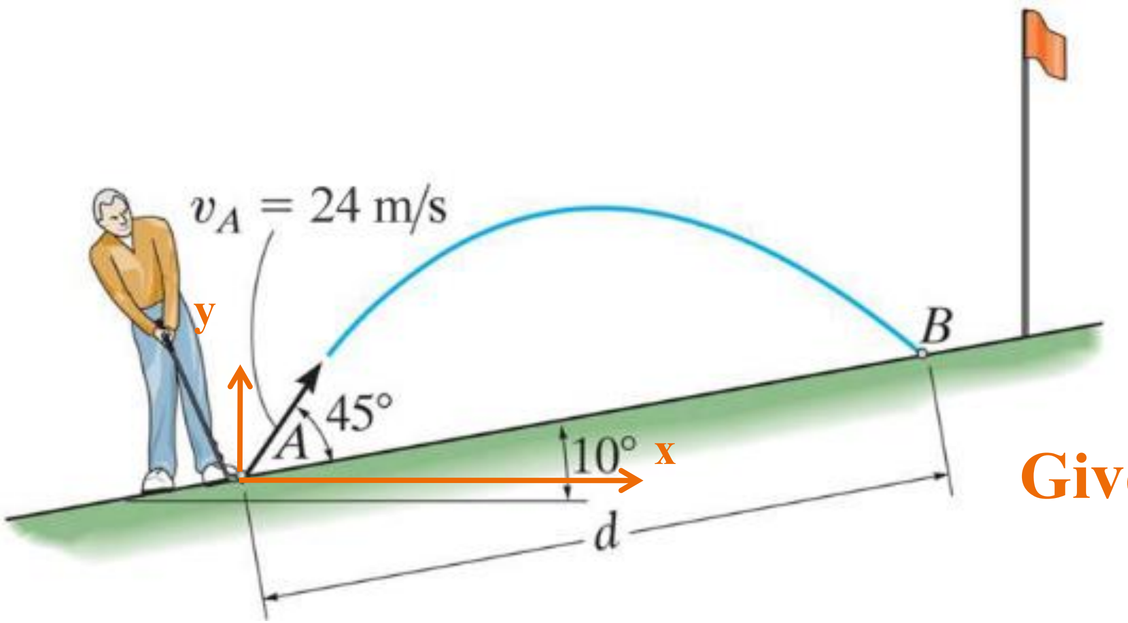
Motion in y-direction:

Using $y_B = y_A + v_{oy}(t_{AB}) - \frac{1}{2} g(t_{AB})^2$

$$-64 = 0 + v_A(\sin 25°)\left\{\frac{88.27}{v_A}\right\} - \frac{1}{2}(9.81)\left\{\frac{88.27}{v_A}\right\}^2$$

$v_A = 19.42$ m/s

Example 4 (T)



Given: The golf ball is struck with a velocity of 24 m/s as shown.

Find: Distance d to where it will land.

Plan: Establish a fixed x, y coordinate system (in this solution, the origin of the coordinate system is placed at A). Apply the kinematic relations in x and y-directions.

Example 4 (T)

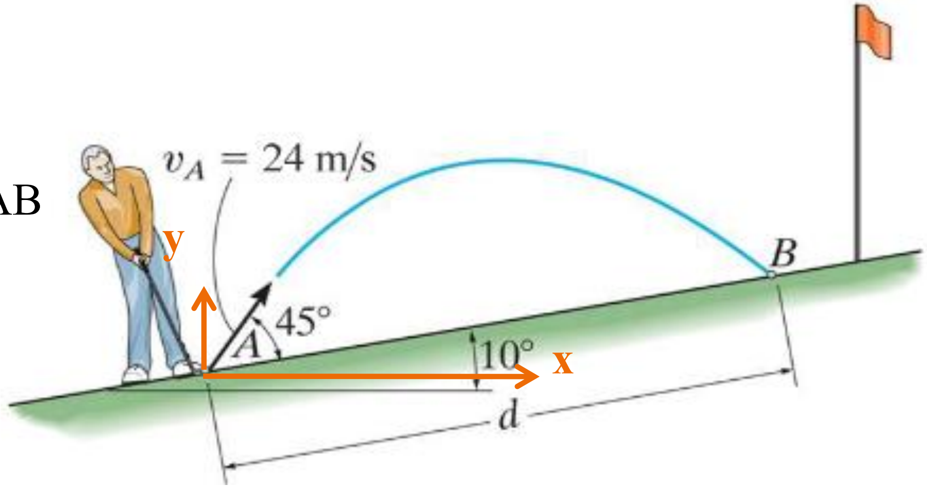
Solution:

<u>Motion in x-direction:</u>

Using $x_B = x_A + v_{ox}(t_{AB})$

$\Rightarrow d \cos 10° = 0 + 24 (\cos 55°)\ t_{AB}$

$$t_{AB} = 0.07154\ d$$



<u>Motion in y-direction:</u>

Using $y_B = y_A + v_{oy}(t_{AB}) - ½\ g(t_{AB})^2$

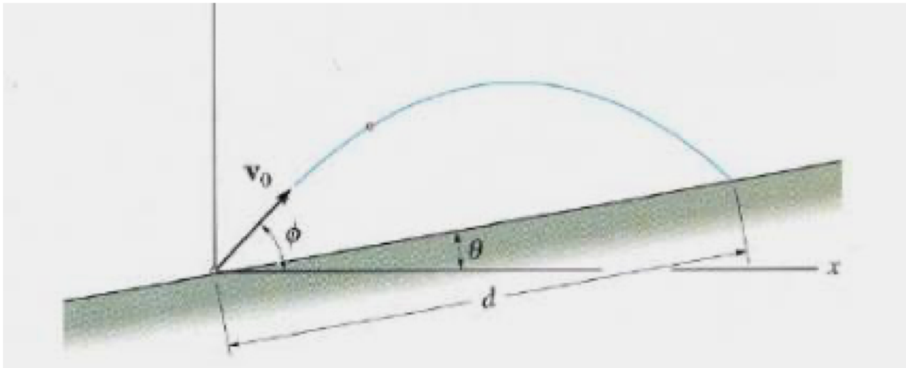
$\Rightarrow d \sin 10° = 0 + 24(\sin 55°)(0.07154\ d) - ½\ (9.81)\ (0.07154\ d)^2$

$\Rightarrow 0 = 1.2328\ d - 0.025104\ d^2$

$$d = 0,\ 49.1 \text{ m}$$

Quiz

1. A projectile is given an initial velocity v_o at an angle ϕ above the horizontal. The velocity of the projectile when it hits the slope is ______________ the initial velocity v_o.



✔ A) less than B) equal to
C) greater than D) none of the above

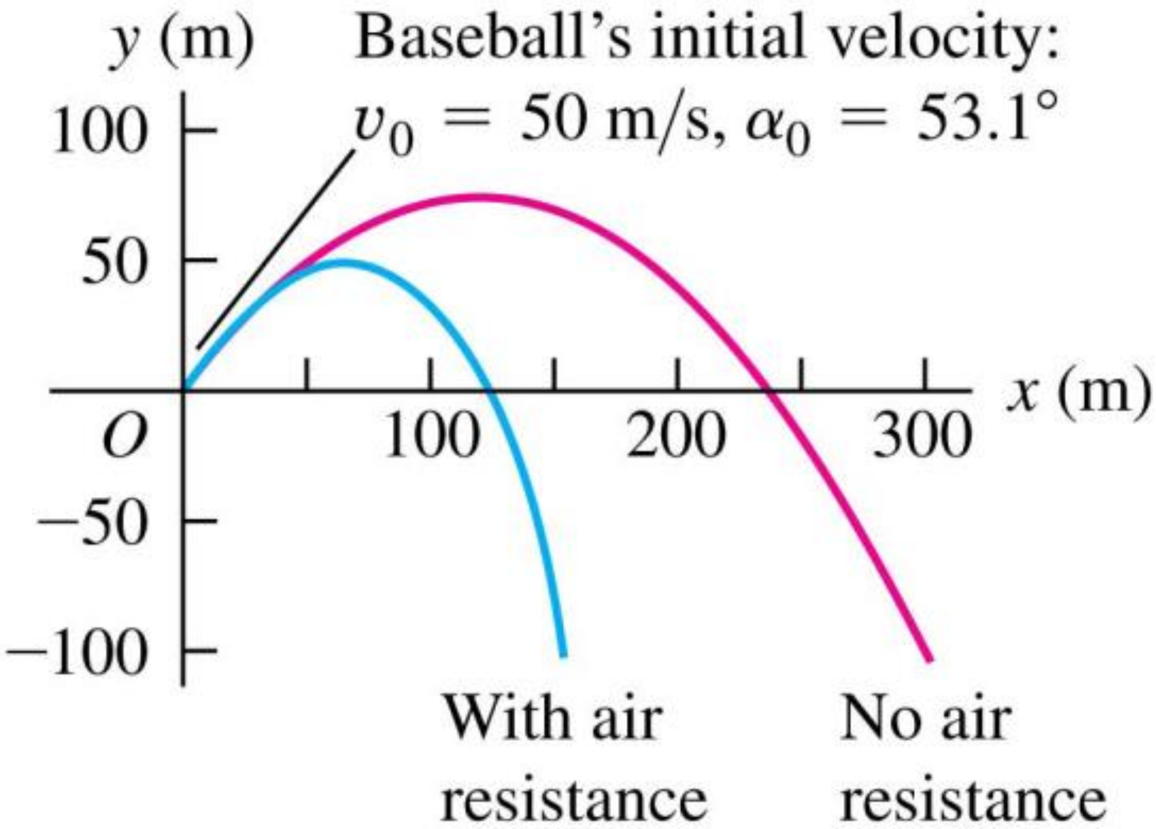
2. A particle has an initial velocity v_o at angle f with respect to the horizontal. The maximum height it can reach is when

A) $\phi = 30°$ B) $\phi = 45°$

C) $\phi = 60°$ ✔ D) $\phi = 90°$

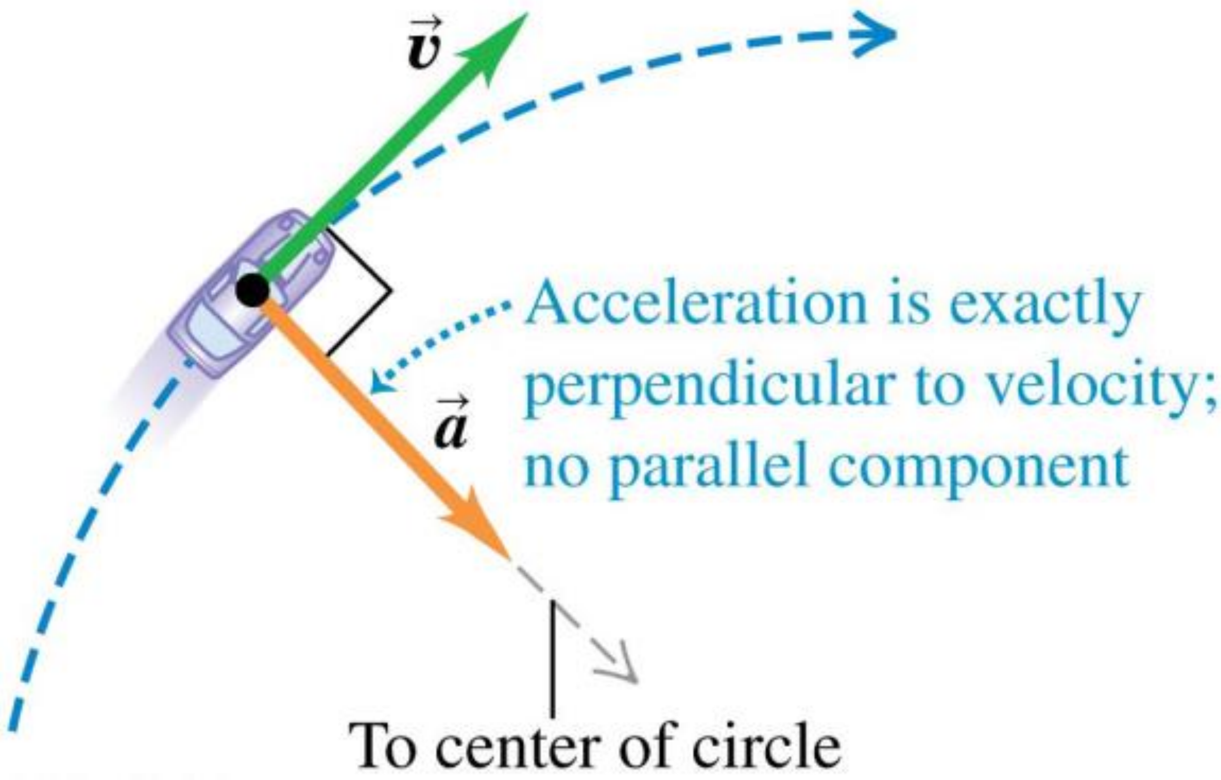
The effects of air resistance

- Calculations become more complicated.
- Acceleration is not constant.
- Effects can be very large.
- Maximum height and range decrease.
- Trajectory is no longer a parabola.



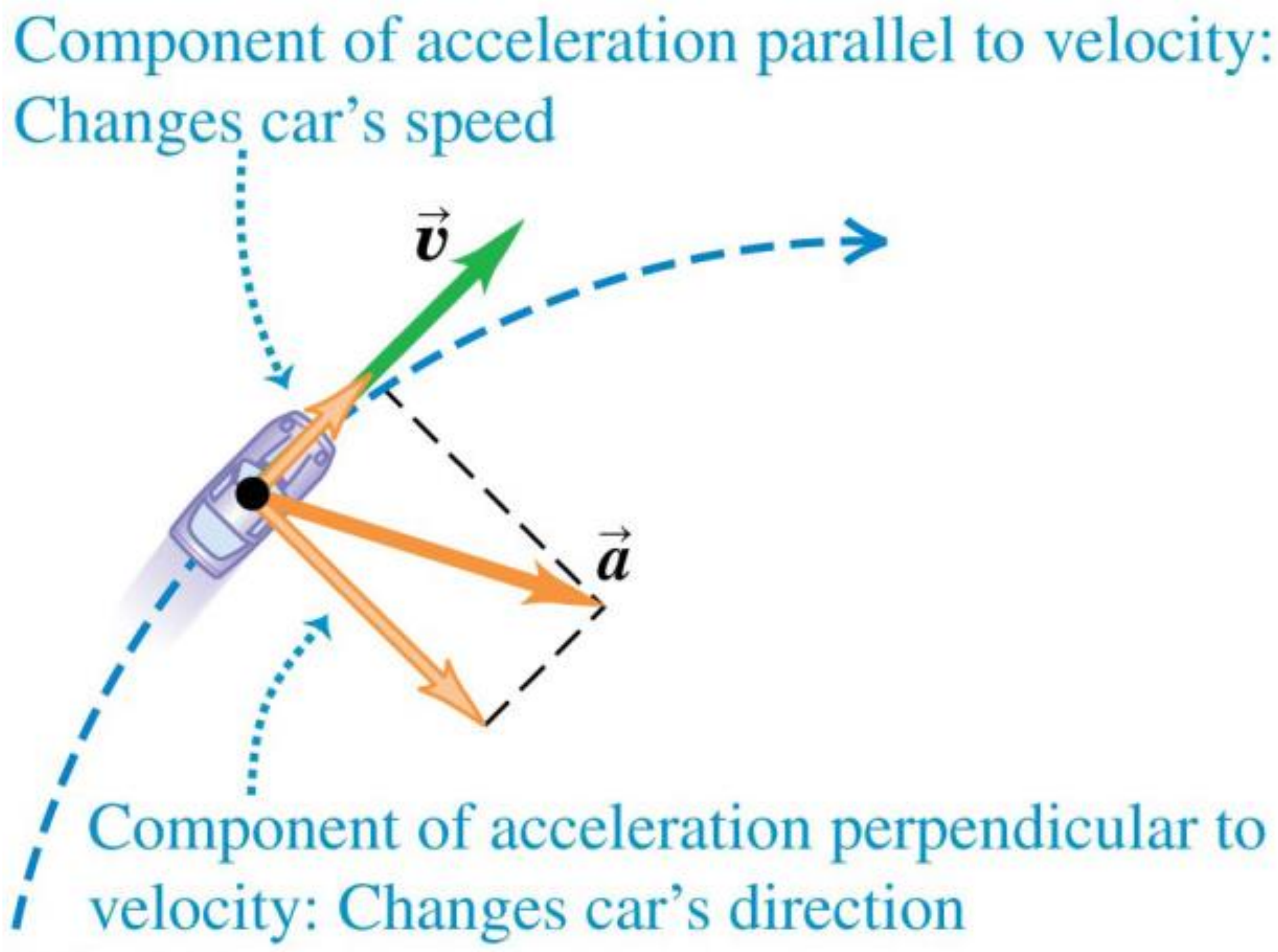
Motion in a circle

- **Uniform circular motion** is constant speed along a circular path.



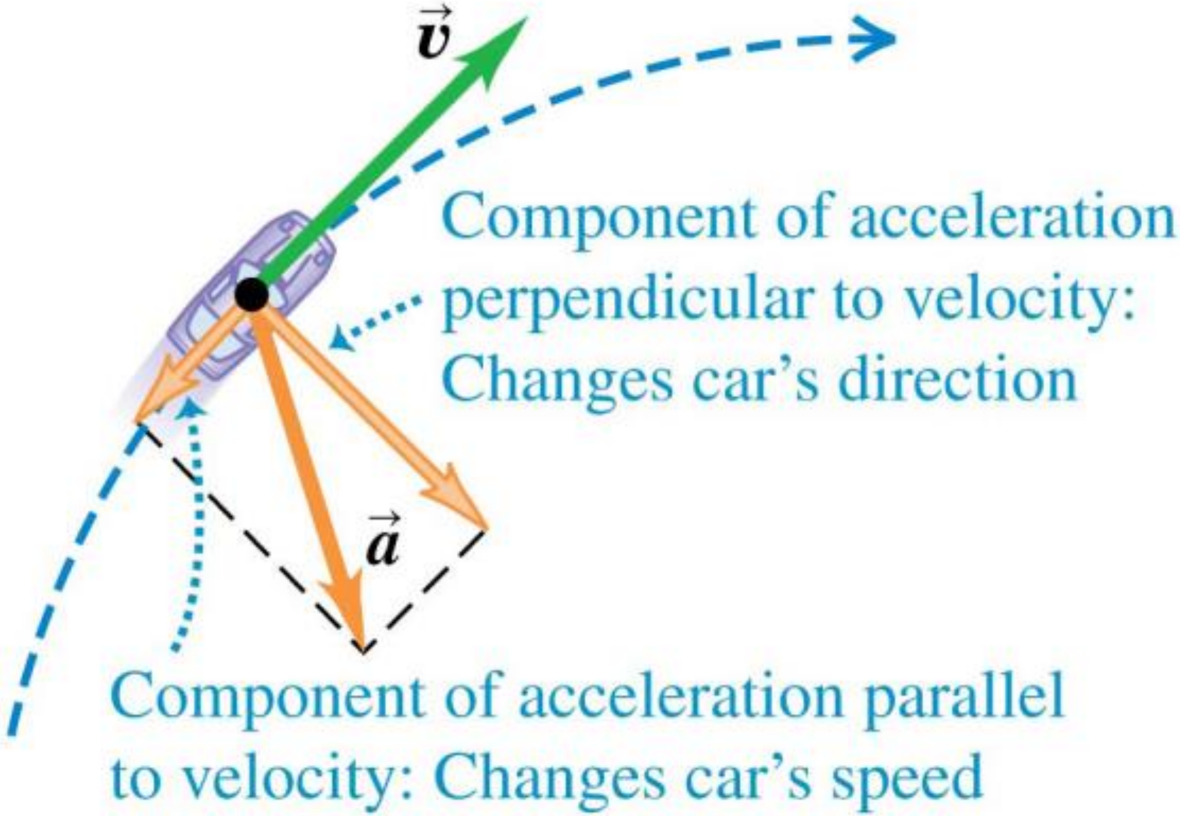
Motion in a circle

- Car speeding up along a circular path



Motion in a circle

- Car slowing down along a circular path



Acceleration for uniform circular motion

$$\frac{|\Delta\vec{v}|}{v_1} = \frac{\Delta s}{R} \qquad \text{or} \qquad |\Delta\vec{v}| = \frac{v_1}{R}\Delta s$$

$$a_{\mathrm{av}} = \frac{|\Delta\vec{v}|}{\Delta t} = \frac{v_1}{R}\frac{\Delta s}{\Delta t}$$

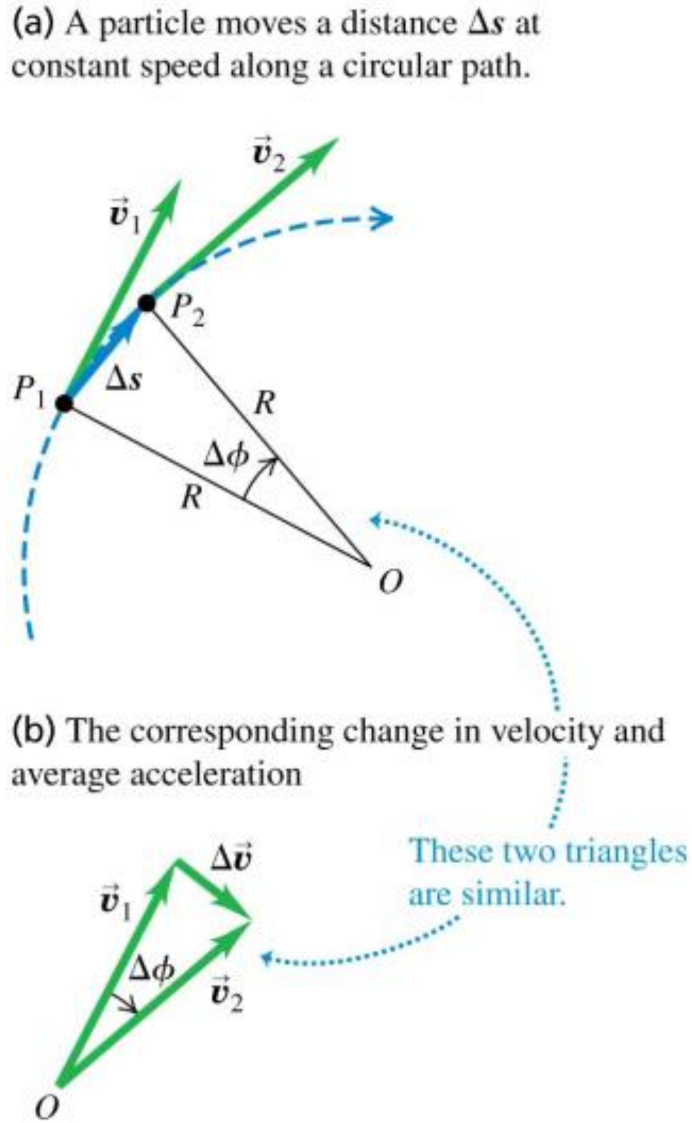
$$a = \lim_{\Delta t \to 0} \frac{v_1}{R}\frac{\Delta s}{\Delta t} = \frac{v_1}{R} \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

Magnitude of acceleration of an object in **uniform circular motion**

$$a_{\mathrm{rad}} = \frac{v^2}{R}$$

Speed of object (v)

Radius of object's circular path (R)

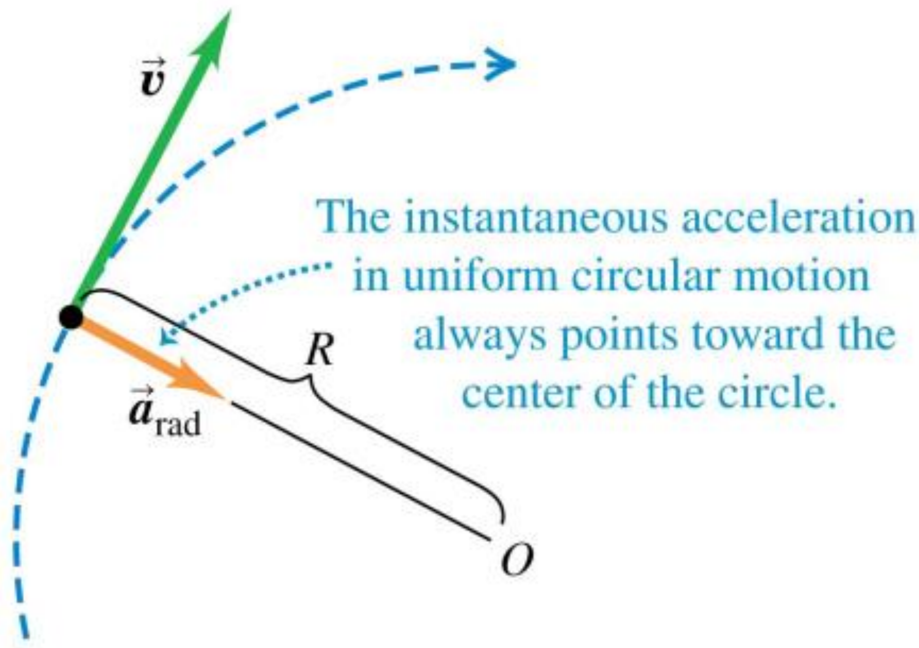


(a) A particle moves a distance Δs at constant speed along a circular path.

(b) The corresponding change in velocity and average acceleration

Acceleration for uniform circular motion

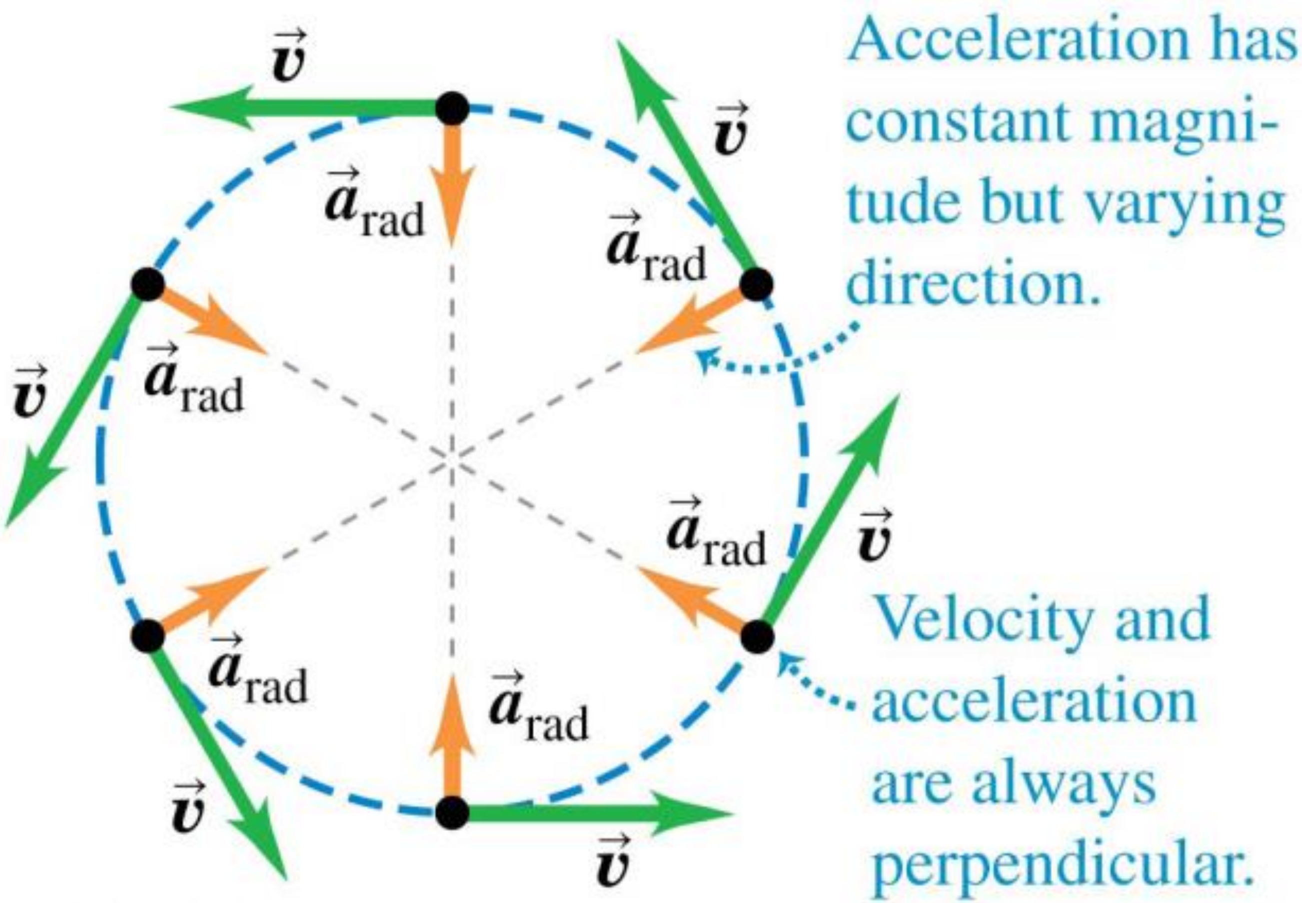
- For uniform circular motion, the instantaneous acceleration always points toward the center of the circle and is called the **centripetal acceleration**.



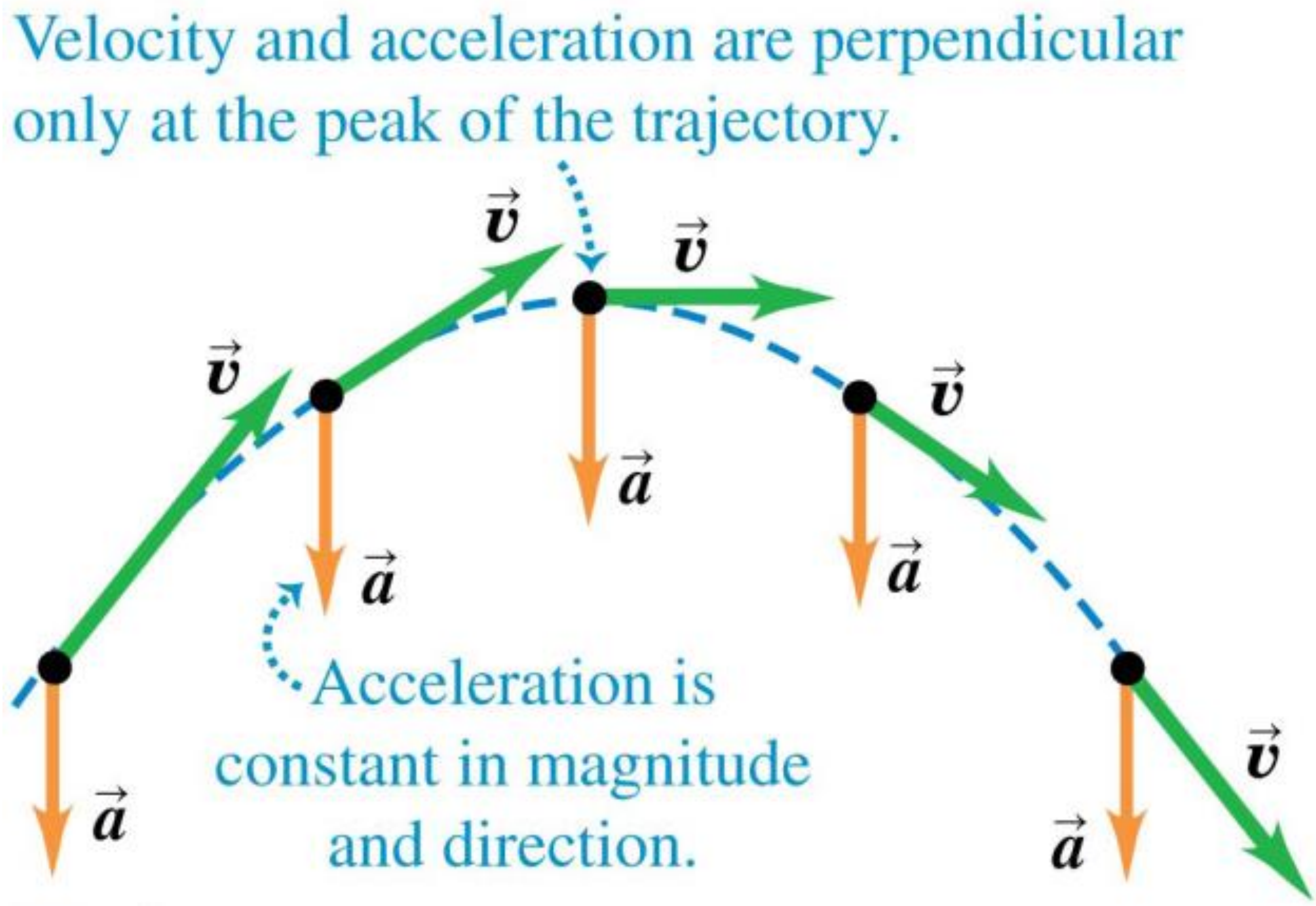
- The magnitude of the acceleration is $a_{rad} = v^2/R$.

- The *period* T is the time for one revolution, and $a_{rad} = 4\pi^2 R/T^2$.

Uniform circular motion

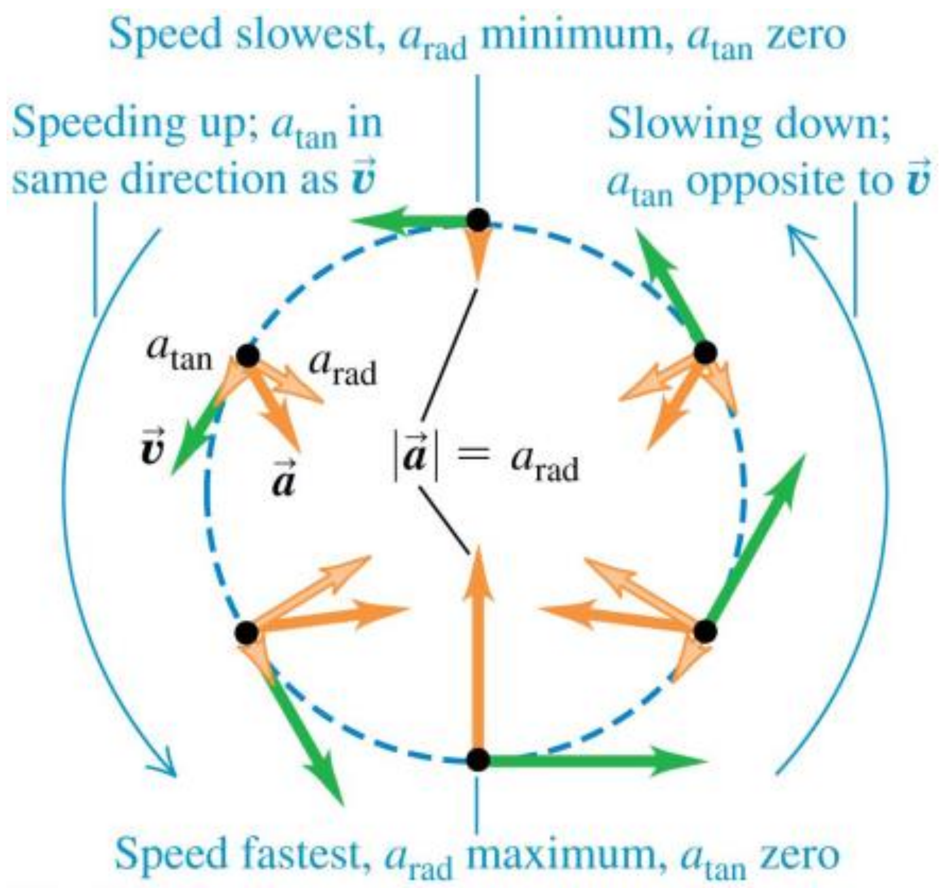


Projectile motion



Nonuniform circular motion

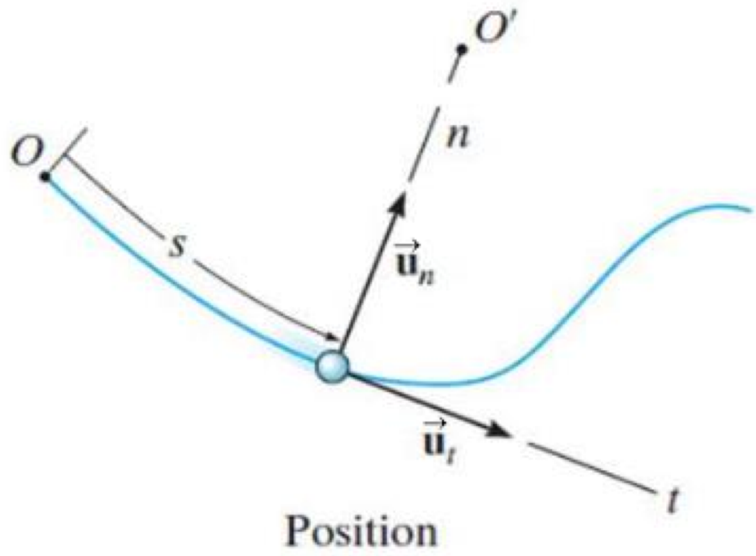
- If the speed varies, the motion is *nonuniform circular motion*.
- The radial acceleration component is still $a_{rad} = v^2/R$, but there is also a tangential acceleration component a_{tan} that is *parallel* to the instantaneous velocity.



Curvilinear motion: n-t coordinate system

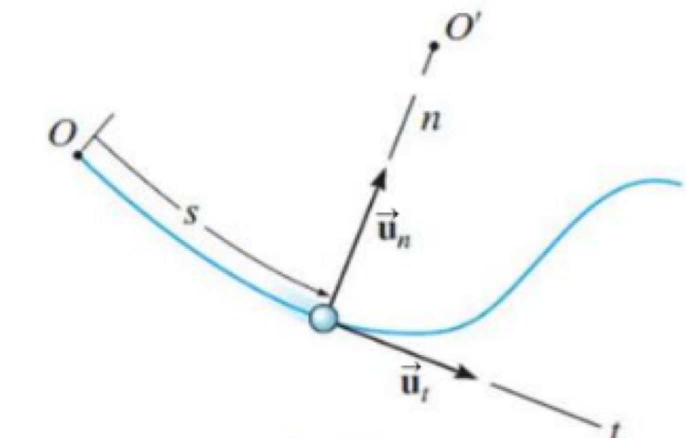
When a particle moves along a curved path, it is sometimes convenient to describe its motion using coordinates other than Cartesian. When the path of motion is known, normal (n) and tangential (t) coordinates are often used.

In the n-t coordinate system, the origin is located on the particle (the origin moves with the particle).



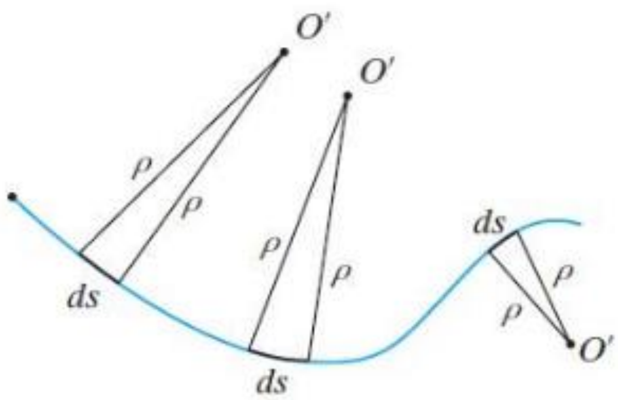
The t-axis is tangent to the path (curve) at the instant considered, positive in the direction of the particle's motion.
The n-axis is perpendicular to the t-axis with the positive direction toward the center of curvature of the curve.

Curvilinear motion: n-t coordinate system



Position

The positive n and t directions are defined by the unit vectors $\vec{u}_n$ and $\vec{u}_t$, respectively.

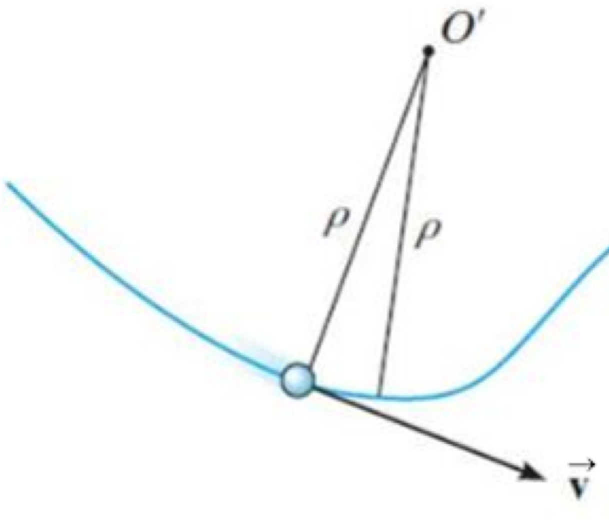


Radius of curvature

The center of curvature, O', always lies on the concave side of the curve.
The radius of curvature, r, is defined as the perpendicular distance from the curve to the center of curvature at that point.

The position of the particle at any instant is defined by the distance, s, along the curve from a fixed reference point.

Velocity in the n-t coordinate system



Velocity

The velocity vector is always tangent to the path of motion (t-direction).

The magnitude is determined by taking the time derivative of the path function, s(t).

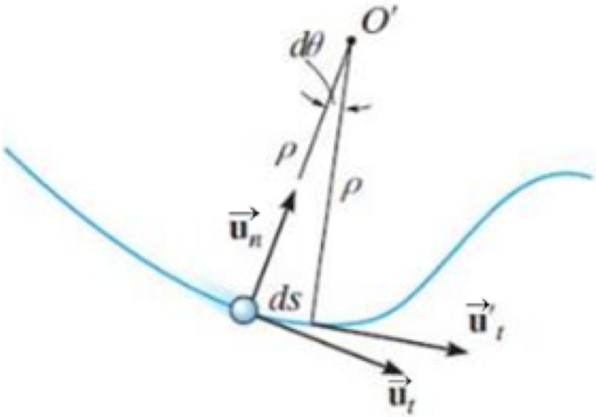
$$\vec{v} = v\,\vec{u}_t \quad \text{where} \quad v = \dot{s} = ds/dt$$

Here v defines the magnitude of the velocity (speed) and $\vec{u}_t$ defines the direction of the velocity vector.

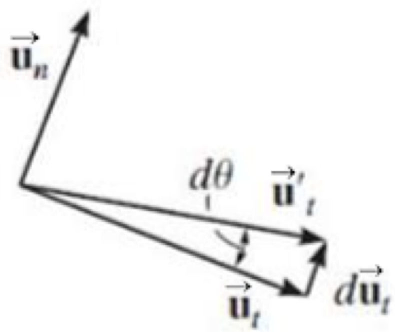
Acceleration in the n-t coordinate system

Acceleration is the time rate of change of velocity:

$$\vec{a} = d\,\vec{v}/dt = d(v\,\vec{u}_t)/dt = \dot{v}\vec{u}_t + \dot{v}\,\vec{u}_t$$



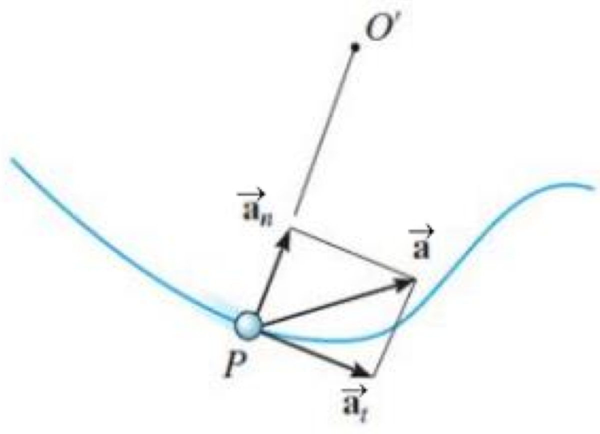
Here $\dot{v}$ represents the change in the magnitude of velocity and $\dot{\vec{u}}_t$ represents the rate of change in the direction of $\vec{u}_t$.



After mathematical manipulation, the acceleration vector can be expressed as:

$$\vec{a} = \mathbf{v}\,\dot{\vec{u}}_t + (v^2/\rho)\,\vec{u}_n = a_t\,\vec{u}_t + a_n\,\vec{u}_n.$$

Acceleration in the n-t coordinate system



Acceleration

So, there are two components to the acceleration vector:

$$\vec{a} = a_t \vec{u}_t + a_n \vec{u}_n$$

- The tangential component is tangent to the curve and in the direction of increasing or decreasing velocity.

$$a_t = \dot{v} \quad \text{or} \quad a_t\, ds = v\, dv$$

- The normal or centripetal component is always directed toward the center of curvature of the curve. $a_n = v^2/\rho$

- The magnitude of the acceleration vector is

$$a = [(a_t)^2 + (a_n)^2]^{1/2}$$

Special cases of motion

There are some special cases of motion to consider.

1) The particle moves along a straight line.

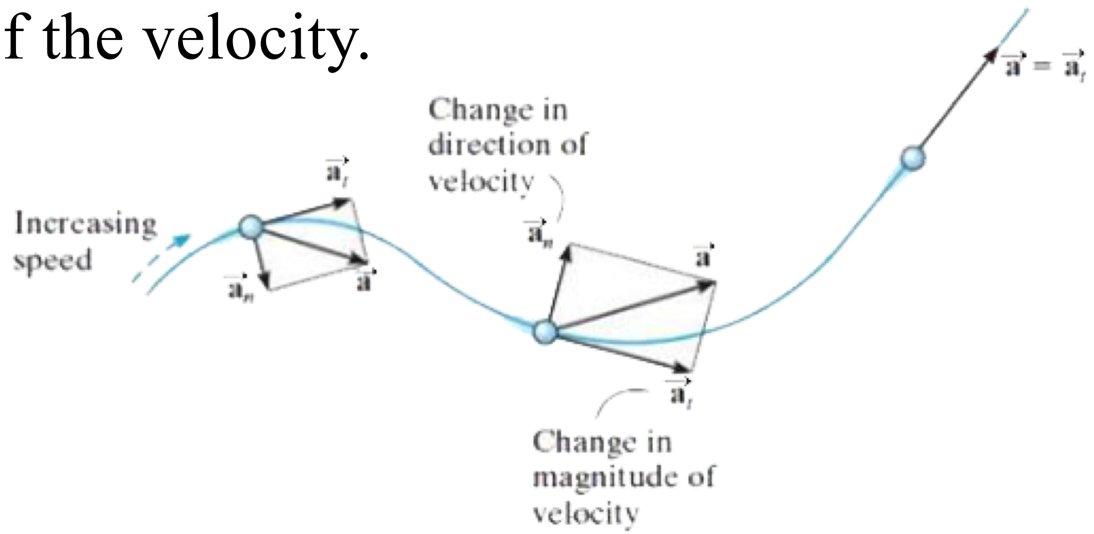
$$r \longrightarrow \infty \quad \Rightarrow \quad a_n = v^2/\rho = 0 \quad \Rightarrow \quad a = a_t = \dot{v}$$

The tangential component represents the time rate of change in the magnitude of the velocity.

2) The particle moves along a curve at constant speed.

$$a_t = \dot{v} = 0 \quad \Rightarrow \quad a = a_n = v^2/\rho$$

The normal component represents the time rate of change in the direction of the velocity.



Special cases of motion

3) The tangential component of acceleration is constant, $a_t = (a_t)_c$.

In this case,

$$s = s_o + v_o t + (1/2)(a_t)_c t^2$$

$$v = v_o + (a_t)_c t$$

$$v^2 = (v_o)^2 + 2(a_t)_c (s - s_o)$$

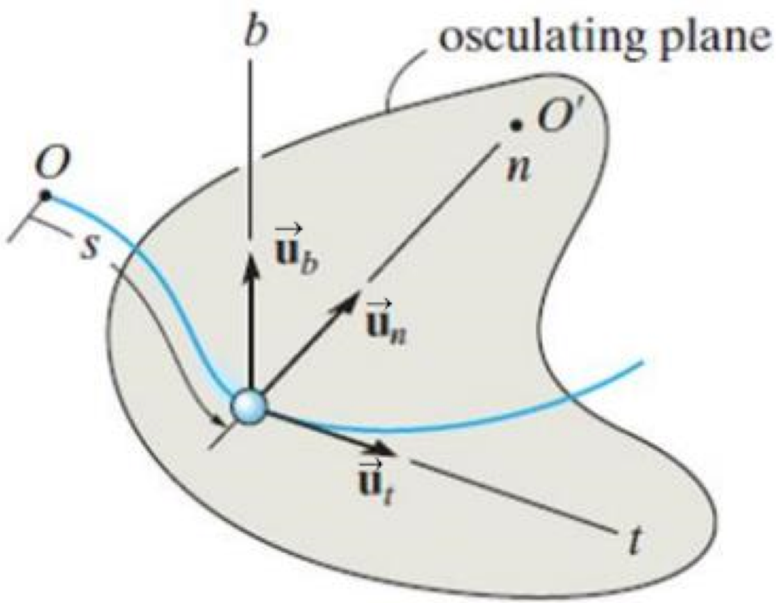
As before, s_o and v_o are the initial position and velocity of the particle at $t = 0$. How are these equations related to projectile motion equations? Why?

4) The particle moves along a path expressed as $y = f(x)$.

The radius of curvature, ρ, at any point on the path can be calculated from

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{\frac{3}{2}}}{|d^2y/dx^2|}$$

Three-dimensional motion



If a particle moves along a space curve, the n and t axes are defined as before. At any point, the t-axis is tangent to the path and the n-axis points toward the center of curvature. The plane containing the n and t axes is called the osculating plane.

A third axis can be defined, called the binomial axis, b. The binomial unit vector, $\vec{u}_b$, is directed perpendicular to the osculating plane, and its sense is defined by the cross product $\vec{u}_b = \vec{u}_t \times \vec{u}_n$.

There is no motion, thus no velocity or acceleration, in the binomial direction.

Quiz

1. If a particle moves along a curve with a constant speed, then its tangential component of acceleration is

   A) positive. B) negative.

   ✔ C) zero. D) constant.

2. The normal component of acceleration represents

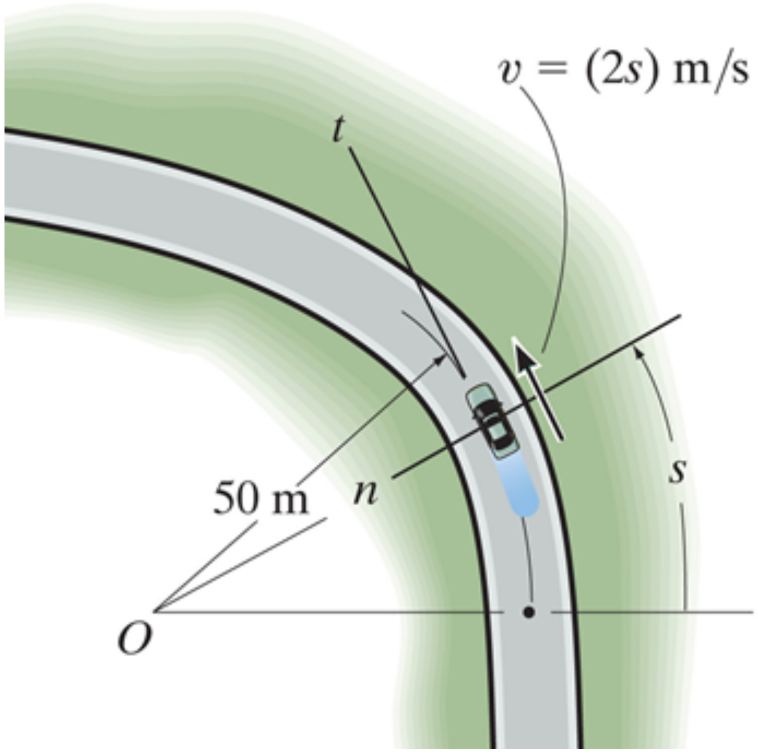
   A) the time rate of change in the magnitude of the velocity.

   ✔ B) the time rate of change in the direction of the velocity.

   C) magnitude of the velocity.

   D) direction of the total acceleration.

Example 1



Given: A car travels along the road with a speed of $v = (2s)$ m/s, where s is in meters.
$\rho = 50$ m

Find: The magnitudes of the car's acceleration at $s = 10$ m.

Plan:

1) Calculate the velocity when $s = 10$ m using $v(s)$.
2) Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.

Example 1

Solution:

1) The velocity vector is $\vec{\boldsymbol{v}} = \text{v}\, \vec{\boldsymbol{u}}_t$, where the magnitude is given by $\text{v} = (2s)$ m/s.

   When $s = 10$ m: $\text{v} = 20$ m/s

2) The acceleration vector is $\vec{\boldsymbol{a}} = a_t\, \vec{\boldsymbol{u}}_t + a_n \vec{\boldsymbol{u}}_n = \dot{\text{v}} \boldsymbol{u}_t + (\text{v}^2/\rho)\, \vec{\boldsymbol{u}}_n$

Tangential component:

Since $a_t = \dot{\text{v}} = d\text{v}/dt = (d\text{v}/ds)\,(ds/dt) = \text{v}\,(d\text{v}/ds)$

where $\text{v} = 2s \quad \Rightarrow \quad a_t = d(2s)/ds\,(\text{v}) = 2\,\text{v}$

At $s = 10$ m: $a_t = 40\ \text{m/s}^2$

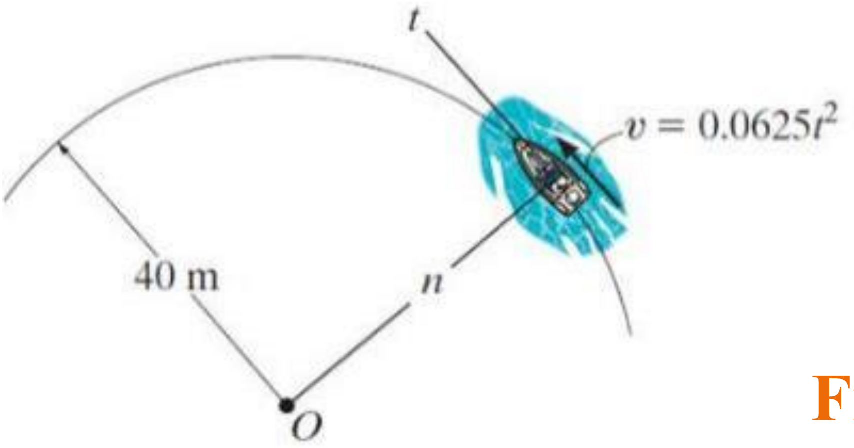
Normal component: $a_n = \text{v}^2/\rho$

When $s = 10$ m: $a_n = (20)^2 / (50) = 8\ \text{m/s}^2$

The magnitude of the acceleration is

$a = [(a_t)^2 + (a_n)^2]^{0.5} = [(40)^2 + (8)^2]^{0.5} = 40.8\ \text{m/s}^2$

Example 2



Given: A boat travels around a circular path, r = 40 m, at a speed that increases with time, $v = (0.0625\ t^2)$ m/s.

Find: The magnitudes of the boat's velocity and acceleration at the instant t = 10 s.

Plan:

The boat starts from rest (v = 0 when t = 0).

1) Calculate the velocity at t = 10 s using v(t).
2) Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.

Example 2

Solution:

1) The velocity vector is $\vec{v} = v\,\vec{u}_t$, where the magnitude is given by $v = (0.0625t^2)$ m/s. At t = 10s:

$$v = 0.0625\,t^2 = 0.0625\,(10)^2 = 6.25 \text{ m/s}$$

2) The acceleration vector is $\vec{a} = a_t\,\vec{u}_t + a_n\,\vec{u}_n = \dot{v}\,\vec{u}_t + (v^2/\rho)\,\vec{u}_n$.

Tangential component: $a_t = \dot{v} = d(0.0625\,t^2)/dt = 0.125\,t \text{ m/s}^2$

At t = 10s: $a_t = 0.125t = 0.125(10) = 1.25 \text{ m/s}^2$

Normal component: $a_n = v^2/\rho \text{ m/s}^2$

At t = 10s: $a_n = (6.25)^2 / (40) = 0.9766 \text{ m/s}^2$

The magnitude of the acceleration is

$$a = [(a_t)^2 + (a_n)^2]^{0.5} = [(1.25)^2 + (0.9766)^2]^{0.5} = 1.59 \text{ m/s}^2$$

Quiz

1. A particle traveling in a circular path of radius 300 m has an instantaneous velocity of 30 m/s and its velocity is increasing at a constant rate of 4 m/s^2. What is the magnitude of its total acceleration at this instant?

   A) 3 m/s^2 B) 4 m/s^2

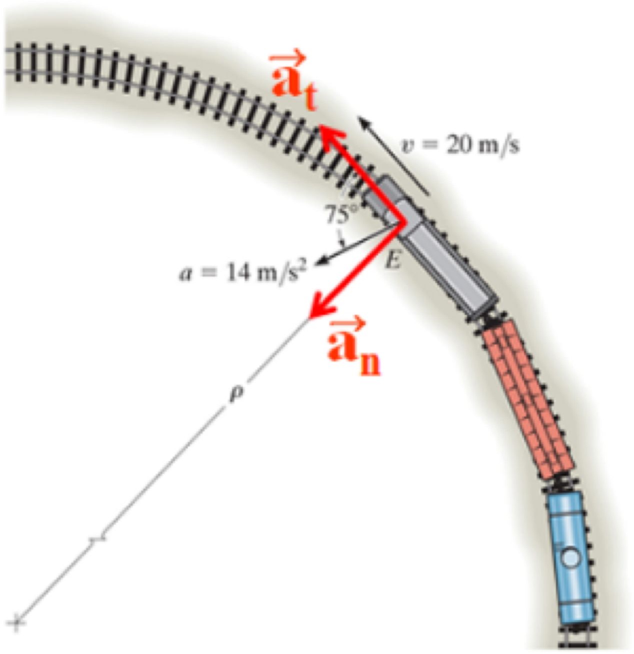
   ✔C) 5 m/s^2 D) -5 m/s^2

2. If a particle moving in a circular path of radius 5 m has a velocity function $v = 4t^2$ m/s, what is the magnitude of its total acceleration at $t = 1$ s?

   A) 8 m/s ✔B) 8.6 m/s

   C) 3.2 m/s D) 11.2 m/s

Example 3 (T)



Given: The train engine at E has a speed of 20 m/s and an acceleration of 14 m/s^2 acting in the direction shown.

Find: The rate of increase in the train's speed and the radius of curvature ρ of the path.

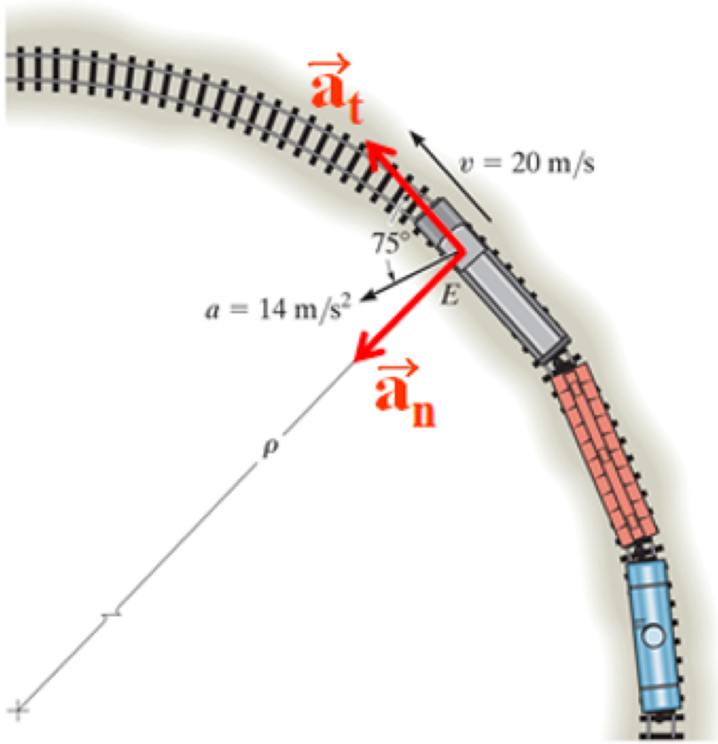
Plan:

1. Determine the tangential and normal components of the acceleration.
2. Calculate $\dot{v}$ from the tangential component of the acceleration.
3. Calculate ρ from the normal component of the acceleration.

Example 3 (T)

Solution:

1) Acceleration



Tangential component :

$a_t = 14\cos(75°) = 3.623\ \text{m/s}^2$

Normal component :

$a_n = 14\sin(75°) = 13.52\ \text{m/s}^2$

2) The tangential component of acceleration is the rate of increase of the train's speed, so

$a_t = \dot{v} = 3.62\ \text{m/s}^2$.

3) The normal component of acceleration is

$a_n = v^2/\rho \quad \Rightarrow 13.52 = 20^2 / \rho$

$\rho = 29.6\ \text{m}$

Quiz

1. The magnitude of the normal acceleration is

   A) proportional to radius of curvature.

   ✔ B) inversely proportional to radius of curvature.

   C) sometimes negative.

   D) zero when velocity is constant.

2. The directions of the tangential acceleration and velocity are always

   A) perpendicular to each other. ✔ B) collinear.

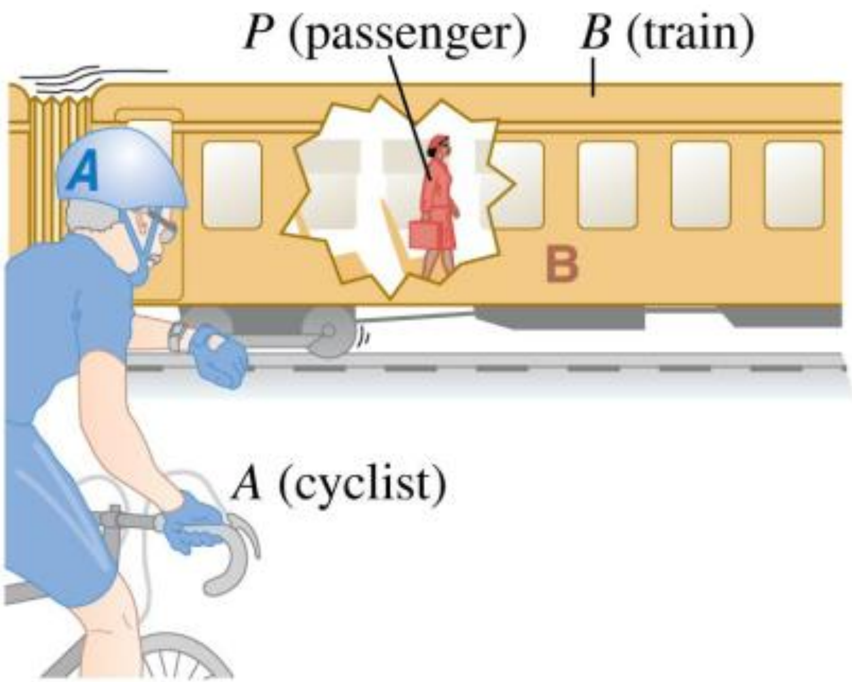
   C) in the same direction. D) in opposite directions.

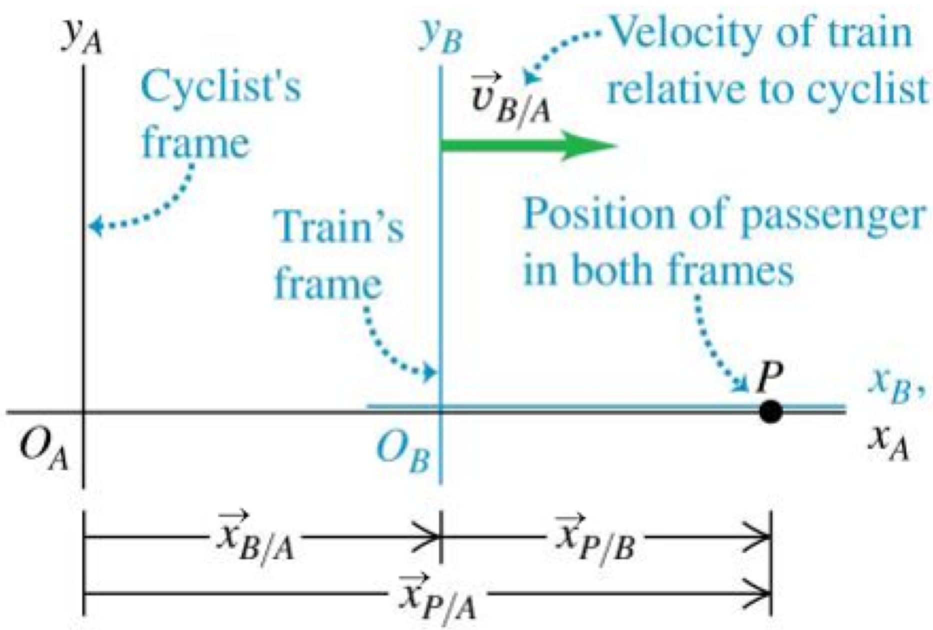
Relative velocity

- The velocity of a moving body seen by a particular observer is called the velocity *relative* to that observer, or simply the **relative velocity**.
- A **frame of reference** is a coordinate system plus a time scale.
- In many situations relative velocity is extremely important.

Relative velocity in one dimension

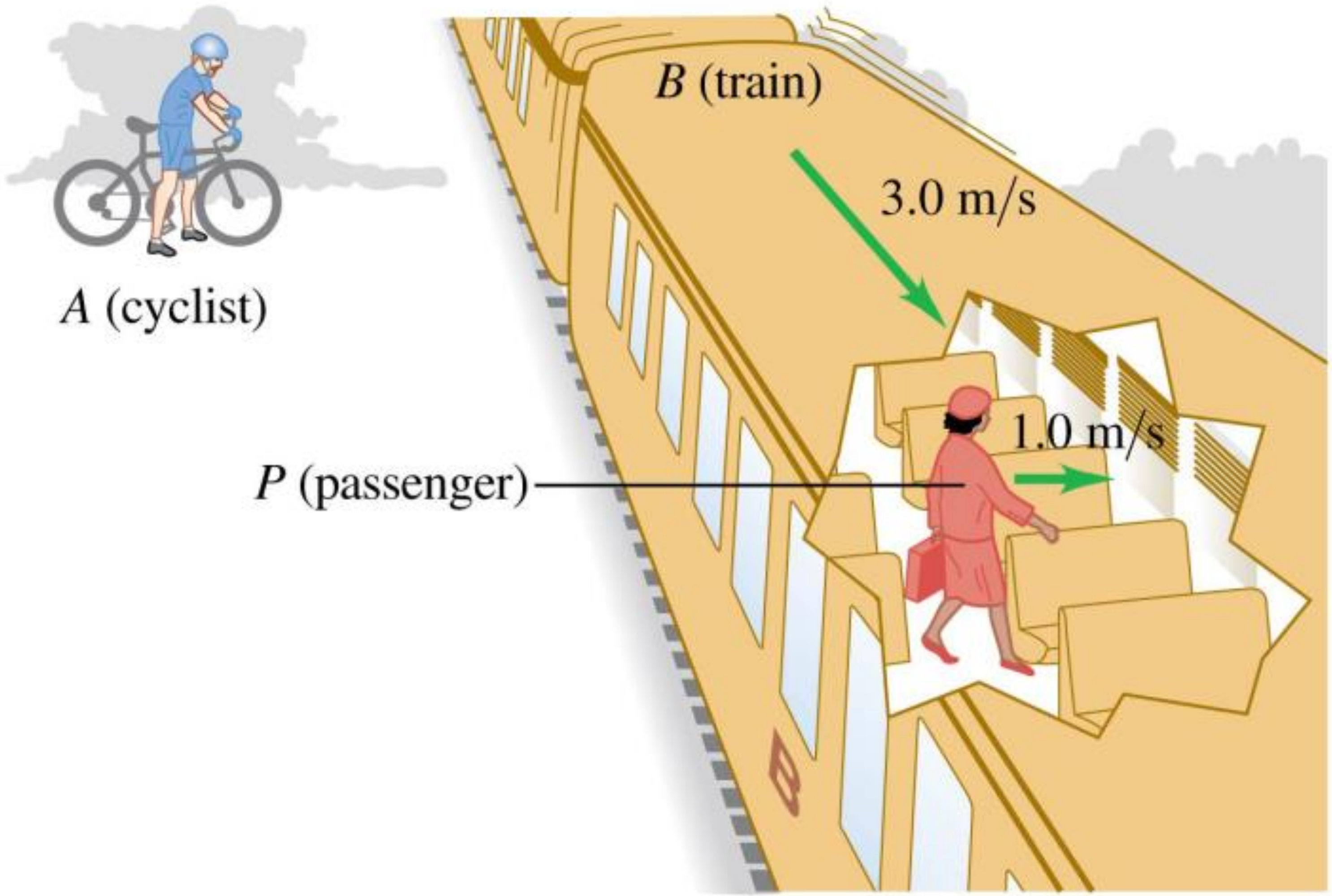
- If point P is moving relative to reference frame A, we denote the velocity of P relative to frame A as $\vec{v}_{P/A}$.
- If P is moving relative to frame B and frame B is moving relative to frame A, then the x-velocity of P relative to frame A is $\vec{v}_{P/A\text{-}x} = \vec{v}_{P/B\text{-}x} + \vec{v}_{B/A\text{-}x}$.



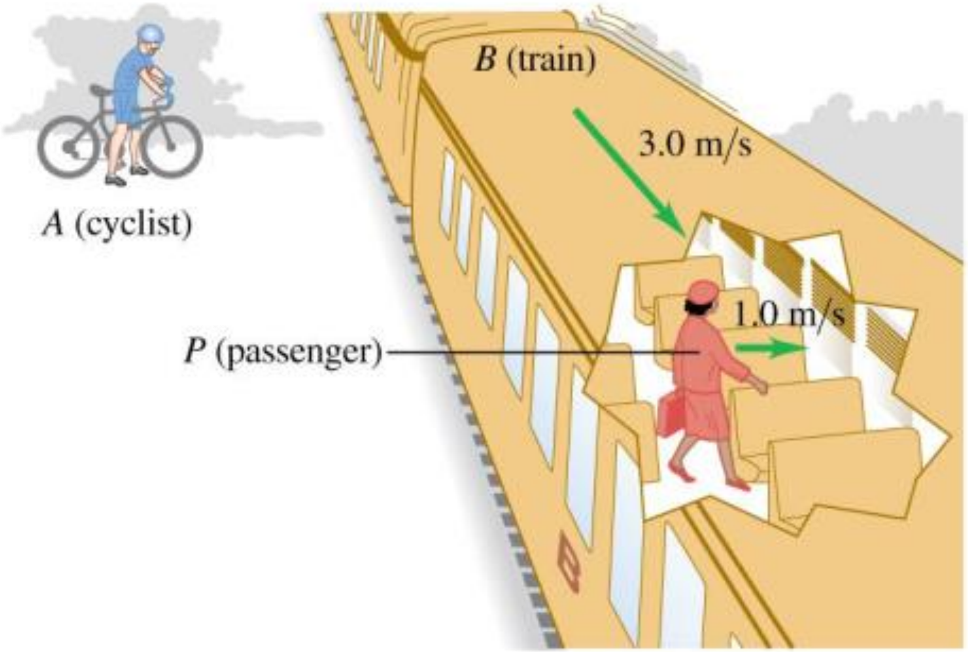


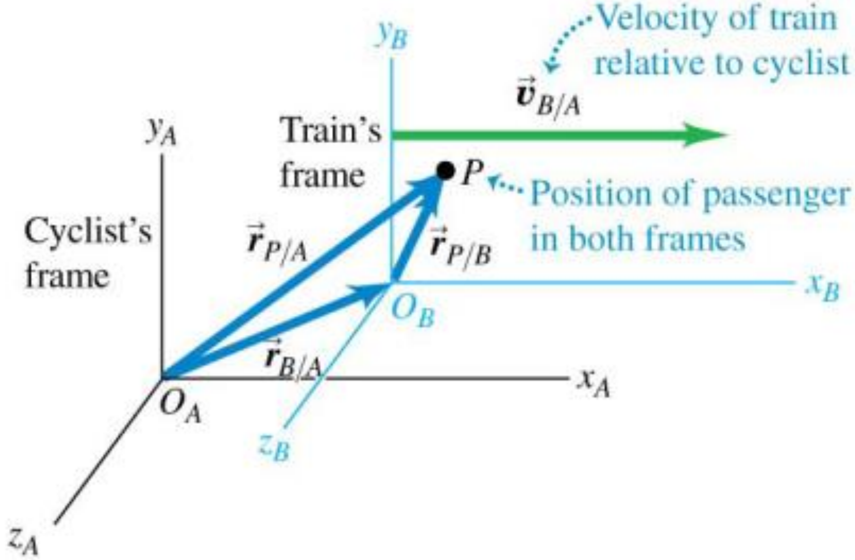
Relative velocity in two or three dimensions

- We extend relative velocity to two or three dimensions by using vector addition to combine velocities.



Relative velocity in two or three dimensions





Relative velocity in space:

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

Velocity of P relative to A

Velocity of P relative to B

Velocity of B relative to A