# Chapter 2 Basic Probability

## 2.1 Sample space and events

*Probability theory focuses on experiments whose outcome is not predictable with certainty (random experiments).*

*Sample space*: The set of all possible outcomes of an experiment, Denoted by *S*.

Example: If the experiment consists of flipping two coins, then the sample space is

$S=\{(H,H),(H,T),(T,H),(T,T)\}$

If one only cares about the number of heads

$S=\{0,1,2\}$

*Event*: Any subset of the sample space. If the outcome of the experiment is one of the members of the event, we say the event has occurred

Example (continued): Suppose *S={(H,H),(H,T),(T,H),(T,T)}*,

Let *E* be the event that a head appears on the first coin, then *E={(H,H),(H,T)}*

Example: Consider the S&P 500 stock index return with the event ”return is less than 1%”

$$S = \{x : -\infty < x < \infty\}, \ E = \{x : x \leq 0.01\}$$

## Relations between events

Let *A* and *B* be any two events defined over the sample space *S*.

*Intersection*: The intersection of *A* and *B*, written as $A \cap B$ or $AB$, is the event whose outcomes belong to both *A* and *B*.

*Union*: The union of *A* and *B*, written as $A \cup B$, is the event whose outcomes belong to either *A* or *B*, or both.

*Complement*: The complement of an event *A* , written as $A^c$, is the event consisting of all the outcomes in the sample space *S* other than those contained in *A*. $\overline{A}$

*Difference:* The difference of A and B, written as *A-B* or *A\B,* is the event whose outcomes belong to A but not B, i.e. $A - B = A \cap B^c$

*Mutually exclusive events*: Two events *A* and *B* are said to be mutually exclusive if they have no outcomes in common – that is, $A \cap B = \emptyset$

Example: A single card is drawn from a deck of 52 cards. Let A be the event that an ace is selected, i.e. A={ace of hearts, ace of diamonds, ace of clubs, ace of spades}; Let B be the event "Heart is drawn", i.e. B={2 of hearts, 3 of hearts, ..., ace of hearts} What is

$A \cap B$ and $A \cup B$ ?

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1 16

If $E_1, E_2, \ldots$ are events, the union of these events, denoted by $\cup_{n=1}^{\infty} E_n$, is defined to be that event which consists of all outcomes that are in $E_n$ for at least one n.

$[0,1]$ $[1,2]$, $[2,3]$ $\cdots$

$[0,\infty)$

Similarly, the intersection of the events, denoted by $\cap_{n=1}^{\infty} E_n$, is defined to be the event consisting of those outcomes that are in all of the events.

## Manipulating events:

Commutative laws:

$$E \cup F = F \cup E \,;\, EF = FE$$

Associative laws:

$$(E \cup F) \cup G = E \cup (F \cup G) \,;\, (EF)G = E(FG)$$

Distributive laws: $(x+y) \times z = x \times z + y \times z$

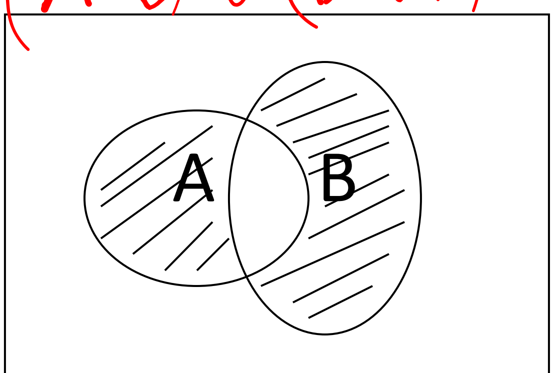
$$(E \cup F) \cap G = (E \cap G) \cup (F \cap G) \,;\, (E \cap F) \cup G = (E \cup G) \cap (F \cup G)$$

Venn diagram: A graphical representation useful for illustrating events.

Example: For two events *A* and *B*, use set operations to express (a) the event that *exactly one* (of the two) occurs; (b) the event that *at most one* (of the two) occurs

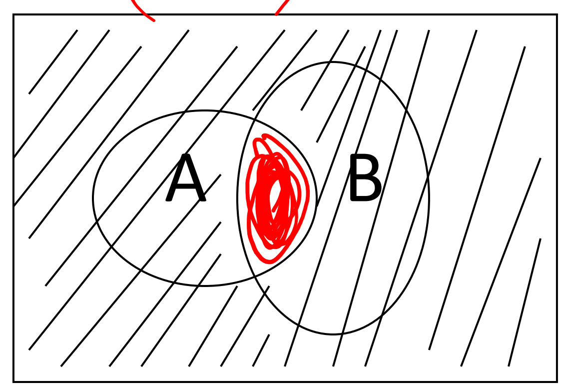
Solution: can be found easily from a Venn diagram.

(a) $(A \setminus B) \cup (B \setminus A)$




$(A \cup B) \setminus (A \cap B)$

(b) $(A \cap B)^c$



## De Morgan's laws:

(a) $(A \cup B)^c = A^c \cap B^c$

(b) $(A \cap B)^c = A^c \cup B^c$

(a') $(\cup_{i=1}^n E_i)^c = \cap_{i=1}^n E_i^c$

(b') $(\cap_{i=1}^n E_i)^c = \cup_{i=1}^n E_i^c$

$E = \{(H,H)\}$, $F = \{(H,H),(T,T)\}$

$\boxed{P(F) \geq P(E)}$

## 2.2 Basic Probability Concepts

Frequency interpretation of probability: *P(E),* the probability of the event *E*, is defined, as the limiting frequency of *E* when an experiment is performed repeatedly , i.e. $P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$

$P(H) = \frac{2}{3}$

$P: \text{event} \to \cancel{\mathbb{R}}\ [0,1]$

$P(T) = \frac{1}{3}$

Example: Experiment: toss a coin. Sample space is *S*={head, tail}. If the experiment is repeated many times, the relative frequency of heads will usually be close to ½. So we define P({head})=1/2

## Axiomatic approach to Probability

A *probability function* (or *probability measure*) on the events in a sample space is a function on the events that satisfies the following three axioms:

$E=\{(H,H)\}$ $P(E)=\frac{1}{3}$

$F=\{(H,T)\}$ $P(F)=\frac{1}{6}$

Axiom 1: For any event $E$, $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$, where $S$ is the sample space

$P((H,H)\cup(H,T))=\frac{1}{3}+\frac{1}{6}$

Axiom 3: For any sequence of mutually exclusive events (that is, events for which $E_iE_j = \emptyset \; i \neq j$ ),

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

## Properties of probability function

1. $P(\emptyset) = 0$

Proof: Consider the sequence of events ,

$$E_1 = S, E_2 = E_3 = \cdots = \emptyset$$

Then, as the events are mutually exclusive and

$S = \cup_{i=1}^{\infty} E_i$ , we have from Axiom 3 that

$$P(\cancel{S}) = \sum_{i=1}^{\infty} P(E_i) = P(\cancel{S}) + \sum_{i=2}^{\infty} P(\emptyset) \qquad = 0$$

implying that $P(\emptyset) = 0$

2. (finite additivity) For any finite sequence of mutually exclusive events $E_1, \ldots, E_n$, $P(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i)$

Proof: Let $E_i = \emptyset, \text{ when } i > n$. The results follows from Axiom 3 combined with the fact $P(\emptyset) = 0$.

$E_1, E_2, \ldots E_n, \phi, \phi, \phi$ —

$P(\underbrace{\bigcup_{i=1}^{\infty} E_i}_{\bigcup_{i=1}^{n} E_i}) = \sum_{i=1}^{\infty} P(E_i)$

$\sum_{i=1}^{n} P(E_i)$

**Example (Application of finite additivity):** If a die is rolled and we suppose that all six sides are equally likely to appear, then we would have

$$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = \frac{1}{6}$$

The probability of rolling an even number would equal

$$P(\{2,4,6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = 1/2$$

$$= \frac{1}{6} \times 3$$

$P(H)+P(T)=1$

**3.** $P(E^c) = 1 - P(E)$

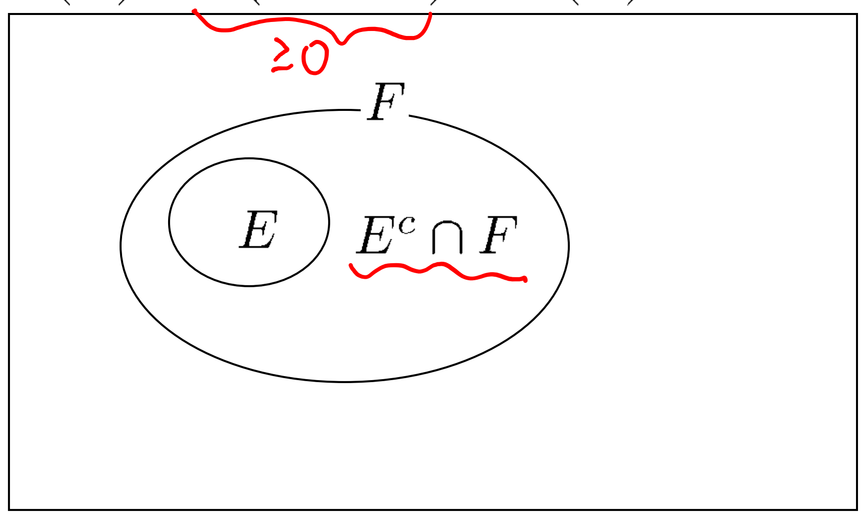
**Proof:** $P(E) + P(E^c) = P(E \cup E^c) = P(S) = 1$

**4** If $E \subset F$, then $P(E) \leq P(F)$

$\boxed{E \subseteq F}$

$E \subsetneq F$

**Proof:** $P(F) = P(E) + \underbrace{P(E^c \cap F)}_{\geq 0} \geq P(E)$



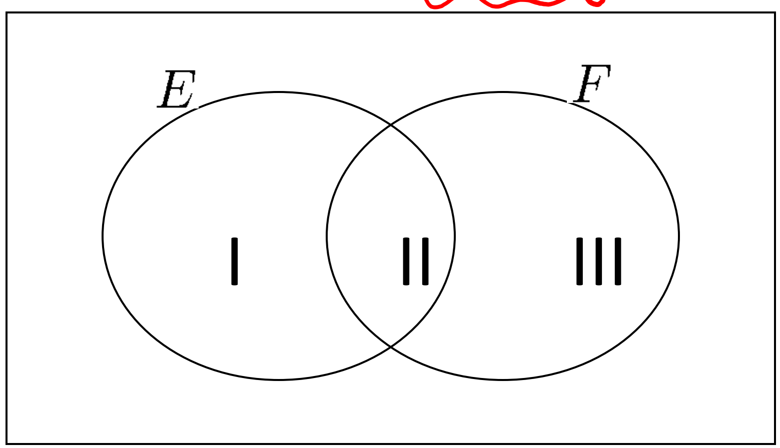

$F = E \cup (F \setminus E)$

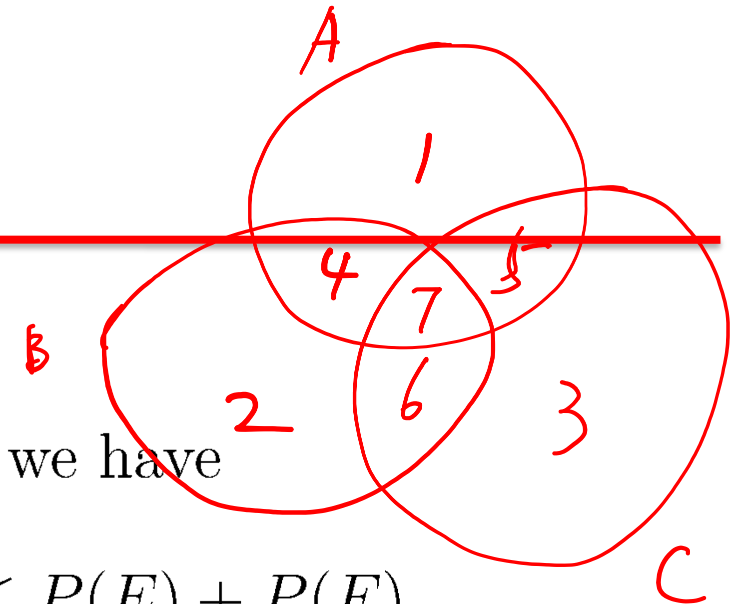
$\mathrm{I} \to E \setminus F$, $\mathrm{II} \to EF$, $\mathrm{III} \to F \setminus E$

5. $P(E \cup F) = P(E) + P(F) - P(EF)$

Proof: $E = \mathrm{I} \cup \mathrm{II}$, $F = \mathrm{II} \cup \mathrm{III}$

$$P(E) + P(F) - P(EF)$$
$$= P(\mathrm{I} \cup \mathrm{II}) + P(\mathrm{II} \cup \mathrm{III}) - P(\mathrm{II})$$
$$= P(\mathrm{I}) + P(\mathrm{II}) + P(\mathrm{II}) + P(\mathrm{III}) - P(\mathrm{II}) = P(\mathrm{I}) + P(\mathrm{II}) + P(\mathrm{III})$$
$$= P(E \cup F)$$






## 6. Boole's inequality

Corollary: For any events $E$ and $F$, we have

$$P(E \cup F) \leq P(E) + P(F)$$

More generally, for any sequence of events $E_i, i = 1, 2, \ldots,$ we have (Boole's inequality):

$P(A \cup B \cup C) = P(1234567)$

$$P(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} P(E_i)$$

$P(A)+P(B)+P(C) = P(1457) + P(2467) + P(3567)$

$- P(4,7) - P(5,7) - P(6,7) + P(7)$

$A \cap B \cap C$

## Extension of Property 5. (*inclusion-exclusion identity*)

$$
\begin{aligned}
& P(E_1 \cup E_2 \cup \cdots \cup E_n) \\
= \ & \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1} E_{i_2}) + \sum_{i_1<i_2<i_3} P(E_{i_1} E_{i_2} E_{i_3}) \\
& + \cdots + (-1)^{n+1} P(E_1 E_2 \cdots E_n)
\end{aligned}
$$

Remark: There is also a similar inclusion-exclusion identity for counting the number of elements in a set (letting $|A|$ denote the number of elements in a set $A$):

$$
\begin{aligned}
& |E_1 \cup E_2 \cup \cdots \cup E_n| \\
= \ & \sum_{i=1}^{n} |E_i| - \sum_{i_1<i_2} |E_{i_1} E_{i_2}| + \sum_{i_1<i_2<i_3} |E_{i_1} E_{i_2} E_{i_3}| \\
& + \cdots + (-1)^{n+1} |E_1 E_2 \cdots E_n|
\end{aligned}
$$

1. For any event $E$, $0 \leq P(E) \leq 1$

2. $P(S) = 1$,( $S$ is the sample space), and $P(\emptyset) = 0$

3. If $E \subset F$, then $P(E) \leq P(F)$

4. For any sequence of **mutually exclusive** events (that is, events for which $E_i E_j = \emptyset$ when $i \neq j$ ),

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

For any sequence of events (Boole's inequality):

$$P(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} P(E_i)$$

5. $P(E^c) = 1 - P(E)$

6. Inclusion-exclusion $P(E \cup F) = P(E) + P(F) - P(EF)$

## Sample spaces having equally likely outcomes

For many experiments, it is natural to assume that all outcomes in the sample space are equally likely to occur. Consider an experiment whose sample space *S* is a finite set, say $S = \{1, 2, \ldots, N\}$ . Then it is often natural to assume that

$$P(\{1\}) = P(\{2\}) = \ldots = P(\{N\}) = \frac{1}{N}$$

By additivity, for any event *E*,

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}$$

Example: If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7?

Solution: There are 36 outcomes in the sample space, all outcomes equally likely.

$$E = \{(i,j) : i + j = 7\} \qquad P(E) = 6/36 = 1/6$$

$P((4,1)) = \frac{1}{36}$

$P(4) \cdot P(1) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

Example: If 3 balls are "randomly drawn" from a bowl containing 6 white and 5 black balls, what is the probability that one of the selected balls is white and the other two black?

$$\frac{\binom{6}{1}\binom{5}{2} \times 3!}{\binom{11}{3} \times 3!}$$

Solution: $$\frac{\binom{6}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{4}{11}$$

$$\frac{11!}{8!}$$

Example: A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women.

Solution:

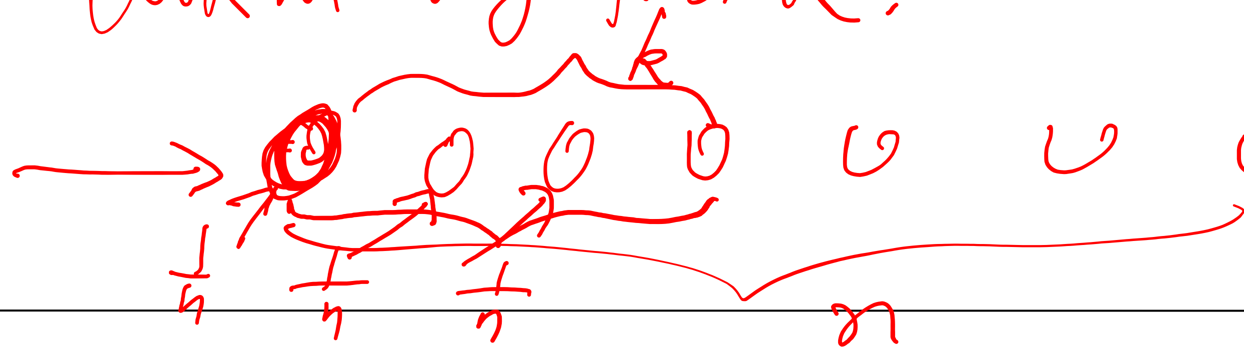
$$\frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}}$$

(1) $\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$

**Example:** An urn contains $n$ balls, of which one is special. If $k$ of these balls are withdrawn one at a time, with each selection being equally likely to be any of the balls that remain at the time, what is the probability that the special ball is chosen?

(2) Select balls 1-by-1 till all $n$ selected

Look at only first $k$.




$\frac{k}{n}$

denom: $n^k$ | $A_i$: i-th group non-empty.

$P(A_1 \cap A_2 \cap \cdots \cap A_n)$, $P(A_1 \cap A_2)$

$\leftrightarrow P(A_1^c \cap A_2^c) = \dfrac{(n-2)^k}{n^k}$

Example: k students can choose (randomly) for themselves which one of the n tutorial groups they want to attend, what is the probability that all tutorial groups have at least one student?

$A_i$: i-th group empty $\quad P(A_1 \cap A_2)$ ✓

$P(A_1^c \cap A_2^c \cdots \cap A_n^c) = P((A_1 \cup A_2 \cdots \cup A_n)^c) = 1 - \boxed{P(A_1 \cup A_2 \cup \cdots A_n)}$ ?

$$P(A_1 \cup \cdots \cup A_n) = \sum P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \cdots (-1)^{n+1} P(A_1 \cap \cdots \cap A_n)$$

$P(A_1) = \dfrac{(n-1)^k}{n^k} = \left(\dfrac{n-1}{n}\right)^k \qquad \sum_{i=1}^{n} P(A_i) = n \cdot \left(\dfrac{n-1}{n}\right)^k$

$P(A_2) = \cdots$

odd term $+$
even $-$

$$\boxed{(-1)^{n+1}}$$

$$P(A_1 \cap A_2) = \left(\frac{n-2}{n}\right)^k, \quad \sum_{i<j} P(A_i \cap A_j) = \binom{n}{2} \cdot \left(\frac{n-2}{n}\right)^k$$

$$\sum_{i<j<k} P(A_i \cap A_j \cap A_k) = \binom{n}{3}\left(\frac{n-3}{n}\right)^k \quad - \; - -$$

$$P(A_1 \cap \cdots \cap A_n) = \left(\frac{n-n}{n}\right)^k = 0 \leftarrow$$

$$\sum P(A_1 \cap \cdots \cap A_{n-1}) = \binom{n}{n-1}\left(\frac{1}{n}\right)^k$$

$$P(A_1 \cup \cdots \cup A_n) = \binom{n}{1}\left(\frac{n-1}{n}\right)^k - \binom{n}{2}\left(\frac{n-2}{n}\right)^k \cdots + (-1)^n \binom{n}{n-1}\left(\frac{1}{n}\right)^k$$

$$\binom{k}{n} n! \; \cancel{\quad} \; n^{k-n}$$

$n=2$, $k=3$ stu 1 2 3

G1: 1 3    G2: 2

G1: 3    G2: 1 2

Example: An entire deck of 52 cards is dealt out to 4 players (so that each player has 13 cards). What is the probability that

(a) One of the players (any one of the 4) receives all 13 spades;

39 cards → 3 players

$$\boxed{\dfrac{4 \times \binom{39}{13,13,13}}{\dfrac{52!}{13!\,13!\,13!\,13!}}}$$

(b) Each player receives 1 ace?

48 cards → 4 players

$$\frac{4! \times \binom{48}{12,12,12,12}}{\left( \quad \cdots \quad \right)}$$

$$\binom{52}{13,13,13,13} \qquad \frac{52!}{(13!)^4}$$

Example (difficult): A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates. If the pairing is done at random, what is the probability that there are no offensive-defensive roommate pairs? ~~What is the probability that there are 2*i* offensive-defensive roommate pairs, *i*=1,2,...,10?~~

| | | | |
|---|---|---|---|
| Supplier | 1 | 2 | 3 |
| order | 2 | 3 | 1 |

n

Example: Three different orders are to be mailed to

n three suppliers. However, an absent-minded secretary gets the orders mixed up, and sends them randomly. If a match refers to the fact that a supplier receives the correct order, find the probability of the event that no matches occur.

Solution:

## 2.3 Conditional Probability

Example: What stocks are cheap? What is the probability that a company will outperform the industry index given that its P/E ratio is the same as industry average?

| | P/E Ratio Relative to Industry Average | | | |
|---|---|---|---|---|
| Performance Relative to Industry Index | Low | Average | High | Total |
| Underperforming | 6% | 11% | 8% | 25% |
| Equally performing | 11% | 19% | 5% | 35% |
| Outperforming | 21% | 15% | 4% | 40% |
| Total | 38% | 45% | 17% | 100% |

Let A={P/E average} B={outperform},

then $P(B|A)=0.15/0.45$, but $P(B)=0.40$

Definition: If an event $F$ has occurred or will occur and we want to know what is the probability of $E$ given $F$ (occurrence of $F$ may influence of occurrence of $E$). It is defined as (suppose $P(F)>0$):

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Example: A student is taking a one-hour-time limit makeup examination. Suppose the probability that the student will finish the exam in less than *x* hours is *x/2*, for all $0 \leq x \leq 1$ .

Given that the student is still working after .75 hours, what is the conditional probability that the full hour is used?

Solution: Let *E* be the event that the student did not finish the exam within one hour, and *F* be the event that the student did not finish within .75 hour.

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{P(E)}{P(F)} = \frac{1/2}{1 - 3/8} = 0.8$$

Multiplication Rule: From the definition of conditional probability $P(E|F) = P(EF)/P(F)$ , we have that

$$P(EF) = P(F)P(E|F)$$

This is a useful formula for computing *P(EF)* when it is easy to compute the conditional probability *P(E|F)*.

For *n* events, we have the multiplication rule:

$$P(E_1 E_2 \cdots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1 E_2) \cdots P(E_n|E_1 \cdots E_{n-1})$$

The Law of Total Probability: Suppose $F_1, F_2, \ldots, F_n$ are mutually exclusive events such that $\cup_{i=1}^{n} F_i = S$

Then, for any event *E*, $P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i)$

One-line proof:

$$P(E) = P(\cup_i (EF_i)) = \sum_i P(EF_i) = \sum_i P(E|F_i)P(F_i)$$

## Bayes' formula:

Suppose $F_1, F_2, \ldots, F_n$ are mutually exclusive events such that $\cup_{i=1}^n F_i = S$. Then, for any event *E*,

$$P(F_j|E) = \frac{P(EF_j)}{P(E)} = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^n P(E|F_i)P(F_i)}$$

Example: Celine is undecided as to whether to take a French course or a chemistry course. She estimates that her probability of receiving an A grade would be ½ in a French course, and 2/3 in a chemistry course. If Celine decides to base her decision on the flip of a fair coin, what is the probability that she gets an A in chemistry.

Example: A coin is flipped twice. Assuming that all four points in the sample space $S=\{(h,h), (h,t), (t,h), (t,t)\}$ are equally likely, what is the conditional probability that both flips land on heads, given that (a) the first flip lands on heads; (b) at least one flip lands on heads.

Example: Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

Example 1000 corporate names, at time of issuance, 335 were assigned speculative-grade ratings, 665 assigned investment-grade ratings. What is the probability that a company defaults within a five-year period?

| | In 5th Year | | |
|---|---|---|---|
| At Issuance | Investment Grade | Speculative Grade | In Default |
| Investment Grade | 94.7% | 5% | 0.3% |
| Speculative Grade | 1.2% | 87.5% | 11.3% |
| In Default | 0% | 0% | 0% |

Example A hedge fund manager gathers the following data about companies:

- The probability that a company becomes an acquisition target during the course of an year is 40%.
- 75% of the companies that became acquisition targets had values of PFCF more than three times the industry average.
- Only 35% of the companies that were not targeted for acquisition had PFCF higher than three times the industry average.

Suppose that a given company has a PFCF higher than three times the industry average. What is the chance that this company becomes a target for acquisition during the course of the year?

Example (part 1). An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. Their statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability .4, whereas this probability decreases to .2 for a non-accident-prone person. If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

$P(an) = 0.26$

Example (part 2). Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is accident prone?

(We want to calculate P({prone}|{accident}) while we know P({accident}|{prone}) )

$$\frac{P(\text{prone} \& \text{acc})}{P(\text{acc})} = \frac{0.3 \times 0.4}{0.26}$$

$$= \frac{6}{13} > 0.3$$

## Solution for Example (part 2):

$$P(\{prone\}|\{accident\})=P(\{accident\}|\{prone\})P(\{prone\})/P(\{accident\})$$

$AB,\ BC,\ AC$

$$P(A\text{ die} \mid \text{say } B) = P(AB \mid \text{say } B) = \frac{\overset{\frac{1}{3}}{P(AB)}\,\overset{1}{P(\text{say } B \mid AB)}}{P(AB)P(\text{say } B \mid AB) + P(AC)\underset{0}{P(\text{say } B \mid AC)} + P(BC)\underbrace{P(\text{say } B \mid BC)}_{\frac{1}{2}}} = \frac{2}{3}$$

Example : (Three prisoner's problem.)

(Note: do not focus too much on this problem if you cannot understand it)

Three prisoners, *A*, *B* and *C* are on death row. The governor decides to pardon one of the three and chooses at random the prisoner to pardon. He informs the warden of his choice but requests that the name be kept secret for a few days.

The next day, *A* tries to get the warden to tell him who had been pardoned. The warden refuses. *A* then asks which of *B* or *C* will be executed. The warden thinks for a while, then tells *A* that *B* is to be executed.

**Warden's reasoning**: Each prisoner has a 1/3 chance of being pardoned. Clearly, either *B* or *C* must be executed, so I have given *A* no information about whether *A* will be pardoned.

***A*'s reasoning**: Given that *B* will be executed, then either *A* or *C* will be pardoned. My chance of being pardoned has risen to 1/2.

Which reasoning is correct?

RR, BB, RB

Example . Suppose that we have 3 cards identical in form except that both sides of the first card are colored red, both sides of the second colored black, and one side of the third card is colored red and the other side is black. The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

$$P(RB \mid \text{see red}) = \frac{\overset{1/3}{P(RB)}\,\overset{1/2}{P(\text{see red} \mid RB)}}{P(RB)P(\text{see red} \mid RB) + \underset{1/3}{P(RR)}\,\underset{1}{P(\text{see red} \mid RR)} + \cancel{P(BB)P(\text{see red} \mid BB)}} = \frac{1}{3}$$

$$\rightarrow P(RR \mid \text{see red}) = \frac{2}{3}$$

$\frac{1}{3} \times \frac{1}{2}$

## Independent Events

Definition: *E* and *F* are said to be ***independent*** if

$P(EF)=P(E)P(F)$ 1

Independence can be expressed in the way $P(E|F)=P(E)$ 2 *if* $P(F)>0$ (knowledge that *E* has occurred does not change the probability of *F*;

Example : Suppose that we toss two fair dice (green and red). Let *E* be the event that the sum of the two dice is 6 and *F* be the event that the green die equals 4. Are *E* and *F* independent?

Solution: $P(E)P(F) = P(\underbrace{\{(1,5),(2,4),(3,3),(4,2),(5,1)\}}_{\frac{5}{36}}) \cdot \frac{1}{6} = 5/216$

$P(EF) = P(\{(4,2)\}) = 1/36$

Dependent.

$P(E) = \frac{6}{36} = \frac{1}{6}$ $\quad$ $P(F) = \frac{1}{6}$,

$P(EF) = P((4,3)) = \frac{1}{36}$

Let *E* be the event that the sum of the two dice is 7 and *F* be the event that the green die equals 4. Then *E* and *F* *are* independent

Proposition Suppose $E$ is independent of $F$. We will now show that $E$ is also independent of $F^c$

Proof:

1 $$P(EF^c) = P(E)P(F^c)$$

$$\Uparrow$$

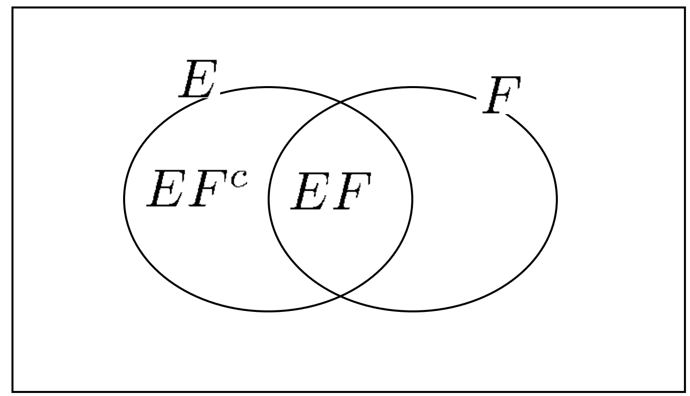
$$P(EF^c) = P(E)(1 - P(F))$$

$$\Uparrow$$

$$P(EF^c) = P(E) - P(E)P(F)$$

$$\Uparrow$$

4 $$P(EF^c) + P(EF) = P(E)$$




By similar reasoning, it follows that if $E$ is independent of $F$, then (i) $E^c$ is independent of F and (ii) $E^c$ is independent of $F^c$

## Independence for more than two events:

Three events *E, F, G* are said to be (mutually) independent if

$$P(EFG)=P(E)P(F)P(G)$$

$$P(EF)=P(E)P(F)\ ;\ P(EG)=P(E)P(G);\ P(FG)=P(F)P(G)$$

Example . An infinite sequence of independent trials is to be performed. Each trial results in a success with probability *p* and a failure with probability *1-p*. What is the probability that

a) At least *1* success occurs in the first *n* trials

b) Exactly *k* successes occur in the first *n* trials

c) All trials result in successes?

$n=5, \quad k=3$ $\quad \binom{5}{3}$

$P(SSFFS), \ P(SFSFS), \ - \ \sim$

$= p \cdot p \cdot (1-p)(1-p) \cdot p$

$= p^3 (1-p)^2$

$\boxed{\binom{5}{3} p^3 (1-p)^2}$

Solution:

a) $1-(1-p)^n$

b) $\boxed{\binom{n}{k}} p^k (1-p)^{n-k}$

c) $p^n \rightarrow \begin{cases} 0 & \text{if } p<1 \\ 1 & \text{if } p=1 \end{cases}$

$P(E)+P(E^c)=1,\ \Rightarrow P(E|F)+P(E^c|F)=1$

$P(E|F)+P(E|F^c)=1$ ? ✗

## P(.|F) is a Probability Function

The conditional probability *P(.|F)* is a probability function on the events in the sample space *S* and satisfies the usual axioms of probability:

(a) $0 \leq P(E|F) \leq 1$

$\frac{P(SF)}{P(F)}$  $P(S \cap F)=P(F)$

(b) $P(S|F)=1$ , $P(F|F)=1$

(c) If $E_i, i=1,2,\ldots$ are mutually exclusive events, then

$$P(\cup_1^\infty E_i|F)=\sum_1^\infty P(E_i|F)$$

Thus, all the formulas we have derived for manipulating probabilities apply to conditional probabilities.

For example, $P(EF)=P(E)P(F|E) \Rightarrow \boxed{P(EF|G)=P(E|G)P(F|EG)}$

## Conditional independence:

Events $E$ and $F$ are *conditionally independent* given $G$ if, given that $G$ occurs, the conditional probability that $E$ occurs is unchanged by information as to whether or not $F$ occurs. More formally, if $\boxed{P(E|FG)=P(E|G)}$ or, equivalently, $P(EF|G)=P(E|G)\,P(F|G)$

Example: An insurance company believes that people can be divided into two classes: those who are accident-prone and those who are not. Their statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability .4, whereas this probability decreases to .2 for a non-accident prone person. 30 percent of the population is accident-prone. Consider a two-year period. *Assume that the event that a person has an accident in the first year is conditionally independent of the event that a person has an accident in the second year given whether or not the person is accident prone.* What is the conditional probability that a randomly selected person will have an accident in the second given that the person had an accident in the first year?

P(2nd acc | 1st acc)

$P(acc) = P(acc \mid prone)P(prone) + \cdots = 0.26$, $P(prone \mid acc) = \frac{6}{13}$

$$P(2nd\ acc \mid 1st\ acc) = P(2nd\ acc \mid \overset{0.4}{prone}, \cancel{1st\ acc})\,P(prone \mid 1st\ acc) \leftarrow \frac{6}{13}$$

$$+ P(2nd\ acc \mid \underset{0.2}{!prone}, \cancel{1st\ acc})\,P(!prone \mid 1st\ acc) \leftarrow \frac{7}{13}$$

$$= 0.29$$

$$\frac{P(1st\ acc\ \&\ 2nd\ acc)}{P(1st\ acc)} = \frac{\overset{0.4\times0.4}{P(1st\ acc\ \&\ 2nd\ acc \mid prone)}\,\overset{0.3}{P(prone)} + \overset{0.2\times0.2}{P(1st\ acc\ \&\ 2nd\ acc \mid !prone)}\,\underset{0.7}{P(!prone)}}{0.26}$$

$$= 0.29$$

**Example (difficult):** Laplace's rule of succession. There are $k+1$ coins in a box. The $i$-th coin will, when flipped, turn up head with probability $i/k$, $i=0,1,\ldots,k$. A coin is randomly selected from the box and is repeatedly flipped. If the first $n$ flips all result in heads, what is the conditional probability that the $(n+1)$-st flip will do likewise?

$$P(n+1\ H \mid n\ Hs)$$

$$P(n\ Hs) = \sum_{i=0}^{k} \frac{1}{k+1}\left(\frac{i}{k}\right)^{n} = \sum_{i=0}^{k} P(i\text{th coin})\, P(n\ Hs \mid i\text{th coin})$$

$$P(n+1\ Hs) =$$

$$= \frac{P(n+1\ Hs)}{P(n\ Hs)}$$

$$P(i\text{th} \mid n\ Hs) = \boxed{\frac{\frac{1}{k+1}\left(\frac{i}{k}\right)^{n}}{P(n\ Hs)}}$$

$$P(n+1\text{th}\ H \mid n\ Hs) = \sum_{i=0}^{k} \boxed{\phantom{x}} \times \left(\frac{i}{k}\right) = \frac{\sum_{i=0}^{k} \frac{1}{k+1} \times \left(\frac{i}{k}\right)^{n} \frac{i}{k}}{\sum_{i=0}^{k} \frac{1}{k+1}\left(\frac{i}{k}\right)^{n}}$$

$$P(E \cap E) = \underbrace{P(E)}_{=P(E)}\,\underbrace{P(E)}_{=P(E)^2} \Rightarrow P(E) = 0 \text{ or } 1$$

1) Can *E* be independent of itself?

1st die, 2nd die, 1st + 3rd

2) If *E* is independent of *F*, and *F* is independent of *G*, is *E* independent of *G*?

3) Is *E* independent of *S* (the whole sample space)?

$$P(\underbrace{E \cap S}_{E}) = P(E) \cdot \underbrace{P(S)}_{1}$$

(a) $P(EF) = P(E) \cdot \underbrace{P(F)}_{0}$ , $P(EF) \le P(E) = 0 \Rightarrow P(EF) = 0$

~~$P(E) = 0 \Rightarrow E = \phi \Rightarrow EF = \phi \Rightarrow P(EF) = 0$~~

# Prove the following:

(a) If $P(E) = 0$, $E$ is independent of any event $F$

(b) If $P(E) = 1$, $E$ is independent of any event $F$

(c) If $P(A|B) = P(A|B^c)$, then $A$ and $B$ are independent.

(d) If $P(B|A) + P(B^c|A^c) = 1$, then $A$ and $B$ are independent.

(e) $P(B|A) \geq 1 - \frac{P(B^c)}{P(A)}$

~~(b) $P(E)=1 \Rightarrow P(E^c)=0$~~

~~$\overset{(a)}{\Rightarrow} E^c$ indep $F$~~

~~$\Rightarrow E$ indep $F$~~

(b) $P(EF) = P(E)P(F)$

$F = EF \cup E^cF$

$P(F) = P(EF) + \underbrace{P(E^cF)}$

$P(E^cF) \le P(E^c) = 0$ ?

(c) $P(AB) = P(A)P(B)$ ?

$\frac{P(AB)}{P(B)} = \frac{P(AB^c)}{P(B^c)} = \frac{P(A) - P(AB)}{1 - P(B)}$

$P(AB)(1 - P(B)) = (P(A) - P(AB))P(B)$

(d) $P(B|A) + P(B^c|A^c) = 1$

$\boxed{1 - P(B^c|A^c) = P(B|A^c)}$

$\Rightarrow P(B|A) = P(B|A^c)$ (2)

(e) $P(AB) \geq P(A) - P(B^c)$ ? (3) $\Leftarrow$ (2) $P(AB) + P(B^c) \geq P(A)$ ? ✓

(1) $P(AB) + P(AB^c) = P(A)$ ✓

$P(5) = \frac{4}{36}$, $P(7) = \frac{6}{36}$, $P(\text{not } 5 \text{ or } 7) = \frac{26}{36}$

**Example :** Independent trials, consisting of rolling a pair of fair dice are performed. What is the probability that an outcome of 5 appears before an outcome of 7 when the outcome of a roll is the sum of the dice?

$$P(5 \text{ bef } 7) = P(\text{1st } 5) + P(\text{1st not } 5 \text{ or } 7, \text{ 2nd } 5)$$

$$+ P(\text{1st, 2nd not } 5 \text{ or } 7, \text{ 3rd } 5) + \cdots$$

$$= \frac{4}{36} + \frac{26}{36} \times \frac{4}{36} + \frac{26}{36} \times \frac{26}{36} \times \frac{4}{36} + \left(\frac{26}{36}\right)^3 \times \frac{4}{36} + \cdots$$

$$= \frac{4}{36} \times \left(1 + \frac{26}{36} + \left(\frac{26}{36}\right)^2 + \left(\frac{26}{36}\right)^3 + \cdots\right) = \frac{2}{5}$$

$$1 + a + a^2 + \cdots + = \frac{1}{1-a} \quad -1 < a < 1$$

$$P(\text{5 bef 7}) = \overbrace{P(\text{5 bef 7} \mid \text{1st 5})}^{1}\, \overbrace{P(\text{1st 5})}^{\frac{4}{36}}$$

$$+ \underbrace{P(\text{5 bef 7} \mid \text{1st 7})}_{0}\, \overbrace{P(\text{1st 7})}^{\frac{6}{36}}$$

$$+ \underbrace{P(\text{5 bef 7} \mid \text{1st not 5 or 7})}_{P(\text{5 bef 7})}\, \overbrace{P(\text{1st not 5 or 7})}^{\frac{26}{36}}$$

$$\underbrace{P(\text{5 bef 7})}_{x}$$

$$x = \frac{4}{36} + x \cdot \frac{26}{36} \Rightarrow x = \frac{2}{5}$$

# (From “Heard on the streets”)

Welcome to my consultation hour. Sit in this chair. Excuse me while I tie your arms and legs to the chair. Thank you. Now we are going to play “Russian roulette.” I have a revolver with six empty chambers. Watch me as I load the weapon with two contiguous rounds (i.e., two bullets side-by-side in the cylindrical barrel). Watch me as I spin the barrel. I am putting the gun against your head. Close your eyes while I pull the trigger. Click! This is your lucky day: you are still alive! Our game differs from regular Russian roulette because I am not going to add any bullets to the barrel before we continue, and I am not going to give you the gun.

My question for you: I am going to shoot at you once more before we talk about your problem. Do you want me to spin the barrel once more, or should I just shoot?