1. (a) $$\int_{-\infty}^{+\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{+\infty} xe^{-x^2}dx$$

$$\int_{-\infty}^{0} xe^{-x^2}dx = \lim_{t\to-\infty} \int_{t}^{0} xe^{-x^2}dx$$
$$= \lim_{t\to-\infty} \left(-\frac{1}{2}e^{-x^2}\right)\Big|_{t}^{0}$$
$$= \lim_{t\to-\infty} \left(-\frac{1}{2} + \frac{1}{2}e^{-t^2}\right) = -\frac{1}{2}$$

and

$$\int_{0}^{+\infty} xe^{-x^2}dx = \lim_{t\to\infty} \int_{0}^{t} xe^{-x^2}dx$$
$$= \lim_{t\to\infty} \left(-\frac{1}{2}e^{-x^2}\right)\Big|_{0}^{t}$$
$$= \lim_{t\to\infty} \left(-\frac{1}{2}e^{-t^2} + \frac{1}{2}\right) = \frac{1}{2}$$

Therefore,

$$\int_{-\infty}^{+\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{+\infty} xe^{-x^2}dx = -\frac{1}{2} + \frac{1}{2} = 0$$

(b) $$\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{\sqrt{3-x}}dx$$
$$= \lim_{t\to-\infty} -2\sqrt{3-x}\Big|_{t}^{0}$$
$$= \lim_{t\to-\infty} \left(-2\sqrt{3} + 2\sqrt{3-t}\right)$$
$$= -2\sqrt{3} + \infty = \infty.$$

Therefore, the improper integral is divergent.

(c) $\int_0^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-} \int_0^t \frac{1}{\sqrt{3-x}}\,dx$

$= \lim_{t\to 3^-} \left(-2\sqrt{3-x}\right)\Big|_0^t$

$= \lim_{t\to 3^-} \left(2\sqrt{3} - 2\sqrt{3-t}\right) = 2\sqrt{3}.$

(d) $\int_0^{+\infty} \frac{1}{x^2}\,dx = \int_0^1 \frac{1}{x^2}\,dx + \int_1^{+\infty} \frac{1}{x^2}\,dx$

$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx$

$= \lim_{t\to 0^+} \left(-\frac{1}{x}\right)\Big|_t^1$ $\Rightarrow \int_0^{+\infty} \frac{1}{x^2}\,dx$ diverges.

$= \lim_{t\to 0^+} \left(-1 + \frac{1}{t}\right) = +\infty$

2. (a) $\frac{2x^2-5x+5}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

$\Rightarrow 2x^2-5x+5 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$

Put $x=1$, we have $2 = -B \Rightarrow B = -2$

Put $x=2$, we have $3 = C$

Put $x=0$, we have $5 = 2A-2B+C \Rightarrow A = 1$.

||

$$\Downarrow$$

The integral can be computed as

$$\int \frac{1}{3x-x^2}\,dx = \frac{1}{3}\int \frac{1}{x}\,dx + \frac{1}{3}\int \frac{1}{3-x}\,dx$$

$$= \frac{1}{3}\ln|x| - \frac{1}{3}\ln|3-x| + C$$

(b) $$\int \frac{3x^4-5x^3+x^2+2x+1}{3x^3-2x^2-x}\,dx = \int x-1\,dx + \int \frac{x+1}{3x^3-2x^2-x}\,dx$$

$$= \frac{x^2}{2} - x + \int \frac{x+1}{3x^3-2x^2-x}\,dx$$

$$\frac{x+1}{3x^3-2x^2-x} = \frac{x+1}{x(3x+1)(x-1)} = \frac{A}{x} + \frac{B}{3x+1} + \frac{C}{x-1}.$$

$\Longrightarrow$ $x+1 = A(3x+1)(x-1) + Bx(x-1) + Cx(3x+1)$

Put $x=0$, $A=-1$

Put $x=1$, $C=\frac{1}{2}$

Put $x=-\frac{1}{3}$, $B=-\frac{3}{2}$.

Then the second integral can be computed as

$$\int \frac{x+1}{3x^3-2x^2-x}\,dx = -\int \frac{1}{x}\,dx - \frac{3}{2}\int \frac{1}{3x+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$

$$= -\ln|x| - \frac{1}{2}\ln|3x+1| + \frac{1}{2}\ln|x-1| + C$$

$$\int \frac{3x^4-5x^3+x^2-2x+1}{3x^3-2x^2-x}\,dx = \frac{x^2}{2} - x = -\ln|x| - \frac{1}{2}\ln|3x+1| + \frac{1}{2}\ln|x-1| + C.$$

(c) $$\frac{x}{(x+1)(x^2+4x+6)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4x+6}$$

$$\Rightarrow \quad x = A(x^2+4x+6) + (Bx+C)(x+1)$$

Put $x=-1$, $\quad A = -\frac{1}{3}$

Put $x=0$, $\quad 0 = 6A+C \Rightarrow C = 2$

Also $\quad B = \frac{1}{3}$.

So $$\int \frac{x}{(x+1)(x^2+4x+6)}\,dx = \int \frac{-\frac{1}{3}}{x+1} + \frac{\frac{1}{3}x+2}{x^2+4x+6}\,dx$$

$$= -\frac{1}{3}\ln|x+1| + \frac{1}{3}\int \frac{x+2+4}{(x+2)^2+2}\,dx$$

$$= -\frac{1}{3}\ln|x+1| + \frac{1}{3}\int \frac{x+2}{(x+2)^2+2}\,dx + \frac{1}{3}\int \frac{4}{(x+2)^2+2}\,dx$$

$$= -\frac{1}{3}\ln|x+1| + \frac{1}{6}\ln\left[(x+2)^2+2\right] + \frac{4}{3\sqrt{2}}\tan^{-1}\left(\frac{x+2}{\sqrt{2}}\right) + C.$$

(d) $$\frac{3x^3-2x-20}{(x^2+3)(2x^2-6x+5)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{2x^2-6x+5}$$

Put $x=0$, we have $5B+3D=-20$

Put $x=1$, we have $A+B+4C+4D=-19$

Put $x=-1$, we have $-13A+13B-4C+4D=-21$

Put $x=2$, we have $2A+B+14C+7D=0$

Solving the equations, we get $A=-1$, $B=2$, $C=5$, $D=-10$.

Then the integral can be computed as

$$\int \frac{3x^3-2x-20}{(x^2+3)(2x^2-6x+5)}dx = \int \frac{-x+2}{x^2+3}dx + \int \frac{5x-10}{2x^2-6x+5}dx$$

$$= -\int \frac{x}{x^2+3}dx + 2\int \frac{1}{x^2+3}dx + \int \frac{5x-\frac{15}{2}}{2x^2-6x+5}dx$$

$$- \frac{5}{2}\int \frac{1}{2x^2-6x+5}dx$$

$$= -\frac{1}{2}\int \frac{d(x^2+3)}{x^2+3} + \frac{2\sqrt{3}}{3}\int \frac{1}{(\frac{x}{\sqrt{3}})^2+1} d\frac{x}{\sqrt{3}}$$

$$+\frac{5}{4}\int \frac{d(2x^2-6x+5)}{2x^2-6x+5}\,dx-\frac{5}{2}\int \frac{1}{(2x-3)^2+1}\,d(2x-3)$$

$$=-\frac{1}{2}\ln|x^2+3|+\frac{2\sqrt{3}}{3}\tan^{-1}\frac{x}{\sqrt{3}}+\frac{5}{4}\ln|2x^2-6x+5|$$

$$-\frac{5}{2}\tan^{-1}(2x-3)+C$$

3(a)

The arc length $= \int_0^5 \sqrt{1+\left(\frac{d}{dx}\left(\frac{1}{3}x^{\frac{3}{2}}\right)\right)^2}\,dx$

$$=\int_0^5 \sqrt{1+\frac{x}{4}}\,dx$$

$$=\frac{1}{\frac{1}{4}}\left.\frac{\left(1+\frac{x}{4}\right)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^5$$

$$=\frac{8}{3}\left(\frac{27}{8}-1\right)=\frac{19}{3}.$$

(b) Note that

$$(y-1)^3=\frac{9}{4}x^2 \Rightarrow y=1+\sqrt[3]{\frac{9}{4}}\,x^{\frac{2}{3}}.$$

The arc length $= \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \sqrt{1+\left(\frac{d}{dx}\left(1+\sqrt[3]{\frac{9}{4}}\,x^{\frac{3}{2}}\right)\right)^2}\,dx$

$$=\int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \sqrt{1+\left(\frac{2}{3}\right)^{\frac{2}{3}}x^{-\frac{2}{3}}}\,dx$$

$$= \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \sqrt{\frac{x^{\frac{2}{3}} + \left(\frac{2}{3}\right)^{\frac{2}{3}}}{x^{\frac{2}{3}}}} \, dx$$

$$= \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \frac{\sqrt{x^{\frac{2}{3}} + \left(\frac{2}{3}\right)^{\frac{2}{3}}}}{x^{\frac{1}{3}}} \, dx$$

$$\overset{y = x^{\frac{2}{3}} + \left(\frac{2}{3}\right)^{\frac{2}{3}}}{=} \int_{\left(\frac{2}{3}\right)^{\frac{2}{3}}}^{\left(\frac{2}{3}\right)^{\frac{2}{3}}(3)} \frac{3}{2}\sqrt{y} \, dy$$

$$= y^{\frac{3}{2}} \Big|_{\left(\frac{2}{3}\right)^{\frac{2}{3}}}^{\left(\frac{2}{3}\right)^{\frac{2}{3}} 3} = \frac{2}{3}\left(3\sqrt{3} - 1\right).$$

(c) The arc length $= \displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$

$$= \int_0^{\frac{\pi}{2}} \sqrt{(t\cos t)^2 + (t\sin t)^2} \, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{t^2} \, dt = \int_0^{\frac{\pi}{2}} t \, dt = \frac{t^2}{2} \Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{8}.$$

4. (a). The surface area $= 2\pi \int_0^2 x^3 \sqrt{1+\left(\frac{d}{dx}x^3\right)^2}\, dx$

$$= 2\pi \int_0^2 x^3 \sqrt{1+9x^4}\, dx$$

$y = 1+9x^4$

$$= \frac{2\pi}{36} \int_1^{145} \sqrt{y}\, dy = \frac{\pi}{18} \left. \frac{y^{\frac{3}{2}}}{\frac{3}{2}} \right|_1^{145}$$

$$= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 1\right).$$

(b) The surface area

$$= \left[ 2\pi \int_1^2 x \sqrt{1+\left(\frac{d}{dx}x\right)^2}\, dx + 2\pi \int_1^2 (2-x) \sqrt{1+\left(\frac{d}{dx}(2-x)\right)^2}\, dx \right]$$

$$+ \left[ 2\pi \int_2^3 (4-x) \sqrt{1+\left(\frac{d}{dx}(4-x)\right)^2}\, dx + 2\pi \int_2^3 (x-2) \sqrt{1+\left(\frac{d}{dx}(x-2)\right)^2}\, dx \right]$$

$$= \left[ 2\sqrt{2}\pi \int_1^2 x\, dx + 2\sqrt{2}\pi \int_1^2 (2-x)\, dx \right]$$

$$+ \left[ 2\sqrt{2}\pi \int_2^3 (4-x)\, dx + 2\sqrt{2}\pi \int_2^3 (x-2)\, dx \right]$$

$$= 2\sqrt{2}\pi \left. \left(\frac{x^2}{2}\right) \right|_1^2 + 2\sqrt{2}\pi \left. \left(2x - \frac{x^2}{2}\right) \right|_1^2 + 2\sqrt{2}\pi \left. \left(4x - \frac{x^2}{2}\right) \right|_2^3$$

$$+ 2\sqrt{2}\pi \left. \left(\frac{x^2}{2} - 2x\right) \right|_2^3$$

$$= 2\sqrt{2}\pi\left(\frac{3}{2}+\frac{1}{2}+\frac{3}{2}+\frac{1}{2}\right) = 8\sqrt{2}\pi.$$