## Chapter 2 Derivatives

Contents



## 2. Derivatives

In the last chapter, we learned limit. Now, we are ready to define the derivative, and do some calculations on it.

### 2.1. **Derivatives and rates of change.** Text section 2.1 Exercise: 3, **7**, 9, 11, **14, 15, 18, 20** 25, 33, 37, 45, **46,** 53

**Definition 2.1.** *The* **derivative of $f$ at a number** $a$*, denoted by* $f'(a)$*, is*

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

*if this limit exists.*

In fact, for function $y = f(x)$, if we substitute $x = a + h$ in the above definition, then we come up another equivalent definition, i.e.

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \tfrac{f(a+h)-f(a)}{h}, \text{ ( original notation)} \\ &= \lim_{x\to a} \tfrac{f(x)-f(a)}{x-a}, \text{ (instantaneous rate of change)} \\ &= \lim_{\Delta x\to 0} \tfrac{\Delta y}{\Delta x}\Big|_a, \text{ ($\Delta$ means change )} \\ &= \tfrac{dy}{dx}\Big|_a, \text{ ( another notation )} \end{aligned}$$

**Ex.**

(1) The *tangent line* to $y = f(x)$ at $P(a, f(a))$ is the line through $P$ with slope

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

(2) Given *position/displacement function* $s = s(t)$, the *instantaneous velocity* at time $t = a$, denoted by $v(a)$ is given by

$$v(a) = \lim_{x \to a} \frac{s(x) - s(a)}{x - a} = s'(a).$$

**Ex.** Find the equation of tangent line for curve $y = x^2$ at $(-1, 1)$.

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting $a + h$ approach $a$ and therefore letting $h$ approach 0. The limit of these average rates of change is called the **(instantaneous) rate of change** of $f$ with respect to $x$ at $x = a$, which (as in the case of velocity) is interpreted as the slope of the tangent to the curve $y = f(x)$ at the point $P(a, f(a))$.

2.2. **The derivative as a function.** Text Section 2.2 Exercise: 3, 19, 23, 29, 35, 41, 47, **52**, 53, **54, 55**

For a function $f : D \to R$, we defined $f'(a)$ for all $a \in D$. Now, we consider $f' : D \to R$ as a function, where $f'(x)$ is the derivative of $f$ at $x$, i.e.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

Sometimes, we can also write

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

**Definition 2.2.** *$f$ is differentiable at $a$ if $f'(a)$ exists. $f$ is differentiable on an open interval $(a, b)$, if it is differentiable at each number in the interval.*

**Ex.** If $f(x) = \sqrt{x}$, find $\dfrac{df}{dx}$ with its domain.

**Ex.** The graph of $f$ is given in Figure 1. Using the fact $f'$ is the slope of tangent,

(1) identify all zeros of $f'$.
(2) sketch the graph of $f'$.

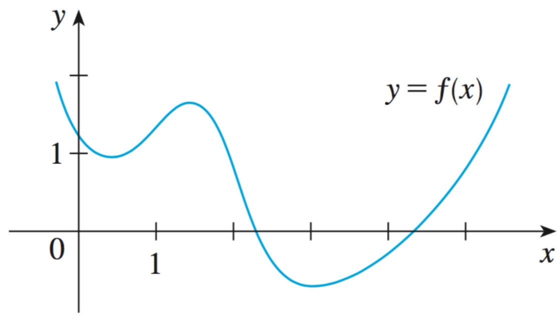


**FIGURE 1**

**Ex.** where is the function $f(x) = |x|$ differentiable?

**Proposition 2.3.** *If $f$ is differentiable at $a$, then $f$ is continuous at $a$.*

**Q.** Justify following statement:
If $f$ is continuous at $a$, then $f$ is differentiable at $a$.
Usually, $f$ is not differentiable at $a$ in one of the following situations: $f$ is

(1) corner at $a$,
(2) discontinuous at $a$,
(3) vertical tangent line at $a$, i.e. $\lim_{x\to a} |f'(x)| = \infty$.

**Ex.** Using definition, show that $\dfrac{dx^n}{dx} = nx^{n-1}$ for all integer $n$.

(**Higher derivatives**) $f'' = (f')'$ is called **second derivative** of $f$ if it exists. Similarly, we define all higher derivatives

$$f^{(3)} = (f'')'; \ \ldots; f^{(n+1)} = (f^{(n)})'.$$

Note that $f^n \neq f^{(n)}$.

(**Motion problem**) Given position function $s = s(t)$, its velocity function is

$$v(t) = s'(t)$$

and acceleration function is

$$a(t) = v'(t) = s''(t)$$

2.3. **Differentiation formulas.** Text Section 2.3 Exercise: **21, 22, 27, 31,** 41, **43, 44, 45**, 49, 59, **62, 63**, 65, **69, 70, 80, 81, 82,** 83, **85,** 89, **95, 98**, 103

Suppose $f'$ and $g'$ exists, then following differentiation formulae hold:

(1) (Power function) For any constant $n$,

$$(x^n)' = nx^{n-1}.$$

(2) (Constant multiple rule) For any constant $c$,

$$(cf)' = cf'$$

(3) (sum rule)

$$(f \pm g)' = f' \pm g'$$

(4) (product rule)

$$(fg)' = f'g + fg'$$

(5) (quotient rule) Given $g \neq 0$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

**Ex.** Prove product rule and quotient rule.

**Ex.** Differentiate:

(1) $\dfrac{d(4\pi^2)}{dx}$

(2) $(x^\pi)'$

(3) $\left(\dfrac{2}{\sqrt{x^3}}\right)'$

**Ex.** Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

**Ex.** Using $f(x) = x^2$ and $g(x) = x$, verify product rule and quotient rule.

**Ex.** Differentiate $y = \dfrac{\sqrt{x}}{1 + x^2}$.

The **Normal line** to a curve $C$ at point $P$ is the line through $P$ that is perpendicular to the tangent line at $P$.

**Ex.** Find equations of the tangent line and normal line to the curve

$$y = \frac{\sqrt{x}}{1+x^2}$$

at the point $(1, \frac{1}{2})$.

2.4. **Derivatives of trigonometric functions.** Text Section 2.4 Exercise: **8**, 9, 13, **31, 32, 36, 38**, 41, 45, 51

**Limit formula**

(1)

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

(2)

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

**Q.** What is

$$\lim_{\theta \to 0} \frac{\cos\theta}{\theta} = ?$$

**Derivatives of trigonometric functions**

(1)

$$(\sin x)' = \cos x$$

(2)

$$(\cos x)' = -\sin x$$

(3)

$$(\tan x)' = \sec^2 x$$

(4)

$$(\csc x)' = -\csc x \cot x$$

(5)

$$(\sec x)' = \sec x \tan x$$

(6)

$$(\cot x)' = -\csc^2 x$$

**Ex.** Let $f(x) = \cos x$. Find $f^{(27)}(x) =?$

**Ex.** Find

$$\frac{d}{dx}(x^2 \sin x)$$

**Ex.** Find

$$\lim_{x \to 0} \frac{\sin 7x}{4x}$$

**Ex.** Find

$$\lim_{x \to 0} x \cot x$$

2.5. **Chain rule.** Text Section 2.5 Exercise: 3, 5, **12, 13, 37, 40, 45**, 47, 61, **68, 71, 76**, 83, **85, 88, 89, 90**

The differential formulas learned before do not work for $F'(x)$, where

$$F(x) = \sqrt{x^2 + 1}.$$

Note that $F(x) = f(g(x))$, where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$.

**Proposition 2.4** (Chain rule)**.** *If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, then $F = f \circ g$ is differentiable at $x$, and*

$$F'(x) = f'(g(x)) \cdot g'(x).$$

In the above, $f'(g(x)) \neq (f(g(x)))'$, but can be understood as

$$f'(g(x)) = \left.\frac{df}{du}\right|_{u=g(x)}.$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Another representation of chain rule is

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

**Ex.** Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

**Ex.** Differentiate

(1)

$$y = \sin(x^2)$$

(2)

$$y = \sin^2 x$$

**Ex.** Differentiate

$$y = (x^3 - 1)^{100}.$$

**Ex.** Differentiate

$$f(x) = \sin(\cos(\tan x)).$$

2.6. **Implicit differentiation.** Text Section 2.6 Exercise: 1, **12, 15, 21, 22, 33, 34, 44, 45, 47**, 51, 53, 57

An **explicit function** is in the form of $y = f(x)$, i.e. $y$ is written explicitly as a function of $x$.

**Example 2.1** (Text excercise 3.6.29)**.** *A lemniscate is a curve given by*

$$2(x^2 + y^2)^2 = 25(x^2 - y^2).$$

*Find an equation of the tangent line to the curve at* $(3, 1)$*.*

Obviously the above equation of lemniscate is not an explicit function. We call such a function as **Implicit function**.

For implicit function, we may use **implicit differentiation**:

(1) Take $\dfrac{d}{dx}$ on both sides of equation.
(2) Solve for $\dfrac{dy}{dx}$.

**Ex.** Solve for Example 2.1, by following two steps of implicit differentiation.

**Ex.** Find slope of tangent line to $x^2 + y^2 = 25$ at the point $(3, 4)$ using following two ways:

(1) using implicit differentiation;
(2) using explicit differentiation.

**Ex.** Find $y''$ if $x^4 + y^4 = 16$.

2.7. **Rates of change and applications.** Text section 2.7 Exercise: 7, 9, **18, 19, 20,** 27, 29

Given $y = f(x)$, the **average rate of change** over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

while **instantaneous rate of change** at $x_1$ is

$$\left.\frac{dy}{dx}\right|_{x=x_1} = \lim_{x_2 \to x_1} \frac{\Delta y}{\Delta x}.$$

2.7.1. *Motion problem.* Let **position function** of a particle moving in a straight line be

$$s = s(t)$$

then **average velocity** over a time period $\Delta t$ is

$$\frac{\Delta s}{\Delta t}$$

**instantaneous velocity** is

$$v(t) = \frac{ds}{dt}$$

**acceleration** is

$$a(t) = v'(t) = s''(t).$$

**Ex.** The position of a particle is given by the equation

$$s(t) = t^3 - 6t^2 + 9t$$

where $t$ is measured in seconds and $s$ in meters.

(1) Find velocity at $t$.
(2) What is the velocity after 2 seconds? After 4 seconds?
(3) When is the particle at rest?
(4) When is the particle moving forward?
(5) Draw a diagram to represent the motion of the particle.
(6) Find the total distance traveled by the particle during the first five seconds.
(7) Find the acceleration at time $t$ and after 4 seconds
(8) Graph the position, velocity, acceleration functions for $0 \leq t \leq 5$.
(9) When is the particle speeding up? when is it slowing down?

2.7.2. *Marginal cost.* Suppose $C(x)$ is the total cost that a company incurs in producing $x$ units of a certain commodity. The function $C$ is called **cost function**.

If the number of items produced is increased from $x_1$ to $x_2$, additional cost is $\Delta C = C(x_2) - C(x_1)$, and **average rate of change in the cost** is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1}.$$

The instantaneous rate of change of the cost w.r.t. number of items produced is called **marginal cost**,

$$\text{marginal cost } = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}.$$

**Ex.** Suppose cost function (in dollars) is given by

$$C(x) = 10000 + 5x + 0.01x^2.$$

(1) Find the marginal cost $C'(x)$
(2) Find $C'(500)$, compare with $C(501) - C(500) = 15.01$

2.8. **Related rates.** Text Section 2.8 Exercise: **2**, 3, 9, 15, 19, **28, 33**, 45

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

**Ex.** Air is being pumped into a spherical balloon so that its volume increases at a rate of $100cm^3/s$. How fast is the radius of the balloon increasing when the diameter is 50 cm?

**Ex.** A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $1m/s$, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3m from the wall?

**Ex.** A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of $2m^3/min$, find the rate at which the water level is rising when the water is 3 m deep.

**Ex.** Car A is traveling west at 90 km/h and car B is traveling north at 100 km/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 60 m and car B is 80 m from the intersection?

**Ex.** A man walks along a straight path at a speed of 1.5 m/s. A searchlight is located on the ground 6 m from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 8 m from the point on the path closest to the searchlight?

**Problem Solving Strategy**

(1) Read the problem carefully.
(2) Draw a diagram if possible.
(3) Introduce notation. Assign symbols to all quantities that are functions of time.
(4) Express the given information and the required rate in terms of derivatives.
(5) Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
(6) Use the Chain Rule to differentiate both sides of the equation with respect to $t$.
(7) Substitute the given information into the resulting equation and solve for the unknown rate.