EE1001 Counting Part I

By Dr. K. F. Tsang

香港城市大學
City University of Hong Kong

Intended Learning Outcomes

- Upon completion of this session, you will be able to:

  ✓Identifying counting problems and solving them with fundamental counting principle.

  ✓Solving permutation and combination problems and identifying their difference.

  ✓Understanding the binomial theorem and its derivation process.

Chapter Contents

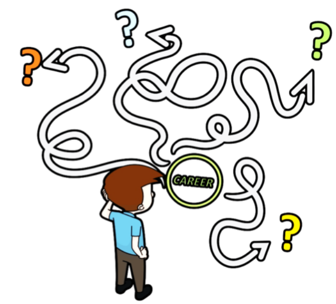
- Counting problem
- Permutation
- Combination
- The Binomial Theorem

1. Counting Problem

Why we need to study this Chapter?

- *Choices, choices, choices*
- People are bombarded with choices daily. Sometimes, it can be hard enough making up your mind about a single decision.
- But what about when you have to make multiple decisions at once?





- This chapter will help you determine how many different possible outcomes there are when you have to make multiple simultaneous decisions.
- *Therefore, you could deal with your situations from a "God's-eye view "*

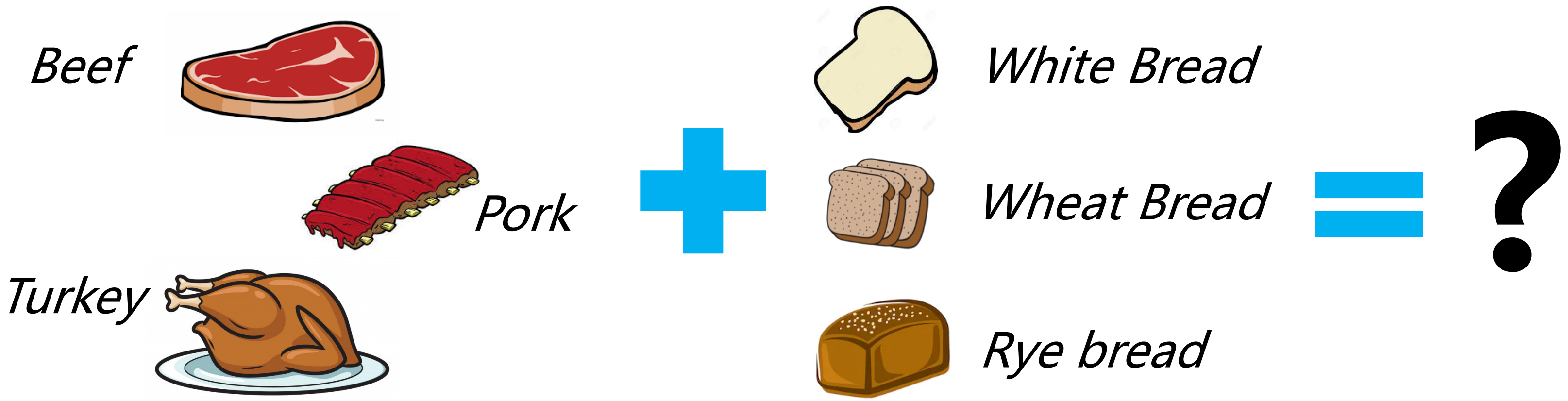
What is a counting problem?

- A counting problem asks ***the number of outcomes*** for a given situation.

E.g.:

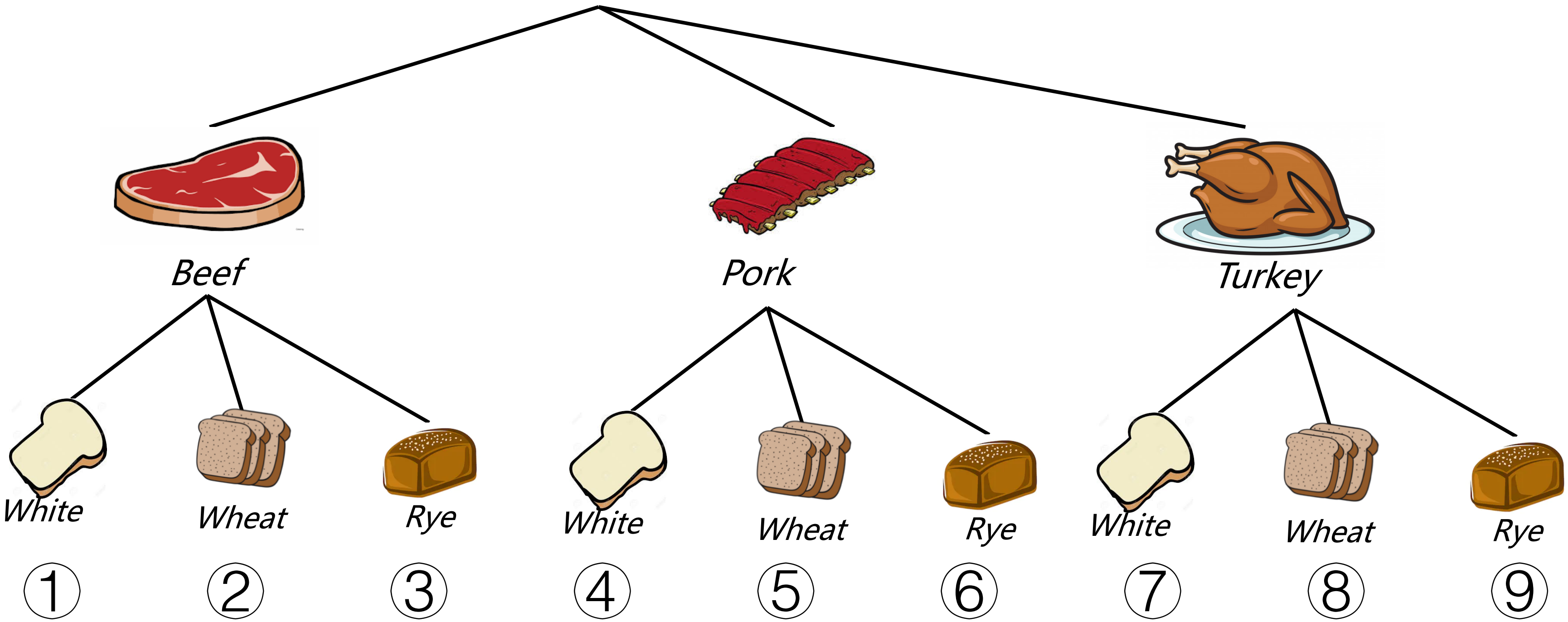
Considering you are making a sandwich. There are three types of meat (beef, pork, turkey) and three types of bread (white, wheat, rye) in your kitchen.

How many choices do you have for the sandwich?



Tree Diagram

- One way to answer this question is to use a *tree diagram.*



Based on the diagram, you can see that there are 9 choices.

What if there are hundreds types of meat and bread?

Fundamental Counting Principle

- If each event in a counting problem can occur in too many ways, the tree diagram is not effective.
- Fundamental Counting Principle:

*Suppose that a counting problem P can be broken into **n** successive ordered events, $S_1, S_2, \ldots S_n$, and suppose that :*

S_1 can occur in r_1 ways.

S_2 can occur in r_2 ways.

……

S_n can occur in r_n ways.

The number of outcomes of P is:

$$r_1 \cdot r_2 \cdots r_n$$

For the sandwich problem, number of possible sandwiches can be directly computed as:

$$\underbrace{3}_{\text{Choices for meat}} \cdot \underbrace{3}_{\text{Choices for Bread}} = 9$$

In-Class Exercises:

- The market has gift wrapping paper and ribbon on sale. The paper comes in red, orange, purple and yellow, and the ribbon in green and gold. If you buy one package of paper and one of ribbon, how many different color combinations can you choose?

Solution: ***4 •2= 8***

- Considering buying fruit trees for a garden. One apple, one orange and one cherry tree is planed. The nursery recommends two varieties of apple, six of orange, and five for cherry. How many possible different groups of trees could be planted?

Solution: ***2 • 6 • 5= 60***

- If the variety of cherry has been decided already, how many choices are left for the other trees?

Solution: ***2 • 6 = 12***

In-Class Exercises:

- Assuming that the standard configuration for a car license is 3 digits followed by 3 letters. For instance: *ABC123*
  - How many different license plates are possible if digits and letters can be repeated?

    Solution: $\boldsymbol{10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 17{,}576{,}000}$

  - How many different license plates are possible if digits and letters CANNOT be repeated?

Solution:

For the first digit, there are still 10 choices;

For the second digit, since one number has been employed by the first digit, there are only (10-1)=9 choice;

Similarly, the third digit only has (9-1)=8 choice;

…

The overall answer is:

$$\boldsymbol{10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24 = 11{,}232{,}000}$$

Generally, this is a ***permutation*** problem.

Daily Application:

Evaluating of the strength of your password

- The password strength is measured by length and complexity.
- The password is usually formed with:

26 Lower-case characters: a-z 10 numbers: 0-8

26 Capital characters: A-Z 34 Symbols: ; . “ $

- Assuming a password with length *n*

- The number of outcomes of passwords with only numbers 10^n
- The number of outcomes of passwords with only numbers and lower-case characters 36^n
- The number of outcomes of passwords with all numbers, lower-case characters, capital characters, numbers, and symbols: 96^n

- ***The most possible outcomes of passwords, the more times hackers will need to crack your passwords!***

→***Longer and more complex will be more secure***

2. Permutation

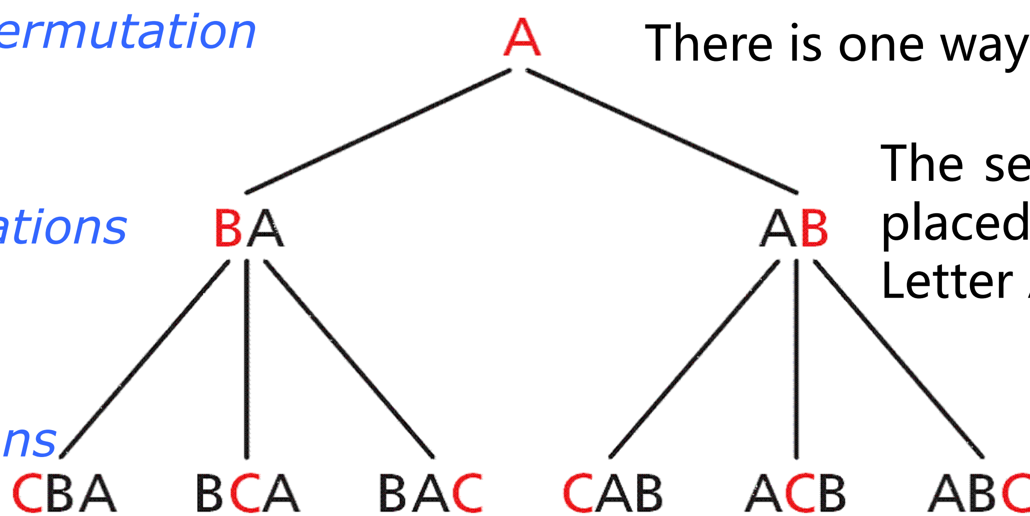
Permutation:

- A permutation is a selection of a group of objects with ***order***.
- The permutations of three letters A,B, and C are CBA, BCA, BAC, CAB, ACB, and ABC

1 permutation

2 permutations

3 permutations



There is one way to arrange one item A.

The second item B can be placed before or after Letter A.

The third item C can be first, second, or third for each order above.

- Therefore, the overall number of permutation is

$$3 \cdot 2 \cdot 1 = 6$$

Factorial notation

- The number of permutations of ***n*** items is:

$$n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 1$$

- Factorial notation is simply a short hand way of writing down some of these products.
- The symbol ***n!*** reads as 'n factorial' and means:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 1$$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

$$1! = 1 = 6$$

- **In this chapter, *0!* Is defined to be 1.**

Permutations of n objects taken r at a time

- Sometimes we may not want to order an entire set of items. Suppose that you want to select and order 3 people from a group of 7. With the Fundamental Counting Principle:

First Person		Second Person		Third Person		*Choosing 3 people from 7 in order*
7 choices	•	6 choices	•	5 choices	=	210 permutations

- This problem can also be regarded as there are 7 total people and 4 whose arrangements do not matter. ***By dividing the total number of arrangements by the number of arrangements that are not used***, the problem can also be solved:

$$\frac{\text{arrangements of 7}}{\text{arrangements of 4}} = \frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 210$$

Permutations of n objects taken r at a time

- With factorials, the number of permutations of *n* objects taken *r* at a time is:

$$
{}_nP_r = \frac{n!}{(n-r)!} = n \bullet (n-1) \bullet (n-2) \bullet \cdots \bullet (n-r+1)
$$

- E.g.

$$
{}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!}
$$

$$
{}_{10}P_5 = \frac{10!}{(10-5)!} = \frac{10!}{5!}
$$

In-Class Exercises:

- You are considering to visit 10 counties. In how many orders can you visit

(a) six of them?

(b) all of them?

Solution:

(a) The number of permutations of 10 objects taken 6 at a time is:

$$ {}_{10}P_6 = \frac{10!}{(10-6)!} = \frac{10!}{4!} = 151,200 $$

(b) The number of permutations of 10 objects taken 6 at a time is:

$$ {}_{10}P_{10} = \frac{10!}{(10-10)!} = \frac{10!}{0!} = 10! = 3,628,800 $$

Permutations with Repetition

- So far we have been finding permutations of distinct objects. In some cases, there may existing repeated objects and we need to find the ***distinguishable permutations***, which refers to the permutations different to each other.

- Considering *n* objects where *k* objects are repeated as:

Object 1 is repeated q_1 times.

Object 2 is repeated q_2 times.

……

Object k is repeated q_k times.

The number of distinguishable permutations is:

$$\frac{n!}{q_1!\bullet q_2!\bullet\cdots\bullet q_k!}$$

Permutations with Repetition

- Find the number of distinguishable permutations of the letters in "*food*".

Solution:

The word "*food*" has 4 letters of which O is repeated 2 times. So, the number of distinguishable permutations is:

$$\frac{4!}{2!} = \frac{24}{2} = 12$$

The 12 distinguishable permutations are:

- food
- fdoo
- fodo
- doof
- dfoo
- dofo
- oodf
- oofd
- odof
- odfo
- ofdo
- ofod

3. Combination

Combination

- Combinations is a technical term meaning '***selections***'. Therefore, the order of selected objects does ***not*** matter.
- For instance, the six permutations of three objects are all same in the combination.

  6 permutations $\rightarrow$ {ABC, ACB, BAC, BCA, CAB, CBA}

  1 combination $\rightarrow$ {ABC}

- The number of combinations of n elements taken r at a time is denoted as ${}_nC_r$ or $\binom{n}{r}$, which is read as 'n choose r'

Relationship between Combination and Permutation

- Recall the number of permutations of n objects taken r at a time $_nP_r$. With the combination, the formation of $_nP_r$ can be regarded as a two step progress:

Step 1: Select the combination $_nC_r$;

Step 2: For each combination, find the number of permutations r!;

(no. of ordered selections = no. of unordered selections × no. of ways of arranging them)

- Then, the relationship between combination and permutation is:

$$_nP_r = {_nC_r} \bullet r! \quad \Rightarrow \quad _nC_r = \frac{_nP_r}{r!} = \frac{n!}{(n-r)!\,r!}$$

In-Class Exercises:

- Write out in factorial notation and hence evaluate:

$$ {}_{7}C_{4} \qquad {}_{7}C_{3} \qquad {}_{7}C_{0} $$

In-Class Exercises:

- Write out in factorial notation and hence evaluate:

$$ {}_7C_4 \qquad {}_7C_3 \qquad {}_7C_0 $$

$$ {}_7C_4 = \frac{7!}{(7-4)!\,4!} = \frac{7!}{3!\,4!} = 35 $$

$$ {}_7C_3 = \frac{7!}{(7-3)!\,3!} = \frac{7!}{4!\,3!} = 35 $$

$$ {}_7C_0 = \frac{7!}{(7-0)!\,0!} = \frac{7!}{7!\,0!} = 1 $$

$$ {}_nC_r = {}_nC_{n-r} $$

This is because every time we selected *r* from *n* objects, the rest *(n-r)* objects is also the combination of selecting *(n-r)* from *n* objects

In-Class Exercises:

- There are 12 different-colored balls in a box. How many ways can we draw a set of 4 balls from the bag?

Solution:

Step 1: The order does not matter. The cubes may be drawn in any order. It is a combination.

Step 2: Use the formula for combinations

$$_{12}C_4 = \frac{12!}{4!(12-4)!} = \frac{12!}{4!(8!)}$$

n = 12 and r = 4

$$= \frac{12 \bullet 11 \bullet 10 \bullet 9 \bullet \cancel{8 \bullet 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}}{(4 \bullet 3 \bullet 2 \bullet 1)(\cancel{8 \bullet 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1})}$$

Divide out common factors.

$$= \frac{12 \bullet 11 \bullet 10 \bullet 9}{4 \bullet 3 \bullet 2 \bullet 1} = \frac{\cancel{12} \bullet 11 \bullet \cancel{10}^{5} \bullet 9}{\cancel{4} \bullet \cancel{3} \bullet \cancel{2} \bullet 1} = 495$$

Application of Permutation and Combination

- One of the most important application of permutation and combination is to compute a ***probability***.

  For example:

  Assuming there are 4 fruits: apple, orange, peach, and banana. You and your three classmates are required to pick one fruit one by one. You are the third to pick the fruit. What is the probability that you can pick your preferred banana?

The number of the permutations of the 4 fruits are: ${}_4P_4 = 24$

The number of the permutations of the 4 fruits that the third is banana are: ${}_3P_3 = 6$

(The number of permutations of others 3 fruits)

The probability of the third is banana is:

$$P = \frac{\text{the number of permunutations with the 3rd is banana}}{\text{the number of permunutations of 4 fruits}} = \frac{6}{24} = 0.25$$

Application of Permutation and Combination

Computation the probability of winning a lottery

In a lottery, assuming there are 50 balls with number from 1 to 50. You are required to pick 6 ball numbers. At the event day, the lottery machine will randomly select six balls. If your numbers are same with the selected number, you will win the lottery.

The combinations of 6balls from 50 balls is ${}_{50}C_6 = 6 = 15890700$

You can only pick one. Then the probability is:

$$P = \frac{1}{15890700} = 0.000006\%$$

4. The Binomial Theorem

Binomial coefficients

- Considering the expansion of $(x + y)^n$ for n=0,1,2....

$$(x+y)^0 = 1 \longleftarrow \text{1 term}$$

$$(x+y)^1 = x + y \longleftarrow \text{2 terms}$$

$$(x+y)^2 = x^2 + 2xy + y^2 \longleftarrow \text{3 terms}$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \longleftarrow \text{4 terms}$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \longleftarrow \text{5 terms}$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \leftarrow \text{6 terms}$$

- Each expansion has (n + 1) terms. (E.g. $(x+y)^{20}$ will have 21 terms)

Binomial coefficients

- Consider the exponents on x and y in each term in the expansion of $(x+y)^n$:

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

- For each term, the exponents of *x* and *y* add up to *n*
- The exponents on x decrease from n to 0.
- The exponents on y increase from 0 to n.

→The *i-th* term of $(x+y)^n$ is a term with $x^{n-i+1}y^{i-1}$

Example: The 5th term of $(x+y)^{10}$ is a term with x^6y^4.

Binomial coefficients

- The coefficients of the binomial expansion are called **binomial coefficients.** The coefficients have ***symmetry***.

$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

- If we write down just the coefficient of each term in the expansion above , we obtain the triangle known as Pascal' s triangle

  - **For each coefficient in the triangle can be obtained by adding the two numbers directly above it.**

$$\begin{array}{l ccccccccccc}
(x+y)^0 & & & & & & 1 & & & & & \\
(x+y)^1 & & & & & 1 & & 1 & & & & \\
(x+y)^2 & & & & \boxed{1} & & \boxed{2} & & 1 & & & \\
 & & & & & & & & 1+2=3 & & & \\
(x+y)^3 & & & 1 & & \boxed{3} & & 3 & & 1 & & \\
(x+y)^4 & & 1 & & 4 & & \boxed{6} & & \boxed{4} & & 1 & \\
 & & & & & & & & & & 6+4=10 & \\
(x+y)^5 & 1 & & 5 & & 10 & & \boxed{10} & & 5 & & 1
\end{array}$$

Binomial coefficients

- Let' s look more closely at how these coefficients are obtained:

$$(x+y)^2 = (x+y)(x+y)$$
$$= x^2 + x\,y + x\,y + y^2$$

- Notice that there are two ways of obtaining a term in x:

  - By choosing 'x' from the first bracket and 'y' from the second, and by choosing 'y' from the first bracket and 'x' from the second.

  - Hence, the coefficient of xy in the expansion of $(x+y)^2$ is $_2C_1$ = 2, the number of ways of choosing one 'x' from two brackets

Binomial coefficients

- Now, considering the expansion of $(x+y)^3$

$$(x+y)^2 = (x+y)(x+y)(x+y)$$

$$= x^3 + \underline{x^2 y} + \underline{x y x} + x y^2 + \underline{y x^2} + y x y + y^2 x + y^3$$

- We see that the terms involving x^2 are: $x^2 y$, xyx and $y\, x^2$ which when added together give $3x^2 y$.
- To obtain the coefficient of $x^2 y$, we are finding the number of ways of selecting two x and one y from the three brackets $(x+y)(x+y)(x+y)$.
- The number of ways of selecting two x from three brackets is:

$${}_3C_2 = \frac{3!}{(3-2)!\,2!} = \frac{3!}{1!\,2!} = 3$$

In-Class Exercises:

- Find the coefficients of x^3y^2, x^4y, and x^5 in the expansion of $(x + y)^5$.

Solution:

For the coefficient of x^3y^2, it is the number of selecting three x from five brackets. Therefore the coefficient of x^3y^2 is ${}_5C_3$=10

For the coefficient of x^4y, it is the number of selecting four x from five brackets. Therefore the coefficient of x^4y is ${}_5C_4$=5

For the coefficient of x^5, it is the number of selecting five x from five brackets. Therefore the coefficient of x^5 is ${}_5C_0$=5

We can check that these are correct by checking with the fifth row of Pascal's triangle.

The Binomial Theorem

- Based on former concepts, it is derived that:

*The coefficient of $x^r y^{n-r}$ in the expansion of $(x + y)^5$ is ${}_nC_r$. Therefore, the expression ${}_nC_r$ is called the **binomial coefficient**.*

- In addition, we can have the ***binomial theorem*** :

★ $$(x+y)^n = x^n + {}_nC_1 x^{n-1} y + {}_nC_2 x^{n-2} y^2 + {}_nC_3 x^{n-3} y^3 + \cdots + {}_nC_{n-1} x y^{n-1} + y^n$$
$$= \sum_{r=0}^{n} {}_nC_r x^r y^{n-r}$$

In-Class Exercises:

- Use the binomial theorem to expand $(x + 2y)^4$.

Solution:

Step 1: determine the number of terms, there are:

$$x^4,\ x^3 2y,\ x^2(2y)^2,\ x^1(2y)^3,\ (2y)^4$$

Step 2: determine the binomial coefficient for each term:

$$_4C_4=1,\ \ _4C_3=4,\ \ _4C_2=6,\ \ _4C_1=4,\ \ _4C_0=1$$

Step 3: write the expansion:

$$(x + 2y)^4 = x^4 + 4x^3 2y + 6x^2(2y)^2 + x^1(2y)^3 + (2y)^4$$

$$= x^4 + 8x^3 y + 24x^2 y^2 + 8xy^3 + 16y^4$$