# MA1300 Self Practice # 10

1. Suppose that a function $f$ is continuous on $[a,b]$ and $f''(x)$ exists for every $x \in (a,b)$. If $f(a) = f(b) = 0$ and $f(c) < 0$ for some point $c \in (a,b)$, prove that there exists some $\xi \in (a,b)$ such that $f''(\xi) > 0$.

2. Suppose that a function $f$ is continuous on $[a,b]$ and the derivatives $f'(a), f'(b)$ exist. If $f(a) = f(b) = 0$ and $f'(a) \cdot f'(b) > 0$, prove that there exists some $\xi \in (a,b)$ such that $f(\xi) = 0$.

3. Let $f$ be a continuous function on $[a,b]$. If it is differentiable on $(a,b)$, and it satisfies

$$f(a) \cdot f(b) > 0, \qquad f(a) \cdot f\left(\frac{a+b}{2}\right) < 0,$$

prove that for every real number $\beta$ there exists some $\xi \in (a,b)$ such that $f'(\xi) = \beta f(\xi)$.

4. Let $f$ be the function given by $f(x) = \frac{x}{x^2-x-2}$. Find $f^{(n)}(x)$ for any positive integer $n$ and $x \neq 2, -1$.

5. Let $f$ be a continuous function on the closed interval $[a,b]$ such that $f(a) = f(b) = 0$ and $f''(x)$ exists for every $x \in (a,b)$. Prove that for every $c \in (a,b)$, there exists some point $\xi \in (a,b)$ such that

$$f(c) = \frac{f''(\xi)}{2}(c-a)(c-b).$$

6. Let $f$ be a second differentiable function on $[0,1]$ satisfying $f(1) = 0$. If we define a function $F$ by $F(x) = x^2 f(x)$, prove that there exists some $\xi \in (0,1)$ such that

$$F''(\xi) = 0.$$

7. Assume that the function $f$ is continuous on $[a,b]$, differentiable on $(a,b)$ and $f'(x) \neq 1$. If $f(a) > a$ and $f(b) < b$, prove that the equation

$$f(x) = x$$

has one and only one root on the interval $(a,b)$.

8. If $f$ and $g$ are differentiable functions on $[a,b]$ such that

$$g'(x) \neq 0,$$

prove that there exists some $\xi \in (a,b)$ such that

$$\frac{f(a) - f(\xi)}{g(\xi) - g(b)} = \frac{f'(\xi)}{g'(\xi)}.$$

9. Assume that the function $f$ is continuous on $[1,2]$, and differentiable on $(1,2)$. If

$$f(1) = \frac{1}{2} \qquad \text{and} \qquad f(2) = 2,$$

prove that there exists some $\xi \in (1, 2)$ such that

$$f'(\xi) = \frac{2f(\xi)}{\xi}.$$

10. If $f$ is a differentiable function on $[0, c]$ such that $f(0) = 0$ and $f'(x)$ is decreasing, prove that for any $a, b$ satisfying the restriction

$$0 < a < b < a + b < c,$$

there holds

$$f(a) + f(b) > f(a + b).$$