Academic Honesty Pledge

I pledge that the answers in this exam/quiz are my own and that I will not seek or obtain an unfair advantage in producing these answers. Specifically,

- I will not plagiarize (copy without citation) from any source;
- I will not communicate or attempt to communicate with any other person during the exam/quiz; neither will I give or attempt to give assistance to another student taking the exam/quiz; and
- I will use only approved devices (e.g., calculators) and/or approved device models.

I understand that any act of academic dishonesty can lead to disciplinary action.

Student ID:____

Signature:____

Date:____

CITY UNIVERSITY OF HONG KONG

Department of Mathematics

Course Code & Title	:	MA1301 Enhanced Calculus and Linear Algebra II
Session	:	Semester B, 2020-2021
Time Allowed	:	Three Hours

This paper has **Three** pages. (including this cover page)

Instructions to candidates:

1. This paper has **Eight** questions.
2. Answer **ALL** questions.
3. Start each main question on a new page.
4. Show all steps.

---

*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable portable battery operated calculator.

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorized materials or aids are found on them.

1. (a) [6 marks] Compute

$$\lim_{n\to\infty}\frac{1}{n^3}\left[\sqrt{2n^2-1}+2\sqrt{2n^2-2^2}+3\sqrt{2n^2-3^2}+\cdots+n\sqrt{2n^2-n^2}\right].$$

(b) [6 marks] Let $f(x)=\displaystyle\int_{x\ln x}^{x^3}\frac{dt}{3+\ln t}$, find $f'(e)$.

(c) [6 marks] Compute $\displaystyle\int (x^2-1)\cos(2x)dx$.

(d) [6 marks] If f and g are inverse functions and f' is continuous. Prove that

$$\int_a^b f(x)dx = bf(b)-af(a)-\int_{f(a)}^{f(b)} g(y)dy.$$

2. [12 marks] For all $x>0$, define $\Gamma(x):=\displaystyle\int_0^\infty t^{x-1}e^{-t}dt$.

(a) Show that the integral is convergent for all $x>0$.

(b) Show that $\Gamma(x+1)=x\Gamma(x)$ for all $x>0$.

(c) Show that $\Gamma(n+1)=n!$ for $n=1,2,3,\cdots$.

3. [12 marks] Assume that two particles A and B repel each other with a force inversely proportional to the cube of the distance between them. Fix the position of the particle A. The work done in moving the particle B from a distance of 10 meters to a distance of 5 meters from the particle A is 48 N·m. Find the work done if we move the particle B from a distance of 10 meters to a distance of 1 meter from the particle A.

4. (a) [5 marks] Find the projection of the vector $\vec{a}=\vec{i}+2\vec{j}-\vec{k}$ on the vector $\vec{b}=-\vec{i}+3\vec{j}+\vec{k}$.

(b) [7 marks] Given the three vectors $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$, which are linearly independent. If $\vec{b}_1=\vec{a}_1+\vec{a}_2$, $\vec{b}_2=\vec{a}_2+\vec{a}_3$, and $\vec{b}_3=\vec{a}_3+\vec{a}_1$, show that the vectors $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3$ are linearly independent.

5. (a) [5 marks] Express $\dfrac{(\sqrt{3}+i)^{15}}{(1-i)^{29}}$ in $x+iy$ form.

(b) [5 marks] Show that the real parts of solutions to the equation $(z-1)^{2021}=z^{2021}$ are all equal to $1/2$.

6. [10 marks] Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

   (a) Find the values of λ such that $|A - \lambda I| = 0$.

   (b) Compute $A^n - 2A^{n-1}$ for $n \geq 2$.

7. [12 marks] Given the linear system

$$\begin{cases} ax - 2y - z = 1, \\ 2x + y + z = b, \\ 10x + 5y + 4z = -1. \end{cases}$$

   For which values of a and b, this system has (a) no solution, (b) a unique solution, (c) infinitely many solutions?

8. Let A and B be non-singular square matrices of order n. Show that

   (a) [2 marks] $(AB)^{-1} = B^{-1}A^{-1}$.

   (b) [3 marks] $(A^T)^{-1} = (A^{-1})^T$. Moreover, if A is symmetric, show that A^{-1} is also symmetric.

   (c) [3 marks] If $\det A = 2$, $\det B = 4$, compute $\det(3B^2A^{-1})$.

- End -