MA1301, Review for Final (2021-22, SemB)

Chapter 1-3

1. (Chapter 1) Definition of Riemann sum and definite integral.
2. (Chapter 1) Comparison properties of the integral .
3. (Chapter 2) Using integration to get Area, Volume, Average value.
4. (Chapter 3) Learn how to solve integrals:
5. (Chapter 3) Definition of improper integral.
6. (Chapter 3) Comparison Test for improper integrals

Chapter 4

1. Arc length of a curve $y = f(x)$:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

2. Area of surface generated by rotating $y = f(x)$ about $y = k$ ($f(x) > k$):

$$A = 2\pi \int_a^b (f(x) - k)\sqrt{1 + [f'(x)]^2} dx.$$

3. Use integrals to calculate moments and the center of mass.

Chapter 5

1. Magnitude of vector $\vec{a}$:

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

where $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$.

2. Change a vector $\vec{a}$ to a unit vector $\vec{n}$ with same direction:

$$\vec{n} = \frac{\vec{a}}{|\vec{a}|}$$

with $|\vec{a}| \neq 0$.

3. Scalar Product:

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

where $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

where $0 \leq \theta \leq \pi$ is the angle between two vectors.
If $\vec{a} \perp \vec{b}$, then $\theta = \pi/2$ and

$$\vec{a} \cdot \vec{b} = 0.$$

If $\vec{a} = \vec{b}$, then

$$\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{a} = |\vec{a}|^2.$$

4. Vector Product:

$$\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin\theta)\vec{n},$$

where $0 \leq \theta \leq \pi$ is the angle between two vectors, and $\vec{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.
If $\vec{a}$ is parallel to $\vec{b}$, then $\theta = 0$ and

$$\vec{a} \times \vec{b} = 0.$$

5. Projection vector of $\vec{a}$ onto $\vec{b}$:

$$Proj_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b}.$$

6. Distance from a point P to a line passing through A and B:

$$d = \sqrt{|\vec{AP}|^2 - |proj_{\vec{AB}}\vec{AP}|^2}.$$

7. Distance from a point D to a plane containing three points A, B and C:

$$d = |proj_{\vec{n}} \vec{AD}|,$$

where $\vec{n} = \vec{AB} \times \vec{AC}$.

8. Distance from a line passing through A and B to a line passing through C and D:

$$d = |proj_{\vec{n}} \vec{AD}|,$$

where $\vec{n} = \vec{AB} \times \vec{CD}$.

9 Area of Triangle ABC:

$$Area = |\vec{AC} \times \vec{AB}|/2.$$

10 Area of Parallelogram formed by $\vec{AB}$ and $\vec{AC}$:

$$Area = |\vec{AC} \times \vec{AB}|.$$

If A, B and C are collinear, then

$$Area = |\vec{AC} \times \vec{AB}| = 0.$$

11 Volume of Parallelepiped formed by A, B, C and D:

$$Volume = |\vec{AB} \cdot (\vec{AC} \times \vec{AD})|.$$

If A, B, C and D are coplanar, then

$$Volume = |\vec{AB} \cdot (\vec{AC} \times \vec{AD})| = 0.$$

12 Definition of Linearly Independent (Linearly dependent). How to check it in R^2 and R^3.

Chapter 6

1. Division between complex numbers

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd+(bc-ad)i}{c^2+d^2}.$$

2. Polar form

$$z = a + bi = r(\cos\phi + i\sin\phi)$$

with the modulus $r = \sqrt{a^2+b^2} \geq 0$ and principle value $(-\pi < \phi \leq \pi)$ of argument can be calculated by following method:

3. Multiplication and division of complex numbers in polar form

$$z_1 z_2 = r_1(\cos\phi_1 + i\sin\phi_1) r_2(\cos\phi_2 + i\sin\phi_2) = r_1 r_2(\cos(\phi_1+\phi_2) + i\sin(\phi_1+\phi_2)),$$

$$z_1/z_2 = \frac{r_1(\cos\phi_1 + i\sin\phi_1)}{r_2(\cos\phi_2 + i\sin\phi_2)} = \frac{r_1}{r_2}(\cos(\phi_1-\phi_2) + i\sin(\phi_1-\phi_2)).$$

Remark: $\phi_1+\phi_2$ and $\phi_1-\phi_2$ may not be principle values.

4. Euler Form

$$z = r(\cos\phi + i\sin\phi) = re^{i\phi}.$$

5. Multiplication and division of complex numbers in Euler form

$$z_1 z_2 = r_1(e^{i\phi_1}) r_2(e^{i\phi_2}) = r_1 r_2 e^{i(\phi_1+\phi_2)},$$

$$z_1/z_2 = \frac{r_1(e^{i\phi_1})}{r_2(e^{i\phi_2})} = \frac{r_1}{r_2} e^{i(\phi_1-\phi_2)}.$$

6 Key examples

$$i = e^{i\pi/2}, \ -1 = e^{i\pi}$$

$$e^{ia} \pm e^{ib} = e^{i(a+b)/2}\left(e^{ia-i(a+b)/2} \pm e^{ib-i(a+b)/2}\right) = e^{i(a+b)/2}\left(e^{i(a-b)/2} \pm e^{-i(a-b)/2}\right)$$

$$2\cos\phi = e^{i\phi} + e^{-i\phi}, \ 2i\sin\phi = e^{i\phi} - e^{-i\phi},$$

7. Relations among three different forms.
8. DeMoivre's Theorem (n, m are integers):

$$z^{n/m} = (r(\cos\phi + i\sin\phi))^{n/m} = (r^n(\cos(n\phi) + i\sin(n\phi)))^{1/m}$$

$$= r^{n/m}\left(\cos\frac{2k\pi + n\phi}{m} + i\sin\frac{2k\pi + n\phi}{m}\right) \text{ for } k = 0, 1, ..., m-1.$$

9. Definition of nth root of unity $w^n = 1$.
10. Application of complex numbers:
identities of trigonometric functions: Binomial Theorem vs. DeMoivre's Theorem
11. Using complex conjugate to obtain roots of polynomials:
If $z = a + bi$ is a root of a polynomial function, the complex conjugate $\bar{z} = a - bi$ is also a root of the function.

Chapter 7

1. Multiplication of matrices A, $m \times p$ matrix, and B, $p \times n$ matrix:

$$AB = \begin{pmatrix} a_{11} & ... & a_{1p} \\ ... & ... & ... \\ a_{i1} & ... & a_{ip} \\ ... & ... & ... \\ a_{m1} & ... & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & ... & b_{1j} & ... & b_{1n} \\ ... & ... & ... & ... & ... \\ b_{p1} & ... & b_{pj} & ... & b_{pn} \end{pmatrix} = C,\ m \times n \text{ matrix}$$

$$= \begin{pmatrix} ... & ... & ... \\ ... & C_{ij} & ... \\ ... & ... & ... \end{pmatrix}$$

where $C_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{ip}b_{pj}$.

2. Transpose of matrix:

$$\begin{pmatrix} a_{11} & ... & a_{1p} \\ ... & ... & ... \\ a_{i1} & ... & a_{ip} \\ ... & ... & ... \\ a_{m1} & ... & a_{mp} \end{pmatrix}^T = \begin{pmatrix} a_{11} & ... & a_{i1} & ... & a_{m1} \\ ... & ... & ... & ... & .... \\ a_{1p} & ... & a_{ip} & ... & a_{mp} \end{pmatrix}$$

3. Definitions of upper (lower) triangular matrix, diagonal matrix, symmetric matrix, anti-symmetric matrix and identity matrix.

4. Determinant of matrix:

(2×2 matrix):

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

(3×3 matrx):

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

($n \times n$ matrix). **5**. Cofactor matrix of A.

6. Definition of inverse matrix.

Inverse of square matrix:

If $\det A \neq 0$, then inverse of A exists (A is non-singular, A is invertible).

If $\det A = 0$, then inverse of A does not exist (A is singular, A is not invertible).

Inverse of $A =$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & ... & A_{1m} \\ ... & ... & .... \\ A_{m1} & ... & A_{mm} \end{pmatrix}^T$$

where A_{ij} is the cofactor of the matrix A.
7. Definition of non-homogeneous system and homogeneous system.
8. A system of linear equations is consistent if the system has at least one solution (one or infinitely many).
A system of linear equations is inconsistent if the system has no solution.
9. Matrix representation of the system:

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \dots \\ b_m \end{pmatrix}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \dots & \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

10. Gaussian Elimination and reduced row echelon form: Definitions of pivot and elementary row operations.
11. In the reduced row echelon form,

- **Case 1**: No solution (inconsistent)
  There is a row $(0\ 0 \dots\ 0|b)$ where $b \neq 0$.

- **Case 2**: Infinitely many solutions (consistent)
  Not **Case 1** and there is a column with no pivot.

- **Case 3**: Only one solution (consistent)
  Not **Case 1** and there is no column with no pivot.

12. Three methods to solve a system of linear equations:

- **Method 1**: By the inverse of Matrix
  Only for square coefficient matrix and **Case 3**.

- **Method 2**: Gaussian Elimination
  For any case.

- **Method 3**: Cramer's Rule
  Only for square coefficient matrix and **Case 3**.

12. Applications of Gaussian Elimination:
a. Finding inverse (Gauss Jordan Method).
b. Checking the linear independency of vectors.