
WEEK 3/4

CHAPTER ONE

- Points and Lines in the Plane

OUTLINE

- **Foundations of Geometry**
- **Points on Cartesian Plane**
- **The gradient/slope of a line**
- **Distance between two points**
- **Internal and external division of a line segment**
- An equation of a line
- Distance from a line to a point
- Parametric equation of a line
- Distance between two points and angle between direct line segments

FOUNDATIONS OF GEOMETRY

- Introduction to Proof
- Axioms, Axiomatic Systems
- Incidence Axioms for Geometry
- Axioms for points, lines and planes

INTRODUCTION TO PROOF

Syllogism : The abstract form

1. All A is B

2. X is A

3. ∴ X is B

EXAMPLE

All whales are mammals

Kimmy is a whale

Kimmy is a mammal.

Remark : Deductive reasoning : the basis for moving from the general to the particular

Inductive reasoning : moving from particular to general

IF-THEN STATEMENTS, CONDITIONALS

$p \to q$ "p implies q". If p then q.

① p : hypothesis

q : conclusion

ie. p = covid.

$\sim p$: no covid.

② Negate a statement ($\sim p$)

③ Converse, Contrapositive.

Given condition $p \to q$

its converse $q \to p$

contrapositive : $\sim q \to \sim p$

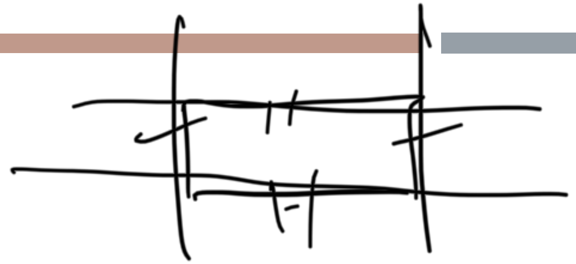
EXAMPLE

Statement: If 2 angles are congruent, then they have the same measure.

Converse: If 2 angles have the same measure, then they are congruent

Inverse: If 2 angles are not congruent, then they do not have the same measure

Contrapositive: If 2 angles don't have the same measure, then they are not congruent.



EXAMPLE

Statement: If a quadrilateral is a rectangle, then it has 2 pairs of parallel sides.

(F) Converse: If a quadrilateral has 2 pairs of parallel sides then it is a rectangle.

Inverse: If a quadrilateral is not a rectangle, then it doesn't have 2 pairs of parallel sides

(T) Contrapositive: If a quadrilateral doesn't have 2 pairs of parallel sides, then it is not a rectangle.

LOGICALLY EQUIVALENT (q characterizes p)

$p \to q$ and $q \to p$

$p \leftrightarrow q$

DIRECT PROOFS

1. p implies q

2. q implies r

3. $\therefore$ p implies r.

Ex. Given $\angle A$ and $\angle B$ are right angles.

Prove: $\angle A$ is congruent to $\angle B$

Proof conclusion	Justifications.
(1) The measures of $\angle A$ and $\angle B$ are each $90°$	Definition (of right angles)
(2) $m\angle A = m\angle B$	Algebra
(3) $\therefore \angle A \cong \angle B$	Definition of congruence.

INDIRECT PROOFS

Prove by contradiction.

Show that p implies q, one assumes p and not q, then proceeds to show that not p follows then this is a contradiction

Ex. Trichotomy property of a real numbers: For any real numbers a and b, either $a<b$, $a=b$ or $a>b$

we want to show $x=0$ then we can show that it is impossible that $x>0$ and $x<0$

AXIOMS, AXIOMATIC SYSTEMS

- An axiomatic system always contains statements which are assumed without proof-the axioms. These axioms are chosen
  (a) for their convenience and efficiency
  (b) fir their consistency and, in some cases, (but not always)
  (c) for their plausibility
- Undefined terms
  Every axiom must, of necessity, contain some terms that have been purposely left without definitions- the undefined terms.

For example, in geometry, the most common undefined terms are “point” and “line.”
In reality, a point is a dot with physical dimension, but ideally in geometry, it has no dimension.
A line is that has length without width.

MODELS FOR AXIOMATIC SYSTEMS

- A Model for an axiomatic system is a realization of the axioms in some mathematical setting. All undefined terms are interpreted, and all the axioms are true.
- Independence and consistency in axiomatic systems
  An axiomatic system must be independent (every axiom is essential, none is a logical consequence of the others).
  and consistent (freedom from contradictions).

EXAMPLE

axiomatic system:

undefined terms: member, committee.

Axioms: 1. Every committee is a collection of at least 2 members

2. Every member is on exactly one committee.

Find 2 distinctly different models for this set of axioms, and discuss how it might be made categorical.

let one model be

members: John, Dave, Robert, Mary, Kathy and Jane.

committee: A: John and Robert

B: Dave and Mary

C: Kathy and Jane.

non-categorical

Another model

members: {a, b, c, d, e, f, g, h}

committees: A: {a, b, c, d}

B: {e, f}

C: {g, h}

⋮

Extension

Axiom: There exist 3 committees and 6 members.

Now categorical

All models will be similar, every model consists of 3 committees w/ 2 members on each.

INCIDENCE AXIOMS FOR GEOMETRY

Let S and T be 2 sets.

1. membership : $x \in S$
2. subset : $S \subseteq T$
3. Intersection : $S \cap T$
4. union : $S \cup T$

universal set : S : all points in space.

a line : ℓ ($\ell \subseteq S$)

a plane : P :
1) a line lie in a plane $\ell \subseteq P$
2) a line pass through a plane at one point $\ell \cap P = \{A\}$
3) a line parallel to a plane : $\ell \cap P = \emptyset$

Incidence axioms : axioms govern how points, lines and plane interact

undefine terms : points, line, plane, space 3D

AXIOMS FOR POINTS, LINES AND PLANES

- Axiom I: Each two distinct points determine a line.

Remark.

1. notation for a line : $\overleftrightarrow{AB}$ given two two distinct points A and B.
2. since $\overleftrightarrow{AB}$ is a set of points,

   $A \in \overleftrightarrow{AB}$ $B \in \overleftrightarrow{AB}$ $\therefore \{A, B\} \subseteq \overleftrightarrow{AB}$
3. The line is unique
4. This axiom doesn't assume the concept of infinite points on on a line.

EXAMPLE TO PROOF A THEOREM BY AXIOM

Theorem: If $C \in \overleftrightarrow{AB}$, $D \in \overleftrightarrow{AB}$ and $C \neq D$ then

$\overleftrightarrow{CD} = \overleftrightarrow{AB}$

proof

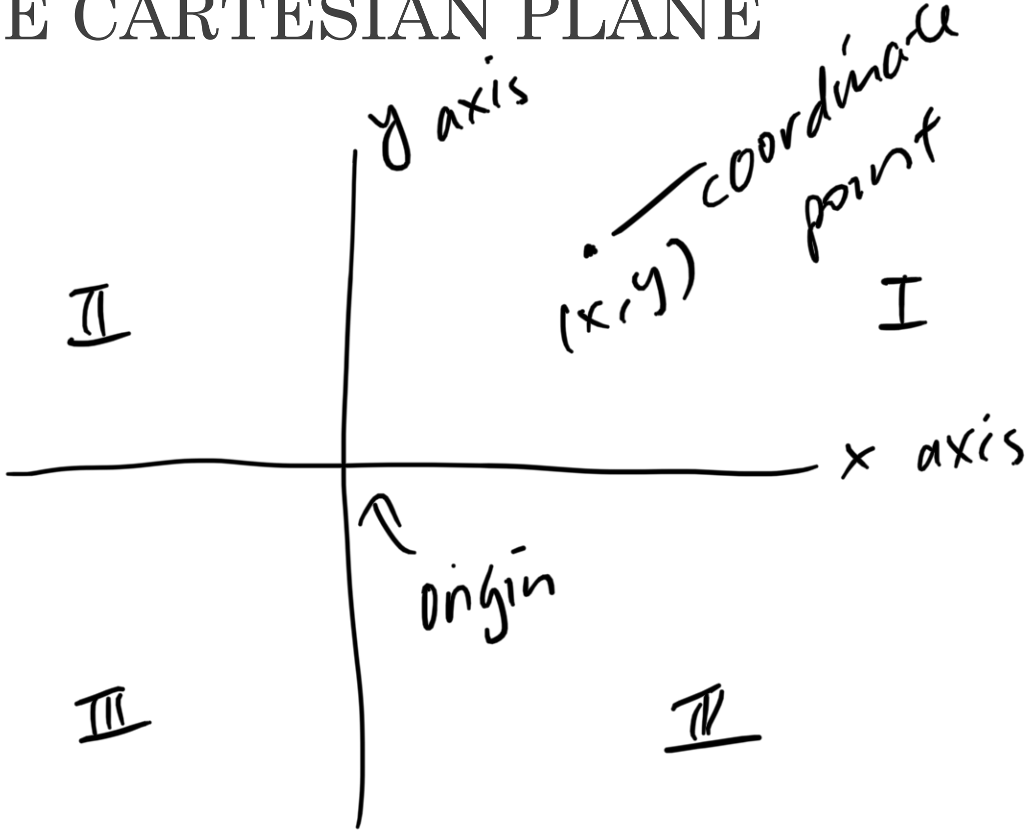
Since $C, D \in \overleftrightarrow{AB}$, so $\overleftrightarrow{AB}$ passes C, D as well.

By Axiom 1, $\overleftrightarrow{CD} = \overleftrightarrow{AB}$.

FAMOUS AXIOMS

- Axiom II: Three noncollinear points determine a (unique) plane.
- Axiom III: If two points lie in a plane, then any line containing those two points lies in that plane.
- Axiom IV: If two distinct planes meet, their intersection is a line.

THE CARTESIAN PLANE



Quadrant I : (+, +)

II : (−, +)

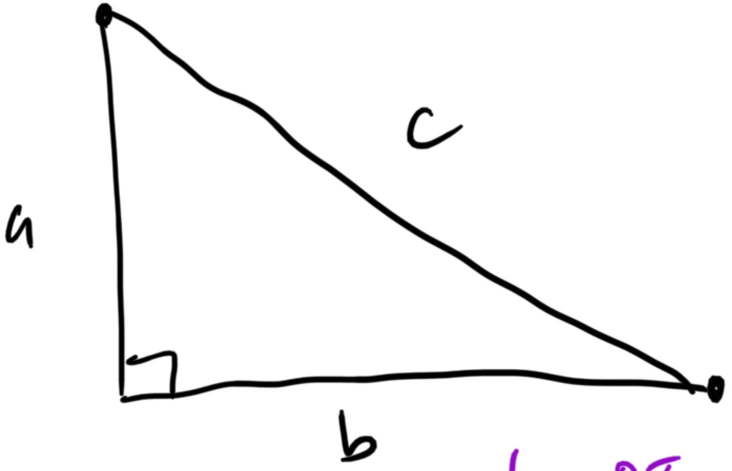
III : (−, −)

IV : (+, −)

LENGTH OF A STRAIGHT LINE

Pythagora's Theorem (PT)

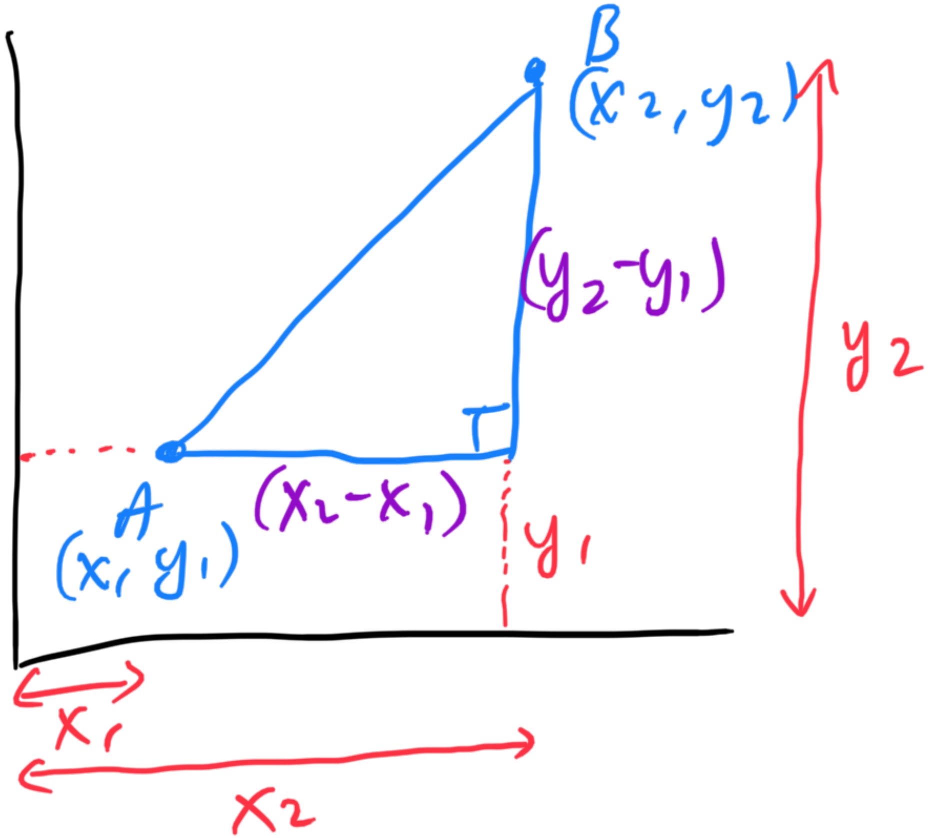
Right angle theorem



$a^2+b^2=c^2$

by PT

$AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$



EXAMPLE

$$\overset{x_1\ \ y_1}{A = (2,3)} \qquad \overset{x_2\ \ y_2}{B = (5,7)}$$

calculate length of AB

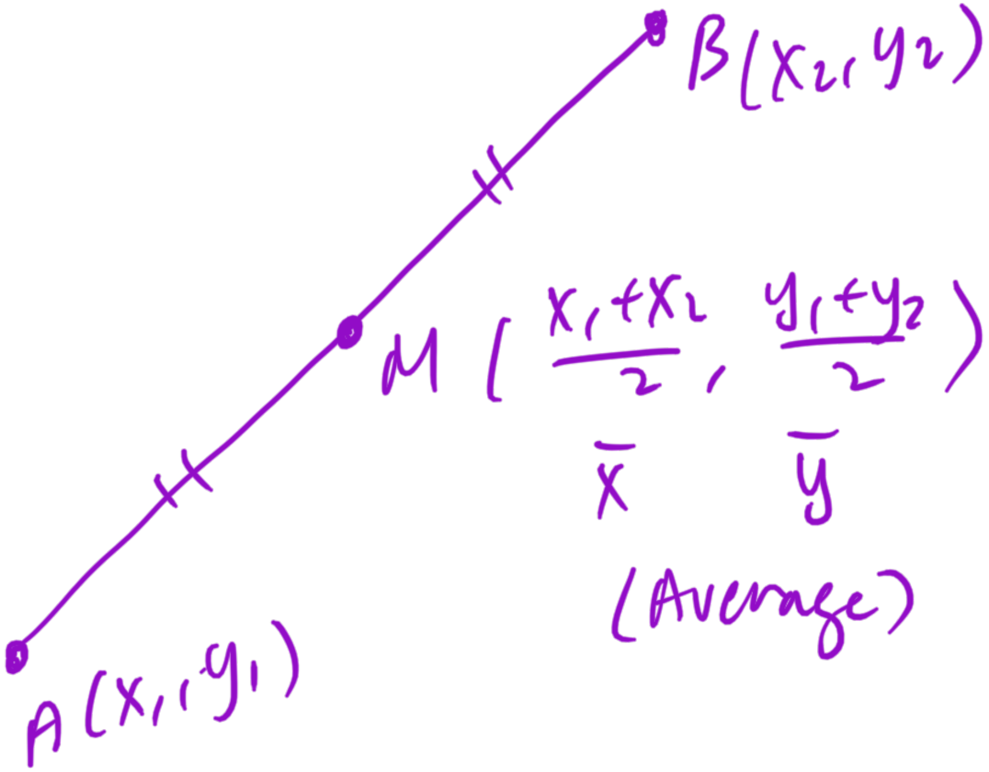
Ans

$$AB = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$= \sqrt{(5-3)^2 + (7-3)^2}$$

$$= \sqrt{9+16} = \sqrt{25} = 5 \text{ units.}$$

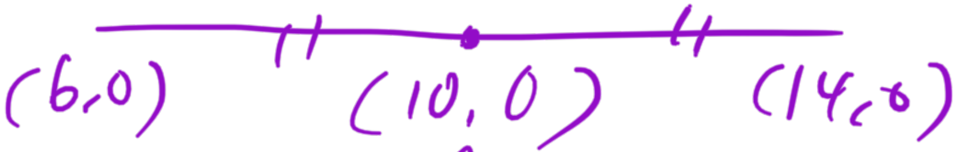
MIDPOINT OF A STRAIGHT LINE



$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$\bar{x}$ $\bar{y}$

(Average)



$$\frac{6+14}{2} = 10$$

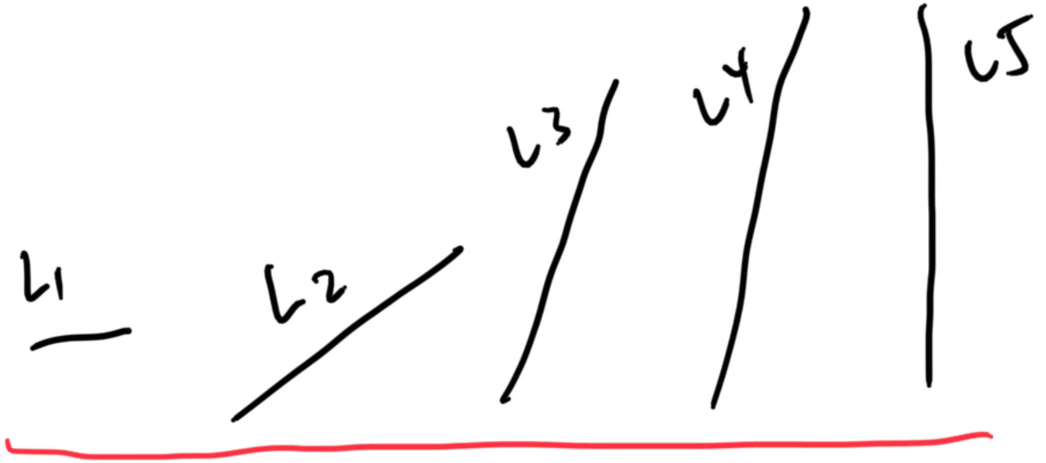
EXAMPLE

If $A = (\overset{x_1}{3}, \overset{y_1}{4})$ and $B = (\overset{x_2}{7}, \overset{y_2}{6})$ calculate the coordinates of midpoint of AB

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{3+7}{2}, \frac{4+6}{2}\right) = (5, 5)$$

(Slope)

GRADIENT OF A STRAIGHT LINE



$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Steepness is increasing from L1 to L5

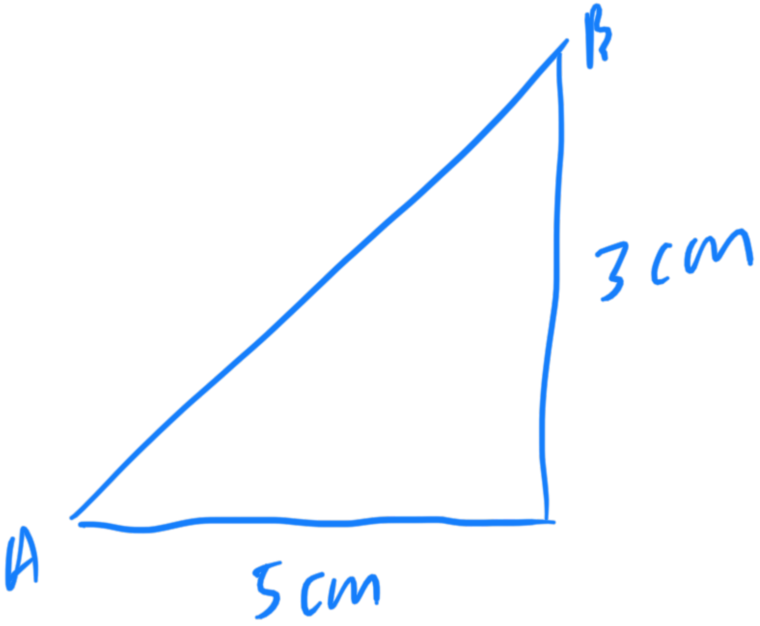
$\left\{\begin{array}{l}\text{Gradient largest at L5} \\ \text{and smallest at L1}\end{array}\right\}$ Absolute value

Ex.

$m = 5$ and -5

Same steepness because same # but different direction

CALCULATING GRADIENT USING MEASUREMENT



$$m = \frac{\text{vertical}}{\text{horizontal}} = \frac{3\text{cm}}{5\text{cm}} = \frac{3}{5}$$

CALCULATING GRADIENT FROM COORDINATES

$A(x_1, y_1)$ and $B(x_2, y_2)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

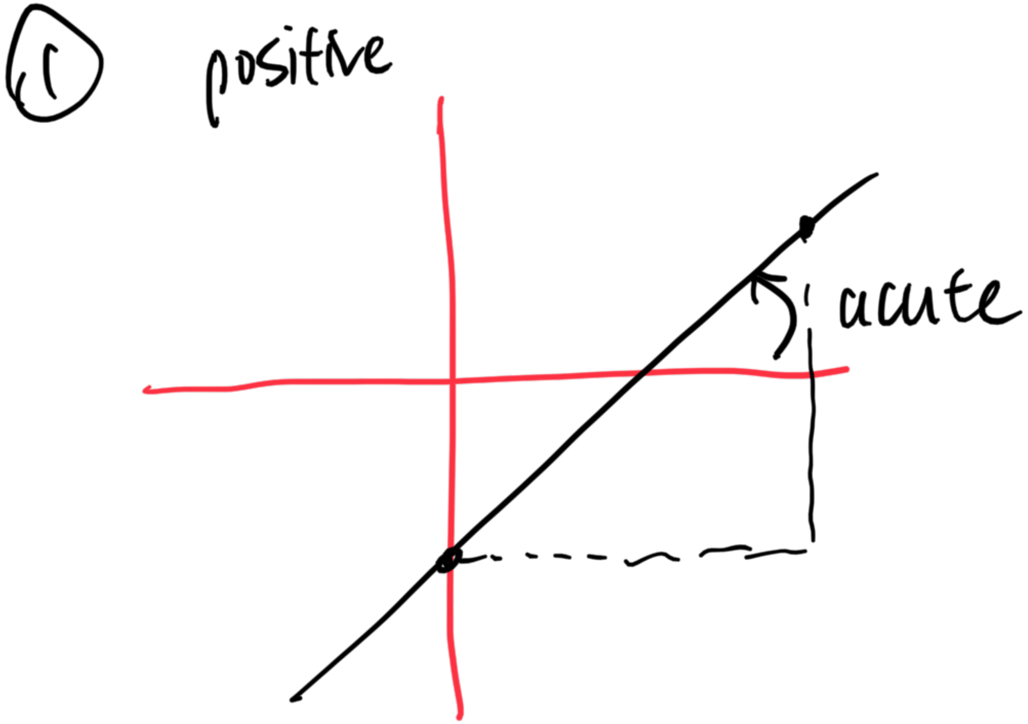
EXAMPLE

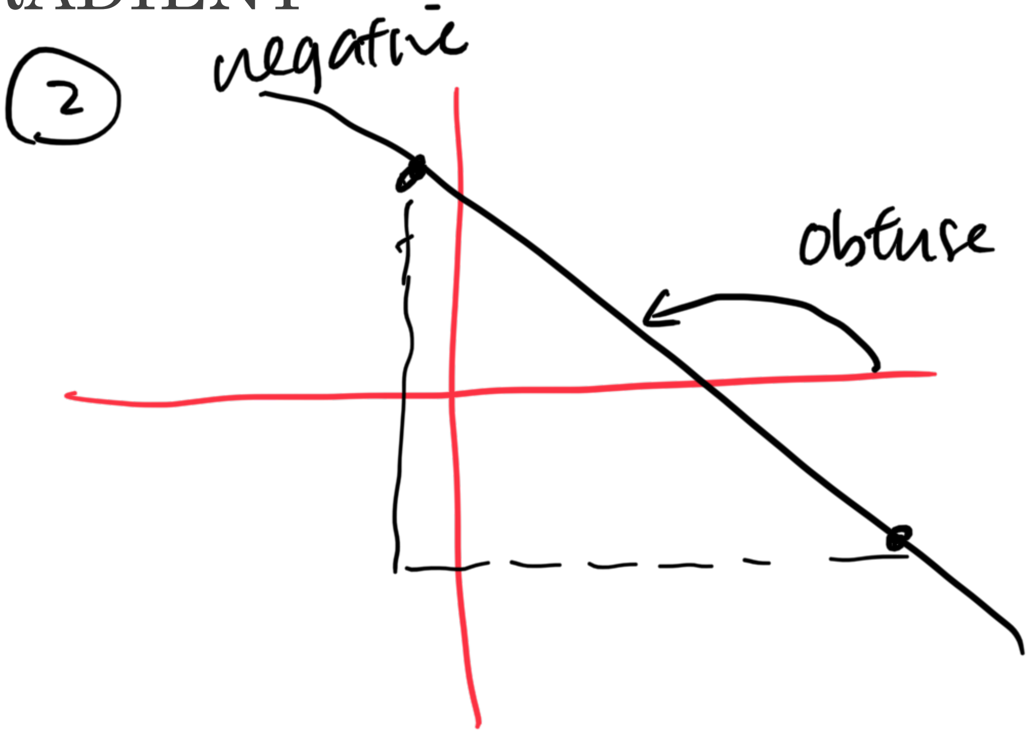
Find gradient points $(1,4)$ and $(6,7)$

$x_1 \; y_1 \qquad x_2 \; y_2$

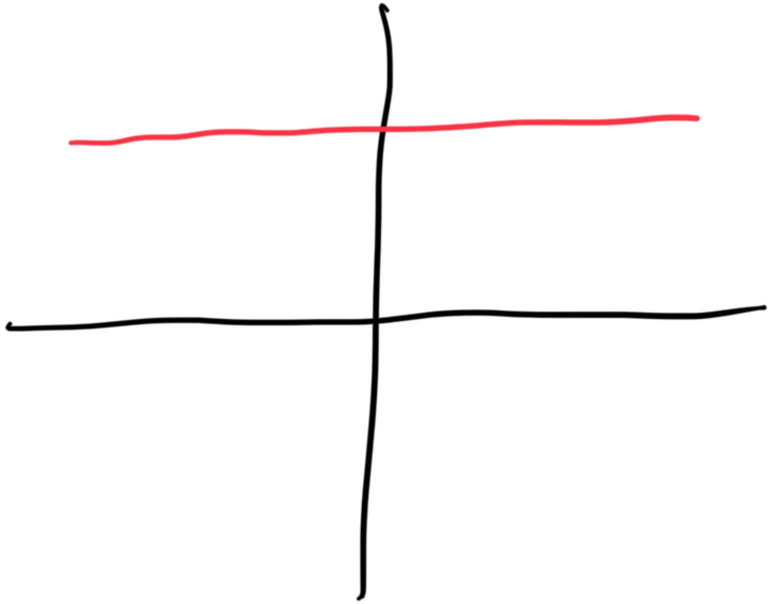
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-4}{6-1} = \frac{3}{5}$$

POSITIVE AND NEGATIVE GRADIENT

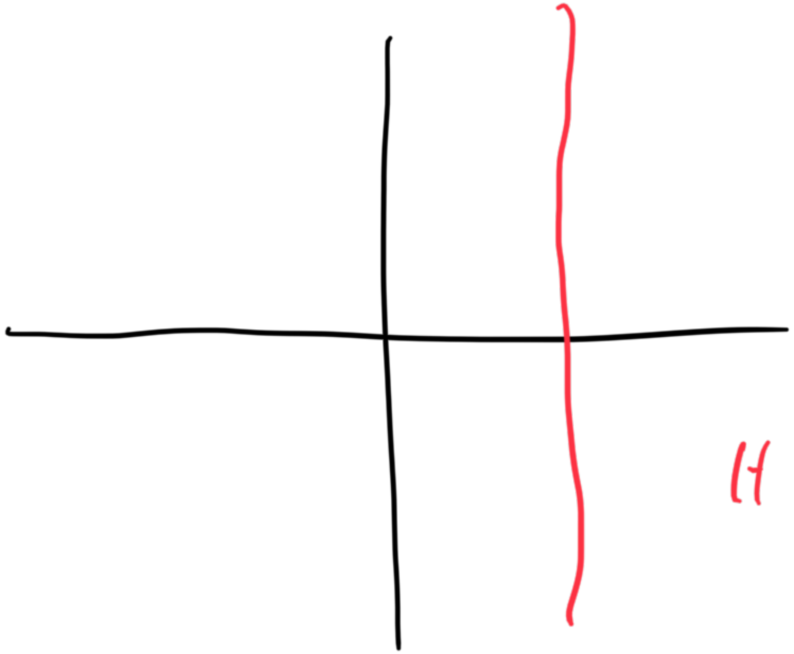




GRADIENT OF HORIZONTAL AND VERTICAL LINES



$m = 0$

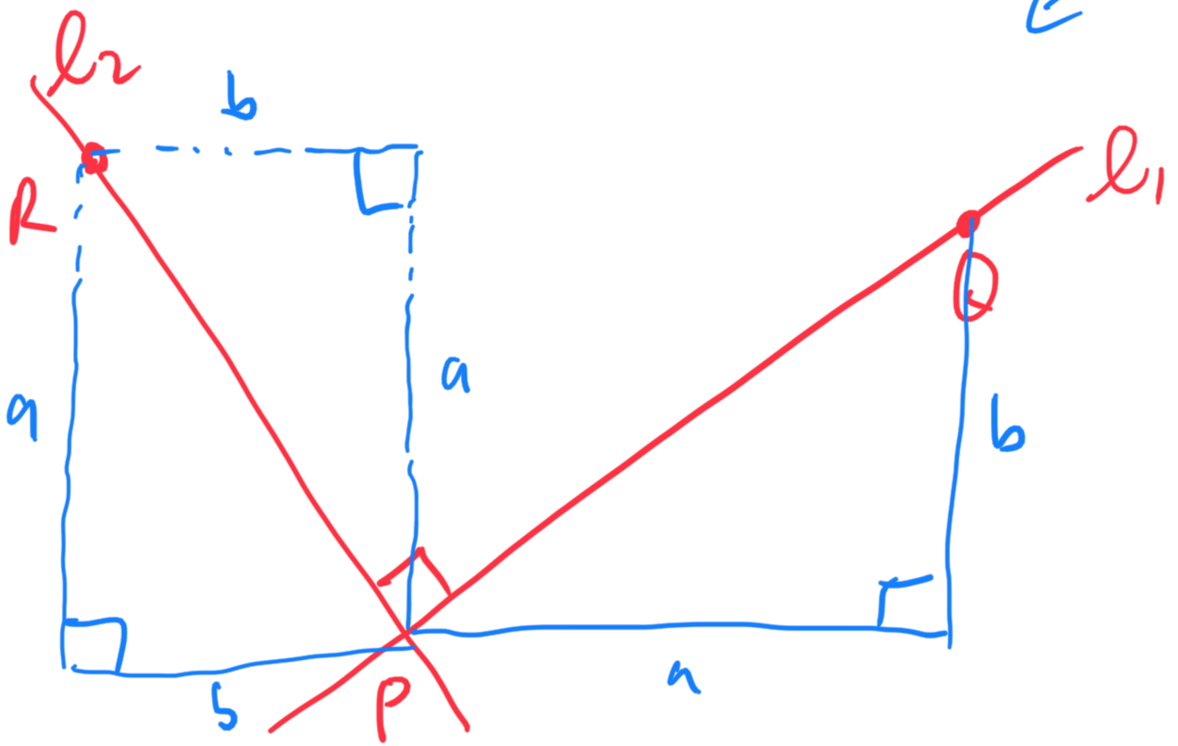


It has "rise" but no run.

$m = \infty$

GRADIENT OF PARALLEL LINES AND PERPENDICULAR LINES

same slope



PQ gradient $= \frac{b}{a}$

PR gradient $= -\frac{a}{b}$

product of gradient

$$\frac{b}{a} \times -\frac{a}{b} = -1$$

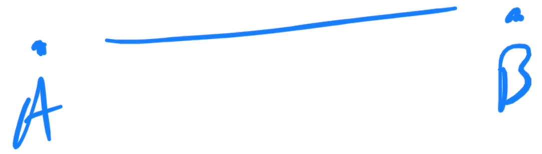
EXAMPLE

$P(0,3)$ $Q(-2,0)$ ← given this points forming a line

1) parallel gradient $= \frac{3}{2}$

2) perpendicular gradient $= -\frac{2}{3}$

DISTANCE BETWEEN TWO POINTS



DISTANCE FORMULA

Recap go to pg. 22.

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

DIVISION OF A LINE SEGMENT

p 38

FIRST CASE

$$\frac{AH}{CD} = \frac{AC}{CB} = \frac{m}{n}$$

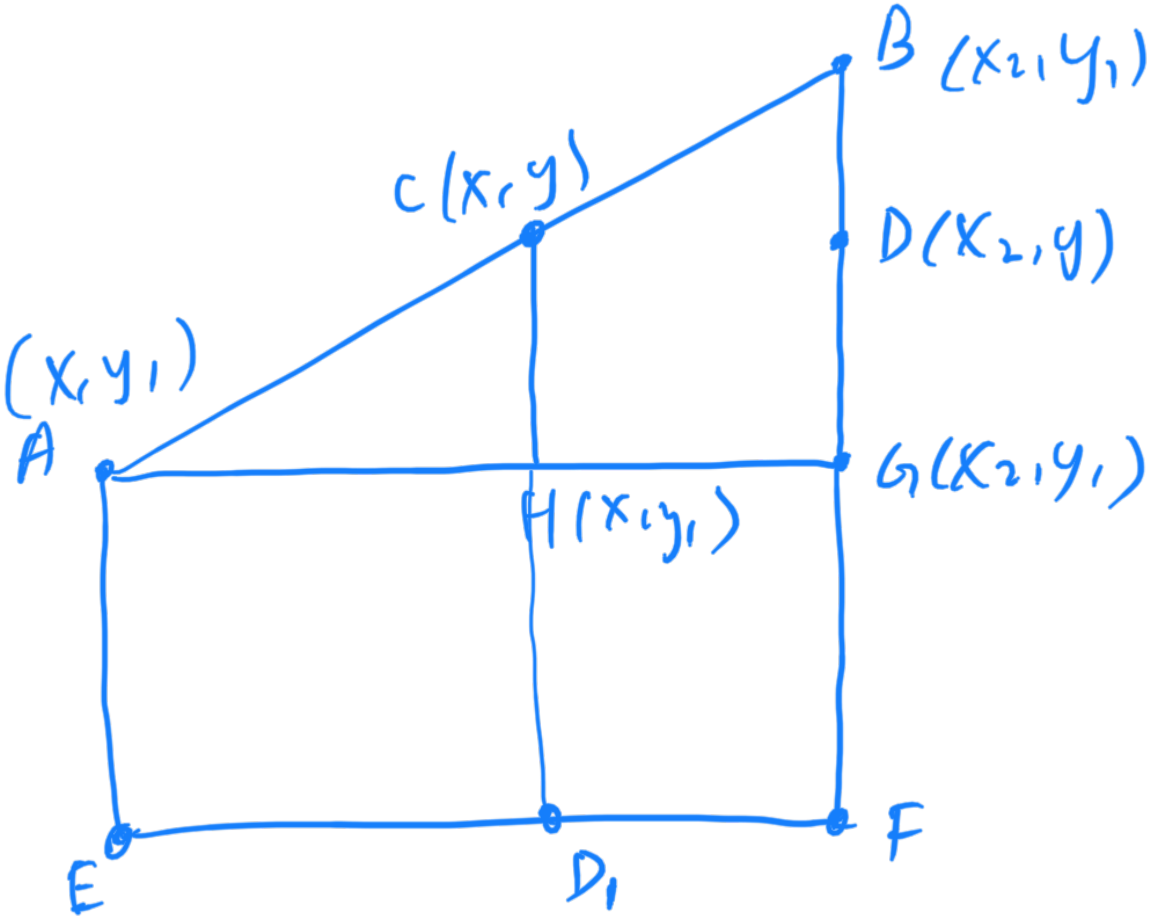
$$\frac{x - x_1}{x_2 - x} = \frac{m}{n}$$

$$n(x - x_1) = m(x_2 - x)$$

$$nx - nx_1 = mx_2 - mx$$

$$nx + mx = mx_2 + nx_1$$

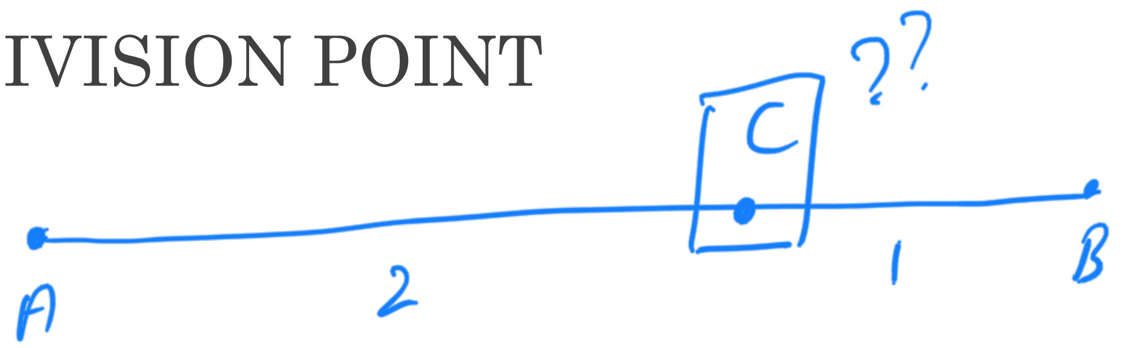
$$(n+m)x = mx_2 + nx_1$$



$$x = \frac{mx_2 + nx_1}{n+m}$$

p37

INTERNAL DIVISION POINT



Suppose that we find a point C on AB which divides AB in the ratio of 2:1

$AC : CB$
$2 : 1$

$\therefore$ C is in AB and it divides AB **internally**.

But how do we find C??

SECOND CASE

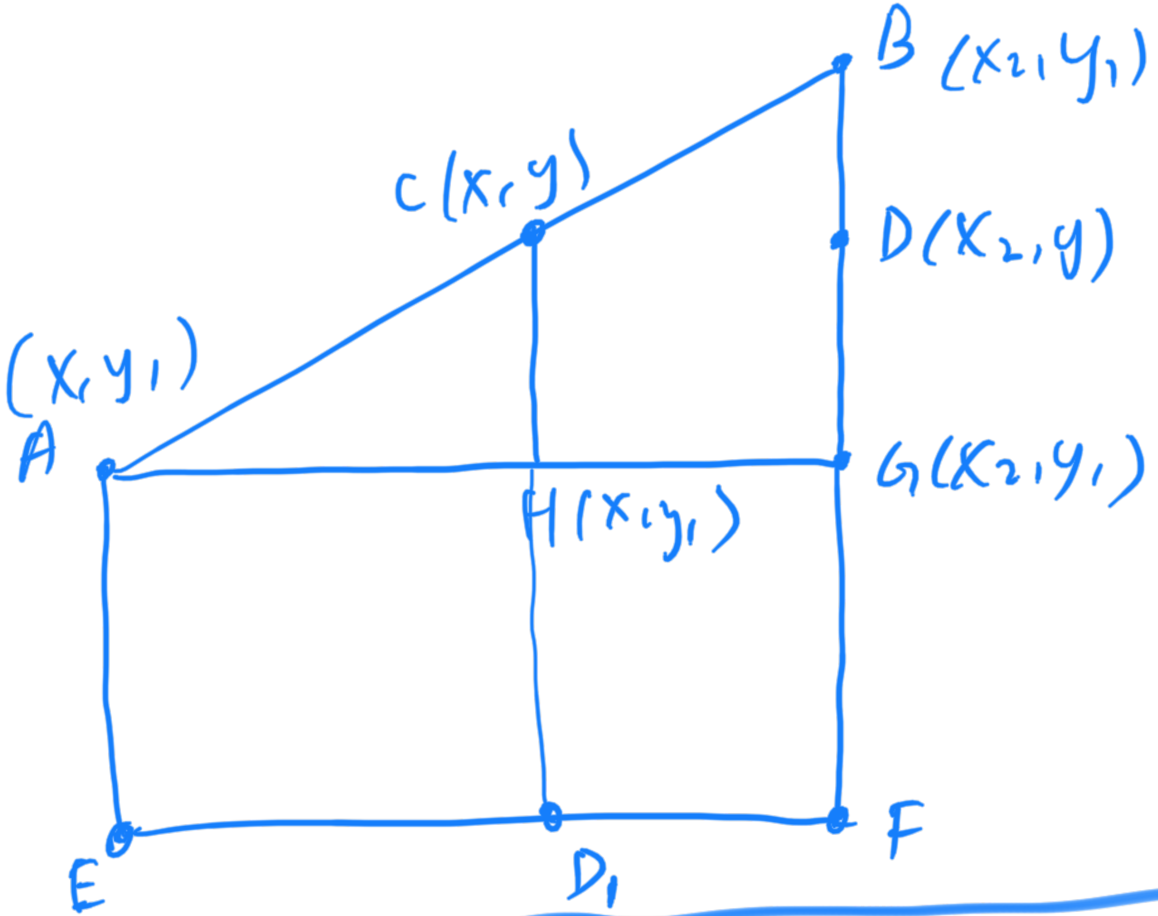
$$\frac{CH}{BD} = \frac{AC}{CB} = \frac{m}{n}$$

$$\frac{y - y_1}{y_2 - y_1} = \frac{m}{n}$$

$$n(y - y_1) = m(y_2 - y)$$

$$ny - ny_1 = my_2 - my$$

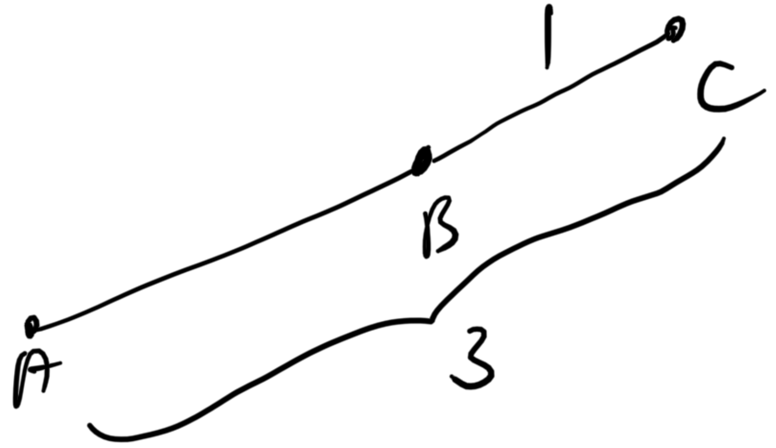
$$y(n+m) = my_2 + ny_1 \qquad y = \frac{my_2 + ny_1}{n+m}$$



$$\boxed{C\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)}$$

EXTERNAL DIVISION POINT

we want to find a point lying outside on AB



AC : CB

3 : 1

$\therefore$ C is externally dividing AB in the ratio 3 : 1

case 1

~~EXAMPLE~~

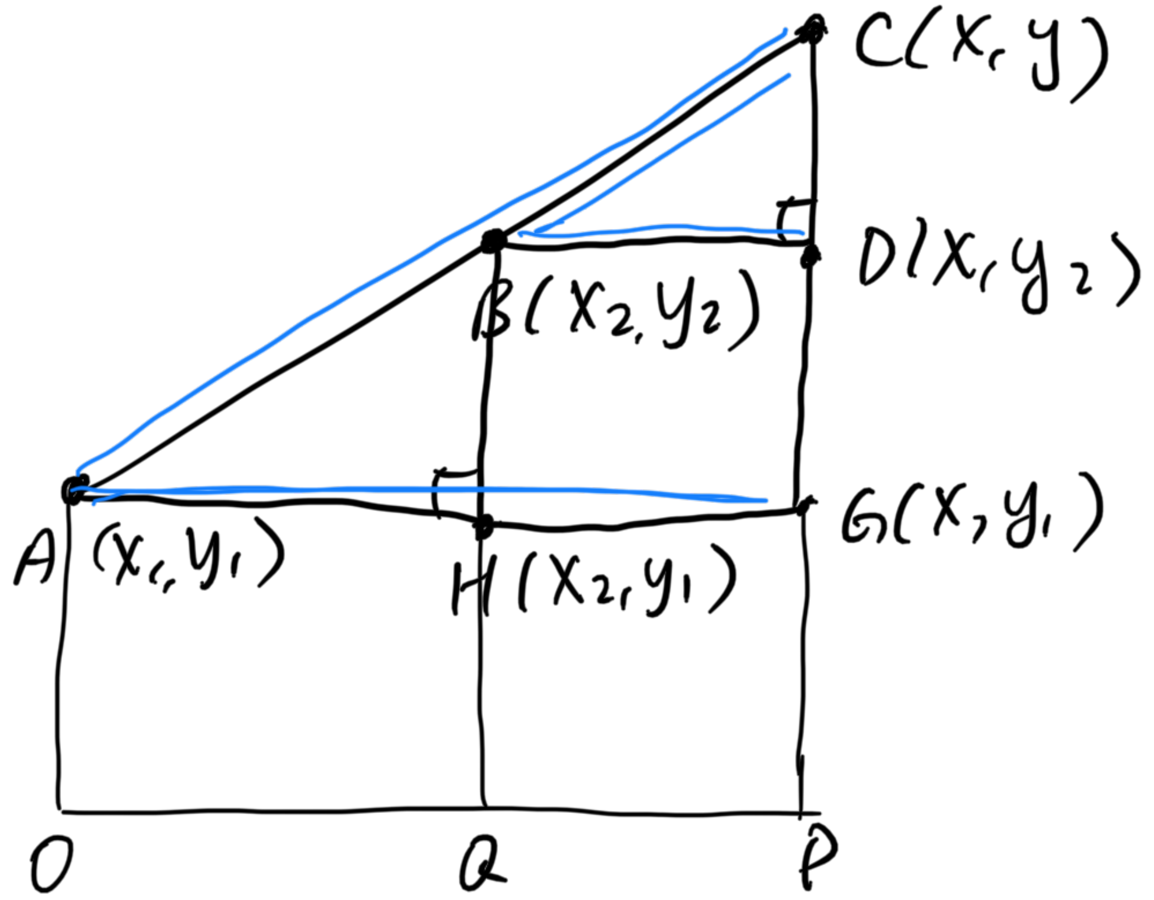
$$\frac{AG}{BD} = \frac{AC}{CB} = \frac{m}{n}$$

$$\frac{x - x_1}{x - x_2} = \frac{m}{n}$$

$$n(x - x_1) = m(x - x_2)$$

$$nx - nx_1 = mx - mx_2$$

$$(n - m)x = nx_1 - mx_2$$



$$x = \frac{nx_1 - mx_2}{n - m}$$

case 2

~~RECAP~~

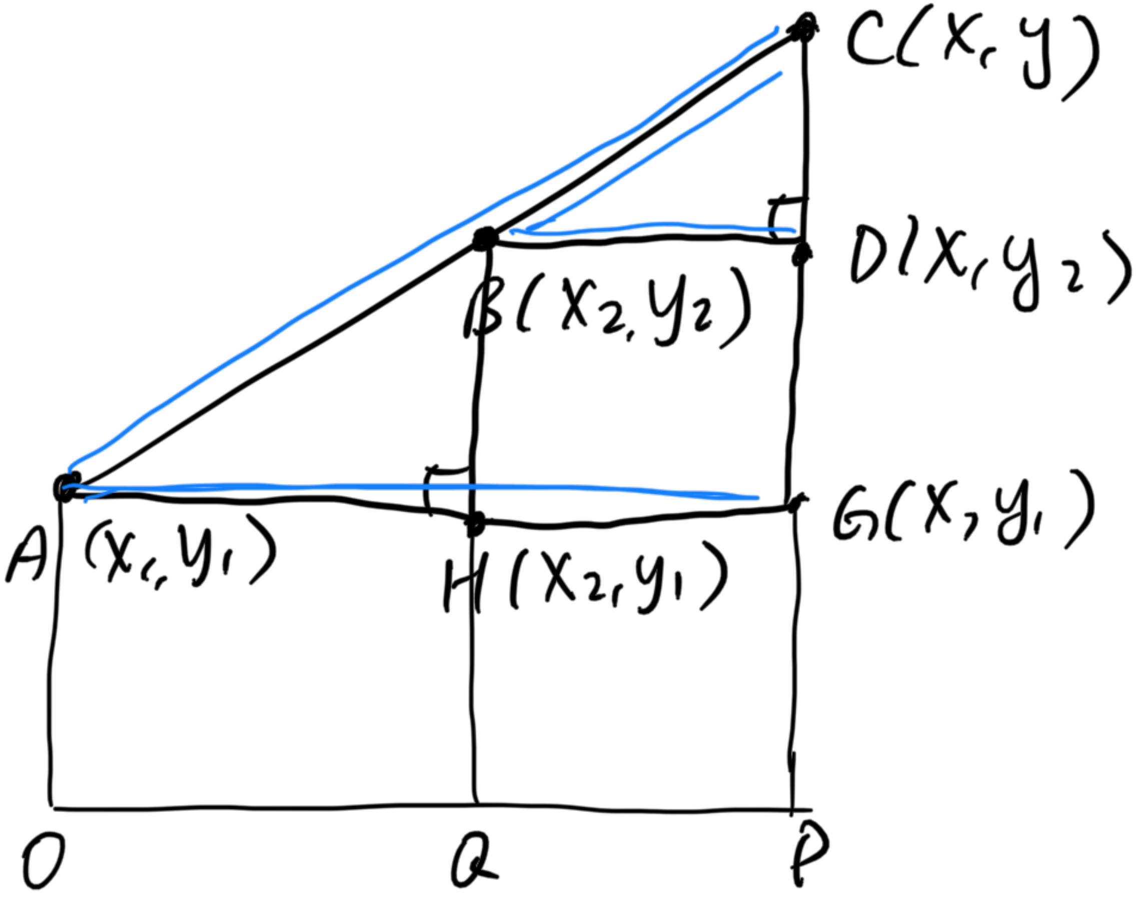
$$\frac{CG}{CD} = \frac{AC}{CB} = \frac{m}{n}$$

$$\frac{y-y_1}{y-y_2} = \frac{m}{n}$$

$$n(y-y_1) = m(y-y_2)$$

$$ny-ny_1 = my-my_2$$

$$(m-n)y = ny_1-my_2$$



$$y = \frac{my_2-ny_1}{m-n}$$

$$\boxed{C\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)}$$

Example 1 used internal

~~LINES ON CARTESIAN PLANE~~

we have $A(-7,4)$, $B(8,9)$ and the ratio $\boxed{m:n=3:2}$

$$x=\frac{mx_2+nx_1}{m+n}=\frac{3(8)+2(-7)}{3+2}=\frac{24-14}{5}=\frac{10}{5}=2$$

$$y=\frac{my_2+ny_1}{m+n}=\frac{3(9)+2(4)}{3+2}=\frac{27+8}{5}=\frac{35}{5}=7$$

$C(2,7)$

Example. 2 Used External

~~VERTICAL AND HORIZONTAL LINES~~

we have $A(-7,4)$, $B(2,7)$ and the ratio $\boxed{m:n = 5:2}$

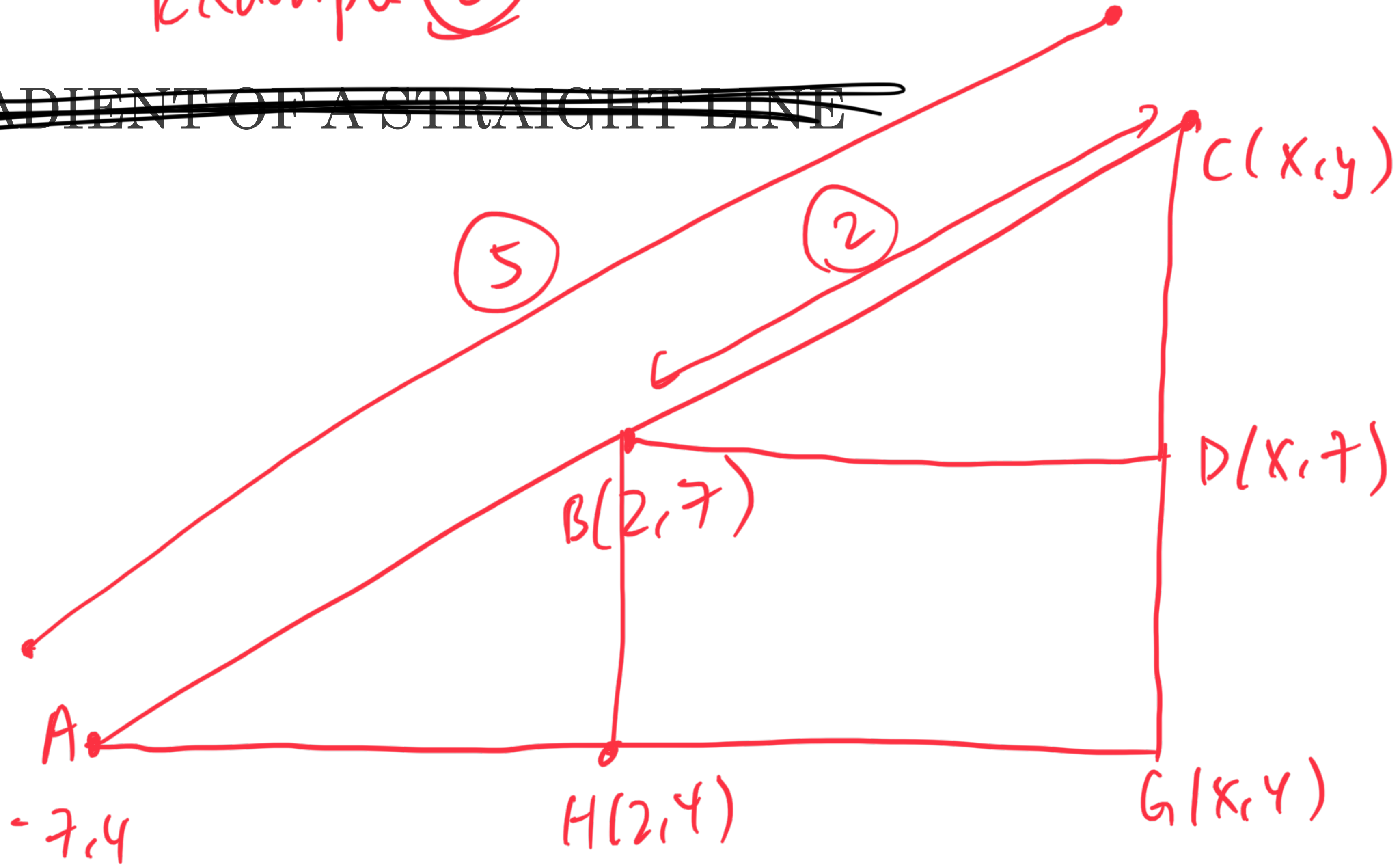
$$x = \frac{mx_2 - nx_1}{m-n} = \frac{5(2)-2(-7)}{5-2} = \frac{10+14}{3} = \frac{24}{3} = 8$$

$$y = \frac{my_2 - ny_1}{m-n} = \frac{5(7)-2(4)}{5-2} = \frac{35-8}{3} = \frac{27}{3} = 9$$

$C = (8, 9)$

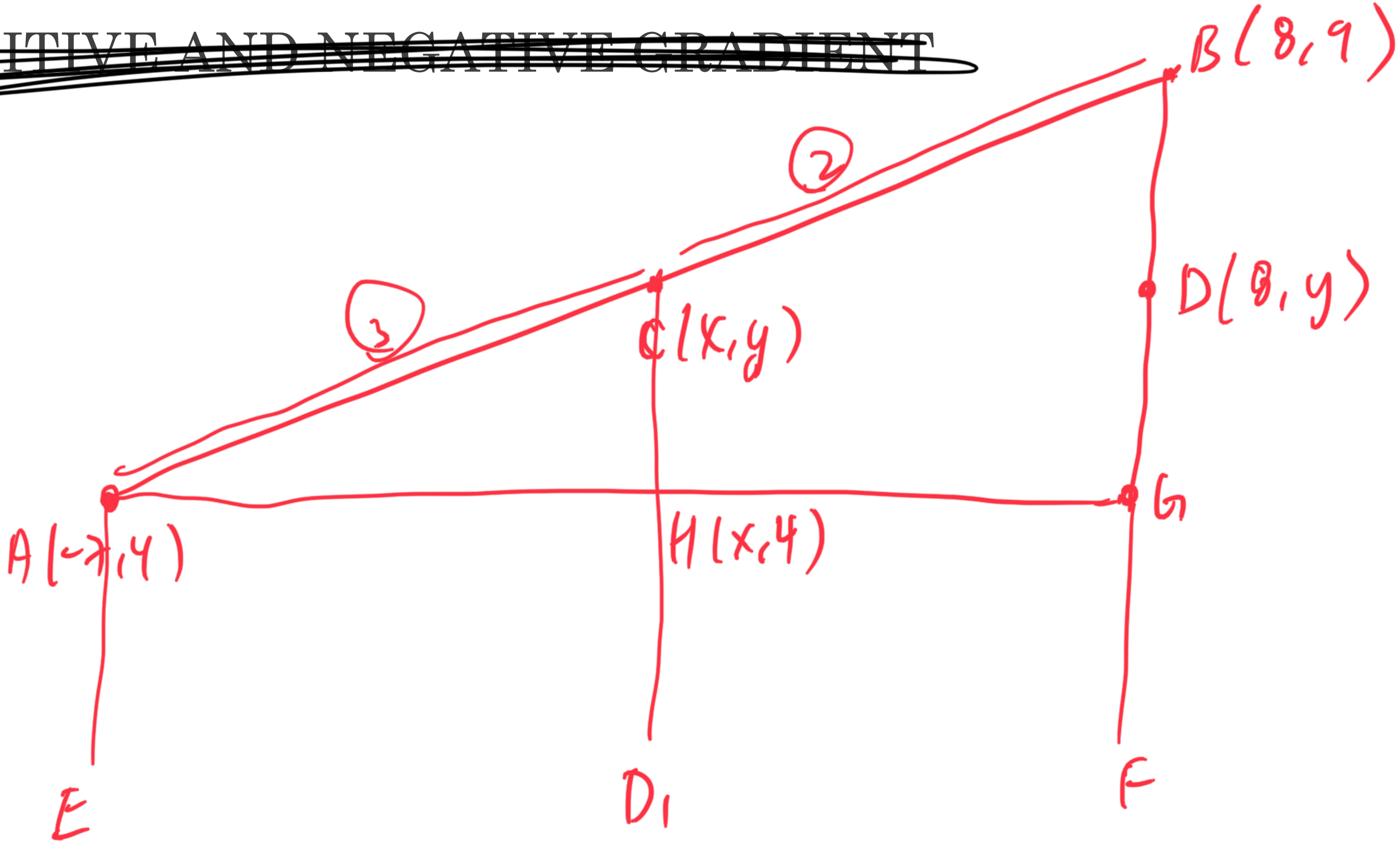
Example 2

~~GRADIENT OF A STRAIGHT LINE~~



Example ①

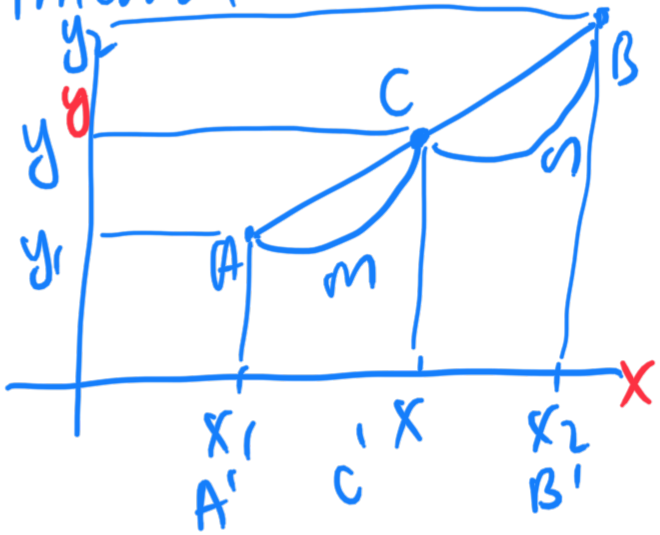
~~POSITIVE AND NEGATIVE GRADIENT~~



Summary

~~CALCULATING GRADIENT~~

Internal

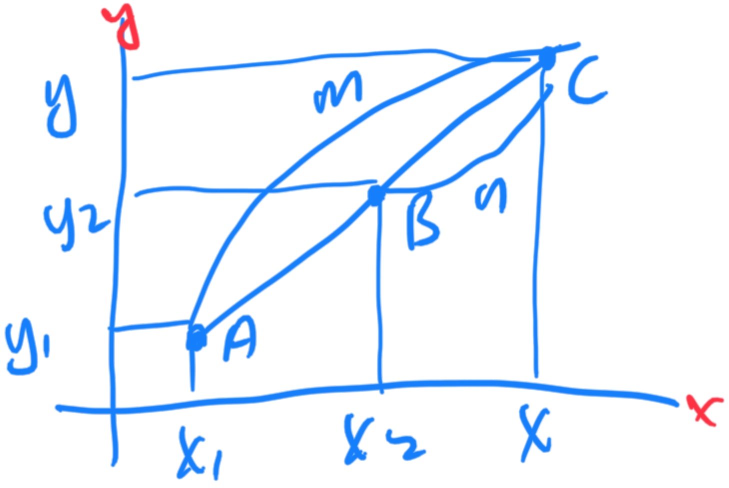


$$(x - x_1) : (x_2 - x)$$
$$m : n$$

$$\therefore x = \frac{nx_1 + mx_2}{m+n}$$

$$y = \frac{ny_1 + my_2}{m+n}$$

External



$$(x - x_1) : (x - x_2)$$
$$m : n$$

$$\therefore x = \frac{mx_2 - nx_1}{m-n}$$

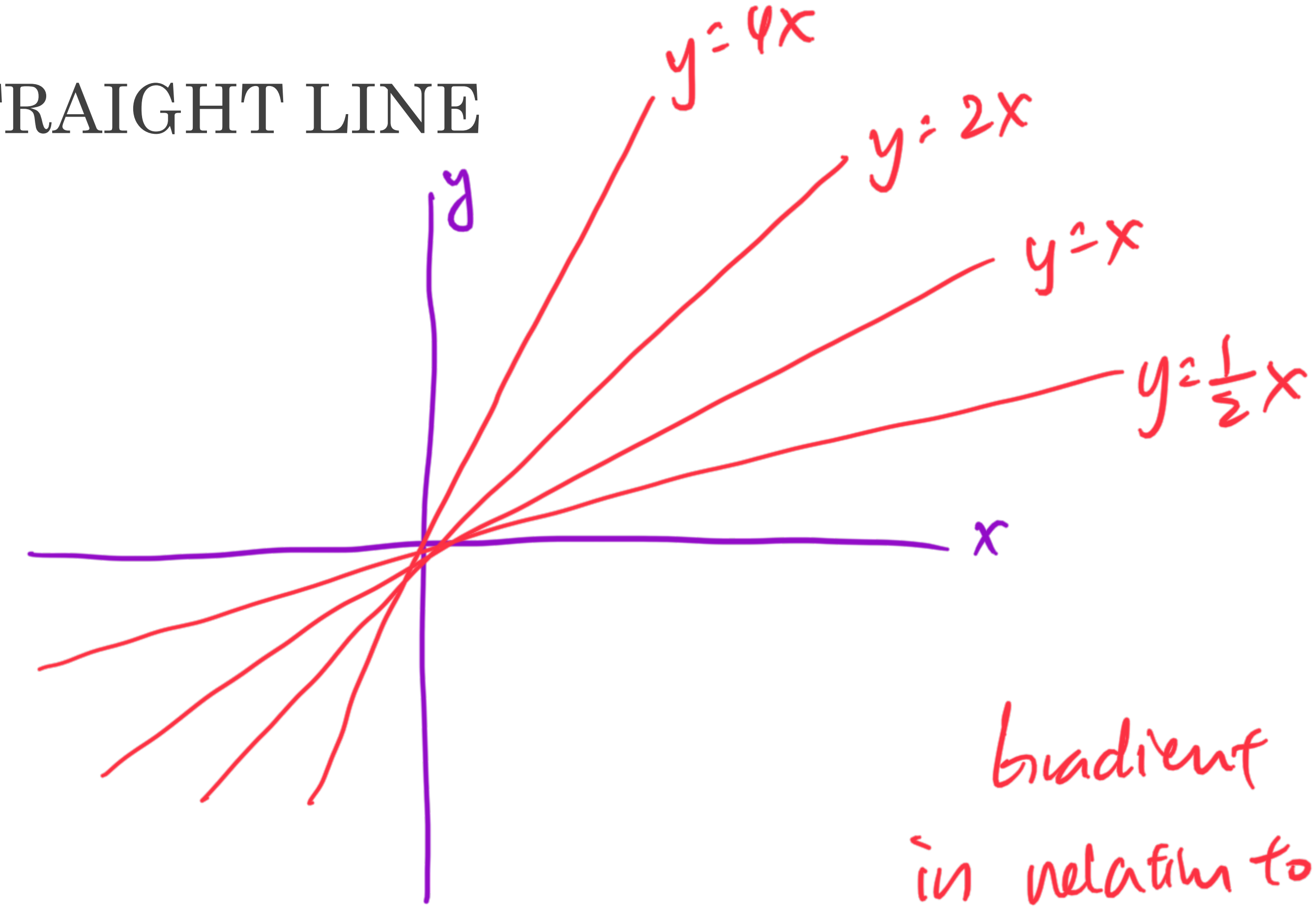
$$y = \frac{my_2 - ny_1}{m-n}$$

~~GRADIENT OF HORIZONTAL AND VERTICAL LINES~~

~~GRADIENT OF PARALLEL AND PERPENDICULAR LINES~~

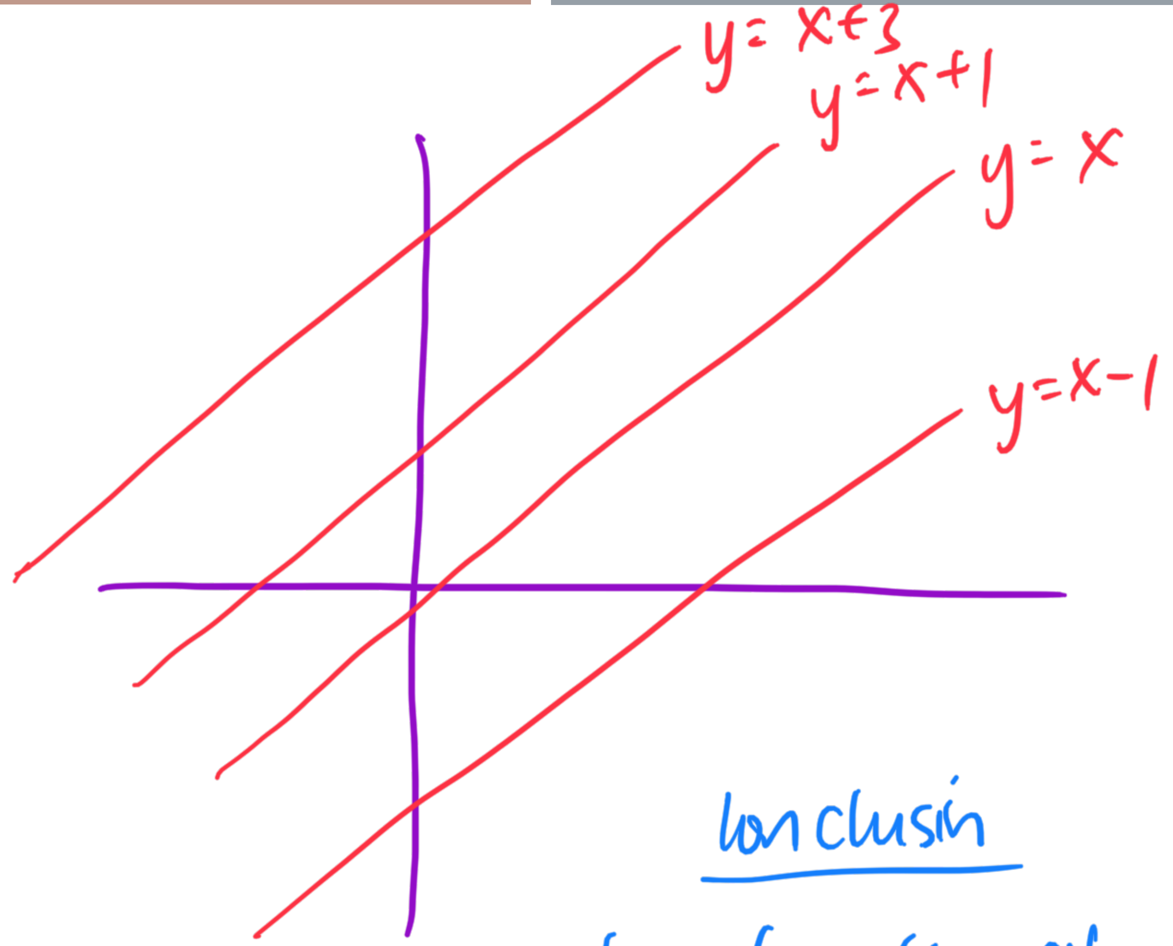
THE EQUATION OF A STRAIGHT LINE

	<u>Gradient</u>
$y = \frac{1}{2}x$	$m = \frac{1}{2}$
$y = x$	$m = 1$
$y = 2x$	$m = 2$
$y = 4x$	$m = 4$



Gradient in relation to steepness.

	intercept
$y = x$	0
$y = x + 1$	1
$y = x + 3$	3
$y = x - 1$	-1



Conclusion

transformation of Gradient, intercept

$y = \underline{m}x + \underline{b}$

EXAMPLE

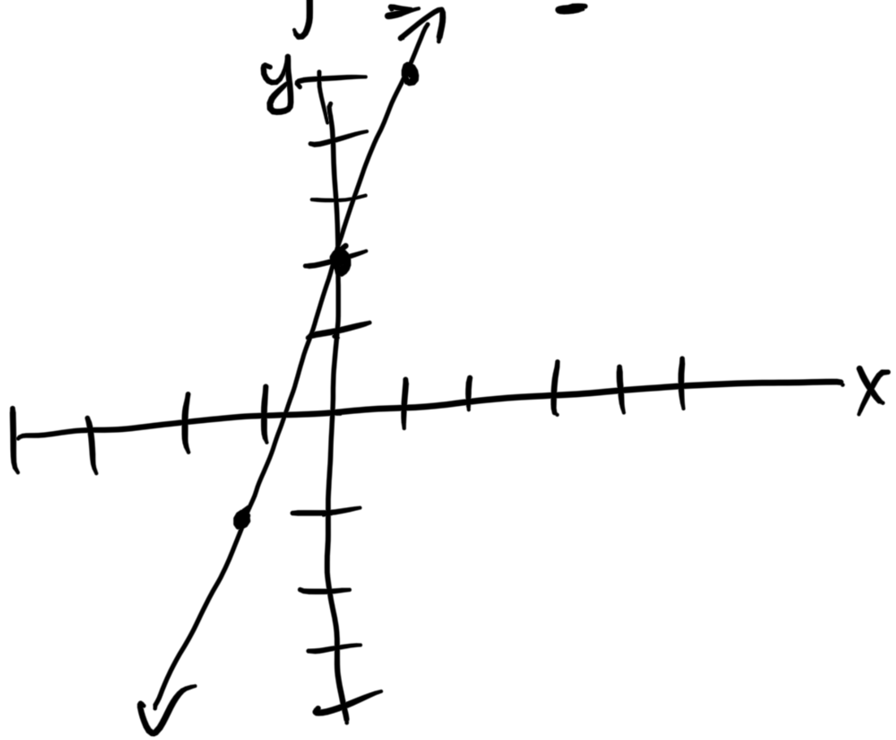
State the gradient and y intercept for the line $y = 3x + 2$

$m = 3$

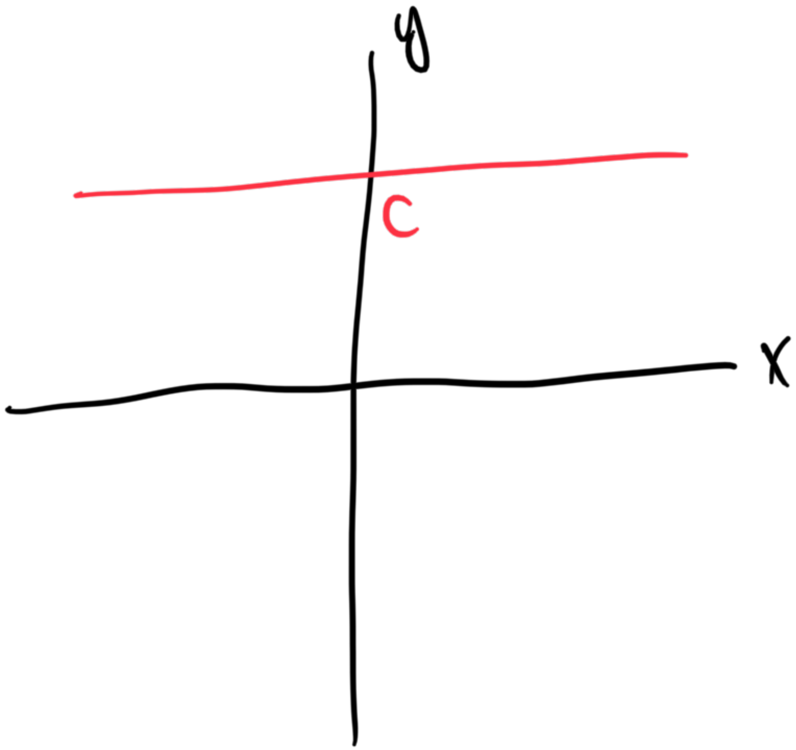
y-intercept = 2

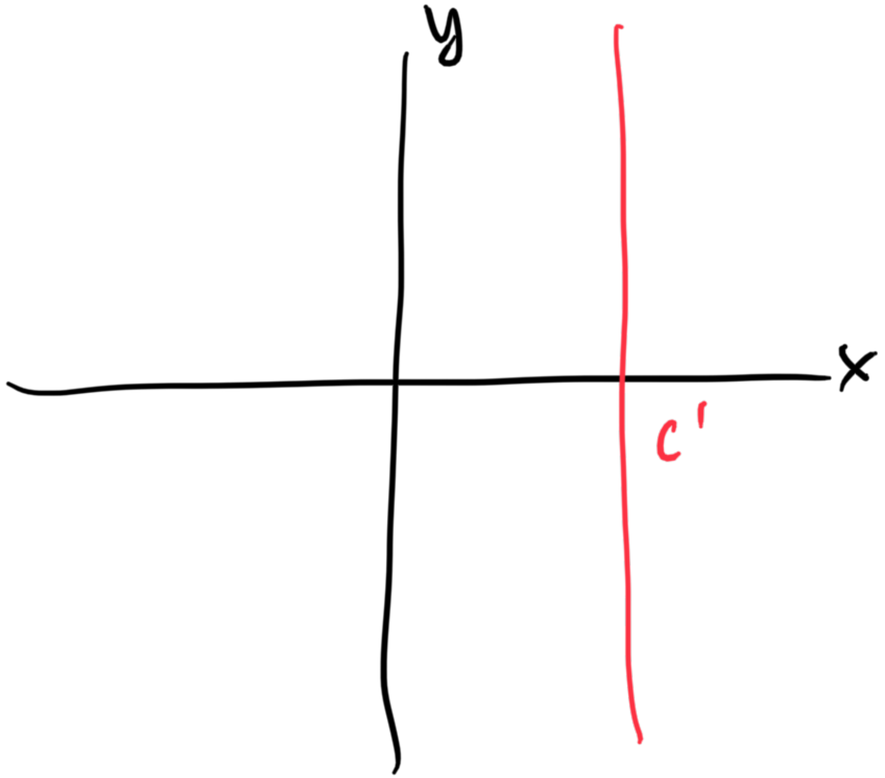
$m = \frac{3}{1} = \frac{\text{rise}}{\text{run}}$

$y = mx + b$



HORIZONTAL AND VERTICAL LINES





1. zero gradient
2. parallel to x axis
3. $y = c$

1. gradient Infinite (undefined)
2. parallel to y-axis
3. $x = c'$