Chapter 9

Rotation of Rigid Bodies

Introduction

- An airplane propeller, a revolving door, a ceiling fan, and a Ferris wheel all involve rotating rigid objects.
- Real-world rotations can be very complicated because of stretching and twisting of the rotating body. But for now we’ll assume that the rotating body is perfectly rigid.

Rigid body motion

There are cases where an object cannot be treated as a particle. In these cases the size or shape of the body must be considered. Rotation of the body about its center of mass requires a different approach.

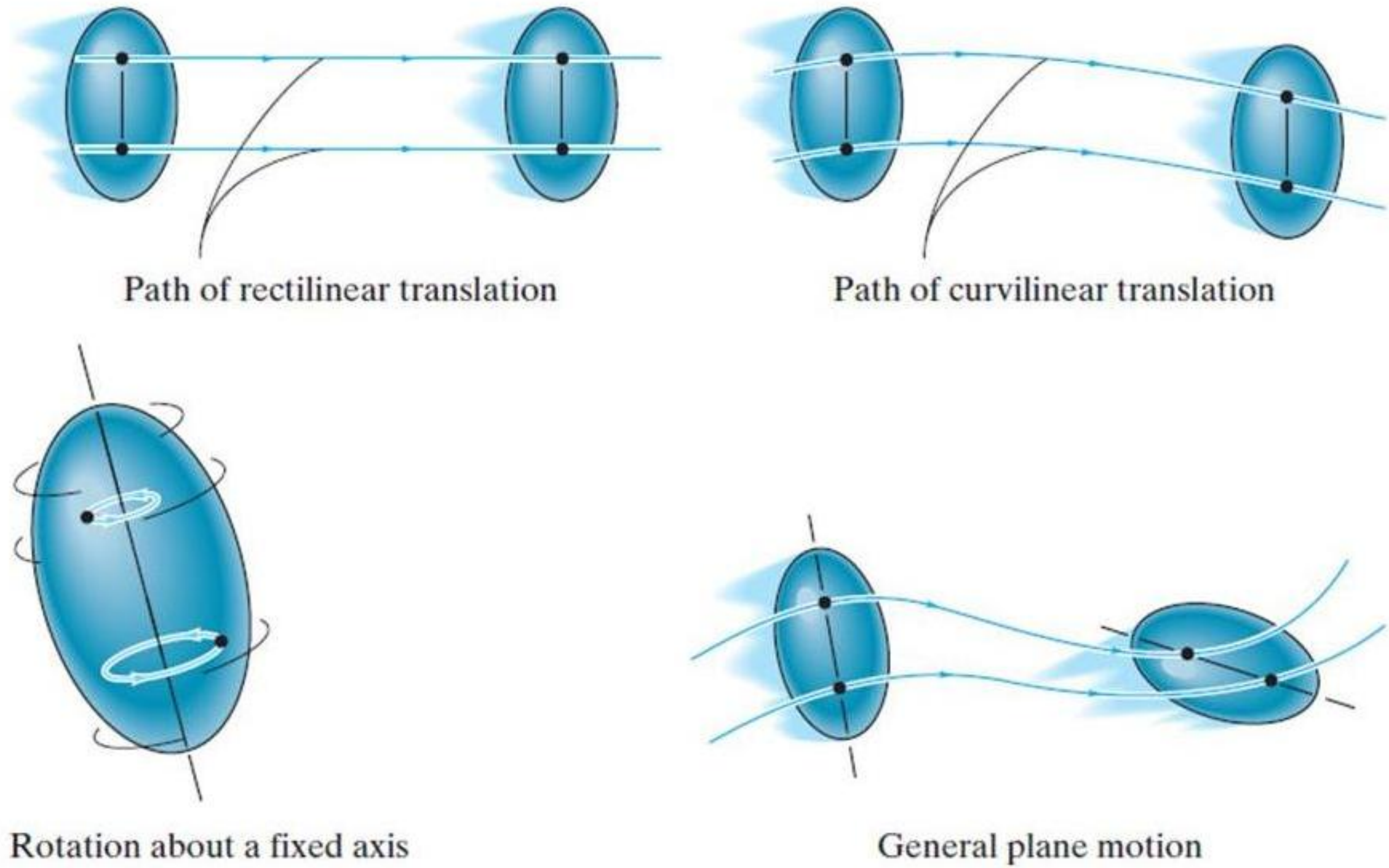
For example, in the design of gears, cams, and links in machinery or mechanisms, rotation of the body is an important aspect in the analysis of motion.

We will now start to study rigid body motion. The analysis will be limited to planar motion.

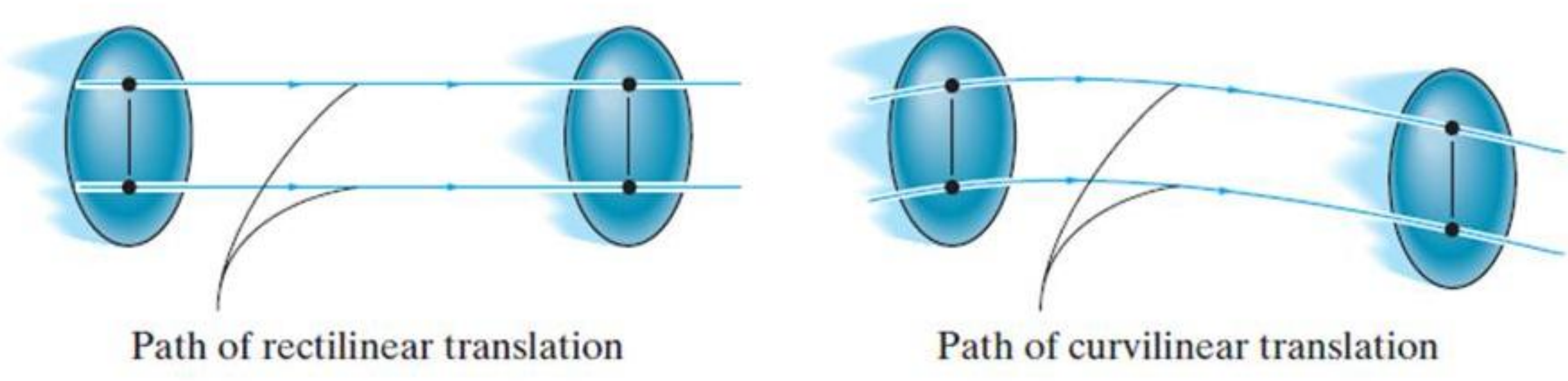
A body is said to undergo planar motion when all parts of the body move along paths equidistant from a fixed plane.

Planar rigid body motion

There are three types of planar rigid body motion.

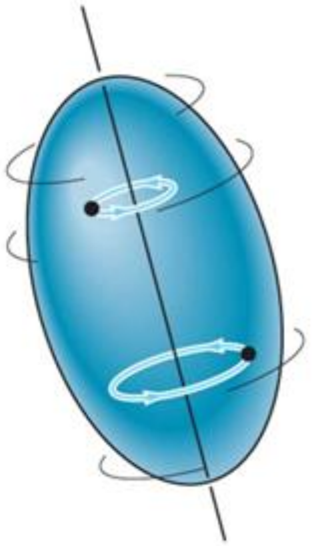


Planar rigid body motion



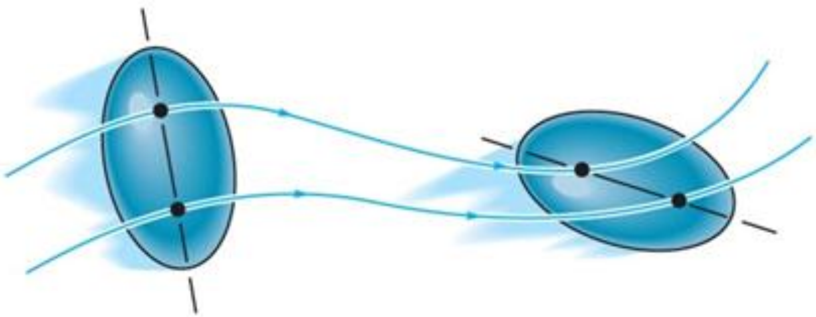
Translation: Translation occurs if every line segment on the body remains parallel to its original direction during the motion. When all points move along straight lines, the motion is called rectilinear translation. When the paths of motion are curved lines, the motion is called curvilinear translation.

Planar rigid body motion



Rotation about a fixed axis

Rotation about a fixed axis: In this case, all the particles of the body, except those on the axis of rotation, move along circular paths in planes perpendicular to the axis of rotation.

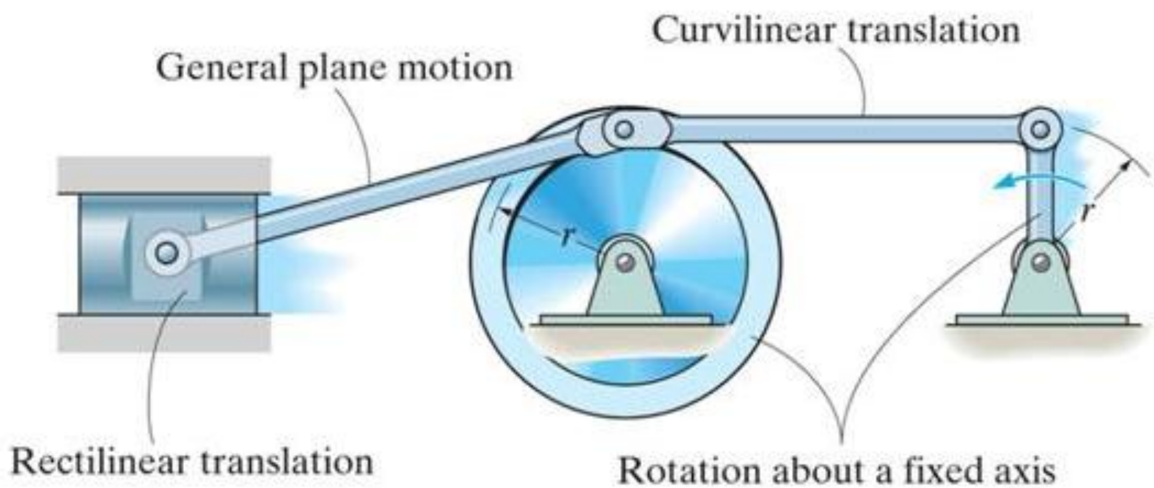


General plane motion

General plane motion: In this case, the body undergoes both translation and rotation. Translation occurs within a plane and rotation occurs about an axis perpendicular to this plane.

Planar rigid body motion

An example of bodies undergoing the three types of motion is shown in this mechanism.

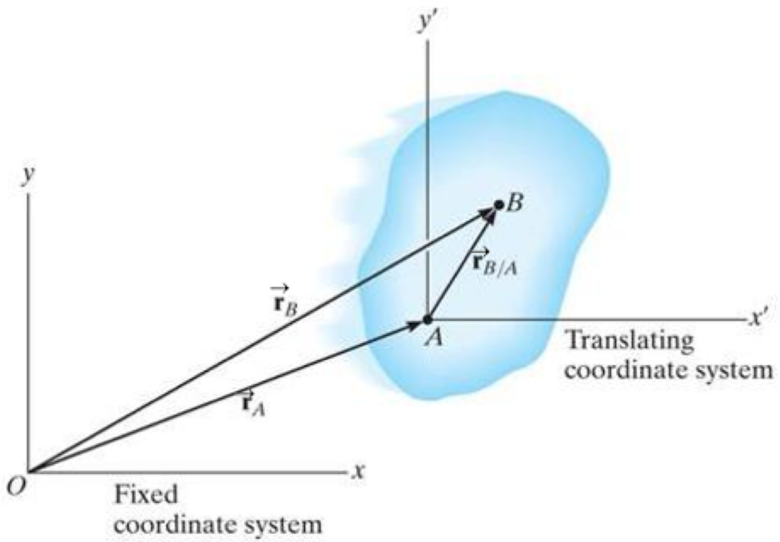


The wheel and crank undergo rotation about a fixed axis. In this case, both axes of rotation are at the location of the pins and perpendicular to the plane of the figure.

The piston undergoes rectilinear translation since it is constrained to slide in a straight line.

The connecting rod undergoes curvilinear translation, since it will remain horizontal as it moves along a circular path.
The connecting rod undergoes general plane motion, as it will both translate and rotate.

Rigid body motion: translation



The positions of two points A and B on a translating body can be related by

$$\vec{\mathbf{r}}_B = \vec{\mathbf{r}}_A + \vec{\mathbf{r}}_{B/A}$$

where $\vec{\mathbf{r}}_A$ & $\vec{\mathbf{r}}_B$ are the absolute position vectors defined from the fixed x-y coordinate system, and $\vec{\mathbf{r}}_{B/A}$ is the relative-position vector between B and A.

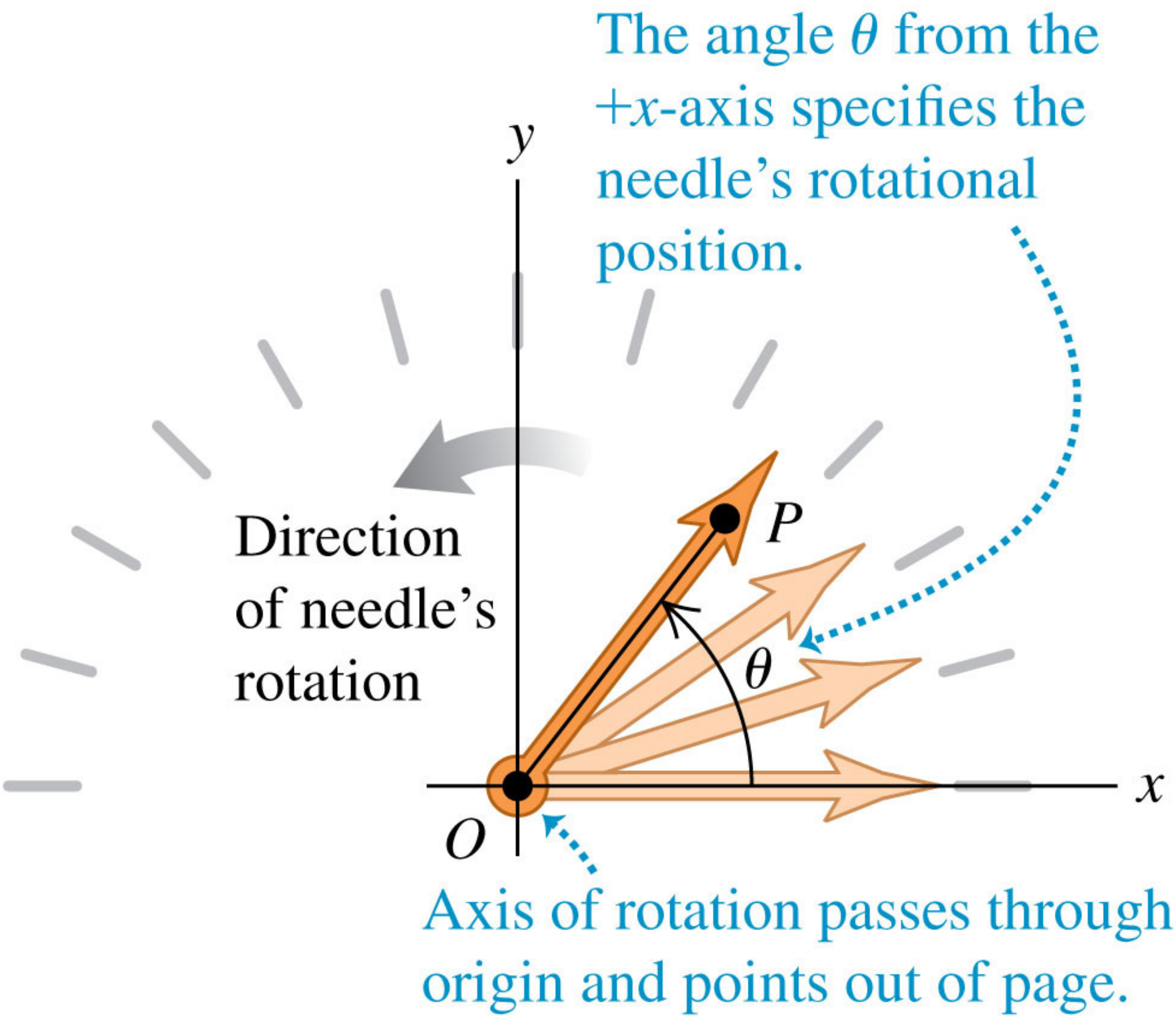
The velocity at B is $\vec{\mathbf{v}}_B = \vec{\mathbf{v}}_A + d\vec{\mathbf{r}}_{B/A}/dt$.

Now $d\vec{\mathbf{r}}_{B/A}/dt = 0$ since $\vec{\mathbf{r}}_{B/A}$ is constant. So, $\vec{\mathbf{v}}_B = \vec{\mathbf{v}}_A$, and by following similar logic, $\vec{\mathbf{a}}_B = \vec{\mathbf{a}}_A$.

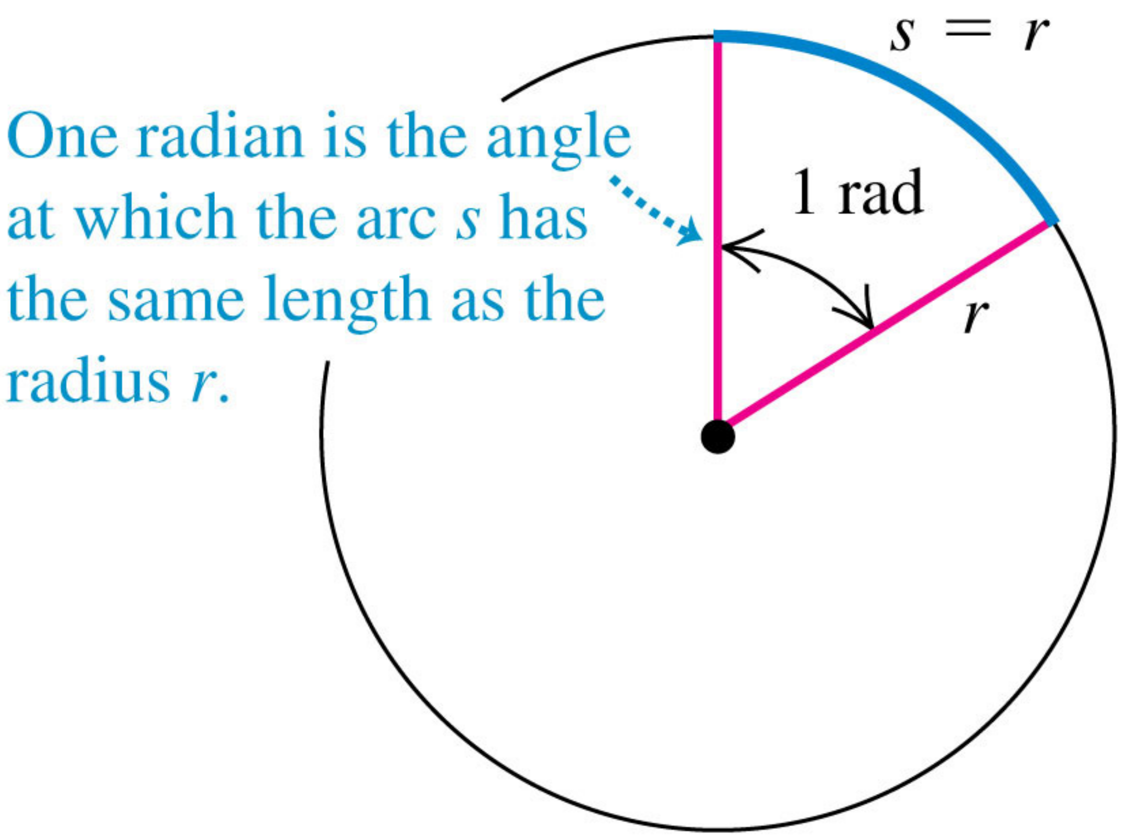
Note, all points in a rigid body subjected to translation move with the same velocity and acceleration.

Angular coordinate

- A car's speedometer needle rotates about a *fixed axis*.



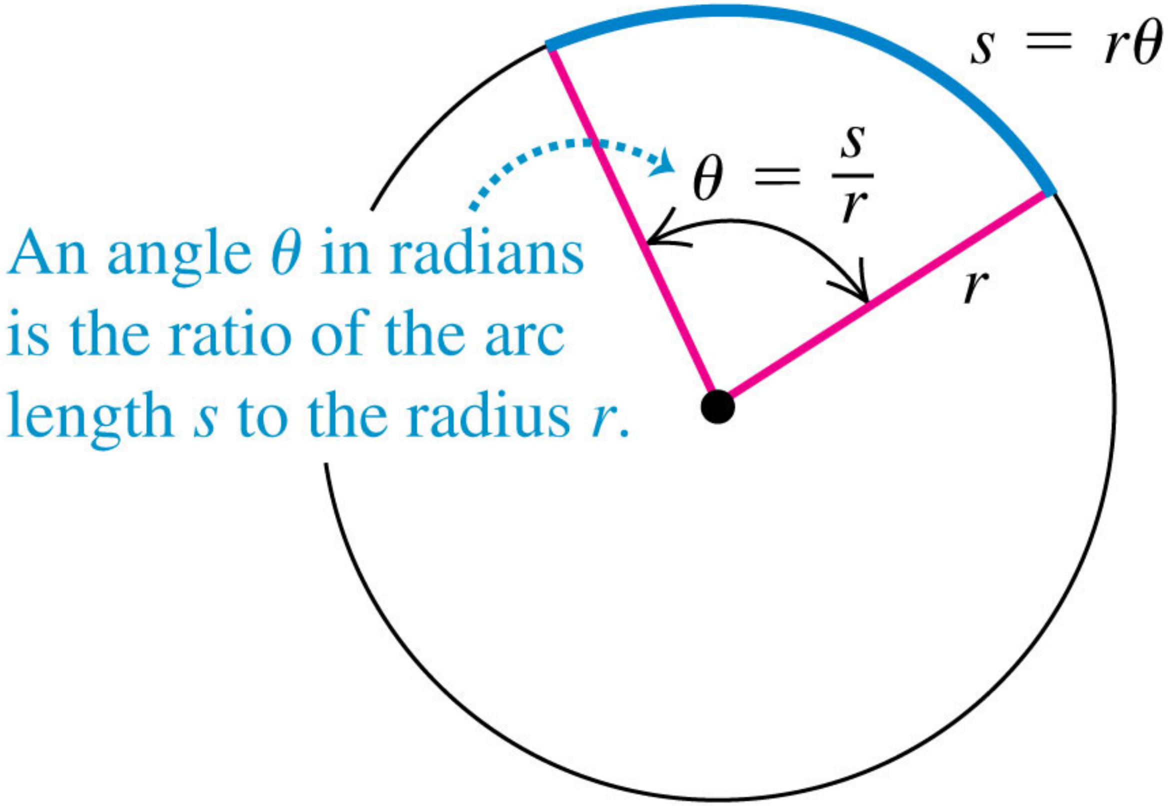
Units of angles



- One complete revolution is $360° = 2\pi$ radians.

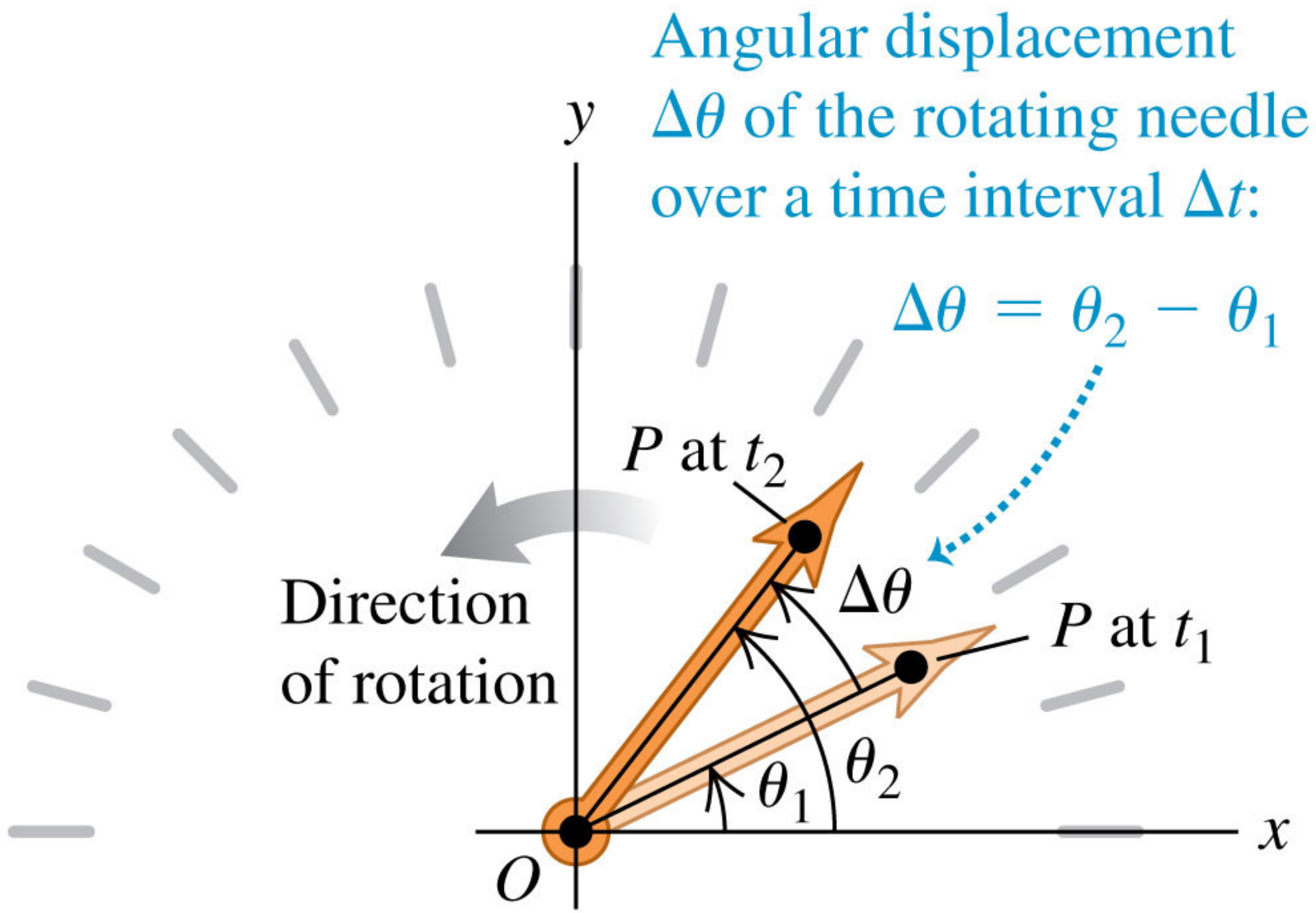
Units of angles

- An angle in radians is $\theta = s/r$, as shown in the figure.



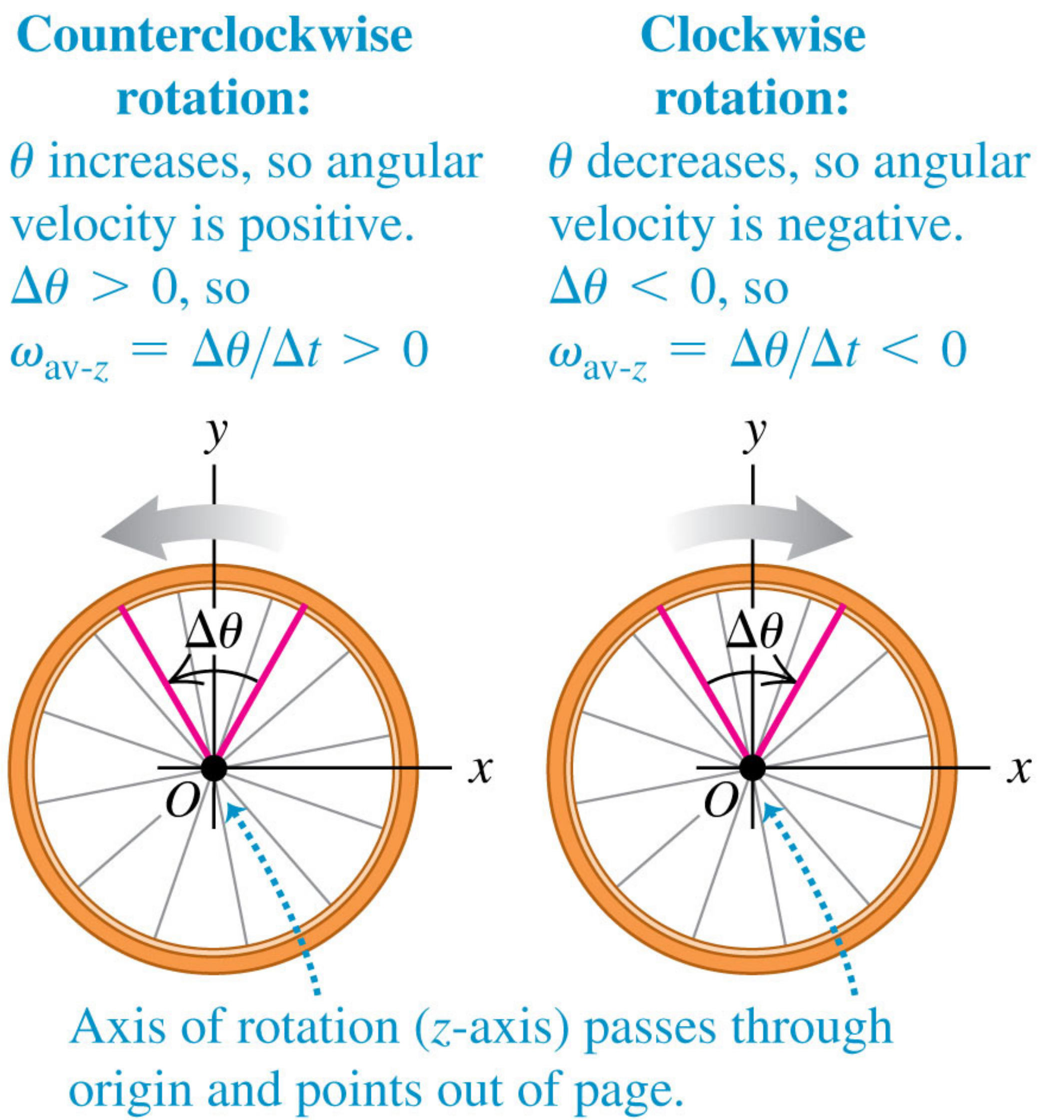
Angular velocity

- The *average angular velocity* of a body is $\omega_{\text{av-}z} = \Delta\theta/\Delta t$.
- The subscript z means that the rotation is about the z-axis.



Angular velocity

- We choose the angle θ to increase in the counterclockwise rotation.



Instantaneous angular velocity

- The **instantaneous angular velocity** is the limit of average angular velocity as $\Delta\theta$ approaches zero:

The **instantaneous angular velocity** of a rigid body rotating around the z-axis ...

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

... equals the limit of the body's average angular velocity as the time interval approaches zero ...

... and equals the instantaneous rate of change of the body's angular coordinate.

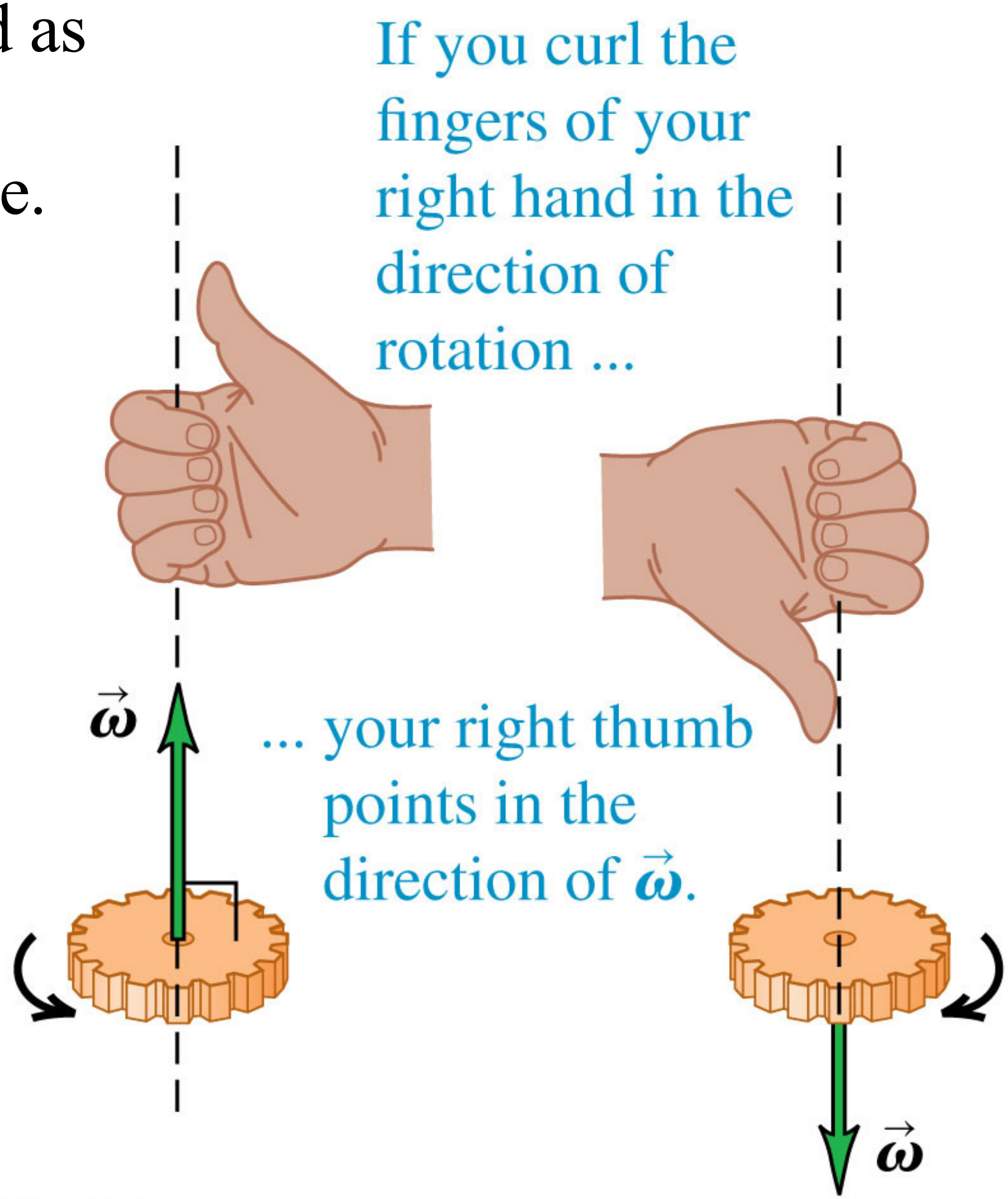
- When we refer simply to "angular velocity," we mean the instantaneous angular velocity, not the average angular velocity.

- The z-subscript means the object is rotating around the z-axis.

- The angular velocity can be positive or negative, depending on the direction in which the rigid body is rotating.

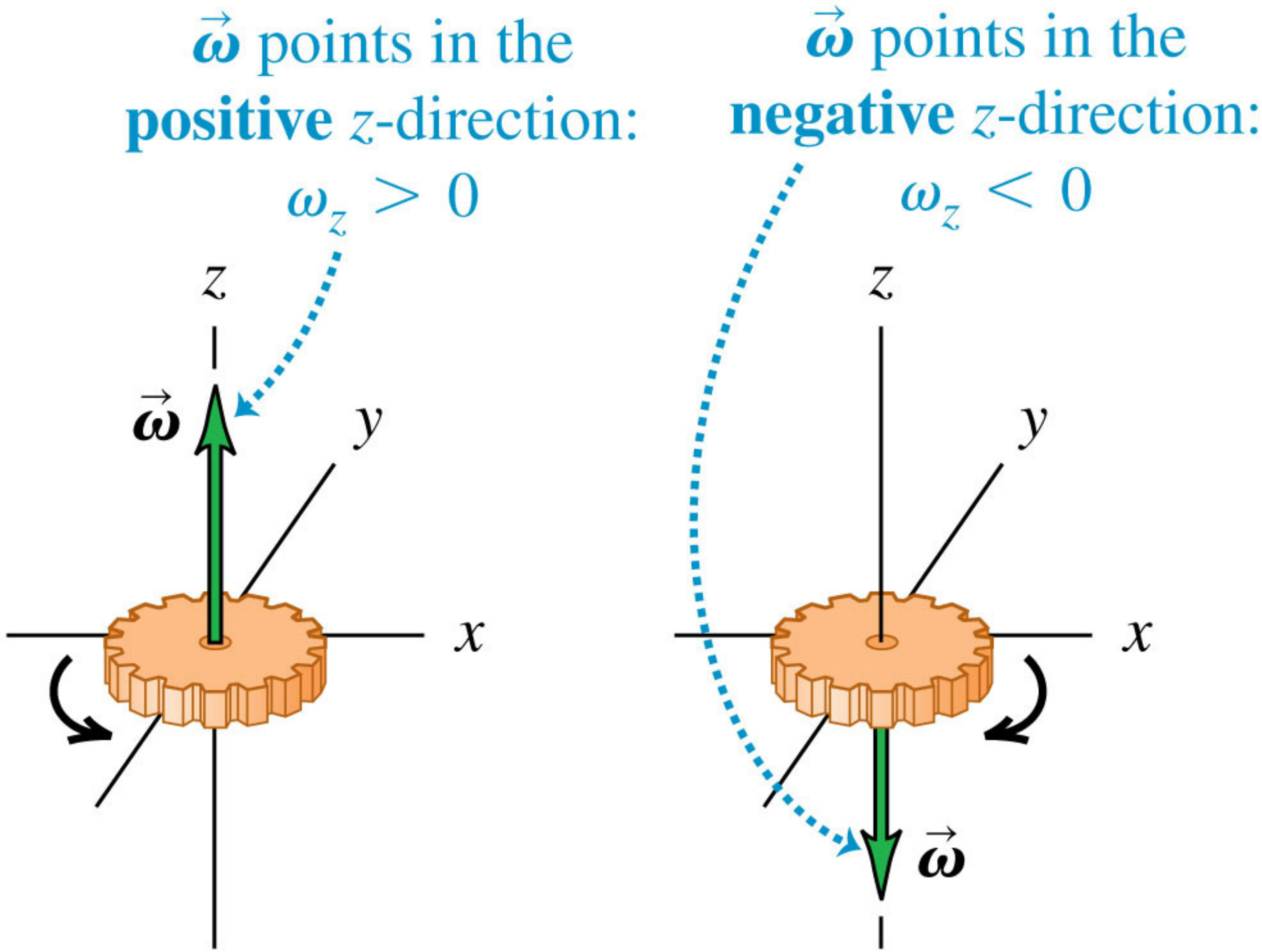
Angular velocity is a vector

- Angular velocity is defined as a vector whose direction is given by the right-hand rule.



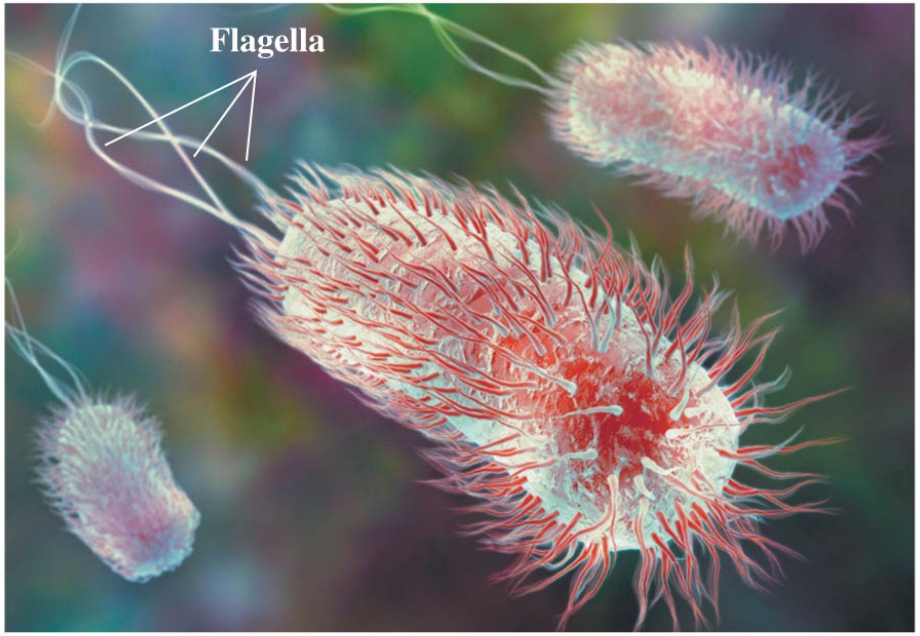
Angular velocity is a vector

- The sign of ω_z for rotation along the z-axis



Rotational motion in bacteria

- *Escherichia coli* bacteria are found in the lower intestines of humans and other warm-blooded animals.

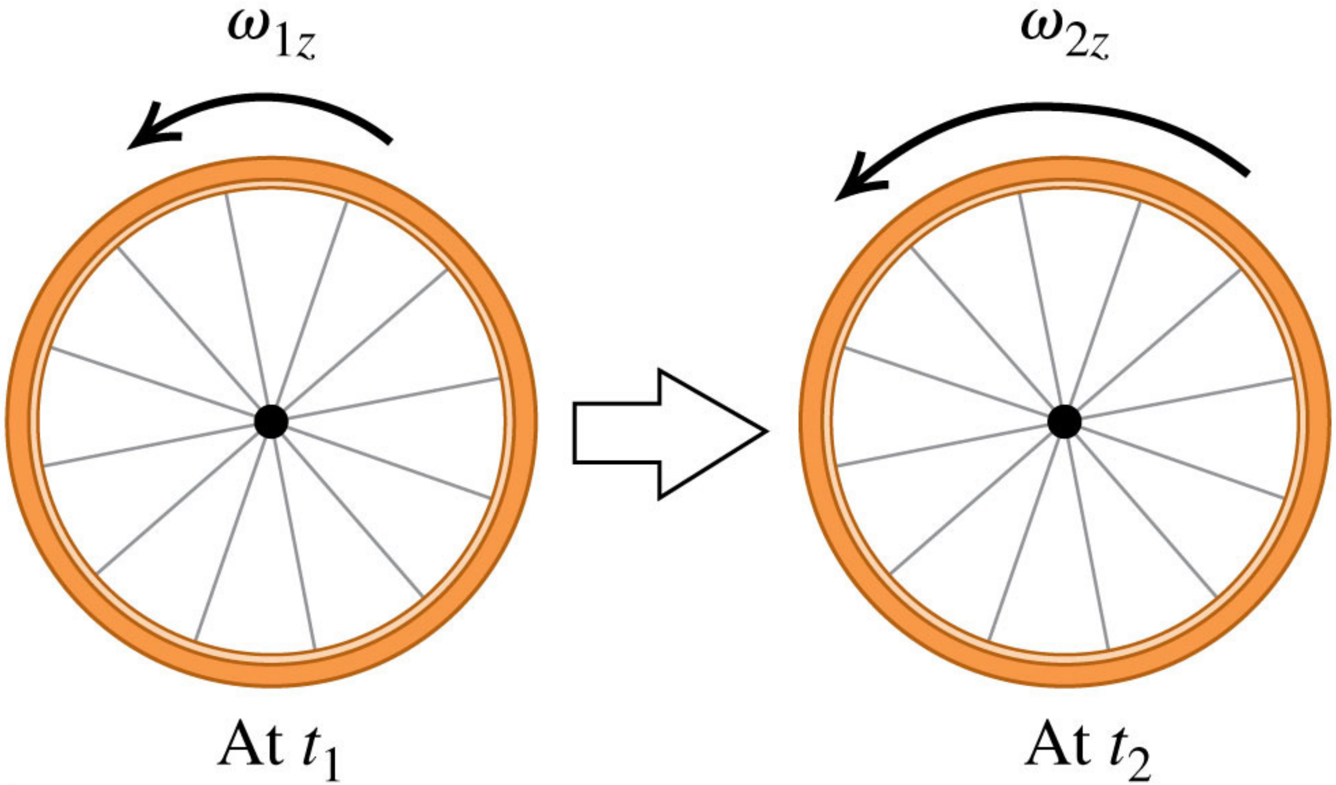


- The bacteria swim by rotating their long, corkscrew-shaped flagella, which act like the blades of a propeller.
- Each flagellum is rotated at angular speeds from 200 to 1000 rev/min (about 20 to 100 rad/s) and can vary its speed to give the flagellum an angular acceleration.

Angular acceleration

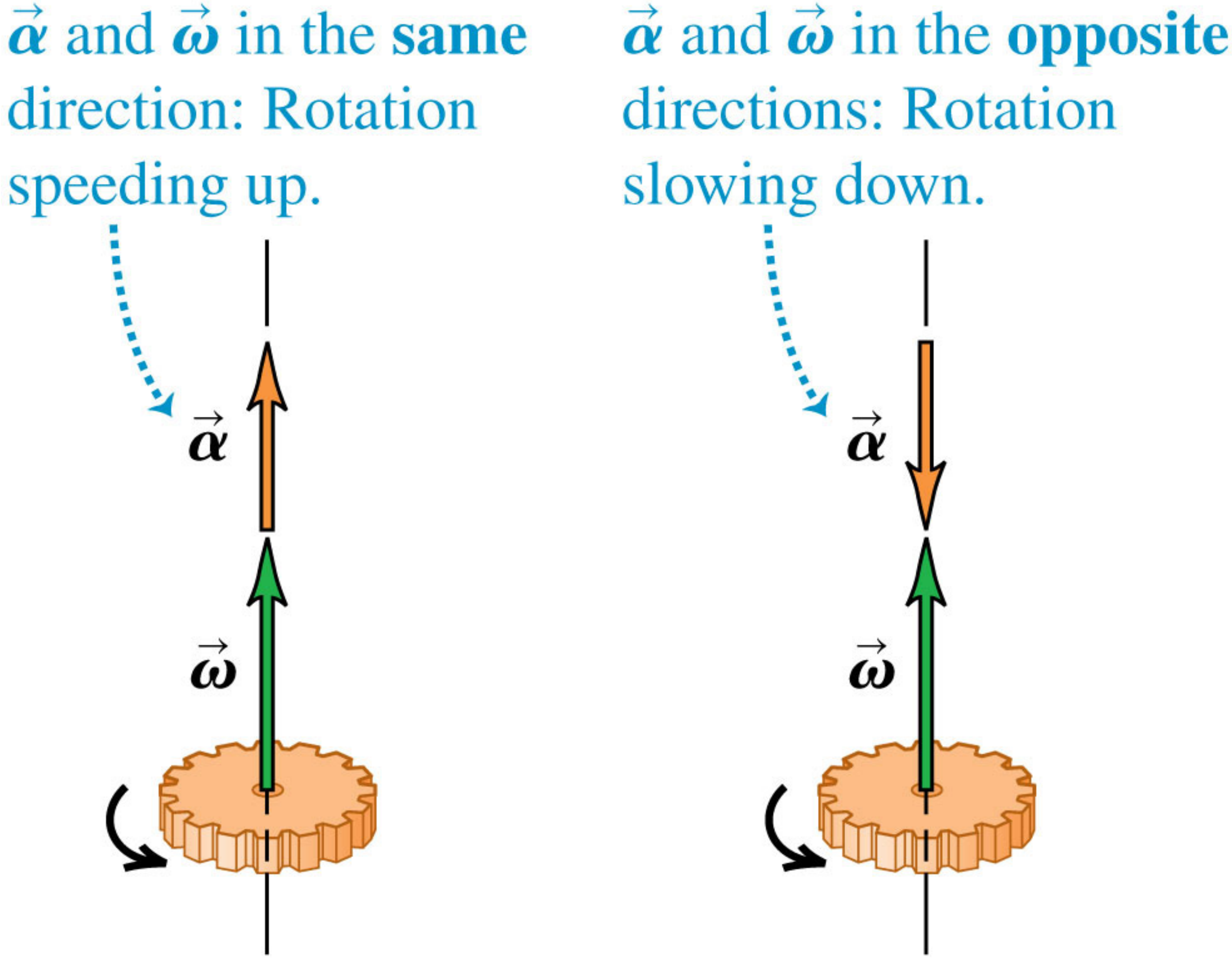
The average angular acceleration is the change in angular velocity divided by the time interval:

$$\alpha_{\text{av-}z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t}$$

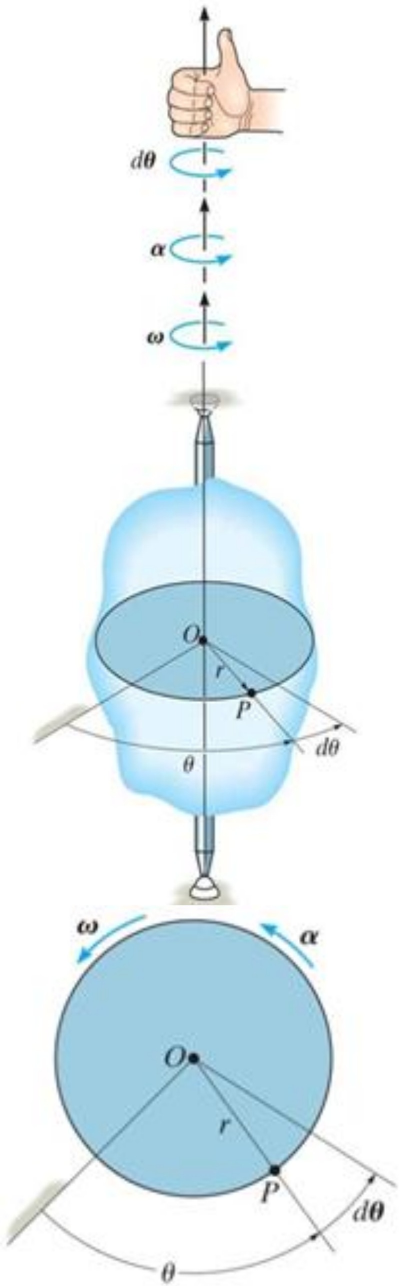


The instantaneous angular acceleration is $\alpha_z = d\omega_z/dt$.

Angular acceleration as a vector



Rotation about a fixed axis



When a body rotates about a fixed axis, any point P in the body travels along a circular path. The angular position of P is defined by θ.

The change in angular position, $d\theta$, is called the angular displacement, with units of either radians or revolutions. They are related by

1 revolution = (2π) radians

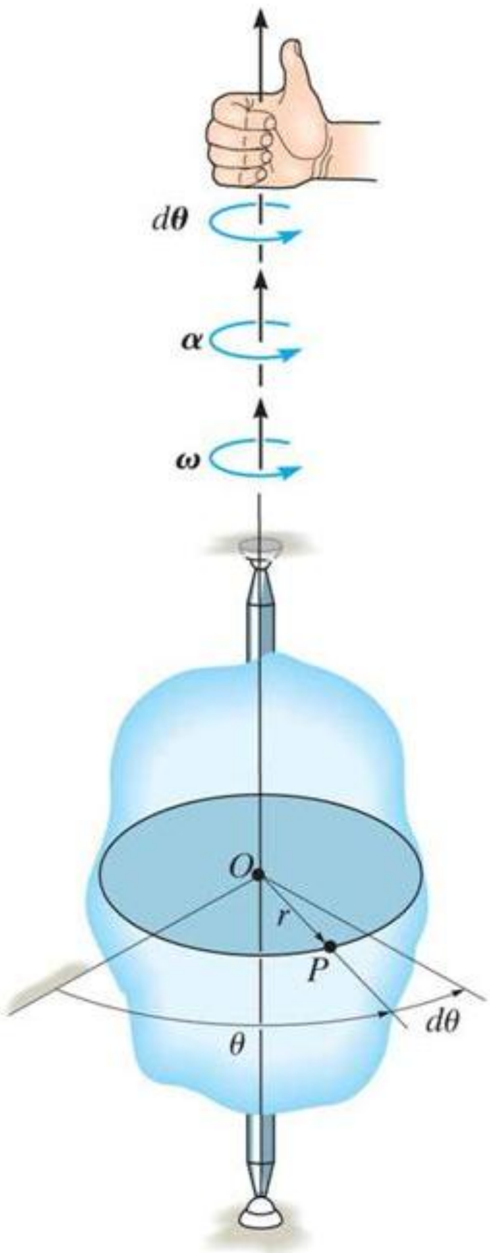
Angular velocity, ω, is obtained by taking the time derivative of angular displacement:

$$\omega = d\theta/dt \text{ (rad/s) } +$$

Similarly, angular acceleration is

$$\alpha = d^2\theta/dt^2 = d\omega/dt \quad \text{or} \quad \alpha = \omega(d\omega/d\theta) \text{ rad/s}^2 +$$

Rotation about a fixed axis



If the angular acceleration of the body is constant, $\alpha = \alpha_C$, the equations for angular velocity and acceleration can be integrated to yield the set of algebraic equations below.

$$\omega = \omega_0 + \alpha_C\, t$$

$$\theta = \theta_0 + \omega_0\, t + 0.5\, \alpha_C\, t^2$$

$$\omega^2 = (\omega_0)^2 + 2\alpha_C\, (\theta - \theta_0)$$

θ_0 and ω_0 are the initial values of the body's angular position and angular velocity. Note these equations are very similar to the constant acceleration relations developed for the rectilinear motion of a particle.

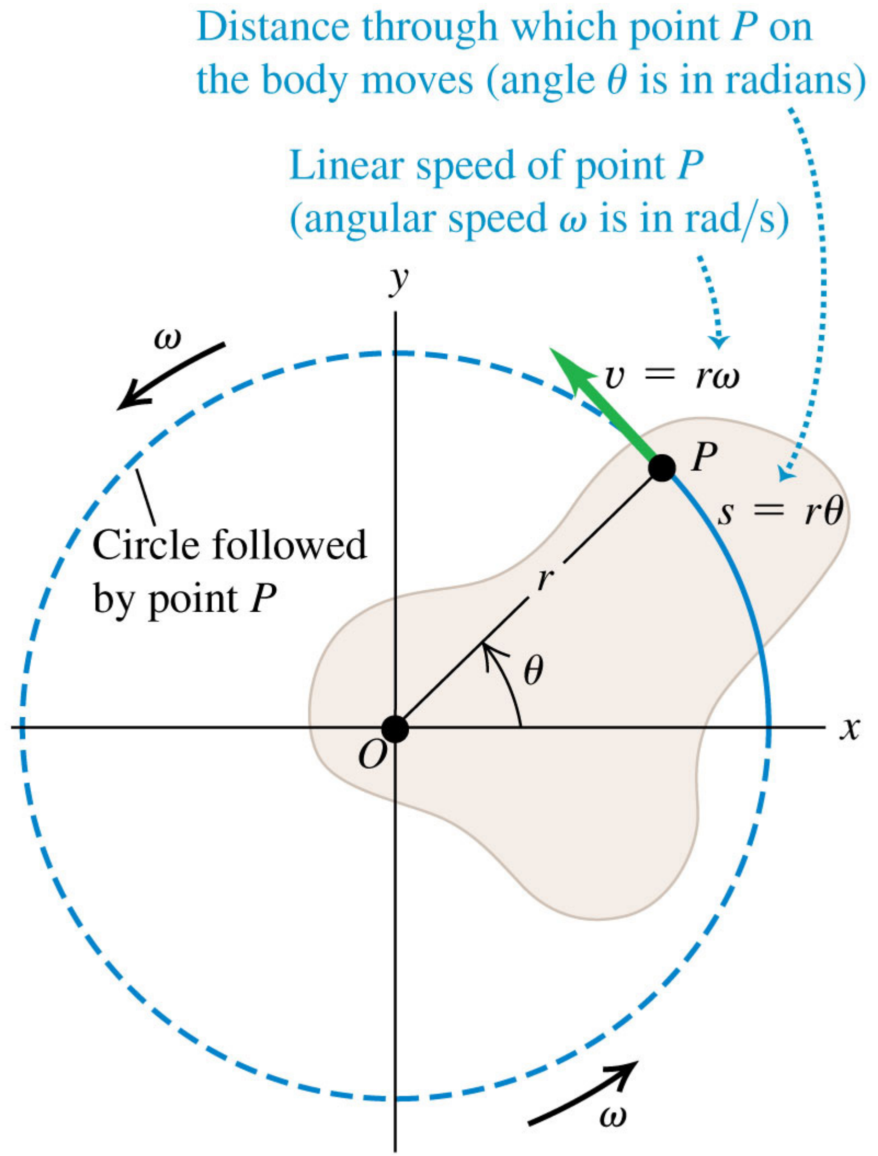
Rotation with constant angular acceleration

- The rotational formulas have the same form as the straight-line formulas, as shown in Table 9.1 below.

Straight-Line Motion with Constant Linear Acceleration	Fixed-Axis Rotation with Constant Angular Acceleration
$a_x = \text{constant}$	$\alpha_z = \text{constant}$
$v_x = v_{0x} + a_x t$	$\omega_z = \omega_{0z} + \alpha_z t$
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$

Relating linear and angular kinematics

- A point at a distance r from the axis of rotation has a linear speed of $v = r\omega$.

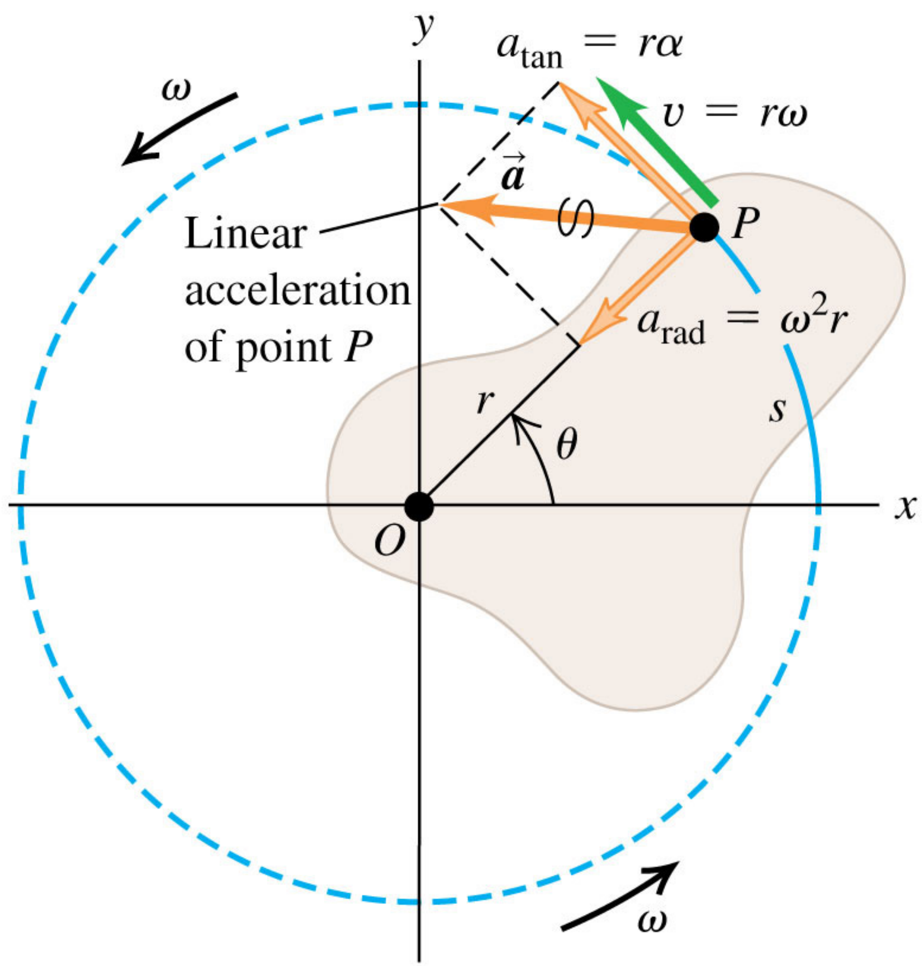


Relating linear and angular kinematics

- For a point at a distance r from the axis of rotation:
  - its tangential acceleration is $a_{\tan} = r\alpha$;
  - its centripetal (radial) acceleration is $a_{\mathrm{rad}} = v^2/r = r\omega$.

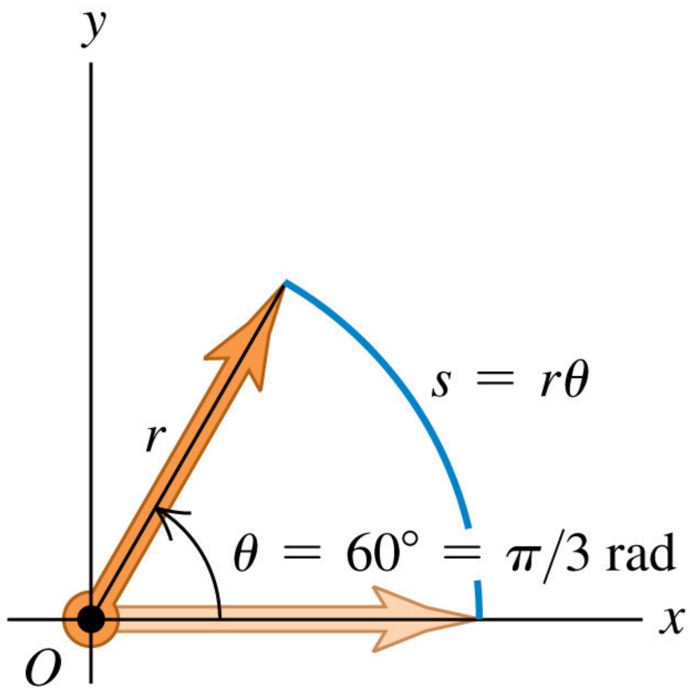
Radial and tangential acceleration components:

- $a_{\mathrm{rad}} = \omega^2 r$ is point P's centripetal acceleration.
- $a_{\tan} = r\alpha$ means that P's rotation is speeding up (the body has angular acceleration).



The importance of using radians, not degrees!

- Always use radians when relating linear and angular quantities.



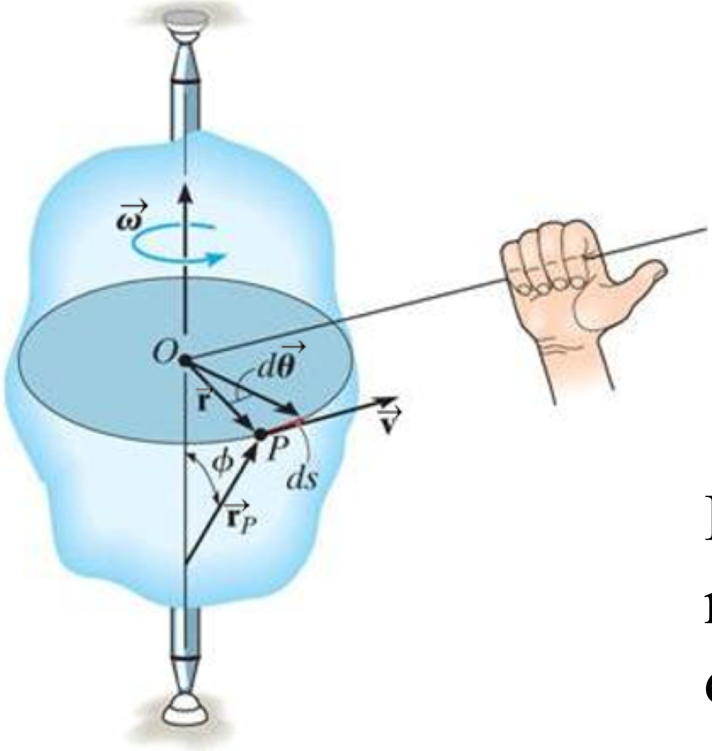
In any equation that relates linear quantities to angular quantities, the angles MUST be expressed in radians ...

RIGHT! ▶ $s = (\pi/3)r$

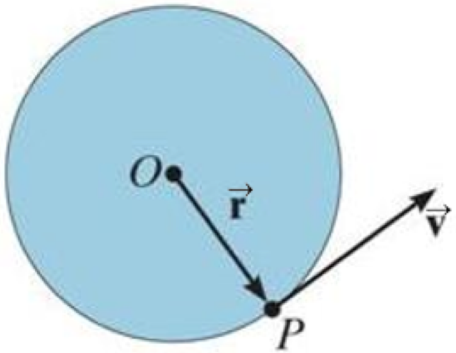
... never in degrees or revolutions.

WRONG ▶ $s = \cancel{60}r$

Rotation: velocity of point P



The magnitude of the velocity of P is equal to ωr. The velocity's direction is tangent to the circular path of P.

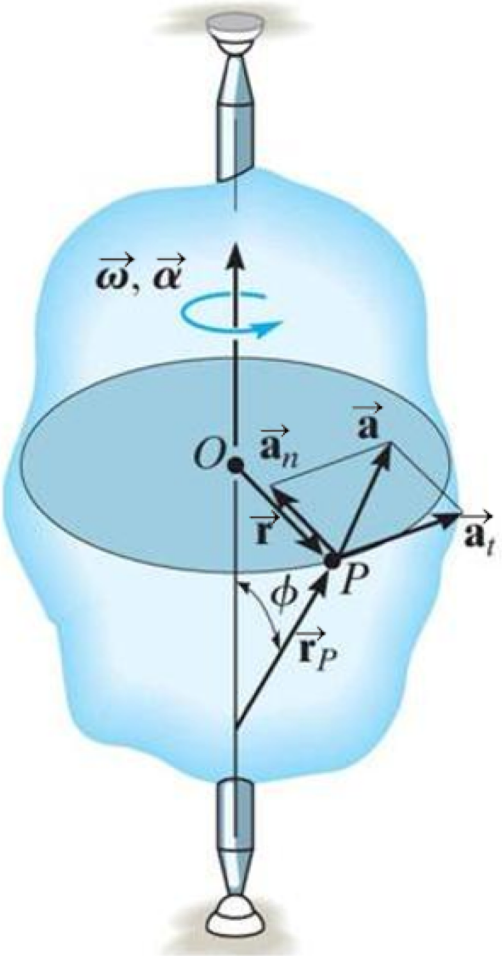


In the vector formulation, the magnitude and direction of $\vec{v}$ can be determined from the cross product of $\vec{\omega}$ and $\vec{r}_p$. Here $\vec{r}_p$ is a vector from any point on the axis of rotation to P.

$$\vec{v} = \vec{\omega} \times \vec{r}_p = \vec{\omega} \times \vec{r}$$

The direction of $\vec{v}$ is determined by the right-hand rule.

Rotation: acceleration of point P

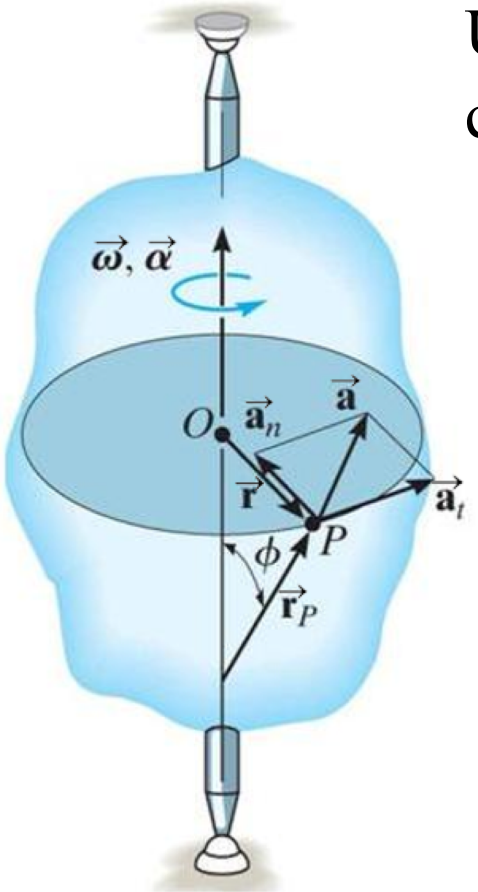


The acceleration of P is expressed in terms of its normal (a_n) and tangential (a_t) components.
In scalar form, these are $a_t = \alpha\ r$ and $a_n = \omega^2\ r$.

The tangential component, a_t, represents the time rate of change in the velocity's magnitude. It is directed tangent to the path of motion.

The normal component, a_n, represents the time rate of change in the velocity's direction. It is directed towards the center of the circular path.

Rotation: acceleration of point P



Using the vector formulation, the acceleration of P can also be defined by differentiating the velocity.

$$\vec{a} = d\vec{v}/dt = d\vec{\omega}/dt \times \vec{r}_P + \vec{\omega} \times d\vec{r}_P/dt$$

$$= \vec{\alpha} \times \vec{r}_P + \vec{\omega} \times (\vec{\omega} \times \vec{r}_P)$$

It can be shown that this equation reduces to

$$\vec{a} = \vec{\alpha} \times \vec{r} - \omega^2 \vec{r} = \vec{a}_t + \vec{a}_n$$

The magnitude of the acceleration vector is $a = \sqrt{(a_t)^2 + (a_n)^2}$

Rotation about a fixed axis: procedure

- Establish a sign convention along the axis of rotation.
- If a relationship is known between any two of the variables (α, ω, θ, or t), the other variables can be determined from the equations:
  $\omega = d\theta/dt \qquad \alpha = d\omega/dt \qquad \alpha\, d\theta = \omega\, d\omega$
- If a is constant, use the equations for constant angular acceleration.
- To determine the motion of a point, the scalar equations
  $v = \omega\, r$, $a_t = \alpha\, r$, $a_n = \omega^2 r$, and $a = \sqrt{(a_t)^2 + (a_n)^2}$ can be used.
- Alternatively, the vector form of the equations can be used (with $\hat{i}$, $\hat{j}$, $\hat{k}$ components).

$$\vec{v} = \vec{\omega} \times \vec{r}_P = \vec{\omega} \times \vec{r}$$

$$\vec{a} = \vec{a}_t + \vec{a}_n = \vec{\alpha} \times \vec{r}_P + \vec{\omega} \times (\vec{\omega} \times \vec{r}_P) = \vec{\alpha} \times \vec{r} - \omega^2 \vec{r}$$

Quiz

1. If a rigid body is in translation only, the velocity at points A and B on the rigid body ________ .

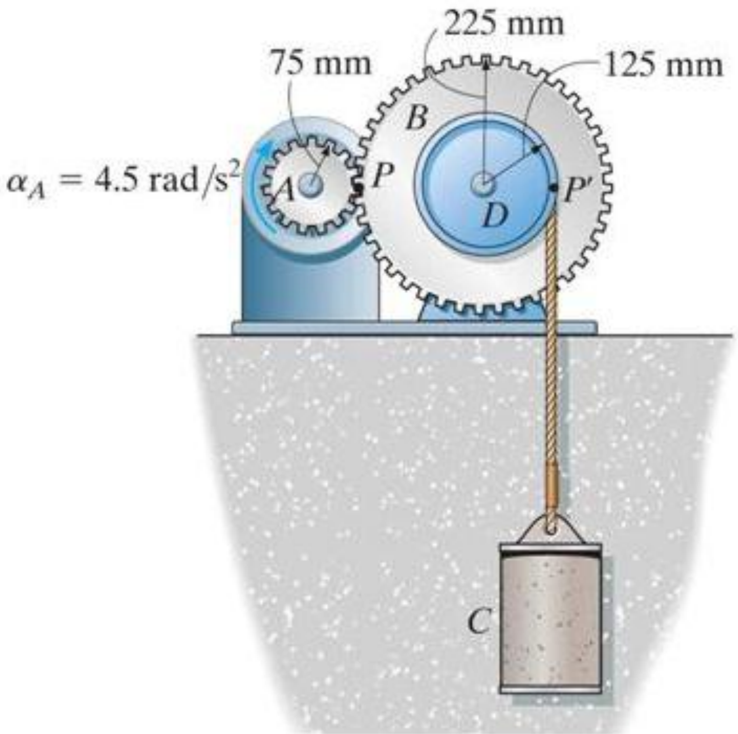
   A) are usually different
   ✔ B) are always the same
   C) depend on their position
   D) depend on their relative position

2. If a rigid body is rotating with a constant angular velocity about a fixed axis, the velocity vector at point P is ________.

   ✔ A) $\omega \times r_p$　　B) $r_p \times \omega$

   C) dr_p/dt　　D) All of the above.

Example



Given: The motor turns gear A with a constant angular acceleration, $\alpha_A = 4.5$ rad/s^2, starting from rest. The cord is wrapped around pulley D which is rigidly attached to gear B.

Find: The velocity of cylinder C and the distance it travels in 3 seconds.

Plan:

1) The angular acceleration of gear B (and pulley D) can be related to α_A.
2) The acceleration of cylinder C can be determined by using the equations of motion for a point on a rotating body since $(a_t)_D$ at point P is the same as a_c.
3) The velocity and distance of C can be found by using the constant acceleration equations.

Example

Solution:

1) Gear A and B will have the same speed and tangential component of acceleration at the point where they mesh. Thus,

$$a_t = \alpha_A r_A = \alpha_B r_B \quad \Rightarrow \quad (4.5)(75) = \alpha_B(225) \quad \Rightarrow \quad \alpha_B = 1.5 \text{ rad/s}^2$$

Since gear B and pulley D turn together, $\alpha_D = \alpha_B = 1.5 \text{ rad/s}^2$

2) Assuming the cord attached to pulley D is inextensible and does not slip, the velocity and acceleration of cylinder C will be the same as the velocity and tangential component of acceleration along the pulley D.

$$a_C = (a_t)_D = \alpha_D\, r_D = (1.5)(0.125) = 0.1875 \text{ m/s}^2 \uparrow$$

Example

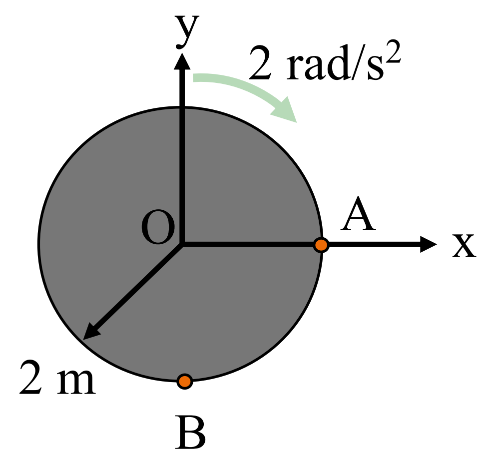
3) Since α_A is constant, α_D and a_C will be constant. The constant acceleration equation for rectilinear motion can be used to determine the velocity and displacement of cylinder C when t = 3 s ($s_0 = v_0 = 0$):

$$v_c = v_0 + a_C\, t = 0 + 0.1875 \text{ m/s}^2 (3 \text{ s}) = 0.563 \text{ m/s} \uparrow$$

$$s_c = s_0 + v_0\, t + (0.5)\, a_C\, t^2$$
$$= 0 + 0 + (0.5)\, 0.1875 \text{ m/s}^2 (3 \text{ s})^2 = 0.844 \text{ m} \uparrow$$

Quiz

1. A disk is rotating at 4 rad/s. If it is subjected to a constant angular acceleration of 2 rad/s^2, determine the acceleration at B.



A) (4 i + 32 j) m/s^2 B) (4 i - 32 j) m/s^2

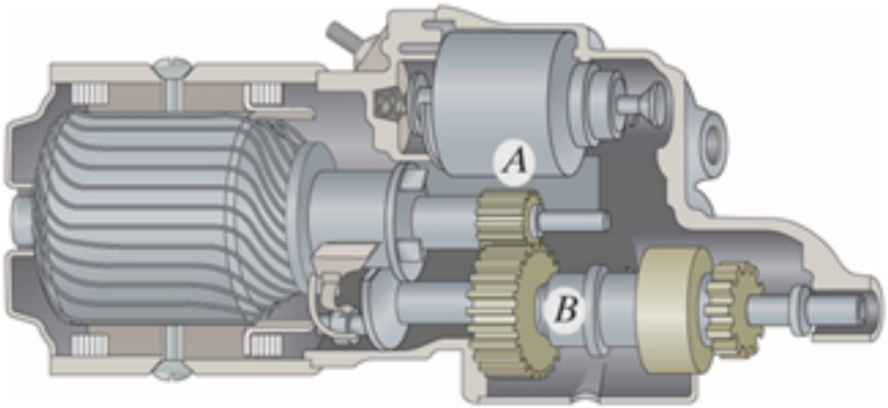
✔ C) (- 4 i + 32 j) m/s^2 D) (- 4 i -32 j) m/s^2

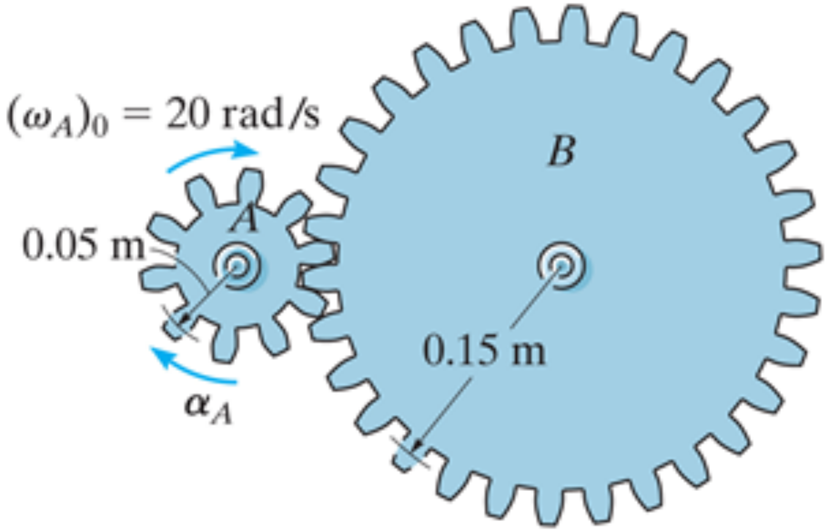
2. A Frisbee is thrown and curves to the right. It is experiencing

A) rectilinear translation. B) curvilinear translation.

C) pure rotation. ✔ D) general plane motion.

Example





Given: Gear A is given an angular acceleration $\alpha_A = 4t^3$ rad/s^2, where t is in seconds, and $(\omega_A)_0 = 20$ rad/s.

Find: The angular velocity and angular displacement of gear B when t = 2 s.

Plan:

1) Apply the kinematic equation of variable angular acceleration to find the angular velocity of gear A.

2) Find the relationship of angular motion between gear A and gear B in terms of time and then use 2 s.

Example

Solution:

1) Motion of Gear A : Applying the kinematic equation

$$\int_{20}^{\omega_A} d\omega_A = \int_0^t \alpha_A dt \;\Rightarrow\; \omega_A - 20 = \int_0^t 4t^3 dt = t^4$$

$$\Rightarrow\; \omega_A = t^4 + 20$$

$$\int_0^{\theta_A} d\theta_A = \int_0^t \omega_A dt$$

$$\Rightarrow\; \theta_A = \int_0^t (t^4 + 20) dt = \frac{1}{5} t^5 + 20\, t$$

When t=2 s, ω_A= 36 rad/s and θ_A= 46.4 rad.

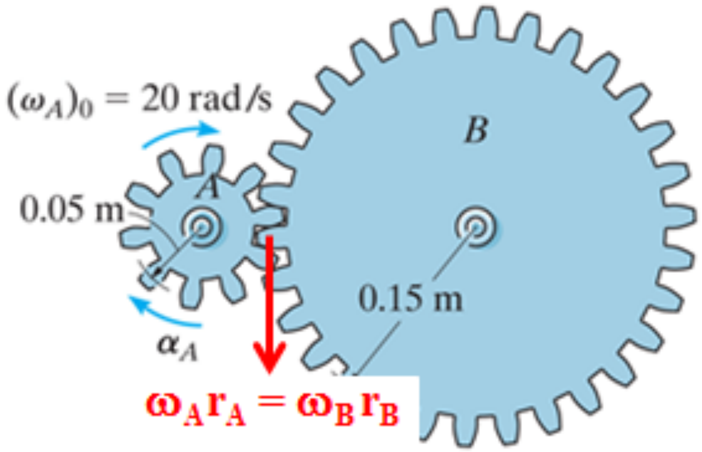
Example

2) Since gear B meshes with gear A,

$$\omega_A \, r_A = \omega_B \, r_B$$

$$\Rightarrow \omega_B = \omega_A \, (r_A / r_B) = \omega_A \, (0.05 / 0.15)$$

Similarly, $\theta_B = \theta_A (0.05 / 0.15)$



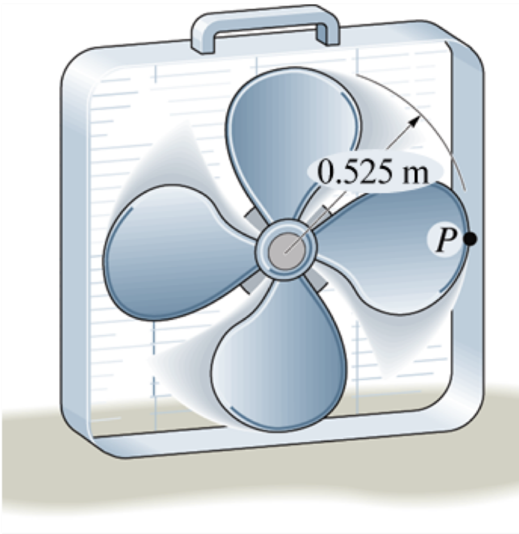
Since $\omega_A = 36$ rad/s and $\theta_A = 46.4$ rad at t=2 s,

$\omega_B = 36\,(0.05 / 0.15) =$ 12 rad/s

$\theta_B = 46.4\,(0.05 / 0.15) =$ 15.5 rad

Quiz

1. The fan blades suddenly experience an angular acceleration of 2 rad/s^2. If the blades are rotating with an initial angular velocity of 4 rad/s, determine the speed of point P when the blades have turned 2 revolutions (when ω = 8.14 rad/s).



✔A) 4.27 m/s B) 5.31 m/s

C) 6.93 m/s D) 8.0 m/s

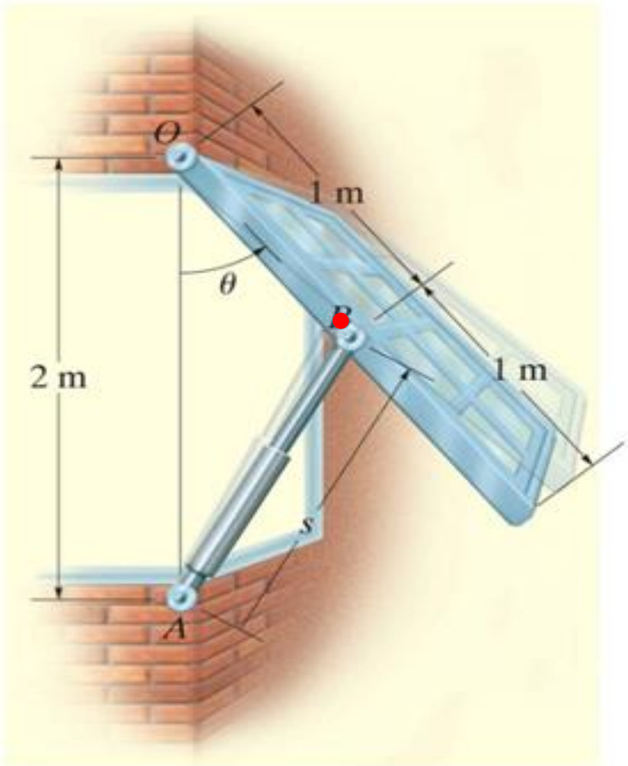
2. Determine the magnitude of the acceleration at P when the blades have turned the 2 revolutions.

A) 0 m/s^2 B) 1.05 m/s^2

C) 34.79 m/s^2 ✔ D) 34.81 m/s^2

Absolute motion analysis

The figure below shows the window using a hydraulic cylinder AB.



The absolute motion analysis method relates the position of a point, B, on a rigid body undergoing rectilinear motion to the angular position, θ, of a line contained in the body.

Once a relationship in the form of $s_B = f(\theta)$ is established, the velocity and acceleration of point B are obtained in terms of the angular velocity and angular acceleration of the rigid body by taking the first and second time derivatives of the position function.

Usually the chain rule must be used when taking the derivatives of the position coordinate equation.

Procedure for absolute motion analysis

The velocity and acceleration of a point undergoing rectilinear motion can be related to the angular velocity and angular acceleration of a line contained within a body using the following procedure.

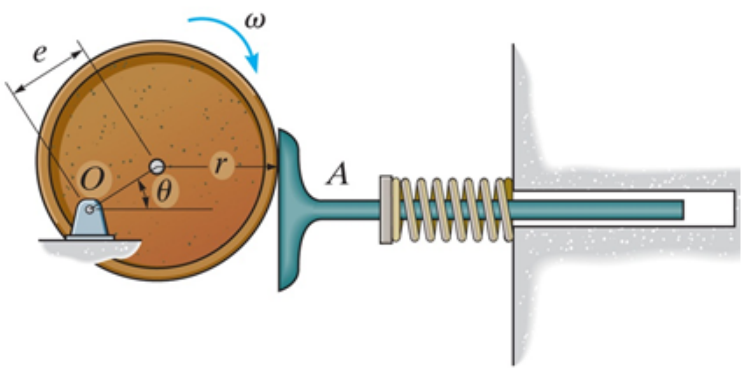
1. Locate point on the body using position coordinate s, which is measured from a fixed origin.
2. From a fixed reference line, measure the angular position θ of a line lying in the body. Using the dimensions of the body, relate s to θ, e.g., $s = f(\theta)$.
3. Take the first time derivative of $s = f(\theta)$ to get a relationship between v and ω.
4. Take the second time derivative to get a relationship between a and α.

Quiz

1. A body subjected to general plane motion undergoes a/an
   A) translation.
   B) rotation.
   ✔ C) simultaneous translation and rotation.
   D) out-of-plane movement.

2. In general plane motion, if the rigid body is represented by a slab, the slab rotates
   ✔ A) about an axis perpendicular to the plane.
   B) about an axis parallel to the plane.
   C) about an axis lying in the plane.
   D) None of the above.

Example 1



Given: A circular cam is rotating clockwise about O with a constant ω.

Find: The velocity and acceleration of the follower rod A as a function of θ.

Plan:

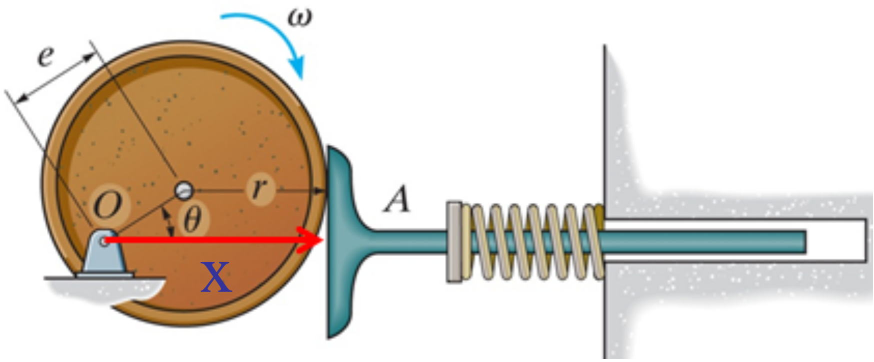
Set the coordinate x to be the distance between O and the rod A. Relate x to the angular position, θ. Then take time derivatives of the position equation to find the velocity and acceleration relationships.

Example 1

Solution:

Relate x, the distance between O and the rod, to θ.

$$x = e \cos\theta + r$$



Take time derivatives of the position to find the velocity and acceleration.

$$\dot{x} = e(-\sin\theta)\dot{\theta} + \dot{r}$$

Since r = constant $\Rightarrow \dot{x} = -e(\sin\theta)\dot{\theta}$

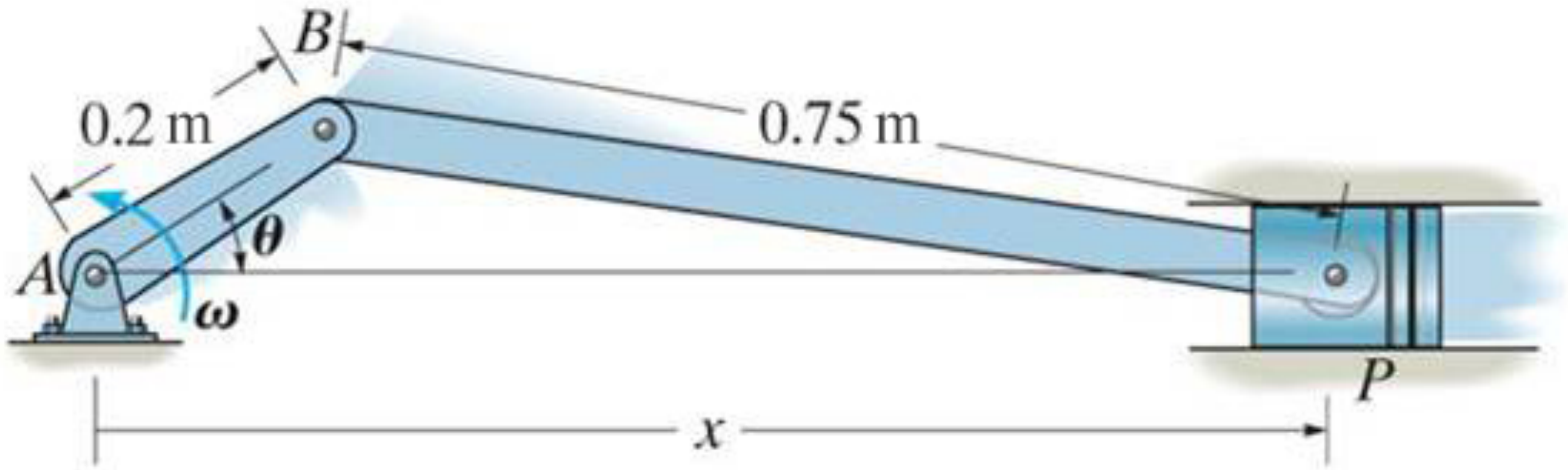
$$\ddot{x} = -e(\cos\theta)\,\dot{\theta}^2 - e(\sin\theta)\ddot{\theta}$$

Since $\dot{\theta}$ = constant $\Rightarrow \ddot{x} = -e(\cos\theta)\dot{\theta}^2$

Notice that the cam is rotating clockwise. $\Rightarrow \dot{\theta} = -\omega$

Therefore, $\dot{x} = e\,\omega(\sin\theta)$ and $\ddot{x} = e\,\omega^2(\cos\theta)$

Example 2

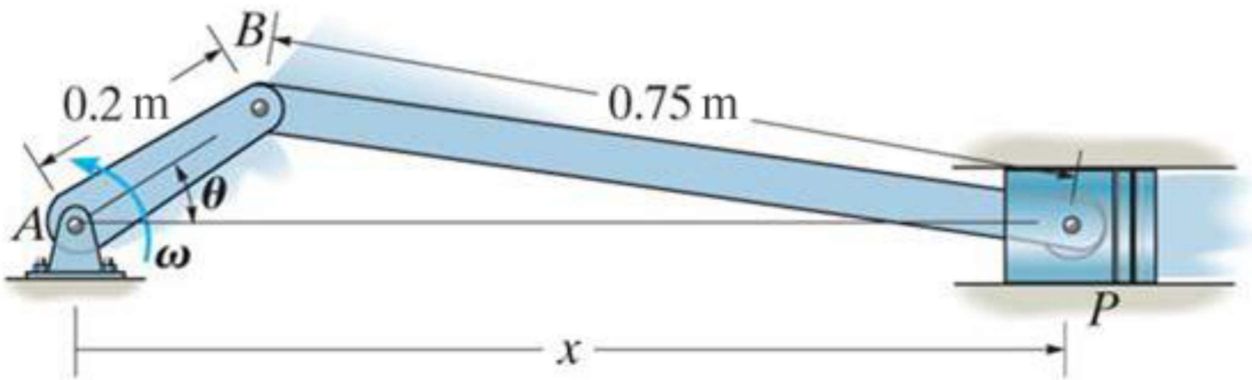


Given: Crank AB rotates at a constant velocity of $\omega = 150$ rad/s .

Find: The velocity of point P when $\theta = 30°$.

Plan: Define x as a function of θ and differentiate with respect to time.

Example 2



Solution:

$$x_P = 0.2\cos\theta + \sqrt{(0.75)^2 - (0.2\sin\theta)^2}$$

$$v_P = -0.2\omega\sin\theta + (0.5)[(0.75)^2 - (0.2\sin\theta)^2]^{-0.5}(-2)(0.2\sin\theta)(0.2\cos\theta)\,\omega$$

$$v_P = -0.2\omega\sin\theta - [0.5(0.2)^2\sin 2\theta\,\omega] / \sqrt{(0.75)^2 - (0.2\sin\theta)^2}$$

At $\theta = 30°$, $\omega = 150$ rad/s and

$v_P = -18.5$ m/s $= 18.5$ m/s $\leftarrow$

Quiz

1. The position, s, is given as a function of angular position, θ, as $s = 10 \sin 2\theta$. The velocity, v, is

   A) $20 \cos 2\theta$
   B) $20 \sin 2\theta$
   ✔C) $20\, \omega \cos 2\theta$
   D) $20\, \omega \sin 2\theta$

2. If $s = 10 \sin 2\theta$, the acceleration, a, is

   A) $20\, \alpha \sin 2\theta$
   ✔B) $20\, \alpha \cos 2\theta - 40\, \omega^2 \sin 2\theta$
   C) $20\, \alpha \cos 2\theta$
   D) $-40\, \alpha \sin 2\theta$

Relative motion analysis

When a body is subjected to general plane motion, it undergoes a combination of translation and rotation.

Point A is called the base point in this analysis. It generally has a known motion. The x'- y' frame translates with the body, but does not rotate. The displacement of point B can be written:

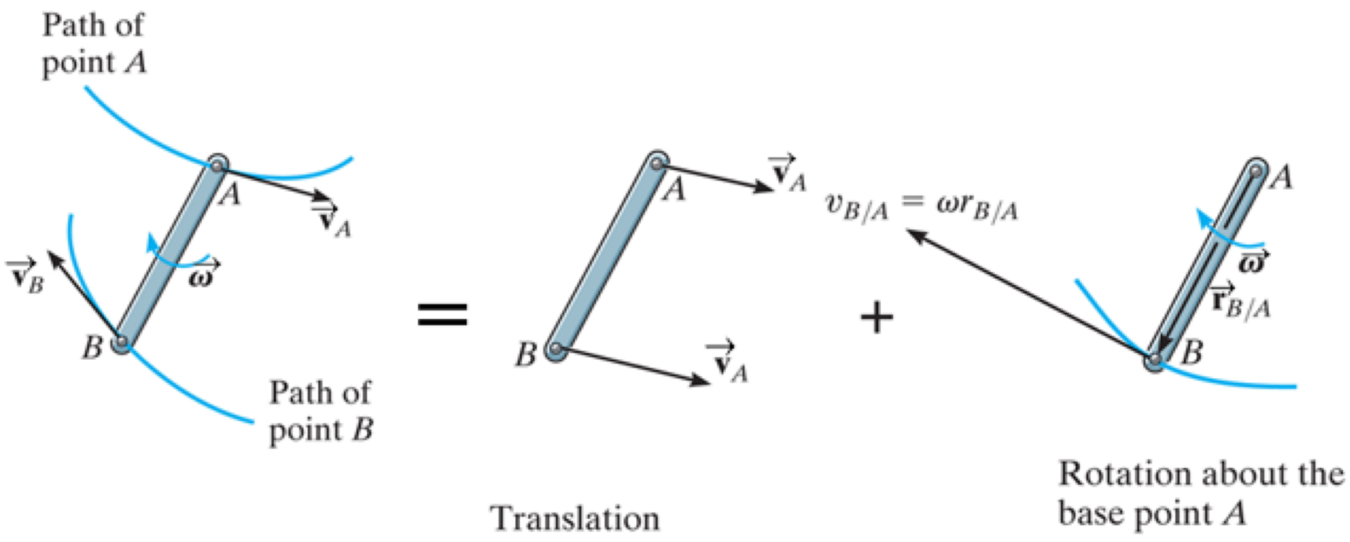
Disp. due to translation

$$d\vec{r}_B = d\vec{r}_A + d\vec{r}_{B/A}$$

Disp. due to translation and rotation

Disp. due to rotation

Relative motion analysis: velocity



The velocity at B is given as : $(dr_B/dt) = (dr_A/dt) + (dr_{B/A}/dt)$ or

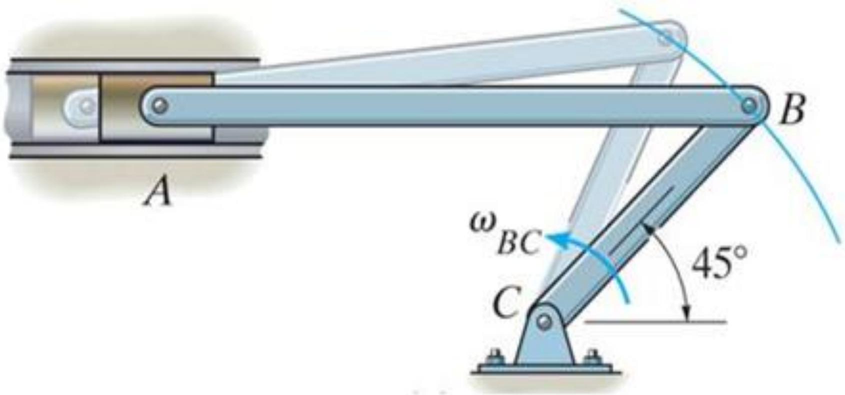
$v_B = v_A + v_{B/A}$

Since the body is taken as rotating about A,

$\vec{v}_{B/A} = d\vec{r}_{B/A}/dt = \vec{\omega} \times \vec{r}_{B/A}$

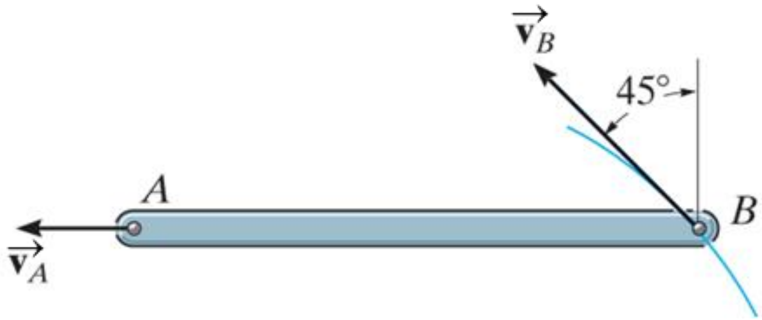
Here $\vec{w}$ will only have a $\hat{k}$ component since the axis of rotation is perpendicular to the plane of translation.

Relative motion analysis: velocity



$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

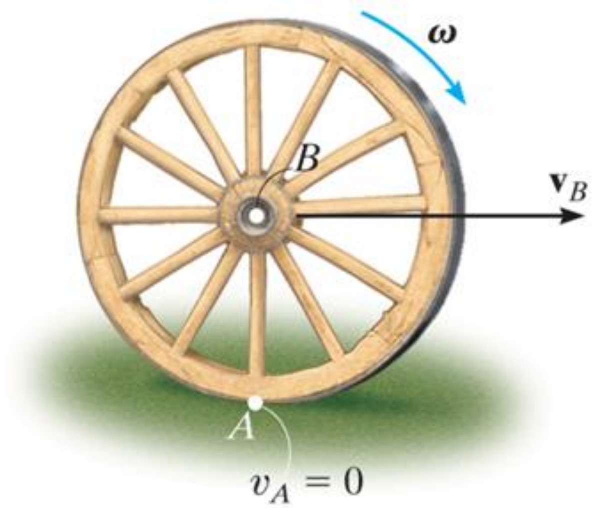
When using the relative velocity equation, points A and B should generally be points on the body with a known motion. Often these points are pin connections in linkages.



For example, point A on link AB must move along a horizontal path, whereas point B moves on a circular path.

The directions of $\vec{v}_A$ and $\vec{v}_B$ are known since they are always tangent to their paths of motion.

Relative motion analysis: velocity



$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

When a wheel rolls without slipping, point A is often selected to be at the point of contact with the ground.

Since there is no slipping, point A has zero velocity.

Furthermore, point B at the center of the wheel moves along a horizontal path. Thus, $\vec{v}_B$ has a known direction, e.g., parallel to the surface.

Procedure for analysis

The relative velocity equation can be applied using scalar x and y component equations or via a Cartesian vector analysis.

Scalar Analysis:

1. Establish the fixed x-y coordinate directions and draw a kinematic diagram for the body. Then establish the magnitude and direction of the relative velocity vector $\vec{v}_{B/A}$.

2. Write the equation $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$. In the kinematic diagram, represent the vectors graphically by showing their magnitudes and directions underneath each term.

3. Write the scalar equations from the x and y components of these graphical representations of the vectors. Solve for the unknowns.

Procedure for analysis

Vector Analysis:

1. Establish the fixed x - y coordinate directions and draw the kinematic diagram of the body, showing the vectors $\vec{v}_A$, $\vec{v}_B$, $\vec{r}_{B/A}$ and $\vec{w}$. If the magnitudes are unknown, the sense of direction may be assumed.

2. Express the vectors in Cartesian vector form (CVN) and substitute them into $\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$. Evaluate the cross product and equate respective $\hat{\imath}$ and $\hat{\jmath}$ components to obtain two scalar equations.

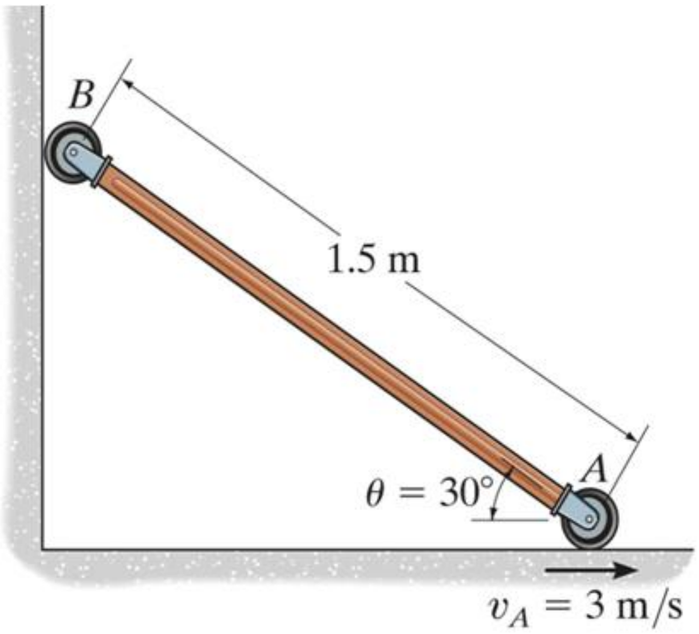
3. If the solution yields a negative answer, the sense of direction of the vector is opposite to that assumed.

Quiz

1. When a relative-motion analysis involving two sets of coordinate axes is used, the x' - y' coordinate system will
   A) be attached to the selected point for analysis.
   B) rotate with the body.
   C) not be allowed to translate with respect to the fixed frame.
   D) None of the above.

2. In the relative velocity equation, $\vec{v}_{B/A}$ is
   A) the relative velocity of B with respect to A.
   B) due to the rotational motion.
   C) $\vec{\omega} \times \vec{r}_{B/A}$.
   D) All of the above.

Example 1



Given: Roller A is moving to the right at 3 m/s.

Find: The velocity of B at the instant $\theta = 30°$.

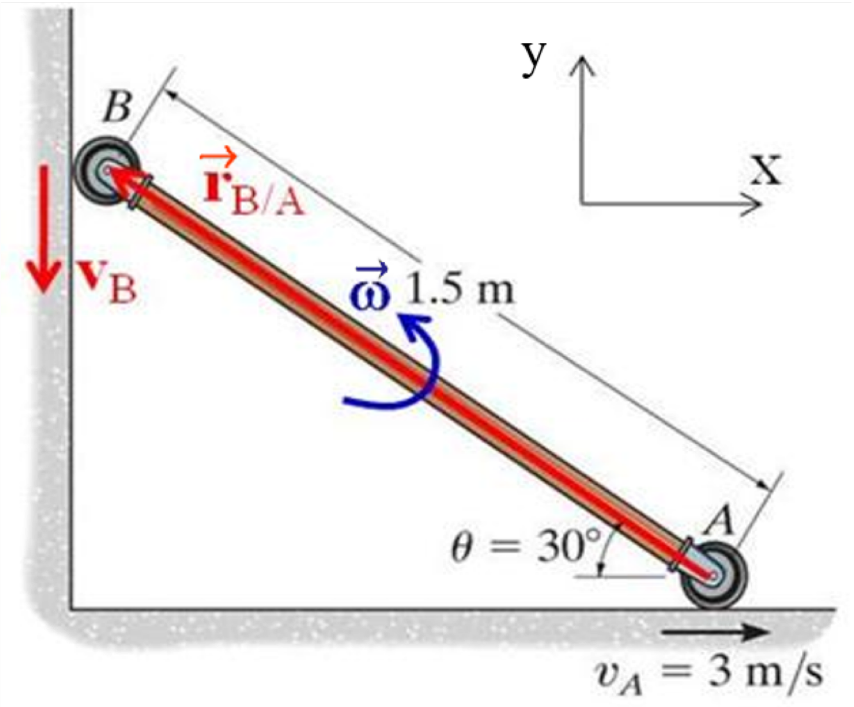
Plan:

1. Establish the fixed x - y directions and draw a kinematic diagram of the bar and rollers.
2. Express each of the velocity vectors for A and B in terms of their $\hat{i}, \hat{j}, \hat{k}$ components and solve $\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$.

Example 1

Solution:

Kinematic diagram:



Express the velocity vectors in CVN

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

$$-v_B\hat{j} = 3\,i + [\,\omega\hat{k} \times$$
$$(-1.5\cos 30°\,\hat{\imath} + 1.5\sin 30°\,\hat{j})]$$

$$-v_B\hat{j} = 3\hat{\imath} - 1.299\,\omega\hat{j} - 0.75\,\omega\,\hat{\imath}$$

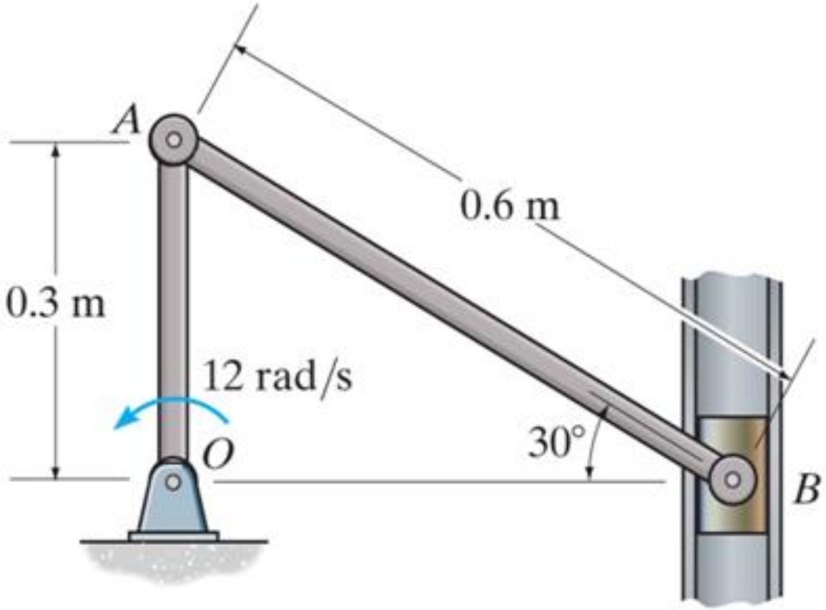
Equating the $\hat{\imath}$ and $\hat{j}$ components gives:

$$0 = 3 - 0.75\,\omega$$
$$-v_B = -1.299\,\omega$$

Solving: $\omega = 4$ rad/s or $\omega = 4$ rad/s $\hat{k}$

$v_B = 5.2$ m/s or $v_B = -5.2$ m/s $\hat{j}$

Example 2



Given: Crank rotates OA with an angular velocity of 12 rad/s.

Find: The velocity of piston B and the angular velocity of rod AB.

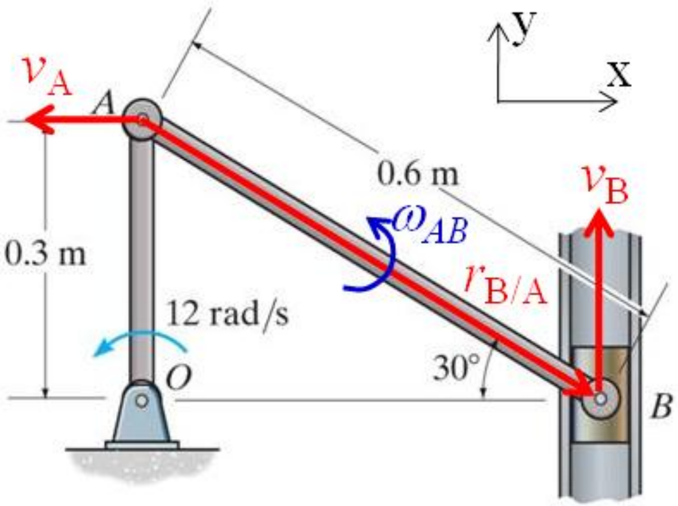
Plan:

Notice that point A moves on a circular path. The directions of $\vec{v}_A$ is tangent to its path of motion. Draw a kinematic diagram of rod AB and use $\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A}$.

Example

Solution:

Kinematic diagram of AB:



Since crack OA rotates with an angular velocity of 12 rad/s, the velocity at A will be:

$\vec{v}_A = -0.3(12)\hat{\imath} = -3.6\,\hat{\imath}$ m/s

Rod AB. Write the relative-velocity equation:

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A}$$

$$\vec{v}_B \boldsymbol{j} = -3.6\hat{\imath} + \omega_{AB}\hat{k} \times (0.6\cos 30°\,\hat{\imath} - 0.6\sin 30°\,\hat{\jmath})$$

$$\vec{v}_B \boldsymbol{j} = -3.6\hat{\imath} + 0.5196\,\omega_{AB}\,\hat{\jmath} + 0.3\,\omega_{AB}\,\hat{\imath}$$

By comparing the i, j components:

i: $0 = -3.6 + 0.3\,\omega_{AB}$ $\Rightarrow$ $\omega_{AB} = 12$ rad/s

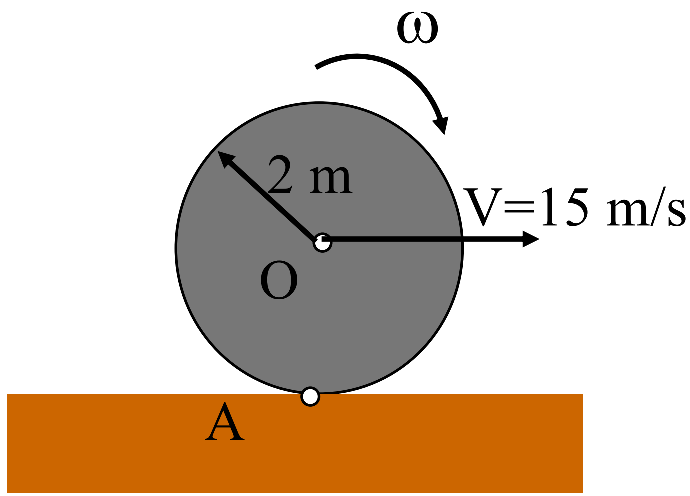
j: $v_B = 0.5196\,\omega_{AB}$ $\Rightarrow$ $v_B = 6.24$ m/s

Quiz

1. If the disk is moving with a velocity at point O of 15 m/s and ω = 2 rad/s, determine the velocity at A.

   A) 0 m/s   B) 4 m/s

   C) 15 m/s   ✔D) 11 m/s



2. If the velocity at A is zero, then determine the angular velocity, ω.

   A) 30 rad/s   B) 0 rad/s

   ✔C) 7.5 rad/s   D) 15 rad/s

Relative motion analysis: acceleration

The equation relating the accelerations of two points on the body is determined by differentiating the velocity equation with respect to time.

$$\frac{d\vec{v}_B}{dt} = \frac{d\vec{v}_A}{dt} + \frac{d\vec{v}_{B/A}}{dt}$$

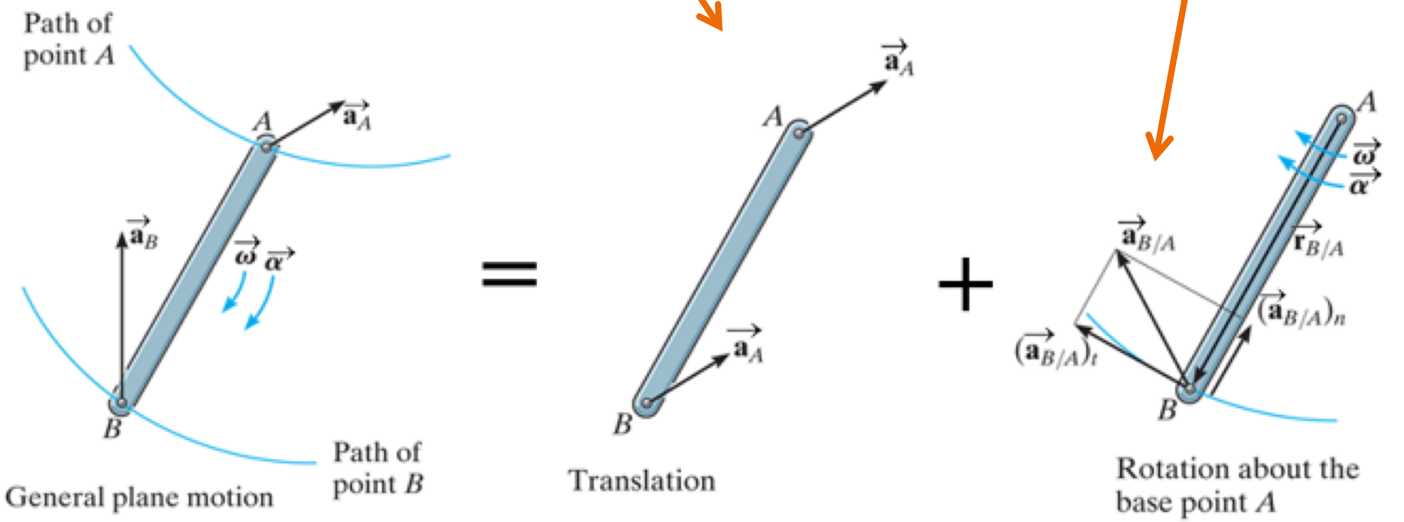
These are absolute accelerations of points A and B. They are measured from a set of fixed x,y axes.

This term is the acceleration of B with respect to A and includes both tangential and normal components.

The result is $\vec{a}_B = \vec{a}_A + (\vec{a}_{B/A})_t + (\vec{a}_{B/A})_n$

Relative motion analysis: acceleration

Graphically: $\vec{a}_B = \vec{a}_A + (\vec{a}_{B/A})_t + (\vec{a}_{B/A})_n$



The relative tangential acceleration component $(\vec{a}_{B/A})_t$ is $(\vec{\alpha} \times \vec{r}_{B/A})$ and perpendicular to $\vec{r}_{B/A}$.

The relative normal acceleration component $(\vec{a}_{B/A})_n$ is $(-\omega^2 \, \vec{r}_{B/A})$ and the direction is always from B towards A.

Relative motion analysis: acceleration

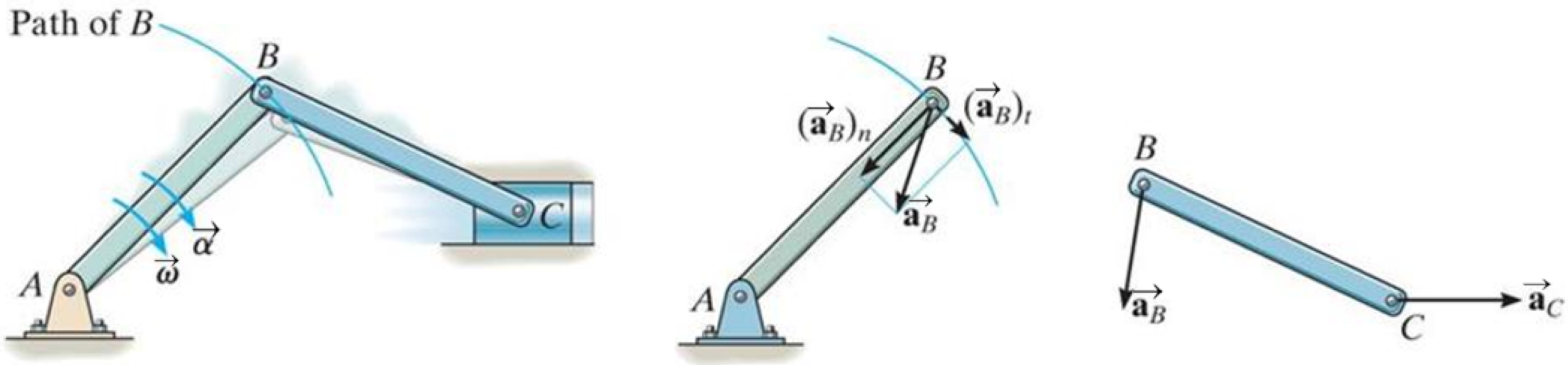
Since the relative acceleration components can be expressed as $(\vec{a}_{B/A})_t = \vec{\alpha} \times \vec{r}_{B/A}$ and $(\vec{a}_{B/A})_n = -\,\omega^2\,\vec{r}_{B/A}$, the relative acceleration equation becomes

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2\,\vec{r}_{B/A}$$

Note that the last term in the relative acceleration equation is not a cross product. It is the product of a scalar (square of the magnitude of angular velocity, ω^2) and the relative position vector, $\vec{r}_{B/A}$.

Application of the relative acceleration eqn

In applying the relative acceleration equation, the two points used in the analysis (A and B) should generally be selected as points which have a known motion, such as pin connections with other bodies.



In this mechanism, point B is known to travel along a circular path, so a_B can be expressed in terms of its normal and tangential components. Note that point B on link BC will have the same acceleration as point B on link AB.

Point C, connecting link BC and the piston, moves along a straight-line path. Hence, a_C is directed horizontally.

Procedure for analysis

1. Establish a fixed coordinate system.

2. Draw the kinematic diagram of the body.

3. Indicate on it $\vec{a}_A$, $\vec{a}_B$, $\vec{\omega}$, $\vec{\alpha}$, and $\vec{r}_{B/A}$. If the points A and B move along curved paths, then their accelerations should be indicated in terms of their tangential and normal components, i.e., $\vec{a}_A = (\vec{a}_A)_t + (\vec{a}_A)_n$ and $\vec{a}_B = (\vec{a}_B)_t + (\vec{a}_B)_n$.

4. Apply the relative acceleration equation:

$$\vec{\mathbf{a}}_B = \vec{\mathbf{a}}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}$$

5. If the solution yields a negative answer for an unknown magnitude, this indicates that the sense of direction of the vector is opposite to that shown on the diagram.

Quiz

1. If two bodies contact one another without slipping, and the points in contact move along different paths, the tangential components of acceleration will be ______ and the normal components of acceleration will be _________.

   A) the same, the same

   ✔ B) the same, different

   C) different, the same

   D) different, different

2. When considering a point on a rigid body in general plane motion,

   ✔ A) It's total acceleration consists of both absolute acceleration and relative acceleration components.

   B) It's total acceleration consists of only absolute acceleration components.

   C) It's relative acceleration component is always normal to the path.

   D) None of the above.