MA1300 Solutions to **Self Practice** # 6

1. Find an equation of the normal line to the parabola $y = x^2 - 5x + 4$ that is parallel to the line $x - 3y = 5$.

Solution: From the equation of the parabola we have $dy/dx = 2x - 5$. Since the slope of the line $x - 3y = 5$ is $1/3$, the slope of the tangent line is the negative reciprocal of $1/3$, namely, -3. We take $2x - 5 = -3$ to give $x = 1$. So the tangent (or intersection) point is $(1, 0)$, and we can use the point-slope form to write the equation of the normal line

$$y - 0 = \frac{1}{3}(x - 1), \qquad \text{or} \qquad y = \frac{x-1}{3}.$$

2. Where does the normal line to the parabola $y = x - x^2$ at the point **(1,0)** ~~(0,1)~~ intersect the parabola a second time? Illustrate with a sketch.

Solution: From the equation of the parabola we have $dy/dx = 1 - 2x$. The slope of the tangent line at $(1, 0)$ is -1, so the slope of the normal line is 1, and thus the equation of the normal line is

$$y = x - 1.$$

Solve the equations

$$\begin{cases} y = x - 1, \\ y = x - x^2, \end{cases}$$

to give $x = \pm 1$. Discarding $x = 1$, we get the other intersect point $(-1, -2)$. The whole picture is illustrated in Figure 1.

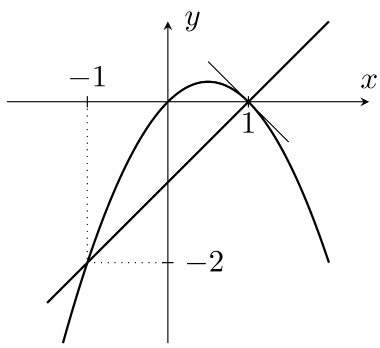

Figure 1: The picture of Problem 2. The parabola $y = x - x^2$ and the normal line.

3.

a Use the Product Rule twice to prove that if f, g, and h are differentiable, then $(fgh)' = f'gh + fg'h + fgh'$.

b Taking $f = g = h$ in part **a**, show that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x).$$

c Use part **b** to differentiate $y = (x^4 + 3x^3 + 17x + 82)^3$

Proof:

a $(fgh)' = ((fg)h)' = (fg)'h + (fg)h' = f'gh + fg'h + fgh'$.

b When $f = g = h$, $(f^3)' = (fgh)' = 3f'f^2$.

c We have

$$\frac{d}{dx}(x^4 + 3x^3 + 17x + 82)^3 = 3(x^4 + 3x^3 + 17x + 82)^2(4x^3 + 9x^2 + 17).$$

4.

a For what values of x is the function $f(x) = |x^2 - 9|$ differentiable? Find a formula for f'.

b Sketch the graphs of f and f'.

Solution: We rewrite f as

$$f(x) = \begin{cases} x^2 - 9 & \text{if } |x| \geq 3, \\ 9 - x^2 & \text{if } |x| < 3. \end{cases}$$

Since

$$\begin{aligned} \lim_{x \to 3^-} \frac{f(x) - f(3)}{x - 3} &= \lim_{x \to 3^-} \frac{9 - x^2 - 0}{x - 3} = -6, \\ \lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3} &= \lim_{x \to 3^+} \frac{x^2 - 9}{x - 3} = 6 \neq -6. \end{aligned}$$

So $f(x)$ is not differentiable at 3, nor similarly, -3. For other real numbers, we have

$$f'(x) = \begin{cases} 2x & \text{if } |x| > 3, \\ -2x & \text{if } |x| < -3. \end{cases}$$

The graphs of f and f' are shown in Figure 2.

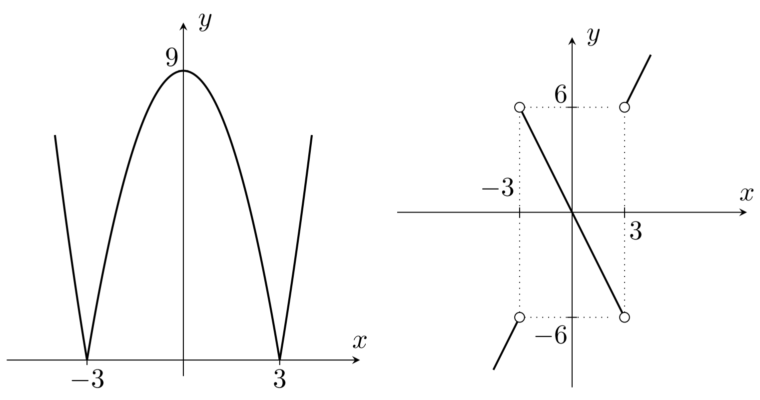

Figure 2: The picture of Problem 4. Left: $y = f(x)$, Right: $y = f'(x)$.

5.

a If $F(x) = f(x)g(x)$, where f and g have derivatives of all orders, show that $F'' = f''g + 2f'g' + fg''$.

b Find similar formulas for F''' and $F^{(4)}$.

c Guess a formula for $F^{(n)}$.

Solution:

a

$$F'' = (F')' = (f'g + g'f)' = f''g + 2f'g' + fg''.$$

b

$$\begin{aligned} F''' &= (f''g + 2f'g' + fg'')' = f'''g + 3f''g' + 3f'g'' + fg''', \\ F^{(4)} &= f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)}. \end{aligned}$$

c

$$F^{(n)} = \sum_{i=1}^{n} \binom{n}{i} f^{(n-i)} g^{(i)},$$

where $\binom{n}{i} := \frac{n!}{i!(n-i)!}$.

6. Differentiate

$$y = u(a \cos u + b \cot u).$$

Solution:

$$\frac{d}{du} u(a \cos u + b \cot u) = a \cos u + b \cot u - au \sin u - bu \csc^2 u.$$

7.

a Use the Quotient Rule to differentiate the function

$$f(x) = \frac{\tan x - 1}{\sec x}.$$

b Simplify the expression for $f(x)$ by writing it in terms of $\sin x$ and $\cos x$, and then find $f'(x)$.

c Show that your answers to parts **a** and **b** are equivalent.

Solution:

a

$$f' = \frac{(\tan^2 x + 1) \sec x - \sec x \tan x (\tan x - 1)}{\sec^2 x} = \frac{\tan x + 1}{\sec x}.$$

b We can rewrite f as $f(x) = \sin x - \cos x$, therefore

$$f'(x) = \cos x + \sin x.$$

c Since

$$\frac{\tan x + 1}{\sec x} = \frac{\sin x + \cos x}{1} = \cos x + \sin x,$$

the answers to part **(a)** and **(b)** are equivalent.

8. Suppose $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let

$$g(x) = f(x)\sin x,$$

and

$$h(x) = \frac{\cos x}{f(x)}.$$

Find **(a)** $g'(\pi/3)$, and **(b)** $h'(\pi/3)$.

Solution: Since

$$g'(x) = f'(x)\sin x + f(x)\cos x,$$

we have

$$g'(\pi/3) = f'(\pi/3)\frac{\sqrt{3}}{2} + f(\pi/3)\frac{1}{2} = 2 - \sqrt{3}.$$

Similarly, because

$$h'(x) = \frac{-f(x)\sin x - f'(x)\cos x}{\cancel{\cos^2 x}\ (f(x))^2},$$

we have

$$h'(\pi/3) = \cancel{4(-4\sqrt{3}/2 + 2/2) = 4 - 8\sqrt{3}}.$$ (-2*sqrt(3) +1)/16

9. An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2\cos t + 3\sin t$, $t \geq 0$, where s is measured in centimeters and t in seconds. (Take the positive direction to be downward.)

a Find the velocity and acceleration at time t.

b Graph the velocity and acceleration functions.

c When does the mass pass through the equilibrium position for the first time?

d How far from its equilibrium position does the mass travel?

e When is the speed the greatest?

Solution:

a Velocity $v = 3\cos t - 2\sin t$, acceleration $a = -2\cos t - 3\sin t$.

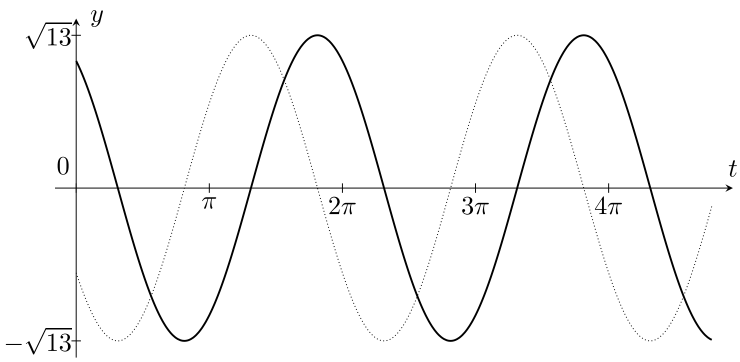

Figure 3: The picture of Problem 9. Solid line: velocity; dotted line: acceleration.

b See Figure 3

c Solve $s = 0$ to obtain $\tan t = -\frac{2}{3}$. For $t > 0$, the smallest solution is $t = \pi - \arctan\frac{2}{3} \approx 2.55(s)$.

d Solve $v = 0$ to obtain $t = \arctan\frac{3}{2}$, which is the time when the mass travels to the position of the maximum distance, namely

$$\left| s\big|_{t=\arctan\frac{3}{2}} \right| = \left| 2 \cdot \frac{2}{\sqrt{13}} + 3 \cdot \frac{3}{\sqrt{13}} \right| = \sqrt{13} \approx 3.61(\text{cm})$$

e The time when the speed is the great is that when $a = 0$. Solve the equation to give $t = (2n + 1)\pi - \arctan\frac{2}{3}$, $n = 0, 1, 2, \cdots$.

10. An Object with mass m is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu m g}{\mu \sin\theta + \cos\theta},$$

where μ is a constant called the *coefficient of friction*.

a Find the rate of change of F with respect to θ.

b When is this rate of change equal to 0?

c If $m = 20$ kg, $g = 9.8$ m/s^2, and $\mu = 0.6$, draw the graph of F as a function of θ and use it to locate the value of θ for which $dF/d\theta = 0$. Is the value consistent with your answer to part **b**?

Solution:

a

$$\frac{dF}{d\theta} = \frac{-\mu m g}{(\mu \sin\theta + \cos\theta)^2}(\mu\cos\theta - \sin\theta).$$

b Solve $\frac{dF}{d\theta} = 0$ to give $\theta = \arctan\mu$.

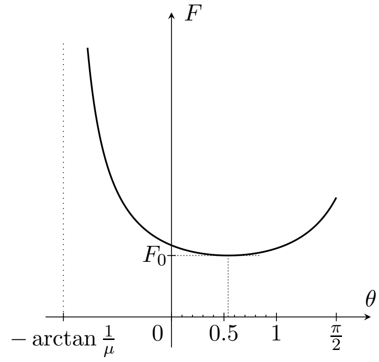

Figure 4: The picture of Problem 10. $F_0 = \frac{0.6\times 20\times 9.8}{\sqrt{0.6^2+1^2}}$.

c The graph is shown in Figure 4. From the graph we see that the θ which makes $dF/d\theta = 0$ is approximately 0.5, while according to **(b)**, $\theta = \arctan 0.6 \approx 0.5404$, consistent.

11. Find the derivative of the function.

$$f(t) = \sqrt[3]{1+\tan t}, \qquad y = \cos(a^3 + x^3), \qquad y = \cot^2(\sin\theta),$$
$$y = \sin(\sin(\sin x)), \qquad y = \cos\sqrt{\sin(\tan \pi x)}.$$

Solution:

$$\begin{aligned}
&\frac{d}{dt}\sqrt[3]{1+\tan t} = \frac{1}{3}(1+\tan t)^{-\frac{2}{3}}(1+\tan^2 t),\\
&\frac{d}{dx}\cos(a^3+x^3) = -3x^2\sin(a^3+x^3),\\
&\frac{d}{d\theta}\cot^2(\sin\theta) = -2\cot(\sin\theta)\cdot\csc^2(\sin\theta)\cdot\cos\theta,\\
&\frac{d}{dx}\sin(\sin(\sin x)) = \cos(\sin(\sin x))\cdot\cos(\sin x)\cdot\cos x,\\
&\frac{d}{dx}\cos\sqrt{\sin(\tan\pi x)} = -\frac{\pi}{2}\sin\sqrt{\sin(\tan\pi x)}\cdot\frac{\cos(\tan\pi x)}{\sqrt{\sin(\tan\pi x)}}(1+\tan^2\pi x).
\end{aligned}$$

12. Suppose f is differentiable on $\mathbb{R}$ and α is a real number. Let $F(x) = f(x^\alpha)$ and $G(x) = [f(x)]^\alpha$. Find expressions for **(a)** $F'(x)$ and **(b)** $G'(x)$.

Solution: $F'(x) = \alpha x^{\alpha-1}f'(x^\alpha)$, $G'(x) = \alpha[f(x)]^{\alpha-1}f'(x)$.

13. If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$, find $F'(0)$.

Solution: $F'(x) = f'(3f(4f(x)))\cdot 3f'(4f(x))\ \cdot 4f'(x)$, so $F'(0) = 2\cdot 3\cdot 2\cdot 4\cdot 2 = 96$.

14. If the equation of motion of a particle is given by $s = A\cos(\omega t+\delta)$, the particle is said to undergo *simple harmonic motion.*

a Find the velocity of the particle at time t.

b When is the velocity 0?

Solution:

a Velocity $v = \frac{ds}{dt} = -A\omega \sin(\omega t + \delta)$.

b $v = 0$ when $\sin(\omega t + \delta) = 0$, or equivalently, $t = \frac{n\pi - \delta}{\omega}$, $n = 0, \pm 1, \pm 2, \cdots$.