Unit 5

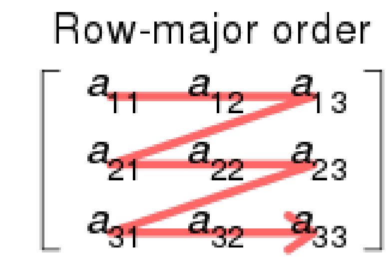
Functions

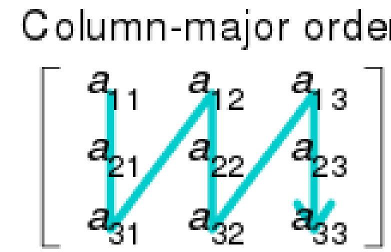
Albert Sung

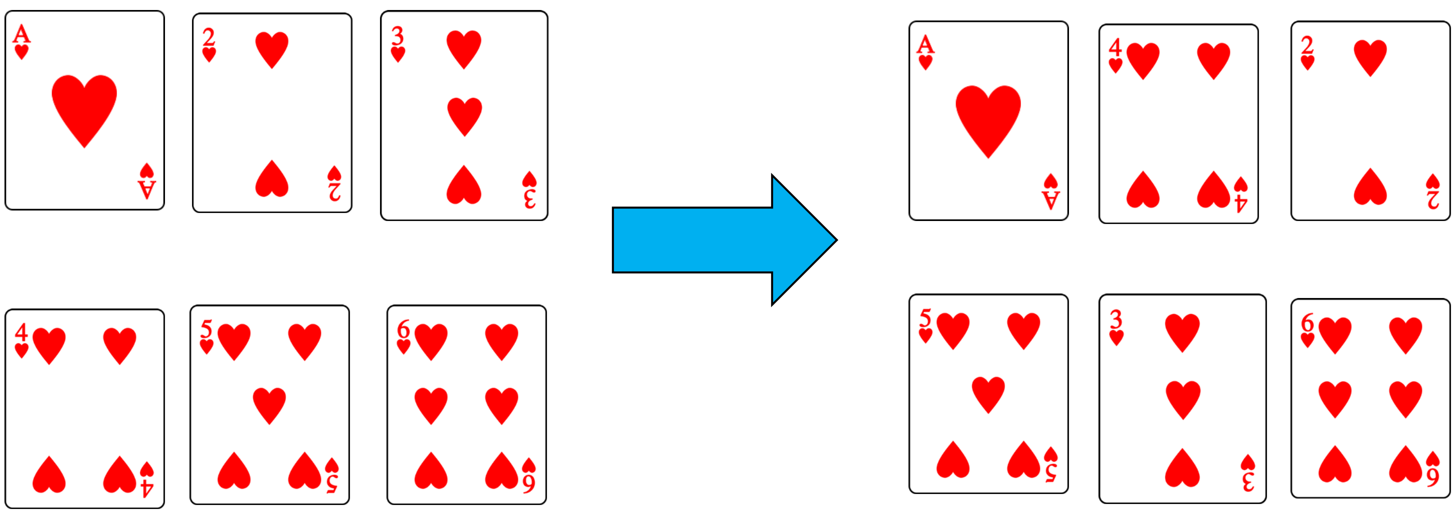
Outline of Unit 5

- 5.1 Basic Concepts
- 5.2 Compositions of Functions
- 5.3 One-to-One and Onto
- 5.4 Permutation Functions

1. Put the 6 cards in a 2×3 matrix in row-major order.
2. Take them out in column-major order.
3. Put them back in row-major order.







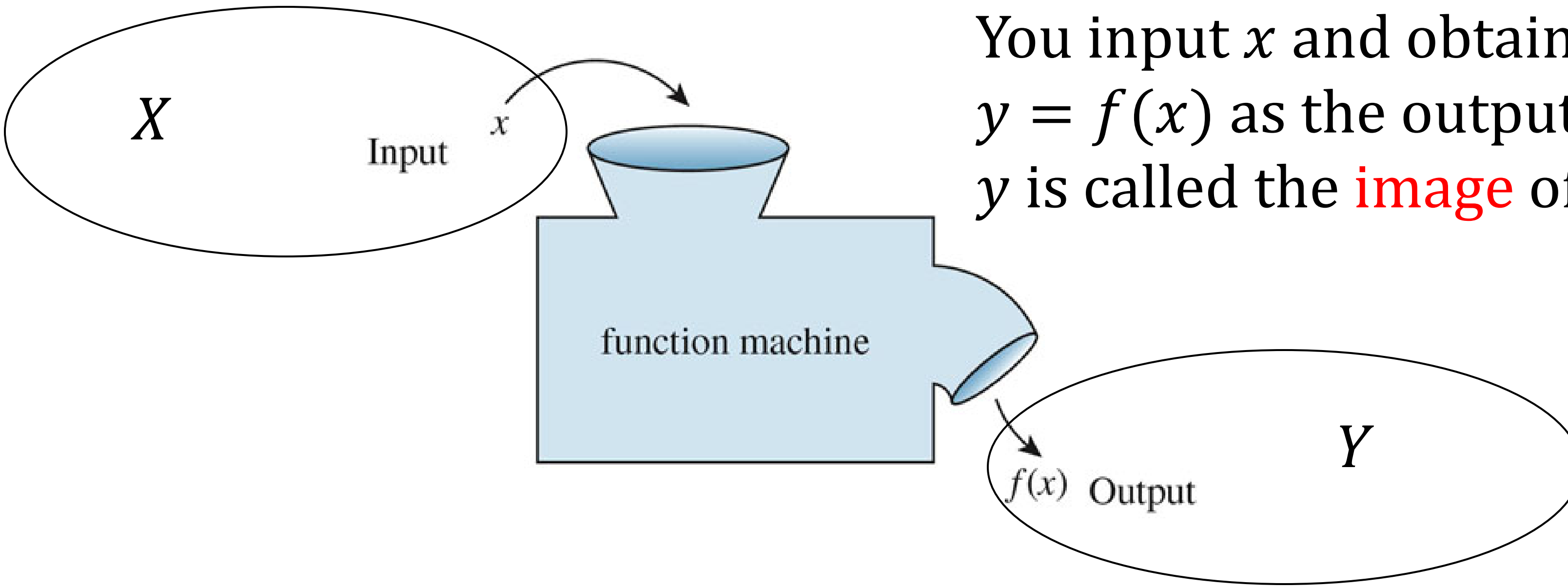
If you repeat the procedure, will the cards be eventually in the original positions?

Unit 5.1

Basic Concepts

Definition of Functions

- A function f from X to Y, denoted by $f: X \rightarrow Y$, (or f maps X to Y) is an assignment of each element of X to exactly one element of Y.
  - X and Y are nonempty sets.



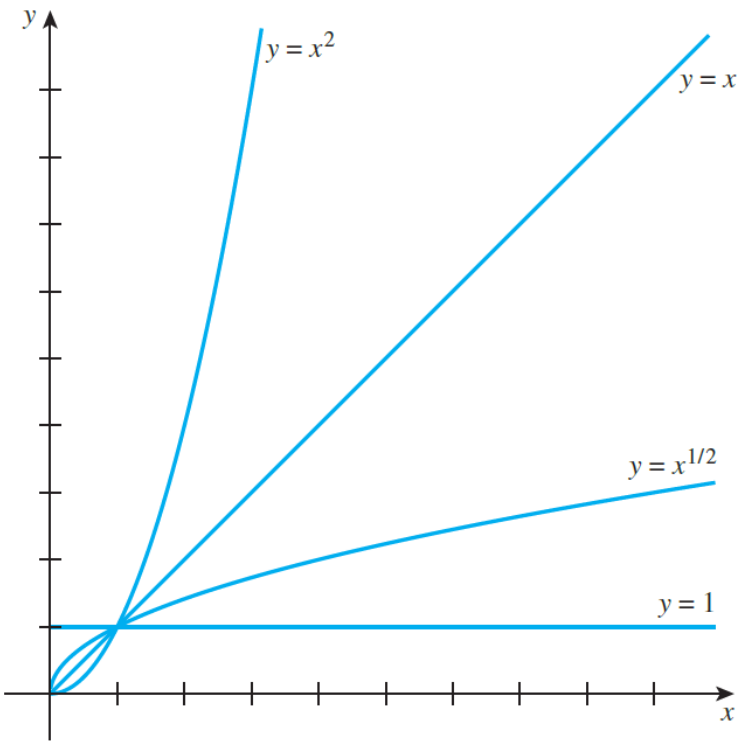
You input x and obtain $y = f(x)$ as the output. y is called the image of x.

Example: Power Functions

- The power function with exponent a is defined as

$$\boxed{p_a(x) = x^a}$$

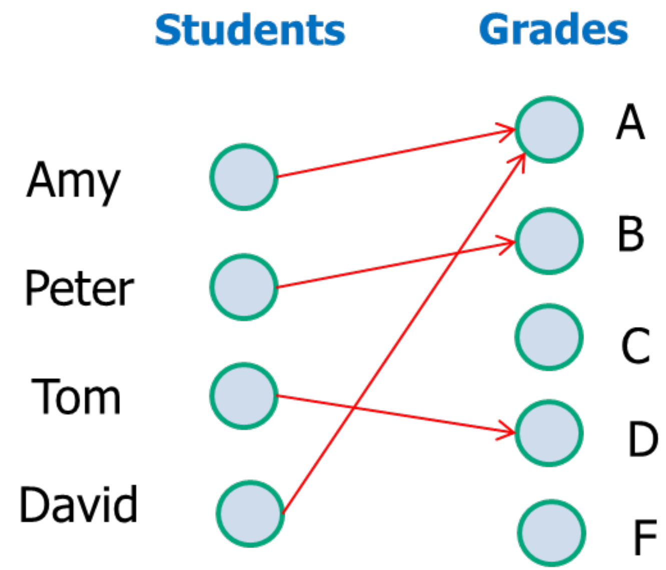
  - For many application, we are concerned with cases where x and a are non-negative.
  - Given *any* input x, there is one and only one corresponding output y.



Graphs of some power functions

Example without Numbers

- It is important to note that the inputs or outputs of a function are *not* necessarily numbers.
- Consider the Grade Assignment Function f which maps a set of students to a set of grades.
  - f assigns each student exactly one grade.



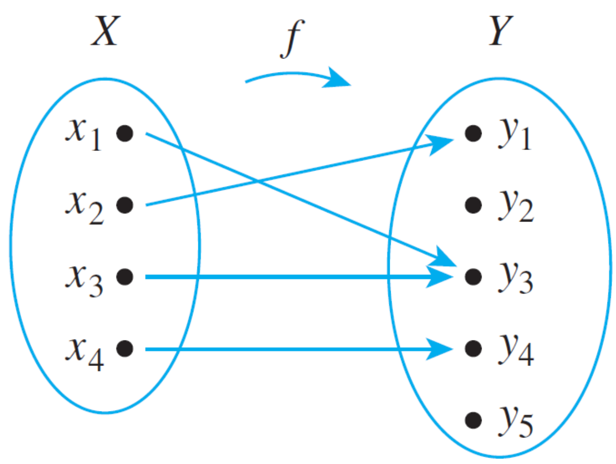
No student is assigned more than one grade.

No student has no grade assigned.

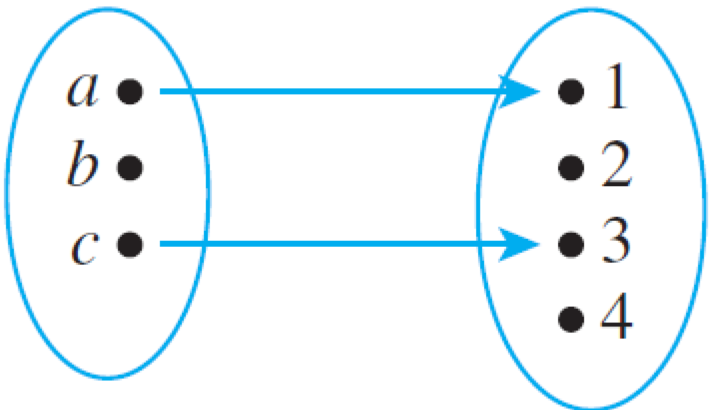
Arrow Diagrams

- A function $f : X \to Y$ can be represented by an arrow diagram.
- An arrow is drawn from each element in X to its corresponding unique element in Y under f.

  - Every element in X points to a unique element in Y.
  - No element of X has two arrows coming out of it.



Are They Functions?



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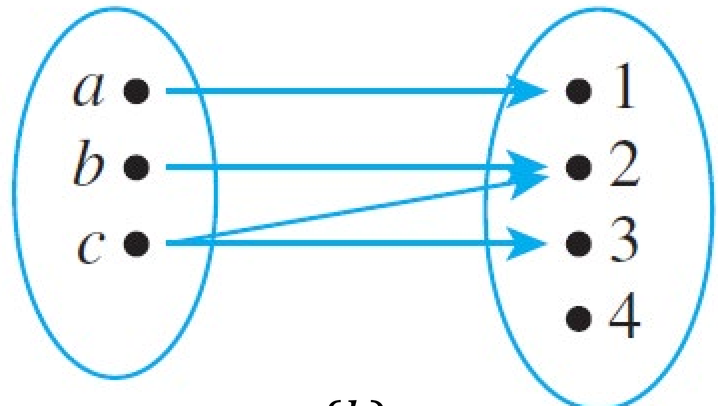

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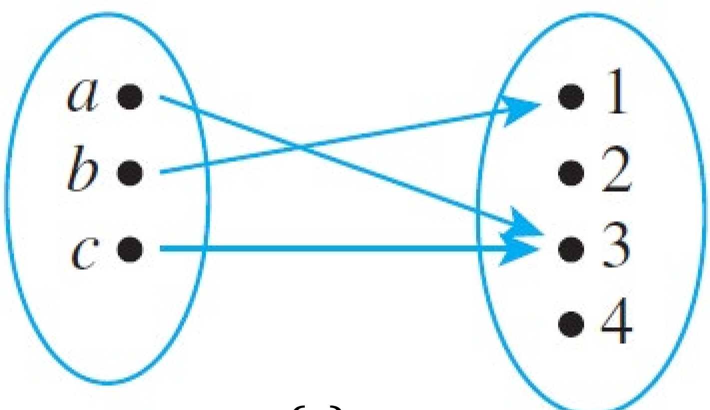
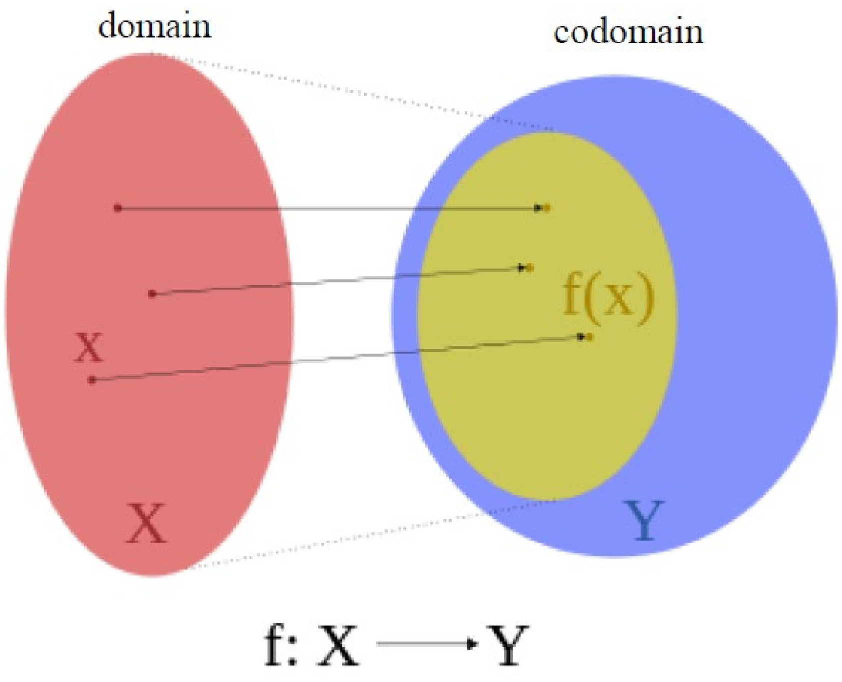


(c)

Domain, Co-Domain, and Range

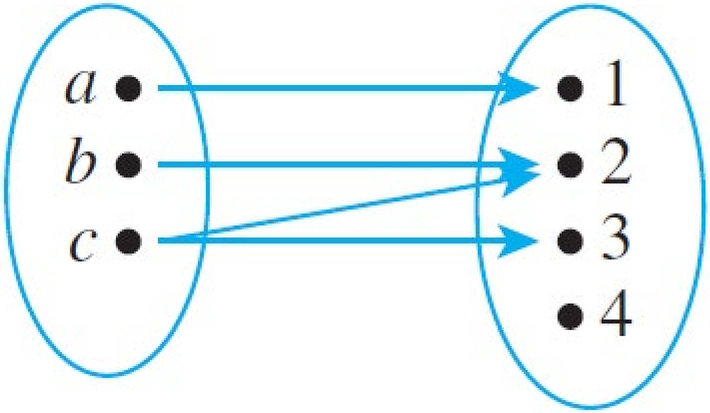
Consider a function $f : X \rightarrow Y$.

- X is called the **domain** of f while Y is called the **co-domain** of f.
- The **range** of f is the set of images of all elements in X.
- Co-domain and range are often confusing.
  - Note: range $\subseteq$ co-domain.



Inverse Image

- Given $y \in Y$, the inverse image of y is the set of all elements $x \in X$ such that $f(x) = y$.
- Example:
  - Inverse image of $1 = \{a\}$
  - Inverse image of $2 = \{b, c\}$
  - Inverse image of $3 = \{c\}$
  - Inverse image of $4 = \Phi$



- Note: Image of x is an element. Inverse image of y is a set.

Classwork

Consider $f: \mathbb{Z} \to \mathbb{Z}$, where $f(x) = 2x$.

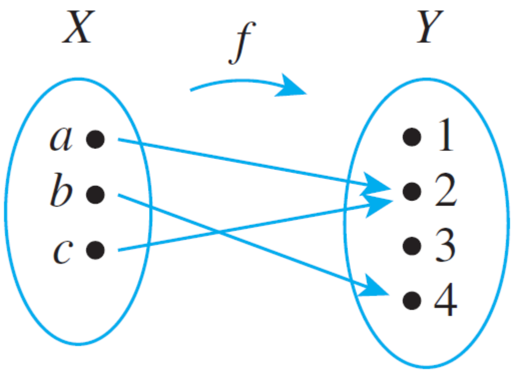
a) What is the domain of f?

b) What is the co-domain of f?

c) What is the range of f?

Classwork

a) What are the domain, co-domain and range of f?



b) What is the image of a under f?

c) What is the inverse image of 2 under f?

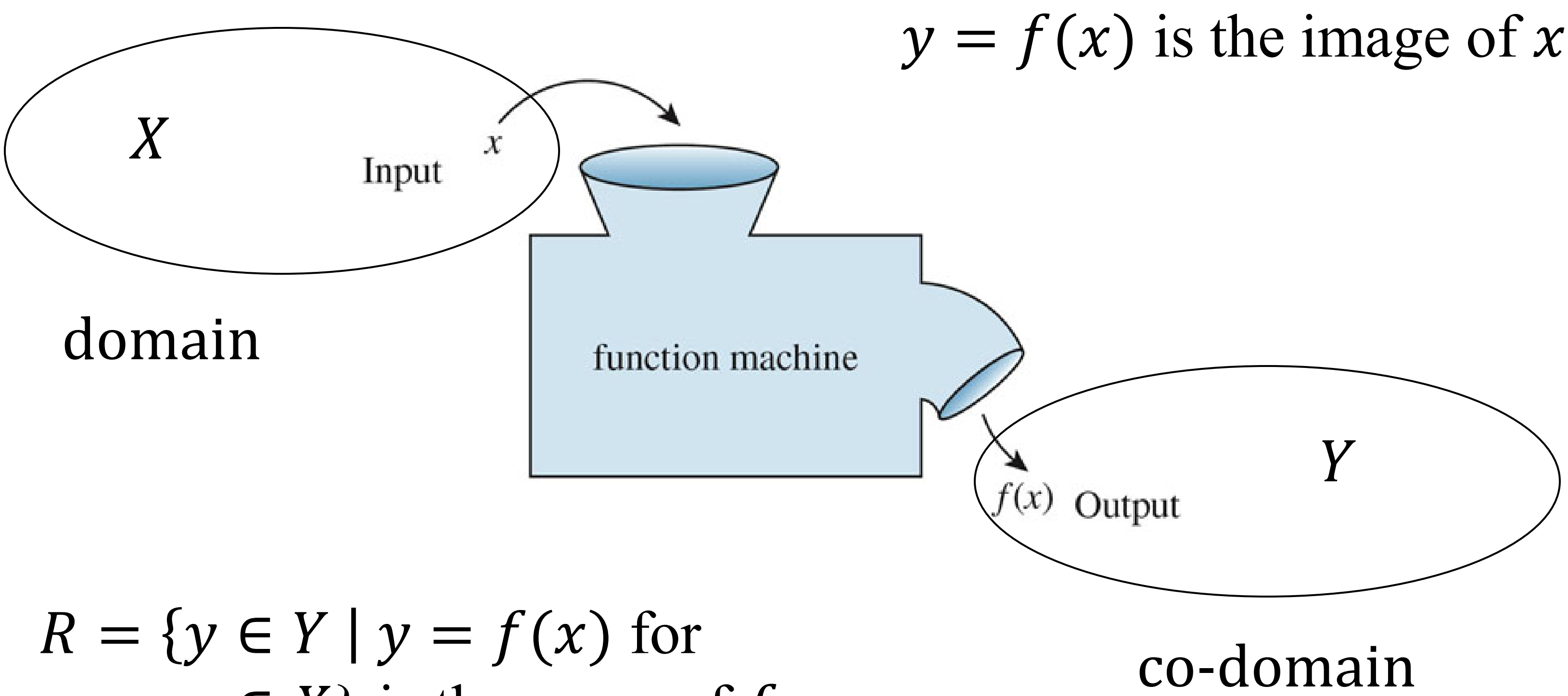
d) What is the inverse image of 3 under f?

Unit 5.2

Composition of Functions

Functions

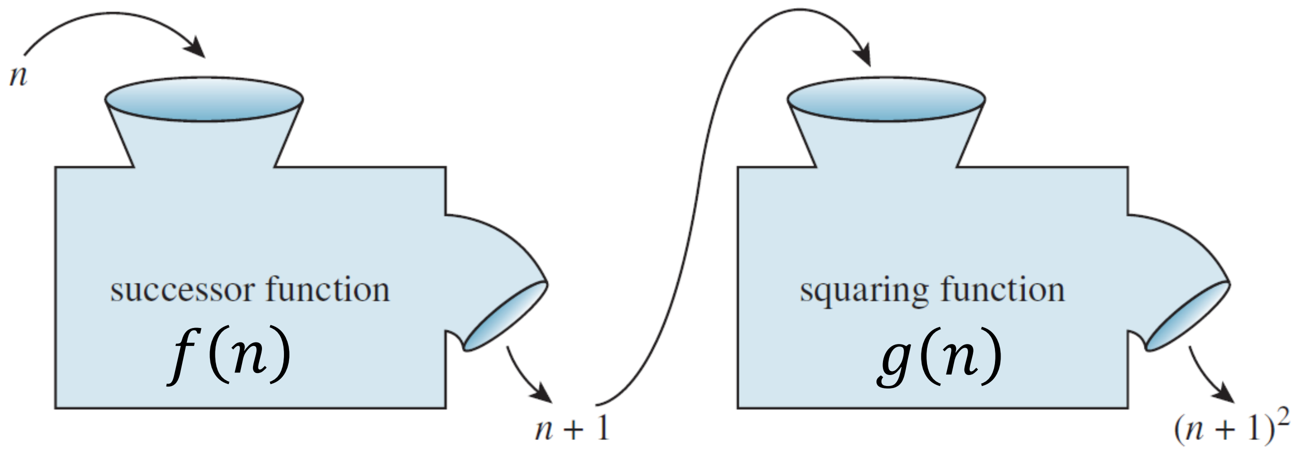
- Consider a function $f: X \to Y$.



$R = \{y \in Y \mid y = f(x)$ for some $x \in X\}$ is the range of f.

Composition of Functions

- If we link two function machines f and g in series, the resultant function is called their composition, denoted by $g \circ f$.



What if we change the order of these two machines? Will we get the same output?

Example

Let $f(n) = n + 1$ and $g(n) = n^2$, where the domains and co-domains of both functions are $\mathbb{Z}$.

a) Find $g \circ f$ and $f \circ g$.

b) Are they equal?

Solution:

a) $(g \circ f)(n) = g\big(f(n)\big) = g(n+1) = (n+1)^2$

$(f \circ g)(n) = f\big(g(n)\big) = f(n^2) = n^2 + 1$

b) No, they are not equal:

$g \circ f \neq f \circ g$ (function composition is not *commutative*)

Is Composition always Possible?

- Not any two functions can be composed.
  - $f(x) = -|x|,\ g(x) = \sqrt{x}$ (domains and codomains $\mathbb{R}$).
  - $(g \circ f)(x) = g\big(f(x)\big) = g(-|x|) = \sqrt{-|x|}$ (undefined!)

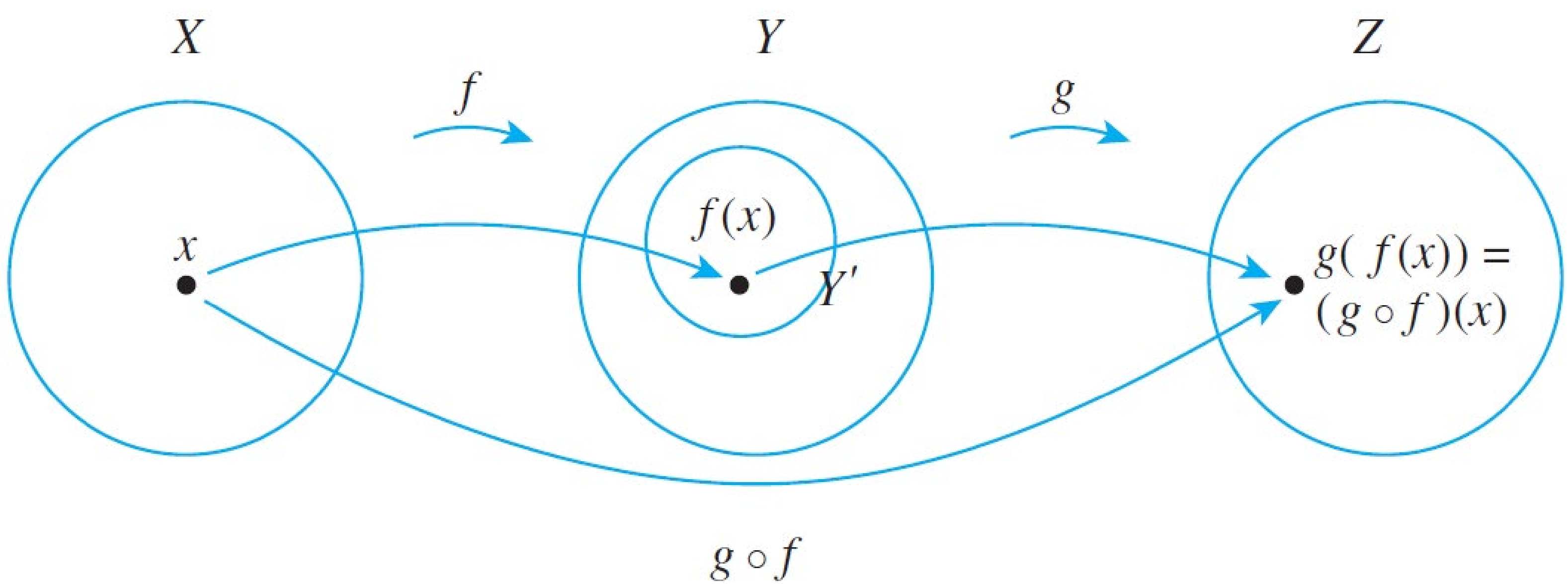
- $g \circ f$ is well defined only if the range of f is a subset of the domain of g.

• Definition

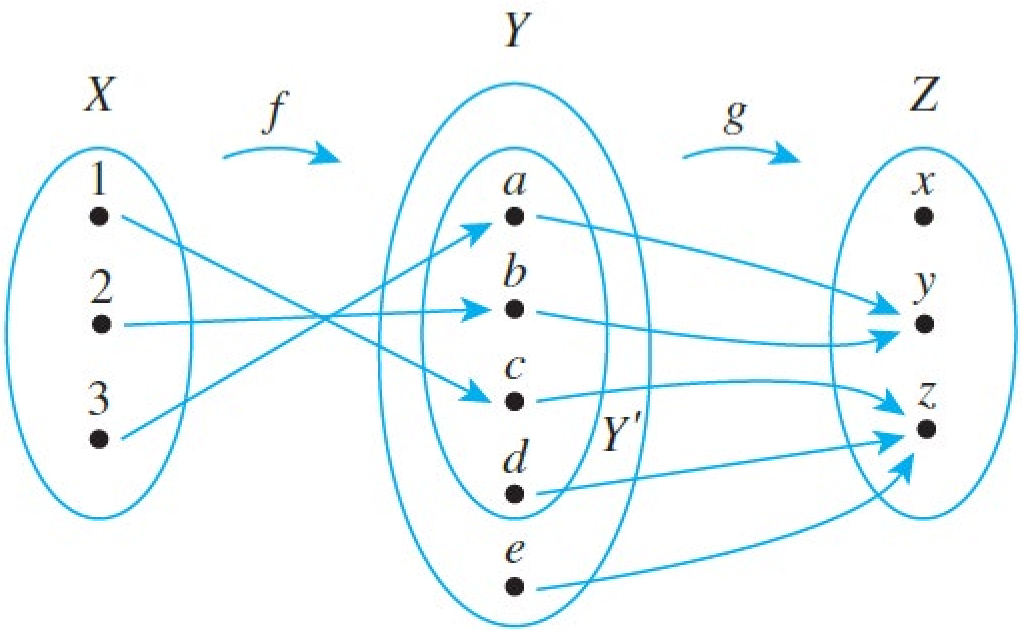
Let $f\colon X \to Y'$ and $g\colon Y \to Z$ be functions with the property that the range of f is a subset of the domain of g. Define a new function $g \circ f\colon X \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in X,$$

where $g \circ f$ is read "g circle f" and $g(f(x))$ is read "g of f of x." The function $g \circ f$ is called the **composition of f and g.**

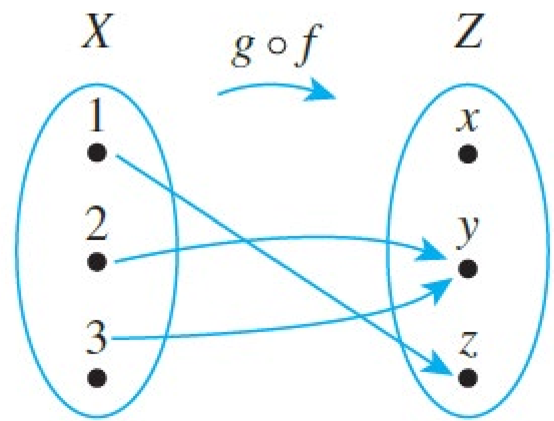


Example



- Draw the arrow diagram for $g \circ f$.
- What is the range of $g \circ f$?

Solution:



Its range is $\{y, z\}$.

Unit 5.3

One-to-One and Onto

One-to-One Function (Injection)

- **Definition**

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

Useful for proof.

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently, $$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

(contrapositive)

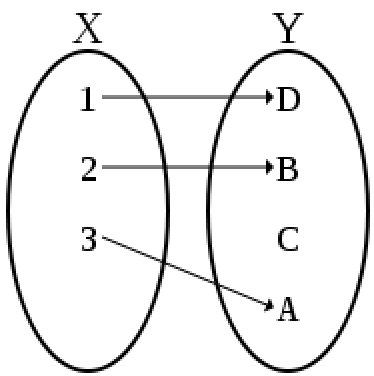
Symbolically,

$$F\colon X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

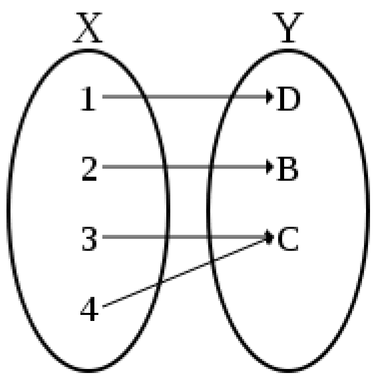
It looks complicated. It's easier (for understanding and for memorization) to use an informal one…

What is an Injection?

- A 1-to-1 function maps distinct elements in its domain to distinct elements in its co-domain.
- Are they injections?



(a)



(b)

Example: Injection

Let $f: [0, \infty) \to \mathbb{R}$, $f(x) = x^2$. Prove f is injective.

Definition: $\forall a, b \in X, f(a) = f(b) \Rightarrow a = b$.

- It means equal output must be from equal input

Proof:

For arbitrary $a, b \in [0, \infty)$, suppose $f(a) = f(b)$.

Therefore, $a^2 = b^2$.

Since both $a \geq 0$ and $b \geq 0$, we must have $a = b$.

Hence, f is injective.

Q.E.D.

Classwork

- Is it injective? Prove or disprove it.
  - (To disprove it, you can simply give a counter-example.)

a) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 4x - 1$ for all $x \in \mathbb{R}$.

b) $g: \mathbb{Z} \to \mathbb{Z}$ such that $g(n) = n^2$ for all $n \in \mathbb{Z}$.

Onto Function (Surjection)

• Definition

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

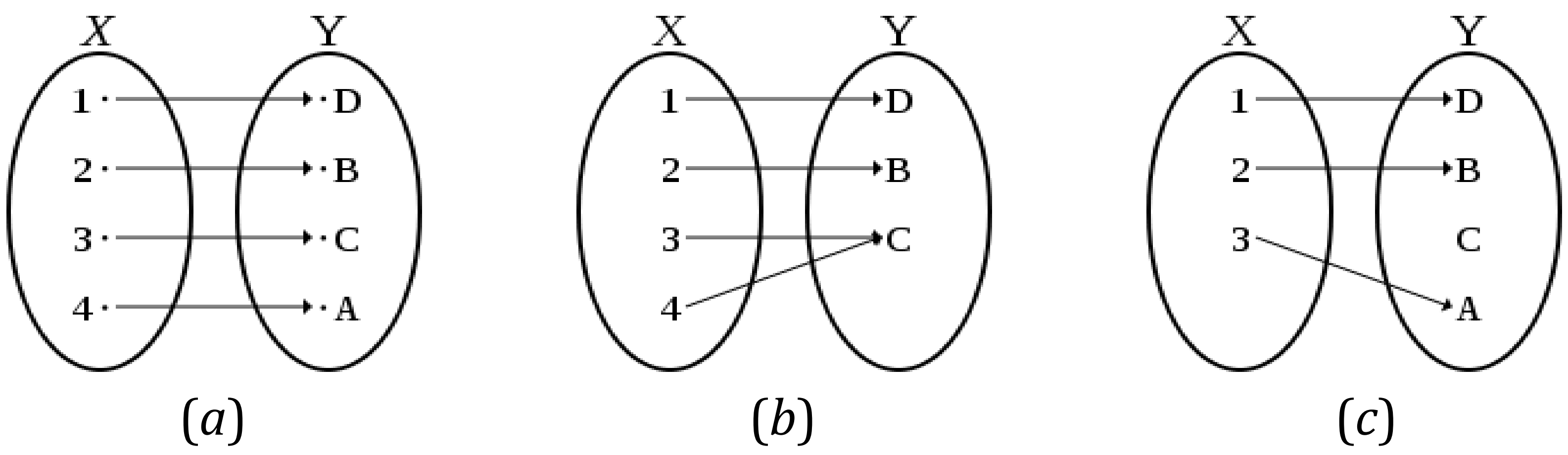
Useful for proof.

$$F: X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Again we also consider an informal one, in the next slide...

What is a Surjection?

- An onto function has its range equal to its co-domain.
  - i.e., every element in its co-domain has one or more inverse images in its domain.
- Are they surjections?



Example: Surjection

Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = mx + c$, where m and c are real numbers. Prove f is surjective.

Definition: $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

Proof:

Consider an arbitrary value $y \in Y$. We want to check if we can find x such that $mx + c = y$.

If $x = \frac{y-c}{m}$, then $f(x) = f\left(\frac{y-c}{m}\right) = m\left(\frac{y-c}{m}\right) + c = y$.

Hence, f is surjective.

Q.E.D.

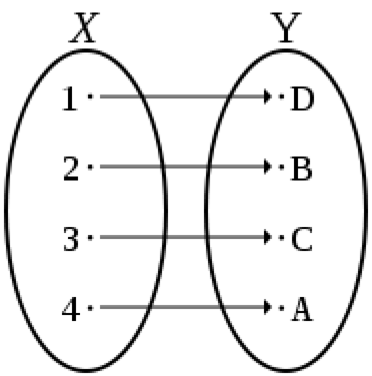
Classwork

- Is it surjective? Prove or disprove it.
  a) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x + 10$ for all $x \in \mathbb{R}$.

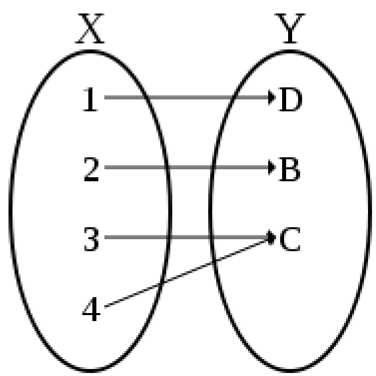
  b) $g : \mathbb{Z} \to \mathbb{Z}$ such that $g(n) = n^2$ for all $n \in \mathbb{Z}$.

What is a Bijection?

- A function is a one-to-one correspondence (or bijection) iff it is both 1-to-1 and onto.
- Are they bijections?



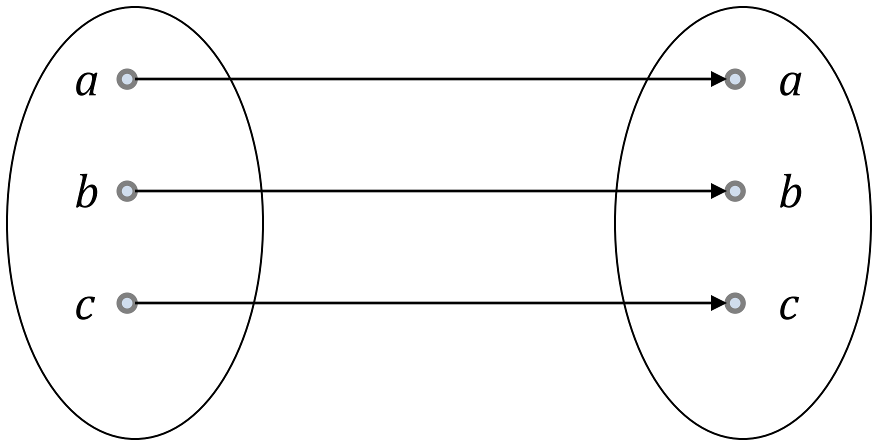
(a)



(b)

Example: Identity Function

- The identity function I_X on a set X is defined as
$$I_X(x) = x \text{ for all } x \in X.$$



- Any identity function is a bijection.

Example

- Let $\wp(\{a, b\})$ be the power set of $\{a, b\}$.
- Let S be the set of all binary strings of length 2, i.e., $S = \{00, 01, 10, 11\}$.
- Let $h\colon \wp(\{a, b\}) \to S$ be defined as follows:

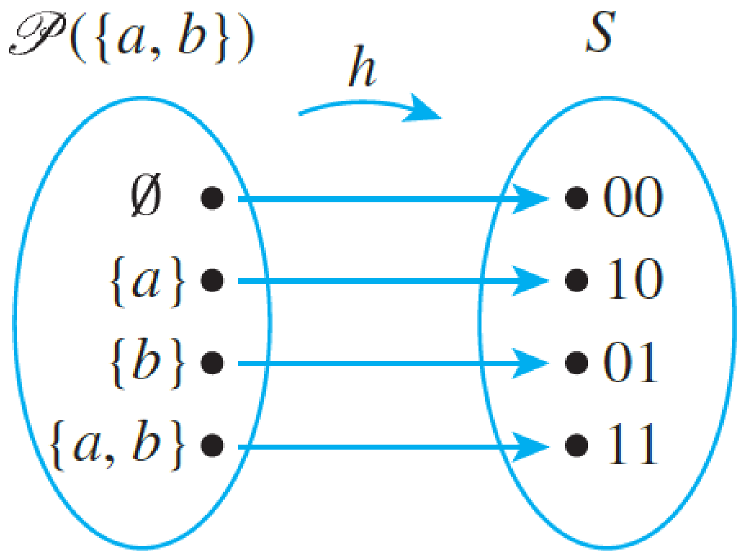
h

Subset A of $\{a, b\}$	**Status of a in A**	**Status of b in A**	**String $h(A)$ in S**
$\emptyset$	not in	not in	00
$\{a\}$	in	not in	10
$\{b\}$	not in	in	01
$\{a, b\}$	in	in	11

- Is h a bijection?

Solution

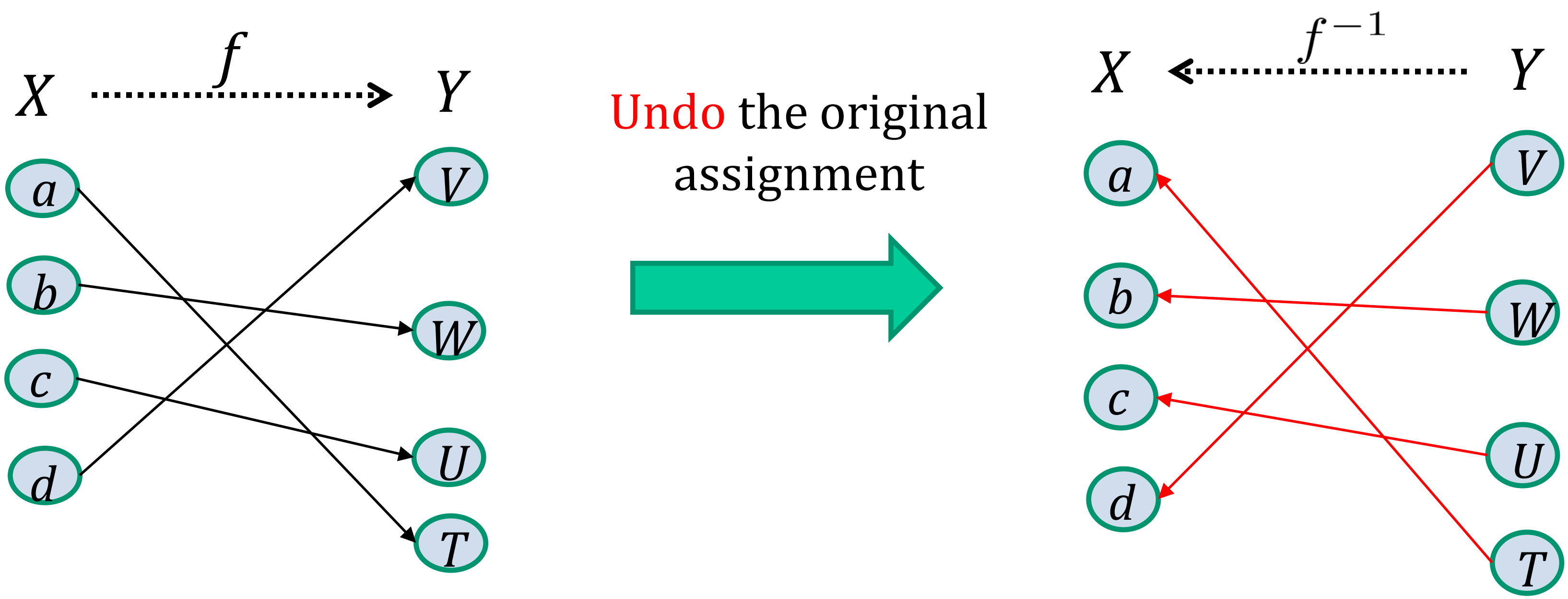
- The arrow diagram is shown below:



- Clearly, it is a bijection.

Inverse Functions

- Given a bijection f, we can "undo" the action of f by defining an inverse function f^{-1}.



f^{-1} is also a bijection.

Example

- Consider $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 4x - 1$. Find the inverse function of f.
  - Note: its inverse function exists because f is bijective.
- Solution:

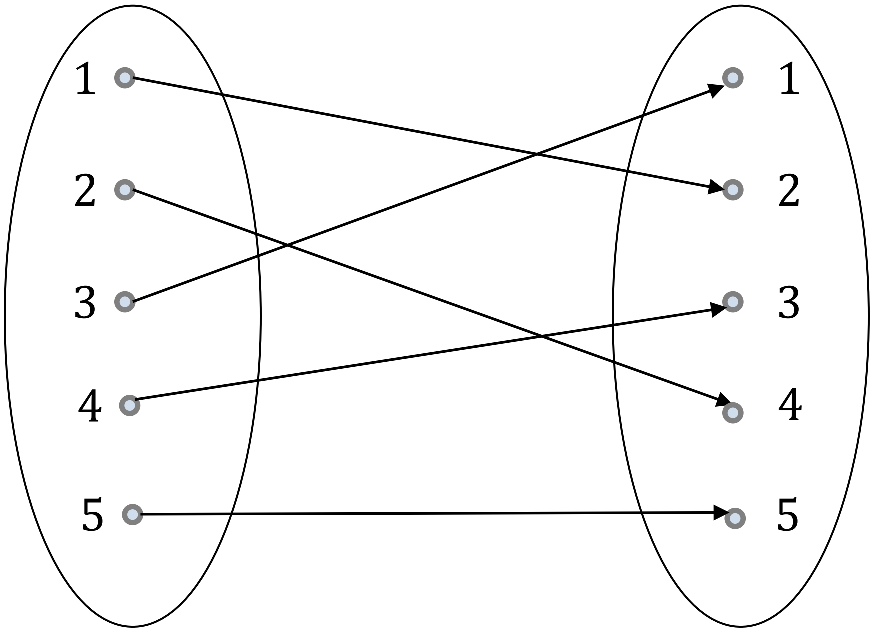
$$
\begin{aligned}
f(x) &= y \\
4x - 1 &= y && \text{by definition of } f \\
x &= \frac{y+1}{4} && \text{by algebra.} \\
f^{-1}(y) &= \frac{y+1}{4}
\end{aligned}
$$

Unit 5.4

Permutation Functions

Permutation Functions

- Let $A = \{1, 2, 3, 4, 5\}$.
- The permutation $12345 \rightarrow 24135$ can be represented by the bijection $f: A \rightarrow A$ shown below:



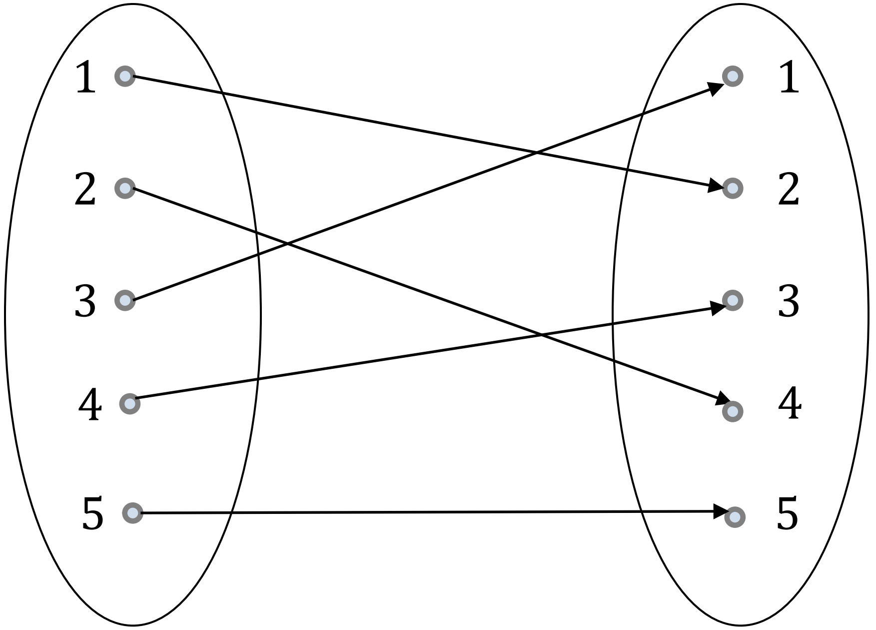
All Permutations on $\{1, 2, \ldots, n\}$

- The set of all permutations on $\{1, 2, \ldots, n\}$ is denoted by S_n.
- Each member of S_n is a bijection which maps from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$.
- S_n contains $n!$ members, which are all possible bijections.
- One of them is the identity function, usually denoted by the Greek letter ι.
  - pronounced as i-o-ta.

Array Representation

- One common way to express a permutation is to use a $2 \times n$ array of integers.
  - The permutation is shown in the second row.
- Example:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$$



$\pi(1) = 2, \pi(2) = 4, \pi(3) = 1,$
$\pi(4) = 3, \pi(5) = 5.$

Cycle Notation

- The permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$ is represented by
$$\pi = (1, 2, 4, 3)(5).$$
- The two lists $(1, 2, 4, 3)$ and (5) are called cycles.
- $(1, 2, 4, 3)$ means that $1 \mapsto 2 \mapsto 4 \mapsto 3 \mapsto 1$.
  - i.e., $\pi(1) = 2, \pi(2) = 4, \pi(4) = 3, \pi(3) = 1$.
- (5) means $5 \mapsto 5$
  - i.e., $\pi(5) = 5$.

Cycle notation is not unique. For example, $(1, 2, 4, 3)(5)$ can also be expressed as $(2, 4, 3, 1)(5)$.

Inverse of a Permutation

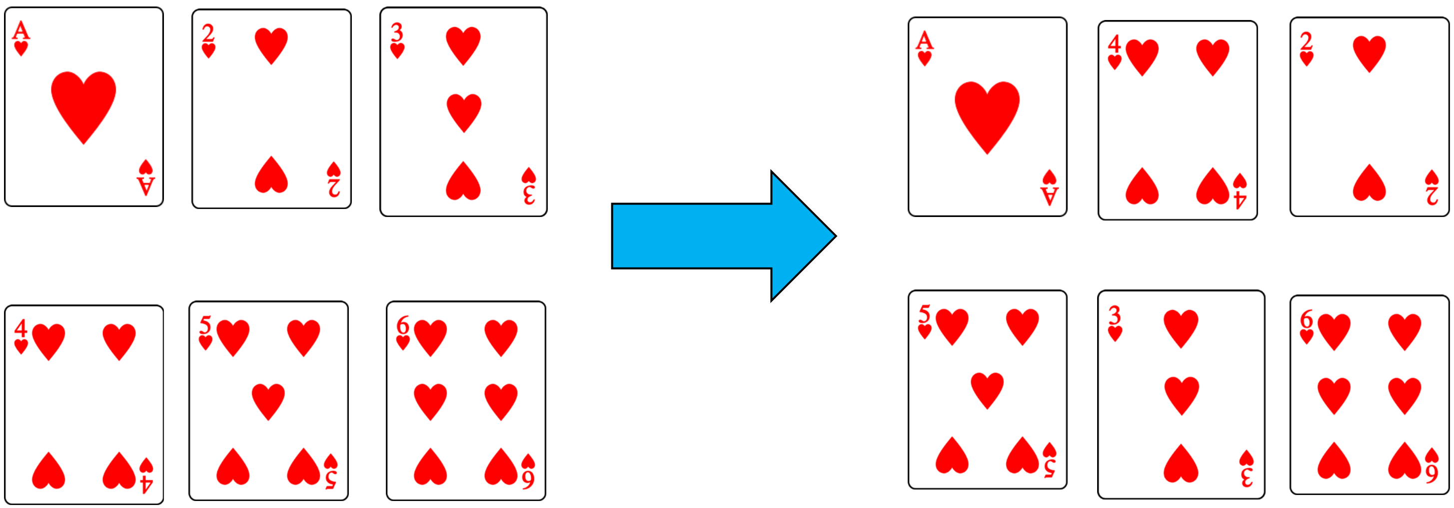
- Since any permutation is a bijection, its inverse exists, and is easy to find.
- If π maps a to b, then π^{-1} maps b to a.

- Example:
  - Let $\pi = (1,2,7,9,8)(5,6,3)(4) \in S_9$.
  - Then $\pi^{-1} = (8,9,7,2,1)(3,6,5)(4)$.

Composition of Permutations

- Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$.
- $\sigma \circ \pi$ means we permute a sequence first by π and then by σ.
- $$\sigma \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$$
  $$= \begin{pmatrix} 2 & 4 & 1 & 3 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$$
  $$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix}$$

  Rearrange the columns of the first matrix.
- In cycle notation, $\sigma \circ \pi = (1, 4, 3, 5)(2)$.



- The permutation is given by $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 \end{pmatrix}$.
- Since 1 and 6 always in the same position, we only need to consider the other four cards.
- Let $\pi = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 \end{pmatrix}$.

The columns of the first matrix have been re-arranged.

$$\pi = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 \end{pmatrix}$$

$$\pi \circ \pi = \begin{pmatrix} 4 & 2 & 5 & 3 \\ 5 & 4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 \end{pmatrix}$$

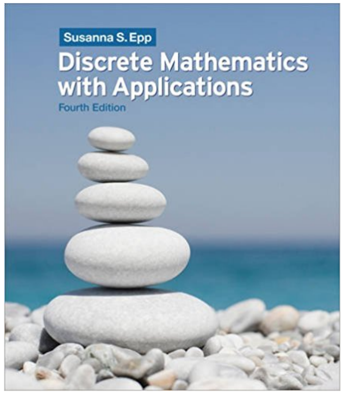
$$\pi \circ \pi \circ \pi = \begin{pmatrix} 5 & 4 & 3 & 2 \\ 3 & 5 & 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 \end{pmatrix}$$

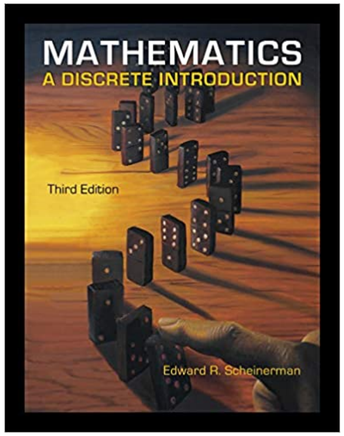
$$\pi \circ \pi \circ \pi \circ \pi = \begin{pmatrix} 3 & 5 & 2 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \end{pmatrix} = \iota$$

Recommended Reading



- Chapter 7, S. S. Epp, *Discrete Mathematics with Applications*, 4th ed., Brooks Cole, 2010.



- Chapter 5, E. R. Scheinerman, *Mathematics: A Discrete Introduction*, 3rd ed., Brooks/Cole, 2013.