Tutorial 4 (Chapter 4 and some discrete problems)

1. Let X be a random variable with probability density function

$$f(x) = \begin{cases} C(1-x^2) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the value of C?

   (b) What is the cumulative distribution function of X?

   (c) Find $E[X]$ and $Var(X)$.

   (d) Find the density function of X^2.

   Solution

   (a) 3/4

   (b)

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{3}{4}(x - \frac{x^3}{3} + \frac{2}{3}) & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

   (c) $E[X] = 0, Var(X) = \int_{-1}^{1} \frac{3}{4}(1-x^2)x^2 = \frac{1}{5}$

   (d) Let $Y = X^2$. Then $P(Y \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$.

   If $y > 1$, $P(Y \leq y) = 1$.

   If $0 < y \leq 1$, $P(Y \leq y) = \frac{3}{2}(\sqrt{y} - \frac{(\sqrt{y})^3}{3})$.

   Taking derivative, the density is

$$f_Y(y) = \begin{cases} \frac{3}{4\sqrt{y}}(1-y) & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (As discussed in class, it is usually better to directly take the derivative of $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$ without plugging in the exact form of F_X first. Here it does not matter too much since we have already calculated F_X in (b).)

2. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   If $E[X] = 3/5$, find a and b.

   Solution From $\int_0^1 a + bx^2 dx = 1$, and $\int_0^1 x(a + bx^2)dx = 3/5$, we have $a = 3/5, b = 6/5$.

3. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x}, x \geq 0$$

   Compute the expected lifetime of such a tube.

   Solution

   The expected lifetime is $\int_0^\infty x^2 e^{-x} dx$. You could use integration by parts (twice) to calculate this, but this is not necessary. Notice the previous integral is just $E[Y^2]$ with $Y \sim exp(1)$, and $E[Y^2] = Var(Y) + (E[Y])^2 = 2$ if you remember the mean and variance for exponentially distributed random varible.

4. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, express the following probability in terms of Φ.

   (a) $P(X > 5)$

   (b) $P(4 < X < 16)$

   (c) $P(X < 8)$

   (d) $P(X > 16)$

   Solution (a)$1 - \Phi(-5/6) = \Phi(5/6)$; (b)$\Phi(1) - \Phi(-1) \approx 0.68$; (c) $\Phi(-1/3)$; (d)$1 - \Phi(1)$

5. The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is

   (a) the probability that a repair exceeds 2 hours?

   (b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

   Solution $(a)e^{-1}, (b)e^{-0.5}$

6. The median of a continuous random variable having distribution function F is that value m such that $F(m) = 1/2$. Find the median of X if X is

   (a) uniformly distributed over (a, b);

   (b) normal with parameters μ, σ^2;

   (c) exponential with rate λ.

   Solution

   (a) $\frac{a+b}{2}$; (b) μ; (c) $\ln 2/\lambda$

7. A random variable X has an absolute value no larger than 1. $P(X = -1) = 1/8$ and $P(X = 1) = 1/4$. Given the event $\{-1 < X < 1\}$ occurs, the probability that X takes a value in an subinterval within $(-1, 1)$ is proportional to the length of the subinterval. Find the cdf of X.

   Solution

   Obviously $F(x) = 0$ if $x < -1$ and $F(x) = 1$ if $x \geq 1$. Also, $P(-1 < X < 1) = 1 - \frac{1}{8} - \frac{1}{4} = \frac{5}{8}$.
   From the assumption given, we have $P(-1 < X \leq x| - 1 < X < 1) = (x + 1)/2, -1 < x < 1$. So $P(-1 < X \leq x) = \frac{5(x+1)}{16}$.
   In summary the cdf is:

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{8} + \frac{5(x+1)}{16} = \frac{7}{16} + \frac{5}{16}x & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

8. Suppose the cumulative distribution function for a continuous random variable X is

$$F(x) = \begin{cases} a & x < 1 \\ bx \ln x + cx + d & 1 \leq x < e \\ d & x \geq e \end{cases}$$

   Determine a, b, c, d and find the density for X.

   Solution

   Note the cdf for a continuous r.v. is not only right continuous, but continuous as well.

First it is obvious $a = 0$ and $d = 1$. From the continuity of F at $x = 1$, we get $c = -1$. From the continuity of F at $x = e$, we get $b = 1$.

The pdf is

$$p(x) = \begin{cases} \ln x & 1 < x < e \\ 0 & \text{otherwise} \end{cases}$$

9. A random variable $X \sim U[0, 5]$(uniform). Observe independently X three times. What is the probability that for at least twice the equation $4x^2 + 4Xx + (X + 2) = 0$ has a real solution.

   Solution

   The equation has a real solution iff (iff means if and only if) $\Delta = (4X)^2 - 16(X + 2) \geq 0$. Solving this we get $X \geq 2$ or $X \leq -1$. And thus the probability that it has a real root is $p = P(X \geq 2) + P(X \leq -1) = 3/5$. And the answer is $Bin(2|3, 3/5) + Bin(3|3, 3/5) = 81/125$

10. Suppose the density for a continuous random variable is $p_X(x) = \frac{1}{\pi(1+x^2)}$, find the density of the random variable $Y = 1 - X^{1/3}$

    Solution Since $X = (1-Y)^3$, $p_Y(y) = p_X((1-y)^3)|[(1-y)^3]'| = \frac{3}{\pi} \frac{(1-y)^2}{1+(1-y)^6}$. (The formula is optional, you can actually use cdf to derive the density as in class.)

11. A certain retailer for a petroleum product sells a random amount X each day. Suppose that X (measured in hundreds of gallons) has the following density function

    $$f(x) = \begin{cases} \frac{3}{8}x^2 & 0 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

    The retailer's profit turns out to be \$5 for each 100 gallons sold if $X \leq 1$, and \$8 per 100 gallons if $X > 1$. Find the retailer's expected profit for any given day.

    Solution

    Let g(X) denote the daily profit. Then,

    $$g(X) = \begin{cases} 5X & 0 \leq X \leq 1 \\ 8X & 1 < X \leq 2 \end{cases}$$

    $E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx = \int_0^1 5x[\frac{3}{8}x^2]dx + \int_1^2 8x[\frac{3}{8}x^2]dx = 375/32$

12. Pick two numbers from $\{1, 2, \ldots, n\}$, find the probability that the sum is even.

    Solution

    The sample space obviously has $\binom{n}{2}$ elements. If n is even, the number of even numbers and odd numbers are both $n/2$. Since sum of two even numbers or two odd numbers is even, the answer is

    $$\frac{2 \cdot \binom{n/2}{2}}{\binom{n}{2}} = \frac{n-2}{2(n-1)}$$

    If n is odd, there are $\frac{n-1}{2}$ even numbers and $\frac{n+1}{2}$ odd numbers in the set $\{1, 2, \ldots, n\}$, and the answer is

    $$\frac{\binom{(n-1)/2}{2} + \binom{(n+1)/2}{2}}{\binom{n}{2}} = \frac{n-1}{2n}$$

13. Pick 4 numbers from $\{0, 1, 2, \ldots, 9\}$ with replacement, and arrange them in the order they are picked. Find the probability for the following events:

(a) The four numbers form a proper integer (i.e. 0 does not appear in the first position);

(b) The four number form a proper even integer;

(c) 0 appears exactly twice;

(d) 0 appears at least once.

Solution

(a) $\frac{9 \cdot 10^3}{10^4} = 0.9$

(b) $\frac{9 \cdot 10^2 \cdot 5}{10^4} = 9/20$

(c) $\frac{\binom{4}{2} 9^2}{10^4}$

(d) $1 - \frac{9^4}{10^4}$

14. A box contains $2n - 1$ white balls and $2n$ black balls. You randomly draw out n of them and find they are all the same color. Find the probability that their color is black.

Solution

Let $A = \{$ those n balls are of the same color$\}$, and $B = \{$ those n balls are all black$\}$.

$$\begin{aligned} P(A) &= \frac{\binom{2n-1}{n} + \binom{2n}{n}}{\binom{4n-1}{n}} \\ P(AB) &= P(B) = \frac{\binom{2n}{n}}{\binom{4n-1}{n}} \\ \text{So } P(B|A) &= \frac{P(AB)}{P(B)} = \frac{\binom{2n}{n}}{\binom{2n-1}{n} + \binom{2n}{n}} \end{aligned}$$

15. We have a batch of products in which 10 are effective and 3 are defective. Randomly pick one at a time, and let X represent the time you get an effective one. Find the distribution of X in the following different situations:

(a) You pick without replacement.

(b) You pick with replacement.

(c) After you pick a defective one, put back an effective one.

Solution

Let $A_i = \{$you pick an effective one at the ith trial$\}$.

(a)

$$P(X = 1) = P(A_1) = 10/13$$

$$P(X = 2) = P(A_1^c)P(A_2|A_1^c) = \frac{3}{13}\frac{10}{12} = 5/26$$

$$P(X = 3) = P(A_1^c)P(A_2|A_1^c)P(A_3|A_1^c A_2^c) = \frac{3}{13}\frac{2}{12}\frac{10}{11} = 5/143$$

Similarly,

$$P(X = 4) \quad = \quad \frac{3}{13}\frac{2}{12}\frac{1}{11}\frac{10}{10} = 1/286$$

(b)

$$P(X = k) = (\frac{3}{13})^{k-1}\frac{10}{13}, k = 1, 2, 3, \ldots$$

(c)

$$P(X = 1) \quad = \quad P(A_1) = 10/13$$

$$P(X = 2) \quad = \quad P(A_1^c)P(A_2|A_1^c) = \frac{3}{13}\frac{11}{13} = 33/169$$

$$P(X = 3) \quad = \quad P(A_1^c)P(A_2|A_1^c)P(A_3|A_1^c A_2^c) = \frac{3}{13}\frac{2}{13}\frac{12}{13} = 72/2197$$

Similarly,

$$P(X = 4) \quad = \quad \frac{3}{13}\frac{2}{13}\frac{1}{13}\frac{13}{13} = 6/2197$$