Chapter 1

Introduction, vectors and basic calculus

The nature of physics

- Physics is an *experimental* science in which physicists seek patterns that relate the phenomena of nature.
- The patterns are called **physical theories**.
- A very well established or widely used theory is called a **physical law** or **principle**.

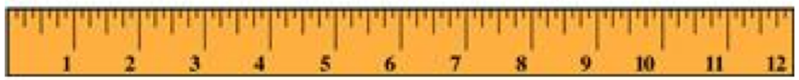
According to legend, Galileo investigated falling objects by dropping them from the Leaning Tower of Pisa, Italy, ...

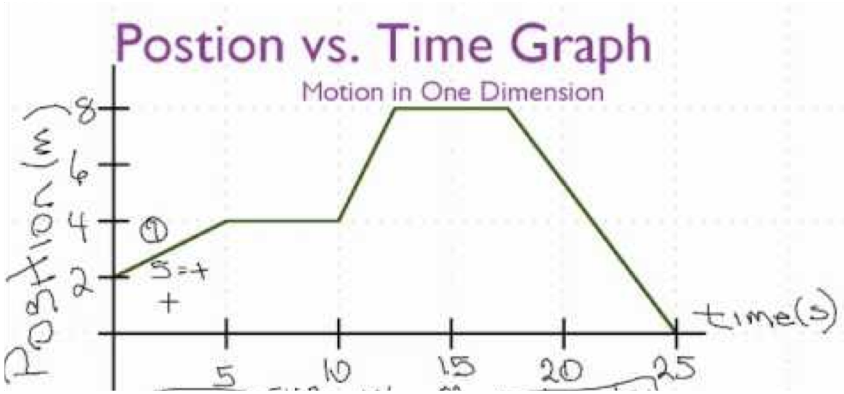
... and he studied pendulum motion by observing the swinging chandelier in the adjacent cathedral.

What is physics?

- For example, the description of motion
- You use a ruler to measure the position and a stop watch to measure the time
- Plot a position and time graph to describe the motion
- The motion can be explained using Newton's law

Newton's law
F=ma



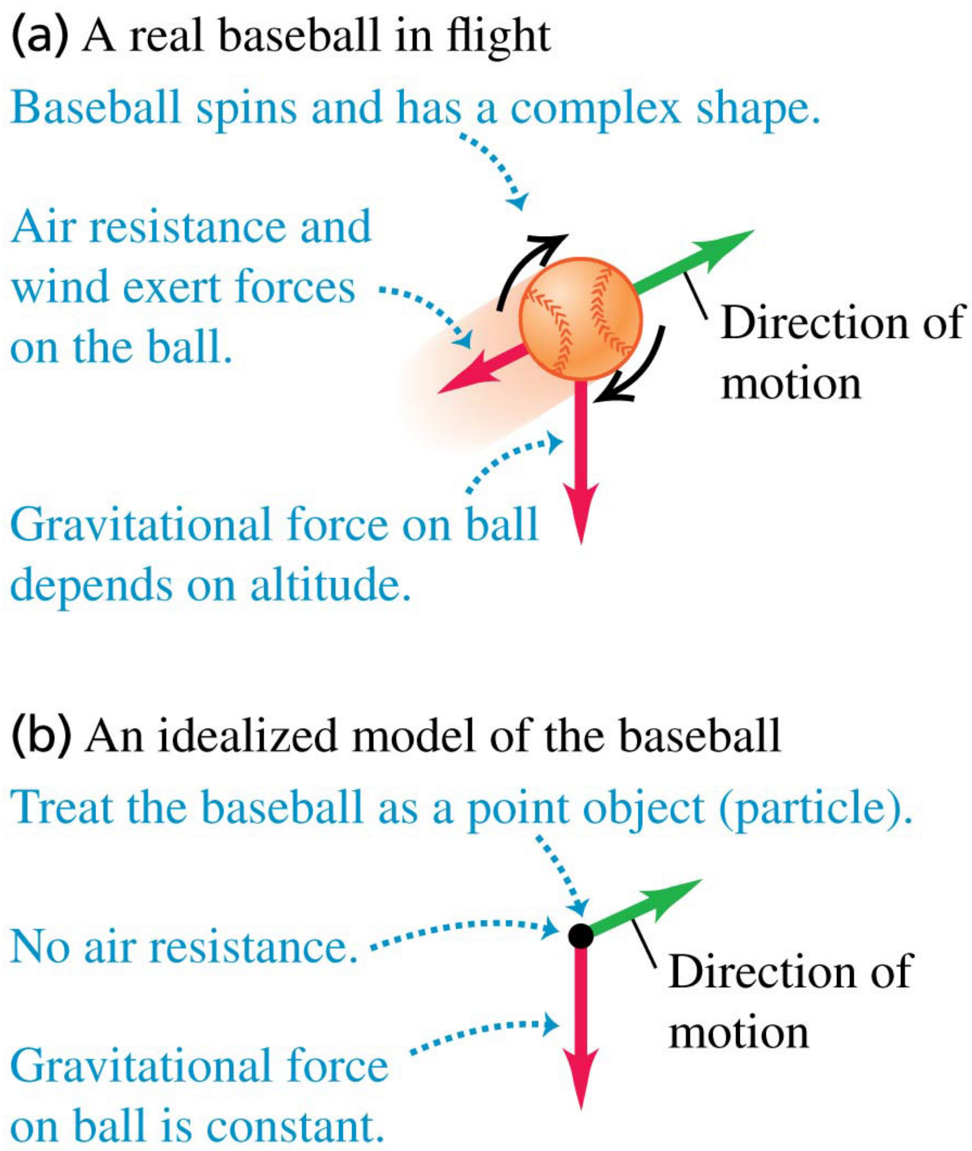


Solving problems in physics

- All of the *Problem-Solving Strategies* and *Examples* in this book will follow these four steps:
- **Identify** the relevant concepts, target variables, and known quantities, as stated or implied in the problem.
- **Set Up** the problem: Choose the equations that you'll use to solve the problem, and draw a sketch of the situation.
- **Execute** the solution: This is where you "do the math."
- **Evaluate** your answer: Compare your answer with your estimates, and reconsider things if there's a discrepancy.

Idealized models

To simplify the analysis of (a) a baseball in flight, we use (b) an idealized model.



Standards and units

- Length, time, and mass are three *fundamental* quantities of physics.
- The *International System* (SI for *Système International*) is the most widely used system of units.
- In SI units, length is measured in *meters*, time in *seconds*, and mass in *kilograms*.

Unit prefixes

- Prefixes can be used to create larger and smaller units for the fundamental quantities. Some examples are:
- 1 μm = 10^{-6} m (size of some bacteria and living cells)
- 1 km = 10^{3} m (a 10-minute walk)
- 1 mg = 10^{-6} kg (mass of a grain of salt)
- 1 g = 10^{-3} kg (mass of a paper clip)
- 1 ns = 10^{-9} s (time for light to travel 0.3 m)

Unit consistency and conversions

- An equation must be *dimensionally consistent*. Terms to be added or equated must *always* have the same units. (Be sure you're adding "apples to apples.")

- Always carry units through calculations.

- Convert to standard units as necessary, by forming a ratio of the same physical quantity in two different units, and using it as a multiplier.

- For example, to find the number of seconds in 3 min, we write:

$$3\ \text{min} = (3\ \cancel{\text{min}})\left(\frac{60\ \text{s}}{1\ \cancel{\text{min}}}\right) = 180\ \text{s}$$

Uncertainty and significant figures

- The uncertainty of a measured quantity is indicated by its number of *significant figures*.
- For multiplication and division, the answer can have no more significant figures than the *smallest* number of significant figures in the factors.
- For addition and subtraction, the number of significant figures is determined by the term having the fewest digits to the right of the decimal point.
- As this train mishap illustrates, even a small percent error can have spectacular results!

Vectors and scalars

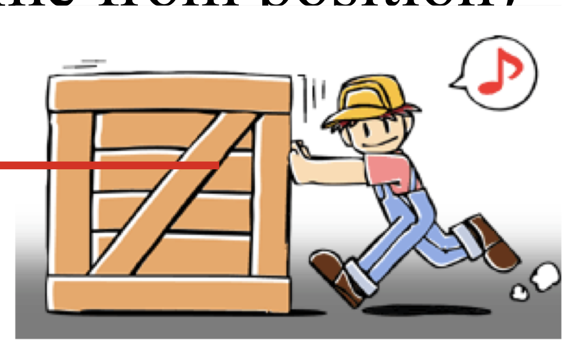
- A **scalar quantity** can be described by a *single number*.
- A **vector quantity** has both a *magnitude* and a *direction* in space.
- A vector quantity is typically represented by an italic letter with an arrow over it: $\vec{A}$
- The magnitude of $\vec{A}$ is written as A or $|\vec{A}|$.
- A typical example of vector: position of Kowloon Tong station relative to the Central station.

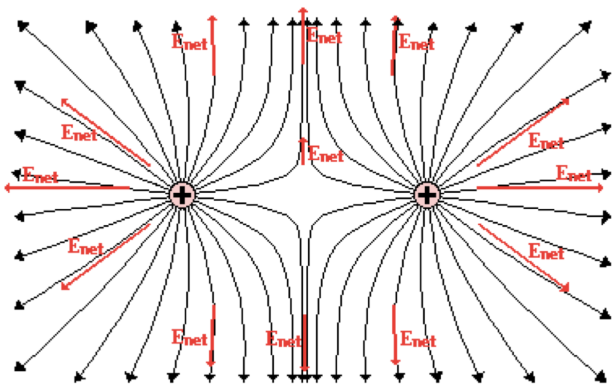
Vector (relative position)

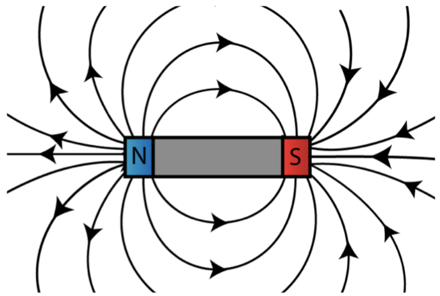
- The relative position of Kowloon Tong with respect to Central is a vector.
- It tells you how to get to Kowloon Tong from central:
- Walk a distance along a certain direction
- If the direction is changed, you go to a wrong place (red arrow)

Other vector physical quantities

- Apart from displacement, the following physical quantities are also vectors
- Velocity, momentum (come from position)
- Angular velocity, angular momentum (come from position)
- Acceleration, force
- electric field, magnetic field
- They are usually related to motion and force

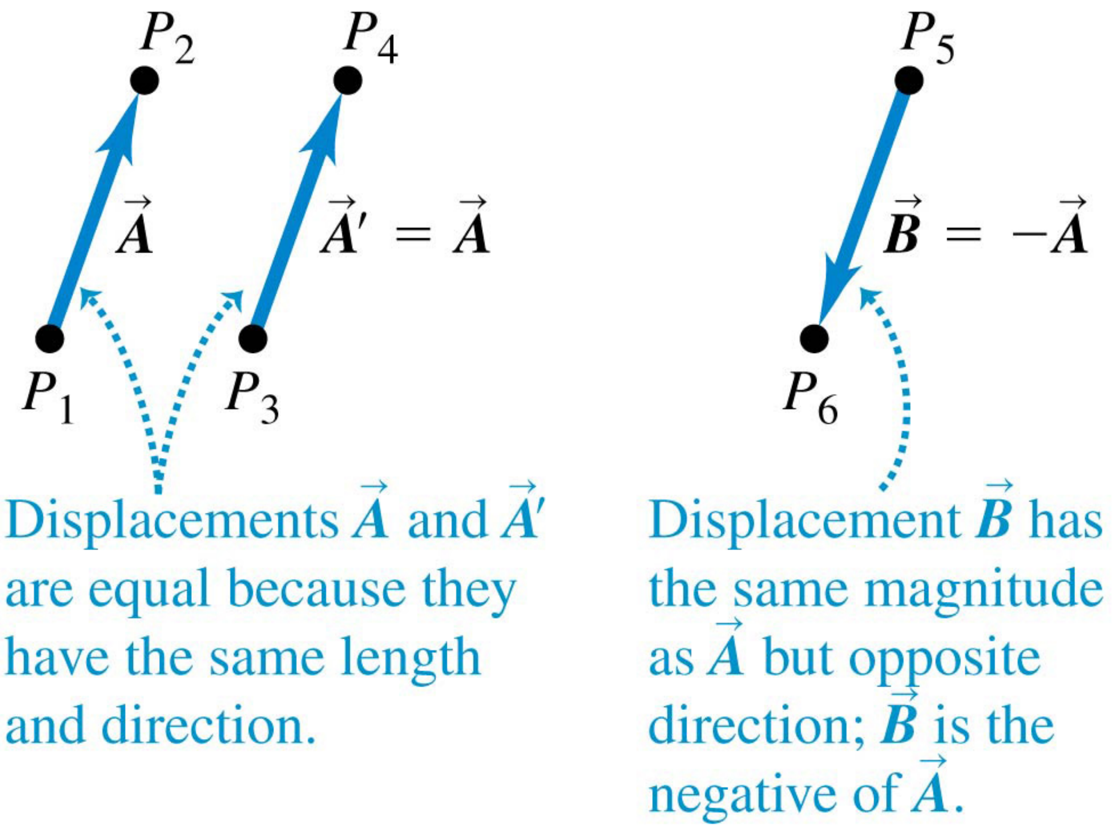






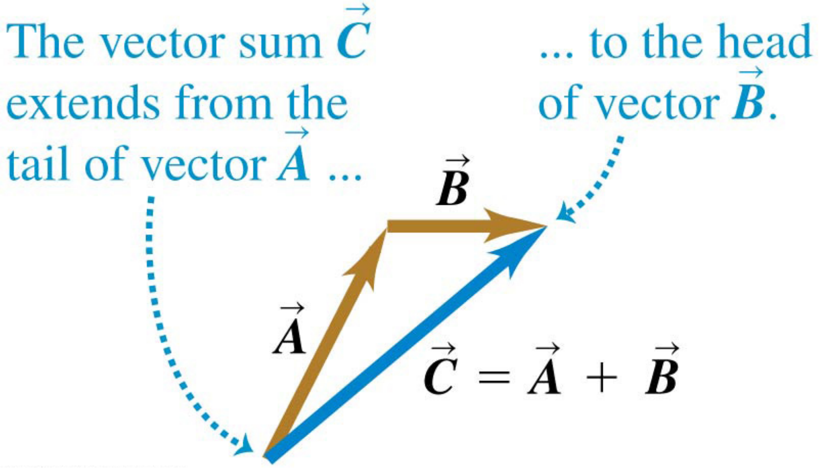
Drawing vectors

- Draw a vector as a line with an arrowhead at its tip.
- The *length* of the line shows the vector's *magnitude*.
- The *direction* of the line shows the vector's *direction*.

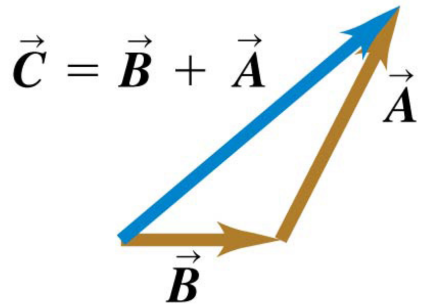


Adding two vectors graphically

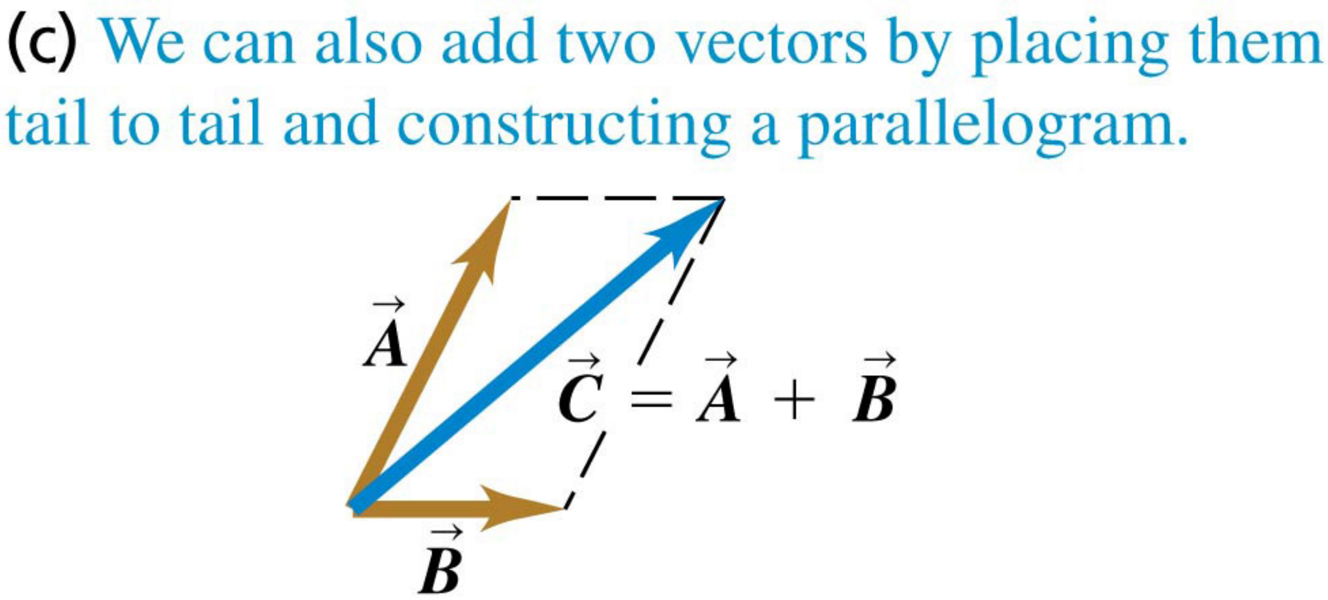
(a) We can add two vectors by placing them head to tail.



(b) Adding them in reverse order gives the same result: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. The order doesn't matter in vector addition.

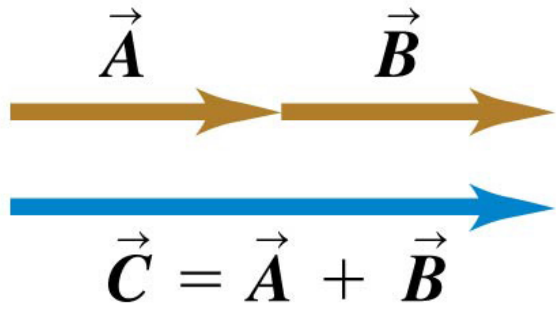


Adding two vectors graphically

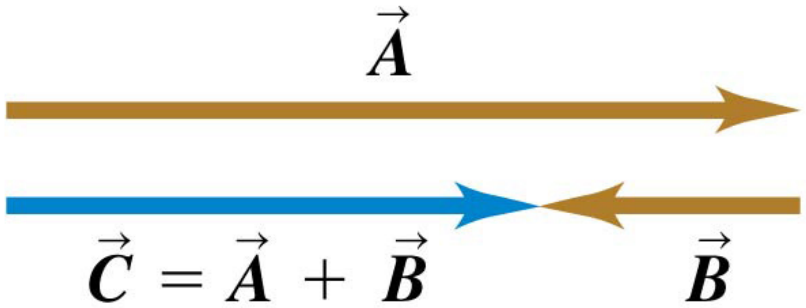


Adding two vectors graphically

(a) Only when vectors $\vec{A}$ and $\vec{B}$ are parallel does the magnitude of their vector sum $\vec{C}$ equal the sum of their magnitudes: $C = A + B$.

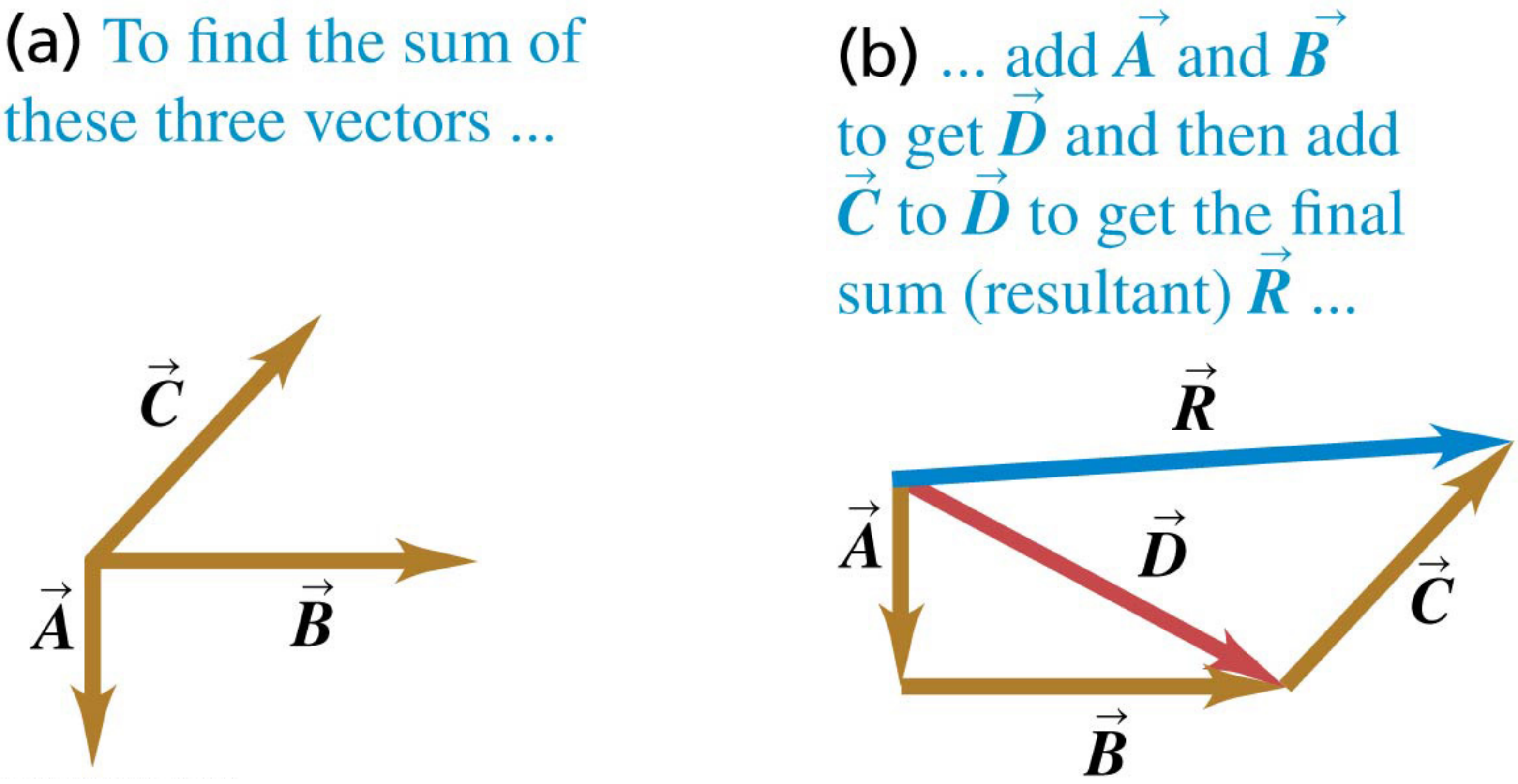


(b) When $\vec{A}$ and $\vec{B}$ are antiparallel, the magnitude of their vector sum $\vec{C}$ equals the *difference* of their magnitudes: $C = |A - B|$.



Adding more than two vectors graphically

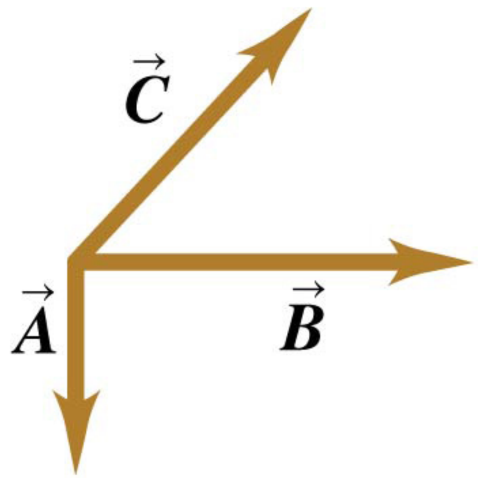
- To add several vectors, use the head-to-tail method.
- The vectors can be added in any order.



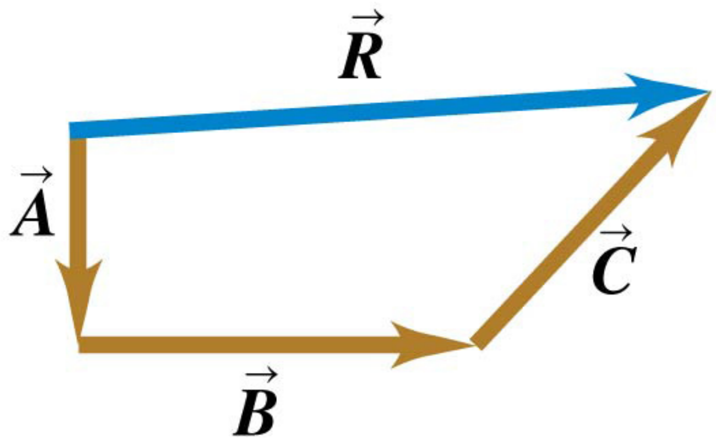
Adding more than two vectors graphically

- To add several vectors, use the head-to-tail method.
- The vectors can be added in any order.

(a) To find the sum of these three vectors ...



(d) ... or add $\vec{A}$, $\vec{B}$, and $\vec{C}$ to get $\vec{R}$ directly ...



Adding more than two vectors graphically

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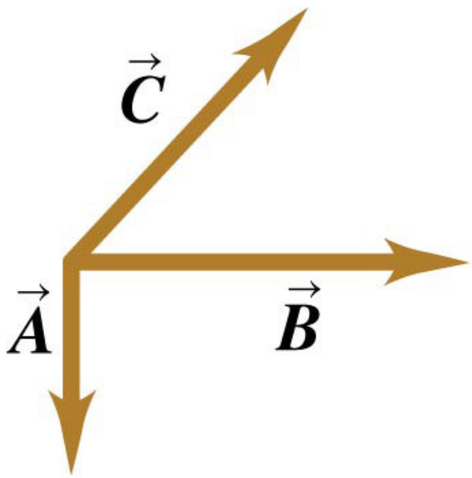
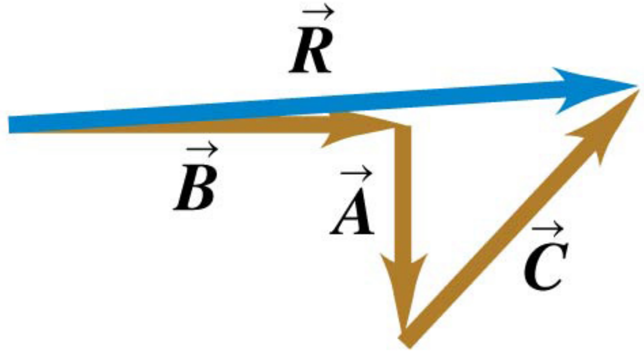
- The vectors can be added in any order.

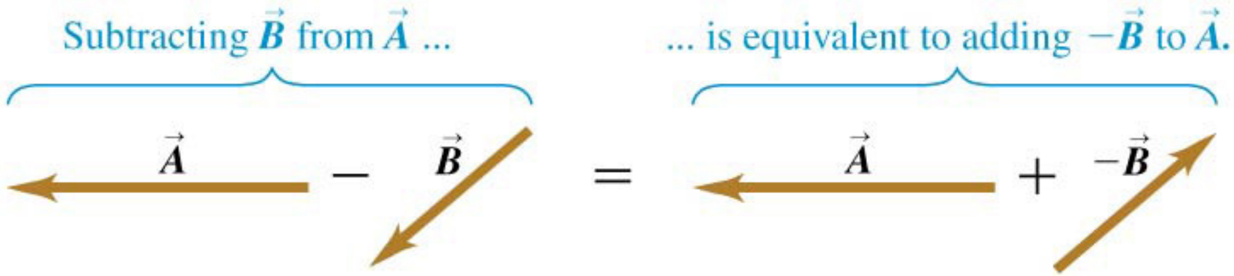
(a) To find the sum of these three vectors ...

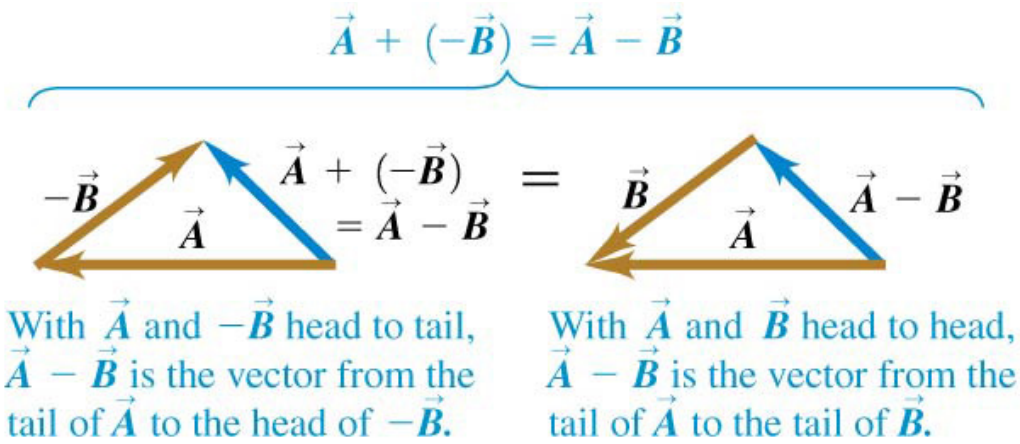


(e) ... or add $\vec{A}$, $\vec{B}$, and $\vec{C}$ in any other order and still get $\vec{R}$.



Subtracting vectors

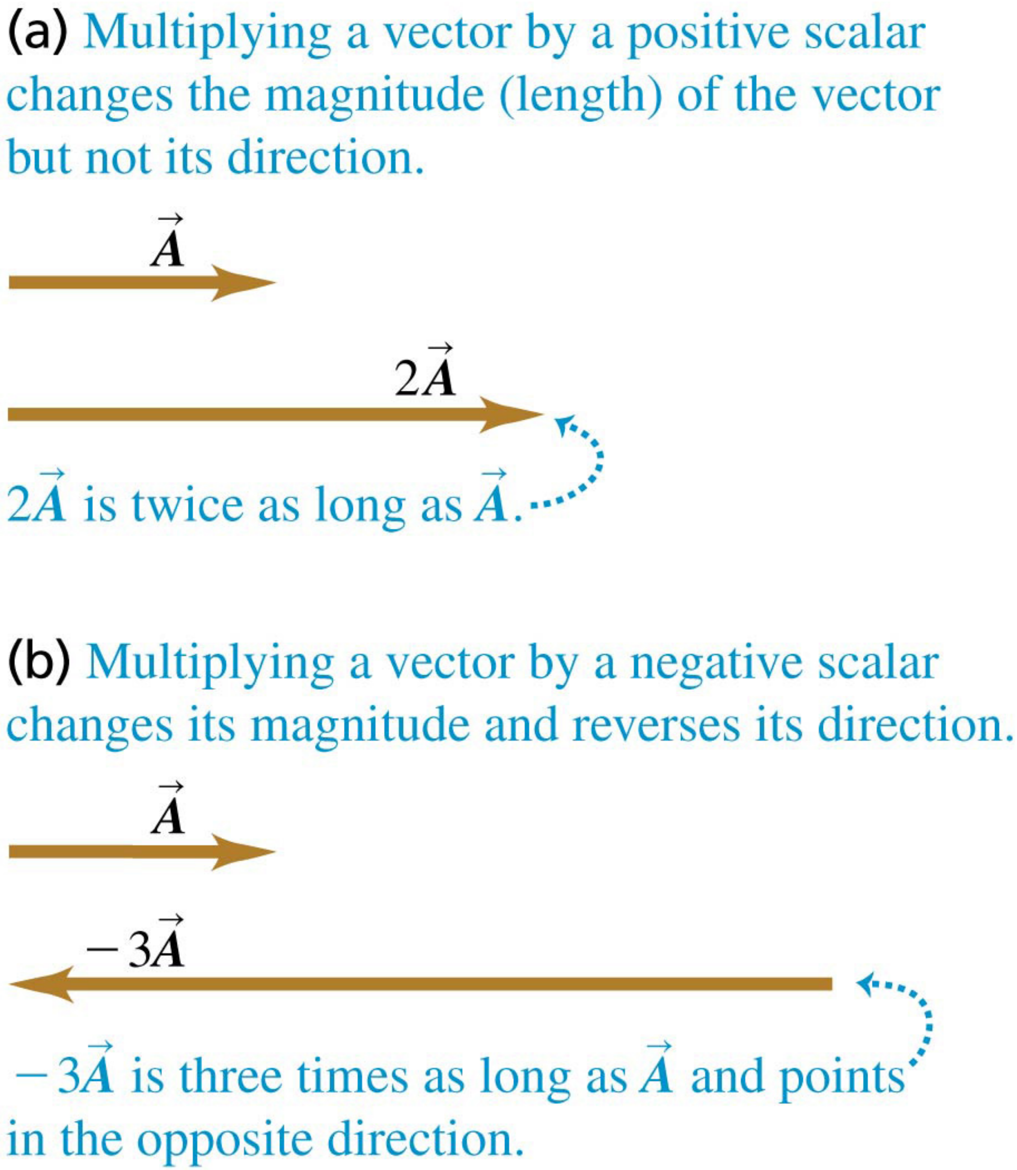




Multiplying a vector by a scalar

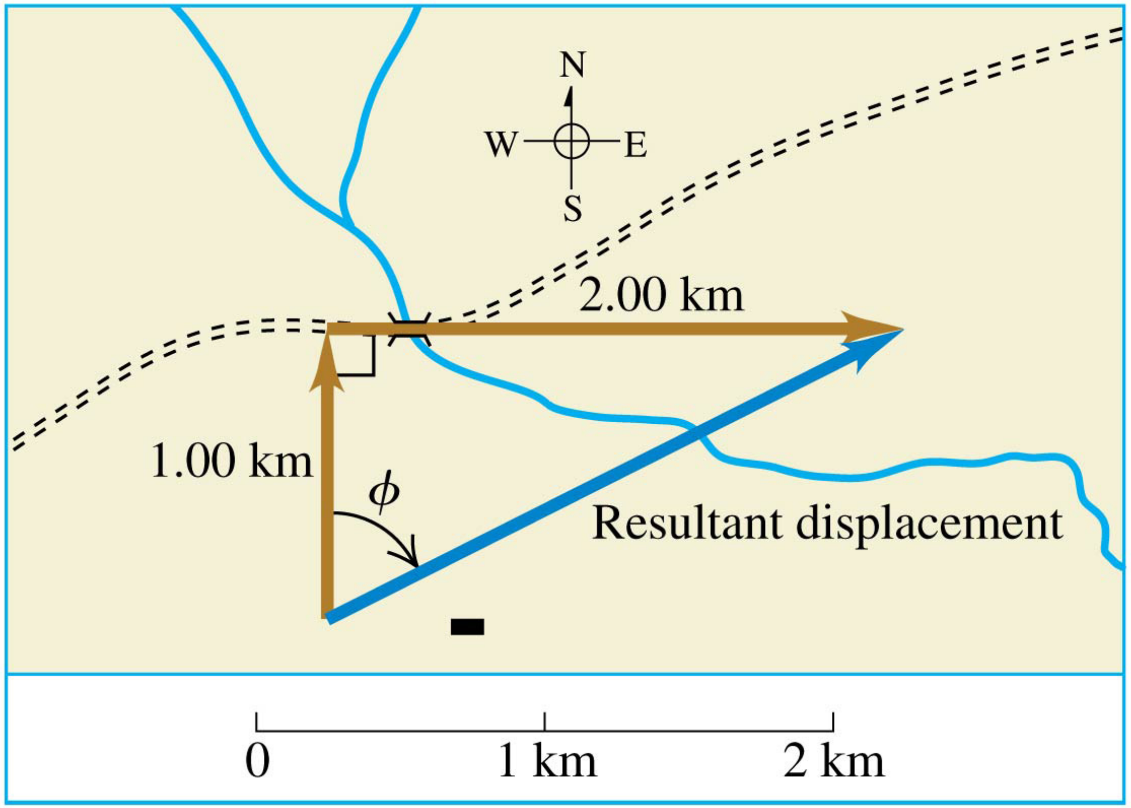
- If c is a scalar, the product $c\vec{A}$ has magnitude $|c|A$.

- The figure illustrates multiplication of a vector by (a) a positive scalar and (b) a negative scalar.



Addition of two vectors at right angles

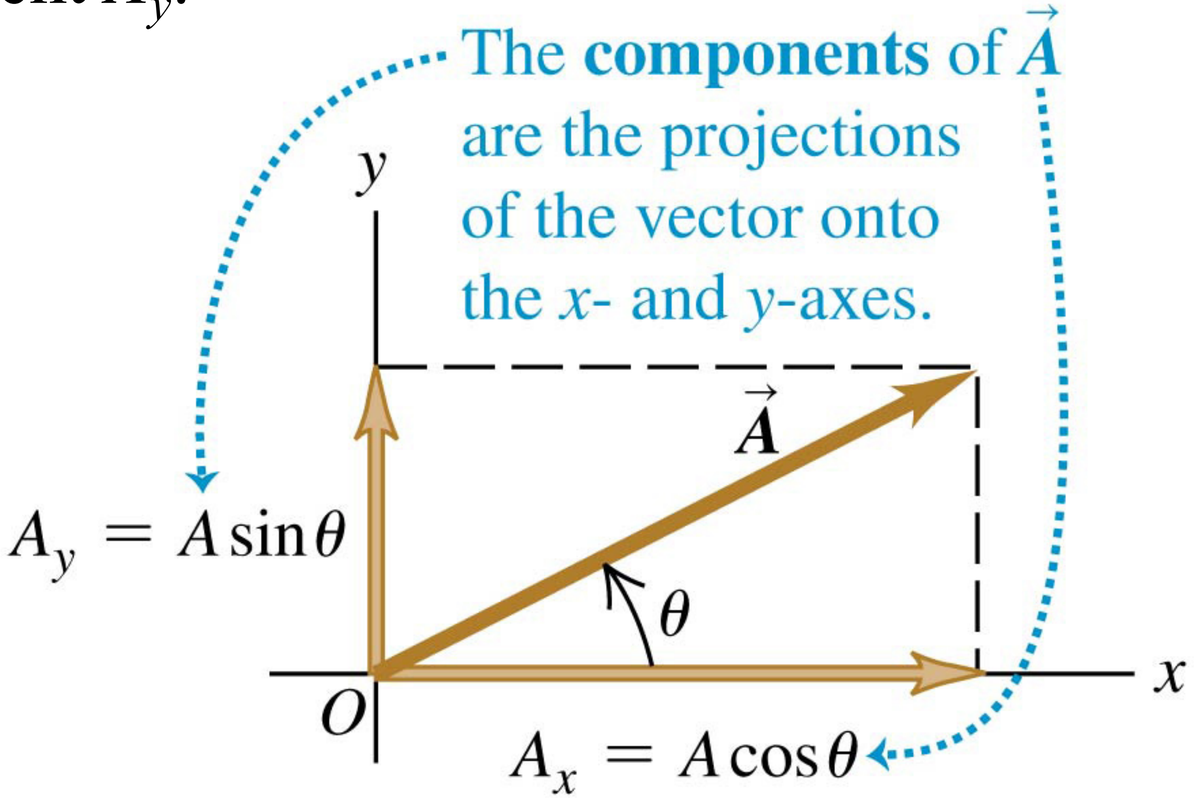
- To add two vectors that are at right angles, first add the vectors graphically.
- Then use trigonometry to find the magnitude and direction of the sum.
- In the figure, a cross-country skier ends up 2.24 km from her starting point, in a direction of 63.4° east of north.



Components of a vector

- Adding vectors graphically provides limited accuracy. Vector components provide a general method for adding vectors.

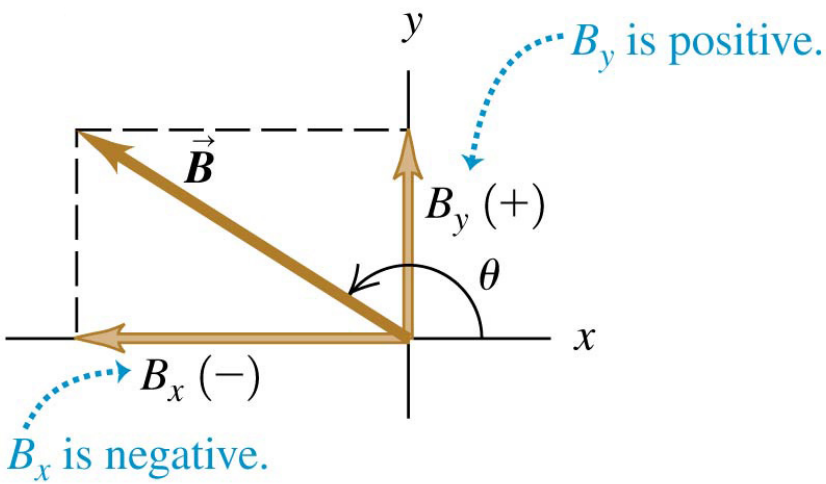
- Any vector can be represented by an x-component A_x and a y-component A_y.

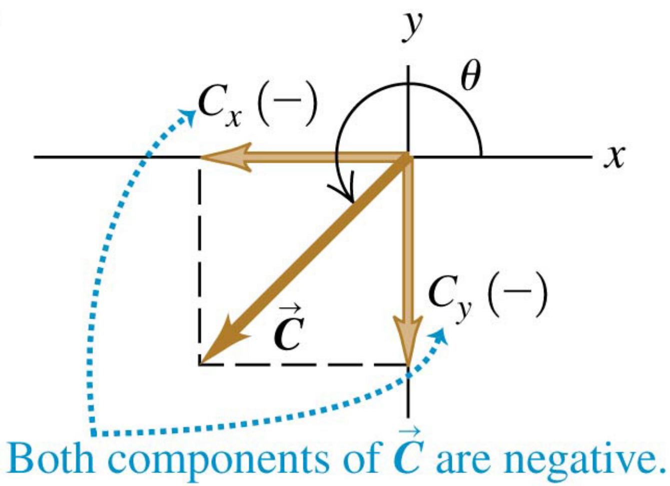


In this case, both A_x and A_y are positive.

Positive and negative components

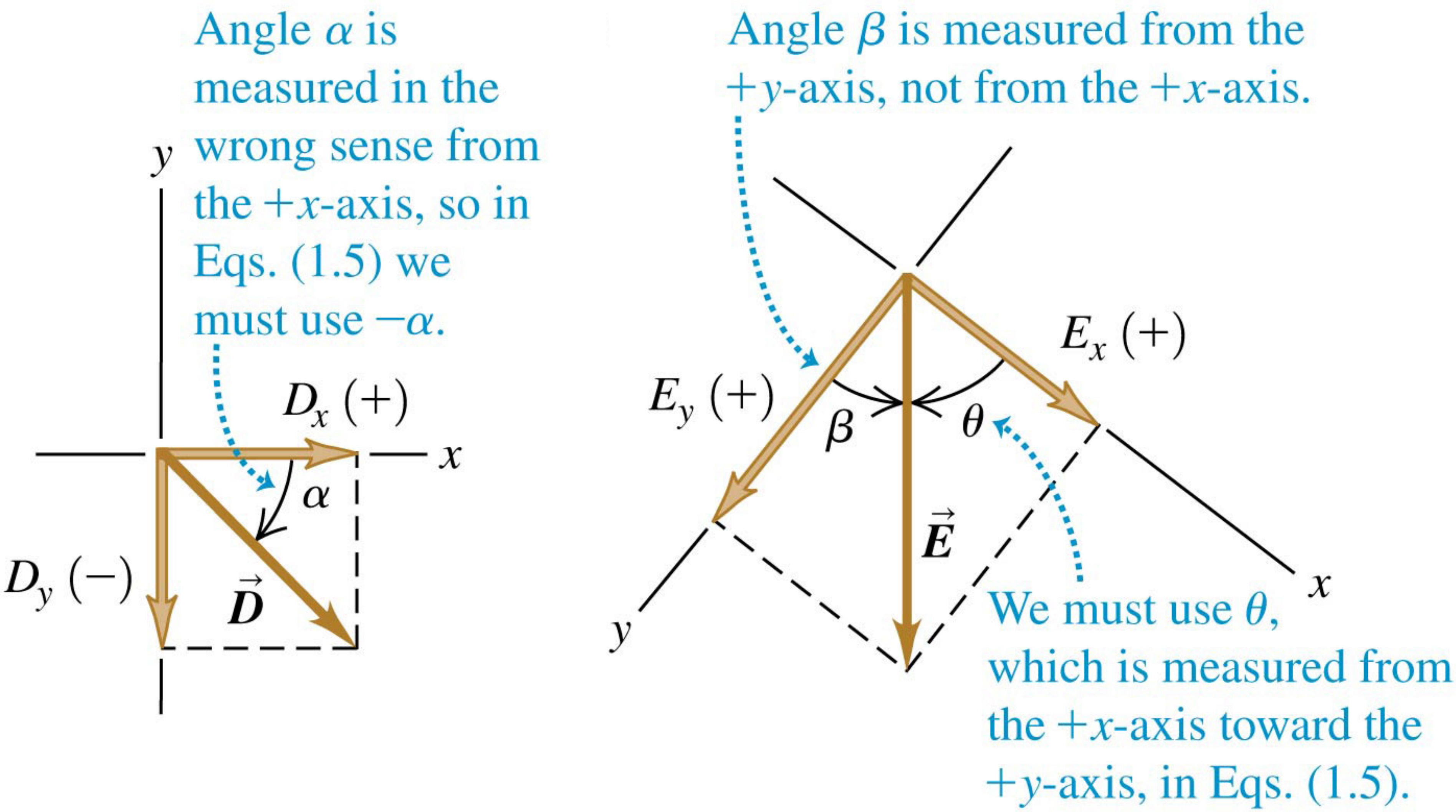
- The components of a vector may be positive or negative numbers, as shown in the figures.





Finding components

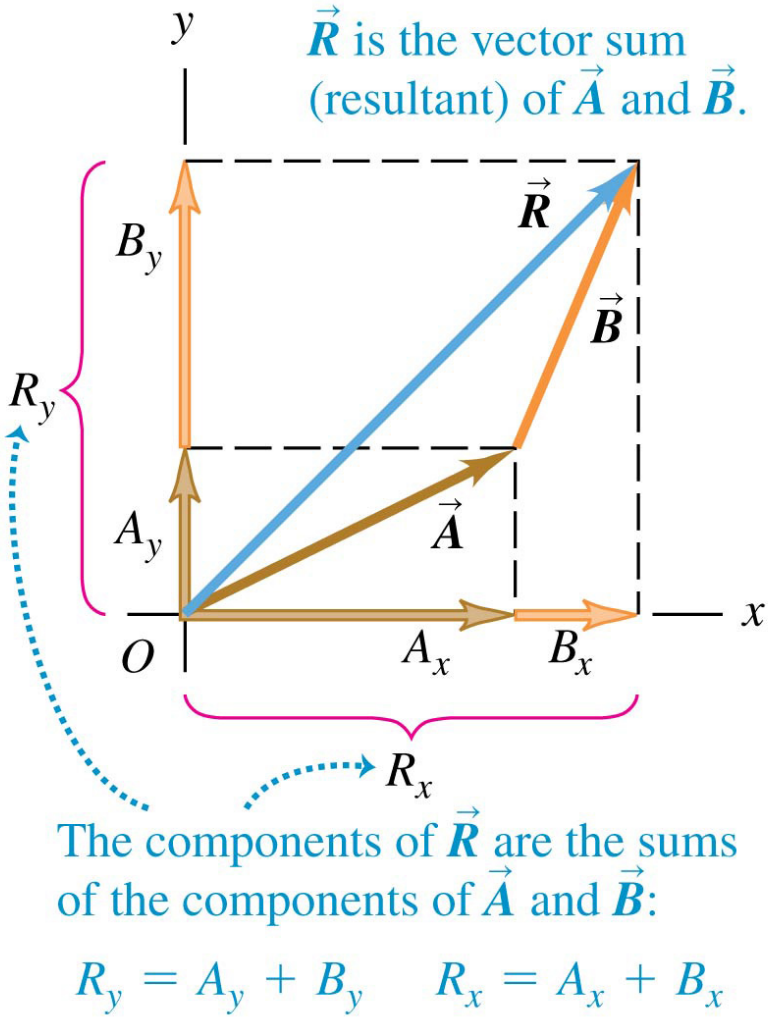
- We can calculate the components of a vector from its magnitude and direction.



Calculations using components

- We can use the components of a vector to find its magnitude and direction: $A = \sqrt{A_x^2 + A_y^2}$ and $\tan\theta = \dfrac{A_y}{A_x}$
- We can use the components of a set of vectors to find the components of their sum:

$$R_x = A_x + B_x + C_x + \cdots, \quad R_y = A_y + B_y + C_y + \cdots$$



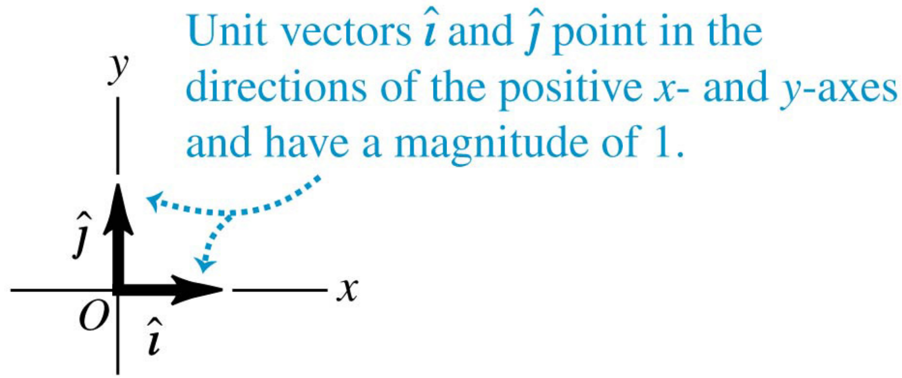
Unit vectors

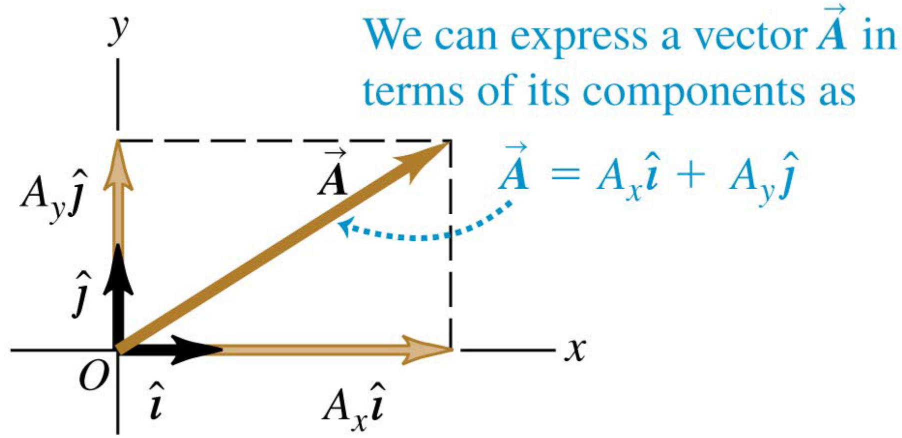
- A **unit vector** has a magnitude of 1 with no units.

- The unit vector $\hat{\boldsymbol{i}}$ points in the $+x$-direction, $\hat{\boldsymbol{j}}$ points in the $+y$-direction, and $\hat{\boldsymbol{k}}$ points in the $+z$-direction.

- Any vector can be expressed in terms of its components as $\vec{\boldsymbol{A}} = A_x\hat{\boldsymbol{i}} + A_y\hat{\boldsymbol{j}} + A_z\hat{\boldsymbol{k}}$.

- The unit vector can also be written as $\hat{x}, \hat{y}, \hat{z}$





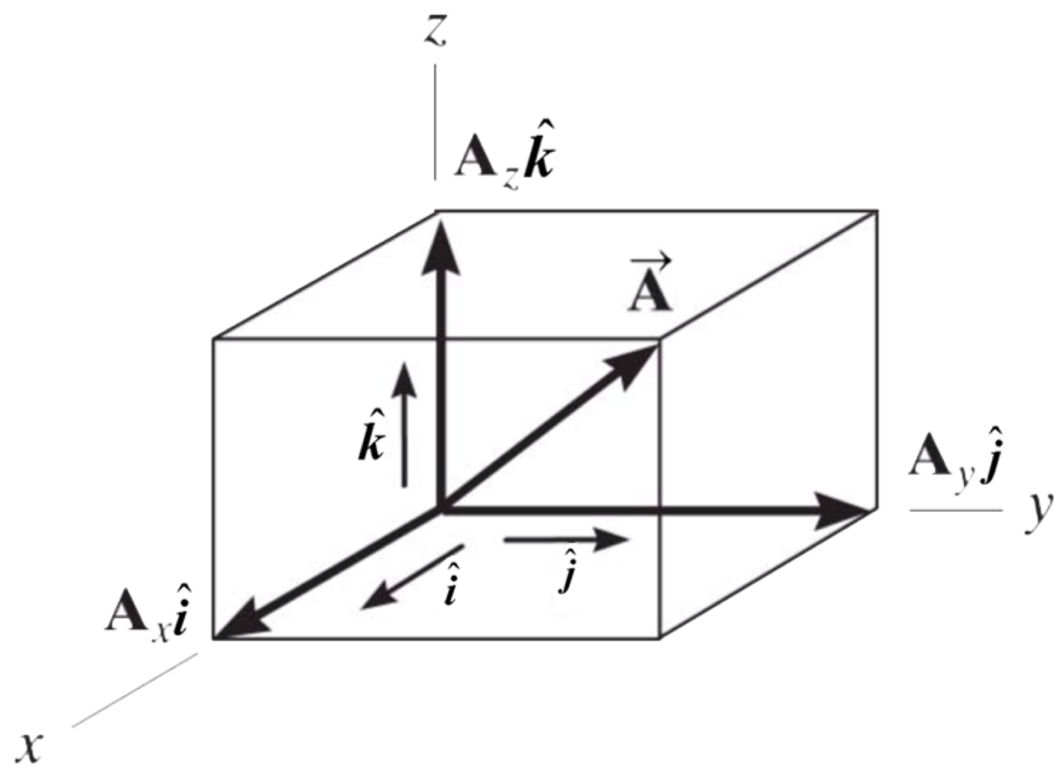
Cartesian vector notation

Vector $\mathbf{A}(\vec{A})$ is formulated by addition of its *x*, *y*, *z* components:

$$\vec{A} = A_x\hat{\boldsymbol{i}} + A_y\hat{\boldsymbol{j}} + A_z\hat{\boldsymbol{k}}.$$

The *magnitude* of **A** is determined from

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$



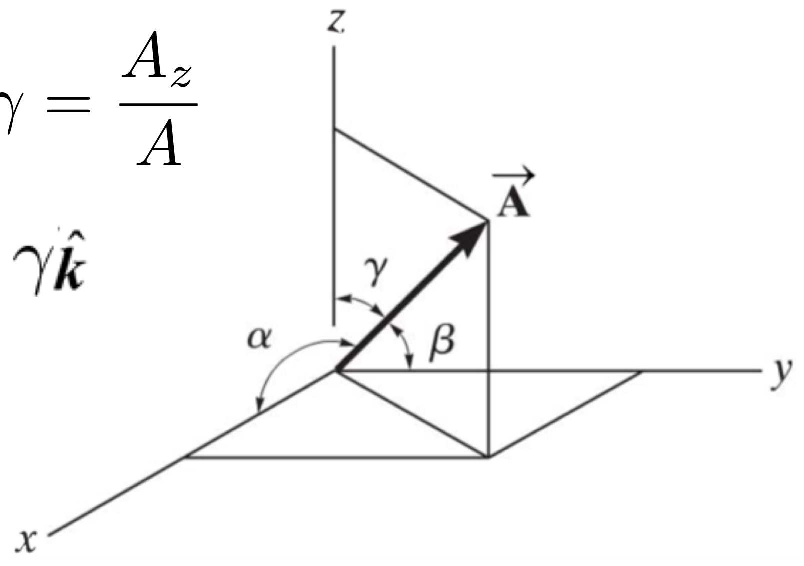
Cartesian vector notation

The *direction* of A is defined in terms of its coordinate direction angles, α, β, γ, measured from the tail of A to the *positive* x, y, z axes. These angles are determined from the *direction cosines* which represent the $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, $\hat{\boldsymbol{k}}$ components of the unit vector $\overrightarrow{\boldsymbol{u}_A}$:

$$\vec{\mathbf{u}}_A = \frac{A_x}{A}\hat{\boldsymbol{i}} + \frac{A_y}{A}\hat{\boldsymbol{j}} + \frac{A_z}{A}\hat{\boldsymbol{k}}$$

so that the direction cosines are

$$\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A}$$

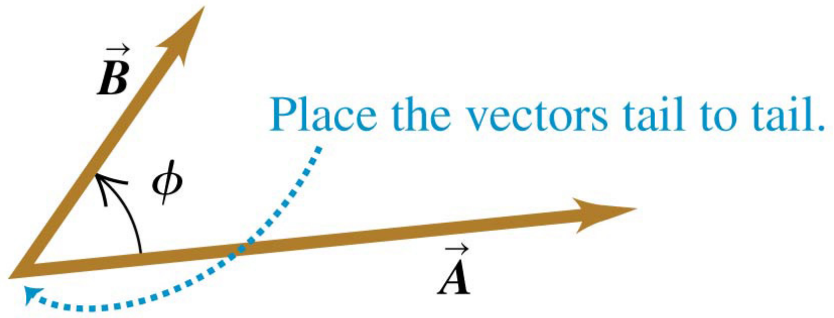


Hence $\vec{\mathbf{u}}_A = \cos\alpha\,\hat{\boldsymbol{i}} + \cos\beta\,\hat{\boldsymbol{j}} + \cos\gamma\hat{\boldsymbol{k}}$

and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

The scalar (dot) product

(a)



Scalar (dot) product of vectors $\vec{A}$ and $\vec{B}$

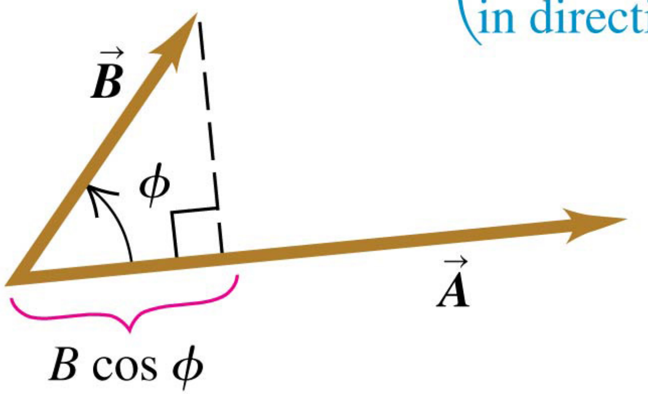
Magnitudes of $\vec{A}$ and $\vec{B}$

$$\vec{A}\cdot\vec{B} = AB\cos\phi = |\vec{A}||\vec{B}|\cos\phi$$

Angle between $\vec{A}$ and $\vec{B}$ when placed tail to tail

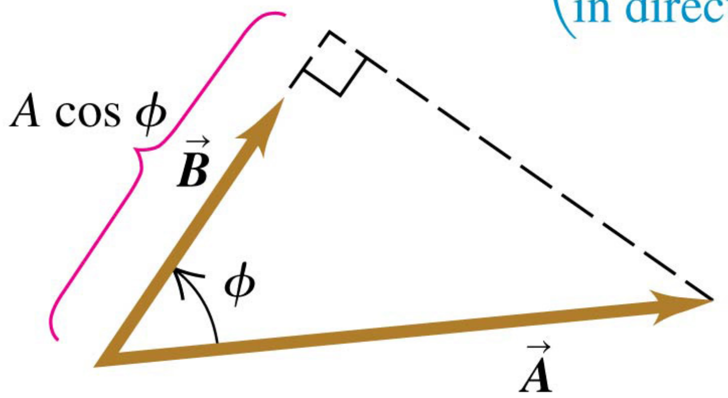
(b) $\vec{A}\cdot\vec{B}$ equals $A(B\cos\phi)$.

$$\underbrace{A}_{\text{(Magnitude of } \vec{A})} \times \underbrace{(B\cos\phi)}_{\text{(Component of } \vec{B} \text{ in direction of } \vec{A})}$$



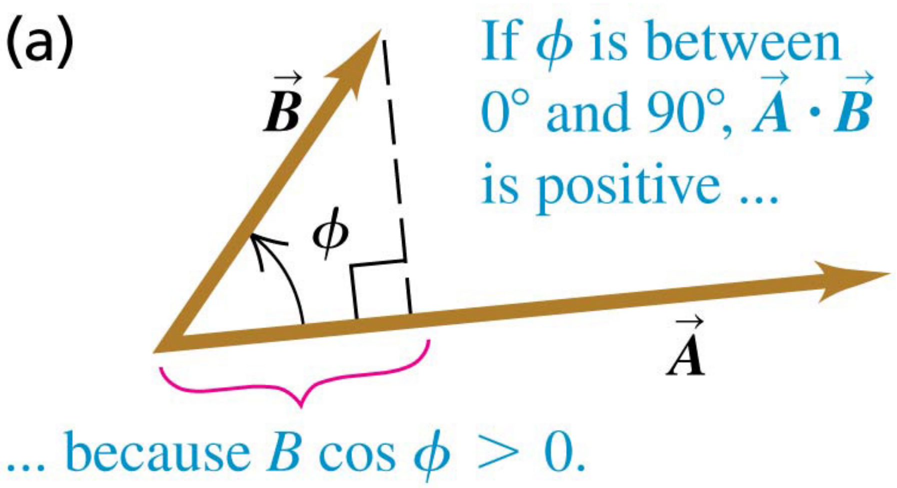
(c) $\vec{A}\cdot\vec{B}$ also equals $B(A\cos\phi)$.

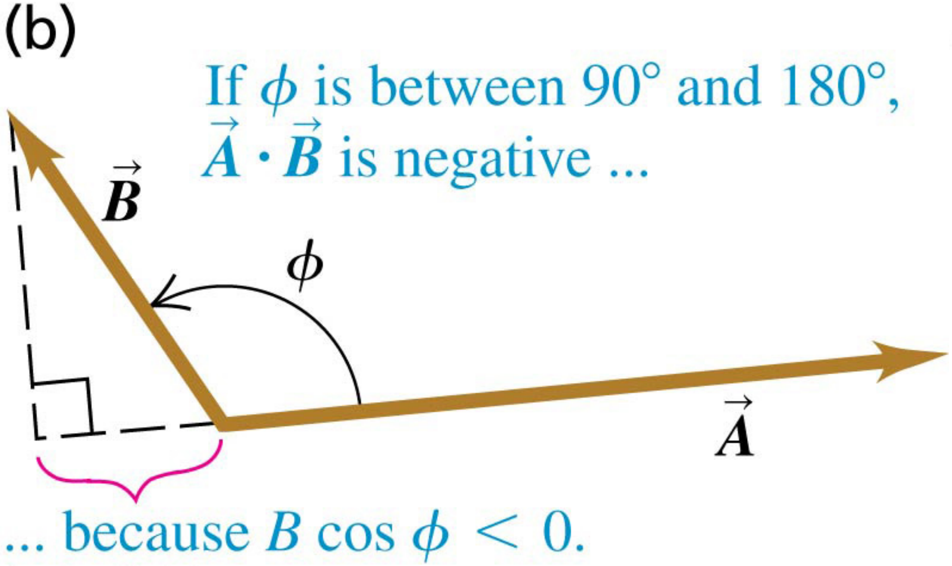
$$\underbrace{B}_{\text{(Magnitude of } \vec{B})} \times \underbrace{(A\cos\phi)}_{\text{(Component of } \vec{A} \text{ in direction of } \vec{B})}$$

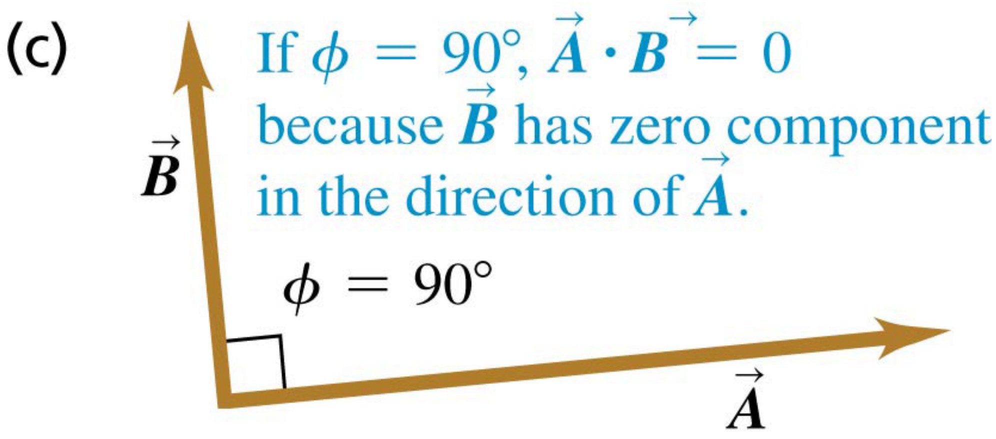


The scalar (dot) product

The scalar product can be positive, negative, or zero, depending on the angle between $\vec{A}$ and $\vec{B}$.







The scalar (dot) product: summary

The dot product of two vectors $\vec{A}$ and $\vec{B}$, which yields a scalar, is defined as

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\theta$$

and reads $\vec{A}$ "dot" $\vec{B}$. The angle θ is formed between the *tails* of $\vec{A}$ and $\vec{B}$ $(0^\circ \leq \theta \leq 180^\circ)$.

The dot product is commutative; i.e.,

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}}$$

The distributive law is valid; i.e.,

$$\vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} + \vec{\mathbf{D}}) = \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} + \vec{\mathbf{A}} \cdot \vec{\mathbf{D}}$$

And scalar multiplication can be performed in any manner, i.e.,

$$m(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}) = (m\vec{\mathbf{A}}) \cdot \vec{\mathbf{B}} = \vec{\mathbf{A}} \cdot (m\vec{\mathbf{B}}) = (\vec{\mathbf{A}} \cdot \vec{\mathbf{B}})m$$

Calculating a scalar product using components

- The dot product between any two Cartesian vectors can be determined, e.g.

$$\hat{\imath}\cdot\hat{\imath} = (1)(1)\cos 0^\circ = 1 \qquad \hat{\imath}\cdot\hat{\jmath} = (1)(1)\cos 90^\circ = 0$$

- If $\vec{A}$ and $\vec{B}$ are expressed in Cartesian component form, then the dot product can be determined from

Scalar (dot) product

Components of $\vec{A}$

$$\overbrace{\vec{A}\bullet\vec{B}}^{\text{of vectors } \vec{A} \text{ and } \vec{B}} = A_xB_x + A_yB_y + A_zB_z$$

Components of $\vec{B}$

- The scalar product of two vectors is the sum of the products of their respective components.

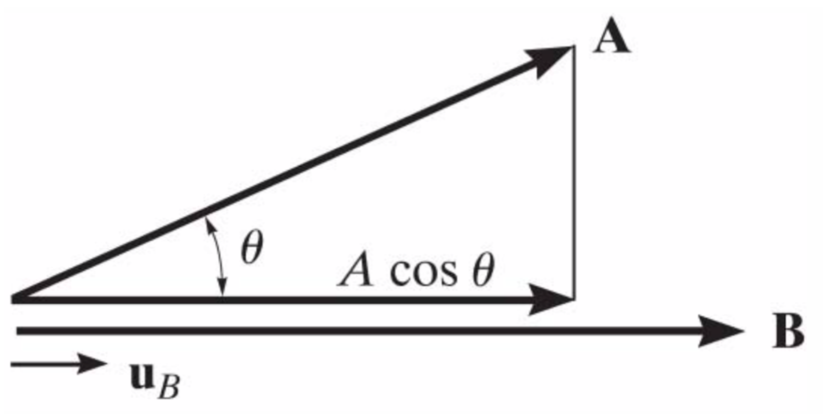
Finding an angle using the scalar product

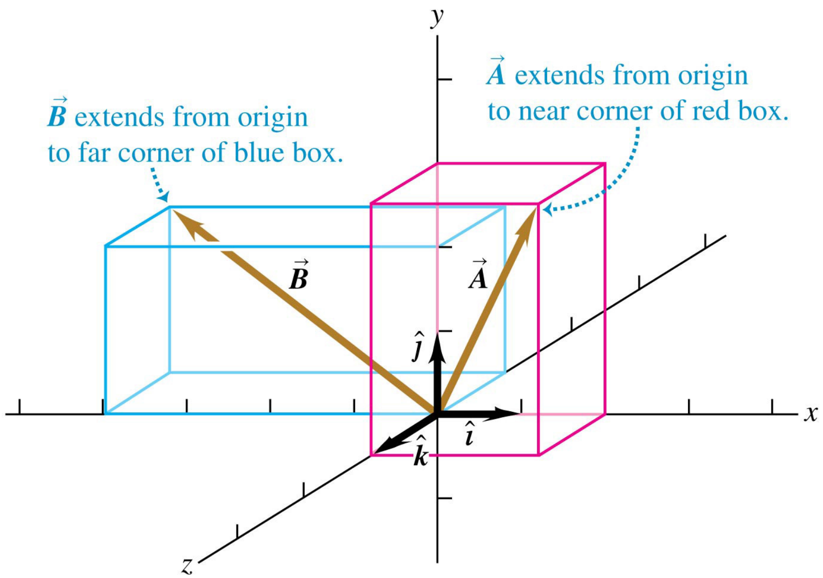
- The dot product may be used to determine the *angle θ formed between two vectors*:

$$\theta = \cos^{-1}\left(\frac{\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}}{AB}\right)$$

- It is also possible to find the component of a vector in a given direction using the dot product. For example, the magnitude of the component (or projection) of vector $\vec{\boldsymbol{A}}$ in the direction of $\vec{\boldsymbol{B}}$, is

$$A\cos\theta = \vec{\mathbf{A}}\cdot\frac{\vec{\mathbf{B}}}{B} = \vec{\mathbf{A}}\cdot\vec{\mathbf{u}}_B$$





The vector (cross) product

If the vector product ("cross product") of two vectors is $\vec{C} = \vec{A} \times \vec{B}$ then:

Magnitude of **vector (cross) product** of vectors $\vec{B}$ and $\vec{A}$

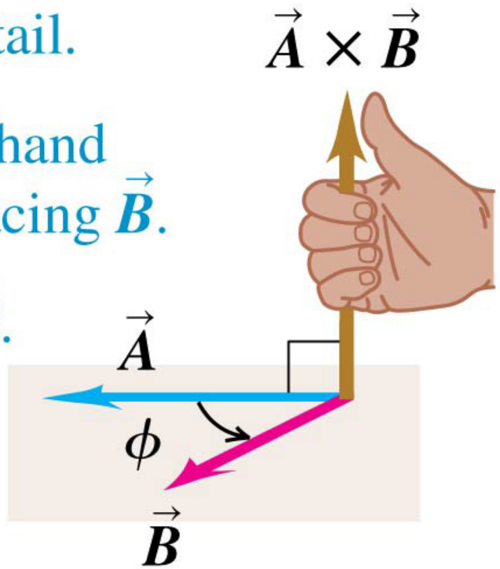
$$C = AB \sin\phi$$

Magnitudes of $\vec{A}$ and $\vec{B}$

Angle between $\vec{A}$ and $\vec{B}$ when placed tail to tail

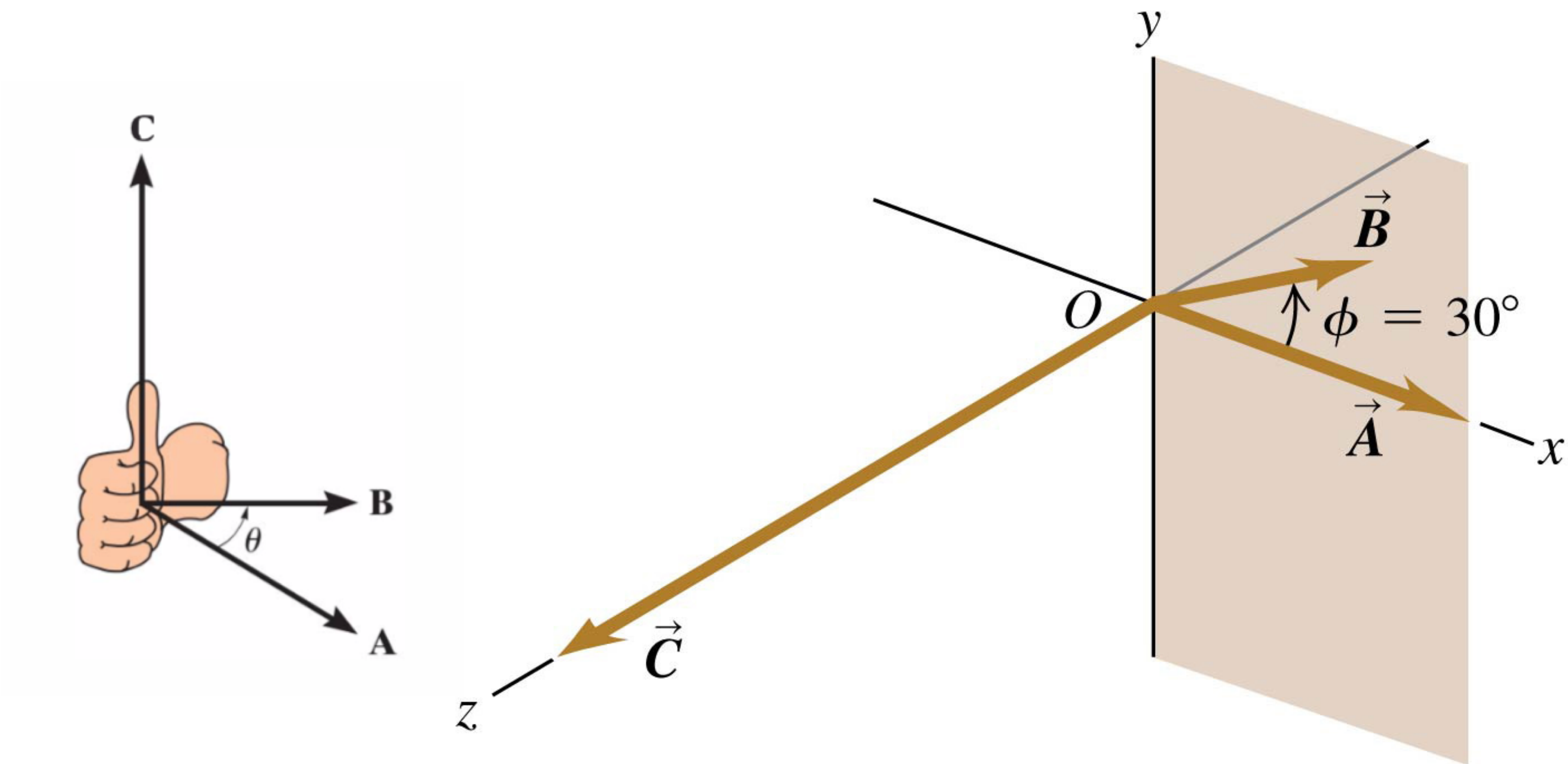
The direction of the vector product can be found using the right-hand rule:

1. Place $\vec{A}$ and $\vec{B}$ tail to tail.
2. Point fingers of right hand along $\vec{A}$, with palm facing $\vec{B}$.
3. Curl fingers toward $\vec{B}$.
4. Thumb points in direction of $\vec{A} \times \vec{B}$.



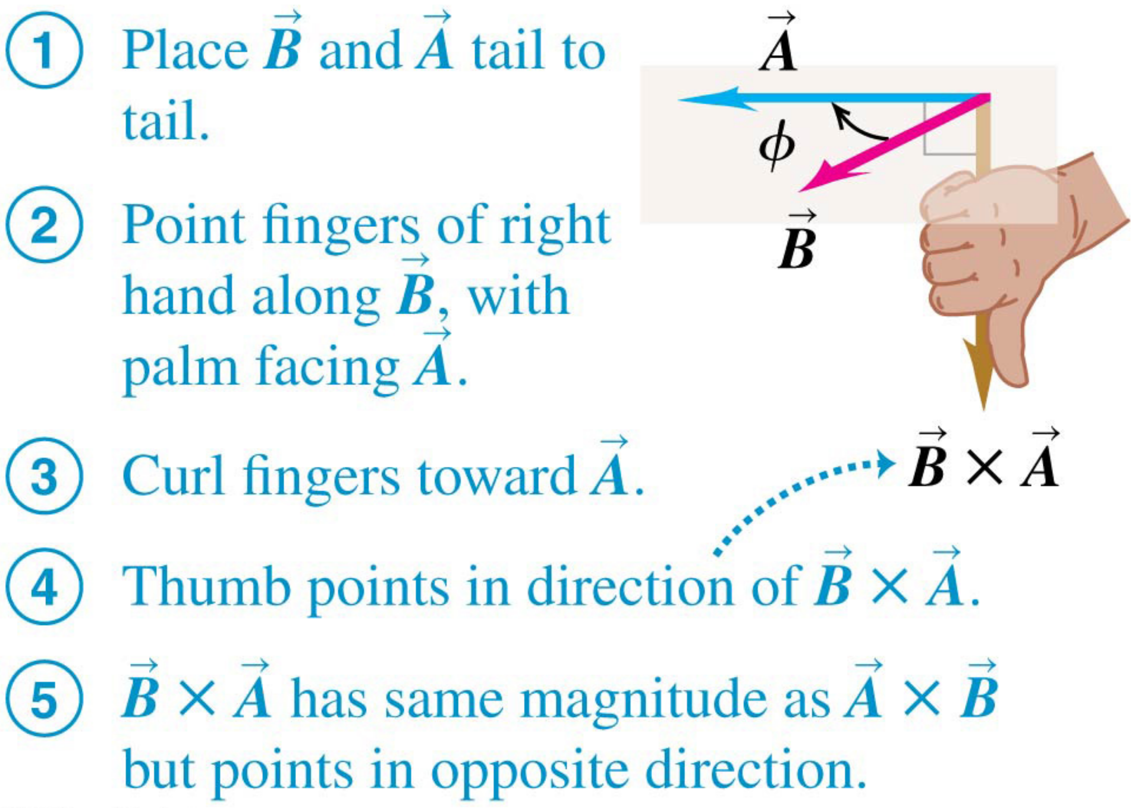
Calculating the vector product

- Use $AB\sin\phi$ to find the magnitude and the right-hand rule to find the direction.



The vector product is anticommutative

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$



The vector (cross) product

The distributive law is valid:

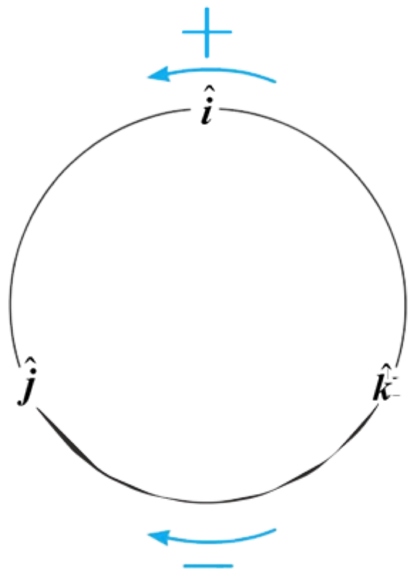
$$\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{D}}) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{D}}$$

And the cross product may be multiplied by a scalar *m* in any manner, i.e.,

$$m(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = (m\vec{\mathbf{A}}) \times \vec{\mathbf{B}} = \vec{\mathbf{A}} \times (m\vec{\mathbf{B}}) = (\vec{\mathbf{A}} \times \vec{\mathbf{B}})m$$

Cross product for any pair of Cartesian unit vectors: For example, to find $\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}}$, the magnitude is $(i)(j)\sin 90° = 1$, and its direction $+\hat{\boldsymbol{k}}$, so $\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}}$.

A simple scheme is shown in the fig. "Crossing" two of the unit vectors in a *counterclockwise* fashion around the circle yields a *positive* third unit vector, e.g. $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}}$. Moving clockwise, a negative unit vector is obtained, e.g. $\hat{\boldsymbol{i}} \times \hat{\boldsymbol{k}} = -\hat{\boldsymbol{j}}$.



The vector (cross) product

If A and B are expressed in Cartesian component form, then the cross product may be evaluated by expanding the determinant

$$\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

which yields

$$\vec{\mathbf{C}} = (A_y B_z - A_z B_y)\hat{\boldsymbol{i}} - (A_x B_z - A_z B_x)\hat{\boldsymbol{j}} + (A_x B_y - A_y B_x)\hat{\boldsymbol{k}}$$

Calculus

- We will give some brief introduction of basic calculus. You will study it in detail in math courses.
    - Differentiation
    - Differentiation of composite functions (chain rule)
    - Integration (indefinite, definite)
    - Differentiation and integration of vector functions
    - Partial differentiation
- There are abundant materials online which you should check out. A helpful one can be found at: https://tutorial.math.lamar.edu/Classes/CalcI/CalcI.aspx (we used some of his examples).

Slope and the derivative

- Consider a function of time x, f(x)
- The derivative (differentiation) of f(x) with respect to t is defined as $$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$
- i.e., divide Δf by Δx and gradually reduce Δx to very small value (the meaning of $\lim\limits_{\Delta x \to 0}$)
- Slope of AC is the derivative
- Slope of AB is $\frac{\Delta f}{\Delta x}$

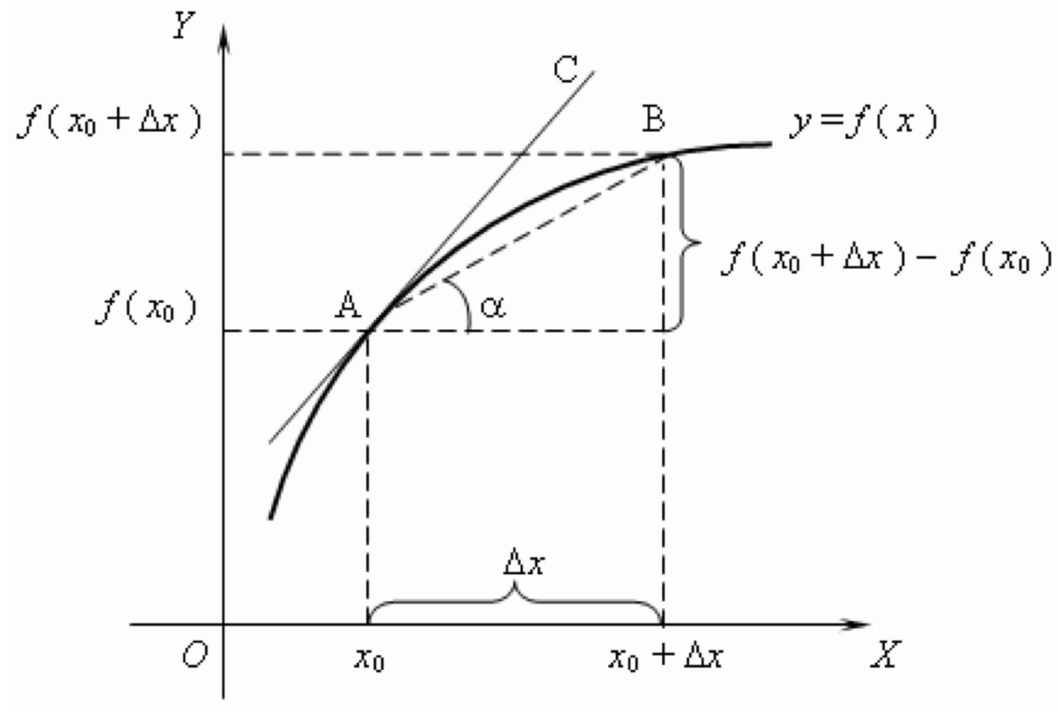


Fig. 1

- If f is distance, then the derivative is the speed

Example

$$y = x^2 - 3$$

$$y' = \lim_{h \to 0} \frac{(x+h)^2 - 3 - (x^2 - 3)}{h}$$

$$y' = \lim_{h \to 0} \frac{\cancel{x^2} + 2x\cancel{h} + \cancel{h^2} - \cancel{x^2}}{\cancel{h}}$$

$$y' = \lim_{h \to 0} 2x + \cancelto{0}{h}$$

$$y' = 2x$$

- Note that for any constant c, $y(x) = x^2 + c$, we will have $y'(x) = 2x$

Notations

- They are many ways to denote the derivative of a function $y = f(x)$
  - $f'(x)$ the derivative of f as a function of x
  - $df(x)/dx$
  - $\frac{d}{dx}f(x)$ the differentiation operator d/dx acting on $f(x)$
  - y' y prime – the derivative of y with respect to x
  - dy/dx *or* $(d/dx)y$
- When the argument is time t, i.e. $y = f(t)$ we sometimes write the time derivative as a "dot" on the top, i.e.

$$\dot{y} = \frac{dy}{dt} = y'(t)$$

Properties of differentiation

$$(f(x) \pm g(x))' = f'(x) \pm g'(x) \quad \text{OR} \quad \frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} \pm \frac{dg}{dx}$$

$$(cf(x))' = cf'(x) \quad \text{OR} \quad \frac{d}{dx}(cf(x)) = c\frac{df}{dx}, \; c \text{ is any number}$$

$$\text{If } f(x) = c \text{ then } f'(x) = 0 \quad \text{OR} \quad \frac{d}{dx}(c) = 0$$

$$\text{If } f(x) = x^n \text{ then } f'(x) = nx^{n-1} \quad \text{OR} \quad \frac{d}{dx}(x^n) = nx^{n-1}, \; n \text{ is any number.}$$

Derivative of simple functions

$$f(x) = c\,,\ \frac{df}{dx} = 0 \quad ; \qquad f(x) = ax\,,\ \frac{df}{dx} = a$$

$$f(x) = x\,,\ \frac{df}{dx} = 1\ ; \quad f(x) = x^2\,,\ \frac{df}{dx} = 2x$$

$$f(x) = x^n\,,\ \frac{df}{dx} = nx^{n-1}\ ; \ f(x) = \sin x,\ \ \frac{df}{dx} = \cos x$$

$$f(x) = \cos x \quad ,\quad \frac{df}{dx} = -\sin x$$

$$f(x) = cy(x)\,, \frac{df}{dx} = c\frac{dy}{dx}$$

Product and quotient rule

- Product rule

If the two functions $f\left(x\right)$ and $g\left(x\right)$ are differentiable (*i.e.* the derivative exist) then the product is differentiable and,

$$(f\,g)' = f'\,g + f\,g'$$

- Quotient rule

If the two functions $f\left(x\right)$ and $g\left(x\right)$ are differentiable (*i.e.* the derivative exist) then the quotient is differentiable and,

$$\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$$

Product rule

- The product rule can be extended to more than two functions, for instance

$$(f\,g\,h)' = f'\,g\,h + f\,g'\,h + f\,g\,h'$$

$$(f\,g\,h\,w)' = f'\,g\,h\,w + f\,g'\,h\,w + f\,g\,h'\,w + f\,g\,h\,w'$$

- Proof:

$$(f\,g\,h)' = ([f\,g]\;h)' = [f\,g]'h + [f\,g]\,h'$$

$$(f\,g\,h)' = \left[f'\,g + f\,g'\right]h + [f\,g]\,h' = f'\,g\,h + f\,g'\,h + f\,g\,h'$$

Composite functions and the chain rule

- If we define

$$f(z) = \sqrt{z} \qquad g(z) = 5z - 8$$

- Then we can write a composite function

$$R(z) = (f \circ g)(z) = f(g(z)) = \sqrt{5z - 8}$$

- We can use the chain rule to differentiate a composite function

Suppose that we have two functions $f(x)$ and $g(x)$ and they are both differentiable.

1. If we define $F(x) = (f \circ g)(x)$ then the derivative of $F(x)$ is,

$$F'(x) = f'(g(x)) \; g'(x)$$

2. If we have $y = f(u)$ and $u = g(x)$ then the derivative of y is,

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Example 1

- differentiate $R(z) = \sqrt{5z - 8}$.

$$f(z) = \sqrt{z} \qquad\qquad g(z) = 5z - 8$$

$$f'(z) = \frac{1}{2\sqrt{z}} \qquad\qquad g'(z) = 5$$

$$\begin{aligned} R'(z) &= f'(g(z))\ \ g'(z) \\ &= f'(5z-8)\ \ g'(z) \\ &= \frac{1}{2}(5z-8)^{-\frac{1}{2}}\ (5) \\ &= \frac{1}{2\sqrt{5z-8}}\ (5) \\ &= \frac{5}{2\sqrt{5z-8}} \end{aligned}$$

Example 1

- Another way to see it:

$$R(z) = \underbrace{(5z-8)}_{\text{inside function}}{}^{\overbrace{\tfrac{1}{2}}}$$

$$R(z) = \underbrace{(5z-8)}_{\text{inside function}}\ \underbrace{{}^{\frac{1}{2}}}_{\substack{\text{outside}\\ \text{function}}}$$

$$R'(z) = \overbrace{\frac{1}{2}\underbrace{(5z-8)}_{\substack{\text{inside function}\\ \text{left alone}}}{}^{-\frac{1}{2}}}^{\substack{\text{derivative of}\\ \text{outside function}}}\underbrace{(5)}_{\substack{\text{derivative of}\\ \text{inside function}}}$$

- In general:

$$F'(x) = \underbrace{f'}_{\substack{\text{derivative of}\\ \text{outside function}}}\underbrace{(g(x))}_{\substack{\text{inside function}\\ \text{left alone}}}\underbrace{g'(x)}_{\substack{\text{times derivative}\\ \text{of inside function}}}$$

Example 2

If $y = x^3$ and $x = t^4$, find $\ddot{y}$, the second derivative of y with respect to time.

SOLUTION.

Using the chain rule, Eq. C–1,

$$\dot{y} = 3x^2\dot{x}$$

To obtain the second time derivative we must use the product rule since x and $\dot{x}$ are both functions of time, and also, for $3x^2$ the chain rule must be applied. Thus, with $u = 3x^2$ and $v = \dot{x}$, we have

$$\begin{aligned} \ddot{y} &= [6x\dot{x}]\dot{x} + 3x^2[\ddot{x}] \\ &= 3x[2\dot{x}^2 + x\ddot{x}] \end{aligned}$$

Since $x = t^4$, then $\dot{x} = 4t^3$ and $\ddot{x} = 12t^2$ so that

$$\begin{aligned} \ddot{y} &= 3(t^4)[2(4t^3)^2 + t^4(12t^2)] \\ &= 132t^{10} \end{aligned}$$

Note that this result can also be obtained by combining the functions, then taking the time derivatives, that is,

$$y = x^3 = (t^4)^3 = t^{12}$$

$$\dot{y} = 12t^{11}$$

$$\ddot{y} = 132t^{10}$$

Example 3

If the path in radical coordinates is given as $r = 5\theta^2$, where θ is a known function of time, find $\ddot{r}$.

SOLUTION

First, using the chain rule then the chain and product rules where $u = 10\theta$ and $v = \dot{\theta}$, we have

$$r = 5\theta^2$$

$$\dot{r} = 10\theta\dot{\theta}$$

$$\ddot{r} = 10[(\dot{\theta})\dot{\theta} + \theta(\ddot{\theta})]$$

$$= 10\dot{\theta}^2 + 10\theta\ddot{\theta}$$

Example 4

If $r^2 = 6\theta^3$, find $\ddot{r}$.

SOLUTION

Here the chain and product rules are applied as follows.

$$r^2 = 6\theta^3$$

$$2r\dot{r} = 18\theta^2\dot{\theta}$$

$$2[(\dot{r})\dot{r} + r(\ddot{r})] = 18[(2\theta\dot{\theta})\dot{\theta} + \theta^2(\ddot{\theta})]$$

$$\dot{r}^2 + r\ddot{r} = 9(2\theta\dot{\theta}^2 + \theta^2\ddot{\theta})$$

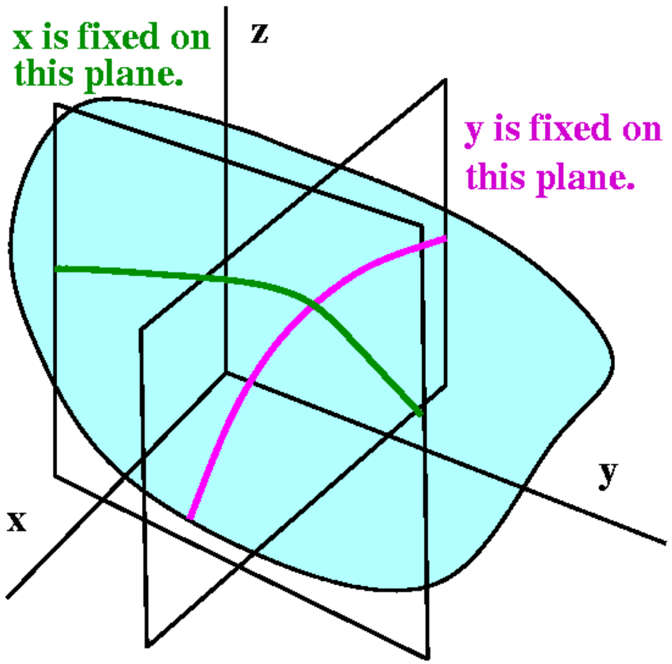
To find $\ddot{r}$ at a specified value of θ which is a known function of time, we can first find $\dot{\theta}$ and $\ddot{\theta}$. Then using these values, evaluate r from the first equation, $\dot{r}$ from the second equation and $\ddot{r}$ using the last equation.

Partial differentiation (partial derivative)

- The function is a function of two variables, x, y

$$z = f(x, y)$$

- Keep one variable x or y constant and differentiate with respect to the other variable y or x.



$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \frac{df}{dx} \text{ assuming y a constant}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \frac{df}{dy} \text{ assuming x a constant}$$

Indefinite integral

- Example: what function did we differentiate to get the following function

$$f(x) = x^4 + 3x - 9$$

  - First term: should be a differentiation of x^5, but a direct differentiation of x^5 gives $5x^4$, we therefore need a prefactor of 1/5, i.e. $x^5/5$.
  - Second term: $3x$ should be the derivative of $3x^2/2$.
  - Third term: 9 should be the derivative of $9x$.

- Altogether we have $F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x$, and one can verify $F'(x) = f(x)$
- However, we can add any constant to $F(x)$, e.g.

$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + 10$$
$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x - 1954$$
$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + \frac{3469}{123}$$
etc.

- In fact, any function of the form (with c being any constant)

$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c,$$

will give $f(x)$ upon differentiating.

Indefinite integral

- Given a function, $f(x)$, an anti-derivative of $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$.
- If $F(x)$ is any anti-derivative of $f(x)$ then the most general anti-derivative of $f(x)$ is called an **indefinite integral** and denoted
$$\int f(x)dx = F(x) + c$$
- $\int$ is called the **integral symbol**, $f(x)$ is called the **integrand**, x is called the **integration variable** and "c" the **constant of integration**.
- The process of finding the indefinite integral is called **integration** or **integrating $f(x)$**. If we need to be specific about the integration variable we say that we are **integrating $f(x)$ with respect to x**.

Properties of indefinite integral

- The integration is linear, i.e.

$$\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$$

$$\int kf(x)dx = k \int f(x)dx$$

k is any number. In particular, when k = -1

$$\int -f(x)dx = -\int f(x)dx$$

Indefinite integral of common functions

$$\int a dx = ax + c$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int \cos x dx = \sin x + c$$

$$\int \sin x dx = -\cos x + c$$

Differentiation and integration of Vector Func

- The rules for differentiation and integration of the sums and products of scalar functions also apply to vector functions. Consider, for example, the two vector functions $\mathbf{A}(s)$ and $\mathbf{B}(s)$. Provided these functions are smooth are continuous for all s, then

$$\frac{d}{ds}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{ds} + \frac{d\mathbf{B}}{ds}$$

$$\int (\mathbf{A}+\mathbf{B})ds = \int \mathbf{A}ds + \int \mathbf{B}ds$$

$$\frac{d}{ds}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{ds}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{ds}$$

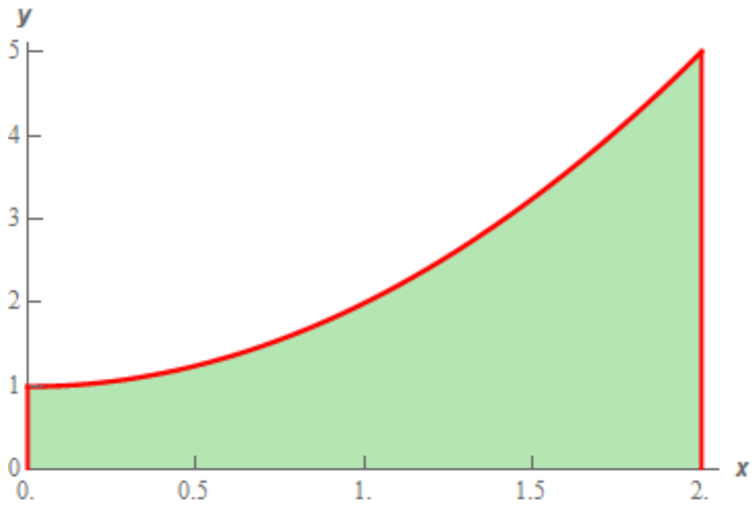
$$\frac{d}{ds}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{ds}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{ds}$$

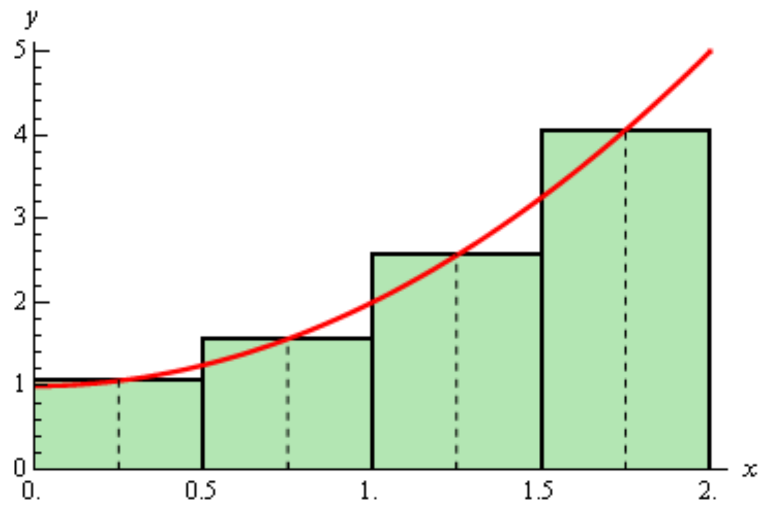
Area problem

- Suppose we want to determine the area under $f(x) = x^2 + 1$ on $[0,2]$ as shown in the fig.

- We can estimate the area by dividing up the interval into n subintervals, each of width $\Delta x = (b - a)/n$. So the are can be estimated as

$$A \approx \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x \qquad A = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$





Definite integral

- Given a function $f(x)$ that is continuous on the interval $[a, b]$ we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*, then the **definite integral** of $f(x)$ from a to b is

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

- Method to calculate: take the indefinite integral of $f(x)$, evaluate at b and a, and take the difference. Example:

$$\int_0^2 x^2 dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{2^3 - 0^3}{3} = \frac{8}{3}$$

(note that the constant c makes no contribution)

Properties of definite integral

- Interchanging the limits gets you a minus sign:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

- If the upper and lower limits are the same then the area will be 0

$$\int_a^a f(x)\,dx = 0$$

- For any number c,

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

- Definite integrals can be broken up across a sum or difference

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Properties of definite integral

- (Note that c does not have to be between a and b)

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

- Changing the integration variable will not affect the answer

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

- For any number c,

$$\int_a^b c\,dx = c(b-a)$$

Fundamental theorem of calculus

- Suppose $f(x)$ is a continuous function on $[a, b]$ and also suppose that $F(x)$ is any anti-derivative for $f(x)$. Then,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

- Example:

$$\int y^2 + y^{-2}\,dy = \frac{1}{3}y^3 - y^{-1} + c$$

$$\begin{aligned}\int_1^2 y^2 + y^{-2}\,dy &= \left(\frac{1}{3}y^3 - \frac{1}{y}\right)\Bigg|_1^2 \\ &= \frac{1}{3}(2)^3 - \frac{1}{2} - \left(\frac{1}{3}(1)^3 - \frac{1}{1}\right) \\ &= \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 \\ &= \frac{17}{6}\end{aligned}$$

Useful mathematical expressions

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Hyperbolic Functions

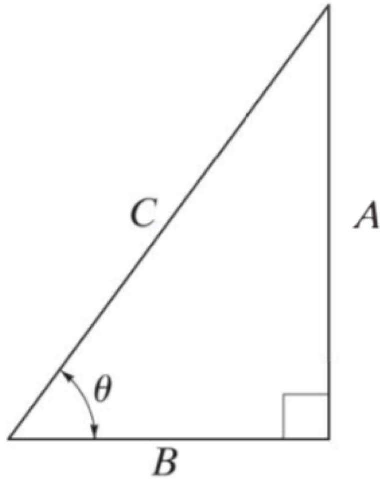
$\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\tanh x = \dfrac{\sinh x}{\cosh x}$

Trigonometric Identities

$\sin\theta = \dfrac{A}{C}$, $\csc\theta = \dfrac{C}{A}$

$\cos\theta = \dfrac{B}{C}$, $\sec\theta = \dfrac{C}{B}$

$\tan\theta = \dfrac{A}{B}$, $\cot\theta = \dfrac{B}{A}$



$\sin^2\theta + \cos^2\theta = 1$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

Useful mathematical expressions

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}}, \sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Power-Series Expansions

$$\sin x = x - \frac{x^3}{3!} + \cdots \qquad \sinh x = x + \frac{x^3}{3!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \cdots \qquad \cosh x = 1 + \frac{x^2}{2!} + \cdots$$

Derivatives

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \frac{d}{dx}(\cot u) = -\csc^2 u\frac{du}{dx}$$

Useful mathematical expressions

$$\frac{d}{dx}(\sec u) = \tan u \sec u \frac{du}{dx}$$

$$\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}$$

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

$$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$$

$$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

Integrals

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int \frac{dx}{a + bx} = \frac{1}{b} \ln(a + bx) + C$$

$$\int \frac{dx}{a + bx^2} = \frac{1}{2\sqrt{-ba}} \ln\left[\frac{a + x\sqrt{-ab}}{a - x\sqrt{-ab}}\right] + C, \ ab < 0$$

$$\int \frac{x\,dx}{a + bx^2} = \frac{1}{2b} \ln(bx^2 + a) + C,$$

$$\int \frac{x^2\,dx}{a + bx^2} = \frac{x}{b} - \frac{a}{b\sqrt{ab}} \tan^{-1} \frac{x\sqrt{ab}}{a} + C, ab > 0$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left[\frac{a + x}{a - x}\right] + C, a^2 > x^2$$

Useful mathematical expressions

$$\int \sqrt{a + bx}\, dx = \frac{2}{3b}\sqrt{(a + bx)^3} + C$$

$$\int x\sqrt{a + bx}\, dx = \frac{-2(2a - 3bx)\sqrt{(a + bx)^3}}{15b^2} + C$$

$$\int x^2\sqrt{a + bx}\, dx = \frac{2(8a^2 - 12abx + 15b^2x^2)\sqrt{(a + bx)^3}}{105b^3} + C$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right] + C, a > 0$$

$$\int x\sqrt{x^2 \pm a^2}\, dx = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3} + C$$

$$\int x^2\sqrt{a^2 - x^2}\, dx = -\frac{x}{4}\sqrt{(a^2 - x^2)^3}$$

$$+ \frac{a^2}{8}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right) + C, a > 0$$

$$\int \sqrt{x^2 \pm a^2}\, dx = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2 \ln\left(x + \sqrt{x^2 \pm a^2}\right)\right] + C$$

Useful mathematical expressions

$$\int x\sqrt{a^2 - x^2}\,dx = -\frac{1}{3}\sqrt{(a^2 - x^2)^3} + C$$

$$\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\ln\left(x + \sqrt{x^2 \pm a^2}\right) + C$$

$$\int \frac{dx}{\sqrt{a + bx}} = \frac{2\sqrt{a + bx}}{b} + C$$

$$\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2} + C$$

$$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}}\ln\left[\sqrt{a + bx + cx^2} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right] + C,\ c > 0$$

$$= \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{-2cx - b}{\sqrt{b^2 - 4ac}}\right) + C,\ c > 0$$

Useful mathematical expressions

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int x \cos(ax) \, dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax) + C$$

$$\int x^2 \cos(ax) \, dx = \frac{2x}{a^2}\cos(ax) + \frac{a^2x^2 - 2}{a^3}\sin(ax) + C$$

$$\int e^{ax} \, dx = \frac{1}{a}e^{ax} + C$$

$$\int xe^{ax} \, dx = \frac{e^{ax}}{a^2}(ax - 1) + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \cosh x \, dx = \sinh x + C$$