MA1300 Solutions to Self Practice # 7

1.

a If n is a positive integer, prove that

$$\frac{d}{dx}(\sin^n x \cos nx) = n \sin^{n-1} x \cos(n+1)x.$$

b Find a formula for the derivative of $y = \cos^n x \cos nx$ that is similar to the on in part **a**.

Solution:

a

$$\begin{aligned}\frac{d}{dx}(\sin^n x \cos nx) &= n \sin^{n-1} x \cos nx \cos x - n \sin^n x \sin nx \\ &= n \sin^{n-1} x(\cos nx \cos x - \sin nx \sin x) \\ &= n \sin^{n-1} x \cos(n+1)x.\end{aligned}$$

b

$$\begin{aligned}\frac{d}{dx}(\cos^n x \cos nx) &= -n \cos^{n-1} x \sin x \cos nx - n \cos^n x \sin nx \\ &= -n \cos^{n-1} x \sin(n+1)x.\end{aligned}$$

2.

a Write $|x| = \sqrt{x^2}$ and use the Chain Rule to show that

$$\frac{d}{dx}|x| = \frac{x}{|x|}.$$

b If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of f and f'. Where is f not differentiable?

c If $g(x) = \sin |x|$, find $g'(x)$ and sketch the graphs of g and g'. Where is g not differentiable?

Solution:

a

$$\frac{d}{dx}|x| = \frac{2x}{2\sqrt{x^2}} = \frac{x}{|x|}.$$

b By the Chain Rule, $f'(x) = \frac{\sin x}{|\sin x|} \cos x$. The graphs of f and f' are shown in Figure 1. f is not differentiable when $\sin x = 0$, or equivalently, at $x = n\pi$, $n = 0, \pm 1, \pm 2, \cdots$.

c By the Chain Rule, $g'(x) = \frac{x}{|x|} \cos |x|$. the graphs of g and g' are shown in Figure 2. g is not differentiable at $x = 0$.

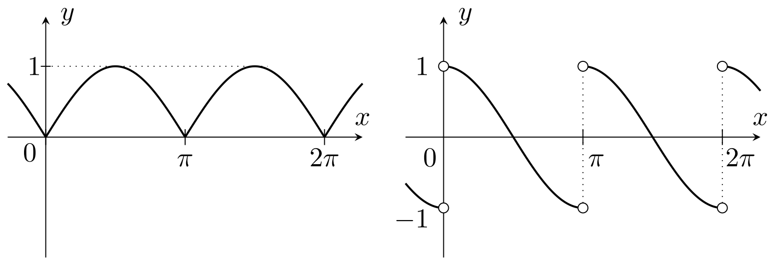

Figure 1: The picture of Problem 2 **b**. Left: $y = f(x)$, right: $y = f'(x)$.

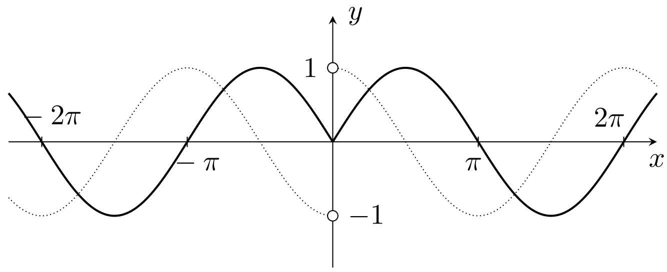

Figure 2: The picture of Problem 2 **c**. Solid line: $y = g(x)$, dotted line: $y = g'(x)$.

3. If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, show that

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}.$$

Proof: By the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Using the Chain Rule again, take the derivatives of both sides to give

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}.$$

4. If $y = f(u)$ and $u = g(x)$, where f and g possess third derivatives, find a formula for d^3y/dx^3 similar to the one given in Exercise 3.

Solution: According to the proof of Problem 3, use the Chain Rule to give

$$\begin{aligned}\frac{d^3y}{dx^3} &= \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3 + 2\frac{d^2y}{du^2}\frac{du}{dx}\frac{d^2u}{dx^2} + \frac{d^2y}{du^2}\frac{d^2u}{dx^2}\frac{du}{dx} + \frac{dy}{du}\frac{d^3u}{dx^3}\\ &= \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3 + 3\frac{d^2y}{du^2}\frac{du}{dx}\frac{d^2u}{dx^2} + \frac{dy}{du}\frac{d^3u}{dx^3}.\end{aligned}$$

5. For the following implicit function relations

$$4x^2 + 9y^2 = 36, \qquad \cos x + \sqrt{y} = 5,$$

a Find y' by implicit differentiation.

b Solve the equation explicitly for y and differentiate to get y' in terms of x.

c Check that your solutions to parts **a** and **b** are consistent by substituting the expression for y into your solution for part **a**.

Solution:

1 For $4x^2 + 9y^2 = 36$, we have

$$8x + 18yy' = 0,$$

so

$$y' = -\frac{4x}{9y}.$$

Solve the equation $4x^2 + 9y^2 = 36$ directly to give

$$y = \pm\frac{\sqrt{36-4x^2}}{3},$$

so

$$\frac{dy}{dx} = \pm\frac{-4x}{3\sqrt{36-4x^2}} = \mp\frac{2x}{3\sqrt{9-x^2}}.$$

Substitute $y = \pm\frac{\sqrt{36-4x^2}}{3}$ into the equation to give

$$\frac{dy}{dx} = -\frac{2x}{9y/2} = -\frac{4x}{9y}.$$

Consistent.

2 For $\cos x + \sqrt{y} = 5$, we have

$$-\sin x + \frac{y'}{2\sqrt{y}} = 0,$$

so

$$y' = 2\sqrt{y}\sin x.$$

Solve$\cos x + \sqrt{y} = 5$ directly to give

$$y = (5 - \cos x)^2,$$

so

$$y' = 2(\cos x - 5)(-\sin x).$$

Since $5 > \cos x$, substitute $5 - \cos x$ for $\sqrt{y}$ to give

$$y' = 2\sqrt{y}\sin x.$$

Consistent.

6. Find dy/dx by implicit differentiation.

$$1 + x = \sin(xy^2), \qquad \tan\frac{x}{y} = x + y.$$

Solution:

1 $1+x=\sin(xy^2)$. Take derivative with respect to x to give

$$1=\cos(xy^2)(y^2+2xyy'),$$

so

$$y'=\left(\frac{1}{\cos(xy^2)}-y^2\right)\frac{1}{2xy}.$$

2 $\tan\frac{x}{y}=x+y$. Take derivative with respect to x to give

$$\left(1+\tan^2\frac{x}{y}\right)\left(\frac{y-y'x}{y^2}\right)=1+y',$$

so

$$y'=\frac{\frac{1}{y}\left(1+\tan^2\frac{x}{y}\right)-1}{1+\frac{x}{y^2}\left(1+\tan^2\frac{x}{y}\right)}.$$

7. If $f(x)+x^2[f(x)]^3=10$ and $f(1)=2$, find $f'(1)$.

Solution: From $f(x)+x^2[f(x)]^3=10$, take derivatives of both side with respect to x to give

$$f'(x)+2x(f(x))^3+3x^2(f(x))^2f'(x)=0.$$

Substitute $x=1$ into the equation to give

$$f'(1)+2\cdot 2^3+3\cdot 2^2f'(1)=0.$$

Therefore $f'(1)=-\frac{16}{13}$.

8. If $g(x)+x\sin g(x)=x^2$, find $g'(0)$.

Solution: Take derivatives of both sides of the equation to give

$$g'(x)+\sin g(x)+x\cos g(x)g'(x)=2x.$$

Substitute 0 for x to give

$$g'(0)=-\sin g(0).$$

Moreover, when x = 0, we have g(0) + 0 sin(g(0)) = 0, then g(0) = 0. Overall, we have g'(0) = -sin(g(0)) = 0.

9.

a The curve with equation $y^2=5x^4-x^2$ is called a **kampyle of Eudoxus**. Find an equation of the tangent line to this curve at the point $(1,2)$.

b Illustrate part **a** by graphing the curve and the tangent line on a common screen. (If your graphing device will graph implicitly defined curves, then use that capability. If not, you can still graph this curve by graphing its upper and lower halves separately.)

Solution: Take derivatives of both sides of the equation with respect to x to give

$$2yy' = 20x^3 - 2x,$$

so

$$y'|_{x=1,y=2} = \frac{20-2}{4} = \frac{9}{2}.$$

Therefore the point-slope form of the tangent line is

$$y - 2 = \frac{9}{2}(x-1).$$

The picture is illustrated in Figure 3

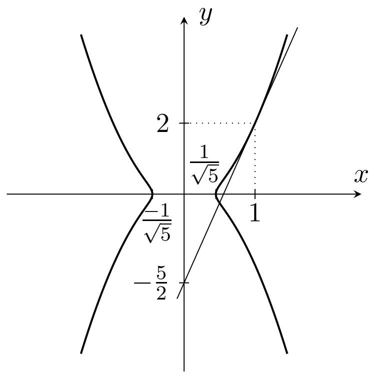

Figure 3: The picture of Problem 9.

10.

a The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, -2)$.

b At what points does this curve have horizontal tangents?

c Illustrate parts **a** and **b** by graphing the curve and the tangent lines on a common screen.

Solution:

a Take derivatives of both sides of the equation with respect to x to give

$$2yy' = 3x^2 + 6x,$$

so $y'|_{x=1,y=-2} = \frac{3+6}{-4} = -\frac{9}{4}$. Therefore the point-slope form of the tangent line is

$$y + 2 = -\frac{9}{4}(x-1).$$

b Let $y' = 0$ in $2yy' = 3x^2 + 6x$ to give $3x^2 + 6x = 0$, that is $x = 0, -2$. For the root $x = -2$, we substitute it into the equation of the curve to give $y = \pm 2$, so at $(2, \pm 2)$, the curve has horizontal tangents. For the root $x = 0$, a substitution into the curve equation leads to $y = 0$. We further check the root by considering the limit

$$\lim_{x\to 0, y\to 0} \frac{3x^2 + 6x}{2y}.$$

Substitute the curve equation $y^2 = x^3 + 3x^2$ into the limit, and restrict $y > 0$, we have

$$\lim_{x\to 0^+, y>0} \frac{3x^2 + 6x}{2\sqrt{x^3 + 3x^2}} = \lim_{x\to 0^+, y>0} \frac{3x + 6}{2\sqrt{x + 3}} = \sqrt{3}.$$

Similarly, $\lim_{x\to 0^-, y>0} \frac{3x^2+6x}{2y} = -\sqrt{3}$, and $\lim_{x\to 0^\pm, y<0} \frac{3x^2+6x}{2y} = \mp\sqrt{3}$. Therefore at $x = y = 0$, the curve has no horizontal tangents.

c The picture is illustrated in Figure 4.

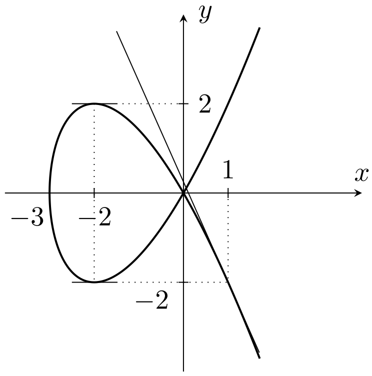

Figure 4: The picture of Problem 10.

11. Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1.$$

Proof: Take derivatives of both sides of the equation with respect to x to give

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0,$$

so the slope at the point (x_0, y_0) is

$$y'|_{x=x_0, y=y_0} = -\frac{b^2 x_0}{y_0 a^2}.$$

Therefore we can write the tangent line by the point-slope form

$$(y - y_0) = -\frac{b^2 x_0}{y_0 a^2}(x - x_0),$$

or

$$\frac{y_0 y}{b^2} + \frac{x_0 x}{a^2} = \frac{y_0^2}{b^2} + \frac{x_0^2}{a^2}.$$

Since (x_0, y_0) locates on the ellipse, we have $\frac{y_0^2}{b^2} + \frac{x_0^2}{a^2} = 1$, and thus the proof is complete.

12. Find an equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at point (x_0, y_0).

Proof: Take derivatives of both sides of the equation with respect to x to give

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0,$$

so the slope at the point (x_0, y_0) is

$$y'|_{x=x_0, y=y_0} = \frac{xb^2}{ya^2},$$

and we can write the tangent line at (x_0, y_0) as

$$y - y_0 = \frac{x_0 b^2}{y_0 a^2}(x - x_0),$$

or

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}.$$

The proof is complete by noticing $\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$.

13. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.

Proof: We write the equation of the circle as

$$x^2 + y^2 = |OP|^2.$$

Take derivatives of both sides of the equation with respect to x to give

$$2x + 2yy' = 0,$$

so the slope of the tangent line at the point $P = (x_0, y_0)$ is $y'|_{x=x_0, y=y_0} = -\frac{x_0}{y_0}$, which is the negative reciprocal of y_0/x_0, the slope of the radius OP. Hence the tangent line is perpendicular to the radius OP.