Chapter 1

Vectors and simple calculus

Objective

vector and calculus

PHY1201 is about mechanics (motion, force) wave (sound, wave on a string and in a pipe) and heat (motion of atoms and molecules). These topics need some knowledge of vector and calculus

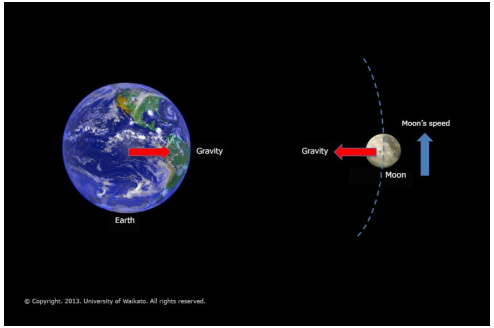
To prepare you for this by providing with some mathematical background, mainly vector and simple calculus

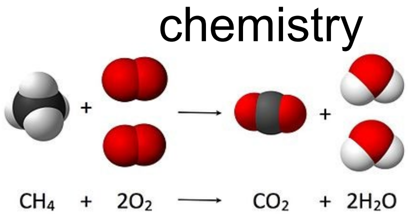
Topics for Chapter 1

- What is physics?
- What is vector? vectors and scalars
- vector components, adding vectors using components
- add vectors graphically
- unit vectors and components
- multiplying vectors
- definition of derivatives (differentiation) of functions
- definition of integration

What is physics? What do we study in PHY1201?

- Its main objective is to understand physical phenomena, not chemical phenomena, not biological phenomena
- physicists try to understand the natural principles and laws behind physical phenomenon
- an *experimental* science: physicists observe/measure physical phenomena using experimental techniques
- Develop theories, models, laws and principles to explain the physical phenomena or experiment results





Categories of Physical Phenomenon, Physics Topics

Mechanics

Heat

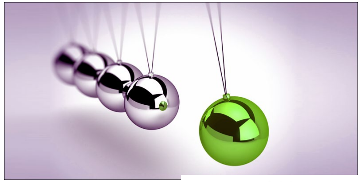
Wave

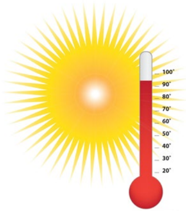
Electricity and magnetism

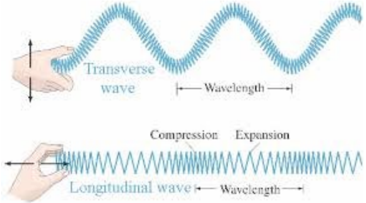
Atoms

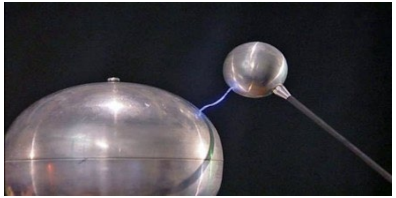
Nuclear

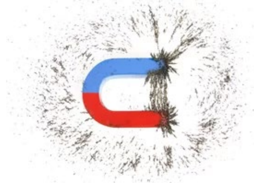
Light

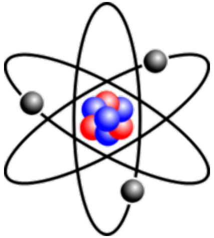


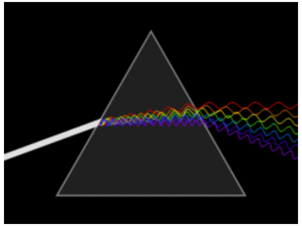












Physics is the foundation of engineering

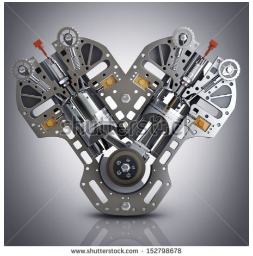
Engineers need to know physics

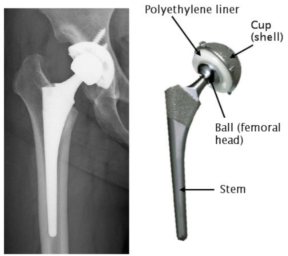
Civil engineer: force, equilibrium in a bridge

Mechanical engineer: motion of the components in a car engine, thermodynamics of car engine motion

Biomedical: force in artificial joint

Electrical & electronics: current in devices ICs, motion of a motor





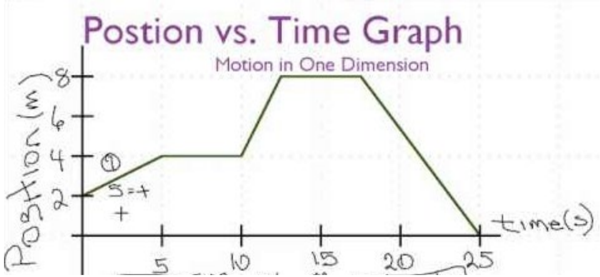
Example of a study of physical phenomenon

- For example, the description of motion: **How fast the cyclist moves? His velocity?**

- You use a tape measure to measure the position and a stop watch to measure the time

- Plot a position and time graph to describe the motion; calculate the velocity=distance travelled/time

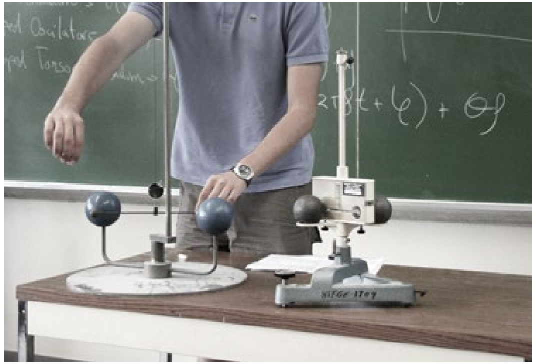
$$v = \Delta x / \Delta t$$



- The motion can be explained using Newton's law

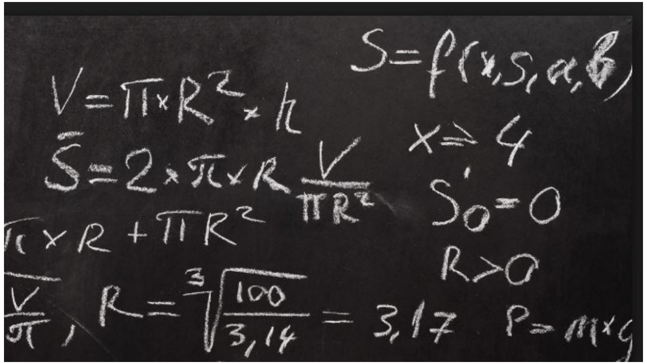
Newton's law
F=ma

Two main components of physics studies



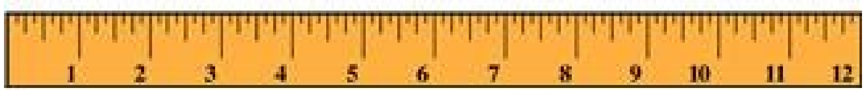
Experiments: observation of physical phenomenon, measurement of physical quantities in the phenomenon

Theory: qualitative explanation, quantitative explanation, in the form of principles and laws using principles and laws



Physical Measurements give you numbers

We use instrument (ruler, stop watch, thermometers) to measure,



Instrument gives you numbers> measurement gives you numbers

Numbers describing physical phenomenon are called Physical quantities

Example: height of this building is 100m, weight of this book is 2kg, time duration is 10 minutes

Physical Quantities	*Units*	*Symbol*
Length	Meter	m
Mass	Kilogram	kg
Time	second	s
Temperature	Kelvin	k
Electric Current	Ampere	A
Luminous Intensity	Candela	cd
Amount of substance	Mole	mol

Different physical quantities have different units

example: “m” represents meters, a unit of length. Mass has unit of kilogram, etc.

Standard Units

- ***Length, time, and mass*** are three ***fundamental quantities*** in Mechanics. Other quantities can be derived from these three:
  speed = distance/time (m/s), momentum = mass*speed (kg.m/s)

  Same physical quantity may have different units and can
  be converted to each other: 1 foot = 0.305 m.
  Numbers are different in different: 10 feet = 3.05 m.

- In Physics, we use SI units (SI for Système International).
- In SI units, length is measured in **meters**, time in **seconds**, and mass in **kilograms**.
- Before numbers are plugged into a formula, units of those quantities must be converted to SI units.

Units of measurement*

Dimension	SI (mks) Unit	Definition
Length	meters (m)	Distance traveled by light in 1/(299,792,458) second
Mass	kilogram (kg)	Mass of a specific platinum-iridium alloy cylinder kept by Intl. Bureau of Weights and Measures at Sèvres, France
Time	seconds (s)	Time needed for a cesium-133 atom to perform 9,192,631,700 complete oscillations

TABLE 1.4

Some Prefixes for Powers of 10 Used with "Metric" (SI and cgs) Units

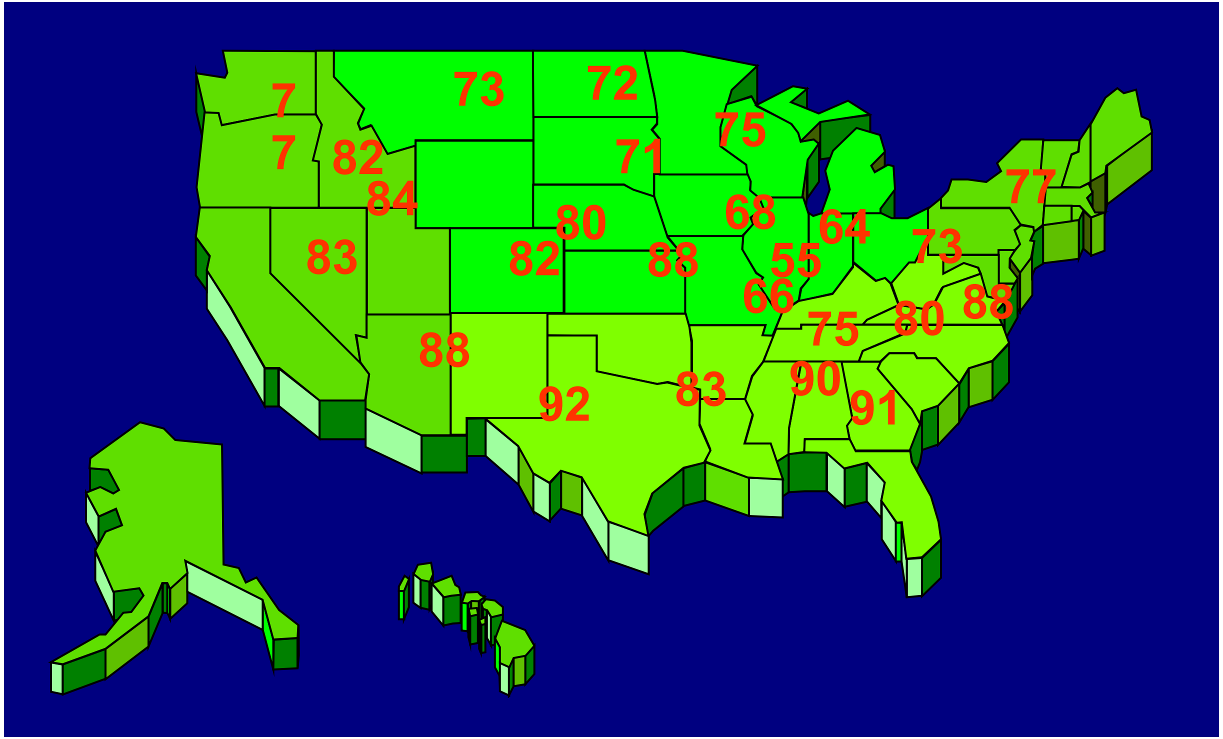
Power	Prefix	Abbreviation
10^{-18}	atto-	a
10^{-15}	femto-	f
10^{-12}	pico-	p
10^{-9}	nano-	n
10^{-6}	micro-	μ
10^{-3}	milli-	m
10^{-2}	centi-	c
10^{-1}	deci-	d
10^{1}	deka-	da
10^{3}	kilo-	k
10^{6}	mega-	M
10^{9}	giga-	G
10^{12}	tera-	T
10^{15}	peta-	P
10^{18}	exa-	E

Vectors and Scalars

physical quantities, such as height, weight, position, velocity, are used to describe physical phenomenon

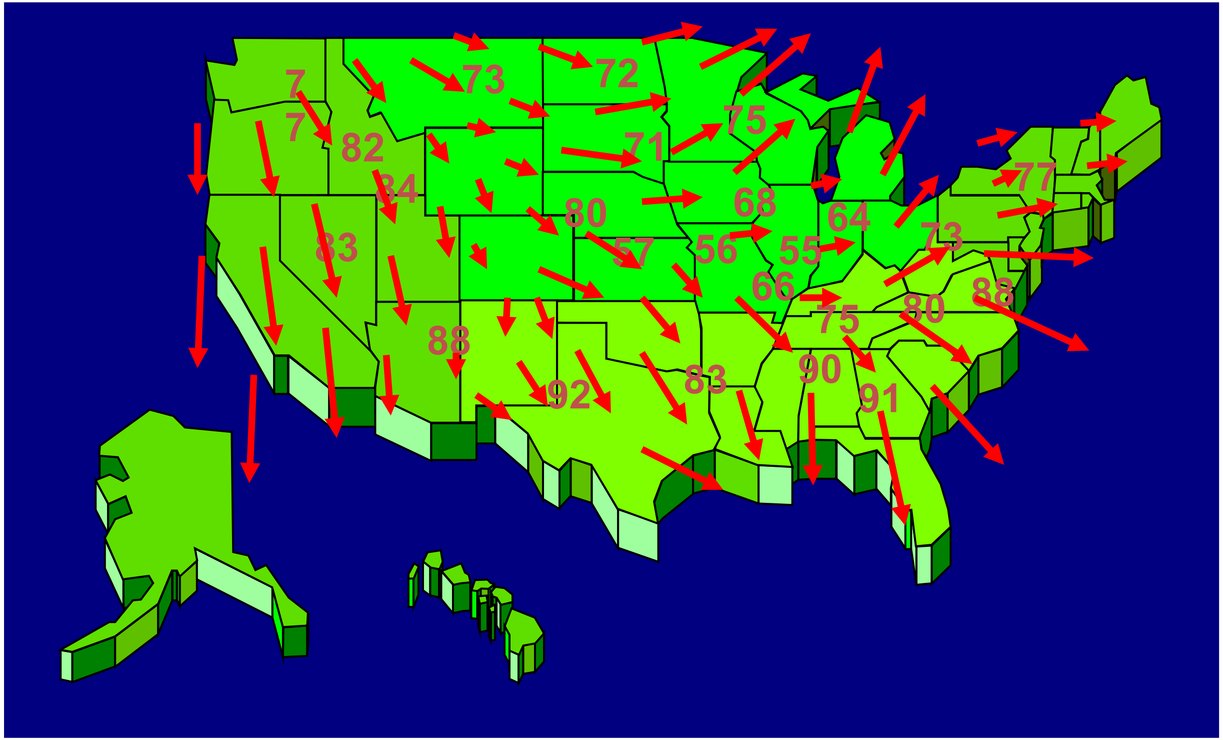
There are two types of physical quantities

A Scalar Distribution of Temperature



Only one number is needed at each location to specify the temperature everywhere (x,y). One number → Scalar T

A Vector Distribution of Wind



It may be more interesting to know which way the wind is blowing and how fast. Magnitude and direction → vector

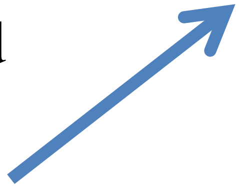
Two kinds of physical quantities: Vector and Scalar

Physical Quantities are represented by numbers only is scalar, such as your weight or your height, temperature

Physical quantities are represented by number and direction are called vectors, such as wind velocity

Vectors, symbols for vectors

- A *vector* is a quantity having both a *magnitude (scalar)* and a *direction*. Graphically it is represented by an arrow



- a vector is usually represented by a letter with an arrow over it: $\vec{A}$.

- Magnitude of $\vec{A}$ is represented by A or $|A|$.

- A typical example of vector: displacement

Consider the position of a Point

One dimensional coordinate

- Consider how to determine the location of a point on a line:

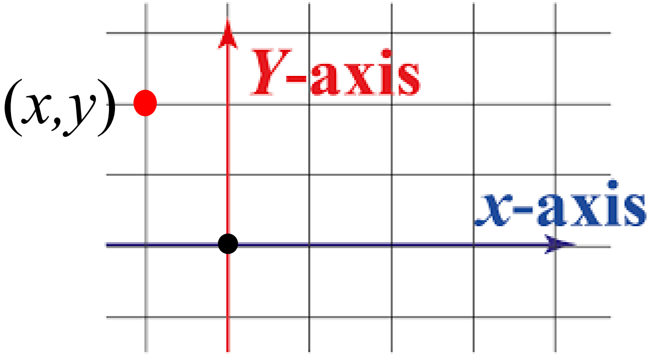
- We set up a coordinate: x-axis with a ruler and an origin

- The origin (O) is the reference point.
- Position can be uniquely determined by one real number ($x = +3$).

negative (−) if on the left of O

Two and three dimensional coordinates

- For a point in a plane, we set up x- and y-coordinate with an origin and two axes

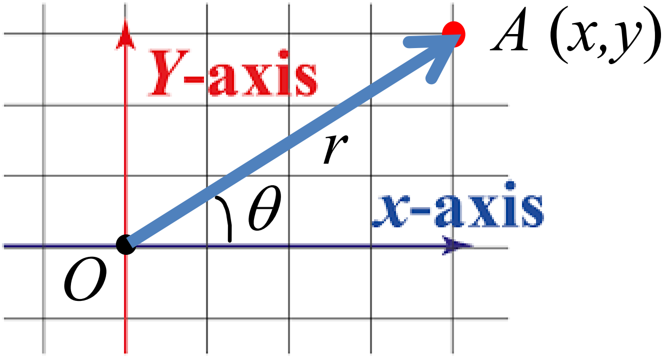


- The position can be uniquely determined by two real numbers (-1, 2)

- We can do the similar with three real number (x,y,z) in 3-dim space

Displacement

- If you go from the origin (O) to one place (A), we can use an arrow to represent it with a size r and direction of an angle θ from x-axis:



- Segment $\overrightarrow{OA}$ is a vector called displacement
- Ex: go to Kowloon Tong station from Central.

An example of Vector: relative position, displacement

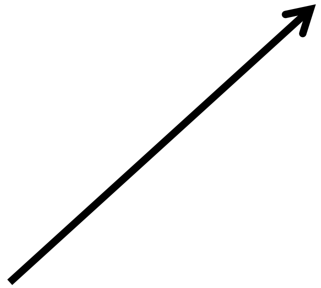
- The straight line from Central to Kowloon Tong is a vector (arrow). This is the Displacement vector that gives the relative position of KLT with C as origin.

- It tells you how to get directly from Central to Kowloon Tong :
- Travel a distance (5km) along a certain direction (10 degree east of North)
- If the direction is changed, you go to a wrong place (red arrow)
- Displacement has a scalar magnitude and direction. So it is a vector

Representing vectors by an arrow or by numbers

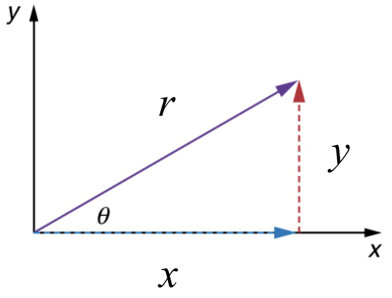
1. A vector is represented graphically by an arrow

- The *length* of the arrow is the vector's *magnitude*.
- The *direction* of the arrow is the vector's *direction*.



2. Representing vector using numbers

- Magnitude can be represented by a number, say r in the figure
- Direction can be represented by the angle made with a fixed direction, angle θ made with the x-axis
- Or equivalently by two numbers $x = r\cos\theta$, $y = r\sin\theta$ (Cartesian coordinates or Polar coordinates)

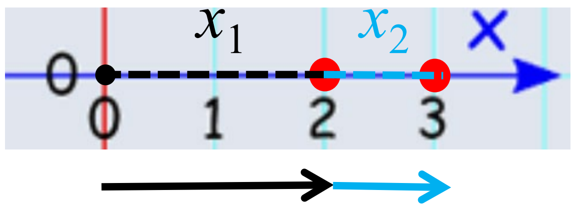


Addition of Vectors

Addition of vectors by the numbers

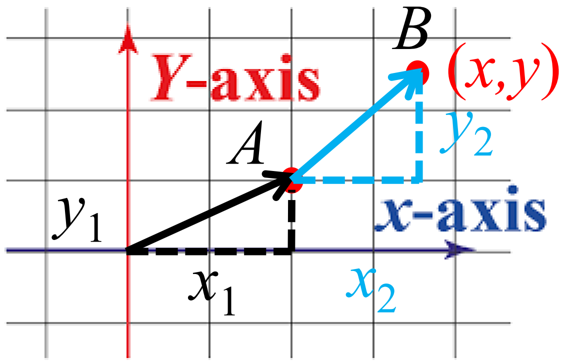
Vectors are like numbers, they can be added

- 1-dim case: $x = x_1 + x_2$



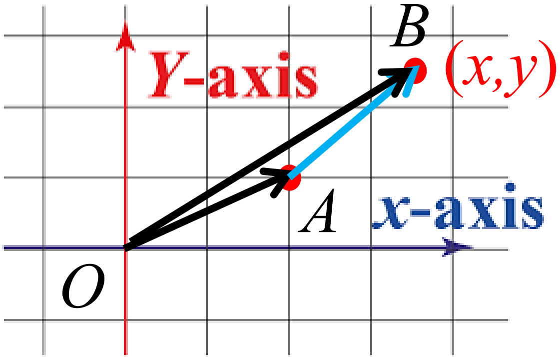
- Likewise, 2-dim case:

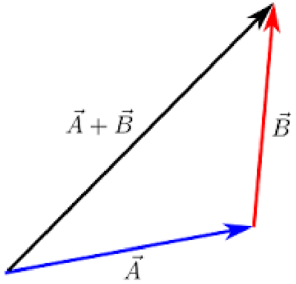
$$x = x_1 + x_2,$$
$$y = y_1 + y_2$$



- There are two numbers (x, y) for each vector. x and y numbers are added separately as though they are not related

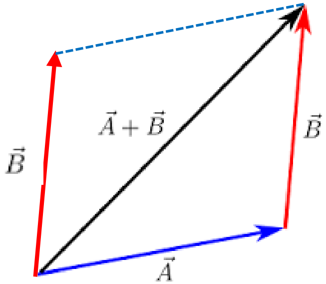
Adding two vectors graphically



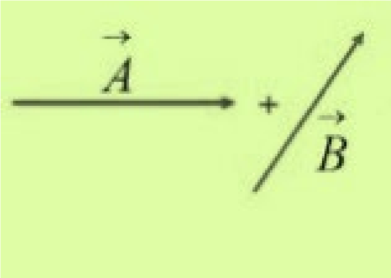


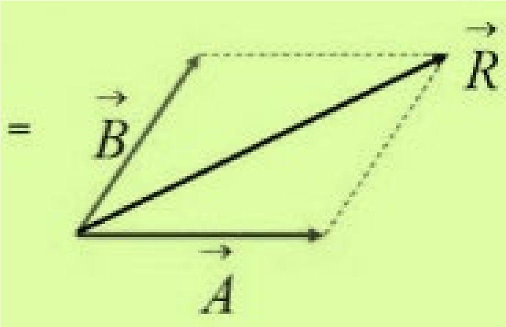
- Two vectors $\overrightarrow{OA} = \vec{A}$, $\overrightarrow{AB} = \vec{B}$ may be added graphically using the *head-to-tail* method to obtain sum $\overrightarrow{OB} = \vec{A} + \vec{B}$.

- We transport $\vec{B}$ while keeping its direction, so that its tail touches the head of $\vec{A}$.



Parallelogram method for adding vectors



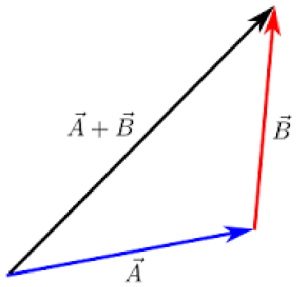


Draw a parallelogram with the two vectors as its sides

The sum of the two vectors (vector R) is the vector pointing to the vertex between vector A and B (Diagonal line between A and B)

Adding more vectors graphically: Head to Tail method

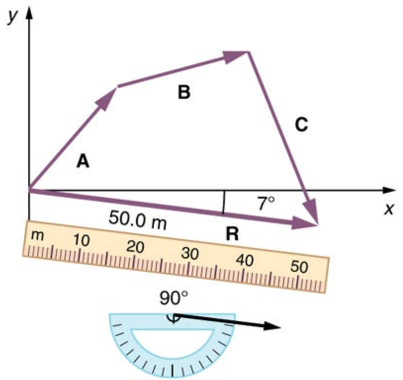
- Two or more vectors may be added graphically using the *head-to-tail* method.



$\vec{C} = \vec{A} + \vec{B}$

Head of A coincides with tail of B

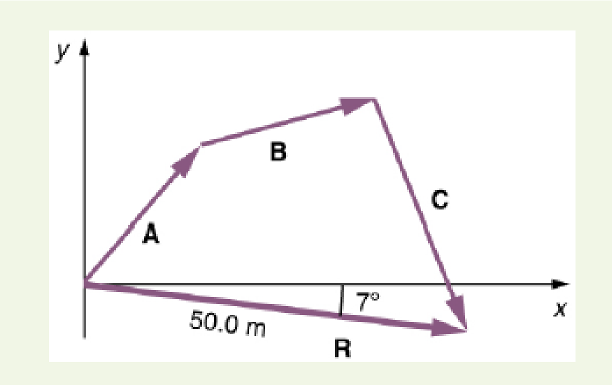
The arrow from tail of A to head of B is the vector C

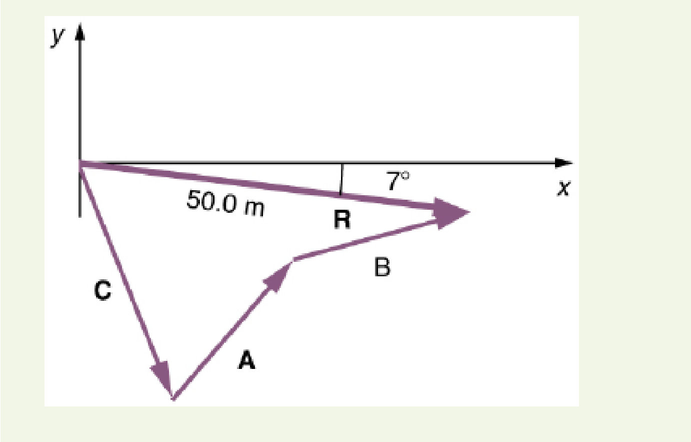


The method can be used to add more than two vectors

Adding vectors: Head to tail method

It is independent on the order, adding in different order gives the same result. This method can be used to find the result of adding displacement vectors (vectors that represent movement). $\vec{R} = \vec{A} + \vec{B} + \vec{C} = \vec{C} + \overrightarrow{A} + \vec{B}$

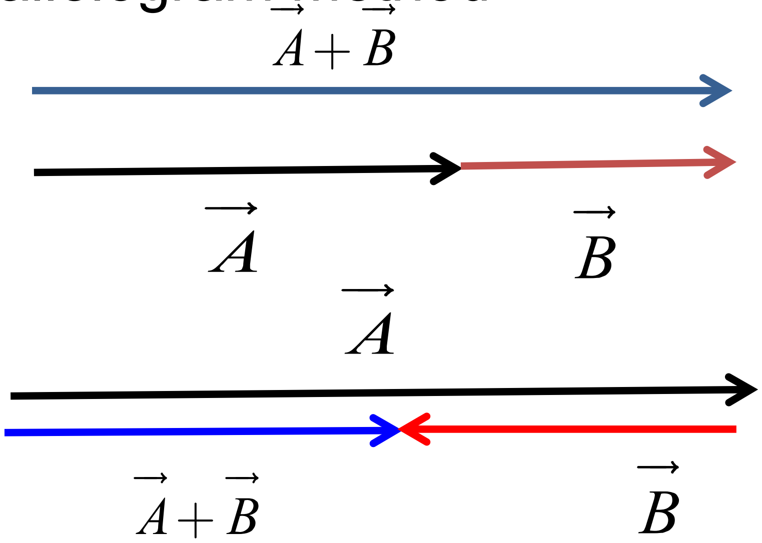




Adding two vectors (collinear vectors)

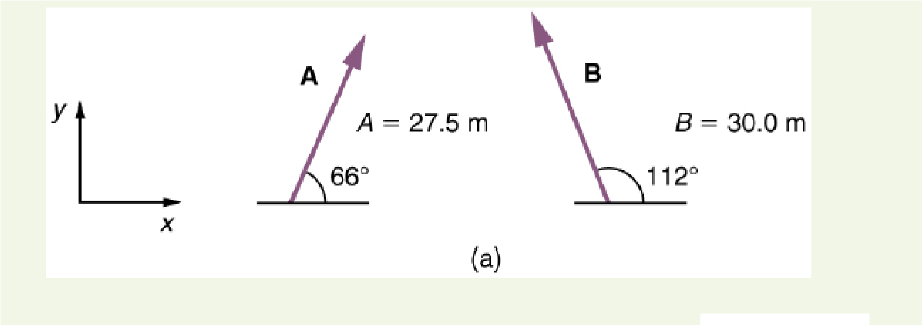
Adding vectors lying on the same straight line

Head to tail method only, cannot use parallelogram method



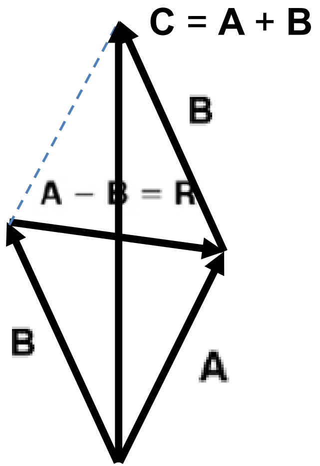
Subtracting vectors ($\vec{C} = \vec{A} - \vec{B}$)

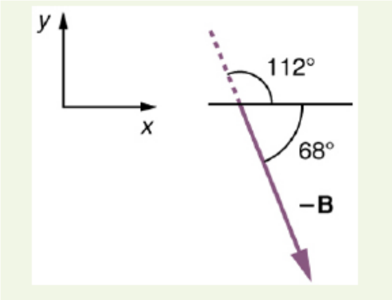
- This shows how to subtract vectors.

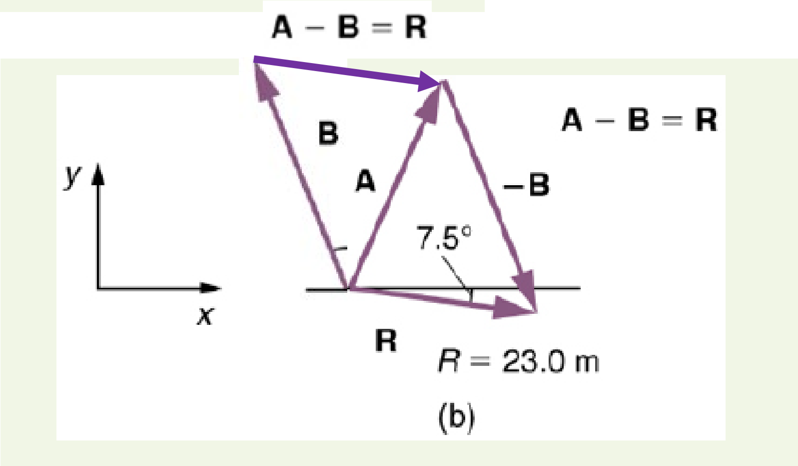


$-\vec{B}$ has the same magnitude as $\vec{B}$ but in opposite direction

$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$



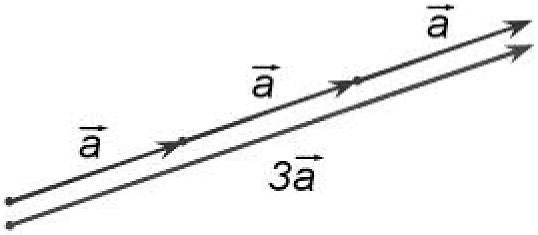




- Tip-to-tip
- **A, B, R** triangle **B + R = A**

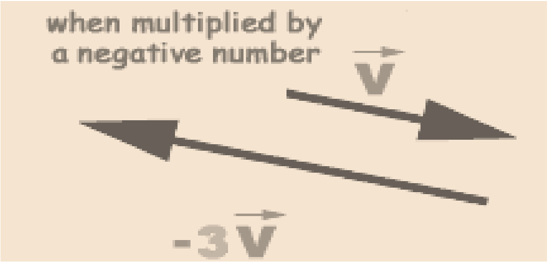
Multiplying a vector by a scalar

- If c is a scalar, the product $c\vec{A}$ has magnitude $|c|A$. direction is unchanged same as $\vec{A}$



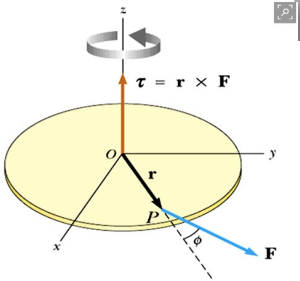
- Multiply by a negative scalar reverse the direction

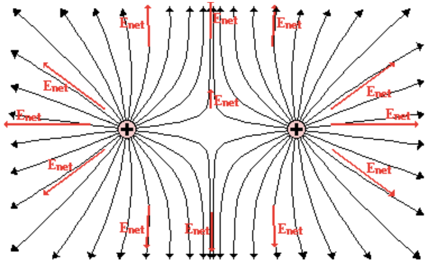
$-\vec{A}$ is opposite in direction to $\vec{A}$

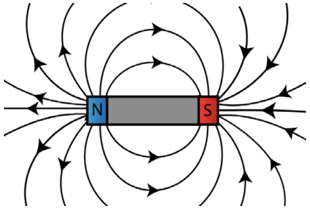


Other vector physical quantities

- Apart from displacement, the following physical quantities are also vectors
- Velocity, momentum (come from position)
- Angular velocity, angular momentum (come from position)
- Acceleration, force (related to position)
- electric field, magnetic field (generate Force)
- They are usually related to motion and force





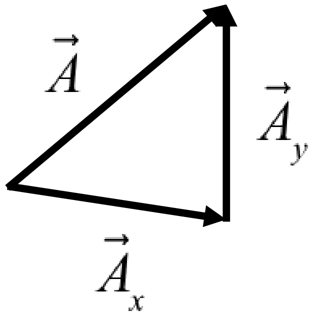


Representing any vector by its components

Representing a vector by its components

Any 2-dim vector can be decomposed or re-written as the sum of two vectors with head-to-tip formation

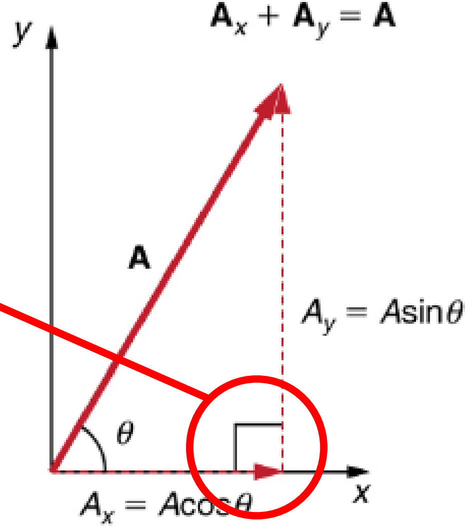
$$\vec{A} = \vec{A}_x + \vec{A}_y$$



The two vectors A_x A_y are the components of the vector **A**.

Usually the components are perpendicular to each other, x-direction is perpendicular to y-direction

The magnitudes of components can be found from trigonometry of right triangle: $\boxed{A_x = A\cos\theta, A_y = A\sin\theta}$



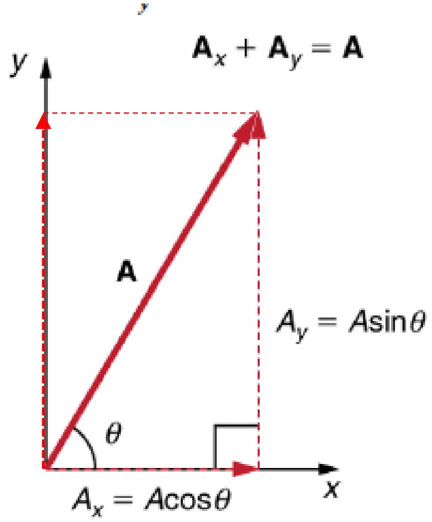
This is similar to the *x*-, *y*-coordinate of displacement vector $\vec{r}$ (x, y): $x = r\cos\theta, y = r\sin\theta$

Example

The vector $\vec{A}$ has a length of 2 and the angle θ made with the x axis is 30°.

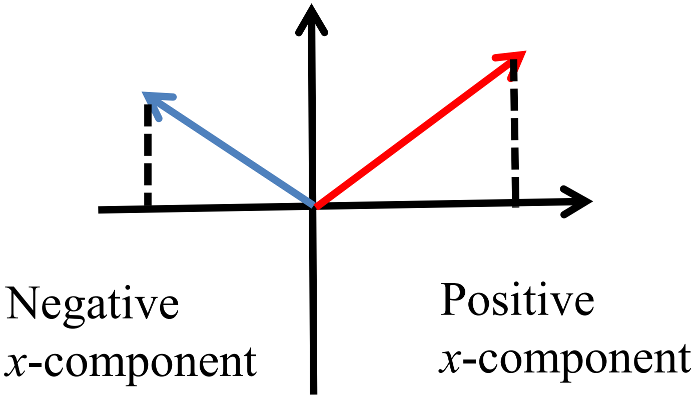
The x component is $A_x = 2 \cos 30^o = 1.732$

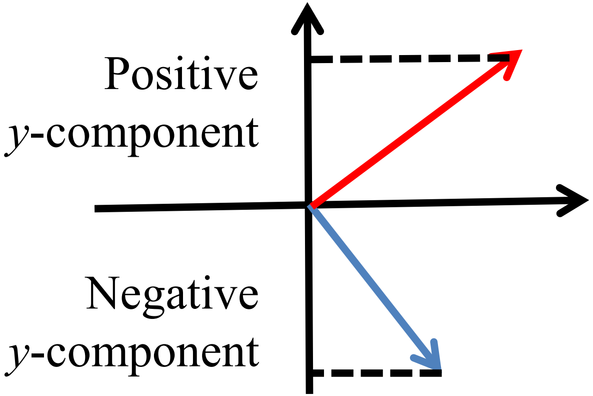
The y component is $A_y = 2 \sin 30^o = 1$.



Positive and negative components

- The components of a vector can be positive or negative numbers, as shown in the figure.





- If the component is point to the negative direction, then the component is a negative number

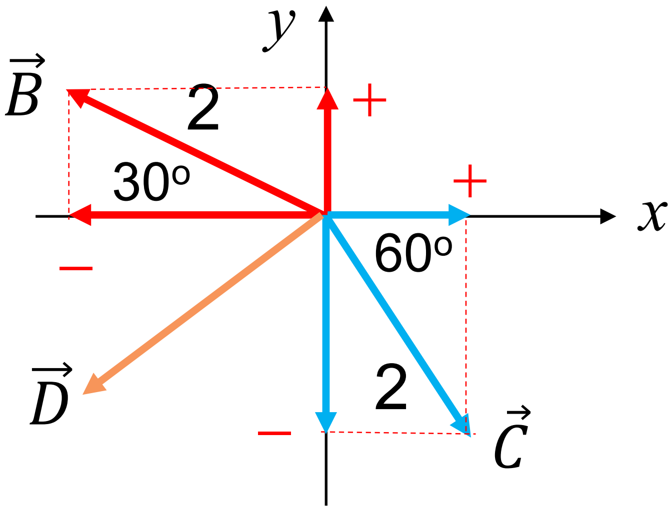
Example

$B_x = -\,2\cos 30^o = -\,1.732$

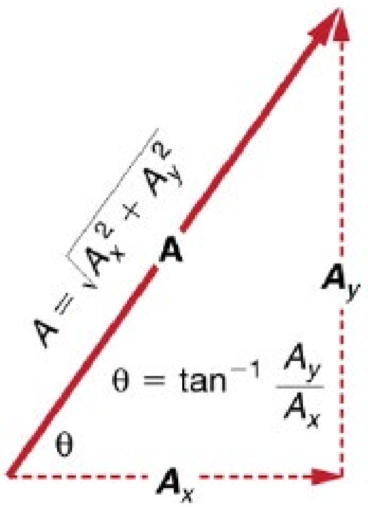
$B_y = +\,2\sin 30^o = +\,1$

$C_x = +\,2\cos 60^o = +\,1$

$C_y = -\,2\sin 60^o = -\,1.732$



Find magnitude and direction from its components



Direction is usually denoted by an angle measured from an axis for example x axis

$$A = \sqrt{A_x^2 + A_y^2} \qquad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

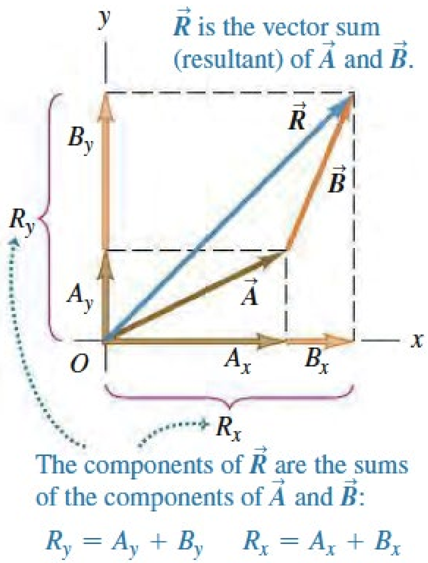
Addition of vector using component

Like displacement vector, you can add any two vectors by their components:

$$\vec{A} = \vec{A}_x + \vec{A}_y \qquad \vec{B} = \vec{B}_x + \vec{B}_y$$

$$\vec{A} + \vec{B} = \vec{R} = \left(\vec{A}_x + \vec{B}_x\right) + \left(\vec{A}_y + \vec{B}_y\right) = \vec{R}_x + \vec{R}_y$$

The vector is the addition of its x component and y component $\vec{R}_x = \vec{A}_x + \vec{B}_x$; $\vec{R}_y = \vec{A}_y + \vec{B}_y$

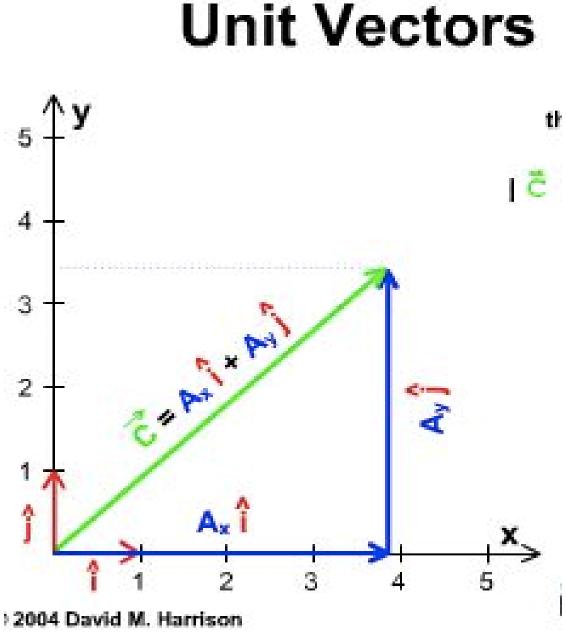


Unit vectors

Vectors along the directions of the components with a unit length are called the unit vectors.

Unit vectors represent the directions of the components

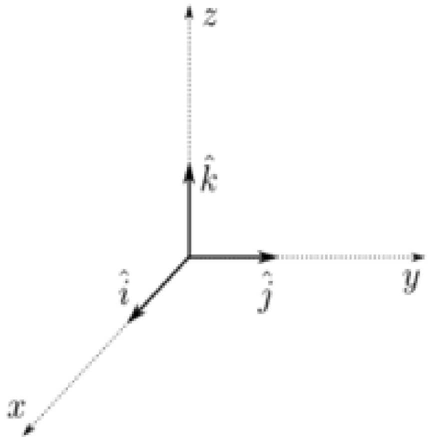
The vector $\vec{C}$ is represented by two components $\vec{A}_x, \vec{A}_y$ along x and y axis and the unit vectors $\hat{\imath}$ *and* $\hat{\jmath}$ represent directions along the x and y axes respectively. We can write:



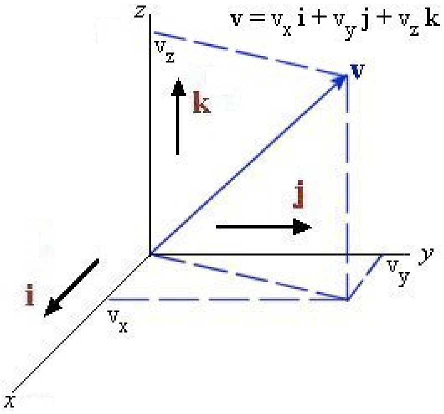
$$\vec{A}_x = A_x\hat{\imath}\ ;\ \vec{A}_y = A_y\hat{\jmath} \qquad \to \vec{C} = A_x\hat{\imath} + A_y\hat{\jmath}$$

Unit vectors (3D)

- For 3 dimension, we need three unit vectors
- The unit vector $\hat{\imath}$ points in the +x-direction, $\hat{\jmath}$ points in the +y-direction, and $\hat{k}$ points in the +z-direction.
- Any vector in 3D can be expressed in terms of its components as $$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$



v_x is the length of the x component of $\vec{\boldsymbol{v}}$
v_y is the length of the y component of $\vec{\boldsymbol{v}}$
v_z is the length of the z component of $\vec{\boldsymbol{v}}$



The end of vector $\vec{\boldsymbol{v}}$ has the coordinates v_x, v_y, v_z, which are the components of the vector $\vec{\boldsymbol{v}}$

Addition using unit vectors

- We can use the unit vector notation in addition as

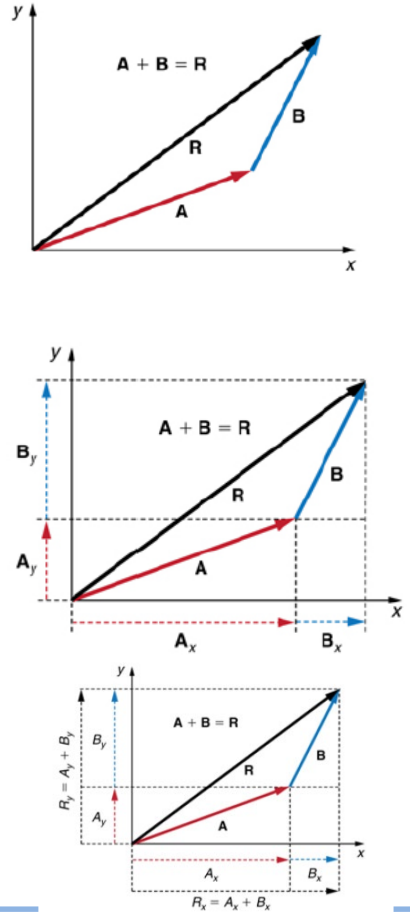
$$\vec{R} = \vec{A} + \vec{B}$$

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$$

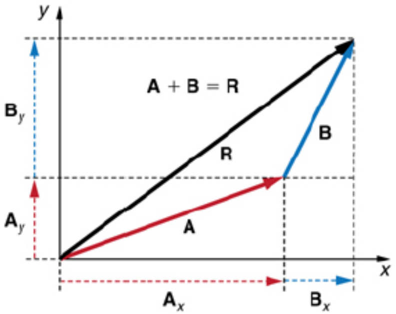
$$\begin{aligned}\vec{R} &= \vec{A} + \vec{B} \\ &= (A_x\hat{\imath} + A_y\hat{\jmath}) + (B_x\hat{\imath} + B_y\hat{\jmath}) \\ &= (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} \\ &= R_x\hat{\imath} + R_y\hat{\jmath}\end{aligned}$$

$$R_x = A_x + B_x \;, \qquad R_y = A_y + B_y$$



More on Vector addition by the numbers

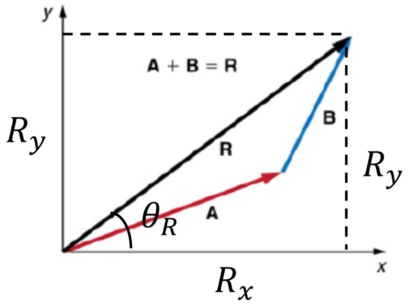
$$\vec{R} = \vec{A} + \vec{B}$$



- Notice that vector addition rule by the numbers only applies to components of the vectors but not to the magnitudes and angles:

$$R_x = A_x + B_x \ , \qquad R_y = A_y + B_y$$

But $\quad R \neq A + B \quad$ and $\quad \theta_R \neq \theta_A + \theta_B$



- The magnitude and angle of can only be calculated by

$$R = \sqrt{R_x^2 + R_y^2} \qquad \theta_R = \arctan(R_y/R_x)$$

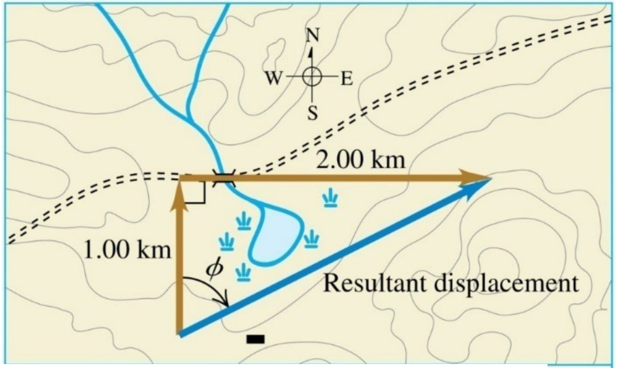
Examples (from university physics, young and freedman, 13th edition pearson)

We may not have time to go through the examples

Read the examples and learn how to apply the knowledge of the chapter. Useful for exam preparation

Addition of two vectors at right angles

- First add the vectors graphically.
- Then use trigonometry to find the magnitude and direction of the sum.
- Follow Example 1.5.)

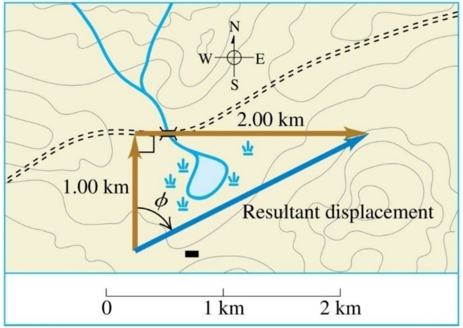


Example 1.5 **Addition of two vectors at right angles**

A cross-country skier skis 1.00 km north and then 2.00 km east on a horizontal snowfield. How far and in what direction is she from the starting point?

SOLUTION

IDENTIFY and SET UP: The problem involves combining two displacements at right angles to each other. In this case, vector addition amounts to solving a right triangle, which we can do using the Pythagorean theorem and simple trigonometry. The target variables are the skier's straight-line distance and direction from her starting point. Figure 1.16 is a scale diagram of the two displacements and the resultant net displacement. We denote the direction from the starting point by the angle ϕ (the Greek letter phi). The displacement appears to be about 2.4 km. Measurement with a protractor indicates that ϕ is about 63°.



EXECUTE: The distance from the starting point to the ending point is equal to the length of the hypotenuse:

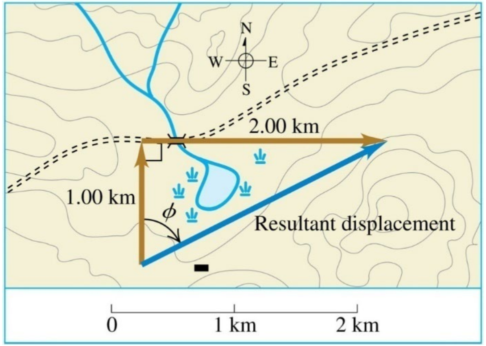
$$\sqrt{(1.00\ \text{km})^2 + (2.00\ \text{km})^2} = 2.24\ \text{km}$$

A little trigonometry (from Appendix B) allows us to find angle ϕ:

$$\tan \phi = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{2.00\ \text{km}}{1.00\ \text{km}}$$

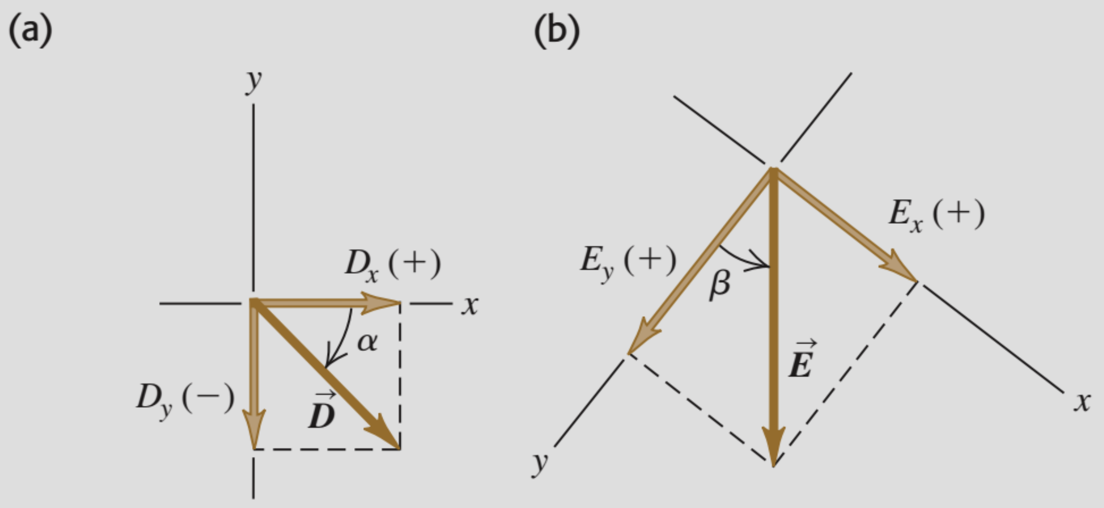
$$\phi = 63.4°$$

We can describe the direction as 63.4° east of north or $90° - 63.4° = 26.6°$ north of east.



EVALUATE: Our answers (2.24 km and $\phi = 63.4^\circ$) are close to our predictions. In the more general case in which you have to add two vectors *not* at right angles to each other, you can use the law of cosines in place of the Pythagorean theorem and use the law of sines to find an angle corresponding to ϕ in this example. (You'll find these trigonometric rules in Appendix B.) We'll see more techniques for vector addition in Section 1.8.

1.19 Calculating the x- and y-components of vectors.



Example 1.6 Finding components

(a) What are the x- and y-components of vector $\vec{D}$ in Fig. 1.19a? The magnitude of the vector is $D = 3.00$ m, and the angle $\alpha = 45°$. (b) What are the x- and y-components of vector $\vec{E}$ in Fig. 1.19b? The magnitude of the vector is $E = 4.50$ m, and the angle $\beta = 37.0°$.

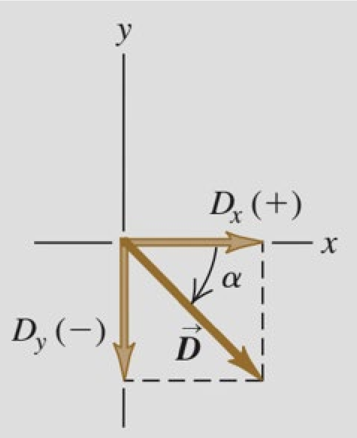
SOLUTION

IDENTIFY and SET UP: We can use Eqs. (1.6) to find the components of these vectors, but we have to be careful: Neither of the angles α or β in Fig. 1.19 is measured from the $+x$-axis toward the $+y$-axis. We estimate from the figure that the lengths of the components in part (a) are both roughly 2 m, and that those in part (b) are 3m and 4 m. We've indicated the signs of the components in the figure.

EXECUTE: (a) The angle α (the Greek letter alpha) between the positive x-axis and $\vec{D}$ is measured toward the *negative* y-axis. The angle we must use in Eqs. (1.6) is $\theta = -\alpha = -45°$. We then find

$$D_x = D\cos\theta = (3.00\text{ m})(\cos(-45°)) = +2.1\text{ m}$$

$$D_y = D\sin\theta = (3.00\text{ m})(\sin(-45°)) = -2.1\text{ m}$$

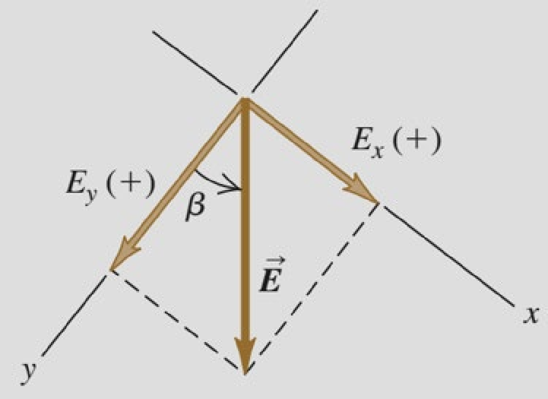


Had you been careless and substituted $+45°$ for θ in Eqs. (1.6), your result for D_y would have had the wrong sign.

(b) The x- and y-axes in Fig. 1.19b are at right angles, so it doesn't matter that they aren't horizontal and vertical, respectively. But to use Eqs. (1.6), we must use the angle $\theta = 90.0° - \beta = 90.0° - 37.0° = 53.0°$. Then we find

$$E_x = E\cos 53.0° = (4.50\text{ m})(\cos 53.0°) = +2.71\text{ m}$$

$$E_y = E\sin 53.0° = (4.50\text{ m})(\sin 53.0°) = +3.59\text{ m}$$



EVALUATE: Our answers to both parts are close to our predictions. But ask yourself this: Why do the answers in part (a) correctly have only two significant figures?

(Ans: because the angle given (45^o) has only 2 significant figures)

Problem-Solving Strategy 1.3 Vector Addition

IDENTIFY *the relevant concepts:* Decide what the target variable is. It may be the magnitude of the vector sum, the direction, or both.

SET UP *the problem:* Sketch the vectors being added, along with suitable coordinate axes. Place the tail of the first vector at the origin of the coordinates, place the tail of the second vector at the head of the first vector, and so on. Draw the vector sum $\vec{R}$ from the tail of the first vector (at the origin) to the head of the last vector. Use your sketch to estimate the magnitude and direction of $\vec{R}$. Select the mathematical tools you'll use for the full calculation: Eqs. (1.6) to obtain the components of the vectors given, if necessary, Eqs. (1.11) to obtain the components of the vector sum, Eq. (1.12) to obtain its magnitude, and Eqs. (1.8) to obtain its direction.

EXECUTE *the solution* as follows:

1. Find the x- and y-components of each individual vector and record your results in a table, as in Example 1.7 below. If a vector is described by a magnitude A and an angle θ, measured from the $+x$-axis toward the $+y$-axis, then its components are given by Eqs. 1.6:

$$A_x = A \cos \theta \qquad A_y = A \sin \theta$$

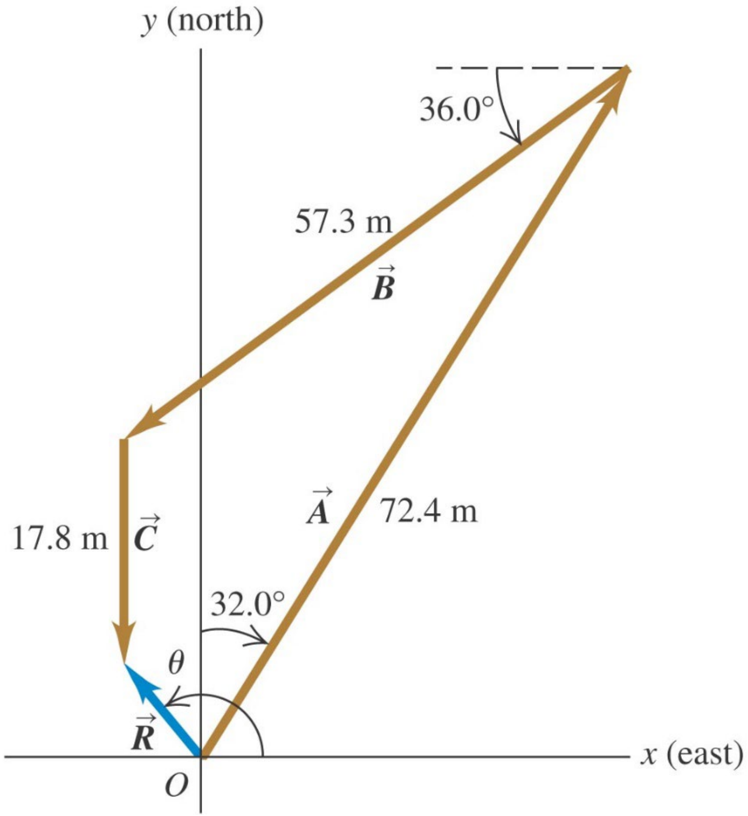
   If the angles of the vectors are given in some other way, perhaps using a different reference direction, convert them to angles measured from the $+x$-axis as in Example 1.6 above.
2. Add the individual x-components algebraically (including signs) to find R_x, the x-component of the vector sum. Do the same for the y-components to find R_y. See Example 1.7 below.
3. Calculate the magnitude R and direction θ of the vector sum using Eqs. (1.7) and (1.8):

$$R = \sqrt{R_x^{\,2} + R_y^{\,2}} \qquad \theta = \arctan \frac{R_y}{R_x}$$

EVALUATE *your answer:* Confirm that your results for the magnitude and direction of the vector sum agree with the estimates you made from your sketch. The value of θ that you find with a calculator may be off by 180°; your drawing will indicate the correct value.

Adding vectors using their components—Figure 1.22

- Follow Examples 1.7 and 1.8.



Example 1.7 Adding vectors using their component

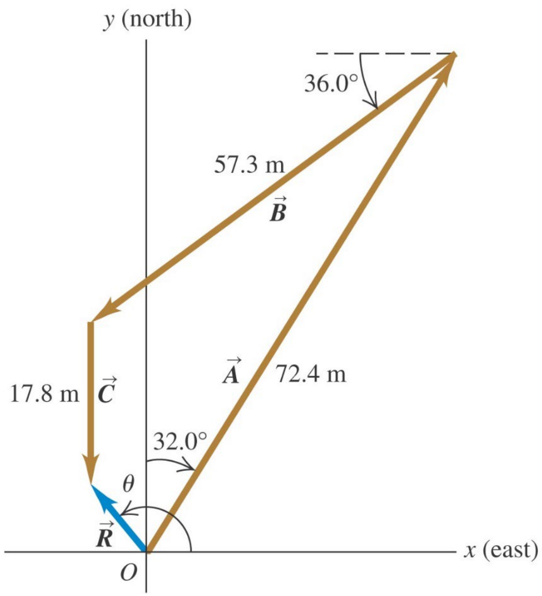
Three players on a reality TV show are brought to the center of a large, flat field. Each is given a meter stick, a compass, a calculator, a shovel, and (in a different order for each contestant) the following three displacements:

$\vec{A}$: 72.4 m, 32.0° east of north

$\vec{B}$: 57.3 m, 36.0° south of west

$\vec{C}$: 17.8 m due south

The three displacements lead to the point in the field where the keys to a new Porsche are buried. Two players start measuring immediately, but the winner first *calculates* where to go. What does she calculate?



SOLUTION

IDENTIFY and SET UP: The goal is to find the sum (resultant) of the three displacements, so this is a problem in vector addition. Figure 1.22 shows the situation. We have chosen the $+x$-axis as

east and the $+y$-axis as north. We estimate from the diagram that the vector sum $\vec{\boldsymbol{R}}$ is about 10 m, 40° west of north (which corresponds to $\theta \approx 130°$).

EXECUTE: The angles of the vectors, measured from the $+x$-axis toward the $+y$-axis, are $(90.0° - 32.0°) = 58.0°$, $(180.0° + 36.0°) = 216.0°$, and $270.0°$, respectively. We may now use Eqs. (1.6) to find the components of $\vec{\boldsymbol{A}}$:

$$A_x = A \cos \theta_A = (72.4 \text{ m})(\cos 58.0°) = 38.37 \text{ m}$$

$$A_y = A \sin \theta_A = (72.4 \text{ m})(\sin 58.0°) = 61.40 \text{ m}$$

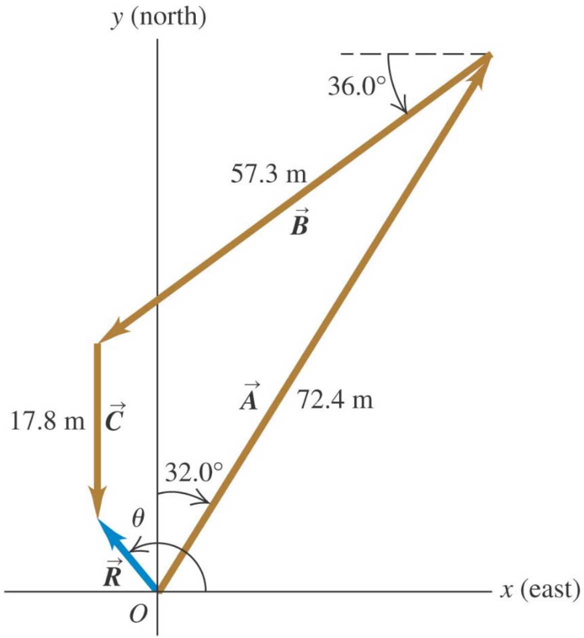
We've kept an extra significant figure in the components; we'll round to the correct number of significant figures at the end of our calculation. The table below shows the components of all the displacements, the addition of the components, and the other calculations.

Distance	Angle	x-component	y-component
$A = 72.4$ m	58.0°	38.37 m	61.40 m
$B = 57.3$ m	216.0°	−46.36 m	−33.68 m
$C = 17.8$ m	270.0°	0.00 m	−17.80 m
		$R_x = -7.99$ m	$R_y = 9.92$ m

$$R = \sqrt{(-7.99\ \text{m})^2 + (9.92\ \text{m})^2} = 12.7\ \text{m}$$

$$\theta = \arctan\frac{9.92\ \text{m}}{-7.99\ \text{m}} = -51°$$

Comparing to Fig. 1.22 shows that the calculated angle is clearly off by 180°. The correct value is $\theta = 180° - 51° = 129°$, or 39° west of north.



EVALUATE: Our calculated answers for R and θ agree with our estimates. Notice how drawing the diagram in Fig. 1.22 made it easy to avoid a 180° error in the direction of the vector sum.

Example 1.9 Using unit vectors

Given the two displacements

$$\vec{D} = (6.00\hat{\imath} + 3.00\hat{\jmath} - 1.00\hat{k})\text{ m} \quad \text{and}$$

$$\vec{E} = (4.00\hat{\imath} - 5.00\hat{\jmath} + 8.00\hat{k})\text{ m}$$

find the magnitude of the displacement $2\vec{D} - \vec{E}$.

SOLUTION

IDENTIFY and SET UP: We are to multiply the vector $\vec{D}$ by 2 (a scalar) and subtract the vector $\vec{E}$ from the result, so as to obtain the vector $\vec{F} = 2\vec{D} - \vec{E}$. Equation (1.9) says that to multiply $\vec{D}$ by 2, we multiply each of its components by 2. We can use Eq. (1.17) to do the subtraction; recall from Section 1.7 that subtracting a vector is the same as adding the negative of that vector.

EXECUTE: We have

$$\vec{F} = 2(6.00\hat{\imath} + 3.00\hat{\jmath} - 1.00\hat{k})\text{ m} - (4.00\hat{\imath} - 5.00\hat{\jmath} + 8.00\hat{k})\text{ m}$$
$$= [(12.00 - 4.00)\hat{\imath} + (6.00 + 5.00)\hat{\jmath} + (-2.00 - 8.00)\hat{k}]\text{ m}$$
$$= (8.00\hat{\imath} + 11.00\hat{\jmath} - 10.00\hat{k})\text{ m}$$

From Eq. (1.12) the magnitude of $\vec{F}$ is

$$F = \sqrt{F_x{}^2 + F_y{}^2 + F_z{}^2}$$
$$= \sqrt{(8.00\text{ m})^2 + (11.00\text{ m})^2 + (-10.00\text{ m})^2}$$
$$= 16.9\text{ m}$$

EVALUATE: Our answer is of the same order of magnitude as the larger components that appear in the sum. We wouldn't expect our answer to be much larger than this, but it could be much smaller.

Vector products

Vector products

- Vectors are like scalar or numbers.
- We have learned how to add two vectors
- How about multiplication of vectors, forming product of two vectors?
- Yes, we can multiply two vectors!
- Actually we have two ways of multiplying vectors, one gives a scalar, another leads to a vector

The scalar product

- Like vector addition, we multiply the components of the two vectors, but then a bit different, we add them up to give a number (scalar):

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$$

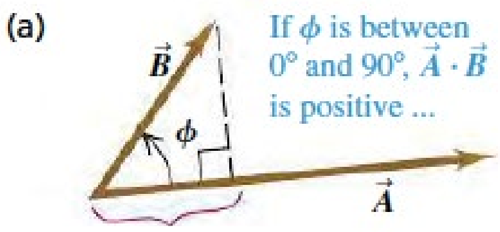
So, it is called scalar product or also called dot product

- For three dimension $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = A_x B_x + A_y B_y + A_z B_z.$
- Later you can see work done is the dot product (scalar product) of force and displacement.

Calculating a scalar product from magnitude and angle

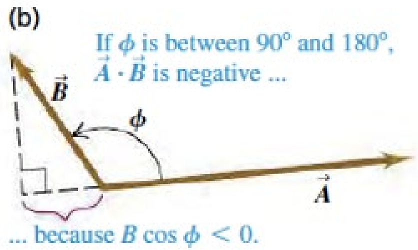
- Using the definition of scalar product, we can prove: $\vec{A} \cdot \vec{B} = AB\cos\phi = AB_{\parallel}$

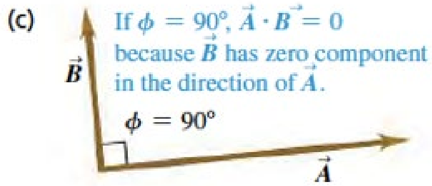
  A, B = size of vectors, ϕ = angle between them



$B_{\parallel} = B\cos\phi$

- It can be positive or negative or zero, depending on $\cos\phi$

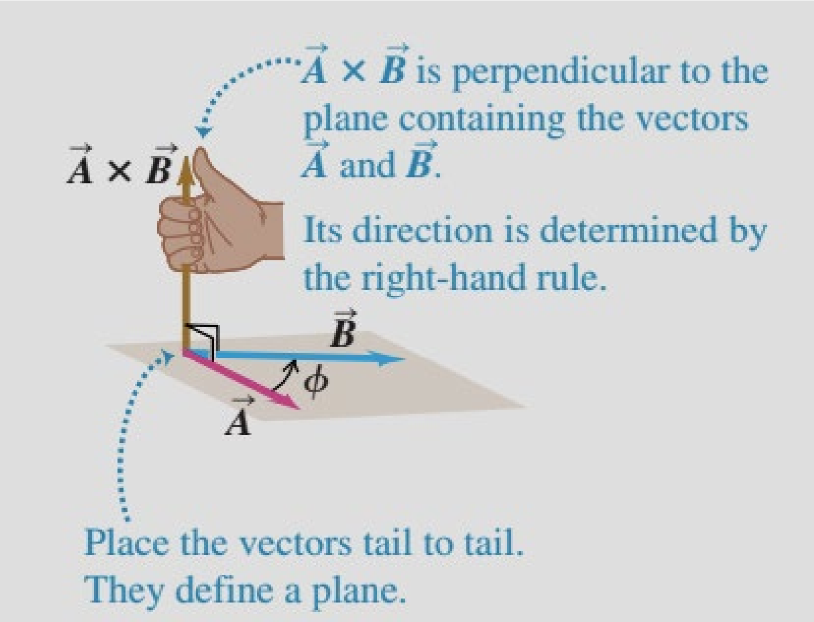




- Later, it is used in calculating work done of a force with a displacement

The vector product

- The vector product ("cross product") of two is a vector .
- The product is a vector $\vec{C} = \vec{A} \times \vec{B}$
- $\vec{C}$ has magnitude $\left|\vec{A} \times \vec{B}\right| = AB \sin\phi$
- $\vec{C}$ is perpendicular to $\vec{A}$ *and* $\vec{B}$ and the *right-hand rule* gives its direction. See the figure



$$\left|\vec{A} \times \vec{B}\right|^2 + \left|\vec{A} \cdot \vec{B}\right|^2 = (AB \sin\phi)^2 + (AB \cos\phi)^2 = A^2 B^2$$

Vector product

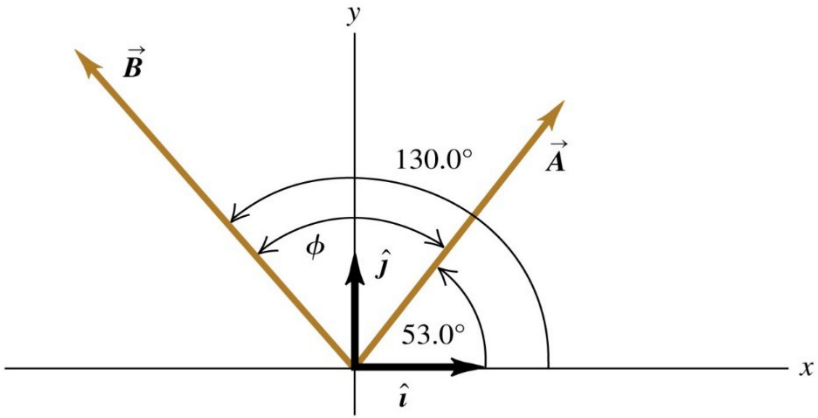
- Later we can see that angular momentum is the vector product of momentum and a position vector
- Torque is the vector product of force and a position vector

Calculating a scalar product

- Example 1.10 shows how to calculate a scalar product in two ways.

Example 1.10 **Calculating a scalar product**

Find the scalar product $\vec{A} \cdot \vec{B}$ of the two vectors in Fig. 1.27. The magnitudes of the vectors are $A = 4.00$ and $B = 5.00$.



SOLUTION

IDENTIFY and SET UP: We can calculate the scalar product in two ways: using the magnitudes of the vectors and the angle between them (Eq. 1.18), and using the components of the vectors (Eq. 1.21). We'll do it both ways, and the results will check each other.

EXECUTE: The angle between the two vectors $\vec{A}$ and $\vec{B}$ is $\phi = 130.0^\circ - 53.0^\circ = 77.0^\circ$, so Eq. (1.16) gives us

$$\vec{A} \cdot \vec{B} = AB\cos\phi = (4.00)(5.00)\cos 77.0^\circ = 4.50$$

To use Eq. (1.19), we must first find the components of the vectors. The angles of $\vec{A}$ and $\vec{B}$ are given with respect to the $+x$-axis and are measured in the sense from the $+x$-axis to the $+y$-axis, so we can use Eqs. (1.5):

$$A_x = (4.00)\cos 53.0^\circ = 2.407$$
$$A_y = (4.00)\sin 53.0^\circ = 3.195$$
$$B_x = (5.00)\cos 130.0^\circ = -3.214$$
$$B_y = (5.00)\sin 130.0^\circ = 3.830$$

As in Example 1.7, we keep an extra significant figure in the components and round at the end. Equation (1.19) now gives us

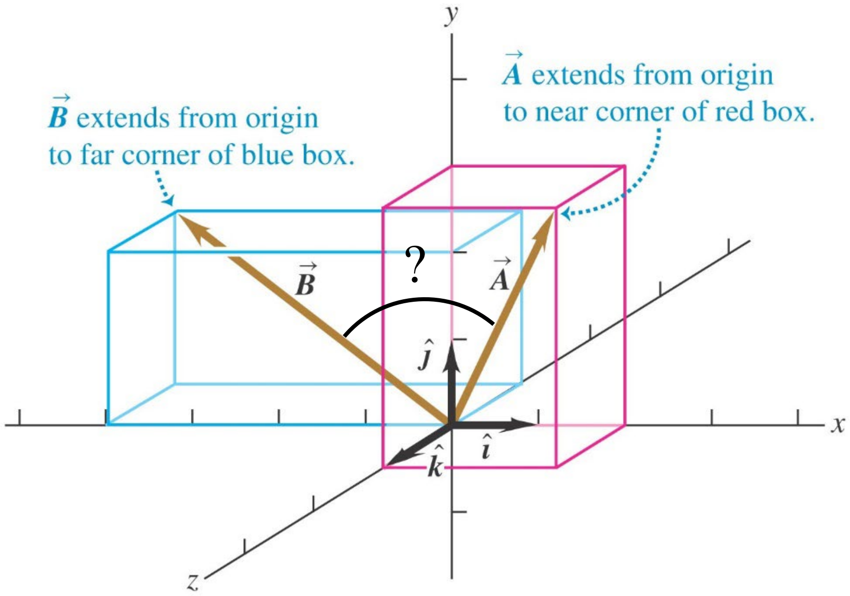
$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

$$= (2.407)(-3.214) + (3.195)(3.830) + (0)(0) = 4.50$$

EVALUATE: Both methods give the same result, as they should.

Finding an angle using the scalar product

- Example 1.11 shows how to use components to find the angle between two vectors.



Example 1.11 Finding an angle with the scalar product

Find the angle between the vectors

$$\vec{A} = 2.00\hat{\imath} + 3.00\hat{\jmath} + 1.00\hat{k} \qquad \text{and}$$
$$\vec{B} = -4.00\hat{\imath} + 2.00\hat{\jmath} - 1.00\hat{k}$$

SOLUTION

IDENTIFY and SET UP: We're given the *x*-, *y*-, and *z*-components of two vectors. Our target variable is the angle ϕ between them (Fig. 1.28). To find this, we'll solve Eq. (1.18), $\vec{A}\cdot\vec{B} = AB\cos\phi$, for ϕ in terms of the scalar product $\vec{A}\cdot\vec{B}$ and the magnitudes A and B. We can evaluate the scalar product using Eq. (1.21),

$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$, and we can find A and B using Eq. (1.7).

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

EXECUTE: We solve Eq. (1.18) for $\cos\phi$ and write $\vec{A}\cdot\vec{B}$ using Eq. (1. 21). Our result is

$$\cos\phi = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{A_xB_x + A_yB_y + A_zB_z}{AB}$$

We can use this formula to find the angle between *any* two vectors $\vec{A}$ and $\vec{B}$. Here we have $A_x = 2.00$, $A_y = 3.00$, and $A_z = 1.00$, and $B_x = -4.00$, $B_y = 2.00$, and $B_z = -1.00$. Thus

$$\begin{aligned}\vec{A}\cdot\vec{B} &= A_xB_x + A_yB_y + A_zB_z \\ &= (2.00)(-4.00) + (3.00)(2.00) + (1.00)(-1.00) \\ &= -3.00\end{aligned}$$

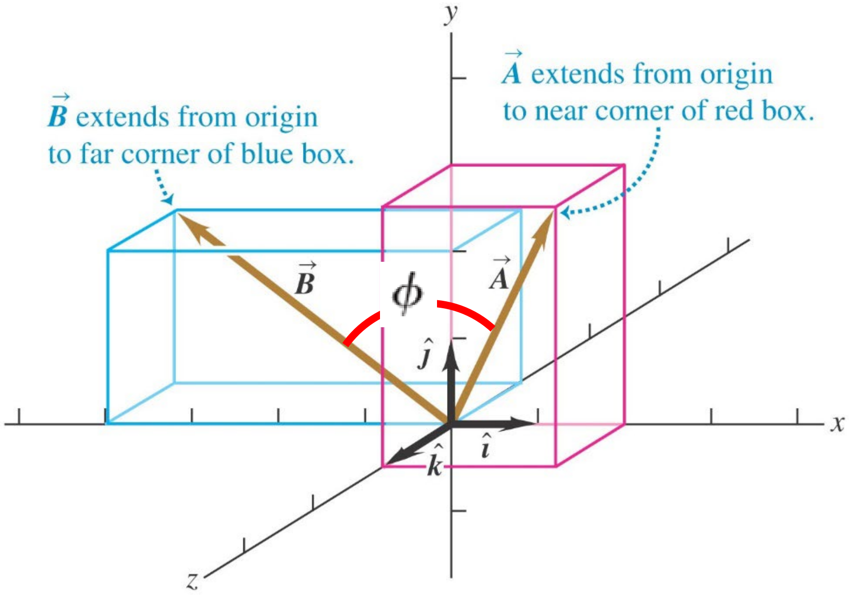
$$\begin{aligned}A &= \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{(2.00)^2 + (3.00)^2 + (1.00)^2} \\ &= \sqrt{14.00}\end{aligned}$$

$$\begin{aligned}B &= \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(-4.00)^2 + (2.00)^2 + (-1.00)^2} \\ &= \sqrt{21.00}\end{aligned}$$

$$\cos\phi = \frac{A_xB_x + A_yB_y + A_zB_z}{AB} = \frac{-3.00}{\sqrt{14.00}\,\sqrt{21.00}} = -0.175$$

$$\phi = 100°$$

EVALUATE: As a check on this result, note that the scalar product $\vec{A} \cdot \vec{B}$ is negative. This means that ϕ is between 90° and 180° (see Fig. 1.26), which agrees with our answer.

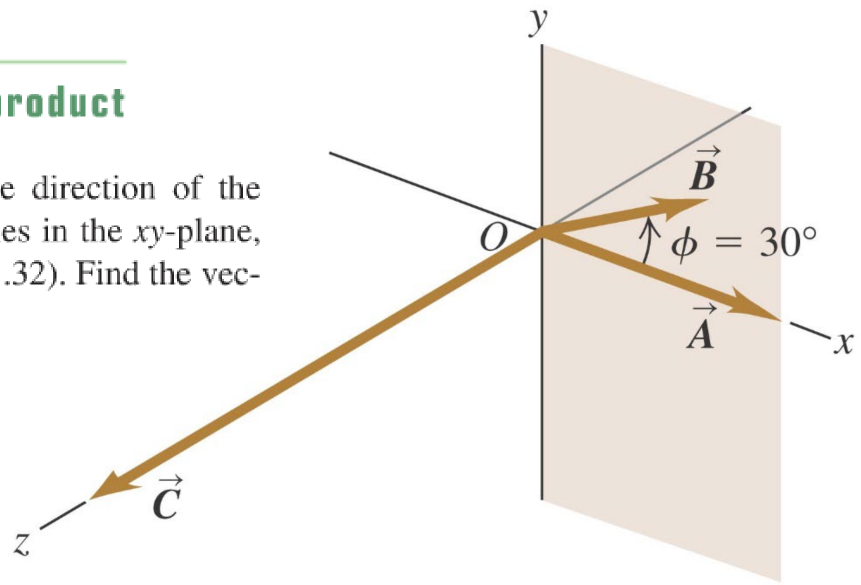


Calculating the vector product—Figure 1.32

- Use $AB\sin\phi$ to find the magnitude and the right-hand rule to find the direction.
- Refer to Example 1.12.

Example 1.12 Calculating a vector product

Vector $\vec{A}$ has magnitude 6 units and is in the direction of the $+x$-axis. Vector $\vec{B}$ has magnitude 4 units and lies in the xy-plane, making an angle of 30° with the $+x$-axis (Fig. 1.32). Find the vector product $\vec{C} = \vec{A} \times \vec{B}$.



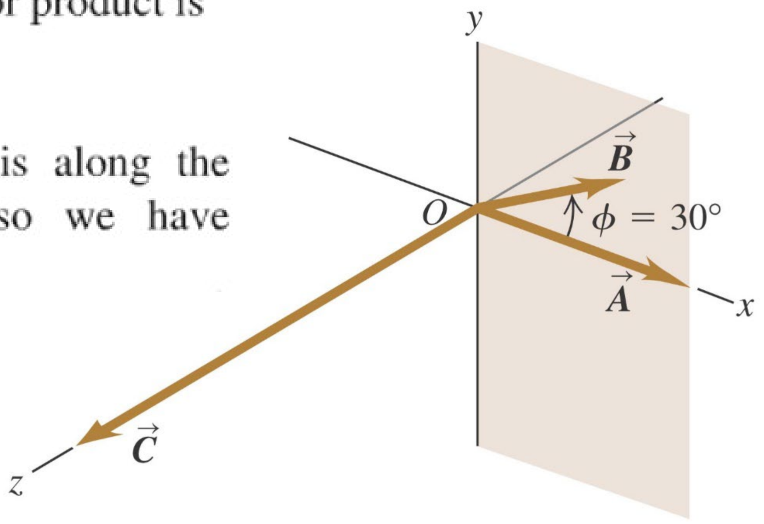
SOLUTION

IDENTIFY and SET UP: We'll find the vector product in two ways, which will provide a check of our calculations. First we'll use Eq. (1.22) and the right-hand rule; then we'll use Eqs. (1.27) to find the vector product using components.

EXECUTE: From Eq. (1.22) the magnitude of the vector product is

$$AB \sin \phi = (6)(4)(\sin 30°) = 12$$

By the right-hand rule, the direction of $\vec{A} \times \vec{B}$ is along the $+z$-axis (the direction of the unit vector $\hat{k}$), so we have $\vec{C} = \vec{A} \times \vec{B} = 12\hat{k}$.



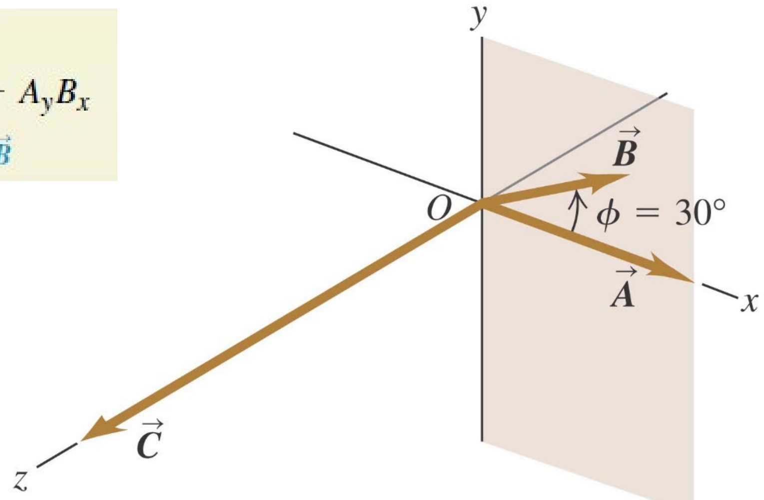
Calculating vector product by component

Eqs. (1.27)

Components of vector (cross) product $\vec{A} \times \vec{B}$

$$C_x = A_yB_z - A_zB_y \qquad C_y = A_zB_x - A_xB_z \qquad C_z = A_xB_y - A_yB_x$$

A_x, A_y, A_z = components of $\vec{A}$ $\qquad$ B_x, B_y, B_z = components of $\vec{B}$



$$A_x = 6 \qquad A_y = 0 \qquad A_z = 0$$
$$B_x = 4\cos 30° = 2\sqrt{3} \qquad B_y = 4\sin 30° = 2 \qquad B_z = 0$$

Then Eqs. (1.27) yield

$$C_x = (0)(0) - (0)(2) = 0$$
$$C_y = (0)(2\sqrt{3}) - (6)(0) = 0$$
$$C_z = (6)(2) - (0)(2\sqrt{3}) = 12$$

Thus again we have $\vec{C} = 12\hat{k}$.

EVALUATE: Both methods give the same result. Depending on the situation, one or the other of the two approaches may be the more convenient one to use.

Calculus

Calculus

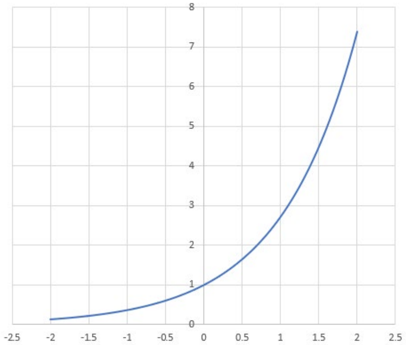
- In mechanics we need to know some calculus for calculating velocity and acceleration
- We need to know how to do differentiation and integration
- In this course, we use a little bit calculus to understand motion, there are some questions in computer exercise. This is the skill I required.

Basic Idea in Calculus – World of Infinitesimally Small (but not zero)

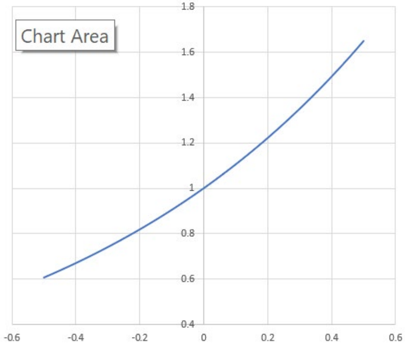
- Let's dive into the small world (magnification in both x and y)

$$y = f(x) = e^x (\mathrm{e} = 2.7183)$$

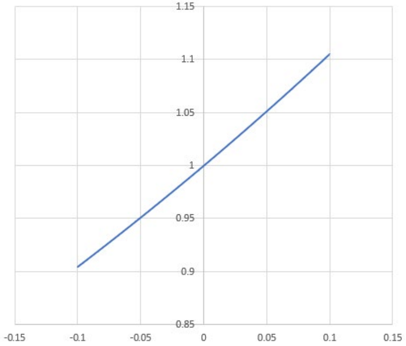
$\Delta x = 4, \Delta y = 8$



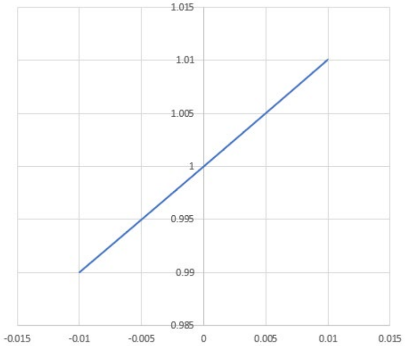
$\Delta x = 0.5, \Delta y = 1$



$\Delta x = 0.2, \Delta y = 0.2$



$\Delta x = 0.02, \Delta y = 0.02$



- The curve becomes a linear line: $\Delta y = a\Delta x \Rightarrow a = \frac{\Delta y}{\Delta x} \rightarrow const.$
- base for differentiation

Differentiation (finding instantaneous rate of change)

Differentiation helps us to find the instantaneous rate of change of a variable with respect to another variable, such as growth rate

For example, y changes with x. y is function of x. The rate of change of y with respect to x represents how fast y changes with x.

Instantaneous means the rate of change at a particular value of x

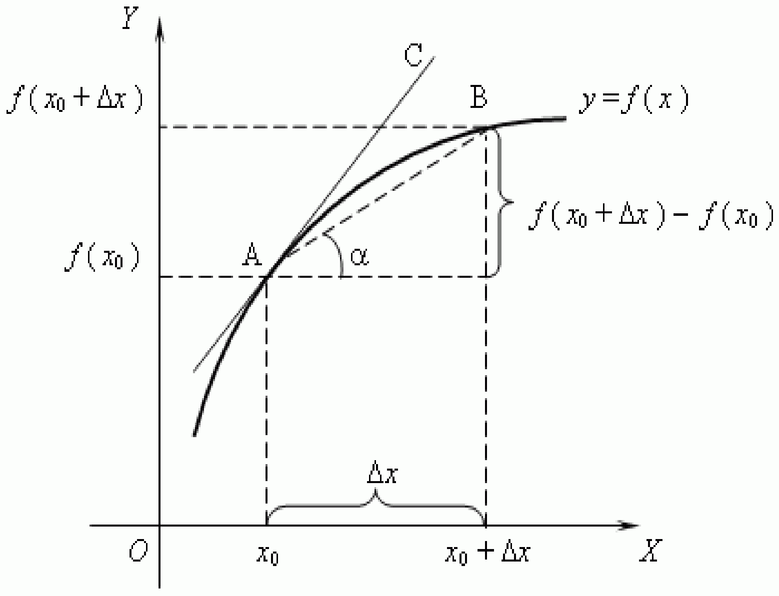


Fig. 1

Average and instantaneous rate of change

- x is changed from x_0 to $x_0+\Delta x$ and y is changed from $f(x_0)$ to $f(x_0+\Delta x)$.
- $\Delta y/\ \Delta x= (f(x_0+\Delta x)-\ f(x_0))/\ \Delta x =\Delta f/\Delta x$ is the average rate of change for interval Δx
- As Δx becomes very small, this rate is a number depending on the location x_0
- The rate of change at point $A(x_0)$ is the instantaneous rate of change.

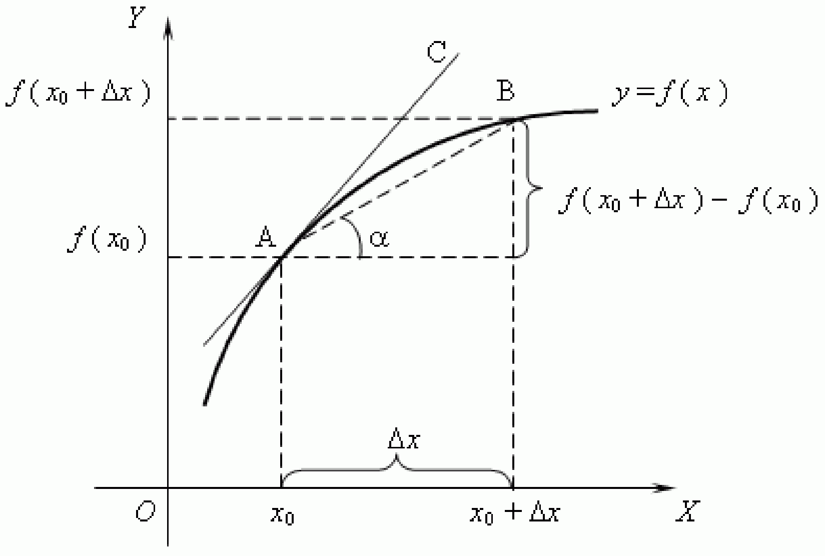


Fig. 1

Simple calculus

- Consider a function of time x, f(x)
- The derivative (differentiation) of f(x) with respect to x is defined as $\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$
- $\Delta f = f(x_0 + \Delta x) - f(x_0)$
- i.e., divide Δf by Δx and gradually reduce Δx to very small value (the meaning of $\lim_{\Delta x \to 0}$)
- Slope of AC is the derivative (instantaneous rate of change) $\frac{df}{dx}$ Slope of AB is $\frac{\Delta f}{\Delta x}$
- When Δx is small, AB becomes AC and Slope of AB=Slope of AC

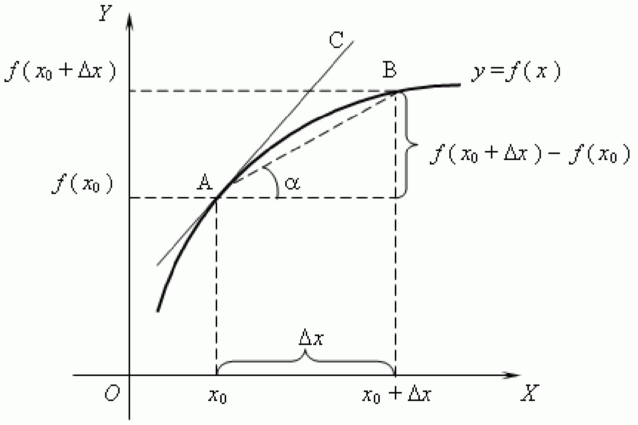


Fig. 1

- If f is distance, then the derivative with respect to time is the speed

Derivative of simple functions

$$f(x)=c\,,\ \frac{df}{dx}=0\quad;\qquad f(x)=ax\,,\ \frac{df}{dx}=a$$

$$f(x)=x\,,\ \frac{df}{dx}=1\ ;\quad \boxed{f(x)=x^2\,,\ \frac{df}{dx}=2x}$$

$$f(x)=x^n\,,\ \frac{df}{dx}=nx^{n-1}\,;\ f(x)=\sin x,\ \frac{df}{dx}=\cos x$$

$$f(x)=\cos x\quad,\ \frac{df}{dx}=-\sin x$$

$$f(x)=cy(x)\,,\frac{df}{dx}=c\frac{dy}{dx}$$

c is constant

Sum rule and chain rule of differentiation

- Derivative of sum of two functions is the sum of the derivatives of the two functions
- For the exercise, you need the following formula and the rule given above.
- In examination, differentiation formula are given. What you need is understand the concepts. Know the calculation in the exercise and you should be able to do the calculation in examination

$$\frac{d(f(x)+g(x))}{dx}=\frac{df}{dx}+\frac{dg}{dx}$$

$$f(x)=2x^2+3x$$

$$\frac{df}{dx}=\frac{d(2x^2)}{dx}+\frac{d(3x)}{dx}=4\mathrm{x}+3$$

$$\frac{df\big(g(x)\big)}{dx}=\frac{df}{dg}\cdot\frac{dg}{dx}$$

$$f(x)=\cos ax\,;f(g)=\cos g\ \ ,g=ax$$

$$\frac{df}{dx}=\frac{d\cos g}{dg}\frac{dg}{dx}=(-\sin ax)\ a$$

$$=-a\sin ax$$

Example

$$\frac{d}{dx}(3x^2 - 2x + 1)$$

$$= 3\frac{d}{dx}(x^2) - 2\frac{d}{dx}(x) + \frac{d}{dx}(1)$$

$$= 3(2x) - 2(1) + (0)$$

$$= 6x - 2$$

$$f(x) = \sin(ax)\,;\, \mathrm{g} = \mathrm{ax}$$

$$\frac{df}{dx} = \frac{d \sin g}{dg}\frac{\mathrm{dg}}{\mathrm{dx}}$$

$$= a \cos ax$$

Partial differentiation (partial derivative)

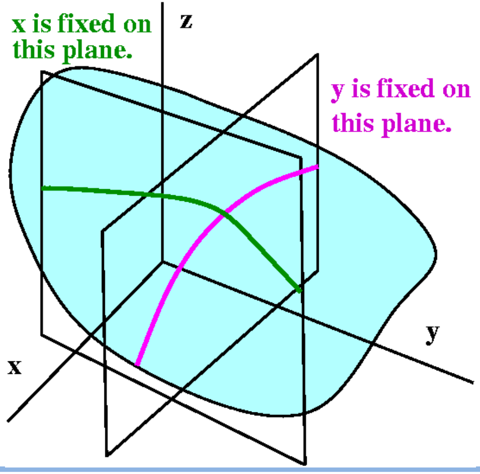
The function is a function of two variables, x, y

$$z = f(x, y)$$

Keep one variable x or y constant and differentiate with respect to the other variable y or x.

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \frac{df}{dx} \quad \text{assuming y a constant}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \frac{df}{dy} \quad \text{assuming x a constant}$$

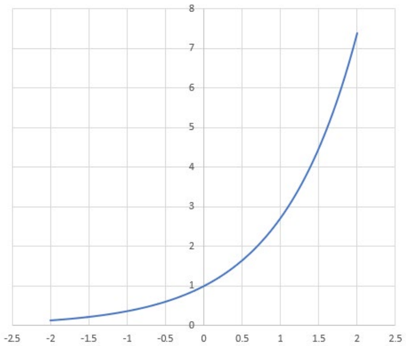


Basic Idea in Calculus – World of Infinitesimally Small (but not zero)

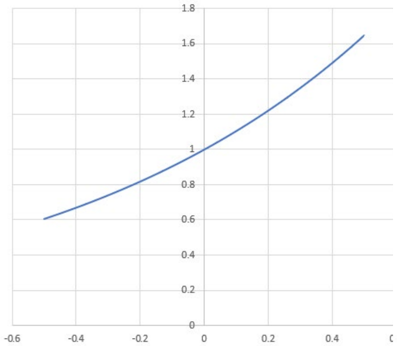
- Let’s dive into the small world (magnification in only x)

$$y = f(x) = e^x (\mathrm{e} = 2.7328)$$

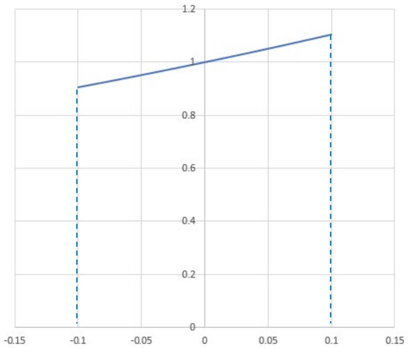
$\Delta x = 4, \Delta y = 8$



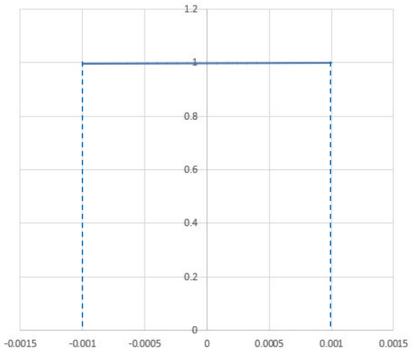
$\Delta x = 0.5, \Delta y = 2$



$\Delta x = 0.2, \Delta y = 1.2$



$\Delta x = 0.002, \Delta y = 1.2$

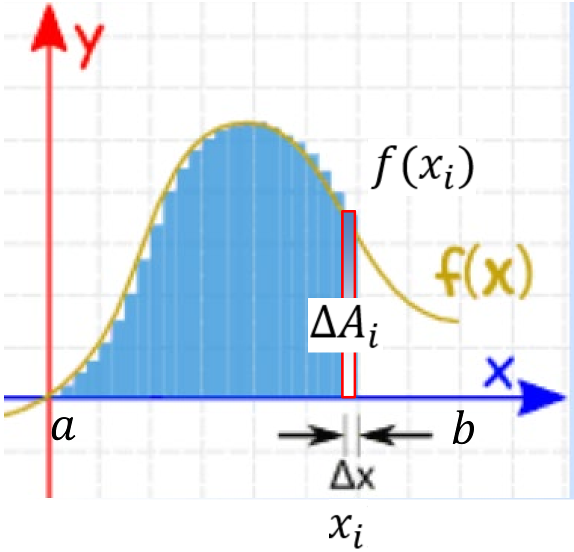


- The curve becomes flat: $f(x) \approx f(x + \Delta x) \Rightarrow Area \approx f(x)\Delta x$
- base for integration

Integral

Integral is related to the area under the curve

We divide the section $[a, b]$ into n small segment $\Delta x = (b - a)/n$



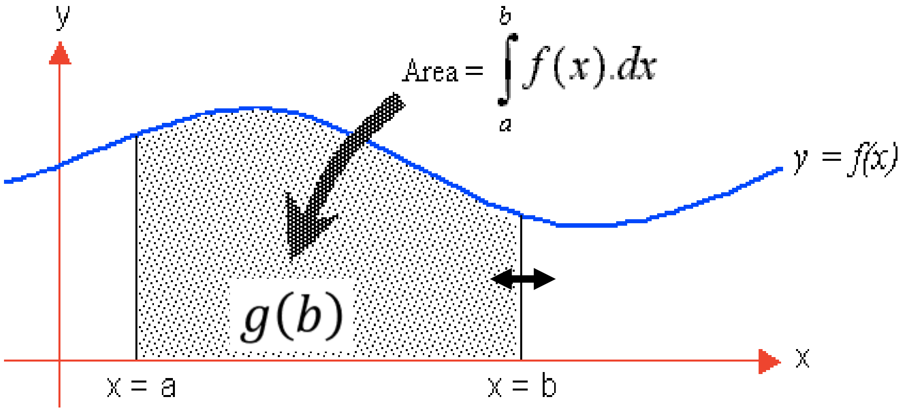
$$\Delta A_i = f(x_i)\Delta x$$

$$Area = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx.$$

Integral – Reverse operation of differentiation

Define a function $g(b)$ as the area from a to b with b as a variable: $g(b)$ is and integral of f with respect to x between a and b

$$g(b) \equiv \int_a^b f(x)\,dx$$



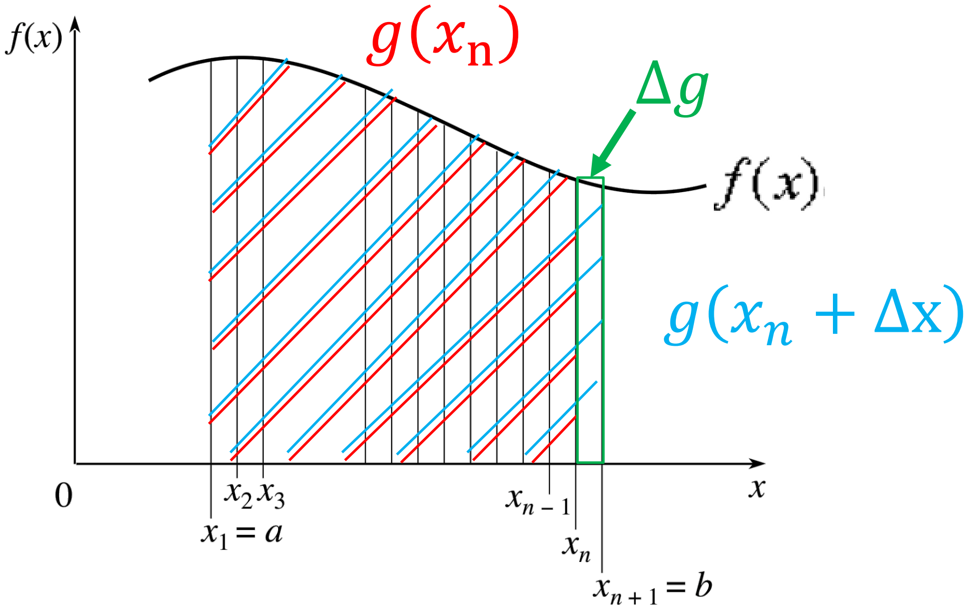
What is the rate of change of $g(b)$ with respect to b?

Integral – Reverse operation of differentiation

$$\frac{dg}{dx} \cong \frac{g(x_n + \Delta x) - g(x_n)}{\Delta x}$$

$$= \frac{\Delta g}{\Delta x} \approx \frac{f(x_n)\Delta x}{\Delta x} = f(x_n)$$

$$\rightarrow \quad \frac{dg}{dx} = f(x)$$



example, $\frac{dx^2}{dx} = 2x$; $\int 2x\, dx = x^2 + c$

arbitrary number related to a

Integration

What you expect to know and understand for integration:

The meaning of integration:

Reverse of differentiation

Area under a curve