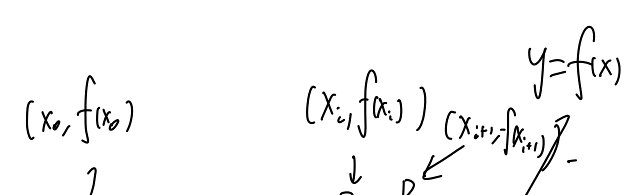
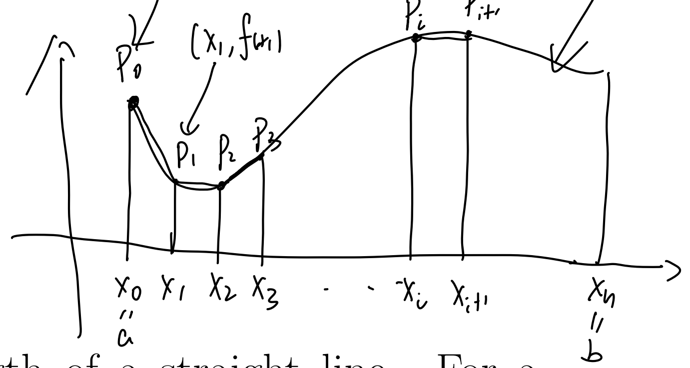
Contents

4. Further Applications of Integration

We have studied techniques of integration and some applications of integrals. In this chapter, we will learn more geometric applications of integration as well as quantities of interest in physics, engineering, economics and biology:

- the length of a curve.
- the area of a surface of revolution.
- hydrostatic force and pressure, moments and centers of mass.





4.1. **Arc length.** Text Section 8.1, Exercise: 11, 15, 25, 33, 37.

We know how to calculate the length of a straight line. For a polygon, we can easily find its length by adding the lengths of the line segments that form the polygon.

Q: How to define the length of a general curve?

The **idea** to find the length of a general curve is:

- to approximate the length by a polygon first.
- to take a limit as the number of segments of the polygon is increased.

Let a curve C be defined by the equation $y = f(x)$, where f is continuous and $a \le x \le b$. Divide the interval $[a, b]$ into n subintervals with equal width $\Delta x = \frac{b-a}{n}$ and endpoints:

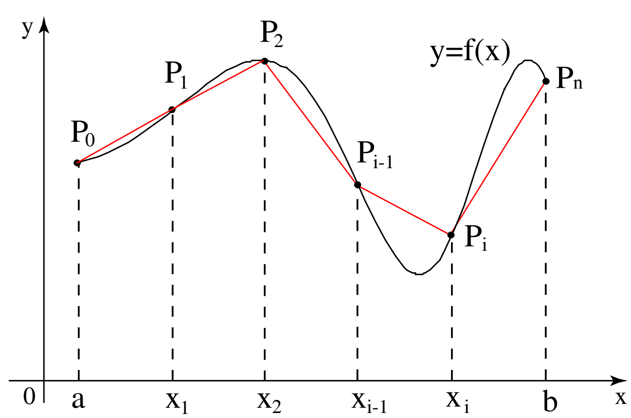
$$x_i = a + i\Delta x, \qquad i = 0, 1, \cdots, n.$$

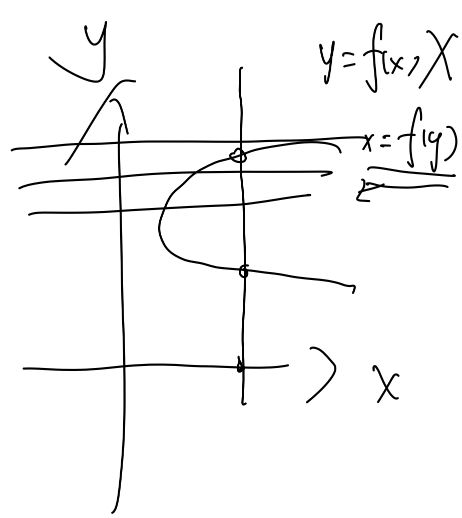
$y = f(x)$

Let

$$y_i = f(x_i),$$

then the points $P_i(x_i, y_i)$ lie on C. These points form a polygon which is an approximation to C.





Therefore, we define the **length** of the curve C as

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

if $\boxed{\text{the limit exists.}}$ $\Leftarrow$ f is continuous.

Let us assume the function f has a continuous derivative (the curve is smooth enough). Note that

(x_{i-1}, y_{i-1}) (x_i, y_i).

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2},$$

where $\boxed{\Delta y_i := y_i - y_{i-1}.}$

By the *Mean Value Theorem for Derivatives*, we can find a number $x_i^* \in [x_{i-1}, x_i]$ such that

$$\underbrace{f(x_i) - f(x_{i-1})}_{=\Delta y_i} = f'(x_i^*)\underbrace{\boxed{(x_i - x_{i-1})}}_{=\Delta x} \Longrightarrow \Delta y_i = f'(x_i^*)\Delta x.$$

Thus we have

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} \\ &= \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x, \qquad \text{since } \Delta x > 0. \end{aligned}$$

So, the formula for the length of the curve is rewritten as

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \underbrace{\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x,}}_{= \int_a^b \sqrt{1+[f'(x)]^2}\, dx}$$

which gives us the following formula:

The Arc Length Formula: If f' is continuous on $[a,b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx, \qquad f'(x) = \frac{dy}{dx} \text{ if } y = f(x).$$

or

$$L = \int_a^b \boxed{\sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx}.$$

If a curve has the equation $x = g(y)$, $c \le y \le d$ and $g'(y)$ is continuous, then the length of the curve is

$$\sqrt{(dx)^2+(dy)^2}$$

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \boxed{\sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy}.$$

$x = g(y)$.

Ex. [Text example 8.1.2] Find the length of the arc of the parabola $y^2 = x$ from $(0,0)$ to $(1,1)$.

Solution: Since $x = y^2$, we have $\frac{dx}{dy} = 2y$, and

$$L = \int_0^1 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1+4y^2}\, dy$$

$y = \frac{1}{2}\tan\theta$

$dy = \frac{1}{2}\sec^2\theta\, d\theta$

$$= \int_0^{\tan^{-1}2} \sqrt{1+4\left(\frac{1}{2}\tan\theta\right)^2}\ \frac{1}{2}\sec^2\theta\, d\theta = \frac{1}{2}\int_0^{\tan^{-1}2} \sec^3\theta\, d\theta$$

$1+4\left(\frac{1}{2}\tan\theta\right)^2 = 1+\tan^2\theta = \sec^2\theta$

P9. Chapter 3

$$\boxed{\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C}$$

$$= \frac{1}{2}\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right)\Bigg|_0^{\tan^{-1}2}$$

$$\frac{1}{2}\left(\frac{1}{2}\sqrt{5}\cdot 2 + \frac{1}{2}\ln|\sqrt{5}+2|\right)$$

$=$

$\frac{\sqrt{5}}{2} + \frac{1}{4}\ln(\sqrt{5}+2)$

$\tan(\tan^{-1} 2) = 2$

$\sec(\tan^{-1} 2) = \sqrt{5}$.

Let $\alpha = \tan^{-1} 2$

$\tan\alpha = 2$

$\sec^2\alpha = 1 + \tan^2\alpha = 5$

$\sec\alpha = \sqrt{5}$

Ex. [Text example 8.1.3]

(a) Set up an integral for the length of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $(2, \frac{1}{2})$.

(b) Use the Simpson's Rule with $n = 10$ to estimate the arc length.

Solution: (a) We have $y = \frac{1}{x}$ and $\frac{dy}{dx} = -\frac{1}{x^2}$. Thus the arc length is

$$L = \int_1^2 \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_1^2 \sqrt{1 + (-\frac{1}{x^2})^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx.$$

(2) Using Simpson's Rule with $a = 1$, $b = 2$, $n = 10$, $\Delta = 0.1$ and $f(x) = \sqrt{1 + \frac{1}{x^4}}$, we have

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx \approx \frac{\Delta x}{3}\Big[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4f(1.5) + \cdots + 2f(1.8) + 4f(1.9) + f(2)\Big]$$

≈ 1.1321.

The Arc Length Function denotes the length from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$:

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

From the above definition, we get $y = f(x)$

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}.$$

We can also rewrite it as

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

$ds = \sqrt{dx^2 + dy^2}$

P_i dx P_{i+1} dy

or a symmetric form

$$(ds)^2 = (dx)^2 + (dy)^2.$$

$y = f(x)$, $(x, f(x))$

$A = \int_D^D \sqrt{1 + (\frac{dy}{dx})^2}\, dx$

$\parallel$

$x = f(y)$, $(f(y), y)$

$= \int_D^D \sqrt{1 + (\frac{dx}{dy})^2}\, dy$

$S(x) = \int_a^b \sqrt{(dx)^2 + (dy)^2}$

$\left[x(t), y(t) \right]$ $= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Ex. [~~Text ex~~ample 8.1.4] Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $P_0(1, 1)$ as the starting point.

Solution $\frac{dy}{dx} = 2x - \frac{1}{8x}$ and the arc length is

$$S(x) = \int_1^x \sqrt{1+\left(2t - \frac{1}{8t}\right)^2}\, dt = \int_1^x \sqrt{1 + 4t^2 - \frac{1}{2} + \frac{1}{64t^2}}\, dt$$

$$= \int_1^x \sqrt{\left(4t^2 + \frac{1}{2} + \frac{1}{64t^2}\right)}\, dt$$

$$= \int_1^x \sqrt{\left(2t + \frac{1}{8t}\right)^2}\, dt$$

$$= \int_1^x 2t + \frac{1}{8t}\, dt$$

$$= \left(t^2 + \frac{1}{8}\ln t\right)\Big|_1^{t=x}$$

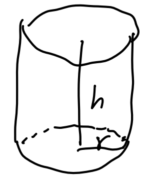
$$= x^2 + \frac{1}{8}\ln x - \left(1 - \frac{1}{8}\ln 1\right)$$

$$= x^2 - 1 + \frac{1}{8}\ln x.$$

4.2. Area of a surface of revolution. Text Section 8.2,
Exercise: 11, 15, 29, 31.

The lateral surface area of a circular cylinder with radius r and height h is

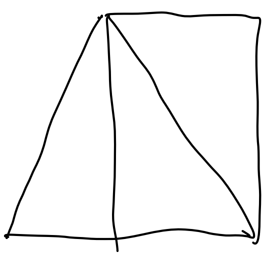
$$A = \boxed{2\pi r h}.$$

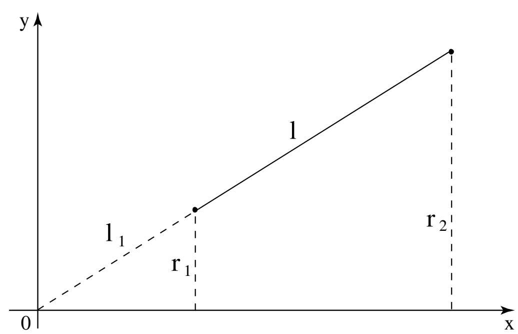


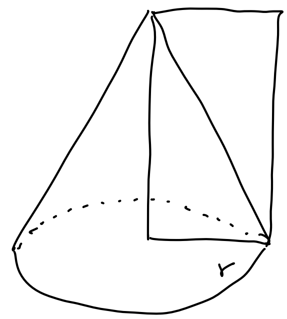
The lateral surface area of a cone with base radius r and slant height l is

$$A = \pi r l.$$

$\pi r l$

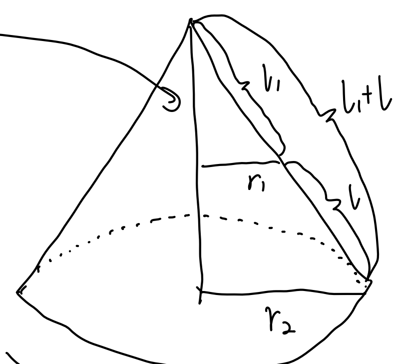






Consider the surface area of the band (or frustum of a cone) with height l and upper and lower radii r_1 and r_2. By subtracting the areas of two cones, we obtain the surface area

$$A = \pi r_2(l_1 + l) - \pi r_1 l_1.$$



From similar triangles, we get

$$\boxed{(r_2-r_1)l_1 = r_1 l} \Leftarrow r_2 l_1 = (l_1+l)r_1 \Leftarrow \frac{l_1}{r_1} = \frac{l_1 + l}{r_2},$$

$\pi l_1(r_2-r_1) + \pi r_2 l$

then

$$A = \pi(r_1 l + r_2 l) = 2\pi r l,$$

$= \pi r_1 l + \pi r_2 l = \pi l(r_1+r_2)$

where $\boxed{r = \frac{1}{2}(r_1 + r_2)}$ is the average radius of the band.

Inspired by the above formula, we can calculate the **surface area of revolution** by dividing the surface S into pieces and approximate each piece by a band.

The surface area of the surface obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$ about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

or

$$S = \int_a^b 2\pi y \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx. \quad = ds$$

If the curved is described as $x = g(y)$, $c \leq y \leq d$, then the formula becomes

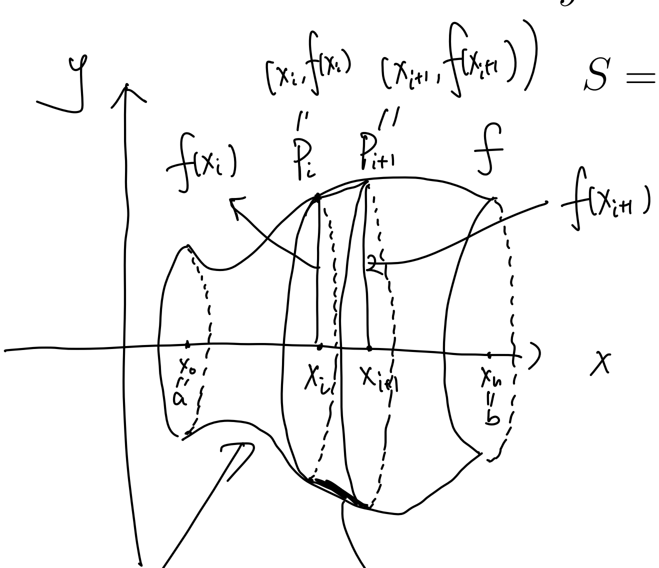
$$S = \int_c^d 2\pi y \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy. \quad = ds$$

Using the notation for arc length, both the above two formulas can be written as

$$S = \int 2\pi y\, ds. \quad \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

For rotation about the y**-axis**, the surface area formula becomes

$$S = \int 2\pi x\, ds.$$



x_i

$$\left(2\pi f(x_i)\sqrt{1 + (f'(x_i))^2}\right)\Delta x$$

$$\parallel$$

$$f(x_i) \qquad \sqrt{(\Delta x)^2 + (f'(x_i)\Delta x)^2}$$

$$\parallel$$

The area of surface → The area of the band: $2\pi \cdot \boxed{\dfrac{f(x_i)+f(x_{i+1})}{2}} \cdot \boxed{\sqrt{(x_{i+1}-x_i)^2+(f(x_{i+1})-f(x_i))^2}}$

Ex. [Text example 8.2.1] The curve $y = \sqrt{4-x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2+y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis.

$$\lim_{n\to\infty}\sum_{i=0}^{n} 2\pi f(x_i)\sqrt{1+(f'(x_i))^2}\,\Delta x = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

Solution $y=\sqrt{4-x^2}$ $-1\le x\le 1 \Rightarrow$ $\dfrac{dy}{dx} = \dfrac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x) = \dfrac{-x}{\sqrt{4-x^2}}$

and thus

$$S = \int_{-1}^{1} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\int_{-1}^{1}\sqrt{4-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\,dx = 2\pi\int_{-1}^{1}\cancel{\sqrt{4-x^2}}\sqrt{\frac{4}{\cancel{4-x^2}}}\,dx$$

$$= 2\pi\int_{-1}^{1} 2\,dx = 4\pi\cdot 2 = 8\pi.$$

Ex. [Text example 8.2.2] The arc of the parabola $y = x^2$ from (1,1) to (2,4) is rotated about the y-axis. Find the area of the resulting surface.

Solution: $S = \displaystyle\int_{D}^{\square} 2\pi x\,\boxed{ds}$

$$= \int_1^2 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_1^2 2\pi x\sqrt{1+4x^2}\,dx$$

$$= 2\pi\int_1^2 x\sqrt{1+4x^2}\,dx$$

$u=1+4x^2$

$$= \frac{2\pi}{8}\int_5^{17}\sqrt{u}\,du = \frac{\pi}{6}\left[17\sqrt{17}-5\sqrt{5}\right]$$

$du = 8x\,dx$

$\Downarrow$

$x\,dx = \frac{1}{8}du$

4.3. **Applications to physics and engineering.** Text Section 8.3

Exercise: 5, 14, 19, 31, 41.

In this section, we will study some applications of integral calculus to physic and engineering. Our strategy is to

- break up the physical quantity into a large number of small parts;
- approximate each small part;
- add the results;
- take the limit;
- evaluate the resulting integral.

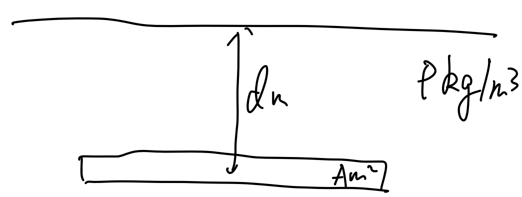
Hydrostatic Force and Pressure

the acceleration of gravity

Physical laws:

Suppose that a thin horizonal plate with A m^2 is submerged in a fluid of density ρ kg/m^3 at a depth d m below the surface of the fluid. The force F exerted by the fluid on the plate is

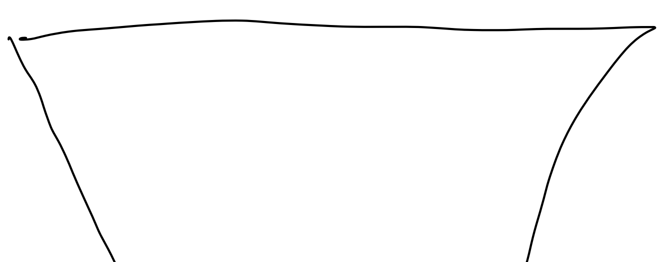
$$F = mg = \rho g A d.$$



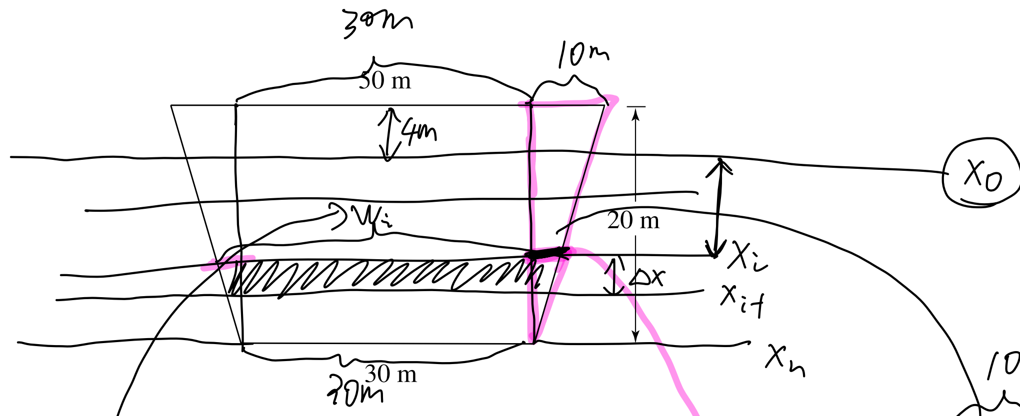
The pressure P on the plate is

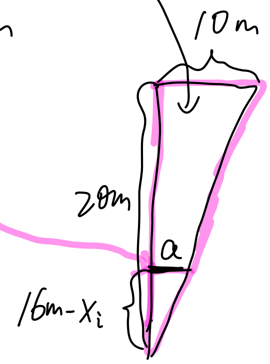
$$P = \frac{F}{A} = \rho g d.$$

An important principal of fluid pressure is that *at any point in a liquid the pressure is the same in all directions.*



Ex. [Text example 8.3.1] Find the force on the dam due to the hydrostatic pressure if the water level is 4 m from the top of the dam.





The width $w_i = 30m + 2a$

$= 30 + 2(8 - \frac{x_i}{2}) = 46 - x_i$

The area of the ith plate: $A_i = w_i \Delta x = (46 - x_i)\Delta x$

The pressure P_i on the ith plate: $P_i = \boxed{1000}\, g\, x_i$ ($= \rho_{water}$)

$\frac{16 - x_i}{a} = \frac{20}{10}$

$\Downarrow$

$a = 8 - \frac{x_i}{2}$

Then $F = \lim_{n\to\infty} \sum_{i=1}^{n} P_i A_i = \lim_{n\to\infty} \sum_{i=1}^{n} 1000\, g\, x_i (46 - x_i)\Delta x$

$= \int_0^{16} 1000\, g\, x(46-x)\, dx$

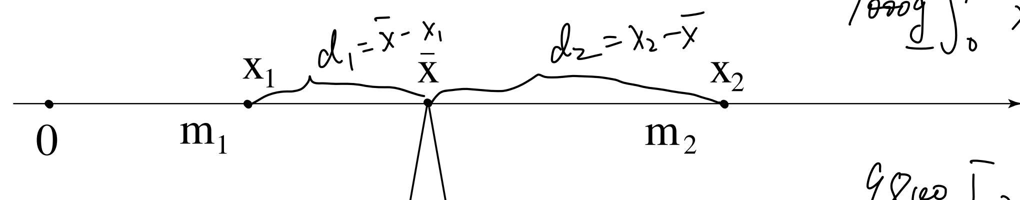
$= 1000g \int_0^{16} x(46-x)\, dx$

$= 9800 \left[23x^2 - \frac{x^3}{3} \right]\Big|_0^{16}$

$= 4.43 \times 10^7\, N.$

Moments and Centers of Mass

Suppose a rod of negligible mass lies along the x-axis with m_1 and m_2. Let us try to locate the center of mass $\bar{x}$.



According to the Law of the Lever, the rod will balance if

$$m_1 d_1 = m_2 d_2,$$

$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$

$(m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2$

where d_1 and d_2 are distances from the fulcrum.

From the above law, the **center of mass** is located at

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

Note that $m_1 x_1$ and $m_2 x_2$ are called the **moments** of the masses m_1 and m_2 with respect to the origin.

More generally, for n particles with masses $m_1, m_2, \cdots, m_n$ located at the points $x_1, x_2, \cdots, x_n$, the center of mass is located at

$$\bar{x} = \frac{M}{m},$$

where

$$m = \sum_{i=1}^{n} m_i$$

is the **total mass of the system** and

$$M = \sum_{i=1}^{n} m_i x_i$$

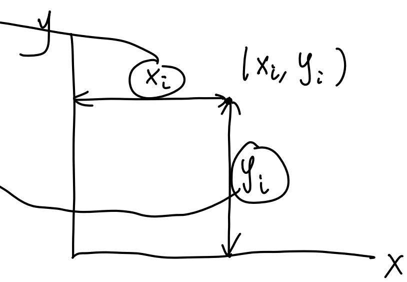
is the **moment of the system about the origin**.

For the two-dimensional case, let us consider a system of n particles with masses $m_1, m_2, \cdots, m_n$ located at the points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ in the xy-plane. Then the center of mass is located at $(\bar{x}, \bar{y})$ with

$$\bar{x} = \frac{M_y}{m}, \quad M_y = \sum_{i=1}^{n} m_i x_i, \qquad \bar{y} = \frac{M_x}{m}, \quad M_x = \sum_{i=1}^{n} m_i y_i$$

where M_x and M_y are the **moments of the system about the** x **and** y **axis, respectively**,

$$M_y = \sum_{i=1}^{n} m_i x_i, \qquad M_x = \sum_{i=1}^{n} m_i y_i.$$



Ex. [Text example 8.3.3] Find the moments and center of mass of the system of objects that have masses 3, 4 and 8 at the points $(-1, 1)$, $(2, -1)$ and $(3, 2)$, respectively.

Solution:

$$M_y = \sum_{i=1}^{3} x_i m_i = (3 \times (-1) + 4 \times 2 + 8 \times 3) = 29$$

$$M_x = \sum_{i=1}^{3} y_i m_i = (3 \times 1 + 4 \times (-1) + 8 \times 2) = 15.$$

and

$$m = \sum_{i=1}^{3} m_i = 3 + 4 + 8 = 15.$$

$$\bar{x} = \frac{M_y}{m} = \frac{29}{15}, \quad \bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1.$$

Consider a flat plate with uniform density ρ that occupies a region $\mathfrak{R}$ of the plane. Assume that $\mathfrak{R}$ lies between the lines $x = a$ and $x = b$, above the x-axis and beneath the graph of f, where f is a continuous function.

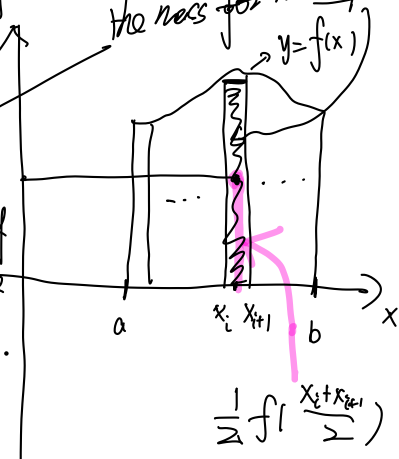
$= \rho \int_a^b x f(x)\,dx$

$M_y = \sum_{i=1}^{n} \boxed{M_{y_i}}$

the mass for the ith plate

$M_y = \lim_{n\to+\infty} \rho \sum_{i=1}^{n} \frac{x_i+x_{i+1}}{2} f(x_i)\Delta x \Longleftarrow M_{y_i} = \frac{x_i+x_{i+1}}{2} \boxed{\rho \boxed{f(x_i)\Delta x}}$

the area of the ith plate



Then the **moment of $\mathfrak{R}$ about the y-axis** is

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i)\Delta x = \rho \int_a^b x f(x) dx.$$

The **moment of $\mathfrak{R}$ about the x-axis** is

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \frac{1}{2}[f(\bar{x}_i)]^2 \Delta x = \rho \int_a^b \frac{1}{2}[f(x)]^2 dx.$$

$M_x = \sum_{i=1}^{n} M_{x_i}$

$M_{x_i} = \frac{1}{2} f\left(\frac{x_i+x_{i+1}}{2}\right) \rho f(x_i)\Delta x$

$M_x = \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{2} f\left(\underset{\substack{\approx \\ x_i}}{\frac{x_i+x_{i+1}}{2}}\right) \rho f(x_i)\Delta x$

$= \int_a^b \frac{1}{2} f(x) \rho f(x)\,dx = \frac{\rho}{2}\int_a^b f(x)^2\,dx$

$\frac{\cancel{\rho}\int_a^b x f(x)\,dx}{\cancel{\rho}\int_a^b f(x)\,dx}$

The **center of mass of the plate** (or **the centroid of $\mathfrak{R}$**) is located at the point $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx, \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx,$$

where $A = \int_a^b f(x)\,dx$ is the area of $\mathfrak{R}$.

Remark: *Note that the location of the centroid is independent of the density ρ.*

If the region $\mathfrak{R}$ lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$, then the centroid of $\mathfrak{R}$ is $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx, \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\,dx,$$

where $A = \int_a^b [f(x) - g(x)]\,dx$ is the area of $\mathfrak{R}$.

Ex. [Text example 8.3.6] Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$.