Tutorial 4 (Chapter 4 and some discrete problems)

1. Let X be a random variable with probability density function

$$f(x) = \begin{cases} C(1-x^2) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the value of C?

   (b) What is the cumulative distribution function of X?

   (c) Find $E[X]$ and $Var(X)$.

   (d) Find the density function of X^2.

2. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   If $E[X] = 3/5$, find a and b.

3. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x}, x \geq 0$$

   Compute the expected lifetime of such a tube.

4. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, express the following probability in terms of Φ.

   (a) $P(X > 5)$

   (b) $P(4 < X < 16)$

   (c) $P(X < 8)$

   (d) $P(X > 16)$

5. The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is

   (a) the probability that a repair exceeds 2 hours?

   (b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

6. The median of a continuous random variable having distribution function F is that value m such that $F(m) = 1/2$. Find the median of X if X is

   (a) uniformly distributed over (a, b);

   (b) normal with parameters μ, σ^2;

   (c) exponential with rate λ.

7. A random variable X has an absolute value no larger than 1. $P(X = -1) = 1/8$ and $P(X = 1) = 1/4$. Given the event $\{-1 < X < 1\}$ occurs, the probability that X takes a value in an subinterval within $(-1, 1)$ is proportional to the length of the subinterval. Find the cdf of X.

8. Suppose the cumulative distribution function for a continuous random variable X is

$$F(x) = \begin{cases} a & x < 1 \\ bx \ln x + cx + d & 1 \leq x < e \\ d & x \geq e \end{cases}$$

Determine a, b, c, d and find the density for X.

9. A random variable $X \sim U[0, 5]$(uniform). Observe independently X three times. What is the probability that for at least twice the equation $4x^2 + 4Xx + (X + 2) = 0$ has a real solution.

10. Suppose the density for a continuous random variable is $p_X(x) = \frac{1}{\pi(1+x^2)}$, find the density of the random variable $Y = 1 - X^{1/3}$

11. A certain retailer for a petroleum product sells a random amount X each day. Suppose that X (measured in hundreds of gallons) has the following density function

$$f(x) = \begin{cases} \frac{3}{8}x^2 & 0 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

The retailer's profit turns out to be \$5 for each 100 gallons sold if $X \leq 1$, and \$8 per 100 gallons if $X > 1$. Find the retailer's expected profit for any given day.

12. Pick two numbers from $\{1, 2, \ldots, n\}$, find the probability that the sum is even.

13. Pick 4 numbers from $\{0, 1, 2, \ldots, 9\}$ with replacement, and arrange them in the order they are picked. Find the probability for the following events:
    (a) The four numbers form a proper integer (i.e. 0 does not appear in the first position);
    (b) The four number form a proper even integer;
    (c) 0 appears exactly twice;
    (d) 0 appears at least once.

14. A box contains $2n - 1$ white balls and $2n$ black balls. You randomly draw out n of them and find they are all the same color. Find the probability that their color is black.

15. We have a batch of products in which 10 are effective and 3 are defective. Randomly pick one at a time, and let X represent the time you get an effective one. Find the distribution of X in the following different situations:
    (a) You pick without replacement.
    (b) You pick with replacement.
    (c) After you pick a defective one, put back an effective one.