MA1300 Solutions to Self Practice # 8

1. If a tank holds 5000 liters of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000\left(1 - \frac{t}{40}\right)^2 \qquad 0 \le t \le 40.$$

Find the rate at which water is draining from the tank after (a) 5 min, (b) 10 min, (c) 20 min, and (d) 40 min. At what time is the water flowing out the fastest? The slowest? Summarize your findings.

Solution: Differentiate V respect to t to get the rate at which water is draining from the tank

$$\frac{dV}{dt} = \frac{10,000}{40}\left(\frac{t}{40} - 1\right).$$

Therefore the rates at the time points **a**$\sim$**d** are

$$\left.\frac{dV}{dt}\right|_{t=5\,\mathrm{min}} = -218.75, \qquad \left.\frac{dV}{dt}\right|_{t=10\,\mathrm{min}} = -187.5, \qquad \left.\frac{dV}{dt}\right|_{t=20\,\mathrm{min}} = -125, \qquad \left.\frac{dV}{dt}\right|_{t=40\,\mathrm{min}} = 0.$$

The water flowing out fastest when $t = 0$, and slowest when $t = 40$. The finding is that the rate water flows out decreases as time passes.

2. The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. Find the current when **(a)** $t = 0.5$ s, and **(b)** $t = 1$ s. (The unit of current is an ampere (1 A = 1 C/s).) At what time is the current lowest?

Solution: Take derivative of Q with respect to t to get the current

$$C = \frac{dQ}{dt} = 3t^2 - 4t + 6.$$

Therefore

$$C|_{t=0.5\,\mathrm{s}} = 4.75 \text{ A}, \qquad C|_{t=1\,\mathrm{s}} = 5 \text{ A}.$$

We rewrite C as

$$C = 3\left(t - \frac{2}{3}\right)^2 + \frac{14}{3},$$

so when $t = \frac{2}{3}$, the current becomes the lowest.

3. Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2},$$

where G is the gravitational constant and r is the distance between the bodies.

a Find dF/dr and explain its meaning. What does the minus sign indicate?

b Suppose it is known that the earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20,000$ km. How fast does this force change when $r = 10,000$ km?

Solution:

a Take derivative of F with respect to r to give

$$\frac{dF}{dr} = -\frac{2GmM}{r^3}.$$

It means that as r increases, the force F decreases. The minus sign indicates a negative direction.

b We let

$$2 = \left.\frac{2GmM}{r^3}\right|_{r=20{,}000\,\text{km}} = \frac{2GmM}{20,000^3},$$

then dividing both sides by 2 to give $GmM = 20,000^3$. It follows that

$$\left.\frac{2GmM}{r^3}\right|_{r=10{,}000\,\text{km}} = \frac{2 \times 20,000^3}{10,000^3} = 16,$$

so the rate the force changes when $r = 10,000$ km is 16 N/km

4.

a If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt.

b Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?

Solution: Differentiate $A = \pi r^2$ with respect to t to get

$$\frac{dA}{dt} = \frac{d\pi r^2}{dr}\frac{dr}{dt} = 2\pi r\frac{dr}{dt}.$$

Substituting $r = 30$ m, we have

$$\left.\frac{dA}{dt}\right|_{r=30\,\text{m}} = 60\pi \times 1 = 60\pi\ (\text{m}^2/\text{s}).$$

So the rate the area of the spill increasing when $r = 30$ m is 60π m^2/s.

5. A kite 50 m above the ground moves horizontally at a speed of 2 m/s. At what rate is the angle between the string and the horizontal decreasing when 100 m of string has been let out?

Solution: Let α be the angle between the string and the horizontal, l be the length of the string. We have

$$\begin{cases} l\sin\alpha = 50 \text{ m}, \\ \frac{d\,l\cos\alpha}{dt} = 2 \text{ m/s}. \end{cases}$$

Substitute the first equation into the second to obtain

$$\frac{d\,(50\cot\alpha)}{dt} = 2, \qquad \text{or} \quad -\csc^2\alpha\frac{d\alpha}{dt} = \frac{1}{25}.$$

When $l = 100$ m, we have $\sin\alpha = \frac{50}{100} = \frac{1}{2}$, so $\csc\alpha = 2$. Therefore

$$\left.\frac{d\alpha}{dt}\right|_{l=100\,\text{m}} = -\frac{1}{100}.$$

So the rate α decreases is $\frac{1}{100}$.

6. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is 600 cm^3, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

Solution: Differentiate both sides of $PV = C$ with respect to t to give

$$V\frac{dP}{dt} + P\frac{dV}{dt} = 0.$$

Substituting $C = 600\ \text{cm}^3$, $P = 150$ kPa, and $\frac{dP}{dt} = 20$ kPa/min, we have

$$\frac{dV}{dt} = -\frac{600 \times 20}{150}\ \text{cm}^3/\text{min} = -80\ \text{cm}^3/\text{min}.$$

So at the instant, the rate the volume decreases is 80 cm^3/min

7. Explain the difference between an absolute minimum and a local minimum.

Solution: One difference is that for a local minimum a of a function f, there might still be some point b in the domain of f such that $f(b) < f(a)$, but if a is an absolute minimum, such a point b never exists.

Another difference is that a local minimum requires an open neighborhood in the domain of f, which is not required for an absolute minimum.

8. Suppose f is a continuous function defined on a closed interval $[a, b]$.

a What theorem guarantees the existence of an absolute maximum value and an absolute minimum value for f?

b What steps would you take to find those maximum and minimum values?

Solution:

a The Extreme Value Theorem.

b The Closed Interval Method: 1. Find the values of f at the critical numbers of f in (a, b). 2. Find the values of f at the endpoints of the interval. 3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

9. Sketch the graph of a function f that is continuous on $[1, 5]$ and has the given properties respectively.

a Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4.

b f has no local maximum or minimum in $(1, 5)$, but 2 and 4 are critical numbers.

Solution: See Figure 1.

10.

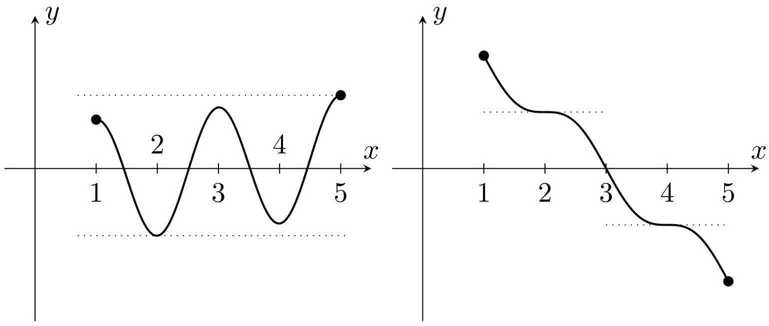

Figure 1: The pictures of Problem 9. Left, **a**; Right, **b**.

a Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.

b Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.

Solution: See Figure 2.

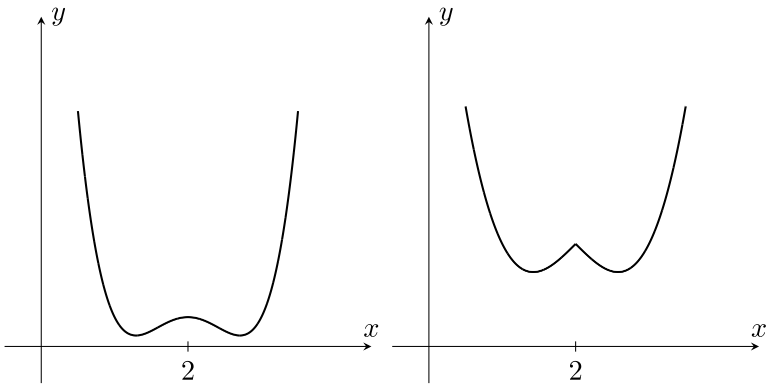

Figure 2: The pictures of Problem 10. Left, **a**; Right, **b**.

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