Chapter 1 Limits

Contents

1. Limits

The entire calculus is built on the concept of limit. In this chapter, we will discuss

- What is limit?
- How to calculate limit?
- Continuity of a function.

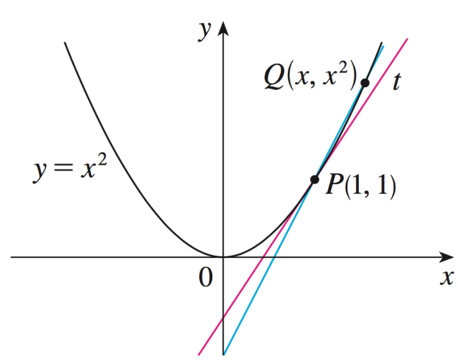
1.1. **Motivation: Tangent and velocity.** Text Sec1.4: 3, 9

In this section, we see how limits arise through two examples.

A **tangent** to a curve is a line that just touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact.

Q: In general, one can not say that a tangent is a line that touch the curve once and only once. Why?

Example 1.1. *Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.*



Proof. [Solution] It's enough to find the slope of the tangent line, denoted by m.

Let's take a point $Q(x, x^2)$ on the curve. We note that, if Q is very close to P, then

$$m_{PQ} \approx m$$

where

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

is the slope of PQ.

Moreover, as $Q \to P$, we have $m_{PQ} \to m$, and we write it as

$$\lim_{Q \to P} m_{PQ} = m$$

or

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = m.$$

Using calculator, we can guess m by taking a series of $x \to 1$ (x is approaching 1). The answer turns out to be ? $\square$

Next, we see how to find velocity from given distance function.

Ex. Suppose a car is moving on a straight line from starting from O to the right. The distance of the car from O after time t seconds is given by

$$s(t) = 4.9t^2 meter.$$

Find the **instantaneous velocity** of the car at $t = 5$ seconds.

Solution. The velocity is a function of t, we denote it by $v(t)$. We want to find $v(5)$.

Approximation of $v(5)$ by **average velocity** in time periods $[5, t]$:

$$v(5) \approx \frac{s(t) - s(5)}{t - 5}, \text{ if } t \text{ is close enough to } 5$$

Limit as $t \to 5$:

$$v(5) = \lim_{t \to 5} \frac{s(t) - s(5)}{t - 5}.$$

Guess what is $v(5)$ by taking a sequence of $t \to 5$? $\square$

1.2. **Limit of a function.** Text Sec1.5: 5, 6, 7, 9, 15, 17, 31, 46

First, we give *not a precise* definition of limit and vertical asymptotes, and try to understand the limit better through several examples.

Definition 1.1. *Limit and one-sided limits are*

(1) *If we can make the values of $f(x)$ arbitrarily close to L by taking $x \neq a$ to be sufficiently close to a (on either side of a), then we write*

$$\lim_{x \to a} f(x) = L.$$

(2) *If we can make the values of $f(x)$ arbitrarily close to L by taking $x < a$ to be sufficiently close to a (on left-hand side of a), then we write*

$$\lim_{x\to a^-} f(x) = L.$$

(3) *If we can make the values of $f(x)$ arbitrarily close to L by taking $x > a$ to be sufficiently close to a (on right-hand side of a), then we write*

$$\lim_{x\to a^+} f(x) = L.$$

From the above definition, we conclude

$$\lim_{x\to a} f(x) = L$$

if and only if

$$\lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.$$

Note. If you observe $f(x) \to L$ by taking a sequence of $x \to a$, then we guess $\lim\limits_{x\to a} f(x) = L$. But, this is only guess. In fact, to prove $\lim\limits_{x\to a} f(x) = L$, you **must** show for any sequence of $x \to a$, $f(x) \to L$ is true. For this purpose, we will later give a mathemtaically precise definition.

Ex. Continued from tangent problem Example 1.1, we need to find

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$

for the slope m.

(1) By taking $x \to 1^+$, we find

$$\frac{x^2-1}{x-1} \to 2,$$

So

$$\lim_{x\to 1^+} \frac{x^2-1}{x-1} = 2.$$

(2) By taking $x \to 1^-$, we also find

$$\frac{x^2-1}{x-1} \to 2,$$

So

$$\lim_{x\to 1^-} \frac{x^2-1}{x-1} = 2.$$

(3) From both sides of $x \to 1$,

$$\frac{x^2-1}{x-1} \to 2,$$

So

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2.$$

Q Graph

$$g(x) = \frac{x^2 - 1}{x - 1}.$$

Ex. Guess

$$\lim_{x \to 0} \sin \frac{\pi}{x}.$$

Ex. Graph $f(x) = [x]$, and find $\lim_{x \to 1} f(x)$.

Ex. The signum (sign) function is defined by

$$sgn(x) = \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ 1 & \text{if } x > 0; \end{cases}$$

Graph this function, and find $\lim_{x \to 0} sgn(x)$.

Ex. Let A be a set of real numbers. Indicator function $I_A(x)$ is piecewisely defined as

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}.$$

Graph functions $I_{[0,1)}(x)$, find

$$\lim_{x \to 0} I_{[0,1)}(x), \text{ and } \lim_{x \to 1} I_{[0,1)}(x).$$

Ex. Show that

$$\lim_{x \to 0} \frac{1}{x^2} = DNE \text{ (Does not exist.)}$$

Definition 1.2. *Infinite limits are defined as follows: As taking $x \to a^-$,*

(1) *if $f(x)$ can be arbitrarily large, then we write*

$$\lim_{x\to a^-} f(x) = \infty.$$

(2) *if $f(x)$ can be arbitrarily small, then we write*

$$\lim_{x\to a^-} f(x) = -\infty.$$

Similarly we can define

$$\lim_{x\to a^+} f(x) = \infty \text{ and } \lim_{x\to a^+} f(x) = -\infty.$$

Moreover, we say

$$\lim_{x\to a} f(x) = \infty \text{ iff } \lim_{x\to a^\pm} f(x) = \infty$$

and

$$\lim_{x\to a} f(x) = -\infty \text{ iff } \lim_{x\to a^\pm} f(x) = -\infty$$

Note.
$\lim_{x\to a} f(x) = \infty$ is a special case of $\lim_{x\to a} f(x) = DNE$.

Definition 1.3. *The line $x = a$ is called a* **vertical asymptote** *of $y = f(x)$ if one of the followings is true:*

$$\lim_{x\to a^+} f(x) = \infty, \quad \lim_{x\to a^-} f(x) = \infty,$$
$$\lim_{x\to a^+} f(x) = -\infty, \quad \lim_{x\to a^-} f(x) = -\infty.$$

Ex. Find vertical asymptotes of

$$f(x) = \tan(x).$$

1.3. **The precise definition of a limit.** Text Section 1.7: Exercise 3, **13**, 17, 25, 31, 37, **39, 41, 42**, 43, 44

Ex. Let $f(x) = 2x$. We know $\lim_{x\to 1} f(x) = 2$. Given $\varepsilon = 0.2$, find a number $\delta > 0$ such that

$$\text{if } 0 < |x-1| < \delta \text{ then } |f(x) - 2| < \varepsilon.$$

Definition 1.4. *A limit is defined as*

(1) $\lim_{x\to a} f(x) = L$, *if for every* $\varepsilon > 0$, *there is a number* $\delta > 0$, *such that*
$$\text{if } 0 < |x-a| < \delta \text{ then } |f(x) - L| < \varepsilon.$$
(2) $\lim_{x\to a^-} f(x) = L$, *if for every* $\varepsilon > 0$, *there is a number* $\delta > 0$, *such that*
$$\text{if } a - \delta < x < a \text{ then } |f(x) - L| < \varepsilon.$$
(3) $\lim_{x\to a^+} f(x) = L$, *if for every* $\varepsilon > 0$, *there is a number* $\delta > 0$, *such that*
$$\text{if } a < x < a + \delta \text{ then } |f(x) - L| < \varepsilon.$$
(4) $\lim_{x\to a} f(x) = \infty$, *if for every* $M > 0$, *there is a number* $\delta > 0$, *such that*
$$\text{if } 0 < |x-a| < \delta \text{ then } f(x) > M.$$

Ex. Prove by definition $\lim_{x\to 1} 2x = 2$.

Ex. Prove by definition
$$\lim_{x\to 0^\pm} \frac{1}{x} = \pm\infty.$$

Ex. Prove by definition $\lim\limits_{x\to 0^+} \sin(\pi/x) = DNE$.

1.4. **Calculating limits with limit laws.** Text Sec1.6: 6, **15**, 18, **19, 26**, 29, **36**, 37, **38,** 39, **40, 41, 42**, **46, 47**, 51, **54**, **59, 62**, 63

Limit Laws. Suppose c is a constant, and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists. Then

(1) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
(2) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
(3) $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
(4) $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
(5) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$.
(6) $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$ for positive integer n
(11) $\lim_{x\to a}[f(x)]^{1/n} = [\lim_{x\to a} f(x)]^{1/n}$ for positive integer n, if both sides are well defined.

Proposition 1.5 (Substitution property)**.** *Suppose f is a combination of the following functions: algebraic, exp, log, trig. Then*

$$\lim_{x\to a} f(x) = f(a), \ \textit{if } f(a) \textit{ is well defined.}$$

Note. Sub property leads to mistakes, typically when $f(a)$ is one of the followings:

$$\frac{0}{0}; \quad \infty - \infty; \quad 1^\infty; \quad 0\cdot\infty, ...$$

Proposition 1.6. *If $f(x) = g(x)$ for every $x \neq a$ in some neighborhood of a, then*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

provided that both limits exist.

Ex. Find $\lim_{x\to -2} \dfrac{x^3+2x^2-1}{5-3x}$.

Followings are some examples of type $\frac{0}{0}$.

Ex. Find $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$, arisen from Example 1.1.

Ex. Find

$$\lim_{t\to 0} \frac{t^2}{\sqrt{t^2+9}-3}.$$

Ex. Find

$$\lim_{x\to 0} \frac{|x|}{x}.$$

Next example is of type $\infty - \infty$.
Ex. Find

$$\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2+t}\right).$$

Next example shows the type of 1^∞
Ex. Natural number is defined as $e = \lim_{x\to 0}(1+x)^{1/x}$. Estimate e by your calculator.

Ex. Can you find two functions f and g, such that $\lim_{x\to 0} f(x)g(x) \neq 0$ while $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = \infty$.

Proposition 1.7 (Comparison result)**.** *If $f(x) \leq g(x)$ in some nbd of a ($x \neq a$), then*

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

provided that both limits exist.

Q. Justify following statement: Let $f(x) < g(x)$, then

$$\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$$

provided that both limits exist.

A straightforward application of comparison result gives

Proposition 1.8 (Squeeze theorem)**.** *If $f(x) \leq g(x) \leq h(x)$ in some nbd of a ($x \neq a$), and*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L.$$

Ex. $\lim\limits_{x\to 0} x^2 \sin(1/x) =?$

1.5. **Continuity.** Text Section 1.8, Exercise 3, 9, **12, 14, 15, 19, 20, 36, 38, 39,** 41, 43, **44, 45, 46, 49, 50, 51, 53,** 59, 60, 63, **65, 66**

Definition 1.9 (Continuity at a point)**.** *Given a function f,*

(1) *f is continuous at a, if $\lim\limits_{x\to a} f(x) = f(a)$. Otherwise, f is said to be discontinuous at a.*
(2) *f is left continuous at a, if $\lim\limits_{x\to a^-} f(x) = f(a)$.*
(3) *f is right continuous at a, if $\lim\limits_{x\to a^+} f(x) = f(a)$.*

Ex. Find all discontinuities of following functions, also identify right/left continuities at discontinuities.

(1) (Jump discontinuity) $f(x) = [x]$
(2) (Removable discontinuity) $f(x) = \dfrac{x^2-1}{x-1}$
(3) (Infinite discontinuity) $f(x) = \dfrac{1}{x^2}$

Ex. **(Funny)** Can you find a function,

- which is nowhere continuous? (Hint: $f = I_Q$, see Exercise 63 of Section 1.8 of Text.)
- which is continuous only at $x = 0$? (Hint: $f(x) = xI_Q(x)$)

Definition 1.10 (Continuity on an interval)**.** *A function f is continuous on an interval, if it is continuous at every number inside the interval, and either right/left continuous at the endpoints.*

Ex. Let $f(x) = \dfrac{x^2-1}{x-1}$. In which interval is f continuous?

$$(-\infty, \infty), \quad [0,2], \quad [0,1], \quad (0,1).$$

By definition, if f is continuous at a, then $\lim_{x\to a} f(x) = f(a)$. In other words, at all continuity points, we can use substitution property to obtain limit at a. (Compare with Proposition 1.5).

Now the question is: what functions are continuous in general?

Proposition 1.11. *If two functions f and g are both continuous at a, then their combinations are also continuous at a, i.e.*

$$f+g, \quad f-g, \quad cf, \quad fg, \quad f/g \text{ if } g(a) \neq 0$$

are all continuous at a. (In the above, c is constant.)

Proposition 1.12. *All algebraic, exp, log, and trig. functions are continuous in their own domain.*

Note. The substitution property Proposition 1.5 is now a direct result from the above two propositions.

Ex. Find $\displaystyle\lim_{x\to\pi} \frac{\sin x}{2+\cos x}$.

Proposition 1.13. *If* $\lim_{x\to a} g(x) = b$ *and* $\lim_{x\to b} f(x) = f(b)$, *then*

$$\lim_{x\to a} f(g(x)) = \lim_{x\to b} f(x) = f(b).$$

Ex. find $\lim\limits_{x\to 1} \sin \dfrac{(x^2-1)\pi}{x-1}$.

Next proposition follows directly from Proposition 1.13.

Proposition 1.14 (Text Theorem 2.5.9)**.** *If g is continuous at a and f is continuous at $g(a)$, then the composition $f \circ g$ is continuous at a.*

Proof. skip □

Next theorem is called **Intermediate Value Theorem**.

Theorem 1.15 (IVT)**.** *Suppose f is continuous on $[a, b]$, and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there exists a number c in (a, b) s.t. $f(c) = N$.*

Ex. Show that there is a root of the equation

$$4x^3 - 6x^2 = -3x + 2$$

between 1 and 2.

Ex. Bob is leaving from A at 7:00am to B. The next morning, he returns from B at 7:00am, taking the same route to A. Show that there is a point on the path that Bob will cross at exactly the same time of day on both days.

More exercises for Chapter 1: Text p.95-p.96, 5-8, 13, 16, 19, **26, 28, 32, 34, 37, 40, 45, 46**