CITY UNIVERSITY OF HONG KONG

Department of Mathematics

Course Code & Title	:	MA1301 Enhanced Calculus and Linear Algebra II
Session	:	Semester B, 2019-2020
Time Allowed	:	Three Hours

This paper has **Four** pages (including this cover page).

Instructions to candidates:

1. This paper has **ten** questions.
2. Answer **ALL** questions.
3. Start each main question on a new page.
4. Show all steps.
5. Submit online (PDF file) in Canvas/MA1301/Assignments.
6. Attach academic honesty pledge with your solutions.
7. Departmental hotline for online exams: 3442-8646.

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*This is an **open-book** examination.*

Candidates are allowed to use the following materials/aids:

Lecture notes, Self-prepared notes, Reference books, Reference eBooks, Non-programmable portable battery operated calculator.

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorized materials or aids are found on them.

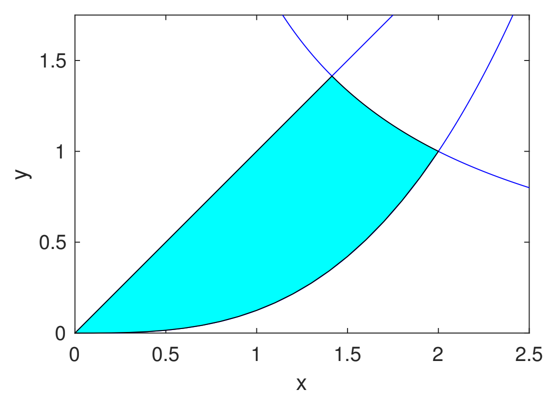
1. Calculate the following limits:

   (a) [8 marks] $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \ln\left(\frac{n\sqrt{k}+k^{3/2}-1}{n\sqrt{k}}\right)$

   (b) [7 marks] $\lim_{x\to 0} \frac{1}{x}\int_{e^{-x}}^{1} \sin(2+t^2)dt$

2. [10 marks] Evaluate the integral $\int_1^5 \frac{dx}{(x^2-2x+5)^2}$.

3. [10 marks] Find the area of the region enclosed by the curves $y = x^3/8$, $y = x$ and $y = 2/x$, as shown in the figure below.



4. For integer $n \geq 0$, let $I_n = \int_0^{+\infty} x^n e^{-x^2} dx$. It is known that $I_0 = \sqrt{\pi}/2$.

   (a) [7 marks] Prove that $I_{n+2} = \frac{n+1}{2} I_n$

   (b) [3 marks] Find I_1, I_2, I_3 and I_4.

5. [10 marks] Determine whether the improper integral below is convergent or divergent. Prove your result.

$$\int_2^{+\infty} \frac{dx}{x(\ln x)^2 + 77\sin^2 x}.$$

6. [5 marks] Find all complex z satisfying $z^3 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$.

7. [10 marks] Let A be the following 3×4 matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 1/2 & 1 & 1/2 \\ 2 & 2 & 1/2 & 3/2 \end{bmatrix}.$$

   (a) [7 marks] Transform A to its reduced row-echelon form by elementary row operations, and write down the elementary matrices corresponding to elementary row operations.

   (b) [3 marks] Solve the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x}$ is a column vector of the unknowns.

8. [10 marks] Let B and C be 3×3 matrices with some known zero entries,

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & 0 \end{bmatrix},$$

   and b_{11}, b_{23}, c_{11}, c_{22} are all nonzero. Let A be a 3×3 matrix satisfying $AB = C$. Prove that there is a nonzero column vector $\mathbf{x}$, such that $A\mathbf{x} = \mathbf{0}$.

9. [10 marks] Let $\{a_n\}$ be a sequence of real numbers, $\{A_1, A_2, A_3, \cdots\}$ be a sequence of matrices given by $A_1 = [a_1]$, $A_2 = \begin{bmatrix} a_1 & 1 \\ 1 & a_2 \end{bmatrix}$, and

$$A_n = \begin{bmatrix} a_1 & 1 & & & & \\ 1 & a_2 & 2 & & & \\ & \frac{1}{2} & a_3 & 3 & & \\ & & \frac{1}{3} & a_4 & \ddots & \\ & & & \ddots & \ddots & n-1 \\ & & & & \frac{1}{n-1} & a_n \end{bmatrix}, \quad \text{for } n \geq 3,$$

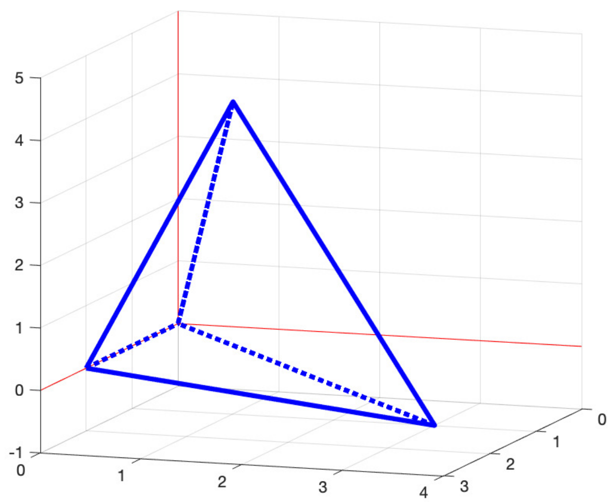
   where all missing entries are 0. Prove that for $n \geq 3$

$$\det(A_n) = a_n \det(A_{n-1}) - \det(A_{n-2}).$$

10. Consider four points $A = (0, 0, 0)$, $B = (2, 0, 0)$, $C = (1, 3, -1)$ and $D = (1, 1, 4)$ in $\mathbb{R}^3$.

    (a) [5 marks] Find a unit vector perpendicular to the plane W that contains the three points A, B and C.

    (b) [5 marks] Find the volume of tetrahedron $ABCD$, as shown in the figure below.



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