Chapter 3 Applications of Differentiation

Contents

3. Applications of Differentiation

So far, we have learned

- limit
- derivative

This section is to apply derivatives in the following applications: curve sketching, optimization.

3.1. **Maximum/Minimum.** Text Section 3.1 Exercise: **1, 2,** 5, **9, 10, 11(a,b), 34, 39, 40, 47, 54, 56, 57, 63, 69, 72**
In this section, we answer how to *find a Maximum/Minimum of a function.* It is important both in curve sketching and optimization.

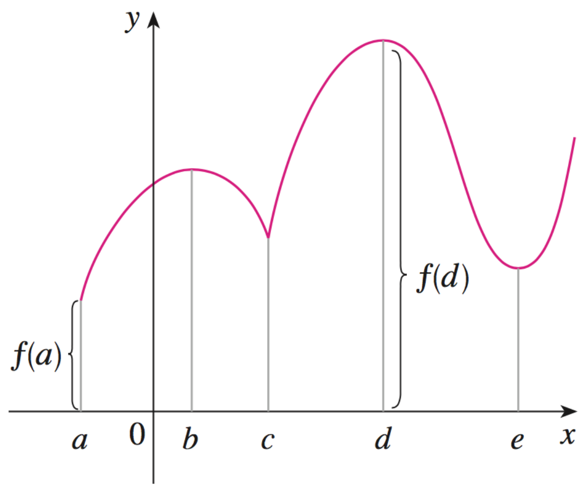
Definition. Given a function $f : D \to \mathbb{R}$ and c is a number in D.

(1) f has an **absolute maximum** (or global maximum) at c if $f(c) \geq f(x)$ for all x in D, and
$$f(c) = \max_{x \in D} f(x)$$
is called the **maximum value** of f on D.
(2) f has an **absolute minimum** (or global minimum) at c if $f(c) \leq f(x)$ for all x in D, and
$$f(c) = \min_{x \in D} f(x)$$
is called the **minimum value** of f on D.
(3) The maximum and minimum values of f are called the **extreme values** of f.
(4) f has a **local maximum** (or relative maximum) at c if $f(c) \geq f(x)$ in some neighborhood (small open interval containing c) of x in D.
(5) f has a **local minimum** (or relative minimum) at c if $f(c) \leq f(x)$ in some neighborhood (small open interval containing c) of x in D.

Note. a local max/min for f in an interval can not be an endpoint by definition.
To find extreme value of f, we need to answer two question:

(1) existence of minimum/maximum

(2) if yes, how to identify minimum/maximum



In the figure above, absolute minimum value is $f(a)$; absolute maximum is $f(d)$; local minimum values are $f(c)$ and $f(e)$; local maximum values are $f(b)$ and $f(d)$.

Ex. Identify all absolute/local maximum/minimums in the following functions.

(1) $f(x) = 1$

(2) $f(x) = x^3$

(3) $f(x) = x^2$ on $D = [-1, 1]$

(4) $f(x) = x^2$ on $D = (-1, 1)$

(5) $f(x) = x(x-1)^2$

From above example, we know maximum/minimum does not exist sometimes. The Extreme Value Theorem (EVT) answers "when it guarantees its existence", i.e. EVT gives sufficient condition for the existence of maximum/minimum.

Theorem 3.1 (The Extreme Value Theorem)**.** *If f is* **continuous** *on a* **closed** *interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c, d \in [a, b]$.*

Ex.
(1) Can you find a counter example of EVT if the continuity is removed?
(2) Can you find a counter example of EVT if the closedness is removed?

Now, we turn to answer how to identify an extreme value.

Theorem 3.2 (Fermat's Theorem)**.** *If f has a local max/min at c, and if $f'(c)$ exists, then $f'(c) = 0$.*

Proof. see P207. □

Given continuous f,
(1) Fermat's theorem is not sufficient condition for local max/min,
Ex. $f(x) = x^3$. $f'(0) = 0$. But $f(0)$ is not local max/min.
(2) Fermat's theorem is not necessary condition for local max/min,
Ex. $f(x) = |x|$. $f(0)$ is min, but $f'(0) \neq 0$.

Proposition 3.3 (Necessary condition for local max/min)**.** *If f has a local max/min at c, then c must be a* **critical number** ($f'(c) =$ either 0 or DNE) .

Finally,
Strategy. To find absolute max/min of continuous f in $[a, b]$, compare all functions values at

critical numbers and **end points**.

Ex. Find aboslute max/min of $f(x) = x^3 - 3x^2 + 1$ in $[-1/2, 4]$.

3.2. **The Mean Value Theorem.** Section 3.2 Exercise: **5**, 17, **18, 19, 21, 23, 27**, 29, **34**

Theorem 3.4 (Rolle's Theorem)**.** *Let function f satisfy*
(1) *f is continuous on $[a, b]$*
(2) *f is differentiable on (a, b)*
(3) *$f(a) = f(b)$*
Then, there is a number c in (a, b) such that $f'(c) = 0$.

Proof. see Page 208 □

Ex. Prove that $x^3 + x - 1 = 0$ has exactly one real root.

Following MVT is generalization of Rolle's Theorem.

Theorem 3.5 (Mean Value Theorem). *Let function f satisfy*

(1) *f is continuous on $[a, b]$*
(2) *f is differentiable in (a, b)*

Then, there is $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. see page 210 □

Ex. If a car traveled 100km in 2 hours, then show that the speedometer must have read 50 km/h at least once.

Ex. Let $f(0) = -3$, $f'(x) \leq 5$ for all x. How large can $f(2)$ possibly be?

Ex. Prove following statements:

(1) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).
(2) If $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists constant c such that $f(x) = g(x) + c$ on (a, b).

Proof. See pages 211-212. □

Ex. Prove $|\sin a - \sin b| \leq |a - b|$ for all a, b.

3.3. **From derivatives to properties of graph.** Section 3.3 Exercise: 1, **11, 14,** 15, 17, **26,** 37, **53, 59, 61, 62,** 65, 67
In this section, we study how we achieve useful information of graph from f' and f'', those are

- Increasing/Decreasing test
- The first derivative test
- Concavity test
- The second derivative test

Q. What is definition of Increasing/Decreasing interval of f?
Increasing/Decreasing Test.

(1) If $f'(x) > 0$ on an interval, then f is increasing on that interval.
(2) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Proof. See Page 214 of Text. □

Ex. Identify I/D intervals for

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5 \tag{3.1}$$

First Derivative Test. Suppose c is a critical number of continuous f,

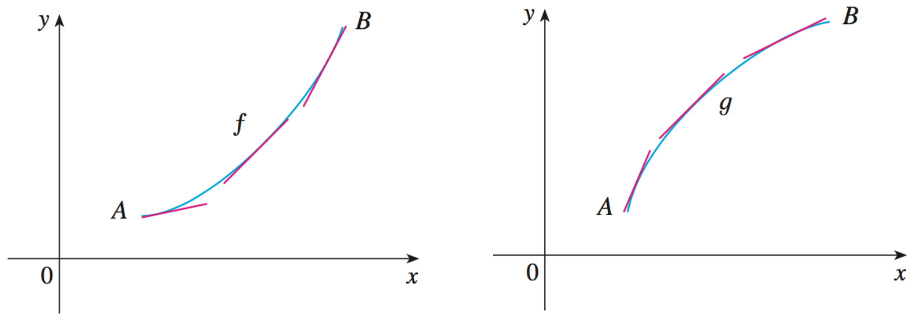
(1) If $f' : + \to -$ at c, then f has a local max at c.
(2) If $f' : - \to +$ at c, then f has a local min at c.
(3) If f' does not change sign at c, then f has no local max/min.

Ex. Find local max/min for f of (3.1).

Definition. If the graph of f lies above (below) all of its tangents on interval I, then it is called **concave upward** (**concave downward**) on I.

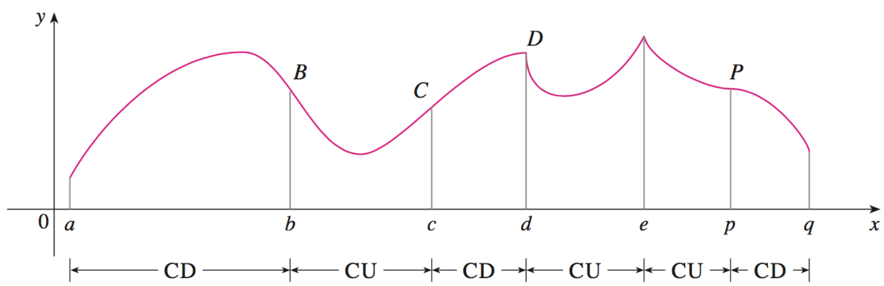
Concavity Test.

(1) If $f'' > 0$ for all x in I, then f is CU on I.
(2) If $f'' < 0$ for all x in I, then f is CD on I.



(a) Concave upward

(b) Concave downward



Definition. A point P of $y = f(x)$ is called **inflection point** if f is continuous at P and concavity is changing at P (either from CU to CD or CD to CU).

Ex. Discuss intervals of I/D and CU/CD of

$$y = x^4 - 4x^3. \tag{3.2}$$

The second derivative test. Suppose f'' is continuous near c,

(1) If $f'(c) = 0$ and $f''(c) > 0$, then f has local min at c.
(2) If $f'(c) = 0$ and $f''(c) < 0$, then f has local max at c.

Ex. Find local max/min of (3.2), using The First/Second derivative tests separately.

Note. Both *first/second derivative tests* gives local max/min. But FDT is more powerful than SDT.

Ex. Find minimum for $f = |x|$. (FDT works here but not SDT)

3.4. **Limits at infinity: Horizontal asymptotes.** Section 3.4 Exercise: **13, 19,** 25, 29, 35, 41, **45**, 55, 57, 59, 69

First, we show the precise definitions:
Definition. Let f be a given function. Then

(1) $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there is a corresponding number N s.t.
$$\text{if } x > N \text{ then } |f(x) - L| < \varepsilon.$$
(2) $\lim_{x\to-\infty} f(x) = L$ if for every $\varepsilon > 0$ there is a corresponding number N s.t.
$$\text{if } x < -N \text{ then } |f(x) - L| < \varepsilon.$$
(3) $\lim_{x\to\infty} f(x) = \infty$ if for every positive M there is a corresponding number N s.t.
$$\text{if } x > N \text{ then } f(x) > M.$$
(4) $\lim_{x\to-\infty} f(x) = \infty$ if for every positive M there is a corresponding number N s.t.
$$\text{if } x < -N \text{ then } f(x) > M.$$

Explanations.

(1) $\lim_{x\to\pm\infty} f(x) = L$ means that $f(x)$ can be made arbitrarily close to L by taking x sufficiently large (small).
(2) $\lim_{x\to\pm\infty} f(x) = \infty$ means that $f(x)$ can be made arbitrarily large by taking x sufficiently large (small).
(3) Similarly we define $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

Ex. Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.
Ex. Find $\lim_{x\to\infty} x^3$, $\lim_{x\to-\infty} x^3$, and $\lim_{x\to\infty} x^2$.

Proposition 3.6. *Let $r > 0$ be a rational number.*

(1)
$$\lim_{x\to\infty} x^{-r} = 0.$$

(2) *If x^r is well defined, then*
$$\lim_{x\to-\infty} x^{-r} = 0.$$

(3)
$$\lim_{x\to\infty} x^{r} = \infty.$$

Ex. Justify following statement:
Let $r > 0$ be rational number, and x^r be well defined, then
$$\lim_{x\to-\infty} x^{r} = -\infty.$$

Strategy. Consider $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$, where
$$P(x) = a_n x^n + \cdots + a_0, \text{ and } Q(x) = b_m x^m + \cdots + b_0$$
are polynomials of degree n and m.

Usually, we first try to factor out the highest order term, i.e.
$$P(x) = a_n x^n (1 + \cdots + a_0 x^{-n}),\ Q(x) = b_m x^m (1 + \cdots + b_0 x^{-m}).$$

Ex. Compute limit for rational functions:

(1) $\lim_{x\to\infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$
(2) $\lim_{x\to\infty} \dfrac{x^2 + x}{3 - x}$.
(3) $\lim_{x\to\infty} x^2 - x$.

Ex. Compute limit:

(1) $\lim_{x\to\infty}(\sqrt{x^2+1}-x)$.

(2) $\lim\limits_{x\to\infty} \sin \dfrac{1}{x}$.

Definition The line $y = L$ is called a **horizontal asymptote** of $y = f(x)$ if

$$\text{either } \lim_{x\to\infty} f(x) = L, \text{ or } \lim_{x\to-\infty} f(x) = L.$$

Ex. Find asymptotes of

$$f(x) = \frac{2x^2+1}{3x-5}.$$

3.5. **Curve sketching.** Section 3.5 Exercise: 5, 17, **27**, 33, **41, 44,** 45, 47

Strategy. Given $y = f(x)$, to sketch the graph, we need to do

(1) Find **Domain** D.
(2) Find x- and y- **intercepts**.
(3) Identify **symmetry**: Is f even/odd/periodic function?
(4) Find horizontal/vertical **asymptotes**.
(5) Identify **I/D intervals** by I/D test.
(6) Identify **local max/min** by either FDT (better) or SDT.
(7) Identify **concavity** and **IP** by Concavity Test.
(8) Sketch the graph.

Ex. Sketch $y = \dfrac{2x^2}{x^2 - 1}$.

Ex. Sketch $y = \dfrac{\cos x}{2 + \sin x}$.

Definition. A **slant asymptote** of $y = f(x)$ is $y = mx + b$ if

$$\lim_{x\to\infty}[f(x) - (mx + b)] = 0 \text{ or } \lim_{x\to-\infty}[f(x) - (mx + b)] = 0.$$

Note. If $f(x) = \dfrac{P(x)}{Q(x)}$ where P and Q are polynomials of deg(P) = deg(Q) + 1, then there is a slant asymptote.

Ex. Sketch $f(x) = \dfrac{x^3}{x^2 + 1}$.

3.6. **Optimization.** Section 3.7 Exercise: **9, 10,** 12, 17, 22, 33, **43,** 53, 58, **67**

Strategy. To solve for optimization problem,

(1) We need to **model** the problem by $y = f(x)$.
(2) The desired quantity is to find **absolute max/min** in **some interval** D, by doing followings:
   (a) FDT or SDT
   (b) compare all critical points and end points

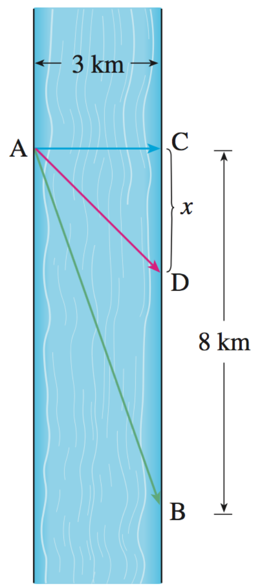
Ex. A farmer has 1200 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions that has the largest area?

Ex. A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to make the can.

Ex. Find the point on $y^2 = 2x$ that is closest to $(1, 4)$.

Ex. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Ex. (see the following figure) A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8km/h, where should he land to reach B as soon as possible? (assume no water speed)



Applications to Business and Economics.

(1) Recall that if $C(x)$, the **cost function**, is the total cost of producing x units of a certain product. **Marginal cost** is $C'(x)$, the rate of change of C w.r.t. x.
(2) Let $p(x)$ be the price per unit that company can charge if it sells x units. Then p is called **demand function** (or **price function**). Usually $p(x)$ *is decreasing function in* x.
(3) If x units are sold and the price per unit is $p(x)$, then the total revenue is

$$R(x) = xp(x).$$

$R(x)$ is called the **Revenue function**. The **marginal revenue function** is $R'(x)$, the rate of change of R w.r.t. x.

(4) If x units are sold, the profit is

$$P(x) = R(x) - C(x)$$

and P is called the **Profit function**. The **marginal profit function** is P', the rate of change in P w.r.t. x.

Ex. A store has been selling 200 DVD burners a week at 350 USD each. A market survey indicates that for each 10 dollar rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

Ex.

(1) Show that if the profit $P(x)$ is a maximum, then the marginal revenue equals the marginal cost.

(2) If $C(x) = 16000+500x-1.6x^2+0.04x^3$ is the cost function and $p(x) = 1700-7x$ is the demand function, find the production level that will maximize profit.