Chapter 5
Steady-State Sinusoidal Analysis

1. Sinusoidal signal.
2. Phasors and complex impedances.
3. Power for ac circuits.
4. AC circuits analysis: Ohm'a law, KCL, KVL, Nodal voltage analysis, Mesh-current analysis, Thévenin and Norton equivalent circuits.
5. Load impedances for maximum power transfer.

Sinusoidal Currents and Voltages

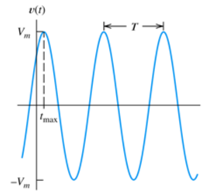

Figure 5.1 A sinusoidal voltage waveform given by $v(t) = V_m \cos(\omega t + \theta)$. Note: Assuming that θ is in degrees, we have $t_{max} = \frac{-\theta}{360} \times T$. For the waveform shown, θ is $-45°$.

V_m is the **peak value**
ω is the **angular frequency**
θ is the **phase angle**
T is the **period**

Relationships

Frequency $\quad f = \frac{1}{T}$

Angular frequency $\quad \omega = \frac{2\pi}{T}$

$$\omega = 2\pi f$$

$$\sin(z) = \cos(z - 90^\circ)$$

Root-Mean-Square Values

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt} \qquad I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt}$$

$$P_{\text{avg}} = \frac{\int_0^T p\,dt}{T} = \frac{\frac{1}{T}\int_0^T v^2 dt}{R} = \frac{V_{\text{rms}}^2}{R}$$

$$P_{\text{avg}} = I_{\text{rms}}^2 R$$

RMS Value of a Sinusoid

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}}$$

This is NOT true for other periodic waveforms such as square waves or triangular waves.

Example 5.1: Find rms value and average power

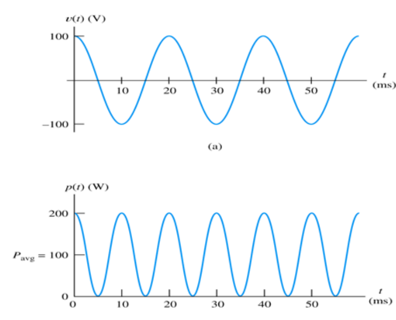

Figure 5.2 Voltage and power versus time for Example 5.1.

Phasor Definition

Time function: $v_1(t) = V_1 \cos(\omega t + \theta_1)$

Phasor: $\mathbf{V}_1 = V_1 \angle \theta_1$

- ω always in radians/sec, θ always in degrees
- Phasor: a complex number that represents the magnitude and phase of a sinusoid
- We will use $j = \sqrt{-1}$

Complex numbers review

Three representations of complex numbers:

$z = x + jy \qquad z = r\angle\phi \qquad z = re^{j\phi}$

where

$r = \sqrt{x^2 + y^2} \qquad \phi = \arctan(\frac{y}{x})$

Eular's identity:

$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$

Adding Sinusoids Using Phasors

Step 1: Determine the phasor for each term.

Step 2: Add the phasors using complex arithmetic.

Step 3: Convert the sum to polar form.

Step 4: Write the result as a time function.

Example 5.3

$$v_1(t) = 20\cos(\omega t - 45^\circ)$$

$$v_2(t) = 10\sin(\omega t + 60^\circ)$$

$$\begin{aligned}\mathbf{V}_s &= \mathbf{V}_1 + \mathbf{V}_2 \\ &= 20\angle -45^\circ + 10\angle -30^\circ \\ &= 14.14 - j14.14 + 8.660 - j5 \\ &= 23.06 - j19.14 \\ &= 29.97\angle -39.7^\circ\end{aligned}$$

Phase Relationships

To determine phase relationships from a phasor diagram, consider the phasors to rotate counterclockwise. Then when standing at a fixed point, if $\mathbf{V}_1$ arrives first followed by $\mathbf{V}_2$ after a rotation of θ, we say that $\mathbf{V}_1$ leads $\mathbf{V}_2$ by θ. Alternatively, we could say that $\mathbf{V}_2$ lags $\mathbf{V}_1$ by θ.

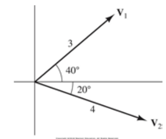

Figure 5.6 Because the vectors rotate counterclockwise, $\mathbf{V}_1$ leads $\mathbf{V}_2$ by 60° (or, equivalently, $\mathbf{V}_2$ lags $\mathbf{V}_1$ by 60°).

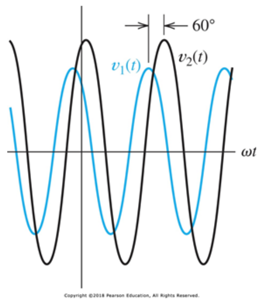

Figure 5.7 The peaks of $v_1(t)$ occur 60° before the peaks of $v_2(t)$. In other words, $v_1(t)$ leads $v_2(t)$ by 60°.

Circuit Analysis

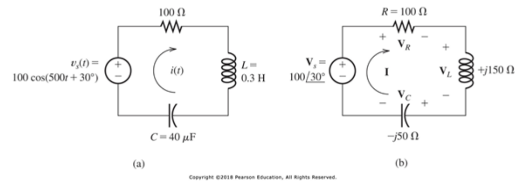

Figure 5.13 Circuit for Example 5.5.

Complex impedance: Inductor

$$i_L = I_m \sin(\omega t + \theta) \qquad v_L = L\frac{di_L}{dt} = \omega L I_m \cos(\omega t + \theta)$$

$$\mathbf{I}_L = I_m \angle \theta - 90^\circ \qquad \mathbf{V}_L = \omega L I_m \angle \theta = V_m \angle \theta$$

$$\mathbf{V}_L = (\omega L \angle 90^\circ) \times \mathbf{I}_L \qquad \mathbf{V}_L = j\omega L \times \mathbf{I}_L$$

$$Z_L = j\omega L = \omega L \angle 90^\circ$$

$$\mathbf{V}_L = Z_L \mathbf{I}_L$$

Complex impedance: Inductor

$$Z_L = j\omega L = \omega L \angle 90^\circ$$

$$\mathbf{V}_L = Z_L \mathbf{I}_L$$

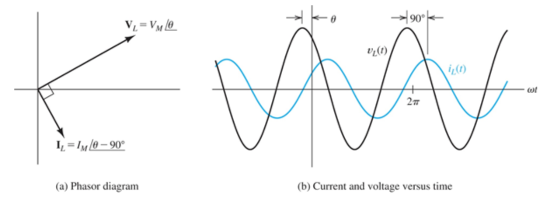

Figure 5.8 Current lags voltage by 90° in a pure inductance.

Complex impedance: Capacitor

$$v_C = V_m \sin(\omega t + \theta)$$

$$i_C = C\frac{dv_C}{dt} = \omega C V_m \cos(\omega t + \theta)$$

$$\mathbf{V}_C = V_m \angle \theta - 90^\circ, \quad \mathbf{I}_C = \omega C V_m \angle \theta = I_m \angle \theta$$

$$\mathbf{I}_C = (\omega C \angle 90^\circ) \times \mathbf{V}_C \qquad \mathbf{V}_C = \frac{1}{\omega C} \angle -90^\circ \times \mathbf{I}_C$$

$$Z_C = \frac{1}{\omega C} \angle -90^\circ = -j\frac{1}{\omega C} = \frac{1}{j\omega C}$$

$$\mathbf{V}_C = Z_C \mathbf{I}_C$$

Complex impedance: Capacitor

$$Z_C = \frac{1}{j\omega C} = \frac{1}{\omega C} \angle -90^\circ$$

$$\mathbf{V}_C = Z_C \mathbf{I}_C$$

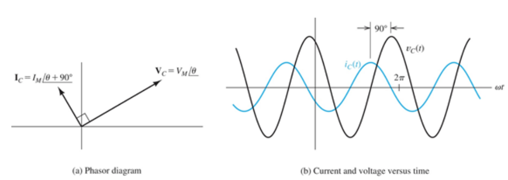

Figure 5.9 Current leads voltage by 90° in a pure capacitance.

Complex impedance: Resistor

$$Z = R \qquad \mathbf{V}_R = R\mathbf{I}_R$$

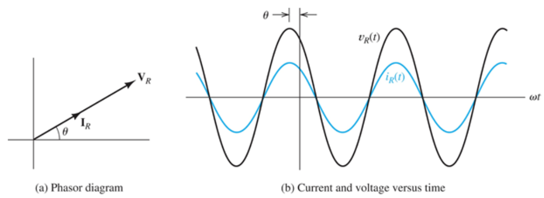

Figure 5.10 For a pure resistance, current and voltage are in phase.

Exercises 5.6-5.8

5.6

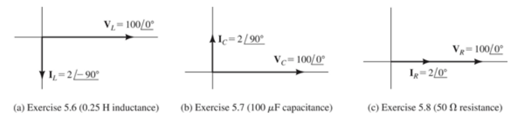

Figure 5.12 Answers for Exercises 5.6, 5.7, and 5.8. The scale has been expanded for the currents compared with the voltages so the current phasors can be easily seen.

Kirchhoff's Laws in Phasor Form

We can apply KVL directly to phasors. The sum of the phasor voltages equals zero for any closed path.

The sum of the phasor currents entering a node must equal the sum of the phasor currents leaving. (KCL)

Circuit Analysis Using Phasors and Impedances

1. Replace the time descriptions of the voltage and current sources with the corresponding phasors. (All of the sources must have the same frequency.)
2. Replace inductances by their complex impedances $Z_L = j\omega L$. Replace capacitances by their complex impedances $Z_C = 1/(j\omega C)$. Resistances have impedances equal to their resistances.
3. Analyze the circuit using any of the techniques studied earlier in Chapter 2, performing the calculations with complex arithmetic.

Example 5.5

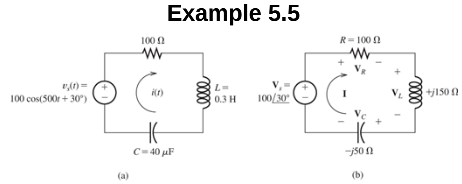

Figure 5.13 Circuit for Example 5.5.

$$Z_L = j\omega L = j150\Omega \qquad Z_C = -j\frac{1}{\omega C} = -j50\Omega$$

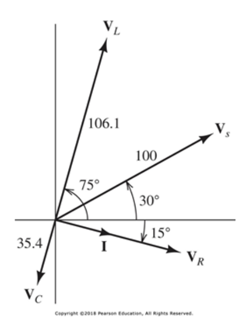

Figure 5.14 Phasor diagram for Example 5.5.

Example 5.6: Series/parallel combination

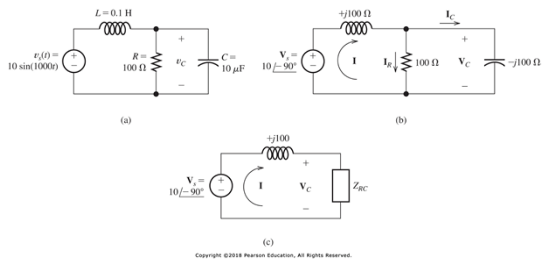

Figure 5.15 Circuit for Example 5.6.

$$Z_{RC} = R // Z_C = \frac{1}{1/R + 1/Z_c} = \frac{1}{1/100 + 1/(-j100)}$$

$$= \frac{1}{0.01 + j0.01} = \frac{1\angle 0°}{0.01414\angle 45°} = 70.7\angle -45° = 50 - j50$$

$$Vc = V_s \frac{Z_{RC}}{Z_L + Z_{RC}}$$

$$= 10\angle -90° \frac{70.71\angle -45°}{j100 + 50 - j50}$$

$$= 10\angle -90° \frac{70.71\angle -45°}{50 + j50}$$

$$= 10\angle -90° \frac{70.71\angle -45°}{70.71\angle 45°}$$

$$= 10\angle -180°$$

$$I_R = \frac{V_C}{R} = 0.1\angle -180°$$

$$I_C = \frac{V_C}{Z_C} = \frac{10\angle -180°}{-j100} = \frac{10\angle -180°}{100\angle -90°} = 0.1\angle -90°$$

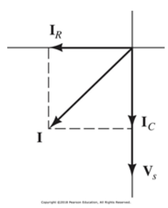

Figure 5.16 Phasor diagram for Example 5.6.

Example 5.7: Node voltage analysis

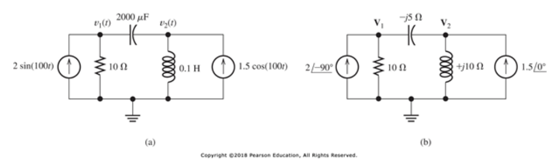

Figure 5.17 Circuit for Example 5.7.

$$(0.1 + j0.2)V_1 - j0.2V_2 = -j2$$
$$-j0.2V_1 + j0.1V_2 = 1.5$$

$$(0.1 - j0.2)V_1 = 3 - j2$$
$$V_1 = \frac{3 - j2}{0.1 - j0.2} = 16.1\angle 29.7°$$

$$v_1(t) = 16.1\cos(100t + 29.7°)$$

Exercise 5.11: Mesh Current analysis

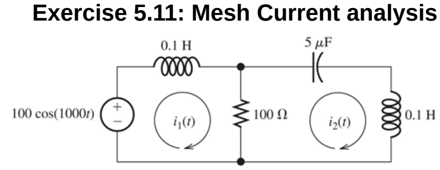

Figure 5.21 Circuit for Exercise 5.11.

$$\mathbf{I}_1 = 1.414\angle -45^\circ \text{ A} \quad \text{and} \quad \mathbf{I}_2 = 1\angle 0^\circ \text{ A.}$$

$$i_1(t) = 1.414\cos(1000t - 45^\circ) \text{ A} \quad \text{and} \quad i_2(t) = \cos(1000t)\text{A.}$$

Power in AC Circuits

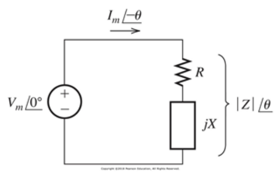

Figure 5.22 A voltage source delivering power to a load impedance $Z = R + jX$.

$$I = \frac{V}{Z} = \frac{V_m \angle 0°}{|Z| \angle \theta°} = I_m \angle -\theta° \qquad I_m = \frac{V_m}{|Z|}$$

Purely resistive circuits

$$p(t) = v(t)i(t)$$
$$= V_m I_m \cos^2(\omega t)$$

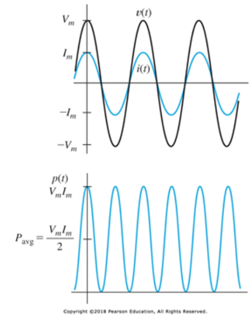

Figure 5.23 Current, voltage, and power versus time for a purely resistive load.

Pure energy-storage elements

$$p(t) = v(t)i(t)$$
$$= \pm V_m I_m \cos(\omega t)\sin(\omega t)$$

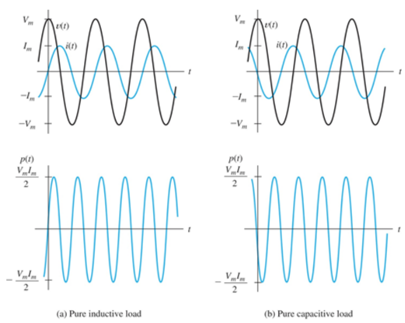

Figure 5.24 Current, voltage, and power versus time for pure energy-storage elements.

AC Power Calculations

Average power $\quad P = V_{\text{rms}} I_{\text{rms}} \cos(\theta) \quad W$

Power factor $\quad \text{PF} = \cos(\theta)$

Power angle $\quad \theta = \theta_v - \theta_i$

Reactive power $\quad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta) \quad VAR$

Apparent power $\quad \text{P}_\text{a} = V_{\text{rms}} I_{\text{rms}} \quad VA$

Relationships

$$P^2 + Q^2 = P_a^2 = (V_{\text{rms}} I_{\text{rms}})^2$$

$$P = I_{\text{rms}}^2 R \qquad P = \frac{V_{R\text{rms}}^2}{R}$$

$$Q = I_{\text{rms}}^2 X \qquad Q = \frac{V_{X\text{rms}}^2}{X}$$

Power Triangle

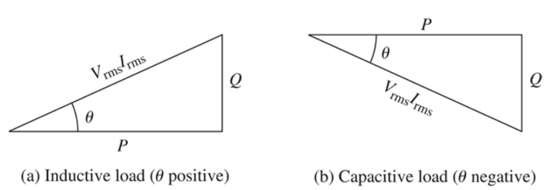

Figure 5.25 Power triangles for inductive and capacitive loads.

Impedance Triangle

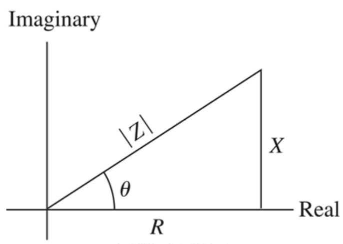

Figure 5.26 The load impedance in the complex plane.

Example 5.9

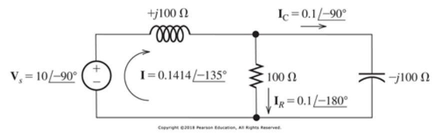

Figure 5.28 Circuit and currents for Example 5.9.

$$\theta = \theta_v - \theta_i = -90^\circ - (-135^\circ) = 45^\circ$$

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Example 5.9 - continued

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Example 5.10

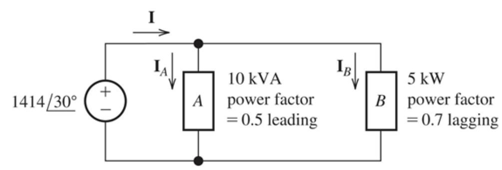

Figure 5.29 Circuit for Example 5.10.

Leading or Lagging: Current versus voltage

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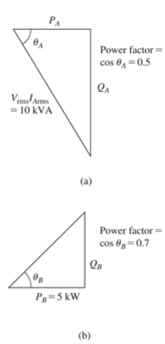

Figure 5.30 Power triangles for loads *A* and *B* of Example 5.10.

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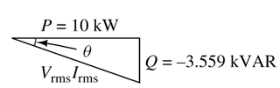
$$P = P_A + P_B = 10kW$$

$$Q = Q_A + Q_B = -3.559kVAR$$

$$\theta = \arctan(\frac{Q}{P}) = -19.59^\circ$$



Power triangle for the source of Example 5.10

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$$P_a = \sqrt{P^2 + Q^2} = 10.61kVA$$

$$I_{rms} = \frac{P_a}{V_{rms}} = \frac{10.61kVA}{1kV} = 10.61A$$

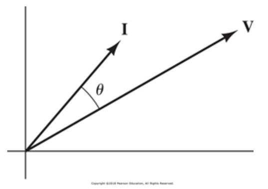

Figure 5.31 Phasor diagram for Example 5.10.

Example 5.11 Power factor correction

$$P = 50kW, \quad f = 60Hz, \quad V_{rms} = 10kV$$

$$PF = 0.6 \qquad \Rightarrow \theta_L = \arccos(0.6) = 53.13°$$

$$Q_L = P\tan(\theta_L) = 66.67kVAR$$

C=? (in Parallel) to make PF=0.9

Example 5.11 Power factor correction

After adding the capacitor

$$\theta_{new} = \arccos(0.9) = 25.84°$$

$$Q_{new} = P\tan(\theta_{new}) = 24.22kVAR$$

$$Q_C = Q_{new} - Q_L = -42.45kVAR$$

$$X_C = \frac{V_{rms}^2}{Q_C} = \frac{(10^4)^2}{-42450} = -2356$$

$$C = \frac{1}{\omega\,|X_C|} = 1.126\mu F$$

Thevenin equivalent circuits

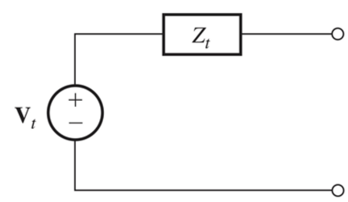

Figure 5.32 The Thévenin equivalent for an ac circuit consists of a phasor voltage source $\mathbf{V}_t$ in series with a complex impedance Z_t.

The Thévenin voltage is equal to the open-circuit phasor voltage of the original circuit.

$$\mathbf{V}_t = \mathbf{V}_{oc}$$

We can find the Thévenin impedance by zeroing the independent sources and determining the impedance looking into the circuit terminals.

The Thévenin impedance equals the open-circuit voltage divided by the short-circuit current.

$$Z_t = \frac{\mathbf{V}_{oc}}{\mathbf{I}_{sc}} = \frac{\mathbf{V}_t}{\mathbf{I}_{sc}}$$

$$\mathbf{I}_n = \mathbf{I}_{sc}$$

Norton equivalent circuits

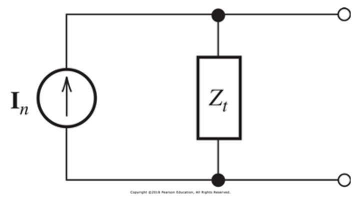

Figure 5.33 The Norton equivalent circuit consists of a phasor current source $\mathbf{I}_n$ in parallel with the complex impedance Z_t.

Example 5.12

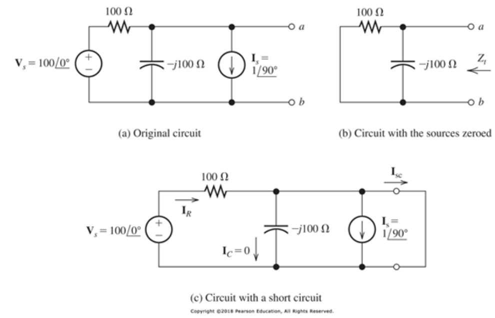

Figure 5.34 Circuit of Example 5.12.

$$Z_t = (100)//(-j100) = \frac{1}{0.01 + j0.01} = 50 - j50\Omega$$

$$I_{sc} = I_R - I_s = \frac{V_s}{100} - I_s = 1 - 1\angle 90° = 1 - j = 1.414\angle -45°$$

$$V_t = I_{sc} Z_t = 1.414\angle -45° \times 70.71\angle -45° = 100\angle -90°$$

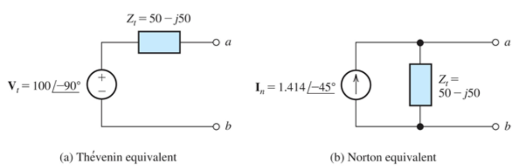

Figure 5.35 Thévenin and Norton equivalents for the circuit of Figure 5.34(a).

Maximum Power Transfer

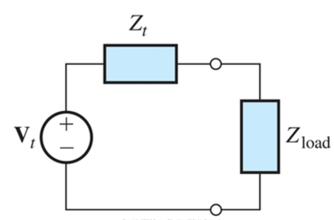

Figure 5.36 The Thévenin equivalent of a two-terminal circuit delivering power to a load impedance.

Maximum Power Transfer Conditions

If the load can take on any complex value, maximum power transfer is attained for a load impedance equal to the complex conjugate of the Thévenin impedance.

Why?

If the load is required to be a pure resistance, maximum power transfer is attained for a load resistance equal to the magnitude of the Thévenin impedance.

Example 5.13

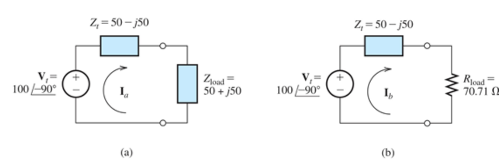

Figure 5.37 Thévenin equivalent circuit and loads of Example 5.13.

$$I = \frac{V_t}{Z_t + Z_{load}} = 1\angle -90°$$

$$P = I_{rms}^2 R_{load} = (\frac{1}{\sqrt{2}})^2 (50) = 25W$$

Example 5.13 (b)

$$R_{load} = |Z_t| = 70.71\Omega$$

$$I = \frac{V_t}{Z_t + Z_{load}} = \frac{100\angle -90^\circ}{50 - j50 + 70.71} = 0.76541\angle -67.5^\circ$$

$$P = I_{rms}^2 R_{load} = \left(\frac{0.7654}{\sqrt{2}}\right)^2 (70.71) = 20.71W$$

Exercise 5.14: find equivalent circuit

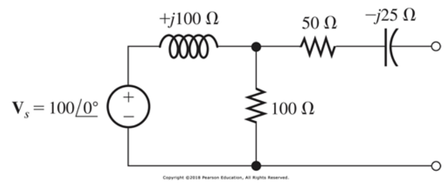

Figure 5.38 Circuit of Exercises 5.14 and 5.15.