Chapter 5. Joint distribution

5.1 Jointly distributed r.v.'s

(Joint) cumulative distribution function (cdf) of *X* and *Y* is defined as the bivariate function

$$F(a,b) = P\{X \leq a, Y \leq b\}, -\infty < a, b < \infty$$

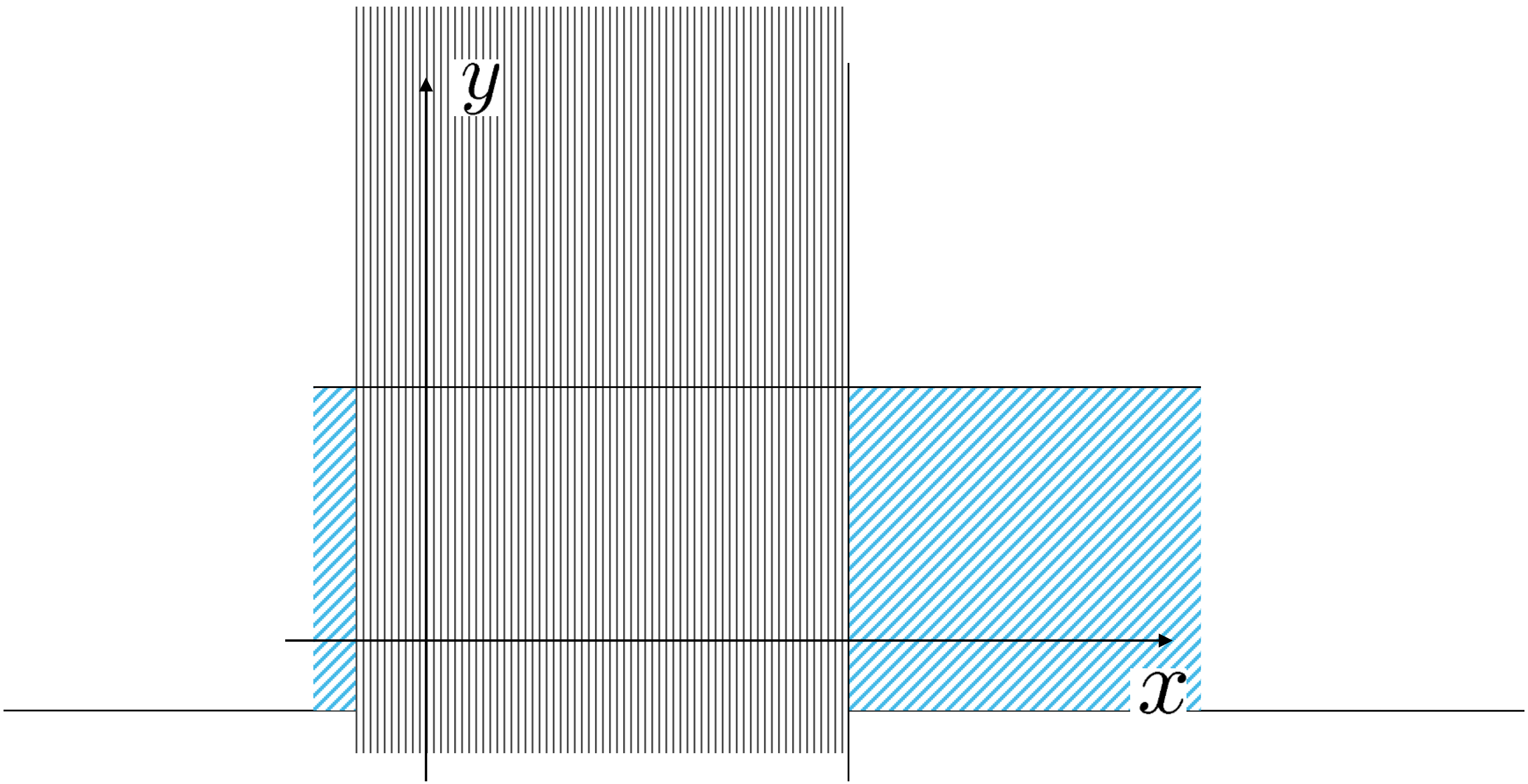
Marginal distributions of X and Y are given by

$$\begin{aligned} F_X(a) &= P\{X \leq a\} = P\{X \leq a, Y \leq \infty\} \\ &= \lim_{b\to\infty} P\{X \leq a, Y \leq b\} = F(a,\infty) \end{aligned}$$

Similarly, $F_Y(b) = F(\infty, b)$

$$P\{X \le a, \text{ or } Y \le b\} = F_X(a) + F_Y(b) - F(a,b)$$

$$P\{a_1 < X \le a_2, b_1 < Y \le b_2\} = F(a_2,b_2) + F(a_1,b_1) - F(a_1,b_2) - F(a_2,b_1)$$



Joint probability mass functions: Let *X* and *Y* be discrete random variables taking on values $x_1, x_2, \ldots,$ and $y_1, y_2, \ldots,$ respectively. The joint probability mass function of (*X*,*Y*) is $p(x_i, y_j) = P\{X = x_i, Y = y_j\}$

The marginal pmf can be computed as

$$p_X(x_i) = \sum_j p(x_i, y_j), p_Y(y_j) = \sum_i p(x_i, y_j)$$

$$P(X=x_i) = \sum_{j=1}^{\infty} P(X=x_i, Y=y_j)$$

Joint probability density function: random variables *X* and *Y* are said to be jointly continuous if there exists a function f(x,y) defined for all real x and y, having the property that for every set $C \subset R^2$ (*C* is a set in the two-dimensional plane)

$$P\{(X,Y) \in C\} = \iint_C f(x,y)\, dx dy$$

f(x,y) is called joint density function of *X* and *Y*.

$$\int_0^1 2y\, dy = 2\int_0^1 y\, dy$$

The marginal pdf for *X* is defined as: $f_X(x) = \int_{-\infty}^{\infty} f(x,y) dy$

$$\int_0^1 x y\, dy = x \cdot \int_0^1 y\, dy = \frac{x}{2}$$

Example : A fair coin is tossed three times independently; let *X* 0, or 1 denote the number of heads on the first toss and *Y* denote the total number of heads. Find the joint probability mass function of *X* and *Y*, together with the marginal pmf of *X* and *Y*.

Solution: The joint and marginal pmf is given in the following table:

$p(\underbrace{X=0, Y=1}_{THT,\ TTH})$

	Y				
X	0	1	2	3	p(x)
0	1/8	1/4	1/8	0	1/2
1	0	1/8	1/4	1/8	1/2
p(y)	1/8	3/8	3/8	1/8	

One of the most important discrete joint distributions is *Multinomial distribution*: A sequence of *n* independent and identical experiments is performed, each resulting in any one of *r* possible outcomes, with respective probabilities $p_1, p_2, \ldots, p_r,\ \sum_{i=1}^{r} p_i = 1.$

Let X_i denote the number of the *n* experiments that result in outcome *i*, then

$$P\{X_1 = n_1, X_2 = n_2, \ldots, X_r = n_r\} = \boxed{\frac{n!}{n_1!n_2!\ldots n_r!}} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$$

$$\boxed{\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots}$$

$$\sum_{i=1}^{r} X_i = n$$

$$\sum n_i = n$$

$$\underbrace{1\ 3\ 2\ 2\ 4\ 5\ 5\ 6}_{n}$$

Example : Suppose that a fair die is rolled 9 times. What is the probability that 1 appears three times, 2 and 3 twice each, 4 and 5 once each and 6 not at all?

Solution: $$\frac{9!}{3!2!2!1!1!0!}(\frac{1}{6})^9$$

Example : The joint density function of *X* and *Y* is given by

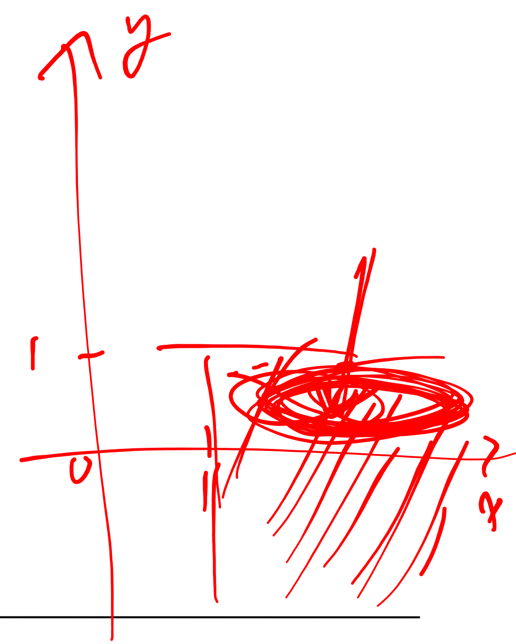
$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Compute (a) *P{X>1,Y<1}* (b) *P{X<a}* (c) *P{X<Y}*

Solution:

(a) $\int_0^1 \underbrace{\int_1^\infty 2e^{-x}e^{-2y}\,dx}\,dy = \int_0^1 2e^{-2y}e^{-1}\,dy$

$\underbrace{2e^{-2y}}\underbrace{\int_1^\infty e^{-x}\,dx}_{e^{-1}}$ $\qquad = e^{-1}(1-e^{-2})$



Example : The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Compute (a) $P\{X>1, Y<1\}$ (b) $P\{X<a\}$ (c) $P\{X<Y\}$

$$\int_0^\infty 2e^{-2y}\,dy = 1$$

Solution:

(b) $P(X<a) = P(X<a, Y<\infty) = \int_0^a \underbrace{\int_0^\infty \boxed{2e^{-x}}\, e^{-2y}\,dy}_{e^{-x}}\,dx$

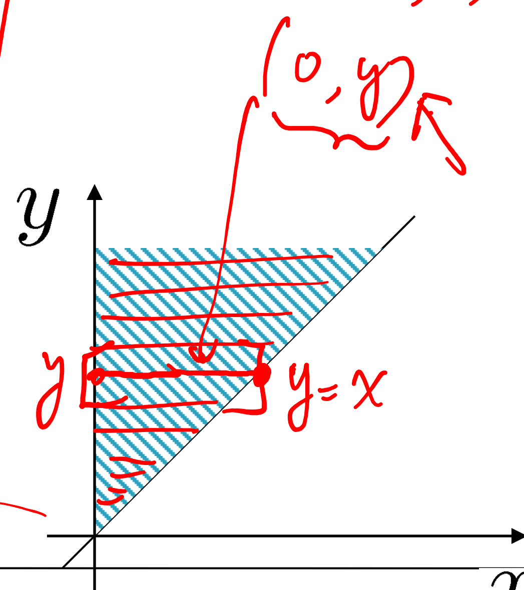
$$= \int_0^a e^{-x}\,dx = e^{-x}\Big|_0^a = 1 - e^{-a}$$

$$\int_0^y e^{-x}\,dx = -e^{-x}\Big|_0^y = 1-e^{-y}$$

(c) $$\underbrace{\int_0^\infty \underbrace{\int_0^y 2e^{-x} e^{-2y}\, dx}_{2e^{-2y}(1-e^{-y})}\, dy}$$

$$= \int_0^\infty 2e^{-2y}(1-e^{-y})\,dy = \frac{1}{3}$$

$$\int_0^\infty \underbrace{\int_x^\infty 2e^{-x} e^{-2y}\, dy}\, dx$$



Independent Random Variables

Definition: X and Y are said to be independent if for any two sets of real number A and B,

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\} \qquad (*)$$

It can be shown that (*) will follow if and only if for all *a* and *b* $F(a,b) = F_X(a)F_Y(b)$

When *X* and *Y* are discrete, it is equivalent to (this you can prove) $p(x,y) = p_X(x)p_Y(y) \quad \text{for all } x, y$

In continuous case, it is equivalent to (this you can prove too) $f(x,y) = f_X(x)f_Y(y) \quad \text{for all } x, y$

Example: A fair die is rolled twice. Let X be the outcome of the first roll, and Z be the sum of the two rolls. Are X and Z independent?

Solution: We showed in a previous example (Chapter 2) that the events *{X=4}* and *{Z=6}* are dependent, while *{X=4}* and *{Z=7}* are independent. Thus.....

X, Z dependent

$$\binom{i+j}{i} = \frac{(i+j)!}{i!\, j!}$$

Example: (try to memorize the result) Suppose that the number of people that enter a post office on a given day is a Poisson random variable with parameter λ. Show that if each person that enters the post office is a male with probability p and a female with probability 1-p, then the number of males and females entering the post office are independent Poisson random variables with respective parameters λp and $\lambda(1-p)$

$$P(X=i, Y=j) = \underbrace{P(X=i, \cancel{Y=j} \mid X+Y=i+j)\, P(X+Y=i+j)}_{P(X=i,\, Y=j,\, X+Y=i+j)}$$

$$= \binom{i+j}{i} p^i (1-p)^j e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!} \stackrel{?}{=} \underbrace{e^{-\lambda p} \frac{(\lambda p)^i}{i!}}_{P(X=i)} \times \underbrace{e^{-\lambda(1-p)} \frac{(\lambda(1-p))^j}{j!}}_{P(Y=j)}$$

$$P(X=i) = \sum_j e^{-\lambda p} \frac{(\lambda p)^i}{i!}\, e^{-\lambda(1-p)} \frac{(\lambda(1-p))^j}{j!} = e^{-\lambda p} \frac{(\lambda p)^i}{i!}$$

Proposition: The continuous (discrete) random variables X and Y are independent if and only if their joint probability density (mass) function can be expressed as

$$f_{XY}(x,y) = h(x)g(y)$$

for all real numbers x,y.

$f(x,y) = 2e^{-2x} \cdot 3e^{-3y}$

Example. If the joint density function of *X* and *Y* is

(a) $f(x,y) = 6e^{-2x}e^{-3y} \quad 0 < x < \infty, 0 < y < \infty$

(b) $f(x,y) = 24xy \quad 0 < x < 1, 0 < y < 1, 0 < x + y < 1$

and equal to 0 otherwise, are *X* and *Y* independent?

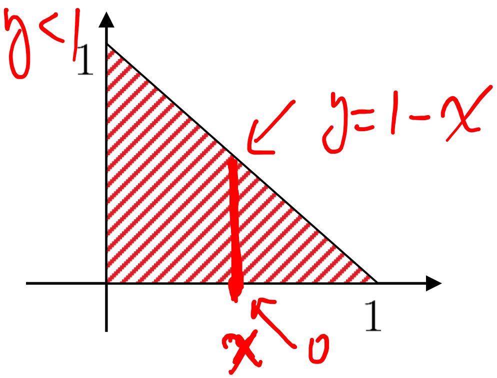
Solution:

(a) Independent

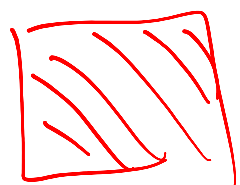
(b) Dependent

$\begin{cases} h(x) = 24x, & 0 < x < 1 \\ g(y) = y, & 0 < y < 1 \end{cases}$

positive domain in (b)



$24xy,$



$f(x) = \int_0^{1-x} 24xy \, dy = 12x(1-x)^2, \quad 0 < x < 1$

$f(y) = 12y(1-y)^2$

Example: Suppose X_1, X_2, X_3 are independent and distributed as $exp(\lambda_1), exp(\lambda_2), exp(\lambda_3)$ respectively. Find the probability that $\min(X_1, X_2, X_3) = X_2$

General Definition of Independence: $X_1, X_2, \ldots, X_n$ are said to be independent if, for all sets $A_1, A_2, \ldots, A_n$,

$$P\{X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n\} = \prod_{i=1}^{n} P\{X_i \in A_i\}$$

Sums of Independent Random Variables

It is often important to calculate the distribution of *X+Y* given the distribution of *X* and *Y* when *X* and *Y* are independent.

$$
\begin{aligned}
F_{X+Y}(a) &= P\{X+Y \leq a\} \\
&= \iint_{x+y\leq a} f_X(x) f_Y(y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f_X(x) f_Y(y)\,dx\,dy \\
&= \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{a-y} f_X(x)\,dx\,dy \\
&= \int_{-\infty}^{\infty} F_X(a-y) f_Y(y)\,dy
\end{aligned}
$$

Differentiating on both sides, we get

$$\begin{aligned} f_{X+Y}(a) &= \frac{d}{da}\int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy \\ &= \int_{-\infty}^{\infty} \frac{d}{da}F_X(a-y)f_Y(y)dy \\ &= \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy \end{aligned}$$

The last integral above is usually called convolution of f_X and f_Y

two fun. g, h

$$\boxed{\int_{-\infty}^{\infty} g(x-y)\,h(y)\,dy}$$

Proposition: If $X_1, X_2, \ldots, X_n$ are independent random variables that are normally distributed with parameters $\mu_i, \sigma_i^2, i = 1, 2, \ldots, n,$ then $\sum_{i=1}^n X_i$ is normally distributed with parameters $\sum_{i=1}^n \mu_i$ and $\sum_{i=1}^n \sigma_i^2$

$$E\left[\sum X_i\right] = \sum E X_i = \sum \mu_i$$

$$Var\left(\sum X_i\right) \stackrel{?}{=} \sum Var(X_i) = \sum \sigma_i^2$$

(Other important results) If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, then X+Y is a Poisson random variable with parameters $\lambda_1 + \lambda_2$

If X and Y are independent binomial random variables with respective parameters (n, p) and (m, p), then X+Y is a binomial random variable with parameters $(n+m, p)$

X: # of suc n experiments. Y: # of suc in m experiment

$$P(X+Y=k) = \sum_{i=0}^{k} P(X=i, Y=k-i) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \binom{m}{k-i} p^{k-i} (1-p)^{m-k+i}$$

$$\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} \stackrel{?}{=} \binom{m+n}{k} \quad \stackrel{?}{=} \binom{m+n}{k} p^k (1-p)^{m+n-k}$$

$$P(X+Y=k) = \sum_{i=0}^{k} P(X=i, Y=k-i) = \sum_{i=0}^{k} \cancel{e^{-\lambda_1}} \frac{\lambda_1^i}{i!} \cancel{e^{-\lambda_2}} \frac{\lambda_2^{k-i}}{(k-i)!}$$

$$\stackrel{?}{=} \cancel{e^{-(\lambda_1+\lambda_2)}} \frac{(\lambda_1+\lambda_2)^k}{k!}$$

need to show: $\displaystyle\sum_{i=0}^{k} \frac{\lambda_1^i}{i!} \frac{\lambda_2^{k-i}}{(k-i)!} \stackrel{?}{=} \frac{(\lambda_1+\lambda_2)^k}{k!}$

$$\sum_{i=0}^{k} \binom{k}{i} \frac{\lambda_1^i \lambda_2^{k-i}}{(\lambda_1+\lambda_2)^k} \stackrel{?}{=} 1$$

$$\sum_{i=0}^{k} \binom{k}{i} \Big(\underbrace{\frac{\lambda_1}{\lambda_1+\lambda_2}}_{p}\Big)^i \Big(\underbrace{\frac{\lambda_2}{\lambda_1+\lambda_2}}_{1-p}\Big)^{k-i} \stackrel{?}{=} 1 \quad \checkmark$$

5.2 Conditional Distributions

Discrete Case

Definition: If *X* and *Y* are jointly distributed discrete random variables, the conditional probability that *X* = *x* given that *Y* = *y* *(conditional pmf)* is

independent

$$p_{X|Y}(x|y) = P\{X = x | Y = y\} = \frac{p(x,y)}{p_Y(y)} = \frac{P_X(x)\cancel{P_Y(y)}}{\cancel{P_Y(y)}}$$

Remark: If *X* and *Y* are independent random variables, then the conditional probability mass function is the same as the unconditional one.

$$\sum_x P_{X|Y}(x|y) = 1$$

$$\sum_y P_{X|Y}(x|y) \stackrel{?}{=}$$

Example : Suppose that *p(x,y),* the joint probability mass function of X and Y, is given by p(0,0)=.4 p(0,1)=.2 p(1,0)=.1 p(1,1)=.3 Calculate the conditional probability mass function of X, given that Y=1

Solution:

$$p(X=1 \mid Y=1) = \frac{p(1,1)}{p_Y(1)} = \frac{0.3}{0.5} = \frac{3}{5}$$

$$p(X=0 \mid Y=1) = \frac{2}{5}$$

Example: If X and Y are independent Poisson random variables with parameters λ_1 and λ_2 respectively, calculate the conditional distribution of X, given that $X+Y=n$.

$\text{Pois}(\lambda_1+\lambda_2)$

Solution:

$$P(X=k \mid X+Y=n) = \frac{P(X=k, \overbrace{\cancel{X+Y=n}}^{Y=n-k})}{P(X+Y=n)}$$

$$= \frac{\cancel{e^{-\lambda_1}} \frac{\lambda_1^k}{k!} \cancel{e^{-\lambda_2}} \frac{\lambda_2^{n-k}}{(n-k)!}}{\cancel{e^{-(\lambda_1+\lambda_2)}} \frac{(\lambda_1+\lambda_2)^n}{n!}} = \frac{n!}{k!(n-k)!} \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} = \binom{n}{k} \cdot \Bigg(\underbrace{\frac{\lambda_1}{\lambda_1+\lambda_2}}_{p}\Bigg)^k \Bigg(\underbrace{\frac{\lambda_2}{\lambda_1+\lambda_2}}_{1-p}\Bigg)^{n-k}$$

$$\frac{n!}{k!(n-k)!} \downarrow \binom{n}{k}$$

$$X \mid X+Y=n \sim \text{Bin}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

$$Y \mid X+Y=n \sim \text{Bin}\left(n, \frac{\lambda_2}{\lambda_1+\lambda_2}\right)$$

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f(x,y)}{f_Y(y)}\,dx = \frac{1}{\cancel{f_Y(y)}} \underbrace{\int_{-\infty}^{\infty} f(x,y)\,dx}_{\cancel{f_Y(y)}} = 1$$

Continuous Case:

Definition: If *X* and *Y* have a joint probability density function *f*(*x*,*y*), then the conditional pdf of *X*, given that *Y*=*y* is defined for all values of *y* such that $f_Y(y) > 0$ by

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \leftarrow P(Y=y) \times$$

$f_0(x|y)$

If X and Y are independent, $f_{X|Y}(x|y) = f_X(x)$

$P(Y=y)$

If *X* and *Y* are jointly continuous, then for any set *A*,

$$P\{X \in A | Y = y\} = \int_A f_{X|Y}(x|y)dx$$

In particular, let $A = (-\infty, a]$, we can define the conditional cdf of *X* given that *Y*=*y* by

$$F_{X|Y}(a|y) = P\{X \leq a | Y = y\}$$

Example : The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} \frac{15}{2}x(2-x-y) & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find f(x|y) $= \dfrac{f(x,y)}{f_Y(y)} = \dfrac{\frac{15}{2}x(2-x-y)}{\frac{15}{2}\int_0^1 x(2-x-y)\,dx} = \dfrac{x(2-x-y)}{\frac{2}{3}-\frac{y}{2}}$

Solution:

$$\int_0^1 2x - x^2 - xy \, dx = x^2 - \frac{x^3}{3} - y\frac{x^2}{2}\Big|_0^1$$

$$= 1 - \frac{1}{3} - \frac{y}{2} = \frac{2}{3} - \frac{y}{2}$$

$$= \frac{6x(2-x-y)}{4-3y}$$

$0 < x, y < 1$

$$\int e^{-\frac{x}{2}}dx = -2e^{-\frac{x}{2}}, \quad \int e^{-\frac{x}{3}}dx = -3e^{-\frac{x}{3}}$$

Example : The joint density function of X and Y is given.

Compute $P\{X>1 / Y=y\}$

$$f(x,y) = \begin{cases} \frac{e^{-x/y}e^{-y}}{y} & 0 < x, y < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_0^\infty e^{-x/y}\frac{e^{-y}}{y}dx = \frac{e^{-y}}{y}\underbrace{\int_0^\infty e^{-\frac{x}{y}}dx}_{-ye^{-\frac{x}{y}}\Big|_0^\infty = y} = e^{-y}$$

$$\boxed{f(x|y) = \frac{1}{y}e^{-\frac{x}{y}}} \quad x>0, \; y>0.$$

$\sim \exp\left(\frac{1}{y}\right)$

$\exp(\lambda)$: $\lambda e^{-\lambda x}$

$$\int_1^\infty \frac{1}{y}e^{-\frac{x}{y}}dx$$

Want $P(X>1)$ then $X \sim \exp\left(\frac{1}{y}\right)$

answer: $e^{-\frac{1}{y}}$

5.3 Joint Distribution of Functions of Random Variables

Let X_1 and X_2 be jointly continuous random variables with joint pdf f_{XY}. We want to compute the density function of $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$

$Y_1 = X_1 + X_2$

$Y_2 = X_1 - X_2$

Assume that the functions $g_1 \text{ and } g_2$ satisfy the following conditions:

1. $y_1 = g_1(x_1, x_2) \text{ and } y_2 = g_2(x_1, x_2)$ can be uniquely solved for $x_1 \text{ and } x_2$ to get, say, $x_1 = h_1(y_1, y_2), x_2 = h_2(y_1, y_2)$
2. The functions $g_1 \text{ and } g_2$ have continuous partial derivatives and the determinant of the following determinant $$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} \neq 0$$

Then $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)|J(x_1, x_2)|^{-1} \text{ where}$

$$x_1 = h_1(y_1, y_2), x_2 = h_2(y_1, y_2)$$

5.4 Expectation, covariance, conditional expectation

Discrete $E[X] = \sum_{x_i} x_i P\{X = x_i\}$

Continuous $E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$

Since E[X] is a weighted average of possible values of X,

$$\text{if } P\{a \le X \le b\} = 1, \text{ then } a \le E[X] \le b.$$

Recall

$$E[g(X)] = \sum_{x_i} g(x_i) p(x_i), E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

A two-dimensional analog is the following:

Proposition.

$$E[g(X,Y)] = \sum_x \sum_y g(x,y)p(x,y)$$

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\,dxdy$$

An important implication is

$$\begin{aligned} E[X+Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)f(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy \\ &= \int_{-\infty}^{\infty} xf_X(x)dx + \int_{-\infty}^{\infty} yf_Y(y)dy \\ &= E[X]+E[Y] \end{aligned}$$

By induction, we have

$$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$$

Example: (Sample mean). Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables having cdf *F* and expected value μ. Such a sequence of r.v.'s is said to be a *random sample* from *F*. The sample mean $\bar{X}$ is defined by $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Compute its expectation. $E \sum X_i = \sum \underbrace{E X_i}_{\mu} = n\mu$

Solution:

$$E \frac{\sum X_i}{n} = \mu$$

Example: (Mean of a negative binomial random variable.)

If independent trials, each having a constant probability p of being a success are performed, determine the expected number of trials required to amass a total of r successes.

Solution: The negative binomial r.v. with parameter r and p is a sum of r i.i.d. geometric(p) r.v.'s. So the expectation of the negative binomial r.v. is the sum of the expectation of r geometric r.v.'s. So the answer is r/p.

Example: Consider the process of randomly tossing identical balls into b bins, numbered 1, 2,..., b. The tosses are independent, and on each toss the ball is equally likely to end up in any bin. The probability that a tossed ball lands in any given bin is $1/b$. Thus, the ball-tossing process is a sequence of Bernoulli trials with a probability $1/b$ of success, where success means that the ball falls in the given bin.

(a) *How many balls fall in a given bin (if a total of n balls are tossed)?*

(b) How many balls must one toss, on the average, until a given bin contains a ball? The number of tosses until the given bin receives a ball follows the geometric distribution with probability $1/b$ and thus the expected number of tosses until success is $1/(1/b) = b$.

(c) How many balls must one toss until every bin contains at least one ball?

Covariance

The covariance between two random variables is a measure of how they are related.

Definition: The covariance between X and Y, denoted by $Cov(X,Y)$, is defined by $Cov(X,Y)=E[(X-E[X])(Y-E[Y])]$.

对 $X=Y$, $Cov(X,X)=E[(X-EX)^2]=VarX$

Interpretation: When $Cov(X,Y)>0$, higher than expected values of X tend to occur together with higher than expected values of Y. When $Cov(X,Y)<0$, higher than expected values of X tend to occur together with lower than expected values of Y.

By expanding the right hand side of the definition of the covariance, we see that

$$
\begin{aligned}
& Cov(X,Y) \\
= \quad & E[(X-E[X])(Y-E[Y])] = E\{XY - \overbrace{E[X]}Y - XE[Y] + \overbrace{E[X]E[Y]}^{Const}\} \\
= \quad & E[XY] - \cancel{E[X]E[Y]} - E[X]E[Y] + \cancel{E[X]E[Y]} = E[XY] - E[X]E[Y]
\end{aligned}
$$

$Var\,X = EX^2 - (EX)^2$

If *X* and *Y* are independent, then

$$
\begin{aligned}
E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy \\
&= E[X]E[Y]
\end{aligned}
$$

$X \in \{-1, 0, 1\}$ $\quad P(Y=1, X=1) = P(X=1) = \frac{1}{3} \neq P(X=1)P(Y=1)$

$Y = X^2$ $\quad EXY = EX^3 = EX = 0, \quad EX \cdot EY = 0$

Definition: If *Cov(X,Y)=0*, we say *X* and *Y* are uncorrelated. If *Cov(X,Y)>0*, we say *X* and *Y* are positively correlated. If *Cov(X,Y)<0*, we say *X* and *Y* are negatively correlated.

So the previous calculation tells us that independence implies uncorrelatedness.

We have

Proposition: If *X* and *Y* are independent, then for any functions of *g* and *h*, g(X) and h(Y) are independent.

Proposition

$$(i) \quad Cov(X,Y) = Cov(Y,X)$$

$$(ii) \quad Cov(X,X) = Var(X)$$

$$(iii) \quad Cov(aX, bY) = abCov(X,Y)$$

$$(\sum X_i)(\sum Y_j) = \sum_i \sum_j X_i Y_j$$

$$(iv) \quad Cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$$

$$(\sum X_i)^2 = \sum X_i^2 + 2\sum_{i<j} X_i X_j$$

$$(v) \quad Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$$

(vi) if $X_1, X_2, \ldots, X_n$ are ~~pairwise~~ independent,

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$$

$\rho(aX, Y)$

对 $X = a$

$Cov(X,Y) = E[\underbrace{(X - EX)}_{0}(Y - EY)]$

$= 0$

ax

Correlation: $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(\underset{ax}{X})Var(Y)}}$

The correlation is always between -1 and 1. If *X* and *Y* are independent, then $\rho(X,Y) = 0$ .but the converse is not true. Generally, the correlation (as well as covariance) is a measure of the degree of linear dependence between *X* and *Y*.

Note that for *a*>0,*b*>0 ,

$$\rho(aX, bY) = \frac{Cov(aX, bY)}{\sqrt{Var(aX)Var(bY)}} = \frac{abCov(X,Y)}{\sqrt{a^2b^2Var(X)Var(Y)}} = \rho(X,Y)$$

$E(X_1-\bar{X})^2$, $Y_i = X_i - \mu$, $\bar{Y} = \frac{\sum Y_i}{n} = \bar{X} - \mu$

$E(Y_1-\bar{Y})^2 = E(X_1-\bar{X})^2$

Example : Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having expected value μ and variance σ^2 . and let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the sample mean. The random variable $S^2 = \sum_{i=1}^{n} \frac{(X_i-\bar{X})^2}{n-1}$ is called the sample variance. Find (a) $E(\bar{X})$ and $Var(\bar{X})$ and (b) $E[S^2] = \sigma^2$

μ

$Var(\sum X_i) = \sum Var(X_i) = n\sigma^2$, $Var\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2} Var(\sum X_i) = \frac{\sigma^2}{n}$

$\boxed{\text{Will show. } ES^2 = \sigma^2}$, pretend $\mu = 0$, $EX_i^2 = Var X_i + (EX_i)^2 = \sigma^2 + \mu^2$

$\sum_{i=1}^{n} E(X_i-\bar{X})^2 = E\left[\sum X_i^2 - 2\underbrace{(\sum X_i)}_{n\bar{X}}\bar{X} + n\bar{X}^2\right] = E\left[\sum X_i^2 - n\bar{X}^2\right]$

$= \sum EX_i^2 - nE\bar{X}^2$

$= n\sigma^2 - n\cdot\frac{\sigma^2}{n} = (n-1)\sigma^2$

Example. Compute the variance of a binomial random variable X with parameters n and p.

Solution: $X \sim Bin(n, p)$, $X = X_1 + \cdots + X_n$

$X_i \sim Ber(p)$

$$\underline{Var(X) = \sum Var(X_i)}$$

$$= np(1-p)$$

$I_A = 1$ if A happens
0 o.w.

$I_A \cdot I_B = I_{A\cap B}$

Example . Let I_A and I_B be indicator variables for the events *A* and *B*. Find $Cov(I_A, I_B) = E\, I_A I_B - E I_A \cdot E I_B$

$= E I_{A\cap B} - E I_A\, E I_B$

$= P(A\cap B) - P(A)P(B)$

Solution:

$E I_A = 1 \cdot P(I_A = 1) + 0 \cdot P(I_A = 0) = P(I_A = 1) = P(A)$

Thus two events are independent if and only if the corresponding indicator variables are uncorrelated. In other words, for indicator variables, independence and uncorrelatedness are equivalent.

Example. Let $X_1, \dots, X_n$ be independent and identically distributed random variables having variance σ^2. Show that $Cov(X_i - \bar{X}, \bar{X}) = 0$

assume $\mu = 0$

$$Cov(X_1 - \bar{X}, \bar{X}) = E[(X_1 - \bar{X})\bar{X}] - \underbrace{E[(X_1 - \bar{X})]}_{0} \cdot E\bar{X}$$

$$E X_1 \bar{X} = E \frac{X_1(X_1 + X_2 + \cdots + X_n)}{n} = E\frac{X_1^2}{n} + \underbrace{E\frac{X_1X_2}{n}}_{0} + \underbrace{\cdots}_{0} = \frac{\sigma^2}{n}$$

because $EX_1X_2 = EX_1 \cdot EX_2 = 0$

$$E\bar{X}^2 = Var\bar{X} = \frac{\sigma^2}{n}$$

$$E[(X_1 - \bar{X})\bar{X}] = \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0$$

Bivariate normal distribution

Definition: The joint density for a bivariate normal distribution is

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} exp\left\{-\frac{1}{2(1-\rho^2)}\left[(\frac{x-\mu_x}{\sigma_x})^2 + (\frac{y-\mu_y}{\sigma_y})^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\}$$

$= f_X(x) f_Y(y)$ if $\rho = 0$

Remarks on bivariate normal random variables (X,Y):

(a) Marginally, $X \sim N(\mu_x, \sigma_x^2), Y \sim N(\mu_y, \sigma_y^2)$

(b) Conditionally, $X|Y = y \sim N(\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y - \mu_y), \sigma_x^2(1 - \rho^2))$

(c) $Cov(X, Y) = \rho\sigma_x\sigma_y$

(d) Linear combinations of X and Y are normal random variables, even though X and Y are not independent when $\rho \neq 0$

(e) Two normal random variables are independent iff they are uncorrelated.

Example: For bivariate normal random variables X and Y with parameters $(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$, find *P(X<Y)*

Solution:

Conditional Expectation

Recall that if X and Y are joint discrete random variables,

It is natural to define, the conditional expectation of *X* given *Y* = *y*

$$E[X|Y=y] = \sum_x xP(X=x|Y=y) = \sum_x xp_{X|Y}(x|y)$$

for continuous random variables:

The conditional expectation of *X*, given that *Y* = *y* , is

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

Example. Let X and Y have the joint pdf

$$f(x,y) = \frac{e^{-x/y}e^{-y}}{y}, 0 < x, y < \infty$$

Find the conditional expectation $E(X \mid Y = y)$.

$$\underline{f(x|y)} = \frac{1}{y} e^{-\frac{x}{y}}, \quad x, y > 0,$$

$$\sim \exp\left(\frac{1}{y}\right)$$

对 $X \sim \exp(\lambda)$

$E[X] = \frac{1}{\lambda}$

$$E[X \mid Y = y] = y, \quad \int_0^\infty \frac{x}{y} e^{-\frac{x}{y}} dx$$

$$\int_0^\infty \frac{x}{2} e^{-\frac{x}{2}} dx$$

Conditional Variance (for your information only)

The conditional variance of $X|Y = y$ is the expected squared difference of the random variable X and its conditional mean, conditioning on the event $Y = y$:

$$Var(X|Y = y) = E[(X - E[X|Y = y])^2|Y = y]$$

Similar to the unconditional case, we can show

$$Var(X|Y = y) = E[X^2|Y = y] - [E(X|Y = y)]^2$$