BMS1901 Calculus for Life Sciences

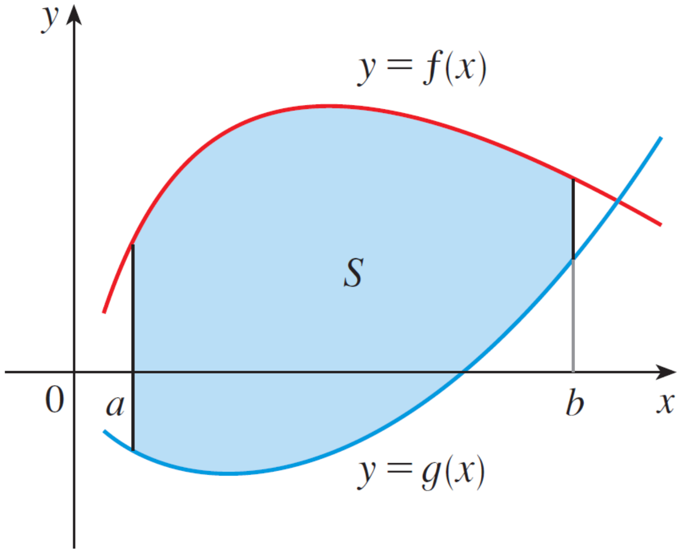
Week 11

Applications of Integrals
Perform separation of variable
Taylor series

Areas Between Curves

Areas Between Curves

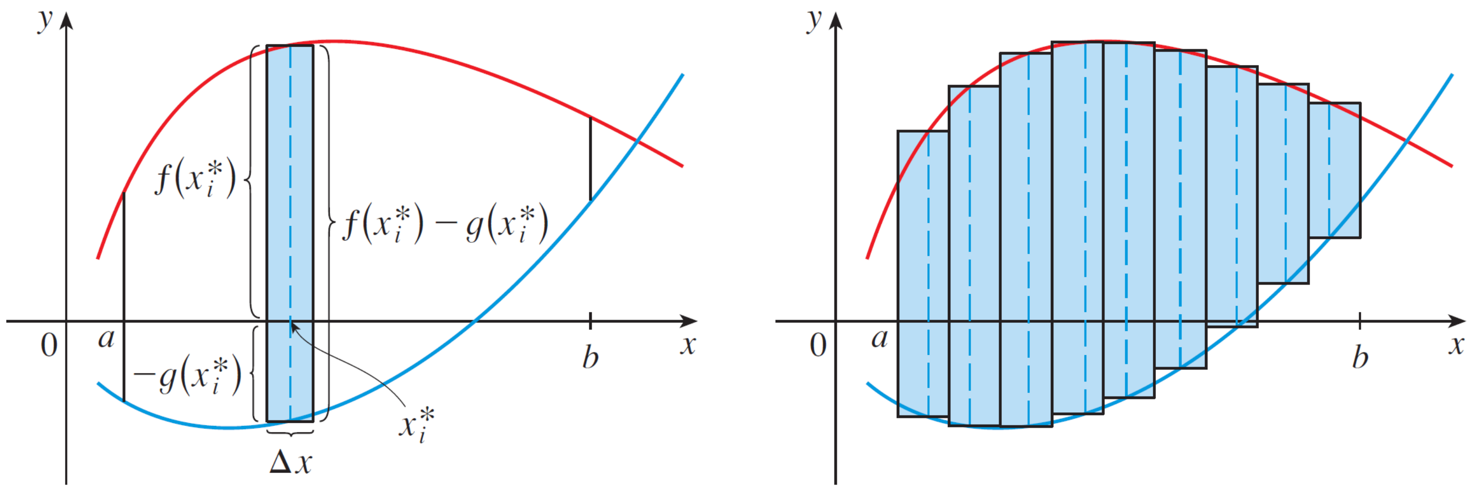
- region S that:
  - lies between two curves $y = f(x)$ and $y = g(x)$ and
  - between the vertical lines $x = a$ and $x = b$
  - f and g are continuous functions
  - $f(x) \geq g(x)$ for all x in $[a, b]$



$S = \{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq f(x)\}$

Areas Between Curves

- divide *S* into *n* strips of equal width
- approximate the *i* th strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$
- take all of the sample points to be right endpoints: $x_i^* = x_i$



(a) Typical rectangle

(b) Approximating rectangles

Areas Between Curves

<u>Riemann sum:</u>

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$$

- ~ to the area of *S*
- approximation may become better as n $\rightarrow \infty$
- define the **area** *A* of the region *S* = limiting value of the sum of the areas of approximating rectangles

(1) $$\boxed{A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x}$$

Areas Between Curves

- limit in (1) = definite integral of $f - g$

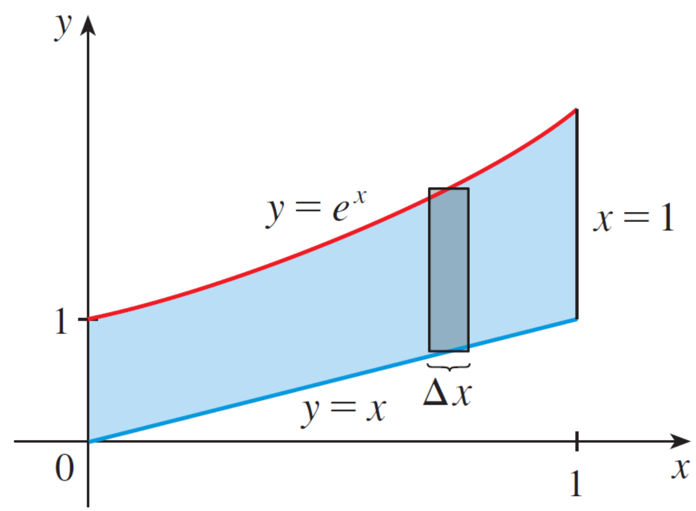
(2) The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Example 1

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Solution:



Example 1 – *Solution*

- upper boundary curve: $y = e^x$
- lower boundary curve: $y = x$

(2) The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

→ formula (2) with $f(x) = e^x$, $g(x) = x$, $a = 0$, and $b = 1$:

$$\begin{aligned} A &= \int_0^1 (e^x - x)\,dx \\ &= e^x - \tfrac{1}{2}x^2 \Big]_0^1 \\ &= e - \tfrac{1}{2} - 1 \\ &= e - 1.5 \end{aligned}$$

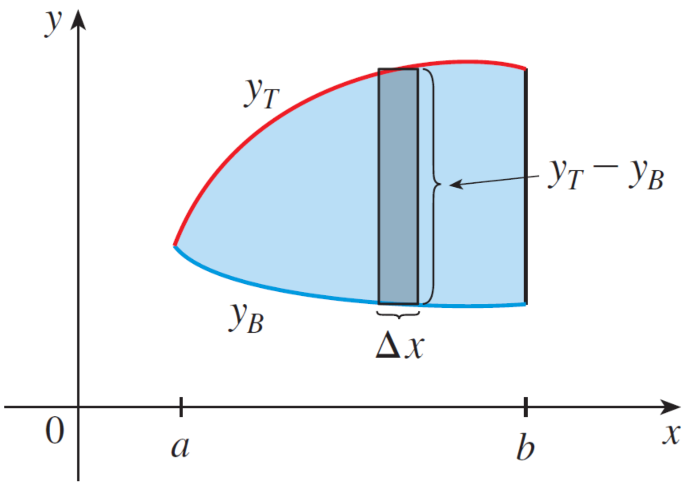
Areas Between Curves

- set up an integral for an area
- sketch the region to identify:
  - top curve y_T
  - the bottom curve y_B
- approximating rectangle

Area of a typical rectangle: $(y_T - y_B)\,\Delta x$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} (y_T - y_B)\,\Delta x = \int_a^b (y_T - y_B)\,dx$$

→summarizes the procedure of adding the areas of all the typical rectangles



Example 2

Find the area of the region enclosed by the parabolas
$y = x^2$ and $y = 2x - x^2$.

Solution:

- find the points of intersection of the parabolas by solving their equations together

- $x^2 = 2x - x^2$ & $2x^2 - 2x = 0$

- $2x(x - 1) = 0$, so $x = 0$ or 1

- points of intersection: (0, 0) and (1, 1).

Example 2 – *Solution*

- top and bottom boundaries:

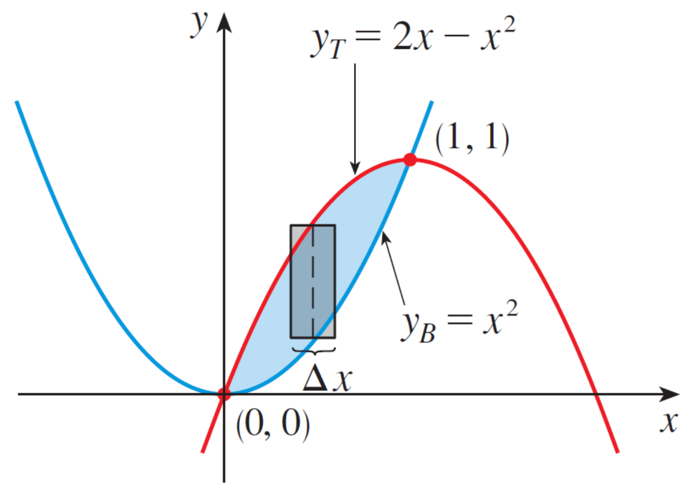
$$y_T = 2x - x^2 \qquad \text{and} \qquad y_B = x^2$$

- area of a typical rectangle:

$$\begin{aligned}(y_T - y_B)\,\Delta x &= (2x - x^2 - x^2)\,\Delta x \\ &= (2x - 2x^2)\,\Delta x\end{aligned}$$

- region lies between $x = 0$ and $x = 1$

- total area: $$A = \int_0^1 (2x - 2x^2)\,dx$$



Example 2 – *Solution*

$$= 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{3}$$

$$2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \cancel{2}\left(\frac{3-2}{\cancel{6}\,3}\right)$$

$$= \frac{1}{3}$$

Average Values

Average Values

- average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

- average value of a function $y = f(x)$, $a \leq x \leq b$
  - dividing the interval $[a, b]$ into n equal subintervals (each with length $\Delta x = (b - a)/n$)
  - choose points $x_1^* \ldots, x_n^*$ in successive subintervals
  - calculate the average of the numbers $f(x_1^*), \ldots, f(x_n^*)$:

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

Average Values

- $\Delta x = (b - a)/n$
- $n = (b - a)/\Delta x$
- average value:

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{\dfrac{b-a}{\Delta x}} = \frac{1}{b-a}\left[f(x_1^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\right]$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

let *n* increase → compute the average value of a large number of closely spaced values

Average Values

The limiting value is

$$\lim_{n \to \infty} \frac{1}{b - a} \sum_{i=1}^{n} f(x_i^*) \, \Delta x = \frac{1}{b - a} \int_a^b f(x) \, dx$$

by the definition of a definite integral.

Therefore we define the **average value of *f*** on the interval [*a*, *b*] as

$$\boxed{f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x) \, dx}$$

Example 1

Find the average value of the function $f(x) = 1 + x^2$ on the Interval [–1, 2].

Solution:

With a = –1 and b = 2 we have

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)\,dx$$

$$= \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2}$$

$$= 2$$

$$\frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[2 + \frac{2^3}{3} - (-1) - \left(\frac{-1}{3}\right)\right]$$

$$= \frac{1}{3}\left[2 + \frac{8}{3} + 1 + \frac{1}{3}\right]$$

$$= \frac{1}{3}\left[\frac{6+8+3+1}{3}\right]$$

$$= \frac{1}{\cancel{3}} \cdot \left[\frac{\cancel{18}^{\,6}}{3}\right]$$

$$= 2$$

Average Values

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

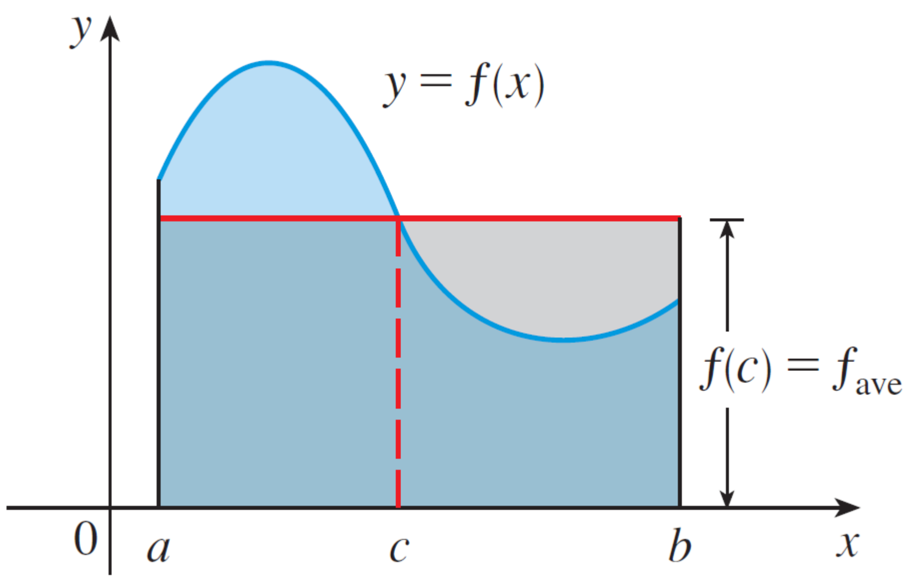
$$f(c) = f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x)\, dx$$

that is,

$$\int_a^b f(x)\, dx = f(c)(b - a)$$

Average Values

- geometric interpretation of MVT for Integrals:
  - *f (positive function):*
    Area of rectangle with base [*a*, *b*] and height *f*(*c*)
    = area of region under the graph of *f* from *a* to *b*



Example 3

Find c for $f(x) = 1 + x^2$ is continuous on the interval $[-1, 2]$ using the Mean Value Theorem for Integrals.

Example 3

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x)\, dx$$

that is, $$\int_a^b f(x)\, dx = f(c)(b - a)$$

- Using MVT for Integrals:

$$\int_{-1}^{2} (1 + x^2)\, dx = f(c)[2 - (-1)]$$

- f_{ave} = 2: (from slide 17)

$$f(c) = f_{\text{ave}} = 2$$

Example 3

$$1 + c^2 = 2 \qquad \text{so} \qquad c^2 = 1$$

two numbers $c = \pm 1$ in the interval $[-1, 2]$ that work in the MVT for Integrals

Separable Equations

Separable Equations

- **Separable equation:** first-order differential equation *dy*/*dt* : factored as a function of *t* times a function of *y*

$$\frac{dy}{dt} = f(t)\, g(y)$$

Separable Equations

$$\frac{dy}{dt} = f(t)\, g(y)$$

- $g(y) \neq 0$:

(1)

$$\frac{dy}{dt} = \frac{f(t)}{h(y)}$$

  - $h(y) = 1/g(y)$
- To solve the equation:

$$h(y)\, dy = f(t)\, dt$$

- y's are on one side
- t's are on the other side

Separable Equations

- integrate both sides:

$$\int h(y)\,dy = \int f(t)\,dt \qquad (2)$$

- defines *y* implicitly as a function of *t*
- solve for *y* in terms of *t*

Separable Equations

<u>Using the Chain Rule</u>:
If *h* and *f* satisfy (2),

(2) $$\int h(y)\,dy = \int f(t)\,dt$$

$$\frac{d}{dt}\left(\int h(y)\,dy\right) = \frac{d}{dt}\left(\int f(t)\,dt\right)$$

so $$\frac{d}{dy}\left(\int h(y)\,dy\right)\frac{dy}{dt} = f(t)$$

and $$h(y)\,\frac{dy}{dt} = f(t)$$

* Equation 1 is satisfied

(1) $$\frac{dy}{dt} = \frac{f(t)}{h(y)}$$

Example 1

(a) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

Solution:

(a) Rewrite the equation in terms of differentials and integrate both sides:

$$y^2 dy = x^2 dx$$

$$\int y^2 dy = \int x^2 dx$$

$$\tfrac{1}{3}y^3 = \tfrac{1}{3}x^3 + C$$

Example 1 – *Solution*

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

$$\frac{1}{3}y^3 + C_1 = \frac{1}{3}x^3 + C_2$$

$$C = C_2 - C_1$$

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

$$y^3 = x^3 + 3C$$

$$y = \sqrt[3]{x^3 + 3C}$$

- C is an arbitrary constant
- Solving for y:

$$y = \sqrt[3]{x^3 + 3C}$$

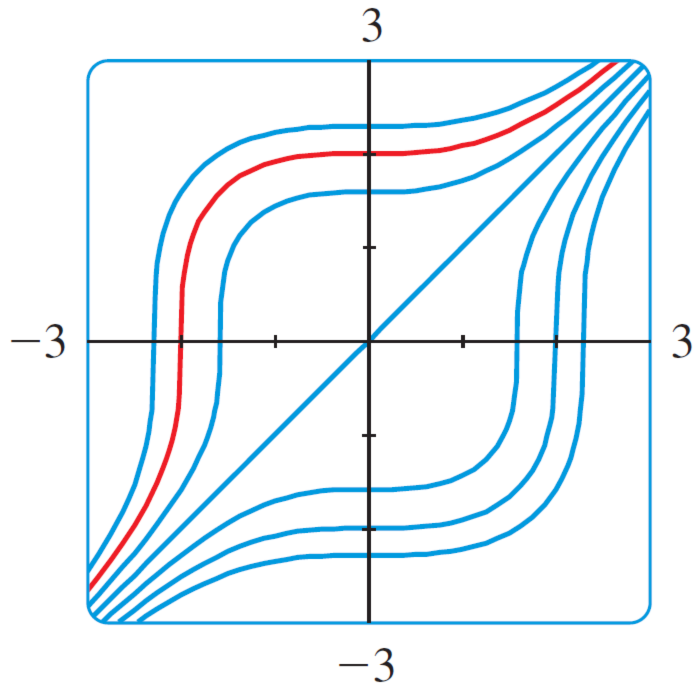
- could leave the solution like this or write it in the form:

$$y = \sqrt[3]{x^3 + K}$$

- $K = 3C$

Example 1 – *Solution*

- Family of solutions:



Graphs of several members of the family of solutions of the differential equation in Example 1. The solution of the initial-value problem in part (b) is shown in red.

Example 1 – *Solution*

(b) put $x = 0$ in the general solution in part (a):

$y(0) = \sqrt[3]{K}$

$y(x=0) = 2$

$\sqrt[3]{K} = 2$

$(\sqrt[3]{K})^3 = 2^3$

$K = 8$

- satisfy the initial condition $y(0) = 2$:

  $\sqrt[3]{K} = 2$

  $K = 8$

- solution of the initial-value problem:

$$y = \sqrt[3]{x^3 + 8}$$

Taylor Polynomials

Taylor Polynomials

- tangent line approximation $L(x)$: best first-degree (linear) approximation to $f(x)$ near $x = a$
  - $f(x)$ and $L(x)$ have the same rate of change (derivative) at a

- second-degree (quadratic) approximation $P(x)$: better approximation than a linear one
  - approximate a curve by a parabola instead of by a straight line

Taylor Polynomials

- Good approximation:

(i) $P(a) = f(a)$ (P and f should have the same value at a.)

(ii) $P'(a) = f'(a)$ (P and f should have the same rate of change at a.)

(iii) $P''(a) = f''(a)$ (The slopes of P and f should change at the same rate at a.)

- → $P(x) = A + B(x - a) + C(x - a)^2$

- → $P'(x) = B + 2C(x - a)$ and $P''(x) = 2C$

$$P(x) = A + B(x-a) + C(x-a)^2$$
$$P'(x) = A + B(x-a) + C(x^2 - 2ax - a^2)$$
$$= B + 2Cx - 2Ca$$
$$= B + 2C(x-a)$$

Taylor Polynomials

Applying (i), (ii), and (iii):

$$P(a) = f(a) \quad \Rightarrow \quad A = f(a)$$

$$P'(a) = f'(a) \quad \Rightarrow \quad B = f'(a)$$

$$P''(a) = f''(a) \quad \Rightarrow \quad 2C = f''(a) \quad \Rightarrow \quad C = \tfrac{1}{2}f''(a)$$

•quadratic function satisfying the three conditions:

(4) $$\boxed{P(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2}$$

- $T_2(x)$ **: second-degree Taylor polynomial of *f* centered at *a***

(i) $P(a) = f(a)$

$$P(a) = A + B\cancelto{0}{(a-a)} + C\cancelto{0}{(a-a)^2} = f(a)$$

$$A = f(a)$$

(ii) $P'(a) = f'(a)$

$$P'(a) = B + 2C\cancelto{0}{(a-a)} = f'(a)$$

$$B = f'(a)$$

(iii) $P''(a) = f''(a)$

$$2C = f''(a)$$

$$C = \frac{1}{2} f''(a)$$

Example 13

Find the second-degree Taylor polynomial $T_2(x)$ centered at $a = 0$ for the function $f(x) = \cos x$. Illustrate by graphing T_2, f, and the linearization $L(x) = 1$.

Solution:

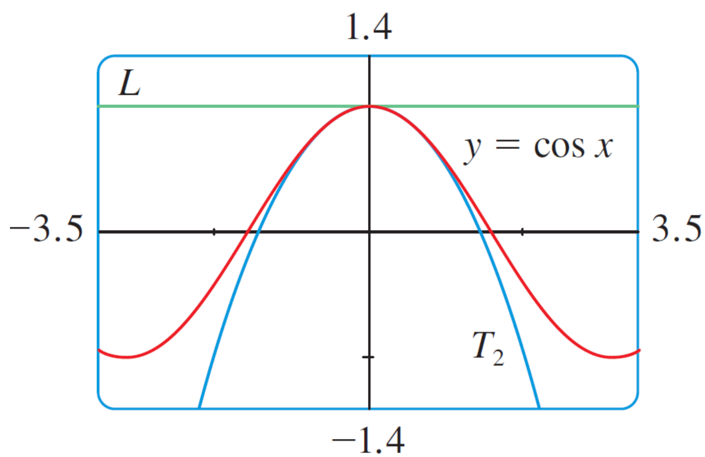
- $f(x) = \cos x$, $f'(x) = -\sin x$, and $f''(x) = -\cos x$
  - second-degree Taylor polynomial centered at 0:

$$\boxed{P(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2} f''(a)(x - a)^2}$$

$$\begin{aligned} T_2(x) &= f(0) + f'(0)x + \tfrac{1}{2} f''(0)x^2 \\ &= 1 + 0 + \tfrac{1}{2}(-1)x^2 \\ &= 1 - \tfrac{1}{2}x^2 \end{aligned}$$

Example 13 – *Solution*

- cosine function + its linear approximation $L(x) = 1$ + its quadratic approximation $T_2(x) = 1 - \frac{1}{2}x^2$ near 0



Figure

- quadratic approximation is much better than the linear one

Taylor Polynomials

- find better approximations with higher-degree polynomials

- *n*th-degree polynomial:

$$T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$$

  - T_n and its first n derivatives have the same values at $x = a$ as f *and* its first n derivatives