EE1001 Counting Part II

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Intended Learning Outcomes

- Upon completion of this session, you will be able to:

  ✓1 Understanding some basic concepts in set theory

  ✓2. Solving problems with the inclusion-exclusion principle

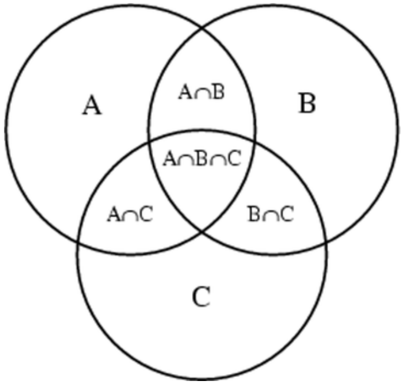
  ✓3. Solving problems with the pigeon-hole principle

Chapter Contents

- Basic Set Theory
- The inclusion-exclusion principle
- The pigeon-hole principle

Why we study Set Theory

- **the common language to speak about mathematics, so learning set theory means learning the common language.**
- Studying set theory, even naively, is the technical spine of how to handle infinite sets. Since modern mathematics is concerned with many infinite sets, larger and smaller, it is a good idea to learn about infinite sets if one wishes to understand mathematical objects better.
- We can easily find the relationship between different things or sets with set theory.



1. Basic Set Theory

Basic Definitions

- **Set:** a set is an unordered collection of objects. For sets, we' ll use variables *S, T, U, …*
- **Element:** The objects used to form a set are called its element. For elements, we' ll use variables *a, b, c …*
- There are ways of describing a set
  - **Explicitly:** listing the elements of a set

    e.g. {a, b, c}, {1, 2, 3}, {Woody, Joe, Paul} are all finite sets
    {1, 2, 3, …} is a way we denote an infinite set

  - **Set builder notation:** For any proposition P(x) over any universe of discourse, {x|P(x)} is the set of all x such that P(x).

    e.g., $\{ x \mid x$ is an integer where $x>0$ and $x<5 \}$

    or $\{ x : x$ is an integer where $x>0$ and $x<5 \}$

Properties of sets

- Sets are inherently ***unordered***.

  {a, b, c} = {a, c, b} = {b, a, c} ={b, c, a} = {c, a, b} = {c, b, a}

- All elements are ***distinct*** (***unequal***): multiple listings make no difference!

  {a, b, c} = {a, a, b, a, b, c, c, c, c}.

- Two sets are declared to be equal ***if and only if*** they contain exactly the same elements.

  The set {1, 2, 3, 4} =

  $\{x \mid x$ is an integer where x>0 and x<5 } =

  $\{x \mid x$ is a positive integer whose square is >0 and <25}

Infinite Sets

- Conceptually, sets may be *infinite* (*i.e.*, not *finite*, without end, unending).
- Symbols for some special infinite sets:
  - **N** = {0, 1, 2, …}   The natural numbers.
  - **Z** = {…, -2, -1, 0, 1, 2, …} The integers.
  - **R** = The "real" numbers, such as 20, 2.14, π
  - $\mathbf{Z}^+$ = {1,2,3,…} set of positive integers

Note: Real number are the numbers that can be represented by an infinite decimal representation, including both **rationa**l, and **irrational** numbers such as π that can be represented as points along an infinitely long number line.

Basic Set Relations

- $x \in S$ ("x is in S") is the proposition that object x is an *element* or *member* of set S.
- $x \notin S$: "x is not in S"
- $\varnothing$ ("null", "the empty set") is the unique set that contains no elements whatsoever.
- $\varnothing = \{\} = \{x | \textbf{False}\}$
- $\varnothing \neq \{\varnothing\}$. Since the right term is a set with an element $\varnothing$.
- $S \subseteq T$ ("S is a ***subset*** of T") means that every element of S is also an element of T.
- $\varnothing$ is a subset of any set. $\varnothing \subseteq S$

Sets Are Objects

- The objects that are elements of a set may themselves be sets.

$$\text{let } S=\{x \mid x \subseteq \{1,2,3\}\}$$

$$\text{then } S=\{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

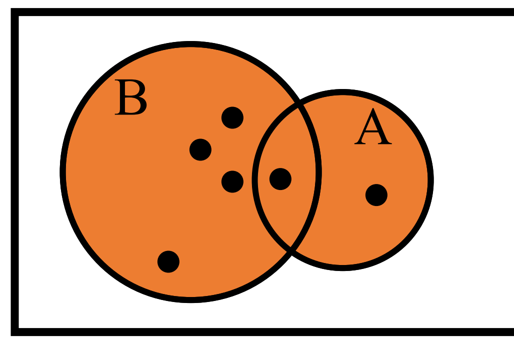
- Note that $\underbrace{1}_{\text{Element 1}} \neq \underbrace{\{1\}}_{\text{Set with one element 1}} \neq \underbrace{\{\{1\}\}}_{\text{Set with one element \{1\}}}$!!!!

Operation Between Set: Union

- For sets A, B, their *union* $A \cup B$ is the set containing all elements that are either in A, **or** ("$\vee$") in B (or, of course, in both).
  - $A \cup B = \{x \mid x \in A \vee x \in B\}$.
- Note that A∪B contains all the elements of A and it contains all the elements of B:
  - $(A \cup B \supseteq A) \wedge (A \cup B \supseteq B)$

e.g.

- {a,b,c}∪{2,3} = {a,b,c,2,3}
- {2,3,5}∪{3,5,7} = {2,3,5,3,5,7} ={2,3,5,7}



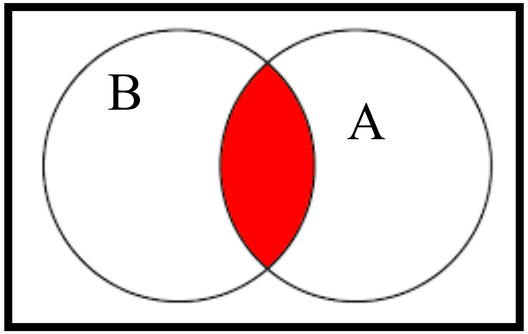
$A \cup B$

Operation Between Set: Intersection

- For sets A, B, their intersection A∩B is the set containing all elements that are simultaneously in A and ("∧") in B.
  - $A \cap B \equiv \{x \mid x \in A \land x \in B\}$
- Note that that $A \cap B$ is a subset of A **and** it is a subset of B:
  - $(A \cap B \subseteq A) \land (A \cap B \subseteq B)$

e.g.

- $\{a,b,c\} \cap \{2,3\} = \varnothing$
- $\{2,4,6\} \cap \{3,4,5\} = \{4\}$



$A \cap B$

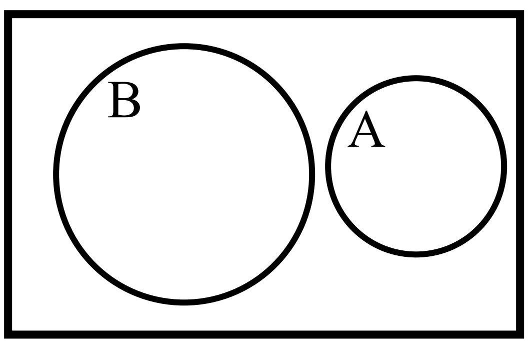
Operation Between Set: Intersection

- Considering A={x : x is a US president};

  B = {x : x is in this room}

  What is $A \cap B$?

$$A \cap B = \{x : x \text{ is a US president in this room}\} = \varnothing$$



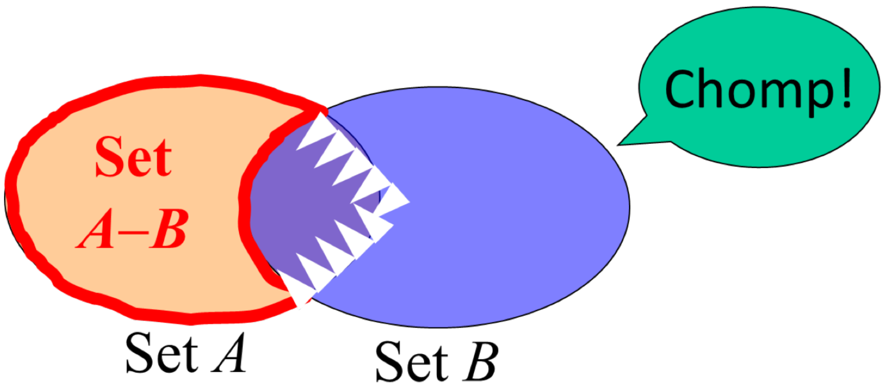
Sets whose intersection is empty are called *disjoint* sets

Operation Between Set: Complement

- Complement of B with respect to A: for sets A, B, the *difference of A and B*, written $A–B$, is the set of all elements that are in A but not B.
  - $A - B :\equiv \{x \mid x \in A \wedge x \notin B\}$

e.g.

- $\{a,b,c\}-\{a,b,c\} = \varnothing$
- $\{2,4,6\}-\{3,4,5\} = \{2,6\}$
- $\{3,4,5\}-\{2,4,6\} = \{3,5\}$



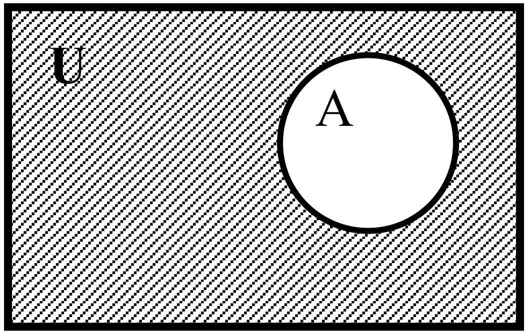
Operation Between Set: Complement

- The *universe of discourse* can itself be considered a set, call it *U*.
- The complement of a set A is:

$$\overline{A} = U - A$$

- Specifically:

$$\overline{\varnothing} = U$$
$$\overline{U} = \varnothing$$



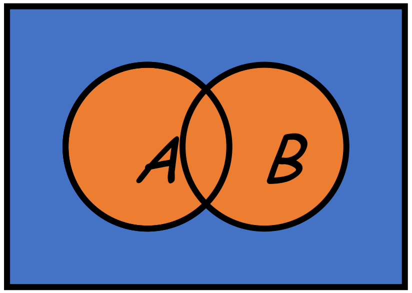
Set Identities

- Identity: $A \cup \varnothing = A \quad A \cap U = A$
- Domination: $A \cup U = U \quad A \cap \varnothing = \varnothing$
- Idempotent: $A \cup A = A = A \cap A$
- Double complement: $\overline{(\overline{A})} = A$
- Commutative: $A \cup B = B \cup A \quad A \cap B = B \cap A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C$
  $A \cap (B \cap C) = (A \cap B) \cap C$

- Specifically:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$



Proof by "diagram" (useful!).

In-Class Exercises:

- Finish the membership Table:

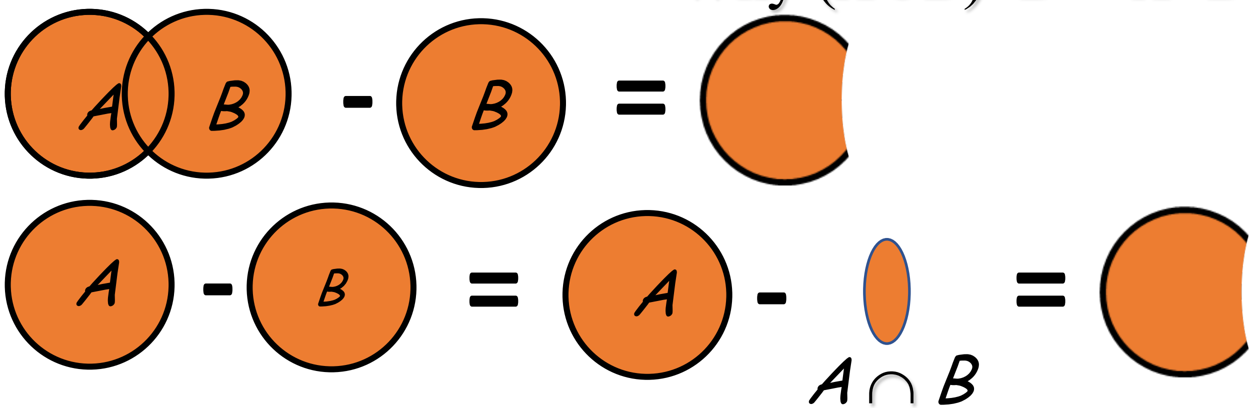
A	B	$A \cup B$	$(A \cup B)-B$	A-B
0	0			
0	1			
1	0			
1	1			

In-Class Exercises:

- Finish the membership Table:

A	B	$A \cup B$	$(A \cup B)-B$	A-B
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

Why $(A \cup B) - B = A - B$?



2. The inclusion-exclusion principle

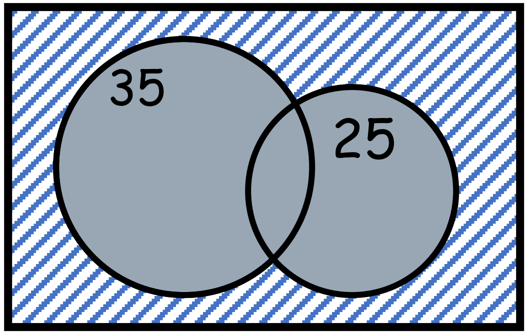
Inclusion/Exclusion

- Considering we have 50 students:

  35 like apple; A

  25 like orange; B

  15 like both; $A \cap B$



How many students don't like the two fruits Neither?

students like orange or apple $A \cap B$

= students only like apple + students only like orange +students like both

= <u>students only like apple +students like both</u> ***(students like apple)*** A

+ <u>students only like orange + students like both</u> ***(students like orange)*** B

- students like both $A \cup B$

=35+25-15=45

$$A \cup B = A + B - A \cap B$$

Students don't like neither = 50-45=5

Principle of Inclusion & Exclusion

- This is so-called Principle of Inclusion & Exclusion:

$$A \cup B = A + B - A \cap B$$

- For three disjoint sets

$$A \cup B = (A-B) + (B-A) + (A \cap B)$$

- We have solved for two finite sets A and B

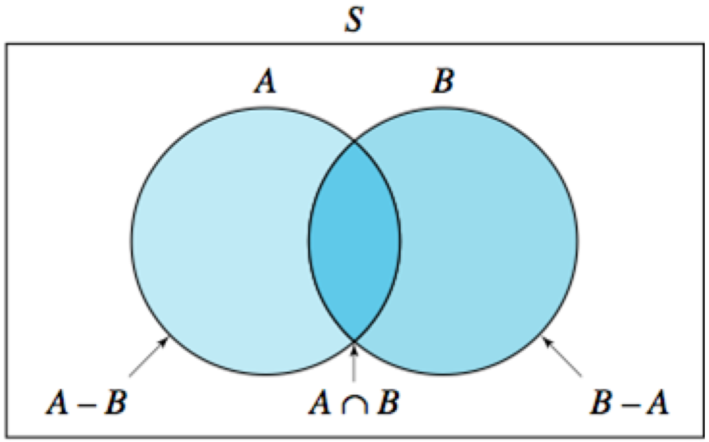
$$A-B = A - (A \cap B) \quad \text{and} \quad B-A = B - (A \cap B)$$

- Hence,

$$A \cup B = (A-B) + (B-A) + (A \cap B)$$

$$= A - (A \cap B) + B - (A \cap B) + (A \cap B)$$

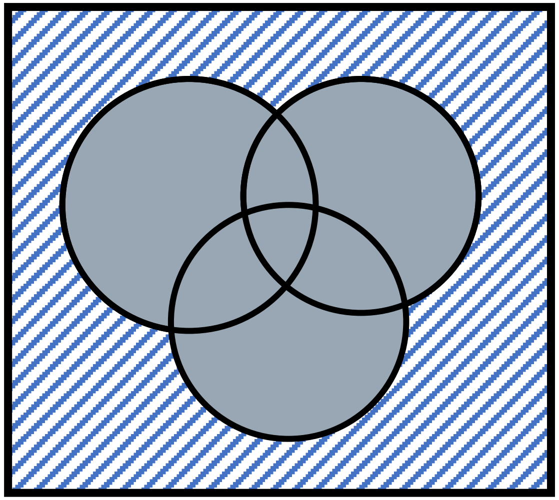
$$= A + B - (A \cap B)$$



Principle of Inclusion & Exclusion

- Similarly, for three sets, we can obtain:

$$A \cup B \cup C = A + B + C - A \cap B - A \cap C - B \cap C + A \cap B \cap C$$



Applications

- How many integers from 1 to 1000 are either multiples of 3 or multiples of 5?

Solutions:

Step 1: Define the sets:

- Let us assume that A = set of all integers from 1 to 1000 that are multiples of 3.
- Let us assume that B = set of all integers from 1 to 1000 that are multiples of 5.
- $A \cup B$ = The set of all integers from 1 to 1000 that are multiples of either 3 or 5.
- $A \cap B$ = The set of all integers that are both multiples of 3 and 5, which also is the set of integers that are multiples of 15.

Step 2: Computer the number in each set: (|A| refers the number of elements in A)

- From 1 to 1000, every third integer is a multiple of 3, each of this multiple can be represented as 3p, for any integer p from 1 through 333, Hence $|A| = 333$.
- Similarly for multiples of 5, each multiple of 5 is of the form 5q for some integer q from 1 through 200. Hence, we have $|B| = 200$.
- To determine the number of multiples of 15 from 1 through 1000, each multiple of 15 is of the form 15r for some integer r from 1 through 66.

Applications

- How many integers from 1 to 1000 are either multiples of 3 or multiples of 5?

Solutions:

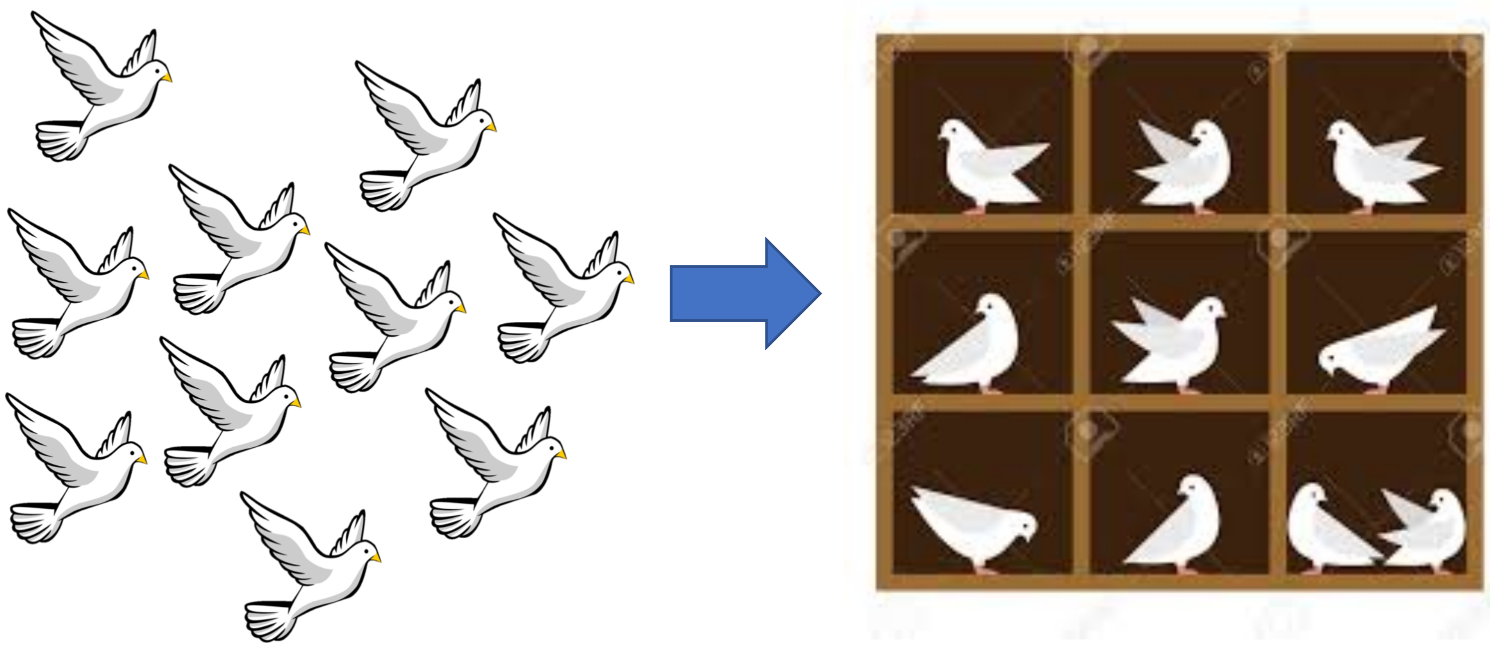
Step 3: To use the inclusion/exclusion principle to obtain $|A \cup B|$

$$
\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B| \\
&= 333 + 200 - 66 \\
&= 467.
\end{aligned}
$$

3. The pigeon-hole principle

The pigeon-hole principle

- Suppose a flock of pigeons fly into a set of pigeonholes to roost;
- If there are more pigeons than pigeonholes, then there must be at least 1 pigeonhole that has more than one pigeon in it
- ***If k+1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects***



The applications of pigeon-hole principle

- Some examples:
- In a group of 13 people, there must be two people with the birthdays in the same month;
- In a group of 367 people, there must be two people with the same birthday.
- In a group of 27 English words, at least two words must start with the same letter

The applications of pigeon-hole principle

- Furthermore:
- If N objects are placed into k boxes, then there is at least one box containing $\lceil N/k \rceil$ objects.
- $\lceil N/k \rceil$ denotes the ceiling function, which is the smallest integer that is not smaller than N/k.

- Examples:
- Among 100 people, there are at least $\lceil 100/12 \rceil = 9$ born on the same month;

In-Class Exercises:

- How many students in a class must there be to ensure that 6 students get the same grade (one of A, B, C, D, or F)?

Solution:

The "boxes" are the grades. Thus, $k = 5$

Thus, we set $\lceil N/5 \rceil = 6$

Lowest possible value for N is 26