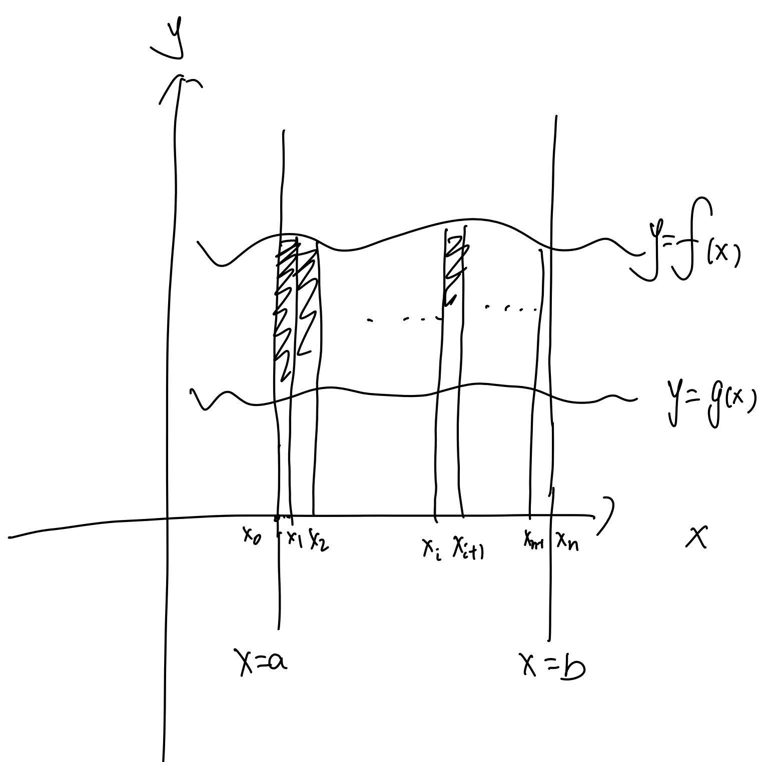
Contents

2. Applications of Integration

This section is to apply integrals in the following applications: computing areas between curves and volumes of solids.

$x_i^* \in [x_{i-1}, x_i]$

$$\sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*) \right) \Delta x$$



2.1. **Areas between curves.** Text Section 5.1,
Exercise: 4, 13, 51, 53.

Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all $x \in [a, b]$.

Using the same idea in Chap. 4, we approximate this region by rectangles and then take the limit of the areas of these rectangles as we increase the number of rectangles. (see Fig 1 and 2 of Page 344)

This approximation is the Riemann sum

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x.$$

Taking the limit as $n \to \infty$, we can also define the **area** as

$$A = \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x.} = \int_a^b (f(x) - g(x))dx$$

Definition 2.1. *The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all $x \in [a, b]$ is*

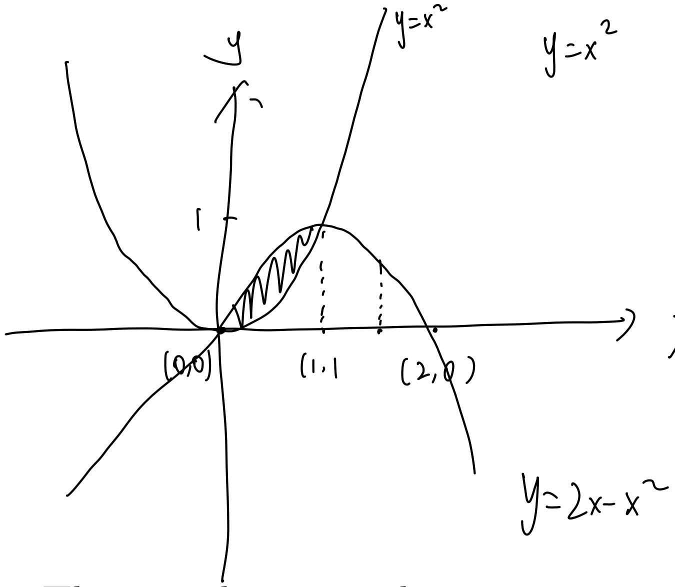
$$A = \int_a^b [f(x) - g(x)]dx.$$

Note that, when $g(x) = 0$, this definition reduces to the previous **Definition 4.1** in Chap 4.

$x^2 = 2x - x^2 \Rightarrow 2x^2 - 2x = 0 \Rightarrow 2x(x-1) = 0$

Ex. [Text example 5.1.2] Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$. $= x(2-x)$

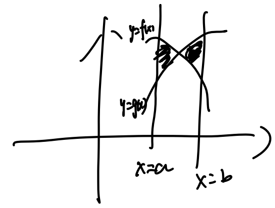
$= 2 - 1 = 1$



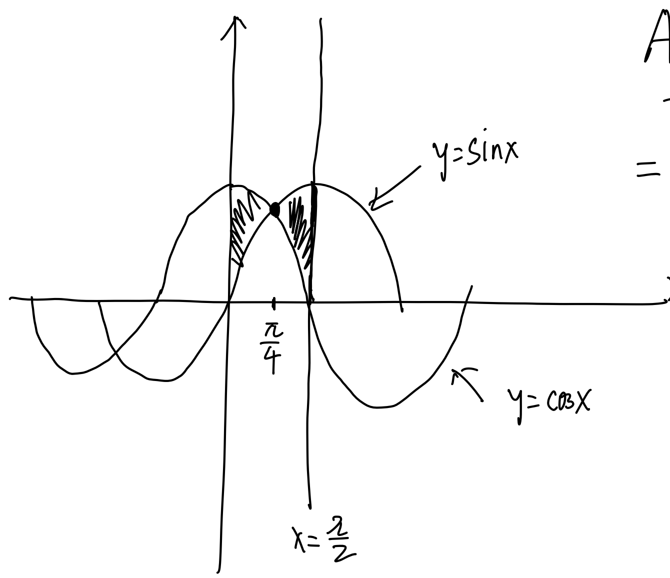
$$A = \int_0^1 (2x - x^2) - x^2\,dx$$
$$= \int_0^1 2x - 2x^2\,dx$$
$$= \left(x^2 - \frac{2}{3}x^3\right)\Big|_{x=0}^{x=1} = \left(1 - \frac{2}{3}\right) - 0 = \frac{1}{3}$$

The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is ($f(x)$ is NOT required to be bigger than $g(x)$ here)

$$\underline{A = \int_a^b |f(x) - g(x)|dx.}$$



Ex. [Text example 5.1.5] Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$.



$$\underline{A = \int_0^{\frac{\pi}{2}} |\sin x - \cos x|\,dx}$$
$$= \int_0^{\frac{\pi}{4}} \cos x - \sin x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x\,dx$$
$$\sin x - (-\cos x)\Big|_0^{\frac{\pi}{4}} + (-\cos x) - \sin x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - [\sin 0 + \cos 0] + \left(-\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)$$

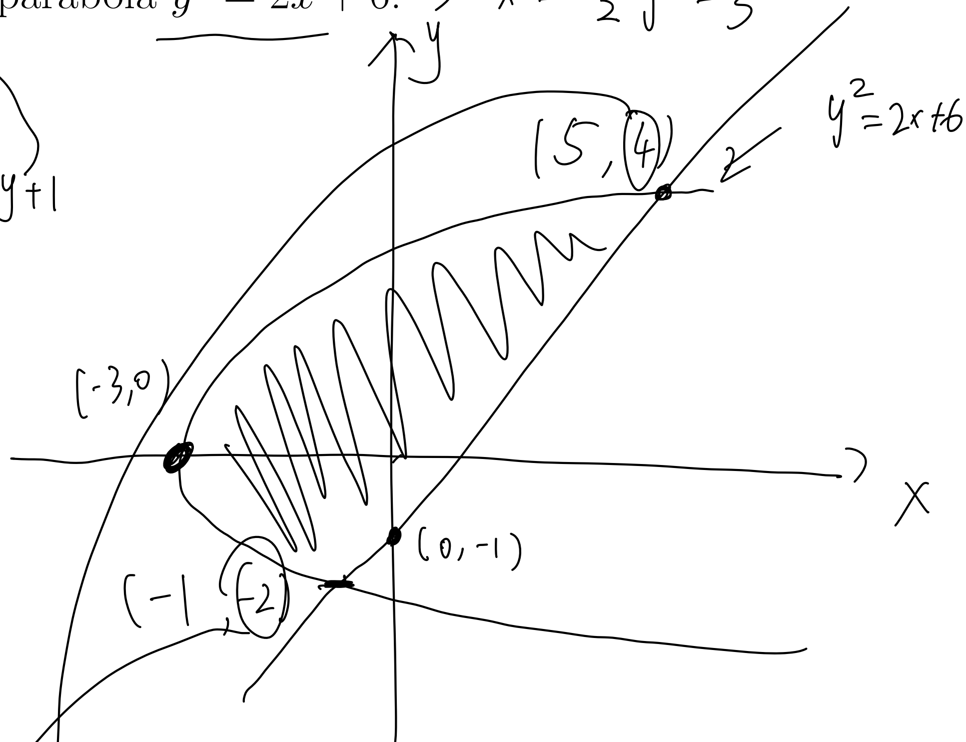
$$= \left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right) - (0+1) + (0-1) - \left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right) = (\sqrt{2}-1) + (\sqrt{2}-1) = 2(\sqrt{2}-1).$$

Some regions are best treated by regarding x as a function of y.

Ex. [Text example 5.1.6] Find the area enclosed by the line $y = x - 1$ $\Rightarrow x = y+1$ and the parabola $y^2 = 2x + 6$. $\Rightarrow \quad x = \frac{1}{2}y^2 - 3$

$$\begin{cases} y^2 = 2\boxed{x}+6 \\ y = x-1 \Rightarrow x = y+1 \end{cases}$$

$$y^2 = 2(1+y)+6$$
$$\Downarrow$$
$$y^2 - 2y - 8 = 0$$
$$(y-4)(y+2) = 0$$
$$\Downarrow$$
$$y = 4, \quad y = -2.$$



$$\int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) \, dy$$

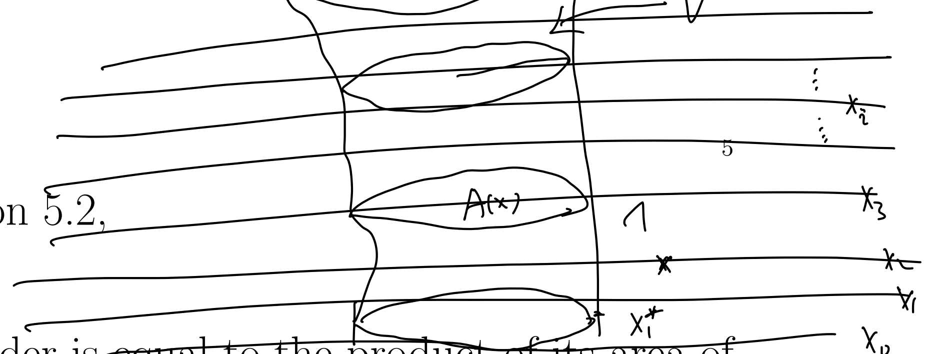
$$= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4 \, dy$$

$$= -\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y \Big|_{y=-2}^{y=4}$$

$$= \left(-\frac{1}{6}\times 64 + \frac{1}{2}\times 16 + 16\right) - \left(-\frac{1}{6}\times(-8) + \frac{1}{2}\times 4 - 8\right)$$

$$\int_0^h A(x)\,dx = ?$$

$\lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x$



2.2. **Volumes.** Text Section 5.2,

Exercise: 7, 9, 49, 61, 65.

The **volume** V of a cylinder is equal to the product of its area of the base A and the height h:

$$V = Ah$$

In particular, if the base is a circle with radius r, the volume of a circular cylinder is

$$V = \pi r^2 h.$$

If the base is a rectangle with length l and width w, the volume of a rectangular box is

$$V = lwh.$$

For a solid S that is not a cylinder, we use the same idea in Sec. 4.1 to calculate the volume. We cut S into pieces and approximate each piece by a cylinder. Then we take the limit of the volumes of these cylinders as we increase the number of cylinders.

Definition 2.2. *Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis is $A(x)$, where A is a continuous function, then the* **volume** *of S is*

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx.$$

Note that, for a cylinder, the cross-sectional area is constant: $A(x) \equiv A$. Then $V = \int_a^b A dx = A(b-a)$, which agrees with the formula $V = Ah$.

$\int_{-r}^{r} \left(\sqrt{r^2-x^2}\right)^2 \pi \, dx$ = ?

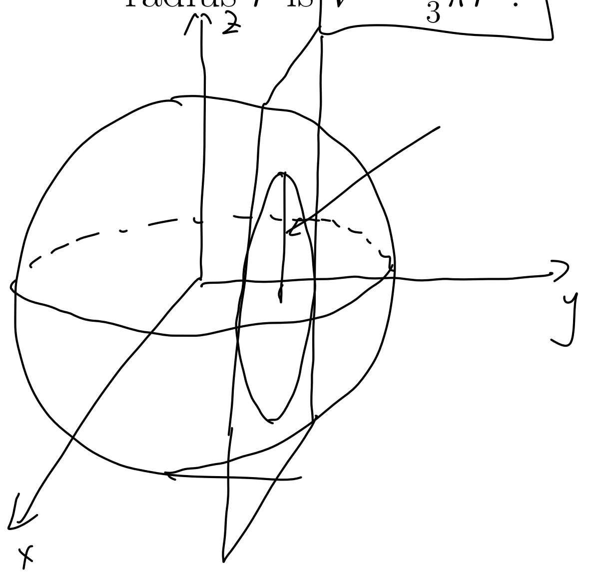
$A(x)$

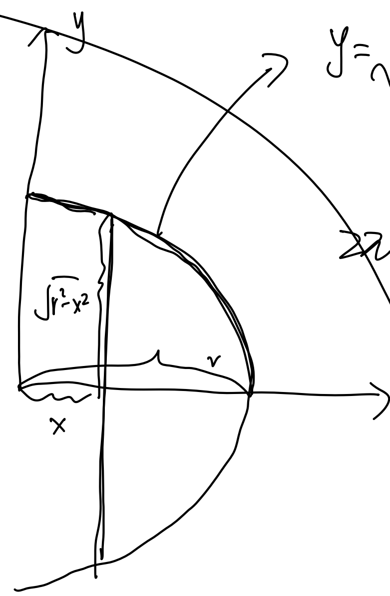
$2\int_0^r (r^2-x^2)\pi \, dx$

$= 2\pi\int_0^r r^2-x^2 \, dx$

$$V = 2\int_0^{r} \boxed{\left(\sqrt{r^2-x^2}\right)^2 \pi}\, dx \qquad 2\pi\left(r^2x - \frac{1}{3}x^3\right)\Big|_{x=0}^{x=r}$$

Ex. [Text example 5.2.1] Show that the volume of a sphere of radius r is $\boxed{V = \frac{4}{3}\pi r^3}$.

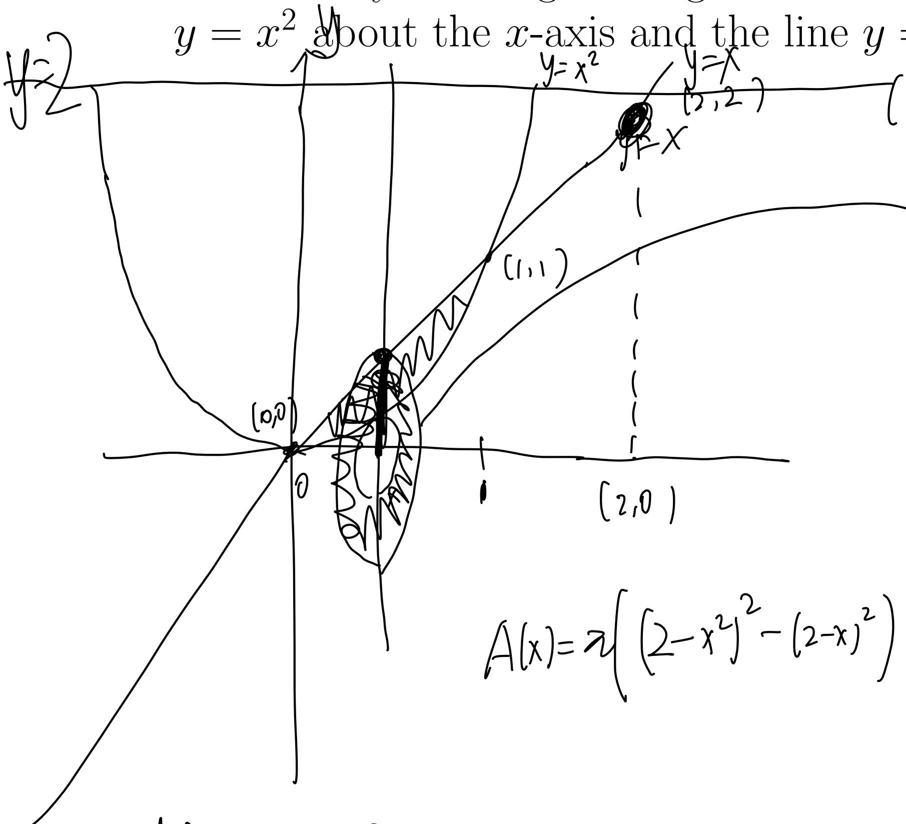




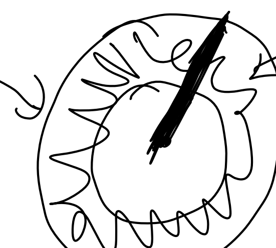
$y = \sqrt{r^2-x^2}$

$$= 2\pi\left(r^3 - \frac{1}{3}r^3\right) - 0 = 2\pi\cdot\frac{2}{3}r^3 = \frac{4}{3}\pi r^3.$$

Ex. [Text example 5.2.4 and 5.2.5] Find the volumes of the solid obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2$ about the x-axis and the line $y = 2$, respectively.



(i) About the x-axis.



$A(x) = \pi\left(x^2 - (x^2)^2\right)$

$$V = \int_0^1 A(x)\,dx$$

$$= \int_0^1 \pi\left(x^2 - x^4\right)dx$$

$$= \pi\int_0^1 x^2 - x^4\, dx$$

$$= \pi\left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^1$$

$A(x) = \pi\left((2-x^2)^2 - (2-x)^2\right)$

(ii) About $y=2$.

$$V = \int_0^1 A(x)\,dx$$

$$= \int_0^1 \pi\left((2-x^2)^2 - (2-x)^2\right)dx$$

$$= \pi \int_0^1 (4-4x^2+x^4-4+4x-x^2)\, dx = \pi\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{2\pi}{15}$$

$$= \pi \int_0^1 4x-5x^2+x^4\, dx = \pi\left(2x^2-\frac{5}{3}x^3+\frac{1}{5}x^5\right)\Big|_{x=0}^{x=1}$$

$$= \pi\left(2-\frac{5}{3}+\frac{1}{5}\right) = \frac{8}{15}\pi$$

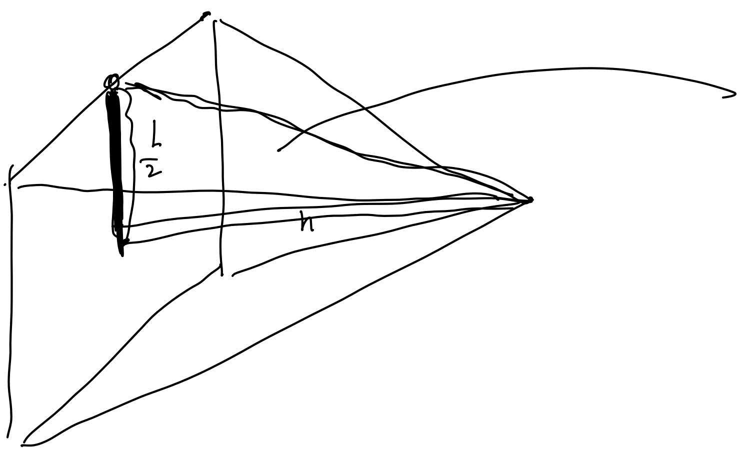
Summary:

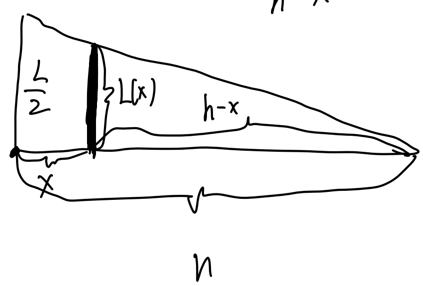
For **solids of revolution**, we find the cross-sectional area $A(x)$ or $A(y)$ in one of the following way:

- If the cross-section is a disk, then
$$A = \pi(\text{radius})^2.$$
- If the cross-section is a washer, then
$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2.$$

Ex. [Text example 5.2.8] Find the volume of a pyramid whose base is a square with side L and whose height is h.

$$\frac{L(x)}{h-x} = \frac{\frac{L}{2}}{h}$$





$$\Downarrow$$

$$L(x) = \frac{L(h-x)}{2h}$$

The length of the square is $2L(x) = \frac{L(h-x)}{h}$

$$A(x) = \left[\frac{L(h-x)}{h}\right]^2$$

$$V = \int_0^h \frac{L^2(h-x)^2}{h^2}\, dx$$

$$= \frac{L^2}{h^2}\int_0^h h^2-2hx+x^2\, dx = \frac{L^2}{h^2}\left(h^2x-hx^2+\frac{1}{3}x^3\right)\Big|_0^h$$

$$= \frac{L^2}{h^2}\left(h^3 - h^3 + \frac{1}{3}h^3\right) - 0 = \frac{L^2 h}{3}.$$

2.3. **Volumes by cylindrical shells.** Text Section 5.3, Exercise: 13, 29, 41, 45.

Consider **a cylindrical shell** with inner radius r_1, outer radius r_2 and height h. Its volume is

$$\begin{aligned} V &= V_2 - V_1 = \pi r_2^2 h - \pi r_1^2 h = \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1) \end{aligned}$$

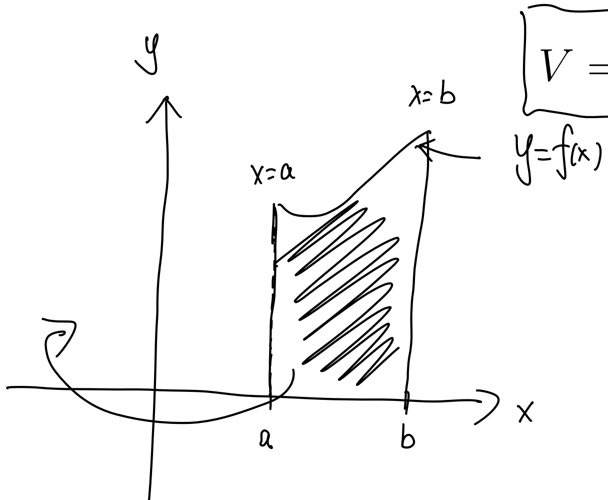
Letting $\Delta r = r_2 - r_1$ (the thickness of the shell) and $r = \frac{1}{2}(r_2 + r_1)$ (the average radius of the shell), the above formula for the volume of a cylindrical shell becomes

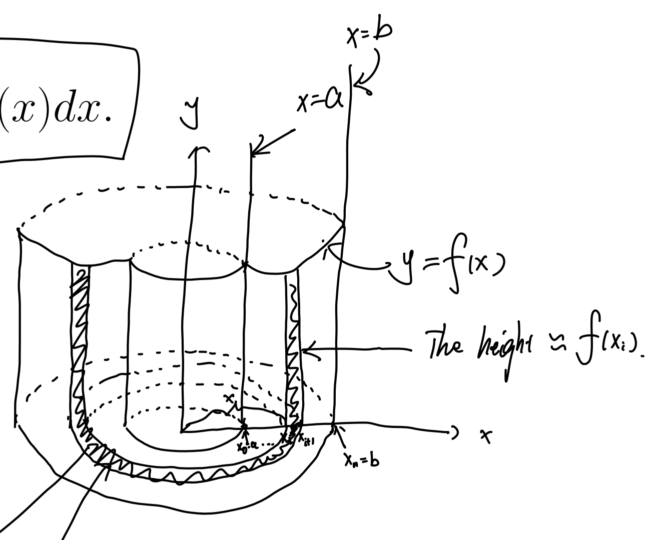
$$V = 2\pi r h \Delta r.$$

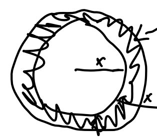
Inspired by this formula, we can calculate volumes of **solids of revolution** in a different way. We cut the solid S into pieces and approximate each piece by a cylindrical shell.

Proposition 2.3. *Let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$ [where $f(x) \geq 0$], $y = 0$, $x = a$ and $x = b$, where $b > a \geq 0$. (see Fig 3 of Page 363) Its volume is given by*

$$\boxed{V = \int_a^b 2\pi x f(x) dx.}$$







The length of the circle

Area of the base $\approx 2\pi x_i \Delta x$

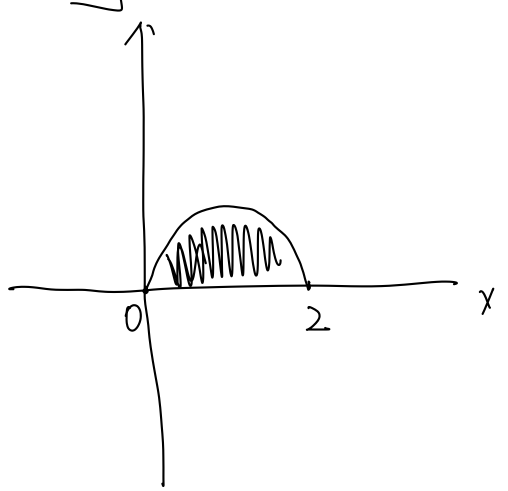
$$V \approx \sum_{i=0}^{n} 2\pi x_i \Delta x f(x_i)$$

$$V = \lim_{n\to\infty} \sum_{i=0}^{n} 2\pi x_i f(x_i)\Delta x$$

$$\int_a^b 2\pi x f(x)\,dx$$

Ex. [Text example 5.3.1] Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

$$y = x^2(2-x)$$



$$V = \int_0^2 2\pi x\left(2x^2 - x^3\right)dx$$

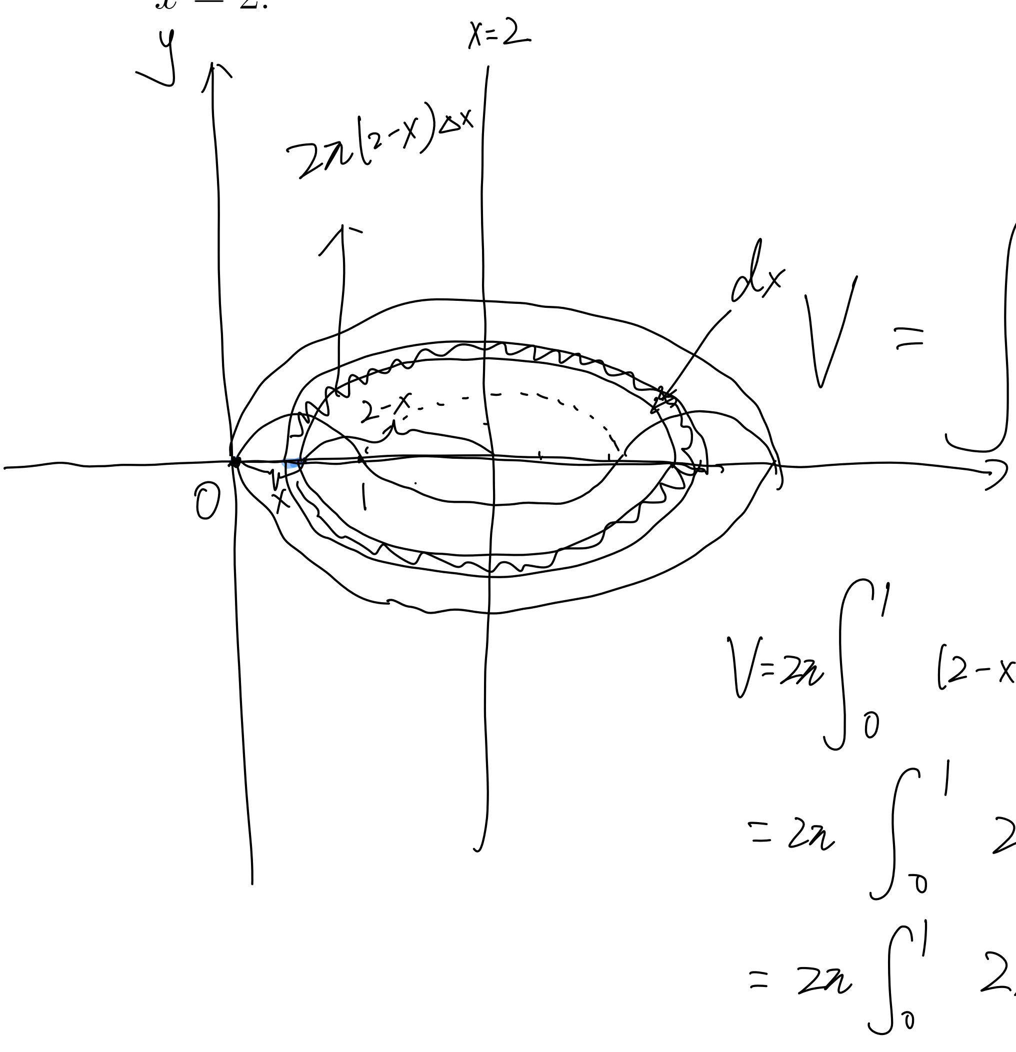
$$= \int_0^2 4\pi x^3 - 2\pi x^4\,dx$$

$$= \pi x^4 - \frac{2}{5}\pi x^5 \Big|_{x=0}^{x=2} = 16\pi - \frac{64}{5}\pi$$

$$\frac{16}{5}\pi$$

$$\begin{array}{r} 320 \\ -256 \\ \hline 64 \end{array}$$

Ex. [Text example 5.3.4] Find the volume of the solid obtained by rotating the region bounded by $\boxed{y = x - x^2}$ and $\boxed{y = 0}$ about the line $x = 2$.



$$V = \int_0^1 (x-x^2)\,2\pi(2-x)\,dx$$

$$V = 2\pi\int_0^1 (2-x)(x-x^2)\,dx$$

$$= 2\pi\int_0^1 2x - 2x^2 - x^2 + x^3\,dx$$

$$= 2\pi\int_0^1 2x - 3x^2 + x^3\,dx$$

$$= 2\pi \left(x^2 - x^3 + \frac{1}{4}x^4\right)\Big|_0^1$$

$$\parallel$$

$$2\pi\,\frac{1}{4} = \frac{\pi}{2}.$$

2.4. Average value of a function. Text Section 5.5, Exercise: 7, 9, 23, 24.

The average value of finitely many numbers $y_1, y_2, \cdots, y_n$ is given by

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}.$$

The average value of a function f on the interval $[a, b]$:

Divide the interval $[a, b]$ into n subintervals with equal width $\Delta x = \frac{b-a}{n}$. We choose sample points $x_i^* \in [x_{i-1}, x_i]$ and calculate the average of the numbers $f(x_1^*), f(x_2^*), \cdots, f(x_n^*)$:

$$\lim_{n\to+\infty} \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{\boxed{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}}{(b-a)/\Delta x} \quad = \sum_{i=1}^{n} f(x_i^*)$$

$$\Delta x = \frac{b-a}{n}$$

$$\Updownarrow$$

$$\boxed{n = \frac{b-a}{\Delta x}}$$

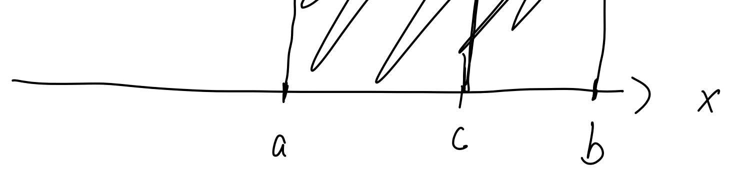
$$= \lim_{n\to+\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

$$= \frac{1}{b-a} \boxed{\lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

By letting $n \to \infty$, the above quantity has a limit, which is defined to be the average.

Definition 2.4. *The average of a function f on the interval $[a, b]$ is*

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)dx.$$



The mean value theorem for integrals

Theorem 2.5. *If f is continuous on $[a, b]$, then there exists a number $c \in [a, b]$ such that*

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)dx,$$

that is

$$\int_a^b f(x)dx = f(c)(b-a).$$

$$\text{Ave} = \frac{\int_{-1}^{2} 1+x^2\,dx}{2-(-1)} = \frac{x+\frac{1}{3}x^3\Big|_{x=-1}^{x=2}}{3} = \frac{2+\frac{8}{3}-(-1-\frac{1}{3})}{3}$$

$$\frac{3+3}{3} = 2$$

Ex. [Text example 5.5.2] Find the average value of $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

The mean value theorem for derivatives.

Thm (Reminder)

Suppose that F is continuously differentiable on $[a,b]$. Then there exists $c \in [a,b]$ such that

$$F(b) - F(a) = F'(c)(b-a).$$

Proof of Thm 2.5: We define $F(x) = \int_a^x f(t)\,dt$

Since f is continuous on $[a,b]$, then $F'(x) = f(x)$ is continuous on $[a,b] \Rightarrow F(x)$ is continuously differentiable on $[a,b]$.

By applying the mean value theorem for derivatives, there exists some $c \in [a,b]$ such that

$$F(b) - F(a) = F'(c)(b-a)$$

$$\int_a^b f(t)\,dt - \underbrace{\boxed{\int_a^a f(t)\,dt}}_{=0} \qquad\qquad f(c)(b-a).$$

$$\boxed{\int_a^b f(x)\,dx}$$