MA1300 Solutions to Self Practice # 12

1. Find the derivative of the function.

$$\text{(a). } y = \cos\left(\frac{1-e^{2x}}{1+e^{2x}}\right),$$

$$\text{(b). } f(t) = \sin^2\left(e^{\sin^2 t}\right),$$

$$\text{(c). } f(x) = \sin(\ln x),$$

$$\text{(d). } h(x) = \ln\left(x + \sqrt{x^2-1}\right),$$

$$\text{(e). } y = x^{\sin x},$$

$$\text{(f). } y = (\sin x)^{\ln x},$$

$$\text{(g). } y = \sin^{-1}(2x+1),$$

$$\text{(h). } g(x) = \sqrt{x^2-1}\sec^{-1} x,$$

$$\text{(i). } y = \cos^{-1}(e^{2x}),$$

$$\text{(j). } h(x) = \ln(\cosh x),$$

$$\text{(k). } y = e^{\cosh(3x)}.$$

Solution.

(a).

$$\begin{aligned}\frac{d}{dx}\cos\left(\frac{1-e^{2x}}{1+e^{2x}}\right) &= -\left(\sin\frac{1-e^{2x}}{1+e^{2x}}\right)\left(\frac{-2e^{2x}(1+e^{2x})-2(1-e^{2x})e^{2x}}{(1+e^{2x})^2}\right)\\ &= \left(\sin\frac{1-e^{2x}}{1+e^{2x}}\right)\frac{4e^{2x}}{(1+e^{2x})^2}.\end{aligned}$$

(b).

$$\frac{d}{dt}\sin^2(e^{\sin^2 t}) = 4\sin(e^{\sin^2 t})\cos((e^{\sin^2 t}))e^{\sin^2 t}\sin t\cos t.$$

(c).

$$\frac{d}{dx}\sin(\ln x) = \cos(\ln x)\frac{1}{x}.$$

(d).

$$\frac{d}{dx}\ln(x+\sqrt{x^2-1}) = \frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{x}{\sqrt{x^2-1}}\right) = \frac{1}{\sqrt{x^2-1}}.$$

(e).

$$\begin{aligned}\frac{d}{dx}x^{\sin x} &= \frac{d}{dx}e^{(\sin x)\ln x} = e^{(\sin x)\ln x}\left(\frac{\sin x}{x}+(\cos x)\ln x\right)\\ &= x^{\sin x}\left(\frac{\sin x}{x}+(\cos x)\ln x\right).\end{aligned}$$

(f).

$$\frac{d}{dx}(\sin x)^{\ln x} = \frac{d}{dx}e^{(\ln x)\ln(\sin x)} = (\sin x)^{\ln x}\left(\frac{\ln(\sin x)}{x}+\frac{(\cos x)\ln x}{\sin x}\right).$$

(g).

$$\frac{d}{dx}\sin^{-1}(2x+1) = \frac{2}{\sqrt{1-(2x+1)^2}} = \frac{1}{\sqrt{-x(x+1)}}.$$

(h).

$$\frac{d}{dx}\sqrt{x^2-1}\sec^{-1}x = \frac{x}{\sqrt{x^2-1}}\sec^{-1}x + \frac{\sqrt{x^2-1}}{|x|\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}}\sec^{-1}x + \frac{1}{|x|}.$$

(i).

$$\frac{d}{dx}\cos^{-1}(e^{2x}) = -\frac{2e^{2x}}{\sqrt{1-e^{4x}}}.$$

(j).

$$\frac{d}{dx}\ln(\cosh x) = \frac{1}{\cosh x}\sinh x = \tanh x.$$

(k).

$$\frac{d}{dx}e^{\cosh(3x)} = 3e^{\cosh(3x)}\sinh(3x).$$

2. Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

$$\begin{aligned}
&\text{(a).}\ \lim_{x\to 0}\frac{e^x-1-x}{x^2},\\
&\text{(b).}\ \lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2},\\
&\text{(c).}\ \lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x},\\
&\text{(d).}\ \lim_{x\to 0}(1-2x)^{1/x}.
\end{aligned}$$

Solution.

(a).

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2} = \lim_{x\to 0}\frac{e^x-1}{2x} = \lim_{x\to 0}\frac{e^x}{2} = \frac{1}{2}.$$

(b).

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2} &= \lim_{x\to 0}\frac{-m\sin mx+n\sin nx}{2x}\\
&= \lim_{x\to 0}\frac{-m^2\cos mx+n^2\cos nx}{2} = \frac{n^2-m^2}{2}.
\end{aligned}$$

(c).

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x} = \lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x} = \lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x} = \lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x} = 2.$$

(d).

$$\begin{aligned}
\lim_{x\to 0}(1-2x)^{1/x} &= \lim_{x\to 0}\exp\left(\frac{1}{x}\ln(1-2x)\right)\\
&= \exp\left(\lim_{x\to 0}\frac{\ln(1-2x)}{x}\right) = \exp\left(\lim_{x\to 0}\frac{\frac{-2}{1-2x}}{1}\right) = e^{-2}.
\end{aligned}$$