MA1300 Solutions to Self Practice # 14

1. Determine whether the series converges or diverges.

$$\text{(a).} \sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}, \quad \text{(b).} \sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}, \quad \text{(c).} \sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}.$$

Solution. (a) Observe that $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ have positive terms. Since $\lim_{n\to\infty} \frac{\frac{n+1}{n\sqrt{n}}}{\frac{1}{\sqrt{n}}} = 1$ and the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (since $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges), we know by the Limit Comparison Test that the series $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$ diverges.

(b) Observe that the terms of $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ are positive for $k \geq 2$. Since $\lim_{k\to\infty} \frac{\frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}}{\frac{1}{k^2}} = 2$ and the p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ with $p = 2 > 1$ converges, we know by the Limit Comparison Test that the series $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ converges.

(c) Observe that $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$ and $\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ have positive terms. Since $\lim_{n\to\infty} \frac{\frac{4^{n+1}}{3^n-2}}{\left(\frac{4}{3}\right)^n} = 4$ and the geometric series $\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ diverges, we know by the Limit Comparison Test that the series $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$ diverges.

2. Test the series for convergence or divergence.

$$\text{(a).} \sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}, \quad \text{(b).} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}, \quad \text{(c).} \sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}, \quad \text{(d).} \sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right).$$

Solution. (a) Since the general term $(-1)^n \frac{3n-1}{2n+1}$ does not converge to zero, we know that the series diverges.

(b) This is an alternating series. Consider the function f defined on $(0, \infty)$ by $f(x) = \frac{x^2}{x^3+4}$. It satisfies $f'(x) = \frac{x(8-x^3)}{(x^3+4)^2} < 0$ for $x > 2$. So the positive term $b_n := \frac{n^2}{n^3+4}$ satisfies $b_n > b_{n+1}$ for $n > 2$ and $\lim_{n\to\infty} b_n = 0$. Then the series converges.

(c) Since $\frac{n^n}{n!} = \frac{n \cdot n \cdots n}{1 \cdot 2 \cdots n} \geq n \to \infty$ as $n \to \infty$, we see that the general term $(-1)^n \frac{n^n}{n!}$ does not converge to zero. Therefore, the series diverges.

(d) Since $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$, we know $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}+\sqrt{n}}$ is an alternating series and the positive term $\frac{1}{\sqrt{n+1}+\sqrt{n}}$ tends to zero monotonically, therefore the series converges.

3. For what values of p is the series convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}.$$

Solution. When $p > 0$, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ is alternating and the positive term $\frac{1}{n^p}$ tends to zero monotonically, therefore the series converges.

When $p \leq 0$, the general term $\frac{(-1)^{n-1}}{n^p}$ does not converge to zero, therefore the series diverges.

4. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\text{(a).}\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}, \quad \text{(b).}\ \sum_{n=1}^{\infty} (-1)^n \frac{(1.1)^n}{n^4}, \quad \text{(c).}\ \sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdots(2n)}{n!}.$$

Solution. (a) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ is a p-series with $p = \frac{1}{4} < 1$, so the series is not absolutely convergent.

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$ is alternating and the positive term $\frac{1}{\sqrt[4]{n}}$ tends to zero monotonically, therefore the series converges. So it is conditionally convergent.

(b) Since $\frac{(1.1)^n}{n^4} = e^{n\ln 1.1 - 4\ln n} = e^{n(\ln 1.1 - 4\frac{\ln n}{n})} \to \infty$ as $n \to \infty$, we know that the general term $(-1)^n \frac{(1.1)^n}{n^4}$ does not converge to zero as $n \to \infty$, so the series diverges.

(c) We have $\frac{2\cdot 4\cdot 6\cdots(2n)}{n!} = \frac{2^n(1\cdot 2\cdot 3\cdots n)}{n!} = 2^n \to \infty$ as $n \to \infty$, we know that the general term $\frac{2\cdot 4\cdot 6\cdots(2n)}{n!}$ does not converge to zero as $n \to \infty$, so the series diverges.

5. Test the series for convergence or divergence.

$$(a)\ \sum_{n=1}^{\infty} (-1)^n \cos(1/n^2), \qquad (b)\ \sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}.$$

Solution. (a) Since $\cos(1/n^2) \to \cos 0 = 1$ as $n \to \infty$, we know that the general term $(-1)^n \cos(1/n^2)$ does not converge to zero as $n \to \infty$, so the series diverges.

(b) This is an alternating series. Consider the function f defined on $(0, \infty)$ by $f(x) = \frac{\sqrt{x}}{x+5}$. It satisfies $f'(x) = \frac{5-x}{2\sqrt{x}(x+5)^2} < 0$ for $x > 5$. So the positive term $b_j := \frac{\sqrt{j}}{j+5}$ satisfies $b_j > b_{j+1}$ for $j > 5$ and $\lim_{j\to\infty} b_j = 0$. Then the series converges.

6. Find the radius of convergence and interval of convergence of the series.

$$(a)\ \sum_{n=1}^{\infty} \frac{x^n}{2n-1}, \quad (b)\ \sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}, \quad (c)\ \sum_{n=1}^{\infty} n!(2x-1)^n.$$

Solution. (a) The general term $a_n = \frac{x^n}{2n-1}$ of the series satisfies $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{2n+1}\right|}{\left|\frac{x^n}{2n-1}\right|} = |x|$. So by the ratio test, the series $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$ converges (absolutely) when $|x| < 1$ and diverges when $|x| > 1$. Hence the radius of convergence of the series is $R = 1$.

To determine the interval of convergence of the series, we consider the end points $x = 1, -1$. When $x = -1$, the series is alternating and the positive term $\frac{1}{2n-1}$ tends to zero monotonically, therefore the series converges.

When $x = 1$, the general term $\frac{1}{2n-1}$ satisfies $\lim_{n\to\infty} \frac{\frac{1}{2n-1}}{\frac{1}{n}} = \frac{1}{2}$. But the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we know by the Limit Comparison Test that the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges. Thus the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$ is $[-1, 1)$.

(b) The general term $a_n = \frac{(2x-1)^n}{5^n\sqrt{n}}$ of the series satisfies $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}\right|}{\left|\frac{(2x-1)^n}{5^n\sqrt{n}}\right|} = \frac{2|x-\frac{1}{2}|}{5}$. So by the ratio test, the series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$ converges (absolutely) when $|x - \frac{1}{2}| < \frac{5}{2}$ and diverges when $|x - \frac{1}{2}| > \frac{5}{2}$. Hence the radius of convergence of the series is $R = \frac{5}{2}$.

To determine the interval of convergence of the series, we consider the end points $x = \frac{1}{2} \pm \frac{5}{2}$. When $x = -2$, the series is alternating and the positive term $\frac{1}{\sqrt{n}}$ tends to zero monotonically, therefore the series converges.

When $x = 3$, the series has positive terms and the general term $a_n = \frac{1}{\sqrt{n}}$ satisfies $a_n \geq \frac{1}{n}$. But the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we know by the Comparison Test that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. Thus interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$ is $[-2, 3)$.

(c) The general term $a_n = n!(2x-1)^n$ of the series satisfies $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|(n+1)!(2x-1)^{n+1}\right|}{|n!(2x-1)^n|} = (n+1)|2x-1|$. So by the ratio test, the series $\sum_{n=1}^{\infty} n!(2x-1)^n$ diverges for any $x \neq \frac{1}{2}$, and converges only at the point $x = \frac{1}{2}$. Hence the radius of convergence of the series is $R = 0$ and the interval of convergence of the series is $[\frac{1}{2}, \frac{1}{2}] = \{\frac{1}{2}\}$.

7. If $f^{(n)}(0) = (n+1)!$ for $n = 0, 1, 2, \ldots$, find the Maclaurin series for f and its radius of convergence.

Solution. According to the definition, the Maclaurin series for f is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} (n+1)x^n$$

To find the radius of convergence of the series, consider the general term $a_n = nx^n$. It satisfies $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|(n+2)x^{n+1}\right|}{|(n+1)x^n|} = |x|$. So by the ratio test, the series $\sum_{n=0}^{\infty} (n+1)x^n$ converges (absolutely) when $|x| < 1$ and diverges when $|x| > 1$. Hence the radius of convergence of the series is $R = 1$.

8. Find the Maclaurin series for f using the definition of a Maclaurin series. Also, find the associated radius of convergence.

$$f(x) = \ln(1+x).$$

Solution. The derivative of f is $f'(x) = \frac{1}{1+x} = (1+x)^{-1}$. So its higher order derivatives are given by

$$f^{(n)}(x) = (-1) \cdot (-2) \cdots (-(n-1))(1+x)^{-n}, \qquad n = 2, 3, \ldots.$$

Hence $f(0) = \ln 1 = 0$, $f'(0) = (1+0)^{-1} = 1$ and $f^{(n)}(0) = (-1)^{n-1}(n-1)!$ for $n = 2, 3, \ldots$. According to the definition, the Maclaurin series for f is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n.$$

To find the radius of convergence of the series, consider the general term $a_n = (-1)^{n-1}\frac{1}{n}x^n$ for $n \in \mathbb{N}$. It satisfies $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|(-1)^n \frac{1}{n+1} x^{n+1}\right|}{\left|(-1)^{n-1} \frac{1}{n} x^n\right|} = |x|$. So by the ratio test, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$ converges (absolutely) when $|x| < 1$ and diverges when $|x| > 1$. Hence the radius of convergence of the series is $R = 1$.

9. Use a Maclaurin series in Table 1 on page 786 of the textbook to obtain the Maclaurin series for the given function

$$f(x) = \begin{cases} \frac{x - \sin x}{x^3} & \text{if } x \neq 0, \\ \frac{1}{6} & \text{if } x = 0. \end{cases}$$

Solution. From the formula for $\sin x$ in Table 1 on page 786 of the textbook, we know that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad x \in (-\infty, \infty).$$

It means that the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ converges and its sum equals to the value $\sin x$ for every $x \in (\infty, \infty)$. It follows by multiplying each term of the series by $\frac{1}{x^3}$ that for every $x \neq 0$, the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-2}}{(2n+1)!}$ also converges and

$$\frac{x}{x^3} + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-2}}{(2n+1)!} = \frac{\sin x}{x^3}.$$

Then for every $x \neq 0$, by multiplying each term of the series by -1 and changing the index $n-1$ by n, the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+3)!}$ converges and

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+3)!} = \frac{x - \sin x}{x^3}.$$

Hence for every $x \neq 0$, the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+3)!}$ converges and equals to $f(x)$. At the point $x = 0$, the series also converges and equals to $f(0) = \frac{1}{6}$. Thus,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+3)!}, \qquad x \in (-\infty, \infty).$$

By Theorem 1 of this section, $f^{(2n)}(0) = \frac{(-1)^n}{(2n+3)(2n+2)(2n+1)}$ and $f^{(2n+1)}(0) = 0$ for $n = 0, 1, \ldots$. Therefore, the Maclaurin series of the function f is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+3)!}$.

10. If $f(x) = (1 + x^3)^{30}$, what is $f^{(58)}(0)$?

Solution. From the formula for $(1 + x)^{30}$ in Table 1 on page 786 of the textbook, we know that

$$(1 + x)^{30} = \sum_{n=0}^{\infty} \binom{30}{n} x^n, \qquad x \in (-1, 1).$$

But for $n \geq 31$, we have $\binom{30}{n} = \frac{30\cdot(30-1)\cdots(30-30)\cdots(30-(n-1))}{n!} = 0$. Hence

$$(1 + x)^{30} = \sum_{n=0}^{30} \binom{30}{n} x^n, \qquad x \in (-1, 1).$$

When $x \in (-1, 1)$, we also have $x^3 \in (-1, 1)$. So

$$f(x) = (1 + x^3)^{30} = \sum_{n=0}^{30} \binom{30}{n} x^{3n}, \qquad x \in (-1, 1).$$

This means that the series $\sum_{n=0}^{30} \binom{30}{n} x^{3n}$ is the Maclaurin series of the function $f(x) = (1 + x^3)^{30}$. It particular, $\frac{f^{(58)}(0)}{58!} = 0$. Hence $f^{(58)}(0) = 0$.

11. Show that the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is not equal to its Maclaurin series.

Proof. We first claim that for $n \in \mathbb{N}$,

$$f^{(n)}(x) = \begin{cases} e^{-1/x^2} P_{3n}(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where P_{3n} is a polynomial of degree $3n$. We prove this claim by mathematical induction. The case $n = 1$ is trivial since for $x \neq 0$, we have $f'(x) = e^{-1/x^2} 2x^{-3}$ while $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{1}{x} e^{-1/x^2} = 0$. Assume that the statement is true for $n = k$. That is,

$$f^{(k)}(x) = \begin{cases} e^{-1/x^2} P_{3k}(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where P_{3k} is a polynomial of degree $3k$. Then we have for $x \neq 0$,

$$f^{(k+1)}(x) = \left(f^{(k)}\right)'(x) = \left(e^{-1/x^2} P_{3k}(\frac{1}{x})\right)'(x) = e^{-1/x^2} 2x^{-3} P_{3k}(\frac{1}{x}) + e^{-1/x^2} \left(P_{3k}\right)'(\frac{1}{x}) \left(-x^{-2}\right).$$

Hence for $x \neq 0$, $f^{(k+1)}(x) = e^{-1/x^2} P_{3k+3}(\frac{1}{x})$, where P_{3k+3} is the polynomial of degree $3k+3$ given by $P_{3k+3}(u) = 2u^3 P_{3k}(u) - u^2 P'_{3k}(u)$. Also, we have

$$f^{(k+1)}(0) = \lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x} = \lim_{x\to 0} \frac{1}{x} e^{-1/x^2} P_{3k}(\frac{1}{x}) = 0.$$

This completes the induction process and proves the statement for every $n \in \mathbb{N}$.

Now we apply the statement. Since $f(0) = 0$ and $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}$, we know that the Maclaurin series of the function f is $\sum_{n=0}^{\infty} 0x^n = 0$. However, the function f is not equal to the zero function. This proves the desired result.

12. Let f be a continuous function on $[a, \infty)$. If f is differentiable on (a, ∞) and $f(a) = 0$, $\lim_{x\to\infty} f(x) = 0$, prove that there exists some $\xi \in (a, \infty)$ such that $f'(\xi) = 0$.

Proof. The statement is trivial if f is identically zero on (a, ∞).

If f is not identically zero on (a, ∞), there exists some $c \in (a, \infty)$ such that $f(c) \neq 0$.

If $f(c) > 0$, since $\lim_{x\to\infty} f(x) = 0$, there exists some $b > c$ such that $f(b) < f(c)$. Take a number Δ such that

$$\max\{0, f(b)\} < \Delta < f(c).$$

Applying the Intermediate Value Theorem to f on the closed intervals $[a, c]$ and $[c, b]$, we know that there exist some $p \in (a, c)$ and $q \in (c, b)$ such that

$$f(p) = \Delta, \qquad f(q) = \Delta.$$

Then applying the mean value theorem to f on the interval $[p, q]$ we know that there exists some $\xi \in (p, q) \subset (a, \infty)$ such that

$$f'(\xi) = \frac{f(q) - f(p)}{q - p} = 0.$$

This proves the statement in the case $f(c) > 0$.

The proof for the case $f(c) < 0$ is similar.

13. Let f be a continuous function on $[0,\infty)$. If f is differentiable on $(0,\infty)$, $f(0)=0$, and f' is increasing on $(0,\infty)$, prove that the function g defined by $g(x)=\frac{f(x)}{x}$ is increasing on $(0,\infty)$.

Proof. We compute

$$g'(x)=\frac{xf'(x)-f(x)}{x^2}=\frac{f'(x)-\frac{f(x)}{x}}{x}, \qquad x\in(0,\infty).$$

To prove that g is increasing on $(0,\infty)$, it is sufficient to show that $f'(x)>\frac{f(x)}{x}$ for each $x\in(0,\infty)$. To this end, let $x\in(0,\infty)$. We apply the mean value theorem to the function f on the interval $[0,x]$, and see that there exists some $\xi\in(0,x)$ such that

$$f'(\xi)=\frac{f(x)-f(0)}{x-0}=\frac{f(x)}{x}.$$

Since the function f' is increasing on $(0,\infty)$ and $0<\xi<x$, we know that $f'(\xi)<f'(x)$. Therefore,

$$\frac{f(x)}{x}=f'(\xi)<f'(x).$$

This proves what we need, and the statement is proved.

14. Prove that the function f defined by $f(x)=\begin{cases}\frac{3-x^2}{2} & \text{if } x\in[0,1]\\ \frac{1}{x} & \text{if } x\in(1,2]\end{cases}$ on the interval $[0,2]$ satisfies the condition of the mean value theorem. Then find a number $\xi\in(0,2)$ such that $f'(\xi)=\frac{f(2)-f(0)}{2-0}$.

Proof. The function f is defined as a piecewise elementary function, so it is continuous on the intervals $[0,1)$ and $(1,2]$. At the point 1, we have $\lim_{x\to1^-}f(x)=\lim_{x\to1}\frac{3-x^2}{2}=\frac{3-1}{2}=1$, $\lim_{x\to1^+}f(x)=\lim_{x\to1}\frac{1}{x}=\frac{1}{1}=1$ and $f(1)=\frac{3-1}{2}=1$. Therefore, the function f is also continuous at the point 1. This prove that f is a continuous function on the closed interval $[0,2]$.

On the open interval $(0,2)$, since the function f is defined as a piecewise elementary function, it is differentiable on the intervals $(0,1)$ and $(1,2)$ and $f'(x)=\begin{cases}-x & \text{if } x\in(0,1),\\ -\frac{1}{x^2} & \text{if } x\in(1,2).\end{cases}$ At the point 1, we have $f'_-(1)=\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^-}\frac{\frac{3-x^2}{2}-1}{x-1}=-1$, and $f'_+(1)=\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^+}\frac{\frac{1}{x}-1}{x-1}=-1$. Therefore, the function f is also differentiable at the point 1 and $f'(1)=-1$. This prove that f is differentiable on the open interval $(0,2)$. Therefore, the function f satisfies the condition of the mean value theorem.

By a simple calculation, $\frac{f(2)-f(0)}{2-0}=\frac{\frac{1}{2}-\frac{3}{2}}{2}=-\frac{1}{2}$. So we can take $\xi=\frac{1}{2}\in(0,2)$ and see that $f'(\xi)=-\frac{1}{2}=\frac{f(2)-f(0)}{2-0}$. This proves the statement.