
EE1001

Foundations of Digital Techniques

Sequences and Series

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Outline

1. Sequences and series
2. Recurrence relations of sequences
3. Arithmetic series and geometric series

Class Intended Learning Outcomes (CILO)

- Understanding the definitions of sequences and series
- Identifying recurrence relations and solving them with iteration method
- Designing and formulating arithmetic and geometric sequences and series

1. Sequences and Series

Sequence and Series

- Sequence: A list of numbers, called **terms**, in a special order.
 - For example,
 - 5, 7, 9, 11, 13
 - 32, 64, 128, 256
- Series: The **sum** of all the **terms** of a sequence.
 - For example,
 - $5 + 7 + 9 + 11 + 13$
 - $32 + 64 + 128 + 256$

List with commas

“Indicated sum”

Finite and Infinite

- The sequence and series can be **finite** or **infinite**

Finite Sequence	Infinite Sequence
Example: 32, 64, 128, 256	Example: 32, 64, 128, 256, ...
General Form: $a_1, a_2, a_3, \dots, a_n$ (n number of terms)	General Form: a_1, a_2, a_3, \dots (infinite number of terms)

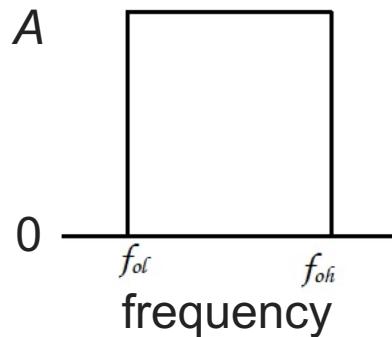
Finite Series	Infinite Series
Example: $32 + 64 + 128 + 256$	Example: $32 + 64 + 128 + 256, \dots$
General Form: $a_1 + a_2 + a_3 + \dots + a_n$ (n number of terms)	General Form: a_1, a_2, a_3, \dots (infinite number of terms)

Why do we learn both finite and infinite calculations?

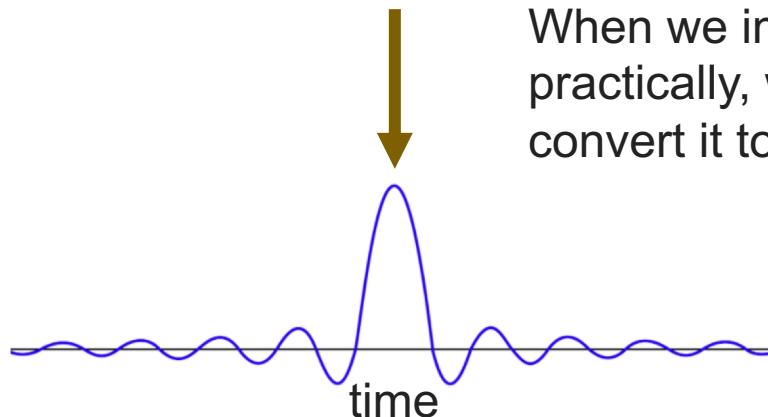
Here gives a real-world engineering example,

Bandpass filters are essential in various engineering applications, such as filtering unwanted signals, limit the transmission bandwidth of the output RF signals, ...

Below is an ideal bandpass filter in frequency domain

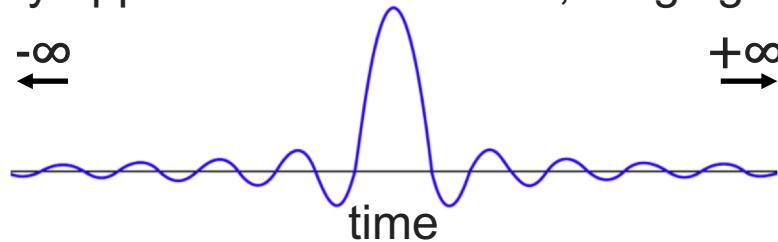


When we implement the filter practically, we need to convert it to time-domain



Why do we learn both finite and infinite calculations?

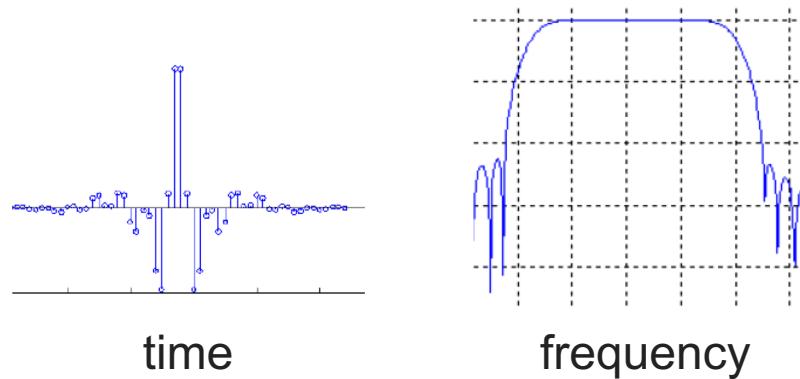
There are many ripples in the waveform, ranging from $-\infty$ to $+\infty$.



Practically, there is impossible to implement something from $-\infty$ time to $+\infty$ time.

Therefore, we need to set a finite range and make the practical filter similar to the ideal one.

Finite bandpass filter using Kaiser window



On the other hand, assuming the ideal cases using infinite condition can **ease calculations** and predict general performance.

General Formula of the n^{th} term

- For sequences and series, we can denote the n^{th} term (or so-called the general term) using superscript notation, i.e., a_n .

The 1st term of a sequence/series is then called a_1

The 2nd term is a_2

The 3rd term is a_3

The $(k+1)^{\text{th}}$ term is a_{k+1} , and so on.

Any term in a sequence/series can be found by
substituting its position number into a given formula for a_n .

General Formula of the n^{th} term

- For instant, the formula for the k^{th} term of a sequence is given by $m_k = 5k + 2$.

Then, $m_1 = 5 \times 1 + 2 = 7$

The 1st term

$$m_2 = 5 \times 2 + 2 = 12$$

The 2nd term

$$m_3 = 5 \times 3 + 2 = 17$$

The 3rd term

...

...

$$m_{(t+2)} = 5 \times (t+2) + 2 = 5t + 12$$

The $(t+2)^{\text{th}}$ term

- The first three terms in the sequence are: 7, 12, 17
- The first $m+2$ terms in the sequence are: 7, 12, 17, ..., $5m+12$

Summation Notation (Σ)

- We can denote a series using the general term (a_n) and summation notation (Σ).
 - The Greek capital letter Σ (sigma) is a summation symbol to represent the sum and abbreviate a series.
 - The summation notation can be expressed as

$N = \text{last value of } n$ \rightarrow $\sum_{n=k}^N a_n = a_k + a_{k+1} + a_{k+2} + \cdots + a_{N-1} + a_N, \forall k \in \mathbb{Z} \text{ and } k \leq N$

$n = \text{index of summation}$ \rightarrow $k = \text{first value of } n$

For example, the series $2 + 4 + 6 + 8 + 10$ or a general term $a_n = 2n$ can be written as

$$\sum_{n=1}^5 a_n \text{ or } \sum_{n=1}^5 2n$$

and it is read as “the sum of the terms a_n or $2n$, as n varies from 1 to 5.”

Product Notation (Π)

- We can indicate repeated multiplication using the general term (a_n) and product notation (Π).
 - The Greek capital letter Π (pi) is a product symbol to the product of sequences.
 - The product notation can be expressed as

$N = \text{last value of } n$
 $n = \text{index of summation}$

$$\prod_{n=k}^N a_n = a_k \times a_{k+1} \times a_{k+2} \times \cdots \times a_{N-1} \times a_N, \forall k \in \mathbb{Z} \text{ and } k \leq N$$

\nearrow \nwarrow
 $k = \text{first value of } n$

For example, the series $2 \times 4 \times 6 \times 8 \times 10$ or a general term $a_n = 2n$ can be written as

$$\prod_{n=1}^5 a_n \text{ or } \prod_{n=1}^5 2n = 2(1) \times 2(2) \times 2(3) \times 2(4) \times 2(5)$$

and it is read as “the product of the terms a_n or $2n$, as n varies from 1 to 5.”

Factorial Notation ($n!$)

- Factorial notation ($n!$)

$$n! = \prod_{k=1}^n k = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n, \forall n \in \mathbb{Z} \text{ and } n > 0$$

- It is also a kind of production notation with the general term of $a_k = k$
- Factorial is frequently used in mathematics, science, and engineering, especially when determining the number of permutations and combinations.
 - We will go into details in the section of “Counting”.

Permutations and Combinations

Number of permutations
(order matters) of n
things taken r at a time:

$$P(n, r) = \frac{n!}{(n-r)!}$$

Number of combinations
(order does not matter) of n
things taken r at a time:

$$C(n, r) = \frac{n!}{(n-r)!r!}$$

Number of different permutations of n
objects where there are n_1 repeated items,
 n_2 repeated items, ... n_k repeated items

$$\frac{n!}{n_1!n_2!...n_k!}$$

Properties of Summation and Product

- If “ $a_n, a_{n+1}, a_{n+2}, \dots$ ” and “ $b_n, b_{n+1}, b_{n+2}, \dots$ ” are sequences of real numbers and c is a constant, then the following equations are true.

$$1. \quad \sum_{n=k}^N a_n + \sum_{n=k}^N b_n = \sum_{n=k}^N (a_n + b_n)$$

$$2. \quad c \times \sum_{n=k}^N a_n = \sum_{n=k}^N c \times a_n$$

$$3. \quad \left(\prod_{n=k}^N a_n \right) \left(\prod_{n=k}^N b_n \right) = \prod_{n=k}^N a_n b_n$$

The summation and product notations can range from $-\infty$ to $+\infty$, i.e.,

$$\sum_{n=-\infty}^{\infty} a_n = \cdots + a_{-1} + a_0 + a_1 + \cdots$$

$$\prod_{n=-\infty}^{\infty} a_n = \cdots \times a_{-1} \times a_0 \times a_1 \times \cdots$$

2. Recurrence Relation

Recurrence Relation

- A **recurrence relation** for the sequence $\{a_n\}$ is a function that defines a_n in terms of one or more of the previous terms of the sequence, i.e., a_0, a_1, \dots, a_{n-1} , for all integers n with $n \geq n_0$, where n_0 is a non-negative integer.

$$\text{e.g., } a_n = a_{n-1} + 2a_{n-2} - 3a_{n-3}$$

- To define a sequence using a recurrence relation, we need
 - the function relating each term to a previous term, and
 - the initial conditions, which is a finite set of start-up values.

For example, express a sequence of $\{-5, -1, 3, 7, \dots\}$ using a recurrence relation

The initial condition is $a_1 = -5$, and the function is $a_{n+1} = a_n + 4$.

$$a_1 = -5 \text{ (Initial condition)}$$

$$a_2 = a_1 + 4 = -5 + 4 = -1$$

$$a_3 = a_2 + 4 = -1 + 4 = 3$$

$$a_4 = a_3 + 4 = 3 + 4 = 7,$$

and so on

In-class Exercises

A1. Express the following sequences using recurrence relation:

a) $\{1, 3, 7, 15, 31, 63, \dots\}$

b) $\{108, 52, 24, 10, 3, -0.5, \dots\}$

A2. Find out the first five terms of the following sequence.

Initial condition: $a_1 = 1, a_2 = 3$

$$a_{n+2} = 1 + 2a_n - a_{n+1}$$

In-class Exercises 1

A1. Express the following sequences using recurrence relation:

a) $\{1, 3, 7, 15, 31, 63, \dots\}$

Initial condition: $a_1 = 1$

$$a_{n+1} = 1 + 2a_n$$

b) $\{108, 52, 24, 10, 3, -0.5, \dots\}$

Initial condition: $a_1 = 108$

$$a_{n+1} = (a_n / 2) - 2$$

A2. Find out the first five terms of the following sequence.

Initial condition: $a_1 = 1, a_2 = 3$

$$a_{n+2} = 1 + 2a_n - a_{n+1}$$

$$a_1 = 1$$

$$a_2 = 3$$

$$n = 1 \quad a_3 = 1 + 2 \times 1 - 3 = 0$$

$$n = 2 \quad a_4 = 1 + 2 \times 3 - 0 = 7$$

$$n = 3 \quad a_5 = 1 + 2 \times 0 - 7 = -6$$

So, the first five terms are $\{1, 3, 0, 7, -6\}$

Iteration Method for Solving Recurrence Relations

- *Iteration* methods (backtracking) is one of the methods of solving recurrence relations
 - Given a recursive expression with initial conditions a_1 , some recurrence relations of a_n can be derived to become an explicit formula
 - For example, $a_n = 2a_{n-1}$, with the initial condition of $a_1=1$, can be derived as follows.

$$\begin{aligned}a_n &= 2a_{n-1} \\&= 2 \times (2a_{n-2}) = 2^2 \times a_{n-2} \\&= 2^2 \times (2a_{n-3}) = 2^3 \times a_{n-3} \\&= \dots \\&= 2^{n-3} \times (2a_2) = 2^{n-2} \times a_2 \\&= 2^{n-2} \times (2a_2) = 2^{n-1} \times a_1 \quad (a_1 = 1) \\&= 2^{n-1}\end{aligned}$$

We can then find the k^{th} term easily by substituting k into n , e.g.,

$$\begin{aligned}a_2 &= 2^{2-1} = 2 \\a_3 &= 2^{3-1} = 4 \\a_{10} &= 2^{10-1} = 512\end{aligned}$$

In-class Exercises 2

A3. Find the explicit formula for the following sequence.

$$a_n = a_{n-1} + 3, \text{ for } n = 2, 3, 4, \dots, \text{ and the initial condition is } a_1 = 2$$

In-class Exercises 2

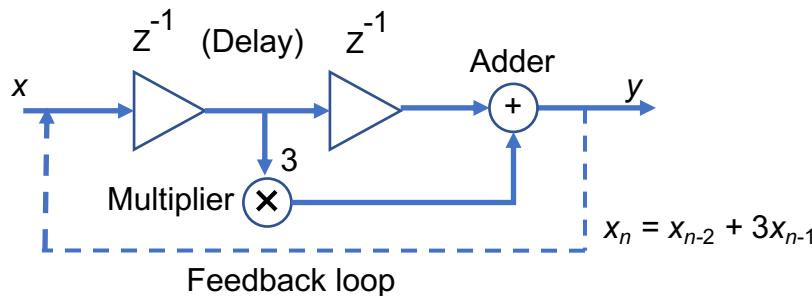
A3. Find the explicit formula for the following sequence.

$a_n = a_{n-1} + 3$, for $n = 2, 3, 4, \dots$, and the initial condition is $a_1 = 2$

$$\begin{aligned}a_n &= a_{n-1} + 3 \\&= a_{n-2} + 3 + 3 = a_{n-2} + 2 \times 3 \\&= a_{n-3} + 2 \times 3 + 3 = a_{n-3} + 3 \times 3 \\&= \dots \\&= a_1 + (n-1) \times 3 \\&= 2 + (n-1) \times 3 = 3n - 1\end{aligned}$$

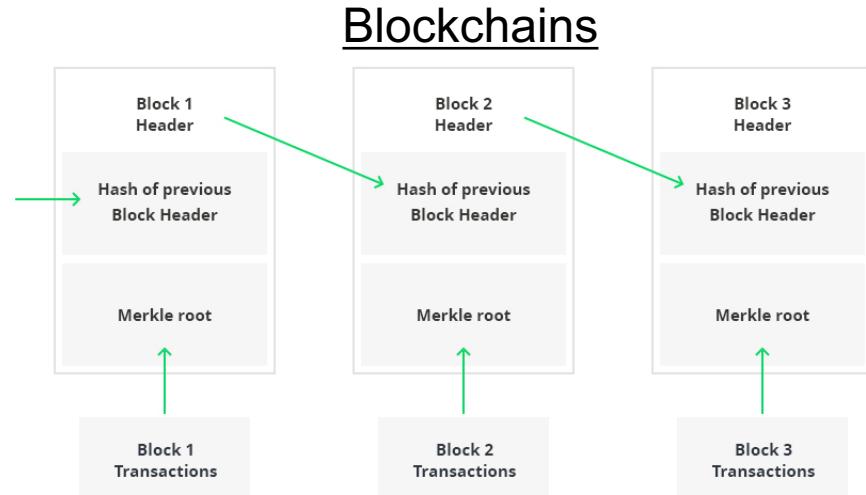
Recurrence Relations in Engineering

Delay and Feedback in Filter Design



Programming

```
Int a, b;  
a = 10; Initial condition:  
a1 = 10  
b = 20;  
  
For (l = 1; l<=10; i++)  
{  
    a = a + 2; an+1 = an + 2  
}
```



$$\text{Block_header}_{k+1} = \text{Block_header}_k + \text{transaction}_k$$

3. Arithmetic Series & Geometric Series

Arithmetic Sequence

- An arithmetic sequence is a sequence in which there is a **common difference d** between two consecutive terms, i.e.,
 - The common difference $d = (a_n - a_{n-1}) = (a_{n-1} - a_{n-2}) = \dots = (a_2 - a_1)$
 - Therefore:
 - 1st term: a_1
 - 2nd term: $a_2 = a_1 + d$
 - 3rd term: $a_3 = a_1 + 2d$
 - ...
- The arithmetic formula of the n^{th} term is as follows.

$$a_n = a_1 + (n-1)d$$

a_1 = the 1st term
 n = the term number
 d = the common difference

Geometric Sequence

- A geometric sequence is a sequence in which there is a **common ratio r** between two consecutive terms, i.e.,
 - The common ratio $r = (a_n - a_{n-1}) = (a_{n-1} - a_{n-2}) = \dots = (a_2 - a_1)$
 - Therefore,
 - 1st term: a_1
 - 2nd term: $a_2 = a_1 \times r$
 - 3rd term: $a_3 = a_1 \times r^2$
 - ...
- The geometric formula of the n^{th} term is as follows.

$$a_n = a_1 \times r^{n-1}$$

a_1 = the 1st term

n = the term number

r = the common difference

Simple Exercise

- A4. Determine the following sequences whether they are arithmetic sequence or geometric sequence
 - 1) {11.5, 25, 38.5, 52}
 - 2) {10, 0.1, 10, 0.1}
 - 3) {5, 10, 20, 60}
- A5. Given {79, 75, 71, 67, 63, ...}, what term number is - 169?

Simple Exercise

- A4. Determine the following sequences whether they are arithmetic sequence or geometric sequence

1) {11.5, 25, 38.5, 52}

Arithmetic sequence $d = 13.5$

2) {10, 0.1, 10, 0.1}

Geometric sequence $r = -1$

3) {5, 10, 20, 60}

Neither

- A5. Given {79, 75, 71, 67, 63, ...}, what term number is - 169?

$$d = 75 - 79 = -4$$

$$a_1 = 79$$

$$a_n = -169$$

$$a_n = a_1 + (n-1)d$$

$$-169 = 79 + (n-1)(-4)$$

$$n = 63$$

Sum of Arithmetic Series

- *Arithmetic series* is an expression formed by adding the terms of an **arithmetic sequence**.
- The sum of a series containing N terms can be denoted by S_N .
 - For example, $S_5 = 1 + 3 + 5 + 7 + 9 = 25$

How about $S_{100} = 1 + 2 + 3 + \dots + 99 + 100$?

Sum of Arithmetic Series

- To find the sum of the first n terms, we can adopt the Gauss' method as follows.

symmetric
$$\begin{cases} S_{100} = 1 + 2 + 3 + \dots + 99 + 100 & \text{From 1}^{\text{st}} \text{ to } 100^{\text{th}} \text{ term} \\ S_{100} = 100 + 99 + 98 + \dots + 2 + 1 & \text{From } 100^{\text{th}} \text{ to } 1^{\text{st}} \text{ term} \end{cases}$$

$$2S_{100} = 101 + 101 + 101 + \dots + 101 + 101$$



100 terms

Therefore, we have

$$2S_{100} = 100 \times 101$$

$$S_{100} = 100 \times 101 / 2 = 5050$$

Sum of Arithmetic Series

- Considering the general form of an arithmetic series having N terms and a common difference of d ,

$$S_N = a + (a + d) + (a + 2d) + \dots + (a + (N - 2)d) + (a + (N - 1)d)$$

$$S_N = a + (a + d) + (a + 2d) + \dots + (f - d) + f$$

$$S_N = f + (f - d) + (f - 2d) + \dots + (a + d) + a$$

$$2S_N = (a + f) + (a + f) + (a + f) + \dots + (a + f)$$

(a denotes the first term)
(f denotes the last term)

$\underbrace{\hspace{10em}}$
 N terms

Therefore, we have $2S_N = N(a + f)$, and

The sum of arithmetic series is

$$S_N = \frac{N}{2}(a + f)$$

$$\text{or } S_N = \frac{N}{2}(2a + (N - 1)d)$$

N = number of terms, a = initial term, f = last term, d = common difference

Example

- Find the sum of the first 23 terms of the arithmetic series

$$-5 + -11 + -17 + -23 + \dots$$

Since we don't know the last term, we adopt the following formula

$$S_N = \frac{N}{2}(2a + (N-1)d)$$

$$d = -11 - (-5) = -6, a = -5, \text{ and } N = 23$$

$$S_{23} = \frac{23}{2}[2(-5) + (23-1)(-6)] = -1633$$

Sum of Geometric Series

- *Geometric series* is an expression formed by adding the terms of a **geometric sequence**.
- The sum of a series containing N terms can be denoted by S_N .
 - For example, $S_5 = 1 + 2 + 4 + 8 + 16 = 56$

How about $\{a, ar, ar^2, ar^3, \dots, ar^N\}$?

Sum of Geometric Series

- Given a finite geometric series with n terms, a common ratio r and the first term a ,

$$S_N = a + ar^1 + ar^2 + \cdots + ar^{N-2} + ar^{N-1}$$

$$S_N = a(1 + r^1 + r^2 + \cdots + r^{N-2} + r^{N-1})$$

$$S_N = \frac{a(1 + r^1 + r^2 + \cdots + r^{N-2} + r^{N-1})(1 - r)}{1 - r}$$

As a result, the general formula of a finite geometric series is

$$S_N = \frac{a(1 - r^N)}{1 - r}$$

N = number of terms, a = initial term, r = common ratio

Sum of Infinite Geometric Series

- If the common ratio $|r| < 1$, the infinite geometric series has the sum:

$$S = \frac{a}{1-r}$$

a = initial term
 r = common ratio

For example, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

$$S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

Exercise

- Given the following geometric series,

$$\{1, -0.25, 0.0625, -0.015625, \dots\}$$

- A8. Find the sum of the first 9 terms

$$r = -0.25, a = 1 \text{ and } N = 9$$

$$S_{10} = 1 \times (1 - (-0.25)^9) / (1 - (-0.25)) = 0.800$$

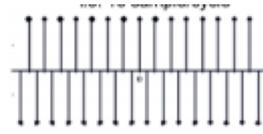
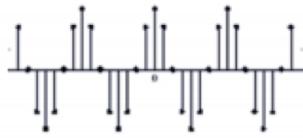
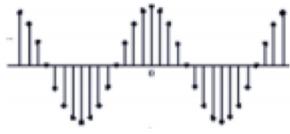
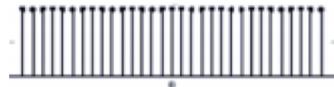
- A9. Find the sum of all terms

$$r = -0.25 \text{ and } |r| < 1$$

$$S = 1 / (1 - (-0.25)) = 0.8$$

Arithmetic and Geometric Series in Engineering

Calculate the energy and power of a discrete time signal



Energy and Power

- The energy of a discrete time signal is defined as

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Example: $\{x[n]\} = \{1, 2, 2, 2, 1\}$

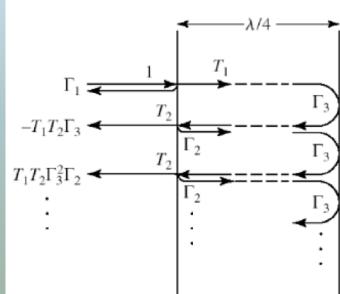
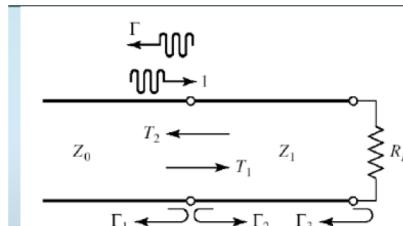
The energy of $x[n]$ is $E_x = 1+4+4+4+1=14$

Example: $x[n] = \exp\{(-0.2 + j0.1\pi)n\} u[n]$

$$E_x = \sum_{n=0}^{\infty} |\exp\{(-0.2 + j0.1\pi)n\}|^2 = \sum_{n=0}^{\infty} e^{-0.4n}$$

Using geometric progression, $E_x = \frac{1}{1-e^{-0.4}} = 3.033$

RF Calculation
(e.g., multiple reflection)



$$\Gamma_1 = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \Gamma_2 = \frac{Z_0 - Z_1}{Z_0 + Z_1} = -\Gamma_1,$$

$$\Gamma_3 = \frac{R_L - Z_1}{R_L + Z_1}$$

$$T_1 = \frac{2Z_1}{Z_1 + Z_0}, T_2 = \frac{2Z_0}{Z_1 + Z_0}$$

$$\Gamma = \Gamma_1 - T_1 T_2 \Gamma_3 + T_1 T_2 \Gamma_2 \Gamma_3^2 - T_1 T_2^2 \Gamma_2^2 \Gamma_3^2 + L$$

$$= \Gamma_1 - T_1 T_2 \Gamma_3 \sum_{n=0}^{\infty} (-\Gamma_2 \Gamma_3)^n$$

$$= \Gamma_1 - \frac{T_1 T_2 \Gamma_3}{1 + \Gamma_2 \Gamma_3} = \frac{\Gamma_1 + \Gamma_1 \Gamma_2 \Gamma_3 - T_1 T_2 \Gamma_3}{1 + \Gamma_2 \Gamma_3}$$

$$\left(\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \right)$$