MA1301 Calculus and Basic Linear Algebra II
Matrix Algebra and System of Linear Algebraic Equations

1. Introduction (p.61 – p.77)

A *matrix* of order $m \times n$ or an $m \times n$ matrix is a rectangular array of numbers having m rows and n columns. It can be written $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & & & . \\ . & & & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (a_{ij})$.

The *mn* numbers $a_{11}, \cdots, a_{mn}$ are called the *elements* (*entries*) of the matrix. The notation a_{ij} denotes the element of A which is in the i^{th} row and j^{th} column.

Example 1

Let $A = \begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & -1 \end{pmatrix}$. What is the element in the second row and first column?

Solution:

$a_{21} = 2$.

□

A matrix with only one row is called a *row matrix* or *row vector* while a matrix with only one column is a *column matrix* or *column vector*. These will be written $\vec{x}^T$ and $\vec{x}$ respectively.

A matrix with *n* rows and *n* columns is a *square matrix* of order $n \times n$. The elements $a_{11}, a_{22}, \cdots, a_{nn}$ are the *diagonal elements*.

Example 2

Consider $B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & -1 \end{pmatrix}$.

Matrix B is a square matrix of order 2×2. Matrix A is not square and is a 2×3 matrix. The diagonal elements B are 1, 4.

□

Two matrices A and B are equal iff they have the same number of rows and columns and $a_{ij} = b_{ij}$ for all i and j.

Example 3

Consider $B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & -1 \end{pmatrix}$. Matrices *A*, *B* are not equal because *A* is a 2×3 matrix, however, *B* is a 2×2 square matrix.

□

Example 4

Consider $A = \begin{pmatrix} x & y \\ 2 & z \end{pmatrix}, B = \begin{pmatrix} 7 & x \\ 8 & y \end{pmatrix}$. Matrices *A*, *B* are not equal because a_{21} of *A* is 2, however, b_{21} of *B* is 8 and they are not equal.

□

Operations among matrices:

(A) Addition

The sum of two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ is the $m \times n$ matrix $C = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$ for $i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$. We write $C = A + B$.

Example 5

Let $A = \begin{pmatrix} x & y \\ 2 & z \end{pmatrix}, B = \begin{pmatrix} 7 & x \\ 8 & y \end{pmatrix}$. Find $C = A + B$.

Solution:

$$C = A + B = \begin{pmatrix} x & y \\ 2 & z \end{pmatrix} + \begin{pmatrix} 7 & x \\ 8 & y \end{pmatrix} = \begin{pmatrix} x+7 & y+x \\ 10 & z+y \end{pmatrix}.$$

□

The *zero matrix* O has all elements zero. For example $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a 2×3 zero matrix.

(B) Scalar multiplication

The *product* of an $m \times n$ matrix $A = (a_{ij})$ with a *scalar* (number) q is $qA = Aq = (qa_{ij})$, that is, every element of *A* is multiplied by *q*.

Example 6

Let $A = \begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & -1 \end{pmatrix}$, find $3A$.

Solution:

$$3A = 3\begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 12 & -6 \\ 6 & 9 & -3 \end{pmatrix}.$$

□

The matrix $-A = \left(-a_{ij}\right)$ is the negation of A.

Properties:

(a) $A + B = B + A$

(b) $A + (B + C) = (A + B) + C$

(c) $A + O = A$

(d) $A + (-A) = O$

(e) $q(A + B) = qA + qB$

(f) $(p + q)A = pA + qA$

(g) $(pq)A = p(qA) = q(pA)$

(h) $1A = A$

The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by interchanging the rows and columns of A.

Example 7

Let $A = \begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & -1 \end{pmatrix}$. Then $A^T = \begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 4 & 3 \\ -2 & -1 \end{pmatrix}$. □

Let A be a real square matrix. It is said to be *symmetric* if $A^T = A$ and *skew-symmetric* if $A^T = -A$.

Example 8

Let $A = \begin{pmatrix} 7 & x \\ 8 & y \end{pmatrix}$. Suppose A is symmetric, find x, y.

Solution:

$A = A^T$, that is, $\begin{pmatrix} 7 & x \\ 8 & y \end{pmatrix} = \begin{pmatrix} 7 & x \\ 8 & y \end{pmatrix}^T = \begin{pmatrix} 7 & 8 \\ x & y \end{pmatrix}$.

Therefore, $x = 8$ and y can be any number.

□

Example 9

Let $A = \begin{pmatrix} x & y \\ 2 & z \end{pmatrix}$. Suppose A is skew-symmetric, find x, y, z.

Solution:

$$A + A^T = \begin{pmatrix} x & y \\ 2 & z \end{pmatrix} + \begin{pmatrix} x & 2 \\ y & z \end{pmatrix} \underset{A^T = -A}{=} A + (-A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2x & 2+y \\ 2+y & 2z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2x = 0 \\ 2+y = 0 \\ 2z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ z = 0 \\ y = -2 \end{cases}$$

So $x = z = 0$ and $y = -2$.

□

A *diagonal matrix* has all elements not on the diagonal zero, that is, $a_{ij} = 0$ if $i \neq j$.

Example 10

Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Observe that A, B and C are all diagonal matrices. In addition, C is also a 2×2 zero matrix.

□

The $n \times n$ *identity matrix* (or *unit matrix*) I or $I_n = (\delta_{ij})$, where $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$, is a diagonal matrix with all its diagonal elements equal to 1.

Example 11

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a 3×3 identity (unit) matrix.

□

(C) Product of two matrices

The *product* of an $m \times n$ matrix A with an $n \times p$ matrix B is the $m \times p$ matrix $C = AB$ with elements

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = (\text{Row } i \text{ of } A) \cdot (\text{Column } j \text{ of } B), \quad i = 1, 2, \cdots, m \; ; \; j = 1, 2, \cdots, p .$$

Note that the matrix multiplication is only possible if the number of columns of A is the same as the number of rows of B, that is, the matrices are *conformable*. The element c_{ij} is the dot product of the i^{th} row of A and the j^{th} column of B considering them as vectors.

Example 12

Let $A = \begin{pmatrix} 1 & 3 & 8 \\ 9 & 6 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 8 & 1 \\ 7 & 2 \\ 5 & 1 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 1\times 8 + 3\times 7 + 8\times 5 & 1\times 1 + 3\times 2 + 8\times 1 \\ 9\times 8 + 6\times 7 + 1\times 5 & 9\times 1 + 6\times 2 + 1\times 1 \end{pmatrix} = \begin{pmatrix} 69 & 15 \\ 119 & 22 \end{pmatrix}.$$

$$BA = \begin{pmatrix} 8 & 1 \\ 7 & 2 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 8 \\ 9 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 8\times 1 + 1\times 9 & 8\times 3 + 1\times 6 & 8\times 8 + 1\times 1 \\ 7\times 1 + 2\times 9 & 7\times 3 + 2\times 6 & 7\times 8 + 2\times 1 \\ 5\times 1 + 1\times 9 & 5\times 3 + 1\times 6 & 5\times 8 + 1\times 1 \end{pmatrix} = \begin{pmatrix} 17 & 30 & 65 \\ 25 & 33 & 58 \\ 14 & 21 & 41 \end{pmatrix}$$

□

The previous example show that AB and BA may not be equal since they may not have the same size. Even if A, B are square matrices of the same order, AB may not equal to BA.

Example 13

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then $AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $BA = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

□

Example 14

Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$, find AB.

Solution:

$$AB=\begin{pmatrix}1&1&2\\2&3&-1\end{pmatrix}\begin{pmatrix}1&0&1\\2&1&0\\-1&1&0\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix} & \begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix} & \begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\\ \begin{pmatrix}2\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix} & \begin{pmatrix}2\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix} & \begin{pmatrix}2\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\end{pmatrix}=\begin{pmatrix}1&3&1\\9&2&2\end{pmatrix}.$$

Suppose $B=\left(\vec{b_1}\middle|\vec{b_2}\middle|\vec{b_3}\right)$ where $\vec{b_1},\vec{b_2},\vec{b_3}$ are the corresponding columns of B. We observe that the first column of AB is $A\vec{b_1}$, the second column of AB is $A\vec{b_2}$, the third column of AB is $A\vec{b_3}$, that is, $AB=\left(A\vec{b_1}\middle|A\vec{b_2}\middle|A\vec{b_3}\right)$.

Then $AB=\left(\begin{pmatrix}1&1&2\\2&3&-1\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix}\middle|\begin{pmatrix}1&1&2\\2&3&-1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}\middle|\begin{pmatrix}1&1&2\\2&3&-1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\right)=\begin{pmatrix}1&3&1\\9&2&2\end{pmatrix}.$

□

Properties for matrix multiplication:

(a) $(qA)B=q(AB)=A(qB)$, normally written as qAB

(b) $A(BC)=(AB)C$, normally written as ABC

(c) $A(B+C)=AB+AC$

(d) $(A+B)C=AC+BC$

Note, however, that

(e) $AB\neq BA$ in general

(f) $AB=O\not\Rightarrow A=O$ or $B=O$
(i.e. It is possible that $A\neq 0$ and $B\neq 0$ but $AB=0$.)

Example 15

$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1&1\\-1&-1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$. However, both $\begin{pmatrix}1&1\\1&1\end{pmatrix}$ and $\begin{pmatrix}1&1\\-1&-1\end{pmatrix}$ are non-zero matrices.

□

2. <u>Determinants</u> (p.128 – p.151)

Every square matrix A has a number associated with it called the *determinant* of A, written as $\det A$ or $|A|$.

For a 2×2 matrix $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$, the determinant of A is defined as $|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$.

For a 3×3 matrix $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}$, the determinant of A is defined as

$$
\begin{aligned}
|A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} \quad \text{(expanded by the first row)} \\
&= a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13} \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix} \\
&= a_{11}\left(a_{22}a_{23}-a_{23}a_{32}\right)-a_{12}\left(a_{21}a_{33}-a_{23}a_{31}\right)+a_{13}\left(a_{21}a_{32}-a_{22}a_{31}\right),
\end{aligned}
$$

where A_{ij} is the *cofactor* of element a_{ij}, and $A_{ij}=(-1)^{i+j}M_{ij}$ where M_{ij} is the *minor* of the element a_{ij} and is the determinant of that *submatrix* of A obtained by deleting the i^{th} row and j^{th} column of A.

$|A|$ above was expanded by the first row although we could have similarly expanded by any row or column to give the same result.

For an $n\times n$ matrix, we have by row: $|A|=a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots+a_{in}A_{in},\quad i=1,2,\cdots,n$, or

by column: $|A|=a_{1j}A_{1j}+a_{2j}A_{2j}+\cdots+a_{nj}A_{nj},\quad j=1,2,\cdots,n$.

<u>Example 16</u>

Find $|A|=\begin{vmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & 3 & 6\end{vmatrix}$.

<u>Solution:</u>

Method 1 (which is valid for 2×2 and 3×3 matrices only.)

$|A|$= 2(2)(6) + 1(5)(0) + 4(3)(3) – 0(2)(4) – 2(3)(5) – (3)(1)(6) = 12

Method 2:

By expanding $|A|$ in the first column, we have

$$|A| = \begin{vmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & 3 & 6 \end{vmatrix} = 2\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix} + 0\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = 2\times(2\times6-3\times5)-3\times(1\times6-3\times4) = 2\times(-3)-3\times(-6) = 12.$$

Method 3:

By expanding $|A|$ in the third row, we have

$$|A| = \begin{vmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & 3 & 6 \end{vmatrix} = 0\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} + 6\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 0(1\times5-2\times4)-3(2\times5-3\times4)+6(2\times2-1\times3) = 12$$

Method 4:

By expanding $|A|$ in the second row, we have

$$|A| = \begin{vmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & 3 & 6 \end{vmatrix} = -3\begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & 4 \\ 0 & 6 \end{vmatrix} - 5\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = -3(1\times6-3\times4)+2(2\times6-0\times4)-5(2\times3-0\times1) = 12$$

Remark: Other methods are available but not shown here.

□

Properties of determinants:

(1) $|A| = |A^T|$, that is, rows and columns may be interchanged.

For example, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

e.g. If $A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$.

We have $|A| = \begin{vmatrix} 1 & 4 \\ -2 & 3 \end{vmatrix} = 1(3)-(-2)(4) = 11$ and $|A^T| = \begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix} = 1(3)-4(-2) = 11$

(2) If all the elements in a row (or column) are zero, then $|A| = 0$.

For example, $\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

e.g. $\begin{vmatrix} 3 & 5 \\ 0 & 0 \end{vmatrix} = 3(0)-0(5) = 0$, $\begin{vmatrix} 1 & 0 \\ -5 & 0 \end{vmatrix} = 1(0)-(-5)(0) = 0$

(3) If corresponding elements in any two rows (or columns) are proportional then $|A| = 0$.

For example, $\begin{vmatrix} a_1 & b_1 & c_1 \\ ma_1 & mb_1 & mc_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

e.g. $\begin{vmatrix} 3 & 0 & 5 \\ 1 & 2 & 3 \\ 10 & 20 & 30 \end{vmatrix} = 0, \begin{vmatrix} 4 & 0 & 2 \\ 2 & 9 & 1 \\ 6 & -3 & 3 \end{vmatrix} = 0$

(4) $|AB| = |A||B|$, both A and B must be square matrices.

e.g Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.

We have $|A| = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$, $|B| = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$, $|AB| = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1$,

that is $|AB| = |A||B|$.

(5) $|A| = 0$ if the rows (or columns) are linearly dependent.

e.g. Consider $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 5 & -3 & 2 \end{bmatrix}$, which is formed by three column vectors $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$.

Observe that $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}$, the three vectors are linearly dependent.

$\therefore$ We have $|A| = 0$.

(6) $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$ where A^{-1} denotes the *inverse* of A (to be discussed later).

e.g. For $A = \begin{bmatrix} 5 & -4 \\ 1 & -3 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \frac{3}{11} & -\frac{4}{11} \\ \frac{1}{11} & -\frac{5}{11} \end{bmatrix}$.

Then $|A| = 5(-3) - 1(-4) = -11$, $|A^{-1}| = \left(\frac{3}{11}\right)\left(-\frac{5}{11}\right) - \frac{1}{11}\left(-\frac{4}{11}\right) = -\frac{1}{11} = \frac{1}{|A|}$

Notice that A is invertible (i.e. the inverse of A exists) iff $|A| \neq 0$.

Properties (7) – (9) reveal how the determinant will change if we perform *elementary row operations* to a matrix:

(7) Interchanging any two rows (or columns) of the matrix A reverses the sign of $|A|$.

e.g. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{vmatrix}$, $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -\begin{vmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{vmatrix}$

(8) The value of a determinant is unchanged if a multiple of one row (or column) is added to another row (or column).

e.g. $\begin{vmatrix} 1 & -2 & 3 \\ 3 & -3 & 5 \\ 2 & 5 & -1 \end{vmatrix} \overset{R_2-3R_1}{=} \begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & -4 \\ 2 & 5 & -1 \end{vmatrix} \overset{R_3-2R_1}{=} \begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & -4 \\ 0 & 9 & -7 \end{vmatrix} \overset{R_3-3R_2}{=} \begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & -4 \\ 0 & 0 & 5 \end{vmatrix} = 1\times3\times5 = 15$

(9) If the elements in any row (or column) are multiplied by a number, then $|A|$ is multiplied by that number, that is, for instance $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

Note that $|kA| = k^n|A| \; (\neq k|A|)$.

e.g. $\begin{vmatrix} -3 & 7 & 2 \\ 4 & 8 & -6 \\ 1 & -3 & 7 \end{vmatrix} = 2\begin{vmatrix} -3 & 7 & 2 \\ 2 & 4 & -3 \\ 1 & -3 & 7 \end{vmatrix} = 2\times224 = 448.$

$\begin{vmatrix} -3 & 7 & 2 \\ 4 & 8 & -6 \\ 3 & -9 & 21 \end{vmatrix} = 2\times3\times\begin{vmatrix} -3 & 7 & 2 \\ 2 & 4 & -3 \\ 1 & -3 & 7 \end{vmatrix} = 2\times3\times224 = 1344.$

Due to properties (7) – (9), we can find the determinant of a matrix by first performing elementary row operations to the matrix so that it is in row echelon form, then we multiply all the elements at the diagonal of the matrix (and the values coming out from performing row operations) to get the corresponding determinant.

i.e. If the matrix is a ***triangular matrix***, then its determinant is equal to the product of all the elements of the diagonal of the matrix.

e.g. $\begin{vmatrix} 3 & 8 & -2 \\ 0 & 5 & 7 \\ 0 & 0 & -4 \end{vmatrix} = 3(5)(-4) = -60$, $\begin{vmatrix} 9 & 0 & 2 \\ 0 & -2 & 6 \\ 0 & 0 & 7 \end{vmatrix} = 9(-2)(7) = -126$, $\begin{vmatrix} 4 & 2 & -4 \\ 0 & 5 & 9 \\ 0 & 0 & 0 \end{vmatrix} = 4(5)(0) = 0$

Determinants are not of great practical use as they are expensive to compute, but are of theoretical value.

Example 17

Without expanding the matrix, find the determinant for each of the following matrices:

(a) $\begin{bmatrix}2&4&-2\\3&6&2\\0&-3&1\end{bmatrix}$ (b) $\begin{bmatrix}3&9&-6\\7&2&-5\\-2&-6&4\end{bmatrix}$ (c) $\begin{bmatrix}3a&-2a&6a\\b&2b&-b\\c&c&2c\end{bmatrix}$ (d) $\begin{bmatrix}1+a&b&c\\a&1+b&c\\a&b&1+c\end{bmatrix}$

Solutions

(a) $$\begin{vmatrix}2&4&-2\\3&6&2\\0&-3&1\end{vmatrix} = 2\begin{vmatrix}1&2&-1\\3&6&2\\0&-3&1\end{vmatrix} \overset{R_2-3R_1}{=} 2\begin{vmatrix}1&2&-1\\0&0&5\\0&-3&1\end{vmatrix} \overset{R2\leftrightarrow R3}{=} -2\begin{vmatrix}1&2&-1\\0&-3&1\\0&0&5\end{vmatrix} = (-2)(1)(-3)(5) = 30$$

(Checking: $\begin{vmatrix}2&4&-2\\3&6&2\\0&-3&1\end{vmatrix} = 2(6)(1)+(4)(2)(0)+(3)(-3)(-2)-0(6)(-2)-2(-3)(2)-(3)(4)(1) = 30$)

(b) $$\begin{vmatrix}3&9&-6\\7&2&-5\\-2&-6&4\end{vmatrix} = 3\begin{vmatrix}1&3&-2\\7&2&-5\\-2&-6&4\end{vmatrix} = 2(3)\begin{vmatrix}1&3&-2\\7&2&-5\\-1&-3&2\end{vmatrix} \overset{R_3+R_1}{=} 2(3)\begin{vmatrix}1&3&-2\\7&2&-5\\0&0&0\end{vmatrix} = 2(3)(0) = 0$$

(Checking: $\begin{vmatrix}3&9&-6\\7&2&-5\\-2&-6&4\end{vmatrix} = 3(2)(4)+(9)(-5)(-2)+(-6)(7)(-6)-(-2)(2)(-6)-3(-6)(-5)-4(7)(9) = 0$)

(c) $$\begin{vmatrix}3a&-2a&6a\\b&2b&-b\\c&c&2c\end{vmatrix} = a\begin{vmatrix}3&-2&6\\b&2b&-b\\c&c&2c\end{vmatrix} = ab\begin{vmatrix}3&-2&6\\1&2&-1\\c&c&2c\end{vmatrix} = abc\begin{vmatrix}3&-2&6\\1&2&-1\\1&1&2\end{vmatrix} \underset{R_3-R_2}{\overset{R_1-3R_2}{=}} abc\begin{vmatrix}0&-8&9\\1&2&-1\\0&-1&3\end{vmatrix}$$

$$\overset{R_1-8R_3}{=} abc\begin{vmatrix}0&0&-15\\1&2&-1\\0&-1&3\end{vmatrix} \overset{R1\leftrightarrow R2}{=} -abc\begin{vmatrix}1&2&-1\\0&0&-15\\0&-1&3\end{vmatrix} \overset{R2\leftrightarrow R3}{=} abc\begin{vmatrix}1&2&-1\\0&-1&3\\0&0&-15\end{vmatrix} = 1(-1)(-15)abc = 15abc$$

(d)

$$\begin{vmatrix}1+a&b&c\\a&1+b&c\\a&b&1+c\end{vmatrix} \underset{\text{add columns 2 and 3 to column 1}}{=} \begin{vmatrix}1+a+b+c&b&c\\1+a+b+c&1+b&c\\1+a+b+c&b&1+c\end{vmatrix} = (1+a+b+c)\begin{vmatrix}1&b&c\\1&1+b&c\\1&b&1+c\end{vmatrix}$$

$$\underset{\text{add }(-1)\text{row 1 to row 2}}{=} (1+a+b+c)\begin{vmatrix}1&b&c\\0&1&0\\1&b&1+c\end{vmatrix} \underset{\text{add }(-1)\text{row 1 to row 3}}{=} (1+a+b+c)\begin{vmatrix}1&b&c\\0&1&0\\0&0&1\end{vmatrix} \cancel{\underset{\text{expand along the first column}}{=} (1+a+b+c)\begin{vmatrix}1&0\\0&1\end{vmatrix}} = 1+a+b+c.$$

□

3. Systems of Linear Equations (p.90 – p.115)

A set of equations of the form $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$ is called a system of *m linear equations* in *n unknowns*, $x_1, x_2, \cdots, x_n$. The numbers a_{ij} are the *coefficients* of the system and are given. The *right-hand side* numbers $b_1, b_2, \cdots, b_m$ are also given. If all the b_i are zero the system is *homogeneous* otherwise it is *inhomogeneous*. A *solution* is a set of numbers $x_1, x_2, \cdots, x_n$ satisfying all *m* equations.

e.g. $\begin{cases} x_1 + 4x_2 - 2x_3 = 3 \\ 2x_1 - 2x_2 + x_3 = 1 \\ 3x_1 + x_2 + 2x_3 = 11 \end{cases}$ is a system of 3 linear equations in 3 unknowns x_1, x_2 and x_3.

the matrix production

The system may be written in *matrix form* $\boxed{A\vec{x} = \vec{b}}$, where

$A \in \mathbb{R}^{m\times n}$ $\quad$ $\in \mathbb{R}^{n\times 1}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

coefficient matrix $\qquad$ solution vector $\qquad$ right-hand side vector

The $m \times (n+1)$ matrix $B = \left(A, \vec{b}\right) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$ is known as the *augmented matrix* of the system.

Illustration

$A \in \mathbb{R}^{m\times n}$

if $m=n$,
$|kA| = k^{n}|A|$.

|A| has no meaning if $m \neq n$.

Referring to the system $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$, (e.g. $\begin{cases} x_1 + x_2 + x_3 - x_4 = 3 \\ 2x_1 - x_2 - 5x_3 + 2x_4 = 0 \\ -x_1 + 2x_2 - x_3 + 3x_4 = 5 \end{cases}$)

The *augmented matrix* is $\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$ (e.g. $\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 2 & -1 & -5 & 2 & 0 \\ -1 & 2 & -1 & 3 & 5 \end{array}\right]$)

and the *coefficient matrix* is $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$. (e.g. $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & -1 & -5 & 2 \\ -1 & 2 & -1 & 3 \end{bmatrix}$)

The solution of a system of linear equations is not changed by
(a) interchanging any two equations,
(b) multiplying any equation by a non-zero constant,
(c) adding a constant multiple of one equation to another equation.

The systems of linear equations can be represented by the corresponding augmented matrices. Operations on equations (for eliminating variables) can be represented by appropriate row operations on the corresponding matrices. If instead of equations we consider the augmented matrix of the system, we may define the following *elementary row operations*:

Type 1: Interchanging any two rows; (Interchange)
Type 2: Multiplying any row by a non-zero constant; (Scaling)
Type 3: Adding a constant multiple of one row to another row. (Replacement)

Two matrices are *row equivalent* if one may be obtained from the other in a finite number of elementary row operations. Clearly two augmented matrices which are row equivalent may represent systems of linear equations with the same solution.

e.g. $\begin{bmatrix} 3 & 0 & 6 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ ($\begin{bmatrix} 3 & 0 & 6 \\ 1 & 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ are row equivalent.)

Gaussian Elimination Method

This method (and variations of it) is the most popular method of solving system of linear equations on a computer.

Example 18

Solve the system $\begin{cases} a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 = b_1^{(1)} \\ a_{21}^{(1)}x_1 + a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 = b_2^{(1)} \\ a_{31}^{(1)}x_1 + a_{32}^{(1)}x_2 + a_{33}^{(1)}x_3 = b_3^{(1)} \end{cases}$ by Gaussian Elimination.

0

0

Solution:

$$\begin{array}{c} \left(\begin{array}{ccc|c} \boxed{a_{11}^{(1)}} \neq 0 & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & b_3^{(1)} \end{array}\right) \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \end{array}$$

$$x_3 = \frac{b_3^{(3)}}{a_{33}^{(3)}}$$

Elimination Stage:

$$\sim \left(\begin{array}{ccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} \neq 0 & a_{23}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & b_3^{(2)} \end{array}\right) \begin{array}{l} r_1 \\ r_2 - m_{21}r_1 \\ r_3 - m_{31}r_1 \end{array} \sim \left(\begin{array}{ccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & b_3^{(3)} \end{array}\right) \begin{array}{l} r_1 \\ r_2 \\ r_3 - m_{32}r_2 \end{array}$$

$$a_{33}^{(3)} x_3 = b_3^{(3)}$$

where $m_{21} = \dfrac{a_{21}^{(1)}}{a_{11}^{(1)}}, m_{31} = \dfrac{a_{31}^{(1)}}{a_{11}^{(1)}}$ and $m_{32} = \dfrac{a_{32}^{(2)}}{a_{22}^{(2)}}$ are the multipliers. The system is now in *upper triangular form* $\begin{cases} a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 = b_1^{(1)} \\ \qquad a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 = b_2^{(2)} \\ \qquad\qquad a_{33}^{(3)}x_3 = b_3^{(3)} \end{cases}$, and we proceed with *back substitution* to find x_1, x_2, x_3 in reverse order: $x_3 = \dfrac{b_3^{(3)}}{a_{33}^{(3)}}$, $x_2 = \dfrac{1}{a_{22}^{(2)}}\left(b_2^{(2)} - a_{23}^{(2)} \cdot \dfrac{b_3^{(3)}}{a_{33}^{(3)}}\right)$ and $x_1 = \dfrac{1}{a_{11}^{(1)}}\left[b_1^{(1)} - a_{13}^{(1)} \cdot \dfrac{b_3^{(3)}}{a_{33}^{(3)}} - \dfrac{a_{12}^{(1)}}{a_{22}^{(2)}}\left(b_2^{(2)} - a_{23}^{(2)} \cdot \dfrac{b_3^{(3)}}{a_{33}^{(3)}}\right)\right]$.

□

Example 19

Solve the system $\begin{cases} x_1 + 4x_2 - 2x_3 = 3 \\ 2x_1 - 2x_2 + x_3 = 1 \\ 3x_1 + x_2 + 2x_3 = 11 \end{cases}$ by Gaussian Elimination.

Solution:

$$\left(\begin{array}{ccc|c} 1 & 4 & -2 & 3 \\ 2 & -2 & 1 & 1 \\ 3 & 1 & 2 & 11 \end{array}\right) \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array}$$

Elimination Stage:

$$\sim\left(\begin{array}{ccc|c} 1 & 4 & -2 & 3 \\ 0 & -10 & 5 & -5 \\ 0 & -11 & 8 & 2 \end{array}\right)\begin{array}{l} r_1 \\ r_2-2r_1 \\ r_3-3r_1 \end{array}\sim\left(\begin{array}{ccc|c} 1 & 4 & -2 & 3 \\ 0 & -10 & 5 & -5 \\ 0 & 0 & 2.5 & 7.5 \end{array}\right)\begin{array}{c} r_1 \\ r_2 \\ r_3-(-11/-10)r_2 \end{array} \Rightarrow \begin{cases} x_1+4x_2-2x_3=3 \\ -10x_2+5x_3=-5 \\ 2.5x_3=7.5 \end{cases}$$

The system is now in *upper triangular form* and we proceed with the *back substitution* to find x_1, x_2, x_3 in reverse order: $x_3=3, x_2=2, x_1=1$.

□

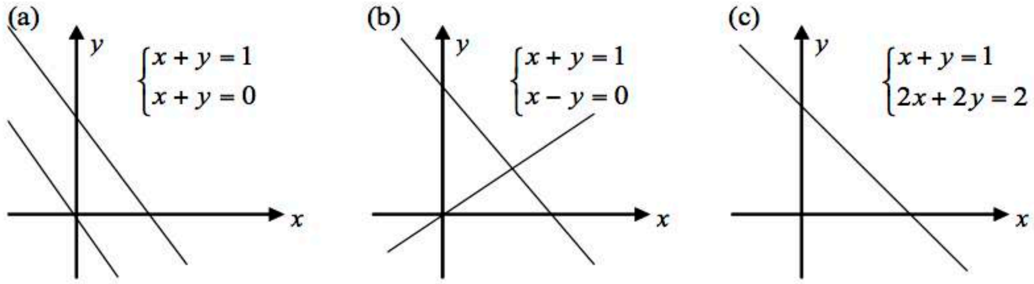
For $n>3$ the method proceeds in the same way, reducing those elements below the diagonal of the coefficient matrix in a column to zero by subtracting multiples of a row.
At an intermediate stage of the elimination we have,

$$\begin{array}{cc} & \text{column } r \\ \text{row } r & \left(\begin{array}{ccc|cccc} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1r}^{(1)} & \cdots & a_{1n}^{(1)} & \vdots\; b_1^{(1)} \\ 0 & a_{22}^{(2)} & & a_{2r}^{(2)} & \cdots & a_{2n}^{(2)} & \vdots\; b_2^{(2)} \\ 0 & 0 & & .. & & & \vdots\;\; \vdots \\ \vdots & & 0 & a_{rr}^{(r)} & \cdots & a_{rn}^{(r)} & \vdots\; b_r^{(r)} \\ \vdots & & 0 & a_{r+1r}^{(r)} & \cdots & \cdots & \vdots\;\; \vdots \\ \vdots & & \vdots & & & \vdots & \vdots\;\; \vdots \\ 0 & \cdots & 0 & a_{nr}^{(r)} & \cdots & a_{nn}^{(r)} & \vdots\; b_n^{(r)} \end{array}\right) \end{array}$$

$a_{rr}^{(r)}$ is known as the *pivot*. $a_{r+1r}^{(r)}$ is reduced to zero by subtracting $(a_{r+1r}^{(r)}/a_{rr}^{(r)})$ times row r from row r+1, and similarly for the other elements $a_{r+2r}^{(r)}, \ldots, a_{nr}^{(r)}$ in the r^{th} column.

If $a_{rr}^{(r)}=0$, then row r cannot be used as the *pivot* and r^{th} is interchanged with some row below it. In the above we have assumed n equations in the n unknowns and that there is a unique solution. There are other possibilities.
If a system of equations has at least one solution, we say the equations are *consistent*, otherwise they are *inconsistent*.

Example 20



no solution (inconsistent)	one solution (consistent)	infinitely many solutions (consistent)

Performing elementary row operations:

$x+y=1$

$$\text{(a)}\quad \left(\begin{array}{cc|c}1&1&1\\1&1&0\end{array}\right)\sim\left(\begin{array}{cc|c}1&1&1\\0&0&-1\end{array}\right)\qquad \text{(b)}\quad \left(\begin{array}{cc|c}1&1&1\\1&-1&0\end{array}\right)\sim\left(\begin{array}{cc|c}1&1&1\\0&-2&-1\end{array}\right)\qquad \text{(c)}\quad \left(\begin{array}{cc|c}1&1&1\\2&2&2\end{array}\right)\sim\left(\begin{array}{cc|c}1&1&1\\0&0&0\end{array}\right).$$

$0x_1+0x_2=0$

In case (a) the last equation gives $0=-1$ which shows that there is no solution. In case (b) there is a unique solution $y=0.5, x=0.5$. In case (c) a row of zeros has been produced which means that *y* may be assigned arbitrarily, say $y=k$ and then $x=1-k$ for any value of *k*.

□

A matrix is said to be in *row echelon form* if

(i) the first *k* rows of it are nonzero rows and the rest are zero rows;

(ii) the *pivot* a_{in_i} of the i^{th} $(i=1,\ldots,k)$ row (the first nonzero entry of each nonzero row) is to the right of the pivots of the preceding rows.

$0 \cdots 0 \ \boxed{\neq 0}$

$0 \ 0 \cdots 0 \ \boxed{\neq 0}$

0

against.

In addition, a matrix is said to be in *reduced row echelon form* if

(iii) the pivot a_{in_i} of each nonzero row is 1;

(iv) it is in row echelon form and the pivots are the only nonzero entries in their columns.

Example 21

The matrices,

$$\begin{pmatrix}1&4&-2\\2&-2&1\\3&1&2\end{pmatrix},\begin{pmatrix}0&4&-2\\2&-2&1\\3&1&2\end{pmatrix},\begin{pmatrix}1&4&-2\\0&0&1\\0&1&2\end{pmatrix}$$ are NOT in row echelon form.

$$\begin{pmatrix}1&4&-2&3\\0&-10&5&-5\\0&0&2.5&7.5\end{pmatrix},\begin{pmatrix}1&1&1\\0&0&-1\end{pmatrix},\begin{pmatrix}1&1&1\\0&-2&-1\end{pmatrix},\begin{pmatrix}1&1&1\\0&0&0\end{pmatrix},\begin{pmatrix}1&2&0&3\\0&0&1&2\\0&0&0&0\end{pmatrix}$$ are in row echelon form.

$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 5 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 5 & -8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ are in reduced row echelon form.

□

Note that the *augmented matrix* $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & -2 & 1 & 0 \\ -2 & 1 & 2 & 0 \\ 3 & 1 & 11 & 0 \end{array}\right]$, the *matrix equation* $\begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 1 \\ -2 & 1 & 2 \\ 3 & 1 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and the *vector equation* $x_1\begin{bmatrix} 1 \\ 4 \\ -2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 1 \\ 2 \\ 11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ all describe the same system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 4x_1 - 2x_2 + x_3 = 0 \\ -2x_1 + x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 11x_3 = 0 \end{cases}.$$

Recall the definition of linearly independence:

A set of vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}$ in R^n are <u>linearly independent</u> *iff* (if and only if) the equation $x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + \ldots + x_k\overrightarrow{v_k} = \boxed{\vec{0}}$ $\in R^n$ has the only solution $x_1 = x_2 = \ldots = x_k = 0$. i.e., the vector equation $x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + \ldots + x_k\overrightarrow{v_k} = \vec{0}$ has <u>only the trivial solution</u> $x_1 = x_2 = \ldots = x_k = 0$.

(i.e. A set of vectors is linearly independent if none of the vectors can be written as a linear combination of the other vectors in the set.)

Therefore, in order to verify whether a set of vectors are linearly independent or not, we can consider solving the equation $x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + \ldots + x_k\overrightarrow{v_k} = \vec{0}$ for $x_1, x_2, \ldots, x_k$ to see whether $x_1 = x_2 = \ldots = x_k = 0$ is the only solution.

<u>Illustration</u>

Question: Given $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 3 \end{bmatrix}$, $\overrightarrow{v_2} = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix}$ and $\overrightarrow{v_3} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 11 \end{bmatrix}$. Are they linearly independent?

Solution

(We shall solve the equation $x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + x_3\overrightarrow{v_3} = \vec{0}$ for x_1, x_2, x_3 to see whether $x_1 = x_2 = x_3 = 0$ is the only solution.)

We shall solve $x_1\begin{bmatrix}1\\4\\-2\\3\end{bmatrix} + x_2\begin{bmatrix}2\\-2\\1\\1\end{bmatrix} + x_3\begin{bmatrix}3\\1\\2\\11\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

(Note that it can be represented as $\begin{bmatrix}x_1\\4x_1\\-2x_1\\3x_1\end{bmatrix} + \begin{bmatrix}2x_2\\-2x_2\\x_2\\x_2\end{bmatrix} + \begin{bmatrix}3x_3\\x_3\\2x_3\\11x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, i.e. $\begin{bmatrix}x_1+2x_2+3x_3\\4x_1-2x_2+x_3\\-2x_1+x_2+2x_3\\3x_1+x_2+11x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$,

which is equivalent to the following homogeneous system: $\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 4x_1 - 2x_2 + x_3 = 0 \\ -2x_1 + x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 11x_3 = 0 \end{cases}$)

Consider the augmented matrix $\left[\begin{array}{ccc|c}1&2&3&0\\4&-2&1&0\\-2&1&2&0\\3&1&11&0\end{array}\right]$.

$$\left[\begin{array}{ccc|c}1&2&3&0\\4&-2&1&0\\-2&1&2&0\\3&1&11&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&2&3&0\\0&-10&-11&0\\0&5&8&0\\0&-5&2&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&2&3&0\\0&0&5&0\\0&5&8&0\\0&0&10&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&2&3&0\\0&5&8&0\\0&0&5&0\\0&0&10&0\end{array}\right] \sim \begin{array}{c}\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}\\ \left[\begin{array}{ccc|c}1&2&3&0\\0&5&8&0\\0&0&5&0\\0&0&0&0\end{array}\right]\end{array} \Rightarrow \begin{array}{r} x_1+2x_2+3x_3=0\\ 5x_2+8x_3=0\\ 5x_3=0\\ 0=0 \end{array}$$

$\therefore$ $x_3 = 0$, $x_2 = 0$ and $x_1 = 0$.

That means the equation $x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + x_3\overrightarrow{v_3} = \vec{0}$ has only trivial solution.

$\therefore$ $\overrightarrow{v_1}, \overrightarrow{v_2}$ and $\overrightarrow{v_3}$ are linearly independent.

Note: 1. The equation $x_1\begin{bmatrix}1\\4\\-2\\3\end{bmatrix} + x_2\begin{bmatrix}2\\-2\\1\\1\end{bmatrix} + x_3\begin{bmatrix}3\\1\\2\\11\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ or $x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + \ldots + x_k\overrightarrow{v_k} = \vec{0}$ is called a ***vector equation***.

2. In general, to verify whether a set of vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}$ are linearly independent or not, we

$\Rightarrow b = 0$ many map.

can consider the augmented matrix formed by putting all the k vectors in the first k columns and the zero vector in the last column. Perform Gaussian Elimination to see whether there is only trivial solution to this $\boxed{\text{homogeneous}}$ system.

3. For the same system $\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 4x_1 - 2x_2 + x_3 = 0 \\ -2x_1 + x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 11x_3 = 0 \end{cases}$, the equation $\begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 1 \\ -2 & 1 & 2 \\ 3 & 1 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is called a ***matrix equation***.

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -10 & -11 \\ & & \end{bmatrix}$

Illustration

Question: Are the vectors $\begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 8 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 14 \\ -8 \end{bmatrix}$ linearly independent?

Solution

Method 1

We shall solve $x_1\begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 8 \\ 4 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 14 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Consider the augmented matrix $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 3 & 8 & 14 & 0 \\ -6 & 4 & -8 & 0 \end{array}\right]$.

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 3 & 8 & 14 & 0 \\ -6 & 4 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 14 & 14 & 0 \\ 0 & -8 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -8 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$x_1 - 2x_2 = 0$ (1)

$x_2 + x_3 = 0$ (2)

$0 = 0$ (3)

Let $x_3 = k$.

by (2) $x_2 = -k$.

by (1) $x_1 = 2k$

Since there is free variable,

∴ The system has infinitely many solutions.

∴ $\vec{v_1}, \vec{v_2}$ and $\vec{v_3}$ are linearly dependent.

infinitely many solutions $\Leftarrow x_1 = 2k, x_2 = -k, x_3 = k$ for any k.

Method 2

$$0 = \boxed{\vec{v_1} \cdot \vec{v_2} \times \vec{v_3}} = \begin{vmatrix} 1 & -2 & 0 \\ 3 & 8 & 14 \\ -6 & 4 & -8 \end{vmatrix} = 1(8)(-8) + (-2)(14)(-6) + 0(3)(4) - (-6)(8)(0) - 1(4)(14) - 3(-2)(-8) = 0$$

∴ $\vec{v_1}, \vec{v_2}$ and $\vec{v_3}$ are coplanar.

∴ $\vec{v_1}, \vec{v_2}$ and $\vec{v_3}$ are linearly dependent.

m vectors $v_1, \cdots, v_m \in \mathbb{R}$

If $m > n$, then

$v_1, \cdots, v_m$ are linearly dependent.

Illustration

Question: Are the vectors $\begin{bmatrix} 5 \\ 2 \\ -6 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -5 \\ -8 \end{bmatrix}$ linearly independent?

Solution

We shall solve $x_1\begin{bmatrix} 5 \\ 2 \\ -6 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ -5 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Consider the augmented matrix $\left[\begin{array}{cccc|c} 5 & -2 & 3 & 0 & 0 \\ 2 & 7 & 1 & -5 & 0 \\ -6 & 4 & 2 & -8 & 0 \end{array}\right]$.

$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 = 0$

$a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = 0$

$a_{33}x_3 + a_{34}x_4 = 0$.

Since the number of rows (which is 3) is smaller than the number of variables (which is 4),

∴ The system will have *at least* 1 free variable.

∴ The system will have infinitely many solutions.

∴ The vectors $\begin{bmatrix} 5 \\ 2 \\ -6 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -5 \\ -8 \end{bmatrix}$ are linearly dependent.

Example 22

$\begin{pmatrix} 1 \\ 4 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \\ 11 \end{pmatrix}$ are linearly independent since $a\begin{pmatrix} 1 \\ 4 \\ -2 \\ 3 \end{pmatrix} + b\begin{pmatrix} 2 \\ -2 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 3 \\ 1 \\ 2 \\ 11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ gives

$$\begin{pmatrix} a + 2b + 3c \\ 4a - 2b + c \\ -2a + b + 2c \\ 3a + b + 11c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} a + 2b + 3c = 0 \\ 4a - 2b + c = 0 \\ -2a + b + 2c = 0 \\ 3a + b + 11c = 0 \end{cases}.$$

Using Gaussian elimination the only solution is $a = b = c = 0$. □

Example 23

Consider the matrix $A = \begin{pmatrix} 1 & 4 & -2 & 3 \\ 0 & -10 & 5 & -5 \\ 0 & 0 & 2.5 & 7.5 \end{pmatrix}$. Show that the row vectors of A $\begin{pmatrix} 1 \\ 4 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -10 \\ 5 \\ -5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2.5 \\ 7.5 \end{pmatrix}$ are linearly independent.

Proof:

$$a\begin{pmatrix}1\\4\\-2\\3\end{pmatrix}+b\begin{pmatrix}0\\-10\\5\\-5\end{pmatrix}+c\begin{pmatrix}0\\0\\2.5\\7.5\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \text{ gives } \begin{cases} a=0\\ 4a-10b=0\\ -2a+5b+2.5c=0\\ 3a-5b+7.5c=0\end{cases}$$

Directly we see that the only solution is $a=b=c=0$.

□

Note that the non-zero rows of a matrix in row echelon form are independent.

The *rank* of a matrix *A*, denoted by rank *A*, is the maximum number of linearly independent row vectors.

Example 24

$$A=\begin{pmatrix}1&4&-2&3\\0&-10&5&-5\\0&0&2.5&7.5\end{pmatrix}, \text{ rank } A=3;\; A=\begin{pmatrix}1&1&1\\2&2&2\end{pmatrix}, \text{ rank } A=1.$$

□

Theorem

Row equivalent matrices have the same rank.

Theorem

Matrices A and A^T have the same rank.

As a result of the theorem, the maximum number of linearly independent row vectors of *A* is the same as the maximum number of linearly independent columns vectors of *A*.

The rank of a matrix *A* may also be defined as the maximum number of linearly independent column vectors of *A*.

Example 25

Show that $\begin{pmatrix}1&4&-2&3\\2&-2&1&1\\3&1&2&11\end{pmatrix}$, $\begin{pmatrix}1&4&-2&3\\0&-10&5&-5\\0&0&2.5&7.5\end{pmatrix}$ both have rank 3.

Proof:

$$\begin{pmatrix}1&4&-2&3\\2&-2&1&1\\3&1&2&11\end{pmatrix}\underset{\substack{r_2-2r_1\\r_3-3r_1}}{\sim}\begin{pmatrix}1&4&-2&3\\0&-10&5&-5\\0&-11&8&2\end{pmatrix}\underset{r_3-\frac{11}{10}r_2}{\sim}\begin{pmatrix}1&4&-2&3\\0&-10&5&-5\\0&0&2.5&7.5\end{pmatrix}$$

We observe $\begin{pmatrix}1&4&-2&3\\0&-10&5&-5\\0&0&2.5&7.5\end{pmatrix}$ has rank 3. Since $\begin{pmatrix}1&4&-2&3\\2&-2&1&1\\3&1&2&11\end{pmatrix}$, $\begin{pmatrix}1&4&-2&3\\0&-10&5&-5\\0&0&2.5&7.5\end{pmatrix}$ are row equivalent. Thus $\begin{pmatrix}1&4&-2&3\\2&-2&1&1\\3&1&2&11\end{pmatrix}$, $\begin{pmatrix}1&4&-2&3\\0&-10&5&-5\\0&0&2.5&7.5\end{pmatrix}$ both have rank 3.

□

This result gives us a practical method of finding the rank of a matrix – we reduce it to an echelon form.

The idea of rank enables us to determine the existence and uniqueness or otherwise of solutions of systems of linear equations.

Theorem

Suppose we have a system of m equations in n unknowns represented by $A\vec{x}=\vec{b}$ with augmented matrix $B=\left(A\middle|\vec{b}\right)$.

The equations have:

(a) a unique solution iff $\text{rank}\,A=\text{rank}\,B=n$;

(b) an infinite number of solutions with number of *parameters* $=n-\text{rank}A$ iff $\text{rank}\,A=\text{rank}\,B<n$;

(c) no solution iff $\text{rank}\,A<\text{rank}\,B$.

Proof:

We perform elementary row operations to reduce B to echelon form (Gaussian elimination).

The results will be illustrated in the case of 5 ($=m$) equations in 4 ($=n$) unknowns.

(a)

$$\begin{pmatrix}x&x&x&x&\vdots&x\\0&x&x&x&\vdots&x\\0&0&x&x&\vdots&x\\0&0&0&x&\vdots&x\\0&0&0&0&\vdots&0\end{pmatrix},\ x\text{ – denotes a non-zero element.}$$

Then $\operatorname{rank} A = \operatorname{rank} B = n$ (= 4 in this case) and the system has a unique solution.

(b)

$$\begin{pmatrix} x & x & x & x & \vdots & x \\ 0 & x & x & x & \vdots & x \\ 0 & 0 & x & x & \vdots & x \\ 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

Here $\operatorname{rank} A = \operatorname{rank} B < n$ ($\operatorname{rank} A = \operatorname{rank} B = 3$), x_4 may be assigned arbitrarily and x_1, x_2 and x_3 determined in terms of x_4. Hence the solution is not unique. More generally if $\operatorname{rank} A = \operatorname{rank} B = r < n$ then $n - r$ unknowns (number of parameters) may be assigned arbitrarily.

(c)

$$\begin{pmatrix} x & x & x & x & \mathrm{M} & x \\ 0 & x & x & x & \mathrm{M} & x \\ 0 & 0 & 0 & 0 & \mathrm{M} & x \\ 0 & 0 & 0 & 0 & \mathrm{M} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{M} & 0 \end{pmatrix}$$

Here $\operatorname{rank} A < \operatorname{rank} B$ $(2 < 3)$ and the system is inconsistent.

Homogeneous Equations

(i) A homogeneous system ($\vec{b} = \vec{0}$) always has $\operatorname{rank} A = \operatorname{rank} B$ and the trivial solution $\vec{x} = \vec{0}$.

If $\operatorname{rank} A < n$ this solution is not unique.

(ii) If there are fewer equations than unknowns ($m < n$) a homogeneous system always has non-trivial solutions, since $\operatorname{rank} A = \operatorname{rank} B \le m < n$.

<u>Example 26</u>

Consider the system $\begin{cases} x_1 + 2x_2 - 3x_3 = -1 \\ 3x_1 - x_2 + 2x_3 = 7 \\ 5x_1 + 3x_2 + ax_3 = b \end{cases}$ for various values of *a* and *b*.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & a & b \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & a+15 & b+5 \end{array}\right) \begin{array}{c} r_1 \\ r_2 - 3r_1 \\ r_3 - 5r_1 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & a+4 & b-5 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 - r_2 \end{array}$$

(i) If $a \ne -4$, then $\operatorname{rank} A = \operatorname{rank} B = 3$ and there is a unique solution.

For example $a = 0,\ b = 9$ gives $x_3 = 1,\ x_2 = \frac{1}{7},\ x_1 = \frac{12}{7}$.

(ii) If $a = -4$ and $b = 5$, then $\operatorname{rank} A = \operatorname{rank} B = 2$ and there are infinitely many solutions with one parameter. Let $x_3 = k$ (say), $x_2 = (11k - 10)$, $x_1 = \dfrac{13-k}{7}$.

(iii) If $a = -4$ and $b \neq 5$, then $\operatorname{rank} A = 2 < 3 = \operatorname{rank} B$ and there is no solution.

□

Cramer's Rule

If $A\vec{x} = \vec{b}$, where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, is a system of n linear equations in n unknowns such that $\det A \neq 0$ then the system has a unique solution.

This solution is $x_1 = \dfrac{\det A_1}{\det A}, x_2 = \dfrac{\det A_2}{\det A}, \cdots, x_n = \dfrac{\det A_n}{\det A}$, where $A_j, j = 1, \mathrm{L}\ , n$ is the matrix obtained by replacing the j^{th} column of A by $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

The above method of finding solution is known as *Cramer's Rule*. This method is only useful when n is small and one set of solution is needed for a vector $\vec{b}$. However, if we have different cases involving different matrices, the method is not too helpful, especially when n is large and the calculation of determinant becomes too complicated.

Example 27

Solve $\begin{cases} x_1 + 4x_2 - 3x_3 = 3 \\ 2x_1 - 2x_2 + x_3 = 1 \\ 3x_1 + x_2 + 2x_3 = 11 \end{cases}$ for $x_1, x_2, \cdots, x_n$ using Cramer's Rule.

Solution:

$$\det A = \det\begin{pmatrix} 1 & 4 & -3 \\ 2 & -2 & 1 \\ 3 & 1 & 2 \end{pmatrix} = \det\begin{pmatrix} 1 & 4 & -3 \\ 0 & -10 & 7 \\ 0 & -11 & 11 \end{pmatrix} = 1\det\begin{pmatrix} -10 & 7 \\ -11 & 11 \end{pmatrix} = (-10)(11) - (7)(-11) = -33$$

Observe that $\det A_1 = -48$, $\det A_2 = -69$, $\det A_3 = -75$. It follows that

$$x_1 = \frac{\det A_1}{\det A} = \frac{-48}{-33} = \frac{16}{11},\ x_2 = \frac{\det A_2}{\det A} = \frac{-69}{-33} = \frac{23}{11}, x_3 = \frac{\det A_3}{\det A} = \frac{-75}{-33} = \frac{25}{11}.$$

□

Efficiency

Cramer's rule may be used for solving $A\vec{x}=\vec{b}$ when *n* is small, say $n=2$ or $n=3$, but is not efficient for large *n*. If we count the number of multiplications and division required for a method, we have for large *n*:

Gaussian elimination – about $n^3/3$; Cramer's rule – about $(n+1)!$

4. Inverse Matrix (p.80 – p.87, p.117 – p.125)

The *inverse* of a square $n\times n$ matrix A is denoted by A^{-1} and is an $n\times n$ matrix such that $AA^{-1}=A^{-1}A=I_n$, where I_n is the $n\times n$ unit matrix. If *A* has an inverse, then *A* is called a *non-singular matrix*, otherwise *A* is a singular *matrix* (i.e. $\det A=0$).

Illustrations

1. Given the matrix $A=\begin{bmatrix}5 & -4\\ 1 & -3\end{bmatrix}$.

   Since $\begin{bmatrix}\frac{3}{11} & -\frac{4}{11}\\ \frac{1}{11} & -\frac{5}{11}\end{bmatrix}\begin{bmatrix}5 & -4\\ 1 & -3\end{bmatrix}=\begin{bmatrix}5 & -4\\ 1 & -3\end{bmatrix}\begin{bmatrix}\frac{3}{11} & -\frac{4}{11}\\ \frac{1}{11} & -\frac{5}{11}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$,

   therefore, $\begin{bmatrix}\frac{3}{11} & -\frac{4}{11}\\ \frac{1}{11} & -\frac{5}{11}\end{bmatrix}$ is the inverse of *A*. We can write $A^{-1}=\begin{bmatrix}\frac{3}{11} & -\frac{4}{11}\\ \frac{1}{11} & -\frac{5}{11}\end{bmatrix}$.

   However, since $\begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix}\begin{bmatrix}5 & -4\\ 1 & -3\end{bmatrix}=\begin{bmatrix}3 & 2\\ 8 & -13\end{bmatrix}\neq\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$,

   therefore, $\begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix}$ is NOT the inverse of *A*.

2. Given the matrix $B=\begin{bmatrix}5 & 2\\ 0 & 0\end{bmatrix}$.

   No matter what matrix $\begin{bmatrix}* & *\\ * & *\end{bmatrix}$ is, $\begin{bmatrix}5 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}* & *\\ * & *\end{bmatrix}=\begin{bmatrix}x & y\\ 0 & 0\end{bmatrix}$, where the second row must be all 0.

   Therefore, there is no matrix such that its product with *B* will be $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

   Therefore, *B* is no inverse.

The inverse of a non-singular $n\times n$ matrix A is defined as

$$A^{-1}=\frac{1}{\det A}\operatorname{adj} A,$$

where adj A is called the adjoint of the matrix A, and is defined as the transpose of the matrix of cofactors of the matrix A, that is, adj $\mathrm{A} = \left(A_{ij}\right)^T$.

Example 28

If A is a square matrix satisfying the equation $A^2 - 3A + I = 0$, show that A is invertible.

Solution:

$A^2 - 3A + I = 0 \Rightarrow 3A - A^2 = I \Rightarrow A\left(3I - AI\right) = \left(3I - AI\right)A = I$. $\therefore A$ is invertible and $A^{-1} = 3I - A$. □

Theorem

If both A, B are square matrix of order n and $AB = I_n$, then $BA = I_n$.

According to the theorem if we have both A, B are square matrix of order n and $AB = I_n$, then $B = A^{-1}$.

Example 29

Find the inverse of the matrix $A = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$.

Solution:

$$\det A = \underset{c_1 \quad c_2 \quad c_3}{\begin{vmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{vmatrix}} = \underset{c_1+3c_2 \quad c_2 \quad c_3+c_2}{\begin{vmatrix} 0 & -1 & 0 \\ 3 & 6 & 1 \\ -1 & -2 & 0 \end{vmatrix}} = -(-1)\begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} = (3)(0) - (1)(-1) = 1 \neq 0$$

The matrix of cofactors of matrix A is given by

$$C = \begin{pmatrix} \begin{vmatrix} 6 & -5 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} -15 & -5 \\ 5 & 2 \end{vmatrix} & \begin{vmatrix} -15 & 6 \\ 5 & -2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 5 & -2 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ 6 & -5 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ -15 & -5 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ -15 & 6 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \end{pmatrix}$$

$$\text{adj } A = C^T = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

$\therefore$ The inverse of matrix A is given by

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}.$$

W

Properties:

(a) If A has an inverse it is unique.

(b) $(AB)^{-1} = B^{-1}A^{-1}$.

(c) For any non-zero real number k, kA is invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$.

(d) $(A^T)^{-1} = (A^{-1})^T$.

(e) The inverse of a symmetric matrix is symmetric.

(f) $|A^{-1}| = |A|^{-1}$.

(g) A^{-1} exists iff rank $A = n$.

(h) A^{-1} exists iff $\det(A) \neq 0$.

(i) $(A^{-1})^{-1} = A$

Although A^{-1} often appears during the manipulation of matrix expressions, it is not usually the case that A^{-1} is needed explicitly, for example, in the solution of the linear equations $A\vec{x} = \vec{b}$. The solution $\vec{x}$ may be expressed as $\vec{x} = A^{-1}\vec{b}$. However, $\vec{x}$ is actually computed using Gaussian elimination and we never need to know A^{-1}.

If A^{-1} is needed, it may be found by performing elementary row operations simultaneously on A and I. We are seeking the matrix X $(= A^{-1})$ such that $AX = I$, $X = A^{-1}I = A^{-1}$. Thus the columns of X are the solutions of $A\vec{x} = \vec{e_j}$, where $\vec{e_j}$ is the j^{th} column of the unit matrix I. Hence we simultaneously solve n sets of linear equations, each with the same matrix A but with a different column of I as the right-hand-side vector.

Example 30

Find the inverse of $A = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ by Gauss-Jordan elimination method.

Solution:

$$\begin{array}{c} \begin{array}{cc} \qquad A \qquad & \qquad I \qquad \end{array} \\ \left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -15 & 6 & -5 & 0 & 1 & 0 \\ 5 & -2 & 2 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \sim \left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & -1/3 & 1/3 & -5/3 & 0 & 1 \end{array}\right) \begin{array}{l} r_1 \\ r_2 + 5r_1 \\ r_3 - \frac{5}{3} r_1 \end{array} : \left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1 \end{array}\right) \begin{array}{l} r_1 \\ r_2 \\ r_3 + \frac{1}{3} r_2 \end{array}$$

We may now carry out 3 separate back substitutions to find the three columns of A^{-1}. Alternatively, but equivalently, we may continue to use row operations to further reduce A to the unit matrix I as follows:

$$\sim \left(\begin{array}{ccc|ccc} 3 & -1 & 0 & 1 & -1 & -3 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1 \end{array}\right) \begin{array}{l} r_1 - 3r_3 \\ r_2 \\ r_3 \end{array} \sim \left(\begin{array}{ccc|ccc} 3 & 0 & 0 & 6 & 0 & -3 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1 \end{array}\right) \begin{array}{l} r_1 + r_2 \\ r_2 \\ r_3 \end{array} \sim \begin{array}{c} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 3 \end{array}\right) \\ \begin{array}{cc} \qquad I \qquad & \qquad A^{-1} \qquad \end{array} \end{array} \begin{array}{l} r_1/3 \\ r_2 \\ 3r_3 \end{array}.$$

W

Check $AA^{-1} = A^{-1}A = I$. Note if $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then $A^{-1} = \dfrac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

And if $A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \cdots & a_{nn} \end{pmatrix}$ is a diagonal matrix with $a_{11}, a_{22}, \cdots, a_{nn} \neq 0$, then

$$A^{-1} = \begin{pmatrix} 1/a_{11} & 0 & \cdots & 0 \\ 0 & 1/a_{22} & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \cdots & 1/a_{nn} \end{pmatrix}.$$