Chapter 8

Momentum, Impulse, and Collisions

Introduction

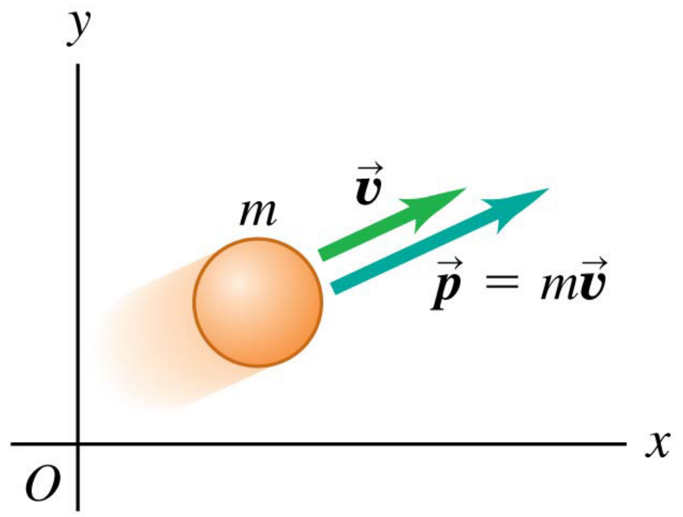
- In many situations, such as a bullet hitting a carrot, we cannot use Newton's second law to solve problems because we know very little about the complicated forces involved.
- In this chapter, we shall introduce *momentum* and *impulse*, and the *conservation of momentum*, to solve such problems.

Momentum and Newton's second law

- The **momentum** of a particle is the product of its mass and its velocity:

$$\vec{p} = m\vec{v}$$

- Newton's second law can be written in terms of momentum:



Momentum $\vec{p}$ is a vector quantity; a particle's momentum has the same direction as its velocity $\vec{v}$.

Newton's second law in terms of momentum: The net force acting on a particle ...

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

... equals the rate of change of the particle's momentum.

Impulse

- The **impulse** of a force is the product of the force and the time interval during which it acts:

Impulse of a **constant net force**

$$\vec{I} = \sum\vec{F}(t_2 - t_1) = \sum\vec{F}\,\Delta t$$

Constant net force

Time interval over which net force acts

- On a graph of $\sum F_x$ versus time, the impulse is equal to the area under the curve:

Impulse of a **general net force** (either constant or varying)

$$\vec{I} = \int_{t_1}^{t_2} \sum\vec{F}\,dt$$

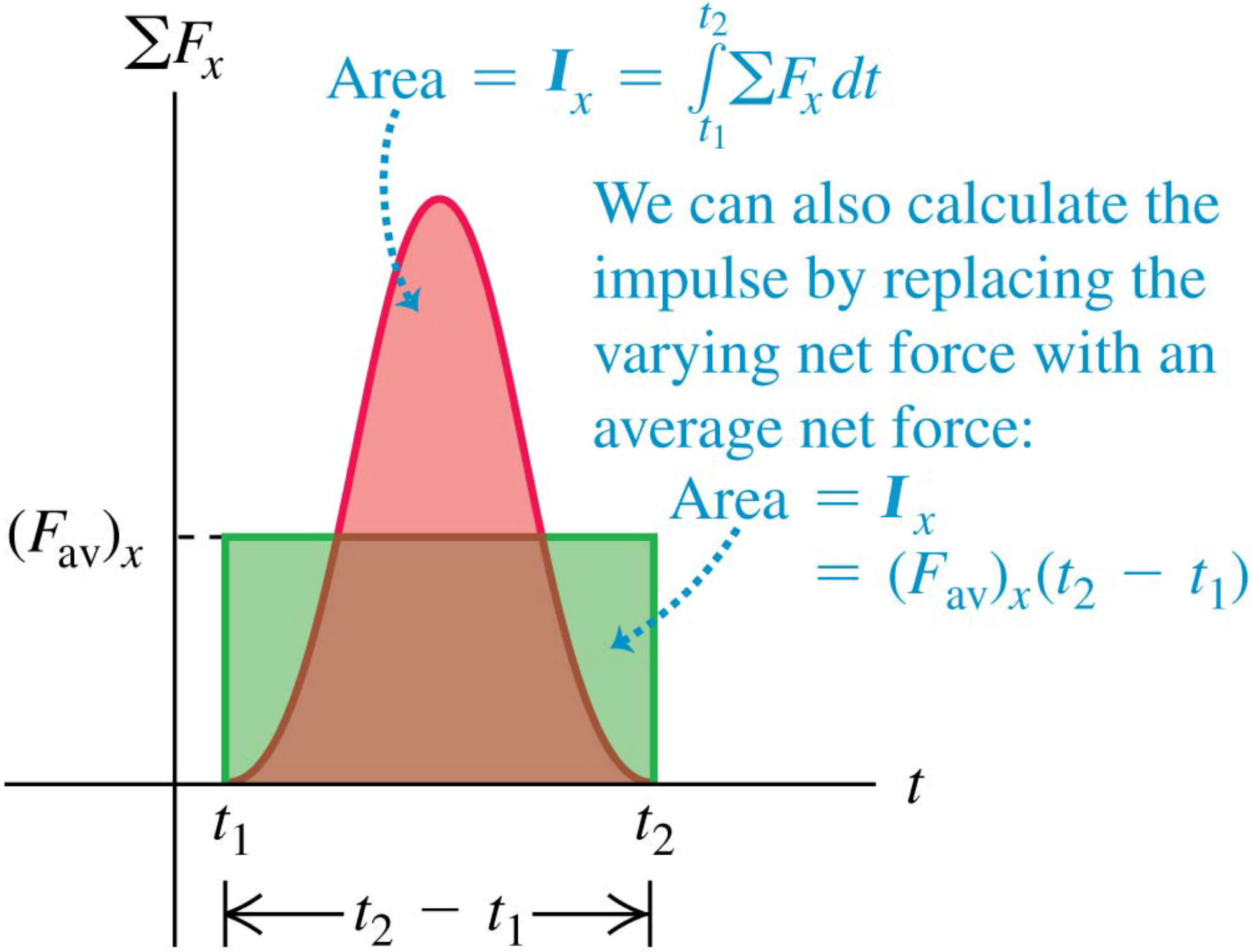
Upper limit = final time

Time integral of net force

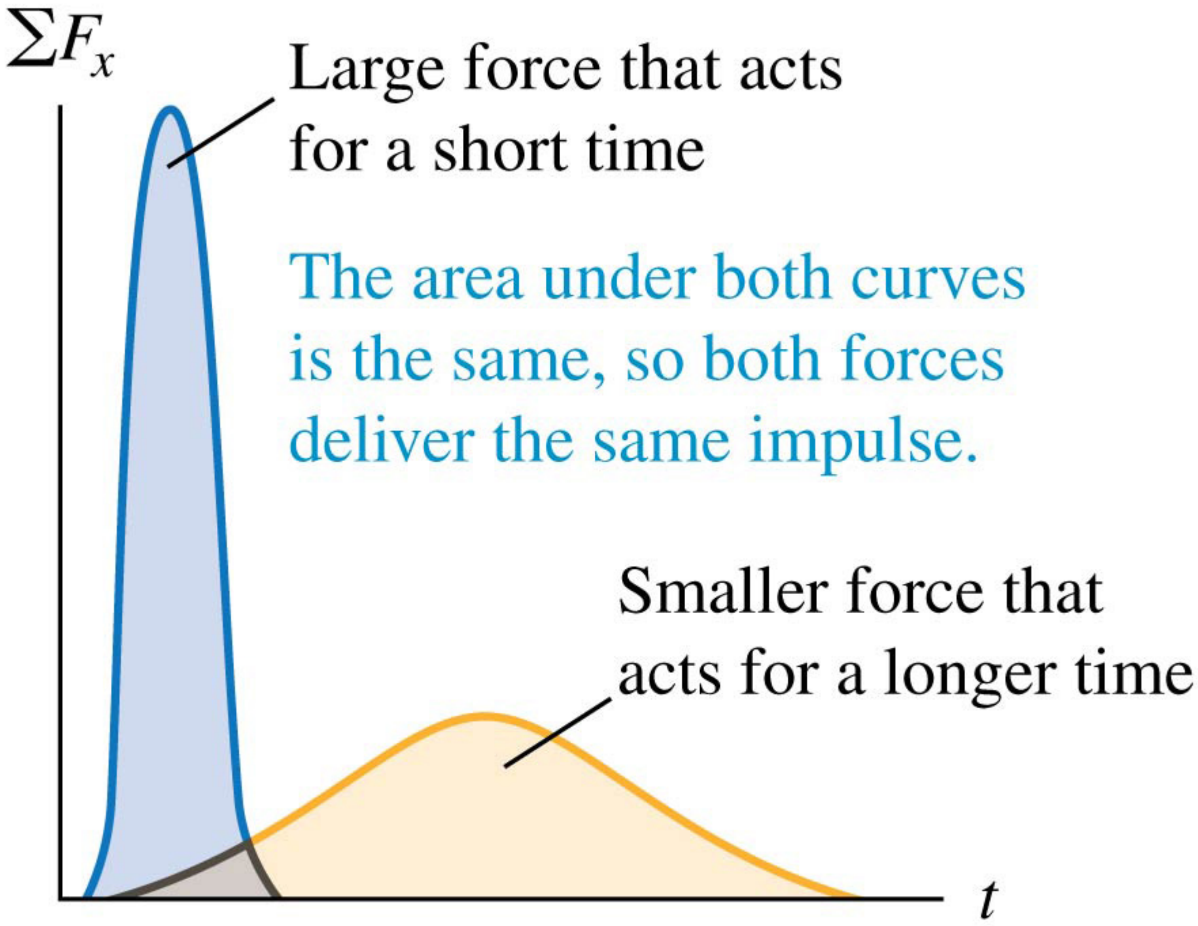
Lower limit = initial time

Impulse

The area under the curve of net force versus time equals the impulse of the net force:



Impulse



ΣF_x
Large force that acts
for a short time
The area under both curves
is the same, so both forces
deliver the same impulse.
Smaller force that
acts for a longer time
t

Impulse and momentum

- **Impulse–momentum theorem**: The change in momentum of a particle during a time interval is equal to the impulse of the net force acting on the particle during that interval:

$$\underbrace{\vec{\boldsymbol{I}}}_{\text{Impulse of net force over a time interval}} = \underbrace{\vec{p}_2}_{\text{Final momentum}} - \underbrace{\vec{p}_1}_{\text{Initial momentum}} = \underbrace{\Delta\vec{p}}_{\text{Change in momentum}}$$

Quiz

You are testing a new car using crash test dummies. Consider two ways to slow the car from 90 km/h to a complete stop:

(i) You let the car slam into a wall, bringing it to a sudden stop.

(ii) You let the car plow into a giant tub of gelatin so that it comes to a gradual halt.

In which case is there a greater *impulse* of the net force on the car?

A. in case (i)

B. in case (ii)

C. The impulse is the same in both cases.

D. not enough information given to decide

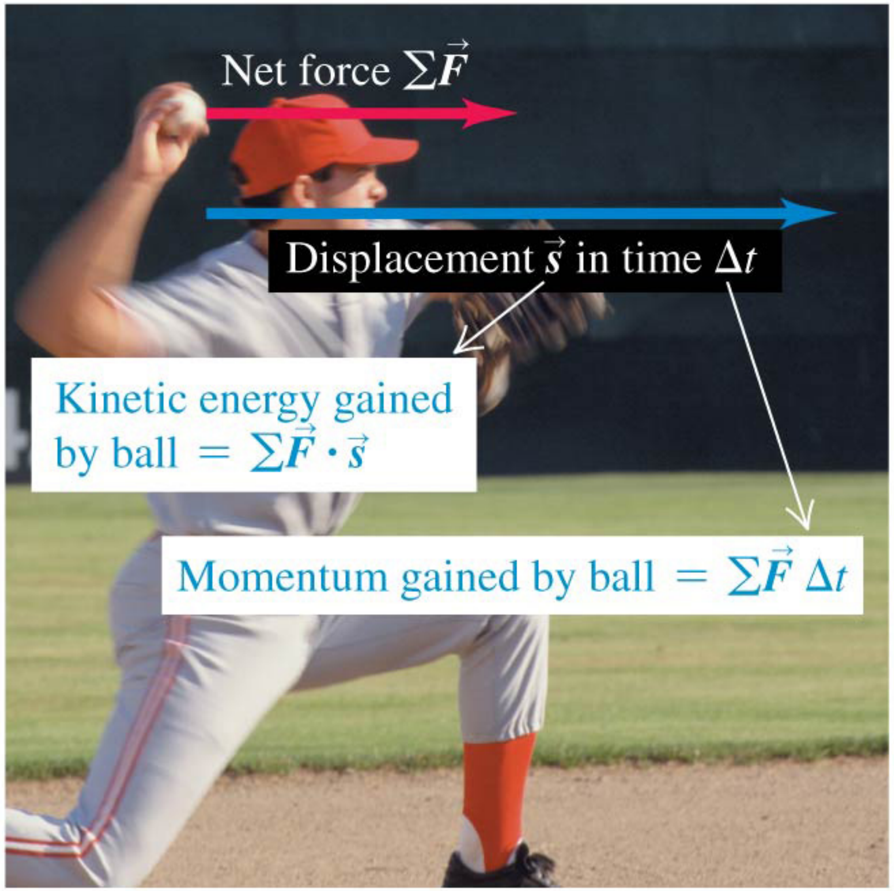
Impulse and momentum

- When you land after jumping upward, your momentum changes from a downward value to zero.
- It's best to land with your knees bent so that your legs can flex.

- You then take a relatively long time to stop, and the force that the ground exerts on your legs is small.
- If you land with your legs extended, you stop in a short time, the force on your legs is larger, and the possibility of injury is greater.

Compare momentum and kinetic energy

- The kinetic energy of a pitched baseball is equal to the work the pitcher does on it (force multiplied by the distance the ball moves during the throw).



- The momentum of the ball is equal to the impulse the pitcher imparts to it (force multiplied by the time it took to bring the ball up to speed).

Principle of linear impulse and momentum

We obtain the principle of impulse and momentum by integrating the equation of motion with respect to time.

It can be applied to problems involving both linear and angular motion.

This principle is useful for solving problems that involve force, velocity, and time.

Principle of linear impulse and momentum

The principle of linear impulse and momentum is obtained by integrating the equation of motion with respect to time. The equation of motion can be written

$$\sum \vec{F} = m\,\vec{\mathbf{a}} = m\,(d\vec{v}/dt)$$

Separating variables and integrating between the limits $\vec{v} = \vec{v}_1$ at $t = t_1$ and $\vec{v} = \vec{v}_2$ at $t = t_2$ results in

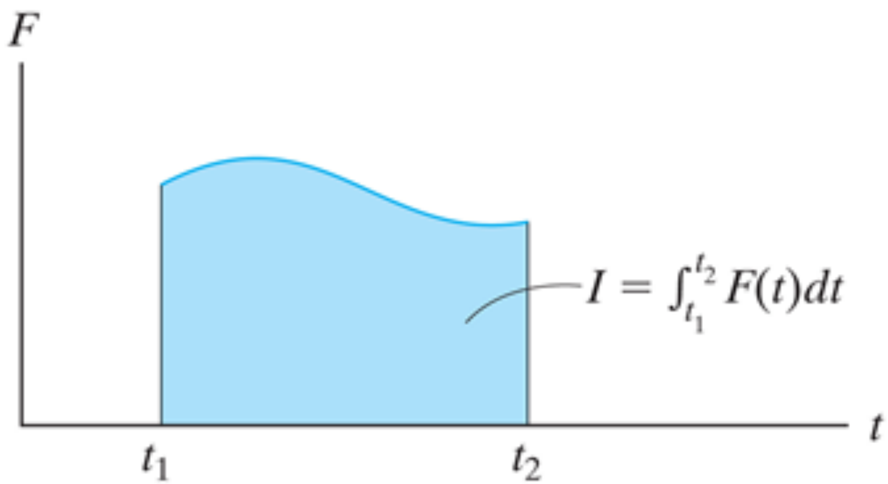
$$\sum \int_{t_1}^{t_2} \vec{F}\,dt = m \int_{v_1}^{v_2} d\vec{v} = m\vec{v}_2 - m\vec{v}_1$$

This equation represents the principle of linear impulse and momentum. It relates the particle's final velocity ($\vec{v}_2$) and initial velocity ($\vec{v}_1$) and the forces acting on the particle as a function of time.

Principle of linear impulse and momentum

Linear momentum: The vector $m\vec{v}$ is called the linear momentum, denoted as $\vec{L}$. This vector has the same direction as $\vec{v}$. The linear momentum vector has units of (kg·m)/s or (slug·ft)/s.

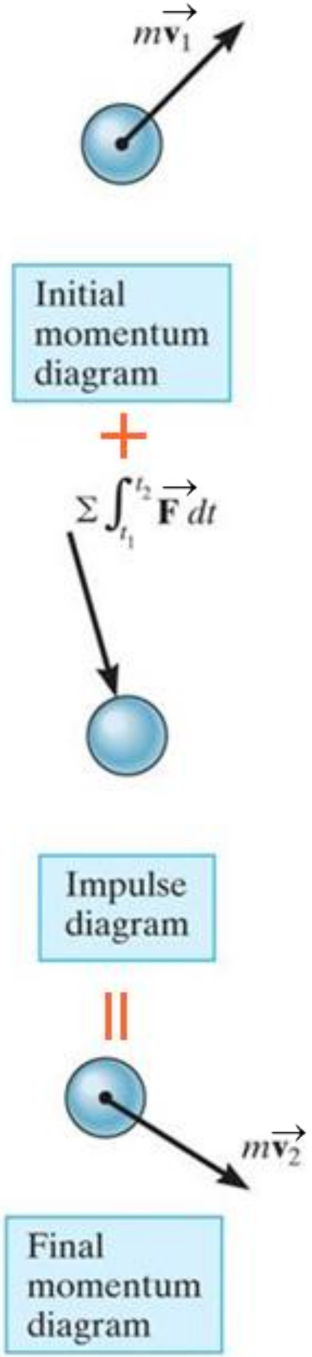
Linear impulse: The integral $\int \vec{F}\,dt$ is the linear impulse, denoted $\vec{I}$. It is a vector quantity measuring the effect of a force during its time interval of action. $\vec{I}$ acts in the same direction as $\vec{F}$ and has units of N·s or lb·s.



The impulse may be determined by direct integration. Graphically, it can be represented by the area under the force versus time curve. If $\vec{F}$ is constant, then

$\vec{I} = \vec{F}\,(t_2 - t_1)$.

Principle of linear impulse and momentum



The principle of linear impulse and momentum in vector form is written as

$$m\vec{v}_1 + \sum\int_{t_1}^{t_2}\vec{F}\,dt = m\vec{v}_2$$

The particle's initial momentum plus the sum of all the impulses applied from t_1 to t_2 is equal to the particle's final momentum.

The two momentum diagrams indicate direction and magnitude of the particle's initial and final momentum, $m\vec{v}_1$ and $m\vec{v}_2$. The impulse diagram is similar to a free-body diagram, but includes the time duration of the forces acting on the particle.

Impulse and momentum: scalar equations

Since the principle of linear impulse and momentum is a vector equation, it can be resolved into its x, y, z component scalar equations:

$$m(v_x)_1 + \sum \int_{t_1}^{t_2} F_x \, dt = m(v_x)_2$$

$$m(v_y)_1 + \sum \int_{t_1}^{t_2} F_y \, dt = m(v_y)_2$$

$$m(v_z)_1 + \sum \int_{t_1}^{t_2} F_z \, dt = m(v_z)_2$$

The scalar equations provide a convenient means for applying the principle of linear impulse and momentum once the velocity and force vectors have been resolved into x, y, z components.

Problem solving

- Establish the x, y, z coordinate system.
- Draw the particle's free-body diagram and establish the direction of the particle's initial and final velocities, drawing the impulse and momentum diagrams for the particle. Show the linear momenta and force impulse vectors.
- Resolve the force and velocity (or impulse and momentum) vectors into their x, y, z components, and apply the principle of linear impulse and momentum using its scalar form.
- Forces as functions of time must be integrated to obtain impulses. If a force is constant, its impulse is the product of the force's magnitude and time interval over which it acts.

Quiz

1. The linear impulse and momentum equation is obtained by integrating the _______ with respect to time.

   A) friction force ✔B) equation of motion

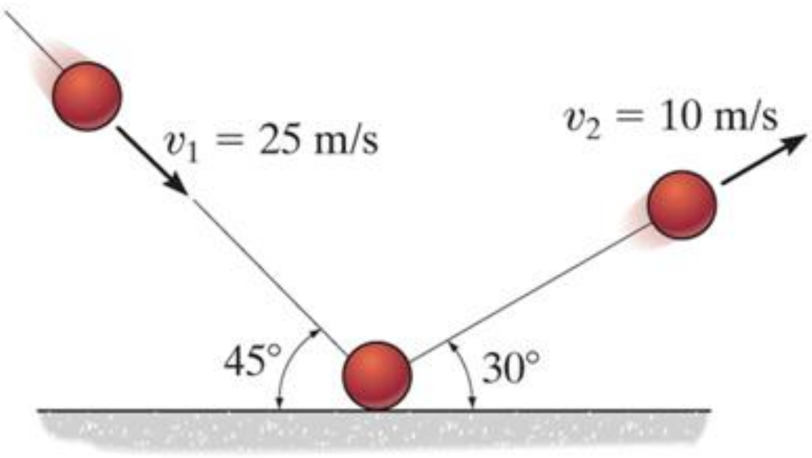
   C) kinetic energy D) potential energy

2. Which parameter is not involved in the linear impulse and momentum equation?

   A) Velocity ✔B) Displacement

   C) Time D) Force

Example



Given: A 0.5-kg ball strikes the rough ground and rebounds with the velocities shown. Neglect the ball's weight during the time it impacts the ground.

Find: The magnitude of impulsive force exerted on the ball.

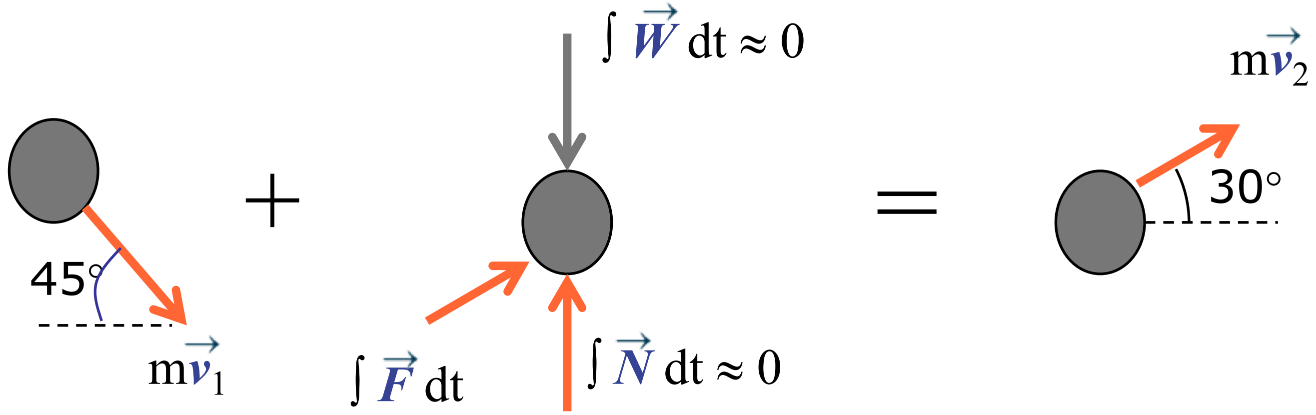
Plan:

1) Draw the momentum and impulse diagrams of the ball as it hits the surface.
2) Apply the principle of impulse and momentum to determine the impulsive force.

Example

Solution:

1) The impulse and momentum diagrams can be drawn as:



The impulse caused by the ball's weight and the normal force $\vec{N}$ can be neglected because their magnitudes are very small as compared to the impulse from the ground.

Example

2) The principle of impulse and momentum can be applied along the direction of motion:

$$m\vec{\mathbf{v}}_1 + \sum \int_{t_1}^{t_2} \vec{\mathbf{F}}\, dt = m\vec{\mathbf{v}}_2$$

$$\Rightarrow \quad 0.5\,(25 \cos 45^\circ\, \hat{\mathbf{i}} - 25 \sin 45^\circ\, \hat{\mathbf{j}}) + \int_{t_1}^{t_2} \sum \vec{\mathbf{F}}\, dt = 0.5\,(10 \cos 30^\circ\, \hat{\mathbf{i}} + 10 \sin 30^\circ\, \hat{\mathbf{j}})$$

The impulsive force vector is

$$\vec{\mathbf{I}} = \int_{t_1}^{t_2} \sum \vec{\mathbf{F}}\, dt = (4.509\, \mathbf{i} + 11.34\, \mathbf{j}\,)\ \mathrm{N{\cdot}s}$$

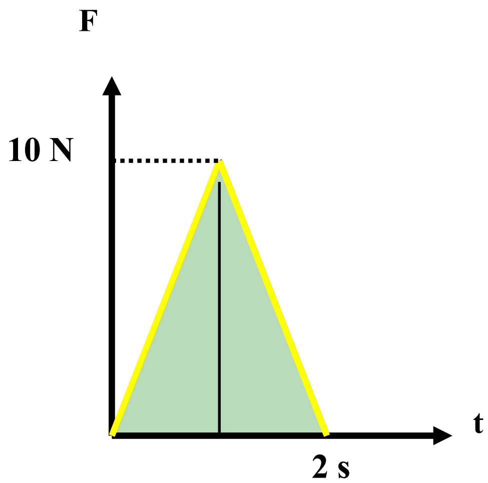
Magnitude: $I = \sqrt{4.509^2 + 11.34^2} = 12.2\ \mathrm{N{\cdot}s}$

Quiz

1. Calculate the impulse due to the force.

   A) 20 kg·m/s ✔ B) 10 kg·m/s

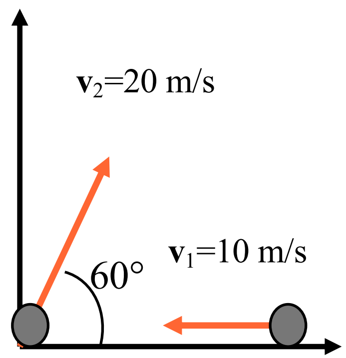
   C) 5 N·s D) 15 N·s



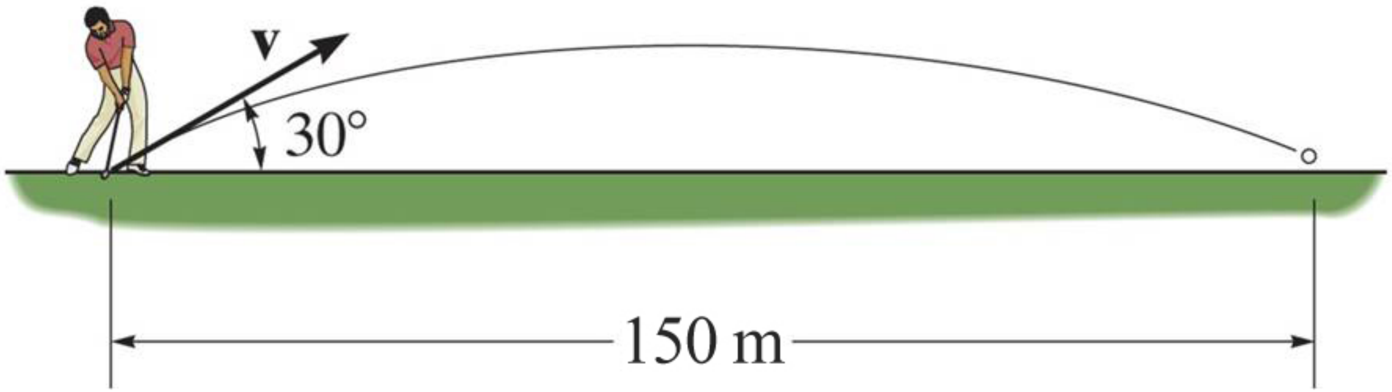
2. A constant force $\vec{F}$ is applied for 2 s to change the particle's velocity from $\vec{v}_1$ to $\vec{v}_2$. Determine the force $\vec{F}$ if the particle's mass is 2 kg.

   A) $(17.3\hat{j})$ N B) $(-10\,i + 17.3\hat{j})$ N

   ✔ C) $(20\hat{i} + 17.3\hat{j})$ N D) $(10\hat{i} + 17.3\hat{j})$ N



Example



Given: A 0.05-kg golf ball is struck by the club and travels along the trajectory shown, $\theta = 30°$, $R = 150$ m.

Assume the club maintains contact with the ball for 0.5 ms.

Find: The average impulsive force exerted on the ball.

Plan:

1) Find $\vec{v}$ using the kinematics equations.
2) Draw the momentum and impulse diagrams of the ball.
3) Apply the principle of impulse and momentum to determine the impulsive force.

Example

Solution:

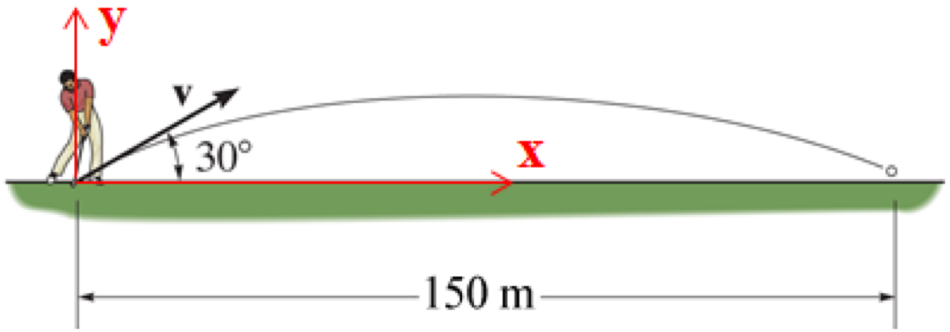
1) Kinematics :

horizontal motion equation :

$x = x_0 + v_x t$

$150 = 0 + v\,(\cos 30°)\,t$

$\Rightarrow t = 150 / (v \cos 30°)$



vertical motion equation :

$y = y_0 + v_y t - 0.5\, g\, t^2$, where $y = 0$, $y_0 = 0$, $v_y = v\,(\sin 30°)$

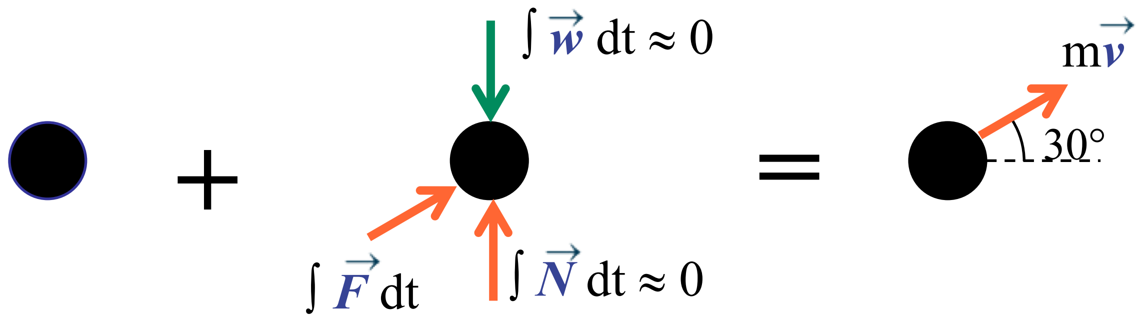
$\Rightarrow 0 = 0 + v\,(\sin 30°)\, t + 0.5\,(-9.81)\, t^2$

$$0 = v(\sin 30°)\left(\frac{150}{v \cos 30°}\right) + 0.5\,(-9.81)\left(\frac{150}{v \cos 30°}\right)^2$$

Solving for v: $v = 41.2$ m/s

Example

2) Draw the momentum and impulse diagrams are



The impulse generated by the weight of the golf ball is very small compared to that generated by the force of the impact. Hence, it and the resultant normal force can be neglected.

Example

3) Now, apply the principle of impulse and momentum to determine the impulsive force.

$$m\,(0) + \sum \int \vec{F}\,dt = m\vec{v}, \quad \text{where } v = 41.2 \text{ m/s}$$

$$\Rightarrow \vec{F}_{avg}\,(0.5)\,10^{-3} = (0.05)\,(41.2 \cos 30^{\circ}\,\hat{i} + 41.2 \sin 30^{\circ}\,\hat{j})$$

The average impulsive force is

$$\vec{F}_{avg} = 4120\,(\cos 30^{\circ}\,\hat{i} + \sin 30^{\circ}\,\hat{j})$$

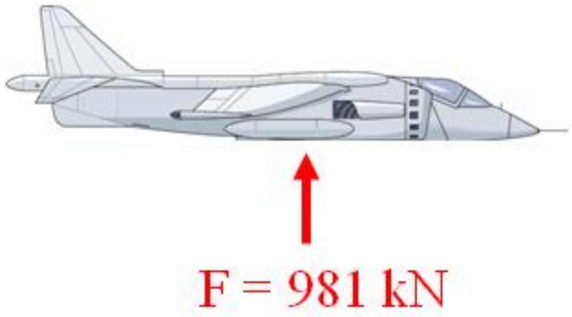
$$= (3568\,\hat{i} + 2060\,\hat{j}) \text{ N}$$

Quiz

1. Jet engines on the 100-Mg VTOL aircraft exert a constant vertical force of 981 kN as it hovers. Determine the net impulse on the aircraft over t = 10 s.

   A) -981 kN·s   B) 0 kN·s

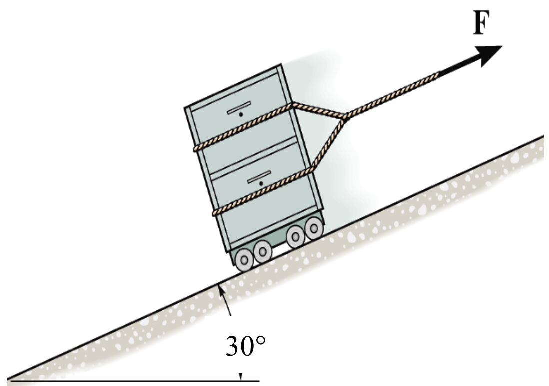
   C) 981 kN·s   D) 9810 kN·s



2. A 100-N(≈10-kg) cabinet is placed on a smooth surface. If a force of a 100 N is applied for 2 s, determine the net impulse on the cabinet during this time interval.

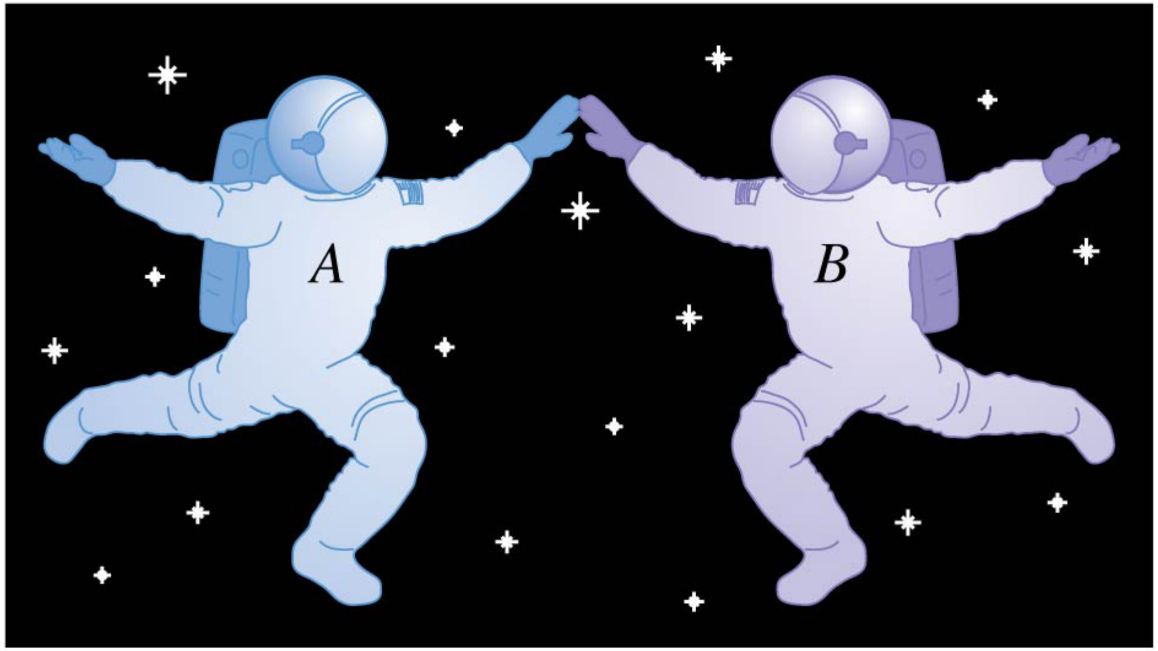
   A) 0 N·s   B) 100 N·s

   C) 200 N·s   D) 300 N·s

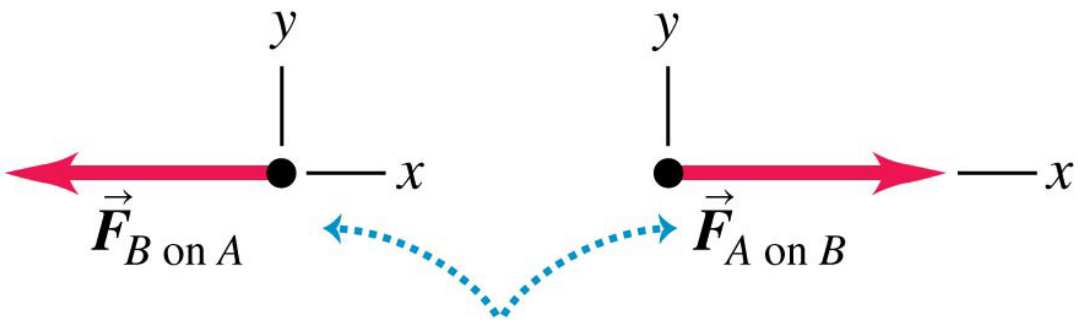


An isolated system

- Two astronauts push each other as they float freely in the zero-gravity environment of space.
- There are *no* external forces; when this is the case, we have an **isolated system**.



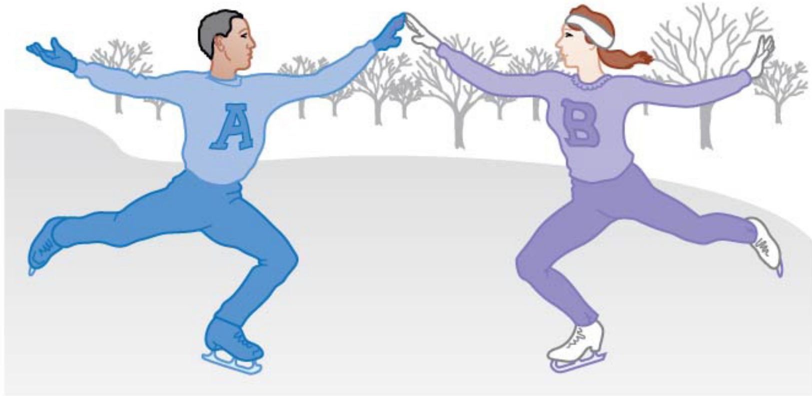
No external forces act on the two-astronaut system, so its total momentum is conserved.



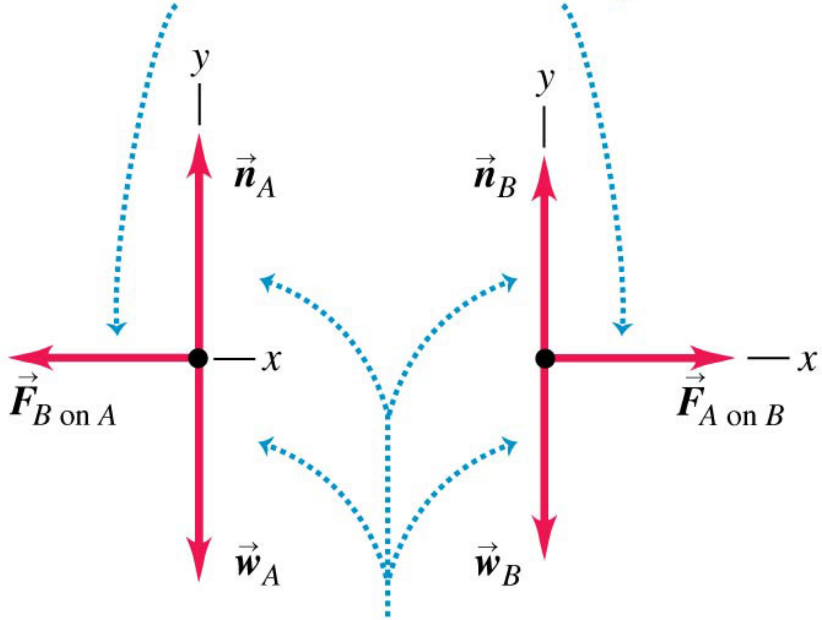
The forces the astronauts exert on each other form an action–reaction pair.

Conservation of momentum

- External forces (the normal force and gravity) act on the skaters shown, but their vector sum is zero.
- Therefore the total momentum of the skaters is conserved.
- **Conservation of momentum**: If the vector sum of the external forces on a system is zero, the total momentum of the system is constant.



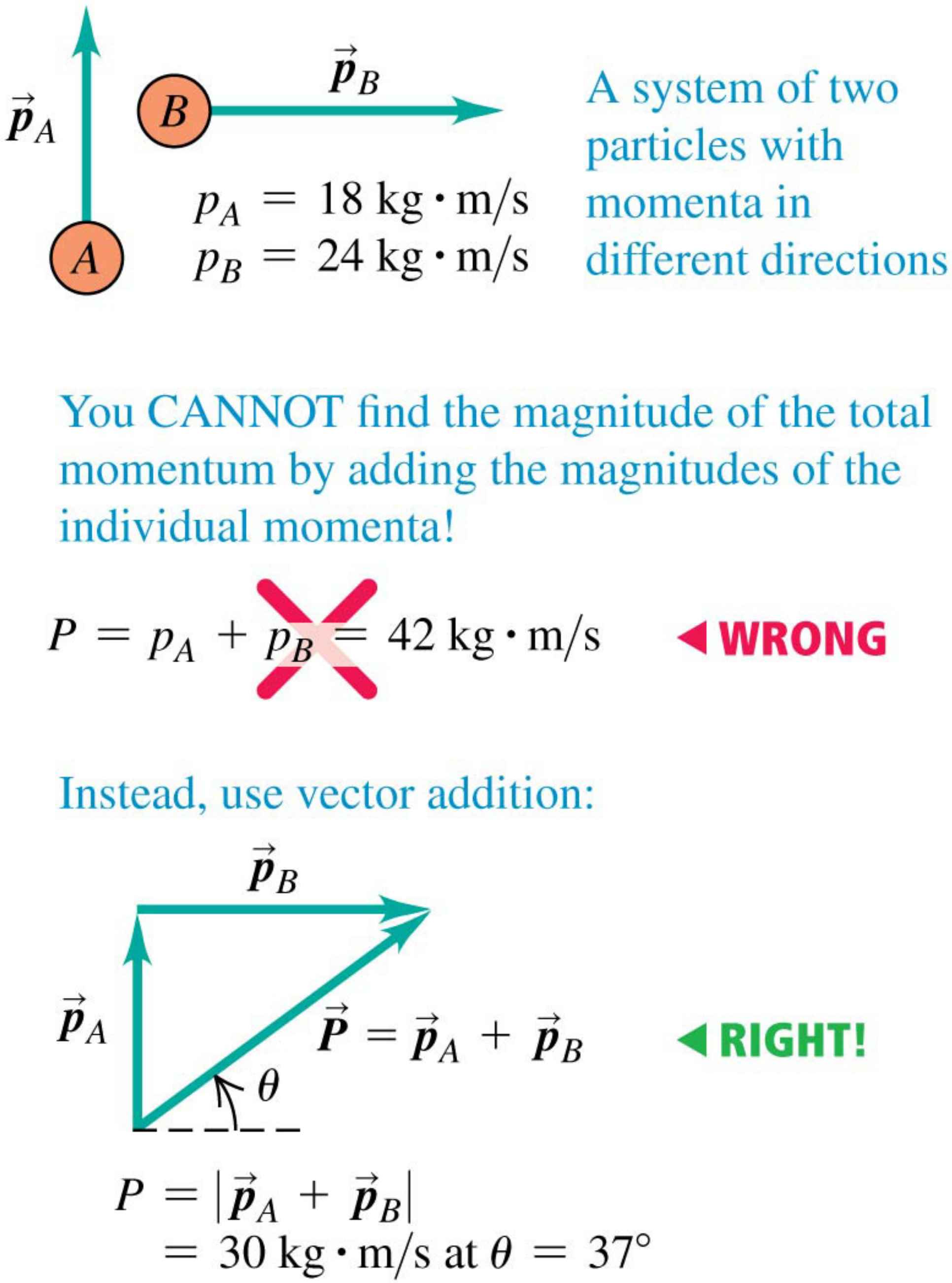
The forces the skaters exert on each other form an action–reaction pair.



Although the normal and gravitational forces are external, their vector sum is zero, so the total momentum is conserved.

Remember that momentum is a vector!

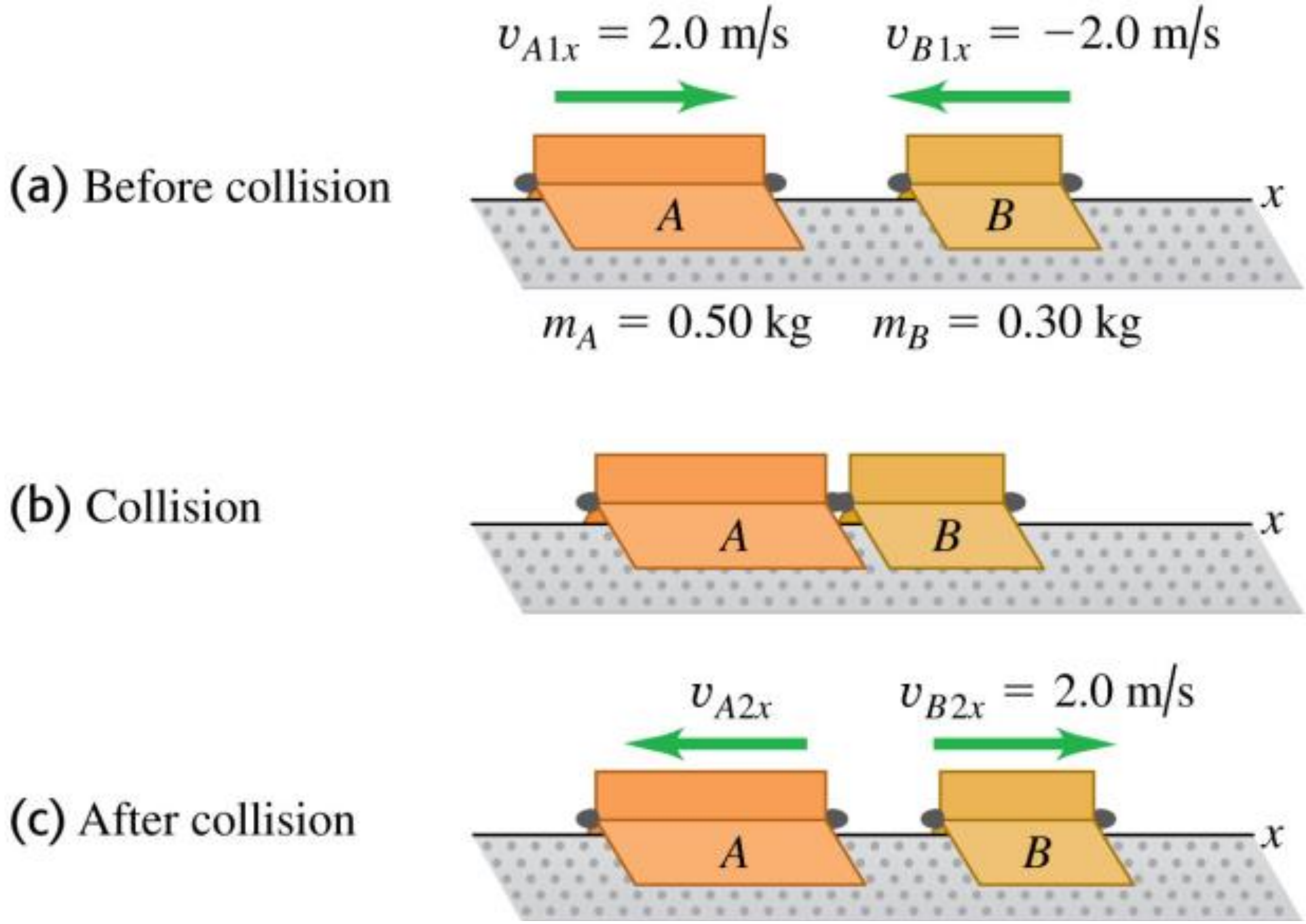
- When applying conservation of momentum, remember that momentum is a vector quantity!
- Use vector addition to add momenta, as shown at the right.



Objects colliding along a straight line

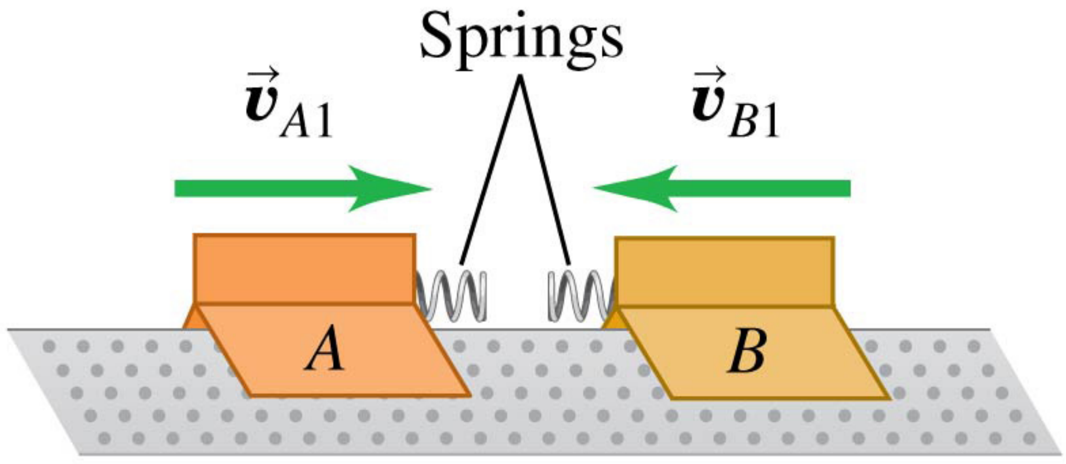
– Two gliders collide on an air trac.

– We can find v_{A2x} and the changes in momenta and velocities of the gliders.



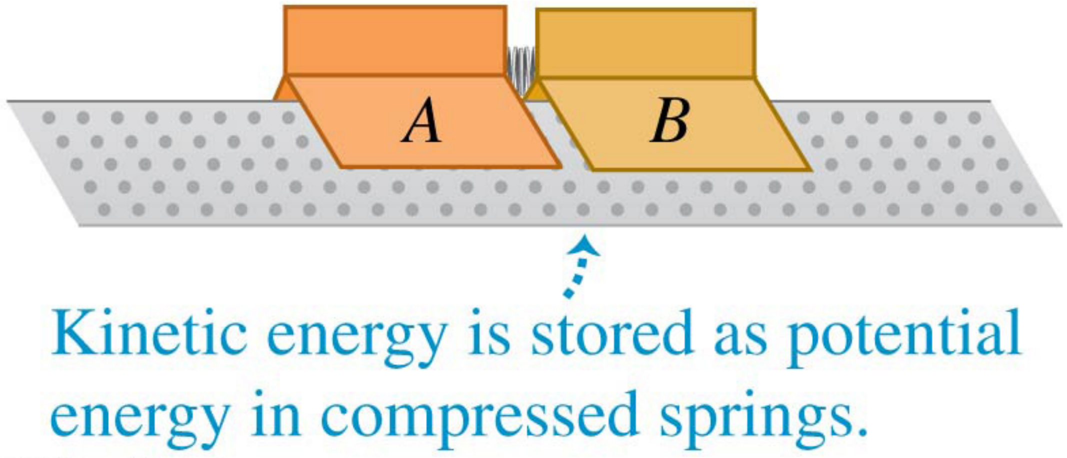
Elastic collisions: Before

- In an *elastic collision*, the total kinetic energy of the system is the same after the collision as before.



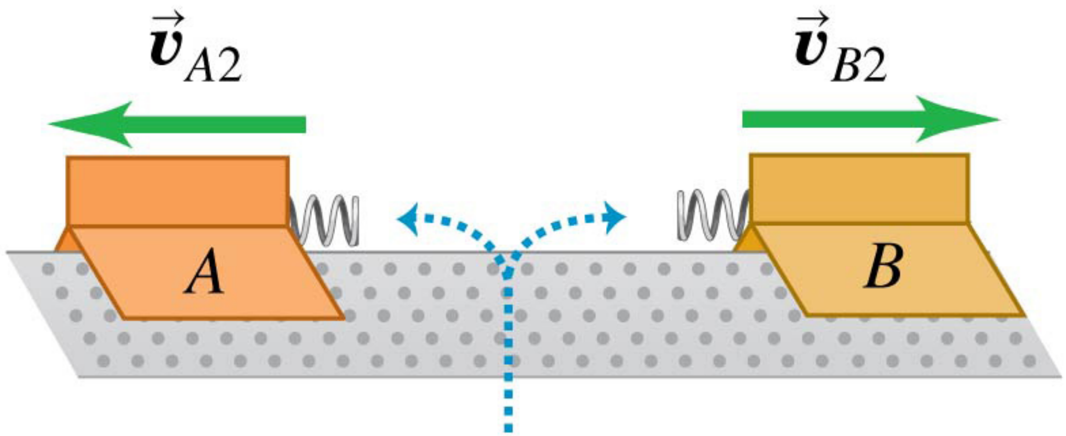
Elastic collisions: During

- In an *elastic collision*, the total kinetic energy of the system is the same after the collision as before.



Elastic collisions: After

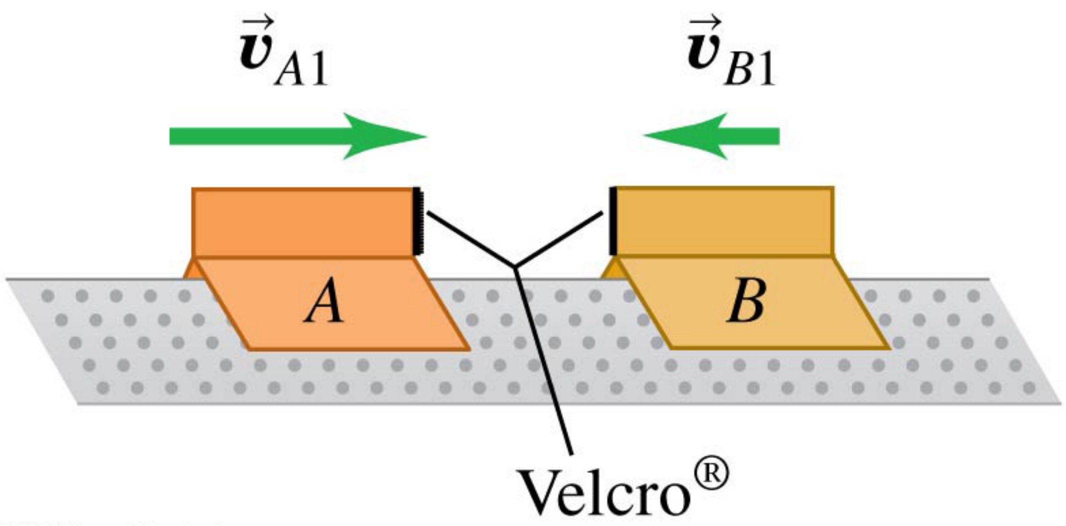
- In an *elastic collision*, the total kinetic energy of the system is the same after the collision as before.



The system of the two gliders has the same kinetic energy after the collision as before it.

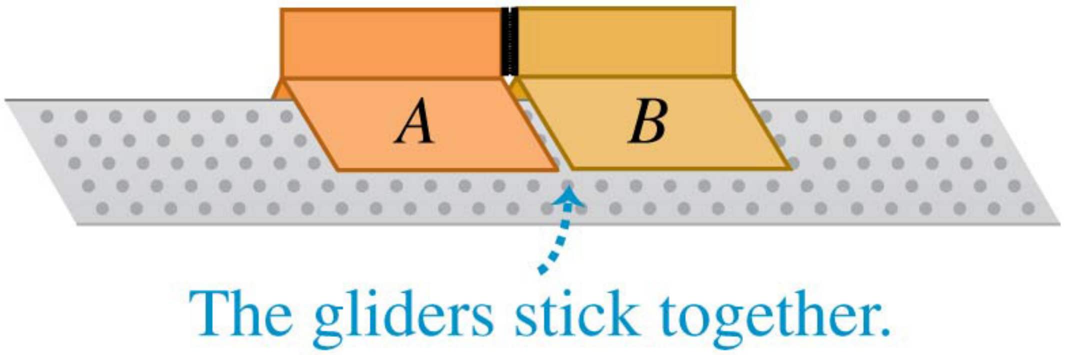
Completely inelastic collisions: Before

- In an *inelastic collision*, the total kinetic energy after the collision is less than before the collision.



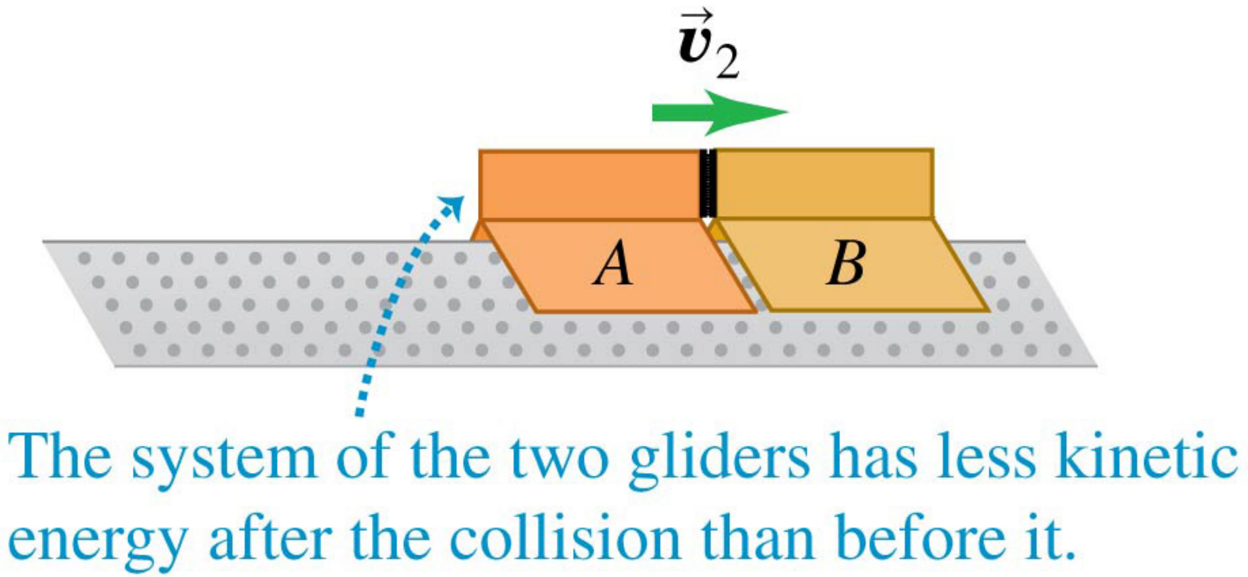
Completely inelastic collisions: During

- In an *inelastic collision*, the total kinetic energy after the collision is less than before the collision.



Completely inelastic collisions: After

- In an *inelastic collision*, the total kinetic energy after the collision is less than before the collision.

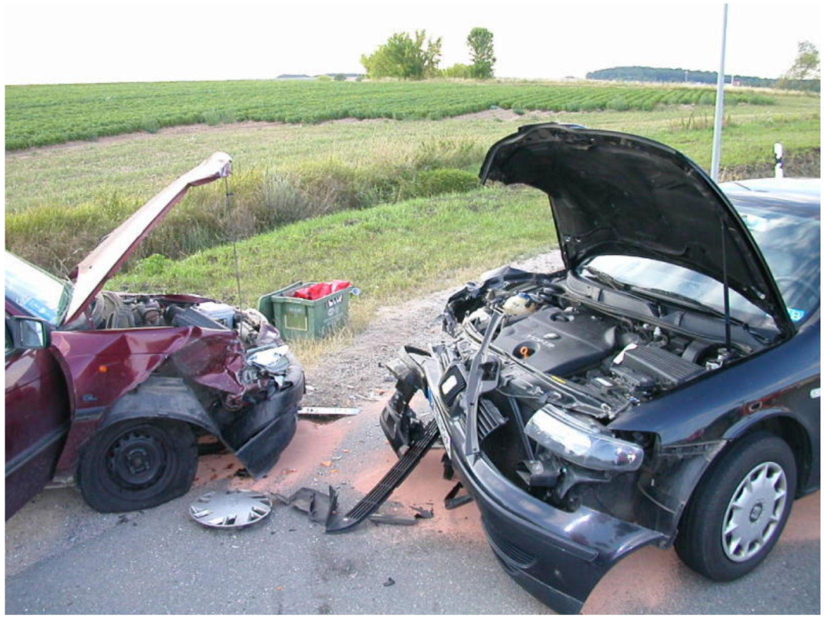


Collisions

- In an *inelastic collision*, the total kinetic energy after the collision is less than before the collision.
- A collision in which the bodies stick together is called a **completely inelastic collision**.
- In *any* collision in which the external forces can be neglected, the total momentum is conserved.
- In elastic collisions *only*, the total kinetic energy before equals the total kinetic energy after.

Some inelastic collisions

- Cars are intended to have inelastic collisions so the car absorbs as much energy as possible.
- Crumple zone (crush space) in automobile absorbs energy during collision by controlled deformation



- This absorbed energy cannot be recovered, since it goes into a permanent deformation of the car.

Elastic collision example

- Billiard balls deform very little when they collide, and they quickly spring back from any deformation they do undergo.

- Hence the force of interaction between the balls is almost perfectly conservative, and the collision is almost perfectly elastic.

Elastic collisions in one dimension

- Let's look at a one-dimensional elastic collision between two bodies *A* and *B*, in which all the velocities lie along the same line.

- We will concentrate on the particular case in which body *B* is at rest before the collision.

- Conservation of kinetic energy and momentum give the result for the final velocities of *A* and *B*

Elastic collisions

- For a collision of two bodies, consider the line of action between them

$$\begin{cases} m_A\vec{v}_{A1} + m_B\vec{v}_{B1} = m_A\vec{v}_{A2} + m_B\vec{v}_{B2} \\ \frac{1}{2}m_A\vec{v}_{A1}^{\,2} + \frac{1}{2}m_B\vec{v}_{B1}^{\,2} = \frac{1}{2}m_A\vec{v}_{A2}^{\,2} + \frac{1}{2}m_B\vec{v}_{B2}^{\,2} \end{cases}$$

- If body B is initially at rest (i.e. $v_{B1} = 0$), solving the above gives

$$\vec{v}_{A2} = \frac{m_A - m_B}{m_A + m_B}\vec{v}_{A1}$$

$$\vec{v}_{B2} = \frac{2m_A}{m_A + m_B}\vec{v}_{A1}$$

- Consider the cases where $m_A = m_B$, $m_A >> m_B$ and $m_A << m_B$

Elastic collisions (with B initially at rest)

$$m_A v_{A1} = m_A v_{A2} + m_B v_{B2}$$

$$\cancel{\tfrac{1}{2}} m_A v_{A1}^2 = \cancel{\tfrac{1}{2}} m_A v_{A2}^2 + \cancel{\tfrac{1}{2}} m_B v_{B2}^2$$

$$m_B v_{B2} = m_A (v_{A1} - v_{A2})$$

$$m_B v_{B2}^2 = m_A (v_{A1}^2 - v_{A2}^2)$$

$$\Longrightarrow \quad v_{B2} = v_{A1} + v_{A2}$$

$$m_A (v_{A1} - v_{A2}) = m_B (v_{A1} + v_{A2})$$

$$(m_A - m_B) v_{A1} = (m_A + m_B) v_{A2}$$

$$v_{A2} = \frac{m_A - m_B}{m_A + m_B} v_{A1} \qquad v_{B2} = \frac{2 m_A}{m_A + m_B} v_{A1}$$

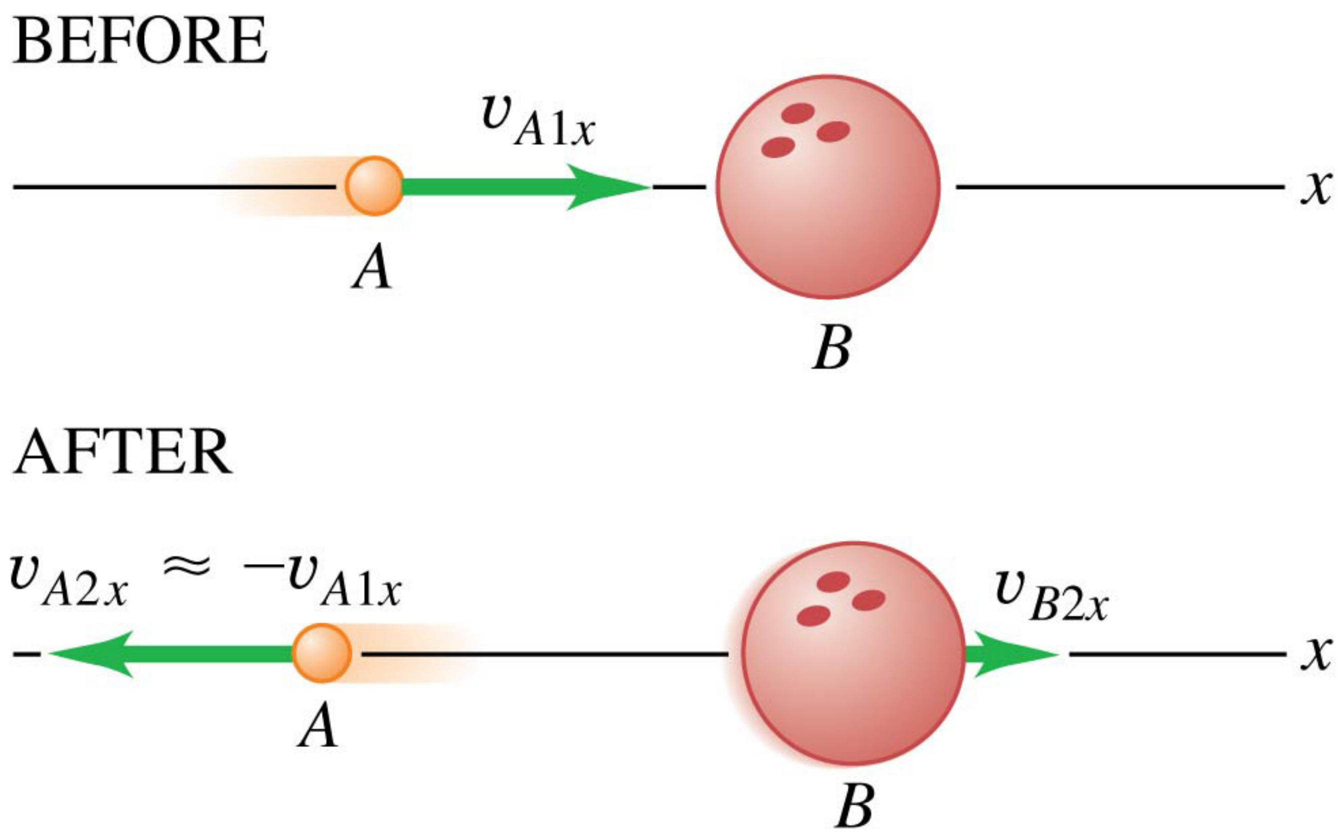
Elastic collisions (with B initially at rest)

$$v_{A2} = \frac{m_A - m_B}{m_A + m_B} v_{A1} \qquad v_{B2} = \frac{2m_A}{m_A + m_B} v_{A1}$$

		init	final	
case I:	$m_A = m_B$			
		v_{A1}	0	(swap vel)
		0	$v_{B2} = v_{A1}$	
case II:	$m_B \gg m_A$	v_{A1}	$v_{A2} \approx -v_{A1}$	
				(bounce back)
		0	$v_{B2} \approx 0$	
case III:	$m_A \gg m_B$	v_{A1}	$v_{A2} \lesssim v_{A1}$	
		0	$v_{B2} \lesssim 2v_{A1}$	

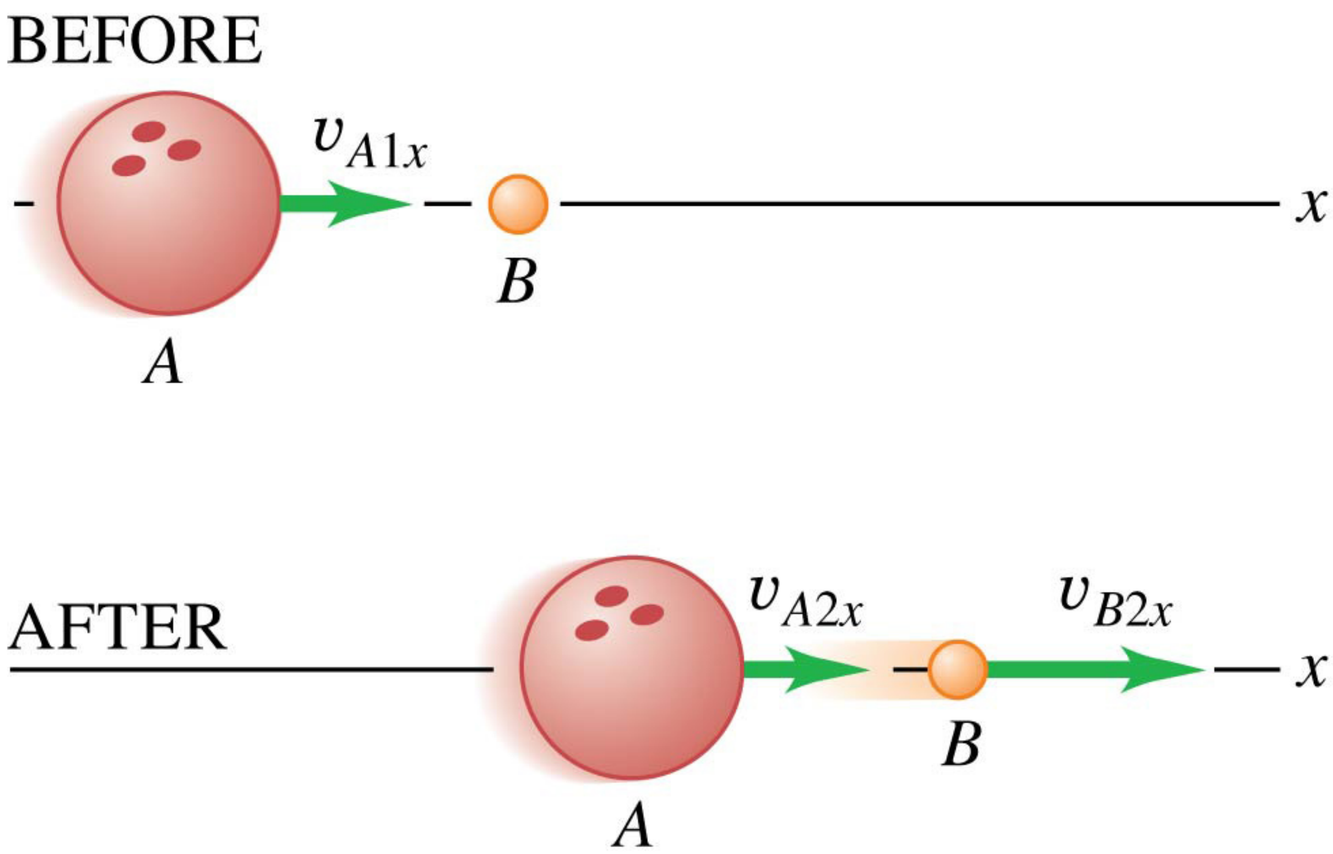
Elastic collisions in one dimension

- When B is much more massive than A, then A reverses its velocity direction, and B hardly moves.



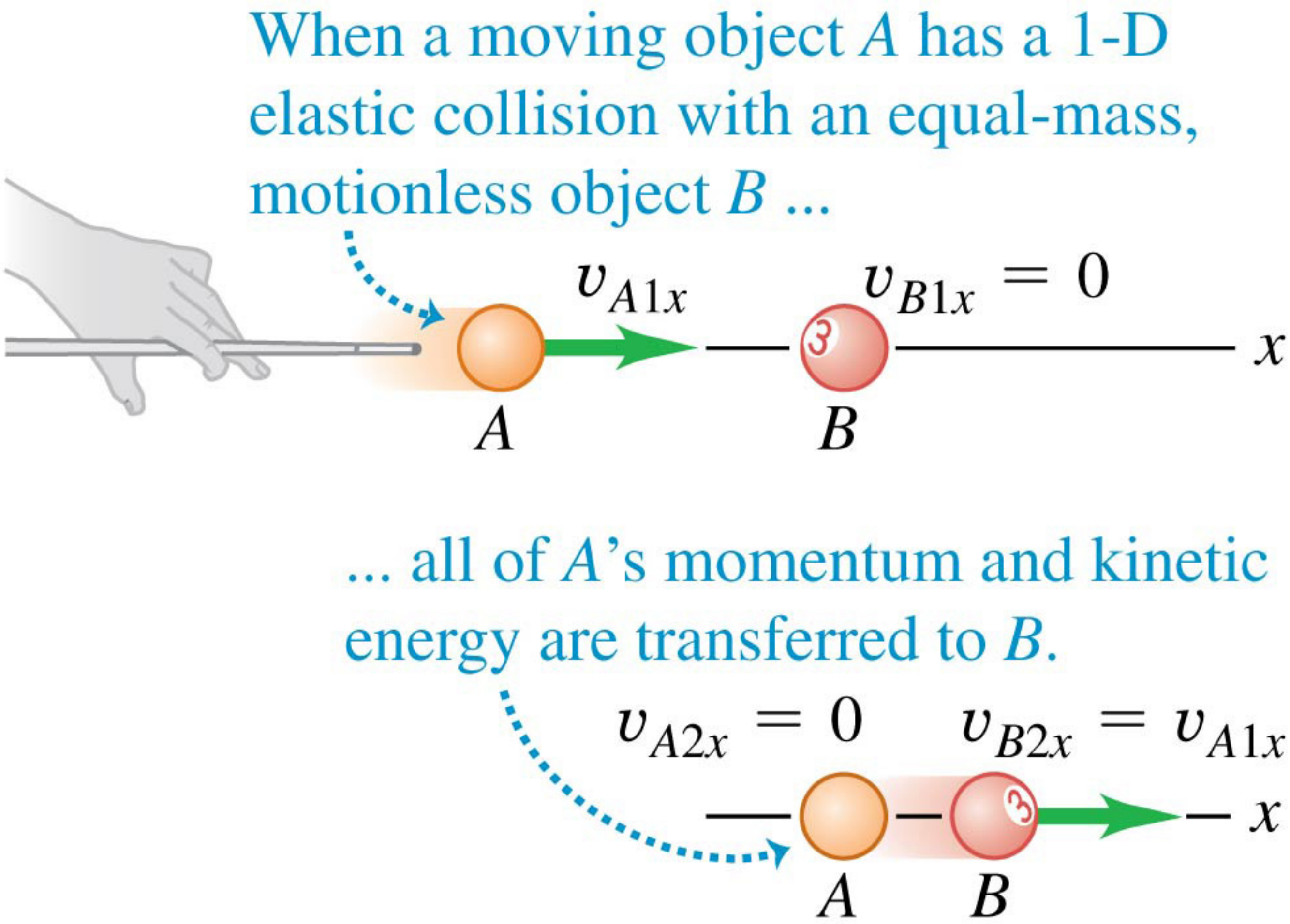
Elastic collisions in one dimension

- When *B* is much less massive than *A*, then *A* slows a little bit, while *B* picks up a velocity of about twice the original velocity of *A*.

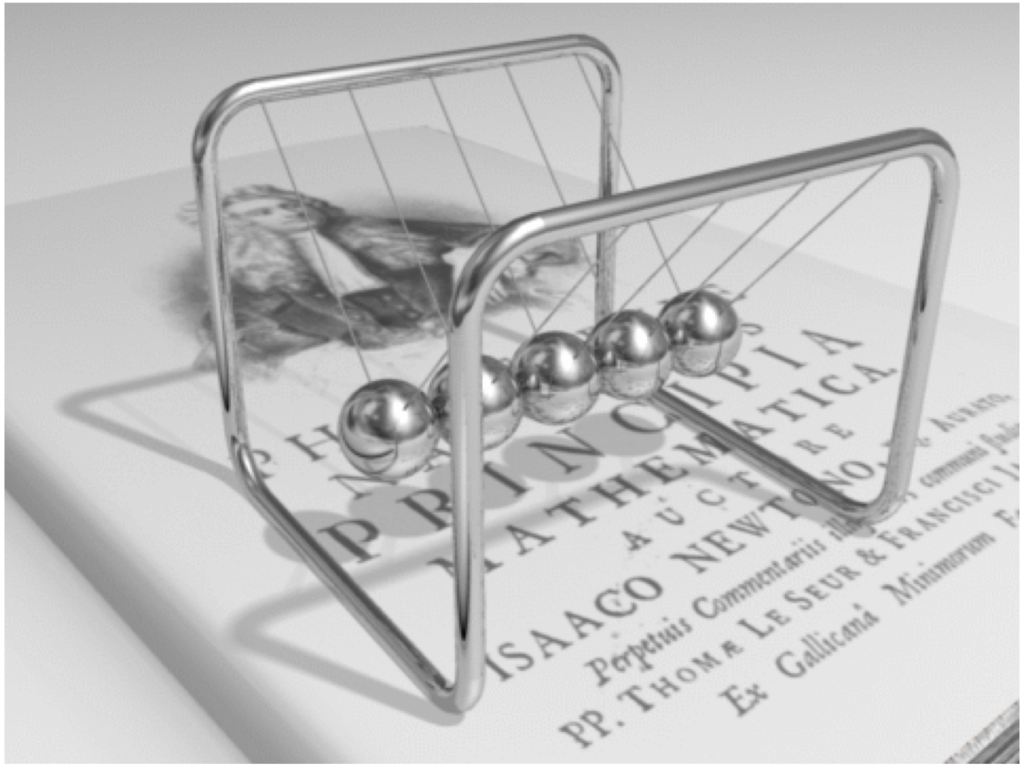


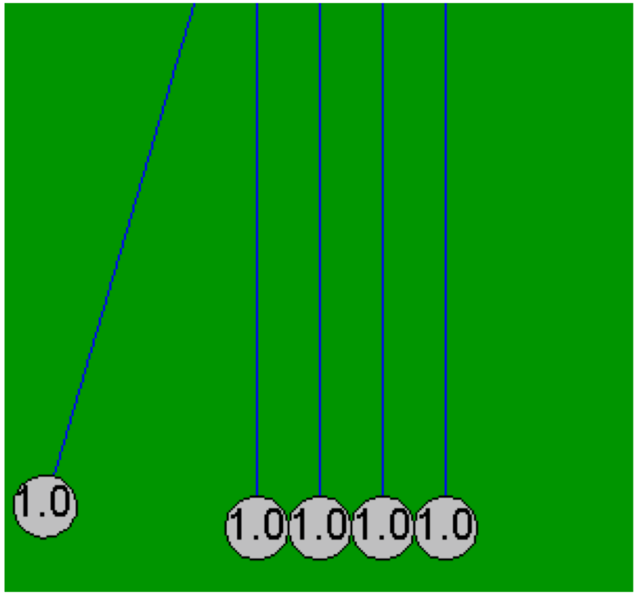
Elastic collisions in one dimension

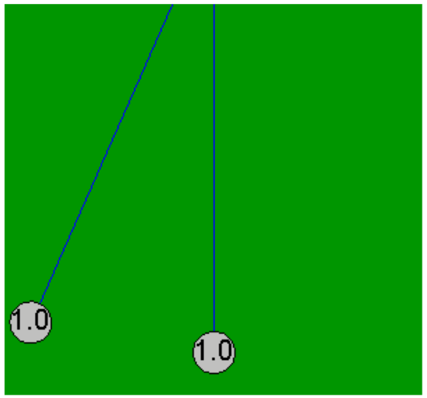
- When A and B have similar masses, then A stops after the collision and B moves with the original speed of A.

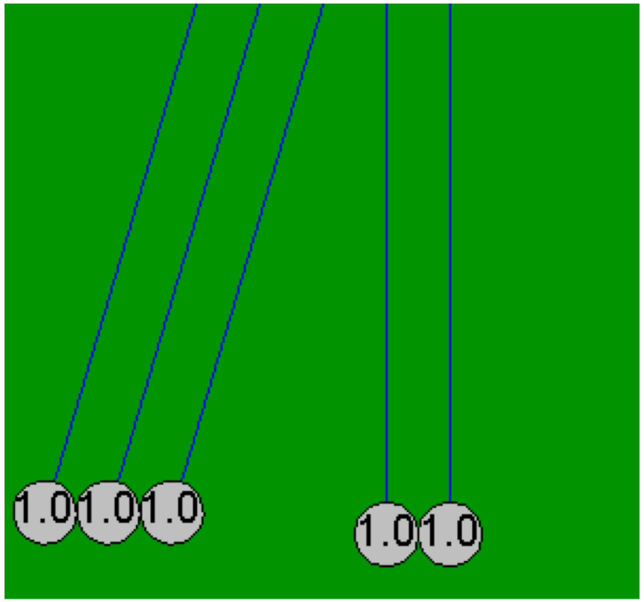


Newton’s cradle









Elastic collisions in 1d (general case)

Before collision: m_A, v_{A1x} $\quad m_B\ v_{B1x}$

After collision: m_A, v_{A2x} $\quad m_B\ v_{B2x}$

momentum conservation $\quad m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$ (

Kinetic energy conservation: $\frac{1}{2} m_A v_{A1x}^2 + \frac{1}{2} m_B v_{B1x}^2 = \frac{1}{2} m_A v_{A2x}^2 + \frac{1}{2} m_B v_{B2x}^2$ (2)

(1) $\Rightarrow$ $m_A (v_{A1x} - v_{A2x}) = m_B (v_{B2x} - v_{B1x})$ (3)

(2) $\Rightarrow$ $m_A (v_{A1x}^2 - v_{A2x}^2) = m_B (v_{B2x}^2 - v_{B1x}^2)$ (4)

$\frac{(4)}{(3)} \Rightarrow v_{A1x} + v_{A2x} = v_{B2x} + v_{B1x}$

(5) $\Rightarrow$ $v_{B2x} = v_{A1x} + v_{A2x} - v_{B1x}$ (6)

Elastic collisions in 1d (general case)

Plug ⑥ into ① $\Rightarrow$ $m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B\left(v_{A1x} + v_{A2x} - v_{B1x}\right)$

$$\Rightarrow (m_A + m_B) v_{A2x} = m_A v_{A1x} + m_B v_{B1x} - m_B v_{A1x} + m_B v_{B1x}$$

$$\Rightarrow v_{A2x} = \frac{m_A v_{A1x} + m_B(2v_{B1x} - v_{A1x})}{m_A + m_B} \quad (7)$$

Plug ⑦ into ⑥ $\Rightarrow$ $v_{B2x} = v_{A1x} - v_{B1x} + \dfrac{m_A v_{A1x} + m_B(2v_{B1x} - v_{A1x})}{m_A + m_B}$

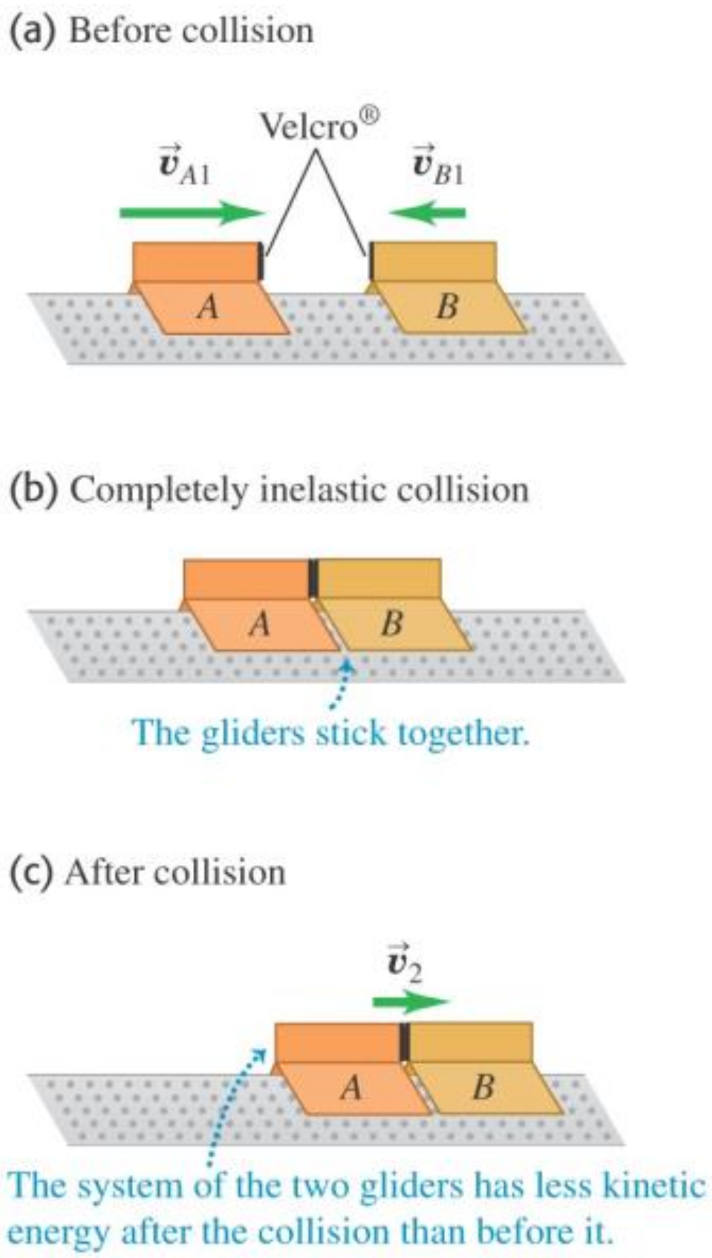
$$= \frac{m_A v_{A1x} - m_A v_{B1x} + \cancel{m_B v_{A1x}} - \cancel{m_B v_{B1x}} + m_A v_{A1x} + m_B(2v_{B1x} - v_{A}}{m_A + m_B}$$

$$= \frac{2m_A v_{A1x} - m_A v_{B1x} + m_A v_{A1x} + m_B v_{B1x}}{m_A + m_B}$$

$$\Rightarrow v_{B2x} = \frac{m_B v_{B1x} + m_A(2v_{A1x} - v_{B1x})}{m_A + m_B} \quad (8)$$

Inelastic collisions

- In an *inelastic collision*, the total kinetic energy after the collision is less than before the collision.
- A collision in which the bodies stick together is called a *completely inelastic collision* (see Figure at the right).
- In *any* collision in which the external forces can be neglected, the total momentum is conserved.



Inelastic collisions

- Conservation of momentum

$$m_A\vec{v}_{A1} + m_B\vec{v}_{B1} = (m_A + m_B)\vec{v}_2$$

- If m_B is initially at rest, i.e. $v_{B1} = 0$

$$\vec{v}_2 = \frac{m_A}{m_A + m_B}\vec{v}_{A1}$$

- Initial and final kinetic energies are

$$KE_1 = \frac{1}{2}m_A\vec{v}_{A1}^2$$

$$KE_2 = \frac{1}{2}(m_A + m_B)\vec{v}_2^2 = \frac{1}{2}\frac{m_A^2}{m_A + m_B}\vec{v}_{A1}^2$$

$$\boxed{\frac{KE_2}{KE_1} = \frac{m_A}{m_A + m_B}}$$

Center of mass

- We can restate the principle of conservation of momentum in a useful way by using the concept of center of mass.
- Suppose we have several particles with masses m_1, m_2, and so on.
- We define the center of mass of the system as the point at the position given by:

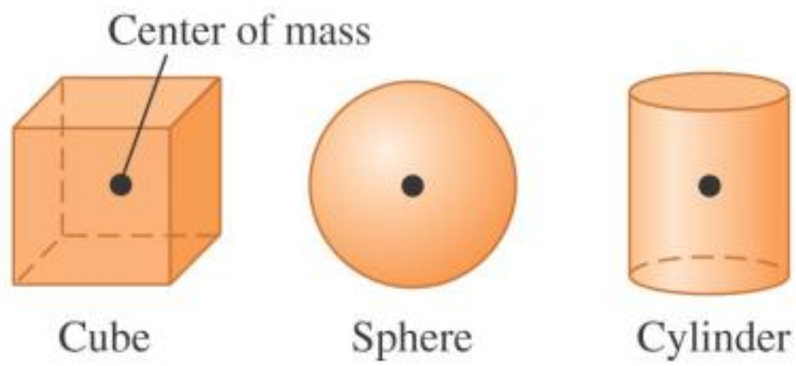
Position vectors of individual particles

Position vector of **center of mass** of a system of particles

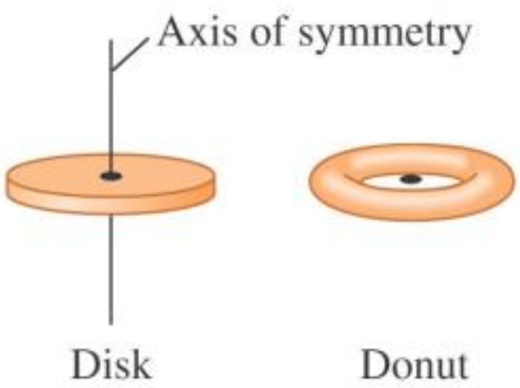
$$\vec{r}_{\text{cm}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i}$$

Masses of individual particles

Center of mass of symmetrical objects



If a homogeneous object has a geometric center, that is where the center of mass is located.

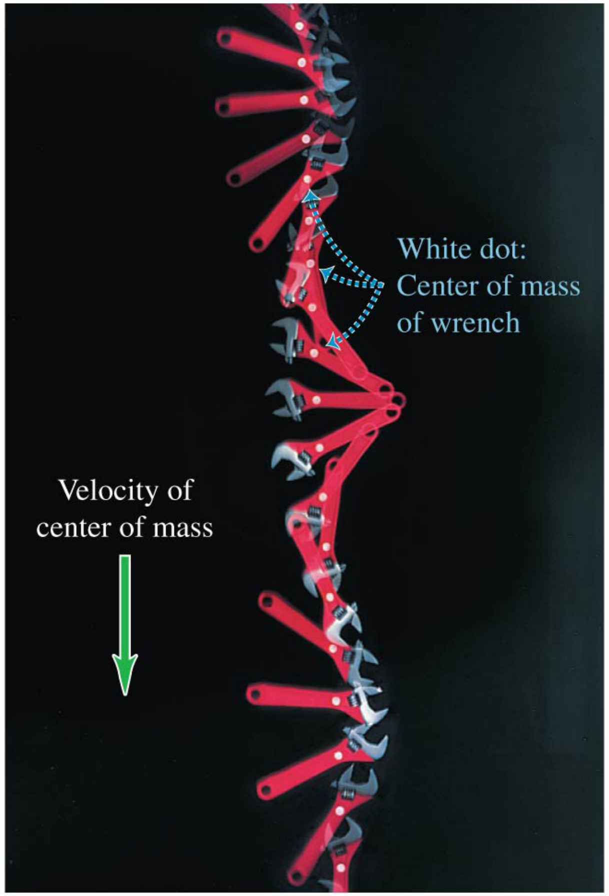


If an object has an axis of symmetry, the center of mass lies along it. As in the case of the donut, the center of mass may not be within the object.

- It is easy to find the center of mass of a homogeneous symmetric object, as shown here.

Motion of the center of mass

- The total momentum of a system is equal to the total mass times the velocity of the center of mass.
- The center of mass of the wrench at the right moves as though all the mass were concentrated there.



Total mass of a system of particles

Momenta of individual particles

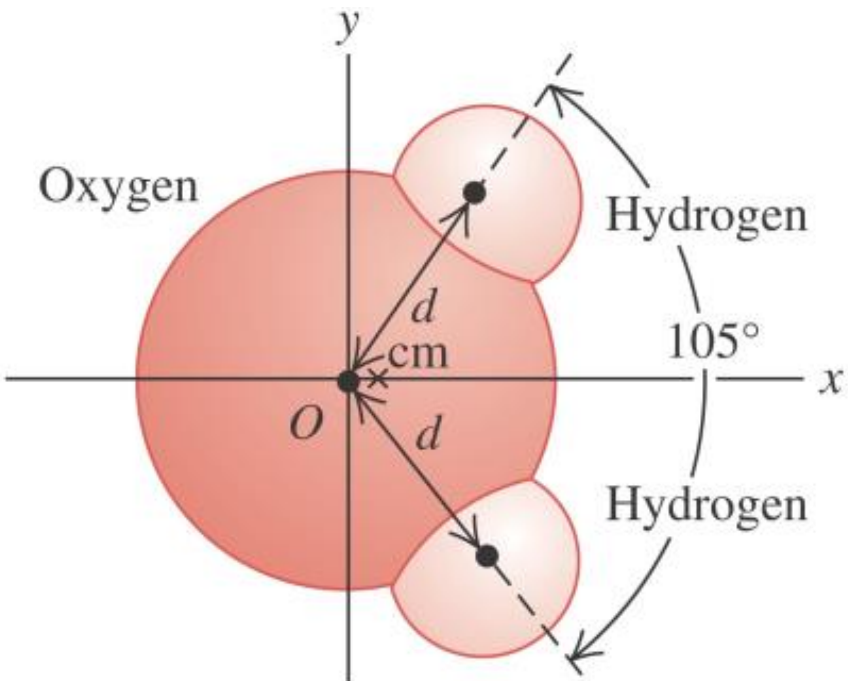
Velocity of center of mass

$$M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots = \vec{P}$$

Total momentum of system

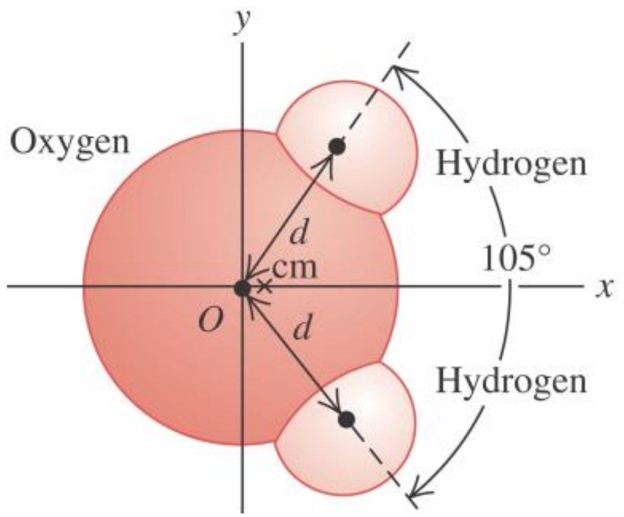
Center of mass of a water molecule

Figure 8.27 shows a simple model of a water molecule. The oxygen-hydrogen separation is $d = 9.57 \times 10^{-11}$ m. Each hydrogen atom has mass 1.0 u, and the oxygen atom has mass 16.0 u. Find the position of the center of mass.



Center of mass of a water molecule

SOLUTION



IDENTIFY and SET UP: Nearly all the mass of each atom is concentrated in its nucleus, whose radius is only about 10^{-5} times the overall radius of the atom. Hence we can safely represent each atom as a point particle. Figure 8.27 shows our coordinate system, with the x-axis chosen to lie along the molecule's symmetry axis. We'll use Eqs. (8.28) to find x_{cm} and y_{cm}.

EXECUTE: The oxygen atom is at $x = 0$, $y = 0$. The x-coordinate of each hydrogen atom is $d\cos(105°/2)$; the y-coordinates are $\pm d\sin(105°/2)$. From Eqs. (8.28),

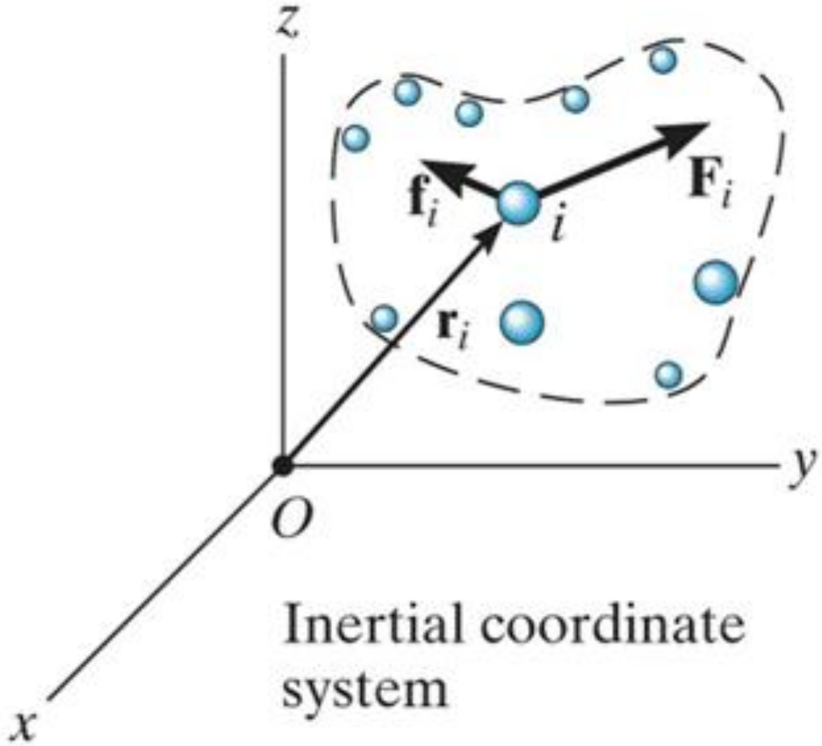
$$x_{cm} = \frac{\begin{bmatrix}(1.0\text{ u})(d\cos 52.5°) + (1.0\text{ u}) \\ \times (d\cos 52.5°) + (16.0\text{ u})(0)\end{bmatrix}}{1.0\text{ u} + 1.0\text{ u} + 16.0\text{ u}} = 0.068d$$

$$y_{cm} = \frac{\begin{bmatrix}(1.0\text{ u})(d\sin 52.5°) + (1.0\text{ u}) \\ \times (-d\sin 52.5°) + (16.0\text{ u})(0)\end{bmatrix}}{1.0\text{ u} + 1.0\text{ u} + 16.0\text{ u}} = 0$$

Substituting $d = 9.57 \times 10^{-11}$ m, we find

$$x_{cm} = (0.068)(9.57 \times 10^{-11}\text{ m}) = 6.5 \times 10^{-12}\text{ m}$$

General formalism



For the system of particles shown, the internal forces f_i between particles always occur in pairs with equal magnitude and opposite directions. Thus the internal impulses sum to zero.

The linear impulse and momentum equation for this system only includes the impulse of external forces.

$$\sum m_i(\vec{\mathbf{v}}_i)_1 + \sum \int_{t_1}^{t_2} \vec{\mathbf{F}}_i \, dt = \sum m_i(\vec{\mathbf{v}}_i)_2$$

Motion of the center of mass

For a system of particles, we can define a "fictitious" center of mass of an aggregate particle of mass m_{tot}, where m_{tot} is the sum ($\sum m_i$) of all the particles. This system of particles then has an aggregate velocity of $\vec{\mathbf{v}}_G = (\sum m_i \vec{\mathbf{v}}_i) / m_{tot}$.

The motion of this fictitious mass is based on motion of the center of mass for the system.

The position vector $\vec{\mathbf{r}}_G = (\sum m_i \vec{\mathbf{r}}_i) / m_{tot}$ describes the motion of the center of mass.

Conservation of linear momentum

When the sum of external impulses acting on a system of objects is zero, the linear impulse-momentum equation simplifies to

$$\sum m_i(\vec{v}_i)_1 = \sum m_i(\vec{v}_i)_2$$

This equation is referred to as the conservation of linear momentum. Conservation of linear momentum is often applied when particles collide or interact. When particles impact, only impulsive forces cause a change of linear momentum.

The sledgehammer applies an impulsive force to the stake. The weight of the stake is considered negligible, or non-impulsive, as compared to the force of the sledgehammer. Also, provided the stake is driven into soft ground with little resistance, the impulse of the ground acting on the stake is considered non-impulsive.

Quiz

1. The internal impulses acting on a system of particles always ____________.

   A) equal the external impulses ✔ B) sum to zero

   C) equal the impulse of weight D) None of the above.

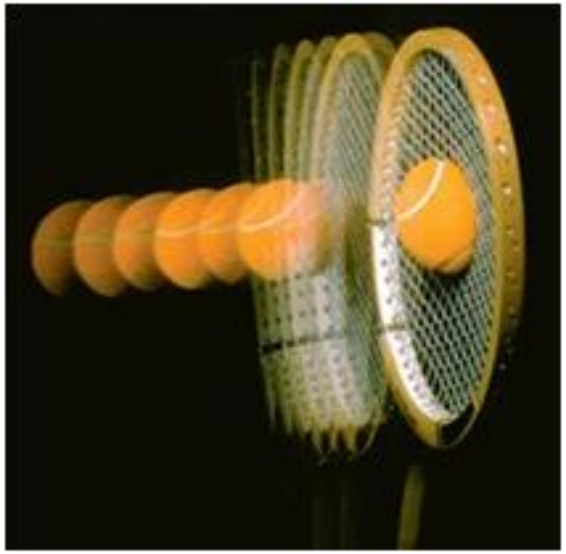
2. If an impulse-momentum analysis is considered during the very short time of interaction, as shown in the picture, weight is a/an ____________.

   A) impulsive force

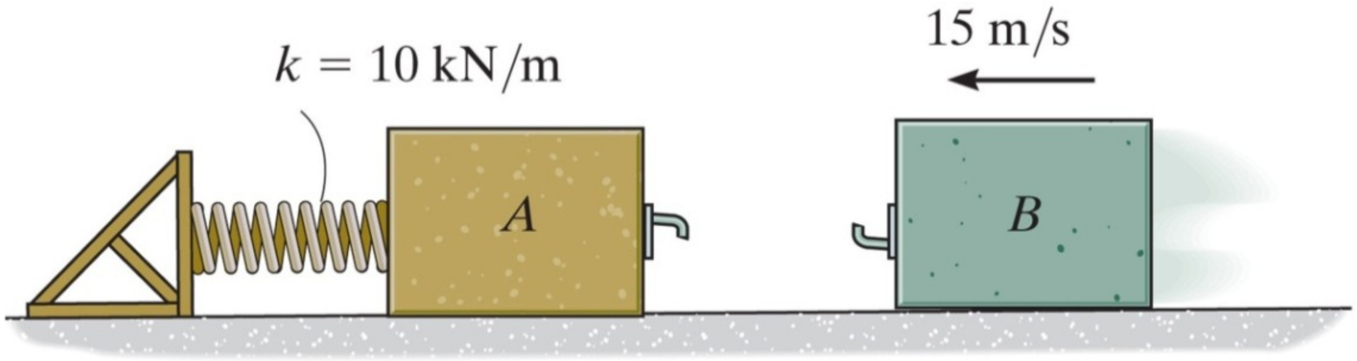
   B) explosive force

   ✔ C) non-impulsive force

   D) internal force



Example 1



Given: Spring constant k = 10 kN/m

m_A = 15 kg, v_A = 0 m/s, m_B = 10 kg, v_B = 15 m/s

The blocks couple together after impact.

Find: The maximum compression of the spring.

Plan:

1) We can consider both blocks as a single system and apply the conservation of linear momentum to find the velocity after impact, but before the spring compresses.

2) Then use the energy conservation to find the compression of the spring.

Example 1

Solution:

1) Conservation of linear momentum

$$\xrightarrow{+} \quad \sum m_i(\vec{v}_i)_0 = \sum m_i(\vec{v}_i)_1$$

$$10\,(-15\hat{i}) = (15+10)\,(v\,\hat{i})$$

$$v = -6 \text{ m/s}$$
$$= 6 \text{ m/s} \leftarrow$$

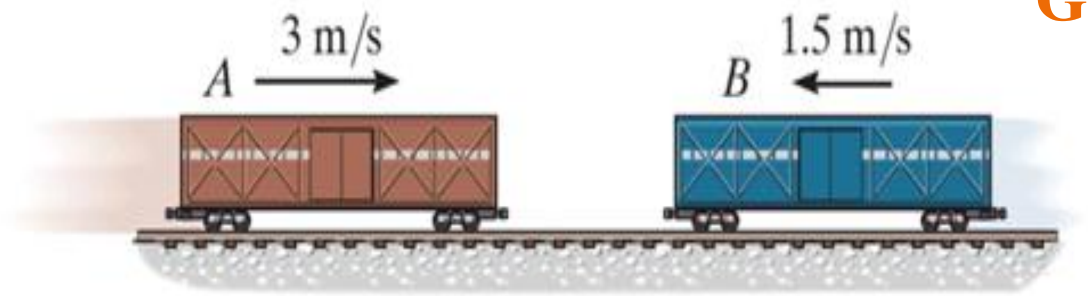
2) Energy conservation equation

$$K_1 + U_1 = K_2 + U_2$$

$$0.5\,(15+10)\,(-6)^2 + 0 = 0 + 0.5\,(10000)\,x^2$$

So the maximum compression of the spring is $x = 0.3$ m.

Example 2



Given: Two rail cars with masses of $m_A = 20$ Mg and $m_B = 15$ Mg and velocities as shown.

Find: The speed of the car A after collision if the cars collide and rebound such that B moves to the right with a speed of 2 m/s. Also find the average impulsive force between the cars if the collision place in 0.5 s.

Plan: Use conservation of linear momentum to find the velocity of the car A after collision (all internal impulses cancel). Then use the principle of impulse and momentum to find the impulsive force by looking at only one car.

Example 2

Solution:

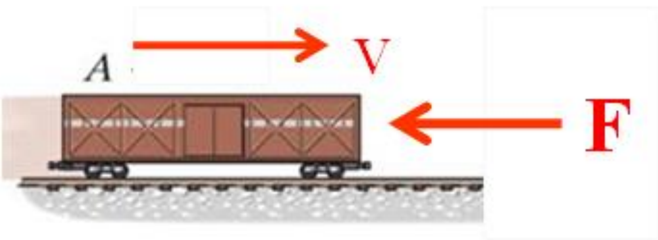
Conservation of linear momentum (x-dir):

$$m_A(v_{A1}) + m_B(v_{B1}) = m_A(v_{A2}) + m_B(v_{B2})$$

$$20000\ (3) + 15000\ (-1.5)$$

$$= (20000)\ v_{A2} + 15000\ (2)$$

$$v_{A2} = 0.375\ \text{m/s}$$



Impulse and momentum on car A (x-dir):

$$m_A\,(v_{A1}) + \int F\,dt = m_A\,(v_{A2})$$

$$20000\ (3) - \int F\,dt = 20000\ (0.375)$$

$$\int F\,dt = 52500\ \text{N}\cdot\text{s}$$

The average force is

$$\int F\,dt = 52500\ \text{N}\cdot\text{s} = F_{avg}(0.5\ \text{sec}); \quad F_{avg} = 105\ \text{kN}$$

Quiz

1) Over the short time span of a tennis ball hitting the racket during a player's serve, the ball's weight can be considered _______________.

✔ A) nonimpulsive

B) impulsive

C) not subject to Newton's second law

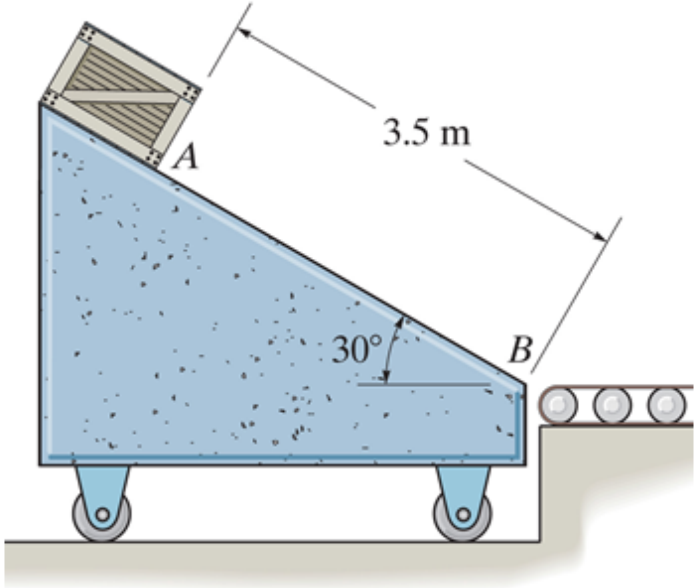
D) Both A and C.

2) A drill rod is used with an air hammer for making holes in hard rock so explosives can be placed in them. How many impulsive forces act on the drill rod during the drilling?

A) None B) One

✔ C) Two D) Three

Example 3



Given: The free-rolling ramp has a mass of 40 kg. The 10-kg crate slides from rest at A, 3.5 m down the ramp to B. Assume that the ramp is smooth, and neglect the mass of the wheels.

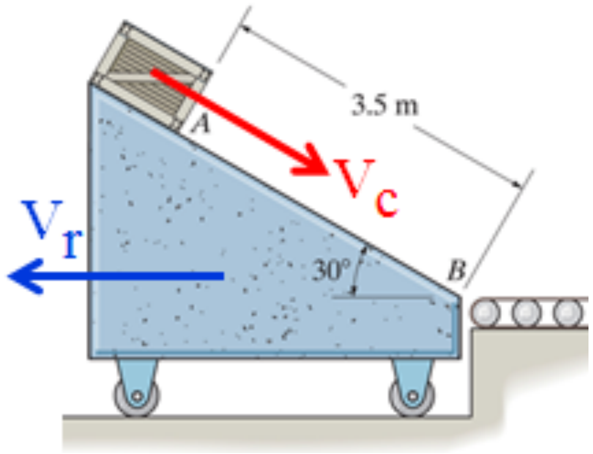
Find: The ramp's speed when the crate reaches *B*.

Plan:

Use the energy conservation equation as well as conservation of linear momentum and the relative velocity equation (you really thought you could safely forget it?) to find the velocity of the ramp.

Example 3

Solution:



To find the relations between $\vec{v}_C$ and $\vec{v}_r$, use conservation of linear momentum:

$$\overset{+}{\rightarrow} \quad 0 = (40)(-v_r) + (10)\, v_{Cx}$$

$$\Rightarrow \quad v_{Cx} = 4\, v_r \qquad (1)$$

Since $\vec{v}_C = \vec{v}_r + \vec{v}_{C/r} \Rightarrow v_{Cx}\hat{\imath} - v_{Cy}\hat{\jmath} = -v_r\hat{\imath} + v_{C/r}\,(\cos 30°\,\hat{\imath} - \sin 30°\,\hat{\jmath})$

$$\Rightarrow \quad v_{Cx} = -\,v_r + v_{C/r}\cos 30° \qquad (2)$$

$$v_{Cy} = v_{C/r}\sin 30° \qquad (3)$$

Eliminating $v_{C/r}$ from Eqs. (2) and (3), and substituting Eq. (1) results in $v_{Cy} = 8.660\, v_r$

Example 3

Then, energy conservation equation can be written ;

$$K_1 + U_1 = K_2 + U_2$$

$$0 + 10\,(9.81)(3.5\ \sin 30°) = 0.5\,(10)(v_C)^2 + 0.5\,(40)(v_r)^2$$

$$\Rightarrow \quad 0 + 10\,(9.81)(3.5\ \sin 30°)$$

$$= 0.5\,(10)\,[(4.0\ v_r)^2 + (8.660\ v_r)^2] + 0.5\,(40)\,(v_r)^2$$

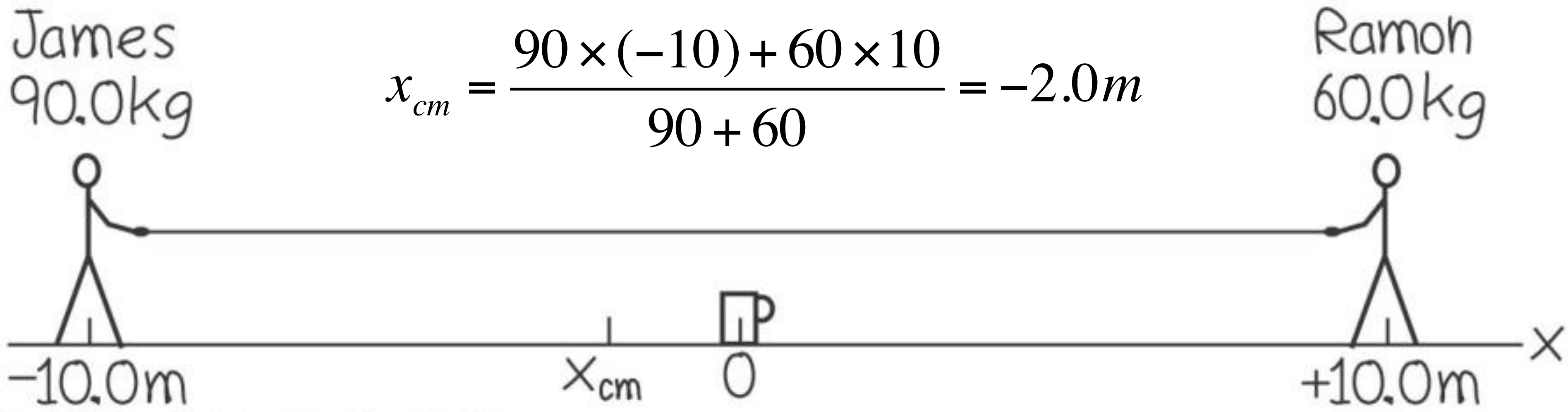
$$\Rightarrow \quad 171.7 = 475.0\,(v_r)^2$$

$v_r = 0.601$ m/s

Example: Tug-of-war on the ice

- A tug-of-war occurs on frictionless ice. When James has moved 6.0 m toward the mug, how far and in what direction has Ramon moved?

$$x_{cm} = \frac{90\times(-10)+60\times10}{90+60} = -2.0m$$

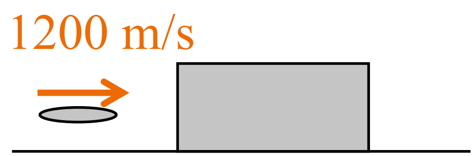


$$x_{cm} = \frac{90\times(-4)+60\times x_{Ramon}}{90+60} = -2.0m$$

$$x_{Ramon} = 1m$$

Quiz

1. The 20-g bullet is fired horizontally at 1200 m/s into the 300-g block resting on a smooth surface. If the bullet becomes embedded in the block, what is the velocity of the block immediately after impact.

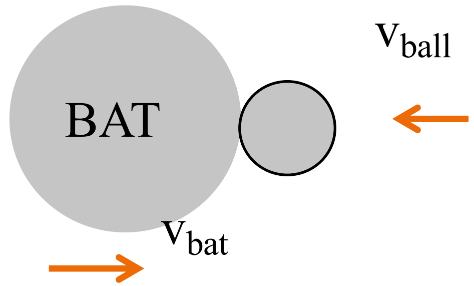


A) 1125 m/s B) 80 m/s

C) 1200 m/s ✔ D) 75 m/s

0.02*1200/0.32=75

2. The 200-g baseball has a horizontal velocity of 30 m/s when it is struck by the bat, B, weighing 900 g, moving at 47 m/s. During the impact with the bat, how many impulses of importance are used to find the final velocity of the ball?

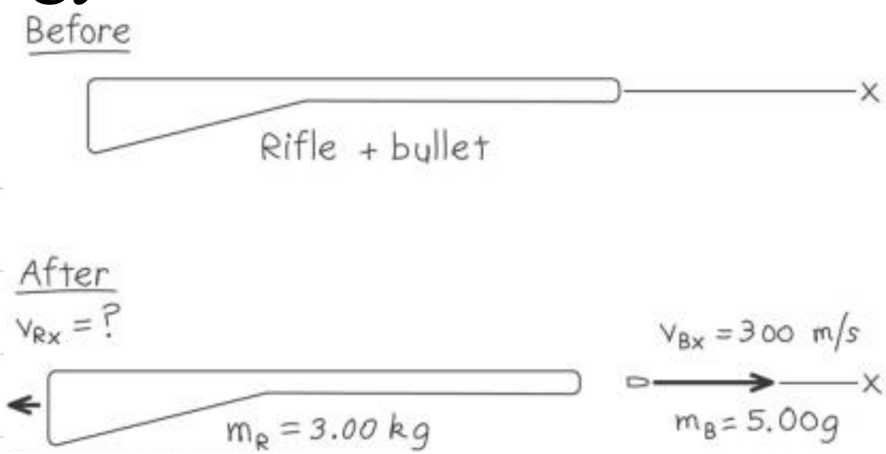


A) Zero ✔ B) One

C) Two D) Three

Example: Recoil of a rifle

- A rifle fires a bullet, causing the rifle to recoil. Find the recoil velocity v_{Rx} of the rifle. What are the final momentum and kinetic energy of the bullet and rifle? Where does the kinetic energy come from?



$$m_B v_{Bx} + m_R v_{Rx} = 0$$

$$v_{Rx} = -\frac{m_B v_{Bx}}{m_R}$$

$$= -\frac{5\times10^{-3} \times 300}{3} = -0.5 \text{ m/s}$$

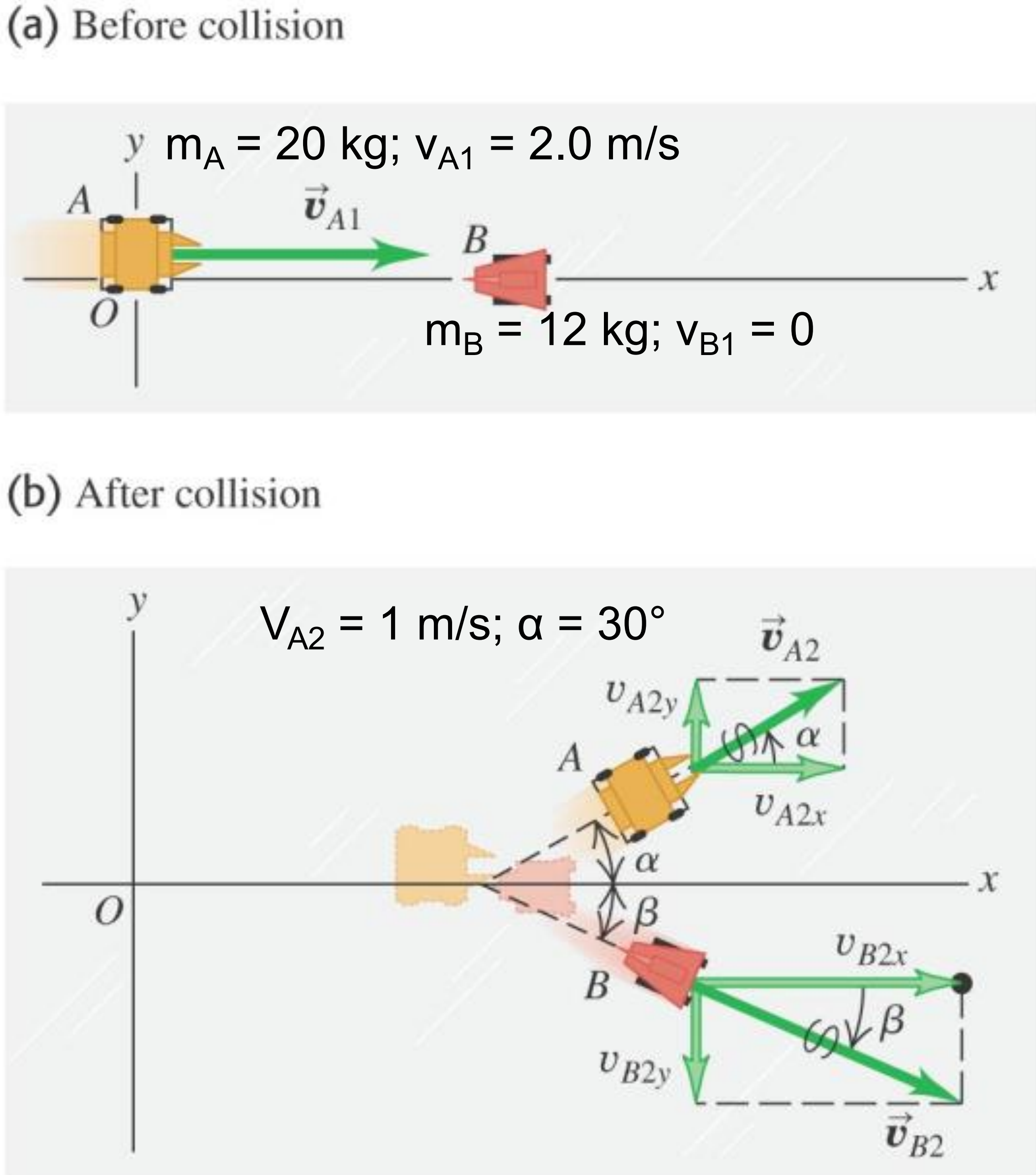
$$P_{Bx} = 1.5 \text{ kg}\cdot\text{m/s}$$

$$P_{Rx} = -1.5 \text{ kg m/s}$$

(verify $P_{Bx} + P_{Rx} = 0$)

A two-dimensional collision

- Two robots collide and go off at different angles.
- What is the final velocity of robot B?



A two-dimensional collision

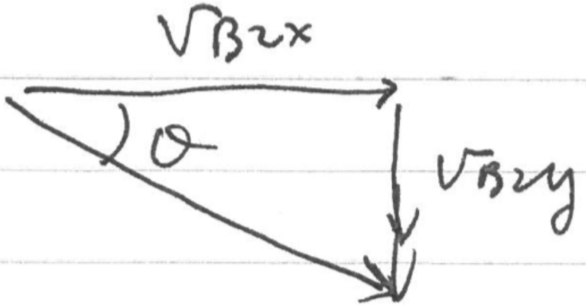
$$m_A v_{A2x} + m_B v_{B2x} = m_A v_A$$

$$20 \times \cos 30^\circ + 12 v_{B2x} = 20 \times 2$$

(1.732/2)

$$v_{B2x} = \frac{20 \times \left(2 - \frac{1.732}{2}\right)}{12} = 1.89\ m/s$$

$$m_A v_{A2y} + m_B v_{B2y} = 0$$



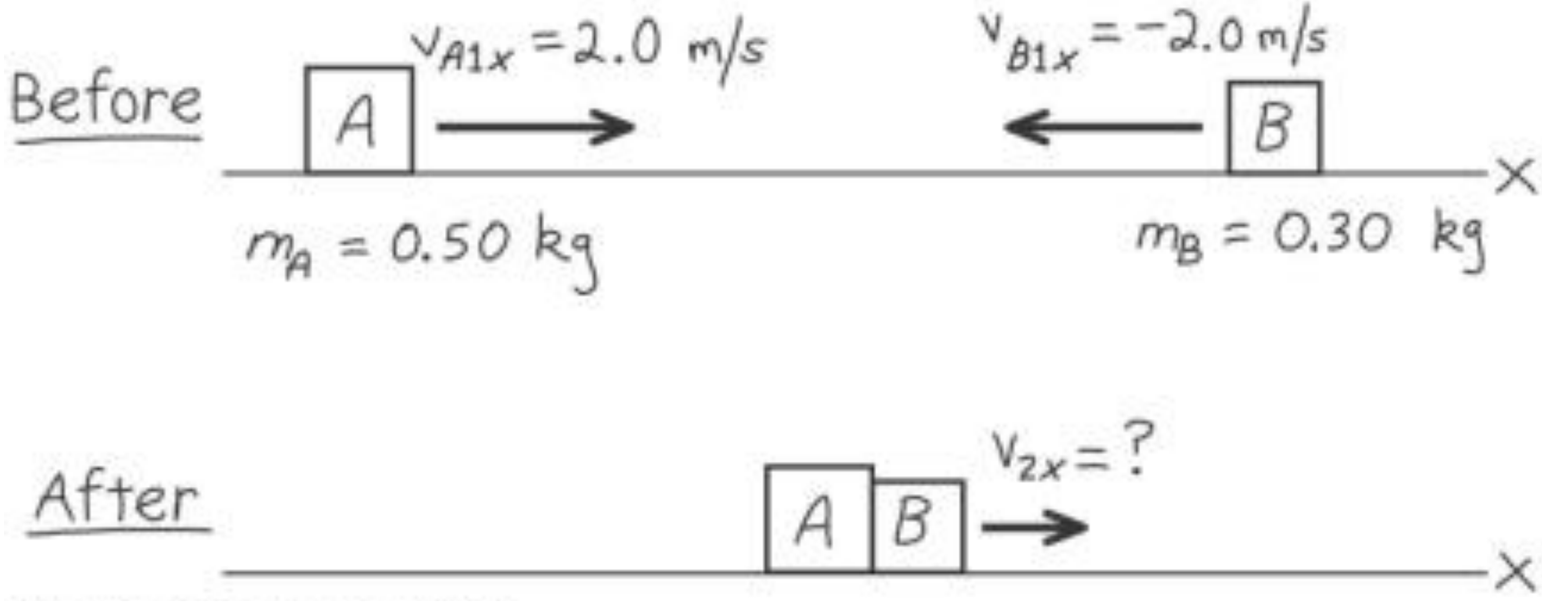
$$v_{B2y} = \frac{-20 \times 0.5}{12} = -0.833\ m/s$$

$$v_{B2} = \sqrt{1.89^2 + 0.833^2} = 2.1\ m/s$$

$$\theta = \mathrm{atan}\frac{-0.833}{1.89} = -24^\circ$$

Example: Inelastic collision

- Follow the figure, which illustrates a completely inelastic collision. Find v2x and change in kinetic energies of the system



Example: Inelastic collision

$$m_A v_A + m_B v_{B1x} = (m_A + m_B) v_{2x}$$

$$v_{2x} = \frac{m_A v_A + m_B v_{B1x}}{m_A + m_B}$$

$$= \frac{0.5 \times 2.0 + 0.30 \times (-2.0)}{0.50 + 0.30}$$

$$= 0.5\ m/s$$

$$KE_1 = \frac{1}{2} m_A v_{A1x}^2 + \frac{1}{2} m_B v_{B1x}^2$$

$$= \frac{1}{2}(0.50)(2.0)^2 + \frac{1}{2}(0.30)(-2.0)^2$$

$$= 1 + 0.6 = 1.6\ J$$

$$KE_2 = \frac{1}{2}(m_A + m_B) v_{2x}^2 = \frac{1}{2} \times 0.8 \times 0.5^2$$

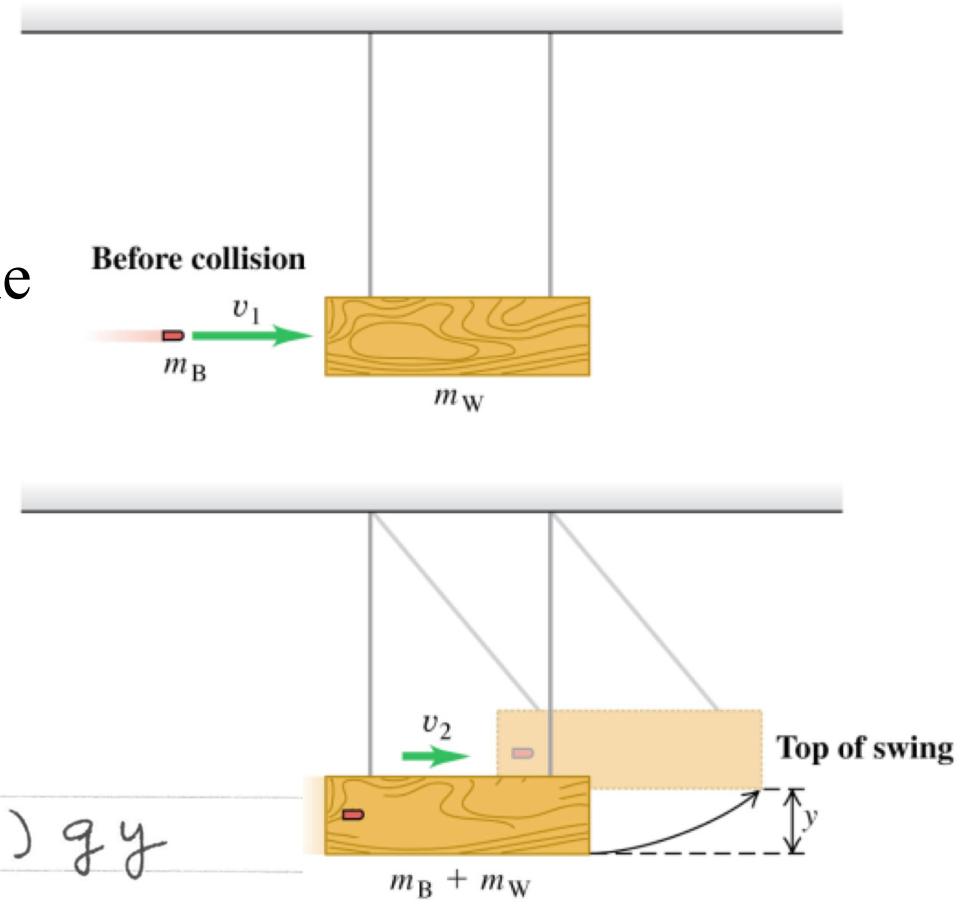
$$= 0.1\ J$$

(compare KE_2 & KE_1 to see energy loss)

The ballistic pendulum

– Ballistic pendulums are used to measure bullet speeds.

– What is the initial speed v_1 of the bullet, in terms of the other variables?



$$v_1 = \frac{m_B + m_w}{m_B}\sqrt{2gy}$$

$$\frac{1}{2}(m_B + m_w)v_2^2 = (m_B + m_w)gy$$

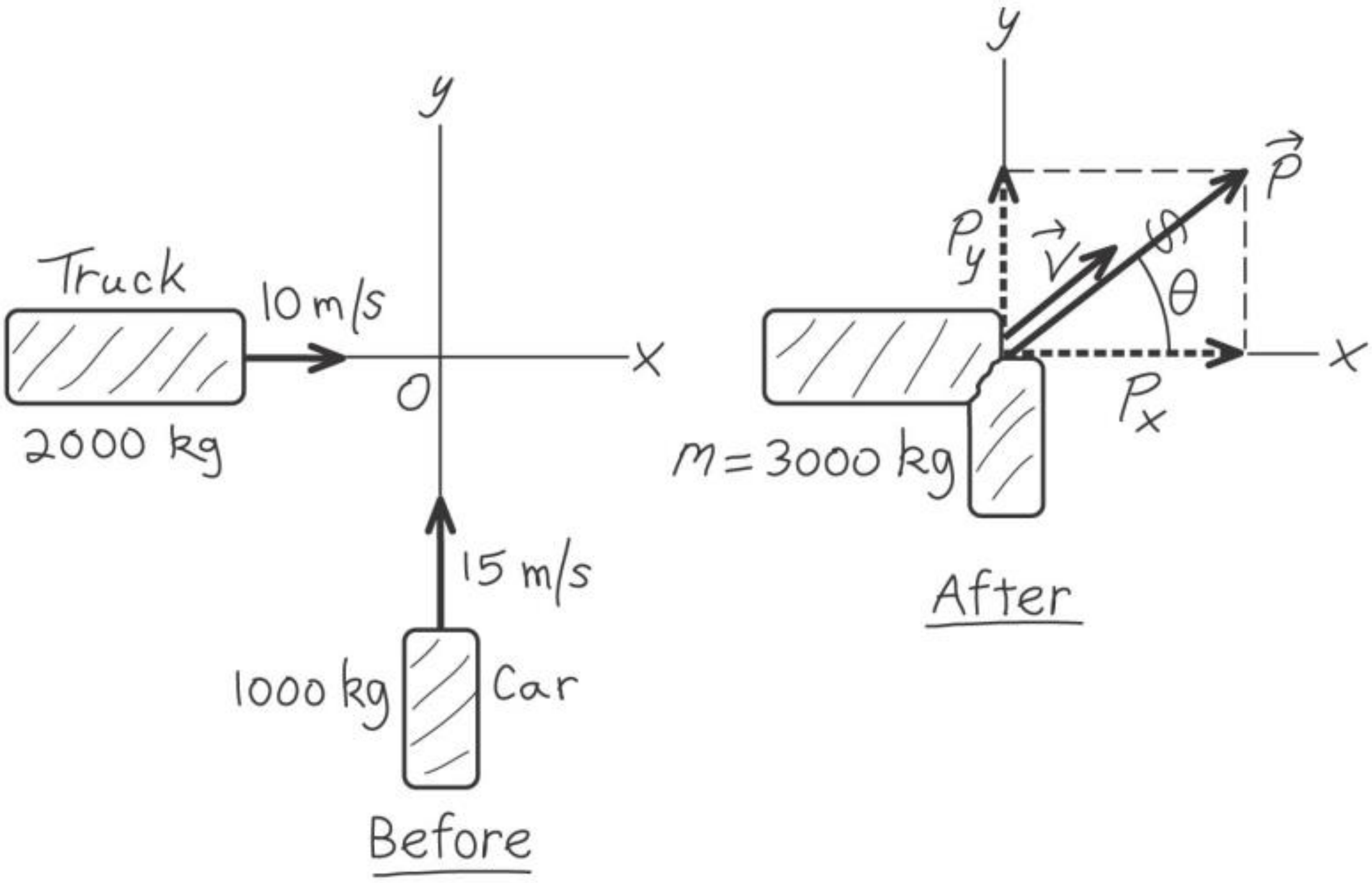
$$v_2 = \sqrt{2gy}$$

$$m_B v_1 = (m_B + m_w)v_2$$

$$v_1 = \frac{m_B + m_w}{m_B}v_2 = \frac{m_B + m_w}{m_B}\sqrt{2gy}$$

An automobile collision

– Two cars traveling at right angles collide. What is the velocity of the wreckage just after impact?



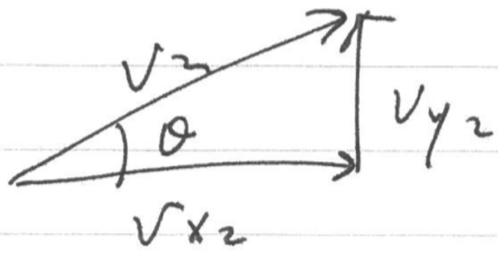
An automobile collision

$\Delta P_x = 0 \qquad m_T v_{x1} = (m_T + m_C) v_{x2}$

$\Delta P_y = 0 \qquad m_C v_{y1} = (m_T + m_C) v_{y2}$

$$v_{x2} = \frac{m_T v_{x1}}{m_T + m_C} = \frac{2000 \cdot 10}{3000} = \frac{20}{3} \simeq 6.7\,m/s$$

$$v_{y2} = \frac{m_C v_{y1}}{m_T + m_C} = \frac{1000 \times 15}{3000} = 5.0\,m/s$$

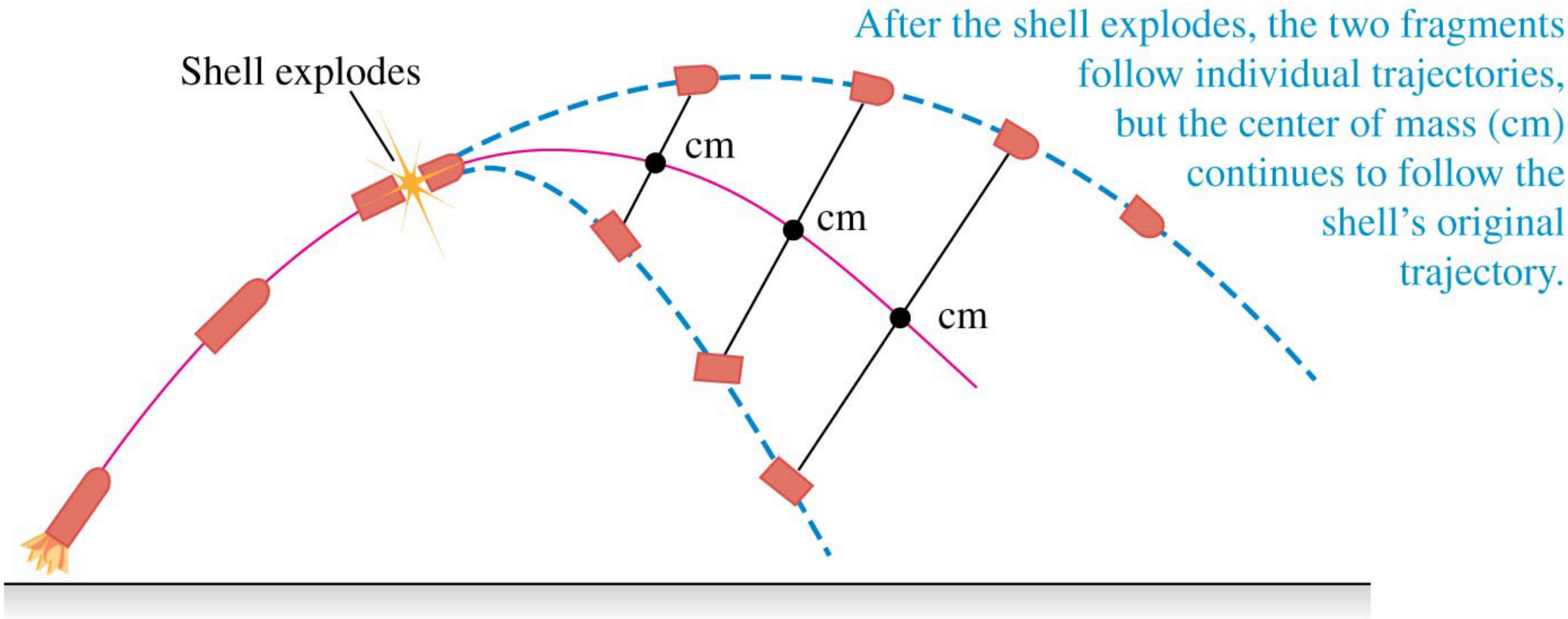


$$v_2 = \sqrt{5.0^2 + 6.7^2} = 8.4\,m/s$$

$$\theta = \operatorname{atan}\frac{5}{6.7} = 37^\circ$$

External forces and center-of-mass motion

- When a body or a collection of particles is acted on by external forces, the center of mass moves as though all the mass were concentrated at that point and it were acted on by a net force equal to the sum of the external forces on the system.



Rocket propulsion

- Why can a rocket function in outer space where “it doesn’t have anything to push against”?

Rocket propulsion

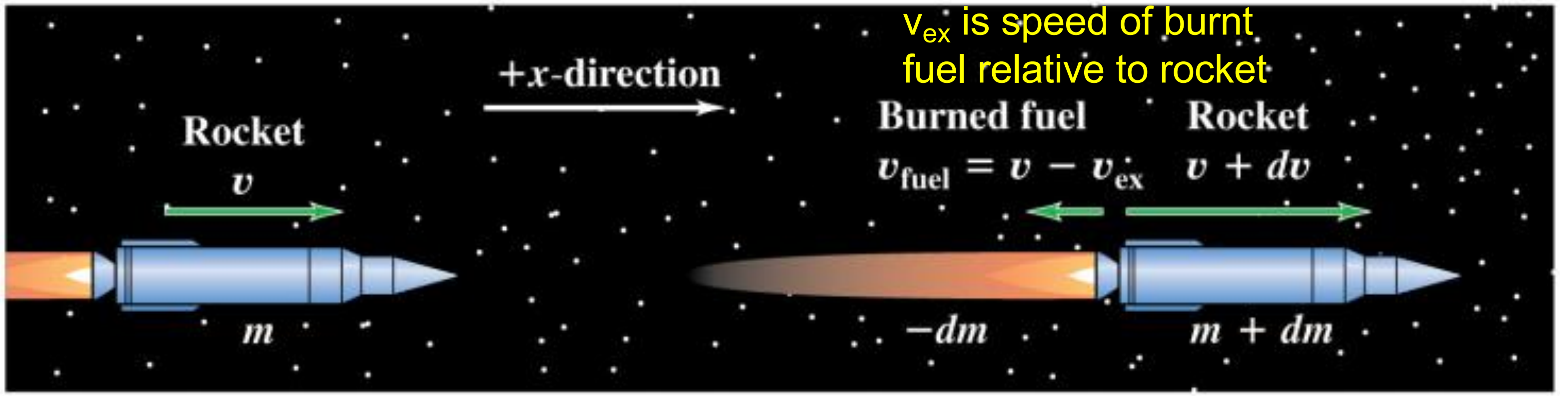
- As a rocket burns fuel, its mass decreases.
- To provide enough thrust to lift its payload into space, this *Atlas V* launch vehicle ejects more than 1000 kg of burned fuel per second at speeds of nearly 4000 m/s.

Rocket propulsion

- Initial total momentum = mv
- After dt, total momentum = (m+dm)(v+dv)+(-dm)(v-v_{ex})
- By conservation of momentum, mdv = -dm v_{ex} – dm dv
- Ignoring dmdv and divide by dt

$$\boxed{m\frac{dv}{dt} = F = -v_{ex}\frac{dm}{dt}}$$

(a) (b)



At time *t*, the rocket has mass m and x-component of velocity v.

At time $t + dt$, the rocket has mass $m + dm$ (where dm is inherently *negative*) and x-component of velocity $v + dv$. The burned fuel has x-component of velocity $v_{fuel} = v - v_{ex}$ and mass $-dm$. (The minus sign is needed to make $-dm$ *positive* because dm is negative.)

Rocket propulsion

- If the exhaust speed is constant, mdv = -dm v_{ex}

$$dv = -v_{ex}\frac{dm}{m}$$

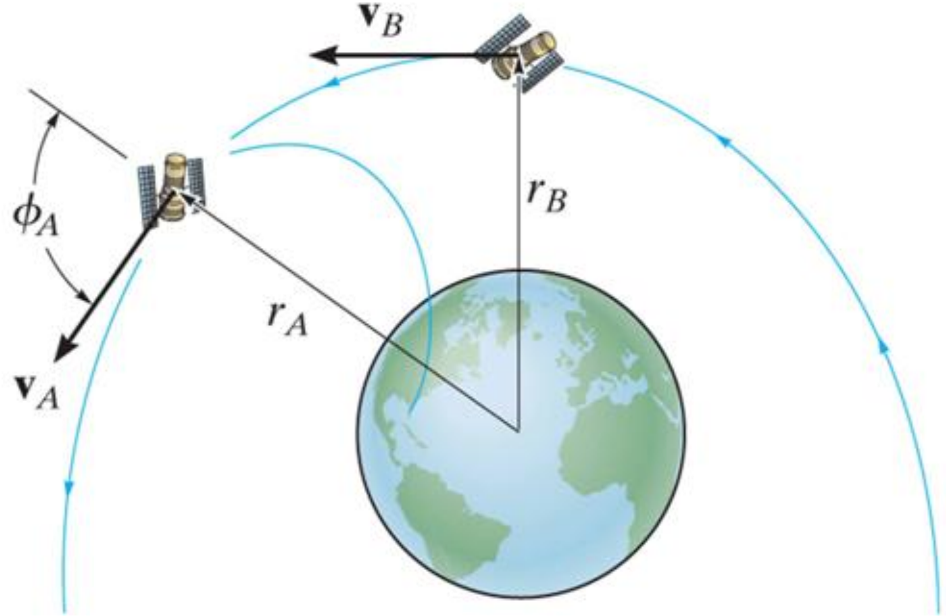
$$\int_{v_0}^{v} dv = -v_{ex}\int_{m_0}^{m}\frac{dm}{m}$$

$$v = v_0 + v_{ex}\ln\frac{m_0}{m}$$

- If $\ln(m_0/m) > 1$, the final speed of the rocket can be larger than v_{ex}.
- Suppose 75% of the initial mass of the rocket is fuel, and v_{ex} is 2400 m/s, find the speed of the rocket, which is initially at rest, after all the fuel is consumed.

$$v = v_0 + v_{ex}\ln\frac{m_0}{m} = 0 + 2400\times\ln\left(\frac{1}{1/4}\right) = 3327\, m/s$$

Angular momentum: introduction

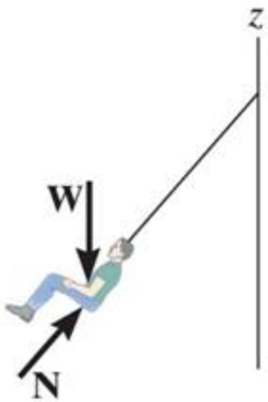


Planets and most satellites move in elliptical orbits. This motion is caused by gravitational attraction forces. Since these forces act in pairs, the sum of the moments of the forces acting on the system will be zero. This means that angular momentum is conserved.

If the angular momentum is constant, does it mean the linear momentum is also constant? Why or why not?

Angular momentum: introduction

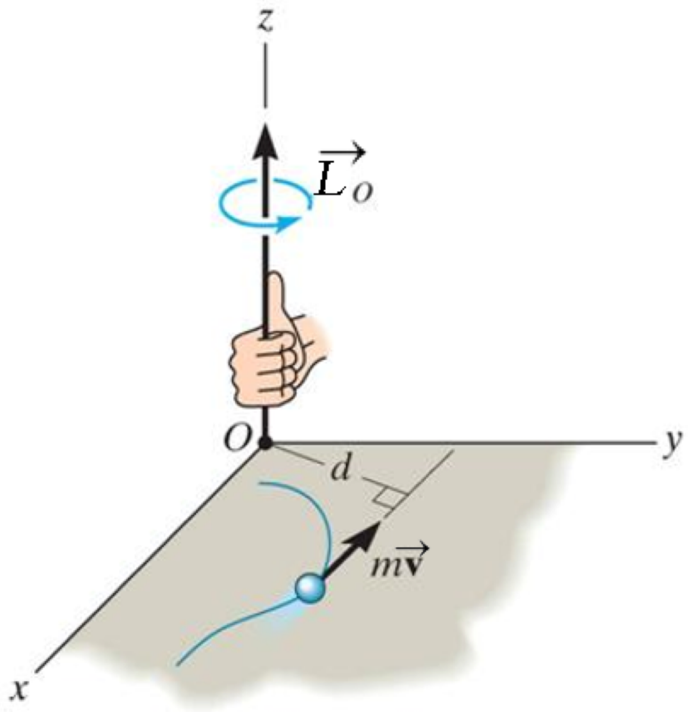
The passengers on the amusement park ride experience conservation of angular momentum about the axis of rotation (the z-axis). As shown on the free-body diagram, the line of action of the normal force, N, passes through the z-axis and the weight's line of action is parallel to it. Therefore, the sum of moments of these two forces about the z-axis is zero.



If the passenger moves away from the z-axis, will his speed increase or decrease? Why?

Angular momentum

The angular momentum of a particle about point O is defined as the "moment" of the particle's linear momentum about O. (In physics typically denoted by $\vec{L}$, but $\vec{H}$ is also used in some books)



$$\vec{L}_o = \vec{r} \times m\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

The magnitude of $\vec{L}_o$ is $(L_o)_z = mvd$

Angular momentum

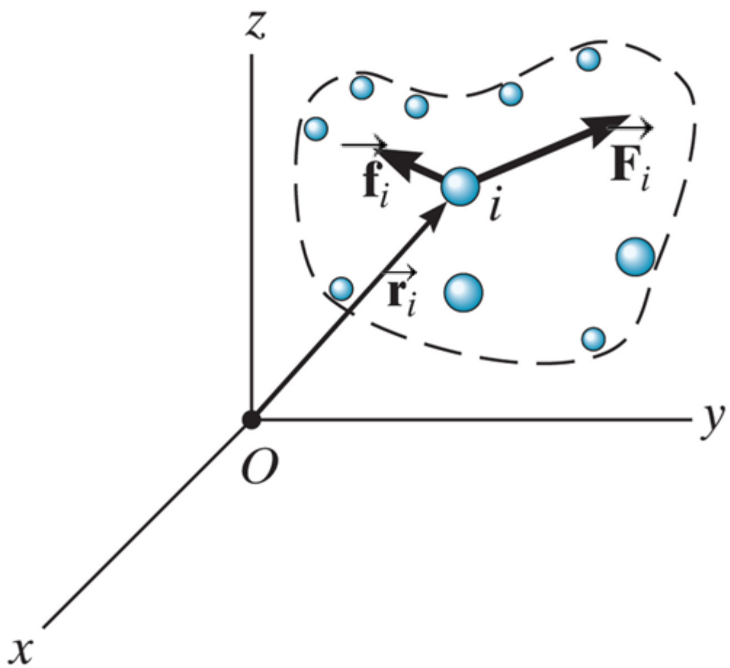
The resultant force acting on the particle is equal to the time rate of change of the particle's linear momentum. Showing the time derivative using the familiar "dot" notation results in the equation

$$\sum \vec{F} = \dot{\vec{p}} = m\dot{\vec{v}}$$

We can prove that the resultant *moment* acting on the particle about point O (also called *torque*, denoted by τ or M)is equal to the time rate of change of the particle's angular momentum about point O or

$$\sum \vec{\tau}_O = \vec{r} \times \vec{F} = \dot{\vec{L}}_O$$

Angular momentum of a system of particles



$$\sum \vec{\boldsymbol{\tau}}_o = r \times \vec{\boldsymbol{F}} = \dot{\vec{\boldsymbol{L}}}_o$$

The same form of the equation can be derived for the system of particles.

The forces acting on the i-th particle of the system consist of a resultant external force $\vec{F}_{\mathrm{i}}$ and a resultant internal force $\vec{f}_{\mathrm{i}}$.

Then, the moments of these forces for the particles can be written as $\sum(\vec{r}_i \times \vec{\boldsymbol{F}}_i) + \sum(\vec{r}_i \times \vec{\boldsymbol{f}}_i) = \sum(\dot{\vec{\boldsymbol{L}}}_i)_o$

The second term is zero since the internal forces occur in equal but opposite collinear pairs. Thus,

$$\sum \vec{\tau}_o = \sum(\vec{r}_i \times \vec{\boldsymbol{F}}_i) = \sum(\dot{\vec{\boldsymbol{L}}}_i)_o$$

Principle of angular impulse and momentum

Considering the relationship between moment and time rate of change of angular momentum

$$\sum \vec{\tau}_o = \dot{\vec{L}}_o = d\vec{L}_o/dt$$

By integrating between the time interval t_1 to t_2

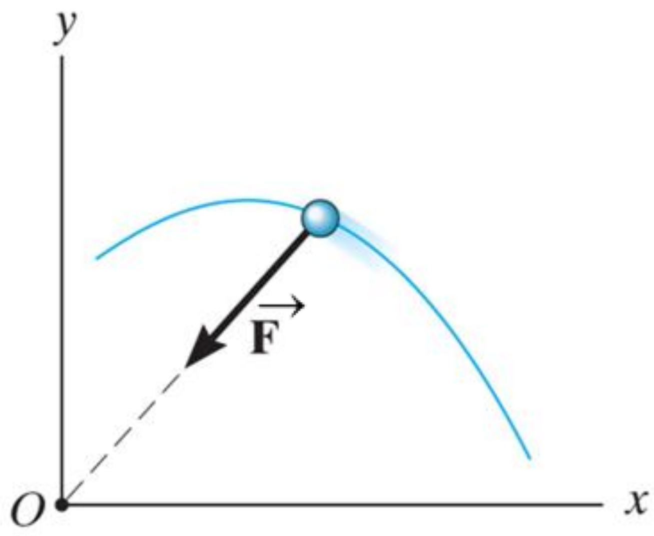
$$\sum \int_{t_1}^{t_2} \vec{\tau}_o \, dt = (\vec{L}_o)_2 - (\vec{L}_o)_1 \quad \text{or} \quad (\vec{L}_o)_1 + \sum \int_{t_1}^{t_2} \vec{\tau}_o \, dt = (\vec{L}_o)_2$$

This equation is referred to as the principle of angular impulse and momentum. The second term on the left side, $\sum\int \tau_o \, dt$, is called the angular impulse. In cases of 2D motion, it can be applied as a scalar equation using components about the z-axis.

Conservation of angular momentum

When the sum of angular impulses acting on a particle or a system of particles is zero during the time t_1 to t_2, the angular momentum is conserved. Thus,

$$(\vec{L}_O)_1 = (\vec{L}_O)_2$$



An example of this condition occurs when a particle is subjected only to a central force. In the figure, the force $\vec{F}$ is always directed toward point O. Thus, the angular impulse of $\vec{F}$ about O is always zero, and angular momentum of the particle about O is conserved.

Quiz

1. Select the correct expression for the angular momentum of a particle about a point.

   A) $\vec{r} \times \vec{v}$ ✔B) $\vec{r} \times (m\,\vec{v})$

   C) $\vec{v} \times \vec{r}$ D) $(m\,\vec{v}) \times \vec{r}$

2. The sum of the moments of all external forces acting on a particle is equal to

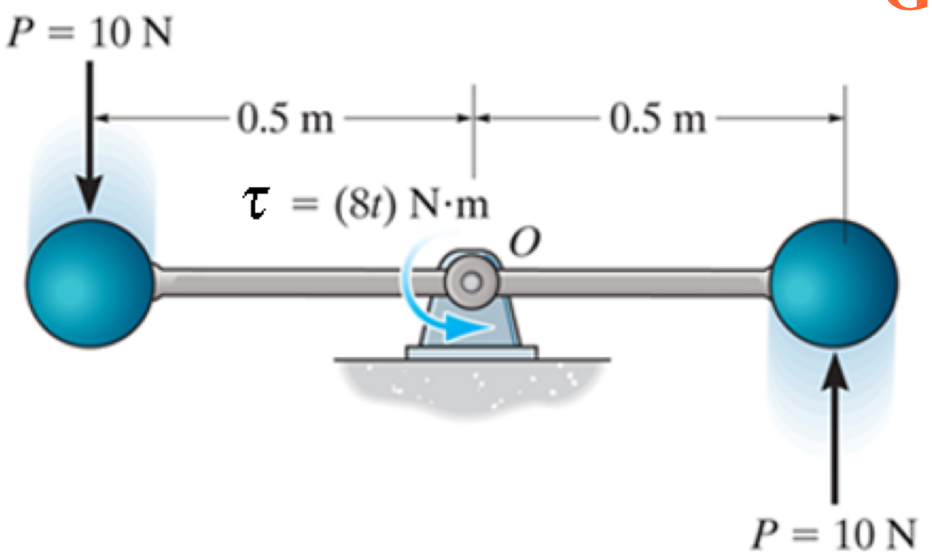
   A) angular momentum of the particle.

   B) linear momentum of the particle.

   ✔ C) time rate of change of angular momentum.

   D) time rate of change of linear momentum.

Example



Given: Two identical 10-kg spheres are attached to the rod, which rotates in the horizontal plane. The spheres are subjected to tangential forces of P = 10 N, and the rod is subjected to a couple moment τ = (8t) N·m, where t is in seconds.

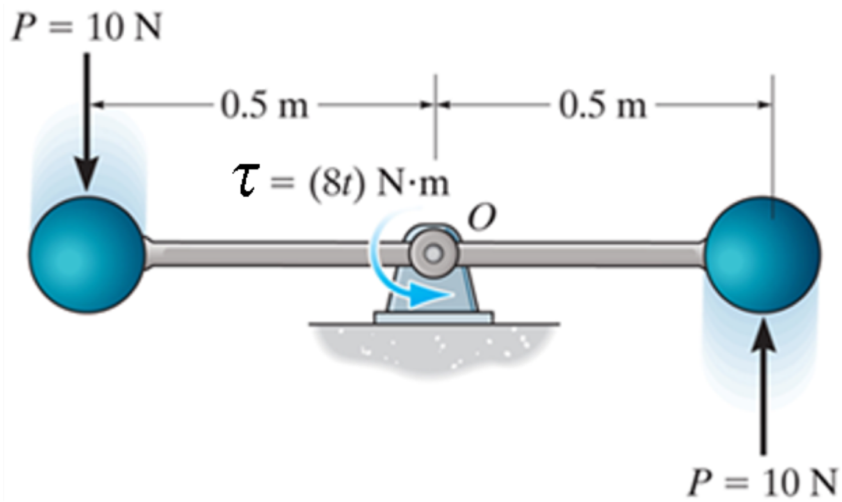
Find: The speed of the spheres at t = 4 s, if the system starts from rest.

Plan:

Apply the principles of conservation of energy and conservation of angular momentum to the system.

Example

Solution:



Conservation of angular momentum :

$$\Sigma(\vec{L}_0)_1 + \Sigma\int_{t1}^{t2} \vec{\tau}_0 dt = \Sigma(\vec{L}_0)_2$$

The above equation about the axis of rotation (z-axis) through O can be written as

$$0 + \int_0^4 8t\,dt + \int_0^4 [2(10)(0.5)]\,dt = 2\,[10\,v\,(0.5)]$$

$$\Rightarrow 4\,(4)^2 + 2\,(5)4 = 10\,v$$

$$\Rightarrow 104 = 10\,v$$

$$v = 10.4\text{ m/s}$$

Quiz

1. If a particle moves in the x - y plane, its angular momentum vector is in the

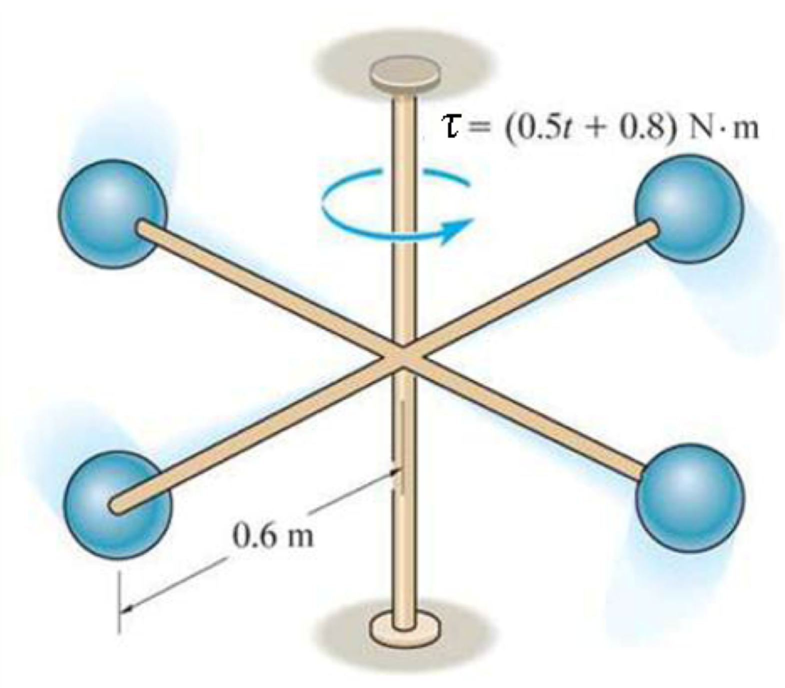
   A) x direction. B) y direction.

   ✔ C) z direction. D) x - y direction.

2. If there are no external impulses acting on a particle

   A) only linear momentum is conserved.
   B) only angular momentum is conserved.
   ✔ C) both linear momentum and angular momentum are conserved.
   D) neither linear momentum nor angular momentum are conserved.

Example



Given: The four 5-kg spheres are rigidly attached to the crossbar frame, which has a negligible weight.
A moment acts on the shaft as shown, $\tau = 0.5t + 0.8$ (N·m).

Find: The velocity of the spheres after 4 seconds, starting from rest.

Plan:

Apply the principle of angular impulse and momentum about the axis of rotation (z-axis).

Example

Solution:

Angular momentum:

$\vec{L}_Z = \vec{r} \times m\vec{v}$ reduces to a scalar equation.

$$(L_Z)_1 = 0 \quad \text{and} \quad (L_Z)_2 = 4\times\{(5)\,(0.6)\,v_2\} = 12\,v_2$$

Angular impulse:

$$\int_{t_1}^{t_2} \tau\, dt = \int_{t_1}^{t_2} (0.5t + 0.8)\, dt = [(0.5/2)\,t^2 + 0.8\,t]\Big|_0^4 = 7.2\ \text{N·m·s}$$

Apply the principle of angular impulse and momentum.

$$0 + 7.2 = 12\,v_2 \quad \Rightarrow \quad v_2 = 0.6\ \text{m/s}$$

Quiz

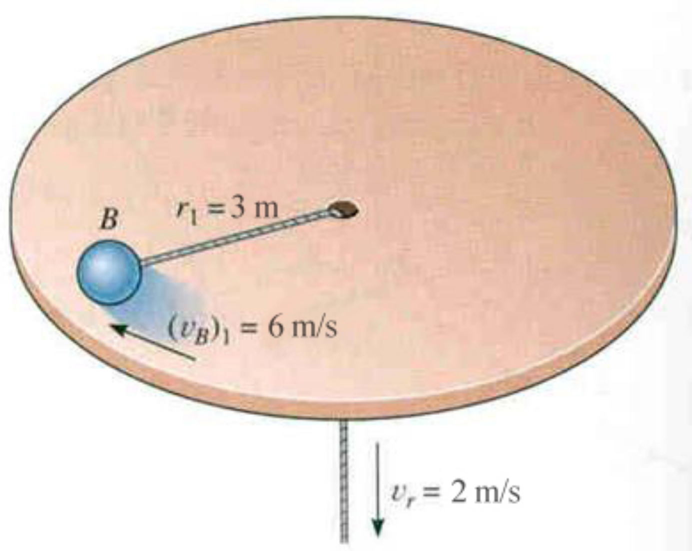
1. A ball is traveling on a smooth surface in a 3-m radius circle with a speed of 6 m/s. If the attached cord is pulled down with a constant speed of 2 m/s, which of the following principles can be applied to solve for the velocity of the ball when r = 2 m?

   A) Conservation of energy

   ✔ B) Conservation of angular momentum

   C) Conservation of linear momentum

   D) Conservation of mass



2. If a particle moves in the z - y plane, its angular momentum vector is in the

   ✔ A) x direction. B) y direction.

   C) z direction. D) z - y direction.