Academic Honesty Pledge

I pledge that the answers in this exam/quiz are my own and that I will not seek or obtain an unfair advantage in producing these answers. Specifically,

- I will not plagiarize (copy without citation) from any source;
- I will not communicate or attempt to communicate with any other person during the exam/quiz; neither will I give or attempt to give assistance to another student taking the exam/quiz; and
- I will use only approved devices (e.g., calculators) and/or approved device models.

I understand that any act of academic dishonesty can lead to disciplinary action.

Student ID:____

Signature:____

Date:____

CITY UNIVERSITY OF HONG KONG

Course code & title	:	MA1301 Enhanced Calculus and Linear Algebra II
Session	:	Semester B 2021/22
Time allowed	:	Three hours

This paper has FOUR pages (including this cover page).

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1. This paper consists of 8 questions.
2. Answer ALL questions.

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*This is a **closed-book** examination.*

Students are allowed to use the following materials/aids:

Calculator

Materials/aids other than those stated above are not permitted. Students will be subject to disciplinary action if any unauthorized materials or aids are found on them.

MA1301
Semester B 2021-22
Final Exam
26/04/2022

Name: ____________________

This exam contains 4 pages (including this page) and 8 questions. Total of points is 100.

Grade Table (for instructor use only)

Question	Points	Score
1	10	
2	15	
3	10	
4	10	
5	15	
6	10	
7	15	
8	15	
Total:	100	

1. (10 points) If $f_{\text{ave}}[a,b]$ denotes the average value of f on the interval $[a,b]$, i.e. $f_{\text{ave}}[a,b] = \frac{1}{b-a}\int_a^b f(x)dx$, and $a < c < b$, show that

$$f_{\text{ave}}[a,b] = \frac{c-a}{b-a} f_{\text{ave}}[a,c] + \frac{b-c}{b-a} f_{\text{ave}}[c,b].$$

2. (15 points) Find the values of p for which the integral converges and evaluate the integral for those values of p.
(i) [5pts]

$$\int_0^1 \frac{1}{x^p} dx;$$

(ii) [5pts]

$$\int_e^\infty \frac{1}{x(\ln x)^p} dx;$$

(iii)[5pts]

$$\int_0^1 x^p \ln x dx.$$

3. (10 points) (a)[5pts] Let L_1 be a line passing through the points $(5, 0, -1)$ and $(6, 2, -2)$. We let L_2 be another line passing through the points $(2, 4, 0)$ and $(3, 3, 1)$. Find the shortest distance between the line L_1 and L_2.
(b)[5pts] Let P_1 be a plane containing the points $A = (3, -2, 0)$, $B = (2, 0, 3)$ and $C = (1, -1, 1)$, find the shortest distance between point $D = (1, 0, -1)$ and the plane P_1.

4. (10 points) Solve the following equation $z^8 - 2\sqrt{3}z^4 + 4 = 0$ in the set of all complex numbers.

5. (15 points) Consider the following system of linear equations

$$\begin{cases} x - 2y + z = 1 \\ x - y + 2z = 2 \\ y + c^2 z = c \end{cases} \tag{1}$$

Find all possible values of c such that the system
(a)[5pts] has unique solution;
(b)[5pts] has infinitely many solutions;
(c)[5pts] has no solution.

6. (10 points) Let z be a complex number with $|z| = 1$ and $z \neq \pm 1$.
(a)[5pts] Show that the complex number $z_0 = \frac{1+z}{1-z}$ is purely imaginary. (i.e. $z_0 = bi$ for some real number b)
(b)[5pts] Using the similar technique, show that if $\arg z \neq k\pi$, (k is integer) then $z_1 = \frac{1+\bar{z}}{1-\bar{z}}$ is also purely imaginary.

7. (15 points) (a)[5pts] We let $z = \cos\theta + i\sin\theta$ be a complex number. By considering the expression $(z - \frac{1}{z})^5$ and using the fact that $z^n - \frac{1}{z^n} = 2i\sin n\theta$, show that

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$

(b)[5pts] By considering the expression $(z - \frac{1}{z})^3(z + \frac{1}{z})^4$ and using the fact that $z^n - \frac{1}{z^n} = 2i\sin n\theta$ and $z^n + \frac{1}{z^n} = 2\cos n\theta$, show that

$$\sin^3\theta\cos^4\theta = -\frac{1}{64}(\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta).$$

(c)[5pts] Compute the integral

$$\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^4\theta d\theta.$$

8. (15 points) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

Using elementary row operations,
(a)[5pts] find the rank of A;
(b)[5pts] find the inverse of A;
(c)[5pts] using (b), solve the solution of linear equations

$$\begin{cases} x+y+z=2 \\ x+2y+3z=0 \\ y+z=0 \end{cases}$$