BMS1901
Final review (Chan)

Chain rule + other rule (e.g. power rule)
Local + absolute max and min (extrema)
Critical number (Fermat’s Theorem)
L’hospital Rule
Application of integrals
Perform separation of variable
Taylor polynomials

The Chain Rule

The Chain Rule

- Outside → Inside

$$\frac{d}{dx}\underbrace{f}_{\text{outer function}}\underbrace{(g(x))}_{\text{evaluated at inner function}} = \underbrace{f'}_{\text{derivative of outer function}}\underbrace{(g(x))}_{\text{evaluated at inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of inner function}}$$

Combining the Chain Rule with Other Rules

Combining the Chain Rule with Other Rules

y = sin *u*
u is a differentiable function of *x*
By the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos u \frac{du}{dx}$$

Thus

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

Combining the Chain Rule with Other Rules

(4) The Power Rule Combined with the Chain Rule If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Example 9

Differentiate $y = (x^3 - 1)^{100}$.

Solution:

- $u = g(x) = x^3 - 1$
- $n = 100$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 1)^{100}$$

$$= 100(x^3 - 1)^{99}\frac{d}{dx}(x^3 - 1)$$

$$= 100(x^3 - 1)^{99} \cdot 3x^2$$

$$= 300x^2(x^3 - 1)^{99}$$

Absolute and Local Extreme Values

Absolute and Local Extreme Values

- Highest point of the function f : (3, 5)

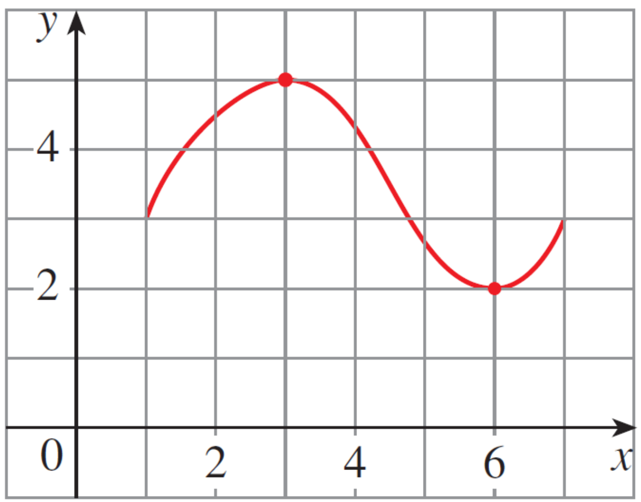


Figure 1

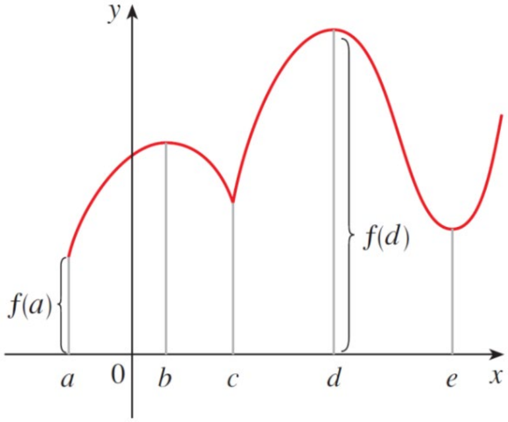
Absolute and Local Extreme Values

(1) Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

Absolute and Local Extreme Values

- **Global** maximum or minimum
- **Extreme values** of *f*



Abs min $f(a)$, abs max $f(d)$
loc min $f(c)$, $f(e)$, loc max $f(b)$, $f(d)$

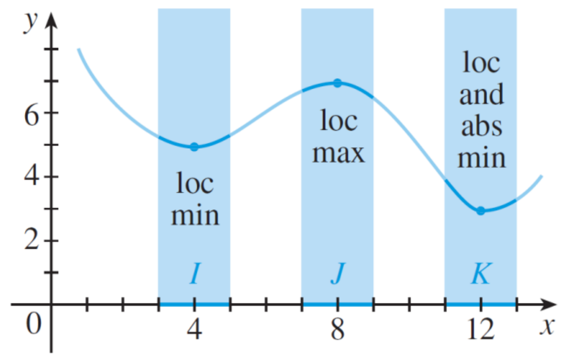
Figure 2

Absolute and Local Extreme Values

(2) Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

Absolute and Local Extreme Values



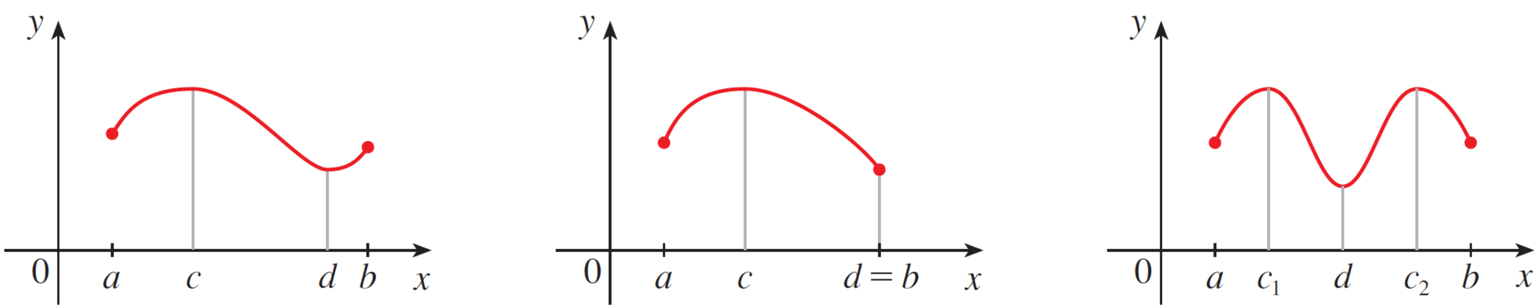
- $f(4) = 5$: local minimum
  - not the absolute minimum
  - $f(x)$ takes smaller values when x is near

- $f(12) = 3$ is both a local minimum and the absolute minimum

- $f(8) = 7$ is a local maximum
  - not the absolute maximum

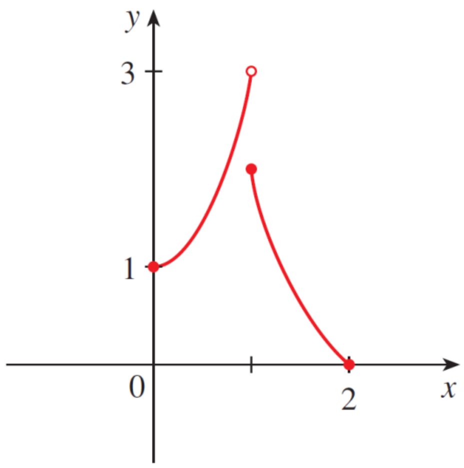
Absolute and Local Extreme Values

(3) The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

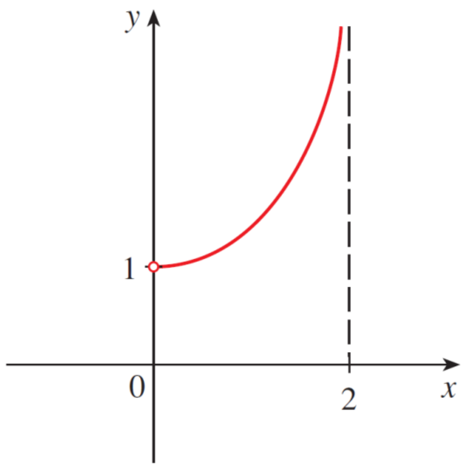


Absolute and Local Extreme Values

- a function need not possess extreme values
  - if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem



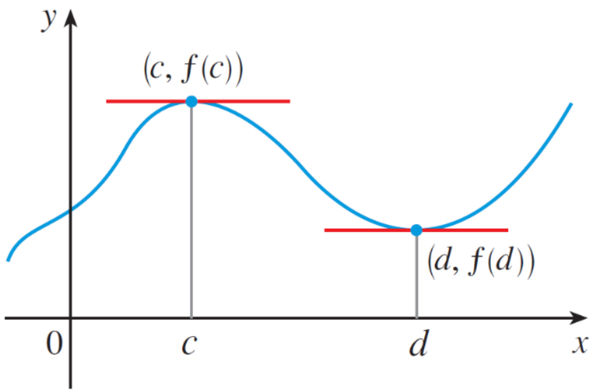
This function has a minimum value $f(2) = 0$, but no maximum value.



This continuous function g has no maximum or minimum.

Fermat’s Theorem

Fermat's Theorem



- function *f:*
  - a local maximum at *c*
  - a local minimum at *d*

Fermat's Theorem

- Derivative: slope of the tangent line
- $f'(c) = 0$ and $f'(d) = 0$

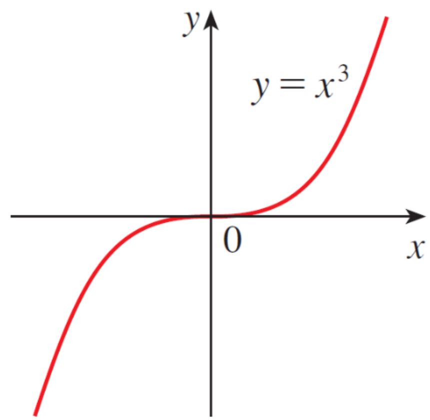
(4) Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Fermat's Theorem

$f(x) = x^3$

- $f'(x) = 3x^2$

  $f'(0) = 0$
- BUT, f has no maximum or minimum at 0



If $f(x) = x^3$, then $f'(0) = 0$ but f has no maximum or minimum.

- $f'(0) = 0$: curve $y = x^3$ has a horizontal tangent at (0, 0)

Fermat's Theorem

- No maximum nor minimum at (0, 0)
- curve crosses its horizontal tangent there
- when $f'(c) = 0$: f doesn't necessarily have a maximum or minimum at c

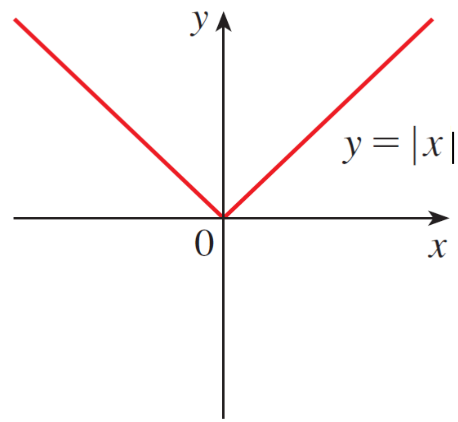
Fermat's Theorem

$f(x) = |x|$

•(local and absolute) minimum value at 0

•Minimum value cannot be found by setting $f'(x) = 0$

- $f'(0)$ does not exist



If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

Fermat's Theorem

- *start* looking for extreme values of f at the numbers c
  - $f'(c) = 0$ or $f'(c)$ does not exist

(5) Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 5

Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

Solution:

The Product Rule gives

$$\begin{aligned} f'(x) &= x^{3/5}(-1) + (4 - x)\left(\tfrac{3}{5}x^{-2/5}\right) \\ &= -x^{3/5} + \frac{3(4 - x)}{5x^{2/5}} \\ &= \frac{-5x + 3(4 - x)}{5x^{2/5}} \\ &= \frac{12 - 8x}{5x^{2/5}} \end{aligned}$$

Example 5

- $f(x) = 4x^{3/5} - x^{8/5}$

- $f'(x) = 0$ if $12 - 8x = 0$
  - $x = \frac{3}{2}$, and $f'(x)$ does not exist when $x = 0$

- Critical numbers are $\frac{3}{2}$ and 0

Fermat's Theorem

- Rephrased Fermat's Theorem:

(6) If f has a local maximum or minimum at c, then c is a critical number of f.

L'Hospital's Rule:
Comparing Rates of Growth

Indeterminate Quotients

Indeterminate Quotients

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x-1)}{(x+1)(x-1)} = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

- both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$
- Limit may or may not exist
- **indeterminate form of type** $\mathbf{\frac{0}{0}}$

- *l'Hospital's Rule*: evaluation of indeterminate forms

Indeterminate Quotients

- L'Hospital's Rule →this type of indeterminate form

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Example 1

Find $\lim_{x \to 1} \frac{\ln x}{x - 1}$.

Solution:

$\lim_{x \to 1} \ln x = \ln 1 = 0$ and $\lim_{x \to 1} (x - 1) = 0$

•l'Hospital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x - 1)}$$

$$= \lim_{x \to 1} \frac{1/x}{1}$$

$$= \lim_{x \to 1} \frac{1}{x} = 1$$

Example 1 – *Solution*

- l'Hospital's Rule: differentiate the numerator and denominator *separately*
  - do ***not*** use the Quotient Rule

Which Functions Grow Fastest?

Which Functions Grow Fastest?

- L'Hospital's Rule: compare the rates of growth of functions

f(*x*) and *g*(*x*)*:* become large as *x* becomes large

$$\lim_{x \to \infty} f(x) = \infty \qquad \text{and} \qquad \lim_{x \to \infty} g(x) = \infty$$

- *f*(*x*) approaches infinity **more quickly** than *g*(*x*) if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$$

- *f*(*x*) approaches infinity **more slowly** than *g*(*x*) if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

Indeterminate Products

Indeterminate Products

$\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$)

$\lim_{x\to a} f(x)g(x)$?

- struggle between *f* and *g*
  - *1) f* wins → limit will be 0
  - *2) g* wins → limit will be ∞ (or $-\infty$)
- compromise : finite nonzero number
- **indeterminate form of type 0 • ∞**

Indeterminate Products

We can deal with it by writing the product *fg* as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞ so that we can use l'Hospital's Rule.

Indeterminate Differences

Indeterminate Differences

$\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$:

$$\lim_{x \to a} [f(x) - g(x)]$$

- **indeterminate form of type** $\infty - \infty$

- Find the limit: difference → quotient
  - common denominator / rationalization / factoring out a common factor→ have an indeterminate form of type $\frac{0}{0}$ or ∞/∞

Example 10

Compute $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$

$\tan x = \dfrac{\sin x}{\cos x}$

$\sec x = \dfrac{1}{\cos x}$

Solution:

- $\sec x \to \infty$ and $\tan x \to \to \infty$ limit is indeterminate

<u>common denominator:</u>

$$\lim_{x \to (\pi/2)^-} (\sec x - \tan x) = \lim_{x \to (\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

- l'Hospital's Rule: $1 - \sin x \to 0$

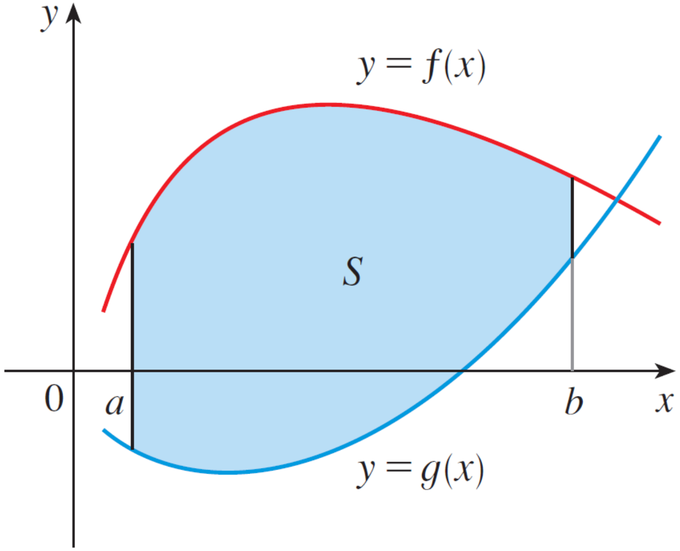
$\cos x \to 0$ as $x \to (\pi / 2)^-$

$$= \lim_{x \to (\pi/2)^-} \frac{1 - \sin x}{\cos x} = \lim_{x \to (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

Areas Between Curves

Areas Between Curves

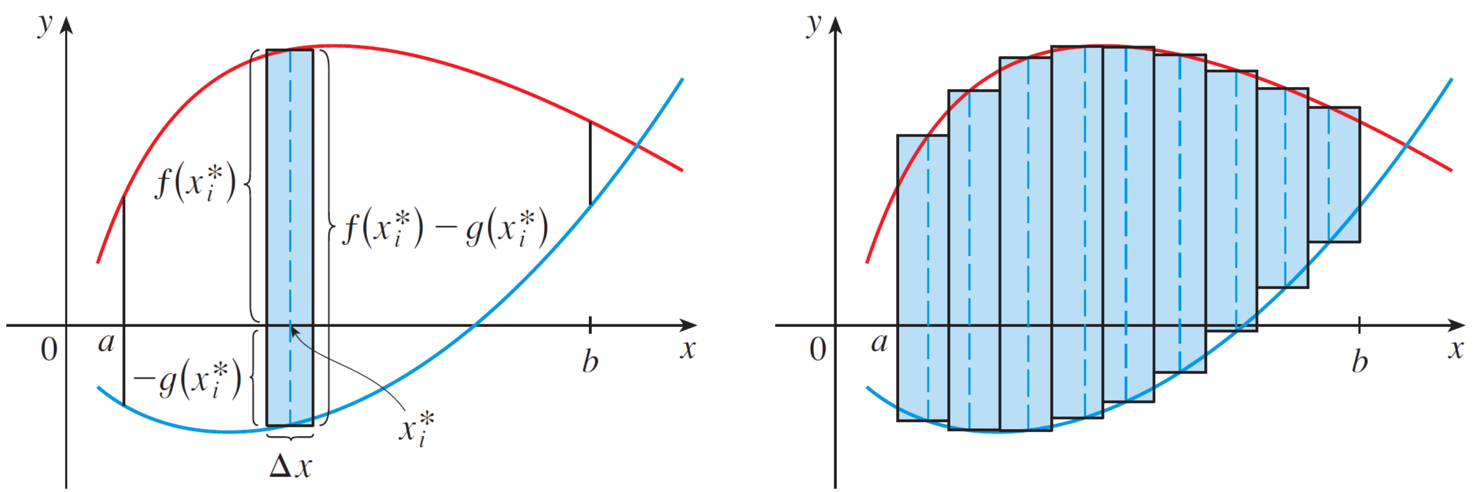
- region S that:
  - lies between two curves $y = f(x)$ and $y = g(x)$ and
  - between the vertical lines $x = a$ and $x = b$
  - f and g are continuous functions
  - $f(x) \geq g(x)$ for all x in $[a, b]$



$S = \{(x, y) \mid a \leq x \leq b,\ g(x) \leq y \leq f(x)\}$

Areas Between Curves

- divide *S* into *n* strips of equal width
- approximate the *i* th strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$
- take all of the sample points to be right endpoints: $x_i^* = x_i$



(a) Typical rectangle

(b) Approximating rectangles

Areas Between Curves

<u>Riemann sum:</u>

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$$

•~ to the area of *S*
•approximation may become better as n $\rightarrow \infty$
•define the **area** *A* of the region *S* = limiting value of the sum of the areas of approximating rectangles

(1)

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x}$$

Areas Between Curves

- limit in (1) = definite integral of $f - g$

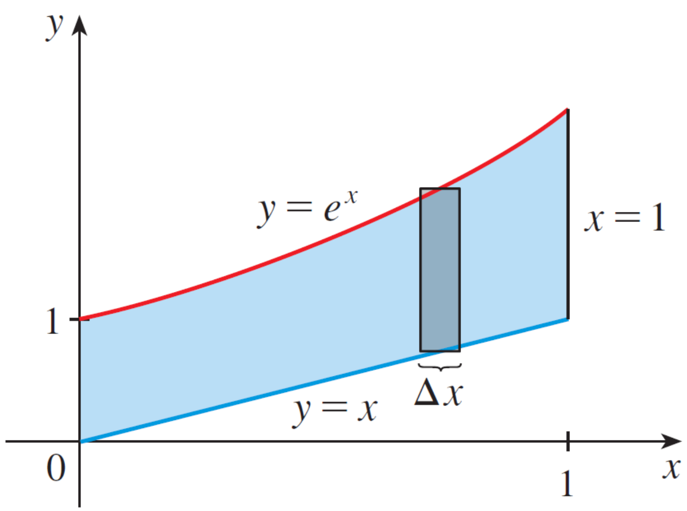
(2) The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Example 1

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Solution:



Example 1 – *Solution*

- upper boundary curve: $y = e^x$
- lower boundary curve: $y = x$

(2) The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

→ formula (2) with $f(x) = e^x$, $g(x) = x$, $a = 0$, and $b = 1$:

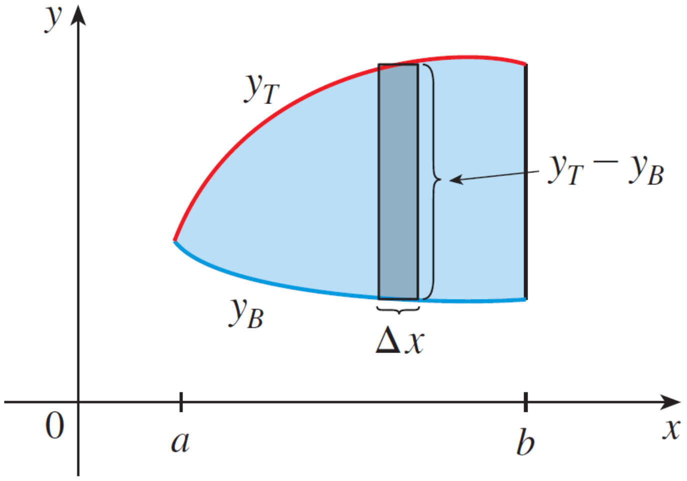
$$\begin{aligned} A &= \int_0^1 (e^x - x)\,dx \\ &= e^x - \tfrac{1}{2}x^2\Big]_0^1 \\ &= e - \tfrac{1}{2} - 1 \\ &= e - 1.5 \end{aligned}$$

Areas Between Curves

- set up an integral for an area
- sketch the region to identify:
  - top curve y_T
  - the bottom curve y_B
- approximating rectangle

Area of a typical rectangle: $(y_T - y_B)\ \Delta x$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} (y_T - y_B)\,\Delta x = \int_a^b (y_T - y_B)\,dx$$



→summarizes the procedure of adding the areas of all the typical rectangles

Average Values

Average Values

- average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

- average value of a function $y = f(x)$, $a \leq x \leq b$
  - dividing the interval $[a, b]$ into n equal subintervals (each with length $\Delta x = (b - a)/n$)
  - choose points $x_1^*, \ldots, x_n^*$ in successive subintervals
  - calculate the average of the numbers $f(x_1^*), \ldots, f(x_n^*)$:

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

Average Values

- $\Delta x = (b - a)/n$
- $n = (b - a)/\Delta x$
- average value:

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{\dfrac{b-a}{\Delta x}} = \frac{1}{b-a}\left[f(x_1^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\right]$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

let *n* increase → compute the average value of a large number of closely spaced values

Average Values

The limiting value is

$$\lim_{n \to \infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \frac{1}{b-a} \int_a^b f(x)\,dx$$

by the definition of a definite integral.

Therefore we define the **average value of *f*** on the interval [*a*, *b*] as

$$\boxed{f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx}$$

Average Values

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x)\, dx$$

that is, $$\int_a^b f(x)\, dx = f(c)(b - a)$$

Separable Equations

Separable Equations

- **Separable equation:** first-order differential equation *dy*/*dt* : factored as a function of *t* times a function of *y*

$$\frac{dy}{dt} = f(t)\,g(y)$$

Separable Equations

$$\frac{dy}{dt} = f(t)\, g(y)$$

- $g(y) \neq 0$:

(1)

$$\frac{dy}{dt} = \frac{f(t)}{h(y)}$$

  - h(y) = 1/g(y)
- To solve the equation:

  $$h(y)\, dy = f(t)\, dt$$

- y's are on one side
- t's are on the other side

Separable Equations

- integrate both sides:

$$\int h(y)\,dy = \int f(t)\,dt \qquad (2)$$

- defines *y* implicitly as a function of *t*
- solve for *y* in terms of *t*

Separable Equations

<u>Using the Chain Rule</u>:
If *h* and *f* satisfy (2),

(2) $$\int h(y)\,dy = \int f(t)\,dt$$

$$\frac{d}{dt}\left(\int h(y)\,dy\right) = \frac{d}{dt}\left(\int f(t)\,dt\right)$$

so $$\frac{d}{dy}\left(\int h(y)\,dy\right)\frac{dy}{dt} = f(t)$$

and $$h(y)\,\frac{dy}{dt} = f(t)$$

* Equation 1 is satisfied

(1) $$\frac{dy}{dt} = \frac{f(t)}{h(y)}$$

Taylor Polynomials

Taylor Polynomials

- tangent line approximation $L(x)$: best first-degree (linear) approximation to $f(x)$ near $x = a$
  - $f(x)$ and $L(x)$ have the same rate of change (derivative) at a

- second-degree (quadratic) approximation $P(x)$: better approximation than a linear one
  - approximate a curve by a parabola instead of by a straight line

Taylor Polynomials

- Good approximation:

  (i) $P(a) = f(a)$ (P and f should have the same value at a.)

  (ii) $P'(a) = f'(a)$ (P and f should have the same rate of change at a.)

  (iii) $P''(a) = f''(a)$ (The slopes of P and f should change at the same rate at a.)

- → $P(x) = A + B(x - a) + C(x - a)^2$

- → $P'(x) = B + 2C(x - a)$ and $P''(x) = 2C$

$$P(x) = A + B(x-a) + C(x-a)^2$$
$$P'(x) = A + B(x-a) + C(x^2 - 2ax - a^2)$$
$$= B + 2Cx - 2Ca$$
$$= B + 2C(x-a)$$

Taylor Polynomials

Applying (i), (ii), and (iii):

$$P(a) = f(a) \qquad \Rightarrow \qquad A = f(a)$$

$$P'(a) = f'(a) \qquad \Rightarrow \qquad B = f'(a)$$

$$P''(a) = f''(a) \qquad \Rightarrow \qquad 2C = f''(a) \qquad \Rightarrow \qquad C = \tfrac{1}{2} f''(a)$$

• quadratic function satisfying the three conditions:

(4) $$\boxed{P(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2} f''(a)(x - a)^2}$$

- $T_2(x)$: **second-degree Taylor polynomial of f centered at a**

(i) $P(a) = f(a)$

$$P(a) = A + B(\overset{\to 0}{a-a}) + C(\overset{\to 0}{a-a})^2 = f(a)$$

$$A = f(a)$$

(ii) $P'(a) = f'(a)$

$$P'(a) = B + 2C(\overset{\to 0}{a-a}) = f'(a)$$

$$B = f'(a)$$

(iii) $P''(a) = f''(a)$

$$2C = f''(a)$$

$$C = \frac{1}{2} f''(a)$$

Example 13

Find the second-degree Taylor polynomial $T_2(x)$ centered at $a = 0$ for the function $f(x) = \cos x$. Illustrate by graphing T_2, f, and the linearization $L(x) = 1$.

Solution:

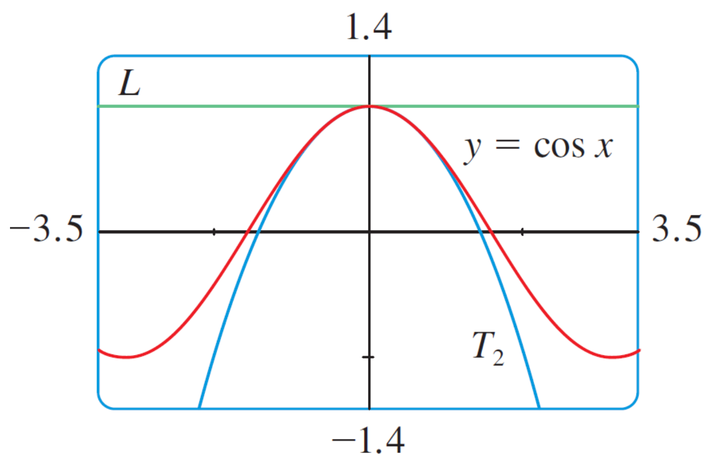
- $f(x) = \cos x$, $f'(x) = -\sin x$, and $f''(x) = -\cos x$
  - second-degree Taylor polynomial centered at 0:

$$\boxed{P(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2} f''(a)(x - a)^2}$$

$$\begin{aligned} T_2(x) &= f(0) + f'(0)x + \tfrac{1}{2} f''(0)x^2 \\ &= 1 + 0 + \tfrac{1}{2}(-1)x^2 \\ &= 1 - \tfrac{1}{2}x^2 \end{aligned}$$

Example 13 – *Solution*

- cosine function + its linear approximation $L(x) = 1$ + its quadratic approximation $T_2(x) = 1 - \frac{1}{2}x^2$ near 0



Figure

- quadratic approximation is much better than the linear one

Taylor Polynomials

- find better approximations with higher-degree polynomials
- *n*th-degree polynomial:

$$T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots + c_n(x - a)^n$$

  - T_n and its first n derivatives have the same values at $x = a$ as f *and* its first n derivatives