Unit 2

Number Systems

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For
Amusement

Why do electronics engineers confuse Christmas with Halloween?

Outline of Unit 2

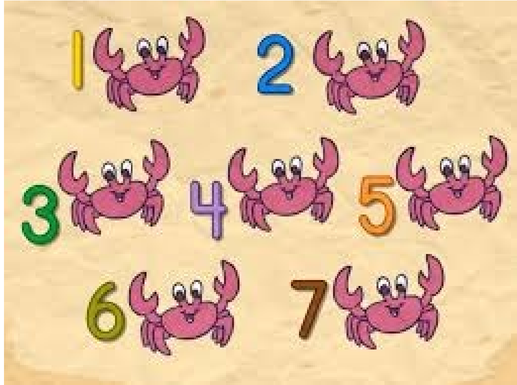
- 2.1 Introduction to Numbers
- 2.2 Number Systems
- 2.3 Real Numbers
- 2.4 Complex Numbers

Unit 2.1

Introduction to Numbers

Numbers

- Why do we have numbers?
- It all starts with counting: 1, 2, 3, ...
- Hence, the set of natural numbers: $\mathbb{N} = \{1, 2, 3, ...\}$.
  - Some people may include 0 to $\mathbb{N}$.
- If 0 is included, it is called whole numbers.

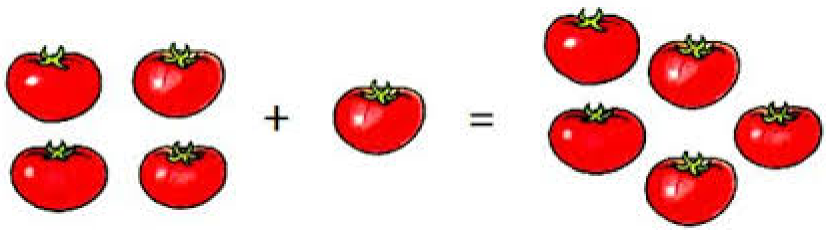


Two dogs

How many dogs?

Addition & Subtraction

- Addition:
  - 4 + 1 = 5



- Subtraction = Inverse of Addition
  - 4 = 5 – 1

- What if 3 – 5 or 3 – 3?
- We need negative numbers and zero.

Multiplication & Division

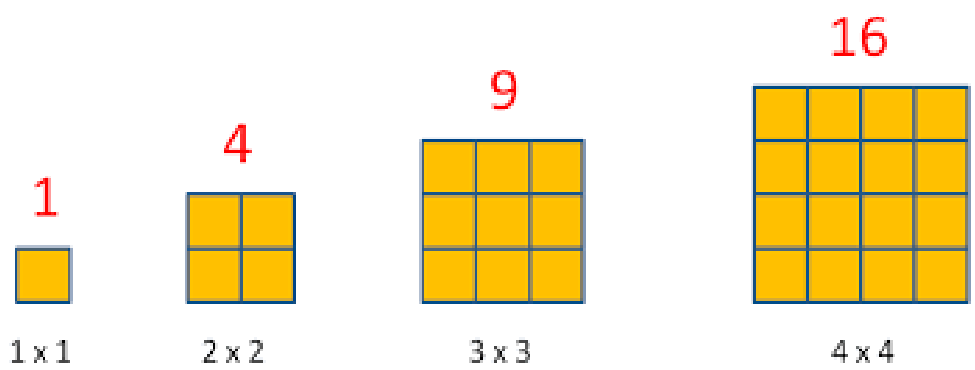
- Multiplication:
  - $6 \times 2 = 12$

- Division: Inverse of Multiplication
  - 6 = 12 / 2

- What if 12 / 5?
- We need rational numbers.

Square Numbers

- A square number (aka. perfect square) is the product of two equal natural numbers.



- What if $\sqrt{-1}$?
- We need complex numbers.

Unit 2.2

Number Systems

A Brief History

- A brief history of numerical systems (5 min):
  - https://www.youtube.com/watch?v=cZH0YnFpjwU

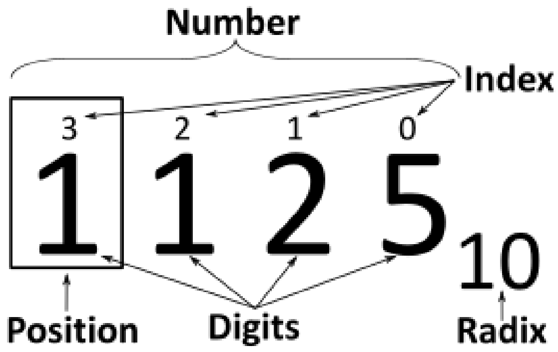
Decimal Number System

- Name
  - “decem” (Latin) means ten.
- Ten Symbols:
  - 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
- Example:
  - 6174 is a special number called Kaprekar’s constant
  - $6174 = (6 * 10^{3)} + (1 * 10^{2}) + (7 * 10^{1}) + (4 * 10^{0})$
  - (3.5 min video) To more about this number:
    - https://www.youtube.com/watch?v=d8TRcZklX_Q

Positional Notation

- The string of digits $(a_{n-1}\ a_{n-2}\ \ldots\ a_2\ a_1\ a_0)_r$

  n : total number of digits in a string
  r : base (or radix) (an integer > 1)
  a_i : digits drawn 0 through r - 1



represents a positive integer N which is given by a power series:

$$N = a_{n-1} r^{n-1} + a_{n-2} r^{n-2} + \ldots + a_2 r^2 + a_1 r + a_0$$

- Kaprekar's constant: $(6174)_{10}$

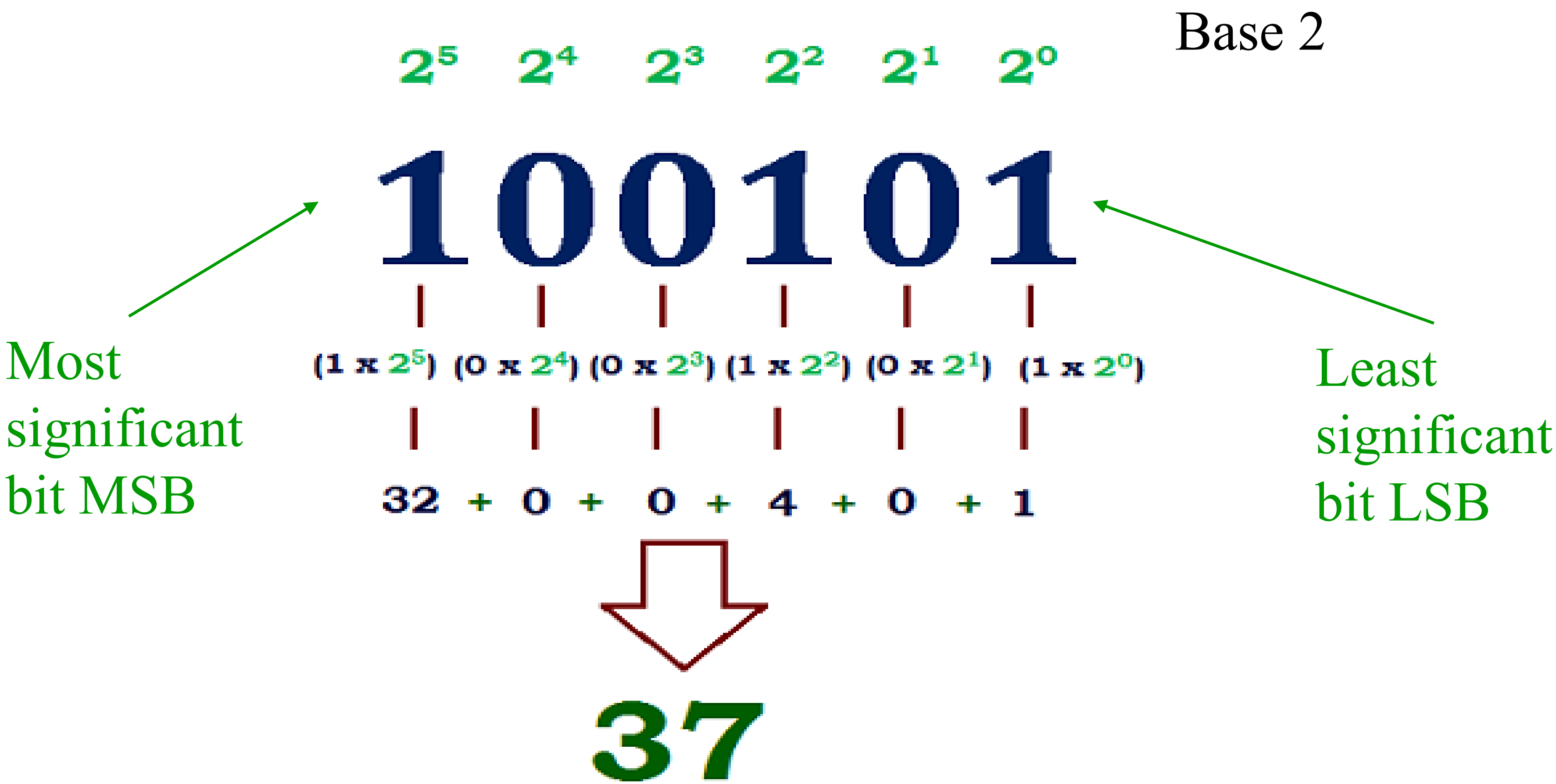
Other Number Systems

- Decimal system (base 10) is for humans.
- Number systems commonly used in the computer world:
  - Binary (base 2)
    - Binary digit is called a bit
  - Octal (base 8)
  - Hexadecimal (base 16)

Binary	Decimal
0	0
1	1
10	2
11	3
100	4
101	5
110	6
111	7

Binary to Decimal Conversion

- Power series expansion:



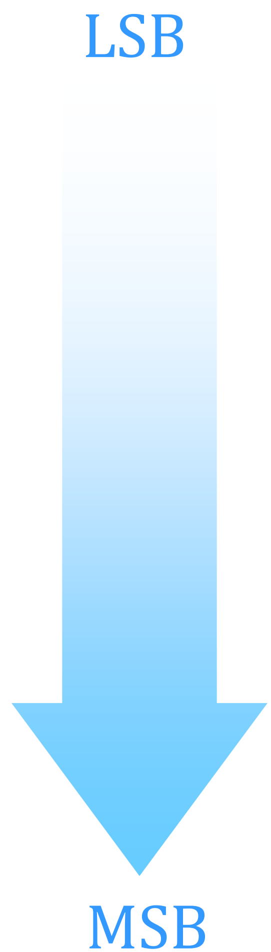
Decimal to Binary Conversion

The Division Method

1. Divide the given positive integer by 2
2. Obtain the quotient q and remainder r
3. Save r as the LSB
4. **Repeat**
5. Divide q by 2
6. Obtain the quotient q and remainder r
7. Save r as the next bit
8. **until** q becomes 0

The Division Method (Example)

	quotient	remainder	
2	7 4 6		
2	3 7 3	… 0	0
2	1 8 6	… 1	10
2	9 3	… 0	010
2	4 6	… 1	1010
2	2 3	… 0	01010
2	1 1	… 1	101010
2	5	… 1	1101010
2	2	… 1	11101010
2	1	… 0	011101010
	0	… 1	1011101010



Hexadecimal Number System

- Sixteen Symbols:
  - 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F

Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

- Why hex?
  - A shorthand notation for binary numbers because it works out nicely for microprocessors of 8, 16, 32, and 64 bits.
  - Every group of 4 bits of a binary number is represented as 1 hex digit
    - e.g., A=1010, B=1011, F = 1111, 0 = 0000, 7 = 0111, etc.

Binary-Hexadecimal Conversion

Every four bits are converted into one hex digit:

Kaprekar's constant:

Add leading zeros to make the length a multiple of 4.

- Binary $(0001\ \ 1000\ \ 0001\ \ 1110)_2$
- Hex $(\ 1\ \ \ 8\ \ \ 1\ \ \ E\ )_{16}$

Converting hex to binary is straightforward.

Decimal-Hexadecimal Conversion

- Hex -> Dec: Power series expansion

$$(181E)_{16} = 1 \times 16^3 + 8 \times 16^2 + 1 \times 16^1 + 14 \times 16^0$$
$$= 4096 + 2048 + 16 + 14$$
$$= 6174$$

- Dec -> Hec: Division Method

6174 / 16 = 385 R 14 — Least significant digit

385 / 16 = 24 R 1

24 / 16 = 1 R 8

1 / 16 = 0 R 1 — Most significant digit

Octal Number Systems

- Eight Symbols:
  - 0, 1, 2, 3, 4, 5, 6, 7
- Every group of 3 bits of a binary number corresponds to 1 octal digit.
- Conversion methods are the same as those for hexadecimal numbers.

Decimal-Octal Conversion

- Oct -> Dec: Power series expansion

$$(14036)_8 = 1 \times 8^4 + 4 \times 8^3 + 0 \times 8^2 + 3 \times 8^1 + 6 \times 8^0$$
$$= 4096 + 2048 + 0 + 24 + 6$$
$$= 6174$$

- Dec -> Oct: Division Method

6174 / 8 = 771 R 6

771 / 8 = 96 R 3

96 / 8 = 12 R 0

12 / 8 = 1 R 4

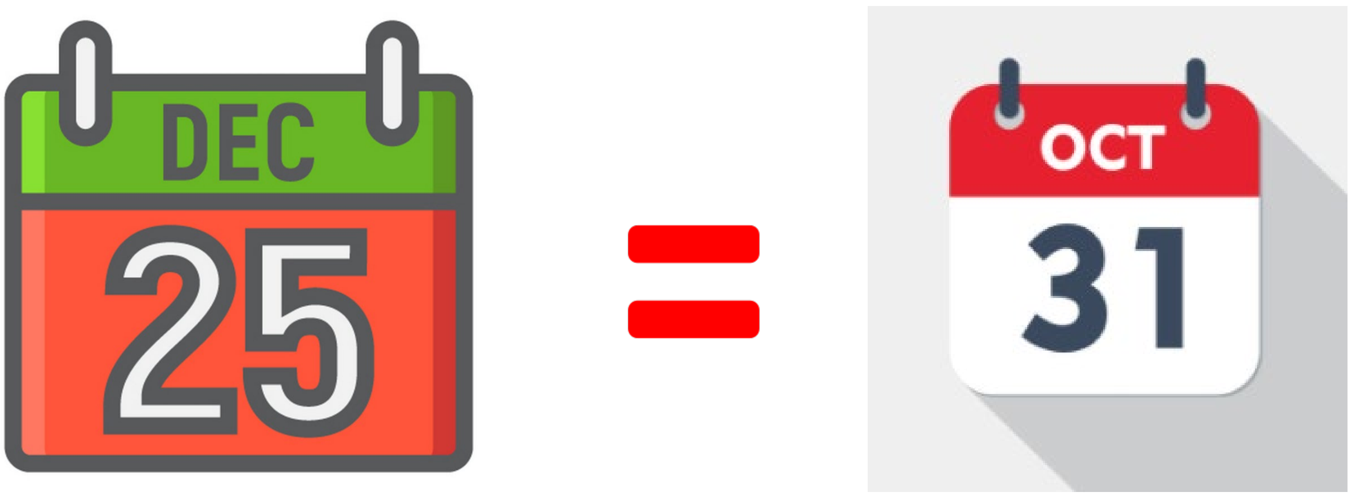
1 / 8 = 0 R 1

Least significant digit

Most significant digit

Christmas vs Halloween

- Why do electronics engineers confuse Christmas with Halloween?



Unit 2.3

Real Numbers

Rational Numbers

- **Definition:**
  - A number r is rational iff there exist integers a and b such that $r = a / b$ and $b \neq 0$.

- Are all numbers rational?

The First Irrational Number

- **Theorem:** $\sqrt{2}$ is Irrational.

- Why does it matter?
  - https://www.youtube.com/watch?v=nT4geKdKVfw (~5 min.)
- How to prove it?

Proof (by Contradiction):

- Suppose $\sqrt{2} = \frac{a}{b}$, where a and b are integers. We can assume that a and b have no common factors.
  - Otherwise, we can cancel out the common factors.
- Squaring both sides, $2 = \frac{a^2}{b^2}$, or $2b^2 = a^2$. Therefore, a^2 is an even number.
- By a result of Unit 1, a is even.
- We can write a as $2k$, where k is an integer. Then, $2b^2 = a^2 = 4k^2$, or $b^2 = 2k^2$. Therefore, b is an even number.
- This contradicts with the assumption that a and b have no common factor.

Q.E.D.

Decimal Representation

- A rational number $\frac{a}{b}$ can be represented in decimal form by long division:
  1) $a \div b \ = \ q_1 \ldots r_1$
  2) $10r_1 \div b = q_2 \ldots r_2$
  3) $10r_2 \div b = q_3 \ldots r_3$

     $\vdots$
- The decimal is either
  - terminating (if $r_i = 0$ in step i), or
  - repeating (since non-zero remainder must be $1, 2, \ldots$, or $b - 1$, it must repeat after at most $b - 1$ steps.
    - Note: 1/7 repeats after 6 steps.

Rational Number	Decimal Representation
1/2	0.5
1/3	0.333333333333...
1/4	0.25
1/5	0.2
1/6	0.166666666666...
1/7	0.142857142857...
1/8	0.125
1/9	0.111111111111...
1/10	0.1

From Decimal to a/b (examples)

- Terminating decimal:
  - $0.125 = \frac{125}{1000} = \frac{1}{8}$.
- Repeating decimal:
  - Let $x = 0.166666666666 \ldots$ (also written as $0.1\overline{6}$)
  - $10x = 1.66666666666 \ldots$ (multiply by 10)
  - $10x - x = 1.5$ (subtract 1st eqn. from 2nd)
  - $9x = 1.5$
  - $x = \frac{15}{90} = \frac{1}{6}$

Does 0.99999... = 1?

a) Yes
b) No
c) Neither yes nor no

Rational vs. Irrational Numbers

- Rational numbers ↔ terminating or repeating decimals.

- Irrational numbers ↔ non-repeating infinite decimals.
  - $\pi = 3.14159265358979323846264338327 9 \ldots$
  - $e = 2.71828182845904523536028747135 2 \ldots$

Euler's Number, *e*

- The discovery of this constant is credited to Jacob Bernoulli in 1683.
- The constant arises from the concept of continuous compound interest:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$$

(see next slide)

- Euler started to use the letter *e* for the constant in 1727 or 1728.

Continuous Compound Interest

Interest rate (initial value = \$1)	Final amount after one year
100% per year	2
50% per 6 months	$\left(1+\frac{1}{2}\right)^2 = 2.25$
25% per 3 months	$\left(1+\frac{1}{4}\right)^4 = 2.44141$
(100/6)% per 2 months	$\left(1+\frac{1}{6}\right)^6 = 2.52163$
(100/12)% per month	$\left(1+\frac{1}{12}\right)^{12} = 2.61304$
$\vdots$	$\vdots$
(100/n)% per 1/n year Compound interest every moment	$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828$

Extension to Negative Exponents

- A number N can be represented by the string below:

$$\Big(\overbrace{a_{n-1}\, a_{n-2} \ldots a_2\, a_1\, a_0}^{n} \,.\, \overbrace{a_{-1}\, a_{-2} \ldots a_{-(m-1)}\, a_{-m}}^{m}\Big)_r \qquad a_i \text{ drawn from 0 through } r-1$$

1) "**.**" is the fractional point
2) The left most digit a_{n-1} is the **M**ost **S**ignificant **D**igit (MSD)
3) The right most digit a_{-m} is the **L**east **S**ignificant **D**igit (LSD)

- N is given by the power series

$$N = \underbrace{\boxed{a_{n-1}r^{n-1} + \ldots + a_1 r^1 + a_0 r^0}}_{\text{Integer part}} + \underbrace{\boxed{a_{-1}r^{-1} + \ldots + a_{-m}r^{-m}}}_{\text{Fractional part}}$$

Fractional Number in Binary

- $(12.6875)_{10}$
- Separate the number into two integers
  - $(12)_{10} = (1100)_2$ (integer part, division-by-2)
  - $(6875)_{10}$ (fractional part, multiplication-by-2)
- For the fractional part,

$$0.6875 \times 2 = 1.3750$$
$$0.3750 \times 2 = 0.7500$$
$$0.7500 \times 2 = 1.5000$$
$$0.5000 \times 2 = 1.0000$$
$$0.6875 = (0.1011)_2$$

MSB

LSB

If the process never ends, you may need to round off at a certain point.

- Combining the two parts, $(12.6875)_{10} = (1100.1011)_2$.

Fractional Number in Octal

- $(12.6875)_{10}$
- Separate the number into two integers
  - $(12)_{10} = (14)_8$ (integer part, division-by-8)
  - $(6875)_{10}$ (fractional part, multiplication-by-8)
- For the fractional part,

$$0.6875 \times 8 = 5.5$$
$$0.5 \times 8 = 4.0$$
$$0.6875 = (0.54)_8$$

MSB

LSB

- Therefore,

$$(12.6875)_{10} = (1100.1011)_2 = (14.54)_8.$$

Fractional Number in Hexadecimal

- $(12.6875)_{10}$
- Separate the number into two integers
  - $(12)_{10} = (C)_{16}$ (integer part, division-by-16)
  - $(6875)_{10}$ (fractional part, multiplication-by-16)
- For the fractional part,

  $0.6875 \times 16 = 11.0$

  $0.6875 = (0.B)_{16}$
- Therefore,

  $(12.6875)_{10} = (1100.1011)_2 = (14.54)_8 = (C.B)_{16}$

Unit 2.4

Complex Numbers

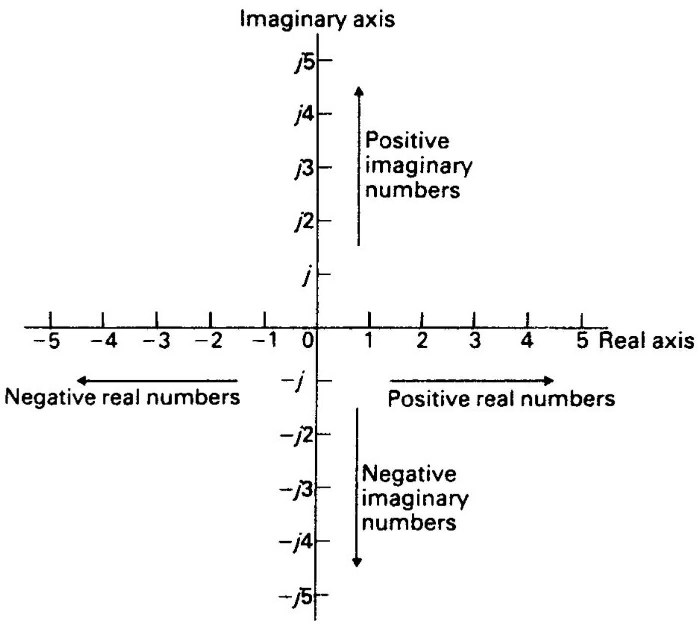
Complex Numbers

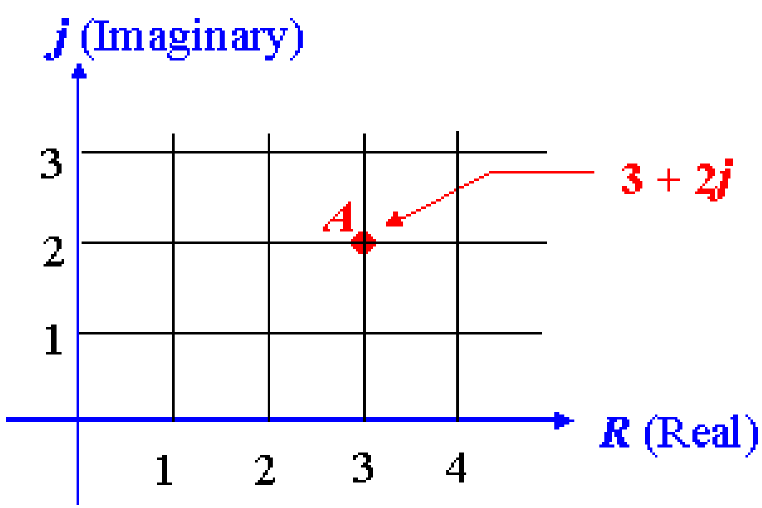
real part	$\mathrm{Re}(x+yi) := x$
imaginary part	$\mathrm{Im}(x+yi) := y$ (Note: It is y, not yi, so $\mathrm{Im}(x+yi)$ is real)
complex conjugate	$\overline{x+yi} := x - yi$ (negate the imaginary component)

- $\sqrt{-1}$ is represented by *i* in mathematics, and usually by *j* in EE (why?).
- Roughly speaking, a set of elements that can perform addition, subtraction, multiplication, and division (excluding division by zero) is called a field.
  - Examples: $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rational numbers), $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers).
    - I assume that you know how to do addition, subtraction, multiplication, and division of complex numbers.

Argand Diagram

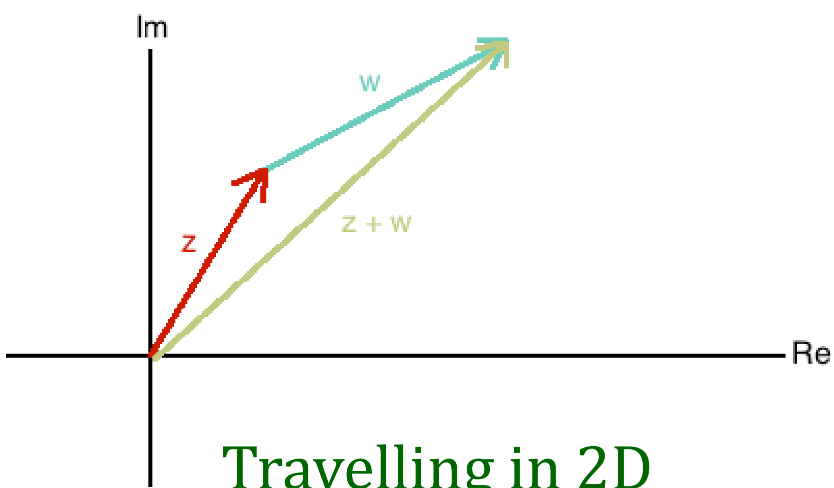
- A real number is a 1-dimensional number, a point on the straight line.
- A complex number is a two-dimensional number, a point on the 2-dimensional plane.



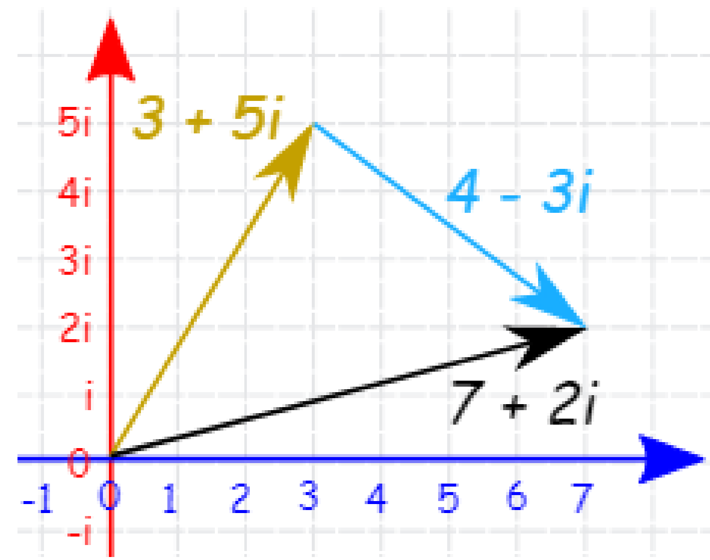


Addition of Complex Numbers

- $(a + bi) + (c + di) = (a + c)i + (b + d)i$
- Geometrically, it looks like this:



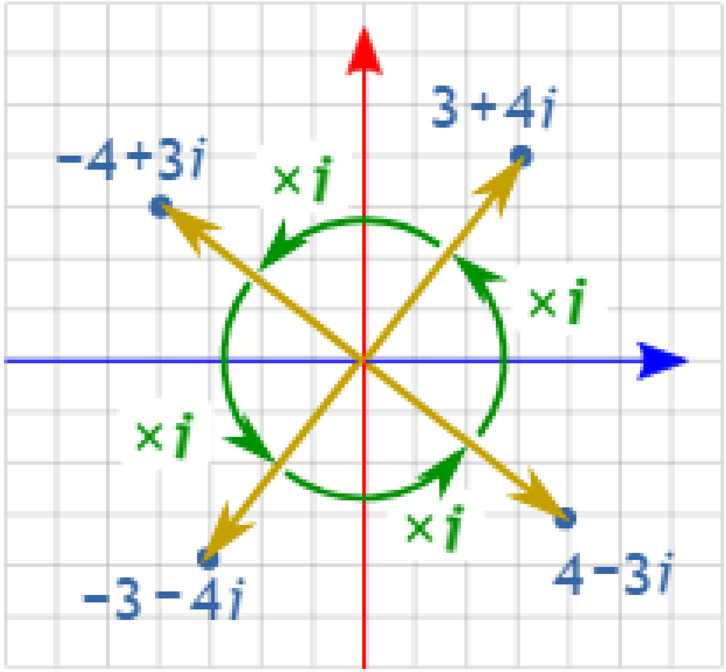
Travelling in 2D



Multiplication by i

- Multiplying z by a real number r is the same as stretching the arrow by a factor of r.
  - Same as real numbers on the number line.
- Multiplying z by i is the same as rotation by 90^0.
- Consider $z = 3 + 4i$.
  - $2z = 6 + 8i$
  - $zi = (3 + 4i)i = -4 + 3i$

Rotating in 2D



Polar Form of $z = a + bi$

- Consider Its modulus is $|z| = r = \sqrt{a^2 + b^2}$.
- Its argument is $\arg(z) = \theta = \tan^{-1}\frac{y}{x}$.
  - The principal value: $-\pi < \theta \le \pi$ (measured in radians, $\pi = 180^0$)

Convert Complex Number from Rectangular Form to Polar (Trigonometric) Form

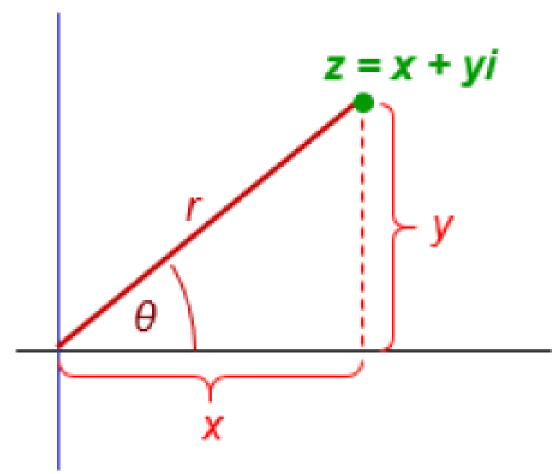
$z = x + yi$ (rectangular form)

$$r = |z| = \sqrt{x^2 + y^2}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

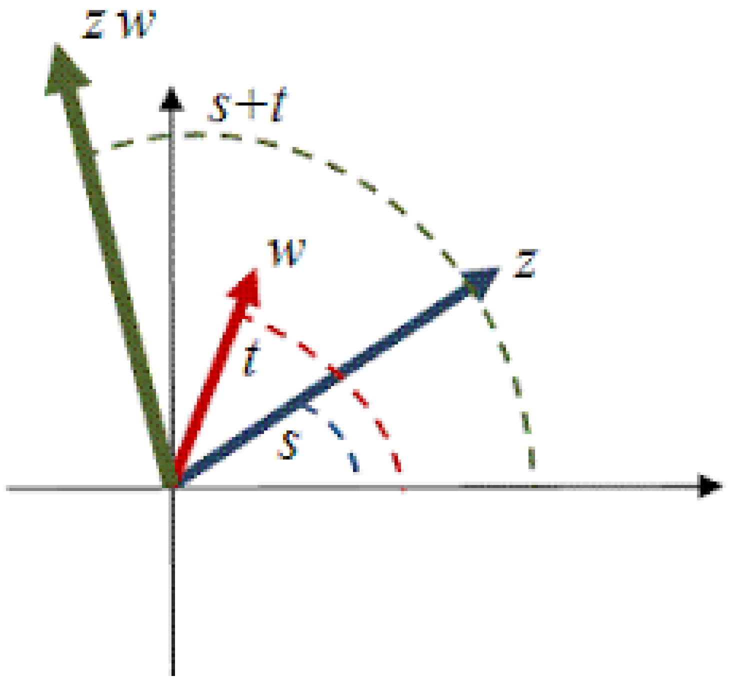
$z = r(\cos\theta + i\sin\theta)$ (polar form)



Multiplication by $a + bi$

- Multiplication can be easily done in polar form:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$



- It can be proved that

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Moduli multiplied Arguments added

de Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$, then for all positive integer n,

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

- It can be obtained by repeated multiplication, or formally proved by mathematical induction (next slide).
- de Moivre's Theorem is also true when n is a rational number (proof omitted).
- de Moivre's Theorem is a simple consequence of Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$ (not considered here).

Proof:

(Base case) It is obviously true for $n = 1$.

(Induction step) Assume it is true for $n = k$, i.e.,

$$z^k = r^k(\cos k\theta + i \sin k\theta).$$

Consider $n = k + 1$.

$z^{k+1} = r^k(\cos k\theta + i \sin k\theta) \times r(\cos\theta + i \sin\theta)$

$= r^{k+1}[(\cos k\theta \cos\theta + \sin k\theta \sin\theta) +$
$i(\sin k\theta \cos\theta + \cos k\theta \sin\theta)]$

$= r^{k+1}[\cos(k+1)\theta + i \sin(k+1)\theta]$

Therefore, it is true for $n = k + 1$.

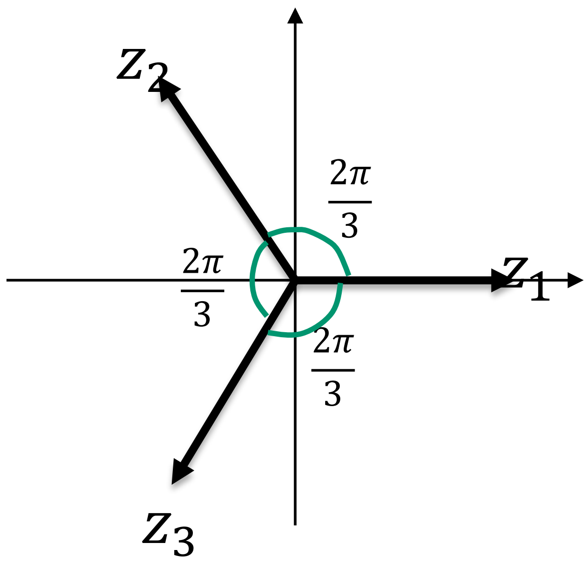
Hence, by induction, the statement is true for all positive integer n.

The Roots of Unity

- For brevity, we write $\operatorname{cis}\theta$ for $\cos\theta + i\sin\theta$.
  - (de Moivre's Theorem) $z^n = r^n \operatorname{cis} n\theta$.
- Let n be a positive integer.
- Consider the equation $x^n = 1$.
- 1 is obviously one of the roots.
- The equation has exactly n roots.
- They are called the n-th roots of unity.

Solve $z^3 = 1$

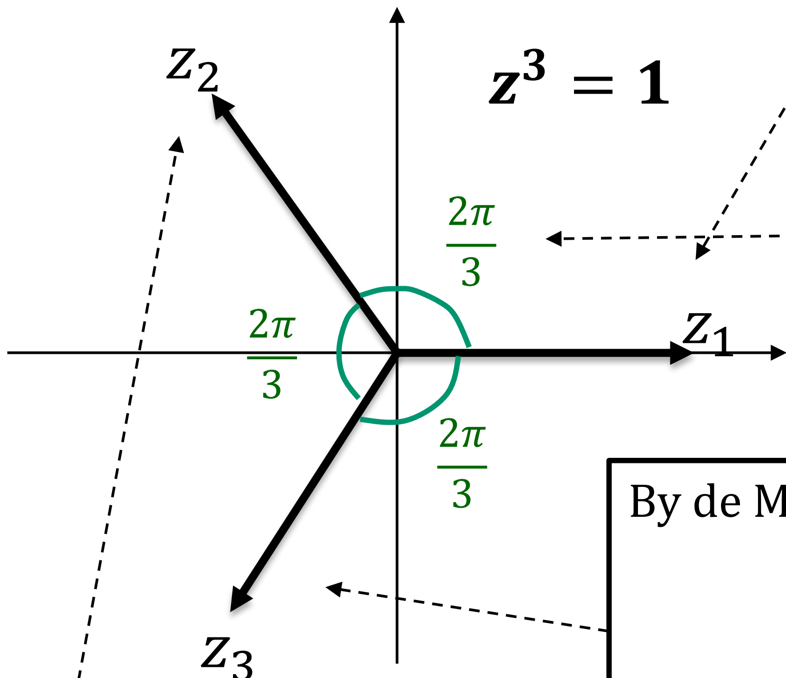
$$z^3 = 1 \text{ cis } 0$$
$$= 1 \text{ cis } 2k\pi \qquad (k \text{ is an integer})$$
$$z = (\text{cis } 2k\pi)^{1/3} \qquad \text{(de Moivre)}$$
$$= \text{cis } \frac{2k\pi}{3}$$
$$k = 0, z_1 = \text{cis } 0 = 1$$
$$k = 1, z_2 = \text{cis}\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$k = 2, z_3 = \text{cis}\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$



These are the cube roots of unity.

Cube Roots of Unity

1 is clearly a root as $1^3 = 1$.



Each time, we add $\frac{2\pi}{3}$ to the argument, but left the modulus untouched.

When $k = 3$, we would have rotated $\frac{2k\pi}{3} = 2\pi$, so end up where we started.

By de Moivre's Theorem, ω^2 would be:

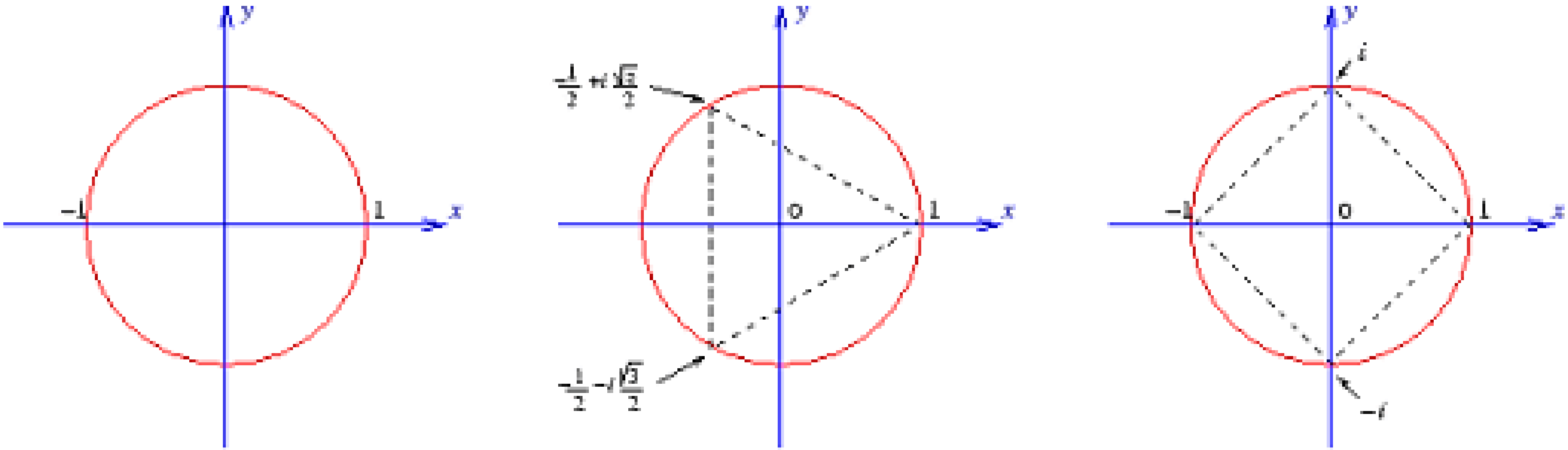
$$\omega^2 = \left(\text{cis}\,\frac{2\pi}{3}\right)^2 = \text{cis}\,\frac{4\pi}{3}$$

i.e., the next root! Therefore, the roots can be represented as $1, \omega, \omega^2$. Note that $\omega^3 = 1$.

Call this root ω

$$\omega = \text{cis}\,\frac{2\pi}{3}$$

n-th Roots of Unity



(c) $n = 2$ (d) $n = 3$ (e) $n = 4$

$$\omega = \operatorname{cis}\frac{2\pi}{n}$$

The roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

Properties

a) Sum of all roots of unity equals 0.

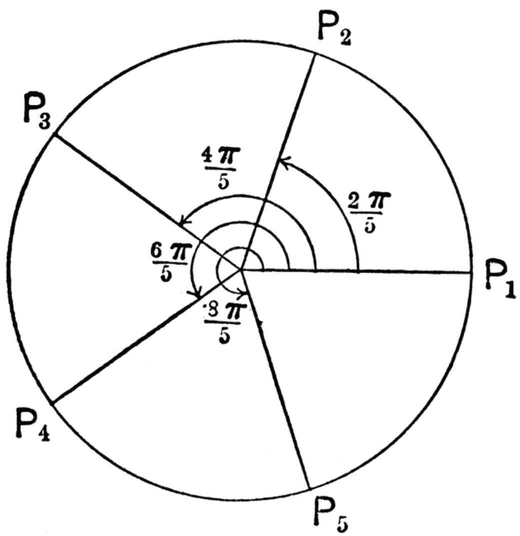
$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = \frac{\omega^n - 1}{\omega - 1} = \frac{1-1}{\omega-1} = 0$$

b) ω^k and ω^{n-k} are complex conjugates.

$$\omega^k = \operatorname{cis}\frac{2k\pi}{n} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$

$$\omega^{n-k} = \operatorname{cis}\frac{2(n-k)\pi}{n} = \operatorname{cis}\left(2\pi - \frac{2k\pi}{n}\right)$$

$$= \cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n}$$



Example

Solve $z^4 = 2 + 2\sqrt{3}\, i$.

Solution:

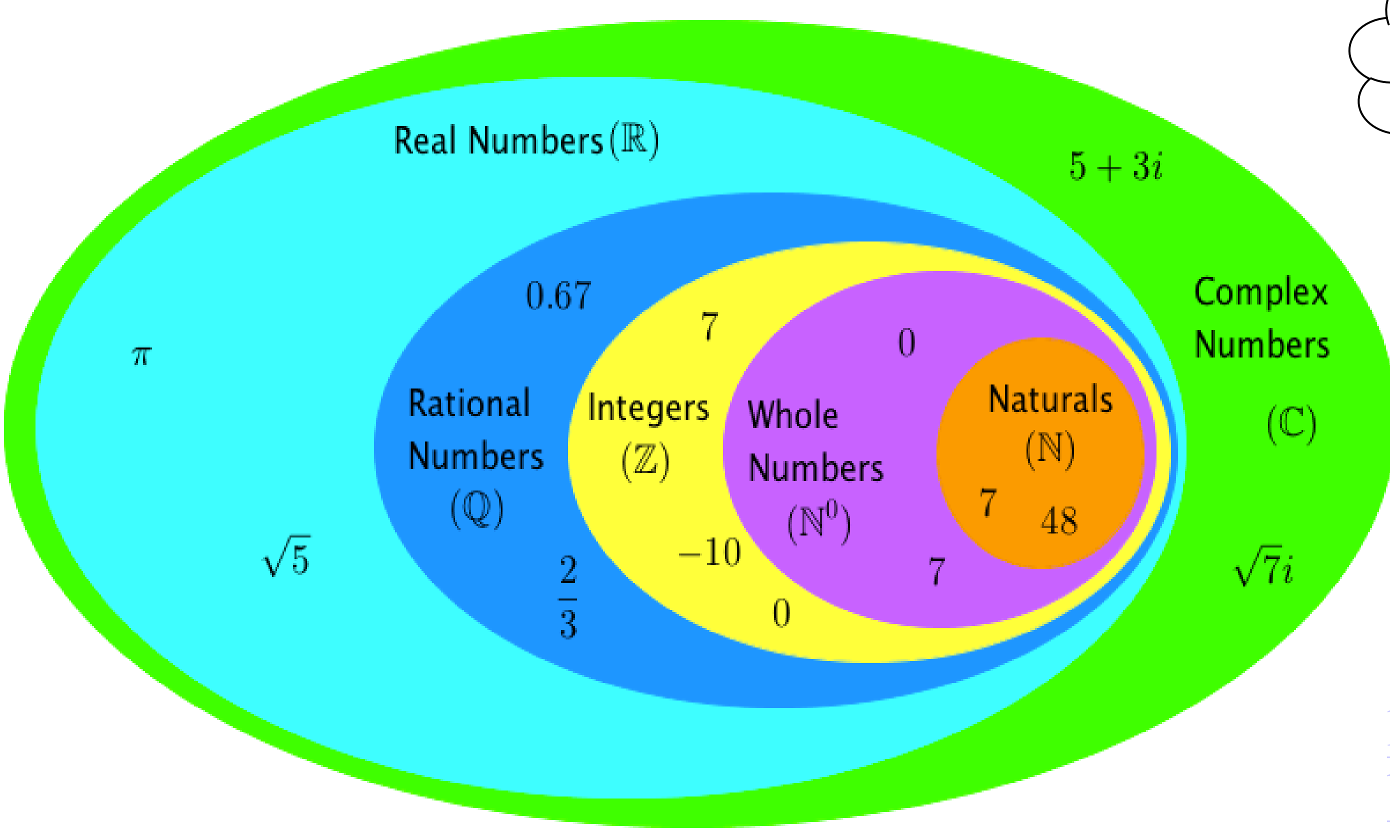
$$\begin{aligned} z^4 &= 4 \operatorname{cis} \frac{\pi}{3} \\ &= 4 \operatorname{cis}\left(\frac{\pi}{3} + 2k\pi\right) \qquad (k\text{: integer}) \\ z &= \sqrt{2} \operatorname{cis}\left(\frac{\pi}{12} + \frac{2k\pi}{4}\right) \end{aligned}$$

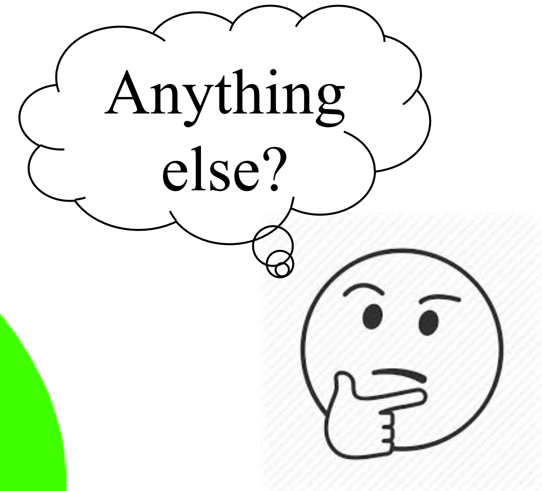
The four roots can be obtained by substituting $k = 0, 1, 2, 3$.

$$k = 0, z = \sqrt{2} \operatorname{cis} \frac{\pi}{12}, \qquad k = 1, z = \sqrt{2} \operatorname{cis} \frac{7\pi}{12}$$

$$k = 2, z = \sqrt{2} \operatorname{cis} \frac{13\pi}{12}, \qquad k = 3, z = \sqrt{2} \operatorname{cis} \frac{19\pi}{12}$$

The World of Numbers





Quaternions…
https://www.youtube.com/watch?v=3BR8tK-LuB0
(12.5 min.)