MA1300 Solutions to Self Practice # 11

1. For the following two functions:

   (a) Find the intervals on which f is increasing or decreasing.

   (b) Find the local maximum and minimum values of f.

   (c) Find the intervals of concavity and the inflection points.

$$f(x) = x^4 - 2x^2 + 3,$$
$$f(x) = \cos^2 x - 2\sin x, \quad 0 \le x \le 2\pi.$$

Solution:

(1) $f(x) = x^4 - 2x^2 + 3$. We have

$$f'(x) = 4x^3 - 4x = 4x(x+1)(x-1).$$

Therefore on intervals $(-\infty, -1)$ and $(0, 1)$, $f'(x) < 0$, hence f is decreasing; on intervals $(-1, 0)$ and $(1, \infty)$, $f'(x) > 0$, hence f is increasing. At points -1 and 1, f' changes from negative to positive, so -1 and 1 are two local minimum points of f, and the corresponding local minimum values are $f(-1) = f(1) = 2$. At point 0, f' changes from positive to negative, so $f(0) = 3$ is a local maximum value of f. We take the second derivative of f

$$f''(x) = 12x^2 - 4 = 4(\sqrt{3}x + 1)(\sqrt{3}x - 1).$$

Therefore on intervals $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$, $f''(x) > 0$, hence f is concave upward; on interval $(-1/\sqrt{3}, 1/\sqrt{3})$, $f''(x) < 0$, hence f is concave downward. Also, we obtain the inflection points $\pm 1/\sqrt{3}$.

(2) $f(x) = \cos^2 x - 2\sin x, \quad 0 \le x \le 2\pi$. We have for $x \in (0, 2\pi)$,

$$f'(x) = -2\cos x \sin x - 2\cos x = -2\cos x(\sin x + 1).$$

We use Table 1 to show the increasing and decreasing intervals of f. So the local maximum value of f is

Interval	$-2\cos x$	$\sin x + 1$	$f'(x)$	f
$(0, \pi/2)$	$-$	$+$	$-$	decreasing
$(\pi/2, 3\pi/2)$	$+$	$+$	$+$	increasing
$(3\pi/2, 2\pi)$	$-$	$+$	$-$	decreasing

Table 1: Problem 0 (2). Increasing and decreasing intervals of f.

$f(3\pi/2) = 2$, and the local minimum value of f is $f(\pi/2) = -2$. We take the second derivative

$$\begin{aligned} f''(x) &= 2\sin^2 x - 2\cos^2 x + 2\sin x = 4\sin^2 x + 2\sin x - 2 \\ &= 2(2\sin x - 1)(\sin x + 1). \end{aligned}$$

We use the following Table 2 to study the concavity of f. Since on two adjacent intervals $(5\pi/6, 3\pi/2)$ and

Interval	$2\sin x - 1$	$\sin x + 1$	$f''(x)$	concavity of f
$(0, \pi/6)$	$-$	$+$	$-$	downward
$(\pi/6, 5\pi/6)$	$+$	$+$	$+$	upward
$(5\pi/6, 3\pi/2)$	$-$	$+$	$-$	downward
$(3\pi/2, 2\pi)$	$-$	$+$	$-$	downward

Table 2: Problem 0 (2). Concavity of f''.

$(3\pi/2, 2\pi)$ f has the same concavity, we now check whether f is concave downward on $(5\pi/6, 2\pi)$. For any point $x_0 \in (5\pi/6, 2\pi)$, the tangent line of f at $(x_0, f(x_0))$ is

$$l(x) = l_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0).$$

For any $x \in (5\pi/6, 2\pi)$, let

$$g(x) = l(x) - f(x) = f(x_0) - f(x) + f'(x_0)(x - x_0)$$

to give $g(x_0) = 0$, and g is differentiable on $(5\pi/6, 2\pi)$ with

$$g'(x) = f'(x_0) - f'(x),$$

hence $g'(x_0) = 0$ and $g'(x)$ itself is differentiable on $(5\pi/6, 2\pi)$. For any $x \in (5\pi/6, 2\pi) \setminus \{x_0\}$, by the Mean Value Theorem, there exists some ξ between x and x_0, and some η between ξ and x_0 such that

$$\begin{aligned} g(x) &= g(x) - g(x_0) = g'(\xi)(x - x_0) = [g'(\xi) - g'(x_0)](x - x_0) \\ &= g''(\eta)(\xi - x_0)(x - x_0) = -f''(\eta)(\xi - x_0)(x - x_0). \end{aligned}$$

Since f'' is **negative** on $(5\pi/6, 2\pi)$, and $\xi - x_0$ has the same sign as $x - x_0$, $g(x) \geq 0$. Therefore we have proved that $f(x) \leq l(x)$ for any $x \in (5\pi/6, 2\pi)$, so f is concave downward on $(5\pi/6, 2\pi)$. The inflection points of f are $\pi/6$ and $5\pi/6$.

2. Suppose $f(3) = 2$, $f'(3) = 1/2$, and $f'(x) > 0$, and $f''(x) < 0$ for all x.

   (a) Sketch a possible graph for f.

   (b) How many solutions does the equation $f(x) = 0$ have? Why?

   (c) Is it possible that $f'(2) = 1/3$? Why?

Solution:

(a) See Figure 1.

(b) The equation $f(x) = 0$ has one and only one solution. First, If $f(x) = 0$ has no root, then by the Intermediate Value Theorem, $f(3) = 2 > 0$ implies $f(-2) > 0$. Since $f''(x) < 0$, $f'(x)$ is decreasing, and so for any $x < 3$, $f'(x) > f'(3) = \frac{1}{2}$. By the Mean Value Theorem, there exist some $\xi \in (-2, 3)$, such that

$$\frac{1}{2} < f'(\xi) = \frac{f(3) - f(-2)}{3 - (-2)} < \frac{2 - 0}{5} < \frac{1}{2},$$

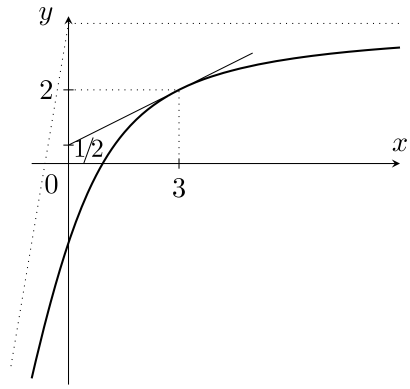

Figure 1: A possible f for Problem 0.

a contradiction. So $f(x) = 0$ has at least one root. Second, if $f(x) = 0$ has at least two roots, we take any two of them: $x_1 < x_2$. By the Mean Value Theorem, there exists some $\eta \in (x_1, x_2)$, such that

$$0 < f'(\eta) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = 0,$$

another contradiction. So f has one and only one root.

(c) We have already shown that for any $x < 3$, $f'(x) > f'(3) = \frac{1}{2}$, therefore it is not possible that $f'(2) = \frac{1}{3}$.

3. Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at -2 and a local minimum value of 0 at 1.

Solution: We take derivative of f.

$$f'(x) = 3ax^2 + 2bx + c. \tag{1}$$

Since f has a local maximum at -2, and a local minimum at 1, Fermat's Theorem implies

$$f'(x) = 3a(x+2)(x-1) = 3ax^2 + 3ax - 6a. \tag{2}$$

Compare the coefficients of $f'(x)$ in (1) and (2) to give $b = \frac{3}{2}a$ and $c = -6a$. Substituting $f(-2) = 3$ and $f(1) = 0$ to give

$$\begin{cases} -8a + 6a + 12a + d = 3, \\ a + \frac{3}{2}a - 6a + d = 0. \end{cases}$$

Solve the equations to give $a = \frac{2}{9}$ and $d = \frac{7}{9}$, so $b = \frac{1}{3}$ and $c = -\frac{4}{3}$. The polynomial is

$$f(x) = \frac{2}{9}x^3 + \frac{1}{3}x^2 - \frac{4}{3}x + \frac{7}{9}.$$

4.

(a) If f and g are positive, increasing, concave upward functions on I, show that the product function fg is concave upward on I.

(b) Show that part (a) remains true if f and g are both decreasing.

(c) Suppose f is increasing and g is decreasing. Show, by giving three examples, that fg may be concave upward, concave downward, or linear. Why doesn't the argument in parts (a) and (b) work in this case?

Proof:

(a) For any $a \in I$, suppose the tangent lines of f and g at a are

$$l_f(x) = \alpha(x-a) + f(a),$$
$$l_g(x) = \beta(x-a) + g(a),$$

respectively, with $\alpha \geq 0$ and $\beta \geq 0$. Then the line

$$l(x) = [f(a)\beta + \alpha g(a)](x-a) + f(a)g(a)$$

goes through $(a, (fg)(a))$. Since both f and g are positive and concave upward, we have for any $x \in I$,

$$\begin{aligned} f(x)g(x) - l(x) &= f(a)(g(x) - \beta(x-a) - g(a)) + g(a)(f(x) - \alpha(x-a) - f(a)) + \\ &\quad f(a)g(a) + f(x)g(x) - f(a)g(x) - f(x)g(a) \\ &= f(a)(g(x) - l_g(x)) + g(a)(f(x) - l_f(x)) + (f(x) - f(a))(g(x) - g(a)) \geq 0. \end{aligned}$$

Therefore l is a tangent line of fg at a, and fg is concave upward on I.

(b) If f and g are both decreasing, using the argument in (a) to give

$$\begin{aligned} f(x)g(x) - l(x) &= f(a)(g(x) - l_g(x)) + g(a)(f(x) - l_f(x)) + (f(x) - f(a))(g(x) - g(a)) \\ &= f(a)(g(x) - l_g(x)) + g(a)(f(x) - l_f(x)) + (f(a) - f(x))(g(a) - g(x)) \geq 0. \end{aligned}$$

Therefore (a) remains true.

(c) Now we let $I = (0, 1)$. Example for fg to be linear

$$f(x) = x^2, \quad g(x) = \frac{1}{x}, \quad \Rightarrow \quad (fg)(x) = x.$$

Example for fg to be concave upward

$$f(x) = x^3, \quad g(x) = \frac{1}{x}, \quad \Rightarrow \quad (fg)(x) = x^2, (fg)''(x) = 2 > 0.$$

Example for fg to be concave downward

$$f(x) = x^2, \quad g(x) = \frac{1}{x^{3/2}}, \quad \Rightarrow \quad (fg)(x) = \sqrt{x}, (fg)''(x) = -\frac{1}{4x^{3/2}} < 0.$$

The argument in parts (a) and (b) does not work here because in that argument, we used the property that $(f(a) - f(x))(g(a) - g(x)) \geq 0$, which holds when both of f and g are increasing or both of them are decreasing. When one of them is increasing while the other is decreasing, the property does not always hold true.

5. Show that $\tan x > x$ for $0 < x < \pi/2$. [Hint: Show that $f(x) = \tan x - x$ is increasing on $(0, \pi/2)$.]

Proof: Let $f(x) = \tan x - x$, so $f(x)$ is continuous on $[0, \pi/2)$, and $f'(x) = 1 + \tan^2 x - 1 > 0$ on $(0, \pi/2)$. By the Mean Value Theorem, for any $0 < x < \pi/2$, there exist some $\xi \in (0, x)$ such that

$$f(x) = f(x) - f(0) = f'(\xi)(x - 0) > 0,$$

which implies $\tan x > x$.

6. Prove that, for all $x > 1$,

$$2\sqrt{x} > 3 - \frac{1}{x}.$$

Proof: Let $f(x) = 2\sqrt{x} - 3 + \frac{1}{x}$ to give $f(1) = 0$, $f(x)$ is continuous on $[1, \infty)$, and $f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2} > 0$ on $(1, \infty)$. By the Mean Value Theorem, for any $x \in (1, \infty)$, there exists some $\xi \in (1, x)$ such that

$$f(x) = f(x) - f(1) = f'(\xi)(x - 1) > 0,$$

so $f(x) > 0$ and thus $2\sqrt{x} > 3 - \frac{1}{x}$.

7. Find the limit.

$$\lim_{t\to\infty} \frac{\sqrt{t} + t^2}{2t - t^2}, \qquad \lim_{x\to\infty} \left(\sqrt{9x^2 + x} - 3x\right).$$

Solution:

$$\lim_{t\to\infty} \frac{\sqrt{t} + t^2}{2t - t^2} = \lim_{t\to\infty} \frac{t^2\left\{\frac{1}{t^{3/2}} + 1\right\}}{-t^2\left\{-\frac{2}{t} + 1\right\}} = -1.$$

$$\lim_{x\to\infty} \left(\sqrt{9x^2 + x} - 3x\right) = \lim_{x\to\infty} = \frac{9x^2 + x - 9x^2}{\sqrt{9x^2 + x} + 3x} = \lim_{x\to\infty} \frac{1}{\sqrt{9 + \frac{1}{x}} + 3} = \frac{1}{6}.$$

8. Find the horizontal asymptotes of the curve and use them, together with concavity and intervals of increase and decrease, to sketch the curve.

$$y = \frac{1 - x}{1 + x}.$$

Solution: Since

$$\lim_{x\to\infty} \frac{1 - x}{1 + x} = \lim_{x\to\infty} \frac{\frac{1}{x} - 1}{\frac{1}{x} + 1} = -1,$$

and

$$\lim_{x\to-\infty} \frac{1 - x}{1 + x} = \lim_{x\to-\infty} \frac{\frac{1}{x} - 1}{\frac{1}{x} + 1} = -1,$$

$y = -1$ is the horizontal asymptote. Take derivatives twice to give

$$y' = \frac{-2}{(x + 1)^2}, \quad y'' = \frac{4}{(1 + x)^3}, \qquad \text{for any } x \neq -1,$$

so on $(-\infty, -1)$, $y' < 0$ and $y'' < 0$, thus y is decreasing and concave downward. On $(-1, \infty)$, $y' < 0$ and $y'' > 0$, so y is decreasing and concave upward. Moreover, $\lim_{x\to-1^+} \frac{1-x}{1+x} = \infty$ and $\lim_{x\to-1^-} \frac{1-x}{1+x} = -\infty$. The curve is sketched in Figure 2.

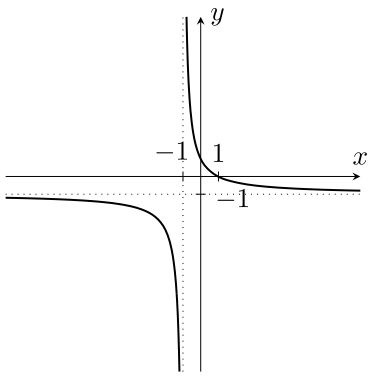

Figure 2: The curve of Problem 0.

9. Use the guidelines of Section 3.5 (A∼H) to sketch the curve.

$$y = \frac{\sqrt{1-x^2}}{x}.$$

Solution:

A The domain is $[-1, 0) \cup (0, 1]$.

B The x-interceptions are ± 1, no y-interception.

C Symmetry. $y(-x) = \frac{\sqrt{1-x^2}}{-x} = -y(x)$, so the function is odd. The function is neither even, nor periodic.

D Asymptotes. No horizontal asymptote. Since $\lim_{x\to 0^+} y = +\infty$, and $\lim_{x\to 0^-} y = -\infty$, $x = 0$ is the vertical asymptote.

E Take derivative

$$y' = \frac{\frac{-2x}{2\sqrt{1-x^2}}x - \sqrt{1-x^2}}{x^2} = -\frac{1}{x^2\sqrt{1-x^2}} < 0,$$

so y is decreasing in both $(-1, 0)$ and $(0, 1)$.

F No local maximum or minimum values.

G Take the second derivative

$$\begin{aligned} y'' &= \frac{1}{x^4(1-x^2)}\left(2x\sqrt{1-x^2} + \frac{-x^3}{\sqrt{1-x^2}}\right) \\ &= \frac{-3x(x+\sqrt{2/3})(x-\sqrt{2/3})}{x^4(1-x^2)^{3/2}}. \end{aligned}$$

Therefore on intervals $(-1, -\sqrt{2/3})$ and $(0, \sqrt{2/3})$, $y'' > 0$ so y is concave upward, and on intervals $(-\sqrt{2/3}, 0)$ and $(\sqrt{2/3}, 1)$, $y'' < 0$ so y is concave downward. The points of inflection are $(-\sqrt{2/3}, -1/\sqrt{2})$ and $(\sqrt{2/3}, 1/\sqrt{2})$.

H The curve is sketched on the Figure 3.

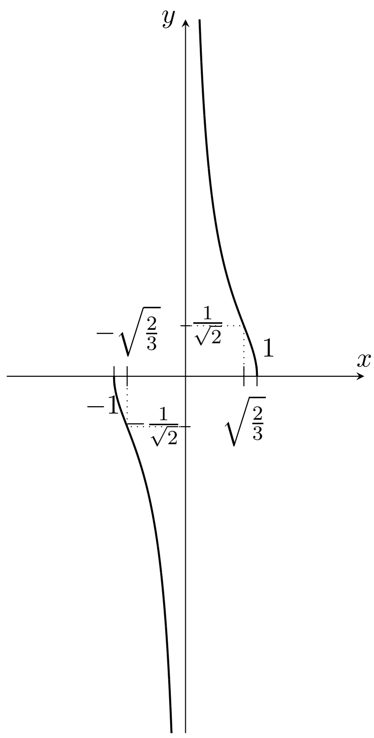

Figure 3: The curve of Problem 0.

10. In the theory of relativity, the mass of a particle is

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}},$$

where m_0 is the rest mass of the particle, m is the mass when the particle moves with speed v relative to the observer, and c is the speed of light. Sketch the graph of m as a function of v.

Solution:

A Domain: $(-c, c)$.

B Since $m > 0$ when $v \in (-c, c)$, there is no v-interception. Let $v = 0$ to give the m-interception m_0.

C m is an even function.

D There is no horizontal asymptotes. Since $\lim_{v \to c^-} m = +\infty = \lim_{v \to -c^+} m$, $v = c$ and $v = -c$ are two vertical asymptotes.

E Take derivative

$$m' = \frac{-m_0}{2(1 - v^2/c^2)^{3/2}} \left(-\frac{2v}{c^2} \right) = \frac{m_0 v}{c^2(1 - v^2/c^2)^{3/2}},$$

so on $(-c, 0)$, m is decreasing, and on $(0, c)$, m is increasing.

F $m' = 0$ when $v = 0$, and m' changes from negative to positive at $v = 0$. So $m(0) = m_0$ is a local minimum.

G Take the second derivative

$$\begin{aligned} m'' &= \frac{m_0 c^2 (1 - v^2/c^2)^{3/2} - m_0 v c^2 \frac{3}{2} (1 - v^2/c^2)^{1/2} \left(-\frac{2v}{c^2} \right)}{c^4 (1 - v^2/c^2)^3} \\ &= \frac{m_0(1 + 2v^2/c^2)}{c^2(1 - v^2/c^2)^{5/2}} > 0, \end{aligned}$$

so on $(-c, c)$ m is concave upward.

H The curve is sketched in Figure 4.

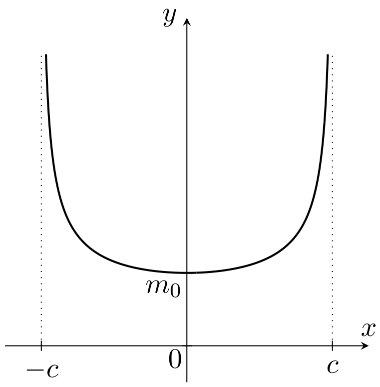

Figure 4: The curve of Problem 0.

11. Coulomb's Law states that the force of attraction between two charged particles is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The figure 5 shows particles with charge 1 located at positions 0 and 2 on a coordinate line and a particle

with charge -1 at a position x between them. It follows from Coulomb's Law that the net force acting on the middle particle is

$$F(x) = -\frac{k}{x^2} + \frac{k}{(x-2)^2}, \qquad 0 < x < 2,$$

where k is a positive constant. Sketch the graph of the net force function. What does the graph say about the force?

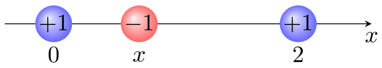

Figure 5: The positions of the charged particles.

Solution:

A Domain: $x \in (0, 2)$.

B There is no F-interception. Let $F = 0$ to give $x = 1$, so 1 is the x-interception.

C $F(x) = -F(2-x)$, so F is symmetric with respect to the point $(1, 0)$.

D There is no horizontal asymptote. Since $\lim_{x\to 0^+} F(x) = -\infty$ and $\lim_{x\to 2^-} F(x) = \infty$, $x = 0$ and $x = 2$ are two vertical asymptotes.

E Take the derivative of F to give

$$F'(x) = \frac{2k}{x^3} - \frac{2k}{(x-2)^3},$$

so on $(0, 2)$, $F'(x) > 0$, thus F is increasing on $(0, 2)$.

F There is no local maximum or minimum.

G Take the second derivative of F to give

$$F''(x) = -\frac{6k}{x^4} + \frac{6k}{(x-2)^4},$$

so on $(0, 1)$, $F''(x) < 0$, and on $(1, 2)$, $F''(x) > 0$. Therefore F is concave downward on $(0, 1)$ and concave upward on $(1, 2)$. The inflection point is $(1, 0)$.

H The curve is sketched in Figure 6

The graph shows that when $0 < x < 1$, the particle with charge -1 experiences an attractive force from left, and as $x \to 0^+$, the force becomes stronger and tends to infinity. When $1 < x < 2$, the particle with charge -1 experiences an attractive force from right, and as $x \to 2^-$, the force becomes stronger and tends to infinity.

12. A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil (measured in appropriate units) is

$$Y = \frac{kN}{1 + N^2},$$

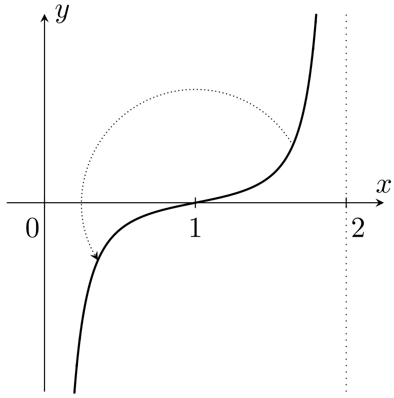

Figure 6: The curve of Problem 0.

where k is a positive constant. What nitrogen level gives the best yield?

Solution: The domain is $N \in (0, \infty)$. Take derivative to give

$$Y' = \frac{k(1+N^2) - 2kN^2}{(1+N^2)^2} = \frac{k(1+N)(1-N)}{(1+N^2)^2},$$

so the only one critical number is $N = 1$, where Y' changes from positive to negative. By the First Derivative Test For Absolute Extreme Values, $Y(1) = \frac{k}{2}$ is a local maximum, and also an absolute maximum. So $N = 1$ gives the best yield.

13. The rate (in mg carbon/m^3/h) at which photosynthesis takes place for a species of phytoplankton is modeled by the function

$$P = \frac{100I}{I^2 + I + 4},$$

where I is the light intensity (measured in thousands of foot-candles). For what light intensity is P a maximum?

Solution: From the background of physics, the domain is $I \in [0, \infty)$. We take the derivative of P to give

$$\begin{aligned} P' &= \frac{100(I^2 + I + 4) - 100I(2I + 1)}{(I^2 + I + 4)^2} \\ &= \frac{-100(I-2)(I+2)}{(I^2 + I + 4)^2}, \end{aligned}$$

so $P' > 0$ on $(0, 2)$ and $P' < 0$ on $(2, \infty)$. By the First Derivative Test For Absolute Extreme Values, $P(2) = 20$ is the absolute maximum, and the corresponding light intensity is $I = 2$.

14. If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2 R}{(R + r)^2}.$$

If E and r are fixed but R varies, what is the maximum value of the power?

Solution: Notice that E and r are positive constants, the function P is differentiable on $\mathbb{R}$ and we have

$$\frac{dP}{dR} = \frac{E^2(R+r)^2 - E^2R2(R+r)}{(R+r)^4}.$$

The only critical number of P is $R = r$. Moreover,

$$\lim_{R\to\infty} \frac{E^2R}{(R+r)^2} = \lim_{R\to\infty} \frac{\frac{E^2}{R}}{(1+\frac{r}{R})^2} = 0$$

and

$$P(0) = 0.$$

So we apply the Closed Interval Method to the function P on the interval $[0,\infty)$, we see that the absolute maximum value of P is $P = \frac{E^2}{4r}$ achieved at the point $R = r$.

15. Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2},$$

where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown in Figure 7. This equation is known as Snell's Law.

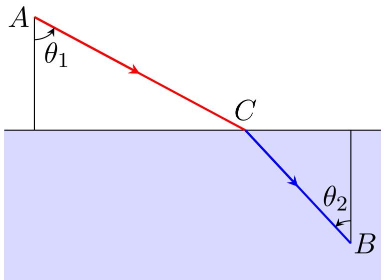

Figure 7: Figure of Problem 0.

Proof: Let the orthogonal projection of points A and B to the surface of the water be E and F respectively, as illustrated in Figure 8.

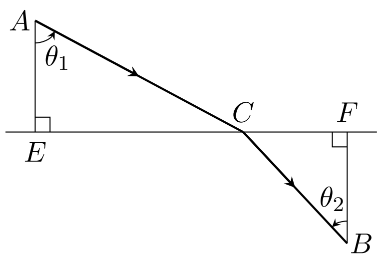

Figure 8: Figure of Problem 0.

If the light choose ACB as the path, the total time cost is

$$T = \frac{AC}{v_1} + \frac{CB}{v_2} = \frac{AE}{v_1\cos\theta_1} + \frac{BF}{v_2\cos\theta_2}.$$

Since $EC + CF = EF$ is a constant, we have

$$\frac{d}{d\theta_1}(AE\tan\theta_1 + FB\tan\theta_2) = \frac{d}{d\theta_1}(EF) = 0. \tag{3}$$

Because AE and FB are constants, we expand (3) to give

$$AE(1+\tan^2\theta_1) + FB(1+\tan^2\theta_2)\frac{d\theta_2}{d\theta_1} = 0,$$

so

$$\frac{d\theta_2}{d\theta_1} = -\frac{AE(1+\tan^2\theta_1)}{FB(1+\tan^2\theta_2)} = -\frac{AE\cos^2\theta_2}{FB\cos^2\theta_1}. \tag{4}$$

We take derivative of T with respect to θ_1 to give

$$\frac{dT}{d\theta_1} = \frac{AE\sin\theta_1}{v_1\cos^2\theta_1} + \frac{BF\sin\theta_2}{v_2\cos^2\theta_2}\frac{d\theta_2}{d\theta_1}. \tag{5}$$

Substituting (4) into (5) to give

$$\frac{dT}{d\theta_1} = \frac{AE\sin\theta_1}{v_1\cos^2\theta_1} - \frac{AE\sin\theta_2}{v_2\cos^2\theta_1}.$$

By Fermat's Theorem, if T achieves minimization then $dT/d\theta_1 = 0$, that is

$$\frac{AE\sin\theta_1}{v_1\cos^2\theta_1} = \frac{AE\sin\theta_2}{v_2\cos^2\theta_1}, \tag{6}$$

since $AE > 0$ and $\theta_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ hence $\cos^2\theta_1 > 0$, (6) implies

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$

===END===