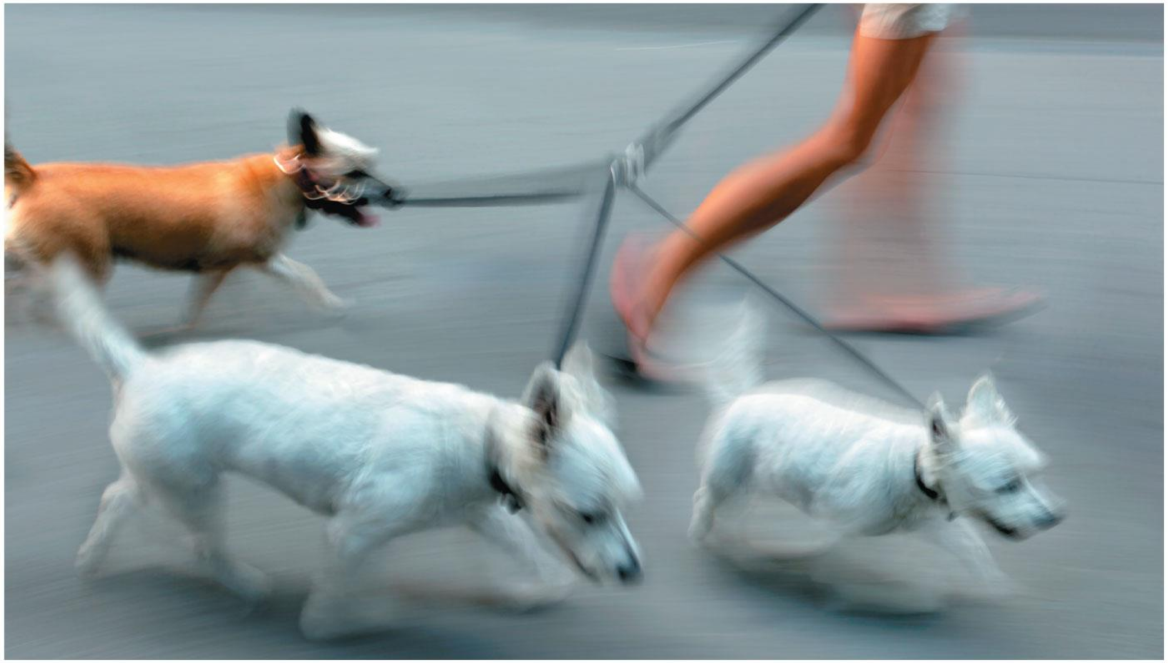
# Chapter 14

## Periodic Motion

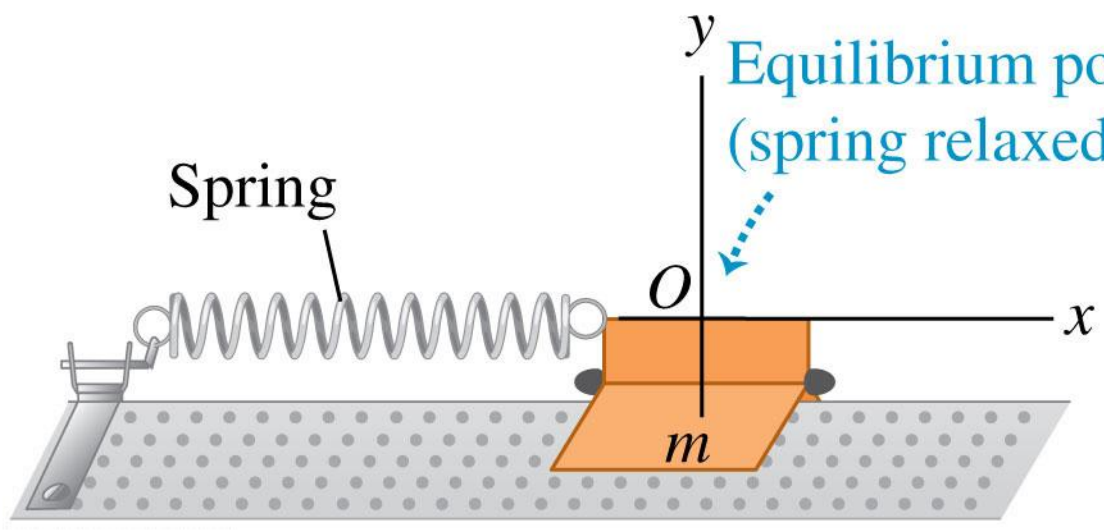
# Introduction

- Why do dogs walk with quicker strides than humans? Does it have anything to do with the characteristics of their legs?
- Many kinds of motion (such as a pendulum, musical vibrations, and pistons in car engines) repeat themselves. We call such behavior **periodic motion** or **oscillation**.

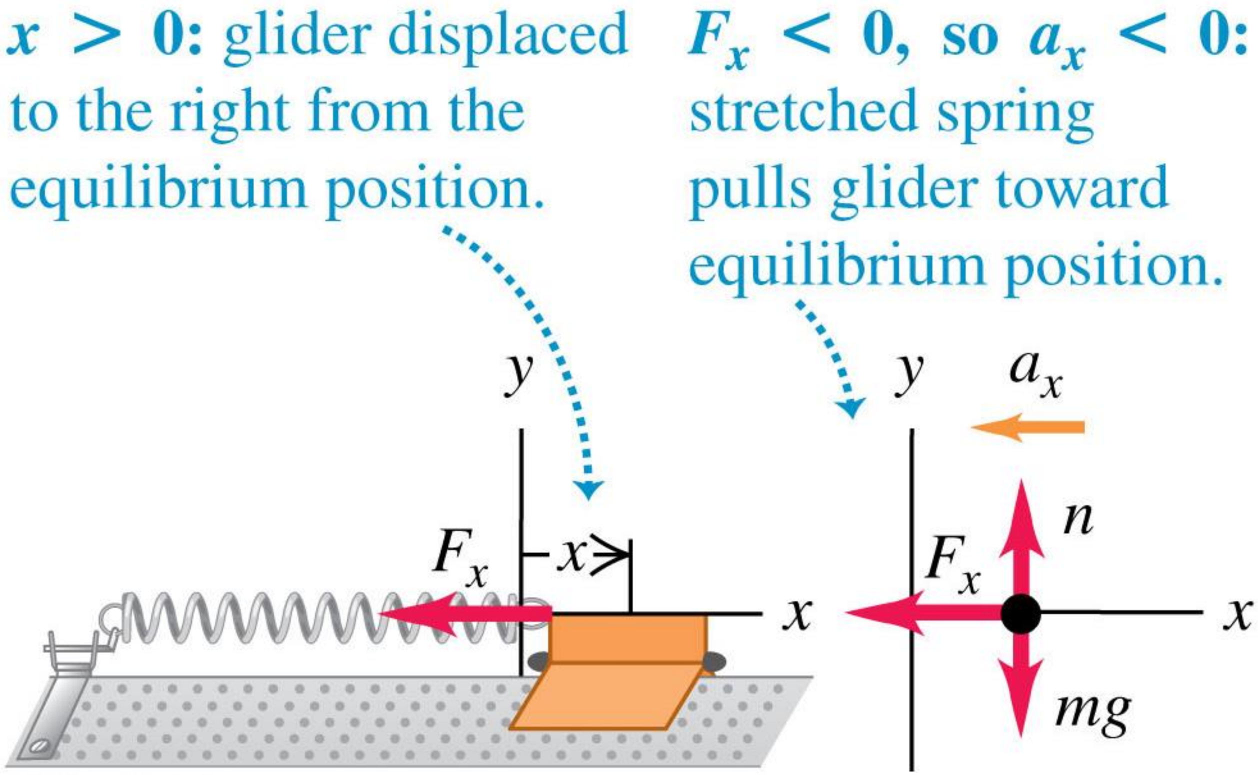


# What causes periodic motion?

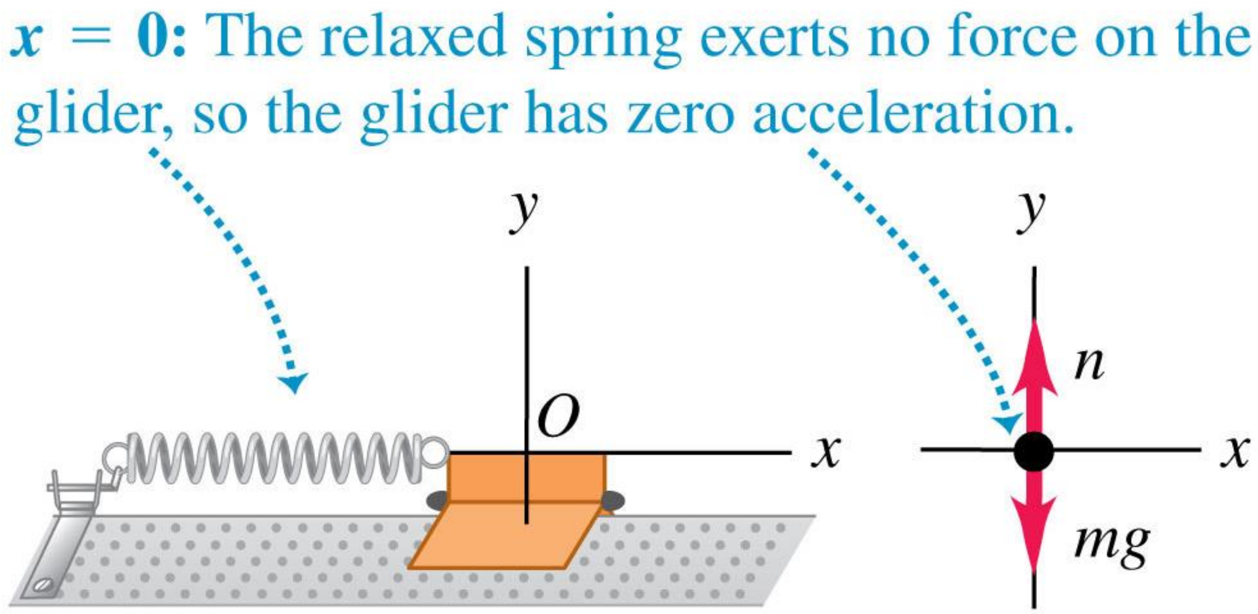
- If a body attached to a spring is displaced from its equilibrium position, the spring exerts a **restoring force** on it, which tends to restore the object to the equilibrium position.
- This force causes **oscillation** of the system, or **periodic motion**.



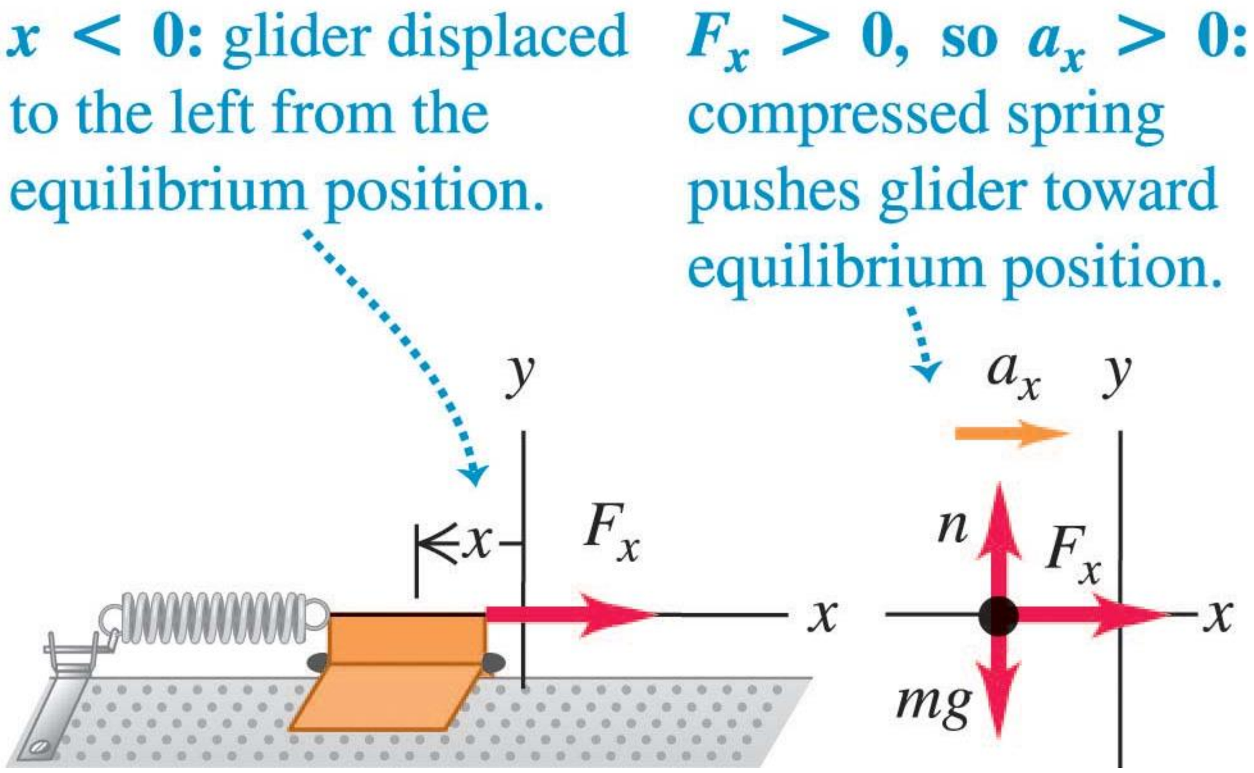
# What causes periodic motion?



# What causes periodic motion?



# What causes periodic motion?



# Characteristics of periodic motion

- The **amplitude**, $A$, is the maximum magnitude of displacement from equilibrium.
- The **period**, $T$, is the time for one cycle.
- The **frequency**, $f$, is the number of cycles per unit time.
- The **angular frequency**, $\omega$, is $2\pi$ times the frequency: $\omega = 2\pi f$.
- The frequency and period are reciprocals of each other: $f = 1/T$ and $T = 1/f$.

# Example 14.1

An ultrasonic transducer used for medical diagnosis oscillates at 6.7 MHz $= 6.7 \times 10^6$ Hz. How long does each oscillation take, and what is the angular frequency?

$$T = \frac{1}{f} = \frac{1}{6.7 \times 10^6\ \text{Hz}} = 1.5 \times 10^{-7}\ \text{s} = 0.15\ \mu\text{s}$$

$$\begin{aligned}\omega &= 2\pi f = 2\pi(6.7 \times 10^6\ \text{Hz}) \\ &= (2\pi\ \text{rad/cycle})(6.7 \times 10^6\ \text{cycle/s}) \\ &= 4.2 \times 10^7\ \text{rad/s}\end{aligned}$$

This is a very rapid vibration, with large $f$ and $\omega$ and small $T$. A slow vibration has small $f$ and $\omega$ and large $T$.

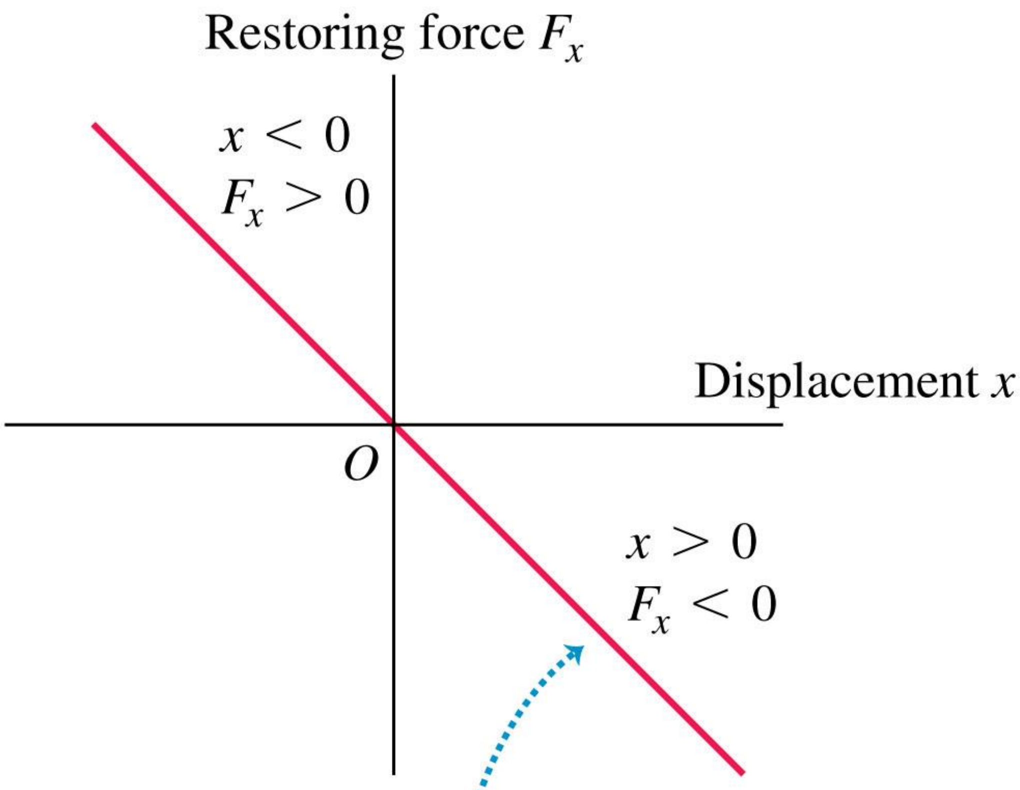
Video solution

# Simple harmonic motion (SHM)

- When the restoring force is *directly proportional* to the displacement from equilibrium, the resulting motion is called **simple harmonic motion** (SHM).
- A body that undergoes SHM is called a **harmonic oscillator.**

$$F_x = -kx$$

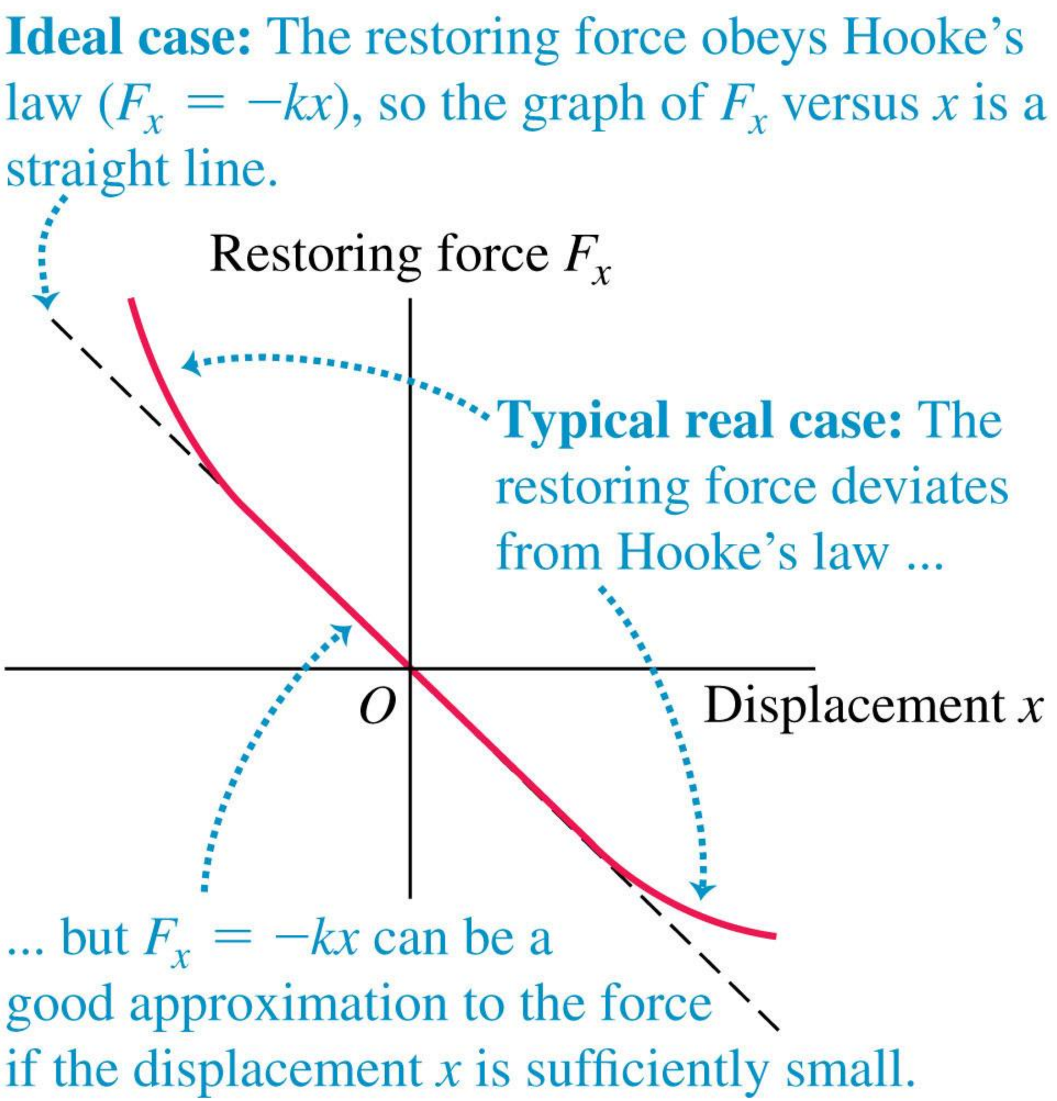
$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$



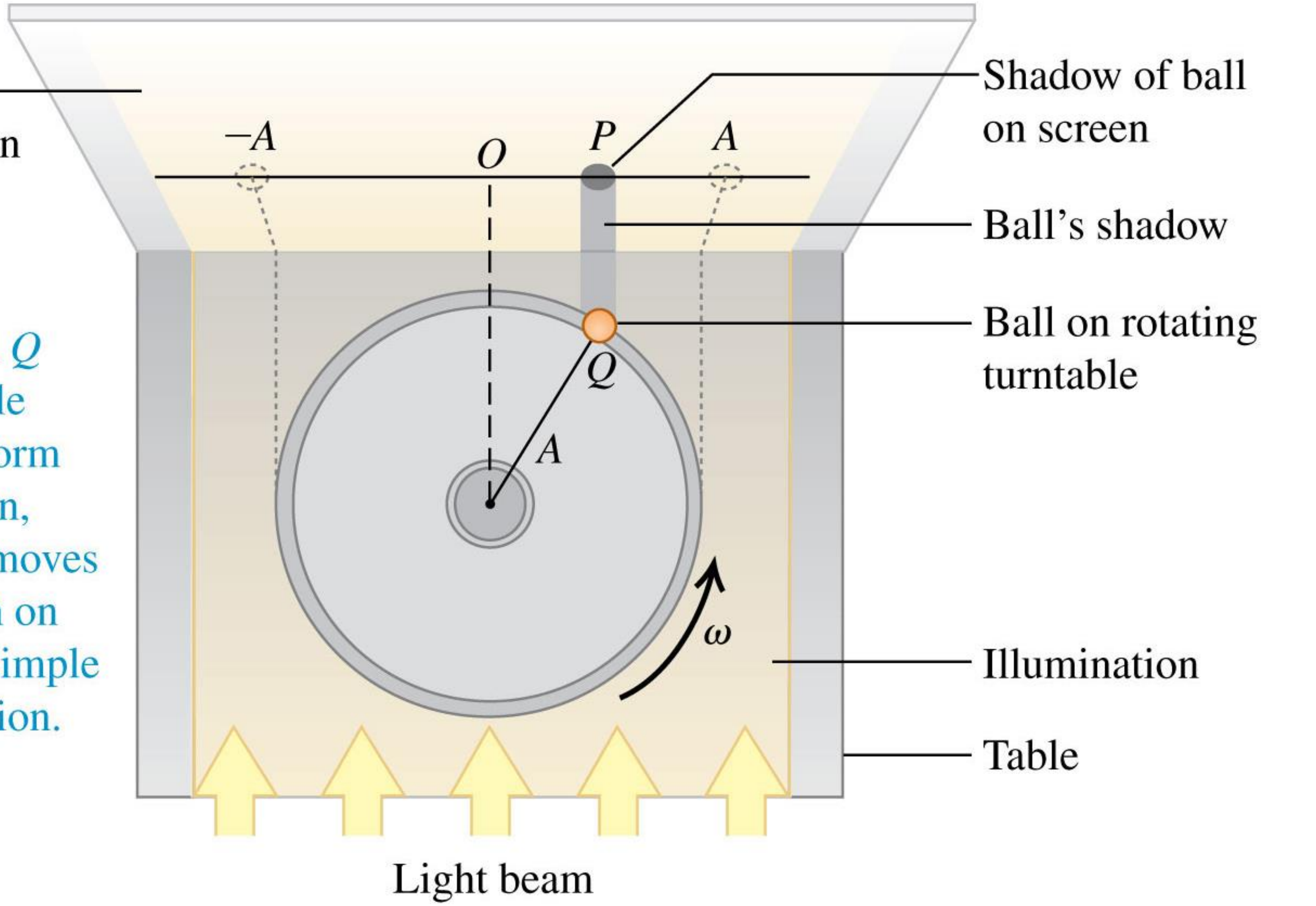
The restoring force exerted by an idealized spring is directly proportional to the displacement (Hooke's law, $F_x = -kx$): the graph of $F_x$ versus $x$ is a straight line.

# Simple harmonic motion (SHM)

- In many systems the restoring force is approximately proportional to displacement if the displacement is sufficiently small.
- That is, if the amplitude is small enough, the oscillations are approximately simple harmonic.

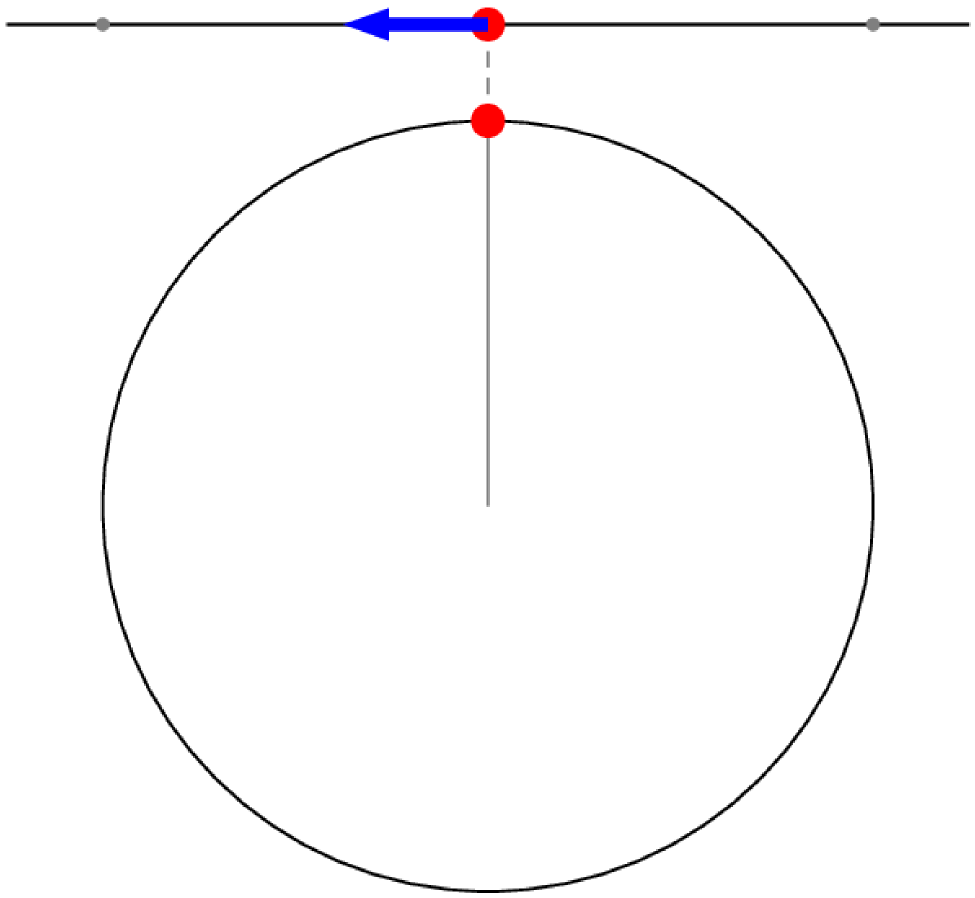


# Simple harmonic motion viewed as a projection



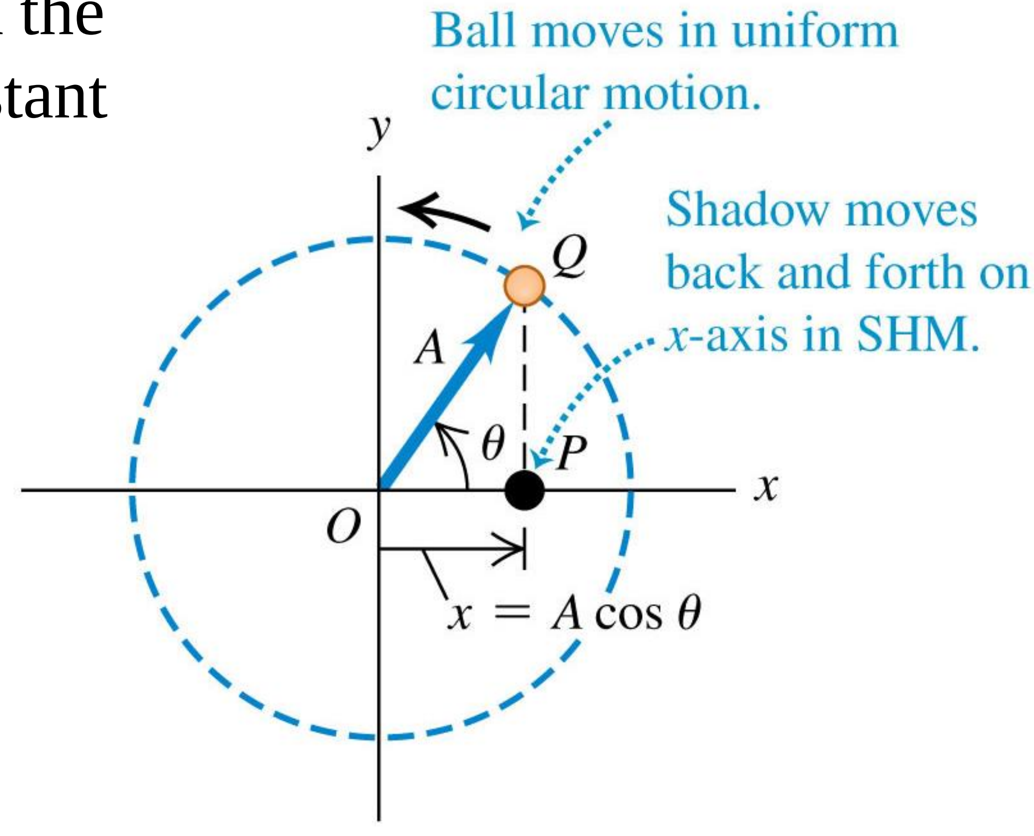

While the ball $Q$ on the turntable moves in uniform circular motion, its *shadow* $P$ moves back and forth on the screen in simple harmonic motion.

# Simple harmonic motion viewed as a projection



# Simple harmonic motion viewed as a projection

- The circle in which the ball moves so that its projection matches the motion of the oscillating body is called the **reference circle**.
- As point $Q$ moves around the reference circle with constant angular speed, vector $OQ$ rotates with the same angular speed.
- Such a rotating vector is called a **phasor**.

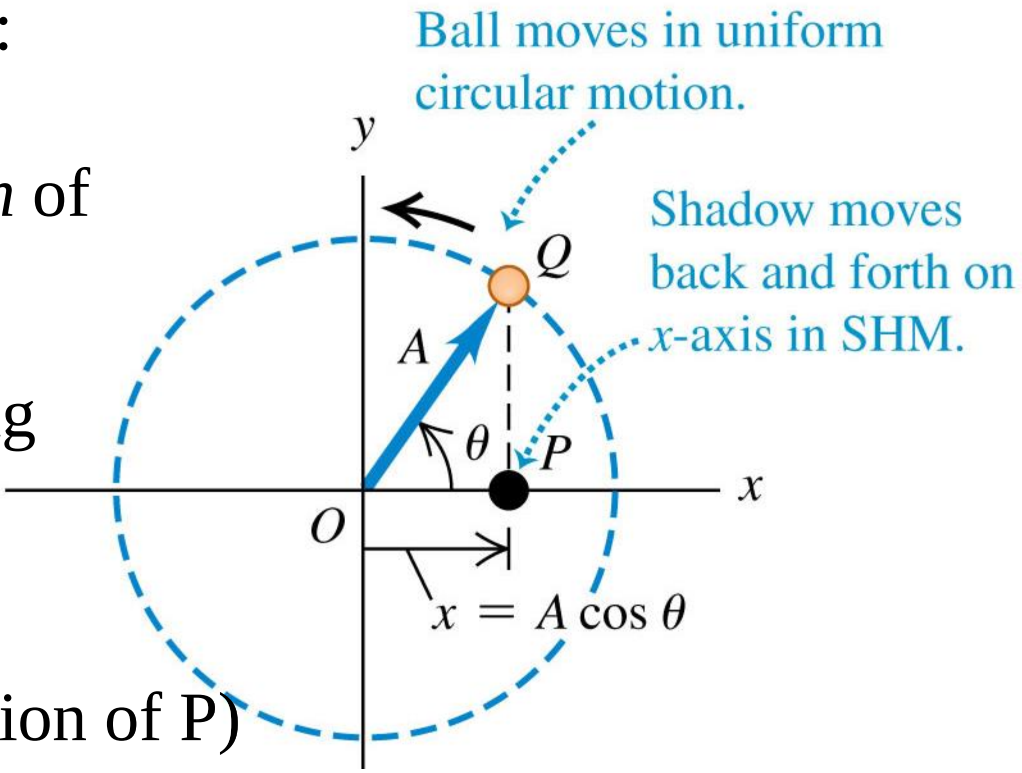


# Simple harmonic motion viewed as a projection

- x-component of the phasor:

  $$x = A\cos\theta$$

  which is also the *projection* of $Q$ onto the x-axis.




- Acceleration of $Q$: (pointing from $Q$ to $O$) $a_Q = \omega^2 A$

- Its projection to the x-axis (which is also the acceleration of P)

  $$a_x = -a_Q\cos\theta = -\omega^2 A\cos\theta$$ , or $$a_x = -\omega^2 x$$

- This is exactly SHM with $\omega^2 = \dfrac{k}{m}$ , or $\omega = \sqrt{\dfrac{k}{m}}$

# Characteristics of SHM

- For a body of mass *m* vibrating by an ideal spring with a force constant *k*:

**Angular frequency for simple harmonic motion**

$$\omega = \sqrt{\frac{k}{m}}$$

Force constant of restoring force ($k$); Mass of object ($m$)

**Frequency for simple harmonic motion**

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Angular frequency ($\omega$); Force constant of restoring force ($k$); Mass of object ($m$)

**Period for simple harmonic motion**

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Mass of object ($m$); Frequency ($f$); Angular frequency ($\omega$); Force constant of restoring force ($k$)

- Check the unit: k (N/m or kg/$s^2$), k/m ($s^{-2}$), ω ($s^{-1}$=rad/s)
- Don't confuse frequency (*f*) and angular frequency (ω).

# Characteristics of SHM

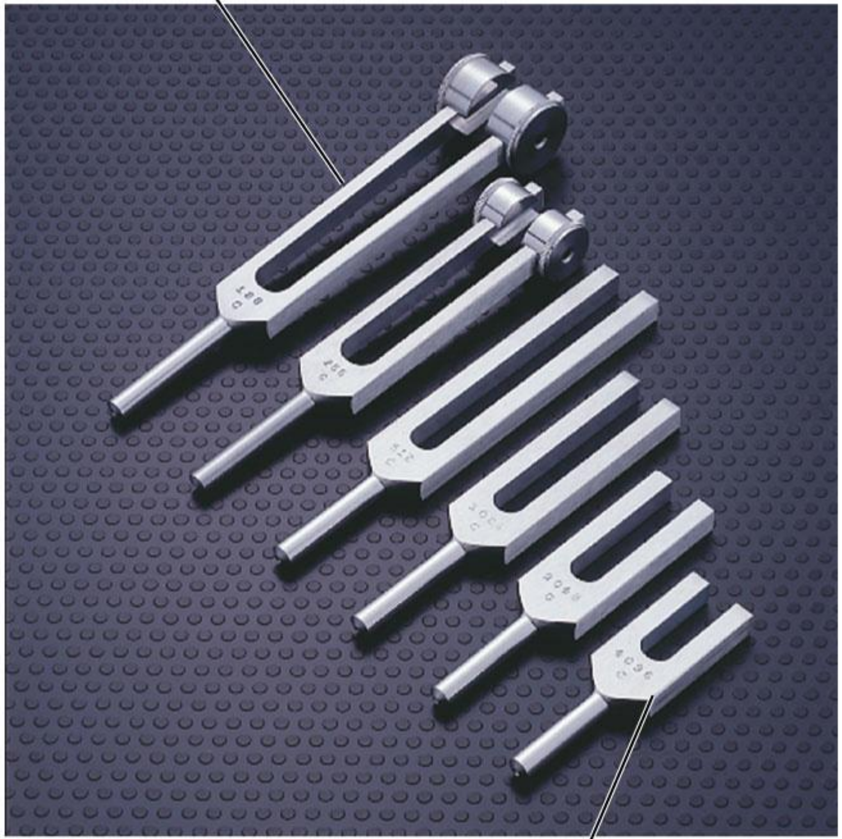
$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

- In SHM, the frequency *f* and the period *T* do not depend on the amplitude *A*.
- For given values of *m* and *k*, the time of one complete oscillation is the same whether the amplitude is large or small.
- Larger *A* means that the body reaches larger values of $|x|$ and is subjected to larger restoring forces. This increases the average speed of the body over a complete cycle; this exactly compensates for having to travel a larger distance, so the same total time is involved.

# Tuning forks

Tines with large mass $m$:
low frequency $f = 128$ Hz



Tines with small mass $m$:
high frequency $f = 4096$ Hz

- The greater the mass $m$ in a tuning fork's tines, the lower the frequency of oscillation, and the lower the pitch of the sound that the tuning fork produces.

# Tuning forks

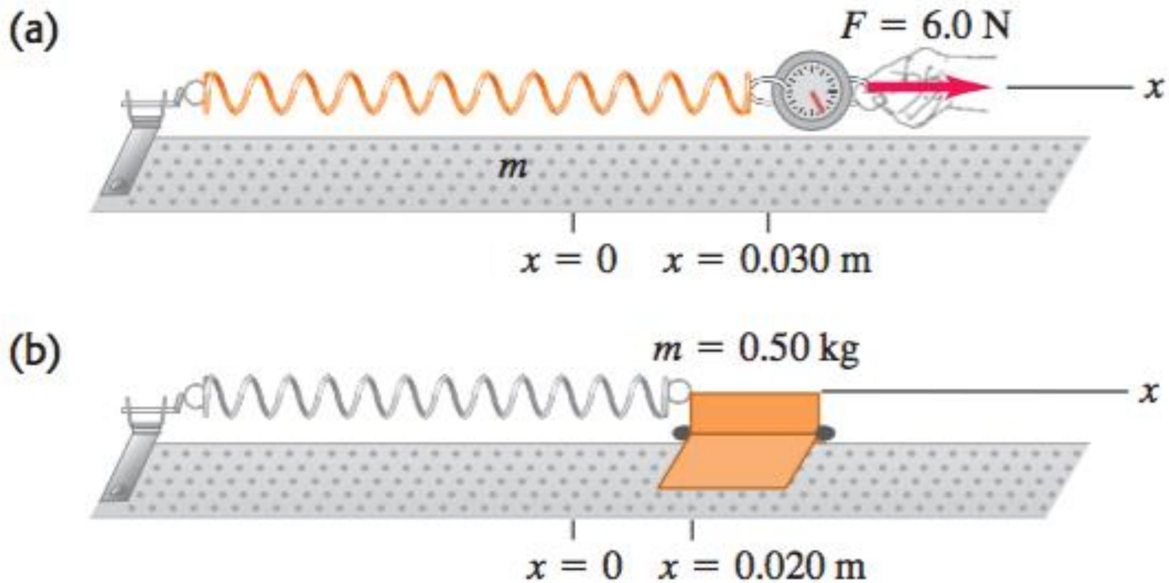

- The oscillations of a tuning fork are essentially SHM, which means that it always vibrates with the same frequency, independent of amplitude.
- This is why a tuning fork can be used as a standard for musical pitch.

# Tuning forks

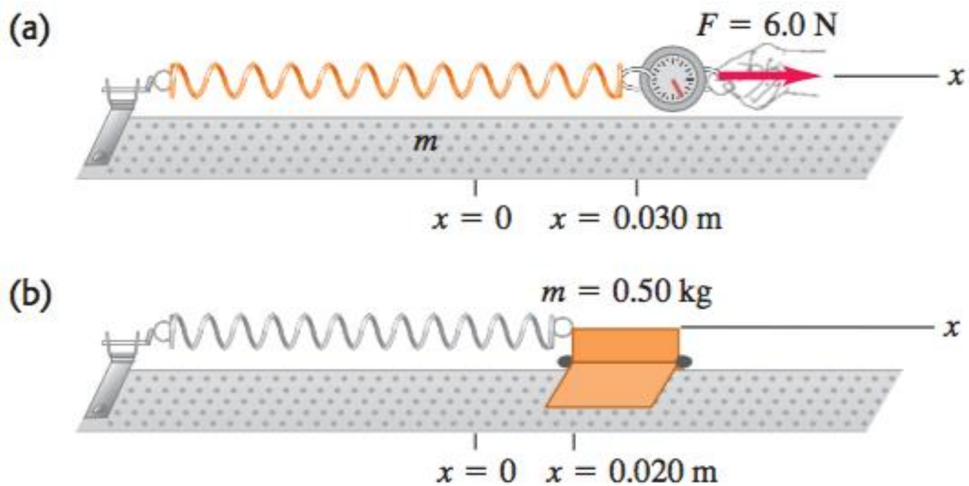

- If it were not SHM, it would be impossible to make familiar types of mechanical and electronic clocks run accurately or to play most musical instruments in tune.
- If you encounter an oscillating body with a period that *does* depend on the amplitude, the oscillation is *not* simple harmonic motion.

# Example 14.2

A spring is mounted horizontally, with its left end fixed. A spring balance attached to the free end and pulled toward the right (Fig. 14.8a) indicates that the stretching force is proportional to the displacement, and a force of 6.0 N causes a displacement of 0.030 m. We replace the spring balance with a 0.50-kg glider, pull it 0.020 m to the right along a frictionless air track, and release it from rest (Fig. 14.8b). (a) Find the force constant $k$ of the spring. (b) Find the angular frequency $\omega$, frequency $f$, and period $T$ of the resulting oscillation.



# Example 14.2




**EXECUTE:** (a) When $x = 0.030$ m, the force the spring exerts on the spring balance is $F_x = -6.0$ N. From Eq. (14.3),

$$k = -\frac{F_x}{x} = -\frac{-6.0\ \text{N}}{0.030\ \text{m}} = 200\ \text{N/m} = 200\ \text{kg/s}^2$$

(b) From Eq. (14.10), with $m = 0.50$ kg,

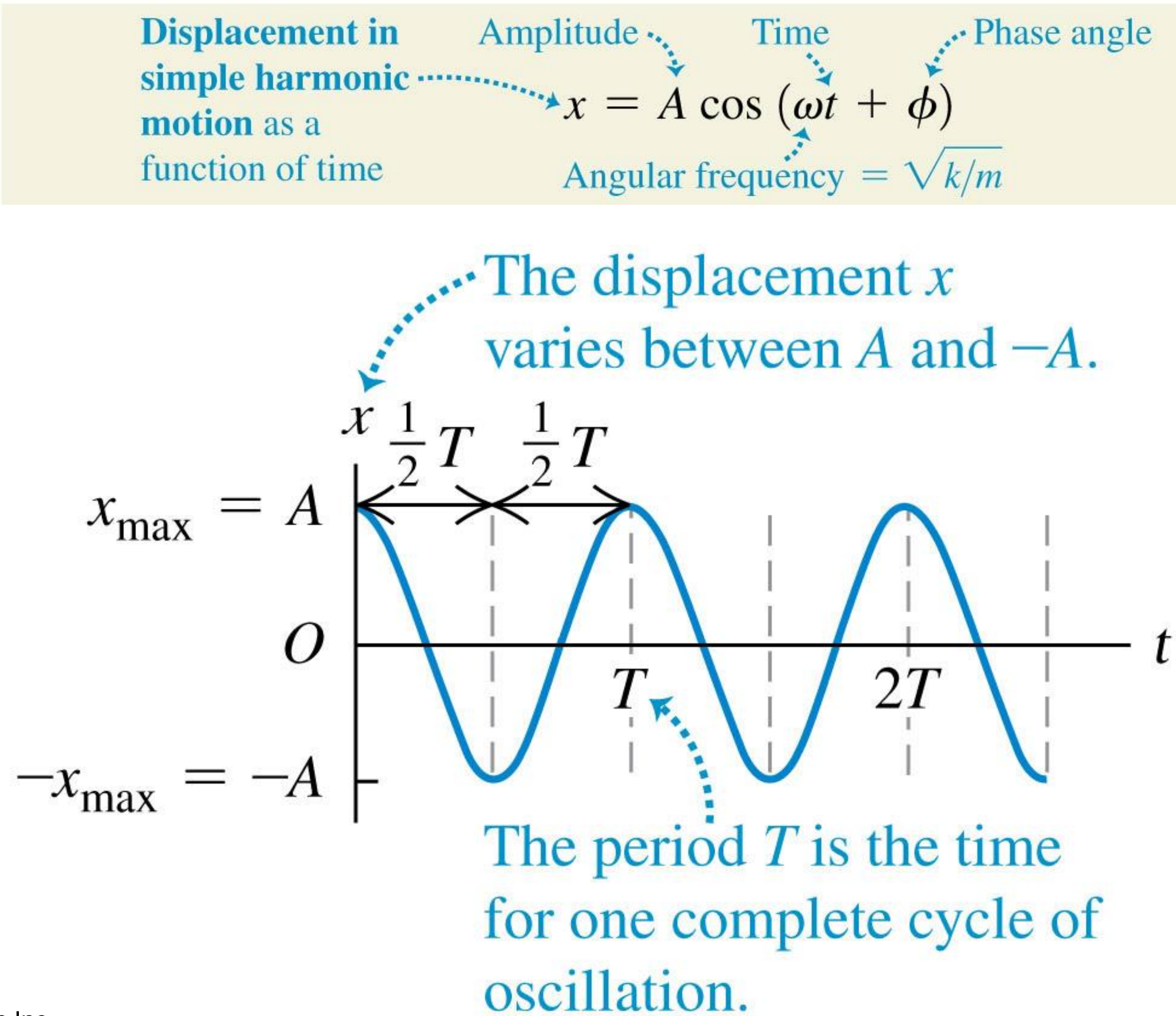
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\ \text{kg/s}^2}{0.50\ \text{kg}}} = 20\ \text{rad/s}$$

$$f = \frac{\omega}{2\pi} = \frac{20\ \text{rad/s}}{2\pi\ \text{rad/cycle}} = 3.2\ \text{cycle/s} = 3.2\ \text{Hz}$$

$$T = \frac{1}{f} = \frac{1}{3.2\ \text{cycle/s}} = 0.31\ \text{s}$$

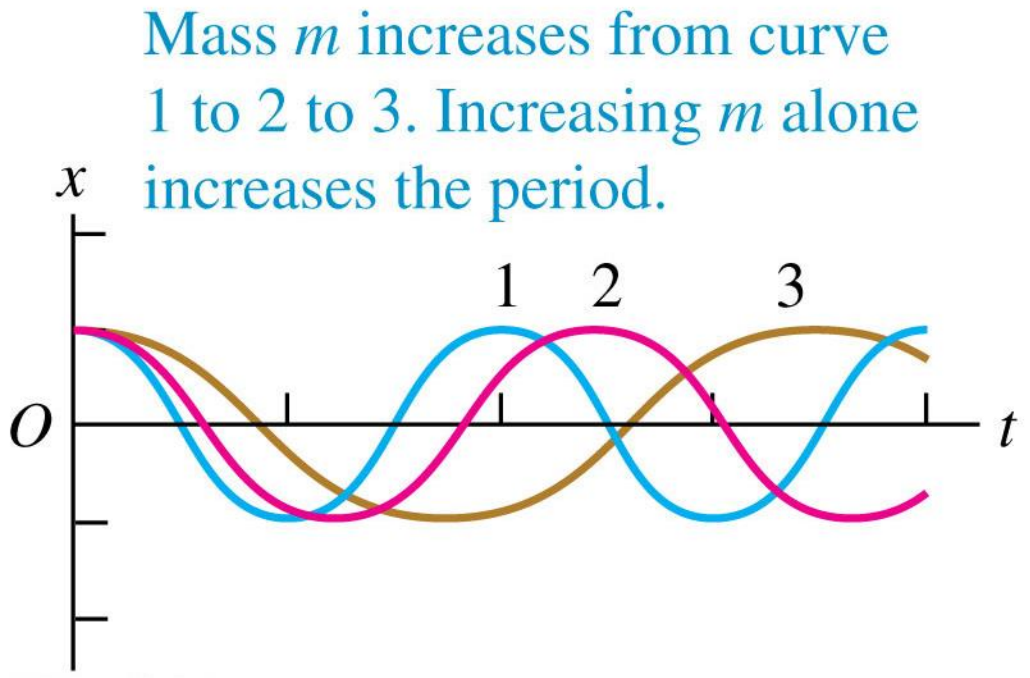
# Displacement as a function of time in SHM

- The displacement for SHM is *a periodic, sinusoidal function of time.*



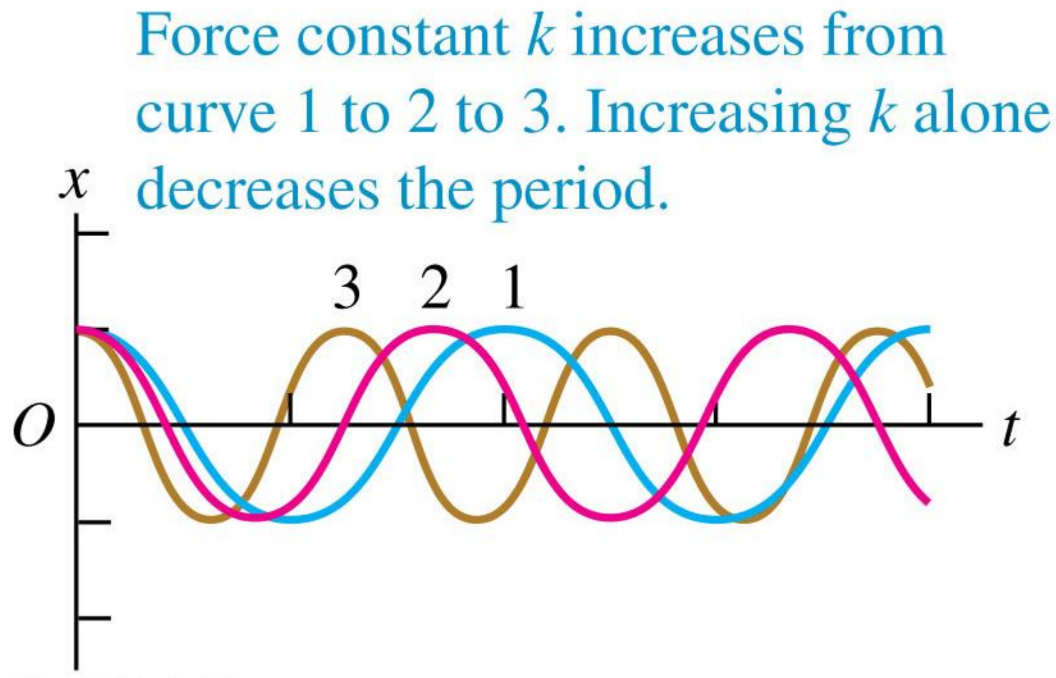
# Displacement as a function of time in SHM

- Increasing *m* with the same *A* and *k* increases the period of the displacement vs time graph.



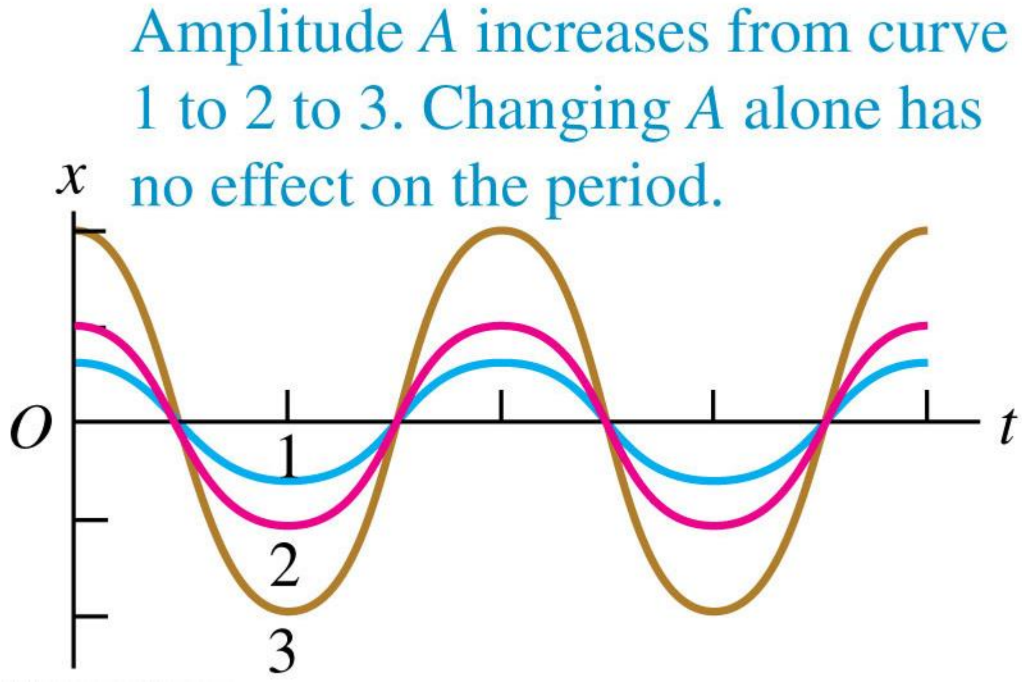
# Displacement as a function of time in SHM

- Increasing *k* with the same *A* and *m* decreases the period of the displacement vs time graph.



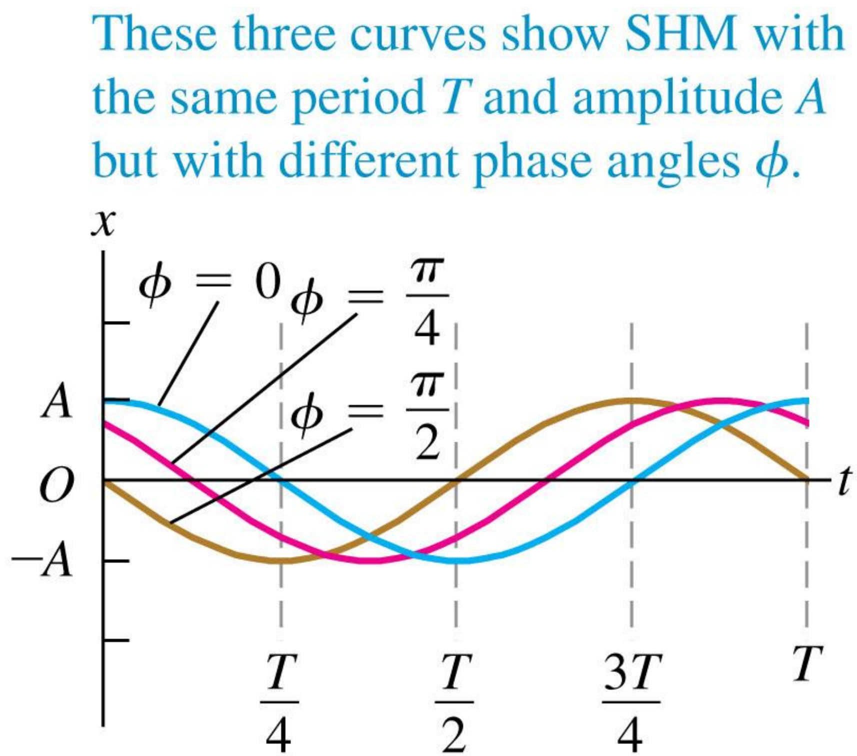
# Displacement as a function of time in SHM

- Increasing $A$ with the same $m$ and $k$ does not change the period of the displacement vs time graph.



# Displacement as a function of time in SHM

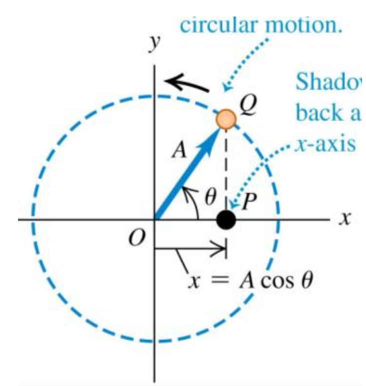
- Increasing $\phi$ with the same $A$, $m$, and $k$ only shifts the displacement vs time graph to the left.



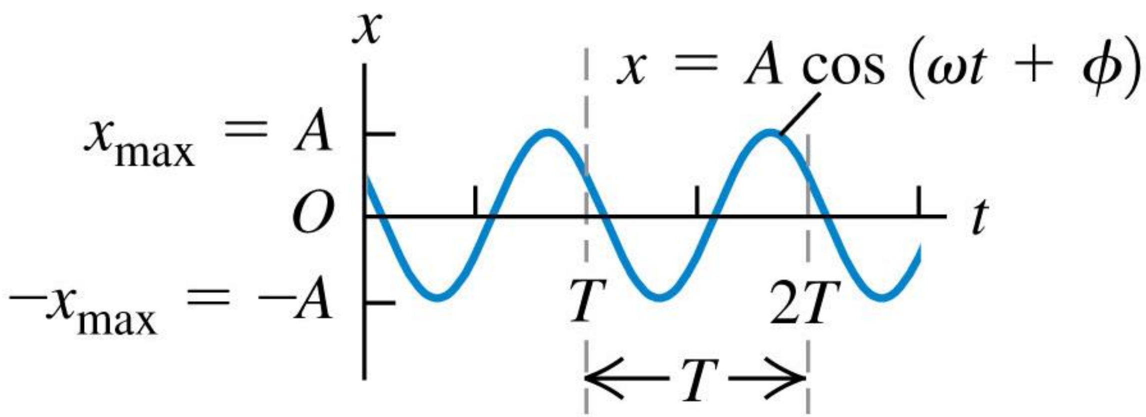

- The constant $\phi$ is called the phase angle. It tells us at what point in the cycle the motion was at $t = 0$ (equivalent to where around the circle the point $Q$ was at $t = 0$)

- We denote the position at $t = 0$ by $x_0$.

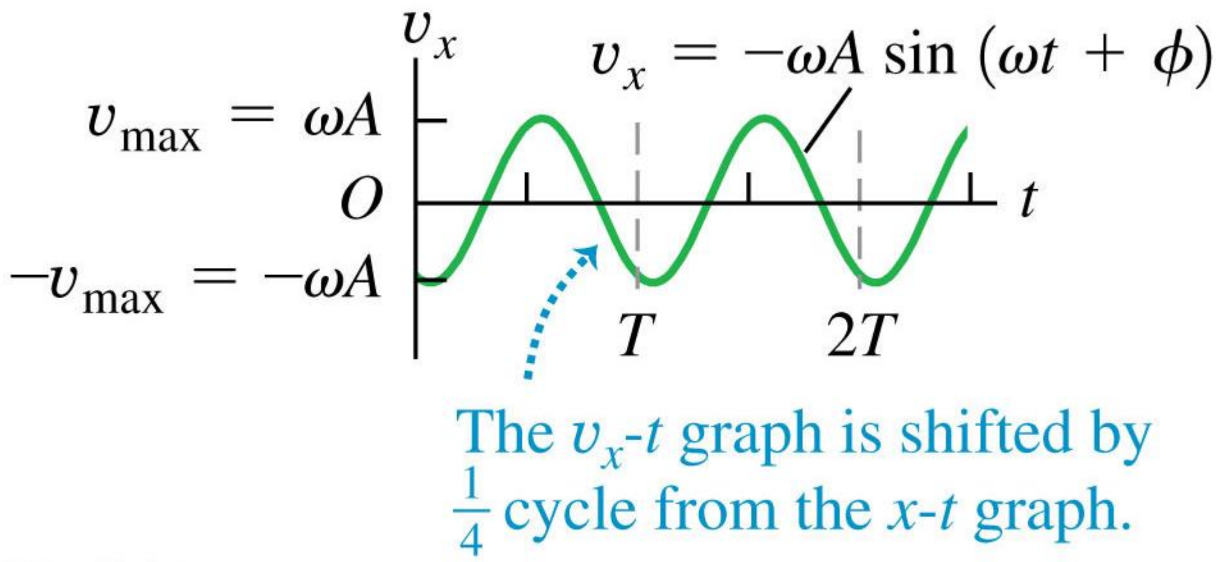
$$x_0 = A\cos\phi$$



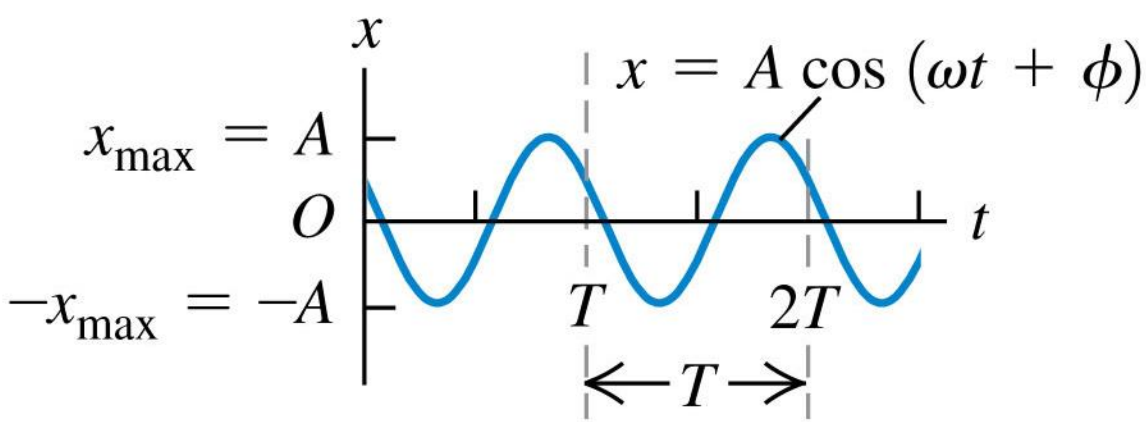
# Graphs of displacement and velocity for SHM



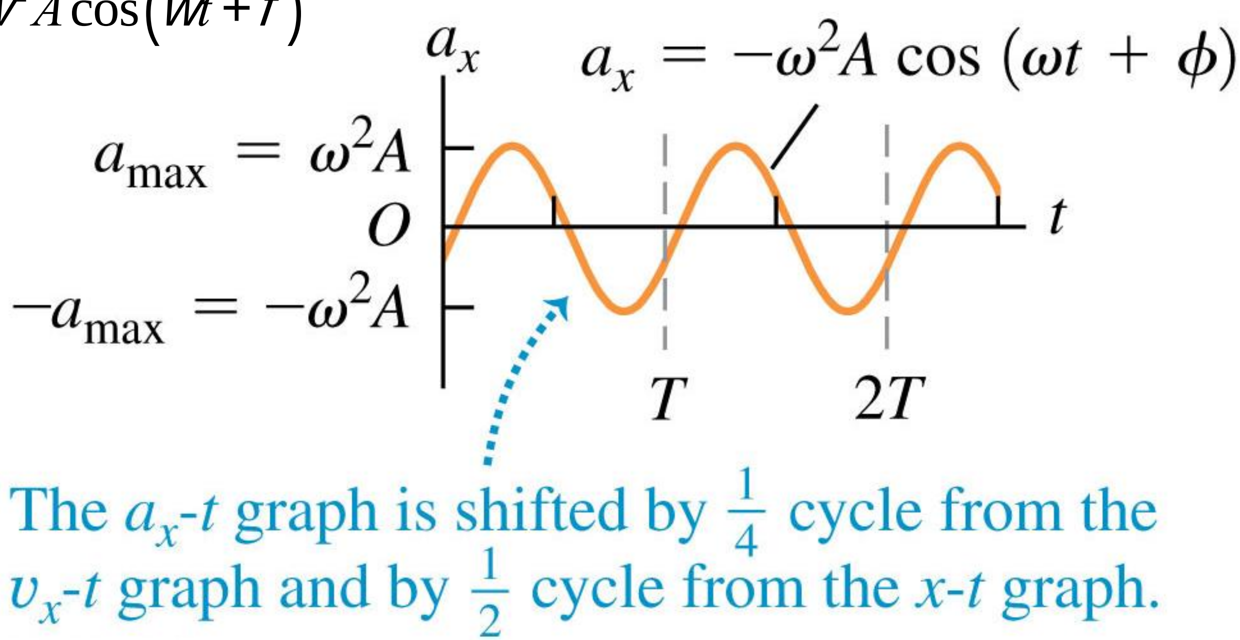

$$v_x = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$



# Graphs of displacement and acceleration for SHM




$$a_x = \frac{dv_x}{dt} = -\omega^2 A\cos(\omega t + \phi)$$



# Equations to describe SHM

$$x = A\cos(\omega t + \phi) \qquad v_x = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$a_x = \frac{dv_x}{dt} = -\omega^2 A\cos(\omega t + \phi)$$

Phase Angle

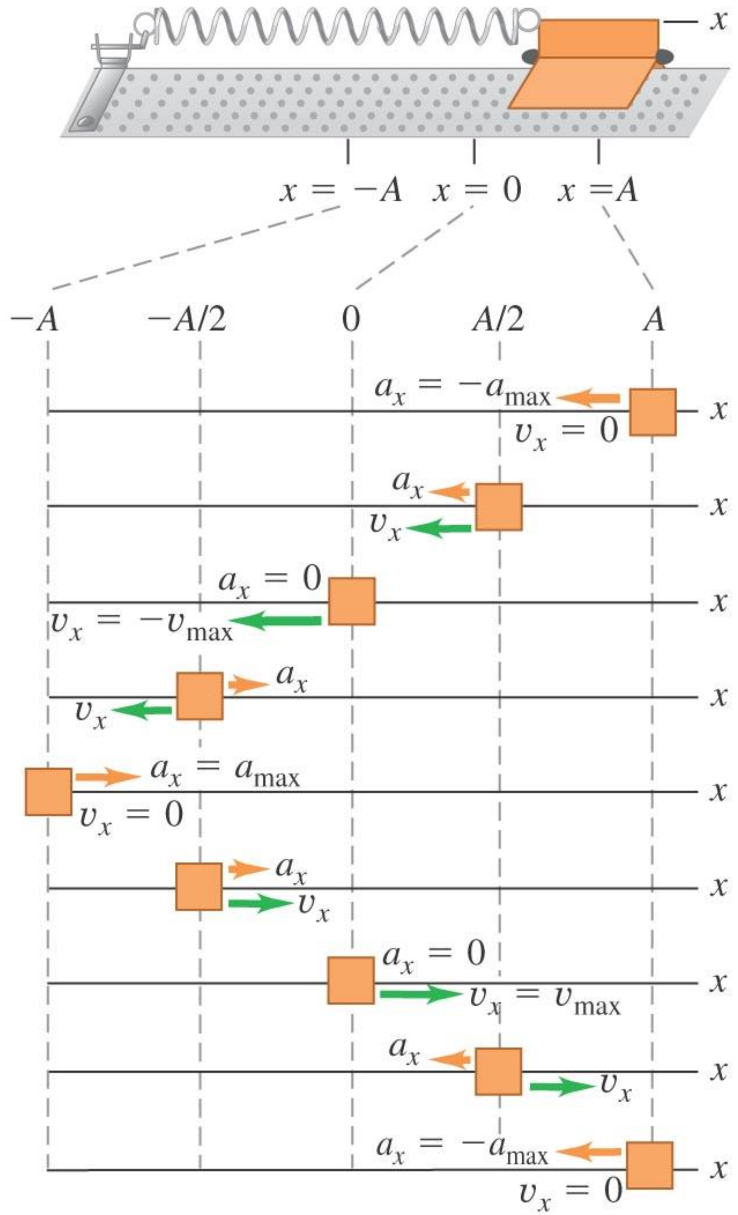
At $t = 0$, $\dfrac{v_{0x}}{x_0} = -\omega \tan\phi$ or, $\phi = \arctan\left(-\dfrac{v_{0x}}{\omega x_0}\right)$

At $t = 0$, $x_0^2 = A^2\cos^2\phi \qquad \dfrac{v_{0x}}{\omega} = -A\sin\phi$

So, $A = \sqrt{x_0^2 + \left(\dfrac{v_{0x}}{\omega}\right)^2}$

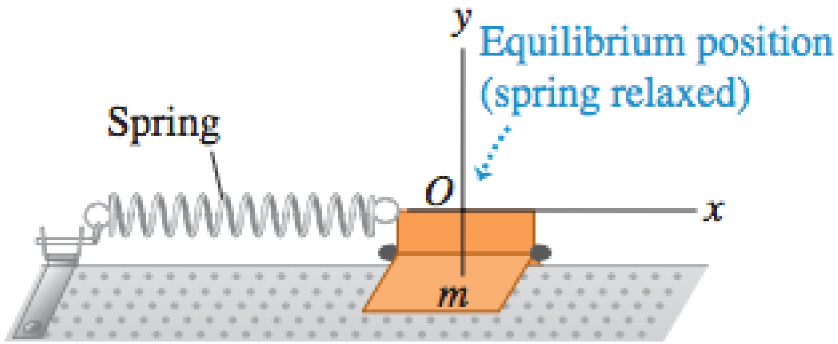
# Behavior of $v_x$ and $a_x$ during one cycle

- Figure 14.13 at the right shows how $v_x$ and $a_x$ vary during one cycle.



# Another Example

**Test Your Understanding of Section 14.1** A body like that shown in Fig. 14.2 oscillates back and forth. For each of the following values of the body's $x$-velocity $v_x$ and $x$-acceleration $a_x$, state whether its displacement $x$ is positive, negative, or zero. (a) $v_x > 0$ and $a_x > 0$; (b) $v_x > 0$ and $a_x < 0$; (c) $v_x < 0$ and $a_x > 0$; (d) $v_x < 0$ and $a_x < 0$; (e) $v_x = 0$ and $a_x < 0$; (f) $v_x > 0$ and $a_x = 0$.



Q14.1

An object on the end of a spring is oscillating in simple harmonic motion. If the amplitude of oscillation is doubled, how does this affect the oscillation period $T$ and the object's maximum speed $v_{max}$?

A. $T$ and $v_{max}$ both double.

B. $T$ remains the same and $v_{max}$ doubles.

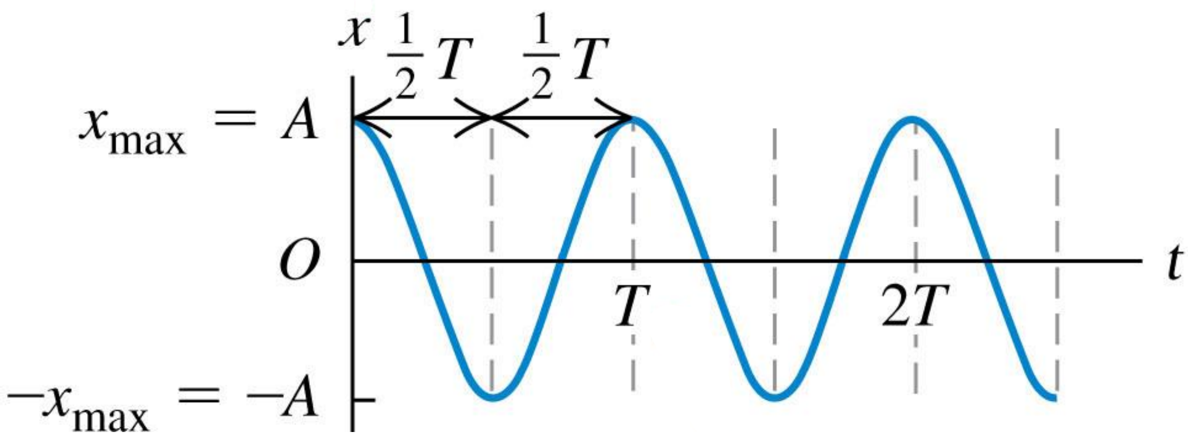
C. $T$ and $v_{max}$ both remain the same.

D. $T$ doubles and $v_{max}$ remains the same.

E. $T$ remains the same and $v_{max}$ increases by a factor of $\sqrt{2}$.

Q14.2

This is an *x*-*t* graph for an object in simple harmonic motion. At which of the following times does the object have the *most negative velocity* $v_x$?




A. $t = T/4$

B. $t = T/2$

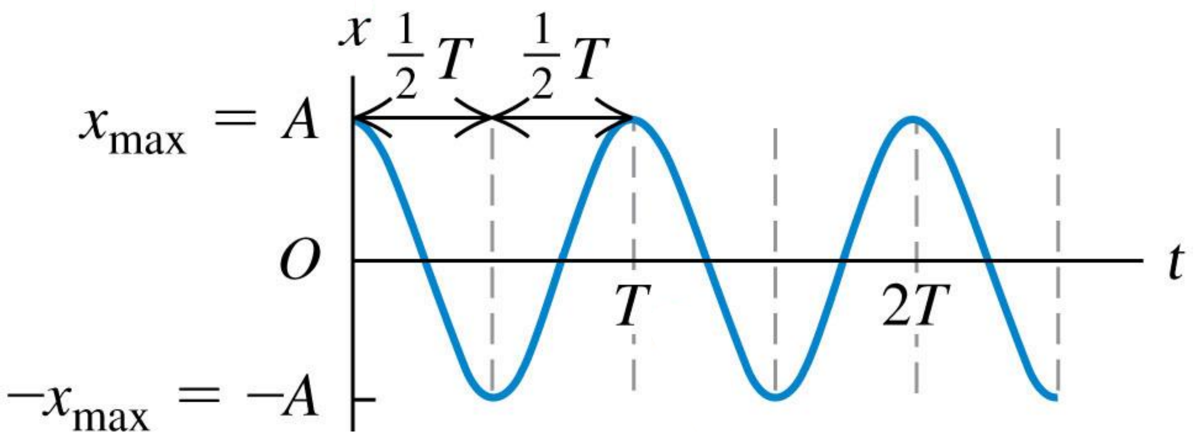
C. $t = 3T/4$

D. $t = T$

E. Two of the above are tied for most negative velocity.

Q14.3

This is an *x*-*t* graph for an object in simple harmonic motion. At which of the following times does the object have the *most negative acceleration* $a_x$?




A. $t = T/4$

B. $t = T/2$

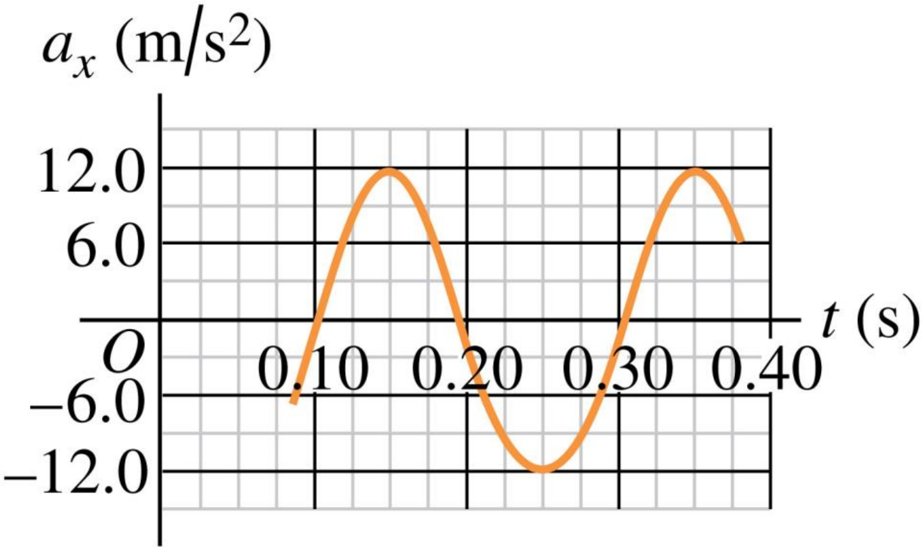
C. $t = 3T/4$

D. $t = T$

E. Two of the above are tied for most negative acceleration.

Q14.4

This is an $a_x$-$t$ graph for an object in simple harmonic motion. At which of the following times does the object have the *most negative displacement x*?




A. $t$ = 0.10 s

B. $t$ = 0.15 s

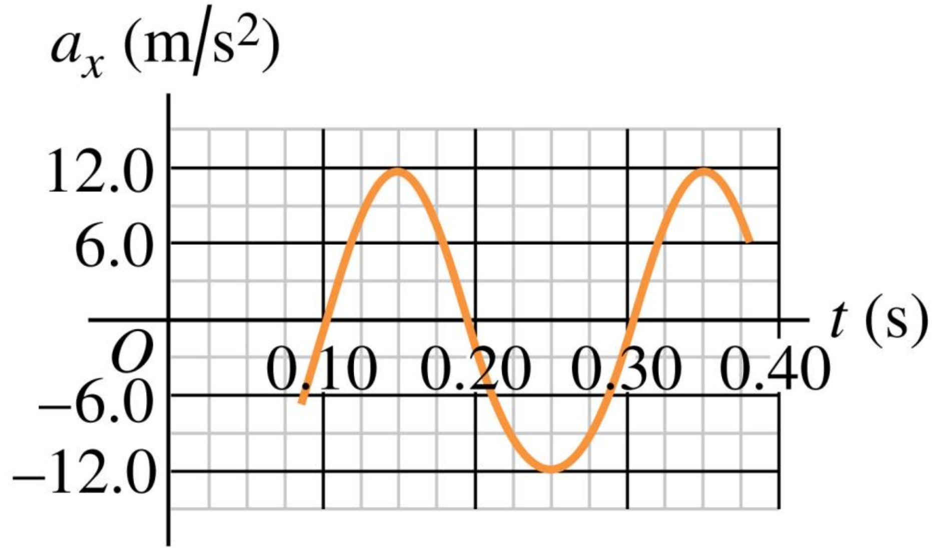
C. $t$ = 0.20 s

D. $t$ = 0.25 s

E. Two of the above are tied for most negative displacement.

Q14.5

This is an $a_x$-$t$ graph for an object in simple harmonic motion. At which of the following times does the object have the *most negative velocity* $v_x$?




A. $t$ = 0.10 s

B. $t$ = 0.15 s

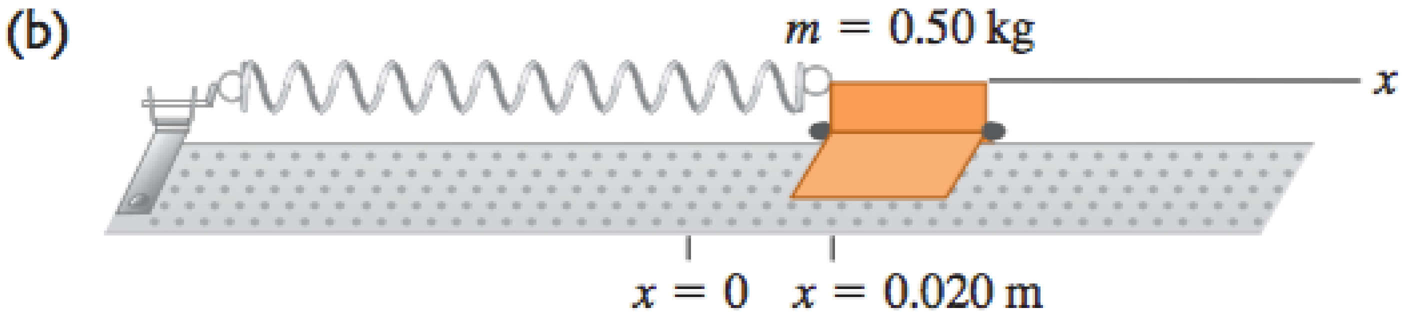
C. $t$ = 0.20 s

D. $t$ = 0.25 s

E. Two of the above are tied for most negative velocity.

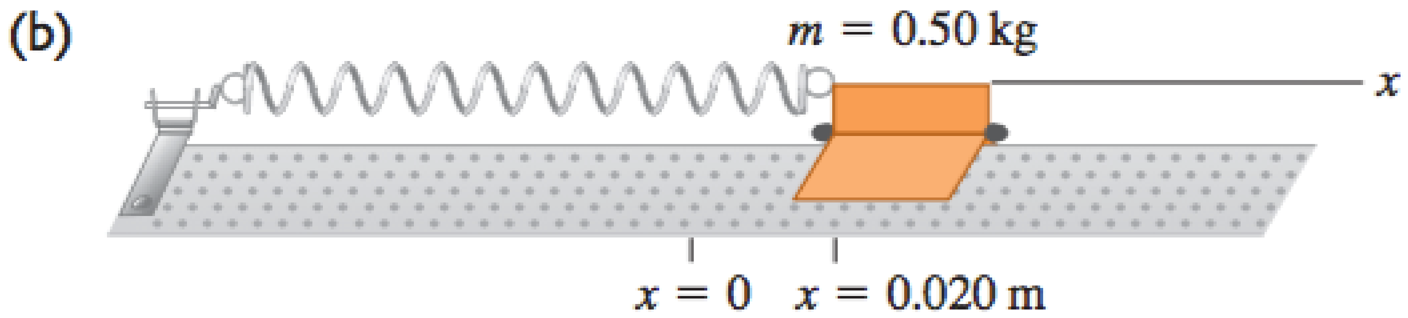
# Example 14.3

$k = 200$ N/m




We give the glider of Example 14.2 an initial displacement $x_0 = +0.015$ m and an initial velocity $v_{0x} = +0.40$ m/s. (a) Find the period, amplitude, and phase angle of the resulting motion. (b) Write equations for the displacement, velocity, and acceleration as functions of time.

# Example 14.3




$k = 200$ N/m, $m = 0.50$ kg, $v_{0x} = +0.40$ m/s

**EXECUTE:** (a) In SHM the period and angular frequency are *properties of the system* that depend only on $k$ and $m$, not on the amplitude, and so are the same as in Example 14.2 ($T = 0.31$ s and $\omega = 20$ rad/s). From Eq. (14.19), the amplitude is

$$A = \sqrt{x_0^2 + \frac{v_{0x}^2}{\omega^2}} = \sqrt{(0.015\ \text{m})^2 + \frac{(0.40\ \text{m/s})^2}{(20\ \text{rad/s})^2}} = 0.025\ \text{m}$$

We use Eq. (14.18) to find the phase angle:

$$\phi = \arctan\left(-\frac{v_{0x}}{\omega x_0}\right)$$

$$= \arctan\left(-\frac{0.40\ \text{m/s}}{(20\ \text{rad/s})(0.015\ \text{m})}\right) = -53° = -0.93\ \text{rad}$$

$$x = (0.025\ \text{m})\cos[(20\ \text{rad/s})t - 0.93\ \text{rad}]$$

$$v_x = -(0.50\ \text{m/s})\sin[(20\ \text{rad/s})t - 0.93\ \text{rad}]$$

$$a_x = -(10\ \text{m/s}^2)\cos[(20\ \text{rad/s})t - 0.93\ \text{rad}]$$

# Energy in SHM

- The total mechanical energy $E = K + U$ is conserved in SHM:

$$E = 1/2\, mv_x^2 + 1/2\, kx^2 = 1/2\, kA^2 = \text{constant}$$

- Verification:

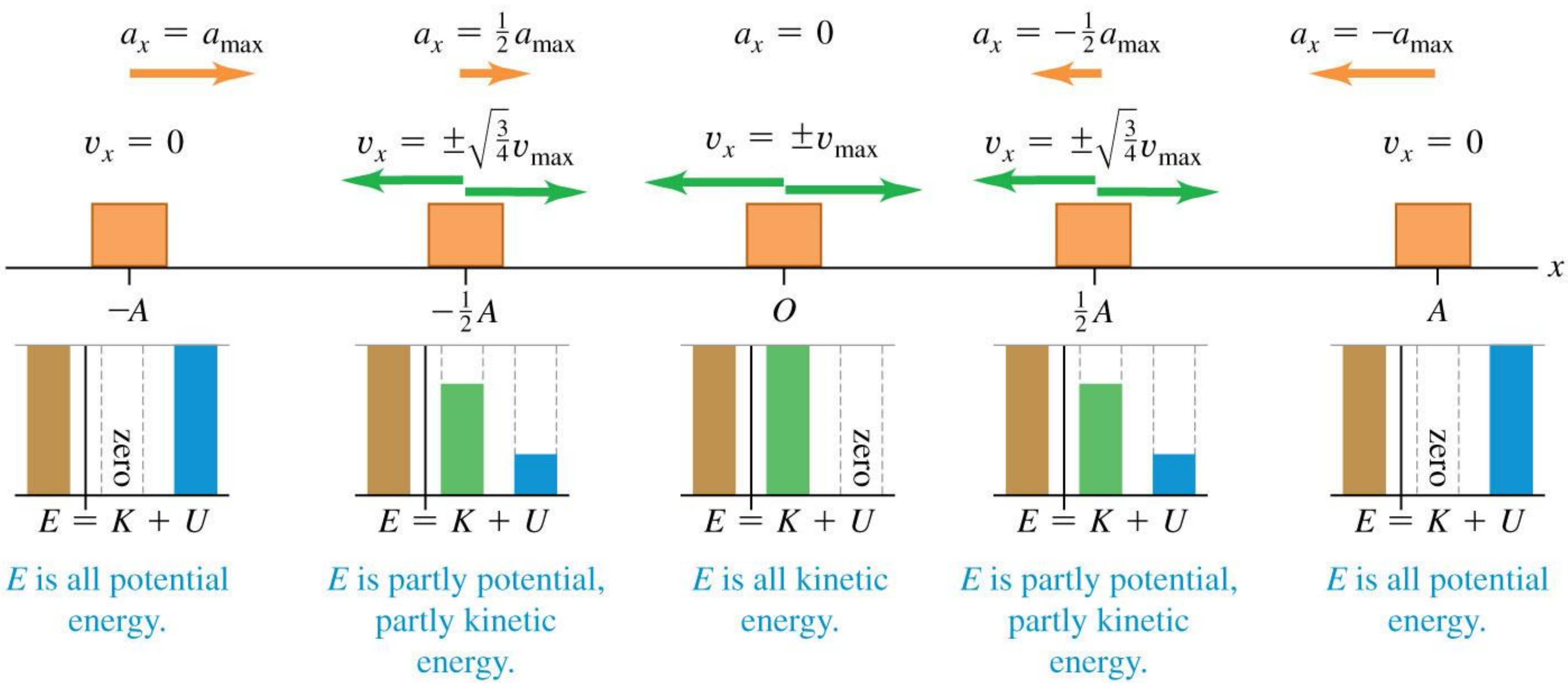
$$\begin{aligned} E &= \tfrac{1}{2}mv_x{}^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}m[-\omega A\sin(\omega t + \phi)]^2 + \tfrac{1}{2}k[A\cos(\omega t + \phi)]^2 \\ &= \tfrac{1}{2}kA^2\sin^2(\omega t + \phi) + \tfrac{1}{2}kA^2\cos^2(\omega t + \phi) \\ &= \tfrac{1}{2}kA^2 \end{aligned}$$

- Reason: All forces are conservative forces.

# Energy in SHM

- The total mechanical energy $E = K + U$ is conserved in SHM:

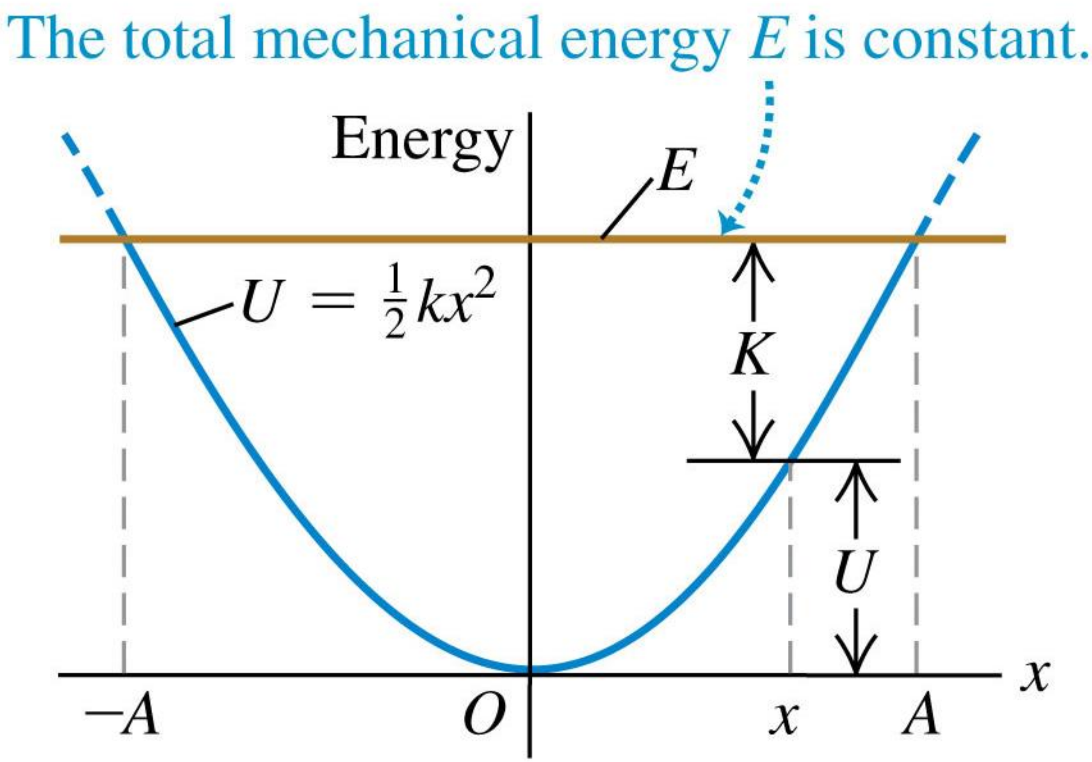
$$E = 1/2\, mv_x^2 + 1/2\, kx^2 = 1/2\, kA^2 = \text{constant}$$




$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2} \qquad v_{max} = \sqrt{\frac{k}{m}}A = wA$$

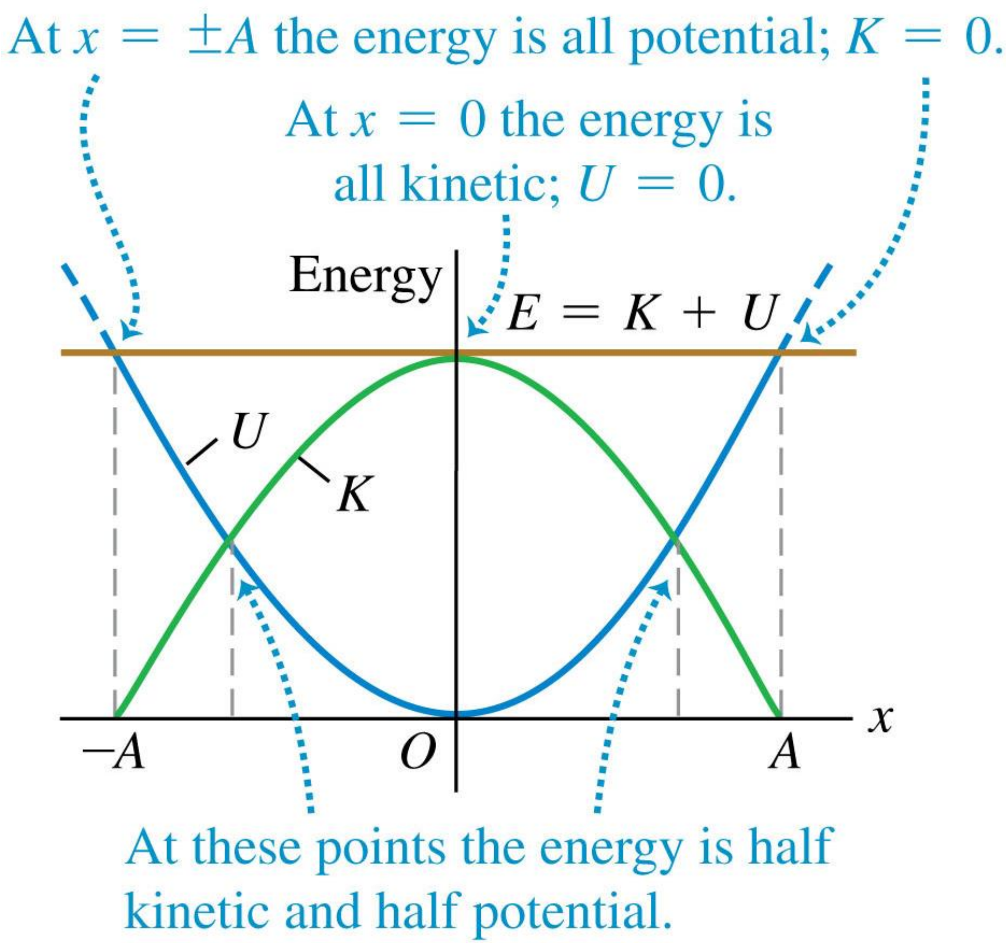
# Energy diagrams for SHM

- The potential energy $U$ and total mechanical energy $E$ for a body in SHM as a function of displacement $x$.



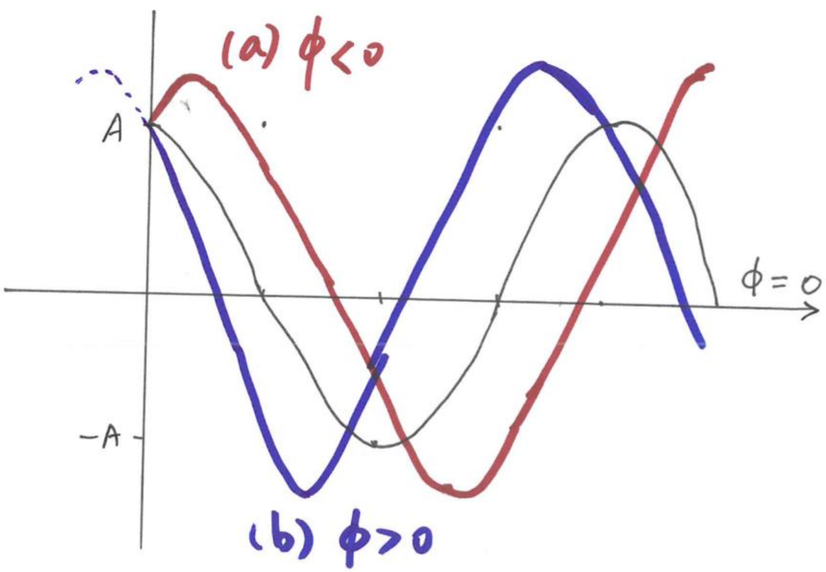
# Energy diagrams for SHM

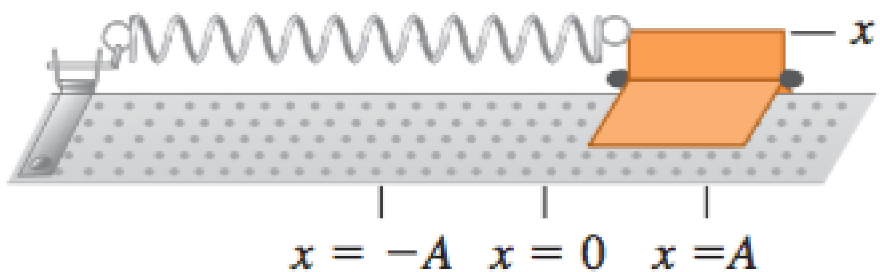
- The potential energy $U$, kinetic energy $K$, and total mechanical energy $E$ for a body in SHM as a function of displacement $x$.

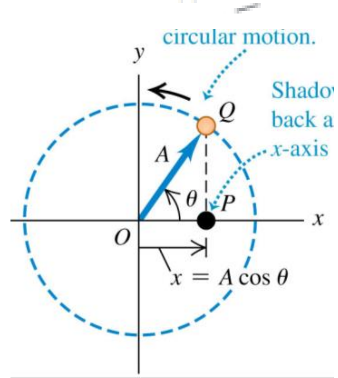


# Another Example

**Test Your Understanding of Section 14.2** A glider is attached to a spring as shown in Fig. 14.13. If the glider is moved to $x = 0.10$ m and released from rest at time $t = 0$, it will oscillate with amplitude $A = 0.10$ m and phase angle $\phi = 0$. (a) Suppose instead that at $t = 0$ the glider is at $x = 0.10$ m and is moving to the right in Fig. 14.13. In this situation is the amplitude greater than, less than, or equal to 0.10 m? Is the phase angle greater than, less than, or equal to zero? (b) Suppose instead that at $t = 0$ the glider is at $x = 0.10$ m and is moving to the left in Fig. 14.13. In this situation is the amplitude greater than, less than, or equal to 0.10 m? Is the phase angle greater than, less than, or equal to zero?

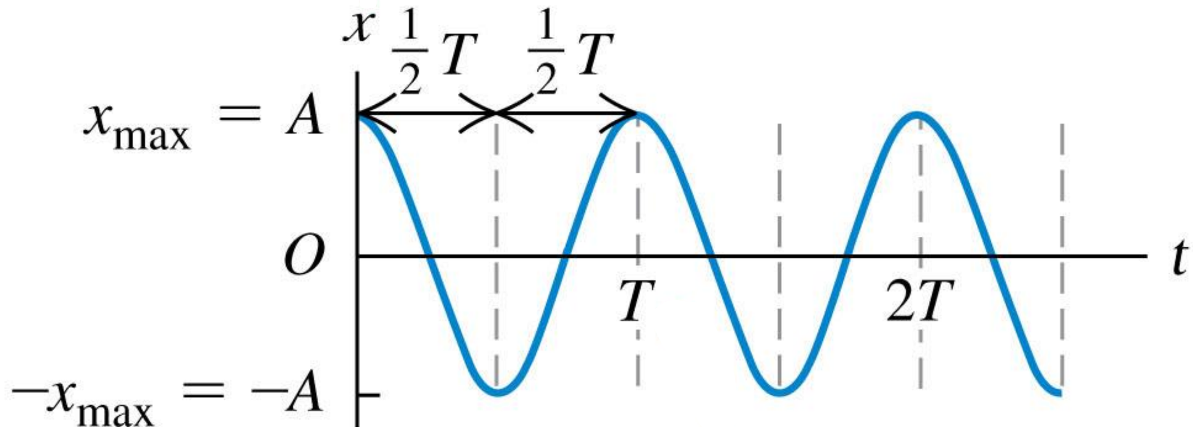


Q14.6

This is an *x*-*t* graph for an object connected to a spring and moving in simple harmonic motion. At which of the following times is the *potential energy* of the spring the greatest?




A. $t = T/8$

B. $t = T/4$

C. $t = 3T/8$

D. $t = T/2$

E. Two of the above are tied for greatest potential energy.

Q14.8

To double the total energy of a mass-spring system oscillating in simple harmonic motion, the amplitude must increase by a factor of

A. 4.

B. $2\sqrt{2} = 2.828.$

C. 2.

D. $\sqrt{2} = 1.414.$

E. $\sqrt[4]{2} = 1.189.$

By what factor will the frequency change due to this amplitude increase?

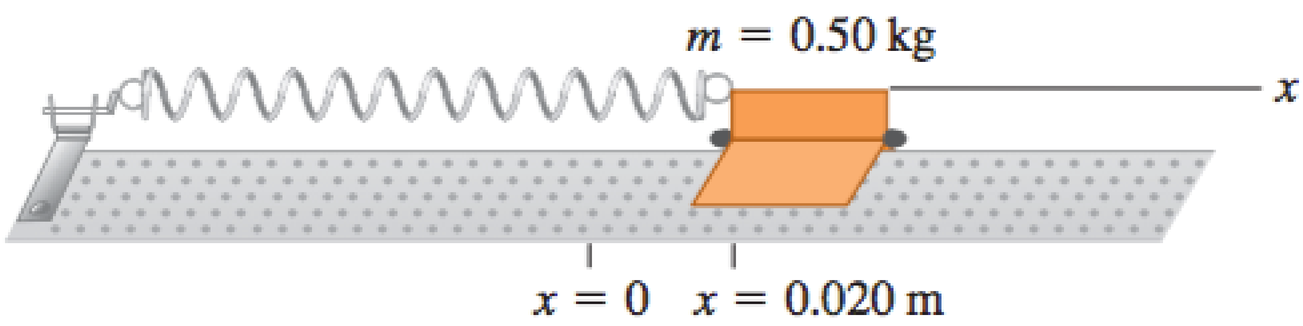
No change

# Example 14.4

(a) Find the maximum and minimum velocities attained by the oscillating glider of Example 14.2. (b) Find the maximum and minimum accelerations. (c) Find the velocity $v_x$ and acceleration $a_x$ when the glider is halfway from its initial position to the equilibrium position $x = 0$. (d) Find the total energy, potential energy, and kinetic energy at this position.

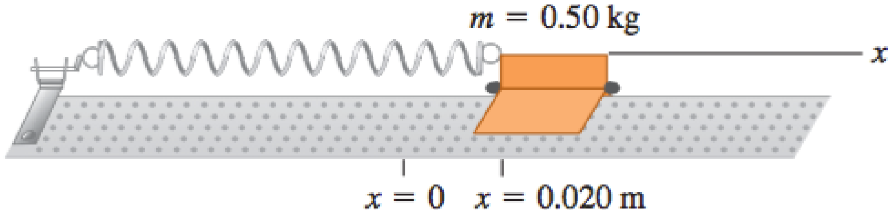
$k = 200$ N/m

(b)



# Example 14.4

$k = 200$ N/m

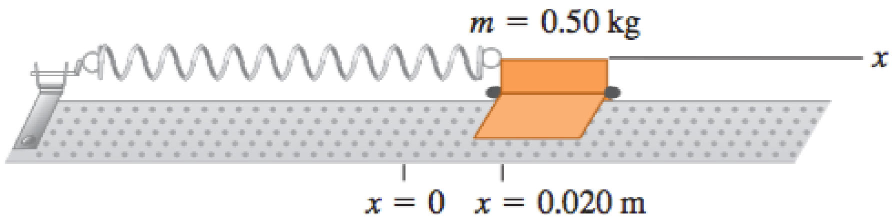



- (a) Conservation of energy: $K + U = E$
- $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = 1/2kA^2$
- $v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$
- Maximum velocity achieved by setting $x = 0$, $v_{\max} = \sqrt{\frac{k}{m}A^2} = 0.40\ \mathrm{m/s}$
- Minimum velocity is most negative velocity, $v_{\min} = -0.40\ \mathrm{m/s}$

- (b) $ma = F = -kx$
- Maximum acceleration is achieved at most negative $x$
- $a_{\max} = -\frac{k}{m}(-A) = 8.0\ \mathrm{m/s^2}$
- $a_{\min} = -8.0\ \mathrm{m/s^2}$

# Example 14.4

$k = 200\ \text{N/m}$

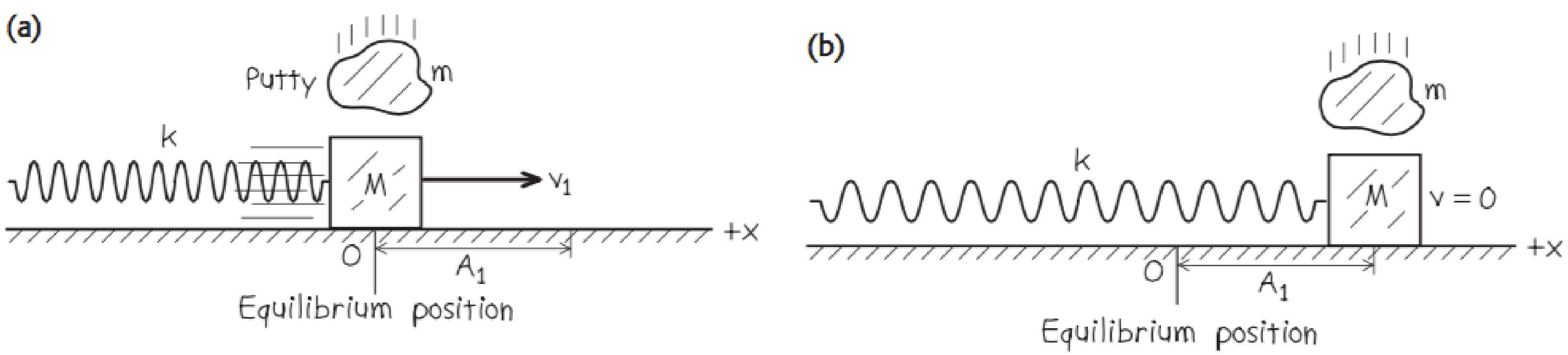



- (c) Conservation of energy: $K + U = E$
- $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$
- $v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$, since $x = 0.01$ m, $v = -0.35\ \text{m/s}$
- $v$ is negative because $x$ is decreasing
- $a = -\frac{k}{m}x = -4.0\ \text{m/s}^2$

- (d) $E = \frac{1}{2}kA^2 = 0.040$ J
- $U = \frac{1}{2}kx^2 = 0.010$ J
- $K = E - U = 0.030$ J

# Energy and Momentum in SHM (Example 14.5)

A block of mass $M$ attached to a horizontal spring with force constant $k$ is moving in SHM with amplitude $A_1$. As the block passes through its equilibrium position, a lump of putty of mass $m$ is dropped from a small height and sticks to it. (a) Find the new amplitude and period of the motion. (b) Repeat part (a) if the putty is dropped onto the block when it is at one end of its path.



# Energy and Momentum in SHM (Example 14.5)

**EXECUTE:** (a) Before the collision the total mechanical energy of the block and spring is $E_1 = \frac{1}{2}kA_1^2$. The block is at $x = 0$, so $U = 0$ and the energy is purely kinetic (Fig. 14.16a). If we let $v_1$ be the speed of the block at this point, then $E_1 = \frac{1}{2}kA_1^2 = \frac{1}{2}Mv_1^2$ and

$$v_1 = \sqrt{\frac{k}{M}}A_1$$

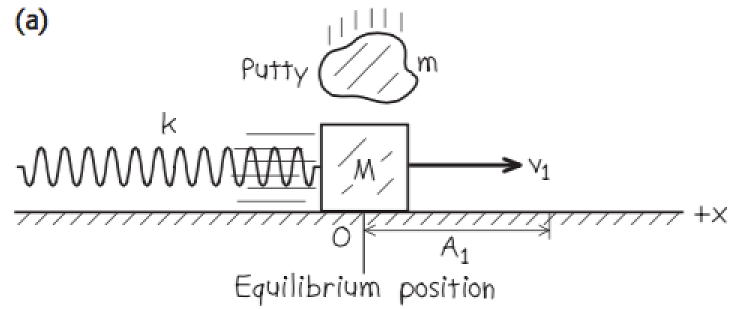
During the collision the $x$-component of momentum of the block–putty system is conserved. (Why?) Just before the collision this component is the sum of $Mv_1$ (for the block) and zero (for the putty). Just after the collision the block and putty move together with speed $v_2$, so their combined $x$-component of momentum is $(M + m)v_2$. From conservation of momentum,

$$Mv_1 + 0 = (M + m)v_2 \qquad \text{so} \qquad v_2 = \frac{M}{M + m}v_1$$

We assume that the collision lasts a very short time, so that the block and putty are still at the equilibrium position just after the collision. The energy is still purely kinetic but is *less* than before the collision:

$$E_2 = \frac{1}{2}(M + m)v_2^2 = \frac{1}{2}\frac{M^2}{M + m}v_1^2$$

$$= \frac{M}{M + m}\left(\frac{1}{2}Mv_1^2\right) = \left(\frac{M}{M + m}\right)E_1$$




Since $E_2 = \frac{1}{2}kA_2^2$, where $A_2$ is the amplitude after the collision, we have

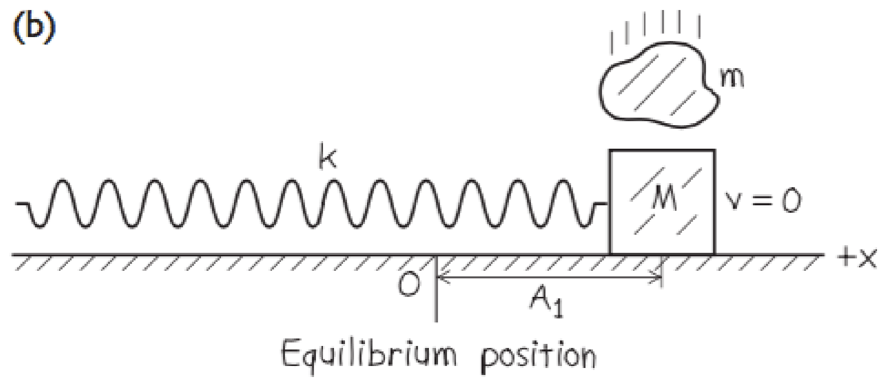
$$\frac{1}{2}kA_2^2 = \left(\frac{M}{M + m}\right)\frac{1}{2}kA_1^2$$

$$A_2 = A_1\sqrt{\frac{M}{M + m}}$$

From Eq. (14.12), the period of oscillation after the collision is

$$T_2 = 2\pi\sqrt{\frac{M + m}{k}}$$

# Energy and Momentum in SHM (Example 14.5)




(b) When the putty falls, the block is instantaneously at rest (Fig. 14.16b). The $x$-component of momentum is zero both before and after the collision. The block and putty have zero kinetic energy just before and just after the collision. The energy is all potential energy stored in the spring, so adding the putty has *no effect* on the mechanical energy. That is, $E_2 = E_1 = \frac{1}{2}kA_1^2$, and the amplitude is unchanged: $A_2 = A_1$. The period is again $T_2 = 2\pi\sqrt{(M+m)/k}$.

**EVALUATE:** Energy is lost in part (a) because the putty slides against the moving block during the collision, and energy is dissipated by kinetic friction. No energy is lost in part (b), because there is no sliding during the collision.