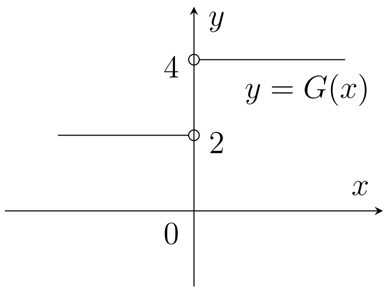
MA1300 Self Practice # 1, Solutions

1. (P21, #45) Find the domain and sketch the graph of the function

$$G(x) = \frac{3x + |x|}{x}.$$

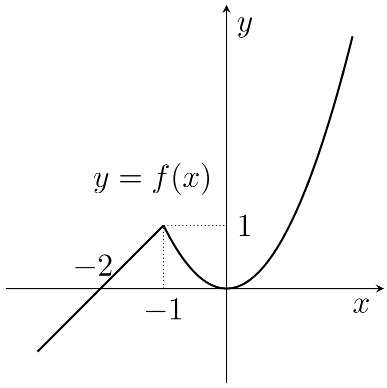
Solution. When $x = 0$, $G(x)$ has no definition; when $x \neq 0$, $G(x)$ is well defined. So the domain of G is $(-\infty, 0) \cup (0, \infty)$. The sketch of G is



2. (P21, #49) Find the domain and sketch the graph of the function

$$f(x) = \begin{cases} x + 2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1. \end{cases}$$

Solution. The domain of f is $\mathbb{R}$. The sketch of f is



3. (P22, #67) In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.

(a) Sketch the graph of the tax rate R as a function of the income I.

(b) How much tax is assessed on an income of \$14,000? On \$26,000?

(c) Sketch the graph of the total assessed tax T as a function of the income I.

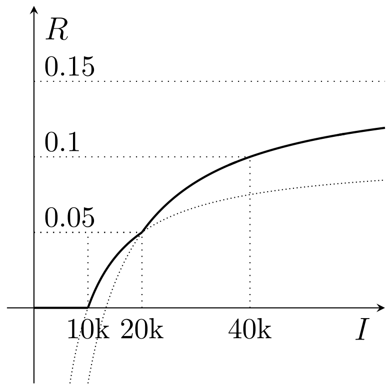
Solution.

(a) R is a piecewise defined function

$$R = \begin{cases} 0 & \text{if } I \le 10,000 \\ \frac{(I-10{,}000)\times 10\%}{I} & \text{if } 10,000 < I \le 20,000 \\ \frac{10{,}000\times 10\% + (I-20{,}000)\times 15\%}{I} & \text{if } I > 20,000 \end{cases}$$

$$= \begin{cases} 0 & \text{if } I \le 10,000 \\ 0.1 - \frac{1{,}000}{I} & \text{if } 10,000 < I \le 20,000 \\ 0.15 - \frac{2{,}000}{I} & \text{if } I > 20,000 \end{cases}.$$

Here is the sketch of R

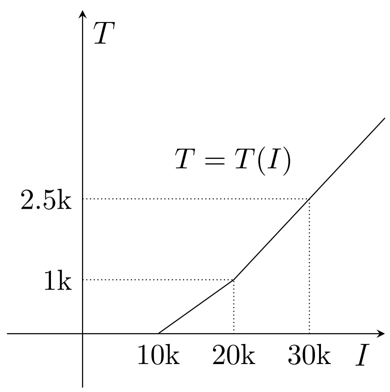


(b) $R(14,000) = 400,\ R(26,000) = 1,900.$

(c)

$$T = \begin{cases} 0 & \text{if } I \le 10,000 \\ (I-10,000) \times 10\% & \text{if } 10,000 < I \le 20,000 \\ 10,000 \times 10\% + (I - 20,000) \times 15\% & \text{if } I > 20,000 \end{cases}$$

$$= \begin{cases} 0 & \text{if } I \le 10,000 \\ 0.1I - 1,000 & \text{if } 10,000 < I \le 20,000 \\ 0.15I - 2,000 & \text{if } I > 20,000 \end{cases}.$$



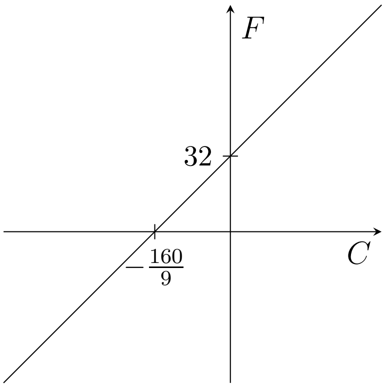
4. (P33, #13) The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the linear function $F = \frac{9}{5}C + 32$.

(a) Sketch a graph of this function.

(b) What is the slope of the graph and what does it represent? What is the F-intercept and what does it represent?

Solution.

(a) We sketch the function below.



(b) Slope $= \frac{9}{5}$, which represents the rate of change of F with respect to C. The F-intercept is 32, which means $0°C$ is equivalent to $32°F$

5. (P34, #15) Biologists have noticed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 112 chirps per minutes at $20°C$ and 180 chirps per minute at $29°C$.

(a) Find a linear equation that models the temperature T as a function of the number of chirps per minute N.

(b) What is the slope of the graph? What does it represent?

(c) If the crickets are chirping at 150 chirps per minute, estimate the temperature.

Solution.

(a) We write $T = aN + b$ to give

$$\begin{cases} 20 = 112a + b \\ 29 = 190a + b \end{cases},$$

so $a = \frac{9}{68}$, $b = \frac{88}{17}$, and $T = \frac{9}{68}N + \frac{88}{17}$.

(b) The slope of the graph is $\frac{9}{68}$, which represents the rate of change of temperature with respect to the number of chirps per minute.

(c) The temperature is $T = \frac{9}{68} \times 150 + \frac{88}{17} \approx 25.0(°C)$.

6. (P43, #26) A variable star is one whose brightness alternately increases and decreases. For the most visible variable star, Delta Cephei, the time between periods of maximum brightness is 5.4 days, the average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude. Find a function that models the brightness of Delta Cephei as a function of time.

Solution.

The function could be

$$B = 4.0 + 0.35 \sin\left(\frac{2\pi t}{5.4}\right).$$

7. (P44, #53) A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.

(a) Express the radius r of this circle as a function of the time t (in seconds).

(b) If A is the area of this circle as a function of the radius, find $A \circ r$ and interpret it.

Solution.

(a) $r = 60t$.

(b) $A \circ r = \underset{3600}{\cancel{360}}\pi t^2$, which is the area of the circular ripple at time t.

8. (P44, #57) The **Heaviside function** is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

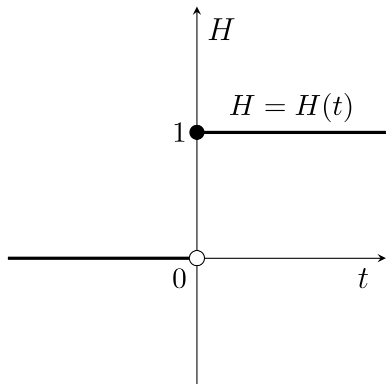
(a) Sketch the graph of the Heaviside function.

(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.

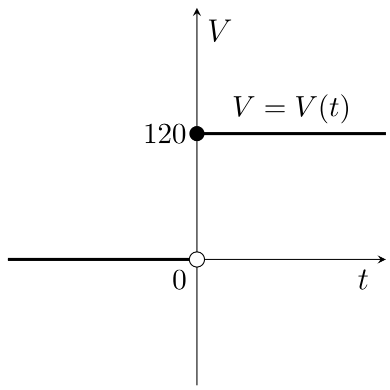
(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

Solution.

(a) Here is the sketch of the function.

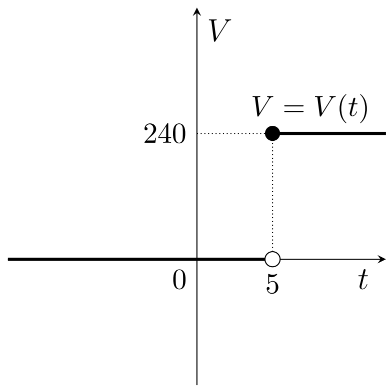


(b) Here is the sketch of V.



$V(t) = 120H(t)$.

(c) Here is the sketch of V.



$V(t) = 240H(t - 5)$.

9. (P44, #58) The Heaviside function defined in the previous exercise can also be used to define the **ramp function** $y = ctH(t)$, which represents a gradual increase in voltage or current in a circuit.

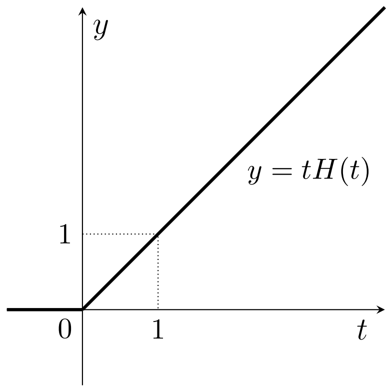
(a) Sketch the graph of the ramp function $y = ctH(t)$.

(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 60$.

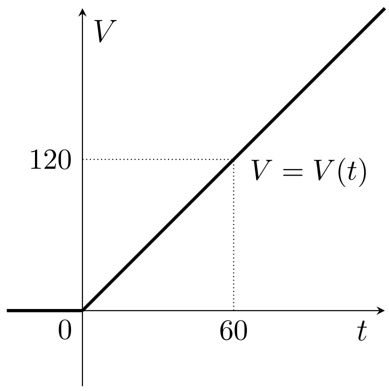
(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 32$.

Solution.

(a) The sketch of $y = tH(t)$:

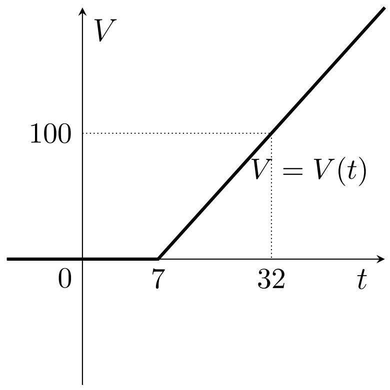


(b) The sketch of V



$V(t) = 2tH(t)$ for $t \leq 60$

(c) The sketch of V



$V(t) = 4(t-7)H(t-7)$ for $t \leq 32$.

10. (P44, #61)

(a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)

(b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

Solution.

$= f(g(x)) = f(2x+1) = (2x+1)^2 + 6$

(a) We write $4x^2 + 4x + 7 = h(x) \cancel{= f(g(x)) = 2g(x) +} 1$ to obtain

$f(x) = x^2 + 6.$

$$\cancel{g(x) = 2x^2 + 2x + 3.}$$

(b) We write $3x^2 + 3x + 2 = h(x) = f(g(x)) = 3g(x) + 5$ to obtain

$$g(x) = x^2 + x - 1.$$

11. (P44, #63) Suppose g is an even function and let $h = f \circ g$. Is h always an even function?
Solution. Yes, since $h(-x) = f(g(-x)) = f(g(x)) = h(x)$.