Academic Honesty Pledge

I pledge that the answers in this exam/quiz are my own and that I will not seek or obtain an unfair advantage in producing these answers. Specifically,

- I will not plagiarize (copy without citation) from any source;
- I will not communicate or attempt to communicate with any other person during the exam/quiz; neither will I give or attempt to give assistance to another student taking the exam/quiz; and
- I will use only approved devices (e.g., calculators) and/or approved device models.

I understand that any act of academic dishonesty can lead to disciplinary action.

Student ID:____

Signature:____

Date:____

CITY UNIVERSITY OF HONG KONG

Course code & title	:	MA3524/3526 Analysis
Session	:	Semester B, 2021-2022
Time Allowed	:	Two hours

This paper has **two** pages. (including this page)

Instructions to candidates:

1. Answer **all** questions.
2. Start each main question on a new page.
3. Show all steps.

Materials, aids & instruments which students are permitted to use during examination:

Non-programmable portable battery operated calculator

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorized materials or aids are found on them.

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1. (a) (15 points) Let $f : (0, \pi) \to \mathbb{R}$ be a bounded continuous function. Show that $\sin(x)f(x)$ is uniformly continuous on $(0, \pi)$.

   (b) (15 points) Let $g : (0, \infty) \to \mathbb{R}$ and $g(x) = \sin(1/x)$. Is g uniformly continuous? Why?

2. (20 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that $f'(a) < f'(b)$. Given a number L such that $f'(a) < L < f'(b)$, show that there exists a value $c \in (a, b)$ such that $f'(c) = L$.

   (Hint: we define a function $g(x) = Lx - f(x)$ for $x \in [a, b]$. Where is the possible maximum of $g(x)$ for $x \in [a, b]$.)

3. (a) (10 points) Let $\{f_n\}$ be a sequence of increasing functions defined on $[0, 1]$ (that is, $f_n(x) \geq f_n(y)$ whenever $1 \geq x \geq y \geq 0$). Suppose $f_n(0) = 0$ or all n, and $\lim_{n\to\infty} f_n(1) = 0$. Show that $\{f_n\}$ converges uniformly to 0.

   (b) (20 points) Assume $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are uniformly convergent sequences of functions on common domain A and $|f_n| + |g_n| \leq M$ for all $n \in \mathbb{N}$, verify that $\{f_n g_n\}_{n=1}^{\infty}$ converges uniformly.

4. (10 points) Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence, and define the set

$$S = \{x \in \mathbf{R} : \text{there exist at most finite number of terms } a_n \text{ satisfying } x < a_n\}.$$

   Show that $s = \inf S$ exists and there exists a subsequences $\{a_{n_k}\}_{k=1}^{\infty}$ converging to s.

5. (10 points) Define that a set $A \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ if, for any two real numbers $a < b$, there exists a point $x \in A$ such that $a < x < b$.

   If $\{O_1, O_2, O_3, ...\}$ is a countable collection of open, dense sets, prove that the intersection $\cap_{n=1}^{\infty} O_n$ is not empty.

   (Hint: applying Nested Interval Property.)

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