Chapter 5 Infinite sequences and series

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5. Infinite sequences and series

So far, we have learned

- limit in Chapter 2,
- derivative in Chapter 3.
- applications of derivatives in Chapter 4.
- derivative on inverse functions, l'hospital's rule in Chapter 6.

In this chapter, we study a little bit advanced application of derivatives on infinite sequences and series, ex. Taylor expansion, ...

Note, we will avoid to involve all parts related to integral (which will be covered in MA1301).

5.1. **Sequences.** Sec 11.1 Exercise: 19, **25, 30**, 35, 43, 57, 61, 67, 71, 79, **80, 82**

In this section, we shall define *sequences*, and observe its *convergence/divergence* behavior.

A **sequence** is a list of numbers written in a definite order:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

and often denoted by $\{a_n\}_{n=1}^{\infty}$.

Ex. The **Fibonacci sequence** $\{a_n\}$ is given by

$$a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2},\ n \geq 3.$$

Definition For a sequence $\{a_n\}$

(1) $\lim_{n\to\infty} a_n = L$, if for every $\varepsilon > 0$ there is an corresponding integer N such that
$$\text{if } n > N \text{ then } |a_n - L| < \varepsilon.$$

(2) $\lim_{n\to\infty} a_n = \infty$, if for every $M > 0$ there is an corresponding integer N such that
$$\text{if } n > N \text{ then } a_n > M.$$

(3) $\lim_{n\to\infty} a_n = -\infty$, if $\lim_{n\to\infty}(-a_n) = \infty$.

We say $\{a_n\}$ **converges** (or is **convergent**) if $\lim_{n\to\infty} a_n = L$ for some number L, otherwise it **diverges**. We may use following result to justify its convergence.

Theorem 5.1. *If $\lim\limits_{x\to\infty} f(x) = L$ exists (or is $\pm\infty$) and $f(n) = a_n$ where n is an integer, then $\lim_{x\to\infty} f(x) = \lim_{n\to\infty} a_n = L$ (or is $\pm\infty$).*

Ex. Find limit.

(1) $\lim\limits_{n\to\infty} \dfrac{\ln n}{n}$.

(2) $\lim\limits_{n\to\infty} \dfrac{n^2}{n+1}$.

Due to the above theorem, we have many similar properties as limit of functions.
Laws of limit If $\{a_n\}$ and $\{b_n\}$ exists and c is constant, then

(1) $\lim\limits_{n\to\infty} c(a_n \pm b_n) = c \lim\limits_{n\to\infty} a_n \pm c \lim\limits_{n\to\infty} b_n$

(2) $\lim\limits_{n\to\infty} (a_n b_n) = \lim\limits_{n\to\infty} a_n \lim\limits_{n\to\infty} b_n$

(3) $\lim\limits_{n\to\infty} (a_n/b_n) = \lim\limits_{n\to\infty} a_n / \lim\limits_{n\to\infty} b_n$, if $\lim\limits_{n\to\infty} b_n \neq 0$

(4) $\lim\limits_{n\to\infty} a_n^p = (\lim\limits_{n\to\infty} a_n)^p$, if $p > 0, a_n > 0$.

Proposition 5.2 (Squeeze theorem)**.** *if $a_n \le b_n \le c_n$ for $n \ge n_0$, and $\lim a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.*

Ex. Prove that, if $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$.

Ex. find $\lim\limits_{n\to\infty} (-1)^n/n$.

Theorem 5.3. *If f is continuous at L, and $\lim_{n\to\infty} a_n = L$, then*

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L).$$

Ex. Find $\lim_{n\to\infty} \sin(\pi/n)$.

Ex. Prove $\lim_{n\to\infty} n!/n^n = 0$. Hint: observe $n!/n^n \le 1/n$, and use squeeze theorem.

Ex. Prove: The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent otherwise, and

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Definition For a sequence $\{a_n\}$

(1) it is called **increasing** if $a_n < a_{n+1}$ for all $n \ge 1$
(2) it is called **decreasing** if $a_n > a_{n+1}$ for all $n \ge 1$
(3) it is called **monotonic** if it is either increasing or decreasing.
(4) it is called **bounded above** if $\exists M > 0$ such that $a_n \le M$ for all $n \ge 1$.
(5) it is called **bounded below** if $\exists M > 0$ such that $a_n > -M$ for all $n \ge 1$.
(6) it is called **bounded** if it is bounded above and below.

Proposition 5.4. (Monotonic Sequence Theorem) *Every bounded monotonic sequence is convergent.*

Ex. Given $\{a_n\}$ by

$$a_1 = 2, a_{n+1} = \frac{1}{2}(a_n + 6)$$

,

(1) prove it's convergent
(2) find its limit

5.2. **Series.** section 11.2 exercise: 17, **23, 30,** 31, 35, **40,** 41, 47, 52, 55, **64,** 65, 73

Given sequence $\{a_n\}$, a **series** is

$$\sum_{n=1}^{\infty} a_n.$$

Consider a new sequence $\{s_n\}$ given by **partial sum**

$$s_n = \sum_{i=1}^{n} a_i.$$

Definition If $\lim_{n\to\infty} s_n = s$ for some s, then the series $\sum a_n$ is called **convergent**, and write

$$\sum_{n=1}^{\infty} a_n = s$$

otherwise, it is **divergent**.

Recall that, a **geometric series** (with **common ratio** r)

$$\sum_{n=1}^{\infty} ar^{n-1}$$

is convergent if $|r| < 1$, and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1.$$

If $r \geq 1$, this is divergent.

Q Do you remember formula for partial sum

$$s_n = \sum_{i=1}^{n} ar^{i-1}$$

Ex. Is $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent?

Ex. Is $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ convergent? if yes, find its sum.

Proposition 5.5. *If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Q If $\lim_{n\to\infty} a_n = 0$, then is $\sum_{n=1}^{\infty} a_n$ convergent?

Ex. Prove **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Proposition 5.5 implies that

Proposition 5.6 (Test for divergences). *If* $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_{n=1}^{\infty} a_n$ *divergent.*

Ex. find $\sum_{n=1}^{\infty}(-1)^n$.

Like laws of limit of sequence, we have

$$\sum_{n=1}^{\infty} c(a_n \pm b_n) = c\sum_{n=1}^{\infty} a_n \pm c\sum_{n=1}^{\infty} b_n$$

given that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge.

5.3. **The comparison tests and p-series test.** sec 11.4 exercise: 1, **5, 12, 15**, 17, 31, 37, 41

Proposition 5.7 (Comparison test). *Suppose* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are with all positive terms.*

(1) *If* $\sum_{n=1}^{\infty} b_n$ *is convergent, and* $a_n \leq b_n$ *for all* n, *then* $\sum_{n=1}^{\infty} a_n$ *is also convergent.*
(2) *If* $\sum_{n=1}^{\infty} b_n$ *is divergent, and* $a_n \geq b_n$ *for all* n, *then* $\sum_{n=1}^{\infty} a_n$ *is also divergent.*

We do not involve any integrals, but select the result of p-series, which will be useful later.

A p-**series** refers to $\sum_{n=1}^{\infty} \frac{1}{n^p}$, for $p > 0$. The proof of the following p-series test in the textbook used integrals, and we will provide an alternative proof here without using integrals.

Proposition 5.8 (p-series test). *The p-series is convergent if $p > 1$ and divergent if $p \leq 1$.*

Proof. Set

$$S_n = \sum_{i=2^n}^{2^{n+1}-1} \frac{1}{i^p}.$$

We can rewrite

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=0}^{\infty} S_n.$$

If $p \le 1$, then $S_n \ge 1/2$, and $\sum_{n=0}^{\infty} S_n = \infty$. If $p \ge 1$, then $S_n \le 2^{n(1-p)}$, which implies $\sum_{n=0}^{\infty} S_n < \infty$, and the conclusion holds since every bounded monotonic sequence is convergent. (see Prop 5.4) □

Note The comparison test is extremely useful combined with p-series test.
Ex. Test $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Proposition 5.9 (The limit comparison test). *Both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are with all positive terms. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

for some $\infty > c > 0$, then either both converge or both diverge.

Ex. Test $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3}{\sqrt{3+n^5}}$.

5.4. **Alternating series.** sec 11.5 exercise: 3, **7, 11**, 13, 17, **19, 20**, **32**

Alternating series is a series whose terms are alternatively positive or negative. ex. $\sum_{n=1}^{\infty}(-1)^n b_n$ with all $b_n > 0$.

Proposition 5.10 (Alternating series test). *If alternating series $\sum_{n=1}^{\infty}(-1)^n b_n$ with all $b_n > 0$ satisfies*

(1) $b_{n+1} \le b_n$ *for all* n
(2) $\lim\limits_{n\to\infty} b_n = 0$

Then, it is convergent.

Ex. Prove: The **alternating harmonic series** $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ is convergent.

Ex. For what values of p, is $\sum_{n=1}^{\infty}(-1)^{n-1}/n^p$ convergent?

5.5. **Absolute convergence and the ratio and root tests.** sec 11.6 exercise: **5, 9,** 13, 19, 21, **29, 30,** 31, 33, 39, 40

Def. Given a series $\sum_{n=1}^{\infty} a_n$

(1) it is called **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.
(2) it is called **conditionally convergent** if it is convergent but not absolutely convergent.

Ex. Show that alternating harmonic series $\sum_{n=1}^{\infty}(-1)^n/n$ is conditionally convergent.

Proposition 5.11. *If a series is absolutely convergent, then it is convergent.*

Proof. see page 757. □

Ex. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \cos n/n^2$ is convergent.

Proposition 5.12 (Ratio Test)**.** *consider* $\sum_{n=1}^{\infty} a_n$.

(1) *If* $\lim_{n\to\infty} |a_{n+1}/a_n| = L < 1$ *then the series is absolutely convergent, therefore convergent.*
(2) *If* $\lim_{n\to\infty} |a_{n+1}/a_n| = L > 1$ *or* $\lim_{n\to\infty} |a_{n+1}/a_n| = \infty$ *then the series is divergent.*
(3) *If* $\lim_{n\to\infty} |a_{n+1}/a_n| = L = 1$ *then the Ratio test is inconclusive.*

Proof. see page 758 □

Ex. Test the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$.

Proposition 5.13 (Root Test)**.** *consider* $\sum_{n=1}^{\infty} a_n$.

(1) *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ *then the series is absolutely convergent, therefore convergent.*
(2) *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ *or* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ *then the series is divergent.*

(3) *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$ *then the Ratio test is inconclusive.*

Ex. Test $\displaystyle\sum_{n=1}^{\infty}(\frac{2n+3}{3n+2})^n$.

Note. Both Ratio and Root Tests are inconclusive when its limit is 1. See next example.

Ex. Both series here give limit of ratio and root as 1. But, there one is convergent, the other is not.

(1) $\sum_{n=1}^{\infty} \frac{1}{n^2}$
(2) $\sum_{n=1}^{\infty} \frac{1}{n}$

5.6. **Strategy for testing series.** sec 11.7 Exercise: 1, 3, 11, 13, **21, 30,** 33, 35

In this section, we summarize tools for testing series:

(1) For p-series of the form $\sum_{n=1}^{\infty} 1/n^p$, it is convergent for $p > 1$ and divergent for $p \leq 1$.
(2) For geometric series of the form $\sum_{n=1}^{\infty} ar^n$, it is convergent for $|r| < 1$ and divergent for $|r| \geq 1$.
(3) If $\sum_{n=1}^{\infty} a_n$ is comparable with p-series or geometric series with all positive terms, then use comparison test. If $\sum_{n=1}^{\infty} a_n$ has some negative terms, then use comparison test for absolute convergence of $\sum_{n=1}^{\infty} |a_n|$.
(4) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent by test of divergence.
(5) For $\sum_{n=1}^{\infty}(-1)^n b_n$, use alternating series test.
(6) If a_{n+1}/a_n can be simplified, use ratio test.
(7) If $a_n = (b_n)^n$, then use root test.

Ex. Determine convergence/divergence.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$.
(2) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+2}$.
(3) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$
(4) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{n^4+1}$.
(5) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$.

(6) $\sum_{n=1}^{\infty} \frac{1}{3^n - n}$.

5.7. **Power series.** sec 11.8 Exercise: 3, **5,** 7, 15, **20, 23**, 29, 37

For simplicity, we define $0^0 = 1$. [1]

A **Power series centered at** a is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots , \tag{5.1}$$

where c_n is called the **coefficient** of nth order term.

In particular, A **Power series** (by default centered at 0) is the form of

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots , \tag{5.2}$$

Ex. For what values of x, is the series $\sum_{n=0}^{\infty} x^n$ convergent/divergent?

The main question is to identify convergent/divergent intervals for a given power series.

Theorem 5.14. *For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three possibilities:*

(1) ($R = 0$) *The series converges only when $x = a$.*
(2) ($R = \infty$) *The series converges for all x.*
(3) ($0 < R < \infty$) *The series converges for $|x-a| < R$ and diverges $|x-a| > R$, for some constant $R > 0$.*

In the above,

(1) R is called the **radius of convergence**.
(2) The **interval of convergence** of a power series is the interval that consists of all x for which the series converges.

Note: In the case (3) of Theorem 5.14, the series may converge or diverge at the end points $x = a \pm R$, which is the major difficulty in identifying the interval of convergence. We illustrate this through following examples.

Ex. Identify radius of convergence and interval of convergence for the following series:

(1) $\sum_{n=0}^{\infty} x^n$

[1] some books argue 0^0 is not well defined, but we skip this discussion

(2) $\displaystyle\sum_{n=0}^{\infty} n!x^n$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

(4) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ (Bessel function of order zero)

5.8. **Representations of functions as power series.** sec 11.9 Exercise: 5, 9, 13, 35, 36, 37

In this section, we study how to represent certain types of functions as sums of power series. In other words, we try to answer the following question:

Q. Given a function f, can you write $f(x)$ into the form of

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad x \in (a-R, a+R)$$

for some suitable $\{c_n\}$, a, and $R > 0$?

5.8.1. *Using geometric series.* Geometric series gives following identity:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1. \tag{5.3}$$

Ex. Find power series with center one of $f(x) = \frac{x^3}{x+2}$, and the interval of convergence. What is $f^{(10)}(1)$?

5.8.2. *By differentiation of power series.*

Theorem 5.15. *If the power series $P(x) := \sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence $R > 0$, i.e. the function $P(x)$ is defined by*

$$P(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad x \in (a-R, a+R).$$

Then, P is differentiable on $(a-R, a+R)$, and

$$P'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

with the same radius of convergence R. In fact, $P(x)$ is infinitely differentiable on $(a-R, a+R)$.

Proof. omitted since it involves advanced concepts (uniform convergence). □

Note: Theorem 5.15 can be understood as sufficient condition under which $\frac{d}{dx}$ and $\sum_{n=0}^{\infty}$ are commutative, i.e.

$$\frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} \frac{d}{dx} c_n(x-a)^n, \quad \text{when } |x-a| < R.$$

Ex. In general, Theorem 5.15 may not be true. For example, let $f_n(x) = \sin(nx)/n^2$. The series $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges for all values of x but the series of derivative $\displaystyle\sum_{n=1}^{\infty} f_n'(x)$ diverges when $x = 2m\pi$, for any integer m.

Note: Theorem 5.15 says the radius of convergence remains the same when a power series is differentiated. But, this does not mean the interval of convergence remains the same.

Ex. Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Find its intervals of convergence for f, f', f''.

Ex. We saw that the Bessel function has $R = \infty$. Use Theorem 5.15, find $J_0'(x)$ and its radius of convergence.

Ex. Express $1/(1-x)^2$ as a power series by differentiating (5.3). What is the interval of convergence?

5.9. **Taylor and Maclaurin series.** sec 11.10 Exercise: **3, 6,** 15, 33, **38,** 55, 57, 59, 63, **72, 74(a)**

In the previous section, using geometric series and differentiation, we can find power series representations for a class of functions. We are going to generalize this result.

Def. Given a smooth function f, the corresponding **Taylor series of f centered at a** is defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In particular, Taylor series of f at centered at 0 is said to be **Maclaurin series**.

We are interested in finding power series representation of a given function f with a center a, i.e. for some $R > 0$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a| < R. \tag{5.4}$$

Then, why are we interested in its Taylor series of a function f? It is because Taylor series is a good candidate of the desired power series of the function f.

Theorem 5.16. *If f has a power series representation of* (5.4)*, then $f(x) = T(x)$ for all $|x-a| < R$, i.e. $c_n = \frac{f^{(n)}(a)}{n!}$.*

Proof. see pages 777-778. □

By Theorem 5.16, to find a power series of a function f, one can actually use the formula to find its Taylor series, *provided that* the function f has a representation of power series. A HUGE question is that if function f always has a power series representation, or more precisely
[Q*] Which functions have power series representation (5.4) in general?
In fact, not every differentiable function can be represented by Taylor series, see following example.
Ex. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$$

(1) By induction, show that

$$f^{(n)}(x) = \begin{cases} \frac{p_n(x)}{q_n(x)} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$$

for some polynomial functions p_n and q_n.

(2) Show that function can not be equal to its Maclaurin series, i.e. $f(x) \neq T(x)$ for all $x \neq 0$.

From the previous example, it is dangerous to claim $f(x) = T(x)$ without justification.

Ex. Find power series representation of e^x at center 0.

Solution. There are two steps involved.

(1) (Find its Maclaurin series for the good candidate, and its radius of convergence) Direct computation leads to its Maclaurin series

$$T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{5.5}$$

By ratio test, $R = \infty$.

(2) (Justify e^x is equal to its Maclaurin series $T(x)$) This step is necessary, and will be provided later on.

□

To answer the question **[Q*]**, we first define n**th degree Taylor polynomial of** f **at** a by

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i, \tag{5.6}$$

and its **remainder** by

$$R_n(x) = f(x) - T_n(x). \tag{5.7}$$

Note that, only if f is equal to Taylor series, following identity is true:

$$R_n(x) = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

Now we have answer:

Proposition 5.17. *If* $\lim_{n\to\infty} R_n(x) = 0$ *for* $|x-a| < R$*, then* f *is equal to its Taylor series on* $|x-a| < R$*.*

Proof. It is directly from the definition of $R_n(x)$ and $T_n(x)$. □

The following is useful to show $\lim\limits_{n\to\infty} R_n(x) = 0$.

Proposition 5.18 (Taylor's Theorem)**.** *If* f *is smooth enough, then there exists* ξ *between* a *and* b *such that*

$$R_n(b) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}.$$

Proof. We will show only the proof of the case $n = 1$, and it can be similarly generalized to arbitrary integer n. We want to show

$$\exists \xi \in (a,b), \text{ s.t. } R_1(b) = \frac{f''(\xi)}{2}(b-a)^2. \tag{5.8}$$

Recall $T_1(x) = f(a) + f'(a)(x-a)$. Define a constant M by

$$M := \frac{R_1(b)}{(b-a)^2}$$

and a function g by

$$g(x) = f(x) - T_1(x) - M(x-a)^2.$$

(1) Note that $g(b) = 0$ and $g(a) = 0$. By MVT, $\exists \xi_1 \in (a, b)$ s.t. $g'(\xi_1) = 0$.
(2) Note that $g'(\xi_1) = 0$ and $g'(a) = 0$. By MVT, $\exists \xi_2 \in (a, \xi_1)$ s.t. $g''(\xi_2) = 0$.

Thus, there exists $\xi \in (a, b)$ s.t. $f''(\xi) - T_1''(\xi) - 2M = 0$. Together with the fact $T_1''(\xi) = 0$, one concludes (5.8). □

Next identity will be also useful to show $\lim_{n\to\infty} R_n(x) = 0$:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0, \ \forall x. \tag{5.9}$$

Ex. Prove (5.9).

Complete Solution for Example on Page 13. First step has been shown before. Here is the complete second step.

(1) (Find its Maclaurin series for the good candidate, and its radius of convergence) Direct computation leads to its Maclaurin series

$$T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{5.10}$$

By ratio test, $R = \infty$.

(2) (Justify e^x is equal to its Maclaurin series $T(x)$) This step is necessary, and will be provided later on. Now we need to show $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. By Proposition 5.17, it's enough to show

$$\lim_{n\to\infty} R_n(x) = 0.$$

Taylor theorem implies that, there exists $\xi \in (0, x)$ s.t.

$$\lim_n |R_n(x)| \le \lim_n \frac{e^\xi}{(n+1)!} |x-a|^{n+1} = 0.$$

This completes the proof.

□

Now we have formula

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \forall x. \tag{5.11}$$

In particular,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}. \tag{5.12}$$

Ex. Maclaurin series are given for functions
(1) $\frac{1}{1-x}$
(2) e^x
(3) $\sin x$
(4) $\cos x$
(5) $\tan^{-1} x$
(6) $(1+x)^k$

Ex. find the intervals of convergence of Maclaurin series for the above functions.

Ex. Find $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$.

Ex. (optional) Find first three nonzero terms in Maclaurin series for
(1) $f(x) = e^x \sin x$
(2) $f(x) = \tan x$ (using $\tan x = \sin x / \cos x$)