Chapter 8. Confidence Interval/Interval estimation

In this part, we will learn how to take into account the uncertainty of some estimate by making statements such as: "I am 90% certain that the value of the parameter is within the interval (-0.5,0.3)".

Confidence and uncertainty in estimation

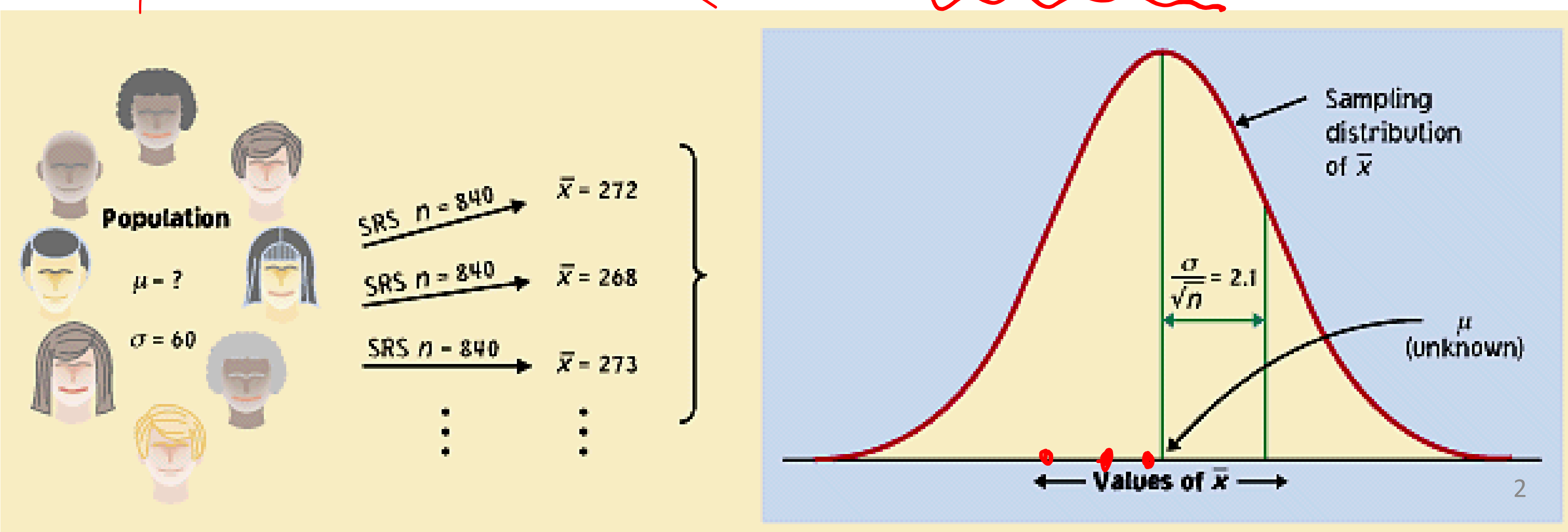
$|\bar{X} - \mu| \le 2\frac{\sigma}{\sqrt{n}} \iff \mu \in \left(\bar{X} - 2\frac{\sigma}{\sqrt{n}}, \bar{X} + 2\frac{\sigma}{\sqrt{n}}\right)$

Although the sample mean, $\bar{x}$ is a unique number for any particular sample, if you pick a different sample you will probably get a different sample mean.

$X_i \sim N(\mu, \sigma^2)$

$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$P\left(\left|\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right| < 2\right) \approx 0.95 \iff P\left(|\bar{X} - \mu| \le \frac{2\sigma}{\sqrt{n}}\right) \approx 0.95$



1. General definition

If $P(\widehat{\theta}_L \leq \theta \leq \widehat{\theta}_U) = 1-\alpha$, $[\widehat{\theta}_L, \widehat{\theta}_U]$ is a (two-sided) confidence interval with confidence level $1-\alpha$.

If $P(\widehat{\theta}_L \leq \theta) = 1-\alpha$, $[\widehat{\theta}_L, \infty]$ is a one-sided confidence interval with confidence level $1-\alpha$.

If $P(\theta \leq \widehat{\theta}_U) = 1-\alpha$, $[-\infty, \widehat{\theta}_U]$ is a one-sided confidence interval with confidence level $1-\alpha$.

2. Confidence interval for proportion

Example: A population consists of 10,000 people; each has a strong opinion for or against some proposition. We wish to know true proportion of the population that is for the proposition. We survey 100 people at random, and the sample proportion is $\hat{p} = \frac{X}{n}$

X: # ppl for prop $\sim Bin(n, p)$

Question: How close is $\hat{p}$ to the true proportion p

In this example, we sample without replacement, and the distribution of $100\widehat{p}$ is hypergeometric – too difficult to work with.

If we sample with replacement, the distribution of $100\widehat{p}$ is $Binomial(100, p)$—easier than hypergeometric.

When the population is large compared to the sample size n, there is little difference between the two. Thus we should expect that $\widehat{p}$ is approximately normal.

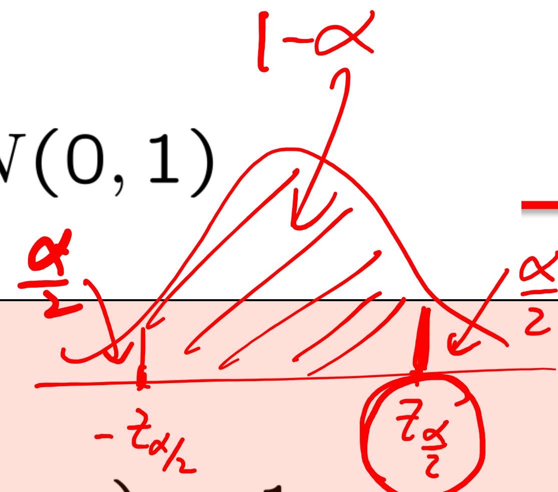
$\hat{p} = \frac{X}{n}$

Since $E[\widehat{p}] = p, Var(\widehat{p}) = p(1-p)/n$, the CLT implies that

$$\frac{\widehat{p} - p}{\sqrt{p(1-p)/n}} \sim N(0, 1)$$

We have:

$$\frac{\widehat{p}-p}{\sqrt{p(1-p)/n}} \sim N(0,1)$$



In particular, if

$$P(-z_{\alpha/2} \leq N(0,1) \leq \underbrace{z_{\alpha/2}}_{1.96}) = 1-\alpha$$

0.95 $\quad P(Z < z_{\frac{\alpha}{2}}) = 1 - \frac{\alpha}{2}$

then

$$P\left(-z_{\alpha/2} \leq \frac{\widehat{p}-p}{\sqrt{p(1-p)/n}} \leq z_{\alpha/2}\right) \approx 1-\alpha \quad \Leftrightarrow$$

$$\Phi(z_{\frac{\alpha}{2}}) = 1 - \frac{\alpha}{2}$$

For example, if $1-\alpha = 0.95$, $z_{\alpha/2} = 1.96$.

To simplify, we replace $SD(\widehat{p}) = \sqrt{p(1-p)/n}$ by standard error $SE(\widehat{p}) = \sqrt{\widehat{p}(1-\widehat{p})/n}$. The central limit theorem still applies with this divisor.

$$\frac{\widehat{p}-p}{SE(\widehat{p})} \approx N(0,1)$$

Rearranging the expression inside the probability,

$$P\left(-z_{\alpha/2} \le \frac{\hat{p}-p}{SE(\hat{p})} \le z_{\alpha/2}\right) \approx 1-\alpha$$

we get

$$\hat{p} - z_{\alpha/2}SE(\hat{p}) \le p \le \hat{p} + z_{\alpha/2}SE(\hat{p})$$

which contains p with approximate probability $1-\alpha$.

The interval is referred to as $1-\alpha$ confidence interval and is often abbreviated $\hat{p} \pm z_{\alpha/2}SE(\hat{p})$. $1-\alpha$ is called the level of confidence.

3. Confidence interval for mean

Chi-squared distribution and Student's t distribution

The $\chi^2(n)$, or χ_n^2 distribution is just the gamma distribution, with $\alpha = n/2$ and $\beta = 1/2$. The integer n is the parameter of the distribution and sometimes called the degree of freedom. If $X \sim \chi^2(n)$, then $E[X] = n, Var(X) = 2n$.

Characterization/Definition: if $Z_i \overset{i.i.d.}{\sim} N(0,1)$, then $\sum_{i=1}^{n} Z_i^2 \sim \chi^2(n)$.

Chi-squared distribution:

Property: if X and Y are independent with χ_n^2 and χ_m^2 distributions, then $X+Y \sim \chi_{n+m}^2$ (this can be proved easily with the above characterization)

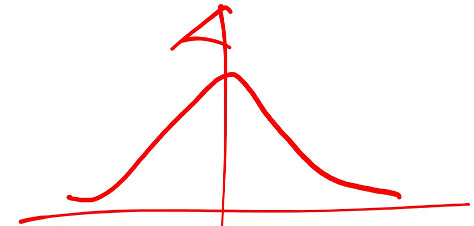
Student's-t distribution:

Characterization:

If $Z \sim N(0,1), X \sim \chi^2_n$, and X and Z are independent, then $\sqrt{n}Z/\sqrt{X} \sim t_n$(or $t(n)$). n is the parameter of the t distribution and called the degrees of freedom like for χ^2 distribution.

$$\frac{1}{\sqrt{n\pi}} e^{-\frac{x^2}{2}}$$

Student's-t distribution:



Density:

$$f(x) = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\sqrt{n\pi}}(1 + x^2/n)^{-(n+1)/2}$$

$$\left(1+\frac{1}{n}\right)^n \to e \qquad \lim_{n\to\infty}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}} \to e^{-\frac{x^2}{2}}$$

Mean and variance:

$$E[X] = 0, Var(X) = \frac{n}{n-2} \text{ if } n > 2$$

Confidence interval for mean

For a random sample $X_1, X_2, \ldots, X_n$, the central limit theorem tell us that for large n

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - \mu}{SD(\bar{X})}$$

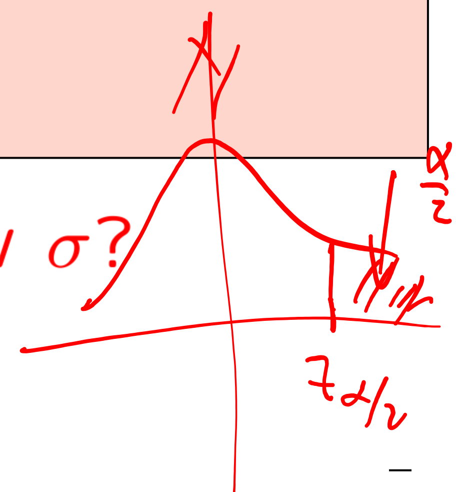
(called Z-statistics) will have an approximately normal distribution. So 95% CI for μ is $\bar{X} \pm 1.96\sigma/\sqrt{n}$.

(Small sample test) When $X_i \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, Z is exactly $N(0, 1)$.

$$P(-z_{\alpha/2} < \frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} < z_{\alpha/2}) \approx 1 - \alpha$$

$$\iff P(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) \approx 1 - \alpha$$

Conclusion: the confidence interval for μ is $[\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}]$



But what happens if we don't know σ?

IDEA: plug in an estimator for σ!

When σ is unknown, we use the standard error, $SE(\bar{X}) = s/\sqrt{n}$, to replace $\sigma/\sqrt{n}$.

$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

Consider

$$T = \frac{\bar{X} - \mu}{s/\sqrt{n}} = \frac{\bar{X} - \mu}{SE(\bar{X})}$$

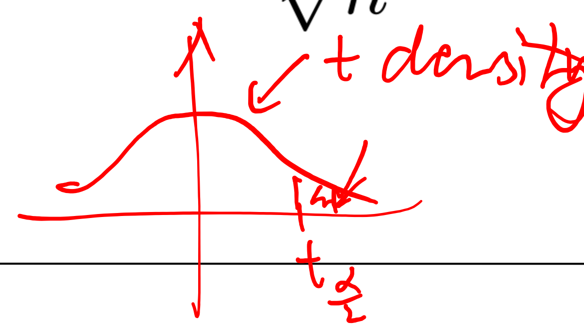
T is still approximately normal when n is large enough. This fact can be used to construct confidence intervals such as a 95% confidence interval as $\bar{X} \pm 1.96 s/\sqrt{n}$.

When n is not large, and X_i are i.i.d. normal, the sampling distribution of T is the t-distribution with $n-1$ degrees of freedom. (Note when n goes to infinity, $t(n-1)$ will be converge to normal distribution, so there is no contradiction)

$\sim t(n-1)$

$$P(-t_{\alpha/2} < T_{n-1} < t_{\alpha/2}) \approx 1-\alpha$$

$$\iff P(\bar{X} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2}\frac{s}{\sqrt{n}}) \approx 1-\alpha$$



Example:

A person has been trained to set the bean grinder so that a 25-second espresso shot results in 2 ounces of espresso. He pours eight shots and measures the amounts to be 1.95, 1.80, 2.10, 1.82, 1.75, 2.01, 1.83, and 1.90 ounces. Find a 90% confidence interval for the mean shot size. Does it include 2.0?

$z_{0.05}$

$$\overline{X} \pm z_{\alpha/2} \frac{s}{\sqrt{8}}$$

$\overline{X} = 1.895$, $z_{\alpha/2} = 1.64$, $t_{7,\frac{\alpha}{2}} = 1.89$

$$s = \sqrt{\frac{1}{7}\sum_{i=1}^{8}(x_i - 1.895)^2}$$

$$\frac{(1.826,\ 1.964)}{(1.816,\ 1.974)}$$

n

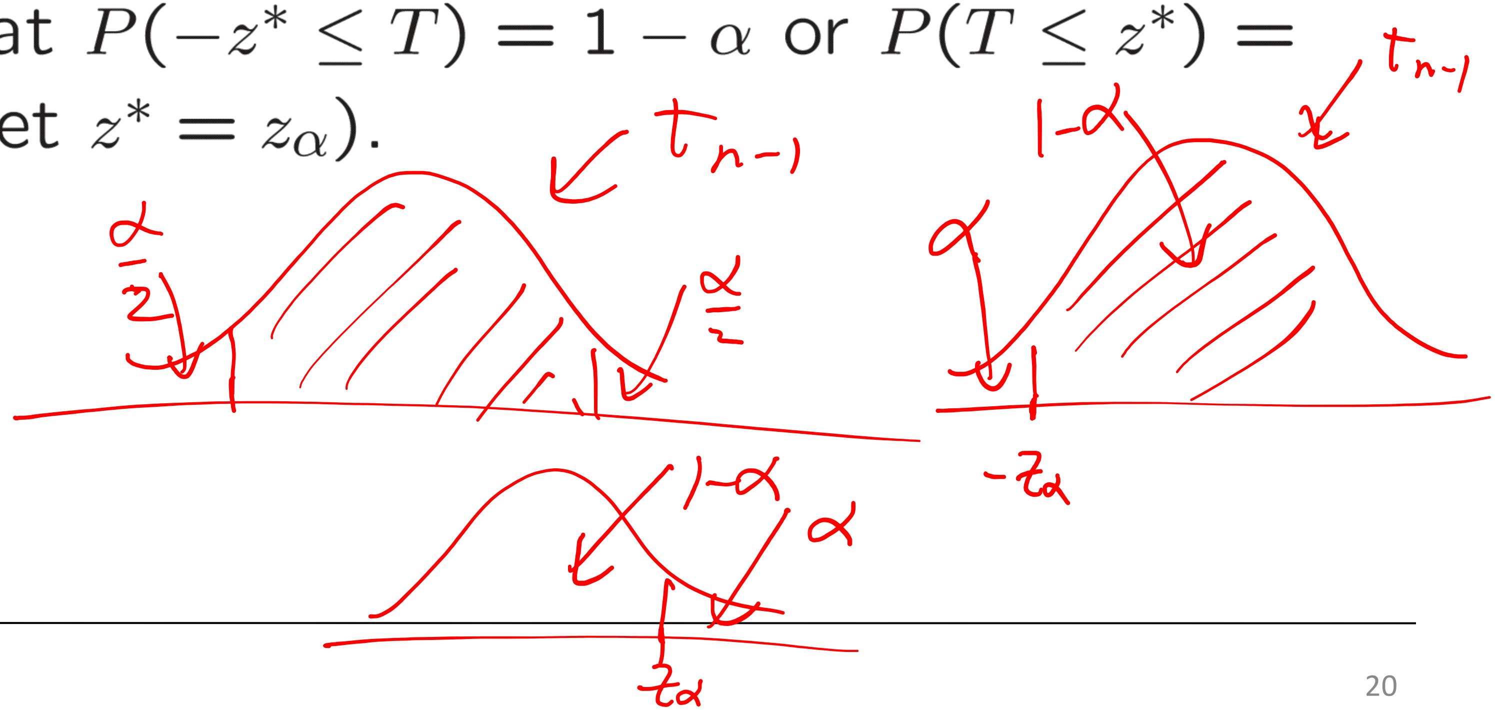
Example: Students in a class of 30 have an average height of 66 inches, with a sample standard deviation of 4 inches. Assume that these heights are normally distributed, and the class can be considered a random sample from the entire college population. What is an 80% interval for the mean height of all the college students?

$\bar{x}$ s μ $1-\alpha$

$$\bar{x} \pm t_{29,\,0.1} \frac{4}{\sqrt{30}}$$

1.31

One-sided CI is obtained when we assign the mass to only one tail. For example, for confidence intervals for the mean, based on the T statistics, these would be found by finding z^* such that $P(-z^* \leq T) = 1 - \alpha$ or $P(T \leq z^*) = 1 - \alpha$.(set $z^* = z_\alpha$).



$[b, \infty)$ $\quad \bar{X} \pm z_\alpha \frac{s}{\sqrt{n}}$ $\quad t_\alpha$

Example:

Find a 90% CI of the form $(-\infty, b]$ for $\bar{X}$.

$$P(T \leq z_\alpha) = 1 - \alpha$$

$$P\left(\frac{\bar{X} - \mu}{s/\sqrt{n}} \leq z_\alpha\right) = 1 - \alpha$$

$$P\left(\bar{X} - \mu \leq z_\alpha \frac{s}{\sqrt{n}}\right) = 1 - \alpha$$

$$P\left(\boxed{\bar{X} - z_\alpha \frac{s}{\sqrt{n}} \leq \mu}\right) = 1 - \alpha$$

$$\left(\bar{X} - z_\alpha \frac{s}{\sqrt{n}}, \infty\right)$$

$$P(-z_\alpha \leq T) = 1 - \alpha$$

$$P\left(-z_\alpha \leq \frac{\bar{X} - \mu}{s/\sqrt{n}}\right) = 1 - \alpha$$

$$P\left(-z_\alpha \frac{s}{\sqrt{n}} \leq \bar{X} - \mu\right) = 1 - \alpha$$

$$P\left(\mu \leq \bar{X} + z_\alpha \frac{s}{\sqrt{n}}\right) = 1 - \alpha$$

$$\left(-\infty, \bar{X} + z_\alpha \frac{s}{\sqrt{n}}\right)$$

$\Phi(z_\alpha) = 1 - \alpha$

The optimal serving temperature for coffee is 180F. Five temperatures are taken of the served coffee: 175,185,170,184, and 175 degrees. Find a 90% CI of the form $(-\infty, b]$ for the mean temperature.

$\alpha = 0.1$

$$\left(-\infty,\ \underbrace{\bar{X} + t_{4,0.1}\frac{s}{\sqrt{5}}}_{182.23}\right)$$