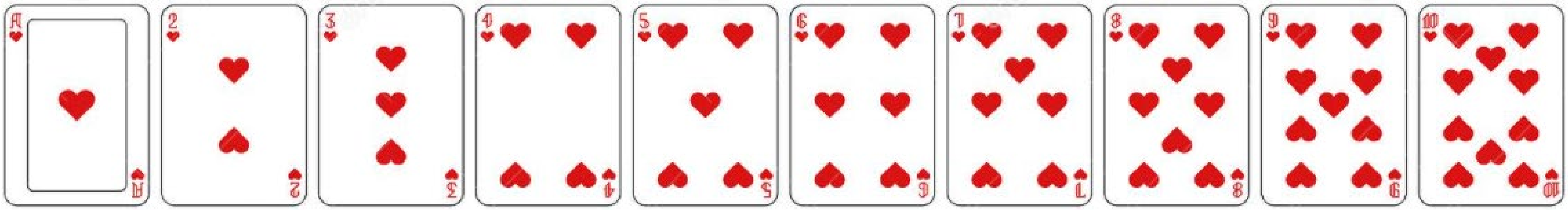
Unit 4

Sets



I don't think so.

After random shuffling, no card will be on its original position.

Who has a higher chance to win?

a) Alice
b) Bob
c) Equal probability

Outline of Unit 4

- 4.1 Basic Concepts
- 4.2 Set Operations
- 4.3 Algebraic Rules
- 4.4 Derangements

Unit 4.1

Basic Concepts

Set

- A set is a collection of objects.
- The cardinality of a set A is defined as the number of elements in the set.
- It is denoted by $|A|$.
  - If $|A|$ is finite, A is called a finite set.
  - Otherwise, A is called an infinite set.
- Example:
  - $A = \{1,2,3,4,5,6,7,8,9,10\}$ (roster notation)
  - $A = \{x \in \mathbb{N}: 1 \leq x \leq 10\}$ (set builder notation)
  - $|A| = 10$

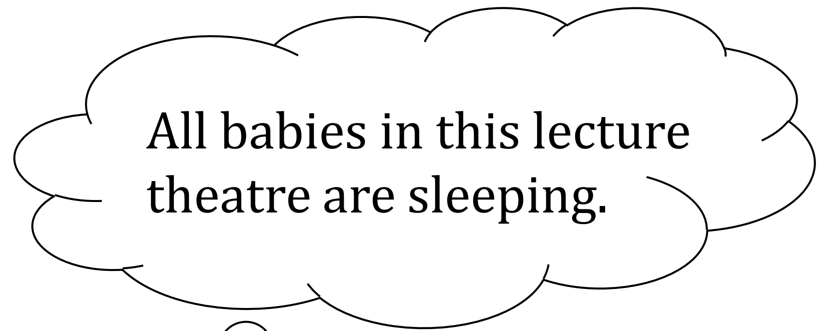
Some Common Sets in Math

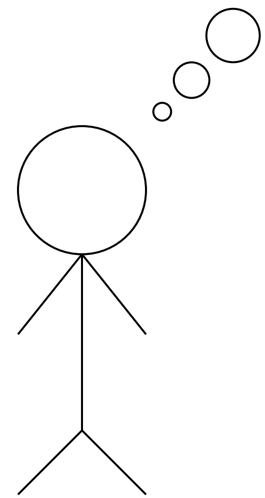
Set	Symbols
Natural Numbers*	$\mathbb{N} = \{1, 2, 3, \dots\}$
Whole Numbers	$\mathbb{N} \cup \{0\}$ or $\mathbb{Z}_{\geq 0}$
Integers	$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
Binary Numbers	$\mathbb{B} = \{0, 1\}$
Rational Numbers	$\mathbb{Q}$
Real Numbers	$\mathbb{R}$
Complex Numbers	$\mathbb{C}$

*In some convention, 0 is included in the set of natural numbers.

The Empty Set

- A set is empty if it contains no elements at all.
- There is only one empty set.
  - If two sets are empty, each set is a subset of the other one, so they are the same set.
- We denote it by ∅.
- Remark:
  - The empty set ∅ is different from the set containing ∅.
    - i.e., $\emptyset \neq \{\emptyset\}$.





True or False?

Vacuous truth

It is universally agreed that given any arbitrary property *P*, all elements of ∅ have property *P*.

Subset

- A is a subset of B, written as $A \subseteq B$, if every member of A is also a member of B. B is then said to be a superset of A.
  - A is a subset of itself, and also a superset of itself.
- A subset A of B is called a proper subset of B if B contains some elements that are not in A.
  - i.e., A is not the same as B.
  - Example: The set of all women is a proper subset of the set of all human beings.
- The empty set is a subset of every other set.

Number of Subsets

- How many subsets does a set have?
- Example: $A = \{1, 2\}$
  - There are four subsets of A: $\emptyset, \{1\}, \{2\}, \{1, 2\}$.
- Example: $A = \{1, 2, 3\}$
  - How many subsets are there?

a) 3

b) 4

c) 6

d) 8

e) 9

Number of Subsets

Theorem: Let A be a finite set and $n = |A|$. The number of subsets of A is 2^n.

Proof:

Let $A = \{a_1, a_2, a_3, \dots, a_n\}$.

To each subset B of A, we can associate a binary sequence of length n.

- For example, if $n = 4$ and $B = \{a_1, a_4\}$, then the binary sequence is 1001.
- For example, 0100 corresponds to $\{a_2\}$.

The number of subsets equals the number of possible binary sequence, which is 2^n.

Q.E.D.

Power Set

- Given a set A, the set of all its subsets, denoted by $\mathcal{P}(A)$, is called the power set of A.
- By the previous result, $|\mathcal{P}(A)| = 2^{|A|}$.
- Example:
  - Suppose $A = \{1, 2, 3\}$.
  - List all subsets of A:

    $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$ and $\{1, 2, 3\}$.
  - Hence,

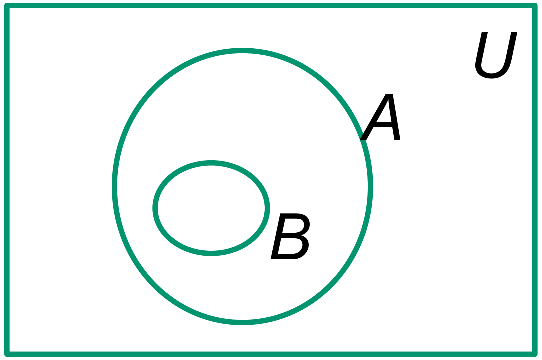
    $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Unit 4.2

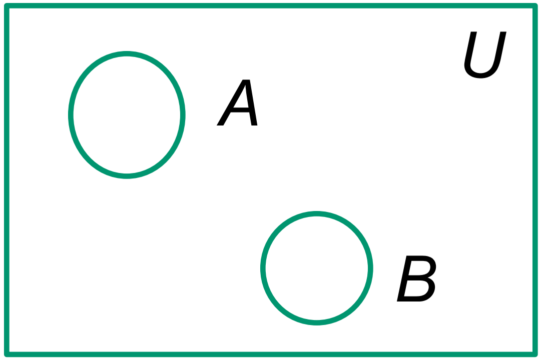
Set Operations

Relationship between Sets

- A universal set U is a set containing everything that we are considering.
- Venn diagram
  - U is represented by a rectangular box.
  - Subsets of U (e. g. A and B) are represented by circles (more precisely, regions inside closed curves).
- A and B are disjoint if they have no elements in common.



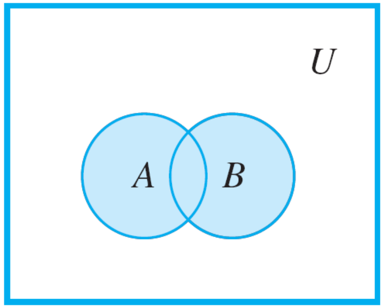
B is a subset of A.



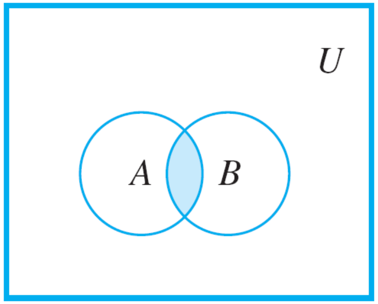
A and B are disjoint.

Three Basic Operations

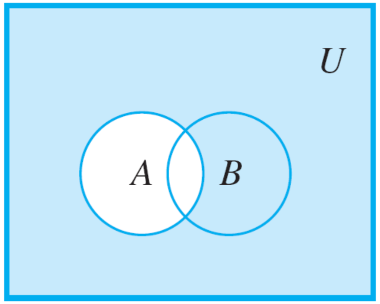
- The union of A and B, denoted by $A \cup B$, is the set of all elements that belong to A or B, (or in both).



- The intersection of A and B, denoted by $A \cap B$, is the set of all elements that are in both A and B.



- The complement of A, denoted by A^c or $\bar{A}$, is the set of all elements in U that do not belong to A.



Algebraic Properties

1. *Commutative Laws:* For all sets A and B,

$$\text{(a)}\ A \cup B = B \cup A \quad \text{and} \quad \text{(b)}\ A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

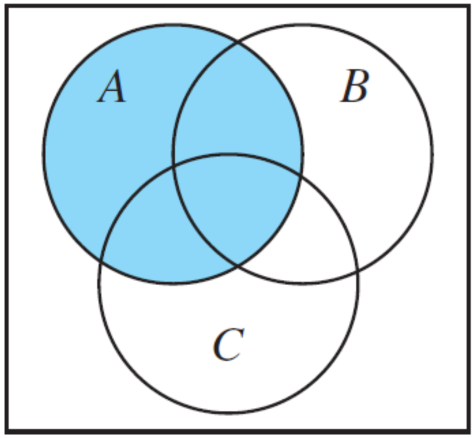
$$\text{(a)}\ (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b)}\ (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets A, B, and C,

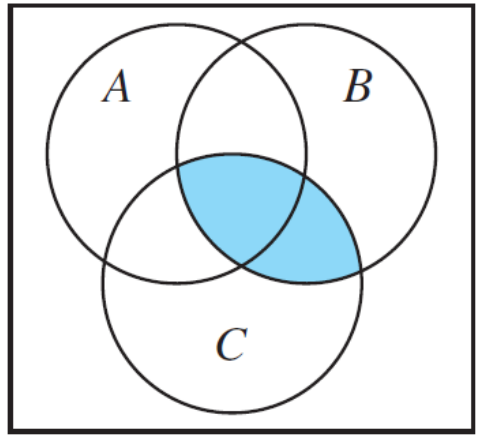
next slide $\longrightarrow$ $\text{(a)}\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$

$$\text{(b)}\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

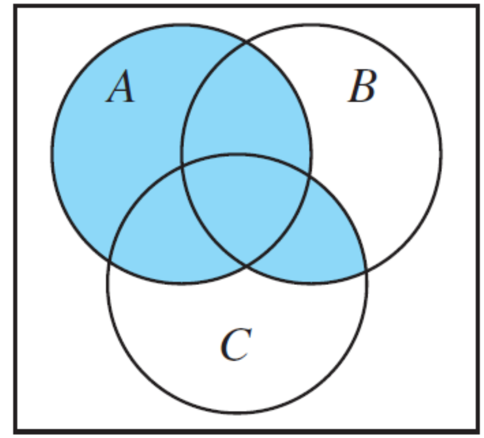
A Distributive Law



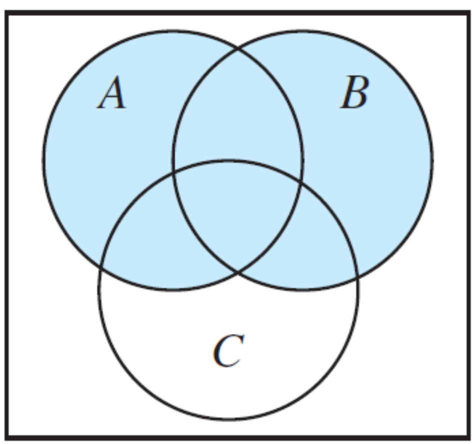
A



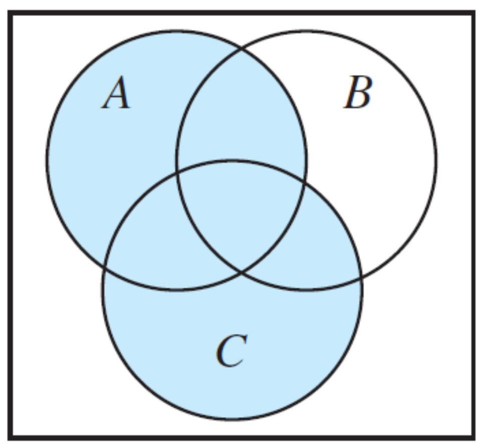
$B \cap C$



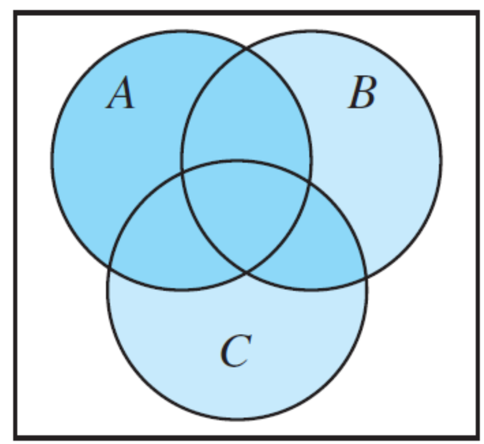
$A \cup (B \cap C)$



$A \cup B$



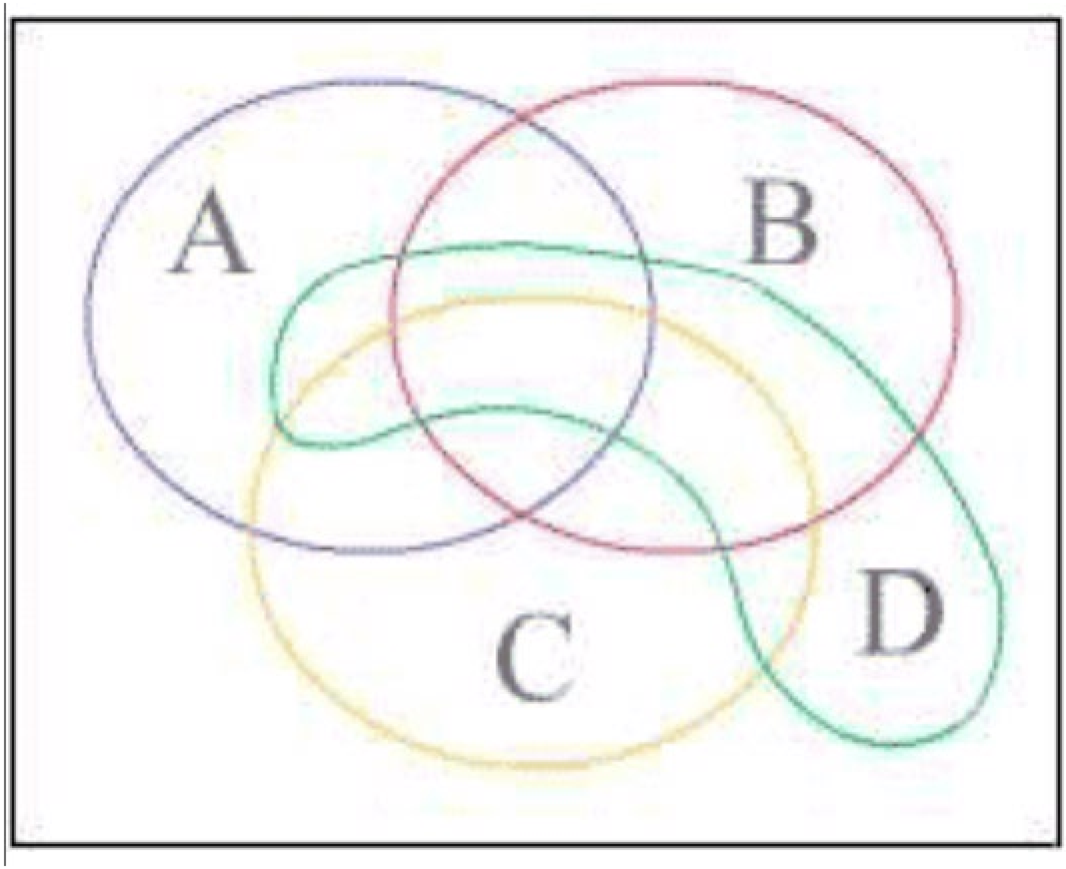
$A \cup C$



$(A \cup B) \cap (A \cup C)$

Four-Variable Venn Diagram

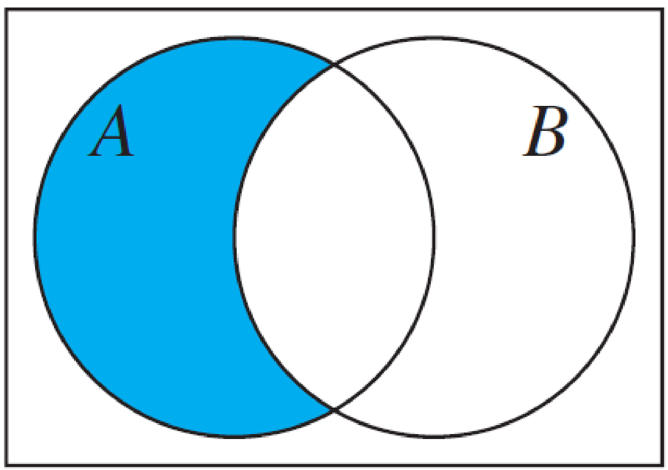
- A possible way to draw it:



Set Difference

- $A - B$ is defined as the set of all elements that are in A but not in B.

$$A - B = A \cap B^c$$



$A - B$

$A - B$ is also denoted as $A \setminus B$.

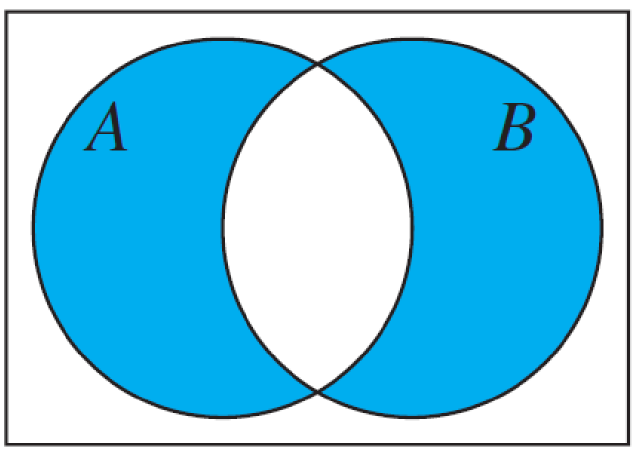
Symmetric Difference

- $A \triangle B$ is the set of all elements "in A but not in B" or "in B but not in A".

$$A \triangle B = (A - B) \cup (B - A)$$

- Venn diagram shows that

$$A \triangle B = (A \cup B) - (A \cap B)$$



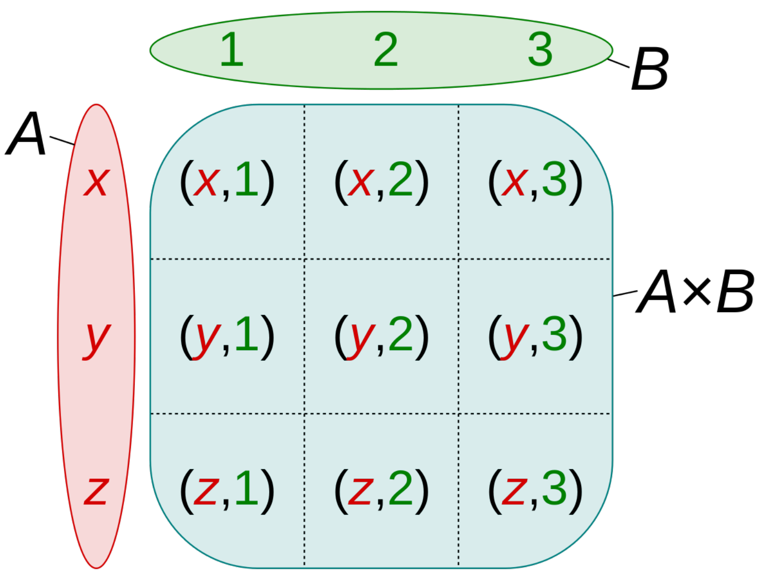
$A \Delta B$

Cartesian Product

- The Cartesian product $A \times B$ of the sets A and B is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

$$A \times B \triangleq \{(a, b) | a \in A \ \wedge b \in B\}.$$

- Example:



Ordered pair:

- The order is important: $(a, b) \neq (b, a)$

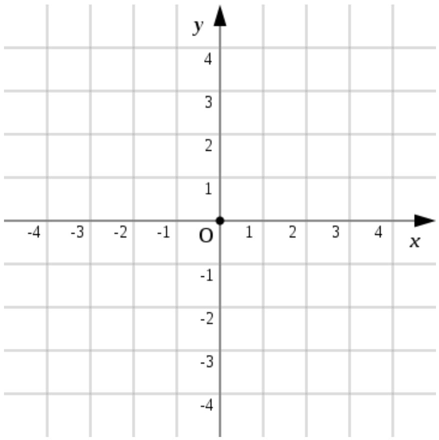
What is $|A \times B|$?

Cartesian Product

- The Cartesian product can be generalized to more than two sets, e.g., $A \times B \times C$.
- If the same set is involved, we write

$$\boxed{\underbrace{A \times A \times \cdots \times A}_{n} = A^n}$$

- For example, the x-y plane is $\mathbb{R}^2$.



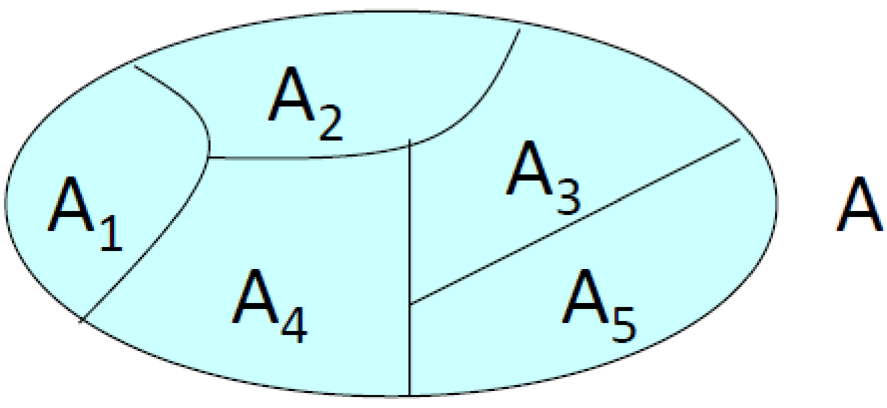
Partition

- A collection of non-empty sets $\{A_1, A_2, \ldots, A_n\}$ is a partition of a set A iff

i. $A = A_1 \cup A_2 \cup \cdots \cup A_n$, and

ii. $A_1, A_2, \ldots, A_n$ are pairwise disjoint, i.e.,

$A_i \cap A_j = \emptyset$ for all $i, j = 1, 2, \ldots, n$ and $i \neq j$.



Examples

- Let $A = \{1, 2, 3, 4, 5, 6\}$.
- $\{\{1, 2\}, \{3\}, \{4, 5, 6\}\}$ is a partition of A.

- Other partitions:
  - $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$
  - $\{\{1, 2, 3\}, \{4\}, \{5, 6\}\}$

    $\vdots$
  - how many are there?

Partition of a set is itself a set.

Bell Numbers

- Consider the set $S_n = \{1, 2, \ldots, n\}$.
- The number of different ways to partition S_n is called the Bell number, denoted by B_n.
  - S_1: $\{\{1\}\}$ is the only partition, so $B_1 = 1$.
  - S_2: $\{\{1\}, \{2\}\}$ and $\{\{1, 2\}\}$ are the partitions, so $B_2 = 2$.
  - S_3: $\{\{1\}, \{2\}, \{3\}\}$, $\{\{1\}, \{2, 3\}\}$, $\{\{2\}, \{1, 3\}\}$, $\{\{3\}, \{1, 2\}\}$, and $\{\{1, 2, 3\}\}$ are the partitions, so $B_3 = 5$.
- How about S_0?
  - S_0 is the empty set $\emptyset$. Its only partition is $\emptyset$, *not* $\{\emptyset\}$.
  - Hence, $B_0 = 1$.

Recurrence Relation

- The Bell numbers satisfy the following recurrence relation:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

(proof omitted)

- Example:
  - $B_3 = \binom{2}{0} B_0 + \binom{2}{1} B_1 + \binom{2}{2} B_2 = 1 + 2 \times 1 + 2 = 5.$

Unit 4.3

Algebraic Rules

How to Prove Set Equalities?

Three methods to prove that $A = B$.

1) To show that each set is a subset of the other, i.e., $A \subseteq B$ and $B \subseteq A$. (omitted)

2) Use the algebraic rules (stated in the next few slides) to construct an algebraic proof.

3) Use membership table (similar to truth table, omitted)

Algebraic Rules

- We have a list of set identities as follows:

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$

$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

Algebraic Rules

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

Algebraic Rules

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B, ← Next slide

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

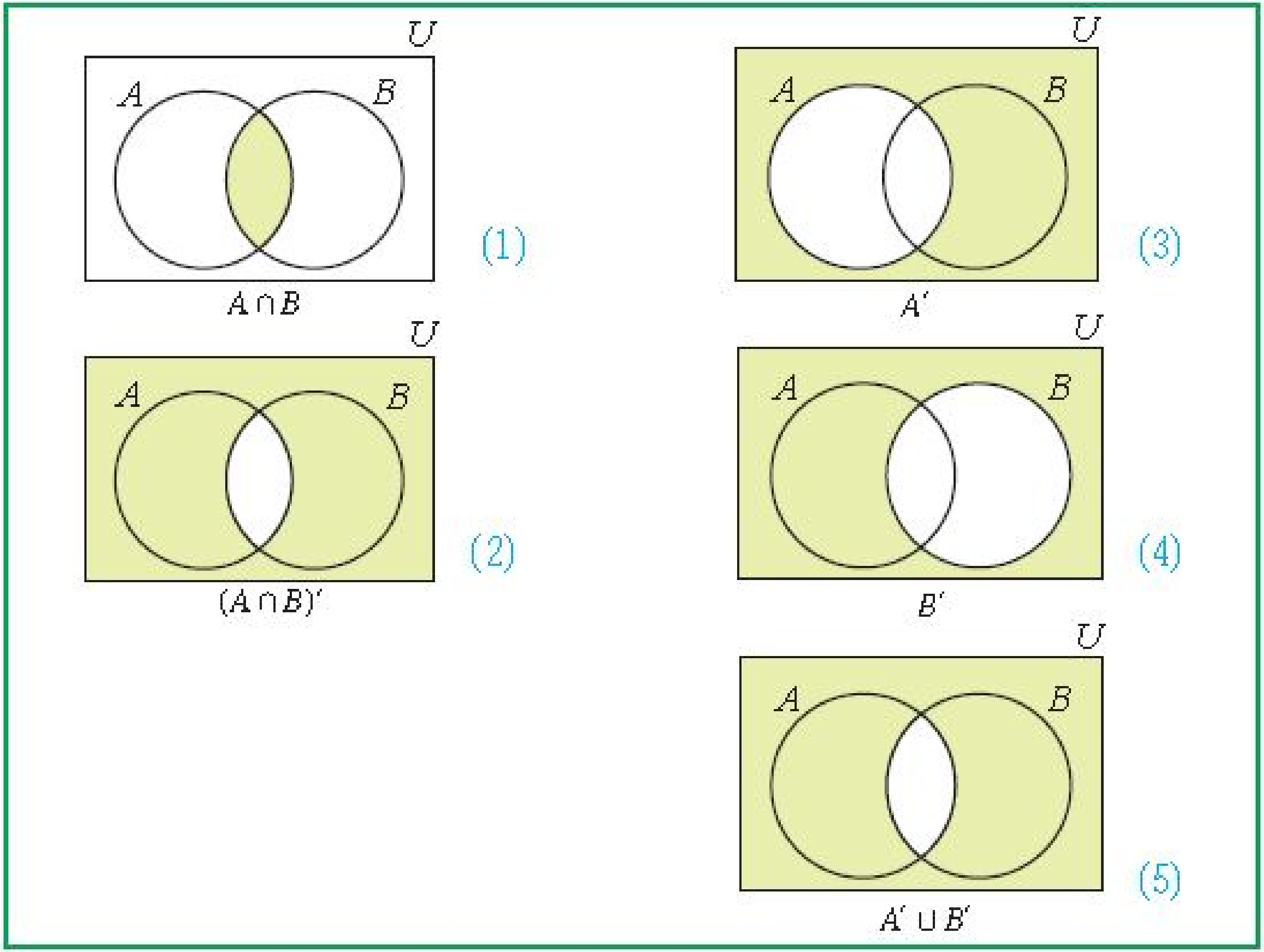
11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

De Morgan's Law: $(A \cap B)^c = A^c \cup B^c$



Example

Construct an algebraic proof that for all sets A and B,

$$(A \cup B) - C = (A - C) \cup (B - C).$$

$$
\begin{aligned}
(A \cup B) - C &= (A \cup B) \cap C^c && \text{by the set difference law} \\
&= C^c \cap (A \cup B) && \text{by the commutative law for } \cap \\
&= (C^c \cap A) \cup (C^c \cap B) && \text{by the distributive law} \\
&= (A \cap C^c) \cup (B \cap C^c) && \text{by the commutative law for } \cap \\
&= (A - C) \cup (B - C) && \text{by the set difference law.}
\end{aligned}
$$

Example

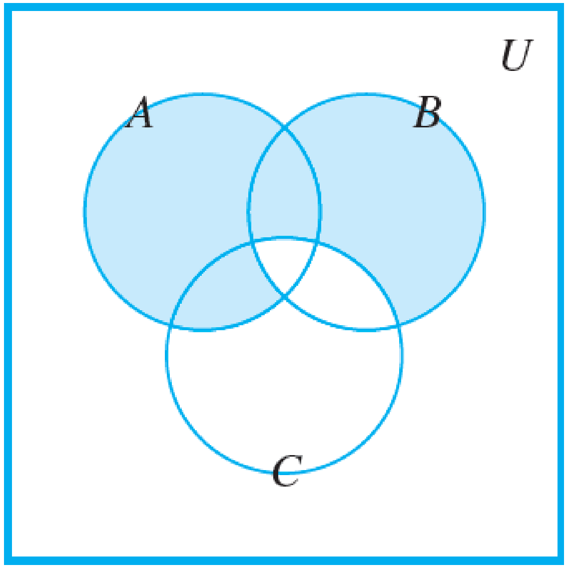
Construct an algebraic proof that for all sets A and B,

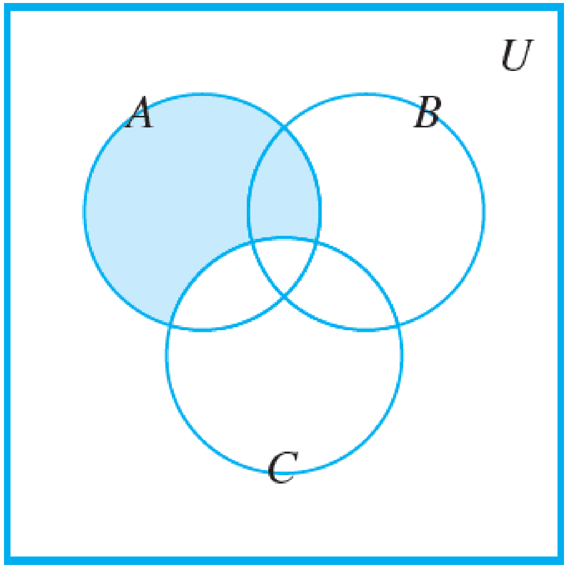
$$A - (A \cap B) = A - B.$$

$$\begin{aligned}
A - (A \cap B) &= A \cap (A \cap B)^c && \text{by the set difference law} \\
&= A \cap (A^c \cup B^c) && \text{by De Morgan's laws} \\
&= (A \cap A^c) \cup (A \cap B^c) && \text{by the distributive law} \\
&= \emptyset \cup (A \cap B^c) && \text{by the complement law} \\
&= (A \cap B^c) \cup \emptyset && \text{by the commutative law for } \cup \\
&= A \cap B^c && \text{by the identity law for } \cup \\
&= A - B && \text{by the set difference law.}
\end{aligned}$$

Example

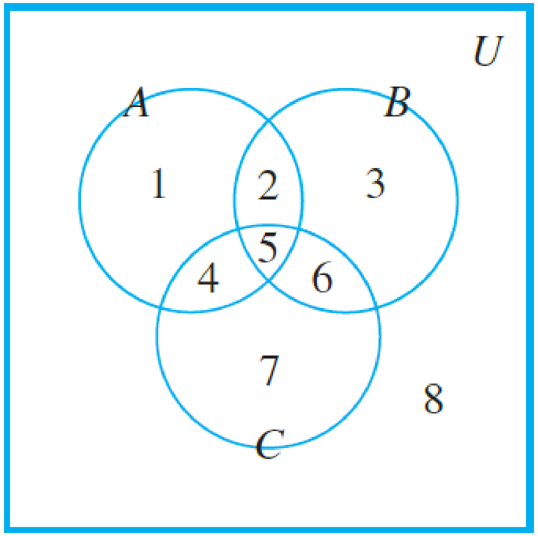
Give a counter-example to disprove that for all sets A, B, and C, $(A - B) \cup (B - C) = A - C$.





How to find a counter-example?

Example (cont'd)



Counterexample 1: Let $A = \{1, 2, 4, 5\}$, $B = \{2, 3, 5, 6\}$, and $C = \{4, 5, 6, 7\}$. Then

$$A - B = \{1, 4\}, \quad B - C = \{2, 3\}, \quad \text{and} \quad A - C = \{1, 2\}.$$

Hence

$$(A - B) \cup (B - C) = \{1, 4\} \cup \{2, 3\} = \{1, 2, 3, 4\}, \quad \text{whereas} \quad A - C = \{1, 2\}.$$

Since $\{1, 2, 3, 4\} \neq \{1, 2\}$, we have that $(A - B) \cup (B - C) \neq A - C$.

Example (cont'd)

- From the Venn diagram, it can be seen that the statement can be disproved if B contains an element that is not in A or in C.
- The following counterexample is simpler:

Counterexample 2: Let $A = \emptyset$, $B = \{3\}$, and $C = \emptyset$. Then

$$A - B = \emptyset, \quad B - C = \{3\}, \quad \text{and} \quad A - C = \emptyset.$$

Hence $\quad (A - B) \cup (B - C) = \emptyset \cup \{3\} = \{3\}, \quad \text{whereas} \quad A - C = \emptyset.$

Since $\{3\} \neq \emptyset$, we have that $(A - B) \cup (B - C) \neq A - C$.

Duality Principle

- The dual of an expression is obtained by interchanging $\cup$ and $\cap$, and interchanging U and $\emptyset$.
  - The dual of $A \cup (A \cap B)$ is $A \cap (A \cup B)$.
  - The dual of $U \cup \emptyset$ is $U \cap \emptyset$.
- The duality principle says that a set identity remains valid if the duals of both sides of the identity is taken.
  - De Morgan: $(A \cap B)^c = A^c \cup B^c$
  - Its Dual: $(A \cup B)^c = A^c \cap B^c$

Why is Duality Principle True?

- An algebraic proof for a set identity is simply a sequence of applying those basic rules.
- Duality holds for the basic algebraic rules, so we can apply duality to every step.

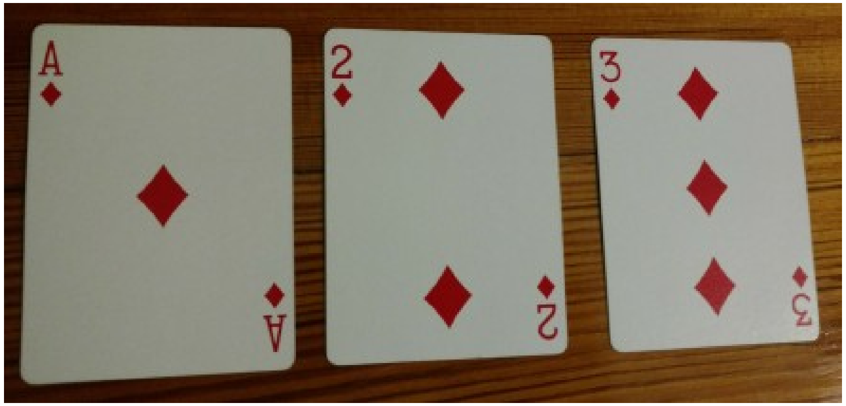
Unit 4.4

Derangements

Example of 3 Cards

- There are 3! = 6 permutations of these cards.

1) A 2 3
2) A 3 2
3) 2 A 3
4) 2 3 A
5) 3 A 2
6) 3 2 A

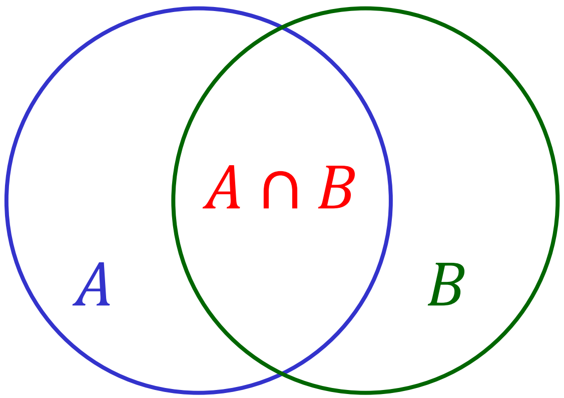


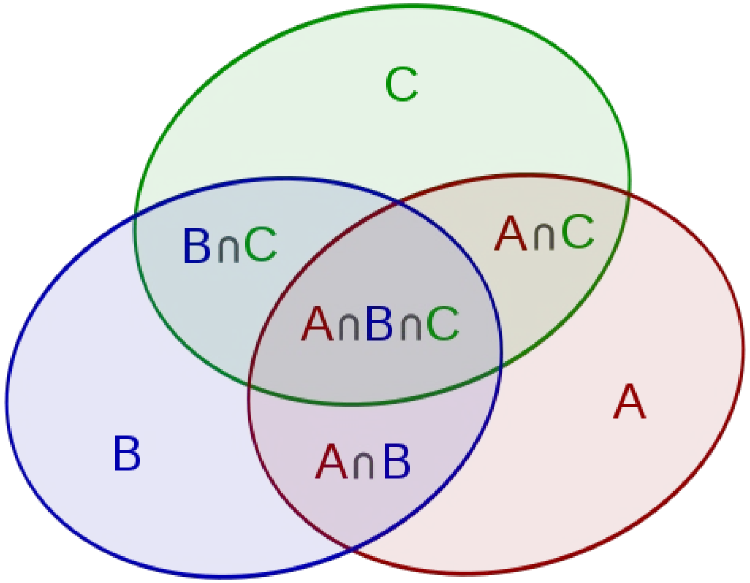
- There are two derangements (4 and 5).

Inclusion-Exclusion Principle

- For finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|$$





$$|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$$

Inclusion-Exclusion Principle

- The general form for n sets is

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j|$$

$$+ \sum_{1 \leqslant i < j < k \leqslant n} |A_i \cap A_j \cap A_k|$$

$$- \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n| .$$

Number of Derangements

- Let there be n cards.
- Let A_i be the set of all permutations such that the i-th position is preserved.
  - A_1 contains permutations of the form 1 _ _ _ _ _ _ _
  - A_2 contains permutations of the form _ 2 _ _ _ _ _ _
- $A_1 \cup A_2 \cup \cdots \cup A_n$ is the set of all permutations that have at least one position preserved.
- No. of derangements

$$!\,n = n! - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} \overbrace{|A_i|}^{(n-1)!} - \sum_{1\leqslant i<j\leqslant n} \overbrace{|A_i \cap A_j|}^{(n-2)!}$$

$$+ \sum_{1\leqslant i<j<k\leqslant n} \overbrace{|A_i \cap A_j \cap A_k|}^{(n-3)!}$$

$$- \cdots + (-1)^{n-1} \overbrace{|A_1 \cap \cdots \cap A_n|}^{(n-n)!=1}.$$

$$= \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \cdots + (-1)^{n-1}$$

$$= n! - \frac{n!}{2!} + \frac{n!}{3!} - \cdots + (-1)^{n-1}$$

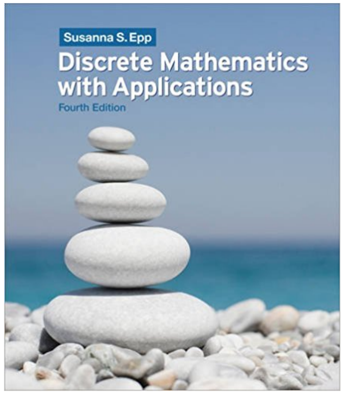
$$!n = n! - |A_1 \cup A_2 \cup \cdots \cup A_n|$$
$$!n = n! - \left(n! - \frac{n!}{2!} + \frac{n!}{3!} - \cdots + (-1)^{n-1}\right)$$

$$\frac{!n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \to e^{-1} \approx 0.37$$

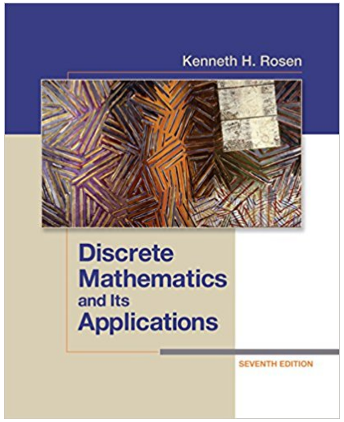
When $n = 10$, the derangement probability is very close to 0.37.

Alice has a higher chance (0.63) to win.

Recommended Reading



- Chapter 6, S. S. Epp, *Discrete Mathematics with Applications*, 4th ed., Brooks Cole, 2010.



- Sections 2.1 and 2.2, K. H. Rosen, *Discrete Mathematics and its Applications*, 7th ed., McGraw-Hill Education, 2011.