EE1001
Foundations of Digital Techniques

Logic – Part 2

Dr. K F Tsang

ee330015@cityu.edu.hk

香港城市大學
City University of Hong Kong

Outline

1. Need for Logic
2. Validity and Soundness of Argument
   - 2.1. Validity of Argument
   - 2.2. Deductive Argument
   - 2.3. Soundness of Argument
3. Propositional Logic
   - 2.1. Propositions and Truth Tables
   - 2.2. Logical Connectives (NOT/AND/OR)
   - 2.3. Tautologies and Contradictions
4. Conditionals
   - 4.1. Conditional Statement (If-then)
   - 4.2. Necessary & Sufficient Conditions

Logic – Part 1

Logic – Part 2

5. From Proposition to Predicate
   - 5.1. Nine Inference Rules to Construct Valid Arguments
6. Predicate Logic
   - 6.1. Universal Quantifier (∀) and Essential Quantifier (∃)
   - 6.2. Negation of Quantification
   - 6.3. Nested Quantification

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Class Intended Learning Outcomes (CILO)

- Understanding the validity and soundness of arguments in a logical way

- Identifying logic problems and solving them with logical methods.

- Designing and formulating logical arguments

Journey of Logic

1. Need for Logic

What is logic?

Why do we study logic?

Where can I find logic in engineering?

2. Validity and Soundness of Argument

How can I start when studying logic?

How can I know whether an argument is logical and valid?

3. Propositional Logic

What can I visualize different conditions (true/false) of propositions?

How can I formulate propositions mathematically?

What can I connect multiple propositions?

4. Conditionals

How to represent the conditional relations between two statements mathematically?

Fulfilling all conditions = success?

5. From Proposition to Predicate

How can I construct a valid argument systematically?

6. Predicate Logic

How to determine whether a statement is true for:
all cases (variables) or
some cases (variables) ?

In the end, you can think logically, analyze problem logically, and deduce a logical conclusion

From Proposition to Predicate

Inference Rules for Propositional Logic

- An argument is valid if its conclusion is a logical consequence of the premises
- Consider an argument in this form:

$$\begin{array}{rc} \text{premises} & p_1, p_2, \ldots, p_n \\ \hline \text{conclusion} & c \end{array}$$

The argument is valid if and only if $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow c$ is a tautology (i.e., always true)

Checking Validity by Truth Table

- To check the validity of an argument, we can always
  1. tabulate the truth values of premises and conclusion,
  2. check whether there is a row in which all premises are true (i.e., critical row) while the conclusion is false.
  3. The argument is valid if no such rows exist.

- Drawback: When there are n proposition variables, there are 2^n rows in the truth table, which grows exponentially in n.

Checking Validity by Truth Table

- $(p \to q) \land p \to q$ is a tautology (i.e., always true)

p	q	$p \to q$	$(p \to q) \land p$	$(p \to q) \land p \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

- There is only one critical row (i.e., the row with all premises true)
- In that row, the conclusion is true
- Therefore, it is a valid argument form

Do we have an efficient way to check the validity?

Inference Rules

- We can use rules of inference to construct valid arguments by taking premises, analyzing their syntax, and returning a conclusion.

- By repeatedly applying inference rules, we can demonstrate the validity of an argument by
  - starting with its premises,
  - taking one tiny valid step at a time, and finally
  - reaching its conclusion.

- We will consider nine elementary inference rules

Inference Rule #1

Modus Ponens
(method of affirming)

$$\begin{array}{c} p \rightarrow q \\ \underline{\quad p \quad} \\ q \end{array}$$

	variables		premises	conclusion
p	***q***	***p → q***	***p***	***q***
T	T	**T**	**T**	**T**
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

- There is only one critical row (i.e., the row with all premises true)
- In that row, the conclusion is true
- Therefore, MP is a valid argument form

modus ponens (Latin) translates to "***mode that affirms***"

Inference Rule #2 and 3

Modus Tollens
(method of denying)

$$\begin{array}{c} p \rightarrow q \\ \sim q \\ \hline \sim p \end{array}$$

If **you attend all classes**, then
you get an A in this course

you didn't get A in this course

You didn't attend all classes

Hypothetical Syllogism
(transitivity)

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline p \rightarrow r \end{array}$$

If you study hard, then **you will attend all classes**

If **you attend all classes**, then you will get an A

If you study hard, then you will get an A

Valid but may not be sound!

Inference Rules #4 to 6

Conjunction

$$\frac{\begin{array}{c} p \\ q \end{array}}{p \wedge q}$$

Simplification

$$\frac{p \wedge q}{p}$$

Absorption

$$\frac{p \rightarrow q}{p \rightarrow (p \wedge q)}$$

If the pandemic is over, then **Vanessa will visit Japan**

If the pandemic is over, then the pandemic is over and **Vanessa will visit Japan**

Need for Absorption

- The main use for Absorption will be in cases where you need to have $p \wedge q$ in order to take further step in the argument.

- Example:

1. $p \rightarrow q$ (Premise)
2. $p \wedge q \rightarrow r$ (Premise)
3. $p \rightarrow (p \wedge q)$ (Absorption 1)
4. $p \rightarrow r$ (HS 3,2)

Apply absorption to step 1

Apply hypothetical syllogism to steps 3 and 2

$$\begin{array}{c} p \rightarrow q \\ (p \wedge q) \rightarrow r \\ \hline p \rightarrow r \end{array}$$

Inference Rules #7 to 9

Addition

$$\frac{p}{p \vee q}$$

Disjunctive Syllogism

$$\begin{array}{c} p \vee q \\ \underline{\sim p} \\ q \end{array}$$

Constructive Dilemma

$$\begin{array}{c} (p \to q) \wedge (r \to s) \\ \underline{p \vee r} \\ q \vee s \end{array}$$

If **all students are in class**, then
the tests will be open-book;
and if **only a few students in class**, then
the test will be closed-book

Either **all students are in class**
or **only a few students in class**

Either the test will be open-book
or the test will be closed-book

Example of Applying Inference Rules

What conclusion can be drawn?

1) If Aunt Mary comes to visit, then Vincent will escape to his bedroom.

2) If Vincent escapes to his bedroom, then his Mom will be displeased.

3) His Mom is not displeased.

Step	Reason
1. $p \to q$	(Premise)
2. $q \to r$	(Premise)
3. $\sim r$	(Premise)
4. $p \to r$	(HS 1,2)
5. $\sim p$	(MT 3,4)

Conclusion: Aunt Mary did not come to visit.

Note: The inference steps may not be unique.

Summary of Inference Rules

Rule of Inference	Tautology	Rule of Inference	Tautology
1. Modus Ponens (method of affirming)	$\dfrac{p \to q \quad p}{q}$ $\quad ((p \to q) \wedge p) \to q$	6. Absorption	$\dfrac{p \to q}{p \to (p \wedge q)}$ $\quad (p \to q) \to (p \to (p \wedge q))$
2. Modus Tollens (method of denying)	$\dfrac{p \to q \quad \sim q}{\sim p}$ $\quad ((p \to q) \wedge (\sim q)) \to \sim p$	7. Addition	$\dfrac{p}{p \vee q}$ $\quad p \to (p \vee q)$
3. Hypothetical Syllogism (transitivity)	$\dfrac{p \to q \quad q \to r}{p \to r}$ $\quad ((p \to q) \wedge (q \to r)) \to (p \to r)$	8. Disjunctive Syllogism	$\dfrac{p \vee q \quad \sim p}{q}$ $\quad ((p \vee q) \wedge (\sim p)) \to (q)$
4. Conjunction	$\dfrac{p \quad q}{p \wedge q}$ $\quad (p \wedge q) \to (p \wedge q)$	9. Constructive Dilemma	$\dfrac{(p \to q) \wedge (r \to s) \quad p \vee r}{q \vee s}$ $\quad (((p \to q) \wedge (r \to s)) \wedge (p \vee r)) \to (q \vee s)$
5. Simplification	$\dfrac{p \wedge q}{p}$ $\quad (p \wedge q) \to p$		

Predicate Logic

Limitation of Propositional Logic

All men are mortal

Socrates was a man

So, Socrates was mortal.

This argument **can't be expressed** with propositional logic.

Why?

Predicates

- In ordinary language, *predicate* refers to the part of a sentence that gives information about the subject

$$\underbrace{\text{Issac Newton}}_{\textbf{subject}}\ \underbrace{\text{is a physicist.}}_{\textbf{predicate}}$$

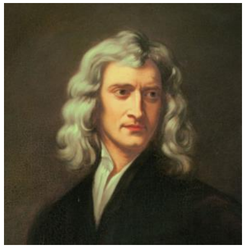
We use $P(x)$ to represent "x is a physicist," where x is a predicate variable.

Predicates

- In logic, a *predicate* is a statement that contains variables and that may be true or false depending on *the values of these variables*

Example:

- $P(x)$ represents "x is a physicist."
- P(Issac Newton) is true.



- P(Ludwig van Beethoven) is false.

Predicate Instantiated

- A *predicate instantiated* (where variables are evaluated in specific values) is a proposition.
  - *P*(Issac Newton) = “Issac Newton is a physicist.”

- The *domain* of a predicate variable is the set of all possible values that the variable may take.
  - The domain of *x* may be

{Issac Newton, Ludwig van Beethoven, William Shakespeare, Albert Einstein}

The Universal Quantifier ∀

- A quantifier tells the amount or quantity.

- The symbol ∀ denotes the *universal quantifier*, which means "given any" or "for all".

- "$\forall x \in D, Q(x)$" is a universal statement
  - It asserts that all elements in D have the property Q.
  - e.g. "$\forall x \in D, x \geq 0$" means all $x \in D$ are non-negative.
  - The domain *D* can be omitted if no ambiguity.

- "$\forall x \in D, Q(x)$" is true iff $Q(x)$ is true for every x in D

Universal Statements

- Example 1
  - Let $P(x,y)$="$\forall x,y \in D, x>y$", where *D* is the set of integers
  - $P(6,2)$ is true, but it doesn't mean that $P(x,y)$ is true
  - $P(3,5)$ is false, a counter-example which shows that $P(x,y)$ is false

- Example 2
  - Let $Q(x)$="$\forall x \in D$, $x^2 \geq x$", where D={1,2,3,4,5}
  - Check that

$$1^2 \geq 1 \quad 2^2 \geq 2 \quad 3^2 \geq 3 \quad 4^2 \geq 4 \quad 5^2 \geq 5$$

  - Hence, $Q(x)$ is true (by the method of exhaustion)

The Existential Quantifier ∃

- The symbol ∃ denotes the *existential quantifier*, which means "there exists".

- "$\exists x \in D, Q(x)$" is an existential statement
  - It asserts that at least one element in *D* has the property *Q*.
  - e.g. "$\exists x \in \boldsymbol{Z}$, $1 < x < 4.5$" means that there is an integer between 1 and 4.5.

- "$\exists x \in D, Q(x)$" is true iff $Q(x)$ is true for some x in D

Existential Statements

- Example 3
  - $\exists m \in \boldsymbol{Z}$ such that $m^2 = m$.
  - It can be shown to be true by the method of case (i.e., giving an example):

    1 is an integer and $1^2 = 1$

- Example 4
  - $\exists m \in \{5,6,7,8,9\}$ such that $m^2 = m$
  - It can be shown to be false by the method of exhaustion

$5^2=25\neq5$ $6^2=36\neq6$ $7^2=49\neq7$ $8^2=64\neq8$ $9^2=81\neq9$

Truth Values

	Statement	When True	When False
Universal Statement	$\forall x \in D, Q(x)$	$Q(x)$ is true for every x	There is one x for which $Q(x)$ is false
Existential Statement	$\exists x \in D, Q(x)$	There is one x for which $Q(x)$ is true	$Q(x)$ is false for every x.

Assume that $D=\{x_1,x_2,\ldots,x_n\}$.

$$\forall x \in D,\ Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n)$$
$$\exists x \in D,\ Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)$$

Universal Conditional Statements

- A universal conditional statement takes the form

$$\forall x \in D, \text{ if } P(x), \text{ then } Q(x)$$

- Example:

$$\forall x \in \boldsymbol{R}, \text{ if } x>2, \text{ then } x^2>4$$

  - Interpretation:
    "For all real numbers *x*, if *x* is greater than 2, then its square is greater than 4."
  - "If a real number is greater than 2, then its square is greater than 4."
    - An implicitly quantified statement, which occurs commonly in mathematics writing.

Negation of Quantification

- The negation of a universal statement is logically equivalent to an existential statement

Statement	When True	When False
$\forall x \in D, Q(x)$	$Q(x)$ is true for every x	There is one x for which $Q(x)$ is false
$\exists x \in D, Q(x)$	There is one x for which $Q(x)$ is true	$Q(x)$ is false for every x.

- Example:

$$\sim[\forall x \in D,\ Q(x)\ ] \equiv\ \exists x \in D \text{ such that } \sim Q(x)$$

Negation of Quantification

- The negation of an existential statement is logically equivalent to a universal statement

Statement	When True	When False
$\forall x \in D, Q(x)$	$Q(x)$ is true for every x	There is one x for which $Q(x)$ is false
$\exists x \in D, Q(x)$	There is one x for which $Q(x)$ is true	$Q(x)$ is false for every x.

- Example:

$$\sim[\exists x \in D \text{ such that } Q(x)] \equiv \forall x \in D, \sim Q(x)$$

Nested Quantification (∀,∃)

- Two quantifiers are nested if one is within the scope of the other.

- "Every smartphone has a function that it will always be installed"

  - Domains: S= {smartphones}, F= {functions}
  - Predicate $L(x,y)$: The smartphone x always install the function y
  - In symbols,

$$\forall x \in S, \exists y \in F, L(x,y)$$

A Closer Look

$$\forall x \in S, \underbrace{\boxed{\exists y \in F, L(x,y)}}_{P(x)}$$

The statement is equivalent to

$$\forall x \in S, P(x)$$

where

$$P(x) = \exists y \in F, L(x,y)$$

Nested Quantification ($\exists,\forall$)

- "There is a book which every EE1001 student reads"
  - Domains: B= {books}, E= {EE1001 students}
  - Predicate $R(x,y)$: The EE1001 student x reads the book y
  - In symbols,

$$\exists y \in B,\ \forall x \in E,\ R(x,y)$$

Nested Quantification (∀,∀) (∃,∃)

Domains: R= {rabbits}, T= {tortoises}

Predicate $F(x,y)$: x is faster than y

"All rabbits are faster than all tortoises"

$\forall x \in R \quad \forall y \in T,\ F(x,y)$

"There is a tortoise which is faster than some rabbits"

$\exists x \in R \quad \exists y \in T,\ F(\mathbf{y,x})$

Negation of Nested Quantification

To find $\sim(\forall x \in S, \exists y \in F, L(x,y))$

$$\sim(\forall x \in S, \exists y \in F, L(x,y)) \equiv \exists x \in S, \sim(\exists y \in F, L(x,y))$$
$$\equiv \exists x \in S, \forall y \in F, \sim L(x,y)$$

Similarly,

$$\sim(\exists y \in B, \forall x \in E, R(x,y)) \equiv \forall y \in B, \sim(\forall x \in E, R(x,y))$$
$$\equiv \forall y \in B, \exists x \in E, \sim R(x,y)$$

Order of Nesting

- Equivalent or not?

$$\forall x \ \ \exists y,\ F(x,y) \quad \stackrel{?}{\equiv} \quad \exists y \ \ \forall x,\ F(x,y)$$

Every student gets an A in some courses $\neq$ There are some courses that all students get A

How about $\forall x$, $\forall y$?
And $\exists x$, $\exists y$?

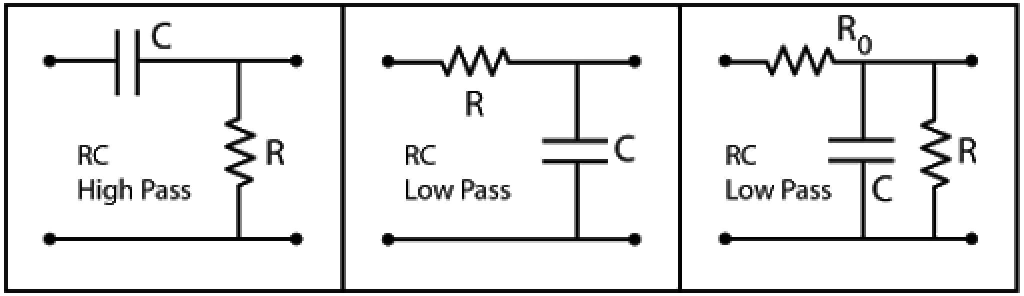
Order of Nesting

- If two (or more) quantifiers are the **same**, their **order doesn't matter**.
- If two quantifiers are **different**, their order **does matter**.

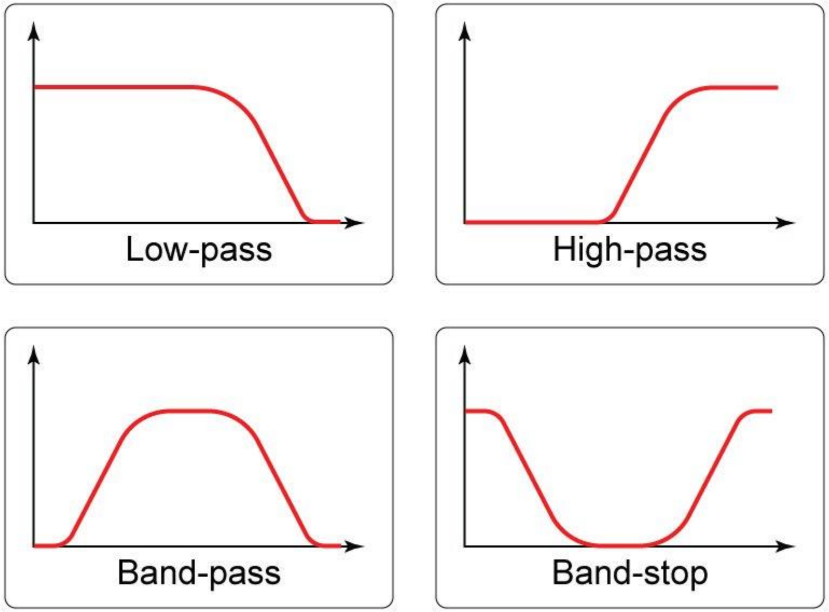
- Example 1
  - Let $P(x,y)$ be $(x+y)^2 = x^2 + 2xy + y^2$.

    $\forall x \in \boldsymbol{R} \quad \forall y \in \boldsymbol{R}, \quad P(x,y)$
    - It means that given any x, no matter how we choose, $P(x,y)$ is true;
    - Given any y, no matter how we choose, $P(x,y)$ is true
- Example 2
  - $\forall x \in \boldsymbol{R}\ \exists y \in \boldsymbol{R}, \quad x>y$
    - Meaning: Given any real number x, we can always find a real number y which is less than x
  - $\exists y \in \boldsymbol{R}\ \forall x \in \boldsymbol{R}, \quad x>y$
    - Meaning: There exists a real number y which is less than all real number x
    - It is false because there is no smallest real number.

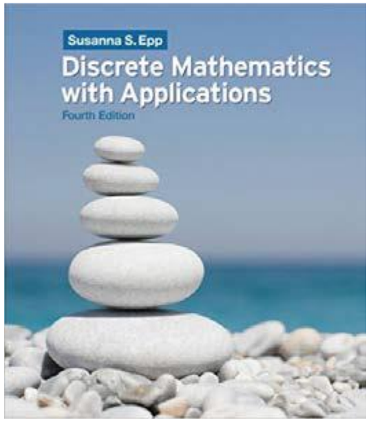
Predicate in Engineering



$\forall x$, $|F(x)| = 1$ for $x < f_{\text{low-cutoff}}$ and $|F(x)| = 0$ for $x > f_{\underline{\text{high}}\text{-cutoff}}$



Recommended Readings



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