Chapter 11

Equilibrium and Elasticity

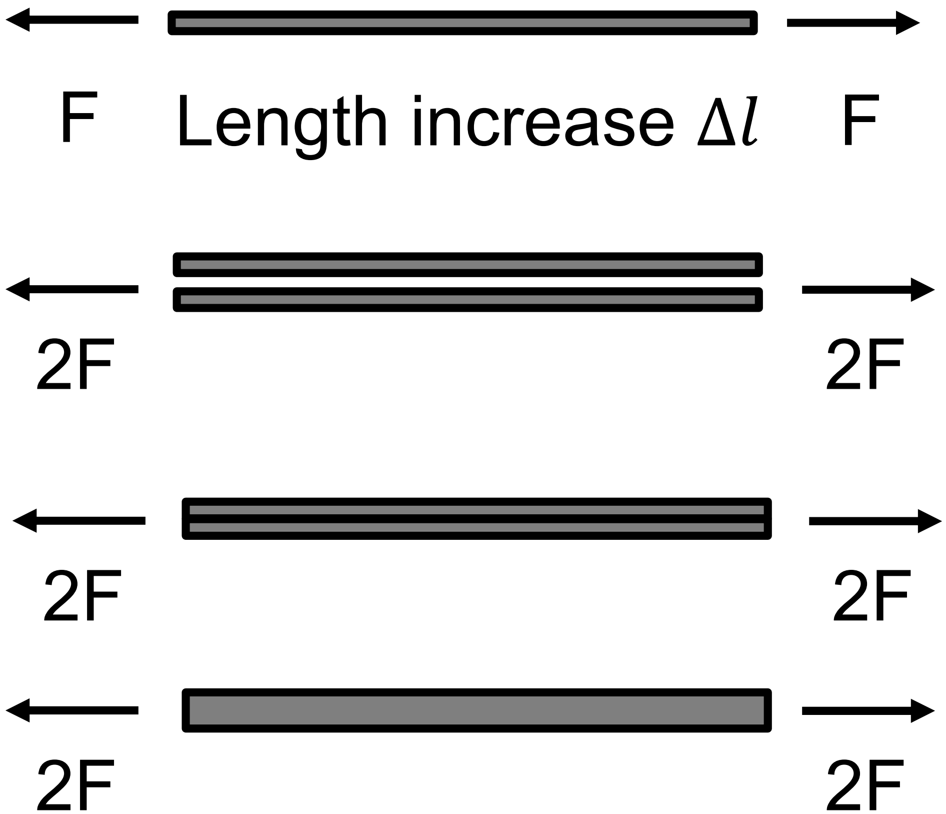
Goals for Chapter 11

- To study the conditions for equilibrium of a body
- To understand center of gravity and how it relates to a body's stability
- To solve problems for rigid bodies in equilibrium
- To analyze situations involving tension, compression, pressure, and shear
- To investigate what happens when a body is stretched so much that it deforms or breaks

Strain, stress, and elastic moduli

- Stretching, squeezing, and twisting a real body causes it to deform. We shall study the relationship between forces and the deformations they cause.

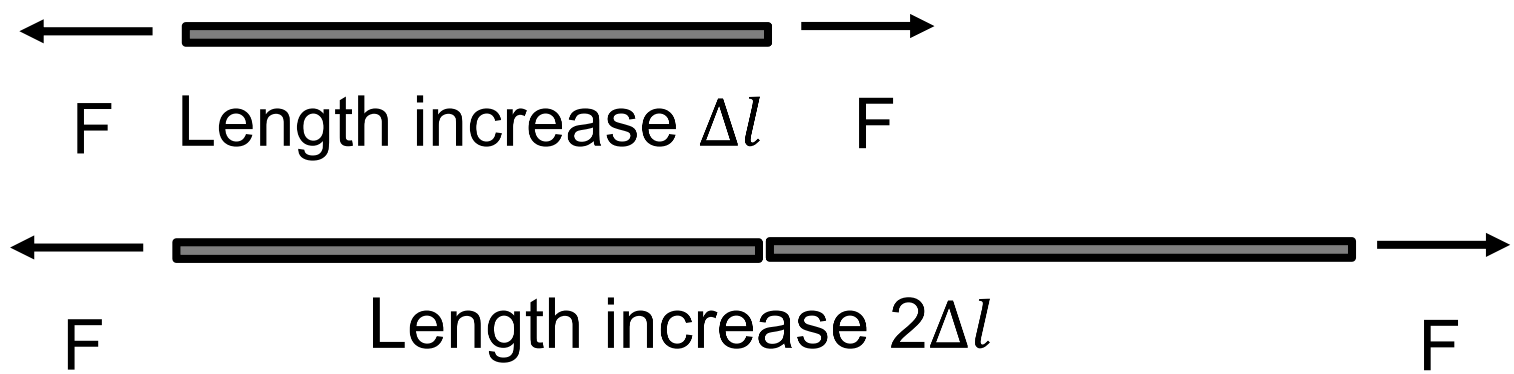
- *Stress* is the force per unit area.



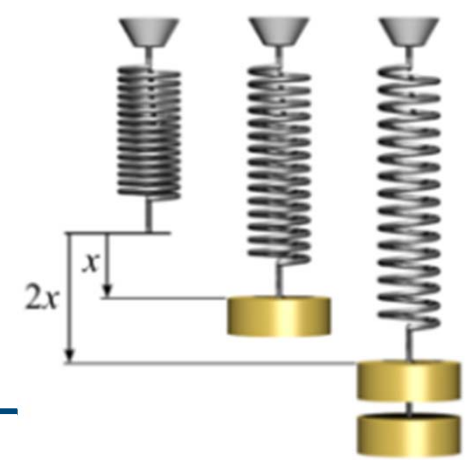
Strain, stress, and elastic moduli

- Stretching, squeezing, and twisting a real body causes it to deform. We shall study the relationship between forces and the deformations they cause.

- *Stress* is the force per unit area.
- *Strain* is the fractional deformation due to the stress.



- *Elastic modulus* is stress divided by strain.
- The proportionality of stress and strain is called (generalized) *Hooke's law*.



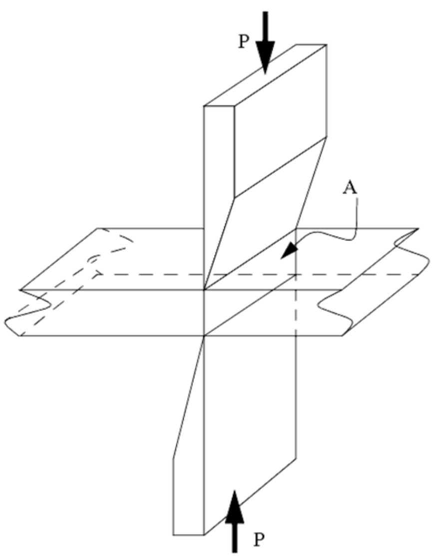
Strain, stress, and elastic moduli

Three types of stress:

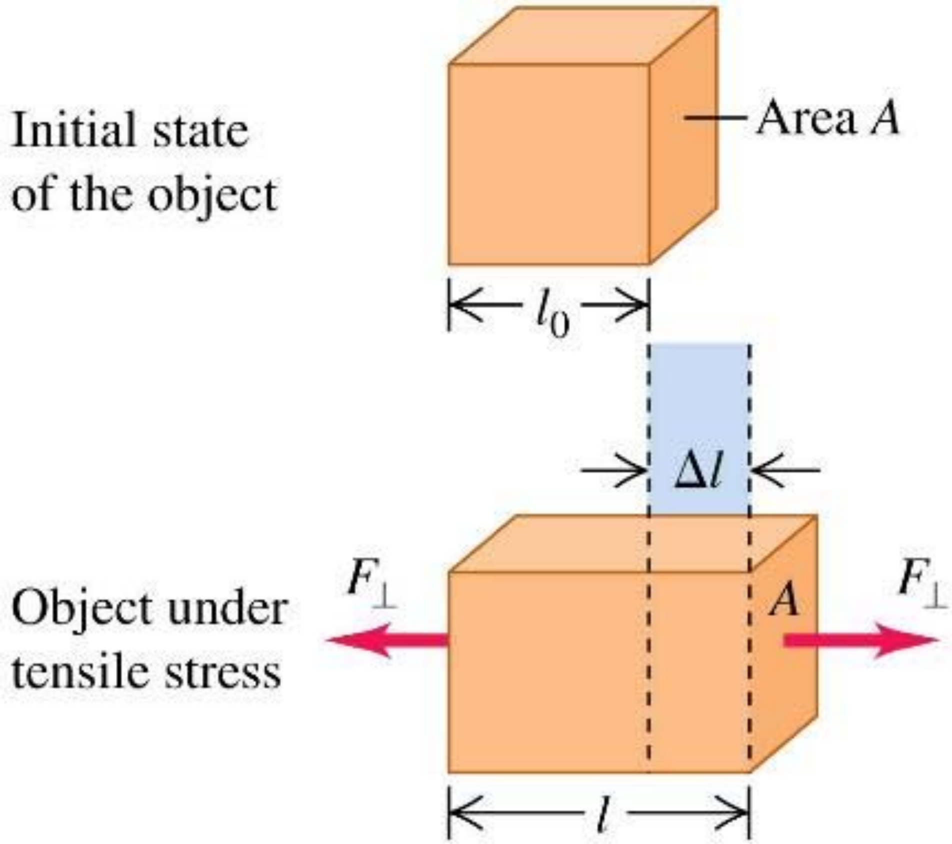
a) Guitar strings under tensile stress, being stretched by forces acting at their ends.

b) A diver under bulk stress, being squeezed from all sides by forces due to water pressure.

c) A ribbon under shear stress, being deformed and eventually cut by forces exerted by the scissors.



Tensile stress and strain



$$\text{Tensile stress} = \frac{F_\perp}{A} \qquad \text{Tensile strain} = \frac{\Delta l}{l_0}$$

- An object in tension.
- The net force and net torque on the object are zero, but the object deforms.
- The tensile stress produces a tensile strain.

Young's modulus

- Experiment shows that for small tensile stress, stress and strain are proportional.
- The corresponding elastic modulus is called **Young's modulus**.

Force applied perpendicular to cross section

Young's modulus for tension

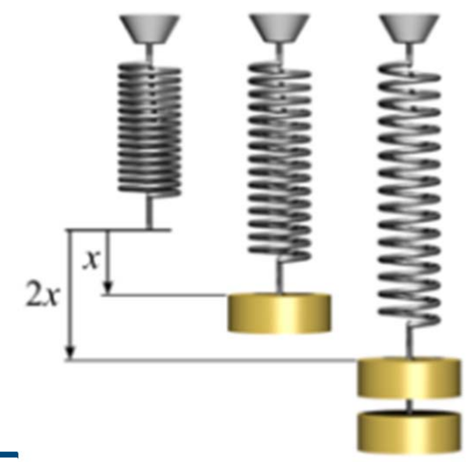
Original length (see Fig. 11.14)

$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_{\perp}/A}{\Delta l / l_0} = \frac{F_{\perp}}{A}\frac{l_0}{\Delta l}$$

Elongation (see Fig. 11.14)

Cross-sectional area of object

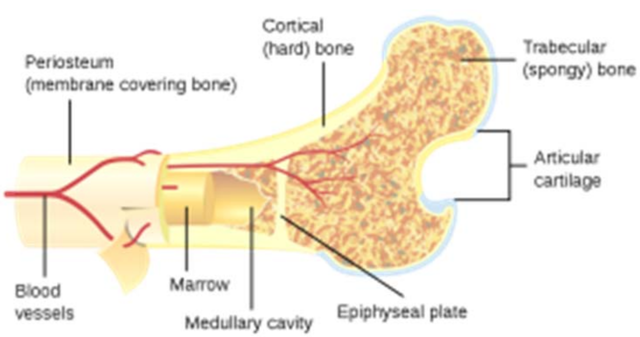
- Compare with original Hooke's law: $F = kx$, where k is the stiffness.
- k is different for two springs of different shape, but same material.
- Y is the same for the same material.



Some values of elastic moduli

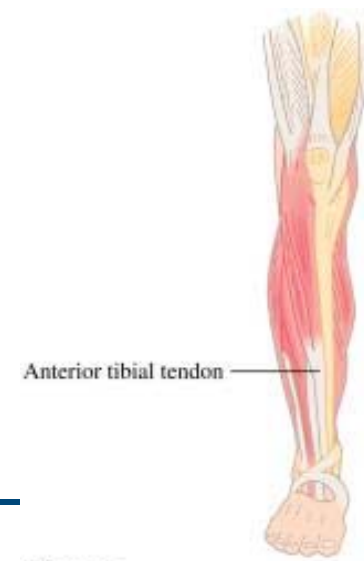
Table 11.1 Approximate Elastic Moduli

Material	Young's Modulus, Y (Pa)	Bulk Modulus, B (Pa)	Shear Modulus, S (Pa)
Aluminum	7.0×10^{10}	7.5×10^{10}	2.5×10^{10}
Brass	9.0×10^{10}	6.0×10^{10}	3.5×10^{10}
Copper	11×10^{10}	14×10^{10}	4.4×10^{10}
Crown glass	6.0×10^{10}	5.0×10^{10}	2.5×10^{10}
Iron	21×10^{10}	16×10^{10}	7.7×10^{10}
Lead	1.6×10^{10}	4.1×10^{10}	0.6×10^{10}
Nickel	21×10^{10}	17×10^{10}	7.8×10^{10}
Steel	20×10^{10}	16×10^{10}	7.5×10^{10}

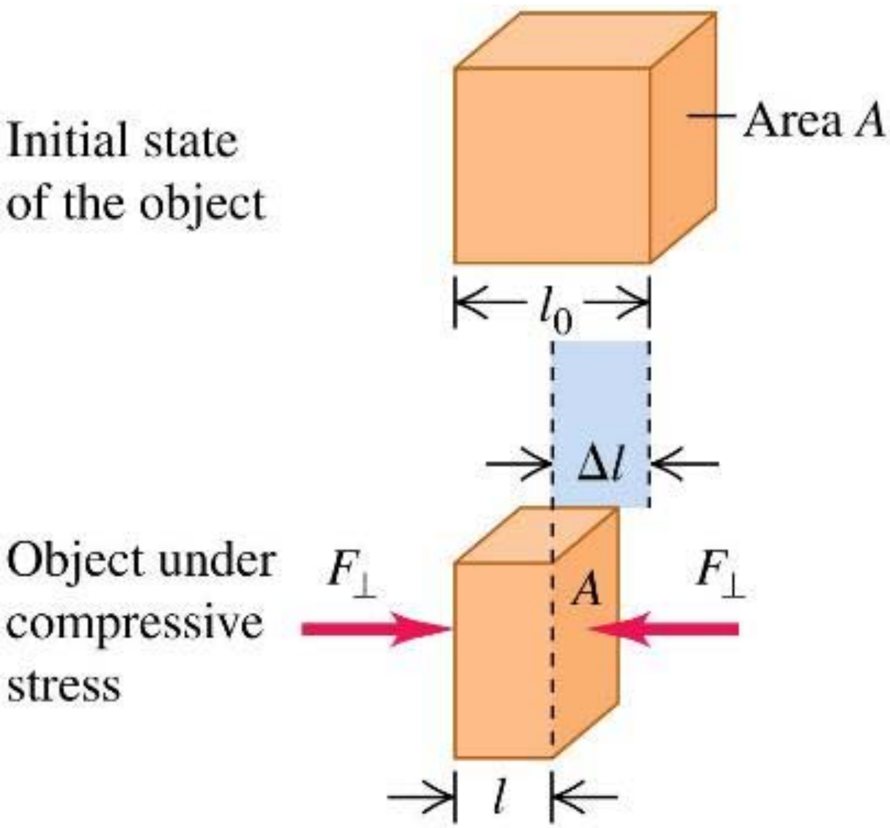


The cortical bone has Young's modulus of the order of 7 - 30 × 10^9 Pa

The anterior tibial tendon has a Young's modulus of 1.2 × 10^9 Pa. It stretches substantially (up to 2.5% of its length) during walking and running.



Compressive stress and strain

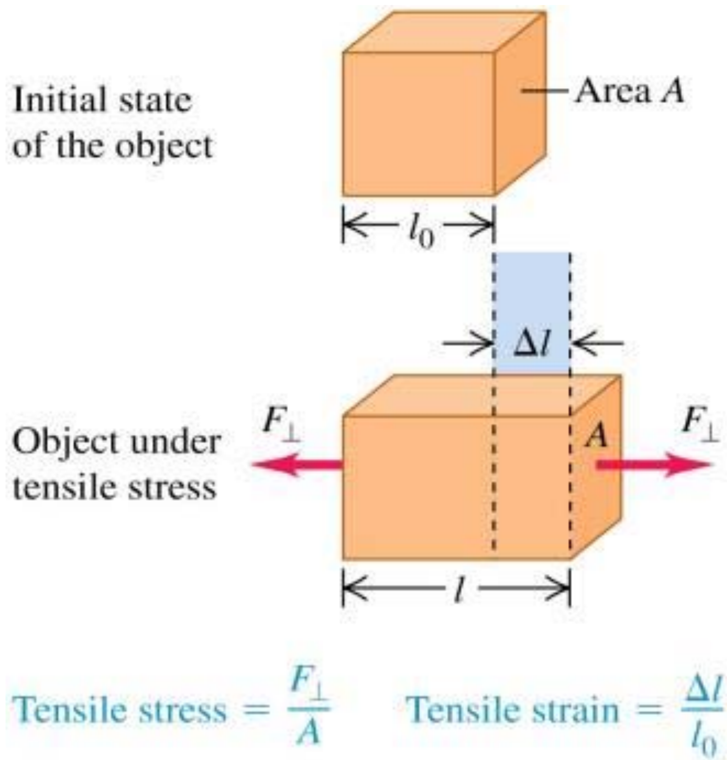


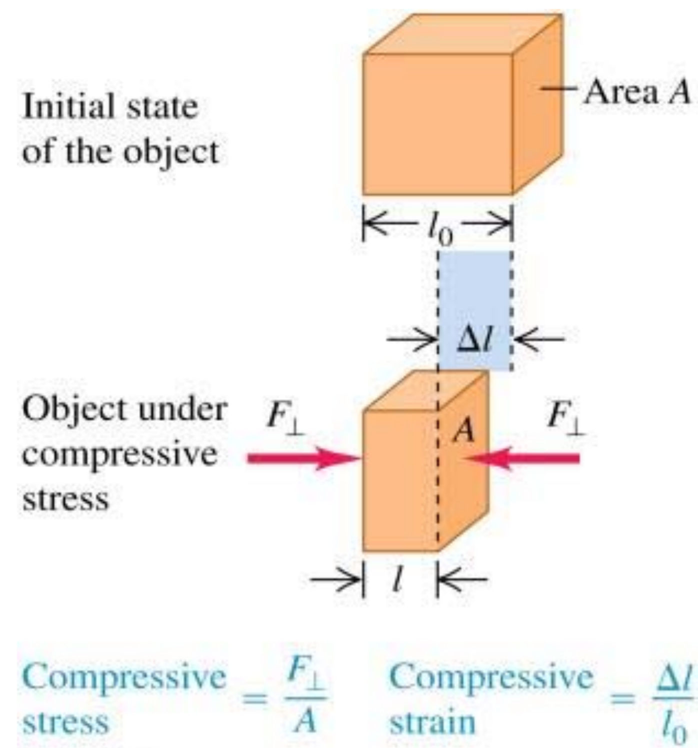
$$\text{Compressive stress} = \frac{F_\perp}{A} \qquad \text{Compressive strain} = \frac{\Delta l}{l_0}$$

- The compressive stress and compressive strain are defined as the opposite of tensile stress and strain.
- Their ratio is also Young’s modulus.

Tensile and compressive stress and strain

- *Tensile stress* = $F_{\perp}/A$ (unit: N/m^2 = Pa) and *tensile strain* = $\Delta l/l_0$.
- The *stress* and *strain* are defined in similar ways for both tensile and compressive cases.
- *Young's modulus* is tensile stress divided by tensile strain, and is also equal to compressive stress divided by compressive strain.



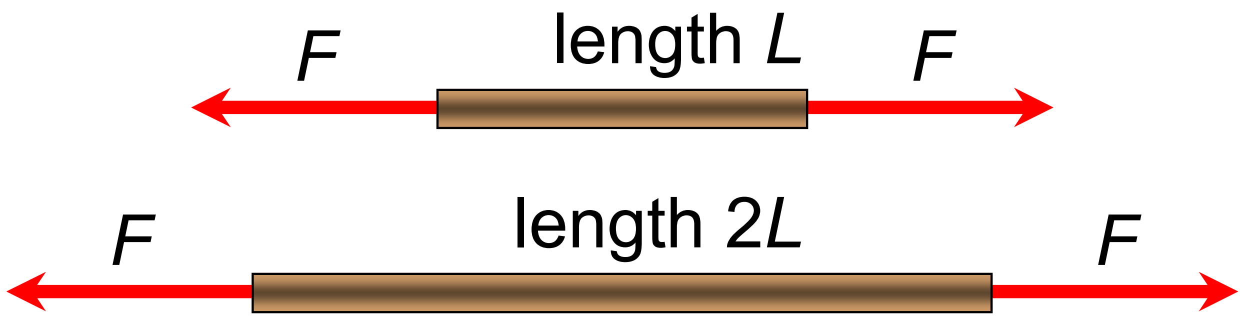


Example

- A Steel rod 2.0 m long has a cross-sectional area of 0.30 cm^2. It is hung by one end from a support, and a 550 kg milling machine is hung from its other end. Determine the stress on the rod and the resulting strain and elongation.
- *Tensile stress* $= \sigma = F_{\perp}/A = 550 \times 9.8/(0.30 \times 10^{-4}) = 1.8 \times 10^{8}$ Pa
- *tensile strain* $= \varepsilon = \Delta l/l_0 = \sigma/E = 1.8 \times 10^{8}/20 \times 10^{10} = 9.0 \times 10^{-4}$
- *Elongation* $= \Delta l = \varepsilon l_0 = 9.0 \times 10^{-4} \times 2.0 = 1.8$ mm

Q11.5

Two rods are made of the same kind of steel and have the same diameter.



A force of magnitude F is applied to the end of each rod. Compared to the rod of length L, the rod of length $2L$ has

A. more stress and more strain.

B. the same stress and more strain.

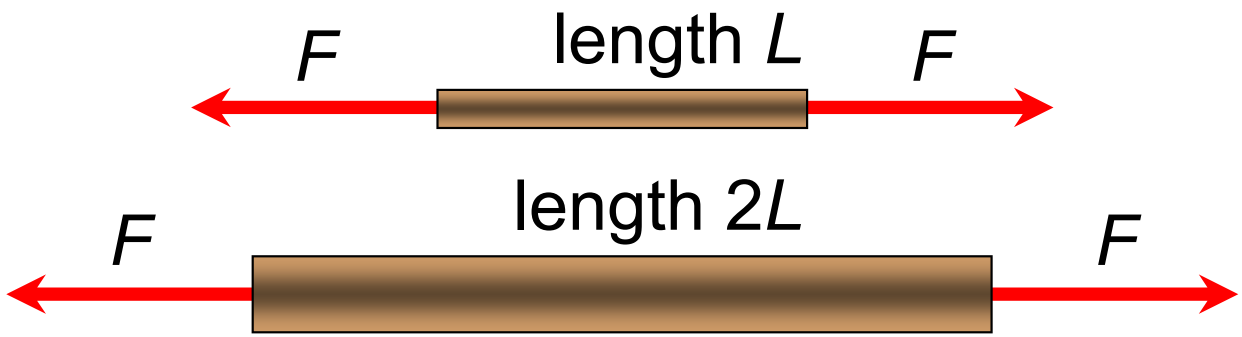
C. the same stress and less strain.

D. less stress and less strain.

E. the same stress and the same strain.

Q11.6

Two rods are made of the same kind of steel. The longer rod has a greater diameter.



A force of magnitude *F* is applied to the end of each rod. Compared to the rod of length *L*, the rod of length 2*L* has

A. more stress and more strain.

B. the same stress and more strain.

C. the same stress and less strain.

D. less stress and less strain.

E. the same stress and the same strain.

Another Example

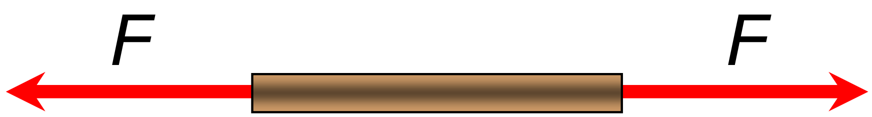
- A copper rod of cross-sectional area 0.500 cm^2 and length 1.00 m is elongated by 2.00×10^{-2} mm, and a steel rod of the same cross-sectional area but 0.100 m in length is elongated by 2.00×10^{-3} mm. Young's Modulus: $Y_{\text{copper}} = 1.1 \times 10^{11}\text{Pa}$, $Y_{\text{steel}} = 2.0 \times 10^{11}\text{Pa}$,
- Which rod has greater tensile *strain*?
- Which rod is under greater tensile *stress*?

Strain:
Copper rod: $2*10^{-2}*10^{-3}/1=2*10^{-5}$
Steel rod: $2*10^{-3}*10^{-3}/0.1=2*10^{-5}$ (same)

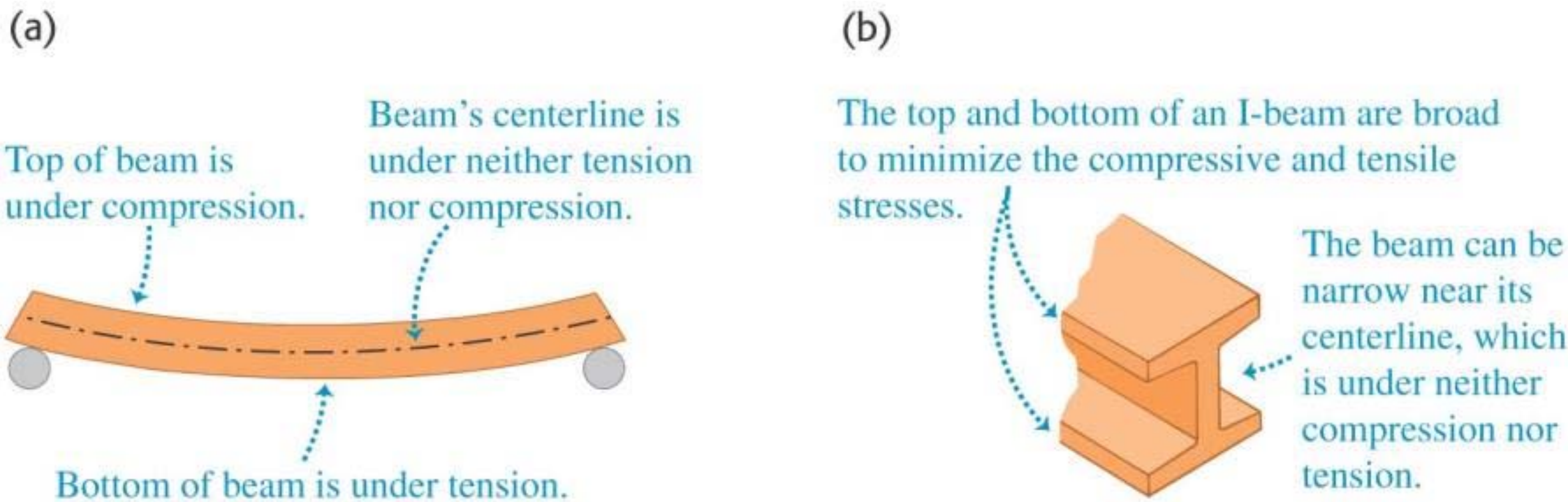
Stress=strain * Y
steel has larger Y, so the steel rod has greater stress.

Tensile stress and strain

- Wire under tension is a simple case.

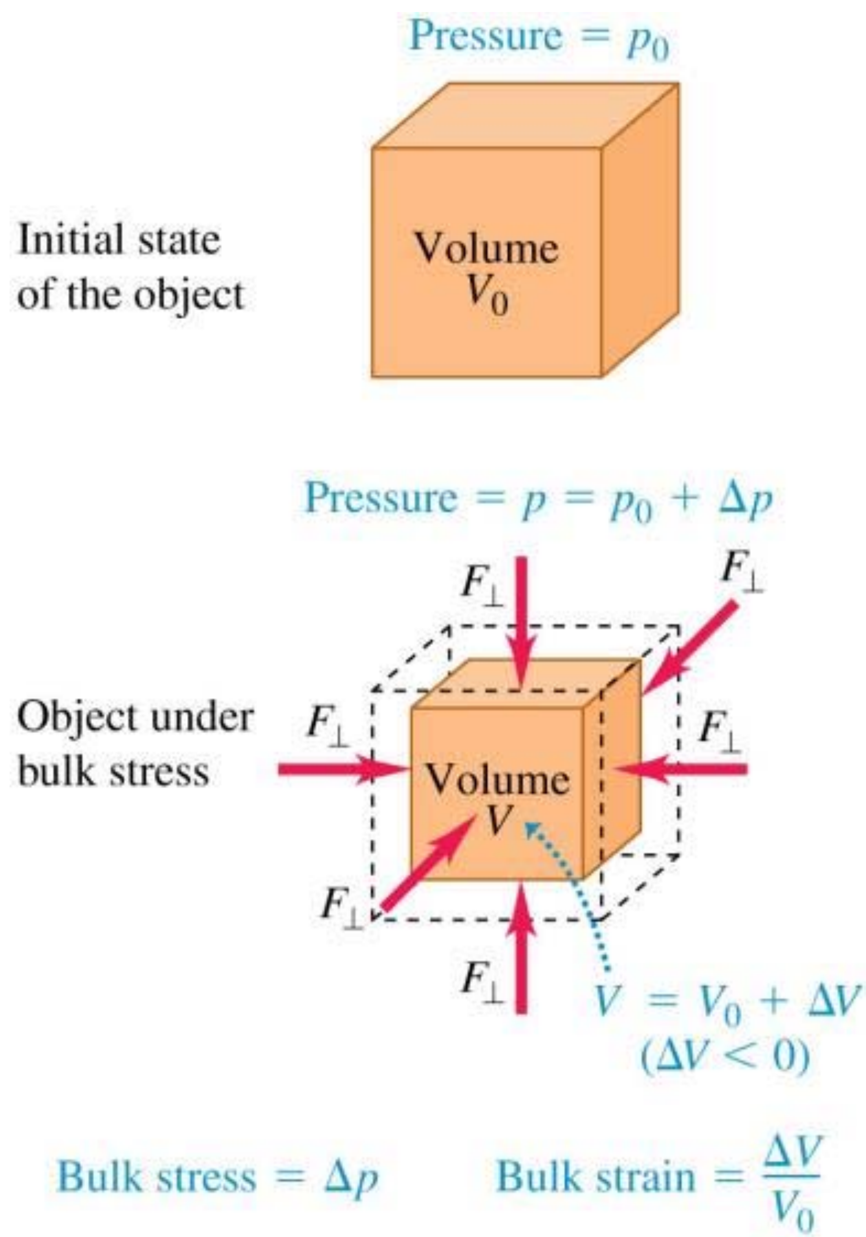


- In many cases, a body can experience both tensile and compressive stress at the same time, as shown below.



Bulk stress and strain

- Pressure in a fluid is force per unit area: $p = F_{\perp}/A$.
- *Bulk stress* is pressure change Δp and *bulk strain* is fractional volume change $\Delta V/V_0$.
- *Bulk modulus* is bulk stress divided by bulk strain and is given by $B = -\Delta p/(\Delta V/V_0)$.



Some values of elastic moduli

Table 11.1 Approximate Elastic Moduli

Material	Young's Modulus, Y (Pa)	Bulk Modulus, B (Pa)	Shear Modulus, S (Pa)
Aluminum	7.0×10^{10}	7.5×10^{10}	2.5×10^{10}
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Compressibility

- The reciprocal of the bulk modulus is called the **compressibility** and is denoted by k.

$$k = \frac{1}{B} = -\frac{\Delta V/V_0}{\Delta p} = -\frac{1}{V_0}\frac{\Delta V}{\Delta p} \qquad \text{(compressibility)}$$

High k, easy to compress.

Low k, difficult to compress.

Example

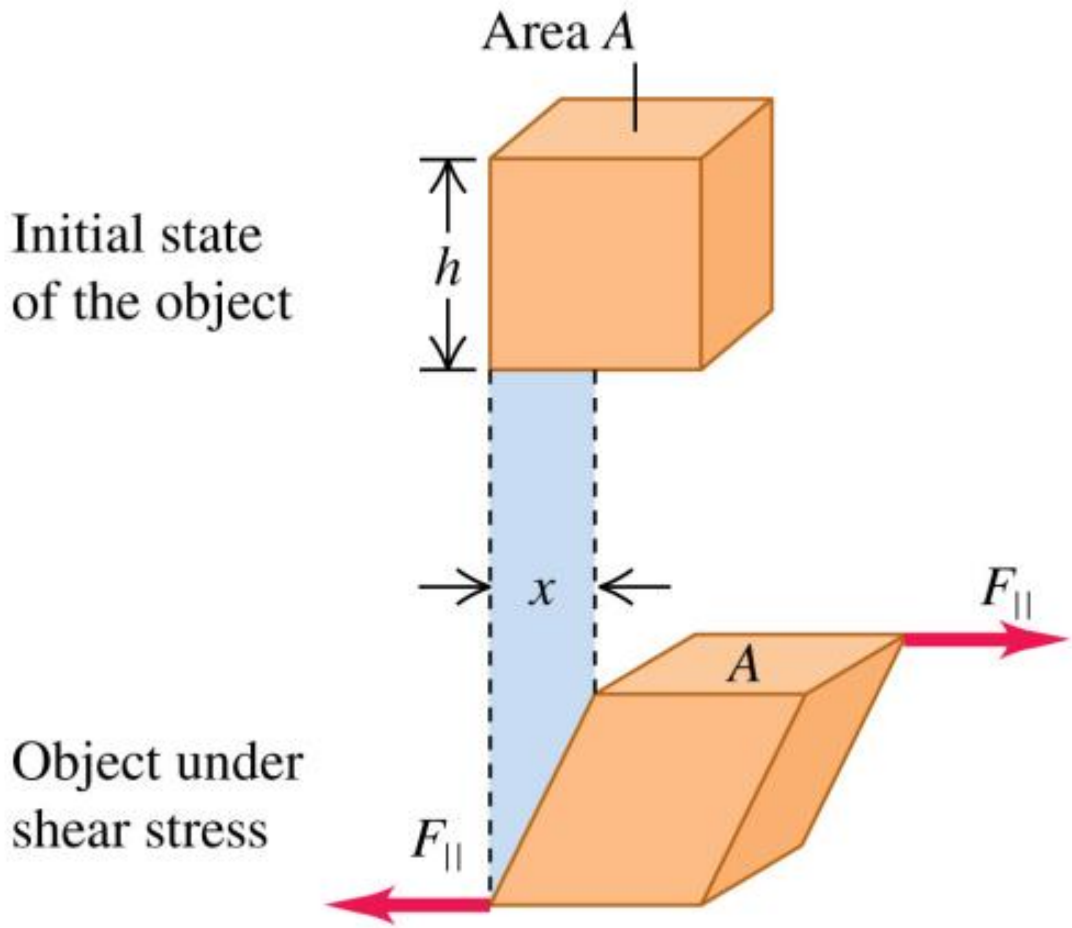
- A hydraulic press contains 0.25 m^3 (250L) of oil. Find the decrease in the volume of the oil when it is subjected to a pressure increase $\Delta p = 1.6 \times 10^7$ Pa (~ 160 atm). The bulk modulus of the oil is $B = 5.0 \times 10^9$ Pa and its compressibility is $k = 1/B = 2.0 \times 10^{-10}$ Pa^{-1}.

- $\Delta V = -V\Delta p/B = 0.25 \times 1.6 \times 10^7/5.0 \times 10^9 = -8.0 \times 10^{-4}$ m^3

- Bulk strain = $\Delta V/V_0 = -8.0 \times 10^{-4}/0.25 = -0.0032$ (or -0.32%)

Shear stress and strain

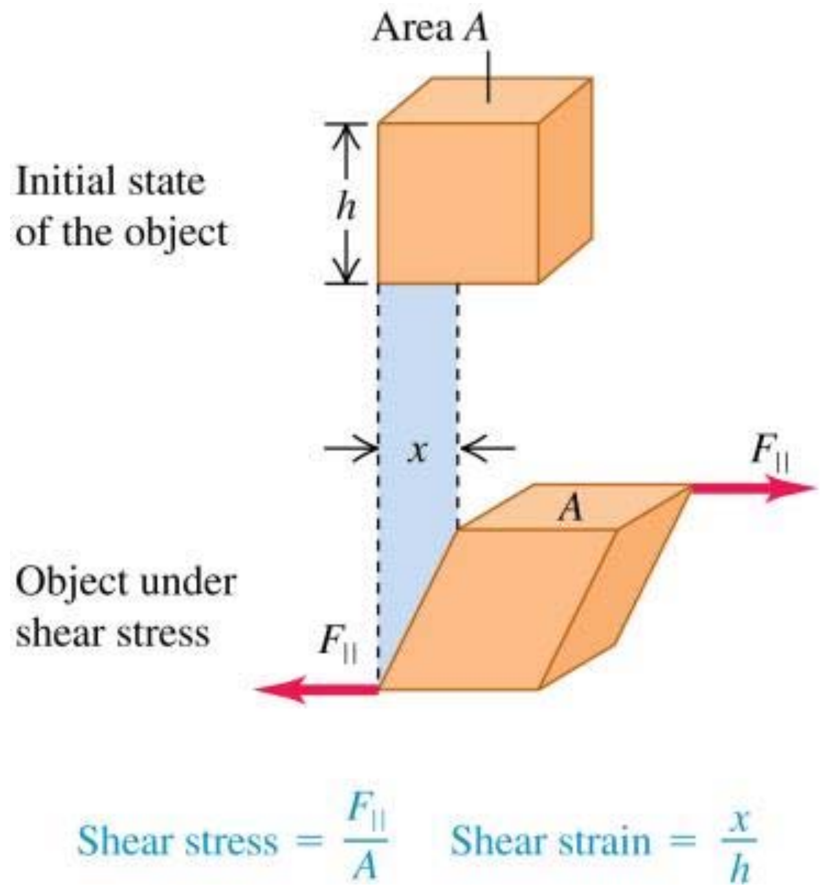
- *Shear stress* is $F_{||}/A$ and *sheer strain* is x/h.
- *Shear modulus* is sheer stress divided by sheer strain, and is given by $S = (F_{||}/A)(h/x)$.



$$\text{Shear stress} = \frac{F_{||}}{A} \qquad \text{Shear strain} = \frac{x}{h}$$

Example

- A brass base plate of an outdoor sculpture experiences shear forces in an earthquake. The plate is 0.80 m square and 0.50 cm thick. What is the force exerted on each of its edges if the resulting displacement x is 0.16 mm? Shear modulus is $S_{\text{brass}} = 3.5 \times 10^{10}$Pa



- $S = (F_{||}/A)(h/x)$
- $F_{||} = (AxS)/h = 0.8^2 \times 0.16 \times 10^{-3} \times 3.5 \times 10^{10}/(0.50 \times 10^{-2})$
  $= 7.2 \times 10^8$ N

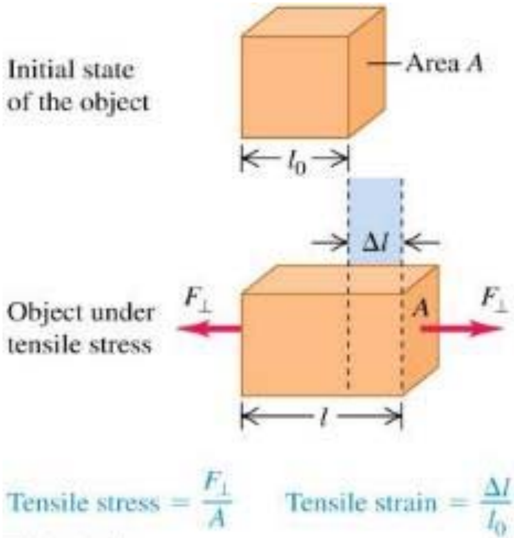
Review

- Stress is force per area
- Strain is unitless deformation, $\Delta l / l_0$, except for the bulk strain, $\Delta V / V_0$

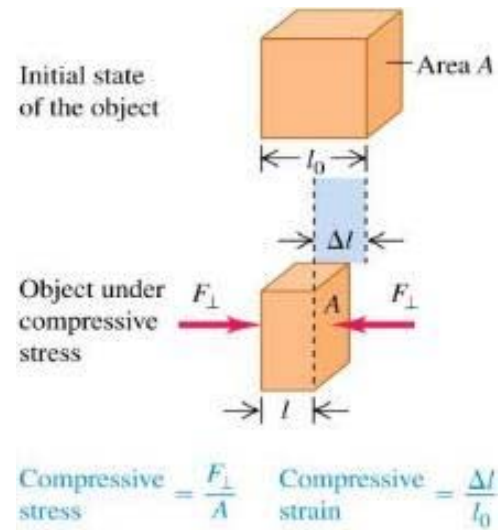
Force direction		Stress name	Strain name	stress/strain
Perpendicular to surface	Compression in one direction	Compressive stress	Compressive strain	Young's modulus
	Elongation in one direction	Tensile stress	Tensile strain	
	Compression in **all directions**	Bulk stress (pressure)	Bulk strain	Bulk modulus
Parallel to surface	Any direction	Shear stress	Shear strain	Shear modulus

Which type of stress we examined today does not satisfy the second condition for equilibrium, $\sum\tau = 0$?

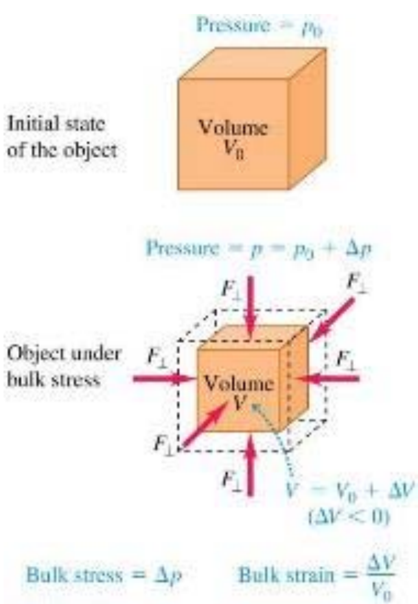
A.



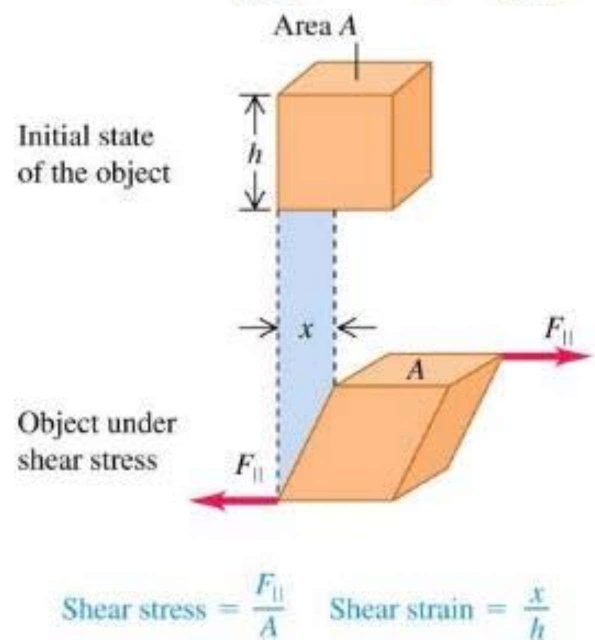
B.



C.



D.



Q11.7

You use a nail to mount a wooden plaque on your wall. Which kind of stress on the nail plays the primary role in keeping the plaque securely attached to the wall?

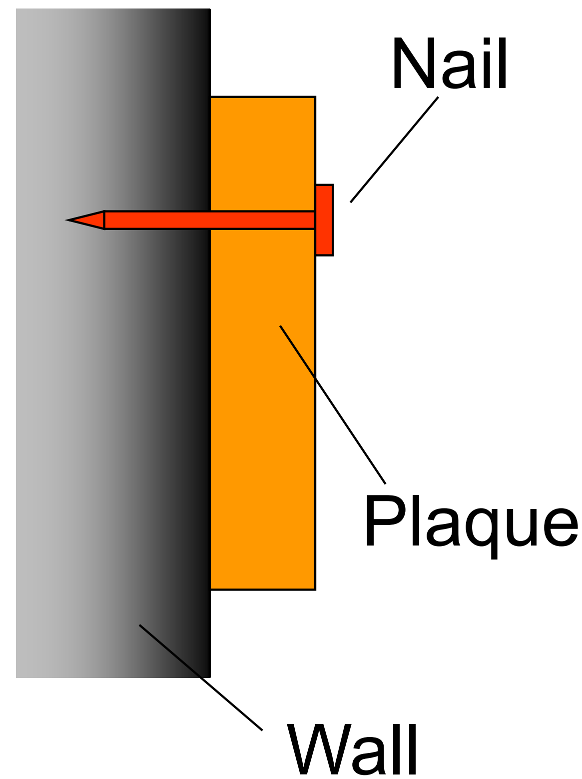
A. Tensile stress

B. Compressive stress

C. Bulk stress

D. Shear stress

E. Plastic stress

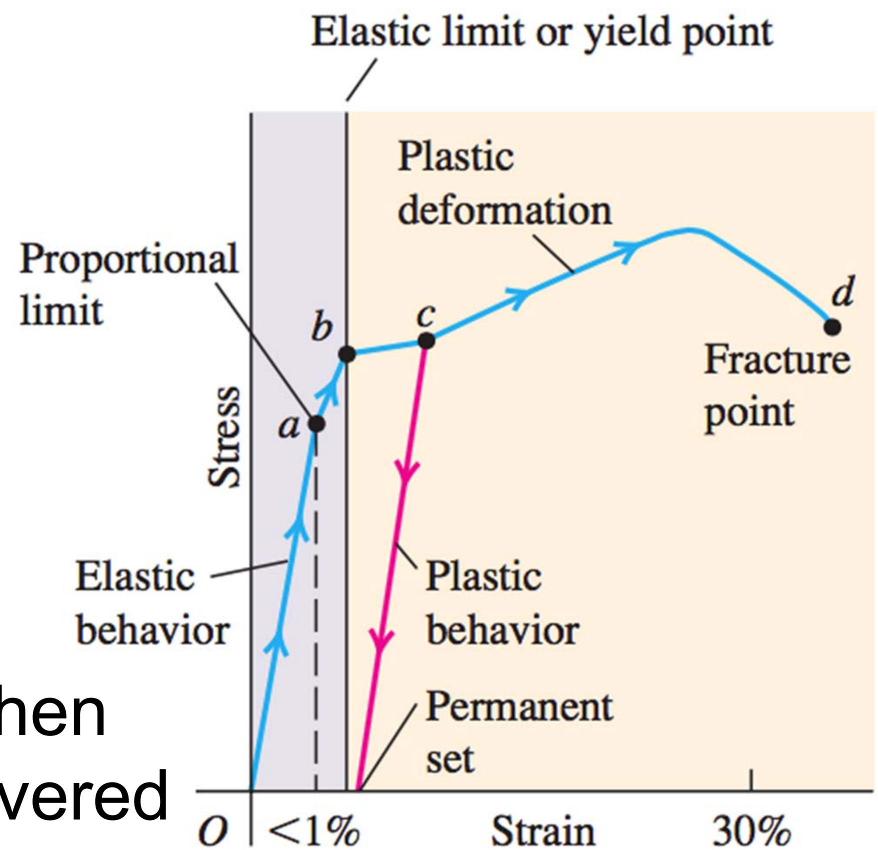


Elasticity and plasticity

- Hooke's law—the proportionality of stress and strain in elastic deformations—has a limited range of validity.
- If you pull, squeeze, or twist anything hard enough, it will bend or break.
- Can we be more precise than that?

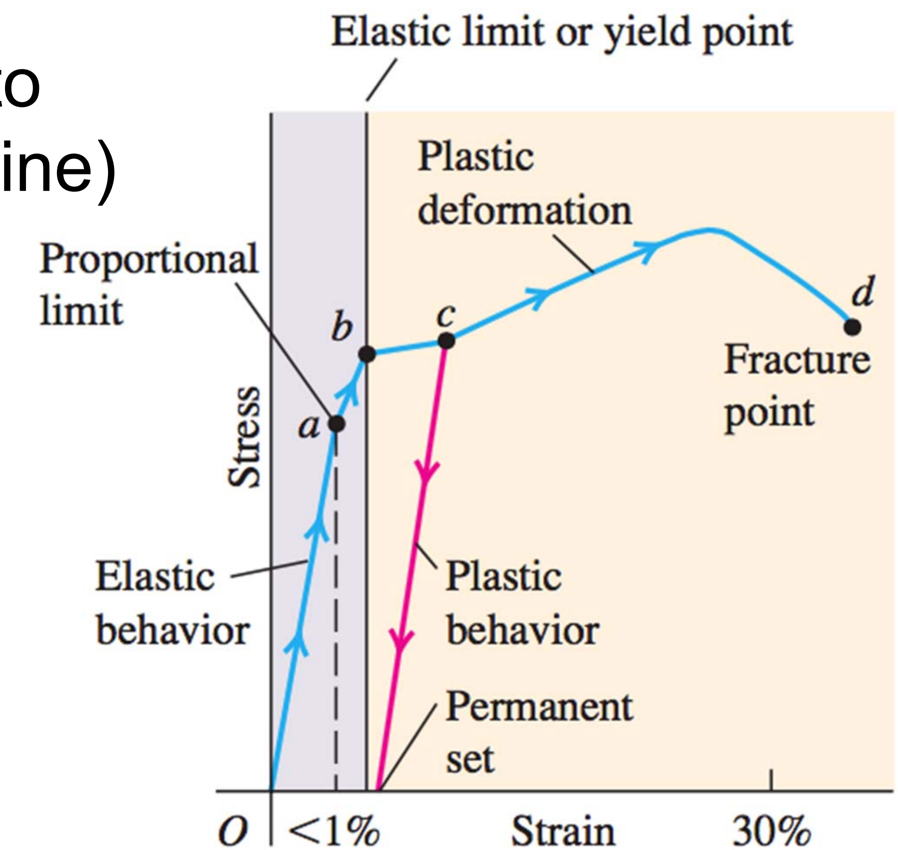
Stress-strain diagram for a ductile metal

- The x-axis is not drawn to scale except strain<1%
- **O->a:** Hooke's law (linear)
  a: Proportional limit
- **a->b**: not proportional, Hooke's law no longer obeyed; but the deformation is still *reversible*, and the forces involved are still conservative (energy put into the material when deforming it is completely recovered when the stress is removed)



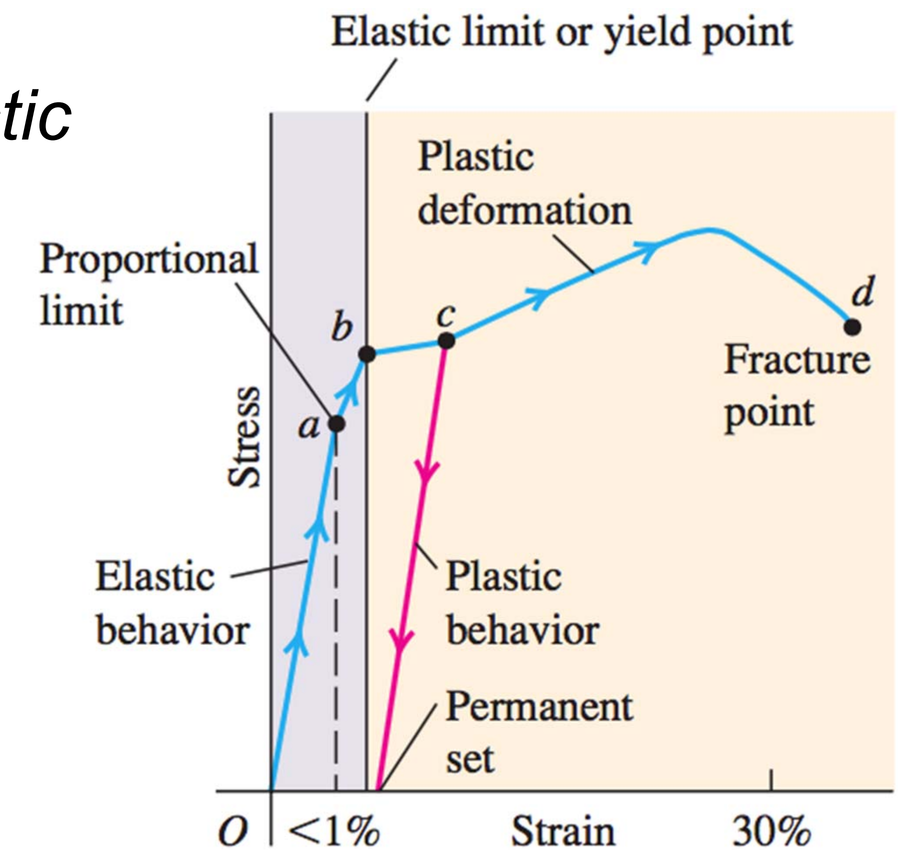
Stress-strain diagram for a ductile metal

- **b->c**: when load removed, the material does not come back to original length (follow the red line)
- Irreversible deformation and *permanent set*
- Further increase of load produces a large increase in strain for a relatively small increase in stress, until a point ***d*** at which *fracture* takes place.



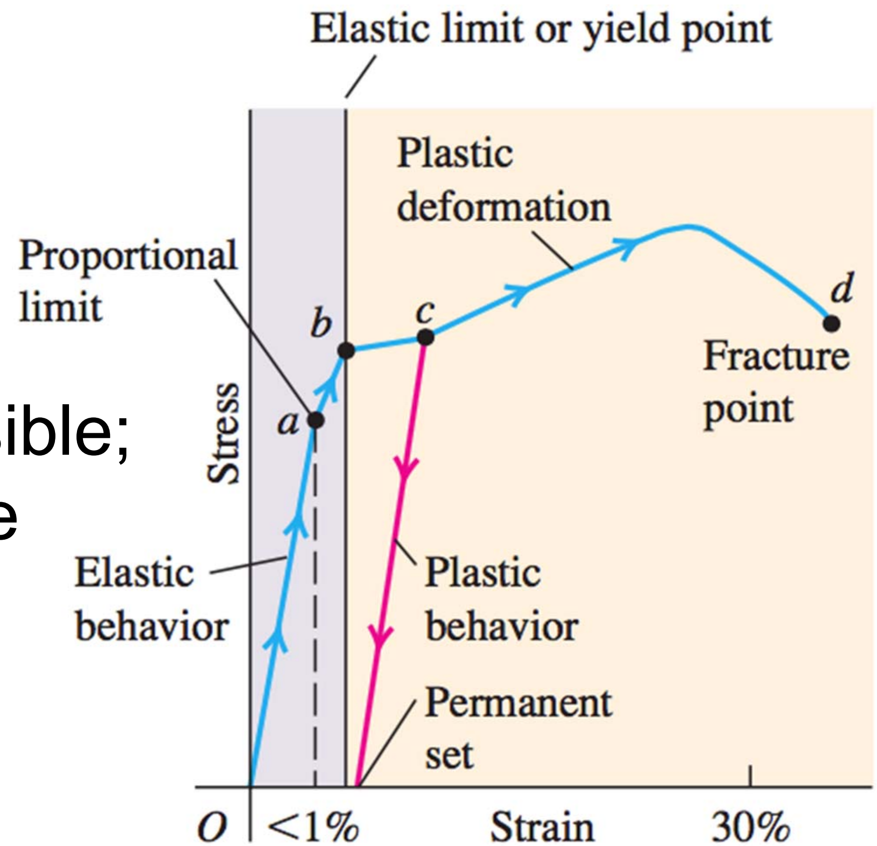
Stress-strain diagram for a ductile metal

- **O->b**: the material shows *elastic behavior*
  b: yield point
  stress at b: elastic limit



Stress-strain diagram for a ductile metal

- **b->d**: the behavior is called *plastic flow* or *plastic deformation.*
- A plastic deformation is irreversible; when the stress is removed, the material does not return to its original state.



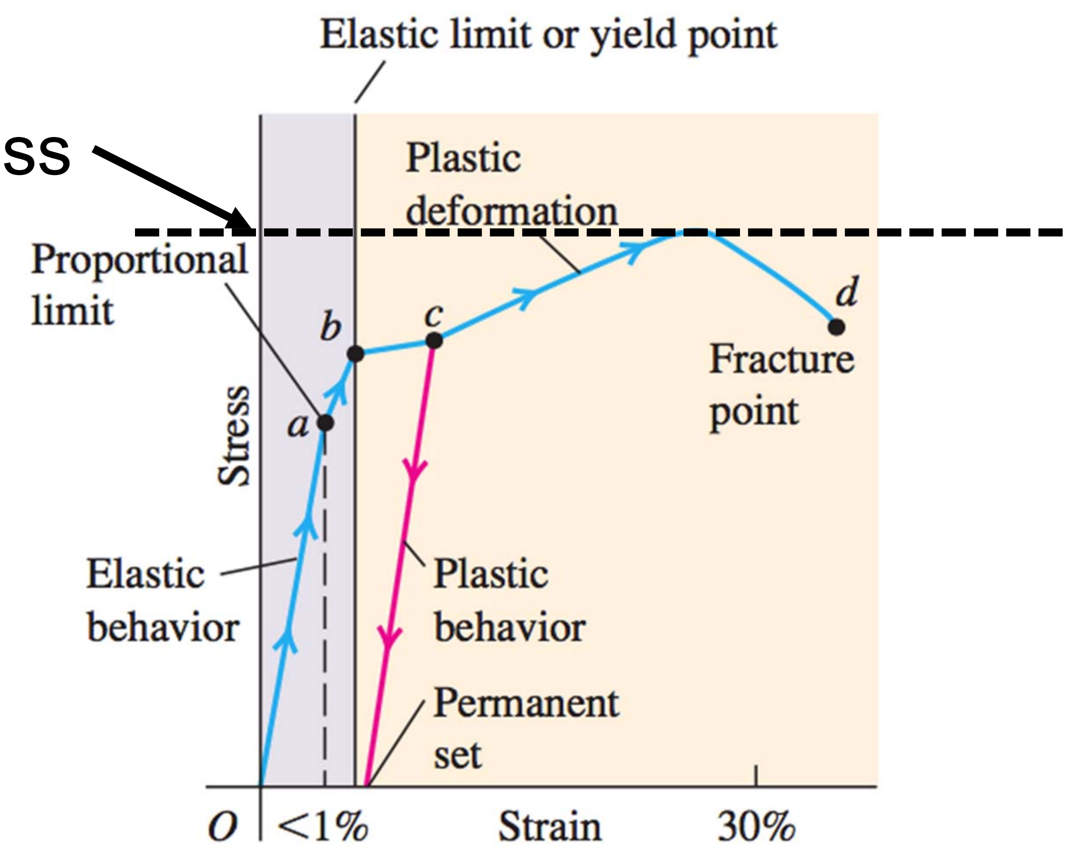
Ductile or brittle?

- Ductile materials: a large amount of plastic deformation takes place between the elastic limit and the fracture point.
- Brittle materials: fracture occurs soon after the elastic limit is passed.
- Clay is ductile.
- Ceramic is brittle.

Breaking stress

- The stress required to cause actual fracture of a material is called the *breaking stress,* the *ultimate strength,* or (for tensile stress) the *tensile strength.*

breaking stress



Breaking stress

- Two materials, such as iron and steel, may have very similar elastic constants but vastly different breaking stresses.

Table 11.3 **Approximate Breaking Stresses**

Material	Breaking Stress (Pa or N/m^2)
Aluminum	2.2×10^8
Brass	4.7×10^8
Glass	10×10^8
Iron	3.0×10^8
Phosphor bronze	5.6×10^8
Steel	$5–20 \times 10^8$

Table 11.1 Approximate Elastic Moduli

Material	Young's Modulus, Y (Pa)	Bulk Modulus, B (Pa)	Shear Modulus, S (Pa)
Iron	21×10^{10}	16×10^{10}	7.7×10^{10}
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Nickel	21×10^{10}	17×10^{10}	7.8×10^{10}
Steel	20×10^{10}	16×10^{10}	7.5×10^{10}

Example

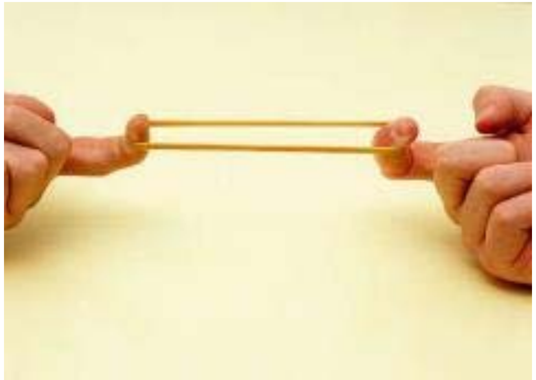
- While parking your car on a crowded street, you accidentally back into a steel post. You pull forward until the car no longer touches the post and then get out to inspect the damage.
- What does your rear bumper look like if the strain in the impact was
  (a) less than at the proportional limit;
  (b) greater than at the proportional limit, but less than at the yield point;
  (c) greater than at the yield point, but less than at the fracture point; and
  (d) greater than at the fracture point?

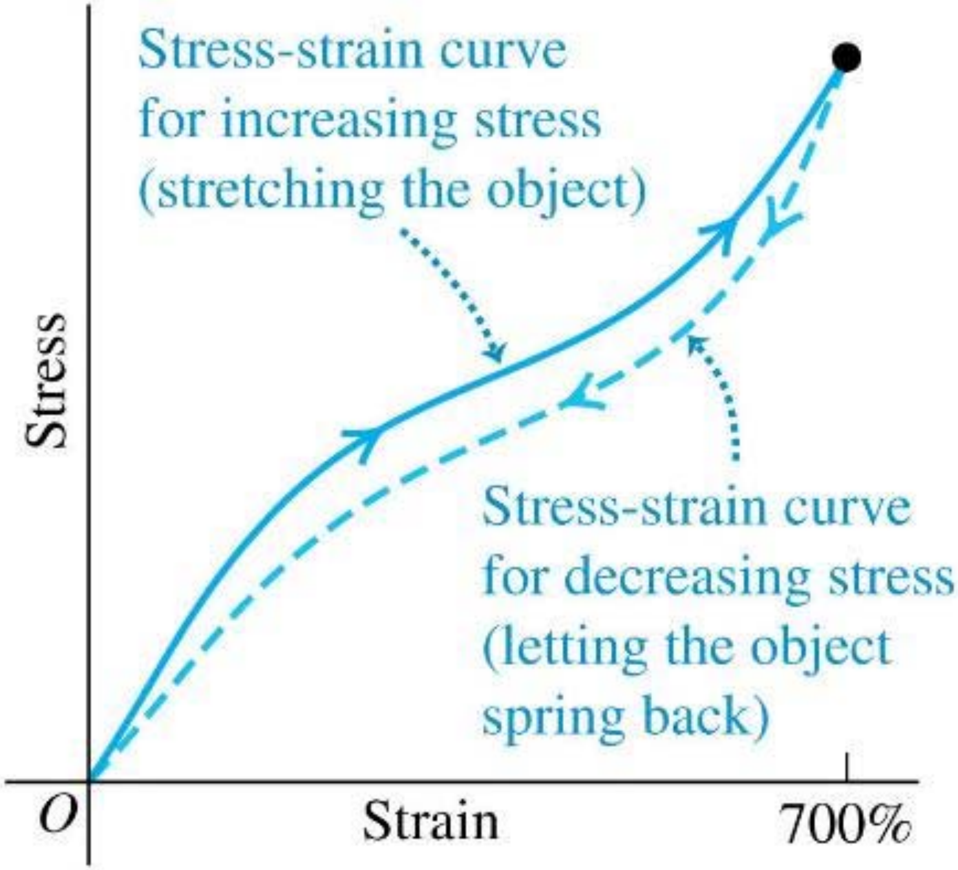
Example

- What does your rear bumper look like if the strain in the impact was
  (a) less than at the proportional limit;
  (b) greater than at the proportional limit, but less than at the yield point;
  (c) greater than at the yield point, but less than at the fracture point; and
  (d) greater than at the fracture point?

Stress-strain diagram for rubber

- The stress is not proportional to the strain, but the behavior is elastic because when the load is removed, the material returns to its original length.
- However, the material follows *different* curves for increasing and decreasing stress.
- This is called ***elastic hysteresis***.

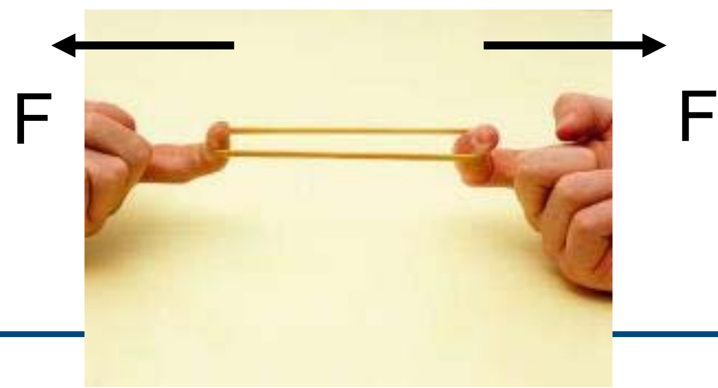


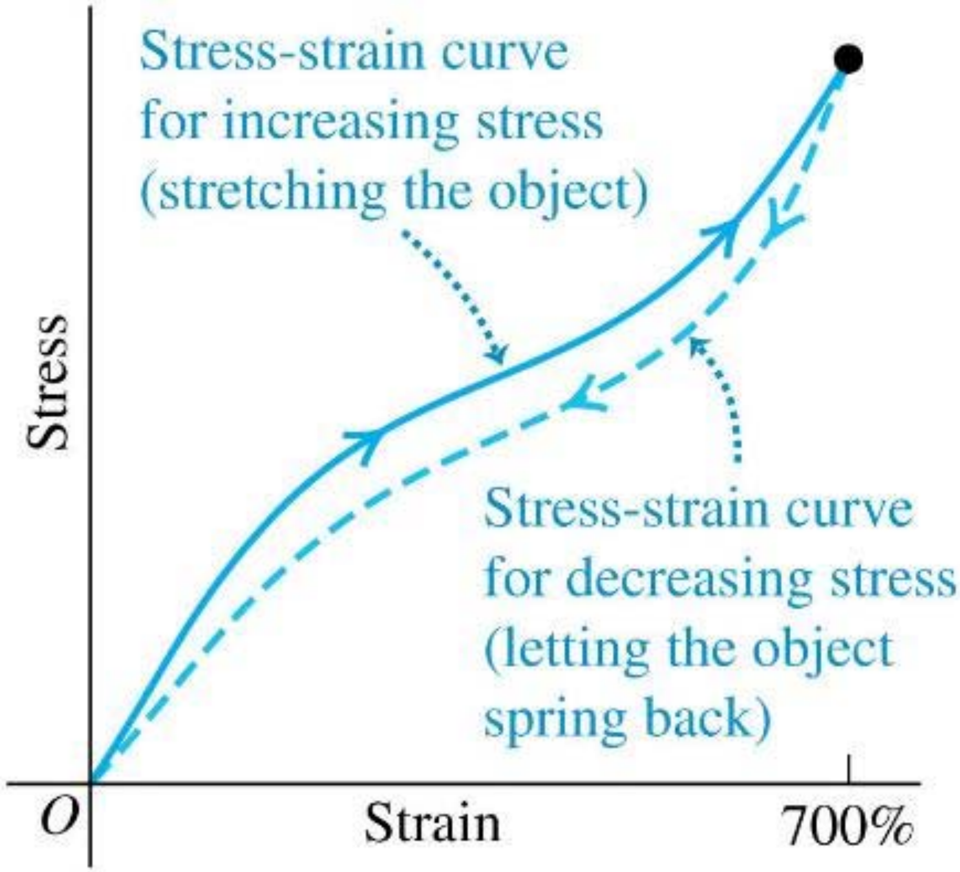


Stress-strain diagram for vulcanized rubber

- The work done by the material when it returns to its original shape is less than the work required to deform it; there are nonconservative forces associated with internal friction.

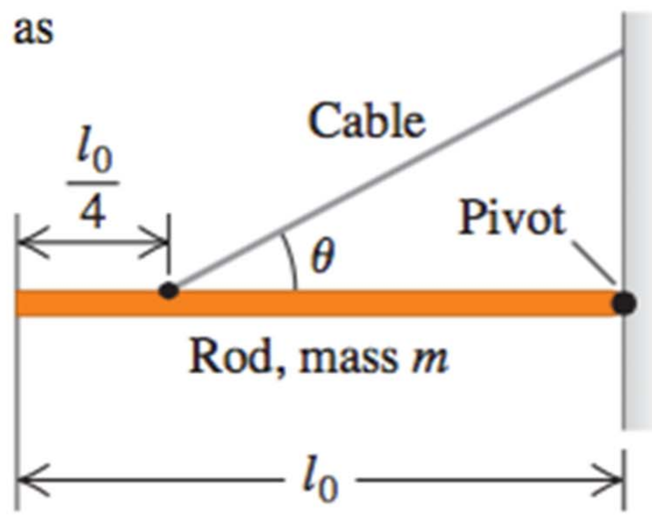
- Rubber with large elastic hysteresis is very useful for absorbing energy, such as in engine mounts and shock-absorber bushings for cars.

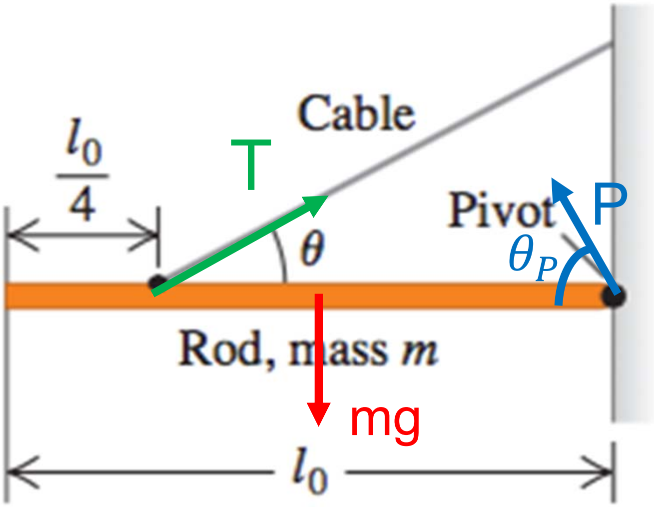




In equilibrium and under stress

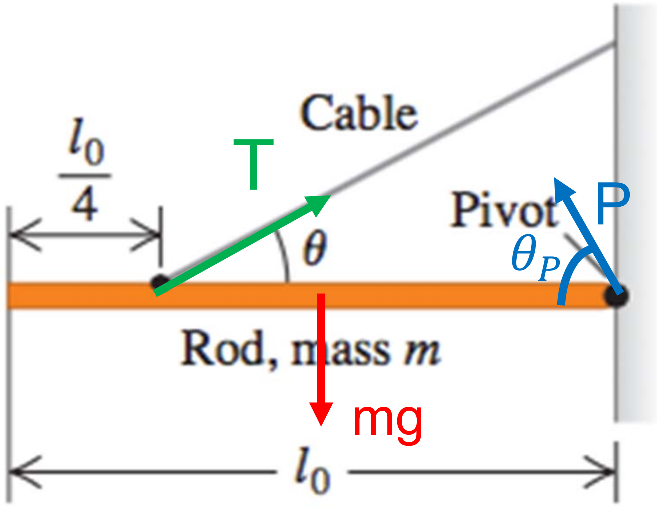
A horizontal, uniform, solid copper rod has an original length l_0, cross-sectional area A, Young's modulus Y, bulk modulus B, shear modulus S, and mass m. It is supported by a frictionless pivot at its right end and by a cable a distance $l_0/4$ from its left end (Fig. 11.20). Both pivot and cable are attached so that they exert their forces uniformly over the rod's cross section. The cable makes an angle θ with the rod and compresses it. (a) Find the tension in the cable. (b) Find the magnitude and direction of the force exerted by the pivot on the right end of the rod. How does this magnitude compare to the cable tension? How does this angle compare to θ? (c) Find the change in length of the rod due to the stresses exerted by the cable and pivot on the rod. (d) By what factor would your answer in part (c) increase if the solid copper rod were twice as long but had the same cross-sectional area?





- Conditions for equilibrium:
- $\sum F_x = T_x - P_x = 0$ (1)
- $\sum F_y = T_y + P_y - mg = 0$ (2)
- $\sum \tau = \frac{3l_0}{4} \times T\sin\theta - \frac{l_0}{2} \times mg = 0$ (3)

- From (3), $T = \frac{2mg}{3\sin\theta}$
- $T_y = T\sin\theta = \frac{2mg}{3}, T_x = T\cos\theta = \frac{2mg}{3\tan\theta}$.
- From (1) and (2), $P_x = \frac{2mg}{3\tan\theta}, P_y = \frac{mg}{3}$
- $P = \sqrt{P_x^2 + P_y^2} = \frac{mg}{3\sin\theta}\sqrt{\sin^2\theta + 4\cos^2\theta}$, $P < T$ since $P_x = T_x, P_y < T_y$
- $\theta_P = \arctan(\frac{P_y}{P_x}) = \arctan(\frac{\tan\theta}{2}) < \arctan\tan\theta = \theta$



A horizontal, uniform, solid copper rod has an original length l_0, cross-sectional area A, Young's modulus Y, bulk modulus B, shear modulus S, and mass m. It is supported by a frictionless pivot at its right end and by a cable a distance $l_0/4$ from its left end (Fig. 11.20).

- Stress= T_x/A, Strain= $\frac{\Delta l}{\frac{3}{4}l_0}$, Y =(stress)/(strain), $T_x = \frac{2mg}{3\tan\theta}$
- $\Delta l = \frac{3}{4}l_0 \times (\text{strain}) = \frac{3}{4}l_0 \times \frac{(\text{stress})}{Y} = \frac{3}{4}l_0 \times \frac{T_x/A}{Y}$
- $\Delta l = \frac{mgl_0}{2AY\tan\theta}$
- If the rod is twice as long, then m and l_0 doubles, so Δl quadrupules.

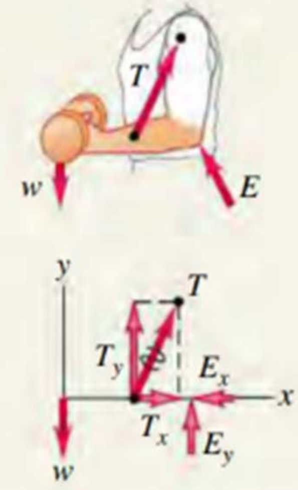
Chapter Summary

Conditions for equilibrium: For a rigid body to be in equilibrium, two conditions must be satisfied. First, the vector sum of forces must be zero. Second, the sum of torques about any point must be zero. The torque due to the weight of a body can be found by assuming the entire weight is concentrated at the center of gravity, which is at the same point as the center of mass if $\vec{g}$ has the same value at all points. (See Examples 11.1–11.4.)

$$\sum \vec{F} = 0 \qquad (11.1)$$

$$\sum \vec{\tau} = 0 \quad \text{about } any \text{ point} \qquad (11.2)$$

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \qquad (11.4)$$

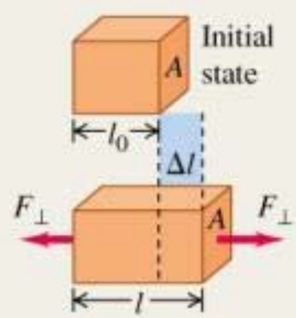


Stress, strain, and Hooke's law: Hooke's law states that in elastic deformations, stress (force per unit area) is proportional to strain (fractional deformation). The proportionality constant is called the elastic modulus.

$$\frac{\text{Stress}}{\text{Strain}} = \text{Elastic modulus} \qquad (11.7)$$

Tensile and compressive stress: Tensile stress is tensile force per unit area, $F_\perp/A$. Tensile strain is fractional change in length, $\Delta l/l_0$. The elastic modulus is called Young's modulus Y. Compressive stress and strain are defined in the same way. (See Example 11.5.)

$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp/A}{\Delta l/l_0} = \frac{F_\perp}{A}\frac{l_0}{\Delta l} \qquad (11.10)$$

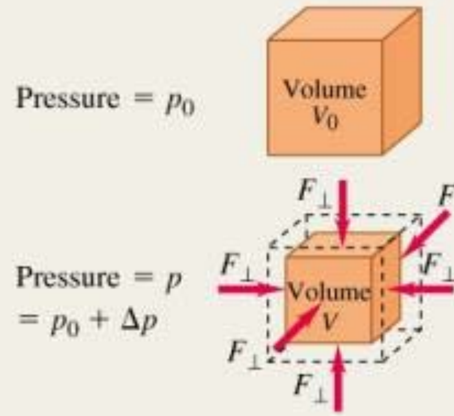


Chapter Summary

Bulk stress: Pressure in a fluid is force per unit area. Bulk stress is pressure change, Δp, and bulk strain is fractional volume change, $\Delta V/V_0$. The elastic modulus is called the bulk modulus, B. Compressibility, k, is the reciprocal of bulk modulus: $k = 1/B$. (See Example 11.6.)

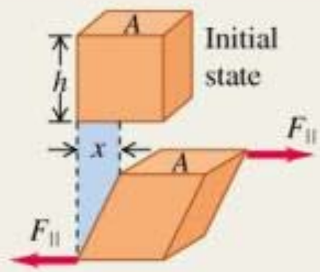
$$p = \frac{F_\perp}{A} \quad (11.11)$$

$$B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\Delta p}{\Delta V/V_0} \quad (11.13)$$



Shear stress: Shear stress is force per unit area, $F_\parallel/A$, for a force applied tangent to a surface. Shear strain is the displacement x of one side divided by the transverse dimension h. The elastic modulus is called the shear modulus, S. (See Example 11.7.)

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_\parallel/A}{x/h} = \frac{F_\parallel}{A}\frac{h}{x} \quad (11.17)$$



The limits of Hooke's law: The proportional limit is the maximum stress for which stress and strain are proportional. Beyond the proportional limit, Hooke's law is not valid. The elastic limit is the stress beyond which irreversible deformation occurs. The breaking stress, or ultimate strength, is the stress at which the material breaks.