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3. Techniques of Integration

In this chapter, we will learn

- integration by parts.
- computing integrals involving trigonometric functions and rational functions.
- approximating integrals numerically.
- improper integrals.

3.1. Integration by parts. Text Section 7.1,
Exercise: 17, 24, 39, 51, 61, 70.

Every differential rule has a corresponding integration rule.

$dg(x) = g'(x)dx$

Differentiation	$\longleftrightarrow$	Integration
the Chain Rule	$\longleftrightarrow$	the Substitution Rule
the Product Rule	$\longleftrightarrow$	Integration by Parts

$\int f(x)\,dg(x)$ $=$ $f(x)g(x) - \int g(x)\,df(x)$

The **formula for integration by parts**:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

$$\int f(x)g'(x)dx + \int g(x)f'(x)dx = f(x)g(x) + C$$

or

$$\cancel{\int udv = uv - \int vdu.}$$

Ex. [Text example 7.1.2] Evaluate $\int \ln x dx$.

$f(x)g'(x) + g(x)f'(x)$

$\|$

$\left(\int f(x)g'(x)dx\right)' + \left(\int g(x)f'(x)dx\right)' = \left(\int f(x)g'(x)dx + \int g(x)f'(x)dx\right)'$

the product rule for derivatives.

$\|?$

$\left(f(x)g(x) + C\right)'$

$\|$

$\left(f(x)g(x)\right)'$

Ex: $\int \underbrace{\boxed{\ln x}}_{f(x)}\, d\underbrace{\boxed{x}}_{g(x)}$ $\quad \frac{1}{x}dx$

$=$

$= x\ln x - \int x \boxed{d\ln x}$

$= x\ln x - \int x \cdot \frac{1}{x}\, dx = x\ln x - \int 1\, dx = x\ln x - x + C.$

Ex. [Text example 7.1.3] Find $\int t^2 e^t dt$.

$\int \boxed{t^2}\boxed{e^t} dt = \int t^2 \boxed{d e^t} = t^2 e^t - \int e^t \boxed{d t^2}$

$f(t)$, $= g'(t)$, $= e^t dt$, $= 2t\,dt$

$g(t) = e^t$

$= t^2e^t - \int e^t\, 2t dt = t^2 e^t - 2 \int \boxed{e^t}\boxed{t} dt = t^2 e^t - 2\int t\, de^t$

$g'(t)$, $f(t)$

$= t^2 e^t - 2\left(te^t - \int e^t dt \right) = t^2 e^t - 2te^t + 2\int e^t dt = t^2 e^t - 2te^t + 2e^t + C$.

Ex. [Text example 7.1.4] Evaluate $\int e^x \sin x dx$.

$\int \boxed{e^x}\boxed{\sin x} dx = \int \sin x\, de^x = e^x \sin x - \int e^x d\sin x$

$g'(x)$, $f(x)$

$g(x) = e^x$

$= e^x \sin x - \int \boxed{e^x}\boxed{\cos x} dx = e^x \sin x - \int \cos x\, de^x$

$g'(x)$, $f(x)$

$= e^x \sin x - \left(e^x \cos x - \int e^x d\cos x \right) = e^x\sin x - \left(e^x \cos x + \int e^x \sin x dx \right)$

$= e^x \sin x - e^x \cos x - \boxed{\int e^x \sin x\, dx}$

Integration by parts for definite integrals:

$$\boxed{\int_a^b f(x)g'(x)dx} = f(x)g(x)\Big|_a^b - \boxed{\int_a^b g(x)f'(x)dx}$$

$\int_a^b f(x)\, dg(x)$, $\int_a^b g(x)\, df(x)$

$$\boxed{\int_a^b f(x)g'(x)\, dx + \int_a^b g(x) f'(x)\, dx} = f(x)g(x)\Big|_a^b$$

$$= \int_a^b f(x)g'(x) + g(x)f'(x)\, dx$$

Ex. [Text example 7.1.5] Calculate $\int_0^1 \tan^{-1} x dx$.

arctan x

$$\int_0^1 \underbrace{\tan^{-1} x}_{f(x)}\, d\boxed{\underbrace{x}_{g(x)}} = x\tan^{-1}x \Big|_0^1 - \int_0^1 x\, d\tan^{-1}x \qquad d\tan^{-1}x = \frac{1}{1+x^2}dx$$

$$= 1\cdot\boxed{\tan^{-1}1} - 0\,\boxed{\tan^{-1}0} - \int_0^1 x\left(\frac{1}{1+x^2}dx\right)$$

$\tan\frac{\pi}{4} = 1 \Rightarrow \frac{\pi}{4} = \tan^{-1}(1)$

$$= 1\cdot\frac{\pi}{4} - 0 - \int_0^1 \frac{x}{1+x^2}dx$$

$$= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{1}{\boxed{1+x^2}}\boxed{d\boxed{(x^2+1)}} = \frac{\pi}{4} - \frac{1}{2}\int_1^2 \frac{1}{u}\,du \qquad u = x^2+1$$

$$= \frac{\pi}{4} - \frac{1}{2}\left(\ln u\Big|_1^2\right) = \frac{\pi}{4} - \frac{1}{2}(\ln 2 - \boxed{\ln 1}) = \frac{\pi}{4} - \frac{\ln 2}{2}.$$

Ex. [Text example 7.1.6] Prove the reduction formula

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx.$$

Solution:

$$\int \sin^n x\, dx = \int \sin^{n-1}x\,\boxed{\sin x\, dx} = \boxed{\int \sin^{n-1}x\, d(-\cos x)} = -\int \sin^{n-1}x\, d\cos x$$

$$= -\sin^{n-1}x\cos x + \int \boxed{\cos x}\,\boxed{d\sin^{n-1}x} = -\sin^{n-1}x\cos x + \int \boxed{(n-1)\sin^{n-2}x}\cos x\,\boxed{\cos x}\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$$

$\sin^2 x + \cos^2 x = 1$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \boxed{\sin^{n-2}x} - \sin^n x\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - \boxed{(n-1)\int \sin^n x\,dx}$$

$$n\int \sin^n x\,dx = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx$$

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$$

$\cos x\,dx = d\sin x$

$df(x) = f'(x)dx$, $f' = \cos x$

3.2. **Trigonometric integrals.** Text Section 7.2,

Exercise: 3, 13, 29, 55, 61, 67.

In this section, we use trigonometric identities to integrate certain combinations of trigonometric functions.

Ex. [Text example 7.2.1] Evaluate $\int \cos^3 x dx$.

$$\int \cos^3 x\,dx = \int \cos^{2\times1+1} x\,dx = \int \boxed{\underset{1-\sin^2 x}{\cos^2 x}}\ \boxed{\underset{d\sin x}{\cos x\,dx}}$$

$\cos x\,dx = du$

$$= \int 1-\sin^2 x\ d\sin x \overset{u=\sin x}{=\!=} \int 1-u^2\,du$$

$$= u - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C,$$

Ex. [Text example 7.2.2] Find $\int \sin^5 x \cos^2 x dx$.

$$\int \sin^5 x \cos^2 x\,dx = \int \sin^4 x \cos^2 x\,(\sin x)\,dx$$

$$= \int (\sin^2 x)^2 \cos^2 x\ \boxed{\sin x\,dx}$$

$\boxed{u = \cos x \qquad -du = \sin x\,dx}$

$$= \int (1-\cos^2 x)^2 \cos^2 x\,[-d(\cos x)]$$

$u = \cos x$

$$= -\int (1-u^2)^2 u^2\,du = -\int (1-2u^2+u^4)\,u^2\,du$$

$$= -\int u^2 - 2u^4 + u^6\,du$$

$$= -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C$$

$$= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C.$$

Ex. [Text example 7.2.4] Find $\int \sin^4 x dx$.

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2 dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x\,dx$$

$$= \frac{1}{4}\left(x - \sin 2x + \int \cos^2 2x\,dx \right)$$

$$= \frac{1}{4}\left[x - \sin 2x + \int \frac{1}{2}(1+\cos 4x)\,dx \right]$$

$= \frac{1}{4}\left[x - \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x\right] + C = \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right] + C.$

Strategy for evaluating $\int \sin^m x \cos^n x dx$

(a) If the power of cosine is odd ($n = 2k+1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$(\cos^2 x)^k = (1-\sin^2 x)^k$

$$\int \sin^m x \boxed{\cos^{2k+1} x} dx = \int \sin^m x \boxed{\cos^{2k} x}\boxed{\cos x}\, dx = d\sin x$$

$$= \int \sin^m x (1 - \sin^2 x)^k \cos x\, dx. = d\sin x$$

Then substitute $u = \sin x$. $\int u^m (1-u^2)^k du$

(b) If the power of sine is odd ($m = 2k+1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \boxed{\sin^{2k+1} x} \cos^n x dx = \int \boxed{\sin^{2k} x} \cos^n x \boxed{\sin x}\, dx$$

$$= \int \boxed{(1 - \cos^2 x)^k} \cos^n x \sin x\, dx. = -d\boxed{\cos x} = u$$

Then substitute $u = \cos x$. [*Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.*]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$\int \sin^{2k_1} x \cos^{2k_2} x\, dx = \int (\sin^2 x)^{k_1} (\cos^2 x)^{k_2} dx$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

Sometimes it is helpful to use the identity

$$\sin x \cos x = \frac{1}{2}\sin 2x.$$

$$\int \sin^{2k_1} x \cos^{2k_2} x\, dx$$

$$= \int (\sin^2 x)^{k_1} (\cos^2 x)^{k_2} dx \qquad u = 2x$$

$$= \frac{1}{2}\int \left(\frac{1}{2}(1-\cos u)\right)^{k_1} \left(\frac{1}{2}(1+\cos u)\right)^{k_2} du$$

$$\int \tan u \, du$$

Ex. [Text example 7.2.5] Evaluate $\int \tan^6 x \sec^4 x dx$.

$$\int \tan^6 x \sec^4 x \, dx = \int \tan^6 x \boxed{\sec^2 x} \boxed{\sec^2 x \, dx} \quad = d\tan x$$

$$= \int \tan^6 x (1 + \tan^2 x) \, d\tan x$$

$$\overset{u=\tan x}{=\!=} \int u^6 (1+u^2) \, du$$

$$= \int u^6 + u^8 \, du = \frac{1}{7} u^7 + \frac{1}{9} u^9 + C$$

$$= \frac{1}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C.$$

Ex. [Text example 7.2.6] Evaluate $\int \tan^5 x \sec^7 x dx$.

(b)

Strategy for evaluating $\int \tan^m x \sec^n x dx$

(a) If the power of secant is even ($n = 2k, k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factor in terms of $\tan x$:

$u = \tan x$

$$\begin{aligned}\int \tan^m x \boxed{\sec^{2k} x} dx &= \int \tan^m x (\sec^2 x)^{k-1} \boxed{\sec^2 x dx} \\ &= \int \tan^m x (\boxed{1 + \tan^2 x})^{k-1} \sec^2 x dx. = d\tan x\end{aligned}$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factor in terms of $\sec x$:

$$\begin{aligned}\int \boxed{\tan^{2k+1} x} \boxed{\sec^n x} dx &= \int (\boxed{\tan^2 x})^k \boxed{\sec^{n-1} x} \boxed{\sec x} \boxed{\tan x} dx \\ &= \int (\boxed{\sec^2 x - 1})^k \boxed{\sec^{n-1} x} \sec x \tan x dx. = d\sec x\end{aligned}$$

Then substitute $u = \sec x$.

$u = \sec x$.

For other cases, we may use integration by parts or the following two formulas:

$$\int \tan x dx = \ln|\sec x| + C$$

and

$$\int \sec x dx = \ln|\sec x + \tan x| + C.$$

Ex. [Text example 7.2.7] Find $\int \tan^3 x dx$.

$$\int \tan^3 x \, dx = \int \frac{\tan^2 x}{\sec x} \boxed{\tan x \sec x \, dx} = d\sec x$$

$$= \int \frac{\sec^2 x - 1}{\sec x} \, d\sec x \overset{u = \sec x}{=\!=} \int \frac{u^2 - 1}{u} du$$

$$= \int u - \frac{1}{u}\, du = \frac{1}{2}u^2 - \ln|u| + C = \frac{1}{2}\sec^2 x - \ln|\sec x| + C$$

$$= \frac{1}{2}(\tan^2 x + 1) - \ln|\sec x| + C$$

$$= \frac{1}{2}\tan^2 x + \frac{1}{2} - \ln|\sec x| + C$$

$$= \frac{1}{2}\tan^2 x - \ln|\sec x| + \underbrace{C + \tfrac{1}{2}}_{C}$$

Ex. [Text example 7.2.8] Find $\int \sec^3 x dx$.

Solution: $\int \sec^3 x\, dx$

$$= \int \sec x \boxed{\sec^2 x\, dx} \quad (= d\tan x)$$

$$= \int \sec x\, d\tan x$$

$$= \sec x \tan x - \int \tan x \boxed{d\sec x} = \sec x \tan x - \int \boxed{\tan^2 x} \sec x\, dx = \sec x \tan x - \int (\sec^2 x - 1)\sec x\, dx$$

($d\sec x = \sec x \tan x\, dx$)

$$= \sec x \tan x - \int \sec^3 x\, dx + \boxed{\int \sec x\, dx}$$

$$2\int \sec^3 x = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$$

Strategy for evaluating (a) $\int \sin mx \cos nx dx$, (b) $\int \sin mx \sin nx dx$, or (c) $\int \cos mx \cos nx dx$

Use the corresponding identity:

$$\begin{aligned}
\text{(a)} \quad \sin A \cos B &= \frac{1}{2}[\sin(A-B) + \sin(A+B)],\\
\text{(b)} \quad \sin A \sin B &= \frac{1}{2}[\cos(A-B) - \cos(A+B)],\\
\text{(c)} \quad \cos A \cos B &= \frac{1}{2}[\cos(A-B) + \cos(A+B)].
\end{aligned}$$

$$\sin(A-B) = \sin A \cos B - \cancel{\cos A \sin B}, \quad \sin(A+B) = \sin A \cos B + \cancel{\cos A \sin B}$$

$$\Downarrow$$

$$\sin(A-B) + \sin(A+B) = 2\sin A \cos B$$

$$\Downarrow$$

$$\boxed{\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))}.$$

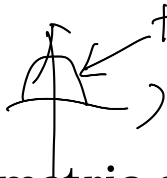
Ex. [Text example 7.2.9] Evaluate $\int \sin 4x \cos 5x dx$.

$$\int \sin 4x \cos 5x\, dx$$

$$= \int \frac{1}{2}[\sin(-x) + \sin 9x]\, dx$$

$$= \frac{1}{2}\int -\sin x + \sin 9x\, dx$$

$$= \frac{1}{2}\cos x + \frac{1}{2}\left(-\frac{1}{9}\cos 9x\right) + C = \frac{1}{2}\cos x - \frac{1}{18}\cos 9x + C.$$



$$\sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)}$$
$$\|$$
$$\sqrt{a^2\cos^2\theta}$$
$$\|$$
$$|a|\cos\theta$$

3.3. **Trigonometric substitution.** Text Section 7.3

Exercise: 7, 13, 25, 31, 43.

Table of trigonometric substitutions

$-a \le x \le a$ $\qquad$ x $\qquad$ $-a \le a\sin\theta \le a$

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta, 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

$x \ge a$ or $x \le -a$

Ex. [Text example 7.3.1] Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}dx$. $\qquad dx = 3\cos\theta\, d\theta$

Solution:

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx \xrightarrow[-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}]{x=3\sin\theta} \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\,3\cos\theta\, d\theta$$

$$= \int \frac{\sqrt{9\cos^2\theta}}{9\sin^2\theta}\,3\cos\theta\, d\theta = \int \frac{\cancel{3}\cos\theta}{\cancel{9}\sin^2\theta}\,\cancel{3}\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

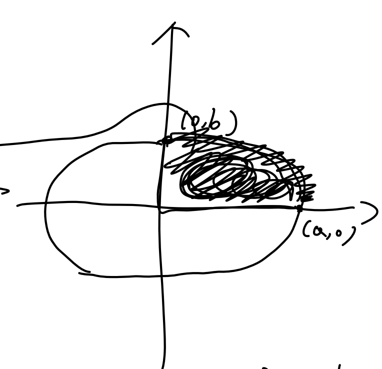
$$= \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta = \int \frac{1}{\sin^2\theta} - 1\, d\theta = \int \csc^2\theta - 1\, d\theta$$

Ex. [Text example 7.3.2] Find the area enclosed by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad a,b>0.$$

Solution: Solving the equation of the ellipse for y, we get

$$y = \pm\frac{b}{a}\sqrt{a^2-x^2} \qquad -a \le x \le a.$$



$$\frac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$\underset{0\le\theta\le\frac{\pi}{2}}{\overset{x=a\sin\theta}{=}} \frac{b}{a}\int_0^{\frac{\pi}{2}} \sqrt{a^2-a^2\sin^2\theta}\,(a\cos\theta)\,d\theta = \frac{b}{\cancel{a}}\int_0^{\frac{\pi}{2}} a^{\cancel{2}}\cos^2\theta\, d\theta$$

$$\|$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$dx = a\cos\theta\, d\theta$

$\sqrt{a^2\cos^2\theta} = a|\cos\theta| = a\cos\theta$

$\overset{\Downarrow}{=} \frac{ab}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$

$= \frac{ab}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{2}}$

$= \frac{ab}{2}\left[\frac{\pi}{2} + \boxed{\frac{1}{2}\sin\pi} - 0 - \boxed{\frac{1}{2}\sin 0}\right] = \frac{ab\pi}{4}$

$\Downarrow$

$A = ab\pi.$

Ex. [Text example 7.3.3] Find $\int \frac{1}{x^2\sqrt{x^2+4}}dx$.

Solution: Let $x = 2\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\Rightarrow$ $\boxed{dx = 2\sec^2\theta\, d\theta}$

Thus, we have $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{4\tan^2\theta\sqrt{4(1+\tan^2\theta)}} \cdot 2\sec^2\theta\, d\theta$

$1+\tan^2\theta = \sec^2\theta$

$= \int \frac{1}{4\tan^2\theta \cdot 2\boxed{|\sec\theta|}} \cdot 2\sec^2\theta\, d\theta = \int \frac{1}{4}\frac{\sec^2\theta}{\tan^2\theta\,\cancel{\sec\theta}}\,d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$

$|\sec\theta| = \sec\theta$

$= \frac{1}{4}\int \frac{\cos^2\theta}{\sin^2\theta} \cdot \frac{1}{\cos\theta} = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

$= \frac{1}{4}\int \frac{d\sin\theta}{\sin^2\theta}$

$\overset{u=\sin\theta}{=} \frac{1}{4}\int \frac{1}{u^2}\,du$

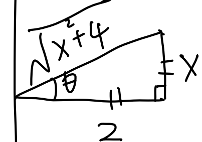
$= -\frac{1}{4}u^{-1} + C$

$= -\frac{1}{4}\frac{1}{\sin\theta} + C.$

$= -\frac{1}{4}\frac{1}{\frac{x}{\sqrt{x^2+4}}} + C$

$= -\frac{\sqrt{x^2+4}}{4x} + C.$

$\tan\theta = \frac{x}{2}$



$\sin\theta = \frac{x}{\sqrt{x^2+4}}$

$dx = \frac{3}{2}\sec^2\theta\, d\theta$

Ex. [Text example 7.3.6] Find $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}dx$.

Solution: $x = \frac{3}{2}\tan\theta \Rightarrow 0 \le \tan\theta \le \sqrt{3} \Rightarrow 0 \le \theta \le \frac{\pi}{3}$

Then $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}dx = \int_0^{\frac{\pi}{3}} \frac{\frac{27}{8}\tan^3\theta}{\left(4\cdot\frac{9}{4}\tan^2\theta + 9\right)^{\frac{3}{2}}} \boxed{\frac{3}{2}\sec^2\theta}\,d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\,\frac{3}{2}\sec^2\theta\, d\theta$

$= \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\tan^3\theta}{\sec\theta}\,d\theta = \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta = \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cos^2\theta}\,d(-\cos\theta)$

Ex. [Text example 7.3.7] Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}dx$.

$= -\frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{1-\cos^2\theta}{\cos^2\theta}\,d\cos\theta \overset{u=\cos\theta}{=} -\frac{3}{16}\int_1^{\frac{1}{2}} \frac{1-u^2}{u^2}\,du$

$= \frac{3}{16}\int_{\frac{1}{2}}^{1} \frac{1-u^2}{u^2}\,du = \frac{3}{16}\int_{\frac{1}{2}}^{1} \frac{1}{u^2} - 1\,du$

$= \frac{3}{16}\left(-\frac{1}{u} - u\right)\Big|_{\frac{1}{2}}^{1}$

$= \frac{3}{16}\left[-1-1-\left(-2-\frac{1}{2}\right)\right]$

$= \frac{3}{32}.$

3.4. **Integration of rational functions by partial fractions.**

Text Section 7.4,

Exercise: 17, 29, 36, 43, 57, 66.

In this section, we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions*.

Consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

$Q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$

where P and Q are polynomials. If the degree of P ($\deg(P)$) is less than the degree of Q ($\deg(Q)$), $\deg(P) < \deg(Q)$, f is called *proper*.

$Q(x) < P(x)$

If f is *improper*, $\deg(P) \geq \deg(Q)$, then we can divide Q into P and get

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \boxed{\frac{R(x)}{Q(x)}},$$

is proper.

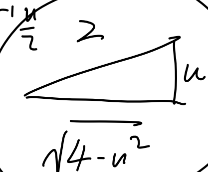
where S and R are also polynomials with $\deg(R) < \deg(Q)$.

Four steps to integrate rational functions:

1. get a proper rational function.
2. factor the denominator $Q(x)$ as far as possible.
3. express the proper rational function as a sum of **partial fractions** of the form
$$\frac{A}{(ax+b)^i} \qquad \text{or} \qquad \frac{Ax+B}{(ax^2+bx+c)^j}.$$
4. integrate the partial fractions.

$\sin\theta = \frac{u}{2}$

$\theta = \sin^{-1}\frac{u}{2}$



$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx = \int \frac{x}{\sqrt{3+1-1-2x-x^2}}\,dx = \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$

$x = u-1 \Leftarrow u = x+1$

$du = dx$

$\int \frac{u-1}{\sqrt{4-u^2}}\,du \quad u = 2\sin\theta, \; -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \quad \int \frac{2\sin\theta - 1}{\sqrt{4-4\sin^2\theta}}\, 2\cos\theta\, d\theta$

$du = 2\cos\theta\, d\theta$ $\quad 2|\cos\theta|$

$$= \int \frac{2\sin\theta - 1}{2\cos\theta}\, 2\cos\theta\, d\theta = \int 2\sin\theta - 1 = -2\cos\theta - \theta + C$$

Four cases depending on $Q(x)$:

$$= -2\frac{\sqrt{4-u^2}}{2} - \sin^{-1}\frac{u}{2} + C$$

$\cos\left(\sin^{-1}x + \frac{1}{2}\right)$

$$= -\sqrt{4-(x+1)^2} - \sin^{-1}\frac{x+1}{2} + C.$$

CASE I. The denominator $Q(x)$ is a product of distinct linear factors, that is,

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

where no factor is repeated.

$Q(x)$ has k distinct real roots.

Then the proper rational function can be rewritten as

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}.$$

Ex. [Text example 7.4.2] Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$.

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}dx = \int x + 1 + \frac{4x}{x^3 - x^2 - x + 1}dx = \int x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}dx$$

The final answer.

$$= \frac{1}{2}x^2 + x + (\ln|x-1|) - \frac{2}{x-1} - \ln|x+1| + C = \boxed{\frac{1}{2}x^2 + x + \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C.}$$

$$\boxed{\frac{4x}{(x-1)^2(x+1)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}}$$

$A_{1,1} = 1,\ A_{1,2} = 2$

$A_2 = -1$

CASE II. The denominator $Q(x)$ is a product of linear factors, some of which are repeated:

$$Q(x) = (a_1x + b_1)^r(a_2x + b_2)\cdots(a_kx + b_k).$$

$$\begin{cases} A_{1,1} + A_2 = 0 \\ A_{1,2} - 2A_2 = 4 \\ -A_{1,1} + A_{1,2} + A_2 = 0 \end{cases}$$

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A_{1,1}}{x-1} + \frac{A_{1,2}}{(x-1)^2} + \frac{A_2}{x+1}$$

$4x = A_{1,1}(x-1)(x+1) + A_{1,2}(x+1) + A_2(x-1)^2$

$= A_{1,1}(x^2-1) + A_{1,2}(x+1) + A_2(x^2-2x+1) = \boxed{(A_{1,1}+A_2)}x^2 + \boxed{(A_{1,2}-2A_2)}x + \boxed{(-A_{1,1}+A_{1,2}+A_2)}$

$A_{1,1}+A_2 = 0$, $A_{1,2}-2A_2 = 4$, $-A_{1,1}+A_{1,2}+A_2 = 0$

Then the proper rational function can be rewritten as

$$\frac{R(x)}{Q(x)} = \frac{A_{1,1}}{a_1x+b_1} + \frac{A_{1,2}}{(a_1x+b_1)^2} + \cdots + \frac{A_{1,r}}{(a_1x+b_1)^r} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}.$$

Ex. [Text example 7.4.4] Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$.

Solution: Step 1: Improper.

$$\begin{array}{r} x+1 \\ x^3-x^2-x+1 \,\big)\, x^4 \qquad -2x^2+4x+1 \\ x^4-x^3-x^2+x \\ \hline x^3-x^2+3x+1 \\ x^3-x^2-x+1 \\ \hline 4x \end{array}$$

$\Rightarrow x^4-2x^2+4x+1 = (x+1)(x^3-x^2-x+1) + 4x$

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = \boxed{x+1} + \frac{4x}{x^3-x^2-x+1}$$

Step 2.

$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A_{11}}{(x-1)} + \frac{A_{12}}{(x-1)^2} + \frac{A_2}{x+1}$$

$x^3-x^2-x+1 = x^2(x-1)-(x-1) = (x^2-1)(x-1) = (x-1)^2(x+1)$

CASE III. The denominator $Q(x)$ contains irreducible quadratic factors, none of which is repeated:

$$Q(x) = (ax^2+bx+c)(a_2x+b_2)\cdots(a_kx+b_k),$$

where $b^2-4ac<0$.

Then the proper rational function can be rewritten as

$$\int \frac{R(x)}{Q(x)}\,dx = \int \boxed{\frac{Ax+B}{ax^2+bx+c}} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}\,dx$$

Then we need to use the formula

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.}$$

$x^2+\ldots$ irreducible

$b^2-4ac = 4^2 - 4\times 4\times 3 < 0 \Rightarrow 4x^2-4x+3$ irreducible.

Ex. [Text example 7.4.6] Evaluate $\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx$. Proper $\Rightarrow$ improper

Solution: By long division: $\frac{4x^2-3x+2}{4x^2-4x+3} = 1 + \boxed{\frac{x-1}{4x^2-4x+3}} = 1 + \frac{x-1}{(4x^2-4x+1)+2} = 1 + \frac{x-1}{(2x-1)^2+2}$

$$\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx = \int 1 + \frac{x-1}{(2x-1)^2+2}\,dx$$

$$= x + \int \frac{x-1}{(2x-1)^2+2}\,dx$$

$u = 2x-1 \Rightarrow x = \frac{u+1}{2} \Rightarrow dx = \frac{1}{2}du$

$$= x + \int \frac{\frac{u}{2}+\frac{1}{2}-1}{u^2+2}\left(\frac{1}{2}du\right) = x + \frac{1}{4}\int \frac{u-1}{u^2+2}du = x + \frac{1}{4}\int \frac{u}{u^2+2}du - \frac{1}{4}\int \frac{1}{u^2+2}du$$

$$x + \frac{1}{8}\ln(u^2+2) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$u = 2x-1$

The final answer: $\boxed{x + \frac{1}{8}\ln\left((2x-1)^2+2\right) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C}$

CASE IV. The denominator $Q(x)$ contains a repeated irreducible quadratic factor:

$$Q(x) = (ax^2+bx+c)^r(a_2x+b_2)\cdots(a_kx+b_k),$$

where $b^2-4ac<0$.

Then the proper rational function can be rewritten as

$$\frac{R(x)}{Q(x)} = \frac{A_{1,1}x+B_1}{ax^2+bx+c} + \frac{A_{1,2}x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_{1,r}x+B_r}{(ax^2+bx+c)^r}$$
$$+\frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}.$$

Ex. [Text example 7.4.8] Evaluate $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx$. $\to 3$, $\to 5$ $\Rightarrow$ proper.

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A_{1,1}x+B_1}{x^2+1} + \frac{A_{1,2}x+B_2}{(x^2+1)^2} + \frac{A_2}{x}$$

$$A_{1,1}=-1,\ B_1=-1,\ A_{1,2}=1,\ B_2=0,\ A_2=1.$$

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx = \int \frac{-x-1}{x^2+1} + \frac{x}{(x^2+1)^2} + \frac{1}{x}\,dx.$$

$$= -\int \frac{x}{x^2+1}dx - \int \frac{1}{x^2+1}dx + \int \frac{x}{(x^2+1)^2}dx + \int \frac{1}{x}dx$$

$$= -\frac{1}{2}\ln|x^2+1| - \tan^{-1}x - \frac{1}{2}\frac{1}{x^2+1} + \ln|x| + C.$$

Rationalizing substitution: some nonrational functions can be changed into rational functions by means of appropriate substitutions.

Ex. [Text example 7.4.9] Evaluate $\int \frac{\sqrt{x+4}}{x}dx$.

$u=\sqrt{x+4}$, $u^2=x+4$, $x=u^2-4$, $dx=2u\,du$

$$\int \frac{\sqrt{x+4}}{x}dx \quad \int \frac{u}{u^2-4}2u\,du = 2\int \frac{u^2}{u^2-4}du$$

$$= 2\int \frac{u^2-4+4}{u^2-4}du = 2\int 1 + \frac{4}{\boxed{u^2-4}}du \qquad u^2-4=(u-2)(u+2)$$

$$\frac{4}{u^2-4} = \frac{A_1}{u-2} + \frac{A_2}{u+2}$$

$$4 = A_1(u+2) + A_2(u-2)$$

$$4 = (A_1+A_2)u + (2A_1-2A_2)$$

$$\begin{cases} A_1+A_2=0 \\ 2(A_1-A_2)=4 \end{cases}$$

$A_1=1$ and $A_2=-1$

$$= 2\left(\int 1\,du + \int \frac{1}{u-2} - \frac{1}{u+2}du\right)$$

$$= 2\left(u + \ln|u-2| - \ln|u+2| + C\right)$$

$$= 2u + 2\ln\left|\frac{u-2}{u+2}\right| + C$$

$$= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C.$$

3.5. **Strategy for integration.** Text Section 7.5, Exercise: 23, 45, 57, 61, 64, 71.

Table of integration formulas:

Constants of integration have been omitted.

$$\int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1), \qquad \int \frac{1}{x} dx = \ln|x|$$

$$\int e^x dx = e^x, \qquad \int a^x dx = \frac{a^x}{\ln a}$$

$$\int \sin x\, dx = -\cos x, \qquad \int \cos x\, dx = \sin x$$

$$\int \sec^2 x\, dx = \tan x, \qquad \int \csc^2 x\, dx = -\cot x$$

$$\int \sec x \tan x\, dx = \sec x, \qquad \int \csc x \cot x\, dx = -\csc x$$

$$\int \sec x\, dx = \ln|\sec x + \tan x|, \qquad \int \csc x\, dx = \ln|\csc x - \cot x|$$

$$\int \tan x\, dx = \ln|\sec x|, \qquad \int \cot x\, dx = \ln|\sin x|$$

$$\int \sinh x\, dx = \cosh x, \qquad \int \cosh x\, dx = \sinh x$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right), \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|, \qquad \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}|$$

Four-step strategy:

1. Simplify the integrand if possible.
2. Look for an obvious substitution.
3. Classify the integrand according to its form.
   (a) Trigonometric functions.
   (b) Rational functions.
   (c) Integration by parts.
   (d) Radicals.
      (i) If $\sqrt{\pm x^2 \pm a^2}$ occurs, use a trigonometric substitution.
      (ii) If $\sqrt[n]{g(x)}$ occurs, use rationalizing substitution $u = \sqrt[n]{g(x)}$.
4. Try again.
   If the first three steps have not produced the answer, remember that there are basically only two methods of integration: substitution and parts.
   (a) Try substitution.
   (b) Try integration by parts.
   (c) Manipulate the integrand.
   (d) Relate the problem to previous problems.
   (e) Use several methods together.

Ex. [Text example 7.5.4] Evaluate $\int \frac{dx}{x\sqrt{\ln x}}$.

$$\int \frac{dx}{x\sqrt{\ln x}} = \int \frac{d\ln x}{\sqrt{\ln x}} \overset{u=\ln x}{=\!=} \int \frac{du}{\sqrt{u}}$$

$$= 2u^{\frac{1}{2}} + C$$

$$= 2(\ln x)^{\frac{1}{2}} + C.$$

Ex. [Text example 7.5.5] Evaluate $\int \sqrt{\frac{1-x}{1+x}} dx$.

$\sin^{-1} x + \sqrt{1-x^2} + C$

$$\int \frac{\sqrt{1-x}}{\sqrt{1+x}} dx = \int \frac{\sqrt{1-x}\sqrt{1-x}}{\sqrt{1+x}\sqrt{1-x}} dx = \int \frac{1-x}{\sqrt{1-x^2}} dx$$

$$= \boxed{\int \frac{1}{\sqrt{1-x^2}} dx} - \boxed{\int \frac{x}{\sqrt{1-x^2}} dx} = \sin^{-1} x + C_1 - \left(-\sqrt{1-x^2} + C_2\right)$$

$= \sin^{-1} x + \sqrt{1-x^2} + (C_1 - C_2)$

$x = \sin u \Rightarrow dx = \cos u \, du$

$$\int \frac{\cos u}{\cos u} du = u + C_1 = \sin^{-1} x + C_1$$

$$-\frac{1}{2}\int \frac{d\,1-x^2}{\sqrt{1-x^2}}$$

$u = 1-x^2$

$$-\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{u} + C_2 = -\sqrt{1-x^2} + C_2$$

Not all functions can be integrated explicitly:

$$\int e^{x^2} dx, \qquad \int \frac{e^x}{x} dx, \qquad \int \sin(x^2) dx$$

$$\int \sqrt{x^3+1} dx, \qquad \int \frac{1}{\ln x} dx, \qquad \int \frac{\sin x}{x} dx.$$

3.6. **Approximate integration.** Text Section 7.7,
Exercise: 1, 11, 29, 47, 49.

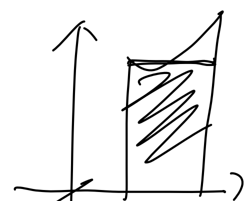
Since some antiderivatives can not be found explicitly, we can not find the **exact** value of a definite integral. For example, see the following two integrals

$$\int_0^1 e^{x^2}dx \qquad \int_{-1}^1 \sqrt{1+x^3}dx.$$

Therefore, we need to find approximate values of these integrals. Recall the definition of definite integrals

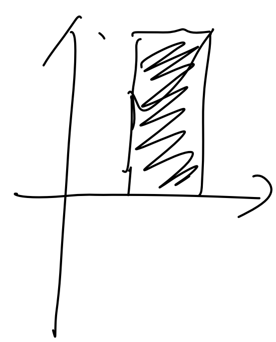
$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x,$$

where x_i^* is any sample points in the i-th subinterval $[x_{i-1}, x_i]$. So we can use the Riemann sum to approximate definite integrals.



Left endpoint approximation:

$$\int_a^b f(x)dx \approx L_n = \sum_{i=1}^n \boxed{f(x_{i-1})\Delta x}, \qquad \Delta x = \frac{b-a}{n}.$$

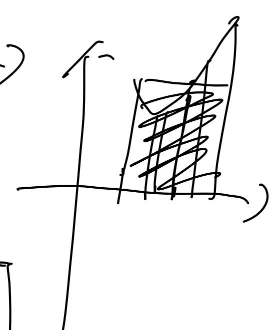


Right endpoint approximation:

$$\int_a^b f(x)dx \approx R_n = \sum_{i=1}^n f(x_i)\Delta x, \qquad \Delta x = \frac{b-a}{n}.$$

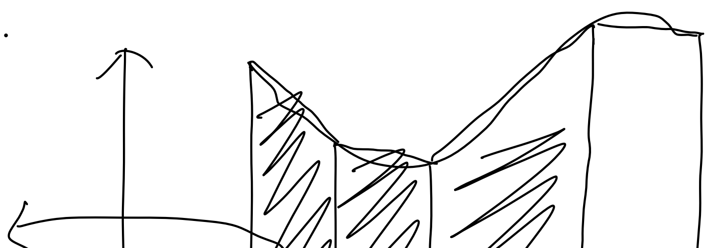
Midpoint Rule:

$$\int_a^b f(x)dx \approx M_n = \sum_{i=1}^n f(\bar{x}_i)\Delta x, \qquad \Delta x = \frac{b-a}{n}.$$

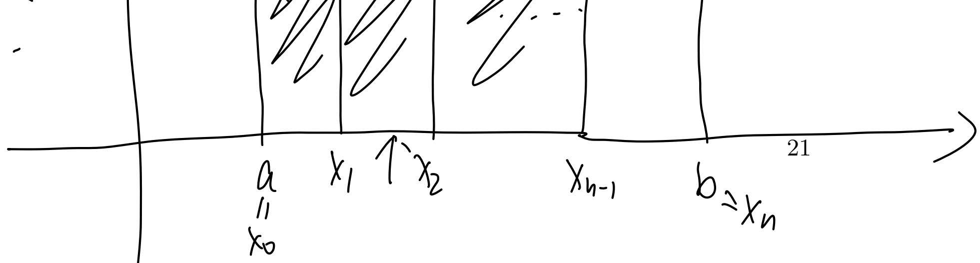


where $\bar{x}_i = \frac{1}{2}(x_{i-1}+x_i)$.

$\Delta x\left(\frac{f(x_0)+f(x_1)}{2}\right) + \Delta x\left(\frac{f(x_1)+f(x_2)}{2}\right)$



$+ \cdots + \Delta x \left(\frac{f(x_{n-1}) + f(x_n)}{2} \right).$



Trapezoidal Rule:

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)],$$

where $\Delta x = \frac{b-a}{n}$.

Ex. [Text example 7.7.1] Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with $n = 5$ to approximate the integral $\int_1^2 \frac{1}{x} dx$.

Solution: (a) $\Delta x = \frac{b-a}{n} = \frac{1}{5} = 0.2.$

$$\int_1^2 \frac{1}{x} dx \approx \boxed{\frac{0.2}{2}} \left[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2) \right]$$

$$= 0.1 \left[\frac{1}{1} + \frac{2}{1.2} + \cdots + \frac{1}{2} \right] \approx 0.695635$$

(b) $\int_1^2 \frac{1}{x} dx \approx \Delta x \left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9) \right] \approx 0.691908$

The **error** in using an approximation is defined to be the amount that needs to be added to the approximation to make it exact:

$$E_T = \int_a^b f(x)dx - T_n \quad \text{and} \quad E_M = \int_a^b f(x)dx - M_n.$$

Approximation to $\int_1^2 \frac{1}{x} dx$:

n	L_n	R_n	T_n	M_n
5	0.745635	0.645635	0.695635	0.691908
10	0.718771	0.668771	0.693771	0.692835
20	0.705803	0.680803	0.693303	0.693069

Corresponding errors:

n	E_L	E_R	E_T	E_M
5	-0.052488	0.047512	-0.002488	0.001239
10	-0.025624	0.024376	-0.000624	0.000312
20	-0.012656	0.012344	-0.000156	0.000078

0 $\quad \frac{1}{10} \quad \to \int_a^b f(x)\,dx \quad \frac{1}{20} \quad \frac{1}{20}$

Error bounds

Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \to 0 \quad \text{and} \quad |E_M| \leq \boxed{\frac{K(b-a)^3}{24\underline{n^2}_{100}}} = 0$$

Ex. [Text example 7.7.3] (a) Use the Midpoint Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2} dx$.

(b) Give an upper bound for the error involved in this approximation.

(a) $\Delta x = \frac{1}{10} = 0.1$

Then $\int_0^1 e^{x^2} dx = \Delta x \left(\underline{f(0.05)} + f(0.15) + \cdots + f(0.95) \right)$

≈ 1.460393

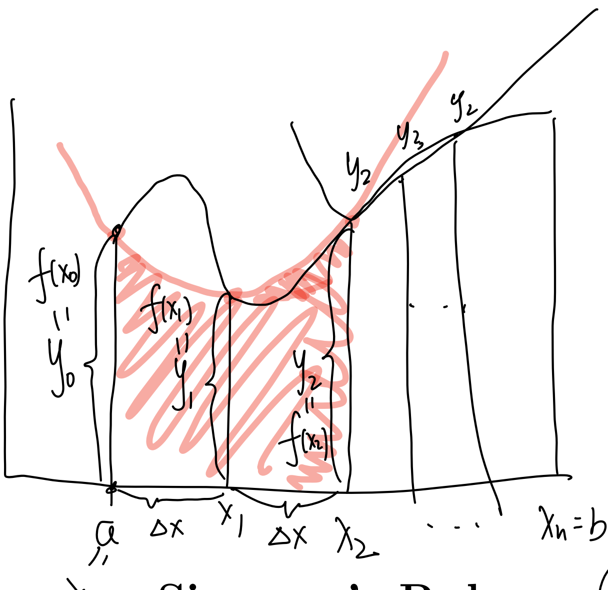
(b) $\underline{f(x) = e^{x^2}} \Rightarrow \underline{f'(x) = 2xe^{x^2}}$ and $f''(x) = (2+4x^2)e^{x^2}$.

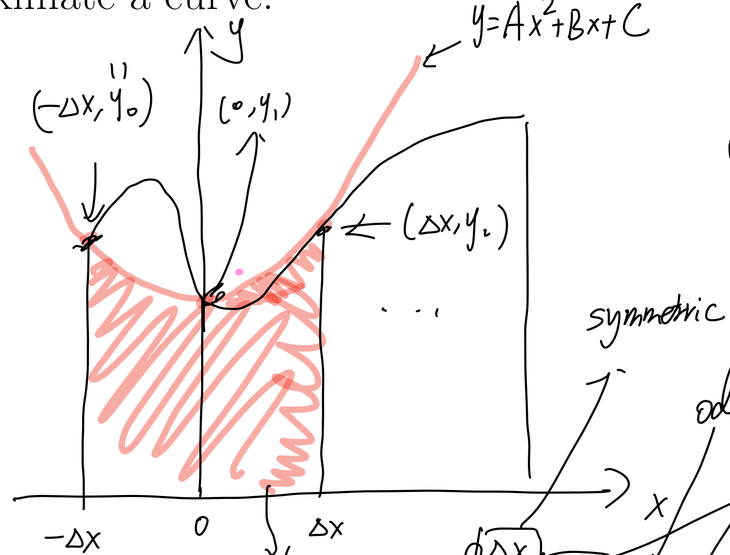
We note that $f''(x) \geq 0$ is increasing on $[0,1]$. $f''(x) = 2e^{x^2} + 2x(2xe^{x^2}) = 2e^{x^2} + 4x^2e^{x^2} = e^{x^2}(2+4x^2)$

$|f''(x)| = f''(x) \leq f''(1) = K \Rightarrow K = e^1(2+4\cdot 1) = 6e \Rightarrow E_m \leq \frac{6e(1)^3}{24\cdot 10^2} = \frac{e}{400} \approx 0.007$

Simpson's Rule

Approximate integration results from using parabolas instead of straight line segments to approximate a curve.





Area below $= \int_{-\Delta x}^{\Delta x} \boxed{Ax^2} + \boxed{Bx} + \boxed{C}\,dx = 2\int_0^{\Delta x} Ax^2 + C\,dx = 2\left(\frac{1}{3}Ax^3 + Cx\right)\Big|_0^{\Delta x} = 2\left(\frac{1}{3}A\Delta x^3 + C\Delta x\right) = \frac{1}{3}(2A\Delta x^3 + 6C\Delta x) = \frac{\Delta x}{3}(2A\Delta x^2 + 6C) = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$

Simpson's Rule

$4y_1 + y_0 + y_2 = 2A(\Delta x)^2 + 2C + 4C = 2A(\Delta x)^2 + 6C$

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)],$$

where n is even and $\Delta x = \frac{b-a}{n}$.

$y_0 = A(-\Delta x)^2 + B(-\Delta x) + C = A(\Delta x)^2 - B\Delta x + C$

$y_1 = C$

$y_2 = A(\Delta x)^2 + B\Delta x + C$

$\boxed{y_0 + 4y_1 + y_2} = 2A\Delta x^2 + 6C$

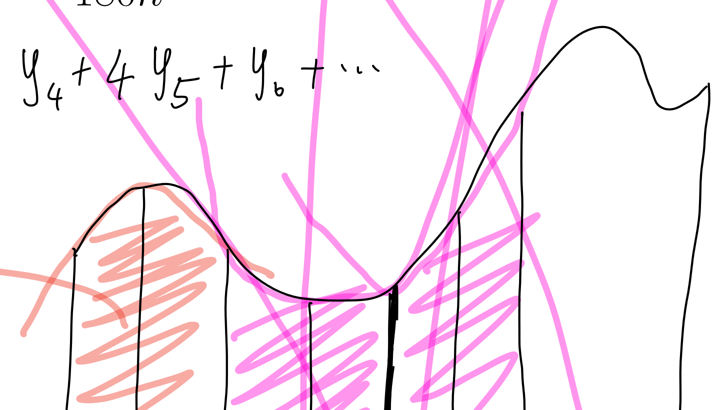
Error Bounds for Simpson's Rule

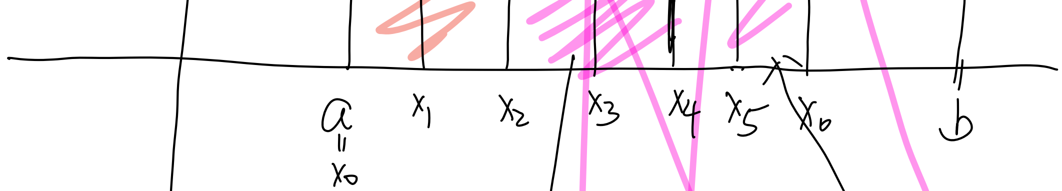
Suppose that $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If E_s is the error involved in using Simpson's Rule, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

$\frac{\Delta x}{3}(y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + y_4 + 4y_5 + y_6 + \cdots$

$\frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$





Ex. [Text example 7.7.7] (a) Use Simpson's Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.

(b) Estimate the error involved in this approximation.

$\frac{\Delta x}{3}(y_4 + 4y_5 + y_6)$.

$\frac{\Delta x}{3}(y_2 + 4y_3 + y_4)$

Solution: (a) $\Delta x = \frac{1}{10} = 0.1$

$e^{0^2} = 1$ $\quad \boxed{4e^{(0.1)^2}}$ $\quad 2e^{(0.2)^2}$

$$\int_0^1 e^{x^2}dx \approx \frac{\Delta x}{3}\Big[\boxed{f(x_0)} + \boxed{4f(x_1)} + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) + 2f(x_8) + 4f(x_9) + f(x_{10})\Big] \approx 1.462681$$

(b). $f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2}$ on $[0,1]$

$\Downarrow$

$f^{(4)}(x) \geq 0$ is increasing on $[0,1]$

$\Downarrow$

$|f^{4}(x)| = f^{(4)}(x) \leq f^{(4)}(1) = (12 + 48 + 16)e^1 = 76e = K$

$\Downarrow$

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} = \frac{76e(1)^5}{180 \cdot 10^4} \approx 0.000115$$

$f''(x) = (2 + 4x^2)e^{x^2}$

$f'''(x) = 8xe^{x^2} + (2+4x^2)e^{x^2}(2x)$

$= [12x + 8x^3]e^{x^2}$

$f^4(x) = (12 + 24x^2)e^{x^2} + [12x + 8x^3]e^{x^2}$ …

$[12x+8x^3]e \; 2x$

$= (12+24x^2)e^{x^2} + [24x^2+16x^4]e^{x^2} = e^{x^2}[16x^4+48x^2+12]$

3.7. **Improper integrals.** Text Section 7.8,
Exercise: 1, 13, 29, 57, 61, 67, 75.

In this section, we will extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$.
In either cases, the integral is called an *improper* integral.

$\int_{-1}^{1} \frac{1}{x} dx$

Type I: $\boxed{\text{Infinite intervals}}$

Definitions of an improper integral of type I:
(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided this limit exists (as a finite number).
(b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

the value is finite.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx.$$

In part (c) any real number a can be used.

$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} \frac{1}{-e^{-t}} = 0$

L'Hospital Rule

Ex. [Text example 7.8.2] Evaluate $\int_{-\infty}^{0} xe^x dx$.

$$\int_{-\infty}^{0} xe^x dx = \lim_{t\to-\infty} \boxed{\int_t^0 xe^x dx} = \lim_{t\to-\infty}\left(-\boxed{te^t} + \boxed{e^t} - 1\right) = -1$$

$$e^{-\infty} = 0$$

$$\int_t^0 xe^x dx = \int_t^0 x\, de^x = xe^x \Big|_t^0 - \int_t^0 e^x dx$$

$$= -te^t - e^x\Big|_t^0$$

$$= -te^t - (1 - e^t) = \boxed{-te^t + e^t - 1}$$

Ex. [Text example 7.8.4] For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p} dx$$

convergent?

$$\int_1^{+\infty} \frac{1}{x^p} dx = \lim_{t\to+\infty} \boxed{\int_1^t x^{-p} dx} = \begin{cases} \text{diverges} & p=1 \\ \text{diverges} & p<1 \\ \frac{1}{p-1} & p>1 \end{cases}$$

$$= \begin{cases} \text{diverges} & p\le 1 \\ \frac{1}{p-1} & p>1 \end{cases}$$

$$\int_1^t \boxed{x^{-p}} dx = \begin{cases} \ln x \Big|_1^t = \ln t - \ln 1 = \ln t & p=1 \\ \frac{x^{-p+1}}{-p+1}\Big|_{x=1}^{x=t} = \frac{t^{1-p}}{1-p} - \frac{1}{1-p} & p\neq 1 \end{cases}$$

for any $t \geq$

$$= \frac{1}{\ }\left(t^{1-p} - 1\right)$$

$1-p$

$1-p>0 \Rightarrow p<1$

Type II: Discontinuous integrands

Definitions of an improper integral of type II:

(a) If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b-}\int_a^t f(x)dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t\to a+}\int_t^b f(x)dx$$

if this limit exists (as a finite number).

The improper integrals $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a<c<b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

$$\int_1^{+\infty}\frac{1}{x^p}dx = \begin{cases} \text{diverges} & p\le 1 \\ \frac{1}{p-1} & p>1 \end{cases}$$

$$\int_0^1 \frac{1}{x^p}dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^p}dx \qquad p>1$$

$$= \begin{cases} \boxed{\lim_{t\to 0^+}\frac{1}{1-p}x^{1-p}\Big|_t^1} & p\neq 1 \\ \boxed{\lim_{t\to 0^+}\ln x\Big|_t^1} & p=1 \end{cases}$$

$$\lim_{t\to 0^+}\frac{1}{1-p}\left(1-\boxed{t^{1-p}}\right) = \begin{cases} \text{diverges} & p>1 \\ \frac{1}{p-1} & p<1 \end{cases}$$

$$\lim_{t\to 0^+}\ln 1 - \ln t = +\infty \quad \text{diverges}$$

diverges $\quad p\ge 1$

$$= \begin{cases} \quad & \\ \frac{1}{1-p} & p<1 \end{cases}$$

Ex. [Text example 7.8.5] Find $\int_2^5 \frac{1}{\sqrt{x-2}}dx$.

$$\int_2^5 \frac{1}{\sqrt{x-2}}dx = \lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}dx = \lim_{t\to 2^+} 2\sqrt{x-2}\Big|_t^5$$

$$= \lim_{t\to 2^+}\left(2\sqrt{3} - 2\sqrt{t-2}\right)$$

$$= 2\sqrt{3}$$

diverges

Ex. [Text example 7.8.7] Evaluate $\int_0^3 \boxed{\frac{1}{x-1}}dx$ if possible.

$$\int_0^3 \frac{1}{x-1}dx = \boxed{\int_0^1 \frac{1}{x-1}dx} + \int_1^3 \frac{1}{x-1}dx \Rightarrow \int_0^3 \frac{1}{x-1}dx$$

diverges.

$$\int_0^1 \frac{1}{x-1}dx = \lim_{t\to 1^-}\int_0^t \frac{1}{x-1}dx = \lim_{t\to 1^-} \ln|x-1|\Big|_0^t$$

$$= \lim_{t\to 1^-} \ln|t-1| - \ln 1 = \lim_{t\to 1^-}\ln|t-1|$$

diverges.

Ex. [Text example 7.8.8] Evaluate $\int_0^1 \ln x\,dx$.

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+}\int_t^1 \ln x\,dx = \lim_{t\to 0^+}\left(x\ln x\Big|_t^1 - \int_t^1 x\,d\ln x\right)$$

$$= \lim_{t\to 0^+}\left(-t\ln t - \int_t^1 1\,dx\right)$$

$$= \lim_{t\to 0^+}\left(-\boxed{t\ln t}^{\to 0} - (1-t)\right)$$

$$\quad \| $$

$$\quad -1$$

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+}\frac{\ln t^{\to -\infty}}{t^{-1}_{\to +\infty}}$$

$$= \lim_{t\to 0^+}\frac{t^{-1}}{-t^{-2}} = \lim_{t\to 0^+} -t$$

$$\|$$

A comparison test for improper integrals

Comparison Theorem:

The crucial assumption

Theorem 3.1. *Suppose that f and g are continuous functions with $\boxed{f(x) \geq g(x)} \boxed{\geq 0}$ for $x \geq a$.*

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent

≥ 0

Ex. [Text example 7.8.9] Show that $\int_0^\infty e^{-x^2} dx$ is convergent.

$\not\geq$ on $[0,+\infty)$

e^{-x}

This is true if $x \geq 1$.

Solution: $\int_0^{+\infty} e^{-x^2} dx = \boxed{\int_0^1 e^{-x^2} dx} + \int_1^{+\infty} e^{-x^2} dx$

Since $f(x) = e^{x^2}$ is continuous on $[0,1]$, then $\int_0^1 e^{-x^2} dx$ is well-defined.

On $[1,+\infty)$, we note that $e^{-x^2} \leq e^{-x}$. Then $\int_1^{+\infty} e^{-x} dx = \lim_{t\to+\infty} \int_1^t e^{-x} dx$

$= \lim_{t\to+\infty} -e^{-x} \Big|_{x=1}^{x=t} = \lim_{t\to+\infty} \left(-e^{-t} - (-e^{-1})\right) = \lim_{t\to+\infty} -e^{-t} + e^{-1} = e^{-1} \Rightarrow \int_1^{+\infty} e^{-x} dx$ converges

$\Downarrow$ by comparison test

$\int_1^{+\infty} e^{-x^2} dx$ converges

At the end, we showed $\int_0^{+\infty} e^{-x^2} dx$ converges.

Ex. [Text example 7.8.10] Show that $\int_1^\infty \frac{1+e^{-x}}{x} dx$ is divergent.

Solution: We note that, on $\boxed{[1,+\infty)}$,

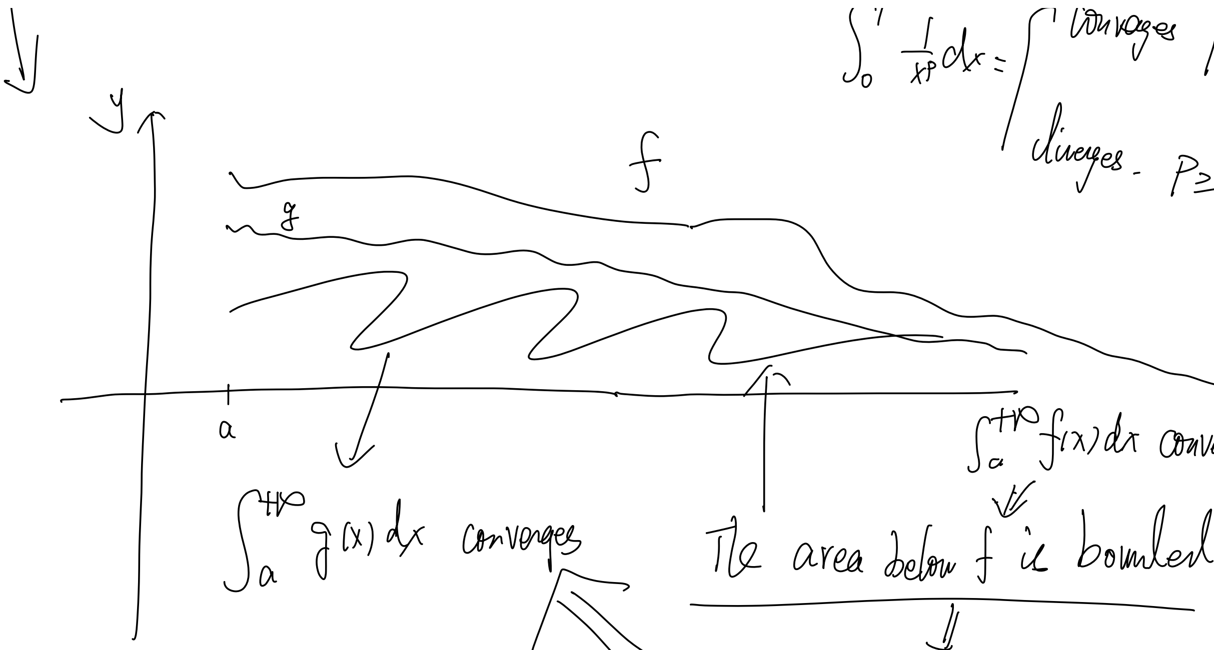
$$\frac{1+e^{-x}}{x} = \frac{1}{x} + \frac{e^{-x}}{x} \geq \frac{1}{x}.$$

Since $\int_1^{+\infty} \frac{1}{x} dx$ diverges, by comparison test

we know that $\int_1^{+\infty} \frac{1+e^{-x}}{x} dx$ diverges.

$\int_1^{+\infty} \frac{1}{1+x} dx$

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{converges} & p<1 \\ \text{diverges} & p\ge 1 \end{cases}$$



$\int_a^{+\infty} f(x)\,dx$ converges

$\Downarrow$

The area below f is bounded.

$\Downarrow$

The area below g is bounded

$\int_a^{+\infty} g(x)\,dx$ converges

Prove or disprove (provide a counter example) the following statement

(a) If f is continuous on $(-\infty, +\infty)$, then

$$\int_{-\infty}^{+\infty} f(x)\, dx = \lim_{t \to +\infty} \int_{-t}^{t} f(x)\, dx.$$

(b) If $f(x) \le g(x)$ and $\int_0^{+\infty} f(x)$ diverges, then

$$\int_0^{+\infty} g(x)\, dx \text{ diverges.}$$