MA2506 Probability and Statistics

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Grading

Final Exam 70%

~~2~~ Quiz/Midterm 30%

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Outlines (subject to change)

1. Combinatorics (counting)
2. Basic concepts on probability
3. Discrete random variables
4. Continuous random variables
5. Joint distribution
6. Limit Theorem
7. Point estimation
8. Confidence interval
9. Hypothesis testing

Chapter 1. Combinatorics

The basic principle of counting:

A Simple Example: A die with six faces is thrown twice, how many different outcomes are possible?

36

Extension to more than two experiments:

Example: How many different 4-place license plates are possible if the first 3 places are occupied by capital letters (A-Z) and the final place is occupied by a number (0-9).

Solution: $26 \times 26 \times 26 \times 10$

abc
acb
bca

1.1 Permutation (i.e. ordered arrangement)

Q: How many different *ordered* arrangements of the 3 letters a,b, and c are possible?

A: Using the basic principle of counting, $3 \times 2 \times 1 = 6$

In general, the number of different permutations of n different objects is n! . $1 \times 2 \times 3 \cdots n$

(This is still the basic principle)

$\rightarrow \boxed{20} \times \boxed{19} \times \boxed{18}$

In how many ways can a person gathering market data select 3 of the 20 households living in a certain apartment complex (order matters)?

<u>Proposition:</u> The number of ordered arrangements or permutations of r objects selected from n distinct objects is given by

$$P_r^n = \underbrace{n(n-1)\cdots(n-r+1)}_{r \text{ terms}} = \boxed{\frac{n!}{(n-r)!}}$$

Permutations when some objects are the same

Q: How many different arrangements with 3 a's and 4 b's are possible?

$a_1\ a_2\ a_3$

$a_2\ a_1\ a_3$

A: $\dfrac{7!}{3!4!}$ $\leftarrow \binom{7}{3} = \binom{7}{4}$ $\dfrac{7!}{6 \cdot 24}$

Explanation: First, 7! permutations are possible if 3 a's and 4 b's are distinguished from each other. Consider one of these permutations: $\overset{a_2}{a_1} b_1 \overset{a_3}{a_2} b_2 b_3 b_4 \overset{a_1}{a_3}$ $3! = 6$

If we now permute the a's (or b's) among themselves, the resulting arrangement is still the same. $4! = 24$

6×24 144

In general, there are $\boxed{\frac{n!}{n_1!n_2!\cdots n_r!},}$ where $\sum_{i=1}^{r} n_i = n$ different arrangements when n_1 of the total n objects are the same, n_2 of the rest of the objects are the same, etc.

r groups

$n_1, n_2, \cdots n_r$

$\sum_{i=1}^{r} n_i = n$

1.2 Combinations

Q: How many different groups of 3 can be selected from the 5 items a,b,c,d,e (that is, we do not care the order of the 3 items selected) → 60

$$\boxed{5 \times 4 \times 3} = \frac{5!}{2!}$$

$$\frac{5 \times 4 \times 3}{3!}$$

A: $$\frac{5 \cdot 4 \cdot 3}{3!} = 10$$

$$\binom{n}{r} = \binom{n}{n-r}$$

In general, the number of possible outcomes of selecting r objects from a pool of n object is

$$\frac{P_r^n}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \boxed{\frac{n!}{r!(n-r)!}} =: \boxed{\binom{n}{r}} \text{ or } C_n^r$$

$n=4$ $(x+y)(x+y)(x+y) \cdots (x+y)$ $\binom{4}{3}$

$\binom{4}{3} x^3 y$

Binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

$y^n, x y^{n-1}, x^2 y^{n-2}, \cdots$

When *n=2*, $(x+y)^2 = x^2 + 2xy + y^2$

When *n=3*, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Remark:

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\frac{n!}{n_1!\,n_2!\cdots}$$

Multinomial coefficients

Consider the problem: A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$ where $\sum_{i=1}^{r} n_i = n$ How many different divisions are possible?

Solution:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}$$

$$\frac{n!}{n_1!\,\cancel{(n-n_1)!}}\;\frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}}\;\frac{\cancel{(n-n_1-n_2)!}}{n_3!\,\cancel{(n-n_1-n_2-n_3)!}}\cdots$$

Notation: $$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}, \text{ when } \sum n_k = n$$

Multinomial theorem:

$n > 0$

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{(n_1,\ldots,n_r):n_i \geq 0, n_1+\ldots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$$

where the sum is over all *nonnegative* integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \ldots + n_r = n.$

task: 1 2 3 4 - 10

A B A A C

How many ways are there to distribute 10 different tasks to 3 persons A, B, and C, with A assigned 3 tasks, B 4 tasks, and C 3 tasks?

$$\frac{10!}{3!\,4!\,3!}$$

$$\binom{10}{3}\binom{7}{4}\underbrace{\binom{3}{3}}_{1}$$

How many ways are there to partition 9 people into a R&D division, a marketing division, and a HR division, with 3 people in each?

Example: What is the number of possible divisions of n **identical** objects into r **distinct** groups of sizes $n_1, \ldots, n_r$?

10 ball

2 8

Example: How many different ways are there to partition *n* persons into at most *r* distinct (may be empty) groups?

Solution:

r^n

$2+0+3=5$

Example (difficult):

How many positive integer solutions are there for the equation: $x_1 + x_2 + \cdots + x_r = n$

Solution: $2+1+2=5$

$x_1+x_2=10$

$x_1+x_2+x_3=10$

$n=5$

$r=3$

$x_1+x_2+x_3=5$

$1+3+1=5$

$\binom{4}{2}$

$0\ 2\ 3$

$1\ 0\ 4$

$$\boxed{\binom{n-1}{r-1}}$$

Example: How many *nonnegative* integer solutions are there for the same equation $x_1 + x_2 + \cdots + x_r = n$ (1)

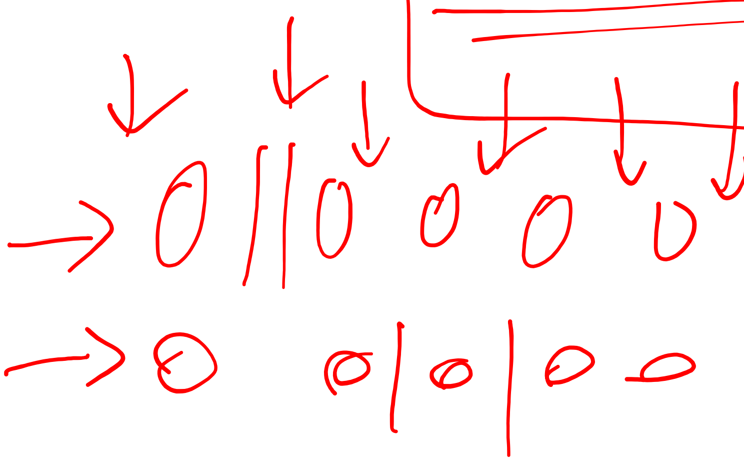
$x_i \geq 0$

Solution:

$y_1 = x_1 + 1,\quad y_2 = x_2 + 1,\quad \ldots \quad y_r = x_r + 1$

$y_1 + y_2 + \cdots + y_r = n + r$ (2) $\quad y_i \geq 1$

$\boxed{\binom{n+r-1}{r-1}}$



n+1 space choose r−1

$\binom{n+r-1}{r-1}$

n, r−1

5 balls 2 bars

$\binom{7}{2}$

Example: There are 10 colored balls, among which 5 balls are of the same color, the other balls are of different colors (so there are 6 distinct colors in total). How many permutations of the 10 balls are possible?

Solution:

$$\frac{10!}{5!\,1!\,1!\,1!\,1!\,1!}$$

$$\binom{10}{5}5! = \frac{10!}{5!}$$

Example:

In how many ways can 3 boys and 3 girls sit together in a row?

$6! = 720$

What if the boys must sit together, so must girls?

BBBGGG ← $3! \times 3!$

GGGBBB

$3! \times 3! \times 2 = 72$

What if only the boys must sit together, not the girls?

→ BGGG GGGB

GBGG

GGBG

$3! \times 3! \times 4 = 144$

What if no two boys can sit together, no two girls can sit together. ?

BGBGBG

GBGBGB

$3! \times 3! \times 2 = 72$