EE1001 Fundamentals of Digital Techniques **Semester A, 2021/22**

Test 2 (Solution)

1. We prove by contraposition. Suppose $\frac{x}{2}$ is rational. Then it can be written as $\frac{p}{q}$, where p and q are integers. Therefore, $x = \frac{2p}{q}$, and $(x+1)^2 = (\frac{2p}{q}+1)^2 = \frac{(2p+q)^2}{q^2}$, which is rational. Hence, by contraposition, the given statement is true.

2. The statement is true for $n = 1$ because $4 > 1$.
   Assume that $4^k > k^2$ for any integers $k \geq 1$.
   Suppose $n = k + 1$. Then $4^{k+1} = 4 \times 4^k > 4k^2$ by induction hypothesis.
   Note that $4k^2 = (k+k)^2 \geq (k+1)^2$ for any positive integers k.
   Hence, $4^{k+1} > 4k^2 \geq (k+1)^2$.
   By mathematical induction, the statement is true for any integers $n \geq 1$.

3. Let $x = 0.\overline{45}$. Then $100x = 45.\overline{45}$. By subtraction, $99x = 45$. Simplifying, $x = \frac{45}{99} = \frac{5}{11}$.

4. 1) Convert to polar form first.

$$\left(\overline{4 - 4\sqrt{3}i}\right)^2 = \left(4 + 4\sqrt{3}i\right)^2$$
$$= \left(8cis\frac{\pi}{3}\right)^2 = 64cis\frac{2\pi}{3}$$

   2) Solve

$$z^3 = 64cis\frac{2\pi}{3}.$$

   We have

$$z = 4cis\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right) \qquad (k\ is\ an\ integer)$$

   3) Obtain 3rd roots by substituting $k = 0, 1, 2$.

$$k = 0, z = 4cis\left(\frac{2\pi}{9}\right),$$
$$k = 1, z = 4cis\left(\frac{8\pi}{9}\right),$$
$$k = 2, z = 4cis\left(\frac{14\pi}{9}\right),$$

5. First, represent the two numbers by two's complement. It is easy to see that $A = 010101_2$.
   Since $17_{10} = 010001_2$, we obtain $B = -17_{10} = 101111_2$ by flipping the bits and adding 1.
   Binary addition of A and B gives 1000100_2, which has seven bits. Dropping the carry gives $A + B = 000100_2 = 4_{10}$.

6. The proof is shown below (step 3 using commutative law can be skipped).

Proof: Let A and B be any sets. Then

$(A - B) \cup (A \cap B)$	$= (A \cap B^c) \cup (A \cap B)$	by the set difference law
	$= A \cap (B^c \cup B)$	by the distributive law
	$= A \cap (B \cup B^c)$	by the commutative law (can be skipped)
	$= A \cap U$	by the complement law
	$= A$	by the identity law

7. 1) Shift exponent of smaller number B by 2 places. (lost final 1)

Exponent (8 bits)	**Fraction** (23 bits)	**Significand**
1000 0001	1000 0000 0000 0000 0000 001	**1**. 1100 0000 0000 0000 0000 001
↓		↓
1000 0011		0. 0**1**11 0000 0000 0000 0000 000

2) Convert negative number B to 2's complement.

Exponent	**(Sign)Fraction**
Significand of -B	0.0**1**11 0000 0000 0000 0000 000
1's Complement of B	1.1000 1111 1111 1111 1111 111
2's Complement of B	1.1001 0000 0000 0000 0000 000

3) Perform addition of A and B in 2's complement.

	1. 0011 0000 0000 0000 0000 000
+	**1**. 1001 0000 0000 0000 0000 000
2's Complement = Significand	(**1**)**0**. 1100 0000 0000 0000 0000 000

(Simply drop the overflow bit)

4) Shift exponent of the result by 1 place.

Exponent (8 bits)	**Significand**
1000 0011	**0**. 1100 0000 0000 0000 0000 000
↓	↓
1000 0010	1. 1000 0000 0000 0000 0000 000

5) Result in IEEE 754 32-bit format is

Sign (1 bit)	**Exponent** (8 bits)	**Fraction** (23 bits)
0	1000 0010	1000 0000 0000 0000 0000 000

which equals to

$$(+1) \times 2^{130-127} \times (1.1)_2 = (12)_{10}.$$

8. The answer is shown below:
   a) No, because A and C are not disjoint.
   b) $A \times (B \cap C) = \{1, 4, 5\} \times \{2, 3\} = \{(1, 2), (1, 3), (4, 2), (4, 3), (5, 2), (5, 3)\}$

9. The answer is shown below:
   a) $\{5, 7, 8\}$
   b) f is not injective because 2 and 3 are distinct elements of the domain but have the same image. (Counter example)
      Besides, f is not surjective because 6 is in Y, but we cannot find an element x in X such that $f(x) = 6$. (Counter example) / Range of f: $\{5,7,8\}$ is not equal to domain of f: Y.

10. $f \diamond g(x) = (3x + 1)^2 + (3x + 1) - 1 = 9x^2 + 9x + 1$
    $g \circ f(x) = 3(x^2 + x - 1) + 1 = 3x^2 + 3x - 2$