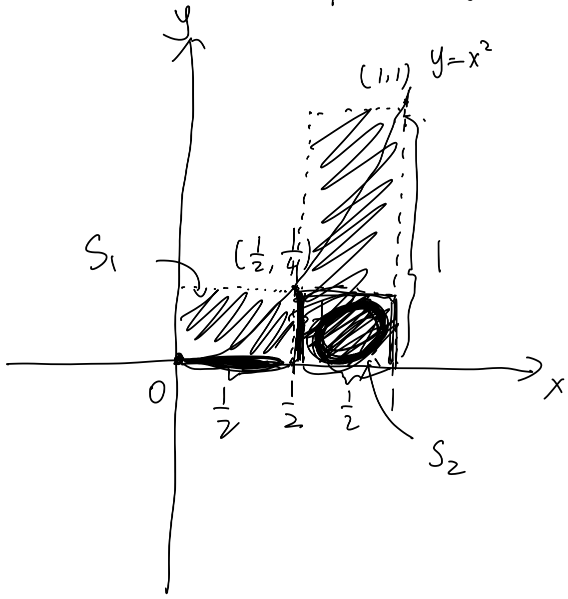
Chapter 1 : Integrals

The area problem: Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

For a region with straight sides, such as a rectangle, a triangle or a polygon, the problem is easy to answer

However, for a region with curved sides, it is hard.

<u>Example</u>: Find the area A under the parabola $y=x^2$ from 0 to 1.



$$S_1^r = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}, \qquad S_2^r = \frac{1}{2} \times 1 = \frac{1}{2}$$

$$\underline{S_1^r + S_2^r} = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}.$$

$$S_1^l = 0 \times \frac{1}{2} = 0, \qquad S_2^l = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}.$$

$$S_1^l + S_2^l = 0 + \frac{1}{8} = \frac{1}{8}.$$

$$\Downarrow$$

$$\frac{1}{8} = S_1^l + S_2^l \le A \le S_1^r + S_2^r = \frac{5}{8}$$

We can apply the idea of the above example to the more general region S. And we define the area A of the region S in the following way:

Def 1.1. The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximaty rectangles:

The area of the first rectangle.

$$A = \boxed{\lim_{n\to+\infty} R_n} = \lim_{n\to+\infty} \boxed{\left[\boxed{f(x_1)\Delta x} + \boxed{f(x_2)\Delta x} + \cdots + f(x_n)\Delta x \right]}$$

where $\Delta x = \frac{b-a}{n}$, $x_0 = a$ and

$$x_i = x_0 + i\Delta x, \quad i = 1, 2, \cdots, n.$$

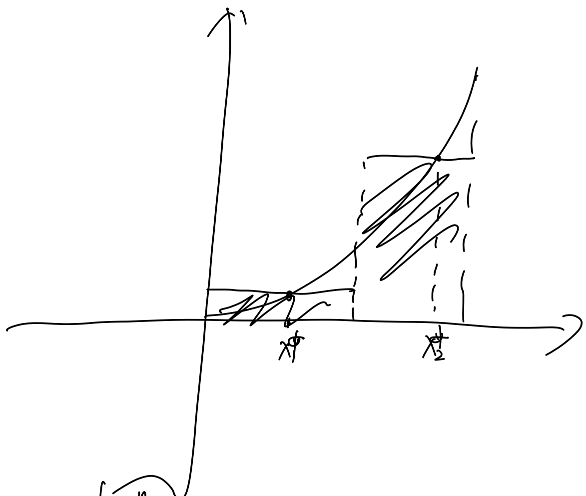
Remark: If we assume that f is continuous in bounded region S, the limit in the above definition always exists. It can be shown that we get the same value if we use left endpts:

$$A = \lim_{n\to+\infty} L_n = \lim_{n\to+\infty} \left[f(x_0)\Delta x + \cdots + f(x_{n-1})\Delta x \right].$$

$\boxed{\text{Cauchy sequence}}$?

or any sample pt x_i^* in the ith subinterval $[x_{i-1}, x_i]$,

$$A = \lim_{n\to+\infty} \underline{\underline{M_n}} = \lim_{n\to+\infty}\left[f(x_1^*)\Delta x + \cdots + f(x_i^*)\Delta x\right].$$



Note: In the notation $\boxed{\sum_{i=m}^{n}}$, Σ is called a summation operator.

Some useful formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

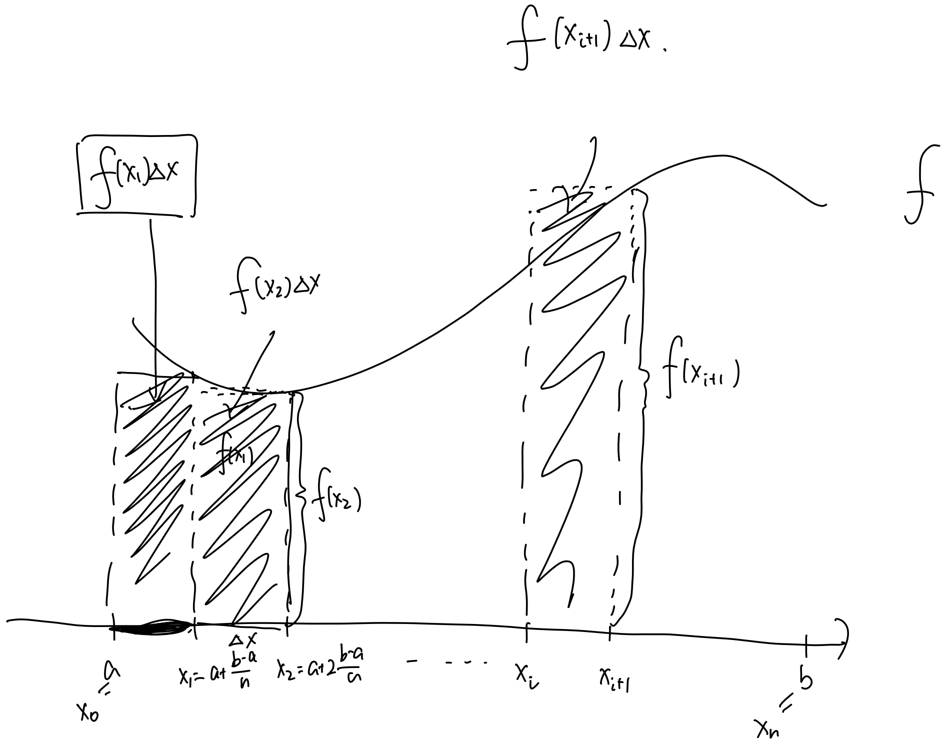
$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i, \quad \underline{\sum_{i=1}^{n}(a_i \pm b_i) = \left(\sum_{i=1}^{n} a_i\right) \pm \left(\sum_{i=1}^{n} b_i\right)}.$$

Q: $\sum_{i=1}^{n}(a_i b_i) \overset{?}{\neq} \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right)$

$a_1 = b_1 = 1$, $a_2 = b_2 = 2$ $\Rightarrow$ $\sum_{i=1}^{2} a_i b_i = 1 + 4 = 5 \neq \left(\sum_{i=1} a_i\right)\left(\sum_{i=1} b_i\right) = 3\times 3 = 9$

$$\sum_{i=1}^{n} \sqrt{a_i} \neq \sqrt{\sum_{i=1}^{n} a_i}$$

$a_1 = 1, a_2 = 4$



The distance problem: Find the distance traveled by an object during the certain time period if the velocity of the object $f(t)$ is known at all times.

Example:

$\frac{27}{3.6}$ m/s $= 7.5$ m/s

9.4 m/s

10.6 m/s

12.8 m/s

Time (s)	$\boxed{0}$	5	10	15	20	25	30
Velocity (km/h)	$\boxed{27}$	34	38	46	51	50	45

12.5 m/s

13.9 m/s

14.2 m/s

During each five seconds the velocity does not change very much, so we can estimate the distance traveled during the time by assuming the velocity is constant

Then the distance can be approximated by

$$7.5\,m/s \times 5s + 9.4\,m/s \times 5s + 10.6\,m/s \times 5s + 12.8\,m/s \times 5s + 14.2\,m/s \times 5s$$
$$+ 13.9\,m/s \times 5s + 12.5\,m/s \times 5s$$
$$= 404.5\,m.$$

The exact displacement d traved is the limit of the following expressions

$$d = \lim_{n\to+\infty} \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \lim_{n\to+\infty} \sum_{i=1}^{n} f(t_i)\,\Delta t$$

if we provide f is continuous.

<u>Def</u> 1.2: If f is a function defined for $a \le x \le b$, we divide the interval $[a,b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$. We let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the endpts of these subintervals and x_i^* be any sample pts in the ith subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\boxed{\int}_{\boxed{a}}^{\boxed{b}} \boxed{f(x)}\,\boxed{dx} = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that the limit exists. If it does exist, we say that f is integrable on $[a,b]$.

integral sign　　the upper limit　　integrand

the lower limit　　the integration variable.

Remark 1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

$$f(x) \not\equiv f(t)$$

Remark 2. The sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

is called a Riemann sum.

$$f(x) = -1, \quad 0 \le x \le 1.$$

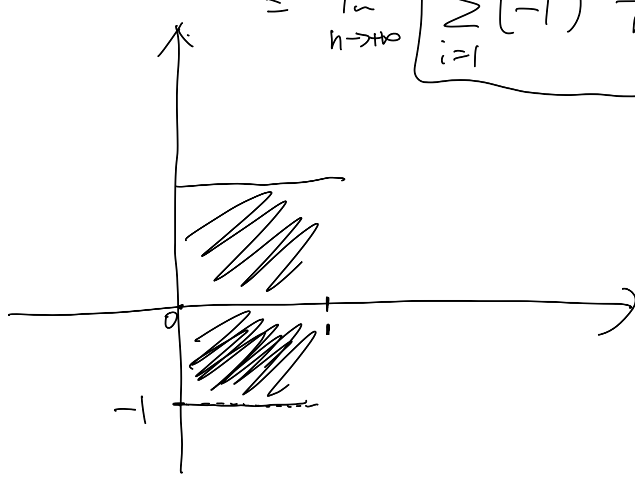
$$\boxed{\int_0^1 f(x)\,dx} = \lim_{n\to+\infty} \sum_{i=1}^{n} \boxed{f(x_i^*)}\,\Delta x$$

$$= \lim_{n\to+\infty} \boxed{\sum_{i=1}^{n} (-1)\frac{1}{n}} = \lim_{n\to+\infty} -\frac{1}{n}\left(\sum_{i=1}^{n} 1\right)$$

$$= \lim_{n\to+\infty} \left(-\frac{1}{\cancel{n}}\right)\cancel{n}$$

$$= \lim_{n\to+\infty} (-1)$$

$$= -1$$



$$\sum_{i=1}^{n} = \overbrace{1+1+\cdots+1}^{n} = n$$

$-1, -1, -1, -1, \cdots -1.$

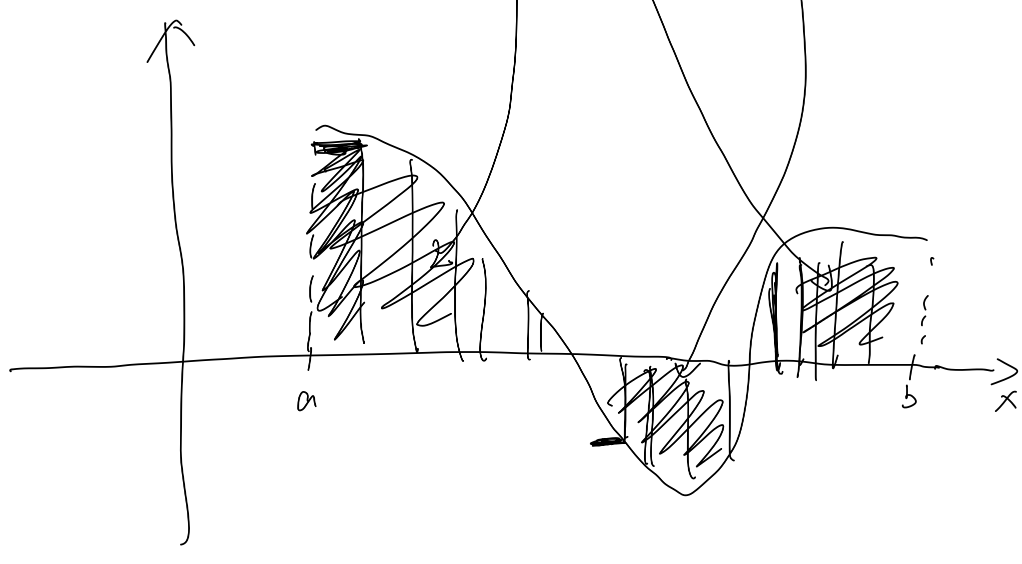
$a_n = -1$ for all n. $\lim_{n\to+\infty} a_n = -1$

A definite integral can be negative. It can be interpreted as a net area

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where $A_1 \geq 0$ is the area of region above axis and below the graph of f,

and $A_2 \geq 0$ is the area of the region below x-axis and the graph of f.



Remark 3: It is not necessary to divide $[a,b]$ into subintervals of equal length:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

where Δx_i is the length of the ith subinterval.

Remark 4: Not all the functions are integrable!!!