MA1300 Solutions to Self Practice # 9

1. Find the critical numbers of the function.

$$g(t) = |3t - 4|, \qquad F(x) = x^{4/5}(x-4)^2, \qquad g(\theta) = 4\theta - \tan\theta.$$

Solution:

a We rewrite $g(t)$ as

$$g(t) = \begin{cases} 3t - 4 & \text{when } t \geq 4/3, \\ 4 - 3t & \text{when } t < 4/3, \end{cases}$$

to give

$$g'(t) = \begin{cases} 3 & \text{when } t > 4/3, \\ -3 & \text{when } t < 4/3, \end{cases}$$

and $g'(t)$ does not exist when $t = 4/3$. Therefore the critical number is $4/3$.

b Since $F(x) = x^{14/5} - 8x^{9/5} + 16x^{4/5}$, we have

$$F'(x) = \frac{14}{5}x^{9/5} - \frac{72}{5}x^{4/5} + \frac{64}{5}x^{-1/5} = \frac{1}{5\sqrt[5]{x}}(14x^2 - 72x + 64).$$

Therefore $F'(x) = 0$ if $14x^2 - 72x + 64 = 0$, that is $x = 4$ or $8/7$, and $F'(x)$ does not exist when $x = 0$. Thus the critical numbers are 4, $8/7$, and 0.

c Take derivative to give

$$g'(\theta) = 4 - 1 - \tan^2\theta,$$

so $g'(\theta) = 0$ if $\tan\theta = \pm\sqrt{3}$, or equivalently, if $\theta = k\pi \pm \frac{\pi}{3}$, where $k = 0, \pm1, \pm2, \cdots$. On the other hand, $g'(\theta)$ does not exist when $\theta = \frac{\pi}{2} + k\pi$, $k = 0, \pm1, \pm2, \cdots$. Thus the critical numbers are $k\pi + \frac{\pi}{2}$ and $k\pi \pm \frac{\pi}{3}$, $k = 0, \pm1, \pm2, \cdots$.

2. Find the absolute maximum and absolute minimum values of f on the given interval.

$$\begin{aligned} &f(x) = 2x^3 - 3x^2 - 12x + 1, \quad [-2, 3]; \\ &f(t) = \sqrt[3]{t}(8 - t), \quad [0, 8]; \\ &f(t) = t + \cot(t/2), \quad [\pi/4, 7\pi/4]. \end{aligned}$$

Solution:

a Since f is continuous on $[-2, 3]$, we use the Closed Interval Method. Take derivative to give

$$f'(x) = 6x^2 - 6x - 12.$$

Since $f'(x)$ exists for all $x \in (-2, 3)$, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = -1$ or 2, both of which lie in $(-2, 3)$. Values of f at critical numbers:

$$f(-1) = 8, \qquad f(2) = -19.$$

Values of f at endpoints of the interval:

$$f(-2) = -3, \qquad f(3) = -8.$$

Therefore, the absolute maximum value is $f(-1) = 8$, and the absolute minimum value is $f(2) = -19$.

b f continuous on $[0, 8]$, so we use the Closed Interval Method. The Product Rule gives

$$f'(t) = \frac{1}{3}t^{-2/3}(8-t) - t^{1/3} = \frac{4t^{-2/3}}{3}(2-t),$$

so $f'(t) = 0$ if $t = 2$, which is in $(0, 8)$, and $f'(t)$ does not exist when $t = 0$, which is not in $(0, 8)$. So the only critical number is 2. Value of f at critical number:

$$f(2) = 6\sqrt[3]{2}.$$

Values of f at endpoints of the interval:

$$f(0) = 0, \qquad f(8) = 0.$$

Therefore, the absolute maximum value is $f(2) = 6\sqrt[3]{2}$, and the absolute minimum value is $f(0) = f(8) = 0$.

c f continuous on $[\pi/4, 7\pi/4]$, so we use the Closed Interval Method. Take derivative to give

$$f'(t) = \frac{2 - \csc^2 \frac{t}{2}}{2}.$$

Since $f'(t)$ exists for all $t \in (\pi/4, 7\pi/4)$, the only critical numbers of f occur when $f'(t) = 0$, that is $\csc \frac{t}{2} = \pm\cancel{1}$ sqrt(2), or equivalently, $t = \pi/2$ or $3\pi/2$. Values of f at critical numbers:

$$f(\frac{\pi}{2}) = 1 + \frac{\pi}{2} \approx 2.57, \qquad f(\frac{3\pi}{2}) = \frac{3\pi}{2} - 1 \approx 3.71.$$

Values of f at endpoints of the interval:

$$f(\frac{\pi}{4}) = \frac{\pi}{4} + \sqrt{2} + 1 \approx 3.20, \qquad f(\frac{7\pi}{4}) = \frac{7\pi}{4} - \sqrt{2} - 1 \approx 3.08.$$

Therefore, the absolute maximum value is $f(\frac{3\pi}{2}) = \frac{3\pi}{2} - 1$, and the absolute minimum value is $f(\frac{\pi}{2}) = 1 + \frac{\pi}{2}$.

3. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b$, $0 \le x \le 1$.

Solution: f continuous on $[0, 1]$, so we use the Closed Interval Method. The Product Rule gives

$$f'(x) = ax^{a-1}(1-x)^b - bx^a(1-x)^{b-1} = x^a(1-x)^b\left(\frac{a}{x} - \frac{b}{1-x}\right).$$

Since $f'(x)$ exists for all $x \in (0, 1)$, the only critical numbers of f occur when $f'(x) = 0$, that is $x = \frac{a}{a+b} \in (0, 1)$. So the function value at the critical number is

$$f(\frac{a}{a+b}) = \frac{a^a b^b}{(a+b)^{a+b}} > 0.$$

Values of f at endpoints of $[0, 1]$:

$$f(0) = 0, \qquad f(1) = 0.$$

Therefore the only local maximum value of f is $f(\frac{a}{a+b}) = \frac{a^a b^b}{(a+b)^{a+b}}$, which is also the absolute maximum value.

4. Between 0°C and 30°C, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given approximately by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3.$$

Find the temperature at which water has its maximum density.

Solution: $V = V(T)$ is continuous, so we use the Closed Interval Method. Take derivative to give

$$V'(T) = -0.06426 + 0.0170086T - 0.0002037T^2.$$

Since $V'(T)$ exists for all $T \in (0, 30)$, the only critical numbers of V occur when $V'(T) = 0$, that is, $T \approx 3.966514624$, or 79.53176716 (out of $(0, 30)$, rejected). The function value at critical number is

$$V(3.966514624) \approx 999.7446746.$$

The values of V at endpoints of $[0, 30]$:

$$V(0) = 999.87, \qquad V(30) \approx 1003.762770.$$

Therefore the temperature at which water has its maximum density is $T \approx 3.966514624$.

5. Prove that the function

$$f(x) = x^{101} + x^{51} + x + 1$$

has neither a local maximum nor a local minimum.

Proof: Since f is differentiable on $\mathbb{R}$, by Fermat's Theorem, if $c \in \mathbb{R}$ is a local maximum or local minimum of f then $f'(c) = 0$, that is $101c^{100} + 51c^{50} + 1 = 0$. But for any $c \in \mathbb{R}$, $101c^{100} + 51c^{50} + 1 > 0$, so f has neither a local maximum nor a local minimum.

6. A cubic function is a polynomial of degree 3; that is, it has the form $f(x) = ax^3 + bx^2 + cx + d$, where $x \neq 0$.

a Show that a cubic function can have two, one or no critical number(s). Give examples and sketches to illustrate the three possibilities.

b How many local extreme values can a cubic function have?

Solution: Since every cubic function f is differentiable on $\mathbb{R}$, the critical numbers occur if the derivative f' vanishes. We see that $f(x) = x^3 - 3x$ has two critical numbers since $f'(x) = 3(x+1)(x-1)$; $f(x) = x^3$ has one critical number since $f'(x) = 3x^2$, and $f(x) = x^3 + 3x$ has no critical number since $f'(x) = 3(x^2 + 1) > 0$ for any $x \in \mathbb{R}$. Figure 1 shows the three possibilities. Since the derivative f' of any cubic function f is a polynomial of degree no more than two, $f' = 0$ has at most two roots. Therefore a cubic function has at most two local extreme values.

7. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

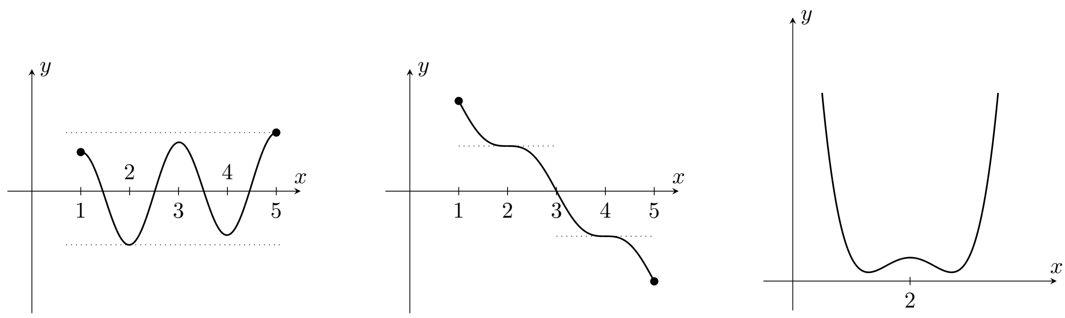

Figure 1: The pictures of Problem 6. Left: $y = x^3 - 3x$, Middle: $y = x^3$, Right: $y = x^3 + 3x$.

Solution: It is easy to see that $f(-1) = 0 = f(1)$. Take derivative to give $f'(x) = -\frac{2}{3}x^{-1/2}$, which never vanishes when $x \in (-1, 1) \setminus \{0\}$. This does not contradict Rolle's Theorem because $f'(0)$ does not exist and thus Rolle's Theorem does not apply.

8. Show that the equation $2x - 1 - sinx = 0$ has exactly one real root.

Proof: Let $f(x) = 2x - 1 - sinx$, then f is continuous on $[-1, 1]$, $f(1) = 1 - \sin 1 > 0$, and $f(-1) = -3 + \sin 1 < 0$. So by the Intermediate Value Theorem there exists some $c \in (-1, 1)$ such that $f(c) = 0$, hence the equation has at least one root c.

If the equation has at least two real roots, we take two of them: $x_1 < x_2$. Since f is differentiable on $\mathbb{R}$, by the Rolle's Theorem, there exists some $\xi \in (x_1, x_2)$ such that $f'(\xi) = 0$, that is,

$$2 - \cos \xi = 0.$$

But for any $x \in \mathbb{R}$, $2 - \cos x > 0$, a contradiction. Therefore the equation has one and only one real root.

9. Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.

Proof: Suppose the equation has two roots x_1, x_2 with $-2 \leq x_1 < x_2 \leq 2$. Let $f(x) = x^3 - 15x + c$, then f is continuous on $[-2, 2]$, and differentiable on $(-2, 2)$. The Rolle's Theorem gives that there exists some $\xi \in (x_1, x_2)$ such that $f'(\xi) = 0$, that is, $3\xi^2 - 15 = 0$, or equivalently, $\xi = \pm\sqrt{5}$, but neither of the two numbers lies in $(-2, 2)$, a contradiction. Therefore the equation has at most one root in $[-2, 2]$.

10.

a Show that a polynomial of degree 3 has at most three real roots.

b Show that a polynomial of degree n has at most n real roots.

Proof: We use mathematical induction. First, it is trivial that a polynomial of degree one has at most one real root. Second, we assume that any polynomial of degree $k = n$ has at most n real roots. Third, when $k = n + 1$, suppose there is a polynomial f of degree $n + 1$ which has at least $n + 2$ roots, we take $n + 2$ roots of f: $x_1 < x_2 < \cdots < x_{n+2}$. Then by Rolle's Theorem, f', which is a polynomial of degree n, has $n + 1$ roots on (x_1, x_2), (x_2, x_3), $\cdots$, (x_{n+1}, x_{n+2}) respectively, a contradiction against our hypothesis. The proof is complete.

11. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

Solution: We claim that $f(4) \geq 16$, and $f(4)$ can be 16. On one hand, if $f(4) < 16$, by the Mean Value Theorem, there exists some $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{f(4) - 10}{3} < 2,$$

a contradiction. So $f(4) \geq 16$. On the other hand, let $f(x) = 2(x-1) + 10$ to give $f(1) = 10$, $f'(x) \equiv 2$, and $f(4) = 16$, which shows the possibility that $f(4)$ can be as small as 16.

12. Show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ if $x > 0$.

Proof: Let $f(x) = \sqrt{1+x} - 1 - \frac{x}{2}$, then f is continuous on $[0, +\infty)$ and differentiable on $(0, +\infty)$. Suppose to the contrary that there exists some $x_0 > 0$ such that $\sqrt{1+x_0} \geq 1 + \frac{1}{2}x_0$, then $f(x_0) \geq 0$. Applying the Mean Value Theorem to the function f on the interval $[0, x_0]$, we know that there exists some $\xi \in (0, x_0)$ such that $f'(\xi) = \frac{f(x_0) - f(0)}{x_0 - 0} = \frac{f(x_0) - 0}{x_0} \geq 0$, but for any $\xi > 0$,

$$f'(\xi) = \frac{1}{2\sqrt{1+\xi}} - \frac{1}{2} < 0,$$

which is a contradiction. So for every $x > 0$, we must have $\sqrt{1+x} < 1 + \frac{1}{2}x$. This completes the proof.

13. A number a is called a **fixed point** of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

Proof: Suppose f has $a_1 < a_2$ as two of its fixed points, then by the Mean Value Theorem, there exist some $\xi \in (a_1, a_2)$ such that

$$f'(\xi) = \frac{f(a_1) - f(a_2)}{a_1 - a_2} = \frac{a_1 - a_2}{a_1 - a_2} = 1,$$

a contradiction. So f has at most one fixed point.

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