MA1300 Solutions to Self Practice # 5

1. Find an equation of the tangent line to the curve at the given point.

$$y = \sqrt{x}, \ (1,1).$$

Solution:

$$\left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{1}{2\sqrt{x}}\right|_{x=1} = \frac{1}{2}, \qquad \text{equation: } y - 1 = \frac{1}{2}(x-1).$$

2. If a rock is thrown upward on the planet Mars with an velocity of 10 m/s, its height (in meters) after t seconds is given by $H = 10t - 1.86t^2$.

a Find the velocity of the rock after one second.

b Find the velocity of the rock when $t = a$.

c When will the rock hit the surface?

d With what velocity will the rock hit the surface?

Solution: velocity $= \frac{dH}{dt} = 10 - 3.72t$. So the velocity after one second is $\frac{dH}{dt}\big|_{t=1} = 6.28$(m/s). The velocity when $t = a$ is $\frac{dH}{dt}\big|_{t=a} = 10 - 3.72a$(m/s). Solve $H = 0$ to give $t = 0$ (extraneous solution, discard), and $t = \frac{10}{1.86} \approx 5.38$. So the time the rock hits the surface is $t = 5.38$(s), when the velocity of the rock is $\frac{dH}{dt}\big|_{t=\frac{10}{1.86}} = -10$(m/s).

3. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 1/t^2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

Solution: $v = \frac{ds}{dt} = -\frac{2}{t^3}$. So $v|_{t=a} = \frac{-2}{a^3}$, $v|_{t=1} = -2$, $v|_{t=2} = -0.25$, and $v|_{t=3} = -\frac{2}{27}$.

4. Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

Solution: The equation takes the form $y - g(5) = g'(5)(x-5)$, that is $y = 4x - 23$.

5. If the tangent line to $y = f(x)$ at $(4,3)$ passes through the point $(0,2)$, find $f(4)$ and $f'(4)$.

Solution: $f(4) = 3$, $f'(4) = \frac{2-3}{0-4} = \frac{1}{4}$.

6. If a cylindrical tank holds $100,000$ liters of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 100,000\left(1 - \frac{t}{60}\right)^2 \qquad 0 \le t \le 60.$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) as a function of t. What are its units? For times $t = 0$, 10, 20, 30, 40, 50 and 60 min, find the flow rate and the amount of water remaining in the tank. Summarize your findings in a sentence or two. At what time is the flow rate the greatest? The least?

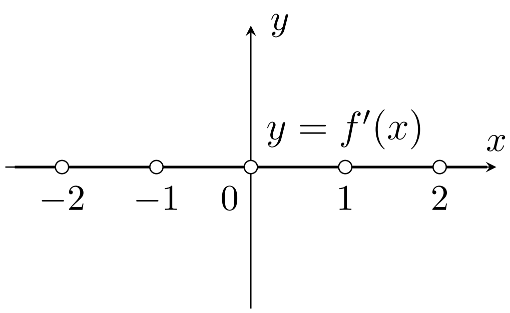
Solution: The rate water flowing out: $\frac{dv}{dt} = \frac{200{,}000}{60}\left(\frac{t}{60} - 1\right)$, the unit is liter/minute.

t (min)	flow rate (L/min)	water remaining (L)
0	$-3,333.33$	$100,000.00$
10	$-2,777.78$	$69,444.44$
20	$-2,222.22$	$44,444.44$
30	$-1,666.67$	$25,000.00$
40	$-1,111.11$	$11,111.11$
50	-555.56	$2,777.78$
60	0.00	0.00

The flow rate and the amount of water remaining in the tank decrease as time goes on. When $t = 0$, the flow rate is the greatest, $t = 60$ the least.

7. Where is the greatest integer function $f(x) = [\![x]\!]$ not differentiable? Find a formula for f' and sketch its graph.

Solution: $f(x)$ is not differentiable when x is integer. $f'(x) = 0$ when x is not an integer. The graph of f' is



8. The **left-hand** and **right-hand derivatives** of f at a are defined by

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h},$$

and

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h},$$

if these limits exist. Then $f'(a)$ exists if and only if these one-sided derivatives exist and are equal.

a Find $f'_-(4)$ and $f'_+(4)$ for the function

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \frac{1}{5-x} & \text{if } x \ge 4. \end{cases}$$

b Sketch the graph of f.

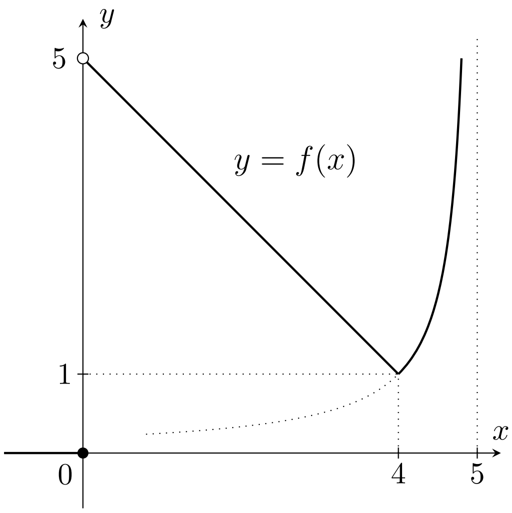
c Where is f discontinuous?

d Where is f not differentiable?

Solution:

a $f'_-(4) = -1$, $f'_+(4) = 1$

b the graph of f:



c,d $\lim_{x\to 0^+} f(x) = 5 \neq f(0)$, $\lim_{x\to 4^-} f(x) = 1 = f(4) = \lim_{x\to x^+} f(x)$. So f is discontinuous at $x = 0$. Therefore f is not differentiable at $x = 0$. Besides this, since $f'_-(4) \neq f'_+(4)$, f is continuous but not differentiable at $x = 4$. When $x = 5$, $f(x)$ is not defined, so it is discontinuous at 5.

9. Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all such x. Prove each of the following.

a The derivative of an even function is an odd function.

b The derivative of an odd function is an even function.

Proof: When f is even,

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = -\lim_{h\to 0} \frac{f(x-h) - f(x)}{-h} = -f'(x).$$

When f is odd,

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} \frac{f(x-h) - f(x)}{-h} = f'(x).$$

10. Differentiate the function.

$$u = \sqrt[5]{t} + 4\sqrt{t^5}, \qquad v = \left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)^2.$$

Solution:

$$\begin{aligned}
\frac{d}{dt}(\sqrt[5]{t} + 4\sqrt{t^5}) &= \frac{1}{5}t^{-4/5} + 10t^{3/2}, \\
\frac{d}{dt}\left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)^2 &= 2\left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)\left(\frac{1}{2\sqrt{x}} - \frac{x^{-4/3}}{3}\right).
\end{aligned}$$

11. Differentiate.

$$F(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3),$$

$$y = \frac{x^3}{1-x^2}, \qquad f(x) = \frac{x}{x + \frac{c}{x}}, \qquad f(x) = \frac{ax+b}{cx+d}.$$

Solution:

$$\begin{aligned}
\frac{d}{dy}\left(\left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)\right) &= \frac{d}{dy}(y^{-1} - 3y^{-3} + 5y - 15y^{-1}) = 5 + 14y^{-2} + 9y^{-4}, \\
\frac{d}{dx}\frac{x^3}{1-x^2} &= \frac{3x^2(1-x^2) + 2x^4}{(1-x^2)^2} = \frac{3x^2 - x^4}{(1-x^2)^2}, \\
\frac{d}{dx}\frac{x}{x + \frac{c}{x}} &= \frac{x + \frac{c}{x} - x\left(1 - \frac{c}{x^2}\right)}{\left(x + \frac{c}{x}\right)^2} = \frac{\frac{2c}{x}}{\left(x + \frac{c}{x}\right)^2}, \\
\frac{d}{dx}\frac{ax+b}{cx+d} &= \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \frac{ad - bc}{(cx+d)^2}.
\end{aligned}$$

12. The general polynomial of degree n has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_n \neq 0$. Find the derivative of P.

Solution: $\frac{dP}{dx} = \sum_{i=1}^{n} i a_i x^{i-1}$.

13. Find the first and second derivatives of the function

$$f(x) = \frac{1}{3-x}.$$

Solution:

$$f'(x) = \frac{1}{(3-x)^2}, \qquad f''(x) = \frac{2}{(3-x)^3}.$$

14. The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds. Find

a the velocity and acceleration as functions of t.

b the acceleration after $2s$, and

c the acceleration when the velocity is 0.

Solution:

a velocity $= \frac{ds}{dt} = 3t^2 - 3$, acceleration $= \frac{d^2s}{dt^2} = 6t$.

b $\left.\frac{d^2s}{dt^2}\right|_{t=2} = 12$.

c When velocity is zero, since $t \geq 0$, $t = 1$. So at that time the acceleration is 6.

15. If $f(x) = \sqrt{x}g(x)$, where $g(4) = 8$, and $g'(4) = 7$, find $f'(4)$.

Solution: $f'(x) = \frac{g(x)}{2\sqrt{x}} + \sqrt{x}g'(x)$, so $f'(4) = \frac{8}{4} + 2 \cdot 7 = 16$.

16. If $h(2) = 4$ and $h'(2) = -3$, find

$$\frac{d}{dx}\left(\frac{h(x)}{x}\right)\bigg|_{x=2}.$$

Solution: $\frac{d}{dx}\left(\frac{h(x)}{x}\right)\Big|_{x=2} = \left.\frac{h'(x)x-h(x)}{x^2}\right|_{x=2} = \frac{-3\cdot 2-4}{4} = -\frac{5}{2}$.

17. Find equations of the tangent lines to the curve

$$y = \frac{x-1}{x+1}$$

that are parallel to the line $x - 2y = 2$.

Solution: $\frac{dy}{dx} = \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$. The slope of the line $x - 2y = 2$ is $\frac{1}{2}$. Let $\frac{1}{2} = \frac{2}{(x+1)^2}$ to give $x = -3, 1$. When $x = -3$, the curve passes through $(-3, 2)$, so the tangent line is $y - 2 = \frac{1}{2}(x + 3)$. When $x = 1$, the curve passes through $(1, 0)$, so the parallel tangent line is $y = \frac{1}{2}(x - 1)$.