Here, we discuss another method using second derivatives $S''(x_i) = M_i (i = 0, 1, \cdots, n)$ to find the expression for spline *S(x)*.

Let $h_i = x_i - x_{i-1}, i = 1, \cdots, n$, $S''(x_i) = C_i''(x_i) = C_{i+1}''(x_i) = M_i (i = 1, \cdots, n-1)$ and $S''(x_0) = C_1''(x_0) = M_0, \text{ and } S''(x_n) = C_n''(x_n) = M_n$. Note that M_i's are unknown (except for type II boundary condition,

Since each C_i is a cubic polynomial, C_i'' is linear.

By Lagrange interpolation, we can interpolate each C_i'' on $[x_{i-1}, x_i]$ since $C_i''(x_{i-1}) = M_{i-1}$ and $C_i''(x_i) = M_i$, the Lagrange form of this interpolating polynomial is:

$$C_i''(x) = M_{i-1}\frac{x_i - x}{h_i} + M_i\frac{x - x_{i-1}}{h_i} \text{ for } x \in [x_{i-1}, x_i].$$

Integrating the above equation twice and using the condition that $C_i(x_{i-1}) = y_{i-1}$ and $C_i(x_i) = y_i$ to determine the constants of integration, we have

$$C_i(x) = M_{i-1}\frac{(x_i - x)^3}{6h_i} + M_i\frac{(x - x_{i-1})^3}{6h_i} + \left(y_{i-1} - \frac{M_{i-1}h_i^2}{6}\right)\frac{x_i - x}{h_i} + \left(y_i - \frac{M_i h_i^2}{6}\right)\frac{x - x_{i-1}}{h_i} \quad \text{for} \quad x \in [x_{i-1}, x_i].$$

This expression gives us the cubic spline *S(x)* if $M_i, i = 0, 1, \cdots, n$ can be determined.

For $i = 1, \cdots, n-1$, when $x \in [x_i, x_{i+1}]$,, we can calculate that

$$C_{i+1}'(x) = -M_i\frac{(x_{i+1} - x)^2}{2h_{i+1}} + M_{i+1}\frac{(x - x_i)^2}{2h_{i+1}} + \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{M_{i+1} - M_i}{6}h_{i+1}.$$

Therefore, $C_{i+1}'(x_i) = -M_i\frac{h_{i+1}}{2} + \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{M_{i+1} - M_i}{6}h_{i+1}.$

Similarly, when $x \in [x_{i-1}, x_i]$,, we can shift the index to obtain

$$C_i'(x) = -M_{i-1}\frac{(x_i - x)^2}{2h_i} + M_i\frac{(x - x_{i-1})^2}{2h_i} + \frac{y_i - y_{i-1}}{h_i} - \frac{M_i - M_{i-1}}{6}h_i.$$

Thus, $C_i'(x_i) = M_i\frac{h_i}{2} + \frac{y_i - y_{i-1}}{h_i} - \frac{M_i - M_{i-1}}{6}h_i.$

Since $C_{i+1}'(x_i) = C_i'(x_i)$, we can derive

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i \quad \text{for} \quad i = 1, 2, \cdots, n-1,$$

where

$$\mu_i = \frac{h_i}{h_i + h_{i+1}}, \quad \lambda_i = 1 - \mu_i = \frac{h_{i+1}}{h_i + h_{i+1}}, \quad \text{and} \quad d_i = 6f[x_{i-1}, x_i, x_{i+1}]$$

and $f[x_{i-1}, x_i, x_{i+1}]$ is a divided difference.

According to different boundary conditions, we can solve the system of equations above to obtain the values of M_i's