GE2256: Game Theory Applications to Business
Lecture 8

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In-class activity: Cournot Duopoly

- Refer to the file *in_class_activity_Cournot_duopoly.pdf* on canvas.
- Each firm chooses an output level in $[0, 50]$.
- The market demand function is given by:

  $$P(Q) = 50 - Q,$$

  for $Q \leqslant 50$; where P is the market price, and Q is total output produced in the market.
- Each firm has a constant marginal cost of 2 per unit of output.

In-class activity: Cournot Duopoly

- In order to find out the Nash equilibrium of the game you can follow the discussion in Lecture 7.
- Each of the players choose a non-negative quantity, q_1 and q_2.
- The market price depends on the actions of both players:

$$P(q_1, q_2) = 50 - q_1 - q_2.$$

- The two players' payoffs are given by

$$u_1(q_1, q_2) = q_1(50 - q_1 - q_2) - 2q_1;$$

$$u_2(q_1, q_2) = q_2(50 - q_1 - q_2) - 2q_2.$$

Best-Response Functions

The first step is to find the best response functions for F1 and F2. The best response function for a firm shows the choice of quantity which provides it with the highest profit, given the quantity of the other firm.

- (CALCULUS METHOD) This is found out by differentiating the profit function u_i w.r.t. q_i, taking the other firm's quantity as given (as a constant) and then setting this $\frac{\partial u_i}{\partial q_i}$ equal to zero (the first order condition, FOC).
- Then rewrite q_i in terms of the other firm's quantity from this FOC.

Best Response

F1's profit is given by

$$u_1(q_1, q_2) = q_1(50 - q_1 - q_2) - 2q_1;$$

Taking q_2 as a constant, differentiate u_1 above w.r.t. q_1 to get the F1's FOC:

$$50 - 2q_1 - q_2 - 2 = 0$$

$$q_1 = \frac{1}{2}(48 - q_2) = \mathscr{B}_1(q_2).$$

Similarly, you can obtain the F2's best-response function to be:

$$q_2 = \frac{1}{2}(48 - q_1) = \mathscr{B}_2(q_1).$$

Best responses via non-calculus method

In this slide, we use the non-calculus method to obtain the best response function for firm 1.

For F1, profits are:

$$u_1(q_1, q_2) = q_1(50 - q_1 - q_2) - 2q_1 = 0 + (48 - q_2)q_1 - q_1^2$$

In terms of our general quadratic function ($Y = A + BX - CX^2$), we have:

$$A = 0, B = (48 - q_2), C = 1$$

Thus,

$$q_1 = \frac{B}{2C} = \frac{48 - q_2}{2}$$

Best responses via non-calculus method

In this slide, we use the non-calculus method to obtain the best response function for firm 2.

For F2, profits are:

$$u_2(q_1, q_2) = q_2(50 - q_1 - q_2) - 2q_2 = 0 + (48 - q_1)q_2 - q_2^2$$

In terms of our general quadratic function ($Y = A + BX - CX^2$), we have:

$$A = 0, B = (48 - q_1), C = 1$$

Thus,

$$q_2 = \frac{B}{2C} = \frac{48 - q_1}{2}$$

Nash Equilibrium

The Nash equilibrium is obtained by solving the system of equations simultaneously:

$$q_1 = \frac{1}{2}(48 - q_2)$$

$$q_2 = \frac{1}{2}(48 - q_1)$$

Substitute q_2 from the second equation to the first to get:

$$q_1 = \frac{1}{2}[48 - \frac{1}{2}(48 - q_1)] = \frac{1}{4}(48) + \frac{1}{4}q_1$$

This gives the following equilibrium quantities:

$$q_1^{NE} = q_2^{NE} = \frac{48}{3} = 16$$

Alternatively, you can use the Cramer's rule to solve the equations simultaneously (see next two slides).

Cournot Model: Cramer's rule

- The two equations are:

$$2q_1 + q_2 = 48$$

$$q_1 + 2q_2 = 48$$

- You can solve the equations simultaneously using the Cramer's rule as well. Write the two equations in compact matrix form:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 48 \\ 48 \end{bmatrix}$$

Cournot Model: Cramer's rule

- We have the following determinants:

$$D_{q_1} = \begin{vmatrix} 48 & 1 \\ 48 & 2 \end{vmatrix} = 2(48) - (48) = 48$$

$$D_{q_2} = \begin{vmatrix} 2 & 48 \\ 1 & 48 \end{vmatrix} = 2(48) - (48) = 48$$

$$D = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$q_1 = \frac{D_{q_1}}{D} = \frac{48}{3} = 16 \qquad q_2 = \frac{D_{q_2}}{D} = \frac{48}{3} = 16$$

In-class activity: Cournot Duopoly

- If both firms merge to become one, then the profit or payoff is given by:

$$u = (50 - Q)Q - 2Q$$

- The first order condition for profit maximization is:

$$\frac{du}{dQ} = \frac{d}{dQ}(50Q - 2Q - Q^2) = 0$$

$$\implies 50 - 2 - 2Q = 0 \implies Q = 24$$

- Total profits under merger:
$(50 - 24)(24) - (2)(24) = (24)(24) = 576$. This is higher than the market profits or sum of firm profits under Nash equilibrium.

In-class activity: Cournot Duopoly

NON-CALCULUS method:

- If both firms merge to become one, then the profit or payoff is given by:

$$u = (50 - Q)Q - 2Q = (50 - 2)Q - Q^2 = 0 + 48Q - Q^2$$

- In terms of our general quadratic function $(Y = A + BX - CX^2)$, we have:

$$A = 0, B = 48, C = 1$$

Thus,

$$Q = \frac{B}{2C} = \frac{48}{2} = 24$$

In-class activity: Cournot Duopoly

- Under Nash equilibrium, each firm makes $(50 - 32)(16) - (2)(16) = (16)(16) = 256$. Total market profits equal 512 as there are two firms.
- It is possible for each firm to obtain exactly half of the higest achievable profits, that is, 288, if each firm produces 12 instead of the NE quantity of 16, even when there is no merger. But, each firm has a temptation to choose a different quantity if the other firm is playing 12 (in other words, choosing 12 is not my best response if the other firm is playing 12). Hence, it would be very difficult to achieve the highest market profits possible in this game.

Bayesian Games

- So far we have discussed games with perfect information in both simultaneous and sequential setting.
- Now we move in to the domain of games with imperfect information.
- In this course, we will only consider simultaneous games with imperfect information. Also, we will restrict ourselves to pure strategies in these games.

Bayesian Game

- A Bayesian game consists of a set of players with each player having a set of actions to choose from.
- Each player can be of different types (chosen by nature). Each player only observes his/her own type.
- The game must also specify the probability distribution of the different types for each player.

Bayesian Game

- A pure strategy for a player in a Bayesian game is a function or decision rule that gives the player's action choice for each realization of his type.
- Payoffs are associated with the outcomes realized from the strategies played by the players. Each player must take into account the expected payoff from choosing a strategy, where the expectation is taken over others' types.

Bayes-Nash equilibrium

- We will still work with the basic notion of Nash equilibrium but with expected payoffs.
- Thus, we will have mutual best response and the objective function will be expected payoffs instead of payoffs.
- The probability distribution of types will be taken into consideration for solving for the equilibrium.
- We will use the name “Bayes-Nash equilibrium (BNE)” for this notion of equilibrium. It must specify what each type of each player must do at the optimum. We discuss several examples/exercises in order to explain the notion of BNE and steps to find the equilibria.

BNE: Example

- Consider the following game of incomplete information.
- Player 1 can choose between U and D. Player 2 can choose between L and R.
- Payoffs depend on the players' types.
- Player 1 can only be of one type which is known by player 2.
- Player 2 may be of types x and y.

BNE: Example

- Player 2 knows his type, but player 1 does not know player 2's type: we have asymmetric incomplete information.
- Player 1 knows that player 2 is of type x with probability 2/3 and of type y with probability 1/3.
- If player 2 is of type x, the following payoff matrix is being played:

	L	R
U	4,3	3,1
D	3,6	2,3

BNE: Example

- If player 2 is of type y, the following payoff matrix is being played:

	L	R
U	3,3	1,6
D	1,1	5,3

- In order to find the BNE, we need to find the optimal strategies for player 1, type x player 2 and type y player 2.

BNE: Example

- First note that for type x player 2, L strictly dominates R: so he will play L.
- For type y player 2, R strictly dominates L: so he will play R.
- Player 1 only has one type and will calculate his expected payoff from choosing U and compare it with the expected payoff from choosing D.
- Expected payoff from U: $(2/3)(4) + (1/3)(1) = 3$.
- Expected payoff from D: $(2/3)(3) + (1/3)(5) = 11/3 > 3$.

BNE: Example

- Player 1 plays D.
- Player 2 plays L if she is of type x and plays R if she is of type y.
- This is the BNE of the game.

Exercise: Public Goods Game

- Suppose there are two people (players) and a public good is provided if at least one is willing to pay the cost of the good.
- Recall that the feature of a public good is that once provided, you cannot exclude the person who didn't contribute towards the provision of the good.
- The good is provided if only one of them contributes or if both contribute.
- Each player differs in the cost of contribution towards the public good.

Exercise: Public Goods Game

- The payoff matrix is as follows (player 1 is row player and player 2 is column player):

	Contribute (C)	Don't Contribute (NC)
Contribute (C)	$1 - c_1, 1 - c_2$	$1 - c_1, 1$
Don't Contribute (NC)	$1, 1 - c_2$	$0, 0$

- Player 1's cost is $c_1 < \frac{1}{2}$ and is known to both players.
- Player 2's cost, c_2 can be of two types: low type (c_{low}) and high type (c_{high}).

Exercise: Public Goods Game

- Only player 2 knows his own type, that is, whether he is of low type or high type. This is the private information.
- Player 1 only knows the probability that player 2 is of certain type: denote the probability of $c_2 = c_{low}$ by p.
- The probability of $c_2 = c_{high}$ is then $(1 - p)$.
- Assume that $0 < c_{low} < 1 < c_{high}$ and $p < \frac{1}{2}$.
- What is the Bayes-Nash equilibrium of this game?

Exercise: Public Goods Game

- We need to find the equilibrium strategy for player 1 and for each type of player 2.
- Note that for the high type of player 2, $c_{high} > 1$. Thus, cost is higher than the benefit. So, not contributing (NC) is a dominant strategy for the player 2 of high type.
- So, type c_{high} of player 2 will play NC. You can write the following payoff matrix for the case when $c_2 = c_{high}$:

	C	NC
C	$1 - c_1, 1 - c_{high}$	$1 - c_1, 1$
NC	$1, 1 - c_{high}$	$0, 0$

Exercise: Public Goods Game

- We can write the payoff matrix for the case when $c_2 = c_{low}$:

	C	NC
C	$1 - c_1, 1 - c_{low}$	$1 - c_1, 1$
NC	$1, 1 - c_{low}$	$0, 0$

- Consider player 2 of type c_{low}. His best response to player 1's choice of C is: NC.
- Best response to player 1's choice of NC is: C.

Exercise: Public Goods Game

- Now consider player 1. There are two actions that he can take: C or NC. We need to find the expected payoffs from each of these two actions.
- Expected payoff from contributing: $1 - c_1$.
- Expected payoff from not contributing: $p(1) + (1-p)(0) = p$. This is because with prob. p player 2 is of low type and the best response of player 2 in that case is C when player 1 does not contribute. With prob. $(1-p)$ player 2 is of high type and in that case, player 2's best reponse is to play NC.

Exercise: Public Goods Game

- Player 1 will contribute if $1 - c_1 > p$.
- Given our assumptions, we have: $c_1 < \frac{1}{2}$. This implies: $1 - c_1 > \frac{1}{2}$.
- We also have: $p < \frac{1}{2}$.
- Thus, $1 - c_1 > p$ holds given our assumptions. So, player 1 contributes.
- The best response of player 2 of low type to player 1 contributing is NC. So, type c_{low} of player 2 does not contribute.

Exercise: Public Goods Game

- So, the Bayes-Nash equilibrium strategies are: C for player 1, NC for low type player 2 and NC for high type player 2.
- The equilibrium payoffs are: $1 - c_1$ for player 1, 1 for player 2 if of low type, 1 for player 2 if of high type.

More Information may Hurt

- If a player in a game has more information and the other players know that she has more information then she may be worse off.
- Consider the following two player Bayesian game where there are two “states of nature”: state A and state B.
- Neither player knows the state.
- If the state is A, then the following game is played:

	L	M	R
U	$1, \frac{1}{2}$	$1, 0$	$1, \frac{3}{4}$
D	$2, 2$	$0, 0$	$0, 3$

More Information may Hurt

- If the state is B, then the following game is played:

	L	M	R
U	$1, \frac{1}{2}$	$1, \frac{3}{4}$	$1, 0$
D	$2, 2$	$0, 3$	$0, 0$

- The probability that state A occurs is 0.5 and the probability that state B occurs is 0.5.
- Each player has the same belief about the occurrence of the states. This is common knowledge.

More Information may Hurt

- Consider player 2. She compares the expected payoffs from L, M and R when computing best respones:
- If player 1 plays U, player 2's best response is to play L:
- Expected payoff from L is $(1/2)(1/2) + (1/2)(1/2) = 0.5$; expected payoff from M is $(0)(1/2) + (3/4)(1/2) = 0.375$ and R is $(3/4)(1/2) + (0)(1/2) = 0.375$.

More Information may Hurt

- If player 1 plays D, player 2's best response is to play L:
- Expected payoff from L is $(1/2)(2) + (1/2)(2) = 2$; expected payoff from M is $(1/2)(0) + (1/2)(3) = 1.5$ and R is $(1/2)(3) + (1/2)(0) = 1.5$.
- We can write down the expected payoffs from the combination of strategies of the two players in the following manner:

	L	M	R
U	$1, \frac{1}{2}$	$1, \frac{3}{8}$	$1, \frac{3}{8}$
D	$2, 2$	$0, \frac{3}{2}$	$0, \frac{3}{2}$

- Notice that regardless of player 1's action choices, player 2's unique BR is to play L. If player 2 plays L, D is the unique BR for player 1. So, the unique equilibrium is (D, L) and each player gets a payoff of 2 in equilibrium.

More Information may Hurt

- Now, consider a variant of this game in which player 2 is informed of the state so that she knows whether the state is A or it is B.
- Player 1 still does not know the state. He has a belief that each state occurs with probability 0.5.
- Player 1 also knows that player 2 knows the state with certainty.
- Now, note that under state A, player 2 has a dominant action of choosing R while under state B, player 2 has a dominant action of choosing M.

More Information may Hurt

- Player 1's expected payoff from choosing U is: $(1/2)(1) + (1/2)(1) = 1$.
- Player 1's expected payoff from choosing D is: $(1/2)(0) + (1/2)(0) = 0$.
- So, player 1 will choose U.
- The equilibrium is: player 1 chooses U, player 2 chooses R if he knows that the state is A and chooses M if he knows that the state is B.

More Information may Hurt

- Equilibrium payoffs are: 1 for player 1 and 0.75 for player 2 irrespective of the state.
- Note that player 2's payoff is 0.75 which is strictly lower than the payoff of 2 in the equilibrium of the game where player 2 didn't know the state.
- More information leads to a lower payoff for player 2, given that the other player knows that player 2 has more information.

References

- Neither the textbook nor the recommended reading has satisfactory discussion on “Bayesian Games”.
- The lecture slide is self-contained. Please read the lecture notes carefully.