Chapter 4 Inverse Functions

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4. Inverse functions

So far, we have learned

- limit
- derivative
- applications of derivatives

In this chapter,

(1) we shall use log and exponential functions to explore the relation of derivatives between inverse functions. We may start with some basic questions, like $(\ln x)'$ and $(e^x)'$.

(2) we will learn powerful l'hospital's rule to solve for limit problem.

Note, we will avoid to involve all parts of this chapter related to integral (which will be covered in subsequent course).

4.1. **Calculus of inverse functions.** Section 6.1 Exercise of Text: 24, 31, 39, 41

We first recall some basics of inverse functions.

A function f is called a **one-to-one** (1-1) function if it never takes on the same value twice, i.e.

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2.$$

Q: How can one identify a 1-1 function?

Horizontal line test. A function is 1-1 if and only if no horizontal line intersects its graph more than once.

Note that, Given a map f, one can identify if f is a function by using *vertical line test*. Now, if it is a function, one can further identify if f is 1-1 function. If a function is 1-1, then we can define its inverse function.

Definition. Let $f : A \to B$ be a 1-1 function with domain A and range B. then its **inverse function** $f^{-1} : B \to A$ is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

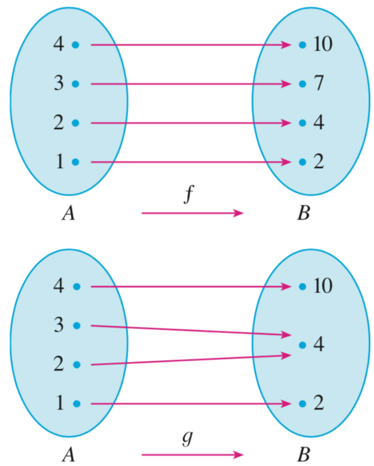

FIGURE 1
f is one-to-one; g is not.

for any $y \in B$.

Here we summarize some properties of inverse functions:

(1) domain of f^{-1} = range of f; domain of f = range of f^{-1};
(2) $f^{-1}(y) = x \Leftrightarrow f(x) = y$;
(3) $f^{-1}(f(x)) = x, \ \forall x \in A$, and $f(f^{-1}(x)) = x, \ \forall x \in B$;
(4) The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Ex. Given $f(x) = \sqrt{-1-x}$,
(1) find domain and range of f and f^{-1}.
(2) sketch graph of f and f^{-1}.
(3) find f^{-1}

Theorem 4.1. *If f is a 1-1 continuous function, then its inverse f^{-1} is also continuous.*

Theorem 4.2. *If f is 1-1 differentiable function, and $f'(f^{-1}(a)) \neq 0$, then the inverse function f^{-1} is differentiable at a and*

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} = \frac{1}{f'(y)}\bigg|_{y=f^{-1}(a)}$$

Intuitively, if f^{-1} is known to be differentiable (need proof), the formula given above is obvious: Since $f(f^{-1}(x)) = x$, taking $\frac{d}{dx}$ both sides using chain rule, we get

$$f'(f^{-1}(x))(f^{-1})'(x) = 1$$

and it implies

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(y)}\Big|_{y=f^{-1}(x)}.$$

On the other hand, if we write $y = f^{-1}(x)$, then $f(y) = x$, and in Leibniz notation, we write

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Ex. Let $f(x) = 2x + \cos x$.

(1) show that f^{-1} exists.
(2) find $(f^{-1})'(1)$

4.2. **Derivatives of exponential function.** Section 6.2 Exercise: 17, 27, 37, **49, 50**, 55

We start with recall of exponential functions.

An **exponential function** is a function of the form

$$f(x) = a^x.$$

We usually assume $a > 0$, and x is all real numbers.

How can we evaluate a^x for any $x \in \mathbb{R}$? (Ex. $2^{\sqrt{2}}$)

(1) For $x = n$ be positive integer, $a^n = a \cdot a \cdots a$ with n many factors; for $x = -n$ be positive integer (n is positive), $a^n = (1/a) \cdot (1/a) \cdots (1/a)$ with n many factors
(2) For $x = p/q$ be rational, $a^{p/q} = \sqrt[q]{a^p}$.
(3) For x be irrational, $a^x = \lim_{\mathbb{Q} \ni r \to x} a^r$.

We summarize basic properties of exponential functions: Let $a, b > 0$.

(1) $f(x) = a^x$ is differentiable everywhere.
(2) $a^{x+y} = a^x a^y,\ a^{x-y} = a^x/a^y,\ (a^x)^y = a^{xy},\ (ab)^x = a^x b^x$.
(3) If $a > 1$, a^x is increasing in x, and

$$\lim_{x\to\infty} a^x = \infty, \quad \lim_{x\to-\infty} a^x = 0.$$

If $0 < a < 1$, a^x is decreasing in x, and

$$\lim_{x\to\infty} a^x = 0, \quad \lim_{x\to-\infty} a^x = \infty.$$

Ex. Let $f(x) = a^x$ for some $a > 0$. Show that

$$f'(x) = f'(0)a^x. \tag{4.1}$$

Hint: Use definition.

The **natural number** e can be defined in different ways in many books. We are going to use following definition from Text:
Def. e is the number such that

$$\lim_{h\to 0}\frac{e^h-1}{h}=1.$$

Using this definition, one can compute e numerically, $e\approx 2.71828\ldots$

$f(x)=e^x$ is called *natural exponential function*. Note that $e>1$ is constant, and thus we have

$$\lim_{x\to\infty}e^x=\infty,\qquad \lim_{x\to-\infty}e^x=0.$$

Using definition, we see

$$(e^x)'\Big|_{x=0}=1.$$

Together with (4.1), we obtain

$$(e^x)'=(e^x)'\Big|_{x=0}\cdot e^x=e^x.$$

Thus, we have following lovely formula

$$(e^x)'=e^x. \tag{4.2}$$

Ex. Find y' if $y=e^{-4x}\sin 5x$.

Ex. Find the absolute maximum of $f(x)=xe^{-x}$.

Ex. Find $\lim\limits_{x\to\infty}\dfrac{e^{2x}}{e^{2x}+1}$.

4.3. **Logarithmic function.** Section 6.3 Exercise: 41, 47, 57, 61

This section is about a brief recall of log functions.

(1) $f(x) = a^x$ is inverse to $f^{-1}(x) = \log_a x$, i.e.
$$\log_a x = y \Leftrightarrow a^y = x$$
(2) $\log_a(a^x) = x, \forall x \in \mathbb{R}; \quad a^{\log_a x} = x, \forall x > 0.$

(3) $\log_a(xy) = \log_a x + \log_a y;$

$\log_a(\frac{x}{y}) = \log_a x - \log_a y;$

$\log_a(x^r) = r \log_a x.$

(4) If $a > 1$, then $\log_a x$ is increasing in x, and
$$\lim_{x\to\infty} \log_a x = \infty, \ \lim_{x\to 0^+} \log_a x = -\infty.$$
If $0 < a < 1$, then $\log_a x$ is decreasing in x, and
$$\lim_{x\to\infty} \log_a x = -\infty, \ \lim_{x\to 0^+} \log_a x = \infty.$$

In particular, log with base e is called **natural logarithm** and denoted by
$$\log_e x \triangleq \ln x$$
for simplicity. So, $\ln e = \log_e e = 1$.

Another popular formula is
$$\log_a x = \frac{\ln x}{\ln a}.$$
In previous section, we learned $(e^x)' = e^x$ by (4.8). Now, we have by chain rule
$$(a^x)' = (e^{x \ln a})' = (e^{x \ln a})(x \ln a)' = a^x \ln a.$$
Now, we have new formula
$$(a^x)' = a^x \ln a. \tag{4.3}$$

4.4. **Derivatives of logarithmic functions.** Sec 6.4 Exercise: **3, 12,** 21, 31, 43, 45, **49, 52,** 53, 91

First formula is
$$(\ln x)' = \frac{1}{x}. \tag{4.4}$$

Ex. Prove (4.4).

Ex. find y' where $y = \ln(x^3 + 1)$

Ex. Find $\dfrac{d}{dx} \ln \dfrac{x+1}{\sqrt{x-2}}$.

Ex. show that

$$(\ln|x|)' = 1/x, \quad \forall x \neq 0. \tag{4.5}$$

The derivative formula of general log function is

$$(\log_a x)' = \frac{1}{x \ln a}. \tag{4.6}$$

Ex. Prove (4.6).

Sometimes, one can simplify calculation of derivatives by following a trick of log differentiation: For $y = f(x)$

(1) Take $\ln y = \ln f(x)$ and simplify the right hand side.
(2) Take $\frac{d}{dx}$ on both sides.
(3) Solve for y'.

Ex. Using above trick, find y' of $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Ex. Show that, using log differentiation trick

$$(x^n)' = nx^{n-1}, \ \forall n \in \mathbb{R}. \tag{4.7}$$

Solution. Since x^n is not necessarily positive, we need to take ln on absolute value of BS, i.e. for $y = x^n$, we have

$$\ln|y| = n \ln|x|.$$

Then follow the rest steps. ☐

Ex. Differentiate $y = x^{\sqrt{x}}$.

Ex. By definition, e is a number satisfying $\lim_{x\to 0}\frac{e^x-1}{x}=1$. Show that, e can written as differently:

$$e=\lim_{x\to 0}(1+x)^{1/x}=\lim_{n\to\infty}(1+\frac{1}{n})^n. \tag{4.8}$$

(From this example, we now have three different representations of e.)

4.5. **Exponential growth and decay.** sec 6.5 Exercise: 5, 9, 19

In this section, we study natural phenomena of quantity dynamics, for example population growth, compounded interest, and so on.

Let's say $y(t)$ be the quantity at time t. It shows that many of quantity change follows relation of the form

$$\frac{dy}{dt}=ky \text{ for some constant } k \tag{4.9}$$

Ex. Show that,

(1) for any constant C, $y(t)=Ce^{kt}$ satisfies (4.9).
(2) show that $C=y(0)$.

Next result shows that $y(t)$ given by above example is the only solution of (4.9)

Theorem 4.3. *The only solution of* (4.9) *is the exponential function*

$$y(t)=y(0)e^{kt}. \tag{4.10}$$

(1) If $k>0$, then $y(t)$ of (4.10) is called **natural grow**; and k is **relative growth rate**.
(2) If $k<0$, then $y(t)$ of (4.10) is called **natural decay**; and k is **relative decay rate**.

Ex.(Population growth) Suppose population follows law of natural growth. Use the fact that population was 2560 million in 1950, and 3040 million in 1960,

(1) find relative growth rate
(2) estimate population in 2020.

Ex. Suppose certain substances decay with $m(t) = m_0 e^{kt}$. **Half life** is the time length $\hat{t}$, such that

$$m(\hat{t}) = \frac{1}{2} m_0.$$

Given that the half-life of radium-226 is 1590 years.

(1) A sample of radium-226 has a mass of 100mg, Find a formula for the mass the sample that remains after t years.
(2) Find mass after 1000 years.
(3) When will the mass be reduced to 30mg.

Compound interest rate. Suppose initially the balance of bank account is $v(0) = 1000$ dollars.

(1) If annual compound interest rate $r = 0.06$, then $v(t) = v(0)(1+r)^t$.
(2) If semi-annual compound interest rate $r = 0.06$, then $v(t) = v(0)(1+\frac{r}{2})^{2t}$.
(3) If nth-annual compound interest rate $r = 0.06$, then $v(t) = v(0)(1+\frac{r}{n})^{nt}$.
(4) If continuous annual compound interest rate $r = 0.06$, then $v(t) = v(0)e^{rt} = \lim_{n\to\infty} v(0)(1+\frac{r}{n})^{nt}$.

Ex.

(1) How long will it take an investment to double in value if the interest rate is 6% compounded continuously?
(2) What is the equivalent annual interest rate?

4.6. **Inverse trigonometric functions.** sec 6.6 Exercise: **25, 26, 29,** 31, 37, 47, 49

We use formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(y)}\Big|_{y=f^{-1}(x)}$$

to find derivatives of inverse trigonometric functions.

(1) Inverse sine function:
   (a) $\sin^{-1} x = y \Leftrightarrow \sin y = x, \text{ for } -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$
   (b) $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}, \ -1 < x < 1.$
(2) Inverse cosine function:
   (a) $\cos^{-1} x = y \Leftrightarrow \cos y = x, \quad \text{for } 0 \le y \le \pi$

(b) $(\cos^{-1} x)' = -\dfrac{1}{\sqrt{1-x^2}}, \ -1 < x < 1.$

(3) Inverse tangent function:

(a) $\tan^{-1} x = y \Leftrightarrow \tan y = x, \quad \text{for } -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}.$

(b) $(\tan^{-1} x)' = \dfrac{1}{1+x^2}.$

For other derivatives of inverse trigonometric functions, like $\csc^{-1} x, \sec^{-1} x$ and $\cot^{-1} x$, are referred to Page 457 of Text.

Ex. Find y' for

(1) $y = \dfrac{1}{\sin^{-1} x}$

(2) $y = x \arctan \sqrt{x}$.

4.7. **Hyperbolic functions.** In this section, hyperbolic functions, like $\sinh x, \cosh x, \tanh x$, are defined, and its derivatives are given, see Section 6.7. Exercise: **33, 35**

Definitions of the hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

Derivatives of the hyperbolic functions:

$$\frac{d}{dx} \sinh x = \cosh x,$$

$$\frac{d}{dx} \cosh x = \sinh x,$$

$$\frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x},$$

Try to prove the derivatives stated above.

4.8. **L'hospital's rule.** sec 6.8 Exercise: 1, 21, **25, 32, 38**, 43, 51, 55, **57**, 81, 83, 89, 91, 93, 99.

Indeterminate form refers to limit problems of the types

$$\frac{\infty}{\infty}, \ \frac{0}{0}, \ 0 \cdot \infty, \ \infty - \infty, \ 0^0, \ \infty^0, \ 1^\infty.$$

One can directly use L'hospital's rule to find limit of $\frac{\infty}{\infty}, \ \frac{0}{0}$.

L'hospital's rule. Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0, \text{ and } \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty, \text{ and } \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have $\frac{0}{0}$ or $\frac{\infty}{\infty}$), Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the RHS exists (or is $\pm\infty$).

Ex. (0/0 type) Find $\lim_{x \to 1} \dfrac{\ln x}{x-1}$. **Ex.** Let $n > 0$.

(1) (∞/∞ type) $\lim_{x \to \infty} \dfrac{e^x}{x^n} =?$
(2) What can you conclude from above problem?
(3) What do you think of $\lim_{x \to \infty} (e^x - x^2 - 2x) =?$

Note. It is very important to verify assumption of L'hospital's rule before you use it, i.e. verify the limit problem is of type 0/0 or ∞/∞.

Ex. What if you use L'hospital's rule for $\lim_{x \to \pi^-} \dfrac{\sin x}{1 - \cos x}$.

For other types of indeterminate forms, we shall try to convert the problem to 0/0 or ∞/∞, then use L'hospital's rule.

Ex. (Type $0 \cdot \infty$) $\lim_{x \to 0^+} x \ln x$

Ex. (Type $\infty - \infty$) $\lim_{x\to(\pi/2)^-}(\sec x - \tan x)$

Ex. (Type 1^∞) $\lim\limits_{x\to 0^+}(1+\sin 4x)^{\cot x}$

Ex. (Type 0^0) $\lim\limits_{x\to 0^+} x^x$.

Ex. (Type ∞^0) $\lim\limits_{x\to\infty} x^{1/x}$

In order to prove L'hospital's rule, one shall use *Cauchy Mean Value Theorem*, which is generalization of MVT.

Theorem 4.4 (Cauchy's Mean Value Theorem)**.** *Suppose that the functions f and g are continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ for all $x \in (a,b)$. Then, there $\exists c \in (a,b)$ s.t.*

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof. Set $h(x) = f(x) - f(a) - \dfrac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]$, then observe $h(a) = h(b) = 0$, and use Rolle's theorem. $\square$

Try to use L'hospital's rule to prove that: suppose that f is continuous at a, and that $f'(x)$ exists for all x in some open interval containing a, except perhaps for $x = a$. Suppose, moreover, that $\lim_{x\to a} f'(x)$ exists. Then $f'(a)$ also exists and

$$f'(a) = \lim_{x\to a} f'(x)$$

. In particular, f' is also continuous at a.

Remark: in general, f' may not be continuous at a if $\lim_{x\to a} f'(x)$ does not exist.