CHAPTER 6
Frequency Response and Resonance

1. Fourier analysis.
2. Filter – Circuits and transfer functions.
3. First-order lowpass or highpass filter circuits and their transfer functions
4. Series and parallel resonant circuits

Fourier Analysis

All real-world signals are sums of sinusoidal components having various frequencies, amplitudes, and phases.

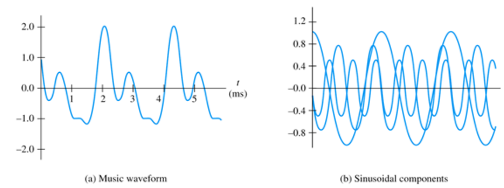


Figure 6.1 The short segment of a music waveform shown in (a) is the sum of the sinusoidal components shown in (b).

Example:

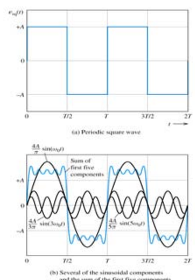


Figure 6.2 A square wave and some of its components.

$$v_{sq}(t) = \frac{4A}{\pi}\sin(\omega_0 t) + \frac{4A}{3\pi}\sin(3\omega_0 t) + \frac{4A}{5\pi}\sin(5\omega_0 t) + \cdots$$

Table 6.1. Frequency Ranges of Selected Signals

Electrocardiogram	0.05 to 100 Hz
Audible sounds	20 Hz to 15 kHz
AM radio broadcasting	540 to 1600 kHz
Video signals (U.S. standards)	Dc to 4.2 MHz
Channel 6 television	82 to 88 MHz
FM radio broadcasting	88 to 108 MHz
Cellular radio	824 to 891.5 MHz
Satellite television downlinks (C-band)	3.7 to 4.2 GHz
Digital satellite television	12.2 to 12.7 GHz

Filters

Filters process the sinusoid components of an input signal differently depending on the frequency of each component.
The goal of the filter is to retain the components in certain frequency ranges and to reject components in other ranges.

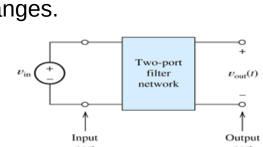


Figure 6.3 When an input signal $v_{in}(t)$ is applied to the input port of a filter, some components are passed to the output port while others are not, depending on their frequencies. Thus, $v_{out}(t)$ contains some of the components of $v_{in}(t)$ but not others. Usually, the amplitudes and phases of the components are altered in passing through the filter.

Filter example

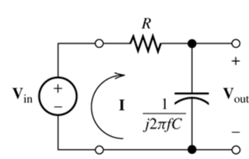


Figure 6.7 A first-order lowpass filter.

Transfer Functions

The **transfer function** *H(f)* of the two-port filter is defined to be the ratio of the phasor output voltage to the phasor input voltage as a function of frequency:

$$H(f) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}$$

The magnitude (or the phase) of the transfer function shows how the amplitude (or the phase) of each frequency component is affected by the filter.

Example 6.1: Find the output

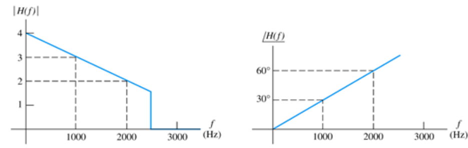


Figure 6.4 The transfer function of a filter. See Examples 6.1 and 6.2.

$$v_{in}(t) = 2\cos(2000\pi t + 40°)$$

$$H(f = 1000) = 3\angle 30°$$

Determining the output of a filter for an input with multiple components:

1. Determine the frequency and phasor representation for each input component.
2. Determine the (complex) value of the transfer function for each component.
3. Obtain the phasor for each output component by multiplying the phasor by the corresponding transfer-function value.
4. Convert the phasors into time functions. Add these time functions to produce the output.

How to determine the transfer function of a filter?

It can be done experimently if the network is not known.

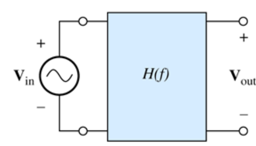


Figure 6.6 To measure the transfer function, we apply a sinusoidal input signal, measure the amplitudes and phases of input and output in steady state, and then divide the phasor output by the phasor input. The procedure is repeated for each frequency of interest.

First-order low pass filters

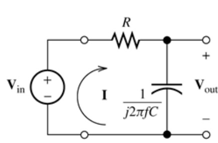


Figure 6.7 A first-order lowpass filter.

$$H(f) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}$$

$$H(f) = \frac{1}{1 + j(f/f_B)} \qquad f_B = \frac{1}{2\pi RC}$$

$$|H(f)| = \frac{1}{\sqrt{1 + (f/f_B)^2}} \qquad \angle H(f) = -\arctan\left(\frac{f}{f_B}\right)$$

Frequency response plot

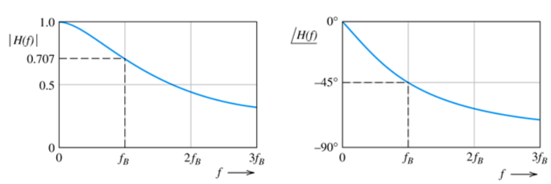


Figure 6.8 Magnitude and phase of the first-order lowpass transfer function versus frequency.

$$|H(f)| = \frac{1}{\sqrt{1 + (f/f_B)^2}} \qquad \angle H(f) = -\arctan\left(\frac{f}{f_B}\right)$$

Example 6.3

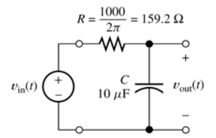


Figure 6.9 Circuit of Example 6.3. The resistance has been picked so the break frequency turns out to be a convenient value.

$$v_{in}(t) = 5\cos(20\pi t) + 5\cos(200\pi t) + 5\cos(2000\pi t)$$

$$v_{out}(t) = ?$$

Half-power frequency: $f_B = \dfrac{1}{2\pi RC} = 100Hz$

$$v_{in}(t) = 5\cos(20\pi t) + 5\cos(200\pi t) + 5\cos(2000\pi t)$$

$$V_{in1} = 5\angle 0°, \quad V_{in2} = 5\angle 0°, \quad V_{in3} = 5\angle 0°$$

$$H(f) = \frac{1}{1 + j(f/f_B)}$$

$$V_{out1} = H(10)V_{in1} = (0.9950\angle -5.71) \times 5\angle 0° = 4.975\angle -5.71°$$

$$v_{out1} = 4.975\cos(20\pi t - 5.71°)$$

$$V_{out2} = H(100)V_{in2} = (0.7071\angle -45°) \times 5\angle 0° = 3.535\angle -45°$$

$$v_{out2} = 3.535\cos(200\pi t - 45°)$$

$$V_{out3} = H(1000)V_{in3} = (0.0995\angle -84.29) \times 5\angle 0° = 0.4975\angle -84.29°$$

$$v_{out3} = 0.4975\cos(2000\pi t - 84.29°)$$

Another low pass filter

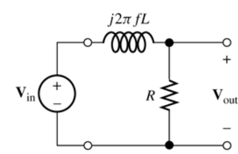


Figure 6.10 Another first-order lowpass filter; see Exercise 6.4.

$$H(f) = \frac{1}{1 + j(f/f_B)} \qquad f_B = \frac{R}{2\pi L}$$

First-order high pass filters

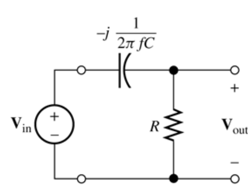


Figure 6.19 First-order highpass filter.

$$H(f) = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = \frac{j(f/f_B)}{1 + j(f/f_B)} \qquad f_B = \frac{1}{2\pi RC}$$

Frequency response plot

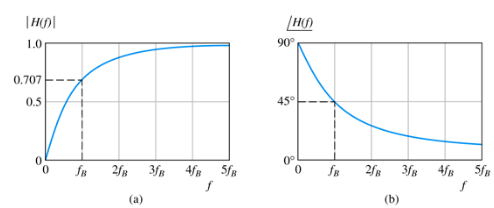


Figure 6.20 Magnitude and phase for the first-order highpass transfer function.

$$|H(f)| = \frac{f/f_B}{\sqrt{1 + (f/f_B)^2}} \qquad \angle H(f) = 90^\circ - \arctan\left(\frac{f}{f_B}\right)$$

Comparison of low-pass and high-pass filters

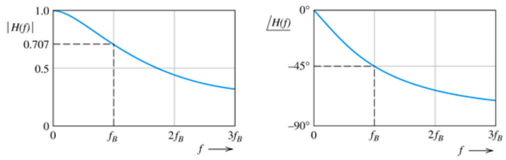
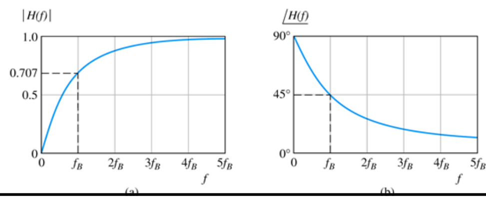


Figure 6.8 Magnitude and phase of the first-order lowpass transfer function versus frequency.



Series Resonance

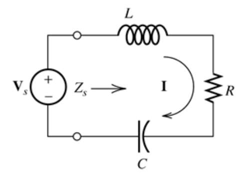


Figure 6.23 The series resonant circuit.

Resonance is a phenomenon that can be observed in mechanical systems and electrical circuits. (Guitar string)

Total Impedance: $$Z_s(f) = R + j2\pi fL - j\frac{1}{2\pi fC}$$

Resonance Condition: Impedance purely resistive

$$Z_s(f) = R + j2\pi fL - j\frac{1}{2\pi fC} = R$$

$$2\pi f_0 L = \frac{1}{2\pi f_0 C}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

Quality Factor: $$Q_s = \frac{2\pi f_0 L}{R}$$

$$Q_s = \frac{1}{2\pi f_0 CR}$$

Total Impedance:

$$Z_s(f) = R\left[1 + jQ_s\left(\frac{f}{f_0} - \frac{f_0}{f}\right)\right]$$

Application of Series Resonant Circuit: Bandpass filter

$$\frac{V_R}{V_s} = \frac{1}{1 + jQ_s(f/f_0 - f_0/f)}$$

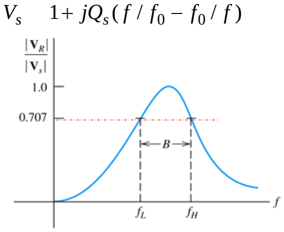


Figure 6.26 The bandwidth B is equal to the difference between the half-power frequencies.

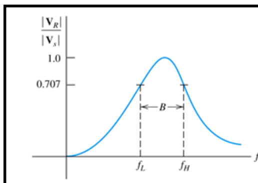
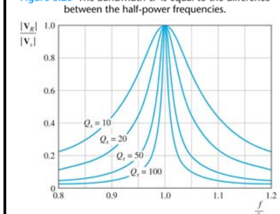


Figure 6.26 The bandwidth B is equal to the difference between the half-power frequencies.

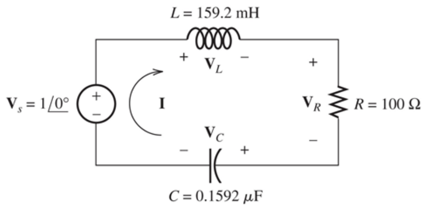


6.25 Plots of the bandpass filter

$$B = f_H - f_L$$

$$B = \frac{f_0}{Q_s}$$

Example 6.6



6.27 Series resonant circuit of Example 6.6

$f_0 = ?, B = ?, f_H = ?, f_L = ?,$

Phasor voltage of each element?

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = 1000Hz, \quad Q_s = \frac{2\pi f_0 L}{R} = 10$$

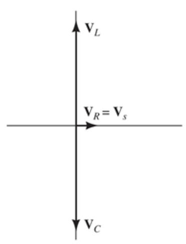
$$B = \frac{f_0}{Q_s} = 100Hz$$

$$f_H \cong f_0 + \frac{B}{2} = 1050Hz, \quad f_L \cong f_0 - \frac{B}{2} = 950Hz$$

$$Z_s = R + Z_L + Z_C = R + j2\pi f_0 L - j\frac{1}{2\pi f_0 C} = R = 100\Omega$$

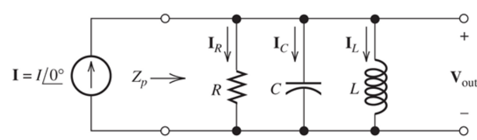
$$I = \frac{V_s}{Z_s} = \frac{1\angle 0°}{100} = 0.01\angle 0°$$

$$V_R = RI = (100)(0.01\angle 0°) = 1\angle 0°$$
$$V_L = Z_L I = (j1000)(0.01\angle 0°) = 10\angle 90°$$
$$V_C = Z_C I = (-j1000)(0.01\angle 0°) = 10\angle -90°$$



6.28 Phasor Diagram for Example 6.6

Parallel resonance



6.29 The parallel resonant circuit

$$Z_p = \frac{1}{(1/R) + j2\pi f C - j(1/2\pi f L)}$$

$$Z_p = \frac{1}{(1/R) + j2\pi f C - j(1/2\pi f L)}$$

Resonance Condition: Impedance purely resistive

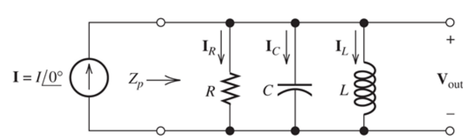
Resonant Frequency: $f_0 = \dfrac{1}{2\pi\sqrt{LC}}$

Quality Factor: $Q_p = \dfrac{R}{2\pi f_0 L}$ $\quad Q_p = 2\pi f_0 CR$

$$Z_p = \frac{R}{1 + jQ_p(f/f_0 - f_0/f)}$$

$$B = f_H - f_L \qquad B = \frac{f_0}{Q_P}$$

Example 6.7: Design



6.29 The parallel resonant circuit

$R = 10k\Omega, \quad f_0 = 1MHz, \quad B = 100kHz, \quad I = 10^{-3}\angle 0°$

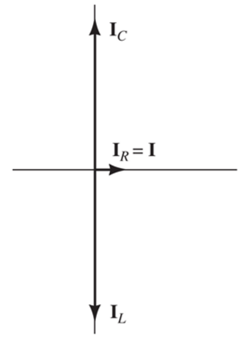
$$Q_p = \frac{f_0}{B} = 10$$

$$L = \frac{R}{2\pi f_0 Q_p} = 159.2\mu H$$

Example 6.7:

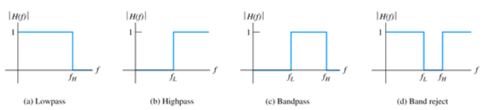
$$C = \frac{Q_p}{2\pi f_0 R} = 159.2\,pF$$

$$V_{out} = IR = (10^{-3}\angle 0°) \times 10^4 = 10\angle 0°$$



6.31 Phasor diagram for Example 6.7

Ideal Filters



Example

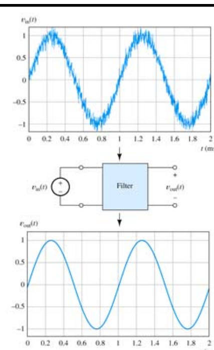
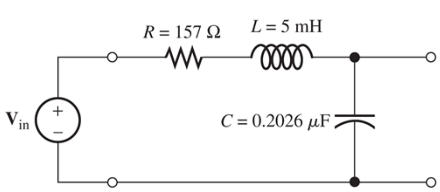


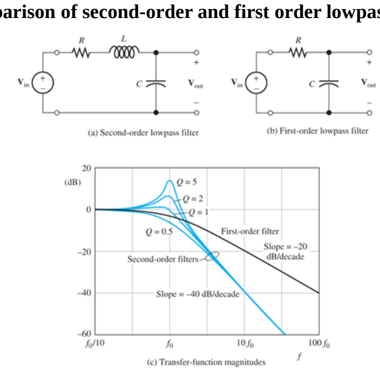
Figure 6.33 The input signal v_{in} consists of a 1-kHz sine wave plus high-frequency noise. By passing v_{in} through an ideal lowpass filter with the proper cutoff frequency, the sine wave is passed and the noise is rejected, resulting in a clean output signal.

Second-order lowpass filter:

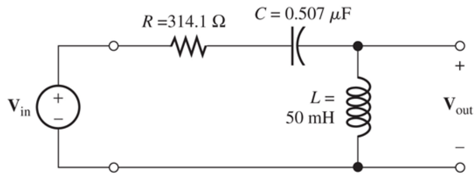


$$H(f) = \frac{V_{out}}{V_{in}} = \frac{-jQ_s(f_0/f)}{1 + jQ_s(f/f_0 - f_0/f)}$$

Comparison of second-order and first order lowpass filters



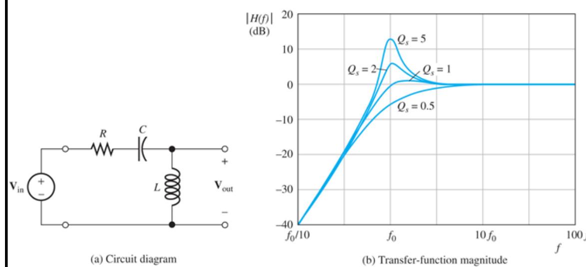
Second-order highpass filter:



6.39 Filter designed for Example 6.9

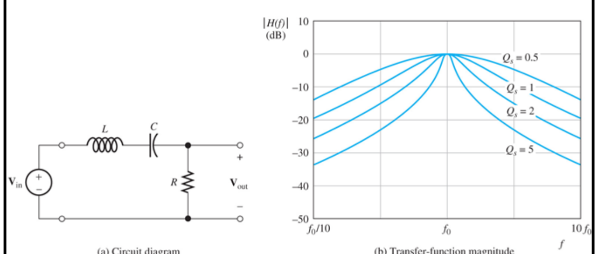
$$H(f) = \frac{V_{out}}{V_{in}} = \frac{jQ_s(f/f_0)}{1 + jQ_s(f/f_0 - f_0/f)}$$

Second-order highpass filter:



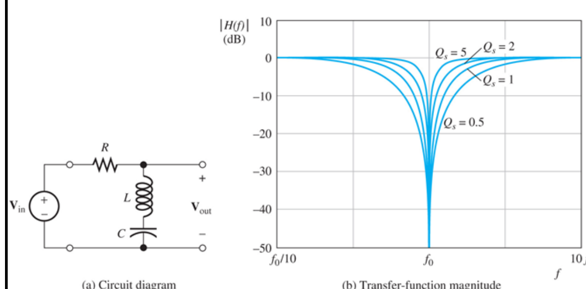
6.36 Second-order highpass filter

Second-order bandpass filter:



6.37 Second-order bandpass filter

Second-order band-reject filter:



6.38 Second-order band-rejector filter