Tutorial 3 (Chapter 3)

1. Suppose that the distribution function of a random variable X is given by

$$F(b) = \begin{cases} 0 & b < 0 \\ \frac{b}{4} & 0 \le b < 1 \\ \frac{1}{2} + \frac{b-1}{4} & 1 \le b < 2 \\ \frac{11}{12} & 2 \le b < 3 \\ 1 & 3 \le b \end{cases}$$

   (i) Find $P(X = i)$, $i = 1, 2, 3$.

   (ii) Find $P(\frac{1}{2} < X < \frac{3}{2})$.

   Solution

   (i)$1/4; 1/6; 1/12$

   (ii)$1/2$

2. If $E[X] = 1$ and $Var(X) = 5$, find

   (a) $E[(2 + X)^2]$;

   (b) $Var(4 + 3X)$.

   Solution

   $Var(X) = E(X^2) - (EX)^2 = 5 \Rightarrow E(X^2) = 6$

   (a) $E[(2 + X)^2] = E(X^2) + 4E(X) + 4 = 6 + 4 + 4 = 14$

   (b) $Var(4 + 3X) = 3^2 Var(X) = 45$

3. Jane takes a multiple-choice exam with 3 possible answers for each of the 5 questions. What is the probability that Jane would get 4 or more correct answers just by guessing.

   Solution

   For any one problem, the probability that Jane gets it right is 1/3. so the probability that Jane get 4 right out of 5 questions is $Bin(4|5, 1/3) + Bin(5|5, 1/3) = \binom{5}{4}(1/3)^4(2/3) + \binom{5}{5}(1/3)^5 \approx 0.045$

4. People enter a gambling casino at a rate of 1 for every 2 minutes. During the time $12:00$ and $12:05$, what is the probability that no one enters the casino?

   Solution

   $Pois(0|2.5) = e^{-2.5}$

5. For a nonnegative integer-valued random variable N, prove that

$$E[N] = \sum_{i=1}^{\infty} P(N \ge i).$$

   Solution

   $\sum_{i=1}^{\infty} P(N \ge i) = \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} P(N = k) = \sum_{k=1}^{\infty} \sum_{i=1}^{k} P(N = k) = \sum_{k=1}^{\infty} kP(N = k) = E(N)$

6. Let X be a random variable having expected value μ and variance σ^2. Find the expected value and variance of $Y = \dfrac{X - \mu}{\sigma}$.

   Solution

   mean is 0, variance is 1.

7. The probabilities of turning up heads for two biased coins are 0.7 and 0.6 respectively. Flip each coin three times.

   (a) What is the probability that same number of heads appears for the two coins.

   (b) What is the probability that more heads appears for the first coin.

   Solution

   Let $A_i = \{i \text{ heads for the 1st coin}\}, B_i = \{i \text{ heads for the 2nd coin}\}$. Then $P(A_i) = \binom{3}{i}0.7^i 0.3^{3-i}$, and $P(B_i) = \binom{3}{i}0.6^i 0.4^{3-i}$.

   (a)

$$\sum_{k=0}^{3} P(A_k B_k) = \sum_{k=0}^{3} P(A_k)P(B_k) \approx 0.321$$

   (b)

$$P(A_1)P(B_0) + P(A_2)[P(B_0) + P(B_1)] + P(A_3)[1 - P(B_3)] \approx 0.436$$

8. Let X be a binomial random variable with parameters n and p. Prove that

$$E[\frac{1}{X+1}] = \frac{1-(1-p)^{n+1}}{(n+1)p}$$

   Solution

$$\begin{aligned}
E[\frac{1}{X+1}] &= \sum_{i=0}^{n} \frac{1}{i+1}\frac{n!}{(n-i)!i!}p^i(1-p)^{n-i} \\
&= \sum_{i=0}^{n} \frac{n!}{(n-i)!(i+1)!}p^i(1-p)^{n-i} \\
&= \frac{1}{(n+1)p}\sum_{i=0}^{n}\binom{n+1}{i+1}p^{i+1}(1-p)^{n-i} \\
&= \frac{1}{(n+1)p}\sum_{j=1}^{n+1}\binom{n+1}{j}p^{j}(1-p)^{n+1-j} \\
&= \frac{1}{(n+1)p}[1-\binom{n+1}{0}(1-p)^{n+1}] \\
&= \frac{1}{(n+1)p}[1-(1-p)^{n+1}]
\end{aligned}$$

9. Let X be a negative binomial random variable with parameters r and p, and let Y be a binomial random variable with parameters n and p. Argue (without computation) that

$$P(X > n) = P(Y < r).$$

   Solution

   Suppose an experiment with success probability p is performed repeatedly. Keep track of the number of successes so far as we perform the experiment. The event $X > n$ is the event that we have not seen r-th success after the n-th trial. Another way of saying this is there are less than r successes in the first n trials, exactly the event $Y < r$.

10. An urn contains one red and one blue ball. At each stage a ball is randomly chosen and then this ball is replaced together with another of the same color. Let X denote the selection number of the first chosen ball that is blue. For instance, if the first selection is red and the second blue, then X is equal to 2.

(a) Find $P(X > i)$, $i \geq 1$

(b) Show $P(X < \infty) = 1$

(c) Find $E[X]$

Solution

(a) $P(X > i) = \frac{1}{2}\frac{2}{3}\cdots\frac{i}{i+1} = \frac{1}{i+1}$

(b) $P(X < \infty) = \lim_{i\to\infty} P(X \leq i) = \lim_i (1 - \frac{1}{i+1}) = 1$

(c) $E[X] = \sum_{i=1}^{\infty} P(X \geq i) = \sum_{i=1}^{\infty} \frac{1}{i} = \infty$