Tutorial 2 (Chapter 2)

1. There are two children in a family. Given one of them is a girl, what is the probability that the other is also a girl (suppose a child being a boy or a girl is equally likely)?

   Solution

   The sample space is $S = \{bb, bg, gb, gg\}$. Then $P(gg|bg \text{ or } gb \text{ or } gg) = 1/3$

2. 9 persons randomly enter 3 different rooms. What is the probability that

   (a) the first room has 3 person?

   (b) every room has 3 persons?

   (c) the first room has 4 person, second room 3 persons, third room 2 persons?

   (d) the three rooms have 4,3,2 persons respectively (but we don't know which room has 4 persons, for example)?

   Solution

   (a) $\frac{\binom{9}{3}2^6}{3^9}$

   (b) $\frac{\binom{9}{3}\binom{6}{3}\binom{3}{3}}{3^9}$ or $\frac{\binom{9}{3,3,3}}{3^9}$

   (c) $\frac{\binom{9}{4}\binom{5}{3}\binom{2}{2}}{3^9}$ or $\frac{\binom{9}{4,3,2}}{3^9}$

   (d) $\frac{\binom{9}{4}\binom{5}{3}\binom{2}{2}3!}{3^9}$ or $\frac{\binom{9}{4,3,2}3!}{3^9}$

3. (a) n persons randomly enter n different rooms, what is the probability that each room has one person?

   (b) n persons randomly enter k different rooms ($n \geq k$), what is the probability that each room has at least one person?

   Solution

   (a) $n!/n^n$

   (b) This is the same as the problem in the lecture slides for Chapter 2 about assigning n students to k tutorial groups.

4. A student is taking a multiple choice exam in which each question has m possible answers, exactly one of which is correct. If the student knows the answer he selects the correct answer. Otherwise he makes a guess at random from the possible answers. Suppose that the probability that he knows the correct answer is 0.7

   (a) What is the probability that on a given question the student gets the correct answer?

   (b) If the student gets the correct answer to a question, what is the probability that he knows the answer?

   Solution

   (a) (Law of Total Probability)

$$\begin{aligned}
& P\{\text{gives correct answer}\} \\
= \; & P\{\text{gives correct answer}|\text{knows the answer}\}P\{\text{knows the answer}\} \\
& +P\{\text{gives correct answer}|\text{does not know the answer}\}P\{\text{does not know the answer}\} \\
= \; & 1 \times 0.7 + \frac{1}{m} \times 0.3 = \frac{7m+3}{10m}
\end{aligned}$$

(b) (Bayes Formula)

$$\begin{aligned}
& P\{\text{knows the answer}|\text{gives correct answer}\} \\
= \; & P\{\text{knows the answer AND gives correct answer}\}/P\{\text{gives correct answer}\} \\
= \; & P\{\text{knows the answer}\}P\{\text{gives correct answer}|\text{knows the answer}\}/P\{\text{gives correct answer}\} \\
= \; & \frac{0.7 \times 1}{\frac{7m+3}{10m}} \\
= \; & \frac{7m}{7m+3}
\end{aligned}$$

5. (Polya's Urn) An urn initially contains 5 white and 7 black balls. Each time a ball is selected its colour is noted and it is replaced in the urn along with 2 other balls of the same colour.

   (a) Compute the probability that the first 2 balls selected are white and the next two black.

   (b) Compute the probability that, of the first 4 balls selected, exactly 2 are black.

   (c) Compute the conditional probability that the first ball was black, given that the second ball drawn was white.

   Solution

   (a)

$$\begin{aligned}
P\{WWBB\} &= P\{1stW\}P\{2ndW|1stW\}P\{3rdB|1stW, 2ndW\}P\{4thB|1stW, 2ndW, 3rdW\} \\
&= \frac{5}{12}\frac{7}{14}\frac{7}{16}\frac{9}{18}
\end{aligned}$$

   (b) OK, there are $\binom{4}{2} = 6$ possible configurations where 2 B's appear. If you try to compute one of these besides $WWBB$ which is from part (a), say you try WBWB, you'll soon realize that the probability is the same. So you should guess all these 6 probabilities are the same, and should try to convince yourself that this is indeed true. The answer for (b) is the answer for (a) multiplied by 6.

   (c) This is an application of Bayes formula.

$$\begin{aligned}
P\{1stB|2ndW\} &= \frac{P\{1stB\}P\{2ndW|1stB\}}{P\{1stB\}P\{2ndW|1stB\} + P\{1stW\}P\{2ndW|1stW\}} \\
&= \frac{\frac{7}{12}\frac{5}{14}}{\frac{7}{12}\frac{5}{14} + \frac{5}{12}\frac{7}{14}} = 1/2
\end{aligned}$$

6. In a bolt factory machines A, B and C manufacture, respectively 25% 35%and 40% of the total. Of their output 5%, 4% and 2% are defective bolts.

   (a) Find the probability that a bolt drawn is defective.

   (b) A bolt is drawn at random from the produce and is found defective. What is the probability that it is manufactured by machine A?

   solution

   (a) (Law of total probability)

$$\begin{aligned}
P\{\text{defective}\} &= P\{\text{defective}|A\}P\{A\} + P\{\text{defective}|B\}P\{B\} + P\{\text{defective}|C\}P\{C\} \\
&= 0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4 = 0.0345
\end{aligned}$$

(b) (Bayes Formula)

$$P(A|\text{defective}) = \frac{P(A)P(\text{defective}|A)}{P(\text{defective})} = \frac{0.25 \times 0.05}{0.345} \approx 0.36$$

7. (a) If $P(A) = 1/2, P(B) = 1/3, P(B|A) = 2/3$, $P(A \cup B) =?$

   (b) If $P(AB) = 0, P(A) = P(B) = a, P(A|B^c) = P(A^c|B^c)$, then $a =?$

   Solution

   (a)$P(AB) = P(A)P(B|A) = 1/3$, and thus $P(A \cup B) = P(A) + P(B) - P(AB) = 1/2$

   (b)$P(A|B^c) = \frac{P(AB^c)}{P(B^c)} = \frac{P(A)-P(AB)}{1-P(A)} = \frac{a}{1-a}$. Since $P(A|B^c) = P(A^c|B^c)$ means $P(A|B^c) = 1/2$, we have $a = 1/3$.

8. Prove or give counterexamples to the following statements:

   (a) If E is independent of F and E is independent of G, then E is independent of $F \cup G$.

   (b) If E is independent of F and E is independent of G and $FG = \phi$, then E is independent of $F \cup G$.

   Solution

   first, (b) is easily proved:

$$\begin{aligned} P(E(F \cup G)) &\overset{(1)}{=} P(EF \cup EG) \overset{(2)}{=} P(EF) + P(EG) \\ &\overset{(3)}{=} P(E)P(F) + P(E)P(G) = P(E)(P(F) + P(G)) \overset{(4)}{=} P(E)P(F \cup G) \end{aligned}$$

   where (1) use De Morgen's law, (2) use finite additivity since EF and EG are mutually exclusive, (3) use independence assumption, (4) use finite additivity again.

   Now you have used $FG = \phi$ in the proof of (b), you might have guessed (a) is not true. One example is as follows:

   Suppose there are two persons, A and B. Each chooses a number, either -1 or 1, with probability 1/2 respectively. Let $E = \{\text{product of the two numbers is 1}\}, F = \{\text{A picks 1}\}, G = \{\text{B picks 1}\}$

9. Show that if A, B and C are three events, (assuming the probabilities of some events are nonzero when appropriate so that we do not need to worry about denominator being zero), and $P(C|AB) = P(C|B)$, then $P(A|BC) = P(A|B)$.

   Solution

   Both two expression means C is independent of A if conditioning on B. To rigorously prove it, use the definition of the conditional probability:

$$P(C|AB) = P(C|B) \iff \frac{P(ABC)}{P(AB)} = \frac{P(BC)}{P(B)} \quad (*)$$

   what we want to prove is$\frac{P(ABC)}{P(BC)} = \frac{P(AB)}{P(B)}$, which is obviously equivalent to (*).

10. A and B flip coins in turn with A starting first. The first person who gets head wins. What is the probability that A wins?

    Solution

Method 1: Let $A_i = \{$A obtains a head in his n-th trial$\}$, $B_i = \{$B obtains a head in his n-th trial$\}$.

$$\begin{aligned} P(\text{A wins}) &= P(A_1) + P(A_1^c B_1^c A_2) + P(A_1^c B_1^c A_2^c B_2^c A_3) + \cdots \\ &= 1/2 + 1/8 + 1/32 + \cdots \\ &= 2/3 \end{aligned}$$

Method 2: (don't worry if you don't understand, this is difficult).

$P(\text{A wins}) + P(\text{B wins}) = 1$

$P(\text{A wins}) = P(\text{A wins}|A_1)P(A_1) + P(\text{A wins}|A_1^c)P(A_1^c) = 1/2 + P(\text{A wins}|A_1^c)/2 = 1/2 + P(\text{B wins})/2$

You can solve the two equations with two unknowns to get $P(\text{A wins}) = 2/3$.

(The reason $P(\text{A wins}|A_1^c) = P(\text{B wins})$ is that if A gets a tail first, then it is B's turn to throw the coin. Now the probability that B wins is the same as the probability that A wins when A is the first person to throw the coin.)

11. A company buys tires from two suppliers-1 and 2. Supplier 1 has a record of delivering tires that contain 10% defectives, where as supplier 2 has a defective rate of only 5%. Suppose that 40% if the current supply came from supplier 1. If a tire taken at random from this supply and observed to be defective, what is the probability that it came from supplier 1?

    Solution

    Let $B_i = \{$a tire comes from supplier $i\}, i = 1, 2$. $A = \{$selected tire is defective$\}$

$$\begin{aligned} P(B_1|A) &= \frac{P(B_1)P(A|B_1)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2)} \\ &= \frac{0.4 \cdot 0.1}{0.4 \cdot 0.1 + 0.6 \cdot 0.05} = 4/7 \end{aligned}$$

12. An individual tried by a 3-judge panel is declared guilty if at least 2 judges cast votes of guilty. Suppose that when the defendant is in fact guilty, each judge will independently vote guilty with probability 0.6; whereas when the defendant is, in fact, innocent, this probability drops to 0.2. If 50% of defendants are guilty, compute the conditional probability that Judge number 3 votes guilty given that Judges 1 and 2 votes guilty.

    Let J_i, $i = 1, 2, 3$ denote the event that Judge i casts a guilty vote. Are these events independent? Are they conditionally independent, i.e. are these events independent given a defendant a defendant is guilty?

    Solution

    Let $J_i = \{$Judge i vote guilty$\}, i = 1, 2, 3$. $G = \{$defendent is in fact guilty$\}$

    If judge 1 and 2 have not voted yet, the question is easy, just use the law of total probability:

$$P(J_3) = P(J_3|G)P(G) + P(J_3|G^c)P(G^c) = 0.6 \cdot 0.5 + 0.2 \cdot 0.5 = 0.4$$

But after judge 1 and 2 voted, we have more information about the defendant (in this case, we would say the defendant is more probable to be guilty since two judges already voted so). We can compute the probability of G given the votes of these 2 judges:

$$\begin{aligned} P(G|J_1J_2) &= \frac{P(J_1J_2|G)P(G)}{P(J_1J_2)} \\ &= \frac{P(J_1J_2|G)P(G)}{P(J_1J_2|G)P(G) + P(J_1J_2|G^c)P(G^c)} \\ &= \frac{0.6 \cdot 0.6 \cdot 0.5}{0.6 \cdot 0.6 \cdot 0.5 + 0.2 \cdot 0.2 \cdot 0.5} = 0.9 \end{aligned}$$

Now the desired probability is

$$P(J_3|J_1J_2) = P(J_3|G, J_1, J_2)P(G|J_1J_2) + P(J_3|G^c, J_1, J_2)P(G^c|J_1J_2)$$

(the above formula is just law of total probability, but everything is conditioned on J_1 and J_2 so it seems more complicated than it really is)

Notice $P(J_3|G, J_1, J_2) = P(J_3|G) = 0.6$ since given the defendant is guilty, the 3 judges vote independently. Similarly, $P(J_3|G^c, J_1, J_2) = P(J_3|G^c) = 0.2$

Final answer is $0.6 \cdot 0.9 + 0.2 \cdot 0.1 = 0.56$

The problem can also be solved more easily by using directly $P(J_3|J_1J_2) = \frac{P(J_1J_2J_3)}{P(J_1J_2)}$, both the numerator and denominator can be found using law of total probability ($P(J_1J_2)$ actually has been calculated in the above).