MA1300 Solution to Self Practice # 10

1. Suppose that a function f is continuous on $[a,b]$ and $f''(x)$ exists for every $x \in (a,b)$. If $f(a) = f(b) = 0$ and $f(c) < 0$ for some point $c \in (a,b)$, prove that there exists some $\xi \in (a,b)$ such that $f''(\xi) > 0$.

Proof. Apply the Mean Value Theorem to the function f on the intervals $[a,c]$ and $[c,b]$ respectively. There exist two points $d_1 \in (a,c)$ and $d_2 \in (c,b)$ such that

$$f'(d_1) = \frac{f(c) - f(a)}{c - a} < 0, \qquad f'(d_2) = \frac{f(b) - f(c)}{b - c} > 0.$$

Then we apply the Mean Value Theorem to the differentiable function f' on the interval $[d_1, d_2]$ and conclude that there exists some point $\xi \in (d_1, d_2) \subset (a,b)$ such that

$$f''(\xi) = (f')'(\xi) = \frac{f'(d_2) - f'(d_1)}{d_2 - d_1} > 0.$$

This proves the desired statement.

2. Suppose that a function f is continuous on $[a,b]$ and the derivatives $f'(a), f'(b)$ exist. If $f(a) = f(b) = 0$ and $f'(a) \cdot f'(b) > 0$, prove that there exists some $\xi \in (a,b)$ such that $f(\xi) = 0$.

Proof. If $f'(a) > 0$, then the assumption $f'(a)f'(b) > 0$ tells us that $f'(b) > 0$. By the definition of derivative, $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} > 0$. So there exists some $x_1 \in (a, \frac{a+b}{2})$ such that $f(x_1) - f(a) = f(x_1) > 0$. On the other hand, since $f'(b) > 0$, by the definition of derivative, $f'(b) = \lim_{x\to b} \frac{f(x)-f(b)}{x-b} > 0$. So there exists some $x_2 \in (\frac{a+b}{2}, b)$ such that $f(x_2) - f(b) = f(x_2) < 0$. By the Intermediate Value Theorem, there exists some $\xi \in (x_1, x_2) \subset (a,b)$ such that $f(\xi) = 0$.

If $f'(a) < 0$, then the assumption $f'(a)f'(b) > 0$ tells us that $f'(b) < 0$. Then the same argument can be used to prove the statement.

3. Let f be a continuous function on $[a,b]$. If it is differentiable on (a,b), and it satisfies

$$f(a) \cdot f(b) > 0, \qquad f(a) \cdot f\left(\frac{a+b}{2}\right) < 0,$$

prove that for every real number β there exists some $\xi \in (a,b)$ such that $f'(\xi) = \beta f(\xi)$.

Proof. Let $\beta \in \mathbb{R}$. Consider the function $F(x) = e^{-\beta x} f(x)$. It is continuous on $[a,b]$ and differentiable on (a,b). By the condition $f(a) \cdot f\left(\frac{a+b}{2}\right) < 0$, we know that $F(a)$ and $F\left(\frac{a+b}{2}\right)$ have different signs. By the Intermediate Value Theorem, there exists some $x_1 \in (a, \frac{a+b}{2})$ such that $F(x_1) = 0$.

The conditions $f(a) \cdot f(b) > 0$, $f(a) \cdot f\left(\frac{a+b}{2}\right) < 0$ also tell us that $f(b) \cdot f\left(\frac{a+b}{2}\right) < 0$. So $F(b)$ and $F\left(\frac{a+b}{2}\right)$ have different signs. By the Intermediate Value Theorem again, there exists some $x_2 \in (\frac{a+b}{2}, b)$ such that $F(x_2) = 0$.

Finally we apply the Rolle Theorem or Mean Value Theorem to the function F on the interval $[x_1, x_2]$, and know that there exists some $\xi \in (x_1, x_2) \subset (a,b)$ such that $F'(\xi) = 0$. But $F'(\xi) = e^{-\beta\xi}(f'(\xi) - \beta f(\xi))$. Therefore, we have $f'(\xi) - \beta f(\xi) = 0$ and hence $f'(\xi) = \beta f(\xi)$. This proves the statement.

4. Let f be the function given by $f(x) = \frac{x}{x^2-x-2}$. Find $f^{(n)}(x)$ for any positive integer n and $x \neq 2, -1$.

Solution. Write $f(x) = \frac{\frac{2}{3}}{x-2} + \frac{\frac{1}{3}}{x+1}$. By mathematical induction we see that for any $a \in \mathbb{R}$, the n-th derivative of the function $\frac{1}{x-a}$ is given by

$$\left(\frac{1}{x-a}\right)^{(n)} = (-1)^n n!(x-a)^{-n-1}, \qquad x \neq a, n \in \mathbb{N}.$$

Applying this expression to $a = 2$ and $a = -1$ yields

$$f^{(n)}(x) = \frac{2}{3}\left(\frac{1}{x-2}\right)^{(n)} + \frac{1}{3}\left(\frac{1}{x+1}\right)^{(n)} = (-1)^n n!\left(\frac{2}{3}(x-2)^{-n-1} + \frac{1}{3}(x+1)^{-n-1}\right).$$

5. Let f be a continuous function on the closed interval $[a,b]$ such that $f(a) = f(b) = 0$ and $f''(x)$ exists for every $x \in (a,b)$. Prove that for every $c \in (a,b)$, there exists some point $\xi \in (a,b)$ such that

$$f(c) = \frac{f''(\xi)}{2}(c-a)(c-b).$$

Proof. Consider the function

$$g(x) = f(x) - \frac{f(c)}{(c-a)(c-b)}(x-a)(x-b).$$

It satisfies $g(a) = g(b) = g(c) = 0$ and

$$g'(x) = f'(x) - \frac{f(c)}{(c-a)(c-b)}(2x-a-b), \qquad g''(x) = f''(x) - 2\frac{f(c)}{(c-a)(c-b)}, \qquad \forall x \in (a,b).$$

The function g is continuous on $[a,c]$ and differentiable on (a,c), so by Rolle's Theorem, there exists some $\xi_1 \in (a,c)$ such that $g'(\xi_1) = 0$. In the same way, g is continuous on $[c,b]$ and differentiable on (c,b), so by Rolle's Theorem, there exists some $\xi_2 \in (c,b)$ such that $g'(\xi_2) = 0$. Finally, since the function $g'(x)$ is continuous on $[\xi_1, \xi_2]$ and differentiable on (ξ_1, ξ_2), by Rolle's Theorem, there exists some $\xi \in (\xi_1, \xi_2)$ such that $(g')'(\xi) = g''(\xi) = 0$. By the expression for $g''(x)$, we have

$$g''(\xi) = f''(\xi) - 2\frac{f(c)}{(c-a)(c-b)} = 0.$$

Hence $f(c) = \frac{f''(\xi)}{2}(c-a)(c-b)$. This proves the desired result.

6. Proof. The condition $f(1) = 0$ implies $F(1) = 1^2 \cdot f(1) = 0$. Also, $F(0) = 0^2 \cdot f(0) = 0$. By the Rolle Theorem, there exists some $\eta \in (0,1)$ such that $F'(\eta) = 0$. On the other hand, the chain rule yields $F'(x) = 2xf(x) + x^2 f'(x)$ which gives $F'(0) = 0$. Then we apply the Rolle Theorem to the function F' on the interval $[0,\eta]$ and know that there exists some $\xi \in (0,\eta) \subset (0,1)$ such that $F''(\xi) = 0$.

7. Proof. We first prove that the equation $f(x) = x$ has at least one root on the interval (a,b). Consider the function $F(x) = f(x) - x$. It is continuous on $[a,b]$, differentiable on (a,b) and satisfies $F(a) = f(a) - a > 0$, $F(b) = f(b) - b < 0$. By the Intermediate Value Theorem, there is some $\xi \in (a,b)$ such that $F(\xi) = 0$. This is a root of the equation $f(x) = x$.

Then we prove that the equation $f(x) = x$ has only one root on the interval (a,b). Suppose to the contrary that the equation has another root $c \neq \xi$ on (a,b). Then c is another zero of F on (a,b). We apply the Rolle Theorem to the function F on the closed interval between ξ and c, and know that there exists some η on the open interval (which is a subset of (a,b)) such that $F'(\eta)=0$. But $F(\eta)=f'(\eta)-1$. So we have $f'(\eta)=1$ which is a contradiction to the assumption $f'(x)\neq 1$. This shows that the equation $f(x)=x$ has only one root on the interval (a,b).

8. Proof. Consider the function F defined by

$$F(x)=f(a)g(x)+g(b)f(x)-f(x)g(x).$$

It is continuous on $[a,b]$ and differentiable on (a,b). It satisfies

$$F(a)=f(a)g(b), \qquad F(b)=f(a)g(b).$$

By the Mean Value Theorem, there is some $\xi\in(a,b)$ such that $F'(\xi)=0$. But

$$F'(\xi)=f(a)g'(\xi)+g(b)f'(\xi)-f'(\xi)g(\xi)-f(\xi)g'(\xi).$$

Therefore,

$$f(a)g'(\xi)+g(b)f'(\xi)-f'(\xi)g(\xi)-f(\xi)g'(\xi)=0.$$

Hence

$$\left(f(a)-f(\xi)\right)g'(\xi)=\left(g(\xi)-g(b)\right)f'(\xi).$$

This yields the desired equality.

9. Proof. Consider the function F defined by $F(x)=\frac{f(x)}{x^2}$. It is continuous on $[1,2]$ and differentiable on $(1,2)$. It also satisfies $F(1)=f(1)=\frac{1}{2}$ and $F(2)=\frac{F(2)}{2^2}=\frac{1}{2}$. So by the Mean Value Theorem, there is some $\xi\in(1,2)$ such that $F'(\xi)=0$. But

$$F'(\xi)=\frac{f'(\xi)\xi^2-2\xi f(\xi)}{\xi^4}.$$

Therefore,

$$\frac{f'(\xi)\xi^2-2\xi f(\xi)}{\xi^4}=0,$$

which implies $f'(\xi)=\frac{2f(\xi)}{\xi}$.

10. Proof. We apply the Mean Value Theorem to the function f on two intervals $[0,a]$ and $[b,a+b]$. We know that there are $\xi_1\in(0,a)$ and $\xi_2\in(b,a+b)$ such that

$$f'(\xi_1)=\frac{f(a)-f(0)}{a-0}=\frac{f(a)}{a}, \qquad f'(\xi_2)=\frac{f(a+b)-f(b)}{a+b-b}=\frac{f(a+b)-f(b)}{a}.$$

Taking the difference yields

$$f(a+b)-f(b)-f(a)=a\left(f'(\xi_2)-f'(\xi_1)\right).$$

Since $\xi_2>b>a>\xi_1$ and $f'(x)$ is decreasing, we know that $f'(\xi_2)-f'(\xi_1)<0$. Hence $f(a+b)-f(b)-f(a)<0$. This proves the desired inequality.