WEEK 12

CHAPTER FIVE

- The Quadric Surfaces

OUTLINE

- Review: ellipsoid, cone,
- Elliptic paraboloid
- Hyperbolic paraboloid
- Hyperboloid of one sheet
- Ruled surfaces

FYI

- https://www.youtube.com/watch?v=ZNbXTUTSVCY&t=1s
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A HISTORY LESSON...

- Rene Descartes (1596-1650)
- Analytic geometry: Algebraic representation of geometry
- First clear formulation of algebraic geometry by combining ideas from branches of mathematics, algebra and geometry
- Descartes: an extraordinary philosopher, mathematician, and scientist who lived in a time when such a person could be all three at once. He is best known for his philosophical maxim, "I think, therefore I am" (Latin: Cogito, ergo sum), which he first wrote in French in his book Discourse on the Method (1637) and later in Latin in his Principles of Philosophy (1644). He is also known for his Cartesian coordinates, which allow the representation of points in two-dimensional space and other mathematical objects in a manner easy to visualize.

STRAIGHT LINE

$f(x) = mx + c$

ALL OTHER SHAPES

parabola

$y = ax^2$

$x = ay^2$

$y^2 = ax$

$x^2 = ay$

circle

$x^2 + y^2 = r^2$

ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

3D TO 2D

3D

$z = x + y$ plane

$z = x^2$

$z = y^2$

$z^2 = x^2 + y^2$

$z^2 = x^2 - y^2$

$z^2 = x^2 + y^2 - 1$

$z^2 = x^2 - y^2 - 1$

$z^2 = x^2 + y^2 + 1$

$z^2 = x^2 - y^2 + 1$

$z^2 = x^2 + y^2 - 1$

$z^2 = x^2 - y^2 - 1$

$z^2 = x^2 + y^2 + 1$

$z^2 = x^2 - y^2 + 1$

QUADRIC SURFACE

- The general form of a quadric surface is

$$\underbrace{Ax^2 + By^2 + Cz^2}_{\text{quadratic}} + \underbrace{Dxy + Exz + Fyz}_{\text{cross terms}} + \underbrace{Gx + Hy + Iz}_{\text{linear}} + J = 0$$

$A, B, \ldots, J$ are constants

- We will go over the definitions, what we need to know about it, starting by understanding what cross-sections of the surface provide us a coordinate plane look like.

STEPS TO SKETCHING EXAMPLE

- We have three different coordinate planes that we can look at: x-y plane, x-z plane, y-z plane.
- Cross-sections are the intersections of the surface with these planes.
- Start by sketching the cross-sections of the surface that is easier right in the middle. Then, [illegible] the rest of the surface.

TRACE

- Both traces are lines
- x-y plane $z = 0$
- x-z plane $y = 0$
- y-z plane $x = 0$

EXAMPLE OF A BASIC FUNCTION

$z^2 = x^2 + y^2$

x-y plane $z = 0$

$x^2 + y^2 = 0$

$x = 0, y = 0$

x-z plane $y = 0$

$z^2 = x^2$

$z = \pm x$

y-z plane $x = 0$

$z^2 = y^2$

$z = \pm y$

EXAMPLE STRAIGHT LINE

$y = mx + c$

$\boxed{y = mx + c}$

x-y plane $z = 0$

EXAMPLE CIRCLE

$x^2 + y^2 = r^2$

$\boxed{x^2 + y^2 = r^2}$

- Cross-section of the cylinder between $(0, 0)$

EXAMPLE ELLIPSOID

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$a = 2, b = 3$

$c = 4$

ellipse

x-y plane $z = 0$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

x-z plane $y = 0$

$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$

y-z plane $x = 0$

$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

EXAMPLE ELLIPTIC CONE

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}, a = b = c$

x-y plane $z = 0$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$

$x = 0, y = 0$

x-z plane $y = 0$

$\frac{x^2}{a^2} = \frac{z^2}{c^2}$

$z = \pm \frac{c}{a} x$

y-z plane $x = 0$

$\frac{y^2}{b^2} = \frac{z^2}{c^2}$

$z = \pm \frac{c}{b} y$

EXAMPLE HYPERBOLOID OF ONE SHEET

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

x-y plane $z = 0$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipse

x-z plane $y = 0$

$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ hyperbola

y-z plane $x = 0$

$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ hyperbola

EXAMPLE HYPERBOLOID OF TWO SHEETS

$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

x-y plane $z = 0$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (hyperbola)

x-z plane $y = 0$

$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ hyperbola

y-z plane $x = 0$

$-\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ no sol

EXAMPLE ELLIPTIC PARABOLOID

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$

x-y plane $z = 0$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$

$x = 0, y = 0$

x-z plane $y = 0$

$\frac{x^2}{a^2} = z$ parabola

y-z plane $x = 0$

$\frac{y^2}{b^2} = z$ parabola

FYI

- The Quadric Surfaces (Video)
- https://www.youtube.com/watch?v=ZNbXTUTSVCY&t=1s
- Hyperbolic Paraboloid
- https://www.youtube.com/watch?v=ZNbXTUTSVCY&t=1s
- Rulled Surfaces (Video)
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RECALL FROM PREVIOUS WEEKS...

- Vector product $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- Scalar product $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- Triple product $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
- Area of parallelogram $= |\vec{a} \times \vec{b}|$
- Volume of parallelepiped $= |\vec{a} \cdot (\vec{b} \times \vec{c})|$

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- Triple product $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
- Area of parallelogram $= |\vec{a} \times \vec{b}|$
- Volume of parallelepiped $= |\vec{a} \cdot (\vec{b} \times \vec{c})|$
- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ so order matters for the cross product
- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ so order doesn't matter for the dot product
- $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ so order doesn't matter for the triple product

SUMMARY

- 2020 Lecture notes
- https://www.youtube.com/watch?v=ZNbXTUTSVCY&t=1s

Lecture

$(0, 0, 0)$ $(0, 0, 1)$ $(0, 1, 0)$ $(1, 0, 0)$

(0) ellipsoid

- $(0, 0, 0)$ center of ellipsoid
- $(0, 0, 0)$ the origin, we can provide cross sections to the coordinate planes easy.
- intercepts: x, y, z axis are the points $(2, 0, 0), (0, 3, 0), (0, 0, 4)$
- Note: all the ellipses are centered at origin $(0, 0, 0)$
- Lecture: see notes of video
- Ellipse: bounded
- Ellipse are symmetric about all coordinate of plane

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \frac{(-x)^2}{a^2} + \frac{(-y)^2}{b^2} + \frac{(-z)^2}{c^2} = 1$$

$$\frac{(-x)^2}{a^2} + \frac{(-y)^2}{b^2} + \frac{z^2}{c^2} = 1$$

Ellipsoid

(1) cone

- intercept: $(0, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$
- Cross: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ x-y plane
- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$ x-y plane
- $\frac{x^2}{a^2} = \frac{z^2}{c^2}$ x-z plane
- $\frac{y^2}{b^2} = \frac{z^2}{c^2}$ y-z plane
- Lecture: elliptical x-y plane
- Bounded: $(0, 0, 0)$ is the center
- Lecture: $(0, 0, 0)$ is the center
- Lecture: $(0, 0, 0)$ origin
- Ellipse is symmetric about all coordinate plane

Hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

intercept: $(0, 0, 0)$, $(0, 0, 0)$

$$x^2 + y^2 = \frac{z^2}{c^2} \quad \text{not perfect, then need}$$

Cross: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ x-y plane

$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ x-z plane

$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ y-z plane

Lecture: Ellipse in the plane parallel to xy plane, xz and yz plane hyperbola

bounded: No

intercepts: $(0, 0, 0)$, $(0, 0, 0)$

Lecture: $(0, 0, 0)$ origin

Hyperboloid of two sheets

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

intercept: $(0, 0, 0)$, $(0, 0, 0)$

$$x^2 = \frac{y^2}{b^2} + \frac{z^2}{c^2} \quad \text{not perfect, then need}$$

Cross: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ x-y plane

$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ x-z plane

$-\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ no sol

Lecture: x-y and x-z plane hyperbola, y-z plane no sol

bounded: No

Elliptic paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z \quad \text{(Lecture)}$$

intercepts: $(0, 0, 0)$

Cross: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$ x-y plane

$\frac{x^2}{a^2} = z$ x-z plane

$\frac{y^2}{b^2} = z$ y-z plane

Lecture: x-y and y-z plane parabola, x-z plane hyperbola

bounded: No

Elliptic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z \quad \text{(Lecture)}$$

intercepts: $(0, 0, 0)$

Cross: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ x-y plane

$\frac{x^2}{a^2} = z$ x-z plane

$-\frac{y^2}{b^2} = z$ y-z plane

Lecture: x-y and y-z plane parabola, x-z plane hyperbola

bounded: No

Ruled surfaces

- Surface is a ruled surface if it can be formed by moving a straight line in space.
- Ruled surfaces: the generator is moving along a directrix.

Elliptic cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- Don't depend on z
- Ellipse if $a \neq b$
- Allow moving along the z-direction.

Hyperbolic cylinder

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

- Hyperbola in xy plane which moving along the z direction.

Hyperbolic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z \quad \text{(Lecture)}$$

intercepts: $(0, 0, 0)$

Cross: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ x-y plane, two lines

$\frac{x^2}{a^2} = z$ x-z plane, up parabola

$-\frac{y^2}{b^2} = z$ y-z plane, down parabola

Lecture: x-y plane two lines, x-z and y-z plane parabola

bounded: No