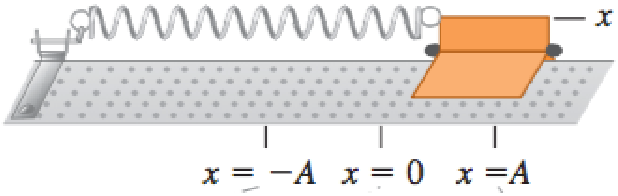
Chapter 14

Periodic Motion

Generalized SHM

- We have been looking at *one* situation in which SHM occurs: a body attached to an ideal horizontal spring.



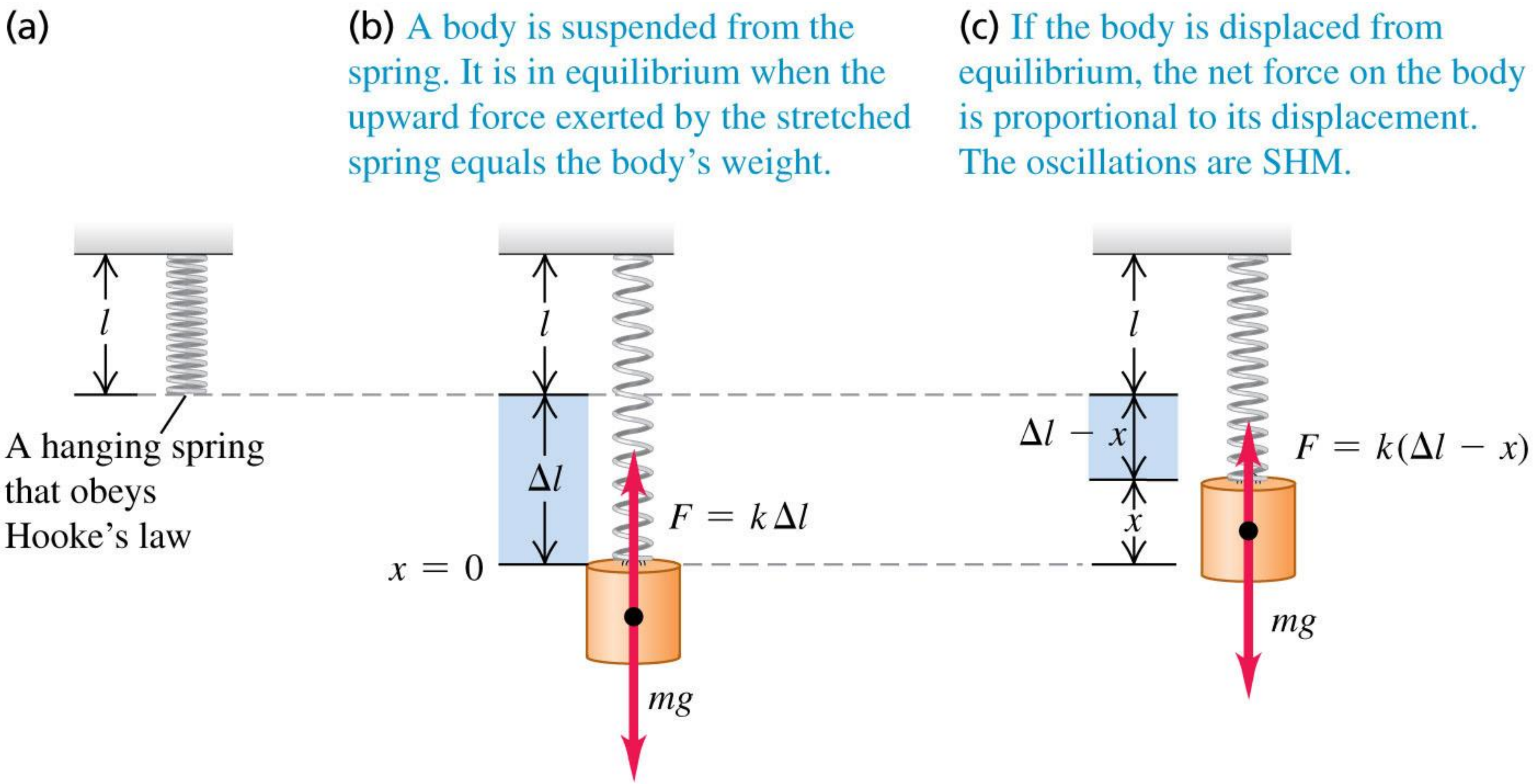
- But SHM can occur in any system in which there is a restoring force that is directly proportional to the displacement from equilibrium, $F_x = -kx$.

- The restoring force will originate in different ways in different situations, so the force constant k has to be found for each case by examining the net force on the system. Once k is found, we can get ω, f and T.

- Let's use these ideas to examine several examples of simple harmonic motion.

Vertical SHM

- If a body oscillates vertically from a spring, the restoring force has magnitude *kx*. Therefore the vertical motion is SHM.



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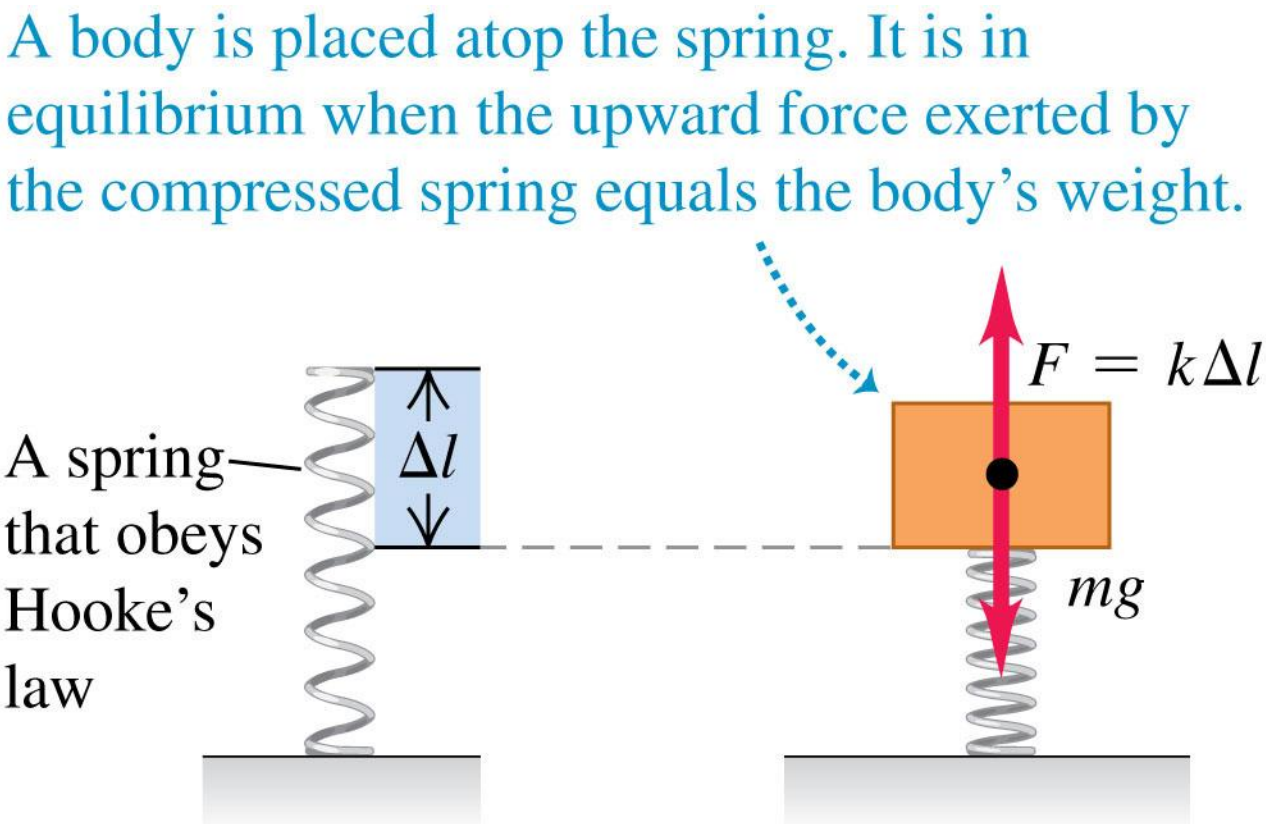
Vertical SHM

- If the weight mg compresses the spring a distance Δl, the force constant is $k = mg/\Delta l$.



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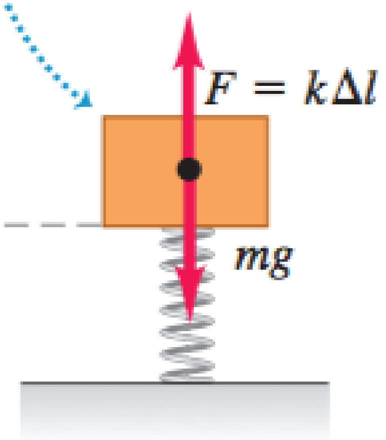
Example 14.6: Vertical SHM in an old car

The shock absorbers in an old car with mass 1000 kg are completely worn out. When a 980-N person climbs slowly into the car at its center of gravity, the car sinks 2.8 cm. The car (with the person aboard) hits a bump, and the car starts oscillating up and down in SHM. Model the car and person as a single body on a single spring, and find the period and frequency of the oscillation.

EXECUTE: When the force increases by 980 N, the spring compresses an additional 0.028 m, and the x-coordinate of the car changes by -0.028 m. Hence the effective force constant (including the effect of the entire suspension) is

$$k = -\frac{F_x}{x} = -\frac{980\ \text{N}}{-0.028\ \text{m}} = 3.5 \times 10^4\ \text{kg/s}^2$$



The person's mass is $w/g = (980\ \text{N})/(9.8\ \text{m/s}^2) = 100$ kg. The *total* oscillating mass is $m = 1000\ \text{kg} + 100\ \text{kg} = 1100$ kg. The period T is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1100\ \text{kg}}{3.5 \times 10^4\ \text{kg/s}^2}} = 1.11\ \text{s}$$

The frequency is $f = 1/T = 1/(1.11\ \text{s}) = 0.90$ Hz.

[Watch the solution on youtube]

Another Example

Test Your Understanding of Section 14.4 A block attached to a hanging ideal spring oscillates up and down with a period of 10 s on earth. If you take the block and spring to Mars, where the acceleration due to gravity is only about 40% as large as on earth, what will be the new period of oscillation? (i) 10 s; (ii) more than 10 s; (iii) less than 10 s.

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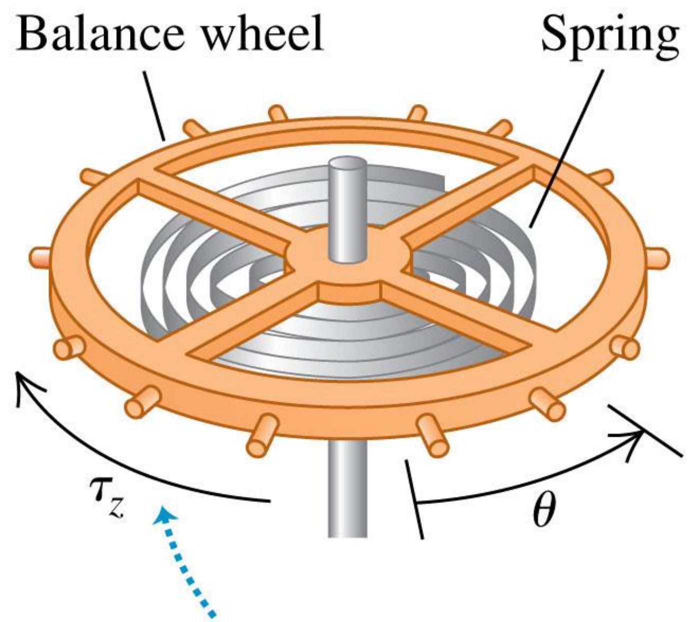
Angular SHM

- A mechanical watch keeps time based on the oscillations of a balance wheel (with moment of inertia I about its axis).
- A coil spring exerts a restoring torque τ_z that is proportional to the angular displacement θ from the equilibrium position.
- $\tau_z = -\kappa\theta$ (κ: torsion constant)
- Define $I = \sum_i m_i(x_i^2 + y_i^2)$ (sum is over all mass elements of the wheel)
- Then $\tau_z = I\alpha_z = Id^2\theta/dt^2$
- This is similar to SHM except it is now *angular*



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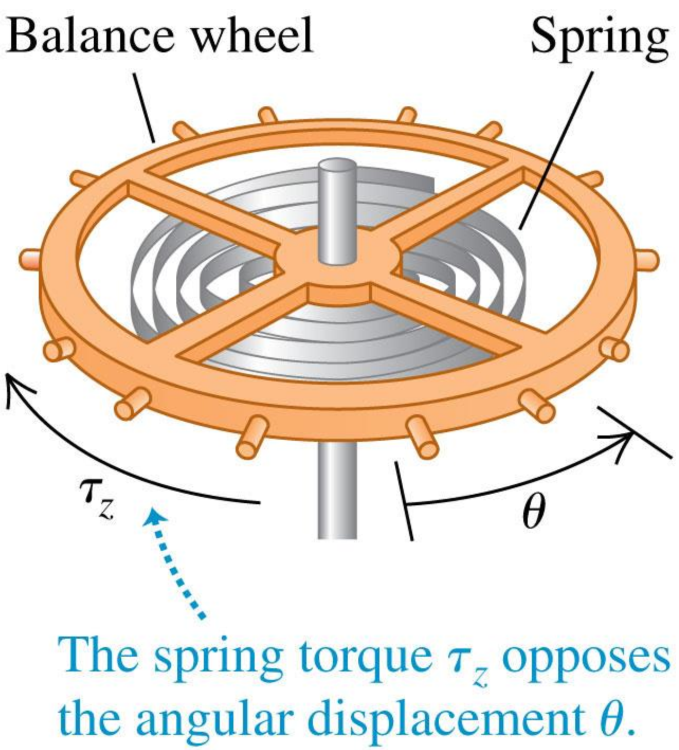
Angular SHM

- *Angular* simple harmonic motion:

$$\omega = \sqrt{\frac{\kappa}{I}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}}$$

- The motion is described by

$$\theta = \Theta \cos(\omega t + \phi)$$



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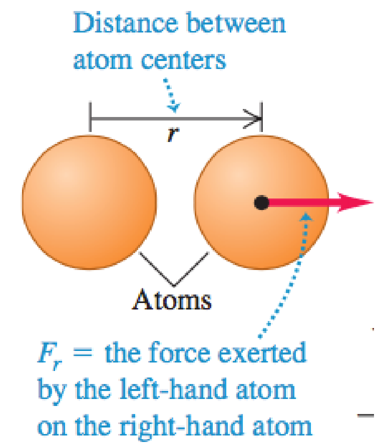
Vibrations of molecules

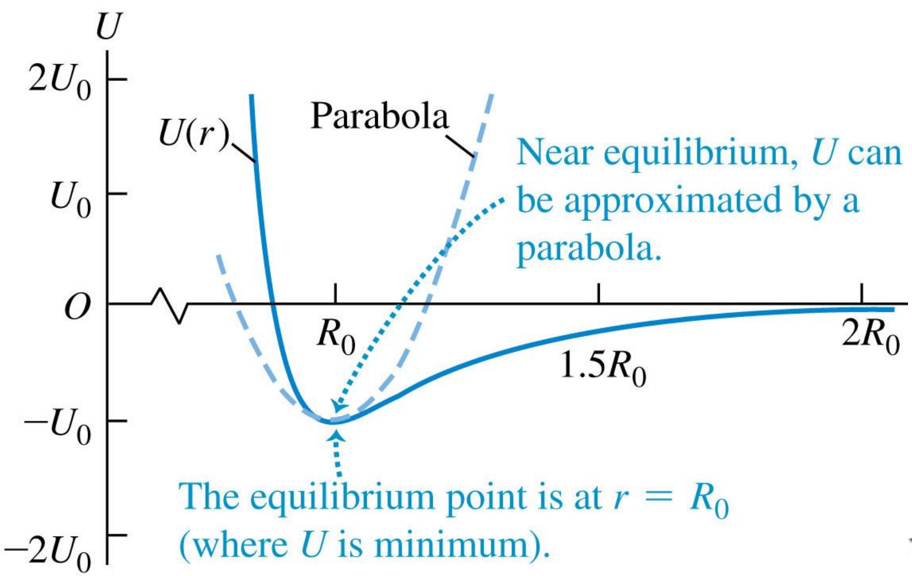
- Two atoms having centers a distance r apart, with the equilibrium point at $r = R_0$.

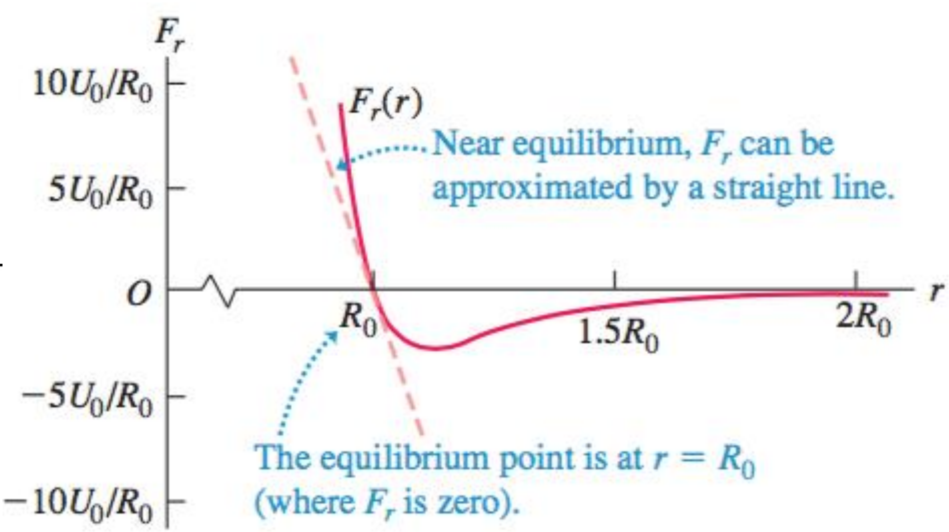


$$U = U_0\left[\left(\frac{R_0}{r}\right)^{12} - 2\left(\frac{R_0}{r}\right)^{6}\right] \qquad F_r = -\frac{dU}{dr}$$

Use $x = r - R_0$, and $(1+u)^n \approx 1 + nu$ for $|u| << 1$

$$F_r \approx -\left(\frac{72U_0}{R_0^2}\right)x$$, for small x ← Eq. for SHM with $k = 72\dfrac{U_0}{R_0^2}$

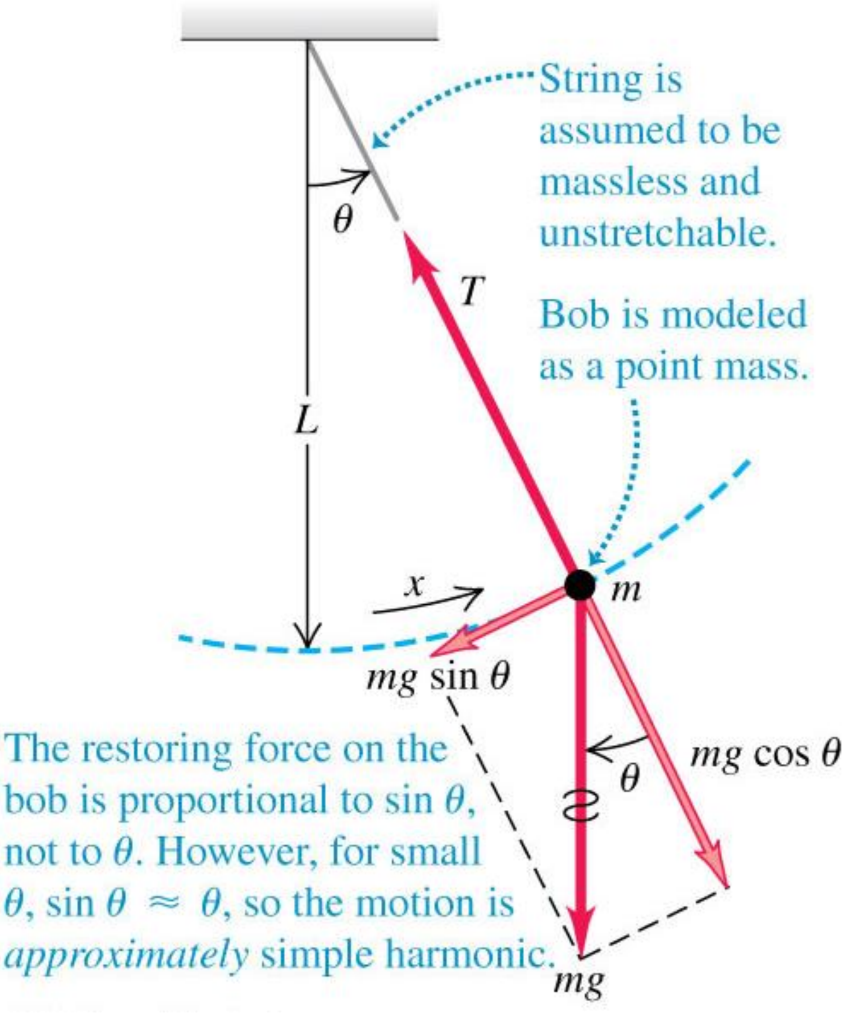




The simple pendulum

- A **simple pendulum** consists of a point mass (the bob) suspended by a massless, unstretchable string.

swing–a real pendulum



$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

The simple pendulum

- If the pendulum swings with a small amplitude θ with the vertical, its motion is simple harmonic.

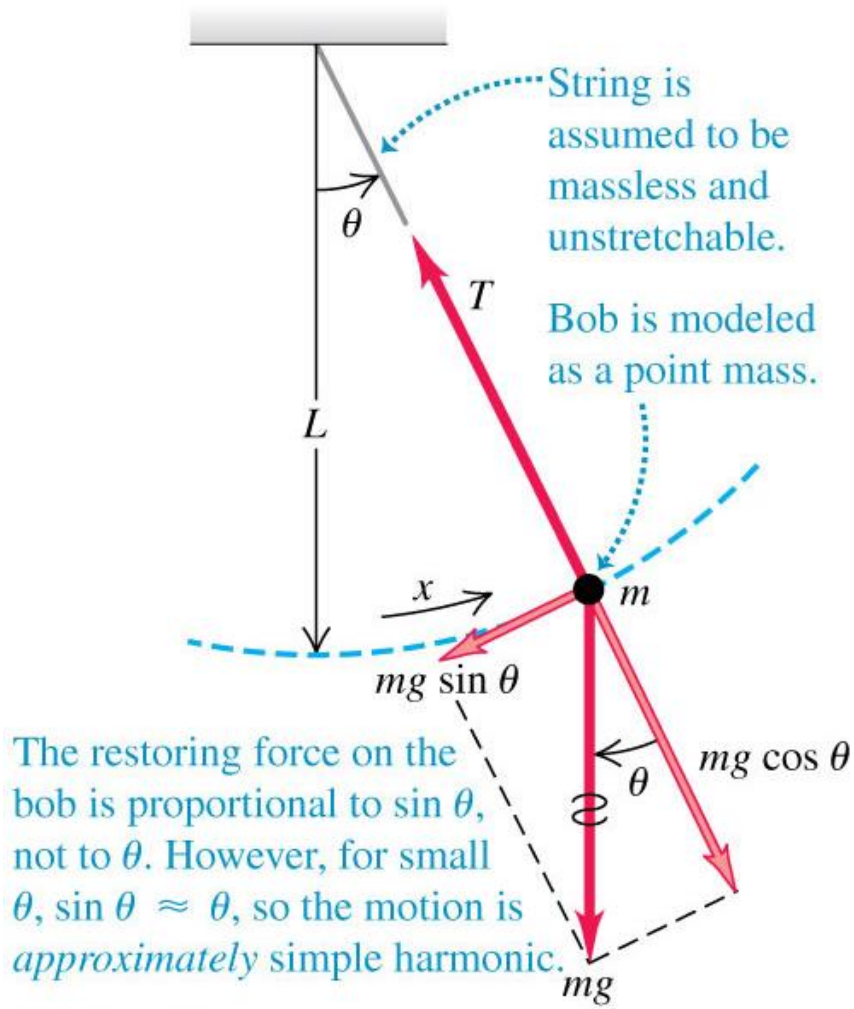
$$F_q = - mg\sin q \text{ » } mgq$$

$$F_q \text{ » } - mg\frac{x}{L}$$

for small x/L

Eq. for SHM with:

$$k = \frac{mg}{L} \qquad w = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{L}}$$



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The simple pendulum

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \sqrt{\frac{g}{L}}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}}$$

(simple pendulum, small amplitude)

- These expressions do not involve the *mass* of the particle. This is because the restoring force, a component of the particle's weight, is proportional to *m*.
- For small oscillations, the period of a pendulum for a given value of *g* is determined entirely by its length. A long pendulum has a longer period than a shorter one. Increasing *g* increases the restoring force, causing the frequency to increase and the period to decrease.

Q14.9

A simple pendulum consists of a point mass suspended by a massless, unstretchable string. If the mass is doubled while the length of the string remains the same, the period of the pendulum

A. becomes four times greater.

B. becomes twice as great.

C. becomes greater by a factor of $\sqrt{2}$.

D. remains unchanged.

E. decreases.

The simple pendulum

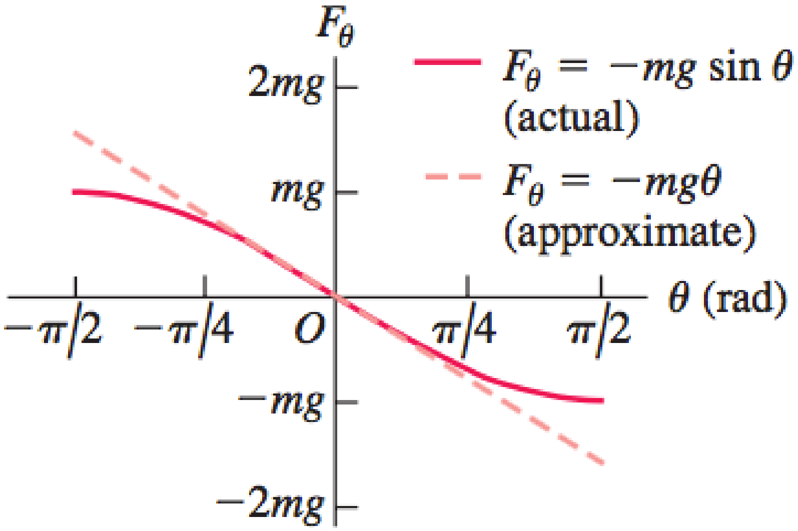
- We emphasize that the motion of a pendulum is only *approximately* simple harmonic. ($\sin\,\theta \approx \theta$)

- When the amplitude is not small, the departures from simple harmonic motion could be substantial. (how small is "small"?) The period can be expressed by an infinite series; when the maximum angular displacement is Θ, the period T is given by

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1^2}{2^2}\sin^2\frac{\Theta}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\sin^4\frac{\Theta}{2} + \cdots\right)$$

- Check: for $\Theta = 15°$, $\sin\Theta = 0.2588, \Theta = 0.2618$ (<1.2% difference)

- $T = T_0\,(1 + 0.00426 + 5.04\text{e-}7 + \ldots)$ (<0.5% difference)



Pendulum as timekeeper

- The usefulness of the pendulum as a timekeeper depends on the period being *very nearly* independent of amplitude, provided that the amplitude is small.
- Thus, as a pendulum clock runs down and the amplitude of the swings decreases a little, the clock still keeps *very nearly* correct time.

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Example 14.8

Find the period and frequency of a simple pendulum 1.000 m long at a location where $g = 9.800\ \mathrm{m/s^2}$.

EXECUTE: From Eqs. (14.34) and (14.1),

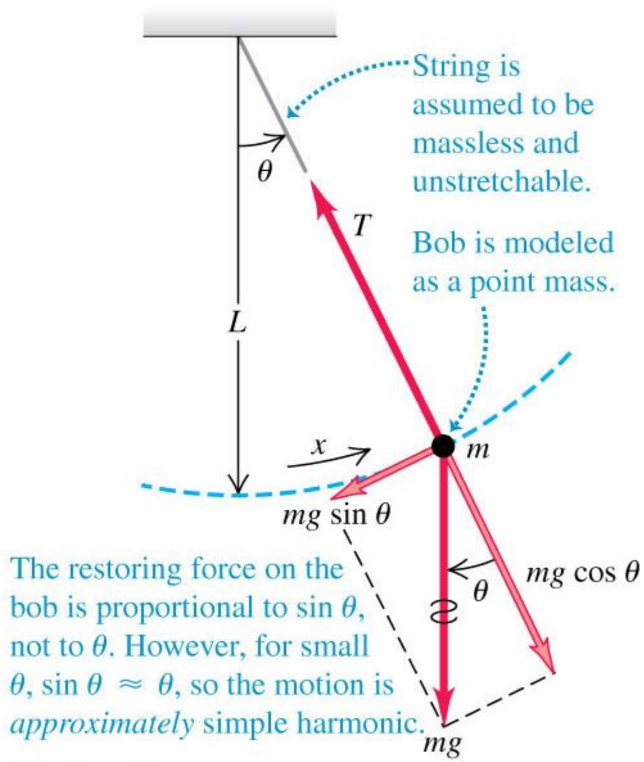
$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{1.000\ \mathrm{m}}{9.800\ \mathrm{m/s^2}}} = 2.007\ \mathrm{s}$$

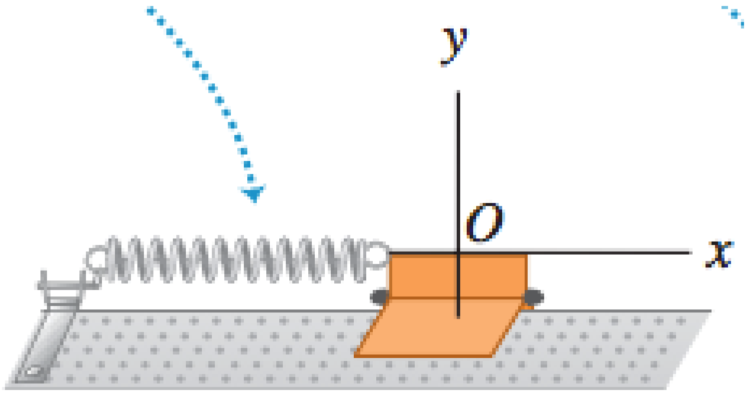
$$f = \frac{1}{T} = \frac{1}{2.007\ \mathrm{s}} = 0.4983\ \mathrm{Hz}$$

EVALUATE: The period is almost exactly 2 s. When the metric system was established, the second was *defined* as half the period of a 1-m simple pendulum. This was a poor standard, however, because the value of g varies from place to place.

Another Example

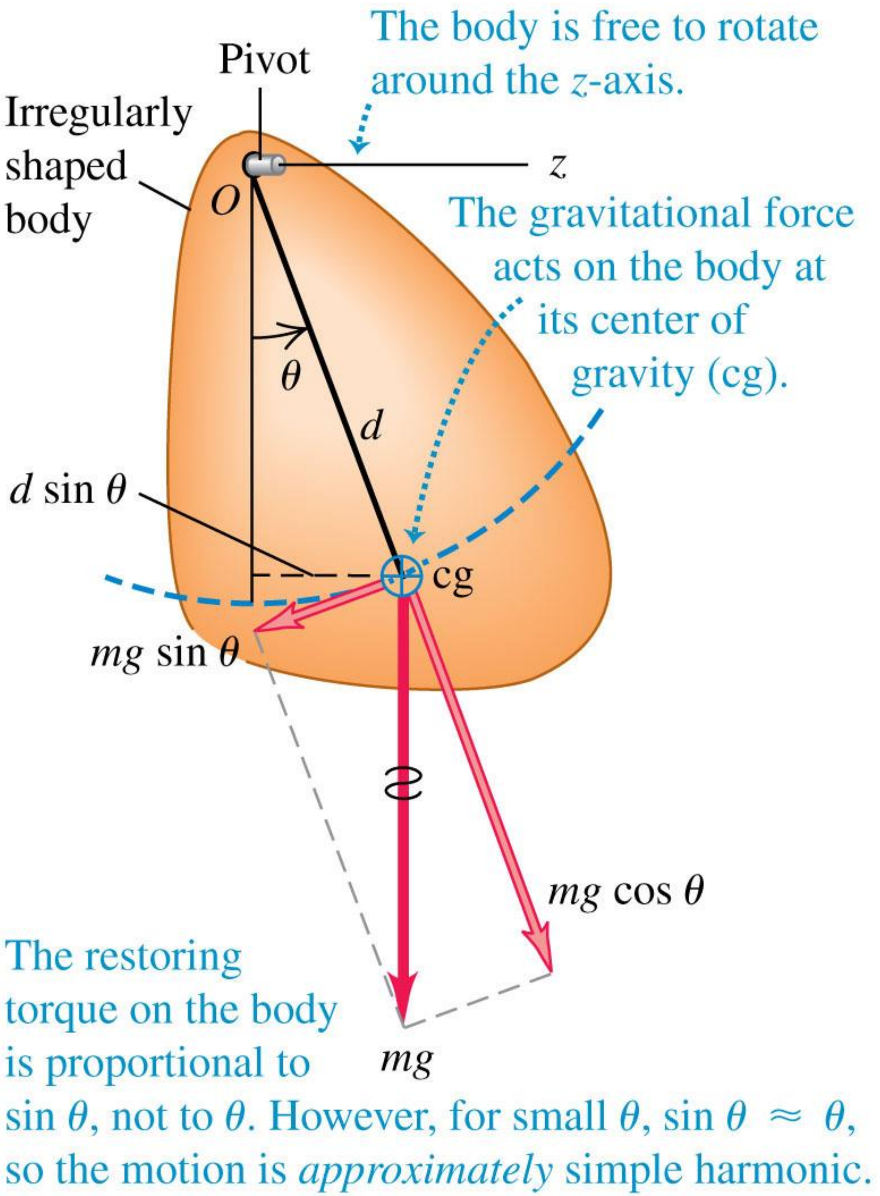
Test Your Understanding of Section 14.5 When a body oscillating on a horizontal spring passes through its equilibrium position, its acceleration is zero (see Fig. 14.2b). When the bob of an oscillating simple pendulum passes through its equilibrium position, is its acceleration zero?





The physical pendulum

- A **physical pendulum** is any real pendulum that uses an extended body instead of a point-mass bob.
- For small amplitudes, its motion is simple harmonic.



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The physical pendulum

Restoring Torque:

$$\tau_z = -(mg)(d\sin\theta) \approx -mgd\theta$$

Use: $\sum \tau_z = I\alpha_z$

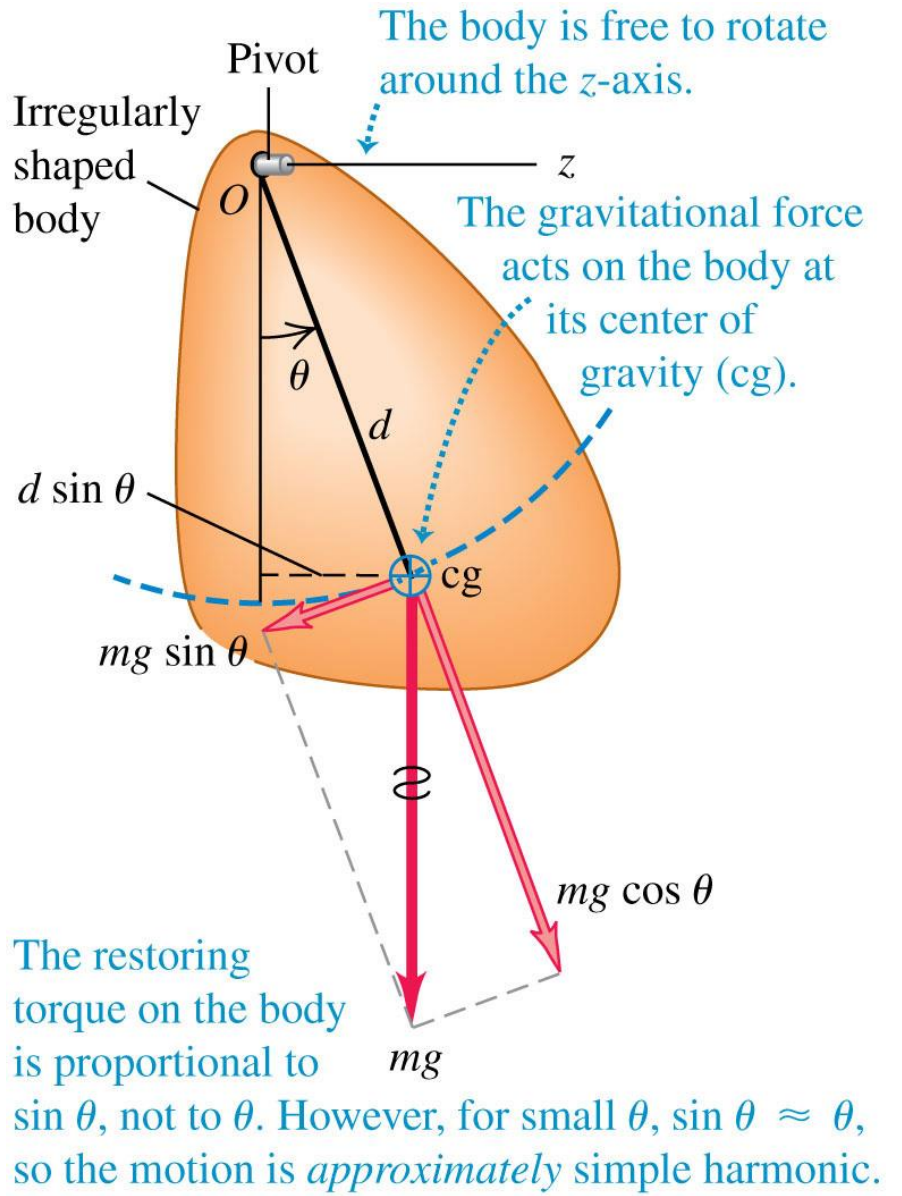
To get: $\dfrac{d^2\theta}{dt^2} = -\dfrac{mgd}{I}\theta$

(Compare with, $a_x = -\dfrac{k}{m}x$)

Angular Frequency for SHM,

$$\omega = \sqrt{\frac{mgd}{I}} \qquad T = 2\pi\sqrt{\frac{I}{mgd}}$$

(physical pendulum, small amplitude)



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Example 14.9

When a uniform rod with length L is pivoted at one end, what is the period of its motion as a pendulum? The moment of inertia of a uniform rod about an axis through one end is $I = ML^2/3$.

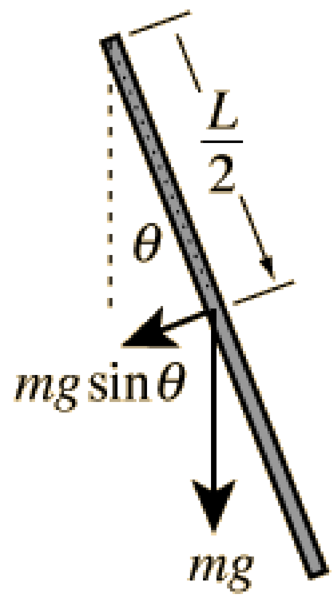
EXECUTE: The moment of inertia of a uniform rod about an axis through one end is $I = \frac{1}{3}ML^2$. The distance from the pivot to the rod's center of gravity is $d = L/2$. Then from Eq. (14.39),

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{MgL/2}} = 2\pi\sqrt{\frac{2L}{3g}}$$

EVALUATE: If the rod is a meter stick ($L = 1.00$ m) and $g = 9.80\ \text{m/s}^2$, then

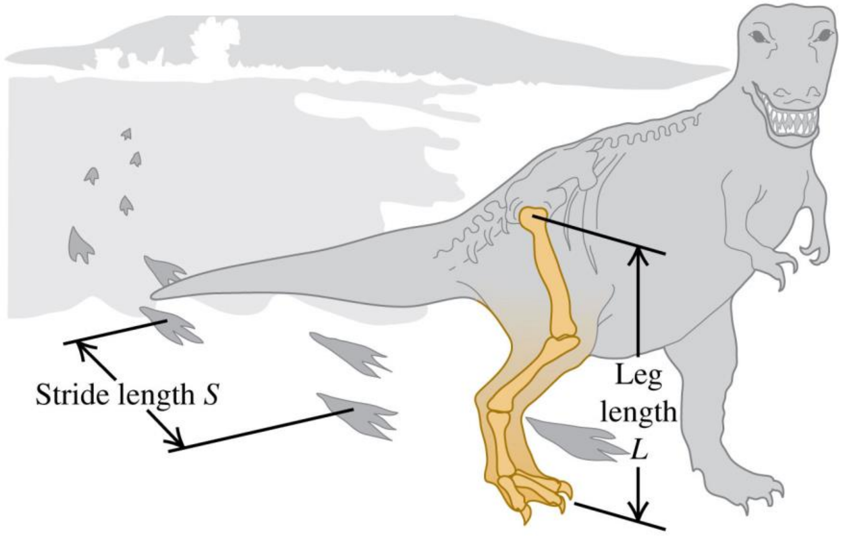
$$T = 2\pi\sqrt{\frac{2(1.00\ \text{m})}{3(9.80\ \text{m/s}^2)}} = 1.64\ \text{s}$$

The period is smaller by a factor of $\sqrt{\frac{2}{3}} = 0.816$ than that of a simple pendulum of the same length (see Example 14.8). The rod's moment of inertia around one end, $I = \frac{1}{3}ML^2$, is one-third that of the simple pendulum, and the rod's cg is half as far from the pivot as that of the simple pendulum. You can show that, taken together in Eq. (14.39), these two differences account for the factor $\sqrt{\frac{2}{3}}$ by which the periods differ.



Example 14.10: *Tyrannosaurus rex* and the physical pendulum

All walking animals, including humans, have a natural walking pace—a number of steps per minute that is more comfortable than a faster or slower pace. Suppose that this pace corresponds to the oscillation of the leg as a physical pendulum. (a) How does this pace depend on the length L of the leg from hip to foot? Treat the leg as a uniform rod pivoted at the hip joint. (b) Fossil evidence shows that *T. rex*, a two-legged dinosaur that lived about 65 million years ago, had a leg length $L = 3.1$ m and a stride length $S = 4.0$ m (the distance from one footprint to the next print of the same foot; see Fig. 14.24). Estimate the walking speed of *T. rex*.



Example 14.10: *Tyrannosaurus rex* and the physical pendulum

EXECUTE: (a) From Example 14.9 the period of oscillation of the leg is $T = 2\pi\sqrt{2L/3g}$, which is proportional to $\sqrt{L}$. Each step takes one-half a period, so the walking pace (in steps per second) is twice the oscillation frequency $f = 1/T$, which is proportional to $1/\sqrt{L}$. The greater the leg length L, the slower the walking pace.

(b) According to our model, *T. rex* traveled one stride length S in a time

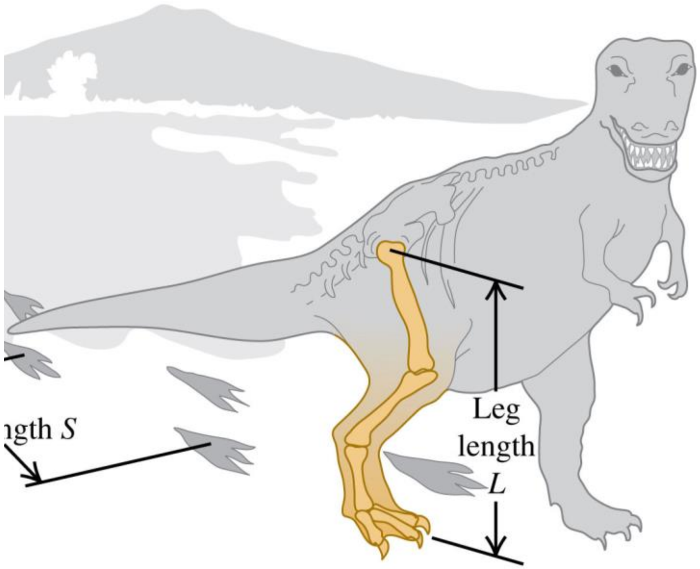
$$T = 2\pi\sqrt{\frac{2L}{3g}} = 2\pi\sqrt{\frac{2(3.1\ \text{m})}{3(9.8\ \text{m/s}^2)}} = 2.9\ \text{s}$$

so its walking speed was

$$v = \frac{S}{T} = \frac{4.0\ \text{m}}{2.9\ \text{s}} = 1.4\ \text{m/s} = 5.0\ \text{km/h}$$

This is roughly the walking speed of an adult human.

EVALUATE: A uniform rod isn't a very good model for a leg. The legs of many animals, including both *T. rex* and humans, are tapered; there is more mass between hip and knee than between knee and foot. The center of mass is therefore less than $L/2$ from the hip; a reasonable guess would be about $L/4$. The moment of inertia is therefore *considerably* less than $ML^2/3$—say, $ML^2/15$. Use the analysis of Example 14.9 with these corrections; you'll get a shorter oscillation period and an even greater walking speed for *T. rex*.



Damped oscillations

- Real-world systems have some dissipative forces that decrease the amplitude.
- The decrease in amplitude is called **damping** and the motion is called **damped oscillation**.
- The mechanical energy of a damped oscillator decreases continuously.

14.25 A swinging bell left to itself will eventually stop oscillating due to damping forces (air resistance and friction at the point of suspension).

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Damped oscillations

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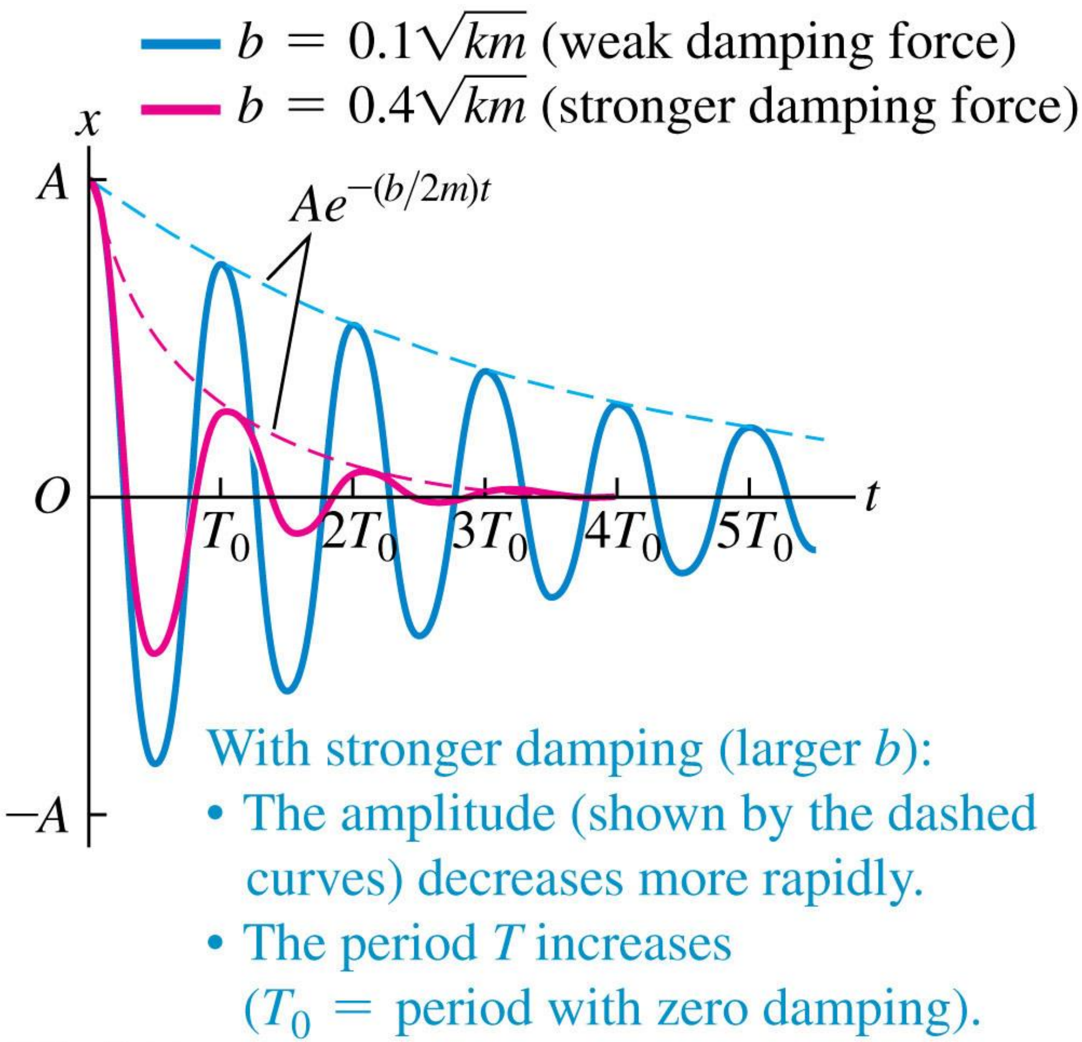

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Damped oscillations

$$\sum F_x = -kx - bv_x$$

$$-kx - bv_x = ma_x$$

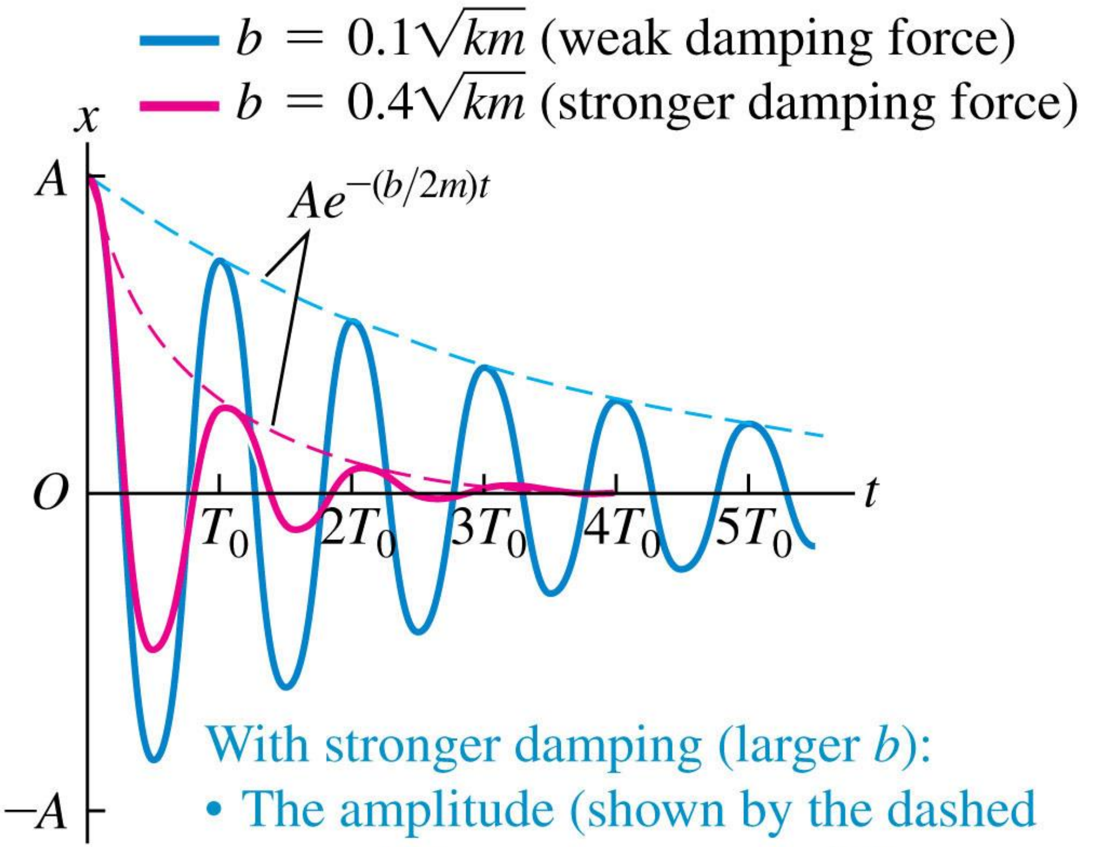
Frictional damping force

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

$$x = Ae^{-(b/2m)t}\cos(\omega' t + \phi)$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

(oscillator with little damping)



With stronger damping (larger b):

- The amplitude (shown by the dashed curves) decreases more rapidly.
- The period T increases (T_0 = period with zero damping).

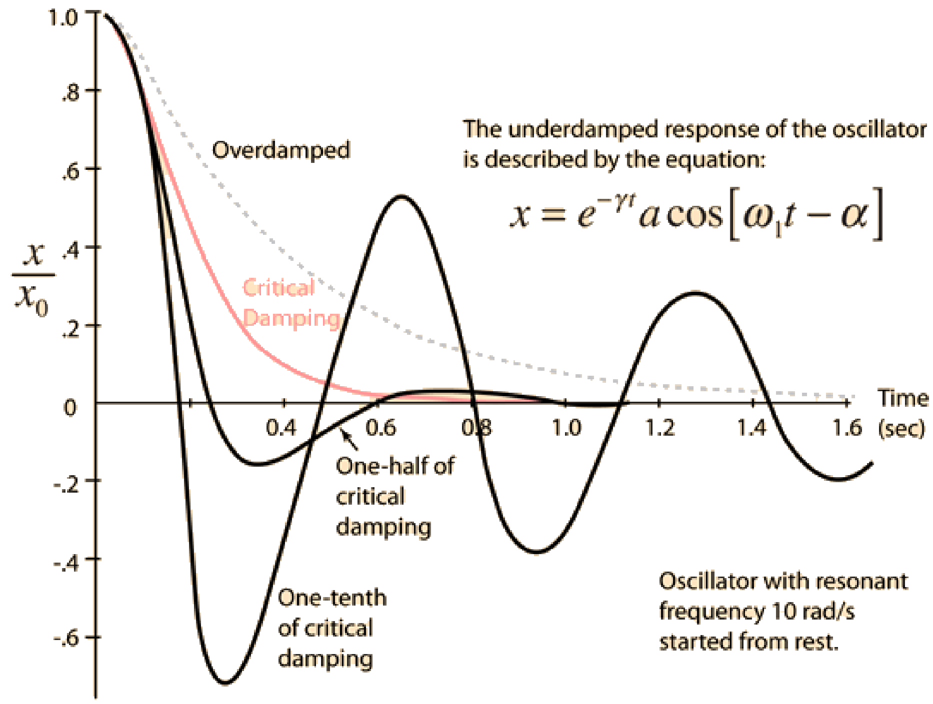
Under-, critical and over-damping

What happens when $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ is not a real number?

When $\omega' = 0$, oscillations disappears. x restores to equilibrium value fastest. (critical damping)

When ω' is imaginary, again there is no oscillation, but the system returns to equilibriums more slowly than with critical damping. (overdamping)

$\omega' > 0$ case is called "underdamping."



Q14.10

The force on a damped oscillator is $F_x = -kx - bv_x$. During its motion, the oscillator loses mechanical energy *most rapidly*

A. when it is at maximum positive displacement.

B. when it is at maximum negative displacement.

C. when it is passing through the equilibrium position.

D. when it is at either maximum positive or maximum negative displacement.

E. Misleading question—the oscillator loses mechanical energy at the same rate at all times during the motion.

Forced oscillations and resonance

- A damped oscillator left to itself will eventually stop moving.
- But we can maintain a constant-amplitude oscillation by applying a force that varies with time in a periodic way.
- We call this additional force a **driving force**.
- If we apply a periodic driving force with angular frequency ω_d to a damped harmonic oscillator, the motion that results is called a forced oscillation or a driven oscillation.

Maximum value of driving force

Amplitude of a forced oscillator

$$A = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}}$$

Damping constant

Force constant of restoring force

Mass

Driving angular frequency

(Remembering this equation is not required)

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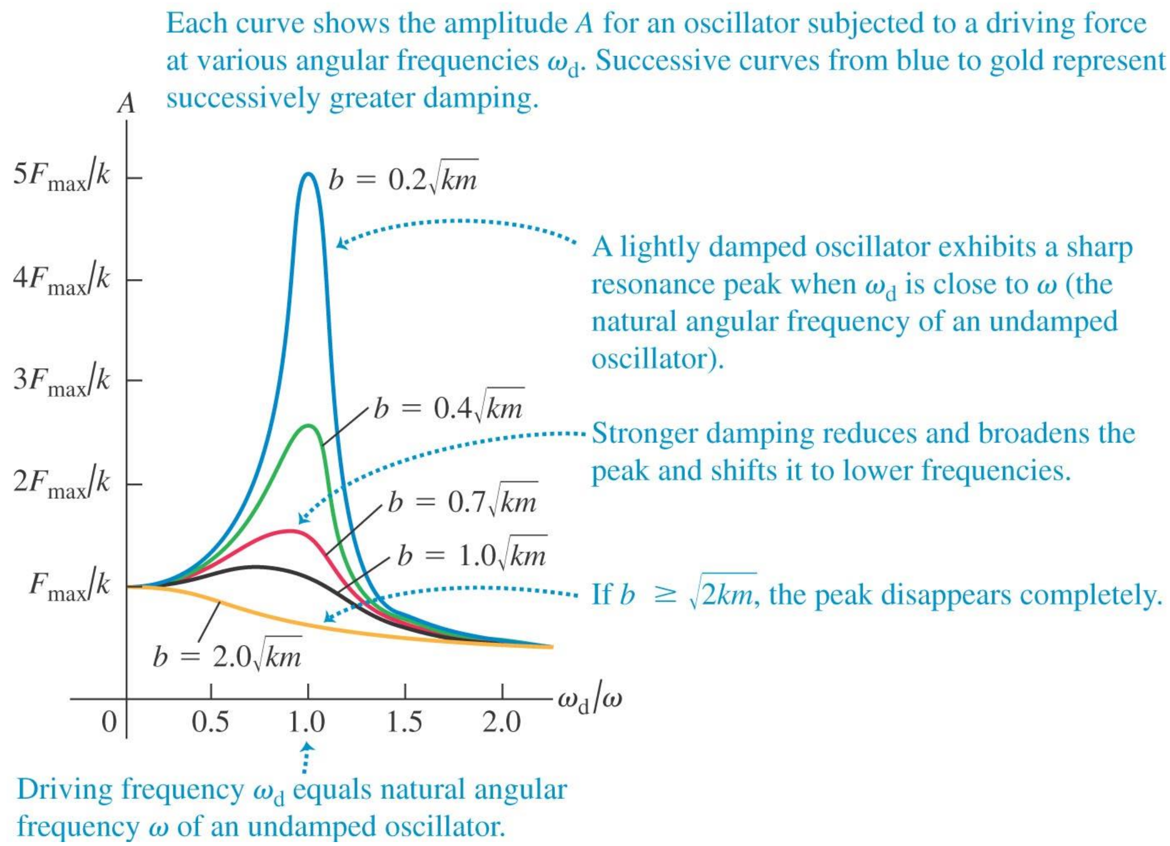
Forced oscillations and resonance

- A *forced oscillation* occurs if a *driving force* acts on an oscillator.
- *Resonance* occurs if the frequency of the driving force is near the *natural frequency* of the system. (See Figure below.)



Resonance can break things

Forced oscillations and resonance

- This lady beetle flies by means of a forced oscillation.
- Unlike the wings of birds, this insect's wings are extensions of its exoskeleton.
- Muscles attached to the inside of the exoskeleton apply a periodic driving force that deforms the exoskeleton rhythmically, causing the attached wings to beat up and down.
- The oscillation frequency of the wings and exoskeleton is the same as the frequency of the driving force.

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Application: Canine Resonance

Unlike humans, dogs have no sweat glands and so must pant in order to cool down. The frequency at which a dog pants is very close to the resonant frequency of its respiratory system. This causes the maximum amount of air to move in and out of the dog and so minimizes the effort that the dog must exert to cool itself.

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Q-RT14.1

Three identical oscillators have the same amplitude *A* and the same angular frequency ω. All three oscillate in simple harmonic motion given by $x = A\cos(\omega t + \phi)$. The three oscillators have different phase angles ϕ, however. **Rank** the three oscillators in order of their *displacement x* at $t = 0$, from most positive to most negative.

A. Oscillator with $\phi = 0$

B. Oscillator with $\phi = \pi/4$

C. Oscillator with $\phi = -\pi/2$

✔ Answer: ABC

Q-RT14.2

Three identical oscillators have the same amplitude *A* and the same angular frequency ω. All three oscillate in simple harmonic motion given by $x = A \cos(\omega t + \phi)$. The three oscillators have different phase angles ϕ, however. **Rank** the three oscillators in order of their *velocity* v_x at $t = 0$, from most positive to most negative.

A. Oscillator with $\phi = 0$

B. Oscillator with $\phi = \pi/4$

C. Oscillator with $\phi = -\pi/2$

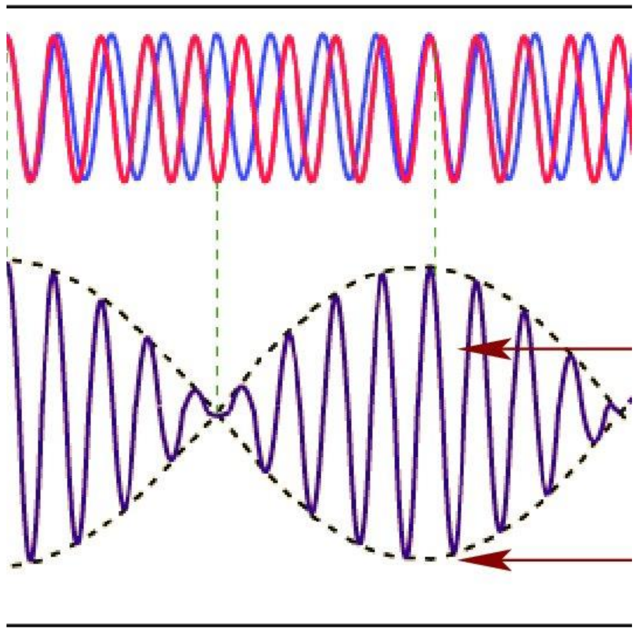
✔ Answer: CAB

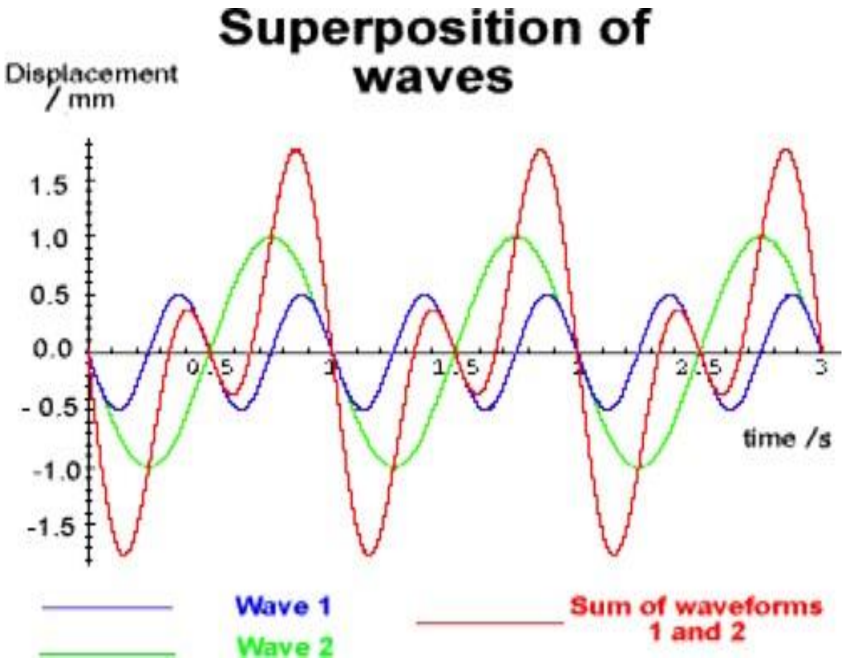
Simple harmonic waves

- Each point on a string is under simple harmonic motion.
- Points motion perpendicular to wave motion $\Longrightarrow$ transverse wave
- Points motion parallel to wave motion $\Longrightarrow$ longitudinal wave

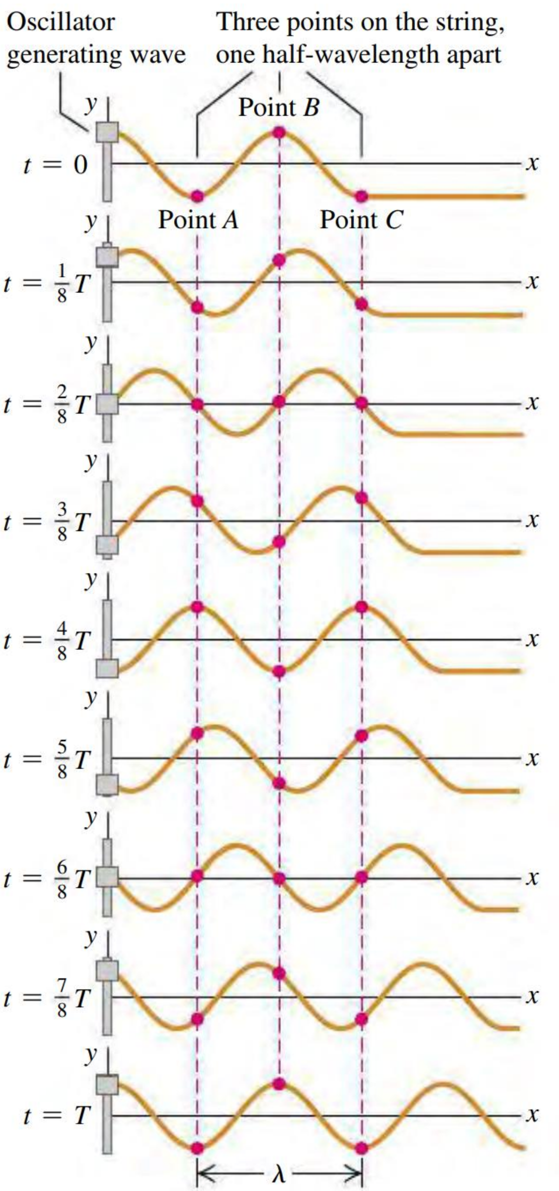
Superposition

- Simple harmonic waves can superpose to form complicated waves.
- In fact, any periodic wave can be represented as a combination of simple harmonic waves.





Formula for simple harmonic waves



- We learned: simple harmonic motion equation is $y = A\cos(\omega t + \phi)$
- Here each point is doing SHM, but has different ϕ. When x increases, ϕ decreases.
- $y(x,t) = A\cos(\omega t - kx)$
- k is called "wave number".
- Just like $\omega = 2\pi/T$, $k = 2\pi/\lambda$.
- Wave travels one wavelength in one period, so wave speed $v = \lambda/T$

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Example problem

- The speed of sound depends on temperature; at 20°C it is 344 m/s. What is the wavelength of a sound wave in air at 20°C if the frequency is 262 Hz (the approximate frequency of middle C on a piano)?

- Answer: $v = \frac{\lambda}{T}$, so

- $\lambda = vT = \frac{v}{f} = \frac{344\text{m/s}}{262\text{ Hz}} = \frac{344\text{m/s}}{262\text{ s}^{-1}} = 1.31\text{m}$

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Example problem

- Transverse waves on a string have wave speed 8.00 m/s, amplitude 0.0700 m, and wavelength 0.320 m. At $t = 0$ the x = 0 end of the string has its maximum upward displacement. (a) Find the frequency, period, and wave number of these waves. (b) Write a wave function describing the wave.

- Answer: (a) $T = \frac{\lambda}{v} = 0.0400$ s, $f = \frac{1}{T} = 25$ Hz, $k = \frac{2\pi}{\lambda} = 19.6$ m^{-1}

- (b) $\omega = 2\pi f = 157\ \text{s}^{-1}$

$$y = A\cos(\omega t - kx)$$
$$= (0.0700\ \text{m})\cos[(157\ \text{s}^{-1})t - (19.6\ \text{m}^{-1})x]$$

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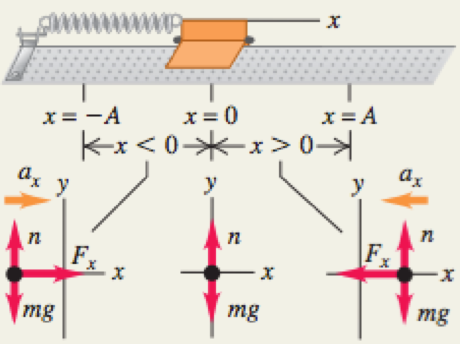
Ch. 14 Summary

Periodic motion: Periodic motion is motion that repeats itself in a definite cycle. It occurs whenever a body has a stable equilibrium position and a restoring force that acts when it is displaced from equilibrium. Period T is the time for one cycle. Frequency f is the number of cycles per unit time. Angular frequency ω is 2π times the frequency. (See Example 14.1.)

$$f = \frac{1}{T} \qquad T = \frac{1}{f} \tag{14.1}$$

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{14.2}$$



Simple harmonic motion: If the restoring force F_x in periodic motion is directly proportional to the displacement x, the motion is called simple harmonic motion (SHM). In many cases this condition is satisfied if the displacement from equilibrium is small. The angular frequency, frequency, and period in SHM do not depend on the amplitude, but only on the mass m and force constant k. The displacement, velocity, and acceleration in SHM are sinusoidal functions of time; the amplitude A and phase angle ϕ of the oscillation are determined by the initial position and velocity of the body. (See Examples 14.2, 14.3, 14.6, and 14.7.)

$$F_x = -kx \tag{14.3}$$

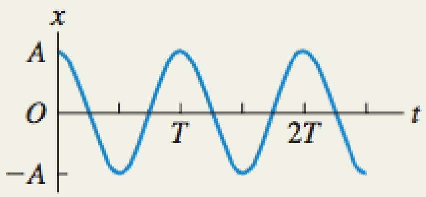
$$a_x = \frac{F_x}{m} = -\frac{k}{m}x \tag{14.4}$$

$$\omega = \sqrt{\frac{k}{m}} \tag{14.10}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{14.11}$$

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \tag{14.12}$$

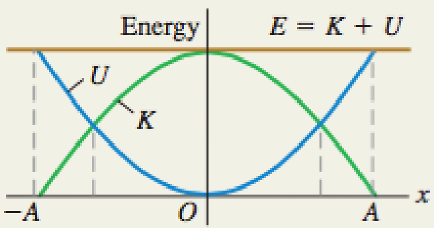
$$x = A\cos(\omega t + \phi) \tag{14.13}$$



Ch. 14 Summary

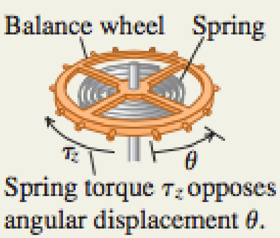
Energy in simple harmonic motion: Energy is conserved in SHM. The total energy can be expressed in terms of the force constant k and amplitude A. (See Examples 14.4 and 14.5.)

$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 = \text{constant} \tag{14.21}$$



Angular simple harmonic motion: In angular SHM, the frequency and angular frequency are related to the moment of inertia I and the torsion constant κ.

$$\omega = \sqrt{\frac{\kappa}{I}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}} \tag{14.24}$$

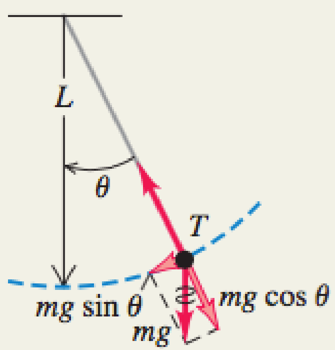


Simple pendulum: A simple pendulum consists of a point mass m at the end of a massless string of length L. Its motion is approximately simple harmonic for sufficiently small amplitude; the angular frequency, frequency, and period then depend only on g and L, not on the mass or amplitude. (See Example 14.8.)

$$\omega = \sqrt{\frac{g}{L}} \tag{14.32}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \tag{14.33}$$

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}} \tag{14.34}$$



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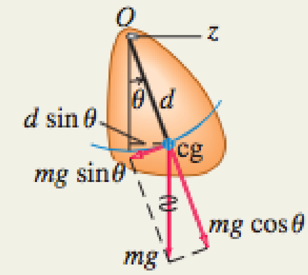
Physical pendulum: A physical pendulum is any body suspended from an axis of rotation. The angular frequency and period for small-amplitude oscillations are independent of amplitude, but depend on the mass m, distance d from the axis of rotation to the center of gravity, and moment of inertia I about the axis. (See Examples 14.9 and 14.10.)

$$\omega = \sqrt{\frac{mgd}{I}} \tag{14.38}$$

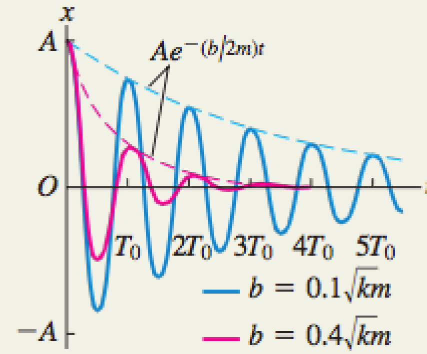
$$T = 2\pi\sqrt{\frac{I}{mgd}} \tag{14.39}$$



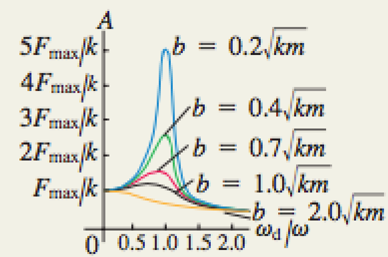
Damped oscillations: When a force $F_x = -bv_x$ proportional to velocity is added to a simple harmonic oscillator, the motion is called a damped oscillation. If $b < 2\sqrt{km}$ (called underdamping), the system oscillates with a decaying amplitude and an angular frequency ω' that is lower than it would be without damping. If $b = 2\sqrt{km}$ (called critical damping) or $b > 2\sqrt{km}$ (called overdamping), when the system is displaced it returns to equilibrium without oscillating.

$$x = Ae^{-(b/2m)t}\cos(\omega' t + \phi) \tag{14.42}$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{14.43}$$



Driven oscillations and resonance: When a sinusoidally varying driving force is added to a damped harmonic oscillator, the resulting motion is called a forced oscillation. The amplitude is a function of the driving frequency ω_d and reaches a peak at a driving frequency close to the natural frequency of the system. This behavior is called resonance.



Simple harmonic wave: Each point on a string is under simple harmonic motion.

$y(x,t) = A\cos(\omega t - kx)$, in which $k = 2\pi/\lambda$