1. (i). Let n be the number of partition, $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\frac{b-a}{n}$ $i=0,\cdots,n$. Then

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x .$$

(ii).

$$\lim_{n\to+\infty}\left[\frac{1}{\sqrt{n^2+n^4}} + \frac{2}{\sqrt{4n^2+n^4}} + \frac{3}{\sqrt{9n^2+n^4}} + \cdots + \frac{n}{\sqrt{n^4+n^4}}\right]$$

$$= \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{i}{\sqrt{i^2n^2+n^4}} = \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{n^2}\frac{i}{\sqrt{(\frac{i}{n})^2+1}}$$

$$= \lim_{n\to+\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)\frac{1}{\sqrt{(\frac{i}{n})^2+1}}\,\frac{1}{n}$$

$$= \int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx$$

(iii)

$$\int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1}\,\Big|_0^1 = \sqrt{2}-1$$

2. (i)

$$\int e^{-x}\cos(3x)\,dx$$

$$= -\int \cos(3x)\,de^{-x}$$

$$= -e^{-x}\cos(3x) + \int e^{-x}\,d\cos(3x)$$

$$= -e^{-x}\cos(3x) - 3\int e^{-x}\sin 3x\,dx$$

$$= -e^{-x}\cos 3x + 3\int \sin 3x\,d(e^{-x})$$

$$= -e^{-x}\cos 3x + 3e^{-x}\sin 3x - 3\int e^{-x}\,d\sin 3x$$

$$= -e^{-x}\cos 3x + 3e^{-x}\sin 3x - 9\int e^{-x}\cos 3x\,dx$$

$$\Downarrow$$

$$10\int e^{-x}\cos 3x\,dx = -e^{-x}\cos 3x + 3e^{-x}\sin 3x + C$$

$$\Downarrow$$

$$\int e^{-x}\cos 3x\,dx = -\frac{1}{10}e^{-x}\cos 3x + \frac{3}{10}e^{-x}\sin 3x + C.$$

(ii)

$$\int \frac{3x}{\sqrt{4x^2+1}}\,dx \overset{u=4x^2+1}{=} \frac{3}{8}\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{3}{4}u^{\frac{1}{2}} + C = \frac{3}{4}\sqrt{4x^2+1} + C$$

(iii) $\displaystyle\int \frac{x^2-5x-5}{(x-2)(x^2+2x+3)}\,dx$

$$\frac{x^2-5x-5}{(x-2)(x^2+2x+3)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+3}.$$

$$\Rightarrow \quad x^2-5x-5 = A(x^2+2x+3)+(Bx+C)(x-2).$$

Put $x=2$, $\quad A=-1$

Put $x=0$, $\quad -5=3A-2C \Rightarrow C=1$

Put $x=1$, $\quad -9=6A+(B+C)(-1) \Rightarrow B=2.$

Thus the integral can be computed as

$$\int \frac{x^2-5x-5}{(x-2)(x^2+2x+3)}\,dx = -\int \frac{1}{x-2}\,dx + \int \frac{2x+1}{x^2+2x+3}\,dx$$

$$= -\ln|x-2| + \int \frac{2x+2}{x^2+2x+3}\,dx - \int \frac{1}{x^2+2x+3}\,dx$$

$$= -\ln|x-2| + \ln|x^2+2x+3|$$

$$-\int \frac{1}{(x+1)^2+2}\,dx$$

$$= -\ln|x-2| + \ln|x^2+2x+3|$$

$$-\frac{1}{2}\int \frac{1}{\left(\frac{x+1}{\sqrt{2}}\right)^2+1}\,dx$$

$$= -\ln|x-2| + \ln|x^2+2x+3| - \frac{\sqrt{2}}{2}\tan\frac{x+1}{\sqrt{2}} + C.$$

(iv) $$\int_0^{+\infty} \frac{x+1}{(9+x^2)^{\frac{3}{2}}}\,dx$$

$$= \lim_{t\to+\infty} \int_0^t \frac{x+1}{(9+x^2)^{\frac{3}{2}}}\,dx$$

$$= \lim_{t\to+\infty} \int_0^t \frac{x}{(9+x^2)^{\frac{3}{2}}}\,dx + \int_0^t \frac{1}{(9+x^2)^{\frac{3}{2}}}\,dx$$

$$\lim_{t\to+\infty}\int_0^t \frac{x}{(9+x^2)}\,dx = \lim_{t\to+\infty} -(x^2+9)^{-\frac{1}{2}}\Big|_0^t$$

$$= 9^{-\frac{1}{2}} = \frac{1}{3}$$

$$\lim_{t\to+\infty}\int_0^t \frac{1}{(x^2+9)^{\frac{3}{2}}}\,dx \overset{x=3\tan\theta}{=} \lim_{t\to+\infty}\int_0^{\arctan\frac{t}{3}} \frac{3\sec^2\theta}{27\sec^3\theta}\,d\theta = \frac{1}{9}\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta$$

$$= \frac{1}{9}$$

Then $$\int_0^{+\infty} \frac{x+1}{(9+x^2)^{\frac{3}{2}}}\,dx = \frac{1}{9} + \frac{1}{3} = \frac{4}{9}.$$

3. The surface area

$$= 2\pi \int_0^4 \sqrt{4-x} \sqrt{1 + \left(\frac{d}{dx}\sqrt{4-x}\right)^2}\, dx$$

$$= 2\pi \int_0^4 \sqrt{4-x} \sqrt{1 + \frac{1}{4(4-x)}}\, dx$$

$$= \pi \int_0^4 \sqrt{17-4x}\, dx$$

$$= \pi \left(-\frac{1}{4} \frac{(17-4x)^{\frac{3}{2}}}{\frac{3}{2}} \right) \Bigg|_0^4 = \frac{\pi}{6}\left(17^{\frac{3}{2}} - 1\right).$$

4 (i)

$$\int_a^b x f'(x)\,dx = x f(x) \Big|_a^b - \int_a^b f(x)\,dx$$

$$= b f(b) - a f(a) - \int_a^b f(x)\,dx$$

$$= b - a$$

(ii)

$$\int_0^1 x(1-x) f''(x)\,dx = x(1-x) f'(x) \Big|_{x=0}^{x=1} - \int_0^1 f'(x)\, d[x(1-x)]$$

$$= 0 - 0 - \int_0^1 f'(x)(1-2x)\,dx$$

$$= \int_0^1 (2x-1) f'(x)\,dx$$

$$= (2x-1) f(x) \Big|_0^1 - \int_0^1 f(x)\,d(2x-1)$$

$$= f(1) - (-f(0)) - 2\int_0^1 f(x)\,dx$$

$$= 2 - 2 = 0.$$