MA1300 Brief Solution of Hand-in Assignment 5

1. Find the limit

$$\lim_{x\to\infty}(1+4/x)^x$$

$$\lim_{x\to 0}\frac{e^x-1}{\tan x}$$

$$\lim_{x\to 0^+} x^2\ln x$$

Ans:

$$\lim_{x\to\infty}(1+4/x)^x = e^4$$

$$\lim_{x\to 0}\frac{e^x-1}{\tan x} = 1$$

$$\lim_{x\to 0^+} x^2\ln x = 0$$

2. Find the derivatives dy/dx

$$y = 3^{x\ln x}$$

$$xe^y = y-1$$

$$y = x^{2x}$$

Ans:

$$y' = 3^{x\ln x}\ln 3(\ln x+1)$$

$$y' = e^y/(1-xe^y)$$

$$y' = x^{2x}(2\ln x+2)$$

3. Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty}\frac{n}{n^3+1}$$

$$\sum_{n=1}^{\infty}\frac{n^3}{5^n}$$

$$\sum_{n=1}^{\infty}\frac{\cos 3n}{1+(1.2)^n}$$

$$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{3n^2+1}$$

Ans:

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1} \text{ converges (by comparison test and p-series test)}$$

$$\sum_{n=1}^{\infty} \frac{n^3}{5^n} \text{ converges (by ratio test)}$$

$$\sum_{n=1}^{\infty} \frac{\cos 3n}{1+(1.2)^n} \text{ converges (by taking absolute value and comparison test)}$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \text{ converges}$$

(by multiplying $\sqrt{n+1}+\sqrt{n-1}$ to the numerator and denominator,

then applying comparison test and p-series test)

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{3n^2+1} \text{ converges (by alternating series test)}$$

4. Find the radius of convergence and interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{2^n}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{3n^2+1}$$

Ans:

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{2^n}$$

The radius of convergence $= \sqrt{2}$ and the interval of convergence $= (-\sqrt{2}, \sqrt{2})$.

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{3n^2+1}$$

The radius of convergence $= 1$ and the interval of convergence $= [-1, 1]$.

5. If $\{a_n\}$ is convergent, using the definition to show that

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n.$$

Ans: Set $\lim\{a_n\} = L$. It means that for all $\epsilon > 0$, there exists an integer $N > 0$ such that

$$\text{if } n > N, \text{ then } |a_n - L| < \epsilon.$$

If $|a_n - L| < \epsilon$ for all $n > N$, then we also have $|a_{n+1} - L| < \epsilon$ for all $n > N$ Overall, for all $\epsilon > 0$, there exists an integer $N > 0$ such that

$$\text{if } n > N, \text{ then } |a_{n+1} - L| < \epsilon.$$

We proved that $\{a_{n+1}\}$ is convergent and

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n.$$

6. If $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, using the definition to show that

$$\lim_{n\to\infty} a_n = L$$

Ans: As $\lim_{n\to\infty} a_{2n} = L$, then for all $\epsilon > 0$, there exists an integer $N_1 > 0$ such that

$$\text{if } n > N_1, \text{ then } |a_{2n} - L| < \epsilon;$$

as $\lim_{n\to\infty} a_{2n+1} = L$, then for all $\epsilon > 0$, there exists an integer $N_2 > 0$ such that

$$\text{if } n > N_2, \text{ then } |a_{2n+1} - L| < \epsilon.$$

So, for all $\epsilon > 0$, there exists an integer $N = 2\max\{N_1, N_2\} + 1 > 0$ such that

$$\text{if } n > N, \text{ then } |a_n - L| < \epsilon.$$

It means that

$$\lim_{n\to\infty} a_n = L$$

7. Show that the sequence defined by

$$a_1 = 1, \ a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $a_n < 3$ for all n. By the Monotonic Sequence Theorem, prove that $\{a_n\}$ is convergent and find its limit.

Ans: The limit is $(3 + \sqrt{5})/2$.

End