Chapter 4. Continuous Random Variables

4.1 Continuous random variables

For continuous random variables, we focus on the *probability density function (pdf)* $f(x)$

Definition: X is said to be a *continuous* random variable if there exists a nonnegative function f, defined for all real $x \in (-\infty, \infty)$, having the property that for any set B of real number, the probability that X will be in a set B is $P\{X \in B\} = \int_B f(x)\,dx$

$f(x)$ is called pdf of X.

Since *X* must take on some value (in this course it is usually assumed that *X* cannot be infinite) , the pdf must satisfy: $1 = P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)dx$

The converse is also true: a nonnegative function *f* is a pdf for some continuous r.v. *X* if and only if $\int_{-\infty}^{\infty} f(x)\, dx = 1$

All probability statements about *X* can be answered by using the pdf, for example:

$$
\begin{aligned}
P\{a \le X \le b\} &= \int_a^b f(x)\,dx \\
P\{X = a\} &= \int_a^a f(x)\,dx = 0 \\
P\{X < a\} &= P\{X \le a\} = \int_{-\infty}^a f(x)\,dx
\end{aligned}
$$

The last line above is the CDF.

Relationship between pdf and cdf: The relationship between the pdf and cdf is expressed by

$$F(a) = P\{X \leq a\} = \int_{-\infty}^{a} f(x)\, dx$$

$$\frac{d}{da} F(a) = f(a)$$

Interpretation of the pdf: Note that

$$P\{a - \frac{\epsilon}{2} \leq X \leq a + \frac{\epsilon}{2}\} = \int_{a-\epsilon/2}^{a+\epsilon/2} f(x)\,dx \sim \epsilon f(a)$$

when ϵ is small and when *f* (·) is continuous at *x*=*a*. So the probability that *X* will be contained in an interval of length ε around the point *a* is approximately $\epsilon f(a)$

Example : The lifetime in hours of a certain kind of radio tube is a random variable having a probability density function given by $f(x) = \begin{cases} 0 & x \leq 100 \\ 100/x^2 & x > 100 \end{cases}$

What is the probability that exactly 2 of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation? Assume that the events $E_i, i = 1, 2, 3, 4, 5,$ that the *i*th such tube will have to be replaced within this time, are independent.

More examples regarding CDF

Example: which of the following is a cdf:

$$(A)\ F(x) = \begin{cases} 0 & x < -2 \\ 1/2 & -2 \leq x < 0 \\ 2 & x \geq 0 \end{cases}$$

$$(B)\ F(x) = \begin{cases} 0 & x < 0 \\ \sin x & 0 \leq x < \pi \\ 1 & x \geq \pi \end{cases}$$

$$(C)\ F(x) = \begin{cases} 0 & x < 0 \\ \sin x & 0 \leq x < \pi/2 \\ 1 & x \geq \pi/2 \end{cases}$$

$$(D)\ F(x) = \begin{cases} 0 & x < 0 \\ x - 1/3 & 0 \leq x < 1/2 \\ 1 & x \geq 1/2 \end{cases}$$

Example: The cumulative distribution function for some random variable X is

$$F(x) = \begin{cases} 0, & x < -1 \\ a, & -1 \le x < 1 \\ \frac{2}{3} - a, & 1 \le x < 2 \\ a + b, & x \ge 2 \end{cases}$$

and $P(X=2)=1/2$. What is the value of a and b?

Example: A continuous random variable X has the following CDF:

$$F(x) = \begin{cases} 0 & x < a \\ x^2 + c & a \leq x \leq b \\ 1 & x > b \end{cases}$$

And $P\{X>1/2\}=3/4$. Find a,b,c.

Example: The cdf of a r.v. is

$$F(x) = \begin{cases} 0 & x < -1 \\ 0.3 & -1 \leq x < 0 \\ 0.6 & 0 \leq x < 1 \\ 0.8 & 1 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

Compute $P(X < 1 | X \neq 0)$

Solution:

Expectation and Variance

Definition: For a discrete random variable, the expected value is $E[X] = \sum_x x P\{X = x\}$

For a continuous random variable X with pdf *f* (*x*),

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

The variance of a continuous random variable is

$$Var(X) = E(X - E[X])^2 = E[X^2] - (E[X])^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - (E[X])^2$$

Proposition If X is a continuous random variable with pdf $f(x)$, then for any real-valued function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

Proposition: If a and b are constants, then

$$\begin{aligned} E[aX + bY] &= aE[X] + bE[Y] \\ Var(aX + b) &= a^2 Var(X) \end{aligned}$$

Example: Find *E[X]* and *Var[X]* when the density function of X is

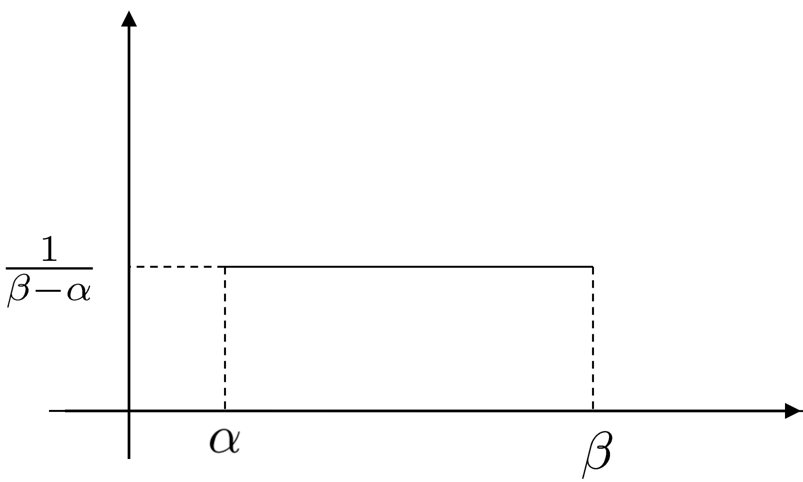
$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

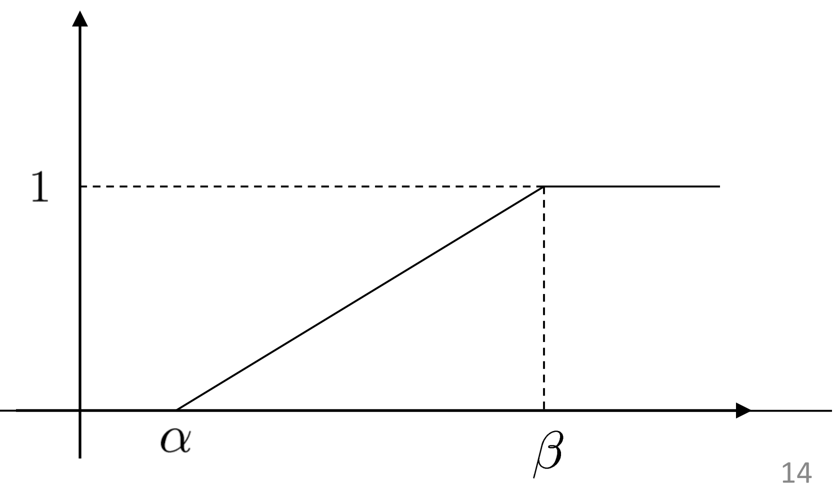
4.2 Special continuous r.v.

Uniform Random Variables

A random variable is said to be uniformly distributed over the interval (α, β) or $[\alpha, \beta]$ if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$



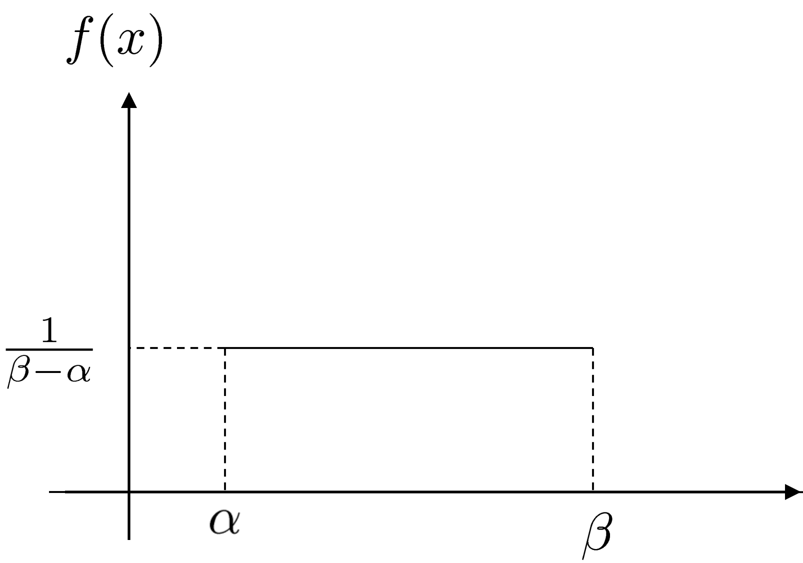


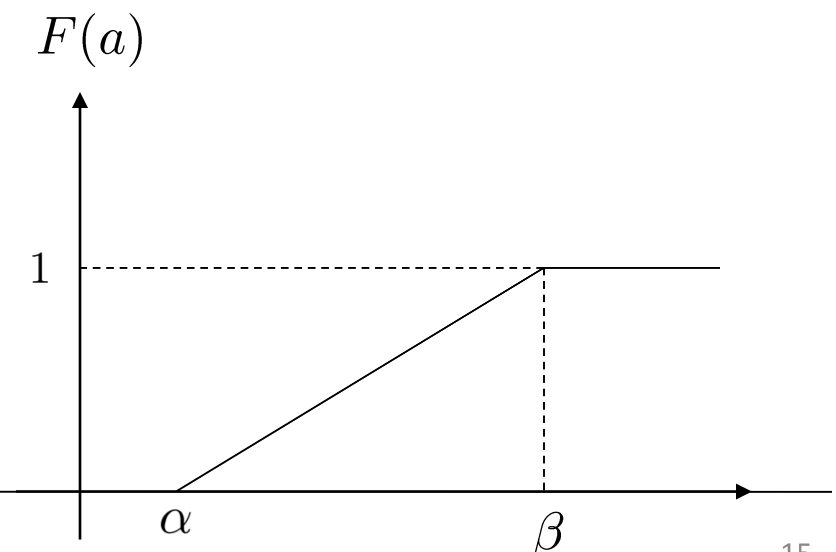
Note: This is a valid pdf because

$$f \geq 0 \text{ and } \int f(x)\,dx = \int_{\alpha}^{\beta} \frac{1}{\beta-\alpha} dx = 1.$$

The CDF of a uniform random variable is

$$F(x) = \begin{cases} 0 & x \leq \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \alpha < x < \beta \\ 1 & x \geq \beta \end{cases}$$





Example : Show that the expectation and variance of a random variable that is uniformly distributed on (α, β) is $E[X] = \frac{\beta+\alpha}{2}, Var(x) = \frac{(\beta-\alpha)^2}{12}$

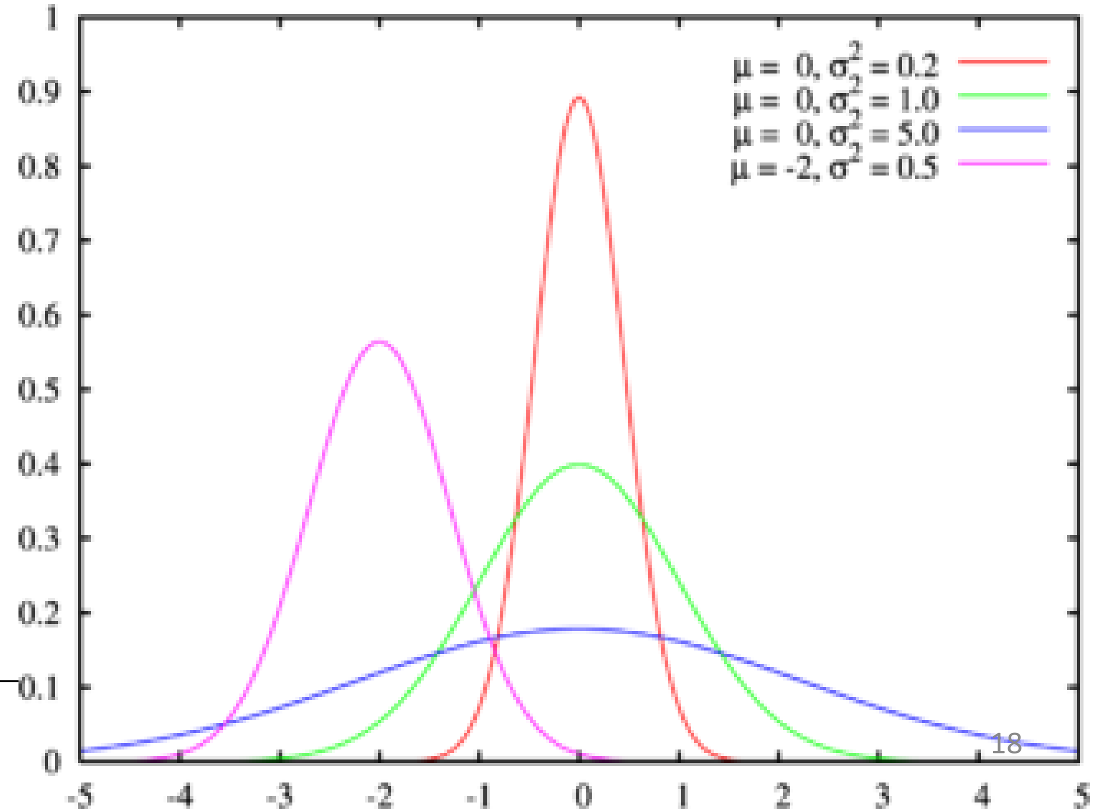
Example : Buses arrive at a specified stop at 15-minute intervals starting at 7 a.m. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that she waits less than 5 minutes for a bus.

Solution:

Normal Random Variables

Definition: We say that *X* is a normal/Gaussian random variable with parameters μ and σ^2 if the pdf of *X* is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$



(1) It can be checked that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}\, dx = 1$ but this is totally nontrivial, which uses very uncommon techniques.

(2) The central limit theorem justifies the use of the normal distribution in many applications. Roughly, the central limit theorem says that if a random variable is the sum of a large number of independent random variables, it is approximately normally distributed.

(3) Sometimes normal distribution is used to model logarithmic return: $X = \log(S_1/S_0)$

Linear transformations of normal random variables:

An important fact about normal random variables is that if *X* is normally distributed with parameters $\mu \text{ and } \sigma^2$, then *Y=aX+b* is normally distributed with parameters $a\mu + b \text{ and } a^2\sigma^2$

An important implication of the preceding result is that if X is normally distributed with parameters $\mu \text{ and } \sigma^2$, then $Z = (X - \mu)/\sigma$ is normally distributed with parameters 0 and 1. Such a random variable is said to be a *standard normal random variable.*

Mean and variance of normal random variables:

We start by finding the mean and variance of the standard normal random variable $Z = (X - \mu)/\sigma$. We have…

$$Var(Z) = E[Z^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx = 1$$

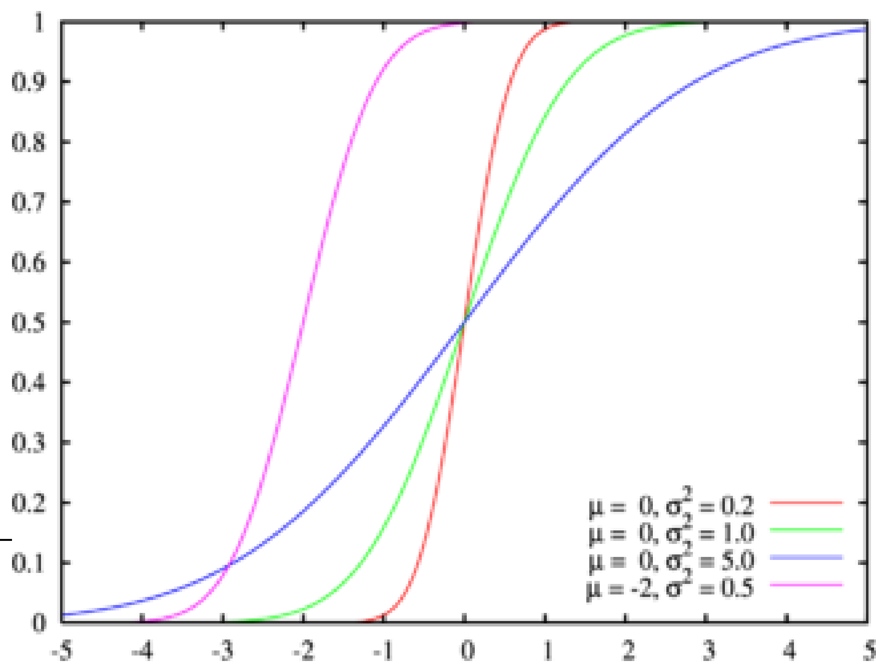
Because $X = \mu + \sigma Z$, we have

$$E[X] = \mu + \sigma E[Z] = \mu \text{ and } Var(X) = \sigma^2 Var(Z) = \sigma^2$$

CDF of a standard normal distribution: $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$

One property of Φ is that $\Phi(-x) = 1 - \Phi(x)$

Proof: Since the distribution of standard Gaussian r.v. is symmetric $P\{X \le -x\} = P\{X \ge x\}$, by the definition of CDF, $P\{X \ge x\} = 1 - \Phi(x)$, $P\{X \le -x\} = \Phi(-x)$



Suppose *X* is a normal r.v. with parameters $\mu \text{ and } \sigma^2$

Since $Z = (X - \mu)/\sigma$ is a standard normal random variable

$$F_X(x) = P\{X \leq x\} = P(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}) = P\{Z \leq \frac{x - \mu}{\sigma}\} = \Phi(\frac{x - \mu}{\sigma})$$

That is, the CDF of a normal r.v. with parameters $\mu \text{ and } \sigma^2$ is $\Phi(\frac{x-\mu}{\sigma})$

Example : If X is normal random variable with parameters $\mu = 3$ and $\sigma = 3$ (i.e. $\sigma^2 = 9$), find $P\{2<X<5\}$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319

$$\Phi(2/3) \approx 0.7486, \quad \Phi(-1/3) = 1 - \Phi(1/3) = 1 - 0.6293$$

It is useful (for statistics part later) to memorize that about 68% of values drawn from a normal distribution are within one standard deviation away from the mean; about 95% of the values are within two standard deviations and about 99.7% lie within 3 standard deviations

i.e.

$$P\{\mu - \sigma < X < \mu + \sigma\} \approx 0.68$$

$$P\{\mu - 2\sigma < X < \mu + 2\sigma\} \approx 0.95$$

$$P\{\mu - 3\sigma < X < \mu + 3\sigma\} \approx 0.997$$

Example: The army is developing a new missile. By observing points of impact, launchers can control the mean of its impact distribution. If the standard deviation of the impact distribution is too large, though, the missile will be ineffective. Suppose the Pentagon requires that at least 95% of the missiles must fall within 1/8 mile of the target when the missiles are aimed properly. Assume the impact distribution is normal. What is the maximum allowable standard deviation?

Example : An expert witness in a paternity suit testifies that the length (in days) of pregnancy is approximately normally distributed with mean 270 and variance 100. The defendant in the suit is able to prove that he was out of the country during a period that began 290 days before the birth of the child and ended 240 days before the birth. If the defendant was, in fact, the father of the child, what is the probability that the mother would have been the very long or very short pregnancy indicated by the testimony?

Exponential Distribution

Definition: A random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is called an exponential random variable with parameter λ

The CDF is $F(a) = \int_0^a \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_0^a = 1 - e^{-\lambda a}, a \geq 0$

Remarks: an exponential random variable describes the time until a specific event occurs when the events occur according to a Poisson process with rate λ , e.g., the amount of time until a telephone call you receive turns out to be a wrong number.

Example : Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = 1/10$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait

(a) more than 10 minutes

(b) between 10 and 20 minutes

Mean and variance of exponential random variable:

$E[X] = 1/\lambda, Var(X) = 1/\lambda^2$

Proof: $E[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x}\, dx$

Using integration by parts, let $\begin{array}{c} u = x^n, dv = \lambda e^{-\lambda x} dx \\ du = nx^{n-1}dx, v = -e^{-\lambda x} \end{array}$

$$
\begin{aligned}
E[X^n] &= \int_0^\infty u dv = uv|_0^\infty - \int_0^\infty v du \\
&= -x^n e^{-\lambda x}|_0^\infty + \int_0^\infty e^{-\lambda x} n x^{n-1}\, dx \\
&= 0 + \frac{n}{\lambda}\int_0^\infty \lambda e^{-\lambda x} x^{n-1} dx \\
&= \frac{n}{\lambda} E[X^{n-1}]
\end{aligned}
$$

$$
\begin{aligned}
E[X] &= 1/\lambda \\
E[X^2] &= \tfrac{2}{\lambda} E[X] = \tfrac{2}{\lambda^2} \\
Var(X) &= E[X^2] - (E[X])^2 = 1/\lambda^2
\end{aligned}
$$

Memorylessness of exponential random variable:

Let X be an exponential random variable with parameter λ

We have for all s,t ≥0,

$$\begin{aligned}
P\{X > s+t | X > t\} &= \frac{P\{X > s+t \cap X > t\}}{P\{X > t\}} \\
&= \frac{P\{X > s+t\}}{P\{X > t\}} = \frac{1 - F(s+t)}{1 - F(t)} \\
&= e^{-\lambda(t+s)}/e^{-\lambda t} = e^{\lambda s} = P\{X > s\}
\end{aligned}$$

Remark: the geometric distribution is also memoryless (in this case s, t >0 are integers)

Example : Consider a post office that is staffed by two clerks. Suppose that when Mr. C enters the post office, he discovers that Mr. A is being served by one of the clerks and Mr. B by the other. Suppose also that C is told that his service will begin as soon as either A or B leaves. If the amount of time that a clerk spends with a customer is exponentially distributed with parameter λ, what is the probability that, of the three customers, C is the last to leave the post office?

Gamma distribution:

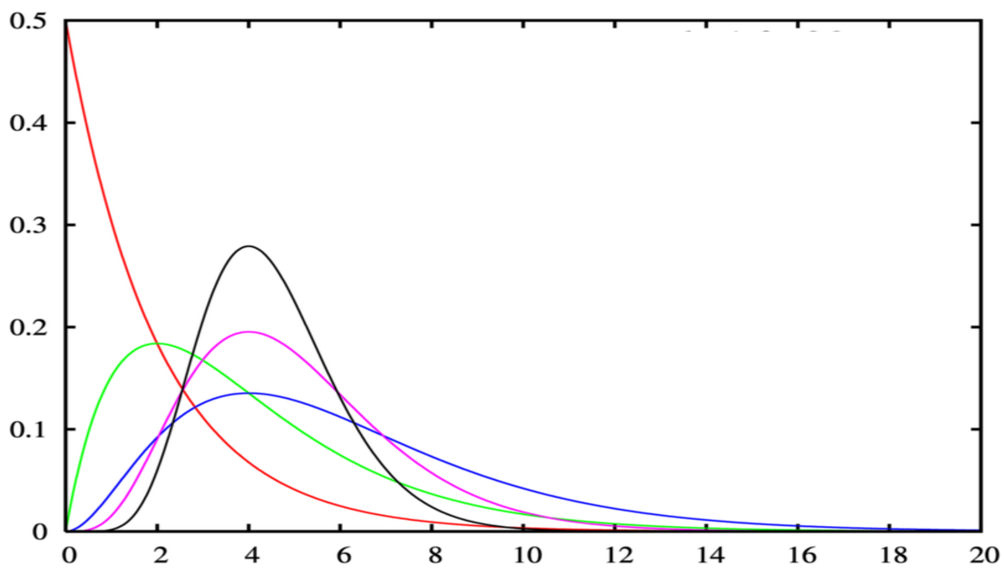
Definition: a random variable X is said to be gamma distributed with parameters (α, λ), denoted as X~Gamma(α, λ) if its pdf is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$

One important property of Γ is that $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

For integer value of α=*n* , Γ(*n*)=(*n-1*)!



Remark:

1) It is trivial that $\int_0^\infty f(x) = 1$ by the definition of Γ.

2) you are not required to memorize the pdf or the definition of the Gamma function. But once I give you the density, you should be able to calculate the moments as below.

Example : Compute the moments of X~Gamma(α,λ)

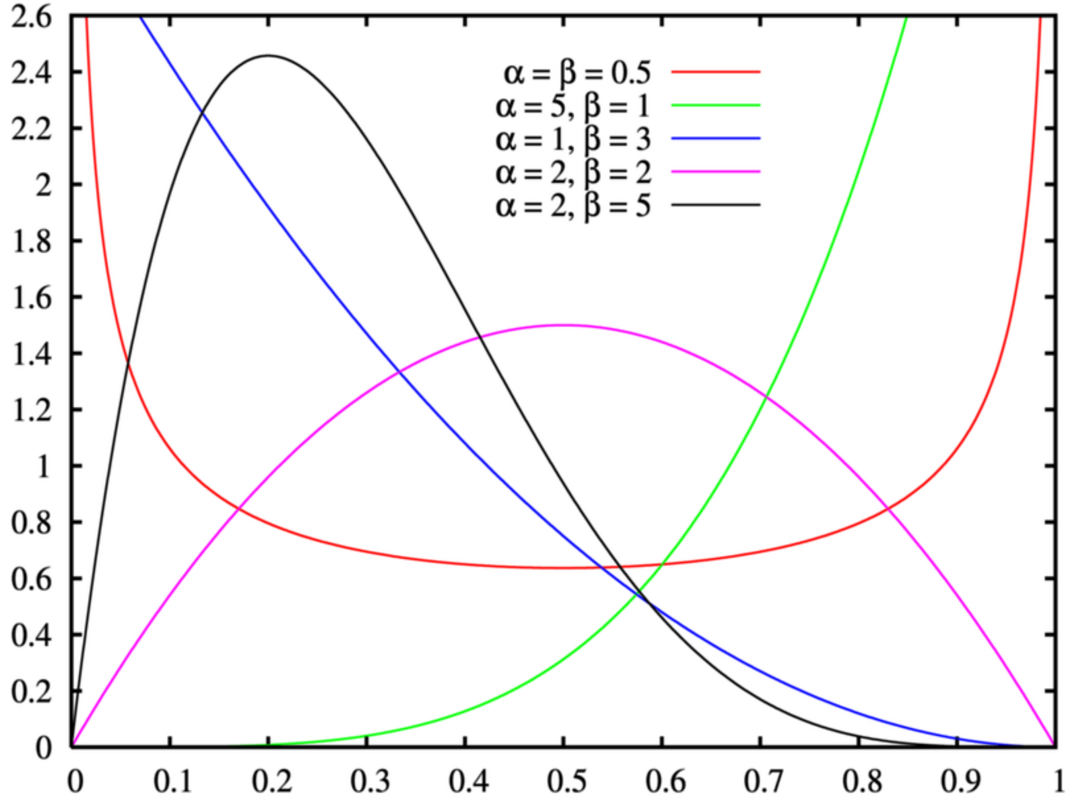
Beta distribution

A r.v. is said to have a beta distribution is its density is given by

$$f(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx$ is called the beta function, so that *f(x)* integrates to 1.

Beta distribution



4.3 Distribution of a function of a random variable

Suppose that we know the distribution of X and we want to find the distribution of $g(X)$. To do so, it is necessary to express the event that $g(X) \leq y$ in terms of X being in some set.

Example : Suppose that a random variable *X* has pdf

$$f_X(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Let *Y=2X+1*, what is the pdf of *Y*?

Example. If X is a random variable with pdf f_X, then the density of $Y = X^2$ can be computed as follows:

(Optional)

Example: Suppose a continuous r.v. has a strictly increasing cumulative distribution function *F.* What is the distribution of *F(X)*?

(Note: "strictly increasing" is not necessary to find the distribution of *F(X),* but it makes the argument simpler)

Solution:

When $0 \leq a \leq 1, P(F(X) \leq a) = P(X \leq F^{-1}(a)) = F(F^{-1}(a)) = a,$

This means $F(X)$ is uniform on $[0, 1]$!

(Optional)

Theorem Let *X* be a continuous random variable with density function f_X . Suppose that *g(x)* is a strictly monotone (increasing or decreasing), differentiable (and thus continuous) function of *x*. Then the random variable *Y* defined by *Y* = *g*(*X*) has a density function given by

$$f_Y(y) = \begin{cases} f_X[g^{-1}(y)]\left|\frac{d}{dy}g^{-1}(y)\right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$