BMS1901 Calculus for Life Sciences

Week 13

Vectors and Matrix Models: linear systems and matrix (Cont.)

Matrix Notation

Matrix Notation

- **Matrix :** a rectangular array of numbers
    - uppercase symbols

- **Size** of a matrix : number of rows and columns

- $m \times n$ matrix : m rows and n columns
- A : $m \times n$ matrix
    - ijth entry of A : entry in the ith row and the jth column
    - a_{ij}

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & \cdots & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{matrix} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Matrix Notation

(1) $$A = \begin{bmatrix} 0 & 7 & 1 \\ 2 & 9 & 2 \end{bmatrix}$$

- 2×3 matrix
- $a_{12} = 7$, $a_{21} = 2$, $a_{23} = 2$

- **square** matrix : number of rows is the same as the number of columns
    - n rows and columns: $n \times n$; has size n

- **transpose** of a matrix : interchanging its rows and columns
    - superscript T

(1)

$$A = \begin{bmatrix} 0 & 7 & 1 \\ 2 & 9 & 2 \end{bmatrix}$$

Matrix Notation

- transpose of the matrix in the previous slide, A^T:

(2)

$$A^T = \begin{bmatrix} 0 & 2 \\ 7 & 9 \\ 1 & 2 \end{bmatrix}$$

- A: $m \times n$ matrix
  - A^T : $n \times m$ matrix

- Vectors: often treated using the notation of matrices
- **row vectors** : components are listed as a row

Matrix Notation

- **row vector,** $[x, y]$: vector with components x and y
- **column vector,** $\begin{bmatrix} x \\ y \end{bmatrix}$**:**placing its components in a column
  - quantify the same thing but they are each used in different contexts in matrix algebra

- row form: $1 \times n$ matrix
- column form: $n \times 1$ matrix
- $\mathbf{v}$: vector in row form
  - $\mathbf{v}^T$: same vector in column form

Matrix Addition and Scalar Multiplication

Matrix Addition and Scalar Multiplication

- Matrix addition: matrices of the same

- A and B : $m \times n$ matrices with entries a_{ij} and b_{ij}
  - $A + B$: new $m \times n$ matrix with entries $a_{ij} + b_{ij}$

- matrix sum $A + B$: adding entries of each matrix

Example 1

Evaluate the following sums, if possible.

$$\text{(a)}\quad M + N, \quad \text{where } M = \begin{bmatrix} 2 & x & 9 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } N = \begin{bmatrix} 92 & 6 & 2 \\ 15 & 3 & 1 \end{bmatrix}$$

$$\text{(b)}\quad X + Y, \quad \text{where } X = \begin{bmatrix} 5 & 3 \\ 7 & 13 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 3 & 9 & 21 \\ 5 & 7 & 6 \end{bmatrix}$$

Solution:

(a) Both are 2×3 matrices → can be added

$$M + N = \begin{bmatrix} 94 & x + 6 & 11 \\ 19 & 8 & 7 \end{bmatrix}.$$

(b) Matrix X : 2×2

Y is 2×3

- not the same size → cannot be added

Matrix Addition and Scalar Multiplication

- Scalar multiplication with matrices works ~ vectors
- A : an $m \times n$ matrix with entries a_{ij}
  c is a scalar
  - product cA : $m \times n$ matrix with entries ca_{ij}
  - multiplying each entry of A by c

- Matrix subtraction : combination of scalar multiplication and matrix addition
- A and B : $m \times n$ matrices with entries a_{ij} and b_{ij}
  - $A - B$: multiplying B by -1 and adding this to A

Matrix Addition and Scalar Multiplication

- Difference $A - B$: subtracting entry b_{ij} from a_{ij}
- matrix subtraction : matrices of the same size

- Matrices A and B are **equal**
  - $A - B = 0$
  - 0 : $m \times n$ matrix of zeros

Properties of Matrix Addition If A, B, and C are $m \times n$ matrices and a and b are scalars, then

1. $A + B = B + A$

2. $A + (B + C) = (A + B) + C$

3. $A + 0 = A$

4. $A + (-A) = 0$

5. $a(A + B) = aA + aB$

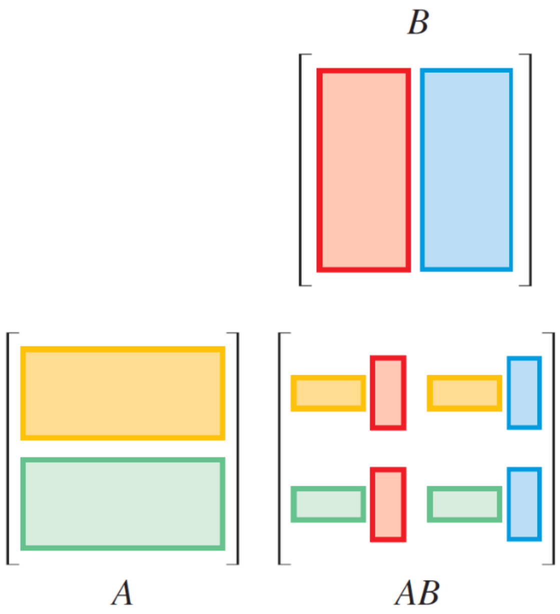
6. $(a + b)A = aA + bA$

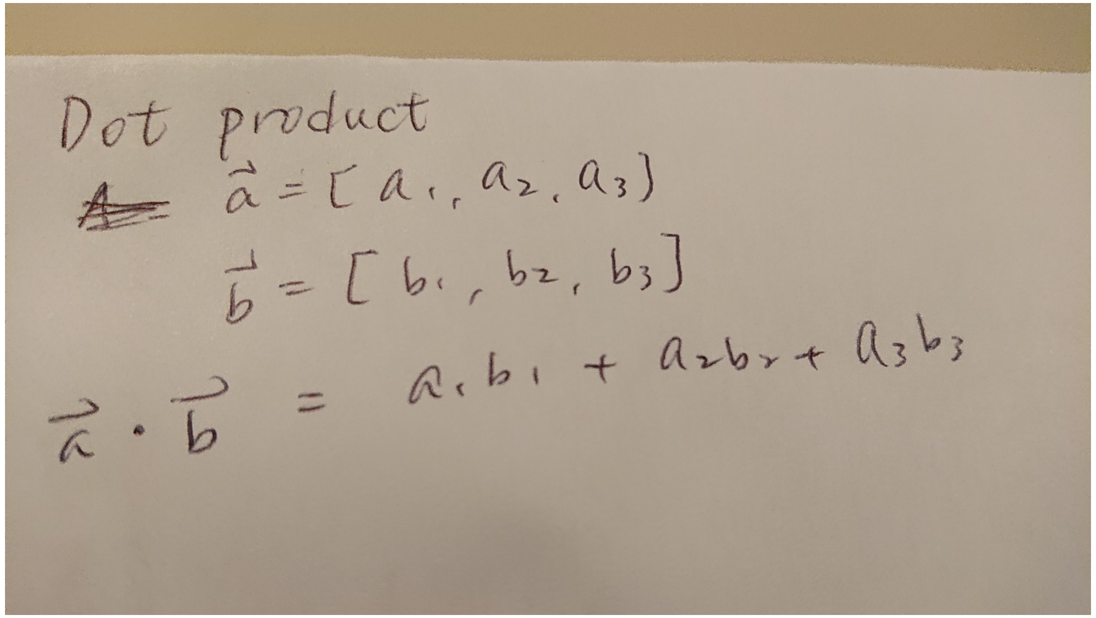
Matrix Multiplication

Matrix Multiplication

Matrix product *AB*

- Matrix *A* : row vectors
- Matrix *B* : column vectors
- *ij*th entry of the resulting product = dot product of the *i*th row of *A* with the *j*th column of *B*





Matrix Multiplication

- A : 2 × 3 matrix
- B : 3 × 2 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

- resulting matrix : 2 × 2

Matrix Multiplication

- A : $m \times n$ matrix
- B : $n \times p$ matrix
- $C = AB$: $m \times p$ matrix
- whose ijth entry : dot product of the ith row of A and jth column of B

$$\underset{A}{\begin{bmatrix} — & \mathbf{r}_1 & — \\ — & \mathbf{r}_2 & — \\ & \vdots & \\ — & \mathbf{r}_m & — \end{bmatrix}} \underset{B}{\begin{bmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_p \\ | & | & & | \end{bmatrix}} = \underset{AB}{\begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \mathbf{r}_1 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{c}_p \\ \mathbf{r}_2 \cdot \mathbf{c}_1 & \mathbf{r}_2 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{c}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{r}_m \cdot \mathbf{c}_1 & \mathbf{r}_m \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_m \cdot \mathbf{c}_p \end{bmatrix}}$$

The ith row of A is indicated by $\boldsymbol{r}_i$.
The jth column of B is indicated by $\boldsymbol{c}_j$.

Matrix Multiplication

- No. of columns of the first matrix ≠ no. of rows of the second matrix → matrix multiplication is not defined

- write the size of the first matrix and size of the second matrix

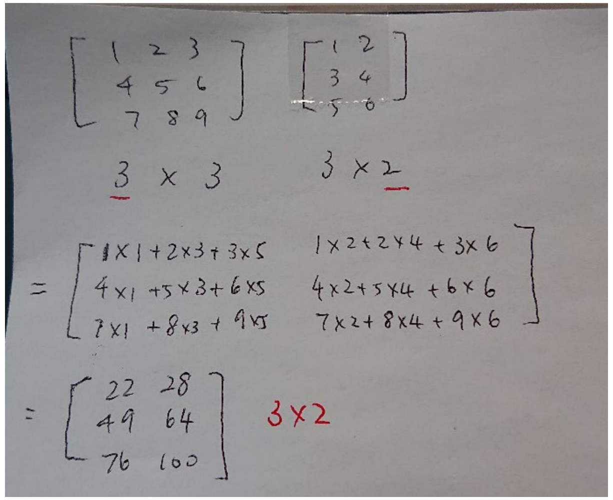
$$\begin{array}{ccl} A & B & = \text{not defined} \\ m \times n & q \times p & \end{array}$$

$$n \neq q$$

$$\begin{array}{cccc} A & B & = & AB \\ m \times n & q \times p & = & m \times p \end{array}$$

$$n = q$$

Matrix Multiplication



- two “inner” numbers are not the same
  - matrix multiplication is not defined
- two “inner” numbers are equal
  - resulting matrix size : two “outer” numbers as

Matrix Multiplication If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then their product $C = AB$ is an $m \times p$ matrix whose entries are given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant p$.

Example 2

Determine each matrix product if it is defined.

(a) *AB* (b) *BA* (c) *AC* (d) *CA*

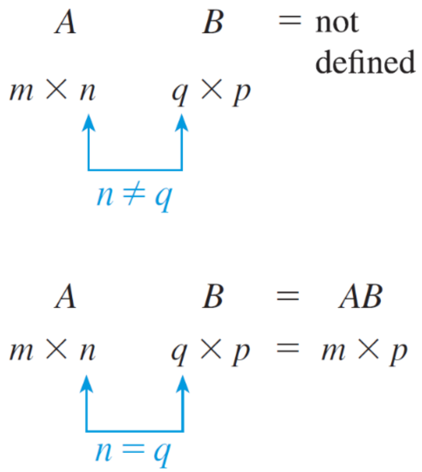
$$A = \begin{bmatrix} 2 & 7 \\ 9 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -7 & 2 \\ 1 & 5 & 9 \end{bmatrix} \qquad C = \begin{bmatrix} 5 & -6 \\ 8 & 2 \end{bmatrix}$$

Solution:

(a) Matrix A : 2 × 2

Matrix B: 2 × 3

- $n = q = 2$
- matrix multiplication is defined
- resulting matrix : 2 × 3



Example 2 – *Solution*

$A = \begin{bmatrix} 2 & 7 \\ 9 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -7 & 2 \\ 1 & 5 & 9 \end{bmatrix} \quad C = \begin{bmatrix} 5 & -6 \\ 8 & 2 \end{bmatrix}$

(a) *AB* (b) *BA* (c) *AC* (d) *CA*

Performing the calculation:

$$\begin{bmatrix} 2 & 7 \\ 9 & -3 \end{bmatrix}\begin{bmatrix} 3 & -7 & 2 \\ 1 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 6+7 & -14+35 & 4+63 \\ 27-3 & -63-15 & 18-27 \end{bmatrix} = \begin{bmatrix} 13 & 21 & 67 \\ 24 & -78 & -9 \end{bmatrix}$$

(b) Matrix B : 2×3

Matrix A : 2×2

- $n \neq q$: $n = 3$ and $q = 2$
  - product is not defined

(c) $n = q = 2$ → resulting matrix is 2×2

Performing the calculation:

$$\begin{bmatrix} 2 & 7 \\ 9 & -3 \end{bmatrix}\begin{bmatrix} 5 & -6 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 10+56 & -12+14 \\ 45-24 & -54-6 \end{bmatrix} = \begin{bmatrix} 66 & 2 \\ 21 & -60 \end{bmatrix}$$

Example 2 – *Solution*

$$A = \begin{bmatrix} 2 & 7 \\ 9 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -7 & 2 \\ 1 & 5 & 9 \end{bmatrix} \qquad C = \begin{bmatrix} 5 & -6 \\ 8 & 2 \end{bmatrix}$$

(a) AB (b) BA (c) AC (d) CA

(d) Using the rule : $n = q = 2$

$$\begin{array}{ccl} A & B & = \text{not defined} \\ m \times n & q \times p & \\ \multicolumn{2}{c}{n \neq q} & \end{array}$$

$$\begin{array}{cccc} A & B & = & AB \\ m \times n & q \times p & = & m \times p \\ \multicolumn{2}{c}{n = q} & & \end{array}$$

- resulting matrix : 2 × 2
- Performing the calculation:

$$\begin{bmatrix} 5 & -6 \\ 8 & 2 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 9 & -3 \end{bmatrix}\begin{bmatrix} 10 - 54 & 35 + 18 \\ 16 + 18 & 56 - 6 \end{bmatrix} = \begin{bmatrix} -44 & 53 \\ 34 & 50 \end{bmatrix}$$

Matrix Multiplication

- **Commutative** multiplication of α and β **:** $\alpha\beta = \beta\alpha$
- matrix multiplication is <u>not commutative</u>
    - $AC \neq CA$

- **Diagonal matrix:** $n \times n$ matrix D
    - all off-diagonal entries are zero
    - $d_{ij} = 0$ for all $i \neq j$

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Example

For an arbitrary 2×2 diagonal matrix D with entries d_{ii}, calculate the matrix DD.

Solution:
Calculating the matrix product:

$$DD = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix} \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix} = \begin{bmatrix} d_{11}^2 & 0 \\ 0 & d_{22}^2 \end{bmatrix}$$

- *DD* is a <u>diagonal matrix</u> with entries: d_{ii}^2

Matrix Multiplication

Identity matrix (*I*):

- a diagonal matrix with all the entries on the diagonal : 1
- play the same role in matrix multiplication that the number 1 plays in regular multiplication

Properties of Matrix Multiplication Suppose A, B, and C are matrices and a and b are scalars. Provided the required matrix multiplications are defined, then

1. $A(BC) = (AB)C$

2. $(aA)(bB) = abAB$

3. $A(B + C) = AB + AC$

4. $(B + C)A = BA + CA$

5. $IA = A, AI = A$

6. $0A = 0, A0 = 0$

Note: Matrix multiplication is *not*, in general, commutative; that is, $AB \neq BA$.

The Inverse of a Matrix

The Inverse of a Matrix

- $AI = A$ *and* $IA = A$ (matrix multiplication)
- $a1 = a$ and $1a = a$ (scalar multiplication)
- *Scalar:* $a^{-1}a = 1$
  - $a^{-1} = 1/a$ and $a \neq 0$

- a^{-1} : when multiplied with $a \rightarrow 1$
- $a = 0 \rightarrow$ no such quantity exists

Example 1

Show that *B* is the inverse of *A*, where

$$A = \begin{bmatrix} 1 & 2 \\ 7 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -\frac{5}{9} & \frac{2}{9} \\ \frac{7}{9} & -\frac{1}{9} \end{bmatrix}$$

Solution:

Calculating the matrix product:

$$BA = \begin{bmatrix} -\frac{5}{9} & \frac{2}{9} \\ \frac{7}{9} & -\frac{1}{9} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} -\frac{5}{9}(1) + \frac{2}{9}(7) & -\frac{5}{9}(2) + \frac{2}{9}(5) \\ \frac{7}{9}(1) - \frac{1}{9}(7) & \frac{7}{9}(2) - \frac{1}{9}(5) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

The Inverse of a Matrix

Definition Suppose that A is an $n \times n$ matrix. If there exists an $n \times n$ matrix B such that

$$AB = BA = I$$

then B is called the **inverse** of A and is denoted by A^{-1}.

- inverse of a matrix exists: unique.
- $AB = I$
  - $BA = I$ as well (and vice versa)
- check only one order of multiplication when finding an inverse
- A has an inverse → A is **invertible** or **nonsingular**
  - Otherwise A : **singular**

The Inverse of a Matrix

The Inverse of a 2 x 2 Matrix Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Then A is invertible and

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

If $a_{11}a_{22} - a_{12}a_{21} = 0$, then A is not invertible (that is, A is singular).

The Inverse of a Matrix

Properties of Matrix Inverses Suppose A and B are both invertible $n \times n$ matrices. Then

1. $(A^{-1})^{-1} = A$

2. $(AB)^{-1} = B^{-1}A^{-1}$

3. A^{-1} is unique.

The Determinant of a Matrix

The Determinant of a Matrix

n × *n* matrix *A*

- assign a scalar quantity : *determinant* (det *A*)
- Scalars: 1 × 1 matrices
  - determinant for matrices of sizes *n* = 1 through *n* = 3 :

The Determinant Suppose A is an $n \times n$ matrix.

1. If $n = 1$, then $\det A = a_{11}$.

2. If $n = 2$, then $\det A = a_{11}a_{22} - a_{12}a_{21}$.

3. If $n = 3$, then
$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

The Determinant of a Matrix

Given the determinant of a matrix:

(1) Theorem If A is an $n \times n$ matrix, then A is invertible if and only if $\det A \neq 0$.

- quantity in the denominator of the formula for the inverse of a 2×2 matrix : determinant

Example 4

Which of the following matrices are invertible?

(a) $A = \begin{bmatrix} 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$ (c) $C = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 10 & 7 & 3 \\ 13 & 5 & 8 \\ 6 & -1 & 7 \end{bmatrix}$

Solution:

(a) matrix A : 1×1 and det $A = 2$ → A: invertible

(b) matrix B : 2×2 and det $B = (2)(9) - (3)(6) = 0$
→ B : not invertible

(c) matrix C is 2×2 and det $C = (5)(1) - (3)(2) = -1$
→ C : invertible

(a) $A = \begin{bmatrix} 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$ (c) $C = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 10 & 7 & 3 \\ 13 & 5 & 8 \\ 6 & -1 & 7 \end{bmatrix}$

Example 4 – *Solution*

(d) matrix D : 3 × 3

$$\det D = (10)(5)(7) + (7)(8)(6) + (3)(13)(-1) - (3)(5)(6) - (10)(8)(-1) - (7)(13)(7)$$

$$= 350 + 336 + (-39) - 90 - (-80) - 637 \quad = 0$$

- D : not invertible

The Determinant Suppose A is an $n \times n$ matrix.

1. If $n = 1$, then $\det A = a_{11}$.
2. If $n = 2$, then $\det A = a_{11}a_{22} - a_{12}a_{21}$.
3. If $n = 3$, then $\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$.