Chapter 0 Functions

Contents

0. Functions

In this chapter, we will briefly recall functions and their properties covered by high school.

0.1. Basic concepts of functions.

Text Sec1.1: 1, 3, 7-10, 38, **45**, **49**, 62, **67**, 73-80.

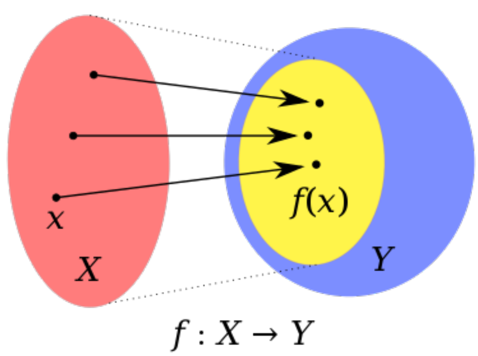
Definition 0.1. *A function f is a rule that assigns to each element x in a set X exactly one element, called $f(x)$, in a set Y.*

Usually, we write a function

$$f : X \to Y, x \mapsto f(x)$$

where

1. $x \in X$, i.e. x belongs to a set X , called the **Domain** (Red region in the figure);
2. $f(x) \in Y$, i.e. $f(x)$ belongs to a set Y, called the **Codomain** (Blue and Yellow regions in the figure);
3. The set of all possible values of $f(x)$ as $x \in X$, called the **Range** (Yellow region in the figure);
4. x is **independent variable**,
5. $f(x)$ is **dependent variable**.



For a function f, its **graph** is the set of points

$$\{(x, f(x)) : x \in D\}$$

in xy-plane. One can also use a table to represent a function.

Ex. Sketch the graph of following two **piecewise defined functions**.

(1) $f(x) = |x|$. i.e. Absolute value of x.
(2) $f(x) = [x]$. i.e. largest integer not greater than x.

Proposition 0.2 (Vertical Line Test)**.** *A curve in the xy-plane is the graph of a function if and only if no vertical line intersects the curve more than once.*

VLT is equivalent to following statements: for any given input x, the output $f(x)$ is determined uniquely. Otherwise, f is not well-defined function.

Symmetry of a function is an important topic.

(1) A function f is **even** if

$$f(-x) = f(x), \forall x \in D.$$

(2) A function f is **odd** if

$$f(-x) = -f(x), \forall x \in D.$$

Monotonicity of a function is another important topic.

(1) A function f is **increasing** on an interval I, if

$$f(x_1) < f(x_2), \forall x_1 < x_2 \text{ in } I.$$

(2) A function f is **decreasing** on an interval I, if

$$f(x_1) > f(x_2), \forall x_1 < x_2 \text{ in } I$$

0.2. **Classification of functions.** Text Sec1.2: 3-4, **13, 15**

A mathematical model is a mathematical description of a real-world phenomenon, and usually represented by a function. In your senior years, you will study some math models for population, demand of product, speed of an object, ...

Some typical functions used for models are

(1) **Polynomial function**

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where a_i are **coefficients**. If $a_n \neq 0$, then the **degree** of $P(x)$ is n.

(a) **Linear model**

$$f(x) = mx + b$$

where m is slope, b is y-intercept.
ex. The relationship between Fahrenheit (F) and Celsius (C) is $F = \frac{9}{5}C + 32$.

(b) **Quadratic model**

$$f(x) = ax^2 + bx + c.$$

(c) **Cubic model** A polynomial with degree 3.

(2) **Power function**

$$f(x) = x^a$$

where a is constant.
Ex. Sketch the graph of power function if a is (1) positive integer; (2) reciprocal of positive integer; (3) -1;

(3) **Rational function**

$$f(x) = \frac{P(x)}{Q(x)}$$

where P, Q are polynomials.
ex. find domain of $f(x) = \frac{2-3x}{x^2-4}$.

(4) **Trigonometric functions** The common trigonometric functions are sin, cos, tan, cot.

(5) **Exponential functions**

$$f(x) = a^x$$

where the base $a \neq 1$ is a positive constant.
ex. Sketch the graph of $y = a^x$ when a is a constant satisfying (1) $a < 1$ (2) $a > 1$.

(6) **Logarithmic functions**

$$f(x) = \log_a x$$

where $a \neq 1$ is a positive constant.
ex. Sketch the graph of $y = \log_a x$ when a is a constant satisfying (1) $a < 1$ (2) $a > 1$.

(7) **Algebraic function** It is a function constructed by polynomials using algebraic operations (such as $+, -, \times, \div, \sqrt[n]{}$). ex. find domain and symmetry of $f(x) = \frac{\sqrt{x^2}+1}{x^3}$

(8) **Transcendental functions** It is a non-algebraic function, including the trig; inverse of trig; exp.; log; and ...
ex. Can you find a Transcendental function not mentioned in the above?

Ex. Classify following functions as one of the types we discussed: poly, power, rational, Trig, exp, log, algebraic, transc.,

(1) $f(x) = 5^x$,
(2) $g(x) = x^5$
(3) $h(x) = \frac{1+x}{1-\sqrt{x}}$
(4) $u(x) = \frac{1+x}{1-x^{1.5}} + x^\pi$.

0.3. **New functions from old functions.** Text Sec1.3: 5, **26**, 29 35, **53, 57, 61** 63, 64

We will discuss two ways of obtaining a new function from old functions:

(1) Shifting, stretching, or reflecting a given function;
(2) Combination/Composition of two given functions

Let $a > 0$ and $b > 1$. Given a function $y = f(x)$, we can obtain a new function using following transformations

(1) $y = f(x) + a$, by shifting $y = f(x)$ a units upward; i.e. $\uparrow_a$
(2) $y = f(x) - a$, by $\downarrow_a$
(3) $y = f(x - a)$, by $\rightarrow_a$
(4) $y = f(x + a)$, by $\leftarrow_a$
(5) $y = bf(x)$ by stretching $y = f(x)$ vertically by a factor of b, i.e. $\updownarrow_b$
(6) $y = \frac{1}{b}f(x)$, by compressing $y = f(x)$ vertically by a factor of b, $\updownarrow_{1/b}$
(7) $y = f(bx)$, by compressing horizontally, $\leftrightarrow_{1/b}$.
(8) $y = f(x/b)$, by stretching horizontally, $\leftrightarrow_b$
(9) $y = -f(x)$, by reflect $y = f(x)$ about x-axis, i.e. R_x.
(10) $y = f(-x)$, by reflect $y = f(x)$ about y-axis, i.e. R_y.

Ex. Using transformation, graph

$$f(x) = 2\sqrt{-x} - 1, \text{and } g(x) = |2\sqrt{-x} - 1|, \text{ for } x \leq 0.$$

Given Two functions of f and g, we may have following combinations: using definition of $+, -, \times, \div$

$$f+g, f-g, fg, f/g$$

Also, **composition** $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x)).$$

Ex. Given $F(x) = \cos^2(x+9)$, try to find functions f, g, h s.t. $F = f \circ g \circ h$.

Remark 0.3. *$f \circ g \neq g \circ f$ in general. Try to give an example.*