MA1300 Midterm, Sections CB1, 18 Oct, 2020

Name: ________________________ Student ID: ________________________

Department/Major: ________________________

- *Please answer all 7 questions, and show all your steps.*
- *L'Hospital rule is not allowed in the exam*
- **NOT to share the questions out after examination**

1. (10 points) Find the (largest possible) domain of the function

$$F(x) = \frac{\sqrt{3-|x|}}{\sqrt{x^2-4}}.$$

Ans: The largest possible domain is $[-3,-2)\bigcup(2,3]$.

2. (10 points) Without using L'Hospital rule, determine if the following limit exists. If the limit exists, find the limit value.

$$\lim_{\phi\to 0} f(|\phi|/\phi)$$

where

$$f(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } x < -1, \\ -1 & \text{if } x > 1. \end{cases}$$

Ans: For $\phi > 0$, $|\phi|/\phi = 1$, then

$$\lim_{\phi\to 0^+} f(|\phi|/\phi) = \lim_{\phi\to 0^+} f(1) = 1.$$

For $\phi < 0$, $|\phi|/\phi = -1$, then

$$\lim_{\phi\to 0^-} f(|\phi|/\phi) = \lim_{\phi\to 0^-} f(-1) = 1.$$

Overall, the limit exists and the limit value is 1.

3. (15 points) Use the precise definition of limit to show that

$$\lim_{x\to 0^+} \frac{\cos(x^2)}{x} \text{ does not exist.}$$

Ans: For any $M > 0$, there exists $\delta = \min\{1, \cos(1)/M\}$ such that if $0 < x < \delta$, then $\frac{\cos(x^2)}{x} > \frac{\cos(1)}{x} > M$.
So, by the definition, we have $\lim_{x\to 0^+} \frac{\cos(x^2)}{x} = \infty$ and it does not exist.

4. a) (15 points) Use the definition of derivative to determine if the following function $f(x)$ is differentiable at $x = 0$.

$$f(x) = \begin{cases} x^2\cos(1/x) & \text{if } x > 0, \\ x\sin(x) & \text{if } x \le 0. \end{cases}$$

b) (15 points) Find an equation of the tangent line to the following curve at the point $(1, 1)$,

$$x^3 + y^3 = xy + 1.$$

Ans: a) Consider

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{f(h) - 0}{h} = \lim_{h\to 0^+} \frac{h^2\cos(1/h)}{h} \quad \lim_{h\to 0^+} h\cos(1/h) = 0.$$

Also,

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{f(h) - 0}{h} = \lim_{h\to 0^-} \frac{h\sin(h)}{h} \quad \lim_{h\to 0^-} \sin(h) = 0.$$

So

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = 0,$$

we conclude that $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.
b) By implicit differentiation, we obtain

$$3x^2 + 3y^2\frac{dy}{dx} = y + x\frac{dy}{dx},$$

When $x = 1$ and $y = 1$,

$$\frac{dy}{dx} = -1.$$

So the equation of the tangent line at the point $(1, 1)$ is

$$y - 1 = -(x - 1).$$

5. (15 points) Let $f(x)$ be a continuous function on an interval $[0, 2]$ such that $f(0) = f(2)$. Prove that there is a number $c \in [1, 2]$ such that $f(c) = f(c - 1)$.

Ans: Let $F(x) = f(x) - f(x - 1)$ for $x \in [1, 2]$. We have $F(1) = f(1) - f(0)$ and $F(2) = f(2) - f(1) = f(0) - f(1)$ (by $f(0) = f(2)$). If $F(1) = 0$, take $c = 1$; if $F(2) = 0$, take $c = 2$. We proved that there is a number $c \in [1, 2]$ such that $f(c) = f(c-1)$ for the two cases above. If $F(1) \times F(2) \neq 0$, we consider

$$F(1) \times F(2) = (f(1) - f(0))(f(0) - f(1)) = -(f(0) - f(1))^2 < 0.$$

As $f(x)$ be a continuous function on an interval $[0, 2]$, $F(x)$ be a continuous function for $x \in [1, 2]$. By IVT, we prove that there is a number $c \in [1, 2]$ such that $F(c) = 0$, then $f(c) = f(c - 1)$.

6. (10 points) Let $f(x)$ be a function at any $x \in \mathbb{R}$.
For any $a, b \in \mathbb{R}$, if $a < b$, then $f(a) < f(b)$. If there exists a value c such that

$$\lim_{x \to c} f(x) \text{ exists,}$$

prove that $f(x)$ is continuous at c.

Ans: For any $x < c$, we have $f(x) < f(c)$, then

$$\lim_{x \to c^-} f(x) \leq f(c).$$

For any $x > c$, we have $f(x) > f(c)$, then

$$\lim_{x \to c^+} f(x) \geq f(c).$$

As

$$\lim_{x \to c} f(x) \text{ exists,}$$

we obtain

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

so

$$\lim_{x \to c} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

We prove that f is continuous at c.

7. (10 points) Suppose that $f(x)$ be a function on $[a, b]$ and there exist a positive value $M > 0$ such that $|f(x)| < M$ for any $x \in [a, b]$. For any values x_1, x_2 with $a \leq x_1 \leq x_2 \leq b$, we have

$$f\left((x_1 + x_2)/2\right) \leq \frac{1}{2}(f(x_1) + f(x_2)).$$

Prove that f is continuous on (a, b).
(Hint: Show that $f(x + \delta) - f(x) \leq \frac{1}{2}(f(x + 2\delta) - f(x))$ if $x, x + 2\delta \in (a, b)$.)

Ans: Set $x_1 = x + 2\delta$ and $x_2 = x$ with $x, x + 2\delta \in (a, b)$, we have

$$f\left(x + \delta\right) \leq \frac{1}{2}(f(x + 2\delta) + f(x)),$$

$$f\left(x + \delta\right) - f(x) \leq \frac{1}{2}(f(x + 2\delta) - f(x)).$$

Follow similar idea, set $x_1 = x + 2^2\delta$ and $x_2 = x$ with $x, x + 2^2\delta \in (a, b)$, we have

$$f\left(x + \delta\right) - f(x) \leq \frac{1}{2}(f(x + 2\delta) - f(x)) \leq \frac{1}{2^2}(f(x + 2^2\delta) - f(x)).$$

In general, for any positive integer n and $x \in (a, b)$, we can set a $\delta_1 > 0$ such that $x - 2^n\delta_1, x + 2^n\delta_1 \in (a, b)$, if $|\delta| < \delta_1$, we have

$$f\left(x + \delta\right) - f(x) \leq \frac{1}{2^n}(f(x + 2^n\delta) - f(x)).$$

For any $x \in (a, b)$ and $\epsilon > 0$, we set an integer n such that $2^{n-1} > M/\epsilon$, so, by the statement above, there exist $\delta_1 > 0$ such that $x - 2^n\delta_1, x + 2^n\delta_1 \in (a, b)$, if $|y - x| < \delta_1$, then $y = x + \delta$ with $|\delta| < \delta_1$ and we have

$$|f(y) - f(x)| \leq \frac{1}{2^n}|f(x + 2^n\delta) - f(x) \leq \frac{1}{2^n}(|f(x + 2^n\delta)| + |f(x)|) \leq \frac{1}{2^n}2M < \epsilon.$$

So

$$\lim_{y \to x} f(y) = \lim_{x \to c^+} f(x) = f(x)$$

We prove that f is continuous on (a, b).

End