Mathematical Foundations for Circuit Analysis

Outline of Mathematical Foundations

- 1. Complex numbers
- 2. Differentiation
- 3. Integration
- 4. Fundamentals to differential equations
- 5. Trigonometry

1. Complex numbers

Complex numbers

- 1.1 Basic idea of complex number
- 1.2 Cartesian Form and Polar Form of Complex Numbers
- 1.3 Euler Form of Complex Number
- 1.4 Arithmetic Operation of Complex Number
- 1.5 Summary of Complex Number

1.1 Basic Idea of Complex Number

- $\sqrt{-1}$ is defined as a number whose square is -1 and is denoted by i , called the **imaginary unit**.
- In circuit analysis, we usually use j to denote the imaginary unit to avoid confusion with current i .
- A complex number z is the sum of a real number and an imaginary number : $z = a + bi$ *(a and b are real numbers)*.
- a is called the ***real part***, denoted by ***Re(z)*** while b is called the ***imaginary part*** , denoted by ***Im(z)***.

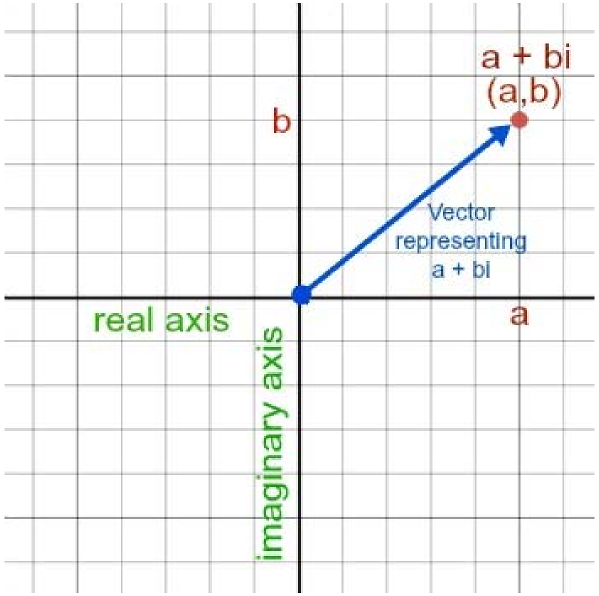
(Notice : Although *bi* is an imaginary number and *b* is a real number, the imaginary part of the complex number is *b*, not *bi* !)

Vector Representation of A Complex Number

- A **vector** has both direction and magnitude.
- Generally, we use the following steps to represent a complex number $a + bi$ with a vector :

1. Enter the point (a, b) inot the complex plane.

2. Draw a directed line segment from the origin of the plane to the point (a, b). This is the vector representing $a + bi$.



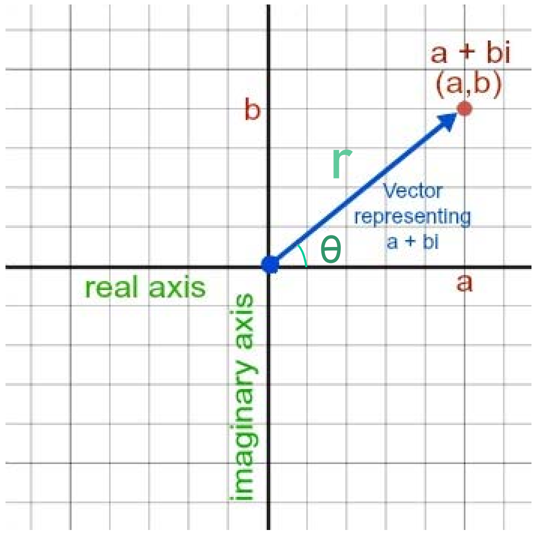
Complex Plane

https://study.com/academy/lesson/representing-complex-numbers-with-vectors.html

Vector Representation of A Complex Number

3. The length of the line segment is known as the magnitude (modulus) r where $r = \sqrt{a^2 + b^2}$

4. The angle between the line segment and the real axis in counterclockwise direction is known as argument θ



Complex Plane

https://study.com/academy/lesson/representing-complex-numbers-with-vectors.html

Examples : Vector Form of Complex Numbers

Representing the following complex numbers with vectors on the complex plane and calculate the modulus and argument of each complex number :

1) $3 + 4j$

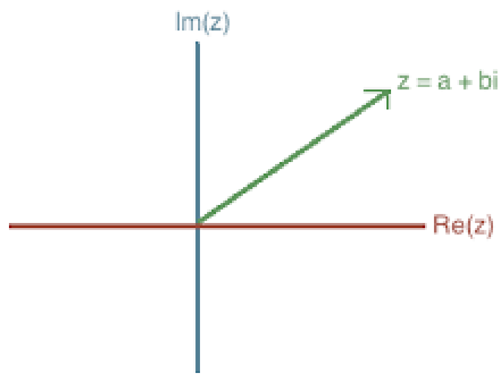
2) $3 - 4j$

3) $1 + j$

4) $1 - j$

5) $6 + 6\sqrt{3}j$

6) $6 - 6\sqrt{3}j$

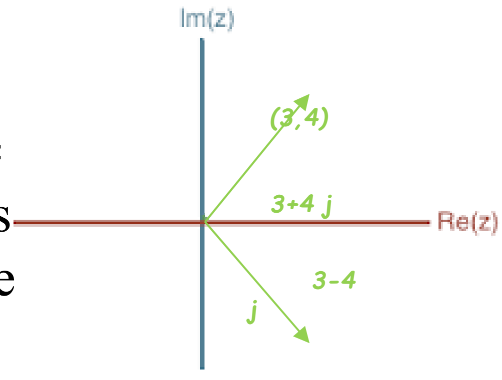


The set of complex numbers over the field of real numbers is a 2-dimensional vector space.

http://mathonline.wikidot.com/dimension-of-a-vector-space

Examples : Vector Form of Complex Numbers

- For $z = 3 + 4j$, *Re(z)* = 3 and *Im(z)* = 4. The cartesian coordinate on the complex plane is *(Re(z) , Im(z))*. Hence enter (3,4) into the plane. The modulus is $r = |z| = \sqrt{3^2 + 4^2} = 5$. The argument is $\theta = \tan^{-1}(\frac{4}{3}) = 53°$
- Likewise, for $z = 3 - 4\,j$, $Re(z) = 3$ and $Im(z) = -4$. The cartesian coordinate on the complex plane is *(Re(z) , Im(z))*. Hence enter (3,-4) into the plane. The modulus is $r = |z| = \sqrt{3^2 + (-4)^2} = 5$. The argument is $\theta = \tan^{-1}(-\frac{4}{3}) = -53°$.
- (*a* + *bj* and *a* – *bj* are know as complex conjugates, which are symmetric about the real axis. Hence, they have the same modulus. The sum of their arguments is always 360°)



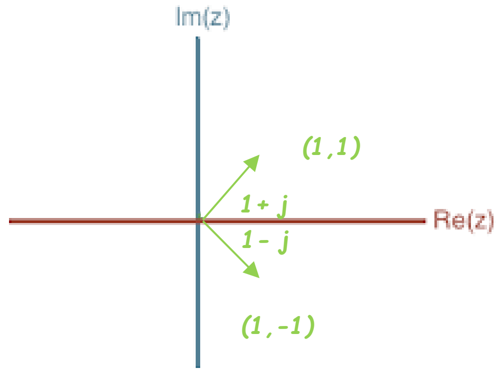
The set of complex numbers over the field of real numbers is a 2-dimensional vector space.

http://mathonline.wikidot.com/dimension-of-a-vector-space

Examples : Vector Form of Complex Numbers

- For $z = 1 + j$, enter (1,1) into the plane. The modulus is $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. The argument is $\theta = \tan^{-1}(1) = 45°$.

- Likely for $z = 1 - j$, enter (1,-1) into the plane. The modulus is $r = |z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. The argument is $\theta = \tan^{-1}(-1) = -45°$.



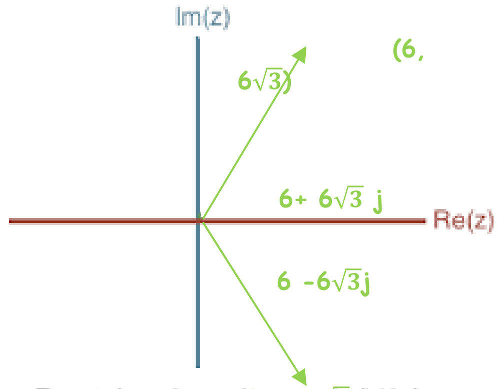
The set of complex numbers over the field of real numbers is a 2-dimensional vector space.

http://mathonline.wikidot.com/dimension-of-a-vector-space

Examples : Vector Form of Complex Numbers

- Now applying the same method to the other four complex numbers.

For $z = 6 + 6\sqrt{3}j$, enter (6, $6\sqrt{3}$) into the plane. The modulus is $r = |z| = \sqrt{6^2 + (6\sqrt{3})^2} = 12$. The argument is $\theta = \tan^{-1}(\sqrt{3}) = 60°$



The set of complex numbers over the field of real numbers is a 2-dimensional vector space.

For $z = 6 - 6\sqrt{3}j$, enter (6, -$6\sqrt{3}$) into the plane. The modulus is $r = |z| = \sqrt{6^2 + (-6\sqrt{3})^2} = 12$. The argument is $\theta = \tan^{-1}(-\sqrt{3}) = -60°$.

http://mathonline.wikidot.com/dimension-of-a-vector-space

1.2 Rectangular Form and Polar Form of Complex Numbers

For the complex number $\boldsymbol{z = x + y\,j}$, we know from the vector on the complex plane that *Re(z)* and *Im(z)* can be represented by $\boldsymbol{x = r}\cos\boldsymbol{\theta}$ and $\boldsymbol{y = r}\sin\boldsymbol{\theta}$, respectively.

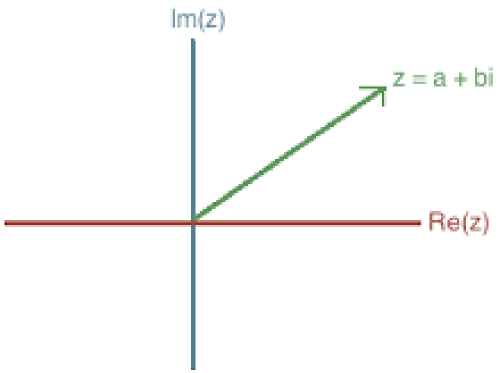
z may be written in the :

- **Rectangular (cartesian) form** where $z = x + yi$
- **polar form** where $z = r(\cos\theta + i\sin\theta)$ which is often abbreviated as $z = r\angle\theta$. In the expression, the symbols r and θ are the modulus and argument of z, respectively.

Examples: Rectangular Form and Polar Form

- Recall the six complex numbers that were mentioned before and express them in their polar forms:

  1) $3 + 4j$
  2) $3 - 4j$
  3) $1 + j$
  4) $1 - j$
  5) $6 + 6\sqrt{3}j$
  6) $6 - 6\sqrt{3}j$



The set of complex numbers over the field of real numbers is a 2-dimensional vector space.

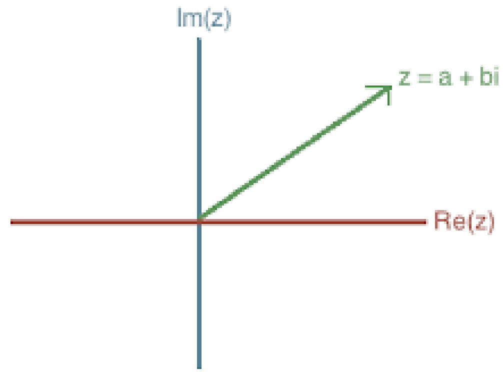
- In polar form, the complex conjugates are expressed in the form of $r\angle\theta$ and $r\angle(-\theta)$

http://mathonline.wikidot.com/dimension-of-a-vector-space

Examples: Rectangular Form and Polar Form

- As their modulus r and arguments have already been calculated in the previous examples, we can express $z = a + bj$ in its polar form of $r\angle\theta$, where r and θ are the modulus and argument, respectively. The complex conjugates can be expressed as $\boldsymbol{r\angle\theta}$ and $\boldsymbol{r\angle(-\theta)}$:

1) $3 + 4\,j = 5\ \angle 53°$	2) $3 - 4\,j = 5\ \angle(-53)°$
3) $1 + j = \sqrt{2}\ \angle 45°$	4) $1 - j = \sqrt{2}\ \angle(-45)°$
5) $6 + 6\sqrt{3}\,j = 12\angle 60°$	6) $6 - 6\sqrt{3}\,j = 12\angle(-60)°$



The set of complex numbers over the field of real numbers is a 2-dimensional vector space.

http://mathonline.wikidot.com/dimension-of-a-vector-space

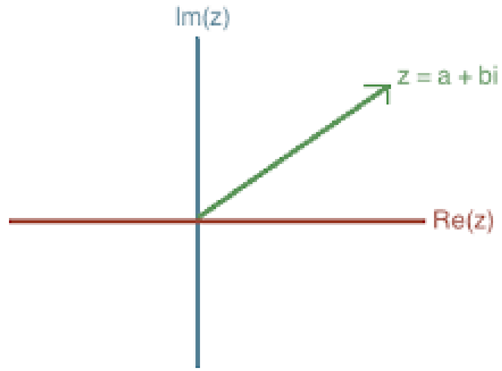
Examples : Cartesian Form and Polar Form

Write the following complex numbers in the rectangular form:

1) $z = 5\angle\theta$ ($\theta = 37°$) 2) $z = 15\angle\theta$ ($\theta = 143°$)

3) $z = 12\angle\theta$ ($\theta = 30°$) 4) $z = 12\angle\theta$ ($\theta = 60°$)

5) $z = 7\sqrt{2}\angle\theta$ ($\theta = 45°$) 6) $z = \frac{9}{2}\sqrt{2}\angle\theta$ ($\theta = 225°$)



The set of complex numbers over the field of real numbers is a 2-dimensional vector space.

http://mathonline.wikidot.com/dimension-of-a-vector-space

Examples : Cartesian Form and Polar Form

For $z = r\angle\theta$,

$$Re(z) = r\cos\theta\ ,\ Im(z) = r\sin\theta.$$

1) $z = 5\angle\theta\ (\ \theta = 37°) = 4 + 3\,j$

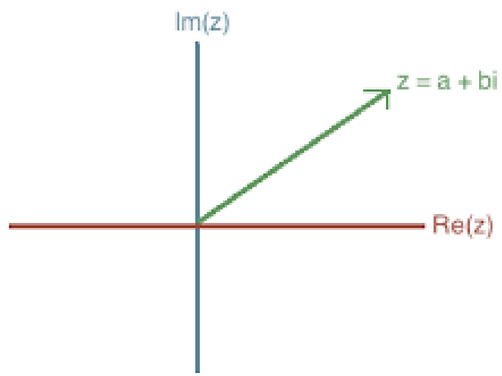
2) $z = 15\angle\theta\ (\ \theta = 143°) = -12 + 9\,j$

3) $z = 12\angle\theta\ (\ \theta = 30°) = 6\sqrt{3} + 6\,j$

4) $z = 12\angle\theta\ (\ \theta = 60°) = 6 + 6\sqrt{3}\,j$

5) $z = 7\sqrt{2}\angle\theta\ (\ \theta = 45°) = 7 + 7\,j$

6) $z = \frac{9}{2}\sqrt{2}\angle\theta\ (\ \theta = 225°) = -\frac{9}{2} - \frac{9}{2}\,j$

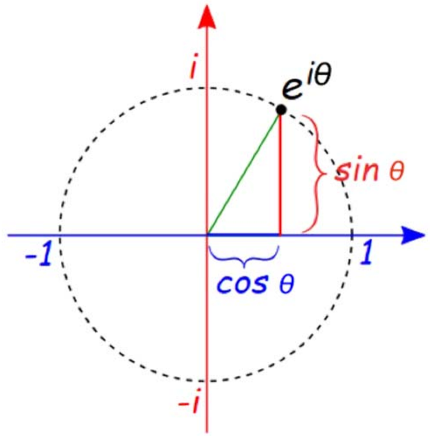


The set of complex numbers over the field of real numbers is a 2-dimensional vector space.

http://mathonline.wikidot.com/dimension-of-a-vector-space

1.3 Euler Form of Complex Number

putting Euler's Formula on that graph produces a circle:



$e^{i\theta}$ produces a circle of radius 1

https://www.mathsisfun.com/algebra/eulers-formula.html

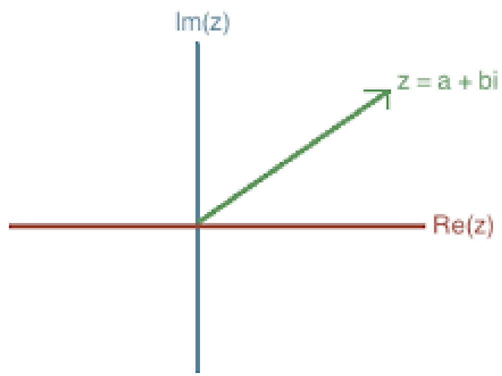
- **Euler's Formula**:
  $$e^{i\theta} = \cos\theta + i\sin\theta$$

- We can turn any point into its polar form of $re^{i\theta}$ (by finding the correct values of θ and r)

Examples: Rectangular Form and Polar Form

Recall the previous six complex numbers and write them in their Euler forms:

1) $3 + 4\,j$
2) $3 - 4\,j$
3) $1 + j$
4) $1 - j$
5) $6 + 6\sqrt{3}\,j$
6) $6 - 6\sqrt{3}\,j$



The set of complex numbers over the field of real numbers is a 2-dimensional vector space.

http://mathonline.wikidot.com/dimension-of-a-vector-space

Examples : Cartesian Form and Polar Form

The essence of writing complex numbers in the Euler form is to find the value of r and θ. The Euler form is simply $z = rei^{\theta}$ where r and θ are the modulus and argument, respectively.

1) $3 + 4\,j = 5e^{\,j(53°)}$

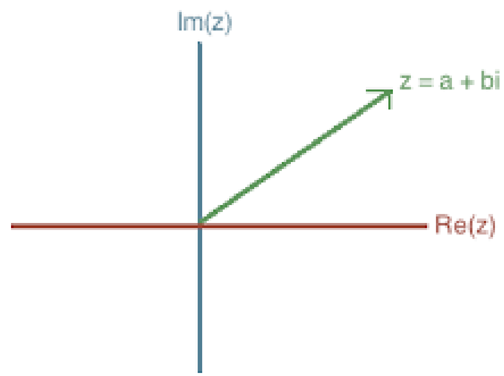
2) $3 - 4\,j = 5e^{\,j(-53°)}$

3) $1 + j = \sqrt{2}e^{\,j(45°)}$

4) $1 - j = \sqrt{2}e^{\,j(-45°)}$

5) $6 + 6\sqrt{3}\,j = 12e^{\,j(60°)}$

6) $6 - 6\sqrt{3}\,j = 12e^{\,j(-60°)}$



The set of complex numbers over the field of real numbers is a 2-dimensional vector space.

http://mathonline.wikidot.com/dimension-of-a-vector-space

1.4 Arithmetic Operation of Complex Number

- The manipulation of basic complex number follows the same rule as the real numbers except that the square of j is negative: $j \cdot j = -1$
- Here are some arithmetic operations:

$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$ **(addition)**

$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$ **(subtraction)**

$z* = x - jy = r\angle - \emptyset = re^{-j\emptyset}$ **(complex conjugate)**

$z_1 z_2 = r_1 r_2 \angle(\emptyset_1 + \emptyset_2)$ **(multiplication)**

$\frac{1}{z} = \frac{1}{r}\angle(-\emptyset)$ **(reciprocal)**

$\sqrt{z} = \sqrt{r}\angle(\emptyset/2)$ **(square root)**

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\emptyset_1 - \emptyset_2)$ **(division)**

Examples : Arithmetic Operation

Compute the following formulas of complex numbers:

1) $(3 + 4\,j)(3 - 4\,j)$

2) $\dfrac{3 + 4\,j}{3 - 4\,j}$

3) $(1 + j) + (1 - j)$

4) $(1 - j) - (1 + j)$

5) $\dfrac{1}{6 + 6\sqrt{3}\,j}$

6) $Z^2 = 6 - 6\sqrt{3}\ j$

Examples : Arithmetic Operation

1) $(3 + 4j)(3 - 4j) = 3\times3 - 3\times4j + 4\times3j - 4\times4j^2 = 9 + 16 = 25$

2) $\frac{3 + 4j}{3 - 4j} = \frac{(3 + 4j)^2}{(3 + 4j)(3 - 4j)} = \frac{9-16+2\times3\times4j}{9+16} = \frac{-7+24j}{25} = -\frac{7}{25} + \frac{24}{25}j$

3) $(1 + j) + (1 - j) = (1+1) + (1-1)j = 2$

4) $(1 - j) - (1 + j) = (1-1) + (-1-1)j = -2j$

5) $\frac{1}{6 + 6\sqrt{3}j} = \frac{6-6\sqrt{3}j}{(6 + 6\sqrt{3}j)(6-6\sqrt{3}j)} = \frac{6-6\sqrt{3}j}{144} = \frac{1-\sqrt{3}j}{24} = \frac{1}{24} - \frac{\sqrt{3}}{24}j$

6) $Z^2=6(1-\sqrt{3}j) = 6 \cdot 2e^{-j\pi/3} \Rightarrow Z = \pm2\sqrt{3}\cdot e^{-j\pi/6} = \pm2\sqrt{3}\cdot(\frac{\sqrt{3}}{2} - j\frac{1}{2})$

$\Rightarrow Z = 3 - \sqrt{3}j$ or $Z = -3 + \sqrt{3}j$

1.5 Summary of Complex Number

- Any complex number z can be written in the form:

  $$Z = a + bi \; (\, a,\, b \text{ are real numbers} \,)$$

  $$Z = r(\cos\theta + i\sin\theta)$$

  $$Z = rei^{\theta}$$

- In the expression, r and θ are the modulus and argument, respectively.
- The arithmetic operations.

2. Differentiation

Differentiation

- 2.1 Introduction
- 2.2 Gradient
- 2.3 Rate of Change
- 2.4 Existence of Derivatives
- 2.5 Common Derivatives
- 2.6 Linearity properties

2.1 Introduction

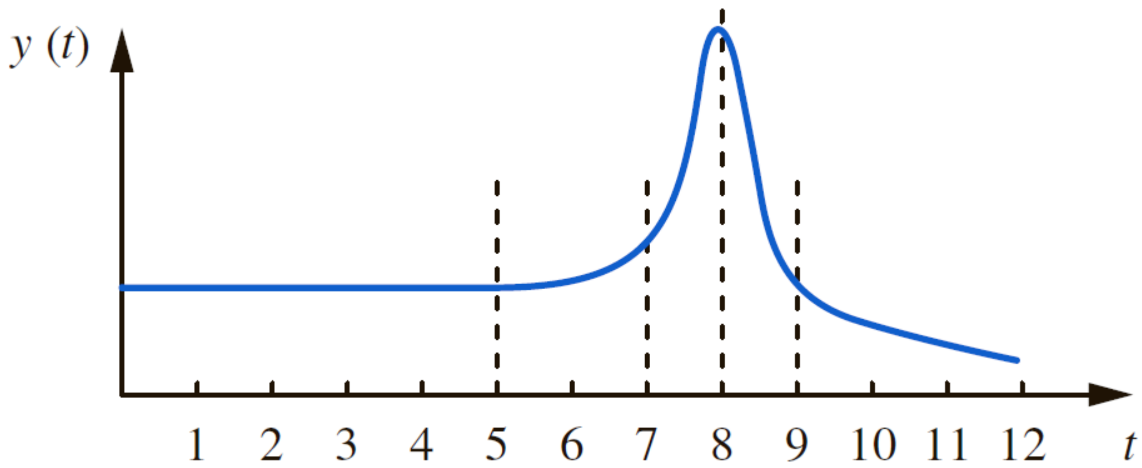
- Differentiation is a mathematical technique for analyzing how a function varies.

- In particular, it determines how rapidly a function is changing at any specific point.

- The function in question may represent the magnetic field of a motor, the voltage across a capacitor, the temperature of a chemical mix, etc., therefore it is often important to know how quickly these quantities change.

2.1 Introduction

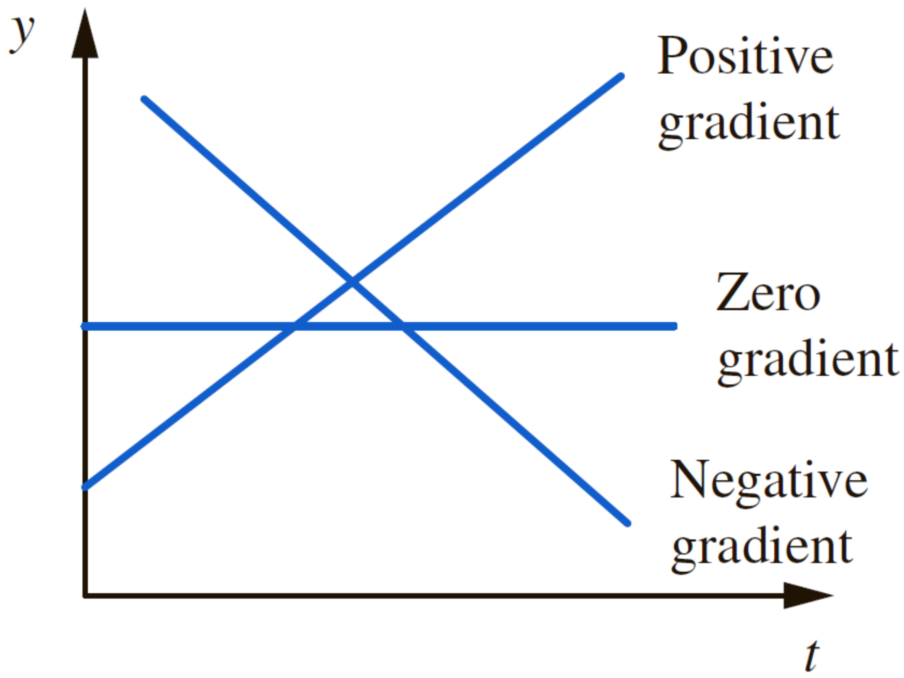
For instance, if the voltage on an electrical supply network is decreasing suddenly due to a short circuit, it needs to detect it and switch off that part.



The function $y(t)$ has different rates of change over time t.

2.2 Gradient

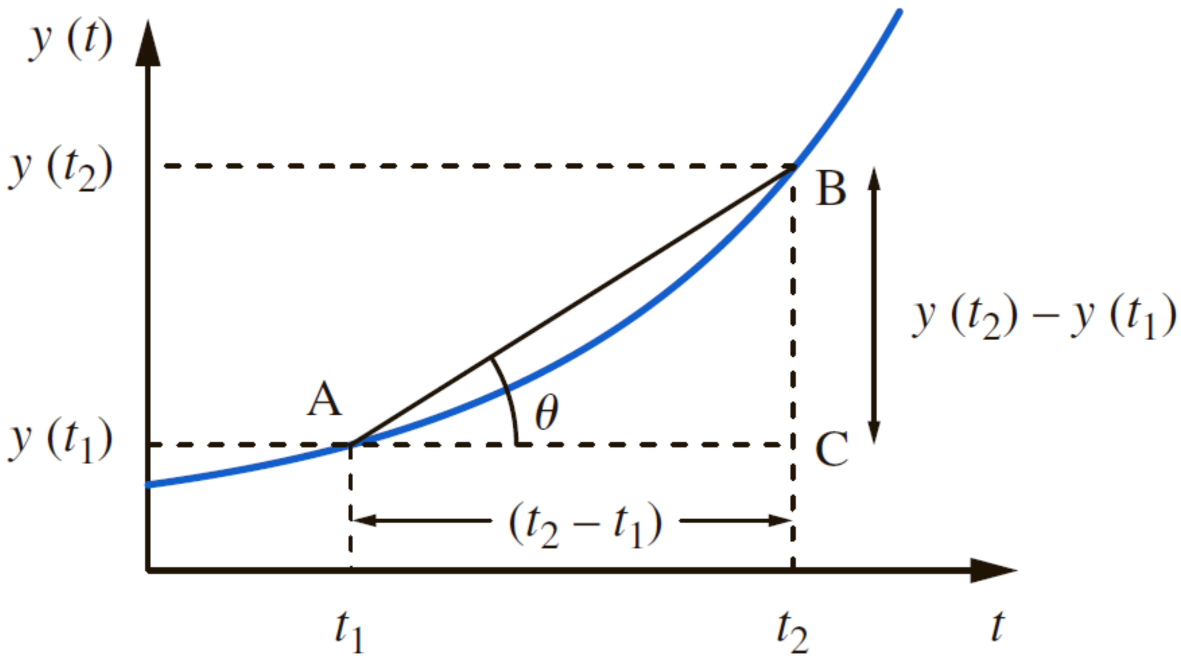
The **gradient** or slope of a line is a measure of its steepness. It can be positive, negative or zero.



Different gradients of different lines

2.2 Gradient

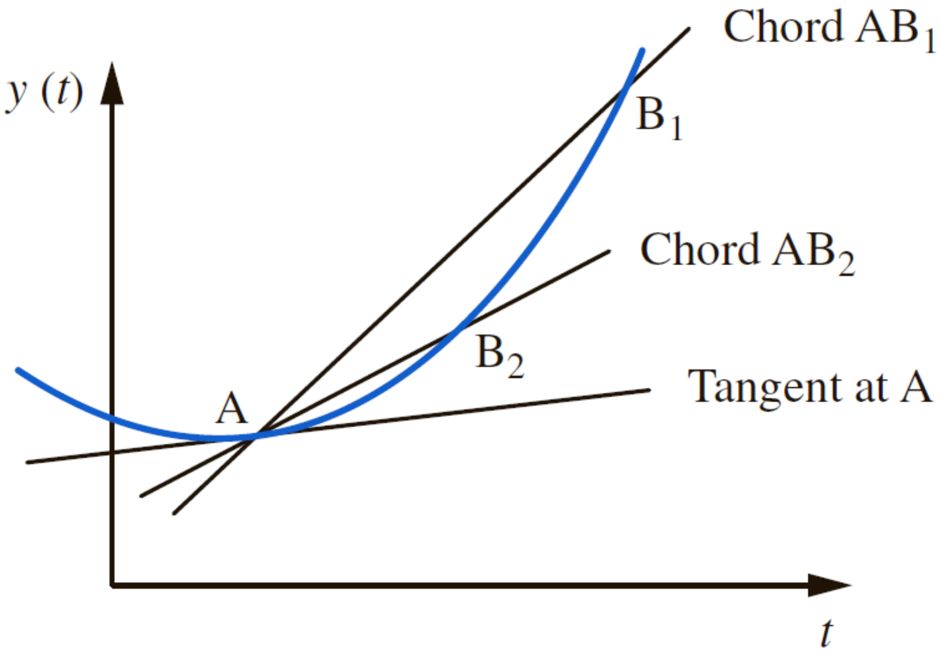
The average rate of change of a function between Point A and Point B is the gradient of the chord AB.



Average rate of change between Points A & B

2.2 Gradient

As Point B approaches Point A from B_1 to B_2 and then to A, the straight line will become a **tangent** to the curve at Point A.

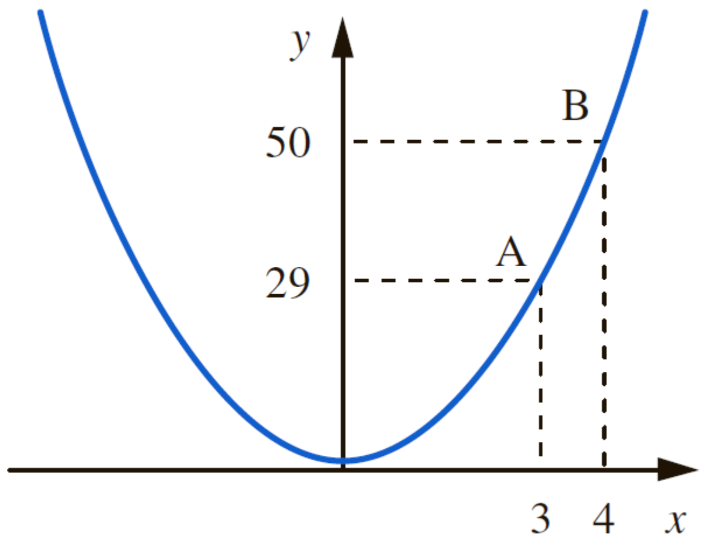


The rate of change of the function at Point A is the slope or gradient of the curve at point A.

2.3 Rate of Change

Example: Given $y = f(x) = 3x^2 + 2$, obtain estimates of the rate of change of y at $x = 3$ by considering the intervals

(a) [3, 4] (b) [3, 3.1] (c) [3, 3.01]



The function: $y = 3x^2 + 2$.

2.3 Rate of Change

Solution:

(a) $y(3) = 3(3)^2 + 2 = 29$; $y(4) = 3(4)^2 + 2 = 50$

Therefore we have Point A (3, 29) and Point B (4, 50) on the curve.

The average rate of change over the interval [3, 4] $= \dfrac{y(4) - y(3)}{4-3} = 21$

This is the gradient of the chord AB and is an estimate of the slope of the tangent at A.

(b) At $x = 3.1$, $y(3.1) = 30.83$,

The average rate of change over the interval [3, 3.1] $= \dfrac{30.83 - 29}{3.1-3} = 18.3$

This is a more accurate estimate of the slope of the tangent at A.

(c) At $x = 3.01$, $y(3.01) = 29.1803$,

The average rate of change over the interval [3, 3.01] $= \dfrac{29.1803 - 29}{3.01 - 3} = 18.03$

This is an even better estimate of the slope of the tangent at A.

2.3 Rate of Change

The rate of change of a function at a particular point has been approximately obtained. We will now develop a **general** method to obtain it accurately.

Suppose we have a function $y(x)$, the average rate of change of y between two points, A(x, $y(x)$) and B($x + \delta x$, $y(x + \delta x)$), is given by

$$\text{average rate of change of } y = \frac{\text{change in } y}{\text{change in } x}$$

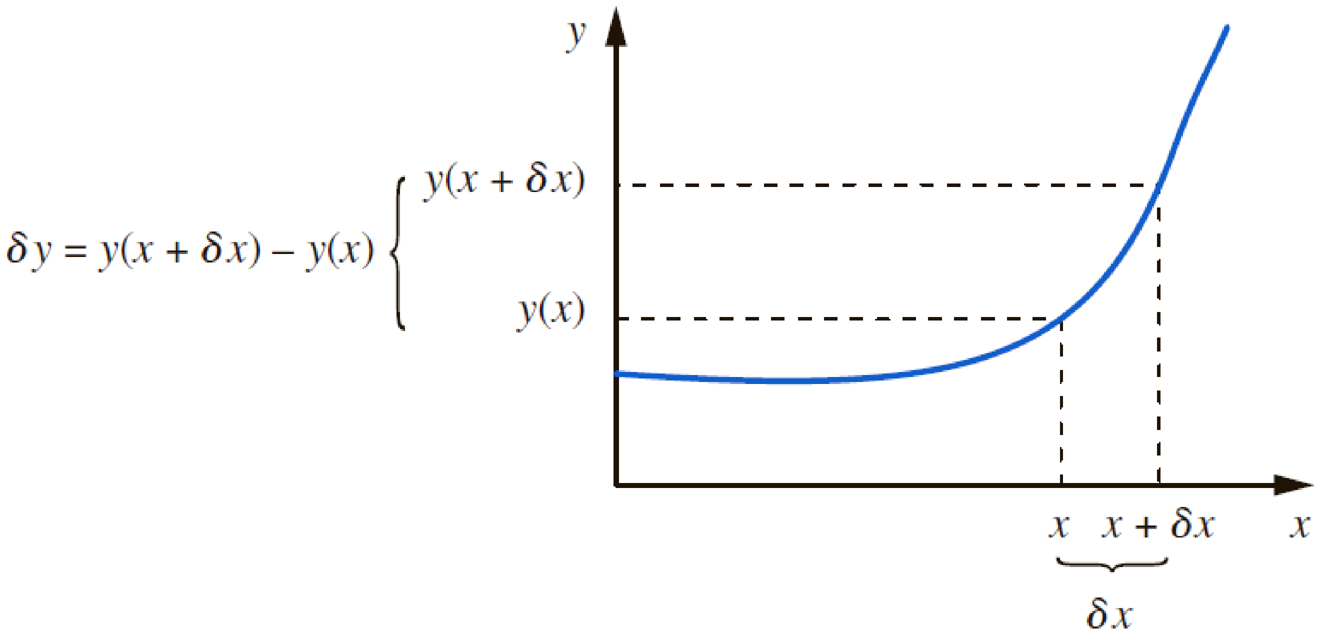
$$= \frac{y(x+\delta x) - y(x)}{(x+\delta x) - x} = \frac{\delta y}{\delta x}$$

Now let δx tend to 0, then Point B will tend to coincide with Point A, i.e.,

$$\text{rate of change of } y = \lim_{\delta x \to 0}\left(\frac{y\big(x + \delta(x)\big) - y(x)}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$$

2.3 Rate of Change

The rate of change of y at a point is found by letting $\delta x \rightarrow 0$.



2.3 Rate of Change

Rate of change of $y(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \left(\frac{y(x+\delta(x)) - y(x)}{\delta x} \right)$

The rate of change of y is called the **derivative** of y. Mathematically,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \right)$$

$\frac{dy}{dx}$ is sometimes denoted by y'

The process of finding y' from y is called **differentiation**.

2.3 Rate of Change

Example: Find the rate of change of $y(x) = 2x^2 + 3x$. Calculate the rates of change of y at $x = 2$ and $x = -3$.

Solution: Given $y(x) = 2x^2 + 3x$

Then

$$y(x+\delta x) = 2(x+\delta x)^2 + 3(x+\delta x)$$
$$= 2x^2 + 4x\delta x + 2(\delta x)^2 + 3x + 3\delta x$$

Hence

$$y(x+\delta x) - y(x) = 2(\delta x)^2 + 4x\delta x + 3\delta x$$

So,

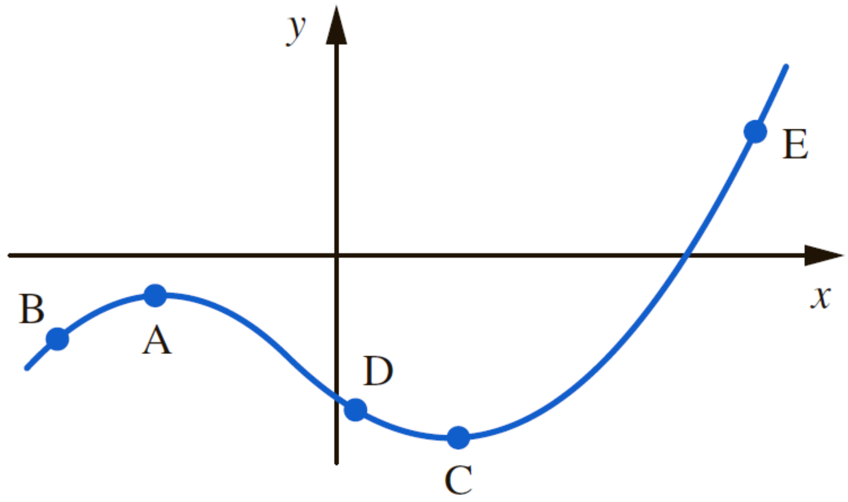
$$\text{rate of change of } y = \lim_{\delta x \to 0}\left(\frac{y(x+\delta x)-y(x)}{\delta x}\right)$$
$$= \lim_{\delta x \to 0}\left(\frac{2(\delta x)^2 + 4x\delta x + 3\delta x}{\delta x}\right)$$
$$= \lim_{\delta x \to 0}(2\delta x + 4x + 3) = 4x + 3$$

When $x = 2$, the rate of change of y is $4(2)+3 = 11$.

When $x = -3$, the rate of change of y is $4(-3) + 3 = -9$.

2.3 Rate of Change

Example: By considering the gradient of the tangent at Points A, B, C, D and E state whether $\frac{dy}{dx}$ is positive, negative, or zero at these points.



Solution:

At A and C the tangent is parallel to the x axis and so $\frac{dy}{dx}$ is zero.

At B and E the tangent has a positive slope and so $\frac{dy}{dx}$ is positive.

At D the tangent has a negative slope and thus $\frac{dy}{dx}$ is negative.

2.4 Existence of Derivatives

So far, we have seen that the derivative, $\frac{\mathrm{d}y}{\mathrm{d}x}$, of a function, $y(x)$, can be viewed either algebraically or geometrically.

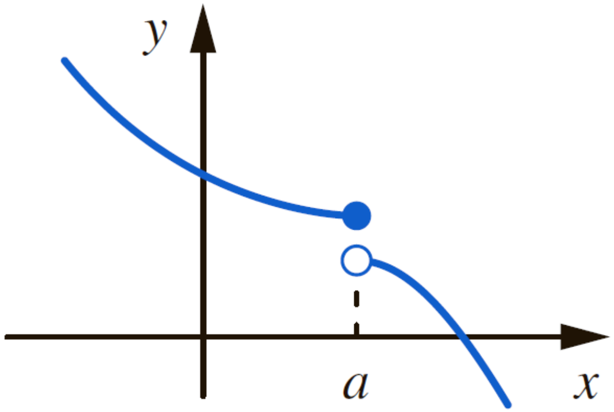
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \left(\frac{y(x + \delta x) - y(x)}{\delta x} \right)$$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \text{rate of change of } y \\ &= \text{slope (gradient) of the tangent of } y \text{ at } x \end{aligned}$$

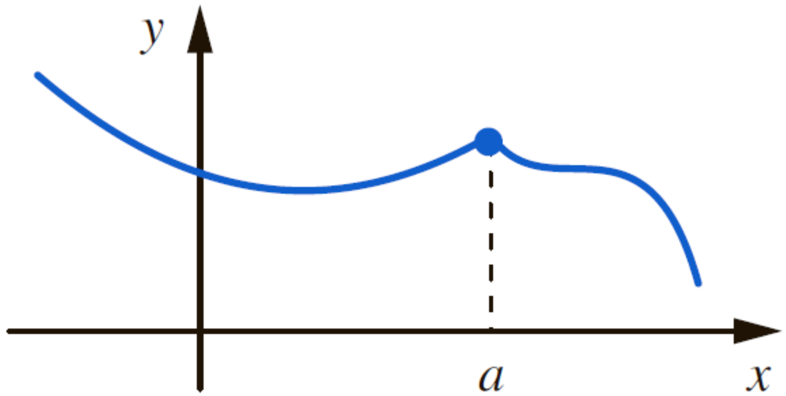
We now discuss briefly the existence of $\frac{\mathrm{d}y}{\mathrm{d}x}$. For some functions, the derivative does **NOT** exist at certain points.

2.4 Existence of Derivatives

Consider the following graphs



(a) The graph has a discontinuity at $x = a$.



(b) The graph is not smooth at $x = a$.

In both cases, it is impossible to draw a tangent at $x = a$, and so $\frac{dy}{dx}$ does not exist at $x = a$. A tangent to either curve at such point does not exist.

2.4 Existence of Derivatives

Example: Sketch the following functions. State the values of t for which the derivative does not exist.

(a) $y = |t|$ (b) $y = \tan t$ (c) $y = 1/t$

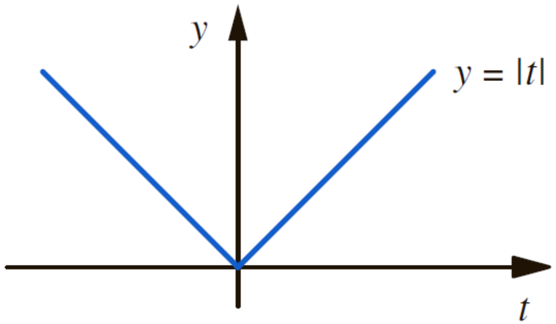
2.4 Existence of Derivatives

Solution:

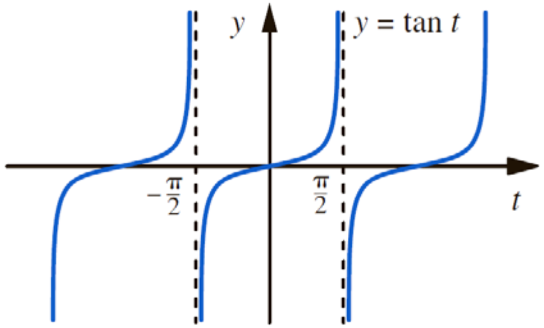
(a) A corner exists at $t = 0$ and so the derivative does not exist at $t = 0$.

(b) There is a discontinuity in tan t when $t = \ldots -3\pi/2, -\pi/2, \pi/2, 3\pi/2, \ldots$ No derivative exists at these points.

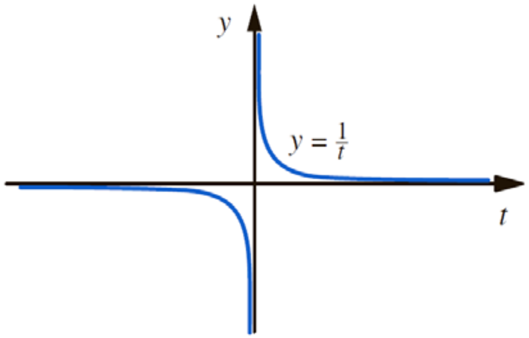
(c) The function has one discontinuity at $t = 0$, and the derivative does not exist at $t = 0$.



(a)



(b)



(c)

2.5 Common Derivatives

Table 1. Derivatives of commonly used functions.

1. $\frac{d}{dx}[cu] = cu'$	**2.** $\frac{d}{dx}[u \pm v] = u' \pm v'$	**3.** $\frac{d}{dx}[uv] = uv' + vu'$
4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$	**5.** $\frac{d}{dx}[c] = 0$	**6.** $\frac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\frac{d}{dx}[x] = 1$	**8.** $\frac{d}{dx}[\lvert u\rvert] = \frac{u}{\lvert u\rvert}(u'), \quad u \neq 0$	**9.** $\frac{d}{dx}[\ln u] = \frac{u'}{u}$
10. $\frac{d}{dx}[e^u] = e^u u'$	**11.** $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$	**12.** $\frac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\frac{d}{dx}[\sin u] = (\cos u)u'$	**14.** $\frac{d}{dx}[\cos u] = -(\sin u)u'$	**15.** $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$	**17.** $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$	**18.** $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1 - u^2}}$	**20.** $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1 - u^2}}$	**21.** $\frac{d}{dx}[\arctan u] = \frac{u'}{1 + u^2}$
22. $\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1 + u^2}$	**23.** $\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{\lvert u\rvert\sqrt{u^2 - 1}}$	**24.** $\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{\lvert u\rvert\sqrt{u^2 - 1}}$
25. $\frac{d}{dx}[\sinh u] = (\cosh u)u'$	**26.** $\frac{d}{dx}[\cosh u] = (\sinh u)u'$	**27.** $\frac{d}{dx}[\tanh u] = (\operatorname{sech}^2 u)u'$
28. $\frac{d}{dx}[\coth u] = -(\operatorname{csch}^2 u)u'$	**29.** $\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u'$	**30.** $\frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \coth u)u'$
31. $\frac{d}{dx}[\sinh^{-1} u] = \frac{u'}{\sqrt{u^2 + 1}}$	**32.** $\frac{d}{dx}[\cosh^{-1} u] = \frac{u'}{\sqrt{u^2 - 1}}$	**33.** $\frac{d}{dx}[\tanh^{-1} u] = \frac{u'}{1 - u^2}$
34. $\frac{d}{dx}[\coth^{-1} u] = \frac{u'}{1 - u^2}$	**35.** $\frac{d}{dx}[\operatorname{sech}^{-1} u] = \frac{-u'}{u\sqrt{1 - u^2}}$	**36.** $\frac{d}{dx}[\operatorname{csch}^{-1} u] = \frac{-u'}{\lvert u\rvert\sqrt{1 + u^2}}$

u **means** $u(x)$

2.5 Common Derivatives

Example: Use Table 1 to find y' when

(a) $y = e^{-7x}$

(b) $y = x^5$

(c) $y = \tan(3x - 2)$

(d) $y = \sin(\omega x + \phi)$

(e) $y = \frac{1}{\sqrt{x}}$

2.5 Common Derivatives

Solution: (a) From Table 1, we find that if

$y = e^{ax}$, then $\quad y' = ae^{ax}$

In this case, a = -7 and so if

$y = e^{-7x}$, then $\quad y' = -7e^{-7x}$

(b) From Table 1, we find that if

$y = x^n$, then $\quad y' = nx^{n-1}$

In this case, n = 5 and so if

$y = x^5$, then $\quad y' = 5x^4$

(c) If $y = \tan(ax+b)$ then $y' = a\sec^2(ax+b)$. In this case, $a = 3$ and $b = -2$. Hence if

$y = \tan(3x-2)$, then $\quad y' = 3\sec^2(3x-2)$

(d) If $y = \sin(ax+b)$ then $y' = a\cos(ax+b)$. Here $a = \omega$ and $b = \phi$, and so if

$y = \sin(\omega x + \phi)$, then $\quad y' = \omega\cos(\omega x + \phi)$

(e) Note that $\frac{1}{\sqrt{x}} = x^{-1/2}$. From Table 1 we find that if $y = x^n$ then $y' = nx^{n-1}$, In this case ,$n = -1/2$ and so if

$y = \frac{1}{\sqrt{x}}$, then $\quad y' = -\frac{1}{2}x^{-\frac{3}{2}}$

2.6 Linearity properties

In mathematical language, differentiation is a **linear operator**. This means that if we wish to differentiate the sum of two functions, we can differentiate each function separately and then simply add the two derivatives, that is

Derivative of $(f + g)$ = derivative of f + derivative of g

This is expressed mathematically as

$$\frac{d}{dx}(f + g) = \frac{\mathrm{df}}{dx} + \frac{dg}{dx}$$

The properties of a linear operator also make the handling of constant factors easy. To differentiate $k\,f$, where k is a constant, we take k times the derivative of f, that is derivative of $(k\,f) = k \times$ derivative of f

Mathematically, we would state:

$$\frac{d}{dx}(kf) = k\frac{df}{dx}$$

2.6 Linearity properties

Example: Differentiate

(a) $3x^2$ (b) $9x$ (c) 7 (d) $3x^2 + 9x + 7$

Solution:

(a) Let $y = 3x^2$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(3x^2\right) \qquad \text{using linearity}$$

$$= 3\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 6x$$

(b) Let $y = 9x$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(9x) \qquad \text{using linearity}$$

$$= 9\frac{\mathrm{d}}{\mathrm{d}x}(x) = 9$$

(c) Let $y = 7$, then $y' = 0$.

(d) Let $y = 3x^2 + 9x + 7$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(3x^2 + 9x + 7\right) \qquad \text{using linearity}$$

$$= 6x + 9$$

2.6 Linearity properties

Example: Use Table 1 and the linearity properties of differentiation to find y' where

(a) $y = 3e^{2x}$

(b) $y = 1/x$

(c) $y = 3\sin(4x)$

(d) $y = \sin(2x) - \cos(5x)$

(e) $y = 3lnx$

(f) $y = ln2x$

(g) $y = 3x^2 + 7x - 5$

2.6 Linearity properties

Solution:

(a) If $y = 3e^{2x}$, then

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(3e^{2x}\right) = 3\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{2x}\right) && \text{using linearity} \\ &= 3\left(2e^{2x}\right) && \text{from Table 1} \\ &= 6e^{2x} \end{aligned}$$

(b) If $y = 1/x$, then

$$\begin{aligned} y' &= -1x^{-2} && \text{from Table 1} \\ &= -\frac{1}{x^2} \end{aligned}$$

(c) If $y = 3\sin 4x$, then

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(3\sin 4x) = 3\frac{\mathrm{d}}{\mathrm{d}x}(\sin 4x) && \text{using linearity} \\ &= 3(4\cos 4x) && \text{from Table 1} \\ &= 12\cos 4x \end{aligned}$$

2.6 Linearity properties

(d) The linearity properties allow us to differentiate each term individually

$$y = \sin 2x - \cos 5x, \qquad \text{then} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(\sin 2x) - \frac{\mathrm{d}}{\mathrm{d}x}(\cos 5x)$$

$$= 2\cos 2x + 5\sin 5x$$

(e) If $y = 3\ln x$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(3\ln x) = 3\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) \qquad \text{using linearity}$$

$$= \frac{3}{x} \qquad \text{from Table 1}$$

(f) If $y = \ln 2x$, then

$$y = \ln 2 + \ln x \qquad \text{using laws of logarithms}$$

and so

$$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x} \qquad \text{since } \ln 2 \text{ is constant}$$

(g) Each term is differentiated:

$$y' = 6x + 7 - 0 = 6x + 7$$

2.7 Chain rule of differentiation

If $y = f(u)$ where $u = u(x)$, then using the chain rule

$$\frac{dy}{dx} = \frac{df}{du}\frac{du}{dx}$$

Example: $y = \sin\left(e^{2x}\right)$, find y'.

Solution: Let $u = e^{2x}$

$$y' = [\frac{d}{du}(\sin u)]\,\frac{du}{dx}$$

$$= (\cos u)\,\frac{d}{dx}(e^{2x})$$

$$= [\cos(e^{2x})]\ (2e^{2x})$$

$$y' = 2e^{2x}\cos(e^{2x})$$

2.8 Product rule of differentiation

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Example: $y = x\sin(2x)$, find y'.

Solution: $y' = x\frac{d}{dx}(\sin 2x) + \sin 2x\frac{d}{dx}(x)$

$= 2x\cos 2x + \sin 2x$

Example: $y = e^{2x}\cos(5x)$, find y'.

Solution: $y' = e^{2x}\frac{d}{dx}(\cos 5x) + \cos 5x\frac{d}{dx}(e^{2x})$

$= -5e^{2x}\sin 5x + 2e^{2x}\cos 5x$

3. Integration

Integration

- 3.1 Introduction
  - 3.1.1 Elementary integration
  - 3.1.2 Definite and indefinite integrals
- 3.2 Techniques of integration
  - 3.2.1 Integration by parts
  - 3.2.2 Integration by substitution
  - 3.2.3 Integration using partial fractions
- 3.3 Applications of integration
  - 3.3.1 Average value of a function
  - 3.3.2 Root mean square values of a function

3.1 Introduction

When a function, $f(x)$, is known we can differentiate it to obtain the derivative, $\frac{df(x)}{dx}$. The reverse process is to obtain the function $f(x)$ from knowledge of its derivative. This process is called **integration**. Thus, **integration is the reverse of differentiation**.

3.1.1 Elementary integration

Considering the problem: *given* $\frac{dy}{dx} = 2x$, *find* $y(x)$.

Solution: The differentiation of the function $y(x) = x^2 + c$ yields

$\frac{dy}{dx} = 2x$ for any c (constant). Hence the solution is $y(x) = x^2 + c$.

Also, since differentiation is the reverse of integration, the solution to this problem can be written as:

$$y = \int 2x\, dx = x^2 + c$$

symbols for integration constant of integration

$$\boxed{\text{In general, if } \frac{dy}{dx} = f(x), \text{ then } y = \int f(x)\, dx}$$

Integration as a linear operator

Integration, like differentiation, is a linear operator. If f and g are two functions of x, then

$$\int (f+g)dx = \int f\,dx + \int g\,dx$$

This states that the integral of a sum of functions is the sum of the integrals of the individual functions. If A is a constant and f is a function of x, then

$$\int Af\,dx = A\int f\,dx$$

Thus, constant factors can be taken through the integral sign.

Integration as a linear operator

(Cont.)

If A and B are constants, and f and g are functions of x, then

$$\int (Af + Bg)dx = A\int f\,dx + B\int g dx$$

These three properties are all consequences of the fact that integration is a linear operator. Note that the first two are special cases of the third.

Example

Given $\frac{dy}{dx} = \cos x - x$, find y.

Solution: First, finding a function which, when differentiated, yields $\cos x - x$. Differentiating $\sin x$ yields $\cos x$, while differentiating $-x^2/2$ yields $-x$. Hence,

$$y = \int (\cos x - x)dx = \sin x - \frac{x^2}{2} + c$$

where c is the constant of integration.

Note: the function to be integrated is known as the **integrand.** In this example, the integrand is $\cos x - x$**.**

Integrals of some common functions

$f(x)$	$\int f(x)\,\mathrm{d}x$	$f(x)$	$\int f(x)\,\mathrm{d}x$
k, constant	$kx + c$	$\cos(ax+b)$	$\dfrac{\sin(ax+b)}{a} + c$
x^n	$\dfrac{x^{n+1}}{n+1} + c \quad n \neq -1$	$\tan x$	$\ln\lvert\sec x\rvert + c$
$x^{-1} = \dfrac{1}{x}$	$\ln\lvert x\rvert + c$	$\tan ax$	$\dfrac{\ln\lvert\sec ax\rvert}{a} + c$
e^x	$\mathrm{e}^x + c$	$\tan(ax+b)$	$\dfrac{\ln\lvert\sec(ax+b)\rvert}{a} + c$
e^{-x}	$-\mathrm{e}^{-x} + c$	$\operatorname{cosec}(ax+b)$	$\dfrac{1}{a}\{\ln\lvert\operatorname{cosec}(ax+b) - \cot(ax+b)\rvert\} + c$
e^{ax}	$\dfrac{\mathrm{e}^{ax}}{a} + c$		
$\sin x$	$-\cos x + c$	$\sec(ax+b)$	$\dfrac{1}{a}\{\ln\lvert\sec(ax+b) + \tan(ax+b)\rvert\} + c$
$\sin ax$	$\dfrac{-\cos ax}{a} + c$		
$\sin(ax+b)$	$\dfrac{-\cos(ax+b)}{a} + c$	$\cot(ax+b)$	$\dfrac{1}{a}\{\ln\lvert\sin(ax+b)\rvert\} + c$
$\cos x$	$\sin x + c$	$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\dfrac{x}{a} + c$
$\cos ax$	$\dfrac{\sin ax}{a} + c$	$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + c$

Note that a, b, n and c are constants. When integrating trigonometric functions, angles must be in radians.

3.1.2 Definite and indefinite integrals

All the integration solutions so far encountered have contained a constant of integration. Such integrals are known as **indefinite integrals**. Integration can be used to determine the area under curves and this gives rise to **definite integrals**.

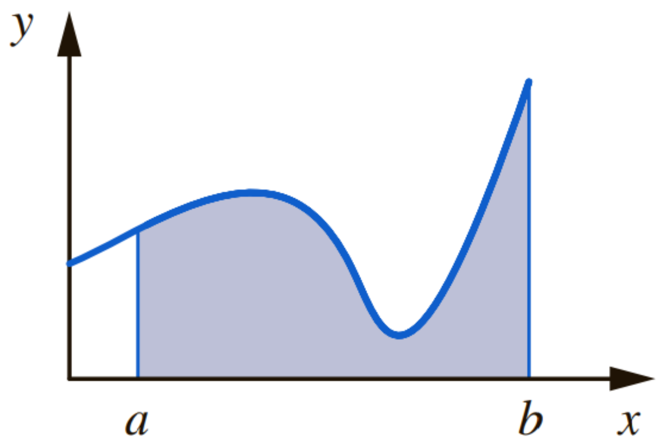


Fig. A

The area under the curve, $y(x)$, between $x = a$ and $x = b$ (see Fig. A) is denoted as

$$\text{Area} = \int_{x=a}^{x=b} y\,dx$$

or more compactly by

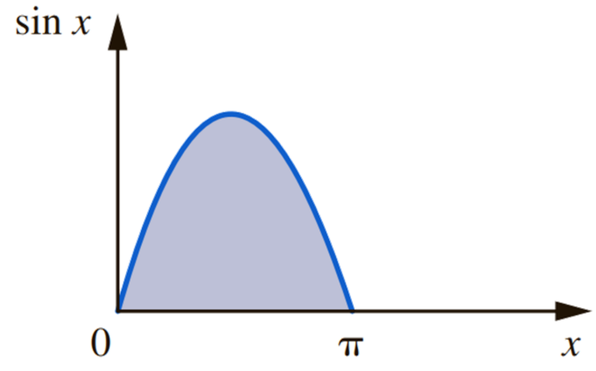
$$\text{Area} = \int_{a}^{b} y\,dx$$

a and b are the **limits** of the integral: **lower** and **upper**, respectively.

Example

Evaluate

$$\int_0^{\pi} \sin x \, dx$$



Figure

Solution Let I represent $\int_0^{\pi} \sin x \, dx$.

$$I = \int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi}$$

The integral is now evaluated at the upper and lower limits. The difference gives the value required.

$$I = (-\cos \pi) - (-\cos 0) = 2$$

Figure shows this area.

3.2.1 Techniques of integration

- **Integration by parts**

This technique is used to integrate a product, and is derived from the product rule for differentiation. Let u and v be functions of x. Then the product rule of differentiation states:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Rearranging we have

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating this equation yields

$$\int u\left(\frac{dv}{dx}\right)dx = \int \frac{d}{dx}(uv)dx - \int v\left(\frac{du}{dx}\right)dx$$

3.2.1 Techniques of integration

Integration by parts

Note that integration and differentiation are inverse processes

$$\int \frac{d}{dx}(uv)dx = uv$$

Hence,

$$\boxed{\int u\left(\frac{dv}{dx}\right)dx = uv - \int v\left(\frac{du}{dx}\right)dx}$$

When dealing with definite integrals the corresponding formula for integration by parts is

$$\boxed{\int_a^b u\left(\frac{dv}{dx}\right)dx = [uv]_a^b - \int_a^b v\left(\frac{du}{dx}\right)dx}$$

Example

Find

$$\int x \sin x dx$$

Solution: We recognize the integrand as a product of the functions x and $\sin x$. Let $u = x$, $\frac{dv}{dx} = \sin x$. Then $\frac{du}{dx} = 1$, $v = -\cos x$. Using the integration by parts formula we get

$$\int x \sin x dx = x(-\cos x) - \int (-\cos x)\, 1 dx = -x \cos x + \sin x + c$$

3.2.2 Techniques of integration

Integration by substitution

This technique is the integral equivalent of the chain rule. It is best illustrated by examples below:

Example 1

Find $\int (3x+1)^{2.5} dx$.

Solution

Let $z = 3x + 1$, so that $\frac{dz}{dx} = 3$, that is $dx = \frac{dz}{3}$. Writing the integral in terms of z, it becomes

$$\int z^{2.5} \frac{1}{3} dz = \frac{1}{3} \int z^{2.5}\, dz = \frac{1}{3}\left(\frac{z^{3.5}}{3.5}\right) + c = \frac{1}{3}\frac{(3x+1)^{3.5}}{3.5} + c$$

3.2.2 Techniques of integration

Example 2

Evaluate $\int_2^3 t\sin(t^2)dt$.

Solution Let $v = t^2$ so $\frac{dv}{dt} = 2t$, that is $dt = \frac{1}{2t}dv$

When changing the integral from one in terms of t to one in terms of v, the limits must also be changed. When $t = 2$, $v = 4$; when $t = 3$, $v = 9$.

Hence, the integral becomes

$$\int_4^9 \frac{\sin v}{2}dv = \frac{1}{2}[-\cos v]_4^9 = \frac{1}{2}[-\cos 9 + \cos 4] = 0.129$$

3.2.3 Techniques of integration

- **Integration using partial fractions**

The technique of expressing a rational function as the sum of its partial fractions. Some expressions which at first sight look impossible to integrate may in fact be integrated when expressed as their partial fractions.

3.2.3 Techniques of integration

- **Integration using partial fractions**

Example 1

Find $\int \frac{1}{x^3+x} dx$.

Solution: First express the integrand in partial fractions:

$$\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Then,

$$1 = A\left(x^2+1\right) + x(Bx+C) = (A+B)x^2 + Cx + A$$

Equating the constant terms: $1 = A$ so that $A = 1$.

Equating the coefficients of x: $0 = C$ so that $C = 0$.

Equating the coefficients of x^2: $0 = A + B$ and hence $B = -1$.

3.2.3 Techniques of integration

- **Integration using partial fractions**

Then **(cont.)**

$$\int \frac{1}{x^3 + x} dx = \int \frac{1}{x} - \frac{x}{x^2 + 1} dx$$

$$= ln|x| - \frac{1}{2} ln\left|x^2 + 1\right| + c$$

$$= ln\left|\frac{x}{\sqrt{x^2+1}}\right| + c$$

3.2.3 Techniques of integration

Integration using partial fractions

Example 2

Evaluate $\int_0^1 \frac{4t^3-2t^2+3t-1}{2t^2+1}\,dt.$

Solution: First express the integrand in partial fractions:

$$\frac{4t^3-2t^2+3t-1}{2t^2+1} = 2t-1+\frac{t}{2t^2+1}$$

Hence,

$$\int_0^1 \frac{4t^3-2t^2+3t-1}{2t^2+1}\,dt = \int_0^1 2t-1+\frac{t}{2t^2+1}\,dt$$

$$= \left[t^2-t+\frac{1}{4}ln\left|2t^2+1\right|\right]_0^1 = 0.275$$

3.3.1 Application of integration

Average value of a function

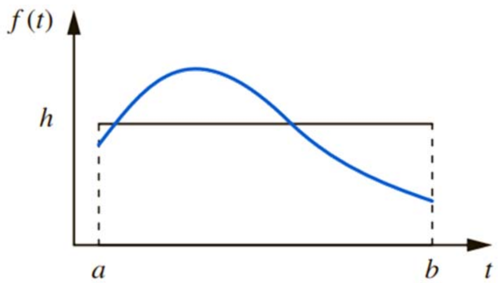
- Suppose *f(t)* is a function defined on $a \le t \le b$. The area, A, under f is given by

$$A = \int_a^b f\,dt$$

- A rectangle with base spanning the interval $[a, b]$ and height h has an area of $h(b - a)$.
  Suppose the height, h, is chosen so that the area under f and the area of the rectangle are equal. This means

$$h = \frac{A}{b-a}$$

h is called the **average value** of the function across the interval $[a, b]$



The area under the curve from $t = a$ to $t = b$ and the area of the rectangle are equal.

Example

Find the average value of $f(t) = t^2$ across

(a) [1,3]

(b) [2,5]

Solution

(a) average value $= \dfrac{\int_1^2 t^2 dt}{3-1} = \dfrac{1}{2}\left[\dfrac{t^3}{3}\right]_1^3 = \dfrac{13}{3}$

(b) average value $= \dfrac{\int_2^5 t^2 dt}{5-2} = \dfrac{1}{3}\left[\dfrac{t^3}{3}\right]_2^5 = 13$

These examples show that if the interval of integration changes then the average value of a function can change.

3.3.2 Application of integration

- **Root mean square value of a function**

➢ If $f(t)$ is defined on $[a, b]$, the **root mean square** (r.m.s.) value is

$$\text{r. m. s.} = \sqrt{\frac{\int_a^b (f(t))^2 dt}{b-a}}$$

Example

Find the r.m.s. value of $f(t) = t^2$ across [1, 3].

Solution

$$\text{r. m. s.} = \sqrt{\frac{\int_1^3 (t^2)^2 dt}{3-1}} = \sqrt{\frac{\int_1^3 t^4 dt}{2}} = \sqrt{\frac{\left[t^5/5\right]_1^3}{2}} = \sqrt{\frac{242}{10}} = 4.92$$

4. Fundamentals to differentiation equations

Fundamentals to differentiation equations

- 4.1 Introduction
- 4.2 Basic definitions: Variables, Order, Linearity, and Solution
- 4.3 First-order equations
  - 4.3.1 Simple equations and separation of variables
  - 4.3.2 Use of an integrating factor
- 4.4 Second-order equations
- 4.5 Constant coefficient equations

4.1 Introduction

The solution of problems concerning the motion of objects, the flow of charged particles, etc., often involves differential equations in the following forms:

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 2x = 3t$$

x can represent distance

$\frac{dx}{dt}$ is speed

$\frac{d^2x}{dt^2}$ represents acceleration

$$\frac{dq}{dt} + 8q = \sin t$$

q can be electric charge

$\frac{dq}{dt}$ is current, rate of flow of charge

4.1 Introduction

Such equations arise out of situations in which change is occurring.

To solve the differential equations as given above, we need to find the functions $x(t), q(t)$.

Solutions to these equations may be **analytical**, i.e., we can write down an answer in terms of common elementary functions such as e^t, $\sin t$ and so on.

Alternatively, the problem may be so difficult that only **numerical methods** are available, which produce approximate solutions.

4.2 Order

- The **order** of a differential equation is the order of its **highest** derivative.
- Examples

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x$$

Second-order differential equation

$$\frac{dx}{dt} = (xt)^5$$

First-order differential equation

- In general, the higher the order of an equation, the more difficult the equation can be solved.

4.2 Linearity

A differential equation is said to be **linear** if:
(1) the dependent variable is of the first order only;
(2) there are no products between the dependent variable and its derivatives; and
(3) there are no non-linear functions of the dependent variable such as sine, exponential, etc.

If an equation is not linear, then it is said to be **non-linear**.
Example of (2): product term $y\frac{dy}{dx}$ causes an equation to be non-linear.

Example of (1) & (3): y^2, $\sin y$, and e^y cause an equation to be non-linear.

4.2 Linearity

Note also that the conditions for linearity are conditions on the dependent variable. The linearity of a differential equation is not determined or affected by the presence of non-linear terms involving the independent variable.

The distinction between a linear and a non-linear differential equation is important because the methods of solution depend upon whether an equation is linear or nonlinear. Furthermore, it is usually easier to solve a linear differential equation.

4.2 Linearity

- **Questions**

  Decide whether or not the following equations are linear:

  (a) $\sin x \dfrac{dy}{dx} + y = x$ (c) $\dfrac{d^2y}{dx^2} + y^2 = 0$

  (b) $\dfrac{dx}{dt} + x = t^3$ (d) $\dfrac{dy}{dx} + \sin y = 0$

- **Solution**

In (a), (c) and (d) the dependent variable is y, and the independent variable is x.
In (b) the dependent variable is x and the independent variable is t.

(a) Linear. (b) Linear (t is not the dependent variable)
(c) Non-linear due to y^2 (d) Non-linear due to $\sin y$

4.3 Solution

The **solution** of a differential equation is a relationship between the dependent and independent variables such that the differential equation is satisfied for all values of the independent variable over a specified domain.

- Example

Verify that $y = Ce^x$ is a solution of the differential equation

$$\frac{dy}{dx} = y$$

Where C is any constant

Solution:

Note that this is a first-order and linear differential equation.

Example

Solution (cont.):

If $y = Ce^x$ then $\frac{dy}{dx} = Ce^x$. For all values of x, $\frac{dy}{dx} = y$ and the equation is satisfied for any constant C.
C is called an **arbitrary constant** and by varying it, all possible solutions can be obtained.

A differential equation can have many different solutions.
The **general solution** embraces all possible solutions.
For example, $y = Ce^x$ is the general solution of $\frac{dy}{dx} = y$

Example

Solution (cont.):

More generally, to determine C we require more information given in the form of a **condition**. For example, if we are told that, at $x = 0$, $y = 4$ then from $y = Ce^x$ we have

$$4 = Ce^0 = C$$

so that $C = 4$. Therefore $y = 4\,e^x$ is the solution of the differential equation which additionally satisfies the condition $y(0) = 4$. This is called a **particular solution**.

4.3 Solution

In general, application of conditions to the general solution yields the **particular solution**.

To obtain a particular solution, the number of given independent conditions must be the same as the number of unknown constants.

Initial conditions: Conditions specified at independent variables $= 0$

4.3.1 First-order equations: simple equations

The simplest first-order equations to deal with are those of the form

$$\frac{dy}{dx} = f(x)$$

where the $f(x)$ r.h.s (right hand side) is a function of the independent variable only. No special treatment is necessary and direct integration yields y as a function of x, that is

$$y = \int f(x)\,dx$$

Simple equations-Example

Find the general solution of $\frac{dy}{dx} = 3\cos 2x$

Solution:

Given that , $\frac{dy}{dx} = 3\cos 2x$

Then $y = \int 3\cos 2x \, dx = \frac{3}{2}\sin 2x + C$

This is the required general solution.

4.3.1 First-order equations: separation of variables

When the function on the r.h.s. of the equation depends upon both independent and dependent. The first-order equations in the form

$$\frac{dy}{dx} = f(x)g(y)$$

form an important class known as **separable equations**.

To obtain a solution we first divide both sides by $g(y)$

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

Integrating both sides with respect to x yields

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

The equation is then said to be **separated**.

Separation of variables-Example

If the last two integrals can be found, we obtain a relationship between y and x, although it is not always possible to write y explicitly in terms of x as the following examples will show.

Solve $\frac{dy}{dx} = \frac{e^{-x}}{y}$

Solution:

Here $f(x) = e^{-x}$ and $g(y) = \frac{1}{y}$.

Multiplication through by y yields

$$y\frac{dy}{dx} = e^{-x}$$

Integration of both sides with respect to x gives

$$\int y\,dy = \int e^{-x}\,dx$$

Separation of variables-Example

Solution (cont.):

So that
$$\frac{y^2}{2} = -e^{-x} + C$$

Note that the constants arising from the two integrals have been combined to give a single constant, C. Finally, we can rearrange this expression to give y in terms of x:

$$y^2 = -2e^{-x} + 2C$$

that is,

$$y = \pm\sqrt{D - 2e^{-x}}$$

Where $D = 2C$.

4.3.2 First-order equations: use of an integrating factor

First-order linear differential equations can always be written in the 'standard' form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are both functions of the independent variable, x, only. In some cases, either of these may be simply constants.

- Examples

$$\frac{dy}{dx} + 3xy = 7x^2 \qquad \frac{dy}{dx} - \frac{2y}{x} = 4e^{-x} \qquad \frac{dy}{dx} - 5y = \sin x$$

$$\mathrm{P}(x) = 3x \qquad \mathrm{P}(x) = -\frac{2}{x} \qquad \mathrm{P}(x) = -5$$

$$Q(x) = 7x^2 \qquad Q(x) = 4e^{-x} \qquad Q(x) = \sin x$$

The integrating factor method

All first-order linear equations, even when they are not exact, can be made exact by multiplying them through by a function known as an **integrating factor**.

Consider again a first-order linear equation in standard form:

$$\frac{dy}{dx} + Py = Q$$

Multiplying both sides by μ yields

$$\mu\frac{dy}{dx} + \mu Py = \mu Q$$

The aim is to make the equation exact. That is, the l.h.s (left hand side) can be written in the form

$$\frac{d}{dx}(\mu y)$$

The integrating factor method

If the l.h.s. is to equal $\frac{d}{dx}(\mu y)$, then we must have

$$\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \mu P y$$

Expanding the l.h.s. using the product rule gives

$$\mu \frac{dy}{dx} + y \frac{d\mu}{dx} = \mu \frac{dy}{dx} + \mu P y$$

which simplifies to

$$y \frac{d\mu}{dx} = \mu P y \qquad \frac{d\mu}{dx} = \mu P$$

$$\mu = e^{\int P dx}$$

The function μ (**integrating factor**) is a function of x only.

The integrating factor method

$$\mu \frac{dy}{dx} + \mu P y = \mu Q$$

With the choice of $\mu = e^{\int P dx}$

$$\frac{d}{dx}(\mu y) = \mu Q$$

This exact equation can be solved by integration to give

$$\mu y = \int \mu Q \, dx$$

$$y = \frac{1}{\mu} \int \mu Q \, dx$$

Integrating factor method-Example

Solve the differential equation $\frac{dy}{dx} + \frac{y}{x} = 1$ using the integrating factor method.

Solution:

Referring to the standard first-order linear equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

we see that $P(x) = \frac{1}{x}$ and $Q(x) = 1$. Using the previous formula for $\mu(x)$, we find

$$\mu(x) = e^{\int (1/x)dx} = e^{\ln x} = x$$

Then with $\mu(x) = x$ and $Q(x) = 1$ we have

$$xy = \int x\,dx = \frac{x^2}{2} + C \qquad y = \frac{x}{2} + \frac{C}{x}$$

4.4 Second-order linear equations

The general form of a second-order linear ordinary differential equation is

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = f(x)$$ **Inhomogeneous equation**

where $p(x)$, $q(x)$, $r(x)$ and $f(x)$ are functions of x only.

An important relative of this equation is

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = 0$$ **Homogeneous equation**

Ignoring the term which is independent of y, all its terms contain y or its derivatives.

4.4 Second-order linear equations

- Questions

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - 1\right)y = e^{-x}$$

$$x \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - 1\right)y = 0$$

What about $p(x)$, $q(x)$, $r(x)$ and $f(x)$ of these equations?

Homogeneous equation? Or Inhomogeneous equation?

4.4 Second-order linear equations-Property

- Property 1

If $y_1(x)$ and $y_2(x)$ are any two linearly independent solutions of a second-order homogeneous equation then the **general solution**, $y_{\mathrm{H}}(x)$, is

$$y_{\mathrm{H}}(x) = Ay_1(x) + By_2(x)$$

where A, B are constants.

- The second-order linear ordinary differential equation has two arbitrary constants in its general solution.
- The functions $y_1(x)$ and $y_2(x)$ are linearly independent if one is not simply a multiple of the other.

4.4 Second-order linear equations-Property

- Property 2

Let $y_{\mathrm{P}}(x)$ be any solution of an inhomogeneous equation and $y_{\mathrm{H}}(x)$ be the general solution of the associated homogeneous equation.
The general solution of the inhomogeneous equation is then given by

$$y(x) = y_{\mathrm{H}}(x) + y_{\mathrm{P}}(x)$$

The function $y_{\mathrm{H}}(x)$ is known as the **complementary function** and $y_{\mathrm{P}}(x)$ is called the **particular integral**.

Second-order linear equations-Example

Verify that $y_1(x) = x$ and $y_2(x) = 1$ both satisfy $\frac{d^2y}{dx^2} = 0$. Write down the general solution of this equation and verify that this indeed satisfies the equation.

Solution:

If $y_1(x) = x$ then $\frac{dy}{dx} = 1$ and $\frac{d^2y}{dx^2} = 0$, so that y_1 satisfies $\frac{d^2y}{dx^2} = 0$.

If $y_2(x) = 1$ then $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, so that y_2 satisfies $\frac{d^2y}{dx^2} = 0$.

Second-order linear equations-Example

Solution (cont.):

From Property 1, the general solution of $\frac{d^2y}{dx^2} = 0$ is

$$\begin{aligned} y_{\mathrm{H}}(x) &= Ay_1(x) + By_2(x) \\ &= Ax + B(1) \\ &= Ax + B \end{aligned}$$

To verify that this satisfies the equation proceed as follows:

$$\frac{dy_{\mathrm{H}}}{dx} = A \qquad \frac{d^2y_{\mathrm{H}}}{dx^2} = 0$$

So $y_{\mathrm{H}}(x)$ satisfies $\frac{d^2y}{dx^2} = 0$

4.5 Constant coefficient equations

Second-order linear equations which have constant coefficients. The general form is

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$ **Inhomogeneous equation**

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$ **Homogeneous equation**

where a, b, c are constants.

4.5 Constant coefficient equations

As stated in Property 2, finding the general solution of $ay'' + by' + cy = f$ is a two-stage process.
The first task is to determine the **complementary function**. This is the general solution of the corresponding homogeneous equation, that is $ay'' + by' + cy = 0$.

- Finding the complementary function

Verify that $y_1 = e^{4x}$ and $y_2 = e^{2x}$ both satisfy the constant coefficient homogeneous equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$$

Write down the general solution of this equation.

4.5 Constant coefficient equations

Solution:

If $y_1 = e^{4x}$, differentiation yields $\frac{dy}{dx} = 4e^{4x}$ and similarly, $\frac{d^2y}{dx^2} = 16e^{4x}$.

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 16e^{4x} - 6(4e^{4x}) + 8e^{4x} = 0$$

So that $y_1 = e^{4x}$ is indeed a solution.

If $y_2 = e^{2x}$, differentiation yields $\frac{dy}{dx} = 2e^{2x}$ and similarly, $\frac{d^2y}{dx^2} = 4e^{2x}$.

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 4e^{2x} - 6\left(2e^{2x}\right) + 8e^{2x} = 0$$

So that $y_2 = e^{2x}$ is also a solution.

4.5 Constant coefficient equations

Solution (cont.):

Now e^{4x} and e^{2x} are **linearly independent functions**,
So, from Property 1 we have

$$y_{\mathrm{H}}(x) = Ay_1(x) + By_2(x) = Ae^{4x} + Be^{2x}$$

As the general solution (**complementary function**) of
$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$

4.5 Constant coefficient equations

- Example

Find values of k so that $y = e^{kx}$ is a solution of

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

Hence state the general solution.

Solution:

For $y = e^{kx}$, differentiation yields $\frac{dy}{dx} = ke^{kx}$ and similarly, $\frac{d^2y}{dx^2} = k^2e^{kx}$.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = k^2e^{kx} - ke^{kx} - 6e^{kx} = 0$$

That is,

$$(k^2 - k - 6)e^{kx} = 0$$

4.5 Constant coefficient equations

Solution (cont.):

The only way this equation can be satisfied for all values of x is if

$$k^2 - k - 6 = (k-3)(k+2) = 0 \qquad \textbf{auxiliary equation}$$

So that $k = 3$ or $k = -2$. If $y = e^{kx}$ is to be a solution of the differential equation k must be either 3 or −2. We therefore have found two solutions

$$y_1(x) = e^{3x} \qquad y_2(x) = e^{-2x}$$

These two functions are **linearly independent** and we can therefore apply Property 1 to give the general solution:

$$y_{\mathrm{H}}(x) = Ay_1(x) + By_2(x) = Ae^{3x} + Be^{-2x}$$

4.5 Constant coefficient equations

As stated in Property 2, finding the general solution of $ay'' + by' + cy = f$ is a two-stage process. Which is the **sum** of the complementary function and a particular integral.

The second task is to determine a **particular integral**. There are a number of advanced techniques available for finding such solutions, but these are beyond the scope of this course.

- Finding a particular integral

Find the general of the equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = e^{2x}$$

Solution:

The complementary function for this equation

$$y_{\mathrm{H}}(x) = Ae^{3x} + Be^{-2x}$$

4.5 Constant coefficient equations

Solution (cont.):

Find a solution of the inhomogeneous problem by trying a function of the same form as that on the r.h.s.

Let us try $y_{\mathrm{P}}(x) = \alpha e^{2x}$, where α is a constant

If $y_{\mathrm{P}}(x) = \alpha e^{2x}$, differentiation yields $\frac{dy_{\mathrm{P}}}{dx} = 2\alpha e^{2x}$ and similarly, $\frac{d^2 y_{\mathrm{P}}}{dx^2} = 4\alpha e^{2x}$.

$$4\alpha e^{2x} - 2\alpha e^{2x} - 6\alpha e^{2x} = e^{2x}$$

So that **the particular integral** $y_{\mathrm{P}}(x) = -\frac{e^{2x}}{4}$.

From Property 2 the **general solution** is

$$y(x) = y_{\mathrm{H}}(x) + y_{\mathrm{P}}(x) = Ae^{3x} + Be^{-2x} - \frac{e^{2x}}{4}$$

4.5 Constant coefficient equations

- Example

  Obtain a particular integral of the equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = x$$

Solution:

In the last example, we found that a fruitful approach was to **assume a solution in the same form** as that on the r.h.s.

Suppose we assume a solution $y_\mathrm{P}(x) = \alpha x$ and proceed to determine α. This approach will actually fail!

If $y_\mathrm{P}(x) = \alpha x$, then $\frac{dy_\mathrm{P}}{dx} = \alpha$ and similarly $\frac{d^2y_\mathrm{P}}{dx^2} = 0$.

$$0 - 6\alpha + 8\alpha x = x$$

No α value can satisfy this equation.

4.5 Constant coefficient equations

Solution (cont.):

So, we try a solution of the form $y_P(x) = \alpha x + \beta$ and proceed to determine α and β.

$$\alpha = \frac{1}{8} \qquad \beta = \frac{3}{32}$$

The required particular integral is $y_P(x) = \frac{1}{8}x + \frac{3}{32}$

- Trial solutions to find the particular integral.

$f(x)$	*Trial solution*
constant	constant
polynomial in x of degree r	polynomial in x of degree r
$\cos kx$	$a\cos kx + b\sin kx$
$\sin kx$	$a\cos kx + b\sin kx$
$a\,e^{kx}$	$\alpha\,e^{kx}$

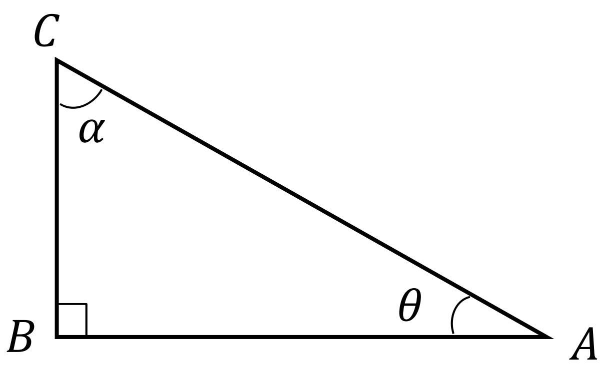
5. Trigonometry

Trigonometry

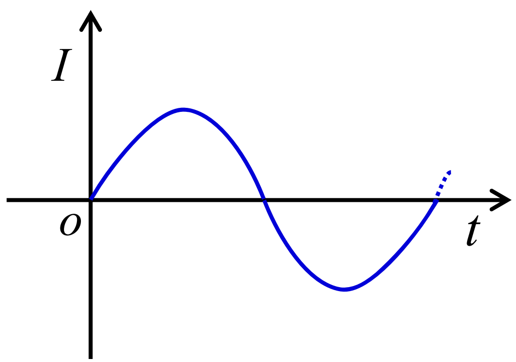
- 5.1 Introduction
- 5.2 Basic Concepts
- 5.3 Trigonometric functions
- 5.4 Trigonometric identities
- 5.5 Modeling waves with Trigonometry
- 5.6 Trigonometric equations

5.1 Introduction

Trigonometry is a branch of mathematics that studies relationships between ***side lengths*** and ***angles*** of triangles. It is an important tool to describe and analyze many ***electrical engineering*** problems, such as alternating currents and voltages, radio waves and so on.



Triangular



Alternating Currents (AC)

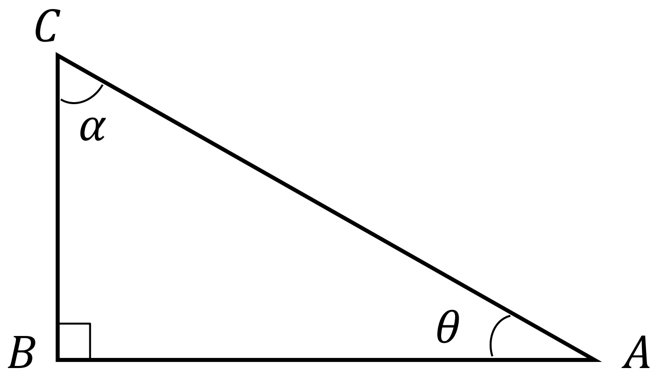
Radio Waves

5.2 Basic Concepts

(1) Degrees and radians:

$$1° = \frac{2\pi}{360} = \frac{\pi}{180} \text{ radians}$$

(2) Trigonometric Ratios



A right-angled triangle, ABC

$$\sin\theta = \frac{\text{side opposite to angle}}{\text{hypotenuse}} = \frac{BC}{AC} = \cos\alpha$$

$$\cos\theta = \frac{\text{side adjacent to angle}}{\text{hypotenuse}} = \frac{AB}{AC} = \sin\alpha$$

$$\tan\theta = \frac{\text{side opposite to angle}}{\text{side adjacent to angle}} = \frac{BC}{AB} = \frac{1}{\tan\alpha}$$

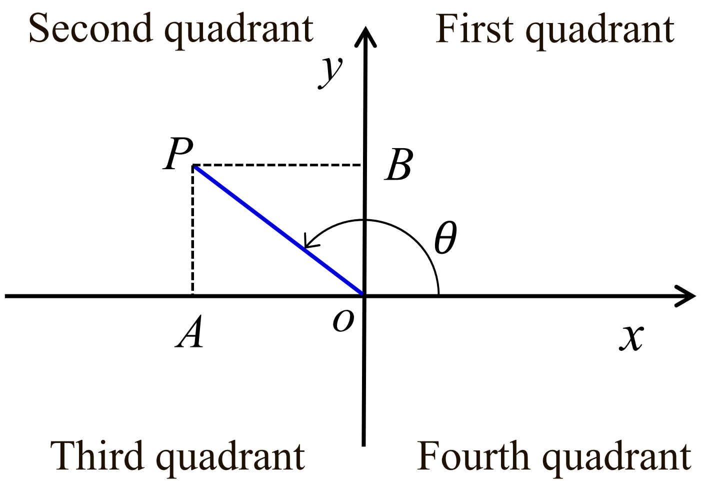
5.2 Basic Concepts

(3) Trigonometry in Cartesian Coordinate

$$\sin\theta = \frac{\text{projection of } OP \text{ onto } y \text{ axis}}{OP} = \frac{OB}{OP}$$

$$\cos\theta = \frac{\text{projection of } OP \text{ onto } x \text{ axis}}{OP} = \frac{OA}{OP}$$

$$\cos\theta = \frac{\text{projection of } OP \text{ onto } y \text{ axis}}{\text{projection of } OP \text{ onto } x \text{ axis}} = \frac{OB}{OA}$$



Examples

(1) Frequently calculations with 30^0, 45^0 and 60^0:

$$\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2},\quad \cos 30° = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2},\quad \tan 30° = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\sin 45° = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}},\quad \cos 45° = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},\quad \tan 45° = \tan\frac{\pi}{4} = 1$$

$$\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},\quad \cos 60° = \cos\frac{\pi}{3} = \frac{1}{2},\quad \tan 60° = \tan\frac{\pi}{3} = \sqrt{3}$$

(2) Find the sign (negative or positive) of the trigonometric ratios in each quadrant:

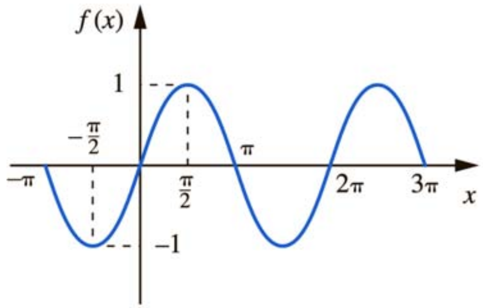
Second quadrant	First quadrant
$\sin\theta > 0$ $\cos\theta < 0$ $\tan\theta < 0$	$\sin\theta > 0$ $\cos\theta > 0$ $\tan\theta > 0$
$\sin\theta < 0$ $\cos\theta < 0$ $\tan\theta > 0$	$\sin\theta < 0$ $\cos\theta > 0$ $\tan\theta < 0$
Third quadrant	Fourth quadrant

y, o, x

5.3 Trigonometric functions

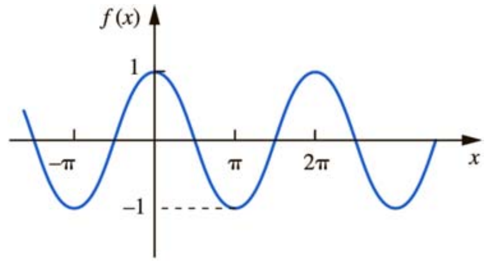
There are three kinds of trigonometric function following directly from the trigonometric ratios:

(1) $f(x) = \sin x$

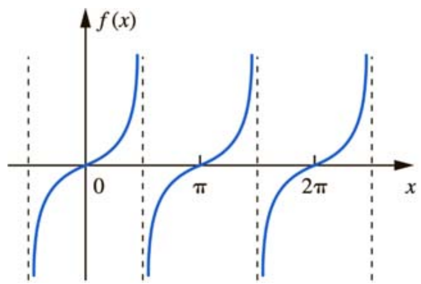


A shifting of π/2

(2) $f(x) = \cos x$



(3) $f(x) = \tan x$



(Notice : In general, we use radians to plot the trigonometric graphs!)

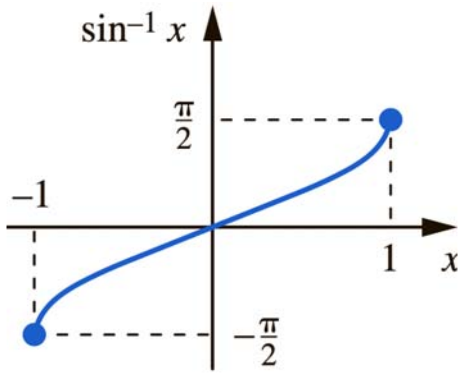
5.3 Trigonometric functions

Observing from the graphs, we can conclude some important properties of the trigonometric functions:

- $-\sin x = \sin(-x)$
- $\cos x = \cos(-x)$
- $-\tan x = \tan(-x)$

The inverse functions of the trigonometric functions are denoted as $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$. For $\sin^{-1} x$, the domain is restricted to $[-\pi/2, \pi/2]$, we write

$$y = \sin^{-1} x, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$



Examples

(1) Find the values of x to meet following equations:

$$\text{(a) } \sin x = 0, \qquad \text{(b) } \sin x = 0.5$$
$$\text{(c) } \cos x = 0, \qquad \text{(d) } \cos x = 0.5$$

Solutions:

(a) $x = n\pi$, ($n=0, \pm1, \pm2\ldots$) (b) $x = \pm\pi/3+(4n\pi + \pi)/2$, ($n=0, \pm1, \pm2\ldots$)

(c) $x = \pi/2 + n\pi$, ($n=0, \pm1, \pm2\ldots$) (d) $x = \pm\pi/3 + 2n\pi$, ($n=0, \pm1, \pm2\ldots$)

(2) Use a scientific calculator to evaluate

$$\text{(a) } \sin^{-1}(0.4139),$$
$$\text{(b) } \sin^{-1}(-0.6001)$$

Solutions:

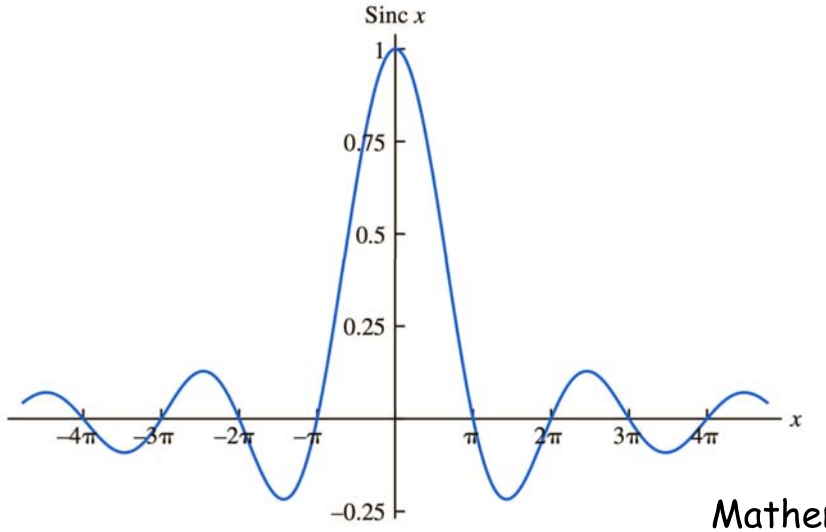
$$\text{(a) } \sin^{-1}(0.4139) = 0.4267,$$
$$\text{(b) } \sin^{-1}(-0.6001) = -0.6436$$

5.3 Trigonometric functions

Another important function in engineering is called the cardinal sine function, Sinc x. It is defined by the sine function. We write

$$\operatorname{Sinc} x = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

its graph is plotted as below



Examples

Find the zero crossing points of the Sinc function :

$$\text{Sinc } x = 0$$

$$\text{Where, Sinc } x = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

Solutions:

$$x = n\pi, \ (n = \pm 1, \pm 2\ , \pm 3\ \ldots)$$

It should be mentioned that those zero crossing points are the same points as sin *x*, except the crossing at $x = 0$.

5.4 Trigonometric identities

Common trigonometric identities.

$$\tan A = \frac{\sin A}{\cos A}$$

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$

$$\cos(\mathrm{A} \pm \mathrm{B}) = \cos \mathrm{A} \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin^2 A + \cos^2 A = 1$$

$$\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \cos^2 A - \sin^2 A$$

$$\sin 2\mathrm{A} = 2\sin A \cos A$$

Examples

Using the identities to simplify the following expressions:

$$\text{(a) } \sin(\pi - \theta)$$

$$\text{(b) } \cos(\pi/2 - \theta)$$

Solutions:

(a) With reference to the identity $\sin(A - B) = \sin \text{A} \cos B - \sin B \cos A$

Putting $A = \pi$ and $B = \theta$, we have

$$\sin(\pi - \theta) = \sin \pi \cos \theta - \sin \theta \cos \pi$$

Because $\sin \pi = 0$, $\cos \pi = -1$, and so

$$\sin(\pi - \theta) = 0 - \sin \theta\,(-1) = \sin \theta.$$

(b) With reference to the identity $\cos(A - B) = \cos \text{A} \cos B + \sin A \sin B$

Putting $A = \pi/2$ and $B = \theta$, we have

$$\cos(\pi/2 - \theta) = \cos \pi/2 \cos \theta + \sin \pi/2 \sin \theta$$

Because $\cos \pi/2 = 0$, $\sin \pi/2 = 1$, and so

$$\cos(\pi - \theta) = 0 + \sin \theta = \sin \theta.$$

5.5 Modeling Waves with Trigonometry

In electrical engineering, we often care about current waves and voltage waves. In fact, those waves can be described with sine or cosine function. They often vary with time (or say time harmonic) and so *t* is the independent variable. They can be described as

$$f(\mathrm{t}) = A \sin(\omega t + \varphi)$$

where, A is the *Amplitude*, ω is the *Angular frequency*, and φ is *phase*.

The *period* T of the waves is

$$T = \frac{2\pi}{\omega}$$

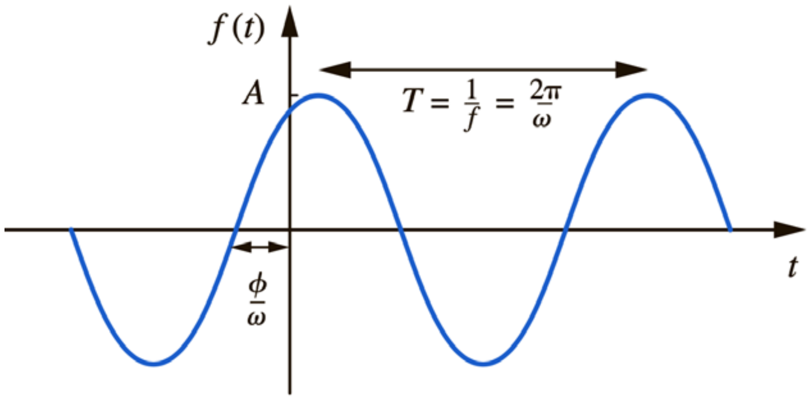
The *frequency* f equals

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

The units of T and f are second and hertz (Hz), respectively.

5.5 Modeling Waves with Trigonometry

The graph of the time variable wave $f(t) = A \sin(\omega t + \varphi)$ is depicted as following



Note from figure that the actual movement of the wave along the time axis is φ/ω, which is called the time displacement.

In many situations, there is a need to combine two or more waves together to form a single wave.

$$f(t) = f_1(t) + f_2(t) + f_3(t) + \ldots$$

5.5 Modeling Waves with Trigonometry

There are occasions where the independent variable is distance, x say, instead of time. Consider the wave

$$y = A \sin(kx + \varphi)$$

As before, A is the amplitude, k is called the wave number. The length of one cycle of the wave, that is the wavelength, denoted λ

$$\lambda = \frac{2\pi}{k}$$

Examples

State the amplitude, period, phase and time displacement of

$$\text{(a) } 3\sin(2t+3)$$

$$\text{(b) } 4\cos\left(\frac{3t+1}{5}\right)$$

$$\text{(c) } 4\sin(t+2)\cos(t+2)$$

Solutions:

(a) Amplitude = 3, period = $2\pi/2 = \pi$, phase = 3, time displacement = 3/2 = 1.5.

(b) Amplitude = 4, period = $\frac{2\pi}{3/5} = \frac{10}{3}\pi$, phase = 1/5, time displacement = 1/3.

(c) Since $4\sin(t+2)\cos(t+2) = 2\sin(2t+4)$, amplitude = 2, period = $2\pi/2 = \pi$, phase = 4, time displacement = 4/2 = 2.

Examples

Express following two waves as a single sine wave:

$$(a)\ 3 \sin 4t, \quad (b)\ 4 \cos 4t$$

Solution:

Let

$$3 \sin 4t + 4 \cos 4t = R \sin(4t + \varphi) = R \cos \varphi \sin 4t + R \sin \varphi \cos 4t$$

Hence

$$R \cos \varphi = 3,\ R \sin \varphi = 4$$

by squaring and adding we obtain

$$(R \cos \varphi)^2 + (R \sin \varphi)^2 = R^2 = 25 \rightarrow R = 5$$

by division we have

$$\tan \varphi = 4/3 \rightarrow \varphi = \tan^{-1} 4/3 = 0.9273$$

Therefore

$$3 \sin 4t + 4 \cos 4t = 5 \sin (4t + 0.9273)$$

5.6 Trigonometric equations

The trigonometric equations can be written in the forms of

$$\sin z = k, \cos z = k \text{ or } \tan z = k$$

where z is the independent variable and k is a constant. As discussed in the example of Part 3, *Trigonometric functions,* there is an infinite number of solutions for the equations.

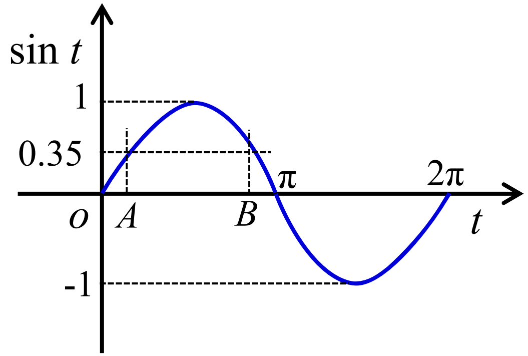
It is necessary to put the solutions to some specific interval.

$$\sin z = k,\ 0 \leq z \leq 2\pi$$

It should be mention that in sine and cosine equations, k should meet the requirement

$$-1 \leq k \leq 1.$$

Examples



Solve the following equation:

$$\sin t = 0.35,\ 0 \le t \le 2\pi.$$

Solution:

From figure, there are two solutions in the interval $0 \le t \le 2\pi$. These are given by points A and B. We have

$$\sin t = 0.35$$

with a scientific calculator, we have

$$t = \sin^{-1} 0.35 = 0.3429$$

This is the solution at point A. From the symmetry of the graph, the second solution is

$$t = \pi - 0.3429 = 2.7987$$

This is the solution at point B. The required solutions are $t = 0.3429, 2.7987$.