MA1300 Self Practice # 3, Solutions

1. (P91, #12, 14) Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

$$(1) \qquad f(x) = x^2 + \sqrt{7-x}, \qquad a = 4.$$
$$(2) \qquad f(t) = \frac{2t - 3t^2}{1+t^3}, \qquad a = 1.$$

Proof. (1), since $\lim_{x\to 4} 7 - x = 3 > 0$, $\lim_{x\to 4} f(x) = 16 + \sqrt{3} = f(4)$. So f is continuous at a. (2), since f is a rational function and $a = 1$ is in the domain of f, so $\lim_{x\to 1} f(t) = f(1)$, and f is continuous at $a = 1$.

2. (P91, #15) Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval:

$$f(x) = \frac{2x+3}{x-2}, \qquad (2, \infty).$$

Proof. Since f is a rational function and $(2, \infty)$ is a subset of the domain of f, we have for any $a \in (2, \infty)$, $\lim_{x\to a} f(x) = f(a)$. So f is continuous on the interval $(2, \infty)$.

For Questions $3 \sim 4$, explain why the function is discontinuous at the given number 1. Sketch the graph of the function.

3. (P91, #19) $f(x) = \begin{cases} 1 - x^2, & \text{if } x < 1 \\ \frac{1}{x}, & \text{if } x \geq 1 \end{cases} \qquad a = 1.$

Solution. Since $\lim_{x\to 1^-} f(x) = 0 \neq 1 = f(1)$, f is not continuous at $a = 1$. The function is sketched in Figure 1.

4. (P91, #20) $f(x) = \begin{cases} \frac{x^2 - x}{x^2 - 1}, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases} \qquad a = 1.$

Solution. Since $\lim_{x\to 1} \frac{x^2-x}{x^2-1} = \lim_{x\to 1} \frac{x(x-1)}{(x+1)(x-1)} = \frac{1}{2} \neq 1 = f(1)$, f is not continuous at $a = 1$. The function is sketched in Figure 1.

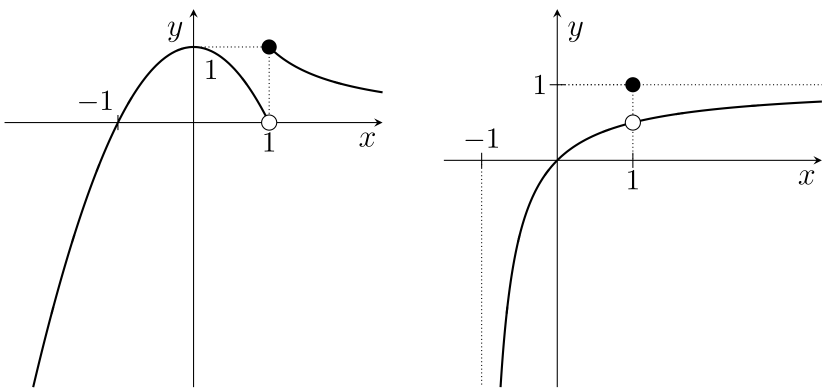

Figure 1: Sketch of the functions in Question 3 and 4. Left: for Question 3, Right, for Question 4.

5. (P91, #36, 38) Use continuity to evaluate the limit.

$$(1) \quad \lim_{x\to\pi} \sin(x + \sin x)$$
$$(2) \quad \lim_{x\to 2} (x^3 - 3x + 1)^{-3}$$

Solution. (1) Since the domain of sine function is $\mathbb{R}$ and $\lim_{x\to\pi}(x+\sin x)=\pi\in\mathbb{R}$, we have

$$\lim_{x\to\pi}\sin(x+\sin x)=\sin(\lim_{x\to\pi}(x+\sin x))=\sin\pi=0.$$

(2) Since $\lim_{x\to 2}(x^3-3x+1)=3$ is within the domain of the function $f(x)=\frac{1}{x^3}$, we have

$$\lim_{x\to 2}(x^3-3x+1)^{-3}=(\lim_{x\to 2}(x^3-3x+1))^{-3}=\frac{1}{27}.$$

6. (P92, #39) Show that f is continuous on $(-\infty,\infty)$:

$$f(x)=\begin{cases} x^2, & \text{if } x<1,\\ \sqrt{x}, & \text{if } x\geq 1.\end{cases}$$

Proof. Since $\lim_{x\to 1^-}x=1>0$, we have $\lim_{x\to 1^-}x^2=1$. Similarly, $\lim_{x\to 1^+}\sqrt{x}=1$. Therefore $\lim_{x\to 1}f(x)=1=f(1)$, and thus f is continuous at $x=1$. Since on $(-\infty,1)$, f is a polynomial, and on $(1,\infty)$, f is a root function, so f is continuous on the whole real line $(-\infty,\infty)$.

7. (P92, #44) The gravitational force exerted by the earth on a unit mass at a distance r from the center of the planet is

$$F(r)=\begin{cases} \frac{GMr}{R^3}, & \text{if } r<R\\ \frac{GM}{r^2}, & \text{if } r\geq R\end{cases}$$

where M is the mass of the earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r? Explain why.

Solution. For $r\geq R>0$, F is a rational function with domain containing $[R,\infty)$, so F is continuous. For $0<r<R$, F is a polynomial and thus continuous. at $r=R$, we have

$$\lim_{r\to R^-}F(r)=\frac{GM}{R^2}=\lim_{r\to R^+}F(r).$$

Therefore F is continuous on $(0,\infty)$.

8. (P92, #45) For what values of the constant c is the function f continuous on $(-\infty,\infty)$?

$$f(x)=\begin{cases} cx^2+2x, & \text{if } x<2,\\ x^3-cx, & \text{if } x\geq 2.\end{cases}$$

Solution. Since $\lim_{x\to 2^-}f(x)=4c+4$ and $\lim_{x\to 2^+}f(x)=8-2c$, let $8-2c=4c+4$ to give $c=2/3$.

9. (P92, #49) If $f(x)=x^2+10\sin x$, show that there is a number c such that $f(c)=1000$.

Proof. Since $f(0)=0<1000$, $f(100)=10,000-10\sin(100)\geq 9990>1000$, and that f is continuous on $(-\infty,\infty)$, by the intermediate value theorem, there exists a constant $c\in(0,100)$ such that $f(c)=1000$.

10. (P92, #50) Suppose f is continuous on $[1,5]$ and the only solutions to the equation $f(x)=6$ are $x=1$ and $x=4$. If $f(2)=8$, explain why $f(3)>6$.

Solution. If $f(3)=6$, there is one more solution $x=3$ to the equation $f(x)=6$, a contradiction. If $f(3)<6$, since f is continuous on $[1,5]$, and $f(2)=8>6$, by the intermediate value theorem, there exists a constant $c\in(2,3)$ such that $f(c)=6$, so $x=c$ is a solution to $f(x)=6$ other than 1 and 4, another contradiction. Therefore $f(3)>6$.