MA1300 Self Practice 4 Solution

1. (P92, #46) Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x < 3 \\ 2x - a + b & \text{if } x \ge 3. \end{cases}$$

Solution. For making f continuous everywhere, sufficient and necessary conditions are

$$\begin{cases} \lim\limits_{x\to 2^+} ax^2 - bx + 3 = \lim\limits_{x\to 2^-} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2^-} x+2, \\ \lim\limits_{x\to 3^-} ax^2 - bx + 3 = \lim\limits_{x\to 3^+} 2x - a + b. \end{cases}$$

Equivalently,

$$\begin{cases} 4 = 4a - 2b + 3, \\ 9a - 3b + 3 = 6 - a + b. \end{cases}$$

Solve the equations to give $a = b = \frac{1}{2}$.

2. (P92, #51, 53) Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

(a) $x^4 + x - 3 = 0, \quad (1, 2)$

(b) $\cos x = x, \quad (0, 1)$

Solution.

(a) Let $f(x) = x^4 + x - 3$ to give $f(1) = -1$, $f(2) = 15$. By the Intermediate Value Theorem, there exists some $c \in (1, 2)$ s.t. $f(c) = 0$, a root.

(b) Let $f(x) = \cos x - x$ to give $f(0) = 1$, $f(1) = \cos 1 - 1 < 0$. By the Intermediate Value Theorem, there exists some $c \in (0, 1)$, such that $f(c) = 0$. Hence the root exists.

3. (P93, #65) Is there a number that is exactly 1 more than its cube?

Solution. Let $f(x) = x - x^3 - 1$ to give $f(0) = -1$ and $f(-2) = 5$. Therefore by the Intermediate Value Theorem, there exists a number $c \in (-2, 0)$ such that $f(c) = 0$, that is, $c = c^3 + 1$, or c is exactly 1 more than its cube.

4. (P93, #66) If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1,1)$.

~~*Solution.* Since $x^3+2x^2-1=(x+1)\left(x+\frac{\sqrt{5}+1}{2}\right)\ldots$ we have $\lim\limits_{x\to\left(\frac{\sqrt{5}-1}{2}\right)^+}\frac{a}{x^3+2x^2-1}=\infty$, $\lim\limits_{x\to\left(\frac{\sqrt{5}-1}{2}\right)^+}\ldots$ is finite, and $\lim\limits_{x\to1^-}\frac{b}{x^3+x-2}=-\infty$. Let $f(x)=\ldots$ $(\frac{\sqrt{5}-1}{2},1)$, and $\lim\limits_{x\to\left(\frac{\sqrt{5}-1}{2}\right)^+}f(x)=\infty$, $\lim\limits_{x\to1^-}f(x)=\ldots$ there exists some $c\in(\frac{\sqrt{5}-1}{2},1)\subset(-1,1)$ such that $\ldots$~~

Solution: Define $F(x)=\frac{a}{x^3+2x^2-1}+\frac{b}{x^3+x-2}$. Since $x^3+2x^2-1=(x+1)\left(x+\frac{\sqrt{5}+1}{2}\right)\left(x-\frac{\sqrt{5}-1}{2}\right)$ and $x^3+x-2=(x-1)(x^2+x+2)$. The function $F(x)$ is well defined and continuous in the domain $(\frac{\sqrt{5}-1}{2},1)$ which is a subset of $(-1,1)$. As we have $\lim\limits_{x\to\left(\frac{\sqrt{5}-1}{2}\right)^+}\frac{a}{x^3+2x^2-1}=+\infty$ and $\lim\limits_{x\to\left(\frac{\sqrt{5}-1}{2}\right)^+}\frac{b}{x^3+x-2}$ exists and is finite, then $\lim\limits_{x\to\left(\frac{\sqrt{5}-1}{2}\right)^+}F(x)=+\infty$; As we have $\lim\limits_{x\to1^-}\frac{b}{x^3+x^2-2}=-\infty$ and $\lim\limits_{x\to1^-}\frac{a}{x^3+2x^2-1}$ exists and is finite, then $\lim\limits_{x\to1^-}F(x)=-\infty$. Then by the definition of limit, we have two small positive values δ_1,δ_2 such that $F(1-\hat{\delta}_1)<-1$ for any positive $\hat{\delta}_1<\delta_1$ and $F(\frac{\sqrt{5}-1}{2}+\hat{\delta}_2)>1$ for any positive $\hat{\delta}_2<\delta_2$. Let $\delta=\min\{\frac{\delta_1}{2},\frac{\delta_2}{2},\left(1-\frac{\sqrt{5}-1}{2}\right)/3\}$, we can obtain that the function $F(x)$ is continuous on an interval $[\frac{\sqrt{5}-1}{2}+\delta,1-\delta]$ and $F(1-\delta)<-1$ and $F(\frac{\sqrt{5}-1}{2}+\delta)>1$. Therefore, by the Intermediate Value Theorem, there exists some $c\in(\frac{\sqrt{5}-1}{2}+\delta_2,1-\delta_1)$ such that $F(c)=0$.

5. (P96, #26, 28, 32, 34, 37) Find the limit.

$$\lim_{x\to-3}\frac{x^2-9}{x^2+2x-3}\qquad\lim_{x\to1^+}\frac{x^2-9}{x^2+2x-3}\qquad\lim_{v\to4^+}\frac{4-v}{|4-v|}$$

$$\lim_{x\to3}\frac{\sqrt{x+6}-x}{x^3-3x^2}\qquad\lim_{x\to0}\frac{1-\sqrt{1-x^2}}{x}$$

Solutions. Since $\frac{x^2-9}{x^2+2x-3}=\frac{(x+3)(x-3)}{(x-1)(x+3)}$, we have

$$\lim_{x\to-3}\frac{x^2-9}{x^2+2x-3}=\frac{-6}{-4}=\frac{3}{2},\qquad\lim_{x\to1^+}\frac{x^2-9}{x^2+2x-3}=-\infty.$$

For the other limits,

$$\lim_{v\to4^+}\frac{4-v}{|4-v|}=\lim_{v\to4^+}\frac{4-v}{v-4}=-1,$$

$$\lim_{x\to3}\frac{\sqrt{x+6}-x}{x^3-3x^2}=\lim_{x\to3}\frac{x+6-x^2}{x^2(x-3)(\sqrt{x+6}+x)}=\lim_{x\to3}\frac{-(x+2)(x-3)}{x^2(x-3)(\sqrt{x+6}+x)}=\frac{-5}{54},$$

$$\lim_{x\to0}\frac{1-\sqrt{1-x^2}}{x}=\lim_{x\to0}\frac{1-1+x^2}{x(1+\sqrt{1-x^2})}=0.$$

6. (P96, #40) Prove that

$$\lim_{x\to0}x^2\cos\frac{1}{x^2}=0.$$

Proof. Since $-x^2\le x^2\cos\frac{1}{x^2}\le x^2$ for any $x\neq0$, and $\lim_{x\to0}(-x^2)=\lim_{x\to0}x^2=0$, the Squeeze Theorem implies that

$$\lim_{x\to0}x^2\cos\frac{1}{x^2}=0.$$

7. (P96, #45) Let

$$f(x)=\begin{cases}\sqrt{-x}, & \text{if } x<0,\\ 3-x, & \text{if } 0\le x\le3,\\ (x-3)^2, & \text{if } x>3.\end{cases}$$

(a) Evaluate each limit, if it exists.

$$\text{(i)}\ \lim_{x\to0^+}f(x)\qquad\text{(ii)}\ \lim_{x\to0^-}f(x)\qquad\text{(iii)}\ \lim_{x\to0}f(x)$$

$$\text{(iv)}\ \lim_{x\to3^-}f(x)\qquad\text{(v)}\ \lim_{x\to3^+}f(x)\qquad\text{(vi)}\ \lim_{x\to3}f(x)$$

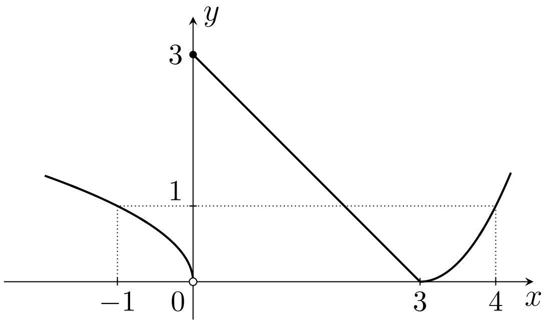
(b) Where is f discontinuous?

(c) Sketch the graph of f.

Solution.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3 - x = 3$$
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sqrt{-x} = 0$$
$$\lim_{x\to 0} f(x) \text{ does not exist, since } \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x).$$
$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 3 - x = 0$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x-3)^2 = 0$$
$$\lim_{x\to 3} f(x) = 0.$$

f is discontinuous at $x = 0$. Here is a graph of the function f.



8. (P96, #46) Let

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \le x \le 2 \\ 2 - x & \text{if } 2 < x \le 3 \\ x - 4 & \text{if } 3 < x < 4 \\ \pi & \text{if } x \ge 4 \end{cases}$$

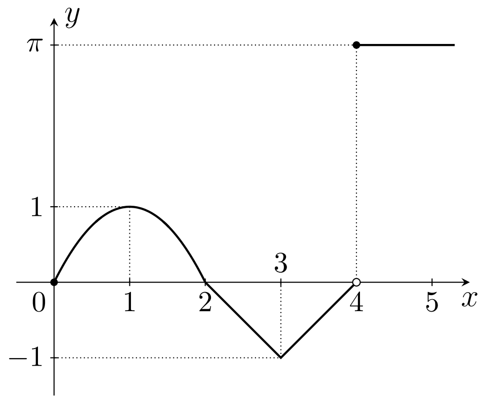
(a) For each of the numbers 2, 3, and 4, discover whether g is continuous from the left, continuous from the right, or continuous at the number.

(b) Sketch the graph of g.

Solution. We have

$$\begin{array}{lll} \lim_{x\to 2^-} g(x) = 0 & \lim_{x\to 2^+} g(x) = 0 & \lim_{x\to 2} g(x) = 0 = g(2) \\ \lim_{x\to 3^-} g(x) = -1 & \lim_{x\to 3^+} g(x) = -1 & \lim_{x\to 3} g(x) = -1 = g(3) \\ \lim_{x\to 4^-} g(x) = 0 & \lim_{x\to 4^+} g(x) = \pi & \lim_{x\to 4} g(x) \text{ does not exist} \end{array}$$

Therefore, g is continuous at 2, 3, and continuous from the right at 4. Here is a graph of the function g.


y
π
1
3
0
1
2
4
5
x
−1