Tutorial 1

1. How many different letter arrangements can be made from the letters

   (a) **M A N G O**

   (b) **N A N Y A N G**?

   Solution:

   (a) $5! = 120$

   (b) $7!/3!2!1!1!$=420

2. For years, telephone area codes in a certain country consisted of a sequence of three digits. The first digit was an integer between 1 and 9; the second digit was either 0 or 1; the third digit was any integer between 2 and 9.

   (a) How many area codes were possible?

   (b) How many area codes starting with 4 were possible?

   Solution:

   (a) $9 \times 2 \times 8 = 144$

   (b) $2 \times 8 = 16$

3. Jimmy has 8 friends, of whom 5 will be invited to a party. How many choices are there if 2 of the friends are feuding and will not attend together?

   Solution:

   One way of thinking is as follows: if the 2 of the 8 cannot attend together, there can be only two possible cases: 1) neither of the two attend; 2) exactly one of these two attend. For case 1), all 5 persons must be selected from the other 6. For case 2), one person is selected from the feuding two persons, and the other 4 persons are selected from those 6 who are not feuding. So the answer is

   $\binom{6}{5} + \binom{2}{1}\binom{6}{4} = 6 + 2 \times 15 = 36$

   Another approach is to first considered the number of choices when none of the friends are feuding. There are $\binom{8}{5}$ choices. Some of the resulting choices contain those 2 feuding friends and thus must be SUBTRACTED. The number of choices where both these 2 feuding friends are chosen is $\binom{6}{3}$. So the answer is

   $\binom{8}{5} - \binom{6}{3} = 56 - 20 = 36$

4. From a group of n people, suppose that we want to choose a committee of k, $k \leq n$, one of whom is to be designated as chairperson.

   (a) By focusing first on the choice of the committee and then on the choice of the chair, argue that there are $\binom{n}{k} k$ possible choices.

   (b) By focusing first on the choice of the nonchair committee members and then on the choice of the chair, argue that there are $\binom{n}{k-1}(n-k+1)$ possible choices.

   (c) By focusing first on the choice of the chair and then on the choice of the other committee members, argue that there are $n\binom{n-1}{k-1}$ possible choices.

   (d) Conclude from parts (a), (b) and (c) that

$$\binom{n}{k} k = \binom{n}{k-1}(n-k+1) = n\binom{n-1}{k-1}$$

(e) Use the factorial definition of $\begin{pmatrix} m \\ r \end{pmatrix}$ to verify the identity in part (d).

5. (a) What is the number of positive integer-valued solutions of

$$x_1 + x_2 + \cdots + x_r = n?$$

(b) What is the number of nonnegative integer-valued solutions of

$$x_1 + x_2 + \cdots + x_r = n$$

for which exactly k of the x_i are equal to 0?

Solution:

(a) $\begin{pmatrix} n-1 \\ r-1 \end{pmatrix}$

(b) $\begin{pmatrix} r \\ k \end{pmatrix} \begin{pmatrix} n-1 \\ n-r+k \end{pmatrix}$ (this is same as $\binom{r}{k}\binom{n-1}{r-k-1}$)

6. A football team produced a record of 8 wins and 4 losses over its season.

(a) How many different arrangements W (win) and L (loss) are possible?

(b) How many different arrangements are possible if there are exactly 3 runs. (A run is a continuous stretch of W's or L's. For example, WWLLLWWLLW has 5 runs.)

(c) What about 4 runs?

Solution:

(a) How can we arrange 8 W's and 4 L's in a row? We choose 4 positions from 12 possible positions in which to put L's. Then the other positions are filled with W's. So the answer is $\binom{12}{4}$

(b) If there are three runs, the only possibilities are 1) W's followed by L's followed by some more W's; 2) L's followed by W's followed by some more L's. For case 1) Suppose we already arranged 8 W's. (There is only one possibility, since the W's are not distinguishable). Then we only need to find a place we can insert 4 L's. The 8 W's create 7 spaces we can choose from, so there are $\binom{7}{1}$ possibilities. Similar reasoning can be applied to case 2). So the answer is

$$\binom{7}{1} + \binom{3}{1} = 10$$

(c) Like (b), there are two groups of arrangements, either you start with W, or start with L. Either way, you want to split 8 W's into two groups, and 4 L's into two groups. There are 7 ways to split 8 W's into 2 groups, and 3 ways to split 4 L's into two groups. So the answer is $2 \times 7 \times 3 = 42$.

7. How many different ways to assign 9 students to 3 tutorial sections, such that

(a) section one has 3 students (some other section may have no students)

(b) each section has 3 students

Solution:

(a) first 3 students are chosen to be in section 1, then each of the other 6 students has two choices: either in section 2 or 3, so the answer is $\binom{9}{3}2^6$

(b) choose 3 students to be in section 1, 3 from the remaining 6 students to be in section 2, the remaining 3 will be in section 3. So the answer is

$$\binom{9}{3}\binom{6}{3}\binom{3}{3}.$$

Another approach is to note that this is just partition 9 different objects into 3 different groups, so using the formula, we immediately get

$$\frac{9!}{3!3!3!}$$