Contents

Preface

- **Textbooks**: Single Variable Calculus, by James Stewart, 7E; Modern Engineering Mathematics, by Glyn James, 4E.

  In this semester, we will cover Chap 4-5, 7-8 of the first book, together with Chap 3-5 of the second book. Upon completion of this course, you should be able to understand integrals and their applications in mathematical modeling, complex numbers, vectors and their properties, matrices and determinants, linear system of equations.

1. Integrals

In the last semester, we have learned limits, derivatives and their applications. Now, we are going to study integrals, and do some calculations on them.

1.1. **Areas and distances.** Text Section 4.1,

Exercise: 2,5,13,23,24.

The area problem: Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

For a region with straight sides, such as a rectangle, a triangle and a polygon, the problem is easy to answer.

For a region with curved sides, we approximate the region S by rectangles and then take the limit of the areas of these rectangles as we increase the number of rectangles.

Ex. [Text example 4.1.1] Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

We can apply the idea of the above example to the more general region S. And we define the area A of the region S in the following way.

Definition 1.1. *The* **area** A *of the region* S *that lies under the graph of the continuous function* f *is the limit of the sum of the areas of approximating rectangles:*

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x],$$

where $\Delta x = \frac{b-a}{n}$, $x_0 = a$ *and*

$$x_i = x_0 + i\Delta x, \qquad i = 1, 2, \cdots, n.$$

If we assume that f is continuous in a bounded region S, the limit in the above definition always exists. It can also be shown that we get the same value if we use left endpoints:

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x],$$

or any **sample points** x_i^* in the i-th subinterval $[x_{i-1}, x_i]$,

$$A = \lim_{n\to\infty} R_n^* = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x].$$

By using the **sigma notation**, the above formulas can be simplified

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Note: In the notation $\sum_{i=m}^{n}$, Σ is called a **summation operator**, i represents the **index of summation**, m is the **lower bound of summation**, and n is the **upper bound of summation**.

Some useful formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, $\quad \sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$.

Q: $\sum_{i=1}^{n}(a_i b_i) = \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i$? $\qquad \sum_{i=1}^{n} \sqrt{a_i} = \sqrt{\sum_{i=1}^{n} a_i}$?

Ex. [Text example 4.1.3] Let A be the area of the region that lies under the graph of $f(x) = \cos x$ between $x = 0$ and $x = b$, where $0 \le b \le \pi/2$.

(a) Using right endpoints, find an expression for A.

(b) Taking the sample points to be midpoints and using four subintervals, estimate the area when $b = \pi/2$.

The distance problem: Find the distance traveled by an object during a certain time period if the velocity of the object $f(t)$ is known at all times. (This is the inverse problem of the velocity problem in Section 1.4.)

Ex. [Text example 4.1.4] Estimate the distance with the speed given in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (km/h)	27	34	38	46	51	50	45

The *exact* displacement d traveled is the *limit* of the following expressions

$$d = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\Delta t = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta t,$$

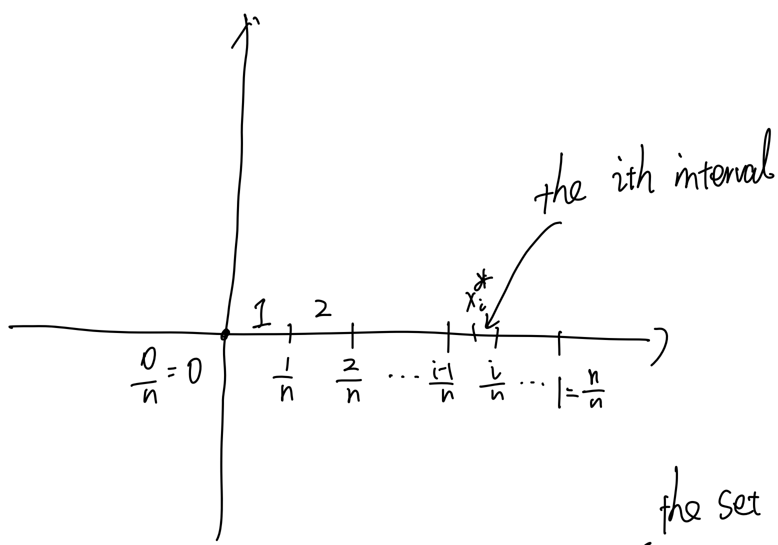
where $\Delta t = \frac{b-a}{n}$, $t_0 = a$ and

$$t_i = t_0 + i\Delta t, \qquad i = 1, 2, \cdots, n.$$

$$\lim_{n\to+\infty} \sum_{i=0}^{n} \underset{\substack{\| \\ 0}}{\overset{\substack{1 \\ \|}}{f}}(\boxed{x_i^*}) \cdot \frac{1}{n}$$

could be Q^c

could be Q



the set of rational numbers

f is not integrable on $[0,1]$ $\Leftarrow$ $f(x) = \begin{cases} 0, & x \in Q \cap [0,1] \\ 1 & x \in [0,1] \setminus Q \end{cases}$ on $[0,1]$

$f(x)$ is nowhere continuous on $[0,1]$.

1.2. **The definite integral.** Text Section 4.2,
Exercise: 5,19,27,33,47,50,57.

Definition 1.2. *If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0(= a), x_1, x_2, \cdots, x_n(= b)$ be the endpoints of these subintervals and x_i^* be any* **sample points** *in the i-th subinterval $[x_{i-1}, x_i]$. Then the* **definite integral of f from a to b** *is*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists. If it does exist, we say that f is **integrable** *on $[a, b]$.*

Note 1. The symbol $\int$ is introduced by Leibniz and called an **integral sign**. In the notation $\int_a^b f(x)dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. The dx simply indicates that the independent variable is x. The procedure of calculating an integral is called **integration**.

Note 2. The definite integral $\int_a^b f(x)dx$ is a number and independent of x:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr.$$

Note 3. The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

is called a **Riemann sum**.

A definite integral can be negative. It can be interpreted as a **net area**:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f. (see Fig 4 of Page 297)

Note 4. It is not necessary to divide $[a, b]$ into subintervals of equal width:

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Note 5. Not all functions are integrable.

Theorem 1.3. *If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the integral $\int_a^b f(x)dx$ exists.*

Theorem 1.4. *If f is continuous on $[a, b]$, then*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

the right end pt

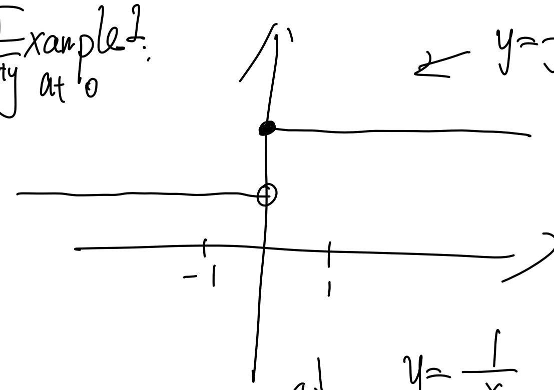
Def: We say that f has a jump discontinuity at $x_0 \in [a,b]$ if $\lim_{x\to x_0^-} f(x)$ exists, $\lim_{x\to x_0^+} f(x)$ exists and $\lim_{x\to x_0^-} f(x) \neq \lim_{x\to x_0^+} f(x)$.

Ex. [Text example 4.2.1] Express

$$\int_0^{\pi} (x^3 + x\sin x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\Delta x$$

as an integral on the interval $[0, \pi]$.

Example 1.



f has a jump discontinuity at 0

By Thm 1.3, f is integrable on $[-1,0]$

Example

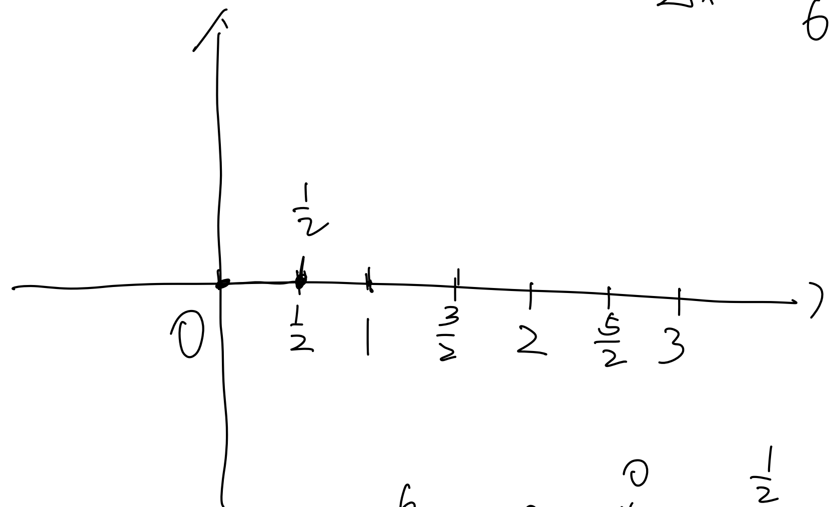
We can not use Thm 1.3 ... f is integrable on $[-1,1]$ f does not to verify whether

In fact f is not integrable on $[-1,0]$. $\Longleftarrow$ have a jump discontinuity at 0 because $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$ do not exist!!

Ex. [Text example 4.2.2] (a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample points to be right endpoints and $a = 0, b = 3$ and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)dx$.

$$\Delta x = \frac{3}{6} = \frac{1}{2}$$



$$\int_0^3 \boxed{\underbrace{(x^3 - 6x)}_{f(x)}} dx \ \boxed{\approx} \ \sum_{i=1}^{6} f(\underbrace{\boxed{a}}_{0} + i\underbrace{\boxed{\Delta x}}_{\frac{1}{2}})\underbrace{\boxed{\Delta x}}_{\frac{1}{2}}$$

approximated by

$$= \sum_{i=1}^{6} f\left(\frac{i}{2}\right)\frac{1}{2}$$

$$= \left(\left(\tfrac{1}{2}\right)^3 - 6\times\tfrac{1}{2}\right)\tfrac{1}{2} + \left(1^3 - 6\times 1\right)\tfrac{1}{2} + \left(\left(\tfrac{3}{2}\right)^3 - 6\times\tfrac{3}{2}\right)\tfrac{1}{2}$$

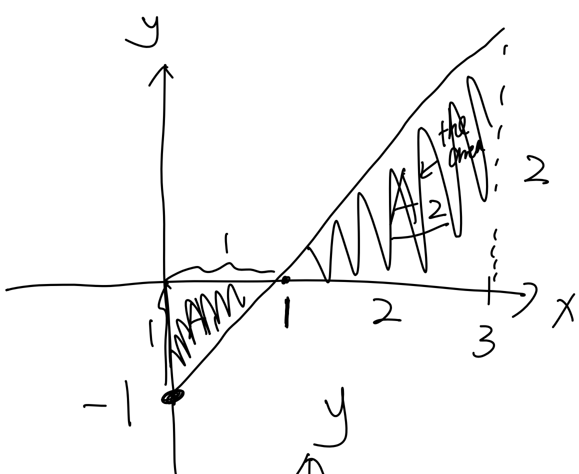
Ex. [Text example 4.2.4] Evaluate the following integrals by interpreting each in terms of areas:

(a) $\int_0^1 \sqrt{1 - x^2}dx$ (b) $\int_0^3 (x - 1)dx$.

$$+\left(2^3 - 6\times 2\right)\tfrac{1}{2} + \left(\left(\tfrac{5}{2}\right)^3 - 6\times\tfrac{5}{2}\right)\tfrac{1}{2}$$

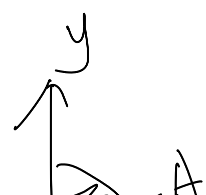
$$+\left(3^3 - 6\times 3\right)\tfrac{1}{2} = ?$$

(b)



$$\int_0^3 (x-1)dx = \underbrace{\boxed{A_2}}_{=2} - \underbrace{\boxed{A_1}}_{=\frac{1}{2}}$$

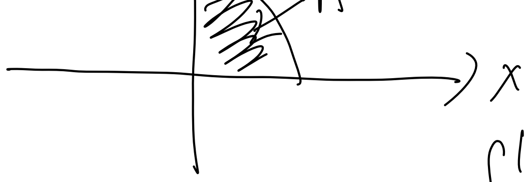
$$= \frac{4}{2} - \frac{1}{2} = \frac{3}{2}.$$



$y = \sqrt{1-x^2}$, $x \in [0,1]$

$\Downarrow$

(a)



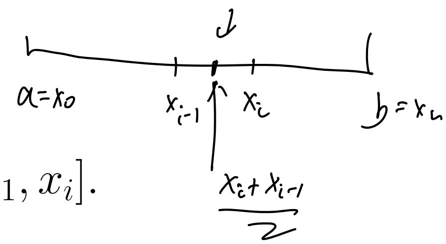
$y^2 = 1 - x^2 \Rightarrow x^2 + y^2 = 1$

The midpoint rule

$\int_0^1 \sqrt{1-x^2}\, dx = A = \frac{1}{4} \times \pi = \frac{\pi}{4}.$

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\overline{x_i})\boxed{\Delta x}$$

the ith interval



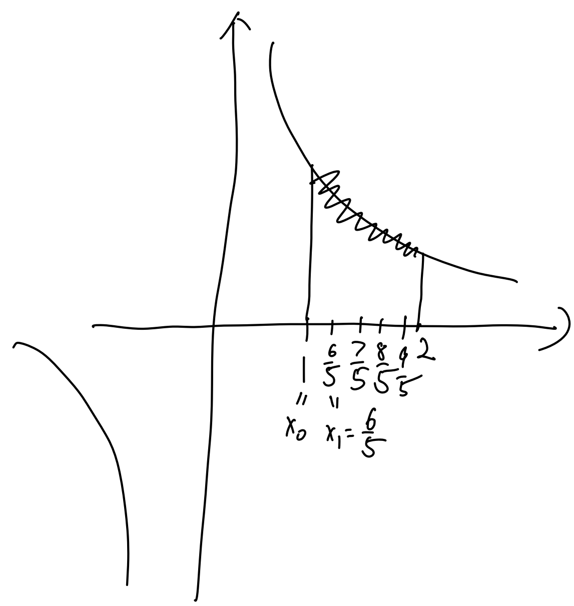
where $\Delta x = \frac{b-a}{n}$ and

$$\boxed{\overline{x_i} = \frac{1}{2}(x_{i-1} + x_i)} = \text{midpoint of } [x_{i-1}, x_i].$$

Ex. [Text example 4.2.5] Use the midpoint rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}dx$.

$$\int_1^2 \boxed{\frac{1}{x}}\, dx \qquad \frac{1}{x} = f(x)$$

$\Delta x = \frac{1}{5}$



$$\approx \sum_{i=1}^{5} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x.$$

where $x_i = \frac{5+i}{5} = 1 + \frac{i}{5}$

$$= \sum_{i=1}^{5} f\left(\frac{1 + \frac{i-1}{5} + 1 + \frac{i}{5}}{2}\right)\Delta x$$

$$= \sum_{i=1}^{5} f\left(\frac{2 + \frac{2i-1}{5}}{2}\right)\frac{1}{5}$$

$$= \frac{1}{\frac{2+\frac{2-1}{5}}{2}} \times \frac{1}{5} + \frac{1}{\frac{2+\frac{4-1}{5}}{2}} \times \frac{1}{5} + \frac{1}{\frac{2+\frac{6-1}{5}}{2}} \times \frac{1}{5}$$

$$+ \frac{1}{\frac{2+\frac{8-1}{5}}{2}} \times \frac{1}{5} + \frac{1}{\frac{2+\frac{10-1}{5}}{2}} \times \frac{1}{5} = ?$$

Properties of the integral

(1) $\displaystyle\int_b^a f(x)dx = \boxed{-\int_a^b f(x)dx.}$ $\Longleftrightarrow$ $\displaystyle\int_a^b f(x)dx + \int_b^a f(x)dx \; \boxed{= 0}$

$\parallel$ from (6)

(2) $\displaystyle\int_a^a f(x)dx = 0.$

$\displaystyle\int_a^a f(x)dx \overset{\text{from (2)}}{=} 0$

(3) $\displaystyle\int_a^b c\,dx = c(b-a)$, where c is any constant.

(4) $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx.$

$\displaystyle\lim_{n\to+\infty} \sum_{i=1}^{n} \left(f(x_i^*) \pm g(x_i^*)\right)\Delta x$

(5) $\displaystyle\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$

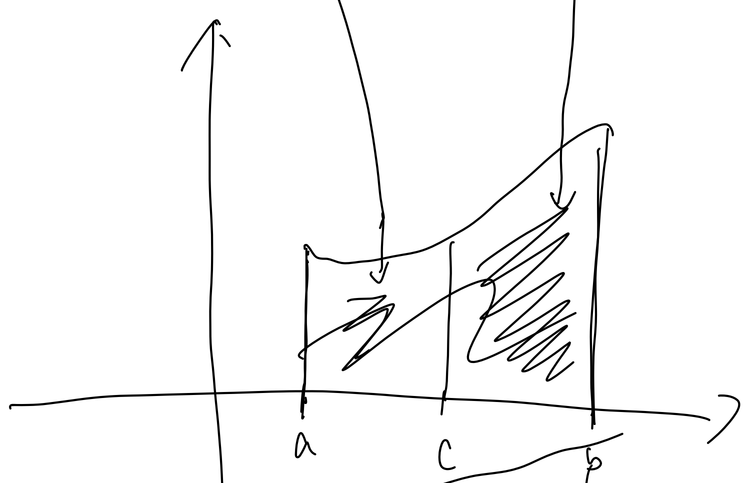
$\displaystyle\parallel \quad \lim_{n\to+\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right) \pm \lim_{n\to+\infty}\left(\sum_{i=1}^{n} g(x_i^*)\Delta x\right)$

$\displaystyle\parallel \quad \int_a^b f(x)dx \pm \int_a^b g(x)dx$

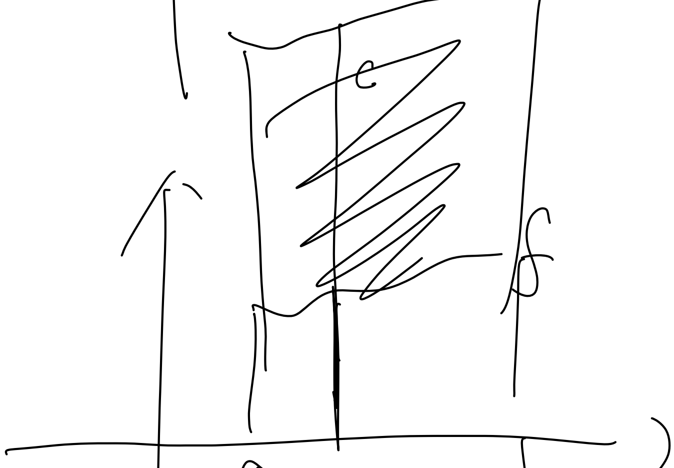
(6) $\displaystyle\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx.$

Ex. [Text example 4.2.7] If it is known that $\int_0^{10} f(x)dx = 17$ and $\int_0^8 f(x)dx = 12$, find $\boxed{\int_8^{10} f(x)dx.}$ $= 5$

$$\boxed{\int_a^b f(x)dx = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$



$$\int_a^b cf(x)dx$$
$$= \lim_{n\to+\infty} \sum_{i=1}^{n} c f(x_i^*)\Delta x$$
$$= \lim_{n\to+\infty} c \sum_{i=1}^{n} f(x_i^*)\Delta x$$
$$= c\boxed{\lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$
$$= c\int_a^b f(x)dx$$



Comparison properties of the integral

(1) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)dx \geq 0$.

(2) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

(3) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0$

$\int_a^b (f(x)-g(x))\,dx \geq 0$

$f(x)-g(x) \geq 0$

Ex. [Text example 4.2.8] Show that $3 \leq \displaystyle\int_1^4 \sqrt{x}dx \leq 6$.

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a)$$

$\int_a^b m\,dx = m(b-a)$

$m = \sqrt{1} = 1$, $\quad M = \sqrt{4} = 2$

$$\int_1^4 m\,dx \leq \int_1^4 \sqrt{x}\,dx \leq \int_1^4 M\,dx$$

$\int_1^4 1\,dx = 3$ $\qquad$ $\int_1^4 2\,dx = 6$

1.3. **The $\boxed{\text{fundamental}}$ theorem of calculus.** Text Section 4.3, Exercise: 3, 9, 17, 37, 47, 51, 69.

Consider the function

a function of x.

$$g(x) = \int_a^x f(t)dt,$$

where f is a continuous function on $[a, b]$ and x varies between a and b. For small h, we can see from Fig 5 in Section 4.3 that the area under the graph of f from x to $x + h$ is approximately equal to the area of the rectangle with height $f(x)$ and width h:

$$g(x+h) - g(x) \approx hf(x) \Rightarrow \frac{g(x+h) - g(x)}{h} \approx f(x).$$

This intuition suggest us the **the first part of the Fundamental Theorem of Calculus (FTC1)**:

Theorem 1.5. *If f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^x f(t)dt, \qquad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on $\boxed{(a, b)}$, and $g'(x) = f(x)$.

Proof. See p312. $\square$

FTC1 can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)dt = f(x),$$

when f is continuous.

Ex. [Text example 4.3.2] Find the derivative of $g(x) = \int_0^x \underbrace{\sqrt{1+t^2}}_{f(t)}\,dt$.

$$g'(x) = \sqrt{1+x^2}$$

Ex. [Text example 4.3.4] Find $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec t\,dt$.

$h(x)$ $\qquad$ $\underset{h(x)}{x^4}$

$$g(x) = \int_1^x \sec t\,dt \Rightarrow g'(x) = \sec x$$

$$h(x) = x^4 \Rightarrow h'(x) = 4x^3$$

$$\frac{d}{dx}\int_1^{x^4} \sec t\,dt = \frac{d}{dx} g(h(x))$$

$$= \boxed{g'(h(x))}\, h'(x)$$

$$= (\sec x^4)\cdot 4x^3$$

The fundamental theorem of calculus, part 2 (FTC2)

Theorem 1.6. *If f is continuous on $[a, b]$, then*

$$\boxed{\int_a^b f(x)dx} = F(b) - F(a),$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Proof. See p315. □

FTC2 can be rewritten as

$$\int_a^b f(x)dx = F(x)\Big|_a^b, \qquad \text{where } F' = f.$$

Ex. [Text example 4.3.6] Find the area under the parabola $y = x^2$ from 0 to 1.

$\Rightarrow g(a) = \int_a^a f(t)dt = 0$

Proof: By FTC I, if we define $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$

Therefore, $\int_a^b f(x)dx = g(b) = g(b) - g(a)$

For any F being antiderivative of f, $F'(x) = f(x) = g'(x)$

for all x. Then $\boxed{F'(x) - g'(x)} = 0$ for all x

$\Downarrow$

$\frac{d}{dx}\left(F(x) - g(x)\right)$

$\Downarrow$

$F(x) - g(x) = C$ for all x.

$g(x) = F(x) - C$ for all x.

Then $\int_a^b f(x)\,dx = g(b) - g(a) = (F(b) - \cancel{C}) - (F(a) - \cancel{C}) = F(b) - F(a)$

Ex. [Text example 4.3.8] What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{3} = \cancel{-\frac{1}{3} - 1} = -\frac{4}{3}.$$

is not correct

$$\boxed{\int_{-1}^{3} \frac{1}{x^2}\,dx = +\infty.}$$

f is discontinuous at $x=0$.

1.4. **Indefinite integrals and the net change theorem.** Text Section 4.4,

Exercise: 3, 9, 29, 41, 46, 49, 55, 72.

Definition 1.7. *An* **indefinite integral** $\boxed{\int f(x)dx}$ *is an anti-derivative of* f*, that is,*

$$\int f(x)dx = \boxed{F(x)} \quad \textit{means} \quad F'(x) = f(x).$$

Note: a definite integral $\int_a^b f(x)dx$ is a number, but an indefinite integral $\int f(x)dx$ is a function (or family of functions).

Table of indefinite integrals

$$\int cf(x)dx = c\int f(x)dx, \qquad \int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

$$\int kdx = kx + C, \qquad \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \sin x\, dx = -\cos x + C, \qquad \int \cos x\, dx = \sin x + C$$

$$\boxed{\int \sec^2 x\, dx = \tan x + C}, \qquad \int \csc^2 x\, dx = -\cot x + C$$

$$\int \sec x \tan x\, dx = \sec x + C, \qquad \int \csc x \cot x\, dx = -\csc x + C$$

where C is an arbitrary constant.

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \cos x - \sin x(-\sin x)}{(\cos x)^2}$$

$$\sec^2 x = \frac{1}{\cos x^2} = \frac{\cos^2 x + \sin^2 x}{(\cos x)^2}$$

Ex. [Text example 4.4.4] Find $\int_0^{12}(x - 12\sin x)dx$.

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

Proof: Suppose that $F(x) = \int f(x)dx \Rightarrow \frac{d}{dx}F(x) = f(x)$

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

$\to x = 12$

$$= \left[\frac{1}{2}x^2 - 12(-\ln 3x)\right]\Bigg|_{x=0}$$

Notation: $f(x)\Big|_{x=a}^{x=b} = f(b) - f(a)$

Ex. [Text example 4.4.5] Evaluate $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}dt$.

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt$$

$$= \int_1^9 2 + t^{\frac{1}{2}} - t^{-2}\, dt = 2t + \frac{2}{3}t^{\frac{3}{2}} - (-1)t^{-1}\Bigg|_{t=1}^{t=9}$$

$= \ ?$

Since $F'(x)$ represent the rate of change of $y = F(x)$ with respect to x, then $F(b) - F(a)$ is the change in y when x changes from a to b. This is the **Net change theorem**

Theorem 1.8. *The integral of a rate of change is the net change*

$$\int_a^b F'(x)dx = F(b) - F(a).$$

$F(x)$ is an anti-derivative of $F'(x)$.

Example 1. If the rate of population is dn/dt, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}dt = n(t_2) - n(t_1)$$

is the net change in population during the time from t_1 to t_2.

Example 2. If a object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$$

is the net change of position, or *displacement*, of the particle during the time period from t_1 to t_2. And the total $\boxed{\text{distance}}$ traveled by the particle is

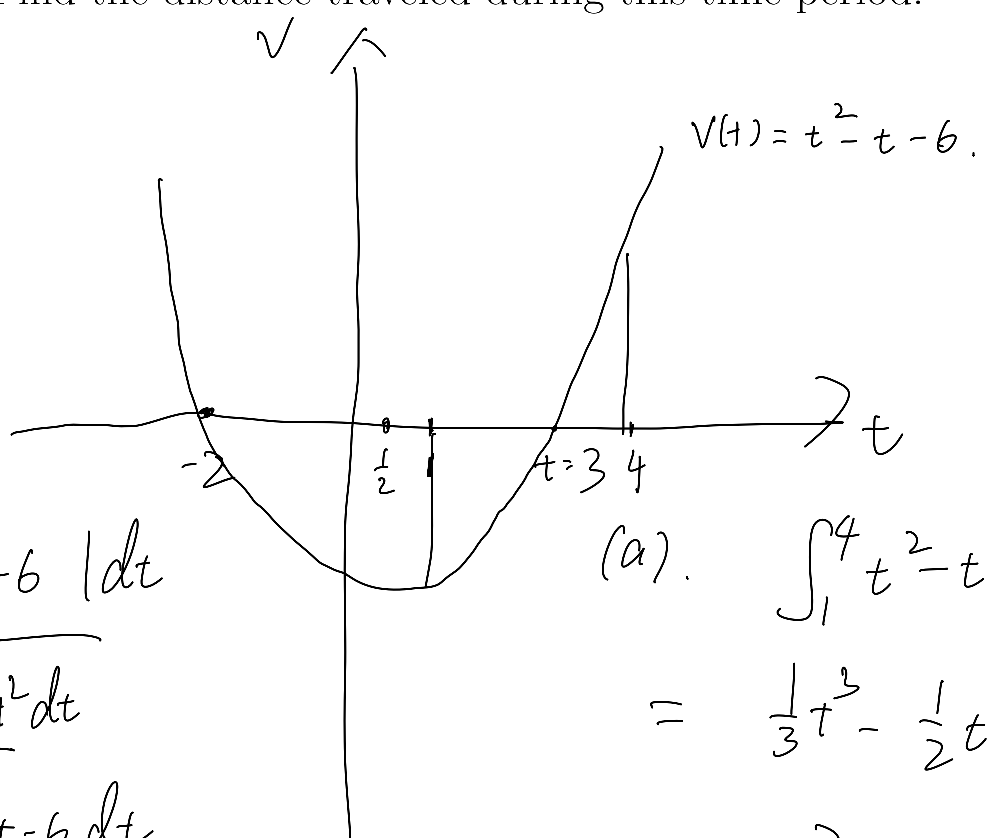
$$\int_{t_1}^{t_2} |v(t)|dt = \text{total distance traveled.}$$

$(t-3)(t+2) = 0$

Ex. [Text example 4.4.6] A particle moves along a line so that its velocity at time t is $v(t) = (t^2 - t - 6)$ m/s.

(a) Find the displacement of the particle during the period $1 \leq t \leq 4$.

(b) Find the distance traveled during this time period.



(b) $\int_1^4 |t^2-t-6|\,dt$

$= \int_1^3 6+t-t^2\,dt$

$+ \int_3^4 t^2-t-6\,dt$

(a). $\int_1^4 t^2-t-6\,dt$

$= \frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t \Big|_{t=1}^{t=4}$

$= ?$

$\boxed{\frac{1}{3}(2x+1)^{\frac{3}{2}}}$

$= \left(6t+\frac{1}{2}t^2-\frac{1}{3}t^3\right)\Big|_{t=1}^{t=3} + \left(\frac{1}{3}t^3-\frac{1}{2}t^2-6t\right)\Big|_{t=3}^{t=4} = ?$

1.5. **The substitution rule.** Text Section 4.5, Exercise: 3, 17, 23, 45, 59, 63, 67, 81.

$\frac{1}{2}\int \sqrt{u}\,du$ // $u=2x+1$

$\frac{1}{2}\int \sqrt{2x+1}\cdot(2x+1)'\,dx$

The substitution rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\boxed{\int f(g(x))g'(x)dx} = \int f(u)du.$$

$f(g(x))g'(x)$

Ex. [Text example 4.5.2] Evaluate $\int \sqrt{2x+1}\,dx$.

Proof: $\frac{d}{dx}\int f(g(x))\,g'(x)\,dx = f(g(x))g'(x).$

$\frac{d}{dx}\int f(u)\,du$ $\qquad F'(u)=f(u)$ $\qquad F'(g(x))g'(x) = F'(u(x))u'(x)$

Define $F(u) = \int f(u)du \Rightarrow \frac{d}{dx}\int f(u)du = \frac{d}{dx}F(u(x))$?

$\frac{1}{5}u^{-1}\Big|_{u=-2}^{u=-7}$

The substitution rule for definite integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$-\frac{1}{5}\int_{-2}^{-7}\frac{1}{u^2}\,du$ // $u=3-5x$

$$\int_a^b \boxed{f(g(x))g'(x)}dx = \boxed{\int_{g(a)}^{g(b)} f(u)du.}$$

Ex. [Text example 4.5.7] Evaluate $\boxed{\int_1^2 \frac{dx}{(3-5x)^2}}$. $= \frac{1}{-5}\int_1^2 \frac{1}{(3-5x)^2}\cdot(3-5x)'\,dx$

Let $F(x)$ be an antiderivative of $f(x) = F'(x)$.

The anti-derivative of $f(g(x))g'(x) = F'(g(x))g'(x)$ is $F(g(x))$

$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\Big|_{x=a}^{x=b} = F(g(b)) - F(g(a))$

$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big|_{u=g(a)}^{u=g(b)} = F(g(b)) - F(g(a))$

$$\int_{g(a)} f(u)\,du = F(u)\Big|_{u=g(a)} = F(g(b)) - F(g(a))$$

Integrals of symmetric functions → is symmetric.

Suppose f is continuous on $\boxed{[-a,a]}$,

(a) If f is even $[f(-x)=f(x)]$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

(b) If f is odd $[f(-x)=-f(x)]$, then $\int_{-a}^{a} f(x)dx = 0$.

Ex. [Text example 4.5.8] Evaluate $\int_{-2}^{2}(\boxed{x^6+1})dx$. $= 2\int_0^2 x^6+1\,dx = 2\left(\frac{1}{7}x^7 + x\right)\Big|_{x=0}^{x=2} = ?$

$$\int_{-a}^{a} f(x)\,dx = \boxed{\int_{-a}^{0} f(x)\,dx} + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

$u(x) = -x$

$$-\int_{-a}^{0} f(\boxed{-u})\,\boxed{u'(x)}\,dx \qquad \int_0^a f(x)\,dx$$

x, $= -1$

$$-\boxed{\int_a^0} f(-u)\,du = \int_0^a f(-u)\,du = \int_0^a f(u)\,du$$

Ex. [Text example 4.5.9] Evaluate $\int_{-1}^{1}\frac{\tan x}{1+x^2+x^4}dx$. $\tan x$ → odd, $1+x^2+x^4$ → even; odd

$= 0$