BMS 1901 Calculus for Life Sciences

Week10

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Study objectives

To understand the Substitution rule

Able to solve an integral using Integration by Parts

Understand how to calculate the outbreak size of an Infectious Disease

References:

Biocalculus by Stewart and Day

Chp5.4 The Substitution Rule

Chp5.5 The Integration by Parts

Chp5.3 The Fundamental Theorem of Calculus

The Substitution Rule

How to Integrate Using U-Substitution

https://www.youtube.com/watch?v=8B31SAk1nD8

Substitution in Indefinite Integrals

Because of the Fundamental Theorem, it’s important to be able to find antiderivatives. But our antidifferentiation formulas don’t tell us how to evaluate integrals such as

$$\int 2x\sqrt{1 + x^2}\, dx \tag{1}$$

Our strategy is to simplify the integral by introducing a new variable. Suppose that we let u be the quantity under the root sign in (1): $u = 1 + x^2$. Then $du/dx = 2x$. We write the equation $du/dx = 2x$ as

$$du = 2x\, dx$$

Substitution in Indefinite Integrals

Then formally, without justifying our calculation, we could write

$$\textbf{(2)} \qquad \int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,2x\,dx = \int \sqrt{u}\,du$$

$$= \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(1+x^2)^{3/2} + C$$

But now we can check that we have the correct answer by using the Chain Rule to differentiate the final function of Equation 2:

$$\frac{d}{dx}\left[\tfrac{2}{3}(1+x^2)^{3/2} + C\right] = \tfrac{2}{3}\cdot\tfrac{3}{2}(1+x^2)^{1/2}\cdot 2x = 2x\sqrt{1+x^2}$$

Substitution in Indefinite Integrals

(4) The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

Notice that if $u = g(x)$, then $du/dx = g'(x)$, which we write in terms of **differentials**: $du = g'(x)\, dx$. This is probably the best way to remember the Substitution Rule.

Example 1

Find $\int x^3 \cos(x^4 + 2)\, dx$.

Solution:
We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3\, dx$, which, apart from the constant factor 4, occurs in the integral.

Thus, using $x^3\, dx = \frac{1}{4}\, du$ and the Substitution Rule, we have

$$\begin{aligned} \int x^3 \cos(x^4 + 2)\, dx &= \int \cos u \cdot \tfrac{1}{4}\, du = \tfrac{1}{4} \int \cos u\, du \\ &= \tfrac{1}{4} \sin u + C \\ &= \tfrac{1}{4} \sin(x^4 + 2) + C \end{aligned}$$

Notice that at the final stage we had to return to the original variable x.

Substitution in Definite Integrals

Substitution in Definite Integrals

When evaluating a *definite* integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Evaluation Theorem.

For instance,

$$\int_0^4 \sqrt{2x+1}\,dx = \int \sqrt{2x+1}\,dx\Big]_0^4$$

$$= \tfrac{1}{3}(2x+1)^{3/2}\Big]_0^4 = \tfrac{1}{3}(9)^{3/2} - \tfrac{1}{3}(1)^{3/2}$$

$$= \tfrac{1}{3}(27-1) = \tfrac{26}{3}$$

Substitution in Definite Integrals

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

(5) The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Example 7

Evaluate $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$.

Solution:

Let $u = 3 - 5x$. Then $du = -5\,dx$, so $dx = -\frac{1}{5}\,du$.

When $x = 1$, $u = -2$ and when $x = 2$, $u = -7$.

Thus

$$\int_1^2 \frac{dx}{(3-5x)^2} = -\frac{1}{5}\int_{-2}^{-7}\frac{du}{u^2}$$

$$= -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7} = \frac{1}{5u}\Bigg]_{-2}^{-7}$$

$$= \frac{1}{5}\left(-\frac{1}{7}+\frac{1}{2}\right) = \frac{1}{14}$$

Integration by Parts

How to do Integration by Parts?

https://www.youtube.com/watch?v=KKg88oSUv0o

Indefinite Integrals

(1) $$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Formula 1 is called the **formula for integration by parts.** It is perhaps easier to remember in the following notation. Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule, the formula for integration by parts becomes

(2) $$\int u\,dv = uv - \int v\,du$$

Example 1

Find $\int x \sin x \, dx$.

Solution 1:
Suppose we choose $f(x) = x$ and $g'(x) = \sin x$.
Then $f'(x) = 1$ and $g(x) = -\cos x$. (For g we can choose *any* antiderivative of g'.)

Thus, using Formula 1, we have

$$\int x \sin x \, dx = f(x)g(x) - \int g(x)f'(x)\, dx$$

$$= x(-\cos x) - \int (-\cos x)\, dx$$

Example 1 – *Solution*

cont'd

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

Solution 2:

Using formula 2, let

$$u = x \qquad dv = \sin x\,dx$$

Then $$du = dx \qquad v = -\cos x$$

and so

$$\int x\sin x\,dx = \int \overbrace{x}^{u}\;\overbrace{\sin x\,dx}^{dv} = \overbrace{x}^{u}\;\overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v}\;\overbrace{dx}^{du}$$

Example 1 – *Solution*

cont'd

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

Definite Integrals

If we combine the formula for integration by parts with the Evaluation Theorem, we can evaluate definite integrals by parts. Evaluating both sides of Formula 1 between *a* and *b*, assuming f' and g' are continuous, and using the Evaluation Theorem, we obtain

(6)
$$\boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx}$$

Average Values

Average Values

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

In general, let's try to compute the average value of a function $y = f(x)$, $a \leq x \leq b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b - a)/n$. Then we choose points $x_1^*, \ldots, x_n^*$ in successive subintervals and calculate the average of the numbers $f(x_1^*), \ldots, f(x_n^*)$:

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

Average Values

Since $\Delta x = (b - a)/n$ we can write $n = (b - a)/\Delta x$ and the average value becomes

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{\dfrac{b - a}{\Delta x}} = \frac{1}{b - a}\left[f(x_1^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\right]$$

$$= \frac{1}{b - a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

If we let n increase, we would be computing the average value of a large number of closely spaced values.

Average Values

The limiting value is

$$\lim_{n \to \infty} \frac{1}{b - a} \sum_{i=1}^{n} f(x_i^*) \, \Delta x = \frac{1}{b - a} \int_a^b f(x) \, dx$$

by the definition of a definite integral.

Therefore we define the **average value of *f*** on the interval [*a*, *b*] as

$$\boxed{f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x) \, dx}$$

Example 1

Find the average value of the function $f(x) = 1 + x^2$ on the Interval [–1, 2].

Solution:

With $a = -1$ and $b = 2$ we have

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)\,dx$$

$$= \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2}$$

$$= 2$$

Average Values

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x)\, dx$$

that is, $$\int_a^b f(x)\, dx = f(c)(b - a)$$

Average Values

The geometric interpretation of the Mean Value Theorem for Integrals is that, for *positive* functions f, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b. (See Figure 2)

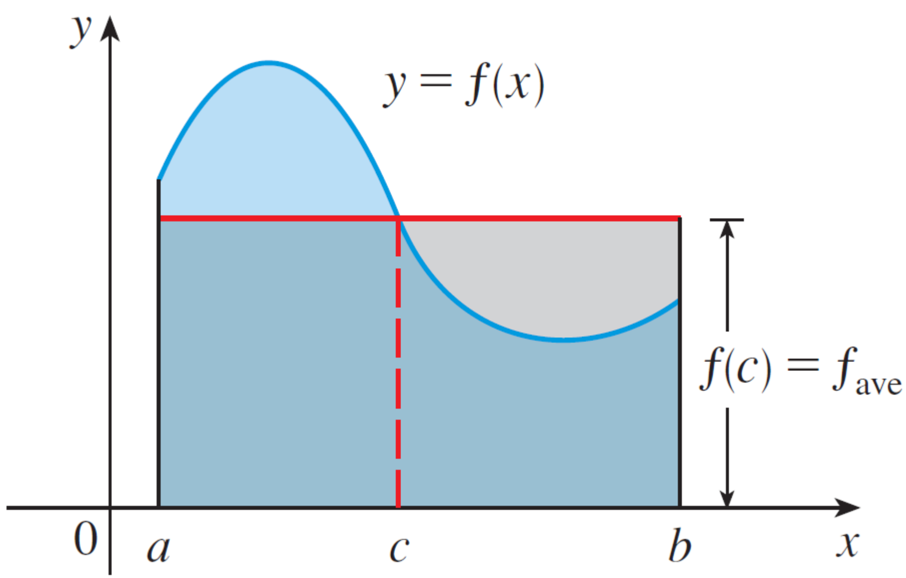

Figure 2

Example 3

Since $f(x) = 1 + x^2$ is continuous on the interval [–1, 2], the Mean Value Theorem for Integrals says there is a number c in [–1, 2] such that

$$\int_{-1}^{2} (1 + x^2)\, dx = f(c)[2 - (-1)]$$

In this particular case we can find c explicitly. $f_{ave} = 2$, so the value of c satisfies

$$f(c) = f_{\text{ave}} = 2$$

Example 3

cont'd

Therefore

$$1 + c^2 = 2 \qquad \text{so} \qquad c^2 = 1$$

So in this case there happen to be two numbers $c = \pm 1$ in the interval $[-1, 2]$ that work in the Mean Value Theorem for Integrals.

Average Values

Examples 1 and 3 are illustrated by Figure 3.

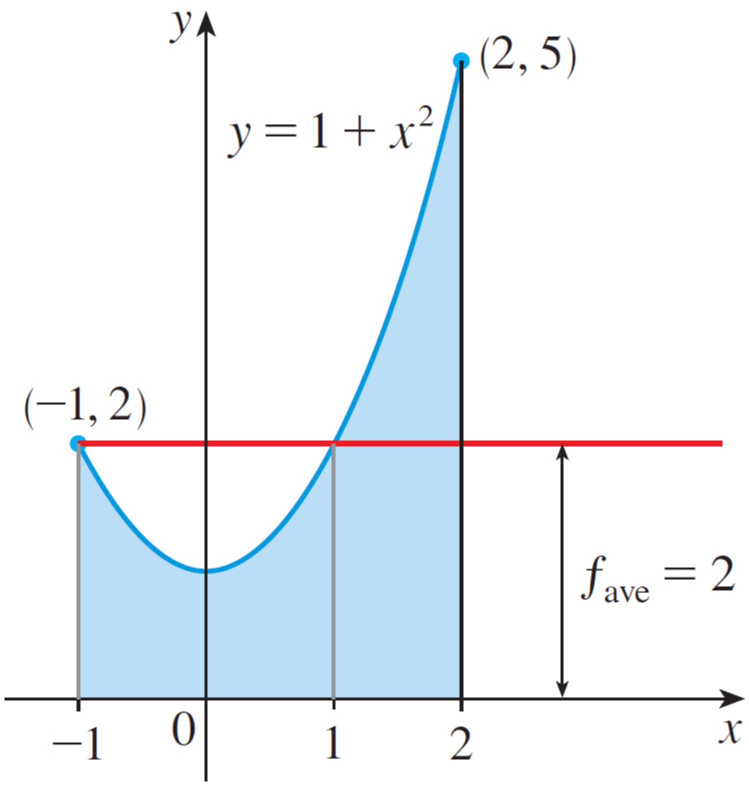

Figure 3

The Outbreak Size of an Infectious Disease

What is an Infectious Disease Model?

https://www.youtube.com/watch?v=XWXqXzAYe4E

https://www.youtube.com/watch?v=NKMHhm2Zbkw

If S(t) is the number of people susceptible to infection at time t and I(t) is the number of people already infected at this time, then a simplified version of the model specifies the derivatives of S and I with respect to time as

$$\frac{dS}{dt} = -\beta SI \tag{1}$$

$$\frac{dI}{dt} = \beta SI - \mu I \tag{2}$$

where β and μ are positive constants and the numbers of susceptible and infected people at time t = 0 are S(0) and I(0), respectively.

The Outbreak Size of an Infectious Disease

1. Use Equation 1 to express the quantity $-\beta\int_0^T S(t)I(t)\,dt$ in terms of the function $S(t)$.
2. Use Equation 1 to express the quantity $-\beta\int_0^T I(t)\,dt$ in terms of the function $S(t)$.
3. Use Equation 2 to express the quantity $\beta\int_0^T S(t)I(t)\,dt - \mu\int_0^T I(t)\,dt$ in terms of the function $I(t)$.
4. Use the results from Problems 1–3 to obtain a single equation that $S(0)$, $S(T)$, $I(T)$, and $I(0)$ must satisfy and that does not involve integrals.

The Outbreak Size of an Infectious Disease

5. It can be shown that $\lim_{T \to \infty} I(T) = 0$ and that $\lim_{T \to \infty} S(T) = S_\infty$, where S_∞ is a positive constant. What equation do you get if you let $T \to \infty$ in your answer to Problem 4?

6. Suppose that the number of people initially infected, $I(0)$, is negligibly small and define $q = \beta S(0)/\mu$ and $A = 1 - S_\infty/S(0)$. Here q is a measure of the transmissibility of the infection and A is the fraction of the original susceptible population $S(0)$ that ultimately gets infected (that is, the outbreak size). Show that your answer to Problem 5 can be written as

$$e^{-qA} = 1 - A \tag{3}$$

Equation 3 is a special case of an equation seen previously in many exercises and examples. For instance, see Example 3.5.13, Exercises 3.5.81, 3.8.32, 3.Review.92, and Example 9.4.6.

References

How to Integrate Using U-Substitution,

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Reminders

Complete Matlab exercises in Lab session

You may practice yourself after installing Matlab on your PC following the instructions

https://www.cityu.edu.hk/csc/deptweb/facilities/central-sw-tah-tc.htm