博弈论作业二

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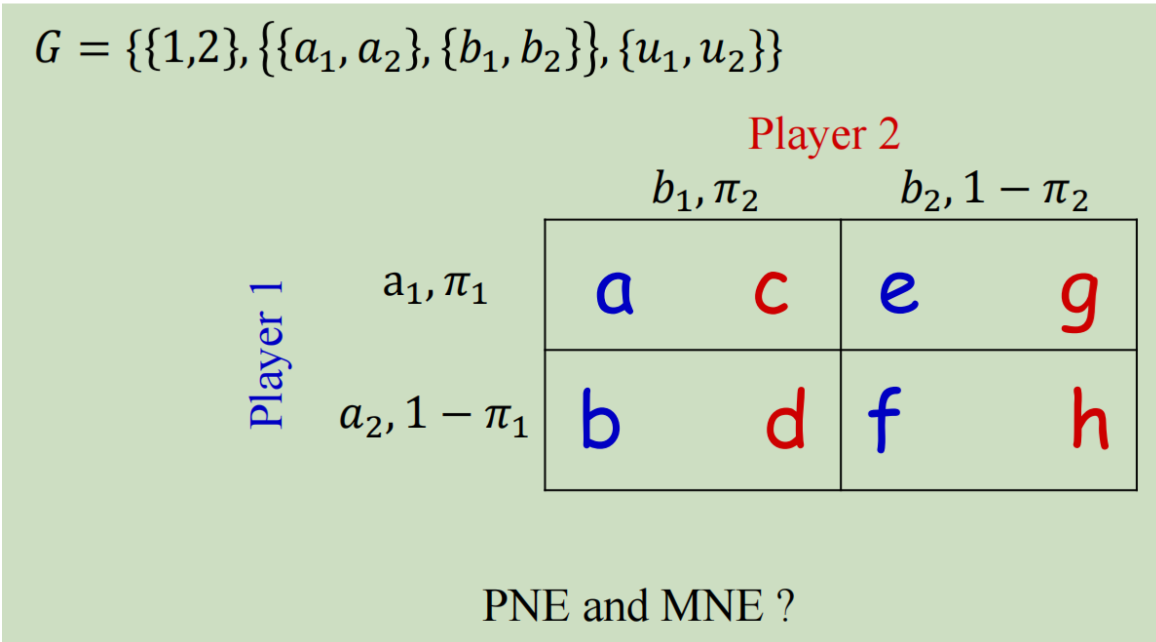
一、PNE&MNE



(1)假如存在PNE

如果是(a_1, b_1)，则需满足：$a > b$, $c > g$

如果是(a_1, b_2)，则需满足：$e > f$, $g > c$

如果是(a_2, b_1)，则需满足：$b > a$, $d > h$

如果是(a_2, b_2)，则需满足：$f > e$, $h > d$

(2)存在MNE

固定玩家1

- 玩家2在b_1上收益期望：$c\pi_1 + d(1-\pi_1)$
- 玩家2在b_2上收益期望：$g\pi_1 + h(1-\pi_1)$

  解得：$\pi_1 = \frac{h-d}{h-d+c-g} = \frac{1}{1+\frac{c-g}{h-d}}$

固定玩家2

- 玩家1在a_1上收益期望：$a\pi_2 + e(1-\pi_2)$
- 玩家1在a_2上收益期望：$b\pi_2 + f(1-\pi_2)$

  解得：$\pi_2 = \frac{f-e}{a-b+f-e} = \frac{1}{1+\frac{a-b}{f-e}}$

则此混合策略玩家1收益期望：

$$
\begin{aligned}
& a\pi_1\pi_2 + e\pi_1(1-\pi_2) + b(1-\pi_1)\pi_2 + f(1-\pi_1)(1-\pi_2) \\
&= (a-b+f-e)\pi_1\pi_2 + (e-f)\pi_1 + (b-f)\pi_1 + f \\
&= (b-f)\pi_1 + f = \pi_1 = \frac{b(h-d)+f(c-g)}{h-d+c-g} \\
&= \frac{b+f\frac{c-g}{h-d}}{1+\frac{c-g}{h-d}}
\end{aligned} \tag{1}
$$

则此混合策略玩家2收益期望：

$$
\begin{aligned}
& c\pi_1\pi_2 + g\pi_1(1-\pi_2) + d(1-\pi_1)\pi_2 + h(1-\pi_1)(1-\pi_2) \\
&= (d-h)\pi_2 + h = \pi_1 = \frac{d+h\frac{a-b}{f-e}}{1+\frac{a-b}{f-e}}
\end{aligned} \tag{2}
$$

因此当$\frac{c-g}{h-d} \geq 0$时，存在$\pi_1 = \frac{1}{1+\frac{a-b}{f-e}} \in [0,1]$

因此当$\frac{a-b}{f-e} \geq 0$时，存在$\pi_2 = \frac{1}{1+\frac{a-b}{f-e}} \in [0,1]$

p.s.当参数不满足上述两个范围中任意之一时，一定已经被（1）PNE中所讨论的情况包含

二、solve NE

	A	B	C
I	0	2	-1
II	-2	0	3
III	1	-3	0

(1)PNE

玩家1：$\max_{a_1\in A_1}\min_{a_2\in A_2} u(a_1,a_2) = -1$

玩家2：$\min_{a_2\in A_2}\max_{a_1\in A_1} u(a_1,a_2) = 1$

1>-1所以没有PNE，只有MNE

(2)MNE

设玩家1的选择概率为$\{p_1,p_2,p_3\}$

玩家2最优收益$max(2p_2-p_3, 3p_3-2p_1, p_1-3p_2)$

玩家1最优收益$min(p_3-2p_2, 2p_1-3p_3, 3p_2-p_1)$

令$v = min(p_3-2p_2, 2p_1-3p_3, 3p_2-p_1)$，原问题等价于线性规划问题

$$
\begin{aligned}
\max \quad & v \\
s.t. \quad & -2p_2 + p_3 \geq v \\
& 2p_1 - 3p_3 \geq v \\
& -p_1 + 3p_2 \geq v \\
& p_1 + p_2 + p_3 = 1 \\
& p_i \geq 0, i = 1,2,3
\end{aligned} \tag{3}
$$

解得$p_1 = 0.5, p_2 = 0.167, p_3 = 0.333, v = 0$

观察原收益矩阵可知$M = -M^T$，因此是二阶零和对称博弈

于是$q_1 = 0.5, q_2 = 0.167, q_3 = 0.333, v = 0$

三、solve NE线性规划

		Player 2			
		r	x	y	z
Player 1	a	1	-2	6	-4
	b	2	-7	2	4
	c	-3	4	-4	-3
	d	-8	3	-2	3

(1)PNE

玩家1：$\max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2) = -4$

玩家2：$\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2) = 2$

所以没有PNE，只有MNE

(2)MNE

设玩家1的选择概率为$\{p_1, p_2, p_3, p_4\}$

玩家1最优收益
$\min(p_1 + 2p_2 - 3p_3 - 8p_4, -2p_1 - 7p_2 + 4p_3 + 3p_4, 6p_1 + 2p_2 - 4p_3 - 2p_4, -4p_1 + 4p_2 - 3p_3 + 3p_4) = v$

问题完全等价于求解线性规划:

$$
\begin{array}{ll}
\max & v \\
\text{s.t.} & p_1+2p_2-3p_3-8p_4 \geq v \\
& -2p_1-7p_2+4p_3+3p_4 \geq v \\
& 6p_1+2p_2-4p_3-2p_4 \geq v \\
& -4p_1+4p_2-3p_3+3p_4 \geq v \\
& p_1+p_2+p_3+p_4=1 \\
& p_i \geq 0, i=1,2,3,4
\end{array} \tag{4}
$$

设玩家2的选择概率为$\{q_1, q_2, q_3, q_4\}$

玩家2最优收益

$v=\max\left(q_1-2q_2+6q_3+4q_4, 2q_1-7q_2+2q_1+4q_4, -3q_1+4q_2-4q_3+3q_4, -8q_1+3q_2-2q_3+3q_4\right)$

问题完全等价于求解线性规划:

$$
\begin{array}{ll}
\min & v \\
\text{s.t.} & q_1-2q_2+6q_3-4q_4 \leq v \\
& 2q_1-7q_2+2q_1+4q_4 \leq v \\
& -3q_1+4q_2-4q_3-3q_4 \leq v \\
& -8q_1+3q_2-2q_3+3q_4 \leq v \\
& \sum_{i=1}^4 q_i=1 \\
& q_i \geq 0, i=1,2,3,4
\end{array} \tag{5}
$$

四、 Proof of Nash Equilibrium

Theorem For two-player zero-sum finite game $G=\{\{1,2\},\{A_1, A_2\}, u\}$, let player 1 select

$$p^* \in \underset{p \in \Delta_1}{\operatorname{argmax}} \min_{q \in \Delta_2} U(p, q),$$

and let player 2 select

$$q^* \in \underset{q \in \Delta_2}{\operatorname{argmin}} \max_{p \in \Delta_1} U(p, q).$$

The mixed strategy outcome (p^*, q^*) is a MNE if and only if

$$\max_{p \in \Delta_1} \min_{q \in \Delta_2} U(p, q)=\min_{q \in \Delta_2} \max_{p \in \Delta_1} U(p, q)$$

$$
\begin{aligned}
\text{已知 } (p^*, q^*) \text{ is a MNE} & \Leftrightarrow U\left(p, q^*\right) \leq U\left(p^*, q^*\right) \leq U\left(p^*, q\right) \\
& \Leftrightarrow U\left(p, q^*\right) \leq U\left(p^*, q\right)
\end{aligned} \tag{6}
$$

(1)必要性

(p^*, q^*) is a MNE，则$\max_{p \in \Delta_1} U\left(p, q^*\right) \leq \min_{q \in \Delta_2} U\left(p^*, q\right)$

又因为

- $q^* \in \operatorname{argmin}_{q \in \Delta_2} \max_{p \in \Delta_1} U(p, q)$

- $p^* \in \operatorname{argmax}_{p\in\Delta_1} \min_{q\in\Delta_2} U(p,q)$
  可将上式可转化成

$\min_{q\in\Delta_2} \max_{p\in\Delta_1} U(p,q) \leq \max_{p\in\Delta_1} \min_{q\in\Delta_2} U(p,q)$

又由定理

$\min_{q\in\Delta_2} \max_{p\in\Delta_1} U(p,q) \geq \max_{p\in\Delta_1} \min_{q\in\Delta_2} U(p,q)$

推导出

$\min_{q\in\Delta_2} \max_{p\in\Delta_1} U(p,q) = \max_{p\in\Delta_1} \min_{q\in\Delta_2} U(p,q)$

(2)充分性

已知$\min_{q\in\Delta_2} \max_{p\in\Delta_1} U(p,q) = \max_{p\in\Delta_1} \min_{q\in\Delta_2} U(p,q)$

$$\begin{aligned} U\left(p,q^*\right) \leq \max_{p\in\Delta_1} U\left(p,q^*\right) = \min_{q\in\Delta_2} \max_{p\in\Delta_1} U(p,q) \\ U\left(p^*,q\right) \geq \min_{q\in\Delta_2} U\left(p^*,q\right) = \max_{p\in\Delta_1} \min_{q\in\Delta_2} U(p,q) \end{aligned} \tag{7}$$

因此$U\left(p,q^*\right) \leq U\left(p^*,q\right)$，这等价于$\left(p^*,q^*\right)$ is a MNE

五、 Proof of Minimax Theorem查资料

The Minmax Theorem For two-player zero-sum finite game $G = \{\{1,2\},\ \{A_1, A_2\}, u\}$, we have

$$\max_{p\in\Delta_1} \min_{q\in\Delta_2} pMq^\top = \min_{q\in\Delta_2} \max_{p\in\Delta_1} pMq^\top .$$

Δ_1,Δ_2是紧且凸的

令$f(p,q) = pMq^T$则，由于乘上矩阵M相当于是多个连续方程的线性组合，所以f是连续函数

$f(\theta x + (1-\theta)y, q) = (\theta x + (1-\theta)y)Mq^T = \theta f(x,q) + (1-\theta)f(y,q) \leq \theta f(x,q) + (1-\theta)f(y,q)$

因此对于固定q，$f(p,q)$对q凹，对于固定p，$f(p,q)$对p凸

因此$\max_{p\in\Delta_1} \min_{q\in\Delta_2} f(p,q) = \max_{p\in\Delta_1} \min_{q\in\Delta_2} f(p,q)$