1、ALC

> 类似普通话，是核心语言

每一个字母代表一或多种表达力

- T表示全集域
- 倒T表示空集

- **Language for $\mathcal{ALC}$ concepts (classes)**
  - concept names $A_0, A_1, \ldots$
  - role names $r_0, r_1, \ldots$
  - the concept $\top$ (often called "thing")
  - the concept $\bot$ (stands for the empty class)
  - the concept constructor $\sqcap$ (often called intersection, conjunction, or simply "and").
  - the concept constructor $\exists$ (often called existential restriction).
  - the concept constructor $\forall$ (often called value restriction).
  - the concept constructor $\sqcup$ (often called union, disjunction, or simply "or").
  - the concept constructor $\neg$ (often called complement or negation).

2、ALC concept递归定义

- All concept names, $\top$ and $\bot$ are $\mathcal{ALC}$ concepts;
- if C is a $\mathcal{ALC}$ concept, then $\neg C$ is a $\mathcal{ALC}$ concept;
- if C and D are $\mathcal{ALC}$ concepts and r is a role names, then

$$(C \sqcap D), \quad (C \sqcup D), \quad \exists r.C, \quad \forall r.C$$

are $\mathcal{ALC}$ concepts.

A **$\mathcal{ALC}$ concept-inclusion** is of the form

$$C \sqsubseteq D,$$

where C, D are $\mathcal{ALC}$ concepts.

- for all可以用否定+存在表示

3、例子

Student $\sqcap$ $\forall$**drinks.tea** (all students who only drink tea).

Person $\sqcap$ $\forall$**hasChild.Male** (everybody whose children are all male);

- 如果有孩子则男，或者没孩子。（如果。。。一定。。。）

- forall语义：要么就没有，要是有就一定怎样怎样

- **Person** $\sqcap$ **∀hasChild.Male** $\sqcap$ **∃hasChild.**$\top$ (everybody who has a child and whose children are all male).
- 那些一定有且仅有男孩的人

4、Interpretation

- interpretation of **complex concepts** in $\mathcal{I}$:

  (C, D are concepts and r a role name)

  - $(\top)^{\mathcal{I}} = \Delta^{\mathcal{I}}$ and $(\bot)^{\mathcal{I}} = \emptyset$
  - $(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
  - $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$ and $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$
  - $(\forall r.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid$ for all $y \in \Delta^{\mathcal{I}}$ with $(x, y) \in r^{\mathcal{I}}$ we have $y \in C^{\mathcal{I}}\}$
  - $(\exists r.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid$ exists $y \in \Delta^{\mathcal{I}}$ such that $(x, y) \in r^{\mathcal{I}}$ and $y \in C^{\mathcal{I}}\}$

注意二元关系没有交换性

Example

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be defined by setting

- $\Delta^{\mathcal{I}} = \{a, b, c, d\}$;
- $A^{\mathcal{I}} = \{b, d\}$, $B^{\mathcal{I}} = \{c\}$;
- $r^{\mathcal{I}} = \{(a, b), (a, c)\}$, $s^{\mathcal{I}} = \{(a, b), (a, d)\}$.

Then

- $(\forall r.A)^{\mathcal{I}} = \{b, c, d\}$, $(\forall s.A)^{\mathcal{I}} = \{a, b, c, d\}$;
- $(\exists r.A \sqcap \forall r.A)^{\mathcal{I}} = \emptyset$, $(\exists s.A \sqcap \forall s.A)^{\mathcal{I}} = \{a\}$;
- $(\exists r.B \sqcap \exists r.A)^{\mathcal{I}} = \{a\}$, $(\exists r.(A \sqcap B))^{\mathcal{I}} = \emptyset$;
- $(\forall r.\neg A)^{\mathcal{I}} = \{b, c, d\}$, $(\forall s.\neg A)^{\mathcal{I}} = \{b, c, d\}$.

1.等价变换equivalent concept

For all interpretations $\mathcal{I}$ and all concepts C, D and roles r the following holds:

- $(\neg\neg C)^{\mathcal{I}} = C^{\mathcal{I}}$;
- $(\forall r.C)^{\mathcal{I}} = (\neg\exists r.\neg C)^{\mathcal{I}}$;
- $(\neg(C \sqcap D))^{\mathcal{I}} = (\neg C \sqcup \neg D)^{\mathcal{I}}$;
- $(\neg(C \sqcup D))^{\mathcal{I}} = (\neg C \sqcap \neg D)^{\mathcal{I}}$;
- $(\neg\exists r.C)^{\mathcal{I}} = (\forall r.\neg C)^{\mathcal{I}}$;
- $(\neg\forall r.C)^{\mathcal{I}} = (\exists r.\neg C)^{\mathcal{I}}$;
- $(C \sqcap \neg C)^{\mathcal{I}} = \bot^{\mathcal{I}} = \emptyset$;
- $(C \sqcup \neg C)^{\mathcal{I}} = \top^{\mathcal{I}} = \Delta^{\mathcal{I}}$.

语义证明法

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(~~C)^{I} = Delta^{I} - (~C)^{I} = Delta^{I} - (Delta^{-} - C^{I}) = Delta^{I} - Delta^{I} + C^{I} = C^{I}
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- 包含inclusion：C is included in D&C is subsumed by D

Tbox术语集合：（termilnology术语）

2.语义蕴含：

左面的解释一定页是右面的解释

构造法证明

Example

Let $\mathcal{T} = \{A \sqsubseteq \exists r.B\}$. Then

$$\mathcal{T} \not\models A \sqsubseteq \forall r.B.$$

To see this, construct an interpretation $\mathcal{I}$ such that

- $\mathcal{I} \models \mathcal{T}$;
- $\mathcal{I} \not\models A \sqsubseteq \forall r.B$.

Let $\mathcal{I}$ be defined by

- $\Delta^{\mathcal{I}} = \{a, b, c\}$;
- $A^{\mathcal{I}} = \{a\}$;
- $r^{\mathcal{I}} = \{(a, b), (a, c)\}$;
- $B^{\mathcal{I}} = \{b\}$.

Then $A^{\mathcal{I}} = \{a\} \subseteq \{a\} = (\exists r.B)^{\mathcal{I}}$ and so $\mathcal{I} \models \mathcal{T}$. But $A^{\mathcal{I}} \not\subseteq \{b, c\} = (\forall r.B)^{\mathcal{I}}$ and so $\mathcal{I} \not\models A \sqsubseteq \forall r.B$.

3.限制定义域（r关系的第一个元素）

$$\exists r.\top \sqsubseteq C$$

- 部分inverse语义用ALC其实可以表达

$$\exists r^{-}.\top \sqsubseteq C$$

$$\top \sqsubseteq \forall r.C$$

证明两个无限类问题每一个都成立：归纳&反证

- 语义等价

$$\text{Vegetable} \sqcap \text{Meat} \sqsubseteq \bot$$

$$\text{Vegetable} \sqsubseteq \neg\text{Meat}.$$

5、ALC扩展--SHOIQ

> 表达力还不够强

传递关系SH

Transitive roles: One can add *transitive*(r) to a TBox to state that the relation r is transitive. Thus,

- $\mathcal{I} \models$ *transitive*(r) if, and only if, $r^{\mathcal{I}}$ is transitive, i.e., for all $x, y, z \in \Delta^{\mathcal{I}}$ such that $(x, y) \in r^{\mathcal{I}}$ and $(y, z) \in r^{\mathcal{I}}$ we have $(x, z) \in r^{\mathcal{I}}$.

Examples

- The role "is part of" is often regarded as transitive.

Role hierarchies: one can add a role inclusion $r \sqsubseteq s$ to a TBox to state that r is included in s. Thus,

- $\mathcal{I} \models r \sqsubseteq s$ iff $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$.

Example:

- hasSon $\sqsubseteq$ hasChild

nominalO

只包含一个元素的集合，与个体individual区分

- 要求C必须的concept，不能是individual，所以需要nominal
- 一定要写存在，但确是任意的语义

In $\mathcal{ALC}$ extended with nominals we can use the expressions $\{a\}$ and $\{a_1, \ldots, a_n\}$ as concepts.
Examples:

- $\exists$**citizen_of**.$\{$**France**$\}$ (citizens of France).
- $\exists$**citizen_of**.$\{$**France**, **Ireland**$\}$ (citizens of France or Ireland).
- $\exists$**has_colour**.$\{$**Green**$\}$ (all green objects).
- $\exists$**student_of**.$\{$**Liverpool_University**$\}$ (students of Liverpool University).
- One can also define the concept **Colour** by giving a list of all colours:

$$\textbf{Colour} \equiv \{\textbf{red}, \textbf{yellow}, \ldots, \textbf{green}\}$$

and give a value restriction for the role **has_colour** by

$$\top \sqsubseteq \forall \textbf{has\_colour}.\textbf{Colour}.$$

逆关系I

- 双元关系换序即可

Inverse roles: If r is a role name, then r^- is a role, called the inverse of r. The interpretation of inverse roles is given by

- $(r^-)^\mathcal{I} = \{(y, x) \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid (x, y) \in r^\mathcal{I}\}$.

r^- can occur in all places in which the role name r can occur.

Examples

- $\exists$**has_child**$^-$.**Gardener** is the class of all objects having a parent who is a gardener.
- $(\geq 3$**parent**$^-$.**Gardener**$)$ is the class of all objects having at least three children who are gardeners.

We have seen inverse roles in DL-Lite. There are no inverse roles in $\mathcal{EL}$. In fact, adding inverse roles to $\mathcal{EL}$ would make reasoning ExpTime-hard.

数字限制Q

Qualified number restrictions: if C is a concept, r a role, and n a number, then

$$(\leq n\ r.C), \quad (\geq n\ r.C)$$

are concepts. If $\mathcal{S}$ is a set, then we denote by $|\mathcal{S}|$ the number of its elements.
The interpretation of qualified number restrictions is given by

- $(\leq n\ r.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid\ |\{y \in \Delta^\mathcal{I} \mid (x, y) \in r^\mathcal{I} \text{ and } y \in C^\mathcal{I}\}| \leq n\ \}$
- $(\geq n\ r.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid\ |\{y \in \Delta^\mathcal{I} \mid (x, y) \in r^\mathcal{I} \text{ and } y \in C^\mathcal{I}\}| \geq n\ \}$

Examples

- $(\geq 3$ **hasChild**.**Male**$)$ is the class of all objects having at least three children who are male.

- 大于等于+小于等于=等于
- 大于等于n至少为1，存在承诺
- 小于等于n至少为0

Unqualified

只能用top，不能用C

We have seen **unqualified** number restrictions in DL-Lite. Recall that unqualified number restrictions are of the form

- $(\leq n\ r\ \top)$, and do not admit qualifications using an arbitrary concept C.

DL-Lite does not admit such qualifications because terminological reasoning would become ExpTime-hard.

图

- 个体是结点，表示domain种一个元素
- 被标记成concept name
- 边是element间的双元关系

6、Bisimulation同构关系

Definition (Bisimulation)

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations. The relation $\bigotimes \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is a bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$ if:

(i) $d_1 \bigotimes d_2$ implies $d_1 \in A^{\mathcal{I}_1}$ iff $d_2 \in A^{\mathcal{I}_2}$, for any $d_1 \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$, and A any concept name;

(ii) $d_1 \bigotimes d_2$ and $(d_1, d_1') \in r^{\mathcal{I}_1}$ implies the existence of $d_2' \in \Delta^{\mathcal{I}_2}$ such that $d_1' \bigotimes d_2'$ and $(d_2, d_2') \in r^{\mathcal{I}_2}$, for any $d_1, d_1' \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$, and r any role name;

(iii) $d_1 \bigotimes d_2$ and $(d_2, d_2') \in r^{\mathcal{I}_2}$ implies the existence of $d_1' \in \Delta^{\mathcal{I}_1}$ such that $d_1' \bigotimes d_2'$ and $(d_1, d_1') \in r^{\mathcal{I}_1}$, for any $d_1 \in \Delta^{\mathcal{I}_1}$, $d_2, d_2' \in \Delta^{\mathcal{I}_2}$, and r any role name;

Given $d_1 \in \Delta^{\mathcal{I}_1}$ and $d_2 \in \Delta^{\mathcal{I}_2}$, we define $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$ if there is a bisimulation $\bigotimes$ between $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $d_1 \bigotimes d_2$, and say that $d_1 \in \mathcal{I}_1$ is bisimilar to $d_2 \in \mathcal{I}_2$.

- **同构性质**

Theorem

If $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$, then the following holds for all $\mathcal{ALC}$ concepts C:

$$d_1 \in C^{\mathcal{I}_1} \text{ if and only if } d_2 \in C^{\mathcal{I}_2}.$$

Proof.

Since $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$, there is a bisimulation $\bigotimes$ between $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $d_1 \bigotimes d_2$. We prove the theorem by induction on the structure of C. Since, up to equivalence, any $\mathcal{ALC}$ concept can be constructed using only the constructors conjunction, negation, and existential quantification, we consider only these constructors in the induction step. The base case is the one where C is a concept name. □

> 证明分别对几个符号成立

Proof.

- Assume that $C = A$. Then $d_1 \in A^{\mathcal{I}_1}$
  if and only if $d_2 \in A^{\mathcal{I}_2}$
  is an immediate consequence of $d_1 \bigotimes d_2$.
- Assume that $C = D \sqcap E$. Then
  $$\begin{aligned} d_1 \in (D \sqcap E)^{\mathcal{I}_1} &\text{ if and only if } d_1 \in D^{\mathcal{I}_1} \text{ and } d_1 \in E^{\mathcal{I}_1}, \\ &\text{ if and only if } d_2 \in D^{\mathcal{I}_2} \text{ and } d_2 \in E^{\mathcal{I}_2}, \\ &\text{ if and only if } d_2 \in (D \sqcap E)^{\mathcal{I}_2}, \end{aligned}$$
  where the first and third equivalences are due to the semantics of conjunction, and the second is due to the induction hypothesis applied to D and E.

□

I1和I2这两个解释中的d1和d2是同构关系

不能说I1和I2这两个解释有同构关系

举例：

L1中有d1属于C，d1’属于D，有关系（d1，d1 ’）

L2中有d2属于C，d2'，d2 ‘’ 属于D，有关系（d2，d2 ’）（d2，d2’ ’）

于是（d1，d1 ’）是同构关系