有趣性质用于证明ALC表达力和可决定性

bisimulation

finite model property

tree model property

每个解释都可以化成有向图

> Lemma引理
>
> Theorem重要定理
>
> Proposition
>
> Corollary直接推论，无需复杂证明

一、Bisimulation

> 最开始出现于状态机，平移到语言学
>
> 判断计算行为相同，互相模拟

方法很自然：在某个结点上，接受相同的所有刺激，反应都相同

> 只能说两个解释之间存在，不能说有

Bisimulation 在 DL 上的定义是：给定两个 DL interpretations I1 and I2，如果 I1 中的一个元素（比如 d1）和 I2 中的一个元素（比如 d2）满足如下关系，则说 I1 中的 d1 与 I2 中的 d2 存在 bisimilar 关系，写作(I1,d1) ~ (I2,d2)：

（1）如果 d1 是某个 concept name A 在 I1 解释下的集合里面的元素，则 d2 也是 A 在 I2 解释下的集合里面的元素，反之亦然；

（2）如果 I1 中存在一个 node f1 与 d1 存在 r 关系：$(d1,f1) \in r^{I1}$，则 I2 中必须也存在一个 node f2 与 d2 存在 r 关系：$(d2,f2) \in r^{I2}$，并且(I1,f1) ~ (I2,f2)；

（3）如果 I2 中存在一个 node f2 与 d2 存在 r 关系：$(d2,f2) \in r^{I2}$，则 I1 中必须也存在一个 node f1 与 d1 存在 r 关系：$(d1,f1) \in r^{I1}$，并且(I2,f2) ~ (I1,f1)。

Theorem 3.2. *If $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$, then the following holds for all $\mathcal{ALC}$ concepts C:*

$$d_1 \in C^{\mathcal{I}_1} \quad \textit{if and only if} \quad d_2 \in C^{\mathcal{I}_2}.$$

- **即所有ALC concept都不能区分两个interpretations上的一对bisimilar元素**

> **C 作用就是分辨两个解释中两个元素的区别，**强制要求d2也要满足C里面限制的式子
>
> 可用于证明表达力强弱，利用限制推矛盾

归纳证明

induction hypothesis

首先假设 D 和 E 已经不能区分 d1 和 d2，再证明则 D ⊓ E 也不能区分 d1 和 d2。

证明才是一个通用的证明。则我们要证明的是：

$$d1 \in (D \sqcap E)^{I1} \text{ iff } d2 \in (D \sqcap E)^{I2}$$

因为是 iff，我们要证明两个方向：

$$d2 \in (D \sqcap E)^{I2} \text{ if } d1 \in (D \sqcap E)^{I1}$$

$$d1 \in (D \sqcap E)^{I1} \text{ if } d2 \in (D \sqcap E)^{I2}$$

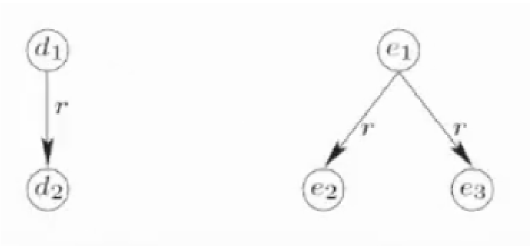
先证明第一个方向，假设 $d1 \in (D \sqcap E)^{I1}$，则根据 ALC 语义，$d1 \in D^{I1}$ and $d1 \in E^{I1}$。我们刚才又假设 D 和 E 已经不能区分 d1 和 d2，所以 $d2 \in D^{I2}$ and $d2 \in E^{I2}$。根据 ALC 语义，我们知道 $d2 \in (D \sqcap E)^{I2}$。第一个方向证明完毕。第二个方向的证明是镜像第一个的版本，我们不再赘述。或者像参考书里面的证明一样，直接使用 iff 关系得到证明。

Proposition 3.3. *$\mathcal{ALCN}$ is more expressive than $\mathcal{ALC}$; that is, there is an $\mathcal{ALCN}$ concept C such that $C \not\equiv D$ holds for all $\mathcal{ALC}$ concepts D.*

举ALCN的例子不能被表达

- 对数目无要求，这个算Bisi

> 本题的证明关键，正是因为同构忽视了数目，所以可以利用同构在数目上找出矛盾
>
> 利用同构说明了数目的描述确实难以ALC表达



Proof. We show that no $\mathcal{ALC}$ concept is equivalent to the $\mathcal{ALCN}$ concept $(\leqslant 1\, r.\top)$. Assume to the contrary that D is an $\mathcal{ALC}$ concept with $(\leqslant 1\, r.\top) \equiv D$. In order to lead this assumption to a contradiction, we consider the interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ depicted in Figure 3.2. Since

$$\rho = \{(d_1, e_1), (d_2, e_2), (d_2, e_3)\}$$

is a bisimulation, we have $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, e_1)$, and thus $d_1 \in D^{\mathcal{I}_1}$ if and only if $e_1 \in D^{\mathcal{I}_2}$. This contradicts our assumption $(\leqslant 1\, r.\top) \equiv D$ since $d_1 \in (\leqslant 1\, r.\top)^{\mathcal{I}_1}$, but $e_1 \notin (\leqslant 1\, r.\top)^{\mathcal{I}_2}$. □

二、Disjoint union

> 多个互不相交的取并集，把多个图组合看成一个图

J：再加一个第二级索引，表示出现在原来的第几个解释中

1、引理

Lemma 3.7. *Let $\mathcal{J} = \biguplus_{\nu \in \mathfrak{N}} \mathcal{I}_\nu$ be the disjoint union of the family $(\mathcal{I}_\nu)_{\nu \in \mathfrak{N}}$ of interpretations. Then we have*

$$d \in C^{\mathcal{I}_\nu} \text{ if and only if } (d, \nu) \in C^{\mathcal{J}}$$

for all $\nu \in \mathfrak{N}$, $d \in \Delta^{\mathcal{I}_\nu}$ and $\mathcal{ALC}$ concept descriptions C.

> 最基础性质，建立I中单概念和J中role关系的桥梁

证明：同构

Proof. It is easy to see that, for all $\nu \in \mathfrak{N}$, the relation

$$\rho = \{(d, (d, \nu)) \mid d \in \Delta^{\mathcal{I}_\nu}\}$$

is a bisimulation between $\mathcal{I}_\nu$ and $\mathcal{J}$. Thus, the bi-implication in the statement of the lemma follows immediately from Theorem 3.2. □

2、定理

Theorem 3.8. *Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox and $(\mathcal{I}_\nu)_{\nu \in \mathfrak{N}}$ a family of models of $\mathcal{T}$. Then its disjoint union $\mathcal{J} = \biguplus_{\nu \in \mathfrak{N}} \mathcal{I}_\nu$ is also a model of $\mathcal{T}$.*

> 原来所有T的model组合后还是model，紧凑性（disjoint union作用）

证明

Proof. Assume that $\mathcal{J}$ is not a model of $\mathcal{T}$. Then there is a GCI $C \sqsubseteq D$ in $\mathcal{T}$ and an element $(d, \nu) \in \Delta^{\mathcal{J}}$ such that $(d, \nu) \in C^{\mathcal{J}}$, but $(d, \nu) \notin D^{\mathcal{J}}$. By Lemma 3.7, this implies $d \in C^{\mathcal{I}_\nu}$ and $d \notin D^{\mathcal{I}_\nu}$, which contradicts our assumption that $\mathcal{I}_\nu$ is a model of $\mathcal{T}$. □

3、推论

> C对于T是可满足的，说明存在解释使C非空且为T的model

- **于是任意可满足的 ALC concept 都存在无限大的模型**

Corollary 3.9. *Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox and C an $\mathcal{ALC}$ concept that is satisfiable with respect to $\mathcal{T}$. Then there is a model $\mathcal{J}$ of $\mathcal{T}$ in which the extension $C^{\mathcal{J}}$ of C is infinite.*

Proof. Since C is satisfiable with respect to $\mathcal{T}$, there is a model $\mathcal{I}$ of $\mathcal{T}$ and an element $d \in \Delta^{\mathcal{I}}$ such that $d \in C^{\mathcal{I}}$. Let $\mathcal{J} = \biguplus_{n \in \mathbb{N}} \mathcal{I}_n$ be the countably infinite disjoint union of $\mathcal{I}$ with itself. By Theorem 3.8, $\mathcal{J}$ is a model of $\mathcal{T}$, and by Lemma 3.7, $(d, n) \in C^{\mathcal{J}}$ for all $n \in \mathbb{N}$. □

三、Finite model property

1、finite

> model有限，一定域有限

Definition 3.10. The interpretation $\mathcal{I}$ is a *model of a concept* C *with respect to a TBox* $\mathcal{T}$ if $\mathcal{I}$ is a model of $\mathcal{T}$ such that $C^{\mathcal{I}} \neq \emptyset$. We call this model *finite* if $\Delta^{\mathcal{I}}$ is finite.

- 若某个ALC对Tbox可满足，则一定会存在finite model，又可以用finite和disjoint union创建infinite

2、size&sub

- If $C = A \in N_C \cup \{\top, \bot\}$, then $\mathsf{size}(C) = 1$ and $\mathsf{sub}(C) = \{A\}$.
- If $C = C_1 \sqcap C_2$ or $C = C_1 \sqcup C_2$, then $\mathsf{size}(C) = 1 + \mathsf{size}(C_1) + \mathsf{size}(C_2)$ and $\mathsf{sub}(C) = \{C\} \cup \mathsf{sub}(C_1) \cup \mathsf{sub}(C_2)$.
- If $C = \neg D$ or $C = \exists r.D$ or $C = \forall r.D$, then $\mathsf{size}(C) = 1 + \mathsf{size}(D)$ and $\mathsf{sub}(C) = \{C\} \cup \mathsf{sub}(D)$.

> 符号也占size
>
> size=concept数+符号数

- 推广到Tbox

$$\mathsf{size}(\mathcal{T}) = \sum_{C \sqsubseteq D \in \mathcal{T}} \mathsf{size}(C) + \mathsf{size}(D) \text{ and } \mathsf{sub}(\mathcal{T}) = \bigcup_{C \sqsubseteq D \in \mathcal{T}} \mathsf{sub}(C) \cup \mathsf{sub}(D).$$

3、引理

size&sub&closed

Lemma 3.11. *Let C be an $\mathcal{ALC}$ concept and $\mathcal{T}$ be an $\mathcal{ALC}$ TBox. Then*

$$|\mathsf{sub}(C)| \leq \mathsf{size}(C) \text{ and } |\mathsf{sub}(\mathcal{T})| \leq \mathsf{size}(\mathcal{T}).$$

We call a set S of $\mathcal{ALC}$ concepts *closed* if $\bigcup\{\mathsf{sub}(C) \mid C \in S\} \subseteq S$. Obviously, if S is the set of subdescriptions of an $\mathcal{ALC}$ concept or TBox, then S is closed.

> sub比较小因为重复的没算

4、S-type

Definition 3.12 (S-type)**.** Let S be a set of $\mathcal{ALC}$ concepts and $\mathcal{I}$ an interpretation. The S*-type* of $d \in \Delta^{\mathcal{I}}$ is defined as

$$t_S(d) = \{C \in S \mid d \in C^{\mathcal{I}}\}.$$

> 是S的子集

> 收集S中所有出现d的内容

- S中所有包含d的概念的集合，共有2^|S|种
- 只要证明S-type有一个元素，即可证明 非空

5、S-filtration

> S上等价关系（非常广义的概念）

作用：消除对于成为model无用的项，合并所有S-type相同的项

- S上与d等价集合

$$[d]_S = \{e \in \Delta^{\mathcal{I}} \mid d \simeq_S e\}.$$

- 等价类（相当于合并同类项）(并不维持同构性质)

$$\begin{array}{rcl} \Delta^{\mathcal{J}} & = & \{[d]_S \mid d \in \Delta^{\mathcal{I}}\}; \\ A^{\mathcal{J}} & = & \{[d]_S \mid \text{there is } d' \in [d]_S \text{ with } d' \in A^{\mathcal{I}}\} \text{ for all } A \in \mathbf{C}; \\ r^{\mathcal{J}} & = & \{([d]_S, [e]_S) \mid \text{there are } d' \in [d]_S, e' \in [e]_S \text{ with } (d', e') \in r^{\mathcal{I}}\} \\ & & \text{for all } r \in \mathbf{R}. \end{array}$$

> 域是集合的域

举例

（1） 先确定 S-type 中的 $S = \{A, B, C, \exists r.C\}$

（2） 再确定初始的解释 I： $\Delta^I = \{a, b, c, d, e, f\}$

$A^I = \{a, b, e\}$ $B^I = \{b, c, d\}$ $C^I = \{c, d, f\}$

$r^I = \{(a, c), (b, d), (e, f)\}$

则 $(\exists r.C)^I = \{a, b, e\}$

（3） 找出各个元素的 S-type：

$t_S(a) = \{A, \exists r.C\}$

$t_S(b) = \{A, B, \exists r.C\}$

$t_S(c) = \{B, C\}$

$t_S(d) = \{B, C\}$

$t_S(e) = \{A, \exists r.C\}$

$t_S(f) = \{C\}$

（4） 对各个元素进行归类：

$[a]_S = \{a, e\}$

$[b]_S = \{b\}$

$[c]_S = \{c, d\}$

$[d]_S = \{c, d\}$

$[e]_S = \{a, e\}$

$[f]_S = \{f\}$

（5） 构建 I 的 S-filtration J：

$\Delta^J = \{\{a, e\}, \{b\}, \{c, d\}, \{f\}\}$

$A^J = \{\{a, e\}, \{b\}\}$ $B^I = \{\{b\}, \{c, d\}\}$ $C^I = \{\{c, d\}, \{f\}\}$

$r^J = \{(\{a, e\}, \{c, d\}), (\{b\}, \{c, d\}), (\{a, e\}, \{f\})\}$

破坏同构的例子

原因：相当于扩充了原来不存在的r关系，产生集合类别（A）的混乱

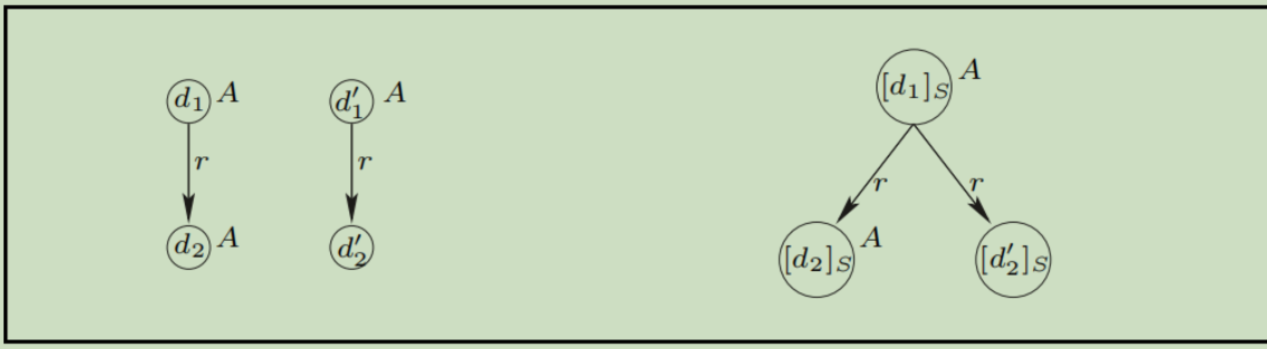


Fig. 3.4. An interpretation $\mathcal{I}$ and its S-filtration $\mathcal{J}$ for $S = \{\top, A, \exists r.\top\}$.

6、引理

> 有限闭包

Lemma 3.15. *Let S be a finite, closed set of $\mathcal{ALC}$ concepts, $\mathcal{I}$ an interpretation and $\mathcal{J}$ the S-filtration of $\mathcal{I}$. Then we have*

$$d \in C^{\mathcal{I}} \quad \text{if and only if} \quad [d]_S \in C^{\mathcal{J}}$$

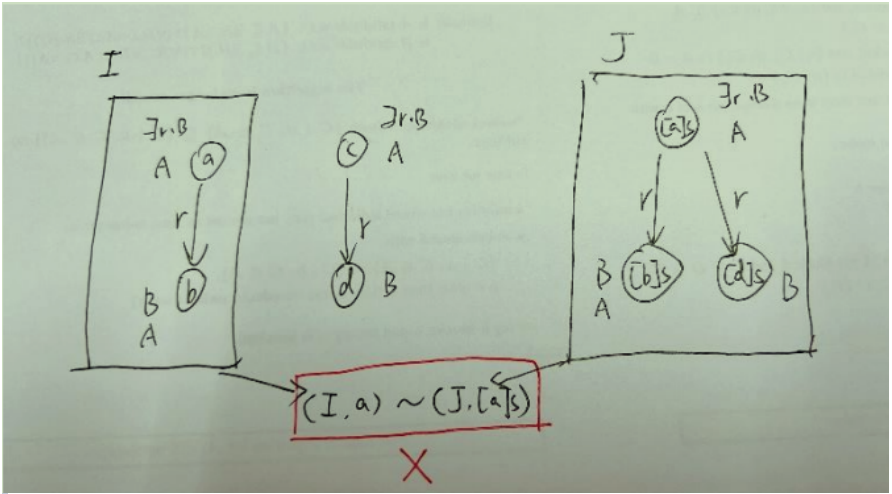
for all $d \in \Delta^{\mathcal{I}}$ and $C \in S$.

Proof. By induction on the structure of C, where we again restrict our attention to concept names, negation, conjunction and existential restriction (see Lemma 2.16):

同构证明失败

想证明 S 中的任何 concept C 都无法区分 d 和[d]S，但实际上 d 和[d]S之间有无 bisimilar

举例：



> (I, a) ~ (J, [a]S)并不成立
>
> 因为J中[a]S有一个 r-successor[d]S，但I中的a却没有这样的r-successor

分符号证明

尽管如此，我们要证明：$d \in C^I$ iff $[d]_S \in C^J$。同样的方法，我们使用归纳法进行证明：我们的 base case 是 $C = A$，其中 A 是一个 concept name，我们要证明 A 无法区分 d 和$[d]_S$：

$$d \in A^I \text{ iff } [d]_S \in A^J$$

因为是 iff，我们要证明两个方向：

$$[d]_S \in A^J \text{ if } d \in A^I$$
$$d \in A^I \text{ if } [d]_S \in A^J$$

先证明第一个方向：如果 $d \in A^I$，则$[d]_S \in A^J$，这是 J 对 A 的解释直接得到的。再证明第二个方向：如果$[d]_S \in A^J$，则根据 J 对 A 的解释，$[d]_S$ 中存在一个元素 d’ 使得 d’ $\in A^I$。又因为 $d =_S$ d’ 且 A 为 S 中一个元素，则可以由 d’ $\in A^I$ 得出 $d \in A^I$。

接下来我们要证明 induction hypothesis，首先 C 是一个 conjunction 的情况：$C = D \sqcap E$。则我们要证明的是：

$$d \in (D \sqcap E)^I \text{ iff } [d]_S \in (D \sqcap E)^J$$

因为是 iff，我们要证明两个方向：

$$[d]_S \in (D \sqcap E)^J \text{ if } d \in (D \sqcap E)^I$$
$$d \in (D \sqcap E)^I \text{ if } [d]_S \in (D \sqcap E)^J$$

先证明第一个方向，假设 $d \in (D \sqcap E)^I$，根据 ALC 语义，$d \in D^I$ 且 $d \in E^I$。又因为 $d \in [d]_S$，则根据 J 的定义，$[d]_S \in D^J$ 且$[d]_S \in D^J$。

我们刚才又假设 D 和 E 已经不能区分 d1 和 d2，所以 $d2 \in D^{I2}$ and $d2 \in E^{I2}$。根据 ALC 语义，我们知道 $d2 \in (D \sqcap E)^{I2}$。第一个方向证明完毕。第二个方向的证明是镜像第一个的版本，我们不再赘述。或者像参考书里面的证明一样，直接使用 iff 关系得到证明。

7、Bounded

> 比finite更强，得到边界
>
> 用算法之前先要证明这个问题的可解决性，需要证明存在有限domain

Theorem 3.16 (Bounded model property). *Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox, C an $\mathcal{ALC}$ concept and $n = \mathsf{size}(\mathcal{T}) + \mathsf{size}(C)$. If C has a model with respect to $\mathcal{T}$, then it has one of cardinality at most 2^n.*

> 至少存在一个解释的域有界

的 S，比如 $S = \{A, B, \exists r.B\}$，对于一个 interpretation I 来说，Δ^I 中的某个元素 d 最多有 2^3 个 S-type，那么$[d]_S$ 最多有 2^3 个，则 Δ^J 里面的也元素最多有 2^3 个。则现在只需要证明 J 和 I 一

8、Decidability

Corollary 3.18 (Decidability). *Satisfiability of $\mathcal{ALC}$ concepts with respect to $\mathcal{ALC}$ TBoxes is decidable.*

9、no finite

Theorem 3.19 (No finite model property). *$\mathcal{ALCIN}$ does not have the finite model property.*

Proof. Let $C = \neg A \sqcap \exists r.A$ and $\mathcal{T} = \{A \sqsubseteq \exists r.A, \top \sqsubseteq (\leqslant 1\, r^-)\}$. We claim that C does not have a finite model with respect to $\mathcal{T}$.

> 关键在于A在existr.A种，嵌套开始
> IN碰到一起会不可决定，所以不做算法研究

四、Tree model property

条件：

- 有唯一根节点
- 唯一父亲

Definition 3.20 (Tree model). Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox and C an $\mathcal{ALC}$ concept description. The interpretation $\mathcal{I}$ is a *tree model* of C with respect to $\mathcal{T}$ if $\mathcal{I}$ is a model of C with respect to $\mathcal{T}$, and the graph

$$\mathcal{G}_\mathcal{I} = \left(\Delta^\mathcal{I}, \bigcup_{r \in \mathbf{R}} r^\mathcal{I}\right)$$

is a tree whose root belongs to $C^\mathcal{I}$.

1、Unravelling算法

> 转化成树状的新解释

- 用每条路径的终止结点代表这条路径（domain中的结点变成path）
- r是两条路径的关系，代表终止结点之间有r关系

Definition 3.21 (Unravelling). Let $\mathcal{I}$ be an interpretation and $d \in \Delta^\mathcal{I}$. The *unravelling of $\mathcal{I}$ at d* is the following interpretation $\mathcal{J}$:

$$\Delta^\mathcal{J} = \{p \mid p \text{ is a } d\text{-path in } \mathcal{I}\},$$
$$A^\mathcal{J} = \{p \in \Delta^\mathcal{J} \mid \underline{\mathsf{end}(p) \in A^\mathcal{I}}\} \text{ for all } A \in \mathbf{C},$$
$$r^\mathcal{J} = \{(p, p') \in \Delta^\mathcal{J} \times \Delta^\mathcal{J} \mid \underline{p' = (p, \mathsf{end}(p'))} \text{ and } \underline{(\mathsf{end}(p), \mathsf{end}(p')) \in r^\mathcal{I}}\}$$
$$\text{for all } r \in \mathbf{R}.$$

In our example, $d_1 = d, e, d \in A^\mathcal{J}$ because $\mathsf{end}(d_1) = d \in A^\mathcal{I}$, and $((d, e, d), (d, e, d, e)) \in r^\mathcal{J}$ because $(d, e) \in r^\mathcal{I}$.

2、路径与终止结点同构

Lemma 3.22. *The relation*

$$\rho = \{(p, \mathsf{end}(p)) \mid p \in \Delta^{\mathcal{J}}\}$$

is a bisimulation between $\mathcal{J}$ *and* $\mathcal{I}$.

proof：还是分三个条件证明

（已收录在专题）其实就是翻译unravelling的定义

Proposition 3.23. *For all* $\mathcal{ALC}$ *concepts* C *and all* $p \in \Delta^{\mathcal{J}}$, *we have*

$$p \in C^{\mathcal{J}} \quad \textit{if and only if} \quad \mathsf{end}(p) \in C^{\mathcal{I}}.$$

We are now ready to show the tree model property of $\mathcal{ALC}$.

- 建立起原解释I和树解释J的联系

同构的证明相当于把集合A层面的联系（同构第一部分）升级到了concept C层面

3、Tree model property

Theorem 3.24 (Tree model property)**.** *$\mathcal{ALC}$ has the* tree model property, *i.e., if* $\mathcal{T}$ *is an* $\mathcal{ALC}$ *TBox and* C *an* $\mathcal{ALC}$ *concept such that* C *is satisfiable with respect to* $\mathcal{T}$, *then* C *has a tree model with respect to* $\mathcal{T}$.

proof

Proof. Let $\mathcal{I}$ be a model of $\mathcal{T}$ and $d \in \Delta^{\mathcal{I}}$ be such that $d \in C^{\mathcal{I}}$. We show that the unravelling $\mathcal{J}$ of $\mathcal{I}$ at d is a tree model of C with respect to $\mathcal{T}$.

1.想证tree model，先证是model（已收录在专题）

2.再证一定有父亲：只有根节点路径长度为1，非根节点一定有父亲

3.再证唯一父亲

(i) To prove that $\mathcal{J}$ is a model of $\mathcal{T}$, consider a GCI $D \sqsubseteq E$ in $\mathcal{T}$, and assume that $p \in \Delta^{\mathcal{J}}$ satisfies $p \in D^{\mathcal{J}}$. We must show $p \in E^{\mathcal{J}}$. By Proposition 3.23, we have $\mathsf{end}(p) \in D^{\mathcal{I}}$, which yields $\mathsf{end}(p) \in E^{\mathcal{I}}$ since $\mathcal{I}$ is model of $\mathcal{T}$. But then Proposition 3.23 applied in the other direction yields $p \in E^{\mathcal{J}}$.

(ii) We show that the graph

$$\mathcal{G}_{\mathcal{J}} = \left(\Delta^{\mathcal{J}}, \bigcup_{r \in N_R} r^{\mathcal{J}}\right)$$

is a tree with root d, where d is viewed as a d-path of length 1. First, note that d is the only d-path of length 1. By definition of

the extensions of roles in $\mathcal{J}$ and the definition of d-paths, all and only d-paths of length > 1 have a predecessor with respect to some role. Consequently, d is the unique node without predecessor, i.e., the root. Assume that p is a d-path of length > 1. Then there is a unique d-path p' such that $p = p', \mathsf{end}(p)$. Thus, p' is the unique d-path with $(p', p) \in E$, which completes our proof that $\mathcal{G}_{\mathcal{J}}$ is a tree with root d.

(iii) It remains to show that the root d of this tree belongs to the extension of C in $\mathcal{J}$. However, this follows immediately by Proposition 3.23 since $d = \mathsf{end}(d)$ and $d \in C^{\mathcal{I}}$.

4、反例

infinite

$$\{A \sqsubseteq \exists r.A\}.$$

ALCO没有Tree model property

We remark that many extensions of $\mathcal{ALC}$, such as $\mathcal{ALCIQ}$, also enjoy the tree model property. However, in the presence of inverse roles, a more liberal definition of trees is needed that also allows edges to be oriented towards the root. An example of a description logic that does not enjoy the tree model property is $\mathcal{ALCO}$: the concept $\{o\} \sqcap \exists r.\{o\}$ can clearly only have a non-empty extension in an interpretation that has a reflexive r-edge.

画图应为：唯一norminal o ，有指向自身的r关系