A is B用\sqsubseteq
adj类限制用\sqcap
作业2存疑题目：3（2），4，5，6（2），8，10

T1：more expressive证明方法

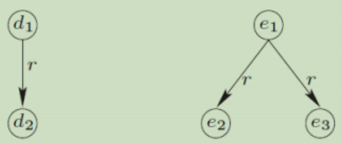
法一：同构

Theorem 3.2. *If* $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$, *then the following holds for all* $\mathcal{ALC}$ *concepts* C:

$$d_1 \in C^{\mathcal{I}_1} \text{ if and only if } d_2 \in C^{\mathcal{I}_2}.$$

- **所有ALC concept都不能区分两个interpretations上的一对bisimilar元素**

以 proposition 3.3 为例，如果想证明：ALCN is more expressive than ALC，只需要构造一个 ALCN concept C 使得 C 能够区分两个 bisimilar 的元素。首先构造两个 pointed interpretations：



(I1,d1)和(I2,e1)。再构造一个 ALCN concept C：<=1r.⊤。显然，<=1r.⊤能够区分 d1 和 e1，因为 e1 ∉ <=1r.⊤。则 C 一定不是 ALC concept，则存在一个 ALCN concept C 使得任意 ALC concept D such that C≠D。其它 DL 语言的表达力与 ALC 的对比也可以参考同样的思路。

法二：反证

同样的构造，不同的证明

仍然可以随意构造解释I

同构关系把对数量的限制转移，造成了与构造事实的矛盾

例题ALCN

Proposition 3.3. *$\mathcal{ALCN}$ is more expressive than $\mathcal{ALC}$; that is, there is an $\mathcal{ALCN}$ concept C such that $C \not\equiv D$ holds for all $\mathcal{ALC}$ concepts D.*

Proof. We show that no $\mathcal{ALC}$ concept is equivalent to the $\mathcal{ALCN}$ concept $(\leqslant 1\, r.\top)$. Assume to the contrary that D is an $\mathcal{ALC}$ concept with $(\leqslant 1\, r.\top) \equiv D$. In order to lead this assumption to a contradiction, we consider the interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ depicted in Figure 3.2. Since

$$\rho = \{(d_1, e_1), (d_2, e_2), (d_2, e_3)\}$$

is a bisimulation, we have $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, e_1)$, and thus $d_1 \in D^{\mathcal{I}_1}$ if and only if $e_1 \in D^{\mathcal{I}_2}$. This contradicts our assumption $(\leqslant 1\, r.\top) \equiv D$ since $d_1 \in (\leqslant 1\, r.\top)^{\mathcal{I}_1}$, but $e_1 \notin (\leqslant 1\, r.\top)^{\mathcal{I}_2}$. □

例题ALCI

Proposition 3.4. *$\mathcal{ALCI}$ is more expressive than $\mathcal{ALC}$; that is, there is an $\mathcal{ALCI}$ concept C such that $C \not\equiv D$ holds for all $\mathcal{ALC}$ concepts D.*

Proof. We show that no $\mathcal{ALC}$ concept is equivalent to the $\mathcal{ALCI}$ concept $\exists r^-.\top$. Assume to the contrary that D is an $\mathcal{ALC}$ concept with $\exists r^-.\top \equiv D$. In order to lead this assumption to a contradiction, we consider the interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ depicted in Figure 3.3.

Since $\rho = \{(d_2, e_2)\}$ is a bisimulation, we have $(\mathcal{I}_1, d_2) \sim (\mathcal{I}_2, e_2)$, and

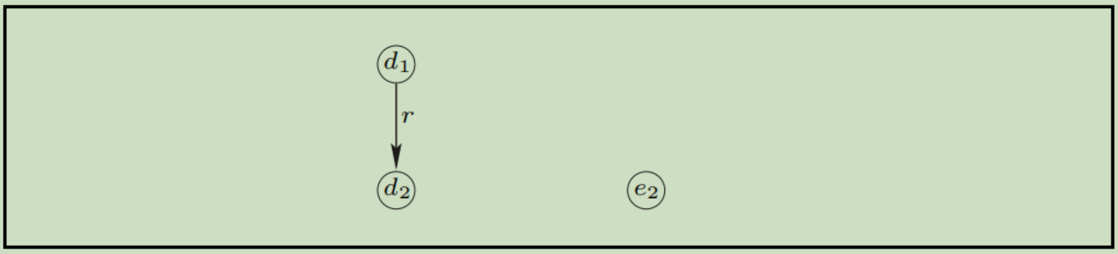


Fig. 3.3. Two more interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ represented as graphs.

thus $d_2 \in D^{\mathcal{I}_1}$ if and only if $e_2 \in D^{\mathcal{I}_2}$. This contradicts our assumption $\exists r^-.\top \equiv D$ since $d_2 \in (\exists r^-.\top)^{\mathcal{I}_1}$, but $e_2 \notin (\exists r^-.\top)^{\mathcal{I}_2}$. □

> 思路：构造同构的两个点，一个有父亲，一个没父亲

例题ALCO

Proposition 3.5. *$\mathcal{ALCO}$ is more expressive than $\mathcal{ALC}$; that is, there is an $\mathcal{ALCO}$ concept C such that $C \not\equiv D$ holds for all $\mathcal{ALC}$ concepts D.*

Proof. We show that no $\mathcal{ALC}$ concept is equivalent to the $\mathcal{ALCO}$ concept $\{a\}$. Using the same pattern as in the previous two proofs, it is enough to show that there are bisimilar elements that can be distinguished by this concept. For this, we consider the interpretation $\mathcal{I}_1$ with $\Delta^{\mathcal{I}_1} = \{d\}$, $a^{\mathcal{I}_1} = d$ and $A^{\mathcal{I}_1} = \emptyset = r^{\mathcal{I}_1}$ for all $A \in \mathbf{C}$ and $r \in \mathbf{R}$; and the interpretation $\mathcal{I}_2$ with $\Delta^{\mathcal{I}_2} = \{e_1, e_2\}$, $a^{\mathcal{I}_2} = e_1$ and $A^{\mathcal{I}_2} = \emptyset = r^{\mathcal{I}_2}$ for all $A \in \mathbf{C}$ and $r \in \mathbf{R}$.

Since $\rho = \{(d, e_2)\}$ is a bisimulation, we have $(\mathcal{I}_1, d) \sim (\mathcal{I}_2, e_2)$, but $d \in \{a\}^{\mathcal{I}_1}$ and $e_2 \notin \{a\}^{\mathcal{I}_2}$. □

无r关系的同构

T2：证明J是Tbox的model

- 反设Tbox中GCI针对I不成立
- 翻译GCI过程举具体例子
- 然后借助已有的I是model和I与J之间的关系进行证明

Theorem 3.8. *Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox and $(\mathcal{I}_\nu)_{\nu \in \mathfrak{N}}$ a family of models of $\mathcal{T}$. Then its disjoint union $\mathcal{J} = \biguplus_{\nu \in \mathfrak{N}} \mathcal{I}_\nu$ is also a model of $\mathcal{T}$.*

Proof. Assume that $\mathcal{J}$ is not a model of $\mathcal{T}$. Then there is a GCI $C \sqsubseteq D$ in $\mathcal{T}$ and an element $(d, \nu) \in \Delta^{\mathcal{J}}$ such that $(d, \nu) \in C^{\mathcal{J}}$, but $(d, \nu) \notin D^{\mathcal{J}}$. By Lemma 3.7, this implies $d \in C^{\mathcal{I}_\nu}$ and $d \notin D^{\mathcal{I}_\nu}$, which contradicts our assumption that $\mathcal{I}_\nu$ is a model of $\mathcal{T}$. □

Proof. Let $\mathcal{I}$ be a model of $\mathcal{T}$ and $d \in \Delta^{\mathcal{I}}$ be such that $d \in C^{\mathcal{I}}$. We show that the unravelling $\mathcal{J}$ of $\mathcal{I}$ at d is a tree model of C with respect to $\mathcal{T}$.

(i) To prove that $\mathcal{J}$ is a model of $\mathcal{T}$, consider a GCI $D \sqsubseteq E$ in $\mathcal{T}$, and assume that $p \in \Delta^{\mathcal{J}}$ satisfies $p \in D^{\mathcal{J}}$. We must show $p \in E^{\mathcal{J}}$. By Proposition 3.23, we have $\mathsf{end}(p) \in D^{\mathcal{I}}$, which yields $\mathsf{end}(p) \in E^{\mathcal{I}}$ since $\mathcal{I}$ is model of $\mathcal{T}$. But then Proposition 3.23 applied in the other direction yields $p \in E^{\mathcal{J}}$.

T3：Bisimulation证明方法

Bisimulation 在 DL 上的定义是：给定两个 DL interpretations I1 and I2，如果 I1 中的一个元素（比如 d1）和 I2 中的一个元素（比如 d2）满足如下关系，则说 I1 中的 d1 与 I2 中的 d2 存在 bisimilar 关系，写作(I1,d1) ~ (I2,d2)：

（1）如果 d1 是某个 concept name A 在 I1 解释下的集合里面的元素，则 d2 也是 A 在 I2 解释下的集合里面的元素，反之亦然；

（2）如果 I1 中存在一个 node f1 与 d1 存在 r 关系：$(d1,f1) \in r^{I1}$，则 I2 中必须也存在一个 node f2 与 d2 存在 r 关系：$(d2,f2) \in r^{I2}$，并且(I1,f1) ~ (I2,f2)；

（3）如果 I2 中存在一个 node f2 与 d2 存在 r 关系：$(d2,f2) \in r^{I2}$，则 I1 中必须也存在一个 node f1 与 d1 存在 r 关系：$(d1,f1) \in r^{I1}$，并且(I2,f2) ~ (I1,f1)。

Lemma 3.22. *The relation*

$$\rho = \{(p, \mathsf{end}(p)) \mid p \in \Delta^{\mathcal{J}}\}$$

is a bisimulation between $\mathcal{J}$ and $\mathcal{I}$.

Proof. By definition of the extensions of concept names in the interpretation $\mathcal{J}$, we have $p \in A^{\mathcal{J}}$ if and only if $\mathsf{end}(p) \in A^{\mathcal{I}}$, and thus Condition (i) of Definition 3.1 is satisfied.

To show that Condition (ii) of Definition 3.1 is also satisfied, we assume that $(p, p') \in r^{\mathcal{J}}$ and $(p, e) \in \rho$. Since $\mathsf{end}(p)$ is the only element of $\Delta^{\mathcal{I}}$ that is ρ-related to p, we have $e = \mathsf{end}(p)$. Thus, we must show that there is an $f \in \Delta^{\mathcal{I}}$ such that $(p', f) \in \rho$ and $(\mathsf{end}(p), f) \in r^{\mathcal{I}}$. We define $f = \mathsf{end}(p')$. Because $(p', \mathsf{end}(p')) \in \rho$, it is thus enough to show $(\mathsf{end}(p), \mathsf{end}(p')) \in r^{\mathcal{I}}$. This is, however, an immediate consequence of the definition of the extensions of roles in $\mathcal{J}$.

To show that Condition (iii) of Definition 3.1 is satisfied, we assume that $(e, f) \in r^{\mathcal{I}}$ and $(p, e) \in \rho$ (i.e., $\mathsf{end}(p) = e$). We must find a path p' such that $(p', f) \in \rho$ and $(p, p') \in r^{\mathcal{J}}$. We define $p' = p, f$. This is indeed a d-path since p is a d-path with $\mathsf{end}(p) = e$ and $(e, f) \in r^{\mathcal{I}}$. In addition, $\mathsf{end}(p') = f$, which shows $(p', f) \in \rho$. Finally, we clearly have $p' = p, \mathsf{end}(p')$ and $(\mathsf{end}(p), \mathsf{end}(p')) \in r^{\mathcal{I}}$ since $\mathsf{end}(p) = e$ and $\mathsf{end}(p') = f$. This yields $(p, p') \in r^{\mathcal{J}}$. □

Definition 3.21 (Unravelling)**.** Let $\mathcal{I}$ be an interpretation and $d \in \Delta^{\mathcal{I}}$. The *unravelling of* $\mathcal{I}$ *at* d is the following interpretation $\mathcal{J}$:

$$
\begin{aligned}
\Delta^{\mathcal{J}} &= \{p \mid p \text{ is a } d\text{-path in } \mathcal{I}\},\\
A^{\mathcal{J}} &= \{p \in \Delta^{\mathcal{J}} \mid \mathsf{end}(p) \in A^{\mathcal{I}}\} \text{ for all } A \in \mathbf{C},\\
r^{\mathcal{J}} &= \{(p, p') \in \Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}} \mid p' = (p, \mathsf{end}(p')) \text{ and } (\mathsf{end}(p), \mathsf{end}(p')) \in r^{\mathcal{I}}\}\\
&\qquad \text{for all } r \in \mathbf{R}.
\end{aligned}
$$

In our example, $d_1 = d, e, d \in A^{\mathcal{J}}$ because $\mathsf{end}(d_1) = d \in A^{\mathcal{I}}$, and $((d, e, d), (d, e, d, e)) \in r^{\mathcal{J}}$ because $(d, e) \in r^{\mathcal{I}}$.

基本不用什么证明，把每一部分要证什么写一遍就完了，别问为啥，问就是由r或者J的定义（狗头）

disjoint union可用于将concept C在某一解释下包含的元素个数翻倍