1、Some interesting properties of $\mathcal{EL}$

- Show that every $\mathcal{EL}$-concept is satisfiable (regardless of the presence of an $\mathcal{EL}$-TBox). That is, for every $\mathcal{EL}$-concept C there exists an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$.
- Show that every $\mathcal{EL}$-TBox is consistent. That is, for every $\mathcal{EL}$-TBox $\mathcal{T}$ there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{T}$.

(1)satisfiable

We give the Interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$:

$\Delta^{\mathcal{I}} = \{a\}$

$A^{\mathcal{I}} = \{a\}$ for all concept name A

$r^{\mathcal{I}} = \{(a,a)\}$ for all role name r

We can prove it by induction on the structure of $\mathcal{EL}$-concept C:

- if $C = \top$, then $C^{\mathcal{I}} = \Delta^{\mathcal{I}} = \{a\}$.
- if $C = A \in \mathbf{C}$, then $C^{\mathcal{I}} = A^{\mathcal{I}} = \{a\}$.
- if $C = D \sqcap E$, then $C^{\mathcal{I}} = D^{\mathcal{I}} \cap E^{\mathcal{I}} = \{a\} \cap \{a\} = \{a\}$.
- if $C = \exists r.\, F$, then $C^{\mathcal{I}} = \{a\}$.

So there exists an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$.

(2)consistent

We continue to use the interpretation $\mathcal{I}$ listed in (1).

Since every $\mathcal{EL} - TBox$ can be transformed into these 4 kinds of normal form:

$$
\begin{gathered}
(sform) A \sqsubseteq B \\
(cform) A_1 \sqcap A_2 \sqsubseteq B \\
(rform) A \sqsubseteq \exists r.\, B \\
(lform) \exists r.\, A \sqsubseteq B
\end{gathered}
$$

From (1) we know that every $\mathcal{EL} - concept$ is satisfiable, then $A, B, A_1, A_2, \exists r.\, A, \exists r.\, B$ are all satisfiable according to $\mathcal{I}$

It's obvious that $\mathcal{I}$ fit all 4 normal forms listed above.(We can prove it by induction on the structure)

So there exists an intterpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{T}$.

2、Reasoning in $\mathcal{EL}$

Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox containing the following (primitive) concept definitions:

$$\mathsf{Bird} \equiv \mathsf{Vertebrate} \sqcap \exists \mathsf{has\_part}.\mathsf{Wing}$$

$$\mathsf{Reptile} \sqsubseteq \mathsf{Vertebrate} \sqcap \exists \mathsf{lays}.\mathsf{Egg}$$

- Compute an $\mathcal{EL}$-TBox $\mathcal{T}'$ in normal form using the pre-processing algorithm given in the lecture.
- Apply the algorithm from the lecture slides deciding whether $A \sqsubseteq_{\mathcal{T}'} B$, where A, B are concept names. Using the normalized TBox $\mathcal{T}'$ as input and explain step-by-step which rules are applied.
- Using the output of the algorithm, decide whether

$$\mathsf{Reptile} \sqsubseteq_{\mathcal{T}'} \mathsf{Vertebrate}$$

$$\mathsf{Vertebrate} \sqsubseteq_{\mathcal{T}'} \mathsf{Bird}$$

(1)pre-processing algorithm

to transform into 4 formal forms

step 1 gives: （注意运算符优先级）

$$\begin{array}{ccc}
\mathrm{Bird} & \sqsubseteq & \mathrm{Vertebrate} \sqcap \exists \mathrm{has_part}.\,\mathrm{Wing} \\
\mathrm{Vertebrate} \sqcap \exists \mathrm{has_part}.\,\mathrm{Wing} & \sqsubseteq & \mathrm{Bird} \\
\mathrm{Reptile} & \sqsubseteq & \mathrm{Vertebrate} \sqcap \exists \mathrm{lays}.\,\mathrm{Egg}
\end{array}$$

step 2 gives:

$$\begin{array}{ccc}
Bird & \sqsubseteq & Vertebrate \\
Bird & \sqsubseteq & \exists has_part.\,Wing \\
Vertebrate \sqcap \exists has_part.\,Wing & \sqsubseteq & Bird \\
Reptile & \sqsubseteq & Vertebrate \\
Reptile & \sqsubseteq & \exists lays.\,Egg
\end{array}$$

step 4 gives:

$$\begin{array}{ccc}
Bird & \sqsubseteq & Vertebrate \\
Bird & \sqsubseteq & \exists has_part.\,Wing \\
X & \sqsubseteq & \exists has_part.\,Wing \\
\exists has_part.\,Wing & \sqsubseteq & X \\
Vertebrate \sqcap X & \sqsubseteq & Bird \\
Reptile & \sqsubseteq & Vertebrate \\
Reptile & \sqsubseteq & \exists lays.\,Egg
\end{array}$$

(2)Deciding intuition algorithm

use 4 rules and 7 axioms above to compute functions **S** and **R**

Initalisition:

$$
\begin{aligned}
S(\text{Bird}) &= \{\text{Bird}\} \\
S(\text{Vertebrate}) &= \{\text{Vertebrate}\} \\
S(\text{Wing}) &= \{\text{Wing}\} \\
S(X) &= \{X\} \\
S(\text{Reptile}) &= \{\text{Reptile}\} \\
S(\text{Egg}) &= \{\text{Egg}\} \\
R(\text{has\_part}) &= \emptyset \\
R(\text{lays}) &= \emptyset
\end{aligned}
$$

rule simpleR:(axiom 1,6)

$$
\begin{aligned}
S(\text{Bird}) &= \{\text{Bird}, \text{Vertebrate}\} \\
S(\text{Reptile}) &= \{\text{Reptile}, \text{Vertebrate}\}
\end{aligned}
$$

rule rightR:(axiom 2,3,7)

$$
\begin{aligned}
R(\text{has\_part}) &= \{(\text{Bird}, \text{Wing}), (X, \text{Wing})\} \\
R(\text{lays}) &= \{(\text{Reptile}, \text{Egg})\}
\end{aligned}
$$

rule leftR:(axiom 4)(要理解)

$$
S(\text{Bird}) = \{\text{Bird}, \text{Vertebrate}, \text{X}\}
$$

final result:

$$
\begin{aligned}
S(\text{Bird}) &= \{\text{Bird}, \text{Vertebrate}, \text{X}\} \\
S(\text{Vertebrate}) &= \{\text{Vertebrate}\} \\
S(\text{Wing}) &= \{\text{Wing}\} \\
S(X) &= \{X\} \\
S(\text{Reptile}) &= \{\text{Reptile}, \text{Vertebrate}\} \\
S(\text{Egg}) &= \{\text{Egg}\} \\
R(\text{has\_part}) &= \{(\text{Bird}, \text{Wing}), (X, \text{Wing})\} \\
R(\text{lays}) &= \{(\text{Reptile}, \text{Egg})\}
\end{aligned}
$$

(3)application

$\text{Reptile} \sqsubseteq_{\mathcal{T}'} \text{Vertebrate}$ is T
$\text{Vertebrate} \sqsubseteq_{\mathcal{T}'} \text{Bird}$ is F

3、 Bisimulation invariance

In the lecture we defined bisimulation for $\mathcal{ALC}$ and showed bisimulation invariance of $\mathcal{ALC}$ (Theorem 3.2).

- Define a notion of "$\mathcal{ALCN}$-bisimulation" that is appropriate for $\mathcal{ALCN}$ in the sense that bisimilar elements satisfy the same $\mathcal{ALCN}$-concepts.
- Use the definition to show that $\mathcal{ALCQ}$ is more expressive than $\mathcal{ALCN}$.

(1)Extend the notion of bisimulation to ALCN:

Let $\mathcal{I}$ and $\mathcal{J}$ be interpretations. The relation $\rho \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$ if

(i) $d \ \rho \ e$ implies

$$
d \in A^{\mathcal{I}} \ if \ and \ only \ if \ e \in A^{\mathcal{J}}
$$

for all $d \in \Delta^{\mathcal{I}}$, $e \in \Delta^{\mathcal{J}}$, and $A \in \mathbf{C}$.

(ii) if $d_1, \cdots, d_n$ are all distinct elements of $\Delta^{\mathcal{I}}$ such that $d\ \rho\ e$ and$(d, d_i) \in r^{\mathcal{I}}$ for $1 \leq i \leq n$ implies the existence of exactly n distinct elements $e_1, \cdots, e_n$ of $\Delta^{\mathcal{J}}$ such that

$$d_i\ \rho\ e_i\ and\ (e, e_i) \in r^{\mathcal{J}} for\ 1 \leq i \leq n$$

for all $d, d_i \in \Delta^{\mathcal{I}}$, $e, \in \Delta^{\mathcal{J}}$, and $r \in \mathbf{R}$.

(iii) if $d_1, \cdots, d_n$ are all distinct elements of $\Delta^{\mathcal{I}}$ such that $d\ \rho\ e$ and$(e, e_i) \in r^{\mathcal{J}}$ for $1 \leq i \leq n$ implies the existence of exactly n distinct elements $d_1, \cdots, d_n$ of $\Delta^{\mathcal{I}}$ such that

$$d_i\ \rho\ e_i\ and\ (d, d_i) \in r^{\mathcal{I}} for\ 1 \leq i \leq n$$

for all $d_1 \in \Delta^{\mathcal{I}}$, $e, e_i, \in \Delta^{\mathcal{J}}$, and $r \in \mathbf{R}$.

Prove

Then we prove that $\mathcal{ALCN}$ is bisimulation invariant for the bisimulation relation on the basis of $\mathcal{ALC}$

For $(\leq nR.\top)$,

$$\begin{aligned} d \in (\leq nR.\top)^{\mathcal{I}} \quad &\Leftrightarrow \quad \exists\, m(\ m \leq n)\ elements\ d_1, \cdots, d_m, (d, d_i) \in R^{\mathcal{I}} \\ &\Leftrightarrow \quad \exists\, m\ elements\ e_1, \cdots, e_m, (e, e_i) \in R^{\mathcal{J}} \\ &\Leftrightarrow \quad e \in (\leq nR.\top)^{\mathcal{J}} \end{aligned}$$

(2)ALCQ is more expressive than ALCN

prove by definition

$\mathcal{ALCN}$ is unqualified number restriction and does not admit qualifications using an arbitrary concept $\mathcal{C}$

Besides, there is no$(\geq nR.\top)$ in $\mathcal{ALCN}$.

Therefore, $\mathcal{ALCQ}$ is more expressive than $\mathcal{ALCN}$

4、Closure under Disjoint Union

Recall Theorem 3.8 from the lecture, which says that the disjoint union of a family of models of an $\mathcal{ALC}$-TBox $\mathcal{T}$ is again a model of $\mathcal{T}$. Note that the disjoint union is only defined for concept and role names.

- Extend the notion of disjoint union to individual names such that the following holds: for any family $(\mathcal{I}_\nu)_{\nu\in\Omega}$ of models of an $\mathcal{ALC}$-knowledge base $\mathcal{K}$, the disjoint union $\biguplus_{\nu\in\Omega}\mathcal{I}_\nu$ is also a model of $\mathcal{K}$.

Theorem 3.8. *Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox and $(\mathcal{I}_\nu)_{\nu\in\mathfrak{N}}$ a family of models of $\mathcal{T}$. Then its disjoint union $\mathcal{J} = \biguplus_{\nu\in\mathfrak{N}}\mathcal{I}_\nu$ is also a model of $\mathcal{T}$.*

Extend the notion of disjoint union to individual names:

Their *disjoint union* $\mathcal{J}$ isdefined as follows:

- $\Delta^{\mathcal{J}} = \{(d,v)|v \in \Omega \text{ and } d \in \Delta^{\mathcal{I}_v}\}$
- $A^{\mathcal{J}} = \{(d,v)|v \in \Omega \text{ and } d \in A^{\mathcal{I}_v}\}$ for all $A \in \mathbf{C}$
- $r^{\mathcal{J}} = \{((d,v),(e,v))|v \in \Omega \text{ and } (d,e) \in r^{\mathcal{I}_v}\}$ for all $r \in \mathbf{R}$
- $a^{\mathcal{J}} = \{(a^{\mathcal{I}_{v_0}}, v_0)|\forall a \in \mathcal{A} \; and \; v_0 \in \Omega\}$

Prove:

Then we prove that its disjoint union $\mathcal{J} = \biguplus_{v\in\Omega} \mathcal{I}_v$ is also a model of $\mathcal{K}$.

From `Theorem 3.8` we know that $\mathcal{J}$ is a model of $\mathcal{T}$.

Assume that $\mathcal{J}$ is not a model of $\mathcal{K}$.

Assume that there is assertion $a : C$ in $\mathcal{K}$ and $(a^{\mathcal{I}_{v_0}}, v_0) \notin C^{\mathcal{J}}$. This implies $a^{\mathcal{I}_{v_0}} \notin C^{\mathcal{I}_{v_0}}$, which contradicts to the assumption of $\mathcal{I}_{v_0}$ is a model of $\mathcal{K}$.

Assume that there is assertion $(a,b) : r$ in $\mathcal{K}$ and $((a^{\mathcal{I}_{v_0}}, v_0),(b^{\mathcal{I}_{v_0}}, v_0)) \notin r^{\mathcal{J}}$. This implies $(a^{\mathcal{I}_{v_0}}, b^{\mathcal{I}_{v_0}}) \notin r^{\mathcal{I}_{v_0}}$, which contradicts to the assumption of $\mathcal{I}_{v_0}$ is a model of $\mathcal{K}$.

5、 Closure under Disjoint Union

Let $\mathcal{K} = \{\mathcal{T}, \mathcal{A}\}$ be a consistent $\mathcal{ALC}$-KB. We write $C \sqsubseteq_{\mathcal{K}} D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for every model $\mathcal{I}$ of $\mathcal{K}$.

- Prove that for all $\mathcal{ALC}$-concepts C and D we have $C \sqsubseteq_{\mathcal{K}} D$ iff $C \sqsubseteq_{\mathcal{T}} D$. Hint: Use the modified definition of disjoint union from the previous exercise.

Lemma 3.7. *Let $\mathcal{J} = \biguplus_{\nu\in\mathfrak{N}} \mathcal{I}_\nu$ be the disjoint union of the family $(\mathcal{I}_\nu)_{\nu\in\mathfrak{N}}$ of interpretations. Then we have*

$$d \in C^{\mathcal{I}_\nu} \quad \textit{if and only if} \quad (d,\nu) \in C^{\mathcal{J}}$$

for all $\nu \in \mathfrak{N}$, $d \in \Delta^{\mathcal{I}_\nu}$ and $\mathcal{ALC}$ concept descriptions C.

$\Leftarrow$:

If $C \sqsubseteq_{\mathcal{T}} D$, then $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for every model $\mathcal{I}$ of $\mathcal{T}$.

Because each model J of $\mathcal{K}$ must be a model of $\mathcal{T}$, so $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$ holds for every model $\mathcal{J}$ of $\mathcal{K}$.

So $C \sqsubseteq_{\mathcal{K}} D$.

$\Rightarrow$:

If $C \sqsubseteq_{\mathcal{K}} D$, then $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for every model $\mathcal{I}$ of $\mathcal{K}$. Assumed $C \not\sqsubseteq_{\mathcal{T}} D$.

So there is a model $\mathcal{I}_1$ of $\mathcal{K}$ satisfying $C^{\mathcal{I}_1} \subseteq D^{\mathcal{I}_1}$, another model $\mathcal{I}_2$ of $\mathcal{T}$ satisfying $C^{\mathcal{I}_2} \not\subseteq D^{\mathcal{I}_2}$.

Let $\mathcal{J}$ be the disjoint union of $\mathcal{I}_1$ and $\mathcal{I}_2$.

Assumed $C^{\mathcal{J}} \not\subseteq D^{\mathcal{J}}$, then there is an element $(d,v) \in C^{\mathcal{J}}$ but $(d,v) \notin D^{\mathcal{J}}$. By `Lemma 3.7`, this implies $d \in C^{\mathcal{I}_1}$ but $d \notin D^{\mathcal{I}_1}$, which contradicts to $C^{\mathcal{I}_1} \subseteq D^{\mathcal{I}_1}$. So $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$.

By `Lemma 3.7`, $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$ implies $\forall$ element d satisfying $(d, v) \in C^{\mathcal{J}}$ also satisfies $(d, v) \in D^{\mathcal{J}}$, which implies $\forall$ element e satisfying $(e, v) \in C^{\mathcal{I}_2}$ also satisfy $(e, v) \in D^{\mathcal{I}_2}$ And thie contradicts to $C^{\mathcal{I}_2} \not\subseteq D^{\mathcal{I}_2}$.

So $C \sqsubseteq_{\mathcal{T}} D$.

6、Finite model property

Let C be an $\mathcal{ALC}$-concept that is satisfiable w.r.t. an $\mathcal{ALC}$-TBox $\mathcal{T}$. Show truth or falsity of the following statement:

- for all $m \geq 1$ there is a finite model $\mathcal{I}_m$ of $\mathcal{T}$ such that $|C^{\mathcal{I}_m}| \geq m$.
- Does it hold if the condition "$|C^{\mathcal{I}_m}| \geq m$" is replaced by "$|C^{\mathcal{I}_m}| = m$"?

(1)True

By the finite model property, there is a finite model $\mathcal{I}$ such that $|C^{\mathcal{I}}| \geq 1$.

Let $\mathcal{I}_m = \biguplus_{v \in 1, \cdots, m} \mathcal{I}$ be the disjoint union of $\mathcal{I}$ itself of m times. So $|C^{I_m}| = m|C^{\mathcal{I}}| \geq m$.

So $\forall m \geq 1$ there is a finite model $\mathcal{I}_m$ of $\mathcal{T}$ such that $|C^{\mathcal{I}_m}| \geq m$.

(2)False

Counter example:

Let $m = 1$, $C = \top$, $\mathcal{T} = \{A \sqsubseteq \exists r.\neg A, \neg A \sqsubseteq \exists r.A\}$

For any model $\mathcal{I}$ of $\mathcal{T}$, assume that $A^{\mathcal{I}} \neq \emptyset$ or $(\neg A)^{\mathcal{I}} \neq \emptyset$.

- If $A^{\mathcal{I}} = \emptyset$, then $(\exists r.A)^{\mathcal{I}} = \emptyset$. From the GCI $\neg A \sqsubseteq \exists r.A$, we know $(\neg A)^{\mathcal{I}} \subseteq (\exists r.A)^{\mathcal{I}}$ then $(\neg A)^{\mathcal{I}} = \emptyset$, which contradicts to $A^{\mathcal{I}} \neq \emptyset$ or $(\neg A)^{\mathcal{I}} \neq \emptyset$.
- If $(\neg A)^{\mathcal{I}} = \emptyset$, then $(\exists r.\neg A)^{\mathcal{I}} = \emptyset$. From the GCI $A \sqsubseteq \exists r.\neg A$, we know $A^{\mathcal{I}} \subseteq (\exists r.\neg A)^{\mathcal{I}}$ then $A^{\mathcal{I}} = \emptyset$, which contradicts to $A^{\mathcal{I}} \neq \emptyset$ or $(\neg A)^{\mathcal{I}} \neq \emptyset$.

Then $|C^{\mathcal{I}}| = |\top^{\mathcal{I}}| = |A^{\mathcal{I}}| + |(\neg A)^{\mathcal{I}}| \geq 1 + 1 = 2$, which contradicts $|C^{\mathcal{I}}| = m = 1$.

So it doesn't hold if the condition "$|C^{\mathcal{I}_m}| \geq m$" is replaced by "$|C^{\mathcal{I}_m}| = m$".

7、 Bisimulation over filtration

Let C be an $\mathcal{ALC}$-concept, $\mathcal{T}$ an $\mathcal{ALC}$-TBox, $\mathcal{I}$ an interpretation and $\mathcal{J}$ its filtration w.r.t. sub$(C) \cup$ sub$(\mathcal{T})$ (see Definition 3.14 for the definition of filtration). Show truth or falsity of the following statement:

- the relation $\rho = \{(\mathrm{d}, [\mathrm{d}]) \mid \mathrm{d} \in \Delta^{\mathcal{I}}\}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.

Definition 3.14 (S-filtration)**.** Let S be a finite set of $\mathcal{ALC}$ concepts and $\mathcal{I}$ an interpretation. We define the equivalence relation $\simeq_S$ on $\Delta^{\mathcal{I}}$ as follows:

$$d \simeq_S e \;\text{ if }\; t_S(d) = t_S(e).$$

The $\simeq_S$*-equivalence class* of $d \in \Delta^{\mathcal{I}}$ is denoted by $[d]_S$, i.e.,

$$[d]_S = \{e \in \Delta^{\mathcal{I}} \mid d \simeq_S e\}.$$

The S*-filtration of* $\mathcal{I}$ is the following interpretation $\mathcal{J}$:

$$\begin{aligned}
\Delta^{\mathcal{J}} &= \{[d]_S \mid d \in \Delta^{\mathcal{I}}\};\\
A^{\mathcal{J}} &= \{[d]_S \mid \text{there is } d' \in [d]_S \text{ with } d' \in A^{\mathcal{I}}\} \text{ for all } A \in \mathbf{C};\\
r^{\mathcal{J}} &= \{([d]_S, [e]_S) \mid \text{there are } d' \in [d]_S, e' \in [e]_S \text{ with } (d', e') \in r^{\mathcal{I}}\}\\
&\quad\; \text{for all } r \in \mathbf{R}.
\end{aligned}$$

False

Counter example:

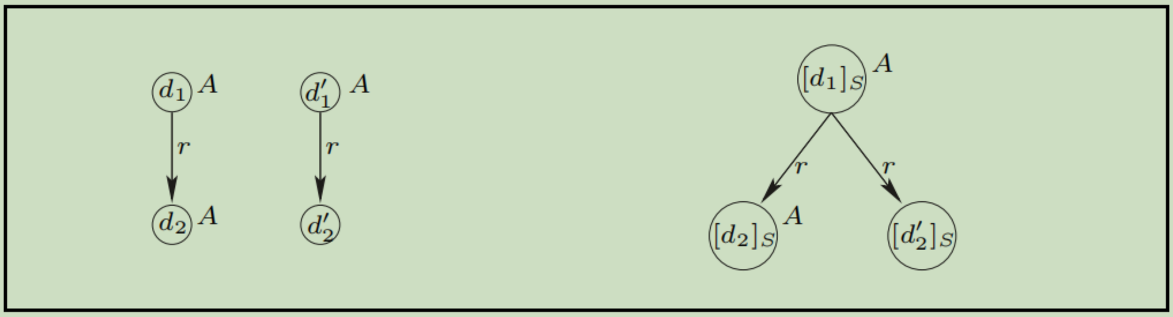


Fig. 3.4. An interpretation $\mathcal{I}$ and its S-filtration $\mathcal{J}$ for $S = \{\top, A, \exists r.\top\}$.

Assume that $S = \{\top, A, \exists r.\top\}$ where $\mathbf{C} = \{A\}$ and $\mathbf{R} = \{r\}$, $\Delta^{\mathcal{I}} = \{d_1, d_2, d'_1, d'_2\}$

Then $\simeq_S$ has 3 equivalence classes: $[d_1]_S = [d_2]_S$, $[d'_1]_S$ and $[d'_2]_S$

We have $(d_1, [d_1]_S) \in \rho$, but $[d_1]_S$ has an r -successor in $\mathcal{J}$ that does not belong to the extension of A , whereas d_1 does not have such an r -successor in $\mathcal{I}$.

8、Bisimulation over ltration

We define "*bisimulations on* $\mathcal{I}$" as bisimulations between an interpretation $\mathcal{I}$ and itself. Let d, e $\in \Delta^{\mathcal{I}}$ be two elements. We write d $\approx_{\mathcal{I}}$ e if they are bisimilar, i.e., if there is a bisimulation ρ on $\mathcal{I}$ such that d ρ e.

- Show that $\approx_{\mathcal{I}}$ is a bisimulation on $\mathcal{I}$.

Consider the interpretation $\mathcal{J}$ defined like the filtration, but with $\approx_{\mathcal{I}}$ instead of $\simeq$.

- Show that $\rho = \{(\mathrm{d}, [\mathrm{d}]_{\approx_{\mathcal{I}}})\} \mid \mathrm{d} \in \Delta^{\mathcal{I}}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.
- Show that, if $\mathcal{I}$ is a model of an $\mathcal{ALC}$-concept C w.r.t. an $\mathcal{ALC}$-TBox $\mathcal{T}$, then so is $\mathcal{J}$.

(1)

condition 1

$d \approx_{\mathcal{I}} e$ implies $d \in A^{\mathcal{I}}$ if and only if $e \in A^{\mathcal{I}}$

(for all $d \in \Delta^{\mathcal{I}}$, $e \in \Delta^{\mathcal{I}}$, and $A \in \mathbf{C}$)

condition 2

$d \approx_{\mathcal{I}} e$ and $(d, d') \in r^{\mathcal{I}}$ implies the existence of $e' \in \Delta^{\mathcal{I}}$ sattisfying $d' \rho\, e'$ and $(e, e') \in r^{\mathcal{I}}$

Then $d' \approx_{\mathcal{I}} e'$ and $(e, e') \in r^{\mathcal{I}}$ because of the definition of $d' \approx_{\mathcal{I}} e'$

(for all $d, d' \in \Delta^{\mathcal{I}}$, $e \in \Delta^{\mathcal{I}}$, and $r \in \mathbf{R}$)

condition 3

similar to condition 2

According to 3 conditions of the definition of BIsimulation, $\approx_{\mathcal{I}}$ is a bisimulation on $\mathcal{I}$.

(2)

condition 1

$(d, [d]_{\approx_{\mathcal{I}}}) \in \rho$ implies $d \in A^{\mathcal{I}}$ if and only if $[d]_{\approx_{\mathcal{I}}} \in A^{\mathcal{J}}$

for all $d \in \Delta^{\mathcal{I}}$, $[d] * \approx *\mathcal{I} \in \Delta^{\mathcal{J}}$, and $A \in \mathbf{C}$.

$\Rightarrow$:

If $d \in A^{\mathcal{I}}$, then we have $d \in [d]_{\approx_{\mathcal{I}}}$ and $d_{\approx \mathcal{I}} d$. We can know $[d]_{\approx_{\mathcal{I}}} \in A^{\mathcal{J}}$ by the definition of $A^{\mathcal{J}}$.

$\Leftarrow$:

If $[d]_{\approx_{\mathcal{I}}} \in A^{\mathcal{J}}$, then we have $d' \in [d]_{\approx_{\mathcal{I}}}$ and $d' \in A^{\mathcal{I}}$. From $d \in [d]_{\approx \mathcal{I}}$ and $d_{\approx \mathcal{I}} d$, we can know that $d_{\approx \mathcal{I}} d'$. However, $d' \in A^{\mathcal{I}}$ if and only if $d \in A^{\mathcal{I}}$.

condition 2

$(d, [d]_{\approx_{\mathcal{I}}}) \in \rho$ and $(d, e) \in r^{\mathcal{I}}$ implies there is $d \in [d]_{\approx_{\mathcal{I}}}$, $e \in [e]_{\approx_{\mathcal{I}}}$ with $(d, e) \in r^{\mathcal{I}}$, which implies the existence of $[e]_{\approx_{\mathcal{I}}} \in \Delta^{\mathcal{J}}$ satisfying$(e, [e]_{\approx_{\mathcal{I}}}) \in \rho$ and $([d]_{\approx - \mathcal{I}}, [e]_{\approx_{\mathcal{I}}}) \in r^{\mathcal{J}}$

for all $d, e \in \Delta^{\mathcal{I}}$, $[d]_{\approx_{\mathcal{I}}} \in \Delta^{\mathcal{J}}$, and $r \in \mathbf{R}$.

condition 3

similar to condition 2

According to 3 conditions of the definition of BIsimulation, we show that $\rho = (d, [d] * \approx *\mathcal{I}) | d \in \Delta^{\mathcal{I}}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.

(3)

$\mathcal{I}$ is a model of an $\mathcal{ALC}$-concept C with respect to an $\mathcal{ALC}$-TBox $\mathcal{T}$.

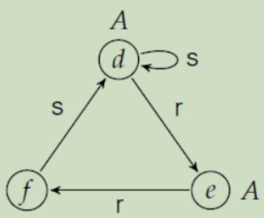
$C^{\mathcal{I}} \neq \emptyset$ implies the existence of $d \in C^{\mathcal{I}}$ and $[d]_{\approx_{\mathcal{I}}} \in C^{\mathcal{J}}$

Let $D \sqsubseteq E$ be a GCI in $\mathcal{T}$, and $[e]_{\approx_\mathcal{I}} \in D^\mathcal{J}$. $e \in D^\mathcal{I}$ implies $e \in E^\mathcal{J}$ since $\mathcal{I}$ is a model of $\mathcal{T}$, which implies $[e]_{\approx_\mathcal{I}} \in E^\mathcal{J}$.

So $\mathcal{J}$ is a model of an $\mathcal{ALC}$-concept C with respect to an $\mathcal{ALC}$-TBox $\mathcal{T}$.

9、Unravelling

Draw the unravelling of the following interpretation $\mathcal{I}$ at d up to depth 5, i.e., restricted to d-paths of length at most 5 (see Definition 3.21):

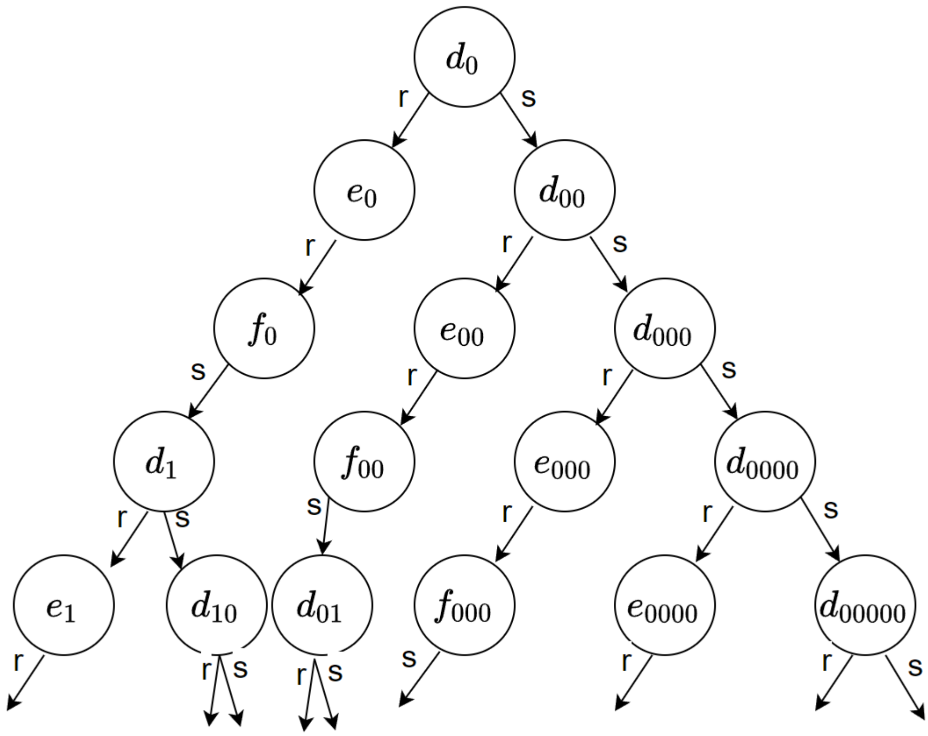


Definition 3.21 (Unravelling). Let $\mathcal{I}$ be an interpretation and $d \in \Delta^\mathcal{I}$. The *unravelling of* $\mathcal{I}$ *at* d is the following interpretation $\mathcal{J}$:

$$\begin{aligned}
\Delta^\mathcal{J} &= \{p \mid \ p \text{ is a } d\text{-path in } \mathcal{I}\},\\
A^\mathcal{J} &= \{p \in \Delta^\mathcal{J} \mid \mathsf{end}(p) \in A^\mathcal{I}\} \text{ for all } A \in \mathbf{C},\\
r^\mathcal{J} &= \{(p, p') \in \Delta^\mathcal{J} \times \Delta^\mathcal{J} \mid p' = (p, \mathsf{end}(p')) \text{ and } (\mathsf{end}(p), \mathsf{end}(p')) \in r^\mathcal{I}\}\\
&\qquad \text{for all } r \in \mathbf{R}.
\end{aligned}$$

In our example, $d_1 = d, e, d \in A^\mathcal{J}$ because $\mathsf{end}(d_1) = d \in A^\mathcal{I}$, and $((d, e, d),\ (d, e, d, e)) \in r^\mathcal{J}$ because $(d, e) \in r^\mathcal{I}$.

Next, we will see that the relation that connects a d-path with its end node is a bisimulation.



10、Tree model property

Show the truth or falsity of the following statement: if $\mathcal{K}$ is an $\mathcal{ALC}$-KB and C an $\mathcal{ALC}$-concept such that C is satisfiable w.r.t. $\mathcal{K}$, then C has a tree model w.r.t. $\mathcal{K}$.

False

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A}), \mathcal{T} = \emptyset, \mathcal{A} = \{a : A, b : B, (a, b) : r, (b, a) : r\}$.

Then $\forall \mathcal{I}$ of such $\mathcal{K}$, $a^{\mathcal{I}}$ and $b^{\mathcal{I}}$ are two distinct elements satisfying $(a^{\mathcal{I}}, b^{\mathcal{I}}), (b^{\mathcal{I}}, a^{\mathcal{I}}) \in r^{\mathcal{I}}$.

So there is a ring "$a \xrightarrow{r} b \xrightarrow{r} a$" for any model $\mathcal{I}$ of such $\mathcal{K}$.

11、Tableau

Apply the Tableau algorithm consistent$(\mathcal{A})$ to the following ABox:

2

$\mathcal{A} = \{(b, a) : r, (a, b) : r, (a, c) : s, (c, b) : s, a : \exists s.A, b : \forall r.((\forall s.\neg A) \sqcup (\exists r.B)), c : \forall s.(B \sqcap (\forall s.\bot))\}$.

If $\mathcal{A}$ is consistent, draw the model generated by the algorithm.

- Initalisition: $\mathcal{A}$ is in NNF, so $\mathcal{A}_0$=$\mathcal{A}$
- An application of $\rightarrow_\exists$ and $a : \exists s. A$ gives

$$\mathcal{A}_1 = \mathcal{A}_0 \cup \{(a, d) : s, d : A\}$$

- An application of $\rightarrow_\forall$ and $b : \forall r. ((\forall s. \neg A) \sqcup (\exists r. B))$ and $(b, a) : r$ gives:

$$\mathcal{A}_2 = \mathcal{A}_1 \cup \{a : (\forall s. \neg A) \sqcup (\exists r. B)\}$$

- An application of $\rightarrow_\sqcup$ and $a : (\forall s. \neg A) \sqcup (\exists r. B)$ gives:
  - Firstly, we can try

$$\mathcal{A}_3 = \mathcal{A}_2 \cup \{a : \forall s. \neg A\}$$

  - An application of $\rightarrow_\forall$ and $a : \forall s. \neg A$ and $(a, c) : s, (a, d) : s$ gives

$$\mathcal{A}_4 = \mathcal{A}_3 \cup \{c : \neg A, d : \neg A\}$$

  - We have abtained a clash because $d : A$ and $d : \neg A$, thus this choice was unsuccessful.
  - Secondly, we can try

$$\mathcal{A}_3^* = \mathcal{A}_2 \cup \{a : \exists r. B\}$$

  - An application of $\rightarrow_\exists$ and $a : \exists r. B$ gives (这里不能新建fresh individual，因为已经有（a，b）：r)

$$\mathcal{A}_4 = \mathcal{A}_3^* \cup \{(a, b) : r, b : B\}$$

- An application of $\rightarrow_\forall$ and $c : \forall s.\, (B \sqcap (\forall s.\, \bot))$ and $(c, b) : s$ gives:

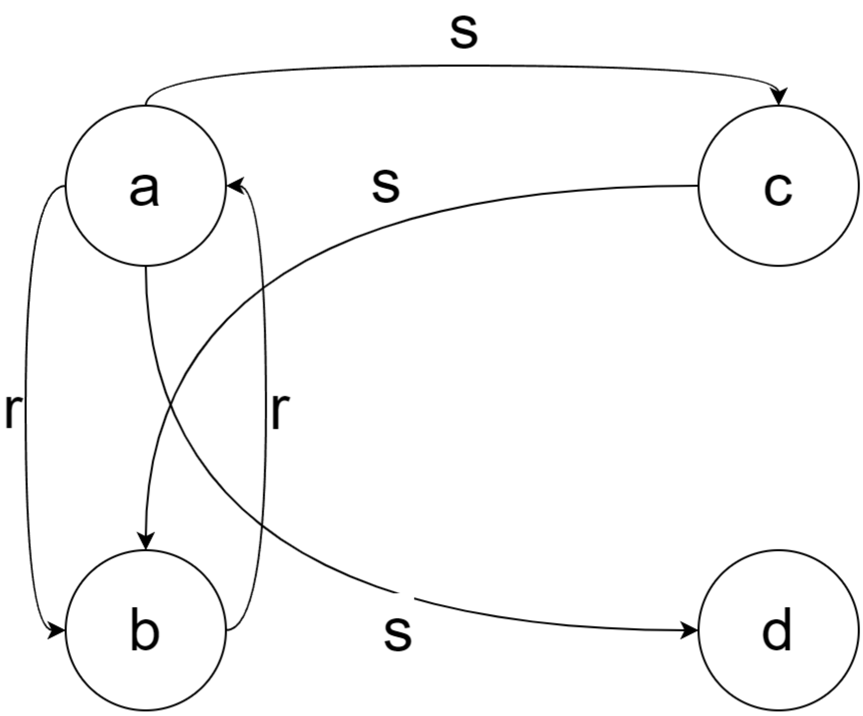
$$\mathcal{A}_5 = \mathcal{A}_4 \cup \{b : B \sqcap (\forall s.\, \bot)\}$$

- An application of $\rightarrow_\sqcap$ and $b : B \sqcap (\forall s.\, \bot)$ gives:

$$\mathcal{A}_6 = \mathcal{A}_5 \cup \{b : B, b : \forall s.\, \bot\}$$

- No rule is applicable to $\mathcal{A}_6$ and it does not contain a clash.

  Thus, $\mathcal{A}$ is consistent.



12、Extension of Tableau algorithm

We consider the concept constructor $\rightarrow$ (implication) with the following semantics:

$$(C \rightarrow D)^\mathcal{I} := \{x \in \Delta^\mathcal{I} \mid x \in C^\mathcal{I} \text{ implies } x \in D^\mathcal{I}\}.$$

To extend consistent$(\mathcal{A})$ to this constructor, we propose two alternatives new expansion rules:

The deterministic $\rightarrow$-rule	
Condition:	$\mathcal{A}$ contains $a : C \rightarrow D$ and $a : C$, but not $a : D$
Action:	$\mathcal{A} \longrightarrow \mathcal{A} \cup \{a : D\}$

The nondeterministic $\rightarrow$-rule	
Condition:	$\mathcal{A}$ contains $a : C \rightarrow D$, but neither $a : \neg C$ nor $a : D$
Action:	$\mathcal{A} \longrightarrow \mathcal{A} \cup \{a : X\}$ for some $X \in \{\neg C, D\}$

For each rule, determine whether the extended algorithm remains terminating, sound, and complete.

Knowledge of implication

$(C \to D)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} | x \in \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \text{ or } x \in D^{\mathcal{I}}\} = (\neg C \sqcup D)^{\mathcal{I}}$

According to NNF, $\neg(C \to D) \equiv C \sqcap \neg D$

The deterministic -rule:

Terminating: True

Omit the part of original proof and add the new proof for $\to$(`Lemma 4.4`)

- The new rule never remove an assertion and add a new assertion of the form $\{a : D\}$. And the size of $sub(A)$ is still bounded by the size of *A*
- The new rule does not add new individual name.
- The new rule only add concept assertions of the form $\{a : D\}$. The depth of each tree in the forest-shaped ABox is bounded by $|\text{sub}(\mathcal{A})|$.

Along with the original properties on the book, these properties ensure that there is a bound on the size of the ABox that can be constructed via rule applications, and thus a bound on the number of recursive applications of expand.

Soundness: False

Conter example:

Let $\mathcal{A} = \{a : (C \sqcup D) \to E, a : C, a : \neg E\}$

Since $a : C \sqcup D \notin \mathcal{A}$, we cannot use the deterministic $\to$-rule or any other rules.

Then no rules are applicable and no clash in it, so the $\text{consistent}(\mathcal{A})$ will return "consistent". However, if we replace $(C \sqcup D) \to E$ with $\neg(C \sqcup D) \sqcup E$ in preprocessing and call the original $\text{consistent}(\mathcal{A})$, clash $a : E, a : \neg E \subseteq \mathcal{A}'$ will be found.

Completeness: True

Omit the part of original proof(`Lemma 4.6`)

- The deterministic $\to$-rule: If $a : C \to D \in \mathcal{A}$ and $a : C \in \mathcal{A}$, then $a^{\mathcal{I}} \in (C \to D)^{\mathcal{I}}$. Thus $a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$ or $a^{\mathcal{I}} \in D^{\mathcal{I}}$ by the semantics of $\to$, but $a^{\mathcal{I}} \in C^{\mathcal{I}}$, so $a^{\mathcal{I}} \in D^{\mathcal{I}}$. Therefore, $\mathcal{I}$ is still a model of $\mathcal{A} \cup a : D$, so $\mathcal{A}$ is still consistent after the rule is applied.

The nondeterministic -rule:

Terminating: True

Omit the part of original proof and add the new proof for $\to$(`Lemma 4.4`)

- The new rule never remove an assertion and add a new assertion of the form $\{a : X\}$. And the size of $sub(A)$ is still bounded by the size of *A*
- The new rule does not add new individual name.
- The new rule only add concept assertions of the form $\{a : X\}$. The depth of each tree in the forest-shaped ABox is bounded by $|\text{sub}(\mathcal{A})|$.

Along with the original properties on the book, these properties ensure that there is a bound on the size of the ABox that can be constructed via rule applications, and thus a bound on the number of recursive applications of expand.

Soundness: True

Omit the part of original proof(`Lemma 4.5`)

Induction Basis: C is a conceptname: by definition of $\mathcal{I}$, if $a : C \in \mathcal{A}'$, then $a^{\mathcal{I}} \in C^{\mathcal{I}}$ as required.

Induction Steps:

- $C = \neg D$: since $\mathcal{A}'$ is clash-free, $a : \neg D \in \mathcal{A}'$ implies that $a : D \in \mathcal{A}'$. Since all concepts in $\mathcal{A}$ are in NNF, D is a concept name. By definition of $\mathcal{I}$, $a^{\mathcal{I}} \notin D^{\mathcal{I}}$, which implies $a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus D^{\mathcal{I}} = C^{\mathcal{I}}$ as required.
- $C = D \rightarrow E$: if $a : D \rightarrow E \in \mathcal{A}'$, then completeness of $\mathcal{A}'$ implies that $a : E \subseteq \mathcal{A}'$ or $a : \dot{\neg} D \subseteq \mathcal{A}'$ (otherwise the nondeterministic $\rightarrow$-rule would be applicable). Thus $a^{\mathcal{I}} \in E^{\mathcal{I}}$ or $a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus D^{\mathcal{I}}$ by induction, and hence $a^{\mathcal{I}} \in (\Delta^{\mathcal{I}} \setminus D^{\mathcal{I}}) \cup E^{\mathcal{I}} = (\neg D \sqcup E)^{\mathcal{I}} = (D \rightarrow E)^{\mathcal{I}}$ by the semantics of $\rightarrow$.

As a consequence, $\mathcal{I}$ satisfies all concept assertions in $\mathcal{A}'$ and thus in $\mathcal{A}$, and it satisfies all role assertions in $\mathcal{A}'$ and thus in $\mathcal{A}$ by definition. Hence $\mathcal{A}$ has a model and thus is consistent.

Completeness: True

Omit the part of original proof(`Lemma 4.6`)

- The nondeterministic $\rightarrow$-rule: If $a : C \rightarrow D \in \mathcal{A}$, then $a^{\mathcal{I}} \in (C \rightarrow D)^{\mathcal{I}}$. Thus $a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$ or $a^{\mathcal{I}} \in D^{\mathcal{I}}$ by the semantics of $\rightarrow$. Therefore, at least one of the ABoxes $\mathcal{A}' \in \exp(\mathcal{A}, \text{nondeterministic } \rightarrow\text{-rule}, a : C \rightarrow D)$ is consistent. Thus, one of the calls of expand is applied to a consistent ABox.

13、Modification of Tableau algorithm

We consider an $\mathcal{ALC}$ TBox $\mathcal{T}$ consisting only of the following two kinds of axioms:

- role inclusions of the form $r \sqsubseteq s$, and
- role disjointness constraints of the form disjoint(r, s).

where r and s are role names. An interpretation $\mathcal{I}$ satisfies these axioms if

- $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$, and
- $r^{\mathcal{I}} \cap s^{\mathcal{I}} = \emptyset$, respectively.

Modify the Tableau algorithm consistent$(\mathcal{A})$ to decide consistency of $(\mathcal{T}, \mathcal{A})$, where $\mathcal{A}$ is an ABox and $\mathcal{T}$ an TBox containing only role inclusions and role disjointness constraints. Show that the algorithm remains terminating, sound, and complete.

Modify

Add a judgement rule of the calsh:

- for role names r and s satisfying $\text{disjoint}(r, s) \subseteq \mathcal{T}$, if $(a, b) : r, (a, b) : s \subseteq \mathcal{A}$

Add a new expansion rule:

- $\sqsubseteq$-rule: if $(a, b) : r \in \mathcal{A}, r \sqsubseteq s \in \mathcal{T}$ and $(a, b) : s \notin \mathcal{A}$, then $\mathcal{A} \rightarrow \mathcal{A} \cup \{(a, b) : s\}$

Termination: True

Omit the part of original proof and add the new proof for $\sqsubseteq$

The number of the new role assertions will be no more than the square of the count of individual names , which is well bounded.

Soundness: True

Let $\mathcal{A}' = \text{consistent}(\mathcal{A})$. Then $\mathcal{A}'$ must be a clash-free and complete ABox.

- $\mathcal{I}$ satisfies each $\text{disjoint}(r, s) \in \mathcal{T}$:
    - Assume $r^{\mathcal{I}} \cap s^{\mathcal{I}} \neq \emptyset$, thus there are a and b satisfying $(a, b) \in r^{\mathcal{I}}$ and $(a, b) \in s^{\mathcal{I}}$. So we can know $(a, b) : r, (a, b) : s \subseteq \mathcal{A}'$, which contradicts $\mathcal{A}'$ is a clash-free ABox.
- $\mathcal{I}$ satisfies each $r \sqsubseteq s \in \mathcal{T}$:
    - Assume there are a and b satisfying $(a, b) \in r^{\mathcal{I}}$ but $(a, b) \in s^{\mathcal{I}}$. Therefore, $(a, b) : r \in \mathcal{A}'$ but $(a, b) : s \notin \mathcal{A}'$, which contradicts $\mathcal{A}'$ is a complete ABox.

Completeness: True

Omit the part of original proof(`Lemma 4.6`)

- The $\sqsubseteq$-rule: if $(a, b) : r \in \mathcal{A}$ and $r \sqsubseteq s \in \mathcal{T}$, then $(a, b) \in r^{\mathcal{I}}$. As $\mathcal{I}$ is a model of $\mathcal{T}$, $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$, thus $(a, b) \in s^{\mathcal{I}}$. Therefore, $\mathcal{I}$ is still a model of $\mathcal{A} \cup (a, b) : s$, so $\mathcal{A}$ is still consistent after the rule is applied.