Construction of Flux Coordinate System

I. CONSTRUCTION OF COORDINATE SYSTEM

Let R, ϕ, Z be conventional right-handed cylindrical coordinates. Let us adopt a normalization scheme in which all lengths are normalized to R_0, all magnetic field-strengths to B_0, and all pressures to B_0^2/μ_0. In the following, all quantities are assumed to be normalized. We can write the equilibrium magnetic field in the form

$$\mathbf{B} = \nabla\phi\times\nabla\psi + g(\psi)\,\nabla\phi = \nabla[\phi - q(\psi)\,\theta]\times\nabla\psi, \tag{1}$$

where the poloidal flux, $\psi(R,Z)$, and the toroidal flux function, $g(\psi)$, are both given, and

$$\nabla\psi\times\nabla\theta\cdot\nabla\phi = \frac{g}{q\,R^2}. \tag{2}$$

The equilibrium is assumed to be up-down symmetric, so that $\psi(R,-Z)=\psi(R,Z)$ for all R and Z. Here, θ is a so-called "straight" poloidal angle, and $q(\psi)$ is the safety-factor. Suppose that the magnetic axis (where $\psi_R = \psi_Z = 0$) lies at $R = R_c$, $Z = 0$. Let $\Psi(R,Z) = \psi(R,Z)/\psi_c$, where $\psi_c = \psi(R_c, 0)$. The previous equation implies that

$$\frac{d\theta}{dl} = \frac{g}{q}\,\frac{1}{|\psi_c|\,R\,\sqrt{\Psi_R^2+\Psi_Z^2}}, \tag{3}$$

where l represents distance along a constant-Ψ surface. The corresponding increments of R and Z are

$$dR = -\frac{\Psi_Z\,dl}{\sqrt{\Psi_R^2+\Psi_Z^2}}, \tag{4}$$

$$dZ = \frac{\Psi_R\,dl}{\sqrt{\Psi_R^2+\Psi_Z^2}}, \tag{5}$$

respectively. Since θ must increase by 2π in a circuit around a given flux surface, we can write

$$\frac{q(\Psi)}{g(\Psi)} = \frac{1}{2\pi\,|\psi_c|}\oint\frac{dl}{R\,\sqrt{\Psi_R^2+\Psi_Z^2}}, \tag{6}$$

which determines $q(\Psi)$. We can then calculate θ from

$$\frac{dR}{d\theta} = -|\psi_c|\,\frac{q(\Psi)}{g(\Psi)}\,R\,\Psi_Z, \tag{7}$$

$$\frac{dZ}{d\theta} = |\psi_c|\,\frac{q(\Psi)}{g(\Psi)}\,R\,\Psi_R. \tag{8}$$

It is convenient to define a new flux-surface label, r, which is such that $r=0$ on the magnetic axis, $r=1$ at the plasma boundary, and

$$\nabla r\times\nabla\theta\cdot\nabla\phi=\frac{1}{\epsilon_\ast^{\,2}\,r\,R^{\,2}}. \tag{9}$$

It follows that

$$\epsilon_\ast=\left(2\,|\psi_c|\int_0^1\frac{q(\Psi)}{g(\Psi)}\,d\Psi\right)^{1/2}, \tag{10}$$

and

$$r(\Psi)=\left(\frac{2\,|\psi_c|}{\epsilon_\ast^{\,2}}\int_\Psi^1\frac{q(\Psi')}{g(\Psi')}\,d\Psi'\right)^{1/2}. \tag{11}$$

Let $R_r=\partial R/\partial r|_\theta$, etc. It follows that

$$\epsilon_\ast^{\,2}\,r\,R=R_\theta\,Z_r-R_r\,Z_\theta, \tag{12}$$

$$|\nabla r|^{-2}=\frac{\epsilon_\ast^{\,4}\,r^2\,R^{\,2}}{R_\theta^{\,2}+Z_\theta^{\,2}}, \tag{13}$$

$$\frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^{\,2}}=-\frac{r\,(R_r\,R_\theta+Z_r\,Z_\theta)}{R_\theta^{\,2}+Z_\theta^{\,2}}. \tag{14}$$

II. DATA REQUIRED BY TOMUHAWC

All lengths in TOMUHAWC are normalized to $\epsilon_*\,R_0$. Let

$$R^{\,2} = \sum_{k=0,\infty} M_k^{(1)}(r)\,\cos(k\,\theta), \tag{15}$$

$$\epsilon_*^{-2}\,|\nabla r|^{\,-2}\,R^{\,-2} = \sum_{k=0,\infty} M_k^{(2)}(r)\,\cos(k\,\theta), \tag{16}$$

$$\epsilon_*^{-2}\,|\nabla r|^{\,-2} = \sum_{k=0,\infty} M_k^{(3)}(r)\,\cos(k\,\theta), \tag{17}$$

$$\epsilon_*^{-2}\,|\nabla r|^{\,-2}\,R^{\,2} = \sum_{k=0,\infty} M_k^{(4)}(r)\,\cos(k\,\theta), \tag{18}$$

$$\epsilon_*^{-2}\,|\nabla r|^{\,-2}\,R^{\,4} = \sum_{k=0,\infty} M_k^{(5)}(r)\,\cos(k\,\theta), \tag{19}$$

$$(r\,\nabla r\cdot\nabla\theta)\,|\nabla r|^{\,-2} = \sum_{k=1,\infty} M_k^{(6)}(r)\,\sin(k\,\theta), \tag{20}$$

$$(r\,\nabla r\cdot\nabla\theta)\,|\nabla r|^{\,-2}\,R^{\,2} = \sum_{k=1,\infty} M_k^{(7)}(r)\,\sin(k\,\theta), \tag{21}$$

$$\epsilon_*^{2}\,|\nabla r|^{\,2} = \sum_{k=0,\infty} M_k^{(8)}(r)\,\cos(k\,\theta), \tag{22}$$

$$R^{\,4} = \sum_{k=0,\infty} M_k^{(9)}(r)\,\cos(k\,\theta). \tag{23}$$

The TOMUHAWC code requires the $M_k^{(1)}(r)$, $M_k^{(2)}(r)$, $M_k^{(3)}(r)$, $M_k^{(4)}(r)$, $M_k^{(5)}(r)$, $M_k^{(6)}(r)$, and $M_k^{(7)}(r)$, as well as $M_0^{(8)}(r)$ and $M_0^{(9)}(r)$. The code also needs $q(r)$, $dq(r)/dr$, and the following six profile functions:

$$p_1(r) = \epsilon_*^{2}\,r^{\,2}, \tag{24}$$

$$p_2(r) = \frac{dg}{d\psi}, \tag{25}$$

$$p_3(r) = \frac{q}{g}\,\frac{dP}{d\psi}, \tag{26}$$

$$p_4(r) = -\frac{d\ln(q/g)}{d\ln r}, \tag{27}$$

$$p_5(r) = \left(\epsilon_*\,r\,\frac{g}{q}\right)^2, \tag{28}$$

$$p_6(r) = \mathit{\Gamma}\,P, \tag{29}$$

where $P(r)$ is the pressure profile, and Γ the plasma ratio of specific heats. All functions are required in the range $0 \leq r \leq 1$.