# The TOMUHAWC Code

## I. COORDINATES

### A. Cylindrical Coordinates

Let $R$, $\phi$, $Z$ be right-handed cylindrical coordinates whose symmetry axis corresponds to the toroidal symmetry axis of the plasma. The Jacobian for these coordinates is

$$(\nabla R\times\nabla\phi\cdot\nabla Z)^{-1} = R. \tag{1}$$

Suppose that $R = R_0$ at the plasma's magnetic axis.

### B. Flux Coordinates

Let $r$, $\theta$, $\phi$ be right-handed flux coordinates. Here, $r$ is a flux surface label with the dimensions of length. Let $r = 0$ correspond to the magnetic axis, $r = 1$ to the plasma boundary, and $r = r_w > 1$ to the location of a perfectly conducting wall surrounding the plasma. The region $1 < r < r_w$ is a vacuum. Furthermore, $\theta$ is a poloidal angle. Let $\theta = 0$ correspond to the inboard midplane. The Jacobian for these coordinates is

$$\mathcal{J}(r,\theta) \equiv (\nabla r\times\nabla\theta\cdot\nabla\phi)^{-1} = \frac{r\,R^2}{R_0}. \tag{2}$$

## II. PLASMA EQUILIBRIUM

### A. Magnetic Field

Consider an axisymmetric tokamak plasma equilibrium. The magnetic field is written

$$\mathbf{B}(r,\theta) = B_0\,R_0\left[f(r)\,\nabla\phi\times\nabla r + g(r)\,\nabla\phi\right], \tag{3}$$

where $B_0$ is the vacuum toroidal magnetic field-strength at the magnetic axis. The toroidal magnetic field-strength is thus

$$B_\phi(r,\theta) = B_0\,\frac{R_0}{R}\,g, \tag{4}$$

where $g(r > 1) = 1$, whereas the poloidal field-strength becomes

$$B_p(r,\theta) = B_0\,\frac{R_0}{R}\,f\,|\nabla r|. \tag{5}$$

Note that $R = R(r,\theta)$.

### B. Safety Factor Profile

The safety factor profile is

$$q(r) = \frac{r\,g}{R_0\,f}. \tag{6}$$

### C. Pressure Profile

Let $p(r)$ be the (unnormalized) equilibrium plasma pressure. The normalized equilibrium pressure profile is written

$$P(r) = \frac{\mu_0\,p(r)}{B_0^2}. \tag{7}$$

### D. Current Density Profile

Given that

$$\mu_0\,\mathbf{J} = \nabla\times\mathbf{B}, \tag{8}$$

the equilibrium poloidal current profile is written

$$\mathbf{J}\cdot\nabla\theta = \frac{B_0}{\mu_0}\,\frac{R_0^2}{R^2}\,\frac{g'}{r}, \tag{9}$$

whereas the toroidal current profile takes the form

$$\mathbf{J}\cdot\nabla\phi = -\frac{B_0}{\mu_0}\,\frac{R_0^2}{R^2}\left(\frac{g\,g'}{R_0\,f} + \frac{R^2}{R_0^2}\,\frac{P'}{R_0\,f}\right). \tag{10}$$

Of course, $\mathbf{J}\cdot\nabla r = 0$. Here, $' \equiv d/dr$.

### E. Grad-Shafranov Equation

Equilibrium force balance,

$$\mathbf{J}\times\mathbf{B} - \nabla p = \mathbf{0}, \tag{11}$$

yields the Grad-Shafranov equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,f\,|\nabla r|^{\,2}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(r\,f\,\nabla r\cdot\nabla\theta\right)+\left(\frac{g\,g'}{f}+\frac{R^{\,2}}{R_0^{\,2}}\,\frac{P'}{f}\right)=0. \tag{12}$$

## F. Toroidal Plasma Current

The equilibrium toroidal plasma current is

$$I_p=\int_0^1\oint\mathbf{J}\cdot\nabla\phi\;\mathcal{J}\,dr\,d\theta. \tag{13}$$

It follows from Eqs. (2), (10), and (12) that

$$I_p=2\pi\,\frac{B_0}{\mu_0\,R_0}\left(\frac{g}{q}\,\langle|\nabla r|^{\,2}\rangle\right)_{r=1}. \tag{14}$$

Here,

$$\langle\cdots\rangle\equiv\oint(\cdots)\,\frac{d\theta}{2\pi} \tag{15}$$

denotes a flux-surface average operator.

## G. Plasma Self-Inductance

The poloidal magnetic energy content of the plasma is

$$\mathcal{E}_p=\frac{1}{2}\,L\,I_p^{\,2}=\int_0^1\oint\oint\frac{B_p^{\,2}}{2\,\mu_0}\,\mathcal{J}\,dr\,d\theta\,d\phi, \tag{16}$$

where $L$ is the plasma self-inductance. The normalized inductance is defined

$$l_i=\frac{2\,L}{\mu_0\,R_0}. \tag{17}$$

Hence, it follows from Eqs. (2), (5), and (6) that

$$\mathcal{E}_p=2\pi^2\,\frac{B_0^{\,2}}{\mu_0\,R_0}\int_0^1\frac{r^3\,g^2}{q^2}\,\langle|\nabla r|^{\,2}\rangle\,dr, \tag{18}$$

and

$$l_i=\frac{2}{\left([(g/q)\,\langle|\nabla r|^{\,2}\rangle]_{r=1}\right)^{\,2}}\int_0^1\frac{r^3\,g^2}{q^2}\,\langle|\nabla r|^{\,2}\rangle\,dr. \tag{19}$$

### H. Beta

The volume averaged plasma pressure is written

$$\langle p\rangle = \left[\int_0^1\oint\oint p\,\mathcal{J}\,dr\,d\theta\,d\phi\right]\left[\int_0^1\oint\oint \mathcal{J}\,dr\,d\theta\,d\phi\right]^{-1}, \tag{20}$$

which reduces to

$$\langle p\rangle = \left[\int_0^1\left\langle\frac{R^2}{R_0^2}\right\rangle p\,r\,dr\right]\left[\int_0^1\left\langle\frac{R^2}{R_0^2}\right\rangle r\,dr\right]^{-1}. \tag{21}$$

The plasma beta is defined

$$\beta = \frac{\langle p\rangle}{(B_0^2/2\,\mu_0)}. \tag{22}$$

Hence,

$$\beta = 2\left[\int_0^1\left\langle\frac{R^2}{R_0^2}\right\rangle P\,r\,dr\right]\left[\int_0^1\left\langle\frac{R^2}{R_0^2}\right\rangle r\,dr\right]^{-1}. \tag{23}$$

### I. Poloidal Beta

The plasma poloidal beta is defined

$$\beta_p = \mathcal{E}_p^{-1}\int_0^1\oint\oint p\,\mathcal{J}\,dr\,d\theta\,d\phi. \tag{24}$$

It follows that

$$\beta_p = 2\,R_0^2\left[\int_0^1\left\langle\frac{R^2}{R_0^2}\right\rangle P\,r\,dr\right]\left[\int_0^1\frac{r^3\,g^2}{q^2}\,\langle|\nabla r|^2\rangle\,dr\right]^{-1}. \tag{25}$$

### J. Normal Beta

The plasma normal beta is defined

$$\beta_N = \frac{\beta(\%)\,a(\mathrm{m})\,B_0(\mathrm{T})}{I_p(\mathrm{MA})}, \tag{26}$$

where $a = \epsilon_0\,R_0$ is the minor radius, and $\epsilon_0$ the inverse aspect-ratio. This expression reduces to

$$\beta_N = 20\,\frac{\beta}{\epsilon_0}\left(\frac{q}{\langle|\nabla r|^2\rangle}\right)_{r=1}. \tag{27}$$

## III. OUTER SOLUTION

### A. Introduction

Consider a small perturbation to the aforementioned equilibrium. The system is divided into an 'outer region' and an 'inner region'. The outer region comprises the vacuum, and all of the plasma except a number of radially thin layers centered on the various internal rational surfaces. The 'inner region' consists of the layers. The perturbation in the outer region is governed by linearized, marginally-stable, ideal-MHD, whereas that in the inner region is governed by resistive-MHD. The overall solution is constructed by asymptotically matching the ideal-MHD solution in the outer region to resistive-MHD layer solutions in the various segments of the inner region.

### B. Governing Equations

The linearized, marginally-stable, ideal-MHD equations that govern the perturbation in the outer region are

$$\delta\mathbf{B} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}), \tag{28}$$

$$\nabla\delta p = \delta\mathbf{J}\times\mathbf{B} + \mathbf{J}\times\delta\mathbf{B}, \tag{29}$$

$$\mu_0\,\delta\mathbf{J} = \nabla\times\delta\mathbf{B}, \tag{30}$$

$$\delta p = -\boldsymbol{\xi}\cdot\nabla p. \tag{31}$$

Here, $\delta\mathbf{B}$, $\delta\mathbf{J}$, and $\delta p$ are the perturbed magnetic field, current density, and pressure, respectively.

### C. Fourier Transformed Equations

Let

$$B_0\,f\,\boldsymbol{\xi}\cdot\nabla r = y(r,\theta)\,\exp(-{\rm i}\,n\,\phi), \tag{32}$$

$$R^2\,\delta\mathbf{B}\cdot\nabla\phi = z(r,\theta)\,\exp(-{\rm i}\,n\,\phi), \tag{33}$$

where $n > 0$ is the toroidal mode number of the perturbation. After considerable algebra, Eqs. (28)–(31) reduce to

$$r\,\frac{\partial}{\partial r}\left[\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y\right] = \frac{\partial}{\partial\theta}\left(Q\,\frac{\partial z}{\partial\theta}\right)+S\,z-\frac{\partial}{\partial\theta}\left[T\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y+U\,y\right], \tag{34}$$

$$\begin{aligned}\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)r\,\frac{\partial z}{\partial r} =& -\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)T^{\ast}\,\frac{\partial z}{\partial\theta}+U\,\frac{\partial z}{\partial\theta}+X\,y\\ &-\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)V\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y+W\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y,\end{aligned} \tag{35}$$

where

$$Q(r,\theta) = \frac{1}{{\rm i}\,n\,|\nabla r|^{\,2}}, \tag{36}$$

$$S(r,\theta) = {\rm i}\,n\,\alpha_\epsilon, \tag{37}$$

$$T(r,\theta) = \frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^{\,2}}-\frac{\alpha_g}{{\rm i}\,n\,|\nabla r|^{\,2}}, \tag{38}$$

$$U(r,\theta) = \frac{\alpha_p}{|\nabla r|^{\,2}}\,\frac{R^{\,2}}{R_0^{\,2}}, \tag{39}$$

$$V(r,\theta) = \frac{1}{|\nabla r|^{\,2}}\left({\rm i}\,n\,\frac{R_0^2}{R^{\,2}}+\frac{\alpha_g^{\,2}}{{\rm i}\,n}\right), \tag{40}$$

$$W(r,\theta) = \frac{2\,\alpha_g\,\alpha_p}{|\nabla r|^{\,2}}\,\frac{R^{\,2}}{R_0^{2}}-r\,\alpha_g', \tag{41}$$

$$X(r,\theta) = {\rm i}\,n\,\alpha_p\left[\frac{\partial}{\partial\theta}\left(T^{\ast}\,\frac{R^{\,2}}{R_0^2}\right)+r\,\frac{\partial}{\partial r}\left(\frac{R^{\,2}}{R_0^2}\right)-\alpha_f\,\frac{R^{\,2}}{R_0^2}-U\,\frac{R^{\,2}}{R_0^2}\right], \tag{42}$$

and

$$\alpha_\epsilon(r) = \frac{r^2}{R_0^2}, \tag{43}$$

$$\alpha_g(r) = \frac{R_0\,g'}{f}, \tag{44}$$

$$\alpha_p(r) = \frac{r\,P'}{f^{\,2}}, \tag{45}$$

$$\alpha_f(r) = \frac{r^2}{f}\,\frac{d}{dr}\left(\frac{f}{r}\right). \tag{46}$$

Let

$$y(r,\theta) = \sum_{j=1,J} y_j(r)\,\exp(\,{\rm i}\,m_j\,\theta), \tag{47}$$

$$z(r,\theta) = \sum_{j=1,J} z_j(r)\,\exp(\,{\rm i}\,m_j\,\theta), \tag{48}$$

where the $m_j$, for $j = 1, J$, are the various coupled poloidal harmonics. Equations (34) and (35) reduce to

$$r\,\frac{d}{dr}\,[(m_j - n\,q)\,y_j] = \sum_{j'=1,J} (B_{jj'}\,z_{j'} + C_{jj'}\,y_{j'})\,, \tag{49}$$

$$(m_j - n\,q)\,r\,\frac{dz_j}{dr} = \sum_{j'=1,J} (D_{jj'}\,z_{j'} + E_{jj'}\,y_{j'})\,, \tag{50}$$

for $j = 1, J$, where

$$B_{jj'}(r) = \frac{1}{2\pi\,\mathrm{i}}\oint \mathrm{e}^{-\mathrm{i}\,m_j\,\theta}\left(\frac{\partial}{\partial\theta}\,Q\,\frac{\partial}{\partial\theta} + S\right)\mathrm{e}^{\mathrm{i}\,m_{j'}\,\theta}\,d\theta, \tag{51}$$

$$C_{jj'}(r) = \frac{1}{2\pi\,\mathrm{i}}\oint \mathrm{e}^{-\mathrm{i}\,m_j\,\theta}\left[-\frac{\partial}{\partial\theta}\,T\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right) - \frac{\partial U}{\partial\theta}\right]\mathrm{e}^{\mathrm{i}\,m_{j'}\,\theta}\,d\theta, \tag{52}$$

$$D_{jj'}(r) = \frac{1}{2\pi\,\mathrm{i}}\oint \mathrm{e}^{-\mathrm{i}\,m_j\,\theta}\left[-\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right)T^*\,\frac{\partial}{\partial\theta} + U\,\frac{\partial}{\partial\theta}\right]\mathrm{e}^{\mathrm{i}\,m_{j'}\,\theta}\,d\theta, \tag{53}$$

$$\begin{aligned} E_{jj'}(r) = &\frac{1}{2\pi\,\mathrm{i}}\oint \mathrm{e}^{-\mathrm{i}\,m_j\,\theta}\left[-\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right)V\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right)\right.\\ &\left. + W\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right) + X\right]\mathrm{e}^{\mathrm{i}\,m_{j'}\,\theta}\,d\theta. \end{aligned} \tag{54}$$

Hence, it follows from Eqs. (36)–(42) that

$$n\,B_{jj'} = m_j\,m_{j'}\,c_{jj'} + n^2\,\alpha_\epsilon\,\delta_{jj'}, \tag{55}$$

$$C_{jj'} = m_j\,(m_{j'} - n\,q)\left(-f_{jj'} + n^{-1}\,\alpha_g\,c_{jj'}\right) - m_j\,\alpha_p\,d_{jj'}, \tag{56}$$

$$D_{jj'} = -\,(m_j - n\,q)\,m_{j'}\left(f_{jj'} + n^{-1}\,\alpha_g\,c_{jj'}\right) + m_{j'}\,\alpha_p\,d_{jj'}, \tag{57}$$

$$\begin{aligned} n^{-1}\,E_{jj'} = &(m_j - n\,q)\,(m_{j'} - n\,q)\left(b_{jj'} - n^{-2}\,\alpha_g^2\,c_{jj'}\right) - (m_{j'} - n\,q)\,n^{-1}\,r\,\alpha_g'\,\delta_{jj'}\\ &+ \alpha_p\left[(m_j - m_{j'})\,g_{jj'} + n^{-1}\,\alpha_g\,(m_j + m_{j'} - 2\,n\,q)\,d_{jj'} + r\,\frac{da_{jj'}}{dr}\right.\\ &\left. -\alpha_f\,a_{jj'} - \alpha_p\,e_{jj'}\right], \end{aligned} \tag{58}$$

where

$$a_{jj'}(r) = \oint \left(\frac{R}{R_0}\right)^2 \exp\left[-\mathrm{i}\,(m_j - m_{j'})\,\theta\right]\frac{d\theta}{2\pi}, \tag{59}$$

$$b_{jj'}(r) = \oint |\nabla r|^{-2}\left(\frac{R}{R_0}\right)^{-2} \exp\left[-\mathrm{i}\,(m_j - m_{j'})\,\theta\right]\frac{d\theta}{2\pi}, \tag{60}$$

$$c_{jj'}(r) = \oint |\nabla r|^{-2} \exp\left[-\mathrm{i}\,(m - m')\,\theta\right]\frac{d\theta}{2\pi}, \tag{61}$$

$$d_{jj'}(r) = \oint |\nabla r|^{-2}\left(\frac{R}{R_0}\right)^{2} \exp\left[-\mathrm{i}\,(m_j - m_{j'})\,\theta\right]\frac{d\theta}{2\pi}, \tag{62}$$

$$e_{jj'}(r) = \oint |\nabla r|^{-2}\left(\frac{R}{R_0}\right)^{4} \exp\left[-\mathrm{i}\,(m_j - m_{j'})\,\theta\right]\frac{d\theta}{2\pi}, \tag{63}$$

$$f_{jj'}(r) = \oint \frac{\mathrm{i}\,r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2} \exp\left[-\mathrm{i}\,(m_j - m_{j'})\,\theta\right]\frac{d\theta}{2\pi}, \tag{64}$$

$$g_{jj'}(r) = \oint \frac{\mathrm{i}\,r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\left(\frac{R}{R_0}\right)^{2} \exp\left[-\mathrm{i}\,(m_j - m_{j'})\,\theta\right]\frac{d\theta}{2\pi}. \tag{65}$$

Let

$$y_j = \frac{\psi_j(r)}{m_j - n\,q}, \tag{66}$$

$$z_j = n\,\frac{Z_j(r)}{m_j - n\,q} - \frac{C_{jj}}{B_{jj}}\,\frac{\psi_j(r)}{m_j - n\,q}. \tag{67}$$

It follows that

$$\delta\mathbf{B}\cdot\nabla r = \mathrm{i}\,\frac{R_0^2}{R^2}\sum_{j=1,J}\frac{\psi_j}{r}\,\exp[\,\mathrm{i}\,(m_j\,\theta - n\,\phi)]. \tag{68}$$

Furthermore, Eqs. (49) and (50) transform to

$$r\,\frac{d\psi_j}{dr} = \sum_{j'=1,J}\frac{L_{jj'}\,Z_{j'} + M_{jj'}\,\psi_{j'}}{m_{j'} - n\,q}, \tag{69}$$

$$(m_j - n\,q)\,r\,\frac{d}{dr}\left(\frac{Z_j}{m_j - n\,q}\right) = \sum_{j'=1,J}\frac{N_{jj'}\,Z_{j'} + P_{jj'}\,\psi_{j'}}{m_{j'} - n\,q}, \tag{70}$$

for $j = 1, J$, where

$$L_{jj'}(r) = n\,B_{jj'}, \tag{71}$$

$$M_{jj'}(r) = C_{jj'} + \lambda_{j'}\,L_{jj'}, \tag{72}$$

$$N_{jj'}(r) = D_{jj'} - \lambda_j\,L_{jj'}, \tag{73}$$

$$\begin{aligned} P_{jj'}(r) = {} & n^{-1}\,E_{jj'} - \lambda_j\,M_{jj'} + \lambda_{j'}\,N_{jj'} + \lambda_j\,\lambda_{j'}\,L_{jj'} \\ & - \lambda_j\,n\,q\,s\,\delta_{jj'} - (m_j - n\,q)\,r\,\lambda_{j'}\,\delta_{jj'}, \end{aligned} \tag{74}$$

with

$$s(r) = \frac{r\,q'}{q}, \tag{75}$$

and

$$\lambda_j(r) = -\frac{C_{jj}}{n\,B_{jj}} = -\left[\frac{m_j\,(m_j - n\,q)\,n^{-1}\,\alpha_g\,c_{jj} - m\,\alpha_p\,d_{jj}}{m_j^2\,c_{jj} + n^2\,\alpha_\epsilon}\right]. \tag{76}$$

Now, for a general (i.e., not necessarily up-down symmetric) plasma equilibrium, it is easily demonstrated that

$$a_{j'j} = a^*_{jj'}, \tag{77}$$

$$b_{j'j} = b^*_{jj'}, \tag{78}$$

$$c_{j'j} = c^*_{jj'}, \tag{79}$$

$$d_{j'j} = d^*_{jj'}, \tag{80}$$

$$e_{j'j} = e^*_{jj'}, \tag{81}$$

$$f_{j'j} = -f^*_{jj'}, \tag{82}$$

$$g_{j'j} = -g^*_{jj'}, \tag{83}$$

for all $j$, $j'$, so that

$$B_{j'j} = B^*_{jj'}, \tag{84}$$

$$C_{j'j} = -D^*_{jj'}, \tag{85}$$

$$D_{j'j} = -C^*_{jj'}, \tag{86}$$

$$E_{j'j} = E^*_{jj'}, \tag{87}$$

and

$$L_{j'j} = L^*_{jj'}, \tag{88}$$

$$M_{j'j} = -N^*_{jj'}, \tag{89}$$

$$N_{j'j} = -M^*_{jj'}, \tag{90}$$

$$P_{j'j} = P^*_{jj'}. \tag{91}$$

It follows from Eqs. (69) and (70) that

$$r\,\frac{d}{dr}\left(\sum_{j=1,J}\frac{Z_j^*\,\psi_j - \psi_j^*\,Z_j}{m_j - n\,q}\right) = 0. \tag{92}$$

The net toroidal electromagnetic torque acting on the region lying within that equilibrium magnetic flux-surface whose label is $r$ takes the form

$$T_\phi(r) = \int_0^r \oint \oint R^2\,\nabla\phi\cdot(\delta\mathbf{J}\times\delta\mathbf{B})\,\mathcal{J}\,dr\,d\theta\,d\phi, \tag{93}$$

which can be shown to reduce to

$$T_\phi(r) = \frac{n\,\pi^2\,R_0}{\mu_0}\,\mathrm{i}\sum_{j=1,J}\frac{Z_j^\ast\,\psi_j-\psi_j^\ast\,Z_j}{m_j-n\,q}. \tag{94}$$

Hence, we deduce that

$$\frac{dT_\phi}{dr} = 0 \tag{95}$$

in the outer region.

### D. Behavior in Vicinity of Plasma Rational Surface

Let there be $K$ rational surfaces in the plasma. Suppose that the $k$th surface is of radius $r_k$, and resonant poloidal mode number $m_k$, where $q(r_k) = m_k/n$, for $k = 1, K$.

#### 1. General Case

Consider the solution of the outer equations, (69) and (70), in the vicinity of the $k$th surface. Let $x = r - r_k$. The most general small-$|x|$ solution of the outer equations can be shown to take the form

$$\psi_j(x) = A_{L\,k}^\pm\,|x|^{\nu_{L\,k}}\,(1+\lambda_{L\,k}\,x+\cdots)+A_{S\,k}^\pm\,\mathrm{sgn}(x)\,|x|^{\nu_{S\,k}}\,(1+\cdots)+A_{C\,k}\,x\,(1+\cdots), \tag{96}$$

$$\begin{aligned}Z_j(x) = {}& A_{L\,k}^\pm\,|x|^{\nu_{L\,k}}(b_{L\,k}+\gamma_{L\,k}\,x+\cdots)+A_{S\,k}^\pm\,\mathrm{sgn}(x)\,|x|^{\nu_{S\,k}}\,(b_{S\,k}+\cdots)\\ &+B_{C\,k}\,x\,(1+\cdots)\end{aligned} \tag{97}$$

if $m_j = m_k$, and

$$\psi_j(x) = A_{L\,k}^\pm\,|x|^{\nu_{L\,k}}\,(a_{kj}+\cdots)+\bar{\psi}_{kj}\,(1+\cdots), \tag{98}$$

$$Z_j(x) = A_{L\,k}^\pm\,|x|^{\nu_{L\,k}}\,(b_{kj}+\cdots)+\bar{Z}_{kj}\,(1+\cdots) \tag{99}$$

if $m_j \neq m_k$. Moreover, the superscripts $^+$ and $^-$ correspond to $x > 0$ and $x < 0$, respectively. Here,

$$\nu_{L\,k} = \frac{1}{2} - \left(\frac{1}{4} + L_{0\,k}\,P_{0\,k}\right)^{1/2}, \tag{100}$$

$$\nu_{S\,k} = \frac{1}{2} + \left(\frac{1}{4} + L_{0\,k}\,P_{0\,k}\right)^{1/2}, \tag{101}$$

$$L_{0\,k} = -\left(\frac{L_{kk}}{m\,s}\right)_{r_k}, \tag{102}$$

$$P_{0\,k} = -\left(\frac{P_{kk}}{m\,s}\right)_{r_k}. \tag{103}$$

Furthermore,

$$b_{L\,k} = \frac{\nu_{L\,k}}{L_{0\,k}}, \tag{104}$$

$$b_{S\,k} = \frac{\nu_{S\,k}}{L_{0\,k}}, \tag{105}$$

$$A_{C\,k} = -\frac{1}{r_k\,P_{0\,k}} \sum_{j=1,J}^{m_j\neq m_k} \frac{1}{m_j - m_k} \left(N_{k\,j}\,\bar{Z}_{kj} + P_{kj}\,\bar{\psi}_{kj}\right)_{r_k}, \tag{106}$$

$$B_{C\,k} = -\frac{1}{r_k\,L_{0\,k}} \sum_{j=1,J}^{m_j\neq m_k} \frac{1}{m_j - m_k} \left(L_{kj}\,\bar{Z}_{kj} + M_{kj}\,\bar{\psi}_{kj}\right)_{r_k} + \frac{A_{C\,k}}{L_{0\,k}}, \tag{107}$$

$$\lambda_{L\,k} = \frac{1}{2\,r_k} \left[\frac{P_{1\,k}\,L_{0\,k}}{\nu_{L\,k}} + T_{1\,k} + \nu_{L\,k} \left(\frac{L_{1\,k}}{L_{0\,k}} - 2\right)\right]_{r_k} - \frac{1}{(m\,s)_{r_k}} \frac{1}{r_k\,\nu_{L\,k}} \sum_{j=1,J}^{m_j\neq m_k} \frac{1}{m_j - m_k} \left(P_{kj}\,L_{kj} - M_{kj}\,N_{kj}\right)_{r_k}, \tag{108}$$

$$\gamma_{L\,k} = \frac{1}{2\,r_k} \left[(1 + \nu_{L\,k}) \left(\frac{P_{1\,k}}{\nu_{L\,k}} + \frac{T_{1\,k}}{L_{0\,k}} - \frac{\nu_{L\,k}}{L_{0\,k}}\right) + P_{0\,k} \left(\frac{L_{1\,k}}{L_{0\,k}} - 1\right)\right]_{r_k} - \frac{1}{(m\,s)_{r_k}} \frac{1}{r_k\,L_{0\,k}} \sum_{j=1,J}^{m_j\neq m_k} \frac{1}{m_j - m_k} \left(P_{kj}\,L_{kj} - M_{kj}\,N_{kj}\right)_{r_k}, \tag{109}$$

$$a_{kj} = \frac{1}{(m\,s)_{r_k}} \left(\frac{N_{kj}}{\nu_{L\,k}} - \frac{L_{kj}}{L_{0\,k}}\right)_{r_k}, \tag{110}$$

$$b_{kj} = \frac{1}{(m\,s)_{r_k}} \left(\frac{M_{kj}}{L_{0\,k}} - \frac{P_{kj}}{\nu_{L\,k}}\right)_{r_k}, \tag{111}$$

and

$$L_{1\,k} = \lim_{x\to 0}\left(\frac{L_{kk}}{m_k - n\,q}\right) - \frac{r_k\,L_{0\,k}}{x}, \tag{112}$$

$$P_{1\,k} = \lim_{x\to 0}\left(\frac{P_{kk}}{m_k - n\,q}\right) - \frac{r_k\,P_{0\,k}}{x}, \tag{113}$$

$$T_{1\,k} = \lim_{x\to 0}\left(\frac{-n\,q\,s}{m_k - n\,q}\right) - \frac{r_k}{x}, \tag{114}$$

The parameters $A_{S\,k}$ and $A_{L\,k}$ are identified from the numerical solution of the outer equations in the vicinity of the rational surface by taking the limits

$$\bar{\psi}_{kj} = \psi_j(r_k+\delta) - a_{kj}\,\psi_k(r_k+\delta), \tag{115}$$

$$\bar{Z}_{kj} = Z_j(r_k+\delta) - b_{kj}\,\psi_k(r_k+\delta), \tag{116}$$

$$\begin{aligned} A^{\pm}_{S\,k} = &\pm \frac{Z_k(r_k \pm |\delta|) - b_{L\,k}\,\psi_k(r_k\pm|\delta|)}{(b_{S\,k}-b_{L\,k})\,|\delta|^{\nu_{S\,k}}} \\ &- \frac{[(B_{C\,k} - b_{L\,k}\,A_{C\,k}) + (\gamma_{L\,k} - b_{L\,k}\,\lambda_{L\,k})\,\psi_k(r_k\pm|\delta|)]\,|\delta|}{(b_{S\,k}-b_{L\,k})\,|\delta|^{\nu_{S\,k}}}, \end{aligned} \tag{117}$$

$$A^{\pm}_{L\,k} = \frac{\psi_k(r_k\pm|\delta|) \mp A^{\pm}_{S\,k}|\delta|^{\nu_{S\,k}} \mp A_{C\,k}\,|\delta|}{(1\pm|\delta|\,\lambda_{L\,k})\,|\delta|^{\nu_{L\,k}}} \tag{118}$$

as $|\delta|\to 0$.

### 2. *Zero Pressure Limit*

In the limit $P_{0\,k}\to 0$, the indices $\nu_{L\,k}$ and $\nu_{S\,k}$ become exactly 0 and 1, respectively. In this case, some of the previous expressions become singular, and a special treatment is required. The most general small-$|x|$ solution of the outer equations takes the form

$$\begin{aligned} \psi_j(x) = &A^{\pm}_{L\,k}\,[1 + \hat{\lambda}_{L\,k}\,x\,(\ln|x| - 1) + \cdots] + A^{\pm}_{S\,k}\,x\,(1+\cdots) + \hat{A}_{C\,k}\,x\,(1+\cdots) \\ &+ A_{D\,k}\,x\,(\ln|x| - 1 + \cdots), \end{aligned} \tag{119}$$

$$Z_j(x) = A^{\pm}_{L\,k}\,(\hat{\gamma}_{L\,k}\,x\,\ln|x| + \cdots) + A^{\pm}_{S\,k}\,x\,(\hat{b}_{S\,k}+\cdots) + B_{D\,k}\,x\,(\ln|x|+\cdots) \tag{120}$$

if $m_j = m_k$, and

$$\psi_j(x) = A^{\pm}_{L\,k}\,(\hat{a}_{kj}\,\ln|x| + \cdots) + \bar{\psi}_{kj}\,(1+\cdots), \tag{121}$$

$$Z_j(x) = A^{\pm}_{L\,k}\,(\hat{b}_{kj}\,\ln|x| + \cdots) + \bar{Z}_{kj}\,(1+\cdots) \tag{122}$$

if $m_j \neq m_k$. Here,

$$\hat{b}_{S\,k} = \frac{1}{L_{0\,k}}, \tag{123}$$

$$\hat{A}_{C\,k} = \frac{1}{r_k} \sum_{j=1,J}^{m_j\neq m_k} \frac{1}{m_j - m_k} \left(L_{kj}\,\bar{Z}_{kj} + M_{kj}\,\bar{\psi}_{kj}\right)_{r_k}, \tag{124}$$

$$A_{D\,k} = \frac{L_{0\,k}}{r_k} \sum_{j=1,J}^{m_j\neq m_k} \frac{1}{m_j - m_k} \left(N_{kj}\,\bar{Z}_{kj} + P_{kj}\,\bar{\psi}_{kj}\right)_{r_k}, \tag{125}$$

$$B_{D\,k} = \frac{A_{D\,k}}{L_{0\,k}}, \tag{126}$$

$$\hat{\lambda}_{L\,k} = \frac{P_{1\,k}\,L_{0\,k}}{r_k} - \frac{1}{(m\,s)_{r_k}}\,\frac{1}{r_k} \sum_{j=1,J}^{m_j\neq m_k} \frac{1}{m_j - m_k}\,(P_{kj}\,L_{kj} - M_{kj}\,N_{kj})_{r_k}, \tag{127}$$

$$\hat{\gamma}_{L\,k} = \frac{P_{1\,k}}{r_k}, \tag{128}$$

$$\hat{a}_{kj} = \frac{1}{(m\,s)_{r_k}}\,(N_{kj})_{r_k}, \tag{129}$$

$$\hat{b}_{kj} = -\frac{1}{(m\,s)_{r_k}}\,(P_{kj})_{r_k}. \tag{130}$$

The parameters $A_{S\,k}$ and $A_{L\,k}$ are identified from the numerical solution of the outer equations in the vicinity of the rational surface by taking the limits

$$\bar{\psi}_{kj} = \psi_j(r_k+\delta) - \hat{a}_{kj}\,\psi_k(r_k+\delta)\,\ln|\delta|, \tag{131}$$

$$\bar{Z}_{kj} = Z_j(r_k+\delta) - \hat{b}_{kj}\,\psi_k(r_k+\delta)\,\ln|\delta|, \tag{132}$$

$$A_{S\,k}^{\pm} = \pm\,\frac{Z_k(r_k\pm|\delta|) \mp [B_{D\,k} + \hat{\gamma}_{L\,k}\,\psi_k(r_k\pm|\delta|)]\,|\delta|\,\ln|\delta|}{|\delta|\,\hat{b}_{S\,k}}, \tag{133}$$

$$A_{L\,k}^{\pm} = \frac{\psi_k(r_k\pm|\delta|) \mp \left[A_{S\,k}^{\pm} + \hat{A}_{C\,k} + A_{D\,k}\,(\ln|\delta|-1)\right]|\delta|}{1 \pm \hat{\lambda}_{L\,k}\,|\delta|\,(\ln|\delta|-1)} \tag{134}$$

as $|\delta| \to 0$.

### E. Behavior in Vicinity of Vacuum Rational Surface

Let there be $L$ rational surfaces in the vacuum region surrounding the plasma (which is characterized by $P=0$ and $g=1$). Suppose that the $l$th surface is of radius $r_l$, and resonant poloidal mode number $m_l$, where $q(r_l) = m_l/n$, for $l = 1, L$.

Consider the solution of the outer equations, (69) and (70), in the vicinity of the $l$th surface. Let $x = r - r_l$. The most general small-$|x|$ solution of the outer equations can be shown to take the form

$$\psi_j(x) = A^{\pm}_{L\,l} + A^{\pm}_{S\,l}\,x + \hat{A}_{C\,l}\,x\mathcal{O}(x^2), \tag{135}$$

$$Z_j(x) = A^{\pm}_{S\,l}\,\hat{b}_{S\,l}\,x + \mathcal{O}(x^2), \tag{136}$$

if $m_j = m_l$, and

$$\psi_j(x) = \bar{\psi}_{jl} + \mathcal{O}(x), \tag{137}$$

$$Z_j(x) = \bar{Z}_{jl} + \mathcal{O}(x), \tag{138}$$

if $m_j \neq m_l$. Here, $\hat{A}_{C\,l}$ and $\hat{b}_{S\,l}$ are defined in the previous subsection. The superscripts $^+$ and $^-$ again correspond to $x > 0$ and $x < 0$, respectively.

The parameters $A^{\pm}_{S\,l}$ and $A^{\pm}_{L\,l}$ can be identified from the numerical solution of the outer equations in the vicinity of the $l$th vacuum rational surface by taking the limits

$$A^{\pm}_{S\,l} = \pm\frac{Z_l(r_l \pm |\delta|)}{|\delta|\,\hat{b}_{S\,l}}, \tag{139}$$

$$A^{\pm}_{L\,l} = \psi_l(r_l \pm |\delta|) \mp |\delta|\,(A^{\pm}_{S\,l} + \hat{A}_{C\,l}) \tag{140}$$

as $|\delta| \to 0$.

### F. Asymptotic Matching Across Plasma Rational Surface

Consider the resistive layer solution in the vicinity of the $k$th plasma rational surface. This solution can be separated into independent tearing and twisting parity components. The even (tearing parity) component is such that $\psi_k(-x) = \psi_k(x)$ throughout the layer, whereas the odd (twisting parity) component is such that $\psi_k(-x) = -\psi_k(x)$. It is helpful to define the quantities

$$A^e_{L\,k} = \frac{1}{2}\,(A^+_{L\,k} + A^-_{L\,k}), \tag{141}$$

$$A^o_{L\,k} = \frac{1}{2}\,(A^+_{L\,k} - A^-_{L\,k}), \tag{142}$$

$$A^e_{S\,k} = \frac{1}{2}\,(A^+_{S\,k} - A^-_{S\,k}), \tag{143}$$

$$A^o_{S\,k} = \frac{1}{2}\,(A^+_{S\,k} + A^-_{S\,k}). \tag{144}$$

The even and odd layer solutions determine the ratios

$$\Delta_k^e = r_k^{\nu_{S\,k}-\nu_{L\,k}}\,\frac{2\,A_{S\,k}^e}{A_{L\,k}^e}, \tag{145}$$

and

$$\Delta_k^o = r_k^{\nu_{S\,k}-\nu_{L\,k}}\,\frac{2\,A_{S\,k}^o}{A_{L\,k}^o}, \tag{146}$$

respectively. Moreover, the net toroidal electromagnetic torque acting on the layer can be shown to take the form

$$\delta T_k = \frac{2\,n\,\pi^2\,R_0}{\mu_0}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L_{kk}}\right)_{r_k}\left[|A_{L\,k}^e|^{\,2}\,{\rm Im}(\Delta_k^e) + |A_{L\,k}^o|^{\,2}\,{\rm Im}(\Delta_k^o)\right]. \tag{147}$$

Let

$$\Psi_k^e = r_k^{\nu_{L\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L_{kk}}\right)_{r_k}^{1/2} A_{L\,k}^e, \tag{148}$$

$$\Delta\Psi_k^e = r_k^{\nu_{S\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L_{kk}}\right)_{r_k}^{1/2} 2\,A_{S\,k}^e, \tag{149}$$

$$\Psi_k^o = r_k^{\nu_{L\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L_{kk}}\right)_{r_k}^{1/2} A_{L\,k}^o, \tag{150}$$

$$\Delta\Psi_k^o = r_k^{\nu_{S\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L_{kk}}\right)_{r_k}^{1/2} 2\,A_{S\,k}^o, \tag{151}$$

The matching conditions become

$$\Delta\Psi_k^e = \Delta_k^e\;\Psi_k^e, \tag{152}$$

$$\Delta\Psi_k^o = \Delta_k^o\;\Psi_k^o. \tag{153}$$

Moreover,

$$\delta T_k = \frac{2\,n\,\pi^2\,R_0}{\mu_0}\,{\rm Im}\left(\Psi_k^{e\,*}\,\Delta\Psi_k^e + \Psi_k^{o\,*}\,\Delta\Psi_k^o\right), \tag{154}$$

and

$$\Psi_k^e - \Psi_k^o = r_k^{\nu_{L\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L_{kk}}\right)_{r_k}^{1/2} A_{L\,k}^-, \tag{155}$$

$$\Delta\Psi_k^e - \Delta\Psi_k^o = -r_k^{\nu_{S\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L_{kk}}\right)_{r_k}^{1/2} 2\,A_{S\,k}^-. \tag{156}$$

### G. Asymptotic Matching Across Vacuum Rational Surface

The vacuum region outside the plasma is characterized by $p = 0$ and $\mathbf{J} = \mathbf{0}$. In this region, Eqs. (28)–(31) yield

$$\delta p = 0, \tag{157}$$

$$\delta\mathbf{J}\times\mathbf{B} = \mathbf{0}, \tag{158}$$

$$\nabla\cdot\delta\mathbf{J} = 0. \tag{159}$$

It follows that

$$\delta\mathbf{J} = \alpha\,\mathbf{B}, \tag{160}$$

where

$$\mathbf{B}\cdot\nabla\alpha = 0. \tag{161}$$

The only single-valued solution of the above equation is

$$\alpha(r,\theta,\phi) = \sum_{l=1,L}\alpha_l\,\delta(r-r_l)\,\exp\left[\,{\rm i}\,(m_l\,\theta - n\,\phi)\right]. \tag{162}$$

We conclude that, in the vacuum region, Eqs. (69) and (70), which are derived from Eqs. (28)–(31), only permit perturbed currents to flow in the immediate vicinity of the vacuum rational surfaces. We can eliminate these unphysical currents by imposing the following matching conditions at such surfaces:

$$A_{L\,l}^{+} = A_{L\,l}^{-}, \tag{163}$$

$$A_{S\,l}^{+} = A_{S\,l}^{-}. \tag{164}$$

These conditions also ensure that zero electromagnetic torque is exerted in the vicinity of a vacuum rational surface.

### H. Toroidal Electromagnetic Torque on Plasma

It follows, from the previous analysis, that the net toroidal electromagnetic torque acting within that equilibrium magnetic flux-surface whose label is $r$ satisfies

$$\frac{dT_\phi}{dr} = \sum_{k=1,K}\delta T_k\,\delta(r-r_k), \tag{165}$$

where

$$\delta T_k = \frac{2\,n\,\pi^2\,R_0}{\mu_0}\,{\rm Im}\,(\Psi_k^{e\,*}\,\Delta\Psi_k^e + \Psi_k^{o\,*}\,\Delta\Psi_k^o)\,. \tag{166}$$

## I. Derivation of Dispersion Relation

Let $\mathbf{y}(r)$ represent the $2J$-dimensional vector of the $\psi_j(r)$ and $Z_j(r)$ functions that satisfy the outer equations, (69) and (70).

Let us launch $J$ linearly independent, well-behaved solution vectors, $\mathbf{y}_j^e(r)$, for $j = 1, J$, from the magnetic axis, $r = 0$, and numerically integrate them to $r = r_w$. The jump conditions imposed at the plasma rational surfaces are

$$\Psi_{k'}^o = 0, \tag{167}$$

$$\Delta\Psi_{k'}^e = 0, \tag{168}$$

for $k' = 1, K$. The jump conditions imposed at the vacuum rational surfaces are

$$A_{L\,l}^+ = A_{L\,l}^-, \tag{169}$$

$$A_{S\,l}^+ = A_{S\,l}^-, \tag{170}$$

for $l = 1, L$.

Next, let us launch a solution vector, $\Delta\mathbf{y}_k^e(r)$, from the $k$th plasma rational surface, and numerically integrate it to $r = r_w$. The jump conditions imposed at the plasma rational surfaces are

$$\Psi_{k'}^o = 0, \tag{171}$$

$$\Delta\Psi_{k'}^e = \delta_{k'k}, \tag{172}$$

for $k' = 1, K$. The jump conditions at the vacuum rational surfaces are specified by Eqs. (169) and (170).

We can form a linear combination of solution vectors,

$$\mathbf{Y}_k^e(r) = \sum_{j=1,J} \alpha_{jk}^e\,\mathbf{y}_j^e + \Delta\mathbf{y}_k^e, \tag{173}$$

and choose the $\alpha_{jk}^e$ so as to ensure that the physical boundary condition at the perfectly conducting wall,

$$\psi_j(r_w) = 0, \tag{174}$$

for $j = 1, J$, is satisfied. By construction, this solution vector is such that

$$\Psi^o_{k'} = 0. \tag{175}$$

$$\Delta\Psi^e_{k'} = \delta_{k'k}, \tag{176}$$

for $k' = 1, K$. Let

$$\Psi^e_{k'} = F^{ee}_{k'k}, \tag{177}$$

$$\Delta\Psi^o_{k'} = F^{oe}_{k'k}, \tag{178}$$

for $k' = 1, K$. We can associate a $\mathbf{Y}^e_k(r)$ with each rational surface in the plasma.

Let us launch $J$ linearly independent, well-behaved solution vectors, $\mathbf{y}^o_j(r)$, for $j = 1, J$, from the magnetic axis, $r = 0$, and numerically integrate them to $r = r_w$. The jump conditions imposed at the plasma rational surfaces are

$$\Psi^e_{k'} = 0, \tag{179}$$

$$\Delta\Psi^o_{k'} = 0, \tag{180}$$

for $k' = 1, K$. The jump conditions at the vacuum rational surfaces are given by Eqs. (169) and (170).

Next, we can launch a solution vector, $\Delta\mathbf{y}^o_k(r)$, from the $k$th plasma rational surface, and integrate it to $r = r_w$. The jump conditions imposed at the plasma rational surfaces are

$$\Psi^e_{k'} = 0, \tag{181}$$

$$\Delta\Psi^o_{k'} = \delta_{k'k}, \tag{182}$$

for $k = 1, K$. The jump conditions at the vacuum rational surfaces are given by Eqs. (169) and (170).

We can form the linear combination of solution vectors,

$$\mathbf{Y}^o_k(r) = \sum_{j=1,J} \alpha^o_{jk}\, \mathbf{y}^o_j + \Delta\mathbf{y}^o_k, \tag{183}$$

and choose the $\alpha^o_{jk}$ so as to satisfy the physical boundary condition at the wall,

$$\psi_j(r_w) = 0, \tag{184}$$

for $j = 1, J$. By construction, this solution vector is such that

$$\Psi^e_{k'} = 0. \tag{185}$$

$$\Delta\Psi^o_{k'} = \delta_{k'k}, \tag{186}$$

for $k' = 1, K$. Let

$$\Psi^o_{k'} = F^{oo}_{k'k}, \tag{187}$$

$$\Delta\Psi^e_{k'} = F^{eo}_{k'k}, \tag{188}$$

for $k' = 1, K$. We can associate a $\mathbf{Y}^o_k(r)$ with each rational surface in the plasma.

The most general well-behaved solution vector that satisfies the physical boundary condition at the wall is written

$$\mathbf{Y}(r) = \sum_{k=1,K} (a_k\,\mathbf{Y}^e_k + b_k\,\mathbf{Y}^o_k), \tag{189}$$

where the $a_k$ and $b_k$ are arbitrary. It follows that

$$\Psi^e_k = \sum_{k'=1,K} F^{ee}_{kk'}\,a_{k'}, \tag{190}$$

$$\Psi^o_k = \sum_{k'=1,K} F^{oo}_{kk'}\,b_{k'}, \tag{191}$$

$$\Delta\Psi^e_k = a_{k'} + \sum_{k=1,K} F^{eo}_{kk'}\,b_{k'}, \tag{192}$$

$$\Delta\Psi^o_k = b_{k'} + \sum_{k=1,K} F^{oe}_{kk'}\,a_{k'}, \tag{193}$$

for $k = 1, K$. Let $\mathbf{\Psi}^e$, $\mathbf{\Psi}^o$, $\mathbf{\Delta\Psi}^e$, and $\mathbf{\Delta\Psi}^o$ be the $K \times 1$ vectors of the $\Psi^e_k$, $\Psi^o_k$, $\Delta\Psi^e_k$, and $\Delta\Psi^o_k$ values, respectively. Let $\mathbf{F}^{ee}$, $\mathbf{F}^{eo}$, $\mathbf{F}^{oe}$, and $\mathbf{F}^{oo}$ be the $K \times K$ matrices of the $F^{ee}_{kk'}$, $F^{eo}_{kk'}$, $F^{oe}_{kk'}$, and $F^{oo}_{kk'}$ values, respectively. Equations (190)–(193) can be combined to give the dispersion relation

$$\begin{pmatrix} \mathbf{\Delta\Psi}^e \\ \mathbf{\Delta\Psi}^o \end{pmatrix} = \begin{pmatrix} \mathbf{E}^e & \mathbf{\Gamma} \\ \mathbf{\Gamma}' & \mathbf{E}^o \end{pmatrix} \begin{pmatrix} \mathbf{\Psi}^e \\ \mathbf{\Psi}^o \end{pmatrix}, \tag{194}$$

where

$$\mathbf{E}^e = (\mathbf{F}^{ee})^{-1}, \tag{195}$$

$$\mathbf{E}^o = (\mathbf{F}^{oo})^{-1}, \tag{196}$$

$$\boldsymbol{\Gamma} = \mathbf{F}^{eo}\,\mathbf{E}^o, \tag{197}$$

$$\boldsymbol{\Gamma}' = \mathbf{F}^{oe}\,\mathbf{E}^e. \tag{198}$$

Now, according to Eqs. (94), (174), and (184),

$$T_\phi(r_w) = 0. \tag{199}$$

In other words, the net toroidal electromagnetic torque acting on the plasma is zero. Hence, it follows from Eqs. (165) and (166) that

$$\boldsymbol{\Psi}^{e\,\dagger}\,\boldsymbol{\Delta\Psi}^e - \boldsymbol{\Delta\Psi}^{e\,\dagger}\,\boldsymbol{\Psi}^e + \boldsymbol{\Psi}^{o\,\dagger}\,\boldsymbol{\Delta\Psi}^o - \boldsymbol{\Delta\Psi}^{o\,\dagger}\,\boldsymbol{\Psi}^o = 0. \tag{200}$$

Thus, making use of the dispersion relation (194), we deduce that

$$\mathbf{E}^{e\,\dagger} = \mathbf{E}^e, \tag{201}$$

$$\mathbf{E}^{o\,\dagger} = \mathbf{E}^o, \tag{202}$$

$$\boldsymbol{\Gamma}' = \boldsymbol{\Gamma}^\dagger. \tag{203}$$

Thus, the dispersion relation can be written

$$\begin{pmatrix} \mathbf{E}^e - \boldsymbol{\Delta}^e & \boldsymbol{\Gamma} \\ \boldsymbol{\Gamma}^\dagger & \mathbf{E}^o - \boldsymbol{\Delta}^o \end{pmatrix} \begin{pmatrix} \boldsymbol{\Psi}^e \\ \boldsymbol{\Psi}^o \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{204}$$

where $\boldsymbol{\Delta}^e$ and $\boldsymbol{\Delta}^o$ are the diagonal $K \times K$ matrices of the $\varDelta_k^e$ and $\varDelta_k^o$ values, respectively. Note that the $\mathbf{E}^e$ and $\mathbf{E}^o$ matrices are Hermitian.

## IV. INNER SOLUTION

### A. Introduction

Let us now consider the resistive-MHD solutions in the various segments of the inner region.

## B. Basic Equations

Let us assume that all perturbed quantities vary in time as $\exp(\mathrm{i}\,\omega\,t)$, where $\omega$ is the error-field frequency. The linearized, resistive-MHD equations that govern perturbed quantities in the inner region are

$$\delta\mathbf{B} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}) - \frac{\eta}{\tilde{\gamma}}\,\nabla\times\delta\mathbf{J}, \tag{205}$$

$$\nabla\delta p = \delta\mathbf{J}\times\mathbf{B} + \mathbf{J}\times\delta\mathbf{B} - \rho\,\tilde{\gamma}^{\,2}\,\boldsymbol{\xi}, \tag{206}$$

$$\mu_0\,\delta\mathbf{J} = \nabla\times\delta\mathbf{B}, \tag{207}$$

$$\delta p = -\boldsymbol{\xi}\cdot\nabla p - \mathit{\Gamma}\,p\,\nabla\cdot\boldsymbol{\xi}. \tag{208}$$

Here,

$$\tilde{\gamma}(r) = \mathrm{i}\,\omega - \mathrm{i}\,n\,\mathit{\Omega}_\phi(r), \tag{209}$$

where $\mathit{\Omega}_\phi(r)$ is the plasma toroidal angular velocity. (Note that we are neglecting the effect of velocity shear in the above equations.) Moreover, $\eta(r)$ and $\rho(r)$ are the plasma resistivity and density profiles, respectively. Finally, $\mathit{\Gamma}$ is the plasma ratio of specific heats.

## C. Layer Equations

Consider the segment of the inner region centered on the $k$th rational surface. It is helpful to define

$$a_0(r) = \left\langle\frac{R^{\,2}}{R_0^{2}}\right\rangle, \tag{210}$$

$$c_0(r) = \left\langle|\nabla r|^{\,-2}\right\rangle, \tag{211}$$

$$d_0(r) = \left\langle|\nabla r|^{\,-2}\frac{R^{\,2}}{R_0^{2}}\right\rangle, \tag{212}$$

$$e_0(r) = \left\langle|\nabla r|^{\,-2}\frac{R^{\,4}}{R_0^{4}}\right\rangle, \tag{213}$$

$$x_0(r) = \langle|\nabla r|^{\,2}\rangle, \tag{214}$$

$$y_0(r) = \left\langle\frac{R^{\,4}}{R_0^{4}}\right\rangle, \tag{215}$$

as well as

$$F_R(r) = \frac{1 + x_0\,\alpha_\epsilon/q^2}{c_0 + \alpha_\epsilon/q^2}, \tag{216}$$

$$F_A(r) = \frac{y_0\,(1 + x_0\,\alpha_\epsilon/q^2) - a_0^2}{f^2\,F_R}, \tag{217}$$

and

$$\omega_A(r) = \frac{B_0}{R_0}\,\frac{n\,s}{\sqrt{\mu_0\,\rho\,F_A}}, \tag{218}$$

$$\omega_R(r) = \frac{\eta\,F_R}{\mu_0\,r^2}, \tag{219}$$

$$S(r) = \frac{\omega_A}{\omega_R}. \tag{220}$$

Here, $\omega_A$ is a typical hydromagnetic frequency, $\omega_R$ a typical resistive diffusion rate, and $S$ the effective magnetic Lundquist number of the plasma. It is assumed that $S \gg 1$.

In the vicinity of the $k$th rational surface, Eqs. (205)–(208) can be shown to reduce to

$$0 = \frac{d^2\varPsi}{dX^2} - H\,\frac{d\varUpsilon}{dX} - Q\,(\varPsi - X\,\varXi), \tag{221}$$

$$0 = Q^{\,2}\,\frac{d^2\varXi}{dX^2} - Q\,X^{\,2}\,\varXi + E\,\varUpsilon + Q\,X\,\varPsi + \varLambda, \tag{222}$$

$$0 = Q\,\frac{d^2\varUpsilon}{dX^2} - X^{\,2}\,\varUpsilon - G\,Q^{\,2}\,\varUpsilon + (G - K\,E)\,Q^{\,2}\,\varXi + X\,\varPsi - K\,Q^{\,2}\,\varLambda, \tag{223}$$

$$0 = H\,\frac{d^2\varLambda}{dX^2} - \frac{d\varLambda}{dX} + F\,\frac{d\varUpsilon}{dX}. \tag{224}$$

Here,

$$x = r - r_k, \tag{225}$$

$$\varPsi(x) = \psi_k(x), \tag{226}$$

$$X = \left(S^{\,1/3}\,\frac{x}{r}\right)_{r_k}, \tag{227}$$

$$Q = \left(S^{\,1/3}\,\frac{\tilde{\gamma}}{\omega_A}\right)_{r_k}, \tag{228}$$

and

$$E = \left[\frac{\alpha_p}{s^2}\,(c_0+\alpha_\epsilon/q^2)\left(-r\,\frac{da_0}{dr}+a_0\,\alpha_f\right)+\frac{a_0\,s}{F_R}\right]_{r_k}, \tag{229}$$

$$F = \left[\frac{\alpha_p^2}{s^2}\,\left([c_0+\alpha_\epsilon/q^2]\,e_0-d_0^2\right)\right]_{r_k}, \tag{230}$$

$$H = \left[\frac{\alpha_p}{s}\left(d_0-\frac{a_0}{F_R}\right)\right]_{r_k}, \tag{231}$$

$$K = \left[\frac{s^2}{\alpha_p^2\,f^2}\,\frac{F_R}{F_A}\right]_{r_k}, \tag{232}$$

$$G = \left[\frac{a_0\,(c_0+\alpha_\epsilon/q^2)}{\Gamma\,P}\,\frac{F_R}{F_A}\right]_{r_k}. \tag{233}$$

Let us write

$$\Psi(X) = \Psi^e(X)+\Psi^o(X)+A_0\,X, \tag{234}$$

$$\Xi(X) = \Xi^e(X)+\Xi^o(X)+A_0, \tag{235}$$

$$\Upsilon(X) = \Upsilon^e(X)+\Upsilon^o(X)+A_0, \tag{236}$$

$$\Lambda(X) = \Lambda^e(X)+\Lambda^o(X)-A_0\,E, \tag{237}$$

where $A_0$ is an arbitrary constant, and $\Psi^e(-X)=\Psi^e(X)$, $\Psi^o(-X)=-\Psi^o(X)$, etc. Equations (221)–(224) can be shown to separate into the following two independent sets of equations:

$$0 = \frac{d^2\Psi^{e,o}}{dX^2}-H\,\frac{d\Upsilon^{o\,e}}{dX}-Q\,(\Psi^{e,o}-X\,\Xi^{o,e}), \tag{238}$$

$$0 = Q^{\,2}\,\frac{d^2\,\Xi^{o,e}}{dX^{\,2}}-Q\,X^{\,2}\,\Xi^{o,e}+(E+F)\,\Upsilon^{o,e}+Q\,X\,\Psi^{e,o}+H\,\frac{d\Psi^{e,o}}{dX}, \tag{239}$$

$$\begin{aligned}0 = {}&Q\,\frac{d^2\Upsilon^{o,e}}{dX^{\,2}}-X^{\,2}\,\Upsilon^{o,e}-Q^{\,2}\,(G+K\,F)\,\Upsilon^{o,e}+Q^{\,2}\,(G-K\,E)\,\Xi^{o,e}\\&-Q^{\,2}\,K\,H\,\frac{d\Psi^{o,e}}{dX},\end{aligned} \tag{240}$$

where

$$\Lambda^{o,e} = H\,\frac{d\Psi^{e,o}}{dX}+F\,\Upsilon^{o,e}. \tag{241}$$

The first set (involving $\Psi^e$) governs tearing parity layer solutions, whereas the second (involving $\Psi^o$) governs twisting parity solutions.

## D. Asymptotic Matching

In the limit $|X| \to \infty$, the asymptotic behavior of the well-behaved solutions of the layer equations, (238)–(241), is such that

$$\Psi^e(X) \to a_L^e\,|X|^{\nu_{L\,k}} + a_S^e\,|X|^{\nu_{S\,k}}, \tag{242}$$

$$\Psi^o(X) \to \mathrm{sgn}(X)\,(a_L^o\,|X|^{\nu_{L\,k}} + a_S^o\,|X|^{\nu_{S\,k}})\,. \tag{243}$$

These solutions are undetermined to an arbitrary multiplicative constant, which means that the ratios $a_S^e/a_L^e$ and $a_S^o/a_L^o$ are fully determined. Here,

$$\nu_{L\,k} = -\frac{1}{2} - \sqrt{\frac{1}{4} - E - F - H}, \tag{244}$$

$$\nu_{S\,k} = -\frac{1}{2} + \sqrt{\frac{1}{4} - E - F - H}. \tag{245}$$

These indices can be shown to be identical to the corresponding indices defined in Section III D. Asymptotic matching to the ideal-MHD solution in the outer region yields

$$\Delta_k^e = S_k^{(\nu_{S\,k}-\nu_{L\,k})/3}\,\hat{\Delta}_k^e, \tag{246}$$

$$\Delta_k^o = S_k^{(\nu_{S\,k}-\nu_{L\,k})/3}\,\hat{\Delta}_k^o, \tag{247}$$

where

$$S_k = S(r_k), \tag{248}$$

and

$$\hat{\Delta}_k^e = \frac{2\,a_S^e}{a_L^e}, \tag{249}$$

$$\hat{\Delta}_k^o = \frac{2\,a_S^o}{a_L^o}. \tag{250}$$

## E. Standard Parameters

The standard hydromagnetic frequency, resistive diffusion rate, and Lundquist number are defined

$$\overline{\omega}_A = \frac{B_0}{R_0}\,\frac{1}{\sqrt{\mu_0\,\rho}}, \tag{251}$$

$$\overline{\omega}_R(r) = \frac{\eta}{\mu_0}, \tag{252}$$

$$\overline{S}(r) = \frac{\overline{\omega}_A}{\overline{\omega}_R}, \tag{253}$$

respectively. Here, $\varpi_A$ is assumed to be a constant (which implies that the plasma mass density is uniform). It follows that

$$\overline{S} = f_S\, S, \tag{254}$$

$$\varpi_A = f_A\, \omega_A, \tag{255}$$

where

$$f_S(r) = \frac{F_A^{1/2}\, F_R}{n\, s\, r^2}, \tag{256}$$

$$f_A(r) = \frac{n\, s}{F_A^{1/2}}. \tag{257}$$

Let

$$\hat{\omega}(r) = \frac{\omega}{\varpi_A}, \tag{258}$$

$$\hat{\Omega}(r) = \frac{\Omega_\phi}{\varpi_A}, \tag{259}$$

$$\hat{\gamma}(r) = \mathrm{i}\,(\hat{\omega} - n\,\hat{\Omega}). \tag{260}$$

It follows that

$$Q = \left[\left(\frac{\bar{S}}{f_S}\right)^{1/3} f_A\,\hat{\gamma}\right]_{r_k}, \tag{261}$$

$$\Delta_k^e = \left(\frac{\bar{S}}{f_s}\right)_{r_k}^{(\nu_{S\,k}-\nu_{L\,k})/3} \hat{\Delta}_k^e, \tag{262}$$

$$\Delta_k^o = \left(\frac{\bar{S}}{f_s}\right)_{r_k}^{(\nu_{S\,k}-\nu_{L\,k})/3} \hat{\Delta}_k^o. \tag{263}$$

## V. ERROR-FIELD RESPONSE THEORY

### A. Error-Field Response

Let us define the solution vectors

$$\tilde{\mathbf{Y}}_k^e(r) = \sum_{k''=1,K} E^e_{k''k}\,\mathbf{Y}^e_{k''}, \tag{264}$$

for $k = 1, K$. It follows, from Section III I, that these vectors are well-behaved solutions of the outer equations, satisfying the physical boundary condition at the wall, and having the properties that

$$\Psi^e_{k'} = \delta_{k'k}, \tag{265}$$

$$\Psi^o_{k'} = 0, \tag{266}$$

$$\Delta\Psi^e_{k'} = E^e_{k'k}, \tag{267}$$

$$\Delta\Psi^o_{k'} = \Gamma^*_{kk'}, \tag{268}$$

for $k' = 1, K$. Let the $\tilde{\psi}^e_{j,k}(r)$ and the $\tilde{Z}^e_{j,k}(r)$ be the elements of the $\tilde{\mathbf{Y}}^e_k(r)$ solution vector, for $j = 1, J$. By construction, we have

$$\tilde{\psi}^e_{j,k}(r_w) = 0, \tag{269}$$

for $j = 1, J$.

Let us define the solution vectors

$$\tilde{\mathbf{Y}}^o_k(r) = \sum_{k''=1,K} E^o_{k''k}\,\mathbf{Y}^o_{k''}, \tag{270}$$

for $k = 1, K$. It follows, from Section III I, that these vectors are well-behaved solutions of the outer equations, satisfying the physical boundary condition at the wall, and having the properties that

$$\Psi^e_{k'} = 0, \tag{271}$$

$$\Psi^o_{k'} = \delta_{k'k}, \tag{272}$$

$$\Delta\Psi^e_{k'} = \Gamma_{k'k}, \tag{273}$$

$$\Delta\Psi^o_{k'} = E^o_{k'k}, \tag{274}$$

for $k' = 1, K$. Let the $\tilde{\psi}^o_{j,k}(r)$ and the $\tilde{Z}^o_{j,k}(r)$ be the elements of the $\tilde{\mathbf{Y}}^o_k(r)$ solution vector, for $j = 1, J$. By construction, we have

$$\tilde{\psi}^o_{j,k}(r_w) = 0, \tag{275}$$

for $j = 1, J$.

The most general well-behaved solution vector in the presence of an error-field is written

$$\mathbf{Y}(r) = \sum_{k=1,K}\left(\Psi^e_k\,\tilde{\mathbf{Y}}^e_k + \Psi^o_k\,\tilde{\mathbf{Y}}^o_k\right) + \mathbf{Y}^x, \tag{276}$$

where $\mathbf{Y}^x(r)$ is the solution vector that describes the ideal (i.e., $\Psi_k^e = \Psi_k^o = 0$ for $k = 1, K$) response of the plasma to the error-field. This solution vector is characterized by

$$\Psi_{k'}^e = 0, \tag{277}$$

$$\Psi_{k'}^o = 0, \tag{278}$$

$$\Delta\Psi_{k'}^e = \chi_{k'}^e, \tag{279}$$

$$\Delta\Psi_{k'}^o = \chi_{k'}^o, \tag{280}$$

for $k' = 0, K$. Thus, in the presence of the error-field, the dispersion relation (194) generalizes to

$$\Delta_k^e\,\Psi_k^e = \sum_{k'=1,K} (E_{kk'}^e\,\Psi_{k'}^e + \Gamma_{kk'}\,\Psi_{k'}^o) + \chi_k^e, \tag{281}$$

$$\Delta_k^o\,\Psi_k^o = \sum_{k'=1,K} (E_{kk'}^o\,\Psi_{k'}^o + \Gamma_{k'k}^*\,\Psi_{k'}^e) + \chi_k^o, \tag{282}$$

for $k = 1, K$.

Let the $\psi_j^x(r)$ and the $Z_j^x(r)$ be the elements of the $\mathbf{Y}^x(r)$ solution vector, for $j = 1, J$. Suppose that

$$\psi_j^x(r_w) = (m_j - n\,q)\,\Xi_j^x, \tag{283}$$

$$Z_j^x(r_w) = \Omega_j^x, \tag{284}$$

for $j = 1, J$. According to Eq. (94),

$$T_\phi(r_w) = -\frac{2\,n\,\pi^2\,R_0}{\mu_0}\,\mathrm{Im}\sum_{j=1,J}\frac{Z_j^*\,\psi_j}{m_j - n\,q}. \tag{285}$$

However,

$$\psi_j(r_w) = [m_j - n\,q(r_w)]\,\Xi_j^x, \tag{286}$$

$$Z_j(r_w) = \sum_{k=1,K}\left[\Psi_k^e\,\tilde{Z}_{j,k}^e(r_w) + \Psi_k^o\,\tilde{Z}_{j,k}^o(r_w)\right] + \Omega_j^x, \tag{287}$$

so

$$T_\phi(r_w) = -\frac{2\,n\,\pi^2\,R_0}{\mu_0}\,\mathrm{Im}\sum_{J=1,J}\sum_{k=1,K}\Xi_j^x\left[\Psi_k^e\,\tilde{Z}_{j,k}^e(r_w) + \Psi_k^o\,\tilde{Z}_{j,k}^o(r_w)\right]^*. \tag{288}$$

Here, we have assumed that

$$\mathrm{Im}\sum_{j=1,J}\Omega_j^{x\,*}\,\Xi_j^x = 0, \tag{289}$$

otherwise the error-field would be able to exert a torque on an ideal plasma (which is unphysical).

It follows from Eqs. (165), (166), (281), (282), as well as the fact that $\mathbf{E}^e$ and $\mathbf{E}^o$ matrices are Hermitian matrices, that

$$T_\phi(r_w) = \sum_{k=1,K} \delta T_k = \frac{2\,n\,\pi^2\,R_0}{\mu_0}\,\mathrm{Im}\sum_{k=1,K}(\Psi_k^{e\,*}\,\chi_k^e + \Psi_k^{o\,*}\,\chi_k^o)\,. \tag{290}$$

Comparison with Eq. (287) reveals that

$$\chi_k^e = -\sum_{j=1,J} \Xi_j^x\,\tilde{Z}_{j,k}^e(r_w), \tag{291}$$

$$\chi_k^o = -\sum_{j=1,J} \Xi_j^x\,\tilde{Z}_{j,k}^o(r_w). \tag{292}$$

Let

$$R_{kk'} = \begin{cases} E^e_{kk'} - \Delta^e_k(\hat{\omega})\,\delta_{kk'} & k=1,K \quad k'=1,K, \\ \Gamma_{k\bar{k}'} & k=1,K \quad \bar{k}'=1,K, \\ \Gamma^*_{k'\bar{k}} & \bar{k}=1,K \quad k=1,K, \\ E^o_{\bar{k}\bar{k}'} - \Delta^o_{\bar{k}}(\hat{\omega})\,\delta_{\bar{k}\bar{k}'} & \bar{k}=1,K \quad \bar{k}'=1,K \end{cases} \tag{293}$$

for $k, k' = 1, 2K$. Here, $\bar{k} = k - K$ and $\bar{k}' = k' - K$. Furthermore, let

$$\Psi_k = \begin{cases} \Psi^e_k & k=1,K, \\ \Psi^o_{\bar{k}} & \bar{k}=1,K \end{cases}, \tag{294}$$

and

$$\chi_k = \begin{cases} \chi^e_k & k=1,K, \\ \chi^o_{\bar{k}} & \bar{k}=1,K \end{cases}, \tag{295}$$

and

$$U_{kj} = \begin{cases} \tilde{Z}^e_{j,k}(r_w) & k=1,K, \\ \tilde{Z}^o_{j,\bar{k}}(r_w) & \bar{k}=1,K \end{cases}. \tag{296}$$

The plasma response to the error-field is governed by Eqs. (281), (282), (290), and (291), which yield

$$\sum_{k'=1,2K} R_{kk'}\,\Psi_{k'} = -\chi_k = \sum_{j=1,J} U_{kj}\,\Xi_j^x. \tag{297}$$

Hence,

$$\Psi_k = \sum_{j=1,J} S^*_{kj}\,\Psi_j^x, \tag{298}$$

where

$$S_{kj} = \sum_{k'=1,2\,K} R^{*\,-1}_{kk'}\,U^{*}_{k'j} \tag{299}$$

Let

$$|S_k| = \left(\sum_{j=1,J} S^{*}_{kj}\,S_{kj}\right)^{1/2}, \tag{300}$$

$$|\Xi^x| = \left(\sum_{j=1,J} \Xi^{x\,*}_{j}\,\Xi^{x}_{j}\right)^{1/2}. \tag{301}$$

For fixed $|\Psi^x|$, $|\Psi_k|$ is maximized when

$$\Psi^x_j \propto S_{kj}. \tag{302}$$

In this case,

$$|\Psi_k| = |S_k|\,|\Xi^x|. \tag{303}$$

Finally, it follows from Eq. (289) that the toroidal torque associated with $\Psi_k$ is

$$\delta T_k = \frac{2\,n\,\pi^2\,R_0}{\mu_0}\,\mathrm{Im}(\Psi^{*}_k\,\chi_k), \tag{304}$$

which yields

$$\delta T_k = \frac{2\,n\,\pi^2\,R_0}{\mu_0}\,\mathrm{Im}\left(\sum_{j=1,J}\sum_{k'=1,2\,K} U^{*}_{kj}\,R^{-1}_{kk'}\,U_{k'j}\right)\,|\Xi^x|^{\,2}. \tag{305}$$