I. VACUUM BOUNDARY CONDITION

A. Toroidal Coordinates

Right-handed orthogonal curvilinear coordinates: η, μ, ϕ. Here, ϕ is toroidal angle.

$$R = \frac{\sinh\mu}{\cosh\mu - \cos\eta}, \tag{1}$$

$$Z = \frac{\sin\eta}{\cosh\mu - \cos\eta}, \tag{2}$$

$$d_1 = [(R+1)^2 + Z^2]^{1/2}, \tag{3}$$

$$d_2 = [(R-1)^2 + Z^2]^{1/2}, \tag{4}$$

$$\cosh\mu = \frac{1}{2}\left(\frac{d_1}{d_2} + \frac{d_2}{d_1}\right), \tag{5}$$

$$\cos\eta = \left(\frac{d_1^2 + d_2^2 - 4}{2\,d_1\,d_2}\right). \tag{6}$$

Scale factors:

$$h_\eta = h_\mu = h = \frac{1}{\cosh\mu - \cos\eta}, \tag{7}$$

$$h_\phi = R = \frac{\sinh\mu}{\cosh\mu - \cos\eta}. \tag{8}$$

B. Magnetic Perturbation

$$\delta\mathbf{B} = \mathrm{i}\,\nabla V, \tag{9}$$

where

$$\nabla^2 V = 0. \tag{10}$$

Let

$$V(\eta,\mu,\phi) = \hat{V}(\eta,\mu)\,\mathrm{e}^{-\mathrm{i}\,n\,\phi}, \tag{11}$$

and

$$\delta\hat{\mathbf{B}} = \mathrm{i}\,\nabla\hat{V}. \tag{12}$$

So,

$$\delta\hat{B}_\eta = \frac{\mathrm{i}}{h}\,\frac{\partial\hat{V}}{\partial\eta}, \tag{13}$$

$$\delta\hat{B}_\mu = \frac{\mathrm{i}}{h}\,\frac{\partial\hat{V}}{\partial\mu}, \tag{14}$$

$$\delta\hat{B}_\phi = \frac{n}{R}\,\hat{V}. \tag{15}$$

C. Current Sheet

Suppose that there is a current sheet at $\mu = \mu_w$. Let

$$\delta\hat{\mathbf{J}} = \int_{\mu_{w-}}^{\mu_{w+}} \delta\hat{\mathbf{j}}\,h\,d\mu, \tag{16}$$

where

$$\delta\hat{\mathbf{j}} = \nabla\times\delta\hat{\mathbf{B}}. \tag{17}$$

Follows that

$$\delta\hat{J}_\eta = [\delta\hat{B}_\phi]_{\mu_{w-}}^{\mu^{w+}} = \frac{n}{R}\,[\hat{V}]_{\mu_{w-}}^{\mu_{w+}}, \tag{18}$$

$$\delta\hat{J}_\mu = 0, \tag{19}$$

$$\delta\hat{J}_\phi = -[\delta\hat{B}_\eta]_{\mu_{w-}}^{\mu^{w+}} = -\frac{\mathrm{i}}{h}\left[\frac{\partial\hat{V}}{\partial\eta}\right]_{\mu_{w-}}^{\mu_{w+}}. \tag{20}$$

Of course,

$$\left[\frac{\partial\hat{V}}{\partial\mu}\right]_{\mu_{w-}}^{\mu_{w+}} = 0. \tag{21}$$

D. Toroidal Torque

Now,

$$\overline{(\delta\mathbf{J}\times\delta\mathbf{B})_\phi} = \overline{\delta J_\eta\,\delta B_\mu} = -\frac{\mathrm{i}\,n}{2\,R\,h}\left\{[\hat{V}]\,\frac{\partial\hat{V}^\ast}{\partial\mu} - [\hat{V}^\ast]\,\frac{\partial\hat{V}}{\partial\mu}\right\}. \tag{22}$$

Thus, net toroidal torque acting on sheet is

$$\hat{T}_\phi = 2\pi\oint R\,\overline{(\delta\mathbf{J}\times\delta\mathbf{B})_\phi}\,h\,R\,d\eta = -\mathrm{i}\,n\,\pi\oint R\left\{[\hat{V}]\,\frac{\partial\hat{V}^\ast}{\partial\mu} - [\hat{V}^\ast]\,\frac{\partial\hat{V}}{\partial\mu}\right\}d\eta. \tag{23}$$

Let

$$z = \cosh\mu. \tag{24}$$

Follows that

$$\hat{T}_\phi = -\mathrm{i}\, n\, \pi\, (z^2 - 1) \oint (z - \cos\eta)^{-1} \left\{ [\hat{V}]\, \frac{\partial \hat{V}^*}{\partial z} - [\hat{V}^*]\, \frac{\partial \hat{V}}{\partial z} \right\} d\eta. \tag{25}$$

Now,

$$[\hat{V}] = \hat{V}_+ - \hat{V}_-, \tag{26}$$

$$\frac{\partial \hat{V}_+}{\partial \mu} = \frac{\partial \hat{V}_-}{\partial \mu}. \tag{27}$$

So,

$$\hat{T}_\phi = -\mathrm{i}\, n\, \pi\, (z^2 - 1) \oint (z - \cos\eta)^{-1} \left\{ \hat{V}_+\, \frac{\partial \hat{V}_+^*}{\partial z} - \hat{V}_+^*\, \frac{\partial \hat{V}_+}{\partial z} - \hat{V}_-\, \frac{\partial \hat{V}_-^*}{\partial z} + \hat{V}_-^*\, \frac{\partial \hat{V}_-}{\partial z} \right\} d\eta. \tag{28}$$

Let

$$\hat{V}(z, \eta) = (z - \cos\eta)^{1/2}\, v(z, \eta). \tag{29}$$

It follows that

$$\hat{T}_\phi = -\mathrm{i}\, n\, \pi\, (z^2 - 1) \oint \left\{ v_+\, \frac{\partial v_+^*}{\partial z} - v_+^*\, \frac{\partial v_+}{\partial z} - v_-\, \frac{\partial v_-^*}{\partial z} + v_-^*\, \frac{\partial v_-}{\partial z} \right\} d\eta. \tag{30}$$

Let

$$v(z, \eta) = \sum_m v_m(z)\, \cos(m\, \eta). \tag{31}$$

It follows that

$$\hat{T}_\phi = -\mathrm{i}\, n\, \pi^2\, (z^2 - 1) \sum_m \left\{ v_{m+}\, \frac{dv_{m+}^*}{dz} - v_{m+}^*\, \frac{dv_{m+}}{dz} - v_{m-}\, \frac{dv_{m-}^*}{dz} + v_{m-}^*\, \frac{dv_{m-}}{dz} \right\}. \tag{32}$$

If

$$v_{m+} = p_m\, P_{m-1/2}^n(z) + q_m\, Q_{m-1/2}^n(z), \tag{33}$$

$$v_{m-} = r_m\, P_{m-1/2}^n(z), \tag{34}$$

then

$$\hat{T}_\phi = -\mathrm{i}\, n\, \pi^2\, (z^2 - 1) \sum_m (q_m\, p_m^* - q_m^*\, p_m) \left[\frac{dP_{m-1/2}^n}{dz}\, Q_{m-1/2}^n - P_{m-1/2}^n\, \frac{dQ_{m-1/2}^n}{dz} \right]. \tag{35}$$

However,
$$\frac{dP^n_{m-1/2}}{dz}\,Q^n_{m-1/2} - P^n_{m-1/2}\,\frac{dQ^n_{m-1/2}}{dz} = \frac{\Gamma(m+1/2+n)}{\Gamma(m+1/2-n)}\,\frac{(-1)^n}{1-z^2}, \tag{36}$$
so
$$\begin{aligned}\hat{T}_\phi &= \mathrm{i}\,n\,\pi^2\,(-1)^n\sum_m (q_m\,p_m^\ast - q_m^\ast\,p_m)\,\frac{\Gamma(m+1/2+n)}{\Gamma(m+1/2-n)}\\ &= -2\,n\,\pi^2\,(-1)^n\sum_m \mathrm{Im}(q_m\,p_m^\ast)\,\frac{\Gamma(m+1/2+n)}{\Gamma(m+1/2-n)}.\end{aligned} \tag{37}$$
Thus, the torque on the plasma is $T_\phi = -\hat{T}_\phi$, giving
$$T_\phi = 2\,n\,\pi^2\,(-1)^n\sum_m \mathrm{Im}(q_m\,p_m^\ast)\,\frac{\Gamma(m+1/2+n)}{\Gamma(m+1/2-n)}. \tag{38}$$

E. Matching to Vacuum Solution

In vacuum the solution vector $\tilde{\mathbf{Y}}^e_k$ has the expansion
$$\hat{V}^e_k = \sum_j p^e_{k,j}\,P_j(\mu,\eta), \tag{39}$$
where
$$P_j(\mu,\eta) = (\cosh\mu - \cos\eta)^{1/2}\,P^n_{m_j-1/2}(\cosh\mu)\,\cos(m_j\,\eta). \tag{40}$$
Likewise, the solution vector $\tilde{\mathbf{Y}}^o_k$ has the expansion
$$\hat{V}^o_k = \sum_j p^o_{k,j}\,P_j(\eta,\mu). \tag{41}$$
Finally, the $\tilde{\mathbf{Y}}^x$ solution vector has the expansion
$$\hat{V}^x = \sum_j \left[p^x_j\,P_j(\eta,\mu) + q^x_j\,Q_j(\eta,\mu)\right], \tag{42}$$
where
$$Q_j(\mu,\eta) = (\cosh\mu - \cos\eta)^{1/2}\,Q^n_{m_j-1/2}(\cosh\mu)\,\cos(m_j\,\eta). \tag{43}$$
The vacuum error-field has the expansion
$$\hat{V}^v = \sum_j q^x_j\,Q_j(\eta,\mu). \tag{44}$$

The general solution vector is

$$\mathbf{Y} = \sum_k (\Psi_k^e\,\mathbf{Y}_k^e + \Psi_k^o\,\mathbf{Y}_k^o) + \mathbf{Y}^x, \tag{45}$$

and has the expansion

$$\hat{V} = \sum_k \left(\Psi_k^e\,\hat{V}_k^e + \Psi_k^o\,\hat{V}_k^o\right) + \hat{V}_x. \tag{46}$$

It follows that

$$\hat{V} = \sum_j (p_j\,P_j + q_j\,Q_j), \tag{47}$$

where

$$p_j = p_j^x + \sum_k (\Psi_k^e\,p_{k,j}^e + \Psi_k^o\,p_{k,j}^o), \tag{48}$$

$$q_j = q_j^x. \tag{49}$$

Thus,

$$\mathrm{Im}(q_j\,p_j^\ast) = \mathrm{Im}\sum_k (\Psi_k^{e\,\ast}\,p_{k,j}^{e\,\ast}\,q_j^x + \Psi_k^{o\,\ast}\,p_{k,j}^{o\,\ast}\,q_j^x). \tag{50}$$

Here, we have assumed that

$$\mathrm{Im}(p_j^{x\,\ast}\,q_j^x) = 0. \tag{51}$$

Thus,

$$T_\phi = 2\,n\,\pi^2\,(-1)^n\,\mathrm{Im}\sum_k\sum_j\,(\Psi_k^{e\,\ast}\,p_{k,j}^{e\,\ast}\,q_j^x + \Psi_k^{o\,\ast}\,p_{k,j}^{o\,\ast}\,q_j^x)\frac{\Gamma(m_j+1/2+n)}{\Gamma(m_j+1/2-n)}. \tag{52}$$

However,

$$T_\phi = -2\,n\,\pi^2\,\mathrm{Im}\sum_k (\Psi_k^{e\,\ast}\chi_k^e + \Psi_k^{o\,\ast}\,\chi_k^o). \tag{53}$$

It follows that

$$\chi_k^e = (-1)^{n+1}\sum_j p_{k,j}^{e\,\ast}\,q_j^x\,\frac{\Gamma(m_j+1/2+n)}{\Gamma(m_j+1/2-n)}, \tag{54}$$

$$\chi_k^o = (-1)^{n+1}\sum_j p_{k,j}^{o\,\ast}\,q_j^x\,\frac{\Gamma(m_j+1/2+n)}{\Gamma(m_j+1/2-n)}. \tag{55}$$

Finally, let

$$\hat{V}^v = \sum_j q_j^v\,(-1)^{n+1}\,\frac{\Gamma(m_j+1/2-n)}{\Gamma(m_j+1/2+n)}\,Q_j(\mu,\eta). \tag{56}$$

It follows that

$$\chi_k^e = \sum_j p_{k,j}^{e\,*}\, q_j^v, \tag{57}$$

$$\chi_k^o = \sum_j p_{k,j}^{o\,*}\, q_j^v. \tag{58}$$