A. Behavior in Vicinity of Rational Surface

Let there be K rational surfaces in the plasma. Suppose that the kth surface has the flux-surface label r_k, and the resonant poloidal mode number m_k, where $q(r_k) = m_k/n$, for $k = 1, K$.

1. General Case

Consider the solution of the outer equations in the vicinity of the kth surface. Let $x = r - r_k$. The most general small-$|x|$ solution of the outer equations can be shown to take the form

$$\psi_j(x) = A^{\pm}_{L\,k}\,|x|^{\nu_{L\,k}}\,(1 + \lambda_{L\,k}\,x + \cdots) + A^{\pm}_{S\,k}\,\mathrm{sgn}(x)\,|x|^{\nu_{S\,k}}\,(1 + \cdots) + A_{C\,k}\,x\,(1 + \cdots), \quad (1)$$

$$Z_j(x) = A^{\pm}_{L\,k}\,|x|^{\nu_{L\,k}}(b_{L\,k} + \gamma_{L\,k}\,x + \cdots) + A^{\pm}_{S\,k}\,\mathrm{sgn}(x)\,|x|^{\nu_{S\,k}}\,(b_{S\,k} + \cdots)$$
$$+ B_{C\,k}\,x\,(1 + \cdots) \quad (2)$$

if $m_j = m_k$, and

$$\psi_j(x) = A^{\pm}_{L\,k}\,|x|^{\nu_{L\,k}}\,(a_{kj} + \cdots) + \bar{\psi}_{kj}\,(1 + \cdots), \quad (3)$$

$$Z_j(x) = A^{\pm}_{L\,k}\,|x|^{\nu_{L\,k}}\,(b_{kj} + \cdots) + \bar{Z}_{kj}\,(1 + \cdots) \quad (4)$$

if $m_j \neq m_k$. Moreover, the superscripts $^+$ and $^-$ correspond to $x > 0$ and $x < 0$, respectively. Here,

$$\nu_{L\,k} = \frac{1}{2} - \sqrt{D_{I\,k}}, \quad (5)$$

$$\nu_{S\,k} = \frac{1}{2} + \sqrt{D_{I\,k}}, \quad (6)$$

$$D_{I\,k} = \frac{1}{4} + L_{0\,k}\,P_{0\,k}, \quad (7)$$

$$L_{0\,k} = -\left(\frac{L_{kk}}{m_k\,s}\right)_{r_k}, \quad (8)$$

$$P_{0\,k} = -\left(\frac{P_{kk}}{m_k\,s}\right)_{r_k}. \quad (9)$$

Furthermore,

$$b_{L\,k} = \frac{\nu_{L\,k}}{L_{0\,k}}, \tag{10}$$

$$b_{S\,k} = \frac{\nu_{S\,k}}{L_{0\,k}}, \tag{11}$$

$$A_{C\,k} = -\frac{1}{r_k\,P_{0\,k}} \sum_{j=1,J}^{m_j\neq m_k} \frac{1}{m_j - m_k} \left(N_{k\,j}\,\bar{Z}_{kj} + P_{kj}\,\bar{\psi}_{kj}\right)_{r_k}, \tag{12}$$

$$B_{C\,k} = -\frac{1}{r_k\,L_{0\,k}} \sum_{j=1,J}^{m_j\neq m_k} \frac{1}{m_j - m_k} \left(L_{kj}\,\bar{Z}_{kj} + M_{kj}\,\bar{\psi}_{kj}\right)_{r_k} + \frac{A_{C\,k}}{L_{0\,k}}, \tag{13}$$

$$\begin{aligned} \lambda_{L\,k} = {}& \frac{1}{2\,r_k}\left[\frac{P_{1\,k}\,L_{0\,k}}{\nu_{L\,k}} + T_{1\,k} + \nu_{L\,k}\left(\frac{L_{1\,k}}{L_{0\,k}} - 2\right)\right]_{r_k} \\ & - \frac{1}{(m_k\,s)_{r_k}}\,\frac{1}{r_k\,\nu_{L\,k}} \sum_{j=1,J}^{m_j\neq m_k} \frac{1}{m_j - m_k}\left(P_{kj}\,L_{kj} - M_{kj}\,N_{kj}\right)_{r_k}, \end{aligned} \tag{14}$$

$$\begin{aligned} \gamma_{L\,k} = {}& \frac{1}{2\,r_k}\left[(1+\nu_{L\,k})\left(\frac{P_{1\,k}}{\nu_{L\,k}} + \frac{T_{1\,k}}{L_{0\,k}} - \frac{\nu_{L\,k}}{L_{0\,k}}\right) + P_{0\,k}\left(\frac{L_{1\,k}}{L_{0\,k}} - 1\right)\right]_{r_k} \\ & - \frac{1}{(m_k\,s)_{r_k}}\,\frac{1}{r_k\,L_{0\,k}} \sum_{j=1,J}^{m_j\neq m_k} \frac{1}{m_j - m_k}\left(P_{kj}\,L_{kj} - M_{kj}\,N_{kj}\right)_{r_k}, \end{aligned} \tag{15}$$

$$a_{kj} = \frac{1}{(m_k\,s)_{r_k}}\left(\frac{N_{kj}}{\nu_{L\,k}} - \frac{L_{kj}}{L_{0\,k}}\right)_{r_k}, \tag{16}$$

$$b_{kj} = \frac{1}{(m_k\,s)_{r_k}}\left(\frac{M_{kj}}{L_{0\,k}} - \frac{P_{kj}}{\nu_{L\,k}}\right)_{r_k}, \tag{17}$$

and

$$L_{1\,k} = \lim_{x\to 0}\left(\frac{L_{kk}}{m_k - n\,q}\right) - \frac{r_k\,L_{0\,k}}{x}, \tag{18}$$

$$P_{1\,k} = \lim_{x\to 0}\left(\frac{P_{kk}}{m_k - n\,q}\right) - \frac{r_k\,P_{0\,k}}{x}, \tag{19}$$

$$T_{1\,k} = \lim_{x\to 0}\left(\frac{-n\,q\,s}{m_k - n\,q}\right) - \frac{r_k}{x}. \tag{20}$$

The parameters $A_{S\,k}$ and $A_{L\,k}$ are identified from the numerical solution of the outer equations in the vicinity of the rational surface by taking the limits

$$\bar{\psi}_{kj} = \psi_j(r_k+\delta) - a_{kj}\,\psi_k(r_k+\delta), \tag{21}$$

$$\bar{Z}_{kj} = Z_j(r_k+\delta) - b_{kj}\,\psi_k(r_k+\delta), \tag{22}$$

$$\begin{aligned} A_{S\,k}^{\pm} = & \pm\,\frac{Z_k(r_k\pm|\delta|) - b_{L\,k}\,\psi_k(r_k\pm|\delta|)}{(b_{S\,k}-b_{L\,k})\,|\delta|^{\nu_{S\,k}}} \\ & - \frac{[(B_{C\,k}-b_{L\,k}\,A_{C\,k}) + (\gamma_{L\,k}-b_{L\,k}\,\lambda_{L\,k})\,\psi_k(r_k\pm|\delta|)]\,|\delta|}{(b_{S\,k}-b_{L\,k})\,|\delta|^{\nu_{S\,k}}}, \end{aligned} \tag{23}$$

$$A_{L\,k}^{\pm} = \frac{\psi_k(r_k\pm|\delta|) \mp A_{S\,k}^{\pm}|\delta|^{\nu_{S\,k}} \mp A_{C\,k}\,|\delta|}{(1\pm|\delta|\,\lambda_{L\,k})\,|\delta|^{\nu_{L\,k}}} \tag{24}$$

as $|\delta|\rightarrow 0$.

2. Zero Pressure Limit

In the limit $D_{I\,k}\rightarrow 1/4$, the indices $\nu_{L\,k}$ and $\nu_{S\,k}$ approach 0 and 1, respectively. Hence, $|x|^{\nu_{L\,k}}\rightarrow 1+\nu_{L\,k}\,\ln|x|$, and $|x|^{\nu_{S\,k}}\rightarrow |x|\,(1-\nu_{L\,k}\,\ln|x|)$. In this situation, Eqs. (3) and (4) become

$$\psi_j(x) = A_{L\,k}^{\pm}\,(\hat{a}_{kj}\,\ln|x| + \cdots) + \bar{\psi}'_{kj}\,(1+\cdots), \tag{25}$$

$$Z_j(x) = A_{L\,k}^{\pm}\,(\hat{b}_{kj}\,\ln|x| + \cdots) + \bar{Z}'_{kj}\,(1+\cdots), \tag{26}$$

where

$$\hat{a}_{kj} = \nu_{L\,k}\,a_{kj} \rightarrow \frac{\nu_{L\,k}}{(m_k\,s)_{r_k}}\left(\hat{N}_{kj} - \frac{L_{kj}}{L_{0\,k}}\right)_{r_k}, \tag{27}$$

$$\hat{b}_{kj} = \nu_{L\,k}\,b_{kj} \rightarrow \frac{\nu_{L\,k}}{(m_k\,s)_{r_k}}\left(\frac{M_{kj}}{L_{0\,k}} - \hat{P}_{jk}\right)_{r_k}, \tag{28}$$

and

$$\bar{\psi}'_{kj} = \bar{\psi}_{kj} + A_{L\,k}^{\pm}\,a_{kj}, \tag{29}$$

$$\bar{Z}'_{kj} = \bar{Z}_{kj} + A_{L\,k}^{\pm}\,b_{kj}. \tag{30}$$

Here, the ratios

$$\hat{N}_{kj} = \frac{N_{kj}}{\nu_{L\,k}}, \tag{31}$$

$$\hat{P}_{kj} = \frac{P_{kj}}{\nu_{L\,k}}, \tag{32}$$

remain finite as $\nu_{L\,k} \to 0$. It follows from Eqs. (12) and (13) that

$$A_{C\,k} \to \frac{L_{0\,k}}{r_k} \sum_{j=1,J}^{m_j \neq m_k} \frac{1}{m_j - m_k} \left(\hat{N}_{kj}\, \bar{Z}'_{kj} + \hat{P}_{kj}\, \bar{\psi}'_{kj} \right)_{r_k}$$

$$+ \frac{A^{\pm}_{L\,k}}{(m_k\, s)_{r_k}} \frac{1}{r_k} \sum_{j=1,J}^{m_j \neq m_k} \frac{1}{m_j - m_k} \left(\hat{P}_{kj}\, L_{kj} - M_{kj}\, \hat{N}_{kj} \right)_{r_k}, \tag{33}$$

$$B_{C\,k} \to -\frac{1}{r_k\, L_{0\,k}} \sum_{j=1,J}^{m_j \neq m_k} \frac{1}{m_j - m_k} \left(L_{kj}\, \bar{Z}'_{kj} + M_{kj}\, \bar{\psi}'_{kj} \right)_{r_k}$$

$$+ \frac{1}{r_k} \sum_{j=1,J}^{m_j \neq m_k} \frac{1}{m_j - m_k} \left(\hat{N}_{kj}\, \bar{Z}'_{kj} + \hat{P}_{kj}\, \bar{\psi}'_{kj} \right)_{r_k}, \tag{34}$$

and from (14) and (15) that

$$\lambda_{L\,k} \to \frac{P_{1\,k}\, L_{0\,k}}{2\, r_k\, \nu_{L\,k}} - \frac{1}{(m_k\, s)_{r_k}} \frac{1}{r_k} \sum_{j=1,J}^{m_j \neq m_k} \frac{1}{m_j - m_k} \left(\hat{P}_{kj}\, L_{kj} - M_{kj}\, \hat{N}_{kj} \right)_{r_k} + \frac{T_{1\,k}}{2\, r_k}, \tag{35}$$

$$\gamma_{L\,k} \to \frac{P_{1\,k}}{2\, r_k\, \nu_{L\,k}} + \frac{1}{2\, r_k} \left(P_{1\,k} + \frac{T_{1\,k}}{L_{0\,k}} \right). \tag{36}$$

Equations (1) and (2) yield

$$\psi_j(x) \to A^{\pm}_{L\,k} \left(1 + \nu_{L\,k}\, \ln|x| + \hat{\lambda}'_{L\,k}\, x\, \ln|x| + \cdots \right) + A_{S\,k}\, x\, (1 + \cdots)$$

$$+ x \left(A^{\pm}_{L\,k}\, \lambda_{L\,k} + A_{C\,k} \right), \tag{37}$$

$$Z_j(x) \to A^{\pm}_{L\,k} \left(\hat{b}_{L\,k} + \hat{\gamma}'_{L\,k}\, x\, \ln|x| + \cdots \right) + A^{\pm}_{S\,k}\, x \left(\hat{b}_{S\,k} + \cdots \right)$$

$$+ x \left(A^{\pm}_{L\,k}\, \gamma_{L\,k} + B_{C\,k} \right), \tag{38}$$

where

$$\hat{\lambda}'_{L\,k} = \nu_{L\,k}\, \lambda_{L\,k} \to \frac{P_{1\,k}\, L_{0\,k}}{2\, r_k}, \tag{39}$$

$$\hat{\gamma}'_{L\,k} = \nu_{L\,k}\, \gamma_{L\,k} \to \frac{P_{1\,k}}{2\, r_k}, \tag{40}$$

$$\hat{b}_{L\,k} = \frac{\nu_{L\,k}}{L_{0\,k}}, \tag{41}$$

$$\hat{b}_{S\,k} = \frac{1}{L_{0\,k}}. \tag{42}$$

Now,

$$A^{\pm}_{L\,k}\,\lambda_{L\,k}+A_{C\,k}=A^{\pm}_{L\,k}\left(\frac{P_{1\,k}\,L_{0\,k}}{2\,r_k\,\nu_{L\,k}}+\frac{T_{1\,k}}{2\,r_k}\right)$$

$$+\frac{L_{0\,k}}{r_k}\sum_{j=1,J}^{m_j\neq m_k}\frac{1}{m_j-m_k}\left(\hat{N}_{kj}\,\bar{Z}'_{kj}+\hat{P}_{kj}\,\bar{\psi}'_{kj}\right)_{r_k}, \tag{43}$$

$$L_{0\,k}\,(A^{\pm}_{L\,k}\,\gamma_{L\,k}+B_{C\,k})=(A^{\pm}_{L\,k}\,\lambda_{L\,k}+A_{C\,k})+A^{\pm}_{L\,k}\,\hat{\lambda}'_{L\,k}$$

$$-\frac{1}{r_k}\sum_{j=1,J}^{m_j\neq m_k}\frac{1}{m_j-m_k}\left(L_{kj}\,\bar{Z}'_{kj}+M_{kj}\,\bar{\psi}'_{kj}\right)_{r_k}. \tag{44}$$

Let

$$A^{\pm}_{S\,k}+L_{0\,k}\left(A^{\pm}_{L\,k}\,\gamma_{L\,k}+B_{C\,k}\right)=\hat{A}^{\pm}_{S\,k}. \tag{45}$$

It follows that

$$\psi_j(x)\rightarrow A^{\pm}_{L\,k}\left[1+\nu_{L\,k}\,\ln|x|+\hat{\lambda}'_{L\,k}\,x\,(\ln|x|-x)+\cdots\right]+\hat{A}^{\pm}_{S\,k}\,x\,(1+\cdots)$$

$$+\hat{A}_{C\,k}\,x\,(1+\cdots), \tag{46}$$

$$Z_j(x)\rightarrow A^{\pm}_{L\,k}\left(\hat{b}_{L\,k}+\hat{\gamma}'_{L\,k}\,x\,\ln|x|+\cdots\right)+\hat{A}^{\pm}_{S\,k}\,x\left(\hat{b}_{S\,k}+\cdots\right), \tag{47}$$

where

$$\hat{A}_{C\,k}=\frac{1}{r_k}\sum_{j=1,J}^{m_j\neq m_k}\frac{1}{m_j-m_k}\left(L_{kj}\,\bar{Z}'_{kj}+M_{kj}\,\bar{\psi}'_{kj}\right)_{r_k}. \tag{48}$$

Expressions (25), (26), (46), and (47) now contain no terms that blow up in the limit $\nu_{L\,k}\rightarrow 0$. Thus, generally speaking, we would expect the asymptotic matching process to yield values of $A^{\pm}_{L\,k}$ and $\hat{A}^{\pm}_{S\,k}$ that remain finite in this limit. It, therefore, follows from Eq. (43)–(45) that

$$A^{\pm}_{S\,k}\rightarrow -A^{\pm}_{L\,k}\,\frac{P_{1\,k}\,L_{0\,k}}{2\,r_k\,\nu_{L\,k}}. \tag{49}$$

in the limit $\nu_{L\,k}\rightarrow 0$.