Complex-Order and Fractional Derivatives: A First Exploration II

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1 Introduction

2 The Field of orders

2.1 more identities and operations

2.2 Taylor series memory

3 More to the Imagination

3.1 The imaginary derivative order polar form

from what we know about D^z we can say that
D^1 and D^{-1} represents a half turn on the $D(i)$ plane
D^i and D^{-i} represents a quarter turn on the $D(i)$ plane
So if we want to get the generalized form of this, we need to use the polar form instead of the Cartesian form, so

$$D^i f(x) = D^{e^{\frac{i\pi}{2}}} f(x) \qquad\qquad D^{-1} f(x) = D^{e^{i\pi}} f(x)$$

and generally speaking, for the unit circle around D^0(the function itself)

$$D^z f(x) = D^{e^{i\theta}} f(x)$$

And to make this formula for the whole complex plane, we write it like this

$$D^z f(x) = D^{re^{i\theta}} f(x)$$

where r is the length from D^0 to the wanted function

3.2 exploring more about Imaginary derivatives

3.3 complex functions with D^z

3.4 Complex derivative of a complex function is complex

3.5 Infinite series and complex derivatives

4 Multi variable D^z

5 functional order $D^{f(\alpha)}$

6 Yes, the whole space is a matrix system

As we have explored in the first paper, the possibility of matrix order derivative with this simple formula

$$D^A = \frac{\Gamma(n+1)}{\Gamma(n-A+1)}x^{n-A} = \frac{\Gamma(n+1)}{\Gamma(n-A+1)}x^n e^{-A\ln(x)} = \frac{\Gamma(n+1)}{\Gamma(n-A+1)}x^n \sum_{k=0}^{\infty} \frac{(-1)^k \ln(x)^k}{k!} A^k$$

But let's refine it a little to make it fully clear by changing all the scalers to matrices

$$D^A = \frac{\Gamma((n+1)I)}{\Gamma((n+1)I-A)}x^{nI-A} = \frac{\Gamma((n+1)I)}{\Gamma((n+1)I-A)}x^{nI} e^{-A\ln(x)}$$

$$= \frac{\Gamma((n+1)I)}{\Gamma((n+1)I-A)}x^{nI} \sum_{k=0}^{\infty} \frac{(-1)^k \ln(x)^k}{k!} A^k$$

For now, to define $\Gamma(A)$ we are going to use eigenvalues, although Cuchy Integrals are better, but for the sake of intuition and understanding, we are going to discuss them in later sections

This means that the formula only works for **square matrices** for now

We can see a simple example with the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; the rotation matrix

for a function $f(x) = x^2$ we are going to take the A-th derivative like this

$$D^A f(x) = \frac{\Gamma(3)}{\Gamma(\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})} x^2 \sum_{n=0}^{\infty} \frac{(-1)^n \ln(x)^n}{n!} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n$$

The matrix A returns back at the $4n$-th powers, and the Gamma expression is $\Gamma(\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix})$, but to continue, we need to find the eigenvalues of this matrix

$$\det\left(\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\left(\begin{pmatrix} 3-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix}\right) \qquad (3-\lambda)(3-\lambda)+1 = 0$$

$$(3-\lambda)^2+1=0 \implies 3-\lambda=\pm i \qquad \lambda=3-i,3+i$$

This equation doesn't have any real roots but only complex roots that are $-3-i$ and $-3+i$
We are going to find the eigenvectors for all roots

$$((3I-A)-\lambda_1 I)v_1=0, \begin{pmatrix} 3-(3-i) & 1 \\ -1 & 3-(3-i) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

then $ix+y=0$ so $y=-ix$, let $x=1$, $y=-i$, the first vector $= v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$
doing the same for the other root, we get the second vector $v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$
We are going to let the Model matrix $P=[v_1|v_2]=\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$ and the diagonal matrix $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} (3-i) & 0 \\ 0 & (3+i) \end{pmatrix}$ so we can calculate $f(3I-A)=Pf(\Lambda)P^{-1}$
Since the original gamma function is in the denominator, we can calculate $\Gamma(3I-A)^{-1}$ to get a direct result. We can so $\Gamma(3I-A)=P\Gamma(\Lambda)^{-1}P^{-1}=$
$\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \frac{1}{\Gamma(3-i)} & 0 \\ 0 & \frac{1}{\Gamma(3+i)} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1} =$

6.1 exploring more about Matrix derivatives

6.2 Term-wise matrix order

7 To the third dimension

7.1 Modeler differentiation

7.2 Set order derivatives

7.3 group order derivatives

7.4 Vector order derivatives

8 convergence and divergence

There are some functions

9 variable order derivatives

since D^α operator extends to linear input, which is α, we can go further and extend it to different inputs as functions

9.1 function order derivatives

The simple function derivative is a function of another variable rather than the differentiated function variable, and the fundamental function is the linear function of alpha $D^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}$, of course, this function is what defines what type of differentiation we are doing, in other words in which direction we are going, as the main linear function α is for derivatives, when multiplied by negatives gives us $-\alpha$ which is for integrals, and there exists αi which is imaginary differentiation, then we have $\alpha + \beta i$ for complex differentiation which is a simple multi variable linear function.
So we know functional orders exist and change in the nature of the function depending on its type, and we can write their general formula like this

$$D^{g(\alpha)} x^n = \frac{\Gamma(n+1)}{\Gamma(n - g(\alpha) + 1)} x^{n-g(\alpha)}$$

We can try a simple example on this formula

$$g(\alpha) = \alpha^2, f(x) = x^2 \qquad D^{g(\alpha)} f(x) = \frac{\Gamma(2+1)}{\Gamma(2 - \alpha^2 + 1)} x^{2-\alpha^2} = \frac{2}{\Gamma(3 - \alpha^2)} x^{2-\alpha^2}$$

if we want to see when this function hits gamma pole we can put it in test

$$3 - \alpha^2 \leq 0 \qquad a^2 \geq 3 \qquad \sqrt{3} \geq a \geq -\sqrt{3} \quad \{3 - a^2 \in \mathbb{Z}\}$$

so this function has gamma poles when $\alpha \in \{\pm\sqrt{3}, \pm 2, \pm\sqrt{5}, \pm\sqrt{6}....\}$, other than that, there will be no gamma pole, so the expression is defined like this

$$D^{g(x)} f(x) = \frac{2}{\Gamma(\alpha^2 - 3)} x^{2-\alpha^2} \quad \{\alpha \in [-\sqrt{3}, \sqrt{3}]\}$$

We can see that in general, the function isn't defined only when

$$n + 1 - g(\alpha) \leq 0 \qquad n + 1 \leq g(\alpha) \quad \{n + 1 - g(\alpha) \in \mathbb{Z}\}$$

another example that we can test on

$$g(\alpha) = \sin(\alpha), f(\alpha) = x^2 \quad D^{g(\alpha)} f(\alpha) = \frac{2}{\Gamma(3 - \sin(\alpha))} x^{2-\sin(\alpha)}$$

This function is actually defined for all $\alpha \in \mathbb{R}$ as the range of this function is between $[-1, 1]$, the expression $D^{g(\alpha)} f(\alpha)$ ranges between the first derivative and the second derivative

9.2 self variable order derivatives

We can define a simple self variable order derivative as

$$D^x x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-x)} x^{n-x}$$

However, we need to refine this expression further.
first we write x^{n-x} as $x^n e^{-x\ln(x)}$ for simpler calculations.
now we can use the series expansion for $e^{-x\ln(x)} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k (\ln(x))^k}{k!}$ then we substitute it back

$$D^x x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-x)} x^n \sum_{k=0}^{\infty} \frac{(-1)^k x^k (\ln(x))^k}{k!}$$

Analyzing this function, we can see that the function reaches it's peaks when $n > x$, as then the Gamma function would be for positives, and the power would be positive, to find where it hits zer,o we can solve for the equation

$$\frac{\Gamma(n+1)}{\Gamma(n+1-x)} x^{n-x} = 0$$

But looking at the function closely, there can only be one reason for it to happen. for any power of x it can never reach zero as there is no number that has a changing power will be zero, and when the numerator is zero of gamma poles, the function itself is non non-differentiable in the first place, leaving only the denominator, it can only reach zero when the gamma function hits zero, so we solve for the equation

$$n+1-x \leq 0 \qquad n+1 \leq x \quad (n+1-x \in \mathbb{Z})$$

only when x is bigger than or equal to $n+1$ and the result is an integer we get a gamma pole which leads to an infinity in the denominator leading to a zero , these are the zeros of the derivative

9.3 Nested differentiation

10 Integrals May I present

10.1 Where is the constant of integration?

As we saw from this whole paper, we can use D^z to differentiate and integrate Real, imaginary, or matrix, yet something seems to be missing.
The constant of integration, something so fundamental in calculus, yet seems to be missing
But here in fractional calculus, things are different; here we have a lot of "Differentiating" forward or backward, 2D or 3D, and a lot of directions that make it hard even to put it somewhere But at first, we need to understand why, not in standard terms, but in this research term
The constant of Integration is something that arises when integrating, that is, because under differentiation, we lose constants and variables according to the order
That is because of the definition of D^z, which transforms functions from one state to another.

take, for example, $f(x) = x^5 + 14$, from a prescriptive of stranded calculus, when we differentiate, constants are gone because they don't have any rate of change, the answer is simply

$$D^1 f(x) = 5x^4$$

But as we discussed, the rate of change isn't a core in the derivative but a side effect. To see the full picture, we need to use the whole Gamma formula with the knowledge that $C = Cx^0$

$$D^1 f(x) = \frac{\Gamma(6)}{\Gamma(5)} x^{5-1} + 14\frac{\Gamma(1)}{\Gamma(0)} x^{0-1} = 5x^4 + \frac{14}{\Gamma(0)} x^{-1}$$

We can see that the denominator has a Gamma pole in it, which leads the whole term to be 0, taking the constant with it
In other words, differentiation, as we discussed, loses memory; the more it differentiates, the more it's gone
Integration is the exact opposite; it retains memory, and we can see that too from the standard calculus approach

$$D^{-1}(x^2) = \frac{x^3}{3} + C$$

We see that here the constant of integration is dependent on logic, the simple logic is that if we differentiate, we lose constants, then integration, the opposite of it must return them
Let's see what happens from the fractional derivatives point of view. First, we assume that there exists a pole $C\frac{\Gamma(1)}{\Gamma(0)}x^{-1}$ that went to zero, which is in the range of possibility

$$D^{-1}(x^2) = \frac{\Gamma(3)}{\Gamma(4)} x^{2-(-1)} + C\frac{\Gamma(1)}{\Gamma(0)}\frac{\Gamma(0)}{\Gamma(1)} x^{-1-(-1)}$$

We can see that both fractions cancel out, leaving only Cx^0, which is C

$$D^{-1}(x^2) = \frac{x^3}{3} + C$$

We can see that this works mathematically here, no matter how many constants may be there, all of them will simply add up under the term x^0
That is, of course, with one problem, there are infinite Gamma poles; whenever the denominator hits a non-positive integer, it hits a pole, we can't really make sure if the gamma pole was at the second derivative, or the tenth, the memory of the functions is wide, and for that we need to define two categories of derivatives

Pure derivatives: these are derivatives that don't hold any memory.
Memory derivatives: these are functions that hold memory and can be separated into three:

1. **Memory stated functions:** functions that have a stated amount of change in order of D^z
2. **Always existed functions:** functions that have an infinite series of memory
3. **Application derivatives:** where the constant exists if and only if it satisfies the required equation or does not make any change to it (Although doubtful since any change in the function will lead to a change in the derivative)

11 Combinations

11.1 the fractional derivative of $x!$

11.2 the fractional derivative of permutations and combinations

12 Differential equations

12.1 Bessel functions

12.2 Hermite polynomials

12.3 Legendre polynomials

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