Cyclic derivatives and functions

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Abstract

This paper presents an independent exploration of cyclic derivatives based on the fractional and complex-order derivatives paper made by the same Author.

Note to Readers: This represents independent rediscovery of classical fractional calculus concepts. I (The Author) present this work as a pedagogical exercise in mathematical exploration rather than novel research.

Background

The Main formula that works for all derivatives and can be used in the Maclaurin series

$$D^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

The more important formulas that are built on this one

$$D^\alpha e^{ax} = a^\alpha e^{ax} \qquad e^{ix} = \cos(x) - i\sin(x)$$

$$e^x = \cosh(x) + \sinh(x) \qquad e^{-x} = \cosh(x) - \sinh(x)$$

$$D^\alpha \sin(x) = \sin(\frac{\alpha\pi}{2} + x) \qquad D^\alpha \cos(x) = \cos(\frac{\alpha\pi}{2} + x)$$

$$D^\alpha \sinh(x) = \frac{e^x - (-1)^\alpha e^{-x}}{2} \qquad D^\alpha \cosh(x) = \frac{e^x + (-1)^\alpha e^{-x}}{2}$$

1 Introduction

from what we have seen in the previous fractional and complex-order derivatives paper, cyclic derivatives are such a big area that deserves its own paper,

2 The foundation

From the fractional and complex-order derivatives paper, we know that

$$\textit{when } a^n = 1, D^n(e^{ax}) = e^{ax} \textit{ with } 2 \times n \textit{ cyclic order}$$

And we can call that the theorem of cyclic derivatives

Theorem 1 *$f(x)$ is a cyclic derivative when $a^n = 1, D^n(e^{ax}) = e^{ax}$ with $2 \times n$ cyclic order*

And in the same paper, we generalized this to hold for any $k \equiv 0 \pmod{n}$

Theorem 2 *$f(x)$ is a lesser cyclic derivative when $a^n = 1, D^n(e^{ax}) = e^{ax}$ with $2 \times k$ cyclic order, where $k \equiv 0 \pmod{n}$*

With a hypothesis that I wish to prove in this paper, that

Hypothesis 1 *For every function cyclic derivatives that can be written in the form e^{ax} where $a^n = 1$ and satisfies the condition $2^n \in \mathbb{Z}^+$, there exists an algebraic perimetric form*

We can already see this for hyperbolic functions, where they can be written in the form $x^2 - y^2 = 1$, and trigonometric functions We also proved some equations

$$D^i e^{-x} = e^{-(x+\pi)} = \cosh(x + \pi) - \sinh(x + \pi)$$

$$D^i \sinh(x) = \frac{e^x - e^{-(x+\pi)}}{2} \qquad D^i \cosh(x) = \frac{e^x + e^{-(x+\pi)}}{2}$$

$$D^i e^{ix} = e^{\frac{-\pi}{2}} \cos(x) + ie^{\frac{-\pi}{2}} \sin(x)$$

$$D^i \sin(x) = \sin(\frac{i\pi}{2} + x) \qquad D^i \cos(x) = \cos(\frac{i\pi}{2} + x)$$

and we also proved that for $\sin(x)$ and $\cos(x)$

3 More about the complex derivatives and known families

from what we know about $\sinh(x)$ and $\cosh(x)$ we can write their formulas in an other way

$$D^\alpha \sinh(x) = \frac{e^x - (-1)^\alpha e^{-x}}{2} = \frac{e^x - (e^{i\pi})^\alpha e^{-x}}{2} = \frac{e^x - (e^{i\pi\alpha})e^{-x}}{2}$$
$$= \frac{e^x - e^{i\pi\alpha - x}}{2}$$

The same goes for $\cosh(x)$

$$D^\alpha \sinh(x) = \frac{e^x - e^{i\pi\alpha - x}}{2} \qquad D^\alpha \cosh(x) = \frac{e^x + e^{i\pi\alpha - x}}{2}$$

We can see that we can't express them as simple forms, since the real value changes differently from the complex value. If we try to apply the derivative operator to e^x with the definition of $1 = e^{2i\pi}$, we get

$$D^\alpha e^x = 1^\alpha e^x = e^{2i0\alpha} e^x = e^x$$

We used that because it's the principal value
For trigonometric functions, we know that D^i represents a half turn before it changes to its integral from the real plane, but there is something to clarify

$$\mathrm{Im}(D^i e^{ix}) = e^{\frac{-\pi}{2}} \sin(x)$$

As we can see, the first i-th derivative acts as a scaler that scales $\sin(x)$ by real value $e^{-\frac{\pi}{2}}$
However, this isn't equal to the $D^i \sin(x)$ as the definition changes from scaling to rotating, like this

$$D^i \sin(x) = \sin(\frac{i\pi}{2} + x) =$$

But since we have proven the multiplication law works in the framework, we can make sure that both are somewhat equal

$${}^iD^i e^{ix} = i^{i\times i} e^{ix} = (e^{\frac{i\pi}{2}})^{-1} e^{ix} = e^{\frac{-i\pi}{2}} e^{ix} = e^{ix-\frac{i\pi}{2}}$$

$$e^{i(x-\frac{\pi}{2})} = \cos(x - \frac{\pi}{2}) + i\sin(x - \frac{\pi}{2}) = \sin(x) - i\cos(x) = D^{-1}e^{ix} = \int e^{ix}$$

and for the sin we proved in the "Complex-order and fractional derivatives: first exploration" paper that the index law works on it, and thus the multiplication law either from here or from the series expansion, so that we can say

$${}^iD^i \sin(x) = \sin(\frac{i \times i\pi}{2} + x) = \sin(\frac{-\pi}{2} + x) = -\cos(x)$$

Thus, both of them work fine, just different prescriptive
We shall call the e^{ix} the complex perspective since it's all about the imaginary unit, and $\sin(x)$ the real perspective, even if there is i in it

3.1 The complex prescriptive

since D^i represents a whole rotation on the $D(i)$ plane, we can get more angles that could help us understand more what happens to the function to "integrate it"
first step is we are going to transform from i to $e^{\frac{i\pi}{2}}$ so we can deal with rotation with radians in circles

$$D^i e^{ax} = D^{e^{\frac{i\pi}{2}}} e^{ax} \qquad D^{-1} e^{ax} = D^{e^{i\pi}} e^{ax}$$

$$D^{e^{i\theta}} e^{ax} = a^{e^{i\theta}} e^{ax} \qquad D^{e^{i\theta}} e^{ix} = i^{e^{i\theta}} e^{ix} = e^{\frac{i\pi e^{i\theta}}{2}} e^{ix} = e^{i(\frac{\pi e^{i\theta}}{2}+x)}$$

$$= \cos(\frac{\pi e^{i\theta}}{2} + x) + i\sin(\frac{\pi e^{i\theta}}{2} + x)$$

So now we can know what happens at the third of rotation or the third root of unity, which is equal to $\frac{\pi}{3}$ in radians, we get

$$D^{e^{\frac{i\pi}{3}}} e^{ix} = e^{i(\frac{\pi e^{\frac{i\pi}{3}}}{2}+x)} = \cos(\frac{\pi e^{\frac{i\pi}{3}}}{2} + x) + i\sin(\frac{\pi e^{\frac{i\pi}{3}}}{2} + x)$$

which after calculating $e^{\frac{i\pi}{3}}$ to be $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ we can then multiply it by $\frac{\pi}{2}$ to get $\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4}$, then we plug it

$$e^{i(\frac{\pi}{4}+i\frac{\pi\sqrt{3}}{4}+x)} = e^{i\frac{\pi}{4}+i^2\frac{\pi\sqrt{3}}{4}+ix} = e^{i(x+\frac{\pi}{4})}e^{-\frac{\pi\sqrt{3}}{4}} = e^{-\frac{\pi\sqrt{3}}{4}}\cos(x+\frac{\pi}{4})+ie^{-\frac{\pi\sqrt{3}}{4}}\sin(x+\frac{\pi}{4})$$

We can see that it scales by a factor of $e^{\frac{-\pi\sqrt{3}}{4}}$ and rotate with a factor of $\frac{\pi}{4}$
let's do the same for two-thirds, the value for $e^{\frac{2i\pi}{3}}$ to be $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$ we can then multiply it again by $\frac{\pi}{2}$ to get $-\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4}$
Plugging it again, we get

$$e^{i(-\frac{\pi}{4}+i\frac{\pi\sqrt{3}}{4}+x)} = e^{-i\frac{\pi}{4}+i^2\frac{\pi\sqrt{3}}{4}+ix} = e^{i(x-\frac{\pi}{4})}e^{-\frac{\pi\sqrt{3}}{4}} = e^{-\frac{\pi\sqrt{3}}{4}}\cos(x-\frac{\pi}{4})+ie^{-\frac{\pi\sqrt{3}}{4}}\sin(x-\frac{\pi}{4})$$

At two-thirds, it rotates with the same value but rotates backwards
Now we have a little information about what happens in the process of integrating such functions
at the first third of the way, it rotates by $\frac{\pi}{4}$ and scales by $e^{-\frac{\pi\sqrt{3}}{4}}$
For Halfway, it doesn't rotate but scales with a factor of $e^{-\frac{\pi}{2}}$
for two-thirds it rotates by $\frac{\pi}{4}$ and scales by $e^{-\frac{\pi\sqrt{3}}{4}}$
This may seem weird at the beginning until we notice that we aren't starting from order 1 or D^1, we are starting from the zero point D^0 or the function itself, so the one-third and two-thirds don't cancel out on rotation, but they rotate to two different directions
The one-third rotates to D^1 and the two-thirds rotate to D^{-1}, while the middle point D^i doesn't rotate but scales because it's not a real derivative or real integral
We can even notice that in the first third we have $\cos(x + \frac{\pi}{4})$ and $\sin(x + \frac{\pi}{4})$, which are both pure half derivatives

$$D^{\frac{1}{2}}\sin(x) = \sin(\frac{\frac{1}{2}\pi}{2}+x) = \sin(\frac{\pi}{4}+x) \qquad D^{\frac{1}{2}}\cos(x) = \cos(\frac{\frac{1}{2}\pi}{2}+x) = \cos(\frac{\pi}{4}+x)$$

and the same happens for the half-integer being rotated by $\frac{\pi}{4}$

3.2 The Real prescriptive

4 exploration into another cyclic derivatives

4.1 the third order cyclic derivatives

From the theorem, we can find the third cyclic derivative to be from the equation $a^3 = 1$, the solutions are going to be denoted by $1, \omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ since one will result in e^x and is fully expected to be here because of Theorem 2, we are going to use ω

$$D^1 e^{\omega x} = \omega e^{\omega x} \qquad D^2 e^{\omega x} = \omega^2 e^{\omega x} \qquad D^3 e^{\omega x} = e^{\omega x}$$

We can call this function the third-order cyclic derivative, which comes between hyperbolic and trigonometric functions
we will name them $\sinh_3$,$\cosh_3$ and $\sinh_3 \mathrm{II}$ We can define them like this

$$D^\alpha \sinh_3(x) = \sinh_3 \mathrm{II}(x) \qquad \alpha \equiv 0 \pmod 3 \qquad D^\alpha \sinh_3 \mathrm{II}(x) = \cosh_3(x) \qquad \alpha \equiv 1 \pmod 3$$

$$D^\alpha \cosh_3(x) = \sinh_3(x) \qquad \alpha \equiv 2 \pmod 3$$

But this isn't the only way to define them, we can also define them with a series
First we find the Maclaurin series for $e^{\omega x}$

$$e^{\omega x} = e^0 + \omega e^0 x + \frac{\omega^2 e^0 x^2}{2!} + \frac{e^0 x^3}{3!} + ... = 1 + \omega x + \frac{\omega^2 x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{\omega^n x^n}{n!}$$

From this, we can divide them into three sums

$$(1 + \frac{x^3}{3!} + \frac{x^6}{6!} + ...) + \omega(x + \frac{x^4}{4!} + \frac{x^7}{7!} + ...) + \omega^2(x^2 + \frac{x^5}{5!} + \frac{x^8}{8!} + ...)$$

$$= \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \omega \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \omega^2 \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

We can see that the first sum can be differentiated 3 times before going back to the first state, which is also for all the other sums, but since sin and sinh all have x^{an+1}, we are going to make the first function to be the second sum, to keep the naming consistnt nothing more
So now we can define them to be

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \qquad \sinh_3 \mathrm{II}(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!} \qquad \cosh_3 = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$$

We can now define an equation that looks and acts like the Euler equation

$$e^{\omega x} = \cosh_3(x) + \omega \sinh_3(x) + \omega^2 \sinh_3 \mathrm{II}(x)$$

We can now try to find one for ω^2

$$e^{\omega^2 x} = \sum_{n=0}^{\infty} \frac{(\omega^2)^n x^n}{n!}$$

at $3n$ we get $\omega^{6n} = 1$ so it's $\cosh_3(x)$, at at $3n+1$ we get $\omega^{6n+2} = \omega^2$ so it's $\sinh_3(x)$ and at $3n$ we get $\omega^{6n+4} = \omega$ so it's $\sinh_3 \mathrm{II}(x)$

$$e^{\omega^2 x} = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \omega^2 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \omega \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

$$e^{\omega^2 x} = \cosh_3(x) + \omega^2 \sinh_3(x) + \omega \sinh_3 \mathrm{II}(x)$$

and for e^x it's quite simple

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

$$e^x = \cosh_3(x) + \sinh_3(x) + \sinh_3 \mathrm{II}(x)$$

like sin and sinh, we can try to find an exponent form for them
First, we begin by adding all of the equations so we have

$$e^x + e^{\omega x} + e^{\omega^2 x} = \cosh_3(x) + \sinh_3(x) + \sinh_3 \mathrm{II}(x)$$

$$+ \cosh_3(x) + \omega \sinh_3(x) + \omega^2 \sinh_3 \mathrm{II}(x) + \cosh_3(x) + \omega^2 \sinh_3(x) + \omega \sinh_3 \mathrm{II}(x)$$

$$= 3\cosh_3(x) + \sinh_3(x)(1 + \omega + \omega^2) + \sinh_3 \mathrm{II}(x)(1 + \omega^2 + \omega)$$

and since we know that $1 + \omega + \omega^2 = 0$

$$e^x + e^{\omega x} + e^{\omega^2 x} = 3\cosh_3(x) \qquad \cosh_3(x) = \frac{e^x + e^{\omega x} + e^{\omega^2 x}}{3}$$

Now we can define the others by differentiating

$$\sinh_3(x) = \frac{e^x + \omega e^{\omega x} + \omega^2 e^{\omega^2 x}}{3} \qquad \sinh_3 \mathrm{II}(x) = \frac{e^x + \omega^2 e^{\omega x} + \omega e^{\omega^2 x}}{3}$$

since these definitions are going to continue with us, we shall call the $e^{ax} = ...$ the **Euler form** and $f(x) = e^x + e^{ax}....$ the **exponentiation form**

4.2 Cyclic derivatives and prime numbers

To understand exactly what is meant by this, we need to see these

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \cosh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$$

As we can see, there is a pattern here, for every function that is k-th cyclic derivative, we can see that its series is $\sum_{n=0}^{\infty} \frac{x^{kn}}{(kn)!}$
But this assumption shortly breaks as we can see for the fourth cyclic derivative

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

To understand more what I am talking about, we may take a look at the fifth cyclic derivative denoted by ϵ, to find the functions we start from the Maclaurin series

$$e^{\epsilon x} = 1 + \epsilon x + \frac{\epsilon^2 x^2}{2!} + \frac{\epsilon^3 x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{\epsilon^n x^n}{n!}$$

And we can find the five functions the same way we made it to the three cyclic derivative functions

$$\cosh_5(x) = \sum_{n=0}^{\infty} \frac{x^{5n}}{(5n)!} \qquad \sinh_5(x) = \sum_{n=0}^{\infty} \frac{x^{5n+1}}{(5n+1)!}$$

$$\sinh_5 \mathrm{II}(x) = \sum_{n=0}^{\infty} \frac{x^{5n+2}}{(5n+2)!} \qquad \sinh_5 \mathrm{III}(x) = \sum_{n=0}^{\infty} \frac{x^{5n+3}}{(5n+3)!} \qquad \sinh_5 \mathrm{IV}(x) = \sum_{n=0}^{\infty} \frac{x^{5n+4}}{(5n+4)!}$$

And since the sum of roots of unity that are over 1 root is zero, we can do the same steps to find that

$$\cosh_5(x) = \frac{e^x + e^{\epsilon x} + e^{\epsilon^2 x} + e^{\epsilon^3 x} + e^{\epsilon^4 x}}{5}$$

and the other functions to be the derivatives of these functions
We can notice that the pattern continued for 5-th cyclic function
So what is the problem with trigonometric ones?
Well, we can see the expansion for $e^{ix}$to see what happens

$$e^{ix} = 1 + ix + -\frac{x^2}{2!} - \frac{ix^3}{3!} + ...$$

We can notice the pattern right there, it's the $-\frac{x^2}{2!}$, this term allows us to either:
- write the sum as four cyclic functions since we will make different additions being $\{1, i, -1, -i\}$
- We write sum as two different cyclic derivatives being $\{1. -1\}$ and $i\{1, -1\}$

In other words, the cyclic derivative family is compisable, reducible with simple algebra
and the reason for that is the **cyclic order**, when it's composite, we can see some roots return, like from order 2 we have $1, -1$ and order 4 we have $1, i, -1, -i$, the $1, -1$ here is back, same for six roots of unity $1, i_1, i_2, -1, i_3, i_4$ (note that i_a here isn't imaginary unit but the a-th root) and we can say that
let gk be all solutions for $a^k = 1$ and gn be for $a^n = 1$, as long as $\frac{k}{n} \in \mathbb{Z}^+, gk \subset$

gn
Thus, for any composite cyclic order, there exists more than 1 way to represent it
which means primes aren't here, so we can write the theorem

Theorem 4.1 (Prime cyclic functions Euler Form) $\forall p \in$ *Primes*,$a^p = 1$ *There exists only one way to represent e^{ax} as a sum of all the cyclic order functions*

From this, we can say that

Theorem 4.2 (Prime cyclic functions exponentiation form Form) $\forall p \in$ *Primes,* $\sinh_p N(x)$ *is a cyclic function; it can be written as this*

$$\sinh_p N(x) = \frac{e^x + a^N e^{ax} + a^{2N} e^{a^2 x} + ... + a^{pN} e^{a^p x}}{p} = \frac{1}{p} \sum_{n=0}^{p-1} a^{n(p-N)} e^{a^n x}$$

4.3 General Cyclic derivatives and Mittag-Leffler connection

As we can see from multiple series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sinh = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

We can see from all these examples that the factorial matches the exponent
Taking the D^α derivative of all gives us

$$D^z e^x = \sum_{n=0}^{\infty} \frac{x^{n-z}}{\Gamma(n-z+1)} \qquad \sinh = \sum_{n=0}^{\infty} \frac{x^{2n+1-z}}{\Gamma(2n+2-z)}$$

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1-z}}{\Gamma(3n+2-z)} \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-z}}{\Gamma(2n-z+1)}$$

If we let $2 - z = \beta$ and let $Cn = \alpha n$ where C is a constant, we see that all of them get the shape

$$D^z f(x) = \sum_{n=0}^{\infty} \frac{x^{\alpha n+\beta}}{\Gamma(\alpha n+\beta)}$$

which is the Mittag-Leffler function, we can see that this happens in all of the functions we know.
That is, of course, except for $\cos(x)$ that will be discussed later

, but we need to generalise it, and we need to generalise the derivative cyclic order in Euler form

$$let\ a\ be\ any\ element\ from\ the\ group\ of\ soluitons\ for\ a^n = 1$$

$$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{a^kx^k}{k!}$$

From here, we can group the sums. Since there exist n roots of unity, we can say that there exist n terms

$$e^{ax} = (1 + \frac{x^n}{n!} + \frac{x^{2n}}{(2n)!} + ...) + (ax + \frac{a^nx^{n+1}}{(n+1)!} + \frac{a^{2n}x^{2n+1}}{(2n+1)!} + ...) + ...$$

$$= \sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!} + \sum_{k=0}^{\infty} \frac{a^{kn+1}x^{kn+1}}{(kn+1)!} + \sum_{k=0}^{\infty} \frac{a^{kn+2}x^{kn+2}}{(kn+2)!} + \sum_{k=0}^{\infty} \frac{a^{kn+3}x^{kn+3}}{(kn+3)!} + ...$$

and since $a^{kn+j} = a^j$, we can take it out as a common factor

$$= \sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!} + a\sum_{k=0}^{\infty} \frac{x^{kn+1}}{(kn+1)!} + a^2\sum_{k=0}^{\infty} \frac{x^{kn+2}}{(kn+2)!} + a^3\sum_{k=0}^{\infty} \frac{x^{kn+3}}{(kn+3)!} + ...$$

We are going to name the first one $\cosh_n(x)$ and the others $\sinh_n I(x)$ and $\sinh_n II(x)$ so on

$$e^{ax} = \cosh_n(x) + a\sinh_n \mathrm{I}(x) + a^2\sinh_n \mathrm{II}(x) + a^3\sinh_n \mathrm{III}(x) + ...a^{n-1}\sinh_n \mathrm{N}(x)$$

Now, if we consider $kn = \alpha n$ and $+C - \alpha = +\beta$ We see that all these functions fall under the Mittag-Leffler formula

$$D^z \cosh_n = \sum_{k=0}^{\infty} \frac{x^{\alpha n}}{\Gamma(\alpha n)} \qquad D^z \sinh_n \mathrm{II}... = \sum_{k=0}^{\infty} \frac{x^{\alpha n+\beta}}{\Gamma(\alpha n+\beta)}$$

From this, we can get the general cyclic derivative sum formula

Theorem 4.3 (Generalized Cyclic Derivative) *Let $f(x)$ be the j-th basis function of the D^n-cyclic system,*
$f(x) = \sum_{k=0}^{\infty} \frac{x^{kn+j}}{(kn+j)!}$*, The z-order derivative is given by* $D^z f(x) = \sum_{k=0}^{\infty} \frac{x^{kn+j-z}}{(kn+j-z)!}$

Theorem 4.4 (Mittag-Leffler Representation) *Every basis function of the D^n-cyclic system is a linear combination of* **n** *distinct Mittag-Leffler functions* $\mathbf{E}_{\mathbf{n},\beta}(\mathbf{x^n})$ *corresponding to the n terms in its series representation.*

Theorem 4.5 (Generalized Euler form) *for every e^{ax} where a stratifies $a^n = 1$, e^{ax} can be written as*

$$e^{ax} = \sum_{j=0}^{n-1} a^j \sinh_n N(x)$$

that is of course if we consider $\cosh_n(x)$ to be the 0-th term

4.4 Odd And Even cyclic derivatives

There are many differences between odd and even cyclic derivatives, and that comes to the roots of unity

4.5 prime cyclic derivatives

the form e^{ix} can be expressed as $\cos(x) + i\sin(x)$, that is because it's composite and can be reducible to it's prime cyclic (2)
as we can see the reason it's reduced from algabric side is the (-1)

$$\cosh = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \cos = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sinh = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \sin = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

if we try another cyclic derivative that has only one prime order factor it will be 9 because it's 3^2
let $a^9 = 1$, with a being from $\{1, \epsilon, \epsilon^2, \omega, \epsilon^4, \epsilon^5, \omega^2, \epsilon^7, \epsilon^8\}$ with $\omega = \epsilon^3$ but we change it just to know it's the third cubic root

$$e^{ax} = 1 + \epsilon x + \frac{\epsilon^2 x^2}{2!} + \frac{\omega x^3}{3!} + \frac{\epsilon^4 x^4}{4!} + \frac{\epsilon^5 x^5}{5!} + \frac{\omega^2 x^6}{2!} + ...$$

we can then make the sum

$$e^{ax} = \sum_{n=0}^{\infty} \frac{\omega^n x^{3n}}{(3n)!} + \sum_{k=0}^{\infty} \frac{\epsilon^{k+1} x^{3k+1}}{(3k+1)!} + \sum_{j=0}^{\infty} \frac{\epsilon^{j+2} x^{3j+2}}{(3j+2)!}$$

we can see the three sums form the shapes of the main three cyclic derivatives functions with the addition of the complex values. but because the cyclic order is odd we don't see any negative ones
to understand more we can try to find the original e^{ix} cyclic functions

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} + i\sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} - \sum_{j=0}^{\infty} \frac{x^{4j+2}}{(4j+2)!} - i\sum_{u=0}^{\infty} \frac{x^{4u+3}}{(4u+3)!}$$

we can actually see how the form sin and cos form

$$\left(\sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} - \sum_{j=0}^{\infty} \frac{x^{4j+2}}{(4j+2)!}\right) + i\left(\sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} - \sum_{u=0}^{\infty} \frac{x^{4u+3}}{(4u+3)!}\right)$$

simply analysing this we can see that the (-1^n) term in the sums come from them being difference between the sums, we can also see that cos only has the order on the x (x^{kn}) because the both sums have a common factor which is the

prime number 2, while sin has the reminder of 1 when dividng that leads to x^{kn+1} term
ultimately we can try to reprsnt the redacted versions of the original functions $\cos, \sin$ with the original functions ($\cosh_4(x), \sinh_4(x), \sinh_4 \mathrm{II}(x), \sinh_4 \mathrm{III}(x)$) using the Euler form

$$e^{ix} = \cos(x) + i\sin(x) = \cosh_4(x) + i\sinh_4(x) - \sinh_4 \mathrm{II}(x) - i\sinh_4 \mathrm{III}(x)$$

then we know that the real equals the real and the imaginary equals the imaginary

$$\cos(x) = \cosh_4(x) + \sinh_4 \mathrm{II}(x) \qquad \sin(x) = -(\sinh_4 \mathrm{I}(x) + \sinh_4 \mathrm{III}(x))$$

to understand more about it we can try to see for 8 order cyclic deriavtives with $a \in \{\epsilon, i, \epsilon^3, -1, \epsilon^5, -i, \epsilon^7, 1\}$ knowing that $\epsilon = e^{\frac{2i\pi}{8}}$

$$e^{ax} = 1 + \epsilon x + \frac{ix^2}{2!} + \frac{\epsilon^3 x}{3!} - \frac{x^4}{4!} + \frac{\epsilon^5 x^5}{5!} - \frac{ix^6}{6!} + \frac{\epsilon^7 x}{7!} + ...$$

$$e^{ax} = \sum_{n_1=0}^{\infty} \frac{x^{8n_1}}{(8n_1)!} + \sum_{n_2=0}^{\infty} \frac{\epsilon x^{8n_2+1}}{(8n_2+1)!} + \sum_{n_3=0}^{\infty} \frac{\epsilon x^{8n_3+1}}{(8n_3+2)!} + ...$$

these are the original functions functions
first of all, we know from the calculations that $\epsilon^5 = -\epsilon, \epsilon^7 = -\epsilon^3$, we can write the expressions as following

$$e^{ax} = \left(\sum_{n_0=0}^{\infty} \frac{x^{8n_0}}{(8n_0)!} - \sum_{n_4=0}^{\infty} \frac{x^{8n_4+4}}{(8n_4+4)!} \right) + \epsilon \left(\sum_{n_1=0}^{\infty} \frac{x^{8n_1+1}}{(8n_1+1)!} - \sum_{n_5=0}^{\infty} \frac{x^{8n_5+5}}{(8n_5+5)!} \right) ...$$

to distinguish original functions from reduced ones, we are going to call the reduced ones with trigonometric names from now on

$$e^{ax} = \cos_8(x) + \epsilon \sin_8(x) + i \sin_8 \mathrm{II}(x) + \epsilon^3 \sin_8 \mathrm{III}(x)$$

we can try to expand the sums to find another simpler sums for them

$$\cos_8(x) = 1 - \frac{x^4}{4!} + \frac{x^8}{8!} + ... \qquad \sin_8(x) = x - \frac{x^5}{5!} + \frac{x^9}{9!} + ...$$

they form the sums

$$\cos_8(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(4n)!} \qquad \sin_8(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{(4k+1)!} \qquad \text{etc...}$$

we can see from this that the number of functions an order n can be reduced to is $\frac{n}{p}$ where p is its prime divisor, that is to be said most functions we have dealt with till now are either primes or have one prime divisor, but we still can prove it

Theorem 4.6 *Every integer order n that has only one prime divisor p can be reduced to $\frac{n}{p}$ functions at max, with each one has different root of unity*

Proof:
Let n be a cyclic order with only p being it's only prime divisor
with the fundamental theorem of arithmetic that states : "every integer greater than 1 is a prime or a product of primes" we can say that $n = p \times q \times ....$
and since p is the only prime divisor we can say that $n = p \times p \times p.... = p^k$ for some integer $k > 1$
we can then define $a^n = 1$ to $a^{p^k} = 1$ with the repeating nature of the roots of unity that only happens when $k|j$ where are both the number roots of unity, we can set m to be the before n roots of unity group that shares the same only prime divisor as p
then we can let $m = p^{k-1}$ with $a^m = 1$ so $a^{p^{k-1}} = 1$
to find the maxmium amount of functions

4.6 Mixture of equations

By now, we know that there exist Prime cyclic functions and composite cyclic functions. The prime cyclic can't be expanded, while the composite ones can
So we can say that $\cosh(x)$ is prime cyclic since its cyclic order is prime (2) and it can't be expanded, so e^{-x} can only be expanded naturally with only $\sinh(x)$ and $\cosh(x)$
on the other hand

5 But why cyclic derivatives?

5.1 Dimension Bender and Hyper operations

Operations are the fundamentals of mathematics. We start with a constant, then succession, which has the definition $S(n) = n + 1$
Repeated succession results addition, which can be defined as $A(n, m) = n+m = \underbrace{S(S(S(S(....n))))}_{m \text{ times}}$, repeating that gives Multiplication $M(n, m) = n \times m$ with the same idea, so one exponentiation and tetration
What brings that here is the properties of exponentiation, we can say that for any constant a, we can do

$$a^n \times a^m = a^{n+m} \qquad (a^n)^m = a^{n \times m}$$

Exponentiation moves other operations up in the hierarchy; it linearises them, "Bending the space around it". We can see that every cyclic derivative is in or can be used to create e^{ax}; an exponentiation form
But then the question arises, why exactly e and not any other base,well we can see it with D^z

$$D^z a^{bx} = b^z a^{bx} \ln(a)^z \Rightarrow D^z e^{bx} = b^z e^{bx} \ln(e)^z = b^z e^{bx}$$

Unlike any other base, e is the only base that stays without remainder, it bends space without any trace, it has the perfect environment for pure cyclic functions to arise like $\sin(x)$ and $\cosh(x)$ as they won't interact with any other change, it allows clean, smooth transformation from point a to point b we can also see that in the expansion of $e^{ax} = \sum_{n=0}^{\infty} \frac{a^n x^n}{n!}$, the simpliest form of expansion for a function

5.2 Analysis of cyclic functions

But now this gives a bigger question: what does this information help us with cyclic derivatives

5.3 Transformtive functions

But unlike cyclic derivatives, this one is problematic
cyclic functions change smoothly from one derivative to another, transformative functions choose to break that in a pretty much big sense, take for example $\sin^{-1}(x)$, from D^z prescriptive, it can be defined as

$$D^z \sin^{-1}(x) = D^z \frac{1}{\sqrt{1-x^2}} \qquad z \geq 0$$

and at 0 it is just $\sin^{-1}(x)$ and for the integrals it seems to be a product of both of them, it just jumps back and forth between faces with no predictable move, that is, unless we go to the complex plane and see the complex definition which is $\sin^{-1}(z) = -i \ln(iz + \sqrt{1-z^2})$
There is a natural logarithm, whic has we know is a problem with D^z
But not all transformative functions directly have $\ln(x)$ in their definitions
for example $\frac{1}{x^n}$ is a transformtive functions because at integration, it directly transforms to another function $\ln(x)$, which isn't in the definition but rather the integral of it
Looking at the big picture, the inverse of x^n it's an inverse function, and nearly all the transformative functions are inverses, and that is for a reason
A function is defined to be an input-to-output machine; many inputs can give the same output, but not the other way around. When one input gives many outputs, it's not an ordinary function in the definition,
When we try to get an inverse out of a function, most of the time that rule is broken. We can see for the simple case x^2 that makes a double input one output system, to define its inverse $\sqrt{x}$ in the real value, we have to sacrifice the other inputs that gave the same outputs, being the negative numbers

5.4 $\ln(x)$ is collapser

6 Cycliation in other ways

from the hyperoperations explanation; cycliation isn't only for simple e^{ax}, which allows us to try different things effectively

6.1 Multi variable cycliation

we can extend the cyclic functions to the 3rd Dimension simply since the defenition is simple (e^{ax} where $a^n = 1$), we can simply change x to $x + y$ to get

$$e^{a(x+y)} \text{ where } a^n = 1$$

this simple formula to extend it to 3D , so

$$e^{-(x+y)} = \cosh(x + y) - \sinh(x + y) \qquad e^{i(x+y)} = \cos(x + y) + i\sin(x + y)$$

infact if we let u be any operation between x, y because we will get $e^{au}$where $a^n =$ 1
this can work and extend to any operation to infinite dimensions, but there is one thing
this extension is linear, we can expect the outcome and result to work right and similer, the one function going upwards at $x > 1$ just becomes a plane going upwards then a cube going upwards
this happens because when we change e^{ax} to e^{au} the reality is similer to the hyperoperations explanation, it's just e^{ax} at the end of it
we can find some true change by changing the rule , since the differentiation of the function is dependent of a^x and it's nature to come back at the end of a full cycle , we can try to change it to e^{ax+by} in the 3D definition
we can say that for this to happen $a^n = b^n = 1$ which is true only when $a = b$
But this doesn't stop us from making a new definition, why not simply make it one-dimensional cyclic?
The idea is simple, we need both to be cyclic but not the same order at the same time, at the end, the reason for its cyclicity is its non-growing nature
We can define it as

$$e^{ax+by} \text{ where } a^n = b^m = 1$$

This definition is n-order cyclic on the x-axis and m-order cyclic on the y-axis, we can try it with $a = 1, b = -1$ to see what happens. First, we expand it

$$e^{x-y} = 1 + (x - y) + \frac{(x - y)^2}{2!} + \frac{(x - y)^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{(x - y)^k}{k!}$$

Of course, this can be treated as $e^{-(y-x)}$ which can be written in sinh and cosh terms

$$e^{x-y} = e^{-(y-x)} = \cosh(y - x) - \sinh(y - x)$$

But we can try something like e^{-x+iy} which is a second cyclic with respect to x but a four cyclic with respect to y
this one can be expanded simply using Euler's formula $e^{i\theta}$ and the definition for e^{-x}

$$e^{-x+iy} = e^{-x}e^{iy} = e^{-x}(\cos(y) + i\sin(y))$$

6.2 Combination of cycliation

6.3 cycliation derivative to a function order

from the original paper we know that we can use D^{α} with the order being the differentiable variable as D^x or $D^{g(x)}$

$$D^x f(x) = D^x e^{ax} = a^x e^{ax}$$

and unlike the monomials we can take the derivative again of this function to be

$$D^x f'^x(x) = a^{2x} e^{ax} + a^x e^{ax} \ln(a)^x$$

we can continue for a third time

$$D^x f''^x(x) = a^{3x} e^{ax} + a^x e^{ax} \ln(a)^{2x} + a^{2x} e^{ax} \ln(x) + a^x e^{ax} \ln(a)^x + a^x e^{ax} \ln(a)^x \ln(\ln(a))^x$$

but the problem comes fro the first derivative, the function isn't cyclic anymore it broke it's rule on expansion thus making this topic declined

6.4 can any function be transformed to a cyclation vatiant?

6.5 x^x is weird

6.6 Cycliation, in imaginary plane system

6.7 Cycliation into the third dimension

6.8 Cycliation, but not in e

With our knowledge, we can try to think of what would happen when we change the base to any other constant a, with the rule being $b^n = 1$, let's try to see what happens at second-order cyclic derivatives ($\sinh(x)$ and $\cosh(x)$) using the Maclaurin series

$$a^{-x} = 1 - x\ln(a) + \frac{x^2 \ln(a)^2}{2!} - \frac{x^3 \ln(a)^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{(-1)^k x^k \ln(a)^k}{k!}$$

we are going to extract the $\sinh(x)$ and $\cosh(x)$ terms from the sum by the negatives and the positives and give them the terms ${}_a \sinh(x)$ and ${}_a \cosh(x)$ t distinguish them from the original ones

$${}_{\mathrm{a}} \cosh = \sum_{k=0}^{\infty} \frac{x^{2k} \ln(a)^{2k}}{(2k)!} \qquad {}_{\mathrm{a}} \sinh = \sum_{k=0}^{\infty} \frac{x^{2k+1} \ln(a)^{2k+1}}{(2k+1)!}$$

then of course, to find the euler form it will be

$$a^{-x} = {}_{\mathrm{a}} \cosh(x) - {}_{\mathrm{a}} \sinh(x)$$

to extend this to any function, we will simply do

6.9 Some properties of other base cyclic derivatives

For the functions ${}_{\mathrm{a}}\cosh(x)$, if we turn the base less than 1, we can observe that it starts to exhibit waves and patterns. When $a = 0.5$, we can see some similarity between these functions and $\cos(x)$
using trial and error, I could make say when $a \approx 0.367879441190$
This number perfectly matches e^{-1},infact if we plugged it directly we can see why

$$ {}_{\mathrm{e^{-1}}}\cosh = \sum_{k=0}^{\infty} \frac{x^{2k}\ln(e^{-1})^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{x^{2k}(-1)^{2k}}{(2k)!} $$

Of course, this is the $\cos(x)$ Maclurin series, and this can give us another angle to understand the connection between composite cyclics and prime cyclics
Composite cyclics can be reduced to prime cyclics with some change, and this change happens to be the change of the base for the cyclic function.

7 The function families of cyclic functions

exploring the families of the functions is intersting , since it may help us understand why such functions behave like this at least from teh point of D^z and being NFRD while they are constructed from FRD functions

7.1 $\tan(x)$ and $\tanh(x)$

from the beginning, we can see some problems
before we even start we realize that ,both $\tan(x)$ and $\tanh(x)$ don't have a "main way" to define them
the stranded definition for them is $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$, but that is only because there exists two functions , and this only happens in hyperbolic functions, and trigonometric functions too because it satisfies $\frac{n}{p} = 2$ because they are the only ones that sat,but anyway we can use many ways with diffrent resaults to define the tangent function(s)
Using the first two functions only
we can try to define $\tan_{\mathrm{n}}(x)$ to use the first two functions of any e^{ax}
so we can write it as $\tanh_{\mathrm{n}}(x) = \frac{\sinh_{\mathrm{n}}(x)}{\cosh_{\mathrm{n}}(x)}$ that is for the non-reduced orignal functions of the expansion and $\tan_{\mathrm{n}}(x) = \frac{\sin_{\mathrm{n}}(x)}{\cos_{\mathrm{n}}(x)}$ to be for the reduced functions we can try it for the 3-order cyclic derivatives

$$ \tanh_3(x) = \frac{\sinh_3(x)}{\cosh_3(x)} = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \left(\sum_{k=0}^{\infty} \frac{x^{3k}}{(kn)!} \right)^{-1} $$

but the problem with this one is there are other functions that aren't put in prespiactive , and the more we go up with the order the more they are which makes this definition unusable
The division between every function and $\cosh_{\mathrm{n}}$ or $\cos_{\mathrm{n}}$

Then we can try to to divide every single function multiplied from the cyclic family

$$\tanh_3 = \frac{\sinh_3(x)}{\frac{\cosh_3(x)}{\sinh_3 \mathrm{II}(x)}} = \frac{\sinh_3(x)\sinh_3 \mathrm{II}(x)}{\cosh_3(x)}$$

and if we continue this we will see that all odd functions from a full cyclic family are on the top while the evens are on the bottm, so if we let $\cosh_{\mathrm{n}} = \sinh_{\mathrm{n}} 0$ we get this formula

$$\tanh_{\mathrm{n}}(x) = \frac{\prod_{K=0}^{n} \sinh_{\mathrm{n}} \mathrm{2K{+}1}(x)}{\prod_{J=0}^{n} \sinh_{\mathrm{n}} \mathrm{2J}(x)}$$

Using the Series

but to actually define $\tanh_{\mathrm{n}}(x)$ we need to observe what the orignal $\tan(x)$ have in that this one needs to have

$\tan(x)$ is equal to $\frac{\cos(x)}{\sin(x)}$: this isn't supposed to be true for all variants, and we will discuss the reason for that later

$\tan(x)$ is an NFRD function with its series representation only: This one is important since a series representation is the represent of a function using x^n, which *According to the original paper* has the change directly happening to them so the best way to define $\tanh_{\mathrm{n}}(x)$ is via the series definition, now we see the Maclaurin series

$$\tan(x) = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} \qquad \tanh(x) = \sum_{n=0}^{\infty} \frac{B_{2n}4^n(1-4^n)}{(2n)!} x^{2n-1}$$

But we can't take these series directly and apply them because they may be only special occians , we need to define the main series using the original functions

We can skip $\tanh(x)$ since it's pretty straightforward in the original functions being $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

for the four original functions of cyclic order 4 $(\cosh_4(x), \sinh_4(x), \sinh_4 \mathrm{II}(x), \sinh_4 \mathrm{III}(x))$

We can use the sin and cos definitions of them being

$$cos(x) = \cosh_4(x) + \sinh_4(x) \qquad \sin(x) = -(\sinh_4 \mathrm{II}(x) + \sinh_4 \mathrm{III}(x))$$

We can then put them in the definition of $\tan(x)$ to get

$$\tan(x) = \frac{-(\sinh_4 \mathrm{II}(x) + \sinh_4 \mathrm{III}(x))}{\cosh_4(x) + \sinh_4(x)}$$

Expanding the series we get

$$\tan(x) = \frac{-(\sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)!} + \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!})}{\sum_{j=0}^{\infty} \frac{x^{4j}}{(4j)!} + \sum_{u=0}^{\infty} \frac{x^{4u+2}}{(4u+2)!}}$$

This shows that for the original functions, it's dividing the second half of the series by the first half we can try to do the same for the 8-order to see the result

$$\tan_8 = \frac{\sin_8(x) + \sin_8 \mathrm{III}(x)}{\cos_8(x) + \sin_8 \mathrm{II}(x)} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)!} + \sum_{j=0}^{\infty} \frac{(-1)^j x^{4j+3}}{(4j+3)!}}{\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{(4k)!} + \sum_{u=0}^{\infty} \frac{(-1)^u x^{4u+2}}{(4u+2)!}}$$

expanding the sums we get

$$\tan_8(x) = \frac{x + \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} + ...}{1 + \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{x^6}{6!} + ...}$$

which is reduced to the regular sum $\sin(x)$ and $\cos(x)$ we get the final result

$$\tan_8(x) = \frac{\sin(x)}{\cos(x)} = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1}$$

8 The inverses of cyclic function families

9 The continuation of the families

9.1 Identities in cyclic derivatives

9.2 Identities between different cyclic derivatives

9.3 Cyclic derivatives and algebraic equations