Complex-Order and Fractional Derivatives: A First Exploration

Faisal Khalid

September 2025

Abstract

This paper presents an independent exploration of fractional and complex-order derivatives, building the framework from algebraic first principles using Gamma function extensions. Without prior exposure to existing fractional calculus literature, I derive formulas for $D^{\alpha}$and D^{z} operators Applied to elementary functions, prove fundamental properties (linearity, Index Law, product rules), and explore geometric interpretations through the "$D(i)$ plane." The work concludes with applications to cyclic derivatives and preliminary extensions to matrix orders.

Note to Readers: This represents independent rediscovery of classical fractional calculus concepts. I (The Author) have since learned that this field has extensive existing literature (Riemann-Liouville, Caputo, etc.) and presents this work as a pedagogical exercise in mathematical exploration rather than novel research.

Part I
Algebraic foundation

1 The Generalized Operator for $f(x) = x^n$

1.1 From Integer to Fractional Order

We begin with the integer derivatives of $f(x) = x^n$:

$$D^k f(x) = n(n-1)\cdots(n-k+1)x^{n-k}$$

Using the identity $n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$, we write:

$$D^k f(x) = \frac{n!}{(n-k)!}x^{n-k}$$

To generalize this for $k \in \mathbb{R}$, we substitute the factorial function with the continuous Gamma function, $\Gamma(z)$. We use the identities $n! = \Gamma(n+1)$ and

$\Gamma(z+1) = z\Gamma(z)$. The α-th derivative (where $\alpha \in \mathbb{R}$) is:

$$D^{\alpha} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

Of course, we can use this formula to get the half-derivative of x^n

$$D^{\frac{1}{2}} = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} x^{1-\frac{1}{2}} = \frac{1}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}}$$

using the rule $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$D^{\frac{1}{2}} = \frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})} x^{\frac{1}{2}} = \frac{1}{\frac{\sqrt{\pi}}{2}} x^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}} x^{\frac{1}{2}}$$

Now, if we take the half-derivative of that half-derivative, it will be

$$\frac{2}{\sqrt{\pi}} D^{\frac{1}{2}} = (\frac{2}{\sqrt{\pi}}) \frac{\Gamma(1/2+1)}{\Gamma(\frac{1}{2}-\frac{1}{2}+1)} x^{\frac{1}{2}-\frac{1}{2}} = (\frac{2}{\sqrt{\pi}}) \frac{\Gamma(\frac{3}{2})}{1} x^0 = (\frac{2}{\sqrt{\pi}})(\frac{\sqrt{\pi}}{2}) = 1$$

We are talking about the half-derivative twice to the same function, giving what a full derivative would give
Of course, such a proof is too simple and doesn't quite give the meaning of a proof; it's just a little confirmation for the time being. The full rigorous proof will be provided later with the properties of the D^z operator
Generalization to Complex Order $z = a + bi$
We now extend the derivative order to the complex number $z = a + bi$:

$$D^{z} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-z}$$

To show the magnitude and phase components, we expand x^{n-z} using the property $x^{a+bi} = x^a e^{b\ln(x)i}$:

$$D^{z} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-a} e^{-b\ln(x)i}$$

With these two formulas, we can use them to find any $\mathbb{C}$ or $\mathbb{R}$ derivatives for x^n
Finding negative order derivatives
we can find the negative derivatives by putting -1 as the α and see what could happen
Putting -1 in the general formula gives the result

$$D^{-1} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-(-1)+1)} x^{n-(-1)} = \frac{\Gamma(n+1)}{\Gamma(n+2)} x^{n+1}$$

and using the $\Gamma(z+1) = z\Gamma(z)$ we can say that

$$D^{-1} f(x) = \frac{\Gamma(n+1)}{(n+1)\Gamma(n+1)} x^{n+1} = \frac{x^{n+1}}{(n+1)}$$

which means that the negative order derivatives are the integrals a function This result unifies the familiar integer derivative, the fractional derivative, and the complex-order derivative into a single, elegant framework.

1.2 the x^{-n} problem

As we have seen, we can apply the past formula to any power of n Whether it's fractional or even complex but problems arise when we try to apply the past formula to x^{-n}

$$D^{\alpha}(x^{-n}) = \frac{\Gamma(0)}{\Gamma(-\alpha)} x^{-n-\alpha}$$

Not only do we have a **Gamma Pole** in the numerator, but also for any value $\alpha \in \mathbb{Z}^{+}$ we also get a Gamma pole in the denominator, which means that this formula can't work, and we need another formula
the m-th formula for x^{-n} is simply

$$\frac{d^m}{dx^m}(x^{-n}) = \frac{(-1)^m \ \ (n)^{(m)}}{x^{n+m}}$$

The $n^{(m)}$ here isn't a power but rather a rising factorial that can be expressed as $n^{(m)} = \frac{(n+m-1)!}{(n-1)!}$ With this knowledge, we can say

$$D^{\alpha}(x^{-n}) = \frac{(-1)^{\alpha} \ \ \frac{\Gamma(n+\alpha)}{\Gamma(n)}}{x^{n+\alpha}} = \frac{(-1)^{\alpha} \ \ \Gamma(n)}{x^{n+\alpha}\Gamma(n+\alpha)}$$

As simple as that, it didn't work with the original x^n formula , but this shows something about fractional derivatives. If we went to find the first integral for both x^n and x^{-n}, we find that

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \int x^{-n} dx = \ln|x| + C$$

Ignoring the integration constant we can find that x^n wasn't the original function for x^{-n} but it actually transformed from $\ln(x)$
not only in complexity only but in dependency , the $D^z(x^n)$ is dependent on the change of n more than x, while $\ln(x)$ is dependent on the change of x
This will be explained later in the **Transformative Functions** section , and this is one of the few cases we use standard integration (other than the Gamma function) in this research

2 Formulas for Other Algebraic Functions

2.1 The General Formula for a^x

Starting with the general integer rule for a^x:

$$D^n(x) = a^x ln(a)^n$$

Substituting α in the place of n gives us

$$D^{\alpha} f(x) = a^{x} \ln(a)^{\alpha}$$

We can see that this simple change was enough for the formula to work by taking the half-derivative twice, and it gives us an order one derivative

$$D^{1/2} f(x) = a^{x} ln(a)^{1/2}$$

Since $ln(a)^{1/2}$ is a constant, we can take it out simply when doing the derivative again

$$D^{1/2}(D^{1/2} f(x)) = ln(a)^{1/2}(D^{1/2} f(x)) = ln(a)^{1/2}(a^{x} ln(a)^{1/2}) = a^{x} ln(a)$$

which is true since our starting function was a^{x} and thus we can say this formula works
also from the same we way we can find that $D^{\alpha} a^{-x} = (-1)^{\alpha} a^{-x} \ln(a)^{\alpha}$ which will be helpful for some serieses later
The Complex Generalization of this formula can be written like the D^{α} formula or like this

$$D^{z} f(x) = a^{x} \ln(a)^{t} e^{bln(ln(a))i}$$

where $z = a + bi$
Of course, we can find the first Anti-derivative of this function by using -1 in the formula

$$D^{-1}(x) = a^{x} ln(a)^{-1} = \frac{a^{x}}{ln(a)}$$

and the first Complex derivative

$$D^{i}(x) = a^{x} ln(a)^{i} = a^{x} e^{ln(ln(x))i}$$

2.2 The General Formula for e^{x}

The function e^{x} is known for its "Unchanging Derivative" because it comes from the $D^{n}(a^{x}) = a^{x} ln(a)^{n}$ and putting $a = e$ we get $D^{n}(e^{x}) = e^{x}$ So this also means there is no change that affects the complex nor the fractional derivatives

$$D^{\alpha} f(x) = e^{x} \qquad\qquad D^{z} f(x) = e^{x}$$

which means the Anti-derivative and the first complex derivative of the function

$$D^{-1}(x) = e^{x} \qquad\qquad D^{i}(x) = e^{x}$$

2.3 The General Formula for e^{ax}

As we saw, there is no change between e^{x} and any of its derivatives. Things change when we consider e^{ax} As we can see, the rule of the first, second, and derivative is

$$D^{1} f(x) = ae^{ax} \qquad D^{2} f(x) = a^{2} e^{ax} \qquad D^{3} f(x) = a^{3} e^{ax}$$

So we can find the formula for the n-th derivative as

$$D^n f(x) = a^n e^{ax}$$

and changing the n to α, we get

$$D^\alpha f(x) = a^\alpha e^{ax}$$

As simple as that, we still have to test it to justify

$$D^{1/2} f(x) = a^{1/2} e^{ax}$$

since $a^{1/2}$ is a constant, we can say that

$$D^{1/2}(D^{1/2} f(x)) = a^{1/2}(D^{1/2} f(x)) = (a^{1/2})(a^{1/2} e^{ax}) = ae^x$$

This confirms that the formula works
and substituting z instead of α, we get the same formula as above that can also be written like that

$$D^z f(x) = a^t e^{bln(a)i+ax}$$

where $z = a + bi$
the first Anti-derivative for e^{ax} is

$$D^{-1} f(x) = a^{-1} e^{ax} = \frac{e^{ax}}{a}$$

and the first complex derivative is

$$D^i f(x) = a^i e^{ax} = e^{ln(a)i+ax}$$

2.4 The problem of $\log_a(x)$

$log_a(x)$ The first derivative of $log_a(x)$ is $\frac{1}{xln(a)}$ and the second derivative is $\frac{-1}{x^2 ln(a)}$ the third derivative is $\frac{2}{x^3 ln(a)}$ lastly the fourth derivative is $\frac{-6}{x^4 ln(a)}$ is We can see the pattern of the n-th derivative

$$D^n f(x) = (-1)^{n+1} \frac{(n-1)!}{x^n ln(a)}$$

and applying the gamma identity $n! = \Gamma(n-1)$ then changing n to α and reversing the x power in the denominator we get

$$D^\alpha f(x) = (-1)^{\alpha+1} \frac{\Gamma(\alpha)}{ln(a)} x^{-\alpha}$$

But it fails, it doesn't work with fractions or negative integers, it only works with positive integers, so what went wrong? My theory is that the problem is fairly simple, the $log_a(x)$ can't be expressed as a Maclurin series and $log_a(x+C)$ can only be expressed to a series with the condition $|x| < C$ or else it won't diverge

GDI theory : For every function that isn't a Function is analytic at x=0 and doesn't converge over $\mathbb{R}$, the derivative formula $D^\alpha f(x)$ works only on $\mathbb{Z}^+$ for that function , we can also call it not fully real differentiable

GDI hypothesis : the function is differentiable in all its valid input values

These can also be rewritten like this:
Let f(x) be an analytic function defined by its Maclaurin series
GDI theory: If f(x) has a singularity at $x = 0$, then the generalized formula for $D^\alpha f(x)$ will contain a singularity at $\alpha = n$ for $n \in \mathbb{Z}^+$, preventing the generalized formula from equaling the expected integer fractional or anti-derivative. – Todo: change this to look better – This will be explained in detail in later sections

3 Properties of the D^z operator

These are properties to identify the nature of it that will help later with the formula and analysis of what I can call the D^z **plane**. More on that later

3.1 General power series rule

Power series are very important tools to analytically describe a function along the $\mathbb{R}$eal or $\mathbb{C}$omplex planes, which is exactly what we need
The general power series definition for a function(Taylor series) is as follows:

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taking the first derivative for both sides, the first term will cancel as it's a constant , the second term is linear, so it will become constant, the third term is quadruple, will become linear, and 2! will cancel the 2 of the power, and so on
we can write is like this

$$D^1 f(x) = f^{(1)}(a) + \frac{f^{(2)}(a)}{1!}(x-a) + \frac{f^{(3)}(a)}{2!}(x-a)^2 ... = \sum_{n=1}^{\infty} \frac{f^{(n+1)}(a)}{n!}(x-a)^n$$

the cancellation happen also because when we take a derivative the power of $(x-a)$ gets down by 1 to the numerator and we divide $n!$ by it leading to $(n-1)!$ which will lead to infinity in the denominator leading to the whole term to be zero so we can skip the first term and start from 1
Taking the third and the fourth derivative gives the same result up to the k-th

derivative

$$D^k f(x) = f^{(k)}(a) + \frac{f^{(k+1)}(a)}{1!}(x-a) + \frac{f^{(k+2)}(a)}{2!}(x-a)^2 ... = \sum_{n=k}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

that is of course for $k \in \mathbb{Z}$ Now, to make it fully fractional, we will put gamma instead of n and use the x^n general formula

$$D^\alpha f(x) = f(a)^{(\alpha)} + \frac{f^{(\alpha+1)}(a)}{\Gamma(2)}(x-a) + \frac{f^{(\alpha+2)}(a)}{\Gamma(3)}(x-a)^2 ... = \sum_{n=0}^{\infty} \frac{f^{(n+\alpha)}(a)}{\Gamma(n+1)} \left[\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}(x-a)^n \right]$$

In the brackets, we can see the general derivative for x^n Canceling the Gammas out, we get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-\alpha+1)}(x-a)^{n-\alpha}$$

for powers of $(x-a)$ lesser than α it will get to ∞ in the denominator,at least when it's a negative integer other than that it will work normal,this happens because of the gamma pole, but taking the limit it will lead to 0 but still doesn't affect the sum , indeed helping us deleting the first $n < \alpha$ terms Of course, for a negative integer derivative, this works too, since it will it will be positive

Note: because of how many terms you take in the differentiation will always be terms that are replaced by them because of the x^n differentiation and the infinite sum,**even if you differentiate it an infinite number of times**, but you are still deleting values We are going to use that knowledge later in the integration and differential equations in the later sections.

We can now let a = 0 to get the important series we need , the Maclaurin series

$$D^\alpha f(x) = f(0) + \frac{f^{(1)}(0)}{\Gamma(2)}(x) + \frac{f^{(2)}(0)}{\Gamma(3)!}(x)^2 ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

3.2 The linearity of D^z operator

For most of the operations with functions and real-life applications, we need to deal with linearity for the D^z operator which is what we are going to prove in this small section **We Must Prove that:**

$$D^z(c_1 f(x) + c_2 g(x)) = c_1 D^z f(x) + c_2 D^z g(x)$$

let's begin with the simple x^n and let $f(x) = x^n, g(x) = x^m$ Firstly, we differentiate them separately

$$D^z(c_1 f(x)) + D^z(c_2 g(x)) = c_1 D^z(f(x)) + c_2 D^z(g(x)) = c_1 \frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-z} + c_2 \frac{\Gamma(m+1)}{\Gamma(m-z+1)} x^{m-z}$$

Let this be statement 1
Now, let's differentiate them together. We get

$$D^z(c_1f(x) + c_2g(x)) = c_1\frac{\Gamma(n+1)}{\Gamma(n-z+1)}x^{n-z} + c_2\frac{\Gamma(m+1)}{\Gamma(m-z+1)}x^{m-z}$$

Let this be statement 2
since **Statement 1** = **Statement 2** we can say that

$$D^z(c_1f(x) + c_2g(x)) = c_1D^zf(x) + c_2D^zg(x)$$

Q.E.D
This is very useful , but to apply it to all functions , which we can do easily with infinite series
in other words
If D^z is linear on basis functions x^n, and $f = \sum c_nx^n$, then:

$$D^z(\sum c_nx^n) = \sum c_nD^z(x^n)$$

by uniform convergence of the series.

3.3 The Index law

The most important property for the formulas is the index law, which is

$$D^{\alpha+\beta}f(x) = D^\alpha(D^\beta(x))$$

we must prove this holds true for every case First, we need to prove it for the simplest function we have, which is x^n Taking the D^z of the function, we get

$$D^\alpha f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha}$$

Now let's apply the D^β with the knowledge that the Gamma functions are constants in the first derivative

$$D^\beta(D^\alpha f(x)) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}D^\beta(x^{n-\alpha}) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}\left[\frac{\Gamma(n-\alpha+1)}{\Gamma(n-\alpha-\beta+1)}x^{n-\alpha-\beta}\right]$$

The Gamma terms cancel out, and we get

$$D^\beta(D^\alpha f(x)) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)}x^{n-\alpha-\beta}$$

Let this be statement 1
Now, if we start from the beginning again, but this time directly substitute $\alpha+\beta$ as O(stands for orders), we get

$$D^Of(x) = \frac{\Gamma(n+1)}{\Gamma(n-O+1)}x^{n-O}$$

if we substitute $O = \alpha + \beta$ back we get

$$D^{\alpha+\beta} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)} x^{n-\alpha-\beta}$$

Let this be statement 2
if we Equal **statement 1** and **statement 2** we get

$$D^{\alpha+\beta} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)} x^{n-\alpha-\beta} = D^{\beta}(D^{\alpha} f(x))$$

Thus, we can say that

$$D^{\alpha+\beta} f(x) = D^{\beta}(D^{\alpha} f(x))$$

Q.E.D
Note: this also works for imaginary numbers $z + w$
This, by itself, is a simple, elegant proof, but it only works for x^n, and applying the same method for each function will be a very large waste of time.
Instead, we can use **Power Series** as they hold for every analytic function, thus one proof will work for every function

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n+1)} (x-a)^n$$

we can immediately see that it's the simple x^n with everything else being a constant to the derivative
Since we've proven the Index Law for x^n, and the power series represents f as a sum of such terms, the Index Law extends to f by linearity of the operator, and see that

$$D^{\alpha+\beta} f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n+1)} \left[\frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)} (x-a)^{n-\alpha-\beta} \right] = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-\alpha-\beta+1)} (x-a)^{n-\alpha-\beta}$$

which means that the Index law works for any analytic function **Q.E.D**

3.4 The Multiplication Law

A logical next step is to prove that there exists a law for multiplying orders of derivatives, as it will help us later to prove and analyze a lot of topics in this research
let's take for example $f(x) = x^6$ We need (for now) to prove that multiplying two order derivatives, for example, 2 and 3, returns the same result
for now we may write the multiplication as $M[D^{\alpha} f(x)]^{\beta}$ just as a placeholder for now
So **we need to prove that**

$$M[D^2 f(x)]^3 = M[D^3 f(x)]^2 = D^6 f(x)$$

evaluating the first expression gives us $D^6 f(x) = \frac{\Gamma(7)}{\Gamma(6-6+1)} x^{6-6} = \Gamma(7)$ a very logical step to do is to use the **Index Law** we just proved
So we start with

$$D^2 f(x) = \frac{\Gamma(7)}{\Gamma(6-2+1)} x^{6-2} = \frac{\Gamma(7)}{\Gamma(5)} x^4$$

$$D^2(D^2 f(x)) = \frac{\Gamma(7)}{\Gamma(5)} D^2 f(x) = \frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(4-2+1)} x^{4-2} = \frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(3)} x^2$$

$$D^2(D^2(D^2 f(x))) = \frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(3)} D^2 f(x) =$$

$$\frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(3)} \frac{\Gamma(3)}{\Gamma(2-2+1)} x^{2-2} = \frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(3)} \times \frac{\Gamma(3)}{\Gamma(1)}$$

As we can see, the orders cancel out perfectly, leaving only $\Gamma(7)$
Let's generalize the idea with α and β following the same index law but with $f(x) = x^n$

$$D^\alpha f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

$$D^\alpha(D^\alpha f(x)) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} D^\alpha f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} \frac{\Gamma(n-\alpha+1)}{\Gamma(n-\alpha-\alpha+1)} x^{n-\alpha-\alpha}$$

$$= \frac{\Gamma(n+1)}{\cancel{\Gamma(n-\alpha+1)}} \frac{\cancel{\Gamma(n-\alpha+1)}}{\Gamma(n-2\alpha+1)} x^{n-2\alpha} = \frac{\Gamma(n+1)}{\Gamma(n-2\alpha+1)} x^{n-2\alpha}$$

Of course, we can do it again

$$D^\alpha(D^\alpha(D^\alpha f(x))) = \frac{\Gamma(n+1)}{\Gamma(n-2\alpha+1)} D^\alpha(D^\alpha f(x)) = \frac{\Gamma(n+1)}{\Gamma(n-2\alpha+1)} \frac{\Gamma(n-2\alpha+1)}{\Gamma(n-2\alpha-\alpha+1)} x^{n-2\alpha-\alpha}$$

$$= \frac{\Gamma(n+1)}{\cancel{\Gamma(n-2\alpha+1)}} \frac{\cancel{\Gamma(n-2\alpha+1)}}{\Gamma(n-3\alpha+1)} x^{n-3\alpha} = \frac{\Gamma(n+1)}{\Gamma(n-3\alpha+1)} x^{n-3\alpha}$$

We can continue like this up to the $\beta - th$ multiplication and get the same result , so from this, we can define

$$M[D^\alpha f(x)]^\beta = \frac{\Gamma(n+1)}{\Gamma(n-\alpha\beta+1)} x^{n-\alpha\beta}$$

and since $\alpha\beta = \gamma$ where γ is the original intended order , then

$$M[D^\alpha]^\beta = D^\gamma$$

Q.E.D
Of course, this works only when $\beta \in \mathbb{N}$ since it's the set we can apply the Multiplication Law in terms of the Index Law
I believe that to ascend it to general multiplication, we may first define it in a better mathematical language as

$Definition: For\ \beta \in \mathbb{N},\ define\ {}^{\beta}D^{\alpha}\ inductively$
1- ${}^{1}D^{\alpha} = D^{\alpha}$
2- ${}^{n+1}D^{\alpha} = D^{\alpha} \circ {}^{n}D^{\alpha}$
By induction, we proved: ${}^{n}D^{\alpha} = D^{n\alpha}\ for\ n \in \mathbb{N}$

And to extend this to $\mathbb{R}/\mathbb{C}$ for general β, we define: ${}^{\beta}D^{\alpha} := D^{\alpha\beta}$
Then we need to verify this satisfies the expected properties: ${}^{\beta_1}({}^{\beta_2}D^{\alpha}) = D^{\beta_1(\beta_2\alpha)} = D^{(\beta_1\beta_2)\alpha} = {}^{\beta_1\beta_2}D^{\alpha}$
${}^{1}D^{\alpha} = D^{\alpha}$
${}^{0}D^{\alpha} = D^{0}$ identity
This makes it a definition extended by continuity/analyticity.

Which also means that

$$M[D^{\alpha}]^{\beta} = M[D^{\beta}]^{\alpha}$$

Of course, that is when the operation itself is in a commutative ring like ($\mathbb{R}$,$\mathbb{C}$) , **for other rings that aren't commutative, this is False** like $\mathbb{H}$

We adopt the notation ${}^{\beta}D^{\alpha}$ to denote the multiplicative action of order β on derivative D^{α}, distinguishing it from composition $D^{\beta} \circ D^{\alpha}$ (which gives $D^{\alpha+\beta}$ by Index Law) and the direct derivative $D^{\alpha\beta}$ but i recommend the first one as it shows we are going from α to β, which may be used later without ruining the first shape of the D^{z} operator
since the Index Law works well for Taylor series, this will also work well

3.5 Operator inverses and identities

The inverse addition order:
From the Index law, we can assume that there exists an equation in which

$$D^{\gamma}D^{\alpha}(f(x)) = f(x)$$

writing $f(x)$ with the geneal D^{z} operator we get

$$D^{\gamma}D^{\alpha}(f(x)) = D^{0}(f(x))$$

We can use the opposite of the operator on the other side, which gives us

$$D^{\alpha}(f(x)) = D^{-\gamma}D^{0}(f(x)) = D^{-\gamma}f(x)$$

This can hold true if and only if $\gamma = -\alpha$ Substituting this back, we get

$$D^{\alpha}(f(x)) = D^{-(-\alpha)}f(x)$$

which is true thus $D^{-\alpha}D^{\alpha}f(x) = f(x)$
You can see this as a **generalization of the fundamental theorem of calculus** as it holds true for any set of numbers, not only integers

The inverse Multiplication order:
From the multiplication law, we can assume that there exists an equation where

$$^{\gamma}D^{\alpha}f(x) = D^{1}f(x)$$

Using the inverse of the multiplication rule, we get

$$D^{\alpha}f(x) = ^{\frac{1}{\gamma}} D^{1}f(x)$$

which, with the multiplication law, we get

$$D^{\alpha}f(x) = D^{\frac{1}{\gamma}}f(x)$$

this only happens if and only if $\gamma = \frac{1}{\alpha}$ Substituting this back, we get

$$D^{\alpha}f(x) = D^{\frac{1}{\frac{1}{\alpha}}}f(x) = D^{\alpha}f(x)$$

which is true, thus

$$D^{\frac{1}{\alpha}}D^{\alpha}f(x) = D^{1}f(x) = D^{\alpha}f(x)$$

inverse Multiplication order theorem: For any analytic function that can be expressed as $\sum_{n=0}^{\infty} a_n x^n$, for the order derivative α there exists an order β such that $^{\beta}D^{\alpha}f(x) = D^{1}f(x)$ and $\beta = \frac{1}{\alpha}$

Of course, there are some things that we can state :

1. The derivative order addition identity: D^0
2. **The derivative order multiplication identity**: D^1
3. **The zeroth-derivative order multiplication property**: $^{0}D^{\alpha}$ and $^{\beta}D^{0}$, or in other words, the function itself is the zero of derivative order multiplication , going to or going from
4. **Every derivative order has its addition inverse and multiplication inverse that lead to the function itself and the first order derivative function**
5. **There are infinitely many ways to get to a function from its derivatives and integrals**

- **Distribution of orders and Order of operations:**

since the whole operation is linear, we can distribute operations on the order, but to prove this, we need to prove that $^{\beta}D^{(\alpha+\gamma)} = {}^{\beta}D^{\alpha} \circ {}^{\beta}D^{\gamma}$
on the left hand side we can let $\sigma = \alpha + \gamma$ so we go:

$$^{\beta}D^{\sigma} = D^{\beta\sigma} = D^{\beta(\alpha+\gamma)} = D^{\beta\alpha+\beta\gamma}$$

with the right hand side we can go

$$ {}^{\beta}D^{\alpha} \circ {}^{\beta}D^{\gamma} = D^{\beta\alpha} \circ D^{\beta\gamma} = D^{\beta\alpha+\beta\gamma} $$

Because both the right-hand side and the left-hand side are equal, we can say that there is order for operations, as multiplication comes before addition, and there is distribution between the order of operations let's take, for example, ${}^{\beta}D^{\alpha} \circ D^{\gamma}$, using what we know to write this as one derivative order, we can write

$$ D^{\beta\alpha}D^{\gamma} = D^{\beta\alpha+\gamma} $$

multiplication then addition
Derivative Order constant We can assume from the integer derivative in the function x^n that there exists a derivative order with a constant, and that will also happen when $\alpha = n$ because then we have x^0, so applying this, we get

$$ D^{\alpha}x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha} = \frac{\Gamma(n+1)}{\Gamma(1)}x^0 = \Gamma(n+1) $$

4 Deriving the Rules of the Fractional and Complex Derivatives

Most of the other functions need us to derive the rules of the fractional Differintegar operator, the α-th derivative for functions like $tan(x)$,$arcsech(x)$,$ln(x)$, etc.. can only be found using power series and product rule

4.1 General product rule

One of the most important and needed formulas in calculus in the product rule let $f(x) = g(x)h(x)$

$$ f'(x) = g(x)h'(x) + g'(x)h(x) $$

$$ f''(x) = g(x)h''(x) + 2\ g'(x)h'(x) + g''(x)h(x) $$

$$ f'''(x) = g(x)h'''(x) + 3\ g'(x)h''(x) + 3\ g''(x)h'(x) + g'''(x)h(x) $$

This is very similar to the binomial theorem; the only difference is it deals with derivatives instead of powers
The general product rule, also known as the **General Leibniz rule**, is

$$ D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k}(f)D^k(g) $$

Of course, whatever the formula for fractional derivatives turns out to be, this is the Integer Value Product Rule
this simple yet elegant formula is what we are going to use for the General

product rule. Before we try to do a simple substitution of α, we need to use the generalized ${}^nC_k$, which means using $n! = \Gamma(n+1)$ in the formula $\frac{n!}{(n-k)!k!}$

$$D^n(fg) = \sum_{k=0}^{n} \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} D^{n-k}(f)D^k(g)$$

This is the same formula, just works the same for positive integers, but let's try to use fractions and ignore the sigma upper term and expand it , for example one one-half expansion will be

$$D^{1/2}(fg) = \frac{\Gamma(\frac{1}{2})}{\Gamma(1\frac{1}{2}-0)\Gamma(1)} D^{1/2}(f)D^0(g) + \frac{\Gamma(\frac{1}{2})}{\Gamma(1\frac{1}{2}-1)\Gamma(2)} D^{-1/2}(f)D^1(g) + ...$$

$$D^{1/2}(fg) = \frac{\Gamma(\frac{1}{2})}{\Gamma(1\frac{1}{2})} D^{1/2}(f)g + D^{-1/2}(f)D^1(g) + \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{1}{2})\Gamma(3)} D^{-3/2}(f)D^2(g) + ...$$

as we can see , it expands to an infinite sum, it will never stop because the lower value never hit the upper value, the main reason the simple form works for integers is that even if the k value goes higher than the n value it will get a negative integer in a Gamma which is a pole and thus equal zero
that means in order for a term to be it must not have the Gamma of integers
So that means we can't find a non-positive integer product rule yet, but anyway, turning back the sum from what we know will be

$$D^\alpha(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} D^{\alpha-k}(f)D^k(g)$$

We can mark this as the general power rule, but we need to make our proof
We can do a simple proof by intuition
First, we do it for 1:

$$D^1(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(2)}{\Gamma(-k+2)\Gamma(k+1)} D^{1-k}(f)D^k(g) = \frac{1}{1}D^1 f(x)g(x) + \frac{1}{1}f(x)D^1 g(x)$$

Then we have the formula for α-th derivative

$$D^\alpha(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} D^{\alpha-k}(f)D^k(g)$$

Then we make two different statements
In the first, we take the α-th derivative and derivative it again

$$D^1(D^\alpha(fg)) = D^1(\sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} D^{\alpha-k}(f)D^k(g)) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} D^1(D^{\alpha-k}(f)D^k(g))$$

We can do this move since only the functions have x in them, like doing it to $\sum_n^\infty a_n x^n$, with the index law we can say that this equals $D^{\alpha+1}$

$$D^{\alpha+1}(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} \left[D^{\alpha-k+1}(f)D^k(g) + D^{\alpha-k}f(x)D^{k+1}g(x) \right]$$

And we can split the sums

$$D^{\alpha+1}(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} D^{\alpha-k+1}(f)D^k(g) + \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} D^{\alpha-k} f(x) D^{k+1} g(x)$$

then in the second sum we can replace k with $k-1$, which makes the sum start from 1

$$\sum_{k=1}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-(k-1)+1)\Gamma((k-1)+1)} D^{\alpha-(k-1)} f(x) D^{(k-1)+1} g(x)$$

$$= \sum_{k=1}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+2)\Gamma(k)} D^{\alpha+1-k} f(x) D^k g(x)$$

But since the zero-th term vanishes because of $\Gamma(0)$ in the declinometer , we can start from there , so the first statement changes to

$$D^{\alpha+1}(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} D^{\alpha-k+1}(f)D^k(g) + \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+2)\Gamma(k)} D^{\alpha-k+1} f(x) D^k g(x)$$

Factoring both sums out, we get

$$D^{\alpha+1}(fg) = \sum_{k=0}^{\infty} \left[\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+2)\Gamma(k)} \right] D^{\alpha-k+1} f(x) D^k g(x)$$

$$= \sum_{k=0}^{\infty} \left[\frac{\Gamma(\alpha-k+2)\Gamma(k)\Gamma(\alpha+1) + \Gamma(\alpha-k+1)\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)\Gamma(\alpha-k+2)\Gamma(k)} \right] D^{\alpha-k+1} f(x) D^k g(x)$$

let's try to simplify things using the gamma identity $\Gamma(x+1) = x\Gamma(x)$

$$= \sum_{k=0}^{\infty} \left[\frac{\Gamma(\alpha+1)((\alpha-k+1)\Gamma(\alpha-k+1)\Gamma(k) + \Gamma(\alpha-k+1)k\Gamma(k))}{\Gamma(\alpha-k+1)k\Gamma(k)(\alpha-k+1)\Gamma(\alpha-k+1)\Gamma(k)} \right] D^{\alpha-k+1} f(x) D^k g(x)$$

$$= \sum_{k=0}^{\infty} \left[\frac{\Gamma(\alpha+1)((\alpha-k+1)\cancel{\Gamma(\alpha-k+1)}\cancel{\Gamma(k)} + \cancel{\Gamma(\alpha-k+1)}k\cancel{\Gamma(k)})}{\cancel{\Gamma(\alpha-k+1)}k\cancel{\Gamma(k)}(\alpha-k+1)\Gamma(\alpha-k+1)\Gamma(k)} \right] D^{\alpha-k+1} f(x) D^k g(x)$$

$$= \sum_{k=0}^{\infty} \left[\frac{\Gamma(\alpha+1)((\alpha-k+1)+k)}{k(\alpha-k+1)\Gamma(\alpha-k+1)\Gamma(k)} \right] D^{\alpha-k+1} f(x) D^k g(x)$$

$$= \sum_{k=0}^{\infty} \left[\frac{\Gamma(\alpha+1)(\alpha+1)}{(\alpha-k+1)\Gamma(\alpha-k+1)k\Gamma(k)} \right] D^{\alpha-k+1} f(x) D^k g(x)$$

using the Gamma identity again, but backwards, gives us

$$= \sum_{k=0}^{\infty} \left[\frac{\Gamma(\alpha+1)((\alpha-k+1)+k)}{k(\alpha-k+1)\Gamma(\alpha-k+1)\Gamma(k)} \right] D^{\alpha-k+1} f(x) D^k g(x)$$

$$= \sum_{k=0}^{\infty} \left[\frac{\Gamma(\alpha+2)}{\Gamma(\alpha-k+2)\Gamma(k+1)} \right] D^{\alpha-k+1} f(x) D^k g(x)$$

We can let this be **statement 1**
then we do the $D^{\alpha+1}$ derivative from the beginning

$$D^{\alpha+1}(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+2)}{\Gamma(\alpha-k+2)\Gamma(k+1)} D^{\alpha+1-k}(f) D^k(g)$$

we can let this be **statement 2**
since we can see that **statement 1 = statement 2**,thus the formula works
Q.E.D
Of course, there will exist some functions that don't have a closed form other than a power series, so we can get a simple series version of the α-th derivative
We can use the original formula, but we replace every function with its series

$$D^{\alpha}(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{\Gamma(j-\alpha+k+1)} x^{(j-\alpha+k)} \sum_{i=0}^{\infty} \frac{g^{(i)}(0)}{\Gamma(i-\alpha+k+1)} x^{(i-\alpha+k)}$$

4.2 General chain rule

The General Chain Rule is hard; we can try any of our normal methods, yet they fail
The easier way is to go with the series option We can get the Maclaurin series with ease

4.2.1 General $f(x^n)$ formula

First we define our functions for simpler solving

$$u(x) = x^n \qquad g(x) = f(u)$$

Then we differentiate for the first three derivatives for $u(x)$

$$u'(x) = nx^{n-1} \qquad u''(x) = n(n-1)x^{n-2} \qquad u'''(x) = n(n-1)(n-2)x^{n-3}$$

then we differentiate $g(x) = f(u)$with respect to u

$$g'(x) = u'f'(u) \qquad g''(x) = u''f'(u) + (u')^2 f''(u)$$

$$g'''(x) = u'''f'(u) + u''u'f''(u) + 2u'u''f''(u) + (u')^3 f'(u)$$

Then we evaluate with the Maclaurin series

$$g(x) = f(0) + u'(0)f'(0)x + \frac{u''(0)f'(0) + (u'(0))^2 f''(0)}{2!} + ...$$

using the chain rule, for the x_1 term we have $nx^{n-1}f'(x^n)$
evaluating at 0 gives us 0 for any term that isn't 1, evaluating at one gives us

$0^0 f'(0)$, so we need to approach it with limits
if we take the limit as x approach zero so we have

$$\lim_{x \to 0} x^0 f'(x^n) = f'(0)$$

So it works fine with the first term, we can do the same for the rest terms, and we get different results based on different multiples of n. If we choose k to be 2, only even terms will show up as other terms equal 0. For n = 3, we get multiples of 3, so on, so forth, and so we get the series

$$f(x^n) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{kn}$$

We can give it a simple check by calculating a random value for a random function like $sin(x^3)$
the series for this function will be $sin(x^3) = \sum_{k=0}^{\infty} \frac{(-1^k)}{(2k+1)!} x^{6n+3}$ we let it equal to $f(x)$ so we can see the difference between both sides

$$\sin(0.365^2) = 0.0486079633377 \qquad f(0.365) = 0.0486079633377$$

We can see that it works fine, now for the α-th derivative, we get a simple

$$D^\alpha f(x^n) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(kn - \alpha + 1)} x^{kn-\alpha}$$

Thus, a simple formula, but something closer to the general Maclaurin series chain rule

4.2.2 General $f(ax)$ formula

Although not quite the difficult formula to get yet, it's easier with a closed form and has big importance in other fields like Fourier transforms
first let $g(x) = f(ax)$, for the first few derivatives we have

$$g'(x) = af'(ax) \qquad g''(x) = a^2 f''(ax) \qquad g'''(x) = a^3 f'''(ax)$$

You probably can guess the rule from the pattern, but to make sure, let's continue with the Maclaurin series

$$g(x) = f(0) + af'(0)x + \frac{a^2 f'(0)}{2!} x^2 + ... = \sum_{n=0}^{\infty} \frac{a^n f^{(n)}(0)}{n!} x^n$$

and from this we can say that $D^n f(ax) = a^n f^{(n)}(ax)$ applying α instead of n will give us

$$D^\alpha f(ax) = a^\alpha f^{(\alpha)}(ax)$$

simple yet needed

4.2.3 General Series form for chain rule

first we find the first derivatives with $h(x) = f(g(x))$
$h'(x) = f'(g(x))g'(x)$, $h''(x) = f'(g(x))(g'(x))^2 + g''(x)f'(g(x))$ Then we do a simple series expansion

$$h(x) = h(0) + h'(0)x + \frac{h''(0)x^2}{2!} + ... = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!}x^n$$

Then we substitute the derivatives

$$f(g(x)) = f(g(0)) + f'(g(0))g'(0)x + \frac{(f'(g(0))(g'(0))^2 + g''(0)f'(g(0)))x^2}{2!} + ...$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(g(0))}{n!}(g(x) - g(0))^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(g(0))}{n!}\left(\sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!}x^k - g(0)\right)^n$$

5 Interpretation of fractional and complex derivative in other functions

5.1 Trigonometric Functions

Deriving the trigonometric functions can be quite tricky , as there exists an n-th derivative formula for them, but it doesn't seem to work as intended

5.1.1 $\sin(x)$ and $\cos(x)$

for $\sin(x)$ there exists a formula which is

$$D^n \sin(x) = \sin(\frac{n\pi}{2} + x)$$

We need to first expand it to it's exponentiation formula

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

then we here can differentiate with ease using our $D^\alpha e^{ax} = a^\alpha e^{ax}$ formula

$$D^\alpha \sin(x) = \frac{i^\alpha e^{ix} - (-i)^\alpha e^{-ix}}{2i}$$

to make it look simpler we can use $i = e^{\frac{i\pi}{2}}$ and $i = e^{\frac{i\pi}{2}+i\pi} = e^{\frac{-i\pi}{2}}$

$$D^\alpha \sin(x) = \frac{(e^{\frac{i\pi}{2}})^\alpha e^{ix} - (e^{\frac{-i\pi}{2}})^\alpha e^{-ix}}{2i} = \frac{(e^{\frac{i\pi\alpha}{2}})e^{ix} - (e^{\frac{-i\pi\alpha}{2}})e^{-ix}}{2i}$$

$$= \frac{e^{\frac{i\pi\alpha}{2}+ix} - e^{\frac{-i\pi\alpha}{2}-ix}}{2i} = \frac{e^{i(\frac{\pi\alpha}{2}+x)} - e^{-i(\frac{\pi\alpha}{2}+x)}}{2i} = \sin(\frac{\alpha\pi}{2} + x)$$

and use the Index Law work here too so $D^\alpha(D^\beta \sin(x)) = \sin(\frac{(n+m)\pi}{2} + x)$ Now we can test it for the half derivative

$$D^{\frac{1}{2}} \sin(x) = \sin(\frac{\frac{1}{2}\pi}{2} + x) = \sin(\frac{\pi}{4} + x)$$

$$D^{\frac{1}{2}}(D^{\frac{1}{2}} \sin(x)) = \sin(\frac{(\frac{1}{2} + \frac{1}{2})\pi}{2} + x) = \sin(\frac{2\pi}{4} + x) = \sin(\frac{\pi}{2} + x)$$

the same also works for $\cos(x)$

$$D^\alpha \cos(x) = \frac{e^{i(\frac{\pi\alpha}{2}+x)} + e^{-i(\frac{\pi\alpha}{2}+x)}}{2i} = \cos(\frac{\alpha\pi}{2} + x)$$

But there is another proof that this works for all real numbers.

from the eular formula $e^{ix} = cos(x) + i\sin(x)$ we can say that

$$\sin(x) = \mathrm{Im}(e^{ix})$$

Taking the alpha-th derivative of both sides

$$D^\alpha \sin(x) = D^\alpha \mathrm{Im}(e^{ix}) = \mathrm{Im}(i^\alpha e^{ix})$$

knowing that $i = e^{i\pi/2}$

$$D^\alpha \sin(x) = \mathrm{Im}(e^{i\pi\alpha/2}e^{ix}) = \mathrm{Im}(e^{i\pi\alpha/2+ix}) = \mathrm{Im}(e^{i(\alpha\pi/2+x)})$$

turning this back to the euler formula will give us

$$D^\alpha \mathrm{Im}(e^{ix}) = \sin(\frac{\alpha\pi}{2} + x)$$

which indeed proves it's true from the same formual we can also get the α-th for $cos(x)$ with the same formula turning this back to the euler formula will give us

$$D^\alpha \mathrm{Re}(e^{ix}) = cos(\frac{\alpha\pi}{2} + x)$$

Now we can write them as

$$D^\alpha \sin(x) = \mathrm{Im}(e^{i(\alpha\pi/2+x)}) \qquad D^\alpha cos(x) = \mathrm{Re}(e^{i(\alpha\pi/2+x)})$$

or

$$D^\alpha \sin(x) = \sin(\frac{\alpha\pi}{2} + x) \qquad D^\alpha cos(x) = cos(\frac{\alpha\pi}{2} + x)$$

And to make it to the complex plane, we can also use these formulas. The negative derivative of these is

$$D^{-1} \sin(x) = \sin(\frac{-\pi}{2}+x) = -cos(x) \qquad D^{-1} cos(x) = cos(\frac{-\pi}{2}+x) = \sin(x)$$

and the first complex derivative of these is

$$D^i \sin(x) = \sin(\frac{i\pi}{2}+x) = \sin(\frac{ln(-1)}{2}+x) \qquad D^i cos(x) = cos(\frac{i\pi}{2}+x) = cos(\frac{ln(-1)}{2}+x)$$

5.1.2 $\tan(x)$ and $\sec(x)$

Finding the alpha-th derivative for $tan(x)$ is quite hard since we didn't get any direct formulas for quotients , and there is no direct integer derivative formula we can plug in and generalize to the Real numbers. We can try to change it a little with some algebra

$$tan(x) = \sin(x)\ (cos(x))^{-1}$$

And then use the general product rule ,but quickly, we can see the problem

$$D^{\alpha}(\sin(x)\ (cos(x))^{-1}) = \sum_{k=0}^{\infty} \frac{\Gamma(0)}{\Gamma(\alpha-k+1)\Gamma(k+1)} D^{\alpha-k}(\sin(x)) D^{k}(cos(x)^{-1})$$

There is a gamma pole, so this solution also fails
We can try using some trig substitution

$$tan(x) = \sin(x)\ (\sqrt{\sin(x)^2+1})^{-1}$$

But this leads to an infinite sum for the product rule and the chain rule that we have proved above. It is impossible to find with the simple algebra we have is very hard to approximate by itself As we can see, there is no simple, elegant closed form for $tan(x)$ in the scientific paper. The reason behind this will be explained later, but we can simply say that it has poles. To analytically see it better, we need the Maclaurin series expansion

$$tan(x) = x + \frac{x^3}{3!} + \frac{2x^5}{15} + ... = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} \qquad where\ |x| < \frac{\pi}{2}$$

If we look closely, we can notice the problem; it has a radius of convergence and that by itself is the problem that will be discussed in detail later All we can do for now is apply the D^{α} to the infinite sum, as it will be the only analytic closed form way for now
We will get:

$$D^{\alpha} tan(x) = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{\Gamma(2n)} \left[\frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1} \right]$$

$$= \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}$$

This infinite series shall work for now
The same also works for $sec(x)$ as it's a quotient so if we tried using the General product rule, we would hit a Gamma pole, so the safest answer for now is to go with the infinite series

$$sec(x) = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{E_{2n}(-1)^n}{(2n)!} x^{2n} \qquad where\ |x| < \frac{\pi}{2}$$

Again, we see the same problem with the radius of convergence
Simply, we apply D^α:

$$D^\alpha sec(x) = \sum_{n=0}^{\infty} \frac{E_{2n}(-1)^n}{\Gamma(2n)} \left[\frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha} \right]$$
$$= \sum_{n=0}^{\infty} \frac{E_{2n}(-1)^n}{\Gamma(2n-\alpha+1)} x^{2n-\alpha}$$

Note: these two work for their radius of convergence only
Note: they also work for the complex derivative

5.1.3 $\csc(x)$ and $\cot(x)$

Now, saying $\csc(x)$ and $\cot(x)$ will work the same as the rest of the trigonometric functions is a bit of a stretch.
since we already know both of their domains aren't $x \in \mathbb{R}$, so they must have some sort of analytical poles and convergence radius that isn't $\mathbb{R}$ in their infinite sums
But if we noticed

$$\csc(x) = \frac{1}{\sin(x)}$$

which means that it has a singularity at $x = 0$, in other words simple Taylor series or a simple Maclurin series won't work, we need the General Laurent series for this one The Laurent series is :

$$\csc(x) = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + ... = \sum_{n=0}^{\infty} \frac{B_{2n}(-1)^{n+1}(2^{2n}-1)}{(2n)!} x^{2n-1} \qquad where\ 0 < |x| < \pi$$

But before plugging the D^z operator into the series, we can notice a little problem in the beginning, the $\frac{1}{x}$ term
Simply plugging in the $D^z(x^n)$ will result in a pole. The simple solution is just to take the linearity of D^z and differentiate the first term alone, and then the rest of the series alone

$$D^\alpha \csc(x) = D^\alpha(x^{-1}) + \sum_{n=1}^{\infty} \frac{B_{2n}(-1)^{n+1}(2^{2n}-1)}{\Gamma(2n)} \left[\frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1} \right]$$
$$= \frac{(-1)^\alpha \Gamma(1+\alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{B_{2n}(-1)^{n+1}(2^{2n}-1)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}$$

of course this is where $0 < |x| < \pi$
The same goes for $\cot(x)$ as it doesn't have any Taylor series but rather a Laurent series.
The Laurent series for $\cot(x)$ is:

$$\cot(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} x^{2n-1} \qquad where\ 0 < |x| < \pi$$

Applying D^α to both sides we get

$$D^\alpha \cot(x) = \frac{(-1)^\alpha \Gamma(1+\alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}$$

Note: this works to complex numbers too

5.2 Hyperbolic Functions

5.2.1 $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$

$\sinh(x)$ is pretty straight forward to get
from the definition

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

differentiating both sides to the α

$$D^\alpha \sinh(x) = \frac{1}{2}(D^\alpha e^x - D^\alpha e^{-x}) = \frac{1}{2}(e^x - (-1)^\alpha e^{-x})$$

We can also do the same for $\cosh(x)$

$$D^\alpha \cosh(x) = \frac{1}{2}(D^\alpha e^x + D^\alpha e^{-x}) = \frac{1}{2}(e^x + (-1)^\alpha e^{-x})$$

but if we change the negative sign in $\sinh(x)$ to $+(-1)$ we turn the derivative to

$$D^\alpha \sinh(x) = \frac{1}{2}(e^x + (-1)^{\alpha+1} e^{-x})$$

which is equal to $D^{\alpha+1}\cosh(x)$ and that is becuase unlike normal $\sin(x)$ and $\cos(x)$ these are the integer integrals and derivatives of their-selves So we can get the negative derivatives to be

$$D^{-1} \sinh(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x) \qquad D^{-1} \cosh(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$$

For the complex derivatives, we can use the formulas from before
However, for $\tanh(x)$ things change , since the definition for it is

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

We can see , it's a quotient of two functions ,which leaves us with nothing but to use its power series
the power series for tanh is:

$$\tanh(x) = x - \frac{x^3}{3!} + \frac{2x^5}{15} - ... = \sum_{n=0}^{\infty} \frac{B_{2n} 4^n (1-4^n)}{(2n)!} x^{2n-1} \qquad where\ |x| < \frac{\pi}{2}$$

As predicted, there will also be a radius of convergence here too
But anyway, we get the D^α with this:

$$D^\alpha tanh(x) = \sum_{n=0}^{\infty} \frac{B_{2n}4^n(1-4^n)}{\Gamma(2n)} \left[\frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1} \right]$$

$$= \sum_{n=0}^{\infty} \frac{B_{2n}4^n(1-4^n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}$$

5.2.2 sech(x) , csch(x) and coth(x)

The rest of the hyperbolic functions shall work the same as the trigonometric functions, in fact, all of the hyperbolic and the trigonometric functions' Laurent/Taylor series look identical with little changes, so finding them won't be that difficult
For sech(x) the Taylor Series is:

$$\operatorname{sech}(x) = 1 - \frac{x^2}{2!} + \frac{5x^4}{4!} - ... = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n} \qquad where\ |x| < \frac{\pi}{2}$$

So Applying D^z will be as simple as sec(x)

$$D^\alpha \operatorname{sech}(x) = \sum_{n=0}^{\infty} \frac{E_{2n}}{\Gamma(2n-\alpha+1)} x^{2n-\alpha} \qquad where\ |x| < \frac{\pi}{2}$$

the same goes for the Laurent series of csch(x) and coth(x)

$$\operatorname{csch}(x) = \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - ... = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{B_{2n}(1-2^{2n})}{(2n)!} x^{2n-1} \qquad where\ 0 < |x| < \pi$$

Of course, it's similar but not identical , anyway, applying D^α gives us

$$D^\alpha \operatorname{csch}(x) = D^\alpha x^{-1} + D^\alpha \sum_{n=0}^{\infty} \frac{B_{2n}(1-2^{2n})}{(2n)!} x^{2n-1}$$

$$= \frac{(-1)^\alpha \Gamma(1+\alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{B_{2n}(1-2^{2n})}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1} \qquad where\ 0 < |x| < \pi$$

and for coth(x):

$$\coth(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n}B_{2n}}{(2n)!} x^{2n-1} \qquad where\ 0 < |x| < \pi$$

and applying D^α operator, we get:

$$D^\alpha \coth(x) = \frac{(-1)^\alpha \Gamma(1+\alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{2^{2n}B_{2n}}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}$$

5.3 The Inverse Trigonometric and Hyperbolic Functions

There is a problem with these functions that makes them special, if we for example tried to take the derivative for $\sin^{-1}(x)$ and the integral of the same function we get this

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \quad -1 < x < 1 \qquad\qquad \int \sin^{-1}(x)dx = x\sin^{-1}(x)+\sqrt{1-x^2}+C$$

As we can see, they are a type of **Transformative functions** which will be discussed in later sections

6 Matrix derivative functions

6.1 Whole matrix derivatives

we start with the simple function x^n, and let our matrix be a simple A matrix, we can right the $D^A x^n$ like this:

$$D^A(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-A+1)} x^{n-A}$$

As we can see, this shape is hard , and a better simplification for it is to use $e^{ln(x)}$, so it will be :

$$D^A(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-A+1)} x^n e^{-\ln(x)A}$$

This is better, as now we can express it as an infinite sum using the Taylor series of the function e^x

$$D^A(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-A+1)} x^n \sum_{k=0}^{\infty} \frac{(-1)^k (\ln x)^k}{k!} A^k$$

where $A^k = \underbrace{A \times A \times A \times ....}_{k \text{ times}}$, and this can be called the simple $D^A x^n$

Let's plug this definition into the Taylor series general formula

$$D^\alpha f(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-\alpha+1)} (x-a)^{n-\alpha}$$

So now it shall be

$$D^A f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-A+1)} (x-a)^{n-A}$$

and thus the Maclaurin series is

$$D^A f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n-A+1)} x^{n-A} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n-A+1)} x^n e^{-\ln(x)A}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n-A+1)} x^n \sum_{k=0}^{\infty} \frac{(-1)^k (\ln x)^k}{k!} A^k$$

Now we know we can do it to any analytical function, and the simple way to put it in any series is to remove the α from the power of and put the term $\sum_{k=0}^{\infty} \frac{(\ln x)^k}{k!} A^k$

6.2 What is the meaning of matrix derivatives

To understand what even is a matrix order derivative, we can try and find a simple matrix order for matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and for the function to be $f(x) = x^2$
So the derivative will be

$$D^A(x^2) = \frac{\Gamma(3)}{\Gamma(3-A)} x^2 \sum_{k=0}^{\infty} \frac{(-1)^k (\ln x)^k}{k!} A^k$$

and this one is simple , since $A = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2I$,we can write the A^k term as $2^k I$,now we can take it out of the summation as it's a constant multiplication forming back to it's original form we get $e^{-2\ln(x)} I$ which is $x^{-2} I$
For the gamma term, we can also do it simply by subtraction. We get $\Gamma(3I - 2I) = \Gamma(I) = I$ So, what we know now we have this expression

$$D^A(x^2) = \frac{2}{I} x^2 \times x^{-2} I = 2I^2 x^{2-2} = 2I$$

So, surprisingly, we get the derivative order matrix

6.3 Matrix order properties

Order Matrix: if we take the derivative A of the function x^n where A is the identity matrix multiplied by n will return $\Gamma(n+1)I$, in other words, any n scaler matrix order derivative will return $\Gamma(n+1)$ times identity matrix
Proof:
let A be a matrix order equal to $\begin{bmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & \alpha & 0 & \cdots & 0 \\ 0 & 0 & \alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha \end{bmatrix}$ and we have function x^α

$$D^A(x^\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-A+1)} x^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k (\ln x)^k}{k!} A^k$$

we know that $A = \alpha \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \alpha I$, the term in the infinite sum will be $A^n = \alpha^n I$, then any number we can treat like scalar matrix as it works the same, so the Gamma term we can write as $\Gamma(\begin{bmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & \alpha & 0 & \cdots & 0 \\ 0 & 0 & \alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha \end{bmatrix} - \begin{bmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & \alpha & 0 & \cdots & 0 \\ 0 & 0 & \alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha \end{bmatrix} + I) = \Gamma(I) = I$

Putting everything together, we get

$$D^A(x^\alpha) = \Gamma(\alpha+1)I^{-1}x^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k(\ln x)^k}{k!}\alpha^k I^k$$

forming the infinite sum back to its original form, we get

$$D^A(x^\alpha) = \Gamma(\alpha+1)Ix^\alpha e^{-\alpha \ln(x)}I = \Gamma(\alpha+1)I^2x^{\alpha-\alpha} = \Gamma(\alpha+1)I$$

Q.E.D

We can consider this as the Matrix version of the **Derivative Order constant** but a proof was needed. Another simpler one is treating these scalar matrices as numbers, then doing the same proof from the section above **Order Addition and Multiplication** Let A,B Be Different matrices with n by n elements Since $A + B = B + A$ then $D^A \circ D^B = D^B \circ D^A$. Thus **Matrix order addition is commutative** , that is because the adoptive nature of D^z with its order field/ring that we have proven in before section

And for the same reason since $AB \neq BA$ then ${}^BD^A \neq {}^AD^B$, Thus Matrix Order Multiplication isn't commutative

7 Functional derivatives

7.1 derivative order as function of itself

The Derivative order being a function is hard, unless we are talking about a function of itself instead of x or y it will be $g(\alpha)$, the formulas for x^n it shall be

$$D^{g(\alpha)}f(x) = \frac{\Gamma(n+1)}{\Gamma(n-g(\alpha)+1)}x^{n-g(\alpha)}$$

This simple formula shall be the way for us to get the Maclaurin and Taylor series

$$D^{g(\alpha)}f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{\Gamma(n - g(\alpha) + 1)}(x - a)^{g(\alpha)}$$

$$D^{g(\alpha)}f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{\Gamma(n - g(\alpha) + 1)}x^{n-g(\alpha)}$$

7.2 derivative order as function of its dependent variable

We can now do the same for functions that are dependent on the function variable

$$D^{g(x)}f(x) = \frac{\Gamma(n + 1)}{\Gamma(n - g(x) + 1)}x^{n-g(\alpha)}$$

same formula here ,yet things change a lot, because that is a one time jump, the next time won't go according to the normal formula since we will have x^x term if not more, so the index law nore the multiplication law work here... in fact no normal formula or law will work after the first one, which is a problem but not a big one since we already can jump one time
There exists a Maclurin series for it

$$D^{g(x)}f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{\Gamma(n - g(x) + 1)}x^{n-g(x)}$$

Part II
D^z Analysis

8 The nature of functions under D^z Field

8.1 Explaining what a fractional derivative is

Taking a positive Integer derivative gives us the rate of change of a function , taking the derivative of that also gives us the rate of change of the derivative function , so on, so forth
Taking the negative Integer derivative is known as the "Area under the curve" or the function in which the original function is the rate of change of it , or simply the "Anti-derivative"
If we tried to explain the D^z operator with the standard definition of integrals and derivatives, we can say that
If the first derivative is the rate of change of the function, the first integral is the area under the curve, then the D^z operator changes from being an Area to the function than the rate of the function itself, but this doesn't make sense, mostly because these are three different things that work and measure differently. How could we explain such things?

The problem becomes bigger and bigger when we say the second derivative, which is the rate of change of the rate of change of the function that gives us information about the function, like when it is at a climax or the function's convexity, or the second integral, which is the 3D volume of the shape.
Then comes the Taylor expansion and the power series of functions which use infinite derivatives, and now we are all out of the meaning of it
even if we thought about it physically , for simple cases like position, its derivative is velocity, and its second derivative is acceleration, which at least changes the unit powers and doesn't make much sense with D^z
In fact, without standers, even if multiple derivatives of integers would somehow make sense, the fractional derivative wouldn't
which is why I suggest another perspective for the derivative, instead of it being the area under the curve or the rate of change , these things are just side effects and the derivative is its actual object by itself
The fractional derivative can be thought of as some change in the being of the function itself, as we can see from the x^n derivative

$$D^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

as we can see ,what really happen is the function get scalled by a certain amount that is $\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}$, and get multiplied by the simple form of it that is $g(x) = x$ raised to the power of the derivative order but from the negative side, if we let $f(x) = x^n$ and $g(x) = x$ and the Scaler S, we can write it like this

$$D^\alpha f(x) = S\ f(x) g(x)^{-\alpha} = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} \frac{f(x)}{g(x)^\alpha}$$

and if we use the Euler product expansion of the $\Gamma(x)$
We can write the function in its simplest, algebraic form with no integrals

$$D^\alpha x^n = \frac{\frac{1}{n+1} \prod_{k=1}^{\infty} \left[\frac{1}{1+\frac{n+1}{k}} \left(1+\frac{1}{k}\right)^{n+1} \right]}{\frac{1}{n-\alpha+1} \prod_{k=1}^{\infty} \left[\frac{1}{1+\frac{n-\alpha+1}{k}} \left(1+\frac{1}{k}\right)^{n-\alpha+1} \right]} \frac{x^n}{x^\alpha}$$

Here, it's the derivative in a simpler form that can show us why we can't over-derivative the function, because we will get a division by zero, but what we can also see is that the function works for non-positive integers
Examining the function's behavior under repeated differentiation, we see that the function becomes increasingly discrete and less smooth. This effect is driven by the scaling factor in the formula, not just by changes in the exponent. Although negative fractional derivatives are defined, the real challenge emerges with non-positive integers, which can lead to undefined expressions.

The meaning of a derivative of complex order extends the concept of fractional derivatives, though our understanding of fractional derivatives is not yet complete. A complex derivative is neither exactly a standard fractional derivative nor simply the rate of change. To explore this idea, consider how the operator affects a basic function like x^n, using the formula

$$D^z(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-z+1)}x^{n-z}$$

For example, taking $z = i$ gives

$$D^i(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-i+1)}x^{n-i}$$

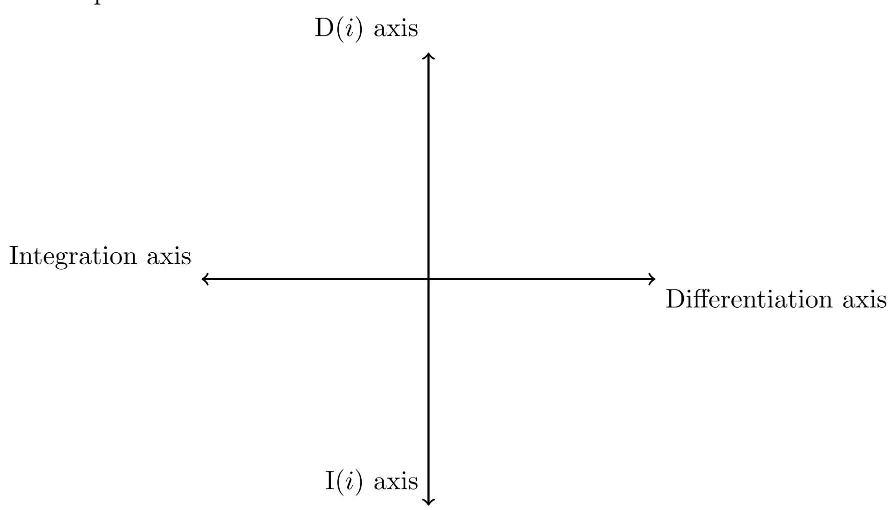
This helps us see the transformation introduced by complex orders. The nature of this will be explored later, but for now, if we try to take another imaginary derivative using the **Multiplication Law** we proved earlier, we see that

$${}^iD^i(x^n) = D^i(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-(i\times i)+1)}x^{n-(i\times i)} = \frac{\Gamma(n+1)}{\Gamma(n-(-1)+1)}x^{n-(-1)} = \frac{\Gamma(n+1)}{\Gamma(n+2)}x^{n+1}$$

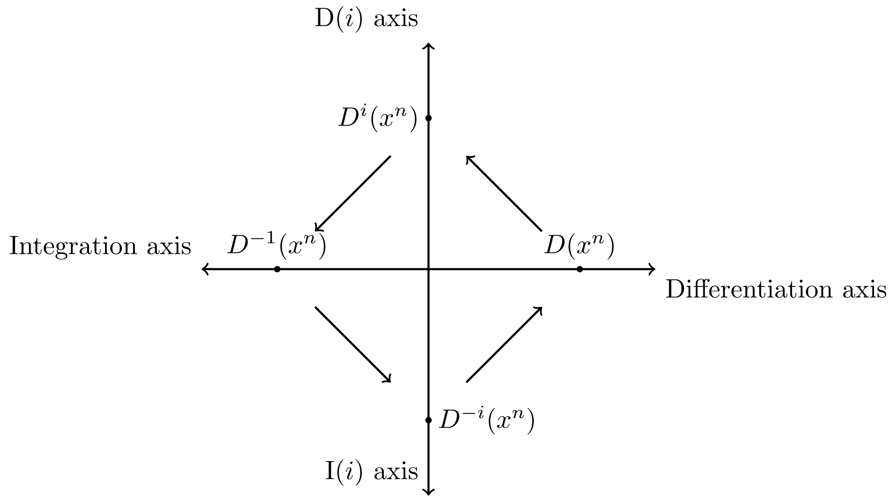
and knowing that $\Gamma(n+1) = n\Gamma(n)$

$${}^iD^i(x^n) = \frac{\Gamma(n+1)}{(n+1)\Gamma(n+1)}x^{n+1} = \frac{x^{n+1}}{n+1}$$

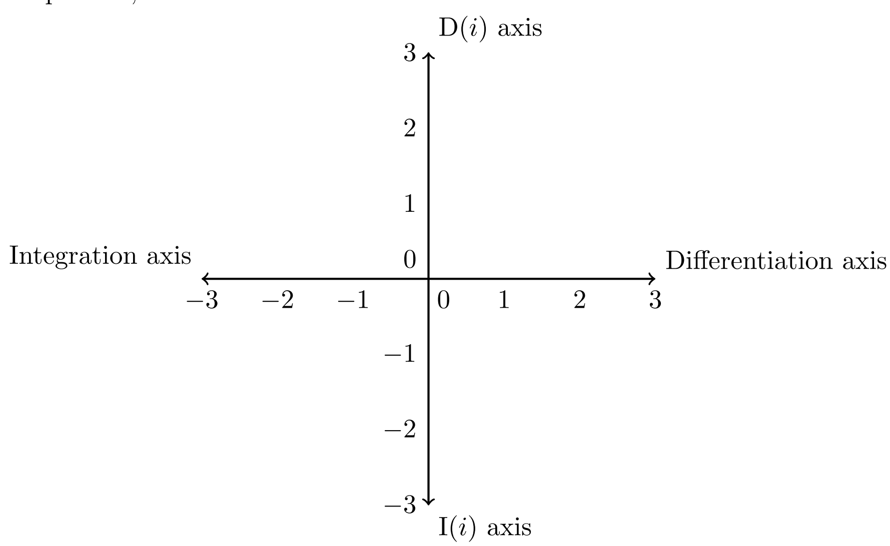
infact this was expected since ${}^iD^i = D^{(i\times i)} = D^{-1}$
The result is kind of weird , clearly it's weird
to see why it's weird (if it's not already), we can use a geometric interpretation
From the information we have right now, we can draw this transformation in the D-I plane



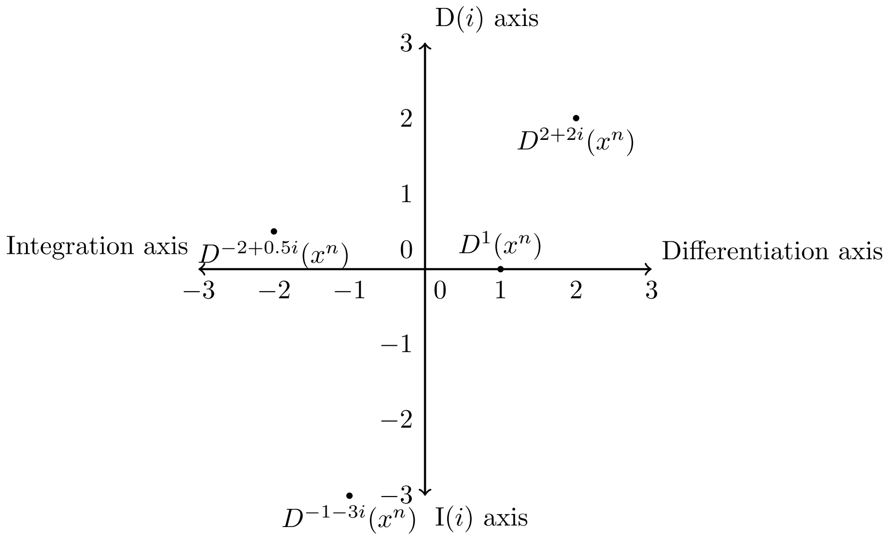
As this map acts in the same way as a complex plane, We map the transformation around it like this



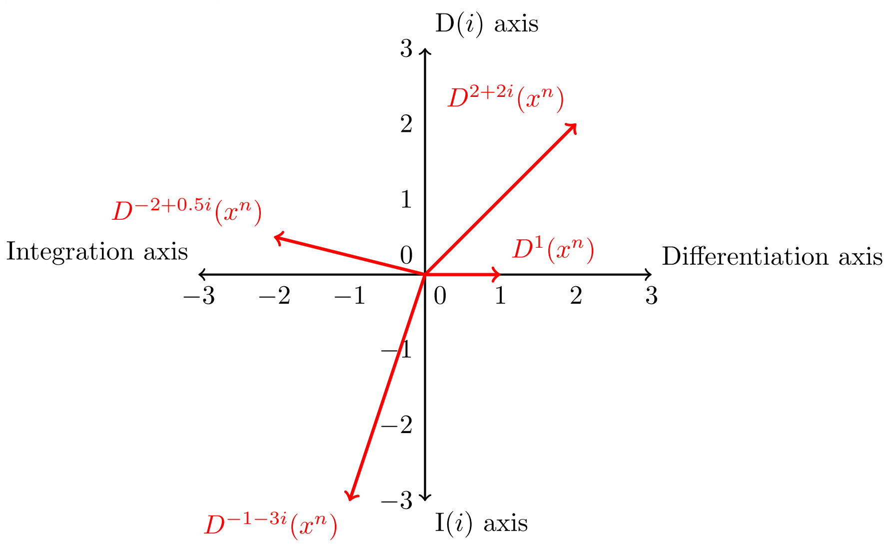
This is how the first derivative acts when using the complex number i
which means also that there exists a way (if not multiple) to represent differentiation and integration geometrically in the same space
We can add more detail to this plane by defining each access as the order of the D operator, so we can write it better like this



With the y-axis being the imaginary order part and the x-axis being the normal part, we can, of course, now express any derivative or integral or any weird result of D^z as points in this space.
that is, of course, for one single function as the input is a variable of the type of function and the output is of variable function, so for now they must stay the same function with the same variables inside of it
so we can now express for example the $D^z(x^n)$ like this



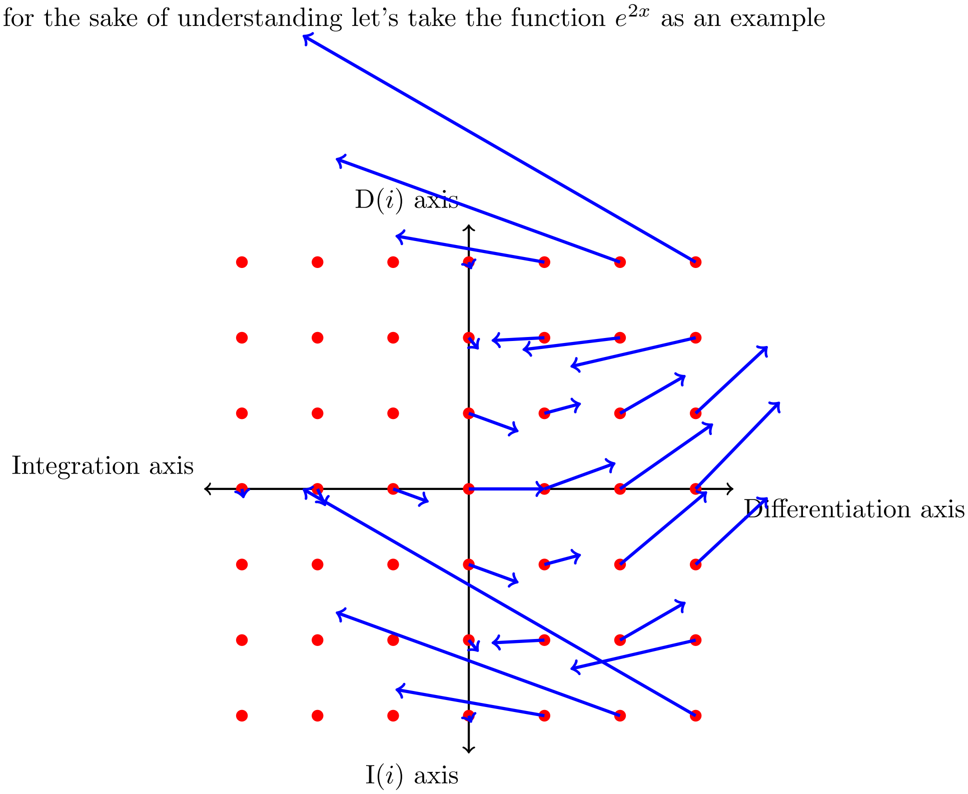
This also allows us to use any root of unity, not only i, like a normal Re-Im plot plane. Of course, we can change the perspective and think of them as vectors in a 2D vector plane
For each derivative, we can draw them as vectors starting from the zero point (the function itself) to the point of derivative order



which means that the vector rules also work fine here with the D^z operator properties as we can see adding and subtracting orders will return another vector using the Index Law, and since we know what the plane is for we can express the orders from $D^{\alpha+\beta i}$ as points $(\alpha, \beta i)$ or vectors $\vec{a} = (\alpha, \beta)$, and supposedly polar (r, θ), of course, if we want to make a vector from a point to another, we can just use the index law that goes in the path to β which also means that there are infinite starting points for vectors to go to the same point, in other words : infinite vectors for any point and to any point

This can be called the **D(i) Plane**, even when dealing with vectors , at the end, this plane isn't really based on the idea of it, but rather an extension to it

This may be helpful , but I suggest another idea that is based on vectors in such a field, and I would like to call it the **D(i) Vector Field**
Instead of using vectors in the plane as beginning and ending points, we will use them as points,arrows, and degrees, sort of a mix of the polar form and the vector form of the plane
1- the point: will represent its head, where it's coming out of
2- the arrow : will the output of the function derivative rate of change, the bigger it is the higher the rate of change
3- the degree: it will represent how the D^z operator is changing the function itself
for the sake of understanding let's take the function e^{2x} as an example



we can now study how the derivatives of e^{2x} work from this vector field
the degrees between the different derivatives First, in the real values, we can see that the more we integrate, the closer we get to the original function , and the more we differentiate, the more the value of the coefficient grows

The degree of the vectors represents how quickly it grows, as we can say it grows by 2^α, so when we differentiate more, the values of outputs of the original function get larger and larger for small inputs, and the opposite happens when we integrate ,
We can define the Scaling of both operations like this

$$\lim_{\alpha\to-\infty}[D^\alpha(e^{2x})] = e^{2x} \qquad \lim_{\alpha\to\infty}[D^\alpha(e^{2x})] = \infty$$

9 Cyclic derivatives

9.1 e^x and Cyclic functions

If we look closely at functions that have **Cyclic-Derivatives**, we can see that there exists a pattern for example

$$D^\alpha e^x = e^x$$

$$D^\alpha \sinh(x) = \frac{1}{2}(e^x - (-1)^\alpha e^{-x}) \qquad D^\alpha \cosh(x) = \frac{1}{2}(e^x + (-1)^\alpha e^{-x})$$

$$D^\alpha \sin(x) = \mathrm{Im}(e^{ix}) \qquad D^\alpha cos(x) = \mathrm{Re}(e^{ix})$$

They all have a connection with e^x in them
The function e^x by itself is a cyclic derivative function; it always returns itself, no matter how many times you differentiate it if we change the function slightly to e^{ax} things now change as it's returns different results

$$when\ a > 1 \quad \lim_{\alpha\to\infty} D^\alpha(e^{ax}) = \infty \qquad when\ a < 1 \quad \lim_{\alpha\to\infty} D^\alpha(e^{\alpha x}) = e^{ax}$$

these are all easy known results of course $when\ a = 1\ \ \lim_{\alpha\to\infty} D^\alpha(e^{\alpha x}) = e^{ax}$
these are all expected to be the result of repeating multiplication of $a^\alpha e^{ax}$
but there is one case for a where it's not any of the above
Let's examine the cyclic derivative functions closely

$$D^\alpha e^x = e^x$$

if we try to write in general e^{ax} formula we get

$$D^\alpha e^{1x} = 1^\alpha e^{1x}$$

which can also explain (with what we already know) why e^x is a repeated derivative of itself
Let's look at hyperbolic functions, which are cyclic derivatives of order 2

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

if we solve for e^{-x} in both functions
we get for $\sinh(x)$

$$2\sinh(x) = e^x - e^{-x} \qquad \therefore 2\sinh(x) - e^x = -e^{-x} \qquad \therefore e^x - 2\sinh(x) = e^{-x}$$

we can do the same for $\cosh(x)$

$$2\cosh(x) = e^x + e^{-x} \qquad \therefore 2\cosh(x) - e^x = e^{-x}$$

Let's add both equations together. We get

$$2e^{-x} = 2\cosh(x) - e^x + e^x - 2\sinh(x) = 2\cosh(x) - 2\sinh(x)$$

Dividing both sides by 2, we get

$$e^{-x} = \cosh(x) - \sinh(x)$$

which is truly an order 2 cyclic derivative function. If we took the first derivative of the function, we would get

$$D(e^{-x}) = D(\cosh(x)) - D(\sinh(x)) =$$

$$-e^{-x} = \sinh(x) - \cosh(x)$$

This is true since if we multiply by -1 in the first expression, we get the same result
Of course, we can express these in terms of e^x like this

$$e^x = \cosh(x) + \sinh(x)$$

which is true since both sides will return the same result, no matter how many times we differentiate them
It looks sort of like the Euler formula, but for hyperbolics
notice that -1 that is in e^{-x} comes from the equation $a^2 = 1, a = \pm 1$ because if we differentiate any of the e^x or e^{-x} expressions twice we get the the same expressions back
If the idea didn't click in yet, let's look at trigonometric
for trigonometric functions, but this time we have a formula ready for us

$$e^{ix} = \cos(x) + i\sin(x)$$

Differentiating both sides, we get

$$ie^{ix} = -\sin(x) + i\cos(x)$$

which again works well ,and if we differentiate again 3 times more, we get back to the same expression
notice that here $a = i$ in fact not only i but any value that satisfies the expression $a^4 = 1$ works as well , and the returning function will be an order 4 cyclic derivative function
The pattern here goes on and on, and this is exactly the missing case for a

$$when\ a^n = 1, D^n(e^{ax}) = e^{ax}\ with\ 2 \times n\ cyclic\ order$$

That is because it holds for both positive and negative values of a, which means that in the case $a^1 = 1$, the order is only once cyclic since it has only positive

values
for case $a^2 = 1$ we get e^{-x} which is 2nd order cyclic derivative
and for case $a^4 = 1$ we get e^{ix} which is 4th order cyclic derivative
This formula helps us generate any order of cyclic derivatives as for example, a 3rd order cyclic derivative will have a as

$$a^3 = 1, a = 1 \ or \ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \ or \ -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

and all of them work exactly as intended, Also, we can see that some cyclic derivatives work correctly under the order of another cyclic derivative, like order 2 (sinh and cosh) under order 4 (sin and cos) This is because in order for $a^n = 1$ to be true isn't only n times, but it works for n and 2n and 3n, so on, because every one of them will lead to 1,$a^n = a^{2n} = a^{3n} = 1$ In other words, the higher cyclic derivatives that work the same will satisfy the property

$$when \ a^n = 1, D^n(e^{ax}) = e^{ax} \ with \ 2 \times k \ cyclic \ order, \ where \ k \equiv 0 \mod n$$

getting back to the basics both of case $a^2 = 1$ and case $a^4 = 1$ can be expanded as trigonometric or hyperbolic functions, not only that but both of them satisfy the condition 2^n and both of them can be expanded algebraically, for hyperbolic it's $x^2 - y^2 = 1$ and $x^2 + y^2 = 1$ for trigonometric functions, and since they are cyclic derivatives they are functions of type FRD, which suggests that they may exist a pattern of function families that are

$$e^{ax} \ where \ a^n = 1 \ that \ satisfies \ the \ condition \ 2^n \in \mathbb{Z}^+$$

and these function families may also have algebraic expression that also suggests it will be from a high-order operator that isn't simple plus or minus
Also, since -1 and i can both be explained geometrically as "rotations," we can also say that $n \ is \ the \ number \ of \ unity \ roots$ which also suggests cyclic order derivatives, but in 3D or 4D using Quaternions and octonions

9.2 infinity cycles between the cyclic

In fact, we can do this now
Let's start with the first imagery derivative for the cyclic functions

$$D^i e^x = e^x \qquad\qquad D^i e^{ax} = a^i e^{ax}$$

The first formula is expected since e^x doesn't change by any means, what is interesting is the second one, as it gives some beautiful results
let $a = -1$ to get the one for the hyperbolic functions

$$D^i e^{-x} = (-1)^i e^{-x} = (e^{i\pi})^i e^{-x} = e^{-\pi} e^{-x} = e^{-x-\pi} = e^{-(x+\pi)}$$

quite interesting, this means that in between $\cosh(x)$ and $\sinh(x)$ lies exactly $e^{-(x+\pi)}$, and from our e^{-x} formula we can say that

$$e^{-(x+\pi)} = \cosh(x+\pi) - \sinh(x+\pi)$$

And the reason I didn't say that it lies between $\sinh(x)$ becoming $\cosh(x)$ nor the other way around is because it's a mix of both of the functions, to find what comes in the middle of any of them, we have to differentiate them by themselves so we get

$$D^i \sinh(x) = \frac{e^x - (-1)^i e^{-x}}{2} = \frac{e^x - e^{-(x+\pi)}}{2}$$

which mean that right in the middle of $\sinh(x)$ transforming to $\cosh(x)$ there exists that real function, the same goes for $\cosh(x)$

$$D^i \cosh(x) = \frac{e^x + (-1)^i e^{-x}}{2} = \frac{e^x + e^{-(x+\pi)}}{2}$$

we can do the same for e^{ix} and see what happens

$$D^i e^{ix} = i^i e^{ix} = (e^{\frac{i\pi}{2}})^i e^{ix} = e^{\frac{-\pi}{2}} e^{ix} = e^{ix-\frac{\pi}{2}}$$

We can expand it with Euler's identity

$$e^{ix-\frac{\pi}{2}} = e^{\frac{-\pi}{2}} \cos(x) + i e^{\frac{-\pi}{2}} \sin(x)$$

Now we can expand this to merge the e terms

$$e^{ix-\frac{\pi}{2}} = e^{\frac{-\pi}{2}} \frac{e^{ix} + e^{-ix}}{2i} + i e^{\frac{-\pi}{2}} \frac{e^{ix} - e^{-ix}}{2i} = e^{\frac{-\pi}{2}} \frac{e^{ix-\frac{\pi}{2}} + e^{-ix-\frac{\pi}{2}}}{2i} + i \frac{e^{ix-\frac{\pi}{2}} - e^{-ix-\frac{\pi}{2}}}{2i}$$

Then we turn it back to trigonometric, we get

$$D^i e^{ix} = \cos(x - \frac{\pi}{2}) + i \sin(x - \frac{\pi}{2})$$

of course, **this is False**, since $e^{\frac{-\pi}{2}}$ acts as a scaler , we can just submit it like this to the equation, but isn't it fascinating that the D^i doesn't act as a position or rotation changer but as a scaler?
As we can see , the i-th derivative represents a half-rotation between sin and $-\cos$ changing, which is expected since it's halfway between them, what wasn't expected is it being a scaler not a rotation nor reflector
Now, let's try getting them from their original rules

$$D^i \sin(x) = \sin(\frac{i\pi}{2} + x)$$

This doesn't mean one of the formulas is wrong, since we still can do ${}^iD^i$ for both of them and get the same results. This simply means that one of them operates on the real plane with scaling and rotation, and the other is on the complex plane, same result, different prescriptive.

10 The Geometric DifferentInetgrel (GDI) theory

10.1 FRD and NFRD

These are two important terms that will be helpful for us
FRD : Full Real Differentiable **NFRD** : Not-Full Real Differentiable
These are two very important terms as they categorize all the functions in D^z since most of the functions are either Integrable and Differentiable or Integrable and not Differentiable, and D^z can work on both sides
FRD Functions are functions that have a clean closed form for it's α-th derivative that isn't a power series (that is, of course, if the function itself isn't defined by a power series)
these are function like x^n with the condition that $\mathrm{Re}(z) \leq n$ and e^x
NFRD functions are function that has only a closed form being a power series, so functions like $\tan(x)$ and other trigonometric functions are NFRD
This happens of course, for a lot of reasons like breaking the GDI first principle
Of course, all cyclic derivatives are FRD since they can be represented as e^{ax}, and all transformative functions are NFRD

10.2 Transformative Functions

They are the type of funcetions that transform depeendence on the derivative, for example

$$\frac{d}{dx}(\cos^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$$

As we can see , when we take the derivative of the function, it changes the dependence from the value of x to the value of the power of x, such functions can be space changers since most of them are based on $\ln(x)$, which will be discussed in future papers

10.3 The inverse functions problem

Inverse functions are a problem, at least for being an FRD function, as it's not possible for most of them, and I have some explanations
Inverse functions are weird compared to the normal functions. We can see some similarities between them and their original functions like x^n and $x^{\frac{1}{n}}$ as they both look identical for odd values because for even numbers we get complex values , but rotated and cut to half, in most of the inverse functions we can find these cuts near always
We can expect this because not every function can be fully inverted
a function is set of inputs that lead to outputs, one output can be gotten by many inputs but not the way around , so when we try to inverse functions like x^2 that has one output for the more than one input we get something that doesn't work in the real number systems, and for this example it's the complex values for negative integers

10.4 The Mirror functions

But that doesn't mean we can't have inverse functions of some sort. Let's look at the formula for x^n

$$D^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

Now let's try to make the numerator go to its limit. Since the Gamma function doesn't work for non-positive integers, we can say that

$$\because n+1 \leq 0 \qquad \therefore n \leq -1 \qquad \therefore -n \geq 1$$

We can then put it in the x^n derivative

$$D^\alpha x^{-n} = \frac{\Gamma(1-n)}{\Gamma(1-n-\alpha)} x^{-n-\alpha} = \frac{\Gamma(1-n)}{\Gamma(1-(n+\alpha))} x^{-(n+\alpha)}$$

This function works exactly like x^n but for negative values of n (-(-1),-(-3)) , it behaves the same for derivatives and Integrals as well

Part III

Applications of D^z

11 Bonus Topics

11.0.1 The zeta function $D^z\zeta(n)$ and it's nature

The Riemann zeta function is one of the most well-known functions in mathematics. Its formula is :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n^{-s}$$

so using the $D^\alpha a^{-x} = (-1)^\alpha a^{-x} \ln(a)^\alpha$ we get

$$D^\alpha \zeta(s) = \sum_{n=1}^{\infty} (-1)^\alpha n^{-s} \ln(n)^\alpha$$

We can put the -1 outside the series since it's not dependent on it
We can now find the first derivative simply

$$D^1\zeta(s) = (-1)\sum_{n=1}^{\infty} n^{-s}\ln(n) = -\sum_{n=1}^{\infty} \frac{\ln(n)}{n^s}$$

And we can find the half-derivative

$$D^{\frac{1}{2}}\zeta(s) = (-1)^{\frac{1}{2}} \sum_{n=1}^{\infty} n^{-s} \ln(n)^{\frac{1}{2}} = i\sum_{n=1}^{\infty} \frac{\sqrt{\ln(n)}}{n^s}$$

Well, this is fascinating
We can also find the i-th derivative

$$D^i\zeta(s) = (-1)^i \sum_{n=1}^{\infty} n^{-s}\ln(n)^i = (e^{i\pi})^i \sum_{n=1}^{\infty} \frac{e^{i\ln(\ln(n))}}{n^s} = e^{-\pi} \sum_{n=1}^{\infty} \frac{e^{i\ln(\ln(n))}}{n^s}$$

12 Fractional and Complex Differential equations

12.1 Matrix equations

We seek all values of n such that for $A = nI$,$D^A = nI$:
First, we need to be reminded of a matrix order property we have proven earlier, that for any n-scalar matrix order derivative will return $\Gamma(n+1)$ times the identity matrix
with this we can say $D^A = \Gamma(n+1)I$ thus we can write the equation as this

$$nI = \Gamma(n+1)I$$

then we use the identity of the Gamma $\Gamma(z+1) = z\Gamma(z)$

$$nI = n\Gamma(n)I$$

dividing both sides by nI returns $1 = \Gamma(n)$
substituting $\Gamma(n) = (n-1)!$ we get $(n-1)! = 1$
This only holds true for 2, 1 as $(1-1)! = 0! = 1$ and $(2-1)! = 1! = 1$
If we try numbers bigger than one, we get $(3-1)! = 2! = 2 \neq 1$ then the function grows beyond it's reach
so the only solutions is where $n \in \{1, 2\}$ does the equation satisfy $D^A = \Gamma(n)I$

13 Applications in real world

As Sir Issac Newton started it based on the motion of everything in the universe , when he didn't find a good way to see it, he created his own way, which we call today calculus, so of course, we are beginning with physics

13.1 Dynamics with D^z operator

dynamics are the beginning of and heart with physics,it's thhe language of motion in it's simplest form , and so we can begin by stating that $f(x)$ is position and the first derivative with respect to time is Velocity , and the second derivative with respect to time is acceleration , that is for integer derivatives , and for fractional derivatives things tend to get weird
let's take. for example the function x^5

$$f(x) = x^5 \qquad v(x) = 5x^4 \qquad a(x) = 20x^3$$

These are the three main dynamical functions for $f(x)$
Now we are going to propose their different functions that work in the middle:

$$
{}_tD^{0.5}f(x) = hf(x) = \frac{120}{\Gamma(5.5)}x^{4.5} \qquad {}_tD^{0.5}f(x) = hv(x) = \frac{120}{\Gamma(4.5)}x^{3.5} \qquad {}_tD^{0.5}f(x) = ha(x) = \frac{120}{\Gamma(3.5)}x^{2.5}
$$

since for every derivative we multiply the time scale unit by 1, we can say that the units for these are $m/\sqrt{s}, m/\sqrt{s^3}, m/\sqrt{s^5}$
As we can see, these are states in between: hf is between position and velocity, hv is between velocity and acceleration let's try some other unusual things like the i-th derivative and $A = 5I$ matrix

$$
D^i f(x) = \frac{120}{\Gamma(6-i)}x^{5-i}\ m/s^i
$$

and that is a present , complex time
Well, it's not fully complex, as we can see this time is just for the system itself, it's for the event , since there is no time squared or square rooted, these are just for the event itself, yet it's weird to see a complex time of the event itself

$$
D^A f(x) = \Gamma(6)I = 120I\ m/s^A
$$

and now Matrix Time , and this one is very special since this isn't an Identified matrix , it can continue forever, or can simply be written as m/s^5 since both are technically the same

Contents