

# Complex-Order Fractional Derivatives: A First Exploration

Faisal Khaled

September 2025

## 1 Introduction

The integer-order derivative  $D^n f(x)$  measures the local rate of change. This paper explores the generalization of the derivative operator to continuous and complex orders,  $D^\alpha f(x)$  and  $D^z f(x)$ , known as Fractional and Complex Calculus.

## 2 The Generalized Operator for $f(x) = x^n$

### 2.1 From Integer to Fractional Order

We begin with the integer derivatives of  $f(x) = x^n$ :

$$D^k f(x) = n(n-1) \cdots (n-k+1)x^{n-k}$$

Using the identity  $n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}$ , we write:

$$D^k f(x) = \frac{n!}{(n-k)!} x^{n-k}$$

To generalize this for  $k \in \mathbb{R}$ , we substitute the factorial function with the continuous Gamma function,  $\Gamma(z)$ . We use the identities  $n! = \Gamma(n+1)$  and  $\Gamma(z+1) = z\Gamma(z)$ . The  $\alpha$ -th derivative (where  $\alpha \in \mathbb{R}$ ) is:

$$D^\alpha f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

of course we can use this formula to get half-derivatave of  $x^n$

$$D^{\frac{1}{2}} = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} x^{1-\frac{1}{2}} = \frac{1}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}}$$

using the rule  $\Gamma(n+1) = n\Gamma(n)$  and  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$D^{\frac{1}{2}} = \frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})} x^{\frac{1}{2}} = \frac{1}{\frac{\sqrt{\pi}}{2}} x^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}} x^{\frac{1}{2}}$$

now if we take the half-derivative of that half-derivative it will be

$$\frac{2}{\sqrt{\pi}} D^{\frac{1}{2}} = \left( \frac{2}{\sqrt{\pi}} \right) \frac{\Gamma(1/2 + 1)}{\Gamma(\frac{1}{2} - \frac{1}{2} + 1)} x^{\frac{1}{2} - \frac{1}{2}} = \left( \frac{2}{\sqrt{\pi}} \right) \frac{\Gamma(\frac{3}{2})}{1} x^0 = \left( \frac{2}{\sqrt{\pi}} \right) \left( \frac{\sqrt{\pi}}{2} \right) = 1$$

we talking the half-derivative twice to the same function gave what a one full derivative would give

of course such a proof is too simple and don't quite give the meaning of a proof, it's just a little confirmation for the time being, the full rigorous proof will be proven later with the properties of the  $D^z$  operator

**Generalization to Complex Order**  $z = a + bi$

We now extend the derivative order to the complex number  $z = a + bi$ :

$$D^z f(x) = \frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-z}$$

To show the magnitude and phase components, we expand  $x^{n-z}$  using the property  $x^{a+bi} = x^a e^{b \ln(x)i}$ :

$$D^z f(x) = \frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-a} e^{-b \ln(x)i}$$

with these two formulas we can use them to find any  $\mathbb{C}$  or  $\mathbb{R}$  derivatives for  $x^n$

**Finding negative order derivatives**

we can find the negative derivatives by putting -1 as the  $\alpha$  and see what could happen

Putting -1 in the general formula gives the result

$$D^{-1} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-(-1)+1)} x^{n-(-1)} = \frac{\Gamma(n+1)}{\Gamma(n+2)} x^{n+1}$$

and using the  $\Gamma(z+1) = z\Gamma(z)$  we can say that

$$D^{-1} f(x) = \frac{\Gamma(n+1)}{(n+1)\Gamma(n+1)} x^{n+1} = \frac{x^{n+1}}{(n+1)}$$

which means that the negative order derivatives are the integrals a function This result unifies the familiar integer derivative, the fractional derivative, and the complex-order derivative into a single, elegant framework.

## 2.2 the $x^{-n}$ problem

as we have seen, we can apply the past formula to any power of  $n$  weather it's fractional or even complex but problems rise when we try to apply the past formula to  $x^{-n}$

$$D^\alpha (x^{-n}) = \frac{\Gamma(0)}{\Gamma(-\alpha)} x^{-n-\alpha}$$

not only do we have a **Gamma Pole** in the numerator but also for any value  $\alpha \in \mathbb{Z}^+$  we also get a Gamma pole in the denominator which means that this formula cant work and we need another formula  
the m-th formula for  $x^{-n}$  is simply

$$\frac{d^m}{dx^m}(x^{-n}) = \frac{(-1)^m (n)^{(m)}}{x^{n+m}}$$

the  $n^{(m)}$  here isn't a power but rather a rising factorial that can be expressed as  $n^{(m)} = \frac{(n+m-1)!}{(n-1)!}$  with this knowledge we can say

$$D^\alpha(x^{-n}) = \frac{(-1)^\alpha \frac{\Gamma(n+\alpha)}{\Gamma(n)}}{x^{n+\alpha}} = \frac{(-1)^\alpha \Gamma(n)}{x^{n+\alpha}\Gamma(n+\alpha)}$$

as simple as that, it didn't work with the original  $x^n$  formula , but this shows something about fractional derivatives , if we went to find the first integral for both  $x^n$  and  $x^{-n}$  we find that

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \int x^{-n} dx = \ln|x| + C$$

Ignoring the integration constant we can find that  $x^n$  wasn't the original function for  $x^{-n}$  but it actually transformed from  $\ln(x)$

not only in complexity only but in dependency , the  $D^z(x^n)$  is dependent on the change of n more than x, while  $\ln(x)$  is dependent on the change of x  
this will be explained later in the **Transformative Functions** section , and this is one of the few cases we use stanederd integration (other than the Gamma function) in this research

### 3 Formulas for Other Algebraic Functions

#### 3.1 The General Formula for $a^x$

Starting with the general integer rule for  $a^x$ :

$$D^n(x) = a^x \ln(a)^n$$

substituting  $\alpha$  in the place of n gives us

$$D^\alpha f(x) = a^x \ln(a)^\alpha$$

we can see that this simple change was enough for the formula to work by taking the half-derivative twice and it gives us order one derivative

$$D^{1/2} f(x) = a^x \ln(a)^{1/2}$$

since  $\ln(a)^{1/2}$  is a constant we can take it out simply when doing the derivative again

$$D^{1/2}(D^{1/2} f(x)) = \ln(a)^{1/2}(D^{1/2} f(x)) = \ln(a)^{1/2}(a^x \ln(a)^{1/2}) = a^x \ln(a)$$

which is true since our starting function was  $a^x$  and thus we can say this formula works

**The Complex Generalization of this formula** can be written like the  $D^\alpha$  formula or like this

$$D^z f(x) = a^x \ln(a)^t e^{b \ln(\ln(a))i}$$

where  $z = a + bi$

of course we can find the first Anti-derivative of this function by using -1 in the formula

$$D^{-1}(x) = a^x \ln(a)^{-1} = \frac{a^x}{\ln(a)}$$

and the first Complex derivative

$$D^i(x) = a^x \ln(a)^i = a^x e^{i \ln(\ln(a))}$$

### 3.2 The General Formula for $e^x$

The function  $e^x$  is known for it's "Unchanging Derivative" because it comes from the  $D^n(a^x) = a^x \ln(a)^n$  and putting  $a = e$  we get  $D^n(e^x) = e^x$  so this also means there is no change affect the complex nor the fractional derivatives

$$D^\alpha f(x) = e^x \quad D^z f(x) = e^x$$

which means the Anti-derivative and the first complex derivative of the function

$$D^{-1}(x) = e^x \quad D^i(x) = e^x$$

### 3.3 The General Formula for $e^{ax}$

as we saw there is not any change between  $e^x$  and any of it's derivatives, things change when we consider  $e^{ax}$  as we can see the rule of the first - second and derivative is

$$D^1 f(x) = a e^{ax} \quad D^2 f(x) = a^2 e^{ax} \quad D^3 f(x) = a^3 e^{ax}$$

so we can find the formula for the n-th derivative as

$$D^n f(x) = a^n e^{ax}$$

and changing the n to  $\alpha$  we get

$$D^\alpha f(x) = a^\alpha e^{ax}$$

as simple as that we still have to test it to justify

$$D^{1/2} f(x) = a^{1/2} e^{ax}$$

since  $a^{1/2}$  is a constant we can say that

$$D^{1/2}(D^{1/2} f(x)) = a^{1/2}(D^{1/2} f(x)) = (a^{1/2})(a^{1/2} e^{ax}) = a e^x$$

this confirms that the formula work and substituting  $z$  instead of  $\alpha$  we get the same formula as above taht can also be written like that

$$D^z f(x) = a^t e^{bln(a)i+ax}$$

where  $z = a + bi$

the first Anti-derivative for  $e^{ax}$  is

$$D^{-1} f(x) = a^{-1} e^{ax} = \frac{e^{ax}}{a}$$

and the first complex derivative is

$$D^i f(x) = a^i e^{ax} = e^{ln(a)i+ax}$$

### 3.4 The problem of $\log_a(x)$

$\log_a(x)$  The first derivative of  $\log_a(x)$  is  $\frac{1}{x \ln(a)}$  and the second derivative is  $\frac{-1}{x^2 \ln(a)}$  the third derivative is  $\frac{2}{x^3 \ln(a)}$  lastly the fourth derivative is  $\frac{-6}{x^4 \ln(a)}$  is we can see the pattren of the n-th derivative

$$D^n f(x) = (-1)^{n+1} \frac{(n-1)!}{x^n \ln(a)}$$

and applying the gamma identity  $n! = \Gamma(n+1)$  then changing  $n$  to  $\alpha$  and reversing the  $x$  power in the denominator we get

$$D^\alpha f(x) = (-1)^{\alpha+1} \frac{\Gamma(\alpha)}{\ln(a)} x^{-\alpha}$$

But it fails, it doesnt work with fractions nor negative integares it only works positave integers, so what went wrong? my theory is that the problem is fairly simple, the  $\log_a(x)$  can't be expressed as a Maclurin series and  $\log_a(x+C)$  can only be expressed to a series with the condition  $|x| < C$  or else it won't diverge

GDI theory : For Every function that isnt Function is analytic at  $x=0$  and doesnt converge over  $\mathbb{R}$  the derivative formula  $D^\alpha f(x)$  works only on  $\mathbb{Z}^+$  for that function , we can also call it not full real differentiable

GDI hypothesis : the function is differentiable in all it's vaild input values

These can also be rewritten like this:

Let  $f(x)$  be an analytic function defined by its Maclaurin series

**GDI theory:** If  $f(x)$  has a singularity at  $x = 0$  , then the generalized formula for  $D^\alpha f(x)$  will contain a singularity at  $\alpha = n$  for  $n \in \mathbb{Z}^+$ , preventing the generalized formula from equaling the expected integer fractional or anti-derivative.

– Todo: change this to look better – this will be explained in detail in later sections

## 4 Properties of the $D^z$ operator

these are properties to identify the nature of it that will help later with the formula and analysis of what i can call the  $D^z$  **plane** More on that later

### 4.1 General power series rule

power serieses are very important tools to analytically describe a function along the Real or Complex planes what is exactly what we need

The general power series definition for a function(Taylor series) is as following :

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

taking the first derivative for both sides, the first term will cancel as it's a constant, the second term is linear so it will become constant, the third term is quadruple will become linear and 2! will cancel the 2 of the power and so on we can write is like this

$$D^1 f(x) = f^{(1)}(a) + \frac{f^{(2)}(a)}{1!}(x-a) + \frac{f^{(3)}(a)}{2!}(x-a)^2 \dots = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(a)}{n!}(x-a)^n$$

taking the third and the forth derivative give the same result up to k-th derivative

$$D^k f(x) = f^{(k)}(a) + \frac{f^{(k+1)}(a)}{1!}(x-a) + \frac{f^{(k+2)}(a)}{2!}(x-a)^2 \dots = \sum_{n=0}^{\infty} \frac{f^{(k+n)}(a)}{n!}(x-a)^n$$

now to make it full fractional we will put gamma instead of n and using the  $x^n$  general formula

$$D^\alpha f(x) = f^{(\alpha)}(a) + \frac{f^{(\alpha+1)}(a)}{\Gamma(2)}(x-a) + \frac{f^{(\alpha+2)}(a)}{\Gamma(3)}(x-a)^2 \dots = \sum_{n=0}^{\infty} \frac{f^{(n+\alpha)}(a)}{\Gamma(n+1)} \left[ \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}(x-a)^n \right]$$

in the brackets we can see the general derivative for  $x^n$  canceling the Gammas out we get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-\alpha+1)}(x-a)^{n-\alpha}$$

for powers of  $(x-a)$  lesser than  $\alpha$  it will get to  $\infty$  in the denominator, this happens because of the gamma pole, but taking the limit it will lead to 0 but still doesn't affect the sum, indeed helping us deleting the first  $n < \alpha$  terms of course for negative integer dervitavte this works too since it will be positive

**Note:** becuase how much terms you take in the diffrentation will alawys come terms that are replace to them because of the  $x^n$  differentiation and the infinite

sum, **even if you differentiate it infinite amount of times** but you are still deleting values we are going to use that knowledge later in the integration and differential equations in later sections

we can now let  $\alpha = 0$  to get the important series we need, the Maclurin series

$$D^\alpha f(x) = f(0) + \frac{f^{(1)}(0)}{\Gamma(2)}(x) + \frac{f^{(2)}(0)}{\Gamma(3)!}(x)^2 \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n - \alpha + 1)} x^{n-\alpha}$$

## 4.2 The linearity of $D^z$ operator

for most of the operations with functions and real life applications we need to deal with linearity for  $D^z$  operator which is what we are going to prove in this small section **We Must Prove that:**

$$D^z(c_1 f(x) + c_2 g(x)) = c_1 D^z f(x) + c_2 D^z g(x)$$

let's begin with the simple  $x^n$  and let  $f(x) = x^n, g(x) = x^m$  firstly we Differentiate them separately

$$D^z(c_1 f(x)) + D^z(c_2 g(x)) = c_1 D^z(f(x)) + c_2 D^z(g(x)) = c_1 \frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-z} + c_2 \frac{\Gamma(m+1)}{\Gamma(m-z+1)} x^{m-z}$$

**Let this be statement 1**

now let's differentiate them together we get

$$D^z(c_1 f(x) + c_2 g(x)) = c_1 \frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-z} + c_2 \frac{\Gamma(m+1)}{\Gamma(m-z+1)} x^{m-z}$$

**Let this be statement 2**

since **Statement 1 = Statement 2** we can say that

$$D^z(c_1 f(x) + c_2 g(x)) = c_1 D^z f(x) + c_2 D^z g(x)$$

**Q.E.D**

## 4.3 The Index law

The most important property for the formulas is the index law that is

$$D^{\alpha+\beta} f(x) = D^\alpha (D^\beta f(x))$$

**we must prove** this holds true for every case first we need to prove it for the simplest function we have which is  $x^n$  taking the  $D^z$  of the function we get

$$D^\alpha f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

now let's apply the  $D^\beta$  with the knowledge that the Gamma functions are constants in the first derivative

$$D^\beta(D^\alpha f(x)) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} D^\beta(x^{n-\alpha}) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} \left[ \frac{\Gamma(n-\alpha+1)}{\Gamma(n-\alpha-\beta+1)} x^{n-\alpha-\beta} \right]$$

the Gamma terms cancel out and we get

$$D^\beta(D^\alpha f(x)) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)} x^{n-\alpha-\beta}$$

**Let this be statement 1**

now if we start from the beginning again but this time directly substitute  $\alpha + \beta$  as  $O$  (stands for orders) we get

$$D^O f(x) = \frac{\Gamma(n+1)}{\Gamma(n-O+1)} x^{n-O}$$

if we substitute  $O = \alpha + \beta$  back we get

$$D^{\alpha+\beta} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)} x^{n-\alpha-\beta}$$

**Let this be statement 2**

if we Equal **statement 1** and **statement 2** we get

$$D^{\alpha+\beta} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)} x^{n-\alpha-\beta} = D^\beta(D^\alpha f(x))$$

thus we can say that

$$D^{\alpha+\beta} f(x) = D^\beta(D^\alpha f(x))$$

**Q.E.D**

**Note:** this also works for imaginary numbers  $z + w$

this by itself is a simple elegant proof but, it only works for  $x^n$  and applying the same method for each function will be very large waste of time

**Instead** we can use **Power Serieses** as they hold for every analytic function thus one proof will work for every function

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n+1)} (x-a)^n$$

we can immediately see that it's the simple  $x^n$  with everything else being a constant to the derivative thus we can apply the Index law safely because of the operator linearity and see that

$$D^{\alpha+\beta} f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n+1)} \left[ \frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)} (x-a)^{n-\alpha-\beta} \right] = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-\alpha-\beta+1)} (x-a)^{n-\alpha-\beta}$$

which means that the Index law works for any analytic function **Q.E.D**

## 5 Deriving the Rules of the Fractional and complex derivatives

most of the other functions need us to derive the rules of the fractional Differ-integar operator, the  $\alpha$ -th derivative for functions like  $\tan(x), \operatorname{arcsech}(x), \ln(x)$  etc.. can only be found using power serieses and product rule

### 5.1 General product rule

One of the most important and needed formulas in calculus in the prdouct rule let  $f(x) = g(x)h(x)$

$$f'(x) = g(x)h'(x) + g'(x)h(x)$$

$$f''(x) = g(x)h''(x) + 2g'(x)h'(x) + g''(x)h(x)$$

$$f'''(x) = g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x)$$

this is very similar to the binomial theorem , the only difference is it deals with derivatives instead of powers

the general product rule also known as the **General Leibniz rule** is

$$D^n(fg) = \sum_{k=0}^n \binom{n}{k} D^{n-k}(f) D^k(g)$$

this simple yet elegant formula is what we are going to use for the General product rule, before we try to do a simple substitution of  $\alpha$  we need to use the generalized  ${}^nC_k$  which means using  $n! = \Gamma(n+1)$  in the formula  $\frac{n!}{(n-k)!k!}$

$$D^n(fg) = \sum_{k=0}^n \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} D^{n-k}(f) D^k(g)$$

this is the same formula just and works the same for positive integers, but let's try to use fractions and ignore the sigma upper term and expand it , for example one half expansion will be

$$D^{1/2}(fg) = \frac{\Gamma(\frac{1}{2})}{\Gamma(1\frac{1}{2}-0)\Gamma(1)} D^{1/2}(f) D^0(g) + \frac{\Gamma(\frac{1}{2})}{\Gamma(1\frac{1}{2}-1)\Gamma(2)} D^{-1/2}(f) D^1(g) + \dots$$

$$D^{1/2}(fg) = \frac{\Gamma(\frac{1}{2})}{\Gamma(1\frac{1}{2})} D^{1/2}(f)g + D^{-1/2}(f)D^1(g) + \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{1}{2})\Gamma(3)} D^{-3/2}(f)D^2(g) + \dots$$

as we can see , it expands to an infinite sum, it will never stop because the lower value never hit the upper value, the main reason the simple form works for integers is that even if the k value goes higher than the n value it will get a negative integer in a Gamma which is a pole and thus equal zero

that means in order for a term to be it must not have the Gamma of integers

so that means we can't find an inetgar product rule yet, but at anyway turning back the sum from what we know will be

$$D^\alpha(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} D^{\alpha-k}(f) D^k(g)$$

we can mark this as the general power rule but we need to use our handy tool the **Index law** to confirm that it works –need to prove it later–

## 5.2 General chain rule

# 6 Interpretation of fractional and complex derivative in other functions

## 6.1 Trigonometric Functions

Deriving the trigonometric functions can be quite tricky , as there exists n-th derivative formula for them but don't seem to work as intended

### 6.1.1 $\sin(x)$ and $\cos(x)$

for  $\sin(x)$  there exists a formula which is

$$D^n \sin(x) = \sin\left(\frac{n\pi}{2} + x\right)$$

and unexpectedly this formula works for fractional or negative values, let  $n = \frac{1}{2}$

$$D^{\frac{1}{2}} \sin(x) = \sin\left(\frac{\frac{1}{2}\pi}{2} + x\right) = \sin\left(\frac{\pi}{4} + x\right)$$

knowing that  $D^n(D^m \sin(x)) = \sin\left(\frac{(n+m)\pi}{2} + x\right)$

$$D^{\frac{1}{2}}(D^{\frac{1}{2}} \sin(x)) = \sin\left(\frac{(\frac{1}{2} + \frac{1}{2})\pi}{2} + x\right) = \sin\left(\frac{2\pi}{4} + x\right) = \sin\left(\frac{\pi}{2} + x\right)$$

but there is another proof that this works for all real numbers.

from the eular formula  $e^{ix} = \cos(x) + i\sin(x)$  we can say that

$$\sin(x) = \text{Im}(e^{ix})$$

taking the alpha-th derivative of both sides

$$D^\alpha \sin(x) = D^\alpha \text{Im}(e^{ix}) = \text{Im}(i^\alpha e^{ix})$$

knowing that  $i = e^{i\pi/2}$

$$D^\alpha \sin(x) = \text{Im}(e^{i\pi\alpha/2} e^{ix}) = \text{Im}(e^{i\pi\alpha/2 + ix}) = \text{Im}(e^{i(\alpha\pi/2 + x)})$$

turning this back to the euler formula will give us

$$D^\alpha \operatorname{Im}(e^{ix}) = \sin\left(\frac{\alpha\pi}{2} + x\right)$$

which indeed proves it's true from the same formula we can also get the  $\alpha$ -th for  $\cos(x)$  with the same formula turning this back to the euler formula will give us

$$D^\alpha \operatorname{Re}(e^{ix}) = \cos\left(\frac{\alpha\pi}{2} + x\right)$$

now we can write them as

$$D^\alpha \sin(x) = \operatorname{Im}(e^{i(\alpha\pi/2+x)}) \quad D^\alpha \cos(x) = \operatorname{Re}(e^{i(\alpha\pi/2+x)})$$

or

$$D^\alpha \sin(x) = \sin\left(\frac{\alpha\pi}{2} + x\right) \quad D^\alpha \cos(x) = \cos\left(\frac{\alpha\pi}{2} + x\right)$$

and to make it to the complex plane we can use these formulas also the negative derivative of these is

$$D^{-1} \sin(x) = \sin\left(\frac{-\pi}{2} + x\right) = -\cos(x) \quad D^{-1} \cos(x) = \cos\left(\frac{-\pi}{2} + x\right) = \sin(x)$$

and the first complex derivative of these is

$$D^i \sin(x) = \sin\left(\frac{i\pi}{2} + x\right) = \sin\left(\frac{\ln(-1)}{2} + x\right) \quad D^i \cos(x) = \cos\left(\frac{i\pi}{2} + x\right) = \cos\left(\frac{\ln(-1)}{2} + x\right)$$

### 6.1.2 $\tan(x)$ and $\sec(x)$

finding the alpha-th derivative for  $\tan(x)$  is quite hard since we didn't get any direct formulas for quotients, and there is no direct integar derivative formula we can plug in and generalize to the Real numbers, we can try to change it a little with some aljabera

$$\tan(x) = \sin(x) (\cos(x))^{-1}$$

and then use the general product rule, but quickly we can see the problem

$$D^\alpha (\sin(x) (\cos(x))^{-1}) = \sum_{k=0}^{\infty} \frac{\Gamma(0)}{\Gamma(\alpha - k + 1) \Gamma(k + 1)} D^{\alpha-k}(\sin(x)) D^k(\cos(x)^{-1})$$

there is a gamma pole so this solution also fails  
we can try using some trig substitution

$$\tan(x) = \sin(x) (\sqrt{\sin(x)^2 + 1})^{-1}$$

but this leads to infinite sum for the product rule and the chain rule that we have proved above it impossible to find with the simple aljabera we have is very hard to approximate by itself as we can see there is no simple elegant closed

form for  $\tan(x)$  in the scientific paper, the reason behind this will be explained later but we can simply say because it has poles, to analytically see it better we need the Maclurin series expansion

$$\tan(x) = x + \frac{x^3}{3!} + \frac{2x^5}{15} + \dots = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} \quad \text{where } |x| < \frac{\pi}{2}$$

if we look closely we can notice the problem, it has a radius of converges and that by itself is the problem that will be discussed in detail later all what we can do for now is applying the  $D^\alpha$  to the infinite sum as it will be the only analytic closed form way for now we will get:

$$\begin{aligned} D^\alpha \tan(x) &= \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{\Gamma(2n)} \left[ \frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1} \right] \\ &= \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1} \end{aligned}$$

this infinite series shall work for now

the same also works for  $\sec(x)$  as it's a quotient so if we tried using the General product rule we will hit a Gamma pole, so the safest answer for now is to go with infinite series

$$\sec(x) = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{E_{2n}(-1)^n}{(2n)!} x^{2n} \quad \text{where } |x| < \frac{\pi}{2}$$

again we see the same problem with radius of convergence simply we apply  $D^\alpha$ :

$$\begin{aligned} D^\alpha \sec(x) &= \sum_{n=0}^{\infty} \frac{E_{2n}(-1)^n}{\Gamma(2n)} \left[ \frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha} \right] \\ &= \sum_{n=0}^{\infty} \frac{E_{2n}(-1)^n}{\Gamma(2n-\alpha+1)} x^{2n-\alpha} \end{aligned}$$

**Note: these two work for their radius of converges only**

**Note: they also work for the complex derivative**

### 6.1.3 $\csc(x)$ and $\cot(x)$

now saying  $\csc(x)$  and  $\cot(x)$  will work the same like the rest of trigonometric functions is a bit of a stretch

since we already know both of their domains aren't  $x \in \mathbb{R}$  so they must have some sort of analytical poles and converges radius that isn't  $\mathbb{R}$  in their infinite

sums

but if we noticed

$$\csc(x) = \frac{1}{\sin(x)}$$

which means that it has a singularity at  $x = 0$ , in other words simple Taylor series nor simple Maclurin series won't work, we need the General Laurent series for this one The Laurent series is :

$$\csc(x) = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \dots = \sum_{n=0}^{\infty} \frac{B_{2n}(-1)^{n+1}(2^{2n}-1)}{(2n)!} x^{2n-1} \quad \text{where } 0 < |x| < \pi$$

but before plugging the  $D^z$  operator to the series we can notice a little problem in the beginning, the  $\frac{1}{x}$  term

simply plugging in the  $D^z(x^n)$  will resault a pole, the simple solution is just to the linearity of  $D^z$  and differentiate the first term alone and then the rest of the series alone

$$\begin{aligned} D^\alpha \csc(x) &= D^\alpha(x^{-1}) + \sum_{n=1}^{\infty} \frac{B_{2n}(-1)^{n+1}(2^{2n}-1)}{\Gamma(2n)} \left[ \frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1} \right] \\ &= \frac{(-1)^\alpha \Gamma(1+\alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{B_{2n}(-1)^{n+1}(2^{2n}-1)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1} \end{aligned}$$

of course this is where  $0 < |x| < \pi$

The same goes for  $\cot(x)$  as it doesn't have any Taylor series but rather Laurent series

The Laurent series for  $\cot(x)$  is:

$$\cot(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} x^{2n-1} \quad \text{where } 0 < |x| < \pi$$

Applying  $D^\alpha$  to both sides we get

$$D^\alpha \cot(x) = \frac{(-1)^\alpha \Gamma(1+\alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}$$

**Note:** this works to complex numbers too

## 6.2 Hyperbolic Functions

### 6.2.1 $\sinh(x)$ , $\cosh(x)$ and $\tanh(x)$

$\sinh(x)$  is pretty straight forward to get from the definition

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

differentiating both sides to the  $\alpha$

$$D^\alpha \sinh(x) = \frac{1}{2}(D^\alpha e^x - D^\alpha e^{-x}) = \frac{1}{2}(e^x - (-1)^\alpha e^{-x})$$

we can also do the same for  $\cosh(x)$

$$D^\alpha \cosh(x) = \frac{1}{2}(D^\alpha e^x + D^\alpha e^{-x}) = \frac{1}{2}(e^x + (-1)^\alpha e^{-x})$$

but if we change the negative sign in  $\sinh(x)$  to  $+(-1)$  we turn the derivative to

$$D^\alpha \sinh(x) = \frac{1}{2}(e^x + (-1)^{\alpha+1} e^{-x})$$

which is equal to  $D^{\alpha+1} \cosh(x)$  and that is because unlike normal  $\sin(x)$  and  $\cos(x)$  these are the integer integrals and derivatives of their-selves so we can get the negative derivatives to be

$$D^{-1} \sinh(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x) \quad D^{-1} \cosh(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$$

for the complex derivatives we can use the formulas from before  
However for  $\tanh(x)$  things change, since the definition for it is

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

we can see, it's a quotient of two functions, which leaves us with nothing but to use its power series  
the power series for  $\tanh$  is:

$$\tanh(x) = x - \frac{x^3}{3!} + \frac{2x^5}{15} - \dots = \sum_{n=0}^{\infty} \frac{B_{2n} 4^n (1 - 4^n)}{(2n)!} x^{2n-1} \quad \text{where } |x| < \frac{\pi}{2}$$

as predicted there will also be radius of convergence here too  
but anyway we get the  $D^\alpha$  with this:

$$\begin{aligned} D^\alpha \tanh(x) &= \sum_{n=0}^{\infty} \frac{B_{2n} 4^n (1 - 4^n)}{\Gamma(2n)} \left[ \frac{\Gamma(2n)}{\Gamma(2n - \alpha + 1)} x^{2n-\alpha-1} \right] \\ &= \sum_{n=0}^{\infty} \frac{B_{2n} 4^n (1 - 4^n)}{\Gamma(2n - \alpha + 1)} x^{2n-\alpha-1} \end{aligned}$$

### 6.2.2 $\operatorname{sech}(x)$ , $\operatorname{csch}(x)$ and $\operatorname{coth}(x)$

The rest of the hyperbolic functions shall work the same as the trigonometric functions, in fact all of the hyperbolic and the trigonometric function's Laurent/Taylor series look identical with little changes, so finding them won't be

that difficult

For  $\text{sech}(x)$  the Taylor Series is:

$$\text{sech}(x) = 1 - \frac{x^2}{2!} + \frac{5x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n} \quad \text{where } |x| < \frac{\pi}{2}$$

So Applying  $D^z$  will be as simple as  $\sec(x)$

$$D^\alpha \text{sech}(x) = \sum_{n=0}^{\infty} \frac{E_{2n}}{\Gamma(2n - \alpha + 1)} x^{2n-\alpha} \quad \text{where } |x| < \frac{\pi}{2}$$

the same goes for the Laurent series of  $\text{csch}(x)$  and  $\text{coth}(x)$

$$\text{csch}(x) = \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - \dots = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{B_{2n}(1 - 2^{2n})}{(2n)!} x^{2n-1} \quad \text{where } 0 < |x| < \pi$$

of course it's similar but not identical , anyway applying  $D^\alpha$  gives us

$$\begin{aligned} D^\alpha \text{csch}(x) &= D^\alpha x^{-1} + D^\alpha \sum_{n=1}^{\infty} \frac{B_{2n}(1 - 2^{2n})}{(2n)!} x^{2n-1} \\ &= \frac{(-1)^\alpha \Gamma(1 + \alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{B_{2n}(1 - 2^{2n})}{\Gamma(2n - \alpha + 1)} x^{2n-\alpha-1} \quad \text{where } 0 < |x| < \pi \end{aligned}$$

and for  $\text{coth}(x)$ :

$$\text{coth}(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1} \quad \text{where } 0 < |x| < \pi$$

and applying  $D^\alpha$  operator we get:

$$D^\alpha \text{coth}(x) = \frac{(-1)^\alpha \Gamma(1 + \alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(2n - \alpha + 1)} x^{2n-\alpha-1}$$

## 6.3 The Inverse Trigonometric and Hyperbolic Functions

## 7 The GDI theory

### 7.1 GDI principles and infinite serieses

### 7.2 FRD and NFRD

### 7.3 Transformative Functions

### 7.4 The inverse functions problem

### 7.5 $D^z$ space

### 7.6 $\ln(x)$ and rotation in $D^z$ space

### 7.7 $e^x$ and Cyclic functions

if we look closely in functions that have **Cyclic-Derivatives** we can see that there exists a patten for example

$$D^\alpha e^x = e^x$$

$$D^\alpha \sinh(x) = \frac{1}{2}(e^x - (-1)^\alpha e^{-x}) \quad D^\alpha \cosh(x) = \frac{1}{2}(e^x + (-1)^\alpha e^{-x})$$

$$D^\alpha \sin(x) = \operatorname{Im}(e^{ix}) \quad D^\alpha \cos(x) = \operatorname{Re}(e^{ix})$$

they all have connection with  $e^x$  in them

– I need to continue this later –

## 8 Going Outside the usual

### 8.1 Explaining what is a fractional derivative

### 8.2 Explaining what it means for a derivative order to be Complex

### 8.3 Matrix derivatives

### 8.4 Function derivatives

## 9 Integration with $D^z$ operator

### 9.1 Deriving the $D^{-z}$ formulas

### 9.2 Fractional and Complex Integration

## 10 Fractional and Complex Differential equations

### 10.1 understanding physics with fractional and complex derivatives

## 11 Applications in real world

## 12 Table of formulas

rule / function	$x^n$	$a^x$	$e^x$	$e^{ax}$	$\log_a(x)$	$\ln(x)$
$D^1 f(x)$	$nx^{n-1}$	$a^x \ln(a)$	$e^x$	$ae^{ax}$		
$D^k f(x)$	$\frac{n!}{(n-k)!} x^{n-k}$	$a^x \ln(a)^k$	$e^x$	$a^k e^{ax}$		
$D^\alpha f(x)$	$\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$	$a^x \ln(a)^\alpha$	$e^x$	$a^\alpha e^{ax}$		
$D^z f(x)$	$\frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-z}$ and $\frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-a} e^{-b \ln(x)i}$		$e^x$			
$D^{-1} f(x)$	$\frac{x^{n+1}}{(n+1)}$	$\frac{a^x}{\ln(a)}$	$e^x$	$\frac{e^{ax}}{a}$		
$D^{-\alpha} f(x)$			$e^x$			
$D^{-z} f(x)$			$e^x$			

  

rule / function	$\sin(x)$	$\cos(x)$	$\tan(x)$	$\sec(x)$	$\csc(x)$	$\cot(x)$
$D^1 f(x)$	$\cos(x)$	$-\sin(x)$				
$D^k f(x)$	$\sin(\frac{k\pi}{2} + x)$	$\cos(\frac{k\pi}{2} + x)$				
$D^\alpha f(x)$	$\sin(\frac{\alpha\pi}{2} + x)$	$\cos(\frac{\alpha\pi}{2} + x)$				
$D^z f(x)$	$\sin(\frac{z\pi}{2} + x)$	$\cos(\frac{z\pi}{2} + x)$				
$D^{-1} f(x)$	$-\cos(x)$	$\sin(x)$				
$D^{-\alpha} f(x)$						
$D^{-z} f(x)$						