Complex-Order Fractional Derivatives: A First Exploration

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1 Introduction

The integer-order derivative $D^n f(x)$ measures the local rate of change. This paper explores the generalization of the derivative operator to continuous and complex orders, $D^\alpha f(x)$ and $D^z f(x)$, known as Fractional and Complex Calculus.

2 The Generalized Operator for $f(x) = x^n$

2.1 From Integer to Fractional Order

We begin with the integer derivatives of $f(x) = x^n$:

$$D^k f(x) = n(n-1)\cdots(n-k+1)x^{n-k}$$

Using the identity $n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$, we write:

$$D^k f(x) = \frac{n!}{(n-k)!}x^{n-k}$$

To generalize this for $k \in \mathbb{R}$, we substitute the factorial function with the continuous Gamma function, $\Gamma(z)$. We use the identities $n! = \Gamma(n+1)$ and $\Gamma(z+1) = z\Gamma(z)$. The α-th derivative (where $\alpha \in \mathbb{R}$) is:

$$D^\alpha f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha}$$

of course we can use this formula to get half-derivtaive of x^n

$$D^{\frac{1}{2}} = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)}x^{1-\frac{1}{2}} = \frac{1}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}}$$

using the rule $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$D^{\frac{1}{2}} = \frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})}x^{\frac{1}{2}} = \frac{1}{\frac{\sqrt{\pi}}{2}}x^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}}x^{\frac{1}{2}}$$

now if we take the half-derivative of that half-derivative it will be

$$\frac{2}{\sqrt{\pi}}D^{\frac{1}{2}} = (\frac{2}{\sqrt{\pi}})\frac{\Gamma(1/2+1)}{\Gamma(\frac{1}{2}-\frac{1}{2}+1)}x^{\frac{1}{2}-\frac{1}{2}} = (\frac{2}{\sqrt{\pi}})\frac{\Gamma(\frac{3}{2})}{1}x^0 = (\frac{2}{\sqrt{\pi}})(\frac{\sqrt{\pi}}{2}) = 1$$

we talking the half-derivative twice to the same function gave what a one full derivative would give , this means that it worked as intended
Generalization to Complex Order $z = a + bi$
We now extend the derivative order to the complex number $z = a + bi$:

$$D^z f(x) = \frac{\Gamma(n+1)}{\Gamma(n-z+1)}x^{n-z}$$

To show the magnitude and phase components, we expand x^{n-z} using the property $x^{a+bi} = x^a e^{b\ln(x)i}$:

$$D^z f(x) = \frac{\Gamma(n+1)}{\Gamma(n-z+1)}x^{n-a}e^{-b\ln(x)i}$$

with these two formulas we can use them to find any $\mathbb{C}$ or $\mathbb{R}$ derivatives for x^n
Finding negative order derivatives
we can find the negative derivatives by putting -1 as the α and see what could happen
Putting -1 in the general formula gives the result

$$D^{-1} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-(-1)+1)}x^{n-(-1)} = \frac{\Gamma(n+1)}{\Gamma(n+2)}x^{n+1}$$

and using the $\Gamma(z+1) = z\Gamma(z)$ we can say that

$$D^{-1} f(x) = \frac{\Gamma(n+1)}{(n+1)\Gamma(n+1)}x^{n+1} = \frac{x^{n+1}}{(n+1)}$$

which means that the negative order derivatives are the integrals a function This result unifies the familiar integer derivative, the fractional derivative, and the complex-order derivative into a single, elegant framework.

3 Formulas for Other Algebraic Functions

3.1 The General Formula for a^x

Starting with the general integer rule for a^x:

$$D^n(x) = a^x ln(a)^n$$

substituting α in the place of n gives us

$$D^\alpha f(x) = a^x \ln(a)^\alpha$$

we can see that this simple change was enough for the formula to work by taking the half-derivative twice and it gives us order one derivative

$$D^{1/2}f(x) = a^x ln(a)^{1/2}$$

since $ln(a)^{1/2}$ is a constant we can take it out simply when doing the derivative again

$$D^{1/2}(D^{1/2}f(x)) = ln(a)^{1/2}(D^{1/2}f(x)) = ln(a)^{1/2}(a^x ln(a)^{1/2}) = a^x ln(a)$$

which is true since our starting function was a^x and thus we can say this formula works

The Complex Generalization of this formula can be written like the D^α formula or like this

$$D^z f(x) = a^x \ln(a)^t e^{bln(ln(a))i}$$

where $z = a + bi$

of course we can find the first Anti-derivative of this function by using -1 in the formula

$$D^{-1}(x) = a^x ln(a)^{-1} = \frac{a^x}{ln(a)}$$

and the first Complex derivative

$$D^i(x) = a^x ln(a)^i = a^x e^{ln(ln(x))i}$$

3.2 The General Formula for e^x

The function e^x is known for it's "Unchanging Derivative" because it comes from the $D^n(a^x) = a^x ln(a)^n$ and putting $a = e$ we get $D^n(e^x) = a^x$ so this also means there is no change affect the complex nor the fractional derivatives

$$D^\alpha f(x) = e^x \qquad D^z f(x) = e^x$$

which means the Anti-derivative and the first complex derivative of the function

$$D^{-1}(x) = e^x \qquad D^i(x) = e^x$$

3.3 The General Formula for e^{ax}

as we saw there isnot any change between e^x and any of it's derivatives , things change when we consider e^{ax} as we can see the rule of the first - second an derivative is

$$D^1 f(x) = ae^{ax} \qquad D^2 f(x) = ae^{ax} \qquad D^3 f(x) = ae^{ax}$$

so we can find the formula for the n-th derivative as

$$D^n f(x) = a^n e^{ax}$$

and changing the n to α we get

$$D^{\alpha} f(x) = a^{\alpha} e^{ax}$$

as simple as that we still have to test it to justify

$$D^{1/2} f(x) = a^{1/2} e^{ax}$$

since $a^{1/2}$ is a constant we can say that

$$D^{1/2}(D^{1/2} f(x)) = a^{1/2}(D^{1/2} f(x)) = (a^{1/2})(a^{1/2} e^{ax}) = ae^{x}$$

this confirms that the formula work and substituting z instead of α we get the same formula as above taht can also be written like that

$$D^{z} f(x) = a^{t} e^{bln(a)i+ax}$$

where $z = a + bi$
the first Anti-derivative for e^{ax} is

$$D^{-1} f(x) = a^{-1} e^{ax} = \frac{e^{ax}}{a}$$

and the first complex derivative is

$$D^{i} f(x) = a^{i} e^{ax} = e^{ln(a)i+ax}$$

3.4 $\log_a(x)$ and $\ln(x)$

$log_a(x)$ The first derivative of $log_a(x)$ is $\frac{1}{xln(a)}$ and the second derivative is $\frac{-1}{x^2 ln(a)}$ the third derivative is $\frac{2}{x^3 ln(a)}$ lastly the fourth derivative is $\frac{-6}{x^4 ln(a)}$ is we can see the pattren of the n-th derivatve

$$D^{n} f(x) = (-1)^{n+1} \frac{(n-1)!}{x^n ln(a)}$$

and applying the gamma identity $n! = \Gamma(n-1)$ then changing n to α and reversing the x power in the denominator we get

$$D^{\alpha} f(x) = (-1)^{\alpha+1} \frac{\Gamma(\alpha-2)}{ln(a)} x^{-\alpha}$$

of course in the same formula we can change the α to z or make change it to look like this

$$D^{z} f(x) = (-1)^{t+1} \frac{\Gamma(t+bi-2)}{ln(a)} x^{-t} e^{-b\pi+biln(x)}$$

$ln(x)$

$$D^{\alpha} f(x) = (-1)^{\alpha+1} \Gamma(\alpha-2) x^{-\alpha}$$
$$D^{z} f(x) = (-1)^{t+1} \Gamma(t+bi-2) x^{-t} e^{-b\pi+biln(x)}$$
$$D^{z} f(x) = (-1)^{t+1} \Gamma(z-2) e^{z(-\pi i+ln(x))}$$

4 Deriving the Rules of the Fractional and complex derivatives

4.1 General product rule

4.2 General chain rule

4.3 General power series rule

5 Interpretation of fractional and complex derivative in other functions

5.1 Trigonometric and Hyperbolic Functions

5.2 Inverse Trigonometric and Inverse Hyperbolic Functions

5.3 The Gamma and error functions

6 Going Outside the usual

6.1 Explaining what is a fractional derivative

6.2 Explaining what it means for a derivative order to be Complex

6.3 Matrix derivatives

6.4 Function derivatives

7 Integration with derivatives

7.1 Fractional and Complex Integration

8 Fractional and Complex Differential equations

8.1 understanding physics with fractional and complex derivatives

8.2 The Physics Problem

9 Table of formulas

rule / function	x^n	a^x	e^x	e^{ax}	$log_a(x)$	$ln(x)$
$D^1 f(x)$	nx^{n-1}	$a^x ln(a)$	e^x	ae^{ax}		
$D^k f(x)$	$\frac{n!}{(n-k)!} x^{n-k}$	$a^x ln(x)^k$	e^x	$a^k e^{ax}$		
$D^\alpha f(x)$	$\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$	$a^x ln(x)^\alpha$	e^x	$a^\alpha e^{ax}$		
$D^z f(x)$	$\frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-z}$ and $\frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-a} e^{-b\ln(x)i}$		e^x			
$D^{-1} f(x)$	$\frac{x^{n+1}}{(n+1)}$	$\frac{a^x}{ln(a)}$	e^x	$\frac{e^{ax}}{a}$		
$D^{-\alpha} f(x)$			e^x			
$D^{-z} f(x)$			e^x			