Cyclic derivatives: The Families , and poles

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1 Introduction

2 The function families of cyclic functions

exploring the families of the functions is intersting , since it may help us understand why such functions behave like this at least from teh point of D^z and being NFRD while they are constructed from FRD functions

2.1 $\tan(x)$ and $\tanh(x)$

from the beginning, we can see some problems
before we even start we realize that ,both $\tan(x)$ and $\tanh(x)$ don't have a "main way" to define them
the stranded definition for them is $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$, but that is only because there exists two functions , and this only happens in hyperbolic functions, and trigonometric functions too because it satisfies $\frac{n}{p} = 2$ because they are the only ones that sat,but anyway we can use many ways with diffrent resaults to define the tangent function(s)
Using the first two functions only
we can try to define $\tan_n(x)$ to use the first two functions of any e^{ax}
so we can write it as $\tanh_n(x) = \frac{\sinh_n(x)}{\cosh_n(x)}$ that is for the non-reduced orignal functions of the expansion and $\tan_n(x) = \frac{\sin_n(x)}{\cos_n(x)}$ to be for the reduced functions
we can try it for the 3-order cyclic derivatives

$$\tanh_3(x) = \frac{\sinh_3(x)}{\cosh_3(x)} = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \left(\sum_{k=0}^{\infty} \frac{x^{3k}}{(kn)!} \right)^{-1}$$

but the problem with this one is there are other functions that aren't put in prespiactive , and the more we go up with the order the more they are which makes this definition unusable
The division between every function and $\cosh_n$ or $\cos_n$
Then we can try to to divide every single function multiplied from the cyclic

family

$$\tanh_3 \; = \; \frac{\sinh_3 (x)}{\frac{\cosh_3 (x)}{\sinh_3 \text{II}(x)}} = \frac{\sinh_3 (x) \sinh_3 \text{II}(x)}{\cosh_3 (x)}$$

and if we continue this we will see that all odd functions from a full cyclic family are on the top while the evens are on the bottom, so if we let $\cosh_n \; = \sinh_n 0$ we get this formula

$$\tanh_n (x) = \frac{\prod_{K=0}^{n} \sinh_n 2\text{K}+1(x)}{\prod_{J=0}^{n} \sinh_n 2\text{J}(x)}$$

Using the Series

but to actually define $\tanh_n (x)$ we need to observe what the orignal $\tan(x)$ have in that this one needs to have

$\tan(x)$ is equal to $\frac{\cos(x)}{\sin(x)}$: this isn't supposed to be true for all variants, and we will discuss the reason for that later

$\tan(x)$ is an NFRD function with its series representation only: This one is important since a series representation is the represent of a function using x^n, which *According to the original paper* has the change directly happening to them so the best way to define $\tanh_n (x)$ is via the series definition, now we see the Maclaurin series

$$\tan(x) = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} \qquad \tanh(x) = \sum_{n=0}^{\infty} \frac{B_{2n}4^n(1-4^n)}{(2n)!} x^{2n-1}$$

But we can't take these series directly and apply them because they may be only special occians , we need to define the main series using the original functions

We can skip $\tanh(x)$ since it's pretty straightforward in the original functions being $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

for the four original functions of cyclic order 4 $(\cosh_4 (x), \sinh_4 (x), \sinh_4 \text{II}(x), \sinh_4 \text{III}(x))$ We can use the sin and cos definitions of them being

$$cos(x) = \cosh_4 (x) + \sinh_4 (x) \qquad \sin(x) = -(\sinh_4 \text{II}(x) + \sinh_4 \text{III}(x))$$

We can then put them in the definition of $\tan(x)$ to get

$$\tan(x) = \frac{-(\sinh_4 \text{II}(x) + \sinh_4 \text{III}(x))}{\cosh_4 (x) + \sinh_4 (x)}$$

Expanding the series we get

$$\tan(x) = \frac{-(\sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)!} + \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!})}{\sum_{j=0}^{\infty} \frac{x^{4j}}{(4j)!} + \sum_{u=0}^{\infty} \frac{x^{4u+2}}{(4u+2)!}}$$

This shows that for the original functions, it's dividing the second half of the series by the first half

we can try to do the same for the 8-order to see the result

$$\tan_8 \; = \; \frac{\sin_8 (x) + \sin_8 \text{III}(x)}{\cos_8 (x) + \sin_8 \text{II}(x)} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)!} + \sum_{j=0}^{\infty} \frac{(-1)^j x^{4j+3}}{(4j+3)!}}{\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{(4k)!} + \sum_{u=0}^{\infty} \frac{(-1)^u x^{4u+2}}{(4u+2)!}}$$

expanding the sums we get

$$\tan_8(x) = \frac{x + \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{1 + \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}$$

which is reduced to the regular sum $\sin(x)$ and $\cos(x)$ we get the final result

$$\tan_8(x) = \frac{\sin(x)}{\cos(x)} = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1 - 4^n)}{(2n)!} x^{2n-1}$$

2.2 $\sec(x)$ and $\operatorname{sech}(x)$

only the first function $\cosh_n$ and it's reduced version $\cos_n$ will make sech_n and $\sec_n$

We know that the series for any original function $\cosh_n$ is equal to $\sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!}$

We can expand the sum and let sech_n be the reportorial of it

$$\operatorname{sech}_n = \frac{1}{1 + \frac{x}{n!} + \frac{x^{2n}}{(2n)!} + \frac{x^{3n}}{(3n)!} + \dots} = \frac{1}{\sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!}}$$

and because of the idenitty $\cos(x)\sec(x) = 1$ we can use euler numbers to any kn in both reduced and orignal series

$$\operatorname{sech}_n = \sum_{k=0}^{\infty} \frac{E_{kn} x^{kn}}{(kn)!} \qquad\qquad \sec_n = \sum_{k=0}^{\infty} \frac{(-1)^k E_{kn} x^{kn}}{(kn)!}$$

2.3 $\csc(x)$ and $\operatorname{csch}(x)$

2.4 $\cot(x)$ and $\coth(x)$

3 The inverses of cyclic function families