Complex-Order and Fractional Derivatives: Cyclic derivatives and functions

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Abstract

This paper presents an independent exploration of cyclic derivatives based on the fractional and complex-order derivatives paper made by the same Author.

Note to Readers: This represents independent rediscovery of classical fractional calculus concepts. I (The Author) present this work as a pedagogical exercise in mathematical exploration rather than novel research.

Background

The Main formula that works for all derivatives and can be used in the Maclaurin series

$$D^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

The more important formulas that are built on this one

$$D^\alpha e^{ax} = a^\alpha e^{ax} \qquad e^{ix} = \cos(x) - i\sin(x)$$

$$e^x = \cosh(x) + \sinh(x) \qquad e^{-x} = \cosh(x) - \sinh(x)$$

$$D^\alpha \sin(x) = \sin(\frac{\alpha\pi}{2} + x) \qquad D^\alpha \cos(x) = \cos(\frac{\alpha\pi}{2} + x)$$

$$D^\alpha \sinh(x) = \frac{e^x - (-1)^\alpha e^{-x}}{2} \qquad D^\alpha \cosh(x) = \frac{e^x + (-1)^\alpha e^{-x}}{2}$$

1 Introduction

from what we have seen in the previous fractional and complex-order derivatives paper, cyclic derivatives are such a big area that deserves its own paper,

2 The foundation

From the fractional and complex-order derivatives paper, we know that

$$\textit{when } a^n = 1, D^n(e^{ax}) = e^{ax} \textit{ with } 2 \times n \textit{ cyclic order}$$

And we can call that the theorem of cyclic derivatives

Theorem 1 *$f(x)$ is a cyclic derivative when $a^n = 1, D^n(e^{ax}) = e^{ax}$ with $2 \times n$ cyclic order*

And in the same paper, we generalized this to hold for any $k \equiv 0 \pmod{n}$

Theorem 2 *$f(x)$ is a lesser cyclic derivative when $a^n = 1, D^n(e^{ax}) = e^{ax}$ with $2 \times k$ cyclic order, where $k \equiv 0 \pmod{n}$*

With a hypothesis that I wish to prove in this paper, that

Hypothesis 1 *For every function cyclic derivatives that can be written in the form e^{ax} where $a^n = 1$ and satisfies the condition $2^n \in \mathbb{Z}^+$, there exists an algebraic perimetric form*

We can already see this for hyperbolic functions, where they can be written in the form $x^2 - y^2 = 1$, and trigonometric functions We also proved some equations

$$D^i e^{-x} = e^{-(x+\pi)} = \cosh(x+\pi) - \sinh(x+\pi)$$

$$D^i \sinh(x) = \frac{e^x - e^{-(x+\pi)}}{2} \qquad D^i \cosh(x) = \frac{e^x + e^{-(x+\pi)}}{2}$$

$$D^i e^{ix} = e^{\frac{-\pi}{2}} \cos(x) + ie^{\frac{-\pi}{2}} \sin(x)$$

$$D^i \sin(x) = \sin(\frac{i\pi}{2} + x) \qquad D^i \cos(x) = \cos(\frac{i\pi}{2} + x)$$

and we also proved that for $\sin(x)$ and $\cos(x)$

3 More about the complex derivatives and known families

from what we know about $\sinh(x)$ and $\cosh(x)$ we can write their formulas in an other way

$$D^\alpha \sinh(x) = \frac{e^x - (-1)^\alpha e^{-x}}{2} = \frac{e^x - (e^{i\pi})^\alpha e^{-x}}{2} = \frac{e^x - (e^{i\pi\alpha})e^{-x}}{2}$$
$$= \frac{e^x - e^{i\pi\alpha - x}}{2}$$

The same goes for $\cosh(x)$

$$D^\alpha \sinh(x) = \frac{e^x - e^{i\pi\alpha - x}}{2} \qquad D^\alpha \cosh(x) = \frac{e^x + e^{i\pi\alpha - x}}{2}$$

We can see that we can't express them as simple forms, since the real value changes differently from the complex value. If we try to apply the derivative operator to e^x with the definition of $1 = e^{2i\pi}$, we get

$$D^\alpha e^x = 1^\alpha e^x = e^{2i0\alpha} e^x = e^x$$

We used that because it's the principal value
For trigonometric functions, we know that D^i represents a half turn before it changes to its integral from the real plane, but there is something to clarify

$$\operatorname{Im}(D^i e^{ix}) = e^{\frac{-\pi}{2}} \sin(x)$$

As we can see, the first i-th derivative acts as a scaler that scales $\sin(x)$ by real value $e^{-\frac{\pi}{2}}$
However, this isn't equal to the $D^i \sin(x)$ as the definition changes from scaling to rotating, like this

$$D^i \sin(x) = \sin(\frac{i\pi}{2} + x)$$

But since we have proven the multiplication law works in the framework, we can make sure that both are somewhat equal

$${}^iD^i e^{ix} = i^{i\times i} e^{ix} = (e^{\frac{i\pi}{2}})^{-1} e^{ix} = e^{\frac{-i\pi}{2}} e^{ix} = e^{ix-\frac{i\pi}{2}}$$

$$e^{i(x-\frac{\pi}{2})} = \cos(x - \frac{\pi}{2}) + i\sin(x - \frac{\pi}{2}) = \sin(x) - i\cos(x) = D^{-1}e^{ix} = \int e^{ix}$$

and for the sin we proved in the "Complex-order and fractional derivatives: first exploration" paper that the index law works on it, and thus the multiplication law either from here or from the series expansion, so that we can say

$${}^iD^i \sin(x) = \sin(\frac{i \times i\pi}{2} + x) = \sin(\frac{-\pi}{2} + x) = -\cos(x)$$

Thus, both of them work fine, then why?

3.1 The rotation in complex rotation

since D^i represents a whole rotation on the $D(i)$ plane, we can get more angles that could help us understand more what happens to the function to "integrate it"
first step is we are going to transform from i to $e^{\frac{i\pi}{2}}$ so we can deal with rotation with radians in circles

$$D^i e^{ax} = D^{e^{\frac{i\pi}{2}}} e^{ax} \qquad D^{-1}e^{ax} = D^{e^{i\pi}} e^{ax}$$

$$D^{e^{i\theta}} e^{ax} = a^{e^{i\theta}} e^{ax} \qquad D^{e^{i\theta}} e^{ix} = i^{e^{i\theta}} e^{ix} = e^{\frac{i\pi e^{i\theta}}{2}} e^{ix} = e^{i(\frac{\pi e^{i\theta}}{2}+x)}$$

$$= \cos(\frac{\pi e^{i\theta}}{2} + x) + i\sin(\frac{\pi e^{i\theta}}{2} + x)$$

So now we can know what happens at the third of rotation or the third root of unity, which is equal to $\frac{\pi}{3}$ in radians, we get

$$D^{e^{\frac{i\pi}{3}}} e^{ix} = e^{i(\frac{\pi e^{\frac{i\pi}{3}}}{2}+x)} = \cos(\frac{\pi e^{\frac{i\pi}{3}}}{2} + x) + i\sin(\frac{\pi e^{\frac{i\pi}{3}}}{2} + x)$$

which after calculating $e^{\frac{i\pi}{3}}$ to be $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ we can then multiply it by $\frac{\pi}{2}$ to get $\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4}$, then we plug it

$$e^{i(\frac{\pi}{4}+i\frac{\pi\sqrt{3}}{4}+x)} = e^{i\frac{\pi}{4}+i^2\frac{\pi\sqrt{3}}{4}+ix} = e^{i(x+\frac{\pi}{4})}e^{-\frac{\pi\sqrt{3}}{4}} = e^{-\frac{\pi\sqrt{3}}{4}}\cos(x+\frac{\pi}{4})+ie^{-\frac{\pi\sqrt{3}}{4}}\sin(x+\frac{\pi}{4})$$

We can see that it scales by a factor of $e^{\frac{-\pi\sqrt{3}}{4}}$ and rotate with a factor of $\frac{\pi}{4}$
let's do the same for two-thirds, the value for $e^{\frac{2i\pi}{3}}$ to be $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$ we can then multiply it again by $\frac{\pi}{2}$ to get $-\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4}$
Plugging it again, we get

$$e^{i(-\frac{\pi}{4}+i\frac{\pi\sqrt{3}}{4}+x)} = e^{-i\frac{\pi}{4}+i^2\frac{\pi\sqrt{3}}{4}+ix} = e^{i(x-\frac{\pi}{4})}e^{-\frac{\pi\sqrt{3}}{4}} = e^{-\frac{\pi\sqrt{3}}{4}}\cos(x-\frac{\pi}{4})+ie^{-\frac{\pi\sqrt{3}}{4}}\sin(x-\frac{\pi}{4})$$

At two-thirds, it rotates with the same value but rotates backwards
Now we have a little information about what happens in the process of integrating such functions
at the first third of the way, it rotates by $\frac{\pi}{4}$ and scales by $e^{-\frac{\pi\sqrt{3}}{4}}$
For Halfway, it doesn't rotate but scales with a factor of $e^{-\frac{\pi}{2}}$
for two-thirds it rotates by $\frac{\pi}{4}$ and scales by $e^{-\frac{\pi\sqrt{3}}{4}}$
This may seem weird at the beginning until we notice that we aren't starting from order 1 or D^1, we are starting from the zero point D^0 or the function itself, so the one-third and two-thirds don't cancel out on rotation, but they rotate to two different directions
The one-third rotates to D^1 and the two-thirds rotate to D^{-1}, while the middle point D^i doesn't rotate but scales because it's not a real derivative or real integral
We can even notice that in the first third we have $\cos(x + \frac{\pi}{4})$ and $\sin(x + \frac{\pi}{4})$, which are both pure half derivatives

$$D^{\frac{1}{2}}\sin(x) = \sin(\frac{\frac{1}{2}\pi}{2}+x) = \sin(\frac{\pi}{4}+x) \qquad D^{\frac{1}{2}}\cos(x) = \cos(\frac{\frac{1}{2}\pi}{2}+x) = \cos(\frac{\pi}{4}+x)$$

and the same happens for the half-integer being rotated by $\frac{\pi}{4}$
More on this will be discussed in later sections

4 exploration into another cyclic derivatives

4.1 the third order cyclic derivatives

From the theorem, we can find the third cyclic derivative to be from the equation $a^3 = 1$, the solutions are going to be denoted by $1, \omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

since one will result in e^x and is fully expected to be here because of Theorem 2, we are going to use ω

$$D^1 e^{\omega x} = \omega e^{\omega x} \qquad D^2 e^{\omega x} = \omega^2 e^{\omega x} \qquad D^3 e^{\omega x} = e^{\omega x}$$

We can call this function the third-order cyclic derivative, which comes between hyperbolic and trigonometric functions
we will name them $\sinh_3$,$\cosh_3$ and $\sinh_3 \mathrm{II}$ We can define them like this

$$D^\alpha \sinh_3(x) = \sinh_3 \mathrm{II}(x) \qquad \alpha \equiv 0 \pmod 3 \qquad D^\alpha \sinh_3 \mathrm{II}(x) = \cosh_3(x) \qquad \alpha \equiv 1 \pmod 3$$

$$D^\alpha \cosh_3(x) = \sinh_3(x) \qquad \alpha \equiv 2 \pmod 3$$

But this isn't the only way to define them, we can also define them with a series
First we find the Maclaurin series for $e^{\omega x}$

$$e^{\omega x} = e^0 + \omega e^0 x + \frac{\omega^2 e^0 x^2}{2!} + \frac{e^0 x^3}{3!} + ... = 1 + \omega x + \frac{\omega^2 x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{\omega^n x^n}{n!}$$

From this, we can divide them into three sums

$$(1 + \frac{x^3}{3!} + \frac{x^6}{6!} + ...) + \omega(x + \frac{x^4}{4!} + \frac{x^7}{7!} + ...) + \omega^2(x^2 + \frac{x^5}{5!} + \frac{x^8}{8!} + ...)$$

$$= \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \omega \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \omega^2 \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

We can see that the first sum can be differentiated 3 times before going back to the first state, which is also for all the other sums, but since sin and sinh all have x^{an+1}, we are going to make the first function to be the second sum, to keep the naming consistnt nothing more
So now we can define them to be

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \qquad \sinh_3 \mathrm{II}(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!} \qquad \cosh_3 = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$$

We can now define an equation that looks and acts like the Euler equation

$$e^{\omega x} = \cosh_3(x) + \omega \sinh_3(x) + \omega^2 \sinh_3 \mathrm{II}(x)$$

We can now try to find one for ω^2

$$e^{\omega^2 x} = \sum_{n=0}^{\infty} \frac{(\omega^2)^n x^n}{n!}$$

at $3n$ we get $\omega^{6n} = 1$ so it's $\cosh_3(x)$, at at $3n+1$ we get $\omega^{6n+2} = \omega^2$ so it's $\sinh_3(x)$ and at $3n$ we get $\omega^{6n+4} = \omega$ so it's $\sinh_3 \mathrm{II}(x)$

$$e^{\omega^2 x} = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \omega^2 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \omega \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

$$e^{\omega^2 x} = \cosh_3(x) + \omega^2 \sinh_3(x) + \omega \sinh_3 \mathrm{II}(x)$$

and for e^x it's quite simple

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

$$e^x = \cosh_3(x) + \sinh_3(x) + \sinh_3 \mathrm{II}(x)$$

like sin and sinh, we can try to find an exponent form for them
First, we begin by adding all of the equations so we have

$$e^x + e^{\omega x} + e^{\omega^2 x} = \cosh_3(x) + \sinh_3(x) + \sinh_3 \mathrm{II}(x)$$

$$+ \cosh_3(x) + \omega \sinh_3(x) + \omega^2 \sinh_3 \mathrm{II}(x) + \cosh_3(x) + \omega^2 \sinh_3(x) + \omega \sinh_3 \mathrm{II}(x)$$

$$= 3\cosh_3(x) + \sinh_3(x)(1 + \omega + \omega^2) + \sinh_3 \mathrm{II}(x)(1 + \omega^2 + \omega)$$

and since we know that $1 + \omega + \omega^2 = 0$

$$e^x + e^{\omega x} + e^{\omega^2 x} = 3\cosh_3(x) \qquad \cosh_3(x) = \frac{e^x + e^{\omega x} + e^{\omega^2 x}}{3}$$

Now we can define the others by differentiating

$$\sinh_3(x) = \frac{e^x + \omega e^{\omega x} + \omega^2 e^{\omega^2 x}}{3} \qquad \sinh_3 \mathrm{II}(x) = \frac{e^x + \omega^2 e^{\omega x} + \omega e^{\omega^2 x}}{3}$$

since these definitions are going to continue with us, we shall call the $e^{ax} = ...$ the **Euler form** and $f(x) = e^x + e^{ax}....$ the **exponentiation form**

4.2 Cyclic derivatives and prime numbers

To understand exactly what is meant by this, we need to see these

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \cosh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$$

As we can see, there is a pattern here, for every function that is k-th cyclic derivative, we can see that its series is $\sum_{n=0}^{\infty} \frac{x^{kn}}{(kn)!}$
But this assumption shortly breaks as we can see for the fourth cyclic derivative

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

To understand more what I am talking about, we may take a look at the fifth cyclic derivative denoted by ϵ, to find the functions we start from the Maclaurin series

$$e^{\epsilon x} = 1 + \epsilon x + \frac{\epsilon^2 x^2}{2!} + \frac{\epsilon^3 x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{\epsilon^n x^n}{n!}$$

And we can find the five functions the same way we made it to the three cyclic derivative functions

$$\cosh_5(x) = \sum_{n=0}^{\infty} \frac{x^{5n}}{(5n)!} \qquad \sinh_5(x) = \sum_{n=0}^{\infty} \frac{x^{5n+1}}{(5n+1)!}$$

$$\sinh_5 \mathrm{II}(x) = \sum_{n=0}^{\infty} \frac{x^{5n+2}}{(5n+2)!} \qquad \sinh_5 \mathrm{III}(x) = \sum_{n=0}^{\infty} \frac{x^{5n+3}}{(5n+3)!} \qquad \sinh_5 \mathrm{IV}(x) = \sum_{n=0}^{\infty} \frac{x^{5n+4}}{(5n+4)!}$$

And since the sum of roots of unity that are over 1 root is zero, we can do the same steps to find that

$$\cosh_5(x) = \frac{e^x + e^{\epsilon x} + e^{\epsilon^2 x} + e^{\epsilon^3 x} + e^{\epsilon^4 x}}{5}$$

and the other functions to be the derivatives of these functions
We can notice that the pattern continued for 5-th cyclic function
So what is the problem with trigonometric ones?
Well, we can see the expansion for $e^{ix}$to see what happens

$$e^{ix} = 1 + ix + -\frac{x^2}{2!} - \frac{ix^3}{3!} + ...$$

We can notice the pattern right there, it's the $-\frac{x^2}{2!}$, this term allows us to either:
- write the sum as four cyclic functions since we will make different additions being $\{1, i, -1, -i\}$
- We write sum as two different cyclic derivatives being $\{1. -1\}$ and $i\{1, -1\}$

In other words, the cyclic derivative family is compisable, reducible with simple algebra
and the reason for that is the **cyclic order**, when it's composite, we can see some roots return, like from order 2 we have $1, -1$ and order 4 we have $1, i, -1, -i$, the $1, -1$ here is back, same for six roots of unity $1, i_1, i_2, -1, i_3, i_4$ (note that i_a here isn't imaginary unit but the a-th root) and we can say that
let gk be all solutions for $a^k = 1$ and gn be for $a^n = 1$, as long as $\frac{k}{n} \in \mathbb{Z}^+, gk \subset gn$
Thus, for any composite cyclic order, there exists more than 1 way to represent it
which means primes aren't here, so we can write the theorem

Theorem 4.1 (Prime cyclic functions Euler Form) $\forall p \in Primes, a^p = 1$
There exists only one way to represent e^{ax} as a sum of all the cyclic order functions

From this, we can say that

Theorem 4.2 (Prime cyclic functions exponentiation form Form) $\forall p \in$ *Primes,* $\sinh_p N(x)$ *is a cyclic function; it can be written as this*

$$\sinh_p N(x) = \frac{e^x + a^N e^{ax} + a^{2N} e^{a^2 x} + ... + a^{pN} e^{a^p x}}{p} = \frac{1}{p} \sum_{n=0}^{p-1} a^{n(p-N)} e^{a^n x}$$

4.3 General Cyclic derivatives and Mittag-Leffler connection

As we can see from multiple series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sinh = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

We can see from all these examples that the factorial matches the exponent
Taking the D^α derivative of all gives us

$$D^z e^x = \sum_{n=0}^{\infty} \frac{x^{n-z}}{\Gamma(n-z+1)} \qquad \sinh = \sum_{n=0}^{\infty} \frac{x^{2n+1-z}}{\Gamma(2n+2-z)}$$

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1-z}}{\Gamma(3n+2-z)} \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-z}}{\Gamma(2n-z+1)}$$

If we let $2-z=\beta$ and let $Cn=\alpha n$ where C is a constant, we see that all of them get the shape

$$D^z f(x) = \sum_{n=0}^{\infty} \frac{x^{\alpha n+\beta}}{\Gamma(\alpha n+\beta)}$$

which is close to the Mittag-Leffler function, we can see that this happens in all of the functions we know.
That is, of course, except for $\cos(x)$ that will be discussed later
, but we need to generalise it, and we need to generalise the derivative cyclic order in Euler form

$$let\ a\ be\ any\ element\ from\ the\ group\ of\ soluitons\ for\ a^n = 1$$

$$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{a^k x^k}{k!}$$

From here, we can group the sums. Since there exist n roots of unity, we can say that there exist n terms

$$e^{ax} = (1 + \frac{x^n}{n!} + \frac{x^{2n}}{(2n)!} + ...) + (ax + \frac{a^n x^{n+1}}{(n+1)!} + \frac{a^{2n} x^{2n+1}}{(2n+1)!} + ...) + ...$$

$$= \sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!} + \sum_{k=0}^{\infty} \frac{a^{kn+1} x^{kn+1}}{(kn+1)!} + \sum_{k=0}^{\infty} \frac{a^{kn+2} x^{kn+2}}{(kn+2)!} + \sum_{k=0}^{\infty} \frac{a^{kn+3} x^{kn+3}}{(kn+3)!} + ...$$

and since $a^{kn+j} = a^j$, we can take it out as a common factor

$$= \sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!} + a\sum_{k=0}^{\infty} \frac{x^{kn+1}}{(kn+1)!} + a^2\sum_{k=0}^{\infty} \frac{x^{kn+2}}{(kn+2)!} + a^3\sum_{k=0}^{\infty} \frac{x^{kn+3}}{(kn+3)!} + ...$$

We are going to name the first one $\cosh_n(x)$ and the others $\sinh_n I(x)$ and $\sinh_n II(x)$ so on

$e^{ax} = \cosh_n(x) + a\sinh_n \mathrm{I}(x) + a^2 \sinh_n \mathrm{II}(x) + a^3 \sinh_n \mathrm{III}(x) + ... a^{n-1} \sinh_n \mathrm{N}(x)$

Now, if we consider $+C - \alpha = +\beta$ in the α-th derivative we get We see that all these functions have similarities with Mittag-Leffler formula, as $D^z \cosh_{\mathrm{n}}$ is itself $E_{\alpha n}$ and $D^z \sinh_{\mathrm{n}}$ is close to $E_{\alpha n, \beta}$

$$D^z \cosh_n = \sum_{k=0}^{\infty} \frac{x^{\alpha kn}}{\Gamma(\alpha kn + 1)} \qquad D^z \sinh_n \mathrm{II}... = \sum_{k=0}^{\infty} \frac{x^{\alpha kn+\beta}}{\Gamma(\alpha kn + \beta + 1)}$$

From this, we can get the general cyclic derivative sum formula

Theorem 4.3 (Generalized Cyclic Derivative) *Let $f(x)$ be the j-th basis function of the D^n-cyclic system,*
$f(x) = \sum_{k=0}^{\infty} \frac{x^{kn+j}}{(kn+j)!}$

Theorem 4.4 (Mittag-Leffler Representation) *Every basis function of the D^n-cyclic system is a linear combination of* **n** *distinct Mittag-Leffler functions* $\mathbf{E}_{\mathbf{n},\beta}(\mathbf{x}^{\mathbf{n}})$ *corresponding to the n terms in its series representation.*

Theorem 4.5 (Generalized Euler form) *for every e^{ax} where a stratifies $a^n = 1$, e^{ax} can be written as*

$$e^{ax} = \sum_{j=0}^{n-1} a^j \sinh_n N(x)$$

that is of course if we consider $\cosh_n(x)$ to be the 0-th term

4.4 Odd And Even cyclic derivatives

There are many differences between odd and even cyclic derivatives, and that comes to the roots of unity

4.5 Generalized D^z cyclic derivatives

there are many yet simple ways to find α-th deriavtive of a cyclic function
General Series
We can use the series expansion for any cyclic function
for any cyclic function with order n denoted as $\sinh_{\mathrm{n}}(x)$ we can say that

$$\sinh_{\mathrm{n}} = \sum_{k=0}^{\infty} \frac{x^{kn+C}}{(kn+C)!}$$

that is of course for reduced or original functions because the reduced functions change will not make change in x power directly
then the α-th derivative is

$$D^{\alpha}\sinh_{\mathrm{n}} = \sum_{k=0}^{\infty}\frac{x^{kn+C-\alpha}}{\Gamma(kn+C+1)}\frac{\Gamma(kn+C+1)}{\Gamma(kn+C+1-\alpha)}$$

canceling the terms we get

$$D^{\alpha}\sinh_{\mathrm{n}} = \sum_{k=0}^{\infty}\frac{x^{kn+C-\alpha}}{\Gamma(kn+C+1-\alpha)}$$

which is the generalized series formula
Deriavtives for reduced functions up til this point we know what is a reduced cyclic deriavtive functions for example e^{ix} is supposed to be four functions but we use commonly two in the euler identity
since the reduced functions are equal to the orignal functions in some sort for example $f(x) = g(x) \pm h(x)$ we can use the linearty of D^z to find the deriavtive of any function based on the known functions
Euler's form
for any cyclic derivatives function , a known thing is they need to equal to e^{ax} where $a^n = 1$ and n is the amount of cycliation, using this backwards we can say that any function $\sinh_{\mathrm{n}}$ can be expressed as the sum between all the roots of unity that are a power of e times x divided by n, at least for orignal functions and prime functions

$$\sinh_{\mathrm{n}} = \frac{1}{n}\sum_{k=0}^{n-1} e^{a^n x}$$

then we can use D^{α} here

$$D^{\alpha}\sinh_{\mathrm{n}} = \frac{1}{n}\sum_{k=0}^{n-1} a^{\alpha}e^{a^n x} = \frac{1}{n}(e^x + \sum_{k=1}^{n-1} a^{\alpha}e^{a^n x})$$

4.6 prime cyclic derivatives decomposition

the form e^{ix} can be expressed as $\cos(x) + i\sin(x)$, that is because it's composite and can be reducible to it's prime cyclic (2)
as we can see the reason it's reduced from algabric side is the (-1)

$$\cosh = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \qquad \cos = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sinh = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \qquad \sin = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

if we try another cyclic derivative that has only one prime order factor it will be 9 because it's 3^2

let $a^9 = 1$, with a being from $\{1, \epsilon, \epsilon^2, \omega, \epsilon^4, \epsilon^5, \omega^2, \epsilon^7, \epsilon^8\}$ with $\omega = \epsilon^3$ but we change it just to know it's the third cubic root

$$e^{ax} = 1 + \epsilon x + \frac{\epsilon^2 x^2}{2!} + \frac{\omega x^3}{3!} + \frac{\epsilon^4 x^4}{4!} + \frac{\epsilon^5 x^5}{5!} + \frac{\omega^2 x^6}{2!} + ...$$

we can then make the sum

$$e^{ax} = \sum_{n=0}^{\infty} \frac{\omega^n x^{3n}}{(3n)!} + \sum_{k=0}^{\infty} \frac{\epsilon^{k+1} x^{3k+1}}{(3k+1)!} + \sum_{j=0}^{\infty} \frac{\epsilon^{j+2} x^{3j+2}}{(3j+2)!}$$

we can see the three sums form the shapes of the main three cyclic derivatives functions with the addition of the complex values. but because the cyclic order is odd we don't see any negative ones

to understand more we can try to find the original e^{ix} cyclic functions

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} + i \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} - \sum_{j=0}^{\infty} \frac{x^{4j+2}}{(4j+2)!} - i \sum_{u=0}^{\infty} \frac{x^{4u+3}}{(4u+3)!}$$

we can actually see how the form sin and cos form

$$\left(\sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} - \sum_{j=0}^{\infty} \frac{x^{4j+2}}{(4j+2)!} \right) + i \left(\sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} - \sum_{u=0}^{\infty} \frac{x^{4u+3}}{(4u+3)!} \right)$$

simply analysing this we can see that the (-1^n) term in the sums come from them being difference between the sums, we can also see that cos only has the order on the x (x^{kn}) because the both sums have a common factor which is the prime number 2, while sin has the reminder of 1 when dividng that leads to x^{kn+1} term

ultimately we can try to reprsnt the redacted versions of the original functions $\cos, \sin$ with the original functions $(\cosh_4(x), \sinh_4(x), \sinh_4 \mathrm{II}(x), \sinh_4 \mathrm{III}(x)\ )$ using the Euler form

$$e^{ix} = \cos(x) + i\sin(x) = \cosh_4(x) + i\sinh_4(x) - \sinh_4 \mathrm{II}(x) - i\sinh_4 \mathrm{III}(x)$$

then we know that the real equals the real and the imaginary equals the imaginary

$$\cos(x) = \cosh_4(x) + \sinh_4 \mathrm{II}(x) \qquad \sin(x) = -(\sinh_4 \mathrm{I}(x) + \sinh_4 \mathrm{III}(x))$$

to understand more about it we can try to see for 8 order cyclic deriavtives with $a \in \{\epsilon, i, \epsilon^3, -1, \epsilon^5, -i, \epsilon^7, 1\}$ knowing that $\epsilon = e^{\frac{2i\pi}{8}}$

$$e^{\epsilon x} = 1 + \epsilon x + \frac{ix^2}{2!} + \frac{\epsilon^3 x^3}{3!} - \frac{x^4}{4!} + \frac{\epsilon^5 x^5}{5!} - \frac{ix^6}{6!} + \frac{\epsilon^7 x}{7!} + ...$$

$$e^{\epsilon x} = \sum_{n_1=0}^{\infty} \frac{x^{8n_1}}{(8n_1)!} + \sum_{n_2=0}^{\infty} \frac{\epsilon x^{8n_2+1}}{(8n_2+1)!} + \sum_{n_3=0}^{\infty} \frac{\epsilon x^{8n_3+1}}{(8n_3+2)!} + ...$$

these are the original functions functions
first of all, we know from the calculations that $\epsilon^5 = -\epsilon, \epsilon^7 = -\epsilon^3$, we can write the expressions as following

$$e^{\epsilon x} = \left(\sum_{n_0=0}^{\infty} \frac{x^{8n_0}}{(8n_0)!} - \sum_{n_4=0}^{\infty} \frac{x^{8n_4+4}}{(8n_4+4)!} \right) + \epsilon \left(\sum_{n_1=0}^{\infty} \frac{x^{8n_1+1}}{(8n_1+1)!} - \sum_{n_5=0}^{\infty} \frac{x^{8n_5+5}}{(8n_5+5)!} \right) ...$$

to distinguish original functions from reduced ones, we are going to call the reduced ones with trigonometric names from now on

$$e^{\epsilon x} = \cos_8(x) + \epsilon \sin_8(x) + i \sin_8 \mathrm{II}(x) + \epsilon^3 \sin_8 \mathrm{III}(x)$$

We can try to expand the sums to find another simpler sum for them

$$\cos_8(x) = 1 - \frac{x^4}{4!} + \frac{x^8}{8!} + ... \qquad \sin_8(x) = x - \frac{x^5}{5!} + \frac{x^9}{9!} + ...$$

they form the sums

$$\cos_8(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(4n)!} \qquad \sin_8(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{(4k+1)!} \qquad \text{etc...}$$

We can also construct an exponentiation form for them
by changing the function root of unity to $e^{\epsilon^3 x}$
We can see from this that the number of functions an order n can be reduced to is $\frac{n}{p}$ where p is its prime divisor, that is to be said most functions we have dealt with till now are either primes or have one prime divisor, but we still can prove it

Theorem 4.6 *Every integer order n that has only one prime divisor p can be reduced to $\frac{n}{p}$ functions at max, with each one has different root of unity*

Proof:
Let n be a cyclic order with only p being it's only prime divisor
with the fundamental theorem of arithmetic that states: "every integer greater than 1 is a prime or a product of primes" we can say that $n = p \times q \times ....$
and since p is the only prime divisor we can say that $n = p \times p \times p.... = p^k$ for some integer $k > 1$
we can then define $a^n = 1$ to $a^{p^k} = 1$ with the repeating nature of the roots of unity that only happens when $k|j$ where are both the number roots of unity, we can set m to be the before n roots of unity group that shares the same only prime divisor as p
then we can let $m = p^{k-1}$ with $a^m = 1$ so $a^{p^{k-1}} = 1$
to find the maximum amount of functions

4.7 Mixture of equations

By now, we know that there exist Prime cyclic functions and composite cyclic functions. The prime cyclic can't be expanded, while the composite ones can

So we can say that $\cosh(x)$ is prime cyclic since its cyclic order is prime (2) and it can't be expanded, so e^{-x} can only be expanded naturally with only $\sinh(x)$ and $\cosh(x)$

On the other hand there exist orders that have more than one prime factor, for example 6-th derivative functions and 10-th order

for the 6-th order roots, we can write them as follow $a \in \{\epsilon, \omega, -1, , \omega^2, \epsilon^5, 1\}$ At first glance, we can see that there are many roots similar that can be used to make functions

first we expand $e^{\epsilon x}$

$$e^{\epsilon x} = 1 + \epsilon x + \frac{\omega x^2}{2!} - \frac{x^3}{3!} + \frac{\omega^2 x^4}{4!} + \frac{\epsilon^5 x^5}{5!} + \frac{x^6}{6!} .... = \sum_{n=0}^{\infty} \frac{\epsilon^n x^n}{n!}$$

first we construct the original functions

$$\cosh_6 = \sum_{n=0}^{\infty} \frac{x^{6n}}{(6n)!} \qquad \sinh_6 = \sum_{n=0}^{\infty} \frac{x^{6n+1}}{(6n+1)!} \qquad etc..,$$

we write $\epsilon = \frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$, we can see that $\epsilon = -\omega^2$, the same goes for $\epsilon^5 = -\omega$, we can then rearrange , the sum to be

$$e^{\epsilon x} = (1 + \frac{\omega x^2}{2!} + \frac{\omega^2 x^4}{4!} + ...) + (-\omega^2 - \frac{x^3}{3!} - \frac{\omega x^5}{5!} - ....) = \sum_{n=0}^{\infty} \frac{\omega^n x^{2n}}{(2n)!} + \sum_{k=0}^{\infty} \frac{(-1)\omega^{2+n} x^{2n+1}}{(2n+1)!}$$

we can let them be $\cos_3$ (x) and $\sin_3$ (x) as they are similer to the $\sin(x), \cos(x)$ But instead of the alternating negative sign we see the omega here,

$$e^{\epsilon x} = \sum_{n=0}^{\infty} \frac{\omega^n x^{2n}}{(2n)!} - \sum_{k=0}^{\infty} \frac{\omega^{2+n} x^{2n+1}}{(2n+1)!}$$

We can rearrange the sum to get different results based on something, and it's negative

$$\begin{aligned} e^{\epsilon x} &= (1 - \frac{x^3}{3!} + \frac{x^6}{6!} + ...) + (-\omega^2 + \frac{\omega^2 x^4}{4!} - \frac{\omega^2 x^7}{7!} - ....) + (\frac{\omega x^2}{2!} - \frac{\omega x^5}{5!} + \frac{\omega x^8}{8!} - ....) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(3n)!} + \omega^2 \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+1}}{(3k+1)!} + \omega \sum_{j=0}^{\infty} \frac{(-1)^j x^{3j+2}}{(3j+2)!} \end{aligned}$$

we can see there exists a negative sign here, even alternating, we can analyse both expressions and see, when we get the amount of 3 functions, we got the alternating -1, when we had two functions, we had the two roots of unity in the series, Note this has 3 ways to expand e^{ax}

We can, for now, assume that this happens because the amount of the functions is the first prime factor, so the root(s) of unity must be from another prime factor

We can try for the 10-th order cyclic derivatives, knowing that, after half of the numbers, we will see numbers going to the negatives of each other
$a \in \{\epsilon, \epsilon^2, \epsilon^3, \epsilon^4, -1, -\epsilon, -\epsilon^2, -\epsilon^3, -\epsilon^4, 1\}$

$$e^{\epsilon x} = 1 + \epsilon x + \frac{\epsilon^2 x^2}{2!} + \frac{\epsilon^3 x^3}{3!} + \frac{\epsilon^4 x^4}{4!} - \frac{x^5}{5!} - \frac{\epsilon x^6}{6!} - \frac{\epsilon^2 x^2}{7!} - \frac{\epsilon^3 x^2}{8!} - \frac{\epsilon^4 x^9}{9!} + \frac{x^{10}}{10!} + ...$$

we have the original functions $\cosh_{10} = \sum_{n=0}^{\infty} \frac{x^{10n}}{(10n)!}$ and the others
We can use the negative and positive values

$$e^{\epsilon x} = (1 - \frac{x^5}{5!} + \frac{x^{10}}{10!} + ...) + \epsilon(x - \frac{x^6}{6!} + \frac{x^{11}}{11!}) + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n}}{(5n)!} + \epsilon \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+1}}{(5n+1)!} + ...$$

We can now rearrange them also as even and odd values

$$e^{\epsilon x} = (1 + \frac{\epsilon^2 x^2}{2!} + \frac{\epsilon^4 x^4}{4!} + \frac{\epsilon^6 x^6}{6!} + ...) + (\epsilon x + \frac{\epsilon^3 x^3}{3!} + \frac{\epsilon^5 x^5}{5!} + \frac{\epsilon^7 x^7}{7!}) = \sum_{n=0}^{\infty} \frac{\epsilon^{2n} x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\epsilon^{2n+1} x^{2n+1}}{(2n+1)!}$$

with that and our knowledge about prime cyclic derivatives we can see the pattern
in both composite cases we could either write them as the original functions or as p-amount of functions with q-roots in them where $p \times q = n$ the order of the cyclic derivative
we can say : generally the amount of ways to divide such expression into different functions are **The amount of divisors of** n, where n is the order of the cyclic
simply because with every factor of such a cyclic order derivative we can make functions that are cyclic also when the original function is cyclic, lesser cyclic derivatives functions like order-3 cyclic derivative functions comes back to their state also when they get derivative 6 times or 9 times making them a family too suitable to reduce the orignal funtions of 6 or 9 cyclic deriavtives
and in prime functions there only exists two factors, 1 and prime number itself, which we can see as 1 represent the e^{ax} itself, and prime represents the original functions that can't be reduced anymore.
so they have 2 families of functions, while order 6 has 4 divisors (1,2,3,6) so we can see 4 forms, the 2-order lesser functions, the 3-order lesser functions, e^{ax} and original functions
this property we can call as "n-order Families" with reduced functions being called "Lesser cyclic derivative function to order-n", while we can call the original functions themselves "Higher cyclic derivative function to order-m" (where m is the factor functions)
and the ways to define these functions are the prime factors of n
we may need a proof, and it's simple one non the less
Proof:
we have a cyclic function $f(x)$ that is order of cyclation is n
from the prime cyclic functions property we know that $f(x)$ is either composite

cyclic or prime cyclic, if $f(x)$ is prime cyclic derivative, we are there , if $f(x)$ is composite cyclic derivative it can be expanded to $g(x)$ functions with roots q such that $a^q = 1$, we keep doing this until $g(x)$ is prime cyclic function, thus it's always true
Q.E.D

4.8 composition of functions

5 But why cyclic derivatives?

5.1 Dimension Bender and Hyper operations

Operations are the fundamentals of mathematics. We start with a constant, then succession, which has the definition $S(n) = n + 1$
Repeated succession results addition, which can be defined as $A(n, m) = n+m = \underbrace{S(S(S(S(....n))))}_{m \text{ times}}$, repeating that gives Multiplication $M(n, m) = n \times m$ with the same idea, so one exponentiation and tetration
What brings that here is the properties of exponentiation, we can say that for any constant a, we can do

$$a^n \times a^m = a^{n+m} \qquad (a^n)^m = a^{n\times m}$$

Exponentiation moves other operations up in the hierarchy; it linearises them, "Bending the space around it". We can see that every cyclic derivative is in or can be used to create e^{ax}; an exponentiation form
But then the question arises, why exactly e and not any other base,well we can see it with D^z

$$D^z a^{bx} = b^z a^{bx} \ln(a)^z \Rightarrow D^z e^{bx} = b^z e^{bx} \ln(e)^z = b^z e^{bx}$$

Unlike any other base, e is the only base that stays without remainder, it bends space without any trace, it has the perfect environment for pure cyclic functions to arise like $\sin(x)$ and $\cosh(x)$ as they won't interact with any other change, it allows clean, smooth transformation from point a to point b we can also see that in the expansion of $e^{ax} = \sum_{n=0}^{\infty} \frac{a^n x^n}{n!}$, the simpliest form of expansion for a function

5.2 Analysis of cyclic functions

But now this gives a bigger question: what does this information help us with cyclic derivatives

5.3 Transformtive functions

But unlike cyclic derivatives, this one is problematic
cyclic functions change smoothly from one derivative to another, transformative

functions choose to break that in a pretty much big sense, take for example $\sin^{-1}(x)$, from D^z prescriptive, it can be defined as

$$D^z \sin^{-1}(x) = D^z \frac{1}{\sqrt{1-x^2}} \qquad z \geq 0$$

and at 0 it is just $\sin^{-1}(x)$ and for the integrals it seems to be a product of both of them, it just jumps back and forth between faces with no predictable move, that is, unless we go to the complex plane and see the complex definition which is $\sin^{-1}(z) = -i\ln(iz + \sqrt{1-z^2})$
There is a natural logarithm, whic has we know is a problem with D^z
But not all transformative functions directly have $\ln(x)$ in their definitions
for example $\frac{1}{x^n}$ is a transformtive functions because at integration, it directly transforms to another function $\ln(x)$, which isn't in the definition but rather the integral of it
Looking at the big picture, the inverse of x^n it's an inverse function, and nearly all the transformative functions are inverses, and that is for a reason
A function is defined to be an input-to-output machine; many inputs can give the same output, but not the other way around. When one input gives many outputs, it's not an ordinary function in the definition,
When we try to get an inverse out of a function, most of the time that rule is broken. We can see for the simple case x^2 that makes a double-input one-output system, to define its inverse $\sqrt{x}$ in the real value, we have to sacrifice the other inputs that give the same outputs, being the negative numbers

5.4 $\ln(x)$ is collapser

First of all, this function isn't analytic; it has a singularity at $x = 0$, and it's a multi-valued function in the complex plane, that is, by itself a problem, but we are going to discuss it later. Instead of $\ln(x)$ we will use $\ln(x+1)$
We first find its n-th derivative

$$f(x) = \ln(x+1) \qquad f'(x) = \frac{1}{x+1} \qquad f''(x) = -\frac{1}{(x+1)^2}$$

this the x^{-n} α-th derivative , so we can assume the formula is simply $D^\alpha \ln(x+1) = \frac{(-1)^{\alpha+1}\Gamma(\alpha)}{(x+1)^\alpha}$, this is supposed to get back $\ln(x+1)$ at $\alpha = 0$

$$D^0 f(x) = \frac{(-1)^1\Gamma(0)}{(x+1)^0} = \frac{-\Gamma(0)}{1} \neq \ln(x+1)$$

We can see that this breaks because of the Gamma pole, but we can try another approach
(Note: this is the Gamma of -1 , using this Gamma definition$\Gamma(n) = \frac{\Gamma(n+1)}{n}$, at -1 we get $\Gamma(-1) = -\frac{\Gamma(0)}{1}$)
we can try to use the Maclaurin series to see the function smoother, then we

can take the derivative and try it

$$\ln(x{+}1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} \qquad D^\alpha \ln(x{+}1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left[\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha}\right] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\Gamma(n)}{\Gamma(n-\alpha+1)}x^{n-\alpha}$$

We can try it for the first derivative

$$D^1 \ln(x{+}1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cancel{\Gamma(n)}}{\cancel{\Gamma(n)}}x^{n-1} = \sum_{n=1}^{\infty}(-1)^{n-1}(x)^{n-1} = 1{-}x{+}x^2{-}x^3{+}... = \frac{1}{x+1}$$

This series works, and we can test it for D^0 to see if it returns the same values or not

$$D^0 f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\Gamma(n)}{\Gamma(n+1)}x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\Gamma(n)}{n\Gamma(n)}x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n = \ln(x{+}1)$$

which is correct, we can even try to see if it works for the first integral

$$D^{-1} f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\Gamma(n)}{\Gamma(n+2)}x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}x^{n+1}$$

Expanding this expression, we get

$$\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \frac{x^6}{30} - ....$$

Now we find the integral for the function

$$f(x) = \int \ln(x+1) = (x+1)(\ln(x+1) - 1)$$

We can then put the Maclaurin expansion in here for $\ln(x+1)$

$$(x+1)(x - \frac{x^2}{2} + \frac{x^3}{3} - ...) - x = (x^2 - \frac{x^3}{2} + \frac{x^4}{3} - ...) + (x - \frac{x^2}{2} + \frac{x^3}{3} - ...) - x$$

We can do some algebraic manipulation here to rearrange the subtractions

$$= (x-x) + (x^2 - \frac{x^2}{2}) + (\frac{x^3}{3} - \frac{x^3}{2}) + ... = (0 + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \frac{x^6}{30} - ....)$$

Both series are equal, thus it works
We can explain that not happen because it never reach a fatal Gamma pole for the sum, and because the sum is infinite, we can continue doing that forever without changing the result. We can get the integral formula in this case to be

$$D^{-\alpha} \ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}\Gamma(n)}{\Gamma(n+\alpha+1)}x^{n+\alpha}$$

6 Cycliation in other ways

from the hyper-operations explanation; cycliation isn't only for simple e^{ax}, which allows us to try different things effectively

6.1 Multi variable cycliation

we can extend the cyclic functions to the 3rd Dimension simply since the defenition is simple (e^{ax} where $a^n = 1$), we can simply change x to $x + y$ to get

$$e^{a(x+y)} \text{ where } a^n = 1$$

this simple formula to extend it to 3D , so

$$e^{-(x+y)} = \cosh(x + y) - \sinh(x + y) \qquad e^{i(x+y)} = \cos(x + y) + i \sin(x + y)$$

infact if we let u be any operation between x, y because we will get $e^{au}$where $a^n = 1$
this can work and extend to any operation to infinite dimensions, but there is one thing
this extension is linear, we can expect the outcome and result to work right and similer, the one function going upwards at $x > 1$ just becomes a plane going upwards then a cube going upwards
this happens because when we change e^{ax} to e^{au} the reality is similer to the hyperoperations explanation, it's just e^{ax} at the end of it
we can find some true change by changing the rule , since the differentiation of the function is dependent of a^x and it's nature to come back at the end of a full cycle , we can try to change it to e^{ax+by} in the 3D definition
we can say that for this to happen $a^n = b^n = 1$ which is true only when $a = b$
But this doesn't stop us from making a new definition, why not simply make it one-dimensional cyclic?
The idea is simple, we need both to be cyclic but not the same order at the same time, at the end, the reason for its cyclicity is its non-growing nature
We can define it as

$$e^{ax+by} \text{ where } a^n = b^m = 1$$

This definition is n-order cyclic on the x-axis and m-order cyclic on the y-axis, we can try it with $a = 1, b = -1$ to see what happens. First, we expand it

$$e^{x-y} = 1 + (x - y) + \frac{(x - y)^2}{2!} + \frac{(x - y)^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{(x - y)^k}{k!}$$

Of course, this can be treated as $e^{-(y-x)}$ which can be written in sinh and cosh terms

$$e^{x-y} = e^{-(y-x)} = \cosh(y - x) - \sinh(y - x)$$

But we can try something like e^{-x+iy} which is a second cyclic with respect to x but a four cyclic with respect to y

this one can be expanded simply using Euler's formula $e^{i\theta}$ and the definition for e^{-x}

$$e^{-x+iy} = e^{-x}e^{iy} = e^{-x}(\cos(y) + i\sin(y))$$

6.2 Combination of cycliation

6.3 cycliation derivative to a function order

from the original paper we know that we can use D^α with the order being the differentiable variable as D^x or $D^{g(x)}$

$$D^x f(x) = D^x e^{ax} = a^x e^{ax}$$

and unlike the monomials we can take the derivative again of this function to be

$$D^x f'^x(x) = a^{2x}e^{ax} + a^x e^{ax}\ln(a)^x$$

we can continue for a third time

$$D^x f''^x(x) = a^{3x}e^{ax} + a^x e^{ax}\ln(a)^{2x} + a^{2x}e^{ax}\ln(x) + a^x e^{ax}\ln(a)^x + a^x e^{ax}\ln(a)^x\ln(\ln(a))^x$$

but the problem comes fro the first derivative, the function isn't cyclic anymore it broke it's rule on expansion thus making this topic declined

6.4 can any function be transformed to a cyclation vatiant?

6.5 x^x is weird

6.6 Cycliation, in imaginary plane system

6.7 Cycliation into the third dimension

6.8 Cycliation, but not in e

With our knowledge, we can try to think of what would happen when we change the base to any other constant a, with the rule being $b^n = 1$, let's try to see what happens at second-order cyclic derivatives ($\sinh(x)$ and $\cosh(x)$) using the Maclaurin series

$$a^{-x} = 1 - x\ln(a) + \frac{x^2\ln(a)^2}{2!} - \frac{x^3\ln(a)^3}{3!} + ... = \sum_{k=0}^{\infty}\frac{(-1)^k x^k\ln(a)^k}{k!}$$

we are going to extract the $\sinh(x)$ and $\cosh(x)$ terms from the sum by the negatives and the positives and give them the terms ${}_a\sinh(x)$ and ${}_a\cosh(x)$ t distinguish them from the original ones

$${}_a\cosh = \sum_{k=0}^{\infty}\frac{x^{2k}\ln(a)^{2k}}{(2k)!} \qquad {}_a\sinh = \sum_{k=0}^{\infty}\frac{x^{2k+1}\ln(a)^{2k+1}}{(2k+1)!}$$

then of course, to find the euler form it will be

$$a^{-x} = {}_{\mathrm{a}}\cosh(x) - {}_{\mathrm{a}}\sinh(x)$$

to extend this to any function, we will simply do

6.9 Some properties of other base cyclic derivatives

For the functions ${}_{\mathrm{a}}\cosh(x)$, if we turn the base less than 1, we can observe that it starts to exhibit waves and patterns. When $a = 0.5$, we can see some similarity between these functions and $\cos(x)$
using trial and error, I could make say when $a \approx 0.367879441190$
This number perfectly matches e^{-1},infact if we plugged it directly we can see why

$${}_{\mathrm{e^{-1}}}\cosh = \sum_{k=0}^{\infty} \frac{x^{2k}\ln(e^{-1})^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{x^{2k}(-1)^{2k}}{(2k)!}$$

Of course, this is the $\cos(x)$ Maclurin series, and this can give us another angle to understand the connection between composite cyclics and prime cyclics
Composite cyclics can be reduced to prime cyclics with some change, and this change happens to be the change of the base for the cyclic function.

7 The function families of cyclic functions

exploring the families of the functions is intersting , since it may help us understand why such functions behave like this at least from teh point of D^z and being NFRD while they are constructed from FRD functions

7.1 $\tan(x)$ and $\tanh(x)$

from the beginning, we can see some problems
before we even start we realize that ,both $\tan(x)$ and $\tanh(x)$ don't have a "main way" to define them
the stranded definition for them is $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$, but that is only because there exists two functions , and this only happens in hyperbolic functions, and trigonometric functions too because it satisfies $\frac{n}{p} = 2$ because they are the only ones that sat,but anyway we can use many ways with diffrent resaults to define the tangent function(s)
Using the first two functions only
we can try to define $\tan_{\mathrm{n}}(x)$ to use the first two functions of any e^{ax}
so we can write it as $\tanh_{\mathrm{n}}(x) = \frac{\sinh_{\mathrm{n}}(x)}{\cosh_{\mathrm{n}}(x)}$ that is for the non-reduced orignal functions of the expansion and $\tan_{\mathrm{n}}(x) = \frac{\sin_{\mathrm{n}}(x)}{\cos_{\mathrm{n}}(x)}$ to be for the reduced functions
we can try it for the 3-order cyclic derivatives

$$\tanh_3(x) = \frac{\sinh_3(x)}{\cosh_3(x)} = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \left(\sum_{k=0}^{\infty} \frac{x^{3k}}{(kn)!} \right)^{-1}$$

but the problem with this one is there are other functions that aren't put in prespiactive , and the more we go up with the order the more they are which makes this definition unusable

The division between every function and $\cosh_{\mathrm{n}}$ **or** $\cos_{\mathrm{n}}$

Then we can try to to divide every single function multiplied from the cyclic family

$$\tanh_3 = \frac{\sinh_3(x)}{\frac{\cosh_3(x)}{\sinh_3 \mathrm{II}(x)}} = \frac{\sinh_3(x)\sinh_3 \mathrm{II}(x)}{\cosh_3(x)}$$

and if we continue this we will see that all odd functions from a full cyclic family are on the top while the evens are on the bottom, so if we let $\cosh_{\mathrm{n}} = \sinh_{\mathrm{n}} 0$ we get this formula

$$\tanh_{\mathrm{n}}(x) = \frac{\prod_{K=0}^{n} \sinh_{\mathrm{n}} \mathrm{2K{+}1}(x)}{\prod_{J=0}^{n} \sinh_{\mathrm{n}} \mathrm{2J}(x)}$$

Using the Series

but to actually define $\tanh_{\mathrm{n}}(x)$ we need to observe what the orignal $\tan(x)$ have in that this one needs to have

$\tan(x)$ is equal to $\frac{\cos(x)}{\sin(x)}$: this isn't supposed to be true for all variants, and we will discuss the reason for that later

$\tan(x)$ is an NFRD function with its series representation only: This one is important since a series representation is the represent of a function using x^n, which *According to the original paper* has the change directly happening to them so the best way to define $\tanh_{\mathrm{n}}(x)$ is via the series definition, now we see the Maclaurin series

$$\tan(x) = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} \qquad \tanh(x) = \sum_{n=0}^{\infty} \frac{B_{2n}4^n(1-4^n)}{(2n)!} x^{2n-1}$$

But we can't take these series directly and apply them because they may be only special occians , we need to define the main series using the original functions

We can skip $\tanh(x)$ since it's pretty straightforward in the original functions being $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

for the four original functions of cyclic order 4 $(\cosh_4(x), \sinh_4(x), \sinh_4 \mathrm{II}(x), \sinh_4 \mathrm{III}(x))$

We can use the sin and cos definitions of them being

$$cos(x) = \cosh_4(x) + \sinh_4(x) \qquad \sin(x) = -(\sinh_4 \mathrm{II}(x) + \sinh_4 \mathrm{III}(x))$$

We can then put them in the definition of $\tan(x)$ to get

$$\tan(x) = \frac{-(\sinh_4 \mathrm{II}(x) + \sinh_4 \mathrm{III}(x))}{\cosh_4(x) + \sinh_4(x)}$$

Expanding the series we get

$$\tan(x) = \frac{-(\sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)!} + \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!})}{\sum_{j=0}^{\infty} \frac{x^{4j}}{(4j)!} + \sum_{u=0}^{\infty} \frac{x^{4u+2}}{(4u+2)!}}$$

This shows that for the original functions, it's dividing the second half of the series by the first half
we can try to do the same for the 8-order to see the result

$$\tan_8 = \frac{\sin_8(x) + \sin_8 \mathrm{III}(x)}{\cos_8(x) + \sin_8 \mathrm{II}(x)} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)!} + \sum_{j=0}^{\infty} \frac{(-1)^j x^{4j+3}}{(4j+3)!}}{\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{(4k)!} + \sum_{u=0}^{\infty} \frac{(-1)^u x^{4u+2}}{(4u+2)!}}$$

expanding the sums we get

$$\tan_8(x) = \frac{x + \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} + ...}{1 + \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{x^6}{6!} + ...}$$

which is reduced to the regular sum $\sin(x)$ and $\cos(x)$ we get the final result

$$\tan_8(x) = \frac{\sin(x)}{\cos(x)} = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1}$$

7.2 $\sec(x)$ and $\mathrm{sech}(x)$

only the first function $\cosh_{\mathrm{n}}$ and it's reduced version $\cos_{\mathrm{n}}$ will make $\mathrm{sech}_{\mathrm{n}}$ and $\sec_{\mathrm{n}}$
We know that the series for any original function $\cosh_{\mathrm{n}}$ is equal to $\sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!}$
We can expand the sum and let $\mathrm{sech}_{\mathrm{n}}$ be the reportorial of it

$$\mathrm{sech}_{\mathrm{n}} = \frac{1}{1 + \frac{x}{n!} + \frac{x^{2n}}{(2n)!} + \frac{x^{3n}}{(3n)!} + ...} = \frac{1}{\sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!}}$$

and because of the idenitty $\cos(x)\sec(x) = 1$ we can use euler numbers to any kn in both reduced and orignal series

$$\mathrm{sech}_{\mathrm{n}} = \sum_{k=0}^{\infty} \frac{E_{kn} x^{kn}}{(kn)!} \qquad \sec_{\mathrm{n}} = \sum_{k=0}^{\infty} \frac{(-1)^k E_{kn} x^{kn}}{(kn)!}$$

7.3 $\csc(x)$ and $\mathrm{csch}(x)$

7.4 $\cot(x)$ and $\coth(x)$

8 The inverses of cyclic function families

9 Analysis of cyclic derivatives

9.1 special derivatives with cyclic derivatives

let's take the derivative of a cyclic function when the order is the root of unity first in Euler's form

$$D^{-1}e^{-x} = -1^{-1}e^{-x} = -e^{-x} = \sinh(x) - \cosh(x)$$

$$D^i e^{ix} = i^i e^{ix} = (e^{\frac{i\pi}{2}})^i e^{ix} = e^{-\frac{\pi}{2}} e^{ix} = e^{-\frac{\pi}{2}} \cos(x) + e^{-\frac{\pi}{2}} i \sin(x)$$

we let $\omega = e^{\frac{2i\pi}{6}}$ which is the sixth root of unity

$$D^\omega e^{\omega x} = \omega^\omega e^{\omega x} = (e^{\frac{2i\pi}{6}})^\omega e^{\omega x} = e^{\frac{\omega 2i\pi}{6} + \omega x} = e^{\omega(\frac{i\pi}{3} + x)}$$

we let $\omega = e^{\frac{2i\pi}{8}}$ which is the eight-th root of unit

$$D^\omega e^{\omega x} = \omega^\omega e^{\omega x} = (e^{\frac{2i\pi}{8}})^\omega e^{\omega x} = e^{\frac{\omega 2i\pi}{6} + \omega x} = e^{\omega(\frac{i\pi}{3} + x)})$$

for the first hyperbolic and trigonometric functions this happen because they are the only ones that satisfy $n^n \in \mathbb{R}$ where $a^n = 1$
we can find the general simple rule
let $a^n = 1$ in the equation e^{ax}

$$D^a e^{ax} = a^a e^{ax}$$

we can express any root of unity using the formula $a = e^{\frac{2ik\pi}{n}}$ where k is the order in the unity circle

$$D^a e^{ax} = (e^{\frac{2ki\pi}{n}})^a e^{ax} = e^{\frac{2aki\pi}{n} + ax} = e^{a(\frac{2ki\pi}{n} + x)}$$

and for normal cases we use k = 1 as it is the principle unit of the circle
we can test it works for -1

$$D^{-1} e^{-x} = e^{-(i\pi + x)} = -e^{-x} \qquad D^i e^{ix} = e^{i(\frac{i\pi}{2} + x)} = e^{-\frac{\pi}{2}} e^{ix}$$

we can see that only for these values this function works as a scaler but for other values it is a rotater;
we can find a smiler formula that is for $D^{\frac{1}{a}}$

$$D^{\frac{1}{a}} e^{ax} = a^{a^{-1}} e^{ax} = (e^{\frac{2ki\pi}{n}})^{\frac{1}{a}} e^{ax} = e^{\frac{2ki\pi}{an} + ax} = e^{a(a^{-1} + x)}$$

and since $a = e^{\frac{2ki\pi}{n}}$ we can say that $\ln(a) = e^{\frac{2ki\pi}{n}}$

$$D^{\frac{1}{a}} e^{ax} = e^{\frac{\ln(a)}{a} + ax} = e^{a(\frac{\ln(a)}{a^2} + x)} = e^{a^{-1}(\frac{2ik\pi}{n}) + ax}$$

of course D^i and D^{-i} alawys returns a scaler of the function that

Theorem 9.1 *D^i and D^{-i} of any cyclic function of order n returns a scaler with the value $\pm\frac{2k\pi}{n}$ times the cyclic function*

Proof: Let $a^n = 1$ thus a can be expresses as $e^{\frac{2ki\pi}{n}}$ where k is the order of root in the roots of unity, we can now take the $\pm i$-th derivative

$$D^{\pm i} e^{ax} = a^{\pm i} e^{ax} = (e^{\frac{2ki\pi}{n}})^{\pm i} e^{ax} = e^{\frac{\mp 2k\pi}{n}} e^{ax}$$

Q.E.D now we can return to the original functions themselves, ;et's try the i-th derivatives of these functions

$$D^i \cosh(x) = \frac{e^x + (-1)^i e^{-x}}{2} = \frac{e^x + (e^{i\pi})^i e^{-x}}{2} = \frac{e^x + (e^{-\pi}) e^{-x}}{2} = \frac{e^x + e^{-(x+\pi)}}{2}$$

with the same formula

$$D^i \sinh(x) = \frac{e^x - e^{-(x+\pi)}}{2}$$

for third cyclic functions we can write them using euler's form

$$D^\alpha \cosh_3 = \frac{e^x + \omega^\alpha e^{\omega x} + \omega^{2\alpha} e^{\omega^2 x}}{3}$$

Now we take the first i-th derivative

$$D^i \cosh_3 = \frac{e^x + \omega^i e^{\omega x} + \omega^{2i} e^{\omega^2 x}}{3} = \frac{e^x + (e^{\frac{2i\pi}{3}})^i e^{\omega x} + (e^{\frac{2i\pi}{3}})^{2i} e^{\omega^2 x}}{3} = \frac{e^x + (e^{\frac{-2\pi}{3}}) e^{\omega x} + (e^{\frac{-4\pi}{3}}) e^{\omega^2 x}}{3}$$

Now, for trigonometric functions we have

$$D^i \cos(x) = \cos(\frac{i\pi}{2} + x) \qquad D^i \sin(x) = \sin(\frac{i\pi}{2} + x)$$

We can see something weird, some functions rotate while others scale, not full scale for the function, but they scale in some sort
We can assume for now that this happens only in original cyclic functions, and we can actually see the reason why
for reduced functions like sin and cos, their Euler form always has some root of unity in all the exponents, while original functions have the term e^x on them, which always returns itself with no change
We can examine more using the expansion of sin

$$\sin = \frac{e^{ix} - e^{-ix}}{2i}$$

plugging the $\frac{i\pi}{2} + x$ term we get

$$\sin(\frac{i\pi}{2} + x) = \frac{e^{i(\frac{i\pi}{2}+x)} - e^{-i(\frac{i\pi}{2}+x)}}{2i} = \frac{e^{\frac{-\pi}{2}+ix} - e^{\frac{\pi}{2}-ix}}{2i}$$

We can see that there is a scaling value, but it's different on each side
Scales that can be represented as clean inputs in their functions can be called "clean scale transforms" like $D^i \sin(x)$, and scales that can be represented using only Euler form or any other way that isn't the original function with a change in it can be called "Middle way scale transforms"
Actually, this can explain the $D^i e^{ix}$ confusion from before.
This transformation, rather than it being clean for its expansion, it's more of a change to the original function with a clean transform, which makes it look like a Middle way scale transformation, we can see by multiplying the i-th derivative to i

$$D^i e^{ix} = e^{-\frac{\pi}{2}} e^{ix} = e^{-\frac{\pi}{2}} \cos(x) + e^{-\frac{\pi}{2}} i \sin(x)$$

$$^i D^i e^{ix} = i^{-1} e^{ix} = -ie^{ix} = e^{\frac{3i\pi}{2}} e^{ix} = e^{i(\frac{3\pi}{2}+x)} = \cos(\frac{3\pi}{2} + x) + i \sin(\frac{3\pi}{2} + x)$$

$$= \sin(x) - i\cos(x)$$

so we can see that this was an expansion problem, the expansion of e^{ax} doesn't necessarily connect toe^{ax} derivative as it works on it's own, it has more functions in it, and it simply doesn't equal, that is at least for complex order, we can see that this happens in fractional order too

$$D^{\frac{1}{2}} e^{ix} = i^{\frac{1}{2}} e^{ix} = (e^{\frac{i\pi}{2}})^{\frac{1}{2}} e^{ix} = e^{\frac{i\pi}{4}} e^{ix} = e^{i(\frac{\pi}{4}+x)}$$

which is equal to $D^{\frac{1}{2}} \sin(x) = \sin(\frac{\pi}{4} + x)$
We can see that this happen explicitly in D^{α} only when $\alpha \in \mathbb{C}$
There are many explanations for this. We can say that this happens because of the change in scaling and rotation between these functions

10 Identities in cyclic derivatives

10.1 The addeition formula

there exists a formula for angles addeition in $\sin(x)$ and $\sinh(x)$ that is

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \qquad \sinh(\alpha+\beta) = \sinh(\alpha)\cosh(\beta)+\cosh(\alpha)\sinh(\beta)$$

we can try to prove the same for third cyclic deiravtives

$$e^{\omega x} = \cosh_3(x) + \omega \sinh_3(x) + \omega^2 \sinh_3 \mathrm{II}(x)$$

$$e^{\omega(\alpha+\beta)} = \cosh_3(\alpha+\beta) + \omega \sinh_3(\alpha+\beta) + \omega^2 \sinh_3 \mathrm{II}(\alpha+\beta)$$

$$e^{\omega\alpha}e^{\omega\beta} = \big(\cosh_3(\alpha)+\omega\sinh_3(\alpha)+\omega^2\sinh_3\mathrm{II}(\alpha)\big)\times\big(\cosh_3(\beta)+\omega\sinh_3(\beta)+\omega^2\sinh_3\mathrm{II}(\beta)\big)$$

after multplying and grouping by the coffiecnts we get a nine long terms equation so for the sake of simplicty we will call this time $\cosh_3(x) = C(x), \sinh_3(x) = S(x), \sinh_3 \mathrm{II}(x) = SS(x)$

$$= \omega\big(C(\alpha)S(\beta)+S(\alpha)C(\beta)+SS(\alpha)SS(\beta)\big)+1\big(C(\alpha)C(\beta)\big)+SS(\alpha)S(\beta)+S(\alpha)SS(\beta)\big)$$

$$+\omega^2\big(C(\alpha)SS(\beta) + S(\alpha)S(\beta) + C(\beta)SS(\alpha)\big)$$

we can now equate both $e^{\omega(\alpha+\beta)}$ and $e^{\omega\alpha}e^{\omega\beta}$, of course we equate with the cofficents in both sides

$$\cosh_3(\alpha+\beta) = C(\alpha)C(\beta) + SS(\alpha)S(\beta) + S(\alpha)SS(\beta)$$

$$\sinh_3(\alpha+\beta) = C(\alpha)S(\beta) + S(\alpha)C(\beta) + SS(\alpha)SS(\beta)$$

$$\sinh_3\mathrm{II}(\alpha+\beta) = C(\alpha)SS(\beta) + S(\alpha)S(\beta) + C(\beta)SS(\alpha)\big)$$

what is worth noting is here $\sinh_3(\alpha+\beta)$ and $\sinh_3\mathrm{II}(\alpha+\beta)$ have some connection in them, because they have swap sums, looking at $\cosh_3(\alpha+\beta)$ we got the $\cosh_3(\alpha)\cosh_3(\beta)$, while this isn't the case in the $\sinh_3(\alpha+\beta)$ and

$\sinh_3 \mathrm{II}(\alpha+\beta)$ case
the double angle formulas may show us more about this

$$\cosh_3(2\alpha) = C(\alpha)^2 + 2SS(\alpha)S(\alpha)$$

$$\sinh_3(2\alpha) = SS(\alpha)^2 + 2C(\alpha)S(\alpha)$$

$$\sinh_3 \mathrm{II}(2\alpha) = S(\alpha)^2 + 2C(\alpha)SS(\alpha)$$

the sums are swapped clearly in this formula, this doesn't happen in trigonometric and hyperbolic functions because there exists two different types of functions the main $\cosh(x)$ and the branch $\sinh(x)$, there is room to swap anything , while third order cyclics have two diffrent functions of the same type (branch) being $\sinh_3(x)$ and $\sinh_3 \mathrm{II}(x)$
we can generalize this formula for any n cyclic derivative
let ϵ be the root of unity for the $n-th$ cyclic function, using the general euler identity

$$e^{\epsilon(\alpha+\beta)} = \sum_{k=0}^{n-1} \epsilon^k \sinh_{\mathrm{n}} \mathrm{K}(\alpha+\beta)$$

(of course when it's zero we get $\cosh_{\mathrm{n}}$)

$$e^{\epsilon\alpha}e^{\epsilon\beta} = \sum_{i=0}^{n-1} \epsilon^i \sinh_{\mathrm{n}} \mathrm{I}(\alpha) \times \sum_{j=0}^{n-1} \epsilon^j \sinh_{\mathrm{n}} \mathrm{J}(\beta)$$

and then we must equalise both sides via coefficients

$$\sum_{k=0}^{n-1} \epsilon^k \sinh_{\mathrm{n}} \mathrm{K}(\alpha+\beta) = \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} \epsilon^{(i+j)} \sinh_{\mathrm{n}} \mathrm{I}(\alpha)\epsilon^k \sinh_{\mathrm{n}} \mathrm{J}(\beta)$$

since ϵ is a root of unity it repeats itself, so we can rewrite them to group the same functions with a root of unity by first letting $k = i + j$

$$\sum_{k=0}^{n-1} \epsilon^k \sinh_{\mathrm{n}} \mathrm{K}(\alpha+\beta) = \sum_{k=0}^{n-1} \epsilon^k \sum_{i+j=k \bmod n}^{n-1} \sinh_{\mathrm{n}} \mathrm{I}(\alpha)\sinh_{\mathrm{n}} \mathrm{J}(\beta)$$

Then we equal the roots to get the final form

$$\sinh_{\mathrm{n}} \mathrm{K}(\alpha+\beta) = \sum_{j=0}^{n-1} \sinh_{\mathrm{n}} \mathrm{K}(\alpha) \sinh_{\mathrm{n}} \text{K-J}(\beta)$$

10.2 The squaring formula

10.3 The Odd and Even

to see if a function is even or odd we need to apply one test
if $f(-x) = f(x)$ it's even, and if $f(-x) = -f(x)$ it's odd, and we can put it in

the test

$$\sinh_{\mathrm{n}} \mathrm{K}(-x) = \sum_{j=0}^{\infty} \frac{(-x)^{(nj+k)}}{(nj+k)!}$$

This only happens when $n + k$ is even, as it will return a positive output

- so all $\cosh_{2\mathrm{n}}(x)$ are even when $n \in \mathbb{N}$
- for $\cosh_{2\mathrm{n}+1}(x)$ we get a mix of positive and negative when $n \in \mathbb{N}$ which isn't even nor odd
- When $\cosh_{\mathrm{n}}(x)$ is even , $\sinh_{\mathrm{n}} 2\mathrm{K}{+}1$ are odd and $\sinh_{\mathrm{n}} 2\mathrm{K}$ is even
- When $\cosh_{\mathrm{n}}(x)$ is odd, $\sinh_{\mathrm{n}} \mathrm{K}$ is , it won't be either odd or even because we would have positive and negative terms

10.4 Identities between different cyclic derivatives

it's known that $\sin(ix) = i\sinh(x)$ and $\cos(ix) = \cosh(x)$, we can see that this happen because of the exponents form

$$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i}$$

multiplying by $\frac{i}{i}$ we get

$$\sin(ix) = \frac{i}{i} \times \frac{e^{-x} - e^{x}}{2i} = i \times \frac{e^{-x} - e^{x}}{-2} = i\frac{e^{x} - e^{-x}}{2} = i\sinh(x)$$

But we can generalise this idea
take a function $\sin_{\mathrm{n}} \mathrm{N}$, we can express such a function with the exponentiation form

$$\sin_{\mathrm{n}} \mathrm{N} =$$

10.5 Cyclic derivatives and algebraic equations

10.6 the periodic nature in cyclic derivatives

sin and cos are known for their periodic nature, as the repeating peaks and troughs of their functions are what we call them periodic for in the first place. While their original functions $\sinh(x)$ and $\cosh(x)$ do not seem to have any periodic nature, this happens because in trigonometric functions there exists a change of sign value $-(1)^n$ that allows such values to behave smoothly going from one positive value to another negative one; this of course because in Euler form of the function e^{ix} we get negative and positive imaginary and real functions, so combining them to result a reduced function make this happen.
but this also means that any reduced function is a periodic function because it involves a changing root of unity.

Theorem 10.1 *Any reduced function is a periodic function, and any original function is a non-periodic function*

And with this, we can try to see periodic functions that are based on any other root of unity.
We can start with the simple 8-order derivative functions

$$e^{\epsilon x} = 1 + \epsilon x + \frac{ix^2}{2!} + \frac{\epsilon^3 x^3}{3!} + \frac{(-1)x^4}{4!} + \frac{\epsilon^5 x^5}{5!} + ...$$

Reducing this function gives us 4 functions with 2 roots of unity being $\{1, -1\}$, which gives them their periodic nature in the real number line, but we can work it the other way around.
We are going to use four roots of unity $\{1, i, -1, -i\}$ to reduce the function to 2 functions that are periodic along the imaginary plane

$$e^{\epsilon x} = (1 + \frac{ix^2}{2!} + \frac{(-1)x^4}{4!} + \frac{(-i)x^6}{6!} + ...) + (\epsilon x + \frac{\epsilon^3 x^3}{3!} + \frac{\epsilon^5 x^5}{5!} + \frac{\epsilon^7 x^7}{7!} + ...)$$

For the first sum, it's clear how it will go, it will be i^n since we start with the 1, but for the second sum, it's not as straight forward.
First we need to remember that $\epsilon^2 = i$ thus $\epsilon = i^{\frac{1}{2}}$, applying it to the sum we get $i^{\frac{1}{2}}x + \frac{i^{\frac{3}{2}}x^3}{3!} + \frac{i^{\frac{5}{2}}x^3}{5!} + ...$; the pattern is $i^{\frac{2n+1}{2}}$
In fact, to make them similar, a better representation would be $i^{\frac{2n}{2}}$, but as we see the twos cancel out, with this information, we can write the function $e^{\epsilon x}$ as

$$e^{\epsilon x} = \sum_{n=0}^{\infty} \frac{i^n x^{2n}}{(2n)!} + \sum_{k=0}^{\infty} \frac{i^{\frac{2k+1}{2}} x^{2k+1}}{(2k+1)!} = \cos_{8|4}(x) + \sin_{8|4}(x)$$

There is a change in notation that is needed; if we create functions out of different roots of unity and different amounts, we would need to make a difference between them.
if the number of reduced functions isn't equal to the number of original four divided by the smallest prime factor (in this case 2), we need to write them like this $\cos_{\mathrm{O|C}}(x)$ where "O" is for the original amount of functions (in this case 8) and "C" is for the chosen root of unity (In this case 4), but if the number of reduced functions is the number of original functions divide by the smallest prime factor we can write it simply as $\cos_{\mathrm{O}}(x)$
The functions we have now behave like normal for real values, being every second terms in $\cos_{8|4}(x)$, for the imaginary terms, we can say that these functions are supposed to act similarly on the imaginary number line, creating a curvy checkerboard look, but for $\sin_{8|4}(x)$ this becomes a little problematic as it will create a weird curvature space that is hard to analyze or tell us any information about.
Instead, I believe a better representation can be made, since $\sin(x)$ and cos use 2 roots of unity on one one-dimensional line (The real number line), we can say that $\cos_{8|4}(x)$ and $\sin_{8|4}(x)$ works on 4 roots of unity one a 3-rd dimension pipe

or cylinder, when there are 2 roots of unity it goes up and down between the roots, while in 4 roots of unity system it needs to go from one root to another, for the roost being 1 on top of the cylinder, i on the right of cylinder, -1 in the bottom and $-i$ on the left, these of course are not in the same y axis or else this would be a circle but they have distance between them like the original trigonometric functions
The lines created between them isn't straight but rather curves creating troughs between every peak (unity root), which make it 3rd arcs that create a sphere when put together, like the arcs making a circle in the trigonometric functions

11 half Cyclic, quartic cyclic, complex cyclic...

11.1 fractional cyclic

fractional cyclic derivatives, unlike other parts in this paper aren't hard to follow we can see a simple example with a half-cyclic function

$$\cosh_{\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{x^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

we can try and use $D^{\frac{1}{2}}$

$$D^{\frac{1}{2}} \cosh_{\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\frac{n}{2}+1)} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}-\frac{1}{2}+1)} x^{\frac{n}{2}-\frac{1}{2}} \right] = \sum_{n=0}^{\infty} \frac{x^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})}$$

we can then expand both functions

$$\cosh_{\frac{1}{2}} = 1 + \frac{x^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} + x + \frac{x^{\frac{3}{2}}}{\Gamma(\frac{3}{2})} + x^2 + \frac{x^{\frac{5}{2}}}{\Gamma(\frac{5}{2})} + \frac{x^3}{2} + ...$$

$$D^{\frac{1}{2}} \cosh_{\frac{1}{2}} = \frac{x^{-\frac{1}{2}}}{\Gamma(-\frac{1}{2})} + 1 + \frac{x^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} + x + \frac{x^{\frac{3}{2}}}{\Gamma(\frac{3}{2})} + x^2 + ...$$

We can see that the derivative process, instead of deleting terms it moves backwards, adding more terms on the left side of the 1, which allows us to write the half derivative like this

$$D^{\frac{1}{2}} \cosh_{\frac{1}{2}} = \frac{x^{-\frac{1}{2}}}{\Gamma(-\frac{1}{2})} + \cosh_{\frac{1}{2}}$$

which breaks the rule of cyclic in these functions for the derivative side, but we can try and see for the integral side what happens

$$D^{-\frac{1}{2}} \cosh_{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\frac{n}{2}+1)} \left[\frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2}+\frac{1}{2}+1)} x^{\frac{n}{2}+\frac{1}{2}} \right] = \sum_{n=0}^{\infty} \frac{1}{\frac{n}{2}\Gamma(\frac{n}{2})} \left[\frac{\frac{\Gamma(\frac{n}{2})}{(\frac{n}{2}-1)}}{\Gamma(\frac{n+3}{2})} x^{\frac{n+1}{2}} \right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{\frac{n}{2}} \frac{1}{(\frac{n}{2}-1)} \frac{1}{\Gamma(\frac{n+3}{2})} x^{\frac{n+1}{2}} = \sum_{n=0}^{\infty} \frac{2}{n} \frac{2}{(n-2)} \frac{1}{\Gamma(\frac{n+3}{2})} x^{\frac{n+1}{2}} = \sum_{n=0}^{\infty} \frac{4x^{\frac{n+1}{2}}}{n(n-2)\Gamma(\frac{n+3}{2})}$$

Expanding this series gives us a gamma pole in zero and two, with it in mind, the series is $-4x + \frac{4x^2}{6} + \frac{4x^{\frac{5}{2}}}{8\Gamma(\frac{7}{2})} + \frac{4x^3}{30} + \frac{4x^{\frac{7}{2}}}{30\Gamma(\frac{9}{2})} + \frac{4x^4}{288} + ...$
the formula for $\cosh_{\frac{1}{2}}$ can be simplified using gamma properties that is $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(\frac{1}{2}+n) = \frac{(2n-1)!!}{(-2)^n}\sqrt{\pi}$
It's also worth mentioning that $\cosh_{\frac{1}{2}}(x)$ is similer to $E_{\frac{1}{2}}$ which is equal to $e^{x^2} erfc(-x)$ but the x powers differs

11.2 negative cyclic

Negative cyclic derivatives unlike for positive integer or fractional ones, aren't fully defined, as we can see for the "-1" order cyclic $\cosh(x)$

$$\cosh_{-1}(x) = \sum_{n=0}^{\infty} \frac{x^{-n}}{\Gamma(-n+1)} = 1 + \frac{x^{-1}}{\Gamma(0)} + \frac{x^{-1}}{\Gamma(-1)} + ...$$

This function shows a lot of gamma poles, in which we can say that this is a straight-up problematic function, which makes it undefined.
We could, for example can take the derivative of the function to add more terms

$$D^1 \cosh_{-1} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(-n+1)} \left[\frac{\Gamma(-n+1)}{\Gamma(-n)} x^{-n-1} \right] = \sum_{n=0}^{\infty} \frac{x^{-n-1}}{\Gamma(-n)}$$

Apparently, differentiating the function gives a worse result, so we could integrate it

$$D^{-1} \cosh_{-1} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(-n+1)} \left[\frac{\Gamma(-n+1)}{\Gamma(-n+2)} x^{-n+1} \right] = \sum_{n=0}^{\infty} \frac{x^{-n+1}}{\Gamma(-n+2)} = x+1+0+0....$$

Unlike differentiation, integration adds more terms
If we, for example, integrate it 5 times we will get the result

$$D^{-5} \cosh_{-1} = \frac{x^5}{5!} + \frac{x^4}{4!} + \frac{x^3}{3!} + \frac{x^2}{2!} + x + 1$$

This function take the form of an inverse e^x expansion, it's logical, as e^x itself is $\cosh_1$
We can see that such functions are sort of a mirror for their positive values, and instead of depending on differentiation, they depend on integration to gain their value so we can say that

$$\lim_{a \to -\infty} D^a \cosh_{-n}(x) = \cosh_n(x)$$

11.3 complex cyclic

from the standard definition, if we tried to define it would be complex, as we can try and replace n with i for the equation to be $a^i = 1$, then we turn it to a form of e powers

$$(e^{\ln(a)})^i = 1 \Rightarrow e^{i\ln(a)} = e^{2i\pi}$$

talking the natural logarithm of both sides we get

$$\ln(e^{i\ln(a)}) = \ln(e^{2i\pi}) \Rightarrow i\ln(a) = 2i\pi \qquad \ln(a) = 2\pi \Rightarrow a = e^{2\pi}$$

But his value is bigger than 1, it's an exponentiation form on which $e^{2\pi x}$ grows exponentially, this happens because of the use of logarithms
instead we will use the $\cosh_{\mathrm{n}}$ and $\sinh_{\mathrm{n}}$ to define such values

$$\cosh_{\mathrm{i}}(x) = \sum_{k=0}^{\infty} \frac{x^{ik}}{\Gamma(ik+1)} \qquad \sinh_{\mathrm{i}}(x) = \sum_{k=0}^{\infty} \frac{x^{ik+1}}{\Gamma(ik+2)}$$

We can check if this is correct by applying the i-th derivative

$$D^i \cosh_{\mathrm{i}} = \sum_{k=0}^{\infty} \frac{1}{\Gamma(ik+1)} \left[\frac{\Gamma(ik+1)}{\Gamma(ik+1-i)} x^{ik-1} \right] = \sum_{k=0}^{\infty} \frac{x^{ik-i}}{\Gamma(ik+1-i)}$$

This is interesting, but before we say we can take the is out because that is an infinite series with the form $ak \pm a$, we need to expand them both first

$$\cosh_{\mathrm{i}} = 1 + \frac{x^i}{\Gamma(i+1)} + \frac{x^{2i}}{\Gamma(i+2)} ... \qquad D^i \cosh_{\mathrm{i}} = \frac{x^{-i}}{\Gamma(1-i)} + 1 + \frac{x^i}{\Gamma(i+1)} + \frac{x^{2i}}{\Gamma(i+2)} ...$$

Both sides are identical except for the first term. What happens here is: since the first term is a constant, it's supposed to go to zero, but since the derivative order is complex, it doesn't reach $\Gamma(0)$, it reaches $\Gamma(1-i)$, which is defined in the Gamma function domain
this function is also the i version of e^x
We can try integrating to see more results

$$D^{-i} \cosh_{\mathrm{i}} = \sum_{k=0}^{\infty} \frac{x^{ik+i}}{\Gamma(ik+1+i)} = \frac{x^i}{\Gamma(1+i)} + 1 + \frac{x^i}{\Gamma(1+i)} + \frac{x^{2i}}{\Gamma(2+i)} + ...$$

as we can see , integration reflects the cyclic derivative from the first point , the more we integrate the more we add similar functions to the left hand side with the middle point being 1, well that is what happens at imaginary differentiation and integration , in Real differentiation and integration we get another results

$$D^1 \cosh_{\mathrm{i}} = \sum_{k=0}^{\infty} \frac{x^{ik-1}}{\Gamma(ik)} = 0 + \frac{x^{i-1}}{\Gamma(i)} + \frac{x^{i-2}}{\Gamma(2i)} + \frac{x^{i-3}}{\Gamma(3i)} + ...$$

(the first zero is because of a gamma pole)
we can see the difference here , a normal derivative result multiples of i in the Gamma function, while an imagery derivative result multiples of i in the power of x
of course we can try now a full complex number

$$D^{1+i}\cosh_{\mathrm{i}} = \sum_{k=0}^{\infty}\frac{x^{ik-1-i}}{\Gamma(ik-i)} = \frac{x^{-1-i}}{\Gamma(-i)} + \frac{x^{-1}}{\Gamma(0)} + \frac{x^{i-1}}{\Gamma(i)} + \frac{x^{2i-1}}{\Gamma(2i)} + ...$$

we can see that this form combines some of the properties from pure complex derivative and pure real derivative.
things like the Gamma pole at zero from the real derivative and multiple of i in both Gamma function and the power
of course the Gamma pole happens in these examples because we have positive 1 in each of them, if we have used any other method
of course, these cycles follow the cyclic derivatives rules, mostly
The N-th cyclic derivative of $\sinh_{\mathrm{i}}$ is $\sinh_{\mathrm{i}}\mathrm{N}(x)$ as any other cyclic derivative function, but in this case, it never reaches a full cycle through normal differentiation, but rather through imaginary order derivatives as we can define $\sinh_{\mathrm{i}}(x)$ to be $D^i\cosh_{\mathrm{i}}(x)$ and to get more variations we simple add more imaginary unit to be $\cosh_{2\mathrm{i}}$ and $\cosh_{3\mathrm{i}}$ so on so forth
As we saw at the beginning of the section, we can't construct an Euler form using the simple formula, and we can see this in act
as we know $e^{ax} = \underbrace{\cosh_{\mathrm{a}} + \sinh_{\mathrm{a}} + \sinh_{\mathrm{a}}\mathrm{II} + ...}_{n-\mathrm{times}}$ where $a^n = 1$, but trying to construct this here is the problem, using the same logic we can write it like this

$$e^{ax} = \cosh_{\mathrm{i}} + \sinh_{\mathrm{i}} + \sinh_{\mathrm{i}}\mathrm{II} + \sinh_{\mathrm{i}}\mathrm{III} + ...$$

this series doesnt stop as we can't simply add something i-times , which leads to the infinte series

$$e^{ax} = \sum_{N=0}^{\infty}\sinh_{\mathrm{i}}\mathrm{N}$$

not only that but we can't get an explanation for the function as we don't know the value of a, but we can use a work around
Since $D^{\alpha}a^{\alpha}e^{axe}$, we can write the exponentiation denominator as an infinite sum, as it's intended to be raised to the power of a, which is a number that we don't know yet. We are simply going to use the Greek letter ξ to denote it

$$\cosh_{\mathrm{i}} = \frac{\sum_{n=0}^{\infty} D^n e^{\xi x}}{\xi} \qquad \sinh_{\mathrm{i}}\mathrm{N} = \frac{\sum_{n=0}^{\infty} D^{n+N} e^{\xi x}}{\xi}$$

I would leave the value of ξ yet to be explored, but I will define it as an object that is connected to / represents in somewhat way the sum from 0 to i
If we try to explain such weird functions, we can simply say that these functions have repeating values in the $D(i)$, unlike normal cyclic functions, which are

repeating on $D(i)$ main number line, in which normal derivatives do exist
But we can still write e^{ax} as an incomplete infinite sum of the first terms in each sum

$$e^{\xi x} = 1 + x + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + ...$$

This is exactly the expansion of Euler's number or as an incomplete sum of the second term

$$e^{\xi x} = \frac{x^i}{\Gamma(i+1)} + \frac{x^{i+1}}{\Gamma(i+2)} + \frac{x^{i+2}}{\Gamma(i+3)} + ...$$

But still, these sums aren't complete

11.4 quaternion cyclic

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