

Complex-Order Fractional Derivatives: A First Exploration

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Introduction

The integer-order derivative $D^n f(x)$ measures the local rate of change. This paper explores the generalization of the derivative operator to continuous and complex orders, $D^\alpha f(x)$ and $D^z f(x)$, known as Fractional and Complex Calculus.

Part I Algebraic foundation

1 The Generalized Operator for $f(x) = x^n$

1.1 From Integer to Fractional Order

We begin with the integer derivatives of $f(x) = x^n$:

$$D^k f(x) = n(n-1)\cdots(n-k+1)x^{n-k}$$

Using the identity $n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$, we write:

$$D^k f(x) = \frac{n!}{(n-k)!} x^{n-k}$$

To generalize this for $k \in \mathbb{R}$, we substitute the factorial function with the continuous Gamma function, $\Gamma(z)$. We use the identities $n! = \Gamma(n+1)$ and $\Gamma(z+1) = z\Gamma(z)$. The α -th derivative (where $\alpha \in \mathbb{R}$) is:

$$D^\alpha f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

of course we can use this formula to get half-derivative of x^n

$$D^{\frac{1}{2}} = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} x^{1-\frac{1}{2}} = \frac{1}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}}$$

using the rule $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$D^{\frac{1}{2}} = \frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})}x^{\frac{1}{2}} = \frac{1}{\frac{\sqrt{\pi}}{2}}x^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}}x^{\frac{1}{2}}$$

now if we take the half-derivative of that half-derivative it will be

$$\frac{2}{\sqrt{\pi}}D^{\frac{1}{2}} = \left(\frac{2}{\sqrt{\pi}}\right)\frac{\Gamma(1/2+1)}{\Gamma(\frac{1}{2}-\frac{1}{2}+1)}x^{\frac{1}{2}-\frac{1}{2}} = \left(\frac{2}{\sqrt{\pi}}\right)\frac{\Gamma(\frac{3}{2})}{1}x^0 = \left(\frac{2}{\sqrt{\pi}}\right)\left(\frac{\sqrt{\pi}}{2}\right) = 1$$

we talking the half-derivative twice to the same function gave what a one full derivative would give

of course such a proof is too simple and don't quite give the meaning of a proof , it's just a little confirmation for the time being, the full rigorous proof will be proven later with the properties of the D^z operator

Generalization to Complex Order $z = a + bi$

We now extend the derivative order to the complex number $z = a + bi$:

$$D^z f(x) = \frac{\Gamma(n+1)}{\Gamma(n-z+1)}x^{n-z}$$

To show the magnitude and phase components, we expand x^{n-z} using the property $x^{a+bi} = x^a e^{b \ln(x)i}$:

$$D^z f(x) = \frac{\Gamma(n+1)}{\Gamma(n-z+1)}x^{n-a}e^{-b \ln(x)i}$$

with these two formulas we can use them to find any \mathbb{C} or \mathbb{R} derivatives for x^n

Finding negative order derivatives

we can find the negative derivatives by putting -1 as the α and see what could happen

Putting -1 in the general formula gives the result

$$D^{-1} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-(-1)+1)}x^{n-(-1)} = \frac{\Gamma(n+1)}{\Gamma(n+2)}x^{n+1}$$

and using the $\Gamma(z+1) = z\Gamma(z)$ we can say that

$$D^{-1} f(x) = \frac{\Gamma(n+1)}{(n+1)\Gamma(n+1)}x^{n+1} = \frac{x^{n+1}}{(n+1)}$$

which means that the negative order derivatives are the integrals a function This result unifies the familiar integer derivative, the fractional derivative, and the complex-order derivative into a single, elegant framework.

1.2 the x^{-n} problem

as we have seen, we can apply the past formula to any power of n weather it's fractional or even complex but problems rise when we try to apply the past formula to x^{-n}

$$D^\alpha(x^{-n}) = \frac{\Gamma(0)}{\Gamma(-\alpha)} x^{-n-\alpha}$$

not only do we have a **Gamma Pole** in the numerator but also for any value $\alpha \in \mathbb{Z}^+$ we also get a Gamma pole in the denominator which means that this formula cant work and we need another formula

the m-th formula for x^{-n} is simply

$$\frac{d^m}{dx^m}(x^{-n}) = \frac{(-1)^m (n)^{(m)}}{x^{n+m}}$$

the $n^{(m)}$ here isn't a power but rather a rising factorial that can be expressed as $n^{(m)} = \frac{(n+m-1)!}{(n-1)!}$ with this knowledge we can say

$$D^\alpha(x^{-n}) = \frac{(-1)^\alpha \frac{\Gamma(n+\alpha)}{\Gamma(n)}}{x^{n+\alpha}} = \frac{(-1)^\alpha \Gamma(n)}{x^{n+\alpha} \Gamma(n+\alpha)}$$

as simple as that, it didn't work with the original x^n formula , but this shows something about fractional derivatives , if we went to find the first integral for both x^n and x^{-n} we find that

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \int x^{-n} dx = \ln|x| + C$$

Ignoring the integration constant we can find that x^n wasn't the original function for x^{-n} but it actually transformed from $\ln(x)$

not only in complexity only but in dependency , the $D^z(x^n)$ is dependent on the change of n more than x, while $\ln(x)$ is dependent on the change of x
this will be explained later in the **Transformative Functions** section , and this is one of the few cases we use stanederd integration (other than the Gamma function) in this research

2 Formulas for Other Algebraic Functions

2.1 The General Formula for a^x

Starting with the general integer rule for a^x :

$$D^n(x) = a^x \ln(a)^n$$

substituting α in the place of n gives us

$$D^\alpha f(x) = a^x \ln(a)^\alpha$$

we can see that this simple change was enough for the formula to work by taking the half-derivative twice and it gives us order one derivative

$$D^{1/2}f(x) = a^x \ln(a)^{1/2}$$

since $\ln(a)^{1/2}$ is a constant we can take it out simply when doing the derivative again

$$D^{1/2}(D^{1/2}f(x)) = \ln(a)^{1/2}(D^{1/2}f(x)) = \ln(a)^{1/2}(a^x \ln(a)^{1/2}) = a^x \ln(a)$$

which is true since our starting function was a^x and thus we can say this formula works

The Complex Generalization of this formula can be written like the D^α formula or like this

$$D^z f(x) = a^x \ln(a)^t e^{bln(\ln(a))i}$$

where $z = a + bi$

of course we can find the first Anti-derivative of this function by using -1 in the formula

$$D^{-1}(x) = a^x \ln(a)^{-1} = \frac{a^x}{\ln(a)}$$

and the first Complex derivative

$$D^i(x) = a^x \ln(a)^i = a^x e^{ln(\ln(x))i}$$

2.2 The General Formula for e^x

The function e^x is known for it's "Unchanging Derivative" because it comes from the $D^n(a^x) = a^x \ln(a)^n$ and putting $a = e$ we get $D^n(e^x) = e^x$ so this also means there is no change affect the complex nor the fractional derivatives

$$D^\alpha f(x) = e^x \quad D^z f(x) = e^x$$

which means the Anti-derivative and the first complex derivative of the function

$$D^{-1}(x) = e^x \quad D^i(x) = e^x$$

2.3 The General Formula for e^{ax}

as we saw there isnot any change between e^x and any of it's derivatives , things change when we consider e^{ax} as we can see the rule of the first - second an derivative is

$$D^1 f(x) = ae^{ax} \quad D^2 f(x) = a^2 e^{ax} \quad D^3 f(x) = a^3 e^{ax}$$

so we can find the formula for the n-th derivative as

$$D^n f(x) = a^n e^{ax}$$

and changing the n to α we get

$$D^\alpha f(x) = a^\alpha e^{ax}$$

as simple as that we still have to test it to justify

$$D^{1/2} f(x) = a^{1/2} e^{ax}$$

since $a^{1/2}$ is a constant we can say that

$$D^{1/2}(D^{1/2} f(x)) = a^{1/2}(D^{1/2} f(x)) = (a^{1/2})(a^{1/2} e^{ax}) = ae^x$$

this confirms that the formula work and substituting z instead of α we get the same formula as above taht can also be written like that

$$D^z f(x) = a^t e^{bln(a)i+ax}$$

where $z = a + bi$

the first Anti-derivative for e^{ax} is

$$D^{-1} f(x) = a^{-1} e^{ax} = \frac{e^{ax}}{a}$$

and the first complex derivative is

$$D^i f(x) = a^i e^{ax} = e^{ln(a)i+ax}$$

2.4 The problem of $\log_a(x)$

$\log_a(x)$ The first derivative of $\log_a(x)$ is $\frac{1}{x\ln(a)}$ and the second derivative is $\frac{-1}{x^2\ln(a)}$ the third derivative is $\frac{2}{x^3\ln(a)}$ lastly the fourth derivative is $\frac{-6}{x^4\ln(a)}$ is we can see the pattern of the n-th derivative

$$D^n f(x) = (-1)^{n+1} \frac{(n-1)!}{x^n \ln(a)}$$

and applying the gamma identity $n! = \Gamma(n - 1)$ then changing n to α and reversing the x power in the denominator we get

$$D^\alpha f(x) = (-1)^{\alpha+1} \frac{\Gamma(\alpha)}{\ln(a)} x^{-\alpha}$$

But it fails, it doesn't work with fractions nor negative integers it only works positave integers, so what went wrong? my theory is that the problem is fairly simple, the $\log_a(x)$ can't be expressed as a Maclurin series and $\log_a(x + C)$ can only be expressed to a series with the condition $|x| < C$ or else it won't diverge

GDI theory : For Every function that isn't Function is analytic at $x=0$ and doesn't converge over \mathbb{R} the derivative formula $D^\alpha f(x)$ works only on \mathbb{Z}^+ for that function , we can also call it not full real differentiable

GDI hypothesis : the function is differentiable in all it's valid input values

These can also be rewritten like this:

Let $f(x)$ be an analytic function defined by its Maclaurin series

GDI theory: If $f(x)$ has a singularity at $x = 0$, then the generalized formula for $D^\alpha f(x)$ will contain a singularity at $\alpha = n$ for $n \in \mathbb{Z}^+$, preventing the generalized formula from equaling the expected integer fractional or anti-derivative.

– Todo: change this to look better – this will be explained in detail in later sections

3 Properties of the D^z operator

these are properties to identify the nature of it that will help later with the formula and analysis of what I can call the D^z plane More on that later

3.1 General power series rule

power serieses are very important tools to analytically describe a function along the Real or Complex planes what is exactly what we need

The general power series definition for a function(Taylor series) is as following :

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

taking the first derivative for both sides, the first term will cancel as it's a constant , the second term is linear so it will become constant, the third term is quadruple will become linear and $2!$ will cancel the 2 of the power and so on we can write is like this

$$D^1 f(x) = f^{(1)}(a) + \frac{f^{(2)}(a)}{1!}(x-a) + \frac{f^{(3)}(a)}{2!}(x-a)^2 \dots = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(a)}{n!}(x-a)^n$$

the cancellation happen also because when we take a derivative the power of $(x-a)$ gets down by 1 to the numerator and we divide $n!$ by it leading to $(n-1)!$ which will lead to infinity in the denominator leading to the whole term to be zero

taking the third and the forth derivative give the same result up to k-th derivative

$$D^k f(x) = f^{(k)}(a) + \frac{f^{(k+1)}(a)}{1!}(x-a) + \frac{f^{(k+2)}(a)}{2!}(x-a)^2 \dots = \sum_{n=0}^{\infty} \frac{f^{(k+n)}(a)}{n!}(x-a)^n$$

now to make it full fractional we will put gamma instead of n and using the x^n general formula

$$D^\alpha f(x) = f(a)^{(\alpha)} + \frac{f^{(\alpha+1)}(a)}{\Gamma(2)}(x-a) + \frac{f^{(\alpha+2)}(a)}{\Gamma(3)}(x-a)^2 \dots = \sum_{n=0}^{\infty} \frac{f^{(n+\alpha)}(a)}{\Gamma(n+1)} \left[\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} (x-a)^n \right]$$

in the brackets we can see the general derivative for x^n canceling the Gammas out we get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-\alpha+1)} (x-a)^{n-\alpha}$$

for powers of $(x-a)$ lesser than α it will get to ∞ in the denominator, at least when it's a negative integer other than that it will work normal, this happens because of the gamma pole, but taking the limit it will lead to 0 but still doesn't affect the sum, indeed helping us deleting the first $n < \alpha$ terms of course for negative integer derivatives this works too since it will be positive

Note: because how much terms you take in the differentiation will always come terms that are replaced to them because of the x^n differentiation and the infinite sum, **even if you differentiate it infinite amount of times** but you are still deleting values we are going to use that knowledge later in the integration and differential equations in later sections

we can now let $a = 0$ to get the important series we need, the Maclaurin series

$$D^\alpha f(x) = f(0) + \frac{f^{(1)}(0)}{\Gamma(2)}(x) + \frac{f^{(2)}(0)}{\Gamma(3)!}(x)^2 \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

3.2 The linearity of D^z operator

for most of the operations with functions and real life applications we need to deal with linearity for D^z operator which is what we are going to prove in this small section **We Must Prove that:**

$$D^z(c_1 f(x) + c_2 g(x)) = c_1 D^z f(x) + c_2 D^z g(x)$$

let's begin with the simple x^n and let $f(x) = x^n, g(x) = x^m$ firstly we Differentiate them separately

$$D^z(c_1 f(x)) + D^z(c_2 g(x)) = c_1 D^z(f(x)) + c_2 D^z(g(x)) = c_1 \frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-z} + c_2 \frac{\Gamma(m+1)}{\Gamma(m-z+1)} x^{m-z}$$

Let this be statement 1

now let's differentiate them together we get

$$D^z(c_1 f(x) + c_2 g(x)) = c_1 \frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-z} + c_2 \frac{\Gamma(m+1)}{\Gamma(m-z+1)} x^{m-z}$$

Let this be statement 2

since **Statement 1 = Statement 2** we can say that

$$D^z(c_1 f(x) + c_2 g(x)) = c_1 D^z f(x) + c_2 D^z g(x)$$

Q.E.D

this is very useful, but to apply it to all functions, which we can do easily

with infinite serieses

in other words

If D^z is linear on basis functions x^n , and $f = \sum c_n x^n$, then:

$$D^z(\sum c_n x^n) = \sum c_n D^z(x^n)$$

by uniform convergence of the series.

3.3 The Index law

The most important property for the formulas is the index law that is

$$D^{\alpha+\beta} f(x) = D^\alpha(D^\beta(x))$$

we must prove this holds true for every case first we need to prove it for the simplest function we have which is x^n taking the D^z of the function we get

$$D^\alpha f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

now let's apply the D^β with the knowledge that the Gamma functions are constants in the first derivative

$$D^\beta(D^\alpha f(x)) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} D^\beta(x^{n-\alpha}) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} \left[\frac{\Gamma(n-\alpha+1)}{\Gamma(n-\alpha-\beta+1)} x^{n-\alpha-\beta} \right]$$

the Gamma terms cancel out and we get

$$D^\beta(D^\alpha f(x)) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)} x^{n-\alpha-\beta}$$

Let this be statement 1

now if we start from the beginning again but this time directly substitute $\alpha + \beta$ as O (stands for orders) we get

$$D^O f(x) = \frac{\Gamma(n+1)}{\Gamma(n-O+1)} x^{n-O}$$

if we substitute $O = \alpha + \beta$ back we get

$$D^{\alpha+\beta} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)} x^{n-\alpha-\beta}$$

Let this be statement 2

if we Equal **statement 1** and **statement 2** we get

$$D^{\alpha+\beta} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)} x^{n-\alpha-\beta} = D^\beta(D^\alpha f(x))$$

thus we can say that

$$D^{\alpha+\beta} f(x) = D^\beta(D^\alpha f(x))$$

Q.E.D

Note: this also works for imaginary numbers $z + w$

this by itself is a simple elegant proof but , it only works for x^n and applying the same method for each function will be very large waste of time

Instead we can use **Power Serieses** as they hold for every analytic function thus one proof will work for every function

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n+1)} (x-a)^n$$

we can immediately see that it's the simple x^n with everything else being a constant to the derivative

Since we've proven the Index Law for x^n , and the power series represents f as a sum of such terms, the Index Law extends to f by linearity of the operator and see that

$$D^{\alpha+\beta} f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n+1)} \left[\frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)} (x-a)^{n-\alpha-\beta} \right] = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-\alpha-\beta+1)} (x-a)^{n-\alpha-\beta}$$

which means that the Index law works for any analytic function **Q.E.D**

3.4 The Multiplication Law

A logical next step is to prove that there exists a law for multiplying orders of derivatives as it will help us later to prove and analysis a lot of topics in this research

let's take for example $f(x) = x^6$ we need (for now) to prove that multiplying two order derivatives for example 2 and 3 return the same result

for now we may write the multiplication as $M[D^\alpha f(x)]^\beta$ just as a placeholder for now

so **we need to prove that**

$$M[D^2 f(x)]^3 = M[D^3 f(x)]^2 = D^6 f(x)$$

evaluating the first expression gives us $D^6 f(x) = \frac{\Gamma(7)}{\Gamma(6-6+1)} x^{6-6} = \Gamma(7)$ a very logical step to do is to use the **Index Law** we just proved so we start with

$$\begin{aligned} D^2 f(x) &= \frac{\Gamma(7)}{\Gamma(6-2+1)} x^{6-2} = \frac{\Gamma(7)}{\Gamma(5)} x^4 \\ D^2(D^2 f(x)) &= \frac{\Gamma(7)}{\Gamma(5)} D^2 f(x) = \frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(4-2+1)} x^{4-2} = \frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(3)} x^2 \\ D^2(D^2(D^2 f(x))) &= \frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(3)} D^2 f(x) = \\ \frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(3)} \frac{\Gamma(3)}{\Gamma(2-2+1)} x^{2-2} &= \frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(3)} \times \frac{\Gamma(3)}{\Gamma(1)} \end{aligned}$$

as we can see the orders cancel out perfectly leaving only $\Gamma(7)$

Let's generalize the idea with α and β following the same index law but with $f(x) = x^n$

$$\begin{aligned} D^\alpha f(x) &= \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha} \\ D^\alpha(D^\alpha f(x)) &= \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} D^\alpha f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} \frac{\Gamma(n-\alpha+1)}{\Gamma(n-\alpha-\alpha+1)} x^{n-\alpha-\alpha} \\ &= \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} \frac{\Gamma(n-\alpha+1)}{\Gamma(n-2\alpha+1)} x^{n-2\alpha} = \frac{\Gamma(n+1)}{\Gamma(n-2\alpha+1)} x^{n-2\alpha} \end{aligned}$$

of course we can do it again

$$\begin{aligned} D^\alpha(D^\alpha(D^\alpha f(x))) &= \frac{\Gamma(n+1)}{\Gamma(n-2\alpha+1)} D^\alpha(D^\alpha f(x)) = \frac{\Gamma(n+1)}{\Gamma(n-2\alpha+1)} \frac{\Gamma(n-2\alpha+1)}{\Gamma(n-2\alpha-\alpha+1)} x^{n-2\alpha-\alpha} \\ &= \frac{\Gamma(n+1)}{\Gamma(n-2\alpha+1)} \frac{\Gamma(n-2\alpha+1)}{\Gamma(n-3\alpha+1)} x^{n-3\alpha} = \frac{\Gamma(n+1)}{\Gamma(n-3\alpha+1)} x^{n-3\alpha} \end{aligned}$$

we can continue like this up till the $\beta - th$ multiplication and get the same result , so from this we can define

$$M[D^\alpha f(x)]^\beta = \frac{\Gamma(n+1)}{\Gamma(n-\alpha\beta+1)} x^{n-\alpha\beta}$$

and since $\alpha\beta = \gamma$ where γ is the original intended order , then

$$M[D^\alpha]^\beta = D^\gamma$$

Q.E.D

of course this works only when $\beta \in \mathbb{N}$ since it's the set we can apply Multiplication Law in terms of Index Law

I believe to ascend it to general multiplication we may first define it in a better mathematical language as

Definition : For $\beta \in \mathbb{N}$, define ${}^\beta D^\alpha$ inductively

$$1- {}^1 D^\alpha = D^\alpha$$

$$2- {}^{n+1} D^\alpha = D^\alpha \circ {}^n D^\alpha$$

By induction, we proved: ${}^n D^\alpha = D^{n\alpha}$ for $n \in \mathbb{N}$

And to extend this to \mathbb{R}/\mathbb{C} for general β , we define: ${}^\beta D^\alpha := D^{\alpha\beta}$

Then we need to verify this satisfies expected properties: ${}^{\beta_1}({}^{\beta_2} D^\alpha) = D^{\beta_1(\beta_2\alpha)} = D^{(\beta_1\beta_2)\alpha} = {}^{\beta_1\beta_2} D^\alpha$

$${}^1 D^\alpha = D^\alpha$$

$${}^0 D^\alpha = D^0$$
 identity

This makes it a definition extended by continuity/analyticity.

Which also means that

$$M[D^\alpha]^\beta = M[D^\beta]^\alpha$$

of course that is when the operation itself is in a commutative ring like (\mathbb{R}, \mathbb{C}) ,
for other rings that isn't commutative this is False like \mathbb{H}

We adopt the notation ${}^\beta D^\alpha$ to denote the multiplicative action of order β on derivative D^α , distinguishing it from composition $D^\beta \circ D^\alpha$ (which gives $D^{\alpha+\beta}$ by Index Law) and the direct derivative $D^{\alpha\beta}$ but i recommend the first one as it shows we are going from α to β which may be used later without ruining the first shape of D^z operator

since the Index Law works well for Taylor series this will also work well

3.5 Operator inverses and identities

The inverse addition order:

from the Index law we can assume that there exists an equation in which

$$D^\gamma D^\alpha(f(x)) = f(x)$$

writing $f(x)$ with the geneal D^z operator we get

$$D^\gamma D^\alpha(f(x)) = D^0(f(x))$$

we can use the oppiste of the opreator in the other side which gives us

$$D^\alpha(f(x)) = D^{-\gamma} D^0(f(x)) = D^{-\gamma} f(x)$$

this can hold true if and only if $\gamma = -\alpha$ substituting this back we get

$$D^\alpha(f(x)) = D^{(-\alpha)} f(x)$$

which is true thus $D^{-\alpha} D^\alpha f(x) = f(x)$

You can see this as a **generalization of the fundamental theorem of calculus** as it hold true for any set of numbers not only integers

The inverse Multiplication order:

from the multiplication law we can assume that there exist an equation where

$${}^\gamma D^\alpha f(x) = D^1 f(x)$$

using the inverse of the multiplication rule we get

$$D^\alpha f(x) = {}^{\frac{1}{\gamma}} D^1 f(x)$$

which with the multiplication law we get

$$D^\alpha f(x) = D^{\frac{1}{\gamma}} f(x)$$

this only happens if and only if $\gamma = \frac{1}{\alpha}$ substituting this back we get

$$D^\alpha f(x) = D^{\frac{1}{\alpha}} f(x) = D^\alpha f(x)$$

which is true thus

$$D^{\frac{1}{\alpha}} D^\alpha f(x) = D^1 f(x) = D^\alpha f(x)$$

of course there are some things that we can state :

The derivative order addition identity: D^0

The derivative order multiplication identity: D^1

The zeroth-derivative order multiplication proprie: ${}^0 D^\alpha$ and ${}^\beta D^0$ or in other words, the function itself is the zero of derivative order multiplication , going to or going from

Every derivative order has it's addition inverse and multiplication inverse that lead to the function itself and the first order derivative function

there are infinity many was to get to a function from it's derivatives and integrals

Distribution of orders and Order of operations:

since the whole operation is linear we can distribute operations on the order, but to prove this we need to prove that ${}^\beta D^{(\alpha+\gamma)} = {}^\beta D^\alpha \circ {}^\beta D^\gamma$
on the left hand side we can let $\sigma = \alpha + \gamma$ so we go:

$${}^\beta D^\sigma = D^{\beta\sigma} = D^{\beta(\alpha+\gamma)} = D^{\beta\alpha+\beta\gamma}$$

with the right hand side we can go

$${}^\beta D^\alpha \circ {}^\beta D^\gamma = D^{\beta\alpha} \circ D^{\beta\gamma} = D^{\beta\alpha+\beta\gamma}$$

because both the right hand side and the left hand side are equal we can say that there is order for operations as multiplication comes before addition and there is distribution between the order operations let's take for example ${}^\beta D^\alpha \circ D^\gamma$, using what we know to write this as one derivative order we can write

$$D^{\beta\alpha} D^\gamma = D^{\beta\alpha+\gamma}$$

multiplication then addition

Order conjugate

Order exponentiation

Order exponentiation(Logarithms of order)

Order Limits

4 Deriving the Rules of the Fractional and complex derivatives

most of the other functions need us to derive the rules of the fractional Differ-integar operator, the α -th derivative for functions like $\tan(x), \text{arcsech}(x), \ln(x)$ etc.. can only be found using power serieses and product rule

4.1 General product rule

One of the most important and needed formulas in calculus is the product rule let $f(x) = g(x)h(x)$

$$\begin{aligned} f'(x) &= g(x)h'(x) + g'(x)h(x) \\ f''(x) &= g(x)h''(x) + 2g'(x)h'(x) + g''(x)h(x) \\ f'''(x) &= g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x) \end{aligned}$$

this is very similar to the binomial theorem, the only difference is it deals with derivatives instead of powers

the general product rule also known as the **General Leibniz rule** is

$$D^n(fg) = \sum_{k=0}^n \binom{n}{k} D^{n-k}(f) D^k(g)$$

this simple yet elegant formula is what we are going to use for the General product rule, before we try to do a simple substitution of α we need to use the generalized nC_k which means using $n! = \Gamma(n+1)$ in the formula $\frac{n!}{(n-k)!k!}$

$$D^n(fg) = \sum_{k=0}^n \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} D^{n-k}(f) D^k(g)$$

this is the same formula just and works the same for positive integers, but let's try to use fractions and ignore the sigma upper term and expand it, for example one half expansion will be

$$D^{1/2}(fg) = \frac{\Gamma(\frac{1}{2})}{\Gamma(1\frac{1}{2}-0)\Gamma(1)} D^{1/2}(f) D^0(g) + \frac{\Gamma(\frac{1}{2})}{\Gamma(1\frac{1}{2}-1)\Gamma(2)} D^{-1/2}(f) D^1(g) + \dots$$

$$D^{1/2}(fg) = \frac{\Gamma(\frac{1}{2})}{\Gamma(1\frac{1}{2})} D^{1/2}(f) g + D^{-1/2}(f) D^1(g) + \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{1}{2})\Gamma(3)} D^{-3/2}(f) D^2(g) + \dots$$

as we can see, it expands to an infinite sum, it will never stop because the lower value never hit the upper value, the main reason the simple form works for integers is that even if the k value goes higher than the n value it will get a negative integer in a Gamma which is a pole and thus equal zero
that means in order for a term to be it must not have the Gamma of integers so that means we can't find an integral product rule yet, but at anyway turning back the sum from what we know will be

$$D^\alpha(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} D^{\alpha-k}(f) D^k(g)$$

we can mark this as the general power rule but we need to use our handy tool the **Index law** to confirm that it works

4.2 General chain rule

The General Chain Rule is hard, we can try any of our normal methods yet they fail

the easier way is to go with the series option we can get the Maclaurin series with ease

4.2.1 General $f(x^n)$ formula

first we define our functions for simpler solving

$$u(x) = x^n \quad g(x) = f(u)$$

then we differentiate for the first three derivatives for $u(x)$

$$u'(x) = nx^{n-1} \quad u''(x) = n(n-1)x^{n-2} \quad u'''(x) = n(n-1)(n-2)x^{n-3}$$

then we differentiate $g(x) = f(u)$ with respect to u

$$g'(x) = u'f'(u) \quad g''(x) = u''f'(u) + (u')^2 f''(u)$$

$$g'''(x) = u'''f'(u) + u''u'f''(u) + 2u'u''f''(u) + (u')^3 f'(u)$$

then we evaluate with the Maclaurin series

$$g(x) = f(0) + u'(0)f'(0)x + \frac{u''(0)f'(0) + (u'(0))^2 f''(0)}{2!} + \dots$$

using the chain rule, for the x_1 term we have $nx^{n-1}f'(x^n)$

evaluating at 0 gives us 0 for any term that isn't 1, evaluating at one gives us $0^0f'(0)$, so we need to approach it with limits

if we take the limit as x approach zero so we have

$$\lim_{x \rightarrow 0} x^0 f'(x^n) = f'(0)$$

so it works fine with the first term, we can do the same for the rest terms and we get diffreant results based on diffrent multiples of n, if we choose k to be 2 only even terms will show up as other terms equal 0, for n = 3 we get multiples of 3 so on so forth, and so we get the series

$$f(x^n) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{kn}$$

we can give it a simple check by calculating random value for a radnom function like $\sin(x^3)$

the series for this function will be $\sin(x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{6n+3}$ we let it equal to $f(x)$ so we can see the difference between both sides

$$\sin(0.365^2) = 0.0486079633377 \quad f(0.365) = 0.0486079633377$$

we can see that it works fine, now for the α -th derivative we get a simple

$$D^\alpha f(x^n) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(kn - \alpha + 1)} x^{kn - \alpha}$$

thus a simple formula but something closer to the general Macluarin series chain rule

4.2.2 General $f(ax)$ formula

Although not quite the difficult formula to get yet it's the easier with a closed form and have big importance in other fields like Fourier transforms
first let $g(x) = f(ax)$, for the first few derivatives we have

$$g'(x) = af'(ax) \quad g''(x) = a^2 f''(ax) \quad g'''(x) = a^3 f'''(ax)$$

you probably can guess the rule from the pattern , but to make sure let's continue with the Macluarin series

$$g(x) = f(0) + af'(0)x + \frac{a^2 f'(0)}{2!} x^2 + \dots = \sum_{n=0}^{\infty} \frac{a^n f^{(n)}(0)}{n!} x^n$$

and from this we can say that $D^n f(ax) = a^n f^{(n)}(ax)$ applying α instead of n will give us

$$D^\alpha f(ax) = a^\alpha f^{(\alpha)}(ax)$$

simple yet needed

4.2.3 General Series form for chain rule

4.2.4 General Closed form for chain rule

the general formula for integral chain rule is called Faà di Bruno's formula and it states

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=0}^n f^{(k)}(g(x)) \times {}^n B_k(g'(x), g''(x), \dots, g^{(n-k+1)}(x))$$

and as we can see there are bunch of problems here ,first if we tried to plug any fraction in place of the n we get an infinite series

5 Interpretation of fractional and complex derivative in other functions

5.1 Trigonometric Functions

Deriving the trigonometric functions can be quite tricky , as there exists n-th derivative formula for them but don't seem to work as intended

5.1.1 $\sin(x)$ and $\cos(x)$

for $\sin(x)$ there exists a formula which is

$$D^n \sin(x) = \sin\left(\frac{n\pi}{2} + x\right)$$

and unexpectedly this formula works for fractional or negative values, let $n = \frac{1}{2}$

$$D^{\frac{1}{2}} \sin(x) = \sin\left(\frac{\frac{1}{2}\pi}{2} + x\right) = \sin\left(\frac{\pi}{4} + x\right)$$

knowing that $D^n(D^m \sin(x)) = \sin\left(\frac{(n+m)\pi}{2} + x\right)$

$$D^{\frac{1}{2}}(D^{\frac{1}{2}} \sin(x)) = \sin\left(\frac{(\frac{1}{2} + \frac{1}{2})\pi}{2} + x\right) = \sin\left(\frac{2\pi}{4} + x\right) = \sin\left(\frac{\pi}{2} + x\right)$$

but there is another proof that this works for all real numbers.

from the euler formula $e^{ix} = \cos(x) + i \sin(x)$ we can say that

$$\sin(x) = \text{Im}(e^{ix})$$

taking the alpha-th derivative of both sides

$$D^\alpha \sin(x) = D^\alpha \text{Im}(e^{ix}) = \text{Im}(i^\alpha e^{ix})$$

knowing that $i = e^{i\pi/2}$

$$D^\alpha \sin(x) = \text{Im}(e^{i\pi\alpha/2} e^{ix}) = \text{Im}(e^{i\pi\alpha/2+ix}) = \text{Im}(e^{i(\alpha\pi/2+x)})$$

turning this back to the euler formula will give us

$$D^\alpha \text{Im}(e^{ix}) = \sin\left(\frac{\alpha\pi}{2} + x\right)$$

which indeed proves it's true from the same formula we can also get the α -th for $\cos(x)$ with the same formula turning this back to the euler formula will give us

$$D^\alpha \text{Re}(e^{ix}) = \cos\left(\frac{\alpha\pi}{2} + x\right)$$

now we can write them as

$$D^\alpha \sin(x) = \text{Im}(e^{i(\alpha\pi/2+x)}) \quad D^\alpha \cos(x) = \text{Re}(e^{i(\alpha\pi/2+x)})$$

or

$$D^\alpha \sin(x) = \sin\left(\frac{\alpha\pi}{2} + x\right) \quad D^\alpha \cos(x) = \cos\left(\frac{\alpha\pi}{2} + x\right)$$

and to make it to the complex plane we can use these formulas also the negative derivative of these is

$$D^{-1} \sin(x) = \sin\left(\frac{-\pi}{2} + x\right) = -\cos(x) \quad D^{-1} \cos(x) = \cos\left(\frac{-\pi}{2} + x\right) = \sin(x)$$

and the first complex derivative of these is

$$D^i \sin(x) = \sin\left(\frac{i\pi}{2} + x\right) = \sin\left(\frac{\ln(-1)}{2} + x\right) \quad D^i \cos(x) = \cos\left(\frac{i\pi}{2} + x\right) = \cos\left(\frac{\ln(-1)}{2} + x\right)$$

5.1.2 $\tan(x)$ and $\sec(x)$

finding the alpha-th derivative for $\tan(x)$ is quite hard since we didn't get any direct formulas for quotients , and there is no direct integral derivative formula we can plug in and generalize to the Real numbers, we can try to change it a little with some algebra

$$\tan(x) = \sin(x) (\cos(x))^{-1}$$

and then use the general product rule ,but quickly we can see the problem

$$D^\alpha(\sin(x) (\cos(x))^{-1}) = \sum_{k=0}^{\infty} \frac{\Gamma(0)}{\Gamma(\alpha - k + 1)\Gamma(k + 1)} D^{\alpha-k}(\sin(x)) D^k(\cos(x)^{-1})$$

there is a gamma pole so this solution also fails
we can try using some trig substitution

$$\tan(x) = \sin(x) (\sqrt{\sin(x)^2 + 1})^{-1}$$

but this leads to infinite sum for the product rule and the chain rule that we have proved above it impossible to find with the simple algebra we have is very hard to approximate by itself as we can see there is no simple elegant closed form for $\tan(x)$ in the scientific paper, the reason behind this will be explained later but we can simply say because it has poles , to analytically see it better we need the Maclaurin series expansion

$$\tan(x) = x + \frac{x^3}{3!} + \frac{2x^5}{15} + \dots = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} \quad \text{where } |x| < \frac{\pi}{2}$$

if we look closely we can notice the problem , it has a radius of convergence and that by itself is the problem that will be discussed in detail later all what we can do for now is applying the D^α to the infinite sum as it will be the only analytic closed form way for now
we will get:

$$\begin{aligned} D^\alpha \tan(x) &= \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{\Gamma(2n)} \left[\frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1} \right] \\ &= \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1} \end{aligned}$$

this infinite series shall work for now

the same also works for $\sec(x)$ as it's a quotient so if we tried using the General product rule we will hit a Gamma pole, so the safest answer for now is to go with infinite series

$$\sec(x) = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{E_{2n}(-1)^n}{(2n)!} x^{2n} \quad \text{where } |x| < \frac{\pi}{2}$$

again we see the same problem with radius of convergence
simply we apply D^α :

$$\begin{aligned} D^\alpha \sec(x) &= \sum_{n=0}^{\infty} \frac{E_{2n}(-1)^n}{\Gamma(2n)} \left[\frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha} \right] \\ &= \sum_{n=0}^{\infty} \frac{E_{2n}(-1)^n}{\Gamma(2n-\alpha+1)} x^{2n-\alpha} \end{aligned}$$

Note: these two work for their radius of converges only

Note: they also work for the complex derivative

5.1.3 $\csc(x)$ and $\cot(x)$

now saying $\csc(x)$ and $\cot(x)$ will work the same like the rest of trigonometric functions is a bit of a stretch

since we already know both of their domains aren't $x \in \mathbb{R}$ so they must have some sort of analytical poles and converges radius that isn't \mathbb{R} in their infinite sums

but if we noticed

$$\csc(x) = \frac{1}{\sin(x)}$$

which means that it has a singularity at $x = 0$, in other words simple Taylor series nor simple Maclaurin series won't work, we need the General Laurent series for this one. The Laurent series is :

$$\csc(x) = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \dots = \sum_{n=0}^{\infty} \frac{B_{2n}(-1)^{n+1}(2^{2n}-1)}{(2n)!} x^{2n-1} \quad \text{where } 0 < |x| < \pi$$

but before plugging the D^z operator to the series we can notice a little problem in the beginning, the $\frac{1}{x}$ term

simply plugging in the $D^z(x^n)$ will result a pole, the simple solution is just to the linearity of D^z and differentiate the first term alone and then the rest of the series alone

$$\begin{aligned} D^\alpha \csc(x) &= D^\alpha(x^{-1}) + \sum_{n=1}^{\infty} \frac{B_{2n}(-1)^{n+1}(2^{2n}-1)}{\Gamma(2n)} \left[\frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1} \right] \\ &= \frac{(-1)^\alpha \Gamma(1+\alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{B_{2n}(-1)^{n+1}(2^{2n}-1)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1} \end{aligned}$$

of course this is where $0 < |x| < \pi$

The same goes for $\cot(x)$ as it doesn't have any Taylor series but rather Laurent series

The Laurent series for $\cot(x)$ is:

$$\cot(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} x^{2n-1} \quad \text{where } 0 < |x| < \pi$$

Applying D^α to both sides we get

$$D^\alpha \cot(x) = \frac{(-1)^\alpha \Gamma(1+\alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}$$

Note: this works to complex numbers too

5.2 Hyperbolic Functions

5.2.1 $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$

$\sinh(x)$ is pretty straight forward to get from the definition

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

differentiating both sides to the α

$$D^\alpha \sinh(x) = \frac{1}{2}(D^\alpha e^x - D^\alpha e^{-x}) = \frac{1}{2}(e^x - (-1)^\alpha e^{-x})$$

we can also do the same for $\cosh(x)$

$$D^\alpha \cosh(x) = \frac{1}{2}(D^\alpha e^x + D^\alpha e^{-x}) = \frac{1}{2}(e^x + (-1)^\alpha e^{-x})$$

but if we change the negative sign in $\sinh(x)$ to $+(-1)$ we turn the derivative to

$$D^\alpha \sinh(x) = \frac{1}{2}(e^x + (-1)^{\alpha+1} e^{-x})$$

which is equal to $D^{\alpha+1} \cosh(x)$ and that is because unlike normal $\sin(x)$ and $\cos(x)$ these are the integer integrals and derivatives of their-selves so we can get the negative derivatives to be

$$D^{-1} \sinh(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x) \quad D^{-1} \cosh(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$$

for the complex derivatives we can use the formulas from before
However for $\tanh(x)$ things change , since the defention for it is

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

we can see , it's a quotient of two functions ,which leaves us with nothing but to use it's power series

the power series for \tanh is:

$$\tanh(x) = x - \frac{x^3}{3!} + \frac{2x^5}{15} - \dots = \sum_{n=0}^{\infty} \frac{B_{2n} 4^n (1 - 4^n)}{(2n)!} x^{2n-1} \quad \text{where } |x| < \frac{\pi}{2}$$

as predicted there will also be radius of converges here too
but at anyway we get the D^α with this:

$$\begin{aligned} D^\alpha \tanh(x) &= \sum_{n=0}^{\infty} \frac{B_{2n} 4^n (1 - 4^n)}{\Gamma(2n)} \left[\frac{\Gamma(2n)}{\Gamma(2n - \alpha + 1)} x^{2n-\alpha-1} \right] \\ &= \sum_{n=0}^{\infty} \frac{B_{2n} 4^n (1 - 4^n)}{\Gamma(2n - \alpha + 1)} x^{2n-\alpha-1} \end{aligned}$$

5.2.2 $\operatorname{sech}(x)$, $\operatorname{csch}(x)$ and $\operatorname{coth}(x)$

The rest of the hyperbolic functions shall work the same as the trigonometric functions, infact all of the hyperbolic and the trigonometric function's Laurent/Taylor seriesses look identical with little changes, so finding them won't be that difficult

For $\operatorname{sech}(x)$ the Taylor Series is:

$$\operatorname{sech}(x) = 1 - \frac{x^2}{2!} + \frac{5x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n} \quad \text{where } |x| < \frac{\pi}{2}$$

So Applying D^z will be as simple as $\sec(x)$

$$D^\alpha \operatorname{sech}(x) = \sum_{n=0}^{\infty} \frac{E_{2n}}{\Gamma(2n - \alpha + 1)} x^{2n-\alpha} \quad \text{where } |x| < \frac{\pi}{2}$$

the same goes for the Laurent series of $\operatorname{csch}(x)$ and $\operatorname{coth}(x)$

$$\operatorname{csch}(x) = \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - \dots = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{B_{2n} (1 - 2^{2n})}{(2n)!} x^{2n-1} \quad \text{where } 0 < |x| < \pi$$

of course it's similar but not identical , anyway applying D^α gives us

$$\begin{aligned} D^\alpha \operatorname{csch}(x) &= D^\alpha x^{-1} + D^\alpha \sum_{n=0}^{\infty} \frac{B_{2n} (1 - 2^{2n})}{(2n)!} x^{2n-1} \\ &= \frac{(-1)^\alpha \Gamma(1 + \alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{B_{2n} (1 - 2^{2n})}{\Gamma(2n - \alpha + 1)} x^{2n-\alpha-1} \quad \text{where } 0 < |x| < \pi \end{aligned}$$

and for $\operatorname{coth}(x)$:

$$\operatorname{coth}(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1} \quad \text{where } 0 < |x| < \pi$$

and applying D^α operator we get:

$$D^\alpha \operatorname{coth}(x) = \frac{(-1)^\alpha \Gamma(1 + \alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(2n - \alpha + 1)} x^{2n-\alpha-1}$$

5.3 The Inverse Trigonometric and Hyperbolic Functions

there is a problem with these function that makes them special, if we for example tried to take the derivative for $\sin^{-1}(x)$ and the integral of the same function we get this

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \quad -1 < x < 1 \quad \int \sin^{-1}(x) dx = x \sin^{-1}(x) + \sqrt{1-x^2} + C$$

as we can see they are a type of **Transformtive functions** which will be discussed in later sections

Part II

D^z Analysis

6 The nature of functions under D^z Field

6.1 Explaining what is a fractional derivative

taking an positive Integer derivative gives us the rate of change of a function , talking the derivative of that also gives us the rate of change the derivative function , so on so forth

taking the negative Integer derivative is known as the "Area under the curve" or the function in which the original function is the rate of change of it , or simply the "Anti-derivative"

if we tried to explain D^z operator with the standard definition of integrals and derivatives we can say that

if the first derivative is the rate of change of the function, the first integral is area under the curve then D^z operator is change from being an Area to the function than the rate of the function itself, but this doesn't make sense mostly because these are three different things that work and measure differently how could we explain such things?

The problem becomes bigger and bigger when we say the second derivative which is the rate of change of the rate of change of the function that gives us information about the function like when is it a climax or function convexity, or the second Integral that is the 3D volume of the shape.

then comes the Taylor expansion and and the power series of functions which use infinite derivatives and now we are all out of the meaning of it

even if we though about it physically , for simple cases like position, it's derivative is velocity and it's second derivative is acceleration which at least change the unit powers and doesnt make much sense with D^z

in fact with out standers even if multiple derivatives integer would somehow make sense, the fractional derivative wouldn't

which why is suggest another prespective for the derivative instead of it being the area under curve or the rate off change , these things are just side effects

and the derivative is it's actual object by itself
the fractional derivative can be thought as

6.2 what it means for a derivative order to be Complex

as we saw how the fractional derivative can say the complex derivative take a weird turn as it actually not a nor rate of change of function...sort of to get an understanding of what i am talking about let's see it on a simple function x^n and see how it works

we know that:

$$D^z(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-z}$$

substituting $z = i$

$$D^i(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-i+1)} x^{n-i}$$

the nature of this will be explored later but for now if we try to take another imaginary derivative using the **Multiplication Law** we proved earlier we see that

$${}^i D^i(x^n) = D^i(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-(i \times i)+1)} x^{n-(i \times i)} = \frac{\Gamma(n+1)}{\Gamma(n-(-1)+1)} x^{n-(-1)} = \frac{\Gamma(n+1)}{\Gamma(n+2)} x^{n+1}$$

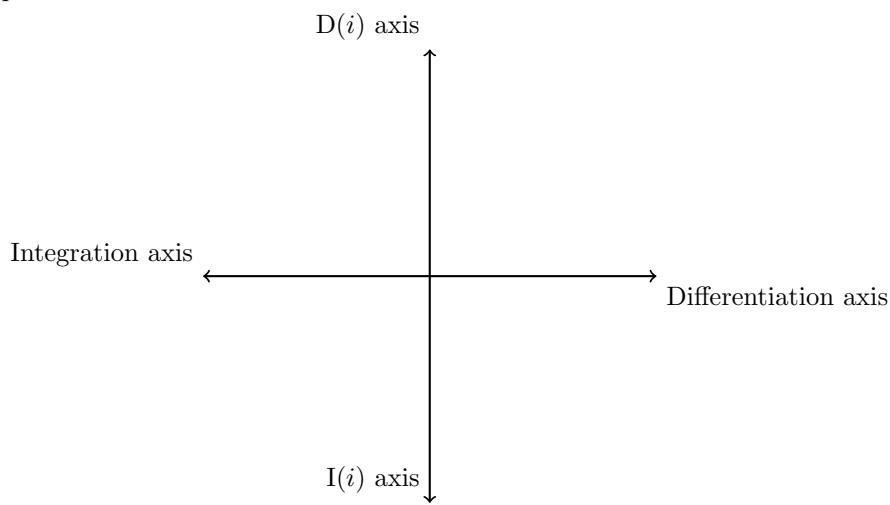
and knowing that $\Gamma(n+1) = n\Gamma(n)$

$${}^i D^i(x^n) = \frac{\Gamma(n+1)}{(n+1)\Gamma(n+1)} x^{n+1} = \frac{x^{n+1}}{n+1}$$

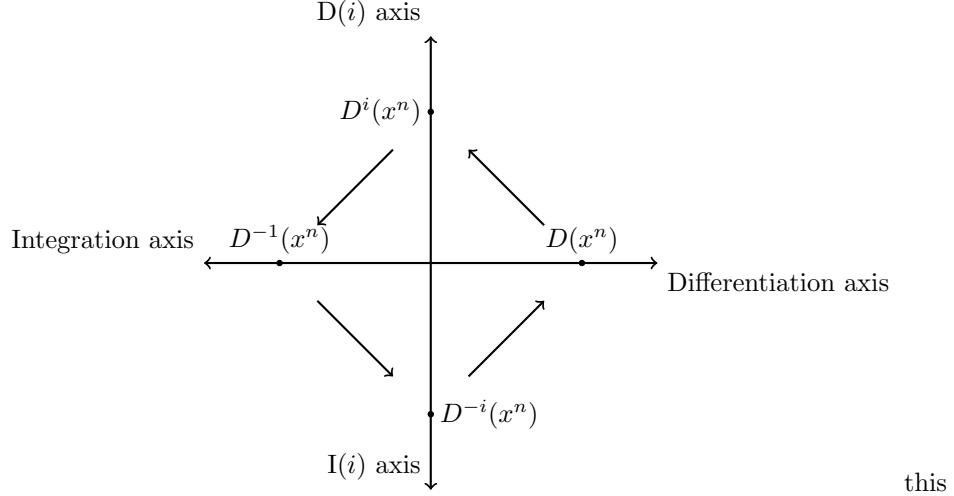
infact this was expected since ${}^i D^i = D^{(i \times i)} = D^{-1}$

the result is kind of weird , clearly it's weird

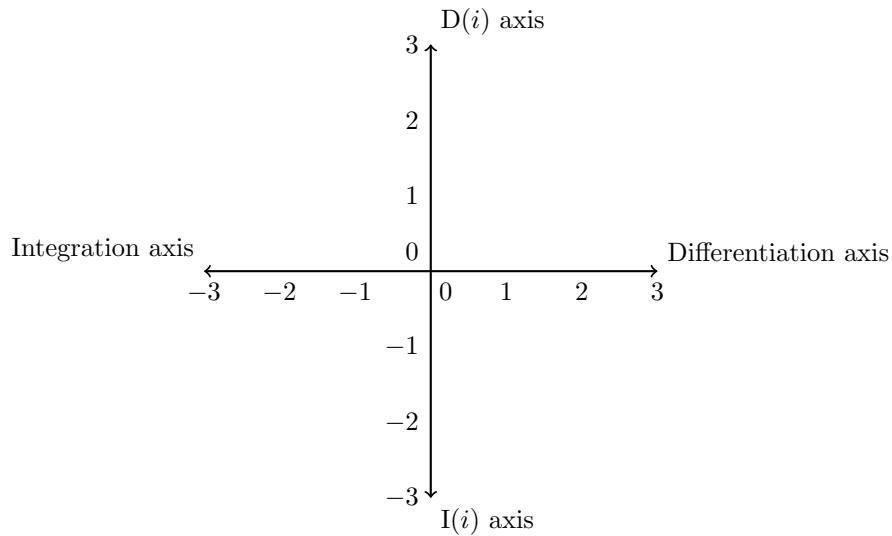
to see why it's weird (if it's not already) we can use a geometric interpretation from the information we have right now we can draw this transformation in D-I plane



as this map acts as the same as complex plane a we map the transformation around it like this like this



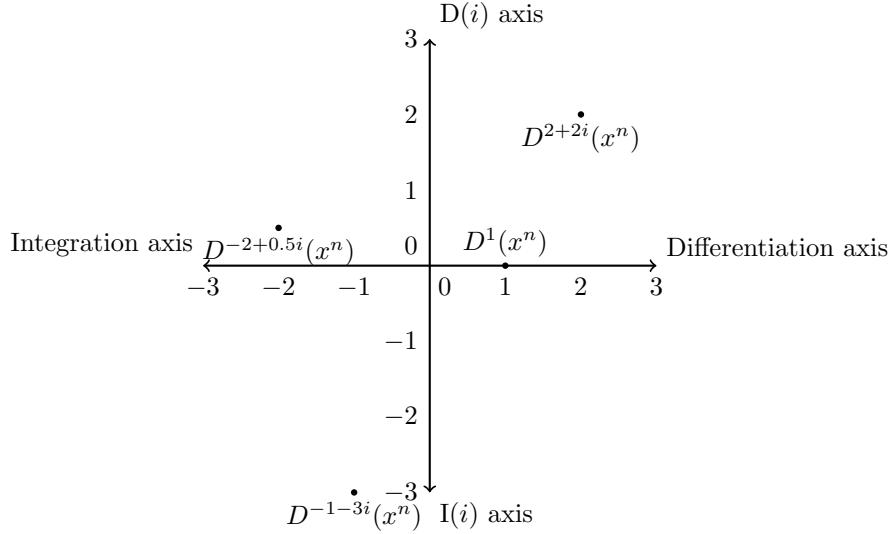
is how the first derivative act when using the complex number i
 which means also that means there exist a way (if not multiple) to represent
 differentiation and integration geometrically in one same space
 we can add more detail to this plane by defining each access as the the order of
 the D opreator so we can write it better like this



with the y-axes being the imaginary order part and the x-axis being the normal part, we can of course now express any derivative or integral or any weird result of D^z as points in this space.

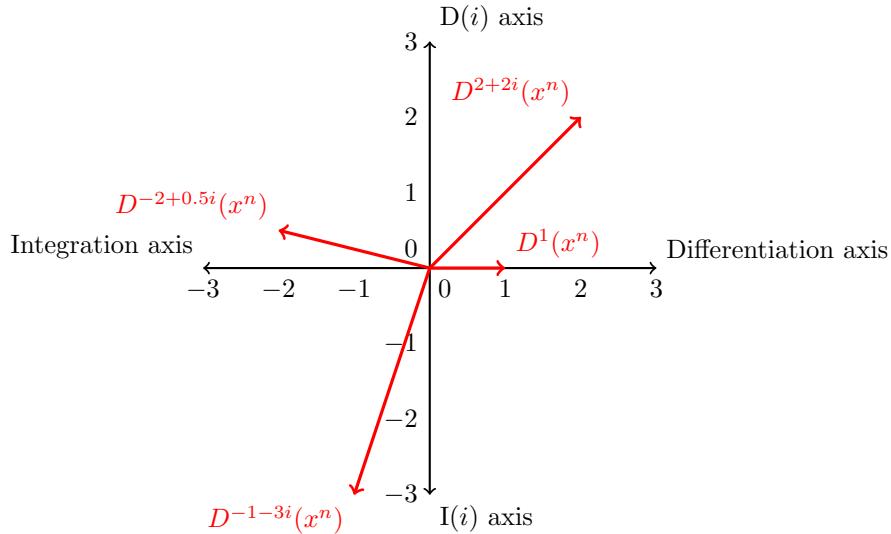
that is, of course for one single function as the input is a variable of type of function and the output is of variable function so for now they must stay the

same function with the same variables inside of it
so we can now express for example the $D^z(x^n)$ like this



this also allows us to use any root of unity not only i like a normal Re-Im plot plane, of course we can change the perspective and think of them as vectors in a 2D vector plane

for each derivative we can draw them as vectors starting from zero point (the function itself) to the point point of derivative order



which means that the vector rules also work here fine with the D^z operator properties as we can see adding and subtracting orders will return another vector using the Index Law, and since we know what the plane is for we can express the orders from $D^{\alpha+\beta i}$ as points $(\alpha, \beta i)$ or vectors $\vec{a} = (\alpha, \beta)$, and supposedly

polar (r, θ) , of course if we want to make a vector from a point to another we can just use the index law that goes in the path to β which also means that there is infinite starting points for vectors to go to the same point, in another words : infinite vectors for any point and to any point

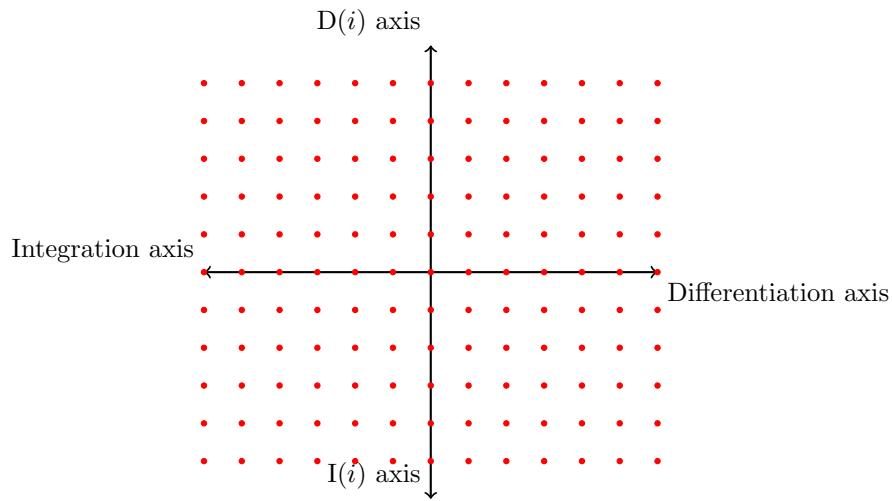
this can be called the **D(i) Plane**, even when dealing with vectors , at the end this plane isn't really based on the idea of it but rather an extension to it

this may helpful , but i suggest another idea that is based on vectors in such field and I would like to call it the **D(i) Vector Field**

instead of using vectors in the plane as beginning and ending points we will use them as points,arrows and degrees, sort of a mix of the polar form and the vector form of the plane

- 1- the point: will represent it's head, where it's coming out of
- 2- the arrow : will be the output of the function derivative rate of change, the bigger it's the higher the rate of change its
- 3- the degree: it will represent how the D^z operator is changing the function itself

for the sake of understanding let's take the function e^{2x} as an example



6.3 Quaternions derivatives

this will be fascinating as it operates on 3D

7 Cyclic derivatives

7.1 e^x and Cyclic functions

if we look closely in functions that have **Cyclic-Derivatives** we can see that there exists a pattern for example

$$\begin{aligned} D^\alpha e^x &= e^x \\ D^\alpha \sinh(x) &= \frac{1}{2}(e^x - (-1)^\alpha e^{-x}) & D^\alpha \cosh(x) &= \frac{1}{2}(e^x + (-1)^\alpha e^{-x}) \\ D^\alpha \sin(x) &= \text{Im}(e^{ix}) & D^\alpha \cos(x) &= \text{Re}(e^{ix}) \end{aligned}$$

they all have connection with e^x in them

the function e^x by itself is a cyclic derivative function , it always returns itself no matter how many times you differentiate it if we change the function slightly to e^{ax} things now change as it's returns different results

$$\text{when } a > 1 \quad \lim_{\alpha \rightarrow \infty} D^\alpha(e^{ax}) = \infty \quad \text{when } a < 1 \quad \lim_{\alpha \rightarrow \infty} D^\alpha(e^{ax}) = e^{ax}$$

these are all easy known results of course when $a = 1$ $\lim_{\alpha \rightarrow \infty} D^\alpha(e^{ax}) = e^{ax}$
 these are all expected to be the result of repeating multiplication of $a^\alpha e^{ax}$
 but there is one case for a where it's not any of the above
 let's examine the cyclic derivative functions closely

$$D^\alpha e^x = e^x$$

if we try to write in general e^{ax} formula we get

$$D^\alpha e^{1x} = 1^\alpha e^{1x}$$

which can also explain (with what we already know) why e^x is a repeated derivative of itself

let's look at hyperbolic functions which are cyclic derivatives of order 2

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

if we solve for e^{-x} in both functions
 we get for $\sinh(x)$

$$2 \sinh(x) = e^x - e^{-x} \quad \therefore 2 \sinh(x) - e^x = -e^{-x} \quad \therefore e^x - 2 \sinh(x) = e^{-x}$$

we can do the same for $\cosh(x)$

$$2 \cosh(x) = e^x + e^{-x} \quad \therefore 2 \cosh(x) - e^x = e^{-x}$$

let's add both equations together we get

$$2e^{-x} = 2 \cosh(x) - e^x + e^x - 2 \sinh(x) = 2 \cosh(x) - 2 \sinh(x)$$

dividing both sides by 2 we get

$$e^{-x} = \cosh(x) - \sinh(x)$$

which is truly an order 2 cyclic derivative function ,if we took the first derivative of the function we get

$$\begin{aligned} D(e^{-x}) &= D(\cosh(x)) - D(\sinh(x)) = \\ -e^{-x} &= \sinh(x) - \cosh(x) \end{aligned}$$

this is true since if we multiplied by -1 in the first expression we get the same result

of course we can express these in terms of e^x like this

$$e^x = \cosh(x) + \sinh(x)$$

which is true since both sides will return the same result no matter how many times we differentiate them

it looks sort of like Euler formula but for hyperbolics

notice that -1 that is in e^{-x} comes from the equation $a^2 = 1, a = \pm 1$ because if we differentiate any of the e^x or e^{-x} expressions twice we get the the same expressions back

if the idea didn't click in yet let's look at trigonometric

for trigonometric functions but this time we have a formula ready for us

$$e^{ix} = \cos(x) + i \sin(x)$$

differentiating both sides we get

$$ie^{ix} = -\sin(x) + i \cos(x)$$

which again works well ,and if we differentiate again 3 times more we get back to the same expression

notice that here $a = i$ infact not only i but any value that satsfies the expression $a^4 = 1$ works as well , and the returning function will be order 4 cyclic derivative function

the pattern here goes on and on and this is exactly the missing case for a

when $a^n = 1, D^n(e^{ax}) = e^{ax}$ with $2 \times n$ cyclic order

that is because it holds for both positive and negative values of a, which means that the case $a^1 = 1$ the order is only once cyclic since it has only positive values for case $a^2 = 1$ we get e^{-x} which is 2nd order cyclic derivative

and for case $a^4 = 1$ we get e^{ix} which is 4th order cyclic derivative

this formula helps us generate any order of cyclic derivatives as for example a 3rd order cyclic derivative will have a as

$$a^3 = 1, a = 1 \text{ or } -\frac{1}{2} + i\frac{\sqrt{3}}{2} \text{ or } -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

and all of them work exactly as intended, also we can see that some cyclic-derivatives work correctly under the order of another cyclic derivatives like order 2 (\sinh and \cosh) under order 4 (\sin and \cos) this is because in order for $a^n = 1$ to be true isn't only n times but it works for n and $2n$ and $3n$ so on because every one of them will lead to $1, a^n = a^{2n} = a^{3n} = 1$ in other words the higher cyclic-derivatives that works the same will satisfy the property

when $a^n = 1, D^n(e^{ax}) = e^{ax}$ with $2 \times k$ cyclic order, where $k \equiv 0 \pmod{n}$

getting back to the basics both of case $a^2 = 1$ and case $a^4 = 1$ can be expanded as trigonometric or hyperbolic functions, not only that but both of them satisfy the condition 2^n and both of them can be expanded algebraically, for hyperbolic it's $x^2 - y^2 = 1$ and $x^2 + y^2 = 1$ for trigonometric functions, and since they are cyclic derivatives they are functions of type FRD, which suggests that they may exist a pattern of function families that are

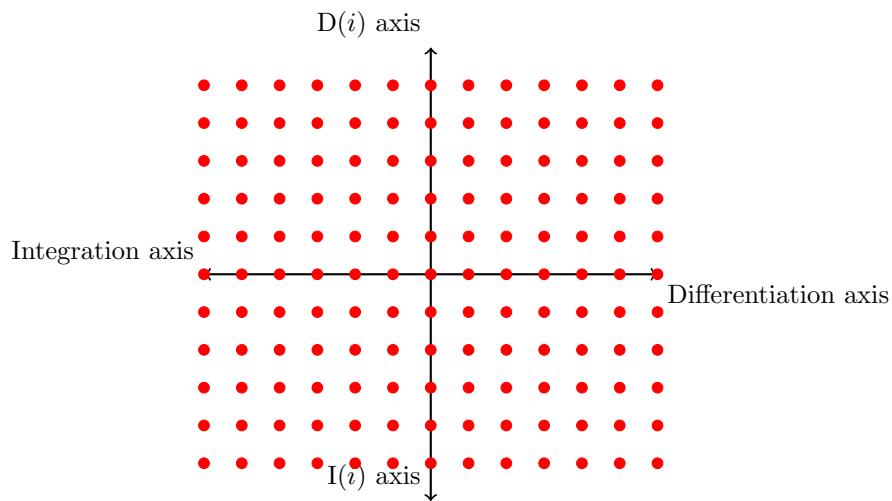
e^{ax} where $a^n = 1$ that satisfies the condition $2^n \in \mathbb{Z}^+$

and these function families may also have algebraic expression that also suggests it will be from high order operator that isn't simple plus or minus also since -1 and i can both be explained geometrically as "rotations" we can also say that n is the number of unity roots which also suggests cyclic order derivatives but in 3D or 4D using Quaternions and octonions

7.2 Geometric Interpretation of cyclic derivatives

infact we can do this now

let's start simple with e^x in the D^z Re-Im Plane



this is expected since e^x is the exact value of itself at any point , even if we tried

to represent them as vectors we will get the same results because the it always begins and ends at any value of the order derivative , in other words there is no direction for the vector to point at, but this plane while may show something but isn't so useful, instead we can use the D(i) vector plane
we can try the same for e^{ax}

7.3 the pattern in cyclic functions powers series

8 The Geometric DifferentInetgrel (GDI) theory

8.1 GDI principles and infinite seriesses

8.2 FRD and NFRD

these are two important terms that will be helpful for us

FRD : Full Real Differentiable **NFRD** : Not-Full Real Differentiable
these are two very important terms as they catagories all the functions in D^z since most of the function are either Integrable and Differentiable or Integrable and not Differentiable and D^z can work on both sides

FRD Functions are functions that have a clean closed form for it's α -th derivative that isn't a power series (that is of course if the function itself isn't defined by a power series)

these are function like x^n with the condition that $\text{Re}(z) \leq n$ and e^x

NFRD function are function that has an only closed form being a power series , so function like $\tan(x)$ and other trigonometric function are NFRD
this happens of course for a lot of reasons like breaking GDI first principle and

8.3 Transformative Functions

They are the type of functions that transform dependence with derivative , for example

$$\frac{d}{dx}(\cos^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$$

as we can see , when we take the derivative of the function it change dependence from the value of x to the value of the power of x

8.4 The inverse functions problem

Inverse functions are a problem , at least for being FRD function as it's not possible for most of them and I have some explanations

Inverse functions are weird compared to the normal functions , we can see some similities between them and their original functions like x^n and $x^{\frac{1}{n}}$ as they both look identical for odd values because for even numbers we get complex values , but rotated and cutted to half, in most of the inverse functions we can find

these cuts near always

we can expect this because not every function can be fully inverted
 a function is set of inputs that lead to outputs, one output can be gotten by
 many inputs but not the way around , so when we try to inverse functions like x^2
 that has one output for the more than one input we get something that doesn't
 work in the real number systems, and for this example it's the complex values
 for negative integers

8.5 D^z space

8.6 $\ln(x)$ and rotation in D^z space

9 Matrix derivative functions

9.1 Whole matrix derivatives

we start with the simple function x^n , and let our matrix be a simple A matrix,
 we can right the $D^A x^n$ like this:

$$D^A(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-A+1)} x^{n-A}$$

as we can see this shape is hard , and a better simplification for it is to use
 $e^{\ln(x)}$ so it will be :

$$D^A(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-A+1)} x^n e^{-\ln(x)A}$$

this is better as now we can express it as infinite sum using the Taylor series of
 function e^x

$$D^A(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-A+1)} x^n \sum_{k=0}^{\infty} \frac{(\ln x)^k}{k!} A^k$$

where $A^k = \underbrace{A \times A \times A \times \dots}_{k \text{ times}}$, and this can be called the simple $D^A x^n$

let's plug in this definition in the Taylor series general formula

$$D^\alpha f(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-\alpha+1)} (x-a)^{n-\alpha}$$

so now it shall be

$$D^A f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-A+1)} (x-a)^{n-A}$$

and thus the Maclaurin series is

$$D^A f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n-A+1)} x^{n-A} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n-A+1)} x^n e^{-\ln(x)A}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n-A+1)} x^n \sum_{k=0}^{\infty} \frac{(\ln x)^k}{k!} A^k$$

now we know we can do it to any analytical function, and the simple way to put it in any series is to remove the α from the power of and put the term $\sum_{k=0}^{\infty} \frac{(\ln x)^k}{k!} A^k$

9.2 What is the meaning of matrix derivatives

10 Functional derivatives

10.1 derivative order as function of itself

10.2 derivative order as function of it's independent variable

Part III

Applications of D^z

11 Bonus Topics

11.1 Differential Geometry Connection

11.2 Fourier Transform Connection Connection

11.3 Category Theory

11.4 Number Theory and D^z operator

11.4.1 Logarithmic functions, function $D^z \ln(x)$ and it's nature

11.4.2 The zeta function $D^z \zeta(n)$ and it's nature

11.4.3 The legendre function and it's nature

11.5 Combinations and D^z operator

11.6 other functions and D^z operator

11.6.1 The Gaussian Integral $D^z e^{-x^2}$ and it's nature

12 Integration with D^z operator

12.1 Deriving the D^{-z} formulas

12.2 Fractional and Complex Integration

13 Fractional and Complex Differential equations

14 Applications in real world

as sir Issac Newton started it based on motion of everything in the universe , when he didn't find good way to see it he created his way which we shall call

today calculus, so of course we are beginning with physics

14.1 Dynamics with D^z operator

dynamics are the beginning of and heart with physics, it's the language of motion in its simplest form, and so we can begin by stating that $f(x)$ is position and the first derivative with respect to time is Velocity, and the second derivative with respect to time is acceleration, that is for integer derivatives, and for fractional derivatives things tend to get weird

let's take for example the function x^5

$$f(x) = x^5 \quad v(x) = 5x^4 \quad a(x) = 20x^3$$

these are the three main dynamical functions for $f(x)$

now we are going to propose three different functions that work in the middle so :

$${}_tD^{0.5}f(x) = hf(x) = \frac{120}{\Gamma(5.5)}x^{4.5} \quad {}_tD^{0.5}f(x) = hv(x) = \frac{120}{\Gamma(4.5)}x^{3.5} \quad {}_tD^{0.5}f(x) = ha(x) = \frac{120}{\Gamma(3.5)}x^{2.5}$$

14.2 Electrodynamics with D^z operator

14.3 Fluid dynamics with D^z operator

14.4 Quantum Mechanics with D^z operator

14.5 Probability and Stochastic Processes with D^z operator

15 Table of formulas

$$\mathbb{R}_t^\beta D_c^\alpha$$

this is the standard notation for this research and it reads:

c : The derivative number in terms of context, default value : 1

α : The derivative order, default value : 1

β : The multiply derivative order, default value : 1

t : the variable derived with respect to, default value : time

\mathbb{R} : the derivative output number set, default value : real numbers

it's read from left down counter clockwise in to out:

the derivative number c of order α to β with respect to t in \mathbb{R}

16 Brief letter and thanks

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