Cyclic derivatives and functions

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Abstract

This paper presents an independent exploration of cyclic derivatives based on the fractional and complex-order derivatives paper made by the same Author.

Note to Readers: This represents independent rediscovery of classical fractional calculus concepts. I (The Author) present this work as a pedagogical exercise in mathematical exploration rather than novel research.

Background

The Main formula that works for all derivatives and can be used in the Maclaurin series

$$D^{\alpha}x^{n} = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha}$$

The more important formulas that are built on this one

$$D^{\alpha}e^{ax} = a^{\alpha}e^{ax} \qquad e^{ix} = \cos(x) - i\sin(x)$$

$$e^{x} = \cosh(x) + \sinh(x) \qquad e^{-x} = \cosh(x) - \sinh(x)$$

$$D^{\alpha}\sin(x) = \sin(\frac{\alpha\pi}{2} + x) \qquad D^{\alpha}\cos(x) = \cos(\frac{\alpha\pi}{2} + x)$$

$$D^{\alpha}\sinh(x) = \frac{e^{x} - (-1)^{\alpha}e^{-x}}{2} \qquad D^{\alpha}\cosh(x) = \frac{e^{x} + (-1)^{\alpha}e^{-x}}{2}$$

1 Introduction

from what we have seen in the previous fractional and complex-order derivatives paper, cyclic derivatives are such a big area that deserves its own paper,

2 The foundation

From the fractional and complex-order derivatives paper, we know that

$$\textit{when } a^n = 1, D^n(e^{ax}) = e^{ax} \textit{ with } 2 \times n \textit{ cyclic order}$$

And we can call that the theorem of cyclic derivatives

Theorem 1 *$f(x)$ is a cyclic derivative when $a^n = 1, D^n(e^{ax}) = e^{ax}$ with $2 \times n$ cyclic order*

And in the same paper, we generalized this to hold for any $k \equiv 0 \pmod n$

Theorem 2 *$f(x)$ is a lesser cyclic derivative when $a^n = 1, D^n(e^{ax}) = e^{ax}$ with $2 \times k$ cyclic order, where $k \equiv 0 \pmod n$*

With a hypothesis that I wish to prove in this paper, that

Hypothesis 1 *For every function cyclic derivatives that can be written in the form e^{ax} where $a^n = 1$ and satisfies the condition $2^n \in \mathbb{Z}^+$, there exists an algebraic perimetric form*

We can already see this for hyperbolic functions, where they can be written in the form $x^2 - y^2 = 1$, and trigonometric functions We also proved some equations

$$D^i e^{-x} = e^{-(x+\pi)} = \cosh(x+\pi) - \sinh(x+\pi)$$

$$D^i \sinh(x) = \frac{e^x - e^{-(x+\pi)}}{2} \qquad D^i \cosh(x) = \frac{e^x + e^{-(x+\pi)}}{2}$$

$$D^i e^{ix} = e^{\frac{-\pi}{2}} \cos(x) + ie^{\frac{-\pi}{2}} \sin(x)$$

$$D^i \sin(x) = \sin(\frac{i\pi}{2} + x) \qquad D^i \cos(x) = \cos(\frac{i\pi}{2} + x)$$

and we also proved that for $\sin(x)$ and $\cos(x)$

3 More about the complex derivatives and known families

from what we know about $\sinh(x)$ and $\cosh(x)$ we can write their formulas in an other way

$$D^\alpha \sinh(x) = \frac{e^x - (-1)^\alpha e^{-x}}{2} = \frac{e^x - (e^{i\pi})^\alpha e^{-x}}{2} = \frac{e^x - (e^{i\pi\alpha})e^{-x}}{2}$$
$$= \frac{e^x - e^{i\pi\alpha - x}}{2}$$

The same goes for $\cosh(x)$

$$D^\alpha \sinh(x) = \frac{e^x - e^{i\pi\alpha - x}}{2} \qquad D^\alpha \cosh(x) = \frac{e^x + e^{i\pi\alpha - x}}{2}$$

We can see that we can't express them as simple forms, since the real value changes differently from the complex value. If we try to apply the derivative operator to e^x with the definition of $1 = e^{2i\pi}$, we get

$$D^\alpha e^x = 1^\alpha e^x = e^{2i0\alpha} e^x = e^x$$

We used that because it's the principal value
For trigonometric functions, we know that D^i represents a half turn before it changes to its integral from the real plane, but there is something to clarify

$$\operatorname{Im}(D^i e^{ix}) = e^{\frac{-\pi}{2}} \sin(x)$$

As we can see, the first i-th derivative acts as a scaler that scales $\sin(x)$ by real value $e^{-\frac{\pi}{2}}$
However, this isn't equal to the $D^i \sin(x)$ as the definition changes from scaling to rotating, like this

$$D^i \sin(x) = \sin(\frac{i\pi}{2} + x) =$$

But since we have proven the multiplication law works in the framework, we can make sure that both are somewhat equal

$$^iD^i e^{ix} = i^{i\times i} e^{ix} = (e^{\frac{i\pi}{2}})^{-1} e^{ix} = e^{\frac{-i\pi}{2}} e^{ix} = e^{ix-\frac{i\pi}{2}}$$

$$e^{i(x-\frac{\pi}{2})} = \cos(x - \frac{\pi}{2}) + i\sin(x - \frac{\pi}{2}) = \sin(x) - i\cos(x) = D^{-1}e^{ix} = \int e^{ix}$$

and for the sin we proved in the "Complex-order and fractional derivatives: first exploration" paper that the index law works on it, and thus the multiplication law either from here or from the series expansion, so that we can say

$$^iD^i \sin(x) = \sin(\frac{i \times i\pi}{2} + x) = \sin(\frac{-\pi}{2} + x) = -\cos(x)$$

Thus, both of them work fine, just different prescriptive
We shall call the e^{ix} the complex perspective since it's all about the imaginary unit, and $\sin(x)$ the real perspective, even if there is i in it

3.1 The complex prescriptive

since D^i represents a whole rotation on the $D(i)$ plane, we can get more angles that could help us understand more what happens to the function to "integrate it"
first step is we are going to transform from i to $e^{\frac{i\pi}{2}}$ so we can deal with rotation with radians in circles

$$D^i e^{ax} = D^{e^{\frac{i\pi}{2}}} e^{ax} \qquad D^{-1} e^{ax} = D^{e^{i\pi}} e^{ax}$$

$$D^{e^{i\theta}} e^{ax} = a^{e^{i\theta}} e^{ax} \qquad D^{e^{i\theta}} e^{ix} = i^{e^{i\theta}} e^{ix} = e^{\frac{i\pi e^{i\theta}}{2}} e^{ix} = e^{i(\frac{\pi e^{i\theta}}{2}+x)}$$

$$= \cos(\frac{\pi e^{i\theta}}{2} + x) + i\sin(\frac{\pi e^{i\theta}}{2} + x)$$

So now we can know what happens at the third of rotation or the third root of unity, which is equal to $\frac{\pi}{3}$ in radians, we get

$$D^{e^{\frac{i\pi}{3}}} e^{ix} = e^{i(\frac{\pi e^{\frac{i\pi}{3}}}{2} + x)} = \cos(\frac{\pi e^{\frac{i\pi}{3}}}{2} + x) + i\sin(\frac{\pi e^{\frac{i\pi}{3}}}{2} + x)$$

which after calculating $e^{\frac{i\pi}{3}}$ to be $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ we can then multiply it by $\frac{\pi}{2}$ to get $\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4}$, then we plug it

$$e^{i(\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4} + x)} = e^{i\frac{\pi}{4} + i^2\frac{\pi\sqrt{3}}{4} + ix} = e^{i(x+\frac{\pi}{4})} e^{-\frac{\pi\sqrt{3}}{4}} = e^{-\frac{\pi\sqrt{3}}{4}} \cos(x+\frac{\pi}{4}) + ie^{-\frac{\pi\sqrt{3}}{4}} \sin(x+\frac{\pi}{4})$$

We can see that it scales by a factor of $e^{\frac{-\pi\sqrt{3}}{4}}$ and rotate with a factor of $\frac{\pi}{4}$
let's do the same for two-thirds, the value for $e^{\frac{2i\pi}{3}}$ to be $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$ we can then multiply it again by $\frac{\pi}{2}$ to get $-\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4}$
Plugging it again, we get

$$e^{i(-\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4} + x)} = e^{-i\frac{\pi}{4} + i^2\frac{\pi\sqrt{3}}{4} + ix} = e^{i(x-\frac{\pi}{4})} e^{-\frac{\pi\sqrt{3}}{4}} = e^{-\frac{\pi\sqrt{3}}{4}} \cos(x-\frac{\pi}{4}) + ie^{-\frac{\pi\sqrt{3}}{4}} \sin(x-\frac{\pi}{4})$$

At two-thirds, it rotates with the same value but rotates backwards
Now we have a little information about what happens in the process of integrating such functions
at the first third of the way, it rotates by $\frac{\pi}{4}$ and scales by $e^{-\frac{\pi\sqrt{3}}{4}}$
For Halfway, it doesn't rotate but scales with a factor of $e^{-\frac{\pi}{2}}$
for two-thirds it rotates by $\frac{\pi}{4}$ and scales by $e^{-\frac{\pi\sqrt{3}}{4}}$
This may seem weird at the beginning until we notice that we aren't starting from order 1 or D^1, we are starting from the zero point D^0 or the function itself, so the one-third and two-thirds don't cancel out on rotation, but they rotate to two different directions
The one-third rotates to D^1 and the two-thirds rotate to D^{-1}, while the middle point D^i doesn't rotate but scales because it's not a real derivative or real integral
We can even notice that in the first third we have $\cos(x + \frac{\pi}{4})$ and $\sin(x + \frac{\pi}{4})$, which are both pure half derivatives

$$D^{\frac{1}{2}} \sin(x) = \sin(\frac{\frac{1}{2}\pi}{2} + x) = \sin(\frac{\pi}{4} + x) \qquad D^{\frac{1}{2}} \cos(x) = \cos(\frac{\frac{1}{2}\pi}{2} + x) = \cos(\frac{\pi}{4} + x)$$

and the same happens for the half-integer being rotated by $\frac{\pi}{4}$

3.2 The Real prescriptive

4 exploration into another cyclic derivatives

4.1 the third order cyclic derivatives

From the theorem, we can find the third cyclic derivative to be from the equation $a^3 = 1$, the solutions are going to be denoted by $1, \omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ since one will result in e^x and is fully expected to be here because of Theorem 2, we are going to use ω

$$D^1 e^{\omega x} = \omega e^{\omega x} \qquad D^2 e^{\omega x} = \omega^2 e^{\omega x} \qquad D^3 e^{\omega x} = e^{\omega x}$$

We can call this function the third-order cyclic derivative, which comes between hyperbolic and trigonometric functions
we will name them $\sinh_3$,$\cosh_3$ and $\sinh_3$ II We can define them like this

$$D^\alpha \sinh_3(x) = \sinh_3\text{II}(x) \qquad \alpha \equiv 0 \pmod 3 \qquad D^\alpha \sinh_3\text{II}(x) = \cosh_3(x) \qquad \alpha \equiv 1 \pmod 3$$

$$D^\alpha \cosh_3(x) = \sinh_3(x) \qquad \alpha \equiv 2 \pmod 3$$

But this isn't the only way to define them, we can also define them with a series
First we find the Maclaurin series for $e^{\omega x}$

$$e^{\omega x} = e^0 + \omega e^0 x + \frac{\omega^2 e^0 x^2}{2!} + \frac{e^0 x^3}{3!} + ... = 1 + \omega x + \frac{\omega^2 x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{\omega^n x^n}{n!}$$

From this, we can divide them into three sums

$$(1 + \frac{x^3}{3!} + \frac{x^6}{6!} + ...) + \omega(x + \frac{x^4}{4!} + \frac{x^7}{7!} + ...) + \omega^2(x^2 + \frac{x^5}{5!} + \frac{x^8}{8!} + ...)$$

$$= \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \omega \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \omega^2 \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

We can see that the first sum can be differentiated 3 times before going back to the first state, which is also for all the other sums, but since sin and sinh all have x^{an+1}, we are going to make the first function to be the second sum, to keep the naming consistnt nothing more
So now we can define them to be

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \qquad \sinh_3\text{II}(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!} \qquad \cosh_3 = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$$

We can now define an equation that looks and acts like the Euler equation

$$e^{\omega x} = \cosh_3(x) + \omega \sinh_3(x) + \omega^2 \sinh_3\text{II}(x)$$

We can now try to find one for ω^2

$$e^{\omega^2 x} = \sum_{n=0}^{\infty} \frac{(\omega^2)^n x^n}{n!}$$

at $3n$ we get $\omega^{6n} = 1$ so it's $\cosh_3(x)$, at at $3n+1$ we get $\omega^{6n+2} = \omega^2$ so it's $\sinh_3(x)$ and at $3n$ we get $\omega^{6n+4} = \omega$ so it's $\sinh_3\text{II}(x)$

$$e^{\omega^2 x} = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \omega^2 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \omega \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

$$e^{\omega^2 x} = \cosh_3(x) + \omega^2 \sinh_3(x) + \omega \sinh_3\text{II}(x)$$

and for e^x it's quite simple

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

$$e^x = \cosh_3(x) + \sinh_3(x) + \sinh_3\text{II}(x)$$

like sin and sinh, we can try to find an exponent form for them
First, we begin by adding all of the equations so we have

$$e^x + e^{\omega x} + e^{\omega^2 x} = \cosh_3(x) + \sinh_3(x) + \sinh_3\text{II}(x)$$

$$+ \cosh_3(x) + \omega \sinh_3(x) + \omega^2 \sinh_3\text{II}(x) + \cosh_3(x) + \omega^2 \sinh_3(x) + \omega \sinh_3\text{II}(x)$$
$$= 3\cosh_3(x) + \sinh_3(x)(1 + \omega + \omega^2) + \sinh_3\text{II}(x)(1 + \omega^2 + \omega)$$

and since we know that $1 + \omega + \omega^2 = 0$

$$e^x + e^{\omega x} + e^{\omega^2 x} = 3\cosh_3(x) \qquad \cosh_3(x) = \frac{e^x + e^{\omega x} + e^{\omega^2 x}}{3}$$

Now we can define the others by differentiating

$$\sinh_3(x) = \frac{e^x + \omega e^{\omega x} + \omega^2 e^{\omega^2 x}}{3} \qquad \sinh_3\text{II}(x) = \frac{e^x + \omega^2 e^{\omega x} + \omega e^{\omega^2 x}}{3}$$

since these definitions are going to continue with us, we shall call the $e^{ax} = ...$ the **Euler form** and $f(x) = e^x + e^{ax}....$ the **exponentiation form**

4.2 Cyclic derivatives and prime numbers

To understand exactly what is meant by this, we need to see these

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \cosh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$$

As we can see, there is a pattern here, for every function that is k-th cyclic derivative we can see that its series is $\sum_{n=0}^{\infty} \frac{x^{kn}}{(kn)!}$
But this assumption shortly breaks as we can see for the fourth cyclic derivative

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

To understand more what I am talking about, we may take a look at the fifth cyclic derivative denoted by ϵ, to find the functions we start from the Maclaurin series

$$e^{\epsilon x} = 1 + \epsilon x + \frac{\epsilon^2 x^2}{2!} + \frac{\epsilon^3 x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{\epsilon^n x^n}{n!}$$

And we can find the five functions the same way we made it to the three cyclic derivative functions

$$\cosh_5(x) = \sum_{n=0}^{\infty} \frac{x^{5n}}{(5n)!} \qquad \sinh_5(x) = \sum_{n=0}^{\infty} \frac{x^{5n+1}}{(5n+1)!}$$

$$\sinh_5\mathrm{II}(x) = \sum_{n=0}^{\infty} \frac{x^{5n+2}}{(5n+2)!} \qquad \sinh_5\mathrm{III}(x) = \sum_{n=0}^{\infty} \frac{x^{5n+3}}{(5n+3)!} \qquad \sinh_5\mathrm{IV}(x) = \sum_{n=0}^{\infty} \frac{x^{5n+4}}{(5n+4)!}$$

And since the sum of roots of unity that are over 1 root is zero, we can do the same steps to find that

$$\cosh_5(x) = \frac{e^x + e^{\epsilon x} + e^{\epsilon^2 x} + e^{\epsilon^3 x} + e^{\epsilon^4 x}}{5}$$

and the other functions to be the derivatives of these functions
We can notice that the pattern continued for 5-th cyclic function
So what is the problem with trigonometric ones?
Well, we can see the expansion for $e^{ix}$to see what happens

$$e^{ix} = 1 + ix + -\frac{x^2}{2!} - \frac{ix^3}{3!} + ...$$

We can notice the pattern right there, it's the $-\frac{x^2}{2!}$, this term allows us to either:
- write the sum as four cyclic functions since we will make different additions being $\{1, i, -1, -i\}$
- We write sum as two different cyclic derivatives being $\{1. -1\}$ and $i\{1, -1\}$

In other words, the cyclic derivative family is compisable, reducible with simple algebra
and the reason for that is the **cyclic order**, when it's composite, we can see some roots return , like from order 2 we have $1, -1$ and order 4 we have $1, i, -1, -i$, the $1, -1$ here is back, same for six roots of unity $1, i_1, i_2, -1, i_3, i_4$ (note that i_a here isn't imaginary unit but the a-th root) and we can say that
let gk be all solutions for $a^k = 1$ and gn be for $a^n = 1$, as long as $\frac{k}{n} \in \mathbb{Z}^+, gk \subset$

gn
Thus, for any composite cyclic order, there exists more than 1 way to represent it
which means primes aren't here, so we can write the theorem

Theorem 4.1 (Prime cyclic functions Euler Form) $\forall p \in Primes, a^p = 1$ *There exists only one way to represent e^{ax} as a sum of all the cyclic order functions*

From this, we can say that

Theorem 4.2 (Prime cyclic functions exponentiation form Form) $\forall p \in Primes$, *$\sinh_p N(x)$ is a cyclic function; it can be written as this*

$$\sinh_p N(x) = \frac{e^x + a^N e^{ax} + a^{2N} e^{a^2 x} + ... + a^{pN} e^{a^p x}}{p} = \frac{1}{p} \sum_{n=0}^{p-1} a^{n(p-N)} e^{a^n x}$$

4.3 General Cyclic derivatives and Mittag-Leffler connection

As we can see from multiple series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sinh = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

We can see from all these examples that the factorial matches the exponent
Taking the D^α derivative of all gives us

$$D^z e^x = \sum_{n=0}^{\infty} \frac{x^{n-z}}{\Gamma(n-z+1)} \qquad \sinh = \sum_{n=0}^{\infty} \frac{x^{2n+1-z}}{\Gamma(2n+2-z)}$$

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1-z}}{\Gamma(3n+2-z)} \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-z}}{\Gamma(2n-z+1)}$$

if we let $2 - z = \beta$ and let $Cn = \alpha n$ where C is a constant we see that all of them get the shape

$$D^z f(x) = \sum_{n=0}^{\infty} \frac{x^{\alpha n+\beta}}{\Gamma(\alpha n + \beta)}$$

which is the Mittag-Leffler function, we can see that this happens in all of the functions we know.
That is, of course, except for $\cos(x)$ that will be discussed later

, but we need to generalize it, and we need to generalize the derivative cyclic order in Euler form

$$let\ a\ be\ any\ element\ from\ the\ group\ of\ soluitons\ for\ a^n = 1$$

$$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{a^kx^k}{k!}$$

From here, we can group the sums. Since there exist n roots of unity, we can say that there exist n terms

$$e^{ax} = (1 + \frac{x^n}{n!} + \frac{x^{2n}}{(2n)!} + ...) + (ax + \frac{a^nx^{n+1}}{(n+1)!} + \frac{a^{2n}x^{2n+1}}{(2n+1)!} + ...) + ...$$

$$= \sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!} + \sum_{k=0}^{\infty} \frac{a^{kn+1}x^{kn+1}}{(kn+1)!} + \sum_{k=0}^{\infty} \frac{a^{kn+2}x^{kn+2}}{(kn+2)!} + \sum_{k=0}^{\infty} \frac{a^{kn+3}x^{kn+3}}{(kn+3)!} + ...$$

and since $a^{kn+j} = a^j$, we can take it out as a common factor

$$= \sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!} + a\sum_{k=0}^{\infty} \frac{x^{kn+1}}{(kn+1)!} + a^2\sum_{k=0}^{\infty} \frac{x^{kn+2}}{(kn+2)!} + a^3\sum_{k=0}^{\infty} \frac{x^{kn+3}}{(kn+3)!} + ...$$

We are going to name the first one $\cosh_n(x)$ and the others $\sinh_n I(x)$ and $\sinh_n II(x)$ so on

$e^{ax} = \cosh_n(x) + a\sinh_n \mathrm{I}(x) + a^2\sinh_n \mathrm{II}(x) + a^3\sinh_n \mathrm{III}(x) + ...a^{n-1}\sinh_n \mathrm{N}(x)$

Now, if we consider $kn = \alpha n$ and $+C - \alpha = +\beta$ We see that all these functions fall for the Mittag-Leffler formula

$$D^z \cosh_n = \sum_{k=0}^{\infty} \frac{x^{\alpha n}}{\Gamma(\alpha n)} \qquad D^z \sinh_n \mathrm{II}... = \sum_{k=0}^{\infty} \frac{x^{\alpha n+\beta}}{\Gamma(\alpha n+\beta)}$$

From this, we can get the general cyclic derivative sum formula

Theorem 4.3 (Generalized Cyclic Derivative) *Let $f(x)$ be the j-th basis function of the D^n-cyclic system,*
$f(x) = \sum_{k=0}^{\infty} \frac{x^{kn+j}}{(kn+j)!}$*, The z-order derivative is given by* $D^z f(x) = \sum_{k=0}^{\infty} \frac{x^{kn+j-z}}{(kn+j-z)!}$

Theorem 4.4 (Mittag-Leffler Representation) *Every basis function of the D^n-cyclic system is a linear combination of* **n** *distinct Mittag-Leffler functions* $\mathbf{E}_{\mathbf{n},\beta}(\mathbf{x}^{\mathbf{n}})$ *corresponding to the n terms in its series representation.*

Theorem 4.5 (Generalized Euler form) *for every e^{ax} where a stratifies $a^n = 1$, e^{ax} can be written as*

$$e^{ax} = \sum_{j=0}^{n-1} a^j \sinh_n N(x)$$

that is of course if we consider $\cosh_n(x)$ to be the 0-th term

4.4 Generalized forms for cyclic derivatives

4.5 Odd And Even cyclic derivatives

4.6 Cyclic derivatives and algebraic equations

4.7 Mixture of equations