Fractional & Complex Derivatives II: Number systems

Faisal Khalid

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Abstract

This paper presents an independent exploration of number systems as orders based on the fractional and complex-order derivatives paper made by the same Author.

Note to Readers: This represents an independent continuation of the rediscovery of classical fractional calculus concepts to other number systems. I (The Author) present this work as a pedagogical exercise in mathematical exploration rather than novel research.

0.1 Background

1 Quaternions

1.1 Quaternion derivatives

since, unlike other number systems we used up to this point, quaternions are non-commutative, which makes it harder for us to work with, but still we can construct it from the main formula

We let $q = a + bi + cj + dk$ or $a + v$ such that $ij = -ji = k, jk = -kj = i$, $ki = -ik = j$ and $i^2 = j^2 = k^2 = -1$

We then substitute it into $D^\alpha x^n$ formula

$$D^q x^n = \frac{\Gamma(n+1)}{\Gamma(n-q+1)} x^{n-q}$$

As we can see, there are two problems

First: we can't raise up to a quaternion value in normal bases, in which we can transform its form to $x^n x^{-q}$ and raise it to Euler's number

$$x^{n-q} = x^n x^{-q} = x^n e^{-q\ln(x)} = x^n \sum_{k=0}^{\infty} \frac{(-1)^k (\ln(x))^k}{k!} q^k$$

We can use the e^{a+v} definition so that we can use the quaternion version of Euler's formula

$$x^n e^{-(a+v)\ln(x)} = x^n e^{-a\ln(x)} e^{-v\ln(x)} = x^{n-a} e^{-\ln(x)v} = x^{n-a} \left(\cos(\ln(x) \, ||v||) - \frac{v}{||v||} \sin(\ln(x) \, ||v||) \right)$$

(we it's supposed to be $-\ln(x)$ in both sides but using trigonometry identities $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, i skipped this step to save space) the second problem is $\Gamma(n - q + 1)$ the standard definition of $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}$ fails because it uses integration , which means we need to define integration for quaternions which as far as I know doesn't have a simple general solution, we could benefit of some algebraic manipulation but it's out of this paper field, instead we can use other detentions for $\Gamma(n)$ that doesn't involve integration we can use Euler's infinite product definition that is used mainly in this research

$$\Gamma(n) = \frac{1}{n} \prod_{k=1}^{\infty} \left[\frac{1}{1 + \frac{n}{k}} \left(1 + \frac{1}{k}\right)^n \right]$$

we then substitute $n = q$

$$\Gamma(q) = \frac{1}{q} \prod_{k=1}^{\infty} \left[\frac{1}{1 + \frac{q}{k}} \left(1 + \frac{1}{k}\right)^q \right]$$

again we use Euler base for the q in the power

$$\Gamma(q) = \frac{1}{q} \prod_{k=1}^{\infty} \left[\frac{1}{1 + \frac{q}{k}} e^{q(1 + \frac{1}{k})} \right] = \frac{1}{q} \prod_{k=1}^{\infty} \left[\frac{1}{1 + \frac{q}{k}} \left(1 + \frac{1}{k}\right)^a \left(\cos\left(||v||(1 + \frac{1}{k})\right) + \frac{v}{||v||} \sin\left(||v||(1 + \frac{1}{k})\right) \right) \right]$$

then we distribute and write $\frac{1}{q}$ as q^{-1} to use the identity $q^{-1} = \frac{\overline{q}}{||q||^2}$

$$\Gamma(q) = \frac{\overline{q}}{||q||^2} \prod_{k=1}^{\infty} \left[\frac{\left(1 + \frac{1}{k}\right)^a}{1 + \frac{q}{k}} \left(\cos\left(||v|| + \frac{||v||}{k}\right) + \frac{v}{||v||} \sin\left((||v|| + \frac{||v||}{k})\right) \right) \right]$$

we can write the expression in another way by unifying the denominators simplifying this expression

$$\Gamma(q) = \frac{\overline{q}}{||q||^2} \prod_{k=1}^{\infty} \left[\frac{\left(\frac{k}{k} + \frac{1}{k}\right)^a}{\frac{k}{k} + \frac{q}{k}} \left(\cos\left(\frac{k||v||}{k} + \frac{||v||}{k}\right) + \frac{v}{||v||} \sin\left(\frac{k||v||}{k} + \frac{||v||}{k}\right) \right) \right]$$

$$= \frac{\overline{q}}{||q||^2} \prod_{k=1}^{\infty} \left[\frac{\left(\frac{k+1}{k}\right)^a}{\frac{k+q}{k}} \left(\cos\left(\frac{||v||(k+1)}{k}\right) + \frac{v}{||v||} \sin\left(\frac{||v||(k+1)}{k}\right) \right) \right]$$

This is the main definition for calculating such values but we can also use Weierstrass's definition

$$\Gamma(n) = \frac{e^{-\gamma n}}{n} \prod_{k=1}^{\infty} (1 + \frac{n}{k})^{-1} e^{\frac{n}{k}} \qquad \Gamma(q) = \frac{e^{-\gamma q}}{q} \prod_{k=1}^{\infty} (1 + \frac{q}{k})^{-1} e^{\frac{q}{k}}$$

we treat $\frac{e^{-\gamma q}}{q}$ as $q^{-1} e^{-\gamma q}$ so we can use the identity $q^{-1} = \frac{\overline{q}}{||q||^2}$)

$$\Gamma(q) = \frac{\overline{q} e^a \left(\cos(\gamma ||v||) - \frac{v}{||v||} \sin(\gamma ||v||) \right)}{||q||^2} \prod \left(1 + \frac{q}{k}\right)^{-1} e^{\frac{a}{k}} \left(\cos(\frac{||v||}{k}) + \frac{v}{||v||} \sin(\frac{||v||}{k}) \right)$$

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we can also use an easier approach with conditions ; we use the $\sin(x)$ reflection formula for gamma function

$$\Gamma(1-n)\Gamma(n) = \frac{\pi}{\sin(\pi n)} \qquad \Gamma(q) = \frac{\pi}{\sin(\pi q)\Gamma(1-q)}$$

but this mean we need to define $\sin(\pi q)$
we will use the exponentiation form as it involves e^q

$$\sin(\pi q) = \frac{e^{i\pi q} - e^{-i\pi q}}{2i} = \frac{e^{i\pi a}\big(\cos(i\pi||v||) + \frac{v}{||v||}\sin(i\pi||v||)\big) - e^{-i\pi a}\big(\cos(i\pi||v||) - \frac{v}{||v||}\sin(i\pi||v||)\big)}{2i}$$

knowing that $\cos(ix) = \cosh(x), \sin(ix) = i\sinh(x)$ we can write the expression like this

$$\sin(\pi q) = \frac{e^{i\pi a}\big(\cosh(\pi||v||) + \frac{v}{||v||}i\sinh(\pi||v||)\big) - e^{-i\pi a}\big(\cosh(\pi||v||) - \frac{v}{||v||}i\sinh(\pi||v||)\big)}{2i}$$

With these definitions, we can finally write the formulas as

$$D^q x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-q)}x^{n-a}\big(\cos(\ln(x)\,||v||) - \frac{v}{||v||}\sin(\ln(x)\,||v||)\big)$$

the formula for the q-th derivative
this is a worked example, let $q = 1 + 1i - 2j + 0k$ and the function $f(x) = x^2$
First, we find the prerequisites; what we know is needed, these are
$||v|| = \sqrt{1^2 + (-2)^2 + 0^2} = \sqrt{5}, \frac{v}{||v||} = \frac{i-2j}{\sqrt{5}} = \frac{\sqrt{5}i - 2\sqrt{5}j}{5}$

$$D^q f(x) = \frac{\Gamma(2+1)}{\Gamma(2+1-(1+i-2j))}x^{2-1}\big(\cos(\ln(x)\sqrt{5}) - \frac{\sqrt{5}i - 2\sqrt{5}j}{5}\sin(\ln(x)\sqrt{5})\big)$$

$$= \frac{2}{\Gamma(2-i+2j)}x\big(\cos(\ln(x)\sqrt{5}) - \frac{\sqrt{5}i - 2\sqrt{5}j}{5}\sin(\ln(x)\sqrt{5})\big)$$

We can then calculate the gamma function using Weierstrass's definition, letting $2 - i + 2j$ be q_2
$\overline{q_2} = 2 + i - 2j, ||v_2|| = \sqrt{1^2 + 2^2} = \sqrt{5}, ||q_2|| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$
separating the two parts of the $\Gamma(q)$, we calculate the fraction first

$$\frac{(2+i-2j)e^2\big(\cos(\gamma\sqrt{5}) - \frac{-\sqrt{5}i+2\sqrt{5}j}{5}\sin(\gamma\sqrt{5})\big)}{9} = \frac{(14.778 + 7.389i - 14.778j)(0.276 - (\frac{-\sqrt{5}i+2\sqrt{5}j}{5})0.961)}{9}$$

$$\approx \frac{(14.778 + 7.389i - 14.778j)(0.276 + 0.430i - 0.860j)}{9} \approx \frac{-11.8067 + 8.3941i - 16.7888j}{9}$$

the first factor of is $\approx -1.3119 + 0.9327i - 1.8654j$, the second factor is

$$\prod_{k=1}^{\infty}\big(1 + \frac{2-i+2j}{k}\big)^{-1}e^{\frac{a}{k}}\big(\cos(\frac{\sqrt{5}}{k}) + \frac{2-i+2j}{\sqrt{5}}\sin(\frac{\sqrt{5}}{k})\big)$$

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We can calculate the first term as an approximation, as it converges rapidly; the other factor is $\approx -0.228 - 0.877i + 1.584j$

so we can calculate $\Gamma(q)$ to be $\approx 4.072 + 0.938i - 1.653j - 0.159k$, and then we apply $q^{-1} = \frac{\bar{q}}{||q||^2}$ and multiply it by 2 for the final shape of the derivative to be

$$D^q f(x) = (0.403 - 0.093i + 0.163j + 0.016k)x\left(\cos(\ln(x)\sqrt{5}) - \frac{\sqrt{5}i - 2\sqrt{5}j}{5}\sin(\ln(x)\sqrt{5})\right)$$

We can calculate some value for the function

$D^q f(0) = 0, D^q f(1) = (0.403 - 0.093i + 0.163j + 0.016k)\ x, D^q f(2) = -0.3584 - 0.3920i + 0.7133j - 0.0192k, D^q f(e) = -1.0772 - 0.2594i + 0.4814j - 0.0474k\ x$

from these results we can say that:

D^q works as a phase and rotation for the function itself, but unlike D^z it works on 4 dimensions instead of 2. We can also see that the change happening to the function is general, not only in the input direction, because at the input there was no k component, but at the final formula, there is

we can also see that for this exact function is an oscillation pattren in the components, as for the first few inputs, we can see that the k, j components goes up a little bit, then goes down slowly, but the j is faster than the k while the i component becomes low when k is high and vice versa

1.2 quaternion rules for other functions derivatives

This section is needed to save time on other sections.
deriavtive of e^x and e^{ax}
We can apply the original rules to them.

$$D^q e^x = e^x \qquad D^q e^{ax} = a^q e^{ax}$$

Of course, we continue for simplifications (here $q = b + v$ because a is used).

$$D^q e^{ax} = a^q e^{ax} = e^{q\ln(a)}e^{ax} = e^{b\ln(a)}e^{v\ln(a)}e^{ax}$$

$$= a^b e^{ax}\left(\cos(\ln(a)||v||) + \frac{v}{||v||}\sin(\ln(a)||v||)\right)$$

Of course, this one is bigger than the usual $a^\alpha e^{ax}$ because, simply unlike real or complex numbers, quaternions are harder to calculate as powers; essintly, both the expansion and composite forms are correct.
derivative of a^x

$$D^q a^x = a^x \ln(a)^q = a^x \ln(a)^b\left(\cos(\ln(\ln(a))||v||) + \frac{v}{||v||}\sin(\ln(\ln(a))||v||)\right)$$

2 Other orders

2.1 Split complex order

2.2 Duel numbers order

Dual numbers are denoted ϵ and defined as $\epsilon^2 = 0, \epsilon \neq 0$; in other words ,an infinitesimal shift. These numbers, in practice, are written in the form $a + b\epsilon$ and known for their property of automatic differentiation
A dual order-derivative can be obtained from this expression

$$D^\epsilon x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon)} x^{n-\epsilon}$$

and the term $x^{-\epsilon}$ can be written as $e^{-\epsilon \ln(x)}$, but unlike other number systems; $e^{b\epsilon} = 1 + b\epsilon$, so the term $x^{-\epsilon} = 1 - \epsilon \ln(x)$, so the ϵ-th derivative can be written like this

$$D^\epsilon x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon)} x^n (1 - \epsilon \ln(x)) = \frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon)} x^n - \epsilon x^n \ln(x)$$

Now, here things get pretty weird, since if we derivative the first expression, we get a derivative that is expected by the Index law, which means that the expansion should also work like this
First, we take the second derivative of the first expression

$$D^\epsilon f^\epsilon(x) = D^\epsilon \left(\frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon)} x^{n-\epsilon} \right) = \frac{\Gamma(n+1)}{\Gamma(n+1-2\epsilon)} x^{n-2\epsilon}$$

Then we differentiate the last expression

$$D^\epsilon f^\epsilon(x) = D^\epsilon \left(\frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon)} x^n - \epsilon x^n \ln(x) \right) = D^\epsilon \left(\frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon)} x^n \right) - D^\epsilon (\epsilon x^n \ln(x))$$

$$= \frac{\Gamma(n+1)^2}{\Gamma(n+1-\epsilon)^2} x^{n-\epsilon} - \sum_{k=0}^{\infty} \frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon-k+1)\Gamma(k+1)} D^{\epsilon-k}(x^n) D^k(\ln(x))$$

The second term can be simplified more

$$\sum_{k=0}^{\infty} \frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon-k+1)\Gamma(k+1)} \frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon+k)} (x^{n-\epsilon+k}) D^k(\ln(x))$$