Fractional & Complex Derivatives II: Number systems

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Abstract

This paper presents an independent exploration of number systems as orders based on the fractional and complex-order derivatives paper made by the same Author.

Note to Readers: This represents an independent continuation of the rediscovery of classical fractional calculus concepts to other number systems. I (The Author) present this work as a pedagogical exercise in mathematical exploration rather than novel research.

0.1 Background

1 Quaternions

1.1 Quaternion derivatives

since, unlike other number systems we used up to this point, quaternions are non-commutative, which makes it harder for us to work with, but still we can construct it from the main formula

We let $q = a + bi + cj + dk$ or $a + v$ such that $ij = -ji = k$,$jk = -kj = i$, $ki = -ik = j$ and $i^2 = j^2 = k^2 = -1$

We then substitute it into $D^\alpha x^n$ formula

$$D^q x^n = \frac{\Gamma(n+1)}{\Gamma(n-q+1)} x^{n-q}$$

As we can see, there are two problems

First: we can't raise up to a quaternion value in normal bases, in which we can transform its form to $x^n x^{-q}$ and raise it to Euler's number

$$x^{n-q} = x^n x^{-q} = x^n e^{-q\ln(x)} = x^n \sum_{k=0}^{\infty} \frac{(-1)^k (\ln(x))^k}{k!} q^k$$

We can use the e^{a+v} definition so that we can use the quaternion version of Euler's formula

$$x^n e^{-(a+v)\ln(x)} = x^n e^{-a\ln(x)} e^{-v\ln(x)} = x^{n-a} e^{-\ln(x)v} = x^{n-a}\big(\cos(\ln(x))\ ||v||) - \frac{v}{||v||} \sin(\ln(x)\ ||v||)\big)$$

(we it's supposed to be $-\ln(x)$ in both sides but using trigonometry identities $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, i skipped this step to save space) the second problem is $\Gamma(n-q+1)$ the standard definition of $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}$ fails because it uses integration , which means we need to define integration for quaternions which as far as I know doesn't have a simple general solution, we could benefit of some algebraic manipulation but it's out of this paper field, instead we can use other detentions for $\Gamma(n)$ that doesn't involve integration
we can use Euler's infinite product definition that is used mainly in this research

$$\Gamma(n) = \frac{1}{n}\prod_{k=1}^{\infty}\left[\frac{1}{1+\frac{n}{k}}\left(1+\frac{1}{k}\right)^n\right]$$

we then substitute $n = q$

$$\Gamma(q) = \frac{1}{q}\prod_{k=1}^{\infty}\left[\frac{1}{1+\frac{q}{k}}\left(1+\frac{1}{k}\right)^q\right]$$

again we use Euler base for the q in the power

$$\Gamma(q) = \frac{1}{q}\prod_{k=1}^{\infty}\left[\frac{1}{1+\frac{q}{k}}e^{q(1+\frac{1}{k})}\right] = \frac{1}{q}\prod_{k=1}^{\infty}\left[\frac{1}{1+\frac{q}{k}}\left(1+\frac{1}{k}\right)^a\left(\cos\left(||v||(1+\frac{1}{k})\right) + \frac{v}{||v||}\sin\left(||v||(1+\frac{1}{k})\right)\right)\right]$$

then we distribute and write $\frac{1}{q}$ as q^{-1} to use the identity $q^{-1} = \frac{\bar{q}}{||q||^2}$

$$\Gamma(q) = \frac{\bar{q}}{||q||^2}\prod_{k=1}^{\infty}\left[\frac{\left(1+\frac{1}{k}\right)^a}{1+\frac{q}{k}}\left(\cos\left(||v|| + \frac{||v||}{k}\right) + \frac{v}{||v||}\sin\left(\left(||v|| + \frac{||v||}{k}\right)\right]\right.$$

we can write the expression in another way by unifying the denominators simplifying this expression

$$\Gamma(q) = \frac{\bar{q}}{||q||^2}\prod_{k=1}^{\infty}\left[\frac{\left(\frac{k}{k}+\frac{1}{k}\right)^a}{\frac{k}{k}+\frac{q}{k}}\left(\cos\left(\frac{k||v||}{k} + \frac{||v||}{k}\right) + \frac{v}{||v||}\sin\left(\frac{k||v||}{k} + \frac{||v||}{k}\right)\right]\right.$$

$$= \frac{\bar{q}}{||q||^2}\prod_{k=1}^{\infty}\left[\frac{\left(\frac{k+1}{k}\right)^a}{\frac{k+q}{k}}\left(\cos\left(\frac{||v||(k+1)}{k}\right) + \frac{v}{||v||}\sin\left(\frac{||v||(k+1)}{k}\right)\right]\right.$$

this is the main definition for calculating such values ut we can also use Weierstrass's definition

$$\Gamma(n) = \frac{e^{-\gamma n}}{n}\prod_{k=1}^{\infty}(1+\frac{n}{k})^{-1}e^{\frac{n}{k}} \qquad \Gamma(q) = \frac{e^{-\gamma q}}{q}\prod_{k=1}^{\infty}(1+\frac{q}{k})^{-1}e^{\frac{q}{k}}$$

we treat $\frac{e^{-\gamma q}}{q}$ as $q^{-1}e^{-\gamma q}$ so we can use the identity $q^{-1} = \frac{\bar{q}}{||q||^2}$)

$$\Gamma(q) = \frac{\bar{q}e^a\left(\cos(\gamma||v||) - \frac{v}{||v||}\sin(\gamma||v||)\right)}{||q||^2}\prod\left(1+\frac{q}{k}\right)^{-1}e^{\frac{a}{k}}\left(\cos(\frac{||v||}{k})+\frac{v}{||v||}\sin(\frac{||v||}{k})\right)$$

we can also use an easier approach with conditions ; we use the $\sin(x)$ reflection formula for gamma function

$$\Gamma(1-n)\Gamma(n)=\frac{\pi}{\sin(\pi n)} \qquad \Gamma(q)=\frac{\pi}{\sin{(\pi q)}\Gamma(1-q)}$$

but this mean we need to define $\sin(\pi q)$
we will use the exponentiation form as it involves e^q

$$\sin(\pi q)=\frac{e^{i\pi q}-e^{-i\pi q}}{2i}=\frac{e^{i\pi a}\big(\cos(i\pi||v||)+\frac{v}{||v||}\sin(i\pi||v||)\big)-e^{-i\pi a}\big(\cos(i\pi||v||)-\frac{v}{||v||}\sin(i\pi||v||)\big)}{2i}$$

knowing that $\cos(ix)=\cosh(x), \sin(ix)=i\sinh(x)$ we can write the expression like this

$$\sin(\pi q)=\frac{e^{i\pi a}\big(\cosh(\pi||v||)+\frac{v}{||v||}i\sinh(\pi||v||)\big)-e^{-i\pi a}\big(\cosh(\pi||v||)-\frac{v}{||v||}i\sinh(\pi||v||)\big)}{2i}$$

With these definitions, we can finally write the formulas as

$$D^qx^n=\frac{\Gamma(n+1)}{\Gamma(n+1-q)}x^{n-a}\big(\cos(\ln(x))\ ||v||)-\frac{v}{||v||}\sin(\ln(x)\ ||v||)\big)$$

the formula for the q-th derivative

1.2 quaternion rules for other functions derivatives

This section is needed to save time on other sections.
deriavtive of e^x and e^{ax}
We can apply the original rules to them.

$$D^qe^x=e^x \qquad D^qe^{ax}=a^qe^{ax}$$

Of course, we continue for simplifications (here $q=b+v$ because a is used).

$$D^qe^{ax}=a^qe^{ax}=e^{q\ln(a)}e^{ax}=e^{b\ln(a)}e^{v\ln(a)}e^{ax}$$
$$=a^be^{ax}\big(\cos(\ln(a)||v||)+\frac{v}{||v||}\sin(\ln(a)||v||)\big)$$

Of course, this one is bigger than the usual $a^\alpha e^{ax}$ because, simply unlike real or complex numbers, quaternions are harder to calculate as powers; essintly, both the expansion and composite forms are correct.
derivative of a^x

$$D^qa^x=a^x\ln(a)^q=a^x\ln(a)^b\big(\cos(\ln(\ln(a))||v||)+\frac{v}{||v||}\sin(\ln(\ln(a))||v||)\big)$$

2 Other orders

2.1 Split complex order

2.2 Duel numbers order

Dual numbers are denoted ϵ and defined as $\epsilon^2 = 0, \epsilon \neq 0$; in other words ,an infinitesimal shift. These numbers, in practice, are written in the form $a + b\epsilon$ and known for their property of automatic differentiation
A dual order-derivative can be obtained from this expression

$$D^{\epsilon}x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon)}x^{n-\epsilon}$$

and the term $x^{-\epsilon}$ can be written as $e^{-\epsilon\ln(x)}$, but unlike other number systems; $e^{b\epsilon} = 1 + b\epsilon$, so the term $x^{-\epsilon} = 1 - \epsilon\ln(x)$, so the ϵ-th derivative can be written like this

$$D^{\epsilon}x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon)}x^n(1-\epsilon\ln(x)) = \frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon)}x^n - \epsilon x^n\ln(x)$$

Now, here things get pretty weird, since if we derivative the first expression, we get a derivative that is expected by the Index law, which means that the expansion should also work like this
First, we take the second derivative of the first expression

$$D^{\epsilon}f^{\epsilon}(x) = D^{\epsilon}(\frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon)}x^{n-\epsilon}) = \frac{\Gamma(n+1)}{\Gamma(n+1-2\epsilon)}x^{n-2\epsilon}$$

Then we differentiate the last expression

$$D^{\epsilon}f^{\epsilon}(x) = D^{\epsilon}(\frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon)}x^n - \epsilon x^n\ln(x)) = D^{\epsilon}(\frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon)}x^n) - D^{\epsilon}(\epsilon x^n\ln(x))$$

$$= \frac{\Gamma(n+1)^2}{\Gamma(n+1-\epsilon)^2}x^{n-\epsilon} - \sum_{k=0}^{\infty}\frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon-k+1)\Gamma(k+1)}D^{\epsilon-k}(x^n)D^k(\ln(x))$$

The second term can be simplified more

$$\sum_{k=0}^{\infty}\frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon-k+1)\Gamma(k+1)}\frac{\Gamma(n+1)}{\Gamma(n+1-\epsilon+k)}(x^{n-\epsilon+k})D^k(\ln(x))$$