Complex-Order and Fractional Derivatives: A First Exploration II

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1 Introduction

2 The Field of orders

before we get through we need to make some claims since in this section we will be working with multiple mathematical sets.
this is the normal definition of D^α

$$D^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

but we need a generalized version
since $x^{n-\alpha}$ may not always be defined directly for groups like $\mathbb{H}$ we can define it like this $\exp((n-\alpha)\ln(x))$ for any definition of the exp() function, which allows us to use the definition needed for any case weather it's $\lim_{n\to\infty}(1+\frac{x}{n})^n$ or the infinite summation definition $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ or any definition
second we can define $\Gamma(x)$ using Euler's product or Weierstrass product for any set of numbers that doesn't have any nice form of integral like $\mathbb{H}$ although the Weierstrass definition is preferred since it has e^x in it which is easier to deal with in most cases.
so we can write the defenition of $D^\alpha x^n$ in the most abstract way like this

$$D^\alpha x^n = \frac{\frac{e^{-\gamma(n+1)}}{n+1}\prod_{k=1}^{\infty}(1+\frac{n+1}{k})^{-1} e^{\frac{n+1}{k}}}{\frac{e^{-\gamma(n+1-\alpha)}}{n+1-\alpha}\prod_{j=1}^{\infty}(1+\frac{n+1-\alpha}{j})^{-1} e^{\frac{n+1-\alpha}{j}}} e^{(n-\alpha)\ln(x)}$$

we can take both the $\frac{e^{-\gamma(n+1)}}{n+1}$ over

$$\frac{e^{-\gamma(n+1-\alpha)}}{n+1-\alpha}$$

out to give us this expression

$$\frac{n+1-\alpha}{e^{-\gamma(n+1-\alpha)}} \times \frac{e^{-\gamma(n+1)}}{n+1} = \frac{n+1-\alpha}{e^{\gamma\alpha}(n+1)} = \frac{n+1}{e^{\gamma\alpha}(n+1)} + \frac{-\alpha}{e^{\gamma\alpha}(n+1)} = e^{-\gamma\alpha}(1-\frac{\alpha}{n+1})$$

for the infinite product; since we have two infinite products with the same index we can combine them

$$\frac{\prod_{k=1}^{\infty}(1+\frac{n+1}{k})^{-1}e^{\frac{n+1}{k}}}{\prod_{j=1}^{\infty}(1+\frac{n+1-\alpha}{j})^{-1}e^{\frac{n+1-\alpha}{j}}} = \prod_{k=1}^{\infty}\frac{(1+\frac{n+1}{k})^{-1}e^{\frac{n+1}{k}}}{(1+\frac{n+1-\alpha}{k})^{-1}e^{\frac{n+1-\alpha}{k}}} = \prod_{k=1}^{\infty} e^{\frac{n+1}{k}-\frac{n+1-\alpha}{k}}\frac{1+\frac{n+1-\alpha}{k}}{1+\frac{n+1}{k}}$$

with the exponent being $e^{\frac{\alpha}{k}}$ at the end we get this expression

$$D^{\alpha}x^{n} = e^{(n-\alpha)\ln(x)-\gamma\alpha}(1-\frac{\alpha}{n+1})\prod_{k=1}^{\infty} e^{\frac{\alpha}{k}}\frac{1+\frac{n+1-\alpha}{k}}{1+\frac{n+1}{k}}$$

This definition proves to us that this framework can wrok with any set of numbers where multiplication and addition are defined. Currently, this is only here for numerical and computational work, but moving forward, we will still use the symbol definition of $\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha}$

2.1 Generalized Index Law and D^z identities

The index law has been proven to work on the operator before, but the proof held for $\alpha \in \mathbb{R}$; therefore, a generalized one is needed for other sets like $\mathbb{C}, \mathbb{H}, ...$

Theorem 2.1 (The Index Law Theorem) *Let $(G,+)$ be a semi group , for any element $\alpha, \beta \in G$*

$$D^{\beta}D^{\alpha} = D^{\beta+\alpha}$$

Proof:

$$D^{\alpha}x^{n} = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha} \qquad D^{\beta}D^{\alpha}x^{n} = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}\frac{\Gamma(n+1-\alpha)}{\Gamma(n+1-\alpha-\beta)}x^{n-\alpha-\beta}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha-\beta)}x^{n-\alpha-\beta}$$

and let this be 1

$$D^{\alpha+\beta}x^{n} = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha-\beta)}x^{n-\alpha-\beta}$$

and let this be 2
since 1 and 2 are equal; hence

$$D^{\beta}D^{\alpha} = D^{\alpha+\beta}$$

this works for when $\alpha, \beta \in \mathbb{R}, \mathbb{C}$ and extend to $\mathbb{H}$ with choice of logarithmic branks and multiplication order.
We define D^{α} initially on monomials and extend linearly; thus, operator equality is determined by its action on x^n We can also turn this observation into an axiom We axiomatize that any family $D^g, g \in G$ where $(G,+)$ satisfying $D^{g_1}D^{g_2} = D^{g_1+g_2}$, $D^0 = I$ where I is the identity operator in most cases $f(x)$ with

linearity ; is a generalized derivative semi group
and it's proven for $\mathbb{R}, \mathbb{C}, \mathbb{H}$ with the Γ formula that such families satisfy these conditions naturally
From here, we can assume some laws are true based on these axioms

- **Commutativity:** $D^{\alpha} D^{\beta} = D^{\beta} D^{\alpha}$
  since $\alpha, \beta \in G$ where $(G, +)$ and commutativity of addition in G exist (G is also known as an abelian group), then

  $$D^{\alpha+\beta} = D^{\beta+\alpha}$$

  Hence, this is true

- **Associativity:** $(D^{\alpha} D^{\beta}) D^{\gamma} = D^{\alpha}(D^{\beta} D^{\gamma})$
  since $\alpha, \beta, \gamma \in G$ where $(G, +)$ then

  $$D^{\alpha+(\beta+\gamma)} = D^{(\alpha+\beta)+\gamma}$$

  Hence, this is true

- **Multiplication law for integer:** ${}^{n}D^{\alpha} = D^{\alpha n}$
  since $\alpha \in G$ where $(G, +)$ and $n \in \mathbb{Z}$, we can treat n as a scaler or as a number depending on G then

  $$\underbrace{D^{\alpha} D^{\alpha} .... D^{\alpha}}_{n \text{ times}} = D^{\alpha+\alpha+....+\alpha} = D^{\alpha n} =^{n} D^{\alpha}$$

  Hence, this is true

- **Inverse order:** $D^{-\alpha} D^{\alpha} = D^{0} = I$ exists in the semi group of orders
  since $\alpha \in G$ where $(G, +)$ is a group(which makes it has Inverse elements) form a group of derivative orders where D^{0} must exist where

  $$D^{\alpha} D^{0} = D^{\alpha} I = D^{\alpha}$$

  Then an inverse must exist in the same group that assures the result of D^{0}

  $$D^{\alpha} D^{-\alpha} = D^{\alpha-\alpha} = D^{0} = I$$

  Hence, this is True

This subsection serves more as an extension of algebraic groups/sets to what was already defined with algebraic manipulation in the first paper, which allows us to extend this operator on the abstract level to any group/set that fulfills the conditions without a proof each time

2.2 The generalized multiplication law

2.3 D^z analytic continuation

We know that $D^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}$ is the main formula for the α-th derivative where $\alpha \in G$ where G is at least defined as a semi group $(G, +)$ known as the general derivatives order semi-group, which includes $\mathbb{R}, \mathbb{C}$ that we are going to work with in this subsection.
and we have derived a formula for D^z that doesn't include any integrals; built on basic arithmetic and constants

$$D^\alpha x^n = e^{(n-\alpha)\ln(x) - \gamma\alpha}(1 - \frac{\alpha}{n+1}) \prod_{k=1}^{\infty} e^{\frac{\alpha}{k}} \frac{1 + \frac{n+1-\alpha}{k}}{1 + \frac{n+1}{k}}$$

where γ is the Euler-Mascheroni constant
We also know that overdifferentiating the formula $D^z x^n$ results in a gamma pole at the denominator, leading to the 0 result whenever it's a gamma of a non-positive integer; for example

$$D^{n+1} x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-n-1)} x^{n-n-1} = \frac{\Gamma(n+1)}{\Gamma(0)} x^{-1} = 0$$

But the same doesn't apply to the product formula

$$D^{n+1} x^n = e^{(n-n-1)\ln(x) - \gamma(n+1)}(1 - \frac{n+1}{n+1}) \prod_{k=1}^{\infty} e^{\frac{n+1}{k}} \frac{1 + \frac{n+1-n-1}{k}}{1 + \frac{n+1}{k}}$$

$$= e^{-1\ln(x) - \gamma(n+1)}(1 - 1) \prod_{k=1}^{\infty} e^{\frac{n+1}{k}} \frac{1 + \frac{0}{k}}{1 + \frac{n+1}{k}} = x^{-1} e^{-\gamma(n+1)}(0) = 0$$

What happens here is that instead of a gamma pole it's a simple multiplication by zero, and the result is correct as $D^n x^n = \frac{\Gamma(n+1)}{\Gamma(1)}$ the next integer derivative is simply $D^1 C$ which is 0
Not only that, but this happens to any n from any semi-group.
we can also skip this by making $\alpha = n + 2$, and this is supposed to return 0 by all means since 0 is a constant and there is supposed to be a gamma pole coming on the way; but here there isn't any gamma function, and we aren't starting from zero

$$D^{n+2} x^n = e^{(n-n-2)\ln(x) - \gamma(n+2)}(1 - \frac{n+2}{n+1}) \prod_{k=1}^{\infty} e^{\frac{n+2}{k}} \frac{1 + \frac{n+1-n-2}{k}}{1 + \frac{n+1}{k}}$$

$$= e^{(-2)\ln(x) - \gamma(n+2)}(1 - \frac{n+1+1}{n+1}) \prod_{k=1}^{\infty} e^{\frac{n+2}{k}} \frac{1 + \frac{-1}{k}}{1 + \frac{n+1}{k}} = x^{-2} e^{-\gamma(n+2)}(-\frac{1}{n+1}) \prod_{k=1}^{\infty} e^{\frac{n+2}{k}} \frac{1 - k^{-1}}{1 + \frac{n+1}{k}}$$

This new function looks weird and unexpected, since in the original definition, this was a gamma pole, but in this function, what happened is n changed from

the power on the x to be the power on e, meaning whichever function x^n we choose, it won't make a change on x but rather on e.
This is sort of similar to a transformtive function, changing the differentiation main independence, like $\sin^{-1}(x)$, is depends mainly on the inside of it when differentiation, but its first derivative $\frac{1}{\sqrt{1-x^2}}$ depends mainly on the power of x
We can test this for x^2 as an example

$$D^4x^2 = e^{(2-4)\ln(x)-4\gamma}(1-\frac{4}{3})\prod_{k=1}^{\infty} e^{\frac{4}{k}}\frac{1+\frac{-1}{k}}{1+\frac{3}{k}} = x^{-2}e^{-4\gamma}(-\frac{1}{3})\prod_{k=1}^{\infty} e^{\frac{4}{k}}\frac{1-k^{-1}}{1+\frac{3}{k}}$$

which is the same value if we used the $\alpha = n+2$ formula.
Continuing this example, the product value turns out to be 0, this is because in the first term $\frac{1-1}{1-3} = 0$, and this is actually a problem that happens in every series where $\alpha = n+2$, as there will always be $1-1$ term in the product
For this function, we can try and regularize it by taking out the factor that is the cause of this, that is the first one.
we let $r = \prod_{k=1}^{1} 1+\frac{3-\alpha}{k} = \prod_{k=1}^{1} 1+\frac{-1}{k} = 1-1 = 0$, then taking the limit as $\alpha \to 4$we get this regularized expression

$$D^4x^2 = \lim_{\alpha\to 4}\frac{D^\alpha x^2}{r} = x^{-2}e^{-4\gamma}(-\frac{1}{3})\lim_{\alpha\to 4}\prod_{k=1}^{\infty} e^{\frac{4}{k}}\frac{1-\frac{3-\alpha}{k}}{1+\frac{3}{k}}$$

this allows us to work our way out, the product converges very slowly to 4.423, to the final result being

$$D^4x^2 = \frac{-0.147}{x^2}$$

doing the same for D^5 we get $\frac{0.024}{x^2}$

2.4 Taylor series memory

3 More to the Imagination

3.1 The imaginary derivative order polar form

from what we know about D^z we can say that
D^1 and D^{-1} represents a half turn on the $D(i)$ plane
D^i and D^{-i} represents a quarter turn on the $D(i)$ plane
So if we want to get the generalized form of this, we need to use the polar form instead of the Cartesian form, so

$$D^i f(x) = D^{e^{\frac{i\pi}{2}}} f(x) \qquad D^{-1}f(x) = D^{e^{i\pi}} f(x)$$

and generally speaking, for the unit circle around D^0(the function itself)

$$D^z f(x) = D^{e^{i\theta}} f(x)$$

And to make this formula for the whole complex plane, we write it like this

$$D^z f(x) = D^{re^{i\theta}} f(x)$$

where r is the length from D^0 to the wanted function

3.2 exploring more about Imaginary derivatives

3.3 complex functions with D^z

3.4 Complex derivative of a complex function is complex

3.5 Infinite series and complex derivatives

4 Multi variable D^z

5 functional order $D^{f(\alpha)}$

6 Yes, the whole space is a matrix system

As we have explored in the first paper, the possibility of matrix order derivative with this simple formula

$$D^A = \frac{\Gamma(n+1)}{\Gamma(n-A+1)} x^{n-A} = \frac{\Gamma(n+1)}{\Gamma(n-A+1)} x^n e^{-A\ln(x)} = \frac{\Gamma(n+1)}{\Gamma(n-A+1)} x^n \sum_{k=0}^{\infty} \frac{(-1)^k \ln(x)^k}{k!} A^k$$

But let's refine it a little to make it fully clear by changing all the scalers to matrices

$$D^A = \frac{\Gamma((n+1)I)}{\Gamma((n+1)I - A)} x^{nI-A} = \frac{\Gamma((n+1)I)}{\Gamma((n+1)I - A)} x^{nI} e^{-A\ln(x)}$$

$$= \frac{\Gamma((n+1)I)}{\Gamma((n+1)I - A)} x^{nI} \sum_{k=0}^{\infty} \frac{(-1)^k \ln(x)^k}{k!} A^k$$

For now, to define $\Gamma(A)$ we are going to use eigenvalues, although Cuchy Integrals are better, but for the sake of intuition and understanding, we are going to discuss them in later sections

This means that the formula only works for **square matrices** for now

We can see a simple example with the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; the rotation matrix

for a function $f(x) = x^2$ we are going to take the A-th derivative like this

$$D^A f(x) = \frac{\Gamma(3)}{\Gamma(\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})} x^2 \sum_{n=0}^{\infty} \frac{(-1)^n \ln(x)^n}{n!} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n$$

The matrix A returns back at the $4n$-th powers, and the Gamma expression is $\Gamma(\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix})$, but to continue, we need to find the eigenvalues of this matrix

$$\det\left(\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\left(\begin{pmatrix} 3-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix}\right) \qquad (3-\lambda)(3-\lambda)+1 = 0$$

$$(3-\lambda)^2 + 1 = 0 \implies 3 - \lambda = \pm i \qquad \lambda = 3-i, 3+i$$

This equation doesn't have any real roots but only complex roots that are $-3-i$ and $-3+i$
We are going to find the eigenvectors for all roots

$$((3I-A)-\lambda_1 I)v_1 = 0, \begin{pmatrix} 3-(3-i) & 1 \\ -1 & 3-(3-i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

then $ix + y = 0$ so $y = -ix$, let $x = 1$, $y = -i$, the first vector $= v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

doing the same for the other root, we get the second vector $v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

We are going to let the Model matrix $P = [v_1|v_2] = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$ and the diagonal matrix $\begin{pmatrix} (3-i) & 0 \\ 0 & (3+i) \end{pmatrix}$ so we can calculate $f(3I-A) = Pf(\Lambda)P^{-1}$
we can calculate P^{-1} using the inverse of 2×2 matrix formula

$$A^{-1} = \frac{1}{\det(A)} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \implies P^{-1} = \frac{1}{\det(P)}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}$$

and the determinant of P is $(1)(i) - (1)(-i) = i + i = 2i$, therefore we have the expression $P^{-1} = \frac{1}{2i}...$, for simplification we multiply the scaler by $\frac{-i}{-i}$

$$P^{-1} = \frac{-i}{2}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (-i)i & (-i)(-1) \\ (-i)i & (-i)1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

Since the original gamma function is in the denominator, we can calculate $\Gamma(3I-A)^{-1}$ to get a direct result. We can so $\Gamma(3I-A) = P\Gamma(\Lambda)^{-1}P^{-1} =$
$$\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} \frac{1}{\Gamma(3-i)} & 0 \\ 0 & \frac{1}{\Gamma(3+i)} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} 0.354+0.492i & 0 \\ 0 & 0.354-0.492i \end{pmatrix}\right]\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0.354+0.492i & 0.354-0.492i \\ -i(0.354+0.492i) & i(0.354-0.492i) \end{pmatrix}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0.354+0.492i & 0.354-0.492i \\ 0.492-0.354i & 0.492+0.354i \end{pmatrix}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0.354+0.492i+0.354-0.492i & i(0.354+0.492i-0.354+0.492i) \\ -i(0.354+0.492i-0.354-0.492i) & 0.354+0.492i+0.354-0.492i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0.708 & -0.984 \\ 0.984 & 0.708 \end{pmatrix}$$

which at the end results $\begin{pmatrix} 0.354 & -0.492 \\ 0.492 & 0.354 \end{pmatrix}$, substituting it back to the original derivative formula we get

$$D^A f(x) = 2 \begin{pmatrix} 0.354 & -0.492 \\ 0.492 & 0.354 \end{pmatrix} x^2 \sum_{n=0}^{\infty} \frac{(-1)^n \ln(x)^n}{n!} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n$$

Note that the infinite sum acts like the imaginary unit (because it's the matrix representation for it), in which we can replace this sum with $\begin{pmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ -\sin(\ln(x)) & \cos(\ln(x)) \end{pmatrix}$
We will test this for some values

$$D^A f(x)\big|_e = \begin{pmatrix} 8.941 & 0.473 \\ -0.473 & 8.941 \end{pmatrix} \qquad D^A f(x)\big|_1 = \begin{pmatrix} 0.708 & -0.984 \\ 0.984 & 0.708 \end{pmatrix}$$

$$D^A f(x)\big|_3 = \begin{pmatrix} 10.785 & 1.647 \\ -1.647 & 10.785 \end{pmatrix}$$

We can see that first of all, and as expected, the matrix order changes the type of the function from a scalar to 2×2 matrix in this example
We can also see that in this example that the main diagonal of the matrix is equal in sign and value to a quicker increasing value rather than the other diagonal
We can also see that for some value of x , the second diagonal reaches zero and changes sign of its elements
This is supposed to be similar to $D^i x^2$ since the matrix we used is isomorphic to the imaginary unit; we know that they have some sort of relationship, nonetheless

6.1 exploring more about Matrix derivatives

6.2 Term-wise matrix order

7 To the third dimension

7.1 Modeler differentiation

7.2 Set order derivatives

7.3 group order derivatives

7.4 Vector order derivatives

8 Structural series analysis

Up till this point in this research series, we have always been using series to identify derivatives, which isn't bad by itself, but the problem arises; these are good for calculation, but we can't get any information about the function from

the series, we can't get identities or similarities or any information at all.
This makes the identity of the function hidden for D^z
And in this section, we will analyze some series and try to make them informative by any means

8.1 The similarities between functions

Functions can be expressed using summation series, which makes them look easy to identify and see the secrets between them.
For example, we have a function $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, this function is responsible for cyclic derivatives and have some key properties like it's natural logarithm is 1 which makes it very good for hyper operations that changes addition and multiplication to powers
On the other side, we have a function that has no connection to e^x, that is $\frac{1}{1-x}$ with its series being $1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n$
These two functions are very similar, although they aren't connected, we can use the D^z operator on both of them to understand more about them

$$D^\alpha e^x = \sum_{n=0}^{\infty} \frac{x^{n-\alpha}}{\Gamma(n-\alpha+1)} \qquad\qquad D^\alpha \frac{1}{1-x} = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha}$$

We can see that both functions are very similar, except that e^x doesn't have a gamma term in the denominator
We can try to find a function in the middle of them constructively
The reason e^x doesn't have a $\Gamma(n+1)$ is that it cancels out with the denominator, so if them were both different slightly, we could see more structure, in which we are asking for this coefficient $\frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-\alpha+1)}$ or something similar of it, and we can find this in ${}^nC_m$, which can be found in the series of $(1+x)^m = \sum_{n=0}^{\infty} {}^mC_n x^n$

$$D^\alpha (1+x)^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} (1+x)^{m-\alpha} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} \sum_{n=0}^{\infty} {}^nC_{m-\alpha} x^n =$$

$$\sum_{n=0}^{\infty} \frac{\Gamma(m-\alpha+1)}{\Gamma(n+1)\Gamma(m-\alpha-n+1)} \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^n = \sum_{n=0}^{\infty} \frac{\Gamma(m+1)}{\Gamma(n+1)\Gamma(m-\alpha-n+1)} x^n$$

This function acts like e^x only when $m = n$ as they will cancel it and leave for use $\frac{x^n}{\Gamma(\alpha+1)}$ and when m approaches ∞ the fraction diverges leaving only x^n acting like $\frac{1}{1-x}$

8.2 construction from functions

8.3 convergence and divergence

There are some functions

8.4 product and fractal representation

9 variable order derivatives

since D^α operator extends to linear input, which is α, we can go further and extend it to different inputs as functions

9.1 function order derivatives

The simple function derivative is a function of another variable rather than the differentiated function variable,. The fundamental function is the linear function of alpha $D^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}$, of course, this function is what defines what type of differentiation we are doing, in other words in which direction we are going, as the main linear function α is for derivatives, when multiplied by negatives gives us $-\alpha$ which is for integrals. There exists αi, which represents imaginary differentiation. Then, we have $\alpha + \beta i$ for complex differentiation, which is a simple multivariable linear function.
So we know functional orders exist and change like the function depending on its type, and we can write their general formula like this

$$D^{g(\alpha)} x^n = \frac{\Gamma(n+1)}{\Gamma(n-g(\alpha)+1)} x^{n-g(\alpha)}$$

We can try a simple example on this formula

$$g(\alpha) = \alpha^2, f(x) = x^2 \qquad D^{g(\alpha)} f(x) = \frac{\Gamma(2+1)}{\Gamma(2-\alpha^2+1)} x^{2-\alpha^2} = \frac{2}{\Gamma(3-\alpha^2)} x^{2-\alpha^2}$$

if we want to see when this function hits gamma pole we can put it in test

$$3 - \alpha^2 \leq 0 \qquad a^2 \geq 3 \qquad \sqrt{3} \geq a \geq -\sqrt{3} \quad \{3 - a^2 \in \mathbb{Z}\}$$

so this function has gamma poles when $\alpha \in \{\pm\sqrt{3}, \pm 2, \pm\sqrt{5}, \pm\sqrt{6} ....\}$, other than that, there will be no gamma pole, so the expression is defined like this

$$D^{g(x)} f(x) = \frac{2}{\Gamma(\alpha^2 - 3)} x^{2-\alpha^2} \quad \{\alpha \in [-\sqrt{3}, \sqrt{3}]\}$$

We can see that in general, the function isn't defined only when

$$n + 1 - g(\alpha) \leq 0 \qquad n + 1 \leq g(\alpha) \quad \{n + 1 - g(\alpha) \in \mathbb{Z}\}$$

another example that we can test on

$$g(\alpha) = \sin(\alpha), f(\alpha) = x^2 \quad D^{g(\alpha)} f(\alpha) = \frac{2}{\Gamma(3 - \sin(\alpha))} x^{2-\sin(\alpha)}$$

This function is actually defined for all $\alpha \in \mathbb{R}$ as the range of this function is between $[-1, 1]$, the expression $D^{g(\alpha)} f(\alpha)$ ranges between the first derivative and the second derivative

9.2 self variable order derivatives

We can define a simple self variable order derivative as

$$D^x x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-x)} x^{n-x}$$

However, we need to refine this expression further.
first we write x^{n-x} as $x^n e^{-x\ln(x)}$ for simpler calculations.
now we can use the series expansion for $e^{-x\ln(x)} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k (\ln(x))^k}{k!}$ then we substitute it back

$$D^x x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-x)} x^n \sum_{k=0}^{\infty} \frac{(-1)^k x^k (\ln(x))^k}{k!}$$

Analyzing this function, we can see that the function reaches it's peaks when $n > x$, as then the Gamma function would be for positives, and the power would be positive, to find where it hits zer,o we can solve for the equation

$$\frac{\Gamma(n+1)}{\Gamma(n+1-x)} x^{n-x} = 0$$

But looking at the function closely, there can only be one reason for it to happen. for any power of x it can never reach zero as there is no number that has a changing power will be zero, and when the numerator is zero of gamma poles, the function itself is non non-differentiable in the first place, leaving only the denominator, it can only reach zero when the gamma function hits zero, so we solve for the equation

$$n + 1 - x \leq 0 \qquad n + 1 \leq x \quad (n + 1 - x \in \mathbb{Z})$$

only when x is bigger than or equal to $n+1$ and the result is an integer we get a gamma pole which leads to an infinity in the denominator leading to a zero , these are the zeros of the derivative

9.3 Nested differentiation

10 Integrals May I present

10.1 Where is the constant of integration?

As we saw from this whole paper, we can use D^z to differentiate and integrate Real, imaginary, or matrix, yet something seems to be missing.
The constant of integration, something so fundamental in calculus, yet seems to be missing
But here in fractional calculus, things are different; here we have a lot of "Differentiating" forward or backward, 2D or 3D, and a lot of directions that make it hard even to put it somewhere But at first, we need to understand why, not

in standard terms, but in this research term
The constant of Integration is something that arises when integrating, that is, because under differentiation, we lose constants and variables according to the order
That is because of the definition of D^z, which transforms functions from one state to another.
take, for example, $f(x) = x^5 + 14$, from a prescriptive of stranded calculus, when we differentiate, constants are gone because they don't have any rate of change, the answer is simply

$$D^1 f(x) = 5x^4$$

But as we discussed, the rate of change isn't a core in the derivative but a side effect. To see the full picture, we need to use the whole Gamma formula with the knowledge that $C = Cx^0$

$$D^1 f(x) = \frac{\Gamma(6)}{\Gamma(5)} x^{5-1} + 14\frac{\Gamma(1)}{\Gamma(0)} x^{0-1} = 5x^4 + \frac{14}{\Gamma(0)} x^{-1}$$

We can see that the denominator has a Gamma pole in it, which leads the whole term to be 0, taking the constant with it
In other words, differentiation, as we discussed, loses memory; the more it differentiates, the more it's gone
Integration is the exact opposite; it retains memory, and we can see that too from the standard calculus approach

$$D^{-1}(x^2) = \frac{x^3}{3} + C$$

We see that here the constant of integration is dependent on logic, the simple logic is that if we differentiate, we lose constants, then integration, the opposite of it must return them
Let's see what happens from the fractional derivatives point of view. First, we assume that there exists a pole $C\frac{\Gamma(1)}{\Gamma(0)}x^{-1}$ that went to zero, which is in the range of possibility

$$D^{-1}(x^2) = \frac{\Gamma(3)}{\Gamma(4)} x^{2-(-1)} + C\frac{\Gamma(1)}{\Gamma(0)} \frac{\Gamma(0)}{\Gamma(1)} x^{-1-(-1)}$$

We can see that both fractions cancel out, leaving only Cx^0, which is C

$$D^{-1}(x^2) = \frac{x^3}{3} + C$$

We can see that this works mathematically here, no matter how many constants may be there, all of them will simply add up under the term x^0
That is, of course, with one problem, there are infinite Gamma poles; whenever the denominator hits a non-positive integer, it hits a pole, we can't really make

sure if the gamma pole was at the second derivative, or the tenth, the memory of the functions is wide, and for that we need to define two categories of derivatives

Pure derivatives: these are derivatives that don't hold any memory.
Memory derivatives: these are functions that hold memory and can be separated into three:

1. **Memory stated functions:** functions that have a stated amount of change in order of D^z
2. **Always existed functions:** functions that have an infinite series of memory
3. **Application derivatives:** where the constant exists if and only if it satisfies the required equation or does not make any change to it (Although doubtful since any change in the function will lead to a change in the derivative)

11 Compression to original fractional derivative work

The author started this work without having prior knowledge of this field, and since the moment I knew it existed, I decided to continue my framework anyway And now, I make this compression between known literature and my framework I will try to justify as many positive and negative aspects in all of them, and forgive me for any mistakes

11.1 Riemann–Liouville integral

One, if not the first, definition of fractional calculus.
The Riemann–Liouville definition shaped how fractional derivatives are now; it works using iterated integration

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt \qquad \{\alpha \in \mathbb{C}, Re(\alpha) > 0, f : \mathbb{R} \to \mathbb{R}\}$$

This definition works for any analytic function that is locally integrable, and a here represents a freedom but fixed point.
I^1 here represents the first Anti-derivative of the function $f(x)$
We can get the fractional derivative $D^{\alpha} f(x)$ from this formula

$$D_x^{\alpha} f(x) = \begin{cases} \frac{d^{\lceil \alpha \rceil}}{dx^{\lceil \alpha \rceil}} I^{\lceil \alpha \rceil - \alpha} f(x) & \alpha > 0 \\ f(x) & \alpha = 0 \\ I^{-\alpha} f(x) & \alpha < 0 \end{cases}$$

the motivation for this definition that started to field was some questions between Leibniz and L'Hôpital in the 17th, but then Joseph Liouville started

working on the idea in 1830s using Cuchy's formula for n-integrals
the fundamental proprieties for this definition hold

$$\frac{d}{dx}I^{\alpha+1}f(x) = I^{\alpha}f(x) \qquad\qquad I^{\alpha}(I^{\beta}f(x)) = I^{\alpha+\beta}f(x)$$

this definition had the propriety of "change between constants"; which means that $I^{\alpha}C$ where C is a constant doesn't always return 0 we can use this to calculate four functions that are going to be used for compression later

$$I^{\alpha}x^{n} = \frac{1}{\Gamma(\alpha)}\int_{a}^{x} t^{n}(x-t)^{\alpha-1}dt = \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}x^{n+\alpha}$$

$$I^{\alpha}e^{ax} = \frac{1}{\Gamma(\alpha)}\int_{a}^{x} e^{at}(x-t)^{\alpha-1}dt = \frac{e^{ax}}{a^{\alpha}}$$

$$I^{\alpha}a^{x} = \frac{1}{\Gamma(\alpha)}\int_{a}^{x} a^{t}(x-t)^{\alpha-1}dt = \frac{a^{x}}{\ln(a)^{\alpha}}$$

$$I^{\alpha}\sin(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x} \sin(t)(x-t)^{\alpha-1}dt = \sin(x-\frac{\alpha\pi}{2})$$

evaluating these function step by step will take so much space and use integrals that are outside the scoop of this paper like the Beta integral; so we are going to use the known results directly

11.2 Caputo definitions and initial conditions

the problem with of constants in the Riemann–Liouville definition

12 Combinations

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12.2 the fractional derivative of permutations and combinations

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