Complex-Order and Fractional Derivatives: A First Exploration II

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1 Introduction

2 The Field of orders

2.1 more identities and operations

2.2 Taylor series memory

3 More to the Imagination

3.1 The imaginary derivative order polar form

from what we know about D^z we can say that
D^1 and D^{-1} represents a half turn on the $D(i)$ plane
D^i and D^{-i} represents a quarter turn on the $D(i)$ plane
So if we want to get the generalized form of this, we need to use the polar form instead of the Cartesian form, so

$$D^i f(x) = D^{e^{\frac{i\pi}{2}}} f(x) \qquad\qquad D^{-1} f(x) = D^{e^{i\pi}} f(x)$$

and generally speaking, for the unit circle around D^0(the function itself)

$$D^z f(x) = D^{e^{i\theta}} f(x)$$

And to make this formula for the whole complex plane, we write it like this

$$D^z f(x) = D^{re^{i\theta}} f(x)$$

where r is the length from D^0 to the wanted function

3.2 exploring more about Imaginary derivatives

3.3 complex functions with D^z

3.4 Complex derivative of a complex function is complex

4 Multi variable D^z

5 functional order $D^{f(\alpha)}$

6 Yes, the whole space is a matrix system

As we have explored in the first paper, the possibility of matrix order derivative with this simple formula

$$D^A = \frac{\Gamma(n+1)}{\Gamma(n-A+1)}x^{n-A} = \frac{\Gamma(n+1)}{\Gamma(n-A+1)}x^n e^{-A\ln(x)} = \frac{\Gamma(n+1)}{\Gamma(n-A+1)}x^n \sum_{k=0}^{\infty} \frac{(-1)^k \ln(x)^k}{k!} A^k$$

But let's refine it a little to make it fully clear by changing all the scalers to matrices

$$D^A = \frac{\Gamma((n+1)I)}{\Gamma((n+1)I-A)}x^{nI-A} = \frac{\Gamma((n+1)I)}{\Gamma((n+1)I-A)}x^{nI} e^{-A\ln(x)} = \frac{\Gamma((n+1)I)}{\Gamma((n+1)I-A)}x^{nI} \sum_{k=0}^{\infty} \frac{(-1)^k \ln(x)^k}{k!} A^k$$

For now, to define $\Gamma(A)$ we are going to use eigenvalues, although Cuchy Integrals are better, but for the sake of intuition and understanding, we are going to discuss them in later sections
This means that the formula only works for **square matrices** for now

6.1 exploring more about Matrix derivatives

6.2 Term-wise matrix order

7 To the third dimension

7.1 Quaternion derivatives

We can't simply define D^q as they have their own unique structure; instead, we can try to do it using matrix representation

$$Q = \begin{pmatrix} a+bi & c-di \\ -c+di & a-bi \end{pmatrix}$$

we can then try put it in the formula

$$D^Q x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-Q)}x^n \sum_{k=0}^{\infty} \frac{(-1)^k \ln(x)^k}{k!} Q^k$$

8 Integrals May I present

8.1 Where is the constant of integration?

As we saw from this whole paper, we can use D^z to differentiate and integrate Real, imaginary, or matrix, yet something seems to be missing.
The constant of integration, something so fundamental in calculus, yet seems to be missing
But here in fractional calculus, things are different; here we have a lot of "Differentiating" forward or backward, 2D or 3D, and a lot of directions that make it hard even to put it somewhere But at first, we need to understand why, not in standard terms, but in this research term
The constant of Integration is something that arises when integrating, that is, because under differentiation, we lose constants and variables according to the order
That is because of the definition of D^z, which transforms functions from one state to another.
take, for example, $f(x) = x^5 + 14$, from a prescriptive of stranded calculus, when we differentiate, constants are gone because they don't have any rate of change, the answer is simply

$$D^1 f(x) = 5x^4$$

But as we discussed, the rate of change isn't a core in the derivative but a side effect. To see the full picture, we need to use the whole Gamma formula with the knowledge that $C = Cx^0$

$$D^1 f(x) = \frac{\Gamma(6)}{\Gamma(5)} x^{5-1} + 14\frac{\Gamma(1)}{\Gamma(0)} x^{0-1} = 5x^4 + \frac{14}{\Gamma(0)} x^{-1}$$

We can see that the denominator has a Gamma pole in it, which leads the whole term to be 0, taking the constant with it
In other words, differentiation, as we discussed, loses memory; the more it differentiates, the more it's gone
Integration is the exact opposite; it retains memory, and we can see that too from the standard calculus approach

$$D^{-1}(x^2) = \frac{x^3}{3} + C$$

We see that here the constant of integration is dependent on logic, the simple logic is that if we differentiate, we lose constants, then integration, the opposite of it must return them
Let's see what happens from the fractional derivatives point of view. First, we assume that there exists a pole $C\frac{\Gamma(1)}{\Gamma(0)}x^{-1}$ that went to zero, which is in the range of possibility

$$D^{-1}(x^2) = \frac{\Gamma(3)}{\Gamma(4)} x^{2-(-1)} + C\frac{\Gamma(1)}{\Gamma(0)}\frac{\Gamma(0)}{\Gamma(1)} x^{-1-(-1)}$$

We can see that both fractions cancel out, leaving only Cx^0, which is C

$$D^{-1}(x^2) = \frac{x^3}{3} + C$$

We can see that this works mathematically here, no matter how many constants may be there, all of them will simply add up under the term x^0
That is, of course, with one problem, there are infinite Gamma poles; whenever the denominator hits a non-positive integer, it hits a pole, we can't really make sure if the gamma pole was at the second derivative, or the tenth, the memory of the functions is wide, and for that we need to define two categories of derivatives

Pure derivatives: these are derivatives that don't hold any memory.
Memory derivatives: these are functions that hold memory and can be separated into three:

1. **Memory stated functions:** functions that have a stated amount of change in order of D^z
2. **Always existed functions:** functions that have an infinite series of memory
3. **Application derivatives:** where the constant exists if and only if it satisfies the required equation or does not make any change to it (Although doubtful since any change in the function will lead to a change in the derivative)

9 Combinations

9.1 the fractional derivative of $x!$

9.2 the fractional derivative of permutations and combinations

10 Differential equations

10.1 Bessel functions

10.2 Hermite polynomials

10.3 Legendre polynomials

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