Complex-Order and Fractional Derivatives: A First Exploration

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Research Overview
This paper presents an independent derivation of fractional and complex-order derivative operators, generalizing D^n to D^α ($\alpha \in \mathbb{R}$) and D^z ($z \in \mathbb{C}$) through algebraic first principles. Without prior exposure to existing fractional calculus literature, I systematically developed formulas, operator properties, and geometric interpretations, later discovering this represents rediscovery of classical results with some novel organizational perspectives.
Key Results 1. Algebraic Foundation (Sections 1-3)

1. Power Function Formula:

$$D^\alpha(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

Derived by replacing factorials in the integer derivative formula with Gamma functions, enabling continuous orders. Fundamental Properties Proven:

2. Linearity: $D^\alpha(c_1 f + c_2 g) = c_1 D^\alpha(f) + c_2 D^\alpha(g)$

3. Index Law: $D^{\alpha+\beta}(f) = D^\alpha(D^\beta(f))$ via Γ-cancellation

4. Multiplication Law: ${}^\beta D^\alpha := D^{\alpha\beta}$ (extended by continuity)

5. General Power Series Rule:

$$D^\alpha f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

Enables term-by-term differentiation of analytic functions.

2. Applications to Elementary Functions (Sections 2, 5)

Function	D^a	Notes
e^{ax}	$a^\alpha e^{ax}$	Clean closed form
$\sin(x)$	$\sin(\frac{\alpha\pi}{2}+x)$	Phase shift interpretation
$\cos(x)$	$\cos(\frac{\alpha\pi}{2}+x)$	Derived via Euler formula
$\sinh(x)$	$\frac{e^x+(-1)^\alpha e^{-x}}{2}$	Involves $(-1)^\alpha$ factor
$\tan(x), \coth(x)$	Power Series Only	No closed form due to poles

Product Rule (Section 4.1):

$$D^\alpha(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} D^{\alpha-k}(f)D^k(g)$$

Converges to Leibniz rule for $\alpha \in \mathbb{Z}^+$.
3. Complex-Order Derivatives (Section 8.3):
Formula for x^n:

$$D^{a+bi}(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-a-bi+1)} x^{n-a} e^{-b\ln(x)i}$$

Geometric Interpretation: Introduced "D(i) plane" where derivative orders map to 2D space:

- **Real axis:** traditional differentiation (positive) / integration (negative)
- **Imaginary axis:** oscillatory transformations

Example: $D^i(x^n)$ acts as rotation in function space
Discovery: Using order multiplication to i on the i-th derivative yields -1-st derivative (integration):

$$D^i x^n = \frac{\Gamma(n+1)}{\Gamma(n-i+1)} \qquad {}^iD^i x^n = \frac{x^{n+1}}{n+1}$$

4. Cyclic Derivatives (Section 9)
Main Result: For e^{ax} where $a^n = 1$, the function returns to itself after n derivatives:

$$D^n(e^{ax}) = e^{ax} \quad \text{when } a^n = 1$$

Examples:

- $a = -1$ ($a^2 = 1$): Hyperbolic functions have 2-cycle ($\sinh\cosh$)
- $a = i$ ($a^4 = 1$): Trig functions have 4-cycle ($\sin \to \cos \to -\sin- \to -\cos \to \sin$)
- General $a^n = 1$: Generates n-cycle families

Geometric Connection: Unity roots correspond to rotational symmetry in the D(i) plane.
Imaginary Derivative Between Cycles:

$$D^i(\sinh(x)) = \frac{e^x - e^{-(x+\pi)}}{2}$$

Represents halfway transformation between sinh and cosh. **Mathematical Gaps Identified:**

1. Convergence: Infinite series convergence conditions stated informally
2. Branch Cuts: Multi-valued functions (x^α for non-integer α) not rigorously addressed
3. Non-Analytic Functions: Functions with singularities at $x = 0$ (log, tan, inverses) require special treatment—proposed "GDI theory" classification but not proven
4. Comparison to Literature: No engagement with Riemann-Liouville vs. Caputo definitions

Potential Applications:

- **Physics:** Half-derivatives appear naturally in viscoelasticity, anomalous diffusion
- **Engineering:** Fractional differential equations for control systems
- **Quantum Mechanics:** Matrix and Complex Order Derivatives Applied to Schrödinger's equation
- **Number Theory:** Applied D^α to Riemann zeta function (Section 11.4.2)

Feedback Requested

1. Mathematical Validity: Are the proofs of Index Law and Multiplication Law rigorous enough?
2. Originality Assessment: Which results (if any) offer new perspectives beyond standard fractional calculus?
3. Directions for Depth: Should I focus on resolving convergence issues, exploring complex orders, or physical applications?
4. Educational Value: Could this serve as pedagogical introduction to fractional calculus for advanced undergraduates?

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