

# Cyclic derivatives and functions

Fire Fox

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## Abstract

This paper presents an independent exploration of cyclic derivatives based on the fractional and complex-order derivatives paper made by the same Author.

**Note to Readers:** This represents independent rediscovery of classical fractional calculus concepts. I (The Author) present this work as a pedagogical exercise in mathematical exploration rather than novel research.

## Background

The Main formula that works for all derivatives and can be used in the Maclaurin series

$$D^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

The more important formulas that are built on this one

$$D^\alpha e^{ax} = a^\alpha e^{ax} \qquad e^{ix} = \cos(x) - i \sin(x)$$

$$e^x = \cosh(x) + \sinh(x) \qquad e^{-x} = \cosh(x) - \sinh(x)$$

$$D^\alpha \sin(x) = \sin\left(\frac{\alpha\pi}{2} + x\right) \qquad D^\alpha \cos(x) = \cos\left(\frac{\alpha\pi}{2} + x\right)$$

$$D^\alpha \sinh(x) = \frac{e^x - (-1)^\alpha e^{-x}}{2} \qquad D^\alpha \cosh(x) = \frac{e^x + (-1)^\alpha e^{-x}}{2}$$

## 1 Introduction

from what we have seen in the previous fractional and complex-order derivatives paper, cyclic derivatives are such a big area that deserves its own paper,

## 2 The foundation

From the fractional and complex-order derivatives paper, we know that

$$\text{when } a^n = 1, D^n(e^{ax}) = e^{ax} \text{ with } 2 \times n \text{ cyclic order}$$

And we can call that the theorem of cyclic derivatives

**Theorem 1**  $f(x)$  is a cyclic derivative when  $a^n = 1, D^n(e^{ax}) = e^{ax}$  with  $2 \times n$  cyclic order

And in the same paper, we generalized this to hold for any  $k \equiv 0 \pmod{n}$

**Theorem 2**  $f(x)$  is a lesser cyclic derivative when  $a^n = 1, D^n(e^{ax}) = e^{ax}$  with  $2 \times k$  cyclic order, where  $k \equiv 0 \pmod{n}$

With a hypothesis that I wish to prove in this paper, that

**Hypothesis 1** For every function cyclic derivatives that can be written in the form  $e^{ax}$  where  $a^n = 1$  and satisfies the condition  $2^n \in \mathbb{Z}^+$ , there exists an algebraic perimetric form

We can already see this for hyperbolic functions, where they can be written in the form  $x^2 - y^2 = 1$ , and trigonometric functions We also proved some equations

$$\begin{aligned} D^i e^{-x} &= e^{-(x+\pi)} = \cosh(x+\pi) - \sinh(x+\pi) \\ D^i \sinh(x) &= \frac{e^x - e^{-(x+\pi)}}{2} & D^i \cosh(x) &= \frac{e^x + e^{-(x+\pi)}}{2} \\ D^i e^{ix} &= e^{\frac{-\pi}{2}} \cos(x) + ie^{\frac{-\pi}{2}} \sin(x) \\ D^i \sin(x) &= \sin\left(\frac{i\pi}{2} + x\right) & D^i \cos(x) &= \cos\left(\frac{i\pi}{2} + x\right) \end{aligned}$$

and we also proved that for  $\sin(x)$  and  $\cos(x)$

## 3 More about the complex derivatives and known families

from what we know about  $\sinh(x)$  and  $\cosh(x)$  we can write their formulas in an other way

$$\begin{aligned} D^\alpha \sinh(x) &= \frac{e^x - (-1)^\alpha e^{-x}}{2} = \frac{e^x - (e^{i\pi})^\alpha e^{-x}}{2} = \frac{e^x - (e^{i\pi\alpha})e^{-x}}{2} \\ &= \frac{e^x - e^{i\pi\alpha-x}}{2} \end{aligned}$$

The same goes for  $\cosh(x)$

$$D^\alpha \sinh(x) = \frac{e^x - e^{i\pi\alpha-x}}{2} \quad D^\alpha \cosh(x) = \frac{e^x + e^{i\pi\alpha-x}}{2}$$

We can see that we can't express them as simple forms, since the real value changes differently from the complex value. If we try to apply the derivative operator to  $e^x$  with the definition of  $1 = e^{2i\pi}$ , we get

$$D^\alpha e^x = 1^\alpha e^x = e^{2i0\alpha} e^x = e^x$$

We used that because it's the principal value

For trigonometric functions, we know that  $D^i$  represents a half turn before it changes to its integral from the real plane, but there is something to clarify

$$\text{Im}(D^i e^{ix}) = e^{\frac{-\pi}{2}} \sin(x)$$

As we can see, the first  $i$ -th derivative acts as a scaler that scales  $\sin(x)$  by real value  $e^{-\frac{\pi}{2}}$

However, this isn't equal to the  $D^i \sin(x)$  as the definition changes from scaling to rotating, like this

$$D^i \sin(x) = \sin\left(\frac{i\pi}{2} + x\right) =$$

But since we have proven the multiplication law works in the framework, we can make sure that both are somewhat equal

$${}^i D^i e^{ix} = i^{i \times i} e^{ix} = (e^{\frac{i\pi}{2}})^{-1} e^{ix} = e^{\frac{-i\pi}{2}} e^{ix} = e^{ix - \frac{i\pi}{2}}$$

$$e^{i(x - \frac{\pi}{2})} = \cos\left(x - \frac{\pi}{2}\right) + i \sin\left(x - \frac{\pi}{2}\right) = \sin(x) - i \cos(x) = D^{-1} e^{ix} = \int e^{ix}$$

and for the sin we proved in the "Complex-order and fractional derivatives: first exploration" paper that the index law works on it, and thus the multiplication law either from here or from the series expansion, so that we can say

$${}^i D^i \sin(x) = \sin\left(\frac{i \times i\pi}{2} + x\right) = \sin\left(\frac{-\pi}{2} + x\right) = -\cos(x)$$

Thus, both of them work fine, just different prescriptive

We shall call the  $e^{ix}$  the complex perspective since it's all about the imaginary unit, and  $\sin(x)$  the real perspective, even if there is  $i$  in it

### 3.1 The complex prescriptive

since  $D^i$  represents a whole rotation on the  $D(i)$  plane, we can get more angles that could help us understand more what happens to the function to "integrate it"

first step is we are going to transform from  $i$  to  $e^{\frac{i\pi}{2}}$  so we can deal with rotation with radians in circles

$$\begin{aligned} D^i e^{ax} &= D^{e^{\frac{i\pi}{2}}} e^{ax} & D^{-1} e^{ax} &= D^{e^{i\pi}} e^{ax} \\ D^{e^{i\theta}} e^{ax} &= a^{e^{i\theta}} e^{ax} & D^{e^{i\theta}} e^{ix} &= i^{e^{i\theta}} e^{ix} = e^{\frac{i\pi e^{i\theta}}{2}} e^{ix} = e^{i(\frac{\pi e^{i\theta}}{2} + x)} \end{aligned}$$

$$= \cos\left(\frac{\pi e^{i\theta}}{2} + x\right) + i \sin\left(\frac{\pi e^{i\theta}}{2} + x\right)$$

So now we can know what happens at the third of rotation or the third root of unity, which is equal to  $\frac{\pi}{3}$  in radians, we get

$$D^{e^{\frac{i\pi}{3}}} e^{ix} = e^{i\left(\frac{\pi e^{\frac{i\pi}{3}}}{2} + x\right)} = \cos\left(\frac{\pi e^{\frac{i\pi}{3}}}{2} + x\right) + i \sin\left(\frac{\pi e^{\frac{i\pi}{3}}}{2} + x\right)$$

which after calculating  $e^{\frac{i\pi}{3}}$  to be  $\frac{1}{2} + i\frac{\sqrt{3}}{2}$  we can then multiply it by  $\frac{\pi}{2}$  to get  $\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4}$ , then we plug it

$$e^{i\left(\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4} + x\right)} = e^{i\frac{\pi}{4} + i^2\frac{\pi\sqrt{3}}{4} + ix} = e^{i\left(x + \frac{\pi}{4}\right)} e^{-\frac{\pi\sqrt{3}}{4}} = e^{-\frac{\pi\sqrt{3}}{4}} \cos\left(x + \frac{\pi}{4}\right) + i e^{-\frac{\pi\sqrt{3}}{4}} \sin\left(x + \frac{\pi}{4}\right)$$

We can see that it scales by a factor of  $e^{-\frac{\pi\sqrt{3}}{4}}$  and rotate with a factor of  $\frac{\pi}{4}$   
let's do the same for two-thirds, the value for  $e^{\frac{2i\pi}{3}}$  to be  $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$  we can then multiply it again by  $\frac{\pi}{2}$  to get  $-\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4}$   
Plugging it again, we get

$$e^{i\left(-\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4} + x\right)} = e^{-i\frac{\pi}{4} + i^2\frac{\pi\sqrt{3}}{4} + ix} = e^{i\left(x - \frac{\pi}{4}\right)} e^{-\frac{\pi\sqrt{3}}{4}} = e^{-\frac{\pi\sqrt{3}}{4}} \cos\left(x - \frac{\pi}{4}\right) + i e^{-\frac{\pi\sqrt{3}}{4}} \sin\left(x - \frac{\pi}{4}\right)$$

At two-thirds, it rotates with the same value but rotates backwards  
Now we have a little information about what happens in the process of integrating such functions

at the first third of the way, it rotates by  $\frac{\pi}{4}$  and scales by  $e^{-\frac{\pi\sqrt{3}}{4}}$

For Halfway, it doesn't rotate but scales with a factor of  $e^{-\frac{\pi}{2}}$

for two-thirds it rotates by  $\frac{\pi}{4}$  and scales by  $e^{-\frac{\pi\sqrt{3}}{4}}$

This may seem weird at the beginning until we notice that we aren't starting from order 1 or  $D^1$ , we are starting from the zero point  $D^0$  or the function itself, so the one-third and two-thirds don't cancel out on rotation, but they rotate to two different directions

The one-third rotates to  $D^1$  and the two-thirds rotate to  $D^{-1}$ , while the middle point  $D^i$  doesn't rotate but scales because it's not a real derivative or real integral

We can even notice that in the first third we have  $\cos\left(x + \frac{\pi}{4}\right)$  and  $\sin\left(x + \frac{\pi}{4}\right)$ , which are both pure half derivatives

$$D^{\frac{1}{2}} \sin(x) = \sin\left(\frac{\frac{1}{2}\pi}{2} + x\right) = \sin\left(\frac{\pi}{4} + x\right) \quad D^{\frac{1}{2}} \cos(x) = \cos\left(\frac{\frac{1}{2}\pi}{2} + x\right) = \cos\left(\frac{\pi}{4} + x\right)$$

and the same happens for the half-integer being rotated by  $\frac{\pi}{4}$

### 3.2 The Real prescriptive

## 4 exploration into another cyclic derivatives

### 4.1 the third order cyclic derivatives

From the theorem, we can find the third cyclic derivative to be from the equation  $a^3 = 1$ , the solutions are going to be denoted by  $1, \omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$  since one will result in  $e^x$  and is fully expected to be here because of Theorem 2, we are going to use  $\omega$

$$D^1 e^{\omega x} = \omega e^{\omega x} \quad D^2 e^{\omega x} = \omega^2 e^{\omega x} \quad D^3 e^{\omega x} = e^{\omega x}$$

We can call this function the third-order cyclic derivative, which comes between hyperbolic and trigonometric functions we will name them  $\sinh_3, \cosh_3$  and  $\sinh_3 \text{II}$  We can define them like this

$$D^\alpha \sinh_3(x) = \sinh_3 \text{II}(x) \quad \alpha \equiv 0 \pmod{3} \quad D^\alpha \sinh_3 \text{II}(x) = \cosh_3(x) \quad \alpha \equiv 1 \pmod{3}$$

$$D^\alpha \cosh_3(x) = \sinh_3(x) \quad \alpha \equiv 2 \pmod{3}$$

But this isn't the only way to define them, we can also define them with a series First we find the Maclaurin series for  $e^{\omega x}$

$$e^{\omega x} = e^0 + \omega e^0 x + \frac{\omega^2 e^0 x^2}{2!} + \frac{e^0 x^3}{3!} + \dots = 1 + \omega x + \frac{\omega^2 x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{\omega^n x^n}{n!}$$

From this, we can divide them into three sums

$$\begin{aligned} & (1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \dots) + \omega(x + \frac{x^4}{4!} + \frac{x^7}{7!} + \dots) + \omega^2(x^2 + \frac{x^5}{5!} + \frac{x^8}{8!} + \dots) \\ &= \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \omega \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \omega^2 \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!} \end{aligned}$$

We can see that the first sum can be differentiated 3 times before going back to the first state, which is also for all the other sums, but since  $\sin$  and  $\sinh$  all have  $x^{an+1}$ , we are going to make the first function to be the second sum, to keep the naming consistent nothing more

So now we can define them to be

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \quad \sinh_3 \text{II}(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!} \quad \cosh_3 = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$$

We can now define an equation that looks and acts like the Euler equation

$$e^{\omega x} = \cosh_3(x) + \omega \sinh_3(x) + \omega^2 \sinh_3 \text{II}(x)$$

We can now try to find one for  $\omega^2$

$$e^{\omega^2 x} = \sum_{n=0}^{\infty} \frac{(\omega^2)^n x^n}{n!}$$

at  $3n$  we get  $\omega^{6n} = 1$  so it's  $\cosh_3(x)$ , at  $3n+1$  we get  $\omega^{6n+2} = \omega^2$  so it's  $\sinh_3(x)$  and at  $3n$  we get  $\omega^{6n+4} = \omega$  so it's  $\sinh_3 \Pi(x)$

$$e^{\omega^2 x} = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \omega^2 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \omega \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

$$e^{\omega^2 x} = \cosh_3(x) + \omega^2 \sinh_3(x) + \omega \sinh_3 \Pi(x)$$

and for  $e^x$  it's quite simple

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

$$e^x = \cosh_3(x) + \sinh_3(x) + \sinh_3 \Pi(x)$$

like  $\sin$  and  $\sinh$ , we can try to find an exponent form for them  
First, we begin by adding all of the equations so we have

$$e^x + e^{\omega x} + e^{\omega^2 x} = \cosh_3(x) + \sinh_3(x) + \sinh_3 \Pi(x)$$

$$\begin{aligned} &+ \cosh_3(x) + \omega \sinh_3(x) + \omega^2 \sinh_3 \Pi(x) + \cosh_3(x) + \omega^2 \sinh_3(x) + \omega \sinh_3 \Pi(x) \\ &= 3 \cosh_3(x) + \sinh_3(x)(1 + \omega + \omega^2) + \sinh_3 \Pi(x)(1 + \omega^2 + \omega) \end{aligned}$$

and since we know that  $1 + \omega + \omega^2 = 0$

$$e^x + e^{\omega x} + e^{\omega^2 x} = 3 \cosh_3(x) \quad \cosh_3(x) = \frac{e^x + e^{\omega x} + e^{\omega^2 x}}{3}$$

Now we can define the others by differentiating

$$\sinh_3(x) = \frac{e^x + \omega e^{\omega x} + \omega^2 e^{\omega^2 x}}{3} \quad \sinh_3 \Pi(x) = \frac{e^x + \omega^2 e^{\omega x} + \omega e^{\omega^2 x}}{3}$$

since these definitions are going to continue with us, we shall call the  $e^{ax} = \dots$  the **Euler form** and  $f(x) = e^x + e^{ax} \dots$  the **exponentiation form**

## 4.2 Cyclic derivatives and prime numbers

To understand exactly what is meant by this, we need to see these

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \cosh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$$

As we can see, there is a pattern here, for every function that is  $k$ -th cyclic derivative we can see that its series is  $\sum_{n=0}^{\infty} \frac{x^{kn}}{(kn)!}$ . But this assumption shortly breaks as we can see for the fourth cyclic derivative

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

To understand more what I am talking about, we may take a look at the fifth cyclic derivative denoted by  $\epsilon$ , to find the functions we start from the Maclaurin series

$$e^{\epsilon x} = 1 + \epsilon x + \frac{\epsilon^2 x^2}{2!} + \frac{\epsilon^3 x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{\epsilon^n x^n}{n!}$$

And we can find the five functions the same way we made it to the three cyclic derivative functions

$$\begin{aligned} \cosh_5(x) &= \sum_{n=0}^{\infty} \frac{x^{5n}}{(5n)!} & \sinh_5(x) &= \sum_{n=0}^{\infty} \frac{x^{5n+1}}{(5n+1)!} \\ \sinh_5 \text{II}(x) &= \sum_{n=0}^{\infty} \frac{x^{5n+2}}{(5n+2)!} & \sinh_5 \text{III}(x) &= \sum_{n=0}^{\infty} \frac{x^{5n+3}}{(5n+3)!} & \sinh_5 \text{IV}(x) &= \sum_{n=0}^{\infty} \frac{x^{5n+4}}{(5n+4)!} \end{aligned}$$

And since the sum of roots of unity that are over 1 root is zero, we can do the same steps to find that

$$\cosh_5(x) = \frac{e^x + e^{\epsilon x} + e^{\epsilon^2 x} + e^{\epsilon^3 x} + e^{\epsilon^4 x}}{5}$$

and the other functions to be the derivatives of these functions

We can notice that the pattern continued for 5-th cyclic function

So what is the problem with trigonometric ones?

Well, we can see the expansion for  $e^{ix}$  to see what happens

$$e^{ix} = 1 + ix + \frac{x^2}{2!} - \frac{ix^3}{3!} + \dots$$

We can notice the pattern right there, it's the  $-\frac{x^2}{2!}$ , this term allows us to either:  
- write the sum as four cyclic functions since we will make different additions being  $\{1, i, -1, -i\}$

- We write sum as two different cyclic derivatives being  $\{1, -1\}$  and  $i\{1, -1\}$

In other words, the cyclic derivative family is compisable, reducible with simple algebra

and the reason for that is the **cyclic order**, when it's composite, we can see some roots return, like from order 2 we have  $1, -1$  and order 4 we have  $1, i, -1, -i$ , the  $1, -1$  here is back, same for six roots of unity  $1, i_1, i_2, -1, i_3, i_4$  (note that  $i_a$  here isn't imagenry unit but the  $a$ -th root) and we can say that let  $gk$  be all solutions for  $a^k = 1$  and  $gn$  be for  $a^n = 1$ , as long as  $\frac{k}{n} \in \mathbb{Z}^+$ ,  $gk \subset$

$gn$

Thus, for any composite cyclic order, there exists more than 1 way to represent it

which means primes aren't here, so we can write the theorem

**Theorem 3**  $\forall n \in Primes, a^n = 1$  *There exists only one way to represent  $e^{ax}$  as a sum of all the  $n$ -th cyclic order functions*

### 4.3 Cyclic derivatives and Mittag-Leffler connection

As we can see from multiple series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \sinh = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$$

#### 4.4 Generalized forms for cyclic derivatives

#### 4.5 Odd And Even cyclic derivatives

#### 4.6 Cyclic derivatives and algebraic equations

#### 4.7 Mixture of equations