Complex-Order Fractional Derivatives: A First Exploration

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1 Introduction

The integer-order derivative $D^n f(x)$ measures the local rate of change. This paper explores the generalization of the derivative operator to continuous and complex orders, $D^\alpha f(x)$ and $D^z f(x)$, known as Fractional and Complex Calculus.

2 The Generalized Operator for $f(x) = x^n$

2.1 From Integer to Fractional Order

We begin with the integer derivatives of $f(x) = x^n$:

$$D^k f(x) = n(n-1)\cdots(n-k+1)x^{n-k}$$

Using the identity $n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$, we write:

$$D^k f(x) = \frac{n!}{(n-k)!}x^{n-k}$$

To generalize this for $k \in \mathbb{R}$, we substitute the factorial function with the continuous Gamma function, $\Gamma(z)$. We use the identities $n! = \Gamma(n+1)$ and $\Gamma(z+1) = z\Gamma(z)$. The α-th derivative (where $\alpha \in \mathbb{R}$) is:

$$D^\alpha f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha}$$

of course we can use this formula to get half-derivtaive of x^n

$$D^{\frac{1}{2}} = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)}x^{1-\frac{1}{2}} = \frac{1}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}}$$

using the rule $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$D^{\frac{1}{2}} = \frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})}x^{\frac{1}{2}} = \frac{1}{\frac{\sqrt{\pi}}{2}}x^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}}x^{\frac{1}{2}}$$

now if we take the half-derivative of that half-derivative it will be

$$\frac{2}{\sqrt{\pi}}D^{\frac{1}{2}} = (\frac{2}{\sqrt{\pi}})\frac{\Gamma(1/2+1)}{\Gamma(\frac{1}{2}-\frac{1}{2}+1)}x^{\frac{1}{2}-\frac{1}{2}} = (\frac{2}{\sqrt{\pi}})\frac{\Gamma(\frac{3}{2})}{1}x^{0} = (\frac{2}{\sqrt{\pi}})(\frac{\sqrt{\pi}}{2}) = 1$$

we talking the half-derivative twice to the same function gave what a one full derivative would give
of course such a proof is too simple and don't quite give the meaning of a proof , it's just a little confirmation for the time being, the full rigorous proof will be proven later with the properties of the D^z operator
Generalization to Complex Order $z = a + bi$
We now extend the derivative order to the complex number $z = a + bi$:

$$D^z f(x) = \frac{\Gamma(n+1)}{\Gamma(n-z+1)}x^{n-z}$$

To show the magnitude and phase components, we expand x^{n-z} using the property $x^{a+bi} = x^a e^{b\ln(x)i}$:

$$D^z f(x) = \frac{\Gamma(n+1)}{\Gamma(n-z+1)}x^{n-a}e^{-b\ln(x)i}$$

with these two formulas we can use them to find any $\mathbb{C}$ or $\mathbb{R}$ derivatives for x^n
Finding negative order derivatives
we can find the negative derivatives by putting -1 as the α and see what could happen
Putting -1 in the general formula gives the result

$$D^{-1}f(x) = \frac{\Gamma(n+1)}{\Gamma(n-(-1)+1)}x^{n-(-1)} = \frac{\Gamma(n+1)}{\Gamma(n+2)}x^{n+1}$$

and using the $\Gamma(z+1) = z\Gamma(z)$ we can say that

$$D^{-1}f(x) = \frac{\Gamma(n+1)}{(n+1)\Gamma(n+1)}x^{n+1} = \frac{x^{n+1}}{(n+1)}$$

which means that the negative order derivatives are the integrals a function
This result unifies the familiar integer derivative, the fractional derivative, and the complex-order derivative into a single, elegant framework.

2.2 the x^{-n} problem

as we have seen, we can apply the past formula to any power of n weather it's fractional or even complex but problems rise when we try to apply the past formula to x^{-n}

$$D^{\alpha}(x^{-n}) = \frac{\Gamma(0)}{\Gamma(-\alpha)}x^{-n-\alpha}$$

not only do we have a **Gamma Pole** in the numerator but also for any value $\alpha \in \mathbb{Z}^+$ we also get a Gamma pole in the denominator which means that this formula cant work and we need another formula
the m-th formula for x^{-n} is simply

$$\frac{d^m}{dx^m}(x^{-n}) = \frac{(-1)^m \ \ (n)^{(m)}}{x^{n+m}}$$

the $n^{(m)}$ here isn't a power but rather a rising factorial that can be expressed as $n^{(m)} = \frac{(n+m-1)!}{(n-1)!}$ with this knowledge we can say

$$D^\alpha(x^{-n}) = \frac{(-1)^\alpha \ \ \frac{\Gamma(n+\alpha)}{\Gamma(n)}}{x^{n+\alpha}} = \frac{(-1)^\alpha \ \ \Gamma(n)}{x^{n+\alpha}\Gamma(n+\alpha)}$$

as simple as that, it didn't work with the original x^n formula , but this shows something about fractional derivatives , if we went to find the first integral for both x^n and x^{-n} we find that

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \int x^{-n} dx = \ln|x| + C$$

Ignoring the integration constant we can find that x^n wasn't the original function for x^{-n} but it actually transformed from $\ln(x)$
not only in complexity only but in dependency , the $D^z(x^n)$ is dependent on the change of n more than x, while $\ln(x)$ is dependent on the change of x
this will be explained later in the **Transformative Functions** section , and this is one of the few cases we use stanederd integration (other than the Gamma function) in this research

3 Formulas for Other Algebraic Functions

3.1 The General Formula for a^x

Starting with the general integer rule for a^x:

$$D^n(x) = a^x ln(a)^n$$

substituting α in the place of n gives us

$$D^\alpha f(x) = a^x \ln(a)^\alpha$$

we can see that this simple change was enough for the formula to work by taking the half-derivative twice and it gives us order one derivative

$$D^{1/2} f(x) = a^x ln(a)^{1/2}$$

since $ln(a)^{1/2}$ is a constant we can take it out simply when doing the derivative again

$$D^{1/2}(D^{1/2} f(x)) = ln(a)^{1/2}(D^{1/2} f(x)) = ln(a)^{1/2}(a^x ln(a)^{1/2}) = a^x ln(a)$$

which is true since our starting function was a^x and thus we can say this formula works

The Complex Generalization of this formula can be written like the D^α formula or like this

$$D^z f(x) = a^x \ln(a)^t e^{bln(ln(a))i}$$

where $z = a + bi$

of course we can find the first Anti-derivative of this function by using -1 in the formula

$$D^{-1}(x) = a^x ln(a)^{-1} = \frac{a^x}{ln(a)}$$

and the first Complex derivative

$$D^i(x) = a^x ln(a)^i = a^x e^{ln(ln(x))i}$$

3.2 The General Formula for e^x

The function e^x is known for it's "Unchanging Derivative" because it comes from the $D^n(a^x) = a^x ln(a)^n$ and putting $a = e$ we get $D^n(e^x) = e^x$ so this also means there is no change affect the complex nor the fractional derivatives

$$D^\alpha f(x) = e^x \qquad D^z f(x) = e^x$$

which means the Anti-derivative and the first complex derivative of the function

$$D^{-1}(x) = e^x \qquad D^i(x) = e^x$$

3.3 The General Formula for e^{ax}

as we saw there isnot any change between e^x and any of it's derivatives , things change when we consider e^{ax} as we can see the rule of the first - second an derivative is

$$D^1 f(x) = ae^{ax} \qquad D^2 f(x) = a^2 e^{ax} \qquad D^3 f(x) = a^3 e^{ax}$$

so we can find the formula for the n-th derivative as

$$D^n f(x) = a^n e^{ax}$$

and changing the n to α we get

$$D^\alpha f(x) = a^\alpha e^{ax}$$

as simple as that we still have to test it to justify

$$D^{1/2} f(x) = a^{1/2} e^{ax}$$

since $a^{1/2}$ is a constant we can say that

$$D^{1/2}(D^{1/2} f(x)) = a^{1/2}(D^{1/2} f(x)) = (a^{1/2})(a^{1/2} e^{ax}) = ae^x$$

this confirms that the formula work and substituting z instead of α we get the same formula as above taht can also be written like that

$$D^z f(x) = a^t e^{bln(a)i+ax}$$

where $z = a + bi$
the first Anti-derivative for e^{ax} is

$$D^{-1} f(x) = a^{-1} e^{ax} = \frac{e^{ax}}{a}$$

and the first complex derivative is

$$D^i f(x) = a^i e^{ax} = e^{ln(a)i+ax}$$

3.4 The problem of $\log_a(x)$

$log_a(x)$ The first derivative of $log_a(x)$ is $\frac{1}{xln(a)}$ and the second derivative is $\frac{-1}{x^2ln(a)}$ the third derivative is $\frac{2}{x^3ln(a)}$ lastly the fourth derivative is $\frac{-6}{x^4ln(a)}$ is we can see the pattren of the n-th derivatve

$$D^n f(x) = (-1)^{n+1} \frac{(n-1)!}{x^n ln(a)}$$

and applying the gamma identity $n! = \Gamma(n-1)$ then changing n to α and reversing the x power in the denominator we get

$$D^\alpha f(x) = (-1)^{\alpha+1} \frac{\Gamma(\alpha)}{ln(a)} x^{-\alpha}$$

But it fails, it doesnt work with fractions nor negative integares it only works positave integers, so what went wrong? my theory is that the problem is fairly simple, the $log_a(x)$ can't be expressed as a Maclurin series and $log_a(x + C)$ can only be expressed to a series with the condition $|x| < C$ or else it won't diverge

GDI theory : For Every function that isnt Function is analytic at x=0 and doesnt converge over $\mathbb{R}$ the derivative formula $D^\alpha f(x)$ works only on $\mathbb{Z}^+$ for that function , we can also call it not full real differentiable

GDI hypothesis : the function is differentiable in all it's vaild input values

These can also be rewritten like this:
Let f(x) be an analytic function defined by its Maclaurin series
GDI theory: If f(x) has a singularity at $x = 0$, then the generalized formula for $D^\alpha f(x)$ will contain a singularity at $\alpha = n$ for $n \in \mathbb{Z}^+$, preventing the generalized formula from equaling the expected integer fractional or anti-derivative. – Todo: change this to look better – this will be explained in detail in later sections

4 Proprties of the D^z operator

these are proprties to identify the nature of it that will help later with the formula and analysis of what i can call the D^z **plane** More on that later

4.1 General power series rule

power seriesses are very important tools to anylaticly describe a function along the $\mathbb{R}$eal or $\mathbb{C}$omplex planes what is exactly what we need
The general power series definition for a function(Talyor series) is as following :

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

taking the first derivative for both sides, the first term will cancel as it's a constant , the second term is linear so it will become constant, the third term is quadruple will become linear and 2! will cancel the 2 of the power and so on we can write is like this

$$D^1 f(x) = f^{(1)}(a) + \frac{f^{(2)}(a)}{1!}(x-a) + \frac{f^{(3)}(a)}{2!}(x-a)^2 ... = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(a)}{n!}(x-a)^n$$

the cancellation happen also because when we take a derivtave the power of $(x-a)$ gets down by 1 to the numerator and we divide $n!$ by it leading to $(n-1)!$ which will lead to infinity in the denominator leading to the whole term to be zero
taking the third and the forth derivative give the same result up to k-th derivative

$$D^k f(x) = f^{(k)}(a) + \frac{f^{(k+1)}(a)}{1!}(x-a) + \frac{f^{(k+2)}(a)}{2!}(x-a)^2 ... = \sum_{n=0}^{\infty} \frac{f^{(k+n)}(a)}{n!}(x-a)^n$$

now to make it full fractional we will put gamma instead of n and using the x^n general formula

$$D^\alpha f(x) = f(a)^{(\alpha)} + \frac{f^{(\alpha+1)}(a)}{\Gamma(2)}(x-a) + \frac{f^{(\alpha+2)}(a)}{\Gamma(3)}(x-a)^2 ... = \sum_{n=0}^{\infty} \frac{f^{(n+\alpha)}(a)}{\Gamma(n+1)} \left[\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}(x-a)^n \right]$$

in the brackets we can see the general derivative for x^n canceling the Gammas out we get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-\alpha+1)}(x-a)^{n-\alpha}$$

for powers of $(x-a)$ lesser than α it will get to ∞ in the denominator,at least when it's a negative integer other than that it will work normal,this happens because of the gamma pole, but taking the limit it will lead to 0 but still doesn't affect the sum , indeed helping us deleting the first $n < \alpha$ terms of course for negative integer dervitavte this works too since it will it will be positive

Note: becuase how much terms you take in the diffrentation will alawys come terms that are replace to them because of the x^n differentiation and the infinite sum,**even if you differentiate it infinite amount of times** but you are still deleting values we are going to use that knowledge later in the integration and differential equations in later sections

we can now let a = 0 to get the important series we need , the Maclurin series

$$D^\alpha f(x) = f(0) + \frac{f^{(1)}(0)}{\Gamma(2)}(x) + \frac{f^{(2)}(0)}{\Gamma(3)!}(x)^2 ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

4.2 The linearity of D^z operator

for most of the opreatiosn with functions and real life applictions we need to deal with linearity for D^z operator whcih is what wwe are going to prove in this small section **We Must Prove that:**

$$D^z(c_1 f(x) + c_2 g(x)) = c_1 D^z f(x) + c_2 D^z g(x)$$

let's beging with the simple x^n and let $f(x) = x^n, g(x) = x^m$ firstly we Differentiate them separately

$$D^z(c_1 f(x)) + D^z(c_2 g(x)) = c_1 D^z(f(x)) + c_2 D^z(g(x)) = c_1 \frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-z} + c_2 \frac{\Gamma(m+1)}{\Gamma(m-z+1)} x^{m-z}$$

Let this be statement 1
now let's differentiate them together we get

$$D^z(c_1 f(x) + c_2 g(x)) = c_1 \frac{\Gamma(n+1)}{\Gamma(n-z+1)} x^{n-z} + c_2 \frac{\Gamma(m+1)}{\Gamma(m-z+1)} x^{m-z}$$

Let this be statement 2
since **Statement 1** = **Statement 2** we can say that

$$D^z(c_1 f(x) + c_2 g(x)) = c_1 D^z f(x) + c_2 D^z g(x)$$

Q.E.D
this is very useful , but to applay it to all functions , which we can do easliy with infinte seriesses
in other words
If D^z is linear on basis functions x^n, and $f = \sum c_n x^n$, then:

$$D^z(\sum c_n x^n) = \sum c_n D^z(x^n)$$

by uniform convergence of the series.

4.3 The Index law

The most important propriete for the formulas is the index law that is

$$D^{\alpha+\beta}f(x) = D^{\alpha}(D^{\beta}(x))$$

we must prove this holds true for every case first we need to prove it for the simplest function we have which is x^n taking the D^z of the function we get

$$D^{\alpha}f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha}$$

now let's apply the D^{β} with the knowledge that the Gamma functions are constants in the first derivative

$$D^{\beta}(D^{\alpha}f(x)) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}D^{\beta}(x^{n-\alpha}) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}\left[\frac{\Gamma(n-\alpha+1)}{\Gamma(n-\alpha-\beta+1)}x^{n-\alpha-\beta}\right]$$

the Gamma terms cancel out and we get

$$D^{\beta}(D^{\alpha}f(x)) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)}x^{n-\alpha-\beta}$$

Let this be statement 1
now if we start from the beginning again but this time directly substitute $\alpha+\beta$ as O(stands for orders) we get

$$D^{O}f(x) = \frac{\Gamma(n+1)}{\Gamma(n-O+1)}x^{n-O}$$

if we substitute $O = \alpha+\beta$ back we get

$$D^{\alpha+\beta}f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)}x^{n-\alpha-\beta}$$

Let this be statement 2
if we Equal **statement 1** and **statement 2** we get

$$D^{\alpha+\beta}f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)}x^{n-\alpha-\beta} = D^{\beta}(D^{\alpha}f(x))$$

thus we can say that

$$D^{\alpha+\beta}f(x) = D^{\beta}(D^{\alpha}f(x))$$

Q.E.D
Note: this also works for imaginary numbers $z+w$
this by itself is a simple elegent proof but , it only works for x^n and applying the same method for each function will be very large waste of time
Instead we can use **Power Seriesses** as they hold for every analytic function

thus one proof will work for every function

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n+1)}(x-a)^n$$

we can immediately see that it's the simple x^n with everything else being a constant to the derivative
Since we've proven the Index Law for x^n, and the power series represents f as a sum of such terms, the Index Law extends to f by linearity of the operator and see that

$$D^{\alpha+\beta} f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n+1)} \left[\frac{\Gamma(n+1)}{\Gamma(n-\alpha-\beta+1)}(x-a)^{n-\alpha-\beta} \right] = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-\alpha-\beta+1)}(x-a)^{n-\alpha-\beta}$$

which means that the Index law works for any analytic function **Q.E.D**

4.4 The Multiplication Law

A logical next step is to prove that there exists a law for multiplying orders of derivatives as it will help us later to prove and analysis a lot of topics in this research
let's take for example $f(x) = x^6$ we need (for now) to prove that multiplying two order derivatives for example 2 and 3 return the same result
for now we may write the multiplication as $M[D^\alpha f(x)]^\beta$ just as a placeholder for now
so **we need to prove that**

$$M[D^2 f(x)]^3 = M[D^3 f(x)]^2 = D^6 f(x)$$

evaluating the first expression gives us $D^6 f(x) = \frac{\Gamma(7)}{\Gamma(6-6+1)} x^{6-6} = \Gamma(7)$ a very logical step to do is to use the **Index Law** we just proved
so we start with

$$D^2 f(x) = \frac{\Gamma(7)}{\Gamma(6-2+1)} x^{6-2} = \frac{\Gamma(7)}{\Gamma(5)} x^4$$

$$D^2(D^2 f(x)) = \frac{\Gamma(7)}{\Gamma(5)} D^2 f(x) = \frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(4-2+1)} x^{4-2} = \frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(3)} x^2$$

$$D^2(D^2(D^2 f(x))) = \frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(3)} D^2 f(x) =$$

$$\frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(3)} \frac{\Gamma(3)}{\Gamma(2-2+1)} x^{2-2} = \frac{\Gamma(7)}{\Gamma(5)} \times \frac{\Gamma(5)}{\Gamma(3)} \times \frac{\Gamma(3)}{\Gamma(1)}$$

as we can see the orders cancel out prefectly leaving only $\Gamma(7)$
Let's generalize the idea with α and β following the same index law but with $f(x) = x^n$

$$D^\alpha f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

$$D^\alpha(D^\alpha f(x)) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} D^\alpha f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} \frac{\Gamma(n-\alpha+1)}{\Gamma(n-\alpha-\alpha+1)} x^{n-\alpha-\alpha}$$

$$= \frac{\Gamma(n+1)}{\cancel{\Gamma(n-\alpha+1)}} \frac{\cancel{\Gamma(n-\alpha+1)}}{\Gamma(n-2\alpha+1)} x^{n-2\alpha} = \frac{\Gamma(n+1)}{\Gamma(n-2\alpha+1)} x^{n-2\alpha}$$

of course we can do it again

$$D^\alpha(D^\alpha(D^\alpha f(x))) = \frac{\Gamma(n+1)}{\Gamma(n-2\alpha+1)} D^\alpha(D^\alpha f(x)) = \frac{\Gamma(n+1)}{\Gamma(n-2\alpha+1)} \frac{\Gamma(n-2\alpha+1)}{\Gamma(n-2\alpha-\alpha+1)} x^{n-2\alpha-\alpha}$$

$$= \frac{\Gamma(n+1)}{\cancel{\Gamma(n-2\alpha+1)}} \frac{\cancel{\Gamma(n-2\alpha+1)}}{\Gamma(n-3\alpha+1)} x^{n-3\alpha} = \frac{\Gamma(n+1)}{\Gamma(n-3\alpha+1)} x^{n-3\alpha}$$

we can continue like this up till the $\beta - th$ multiplication and get the same resault , so from this we can define

$$M[D^\alpha f(x)]^\beta = \frac{\Gamma(n+1)}{\Gamma(n-\alpha\beta+1)} x^{n-\alpha\beta}$$

and since $\alpha\beta = \gamma$ where γ is the original intended order , then

$$M[D^\alpha]^\beta = D^\gamma$$

Q.E.D
of course this works only when $\beta \in \mathbb{N}$ since it's the set we can apply Multiplication Law in terms of Index Law
I believe to ascend it to general multiplication we may first define it in a better mathematical language as
$Definition : For\ \beta \in \mathbb{N},\ define\ {}^\beta D^\alpha\ inductively$
1- ${}^1D^\alpha = D^\alpha$
2- ${}^{n+1}D^\alpha = D^\alpha \circ {}^nD^\alpha$
By induction, we proved: ${}^nD^\alpha = D^{n\alpha}\ for\ n \in \mathbb{N}$

And to extend this to $\mathbb{R}/\mathbb{C}$ for general β, we define: ${}^\beta D^\alpha := D^{\alpha\beta}$
Then we need to verify this satisfies expected properties: ${}^{\beta_1}({}^{\beta_2}D^\alpha) = D^{\beta_1(\beta_2\alpha)} = D^{(\beta_1\beta_2)\alpha} = {}^{\beta_1\beta_2}D^\alpha$
${}^1D^\alpha = D^\alpha$
${}^0D^\alpha = D^0$ identity
This makes it a definition extended by continuity/analyticity.

Which also means that

$$M[D^\alpha]^\beta = M[D^\beta]^\alpha$$

of course that is when the operation itself is in a commutative ring like ($\mathbb{R}$,$\mathbb{C}$) , **for other rings that isn't commutative this is False** like $\mathbb{H}$

We adopt the notation ${}^\beta D^\alpha$ to denote the multiplicative action of order β on derivative D^α, distinguishing it from composition $D^\beta \circ D^\alpha$ (which gives $D^{\alpha+\beta}$

by Index Law) and the direct derivative $D^{\alpha\beta}$ but i recommend the first one as it shows we are going from α to β which may be used later without ruining the first shape of D^z operator
since the Index Law works well for Taylor series this will also work well
of course there are some things that we can state :
The derivative order addition identity: D^0
The derivative order multiplication identity: D^1
The zeroth-derivative order multiplication proprtie: D^0 or in other words, the function itself is the zero of derivative order multiplication , going to or going from

5 Deriving the Rules of the Fractional and complex derivatives

most of the other functions need us to derive the rules of the fractional Differintegar operator, the α-th derivative for functions like $tan(x)$,$arcsech(x)$,$ln(x)$ etc.. can only be found using power serieses and product rule

5.1 General product rule

One of the most important and needed formulas in calculus in the prdouct rule
let $f(x) = g(x)h(x)$

$$f'(x) = g(x)h'(x) + g'(x)h(x)$$
$$f''(x) = g(x)h''(x) + 2\ g'(x)h'(x) + g''(x)h(x)$$
$$f'''(x) = g(x)h'''(x) + 3\ g'(x)h''(x) + 3\ g''(x)h'(x) + g'''(x)h(x)$$

this is very similar to the binomial theorem , the only difference is it deals with derivatives instead of powers
the general product rule also known as the **General Leibniz rule** is

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k}(f)D^k(g)$$

this simple yet elegant formula is what we are going to use for the General product rule, before we try to do a simple substitution of α we need to use the generalized ${}^nC_k$ which means using $n! = \Gamma(n+1)$ in the formula $\frac{n!}{(n-k)!k!}$

$$D^n(fg) = \sum_{k=0}^{n} \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} D^{n-k}(f)D^k(g)$$

this is the same formula just and works the same for positive integers, but let's try to use fractions and ignore the sigma upper term and expand it , for example one half expansion will be

$$D^{1/2}(fg) = \frac{\Gamma(\frac{1}{2})}{\Gamma(1\frac{1}{2}-0)\Gamma(1)} D^{1/2}(f)D^0(g) + \frac{\Gamma(\frac{1}{2})}{\Gamma(1\frac{1}{2}-1)\Gamma(2)} D^{-1/2}(f)D^1(g) + ...$$

$$D^{1/2}(fg) = \frac{\Gamma(\frac{1}{2})}{\Gamma(1\frac{1}{2})} D^{1/2}(f)g + D^{-1/2}(f)D^1(g) + \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{1}{2})\Gamma(3)} D^{-3/2}(f)D^2(g) + ...$$

as we can see , it expands to an infinite sum, it will never stop because the lower value never hit the upper value, the main reason the simple form works for integers is that even if the k value goes higher than the n value it will get a negative integer in a Gamma which is a pole and thus equal zero
that means in order for a term to be it must not have the Gamma of integers so that means we can't find an inetgar product rule yet, but at anyway turning back the sum from what we know will be

$$D^\alpha(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} D^{\alpha-k}(f)D^k(g)$$

we can mark this as the general power rule but we need to use our handy tool the **Index law** to confirm that it works –need to prove it later–

6 Interpretation of fractional and complex derivative in other functions

6.1 Trigonometric Functions

Deriving the trigonometric functions can be quite tricky , as there exists n-th derivative formula for them but don't seem to work as intended

6.1.1 $\sin(x)$ and $\cos(x)$

for $\sin(x)$ there exists a formula which is

$$D^n \sin(x) = \sin(\frac{n\pi}{2} + x)$$

and unexpectedly this formula works for fractional or negative values, let $n = \frac{1}{2}$

$$D^{\frac{1}{2}} \sin(x) = \sin(\frac{\frac{1}{2}\pi}{2} + x) = \sin(\frac{\pi}{4} + x)$$

knowing that $D^n(D^m \sin(x)) = \sin(\frac{(n+m)\pi}{2} + x)$

$$D^{\frac{1}{2}}(D^{\frac{1}{2}} \sin(x)) = \sin(\frac{(\frac{1}{2}+\frac{1}{2})\pi}{2} + x) = \sin(\frac{2\pi}{4} + x) = \sin(\frac{\pi}{2} + x)$$

but there is another proof that this works for all real numbers.

from the eular formula $e^{ix} = cos(x) + i\sin(x)$ we can say that

$$\sin(x) = \text{Im}(e^{ix})$$

taking the alpha-th derivative of both sides

$$D^\alpha \sin(x) = D^\alpha \text{Im}(e^{ix}) = \text{Im}(i^\alpha e^{ix})$$

knowing that $i = e^{i\pi/2}$

$$D^\alpha \sin(x) = \mathrm{Im}(e^{i\pi\alpha/2}e^{ix}) = \mathrm{Im}(e^{i\pi\alpha/2+ix}) = \mathrm{Im}(e^{i(\alpha\pi/2+x)})$$

turning this back to the euler formula will give us

$$D^\alpha \mathrm{Im}(e^{ix}) = \sin(\frac{\alpha\pi}{2} + x)$$

which indeed proves it's true from the same formual we can also get the α-th for $cos(x)$ with the same formula turning this back to the euler formula will give us

$$D^\alpha \mathrm{Re}(e^{ix}) = cos(\frac{\alpha\pi}{2} + x)$$

now we can write them as

$$D^\alpha \sin(x) = \mathrm{Im}(e^{i(\alpha\pi/2+x)}) \qquad D^\alpha cos(x) = \mathrm{Re}(e^{i(\alpha\pi/2+x)})$$

or

$$D^\alpha \sin(x) = \sin(\frac{\alpha\pi}{2} + x) \qquad D^\alpha cos(x) = cos(\frac{\alpha\pi}{2} + x)$$

and to make it to the complex plane we can use these formulas also the negative derivative of these is

$$D^{-1} \sin(x) = \sin(\frac{-\pi}{2}+x) = -cos(x) \qquad D^{-1} cos(x) = cos(\frac{-\pi}{2}+x) = \sin(x)$$

and the first complex derivative of these is

$$D^i \sin(x) = \sin(\frac{i\pi}{2}+x) = \sin(\frac{ln(-1)}{2}+x) \qquad D^i cos(x) = cos(\frac{i\pi}{2}+x) = cos(\frac{ln(-1)}{2}+x)$$

6.1.2 $\tan(x)$ and $\sec(x)$

finding the alpha-th derivative for $tan(x)$ is quite hard since we didn't get any direct formulas for quotients , and there is no direct integar derivative formula we can plug in and generalize to the Real numbers, we can try to change it a little with some algabera

$$tan(x) = \sin(x)\ (cos(x))^{-1}$$

and then use the general product rule ,but quickly we can see the problem

$$D^\alpha(\sin(x)\ (cos(x))^{-1}) = \sum_{k=0}^{\infty} \frac{\Gamma(0)}{\Gamma(\alpha-k+1)\Gamma(k+1)} D^{\alpha-k}(\sin(x)) D^k(cos(x)^{-1})$$

there is a gamma pole so this soltuion also fails
we can try using some trig substation

$$tan(x) = \sin(x)\ (\sqrt{\sin(x)^2+1})^{-1}$$

but this leads to infinite sum for the product rule and the chain rule that we have proved above it impossible to find with the simple algabera we have is very hard to approximate by itself as we can see there is no simple elegent closed form for $tan(x)$ in the scientific paper, the reason behind this will be explained later but we can simply say becuase it has poles , to analytically see it better we need the Maclurin series expansion

$$tan(x) = x + \frac{x^3}{3!} + \frac{2x^5}{15} + ... = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} \qquad where\ |x| < \frac{\pi}{2}$$

if we look closely we can notice the problem , it has a radius of converges and that by itself is the problem that will be discussed in detail later all what we can do for now is applying the D^α to the infinite sum as it will be the only analytic closed form way for now
we will get:

$$D^\alpha tan(x) = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{\Gamma(2n)} \left[\frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}\right]$$

$$= \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}$$

this infinite series shall work for now
the same also works for $sec(x)$ as it's a quotient so if we tried using the General product rule we will hit a Gamma pole, so the safest answer for now is to go with infinite series

$$sec(x) = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{E_{2n}(-1)^n}{(2n)!} x^{2n} \qquad where\ |x| < \frac{\pi}{2}$$

again we see the same problem with radius of convergence
simply we apply D^α:

$$D^\alpha sec(x) = \sum_{n=0}^{\infty} \frac{E_{2n}(-1)^n}{\Gamma(2n)} \left[\frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha}\right]$$

$$= \sum_{n=0}^{\infty} \frac{E_{2n}(-1)^n}{\Gamma(2n-\alpha+1)} x^{2n-\alpha}$$

Note: these two work for their radius of converges only
Note: they also work for the complex derivative

6.1.3 $\csc(x)$ and $\cot(x)$

now saying $\csc(x)$ and $\cot(x)$ will work the same like the rest of trigonometric functions is a bit of a stretch
since we already know both of their domains aren't $x \in \mathbb{R}$ so they mus t have

some sort of analytical poles and converges radius that isnt $\mathbb{R}$ in their infinte sums
but if we noticed

$$\csc(x) = \frac{1}{\sin(x)}$$

which means that it has a singularity at $x = 0$, in other words simple Taylor series nor simple Maclurin series won't work, we need the General Laurent series for this one The Laurent series is :

$$\csc(x) = \frac{1}{x}+\frac{x}{6}+\frac{7x^3}{360}+... = \sum_{n=0}^{\infty} \frac{B_{2n}(-1)^{n+1}(2^{2n}-1)}{(2n)!} x^{2n-1} \qquad where\ 0 < |x| < \pi$$

but before plugging the D^z operator to the series we can notice a little problem in the begnning, the $\frac{1}{x}$ term
simply plugging in the $D^z(x^n)$ will resault a pole, the simple solution is just to the linearity of D^z and differentiate the first term alone and then the rest of the series alone

$$D^\alpha \csc(x) = D^\alpha(x^{-1}) + \sum_{n=1}^{\infty} \frac{B_{2n}(-1)^{n+1}(2^{2n}-1)}{\Gamma(2n)} \left[\frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}\right]$$

$$= \frac{(-1)^\alpha \Gamma(1+\alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{B_{2n}(-1)^{n+1}(2^{2n}-1)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}$$

of course this is where $0 < |x| < \pi$
The same goes for $\cot(x)$ as it doesn't have any Taylor series but rather Laurent series
The Laurent series for $\cot(x)$ is:

$$\cot(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} x^{2n-1} \qquad where\ 0 < |x| < \pi$$

Applying D^α to both sides we get

$$D^\alpha \cot(x) = \frac{(-1)^\alpha \Gamma(1+\alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}$$

Note: this works to complex numbers too

6.2 Hyperbolic Functions

6.2.1 $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$

$\sinh(x)$ is pretty straight forward to get
from the definition

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

differentiating both sides to the α

$$D^\alpha \sinh(x) = \frac{1}{2}(D^\alpha e^x - D^\alpha e^{-x}) = \frac{1}{2}(e^x - (-1)^\alpha e^{-x})$$

we can also do the same for $\cosh(x)$

$$D^\alpha \cosh(x) = \frac{1}{2}(D^\alpha e^x + D^\alpha e^{-x}) = \frac{1}{2}(e^x + (-1)^\alpha e^{-x})$$

but if we change the negative sign in $\sinh(x)$ to $+(-1)$ we turn the derivative to

$$D^\alpha \sinh(x) = \frac{1}{2}(e^x + (-1)^{\alpha+1} e^{-x})$$

which is equal to $D^{\alpha+1}\cosh(x)$ and that is becuase unlike normal $\sin(x)$ and $\cos(x)$ these are the integer integrals and derivatves of their-selves so we can get the negative derivatives to be

$$D^{-1}\sinh(x) = \frac{1}{2}(e^x+e^{-x}) = \cosh(x) \qquad D^{-1}\cosh(x) = \frac{1}{2}(e^x-e^{-x}) = \sinh(x)$$

for the complex derivatives we can use the formulas from before
However for $\tanh(x)$ things change , since the defention for it is

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

we can see , it's a quotient of two functions ,which leaves us with nothing but to use it's power series
the power series for tanh is:

$$\tanh(x) = x - \frac{x^3}{3!} + \frac{2x^5}{15} - ... = \sum_{n=0}^{\infty} \frac{B_{2n}4^n(1-4^n)}{(2n)!} x^{2n-1} \qquad where\ |x| < \frac{\pi}{2}$$

as predicted there will also be radius of converges here too
but at anyway we get the D^α with this:

$$D^\alpha tanh(x) = \sum_{n=0}^{\infty} \frac{B_{2n}4^n(1-4^n)}{\Gamma(2n)} \left[\frac{\Gamma(2n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}\right]$$

$$= \sum_{n=0}^{\infty} \frac{B_{2n}4^n(1-4^n)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}$$

6.2.2 $\operatorname{sech}(x)$, $\operatorname{csch}(x)$ and $\coth(x)$

The rest of the hyperbolic functions shall work the same as the trigonometric functions, infact all of the hyperbolic and the trigonometric function's Laurent/Taylor seriesses look identical with little changes, so finding them won't be

that diffcult
For $\operatorname{sech}(x)$ the Taylor Series is:

$$\operatorname{sech}(x) = 1 - \frac{x^2}{2!} + \frac{5x^4}{4!} - ... = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n} \qquad \textit{where } |x| < \frac{\pi}{2}$$

So Applaying D^z will be as simple as $\sec(x)$

$$D^\alpha \operatorname{sech}(x) = \sum_{n=0}^{\infty} \frac{E_{2n}}{\Gamma(2n-\alpha+1)} x^{2n-\alpha} \qquad \textit{where } |x| < \frac{\pi}{2}$$

the same goes for the Laurent series of $\operatorname{csch}(x)$ and $\coth(x)$

$$\operatorname{csch}(x) = \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - ... = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{B_{2n}(1-2^{2n})}{(2n)!} x^{2n-1} \qquad \textit{where } 0 < |x| < \pi$$

of course it's similar but not identical , anyway applying D^α gives us

$$D^\alpha \operatorname{csch}(x) = D^\alpha x^{-1} + D^\alpha \sum_{n=0}^{\infty} \frac{B_{2n}(1-2^{2n})}{(2n)!} x^{2n-1}$$

$$= \frac{(-1)^\alpha \Gamma(1+\alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{B_{2n}(1-2^{2n})}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1} \qquad \textit{where } 0 < |x| < \pi$$

and for $\coth(x)$:

$$\coth(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1} \qquad \textit{where } 0 < |x| < \pi$$

and applaying D^α operator we get:

$$D^\alpha \coth(x) = \frac{(-1)^\alpha \Gamma(1+\alpha)}{x^{1+\alpha}} + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(2n-\alpha+1)} x^{2n-\alpha-1}$$

6.3 The Inverse Trigonometric and Hyperbolic Functions

there is a problem with these function that makes them special, if we for example tried to take the derivative for $\sin^{-1}(x)$ and the integral of the same function we get this

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \quad -1 < x < 1 \qquad \int \sin^{-1}(x)dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C$$

as we can see they are a type of **Transformtive functions** which will be discussed in later sections

7 The nature of functions under D^z Field

7.1 Explaining what is a fractional derivative

taking an positive Integer derivative gives us the rate of change of a function , talking the derivative of that also gives us the rate of change the derivative function , so on so forth
taking the negative Integer derivative is known as the "Area under the curve" or the function in which the original function is the rate of change of it , or simply the "Anti-derivative"
the fractional derivative can be thought as

7.2 what it means for a derivative order to be Complex

as we saw how the fractional derivative can say the complex derivative take a weird turn as it actually not a nor rate of change of function...sort of
to get an understanding of what i am talking about let's see it on a simple function x^n and see how it works
we know that:

$$D^z(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-z+1)}x^{n-z}$$

substituting $z = i$

$$D^i(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-i+1)}x^{n-i}$$

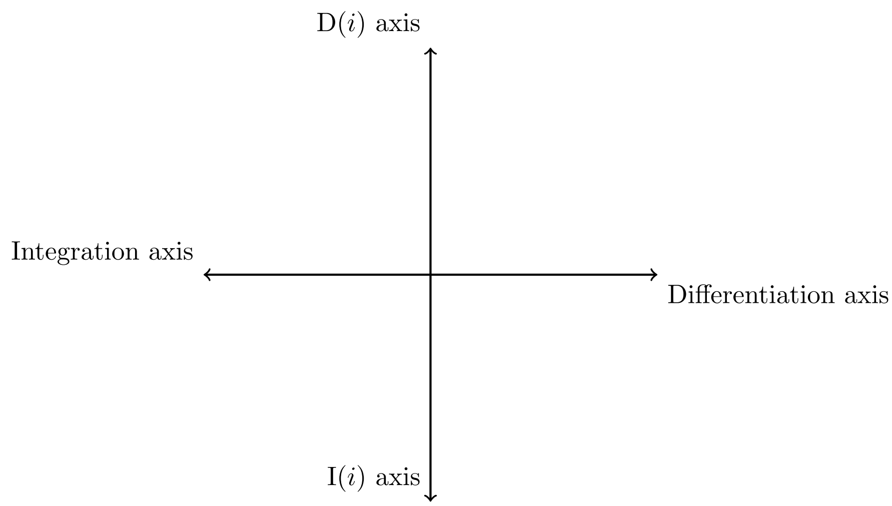
the nature of this will be explored later but for now if we try to take another imaginary derivative using the **Multiplication Law** we proved earlier we see that

$$^iD^i(x^n) = D^i(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-(i\times i)+1)}x^{n-(i\times i)} = \frac{\Gamma(n+1)}{\Gamma(n-(-1)+1)}x^{n-(-1)} = \frac{\Gamma(n+1)}{\Gamma(n+2)}x^{n+1}$$

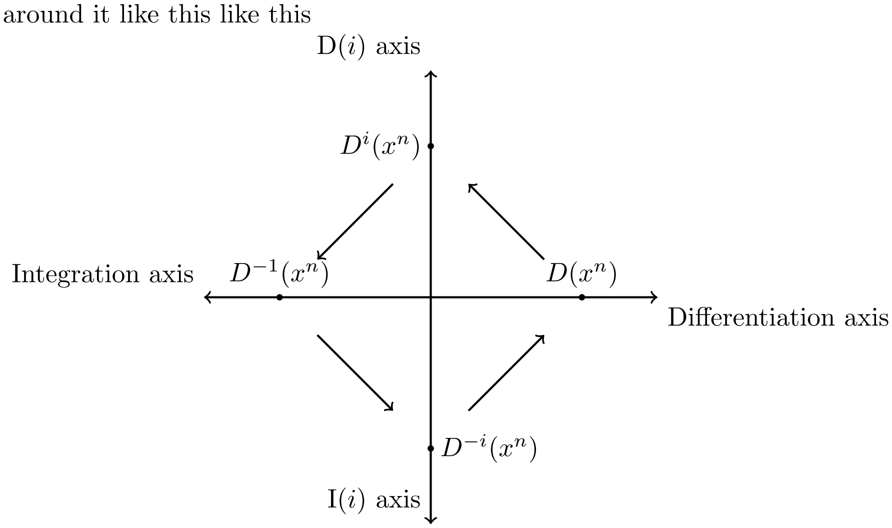
and knowing that $\Gamma(n+1) = n\Gamma(n)$

$$^iD^i(x^n) = \frac{\Gamma(n+1)}{(n+1)\Gamma(n+1)}x^{n+1} = \frac{x^{n+1}}{n+1}$$

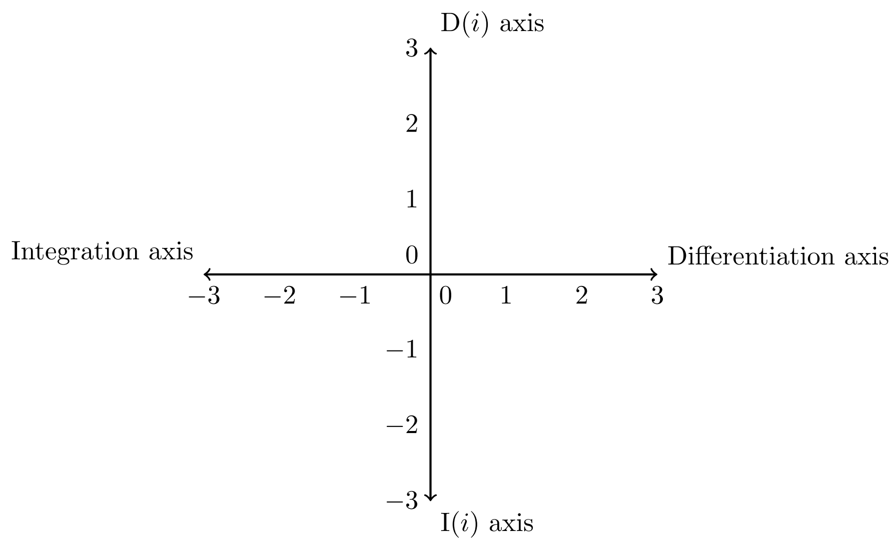
infact this was expected since $^iD^i = D^{(i\times i)} = D^{-1}$
the result is kind of weird , clearly it's weird
to see why it's weird (if it'snot already) we can use a geometric interpretation from the information we have right now we can draw this transformation in D-I plane



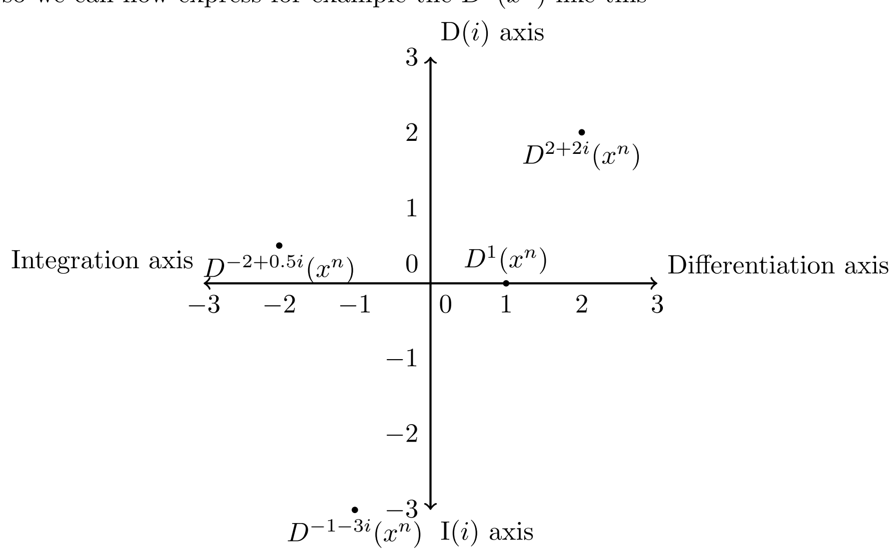
as this map acts as the same as complex plane a we map the transformation around it like this like this



this is how the first derivative act when using the complex number i
which means also that means there exist a way (if not multiple) to represent differentiation and integration geometrically in one same space
we can add more detail to this plane by defining each access as the the order of the D opreator so we can write it better like this

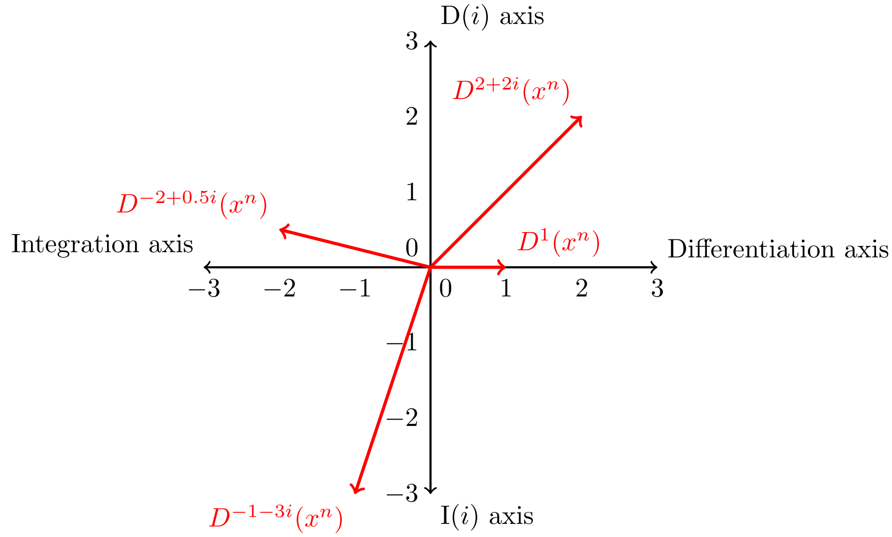


with the y-axes being the imaginary order part and the x-axis being the normal part, we can of course now express any derivative or integral or any weird result of D^z as points in this space.
that is, of course for one single function as the input is a variable of type of function and the output is of variable function so for now they must stay the same function with the same variables inside of it
so we can now express for example the $D^z(x^n)$ like this



this also allows us to use any root of unity not only i like a normal Re-Im plot plane, of course we can change the perspective and and think of them as vectors in a 2D vector field
for each derivative we can draw them as vectors starting from zero point (the

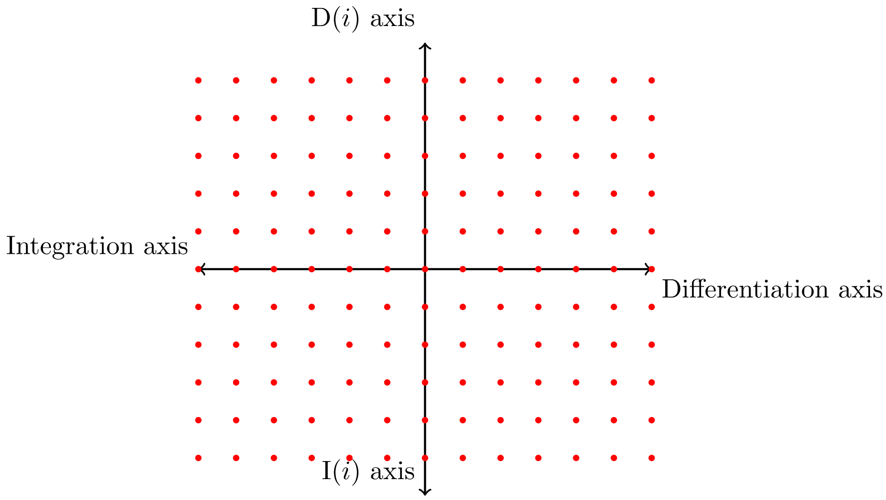
function itself) to the point point of derivative order



which means that the vector rules also work here fine with the D^z operator properties as we can see adding and subtracting orders will return another vector using the Index Law, and since we know what the plane is for we can express the orders from $D^{\alpha+\beta i}$ as points $(\alpha, \beta i)$ or vectors $\vec{a} = (\alpha, \beta)$, and supposedly polar (r, θ), of course if we want to make a vector from a point to another we can just use the index law that goes in the path to β which also means that there is infinite starting points for vectors to go to the same point, in another words : infinite vectors for any point and to any point

this can be called the **D(i) Plane**, even when dealing with vectors , at the end this plane isnt really based on the idea of it but rather an extension to it

this may helpful , but i suggest another idea that is based on vectors in such field and I would like to call it the **D(i) Vector Field**
isntead of using vectors in the plane as beginning and ending points we will use them as points,arrows and degrees, sort of a mix of the polar form and the vector form of the plane
1- the point: will represent it's head, where it's coming out of
2- the arrow : will the output of the function derivative rate of change, the bigger it's the higher the rate of change its
3- the degree: it will represent how the D^z operator is changing the function itself
for the sake of understanding let's take the function e^{2x} as an example



7.3 Quaternions derivatives

this will be fascinating as it operates on 3D

8 The Geometric DifferentInetgrel (GDI) theory

8.1 The weird similarity of derivative and

8.2 e^x and Cyclic functions

if we look closely in functions that have **Cyclic-Derivatives** we can see that there exists a pattren for example

$$D^\alpha e^x = e^x$$

$$D^\alpha \sinh(x) = \frac{1}{2}(e^x - (-1)^\alpha e^{-x}) \qquad D^\alpha \cosh(x) = \frac{1}{2}(e^x + (-1)^\alpha e^{-x})$$

$$D^\alpha \sin(x) = \mathrm{Im}(e^{ix}) \qquad D^\alpha cos(x) = \mathrm{Re}(e^{ix})$$

they all have connection with e^x in them
the function e^x by itself is a cyclic derivative function , it always return itself no matter how many times you differentiate it if we change the function slightly to e^{ax} things now change as it's returns different results

$$when\ a > 1 \quad \lim_{\alpha\to\infty} D^\alpha(e^{ax}) = \infty \qquad\qquad when\ a < 1 \quad \lim_{\alpha\to\infty} D^\alpha(e^{\alpha x}) = e^{ax}$$

these are all easy known results of course $when\ a = 1 \ \ \lim_{\alpha\to\infty} D^\alpha(e^{\alpha x}) = e^{ax}$
these are all expected to be the result to repeating multiplication of $a^\alpha e^{ax}$

but there is one case for a where it's not any of the above
let's examine the cyclic derivative functions closely

$$D^\alpha e^x = e^x$$

if we try to write in general e^{ax} formula we get

$$D^\alpha e^{1x} = 1^\alpha e^{1x}$$

which can also explain (with what we already know) why e^x is a repeated derivative of itself
let's look at hyperbolic functions which are cyclic derivatives of order 2

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

if we solve for e^{-x} in both functions
we get for $\sinh(x)$

$$2\sinh(x) = e^x - e^{-x} \qquad \therefore 2\sinh(x) - e^x = -e^{-x} \qquad \therefore e^x - 2\sinh(x) = e^{-x}$$

we can do the same for $\cosh(x)$

$$2\cosh(x) = e^x + e^{-x} \qquad \therefore 2\cosh(x) - e^x = e^{-x}$$

let's add both equations together we get

$$2e^{-x} = 2\cosh(x) - e^x + e^x - 2\sinh(x) = 2\cosh(x) - 2\sinh(x)$$

dividing both sides by 2 we get

$$e^{-x} = \cosh(x) - \sinh(x)$$

which is truly an order 2 cyclic derivative function ,if we took the first derivative of the function we get

$$D(e^{-x}) = D(\cosh(x)) - D(\sinh(x)) =$$

$$-e^{-x} = \sinh(x) - \cosh(x)$$

this is true since if we multiplied by -1 in the first expression we get the same result
of course we can express these in terms of e^x like this

$$e^x = \cosh(x) + \sinh(x)$$

which is true since both sides will return the same result no matter how many times we differentiate them
it looks sort of like Euler formula but for hyperbolics
notice that -1 that is in e^{-x} comes from the equation $a^2 = 1, a = \pm 1$ because if we differentiate any of the e^x or e^{-x} expressions twice we get the the same

expressions back
if the idea didn't click in yet let's look at trigonometric
for trigonometric functions but this time we have a formula ready for us

$$e^{ix} = \cos(x) + i\sin(x)$$

differentiating both sides we get

$$ie^{ix} = -\sin(x) + i\cos(x)$$

which again works well ,and if we differentiate again 3 times more we get back to the same expression
notice that here $a = i$ infact not only i but any value that satsifies the expression $a^4 = 1$ works as well , and the returning function will be order 4 cyclic derivative function
the pattren here goes on and on and this is exactly the missing case for a

$$when\ a^n = 1, D^n(e^{ax}) = e^{ax}\ with\ 2 \times n\ cyclic\ order$$

that is because it holds for both positive and negative values of a, which means that the case $a^1 = 1$ the order is only once cyclic since it has only positive values
for case $a^2 = 1$ we get e^{-x} which is 2nd order cyclic derivative
and for case $a^4 = 1$ we get e^{ix} which is 4th order cyclic derivative
this formula helps us generate any order of cyclic derivatives as for example a 3rd order cyclic derivatve will have a as

$$a^3 = 1, a = 1\ or\ -\frac{1}{2} + i\frac{\sqrt{3}}{2}\ or\ -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

and all of them work exactly as intended, also we can see that some cyclic-derivatives work correctly under the order of another cyclic derivatives like order 2 (sinh and cosh) under order 4 (sin and cos) this is because in order for $a^n = 1$ to be true isn't only n times but it works for n and 2n and 3n so on because every one of them will lead to 1,$a^n = a^{2n} = a^{3n} = 1$ in other words the higher cyclic-derivatives that works the same will satsify the property

$$when\ a^n = 1, D^n(e^{ax}) = e^{ax}\ with\ 2 \times k\ cyclic\ order,\ where\ k \equiv 0 \mod n$$

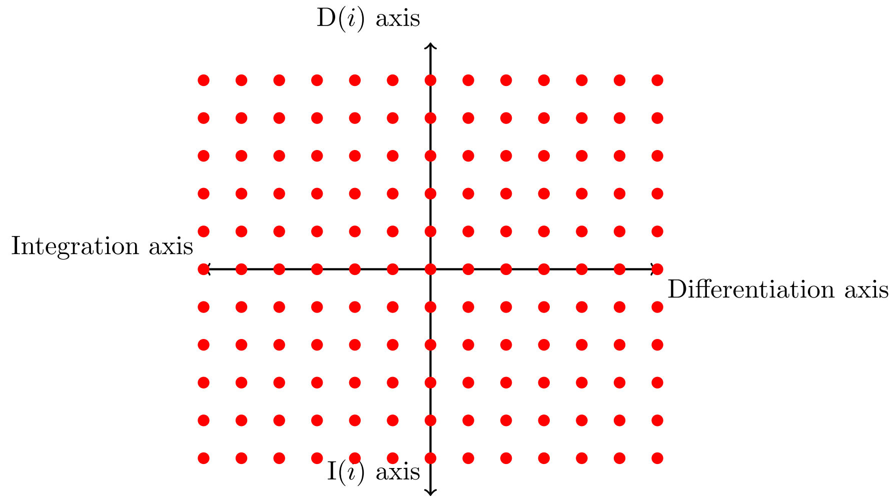
getting back to the basics both of case $a^2 = 1$ and case $a^4 = 1$ can be expanded as trigonometric or hyperbolic functions, not only that but both of them satisfy the condition 2^n and both of them can be expanded algebraically, for hyperbolic it's $x^2 - y^2 = 1$ and $x^2 + y^2 = 1$ for trigonometric functions, and since they are cyclic derivatives they are functions of type FRD, which suggests that they may exist a pattren of function families that are

$$e^{ax}\ where\ a^n = 1\ that\ satisfies\ the\ condition\ 2^n \in \mathbb{Z}^+$$

and these function families may also have algebraic expression that also suggests it will be from high order operator that isn't simple plus or minus also since -1 and i can both be explained geometrically as "rotations" we can also say that *n is the number of unity roots* which also suggests cyclic order derivatives but in 3D or 4D using Quaternions and octonions

8.3 Geometric Interpretation of cyclic derivatives

infact we can do this now
let's start simple with e^x in the D^z Re-Im Plane



this is expected since e^x is the exact value of itself at any point , even if we tried to represent them as vectors we will get the same results because the it always begins and ends at any value of the order derivative , in other words there is no direction for the vector to point at, but this plane while may show something but isnt so useful, instead we can use the D(i) vector plane
we can try the same for e^{ax}

8.4 GDI principles and infinite seriesses

8.5 FRD and NFRD

these are two important terms that will be helpful for us
FRD : Full Real Differentiable **NFRD** : Not-Full Real Differentiable
these are two very important terms as they catagories all the functions in D^z since most of the function are either Integrable and Differentiable or Integrable and not Differentiable and D^z can work on both sides
FRD Functions are functions that have a clean closed form for it's α-th derivative that isn't a power series (that is of course if the function itself isn't defined by a power series)
these are function like x^n with the condition that $\text{Re}(z) \leq n$ and e^x
NFRD function are function that has an only closed form being a power series

, so function like $\tan(x)$ and other trigonometric function are NFRD
this happens of course for a lot of reasons like breaking GDI first principle and

8.6 Transformative Functions

They are the type of functions that transform dependence with derivative , for example

$$\frac{d}{dx}(\cos^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$$

as we can see , when we take the derivative of the function it change dependence from the value of x to the value of the power of x

8.7 The inverse functions problem

Inverse functions are a problem , at least for being FRD function as it's not possible for most of them and I have some explanations
Inverse functions are weird compared to the normal functions , we can see some similties between them and their original functions like x^n and $x^{\frac{1}{n}}$ as they both look identical for odd values because for even numbers we get complex values , but rotated and cutted to half, in most of the inverse functions we can find these cuts near always
we can expect this because not every function can be fully inversed
a function is set of inputs that lead to outputs, one output can be gotten by many inputs but not the way around , so when we try to inverse functions like x^2 that has one output for the more than one input we get something that doesnt work in the real number systems, and for this example it's the complex values for negative integers

8.8 D^z space

8.9 $ln(x)$ and rotation in D^z space

9 Matrix derivative functions

9.1 Whole matrix derivatives

we start with the simple function x^n, and let our matrix be a simple A matrix, we can right the $D^A x^n$ like this:

$$D^A(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-A+1)} x^{n-A}$$

as we can see this shape is hard , and a better simplification for it is to use $e^{ln(x)}$ so it will be :

$$D^A(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-A+1)} x^n e^{-\ln(x)A}$$

this is better as now we can express it as infinite sum using the Taylor series of function e^x

$$D^A(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-A+1)} x^n \sum_{k=0}^{\infty} \frac{(\ln x)^k}{k!} A^k$$

where $A^k = \underbrace{A \times A \times A \times ....}_{k \text{ times}}$, and this can be called the simple $D^A x^n$

let's plug in this definition in the Taylor series general formula

$$D^\alpha f(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-\alpha+1)} (x-a)^{n-\alpha}$$

so now it shall be

$$D^A f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(n-A+1)} (x-a)^{n-A}$$

and thus the Maclaurin series is

$$D^A f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n-A+1)} x^{n-A} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n-A+1)} x^n e^{-\ln(x)A}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n-A+1)} x^n \sum_{k=0}^{\infty} \frac{(\ln x)^k}{k!} A^k$$

now we know we can do it to any analytical function, and the simple way to put it in any series is to remove the α fro mthe power of and put the term $\sum_{k=0}^{\infty} \frac{(\ln x)^k}{k!} A^k$

9.2 What is the meaning of matrix derivatives

10 Functional derivatives

10.1 derivative order as function of itself

10.2 derivative order as function of it's independent variable

11 Bonus Topics

11.1 General chain rule

11.2 Differential Geometry Connection

11.3 Fourier Transform Connection Connection

11.4 Category Theory

11.5 Number Theory and D^z operator

11.5.1 Logarithmic functions, function $D^z Ln(x)$ and it's nature

11.5.2 The zeta function $D^z \zeta(n)$ and it's nature

11.5.3 The legnedre function and it's nature

11.6 Combinations and D^z operator

11.7 other functions and D^z operator

11.7.1 The Gaussian Integral $D^z e^{-x^2}$ and it's nature

12 Integration with D^z operator

12.1 Deriving the D^{-z} formulas

12.2 Fractional and Complex Integration

13 Fractional and Complex Differential equations

14 Applications in real world

14.1 Dynamics with D^z operator

14.2 Electrodynamics with D^z operator

14.3 Fluid dynamics with D^z operator

14.4 Quantum Mechanics with D^z operator

14.5 Probability and Stochastic Processes with D^z operator

15 Table of formulas

$$\mathbb{R}\ {}_{t}^{\beta}D_{c}^{\alpha}$$

this is the stander notation for this research and it reads:

c : The derivatve number in terms of context , default value :1
α : The derivative order , default value : 1
β : The multiply derivative order , default value : 1
t : the variable derived with repsect to , default value : time
$\mathbb{R}$: the derivative output number set , default value : real numbers
it's read from left down courner counter clockwise in to out:
the derivtave number c of order α to β with respect to t in $\mathbb{R}$

16 Brief letter and thanks