

Complex-Order and Fractional Derivatives: Cyclic derivatives and functions

Faisal Khalid

November 2025

Abstract

This paper presents an independent exploration of cyclic derivatives based on the fractional and complex-order derivatives paper made by the same Author.

Note to Readers: This represents independent rediscovery of classical fractional calculus concepts. I (The Author) present this work as a pedagogical exercise in mathematical exploration rather than novel research.

Background

The Main formula that works for all derivatives and can be used in the Maclaurin series

$$D^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

The more important formulas that are built on this one

$$D^\alpha e^{ax} = a^\alpha e^{ax} \quad e^{ix} = \cos(x) - i \sin(x)$$

$$e^x = \cosh(x) + \sinh(x) \quad e^{-x} = \cosh(x) - \sinh(x)$$

$$D^\alpha \sin(x) = \sin\left(\frac{\alpha\pi}{2} + x\right) \quad D^\alpha \cos(x) = \cos\left(\frac{\alpha\pi}{2} + x\right)$$

$$D^\alpha \sinh(x) = \frac{e^x - (-1)^\alpha e^{-x}}{2} \quad D^\alpha \cosh(x) = \frac{e^x + (-1)^\alpha e^{-x}}{2}$$

1 Introduction

from what we have seen in the previous fractional and complex-order derivatives paper, cyclic derivatives are such a big area that deserves its own paper,

2 The foundation

From the fractional and complex-order derivatives paper, we know that

$$\text{when } a^n = 1, D^n(e^{ax}) = e^{ax} \text{ with } 2 \times n \text{ cyclic order}$$

And we can call that the theorem of cyclic derivatives

Theorem 1 $f(x)$ is a cyclic derivative when $a^n = 1, D^n(e^{ax}) = e^{ax}$ with $2 \times n$ cyclic order

And in the same paper, we generalized this to hold for any $k \equiv 0 \pmod{n}$

Theorem 2 $f(x)$ is a lesser cyclic derivative when $a^n = 1, D^n(e^{ax}) = e^{ax}$ with $2 \times k$ cyclic order, where $k \equiv 0 \pmod{n}$

With a hypothesis that I wish to prove in this paper, that

Hypothesis 1 For every function cyclic derivatives that can be written in the form e^{ax} where $a^n = 1$ and satisfies the condition $2^n \in \mathbb{Z}^+$, there exists an algebraic perimetric form

We can already see this for hyperbolic functions, where they can be written in the form $x^2 - y^2 = 1$, and trigonometric functions We also proved some equations

$$\begin{aligned} D^i e^{-x} &= e^{-(x+\pi)} = \cosh(x+\pi) - \sinh(x+\pi) \\ D^i \sinh(x) &= \frac{e^x - e^{-(x+\pi)}}{2} & D^i \cosh(x) &= \frac{e^x + e^{-(x+\pi)}}{2} \\ D^i e^{ix} &= e^{\frac{-\pi}{2}} \cos(x) + ie^{\frac{-\pi}{2}} \sin(x) \\ D^i \sin(x) &= \sin\left(\frac{i\pi}{2} + x\right) & D^i \cos(x) &= \cos\left(\frac{i\pi}{2} + x\right) \end{aligned}$$

and we also proved that for $\sin(x)$ and $\cos(x)$

3 More about the complex derivatives and known families

from what we know about $\sinh(x)$ and $\cosh(x)$ we can write their formulas in an other way

$$\begin{aligned} D^\alpha \sinh(x) &= \frac{e^x - (-1)^\alpha e^{-x}}{2} = \frac{e^x - (e^{i\pi})^\alpha e^{-x}}{2} = \frac{e^x - (e^{i\pi\alpha})e^{-x}}{2} \\ &= \frac{e^x - e^{i\pi\alpha-x}}{2} \end{aligned}$$

The same goes for $\cosh(x)$

$$D^\alpha \sinh(x) = \frac{e^x - e^{i\pi\alpha-x}}{2} \quad D^\alpha \cosh(x) = \frac{e^x + e^{i\pi\alpha-x}}{2}$$

We can see that we can't express them as simple forms, since the real value changes differently from the complex value. If we try to apply the derivative operator to e^x with the definition of $1 = e^{2i\pi}$, we get

$$D^\alpha e^x = 1^\alpha e^x = e^{2i0\alpha} e^x = e^x$$

We used that because it's the principal value

For trigonometric functions, we know that D^i represents a half turn before it changes to its integral from the real plane, but there is something to clarify

$$\text{Im}(D^i e^{ix}) = e^{\frac{-\pi}{2}} \sin(x)$$

As we can see, the first i -th derivative acts as a scaler that scales $\sin(x)$ by real value $e^{-\frac{\pi}{2}}$

However, this isn't equal to the $D^i \sin(x)$ as the definition changes from scaling to rotating, like this

$$D^i \sin(x) = \sin\left(\frac{i\pi}{2} + x\right)$$

But since we have proven the multiplication law works in the framework, we can make sure that both are somewhat equal

$${}^i D^i e^{ix} = i^{i \times i} e^{ix} = (e^{\frac{i\pi}{2}})^{-1} e^{ix} = e^{\frac{-i\pi}{2}} e^{ix} = e^{ix - \frac{i\pi}{2}}$$

$$e^{i(x - \frac{\pi}{2})} = \cos\left(x - \frac{\pi}{2}\right) + i \sin\left(x - \frac{\pi}{2}\right) = \sin(x) - i \cos(x) = D^{-1} e^{ix} = \int e^{ix}$$

and for the sin we proved in the "Complex-order and fractional derivatives: first exploration" paper that the index law works on it, and thus the multiplication law either from here or from the series expansion, so that we can say

$${}^i D^i \sin(x) = \sin\left(\frac{i \times i\pi}{2} + x\right) = \sin\left(\frac{-\pi}{2} + x\right) = -\cos(x)$$

Thus, both of them work fine, then why?

3.1 The rotation in complex rotation

since D^i represents a whole rotation on the $D(i)$ plane, we can get more angles that could help us understand more what happens to the function to "integrate it"

first step is we are going to transform from i to $e^{\frac{i\pi}{2}}$ so we can deal with rotation with radians in circles

$$\begin{aligned} D^i e^{ax} &= D^{e^{\frac{i\pi}{2}}} e^{ax} & D^{-1} e^{ax} &= D^{e^{i\pi}} e^{ax} \\ D^{e^{i\theta}} e^{ax} &= a^{e^{i\theta}} e^{ax} & D^{e^{i\theta}} e^{ix} &= i^{e^{i\theta}} e^{ix} = e^{\frac{i\pi e^{i\theta}}{2}} e^{ix} = e^{i(\frac{\pi e^{i\theta}}{2} + x)} \\ &= \cos\left(\frac{\pi e^{i\theta}}{2} + x\right) + i \sin\left(\frac{\pi e^{i\theta}}{2} + x\right) \end{aligned}$$

So now we can know what happens at the third of rotation or the third root of unity, which is equal to $\frac{\pi}{3}$ in radians, we get

$$D^{e^{\frac{i\pi}{3}}} e^{ix} = e^{i(\frac{\pi e^{\frac{i\pi}{3}}}{2} + x)} = \cos(\frac{\pi e^{\frac{i\pi}{3}}}{2} + x) + i \sin(\frac{\pi e^{\frac{i\pi}{3}}}{2} + x)$$

which after calculating $e^{\frac{i\pi}{3}}$ to be $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ we can then multiply it by $\frac{\pi}{2}$ to get $\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4}$, then we plug it

$$e^{i(\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4} + x)} = e^{i\frac{\pi}{4} + i^2\frac{\pi\sqrt{3}}{4} + ix} = e^{i(x + \frac{\pi}{4})} e^{-\frac{\pi\sqrt{3}}{4}} = e^{-\frac{\pi\sqrt{3}}{4}} \cos(x + \frac{\pi}{4}) + i e^{-\frac{\pi\sqrt{3}}{4}} \sin(x + \frac{\pi}{4})$$

We can see that it scales by a factor of $e^{-\frac{\pi\sqrt{3}}{4}}$ and rotate with a factor of $\frac{\pi}{4}$
let's do the same for two-thirds, the value for $e^{\frac{2i\pi}{3}}$ to be $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$ we can then multiply it again by $\frac{\pi}{2}$ to get $-\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4}$
Plugging it again, we get

$$e^{i(-\frac{\pi}{4} + i\frac{\pi\sqrt{3}}{4} + x)} = e^{-i\frac{\pi}{4} + i^2\frac{\pi\sqrt{3}}{4} + ix} = e^{i(x - \frac{\pi}{4})} e^{-\frac{\pi\sqrt{3}}{4}} = e^{-\frac{\pi\sqrt{3}}{4}} \cos(x - \frac{\pi}{4}) + i e^{-\frac{\pi\sqrt{3}}{4}} \sin(x - \frac{\pi}{4})$$

At two-thirds, it rotates with the same value but rotates backwards
Now we have a little information about what happens in the process of integrating such functions

at the first third of the way, it rotates by $\frac{\pi}{4}$ and scales by $e^{-\frac{\pi\sqrt{3}}{4}}$

For Halfway, it doesn't rotate but scales with a factor of $e^{-\frac{\pi}{2}}$

for two-thirds it rotates by $\frac{\pi}{4}$ and scales by $e^{-\frac{\pi\sqrt{3}}{4}}$

This may seem weird at the beginning until we notice that we aren't starting from order 1 or D^1 , we are starting from the zero point D^0 or the function itself, so the one-third and two-thirds don't cancel out on rotation, but they rotate to two different directions

The one-third rotates to D^1 and the two-thirds rotate to D^{-1} , while the middle point D^i doesn't rotate but scales because it's not a real derivative or real integral

We can even notice that in the first third we have $\cos(x + \frac{\pi}{4})$ and $\sin(x + \frac{\pi}{4})$, which are both pure half derivatives

$$D^{\frac{1}{2}} \sin(x) = \sin(\frac{\frac{1}{2}\pi}{2} + x) = \sin(\frac{\pi}{4} + x) \quad D^{\frac{1}{2}} \cos(x) = \cos(\frac{\frac{1}{2}\pi}{2} + x) = \cos(\frac{\pi}{4} + x)$$

and the same happens for the half-integer being rotated by $\frac{\pi}{4}$

More on this will be discussed in later sections

4 exploration into another cyclic derivatives

4.1 the third order cyclic derivatives

From the theorem, we can find the third cyclic derivative to be from the equation $a^3 = 1$, the solutions are going to be denoted by $1, \omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

since one will result in e^x and is fully expected to be here because of Theorem 2, we are going to use ω

$$D^1 e^{\omega x} = \omega e^{\omega x} \quad D^2 e^{\omega x} = \omega^2 e^{\omega x} \quad D^3 e^{\omega x} = e^{\omega x}$$

We can call this function the third-order cyclic derivative, which comes between hyperbolic and trigonometric functions we will name them \sinh_3, \cosh_3 and $\sinh_3 \text{ II}$ We can define them like this

$$D^\alpha \sinh_3(x) = \sinh_3 \text{ II}(x) \quad \alpha \equiv 0 \pmod{3} \quad D^\alpha \sinh_3 \text{ II}(x) = \cosh_3(x) \quad \alpha \equiv 1 \pmod{3}$$

$$D^\alpha \cosh_3(x) = \sinh_3(x) \quad \alpha \equiv 2 \pmod{3}$$

But this isn't the only way to define them, we can also define them with a series First we find the Maclaurin series for $e^{\omega x}$

$$e^{\omega x} = e^0 + \omega e^0 x + \frac{\omega^2 e^0 x^2}{2!} + \frac{e^0 x^3}{3!} + \dots = 1 + \omega x + \frac{\omega^2 x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{\omega^n x^n}{n!}$$

From this, we can divide them into three sums

$$(1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \dots) + \omega(x + \frac{x^4}{4!} + \frac{x^7}{7!} + \dots) + \omega^2(x^2 + \frac{x^5}{5!} + \frac{x^8}{8!} + \dots)$$

$$= \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \omega \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \omega^2 \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

We can see that the first sum can be differentiated 3 times before going back to the first state, which is also for all the other sums, but since \sin and \sinh all have x^{an+1} , we are going to make the first function to be the second sum, to keep the naming consistent nothing more

So now we can define them to be

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \quad \sinh_3 \text{ II}(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!} \quad \cosh_3 = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$$

We can now define an equation that looks and acts like the Euler equation

$$e^{\omega x} = \cosh_3(x) + \omega \sinh_3(x) + \omega^2 \sinh_3 \text{ II}(x)$$

We can now try to find one for ω^2

$$e^{\omega^2 x} = \sum_{n=0}^{\infty} \frac{(\omega^2)^n x^n}{n!}$$

at $3n$ we get $\omega^{6n} = 1$ so it's $\cosh_3(x)$, at $3n+1$ we get $\omega^{6n+2} = \omega^2$ so it's $\sinh_3(x)$ and at $3n+2$ we get $\omega^{6n+4} = \omega$ so it's $\sinh_3 \text{ II}(x)$

$$e^{\omega^2 x} = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \omega^2 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \omega \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

$$e^{\omega^2 x} = \cosh_3(x) + \omega^2 \sinh_3(x) + \omega \sinh_3 \Pi(x)$$

and for e^x it's quite simple

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} + \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

$$e^x = \cosh_3(x) + \sinh_3(x) + \sinh_3 \Pi(x)$$

like sin and sinh, we can try to find an exponent form for them

First, we begin by adding all of the equations so we have

$$e^x + e^{\omega x} + e^{\omega^2 x} = \cosh_3(x) + \sinh_3(x) + \sinh_3 \Pi(x)$$

$$+ \cosh_3(x) + \omega \sinh_3(x) + \omega^2 \sinh_3 \Pi(x) + \cosh_3(x) + \omega^2 \sinh_3(x) + \omega \sinh_3 \Pi(x)$$

$$= 3 \cosh_3(x) + \sinh_3(x)(1 + \omega + \omega^2) + \sinh_3 \Pi(x)(1 + \omega^2 + \omega)$$

and since we know that $1 + \omega + \omega^2 = 0$

$$e^x + e^{\omega x} + e^{\omega^2 x} = 3 \cosh_3(x) \quad \cosh_3(x) = \frac{e^x + e^{\omega x} + e^{\omega^2 x}}{3}$$

Now we can define the others by differentiating

$$\sinh_3(x) = \frac{e^x + \omega e^{\omega x} + \omega^2 e^{\omega^2 x}}{3} \quad \sinh_3 \Pi(x) = \frac{e^x + \omega^2 e^{\omega x} + \omega e^{\omega^2 x}}{3}$$

since these definitions are going to continue with us, we shall call the $e^{ax} = \dots$ the **Euler form** and $f(x) = e^x + e^{ax} \dots$ the **exponentiation form**

4.2 Cyclic derivatives and prime numbers

To understand exactly what is meant by this, we need to see these

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \cosh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$$

As we can see, there is a pattern here, for every function that is k -th cyclic derivative, we can see that its series is $\sum_{n=0}^{\infty} \frac{x^{kn}}{(kn)!}$

But this assumption shortly breaks as we can see for the fourth cyclic derivative

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

To understand more what I am talking about, we may take a look at the fifth cyclic derivative denoted by ϵ , to find the functions we start from the Maclaurin series

$$e^{\epsilon x} = 1 + \epsilon x + \frac{\epsilon^2 x^2}{2!} + \frac{\epsilon^3 x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{\epsilon^n x^n}{n!}$$

And we can find the five functions the same way we made it to the three cyclic derivative functions

$$\begin{aligned}\cosh_5(x) &= \sum_{n=0}^{\infty} \frac{x^{5n}}{(5n)!} & \sinh_5(x) &= \sum_{n=0}^{\infty} \frac{x^{5n+1}}{(5n+1)!} \\ \sinh_5 \text{ II}(x) &= \sum_{n=0}^{\infty} \frac{x^{5n+2}}{(5n+2)!} & \sinh_5 \text{ III}(x) &= \sum_{n=0}^{\infty} \frac{x^{5n+3}}{(5n+3)!} & \sinh_5 \text{ IV}(x) &= \sum_{n=0}^{\infty} \frac{x^{5n+4}}{(5n+4)!}\end{aligned}$$

And since the sum of roots of unity that are over 1 root is zero, we can do the same steps to find that

$$\cosh_5(x) = \frac{e^x + e^{\epsilon x} + e^{\epsilon^2 x} + e^{\epsilon^3 x} + e^{\epsilon^4 x}}{5}$$

and the other functions to be the derivatives of these functions

We can notice that the pattern continued for 5-th cyclic function

So what is the problem with trigonometric ones?

Well, we can see the expansion for e^{ix} to see what happens

$$e^{ix} = 1 + ix + \frac{x^2}{2!} - \frac{ix^3}{3!} + \dots$$

We can notice the pattern right there, it's the $-\frac{x^2}{2!}$, this term allows us to either:

- write the sum as four cyclic functions since we will make different additions being $\{1, i, -1, -i\}$

- We write sum as two different cyclic derivatives being $\{1, -1\}$ and $i\{1, -1\}$

In other words, the cyclic derivative family is compisable, reducible with simple algebra

and the reason for that is the **cyclic order**, when it's composite, we can see some roots return, like from order 2 we have $1, -1$ and order 4 we have $1, i, -1, -i$, the $1, -1$ here is back, same for six roots of unity $1, i_1, i_2, -1, i_3, i_4$ (note that i_a here isn't imaginary unit but the a -th root) and we can say that

let gk be all solutions for $a^k = 1$ and gn be for $a^n = 1$, as long as $\frac{k}{n} \in \mathbb{Z}^+$, $gk \subset gn$

Thus, for any composite cyclic order, there exists more than 1 way to represent it

which means primes aren't here, so we can write the theorem

Theorem 4.1 (Prime cyclic functions Euler Form) $\forall p \in \text{Primes}, a^p = 1$
There exists only one way to represent e^{ax} as a sum of all the cyclic order functions

From this, we can say that

Theorem 4.2 (Prime cyclic functions exponentiation form Form) $\forall p \in \text{Primes}$, $\sinh_p N(x)$ is a cyclic function; it can be written as this

$$\sinh_p N(x) = \frac{e^x + a^N e^{ax} + a^{2N} e^{a^2 x} + \dots + a^{pN} e^{a^p x}}{p} = \frac{1}{p} \sum_{n=0}^{p-1} a^{n(p-N)} e^{a^n x}$$

4.3 General Cyclic derivatives and Mittag-Leffler connection

As we can see from multiple series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \sinh = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \quad \cos(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

We can see from all these examples that the factorial matches the exponent. Taking the D^α derivative of all gives us

$$D^z e^x = \sum_{n=0}^{\infty} \frac{x^{n-z}}{\Gamma(n-z+1)} \quad \sinh = \sum_{n=0}^{\infty} \frac{x^{2n+1-z}}{\Gamma(2n+2-z)}$$

$$\sinh_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1-z}}{\Gamma(3n+2-z)} \quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-z}}{\Gamma(2n-z+1)}$$

If we let $2-z = \beta$ and let $Cn = \alpha n$ where C is a constant, we see that all of them get the shape

$$D^z f(x) = \sum_{n=0}^{\infty} \frac{x^{\alpha n + \beta}}{\Gamma(\alpha n + \beta)}$$

which is the Mittag-Leffler function, we can see that this happens in all of the functions we know.

That is, of course, except for $\cos(x)$ that will be discussed later, but we need to generalise it, and we need to generalise the derivative cyclic order in Euler form

let a be any element from the group of solutions for $a^n = 1$

$$e^{ax} = 1 + ax + \frac{a^2 x^2}{2!} + \frac{a^3 x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{a^k x^k}{k!}$$

From here, we can group the sums. Since there exist n roots of unity, we can say that there exist n terms

$$e^{ax} = (1 + \frac{x^n}{n!} + \frac{x^{2n}}{(2n)!} + \dots) + (ax + \frac{a^n x^{n+1}}{(n+1)!} + \frac{a^{2n} x^{2n+1}}{(2n+1)!} + \dots) + \dots$$

$$= \sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!} + \sum_{k=0}^{\infty} \frac{a^{kn+1} x^{kn+1}}{(kn+1)!} + \sum_{k=0}^{\infty} \frac{a^{kn+2} x^{kn+2}}{(kn+2)!} + \sum_{k=0}^{\infty} \frac{a^{kn+3} x^{kn+3}}{(kn+3)!} + \dots$$

and since $a^{kn+j} = a^j$, we can take it out as a common factor

$$= \sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!} + a \sum_{k=0}^{\infty} \frac{x^{kn+1}}{(kn+1)!} + a^2 \sum_{k=0}^{\infty} \frac{x^{kn+2}}{(kn+2)!} + a^3 \sum_{k=0}^{\infty} \frac{x^{kn+3}}{(kn+3)!} + \dots$$

We are going to name the first one $\cosh_n(x)$ and the others $\sinh_n I(x)$ and $\sinh_n II(x)$ so on

$$e^{ax} = \cosh_n(x) + a \sinh_n I(x) + a^2 \sinh_n II(x) + a^3 \sinh_n III(x) + \dots a^{n-1} \sinh_n N(x)$$

Now, if we consider $+C - \alpha = +\beta$ in the α -th derivative we get We see that all these functions fall under the Mittag-Leffler formula

$$D^z \cosh_n = \sum_{k=0}^{\infty} \frac{x^{\alpha kn}}{\Gamma(\alpha kn + 1)} \quad D^z \sinh_n II \dots = \sum_{k=0}^{\infty} \frac{x^{\alpha n + \beta}}{\Gamma(\alpha n + \beta + 1)}$$

From this, we can get the general cyclic derivative sum formula

Theorem 4.3 (Generalized Cyclic Derivative) *Let $f(x)$ be the j -th basis function of the D^n -cyclic system,*

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{kn+j}}{(kn+j)!}$$

Theorem 4.4 (Mittag-Leffler Representation) *Every basis function of the D^n -cyclic system is a linear combination of n distinct Mittag-Leffler functions $E_{n,\beta}(\mathbf{x}^n)$ corresponding to the n terms in its series representation.*

Theorem 4.5 (Generalized Euler form) *for every e^{ax} where a stratifies $a^n = 1$, e^{ax} can be written as*

$$e^{ax} = \sum_{j=0}^{n-1} a^j \sinh_n N(x)$$

that is of course if we consider $\cosh_n(x)$ to be the 0-th term

4.4 Odd And Even cyclic derivatives

There are many differences between odd and even cyclic derivatives, and that comes to the roots of unity

4.5 Generalized D^z cyclic derivatives

there are many yet simple ways to find α -th derivative of a cyclic function

General Series

We can use the series expansion for any cyclic function

for any cyclic function with order n denoted as $\sinh_n(x)$ we can say that

$$\sinh_n = \sum_{k=0}^{\infty} \frac{x^{kn+C}}{(kn+C)!}$$

that is of course for reduced or original functions because the reduced functions change will not make change in x power directly
then the α -th derivative is

$$D^\alpha \sinh_n = \sum_{k=0}^{\infty} \frac{x^{kn+C-\alpha}}{\Gamma(kn+C+1)} \frac{\Gamma(kn+C+1)}{\Gamma(kn+C+1-\alpha)}$$

canceling the terms we get

$$D^\alpha \sinh_n = \sum_{k=0}^{\infty} \frac{x^{kn+C-\alpha}}{\Gamma(kn+C+1-\alpha)}$$

which is the generalized series formula

Deriavtives for reduced functions up til this point we know what is a reduced cyclic deriavtive functions for example e^{ix} is supposed to be four functions but we use commonly two in the euler identity

since the reduced functions are equal to the orignal functions in some sort for example $f(x) = g(x) \pm h(x)$ we can use the linearty of D^z to find the deriavtive of any function based on the known functions

Euler's form

for any cyclic derivatives function , a known thing is they need to equal to e^{ax} where $a^n = 1$ and n is the amount of cycliation, using this backwards we can say that any function \sinh_n can be expressed as the sum between all the roots of unity that are a power of e times x divided by n , at least for orignal functions and prime functions

$$\sinh_n = \frac{1}{n} \sum_{k=0}^{n-1} e^{a^n x}$$

then we can use D^α here

$$D^\alpha \sinh_n = \frac{1}{n} \sum_{k=0}^{n-1} a^\alpha e^{a^n x} = \frac{1}{n} (e^x + \sum_{k=1}^{n-1} a^\alpha e^{a^n x})$$

4.6 prime cyclic derivatives decomposition

the form e^{ix} can be expressed as $\cos(x) + i \sin(x)$, that is because it's composite and can be reducible to it's prime cyclic (2)

as we can see the reason it's reduced from algabric side is the (-1)

$$\begin{aligned} \cosh &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} & \cos &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \sinh &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} & \sin &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned}$$

if we try another cyclic derivative that has only one prime order factor it will be 9 because it's 3^2

let $a^9 = 1$, with a being from $\{1, \epsilon, \epsilon^2, \omega, \epsilon^4, \epsilon^5, \omega^2, \epsilon^7, \epsilon^8\}$ with $\omega = \epsilon^3$ but we change it just to know it's the third cubic root

$$e^{ax} = 1 + \epsilon x + \frac{\epsilon^2 x^2}{2!} + \frac{\omega x^3}{3!} + \frac{\epsilon^4 x^4}{4!} + \frac{\epsilon^5 x^5}{5!} + \frac{\omega^2 x^6}{2!} + \dots$$

we can then make the sum

$$e^{ax} = \sum_{n=0}^{\infty} \frac{\omega^n x^{3n}}{(3n)!} + \sum_{k=0}^{\infty} \frac{\epsilon^{k+1} x^{3k+1}}{(3k+1)!} + \sum_{j=0}^{\infty} \frac{\epsilon^{j+2} x^{3j+2}}{(3j+2)!}$$

we can see the three sums form the shapes of the main three cyclic derivatives functions with the addition of the complex values. but because the cyclic order is odd we don't see any negative ones

to understand more we can try to find the original e^{ix} cyclic functions

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} + i \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} - \sum_{j=0}^{\infty} \frac{x^{4j+2}}{(4j+2)!} - i \sum_{u=0}^{\infty} \frac{x^{4u+3}}{(4u+3)!}$$

we can actually see how the form sin and cos form

$$\left(\sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} - \sum_{j=0}^{\infty} \frac{x^{4j+2}}{(4j+2)!} \right) + i \left(\sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} - \sum_{u=0}^{\infty} \frac{x^{4u+3}}{(4u+3)!} \right)$$

simply analysing this we can see that the (-1^n) term in the sums come from them being difference between the sums, we can also see that cos only has the order on the x (x^{kn}) because the both sums have a common factor which is the prime number 2, while sin has the reminder of 1 when dividing that leads to x^{kn+1} term

ultimately we can try to reprsnt the redacted versions of the original functions cos, sin with the original functions ($\cosh_4(x)$, $\sinh_4(x)$, $\sinh_4 \text{II}(x)$, $\sinh_4 \text{III}(x)$) using the Euler form

$$e^{ix} = \cos(x) + i \sin(x) = \cosh_4(x) + i \sinh_4(x) - \sinh_4 \text{II}(x) - i \sinh_4 \text{III}(x)$$

then we know that the real equals the real and the imaginary equals the imaginary

$$\cos(x) = \cosh_4(x) + \sinh_4 \text{II}(x) \quad \sin(x) = -(\sinh_4 \text{I}(x) + \sinh_4 \text{III}(x))$$

to understand more about it we can try to see for 8 order cyclic derivatives with $a \in \{\epsilon, i, \epsilon^3, -1, \epsilon^5, -i, \epsilon^7, 1\}$ knowing that $\epsilon = e^{\frac{2i\pi}{8}}$

$$e^{\epsilon x} = 1 + \epsilon x + \frac{ix^2}{2!} + \frac{\epsilon^3 x}{3!} - \frac{x^4}{4!} + \frac{\epsilon^5 x^5}{5!} - \frac{ix^6}{6!} + \frac{\epsilon^7 x}{7!} + \dots$$

$$e^{\epsilon x} = \sum_{n_1=0}^{\infty} \frac{x^{8n_1}}{(8n_1)!} + \sum_{n_2=0}^{\infty} \frac{\epsilon x^{8n_2+1}}{(8n_2+1)!} + \sum_{n_3=0}^{\infty} \frac{\epsilon x^{8n_3+1}}{(8n_3+2)!} + \dots$$

these are the original functions functions

first of all, we know from the calculations that $\epsilon^5 = -\epsilon, \epsilon^7 = -\epsilon^3$, we can write the expressions as following

$$e^{\epsilon x} = \left(\sum_{n_0=0}^{\infty} \frac{x^{8n_0}}{(8n_0)!} - \sum_{n_4=0}^{\infty} \frac{x^{8n_4+4}}{(8n_4+4)!} \right) + \epsilon \left(\sum_{n_1=0}^{\infty} \frac{x^{8n_1+1}}{(8n_1+1)!} - \sum_{n_5=0}^{\infty} \frac{x^{8n_5+5}}{(8n_5+5)!} \right) \dots$$

to distinguish original functions from reduced ones, we are going to call the reduced ones with trigonometric names from now on

$$e^{\epsilon x} = \cos_8(x) + \epsilon \sin_8(x) + i \sin_8 \text{II}(x) + \epsilon^3 \sin_8 \text{III}(x)$$

we can try to expand the sums to find another simpler sums for them

$$\cos_8(x) = 1 - \frac{x^4}{4!} + \frac{x^8}{8!} + \dots \quad \sin_8(x) = x - \frac{x^5}{5!} + \frac{x^9}{9!} + \dots$$

they form the sums

$$\cos_8(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(4n)!} \quad \sin_8(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{(4k+1)!} \quad \text{etc...}$$

we can see from this that the number of functions an order n can be reduced to is $\frac{n}{p}$ where p is its prime divisor, that is to be said most functions we have dealt with till now are either primes or have one prime divisor, but we still can prove it

Theorem 4.6 *Every integer order n that has only one prime divisor p can be reduced to $\frac{n}{p}$ functions at max, with each one has different root of unity*

Proof:

Let n be a cyclic order with only p being it's only prime divisor

with the fundamental theorem of arithmetic that states: "every integer greater than 1 is a prime or a product of primes" we can say that $n = p \times q \times \dots$

and since p is the only prime divisor we can say that $n = p \times p \times p \dots = p^k$ for some integer $k > 1$

we can then define $a^n = 1$ to $a^{p^k} = 1$ with the repeating nature of the roots of unity that only happens when $k|j$ where are both the number roots of unity, we can set m to be the before n roots of unity group that shares the same only prime divisor as p

then we can let $m = p^{k-1}$ with $a^m = 1$ so $a^{p^{k-1}} = 1$

to find the maximum amount of functions

4.7 Mixture of equations

By now, we know that there exist Prime cyclic functions and composite cyclic functions. The prime cyclic can't be expanded, while the composite ones can
So we can say that $\cosh(x)$ is prime cyclic since its cyclic order is prime (2) and

it can't be expanded, so e^{-x} can only be expanded naturally with only $\sinh(x)$ and $\cosh(x)$

On the other hand there exist orders that have more than one prime factor, for example 6-th derivative functions and 10-th order

for the 6-th order roots, we can write them as follow $a \in \{\epsilon, \omega, -1, \omega^2, \epsilon^5, 1\}$ At first glance, we can see that there are many roots similar that can be used to make functions

first we expand $e^{\epsilon x}$

$$e^{\epsilon x} = 1 + \epsilon x + \frac{\omega x^2}{2!} - \frac{x^3}{3!} + \frac{\omega^2 x^4}{4!} + \frac{\epsilon^5 x^5}{5!} + \frac{x^6}{6!} \dots = \sum_{n=0}^{\infty} \frac{\epsilon^n x^n}{n!}$$

first we construct the original functions

$$\cosh_6 = \sum_{n=0}^{\infty} \frac{x^{6n}}{(6n)!} \quad \sinh_6 = \sum_{n=0}^{\infty} \frac{x^{6n+1}}{(6n+1)!} \quad etc.,$$

we write $\epsilon = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, we can see that $\epsilon = -\omega^2$, the same goes for $\epsilon^5 = -\omega$, we can then rearrange , the sum to be

$$e^{\epsilon x} = (1 + \frac{\omega x^2}{2!} + \frac{\omega^2 x^4}{4!} + \dots) + (-\omega^2 - \frac{x^3}{3!} - \frac{\omega x^5}{5!} - \dots) = \sum_{n=0}^{\infty} \frac{\omega^n x^{2n}}{(2n)!} + \sum_{k=0}^{\infty} \frac{(-1)\omega^{2+n} x^{2n+1}}{(2n+1)!}$$

we can let them be $\cos_3(x)$ and $\sin_3(x)$ as they are similar to the $\sin(x), \cos(x)$ But instead of the alternating negative sign we see the omega here,

$$e^{\epsilon x} = \sum_{n=0}^{\infty} \frac{\omega^n x^{2n}}{(2n)!} - \sum_{k=0}^{\infty} \frac{\omega^{2+n} x^{2n+1}}{(2n+1)!}$$

We can rearrange the sum to get different results based on something, and it's negative

$$\begin{aligned} e^{\epsilon x} &= (1 - \frac{x^3}{3!} + \frac{x^6}{6!} + \dots) + (-\omega^2 + \frac{\omega^2 x^4}{4!} - \frac{\omega^2 x^7}{7!} - \dots) + (\frac{\omega x^2}{2!} - \frac{\omega x^5}{5!} + \frac{\omega x^8}{8!} - \dots) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(3n)!} + \omega^2 \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+1}}{(3k+1)!} + \omega \sum_{j=0}^{\infty} \frac{(-1)^j x^{3j+2}}{(3j+2)!} \end{aligned}$$

we can see there exists a negative sign here, even alternating, we can analyse both expressions and see, when we get the amount of 3 functions, we got the alternating -1 , when we had two functions, we had the two roots of unity in the series, Note this has 3 ways to expand e^{ax}

We can, for now, assume that this happens because the amount of the functions is the first prime factor, so the root(s) of unity must be from another prime factor

We can try for the 10-th order cyclic derivatives, knowing that, after half of the

numbers, we will see numbers going to the negatives of each other
 $a \in \{\epsilon, \epsilon^2, \epsilon^3, \epsilon^4, -1, -\epsilon, -\epsilon^2, -\epsilon^3, -\epsilon^4, 1\}$

$$e^{\epsilon x} = 1 + \epsilon x + \frac{\epsilon^2 x^2}{2!} + \frac{\epsilon^3 x^3}{3!} + \frac{\epsilon^4 x^4}{4!} - \frac{x^5}{5!} - \frac{\epsilon x^6}{6!} - \frac{\epsilon^2 x^2}{7!} - \frac{\epsilon^3 x^2}{8!} - \frac{\epsilon^4 x^9}{9!} + \frac{x^{10}}{10!} + \dots$$

we have the original functions $\cosh_{10} = \sum_{n=0}^{\infty} \frac{x^{10n}}{(10n)!}$ and the others
 We can use the negative and positive values

$$e^{\epsilon x} = (1 - \frac{x^5}{5!} + \frac{x^{10}}{10!} + \dots) + \epsilon(x - \frac{x^6}{6!} + \frac{x^{11}}{11!} + \dots) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n}}{(5n)!} + \epsilon \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+1}}{(5n+1)!} + \dots$$

We can now rearrange them also as even and odd values

$$e^{\epsilon x} = (1 + \frac{\epsilon^2 x^2}{2!} + \frac{\epsilon^4 x^4}{4!} + \frac{\epsilon^6 x^6}{6!} + \dots) + (\epsilon x + \frac{\epsilon^3 x^3}{3!} + \frac{\epsilon^5 x^5}{5!} + \frac{\epsilon^7 x^7}{7!} + \dots) = \sum_{n=0}^{\infty} \frac{\epsilon^{2n} x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\epsilon^{2n+1} x^{2n+1}}{(2n+1)!}$$

with that and our knowledge about prime cyclic derivatives we can see the pattern

in both composite cases we could either write them as the original functions or as p-amount of functions with q-roots in them where $p \times q = n$ the order of the cyclic derivative

we can say : generally the amount of ways to divide such expression into different functions are **The amount of divisors of n** , where n is the order of the cyclic

simply because with every factor of such a cyclic order derivative we can make functions that are cyclic also when the original function is cyclic, lesser cyclic derivatives functions like order-3 cyclic derivative functions comes back to their state also when they get derivative 6 times or 9 times making them a family too suitable to reduce the original functions of 6 or 9 cyclic derivatives

and in prime functions there only exists two factors, 1 and prime number itself, which we can see as 1 represent the e^{ax} itself, and prime represents the original functions that can't be reduced anymore.

so they have 2 families of functions, while order 6 has 4 divisors (1,2,3,6) so we can see 4 forms, the 2-order lesser functions, the 3-order lesser functions, e^{ax} and original functions

this property we can call as "n-order Families" with reduced functions being called "Lesser cyclic derivative function to order-n", while we can call the original functions themselves "Higher cyclic derivative function to order-m" (where m is the factor functions)

and the ways to define these functions are the prime factors of n

we may need a proof, and it's simple one non the less

Proof:

we have a cyclic function $f(x)$ that is order of cyclation is n

from the prime cyclic functions property we know that $f(x)$ is either composite cyclic or prime cyclic, if $f(x)$ is prime cyclic derivative, we are there, if $f(x)$

is composite cyclic derivative it can be expanded to $g(x)$ functions with roots q such that $a^q = 1$, we keep doing this until $g(x)$ is prime cyclic function, thus it's always true

Q.E.D

4.8 composition of functions

5 But why cyclic derivatives?

5.1 Dimension Bender and Hyper operations

Operations are the fundamentals of mathematics. We start with a constant, then succession, which has the definition $S(n) = n + 1$

Repeated succession results addition, which can be defined as $A(n, m) = n + m = \underbrace{S(S(S(...n)))}_{m \text{ times}}$, repeating that gives Multiplication $M(n, m) = n \times m$ with

the same idea, so one exponentiation and tetration

What brings that here is the properties of exponentiation, we can say that for any constant a , we can do

$$a^n \times a^m = a^{n+m} \quad (a^n)^m = a^{n \times m}$$

Exponentiation moves other operations up in the hierarchy; it linearises them, "Bending the space around it". We can see that every cyclic derivative is in or can be used to create e^{ax} ; an exponentiation form

But then the question arises, why exactly e and not any other base, well we can see it with D^z

$$D^z a^{bx} = b^z a^{bx} \ln(a)^z \Rightarrow D^z e^{bx} = b^z e^{bx} \ln(e)^z = b^z e^{bx}$$

Unlike any other base, e is the only base that stays without remainder, it bends space without any trace, it has the perfect environment for pure cyclic functions to arise like $\sin(x)$ and $\cosh(x)$ as they won't interact with any other change, it allows clean, smooth transformation from point a to point b we can also see that in the expansion of $e^{ax} = \sum_{n=0}^{\infty} \frac{a^n x^n}{n!}$, the simplest form of expansion for a function

5.2 Analysis of cyclic functions

But now this gives a bigger question: what does this information help us with cyclic derivatives

5.3 Transformative functions

But unlike cyclic derivatives, this one is problematic
cyclic functions change smoothly from one derivative to another, transformative

functions choose to break that in a pretty much big sense, take for example $\sin^{-1}(x)$, from D^z prescriptive, it can be defined as

$$D^z \sin^{-1}(x) = D^z \frac{1}{\sqrt{1-x^2}} \quad z \geq 0$$

and at 0 it is just $\sin^{-1}(x)$ and for the integrals it seems to be a product of both of them, it just jumps back and forth between faces with no predictable move, that is, unless we go to the complex plane and see the complex definition which is $\sin^{-1}(z) = -i \ln(iz + \sqrt{1-z^2})$

There is a natural logarithm, which we know is a problem with D^z

But not all transformative functions directly have $\ln(x)$ in their definitions for example $\frac{1}{x^n}$ is a transformative function because at integration, it directly transforms to another function $\ln(x)$, which isn't in the definition but rather the integral of it

Looking at the big picture, the inverse of x^n it's an inverse function, and nearly all the transformative functions are inverses, and that is for a reason

A function is defined to be an input-to-output machine; many inputs can give the same output, but not the other way around. When one input gives many outputs, it's not an ordinary function in the definition,

When we try to get an inverse out of a function, most of the time that rule is broken. We can see for the simple case x^2 that makes a double-input one-output system, to define its inverse \sqrt{x} in the real value, we have to sacrifice the other inputs that give the same outputs, being the negative numbers

5.4 $\ln(x)$ is collapse

First of all, this function isn't analytic; it has a singularity at $x = 0$, and it's a multi-valued function in the complex plane, that is, by itself a problem, but we are going to discuss it later. Instead of $\ln(x)$ we will use $\ln(x+1)$

We first find its n-th derivative

$$f(x) = \ln(x+1) \quad f'(x) = \frac{1}{x+1} \quad f''(x) = -\frac{1}{(x+1)^2}$$

this the x^{-n} α -th derivative, so we can assume the formula is simply $D^\alpha \ln(x+1) = \frac{(-1)^{\alpha+1} \Gamma(\alpha)}{(x+1)^\alpha}$, this is supposed to get back $\ln(x+1)$ at $\alpha = 0$

$$D^0 f(x) = \frac{(-1)^1 \Gamma(0)}{(x+1)^0} = \frac{-\Gamma(0)}{1} \neq \ln(x+1)$$

We can see that this breaks because of the Gamma pole, but we can try another approach

(Note: this is the Gamma of -1 , using this Gamma definition $\Gamma(n) = \frac{\Gamma(n+1)}{n}$, at -1 we get $\Gamma(-1) = -\frac{\Gamma(0)}{1}$)

we can try to use the Maclaurin series to see the function smoother, then we

can take the derivative and try it

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad D^\alpha \ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha} \right] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \Gamma(n)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

We can try it for the first derivative

$$D^1 \ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cancel{\Gamma(n)}}{\cancel{\Gamma(n)}} x^{n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1} = 1 - x + x^2 - x^3 + \dots = \frac{1}{x+1}$$

This series works, and we can test it for D^0 to see if it returns the same values or not

$$D^0 f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \Gamma(n)}{\Gamma(n+1)} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \Gamma(n)}{n \Gamma(n)} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = \ln(x+1)$$

which is correct, we can even try to see if it works for the first integral

$$D^{-1} f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \Gamma(n)}{\Gamma(n+2)} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)} x^{n+1}$$

Expanding this expression, we get

$$\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \frac{x^6}{30} - \dots$$

Now we find the integral for the function

$$f(x) = \int \ln(x+1) = (x+1)(\ln(x+1) - 1)$$

We can then put the Maclaurin expansion in here for $\ln(x+1)$

$$(x+1)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) - x = \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \dots\right) + \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) - x$$

We can do some algebraic manipulation here to rearrange the subtractions

$$= (x - x) + \left(x^2 - \frac{x^2}{2}\right) + \left(\frac{x^3}{3} - \frac{x^3}{2}\right) + \dots = \left(0 + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \frac{x^6}{30} - \dots\right)$$

Both series are equal, thus it works

We can explain that not happen because it never reach a fatal Gamma pole for the sum, and because the sum is infinite, we can continue doing that forever without changing the result. We can get the integral formula in this case to be

$$D^{-\alpha} \ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} \Gamma(n)}{\Gamma(n+\alpha+1)} x^{n+\alpha}$$

6 Cycliation in other ways

from the hyper-operations explanation; cycliation isn't only for simple e^{ax} , which allows us to try different things effectively

6.1 Multi variable cycliation

we can extend the cyclic functions to the 3rd Dimension simply since the definition is simple (e^{ax} where $a^n = 1$), we can simply change x to $x + y$ to get

$$e^{a(x+y)} \text{ where } a^n = 1$$

this simple formula to extend it to 3D , so

$$e^{-(x+y)} = \cosh(x+y) - \sinh(x+y) \quad e^{i(x+y)} = \cos(x+y) + i \sin(x+y)$$

infact if we let u be any operation between x, y because we will get e^{au} where $a^n = 1$

this can work and extend to any operation to infinite dimensions, but there is one thing

this extension is linear, we can expect the outcome and result to work right and similar, the one function going upwards at $x > 1$ just becomes a plane going upwards then a cube going upwards

this happens because when we change e^{ax} to e^{au} the reality is similar to the hyperoperations explanation, it's just e^{ax} at the end of it

we can find some true change by changing the rule , since the differentiation of the function is dependent of a^x and it's nature to come back at the end of a full cycle , we can try to change it to e^{ax+by} in the 3D definition

we can say that for this to happen $a^n = b^n = 1$ which is true only when $a = b$ But this doesn't stop us from making a new definition, why not simply make it one-dimensional cyclic?

The idea is simple, we need both to be cyclic but not the same order at the same time, at the end, the reason for its cyclicity is its non-growing nature

We can define it as

$$e^{ax+by} \text{ where } a^n = b^m = 1$$

This definition is n-order cyclic on the x-axis and m-order cyclic on the y-axis, we can try it with $a = 1, b = -1$ to see what happens. First, we expand it

$$e^{x-y} = 1 + (x-y) + \frac{(x-y)^2}{2!} + \frac{(x-y)^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{(x-y)^k}{k!}$$

Of course, this can be treated as $e^{-(y-x)}$ which can be written in sinh and cosh terms

$$e^{x-y} = e^{-(y-x)} = \cosh(y-x) - \sinh(y-x)$$

But we can try something like e^{-x+iy} which is a second cyclic with respect to x but a four cyclic with respect to y

this one can be expanded simply using Euler's formula $e^{i\theta}$ and the definition for e^{-x}

$$e^{-x+iy} = e^{-x}e^{iy} = e^{-x}(\cos(y) + i\sin(y))$$

6.2 Combination of cycliation

6.3 cycliation derivative to a function order

from the original paper we know that we can use D^α with the order being the differentiable variable as D^x or $D^{g(x)}$

$$D^x f(x) = D^x e^{ax} = a^x e^{ax}$$

and unlike the monomials we can take the derivative again of this function to be

$$D^x f'(x) = a^{2x} e^{ax} + a^x e^{ax} \ln(a)^x$$

we can continue for a third time

$$D^x f''(x) = a^{3x} e^{ax} + a^x e^{ax} \ln(a)^{2x} + a^{2x} e^{ax} \ln(x) + a^x e^{ax} \ln(a)^x + a^x e^{ax} \ln(a)^x \ln(\ln(a))^x$$

but the problem comes from the first derivative, the function isn't cyclic anymore it broke its rule on expansion thus making this topic declined

6.4 can any function be transformed to a cycliation variant?

6.5 x^x is weird

6.6 Cycliation, in imaginary plane system

6.7 Cycliation into the third dimension

6.8 Cycliation, but not in e

With our knowledge, we can try to think of what would happen when we change the base to any other constant a , with the rule being $b^n = 1$, let's try to see what happens at second-order cyclic derivatives ($\sinh(x)$ and $\cosh(x)$) using the Maclaurin series

$$a^{-x} = 1 - x \ln(a) + \frac{x^2 \ln(a)^2}{2!} - \frac{x^3 \ln(a)^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^k \ln(a)^k}{k!}$$

we are going to extract the $\sinh(x)$ and $\cosh(x)$ terms from the sum by the negatives and the positives and give them the terms ${}_a \sinh(x)$ and ${}_a \cosh(x)$ to distinguish them from the original ones

$${}_a \cosh = \sum_{k=0}^{\infty} \frac{x^{2k} \ln(a)^{2k}}{(2k)!} \quad {}_a \sinh = \sum_{k=0}^{\infty} \frac{x^{2k+1} \ln(a)^{2k+1}}{(2k+1)!}$$

then of course, to find the euler form it will be

$$a^{-x} = {}_a \cosh(x) - {}_a \sinh(x)$$

to extend this to any function, we will simply do

6.9 Some properties of other base cyclic derivatives

For the functions ${}_a \cosh(x)$, if we turn the base less than 1, we can observe that it starts to exhibit waves and patterns. When $a = 0.5$, we can see some similarity between these functions and $\cos(x)$

using trial and error, I could make say when $a \approx 0.367879441190$

This number perfectly matches e^{-1} , infact if we plugged it directly we can see why

$$e^{-1} \cosh = \sum_{k=0}^{\infty} \frac{x^{2k} \ln(e^{-1})^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{x^{2k} (-1)^{2k}}{(2k)!}$$

Of course, this is the $\cos(x)$ Maclurin series, and this can give us another angle to understand the connection between composite cyclics and prime cyclics Composite cyclics can be reduced to prime cyclics with some change, and this change happens to be the change of the base for the cyclic function.

7 The function families of cyclic functions

exploring the families of the functions is intersting , since it may help us understand why such functions behave like this at least from teh point of D^z and being NFRD while they are constructed from FRD functions

7.1 $\tan(x)$ and $\tanh(x)$

from the beginning, we can see some problems

before we even start we realize that ,both $\tan(x)$ and $\tanh(x)$ don't have a "main way" to define them

the stranded definition for them is $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$, but that is only because there exists two functions , and this only happens in hyperbolic functions, and trigonometric functions too because it satisfies $\frac{n}{p} = 2$ because they are the only ones that sat,but anyway we can use many ways with diffrent resaults to define the tangent function(s)

Using the first two functions only

we can try to define $\tan_n(x)$ to use the first two functions of any e^{ax}

so we can write it as $\tanh_n(x) = \frac{\sinh_n(x)}{\cosh_n(x)}$ that is for the non-reduced orignal functions of the expansion and $\tan_n(x) = \frac{\sin_n(x)}{\cos_n(x)}$ to be for the reduced functions we can try it for the 3-order cyclic derivatives

$$\tanh_3(x) = \frac{\sinh_3(x)}{\cosh_3(x)} = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \left(\sum_{k=0}^{\infty} \frac{x^{3k}}{(kn)!} \right)^{-1}$$

but the problem with this one is there are other functions that aren't put in prespiactive , and the more we go up with the order the more they are which makes this definition unusable

The division between every function and \cosh_n or \cos_n

Then we can try to to divide every single function multiplied from the cyclic family

$$\tanh_3 = \frac{\sinh_3(x)}{\frac{\cosh_3(x)}{\sinh_3 \Pi(x)}} = \frac{\sinh_3(x) \sinh_3 \Pi(x)}{\cosh_3(x)}$$

and if we continue this we will see that all odd functions from a full cyclic family are on the top while the evens are on the bottm, so if we let $\cosh_n = \sinh_n 0$ we get this formula

$$\tanh_n(x) = \frac{\prod_{K=0}^n \sinh_n 2K+1(x)}{\prod_{J=0}^n \sinh_n 2J(x)}$$

Using the Series

but to actually define $\tanh_n(x)$ we need to observe what the original $\tan(x)$ have in that this one needs to have

$\tan(x)$ is equal to $\frac{\cos(x)}{\sin(x)}$: this isn't supposed to be true for all variants, and we will discuss the reason for that later

$\tan(x)$ is an NFRD function with its series representation only: This one is important since a series representation is the represent of a function using x^n , which *According to the original paper* has the change directly happening to them so the best way to define $\tanh_n(x)$ is via the series definition, now we see the Maclaurin series

$$\tan(x) = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} \quad \tanh(x) = \sum_{n=0}^{\infty} \frac{B_{2n}4^n(1-4^n)}{(2n)!} x^{2n-1}$$

But we can't take these series directly and apply them because they may be only special occians , we need to define the main series using the original functions

We can skip $\tanh(x)$ since it's pretty straightforward in the original functions being $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

for the four original functions of cyclic order 4 ($\cosh_4(x)$, $\sinh_4(x)$, $\sinh_4 \Pi(x)$, $\sinh_4 \text{III}(x)$)

We can use the sin and cos definitions of them being

$$\cos(x) = \cosh_4(x) + \sinh_4(x) \quad \sin(x) = -(\sinh_4 \Pi(x) + \sinh_4 \text{III}(x))$$

We can then put them in the definition of $\tan(x)$ to get

$$\tan(x) = \frac{-(\sinh_4 \Pi(x) + \sinh_4 \text{III}(x))}{\cosh_4(x) + \sinh_4(x)}$$

Expanding the series we get

$$\tan(x) = \frac{-(\sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)!} + \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!})}{\sum_{j=0}^{\infty} \frac{x^{4j}}{(4j)!} + \sum_{u=0}^{\infty} \frac{x^{4u+2}}{(4u+2)!}}$$

This shows that for the original functions, it's dividing the second half of the series by the first half

we can try to do the same for the 8-order to see the result

$$\tan_8 = \frac{\sin_8(x) + \sin_8 \text{III}(x)}{\cos_8(x) + \sin_8 \text{II}(x)} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)!} + \sum_{j=0}^{\infty} \frac{(-1)^j x^{4j+3}}{(4j+3)!}}{\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{(4k)!} + \sum_{u=0}^{\infty} \frac{(-1)^u x^{4u+2}}{(4u+2)!}}$$

expanding the sums we get

$$\tan_8(x) = \frac{x + \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{1 + \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}$$

which is reduced to the regular sum $\sin(x)$ and $\cos(x)$ we get the final result

$$\tan_8(x) = \frac{\sin(x)}{\cos(x)} = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1}$$

7.2 $\sec(x)$ and $\text{sech}(x)$

only the first function \cosh_n and it's reduced version \cos_n will make sech_n and \sec_n

We know that the series for any original function \cosh_n is equal to $\sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!}$

We can expand the sum and let sech_n be the reportorial of it

$$\text{sech}_n = \frac{1}{1 + \frac{x}{n!} + \frac{x^{2n}}{(2n)!} + \frac{x^{3n}}{(3n)!} + \dots} = \frac{1}{\sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!}}$$

and because of the identity $\cos(x)\sec(x) = 1$ we can use euler numbers to any kn in both reduced and original series

$$\text{sech}_n = \sum_{k=0}^{\infty} \frac{E_{kn} x^{kn}}{(kn)!} \quad \sec_n = \sum_{k=0}^{\infty} \frac{(-1)^k E_{kn} x^{kn}}{(kn)!}$$

7.3 $\csc(x)$ and $\text{csch}(x)$

7.4 $\cot(x)$ and $\text{coth}(x)$

8 The inverses of cyclic function families

9 The continuation of the families

9.1 special derivatives with cyclic derivatives

let's take the derivative of a cyclic function when the order is the the root of unit

first in Euler's form

$$D^{-1}e^{-x} = -1^{-1}e^{-x} = -e^{-x} = \sinh(x) - \cosh(x)$$

$$D^i e^{ix} = i^i e^{ix} = (e^{\frac{i\pi}{2}})^i e^{ix} = e^{-\frac{\pi}{2}} e^{ix} = e^{-\frac{\pi}{2}} \cos(x) + e^{-\frac{\pi}{2}} i \sin(x)$$

we let $\omega = e^{\frac{2i\pi}{6}}$ which is the sixth root of unity

$$D^\omega e^{\omega x} = \omega^\omega e^{\omega x} = (e^{\frac{2i\pi}{6}})^\omega e^{\omega x} = e^{\frac{\omega 2i\pi}{6} + \omega x} = e^{\omega(\frac{i\pi}{3} + x)}$$

we let $\omega = e^{\frac{2i\pi}{8}}$ which is the eight-th root of unit

$$D^\omega e^{\omega x} = \omega^\omega e^{\omega x} = (e^{\frac{2i\pi}{8}})^\omega e^{\omega x} = e^{\frac{\omega 2i\pi}{8} + \omega x} = e^{\omega(\frac{i\pi}{4} + x)}$$

for the first hyperbolic and trigonometric functions this happen because they are the only ones that satisfy $n^n \in \mathbb{R}$ where $a^n = 1$

we can find the general simple rule

let $a^n = 1$ in the equation e^{ax}

$$D^a e^{ax} = a^a e^{ax}$$

we can express any root of unity using the formula $a = e^{\frac{2ik\pi}{n}}$ where k is the order in the unity circle

$$D^a e^{ax} = (e^{\frac{2ik\pi}{n}})^a e^{ax} = e^{\frac{2aik\pi}{n} + ax} = e^{a(\frac{2ik\pi}{n} + x)}$$

and for normal cases we use k = 1 as it is the principle unit of the circle

we can test it works for -1

$$D^{-1} e^{-x} = e^{-(i\pi+x)} = -e^{-x} \quad D^i e^{ix} = e^{i(\frac{i\pi}{2}+x)} = e^{-\frac{\pi}{2}} e^{ix}$$

we can see that only for these values this function works as a scaler but for other values it is a rotater;

we can find a smiler formula that is for $D^{\frac{1}{a}}$

$$D^{\frac{1}{a}} e^{ax} = a^{a^{-1}} e^{ax} = (e^{\frac{2ki\pi}{n}})^{\frac{1}{a}} e^{ax} = e^{\frac{2ki\pi}{an} + ax} = e^{a(a^{-1} + x)}$$

and since $a = e^{\frac{2ki\pi}{n}}$ we can say that $\ln(a) = \frac{2ki\pi}{n}$

$$D^{\frac{1}{a}} e^{ax} = e^{\frac{\ln(a)}{a} + ax} = e^{a(\frac{\ln(a)}{a^2} + x)} = e^{a^{-1}(\frac{2ki\pi}{n} + x) + ax}$$

of course D^i and D^{-i} alawys returns a scaler of the function that

Theorem 9.1 D^i and D^{-i} of any cyclic function of order n returns a scaler with the value $\pm \frac{2k\pi}{n}$ times the cyclic function

Proof: Let $a^n = 1$ thus a can be expresses as $e^{\frac{2ki\pi}{n}}$ where k is the order of root in the roots of unity, we can now take the $\pm i$ -th derivative

$$D^{\pm i} e^{ax} = a^{\pm i} e^{ax} = (e^{\frac{2ki\pi}{n}})^{\pm i} e^{ax} = e^{\frac{\mp 2k\pi}{n}} e^{ax}$$

Q.E.D now we can return to the original functions themselves, ;et's try the i -th derivatives of these functions

$$D^i \cosh(x) = \frac{e^x + (-1)^i e^{-x}}{2} = \frac{e^x + (e^{i\pi})^i e^{-x}}{2} = \frac{e^x + (e^{-\pi}) e^{-x}}{2} = \frac{e^x + e^{-(x+\pi)}}{2}$$

with the same formula

$$D^i \sinh(x) = \frac{e^x - e^{-(x+\pi)}}{2}$$

for third cyclic functions we can write them using euler's form

$$D^\alpha \cosh_3 = \frac{e^x + \omega^\alpha e^{\omega x} + \omega^{2\alpha} e^{\omega^2 x}}{3}$$

Now we take the first i -th derivative

$$D^i \cosh_3 = \frac{e^x + \omega^i e^{\omega x} + \omega^{2i} e^{\omega^2 x}}{3} = \frac{e^x + (e^{\frac{2i\pi}{3}})^i e^{\omega x} + (e^{\frac{2i\pi}{3}})^{2i} e^{\omega^2 x}}{3} = \frac{e^x + (e^{\frac{-2\pi}{3}})^i e^{\omega x} + (e^{\frac{-4\pi}{3}})^i e^{\omega^2 x}}{3}$$

Now, for trigonometric functions we have

$$D^i \cos(x) = \cos(\frac{i\pi}{2} + x) \quad D^i \sin(x) = \sin(\frac{i\pi}{2} + x)$$

We can see something weird, some functions rotate while others scale, not full scale for the function, but they scale in some sort

We can assume for now that this happens only in original cyclic functions, and we can actually see the reason why

for reduced functions like sin and cos, their Euler form always has some root of unity in all the exponents, while original functions have the term e^x on them, which always returns itself with no change

We can examine more using the expansion of sin

$$\sin = \frac{e^{ix} - e^{-ix}}{2i}$$

plugging the $\frac{i\pi}{2} + x$ term we get

$$\sin(\frac{i\pi}{2} + x) = \frac{e^{i(\frac{i\pi}{2} + x)} - e^{-i(\frac{i\pi}{2} + x)}}{2i} = \frac{e^{\frac{-\pi}{2} + ix} - e^{\frac{\pi}{2} - ix}}{2i}$$

We can see that there is a scaling value, but it's different on each side

Scales that can be represented as clean inputs in their functions can be called "clean scale transforms" like $D^i \sin(x)$, and scales that can be represented using only Euler form or any other way that isn't the original function with a change in it can be called "Middle way scale transforms"

Actually, this can explain the $D^i e^{ix}$ confusion from before.

This transformation, rather than it being clean for its expansion, it's more of a change to the original function with a clean transform, which makes it look like a Middle way scale transformation, we can see by multiplying the i -th derivative to i

$$D^i e^{ix} = e^{-\frac{\pi}{2}} e^{ix} = e^{-\frac{\pi}{2}} \cos(x) + e^{-\frac{\pi}{2}} i \sin(x)$$

$$i D^i e^{ix} = i^{-1} e^{ix} = -i e^{ix} = e^{\frac{3i\pi}{2}} e^{ix} = e^{i(\frac{3\pi}{2} + x)} = \cos(\frac{3\pi}{2} + x) + i \sin(\frac{3\pi}{2} + x)$$

$$= \sin(x) - i \cos(x)$$

so we can see that this was an expansion problem, the expansion of e^{ax} doesn't necessarily connect to e^{ax} derivative as it works on it's own, it has more functions in it, and it simply doesn't equal, that is at least for complex order, we can see that this happens in fractional order too

$$D^{\frac{1}{2}} e^{ix} = i^{\frac{1}{2}} e^{ix} = (e^{\frac{i\pi}{2}})^{\frac{1}{2}} e^{ix} = e^{\frac{i\pi}{4}} e^{ix} = e^{i(\frac{\pi}{4} + x)}$$

which is equal to $D^{\frac{1}{2}} \sin(x) = \sin(\frac{\pi}{4} + x)$

We can see that this happen explicitly in D^α only when $\alpha \in \mathbb{C}$

There are many explanations for this. We can say that this happen because of the change in scaling and rotation between these functions

9.2 Identities in cyclic derivatives

9.3 Identities between different cyclic derivatives

9.4 Cyclic derivatives and algebraic equations

9.5 half Cyclic, quartic cyclic, complex cyclic...

from the standard definition, if we tried to define it would be complex, as we can try and replace n with i for the equation to be $a^i = 1$, then we turn it to a form of e powers

$$(e^{\ln(a)})^i = 1 \Rightarrow e^{i \ln(a)} = e^{2i\pi}$$

talking the natural logarithm of both sides we get

$$\ln(e^{i \ln(a)}) = \ln(e^{2i\pi}) \Rightarrow i \ln(a) = 2i\pi \quad \ln(a) = 2\pi \Rightarrow a = e^{2\pi}$$

But his value is bigger than 1, it's an exponentiation form on which $e^{2\pi x}$ grows exponentially, this happens because of the use of logarithms instead we will use the \cosh_n and \sinh_n to define such values

$$\cosh_i(x) = \sum_{k=0}^{\infty} \frac{x^{ik}}{\Gamma(ik+1)} \quad \sinh_i(x) = \sum_{k=0}^{\infty} \frac{x^{ik+1}}{\Gamma(ik+2)}$$

we can check if this is correct by applying the i -th deriavtive

$$D^i \cosh_i = \sum_{k=0}^{\infty} \frac{1}{\Gamma(ik+1)} \left[\frac{\Gamma(ik+1)}{\Gamma(ik+1-i)} x^{ik-1} \right] = \sum_{k=0}^{\infty} \frac{x^{ik-i}}{\Gamma(ik+1-i)}$$

this is interesting, but before we say we can take the i s out because that is an infinite series with the form $ak \pm a$, we need to expand them both first

$$\cosh_i = 1 + \frac{x^i}{\Gamma(i+1)} + \frac{x^{2i}}{\Gamma(i+2)} \dots \quad D^i \cosh_i = \frac{x^{-i}}{\Gamma(1-i)} + 1 + \frac{x^i}{\Gamma(i+1)} + \frac{x^{2i}}{\Gamma(i+2)} \dots$$

Both sides are identical except for the first term. What happens here is: since the first term is a constant, it's supposed to go to zero, but since the derivative order is complex, it doesn't reach $\Gamma(0)$, it reaches $\Gamma(1-i)$, which is defined in the Gamma function domain

We can try integrating to see more results

$$D^{-i} \cosh_i = \sum_{k=0}^{\infty} \frac{x^{ik+i}}{\Gamma(ik+1+i)} = \frac{x^i}{\Gamma(1+i)} + 1 + \frac{x^i}{\Gamma(1+i)} + \frac{x^{2i}}{\Gamma(2+i)} + \dots$$

as we can see, integration reflects the cyclic derivative from the first point, the more we integrate the more we add similar functions to the left hand side with the middle point being 1, well that is what happens at imaginary differentiation and integration, in Real differentiation and integration we get another results

$$D^1 \cosh_i = \sum_{k=0}^{\infty} \frac{x^{ik-1}}{\Gamma(ik)} = 0 + \frac{x^{i-1}}{\Gamma(i)} + \frac{x^{i-2}}{\Gamma(2i)} + \frac{x^{i-3}}{\Gamma(3i)} + \dots$$

(the first zero is because of a gamma pole)

we can see the difference here, a normal derivative result multiples of i in the Gamma function, while an imagery derivative result multiples of i in the power of x

of course we can try now a full complex number

$$D^{1+i} \cosh_i = \sum_{k=0}^{\infty} \frac{x^{ik-1-i}}{\Gamma(ik-i)} = \frac{x^{-1-i}}{\Gamma(-i)} + \frac{x^{-1}}{\Gamma(0)} + \frac{x^{i-1}}{\Gamma(i)} + \frac{x^{2i-1}}{\Gamma(2i)} + \dots$$

we can see that this form combines some of the properties from pure complex derivative and pure real derivative.

things like the Gamma pole at zero from the real derivative and multiple of i in both Gamma function and the power

of course the Gamma pole happens in these examples because we have positive 1 in each of them, if we have used any other