Generalizing Wang's (2012) Model for Variable Foliage Distribution

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Canopy_Flow_Model_VF.m

Wang's (2012) canopy flow model, which is applicable to uniform foliage distribution only, is a solution to the following momentum balance equation:

$$-\frac{d}{dz}\left(\kappa s_h u_* z \frac{dU(z)}{dz}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial x} - C_L a_0 U(z) U_h \tag{1}$$

This solution after non-dimensionalizing is

$$\frac{U(z)}{U_h} = \frac{C_1}{U_h} I_0\left(2\sqrt{A\frac{z}{h}}\right) + \frac{C_2}{U_h} K_0\left(2\sqrt{A\frac{z}{h}}\right) + \frac{U_p}{U_h} \tag{2}$$

which he further simplifies by parameterizing A as $A = 4.52FAI + 0.62FAI^2$, thereby eliminating the foliage drag coefficient C_L from his original solution. This is an extremely important simplification and will be preserved when generalizing the model to variable foliage distribution. Expressing the model in terms of non-dimensional variables simplifies the computation and the computer code, so I will adopt it from here on out. For distance the normalization factor is h, the canopy height, and for canopy wind speed it is U_h, the wind speed at h. Therefore and with no loss in generality, the upper boundary condition on the solution is written as $U(1)/U_h = 1$, the lower boundary condition, $U(z_0/h)/U_h = 0$, remains the same.

To generalize Wang's solution I will model the canopy as a set of discrete layers, each with a uniform foliage distribution that is constant throughout each individual layer, but that differ from one layer to the next. (See *Massman* 1982 for examples of this approach to representing the canopy foliage distribution.) Modeling the canopy as a set layers of uniform foliage distribution means that Equation (2) is appropriate for each layer, but that C_1, C_2, and A are different with each layer. What follows (beginning on the next page) is how I generalized Wang's (2012) model to this case of a canopy with variable foliage distribution.

The wind speed at the top of the canopy $z/h = 1$ (the first node, $j = 1$), is

$$C_1 I_0(1, A_1) + D_1 K_0(1, A_1) + U_p = 1$$

where $I_0(1, A_1) = I_0(2\sqrt{A_1})$, $K_0(1, A_1) = K_0(2\sqrt{A_1})$, and $A_1 = (4.52 + 0.62 FAI_1) FAI_1$; here FAI_1 is the total FAI times the fractional amount of the total FAI that is contained in the first layer. Note I have dropped the normalization constant U_h, which will now be taken as understood.

At the next node and all interior nodes I will impose two conditions: (1) that the wind speed profile be continuous across the node and (2) that the momentum flux is also continuous across the node. This yields for $2 \leq j \leq N - 1$ (where N = total number of nodes, which will always be one less that the total number of canopy layers)

$$C_{j-1} I_0(z_j/h, A_{j-1}) + D_{j-1} K_0(z_j/h, A_{j-1}) = C_j I_0(z_j/h, A_j) + D_j K_0(z_j/h, A_j)$$

and

$$\sqrt{A_{j-1}} \left[C_{j-1} I_1(z_j/h, A_{j-1}) - D_j K_1(z_j/h, A_{j-1}) \right] = \sqrt{A_j} \left[C_j I_1(z_j/h, A_j) - D_j K_1(z_j/h, A_j) \right]$$

The lower boundary condition is imposed at the bottom node (at z_0/h and $j = N$). This yields

$$C_N I_0(z_0/h, A_{N-1}) + D_N K_0(z_0/h, A_{N-1}) + U_p = 0$$

The model for the wind speed profile now requires solving for the coefficients C_j and D_j by inverting a matrix. This is done on the next page.

$$
\begin{bmatrix}
I_0^{[11]} & K_0^{[11]} & 0 & 0 & 0 & 0 & & 0 \\
I_0^{[21]} & K_0^{[21]} & -I_0^{[22]} & -K_0^{[22]} & 0 & 0 & & 0 \\
\mathcal{I}_1^{[21]} & -\mathcal{K}_1^{[21]} & -\mathcal{I}_1^{[22]} & \mathcal{K}_1^{[22]} & 0 & 0 & & 0 \\
0 & 0 & I_0^{[32]} & K_0^{[32]} & -I_0^{[33]} & -K_0^{[33]} & & 0 \\
0 & 0 & \mathcal{I}_1^{[32]} & -\mathcal{K}_1^{[32]} & -\mathcal{I}_1^{[33]} & \mathcal{K}_1^{[33]} & & 0 \\
 & & & & & & & \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & & 0 \\
\vdots & \vdots & & & & & & \vdots \\
0 & 0 & & & I_0^{[N-1\,N-2]} & K_0^{[N-1\,N-2]} & -I_0^{[N-1\,N-1]} & -K_0^{[N-1\,N-1]} \\
0 & 0 & & & \mathcal{I}_1^{[N-1\,N-2]} & -\mathcal{K}_1^{[N-1\,N-2]} & -\mathcal{I}_1^{[N-1\,N-1]} & \mathcal{K}_1^{[N-1\,N-1]} \\
0 & 0 & & & 0 & 0 & I_0^{[N\,N-1]} & K_0^{[N\,N-1]}
\end{bmatrix} \bullet
$$

$$
\begin{bmatrix}
C_1 \\ D_1 \\ C_2 \\ D_2 \\ \vdots \\ \vdots \\ C_{N-1} \\ D_{N-1} \\ C_N \\ D_N
\end{bmatrix}
=
\begin{bmatrix}
1 - U_p \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ 0 \\ -U_p
\end{bmatrix}
$$

where $I_0^{[nm]} = I_0(z_n/h, A_m)$ and $\mathcal{I}_0^{[nm]} = \sqrt{A_m} I_0(z_n/h, A_m)$ and similarly for $K_0^{[nm]}$, $\mathcal{K}_0^{[nm]}$, $I_1^{[nm]}$, $\mathcal{I}_1^{[nm]}$, $K_1^{[nm]}$, and $\mathcal{K}_1^{[nm]}$. The $\bullet$ signifies matrix multiplication.

Wang's (2012) analytical solution results on the assumption of uniform (constant) foliage distribution. As I have shown it is possible to adapt his solution to variable foliage distribution, but only in a piecewise continuous manner, where each piece or layer of foliage is uniform. Nonetheless, it is natural to wonder if there are analytical solutions for other foliage distribution functions. I will take that up next.

<u>Other Analytical Solutions for Canopy Wind Speed?</u>

Equation (1) can be written (non-dimensionally) as follows:

$$\frac{d}{dz'}\left(z'\frac{dU'}{dz'}\right) - RU' = U'_p \tag{3}$$

or

$$z'\frac{d^2U'}{dz'^2} + \frac{dU'}{dz'} - RU' = U'_p$$

where $z' = z/h$, $U' = U/U_h$, $R = (ha_0C_LU_h)/(\kappa s_h u_*)$ and $U'_p = \frac{h}{\kappa s_h u_* U_h}\frac{1}{\rho}\frac{\partial P}{\partial x}$. So the question is "Is there a solution to this equation for $R = R(z')$, or to be more precise for $a_0 = a_0(z')$?". The answer is yes, but only for a canopy that has it's maximum foliage at the top of the canopy and that decreases linearly to some finite value at the z_0/h. In other words, if I replace a_0 in Equation (3) by $2z'a_0$, there is an analytical solution. (Note: the factor of 2 preserves the total FAI when going from a canopy with foliage distributed uniformly to one with foliage decreases linearly with distance from the canopy top.) For this rather unique case the solution for the canopy wind speed profile is

$$U'(z') = C_1 I_0\left(\sqrt{R}\,z'\right) + C_2 K_0\left(\sqrt{R}\,z'\right)$$

which is the particular solution ($U'_p = 0$) to

$$z'\frac{d^2U'}{dz'^2} + \frac{dU'}{dz'} - Rz'U' = U'_p$$

where $R = (2ha_0C_LU_h)/(\kappa s_h u_*)$ and $a_0h = FAI$. Beyond this I was unable to find an analytical solution to any other continuous foliage distribution.