Modal Intervals

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Part 1: Background literature

Covering

1. Alexandre Goldsztejn et al. - *Modal Intervals Revisited: a mean-value extension to generalized intervals* (GCP 2005)
2. Alexandre Goldsztejn - *Modal Intervals Revisited Part 1: A Generalized Interval Natural Extension* (Reliable Computing 2012)
3. Miguel A. Sainz et al. - *Modal Interval Analysis: New Tools for Numerical Information* (Lecture Notes in Mathematics, Springer 2014)

Notable mention: Edgar Kaucher *Interval Analysis in the Extended Interval Space in IR* (1980)

Questions

Intervals introduce variables representing uncertainty or variation in a model.
Goal: Given function $f : \mathbb{R}^n \to \mathbb{R}$ and an interval domain $[\boldsymbol{x}] \subset \mathbb{R}^n$ we want to estimate $f([\boldsymbol{x}]) = \textbf{range}(f, [\boldsymbol{x}])$.
Outer approximations $\quad f([\boldsymbol{x}]) \subseteq [f]([\boldsymbol{x}])$
Inner approximations $\quad]f[([\boldsymbol{x}]) \subseteq f([\boldsymbol{x}])$

1. Outer approximations over-estimate due to amplification of dependence
2. Inner approximations are hard to compute

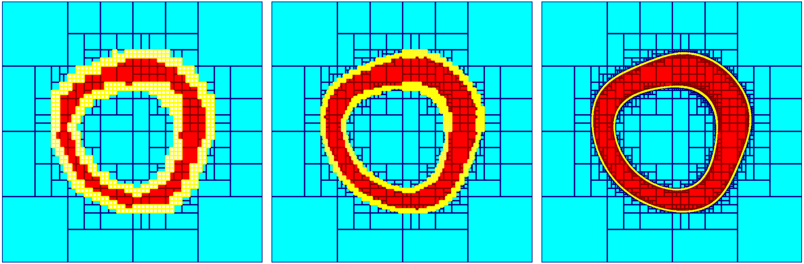
Concept: amplification of dependence

$f(x) = x - x$

Classical interval extension fR:
$fR([1,2]) = [1,2] - [1,2] = \{x - y : x, y \in [1,2]\} = [-1,1]$

Range:
$f([1,2]) = \{x - x : x \in [1,2]\} = [0,0]$

Concept: inner-outer approximations



Example: formulating problems with intervals

Problem 1: We have a power line with length between 60 to 70km. We want to be able to extend it to any length between 100 and 120km. What are the cable lengths should we should procure to make this possible? Possible answer: $[30, 60]$km

$$(\forall a \in [60, 70]\mathrm{km})(\forall t \in [100, 120]\mathrm{km})(\exists x \in [30, 60]\mathrm{km})(a + x = t)$$

Letting $f(a, t) = t - a$, we say $[30, 60]$km is $(f, ([60, 70], [100, 120])\mathrm{km}^2)$-interpretable

Example: formulating problems with intervals

Problem 2: We have a power line with length between 60 to 70km. We want to extend it to some arbitrary length between 100 and 120km. What cable lengths can we procure to make this possible? Possible answer: $[40, 50]$km

$(\forall a \in [60, 70]\text{km})(\forall x \in [40, 50]\text{km})(\exists t \in [100, 120]\text{km})(a + x = t)$

Letting $f(a, t) = t - a$, we say $[50, 40]$km is $(f, ([60, 70], [120, 100])\text{km}^2)$-interpretable.

Concepts: two desciptions of modal intervals

Description 1:
Real interval set $\mathbb{IR}$ contains intervals $[a, b]$, $a \leq b$.
Modal interval set $\mathbb{KR}$ contains intervals $[a, b]$, $a, b \in \mathbb{R}$

Description 2:
A modal interval $A = (A', Q_A)$ is an element of the cartesian product $\mathbb{IR} \times \{\forall, \exists\}$.
Real interval set $\mathbb{IR} \cong \mathbb{IR} \times \{\forall\}$

Concepts: two desciptions of modal intervals

The two definitions are equivalent:

For $[a, b] \in \mathbb{KR}$,
$[a, b] =$
$$\begin{cases} ([a, b]', \exists) \text{ if } a \leq b \text{ (proper)} \\ ([b, a]', \forall) \text{ if } a > b \text{ (improper)} \end{cases}$$

For real numbers $[a, a]$, $\forall, \exists$ quantifies are interchangable.

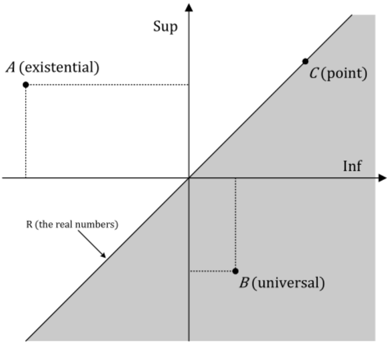


Figure 1: (Inf, Sup)-diagram

Operations

$\mathrm{Sup}([a, b]) = b$
$\mathrm{Inf}([a, b]) = a$
$\mathrm{dual}([a, b]) = [b, a]$
$\mathrm{wid}([a, b]) = b - a$

metric: $d([a_1, b_1], [a_2, b_2]) = \max(|a_1 - a_2|, |b_1 - b_2|)$

$\mathrm{prop}([a, b]) = [\min(a, b), \max(a, b)]$

We denote interval variables $[x] = [a, b]$

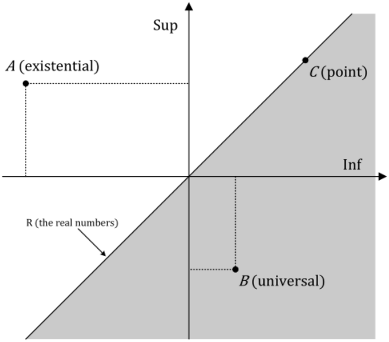


Figure 1: (Inf, Sup)-diagram

Definition: k-dimensional modal intervals

k-dimensional modal intervals $[\boldsymbol{x}] = ([x_1], \ldots, [x_k]) \in \mathbb{KR}^k$

Modal interval matrixes are defined similarly.

We can partition interval arrays into proper-improper components $[\boldsymbol{x}] = ([\boldsymbol{x}_{\mathcal{P}}], [\boldsymbol{x}_{\mathcal{I}}])$ where $\mathcal{P} = \{k : [x_k] \in \mathbb{IR}\}$ and $\mathcal{I} = \{k : [x_k] \notin \mathbb{IR}\}$.

Definition: $(f, [\boldsymbol{x}])$-interpretability

Given a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ and $[\boldsymbol{x}] \in \mathbb{KR}^n$ we want to compute $[z] \in \mathbb{KR}$ such that the following quantified proposition is true:

$$(\forall \boldsymbol{x}_{\mathcal{P}} \in [\boldsymbol{x}_{\mathcal{P}}])(Q_z z \in [z])(\exists \boldsymbol{x}_{\mathcal{I}} \in [\boldsymbol{x}_{\mathcal{I}}])(f(\boldsymbol{x}) = z)$$

where $\mathcal{P} = \{k : [x_k] \in \mathbb{IR}\}$, $\mathcal{I} = \{k : [x_k] \notin \mathbb{IR}\}$ and $Q_z = \exists$ if $[z] \in \mathbb{IR}$, $Q_z = \forall$ otherwise.

We call $[z]$ an interval that is $(f, [\boldsymbol{x}])$-interpretable.

Example: $(f, [\boldsymbol{x}])$-interpretability

Given a continuous function $f : \mathbb{R}^n \to \mathbb{R}$, $[\boldsymbol{x}] \in \mathbb{KR}^n$, $[z] \in \mathbb{KR}$:

If $[\boldsymbol{x}] \in \mathbb{IR}^n$ and $[z] \in \mathbb{IR}$ then $(\exists z \in [z])(\forall \boldsymbol{x} \in [\boldsymbol{x}])(f(\boldsymbol{x}) = z)$
so $[z]$ is an outer approximation

If $[\boldsymbol{x}] \notin \mathbb{IR}^n$ and $[z] \notin \mathbb{IR}$ then $(\forall z \in [z])(\exists \boldsymbol{x} \in [\boldsymbol{x}])(f(\boldsymbol{x}) = z)$
making $[z]$ an inner approximation

Definition: subsets - comparing $(f, [\boldsymbol{x}])$-interpretable intervals

Given a continuous function $f : \mathbb{R}^n \to \mathbb{R}$, $[\boldsymbol{x}] \in \mathbb{KR}^n$, $[z], [z'] \in \mathbb{KR}$ are intervals that are $(f, [\boldsymbol{x}])$-interpretable.

If $[z'] \subseteq [z]$ then $[z']$ is more accurate than $[z]$

$$A \subseteq B \;\equiv \begin{cases} A' \subseteq B' \;\text{ if } A, B \in \mathbb{IR} \\ A' \supseteq B' \;\text{ if } A, B \notin \mathbb{IR} \\ A' \cap B' \neq \varnothing \;\text{ if } A \notin \mathbb{IR}, B \in \mathbb{IR} \\ A' = B' = [a, a]' \;\text{ if } A \in \mathbb{IR}, B \notin \mathbb{IR} \end{cases}$$

Question: AE-extensions

We call $[z]$ an interval that is $(f, [\boldsymbol{x}])$-interpretable... but how do we reliably obtain $[z]$?

Given a continuous function $f : \mathbb{R}^n \to \mathbb{R}$, we wish to obtain an AE-extension, that is a function $f : \mathbb{KR}^n \to \mathbb{KR}$ such that $\forall [\boldsymbol{x}] \in \mathbb{KR}^n, f([\boldsymbol{x}])$ is $(f, [\boldsymbol{x}])$-interpretable.

Question: AE-extensions

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Can we construct f from *elementary functions*?

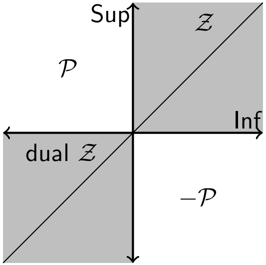
Extending elementary functions

Addition: $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$
Subtraction: $[a_1, a_2] - [b_1, b_2] = [b_1 - a_1, b_2 - a_2]$

Table 1: Kaucher multiplication

$\boldsymbol{x} \times \boldsymbol{y}$	$\boldsymbol{y} \in \mathcal{P}$	$\boldsymbol{y} \in \mathcal{Z}$	$\boldsymbol{y} \in -\mathcal{P}$	$\boldsymbol{y} \in$ dual $\mathcal{Z}$
$\boldsymbol{x} \in \mathcal{P}$	$[\underline{x}\,\underline{y}, \overline{x}\,\overline{y}]$	$[\overline{x}\,\underline{y}, \overline{x}\,\overline{y}]$	$[\overline{x}\,\underline{y}, \underline{x}\,\overline{y}]$	$[\underline{x}\,\underline{y}, \underline{x}\,\overline{y}]$
$\boldsymbol{x} \in \mathcal{Z}$	$[\underline{x}\,\overline{y}, \overline{x}\,\overline{y}]$	$[\min\{\underline{x}\,\overline{y}, \overline{x}\,\underline{y}\}, \max\{\underline{x}\,\underline{y}, \overline{x}\,\overline{y}\}]$	$[\overline{x}\,\underline{y}, \underline{x}\,\underline{y}]$	0
$\boldsymbol{x} \in -\mathcal{P}$	$[\underline{x}\,\overline{y}, \overline{x}\,\underline{y}]$	$[\underline{x}\,\overline{y}, \underline{x}\,\underline{y}]$	$[\overline{x}\,\overline{y}, \underline{x}\,\underline{y}]$	$[\overline{x}\,\overline{y}, \overline{x}\,\underline{y}]$
$\boldsymbol{x} \in$ dual $\mathcal{Z}$	$[\underline{x}\,\underline{y}, \overline{x}\,\underline{y}]$	0	$[\overline{x}\,\overline{y}, \underline{x}\,\overline{y}]$	$[\max\{\underline{x}\,\underline{y}, \overline{x}\,\overline{y}\}, \min\{\underline{x}\,\overline{y}, \overline{x}\,\underline{y}\}]$

where $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{KR} \mid 0 \le \underline{x} \wedge 0 \le \overline{x}\}$, $-\mathcal{P} = \{\boldsymbol{x} \in \mathbb{KR} \mid 0 \ge \underline{x} \wedge 0 \ge \overline{x}\}$, $\mathcal{Z} = \{\boldsymbol{x} \in \mathbb{KR} \mid \underline{x} \le 0 \le \overline{x}\}$ and dual $\mathcal{Z} = \{\boldsymbol{x} \in \mathbb{KR} \mid \underline{x} \ge 0 \ge \overline{x}\}$.



Properties of elementary functions

Additive identity : $[a_1, a_2] + [0, 0] = [a_1, a_2] = [0, 0] + [a_1, a_2]$
Additive inverse:
$[a_1, a_2] + -\text{dual}([a_1, a_2]) = [a_1, a_2] + [-a_1, -a_2] = [0, 0]$

Multiplicative identity: $[a_1, a_2] * [1, 1] = [a_1, a_2] = [1, 1] * [a_1, a_2]$
Multiplicative inverse: if $0 \notin \text{prop}([a_1, a_2])$ then
$[a_1, a_2] * \frac{1}{\text{dual}([a_1, a_2])} = [a_1, a_2] * \left[\frac{1}{a_2}, \frac{1}{a_1}\right] = [1, 1]$

$(\mathbb{KR}, +, \times)$ is *not* a Ring
$A(B + C) \subseteq AB + AC$ if A is proper, and $A(B + C) \supseteq AB + AC$ if A is improper.

Theorem: JM-commutivity

Every one variable continuous function $f : \mathbb{R} \to \mathbb{R}$ is JM-commutable: for $A \in \mathbb{KR}$

$$f(A) = \begin{cases} [\min_{x \in A} f(x), \max_{x \in A} f(x)] & \text{if } A \text{ is proper} \\ [\max_{x \in A} f(x), \min_{x \in A} f(x)] & \text{if } A \text{ is improper} \end{cases}$$

Examples:

$$sin(A) = \begin{cases} [\min_{x \in A} sin(x), \max_{x \in A} sin(x)] & \text{if } A \text{ is proper} \\ [\max_{x \in A} sin(x), \min_{x \in A} sin(x)] & \text{if } A \text{ is improper} \end{cases}$$

Example: JM-commutivity

$$X^n = \underline{\text{if}}\ n \text{ is odd } \underline{\text{then}}\ [x_1^n, x_2^n],$$

$$\underline{\text{if}}\ n \text{ is even } \underline{\text{then}}$$

$$\underline{\text{if}}\ (x_1 \geq 0,\ x_2 \geq 0)\ \underline{\text{then}}\ [x_1^n, x_2^n],$$

$$\underline{\text{if}}\ (x_1 < 0,\ x_2 < 0)\ \underline{\text{then}}\ [x_2^n, x_1^n],$$

$$\underline{\text{if}}\ (x_1 < 0,\ x_2 \geq 0)\ \underline{\text{then}}\ [0, \max(|x_1|^n, |x_2|^n)],$$

$$\underline{\text{if}}\ (x_1 \geq 0,\ x_2 < 0)\ \underline{\text{then}}\ [\max(|x_1|^n, |x_2|^n), 0].$$

$$|X| = \underline{\text{if}}\ x_1 \geq 0,\ x_2 \geq 0\ \underline{\text{then}}\ [x_1, x_2],$$

$$\underline{\text{if}}\ x_1 < 0,\ x_2 < 0\ \underline{\text{then}}\ [|x_2|, |x_1|],$$

$$\underline{\text{if}}\ x_1 < 0,\ x_2 \geq 0\ \underline{\text{then}}\ [0, \max(|x_1|, |x_2|)],$$

$$\underline{\text{if}}\ x_1 \geq 0,\ x_2 < 0\ \underline{\text{then}}\ [\max(|x_1|, |x_2|), 0].$$

AE-extensions: general interval extension vs. mean value extension

Let $f : \mathbb{R}^n \to \mathbb{R}$, $[\boldsymbol{x}] \in \mathbb{KR}^n$

General interval extension:
If $f(x_1, \ldots, x_n)$ can be expressed in elementary functions where each variable appears only once, then $f([\boldsymbol{x}])$ calculated by replacing each x_i by $[x_i]$ is $(f, [\boldsymbol{x}])$-interpretable.

Mean value extension:
Let $\boldsymbol{\Delta} \in \mathbb{IR}^n$ such that $\left\{ \frac{\partial f}{\partial x_i}(x) : x \in \mathrm{prop}([\boldsymbol{x}]) \right\} \subseteq \Delta_i$.
Then $f^{\mathrm{MV}}([\boldsymbol{x}]) := f(\tilde{x}) + \boldsymbol{\Delta} \cdot ([\boldsymbol{x}] - \tilde{x})$ is $(f, [\boldsymbol{x}])$-interpretable for all $\tilde{x} \in \mathrm{prop}([\boldsymbol{x}])$.

(Δ can be interval approx. of a gradient or Jacobian)

Order of convergence for AE-extensions

How do these two kinds of extensions compare?

Definition: order of convergence α
The AE-extension for f has an order of convergence $\alpha \in \mathbb{R}, \alpha > 0$ if and only if there exists a minimal AE-extension $h : \mathbb{IR}^n \to \mathbb{IR}$ for f such that $\forall [\boldsymbol{y}] \in \mathbb{IR}^n, \exists \gamma > 0, \forall [x] \in \boldsymbol{K}[\boldsymbol{y}]$ we have

$$||\text{wid}(h([\boldsymbol{x}])) - \text{wid}(f([\boldsymbol{x}]))|| \leq \gamma ||\text{wid}([\boldsymbol{x}])||^{\alpha}$$

1. General interval extension: linear order ($\alpha = 1$)
2. Mean value extension: quadratic order ($\alpha = 2$)
3. a minimal AE-extension h of f always exists, but may not be feasible.
4. $\boldsymbol{K}[\boldsymbol{y}] = \{[a, b] : a, b \in [\boldsymbol{y}]\}$

Part 2: Inner Approximated Reachability Analysis

Covering:

~~Eric Goubault, Olivier Mullier and Sylvie Putot (French Alternative Energies and Atomic Energy Commission (CEA)), Michel Kieffer (French National Center for Scientific Research (CNRS)) *Inner Approximated Reachability Analysis* (HSCC 2014)~~

Eric Goubault and Sylvie Putot (French Alternative Energies and Atomic Energy Commission (CEA)) *Forward Inner-Approximated Reachability of Non-Linear Continuous Systems* (HSCC 2017)

Problem

Problem specification: an autonomous ODE $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is infinitely differentiable, a time interval $[0, \tau]$ and initial conditions $\left[\boldsymbol{x}^{(0)}\right] \in \mathbb{IR}$

Goal: Find tight inner approximation of reachable states for all initial conditions in $\left[\boldsymbol{x}^{(0)}\right]$ over time interval.

Solution

Given partitions $[t_j, t_{j+1}]$ of $[0, \tau]$...

Use automatic differentiation to obtain the coefficients of the k-th order taylor approximation for $\boldsymbol{x}(t)$:

1. $\boldsymbol{f}^{[1]} = \boldsymbol{f}$
2. $\boldsymbol{f}^{[i+1]} = J(\boldsymbol{f}^{[i]}) \cdot \boldsymbol{f}$
3. Aproximate $\left[\boldsymbol{r}^{(j+1)}\right]$
4. For $t \in [t_j, t_{j+1}]$, $\boldsymbol{x}(t, t_j, \left[\boldsymbol{x}^{(j)}\right]) =$ $\left[\boldsymbol{x}^{(j)}\right] + \sum_{i=0}^{k-1} (t - t_j)^i \frac{\boldsymbol{f}^{[i]}(\left[\boldsymbol{x}^{(j)}\right])}{i!} + (t - t_j)^k \frac{\boldsymbol{f}^{[k]}(\left[\boldsymbol{r}^{(j+1)}\right])}{k!}$
5. $\left[\boldsymbol{x}^{(j+1)}\right] = \boldsymbol{x}(t_{j+1}, t_j, \left[\boldsymbol{x}^{(j)}\right])$

Solution

Starting from a point $\tilde{x}^{(0)} \in \left[\boldsymbol{x}^{(0)}\right]$ initialize $\left[\tilde{\boldsymbol{x}}^{(0)}\right] = \tilde{x}^{(0)}$

1. Aproximate $\left[\tilde{\boldsymbol{r}}^{(j+1)}\right]$
2. For $t \in [t_j, t_{j+1}]$, $\tilde{\boldsymbol{x}}(t, t_j, \left[\tilde{\boldsymbol{x}}^{(j)}\right]) =$
   $\left[\tilde{\boldsymbol{x}}^{(j)}\right] + \sum_{i=0}^{k-1}(t - t_j)^i \frac{\boldsymbol{f}^{[i]}(\left[\tilde{\boldsymbol{x}}^{(j)}\right])}{i!} + (t - t_j)^k \frac{\boldsymbol{f}^{[k]}(\left[\tilde{\boldsymbol{r}}^{(j+1)}\right])}{k!}$
3. $\left[\tilde{\boldsymbol{x}}^{(j+1)}\right] = \tilde{\boldsymbol{x}}(t_{j+1}, t_j, \left[\tilde{\boldsymbol{x}}^{(j)}\right])$

Solution

Obtain taylor approximation of the Jacobian of $\boldsymbol{x}(t)$. Let

$$\mathrm{Jac}_{\boldsymbol{x}} f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

1. Aproximate $\left[\boldsymbol{R}^{(j+1)}\right]$
2. For $t \in [t_j, t_{j+1}]$, $\boldsymbol{J}(t, t_j, \left[\boldsymbol{x}^{(j)}\right]) = \left[\boldsymbol{J}^{(j)}\right] + \sum_{i=0}^{k-1} (t - t_j)^i \frac{\mathrm{Jac}_{\boldsymbol{x}} \boldsymbol{f}^{[i]}(\left[\boldsymbol{J}^{(j)}\right])}{i!} \cdot \left[\boldsymbol{J}^{(j)}\right] + (t - t_j)^k \frac{\mathrm{Jac}_{\boldsymbol{x}} \boldsymbol{f}^{[k]}(\left[\boldsymbol{r}^{(j+1)}\right])}{k!} \cdot \left[\boldsymbol{R}^{(j+1)}\right]$
3. $\left[\boldsymbol{J}^{(j+1)}\right] = \boldsymbol{J}(t_{j+1}, t_j, \left[\boldsymbol{x}^{(j)}\right])$

Solution

Recall mean-value AE-extension:
Let $f : \mathbb{R}^n \to \mathbb{R}$, $[\boldsymbol{x}] \in \mathbb{KR}^n$
and $\boldsymbol{\Delta} \in \mathbb{IR}^n$ such that $\left\{\frac{\partial f}{\partial x_i}(x) : x \in \text{prop}([\boldsymbol{x}])\right\} \subseteq \Delta_i$.
Then $f^{\text{MV}}([\boldsymbol{x}]) := f(\tilde{x}) + \boldsymbol{\Delta}([\boldsymbol{x}] - \tilde{x})$ is $(f, [\boldsymbol{x}])$-interpretable for all $\tilde{x} \in \text{prop}([\boldsymbol{x}])$.

Define:
$]\boldsymbol{x}[(t, t_j) := \tilde{\boldsymbol{x}}(t, t_j, \left[\tilde{\boldsymbol{x}}^{(j)}\right]) + \boldsymbol{J}(t, t_j, \left[\boldsymbol{x}^{(j)}\right])(\text{dual}\left[\boldsymbol{x}^{(0)}\right] - \tilde{\boldsymbol{x}}^{(0)})$

If it is improper for given t, then $]\boldsymbol{x}[(t, t_j)$ is an inner approximation.

Solution: Vanderpoll