M 29

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LEAVING CERTIFICATE EXAMINATION, 2000

MATHEMATICS — HIGHER LEVEL — PAPER 1 (300 marks)

THURSDAY, 8 JUNE — MORNING, 9.30 to 12.00

Attempt **SIX QUESTIONS** (50 marks each).

Marks may be lost if all necessary work is not clearly shown.

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1. (a) Show that the following simplifies to a constant when $x \neq 2$

$$\frac{3x-5}{x-2}+\frac{1}{2-x}.$$

(b) $f(x)=ax^3+bx^2+cx+d$ where $a,\ b,\ c,\ d \in \mathbf{R}$.

If k is a real number such that $f(k)=0$, prove that $x-k$ is a factor of $f(x)$.

(c) $(x-t)^2$ is a factor of $x^3+3px+c$.

Show that

(i) $p=-t^2$

(ii) $c=2t^3$.

2. (a) Solve for x, y, z

$$\begin{aligned} 3x-y+3z&=1\\ x+2y-2z&=-1\\ 4x-y+5z&=4. \end{aligned}$$

(b) Solve $x^2-2x-24=0$.

Hence, find the values of x for which

$$\left(x+\frac{4}{x}\right)^2-2\left(x+\frac{4}{x}\right)-24=0, \qquad x \in \mathbf{R},\ x \neq 0.$$

(c) **(i)** Express a^4-b^4 as a product of three factors.

(ii) Factorise $a^5-a^4b-ab^4+b^5$.

Use your results from **(i)** and **(ii)** to show that

$$a^5+b^5>a^4b+ab^4$$

where a and b are positive unequal real numbers.

3. **(a)** Given that A = $\begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & 1 \\ -5 & -2 \end{pmatrix}$, find $B^{-1}A$.

(b) **(i)** Simplify $\left(\dfrac{-2+3i}{3+2i}\right)$ and hence, find the value of $\left(\dfrac{-2+3i}{3+2i}\right)^9$ where $i^2 = -1$.

(ii) Find the two complex numbers $a + ib$ such that

$$(a+ib)^2 = 15 - 8i\,.$$

(c) Use De Moivre's theorem

(i) to prove that $\cos 3q = 4\cos^3 q - 3\cos q$

(ii) to express $\left(-\sqrt{3} - i\right)^{10}$ in the form $2^n\left(1 - i\sqrt{k}\right)$ where $n,\ k \in \mathbf{N}$.

4. **(a)** The first three terms of a geometric sequence are

$$2x-4,\quad x+1,\quad x-3\,.$$

Find the two possible values of x.

(b) Given that

$$u_n = \frac{1}{2}\left(4^n - 2^n\right)$$

for all integers n, show that

$$u_{n+1} = 2u_n + 4^n\,.$$

(c) **(i)** Given that $g(x) = 1 + 2x + 3x^2 + 4x^3 \ldots$ where $-1 < x < 1$, show that

$$g(x) = \frac{1}{(1-x)^2}\,.$$

(ii) $P(n) = u_1 u_2 u_3 u_4 \ldots\ u_n$ where

$u_k = ar^{k-1}$ for $k = 1, 2, 3, \ldots, n$ and $a, r \in \mathbf{R}$.

Write $P(n)$ in the form $a^n r^{f(n)}$ where $f(n)$ is a quadratic expression in n.

5. **(a)** Express the recurring decimal $1.\dot{2}$ in the form $\frac{a}{b}$ where $a,\ b \in \mathbf{N}$.

(b) Prove by induction that $n! > 2^n,\ n \in \mathbf{N},\ n \geq 4$.

(c) **(i)** Solve for x

$$2\log_9 x = \frac{1}{2} + \log_9(5x+18), \quad x > 0.$$

(ii) Solve for x

$$3e^x - 7 + 2e^{-x} = 0.$$

6. **(a)** Differentiate with respect to x

(i) $(1+5x)^3$

(ii) $\frac{7x}{x-3},\quad x \neq 3.$

(b) **(i)** Prove, from first principles, the product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

where $u = u(x)$ and $v = v(x)$.

(ii) Given $y = \sin^{-1}(2x-1)$, find $\frac{dy}{dx}$ and calculate its value at $x = \frac{1}{2}$.

(c) $f(x) = \frac{1}{x+1}$ where $x \in \mathbf{R},\ x \neq -1.$

(i) Find the equations of the asymptotes of the graph of $f(x)$.

(ii) Prove that the graph of $f(x)$ has no turning points or points of inflection.

(iii) If the tangents to the curve at $x = x_1$ and $x = x_2$ are parallel and if $x_1 \neq x_2$, show that

$$x_1 + x_2 + 2 = 0.$$

7. **(a)** Find the slope of the tangent to the curve

$$x^2 - xy + y^2 = 1 \quad \text{at the point } (1,0).$$

(b) The parametric equations of a curve are

$$x = \cos^3 t \quad \text{and} \quad y = \sin^3 t, \quad 0 \le t \le \frac{p}{2}.$$

(i) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in terms of t.

(ii) Hence, find integers a and b such that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{a}{b}(\sin 2t)^2 .$$

(c) $f(x) = \dfrac{\ln x}{x}$ where $x > 0$.

(i) Show that the maximum of $f(x)$ occurs at the point $\left(e, \frac{1}{e}\right)$.

(ii) Hence, show that $x^e \le e^x$ for all $x > 0$.

8. **(a)** Find **(i)** $\displaystyle\int (x^2 + 2)\,dx$ **(ii)** $\displaystyle\int e^{3x}\,dx.$

(b) Evaluate **(i)** $\displaystyle\int_0^{\frac{p}{2}} \sin^2 3q \; dq$ **(ii)** $\displaystyle\int_0^1 \frac{x}{x^2+4}\,dx .$

(c) **(i)** Find the value of the real number p given that

$$\int_2^p \frac{dx}{x^2 - 4x + 5} = \frac{p}{4}.$$

(ii) The region bounded by the curve $y = x^2$ and the line $y = 4$ is divided into two regions of equal area by the line $y = k$.

Show that $k^3 = 16$.