MAT157 LEC0101 Notes

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1 Course Information

1.1 General Info

- Lectures are recorded and will be posted on Quercus
- There is a Piazza for this course, accessible from Quercus (not mandatory, purely for discussions)
- Homework will be submitted and graded through Crowdmark

1.2 Grading

Course grade is based on weekly homework assignments and biweekly quizzes. Quizzes will be held during lecture time, and will be announced.

1.3 Prerequisites

- Set theory basics ($A \cap B$, $A \cup B$, ...)
- Trigonometry (sin, cos, ...)

1.4 What is MAT157 about?

Goals of the course:

1. To teach mathematical writing/language
   (a) Definitions
   (b) Theorems
   (c) Proofs
2. To teach calculus:
   (a) Theory of functions of one real variable (MAT257 will teach multivariable calculus)
   (b) Continuity
   (c) Differentiation
   (d) Integration
   (e) Sequences and series of numbers and functions

2 Proofs

2.1 Example: Pythagorean Theorem

Pythagorean Theorem states that in a right triangle, $a^2 + b^2 = c^2$, where a and b are the side lengths, and c is the hypotenuse of the triangle.

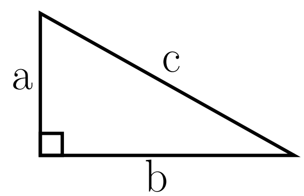

Figure 1: A right triangle with side lengths a and b, and a hypotenuse of c.

The Pythagorean theorem can be proved visually. Consider the following figure, consisting of four congruent triangles enclosed by a square of side lengths $a + b$.

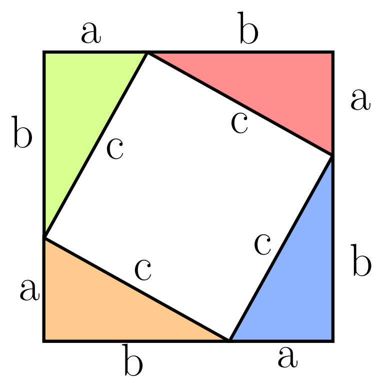

Figure 2: The area of the inner square is c^2.

This figure shows us that the white area is equal to c^2.

By rearranging the four triangles, we get:

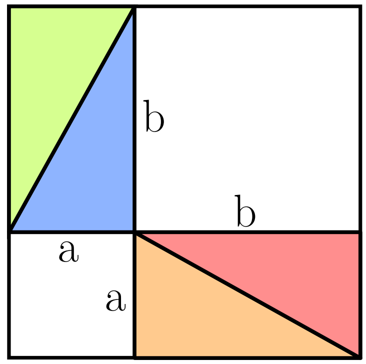

Figure 3: A rearrangement of the previous figure.

This figure shows us that the white area is now equal to $a^2 + b^2$, proving that $a^2 + b^2 = c^2$.

There are, however, there are some issues with this proof. For example, areas and lengths are not defined, and neither are their properties. What axioms are allowed? To make this an actually rigorous proof, axioms of Euclidian geometry need to be stated.

Additionally, it's easy to cheat with picture-based proofs by drawing a picture and proving a bunch of nonsense. For example, consider the following figure:

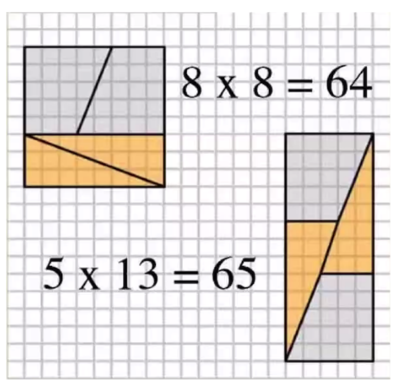

Figure 4: Can you see the issue?

2.2 Proving irrationality of $\sqrt{2}$

Proof. Prove that $\sqrt{2}$ is irrational.

We will prove this by proof with contradiction. Suppose that $\sqrt{2} = \frac{p}{q}, p, q \in \mathbb{N}$. We may assume that q is as small as possible, i.e. we take the smallest q where $q \cdot \sqrt{2} = p, p \in \mathbb{N}$. In particular, p, q are not both even (divisible by 2).

$$\begin{aligned}
&\sqrt{2} = \frac{p}{q} \\
&\Rightarrow \sqrt{2} \cdot q = p \\
&\Rightarrow 2 \cdot q^2 = p^2 \\
&\Rightarrow p^2 \text{ is even} \\
&\Rightarrow p \text{ is even} \\
&\Rightarrow p = 2k, k \in \mathbb{N}
\end{aligned}$$

We can then say that:

$$\begin{aligned}
&2 \cdot q^2 = p^2 \\
&\Rightarrow 2 \cdot q^2 = (2k)^2 = 4k^2 \\
&\Rightarrow q^2 = 2k^2 \\
&\Rightarrow q^2 \text{ is even} \\
&\Rightarrow q \text{ is even}
\end{aligned}$$

So, p and q are both even, which is a contradiction! Therefore $\sqrt{2}$ is irrational, which is what we were trying to prove. $\square$

There are still some issues with this proof, such as the fact that the implication was made that every natural number is either even or odd.