MAT240 Lecture Notes

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1 Injection, Surjection, Bijection

The map $f : X \to Y$ is:

- **injective** when different inputs result in different outputs, i.e. if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

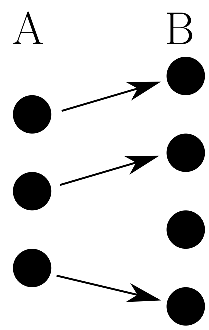

Figure 1: An example of a strictly injective map between A and B.

- **surjective** when all possible outputs are achieved, i.e. $\forall y \in Y, \exists x \in X$ s.t. $f(x) = y$.

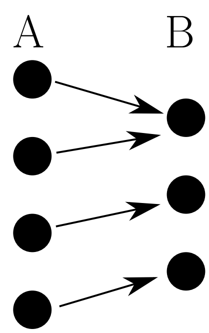

Figure 2: An example of a strictly surjective map between A and B.

- **bijective** when f is both injective and surjective.

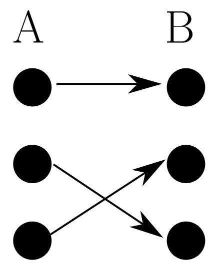

Figure 3: An example of a bijective map between A and B.

When the domain of a map is equal to its codomain, i.e.

$$g : X \to X,$$

there is a specific map called the **identity map** $\mathrm{id}_X = \mathrm{I}_X$ where $\forall x \in X, g(x) = x$.

2 Composition of Maps

The **composition** of $f : X \to Y$ and $g : Y \to Z$ (defined only if the the codomain of f is equal to the domain of g) is a map $g \circ f : X \to Z$ which takes $x \in X$ to $g(f(x)) \in Z$.

For example, the composition of maps $f : X \to Y$ and $g : Y \to Z$ can be written as

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

can be written as

$$X \xrightarrow{g \circ f} Z.$$

Composition is associative, i.e.:

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Keep in mind that composition is usually NOT commutative, i.e. $f \circ g$ is usually not equal to $g \circ f$. In fact, one composition may not even be defined.

Category of sets: **sets**, **maps** between sets, and **composition** of maps.

Deliberate confusion

Purists draw their map in reverse. Consider a map $Z \xleftarrow{g} Y \xleftarrow{f} X$. This is simply $Z \xleftarrow{g \circ f} X$, where the composition of the maps $g \circ f$ follows the correct order of the illustration.

3 Inverses of maps

The **inverse** of a map $f : X \to Y$ is, by definition, a map $g : Y \to X$ s.t. $g \circ f = \mathrm{I}_X$ and $f \circ g = \mathrm{I}_Y$.

A map may not have an inverse. But, if it does exist, we call it f^{-1} (keep in mind that this has nothing to do with $\frac{1}{f}$).

Bijective maps are clear examples of maps that have an inverse, as the mappings from the domain to the codomain can just be "followed" in the opposite direction.

A map $f : X \to X$ is a **self-inverse** when $f(f(x)) = x$.

The bijections from a set X to itself $\mathrm{Bij}(X, X)$ is a very special set of maps:

- any two of them may be composed to give a third, giving an associative multiplication on $\mathrm{Bij}(X, X)$
- $\mathrm{Bij}(X, X)$ contains a special element I_X which acts as a multiplicative identity
- every element in $\mathrm{Bij}(X, X)$ has an inverse

$\Rightarrow$ $(\mathrm{Bij}(X, X), \circ, \mathrm{I}_X)$ is a **group** called S_X, the permutations (symmetries) of the set X.

4 Subsets

Given a set Y, a subset $X \subseteq Y$ is a set comprising some (possibly none, or all) of the elements of Y.

We can think of X as the elements of Y satisfying some constraint. For example: $\mathbb{Z}$ is the set of integers. Let $X \subseteq \mathbb{Z}, X = \{n \in \mathbb{Z} : \text{ n is odd and } 1 \leq n \leq 6\} = \{1, 3, 5\}$.

4.1 Restrictions

If $Y \xrightarrow{f} Z$ and $X \subseteq Y$ then the **restriction** of f to X is a map $f|_X : X \to Z$, where $x \in X \mapsto f(x) \in Z$.

4.2 Power set

The set of all subsets is called $\mathcal{P}(x)$, the **power set** of X.

For example:

$\mathcal{P}(\{R, G, B\}) = \{\emptyset, \{R, G, B\}, \{R\}, \{G\}, \{B\}, \{R, G\}, \{R, B\}, \{G, B\}\}$,

is a set of cardinality 8.

4.3 Basic operations on subsets $X \subseteq Y$

- The **union** of subsets X_1 and X_2 is denoted as

$$X_1 \cup X_2 = \{x \in Y : x \in X_1, \textbf{ or } x \in X_2\}$$

- The **intersection** of subsets X_1 and X_2 is denoted as

$$X_1 \cap X_2 = \{x \in Y : x \in X_1, \textbf{ and } x \in X_2\}$$

Given two elements of $\mathcal{P}(x)$, you can use the union and intersection to get another element of $\mathcal{P}(x)$.

Sidenote: the union and intersections are "binary operations" on $\mathcal{P}(x)$.

4.4 Subsets coming from a map $f : X \to Y$

The **image** of f, $\mathrm{Im}(f)$, is the subset of Y defined by

$$\mathrm{Im}(f) = \{y \in Y : \exists x \in X \text{ with } f(x) = y\}.$$

If $f : X \to Y$ and we fix an element $y \in Y$ then its **pre-image** is

$$f^{-1}(y) = \{x \in X : f(x) = y\}.$$

Keep in mind that in this case $f^{-1}(y)$ does not refer to the inverse.

This defines a **partition** of the domain into subsets labeled by the pre-images of the elements of the image.

For a map $f : X \to Y$, the partitions of the domain X are the set

$$\{f^{-1}(y) \subseteq X : y \in \mathrm{Im}(f)\}.$$