MAT240 Lecture 3 Notes

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Preface (pre-class questions)

Let $Y_1, Y_2 \in X$. $Y_1 \subseteq Y_2$ and $Y_2 \subseteq Y_1 \implies Y_1 = Y_2$.

Let $f : X \to Y$ and $g : X \to Y$. $\forall x \in X, f(x) = g(x) \implies f = g$.

1 Partition from pre-image

Any time you have a map, it automatically gives a partition of the domain.

Given a map $f : X \to Y$, we obtain:

1. $\mathrm{Im}(f) = \{y \in Y : \exists x \in X \text{ with } y = f(x)\}$
2. Partition of X into preimages:

$$P = \{f^{-1}(y) : y \in \mathrm{Im}(f)\}$$

This is a partition of X labelled by points in $\mathrm{Im}(f)$.

Note:

1. There is a natural map from X to P (that is, it can be defined without any additional data):

$$X \xrightarrow{\pi} P$$
$$x \mapsto f^{-1}(f(x))$$

   Note that here, f^{-1} represents the pre-image.

   This is a **surjective** map!

2. There is a natural map

$$P \to \mathrm{Im}(f)$$
$$f^{-1}(y) \mapsto y$$

   Note that this map is a bijection.

3. There is a natural map

$$P \xrightarrow{j} Y$$
$$f^{-1}(y) \mapsto y$$

   Note that this is an injective map!

4. If we compose, we get $x \mapsto f(x)$

$$X \xrightarrow{\pi} P \xrightarrow{j} Y$$

   or simply

$$X \xrightarrow{f} Y$$

   $f = j \circ \pi$ is the **factorization** of f into a surjection followed by injection.

Proposition

Any map $f : X \to Y$ may be factorized $f = j \circ \pi$, where π is surjective and j is injective.

2 Representing Standard Sets with Matrices

The **standard set** of n elements $n = 0, 1, 2, ...$ is defined as

$$B_n = \{1, 2, 3, ...n\}, \text{ where } B_0 = \emptyset$$

One can sketch a map $B_4 \to B_3$ as follows:

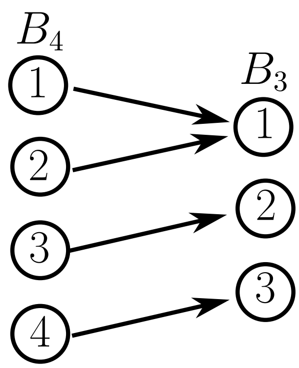

Figure 1: A possible representation of a map $B_4 \to B_3$

$f : B_4 \to B_3$ can also be represented as a 4 by 3 matrix, where the columns represent the domain, and the rows represent the codomain:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This representation uses the fact that B_n has a preferred order (we put the domain and codomain in a specific order).

Note that if we permute the codomain, then the rows would permute. That is, if you permute the codomain, this gives a permutation of the rows. If you permute the domain, this gives a permutation of the columns.

3 Cartesian Product

Given sets X and Y, their (Cartesian) product $X \times Y$ is defined as follows:

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}.$$

Note that (x, y) represents an ordered pair (an ordered pair like $(1, 2) \neq (2, 1)$).

For example, consider $P = R, G, B$.

$$B_2 \times P = \{(1, R), (2, R), (1, G), (2, G), (1, B), (2, B)\}.$$

Similarly, if $X_1, ...X_k$ are sets,

$$X_1 \times X_2 \times ... \times X_k = \{(x_1, ...x_k) : x_i \in X_y \forall i\}.$$

Note that $(x_1, ...x_k)$ represents a k**-tuple**, or **list** of length k.

A shorthand convention for writing $X_1 \times X_2 \times ... \times X_k = \prod_{i=1}^{k} X_i$.

A special case is that $\underbrace{X \times ... \times X}_{k \text{ times}} = X^k$.

Definition: The **graph** of a map $f : X \to Y$ is the subset

$$\text{Graph}(f) = \Gamma_f = \{(x, y) \in X \times Y : y = f(x)\}$$