Example Epsilon-Delta Proof - Anatoly Zavyalov

Suppose we want to prove the statement, for some $a \in \mathbb{R}$,

$$\lim_{x \to a} x^2 = a^2.$$

Equivalently, we want to show that

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x \in \mathbb{R}, |x-a| < \delta \implies \left|x^2 - a^2\right| < \epsilon.$$

Proof. Let $\epsilon > 0$ be given. Let $\delta = \min\left\{1, \dfrac{\epsilon}{2|a|+1}\right\}$. Let $x \in \mathbb{R}$, and suppose that $|x-a| < \delta$. We know that $|x| - |a| < |x-a|$ by the triangle inequality, so

$$\begin{aligned}
&|x| - |a| < |x-a| < \delta \\
\implies &|x| < |a| + \delta \\
\implies &|x| < |a| + \delta < |a| + 1 && \text{as } \delta < 1 \\
\implies &|x| < |a| + 1.
\end{aligned}$$

Next, we know

$$\begin{aligned}
&\delta < \frac{\epsilon}{2|a|+1} \\
\implies &\delta\,(2|a|+1) = \delta\,(|a|+|a|+1) < \epsilon \\
\implies &\delta\,(|x|+|a|) < \delta\,(|a|+|a|+1) < \epsilon \\
\implies &\delta\,(|x+a|) < \delta\,(|x|+|a|) < \epsilon && \text{by the triangle inequality} \\
\implies &|x-a||x+a| < \epsilon && \text{as } |x-a| < \delta \\
\implies &|x^2-a^2| < \epsilon,
\end{aligned}$$

as desired. Since $x \in \mathbb{R}$ was chosen arbitrarily, we know that

$$\forall x \in \mathbb{R}, |x-a| < \delta \implies \left|x^2 - a^2\right| < \epsilon.$$

Hence,

$$\exists \delta > 0 : \forall x \in \mathbb{R}, |x-a| < \delta \implies \left|x^2 - a^2\right| < \epsilon.$$

Since $\epsilon > 0$ was also chosen arbitrarily, we know that

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x \in \mathbb{R}, |x-a| < \delta \implies \left|x^2 - a^2\right| < \epsilon,$$

which is what we wanted to show.

$\square$