Chapter 1 of Friedberg's Linear Algebra, 4th Ed.

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Section 1.1

1. Determine whether the vectors emanating from the origin and terminating at the following pairs of points are parallel.

   (a) $(3,1,2)$ and $(6,4,2)$
   The vectors are not parallel, as there exists no λ such that $\lambda(3,1,2) = (6,4,2)$.

   (b) $(-3,1,7)$ and $(9,-3,-21)$
   The vectors are parallel, as $\lambda = -3$ satisfies $\lambda(-3,1,7) = (9,-3,-21)$.

   (c) $(5,-6,7)$ and $(-5,6,-7)$
   These vectors are parallel, as $\lambda = -1$ satisfies $\lambda(5,-6,7) = (-5,6,-7)$.

   (d) $(2,0,-5)$ and $(5,0,-2)$
   These vectors are not parallel, as there exists no λ such that $\lambda(2,0,-5) = (5,0,2)$.

2. Find the equations of the lines through the following pairs of points in space.

   (a) $(3,-2,4)$ and $(-5,7,1)$
   The endpoint of the vector emanating from the origin and having the same direction as the vector beginning at $(3,-2,4)$ and terminating at $(-5,7,1)$ has coordinates $(-5,7,1)-(3,-2,4) = (-8,9,-3)$.
   Hence the equation of the line is

$$x = (3,-2,4) + t(-8,9,-3).$$

   (b) $(2,4,0)$ and $(-3,-6,0)$
   Using a process similar to (a), we find that the equation of the line is

$$x = (2,4,0) + t(-5,-10,0)$$

   (c) $(3,7,2)$ and $(3,7,-8)$
   Using a process similar to (a), we find that the equation of the line is

$$x = (3,7,2) + t(0,0,-10)$$

   (d) $(-2,-1,5)$ and $(3,9,7)$
   Using a process similar to (a), we find that the equation of the line is

$$x = (-2,-1,5) + t(5,10,2)$$

3. Find the equations of the planes containing the following points in space.

   (a) $(2, -5, -1), (0, 4, 6)$ and $(-3, 7, 1)$
   First, we find the endpoint of the vector emanating from the origin and having the same direction as the vector beginning at $(2, -5, -1)$ and ending at $(0, 4, 6)$:

$$(0, 4, 6) - (2, -5, -1) = (-2, 9, 7)$$

   Next, we do the same for $(2, -5, -1)$ and $(-3, 7, 1)$:

$$(-3, 7, 1) - (2, -5, -1) = (-5, 12, 2)$$

   Thus, the equation of the plane is

$$x = (2, -5, -1) + s(-2, 9, 7) + t(-5, 12, 2)$$

   (b) $(3, -6, 7)$, $(-2, 0, -4)$ and $(5, -9, -2)$
   Using methods similar to (a), we find that the equation of the plane is

$$x = (3, -6, 7) + s(-5, 6, -11) + t(2, -3, -9)$$

   (c) $(-8, 2, 0), (1, 3, 0)$, $(6, -5, 0)$
   Using methods similar to (a), we find that the equation of the plane is

$$x = (-8, 2, 0) + s(9, 1, 0) + t(14, -7, 0)$$

   (d) $(1, 1, 1), (5, 5, 5), (-6, 4, 2)$ Using methods similar to (a), we find that the equation of the plane is

$$x = (1, 1, 1) + s(5, 5, 5) + t(-6, 4, 2)$$

   However, since $(1, 1, 1)$ and $(5, 5, 5)$ are parallel, we can reduce this equation to

$$x = s(1, 1, 1) + t(-6, 4, 2)$$

4. What are the coordinates of the vector 0 in the Euclidean plane that satisfies property 3 on page 3? Justify your answer.

   The coordinates of the vector 0 is $(0, 0, 0)$, as for any $x = (a, b, c)$, we find that

$$x + 0 = (a, b, c) + (0, 0, 0) = (a + 0, b + 0, c + 0) = (a, b, c) = x.$$

5. Prove that if the vector x emanates from the origin of the Euclidean plane and terminates at the point, with coordinates (a_1, a_2), then the vector tx that emanates from the origin terminates at the point with coordinates (ta_1, ta_2).

   Proof. We see that for $x = (a_1, a_2)$,

$$tx = t(a_1, a_2) = (ta_1, ta_2).$$

□

6. Show that the midpoint of the line segment joining the points (a, b) and (c, d) is $((a+c)/2, (b+d)/2)$.

Proof. The midpoint of (a, b) and (c, d) is, the average of the arguments of the vector in the corresponding argument. That is, the first argument of the midpoint vector is the average of the first arguments of the vectors, and similarly for the second argument.

Thus, the midpoint is $(\frac{a+b}{2}, \frac{b+d}{2})$, which is what we wanted to show.

□

7. Prove that the diagonals of a parallelogram bisect each other.

Proof. Let $x = (a_1, a_2), y = (b_1, b_2)$ be any two vectors.

We can create a parallelogram using these vectors, as in Figure 1.2 (a).

In Figure 1.2 (a), we see that one diagonal is the line segment joining the points $(0, 0)$ and $x + y = (a_1 + b_1, a_2 + b_2)$.

By Question 6, we can find the midpont between these points:

$$\frac{1}{2}(x + y) = \left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}\right)$$

Similarly, we can find the midpoint of the other diagonal, which is the line segment joining the points (a_1, a_2) and (b_1, b_2), which is $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right)$, which is the midpoint of the first diagonal.

This shows that these diagonals intersect at their midpoints, thus bisecting each other. □

Section 1.2

1. Label the following statements as true or false.

   (a) Every vector space contains a zero vector: **TRUE**

   (b) A vector space may have more than one zero vector: **FALSE**

   (c) In any vector space, $ax = bx$ implies that $a = b$: **FALSE** (consider $x = 0$)

   (d) In any vector space, $ax = ay$ implies that $x = y$: **FALSE** (consider $a = 0$)

   (e) A vector in $\mathbb{F}^n$ may be regarded as a matrix in $M_{n\times 1}(\mathbb{F})$: **TRUE**

   (f) An $m \times n$ matrix has m columns and n rows: **FALSE** (m rows and n columns)

   (g) In $\mathcal{P}(\mathbb{F})$, only polynomials of the same degree may be added: **TRUE**

   (h) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n: **TRUE**

   (i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n: **TRUE**

   (j) A nonzero scalar of $\mathbb{F}$ may be considered to be a polynomial in $\mathcal{P}(\mathbb{F})$ having degree zero: **TRUE**

   (k) Two functions in $\mathcal{F}(S, \mathbb{F})$ are equal if and only if they have the same value at each element of S: **FALSE** (they are equal if $f(s) = g(s)\ \forall s \in S$)

2. Write the zero vector of $M_{3\times 4}(\mathbb{F})$.

   The zero vector is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3. If

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

what are M_{13}, M_{21}, and M_{22}?

$M_{13} = 3, M_{21} = 4, M_{22} = 5$.

4. Perform the indicated operations.

(a)

$$\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 5 \\ -5 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 3 & 2 \\ -4 & 3 & 9 \end{pmatrix}$$

(b)

$$\begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix} + \begin{pmatrix} 7 & -5 \\ 0 & -3 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -5 \\ 3 & 8 \end{pmatrix}$$

(c)

$$4 \begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 8 & 20 & -12 \\ 4 & 0 & 28 \end{pmatrix}$$

(d)

$$-5 \begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} 30 & -20 \\ -15 & 10 \\ -5 & -40 \end{pmatrix}$$

(e)

$$(2x^4 - 7x^3 + 4x + 3) + (8x^3 + 2x^2 - 6x + 7) = 2x^4 + x^3 + 2x^2 - 2x + 10$$

(f)

$$(-3x^3 + 7x^2 + 8x - 6) + (2x^3 - 8x + 10) = -x^3 + 7x^2 + 4$$

(g)

$$5(2x^7 - 6x^4 + 8x^2 - 3x) = 10x^7 - 30x^4 + 40x^2 - 15x$$

(h)

$$3(3x^5 - 2x^3 + 4x + 2) = 9x^5 - 6x^3 + 12x + 6$$

5. Refer to textbook for question. I'm not going to type all of this out.

The matrix for the upstream crossings is

$$\begin{pmatrix} 8 & 3 & 1 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix},$$

and the matrix for the downstream crossings is

$$\begin{pmatrix} 9 & 1 & 4 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

The sum of the matrices is

$$\begin{pmatrix} 17 & 4 & 5 \\ 6 & 0 & 0 \\ 4 & 1 & 0 \end{pmatrix},$$

which indeed gives the total number of crossings categorized by trout species and season.

6. Again, refer to the textbook for the question.

The matrix for the data is

$$M = \begin{pmatrix} 4 & 2 & 1 & 3 \\ 5 & 1 & 1 & 4 \\ 3 & 1 & 2 & 6 \end{pmatrix}.$$

It is clear to see that doubling the stock will double the matrix M, thus resulting in $2M$:

$$2M = \begin{pmatrix} 8 & 4 & 2 & 6 \\ 10 & 2 & 2 & 8 \\ 6 & 2 & 4 & 12 \end{pmatrix}.$$

$2M - A$ represents the data of all the items **sold** over May and June, categorized by type of suite and style.

We see that

$$2M - A = \begin{pmatrix} 8 & 4 & 2 & 6 \\ 10 & 2 & 2 & 8 \\ 6 & 2 & 4 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 3 & 1 & 2 \\ 6 & 2 & 1 & 5 \\ 1 & 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 & 4 \\ 4 & 0 & 1 & 3 \\ 5 & 2 & 1 & 9 \end{pmatrix}.$$

To find the total number of suites sold during the June sale, we simply add all of the entries of the matrix $2M - A$ to get

$$3 + 1 + 1 + 4 + 4 + 0 + 1 + 3 + 5 + 2 + 1 + 9 = 34.$$

Thus, 34 suites were sold.

7. Let $S = \{0, 1\}$ and $\mathbb{F} = \mathbb{R}$. In $\mathcal{F}(S, \mathbb{R})$, show that $f = g$ and $f + g = h$, where $f(t) = 2t + 1, g(t) = 1 + 4t - 2t^2$, and $h(t) = 5^t + 1$..

Proof. Note that

$$f(0) = 2(0) + 1 = 1 = g(0) = 1 + 4(0) - 2(0)^2$$

and

$$f(1) = 2(1) + 1 = 1 = g(1) = 1 + 4(1) - 2(1)^2,$$

thus $f = g$.

Next, note that

$$\begin{aligned} (f + g)(0) &= f(0) + g(0) \\ &= 1 + 1 = 0 \\ &= h(0) = 5^0 + 1 = 1 + 1 \end{aligned}$$

and

$$\begin{aligned} (f + g)(1) &= f(1) + g(1) \\ &= 1 + 1 = 0 \\ &= h(1) = 5^1 + 1 = 6 = 0. \end{aligned}$$

Thus, $f + g = h$.

□

8. In any vector space V, show that $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in \mathbb{F}$.

Proof. Let V be a vector space, and let $x, y \in V$ and $a, b \in \mathbb{F}$.

We see that

$$\begin{aligned}
(a+b)(x+y) &= (a+b)x + (a+b)y && \text{as } (a+b) \in \mathbb{F} \text{ and by VS 7} \\
&= ax + bx + ay + by && \text{by VS 8}
\end{aligned}$$

which is what we wanted to show.

□

9. **Corollary 1.** The vector 0 described in (VS 3) is unique.

Proof. Let $a, b \in V$ such that $x + a = x$ and $x + b = x$ for all $x \in V$. We see that

$$\begin{aligned}
a &= a + b \\
&= b + a && \text{by VS 1} \\
&= b
\end{aligned}$$

which is what we wanted to show. □

Corollary 2. The vector y described in (VS 4) is unique.

Proof. Let $x \in V$. Let $y, z \in V$ such that $x + y = 0$, $x + z = 0$. We see that

$$\begin{aligned}
y &= y + 0 \\
&= y + (x + z) \\
&= (y + x) + z && \text{by VS 2} \\
&= 0 + z \\
&= z, && \text{by VS 3}
\end{aligned}$$

which is what we wanted to show.

□

10. Let V denote the set of all differentiable real-valued functions defined on the real line. Prove that V is a vector space with the operations of addition and scalar multiplication defined in Example 3.

Proof. Let $f, g, h \in V$.

First, we see that $f + g \in V$, as the sum of two differentiable functions is still differentiable:

$$(f + g)' = f' + g'$$

Similarly, for any $c \in \mathbb{R}$, we see that $cf \in V$, as $(cf)' = c(f')$, which is differentiable.

(As an aside, one may think of this as the derivative map D being **linear**, as it satisfies $D(f + cg) = D(f) + cD(g)$ for any $c \in \mathbb{R}$, $f, g \in V$.)

To verify VS 1, we see that for all $x \in \mathbb{R}$,

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x) \\
&= g(x) + f(x) && \text{by commutativity in } \mathbb{R} \\
&= (g + f)(x)
\end{aligned}$$

Thus, $f + g = g + f$.

Next, to verify VS 2, we see that for any $x \in \mathbb{R}$,

$$\begin{aligned}
((f+g)+h)(x) &= (f+g)(x) + h(x) \\
&= (f(x)+g(x)) + h(x) \\
&= f(x) + (g(x)+h(x)) && \text{associativity of addition in } \mathbb{R} \\
&= f(x) + (g+h)(x) \\
&= (f+(g+h))(x),
\end{aligned}$$

thus $(f+g)+h = f+(g+h)$.

For VS 3, the zero element is 0, the map that sends everything to 0, as, for any $x \in \mathbb{R}$,

$$(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x).$$

For VS 4, we see that $-f$ is an inverse element of f, as for any $x \in \mathbb{R}$,

$$(f-f)(x) = f(x) - f(x) = 0 = 0(x),$$

so $f - f = 0$.

For VS 5, we see that for any $x \in \mathbb{R}$,

$$(1f)(x) = 1f(x) = f(x).$$

For VS 6, we see that for any $x \in \mathbb{R}$ and $a, b \in \mathbb{R}$,

$$((ab)f)(x) = abf(x) = a(bf)(x).$$

For VS 7, we see that for any $a, x \in \mathbb{R}$,

$$a(f+g)(x) = a(f(x)+g(x)) = af(x) + ag(x)$$

For VS 8, we see that for any $a, b, x \in \mathbb{R}$,

$$((a+b)f)(x) = (a+b)f(x) = af(x) + bf(x) \qquad \text{by distributivity in } \mathbb{R}$$

Thus, V is a vector space. □

11. Let $V = \{0\}$ consist of a single vector 0 and define $0 + 0 = 0$ and $c0 = 0$ for each scalar $c \in \mathbb{F}$. Prove that V is a vector space over $\mathbb{F}$. (V is called the **zero vector space**.)

We know that addition and scalar multiplication is closed, so we just have to show that V satisfies VS 1-8.

We see that $0 + 0 = 0 + 0$, satisfying VS 1.

Next, $(0+0)+0 = 0+0 = 0+(0+0)$, satisfying VS 2.

$0 + 0 = 0$, satisfying VS 3 and 4.

$1 \cdot 0 = 0$, satisfying VS 5.

For VS 6, we see that for any $a, b \in \mathbb{F}$,

$$(ab)0 = a(b0) = 0.$$

For VS 7, we see that for any $a \in \mathbb{F}$,

$$a(0+0) = a(0) = 0 = a(0) + a(0).$$

For VS 8, we see that for any $a, b \in \mathbb{F}$,

$$(a+b)(0) = 0 = 0 + 0 = a(0) + b(0).$$

Thus, V is a vector space.

12. A real-valued function f defined on the real line is called an **even function** if $f(-t) = f(t)$ for each real number t. Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

Proof. Let V be the set described above. Let $f, g \in V$. We see that for any $t \in \mathbb{R}$,

$$(f+g)(-t) = f(-t) + g(-t) = f(t) + g(t) = (f+g)(t),$$

so $f + g \in V$. Next, for any $c \in \mathbb{R}$,

$$(cf)(-t) = c(f(-t)) = c(f(t)) = (cf)(t),$$

so $cf \in V$.

We can show that V satisfies VS 1 - 8 as done in Question 10.

Note that for VS 3, $0(t) = 0 = 0(-t)$.

Additionally, for VS 4, the additive inverse $-f$ of some $f \in V$ is also in V, as for any $t \in \mathbb{R}$,

$$(-f)(-t) = -(f(-t)) = -(f(t)) = (-f)(t).$$

Thus, V is a vector space.

□

13. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \text{ and } c(a_1, a_2) = (ca_1, a_2).$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

V is not a vector space, as it does not satisfy VS 8, which we will prove below.

Proof. Let $a, b \in \mathbb{R}$ and $(x, y) \in V$.

We see that

$$(a+b)(x, y) = ((a+b)x, y),$$

but

$$a(x, y) + b(x, y) = (ax, y) + (bx, y) = (ax + bx = y^2)$$

Thus, VS 8 is not satisfied. □

14. Let $V = \{(a_1, a_2, ..., a_n) : a_i \in \mathbb{C} \text{ for } i = 1, 2, ..., n\}$; so V is a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?

It is still a vector space, as all properties still hold over $\mathbb{R}$ (this can be seen as any element $a \in \mathbb{R}$ being represented as $a + 0i \in \mathbb{C}$). Note that the additive identity is simply $0 \in \mathbb{R}$ (which is $0 + 0i \in \mathbb{C}$).

Additionally, note that V is still closed under addition and scalar multiplication.

15. Let $V = \{(a_1, a_2, ..., a_n) : a_i \in \mathbb{R} \text{ for } i = 1, 2, ..., n\}$; so V is a vector space over $\mathbb{R}$ by Example 1. Is V a vector space over $\mathbb{C}$ with the operations of coordinatewise addition and multiplication?

    V is no longer a vector space, as it is not closed under scalar multiplication. For example, if $c \in \mathbb{C} \setminus \mathbb{R}$ and $x \in V$, we see that $cx \notin V$, as not all coordinates of cx are in $\mathbb{R}$.

16. Let V denote the set of all $m \times n$ matrices with entries in $\mathbb{R}$; so V is a vector space over $\mathbb{R}$ by Example 2. Is V a vector space over $\mathbb{Q}$ with the usual definitions of matrix addition and scalar multiplication?

    V is still a vector space, as the elements 0 and 1 are in $\mathbb{Q}$, so VS 3 and 5 still hold. VS 6, 7, 8 hold since $\mathbb{Q} \subsetneq \mathbb{R}$, and VS 1, 2, 4 hold irregardless of the field that V is over.

17. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{F}\}$, where $\mathbb{F}$ is a field. Define addition of elements of V coordinatewise, and for $c \in \mathbb{F}$ and $(a_1, a_2) \in V$, define

    $$c(a_1, a_2) = (a_1, 0).$$

    Is V a vector space over $\mathbb{F}$ with these operations? Justify your answer.

    V is not a vector space, as it trivially does not satisfy VS 5, as for any $(a_1, a_2) \in V$, we see that $\forall c \in \mathbb{F}$, $c(a_1, a_2) = (a_1, 0)$, which is not always equal to (a_1, a_2) (as $\mathbb{F}$ is a field, it must contain at least two elements).

18. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

    $$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

    Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

    V is not a vector space over $\mathbb{R}$, as it does not satisfy VS 1. For any $(a_1, a_2), (b_1, b_2) \in V$, we see that

    $$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$$

    but

    $$(b_1, b_2) + (a_1, a_2) = (b_1 + 2a_1, b_2 + 3a_2),$$

    which are not equal.

19. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. Define addition of elements of V coordinatewise, and for $(a_1, a_2) \in V$ and $c \in \mathbb{R}$, define

    $$c(a_1, a_2) = \begin{cases} (0, 0) & \text{if } c = 0 \\ \left(ca_1, \frac{a_2}{c}\right) & \text{if } c \neq 0. \end{cases}$$

    Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

    V is not a vector space over $\mathbb{R}$ (but it is a vector space over $\mathbb{Z}_2$!), as it does not satisfy VS 8.

    For any nonzero $a, b \in \mathbb{F}$ such that $a + b \neq 0$ and $(x_1, y_1) \in V$, we have that

    $$(a + b)(x_1, y_1) = ((a + b)x_1, \frac{y_1}{a + b})$$

    but

    $$a(x_1, y_1) + b(x_1, y_1) = (ax_1, \frac{y_1}{a}) + (bx_1, \frac{y_1}{b}) = (ax_1 + bx_1, \frac{y_1}{a} + \frac{y_1}{b}),$$

    where the second coordinates are not always equal.

20. Let V be the set of sequences $\{a_n\}$ of real numbers. For $\{a_n\}, \{b_n\} \in V$ and any real number t, define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \text{ and } t\{a_n\} = \{ta_n\}.$$

Prove that, with these operations, V is a vector space over $\mathbb{R}$.

Proof. Let $\{a_n\}, \{b_n\}, \{c_n\} \in V$ and $a, b \in \mathbb{R}$.

First, note that V is closed under addition and scalar multiplication, as they both create new sequences of real numbers. For addition, the nth element of the new sequence is the sum of the nth elements of the two sequences. For scalar multiplication, the nth element of the new sequence is the product of the nth element of the original sequence by the scalar.

VS 1 holds, as

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} = \{b_n + a_n\} = \{b_n\} + \{a_n\}$$

VS 2 holds, as

$$(\{a_n\} + \{b_n\}) + \{c_n\} = \{a_n + b_n\} + \{c_n\} = \{a_n + b_n + c_n\} = \{a_n\} + \{b_n + c_n\} = \{a_n\} + (\{b_n\} + \{c_n\})$$

VS 3 holds, as there is an additive identity 0 which is a sequence of 0s as every element. We see that

$$\{a_n\} + 0 = \{a_n + 0\} = \{a_n\}.$$

VS 4 holds, as for $\{a_n\}$ there is an additive inverse $\{-a_n\}$ consisting of the negatives of the elements of $\{a_n\}$, where

$$\{a_n\} + \{-a_n\} = \{a_n - a_n\} = 0.$$

VS 5 holds, as we have that

$$1\{a_n\} = \{1a_n\} = \{a_n\}$$

VS 6 holds, as

$$(ab)\{a_n\} = \{aba_n\} = a\{ba_n\} = a(b(\{a_n\}))$$

VS 7 holds, as

$$a(\{a_n\} + \{b_n\}) = a\{a_n + b_n\} = \{a(a_n + b_n)\} = \{aa_n + ab_n\} = \{aa_n\} + \{ab_n\} = a\{a_n\} + a\{b_n\}$$

VS 8 holds, as

$$(a + b)\{c_n\} = \{(a + b)c_n\} = \{ac_n + bc_n\} = \{ac_n\} + \{bc_n\} = a\{c_n\} + b\{c_n\}$$

Thus, V is indeed a vector space.

$\square$

21. Let V and W be vector spaces over a field $\mathbb{F}$. Let

$$Z = \{(v, w) : v \in V \text{ and } w \in W\}.$$

Prove that Z is a vector space over $\mathbb{F}$ with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1).$$

Proof. I really want to go to sleep right now, so I will only verify the closure under addition and multiplication and make a few remarks about some of the properties.

Z is closed under addition, as for any $(v_1, w_1), (v_2, w_2) \in Z$, we have that

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2).$$

We know that because V and W are vector spaces, they are closed under addition. Thus, $v_1 + v_2 \in V$ and $w_1 + w_2 \in W$.

Similarly, Z is closed under scalar multiplication.

For VS 3, the zero element is $(0_V, 0_W)$, where 0_V is the additive identity of V and 0_W is the additive identity of W. The reader can verify that this is the case.

The rest of the properties should be trivial to show.

□

22. How many matrices are there in the vector space $M_{m\times n}(\mathbb{Z}_2)$?

    There are 2^{mn} such matrices, as for every a_{ij} (where $1 \leq i \leq m$ and $1 \leq j \leq n$), we have 2 possible values (0 or 1). Since there are mn such elements, there are 2^{mn} total matrices.

Section 1.3

1. Label the following statements as true or false.

    (a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V: **TRUE**

    (b) The empty set is a subspace of every vector space: **FALSE**, as it does not contain the zero element.

    (c) If V is a vector space other than the zero vector space then V contains a subspace W such that $W \neq V$: **TRUE** ($W = 0$)

    (d) The intersection of any two subsets of V is a subspace of V: **FALSE**, as the two subsets must be subspaces of V in order for their intersection to also be a subspace.

    (e) An $n \times n$ diagonal matrix can never have more than n nonzero entries: **TRUE**

    (f) The trace of a square matrix is the product of its diagonal entries: **FALSE**, it is the sum of the diagonal entries.

    (g) Let W be the xy-plane in $\mathbb{R}^3$; that is, $W = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$. Then $W = \mathbb{R}^2$: **FALSE**, as $\mathbb{R}^2$ is defined as a set containing two-tuples, while W contains three-tuples. While an isomorphism (discussed in later chapters) between the two vector spaces (over $\mathbb{R}$) does exist, the sets are not equal.

2. Determine the transpose of each other matrices that follow. In addition, if the matrix is square, compute its trace.

    (a)

    $$\begin{pmatrix} -4 & 2 \\ 5 & -1 \end{pmatrix}$$

    The transpose is

    $$\begin{pmatrix} -4 & 5 \\ 2 & -1 \end{pmatrix},$$

    and the trace is $-4 - 1 = -5$.

(b)

$$\begin{pmatrix} 0 & 8 & -6 \\ 3 & 4 & 7 \end{pmatrix}$$

The transpose is

$$\begin{pmatrix} 0 & 3 \\ 8 & 4 \\ -6 & 7 \end{pmatrix}$$

(c)

$$\begin{pmatrix} -3 & 9 \\ 0 & -2 \\ 6 & 1 \end{pmatrix}$$

The transpose is

$$\begin{pmatrix} -3 & 0 & 6 \\ 9 & -2 & 1 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 10 & 0 & -8 \\ 2 & -4 & 3 \\ -5 & 7 & 6 \end{pmatrix}$$

The transpose is

$$\begin{pmatrix} 10 & 2 & -5 \\ 0 & -4 & 7 \\ -8 & 3 & 6 \end{pmatrix}.$$

The trace is $10 - 4 + 6 = 12$.

(e)

$$\begin{pmatrix} 1 & -1 & 3 & 5 \end{pmatrix}$$

The transpose is

$$\begin{pmatrix} 1 \\ -1 \\ 3 \\ 5 \end{pmatrix}$$

(f)

$$\begin{pmatrix} -2 & 5 & 1 & 4 \\ 7 & 0 & 1 & -6 \end{pmatrix}$$

The transpose is

$$\begin{pmatrix} -2 & 7 \\ 5 & 0 \\ 1 & 1 \\ 4 & -6 \end{pmatrix}$$

(g)

$$\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

The transpose is

$$\begin{pmatrix} 5 & 6 & 7 \end{pmatrix}$$

(h)

$$\begin{pmatrix} -4 & 0 & 6 \\ 0 & 1 & -3 \\ 6 & -3 & 5 \end{pmatrix}$$

The transpose is

$$\begin{pmatrix} -4 & 0 & 6 \\ 0 & 1 & -3 \\ 6 & -3 & 5 \end{pmatrix} \qquad \text{(it's symmetric!)}$$

The trace is $-4 + 1 + 5 = 2$.

3. Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in M_{m\times n}(\mathbb{F})$ and any $a, b \in \mathbb{F}$.

Lemma 1: For any $A, B \in M_{m\times n}(\mathbb{F})$, $(A + B)^t = A^t + B^t$.

Proof. Let

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}, \ B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix},$$

where $a_{ij}, b_{ij} \in \mathbb{F}$ for $1 \le i \le m, 1 \le j \le n$ (for brevity, these constraints on i, j will be assumed from now on).

We see that

$$\begin{aligned} A^t + B^t &= \begin{pmatrix} a_{11} & \dots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & \dots & b_{m1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \dots & b_{mn} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} + b_{11} & \dots & a_{m1} + b_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} + b_{1n} & \dots & a_{mn} + b_{mn} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{pmatrix}^t \\ &= (A + B)^t, \end{aligned}$$

which is what we wanted to show. $\square$

Lemma 2: For any $A \in M_{m\times n}(\mathbb{F})$ and any $c \in \mathbb{F}$, $cA^t = (cA)^t$.

Proof. Let $c \in \mathbb{F}$, and

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix},$$

where $a_{ij} \in \mathbb{F}$ for $1 \le i \le m, 1 \le j \le n$.

We see that

$$
\begin{aligned}
c(A)^t &= c\begin{pmatrix} a_{11} & \dots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{mn} \end{pmatrix} \\
&= \begin{pmatrix} ca_{11} & \dots & ca_{m1} \\ \vdots & \ddots & \vdots \\ ca_{1n} & \dots & ca_{mn} \end{pmatrix} \\
&= \begin{pmatrix} ca_{11} & \dots & ca_{1m} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \dots & ca_{mn} \end{pmatrix}^t \\
&= (cA)^t,
\end{aligned}
$$

which is what we wanted to show. □

Now, to prove the actual claim.

Proof. Let $A, B \in M_{m\times n}(\mathbb{F})$, $a, b \in \mathbb{F}$. We see that

$$
\begin{aligned}
(aA + bB)^t &= (aA)^t + (bB)^t && \text{by } \textbf{Lemma 1} \\
&= aA^t + bB^t, && \text{by } \textbf{Lemma 2}
\end{aligned}
$$

which is what we wanted to show. □

4. Prove that $(A^t)^t = A$ for each $A \in M_{m\times n}(\mathbb{F})$.

Proof. Let

$$
A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}.
$$

We see that

$$
\begin{aligned}
(A^t)^t &= \begin{pmatrix} a_{11} & \dots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{mn} \end{pmatrix}^t \\
&= \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{mn} \end{pmatrix} \\
&= A,
\end{aligned}
$$

which is what we wanted to show. □

5. Prove that $A + A^t$ is symmetric for any square matrix A.

Proof. Let

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}. \qquad \text{where } n \in \mathbb{N} \text{ (}\mathbb{N}\text{ does not contain 0)}$$

First, we find

$$A + A^t = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} + \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} + a_{11} & \dots & a_{1n} + a_{n1} \\ \vdots & \ddots & \vdots \\ a_{n1} + a_{1n} & \dots & a_{nn} + a_{nn} \end{pmatrix}.$$

We see that

$$(A + A^t)^t = \begin{pmatrix} a_{11} + a_{11} & \dots & a_{n1} + a_{1n} \\ \vdots & \ddots & \vdots \\ a_{1n} + a_{n1} & \dots & a_{nn} + a_{nn} \end{pmatrix} = A + A^t.$$

□

6. Prove that $\text{tr}(aA + bB) = a\text{tr}(A) + b\text{tr}(B)$ for any $A, B \in M_{n\times n}(\mathbb{F})$.

Proof. Let

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \ B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix},$$

and $c, d \in \mathbb{F}$ (I avoided using a, b for obvious reasons).
First, we know that

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii}, \ \text{tr}(B) = \sum_{i=1}^{n} b_{ii}.$$

Next, we find that

$$cA + dB = \begin{pmatrix} ca_{11} + db_{11} & \dots & ca_{1n} + db_{1n} \\ \vdots & \ddots & \vdots \\ ca_{n1} + db_{n1} & \dots & ca_{nn} + db_{nn} \end{pmatrix}.$$

From this, we see that

$$\text{tr}(cA + dB) = \sum_{i=1}^{n} ca_{ii} + db_{ii} = c\sum_{i=1}^{n} a_{ii} + d\sum_{i=1}^{n} b_{ii} = c\text{tr}(A) + d\text{tr}(B),$$

which is what we wanted to show. □

7. Prove that diagonal matrices are symmetric matrices.

Proof. By definition, diagonal matrices have zeroes everywhere except (possibly) on the diagonal. Thus, the transpose of a diagonal matrix will have the same entries on the diagonals and still have zeroes everywhere else.

For example, consider this matrix:

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

Clearly, it and its transpose will be equal, as the diagonal entries will remain the same and all other entries will remain zeroes. □

8. Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

   (I will rephrase the sets provided to be more succinct.)

   (a)
   $$W_1 = \{(3a, a, -a) : a \in \mathbb{R}\}.$$

   W_1 is a subspace of $\mathbb{R}^3$.

   Proof. Clearly, $(0,0,0) = (3 \cdot 0, 0, -0) \in W_1$.
   For any $(3a, a, -a), (3b, b, -b) \in W_1$, we see that

   $$(3a, a, -a) + (3b, b, -b) = (3(a+b), a+b, -(a+b)) \in W_1,$$

   so W_1 is closed under addition. For any $c \in \mathbb{R}$ and $(3a, a, -a) \in W_1$, we see that

   $$c(3a, a, -a) = (3(ca), ca, -(ca)) \in W_1,$$

   so W_1 is closed under scalar multiplication.
   Thus, W_1 is a subspace of $\mathbb{R}^3$. □

   (b)
   $$W_2 = \{(a+2, b, a) : a, b \in \mathbb{R}\}$$

   W_2 is not a subspace of $\mathbb{R}^3$.

   Proof. It does not contain $(0,0,0)$, as the first argument is always 2 bigger than the third. Thus, it is not a subspace of $\mathbb{R}^3$. □

   (c)
   $$W_3 = \{(a, b, 7b - 2a) : a, b \in \mathbb{R}^3\}$$

   Proof. It is clear that $(0,0,0) = (0, 0, 7(0) - 2(0)) \in W_3$.
   Next, we see that for any $(a, b, 7b - 2a), (c, d, 7c - 2d) \in W_3$,

   $$(a, b, 7b - 2a) + (c, d, 7d - 2c) = (a + c, b + d, 7(b + d) - 2(a + c)) \in W_3,$$

   so W_3 is closed under addition.
   Next, for any $c \in \mathbb{R}$, $(a, b, 7b - 2a) \in W_3$,

   $$c(a, b, 7b - 2a) = (ac, bc, 7bc - 2ac) \in W_3,$$

   so W_3 is closed under multiplication.
   Thus, W_3 is a subspace of $\mathbb{R}^3$. □

(d)

$$W_4 = \{(a, b, a - 4b) : a, b \in \mathbb{R}\}$$

Proof. It is clear that $(0,0,0) = (0,0,0-4(0)) \in W_4$.
The reader can verify that W_4 is closed under addition and scalar multiplication, in a similar process as the previous proofs.
Thus, W_4 is a subspace of $\mathbb{R}^3$. □

(e)

$$W_5 = \{(3b - 2a + 1, a, b) : a, b \in \mathbb{R}^3\}$$

W_5 is not a subspace of $\mathbb{R}^3$.

Proof. W_5 does not contain $(0,0,0)$. This is because for the second and third coordinates to be zero, $a = b = 0$). However, this results in the first coordinate being equal to 1. □

(f)

$$W_6 = \{(a, b, c) \in \mathbb{R}^3 : 5a^2 - 3b^2 + 6c^2 = 0\}$$

W_6 is not a subspace of $\mathbb{R}^3$.

Proof. We will prove that W_6 is not closed under addition by counterexample.
Suppose that $a = \sqrt{\frac{6}{5}}, b = 2, c = 1$, and $x = \sqrt{\frac{21}{5}}, y = 3, z = 1$.
We see that

$$5(\sqrt{\frac{6}{5}})^2 - 3(2)^2 + 6(1)^2 = 6 - 12 + 6 = 0,$$

so $(\sqrt{\frac{6}{5}}, 2, 1) \in W_6$. Similarly,

$$5(\sqrt{\frac{21}{5}})^2 - 3(3)^2 + 6(1)^2 = 21 - 27 + 6 = 0,$$

so $(\sqrt{\frac{21}{5}}, 3, 1) \in W_6$. However, when we add

$$(\sqrt{\frac{6}{5}}, 2, 1) + (\sqrt{\frac{21}{5}}, 3, 1) = (\sqrt{\frac{6}{5}} + \sqrt{\frac{21}{5}}, 5, 2),$$

we see that

$$5(\sqrt{\frac{6}{5}} + \sqrt{\frac{21}{5}})^2 - 3(5)^2 + 6(2)^2 = 27 + 6\sqrt{14} - 75 + 24 \neq 0,$$

so W_6 is not closed under addition, and is therefore not a subspace of $\mathbb{R}^3$. □

9. Let W_1, W_3 and W_4 be as in Exercise 8. Describe $W_1 \cap W_3$, $W_1 \cap W_4$, and $W_3 \cap W_4$, and observe that each is a subspace of $\mathbb{R}^3$.

$$W_1 \cap W_3 = \{(0,0,0)\}.$$
$$W_1 \cap W_4 = \{(0,0,0)\}.$$
$$W_3 \cap W_4 = \{(\frac{11a}{3}, a, b) : a, b \in \mathbb{R}^3\}.$$

Note that by Theorem 1.4, these are all subspaces of $\mathbb{R}^3$.

10. Prove that $W_1 = \{(a_1, a_2, ..., a_n) \in \mathbb{F}^n : a_1 + a_2 + ... + a_n = 0\}$ is a subspace of $\mathbb{F}^n$, but $W_2 = \{(a_1, a_2, ..., a_n) \in \mathbb{F}^n : a_1 + a_2 + ... + a_n = 1\}$ is not.

Proof. W_1 trivially contains the zero vector, and, by inspection, it is closed under addition (as$a_1 + ... + a_n = 0$ and $b_1 + ... + b_n = 0$ implies $(a_1 + b_1) + ... + (a_n + b_n) = 0$) and scalar multiplication. Thus, it is a subspace of $\mathbb{F}^n$.

W_2 does not contain the zero vector, so it is not a subspace of $\mathbb{F}^n$.

□

11. Is the set $W = \{f(x) \in \mathcal{P}(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$ a subspace of $\mathcal{P}(F)$ if $n \geq 1$?

Yes, W is a subspace of $\mathcal{P}(F)$ if $n \geq 1$.

Proof. W, by definition, contains the zero polynomial.

It is also closed under addition, as the sum of two polynomials of degree n results in a polynomial of degree n. Similarly, it is closed under scalar multiplication, as a polynomial of degree n multiplied by a scalar results in a polynomial of degree n.

It also goes without mentioning that adding the zero polynomial to any polynomial will not affect its degree.

□

12. An $m \times n$ matrix A is called **upper triangular** if all entries lying below the diagonal entries are zero, that is, if $A_{ij} = 0$ whenever $i > j$. Prove that the upper triangular matrices form a subspace of $M_{m\times n}(\mathbb{F})$.

(This proof is by inspection, but can be rigorized with actual matrices which I am too lazy to write out. Reader, do it yourself if you really feel like it. Is anyone even reading this?)

Proof. The $m \times n$ with zeroes everywhere is an upper triangular matrix.

The sum of any two upper triangular matrices will result in another upper triangular matrix, as the elements below the diagonal will remain zero.

Similarly, the scalar multiple of any upper triangular matrix will remain upper triangular (as the elements below the diagonal remain zero).

Thus, the $m \times n$ upper triangular matrices form a subspace of $M_{m\times n}(\mathbb{F})$.

□

13. Let S be a nonempty set and F a field. Prove that for any $s_0 \in S$, $\{f \in \mathcal{F}(S, \mathbb{F}) : f(s_0) = 0\}$, is a subspace of $\mathcal{F}(S, \mathbb{F})$.

Proof. The zero function is clearly in the set, as $0(s_0) = 0$ for any $s_0 \in S$.

Let f, g be two functions such that $f(s_0) = g(s_0) = 0$.

We see that

$$(f + g)(s_0) = f(s_0) + g(s_0) = 0 + 0 = 0,$$

so the set is closed under addition.

Similarly, for any $c \in \mathbb{F}$,

$$(cf)(s_0) = cf(s_0) = c(0) = 0,$$

so the set is closed under scalar multiplication.

Thus, the set is a subspace of $\mathcal{F}(S, \mathbb{F})$.

$\square$

14. Let S be a nonempty set and $\mathbb{F}$ a field. Let $\mathcal{C}(S, \mathbb{F})$ denote the set of all functions $f \in \mathcal{F}(S, \mathbb{F})$ such that $f(s) = 0$ for all but a finite number of elements of S. Prove that $\mathcal{C}(), \mathbb{F})$ is a subspace of $\mathcal{F}(S, \mathbb{F})$.

Proof. Let $W = \mathcal{C}(S, \mathbb{F})$.

Clearly the function $0 : S \to \mathbb{F}, s \mapsto 0$ is in W, as it is 0 for all but a finite number (0) of elements of S.

If S is finite, then W is trivially closed under addition and scalar multiplication, as the sum of any two elements or the scalar multiple of any element of W satisfies the condition. Therefore, if S is finite, W is a subspace of $\mathcal{F}(S, \mathbb{F})$.

Now, assume that S has an infinite number of elements.

For any $f, g \in W$, we see that, for some $s \in S$, if $(f + g)(s) \neq 0$, there are only a finite number of such $s \in S$, as this can only occur when $f(s) \neq 0$ or $g(s) \neq 0$ (only a finite number of $s \in S$ satisfy this).

Thus, $f + g \in W$.

For any $f \in W$ and $c \in \mathbb{F}$, clearly $(cf)(s) = cf(s) \neq 0$ for only a finite number of elements of S (if $c = 0$, then that number is 0).

Thus, W is closed under addition and scalar multiplication, so it is a subspace of $\mathcal{F}(S, \mathbb{F})$.

$\square$

15. Is the set of all differentiable real-valued functions defined on $\mathbb{R}$ a subspace of $\mathcal{C}(\mathbb{R})$?

Yes.

Proof. The zero function $0 : \mathbb{R} \to \mathbb{R}, x \mapsto 0$ is differentiable, so it is in the set.

Next, the sum of two differentiable real-valued functions is differentiable and remains real-valued, so the set is closed under addition.

Lastly, the scalar multiple of a differentiable real-valued function is differentiable and real-valued, so the set is closed under scalar multiplication.

Thus, it is a subspace of $\mathcal{C}(\mathbb{R})$.

$\square$

16. Let $\mathcal{C}^n(\mathbb{R})$ denote the set of all real-valued functions defined on the real line that have a continuous nth derivative. Prove that $\mathcal{C}^n(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Proof. Let $W = \mathcal{C}^n(\mathbb{R})$, and let $n \in \mathbb{N}$.

Clearly, the zero function is in W, as it has a continuous nth derivative.

Let $f, g \in W$ and $c \in \mathbb{R}$.

We see that the nth derivative of $f + g$ is equal to $f^{(n)} + g^{(n)}$ which is continuous (as the sum of two continuous functions is still continuous), so W is closed under addition.

We also see that the nth derivative of cf is equal to $c(f^{(n)})$, which is continuous (as the scalar multiple of any continuous function remains continuous).

Thus, W contains the zero element, and is closed under both scalar multiplication and addition. Thus, it is a subspace of $\mathcal{F}(\mathbb{R},\mathbb{R})$.

$\square$

17. Prove that a subset W of a vector space V is a subspace of V if and only if $W \neq \varnothing$, and, whenever $a \in \mathbb{F}$ and $x, y \in W$, then $ax \in W$ and $x + y \in W$.

Proof. First, we see that $0 \in W$, as $0 \in \mathbb{F}$ and for any $x \in W$ (which exists because $W \neq \varnothing$), $0x = 0 \in W$.

By definition, W is closed under addition and scalar multiplication.

Thus, W is a subspace of V.

$\square$

18. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof. For the first direction, assume that $W_1 \cup W_2$ is a subspace of V.

We want to show that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

We will prove this by providing a counterexample to the negation.

That is, we will disprove $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$ by counterexample.

Now, suppose that $V = \mathbb{R}^2, W_1 = \{(0, a) : a \in \mathbb{R}\}, W_2 = \{(a, 0) : a \in \mathbb{R}\}$.

V is obviously a vector space, and the reader can verify that W_1 and W_2 are indeed subspaces of V (as they, even by inspection, satisfy the three conditions to be a subspace).

We also see that $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$.

However, when we pick $(0, 1) \in W_1$ and $(1, 0) \in W_2$, we see that $(0, 1) + (1, 0) = (1, 1)$ is in neither sets. However, since $W_1 \cup W_2$ was assumed to be a subspace and therefore closed under addition, this is a contradiction!

Thus, $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$ is false, so $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

For the other direction, assume that $W_1 \subseteq W_2$. We see that $W_1 \cup W_2 = W_2$, which is a subspace. Similarly, for $W_2 \subseteq W_1$, $W_1 \cup W_2 = W_1$, which is again a subspace.

$\square$

19. Prove that if W is a subspace of a vector space V and $w_1, w_2, ..., w_n$ are in W, then $a_1w_1 + a_2w_2 + ... + a_nw_n \in W$ for any scalars $a_1, a_2, ..., a_n$.

Proof. Since W is a subspace, it is closed under addition and scalar multiplication.

Thus, $a_1w_1, ..., a_nw_n \in W$ (by closure under scalar multiplication).

Next, since W is closed under addition, we know $a_1w_1 + ... + a_nw_n \in W$, which is what we wanted to show.

$\square$

20. Show that the set of convergent sequences $\{a_n\}$ (i.e., those for which $\lim_{n\to\infty} a_n$ exists) is a subspace of the vector space V in Exercise 20 of Section 1.2.

Proof. Let W be the set described in the question.

The zero sequence is clearly in W, as it converges to 0 as $n \to \infty$.

For any $\{a_n\}, \{b_n\} \in W$, it is clear that $\{a_n+b_n\}$ will converge (specifically, to $\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$).

Similarly, for any $c \in \mathbb{F}$, $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$, so W is closed under addition and scalar multiplication.

Thus, W is a subspace of V.

□

21. Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be fields. A function $g \in \mathcal{F}(\mathbb{F}_1, \mathcal{F}_2)$ is called an **even function** if $g(-t) = g(t)$ for each $t \in \mathbb{F}_1$ and is called an **odd function** if $g(-t) = -g(t)$ for each $t \in \mathbb{F}_1$. Prove that the set of all even functions in $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$ and the set of all odd functions in $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$ are subspaces of $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$.

Proof. Let E, O denote the sets of even and odd functions of $\mathcal{F}(\mathbb{F}_1, \mathcal{F}_2)$, respectively.

Clearly, the zero function is in both E and 0.

For any $f, g \in E$ and any $t \in \mathbb{F}_1$, we see that

$$(f+g)(t) = f(t) + g(t) = f(-t) + g(-t) = (f+g)(-t),$$

so $f + g \in E$.

Similarly, for any $f, g \in O$ and any $t \in \mathbb{F}_1$, we see that

$$(f+g)(t) = f(t) + g(t) = -f(-t) - g(-t) = -(f+g)(-t),$$

so $f + g \in O$.

Lastly, let $c \in \mathbb{F}_2$.

For any $f \in E$,

$$(cf)(t) = c(f(t)) = c(f(-t)) = (cf)(-t),$$

so $cf \in E$.

For any $g \in O$,

$$(cg)(t) = c(g(t)) = c(-g(-t)) = -(cg)(-t),$$

so $cg \in O$.

Thus, E, O are subspaces of $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$, which is what we wanted to show.

□

22. Let W_1 and W_2 be subspaces of a vector space V.

   (a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

   Proof. $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$ is trivially true, as $W_1 + W_2$ contains all elements of the form $x + 0 = x$ for $x \in W_1$ and $y + 0 = y$ for $y \in W_2$.
   We also see that $0 + 0 = 0 \in W_1 + W_2$.
   For any $(x+y), (a+b) \in W_1+W_2$ (where $a, x \in W_1$ and $b, y \in W_2$), we see that $(x+y)+(a+b) = (a+x) + (y+b) \in W_1 + W_2$.
   Similarly, for any $c \in \mathbb{F}$ and $a + b \in W_1 + W_2$, $c(a+b) = ca + cb \in W_1 + W_2$.
   Thus, $W_1 + W_2$ is a subspace of V. □

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Proof. Let U be the subspace of V that contains both W_1 and W_2.
Let $x \in W_1, y \in W_2$. We see that $x, y \in U$, so, since U is a vector space, $x + y \in U$. Thus, $W_1 + W_2 \subseteq U$.

□