MAT157 Lecture 3 Notes

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1 Set Theory

- DO NOT use $\therefore$ or $\because$! They are not used in this course, or just generally in mathematics.
- $X \cap Y$ represents the intersection between set X and Y.
- $X \cup Y$ represents the union between set X and Y.
- $\emptyset$ represents an empty set.
- $X \subseteq Y$ means that X is a subset of Y, and possibly equal to Y (we will avoid writing $X \subset Y$ as to avoid confusion).
  - For X a propert subset of Y, we will write $X \subseteq Y, X \neq Y$ or $X \subsetneq Y$.
- $\#X$ or represents the **cardinality** (number of elements) in the set X (some authors use $|X|$).
- To describe a set, sometimes we list elements: $\{1, 2, 3\}$, $1, 2, ..., 1000$, or $\{x \in \mathbb{R} | x > 0\}$, or $\{x \in \mathbb{R} : x > 0\}$
- Something like $\{x | x > 0\}$ is vague, as we do not state what x belongs to.
- A set like $\{1, 2, 3\} = \{1, 1, 2, 3, 2, 3\}$, as duplicate elements are ignored.
- Infinite intersections/unions. Consider $X_t = \{x \in \mathbb{R} | 1 + \frac{1}{t} < x < 2 + \frac{1}{t}\}$, where $t > 0$ and $t \in \mathbb{R}$

$$\bigcap_{t>0} X_t = \emptyset$$

$$\bigcap_{t>1} X_t = \{2\}$$

$$\bigcup_{t>0} X_t = \{x | x > 1\}$$

2 Logic

You can use symbols like $\Rightarrow, \Leftarrow, \Leftrightarrow, \forall, \exists, ...$, (assuming that you know what they mean) but don't overuse!

Let $A, B, ...$ true/false statements.

'$A \implies B$' means 'A implies B'. Something more accurate is 'if A, then B'. This is, itself, a true/false statement, which is true if A is true and B is true, or if A is false (this is called a **vacuous truth**).

2.0.1 Example

The moon is made of cheese $\Rightarrow \sqrt{2}$ is rational. This is a true statement, as the moon is **not** made of cheese (trust me!).

The moon is made of cheese $\Rightarrow \sqrt{2}$ is irrational. This is also true for the exact same reason as the previous statement (note that this is not a valid proof that $\sqrt{2}$ is irrational).

2.0.2 Back to abstraction

In short, $A \Rightarrow B$ is true if A is false or B is true.

$A \Leftarrow B$ has the same true/false meaning as $B \Rightarrow A$.

$A \Leftrightarrow B$ means $A \Rightarrow B$ and $B \Rightarrow A$. In this case, we say that 'A and B are equivalent' or 'A if and only if B'.

2.1 Negations

'$\neg A =$ not A' is the statement which is true exactly when A is false.

$$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$$
$$\neg(A \implies B) \Leftrightarrow \neg(\neg A \text{ or } B) \Leftrightarrow (A \text{ and } \neg B)$$

2.2 'For all' and 'Exists' Symbols

$\forall$ means 'for all'. $\exists$ means 'there exists'.

Example

Given $S \subseteq \mathbb{R}$, consider

$$A = \forall \epsilon > 0, \forall x \in S, \exists n \in \mathbb{N} : x^n < \epsilon$$

Statements like this are to be left from left to right.

Negation of $\forall$, $\exists$

$$\neg(\forall a : P(a)) \Leftrightarrow \exists a : \neg P(a)$$
$$\neg(\exists a : P(a)) \Leftrightarrow \forall a : \neg P(a)$$

The negation of the above statement A will be:

$$\neg A = \exists \epsilon > 0, \exists x \in S, \forall n \in \mathbb{N} : x^n \geq \epsilon$$