MAT157 Lecture Notes

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1 Course Marking Scheme

- There will be no exams due to COVID-19.
- Homework (weekly) will count for 60% of your grade.
- Quizzes (biweekly) will count for 40% of your grade.

2 More proofs that $\sqrt{2}$ is irrational

Note that $1 < \sqrt{2} < 2$ (from $1 < 2 < 4$).
This can be written as $0 < \sqrt{2} - 1 < 1$ **(*)**.

Proof #2. Suppose that $\sqrt{2}$ is rational.
Write $\sqrt{2} = \frac{p}{q}$ with $p, q, \in \mathbb{N}$, with q as small as possible.
That is, q is the smallest number s.t. $q \cdot \sqrt{2} \in \mathbb{Z}$.
By **(*)**, multiplied by q:

$$0 < p - q < q.$$

So, $p - q$ is a natural number less than q.
We have

$$(p - q) \cdot \sqrt{2} = (q \cdot \sqrt{2} - q) \cdot \sqrt{2} = 2q - \sqrt{2}q = 2q - p \in \mathbb{Z}.$$

This is a contradiction, as we have found a number less than q (that is, $p - q$) whose product with $\sqrt{2}$ is in $\mathbb{Z}$. $\square$

Proof #3. Suppose that $\sqrt{2} = \frac{p}{q}$ with $p, q, \in \mathbb{N}$. Let $x = \sqrt{2} - 1$ by **(*)**.

$$0 < x < 1$$

Choose $n \in \mathbb{N}$, s.t. $x^n < \frac{1}{q}$.
But $(\sqrt{2} - 1)^n = a \cdot \sqrt{2} + b$ with a, b being certain integers.

$$\begin{aligned} q \cdot x^n &= q(a \cdot \sqrt{2} + b) \\ &= pa + qb && \text{since } q \cdot \sqrt{2} = p \\ &\in \mathbb{Z}. \end{aligned}$$

But $0 < x^n < \frac{1}{q}$, so $0 < qx^n < 1$.
This is a contradiction, as there are no integers between 0 and 1.

$\square$

Prime numbers

A prime number $p \in \mathbb{N}$ is a natural number larger than 1 that is only divisible by 1 and p itself. A prime factorization of a natural number $n \in \mathbb{N}$ is given by prime numbers $p_1 \leq p_2 \leq ... \leq p_r$ s.t. $n = p_1 \cdot p_2 \cdot ... \cdot p_r$.

E.g.: $153 = 3 \cdot 3 \cdot 17$.

Theorem 1. *Every $n \in \mathbb{N}, n > 1$ has a unique prime factorization.*

This is not so easy to prove, so it will not be proved here, but rather in one of the handouts.

Proof #4. Suppose $\sqrt{2} = \frac{p}{q}$.

Then $q \cdot \sqrt{2} = p$, so $q^2 \cdot 2 = p^2$.

On the left hand side, the prime factor 2 appears an **odd** number of times, while on the right hand side, it appears an **even** number of times, which is a contradiction. □

More detail

Let $\mathrm{Mult}_2(n)$ be the number of times the prime factor 2 appears in $n \in \mathbb{N}$.

$$\mathrm{Mult}_2(n \cdot m) = \mathrm{Mult}_2(n) + \mathrm{Mult}_2(m).$$

So, $\mathrm{Mult}_2(n^2) = 2 \cdot \mathrm{Mult}_2(n)$.

Example

Proof. Show $\sqrt[7]{2}$ is irrational.

Suppose $\sqrt[7]{2} = \frac{p}{q}$. Then $2 \cdot q^7 = p^7$.

The number of times that the prime factor 2 appears on the right side is divisible by 7, while on the left side it is not divisible by seven.

This is a contradiction. □

Explanation

On the left side,

$$\mathrm{Mult}_2(2 \cdot q^7) = \mathrm{Mult}_2(2) + \mathrm{Mult}_2(q^7) = 1 + 7\mathrm{Mult}_2(q).$$

On the right side,

$$\mathrm{Mult}_2(p^7) = 7\mathrm{Mult}_2(p).$$