MAT157 Lecture 4 Notes

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1 Cartesian Products of Sets

$$X \times Y = \{(x, y) \text{ with } x \in X, y \in Y\}.$$

Another example:

$$X \times Y \times Z = \{(x, y, z) \text{ with } x \in X, y \in Y, z \in Z\}.$$

2 Field Axioms

A set F with two operations

$$+ : F \times F \to F$$
$$(a, b) \mapsto a + b$$

$$\cdot : F \times F \to F$$
$$(a, b) \mapsto a \cdot b$$

is called a **field** if the following "axioms" are satisfied:

Axioms concerning addition

P1 - Associativity

For all $a, b, c, \in F$:

$$a + (b + c) = (a + b) + c$$

P2 - Neutral Element

There exists an element $\mathbf{0} \in \mathbf{F}$ s.t. for all $a \in F$:

$$a + \mathbf{0} = a = \mathbf{0} + a$$

P3 - Inverse Element

For every $a \in F$, there exists $(-a) \in F$ s.t.

$$a + (-a) = \mathbf{0} = (-a) + a$$

P4 - Commutavity

For every $a, b, \in F$:

$$a + b = b + a$$

Axioms concerning multiplication

P5 - Associativity

For all $a, b, c \in F$:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

P6 - Neutral Element

There exists an element $\mathbf{1} \in F, \mathbf{1} \neq \mathbf{0}$, s.t. for all $a \in F$:

$$a \cdot \mathbf{1} = a = \mathbf{1} = a$$

P7 - Inverse Element

For every $a \in F$ with $a \neq \mathbf{0}$, there exists $a^{-1} \in F$ s.t.

$$a \cdot a^{-1} = \mathbf{1} = a^{-1} \cdot a$$

P8 - Commutativity

For all $a, b, \in F$:

$$a \cdot b = b \cdot a$$

Axiom(s) linking addition and multiplication

P9 - Distributive Law

For all $a, b, c \in F$:

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

Example

Let $F = \{\epsilon, \omega\}$, where $\epsilon \neq \omega$. Addition and multiplication between ϵ and ω are defined as following:

$$\epsilon + \epsilon = \epsilon$$
$$\epsilon + \omega = \omega$$
$$\omega + \omega = \epsilon$$
$$\omega + \epsilon = \omega$$

$$\epsilon \cdot \epsilon = \epsilon$$
$$\epsilon \cdot \omega = \epsilon$$
$$\omega \cdot \omega = \epsilon$$
$$\omega \cdot \epsilon = \omega$$

This is a field, with $\mathbf{0} = \epsilon, \mathbf{1} = \omega$.
For example, we also know:

$$(-\omega) = \omega$$

This is the field $F = \{\mathbf{0}, \mathbf{1}\}$.