

## Example Epsilon-Delta Proof - Anatoly Zavyalov

Suppose we want to prove the statement, for some  $a \in \mathbb{R}$ ,

$$\lim_{x \rightarrow a} x^2 = a^2.$$

Equivalently, we want to show that

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x \in \mathbb{R}, |x - a| < \delta \implies |x^2 - a^2| < \epsilon.$$

*Proof.* Let  $\epsilon > 0$  be given.

$$\text{Let } \delta = \min \left\{ 1, \frac{\epsilon}{2|a| + 1} \right\}.$$

Let  $x \in \mathbb{R}$ , and suppose that  $|x - a| < \delta$ .

We know that  $|x| - |a| < |x - a|$  by the triangle inequality, so

$$\begin{aligned} |x| - |a| &< |x - a| < \delta \\ \implies |x| &< |a| + \delta \\ \implies |x| &< |a| + \delta < |a| + 1 && \text{as } \delta < 1 \\ \implies |x| &< |a| + 1. \end{aligned}$$

Next, we know

$$\begin{aligned} \delta &< \frac{\epsilon}{2|a| + 1} \\ \implies \delta (2|a| + 1) &= \delta (|a| + |a| + 1) < \epsilon \\ \implies \delta (|x| + |a|) &< \delta (|a| + |a| + 1) < \epsilon \\ \implies \delta (|x + a|) &< \delta (|x| + |a|) < \epsilon && \text{by the triangle inequality} \\ \implies |x - a||x + a| &< \epsilon && \text{as } |x - a| < \delta \\ \implies |x^2 - a^2| &< \epsilon, \end{aligned}$$

as desired.

Since  $x \in \mathbb{R}$  was chosen arbitrarily, we know that

$$\forall x \in \mathbb{R}, |x - a| < \delta \implies |x^2 - a^2| < \epsilon.$$

Hence,

$$\exists \delta > 0 : \forall x \in \mathbb{R}, |x - a| < \delta \implies |x^2 - a^2| < \epsilon.$$

Since  $\epsilon > 0$  was also chosen arbitrarily, we know that

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x \in \mathbb{R}, |x - a| < \delta \implies |x^2 - a^2| < \epsilon,$$

which is what we wanted to show. □