Example Epsilon-Delta Proof - Anatoly Zavyalov

Suppose we want to prove the statement, for some $a \in \mathbb{R}$,

$$\lim_{x \to a} x^2 = a^2.$$

Equivalently, we want to show that

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x \in \mathbb{R}, |x - a| < \delta \implies \left| x^2 - a^2 \right| < \epsilon.$$

Proof. Let $\epsilon > 0$ be given. Let $\delta = \min \left\{ 1, \dfrac{\epsilon}{2|a| + 1} \right\}$. Let $x \in \mathbb{R}$, and suppose that $|x - a| < \delta$. We know that $|x| - |a| < |x - a|$ by the triangle inequality, so

$$\begin{aligned} & |x| - |a| < |x - a| < \delta \\ \implies & |x| < |a| + \delta \\ \implies & |x| < |a| + \delta < |a| + 1 && \text{as } \delta < 1 \\ \implies & |x| < |a| + 1. \end{aligned}$$

Next, we know

$$\begin{aligned} & \delta < \frac{\epsilon}{2|a| + 1} \\ \implies & \delta \left(2|a| + 1 \right) = \delta \left(|a| + |a| + 1 \right) < \epsilon \\ \implies & \delta \left(|x| + |a| \right) < \delta \left(|a| + |a| + 1 \right) < \epsilon \\ \implies & \delta \left(|x + a| \right) < \delta \left(|x| + |a| \right) < \epsilon && \text{by the triangle inequality} \\ \implies & |x - a||x + a| < \epsilon && \text{as } |x - a| < \delta \\ \implies & |x^2 - a^2| < \epsilon, \end{aligned}$$

as desired. Since $x \in \mathbb{R}$ was chosen arbitrarily, we know that

$$\forall x \in \mathbb{R}, |x - a| < \delta \implies \left| x^2 - a^2 \right| < \epsilon.$$

Hence,

$$\exists \delta > 0 : \forall x \in \mathbb{R}, |x - a| < \delta \implies \left| x^2 - a^2 \right| < \epsilon.$$

Since $\epsilon > 0$ was also chosen arbitrarily, we know that

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x \in \mathbb{R}, |x - a| < \delta \implies \left| x^2 - a^2 \right| < \epsilon,$$

which is what we wanted to show.

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