Matthew Koster's Big, Fat, Dumb List of Problems

Solutions by Anatoly Zavyalov

December 17, 2020

Problems 1.

1. *Proof.* For $p(x)$ to split, it must be able to be written in the form

$$p(x) = (x - a_1)(x - a_2),$$

where $a_1, a_2 \in \mathbb{R}$. Expanding this, we get

$$p(x) = x^2 - (a_1 + a_2)x + a_1 a_2.$$

By equating coefficients, we see that

$$p(x) = x^2 + 1 = x^2 - (a_1 + a_2)x + a_1 a_2 \iff a_1 + a_2 = 0, a_1 \cdot a_2 = 1$$

Solving this system of equations results in $a_1 = i, a_2 = -i$, which are not in $\mathbb{R}$. Thus, $p(x)$ cannot be split.

□

ALTERNATE PROOF:

Proof. Note that $\forall x \in \mathbb{R}$, $x^2 + 1 > 0$. Thus, there are no real solutions to $x^2 + 1 = 0$, so $p(x) = x^2 + 1$ cannot be written as $(x - a_1)(x - a_2)$ for $a_1, a_2 \in \mathbb{R}$.

□

2. *Proof.* For $p(x)$ to split, it must be able to be written in the form

$$p(x) = (x - a_1)(x - a_2),$$

where $a_1, a_2 \in \mathbb{F}_2$.

Expanding this, we get

$$p(x) = x^2 - (a_1 + a_2)x + a_1 a_2.$$

By equating coefficients, we see that

$$p(x) = x^2 + x + 1 = x^2 - (a_1 + a_2)x + a_1 a_2 \iff a_1 + a_2 = -1, a_1 \cdot a_2 = 1$$

By glancing at it for no more than a fraction of a second, we see that no $a_1, a_2 \in \mathbb{F}_2$ exist that are solutions to this system of equations.

□

3. *Proof.* We know that

$$T = P^{-1}JP$$

where J is a matrix in Jordan form.

By definition, any matrix in Jordan form is also upper triangular. Thus, T is similar to an upper triangular matrix.

$\square$

4. *Proof.* By Axler 8.60, any operator A over a finite-dimensional complex vector space V has a basis that is a Jordan basis. That is, A can be put into Jordan canonical form. By 3., A is similar to an upper triangular matrix.

$\square$

5. *Proof.* Let A be an upper triangular matrix, which we will define

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,n} \\ 0 & 0 & \ddots & \\ 0 & 0 & \dots & a_{n,n} \end{bmatrix}.$$

We will prove this by induction. The base case ($k = 1$) is true by definition.

Now, assume that, for some $k \in \mathbb{N}$, the diagonal entries of A^k are $a_{1,1}^k, a_{2,2}^k, ..., a_{n,n}^k$.

We see that

$$\begin{aligned} A^{k+1} &= A \cdot A^k \\ &= \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,n} \\ 0 & 0 & \ddots & \\ 0 & 0 & \dots & a_{n,n} \end{bmatrix} \cdot \begin{bmatrix} a_{1,1}^k & * & \dots & * \\ 0 & a_{2,2}^k & \dots & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \dots & a_{n,n}^k \end{bmatrix} && \text{induction hypothesis} \\ &= \begin{bmatrix} a_{1,1}^{k+1} & * & \dots & * \\ 0 & a_{2,2}^{k+1} & \dots & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \dots & a_{n,n}^{k+1} \end{bmatrix}, \end{aligned}$$

as, after matrix multiplication, for the ith diagonal entry, only the $a_{i,i} \cdot a_{i,i}^k$ term will be possibly non-zero.

Thus, by induction, A^k has diagonal entries $a_1^k, a_2^k, ...a_n^k$ for any $k \in \mathbb{N}$.

$\square$

Problems 2.

1. *Proof.*

$$e^U = I + U + \frac{1}{2!}U^2 + \frac{1}{3!}U^3 + ...$$

$$= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} + \begin{bmatrix} a_1 & * & \dots & * \\ 0 & a_2 & \dots & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \dots & a_n \end{bmatrix} + \begin{bmatrix} \frac{1}{2!}a_1^2 & * & \dots & * \\ 0 & \frac{1}{2!}a_2^2 & \dots & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \dots & \frac{1}{2!}a_n^2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3!}a_1^3 & * & \dots & * \\ 0 & \frac{1}{3!}a_2^3 & \dots & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \dots & \frac{1}{3!}a_n^3 \end{bmatrix} + ...$$

by 5. from **problems 1.**

$$= \begin{bmatrix} 1 + a_1 + \frac{1}{2!}a_1^2 + \frac{1}{3!}a_1^3 + ... & * & \dots & * \\ 0 & 1 + a_2 + \frac{1}{2!}a_2^2 + \frac{1}{3!}a_2^3 + ... & \dots & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \dots & 1 + a_n + \frac{1}{2!}a_n^2 + \frac{1}{3!}a_n^3 + ... \end{bmatrix}$$

$$= \begin{bmatrix} e^{a_1} & * & \dots & * \\ 0 & e^{a_2} & \dots & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \dots & e^{a_n} \end{bmatrix},$$

which is what we wanted to show.

□

2. **Lemma:** For any $A, Q \in M_n(\mathbb{F})$ where Q is invertible, $(QAQ^{-1})^n = QA^nQ^{-1}$ for any $n \in \mathbb{N}$.

Proof. Let $A, Q \in M_n(\mathbb{F})$ such that Q is invertible.

We will prove this by induction. The base case $(n = 1)$ trivially holds.

Assume that for some $k \in \mathbb{N}$, $(QAQ^{-1})^k = QA^kQ^{-1}$.

We want to show that $(QAQ^{-1})^{k+1} = QA^{k+1}Q^{-1}$.

We see that

$$\begin{aligned} (QAQ^{-1})^{k+1} &= (QAQ^{-1})^k \cdot QAQ^{-1} \\ &= QA^kQ^{-1} \cdot QAQ^{-1} && \text{by induction hypothesis} \\ &= QA^k(Q^{-1} \cdot Q)AQ^{-1} \\ &= QA^kIAQ^{-1} \\ &= QA^{k+1}Q^{-1}. \end{aligned}$$

Thus, by induction hypothesis, $(QAQ^{-1})^n = QA^nQ^{-1}$ for any $n \in \mathbb{N}$. □

Now, onto the actual question.

Proof. We see that

$$\begin{aligned}
exp(QAQ^{-1}) &= I + QAQ^{-1} + \frac{1}{2!}(QAQ^{-1})^2 + \frac{1}{3!}(QAQ^{-1})^3 + ... \\
&= I + QAQ^{-1} + \frac{1}{2!}QA^2Q^{-1} + \frac{1}{3!}QA^3Q^{-1} + ... && \text{by lemma} \\
&= QQ^{-1} + QAQ^{-1} + \frac{1}{2!}QA^2Q^{-1} + \frac{1}{3!}QA^3Q^{-1} + ... \\
&= Q(Q^{-1} + AQ^{-1} + \frac{1}{2!}A^2Q^{-1} + \frac{1}{3!}A^3Q^{-1} + ...) \\
&= Q(I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + ...)Q^{-1} \\
&= Qe^AQ^{-1},
\end{aligned}$$

which is what we wanted to show.

□

3. *Proof.* By question 4 from **problems 1.**, we know that A is similar to an upper triangular matrix, that is,

$$A = P^{-1}UP,$$

where U is upper triangular.

We know that because the trace is independent of the basis, $\text{trace}(A) = \text{trace}(P^{-1}UP) = \text{trace}(U) = 0$.

Next, we know that

$$e^A = \exp(P^{-1}UP) = P^{-1}e^UP,$$

so

$$\begin{aligned}
\det(e^A) &= \det(P^{-1}e^UP) \\
&= \det(P^{-1}) \cdot \det(e^U) \cdot \det(P) \\
&= \det(e^U) && \text{as det is independent of basis}
\end{aligned}$$

By Axler 10.35, the determinant of an upper triangular matrix is equal to the product of the diagonal entries. Thus,

$$\begin{aligned}
\det(e^U) &= e^{a_1} \cdot e^{a_2} \cdot e^{a_3} \cdot ... \cdot e^{a_n} && \text{where } a_1, ..., a_n \text{ are the diagonal entries of } U \\
&= e^{a_1+a_2+a_3+...+a_n} \\
&= e^{\text{trace}(U)} = e^0 = 1,
\end{aligned}$$

which is what we wanted to show. □

Problems 3.

1. *Proof.* First, we show that the bracket is bilinear. To show that the bracket is linear in its first argument, we must show that

$$[\lambda X + Y, Z] = \lambda[X, Z] + [Y, Z] \qquad \text{where } \lambda \in \mathbb{R}, \text{ and } X, Y, Z \in \mathfrak{gl}(n; \mathbb{R})$$

We see that for some $\lambda \in \mathbb{R}$ and $X, Y, Z \in \mathfrak{gl}(n; \mathbb{R})$,

$$\begin{aligned} [\lambda X + Y, Z] &= (\lambda X + Y)Z - Z(\lambda X + Y) \\ &= \lambda XZ + YZ - \lambda ZX - ZY \\ &= \lambda(XZ - ZX) + (YZ - ZY) \\ &= \lambda[X, Z] + [Y, Z]. \end{aligned}$$

Next, to show that the bracket is linear in its second argument, we must show that

$$[X, \lambda Y + Z] = \lambda[X, Y] + [X, Z] \qquad \text{where } \lambda \in \mathbb{R}, \text{ and } X, Y, Z \in \mathfrak{gl}(n; \mathbb{R})$$

We see that for some $\lambda \in \mathbb{R}$ and $X, Y, Z \in \mathfrak{gl}(n; \mathbb{R})$,

$$\begin{aligned} [X, \lambda Y + Z] &= X(\lambda Y + Z) - (\lambda Y + Z)X \\ &= \lambda XY + XZ - \lambda YZ - ZX \\ &= \lambda(XY - YX) + (XZ - ZX) \\ &= \lambda[X, Y] + [X, Z] \end{aligned}$$

Thus, the bracket is bilinear.

Next, we see that the bracket satisfies the first identity, as, for any $X \in \mathfrak{gl}(n; \mathbb{R})$,

$$[X, X] = XX - XX = 0.$$

Lastly, we see that it satisfies the Jacobi identity, as for any $X, Y, Z \in \mathfrak{gl}(n; \mathbb{R})$,

$$\begin{aligned} &[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] \\ &= (X[Y, Z] - [Y, Z]X) + (Z[X, Y] - [X, Y]Z) + (Y[Z, X] - [Z, X]Y) \\ &= X(YZ - ZY) - (YZ - ZY)X + Z(XY - YX) - (XY - YX)Z + Y(ZX - XZ) - (ZX - XZ)Y \\ &= XYZ - XZY - YZX + ZYX + ZXY - ZYX - XYZ + YXZ + YZX - YXZ - ZXY + XZY \\ &= (XYZ - XYZ) + (XZY - XZY) + (YZX - YZX) \\ &+ (ZYX - ZYX) + (ZXY - ZXY) + (YXZ - YXZ) \\ &= 0, \end{aligned}$$

which is what we wanted to show.

With but a brief glance, even a newborn baby could tell you that the dimension of $\mathfrak{gl}(n; \mathbb{R})$ is n^2, and the newborn baby would be correct, as $\mathfrak{gl}(n; \mathbb{R})$ is the space of all n-by-n matrices with entries in $\mathbb{R}$. Good job, baby. $\square$

2. *Proof.* In 1., we already showed that $M_n(\mathbb{R})$ together with the bracket $[X,Y] = XY - YX$ is a real Lie algebra. Next, we see that for any $X, Y \in \mathfrak{sl}(n;\mathbb{R})$, we have that $[X,Y] \in \mathfrak{sl}(n;\mathbb{R})$, as

$$\operatorname{trace}(XY - YX) = \operatorname{trace}(XY) - \operatorname{trace}(YX) = \operatorname{trace}(XY) - \operatorname{trace}(XY) = 0.$$

Thus, $\mathfrak{sl}(n;\mathbb{R})$ with the bracket $[X,Y] = XY - YX$ is a real Lie algebra.

To find the dimension of $\mathfrak{sl}(n;\mathbb{R})$ we first note that

$$\operatorname{trace} : M_n(\mathbb{R}) \to \mathbb{R}$$

is a linear map (which can be easily verified by the reader), so we can use the rank-nullity theorem:

$$\begin{aligned} \dim M_n(\mathbb{R}) &= \dim \operatorname{range}(\operatorname{trace}) + \dim \operatorname{null}(\operatorname{trace}) \\ \implies n^2 &= 1 + \dim \operatorname{null}(\operatorname{trace}) \\ \implies \dim \operatorname{null}(\operatorname{trace}) &= n^2 - 1. \end{aligned}$$

Since $\mathfrak{sl}(n;\mathbb{R}) = \operatorname{null}(\operatorname{trace})$, we conclude that $\dim \mathfrak{sl}(n;\mathbb{R}) = n^2 - 1$ □

3. *Proof.* First, we want to show that the bracket is bilinear. First, we want to show that the bracket is linear in the first argument, that is

$$[\lambda X + Y, Z] = \lambda[X,Z] + [Y,Z] \qquad \text{for } X, Y, Z \in V, \lambda \in \mathbb{F}$$

This is obviously true, as both sides equal zero.

Similarly, the bracket is linear in the second argument. Thus, it is bilinear.

Next, the bracket trivially satisfies both identities ($[X,X] = 0$ for $X \in V$, and the Jacobi identity).

Thus, V together with the bracket $[X,Y] = 0$ is a Lie algebra.

□

4. *Proof.* We know that, from question 1., $M_n(\mathbb{R})$ equipped with the bracket $[X,Y] = XY - YX$ is a Lie algebra. Thus, it is sufficient to show that $[X,Y] \in \mathfrak{so}(n;\mathbb{R})$ for any $X, Y \in \mathfrak{so}(n;\mathbb{R})$.

We see that

$$\begin{aligned} ([X,Y])^T &= (XY - YX)^T \\ &= (XY)^T - (YX)^T && \text{as } (A+B)^T = A^T + B^T \ \forall A, B \in M_n\mathbb{F} \\ &= Y^T X^T - X^T Y^T \\ &= -Y(-X) - (-X)(-Y) \\ &= -(XY - YX) \\ &= -[X,Y], \end{aligned}$$

which is what we wanted to show.

To find the dimension of $\mathfrak{so}(n;\mathbb{R})$, we will find a basis.

We claim that a basis of $\mathfrak{so}(n;\mathbb{R})$ is $\{E_{i,j} - E_{j,i} : 1 \le i < j \le n\}$, where $E_{i,j}$ is the matrix with 1 at the (i,j) entry and 0s elsewhere.

By inspection, this set is clearly linearly independent. It also spans all of $\mathfrak{so}(n;\mathbb{R})$, as, for any $A \in \mathfrak{so}(n;\mathbb{R})$,

$$A = \begin{bmatrix} 0 & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ -a_{1,2} & 0 & a_{2,3} & \dots & a_{2,n} \\ -a_{1,3} & -a_{2,3} & 0 & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1,n} & -a_{2,n} & -a_{3,n} & \dots & 0 \end{bmatrix} \qquad \text{(one can easily check that } A^T = -A\text{)}$$
$$= a_{1,2}(E_{1,2} - E_{2,1}) + a_{1,3}(E_{1,3} - E_{3,1}) + ... + a_{1,n}(E_{1,n} - E_{n,1})$$
$$+ a_{2,3}(E_{2,3} - E_{3,2}) + ... + a_{n-1,n}(E_{n-1,n} - E_{n,n-1}).$$

Thus, the claimed basis is indeed a basis.

It is also clear to see that the cardinality of this basis is $\binom{n}{2} = \frac{n(n-1)}{2}$, which is thus the dimension of $\mathfrak{so}(n;\mathbb{R})$. □

5. *Proof.* In 1., we showed that $M_n(\mathbb{R})$ equipped with this bracket is a real Lie algebra. Thus, we must show that $[X,Y] \in H(3;\mathbb{R})$ for any $X,Y \in H(3;\mathbb{R})$.

   We see that, for some $X,Y \in H(3;\mathbb{R})$ where

$$X = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{where } a,b,c,x,y,z \in \mathbb{R}$$

   we have

$$\begin{aligned} [X,Y] &= XY - YX \\ &= \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & az \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & cx \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & az - cx \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

   which is clearly in $H(3;\mathbb{R})$.

   Lastly, to find the dimension, we see that a basis of $H(3;\mathbb{R})$ is $(E_{1,2}, E_{1,3}, E_{2,3})$, which can be easily verified by the reader. This is a 3-tuple, so $\dim H(3;\mathbb{R}) = 3$.

   □

6. *Proof.* First, we will show that the bracket $[X,Y] = X \times Y$ is bilinear.

   We see that, by properties of the cross product (namely distributivity over addition), for some $X,Y,Z \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$, we have

$$[\lambda X + Y, Z] = (\lambda X + Y) \times Z = \lambda(X \times Z) + Y \times Z = \lambda[X,Z] + [Y,Z],$$

   showing that the bracket is linear in the first argument. Similarly, it can be shown that the bracket is linear in the second argument.

Thus, the bracket is bilinear.

Next, we see that the bracket satisfies the first identity, that is,

$$[X, X] = X \times X = 0 \qquad \text{for some } X \in \mathbb{R}^3$$

To see why $X \times X = 0$, we recall the fact that the cross product is anticommutitative, thus

$$X \times X = -X \times X \iff 2X \times X = 0 \iff X \times X = 0.$$

Lastly, we verify the Jacobi identity: for any $X, Y, Z \in \mathbb{R}^3$, we have

$$\begin{aligned}
&[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] \\
&= X \times (Y \times Z) + Z \times (X \times Y) + Y \times (Z \times X) \\
&= (X \cdot Z)Y - (Y \cdot Z)X + (Z \cdot Y)X - (Z \cdot X)Y + (Y \cdot X)Z - (Y \cdot Z)X \quad \text{by Lagrange's formula} \\
&= (X \cdot Z)Y - (X \cdot Z)Y + (Z \cdot Y)X - (Z \cdot Y)X + (Y \cdot X)Z - (Y \cdot X)Z \\
&\qquad \text{by commutativity of the dot product} \\
&= 0,
\end{aligned}$$

which is what we wanted to show. □

7. *Proof.* First, we establish an isomorphism between $\mathbb{R}^3$ and $\mathfrak{so}(3, \mathbb{R})$.

Define a map

$$T : \mathbb{R}^3 \to \mathfrak{so}(3, \mathbb{R}),\ T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & x & y \\ -x & 0 & -z \\ -y & z & 0 \end{bmatrix}.$$

The routine check that this map is linear is, as always, left to the reader (this is easy to see by inspection).

Define another map

$$S : \mathfrak{so}(3, \mathbb{R}) \to \mathbb{R}^3,\ S \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ -z \end{bmatrix}.$$

Again, one can check that this is linear.

We also see that this is the inverse of T, which can be verified by inspection.

Thus, $\mathbb{R}^3$ and $\mathfrak{so}(3, \mathbb{R})$ are isomorphic.

Lastly, we must show that $[X, Y]_{\mathbb{R}^3} = [T(X), T(Y)]_{\mathfrak{so}(3,\mathbb{R})}$ for $X, Y \in \mathbb{R}$.

Let

$$X = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix},\ Y = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$

We have that

$$
\begin{aligned}
T([X,Y]_{\mathbb{R}^3}) &= T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) \\
&= T\begin{bmatrix} y_1z_2 - z_1y_2 \\ z_1x_2 - x_1z_2 \\ x_1y_2 - y_1x_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & y_1z_2 - z_1y_2 & z_1x_2 - x_1z_2 \\ z_1y_2 - y_1z_2 & 0 & y_1x_2 - x_1y_2 \\ x_1z_2 - z_1x_2 & x_1y_2 - y_1x_2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & x_1 & y_1 \\ -x_1 & 0 & -z_1 \\ -y_1 & z_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & x_2 & y_2 \\ -x_2 & 0 & -z_2 \\ -y_2 & z_2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & x_2 & y_2 \\ -x_2 & 0 & -z_2 \\ -y_2 & z_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & x_1 & y_1 \\ -x_1 & 0 & -z_1 \\ -y_1 & z_1 & 0 \end{bmatrix} \\
&= [T(X), T(Y)]_{\mathfrak{so}(3,\mathbb{R})},
\end{aligned}
$$

which is what we wanted to show. $\square$

8. Matt said I can skip this question.

9. *Proof.* First, note that

$\square$