Term Paper

MA473

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1 Introduction

Based on the following assumptions Black and Scholes derived the Black Scholes formula

1. The asset price follows geometric Brownian motion i.e. $dS = \mu S dt + \sigma S dW$.
2. The risk-free interest rate is r is constant until expiration date.
3. Investors can borrow and lend at risk-free rates.
4. The stock doesn't pay dividends.
5. The market is frictionless (no tax/transaction costs and all securities are divisible).
6. No risk-free arbitrage opportunity exists.
7. Security trading is continuous.
8. Option is a European option.

Based on above, the Black-Scholes PDE is given by:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

with terminal conditions:

1. For European call option:

$$V(0,t) = 0 \qquad \lim_{S \to +\infty} V(S,t) = S \qquad V(S,T) = \max(S-K,0)$$

2. For European put option

$$V(0,t) = Ke^{-r(T-t)} \qquad \lim_{S \to +\infty} V(S,t) = 0 \qquad V(S,T) = \max(K-S,0)$$

The solution can be found analytically to be:

1. For European call: $V(S,t) = SN(d_1) - KN(d_2)e^{-r(T-t)}$

2. For European put: $V(S,t) = SN(-d_2)\, e^{-r(T-t)} - SN(-d_1)$

where,

$$\begin{aligned}
N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy \\
d_1 &= \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}$$

2 Space discretization

2.1 Fourth order central finite difference

We write the formula as:

$$\frac{\partial V}{\partial S} = AV + b_1 \qquad \frac{\partial^2 V}{\partial S^2} = BV + b_2$$

where

$$A = \frac{1}{12h}\begin{bmatrix} -10 & 18 & -6 & 1 & & \\ -8 & 0 & 8 & -1 & & \\ 1 & -8 & 0 & 8 & & \\ & & \ddots & \ddots & \ddots & \\ & & 1 & \dots & 0 & 8 \\ & & 1 & -6 & 18 & -10 \end{bmatrix} \quad B = \frac{1}{12h^2}\begin{bmatrix} -15 & -4 & 14 & -6 & 1 & \\ 16 & -30 & 16 & -1 & & \\ -1 & 16 & -30 & 16 & \ddots & \\ & & & \ddots & \ddots & \\ & & -4 & \cdots & -30 & 16 \\ & 1 & -6 & 14 & -4 & -15 \end{bmatrix}$$

$$\begin{aligned}
b_1 &= \frac{1}{12h}\begin{pmatrix} -3V_0 & V_0 & 0 & \cdots & -V_N & -3V_N \end{pmatrix}^T \\
b_2 &= \frac{1}{12h^2}\begin{pmatrix} 10V_0 & -V_0 & 0 & \cdots & -V_N & 10V_N \end{pmatrix}^T
\end{aligned}$$

Thus the BS PDE can be written as

$$\frac{dV}{dt} = -PV - Q \text{ where } P = \frac{1}{2}\sigma^2 S^2 B + rSA - rI \text{ and } Q = \frac{1}{2}\sigma^2 S^2 b_2 + rSb_1$$

2.2 Fourth order compact finite difference scheme

$$\frac{\partial V}{\partial t} = -LV$$

where

$$\begin{aligned}
L &= \frac{1}{2}\sigma^2 S^2 D^2 + rSD - rI \\
D &= F^{-1}G \\
D^2 &= U^{-1}W
\end{aligned}$$

$$F = \begin{bmatrix} 1 & 3 & 0 & & \\ \frac{1}{4} & 1 & \frac{1}{4} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{4} & 1 & \frac{1}{4} \\ & & 0 & 3 & 1 \end{bmatrix} \quad G = \begin{bmatrix} -\frac{17}{6h} & \frac{3}{2h} & \frac{3}{2h} & -\frac{1}{6h} & & & \\ -\frac{3}{4h} & 0 & \frac{3}{4h} & & & & \\ & -\frac{3}{4h} & 0 & \frac{3}{4h} & & & \\ & & \ddots & \ddots & & & \\ & & & & -\frac{3}{4h} & 0 & \frac{3}{4h} \\ & & & \frac{1}{6h} & -\frac{3}{2h} & -\frac{3}{2h} & \frac{17}{6h} \end{bmatrix}$$

and

$$U = \begin{bmatrix} 1 & 10 & 0 & & & \\ \frac{1}{10} & 1 & \frac{1}{10} & & & \\ & \frac{1}{10} & 1 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 1 & \frac{1}{10} \\ & & & 0 & 10 & 1 \end{bmatrix}$$

$$\mathrm{W} = \begin{bmatrix} \frac{145}{12h^2} & -\frac{76}{3h^2} & \frac{29}{2h^2} & -\frac{4}{3h^2} & \frac{1}{12h^2} & \\ \frac{6}{5h^2} & -\frac{12}{5h^2} & \frac{6}{5h^2} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{6}{5h^2} & -\frac{12}{5h^2} & \frac{6}{5h^2} \\ & \frac{1}{12h^2} & -\frac{4}{3h^2} & \frac{29}{2h^2} & -\frac{76}{3h^2} & \frac{145}{12h^2} \end{bmatrix}$$

3 Time discretization

3.1 Crank Nicolson method

$$\text{For } \frac{\partial V}{\partial t} = AV + b$$
$$\left(I - \frac{\delta A}{2}\right) V^{(j+1)} = \left(I + \frac{\delta A}{2}\right) V^{(j)} + b$$

3.2 BDF4 method

For $\frac{du}{dt} = Au + b$, the scheme is given by

$$\left(\frac{25}{12} I - \delta A\right) u^{(j+1)} = 4u^{(j)} - 3u^{(j-1)} + \frac{4}{3} u^{(j-2)} - \frac{1}{4} u^{(j-3)} + \delta b^{(j+1)}$$

3.3 Grid refinement

Transform

$$y = \psi(S) = \frac{\sinh^{-1}(\xi(S-K)) - c_1}{c_2 - c_1}$$
$$S = \psi^{-1}(y) = \varphi(y) = \frac{1}{\xi} \sinh\left(c_2 y + c_1(1-y)\right) + K$$

where

$$c_1 = \sinh^{-1}\left(\xi\left(S_{\min} - K\right)\right) \qquad c_2 = \sinh^{-1}\left(\xi\left(S_{\max} - K\right)\right)$$

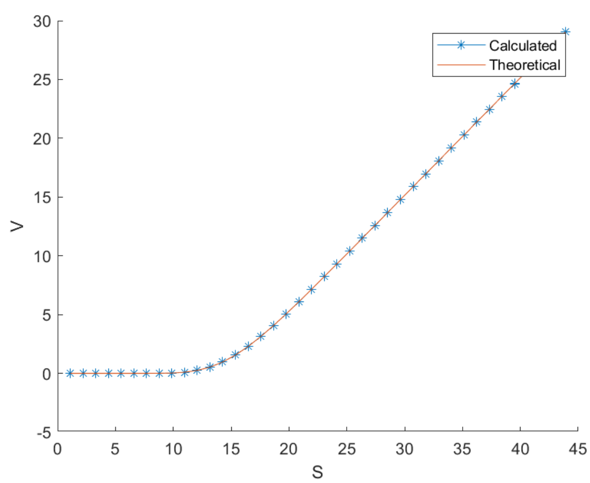
The Black-Scholes PDE transforms to:

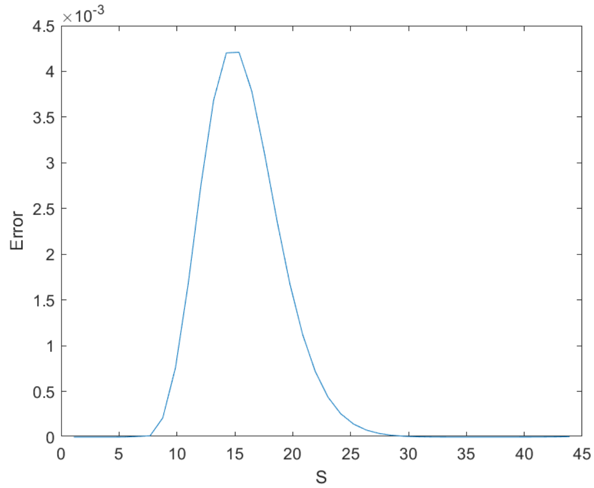
$$\frac{\partial V}{\partial t} = \frac{1}{2}\sigma^2 \frac{\varphi(y)^2}{J(y)^2}\frac{\partial^2 V}{\partial y^2} + \left(r\frac{\varphi(y)}{J(y)} - \frac{1}{2}\sigma^2\frac{\varphi(y)^2}{J(y)^3}H(y)\right)\frac{\partial V}{\partial y} - rV$$

4 Results

4.1 Compact Finite Difference + BDF4

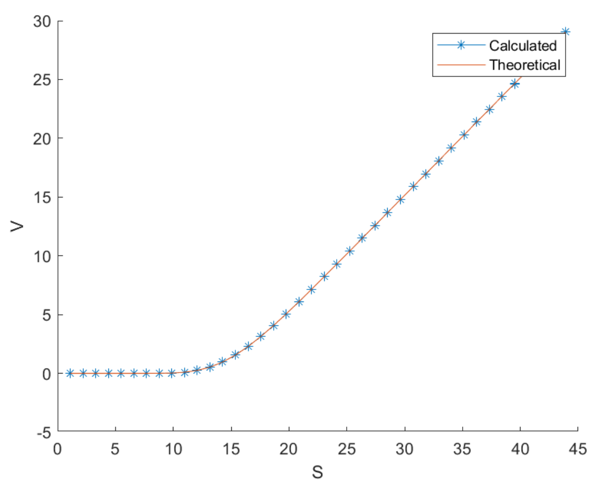
Grid Size	Error	Order
10×10	0.031667	
20×20	0.009247	1.775868
40×40	0.000782	3.563366

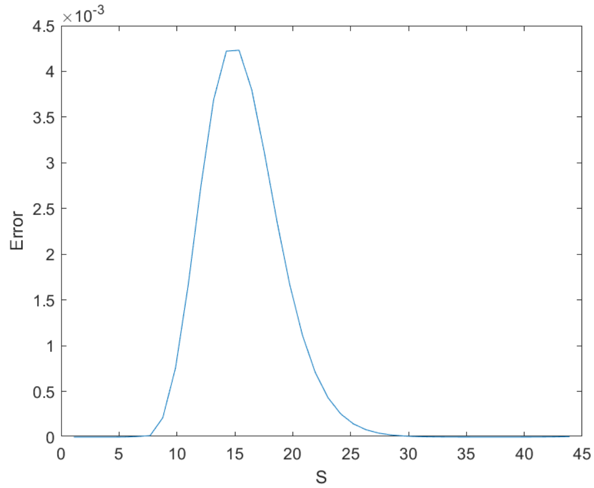




4.2 Compact Finite Difference + Crank Nicolson

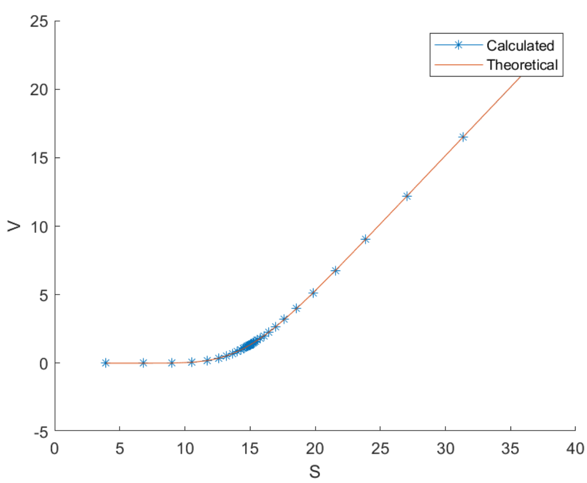
Grid Size	Error	Order
10×10	0.032057	
20×20	0.009247	1.793653
40×40	0.000782	3.563347

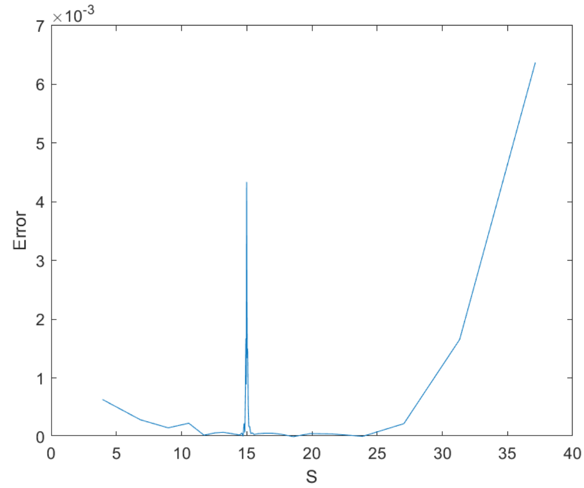




4.3 Finite Difference + Crank Nicolson + Grid Refinement

Grid Size	Error	Order
10×10	0.017660	
20×20	0.005036	1.810200
40×40	0.000613	3.039027





4.4 Compact Finite Difference + BDF4 + Grid Refinement

Grid Size	Error	Order
10×10	0.015779	
20×20	0.003903	2.015336
40×40	0.000256	3.930728