CHAPTER 10

Linear Programming

In this chapter, we shall give the notion of linear programming (LP) and then study in detail the formulation of linear programming problems (LPP), and the solution of LPP using graphical method and simplex method. In sequence, we shall study the construction of the dual of the LPP and the optimization of LPP by dual simplex method. We shall conclude this chapter with the study of the solution of transportation problems.

10.1 Linear Programming

Linear programming was first introduced by Leonid Kantorovich in 1939. He developed the earliest linear programming problems (LPP) that were used by the army during the Second World War (WW II) in order to reduce the costs of the army and increase the efficiency in the battlefield. The method was a secret, because of its use in war-time strategies, until 1947 when George B. Dantzig published the simplex method and John von Neuman developed the theory of duality. After WW II, many industries began adopting linear programming for its usefulness in planning optimization.

Linear programming was developed as a discipline in the 1940s, motivated initially by the need to solve complex planning problems in wartime operations. The founders of linear programming are generally regarded as George B. Dantzig, who devised the simplex method in1947, and John Von Neumann, who established the theory of duality that same year. The Nobel Prize in economics was awarded in 1975 to the mathematician Leonid Kantorovich (USSR) and the economist Tjalling Koopmans (USA) for their contributions to the theory of optimal allocation of resources, in which linear programming played a key role.

The term *programming* means planning to maximize profit or minimize cost or minimize loss or minimum use of resources or minimizing time etc. Such problems are called optimization problems. The term *linear* means that all equations or inequations involved are linear. Linear programming is the most popular mathematical technique which involves the limited resources in an optimal manner.

Any problem in which we apply linear programming is called a linear programming problem (LPP). Consider a general LPP with n variables and m constraints.

(Max/Min) $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots. + a_{1n}\,x_n\,(\le, =, \ge)\;b_1 \\ a_{21}x + a_{22}x_2 + \ldots + a_{2n}x_n(\le, =, \ge)\;b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots. + a_{mn}x_n(\le, =, \ge)\;b_m \end{array}\right\} \qquad (10.1)$$

where $x_1, x_2, \ldots, x_n \ge 0$. (10.2)

The linear function z which is to be optimised (maximised or minimised) is called the **objective function of LPP**. The variables used in equations or inequations (10.1), that is, $x_1, x_2, \ldots, x_n$, are known as **decision variables**. Equations or inequations (10.1) are called the **constraints of the LPP**. The set of the inequations (10.1) is known as the **non-negative restriction** of the given LPP. The optimal value is a maximum or minimum value of an objective function to be calculated in an LPP.

A set of values of the decision variables $x_1, x_2, \ldots, x_n$ which satisfy all the constraints of the LPP is called the **solution** to that problem.

Any solution of an LPP which satisfies the non-negative restrictions of the problem is called the **feasible solution** of that problem.

Any feasible solution of an LPP which optimises the objective function of the LPP is called the **optimal solution** of that problem. Also, the set of all feasible solutions of an LPP is called the **feasible region** of that problem. A region is said to be convex if the line segment joining any two arbitrary points of the region lies entirely within the region. The feasible region of an LPP is always a polyhedral convex region.

10.2 Formulation of LPP

The formulation of the LPP is the first and the most important step in obtaining the solution of an LPP. To formulate a problem into LPP, we need to identify the decision variables involved, establish the relationship between them and then construct the objective function and the constraints. This is illustrated with the help of the following examples.

EXAMPLE 10.1: A manufacturer of patent medicines is preparing a production plan on medicines A and B. There are sufficient raw materials available to make 20,000 bottles of A and 40,000 bottles of B, but there are only 45,000 bottles into which either of the medicines can be put. Further, it takes 3h to prepare enough material to fill 1000 bottles of A, it takes 1h to prepare enough material to fill 1000 bottles of B and there are 66h available for this operation. The profit is Rs. 8 per bottle for A and Rs. 7 per bottle for B. The manufacturer wants to schedule his production in order to maximise his profit. Formulate this problem as an LP model.

Solution: Let x and y denote the number of bottles of type A and type B medicines, respectively.

Total profit (in Rs.) is $Z = 8x + 7y$.

Raw material content are $x \leq 20{,}000$, $y \leq 40{,}000$. Since only 45,000 bottles are available, we have

$$x + y < 45{,}000.$$

It takes 3h to prepare enough material to fill 1000 bottles of type A. Thus, the number of hours required to prepare enough material to fill x bottles of type A $= \frac{3x}{1000}$. Similarly, the number of hours required to prepare enough material to fill y bottles of type B $= \frac{y}{1000}$. Since the total number of hours available for this operation is 66h, we have

$$\frac{3x}{1000} + \frac{y}{1000} \leq 66 \text{ or } 3x + y \leq 66000.$$

Obviously, $x \geq 0$, $y \geq 0$.

The LP model of the problem is given by

$$\text{maximise } Z = 8x + 7y$$

subject to the constraints $x \leq 20{,}000$, $y \leq 40{,}000$, $x + y \leq 45{,}000$, $3x + y \leq 66000$, $x \geq 0$, $y \geq 0$.

EXAMPLE 10.2: A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 9750 to invest and has space for at most 30 items. A fans costs him Rs. 480 and a sewing machine Rs. 360. His expectation is that he can sell a fan at a profit of Rs. 35 and a sewing machine at a profit of Rs. 24. Assume that he can sell all the items that he buys. Formulate this problem as an LPP so that he can maximise his profit.

Solution: Let x and y denote the number of fans and sewing machines respectively.

Total profit (in Rs) is $Z = 35x + 24y$.

Since only 30 items are available, we have

$$x + y \leq 30.$$

A fan cost = Rs. 480 and a sewing machine cost = Rs. 36 and he has only Rs. 9750 to invest. We have

$$480x + 360y \leq 9750$$

$$16x + 12y \leq 325.$$

Obviously, $x \geq 0$, $y \geq 0$.

The LP model of the problem is given by

$$\text{maximise } Z = 35x + 24y$$

subject to the constraints $16x + 12y \leq 325$, $x + y \leq 30$, $x \geq 0$, $y \geq 0$.

EXAMPLE 10.3: A manufacturer produces two types of items M_1 and M_2. Each M_1 requires 4h of grinding and 2h of polishing, whereas each M_2 item requires 2h of grinding and 5h of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works for 40h a week and each polisher works for 60h a week. The profit on M_1 item in Rs. 3 and that on M_2 item is Rs. 4. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of items so that he may make the maximum profit in a week?

Solution: Let x and y denote the number of items of type M_1 and M_2 respectively to be produced per week. Then the weekly profit (in Rs) is

$$Z = 3x + 4y. \quad \text{(i)}$$

To produce these number of items, the total number of grinding hours needed per week

$$= 4x + 2y$$

and the total number of polishing hours needed per week

$$= 2x + 5y.$$

Since the number of grinding hours (2 grinders having each 40h per week) is not more than 80h and the number of polishing hours (3 polishers having each 60h per week) is not more than 180h, Therefore

$$4x + 2y \leq 80 \quad \text{(ii)}$$

and $$2x + 5y \leq 180. \quad \text{(iii)}$$

Since negative number of items is not produced, obviously we must have

$$x \geq 0 \text{ and } y \geq 0. \quad \text{(iv)}$$

Hence this allocation problem is, to find x, y which is given

maximize $Z = 3x + 4y$

subject to the constraints $4x + 2y \leq 80$, $2x + 5y \leq 180$, $x, y \geq 0$.

EXERCISE 10.1

1. An aeroplane can carry a maximum of 200 passengers. A profit of Rs. 400 is made on each first-class ticket and a profit of Rs. 300 is made on each economy-class ticket. The airline reserves at least 20 seats for the first class. However, at least 4 times as many passengers prefer to travel by the economy class than by the first class. How many tickets of each class must be sold in order to maximize the profit for the airline? Formulate the problem as an LP model.

2. A firm manufactures 3 products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4, respectively. The firm has two machines M_1 and M_2, and the following is the required processing time in minutes for each machine on each product.

	A	B	C
M_1	4	3	5
M_2	2	2	4

Machines M_1 and M_2 have 2000 and 2500 machine-minutes respectively. The firm must manufacture 100 A's, 200 B's and 50C's, but not more than 150 A's. Set up an LPP to maximize the profit.

3. A firm manufactures headache pills in two sizes A and B. Size A contains 2 grains of aspirin, 5 gains of bicarbonate and 1 grain of codeine. Size B contains 1 grain of aspirin, 8 grains of bicarbonate and 6 grains of codeine. It is found by users that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine to provide immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate as a standard LPP.
4. A housewife wishes to mix two types of food X and Y in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food X costs Rs. 60 per kg and food Y costs Rs. 80 per kg. Food X contains 3 units per kg of vitamin A and 5 units per kg of vitamin B while food Y contains 4 units per kg of vitamin A and 2 units per kg of vitamin B. Formulate the above problem as an LPP to minimise the cost of mixture.
5. A firm produces an alloy with the following specifications:

   (i) specific gravity ≤ 0.97, (ii) chromium content $\geq 15\%$,

   (iii) melting temperature $\geq 494°C$.

   The alloy requires three raw materials A, B and C, whose properties are as follows;

Property	Properties of raw material		
	A	B	C
Specific gravity	0.94	1.00	1.05
Chromium	10%	15%	17%
Melting Point	470°C	500°C	520°C

   Find the values of A, B and C to be used to make 1 tonne of alloy of the desired properties, keeping the raw materials costs at the minimum when they are Rs. 105/tonne for A, Rs. 245/tonne for B and Rs. 165/tonne for C. Formulate the problem as an LPP.
6. A dairy feed company may purchase and mix one or more of three types of grains containing different amounts of nutritional elements. The data is given in the table below. The production manager specifies that any feed mix for his livestock must meet at least minimum nutritional requirements and seeks the least costly among all three mixes.

Item		**One unit weight of**			**Minimum Requirement**
		Grain I	**Grain II**	**Grain III**	
Nutritional	A	2	3	7	1250
	B	1	1	0	250
Ingredients	C	5	3	0	900
	D	6	25	1	232.5
Cost per weight of		41	35	96	

   Formulate the problem as a LP model.

ANSWER

1. Max. $Z = 400x + 300y$; subject to $x + y \leq 200$, $x \geq 20$, $y \geq 4x$, $x \geq 0$, $y \geq 0$.
2. Max. $Z = 3x_1 + 2x_2 + 4x_3$; subject to $4x_1 + 3x_2 + 5x_3 \leq 2000$, $2x_1 + 2x_2 + 4x_3 + \leq 2500$, $100 \leq x_1 \leq 150$, $200 \leq x_2$, $50 \leq x_3$.
3. Min $Z = x + y$; subject to $2x + y \geq 12$, $5x + 8y \geq 74$, $x + 6y \geq 24$, and $x \geq 0$, $y \geq 0$.
4. Min $Z = 60x + 80y$; subject to $3x + 4y \geq 8$, $5x + 2y \geq 11$, $x \geq 0, y \geq 0$.
5. Min $Z = 100x_1 + 250x_2 + 160x_3$; subject to $0.94x_1 + x_2 + 1.04x_3 \leq 0.98$, $10x_1 + 15x_2 + 17x_3 \geq 14{,}470x_1 + 500x_2 + 520x_3 \geq 495$, $x_1 + x_2 + x_3 = 1$ and x_1, x_2, $x_3 \geq 0$
6. Min. $Z = 41x_1 + 35x_2 + 96x_3$; subject to $2x_1 + 3x_2 + 7x_3 \geq 1250$, $x_1 + x_2 \geq 250$, $5x_1 + 3x_2 \geq 900$, $6x_1 + 25x_2 + x_3 \geq 232.5$ and x_1, x_2, $x_3 \geq 0$.

10.3 Graphical Method

If an LPP contains only two variables, then we can solve the given problem by graphical method.

The following steps will be helpful in solving the LPP by graphical method.

Step-I: Formulate the given problem as a linear programming problem (LPP).

Step-II: Convert inequations into equations, plot the given constraints as equation on the $x_1x_2(xy)$ coordinate plane and find the feasible region formed by them.

Step-III: Find all the corner points of the feasible region that satisfy all the constraints and determine the value of the objective function at each corner point. The desired optimal solution to the problem is obtained from that corner point which gives the optimal value of the objective function.

The following examples will be helpful in solving the LPP by graphical method.

EXAMPLE 10.4: Solve the LPP:

Maximize $Z = 3x + 4y$; subject to $4x + 2y \leq 80$, $2x + 5y \leq 180$, $x \geq 0$, $y \geq 0$ by the graphical method.

Solution: We have

Maximize	$Z = 3x + 4y.$	(i)
Subject to	$4x + 2y \leq 80,$	(ii)
	$2x + 5y \leq 180,$	(iii)
and	$x \geq 0$ and $y \geq 0.$	(iv)

First of all we change the inequations (ii), (iii) and (iv) into equations, we have

$$4x + 2y = 80,\ 2x + 5y = 180,\ x = 0 \text{ and } y = 0 \qquad \text{(v)}$$

For $4x + 2y = 80$,

Put $y = 0$, then $x = 20$ and put $x = 0$, then $y = 40$

For $2x + 5y = 180$,

Put $y = 0$, then $x = 90$ and put $x = 0$, then $y = 36$ and $x = 0$, $y = 0$.

Now, plot the lines given by (v), as shown in figure 10.1.

From (iv) we see that the values of (x,y) lie in the first quadrant only.

Any point on or below the line $4x + 2y = 80$ satisfies (ii) and any point on or below the line $2x + 5y = 180$ satisfies (iii). This shows that the desired point (x,y) must be somewhere in the shaded region OAPD, which is known as the solution of space or the region of the feasible solution for the given problem. Its vertices are O (0,0), A (20,0), B (2.5, 35) and D (0,36).

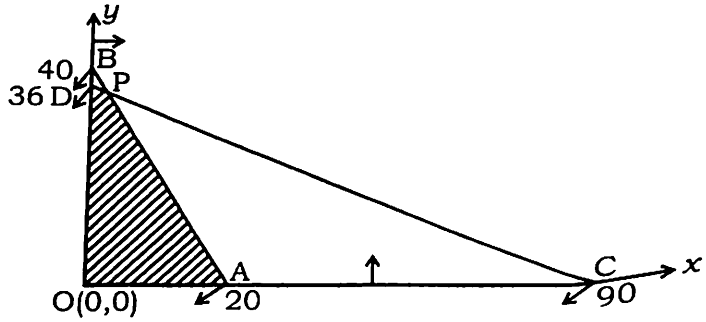

Fig. 10.1

The values of the objective function (i) at these points are

$Z(O) = 0$, $Z(A) = 3(20) + 4(0) = 60$, $Z(B) = 3(2.5) + 4(35) = 147.5$

and $Z(D) = 3(0) + 4(36) = 144$.

Thus, the maximum value of Z is 147.5 and it occurs at B. Hence the optimal solution of the given LPP is $x = 2.5$, $y = 35$ and $Z_{max} = 147.5$.

EXAMPLE 10.5: A company making cold drinks has two bottling plants located at town T_1 and T_2. Each plant produces three drinks A,B and C and their production capacity per day is as follows:

Cold drinks	Plant at	
	T_1	T_2
A	6000	2000
B	1000	2500
C	3000	3000

The marketing department of the company forecasts a demand of 80,000 bottles of A, 22,000 bottles of B and 40,000 bottles of C during the month of June. The operating costs per day of plants at T_1 and T_2 are Rs. 6,000 and Rs. 4,000, respectively. Find (graphically) the number of days for which each plant must be run in June so as to minimise the operating costs while meeting the market demand.

Solution: Let the plants at T_1 and T_2 be run for x and y days respectively. Then the objective is to minimise the operation costs, that is,

min. Z (in Rs.) $= 6000x + 4000y$.

Since the company forecasts a demand of 80,000 bottles of A, we have

$$6000x + 2000y \geq 80{,}000 \text{ or } 3x + y \geq 40.$$

Also, for B, we have

$$1000x + 2500y \geq 22000 \text{ or } x + 2.5y \geq 22.$$

For C, we have

$$3000x + 3000y \geq 40{,}000 \text{ or } x + y \geq \frac{40}{3}.$$

and $x, y \geq 0.$

Thus, the LPP mode of the problem is given by

Min. $Z = 6000x + 4000y.$ (i)

Subject to $3x + y \geq 40,$ (ii)

$x + 2.5\,y \geq 22,$ (iii)

$x + y \geq \frac{40}{3},$ (iv)

and $x \geq 0,\ y \geq 0.$ (v)

First of all we change the inequations (ii), (iii),(iv) and (v) into equations, we have

$$3x + y = 40,\ x + 2.5y = 22,\ x + y = \frac{40}{3},\ x = 0 \text{ and } y = 0. \quad \text{(vi)}$$

From (vi), we have

For $3x + y = 40,$

put $y = 0$, then $x = 13.3$ and put $x = 0$, then $y = 40$

For $x + 2.5y = 22,$

put $y = 0$, then $x = 22$ and put $x = 0$, then $y = 8.8$

For $x + y = \frac{40}{3},$

put $y = 0$, then $x = 13.3$ and put $x = 0$, then $y = 13.3$. Also, $x = 0$, $y = 0$.

Now, plot the lines given by (vi), as shown in figure 10.2.

From (v), we see that the values of (x, y) lie in the first quadrant only.

The solution space satisfying the constraints (ii) to (v) is shown shaded in figure 10.2. From the figure 10.2, we observe that the solution space is unbounded. The constraint (iv) is dominated by the constraints (ii) and (iii) and, hence, does not affect the solution space. Such a constraint (iv) is called a redundant constraint.

The vertices of the convex region ABC are A (22, 0), B (12, 4) and C (0, 40).

The values of the objective function (i) at these points are

$Z(A) = 1{,}32{,}000$, $Z(B) = 88{,}000$,

$Z(C) = 1{,}60{,}000$.

Thus, the minimum value of Z is Rs. 88,000 and it occurs at B. Hence the solution to the problem is $x = 12$ days, $y = 4$ days and $Z_{min} =$ Rs. 88,000.

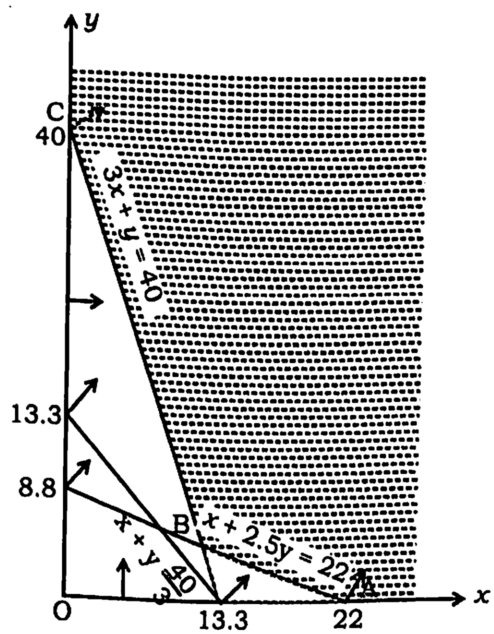

Fig. 10.2

10.4 Some Exceptional Cases

The constraints, generally, give the region of feasible solution which may be bounded or unbounded. We observed that the optimal solution of a problem involving two variables and having a finite solution existed at a corner point of the feasible solution. In fact, this is true for all LP problems for which solutions exist. Thus, we conclude that if there exists an optimal solution of the LPP, then it will be at one of the corners of the solution space.

In the previous section, we have studied only those examples where the optimal solution was unique. But it is not always so. In fact, LPP may have (i) infeasible solution, (ii) unbounded solution, (iii) multiple optimal solution, (iv) no solution.

We now give some examples based on some exceptional cases which are helpful for the readers.

Example 10.6: Using graphical method, solve the following LPP

Maximize $Z = 2x + 3y$.

Subject to $x - y \le 2,\ x + y \ge 4$, and $x, y \ge 0$.

Solution: We have

Maximize $Z = 2x + 3y$. (i)

Subject to $x - y \le 2$, (ii)

$x + y \ge 4$, (iii)

and $x, y \ge 0$. (iv)

First of all, we change the inequation (ii), (iii) and (iv) into equations, we have

$x - y = 2,\ x + y = 4,\ x = 0$ and $y = 0$. (v)

For $x - y = 2$,

Put $y = 0$, then $x = 2$ and put $x = 0$, then $y = -2$.

For $x + y = 4$,

Put $y = 0$, then $x = 4$ and $x = 0$, then $y = 4$. Also $x = 0$, $y = 0$.

Now, plot the lines given by (v) as shown in figure 10.3

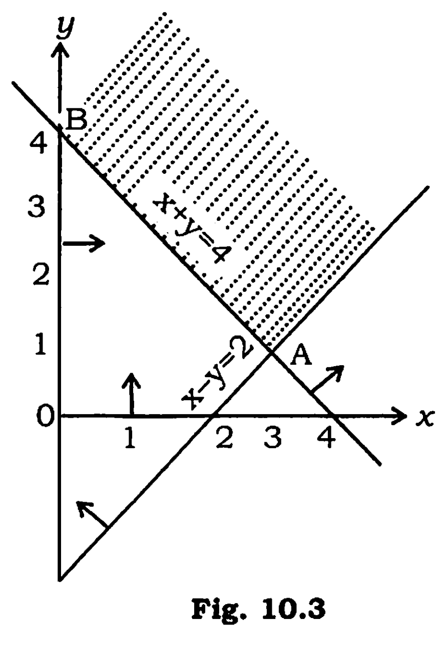

Fig. 10.3

From (iv), we see that the values of (x, y) lie in the first quadrant only. The solution space satisfying the constraints (ii) and (iii) is the convex region shown shaded in figure 10.3. From the figure 10.3, we observe that the solution space is unbounded. The vertices (or corner points) of the feasible region (in the finite plane) are A (3,1) and B (0,4). The values of the objective function (i) at these points are

$$Z(A) = 2(3) + 3(1) = 9 \text{ and } Z(B) = 2(0) + 3(4) = 12.$$

But there are points in the convex region for which Z will have much higher values. For instance, the point (5,5) lies in the shaded region and, here, the value of Z is 25. In fact, the maximum value of Z occurs at infinity. Thus, the problem has an unbounded solution.

EXAMPLE 10.7: Using graphical method, solve the following LPP:

Maximize $Z = 5x + 12y$,

Subject to $2x + 3y \leq 18, x + y \geq 10$, and $x, y \geq 0$.

Solution: We have

Maximize $Z = 5x + 12y$. (i)

Subject to $2x + 3y \leq 18$, (ii)

$x + y \geq 10$, (iii)

and $x, y \geq 0$. (iv)

First of all, we change the inequations (ii), (iii) and (iv) into equations, we have

$2x + 3y = 18$, $x + y = 10$, $x = 0$ and $y = 0$. (v)

From (v), we have

For $2x + 3y = 18$,

put $y = 0$, then $x = 9$ and put $x = 0$, then $y = 6$.

For $x + y = 10$,

put $y = 0$, then $x = 10$ and put $x = 0$, then $y = 10$. Also, $x = 0$, $y = 0$.

Now, plot the lines given by (v) as shown in figure 10.4.

From (iv), we see that the values of (x, y) lie in the first quadrant only. The two solution spaces, one satisfying (ii), and the other satisfying (iii) are shown shaded in figure 10.4. From figure 10.4, it is clear that there is no point in the first quadrant satisfying all the constraints. Hence, there is no feasible solution to the problem because of the conflicting constraints.

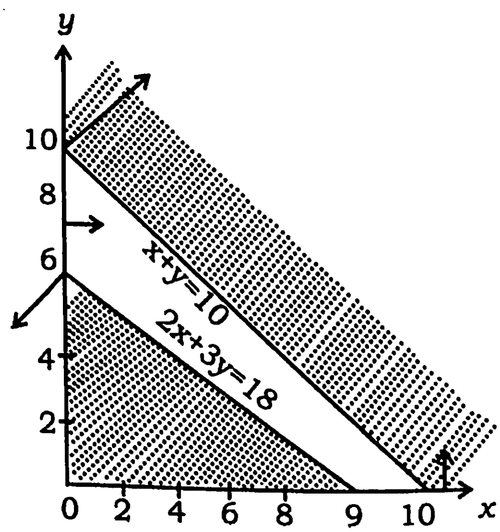

Fig. 10.4

EXAMPLE 10.8: Using graphical method, solve the following LPP: Maximize $Z = 2.5x + y$, subject to $3x + 5y \leq 15$, $5x + 2y \leq 10$, and $x, y \geq 0$.

Solution: We have

Maximize $Z = 2.5x + y$. (i)

Subject to $3x + 5y \leq 15$, (ii)

$5x + 2y \leq 10$, (iii)

and $x, y \geq 0$. (iv)

First of all we change the inequation (ii), (iii) and (iv) into equations, we have

$3x + 5y = 15$, $5x + 2y = 10$, $x = 0$ and $y = 0$. (v)

For $3x + 5y = 15$,

put $y = 0$, then $x = 5$ and put $x = 0$, then $y = 3$.

For $5x + 2y = 10$,

put $y = 0$, then $x = 2$ and put $x = 0$, then $y = 5$,

and $x = 0$, $y = 0$.

Now, plot the lines given by (v) as shown in figure 10.5.

From (iv), we see that the values of (x,y) lie in the first quadrant only. The solution space satisfying the constraints (ii) to (iv) is shown by shaded area OABC in figure 10.5. From figure 10.5, we observe that the corner points of the feasible region are A (2, 0), B $\left(\frac{20}{19}, \frac{45}{19}\right)$.

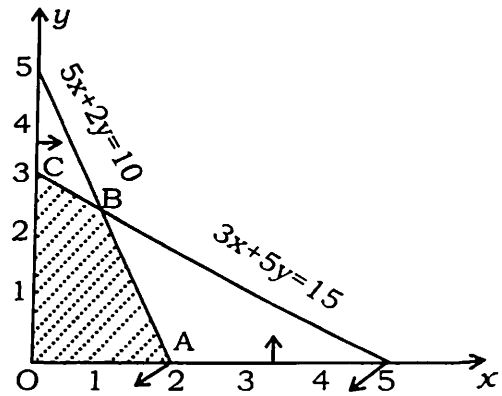

Fig. 10.5

The values of the objective function at these points are Z(A) = 2.5(2) + 0 = 5 and Z(B) = $2.5\left(\frac{20}{19}\right) + \frac{45}{19} = 5$. So, we conclude that all the points on the line segment AB give the same optimal value Z = 5.

The optimal value is unique but there are infinite numbers of optimal solution. Every point on AB correspond to an optimal solution.

EXERCISE 10.2

1. Using graphical method, solve the following LP problems;
   (i) Maximize $Z = 3x + 2y$,
   Subject to constraints $x + 2y \leq 10$, $3x + y \leq 15$, and $x, y \geq 0$.
   (ii) Maximize $Z = 2x + 3y$,
   Subject to constraints $x + y \leq 30$, $y \geq 30$, $0 \leq y \leq 12$, $x - y \geq 0$, and $0 \leq x \leq 20$.
   (iii) Minimize and maximize $Z = x + 2y$,
   Subject to constraints $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$, and $x, y \geq 0$.
   (iv) Maximize $Z = 5x + 3y$,
   Subject to constraints $x + y \leq 300$, $2x + y \leq 360$, and $x, y \geq 0$.
   (v) Minimize $Z = 5x + 4y$,
   Subject to constraints $80x + 100y \geq 88$, $40x + 30y \geq 36$, and $x, y \geq 0$.
   (vi) Minimize and maximize $Z = 5x + 10y$
   Subject to constraints $x + 2y \leq 120$, $x + y \geq 60$, $x - 2y \geq 0$ and $x, y \geq 0$
2. A manufacturer has 3 machines installed in his factory. Machines I and II are capable of being operated for at most 12h, whereas machine III must be operated for at least 5h a day. He produced only two items, each requiring the use of the three machines. The numbers of hours required for producing 1 unit of each of the items A and B on the three machines are given in the following table:

Items	Number of hours required on the machines		
	I	II	III
A	1	2	1
B	2	1	5/4

He makes a profit of Rs. 60 on item A and Rs. 40 on item B. Assuming that he can sell all that he produces, how many of each item should he produce so as to maximize his profit? Solve the LP problem graphically.

3. Solve graphically the following LPP:
Maximize $Z = 4x + 3y$,
Subject to constraints $x - y \le -1, -x + y \le 0$, and $x, y \ge 0$.

4. If a young man rides his motorcycle at 25 km/hour he has to spend Rs. 2 per km on petrol. If he rides at a faster speed of 40 km/hour, the petrol cost increases at Rs. 5 per km. He has Rs. 100 to spend on petrol and wishe to find what is the maximum distance he can travel within one hour? Express this as an LPP and solve it graphically.

5. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of 1kg food is given below:

Food	Vitamin A	Vitamin B	Vitamin C	Cost per kg
X	1	2	3	16
Y	2	2	1	20
Mixture	10	12	8	

1 kg of food X costs Rs. 16 and 1 kg of food Y costs Rs. 20. Find the least cost of the mixture which will produced the required diet.

6. An oil company required 13,000, 20,000 and 15,000 barrels of high-grade, medium-grade and low-grade oil respectively. Refinery P produced 100, 300 and 200 barrels per day of high-, medium- and low-grade oil respectively whereas Refinery Q produces 200, 400 and 100 barrels per day of high-, medium- and low-grade oil respectively. If P costs Rs. 400 per day and Q costs Rs. 300 per day to operate, how many days should each be run to minimize the cost of requirement?

ANSWER

1. (i) $x = 4$, $y = 3$ and $Z_{max} = 18$
(ii) $x = 18$, $y = 12$ and $Z_{max} = 72$
(iii) $x = 0$, $y = 200$ and $Z_{max} = 400$; $x = 20$, $y = 40$ and $Z_{min} = 100$
(iv) $x = 60$, $y = 240$ and $Z_{max} = 1020$
(v) $x = 0.6$, $y = 0.4$ and $Z_{min} = 4.6$
(vi) $x = 60$, $y = 0$ and $Z_{min} = 300$; $x = 60$, $y = 30$ and $Z_{max} = 600$.
2. $x = 4$ units of item A, $y = 4$ units of item B and maximum profit, that is, Z_{max} = Rs. 400.
3. The solution does not exist.
4. $x = \frac{50}{3}$, $y = \frac{40}{3}$ and $Z_{max} = 30$.
5. $x = 2$, $y = 4$ and the least cost of the mixture, that is Z_{min} = Rs. 112.

6. Refinery P will run for $x = \frac{170}{3}$ day and refinery Q will run for $y = \frac{110}{3}$ days and the minimum cost, that is, Z_{min} = Rs. 33,666.67.

10.5 Simplex Method

In the previous section, we have discussed how to solve LP problems involving two variables x and y graphically. If more than two variables are involved in an LP problem, then the graphical method fails to give a solution of the given problems. In such a case, we can solve the problems involving more than two variables by algebraic method. This algebraic method is known as the simplex method.

The concept of the simplex method is similar to the graphical method. The simplex method consists of a number of steps. In the first step, we get the value of Z, which is equal to the value at one vertex by graphical solution. In the next step, the value of Z will be better than the previous one and is equal to at the next adjoining vertex and so on. Since the number of vertices is finite, the simplex method also consists of finite number of steps to get the optimal solution.

Here we shall give some important definitions which are much useful in solving an LPP by simplex method.

The general LPP can also be put in the following form:

Maximize (or minimize) $Z = c_1 x_1 + c_2x_2 + \ldots.. + c_nx_n$. (10.3)

Subject to the constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots.. + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots. + a_{2n}x_n \le b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots. + a_{mn}x_n \le b_m \end{array}\right\} \quad (10.4)$$

where $x_1, x_2, \ldots.., x_n \ge 0$.

The relation (10.4) can be written as

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i;\ (i = 1, 2, \ldots, m), \quad (10.5)$$

that is, the constraints of the general LPP are of less than or equal to type.

Also, if the constraints of the general LPP are of greater than or equal to type, that is,

$$\sum_{j=1}^{n} a_{ij}x_i \ge b_i;\ (i = m, m + 1, \ldots). \quad (10.6)$$

If the constraints of a general LPP are of the form

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i;\ i = 1, 2, \ldots.m.$$

Then the non-negative variables s_i which satisfy

$$\sum_{j=1}^{n} a_{ij}x_j + s_i b_i;\ i = 1, 2, \ldots\ldots, m$$

are called **slack variables**.

Again, if the constraints of a general LPP are of the form

$$\sum_{j=1}^{n} a_{ij}x_i \geq b_i;\ i = m, m+1, \ldots$$

then the non-negative variables s_i which satisfy

$$\sum_{j=1}^{n} a_{ij}x_i - s_i = b_i;\ i = m, m+1, \ldots$$

are called **surplus variables**.

A set of values $x_1, x_2, \ldots, x_n$ which satisfy the constraints (10.4) of the LPP is called its solution.

Any solution of an LPP which satisfies the non-negativity restrictions of the problem is called its feasible solution.

Any feasible solution which maximizes (or minimizes) the objective of the LPP is called its optimal solution.

The general LPP can also be put in the following form:

Max $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to the constraints

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i;\ i = 1, 2, \ldots, m$$

where $x_j \geq 0, j = 1, 2, \ldots, n$, by making some elementary transformation. This form of the LPP is called its **canonical form** and has the following characteristics

(i) The objective function is of the maximization type.

(ii) All the constraints are of less than and equal to type.

(iii) All the variables are non-negative.

The canonical form is a format for an LPP which finds its use in the duality theory.

The general LPP can also be put in the following form:

Maximize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to the constraints

$$\sum_{j=1}^{n} a_{ij}\, x_j = b_i;\ i = 1, 2, \ldots, m$$

where $x_j \geq 0; j = 1, 2, \ldots, n$. This form of the LPP is called its **standard form** and has the the following characteristics:

(i) The objective function is of the maximization type.

(ii) All the constraints should be expressed as equations by adding (or subtracting) the slack (or surplus) variables to the L.H.S. of such constraints.

(iii) The R.H.S. of each constraint should be non- negative.

(iv) All variables are non-negative.

It is noted that Min Z = –Max (–Z) and the objective function can always be expressed in the maximization type.

The following examples will be helpful in solving the LPP by simplex method.

EXAMPLE 10.9: Convert the following LPP to the standard form

Maximize $Z = 2x_1 + 4x_2 + 8x_3$

Subject to constraints

$$7x_1 - 4x_2 \le 6,\ 4x_1 + 3x_2 + 6x_3 \ge 15,\ 3x_1 + 2x_3 \le 4;\ x_1,\ x_2 \ge 0.$$

Solution: The given LPP is

Maximize $Z = 2x_1 + 4x_2 + 8x_3$. (i)

Subject to constraints

$$7x_1 - 4x_2 \le 6,\ 4x_1 + 3x_2 + 6x_3 \ge 15,\ 3x_1 + 2x_3 \le 4;\ x_1 x_2 \ge 0 \quad \text{(ii)}$$

As x_3 is unrestricted, let $x_3 = x'_3 - x''_3$, where $x'_3,\ x''_3 \ge 0$. Now, the given constraints (ii) can be expressed as

$$7x_1 - 4x_2 \le 6,\ 4x_1 + 3x_2 + 6x'_3 - 6x''_3 \ge 15,\ 3x_1 + 2x'_3 - 2x''_3 \le 4,\ x_1,\ x_2,\ x_3',\ x''_3 \ge 0$$

Introducing the slack/surplus variables, the problem in standard form becomes

Maximize $Z = 2x_1 + 4x_2 + 8x'_3 - 8x''_3 + 0s_1 + 0s_2 + 0s_3$

Subject to constraints

$$7x_1 - 4x_2 + s_1 = 6,\ 4x_1 + 3x_2 + 6x' - 6x_3'' + s_2 = 15,\ 3x_1 + 2x'_3 - 2x''_3 + s_3 = 4;$$

$x_1,\ x_2,\ x_3',x_3'',\ s_1,\ s_2,\ s_3 \ge 0.$

which is the required result.

EXAMPLE 10.10: Express the following problem in the standard form

Minimize $Z = 3x_1 + 4x_2$

Subject to $2x_1 - x_2 - 3x_3 = -4,\ 3x_1 + 5x_2 + x_4 = 10,\ x_1 - 4x_2 = 12;\ x_1,\ x_3,\ x_4 \ge 0.$

Solution: The given LPP is

Minimize $Z = 3x_1 + 4x_2$

Subject to constraints

$$2x_1 - x_2 - 3x_3 = -4,\ 3x_1 + 5x_2 + x_4 = 10,\ x_1 - 4x_2 = 12;\ x_1,\ x_3,\ x_4 \ge 0.$$

Here x_3 and x_4 are slack/surplus variables and x_1 and x_2 are the decision variables. As x_2 is unrestricted, let $x_2 = x'_2 - x''_2$, where $x'_2,\ x''_2 \ge 0$. Therefore the problem in standard form is

Maximize $Z' (= -Z) = -3x_1 - 4x'_2 + 4x''_2$

Subject to constraints

$$-2x_1 + x'_2 - x''_2 + 3x_3 = 4,\ 3x_1 + 5x'_2 - 5x'_2 + x_4 = 10,\ x_1 - 4x'_2 + 4x''_2 = 12,$$

where $x_1,\ x'_2,\ x''_2,\ x_3,\ x_4 \ge 0$.

EXAMPLE 10.11: Find all the basic solutions to the following problem:

Max $Z = x_1 + 3x_2 + 3x_3$.

Subject to constraints $x_1 + 2x_2 + 3x_3 = 4$, $2x_1 + 3x_2 + 5x_3 = 7$; $x_1, x_2, x_3 \geq 0$.

which of the basic solutions are (i) non-degenerate basic feasible? and (ii) optimal basic feasible?

Solution: The given LPP is

Max $Z = x_1 + 3x_2 + 3x_3$.

Subject to constraints $x_1 + 2x_2 + 3x_3 = 4$, $2x_1 + 3x_2 + 5x_3 = 7$; $x_1, x_2, x_3 > 0$.

Since there are 3 variables and 2 constraints, a basic solution can be obtained by setting $3 - 2 = 1$ variable equal to zero and then solving the resulting equations. Also, the total number of basic solutions is ${}^3C_2 = 3$.

No. of basic solutions	Basic Variables	Non-basic variables	Values of basic variables	Is the solution feasible? (are all $x_j > 0$?)	Value of Z	Is the solution degenerate?
1	x_1, x_2	$x_3 = 0$	$x_1 + 2x_2 = 4$, $2x_1 + 3x_2 = 7$ $\therefore x_1 = 2, x_2 = 1$	Yes	5	No
2	x_1, x_3	$x_2 = 0$	$x_1 + 3x_3 = 4$, $2x_1 + 5x_3 = 7$ $\therefore x_1 = 1, x_3 = 1$	Yes	4	No
3	x_2, x_3	$x_1 = 0$	$2x_2 + 3x_3 = 4$, $3x_2 + 5x_3 = 7$ $\therefore x_2 = -1, x_3 = 2$	No	3	Yes

(i) The first two solutions are non-degenerate basic feasible solutions.

(ii) The first solution is optimal and $Z_{max} = 5$.

EXAMPLE 10.12: Find an optimal solution to the following LPP by computing all the basic solutions and then finding one that maximizes the object function.

Max $Z = 2x_1 + 3x_2 + '4x_3 + 7x_4$.

Subject to constraints

$2x_1 + 3x_2 - x_3 + 4x_4 = 8$, $x_1 - 2x_2 + 6x_3 - 7x_4 = -3$; $x_1, x_2, x_3, x_4 \geq 0$.

Solution: The given LPP is

Max $Z = 2x_1 + 3x_2 + 4x_3 + 7x_4$.

Subject to constraints

$$2x_1 + 3x_2 - x_3 + 4x_4 = 8,\ x_1 - 2x_2 + 6x_3 - 7x_4 = -3;\ x_1, x_2, x_3, x_4 \geq 0.$$

Since there are 4 variables and 2 constraints, a basic solution can be obtained by setting any 2 ($4 - 2 = 2$) variables equal to zero and then solving the resulting equations. Also, the total number of basic solutions is ${}^4C_2 = 6$.

No. of basic solutions	Basic Variables	Non-basic variables	Values of basic variables	Is the solution feasible? (are all $x_j > 0$?)	Value of Z	Is the solution degenerate?
1	x_1, x_2	$x_3, x_4 = 0$	$2x_1 + 3x_2 = 8,\ x_1 - 2x_2 = -3$ $\therefore\ x_1 = 1,\ x_2 = 2$	Yes	8	No
2	x_1, x_3	$x_2 x_4 = 0$	$2x_1 - x_3 = 8,\ x_1 + 6x_3 = -3$ $\therefore\ x_1 = 45/13,\ x_3 = -14/13$	No	–	–
3	x_1, x_4	$x_2, x_3 = 0$	$2x_1 + 4x_4 = 8,\ x_1 - 7x_4 = -3$ $\therefore x_1 = 29/9,\ x_4 = 7/9$	Yes	10.3	No
4	x_2, x_3	$x_1, x_4 = 0$	$3x_2 - x_3 = 8,\ -2x_2 + 6x_3 = -3$ $\therefore x_2 = 45/16,\ x_3 = 7/16$	Yes	10.2	No
5	x_2, x_4	$x_1, x_3 = 0$	$3x_2 + 4x_4 = 8,\ -2x_2 - 7x_4 = -3$ $\therefore\ x_2 = 132/39,\ x_4 = -7/13$	No	–	–
6	x_3, x_4	$x_1, x_2 = 0$	$-x_3 + 4x_4 = 8,\ 6x_3 - 7x_4 = -3$ $\therefore\ x_3 = 44/17,\ x_4 = 45/17$	Yes	28.9	Yes

Hence, the optimal basic feasible solution is

$$x_1 = 0,\ x_2 = 0,\ x_3 = \frac{44}{17},\ x_4 = \frac{45}{17} \text{ and maximum value of } Z = 28.9\ .$$

Example 10.13: Solve by simplex method the following LP problem:

Max $Z = 4x_1 + 10x_2$

subject to the constraints

$2x_1 + x_2 \le 50,\ 2x_1 + 5x_2 \le 100,\ 2x_1 + 3x_2 \le 90$ and $x_1, x_2 \ge 0$.

Solution: The given LP problem is

Max $Z = 4x_1 + 10x_2$. (i)

Subject to constraints

$2x_1 + x_2 \le 50,$ (ii)

$2x_1 + 5x_2 \le 100,$ (iii)

$2x_1 + 3x_2 \le 90,$ (iv)

and $x_1, x_2 \ge 0.$ (v)

From (i) to (v), we see that the problem being of the maximization type and all b's being ≤ 0, so we express the problem in the standard form by adding slack variables s_1, s_2 and s_3 (one for each constraint). The standard form of the problem is

Max $Z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to

$$2x_1 + x_2 + s_1 = 50,\ 2x_1 + 5x_2 + s_2 = 100,\ 2x_1 + 3x_2 + s_3 = 90, \quad \text{(vi)}$$

where $x_1, x_2, s_1, s_2, s_3 \geq 0$.

Since we have 3 constraints and 5 variables, a solution is obtained by setting $5 - 3 = 2$ variables equal to zero and solving for the remaining 3 variables.

An initial basic feasible solution is obtained by setting $x_1 = x_2 = 0$ (non-basic). Now, substituting $x_1 = x_2 = 0$ in (vi), we have

$s_1 = 50,\ s_2 = 100$ and $s_3 = 90$ (basic) .

Since all s_1, s_2 and s_3 are positive, therefore the basic solution is also feasible and non-degenerate. Thus, the basic feasible solution is $x_1 = x_2 = 0$ (non-basic) and $s_1 = 50,\ s_2 = 100,\ s_3 = 90$ (basic).

The above information is conveniently expressed in table-I as follow:

Table-I

$C_j \rightarrow$			4	10	0	0	0	Ratio
C_B	Basis	Solution ($b = X_B$)	x_1	x_2	s_1	s_2	s_3	X_B/x_2
0	s_1	50	2	1	1	0	0	50
0	s_2	100	2	5	0	1	0	20 $\rightarrow$
0	s_3	90	2	3	0	0	1	30
Z = 0		Z_j	0	0	0	0	0	
		$C_j = c_j - Z_j$	4	10 $\uparrow$	0	0	0	

Since C_j is positive under x_1 and x_2 columns, so the initial basic feasible solution is not optimal and can be improved.

From table-I, we see that x_2 is the incoming variable as its incremental contribution C_j (= 10) is maximum. The column in which it appears is the key column (shown by an arrow $\uparrow$) .

Dividing each element of X_B column by the corresponding element of the key column, we find that the smallest positive ratio is 20. The row corresponding to the smallest positive ratio (that is, 20) is the key row. So, s_2 is the outgoing variable (shown by an arrow $\rightarrow$). The element at the intersection of the key row and the key column (that is, 5) is the key element.

Having decided that x_2 is to enter the solution, we have tried to find as to what maximum value x_2 can have without violating the constraints. So, removing s_2, the new basic will contain s_1, x_2 and s_3 as the basic variables.

To transform the initial set of equations with a basic feasible solution into an equivalent set of equations with a different basic feasible solution, we make the key element unit. Divide the elements of the key row by the key element (that is, 5) to obtain new values of the elements in this row. Thus, the key row

Old row 2 0 s_2 100 2 5 0 1 0 0
is replaced by the new row

New row 2 10 x_2 2 $\frac{2}{5}$ 1 0 $\frac{1}{5}$ 0 0.

Then, to make all other elements in the key column zero, we subtract proper multiples of the new key row from the other rows. Here we subtract the elements of the new key row from the first row and 3 times the element of the new key row from the third row. These become the second and third rows of the next table, respectively.

The key column entry in row 1 is 1. Thus, row 1 – 1 × new row 2 gives

$$50 - 1 \times 20 = 30, \quad 2 - 1 \times \frac{2}{5} = \frac{8}{5}, \quad 1 - 1 \times 1 = 0.$$

$$1 - 1 \times 0 = 1, \quad 0 - 1 \times \frac{1}{5} = -\frac{1}{5}, \quad 0 - 1 \times 0 = 0.$$

The key column entry in row 3 is 3. Thus, row 3 – 3 × new row 2 gives

$$90 - 3 \times 20 = 30, \quad 2 - 3 \times \frac{2}{5} = \frac{4}{5}, \quad 3 - 3 \times 1 = 0.$$

$$0 - 3 \times 0 = 0, \quad 0 - 3 \times \frac{1}{5} = -\frac{3}{5}, \quad 1 - 3 \times 0 = 1.$$

We also change the corresponding value under the C_B column from 0 to 10, while replacing s_2 by x_2 under the basis. Thus, the second basic feasible solution is given by table-II.

Table-II

$C_j \rightarrow$			4	10	0	0	0
C_B	Basis	Solution ($b = X_B$)	x_1	x_2	s_1	s_2	s_3
0	s_1	30	8/5	0	1	–1/5	0
10	x_2	20	2/5	1	0	1/5	0
0	s_3	30	4/5	0	0	–3/5	1
Z = 200		Z_j	4	10	0	2	0
		$C_j = c_j - Z_j$	0	0	0	–2	0

As C_j is either zero or negative under all columns, the table-II gives the optimal basic feasible solution. The optimal solution is

$x_1 = 0$, $x_2 = 20$, and maximum Z = 200.

Note : Z = 0 (30) + 10 (20) + 0 (30) = 200 entries

$x_1 = 0\,(\frac{8}{5}) + 10\,(\frac{2}{5}) + 0\,(\frac{4}{5}) = 4,\ x_2 = 0\,(0) + 10(1) + 0(0) = 10,$

$s_1 = 0\,(-\frac{1}{5}) + 10\,(0) + 0\,(0) = 0,\ s_2 = 0\,(-\frac{1}{5}) + 10\,(\frac{1}{5}) + 0\,(-\frac{3}{5}) = 2,$

$s_3 = 0\,(0) + 10\,(0) + 0\,(1) = 0.$

EXAMPLE 10.14: Using simplex method, solve the following LP problem:
Maximize $Z = 3x_1 + 5x_2 + 4x_3$
Subject to the constraints $2x_1 + 3x_2 + \leq 8,\ 2x_2 + 5x_3 \leq 10,$
$3x_1 + 2x_2 + 4x_3 \leq 15$ and $x_1, x_2, x_3 \geq 0$.

Solution: The given LP problem is

Max. $Z = 3x_1 + 5x_2 + 4x_3.$ (i)

Subject to constraints

$2x_1 + 3x_2 \leq 8,$ (ii)

$2x_2 + 5x_3 \leq 10,$ (iii)

$3x_1 + 2x_2 + 4x_3 \leq 15,$ (iv)

and $x_1, x_2, x_3 \geq 0$ (v)

From (i) to (v), we see that the problem being of the maximization type and all b's being ≤ 0, we express the problem in the standard form by adding slack variables s_1, s_2 and s_3 (one for each constraint). The standard form of the problem is

Max. $Z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3.$

Subject to

$2x_1 + 3x_2 + s_1 = 8,\ 2x_2 + 5x_3 + s_2 = 10,\ 3x_1 + 2x_2 + 4x_3 + s_3 = 15,$ (vi)

where, $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.

Since we have 3 constraints and 6 variables, a solution is obtained by setting $6 - 3 = 3$ variables equal to zero and solving for the remaining 3 variables.

An initial basic feasible solution is obtained by setting $x_1 = x_2 = x_3 = 0$, so that $s_1 = 8$, $s_2 = 10$ and $s_3 = 15$.

The initial basic feasible solution is given in table-I as follows :

Table-I

$C_j \rightarrow$			3	5	4	0	0	0	Ratio
C_B	Basis	Solution ($b = X_B$)	x_1	x_2	x_3	s_1	s_2	s_3	X_B / x_2
0	s_1	8	2	3	0	1	0	0	8/3 $\rightarrow$
0	s_2	10	0	2	5	0	1	0	5
0	s_3	15	3	2	4	0	0	1	15/2
Z = 0		Z_j	0	0	0	0	0	0	0
		$C_j = c_j - Z_j$	3	5 $\uparrow$	4	0	0	0	

Since C_j is positive under x_1, x_2, and x_3 columns, so the initial basic feasible solution is not optimal and can be improved.

From table-I, we see that x_2 is incoming variable as its incremental contribution C_j (= 5) is maximum. The column in which it appears is the key column (shown by an arrow ↑).

Dividing each element of X_B column by the corresponding element of the key column, we find that the smallest positive ratio is $\frac{8}{3}$. The row corresponding to the smallest positive ratio (that is $\frac{8}{3}$) is the key row. So s_1 is the outgoing variable (shown by an arrow →).

The element at the intersection of the key row and the key column (that is, 3) is the key element.

Having decided that x_2 is to enter the solution, we have tried to find as to what maximum value x_2 can have without violating the constraints. So removing s_1, the new basis will contain x_2, s_2 and s_3 as the basic variables.

Drop s_1 and introduce x_2 with its associated value 5 under the C_B column.

Convert the key element (that is, 3) to unity and make all other elements of the key column zero. Then, the second basic feasible solution is given in table-II as follows:

Table-II

$C_j \to$			3	5	4	0	0	0	Ratio
C_B	Basis	Solution ($b = X_B$)	x_1	x_2	x_3	s_1	s_2	s_3	X_B/x_3
5	x_2	8/13	2/3	1	0	1/3	0	0	-
0	s_2	14/3	-4/3	0	5	-2/3	1	0	14/15 →
0	s_3	29/3	5/3	0	4	-2/3	0	1	29/12
Z=40/3		Z_j	10/3	5	0	5/3	0	0	
		$C_j = c_j - Z_j$	-1/3	0	4 ↑	-5/3	0	0	

Since C_j row has a positive entry, the initial basic feasible solution is not optimal and can be improved. From table-II, we see that the incoming variable is x_3 and the outgoing basic variable is s_2. Also, 5 is the key element.

Drop s_2 and introduce x_3 with its associated value 4 under the C_B column. Convert the key element to unity and make all other elements of the key column zero. Then, the third basic feasible solution is given in table-III as follows:

Table-III

$C_j \rightarrow$			3	5	4	0	0	0	Ratio
C_B	Basis	Solution (b=X_B)	x_1	x_2	x_3	s_1	s_2	s_3	X_B/x_1
5	x_2	8/3	2/3	1	0	1/3	0	0	4
4	x_3	14/15	–4/15	0	1	–2/15	–4/5	0	-
0	s_3	89/15	41/15	0	0	–2/15	–4/5	1	→ 89/41
Z=256/15		Z_j	34/15	5	4	17/15	4/5	0	
		$C_j = c_j - Z_j$	11/15 ↑	0	0	–17/15	–4/5	0	

Since C_j row has a positive entry, the initial basic feasible solution is not optimal, and can be improved.

From table-III, we see that the incoming variable is x_1 and the outgoing basic variable is s_3. Also, $\frac{41}{15}$ is the key element.

Drop s_3 and introduce x_1 with its associated value 3 under the C_B column. Convert the key element to unity and make all other elements of the key column zero. Then, the fourth basic feasible solution is given in table-IV as follows.

Table-IV

$C_j \rightarrow$			3	5	4	0	0	0	Ratio
C_B	Basis	Solution (b=X_B)	x_1	x_2	x_3	s_1	s_2	s_3	
5	x_2	50/41	0	1	0	15/41	8/41	–10/41	
4	x_3	62/41	0	0	1	–6/41	5/41	4/41	
3	x_1	89/41	1	0	0	–2/41	–12/41	15/41	
Z=765/41		Z_j	3	5	4	45/41	24/41	11/41	
		$C_j = c_j - Z_j$	0	0	0	–45/41	–24/41	–11/41	

Since all the entries in the C_j row are either negative or zero, the optimal solution is obtained with $x_1 = \frac{89}{41}$, $x_2 = \frac{50}{4}$, $x_3 = \frac{62}{41}$, and max $Z = \frac{765}{41}$.

EXAMPLE 10.15: Using simplex method, solve the following LP problem:

Minimize $Z = x_1 - 3x_2 + 3x_3$

Subject to the constraints $3x_1 - x_2 + 2x_3 \leq 7$, $2x_1 + 4x_2 \geq -12$, $-4x_1 + 3x_2 + 8x_3 \leq 10$ and $x_1, x_2, x_3 \geq 0$.

Solution: The given LP problem is

Min. $Z = x_1 - 3x_2 + 3x_3$ (i)

Subject to the constraints

$3x_1 - x_2 + 2x_3 \leq 7,$ (ii)

$2x_1 + 4x_2 \geq -12,$ (iii)

$-4x_1 + 3x_2 + 8x_3 \leq 10$ (iv)

and $x_1, x_2, x_3 \geq 0$ (v)

From (i) to (v), we see that the problem is of the minimization type. Converting it to the maximization type, we have

Max. $Z' = -x_1 + 3x_2 - 3x_3.$ (vi)

As the right hand side of (iii) is negative, we write it as

$-2x_1 - 4x_2 \leq 12.$ (vii)

Again, from (vi), (ii), (vii), (iv) and (v), we see that the problem being of the maximization type and all b's being ≤ 0, we express the problem in the standard form by adding slack variables s_1, s_2 and s_3 (one for each constraint). The standard form of the problem is

Max. $Z' = -x_1 + 3x_2 - 3x_3$

Subject to the constraints

$3x_1 - x_2 + 2x_3 + s_1 = 7, -2x_1 - 4x_2 + s_2 = 12, -4x_1 + 3x_2 + 8x_3 + s_3 = 10$ (viii)

where $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.

Since we have 3 constraints and 6 variables, a solution is obtained by setting $6 - 3 = 3$ variables equal to zero and solving for remaining variables.

An initial basic feasible solution is obtained by setting $x_1 = x_2 = x_3 = 0$, so that $s_1 = 7$, $s_2 = 12$ and $s_3 = 10$.

The initial basic feasible solution is given in table-I as follows:

Table-I

$C_j \rightarrow$			-1	3	-3	0	0	0	Ratio
C_B	Basis	Solution ($b=X_B$)	x_1	x_2	x_3	s_1	s_2	s_3	X_B/x_2
0	s_1	7	3	-1	2	1	0	0	$7/(-1) = -7$
0	s_2	12	-2	-4	0	0	1	0	$12/(-4) = -3$
0	s_3	10	-4	3	8	0	0	1	$10/3 = 10/3 \rightarrow$
$Z' = 0$		Z_j	0	0	0	0	0	0	
		$C_j = c_j - Z_j$	-1	$3\uparrow$	-3	0	0	0	

Since C_j is positive under x_2, so the initial basic feasible solution is not optimal and can be improved. From table-I, we see that x_2 is incoming variable, and s_3 is the outgoing variable. Also, 3 is the key element.

Drop s_3 and introduce x_2 with its associated value 3 under C_B column.Convert the key element to unity and make all other elements of the key column zero. Then, the second basic feasible solution is given in table-II as follows:

Table-II

$C_j \rightarrow$			-1	3	-3	0	0	0	Ratio
C_B	Basis	Solution ($b = X_B$)	x_1	x_2	x_3	s_1	s_2	s_3	X_B/x_1
0	s_1	31/3	5/3	0	14/3	1	0	1/3	31/5 →
0	s_2	76/3	– 22/3	0	32/3	0	1	4/3	– 38/11
3	x_2	10/3	– 4/3	1	8.3	0	0	1/3	– 5/2
		Z_j	– 4	3	8	0	0	1	
		$C_j = c_j - Z_j$	3 ↑	0	– 11	0	0	– 1	

Since C_j is positive under x_1, so the initial basic feasible solution is not optimal and can be improved.

From table-II, we see that x_1 is the incoming variable and s_1 is the outgoing variable. Also, $\frac{5}{3}$ is the key element.

Drop s_1 and introduce x_1 with its associated value 1 under C_B column. Convert the key element to unity and make all other elements of the key column zero. Then, the third basic feasible solution is given in table-III as follows:

Table-III

$C_j \rightarrow$			–1	3	–3	0	0	0	Ratio
C_B	Basis	Solution ($b = X_B$)	x_1	x_2	x_3	s_1	s_2	s_3	
– 1	x_1	31/5	1	0	14/5	3/5	0	1/5	
0	s_2	354/5	0	0	156/5	22/5	1	14/5	
3	x_2	58/5	0	1	32/5	4/5	0	3/5	
		Z_j	– 1	3	82/5	9/5	0	8/5	
		$C_j = c_j - Z_j$	0	0	– 97/5	– 9/5	0	– 8/5	

Now, since each $C_j \le 0$, therefore it gives the optimal solution $x_1 = \frac{31}{5}$, $x_2 = \frac{58}{5}$,

$x_3 = 0$ (Non-basic) and $Z'_{max} = \frac{143}{5}$ or Z_{min}. $-\frac{143}{5}$.

Example 10.16: Using simplex method, solve the following LP problem:

Maximize $Z = 107x_1 + x_2 + 2x_3$

Subject to the constraints $14x_1 + x_2 - 6x_3 + 3x_4 = 7$,

$16x_1 + \frac{1}{2}x_2 - 6x_3 \le 5, 3x_1 - x_2 - x_3 \le 0$ and $x_1, x_2, x_3, x_4 \ge 0$.

Solution: The given LP problem is

Max. $Z = 107x_1 + x_2 + 2x_3$. (i)

Subject to the constraints

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7 \text{ or } \frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 = \frac{7}{3} \quad \text{(ii)}$$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 \le 5, \quad \text{(iii)}$$

$$3x_1 - x_2 - x_3 \le 0, \quad \text{(iv)}$$

and $x_1, x_2, x_3, x_4 \ge 0$.

From (i) to (v), we see that the problem is of the maximization type and all b's being ≤ 0. So we express the problem in the standard form by adding slack variables s_1 and s_2 (one for each constraint (iii) and (iv)). Here x_4 is a slack variable. The standard form of the problem is

Max. $Z = 107x_1 + x_2 + 2x_3 + 0x_4 + 0s_1 + 0s_2$

Subject to the constraints

$$\left.\begin{aligned} \frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 &= \frac{7}{3} \\ 16x_1 + \frac{1}{2}x_2 - 6x_3 + s_1 &= 5 \\ 3x_1 - x_2 - x_3 + s_2 &= 0 \end{aligned}\right\} \quad \text{(vi)}$$

where $x_1, x_2, x_3, x_4, s_1, s_2 \ge 0$.

Since we have 3 constraints and 6 variables, a solution is obtained by setting $6 - 3 = 3$ variables equal to zero and solving for the remaining 3 variables.

An initial basic feasible solution is obtained by setting $x_1 = x_2 = x_3 = 0$ (non-basic), so that $x_4 = \frac{7}{3}$, $s_1 = 5$ and $s_2 = 0$ (basic)

The initial basic feasible solution is given in table-I as follows:

Table-I

$C_j \rightarrow$			107	1	2	0	0	0	Ratio
C_B	Basis	Solution ($b = X_B$)	x_1	x_2	x_3	x_4	s_1	s_2	X_B/x_1
0	x_4	7/3	14/3	1/3	–2	1	0	0	1/2
0	s_1	5	16	1/2	–6	0	1	0	5/16
0	s_2	0	3	–1	–1	0	0	1	$0 \rightarrow$
Z = 0		Z_j	0	0	0	0	0	0	
		$C_j = c_j - Z_j$	107 ↑	1	2	0	0	0	

Since C_j is positive under some column, so the initial basic feasible solution is not optimal and can be improved.

From table-I, we see that x_1 is the incoming variable and s_2 is the outgoing variable. Also, 3 is the key element.

Drop s_2 and introduce x_1 with its associated value 107 under C_B column. Convert the key element to unity and make all other elements of the key column zero. Then, the second basic feasible solution is given in table-II as follows:

Table-II

$C_j \rightarrow$			107	1	2	0	0	0	Ratio
C_B	Basis	Solution ($b = X_B$)	x_1	x_2	x_3	x_4	s_1	s_2	X_B/x_3
0	x_4	7/3	0	17/9	–4/9	1	0	–14/9	–21/4
0	s_1	5	0	35/6	–2/3	0	1	–16/3	–15/2
107	x_1	0	1	–1/3	–1/3	0	0	1/3	$0 \rightarrow$
Z = 0		Z_j	107	–107/3	–107/3	0	0	107/3	
		$C_j = c_j - Z_j$	0	110/3	113/3 ↑	0	0	–107/3	

Since C_j is positive under some column, so the initial basic feasible solution is not optimal. Here $\frac{113}{3}$ being the largest positive value of C_j, x_3 is the incoming variable. But all the values of ratios ≤ 0, So, there is no outgoing variable. Hence x_3 will not enter the basis. This indicates that the solution to the problem is unbounded.

EXERCISE 10.3

1. Express the following LP problems in the standard form.
   (i) Maximize $Z = 3x_1 + 5x_2 + 7x_3$
   Subject to the constraints $6x_1 - 4x_2 \le 5$, $3x_1 + 2x_2 + 5x_3 \ge 11$,
   $4x_1 + 3x_3 \le 2$; $x_1, x_2, x_3 \ge 0$.
   (ii) Maximize $Z = 3x_1 + 4x_2 + 5x_3$
   Subject to the constraints $2x_1 + 7x_2 + x_3 \le 10$, $5x_1 + 9x_2 + 4x_3 \ge 20$,
   $8x_1 + 15x_3 \le 30$; $x_1, x_2, x_3 \ge 0$.
   (iii) Maximize $Z = 2x_1 + 5x_2 + 7x_3$
   Subject to the constraints $5x_1 - 3x_2 \le 4$, $7x_1 + 6x_2 + 9x_3 \ge 15$,
   $8x_1 + 6x_3 \le 5$; $x_1, x_2, x_3 \ge 0$.
   (iv) Minimize $Z = x_1 - 3x_2 + 3x_3$
   Subject to the constraints $3x_1 - x_2 + 2x_3 \le 7$, $2x_1 + 4x_2 \ge -12$,
   $-4x_1 + 3x_2 + 8x_3 \le 10$; $x_1, x_2, x_3 \ge 0$.
   (v) Minimize $Z = 5x_1 + 8x_2$
   Subject to the constraints $3x_1 + 7x_2 + x_3 = 9$, $4x_1 - 2x_2 - x_4 = -15$,
   $2x_1 - 3x_2 + x_5 = 8$; $x_1, x_2, x_3, x_4, x_5 \ge 0$.
   (vi) Maximize $Z = 3x_1 + 5x_2 + 8x_3$
   Subject to the constraints $2x_1 - 5x_2 \le 6$, $3x_1 + 2x_2 + x_3 \ge 5$,
   $3x_1 + 4x_3 \le 3$; $x_1, x_2, x_3 \ge 0$.
   (vii) Maximize $Z = 3x_1 + 2x_2 + 5x_3$
   Subject to the constraints $-5x_1 + 2x_2 \le 5$, $2x_1 + 3x_2 + 4x_3 \ge 7$,
   $2x_1 + 5x_3 \le 3$; $x_1, x_2, x_3 \ge 0$.
2. Find all the basic solutions of the following system of equations identifying in each case the basic and non-basic variables: $2x_1 + x_2 + 4x_3 = 11$, $3x_1 + x_2 + 5x_3 = 14$. Investigate whether the basic solutions are degenerate basic solutions or not. Hence find the basic feasible solution of the system.
3. Find all the basic solutions to the following LP problem:
   Maximize $Z = x_1 + 3x_2 + 3x_3$
   Subject to the constraints $x_1 + 2x_2 + 3x_3 = 4$, $2x_1 + 3x_2 + 5x_3 = 7$ and $x_1, x_2, x_3 \ge 0$.
   which of the basic solutions are (a) non-degenerate basic feasible, (b) optimal basic feasible?
4. Find all the basic solutions of the following system of linear equations:
   $x_1 + 2x_2 + x_3 = 4$, $2x_1 + x_2 + 5x_3 = 5$
5. Show that the following system of linear equations has two degenerate feasible basic solutions and that a non-degenerate basic solution is not feasible:
   $2x_1 + x_2 - x_3 = 2$, $3x_1 + 2x_2 + x_3 = 3$.
6. Find an optimal solution to the following LPP by computing all the basic solutions and then finding the one that maximizes the objective function:
   Maximize $Z = 2x_1 + 3x_2 + 4x_3 + 7x_4$
   Subject to the constraints $2x_1 + 3x_2 - x_3 + 4x_4 = 8$,
   $x_1 - 2x_2 + 6x_3 - 7x_4 = -3$; $x_1, x_2, x_3, x_4 \ge 0$.

7. Using simplex method, solve the following LP problem:
   Maximize $Z = 5x_1 + 3x_2$
   Subject to the constraints $x_1 + x_2 \le 2$, $5x_1 + 2x_2 \le 10$, $3x_1 + 8x_2 \le 12$; $x_1, x_2, \ge 0$.
8. Solve by using simplex method:
   Maximize $Z = 3x_1 + 4x_2$
   Subject to $x_1 + x_2 \le 450$, $2x_1 + x_2 \le 600$; $x_1, x_2, \ge 0$.
9. Using simplex method, solve the following LP problem:
   Maximize $Z = 2x_1 + 5x_2$
   Subject to $x_1 + 4x_2 \le 24$, $3x_1 + x_2 \le 21$, $x_1 + x_2 \le 9$, $x_1 + x_2 \ge 0$; $x_1, x_2 \ge 0$.
10. A firm produces 3 products which are processed on 3 machines. The relevant data is given below:

Machine	Time per unit (minutes)			Machine capacity (minutes/day)
	Product A	Product B	Product C	
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

The profit per unit for products A, B and C is Rs. 4, Rs. 3 and Rs. 6 respectively. Determine the daily number of units to be manufactured for each product. Assume that all the units produced are consumed in the market.

11. Following data are available for a firm which manufactures 3 items A, B and C:

Product	Time required in hours		Profit (in Rs.)
	Assembly	Finishing	
A	10	2	80
B	4	5	60
C	5	4	30
Firm's Capacity	2000	1009	

Express the above data in the form of LP problem to maximize the profit from production and solve it by simplex method.

12. Solve by simplex method the following LP problem:
    Maximize $Z = x_1 - 3x_2 + 3x_3$
    Subject to $3x_1 - x_2 + 2x_3 \le 7$, $2x_1 + 4x_2 \ge -12$, $-4x_1 + 3x_2 + 8x_3 \le 10$; $x_1, x_2, x_3 \ge 0$.
13. Solve by simplex method the following LP problem:
    Maximize $Z = x_1 + x_2$
    Subject to $x_1 - 2x_2 \ge -8$, $3x_1 + x_2 \le 11$; $x_1, x_2 \ge 0$.
14. Solve by simplex method the following LP problem:
    Maximize $Z = 2x_1 + 3x_2 + 10x_3$
    Subject to $x_1 + 2x_3 = 0$, $x_2 + x_3 = 1$; $x_1, x_2, x_3 \ge 0$.

ANSWER

1. (i) Max. $Z = 3x_1 + 5x_2 + 7x_3' - 7x_3''$
Subject to $6x_1 - 4x_2 + s_1 = 5$, $3x_1 + 2x_2 + 5x'_3 - 5x_3'' - s_2 = 11$,
$4x_1 + 3x'_3 - 3x''_3 + s_3 = 2$; $x_1, x_2, x'_3, x''_3, s_1, s_2, s_3 \geq 0$.
(ii) Max. $Z = 3x_1 + 4x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$;
Subject to $2x_1 + 7x_2 + x_3 + s_1 = 10$, $5x_1 + 9x_2 + 4x_3 - s_2 = 20$,
$8x_1 + 15x_3 + s_3 = 30$; $x_1, x_2, x_2, s_1, s_2, s_3 \geq 0$.
(iii) Max. $Z = 3x_1 + 5x_2 + 7x'_3 - 7x''_3$
Subject to $6x_1 - 4x_2 + s_1 = 5$, $3x_1 + 2x_2 + 5x_3' - x_3'' - s_2 = 11$,
$4x_1 + 3x'_3 - 3x''_3 + s_3 = 2$; $x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \geq 0$.
(iv) Max. $Z^* (= -Z) = -x_1 + 3x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_3$,
Subject to $3x_1 - x_2 + 2x_3 + s_1 = 7$, $-2x_1 - 4x_2 + s_2 = 12$,
$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$; $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.
(v) Max. $Z^* (= -Z) = -5x_1 - 8x'_2 + 8x''_2 + 0x_3 + 0x_4 + 0x_5$;
Subject to $3x_1 + 7x'_2 - 7x''_2 + x_3 = 9$, $-4x_1 + 2x'_2 - 2x''_2 + x_4 = 15$,
$2x_1 - 3x'_2 + 3x''_2 + x_5 = 8$; $x_1, x'_2, x''_2, x_3, x_4, x_5 \geq 0$.
(vi) Max. $Z = 3x_1 + 5x_2 + 8x_3$;
Subject to $2x_1 - 5x_2 + s_1 = 6$, $3x_1 + 2x_2 + x_3 - s_2 = 5$, $3x_1 + 4x_3 + s_3 = 3$;
$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.
(vii) Min. $Z = 3x_1 + 2x_2 + 5x_3$;
Subject to $-5x_1 + 2x_2 + s_1 = 5$, $2x_1 + 3x_2 + 4x_3 - s_2 = 7$, $2x_1 + 5x_3 + s_3 = 3$;
$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.
2. The basic feasible solutions are
(i) $x_1 = 3, x_2 = 5, x_3 = 0$; (ii) $x_1 = \frac{1}{2}, x_2 = 0, x_3 = \frac{5}{2}$,
which are also non-degenerate basic solutions.
3. Basic solutions are
(i) $x_1 = 2, x_2 = 1$ (Basic), and $x_3 = 0$;
(ii) $x_1 = x_3 = 1$ (Basic), and $x_2 = 0$;
(iii) $x_2 = -1$ $x_3 = 2$ (Basic), and $x_1 = 0$;
(a) First two solutions are non-degenerate basic feasible solutions;
(b) First solution is optimal and $Z_{max} = 5$.
4. (i) $x_1 = 2, x_3 = 1$ (Basic); $x_3 = 0$ (Non-basic);
(ii) $x_1 = 5, x_3 = -1$ (Basic); $x_2 = 0$ (Non-basic);
(iii) $x_2 = \frac{5}{3}, x_3 = \frac{2}{3}$ (Basic); $x_1 = 0$ (Non-basic). All the three basic solutions are non-degenerate.

6. The optimal basic feasible solution is $x_1 = 0$, $x_2 = 0$, $x_3 = \frac{44}{17}$, $x_4 = \frac{45}{17}$, and

$$Z_{max} = \frac{491}{17}.$$

7. Optimal solution is $x_1 = 2$, $x_2 = 0$, and $Z_{max} = 10$.
8. Optimal solution is $x_1 = 0$, $x_2 = 45$, and $Z_{max} = 1800$.
9. Optimal solution is $x_1 = 4$, $x_2 = 5$, $s_2 = 4$, and $Z_{max} = 33$.
10. Optimal solution is $x_1 = 0$, $x_2 = \frac{380}{9}$, $x_3 = \frac{470}{3}$, and $Z_{max} = \frac{3200}{3} = 1066.67$.
11. Optimal solution is $x_1 = 142$, $x_2 = 145$, $x_3 = 0$, and $Z_{max} = 20,060$.
12. Optimal solution is $x_1 = \frac{31}{5}$, $x_2 = \frac{58}{5}$, and $Z'_{max} = \frac{143}{5}$ or $Z_{min} = -\frac{143}{5}$.
13. Optimal solution is $x_1 = 2$, $x_2 = 5$, and $Z_{max} = 7$.
14. The solution to the problem is unbounded.

10.6. Theory of Duality

A duality is a situation in which two opposite ideas or feelings exist at the same time. The duality theory is one of the most interesting concepts in linear programming. Every given LPP has associated with it another LPP involving the same data and closely related optimal solutions. The given problem is called primal and the associated problem is called dual. If the primal problem requires maximization, then the dual problem is one of the minimizing problem and *vice versa*.

If the number of constraints in the primal problem is greater than the number of variables, then in the dual problem, the number of constraints is less than the number of variables. It is easier to solve the dual with less number of constraints as the labour of computation is considerably reduced. Hence, we prefer to solve the dual, and from the solution of the dual, we can obtain the solution of the given primal.

10.6.1 Formulation of Dual Problem

Consider the following LPP:

Maximize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$$

where $x_1, x_2, \dots, x_n \ge 0$.

To formulate the dual problem, we follow the following rules:

1. The maximization problem in the primal become the minimization problem in the dual and *vice versa*.
2. The ($\le$) type of constraints in the primal become the ($\ge$) type of constraints in the dual and *vice versa*.
3. The coefficients $c_1, c_2, \dots, c_n$ in the objective function of the primal become

$b_1, b_2, \ldots, b_m$ in the objective function of the dual.

4. The constraints $b_1, b_2, \ldots, b_m$ in the constraints of the primal become $c_1, c_2, \ldots, c_n$ in the constraints of the dual.
5. If the primal has n variables and m constraints, then the dual will have m variables and n constraints. Thus, the body matrix of the dual is obtained by the transpose of the body matrix of the primal problem and *vice versa*.
6. The variables in both the primal and the dual are non-negative.

Using the above rules, the dual problem will be of the form

Minimize $W = b_1y_2 + b_2y_2 + \ldots + b_my_m$

Subject to constraints

$a_{11}y_1 + a_{12}y_2 + \ldots + a_{m1}y_m \geq c_1$

$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_my \geq c_2$

....

....

$a_{1n}y_1 + a_{2n}y_2 + \ldots, + a_{mn}y_m \leq c_n$

where $y_1, y_2, \ldots y_m \geq 0$.

The following example makes the above method clear.

EXAMPLE 10.17: Write the dual of the following LP problem:

Minimize $Z = 2x_1 + 4x_2 + 3x_3$

Subject to the constraints

$3x_1 + 4x_2 + x_3 \geq 11, -2x_1 - 3x_2 + 2x_3 \leq -7,$

$x_1 - 2x_2 - 3x_3 \leq -1, 3x_1 + 2x_2 + 2x_3 \geq 5; x_1, x_2, x_3 \geq 0.$

Solution: The given LPP is

$$\text{Minimize } Z = 2x_1 + 4x_2 + 3x_3. \quad \text{(i)}$$

Subject to the constraints

$$3x_1 + 4x_2 + x_3 \geq 11, \quad \text{(ii)}$$

$$-2x_1 - 3x_2 + 2x_3 \leq -7 \text{ or } 2x_1 + 3x_2 - 2x_3 \geq 7, \quad \text{(iii)}$$

$$x_1 - 2x_2 - 3x_3 \leq -1 \text{ or } -x_1 + 2x_2 + 3x_3 \geq 1, \quad \text{(iv)}$$

$$\text{and} \quad 3x_1 + 2x_2 + 2x_3 \geq 5, \quad \text{(v)}$$

where $x_1, x_2, x_3 \geq 0$.

From (i) to (v), we see that the problem is of the minimization type and all the constraints are of ($\geq$) type.

Let y_1, y_2, y_3 and y_4 be the dual variables associated with the above four constraints. Then, the dual problem is given by

Maximize $W = 11y_1 + 7y_2 + y_3 + 5y_4$

Subject to the constraints

$3y_1 + 2y_2 - y_3 + 3y_4 \leq 2, 4y_1 + 3y_2 + 2y_3 + 2y_4 \leq 4,$ and $y_1 - 2y_2 + 3y_3 + 2y_4 \leq 3,$ where $y_1, y_2, y_3, y_4 \geq 0$.

10.6.2 Formulation of Dual Problem When the Primal has Equality Constraints

Consider the following LPP:

Maximize $Z = c_1x_1 + c_2x_2$

Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 = b_1,\ a_{21}x_1 + a_{22}x_2 \le b_2,\ x_1, x_2 \ge 0.$$

The equality constraints can be written as

$$a_{11}x_1 + a_{12}x_2 \le b_1, \text{ and } a_{11}x_1 + a_{12}x_2 \ge b_1,$$

or $$a_{11}x_1 + a_{12}x_2 \le b_1, \text{ and } -a_{11}x_1 - a_{12}x_2 \le -b_1.$$

Now, the above problem can be written as

Maximize $Z = c_1x_1 + c_2x_2$.

Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 \le b_1,\ -a_{11}x_1 - a_{12}x_2 \le -b_1,\ a_{21}x_1 + a_{22}x_2 \le b_2,$$

where $x_1, x_2 \ge 0$.

Now, construct the dual using y'_1, y''_1, y_2 as the dual variables. Then, the dual problem becomes

Minimize $W = b_1(y'_1 - y''_1) + b_2y_2$.

Subject to the constraints

$$a_{11}(y'_1 - y''_1) + a_{21}y_2 \ge c_1,\ a_{12}(y'_1 - y''_1) + a_{22}y_2 \ge c_2;\ y'_1, y''_1, y_2 \ge 0.$$

The term $(y'_1 - y''_1)$ occurs in the objective function as well as in all the constraints of the dual. This will always happen whenever the primal has an equality constraint. Since y'_1 and y''_1 are non-negative variables, their difference

$y'_1 - y''_1\ (= y_1)$ becomes unrestricted in sign.

Now, the above dual problem reduces to

Minimize $W = b_1y_1 + b_2y_2$.

Subject to the constraints

$a_{11}y_1 + a_{21}y_2 \ge c_1,\ a_{12}y_1 + a_{22}y_2 \ge c_2,\ y_1$ unrestricted in sign, $y_2 \ge 0$.

In general, consider the primal problem

Maximize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_2$.

Subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\ldots\ldots \\ &\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

where $x_1, x_2, \ldots\ldots, x_n \ge 0$. Then the dual of the above problem is

Minimize $W = b_1y_1 + b_2y_2 + \ldots + b_my_m$.

Subject to the constraints

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m &\ge c_1 \\ a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m &\ge c_2 \\ &\ldots\ldots \\ &\ldots\ldots \\ a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m &\ge c_n \end{aligned}$$

where $y_1, y_2, \ldots\ y_m$, are unrestricted in sign.

The following example makes the above method clear.

EXAMPLE 10.18: Write the dual of the problem

Maximize $Z = 4x_1 + 9x_2 + 2x_3$.

Subject to the constraints $2x_1 + 3x_2 + 2x_3 \le 7,\ 3x_1 - 2x_2 + 4x_3 = 5;\ x_1, x_2, x_3 \ge 0$.

Solution: The given LPP is

Maximize $Z = 4x_1 + 9x_2 + 2x_3$. (i)

Subject to the constraints

$2x_1 + 3x_2 + 2x_3 \leq 7$ (ii)

$3x_1 - 2x_2 + 4x_3 = 5$ (iii)

Let y_1 and y_2 be the dual variables associated with the first and second constraints. Then the dual problem is

Minimize $W = 7y_1 + 5y_2$.

Subject to the constraints

$2y_1 + 3y_2 \leq 4,\ 3y_1 - 2y_2 \leq 9,\ \ 2y_1 + 4y_2 \leq 2,$

where $y_1 \geq 0$, y_2 is unrestricted in sign.

10.6.3 Fundamental Theorem of Duality

The fundamental theorem of duality is stated as follows:

If the primal and the dual problems have feasible solutions, then both have optimal solutions and the optimal value of the objective function to the primal is equal to the optimal value of the objective function to the dual problem, that is, Max Z = Min W.

This theorem makes it clear that an optimal solution to the primal problem can directly be found from that to the dual problem and *vice versa*.

Suppose we have found an optimal solution to the dual problem using the simplex method. The rules to obtain an optimal solution to the primal problem from that to the dual problem are as follows:

1. If the primal variable corresponds to a slack variable in the dual problem, then the optimal value of the primal is directly obtained from the coefficient of the slack variable, with changed sign in the C_j row of the final simplex table of the dual.
2. If the primal variable corresponds to an artificial variable in the dual problem, then the optimal value of the primal is directly obtained from the co-efficient of the artificial variable, with changed sign, in the C_j row of the final simplex table of the dual, after deleting the constant M and *vice versa*.

Note that we can obtain an optimal solution to the dual problem from that to the primal problem.

If the primal problem has an unbounded solution, then the dual problem will not have a feasible solution and *vice versa*.

The following examples will be helpful in solving the LP problems based on duality principle of the LPP.

EXAMPLE 10.19: Construct the dual of the following problem and solve both the primal and dual:

Maximize $Z = 2x_1 + x_2$.

Subject to the constraints $-x_1 + 2x_2 \leq 2,\ x_1 + x_2 \leq 4,\ x_1 \leq 3;\ x_1, x_2 \geq 0$.

Solution: The given LP problem is

Maximize $Z = 2x_1 + x_2$. (i)

Subject to $-x_1 + 2x_2 \leq 2$, (ii)

$x_1 + x_2 \leq 4$, and (iii)
$x_2 \leq 3$,
where $x_1, x_2 \geq 4$ (iv)

For Primal Problem: From the given problem as in equations [(i to iv)], we see that there are only two variables involved. It is convenient to solve the problem graphically.

In the x_1, x_2 plane, the three constraints [(ii to iv)] are shown in the shaded region OABCD of figure 10.6. The corner points of the region OABCD are O (0, 0), A(3, 0), B(3, 1), C(2, 2), and D(0, 1). Hence, the values of the objective function at these conrer points are Z(0) = 0, Z(A) = 6, Z(B) = 7, Z(C) = 6 and Z(D) = 1. Hence, the optimal solution is $x_1 = 3$, $x_2 = 1$ and max Z = 7.

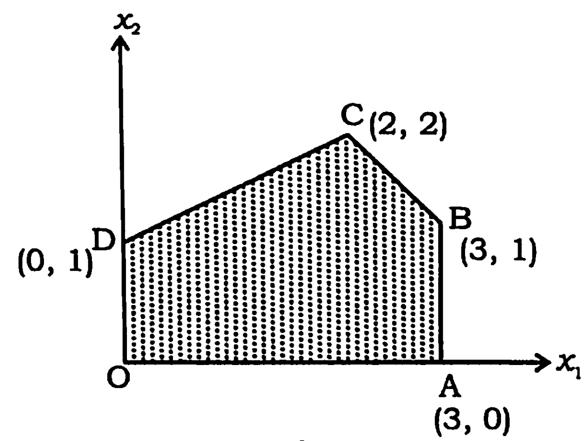

Fig. 10.6

For Dual Problem: The dual problem of the primal becomes.

Minimize $W = 2y_1 + 4y_2 + 3y_3$.

Subject to $-y_1 + y_2 + y_3 \geq 2$, $2y_1 + y_2 \geq 1$; $y_1, y_2 \geq 0$.

Introducing the slack and the artificial variables, the dual problem in the standard form is given as follows:

Max. $W' = -2y_1 - 4y_2 - 3y_1 + 0s_1 + 0s_2 - MA_1 - MA_2$

Subject to

$-y_1 + y_2 + y_3 - s_1 + A_1 + 0.A_2 = 2$, and

$2y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1$

where $y_1, y_2, y_3, s_1, s_2, A_1, A_2 \geq 0$

For finding initial basic feasible solution, setting the non-basic variables y_1, y_2, y_3, s_1, s_2 each equal to zero, we get the initial basic feasible solution as

$y_1 = y_2 = y_3 = s_1 = s_2 = 0$ (non-basic); $A_1 = 2$, $A_2 = 1$ (basic). Therefore, the initial simplex table is given in table-I as follows:

Table-I

$C_j \rightarrow$			−2	−4	−3	0	0	−M	−M	Ratio
C_B	Basis	Solution ($b = y_B$)	y_1	y_2	y_3	s_1	s_2	A_1	A_2	y_B/y_2
−M	A_1	2	−1	1	1	−1	0	1	0	2/1=2
−M	A_2	1	2	1	0	0	−1	0	1	1/1=1 →
W' = −3M		W'_j	−M	−2M	−M	M	M	−M	−M	
		$C_j = c_j - W'_j$	M−2	2M−4 ↑	M−3	−M	−M	0	0	

Since C_j is positive under some columns, the initial solution is not optimal and we proceed to the next step. Introduce y_2 and drop A_2. Then the new simplex table is given in table-II as follows:

Table-II

$C_j \rightarrow$			-2	-4	-3	0	0	$-M$	$-M$	Ratio
C_B	Basis	Solution ($b = Y_B$)	y_1	y_2	y_3	s_1	s_2	A_1	A_2	y_B/y_3
$-M$	A_1	1	-3	0	1	-1	1	1	-1	$1/1=1 \rightarrow$
-4	y_2	1	2	1	0	0	-1	0	1	$1/0 = \infty$
$W' = M-4$		W'_j	$3M-8$	-4	$-M$	M	$4-M$	$-M$	$M-4$	
		$C_j = c_j - W'_j$	$6-3M$	0	$M-3 \uparrow$	$-M$	$M-4$	0	$4-2M$	

Since C_j is positive under some columns, this solution is not optimal and we proceed further.

Now, introduce y_3 and drop A_1. Then the revised simplex table is given in table-III as follows:

Table-III

$C_j \rightarrow$			-2	-4	-3	0	0	$-M$	$-M$
C_B	Basis	Solution ($b = y_B$)	y_1	y_2	y_3	s_1	s_2	A_1	A_2
-3	y_2	1	-3	0	1	-1	1	1	-1
-4	y_3	1	2	1	0	0	-1	0	1
$w' = -7$		W'_j	1	-4	-3	3	1	-3	-1
		$C_j = c_j - W'_j$	-3	0	0	-3	-1	$3-M$	$1-M$

Since all $C_j \leq 0$, the optimal solution is attained. Thus, an optimal solution to the dual problem is $y_1 = 0$, $y_2 = 1$, $y_3 = 1$ and Min W = – Max W' = 7.

To derive the optimal basic feasible solution to the primal problem, we note that the primal variables x_1 and x_2 correspond to the artificial dual variables A_1 and A_2 respetively. In the final simplex table (= table-III) of the dual problem, C_j corresponding to A_1 and A_2, with changed sign are 3 and 1 respectively after ignoring M. Thus, $x_1 = 3$ and $x_2 = 1$. Hence, an optimal basic feasible solution to the given primal is $x_1 = 3$, $x_2 = 1$ and Max Z = 7.

EXAMPLE 10.20: Using the duality principle, solve the following LP problem:
Minimize $Z = 0.7x_1 + 0.5x_2$.
Subject to the constraints $x_1 \geq 4$, $x_2 \geq 6$, $x_1 + 2x_2 \geq 20$, $2x_1 + x_2 \geq 18$; $x_1, x_2 \geq 0$.

Solution: The given LP problem is
Minimize $Z = 0.7x_1 + 0.5x_2$.
Subject to the constraints
$$x_1 \geq 4,\ x_2 \geq 6,\ x_1 + 2x_2 \geq 20, \text{ and } 2x_1 + x_2 \geq 18,$$
where $x_1, x_2 \geq 0$.
The dual of the given problem is
Max. $W = 4y_1 + 6y_2 + 20y_3 + 18y_4$.
Subject to $y_1 + y_3 + 2y_4 \leq 0.7$, $y_2 + 2y_3 + y_4 \leq 0.5$, $y_1, y_2, y_3, y_4 \geq 0$.
Introducing the slack variables, the dual problem in the standard form is given as follows:
Max. $W = 4y_1 + 6y_2 + 20y_3 + 18y_4 + 0s_1 + 0s_2$.
Subject to
$$y_1 + 0y_2 + y_3 + 2y_4 + s_1 + 0s_2 = 0.7, \text{ and}$$
$$0y_1 + y_2 + 2y_3 + y_4 + 0s_1 + s_2 = 0.5$$
where $y_1, y_2, y_3, y_4 \geq 0$.

Setting the non-basic variables y_1, y_2, y_3, y_4 each equal to zero, we get the initial basic feasible solution as $y_1 = y_2 = y_3 = y_4 = 0$ (non-basic); $s_1 = 0.7$, $s_2 = 0.5$ (basic).

Since the basic variables $s_1, s_2 \geq 0$, the initial basic solution is feasible and non-degenerate. Therefore, the initial simplex table is given in table-I as follows:

Table-I

$C_j \rightarrow$			4	6	20	18	0	0	Ratio
C_B	Basis	Solution ($b = y_B$)	y_1	y_2	y_3	y_4	s_1	s_2	y_B/y_3
0	s_1	0.7	1	0	1	2	1	0	0.7/1
0	s_2	0.5	0	1	2	1	0	1	0.5/2 $\rightarrow$
W = 0		Z_j	0	0	0	0	0	0	
		$C_j = c_j - Z_j$	4	6	20 $\uparrow$	18	0	0	

Since C_j is positive under some columns, the initial solution is not optimal and we proceed to the next step. Introduce y_3 and drop s_2. Then the new simplex table is given in table-II as follows:

Table-II

$C_j \rightarrow$			4	6	20	18	0	0	Ratio
C_B	Basis	Solution ($b = y_B$)	y_1	y_2	y_3	y_4	s_1	s_2	y_B/y_4
0	s_1	9/20	1	−1/2	0	3/2	1	−1/2	3/10 →
20	y_3	1/4	0	1/2	1	1/2	0	1/2	1/2
W = 5		Z_j	0	10	20	10	0	10	
		$C_j = c_j - Z_j$	4	−4	0	8 ↑	0	−10	

Since C_j is positive under some columns, the initial solution is not optimal and we proceed to the next step. Introduce y_4 and drop s_1. Then the new simplex table is given in table-III as follows:

Table-III

$C_j \rightarrow$			4	6	20	18	0	0
C_B	Basis	Solution ($b = y_B$)	y_1	y_2	y_3	y_4	s_1	s_2
18	y_4	3/10	2/3	−1/3	0	1	2/3	−1/3
20	y_3	1/10	−1/3	2/3	1	0	−1/3	2/3
W = 74/10 =7.4		Z_j	16/3	22/3	20	18	16/3	22/3
		$C_j = c_j - Z_j$	−4/3	−4/3	0	0	−16/3	−22/3

Since all $C_j \leq 0$, the optimal solution is attained. Thus, an optimal solution to the dual problem is $y_1 = 0$, $y_2 = 0$, $y_3 = 20$, $y_4 = 18$, and max W = 7.4.

To derive the optimal basic feasible solution to the primal problem, we note that the primal variables x_1 and x_2 correspond to the slack starting dual variables s_1 and s_2 respectively. In the final simplex table (= table-III) of the dual problem, C_j corresponding to s_1 and s_2, with changed sign are $\frac{16}{3}$ and $\frac{22}{3}$ respectively. Thus, $x_1 = \frac{16}{3}$ and $x_2 = \frac{22}{3}$. Hence, an optimal basic feasible solution to the given primal is $x_1 = \frac{16}{3}$, $x_2 = \frac{22}{3}$ and min Z = 7.4.

EXERCISE 10.4

1. Write the dual of the following LP problems:

(i) Minimize $Z = 3x_1 - 2x_2 + 4x_3$
Subject to $3x_1 + 5x_2 + 4x_3 \geq 7$, $6x_1 + x_2 + 3x_3 \geq 4$, $7x_1 - 2x_2 - x_3 \leq 10$, $x_1 - 2x_2 + 5x_3 \geq 3$, $4x_1 + 7x_2 - 2x_3 \geq 2$; $x_1, x_2, x_3 \geq 0$.

(ii) Minimize $Z = 3x_1 - 3x_2 + x_3$
Subject to $2x_1 - 3x_2 + x_3 \leq 5$, $4x_1 - 2x_2 \geq 9$, $-8x_1 - 4x_2 + 3x_3 = 8$; $x_1, x_2 \geq 0$, and x_3 is unrestricted.

(iii) Maximize $Z = 5x_1 + 3x_2$
Subject to $x_1 + x_2 \leq 2$, $5x_1 + 2x_2 \leq 10$, $3x_1 + 8x_2 \leq 12$; $x_1, x_2 \geq 0$.

(iv) Maximize $Z = 10x_1 + 13x_2 + 19x_3$
Subject to $6x_1 + 5x_2 + 3x_3 \leq 26$, $4x_1 + 2x_2 + 5x_3 \leq 7$; $x_1, x_2, x_3 \geq 0$.

(v) Maximize $Z = 4x_1 + 9x_2 + 2x_3$
Subject to $2x_1 + 3x_2 + 2x_3 \leq 7$, $3x_1 - 2x_2 + 4x_3 = 5$; $x_1, x_2, x_3 \geq 0$.

(vi) Minimize $Z = 2x_1 + 4x_2 + 3x_3$
Subject to $3x_1 + 4x_2 + x_3 \geq 11$, $-2x_1 - 3x_2 + 2x_3 \leq -7$, $x_1 - 2x_2 - 3x_3 \leq -1$, $3x_1 + 2x_2 + 2x_3 \geq 5$; $x_1, x_2, x_3 \geq 0$.

(vii) Maximize $Z = 3x_1 + 16x_2 + 7x_3$
Subject to $x_1 - x_2 + x_3 \geq 3$, $-3x_1 + 2x_3 \leq 1$; $2x_1 + x_2 - x_3 = 4$; $x_1, x_2, x_3 \geq 0$.

(viii) Maximize $Z = 3x_1 + 16x_2 + 7x_3$
Subject to $x_1 - x_2 + x_3 \geq 3$, $-3x_1 + 2x_3 \leq 1$, $2x_1 + x_2 - x_3 = 4$; $x_1, x_2, x_3 \geq 0$.

2. Obtain the dual problem of the following LP problem:
Maximize $f(x) = 2x_1 + 5x_2 + 6x_3$
Subject to $5x_1 + 6x_2 - x_3 \leq 3$, $-2x_1 + x_2 + 4x_3 \leq 4$, $x_1 - 5x_2 + 3x_3 \leq 1$, $-3x_1 - 3x_2 + 7x_3 \leq 6$; $x_1, x_2, x_3 \geq 0$.
Also, verify that the dual of the dual problem is the primal problem.

ANSWER

1. (i) Max. $W = 7y_1 + 4y_2 + 10y_3 + 3y_4 + 2y_5$ Subject to $3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3$, $5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$, $4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$; $y_1, y_2, y_3, y_4, y_5 \geq 0$.

(ii) Max. $W = -5y_1 + 9y_2 + 8y_3$ Subject to $-2y_1 + 4y_2 - 8y_3 \leq 3$, $3y_1 - 2y_2 + 4y_3 \leq -2$, $-y_1 + 3y_3 = 1$; $y_1, y_2 \geq 0$, y_3 is unrestricted.

(iii) Min. $W = 2y_1 + 10y_2 + 12y_3$ Subject to $y_1 + 5y_2 + 3y_3 \geq 5$, $y_1 + 2y_2 + 8y_3 \geq 3$; $y_1, y_2, y_3 \geq 0$.

(iv) Min. $W = 26y_1 + 7y_2$ Subject to $6y_1 + 4y_2 \geq 10$, $5y_1 + 2y_2 \geq 13$, $3y_1 + 5y_2 \geq 19$; $y_1; y_2 \geq 0$.

(v) Min. $W = 7y_1 + 5y_2$ Subject to $2y_1 + 3y_2 \leq 4$, $3y_1 - 2y_2 \leq 9$, $2y_1 + 4y_2 \leq 2$; $y_1 \geq 0$, y_2 is unrestricted in sign.

(vi) Max. $W = 11y_1 + 7y_2 + y_3 + 5y_4$ Subject to $3y_1 + 2y_2 - y_3 + 3y_4 \leq 2$, $4y_1 + 3y_2 + 2y_3 + 2y_4 \leq 4$, $y_1 - 2y_2 + 3y_3 + 2y_4 \leq 3$; $y_1, y_2, y_3, y_4 \geq 0$.

(vii) Min. $W = -3y_1 + y_2 + 4y_3$ Subject to $y_1 + 3y_2 - 2y_3 \leq -3$, $y_1 + y_3 \geq 16$, $y_1 - 2y_2 + y_3 \leq -7$; $y_1, y_2 \geq 0$, y_3 is unrestricted in sign.

(viii) Min. $W = -3y_1 + y_2 + 4y_3$ Subject to $y_1 + 3y_2 - 2y_3 \le -3$, $y_1 + y_3 \ge 16$, $y_1 - 2y_2 + y_3 \le -7$; $y_1, y_2 \ge 0$. y_3 is unrestricted in sign.

2. Min. $y = 3y_1 + 4y_2 + y_3 + 6y_4$ subject to $5y_1 - 2y_2 + y_3 - 3y_4 \ge 2$, $6y_1 + y_2 - 5y_3 - 3y_4 \ge 5$, $-y_1 + 4y_2 + 3y_3 + 7y_4 \ge 6$; $y_1, y_2, y_3, y_4 \ge 0$.

10.7 Dual Simplex Method

The dual simplex method is similar to the regular simplex method. The only difference lies in the criterion used for selecting the incoming and outgoing variables. In the dual simplex method, we first determine the outgoing variable and then the incoming variable while in the case of the regular simplex method, the reverse is done. Working procedure for dual simplex method are given as follows:

Step I: Check whether the objective function is to be maximized or minimized, then convert it into maximization form.

Step II: (i) Convert ($\ge$) type constraints, if any, into ($\le$) type by multiplying such constraint by -1.

(ii) Express the problem in standard form by introducing slack variables.

Step III: Find the initial basic solution and express this information in the form of dual simplex table.

Step IV: Compute $C_j = c_j - Z_j$.

(i) If all $C_j \le 0$ and at least one $b_i < 0$, then proceed to the next step.

(ii) If all $C_j \le 0$ and all $b_i \ge 0$, then the optimal basic feasible solution has been obtained.

(iii) If any $C_j > 0$, then the method fails.

Step V: Select the outgoing variable. Select the row that contains the most negative b_i. This row is the key row and the corresponding basic variable is the outgoing variable.

Step VI: Test the nature of key row elements:

(i) If all the elements of the key row are positive or zero, then the problem does not have a feasible solution.

(ii) If at least one element in the key row is negative, then determine the ratios of the corresponding elements of the C_j row to these elements. Then, select the smallest of these ratios (Ignore the ratio associated with positive or zero element of the key row). The corresponding column is the key column and the associated variable is the incoming variable.

Step VII: Iterate towards optimal feasible solution. Encircle the key element and make it unity. Perform row operations as in the regular simplex method and repeat iterations until either an optimal feasible solution is attained or there is an indication of the non-existence of a feasible solution.

The following examples make the dual simplex method clear.

EXAMPLE 10.21: Using the dual simplex method, solve the following LP problems:

Maximizer $Z = -3x_1 - x_2$

Subject to $x_1 + x_2 \ge 1$, $2x_1 + 3x_2 \ge 2$; $x_1, x_2 \ge 0$.

Solution: The given LP problem is

Maximize $Z = -3x_1 - x_2$

Subject to $x_1 + x_2 \geq 1, 2x_1 + 3x_2 \geq 2; x_1, x_2 \geq 0.$

Converting ($\geq$) type constraints into ($\leq$) type, the given LPP takes the form

Max. $Z = -3x_1 - x_2$

Subject to $-x_1 - x_2 \leq -1, -2x_1 - 3x_2 \leq -2; x_1, x_2 \geq 0.$

Express the LPP in standard form by introducing slack variables s_1, s_2 (one for each constraint). Thus the problem becomes

Max. $Z = -3x_1 - x_2 + 0s_1 + 0s_2$

Subject to $-x_1 - x_2 + s_1 = -1, -2x_1 - 3x_2 + s_2 = -2$

where $x_1, x_2, s_1, s_2 \geq 0$.

To find initial basic feasible solution, set the non-basic vartiables $x_1 = x_2 = 0$, so that $s_1 = -1$, $s_2 = -2$. The initial dual simplex table is given in table-I as follows:

Table-I

$C_j \rightarrow$			-3	-1	0	0	Ratio
C_B	Basis	Solution ($b = y_B$)	x_1	x_2	s_1	s_2	X_B/x_2
0	s_1	-1	-1	-1	1	0	1
0	s_2	-2	-2	-3	0	1	2/3 →
Z = 0		Z_j	0	0	0	0	
		$C_j = c_j - Z_j$	-3	-1 ↑	0	0	

Since all $C_j \leq 0$ and $b_i \leq 0$, the initial solution is optimal but infeasible and we proceed to the next step.

The negative and numerically largest b_i is $b_2 = -2$. Thus, the second row is the key row and s_2 is the outgoing variable. Also, – 3 is the key element and the x_2 column is the key column. Drop s_2 and introduce s_2 with its associated value –1 under the C_B column. Convert the key element to unity and make all other elements of the key column zero. The second basic feasible solution is given in table-II as follows :

Table-II

$C_j \rightarrow$			-3	-1	0	0
C_B	Basis	Solution ($b = X_B$)	x_1	x_2	s_1	s_2
0	s_1	-1/3	-1/3	0	1	-1/3
-1	x_2	2/3	2/3	1	0	-1/3
Z = -2/3		Z_j	-2/3	-1	0	1/3
		$C_j = c_j - Z_j$	-7/3	0	0	-1/3 ↑

Since all $C_j \leq 0$ but $b_1 = -\frac{1}{3} < 0$, the current solution is is not optimal. Therefore, an improvement in the value of Z is possible and we proceed to the next step.

The first row is the key row and s_1 is the outgoing variable. The ratios of the elements in the C_j row to the corresponding negative elements of the key row are

$$\frac{\frac{-7}{3}}{\frac{-1}{3}} = 7, \quad \frac{\frac{-1}{3}}{\frac{-1}{3}} = 1.$$

Since the smallest ratio is 1, the s_2 column is the key column and s_2 is the incoming variable. The key element is $-\frac{1}{3}$, shown in circle.

Drop s_1 and introduce s_2. Convert the key element to unity and make all other elements of the key column zero. The third basic feasible solution is given in table-III as follows :

Table-III

$C_j \rightarrow$			−3	−1	0	0
C_B	Basis	solution ($b = X_B$)	x_1	x_2	s_1	s_2
0	s_2	1	1	0	−3	1
−1	x_2	1	1	1	−1	0
Z = −1		Z_j	−1	−1	1	0
		$C_j = c_j - Z_j$	−2	0	−1	0

Here, all $C_j \leq 0$ and all $b_i > 0$. The optimal feasible solution has been attained. Thus, the optimal feasible solution is $x_1 = 0$, $x_2 = 1$, and max. Z = −1.

EXAMPLE 10.22: Using the dual simplex method, solve the following LP problem:

Minimize $Z = 2x_1 + 2x_2 + 4x_3$

Subject to $2x_1 + 3x_2 + 5x_3 \geq 2$, $3x_1 + x_2 + 7x_3 \leq 3$, $x_1 + 4x_2 + 6x_3 \leq 5$; $x_1, x_2, x_3 \geq 0$.

Solution: The given LP problem is

Minimize $Z = 2x_1 + 2x_2 + 4x_3$.

Subject to $2x_1 + 3x_2 + 5x_3 \geq 2$, $3x_1 + x_2 + 7x_3 \leq 3$, $x_1 + 4x_2 + 6x_3 \leq 5$; $x_1, x_2, x_3 \geq 0$, which on converting the problem maximization and the first constraint into ($\leq$) type gives

Maximize $Z' = -2x_1 - 2x_2 - 4x_3$

Subject to $-2x_1 - 3x_2 - 5x_3 \leq -2$, $3x_1 + x_2 + 7x_3 \leq 3$, $x_1 + 4x_2 + 6x_3 \leq 5$; $x_1, x_2, x_3 > 0$

Express the LPP in standard form by introducing slack variables s_1, s_2, s_3 (one for each constraint). Thus, the problem becomes

Max $Z' = -2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3$
Subject to $-2x_1 - 3x_2 - 5x_3 + s_1 = -2$, $3x_1 + x_2 + 7x_3 + s_2 = 3$
$x_1 + 4x_2 + 6x_3 + s_3 = 5$; $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.

To find the initial basic feasible solution, set the non-basic variables $x_1 = x_2 = x_3 = 0$, so that $s_1 = -2$, $s_2 = 3$ and $s_3 = 5$ (basic). The initial dual simplex table is given in table-I as follows:

Table-I

$C_j \rightarrow$			-2	-2	-4	0	0	0
C_B	Basis	Solution ($b = X_B$)	x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	-2	-2	-3	-5	1	0	0
0	s_2	3	3	1	7	0	1	0
0	s_3	5	1	4	6	0	0	1
$Z' = 0$		Z_j	0	0	0	0	0	0
		$C_j = c_j - Z'_j$	-2	-2↑	-4	0	0	

Since all C_j values are less than or equal to zero and $b_1 = -2$, the initial solution is optimal but infeasible. Since $b_1 < 0$, the first row is the key row and s_1 is the outgoing variable.

Compute the ratios of the elements of the C_j row to the corresponding negative elements of the key row. These ratios are $\frac{-2}{-2} = 1, \frac{-2}{-3} = 0.67, \frac{-4}{-5} = 0.8$. Since 0.67 is the smallest ratio, the x_2 column is the key column. The key element is –3.

Drop s_1 and introduce x_2 with its associate value –2 under the C_B column. The revised dual simplex table is given in table-II as follows:

Table-II

$C_j \rightarrow$			-2	-2	-4	0	0	0
C_B	Basis	Solution ($b = X_B$)	x_1	x_2	x_3	s_1	s_2	s_3
-2	x_2	2/3	2/3	1	5/3	-1/3	0	0
0	s_2	7/3	7/3	0	16/3	1/3	1	0
0	s_3	7/3	-5/3	0	-2/3	4/3	0	1
$Z' = -4/3$		Z'_j	-4/3	-2	-10/3	2/3	0	0
		$C_j = c_j - Z'_j$	-2/3	0	-2/3	-2/3	0	0

Since all $C_j < 0$ and all b_i' s are greater than 0, this solution is optimal and feasible.

Hence, the optimal solution is $x_1 = 0$, $x_2 = \frac{2}{3}$, $x_3 = 0$, and max $Z' = -\frac{4}{3}$, that is, min

$Z = \frac{4}{3}$.

EXERCISE 10.5

1. Using dual simplex method, solve the following LP problems :
   (i) Maximize $Z = -4x_1 - 6x_2 - 18x_3$
   Subject to $x_1 + 3x_3 \geq 3$, $x_2 + 2x_3 \geq 5$; $x_1, x_2, x_3 \geq 0$.
   (ii) Maximize $Z = -3x_1 - 2x_2$
   Subject to $x_1 + x_2 \geq 1$, $x_1 + x_2 \leq 7$, $x_1 + 2x_2 \geq 3$; $x_1, x_2 \geq 0$.
   (iii) Minimize $Z = x_1 + 2x_2 + x_3 + 4x_4$
   Subject to $2x_1 + 4x_2 + 5x_3 + x_4 \geq 10$, $3x_1 - x_2 + 7x_3 - 2x_4 \geq 2$, $5x_1 + 2x_2 + x_3 + 6x_4 \geq 15$; $x_1, x_2, x_3, x_4 \geq 0$.
   (iv) Minimize $Z = 2x_1 + x_2$,
   Subject to $3x_1 + x_2 \geq 3$, $4x_1 + 3x_2 \geq 6$, $x_1 + 2x_2 \leq 3$; $x_1, x_2 \geq 0$.
   (v) Minimize $Z = 20\, x_1 + 16x_2$
   Subject to $x_1 + x_2 \geq 12$, $2x_1 + x_2 \geq 17$, $x_1 \geq \frac{5}{2}$, $x_2 \geq 6$; $x_1, x_2 \geq 0$.
   (vi) Maximize $Z = -\, 3x_1 - 2x_2$,
   Subject to $x_1 + x_2 \geq 1$, $x_1 + x_2 \leq 7$, $x_1 + 2x_2 \geq 10$, $x_2 \geq 3$; $x_1, x_2 \geq 0$.
   (vii) Maximize $Z = 3x_1 - x_2$
   Subject to $x_1 + x_2 \geq 1$, $2x_1 + 3x_2 \geq 2$; $x_1, x_2 \geq 0$.
   (viii) Minimize $Z = x_1 + 2x_2 + 3x_3$
   Subject to $2x_1 - x_2 + x_3 \geq 4$, $x_1 + x_2 + 2x_3 \leq 8$, $x_2 - x_3 \geq 2$; $x_1, x_2, x_3 > 0$.

ANSWER

1. (i) $x_1 = 0, x_2 = 3, x_3 = 1$; Max. $Z = -36$
   (ii) $x_1 = 4, x_2 = 3$, Max. $Z = -18$
   (iii) $x_1 = \frac{65}{23}$, $x_2 = 0$, $x_3 = \frac{20}{23}$, $x_4 = 0$, Min. $Z = \frac{215}{23}$
   (iv) $x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$, Min. $Z = \frac{12}{5}$
   (v) $x_1 = 5, x_2 = 7$, Min. $Z = 212$
   (vi) $x_1 = 4, x_2 = 3$, Max. $Z = -18$
   (vii) $x_1 = 0, x_2 = 1$, Min. $Z = -1$
   (viii) $x_1 = 6, x_2 = 2, x_3 = 0$, Min. $Z = 10$.

10.8 Transportation Problem

Transportation problem is a special kind of LPP in which goods are transported from a set of sources to a set of destinations subject to the supply and demand of the sources and destination respectively such that the total cost of transportation is minimized. It is also sometimes called the Hitchcock problem.

There are two types of transportation problems. One is balanced: when both supply and demand are equal then the problem is said to be a balanced transportation problem; and the other is unbalanced: when the supply and the demand are not equal then it is said to be an unbalanced tranportation problem. In this type of problem, either a dummy row or a dummy column is added according to the requirement to make it a balanced problem. Then it can be solved similar to the balanced problem.

Also, a special class of LPP in which our aim is to transport a single product from various production units, called the origins, to different locations, called the destinations, at a minimum cost is called a transportation problem.

10.8.1 Formulation of a Transportation Problem

Suppose there are m origins (plant locations) and n destinations (distribution centres). Let a_i be the quantity of the product available at the i^{th} origin and let b_i be the quantity of the product required at the j^{th} destination. Suppose that the cost of the transportation of one unit of the product from the i^{th} origin to j^{th} destination is C_{ij}. Our objective is to determine the number of units to be transported from the i^{th} origin to the j^{th} destination so that the total transportation cost is minimum.

Let x_{ij} be the number of units trasported from i^{th} origin to the j^{th} destination. Then, the general transportation problem is given by

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}.$$

Subject to the constraints

$$x_{i1} + x_{i2} + \ldots + x_{in} = a_i, \text{ for } i^{th} \text{ origin } (i = 1, 2, \ldots, m), \text{ and}$$

$$x_{1j} + a_{2j} + \ldots + x_{mj} = b_j, \text{ for } j^{th} \text{ destination } (j = 1, 2, \ldots, n)$$

where $x_{ij} \geq 0$

Some useful definitions which may be helpful in solving the transportation problems are given one by one as follows:

1. The two sets of constraints will be consistent if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

which is the condition for a transportation problem to have a feasible solution. The problems which satisfy this condition are called balanced transportation problems.

2. A feasible solution to a transportation problem is said to be a basic feasible solution if it contains at most $(m + n - 1)$ strictly positive allocations, otherwise the solution will be degenerate. If the total number of positive (non-zero) allocations is exactly $(m + n - 1)$, then the basic feasible solution is said to be non-degenerate.
3. A feasible solution which minimizes the transportation cost is called an optimal solution. This problem is explicitly represented in the transportation table-I.

The mn squares shown above in table-I are called cells. The transportation cost c_{ij} of one unit from the i^{th} origin to the j^{th} destination is shown in the lower-right side of the $(i, j)^{th}$ cell. The quantity x_{ij} transported from the i^{th} origin to the j^{th} destination is displayed in the small square at the upper-left side of the $(i, j)^{th}$ cell. The supplies a_i are shown as the column on the RHS of mn cell while the demand b_j are shown as a row below the mn cell. The various a_i's and b_j's are called rim requirements. A feasible solution to a transportation problem is a set of positive allocations x_{ij} that satisfy the rim requirements. The feasibility of a solution can be verified by summing up the values of x_{ij} along the rows and down the columns.

Table-I

Distribution centres (destination)

		1	2		j		n	Supply
	1	c_{11}	c_{12}		c_{1j}		c_{1n}	a_1
	2	c_{21}	c_{22}		c_{2j}		c_{2n}	a_2
Plants								⋮
(Origins)	i	c_{i1}	c_{i2}		c_{ij}		c_{in}	a_i
								⋮
	m	c_{m1}	c_{m2}		c_{mj}		c_{mn}	a_m
Demand		b_1	b_2	………	b_j	………	b_n	$\sum a_i = \sum b_j$

The special characteristics of a transportation problem are as follows:

(i) The coefficients of all x_{ij} in the constraints are unity, and

(ii) The total supply $\sum a_i$ = total demand $\sum b_j$

Note that if the objective function and the constraints are all linear, the problem can be solved by simplex method. But the number of variables being large, there will be too many calculations. However, the coefficient of all x_{ij} in the constraints being unity, we can look for some technique which is easier than the simplex method.

10.8.2 Working Procedure for Transportation Problem

Following are the steps in the working procedure for transportation problem:

Step I. Construct the transportation table. Express the supply from the origins a_i, demand at destinations b_j and the transportation cost C_{ij} per unit in the form of a matrix, which is called the transportation table. If the supply and demand are equal, the problem is balanced.

Step II. Find the initial basic feasible solution. We find an initial allocation which satisfies the demand at each project site without violating the capacities of the plants (origins) and also meeting the non-negativity restrictions. There are various methods to find initial allocations such as the North-West corner rule, the row minima method, the least cost method and Vogel's approximation method (VAM). In VAM, we consider the least cost C_{ij} as well as the costs that just exceed the least cost C_{ij}. Thus, the initial solution obtained from this method is better than that obtained from the other methods. We will restrict ourselves to the VAM method only, which consists of the following steps.

(i) Find the difference between the least and the next least costs in each row and display it in brackets to the right of that row. In a similar way, display in brackets the difference between the least and the next least cost in each column below that column.

(ii) Identify the row or column with the largest difference among all the rows and columns and allocate as much as possible under the rim requirements, to the lowest-cost cell in that row or column. In case of a tie, allocate to the cell associated with the lower cost.

If the greatest difference corresponds to the i^{th} row and C_{ij} is the lowest cost in the i^{th} row, allocate as much as possible, that is, $\min(a_i, b_j)$ in the $(i, j)^{th}$ cell and cross off the i^{th} row or the j^{th} column.

(iii) Recalculate the row and column differences for the reduced table and go to that previous step.

(iv) Continue the process till all the rim requirements are satisfied. Note the solution in the upper-left corner of small squares of the basic cells.

Step III: Test for optimality: The number of allocations should be $m + n - 1$ in the solution obtained above, otherwise the basic solution degenerates. To test the optimality of the basic feasible solution, we apply the modified distribution (MODI) method and examine each unoccupied cell to determine whether making an allocation in it reduces the total transportation cost and then repeat this procedure until the lowest possible transportation cost is obtained. The steps involved in this method are as follows:

(i) Note the numbers u_i along the left and v_j along the top of the cost matrix such that their sums equal to the original costs of the occupied cell, that is, solve the equations $[u_i + v_j = c_{ij}]$, starting initially with some $u_i = 0$

(ii) Calculate the net evaluations $\omega_{ij} = u_i + v_j - c_{ij}$ for all the empty (unoccupied) cells and enter them in the upper-right hand corners of the corresponding cells.

(iii) Test the sign of each ω_{ij}. If all $\omega_{ij} \leq 0$, then the current basic feasible solution is optimal. If at least one of ω_{ij} is positive, then this solution is not optimal and we proceed to step IV.

Step IV: Iterate towards optimal solution:

(i) Choose the unoccupied cell with the largest positve ω_{ij} and mark θ inside this cell and call it as θ-cell. In case of the tie, select the one with lower original cost and mark θ on it.

(ii) Draw a closed path consisting of horizontal and vertical lines beginning and ending at θ-cell and having its other corners at the allocated cells.

(iii) Add and subtract θ alternately to and from the transition cells of the loop, subject to rim requirements. Assign a maximum value to θ in such a way that one basic variable becomes zero and the other basic variables remain non-negative. Now, the basic cell whose allocation has been reduced to zero leaves the basis.

Step V: Return to step III and repeat the process until an optimal basic feasible solution is obtained.

The following examples make the above method clear:

EXAMPLE 10.23: Solve the following transportation problem:

		Destination					
		A	**B**	**C**	**D**		
Source	I	21	16	25	13	11	**Availability**
	II	17	18	14	23	13	
	III	32	27	18	41	19	
	Requirement	6	10	12	15	43	

Solution: The given transportation problem is

		Destination					
		A	**B**	**C**	**D**		
Source	I	21	16	25	13	11	**Availability**
	II	17	18	14	23	13	
	III	32	27	18	41	19	
	Requirement	6	10	12	15	43	

From the given problem, we see that the total availability and the total requirement being the same, that is, 43, the problem is balanced. By VAM, the difference between the smallest and the next smallest cost in each row and each column are computed and written within brackets against the respective rows and columns of table-1.

Table-I

21	16	25	$\boxed{11}$ 13	11, (16 − 13 = 3)
17	18	14	23	13, (17 − 14 = 3)
32	27	18	41	19, (27 − 18 = 9)
6	10	12	15	
(21−17 =4)	(18−16 =2)	(18−14 =4)	(23−13 =10)	

From table-I, we see that the largest difference is 10, which is associated with the 4th column. Since c_{14} (= 13) is the minimum cost, we allocate x_{14} = min (11, 15) = 11. This exhausts the availability of the 1st row and, therefore, we cross it and write table-II as follows :

Table-II

17	18	14	$\boxed{4}$ 23	13, (17 − 14 = 3)
32	27	18	41	19, (27 − 18 = 9)
6	10	12	4=(15−11)	
(32−17 =15)	(27−18 =9)	(18−14 =4)	(41−23 =18)	

From table-II, we see that the largest difference is 18, which is associated with the 4th column. Since c_{14} (= 23) is the minimum cost, we allocate x_{14} = min (13, 4) = 4. This exhausts the availability of the 4th column. The reduced table is written in table-III.

Table-III

17	18	14	$\boxed{4}$ 23	13, (17 − 14 = 3)
32	27	18	41	19, (27 − 18 = 9)
6	10	12	4=(15−11)	
(32−17 =15)	(27−18 =9)	(18−14 =4)	(41−23 =18)	

From table-III, we see that the largest difference is 15, which is asociated with the 1st column. In this column, the 1st row corresponds to the lowest cost, that is, c_{11} = 17. So allocate to (1, 1) box with the min (6, 9) = 6 . This exhausts the availability of the 1st column. The reduced table is written in table-IV.

Table-IV

$\boxed{3}$ 18	14	3 = 9 – 6, (18 – 14 = 4)
27	18	19, (27 – 18 = 9)
10 (27–18=9)	12 (18–14=4)	

From table-IV, we see that the largest difference is 9, which occurs at 2 places. So there is a tie here. The costs on both largest differences are identical-choose any one. Allocate (1, 1) box with min (3, 10) = 3 . This exhausts the availability of the 1st row. The reduced table is written in table-V.

Table-V

	$\boxed{7}$ 27	18	19, (27 – 18 = 9)
	10 – 3 = 7 (27 – 0 = 27)	12 (18 – 0 = 18)	

From table-V, we see that the largest difference is 27, which is associated with the 1st column. The 1st column is exhausted. The reduced table is written in table-VI.

Table-VI

$\boxed{12}$ 18	19 – 7 = 12
12	

From table-VI, we see that minimum of 12 and 12 is 12 and this is written in this box.

Finally the initial basic feasible solution is given in table VII.

Table-VII

21	16	25	$\boxed{11}$ 13
$\boxed{6}$ 17	$\boxed{3}$ 18	14	$\boxed{4}$ 23
32	$\boxed{7}$ 27	$\boxed{12}$ 18	41

Checking of optimality

For checking of optimality, we apply Modi method. The number of allocations = m + n – 1 (i.e., 6)

(i) Taking u for row and v for column, we have the following equations:
$u_2 + v_1 = 17$, $u_2 + v_2 = 18$, $u_3 + v_2 = 27$, $u_3 + v_3 = 18$, $u_1 + v_4 = 13$, $u_2 + v_4 = 23$.
Let $u_2 = 0$ in the above relations. Then $v_1 = 17$, $v_2 = 18$, $u_3 = 9$, $v_3 = 9$, $v_4 = 23$, $u_1 = -10$

(ii) Net evaluation $\omega_{ij} = (u_i + v_j) - c_{ij}$ for all empty cells are

$$\omega_{11} = u_1 + v_1 - c_{11} = -10 + 17 - 21 = -14,$$
$$\omega_{12} = u_1 + v_2 - c_{12} = -10 + 18 - 16 = -8,$$
$$\omega_{13} = u_1 + v_3 - c_{13} = -10 + 9 - 25 = -26,$$

$\omega_{23} = u_2 + v_3 - c_{23} = 0 + 9 - 14 = -5,$
$\omega_{31} = u_3 + v_1 - c_{31} = 9 + 17 - 32 = -6,$
$\omega_{34} = u_3 + v_4 - c_{34} = 9 + 23 - 41 = -9.$

Since all the net evaluations are negative, so the current solution is optimal. Hence the optimal allocation is given by

$x_{14} = 11, x_{21} = 6, x_{22} = 3, x_{24} = 4, x_{32} = 7$ and $x_{33} = 12,$

which are represented in table-VIII.

Table-VIII

$u_i \backslash v_j$	17	18	9	23
−10	(–) 21	(–) 16	(–) 25	$\boxed{11}$ 13
0	$\boxed{6}$ 17	$\boxed{3}$ 18	(–) 14	$\boxed{4}$ 23
9	(–) 32	$\boxed{7}$ 27	$\boxed{12}$ 18	(–) 41

(iii)The optimal (minimum transportation cost)

$= (11 \times 13) + (6 \times 17) + (3 \times 18) + (4 \times 23) + (7 \times 27) + (12 \times 18)$
$= 143 + 102 + 54 + 92 + 189 + 216 =$ Rs. 796.

EXAMPLE 10.24: A company has 3 cement factories located in cities 1, 2, 3 which supply cement to 4 projects located in towns 1, 2, 3, 4. Each plant can supply 6, 1, 10 truckloads of cement daily respectively, and the daily cement requirements of the projects are respectively 7, 5, 3, 2 truckloads. The transportation costs per truckload of cement (in hundreds of rupees) from each plant to each project site are as follows:

Project sites

		1	**2**	**3**	**4**
Factories	1	2	3	11	7
	2	1	0	6	1
	3	5	8	15	9

Determine the optimal distribution for the company so as to minimize the total transportation cost.

Solution: The given transportation problem is

Project sites

		1	**2**	**3**	**4**
Factories	1	2	3	11	7
	2	1	0	6	1
	3	5	8	15	9

Also, the supply and the demand are 6, 1, 10 and 7, 5, 3, 2 respectively. The transportation table is written in table-1 as follows:

Table-I

Project sites						
		1	**2**	**3**	**4**	**Supply**
Factories	1	2	3	11	7	6
	2	1	0	6	1	1
	3	5	8	15	9	10
Demand		7	5	3	2	17

From table-I, we see that the supply being equal to the demand, that is 17, the problem is balanced.

Using VAM, the initial basic feasible solution is shown in table-II.

Table-II

$\boxed{1}$ 2	$\boxed{5}$ 3	11	7	6
1	0	6	$\boxed{1}$ 1	1
$\boxed{6}$ 5	8	$\boxed{3}$ 15	$\boxed{1}$ 9	10
7	5	3	2	

From table-II, we have transportation cost

$= (1 \times 2 + 5 \times 3 + 1 \times 1 + 6 \times 5 + 3 \times 15 + 1 \times 9)$ times 100 = Rs. 10,200.

Checking of optimality

For checking of optimality, we apply Modi-method, the number of allocations = $(m + n - 1)$, that is, 6.

(i) Taking u for row and v for column, we have the following equations.

$$u_1 + v_1 = 2,\ u_1 + v_2 = 3,\ u_2 + v_4 = 1,\ u_3 + v_1 = 5,\ u_3 + v_3 = 15,\ u_3 + v_4 = 9$$

Let $u_1 = 0$ in the above relations. Then

$$v_1 = 2,\ v_2 = 3,\ u_3 = 3,\ v_3 = 12,\ v_4 = 6,\ u_2 = -5$$

(ii) Net evaluation $\omega_{ij} = (u_i + v_j) - c_{ij}$ for all empty cells are

$$\omega_{13} = u_1 + v_3 - c_{13} = 0 + 12 - 11 = 1,$$
$$\omega_{14} = u_1 + v_4 - c_{14} = 0 + 6 - 7 = -1,$$
$$\omega_{21} = u_2 + v_1 - c_{12} = -5 + 2 - 1 = -4,$$
$$\omega_{22} = u_2 + v_2 - c_{22} = -5 + 3 - 0 = -2,$$
$$\omega_{23} = u_2 + v_3 - c_{23} = -5 + 12 - 6 = 1,$$
$$\omega_{32} = u_3 + v_2 - c_{23} = 3 + 3 - 8 = -2.$$

Since some ω_{ij} are positive, so this solution is not optimal and we proceed further.

The values of ω_{ij} are given in table III.

Table-III

$u_i \backslash v_j$	2	3	12	
0	$\boxed{1}$ 2	$\boxed{5}$ 3	(+) 11	(–) 7
–5	(–) 1	(–) 0	(+) 6	$\boxed{1}$ 1
3	$\boxed{6}$ 5	(–) 8	$\boxed{3}$ 15	$\boxed{1}$ 9

Iterate towards optimal solution

First iteration

(a) Next basic feasible solution:

(i) Choose the unoccupied cell with maximum ω_{ij}. In case of a tie, select the one with lower original cost.
In tableIIII cells (1, 3) and (2, 3) both have $\omega_{ij} = 1$, and out of these, cell (2, 3) has lower original cost 6, therefore we take this as the next basic cell and note θ in it.

(ii) Draw a closed path beginning and ending at θ-cell. Add and subtract θ, alternately, to and from the transition cells of the loop subject to the rim requirements. Assign a maximum value to θ so that one basic variable becomes zero and the other variable remains ≥ 0. Now, the basic cell whose allocation has been reduced to zero leaves the basis. This gives the second basic feasible solution, which is shown in table IV.

Table-IV

$\boxed{1}$ 2	$\boxed{5}$ 3	11	7
1	0	$\boxed{\theta=1}$ 6	$\boxed{1-1=0}$ 1
$\boxed{6}$ 5	8	$\boxed{3-1=2}$ 15	$\boxed{1+1=2}$ 9

The total transportation cost of this revised solution

$= \text{Rs. } (1 \times 2 + 5 \times 3 + 1 \times 6 + 6 \times 5 + 2 \times 15 + 2 \times 9)$ times 100

$= \text{Rs. } 10{,}100.$

(b) Test of optimality:

Since the number of allocations is equal to $(m + n - 1)$, that is, 6, we can apply the Modi-method. $u_1 + v_1 = 2$, $u_1 + v_2 = 3$, $u_2 + v_3 = 6$, $u_3 + v_1 = 5$, $u_3 + v_3 = 15$, $u_3 + v_4 = 9$

Let $u_1 = 0$ in the above relations. Then

$v_1 = 2,\ v_2 = 3,\ u_3 = 3,\ v_3 = 12,\ u_2 = -6,\ v_4 = 6$

Now, net evaluation $w_{ij} = (u_i + v_j) - c_{ij}$ for the unoccupied cells are

$\omega_{13} = u_1 + v_3 - c_{13} = 0 + 12 - 11 = 1,$

$\omega_{14} = u_1 + v_4 - c_{14} = 0 + 6 - 7 = -1,$

$\omega_{21} = u_2 + v_1 - c_{21} = -6 + 2 - 1 = -5,$

$$\omega_{22} = u_2 + v_2 - c_{22} = -6 + 3 - 0 = -3,$$
$$\omega_{24} = u_2 + v_4 - c_{24} = -6 + 6 - 1 = -1, \text{ and}$$
$$\omega_{32} = u_3 + v_2 - c_{32} = 3 + 3 - 8 = -2.$$

The values of ω_{ij} are given in table-V.

Table-V

$u_i \backslash v_j$	2	3	12	6
0	$\boxed{1}$ 2	$\boxed{5}$ 3	(1) 11	(−1) 7
−6	(−5) 1	(−3) 0	$\boxed{1}$ 6	(−1) 1
3	$\boxed{6}$ 5	(−2) 8	$\boxed{2}$ 15	$\boxed{2}$ 9

Since, some w_{ij} (i.e., $\omega_{13} = 1$) are positive, so this solution is not optimal and we proceed further.

Second Iteration

(a) Next basic feasible solution: In the second basic feasible solution, introduce the cell (1, 3) taking $\theta = 1$ and drop the cell (1, 1) giving table-VI. Thus, we obtain the third basic feasible solution in table-VII.

Table-VI

$\boxed{1-1}$ 2	5 3	$\boxed{\theta=1}$ 11	7
1	0	$\boxed{1}$ 6	1
$\boxed{6+1}$ 5	8	$\boxed{2-1}$ 15	$\boxed{2}$ 9

Table-VII

2	$\boxed{5}$ 3	$\boxed{1}$ 11	7
1	0	$\boxed{1}$ 6	1
$\boxed{7}$ 5	8	$\boxed{1}$ 15	$\boxed{2}$ 9

(b) Test of optimality: Since the number of allocations in table-VII = $m + n - 1$ (i.e. 6), we can apply the Modi method. Similarly, the net evaluation ω_{ij} for the table-VII is shown in the table-VIII. Since all $\omega_{ij} \le 0$, this basic feasible solution is optimal. Hence, the optimal transportation cost = Rs. $(5 \times 3 + 1 \times 11 + 1 \times 6 + 7 \times 5 + 1 \times 15 + 2 \times 9)$ times 100 = Rs. 10,000.

Table-VIII

$u_i \backslash v_j$	1	3	11	5
0	(−1) 2	$\boxed{5}$ 3	$\boxed{1}$ 11	(−2) 7
−5	(−5) 1	(−2) 0	$\boxed{1}$ 6	(−1) 1
4	$\boxed{7}$ 5	(−1) 8	$\boxed{1}$ 15	$\boxed{2}$ 9

10.8.3 Degeneracy (Transportation Problem)

When the number of basic cells in a transportation problem of m rows and n columns is less than $m + n - 1$, the basic solution degenerates. To remove degeneracy, we assign a small positive value $\in$ to one or more of the unoccupied cells so that the number of occupied cells becomes $m + n - 1$. The cells containing $\in$ are then treated like other basic cells and the problem is solved in the usual manner. The $\in$'s are kept till the optimum solution is obtained. It may be removed by letting $\in \to 0$ once, we get the optimal solution.

The following example clears the above concept.

EXAMPLE 10.25: Solve the following transportation problem:

		Destination				
		1	2	3	4	Supply
Source	1	10	20	5	7	10
	2	13	9	12	8	20
	3	4	5	7	9	30
	4	14	7	1	0	40
	5	3	12	5	19	50
Demand		60	60	20	10	150

Solution: From the given problem, we construct the transportation table. The total supply and total demand are equal (that is, 150), so the transportation table is balanced.

Using VAM method, the initial basic feasible solution is as shown in table-I.

Table-II

$\boxed{10}$ 10	20	5	7
13	$\boxed{2}$ 9	12	8
$\boxed{\in}$ 4	$\boxed{30}$ 5	7	9
14	$\boxed{10}$ 7	$\boxed{20}$ 1	$\boxed{10}$ 0
$\boxed{50}$ 3	12	5	19

Table-I

$\boxed{10}$ 10	20	5	7
13	$\boxed{20}$ 9	12	8
4	$\boxed{30}$ 5	7	9
14	$\boxed{10}$ 7	$\boxed{20}$ 1	$\boxed{10}$ 0
$\boxed{50}$ 3	12	5	19

Since the number of non-negative allocations is 7, which is less than $m + n - 1 = 8$, the basic solution degenerates. To remove this degeneracy, allocate a very small quantity $\in$ to the unoccupied cell (3, 1) so that the number of occupied cells become $m + n - 1$. Hence the non-degenerate basic feasible is as shown in table-II.

From table-II we have for occupied cells (u_i and v_j denote the rows and columns respectively.)

$$u_1 + v_1 = 10,\ u_2 + v_2 = 9,\ u_3 + v_1 = 4,\ u_3 + v_2 = 5,$$
$$u_4 + v_2 = 7,\ u_4 + v_3 = 1,\ u_4 + v_4 = 0,\ u_5 + v_1 = 3,$$

Let $u_4 = 0$ in the above relation. Then

$$v_2 = 7,\ v_3 = 1,\ v_4 = 0,\ u_2 = 2,\ u_3 = -2,\ v_1 = 6,\ u_5 = -3,\ u_1 = 4.$$

Now, net evalution $\omega_{ij} = (u_j + v_j) - c_{ij}$ for unoccupied cells are

$$\omega_{12} = u_1 + v_2 - c_{12} = 4 + 7 - 20 = -9,$$
$$\omega_{13} = u_1 + v_3 - c_{13} = 4 + 1 - 5 = 0,$$
$$\omega_{14} = u_1 + v_4 - c_{14} = 4 + 0 - 7 = -3,$$
$$\omega_{21} = u_2 + v_1 - c_{21} = 2 + 7 - 13 = -4,$$
$$\omega_{23} = u_2 + v_3 - c_{23} = 2 + 1 - 12 = -9,$$
$$\omega_{24} = u_2 + v_4 - c_{24} = 2 + 0 - 8 = -6,$$
$$\omega_{33} = u_3 + v_3 - c_{33} = -2 + 1 - 7 = -8,$$
$$\omega_{34} = u_3 + v_4 - c_{34} = -2 + 0 - 9 = -11,$$
$$\omega_{41} = u_4 + v_1 - c_{41} = 0 + 6 - 14 = -8,$$
$$\omega_{52} = u_5 + v_2 - c_{52} = -3 + 7 - 12 = -8,$$
$$\omega_{53} = u_5 + v_3 - c_{53} = -3 + 1 - 5 = -7,$$
$$\omega_{54} = u_5 + v_4 - c_{54} = -3 + 0 - 19 = -22.$$

Since all $\omega_{ij} \leq 0$, so the solution is optimal. Hence the optimal allocation is given by

$x_{11} = 10$, $x_{22} = 20$, $x_{32} = 30$; $x_{42} = 10$, $x_{43} = 20$, $x_{44} = 10$, $x_{51} = 50$ which are represented in table-III.

Table-III

$u_i \backslash v_j$	6	7	1	0
4	$\boxed{10}$ 10	(-9) 20	(0) 5	(-3) 7
2	(-5) 13	$\boxed{20}$ 9	(-9) 12	(-6) 8
-2	$\boxed{\in}$ 4	$\boxed{30}$ 5	(-8) 7	(-11) 9
0	(-8) 14	$\boxed{10}$ 7	$\boxed{20}$ 1	$\boxed{10}$ 0
-3	$\boxed{50}$ 3	(-8) 12	(-7) 5	(-22) 19

The optimal transportation cost

$= \text{Rs. } (10 \times 10 + 20 \times 9 + 30 \times 5 + 10 \times 7 + 20 \times 1 + 10 \times 0 + 50 \times 3 + 4\in)$

$= \text{Rs. } 670 + 4\in = \text{Rs. } 670 \text{ as} \in \to 0.$

EXERCISE 10.6

1. Solve the following transportation problem.

(i)

	X	Y	Z	Availability	
A	8	7	5	60	
B	6	8	9	70	
C	9	6	5	80	
Requirement	50	80	80	210	Balance

(ii)

	D_1	D_2	D_3	D_4	Supply
A	6	4	1	5	14
B	8	9	2	7	16
C	4	3	6	2	5
Demand	6	10	15	4	35

(iii)

	D_1	D_2	D_3	D_4	D_5	Supply
A	3	2	3	4	1	100
B	4	1	2	4	2	125
C	1	0	5	3	2	75
Demand	100	60	40	75	25	300

(iv)

	D_1	D_2	D_3	D_4	Supply
A	1	2	3	4	6
B	4	3	2	0	8
C	0	2	2	1	10
Demand	4	6	8	6	24

(v)

Suppliers → Consumers ↓	A	B	C	Available
I	6	8	4	14
II	4	9	8	12
III	1	2	6	5
Requirement	6	10	15	31

2. Obtain an initial basic feasible solution to the following transportation problem:

		D	E	F	G	
From	A	11	13	17	14	250
	B	16	18	14	10	300
	C	21	24	13	10	400
		200	225	275	250	

3. A company is spending Rs. 1,000 on transportation of its unit's plants to four distribution centres. The supply and demand of units, with unit cost of transportation, are as follows:

Plants	D_1	D_2	D_3	D_4	Availability
P_1	19	30	50	12	7
P_2	70	30	40	60	10
P_3	40	10	60	20	18
Requirements	5	8	7	15	

What can be maximum saving by optimal sheduling?

4. A company has three plants at locations A, B and C which supply to warehouses located as D, E, F, G and H. The monthly plant capacities are 800, 500 and 900 units respectively. The monthly warehouse requirements are 400, 400, 500, 400 and 800 units, respectively. The unit transportation cost in rupees is given below:

To

From		D	E	F	G	H
	A	5	8	6	6	3
	B	4	7	7	6	6
	C	8	4	6	6	3

Determine an optimum distribution for the company in order to minimize the total transportion cost.

5. Obtain an optimum basic feasible solution to the following transportation problem:

To

From				Available
	7	3	4	2
	2	1	3	3
	3	4	6	5
Demand	4	1	5	10

6. Consider four bases of operations B_i and three targets T_j. The tonnes of bombs per aircraft from any base that can be delivered to any target are given in the following table:

B_i \ T_j	1	2	3
1	8	6	5
2	6	6	6
3	10	8	4
4	8	6	4

The daily sortie capability of each of the four bases is 150 sorties per day. The daily requirement in sorties over each target is 200. Find the allocation of sorties from each base to each target which maximizes the total tonnage over all the three targets.

7. A company has factories F_1, F_2, F_3 which supply warehouses at W_1, W_2 and W_3. The weekly factory capacities, weekly warehouse requirements and unit shipping costs (in rupees) are as follows:

Factories	Warehouses			Supply
	W_1	W_2	W_3	
F_1	16	20	12	200
F_2	14	8	18	160
F_3	26	24	16	90
Demand	180	120	150	450

Determine the optimal distribution for this company to minimize shipping costs.

ANSWER

1. (i) $x_{13} = 60$, $x_{21} = 50$, $x_{22} = 20$, $x_{32} = 60$, $x_{33} = 20$; Total costs = Rs. 1220
   (ii) $x_{11} = 4$, $x_{12} = 10$, $x_{21} = 1$, $x_{23} = 1$, $x_{34} = 4$; Minimum cost = Rs. 114.
   (iii) $x_{11} = 25$, $x_{14} = 50$, $x_{15} = 25$, $x_{22} = 60$, $x_{23} = 4$, $x_{24} = 25$, $x_{31} = 75$; Minimum cost = Rs. 615
   (iv) $x_{12} = 6$, $x_{23} = 2$, $x_{31} = 4$, $x_{33} = 6$, $x_{11} = 25$, $x_{14} = 50$, $x_{11} = 25$, $x_{22} = 60$, $x_{23} = 40$, $x_{24} = 25$, $x_{31} = 7$; Minimum cost = Rs. 28
   (v) $x_{13} = 14$, $x_{21} = 6$, $x_{22} = 5$, $x_{23} = 1$, $x_{32} = 5$; Mminimum cost = Rs. 143
2. $x_{11} = 200$, $x_{12} = 50$, $x_{22} = 175$, $x_{24} = 125$, $x_{33} = 275$, $x_{34} = 125$; Min. cost = Rs. 1205
3. $x_{11} = 5$, $x_{14} = 2$, $x_{22} = 3$, $x_{23} = 7$, $x_{32} = 5$, $x_{34} = 13$; minimum cost = Rs. 799 and maximum saving = Rs. 201.
4. $x_{13} = 0$, $x_{15} = 800$, $x_{21} = 400$, $x_{24} = 100$, $x_{32} = 4$, $x_{33} = 200$, $x_{34} = 300$, $x_{43} = 300$; min. cost = Rs. 9200.
5. $x_{13} = 2$, $x_{22} = 1$, $x_{23} = 2$, $x_{31} = 1$; min. cost = Rs. 33
6. $x_{11} = 50$, $x_{12} = 100$, $x_{21} = 150$, $x_{33} = 150$, $x_{42} = 100$, $x_{43} = 50$, min. tonnage = Rs. 3300.
7. $x_{11} = 140$, $x_{13} = 60$, $x_{21} = 40$, $x_{22} = 120$, $x_{33} = 90$; min. cost = Rs. 5920.

10.8.4 Assignment Problem

A special type of transportation in which our aim is to assign the number of origins to an equal number of destinations at a minimum cost (or maximum profit) is called an assignment problem. An assignment problem can be formulated.

There are n new machines M_i ($i = 1, 2, ..., n$) which are to be set up in a machine shop. There are n vacant spaces S_j ($j = 1, 2, ..., n$) available. The cost of