Problem 1.

a) Find the general solution of the following equation (5 pts.)

$$D(D^2+9)(D^2-9)y=0$$

$y = C_1 + C_2 \cdot \cos 3t + C_3 \cdot \sin 3t + C_4 \cdot e^{3t} + C_5 \cdot e^{-3t}$

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b) Determine a suitable form for a particular solution of the following equation (Do not calculate the coefficients) (10 pts.)

$$D(D^2+9)(D^2-9)y = 2t^2 + 5e^{3t} + \sin(3t) + 3e^{3t}\cos(2t)$$

Annihilator: $D^3 \cdot (D-3) \cdot (D^2+9) \cdot \underbrace{(D-(3+2i))(D-(3-2i))}_{(D^2-6D+13)}$ 4

$\Rightarrow D^4 \cdot (D-3)^2 \cdot (D+3)(D^2+9)^2(D^2-6D+13)\,y = 0$

$\Rightarrow y = \boxed{C_1} + C_2 t + C_3 t^2 + C_4 t^3 \boxed{+ C_5 \cdot e^{3t}} + C_6\, t \cdot e^{3t} \boxed{+ C_7 \cdot e^{-3t}}$
$\boxed{+ C_8 \cdot \cos 3t} + C_9 \cdot t \cdot \cos 3t \boxed{+ C_{10} \cdot \sin 3t} + C_{11} \cdot t \cdot \sin 3t$
$+ C_{12} \cdot e^{3t} \cdot \cos 2t + C_{13} \cdot e^{3t} \cdot \sin 2t$

$y_p = A \cdot t + B \cdot t^2 + C \cdot t^3 + D \cdot t \cdot e^{3t} + E \cdot t \cdot \cos 3t$
$+ F \cdot t \cdot \sin 3t + G \cdot e^{3t} \cdot \cos 2t + H \cdot e^{3t} \sin 2t$ 6

(-1) for missing terms

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Problem 2. Determine the first 4 terms of the Power Series solution about $x_0 = 0$ of the following initial-value Problem

$$y'' - xy' - (x+2)y = 0\ , \qquad y(0) = 1,\ y'(0) = 1$$

Find the radius of convergence of this Power Series and state the theorem you use to find this radius. (20 pts.)

$$y = a_0 + a_1 x + a_2 x^2 + \text{—} = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n \cdot (n)(n-1) \cdot x^{n-2}$$

$$y''(0) - 0 \cdot y'(0) - (0+2) \cdot y(0) = 0$$

$$y''(0) = 2(y(0)) = 2$$

$$a_n = \frac{y^{(n)}(0)}{n!}$$

Take derivative:

$$y'''(x) - y'(x) - x \cdot y''(x) - y(x) - (x+2)y'(x) = 0$$

$$y'''(0) = 1 + 1 + 2 = 4$$

$$a_0 = y(0) = 1$$

$$a_1 = y'(0) = 1$$

$$a_2 = \frac{y''(0)}{2!} = \frac{2}{2} = 1$$

$$a_3 = \frac{y'''(0)}{3!} = \frac{4}{6} = \frac{2}{3}$$

$$y = 1 + 1 \cdot x + 1 \cdot x^2 + \frac{2}{3} \cdot x^3 + \text{— —}$$

If $P(x) \cdot y'' + Q(x) \cdot y' + R(x) \cdot y = 0$

Then, if $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are analytic at x_0, then the radius of convergency is at least as the minimum of the radii of convergence of p and q.

$p(x) = -x \qquad q(x) = -(x+2)$

The radius of conv. is ∞ for both of series.

So, $R = \infty$.

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Problem 3. Find the general terms of two linearly independent Power Series solutions of the following differential equation about the point $x_0 = 0$. (25 pts.)

$$y'' + x^2y' + xy = 0$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1}, \quad x^2y' = \sum_{n=1}^{\infty} n a_n x^{n+1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = y'' + x^2y' + xy = 2a_2 + (6a_3 + a_0)x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + n a_{n-1}]x^n$$

So $a_2 = 0$, $a_3 = -\frac{a_0}{6}$ and for $n \geq 2$ $a_{n+2} = \frac{-n}{(n+2)(n+1)} a_{n-1}$

e.g. $a_4 = \frac{-2}{3 \cdot 4} a_1 = \frac{-2^2}{4!} a_1$

$a_5 = \frac{-3}{4 \cdot 5} a_2 = 0$

$a_6 = \frac{-4}{5 \cdot 6} a_3 = \frac{4}{6 \cdot 5 \cdot 6} a_0 = \frac{4^2}{6!} a_0$

$a_7 = \frac{-5}{6 \cdot 7} a_4 = \frac{2^2 \cdot 5^2}{7!} a_1$

Therefore

$$y = a_0 + a_1 x - \frac{1}{3!} a_0 x^3 - \frac{2^2}{4!} a_1 x^4 + \frac{4^2}{6!} a_0 x^6 + \cdots$$

$$= a_0\left(1 - \frac{1}{3!}x^3 + \frac{4^2}{6!}x^6 - \frac{4^2 \cdot 7^2}{9!}x^9 + \cdots\right) + a_1\left(x - \frac{2^2}{4!}x^4 + \frac{2^2 \cdot 5^2}{7!}x^7 - \frac{2^2 \cdot 5^2 \cdot 8^2}{10!}x^{10} + \cdots\right)$$

Two lin. indep P.S. solutions are

$$y_1 = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1^2 4^2 \cdots (3k-2)^2}{(3k)!} x^{3k}, \quad y_2 = \sum_{k=0}^{\infty} \frac{(-1)^k 2^2 5^2 \cdots (3k-1)^2}{(3k+1)!} x^{3k+1}$$

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Problem 4. Solve the following initial-value problem. (15 pts.)

$$x^2y'' - 5xy' + 13y = 0\ ,\ y(1) = 2\ ,\ y'(1) = 0$$

This is an Euler type of differential equation. So plugging in $y = x^r$, we get $(y' = r \cdot x^{r-1};\ y'' = r(r-1)x^{r-2})$

$x^r(r(r-1) - 5r + 13) = 0 \Rightarrow r^2 - 6r + 13 = 0\ (x > 0)$

$r_{1,2} = 3 \pm 2i$

$\Rightarrow$ homogeneous solutions are $y_1 = x^3\cos(2\ln x)$

$y_2 = x^3\sin(2\ln x)$

$y = c_1y_1 + c_2y_2 = c_1x^3\cos(2\ln x) + c_2x^3\sin(2\ln x)$

$y'(x) = \left(3x^2\cos(2\ln x) - x^3\sin(2\ln x)\cdot\frac{2}{x}\right)\cdot c_1$

$+ \left(3x^2\sin(2\ln x) + x^3\cos(2\ln x)\cdot\frac{2}{x}\right)\cdot c_2$

$y(1) = c_1\cdot\cos(0) + c_2\sin(0) = \boxed{c_1 = 2}$

$y'(1) = 3c_1 + 2c_2 = 0 \Rightarrow 2c_2 = -6$

$\boxed{c_2 = -3}$

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Problem 5. Solve the following initial-value problem using the Laplace transform
(Hint: You may use the table on the last page of the exam) (25 pts.)

$$y'' - 4y' + 5y = 10e^{-t}, \quad y(0) = 0,\ y'(0) = 4$$

$\mathcal{L}(y''-4y'+5y) = \mathcal{L}(10e^{-t})$

$\mathcal{L}(y''-4y'+5y) = \mathcal{L}(y'') - 4\mathcal{L}(y') + 5\mathcal{L}(y)$

$= [s^2\mathcal{L}(y) - sy(0) - y'(0)] - 4[s\mathcal{L}(y) - y(0)] + 5\mathcal{L}(y)$

Derivative Formula $= (s^2-4s+5)\mathcal{L}(y) - 4$

$\mathcal{L}(10e^{-t}) = 10\mathcal{L}(e^{-t}) = \frac{10}{s+1}$

$\Rightarrow \mathcal{L}(y) = \frac{10}{(s+1)(s^2-4s+5)} + \frac{4}{s^2-4s+5}$

$\frac{10}{(s+1)(s^2-4s+5)} = \frac{a}{s+1} + \frac{bs+c}{s^2-4s+5} \Rightarrow a(s^2-4s+5) + (bs+c)(s+1) = 10$

$\Rightarrow (a+b)s^2 + (b+c-4a)s + (c+5a) = 10$

$\Rightarrow \left.\begin{array}{l} a+b=0 \\ b+c-4a=0 \\ c+5a=10 \end{array}\right\} \quad \begin{array}{l} a=1 \\ b=-1 \\ c=5 \end{array}$

$\Rightarrow \mathcal{L}(y) = \frac{1}{s+1} + \frac{-s+5}{s^2-4s+5} + \frac{4}{s^2-4s+5} = \frac{1}{s+1} + \frac{-s+9}{s^2-4s+5}$

$= \frac{1}{s+1} - \frac{(s-2)}{(s-2)^2+1^2} + \frac{7}{(s-2)^2+1} = \mathcal{L}(e^{-t}) - \mathcal{L}(e^{2t}\cos t) + \mathcal{L}(7e^{2t}\sin t)$

$= \mathcal{L}(e^{-t} - e^{2t}\cos t + 7e^{2t}\sin t) \Rightarrow \boxed{y(t) = e^{-t} - e^{2t}\cos t + 7e^{2t}\sin t}$

1