MATH 204
REVIEW QUESTIONS – II

Section 3.6:

Q1) Solve the given differential equation.

$$y'' + 4y' + 4y = t^{-2} \cdot e^{-2t} \quad , t > 0$$

Solution:

See that the characteristic equation is: $r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$r_1 = r_2 = -2$, i.e.,

Solution to the homogeneous equation is:

$$y_c(t) = c_1 \cdot e^{-2t} + c_2 \cdot t \cdot e^{-2t} \quad , W(e^{-2t}, te^{-2t}) = e^{-4t} > 0$$

Let $t_0 = 1$, $\quad u_1(t) = -\int_{t_0}^{t} \frac{y_2(s) g(s)}{W(y_1, y_2)(s)} ds$ & $u_2(t) = \int_{t_0}^{t} \frac{y_1(s) g(s)}{W(y_1, y_2)(s)} ds$

$$u_1(t) = -\int_1^t \frac{s \cdot e^{-2s} \cdot (s^{-2} e^{-2s})}{e^{-4s}} ds = -\int_1^t s^{-1} ds = -\left[\ln s\right]_{s=1}^{s=t} = -\ln t$$

$$u_2(t) = \int_1^t \frac{e^{-2s} (s^{-2} e^{-2s})}{e^{-4s}} ds = \int_1^t s^{-2} ds = \left[-s^{-1}\right]_{s=1}^{s=t} = -\frac{1}{t} + 1$$

Hence, a particular solution to the nonhomogeneous equation is

$Y(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$

$Y(t) = (-\ln t)\, e^{-2t} + \left(-\frac{1}{t} + 1\right) \cdot t \cdot e^{-2t}$, so the general solution becomes

$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} - e^{-2t} \ln t$$

Section 4.2:

Q1) Find the general solution of the given differential equation.

$$y^{(4)} - 5y'' + 4y = 0$$

Solution:

Characteristic equation: $r^4 - 5r^2 + 4 = 0$

$$(r^2 - 4)(r^2 - 1) = 0$$

$$r_1 = 1 \quad r_2 = -1 \quad r_3 = 2 \quad r_4 = -2$$

So the general solution of the homogenous equation is:

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-2t}$$

Section 4.3:

Q1) Determine the general solution of the given differential equation

$$y''' + y'' + y' + y = e^{-t} + 4t$$

Solution:

Characteristic equation: $r^3 + r^2 + r + 1 = 0$

$$r^2(r+1) + r + 1 = 0$$

$$(r^2+1)(r+1) = 0$$

$$r_1 = -1 \quad r_{2,3} = 0 \mp i$$

$$\lambda = 0 \quad \mu = 1$$

so that

$$y_c(t) = c_1 \cdot e^{-t} + c_2 \sin t + c_3 \cos t .$$

See that $g(t) = \underbrace{e^{-t}}_{g_1(t)} + \underbrace{4t}_{g_2(t)}$

Assuming the form $Y(t) = Y_1(t) + Y_2(t)$,

Say $Y_1(t) = (Ae^{-t})t$

$$Y_1'(t) = Ae^{-t} - Ate^{-t}$$
$$Y_1''(t) = -2Ae^{-t} + Ate^{-t}$$
$$Y_1'''(t) = 3Ae^{-t} - Ate^{-t}$$

Substituting,

$$2Ae^{-t} = e^{-t} \Rightarrow A = 1/2$$

Say $Y_2(t) = Bt + C$, substituting,

$$Y_2'(t) = B$$
$$Y_2''(t) = Y_2'''(t) = 0$$

$$Bt + B + C = 4t$$
$$B = 4, \quad C = -4$$

$$Y(t) = \frac{1}{2}te^{-t} + 4t - 4,$$

The general solution becomes

$$y(t) = c_1 e^{-t} + c_2 \sin t + c_3 \cos t + \frac{1}{2}te^{-t} + 4t - 4$$

Section 5.2:

Q1) $y'' + xy' + 2y = 0$, $x_0 = 0$. (Solve with power series.)
(a) Find the recurrence relation (b) Find the power series sol'ns
(c) Show that sol'ns in (b) form a fundamental set of sol'ns.

Solution:

We assume that the solution, y, has a power series expansion around $x_0 = 0$, i.e, $y = \sum_{n=0}^{\infty} a_n \cdot x^n$

See that $y' = \sum_{n=1}^{\infty} a_n . n . x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n . n . (n-1) . x^{n-2}$

Substituting,

$$\sum_{n=2}^{\infty} a_n . n . (n-1) . x^{n-2} + x . \sum_{n=1}^{\infty} a_n . n . x^{n-1} + 2 \sum_{n=0}^{\infty} a_n . x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2} . (n+2)(n+1) . x^n + \underbrace{\sum_{n=1}^{\infty} a_n . n . x^n}_{(n=0 \text{ de alabilir})} + 2 \sum_{n=0}^{\infty} a_n . x^n = 0$$

For $n=0$, $2a_2 + 2a_0 = 0$, $a_2 = -a_0$

For $n \geq 1$, $a_{n+2}(n+2)(n+1) + a_n (n+2) = 0$

(a) Recurrence relation: $\boxed{a_{n+2} = -\frac{a_n}{n+1}}$, for $n \geq 0$.

(b) See that for even indices, $a_2 = \frac{-a_0}{1}$, $a_4 = \frac{-a_2}{3} = \frac{a_0}{1.3}$, $a_6 = \frac{-a_4}{5} = \frac{-a_0}{1.3.5}$

$$a_{2k} = -\frac{a_{2k-2}}{2k-1} = \frac{a_{2k-4}}{(2k-3)(2k-1)} = \dots = \frac{(-1)^k . a_0}{1.3.5\dots(2k-1)} \quad k \geq 1$$

and for odd indices

$$a_{2k+1} = -\frac{a_{2k-1}}{2k} = \frac{a_{2k-3}}{(2k-2)2k} = \dots = \frac{(-1)^k a_1}{2.4.6\dots(2k)}, \quad k \geq 1$$

$a_3 = \frac{-a_1}{2}$, $a_5 = \frac{-a_3}{4} = \frac{a_1}{2.4}$, $a_7 = \frac{-a_5}{6} = \frac{-a_1}{2.4.6}$

$$y(x) = a_0 \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1.3.5\ldots(2n-1)}\right) + a_1 \left(\underbrace{x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2.4.6\ldots(2n)}}_{y_2(x)} \right)$$

So $y(x) = a_0 + a_1 \cdot x + (-a_0)\, x^2 + \left(-\frac{a_1}{2}\right) x^3 + \ldots$, therefore, there are two linearly independent solutions:

$$y_1(x) = 1 - x^2 + \frac{x^4}{1.3} - \frac{x^6}{1.3.5} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot x^{2n}}{1.3.5\ldots(2n-1)}$$

$$y_2(x) = x - \frac{x^3}{2} + \frac{x^5}{2.4} - \frac{x^7}{2.4.6} + \ldots = x + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{2.4.6\ldots(2n)}$$

with $y(x) = a_0 \cdot y_1(x) + a_1 \cdot y_2(x)$.

(c)

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}_{x=0} = 1 + f(x)\Big|_{x=0},$$

where $f(x)$ is a function that has only powers of x, without any constant, i.e, $f(0) = 0$.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$y_1' = -2x + \frac{4}{3}x^3 - \frac{2}{5}x^5 + \ldots$$

$$y_2' = 1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \ldots$$

$W(y_1, y_2)(0) = 1 \neq 0$, i.e, y_1 & y_2 are linearly independent, constitute a fundamental set of solutions.

Section 5.3:

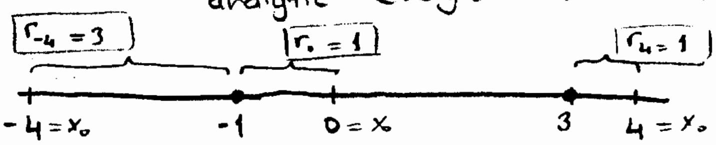
Q1) Determine a lower bound for the radius of convergence of series solutions about each given point x_0, for the given d.e.

$(x^2 - 2x - 3)\, y'' + x \cdot y' + 4y = 0$; $x_0 = 4$, $x_0 = -4$, $x_0 = 0$

Solution: (Thm 5.3.1)

$$y'' + \underbrace{\frac{x}{(x-3)(x+1)}} \cdot y' + \underbrace{\frac{4}{(x-3)(x+1)}}\, y = 0$$

analytic everywhere, except $x = 3$, $x = -1$



- for $x_0 = -4$: $r \geq 3$
- for $x_0 = 0$: $r \geq 1$
- for $x_0 = 4$: $r \geq 1$

Section 5.4:

(3) Determine the general solution of the given d.e. that is valid in any interval not including the singular point.

$$x^2 . y'' - 3x . y' + 4y = 0$$

Solution:

Assuming $y = x^r$, $y' = r . x^{r-1}$, $y'' = r.(r-1).x^{r-2}$

$$x^2 . r.(r-1).x^{r-2} - 3x.r.x^{r-1} + 4.x^r = 0$$

$(r^2 - 4r + 4)\, x^r = 0$, for all $x \in \mathbb{R}$.

Indicial equation: $r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r_1 = r_2 = 2$

$$y(x) = c_1 . x^2 + c_2 . \ln|x| . x^2 \qquad x \neq 0$$

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(6) $(x-1)^2 y'' + 8(x-1) y' + 12 y = 0$

Solution:

$y = (x-1)^r \Rightarrow y' = r.(x-1)^{r-1}$, $y'' = r(r-1).(x-1)^{r-2}$

Ch. eqn: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r_1 = r_2 = 1$

So $y(x) = c_1 . x + c_2 . \ln|x| . x$, $x \neq 0$

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Section 6.2:

Q1) Find the inverse Laplace transform of the given function.

$$F(s) = \frac{8s^2 - 4s + 12}{s.(s^2+4)}$$

Solution:

$$\frac{8s^2-4s+12}{s.(s^2+4)} = \frac{A}{s} + \frac{Bs}{s^2+4} + \frac{C}{s^2+4}$$

$$As^2 + 4A + Bs^2 + Cs = 8s^2 - 4s + 12 \qquad \begin{cases} A = 3 \\ B = 5 \\ C = -4 \end{cases}$$

$$F(s) = 3.\frac{1}{s} + 5.\frac{s}{s^2+4} - 2.\frac{2}{s^2+4}$$

So that $\mathcal{L}^{-1}\{F(s)\} = 3.\mathcal{L}^{-1}\{\frac{1}{s}\} + 5.\mathcal{L}^{-1}\{\frac{s}{s^2+4}\} - 2.\mathcal{L}^{-1}\{\frac{2}{s^2+4}\}$

$$= 3 + 5\cos 2t - 2\sin 2t.$$

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Q2) Use the Laplace transform to solve the given initial value problem.

$$y'' + 3y' + 2y = 0; \quad y(0) = 1, \quad y'(0) = 0$$

Solution:

$$\mathcal{L}\{y''\} = s^2.\mathcal{L}\{y\} - s\,y(0) - y'(0) = s^2.F(s) - s$$

$$\mathcal{L}\{y'\} = s\,\mathcal{L}\{y\} - y(0) = s\,F(s) - 1$$

$$\mathcal{L}\{y\} = F(s)$$

Taking the Laplace transform, we get

$$s^2 . F(s) - s + 3sF(s) - 3 + 2F(s) = 0$$

$$\downarrow$$

$$F(s) . (s^2 + 3s + 2) = s+3$$

$$F(s) = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$As + A + Bs + 2B = s+3$$

$$\left.\begin{array}{l} A+B=1 \\ A+2B=3 \end{array}\right\} \quad B=2 \qquad A=-1$$

$$F(s) = \frac{2}{s+1} - \frac{1}{s+2} \text{, so}$$

$$\mathcal{L}^{-1}\{F(s)\} = y(t) = 2.e^{-t} - e^{-2t}$$