Section 7.9 – Problem 1 :

$$x' = \underbrace{\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}}_{A} x + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

First of all, note that there are many ways to solve this problem. But, if A is diagonalizable then we'll use method of diagonalization provided in textbook, page 440.

eigenvalues: $0 = \det(A - \lambda I)$

$$0 = \det\begin{pmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{pmatrix} = (2-\lambda)(-2-\lambda) - 3\cdot(-1)$$

$$0 = -4 - 2\lambda + 2\lambda + \lambda^2 + 3$$

$$0 = \lambda^2 - 0\lambda + 3 - 4 = \lambda^2 - 1$$

$$0 = (\lambda+1)(\lambda-1)$$

So $\lambda_1 = -1$ and $\lambda_2 = 1$ $(\lambda_1 \neq \lambda_2)$

Hence A is diagonalizable.

corresponding eigenvectors are

$(A - \lambda_1 I)v_1 = 0$ and $(A - \lambda_2 I)v_2 = 0$

So that $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

forming the matrix $T = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}$

from $x' = Ax + g(t)$, where $g(t) = \begin{pmatrix} e^t \\ t \end{pmatrix}$.

letting $x = Ty$,

where y is a new variable, substitute

in the equation to get

$$(Ty)' = A(Ty) + g(t)$$

$$Ty' = ATy + g(t)$$, now multiply both side by T^{-1}

$$\underbrace{T^{-1}T}_{I} y' = \underbrace{T^{-1}AT}_{D} y + T^{-1}g(t)$$

$$y' = Dy + T^{-1}(g(t))$$

where D is a diagonal matrix with diagonal entries λ_1 and λ_2

i.e $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

from $T = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix} \Rightarrow, T^{-1} = \frac{1}{2} \cdot \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix}$

so eqn,

$$y' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} y + \frac{1}{2}\begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} e^t \\ t \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} e^t + 3t \\ -e^t - t \end{pmatrix}$$

$$y_1' = -y_1 + \frac{1}{2}(e^t + 3t)$$

$$y_2' = y_2 + \frac{1}{2}(-e^{+t} - t)$$

But we already know how to solve this eqns,

$$y_1' + y_1 = \frac{1}{2}\left(e^t + 3t\right)$$

$$e^t\left(y_1' + y_1\right) = \frac{1}{2}\left(e^{2t} + 3te^t\right)$$

$$\left(y_1 \cdot e^t\right)' = \frac{1}{2}e^{2t} + \frac{3}{2}te^t$$

$$e^t \cdot y_1 = \int\left(\frac{1}{2}e^{2t} + \frac{3}{2}te^t\right)dt$$

$$= \frac{1}{2}\cdot\frac{1}{2}e^{2t} + \frac{3}{2}\left(te^t - e^t\right) + c_1$$

$$\boxed{y_1(t) = \frac{1}{4}\cdot e^t + \frac{3}{2}\cdot(t-1) + c_1 \cdot e^{-t}}$$

And, similarly

$$y_2' - y_2 = -\frac{1}{2}\cdot\left(e^t + t\right)$$

$$e^{-t}\left(y_2' - y_2\right) = -\frac{1}{2}\left(1 + te^{-t}\right)$$

$$\left(y_2 \cdot e^{-t}\right)' = -\frac{1}{2}\left(1 + te^{-t}\right)$$

$$y_2(t) \cdot e^{-t} = \int\left(-\frac{1}{2} - \frac{1}{2}te^{-t}\right)dt$$

$$y_2 \cdot e^{-t} = -\frac{1}{2}t - \frac{1}{2}\cdot\left(-te^{-t} - e^{-t}\right) + c_2$$

$$\boxed{y_2(t) = -\frac{1}{2}te^t + \frac{1}{2}\cdot(t-1) + c_2 \cdot e^t}$$

Finally, we write the solution in terms of the original variables.

$$X = Ty$$

$$= \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} -y_1 - 3y_2 \\ y_1 + y_2 \end{pmatrix} = y_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$X(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix} te^{t}$$
$$+ \begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix} e^{t} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} //$$

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Section 7.9 - Problem 3:

$$x' = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix} x + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

$$0 = \det(A - \lambda I) = (2-\lambda)\cdot(-2-\lambda) - 1\cdot(-5)$$

$$0 = -4 - 2\lambda + 2\lambda + \lambda^2 + 5$$

$$0 = \lambda^2 + 1$$

so $\lambda_1 = -i$ and $\lambda_2 = i$.

from $(A - \lambda_1 I)\vartheta_1 = 0$

$$\begin{pmatrix} 2+i & 1 \\ -5 & -2+i \end{pmatrix} \vartheta_1 = 0 \quad \text{gives} \quad \vartheta_1 = \begin{pmatrix} -1 \\ 2+i \end{pmatrix}$$

and $\vartheta_2 = \begin{pmatrix} -1 \\ 2-i \end{pmatrix}$ conjugate of ϑ_1.

So, $x^{(1)}(t) = \vartheta_1 \cdot e^{-it} = \begin{pmatrix} -1 \\ 2+i \end{pmatrix} e^{-it} = \begin{pmatrix} -1 \\ 2+i \end{pmatrix} (\cos t - i \sin t)$

$$= \begin{pmatrix} -\cos t + i \cdot \sin t \\ 2\cos t + \sin t + i \cdot (\cos t - 2 \cdot \sin t) \end{pmatrix}$$

$$x^{(1)}(t) = \begin{pmatrix} -\cos t \\ 2\cos t + \sin t \end{pmatrix} + i \cdot \begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix}$$

Hence a set of real-valued solutions are

$$u(t) = \begin{pmatrix} -\cos t \\ 2\cos t + \sin t \end{pmatrix} \quad \text{and} \quad \vartheta(t) = \begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix}$$

$W(u, \vartheta)(t) = -1 \neq 0$ u, ϑ are linearly independent.

u and ϑ are hom. solutions of $x' = Ax$.

As for non-homogeneous part

let $x_p(t) = \sin t \cdot \alpha + \cos t \cdot \beta + t \cdot \sin t \, \gamma + t \cos t \cdot \theta$
(guess)

where $\alpha, \beta, \gamma, \theta \in \mathbb{R}^2$.

plugging in the equation we obtain that

$\alpha = \begin{pmatrix} 1 \\ -5/2 \end{pmatrix}$, $\beta = \begin{pmatrix} -1/2 \\ 2 \end{pmatrix}$, $\gamma = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $\theta = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Hence,

$x(t) = c_1 \cdot u(t) + c_2 \cdot v(t) + x_p(t)$.

Section 7.9- Problem 4:

$$x' = \underbrace{\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}}_{A} x + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

$$0 = \det(A - \lambda I) = (1-\lambda)(-2-\lambda) - 4 \cdot 1$$

$$0 = -2 - \lambda + 2\lambda + \lambda^2 - 4$$

$$0 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

So $\lambda_1 = -3$ and $\lambda_2 = 2$

and corresponding eigenvalues are

$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = v_1$ and $v_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

So, $x_h(t) = c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t}$

As for non-hom. part, guess particular solution

$x_p(t) = e^{-2t} \cdot \alpha + e^{t} \cdot \beta$ where $\alpha, \beta \in \mathbb{R}^2$.

plug in the eqn to obtain,

$\alpha = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

thus, $x_p(t) = e^{-2t} \cdot \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} + e^{t} \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

finally, the general solution is,

$x(t) = x_h(t) + x_p(t).$

6. The eigenvalues of the coefficient matrix are $r_1 = 0$ and $r_2 = -5$. It follows that the solution of the homogeneous equation is

$$\mathbf{x}_c = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2e^{-5t} \\ e^{-5t} \end{pmatrix}.$$

The coefficient matrix is symmetric. Hence the system is diagonalizable. Using the normalized eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \qquad \mathbf{T}^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

Setting $\mathbf{x}=\mathbf{T}\mathbf{y}$, and $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$y_1' = \frac{5+4t}{\sqrt{5}\,t}$$
$$y_2' = -5y_2 + \frac{2}{\sqrt{5}}.$$

The solutions are readily obtained as

$$y_1(t) = \sqrt{5}\,\ln t + \frac{4}{\sqrt{5}}t + c_1 \;\text{ and }\; y_2(t) = c_2\,e^{-5t} + \frac{2}{5\sqrt{5}}.$$

Transforming back to the original variables, we have $\mathbf{x}=\mathbf{T}\mathbf{y}$, with

$$\mathbf{x} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} y_1(t) + \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} y_2(t).$$

Hence the general solution is

$$\mathbf{x} = k_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} -2e^{-5t} \\ e^{-5t} \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ln t + \frac{4}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \frac{2}{25} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

9. Note that the coefficient matrix is symmetric. Hence the system is diagonalizable. The eigenvalues and eigenvectors are given by

$$r_1 = -\frac{1}{2}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix}1\\1\end{pmatrix} \text{ and } r_2 = -2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix}1\\-1\end{pmatrix}.$$

Using the normalized eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \qquad \mathbf{T}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}.$$

Setting $\mathbf{x}=\mathbf{T}\mathbf{y}$, and $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$\begin{aligned} y_1' &= -\frac{1}{2}y_1 + 2\sqrt{2}\,t + \frac{1}{\sqrt{2}}e^t \\ y_2' &= -2y_2 + 2\sqrt{2}\,t - \frac{1}{\sqrt{2}}e^t\,. \end{aligned}$$

Using any elementary method for first order linear equations, the solutions are

$$\begin{aligned} y_1(t) &= k_1 e^{-t/2} + \frac{\sqrt{2}}{3}e^t - 8\sqrt{2} + 4\sqrt{2}\,t \\ y_2(t) &= k_2 e^{-2t} - \frac{1}{3\sqrt{2}}e^t - \frac{1}{\sqrt{2}} + \sqrt{2}\,t\,. \end{aligned}$$

Transforming back to the original variables, $\mathbf{x}=\mathbf{T}\mathbf{y}$, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t/2} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-2t} - \frac{1}{2}\begin{pmatrix}17\\15\end{pmatrix} + \begin{pmatrix}5\\3\end{pmatrix}t + \frac{1}{6}\begin{pmatrix}1\\3\end{pmatrix}e^t.$$