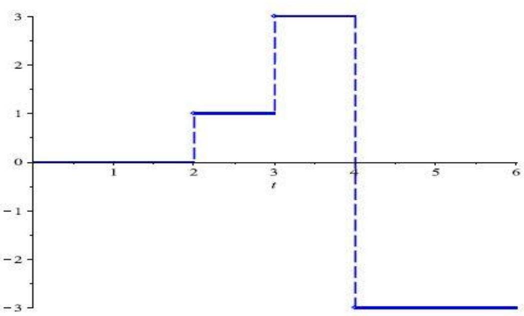
6.3

1.



Problem 2: $g(t) = (t-3)u_2(t) - (t-4)u_3(t)$

$$u_2(t) = \begin{cases} 0, & t<2 \\ 1, & t \geq 2 \end{cases}$$

$$u_3(t) = \begin{cases} 0, & t<3 \\ 1, & t \geq 3 \end{cases}$$

So the critical points are $t=2$ and $t=3$.

if $t<2$: Then,

$g(t) = (t-3)\cdot 0 - (t-4)\cdot 0 = 0$

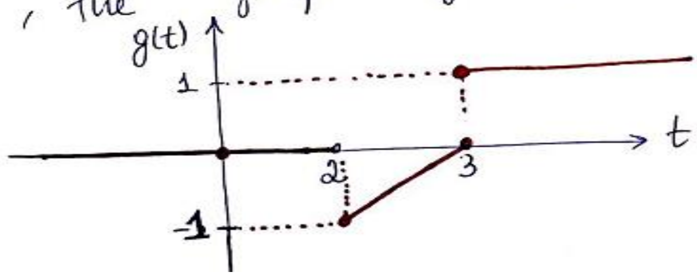
if $2 \leq t < 3$: Then,

$g(t) = (t-3)\cdot 1 - (t-4)\cdot 0 = t-3$

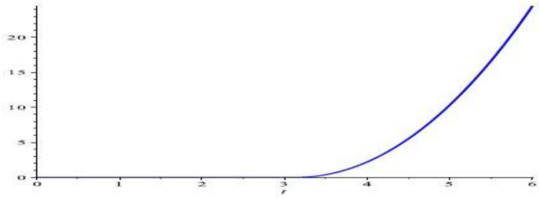
if $3 \leq t$: Then,

$g(t) = (t-3)\cdot 1 - (t-4)\cdot 1 = 1$

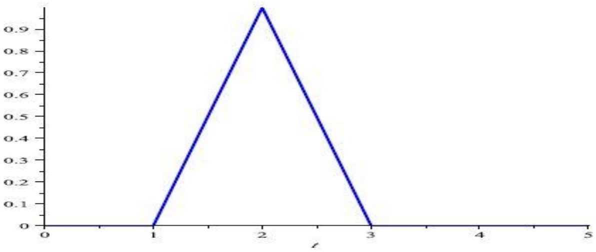
So, the graph of $g(t)$ for $t \geq 0$ is



3.



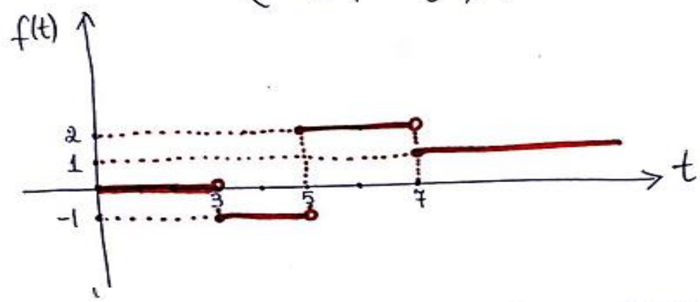
6.



Problem 7 :

$$f(t) = \begin{cases} 0, & 0 \le t < 3 \\ -1, & 3 \le t < 5 \\ 2, & 5 \le t < 7 \\ 1, & t \geq 7 \end{cases}$$

(a)

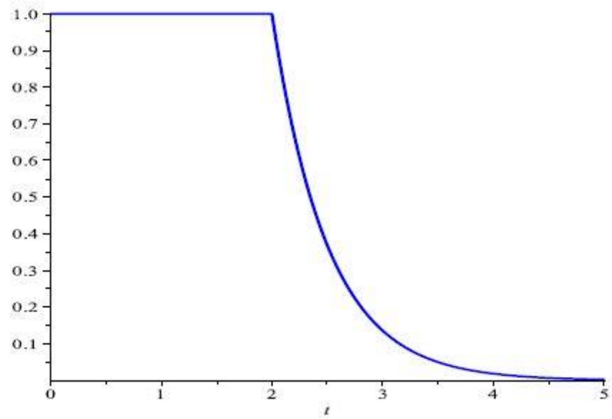


(b) we start with the function $f_1(t) = 0$, which agrees with $f(t)$ on $[0,3)$. To produce the negative jump of 1 units at $t=3$ corresponds to adding $-1 \cdot u_3(t)$, which gives

$f_2(t) = 0 - u_3(t)$, which agrees with $f(t)$ on $[0,5)$. Now, to produce the jump of 3 units at $t=5$, we add $3 \cdot u_5(t)$ to $f_2(t)$, obtaining $f_3(t) = -u_3(t) + 3 \cdot u_5(t)$ which agrees with $f(t)$ on $[0,7)$. Finally, produce the negative jump of 1 unit at $t=7$ is adding $-1 \cdot u_7(t)$ to $f_3(t)$ to obtain t,

$$\boxed{f(t) = -u_3(t) + 3 \cdot u_5(t) - u_7(t)}$$ //

9.(a)



(b) $f(t) = 1 + (e^{-2(t-2)} - 1)u_2(t)$.

13. Using the Heaviside function, we can write $f(t) = (t-2)^3\, u_2(t)$. The Laplace transform has the property that $\mathcal{L}\,[\, u_c(t) f(t-c)] = e^{-cs}\mathcal{L}\,[f(t)]$. Hence

$$\mathcal{L}\left[\,(t-2)^3\, u_2(t)\right] = \frac{6\, e^{-2s}}{s^4}.$$

15. The function can be expressed as $f(t) = (t-\pi)\,[u_\pi(t) - u_{2\pi}(t)]$. Before invoking the translation property of the transform, write the function as

$$f(t) = (t-\pi)\, u_\pi(t) - (t-2\pi)\, u_{2\pi}(t) - \pi\, u_{2\pi}(t)\,.$$

It follows that

$$\mathcal{L}\,[\, f(t)] = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-2\pi s}}{s}.$$

17. Before invoking the translation property of the transform, write the function as

$$f(t) = (t-2)\, u_2(t) - 2\, u_2(t) - (t-3)\, u_3(t) - u_3(t)\,.$$

It follows that

$$\mathcal{L}\,[\, f(t)] = \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}.$$

19. Using the fact that $\mathcal{L}\left[e^{at} f(t)\right] = \mathcal{L}\left[f(t)\right]_{s \to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{3!}{(s-5)^4}\right] = t^3 e^{5t}.$$

22. The inverse transform of the function $2/(s^2-4)$ is $f(t) = \sinh 2t$. Using the translation property of the transform,

$$\mathcal{L}^{-1}\left[\frac{2\,e^{-4s}}{s^2-4}\right] = \sinh\left(2(t-4)\right) \cdot u_4(t).$$

24. Write the function as

$$F(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{e^{-4s}}{s}.$$

It follows from the translation property of the transform, that

$$\mathcal{L}^{-1}\left[\frac{e^{-s} + e^{-2s} - e^{-3s} + e^{-4s}}{s}\right] = u_1(t) + u_2(t) - u_3(t) + u_4(t).$$

25.(a) By definition of the Laplace transform,

$$\mathcal{L}\left[\,f(ct)\right] = \int_0^\infty e^{-st} f(ct)dt.$$

Making a change of variable, $\tau = ct$, we have

$$\mathcal{L}\left[\,f(ct)\right] = \frac{1}{c}\int_0^\infty e^{-s(\tau/c)} f(\tau)d\tau = \frac{1}{c}\int_0^\infty e^{-(s/c)\tau} f(\tau)d\tau.$$

Hence $\mathcal{L}\left[\,f(ct)\right] = (1/c)\,F(s/c)$, where $s/c > a$.

(b) Using the result in part (a),

$$\mathcal{L}\left[f\left(\frac{t}{k}\right)\right] = k\,F(ks).$$

Hence

$$\mathcal{L}^{-1}\left[F(ks)\right] = \frac{1}{k} f\left(\frac{t}{k}\right).$$

(c) From part (b), $\mathcal{L}^{-1}\left[F(as)\right] = (1/a)f(t/a)$ Note that $as + b = a(s + b/a)$. Using the fact that $\mathcal{L}\left[e^{ct} f(t)\right] = \mathcal{L}\left[f(t)\right]_{s \to s-c}$,

$$\mathcal{L}^{-1}\left[F(as+b)\right] = e^{-bt/a}\,\frac{1}{a} f\left(\frac{t}{a}\right).$$

6.4

Problem 1 : $y'' + 9y = f(t)$

$y(0) = 0, \ y'(0) = 1$

$$f(t) = \begin{cases} 1 \ , & 0 \le t < 3\pi \\ 0 \ , & 3\pi \le t < \infty . \end{cases}$$

Taking the Laplace transform of both sides of the ODE, we obtain

$$\mathcal{L}(y'') = s^2 \cdot Y(s) - s y(0) - y'(0)$$

$$\mathcal{L}(y') = s \cdot Y(s) - y(0)$$

$$\mathcal{L}(y) = Y(s)$$

$$\mathcal{L}(y'') + 9 \cdot \mathcal{L}(y) = \mathcal{L}(f(t))$$

$$s^2 \cdot Y(s) - \cancel{s \cdot y(0)}^{\,0} - \cancel{y'(0)}^{\,1} + 9 \cdot Y(s) = \mathcal{L}(f(t))$$

$$(s^2+9) \cdot Y(s) - 1 = \frac{1}{s} - \frac{1}{s} \cdot e^{-3\pi s}$$

Since, $\mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st} dt = \int_0^{3\pi} 1 e^{-st} dt + \int_{3\pi}^{\infty} 0 \cdot e^{-st} dt$

$$= \int_0^{3\pi} e^{-st} dt = \frac{1}{s} - \frac{1}{s} e^{-3\pi s} .$$

Thus,

$$Y(s) = \frac{1}{s^2+9} + \frac{1}{s(s^2+9)} - \frac{1}{s\cdot(s^2+9)}\cdot e^{-3\pi s}$$

observe that,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right) = \frac{1}{3}\cdot \sin(3t)$$

and $$\frac{1}{s\cdot(s^2+9)} = \frac{1}{9}\left(\frac{s}{s^2} - \frac{s}{s^2+9}\right) = \frac{1}{9}\cdot\frac{1}{s} - \frac{1}{9}\cdot\frac{s}{s^2+9}$$

So that,

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+9)}\right) = \mathcal{L}^{-1}\left(\frac{1}{9\cdot s}\right) - \mathcal{L}^{-1}\left(\frac{s}{9\cdot(s^2+9)}\right)$$

$$= \frac{1}{9}\cdot 1 - \frac{1}{9}\cdot\cos(3t)$$

And, by Theorem 6.3.1

$$\mathcal{L}^{-1}\left(\frac{e^{-3\pi s}}{s(s^2+9)}\right) = \mathcal{L}^{-1}\left(\frac{e^{-3\pi s}}{9s}\right) - \mathcal{L}^{-1}\left(\frac{s\cdot e^{-3\pi s}}{9(s^2+9)}\right)$$

$$= \left(\frac{1}{9} - \frac{1}{9}\cos(3(t-3\pi))\right)u_{3\pi} = \left(\frac{1}{9} + \frac{1}{9}\cos 3t\right)u_{3\pi}$$

Thus,

$$y(t) = \mathcal{L}^{-1}(Y(s))$$

$$= \frac{1}{3}\sin 3t + \left(\frac{1}{9} - \frac{1}{9}\cos 3t\right) + \left(\frac{1}{9} + \frac{1}{9}\cos 3t\right)u_{3\pi}$$

Problem 3 : $y'' + 4y = \sin t - u_{2\pi}(t) \cdot \sin(t-2\pi)$

$0 = y(0), \; \& \; y'(0) = 0$

Taking the Laplace transformation of both sides of the ODE, we obtain

$$\mathcal{L}(y'') + 4\,\mathcal{L}(y) = \mathcal{L}\left(\sin t - u_{2\pi}(t)\sin(t-2\pi)\right)$$

$$\left[s^2 \cdot Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0}\right] + 4 \cdot Y(s) = \frac{1}{s^2+1} - e^{2\pi s} \cdot \frac{1}{s^2+1}$$

$$Y(s) \cdot (s^2+4) = \frac{1}{s^2+1} - \frac{e^{2\pi s}}{s^2+1}$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)} - \frac{e^{2\pi s}}{(s^2+1)(s^2+4)}$$

observe that,

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

So that,

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)(s^2+4)}\right) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{1}{3} \cdot \sin t - \frac{1}{6} \cdot \sin 2t$$

and,

$$\mathcal{L}^{-1}\left(\frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}\right) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{e^{-2\pi s}}{s^2+1}\right) - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{e^{-2\pi s}}{s^2+4}\right)$$

$$= \frac{1}{3} u_{2\pi}(t) \cdot \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3} u_{2\pi}(t) \cdot \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{1}{3} \sin(t-2\pi) u_{2\pi}(t) - \frac{1}{3} u_{2\pi}(t) \frac{1}{2} \sin 2(t-2\pi)$$

$$= \left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right) u_{2\pi}(t)$$

Thus,

$$y(t) = \left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right) + \left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right) u_{2\pi}(t)$$

$$= \frac{1}{6}\left(2\sin t - \sin 2t\right) + \frac{1}{6}\left(2\sin t - \sin 2t\right) u_{2\pi}(t)$$

OR,

$$y(t) = \frac{1}{6}\left(1 - u_{2\pi}(t)\right) \cdot \left(2\sin t - \sin 2t\right) //$$

5.(a) Let $f(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 3\,[s\,Y(s) - \,y(0)] + 2\,Y(s) = \mathcal{L}\,[f(t)]\,.$$

Applying the initial conditions,

$$s^2\,Y(s) + 3s\,Y(s) + 2\,Y(s) - s - 3 = \mathcal{L}\,[f(t)]\,.$$

The transform of the forcing function is

$$\mathcal{L}\,[f(t)] = \frac{1}{s} - \frac{e^{-10s}}{s}\,.$$

Solving for the transform,

$$Y(s) = \frac{s+3}{s^2+3s+2} + \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right], \quad \frac{s+3}{s^2+3s+2} = \frac{2}{s+1} - \frac{1}{s+2}.$$

Hence

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+3s+2)}\right] = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}\,.$$

Based on Theorem 6.3.1,

$$\mathcal{L}^{-1}\left[\frac{e^{-10s}}{s(s^2+3s+2)}\right] = \frac{1}{2}\left[1 + e^{-2(t-10)} - 2e^{-(t-10)}\right]u_{10}(t)\,.$$

Hence the solution of the IVP is

$$y(t) = 2e^{-t} - e^{-2t} + \frac{1}{2}\,[1 - u_{10}(t)] + \frac{e^{-2t}}{2} - e^{-t} - \frac{1}{2}\left[e^{-(2t-20)} - 2e^{-(t-10)}\right]u_{10}(t)\,.$$

7.(a) Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2\,Y(s) + Y(s) - 2s = \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{2s}{s^2+1} + \frac{e^{-3\pi s}}{s(s^2+1)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

Hence

$$Y(s) = \frac{2s}{s^2+1} + e^{-3\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right].$$

Taking the inverse transform, the solution of the IVP is

$$y(t) = 2\cos\,t + [1 - \cos(t - 3\pi)]\,u_{3\pi}(t) = 2\cos\,t + [1 + \cos\,t\,]\,u_{3\pi}(t)\,.$$

11.(a) Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) + 4Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2 Y(s) + 4Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-\pi s}}{s(s^2+4)} - \frac{e^{-3\pi s}}{s(s^2+4)}.$$

Using partial fractions,

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right].$$

Taking the inverse transform, and applying Theorem 6.3.1,

$$\begin{aligned} y(t) &= \frac{1}{4}\left[1 - \cos(2t - 2\pi)\right] u_\pi(t) - \frac{1}{4}\left[1 - \cos(2t - 6\pi)\right] u_{3\pi}(t) \\ &= \frac{1}{4}\left[u_\pi(t) - u_{3\pi}(t)\right] - \frac{1}{4}\cos 2t \cdot \left[u_\pi(t) - u_{3\pi}(t)\right]. \end{aligned}$$

6.6

By definition of Convolution,

$$\begin{aligned} ((f*g)*h)\ (u) &= \int_{\mathbb{R}} (f*g)\ (x)\, h\,(u-x)\, dx \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f\,(y)\, g\,(x-y)\, dy\right] h\,(u-x)\, dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f\,(y)\, g\,(x-y)\, h\,(u-x)\, dy\, dx. \end{aligned}$$

By Fubini's theorem we can switch the integration,

$$\begin{aligned} ((f*g)*h)\ (u) &= \int_{\mathbb{R}} \int_{\mathbb{R}} f\,(y)\, g\,(x-y)\, h\,(u-x)\, dx\, dy \\ &= \int_{\mathbb{R}} f\,(y) \left[\int_{\mathbb{R}} g\,(x-y)\, h\,(u-x)\, dx\right] dy. \end{aligned}$$

Look at the inner integral, by translation invariant

$$\int_{\mathbb{R}} g(x-y)\,h(u-x)\,dx = \int_{\mathbb{R}} g((x+y)-y)\,h(u-(x+y))\,dx$$
$$= \int_{\mathbb{R}} g(x)\,h((u-y)-x)\,dx$$
$$= (g*h)\ (u-y)\,.$$

So we have shown that

$$((f*g)*h)\ (u) = \int_{\mathbb{R}} f(y)\ (g*h)\ (u-y)\,dy,$$

which by definition is $(f*(g*h))\ (u)$. Hence convolution is associative

3. It follows directly that

$$(f * f)(t) = \int_0^t \cos(t-\tau)\,\cos(\tau)\,d\tau = \frac{1}{2}\int_0^t [\cos(t-2\tau)+\cos(t)]d\tau = \frac{1}{2}(\sin t + t\cos t).$$

The range of the resulting function is $\mathbb{R}$.

5. We have $\mathcal{L}\,[e^{-t}] = 1/(s+1)$ and $\mathcal{L}\,[\sin 2t] = 2/(s^2+4)$. Based on Theorem 6.6.1,

$$\mathcal{L}\left[\int_0^t e^{-(t-\tau)}\sin(2\tau)\,d\tau\right] = \frac{1}{s+1}\cdot\frac{2}{s^2+4} = \frac{2}{(s+1)(s^2+4)}\,.$$

7. We have $f(t) = (g * h)(t)$, in which $g(t) = \sin t$ and $h(t) = \cos 2t$. The transform of the convolution integral is

$$\mathcal{L}\left[\int_0^t g(t-\tau)h(\tau)\,d\tau\right] = \frac{1}{s^2+1}\cdot\frac{s}{s^2+4} = \frac{s}{(s^2+1)(s^2+4)}\,.$$

Problem 8 : $F(s) = \dfrac{1}{s^4(s^2+4)}$

note that,

$\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{1}{6}\cdot t^3$ and

$\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\cdot \sin 2t$

Based on the convolution theorem,

$$\mathcal{L}^{-1}\left(\frac{1}{s^4}\cdot\frac{1}{s^2+4}\right) = \frac{1}{6}\cdot\frac{1}{2}\cdot\int_0^t (t-\tau)^3\cdot\sin 2\tau\,d\tau$$

10. We first note that

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^3}\right] = \frac{1}{2}t^2\, e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin\, 2t\,.$$

Based on the convolution theorem,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{(s+1)^3(s^2+4)}\right] &= \frac{1}{4}\int_0^t (t-\tau)^2 e^{-(t-\tau)}\sin\, 2\tau\, d\tau \\ &= \frac{1}{4}\int_0^t \tau^2\, e^{-\tau}\, \sin(2t-2\tau)\, d\tau\,.\end{aligned}$$

13. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\, Y(s) - s - 1 + \omega^2\, Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{s+1}{s^2+\omega^2} + \frac{G(s)}{s^2+\omega^2}.$$

As shown in a related situation, Problem 11,

$$\mathcal{L}^{-1}\left[\frac{G(s)}{s^2+\omega^2}\right] = \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))\, g(\tau)\, d\tau\,.$$

Hence the solution of the IVP is

$$y(t) = \cos(\omega t) + \frac{1}{\omega}\sin(\omega t) + \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))\, g(\tau)\, d\tau\,.$$

15. The transform of the ODE (given the specified initial conditions) is

$$4s^2\, Y(s) + 4s\, Y(s) + 17Y(s) - 4 = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{G(s)+4}{4s^2+4s+17}.$$

First write

$$\frac{1}{4s^2+4s+17} = \frac{\frac{1}{4}}{(s+\frac{1}{2})^2+4}\,.$$

Based on the elementary properties of the Laplace transform,

$$\mathcal{L}^{-1}\left[\frac{1}{4s^2+4s+17}\right] = \frac{1}{8}e^{-t/2}\sin\, 2t\,.$$

Applying the convolution theorem, the solution of the IVP is

$$y(t) = \frac{1}{2}e^{-t/2}\sin 2t + \frac{1}{8}\int_0^t e^{-(t-\tau)/2}\,\sin\, 2(t-\tau)\, g(\tau)\, d\tau\,.$$

17. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) - s + 2 + 4\,[s\,Y(s) - 1] + 4Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{s+2}{(s+2)^2} + \frac{G(s)}{(s+2)^2}.$$

We can write

$$\frac{s+2}{(s+2)^2} = \frac{1}{s+2}.$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}.$$

Based on the convolution theorem, the solution of the IVP is

$$y(t) = e^{-2t} + \int_0^t (t-\tau)e^{-2(t-\tau)}g(\tau)\,d\tau\,.$$

19. The transform of the ODE (given the specified initial conditions) is

$$s^4 Y(s) - Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{G(s)}{s^4-1}.$$

First write

$$\frac{1}{s^4-1} = \frac{1}{2}\left[\frac{1}{s^2-1} - \frac{1}{s^2+1}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^4-1}\right] = \frac{1}{2}\left[\sinh t - \sin t\right]\,.$$

Based on the convolution theorem, the solution of the IVP is

$$y(t) = \frac{1}{2}\int_0^t \left[\sinh(t-\tau) - \sin(t-\tau)\right] g(\tau)\,d\tau\,.$$