SAMPLE QUESTIONS (CHAPTER 3)

Section 3.1:

Q1) Find the solution of the given initial value problem, and describe its behavior as t increases.

$$y'' + 8y' - 9y = 0 \quad , \quad y(1) = 1, \quad y'(1) = 0$$

$$y = e^{rt} \quad , \quad y' = r.e^{rt} \quad , \quad y'' = r^2 e^{rt}$$

Placing in the equation, we obtain

$$r^2 e^{rt} + 8r\,e^{rt} - 9e^{rt} = 0 \quad , \text{ i.e,}$$

$$e^{rt}\,(r^2 + 8r - 9) = 0$$

e^{rt} is nonzero for any $t \in \mathbb{R}$, so we obtain the characteristic equation:

$$r^2 + 8r - 9 = 0$$

$$\begin{matrix} r & +9 \\ r & -1 \end{matrix}$$

$$(r-1)(r+9) = 0$$

two roots:

$$\begin{cases} r_1 = 1 \\ r_2 = -9 \end{cases}$$

So we have two solutions:

$$y_1(t) = e^t \quad , \quad y_2(t) = e^{-9t}$$

Solutions have the general form:

$$y(t) = c_1 . e^t + c_2 . e^{-9t}$$

But we have initial conditions, so there'll be a unique solution.

$$y(1) = c_1 . e^1 + c_2 . e^{-9} = 1$$

$$y'(t) = c_1 e^t - 9 c_2 e^{-9t}$$

$$y'(1) = c_1 . e^1 - 9 c_2 e^{-9} = 0$$

So $10 c_2 e^{-9} = 1 \Rightarrow c_2 = \frac{e^9}{10}$

and $c_1 = \frac{9}{10e}$

so that

$$y(t) = \frac{9}{10} e^{t-1} + \frac{1}{10} e^{9(1-t)}$$

As t increases, second part of the function will get closer to 0, and the first part will get bigger and bigger. As $t \to \infty$, $y(t)$ diverges.

Section 3.2:

Q1) Determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution.

$$(x-3)y'' + xy' + (\ln|x|)y = 0, \quad y(1) = 0, \quad y'(1) = 1.$$

First we write the equation in the form:

$$y'' + \frac{x}{x-3}y' + \frac{\ln|x|}{x-3}y = 0$$

See that $\ln|x|$ does not exist for $x=0$, also for $x=3$, we have discontinuity for coefficient functions.

As the initial condition point, $t_0 = 1$ is in the interval $(0,3)$, we choose that interval, for the unique, twice differentiable solution.

Q2) Verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

$$x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x>0;$$

$$y_1(x) = x, \qquad y_2(x) = xe^x$$

$$\left.\begin{array}{l} y_1(x) = x \\ y_1'(x) = 1 \\ y_1''(x) = 0 \end{array}\right\}$$

Subsituting,

$$x^2 . 0 - x(x+2).1 + (x+2)x = -x^2 - 2x + x^2 + 2x = 0$$

✓ (y_1 is a solution)

$$\left.\begin{array}{l} y_2(x) = xe^x \\ y_2'(x) = e^x + xe^x \\ y_2''(x) = 2e^x + xe^x \end{array}\right\}$$

Subsituting,

$$2x^2e^x + x^3e^x - (x^2+2x)(e^x + xe^x) + (x+2)xe^x = 2x^2e^x + x^3e^x - x^2e^x - x^3e^x - 2xe^x - 2x^2e^x + x^2e^x + 2xe^x = 0$$

✓ (y_2 is a solution)

Now we'll check their Wronskian,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$= x(e^x + xe^x) - x.e^x.1$$

$$= xe^x + x^2e^x - xe^x$$

$$= x^2 e^x$$

For $x > 0$, $W(x) = x^2e^x \neq 0$. So we have y_1 & y_2 constituting a fundamental set of solutions.

Section 3.3:

Q1) Find the solution of the given initial value problem.

$$y'' - 6y' + 13y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2.$$

The equation above has the characteristic equation:

$$r^2 - 6r + 13 = 0$$, which has the roots:

$$r_{1,2} = \frac{6 \mp \sqrt{36 - 4.13.1}}{2} = 3 \mp 2i$$

$$r_1 = 3 + 2i \qquad r_2 = 3 - 2i$$

See that $\lambda = 3$, $\mu = 2$.

$$y(t) = c_1 . e^{3t} . \cos 2t + c_2 . e^{3t} \sin 2t$$

$$y(\pi/2) = c_1 . e^{3\pi/2} \cos\pi + c_2 . e^{3\pi/2} \sin\pi$$

$$- c_1 e^{3\pi/2} = 0, \quad \boxed{c_1 = 0}$$

$$y'(t) = c_2 (3e^{3t} \sin 2t + 2 e^{3t} \cos 2t)$$

$$y'(\pi/2) = c_2 (3e^{3\pi/2} \sin\pi + 2 e^{3\pi/2} \cos\pi)$$

$$-2 \cdot c_2 \cdot e^{3\pi/2} = 2$$

$$\boxed{c_2 = -e^{-3\pi/2}}$$

So that

$$y(t) = -e^{3t - 3\pi/2} \sin 2t$$

Section 3.4:

Q1) Solve the given initial value problem.

$$9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1$$

See that the characteristic equation is

$$9r^2 - 12r + 4 = 0.$$

$$\begin{matrix} 3r & -2 \\ 3r & -2 \end{matrix}$$

and we have

$$r_1 = r_2 = \frac{2}{3}$$

Thus the general solution

$$y(t) = c_1 \cdot e^{2t/3} + c_2 \cdot t \cdot e^{2t/3}$$

$$y(0) = c_1 + c_2 \cdot 0 = 2 \quad , \text{ i.e,} \quad c_1 = 2$$

$$y'(t) = \frac{2c_1}{3} e^{2t/3} + c_2 \left(e^{2t/3} + \frac{2t}{3} e^{2t/3} \right)$$

$$y'(0) = \frac{2c_1}{3} + c_2 = -1 \quad , \text{ i.e,} \quad c_2 = \frac{-7}{3}$$

And the unique solution to the initial value problem is:

$$\boxed{y(t) = 2e^{2t/3} - \frac{7}{3} t \cdot e^{2t/3}}$$

Q2) Use the method of reduction of order to find a second solution of the given differential equation.

$$t^2 y'' - t(t+2) y' + (t+2) y = 0 \quad , t > 0;$$

and $y_1(t) = t$.

First, we set $y_2(t) = v(t) \cdot t$.

$$y_2'(t) = v(t) + t \cdot v'(t)$$

$$y_2''(t) = 2v'(t) + t \cdot v''(t)$$

Substituting,

$$2t^2 v'(t) + t^3 v''(t) - t^2 v(t) - t^3 v'(t) - 2t v(t) - 2t^2 v'(t)$$

$$+ t^2 v(t) + 2v(t) t = 0$$

and that leads to

$$t^3 v''(t) - t^3 v'(t) = 0, \quad \text{or, equivalently,}$$

$$v''(t) - v'(t) = 0 \quad (\text{recall that } t>0).$$

Let $w = v'$, then

$$w' - w = 0$$

this has the immediate solution $w(t) = e^t$

so that, again, $v(t) = e^t$.

Therefore $y_2(t) = t \cdot v(t) = t \cdot e^t$

Section 3.5:

Find the solution of the given initial value problem.

Q1) $y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1.$

First, we solve the homogenous equation,

$$r^2 + r - 2 = 0 = (r+2)(r-1)$$

$$r_1 = 1 \quad r_2 = -2$$

$$y_c(t) = c_1 \cdot e^t + c_2 e^{-2t} \quad (\text{complementary solution})$$

Set $Y(t) = At + B$

$Y'(t) = A$

$Y''(t) = 0$

Subsituting,

$$0 + A - 2(At + B) = 2t$$

$$(-2A)t + (A - 2B) = 2t$$

$$A = -1$$

$$B = -1/2$$

$$Y(t) = -t - 1/2$$

General solution of the nonhomogenous equation

$$y(t) = c_1 e^t + c_2 e^{-2t} - t - 1/2$$

$$y'(t) = c_1 e^t - 2c_2 e^{-2t} - 1$$

$y(0) = c_1 + c_2 - 1/2 = 0 \Rightarrow c_1 + c_2 = 1/2$

$y'(0) = c_1 - 2c_2 - 1 = 1 \Rightarrow c_1 - 2c_2 = 2$

$$c_2 = -1/2$$

$$c_1 = 1$$

So, the solution of the initial value problem

$$\boxed{y(t) = e^t - \frac{1}{2} e^{-2t} - t - 1/2}$$

Q2) $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$

$$r^2 - 2r + 1 = 0 \qquad r_1 = r_2 = 1$$

$$(r-1)(r-1) = 0$$

So we have the solutions

$$y_1(t) = e^t \qquad y_2(t) = te^t$$

te^t is a solution for the homogeneous equation, so we set:

$$Y_1(t) = At^3e^t + Bt^2e^t$$

$$Y_1'(t) = 3At^2e^t + At^3e^t + 2Bte^t + Bt^2e^t$$

$$Y_1''(t) = 6Ate^t + 6At^2e^t + At^3e^t + 2Be^t + 4Bte^t$$
$$+ Bt^2e^t$$

Substituting,

$$At^3e^t + 6At^2e^t + Bt^2e^t + 6Ate^t + 4Bte^t + 2Be^t - 6At^2e^t - 2At^3e^t$$
$$- 4Bte^t - 2Bt^2e^t + At^3e^t + Bt^2e^t = te^t$$

$$6Ate^t + 2Be^t = t.e^t$$

$$A = 1/6 \qquad B = 0$$

$$Y_1(t) = \frac{t^3e^t}{6}$$

$$\left.\begin{aligned} Y_2(t) &= A, \quad A \in \mathbb{R} \\ Y_2'(t) &= 0 \\ Y_2''(t) &= 0 \end{aligned}\right\} \text{ Subsituting,}$$

$$0 - 2.0 + A = 4$$

$$A = 4$$

$$Y_2(t) = 4, \text{ therefore}$$

$$Y(t) = \frac{t^3 e^t}{6} + 4$$

Thus

$$y(t) = c_1 e^t + c_2 t e^t + \frac{t^3 e^t}{6} + 4$$

$$y(0) = c_1 + 4 = 1 \Rightarrow c_1 = -3$$

$$y'(t) = c_1 e^t + c_2 e^t + c_2 t e^t + \frac{t^2 e^t}{2} + \frac{t^3 e^t}{6}$$

$$y'(0) = c_1 + c_2 = 1$$

$$\Rightarrow c_2 = 4$$

So the solution of the initial value problem:

$$\boxed{y(t) = -3e^t + 4te^t + \frac{t^3 e^t}{6} + 4}$$

Q3) $y'' + 4y = t^2 + 3e^t \,, \quad y(0) = 0, \quad y'(0) = 2$

$$r^2 + 4 = 0 \qquad r_{1,2} = 0 \mp 2i$$

$$r^2 = -4$$

$$\lambda = 0 \qquad \mu = 2$$

$$y_c(t) = c_1 . \cos 2t + c_2 . \sin 2t$$

$$\left.\begin{aligned} Y_1(t) &= At^2 + Bt + C \\ Y_1'(t) &= 2At + B \\ Y_1''(t) &= 2A \end{aligned}\right\} \qquad \begin{aligned} &2A + 4At^2 + 4Bt + 4C = t^2 \\ &B = 0 \quad A = 1/4 \quad C = -1/8 \end{aligned}$$

$$Y_1(t) = t^2/4 - 1/8$$

$$Y_2(t) = de^t = Y_2'(t) = Y_2''(t)$$

$$de^t + 4d\,e^t = 5de^t = 3e^t \Rightarrow d = \frac{3}{5}$$

$$Y_2(t) = \frac{3}{5} e^t$$

So the general solution of the nonhomogeneous equation,

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + 3e^t/5 + t^2/4 - 1/8$$

$$y(0) = c_1 + \frac{3}{5} - \frac{1}{8} = 0 \Rightarrow \boxed{c_1 = \frac{-19}{40}}$$

$$y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t + 3e^t/5 + t/2$$

$$y'(0) = 2c_2 + 3/5 = 2 \Rightarrow \boxed{c_2 = \frac{7}{10}}$$

Hence $y(t) = \frac{-19}{40} \cos 2t + \frac{7}{10} \sin 2t + \frac{3}{5} e^t + \frac{t^2}{4} - \frac{1}{8}$

(94) $y'' + 4y = 2\sin 2t, \quad y(0) = 2, \quad y'(0) = -1$

$r^2 + 4 = 0 \quad \Rightarrow \quad y_c(t) = C_1 \cos 2t + C_2 \sin 2t$

Let $Y(t) = \underline{A \cos 2t} + \underline{B \sin 2t}$ won't work, as these two are solutions of the homogeneous equation.

Set $Y(t) = At \cos 2t + Bt \sin 2t$

$Y'(t) = -2At \sin 2t + A\cos 2t + 2Bt \cos 2t + B \sin 2t$

$Y''(t) = -4At \cos 2t - 4A \sin 2t - 4Bt \sin 2t + 4B \cos 2t$

$-4At \cos 2t - 4Bt \sin 2t - 4A \sin 2t + 4B \cos 2t + 4At \cos 2t + 4Bt \sin 2t = 2 \sin 2t$

$-4A \sin 2t + 4B \cos 2t = 2 \sin 2t$

$B = 0$

$A = -1/2$

So $Y(t) = -t \cos 2t / 2$

$y(t) = C_1 \cos 2t + C_2 \sin 2t - t \cos 2t / 2$

$C_1 = 2 \qquad C_2 = -1/4$

$\boxed{y(t) = 2\cos 2t - \sin 2t / 4 - t \cdot \cos 2t / 2}$