Math 204: Midterm Exam # 1
Spring 2018

- Write your name and Student ID number in the space provided below and sign.

Last Name, First Name:	
Student ID Number:	
Signature:	KEY

- Mark the section you are registered below.

  - ☐ Section 1 (Mon. & Wed. 14:30-15:45. Instructor: Hasan İnci)
  - ☐ Section 2 (Tue. & Thu. 16:00-17:15. Instructor: Tolga Etgü)
  - ☐ Section 3 (Tue. & Thu. 13:00-14:15. Instructor: Tolga Etgü)

- You have 90 minutes.
- ***You must show all your work to receive full credit.***

To be filled by the grader:

Problem 1:	
Problem 2:	
Problem 3:	
Problem 4:	
Problem 5:	
Problem 6:	
Problem 7:	
Total Grade:	

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Problem 1. Solve the following initial-value problems.

a) (12 pts.) $t^2y' + ty = t^4 + 3t^2\ ,\ y(2) = 9$

$$\left.\begin{array}{l} y' + \frac{1}{t}\cdot y = t^2+3 \\ \mu = e^{\int \frac{1}{t} dt} \Rightarrow \mu = t \end{array}\right\} \Rightarrow \underbrace{ty' + y}_{(y\cdot t)'} = t^3 + 3t$$

$$ty = \int t^3 + 3t\,dt = \frac{t^4}{4} + \frac{3t^2}{2} + C$$

General Solution: $y = \frac{t^3}{4} + \frac{3t}{2} + \frac{C}{t}$

$y(2) = 9 \Rightarrow 9 = 2 + 3 + \frac{C}{2} \Rightarrow C = 8$

Unique solution of the IVP: $\boxed{y = \frac{t^3}{4} + \frac{3t}{2} + \frac{8}{t}}$

b) (10 pts.) $yy' = 4x(y^2 + 1)\ ,\ y(0) = -1$

Separable equation: $\displaystyle\int \frac{y\,dy}{y^2+1} = \int 4x\,dx$

$$\frac{1}{2}\ln(y^2+1) = 2x^2 + C$$

$$y^2 + 1 = A\cdot e^{4x^2}$$

$$y = \mp\sqrt{A\cdot e^{4x^2} - 1}$$

Applying initial condition: $y(0) = -1 \Rightarrow -1 = -\sqrt{A-1} \Rightarrow A = 2$

Solution of the IVP: $\boxed{y = -\sqrt{2e^{4x^2} - 1}}$

Problem 2. What is the largest interval on which the following initial-value problem has a unique solution? (10 pts.)

$$(t^2 + 2t - 15)y' + (\sin^2 t)y = \cos^3 t\,, \quad y(2) = 1$$

$$y' + \frac{\sin^2 t}{(t+5)(t-3)}y = \frac{\cos^3 t}{(t+5)(t-3)}$$

The largest open interval containing $t_0 = 2$ on which $\frac{\sin^2 t}{(t+5)(t-3)}$ and $\frac{\cos^3 t}{(t+5)(t-3)}$ are cont. is $(-5,3)$

Therefore, the IVP has a unique solution on: $(-5,3)$

Problem 3. Find all the solutions of the following equation. (12 pts.)

$$(x + y)^2 dx + (2xy + x^2 - 1)dy = 0$$

$$\frac{\partial((x+y)^2)}{\partial y} = 2(x+y) \qquad \frac{\partial(2xy+x^2-1)}{\partial x} = 2(x+y)$$

Hence, the equation is exact.

$$F(x,y) = \int 2xy + x^2 - 1\,dy = xy^2 + x^2y - y + g(x)$$

So $(x+y)^2 = \frac{\partial F}{\partial x} = y^2 + 2xy + g'(x)$

$$g'(x) = x^2 \Rightarrow g(x) = \frac{x^3}{3} + C$$

Hence, the general solution is:

$$xy^2 + x^2y - y + \frac{x^3}{3} = C$$

Problem 4. Solve the following initial-value problem. (10 pts.)

$$y'' + 16y = 0\ ,\ y(0) = 2\ ,\ y'(0) = -4$$

Characteristic Equation: $r^2+16=0$

$r_{1,2} = \mp 4i$

General solution: $y = c_1 \cos 4t + c_2 \sin 4t$

$y' = -4c_1 \sin 4t + 4c_2 \cos(4t)$

$y(0) = 2 \Rightarrow c_1 = 2$

$y'(0) = -4 \Rightarrow 4c_2 = -4$

$c_2 = -1$

The solution of the IVP:

$y = 2\cos 4t - \sin 4t$

Problem 5. Verify that $y(t) = t$ is a solution and solve the following equation. (12 pts.)

Hint: Look for a solution of the form $y(t) = v(t) \cdot t$

$$t^2 y'' + t(t-2)y' - (t-2)y = 0\ ,\ t > 0\ .$$

$$\left.\begin{array}{l} \text{If } y = t \\ \Rightarrow y' = 1 \\ y'' = 0 \end{array}\right\} \quad t^2 \cdot 0 + t(t-2) - (t-2)t = 0$$

So, $y_1(t) = t$ is a solution.

$y_2 = v \cdot t \Rightarrow y_2' = v't + v,\ y_2'' = v''t + 2v'$

Plugging into the equation:

$0 = t^2(v''t + 2v') + t(t-2)(v't+v) - (t-2)vt$

$0 = t^3 v'' + t^3 v' \Rightarrow v'' + v' = 0$

$$\left.\begin{array}{l} \text{Let } w = v' \\ w' = v'' \end{array}\right\} \quad \begin{array}{l} w' + w = 0 \\ \dfrac{dw}{w} = -dt \Rightarrow w = e^{-t} \Rightarrow v' = e^{-t} \end{array}$$

$\Rightarrow v = -e^{-t}$ (or $v = e^{-t}$)

Therefore, $y_2 = t \cdot e^{-t}$ is a solution

General Solution:

$$W(y_1, y_2) = \begin{vmatrix} t & te^{-t} \\ 1 & -te^{-t} + e^{-t} \end{vmatrix} = -t^2 e^{-t} \neq 0 \text{ since } t > 0 \Rightarrow \boxed{y = c_1 t + c_2 t e^{-t}}$$

Problem 6. a) Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are solutions, and also verify that they form a fundamental set of solutions of the following equation. (10 pts.)

$$y'' - 2t^{-2}y = 0 \ , \ t > 0$$

$$\left.\begin{aligned} y_1 &= t^2 \\ y_1' &= 2t \\ y_1'' &= 2 \end{aligned}\right\} \Rightarrow 2 - 2t^{-2} \cdot t^2 = 0 \ \checkmark$$

y_1 is a solution

$$\left.\begin{aligned} y_2 &= t^{-1} \\ y_2' &= -t^{-2} \\ y_2'' &= 2t^{-3} \end{aligned}\right\} \Rightarrow 2t^{-3} - 2t^{-2}t^{-1} = 0 \ \checkmark$$

y_2 is a solution

$$W(y_1, y_2) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -1 - 2 = -3 \neq 0$$

$\Rightarrow \{y_1, y_2\}$ form a fundamental set of solutions.

b) Find all the solutions of the following equation. (12 pts.)

$$y'' - 2t^{-2}y = 3t^{-1} - t^{-2} \ , \ t > 0$$

<u>variation of parameters:</u>

$y = u_1 t^2 + u_2 t^{-1}$, where

$$u_1 = \int \frac{-(3t^{-1} - t^{-2})t^{-1}}{-3} \, dt = \frac{t^{-1}}{-1} - \frac{t^{-2}}{-6} + c_1$$

$$u_2 = \int \frac{(3t^{-1} - t^{-2})t^2}{-3} \, dt = \frac{-t^2}{2} + \frac{t}{3} + c_2$$

So the general solution is:

$$y = \left(-t^{-1} + \frac{t^{-2}}{6} + c_1\right)t^2 + \left(\frac{-t^2}{2} + \frac{t}{3} + c_2\right)t^{-1}$$

OR $y = c_1 t^2 + c_2 t^{-1} - \frac{3t}{2} + \frac{1}{2}$

Problem 7. Suppose that $p(t)$ and $q(t)$ are continuous on an open interval I, (12 pts.) and y_1 and y_2 are solutions of

$$y'' + p(t)y' + q(t)y = 0$$

on I such that $y_1'(t_0) = y_2'(t_0) = 0$ for a point t_0 in I. Prove that the equation above has a solution on I which is not of the form $c_1y_1 + c_2y_2$, where c_1 and c_2 are constants. State the existence theorem you use in the proof.

Observe that $W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ 0 & 0 \end{vmatrix}$

$= 0$

So, $\{y_1, y_2\}$ is not a fundamental set of solutions.

On the other hand, let y_3 be the unique solution of the IVP:

$y'' + p(t)y' + q(t)y = 0$
$y(t_0) = 0$
$y'(t_0) = 1$

The existence and uniqueness theorem:
If p, q are continuous on an open interval I containing t_0, then the IVP
$y'' + py' + qy = 0$, $y(t_0) = A$, $y'(t_0) = B$ has a unique solution on I.

$y_3 \neq c_1y_1 + c_2y_2$ for any constants c_1 and c_2.
Since otherwise $y_3'(t_0) = 1$ would be $c_1y_1'(t_0) + c_2y_2'(t_0) = c_1 \cdot 0 + c_2 \cdot 0 = 0$.