3.6

1. The solution of the homogeneous equation is $y_c(t) = c_1e^{2t} + c_2e^{3t}$. The functions $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$ form a fundamental set of solutions. The Wronskian

of these functions is $W(y_1, y_2) = e^{5t}$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{e^{3t}(4e^t)}{W(t)} dt = 4e^{-t} \quad \text{and} \quad u_2(t) = \int \frac{e^{2t}(4e^t)}{W(t)} dt = -2e^{-2t}.$$

Hence the particular solution is $Y(t) = 4e^t - 2e^t = 2e^t$.

3. The solution of the homogeneous equation is $y_c(t) = c_1 e^{-t} + c_2 t e^{-t}$. The functions $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-2t}$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{te^{-t}(6e^{-t})}{W(t)} dt = -3t^2 \quad \text{and} \quad u_2(t) = \int \frac{e^{-t}(6e^{-t})}{W(t)} dt = 6t.$$

Hence the particular solution is $Y(t) = -3t^2 e^{-t} + 6t^2 e^{-t} = 3t^2 e^{-t}$.

4. The functions $y_1(t) = e^{t/2}$ and $y_2(t) = te^{t/2}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^t$. First write the equation in standard form, so that $g(t) = 2e^{t/2}$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{te^{t/2}(2e^{t/2})}{W(t)} dt = -t^2 \quad \text{and} \quad u_2(t) = \int \frac{e^{t/2}(2e^{t/2})}{W(t)} dt = 2t.$$

Hence the particular solution is $Y(t) = -t^2 e^{t/2} + 2t^2 e^{t/2} = t^2 e^{t/2}$.

6. The solution of the homogeneous equation is $y_c(t) = c_1 \cos 3t + c_2 \sin 3t$. The two functions $y_1(t) = \cos 3t$ and $y_2(t) = \sin 3t$ form a fundamental set of solutions, with $W(y_1, y_2) = 3$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{\sin 3t(9 \sec^2 3t)}{W(t)} dt = -\csc 3t$$

$$u_2(t) = \int \frac{\cos 3t(9 \sec^2 3t)}{W(t)} dt = \ln(\sec 3t + \tan 3t),$$

since $0 < t < \pi/6$. Hence $Y(t) = -1 + (\sin 3t) \ln(\sec 3t + \tan 3t)$. The general solution is given by

$$y(t) = c_1 \cos 3t + c_2 \sin 3t + (\sin 3t) \ln(\sec 3t + \tan 3t) - 1\,.$$

7. The functions $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-4t}$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{te^{-2t}(2t^{-2}e^{-2t})}{W(t)} dt = -2 \ln t \quad \text{and} \quad u_2(t) = \int \frac{e^{-2t}(2t^{-2}e^{-2t})}{W(t)} dt = -2/t.$$

Hence the particular solution is $Y(t) = -2e^{-2t}\ln t - 2e^{-2t}$. Since the second term is a solution of the homogeneous equation, the general solution is given by

$$y(t) = c_1 e^{-2t} + c_2 te^{-2t} - 2e^{-2t}\ln t.$$

8. The solution of the homogeneous equation is $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$. The two functions $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$ form a fundamental set of solutions, with $W(y_1, y_2) = 2$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{\sin 2t(3 \csc 2t)}{W(t)} dt = -3t/2$$

$$u_2(t) = \int \frac{\cos 2t(3 \csc 2t)}{W(t)} dt = \frac{3}{4}\ln(\sin 2t),$$

since $0 < t < \pi/2$. Hence $Y(t) = -(3/2)t\cos 2t + (3/4)(\sin 2t)\ln(\sin 2t)$. The general solution is given by

$$y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{3}{2}t\cos 2t + \frac{3}{4}(\sin 2t)\ln(\sin 2t).$$

9. The functions $y_1(t) = \cos(t/2)$ and $y_2(t) = \sin(t/2)$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = 1/2$. First write the ODE in standard form, so that $g(t) = 2\sec(t/2)$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{\cos(t/2)\,[2\sec(t/2)]}{W(t)} dt = 8\ln(\cos(t/2))$$

$$u_2(t) = \int \frac{\sin(t/2)\,[2\sec(t/2)]}{W(t)} dt = 4t.$$

The particular solution is $Y(t) = 8\cos(t/2)\ln(\cos(t/2)) + 4t\sin(t/2)$. The general solution is given by

$$y(t) = c_1 \cos(t/2) + c_2 \sin(t/2) + 8\cos(t/2)\ln(\cos(t/2)) + 4t\sin(t/2).$$

10. The solution of the homogeneous equation is $y_c(t) = c_1 e^t + c_2 te^t$. The functions $y_1(t) = e^t$ and $y_2(t) = te^t$ form a fundamental set of solutions, with $W(y_1, y_2) = e^{2t}$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{te^t(e^t)}{W(t)(1+t^2)} dt = -\frac{1}{2}\ln(1+t^2)$$

$$u_2(t) = \int \frac{e^t(e^t)}{W(t)(1+t^2)} dt = \arctan t.$$

The particular solution is $Y(t) = -(1/2)e^t \ln(1+t^2) + te^t \arctan(t)$. Hence the general solution is given by

$$y(t) = c_1 e^t + c_2 te^t - \frac{1}{2}e^t \ln(1+t^2) + te^t \arctan(t).$$

12. The functions $y_1(t) = \cos 3t$ and $y_2(t) = \sin 3t$ form a fundamental set of solutions, with $W(y_1, y_2) = 3$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\frac{1}{3}\int_{t_0}^{t} g(s)\,\sin 3s\, ds \quad \text{and} \quad u_2(t) = \frac{1}{3}\int_{t_0}^{t} g(s)\,\cos 3s\, ds.$$

Hence the particular solution is

$$Y(t) = -\frac{1}{3}\cos 3t\int_{t_0}^{t} g(s)\,\sin 3s\, ds + \frac{1}{3}\sin 3t\int_{t_0}^{t} g(s)\,\cos 3s\, ds\,.$$

Note that $\sin 3t\,\cos 3s - \cos 3t\,\sin 3s = \sin(3t - 3s)$. It follows that

$$Y(t) = \frac{1}{3}\int_{t_0}^{t} g(s)\sin(3t - 3s)ds\,.$$

The general solution of the differential equation is given by

$$y(t) = c_1\cos 3t + c_2\sin 3t + \frac{1}{3}\int_{t_0}^{t} g(s)\sin(3t - 3s)ds\,.$$

13. Note first that $p(t) = 0\,, q(t) = -2/t^2$ and $g(t) = (4t^2 - 3)/t^2$. The functions $y_1(t)$ and $y_2(t)$ are solutions of the homogeneous equation, verified by substitution. The Wronskian of these two functions is $W(y_1,y_2) = -3$. Using the method of variation of parameters, the particular solution is $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{t^{-1}(4t^2 - 3)}{t^2\, W(t)} dt = t^{-2}/2 + 4\ln t/3$$

$$u_2(t) = \int \frac{t^2(4t^2 - 3)}{t^2\, W(t)} dt = -4t^3/3 + t.$$

Therefore $Y(t) = 1/2 + 4t^2\ln t/3 - 4t^2/3 + 1\,.$

15. Observe that $g(t) = t\, e^{2t}$. The functions $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions. The Wronskian of these two functions is $W(y_1,y_2) = t\, e^t$. Using the method of variation of parameters, the particular solution is $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{e^t(t\, e^{2t})}{W(t)} dt = -e^{2t}/2 \quad \text{and} \quad u_2(t) = \int \frac{(1+t)(t\, e^{2t})}{W(t)} dt = t\, e^t.$$

Therefore $Y(t) = -(1+t)e^{2t}/2 + t\, e^{2t} = -e^{2t}/2 + t\, e^{2t}/2\,.$

16. Observe that $g(t) = 2(1-t)\, e^{-t}$. Direct substitution of $y_1(t) = e^t$ and $y_2(t) = t$ verifies that they are solutions of the homogeneous equation. The Wronskian of the two solutions is $W(y_1,y_2) = (1-t)\, e^t$. Using the method of variation of parameters, the particular solution is $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{2t(1-t)e^{-t}}{W(t)} dt = te^{-2t} + e^{-2t}/2$$

$$u_2(t) = \int \frac{2(1-t)}{W(t)} dt = -2\,e^{-t}.$$

Therefore $Y(t) = te^{-t} + e^{-t}/2 - 2t\,e^{-t} = -te^{-t} + e^{-t}/2$.

17. Note that $g(x) = \ln x$. The functions $y_1(x) = x^2$ and $y_2(x) = x^2 \ln x$ are solutions of the homogeneous equation, as verified by substitution. The Wronskian of the solutions is $W(y_1,y_2) = x^3$. Using the method of variation of parameters, the particular solution is $Y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$, in which

$$u_1(x) = -\int \frac{x^2 \ln x(\ln x)}{W(x)} dx = -(\ln x)^3/3$$

$$u_2(x) = \int \frac{x^2(\ln x)}{W(x)} dx = (\ln x)^2/2.$$

Therefore $Y(x) = -x^2(\ln x)^3/3 + x^2(\ln x)^3/2 = x^2(\ln x)^3/6$.

19. First write the equation in standard form. Note that the forcing function becomes $g(x)/(1-x)$. The functions $y_1(x) = e^x$ and $y_2(x) = x$ are a fundamental set of solutions, as verified by substitution. The Wronskian of the solutions is $W(y_1,y_2) = (1-x)e^x$. Using the method of variation of parameters, the particular solution is $Y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$, in which

$$u_1(x) = -\int_{x_0}^{x} \frac{\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau \quad \text{and} \quad u_2(x) = \int_{x_0}^{x} \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau.$$

Therefore

$$Y(x) = -e^x \int_{x_0}^{x} \frac{\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau + x \int_{x_0}^{x} \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau =$$

$$= \int_{x_0}^{x} \frac{(xe^\tau - e^x\tau)g(\tau)}{(1-\tau)^2 e^\tau} d\tau.$$

20. First write the equation in standard form. The forcing function becomes $g(x)/x^2$. The functions $y_1(x) = x^{-1/2}\sin x$ and $y_2(x) = x^{-1/2}\cos x$ are a fundamental set of solutions. The Wronskian of the solutions is $W(y_1,y_2) = -1/x$. Using the method of variation of parameters, the particular solution is $Y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$, in which

$$u_1(x) = \int_{x_0}^{x} \frac{\cos\tau\,(g(\tau))}{\tau\sqrt{\tau}} d\tau \quad \text{and} \quad u_2(x) = -\int_{x_0}^{x} \frac{\sin\tau\,(g(\tau))}{\tau\sqrt{\tau}} d\tau.$$

Therefore

$$Y(x) = \frac{\sin x}{\sqrt{x}} \int_{x_0}^{x} \frac{\cos\tau\,(g(\tau))}{\tau\sqrt{\tau}} dt - \frac{\cos x}{\sqrt{x}} \int_{x_0}^{x} \frac{\sin\tau\,(g(\tau))}{\tau\sqrt{\tau}} d\tau =$$

$$= \frac{1}{\sqrt{x}} \int_{x_0}^{x} \frac{\sin(x-\tau)\,g(\tau)}{\tau\sqrt{\tau}} d\tau.$$

21. Let $y_1(t)$ and $y_2(t)$ be a fundamental set of solutions, and $W(t) = W(y_1, y_2)$ be the corresponding Wronskian. Any solution, $u(t)$, of the homogeneous equation is

a linear combination $u(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$. Invoking the initial conditions, we require that

$$\begin{aligned} y_0 &= \alpha_1 \, y_1(t_0) + \alpha_2 \, y_2(t_0) \\ y_0' &= \alpha_1 \, y_1'(t_0) + \alpha_2 \, y_2'(t_0) \end{aligned}$$

Note that this system of equations has a unique solution, since $W(t_0) \neq 0$. Now consider the nonhomogeneous problem, $L\,[v] = g(t)$, with homogeneous initial conditions. Using the method of variation of parameters, the particular solution is given by

$$Y(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)\, g(s)}{W(s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)\, g(s)}{W(s)} ds\,.$$

The general solution of the IVP (iii) is

$$v(t) = \beta_1 y_1(t) + \beta_2 y_2(t) + Y(t) = \beta_1 y_1(t) + \beta_2 y_2(t) + y_1(t) u_1(t) + y_2(t) u_2(t)$$

in which u_1 and u_2 are defined above. Invoking the initial conditions, we require that

$$\begin{aligned} 0 &= \beta_1 y_1(t_0) + \beta_2 y_2(t_0) + Y(t_0) \\ 0 &= \beta_1 y_1'(t_0) + \beta_2 y_2'(t_0) + Y'(t_0) \end{aligned}$$

Based on the definition of u_1 and u_2, $Y(t_0) = 0$. Furthermore, since $y_1 u_1' + y_2 u_2' = 0$, it follows that $Y'(t_0) = 0$. Hence the only solution of the above system of equations is the trivial solution. Therefore $v(t) = Y(t)$. Now consider the function $y = u + v$. Then $L\,[y] = L\,[u + v] = L\,[u] + L\,[v] = g(t)$. That is, $y(t)$ is a solution of the nonhomogeneous problem. Further, $y(t_0) = u(t_0) + v(t_0) = y_0$, and similarly, $y'(t_0) = y_0'$. By the uniqueness theorems, $y(t)$ is the unique solution of the initial value problem.

23.(a) A fundamental set of solutions is $y_1(t) = \cos\, t$ and $y_2(t) = \sin\, t$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = 1$. By the result in Problem 22,

$$\begin{aligned} Y(t) \;&= \int_{t_0}^{t} \frac{\cos(s)\, \sin(t) - \cos(t)\, \sin(s)}{W(s)} g(s) ds \\ &= \int_{t_0}^{t} [\cos(s)\, \sin(t) - \cos(t)\, \sin(s)]\, g(s) ds\,. \end{aligned}$$

Finally, we have $\cos(s)\, \sin(t) - \cos(t)\, \sin(s) = \sin(t - s)$.

(b) Using Problem 21 and part (a), the solution is

$$y(t) = y_0 \cos t + y_0' \sin t + \int_0^t \sin(t - s) g(s) ds\,.$$

24. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = e^{bt}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = (b - a) e^{(a+b)t}$. By the result in Problem 22,

$$Y(t) = \int_{t_0}^{t} \frac{e^{as} e^{bt} - e^{at} e^{bs}}{W(s)} g(s) ds = \frac{1}{b - a} \int_{t_0}^{t} \frac{e^{as} e^{bt} - e^{at} e^{bs}}{e^{(a+b)s}} g(s) ds\,.$$

Hence the particular solution is

$$Y(t) = \frac{1}{b-a}\int_{t_0}^{t}\left[e^{b(t-s)} - e^{a(t-s)}\right] g(s)ds\,.$$

26. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = te^{at}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = e^{2at}$. By the result in Problem 22,

$$Y(t) = \int_{t_0}^{t}\frac{te^{as+at} - s\,e^{at+as}}{W(s)}g(s)ds = \int_{t_0}^{t}\frac{(t-s)e^{as+at}}{e^{2as}}g(s)ds\,.$$

Hence the particular solution is

$$Y(t) = \int_{t_0}^{t}(t-s)e^{a(t-s)}g(s)ds\,.$$

27. The form of the kernel depends on the characteristic roots. If the roots are real and distinct,

$$K(t-s) = \frac{e^{b(t-s)} - e^{a(t-s)}}{b-a}\,.$$

If the roots are real and identical,

$$K(t-s) = (t-s)e^{a(t-s)}\,.$$

If the roots are complex conjugates,

$$K(t-s) = \frac{e^{\lambda(t-s)}\ \sin\mu(t-s)}{\mu}\,.$$

28. Let $y(t) = v(t)y_1(t)$, in which $y_1(t)$ is a solution of the homogeneous equation. Substitution into the given ODE results in

$$v''y_1 + 2v'y_1' + vy_1'' + p(t)\left[v'y_1 + vy_1'\right] + q(t)vy_1 = g(t)\,.$$

By assumption, $y_1'' + p(t)y_1 + q(t)y_1 = 0$, hence $v(t)$ must be a solution of the ODE

$$v''y_1 + \left[2y_1' + p(t)y_1\right]v' = g(t)\,.$$

Setting $w = v'$, we also have $w'y_1 + \left[2y_1' + p(t)y_1\right]w = g(t)$.

30. First write the equation as $y'' + 7t^{-1}y + 5t^{-2}\,y = 3t^{-1}$. As shown in Problem 28, the function $y(t) = t^{-1}v(t)$ is a solution of the given ODE as long as v is a solution of

$$t^{-1}\,v'' + \left[-2t^{-2} + 7t^{-2}\right]v' = 3t^{-1}\,,$$

that is, $v'' + 5t^{-1}\,v' = 3$. This ODE is linear and first order in v'. The integrating factor is $\mu = t^5$. The solution is $v' = t/2 + c\,t^{-5}$. Direct integration now results in $v(t) = t^2/4 + c_1 t^{-4} + c_2$. Hence $y(t) = t/4 + c_1 t^{-5} + c_2 t^{-1}$.

31. Write the equation as $y'' - t^{-1}(1+t)y + t^{-1}\,y = t\,e^{2t}$. As shown in Problem 28, the function $y(t) = (1+t)v(t)$ is a solution of the given ODE as long as v is a solution of

$$(1+t)\,v'' + \left[2 - t^{-1}(1+t)^2\right]v' = t\,e^{2t},$$

that is,

$$v'' - \frac{1+t^2}{t(t+1)}\, v' = \frac{t}{t+1} e^{2t}.$$

This equation is first order linear in v', with integrating factor $\mu = t^{-1}(1+t)^2 e^{-t}$. The solution is $v' = (t^2 e^{2t} + c_1 t e^t)/(1+t)^2$. Integrating, we obtain $v(t) = e^{2t}/2 - e^{2t}/(t+1) + c_1 e^t/(t+1) + c_2$. Hence the solution of the original ODE is $y(t) = (t-1)e^{2t}/2 + c_1 e^t + c_2(t+1)$.

32. Write the equation as $y'' + t(1-t)^{-1} y - (1-t)^{-1}\, y = 2(1-t)\, e^{-t}$. The function $y(t) = e^t v(t)$ is a solution to the given ODE as long as v is a solution of

$$e^t v'' + \left[2e^t + t(1-t)^{-1} e^t\right] v' = 2(1-t)\, e^{-t},$$

that is, $v'' + [(2-t)/(1-t)]\, v' = 2(1-t)\, e^{-2t}$. This equation is first order linear in v', with integrating factor $\mu = e^t/(t-1)$. The solution is

$$v' = (t-1)(2e^{-2t} + c_1 e^{-t}).$$

Integrating, we obtain $v(t) = (1/2 - t)e^{-2t} - c_1 t e^{-t} + c_2$. Hence the solution of the original ODE is $y(t) = (1/2 - t)e^{-t} - c_1 t + c_2 e^t$.