2.4

3. The function $\tan t$ is discontinuous at odd multiples of $\pi/2$. Since $3\pi/2 < 2\pi < 5\pi/2$, the initial value problem has a unique solution on the interval $(3\pi/2, 5\pi/2)$.

5. $p(t) = 2t/(16 - t^2)$ and $g(t) = 3t^2/(16 - t^2)$. These functions are discontinuous at $x = \pm 4$. The initial value problem has a unique solution on the interval $(-4, 4)$.

7. The function $f(t, y)$ is continuous everywhere on the plane, except along the straight line $y = -2t/5$. The partial derivative $\partial f/\partial y = -16t/(2t + 5y)^2$ has the same region of continuity.

9. The function $f(t, y)$ is discontinuous along the coordinate axes, and on the hyperbola $t^2 - y^2 = 1$. Furthermore,

$$\frac{\partial f}{\partial y} = \frac{\pm 1}{y(1 - t^2 + y^2)} - 2\frac{y \ln |ty|}{(1 - t^2 + y^2)^2}$$

has the same points of discontinuity.

2.6

1. $M(x, y) = 4x + 3$ and $N(x, y) = 6y - 1$. Since $M_y = N_x = 0$, the equation is exact. Integrating M with respect to x, while holding y constant, yields $\psi(x, y) = 2x^2 + 3x + h(y)$. Now $\psi_y = h'(y)$, and equating with N results in the possible function $h(y) = 3y^2 - y$. Hence $\psi(x, y) = 2x^2 + 3x + 3y^2 - y$, and the solution is defined implicitly as $2x^2 + 3x + 3y^2 - y = c$.

11. $M(x, y) = x \ln y + xy$ and $N(x, y) = y \ln x + xy$. Note that $M_y \neq N_x$, and hence the differential equation is not exact.

18. Observe that $(M(x))_y = (N(y))_x = 0$.

2.4. 1) Determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

$$(t-5)y' + \ln(t)\, y = 2t \quad , \; y(1) = 2$$

Rewrite the differential equation as $y' + \frac{\ln(t)}{t-5} y = 2t$

The coefficient $\frac{\ln(t)}{t-5}$ is continuous where $t>0, t\neq 5$.
Since the initial condition is specified at $t=1$, Theorem 2.4.1 assures the existence of a unique solution on the interval $0<t<5$.

11) State where in ty-plane the hypotheses of Theorem 2.4.2 are satisfied.

$$\frac{dy}{dt} = \frac{2+t^3}{3y-y^2} \quad , \quad y' = \frac{2+t^3}{y(3-y)} = f(t,y)$$

The function $f(t,y)$ is continuous everywhere except $y=0$ & $y=3$. The partial derivative, $\frac{\partial f}{\partial y}$ has the same region of continuity.

13) Solve the IVP and determine how the interval in which the solution exists depends on the initial value y_0.

$y' = -2t/y \, , \quad y(0) = y_0$.

The equation is seperable, with $y\,dy = -2t\,dt$.
Integrating both sides, the solution is given by
$y^2(t) = -2t^2 + y_0^2 \quad , \quad y(t) = \mp\sqrt{-2t^2+y_0^2}$.
If $y_0 \neq 0$, the solution exists as long as $|t| < y_0/2$

22) a) Verify that both $y_1(t) = 1-t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem

$$y' = \frac{-t + \sqrt{t^2+4y}}{2}, \quad y(2) = -1.$$

where are these solutions valid?

Insert the solutions in IVP, observe that $y_1(t)$ is a solution for $t \geq 2$; $y_2(t)$ is a solution for all t.

b) Explain why the existence of two solutions of the given problem does not contradict the uniqueness pat of theorem 2.4.2.

Because $f_y = \dfrac{\partial f}{\partial y} = \dfrac{1}{\sqrt{t^2+4y}}$ is not continuous at $(2,-1)$ (inial value).

c) Show that $y = ct + c^2$, where c is an arbitrary contant, satisfies the differential equation in part (a) for $t \geq -2c$. If $c = -1$ the initial condition is also satisfied, and the solution $y = y_1(t)$ is obtained. Show that there is no choice of c that gives the second solution $y = y_2(t)$.

Insert the solution in IVP, observe the expression with the square root is $\sqrt{t^2+4ct+4c^2} = \sqrt{(t+2c)^2}$

Thus $t + 2c \geq 0 \Rightarrow t \geq -2c$ then equation holds

If $c = -1$ then $y(t) = -t+1$ satisfies $y(2) = -1$, the initial condition.

$y(t) = ct + c^2 = y_2(t) = -t^2/4 \Rightarrow ct + c^2 = -t^2/4$

$c = -t/2 \rightarrow$ not constant

KOÇ UNIVERSITY

HW2

Section 2.6 – Problem 3 : $\underbrace{(6x^2-2xy+4)}_{M} + \underbrace{(6y^2-x^2+2)}_{N}y'=0$

by theorem 2.6.1, page 96 the eqn is exact if and only if

$$M_y = N_x$$

$M_y = -2x$ and $N_x = -2x$ so eqn is exact.

$\Rightarrow \exists$ $\psi(x,y)$ such that

$$M(x,y) = \psi_x(x,y) \quad \text{and} \quad N(x,y) = \psi_y(x,y)$$

~~From $M_y = -2x$ we have that~~

~~$\psi(x,y) = \int M_y\, dy = \int (-2x)\, dy = -2xy$.~~

So, from $M = \psi_x$

$$\psi(x,y) = \int M\,dx = \int (6x^2-2xy+4)\,dx = 2x^3 - x^2y + 4x + h(y)$$

and from $N = \psi_y$

$$\psi_y = \frac{d}{dy}\left(2x^3 - x^2y + 4x + h(y)\right) = N = 6y^2 - x^2 + 2$$

$$\cancel{-x^2} + h'(y) = 6y^2 \cancel{- x^2} + 2$$

$$\frac{d}{dy}h(y) = 6y^2 + 2 \quad \text{so} \quad h(y) = 2y^3 + 2y$$

$$\psi(x,y) = 2x^3 - x^2y + 4x + 2y^3 + 2y$$

the solution is $\psi(x,y) = c$ i.e $\boxed{2x^3 + 2y^3 - x^2y + 4x + 2y = c}$

Section 2.6 - Problem 5 ; $\dfrac{dy}{dx} = - \dfrac{ax+by}{bx+cy}$

$$y'\underbrace{(bx+cy)}_{N(x,y)} + \underbrace{(ax+by)}_{M(x,y)} = 0$$

$M_y(x,y) = b$ and $N_x(x,y) = b$ so the eqn is exact since $M_y = N_x$

$\exists\ \psi(x,y)$ such that

$\psi_x = M$ and $\psi_y = N$

From $M = \psi_x$ we have,

$$\psi(x,y) = \int M(x,y)\,dx = \int (ax+by)\,dx = \frac{ax^2}{2} + byx + h(y)$$

and from $\psi_y = N$

$$N = bx + cy = \psi_y = \frac{d}{dy}\left(\frac{ax^2}{2} + byx + h(y)\right)$$

$$\cancel{bx} + cy = \cancel{bx} + h'(y)$$

$$h(y) = \frac{cy^2}{2}$$ so that,

$$\psi(x,y) = \frac{ax^2}{2} + byx + \frac{cy^2}{2}$$ Thus, the sol. is of the form,

$$\psi = \boxed{\frac{ax^2}{2} + byx + \frac{cy^2}{2} = C_0}\ ,\ C_0 \in \mathbb{R}.$$

KOÇ UNIVERSITY

Section 2.6 – Problem 7 : $\underbrace{(e^x \sin y - 3y\sin x)}_{M(x,y)} + \underbrace{(e^x \cos y + 3\cos x)}_{N(x,y)} y' = 0$

$M_y = e^x.\cos y - 3\sin x$ and

$N_x = e^x \cos y - 3\sin x$ so $M_y = N_x$ eqn is exact.

then $\exists$ $\psi(x,y)$ such that,

$\psi_x = M$ and $\psi_y = N$

from $M = \psi_x$

$\psi(x,y) = \int M(x,y)dx = \int (e^x \sin y - 3y\sin x)dx$

$= e^x \sin y + 3y\cos x + h(y)$

and from $\psi_y = N$

$N = \frac{d}{dy}\psi(x,y) = \frac{d}{dy}\left(e^x \sin y + 3y\cos x + h(y)\right)$

$\cancel{e^x \cos y} + \cancel{3\cos x} = \cancel{e^x \cos y} + \cancel{3\cos x} + \frac{d}{dy}h(y)$

$0 = \frac{d}{dy}h(y)$, $h(y) = c_1$

thus, $\boxed{\psi(x,y) = e^x \cdot \sin y + 3y\cos x + c_1}$

the solution is of the form $\psi(x,y) = c_2$

$e^x \sin y + 3y\cos x + c_1 = c_2$ OR call $c = c_2 - c_1$

$\boxed{e^x \sin y + 3y\cos x = c}$

KOÇ
UNIVERSITY

Section 2.6 - Problem 15: $\quad \underbrace{(xy^2+bx^2y)}_{M(x,y)}+\underbrace{(x+y)x^2}_{N(x,y)}y'=0$

$$M_y=\cancel{2xy}+bx^2 \;=\; N_x=3x^2\cancel{+2xy}$$

eqn is exact iff $\quad bx^2=3x^2$ i.e $\boxed{b=3}$

So $\quad M_y=2xy+3x^2$ and $N_x=3x^2+2xy$

then $\exists\,\psi(x,y)$ such that

$$\psi_x=M \quad \text{and} \quad \psi_y=N$$

from $M=\psi_x$ we have that,

$$\psi(x,y)=\int M(x,y)\,dx=\int(xy^2+3x^2y)\,dx=\frac{x^2y^2}{2}+x^3y+h(y)$$

$$\psi(x,y)=\frac{1}{2}x^2y^2+x^3y+h(y)$$

from $\quad \psi_y=N$

$$N=x^3+yx^2=\frac{d}{dy}\psi(x,y)=\frac{d}{dy}\left(\frac{1}{2}x^2y^2+x^3y+h(y)\right)$$

$$\cancel{x^3}+\cancel{yx^2}=\cancel{x^2y}+\cancel{x^3}+h'(y)$$

$$0=h'(y)\;\rightarrow\; h(y)=c_1$$

the sol. is of the form $\quad \psi(x,y)=c_2$

OR $\quad \psi(x,y)=\frac{1}{2}x^2y^2+x^3y+c_1=c_2 \quad$ i.e $\boxed{\frac{1}{2}x^2y^2+x^3y=\overbrace{c_2-c_1}^{\tilde{C}}}$

Section 2.6 – Problem 25: find integrating factor and solve it.

$$\underbrace{(3x^2y+2xy+y^3)}_{M}+\underbrace{(x^2+y^2)}_{N}y'=0.$$

from equation (27) (textbook, page 99, section 2.6) we have that,

$$\frac{d\mu}{dx}=\frac{M_y-N_x}{N}\mu$$

where $M_y = 3x^2+2x+3y^2$, $N_x = 2x$

$$\Rightarrow \frac{d\mu}{dx}=\frac{(3x^2+2x+3y^2)-2x}{x^2+y^2}\mu = 3\mu$$

$$\frac{d\mu}{\mu}=3dx \rightarrow \text{integreate,}$$

$$\ln(\mu)=3x \quad \text{OR,} \quad \mu=e^{3x}.$$

Multiply the equation by e^{3x}

$$\underbrace{e^{3x}\cdot(3x^2y+2xy+y^3)}_{M}+\underbrace{e^{3x}(x^2+y^2)}_{N}y'=0$$

$M_y = N_x$ eqn is exact.

$\exists\psi(x,y)$ such that $\psi_x = M$ and $\psi_y = N$

from which $\psi(x,y)=\boxed{(3x^2y+y^3)e^{3x}=c}$

Implicit solution.

Section 2.6 - Problem 27 : Find an integrating factor and solve the given equation.

$$\underbrace{1}_{M} + \underbrace{(x/y - \cos y)}_{N} \cdot y' = 0$$

$$M_y = 0 \quad \text{and} \quad N_x = \frac{1}{y}$$

From equation (26) (textbook, page 99, Section 2.6) the integrating factor μ satisfies,

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

Taking $\mu_x = 0$ (integrating factor depends only on y),

$$M \cdot \mu_y + \left(0 - \frac{1}{y}\right)\mu = 0$$

$$1 \cdot \mu_y = \frac{1}{y}\mu \quad \text{OR} \quad \frac{d\mu}{dy} = \frac{\mu}{y}$$

$$\Rightarrow \frac{d\mu}{\mu} = \frac{dy}{y} \quad \text{so that,}$$

$$\ln(\mu) = \ln(y) \quad \text{OR} \quad \mu = y$$

Multiply the equation by y to get,

$$\underbrace{y}_{M} + \underbrace{(x - y\cos y)}_{N} y' = 0 \quad \text{so that eqn is exact.}$$

$$M_y = N_x$$

$\Rightarrow \exists \psi(x,y)$ such that $\psi_x = M$ and $\psi_y = N$.

Solving for ψ we obtain,

$$\boxed{xy - y\sin y - \cos y = C}$$

implicit solution.