3.5

3. The characteristic equation for the homogeneous problem is $r^2 - r - 2 = 0$, with roots $r = -1, 2$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{2t}$. Set $Y = At^2 + Bt + C$. Substitution into the given differential equation, and comparing the coefficients, results in the system of equations $-2A = 4$, $-2A - 2B = 0$ and $2A - B - 2C = -3$. Hence $Y = -2t^2 + 2t - 3/2$. The general solution is $y(t) = y_c(t) + Y$.

5. The characteristic equation for the homogeneous problem is $r^2 - 2r - 3 = 0$, with roots $r = -1, 3$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{3t}$. Note that the assignment $Y = Ate^{-t}$ is not sufficient to match the coefficients. Try $Y = Ate^{-t} + Bt^2 e^{-t}$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $-4A + 2B = 0$ and $-8B = -6$. This implies that $Y = (3/8)te^{-t} + (3/4)t^2 e^{-t}$. The general solution is $y(t) = y_c(t) + Y$.

6. The characteristic equation for the homogeneous problem is $r^2 + 2r = 0$ with roots $r = 0, -2$. Hence $y_c = c_1 + c_2 e^{-2t}$. Note that the assignment $Y = A + B\sin 2t$ is not sufficient to match the coefficients. Try $Y = At + B\sin 2t + C\cos 2t$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $-4(B+C) = 4$ and $-4(C-B) = 0$. This implies that $Y = \frac{5}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$. The general solution is $y(t) = c_1 + c_2 e^{-2t} + \frac{5}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$.

9. The characteristic equation for the homogeneous problem is $2r^2 + 3r + 1 = 0$, with roots $r = -1, -1/2$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{-t/2}$. To simplify the analysis, set $g_1(t) = t^2$ and $g_2(t) = 3\sin t$. Based on the form of g_1, set $Y_1 = A + Bt + Ct^2$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $A + 3B + 4C = 0$, $B + 6C = 0$, and $C = 1$. Hence we obtain $Y_1 = 14 - 6t + t^2$. On the other hand, set $Y_2 = D\cos t + E\sin t$. After substitution into the ODE, we find that $D = -3/10$ and $E = 9/10$. The general solution is $y(t) = y_c(t) + Y_1 + Y_2$.

17. The characteristic equation for the homogeneous problem is $r^2 - 2r + 1 = 0$, with a double root $r = 1$. Hence $y_c(t) = c_1 e^t + c_2 t\, e^t$. Consider $g_1(t) = t\, e^t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1 = At^2 e^t + Bt^3 e^t$ (the first term is not sufficient for a match). Upon substitution, we obtain $Y_1 = t^3 e^t/6$. By inspection, $Y_2 = 4$. Hence the general solution is $y(t) = c_1 e^t + c_2 t\, e^t + t^3 e^t/6 + 4$. Invoking the initial conditions, we require that $c_1 + 4 = 2$ and $c_1 + c_2 = 1$. Hence $c_1 = -2$ and $c_2 = 3$.

13. The characteristic equation for the homogeneous problem is $r^2 + r = 0$ with roots $r = -\frac{1}{2} \pm \frac{\sqrt{15}}{2} i$. Hence $y = c_1 e^{-t/2} \cos \sqrt{152} t + c_2 e^{-t/2} \sin \sqrt{152} t$. Set $Y = Ae^t + Be^{-t}$. Substitution into the differential equation, and comparing the coefficients, we have $A = \frac{1}{3}$ and $B = -\frac{1}{2}$. The general solution is $y(t) = c_1 e^{-t/2} \cos \frac{\sqrt{15}}{2} t + c_2 e^{-t/2} \sin \frac{\sqrt{15}}{2} t + \frac{1}{3} e^t - \frac{1}{2} e^{-t}$.

21. a) Note that the assignment $Y = A_0 t^4 + A_1 t^3 + A_2 t^2 + A_3 t + A_4 + (Bt^2 + B_1 t + B_2) e^{-3t} + D \sin 3t + E \cos 3t$ is not sufficient to match the coefficients. Try $Y = t(A_0 t^4 + A_1 t^3 + A_2 t^2 + A_3 t + A_4) + t(Bt^2 + B_1 t + B_2) e^{-3t} + D \sin 3t + E \cos 3t$.

3.6

2. The solution of the homogeneous equation is $y_c(t) = c_1 e^{2t} + c_2 e^{-t}$. The functions $y_1(t) = e^{2t}$ and $y_2(t) = e^{-t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1; y_2) = -3e^t$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$, in which $u_1(t) = -\int \frac{e^{-t}(4e^{-t}}{W(t)} dt = -\frac{4}{9} e^{-3t}$ and $u_2(t) = \int \frac{e^{2t}(4e^{-t}}{W(t)} dt = -\frac{4}{3} t$. Hence the particular solution is $Y(t) = -\frac{4}{9} e^{-t} - \frac{4}{3} t e^{-t}$.

5. The solution of the homogeneous equation is $y_c(t) = c_1 \cos t + c_2 \sin t$. The functions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1; y_2) = 1$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$, in which $u_1(t) = -\int \frac{\sin t 2 \tan t}{W(t)} dt = \sin t - 2 \ln(\tan t + \sec t)$ and $u_2(t) = \int \frac{\cos t 2 \tan t}{W(t)} dt = -2 \cos t$. This implies that $Y = -2 \cos t \ln(\tan t + \sec t)$. Thus the general solution is $y(t) = c_1 \cos t + c_2 \sin t - 2 \cos t \ln(\tan t + \sec t)$.

6. The solution of the homogeneous equation is $y_c(t) = c_1 \cos 3t + c_2 \sin 3t$. The two functions $y_1(t) = \cos 3t$ and $y_2(t) = \sin 3t$ form a fundamental set of solutions, with $W(y_1, y_2) = 3$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{\sin 3t(9 \sec^2 3t)}{W(t)} dt = -\csc 3t$$

$$u_2(t) = \int \frac{\cos 3t(9 \sec^2 3t)}{W(t)} dt = \ln(\sec 3t + \tan 3t),$$

since $0 < t < \pi/6$. Hence $Y(t) = -1 + (\sin 3t) \ln(\sec 3t + \tan 3t)$. The general solution is given by

$$y(t) = c_1 \cos 3t + c_2 \sin 3t + (\sin 3t) \ln(\sec 3t + \tan 3t) - 1\,.$$

13. Note first that $p(t) = 0\,, q(t) = -2/t^2$ and $g(t) = (4t^2 - 3)/t^2$. The functions $y_1(t)$ and $y_2(t)$ are solutions of the homogeneous equation, verified by substitution. The Wronskian of these two functions is $W(y_1,y_2) = -3$. Using the method of variation of parameters, the particular solution is $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{t^{-1}(4t^2 - 3)}{t^2\, W(t)} dt = t^{-2}/2 + 4 \ln t/3$$

$$u_2(t) = \int \frac{t^2(4t^2 - 3)}{t^2\, W(t)} dt = -4t^3/3 + t.$$

Therefore $Y(t) = 1/2 + 4t^2 \ln t/3 - 4t^2/3 + 1\,.$

15. Observe that $g(t) = t\, e^{2t}$. The functions $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions. The Wronskian of these two functions is $W(y_1,y_2) = t\, e^t$. Using the method of variation of parameters, the particular solution is $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$u_1(t) = -\int \frac{e^t(t\, e^{2t})}{W(t)} dt = -e^{2t}/2 \quad \text{and} \quad u_2(t) = \int \frac{(1+t)(t\, e^{2t})}{W(t)} dt = t\, e^t.$$

Therefore $Y(t) = -(1+t)e^{2t}/2 + t\, e^{2t} = -e^{2t}/2 + t\, e^{2t}/2\,.$

17. Note that $g(x) = \ln x$. The functions $y_1(x) = x^2$ and $y_2(x) = x^2 \ln x$ are solutions of the homogeneous equation, as verified by substitution. The Wronskian of the solutions is $W(y_1, y_2) = x^3$. Using the method of variation of parameters, the particular solution is $Y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$, in which

$$u_1(x) = -\int \frac{x^2 \ln x(\ln x)}{W(x)} dx = -(\ln x)^3/3$$

$$u_2(x) = \int \frac{x^2(\ln x)}{W(x)} dx = (\ln x)^2/2.$$

Therefore $Y(x) = -x^2(\ln x)^3/3 + x^2(\ln x)^3/2 = x^2(\ln x)^3/6$.