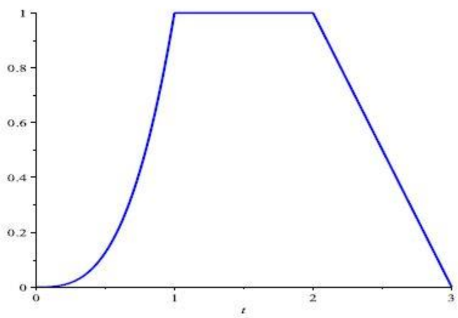
6.1

3.



The function $f(t)$ is continuous.

7. Integration is a linear operation. It follows that

$$\int_0^A \cosh bt \cdot e^{-st} dt = \frac{1}{2}\int_0^A e^{bt} \cdot e^{-st} dt + \frac{1}{2}\int_0^A e^{-bt} \cdot e^{-st} dt =$$

$$= \frac{1}{2}\int_0^A e^{(b-s)t} dt + \frac{1}{2}\int_0^A e^{-(b+s)t} dt \,.$$

Hence

$$\int_0^A \cosh bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1-e^{(b-s)A}}{s-b}\right] + \frac{1}{2}\left[\frac{1-e^{-(b+s)A}}{s+b}\right].$$

Taking a limit, as $A \to \infty$,

$$\int_0^\infty \cosh bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1}{s-b}\right] + \frac{1}{2}\left[\frac{1}{s+b}\right] = \frac{s}{s^2-b^2} \,.$$

Note that the above is valid for $s > |b|$.

11. Using the linearity of the Laplace transform,

$$\mathcal{L}\left[\sin bt\right] = \frac{1}{2i}\mathcal{L}\left[e^{ibt}\right] - \frac{1}{2i}\mathcal{L}\left[e^{-ibt}\right].$$

Since

$$\int_0^\infty e^{(a+ib)t} e^{-st} dt = \frac{1}{s-a-ib} \,,$$

we have

$$\int_0^\infty e^{\pm\, ibt}\, e^{-st} dt = \frac{1}{s \mp ib} \,.$$

Therefore

$$\mathcal{L}\left[\sin bt\right] = \frac{1}{2i}\left[\frac{1}{s\,-\,ib} - \frac{1}{s\,+\,ib}\right] = \frac{b}{s^2+b^2} \,.$$

The formula holds for $s > 0$.

15. Integrating by parts,

$$\int_0^A te^{at} \cdot e^{-st} dt = -\frac{t\, e^{(a-s)t}}{s-a}\bigg|_0^A + \int_0^A \frac{1}{s-a} e^{(a-s)t} dt =$$

$$= \frac{1 - e^{A(a-s)} + A(a-s)e^{A(a-s)}}{(s-a)^2}\,.$$

Taking a limit, as $A \to \infty$,

$$\int_0^\infty te^{at} \cdot e^{-st} dt = \frac{1}{(s-a)^2}\,.$$

Note that the limit exists as long as $s > a$.

17. Observe that $t \sinh at = (t\, e^{at} - t\, e^{-at})/2$. For any value of c,

$$\int_0^A t\, e^{ct} \cdot e^{-st} dt = -\frac{t\, e^{(c-s)t}}{s-c}\bigg|_0^A + \int_0^A \frac{1}{s-c} e^{(c-s)t} dt =$$

$$= \frac{1 - e^{A(c-s)} + A(c-s)e^{A(c-s)}}{(s-c)^2}\,.$$

Taking a limit, as $A \to \infty$,

$$\int_0^\infty te^{ct} \cdot e^{-st} dt = \frac{1}{(s-c)^2}\,.$$

Note that the limit exists as long as $s > |c|$. Therefore,

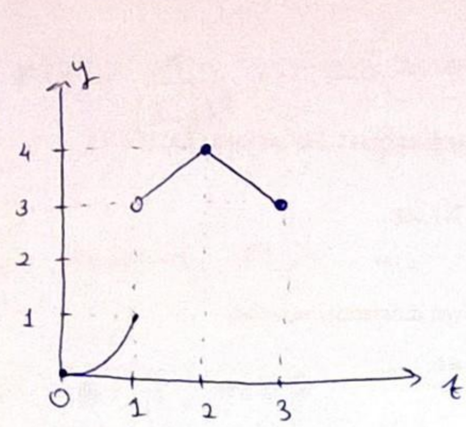
$$\int_0^\infty t \sinh at \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1}{(s-a)^2} - \frac{1}{(s+a)^2}\right] = \frac{2as}{(s-a)^2(s+a)^2}\,.$$

23. Using the definition of the Laplace transform and Problem 22, we get that

$$\mathcal{L}\,[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^3 e^{-st} t\, dt + \int_3^\infty e^{-st}\, dt =$$

$$= -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{e^{-3s}}{s} = -\frac{(2s+1)e^{-3s}}{s^2} + \frac{1}{s^2}.$$

1



f is piecewise continuous on the interval $0 \leq t \leq 3$.

5 a) $f(t) = t$. $\quad F(s) = \int_0^\infty e^{-st} \cdot t \, dt$

integral by parts ↳

$$= \lim_{A \to \infty} - t \cdot \frac{e^{-st}}{s} \Big]_0^A + \lim_{A \to \infty} \int_0^A \frac{e^{-st}}{s} dt$$

$$= 0 + \frac{1}{s^2} = \frac{1}{s^2}. \qquad s > 0.$$

b) $f(t) = t^2$. $\quad F(s) = \int_0^\infty e^{-st} \, t^2 \, dt$ ↓ integral by parts.

$$= \frac{2}{s^3}.$$

c) $f(t) = t^n$, $\quad F(s) = \int_0^\infty e^{-st} t^n \, dt$ ↓ observe ~~deduce~~ from previous results.

$$= \frac{n!}{s^{n+1}}, \quad s > 0.$$

16. $f(t) = t \cdot \cos(at)$. we know; $\cos(at) = (e^{iat} + e^{-iat})/2$.

$$F(s) = \frac{1}{2} \left[\int_0^\infty t \, e^{iat - st} dt + \int_0^\infty t \, e^{-iat - st} dt \right]$$

$$= \frac{1}{2} \left[\int_0^\infty t \, e^{(ia - s)t} + \int_0^\infty t \cdot e^{-(ia + s)t} \right].$$

$$= \frac{1}{2}\left[\frac{1}{(ia-s)^2} + \frac{1}{(ia+s)^2}\right], \text{ from } (5.a).$$

$$= \frac{s^2 - a^2}{(a^2+s^2)^2}.$$

21) $f(t) = \begin{cases} 1, & 0 \le t < 2\pi \\ 0, & 2\pi \le t < \infty. \end{cases}$

Using the fact that $f(t)=0$ when $t \geq 2\pi$ and I by parts:

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^{\infty} e^{-st}\,dt = \int_0^{2\pi} e^{-st}\,dt$$

$$= \left.\frac{-e^{-st}}{s}\right]_0^{2\pi} = \frac{1-e^{-2\pi s}}{s}.$$

6.2

6. Using partial fractions,

$$\frac{2s+1}{s^2-4} = \frac{1}{4}\left[\frac{5}{s-2} + \frac{3}{s+2}\right].$$

Hence $\mathcal{L}^{-1}\left[Y(s)\right] = (5e^{2t} + 3e^{-2t})/4$. Note that we can also write

$$\frac{2s+1}{s^2-4} = 2\,\frac{s}{s^2-4} + \frac{1}{2}\frac{2}{s^2-4}.$$

8. Using partial fractions,

$$\frac{8s^2-6s+12}{s(s^2+4)} = 3\,\frac{1}{s} + 5\,\frac{s}{s^2+4} - 3\,\frac{2}{s^2+4}.$$

Hence $\mathcal{L}^{-1}\left[Y(s)\right] = 3 + 5\,\cos 2t - 3\,\sin 2t$.

10. Note that the denominator $s^2 + 2s + 10$ is irreducible over the reals. Completing the square, $s^2 + 2s + 10 = (s+1)^2 + 9$. Now convert the function to a rational function of the variable $\xi = s + 1$. That is,

$$\frac{2s-5}{s^2+2s+10} = \frac{2(s+1)-7}{(s+1)^2+9}.$$

We find that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+9} - \frac{7}{\xi^2+9}\right] = 2\,\cos 3t - \frac{7}{3}\,\sin 3t\,.$$

Using the fact that $\mathcal{L}\left[e^{at}f(t)\right] = \mathcal{L}\left[f(t)\right]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{2s-5}{s^2+2s+10}\right] = e^{-t}(2\,\cos 3t - \frac{7}{3}\,\sin 3t)\,.$$

13. Taking the Laplace transform of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) - 2\,[s\,Y(s) - y(0)] + 2Y(s) = 0\,.$$

Applying the initial conditions,

$$s^2 Y(s) - 2s\,Y(s) + 2Y(s) - s + 1 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s-1}{s^2 - 2s + 2}\,.$$

Since the denominator is irreducible, write the transform as a function of $\xi = s - 1$. That is,

$$\frac{s-1}{s^2-2s+2} = \frac{s-1}{(s-1)^2+1}\,.$$

First note that

$$\mathcal{L}^{-1}\left[\frac{\xi}{\xi^2+1}\right] = \cos t\,.$$

Using the fact that $\mathcal{L}\,[e^{at}f(t)] = \mathcal{L}\,[f(t)]_{s \to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{s-1}{s^2-2s+2}\right] = e^t \cos t\,.$$

Hence $y(t) = e^t \cos t$.

18. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0) - Y(s) = 0\,.$$

Applying the initial conditions,

$$s^4 Y(s) - Y(s) - s^3 - s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2-1}\,.$$

By inspection, it follows that $y(t) = \mathcal{L}^{-1}\,[Y(s)] = \cosh t$.

19. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0) - 4\,Y(s) = 0\,.$$

Applying the initial conditions,

$$s^4 Y(s) - 4Y(s) - s^3 + 2s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2+2}\,.$$

It follows that $y(t) = \mathcal{L}^{-1}\,[Y(s)] = \cos\sqrt{2}\,t$.

21. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) - 2\,[s\,Y(s) - y(0)] + 2\,Y(s) = \frac{s}{s^2+1}\,.$$

Applying the initial conditions,

$$s^2 Y(s) - 2s\,Y(s) + 2\,Y(s) - s + 1 = \frac{s}{s^2+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s}{(s^2-2s+2)(s^2+1)} + \frac{s-1}{s^2-2s+2}\,.$$

Using partial fractions on the first term,

$$\frac{s}{(s^2-2s+2)(s^2+1)} = \frac{1}{5}\left[\frac{s-2}{s^2+1} - \frac{s-4}{s^2-2s+2}\right].$$

Thus we can write

$$Y(s) = \frac{1}{5}\,\frac{s}{s^2+1} - \frac{2}{5}\,\frac{1}{s^2+1} + \frac{1}{5}\,\frac{4s-1}{s^2-2s+2}\,.$$

For the last term, we note that $s^2 - 2s + 2 = (s-1)^2 + 1$. So that

$$\frac{4s-1}{s^2-2s+2} = \frac{4(s-1)+3}{(s-1)^2+1}\,.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{4\,\xi}{\xi^2+1} + \frac{3}{\xi^2+1}\right] = 4\,\cos\,t + 3\sin\,t\,.$$

Based on the translation property of the Laplace transform,

$$\mathcal{L}^{-1}\left[\frac{4s-1}{s^2-2s+2}\right] = e^t(4\,\cos\,t + 3\sin\,t).$$

Combining the above, the solution of the IVP is

$$y(t) = \frac{1}{5}\,\cos\,t - \frac{2}{5}\,\sin\,t + \frac{1}{5}\,e^t(4\,\cos\,t + 3\sin\,t)\,.$$

23. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) + 2\,[s\,Y(s) - y(0)] + Y(s) = \frac{4}{s+1}\,.$$

Applying the initial conditions,

$$s^2 Y(s) + 2s\,Y(s) + Y(s) - 2s - 3 = \frac{4}{s+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2}\,.$$

First write

$$\frac{2s+3}{(s+1)^2} = \frac{2(s+1)+1}{(s+1)^2} = \frac{2}{s+1} + \frac{1}{(s+1)^2}\,.$$

We note that

$$\mathcal{L}^{-1}\left[\frac{4}{\xi^3} + \frac{2}{\xi} + \frac{1}{\xi^2}\right] = 2\,t^2 + 2 + t\,.$$

So based on the translation property of the Laplace transform, the solution of the IVP is

$$y(t) = 2\,t^2 e^{-t} + t\,e^{-t} + 2\,e^{-t}.$$

(2) $F(s) = \dfrac{5}{(s-1)^3}$, We know from Table 6.21. that

$f(t) = t^n e^{at}$ then $F(s) = \dfrac{n!}{(s-a)^{n+1}}$. $s>a$.

then rewrite $F(s)$ as. $\dfrac{5}{2} \cdot \dfrac{2!}{(s-1)^{2+1}}$.

Then $a = 1$, $n = 2$. Thus $f(t) = \frac{5}{2} t^2 e^t$.

(4) $F(s) = \dfrac{2s}{s^2 - s - 6} = \dfrac{6}{5} \cdot \dfrac{1}{(s-3)} + \dfrac{4}{5} \cdot \dfrac{1}{s+2}$

Then from Table 6.21 $\Rightarrow f(t) = \frac{6}{5} e^{3t} + \frac{4}{5} e^{-2t}$.

(15) $y'' - 2y' + 4y = 0$, $y(0) = 3$, $y'(0) = 0$.

Taking the laplace transform of the ODE, we have:

$s^2 Y(s) - s\,y(0) - y'(0) - 2[s\,Y(s) - y(0)] + 4\,Y(s) = 0$.

Applying the initial conditions:

$s^2 Y(s) - 2s\,Y(s) + 4\,Y(s) - 3s + 6 = 0$.

$\Rightarrow Y(s) = \dfrac{3s - 6}{s^2 - 2s + 4} =$

$= \dfrac{3\,(s-1)}{(s-1)^2 + (\sqrt{3})^2} - \dfrac{3}{(s-1)^2 + (\sqrt{3})^2}$. from table 6.21

$\Rightarrow y(t) = 3e^t \cos\sqrt{3}t - e^t \sin\sqrt{3}t$ //