SAMPLE QUESTIONS (CHAPTER 2)

Section 2.1:

Q1) Find the solution of the given initial value problem.

$$y' - 4y = e^{4t}, \quad y(0) = 2$$

$p(t) = -4 \qquad \mu(t) = e^{\int p(t)\,dt} = e^{-4t}$

Multiplying the equation with $\mu(t)$,

$$e^{-4t} \cdot y' - 4e^{-4t} y = 1.$$

$(y \cdot e^{-4t})' = 1$, integrating both sides,

$$y(t) \cdot e^{-4t} = t + c, \quad c \in \mathbb{R}$$

$$y(t) = t \cdot e^{4t} + c \cdot e^{4t}$$

$y(0) = 0 + c = 2$, so we have

the solution of the initial value problem,

$$\boxed{y(t) = (t+2)\, e^{4t}}$$

Q2) Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \quad y(0) = y_0$$

remains finite as $t \to \infty$.

Clearly, $\mu(t) = e^{-t}$

$$(y \cdot e^{-t})' = e^{-t} + 3e^{-t}\sin t \text{ , integrating,}$$

$$y \cdot e^{-t} = -e^{-t} + 3\int e^{-t}\sin t\, dt$$

We use integration by parts twice to calculate the integral,

$$(\star) \quad \int e^{-t}\sin t \, dt = -e^{-t}\sin t + \int e^{-t}\cos t\, dt$$

$u = \sin t \quad dv = e^{-t}dt$

$du = \cos t\, dt \quad V = -e^{-t}$

$$(\star\star) \quad \int e^{-t}\cos t\, dt = -e^{-t}\cos t - \int e^{-t}\sin t\, dt$$

$u = \cos t \quad dv = e^{-t}dt$

$du = -\sin t\, dt \quad V = -e^{-t}$

Subsituting $(\star\star)$ in $(\star)$, we get

$$\int e^{-t}\sin t = -e^{-t}\frac{(\cos t + \sin t)}{2} + c \text{ , } c \in \mathbb{R}$$

$$y(t).e^{-t} = -e^{-t} - 3e^{-t}\frac{(\cos t + \sin t)}{2} + d, \quad d \in \mathbb{R}$$

$$y(t) = -1 - \frac{3}{2}(\cos t + \sin t) + d.e^{t}$$

Placing $t = 0$,

$$y(0) = -1 - \frac{3}{2} + d = y_0$$

$$d = y_0 + 5/2$$

Here is our unique solution to the initial value problem

$$y(t) = -1 - \frac{3}{2}(\cos t + \sin t) + (y_0 + 5/2)\, e^{t},$$

as $t \to \infty$, cost & sint remain bounded, but e^t diverge to positive infinity. So if we want finite values, we need to eliminate this term, i.e, choose $\boxed{y_0 = -5/2}$.

Section 2.2:

Q1) Solve the given differential equation. $xy' = (1-y^2)^{1/2}$

First, we notice that the equation is seperable.

$(1-y^2)^{-1/2}\, dy = 1/x\, dx$, integrating both sides,

$\arcsin y = \ln x + c, \quad c \in \mathbb{R}$

So that $\quad y(x) = \sin(\ln x + c), \quad c \in \mathbb{R}$

Q2) $\sin 2x\,dx + \cos 3y\,dy = 0$, $y(\pi/2) = \pi/3$

$\cos 3y\,dy = -\sin 2x\,dx$, the equation is seperable,

integrating both sides,

$$\frac{\sin 3y}{3} = \frac{\cos 2x}{2} + c, \qquad c \in \mathbb{R}$$

Using the initial condition,

$$\frac{\sin \pi}{3} = \frac{\cos \pi}{2} + c \implies c = \frac{1}{2}$$

$$\sin 3y = \frac{3}{2}\cos 2x + \frac{3}{2}$$

$$y(x) = \frac{\arcsin\left(\frac{3}{2}(\cos 2x + 1)\right)}{3}$$

Q3). Solve the equation

$$\frac{dy}{dx} = \frac{ay+b}{cy+d},$$

where a, b, c, d are constants.

See that we have seperability,

$$\frac{cy+d}{ay+b}\, dy = 1 .\, dx$$

$$\left(\frac{cy}{ay+b} + \frac{d}{ay+b} \right) dy = 1\, dx,$$ integrating both sides,

$$\frac{c}{a^2} \cdot \left(ay - b . \ln(ay+b)\right) + \frac{d}{a} \ln(ay+b) = x + r, \quad r \in \mathbb{R}$$

Hence we have the implicit form of the solution.

Section 2.4:

Q1) Solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

$$y' + y^3 = 0, \qquad y(0) = y_0$$

$$\frac{dy}{dx} = -y^3 \qquad -y^{-3}\, dy = 1 . dx$$

$$\frac{1}{2} y^{-2} = x + c, \quad c \in \mathbb{R}$$

$$\frac{1}{2x+d} = y^2, \quad d \in \mathbb{R},$$

Putting the initial condition, $d = \frac{1}{y_0^2}$, moreover

$$y(x) = \pm \frac{y_0}{\sqrt{2y_0^2 x + 1}}$$, and it exists as long as

$$2y_0^2 x + 1 > 0$$

$$2y_0^2 x > -1$$

If $y_0 = 0 \Rightarrow$ Solution exists for all $x \in \mathbb{R}$. (It's $y(x)=0$)

If $y_0 \neq 0 \Rightarrow$ Solution exists for all $x > \frac{-1}{2y_0^2}$

Q2) Bernoulli Equations: $y' + p(t)y = q(t)y^n$, $n \in \mathbb{N}$

(a) Solve Bernoulli's equation when $n=0$; when $n=1$.

$n=0 \Rightarrow y' + p(t)y = q(t)$, i.e., usual linear 1st order O.D.E.

$n=1 \Rightarrow y' + (p(t) - q(t))y = 0$, i.e, usual linear 1st order O.D.E.

(b) Show that if $n \neq 0,1$, then the subsitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation. (Leibniz, 1696)

$v = y^{1-n} \Rightarrow v' = (1-n).\, y^{-n}.\, y' \Rightarrow y' = \frac{v'.\, y^n}{1-n}$

$v = y^{1-n} \Rightarrow y = v.\, y^n$

Substituting, we have

$$y' + p(t)y = q(t)\, y^n$$

$$\frac{v' \cdot y^n}{(1-n)} + p(t) \cdot v \cdot y^n = q(t) \cdot y^n$$, or, equivalently,

$$v' + (1-n) \cdot p(t) \cdot v = (1-n)\, q(t),$$ a first order linear equation with respect to v.

Q3) Solve the Bernoulli equation:

$$y' = \varepsilon y - \sigma y^3, \quad \varepsilon > 0, \quad \sigma > 0.$$

$$y' - \varepsilon y = -\sigma y^3$$, $n = 3$, so let $v = y^{1-n} = y^{-2}$

$$v' = -2 \cdot y^{-3} \cdot y' \Rightarrow y' = \frac{-v' \cdot y^3}{2}$$ $\qquad y = v \cdot y^3$

Substituting,

$$\frac{-v' \cdot y^3}{2} - \varepsilon \cdot v \cdot y^3 = -\sigma y^3$$, or, equivalently,

$$v' + 2\varepsilon v = 2\sigma, \quad \mu(t) = e^{2\varepsilon t}$$

$$\left(v \cdot e^{2\varepsilon t}\right)' = e^{2\varepsilon t}\, 2\sigma$$, integrating both sides

$$v \cdot e^{2\varepsilon t} = \frac{\sigma}{\varepsilon} \cdot e^{2\varepsilon t} + c, \quad c \in \mathbb{R}$$

$$v(t) = \frac{\sigma}{\varepsilon} + c\, e^{-2\varepsilon t}$$ Recall that $v = y^{-2}$, i.e.,

$$v(t) = \frac{\sigma\, e^{2\varepsilon t} + \varepsilon c}{\varepsilon\, e^{2\varepsilon t}} \qquad y = \sqrt{1/v}$$

So $$y(t) = \sqrt{\frac{\varepsilon\, e^{2\varepsilon t}}{\sigma\, e^{2\varepsilon t} + \varepsilon . c}}, \text{ where } \varepsilon, \sigma > 0, \; c \in \mathbb{R}.$$

Section 2.6:

Q1) Solve the following equation.

$$(y/x + 6x)\,dx + (\ln x - 2)\,dy = 0, \quad x > 0$$

Let $M(x,y) = y/x + 6x$ $\quad N(x,y) = \ln x - 2$

$$\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x}$$, so the equation is exact.

There exists a function $\psi(x,y)$ such that $\frac{\partial \psi}{\partial x} = M$, $\frac{\partial \psi}{\partial y} = N$ and $\psi = c$, $c \in \mathbb{R}$ gives the solution.

$$\psi = \int M\,dx = y.\ln x + 3x^2 + g(y),$$ g is a fn. of y.

$$\frac{\partial \psi}{\partial y} = \ln x + g'(y) = N = \ln x - 2$$ So $g(y) = -2y + d$, $d \in \mathbb{R}$

Hence we have

$$\Psi(x,y) = y \cdot \ln x + 3x^2 - 2y + d, \quad d \in \mathbb{R}$$

and we have

$$y \cdot \ln x + 3x^2 - 2y = c, \quad c \in \mathbb{R}$$

$$y(\ln x - 2) = c - 3x^2$$

$$\boxed{y(x) = \frac{c - 3x^2}{\ln x - 2}}$$

Q2) Solve the following equation.

$$(x+2)\sin y \, dx + x \cdot \cos y \, dy = 0$$

Let $M(x,y) = (x+2)\sin y$ $\quad N(x,y) = x \cdot \cos y$

$$\frac{\partial M}{\partial y} = (x+2)\cos y \neq \frac{\partial N}{\partial x} = \cos y \text{ , not exact.}$$

We will try to find an integrating factor to make this exact. If the integrating factor will depend only on x, it will satisfy

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu,$$

if it will depend only on y, it will satisfy

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu.$$

Try

$$\frac{d\mu}{dx} = \underbrace{\frac{(x+2)\cos y - \cos y}{x . \cos y}}_{\frac{x+1}{x}} \mu$$

Solve the 1^{st} order o.d.e,

$$\mu' - \left(\frac{x+1}{x}\right)\mu = 0$$, the integrating factor, λ,

$$\lambda = e^{-\int \frac{x+1}{x} dx} = e^{-\int 1 \, dx - \int \frac{1}{x} dx} = x^{-1} e^{-x} = \frac{1}{x e^x}$$

$$\left(\mu . \frac{1}{x . e^x}\right)' = 0 \qquad \boxed{\mu(x) = x . e^x}$$ (without loss of generality choose $c = 1$

Multiplying with μ, the equation becomes

$$\underbrace{x(x+2) e^x \sin y}_{M(x,y)} \, dx + \underbrace{x^2 e^x \cos y}_{N(x,y)} \, dy = 0$$

$$M_y = (x^2+2x) e^x \cos y = N_x = (x^2 e^x + 2x e^x) \cos y$$

Equation is exact now. So there is such Ψ.

$\Psi = \int N dy = x^2 e^x \sin y + g(x)$, g a fn. of only x.

$\Psi_x = (x^2+2x) e^x . \sin y + g'(x)$, so $g'(x) = 0 \Rightarrow g(x) = c, c \in \mathbb{R}$

Thus we have the solution given by

$$x^2 e^x \sin y = d, \quad d \in \mathbb{R}$$

$$\sin y = d x^{-2} e^{-x}$$

$$\boxed{y(x) = \arcsin\left(\frac{d}{x^2 e^x}\right), \quad d \in \mathbb{R}}$$

is the solution.