Boyce/DiPrima 10th ed, Ch 3.1: 2nd Order Linear Homogeneous Equations-Constant Coefficients

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- A **second order ordinary differential equation** has the general form
  $$y'' = f(t, y, y')$$
  where f is some given function.
- This equation is said to be **linear** if f is linear in y and y':
  $$y'' = g(t) - p(t)y' - q(t)y$$
  Otherwise the equation is said to be **nonlinear**.
- A second order linear equation often appears as
  $$P(t)y'' + Q(t)y' + R(t)y = G(t)$$
- If $G(t) = 0$ for all t, then the equation is called **homogeneous**. Otherwise the equation is **nonhomogeneous**.

Homogeneous Equations, Initial Values

- In Sections 3.5 and 3.6, we will see that once a solution to a homogeneous equation is found, then it is possible to solve the corresponding nonhomogeneous equation, or at least express the solution in terms of an integral.
- The focus of this chapter is thus on homogeneous equations; and in particular, those with constant coefficients:
  $$ay'' + by' + cy = 0$$
  We will examine the variable coefficient case in Chapter 5.
- Initial conditions typically take the form
  $$y(t_0) = y_0, \;\; y'(t_0) = y_0'$$
- Thus solution passes through (t_0, y_0), and slope of solution at (t_0, y_0) is equal to y_0'.

Example 1: Infinitely Many Solutions (1 of 3)

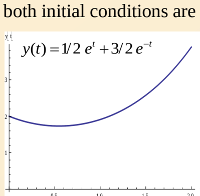
- Consider the second order linear differential equation
  $$y'' - y = 0$$
- Two solutions of this equation are
  $$y_1(t) = e^t, \quad y_2(t) = e^{-t}$$
- Other solutions include
  $$y_3(t) = 3e^t, \quad y_4(t) = 5e^{-t}, \quad y_5(t) = 3e^t + 5e^{-t}$$
- Based on these observations, we see that there are infinitely many solutions of the form
  $$y(t) = c_1 e^t + c_2 e^{-t}$$
- It will be shown in Section 3.2 that all solutions of the differential equation above can be expressed in this form.

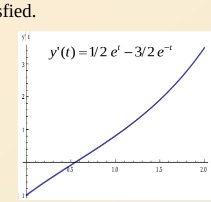
Example 1: Initial Conditions (2 of 3)

- Now consider the following initial value problem for our equation:
  $$y'' - y = 0, \quad y(0) = 2, \; y'(0) = -1$$
- We have found a general solution of the form
  $$y(t) = c_1 e^t + c_2 e^{-t}$$
- Using the initial equations,
  $$\left.\begin{array}{l} y(0) = c_1 + c_2 = 2 \\ y'(0) = c_1 - c_2 = -1 \end{array}\right\} \Rightarrow c_1 = 1/2, \; c_2 = 3/2$$
- Thus
  $$y(t) = 1/2 \; e^t + 3/2 \; e^{-t}$$

Example 1: Solution Graphs (3 of 3)

- Our initial value problem and solution are
  $$y'' - y = 0, \quad y(0) = 2, \; y'(0) = -1 \quad \Rightarrow \quad y(t) = 1/2\, e^t + 3/2\, e^{-t}$$
- Graphs of both y(t) and y'(t) are given below. Observe that both initial conditions are satisfied.





Characteristic Equation

- To solve the 2nd order equation with constant coefficients,
  $$ay'' + by' + cy = 0,$$
  we begin by assuming a solution of the form $y = e^{rt}$.
- Substituting this into the differential equation, we obtain
  $$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$
- Simplifying,
  $$e^{rt}(ar^2 + br + c) = 0$$
  and hence
  $$ar^2 + br + c = 0$$
- This last equation is called the **characteristic equation** of the differential equation.
- We then solve for r by factoring or using quadratic formula.

General Solution

- Using the quadratic formula on the characteristic equation
$$ar^2 + br + c = 0,$$
we obtain two solutions, r_1 and r_2.
- There are three possible results:
  - The roots r_1, r_2 are real and $r_1 \neq r_2$.
  - The roots r_1, r_2 are real and $r_1 = r_2$.
  - The roots r_1, r_2 are complex.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- In this section, we will assume r_1, r_2 are real and $r_1 \neq r_2$.
- In this case, the general solution has the form
$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Initial Conditions

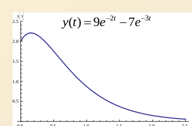
- For the initial value problem
$$ay'' + by' + cy = 0, \quad y(t_0) = y_0, \; y'(t_0) = y_0',$$
we use the general solution
$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$
together with the initial conditions to find c_1 and c_2. That is,
$$\left.\begin{aligned} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} &= y_0 \\ c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} &= y_0' \end{aligned}\right\} \Rightarrow c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \; c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}$$
- Since we are assuming $r_1 \neq r_2$, it follows that a solution of the form $y = e^{rt}$ to the above initial value problem will always exist, for any set of initial conditions.

Example 2 (General Solution)

- Consider the linear differential equation
$$y'' + 5y' + 6y = 0$$
- Assuming an exponential solution leads to the characteristic equation:
$$y(t) = e^{rt} \;\Rightarrow\; r^2 + 5r + 6 = 0 \;\Leftrightarrow\; (r+2)(r+3) = 0$$
- Factoring the characteristic equation yields two solutions: $r_1 = -2$ and $r_2 = -3$
- Therefore, the general solution to this differential equation has the form
$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

Example 3 (Particular Solution)

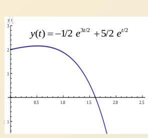
- Consider the initial value problem
$$y'' + 5y' + 6y = 0, \; y(0) = 2, \; y'(0) = 3$$
- From the preceding example, we know the general solution has the form: $\quad y(t) = c_1 e^{-2t} + c_2 e^{-3t}$
- With derivative: $\quad y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$
- Using the initial conditions:
$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ -2c_1 - 3c_2 &= 3 \end{aligned}\right\} \Rightarrow c_1 = 9, \; c_2 = -7$$



- Thus $\quad y(t) = 9e^{-2t} - 7e^{-3t}$

Example 4: Initial Value Problem

- Consider the initial value problem
$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \; y'(0) = 1/2$$
- Then
$$y(t) = e^{rt} \;\Rightarrow\; 4r^2 - 8r + 3 = 0 \;\Leftrightarrow\; (2r-3)(2r-1) = 0$$
- Factoring yields two solutions, $r_1 = 3/2$ and $r_2 = 1/2$
- The general solution has the form
$$y(t) = c_1 e^{3t/2} + c_2 e^{t/2}$$
- Using initial conditions:
$$\left.\begin{aligned} c_1 + c_2 &= 2 \\ 3/2\, c_1 + 1/2\, c_2 &= 1/2 \end{aligned}\right\} \Rightarrow c_1 = -1/2, \; c_2 = 5/2$$

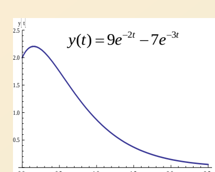


- Thus $\quad y(t) = -1/2\, e^{3t/2} + 5/2\, e^{t/2}$

Example 5: Find Maximum Value

- For the initial value problem in Example 3, to find the maximum value attained by the solution, we set $y'(t) = 0$ and solve for t:

$$\begin{aligned} y(t) &= 9e^{-2t} - 7e^{-3t} \\ y'(t) &= -18e^{-2t} + 21e^{-3t} \overset{set}{=} 0 \\ 6e^{-2t} &= 7e^{-3t} \\ e^{t} &= 7/6 \\ t &= \ln(7/6) \\ t &\approx 0.1542 \\ y &\approx 2.204 \end{aligned}$$



Boyce/DiPrima 10th ed, Ch 3.2: Fundamental Solutions of Linear Homogeneous Equations

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- Let p, q be continuous functions on an interval $I = (\alpha, \beta)$, which could be infinite. For any function y that is twice differentiable on I, define the differential operator L by

$$L[y] = y'' + p\,y' + q\,y$$

- Note that $L[y]$ is a function on I, with output value

$$L[y](t) = y''(t) + p(t)\,y'(t) + q(t)\,y(t)$$

- For example,

$$p(t) = t^2,\ q(t) = e^{2t},\ y(t) = \sin(t),\ \ I = (0, 2\pi)$$
$$L[y](t) = -\sin(t) + t^2\cos(t) + 2e^{2t}\sin(t)$$

Differential Operator Notation

- In this section we will discuss the second order linear homogeneous equation $L[y](t) = 0$, along with initial conditions as indicated below:

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$
$$y(t_0) = y_0,\ y'(t_0) = y_1$$

- We would like to know if there are solutions to this initial value problem, and if so, are they unique.
- Also, we would like to know what can be said about the form and structure of solutions that might be helpful in finding solutions to particular problems.
- These questions are addressed in the theorems of this section.

Theorem 3.2.1 (Existence and Uniqueness)

- Consider the initial value problem

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$
$$y(t_0) = y_0,\ y'(t_0) = y_0'$$

- where p, q, and g are continuous on an open interval I that contains t_0. Then there exists a unique solution $y = \phi(t)$ on I.

- Note: While this theorem says that a solution to the initial value problem above exists, it is often not possible to write down a useful expression for the solution. This is a major difference between first and second order linear equations.

Example 1

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$
$$y(t_0) = y_0,\ y'(t_0) = y_1$$

- Consider the second order linear initial value problem

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0,\ y(1) = 2,\ \ y'(1) = 1$$

- Wrting the differential equation in the form :

$$y'' + p(t)y' + q(t)y = g(t)$$

- $p(t) = 1/(t-3),\ q(t) = -(t+3)/(t(t-3))$ and $g(t) = 0$
- The only points of discontinuity for these coefficients are $t = 0$ and $t = 3$. So the longest open interval containing the initial point $t = 1$ in which all the coefficients are continuous is $0 < t < 3$
- Therefore, the longest interval in which Theorem 3.2.1 guarantees the existence of the solution is $0 < t < 3$

Example 2

- Consider the second order linear initial value problem

$$y'' + p(t)y' + q(t)y = 0,\ y(0) = 0,\ \ y'(0) = 0$$

where p, q are continuous on an open interval I containing t_0.

- In light of the initial conditions, note that $y = 0$ is a solution to this homogeneous initial value problem.

- Since the hypotheses of Theorem 3.2.1 are satisfied, it follows that $y = 0$ is the only solution of this problem.

Theorem 3.2.2 (Principle of Superposition)

- If y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

then the linear combination $c_1y_1 + y_2c_2$ is also a solution, for all constants c_1 and c_2.

- To prove this theorem, substitute $c_1y_1 + y_2c_2$ in for y in the equation above, and use the fact that y_1 and y_2 are solutions.
- Thus for any two solutions y_1 and y_2, we can construct an infinite family of solutions, each of the form $y = c_1y_1 + c_2\,y_2$.
- Can all solutions can be written this way, or do some solutions have a different form altogether? To answer this question, we use the Wronskian determinant.

The Wronskian Determinant (1 of 3)

- Suppose y_1 and y_2 are solutions to the equation
$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$
- From Theorem 3.2.2, we know that $y = c_1y_1 + c_2y_2$ is a solution to this equation.
- Next, find coefficients such that $y = c_1y_1 + c_2y_2$ satisfies the initial conditions
$$y(t_0) = y_0,\ y'(t_0) = y_0'$$
- To do so, we need to solve the following equations:
$$c_1y_1(t_0) + c_2y_2(t_0) = y_0$$
$$c_1y_1'(t_0) + c_2y_2'(t_0) = y_0'$$

$$c_1y_1(t_0) + c_2y_2(t_0) = y_0$$
$$c_1y_1'(t_0) + c_2y_2'(t_0) = y_0'$$

The Wronskian Determinant (2 of 3)

- Solving the equations, we obtain
$$c_1 = \frac{y_0y_2'(t_0) - y_0'y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}$$
$$c_2 = \frac{-y_0y_1'(t_0) + y_0'y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}$$
- In terms of determinants:
$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}},\quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

The Wronskian Determinant (3 of 3)

- In order for these formulas to be valid, the determinant W in the denominator cannot be zero:
$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W},\quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W}$$
$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$$
- W is called the **Wronskian determinant**, or more simply, the Wronskian of the solutions y_1 and y_2. We will sometimes use the notation
$$W(y_1, y_2)(t_0)$$

Theorem 3.2.3

- Suppose y_1 and y_2 are solutions to the equation
$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$
with the initial conditions
$$y(t_0) = y_0,\ y'(t_0) = y_0'$$
Then it is always possible to choose constants c_1, c_2 so that
$$y = c_1y_1(t) + c_2y_2(t)$$
satisfies the differential equation and initial conditions if and ony if the Wronskian
$$W = y_1y_2' - y_1'y_2$$
is not zero at the point t_0

Example 3

- In Example 2 of Section 3.1, we found that
$$y_1(t) = e^{-2t} \text{ and } y_2(t) = e^{-3t}$$
were solutions to the differential equation
$$y'' + 5y' + 6y = 0$$
- The Wronskian of these two functions is
$$W = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}$$
- Since W is nonzero for all values of t, the functions y_1 and y_2 can be used to construct solutions of the differential equation with initial conditions at any value of t

Theorem 3.2.4 (Fundamental Solutions)

- Suppose y_1 and y_2 are solutions to the equation
$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0.$$
Then the family of solutions
$$y = c_1y_1 + c_2y_2$$
with arbitrary coefficients c_1, c_2 includes every solution to the differential equation if an only if there is a point t_0 such that $W(y_1,y_2)(t_0) \neq 0$, .
- The expression $y = c_1y_1 + c_2y_2$ is called the **general solution** of the differential equation above, and in this case y_1 and y_2 are said to form a **fundamental set of solutions** to the differential equation.

Example 4

- Consider the general second order linear equation below, with the two solutions indicated:

$$y'' + p(t)\,y' + q(t)\,y = 0$$

- Suppose the functions below are solutions to this equation:

$$y_1 = e^{r_1 t},\ y_2 = e^{r_2 t},\quad r_1 \neq r_2$$

- The Wronskian of y_1 and y_2 is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)e^{(r_1+r_2)t} \neq 0 \text{ for all } t.$$

- Thus y_1 and y_2 form a fundamental set of solutions to the equation, and can be used to construct all of its solutions.
- The general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Example 5: Solutions (1 of 2)

- Consider the following differential equation:

$$2t^2 y'' + 3t\,y' - y = 0,\ t > 0$$

- Show that the functions below are fundamental solutions:

$$y_1 = t^{1/2},\ y_2 = t^{-1}$$

- To show this, first substitute y_1 into the equation:

$$2t^2\left(\frac{-t^{-3/2}}{4}\right) + 3t\left(\frac{t^{-1/2}}{2}\right) - t^{1/2} = \left(-\frac{1}{2} + \frac{3}{2} - 1\right)t^{1/2} = 0$$

- Thus y_1 is a indeed a solution of the differential equation.
- Similarly, y_2 is also a solution:

$$2t^2\left(2t^{-3}\right) + 3t\left(-t^{-2}\right) - t^{-1} = (4 - 3 - 1)t^{-1} = 0$$

Example 5: Fundamental Solutions (2 of 2)

- Recall that

$$y_1 = t^{1/2},\ y_2 = t^{-1}$$

- To show that y_1 and y_2 form a fundamental set of solutions, we evaluate the Wronskian of y_1 and y_2:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -t^{-3/2} - \frac{1}{2}t^{-3/2} = -\frac{3}{2}t^{-3/2} = -\frac{3}{2\sqrt{t^3}}$$

- Since $W \neq 0$ for $t > 0$, y_1, y_2 form a fundamental set of solutions for the differential equation

$$2t^2 y'' + 3t\,y' - y = 0,\ t > 0$$

Theorem 3.2.5: Existence of Fundamental Set of Solutions

- Consider the differential equation below, whose coefficients p and q are continuous on some open interval I:

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

- Let t_0 be a point in I, and y_1 and y_2 solutions of the equation with y_1 satisfying initial conditions

$$y_1(t_0) = 1,\ y_1'(t_0) = 0$$

  and y_2 satisfying initial conditions

$$y_2(t_0) = 0,\ y_2'(t_0) = 1$$

- Then y_1, y_2 form a fundamental set of solutions to the given differential equation.

Example 6: Apply Theorem 3.2.5 (1 of 3)

- Find the fundamental set specified by Theorem 3.2.5 for the differential equation and initial point

$$y'' - y = 0,\quad t_0 = 0$$

- In Section 3.1, we found two solutions of this equation:

$$y_1 = e^t,\ y_2 = e^{-t}$$

  The Wronskian of these solutions is $W(y_1, y_2)(t_0) = -2 \neq 0$ so they form a fundamental set of solutions.
- But these two solutions do not satisfy the initial conditions stated in Theorem 3.2.5, and thus they do not form the fundamental set of solutions mentioned in that theorem.
- Let y_3 and y_4 be the fundamental solutions of Thm 3.2.5.

$$y_3(0) = 1,\ y_3'(0) = 0;\quad y_4(0) = 0,\ y_4'(0) = 1$$

Example 6: General Solution (2 of 3)

- Since y_1 and y_2 form a fundamental set of solutions,

$$y_3 = c_1 e^t + c_2 e^{-t},\quad y_3(0) = 1, y_3'(0) = 0$$

$$y_4 = d_1 e^t + d_2 e^{-t},\quad y_4(0) = 0, y_4'(0) = 1$$

- Solving each equation, we obtain

$$y_3(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh(t),\quad y_4(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh(t)$$

- The Wronskian of y_3 and y_4 is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} = \cosh^2 t - \sinh^2 t = 1 \neq 0$$

- Thus y_3, y_4 form the fundamental set of solutions indicated in Theorem 3.2.5, with general solution in this case

$$y(t) = k_1 \cosh(t) + k_2 \sinh(t)$$

Example 6: Many Fundamental Solution Sets (3 of 3)

- Thus
$$S_1 = \{e^t, e^{-t}\}, \quad S_2 = \{\cosh t, \sinh t\}$$
both form fundamental solution sets to the differential equation and initial point
$$y'' - y = 0, \quad t_0 = 0$$

- In general, a differential equation will have infinitely many different fundamental solution sets. Typically, we pick the one that is most convenient or useful.

Theorem 3.2.6

Consider again the equation (2):

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

where p and q are continuous real-valued functions. If $y = u(t) + iv(t)$ is a complex-valued solution of Eq. (2), then its real part u and its imaginary part v are also solutions of this equation.

Theorem 3.2.7 (Abel's Theorem)

- Suppose y_1 and y_2 are solutions to the equation
$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$
where p and q are continuous on some open interval I. Then the $W(y_1,y_2)(t)$ is given by
$$W(y_1, y_2)(t) = ce^{-\int p(t)dt}$$
where c is a constant that depends on y_1 and y_2 but not on t.

- Note that $W(y_1,y_2)(t)$ is either zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

Example 7 Apply Abel's Theorem

- Recall the following differential equation and its solutions:
$2t^2 y'' + 3t\,y' - y = 0, \ t > 0$ with solutions $y_1 = t^{1/2}, y_2 = t^{-1}$
- We computed the Wronskian for these solutions to be
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{3}{2} t^{-3/2} = -\frac{3}{2\sqrt{t^3}}$$
- Writing the differential equation in the standard form
$$y'' + 3/(2t)\,y' - 1/(2t^2)\,y = 0, \ t > 0$$
- So $p(t) = 3/(2t)$ and the Wronskian given by Thm.3.2.6 is
$$W(y_1, y_2)(t) = ce^{-\int 3/(2t)dt} = ce^{(-3/2\,\ln t\,)} = ct^{-3/2}$$
- This is the Wronskian for any pair of fundamental solutions. For the solutions given above, we must let c = -3/2

Summary

- To find a general solution of the differential equation
$$y'' + p(t)\,y' + q(t)\,y = 0, \ \alpha < t < \beta$$
we first find two solutions y_1 and y_2.
- Then make sure there is a point t_0 in the interval such that $W(y_1, y_2)(t_0) \neq 0$.
- It follows that y_1 and y_2 form a fundamental set of solutions to the equation, with general solution $y = c_1 y_1 + c_2 y_2$.
- If initial conditions are prescribed at a point t_0 in the interval where $W \neq 0$, then c_1 and c_2 can be chosen to satisfy those conditions.

Boyce/DiPrima 10th ed, Ch 3.3: Complex Roots of Characteristic Equation

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- Recall our discussion of the equation
$$ay'' + by' + cy = 0$$
where a, b and c are constants.
- Assuming an exponential soln leads to characteristic equation:
$$y(t) = e^{rt} \quad \Rightarrow \quad ar^2 + br + c = 0$$
- Quadratic formula (or factoring) yields two solutions, r_1 & r_2:
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
- If $b^2 - 4ac < 0$, then complex roots: $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$
- Thus
$$y_1(t) = e^{(\lambda + i\mu)t}, \ y_2(t) = e^{(\lambda - i\mu)t}$$

Euler's Formula; Complex Valued Solutions

- Substituting it into Taylor series for e^t, we obtain **Euler's formula**:

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!} = \cos t + i \sin t$$

- Generalizing Euler's formula, we obtain

$$e^{i\mu t} = \cos \mu t + i \sin \mu t$$

- Then

$$e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}\left[\cos \mu t + i \sin \mu t\right] = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$

- Therefore

$$y_1(t) = e^{(\lambda + i\mu)t} = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$
$$y_2(t) = e^{(\lambda - i\mu)t} = e^{\lambda t} \cos \mu t - i e^{\lambda t} \sin \mu t$$

Real Valued Solutions

- Our two solutions thus far are complex-valued functions:

$$y_1(t) = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$
$$y_2(t) = e^{\lambda t} \cos \mu t - i e^{\lambda t} \sin \mu t$$

- We would prefer to have real-valued solutions, since our differential equation has real coefficients.
- To achieve this, recall that linear combinations of solutions are themselves solutions:

$$y_1(t) + y_2(t) = 2e^{\lambda t} \cos \mu t$$
$$y_1(t) - y_2(t) = 2ie^{\lambda t} \sin \mu t$$

- Ignoring constants, we obtain the two solutions

$$y_3(t) = e^{\lambda t} \cos \mu t, \quad y_4(t) = e^{\lambda t} \sin \mu t$$

Real Valued Solutions: The Wronskian

- Thus we have the following real-valued functions:

$$y_3(t) = e^{\lambda t} \cos \mu t, \quad y_4(t) = e^{\lambda t} \sin \mu t$$

- Checking the Wronskian, we obtain

$$W = \begin{vmatrix} e^{\lambda t} \cos \mu t & e^{\lambda t} \sin \mu t \\ e^{\lambda t}(\lambda \cos \mu t - \mu \sin \mu t) & e^{\lambda t}(\lambda \sin \mu t + \mu \cos \mu t) \end{vmatrix}$$
$$= \mu e^{2\lambda t} \neq 0$$

- Thus y_3 and y_4 form a fundamental solution set for our ODE, and the general solution can be expressed as

$$y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$$

Example 1 (1 of 2)

- Consider the differential equation

$$y'' + y' + 9.25y = 0$$

- For an exponential solution, the characteristic equation is

$$y(t) = e^{rt} \Rightarrow r^2 + r + 1 = 0 \Leftrightarrow r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

- Therefore, separating the real and imaginary components,

$$\lambda = -1/2, \ \mu = 3$$

and thus the general solution is

$$y(t) = c_1 e^{-t/2} \cos(3t) + c_2 e^{-t/2} \sin(3t) = e^{-t/2}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

Example 1 (2 of 2)

- Using the general solution just determined

$$y(t) = e^{-t/2}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

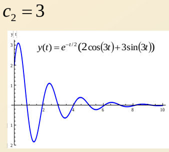
- We can determine the particular solution that satisfies the initial conditions $y(0) = 2$ and $y'(0) = 8$
- So

$$\left.\begin{aligned} y(0) &= c_1 = 2 \\ y'(0) &= -1/2\, c_1 + 3c_2 = 8 \end{aligned}\right\} \Rightarrow c_1 = 2, \ c_2 = 3$$

- Thus the solution of this IVP is

$$y(t) = e^{-t/2}\left(2\cos(3t) + 3\sin(3t)\right)$$

- The solution is a decaying oscillation



Example 2

- Consider the initial value problem

$$16y'' - 8y' + 145y = 0, \qquad y(0) = -2, \quad y'(0) = 1$$

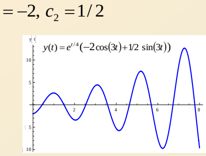
- Then $y(t) = e^{rt} \Rightarrow 16r^2 - 8r + 145 = 0 \Leftrightarrow r = \frac{1}{4} \pm 3i$
- Thus the general solution is $y(t) = c_1 e^{t/4} \cos(3t) + c_2 e^{t/4} \sin(3t)$
- And

$$\left.\begin{aligned} y(0) &= c_1 = -2 \\ y'(0) &= -1/4\, c_1 + 3c_2 = 1 \end{aligned}\right\} \Rightarrow c_1 = -2, \ c_2 = 1/2$$

- The solution of the IVP is

$$y(t) = e^{t/4}\left(-2\cos(3t) + 1/2 \sin(3t)\right)$$

- The solution is displays a growing oscillation



Example 3

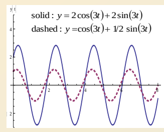
- Consider the equation

$$y'' + 9y = 0$$

- Then $y(t) = e^{rt} \;\Rightarrow\; r^2 + 9 = 0 \;\Leftrightarrow\; r = \pm 3i$
- Therefore $\lambda = 0,\ \mu = 3$
- and thus the general solution is

$$y(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

- Because $\lambda = 0$, there is no exponential factor in the solution, so the amplitude of each oscillation remains constant. The figure shows the graph of two typical solutions



Boyce/DiPrima 10th ed, Ch 3.4: Repeated Roots; Reduction of Order

Elementary Differential Equations and Boundary Value Problems, 10th edition, by William E. Boyce and Richard C. DiPrima, ©2013 by John Wiley & Sons, Inc.

- Recall our 2nd order linear homogeneous ODE

$$ay'' + by' + cy = 0$$

- where *a*, *b* and *c* are constants.
- Assuming an exponential soln leads to characteristic equation:

$$y(t) = e^{rt} \;\Rightarrow\; ar^2 + br + c = 0$$

- Quadratic formula (or factoring) yields two solutions, r_1 & r_2:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- When $b^2 - 4ac = 0$, $r_1 = r_2 = -b/2a$, since method only gives one solution:

$$y_1(t) = ce^{-bt/2a}$$

Second Solution: Multiplying Factor *v*(*t*)

- We know that

$$y_1(t) \text{ a solution} \Rightarrow y_2(t) = cy_1(t) \text{ a solution}$$

- Since y_1 and y_2 are linearly dependent, we generalize this approach and multiply by a function *v*, and determine conditions for which y_2 is a solution:

$$y_1(t) = e^{-bt/2a} \text{ a solution} \Rightarrow \text{ try } y_2(t) = v(t)e^{-bt/2a}$$

- Then

$$y_2(t) = v(t)e^{-bt/2a}$$

$$y_2'(t) = v'(t)e^{-bt/2a} - \frac{b}{2a}v(t)e^{-bt/2a}$$

$$y_2''(t) = v''(t)e^{-bt/2a} - \frac{b}{2a}v'(t)e^{-bt/2a} - \frac{b}{2a}v'(t)e^{-bt/2a} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}$$

$$ay'' + by' + cy = 0$$

Finding Multiplying Factor *v*(*t*)

- Substituting derivatives into ODE, we seek a formula for *v*:

$$e^{-bt/2a}\left\{a\left[v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\right] + b\left[v'(t) - \frac{b}{2a}v(t)\right] + cv(t)\right\} = 0$$

$$av''(t) - bv'(t) + \frac{b^2}{4a}v(t) + bv'(t) - \frac{b^2}{2a}v(t) + cv(t) = 0$$

$$av''(t) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v(t) = 0$$

$$av''(t) + \left(\frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a}\right)v(t) = 0 \;\Leftrightarrow\; av''(t) + \left(\frac{-b^2}{4a} + \frac{4ac}{4a}\right)v(t) = 0$$

$$av''(t) - \left(\frac{b^2 - 4ac}{4a}\right)v(t) = 0$$

$$v''(t) = 0 \;\Rightarrow\; v(t) = k_3 t + k_4$$

General Solution

- To find our general solution, we have:

$$\begin{aligned} y(t) &= k_1 e^{-bt/2a} + k_2 v(t)e^{-bt/2a} \\ &= k_1 e^{-bt/2a} + (k_3 t + k_4)e^{-bt/2a} \\ &= c_1 e^{-bt/2a} + c_2 te^{-bt/2a} \end{aligned}$$

- Thus the general solution for repeated roots is

$$y(t) = c_1 e^{-bt/2a} + c_2 te^{-bt/2a}$$

Wronskian

- The general solution is

$$y(t) = c_1 e^{-bt/2a} + c_2 te^{-bt/2a}$$

- Thus every solution is a linear combination of

$$y_1(t) = e^{-bt/2a},\ y_2(t) = te^{-bt/2a}$$

- The Wronskian of the two solutions is

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} e^{-bt/2a} & te^{-bt/2a} \\ -\frac{b}{2a}e^{-bt/2a} & \left(1 - \frac{bt}{2a}\right)e^{-bt/2a} \end{vmatrix} \\ &= e^{-bt/a}\left(1 - \frac{bt}{2a}\right) + e^{-bt/a}\left(\frac{bt}{2a}\right) \\ &= e^{-bt/a} \neq 0 \quad \text{for all } t \end{aligned}$$

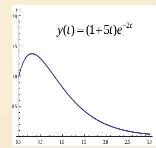
- Thus y_1 and y_2 form a fundamental solution set for equation.

Example 1 (1 of 2)

- Consider the initial value problem
$$y'' + 4y' + 4y = 0$$
- Assuming exponential soln leads to characteristic equation:
$$y(t) = e^{rt} \Rightarrow r^2 + 4r + 4 = 0 \Leftrightarrow (r+2)^2 = 0 \Leftrightarrow r = -2$$
- So one solution is $y_1(t) = e^{-2t}$ and a second solution is found:
$$y_2(t) = v(t)e^{-2t}$$
$$y_2'(t) = v'(t)e^{-2t} - 2v(t)e^{-2t}$$
$$y_2''(t) = v''(t)e^{-2t} - 4v'(t)e^{-2t} + 4v(t)e^{-2t}$$
- Substituting these into the differential equation and simplifying yields $v''(t) = 0,\ v'(t) = k_1,\ v(t) = k_1 t + k_2$ where c_1 and c_2 are arbitrary constants.

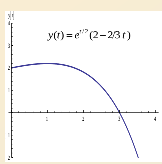
Example 1 (2 of 2)

- Letting $k_1 = 1$ and $k_2 = 0$, $v(t) = t$ and $y_2(t) = te^{-2t}$
- So the general solution is
$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$
- Note that both y_1 and y_2 tend to 0 as $t \to \infty$ regardless of the values of c_1 and c_2
- Using initial conditions
$y(0) = 1$ and $y'(0) = 3$
$$\left.\begin{array}{rcl} c_1 & = & 1 \\ -2c_1 + c_2 & = & 3 \end{array}\right\} \Rightarrow c_1 = 1,\ c_2 = 5$$
- Therefore the solution to the IVP is $y(t) = e^{-2t} + 5te^{-2t}$



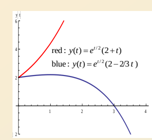
Example 2 (1 of 2)

- Consider the initial value problem
$$y'' - y' + 0.25y = 0,\quad y(0) = 2,\quad y'(0) = 1/3$$
- Assuming exponential soln leads to characteristic equation:
$$y(t) = e^{rt} \Rightarrow r^2 - r + 0.25 = 0 \Leftrightarrow (r - 1/2)^2 = 0 \Leftrightarrow r = 1/2$$
- Thus the general solution is
$$y(t) = c_1 e^{t/2} + c_2 t e^{t/2}$$
- Using the initial conditions:
$$\left.\begin{array}{rcl} c_1 & = & 2 \\ \frac{1}{2}c_1 + c_2 & = & \frac{1}{3} \end{array}\right\} \Rightarrow c_1 = 2,\ c_2 = -\frac{2}{3}$$
- Thus $y(t) = 2e^{t/2} - \frac{2}{3} t e^{t/2}$



Example 2 (2 of 2)

- Suppose that the initial slope in the previous problem was increased
$$y(0) = 2,\ y'(0) = 2$$
- The solution of this modified problem is
$$y(t) = 2e^{t/2} + te^{t/2}$$
- Notice that the coefficient of the second term is now positive. This makes a big difference in the graph, since the exponential function is raised to a positive power: $\lambda = 1/2 > 0$



Reduction of Order

- The method used so far in this section also works for equations with nonconstant coefficients:
$$y'' + p(t)y' + q(t)y = 0$$
- That is, given that y_1 is solution, try $y_2 = v(t)y_1$:
$$y_2(t) = v(t)y_1(t)$$
$$y_2'(t) = v'(t)y_1(t) + v(t)y_1'(t)$$
$$y_2''(t) = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t)$$
- Substituting these into ODE and collecting terms,
$$y_1 v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0$$
- Since y_1 is a solution to the differential equation, this last equation reduces to a first order equation in v':
$$y_1 v'' + (2y_1' + py_1)v' = 0$$

Example 3: Reduction of Order (1 of 3)

- Given the variable coefficient equation and solution y_1,
$$2t^2 y'' + 3ty' - y = 0,\quad t > 0;\quad y_1(t) = t^{-1},$$
use reduction of order method to find a second solution:
$$y_2(t) = v(t)\,t^{-1}$$
$$y_2'(t) = v'(t)\,t^{-1} - v(t)\,t^{-2}$$
$$y_2''(t) = v''(t)\,t^{-1} - 2v'(t)\,t^{-2} + 2v(t)\,t^{-3}$$
- Substituting these into the ODE and collecting terms,
$$2t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) - vt^{-1} = 0$$
$$\Leftrightarrow 2v''t - 4v' + 4vt^{-1} + 3v' - 3vt^{-1} - vt^{-1} = 0$$
$$\Leftrightarrow 2tv'' - v' = 0$$
$$\Leftrightarrow 2tu' - u = 0, \text{ where } u(t) = v'(t)$$

Example 3: Finding $v(t)$ (2 of 3)

- To solve
  $$2tu' - u = 0, \quad u(t) = v'(t)$$
  for u, we can use the separation of variables method:
  $$2t\frac{du}{dt} - u = 0 \quad \Leftrightarrow \quad \int \frac{du}{u} = \int \frac{1}{2t}dt \quad \Leftrightarrow \quad \ln|u| = 1/2 \, \ln|t| + C$$
  $$\Leftrightarrow \quad |u| = |t|^{1/2} e^C \quad \Leftrightarrow \quad u = ct^{1/2}, \quad \text{since } t > 0.$$
- Thus
  $$v' = ct^{1/2}$$
  and hence
  $$v(t) = 2/3\, ct^{3/2} + k$$

Example 3: General Solution (3 of 3)

- Since $v(t) = 2/3\, ct^{3/2} + k$
  $$y_2(t) = \left(2/3\, ct^{3/2} + k\right)t^{-1} = 2/3\, ct^{1/2} + k\, t^{-1}$$
- Recall $y_1(t) = t^{-1}$
- So we can neglect the second term of y_2 to obtain
  $$y_2(t) = t^{1/2}$$
- The Wronskian of $y_1(t)$ and $y_2(t)$ can be computed
  $$W(y_1, y_2)(t) = 3/2\, t^{-3/2} \neq 0, \quad t > 0$$
- Hence the general solution to the differential equation is
  $$y(t) = c_1 t^{-1} + c_2 t^{1/2}$$

Boyce/DiPrima 10th ed, Ch 3.5: Nonhomogeneous Equations;Method of Undetermined Coefficients

- Recall the nonhomogeneous equation
  $$y'' + p(t)y' + q(t)y = g(t)$$
  where p, q, g are continuous functions on an open interval I.
- The associated homogeneous equation is
  $$y'' + p(t)y' + q(t)y = 0$$
- In this section we will learn the method of undetermined coefficients to solve the nonhomogeneous equation, which relies on knowing solutions to the homogeneous equation.

Theorem 3.5.1

- If Y_1 and Y_2 are solutions of the nonhomogeneous equation
  $$y'' + p(t)y' + q(t)y = g(t)$$
  then $Y_1 - Y_2$ is a solution of the homogeneous equation
  $$y'' + p(t)y' + q(t)y = 0$$
- If, in addition, $\{y_1, y_2\}$ forms a fundamental solution set of the homogeneous equation, then there exist constants c_1 and c_2 such that
  $$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

Theorem 3.5.2 (General Solution)

- The general solution of the nonhomogeneous equation
  $$y'' + p(t)y' + q(t)y = g(t)$$
  can be written in the form
  $$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$
  where y_1 and y_2 form a fundamental solution set for the homogeneous equation, c_1 and c_2 are arbitrary constants, and $Y(t)$ is a specific solution to the nonhomogeneous equation.

Method of Undetermined Coefficients

- Recall the nonhomogeneous equation
  $$y'' + p(t)y' + q(t)y = g(t)$$
  with general solution
  $$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$
- In this section we use the method of **undetermined coefficients** to find a particular solution Y to the nonhomogeneous equation, assuming we can find solutions y_1, y_2 for the homogeneous case.
- The method of undetermined coefficients is usually limited to when p and q are constant, and $g(t)$ is a polynomial, exponential, sine or cosine function.

Example 1: Exponential $g(t)$

- Consider the nonhomogeneous equation
  $$y'' - 3y' - 4y = 3e^{2t}$$
- We seek Y satisfying this equation. Since exponentials replicate through differentiation, a good start for Y is:
  $$Y(t) = Ae^{2t} \Rightarrow Y'(t) = 2Ae^{2t},\ Y''(t) = 4Ae^{2t}$$
- Substituting these derivatives into the differential equation,
  $$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$$
  $$\Leftrightarrow\ -6Ae^{2t} = 3e^{2t} \quad \Leftrightarrow\ A = -1/2$$
- Thus a particular solution to the nonhomogeneous ODE is
  $$Y(t) = -\frac{1}{2}e^{2t}$$

Example 2: Sine $g(t)$, First Attempt (1 of 2)

- Consider the nonhomogeneous equation
  $$y'' - 3y' - 4y = 2\sin t$$
- We seek Y satisfying this equation. Since sines replicate through differentiation, a good start for Y is:
  $$Y(t) = A\sin t \Rightarrow Y'(t) = A\cos t,\ Y''(t) = -A\sin t$$
- Substituting these derivatives into the differential equation,
  $$-A\sin t - 3A\cos t - 4A\sin t = 2\sin t$$
  $$\Leftrightarrow\ (2+5A)\sin t + 3A\cos t = 0$$
  $$\Leftrightarrow\ c_1\sin t + c_2\cos t = 0$$
- Since $\sin(x)$ and $\cos(x)$ are not multiples of each other, we must have $c_1 = c_2 = 0$, and hence $2 + 5A = 3A = 0$, which is impossible.

$$y'' - 3y' - 4y = 2\sin t$$

Example 2: Sine $g(t)$, Particular Solution (2 of 2)

- Our next attempt at finding a Y is
  $$Y(t) = A\sin t + B\cos t$$
  $$\Rightarrow Y'(t) = A\cos t - B\sin t,\ Y''(t) = -A\sin t - B\cos t$$
- Substituting these derivatives into ODE, we obtain
  $$(-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) = 2\sin t$$
  $$\Leftrightarrow\ (-5A+3B)\sin t + (-3A-5B)\cos t = 2\sin t$$
  $$\Leftrightarrow\ -5A+3B = 2,\ -3A-5B = 0$$
  $$\Leftrightarrow\ A = -5/17,\ B = 3/17$$
- Thus a particular solution to the nonhomogeneous ODE is
  $$Y(t) = \frac{-5}{17}\sin t + \frac{3}{17}\cos t$$

Example 3: Product $g(t)$

- Consider the nonhomogeneous equation
  $$y'' - 3y' - 4y = -8e^t\cos 2t$$
- We seek Y satisfying this equation, as follows:
  $$Y(t) = Ae^t\cos 2t + Be^t\sin 2t$$
  $$Y'(t) = Ae^t\cos 2t - 2Ae^t\sin 2t + Be^t\sin 2t + 2Be^t\cos 2t$$
  $$= (A+2B)e^t\cos 2t + (-2A+B)e^t\sin 2t$$
  $$Y''(t) = (A+2B)e^t\cos 2t - 2(A+2B)e^t\sin 2t + (-2A+B)e^t\sin 2t$$
  $$+ 2(-2A+B)e^t\cos 2t$$
  $$= (-3A+4B)e^t\cos 2t + (-4A-3B)e^t\sin 2t$$
- Substituting these into the ODE and solving for A and B:
  $$A = \frac{10}{13},\ B = \frac{2}{13} \quad \Rightarrow Y(t) = \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t$$

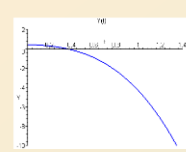
Discussion: Sum $g(t)$

- Consider again our general nonhomogeneous equation
  $$y'' + p(t)y' + q(t)y = g(t)$$
- Suppose that $g(t)$ is sum of functions:
  $$g(t) = g_1(t) + g_2(t)$$
- If Y_1, Y_2 are solutions of
  $$y'' + p(t)y' + q(t)y = g_1(t)$$
  $$y'' + p(t)y' + q(t)y = g_2(t)$$

  respectively, then $Y_1 + Y_2$ is a solution of the nonhomogeneous equation above.

Example 4: Sum $g(t)$

- Consider the equation
  $$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t\cos 2t$$
- Our equations to solve individually are
  $$y'' - 3y' - 4y = 3e^{2t}$$
  $$y'' - 3y' - 4y = 2\sin t$$
  $$y'' - 3y' - 4y = -8e^t\cos 2t$$



- Our particular solution is then
  $$Y(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t$$

Example 5: First Attempt (1 of 3)

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = 2e^{-t}$$

- We seek Y satisfying this equation. We begin with

$$Y(t) = Ae^{-t} \Rightarrow Y'(t) = -Ae^{-t},\ Y''(t) = Ae^{-t}$$

- Substituting these derivatives into differential equation,

$$(A + 3A - 4A)e^{-t} = 2e^{-t}$$

- Since the left side of the above equation is always 0, no value of A can be found to make $Y(t) = Ae^{-t}$ a solution to the nonhomogeneous equation.
- To understand why this happens, we will look at the solution of the corresponding homogeneous differential equation

Example 5: Homogeneous Solution (2 of 3)

- To solve the corresponding homogeneous equation:

$$y'' - 3y' - 4y = 0$$

- We use the techniques from Section 3.1 and get

$$y_1(t) = e^{-t} \text{and } y_2(t) = e^{4t}$$

- Thus our assumed particular solution $Y(t) = Ae^{-t}$ solves the homogeneous equation instead of the nonhomogeneous equation.
- So we need another form for $Y(t)$ to arrive at the general solution of the form:

$$y(t) = c_1 e^{-t} + c_2 e^{4t} + Y(t)$$

$$y'' - 3y' - 4y = 2e^{-t}$$

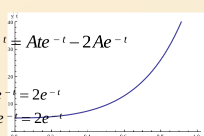
Example 5: Particular Solution (3 of 3)

- Our next attempt at finding a $Y(t)$ is:

$$Y(t) = Ate^{-t}$$

$$Y'(t) = Ae^{-t} - Ate^{-t}$$

$$Y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t} = Ate^{-t} - 2Ae^{-t}$$



$y(t) = 4e^{-t} + e^{4t} - 2/5\,te^{-t}$

- Substituting these into the ODE,

$$Ate^{-t} - 2Ae^{-t} - 3Ae^{-t} + 3Ate^{-t} - 4Ate^{-t} = 2e^{-t}$$

$$0 \cdot Ate^{-t} - 5Ate^{-t} = -5Ate^{-t} = 2e^{-t}$$

$$\Rightarrow A = -2/5$$

$$\Rightarrow Y(t) = -\frac{2}{5}te^{-t}$$

- So the general solution to the IVP is

$$y(t) = c_1 e^{-t} + c_2 e^{4t} - 2/5\,te^{-t}$$

Summary – Undetermined Coefficients (1 of 2)

- For the differential equation

$$ay'' + by' + cy = g(t)$$

where a, b, and c are constants, if $g(t)$ belongs to the class of functions discussed in this section (involves nothing more than exponential functions, sines, cosines, polynomials, or sums or products of these), the method of undetermined coefficients may be used to find a particular solution to the nonhomogeneous equation.
- The first step is to find the general solution for the corresponding homogeneous equation with $g(t) = 0$.

$$y_C(t) = c_1 y_1(t) + c_2 y_2(t)$$

Summary – Undetermined Coefficients (2 of 2)

- The second step is to select an appropriate form for the particular solution, $Y(t)$, to the nonhomogeneous equation and determine the derivatives of that function.
- After substituting $Y(t)$, $Y'(t)$, and $Y''(t)$ into the nonhomogeneous differential equation, if the form for $Y(t)$ is correct, all the coefficients in $Y(t)$ can be determined.
- Finally, the general solution to the nonhomogeneous differential equation can be written as

$$y_{gen}(t) = y_C(t) + Y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

Boyce/DiPrima 10th ed, Ch 3.6: Variation of Parameters

Elementary Differential Equations and Boundary Value Problems, 10th edition, by William E. Boyce and Richard C. DiPrima, ©2013 by John Wiley & Sons, Inc

- Recall the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q, g are continuous functions on an open interval I.
- The associated homogeneous equation is

$$y'' + p(t)y' + q(t)y = 0$$

- In this section we will learn the **variation of parameters** method to solve the nonhomogeneous equation. As with the method of undetermined coefficients, this procedure relies on knowing solutions to the homogeneous equation.
- Variation of parameters is a general method, and requires no detailed assumptions about solution form. However, certain integrals need to be evaluated, and this can present difficulties.

Example 1: Variation of Parameters (1 of 6)

- We seek a particular solution to the equation below.
$$y'' + 4y = 3\csc t$$
- We cannot use the undetermined coefficients method since $g(t)$ is a quotient of $\sin t$ or $\cos t$, instead of a sum or product.
- Recall that the solution to the homogeneous equation is
$$y_C(t) = c_1\cos 2t + c_2\sin 2t$$
- To find a particular solution to the nonhomogeneous equation, we begin with the form
$$y(t) = u_1(t)\cos 2t + u_2(t)\sin 2t$$
- Then
$$y'(t) = u_1'(t)\cos 2t - 2u_1(t)\sin 2t + u_2'(t)\sin 2t + 2u_2(t)\cos 2t$$
- or
$$y'(t) = -2u_1(t)\sin 2t + 2u_2(t)\cos 2t + u_1'(t)\cos 2t + u_2'(t)\sin 2t$$

Example 1: Derivatives, 2nd Equation (2 of 6)

- From the previous slide,
$$y'(t) = -2u_1(t)\sin 2t + 2u_2(t)\cos 2t + u_1'(t)\cos 2t + u_2'(t)\sin 2t$$
- Note that we need two equations to solve for u_1 and u_2. The first equation is the differential equation. To get a second equation, we will require
$$u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0$$
- Then
$$y'(t) = -2u_1(t)\sin 2t + 2u_2(t)\cos 2t$$
- Next,
$$y''(t) = -2u_1'(t)\sin 2t - 4u_1(t)\cos 2t + 2u_2'(t)\cos 2t - 4u_2(t)\sin 2t$$

Example 1: Two Equations (3 of 6)

- Recall that our differential equation is
$$y'' + 4y = 3\csc t$$
- Substituting y'' and y into this equation, we obtain
$$-2u_1'(t)\sin 2t - 4u_1(t)\cos 2t + 2u_2'(t)\cos 2t - 4u_2(t)\sin 2t$$
$$+ 4\left(u_1(t)\cos 2t + u_2(t)\sin 2t\right) = 3\csc t$$
- This equation simplifies to
$$-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t$$
- Thus, to solve for u_1 and u_2, we have the two equations:
$$-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t$$
$$u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0$$

Example 1: Solve for u_1' (4 of 6)

- To find u_1 and u_2, we first need to solve for u_1' and u_2'
$$-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t$$
$$u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0$$
- From second equation,
$$u_2'(t) = -u_1'(t)\frac{\cos 2t}{\sin 2t}$$
- Substituting this into the first equation,
$$-2u_1'(t)\sin 2t + 2\left[-u_1'(t)\frac{\cos 2t}{\sin 2t}\right]\cos 2t = 3\csc t$$
$$-2u_1'(t)\sin^2(2t) - 2u_1'(t)\cos^2(2t) = 3\csc t\sin 2t$$
$$-2u_1'(t)\left[\sin^2(2t) + \cos^2(2t)\right] = 3\left[\frac{2\sin t\cos t}{\sin t}\right]$$
$$u_1'(t) = -3\cos t$$

Example 1: Solve for u_1 and u_2 (5 of 6)

- From the previous slide,
$$u_1'(t) = -3\cos t, \quad u_2'(t) = -u_1'(t)\frac{\cos 2t}{\sin 2t}$$
- Then
$$u_2'(t) = 3\cos t\left[\frac{\cos 2t}{\sin 2t}\right] = 3\cos t\left[\frac{1 - 2\sin^2 t}{2\sin t\cos t}\right] = 3\left[\frac{1 - 2\sin^2 t}{2\sin t}\right]$$
$$= 3\left[\frac{1}{2\sin t} - \frac{2\sin^2 t}{2\sin t}\right] = \frac{3}{2}\csc t - 3\sin t$$
- Thus
$$u_1(t) = \int u_1'(t)dt = \int -3\cos t\,dt = -3\sin t + c_1$$
$$u_2(t) = \int u_2'(t)dt = \int\left(\frac{3}{2}\csc t - 3\sin t\right)dt = \frac{3}{2}\ln\left|\csc t - \cot t\right| + 3\cos t + c_2$$

Example 1: General Solution (6 of 6)

- Recall our equation and homogeneous solution y_C:
$$y'' + 4y = 3\csc t, \quad y_C(t) = c_1\cos 2t + c_2\sin 2t$$
- Using the expressions for u_1 and u_2 on the previous slide, the general solution to the differential equation is
$$y(t) = u_1(t)\cos 2t + u_2(t)\sin 2t + y_C(t)$$
$$= -3\sin t\cos 2t + \frac{3}{2}\ln\left|\csc t - \cot t\right|\sin 2t + 3\cos t\sin 2t + y_C(t)$$
$$= 3\left[\cos t\sin 2t - \sin t\cos 2t\right] + \frac{3}{2}\ln\left|\csc t - \cot t\right|\sin 2t + y_C(t)$$
$$= 3\left[2\sin t\cos^2 t - \sin t\left(2\cos^2 t - 1\right)\right] + \frac{3}{2}\ln\left|\csc t - \cot t\right|\sin 2t + y_C(t)$$
$$= 3\sin t + \frac{3}{2}\ln\left|\csc t - \cot t\right|\sin 2t + c_1\cos 2t + c_2\sin 2t$$

Summary

$$y'' + p(t)y' + q(t)y = g(t)$$
$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

- Suppose y_1, y_2 are fundamental solutions to the homogeneous equation associated with the nonhomogeneous equation above, where we note that the coefficient on y'' is 1.
- To find u_1 and u_2, we need to solve the equations
$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$
$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$
- Doing so, and using the Wronskian, we obtain
$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)}, \quad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}$$
- Thus
$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\,dt + c_1, \quad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\,dt + c_2$$

Theorem 3.6.1

- Consider the equations
$$y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$
$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$
- If the functions p, q and g are continuous on an open interval I, and if y_1 and y_2 are fundamental solutions to Eq. (2), then a particular solution of Eq. (1) is
$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\,dt$$
and the general solution is
$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$$