

Boyce/DiPrima 10th ed, Ch 7.1: Introduction to Systems of First Order Linear Equations

Elementary Differential Equations and Boundary Value Problems, 10th edition, by William E. Boyce and Richard C. DiPrima, ©2013 by John Wiley & Sons, Inc.

- A system of simultaneous first order ordinary differential equations has the general form

$$x_1' = F_1(t, x_1, x_2, \dots, x_n)$$

$$x_2' = F_2(t, x_1, x_2, \dots, x_n)$$

$$\vdots$$

$$x_n' = F_n(t, x_1, x_2, \dots, x_n)$$

where each x_k is a function of t . If each F_k is a linear function of x_1, x_2, \dots, x_n , then the system of equations is said to be **linear**, otherwise it is **nonlinear**.

- Systems of higher order differential equations can similarly be defined.

Example 1

- The motion of a certain spring-mass system from Section 3.7 was described by the differential equation

$$u''(t) + 0.125u'(t) + u(t) = 0$$

- This second order equation can be converted into a system of first order equations by letting $x_1 = u$ and $x_2 = u'$. Thus

$$x_1' = x_2$$

$$x_2' + 0.125x_2 + x_1 = 0$$

or

$$x_1' = x_2$$

$$x_2' = -x_1 - 0.125x_2$$

Nth Order ODEs and Linear 1st Order Systems

- The method illustrated in the previous example can be used to transform an arbitrary n th order equation

$$y^{(n)} = F(t, y, y', y'', \dots, y^{(n-1)})$$

into a system of n first order equations, first by defining

$$x_1 = y, x_2 = y', x_3 = y'', \dots, x_n = y^{(n-1)}$$

Then

$$x_1' = x_2$$

$$x_2' = x_3$$

$$\vdots$$

$$x_{n-1}' = x_n$$

$$x_n' = F(t, x_1, x_2, \dots, x_n)$$

Solutions of First Order Systems

- A system of simultaneous first order ordinary differential equations has the general form

$$x_1' = F_1(t, x_1, x_2, \dots, x_n)$$

$$\vdots$$

$$x_n' = F_n(t, x_1, x_2, \dots, x_n).$$

It has a **solution** on $I: \alpha < t < \beta$ if there exists n functions

$$x_1 = \phi_1(t), x_2 = \phi_2(t), \dots, x_n = \phi_n(t)$$

that are differentiable on I and satisfy the system of equations at all points t in I .

- Initial conditions may also be prescribed to give an IVP:

$$x_1(t_0) = x_1^0, x_2(t_0) = x_2^0, \dots, x_n(t_0) = x_n^0$$

Theorem 7.1.1

- Suppose F_1, \dots, F_n and $\partial F_1 / \partial x_1, \dots, \partial F_1 / \partial x_n, \dots, \partial F_n / \partial x_1, \dots, \partial F_n / \partial x_n$ are continuous in the region R of t, x_1, x_2, \dots, x_n -space defined by $\alpha < t < \beta, \alpha_1 < x_1 < \beta_1, \dots, \alpha_n < x_n < \beta_n$, and let the point

$$(t_0, x_1^0, x_2^0, \dots, x_n^0)$$

be contained in R . Then in some interval $(t_0 - h, t_0 + h)$ there exists a unique solution

$$x_1 = \phi_1(t), x_2 = \phi_2(t), \dots, x_n = \phi_n(t)$$

that satisfies the IVP.

$$x_1' = F_1(t, x_1, x_2, \dots, x_n)$$

$$x_2' = F_2(t, x_1, x_2, \dots, x_n)$$

$$\vdots$$

$$x_n' = F_n(t, x_1, x_2, \dots, x_n)$$

Linear Systems

- If each F_k is a linear function of x_1, x_2, \dots, x_n , then the system of equations has the general form

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \dots + p_{1n}(t)x_n + g_1(t)$$

$$x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \dots + p_{2n}(t)x_n + g_2(t)$$

$$\vdots$$

$$x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \dots + p_{nn}(t)x_n + g_n(t)$$

- If each of the $g_k(t)$ is zero on I , then the system is **homogeneous**, otherwise it is **nonhomogeneous**.

Theorem 7.1.2

- Suppose $p_{11}, p_{12}, \dots, p_{mn}, g_1, \dots, g_n$ are continuous on an interval $I: \alpha < t < \beta$ with t_0 in I , and let $x_1^0, x_2^0, \dots, x_n^0$ prescribe the initial conditions. Then there exists a unique solution

$$x_1 = \phi_1(t), x_2 = \phi_2(t), \dots, x_n = \phi_n(t)$$

that satisfies the IVP, and exists throughout I .

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \dots + p_{1n}(t)x_n + g_1(t)$$

$$x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \dots + p_{2n}(t)x_n + g_2(t)$$

\vdots

$$x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \dots + p_{nn}(t)x_n + g_n(t)$$

Boyce/DiPrima 10th ed, Ch 7.2: Review of Matrices

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- For theoretical and computational reasons, we review results of matrix theory in this section and the next.
- A **matrix** \mathbf{A} is an $m \times n$ rectangular array of elements, arranged in m rows and n columns, denoted

$$\mathbf{A} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- Some examples of 2×2 matrices are given below:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 3-2i \\ 4+5i & 6-7i \end{pmatrix}$$

Transpose

- The **transpose** of $\mathbf{A} = (a_{ij})$ is $\mathbf{A}^T = (a_{ji})$.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

- For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow \mathbf{A}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow \mathbf{B}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Conjugate

- The **conjugate** of $\mathbf{A} = (a_{ij})$ is $\bar{\mathbf{A}} = (\bar{a}_{ij})$.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \bar{\mathbf{A}} = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} \\ \bar{a}_{21} & \bar{a}_{22} & \cdots & \bar{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{m1} & \bar{a}_{m2} & \cdots & \bar{a}_{mn} \end{pmatrix}$$

- For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2+3i \\ 3-4i & 4 \end{pmatrix} \Rightarrow \bar{\mathbf{A}} = \begin{pmatrix} 1 & 2-3i \\ 3+4i & 4 \end{pmatrix}$$

Adjoint

- The **adjoint** of \mathbf{A} is $\bar{\mathbf{A}}^T$, and is denoted by \mathbf{A}^*

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \mathbf{A}^* = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{m1} \\ \bar{a}_{12} & \bar{a}_{22} & \cdots & \bar{a}_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & \bar{a}_{mn} \end{pmatrix}$$

- For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2+3i \\ 3-4i & 4 \end{pmatrix} \Rightarrow \mathbf{A}^* = \begin{pmatrix} 1 & 3+4i \\ 2-3i & 4 \end{pmatrix}$$

Square Matrices

- A **square matrix** \mathbf{A} has the same number of rows and columns. That is, \mathbf{A} is $n \times n$. In this case, \mathbf{A} is said to have order n .

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Vectors

- A **column vector** \mathbf{x} is an $n \times 1$ matrix. For example,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

- A **row vector** \mathbf{y} is a $1 \times n$ matrix. For example,

$$\mathbf{y} = (1 \ 2 \ 3)$$

- Note here that $\mathbf{y} = \mathbf{x}^T$, and that in general, if \mathbf{x} is a column vector \mathbf{x} , then \mathbf{x}^T is a row vector.

The Zero Matrix

- The **zero matrix** is defined to be $\mathbf{0} = (0)$, whose dimensions depend on the context. For example,

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots$$

Matrix Equality

- Two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are **equal** if $a_{ij} = b_{ij}$ for all i and j . For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow \mathbf{A} = \mathbf{B}$$

Matrix – Scalar Multiplication

- The product of a matrix $\mathbf{A} = (a_{ij})$ and a constant k is defined to be $k\mathbf{A} = (ka_{ij})$. For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow -5\mathbf{A} = \begin{pmatrix} -5 & -10 & -15 \\ -20 & -25 & -30 \end{pmatrix}$$

Matrix Addition and Subtraction

- The **sum** of two $m \times n$ matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ is defined to be $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$. For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \Rightarrow \mathbf{A} + \mathbf{B} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

- The **difference** of two $m \times n$ matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ is defined to be $\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})$. For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \Rightarrow \mathbf{A} - \mathbf{B} = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix}$$

Matrix Multiplication

- The **product** of an $m \times n$ matrix $\mathbf{A} = (a_{ij})$ and an $n \times r$ matrix $\mathbf{B} = (b_{ij})$ is defined to be the matrix $\mathbf{C} = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

- Examples (note \mathbf{AB} does not necessarily equal \mathbf{BA}):

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \Rightarrow \mathbf{AB} = \begin{pmatrix} 1+4 & 3+8 \\ 3+8 & 9+16 \end{pmatrix} = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix}$$

$$\Rightarrow \mathbf{BA} = \begin{pmatrix} 1+9 & 2+12 \\ 2+12 & 4+16 \end{pmatrix} = \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \\ 0 & -1 \end{pmatrix} \Rightarrow \mathbf{CD} = \begin{pmatrix} 3+2+0 & 0+4-3 \\ 12+5+0 & 0+10-6 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 17 & 4 \end{pmatrix}$$

Example 1: Matrix Multiplication

- To illustrate matrix multiplication and show that it is not commutative, consider the following matrices:

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

- From the definition of matrix multiplication we have:

$$AB = \begin{pmatrix} 2-2+2 & 1+2-1 & -1+1 \\ 2-2 & -2+1 & -1 \\ 4+1+2 & 2-1-1 & -2+1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & -1 & -1 \\ 7 & 0 & -1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2-2 & -4+2-1 & 2-1-1 \\ 1 & -2-2 & 1+1 \\ 2+2 & -4-2+1 & 2+1+1 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 0 \\ 1 & -4 & 2 \\ 4 & -5 & 4 \end{pmatrix} \neq AB$$

Vector Multiplication

- The **dot product** of two $n \times 1$ vectors \mathbf{x} & \mathbf{y} is defined as

$$\mathbf{x}^T \mathbf{y} = \sum_{k=1}^n x_k y_k$$

- The **inner product** of two $n \times 1$ vectors \mathbf{x} & \mathbf{y} is defined as

$$(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \bar{\mathbf{y}} = \sum_{k=1}^n x_k \bar{y}_k$$

- Example:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -1 \\ 2-3i \\ 5+5i \end{pmatrix} \Rightarrow \mathbf{x}^T \mathbf{y} = (1)(-1) + (2)(2-3i) + (3i)(5+5i) = -12 + 9i$$

$$\Rightarrow (\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \bar{\mathbf{y}} = (1)(-1) + (2)(2+3i) + (3i)(5-5i) = 18 + 2li$$

Vector Length

- The **length** of an $n \times 1$ vector \mathbf{x} is defined as

$$\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2} = \left[\sum_{k=1}^n x_k \bar{x}_k \right]^{1/2} = \left[\sum_{k=1}^n |x_k|^2 \right]^{1/2}$$

- Note here that we have used the fact that if $x = a + bi$, then

$$x \cdot \bar{x} = (a + bi)(a - bi) = a^2 + b^2 = |x|^2$$

- Example:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3+4i \end{pmatrix} \Rightarrow \|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2} = \sqrt{(1)(1) + (2)(2) + (3+4i)(3-4i)}$$

$$= \sqrt{1+4+(9+16)} = \sqrt{30}$$

Orthogonality

- Two $n \times 1$ vectors \mathbf{x} & \mathbf{y} are **orthogonal** if $(\mathbf{x}, \mathbf{y}) = 0$.

- Example:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 11 \\ -4 \\ -1 \end{pmatrix} \Rightarrow (\mathbf{x}, \mathbf{y}) = (1)(11) + (2)(-4) + (3)(-1) = 0$$

Identity Matrix

- The multiplicative **identity matrix** \mathbf{I} is an $n \times n$ matrix given by

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- For any square matrix \mathbf{A} , it follows that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.
- The dimensions of \mathbf{I} depend on the context. For example,

$$\mathbf{AI} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \mathbf{IB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Inverse Matrix

- A square matrix \mathbf{A} is **nonsingular**, or **invertible**, if there exists a matrix \mathbf{B} such that that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. Otherwise \mathbf{A} is **singular**.
- The matrix \mathbf{B} , if it exists, is unique and is denoted by \mathbf{A}^{-1} and is called the **inverse** of \mathbf{A} .
- It turns out that \mathbf{A}^{-1} exists iff $\det \mathbf{A} \neq 0$, and \mathbf{A}^{-1} can be found using **row reduction** (also called Gaussian elimination) on the augmented matrix $(\mathbf{A}|\mathbf{I})$, see example on next slide.
- The three elementary row operations:
 - Interchange two rows.
 - Multiply a row by a nonzero scalar.
 - Add a multiple of one row to another row.

Example 2: Finding the Inverse of a Matrix (1 of 2)

- Use row reduction to find the inverse of the matrix \mathbf{A} below, if it exists.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

- Solution: If possible, use elementary row operations to reduce $(\mathbf{A}|\mathbf{I})$,

$$(\mathbf{A}|\mathbf{I}) = \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array} \right),$$

such that the left side is the identity matrix, for then the right side will be \mathbf{A}^{-1} . (See next slide.)

Example 2: Finding the Inverse of a Matrix (2 of 2)

$$\begin{aligned} (\mathbf{A}|\mathbf{I}) &= \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -3 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array} \right) \\ &\rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array} \right) \\ &\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/10 & -1/10 & 3/10 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array} \right) \end{aligned}$$

• Thus $\mathbf{A}^{-1} = \begin{pmatrix} 7/10 & -1/10 & 3/10 \\ 1/2 & -1/2 & 1/2 \\ -4/5 & 2/5 & -1/5 \end{pmatrix}$

Matrix Functions

- The elements of a matrix can be functions of a real variable. In this case, we write

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{pmatrix}$$

- Such a matrix is continuous at a point, or on an interval (a, b) , if each element is continuous there. Similarly with differentiation and integration:

$$\frac{d\mathbf{A}}{dt} = \left(\frac{da_{ij}}{dt} \right), \quad \int_a^b \mathbf{A}(t) dt = \left(\int_a^b a_{ij}(t) dt \right)$$

Example & Differentiation Rules

- Example:

$$\begin{aligned} \mathbf{A}(t) &= \begin{pmatrix} 3t^2 & \sin t \\ \cos t & 4 \end{pmatrix} \Rightarrow \frac{d\mathbf{A}}{dt} = \begin{pmatrix} 6t & \cos t \\ -\sin t & 0 \end{pmatrix} \\ &\Rightarrow \int_0^\pi \mathbf{A}(t) dt = \begin{pmatrix} \pi^3 & 0 \\ -1 & 4\pi \end{pmatrix} \end{aligned}$$

- Many of the rules from calculus apply in this setting. For example:

$$\begin{aligned} \frac{d(\mathbf{CA})}{dt} &= \mathbf{C} \frac{d\mathbf{A}}{dt}, \quad \text{where } \mathbf{C} \text{ is a constant matrix} \\ \frac{d(\mathbf{A} + \mathbf{B})}{dt} &= \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt} \\ \frac{d(\mathbf{AB})}{dt} &= \left(\frac{d\mathbf{A}}{dt} \right) \mathbf{B} + \mathbf{A} \left(\frac{d\mathbf{B}}{dt} \right) \end{aligned}$$

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- A system of n linear equations in n variables,

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2$$

\vdots

$$a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n,$$

can be expressed as a matrix equation $\mathbf{Ax} = \mathbf{b}$:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

- If $\mathbf{b} = \mathbf{0}$, then system is **homogeneous**; otherwise it is **nonhomogeneous**.

Nonsingular Case

- If the coefficient matrix \mathbf{A} is nonsingular, then it is invertible and we can solve $\mathbf{Ax} = \mathbf{b}$ as follows:

$$\mathbf{Ax} = \mathbf{b} \Rightarrow \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \Rightarrow \mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b} \Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

- This solution is therefore unique. Also, if $\mathbf{b} = \mathbf{0}$, it follows that the unique solution to $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$.
- Thus if \mathbf{A} is nonsingular, then the only solution to $\mathbf{Ax} = \mathbf{0}$ is the trivial solution $\mathbf{x} = \mathbf{0}$.

Linear Dependence and Independence

- A set of vectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$ is **linearly dependent** if there exists scalars c_1, c_2, \dots, c_n , not all zero, such that

$$c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} + \dots + c_n \mathbf{x}^{(n)} = \mathbf{0}$$

- If the only solution of

$$c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} + \dots + c_n \mathbf{x}^{(n)} = \mathbf{0}$$

is $c_1 = c_2 = \dots = c_n = 0$, then $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$ is **linearly independent**.

Example 3: Linear Dependence (1 of 2)

- Determine whether the following vectors are linear dependent or linearly independent.

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{x}^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}$$

- We need to solve

$$c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} + c_3 \mathbf{x}^{(3)} = \mathbf{0}$$

or

$$c_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example 3: Linear Dependence (2 of 2)

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{x}^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}$$

- We can reduce the augmented matrix $(\mathbf{A}|\mathbf{b})$, as before.

$$(\mathbf{A}|\mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 2 & 1 & 1 & 0 \\ -1 & 3 & -11 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 5 & 15 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{aligned} c_1 + 2c_2 - 4c_3 &= 0 \\ \rightarrow c_2 - 3c_3 &= 0 \rightarrow c_2 = c_3 \end{aligned} \quad \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \text{ where } c_3 \text{ can be any number}$$

- So, the vectors are linearly dependent: if $c_1 = -1, 2\mathbf{x}^{(1)} - 3\mathbf{x}^{(2)} - \mathbf{x}^{(3)} = \mathbf{0}$
- Alternatively, we could show that the following determinant is zero:

$$\det(\mathbf{x}_j) = \begin{vmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{vmatrix} = 0$$

Linear Independence and Invertibility

- Consider the previous two examples:
 - The first matrix was known to be nonsingular, and its column vectors were linearly independent.
 - The second matrix was known to be singular, and its column vectors were linearly dependent.
- This is true in general: the columns (or rows) of \mathbf{A} are linearly independent iff \mathbf{A} is nonsingular iff \mathbf{A}^{-1} exists.
- Also, \mathbf{A} is nonsingular iff $\det(\mathbf{A}) \neq 0$, hence columns (or rows) of \mathbf{A} are linearly independent iff $\det(\mathbf{A}) \neq 0$.
- Further, if $\mathbf{A} = \mathbf{BC}$, then $\det(\mathbf{C}) = \det(\mathbf{A})\det(\mathbf{B})$. Thus if the columns (or rows) of \mathbf{A} and \mathbf{B} are linearly independent, then the columns (or rows) of \mathbf{C} are also.

Linear Dependence & Vector Functions

- Now consider vector functions $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \dots, \mathbf{x}^{(n)}(t)$, where

$$\mathbf{x}^{(k)}(t) = \begin{pmatrix} x_1^{(k)}(t) \\ x_2^{(k)}(t) \\ \vdots \\ x_m^{(k)}(t) \end{pmatrix}, \quad k=1,2,\dots,n, \quad t \in I = (\alpha, \beta)$$

- As before, $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \dots, \mathbf{x}^{(n)}(t)$ is **linearly dependent** on I if there exists scalars c_1, c_2, \dots, c_n , not all zero, such that

$$c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) + \dots + c_n \mathbf{x}^{(n)}(t) = \mathbf{0}, \quad \text{for all } t \in I$$

- Otherwise $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \dots, \mathbf{x}^{(n)}(t)$ is **linearly independent** on I . See text for more discussion on this.

Eigenvalues and Eigenvectors

- The eqn. $\mathbf{Ax} = \mathbf{y}$ can be viewed as a linear transformation that maps (or transforms) \mathbf{x} into a new vector \mathbf{y} .
- Nonzero vectors \mathbf{x} that transform into multiples of themselves are important in many applications.
- Thus we solve $\mathbf{Ax} = \lambda \mathbf{x}$ or equivalently, $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$.
- This equation has a nonzero solution if we choose λ such that $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$.
- Such values of λ are called **eigenvalues** of \mathbf{A} , and the nonzero solutions \mathbf{x} are called **eigenvectors**.

Example 4: Eigenvalues (1 of 3)

- Find the eigenvalues and eigenvectors of the matrix \mathbf{A} .

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

- Solution: Choose λ such that $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$, as follows.

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \det \left(\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\ &= \det \begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix} \\ &= (3-\lambda)(-2-\lambda) - (-1)(4) \\ &= \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) \\ \Rightarrow \lambda &= 2, \lambda = -1 \end{aligned}$$

Example 4: First Eigenvector (2 of 3)

- To find the eigenvectors of the matrix \mathbf{A} , we need to solve $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda = 2$ and $\lambda = -1$.

- Eigenvector for $\lambda = 2$: Solve

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0} \Leftrightarrow \begin{pmatrix} 3-2 & -1 \\ 4 & -2-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and this implies that $x_1 = x_2$. So

$$\mathbf{x}^{(1)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}, c \text{ arbitrary} \rightarrow \text{choose } \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Example 4: Second Eigenvector (3 of 3)

- Eigenvector for $\lambda = -1$: Solve

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0} \Leftrightarrow \begin{pmatrix} 3+1 & -1 \\ 4 & -2+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and this implies that $x_2 = 4x_1$. So

$$\mathbf{x}^{(2)} = \begin{pmatrix} x_1 \\ 4x_1 \end{pmatrix} = c \begin{pmatrix} 1 \\ 4 \end{pmatrix}, c \text{ arbitrary} \rightarrow \text{choose } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Normalized Eigenvectors

- From the previous example, we see that eigenvectors are determined up to a nonzero multiplicative constant.
- If this constant is specified in some particular way, then the eigenvector is said to be **normalized**.
- For example, eigenvectors are sometimes normalized by choosing the constant so that $\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2} = 1$.

Algebraic and Geometric Multiplicity

- In finding the eigenvalues λ of an $n \times n$ matrix \mathbf{A} , we solve $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$.
- Since this involves finding the determinant of an $n \times n$ matrix, the problem reduces to finding roots of an n th degree polynomial.
- Denote these roots, or eigenvalues, by $\lambda_1, \lambda_2, \dots, \lambda_n$.
- If an eigenvalue is repeated m times, then its **algebraic multiplicity** is m .
- Each eigenvalue has at least one eigenvector, and a eigenvalue of algebraic multiplicity m may have q linearly independent eigenvectors, $1 \leq q \leq m$, and q is called the **geometric multiplicity** of the eigenvalue.

Eigenvectors and Linear Independence

- If an eigenvalue λ has algebraic multiplicity 1, then it is said to be **simple**, and the geometric multiplicity is 1 also.
- If each eigenvalue of an $n \times n$ matrix \mathbf{A} is simple, then \mathbf{A} has n distinct eigenvalues. It can be shown that the n eigenvectors corresponding to these eigenvalues are linearly independent.
- If an eigenvalue has one or more repeated eigenvalues, then there may be fewer than n linearly independent eigenvectors since for each repeated eigenvalue, we may have $q < m$. This may lead to complications in solving systems of differential equations.

Example 5: Eigenvalues (1 of 5)

- Find the eigenvalues and eigenvectors of the matrix \mathbf{A} .

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

- Solution: Choose λ such that $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$, as follows.

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \\ &= -\lambda^3 + 3\lambda + 2 \\ &= (\lambda - 2)(\lambda + 1)^2 \\ \Rightarrow \lambda_1 &= 2, \lambda_2 = -1, \lambda_3 = -1 \end{aligned}$$

Example 5: First Eigenvector (2 of 5)

- Eigenvector for $\lambda = 2$: Solve $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$, as follows.

$$\begin{aligned} \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} 1x_1 & -1x_3 & = 0 \\ & 1x_2 & -1x_3 & = 0 \\ & & 0x_3 & = 0 \end{matrix} \\ \rightarrow \mathbf{x}^{(1)} &= \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, c \text{ arbitrary} \rightarrow \text{choose } \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Example 5: 2nd and 3rd Eigenvectors (3 of 5)

- Eigenvector for $\lambda = -1$: Solve $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$, as follows.

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} 1x_1 + 1x_2 + 1x_3 & = 0 \\ & 0x_2 & = 0 \\ & & 0x_3 & = 0 \end{matrix} \\ \rightarrow \mathbf{x}^{(2)} &= \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \text{ where } x_2, x_3 \text{ arbitrary} \\ \rightarrow \text{choose } \mathbf{x}^{(2)} &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Example 5: Eigenvectors of \mathbf{A} (4 of 5)

- Thus three eigenvectors of \mathbf{A} are

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

where $\mathbf{x}^{(2)}, \mathbf{x}^{(3)}$ correspond to the double eigenvalue $\lambda = -1$.

- It can be shown that $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}$ are linearly independent.
- Hence \mathbf{A} is a 3×3 **symmetric matrix** ($\mathbf{A} = \mathbf{A}^T$) with 3 real eigenvalues and 3 linearly independent eigenvectors.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Example 5: Eigenvectors of \mathbf{A} (5 of 5)

- Note that we could have chosen

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{x}^{(3)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

- Then the eigenvectors are orthogonal, since

$$(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = 0, (\mathbf{x}^{(1)}, \mathbf{x}^{(3)}) = 0, (\mathbf{x}^{(2)}, \mathbf{x}^{(3)}) = 0$$

- Thus \mathbf{A} is a 3×3 symmetric matrix with 3 real eigenvalues and 3 linearly independent orthogonal eigenvectors.

Hermitian Matrices

- A **self-adjoint**, or **Hermitian** matrix, satisfies $\mathbf{A} = \mathbf{A}^*$, where we recall that $\mathbf{A}^* = \overline{\mathbf{A}}^T$.
- Thus for a Hermitian matrix, $a_{ij} = \overline{a_{ji}}$.
- Note that if \mathbf{A} has real entries and is symmetric (see last example), then \mathbf{A} is Hermitian.
- An $n \times n$ Hermitian matrix \mathbf{A} has the following properties:
 - All eigenvalues of \mathbf{A} are real.
 - There exists a full set of n linearly independent eigenvectors of \mathbf{A} .
 - If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are eigenvectors that correspond to different eigenvalues of \mathbf{A} , then $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are orthogonal.
 - Corresponding to an eigenvalue of algebraic multiplicity m , it is possible to choose m mutually orthogonal eigenvectors, and hence \mathbf{A} has a full set of n linearly independent orthogonal eigenvectors.

Boyce/DiPrima 10th ed, Ch 7.4: Basic Theory of Systems of First Order Linear Equations

Elementary Differential Equations and Boundary Value Problems, 10th edition, by William E. Boyce and Richard C. DiPrima, ©2013 by John Wiley & Sons, Inc.

- The general theory of a system of n first order linear equations

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \dots + p_{1n}(t)x_n + g_1(t)$$

$$x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \dots + p_{2n}(t)x_n + g_2(t)$$

\vdots

$$x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \dots + p_{nn}(t)x_n + g_n(t)$$

parallels that of a single n th order linear equation.

- This system can be written as $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$, where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}, \quad \mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{pmatrix}$$

Vector Solutions of an ODE System

- A vector $\mathbf{x} = \phi(t)$ is a solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ if the components of \mathbf{x} ,

$$x_1 = \phi_1(t), x_2 = \phi_2(t), \dots, x_n = \phi_n(t),$$

satisfy the system of equations on $I: \alpha < t < \beta$.

- For comparison, recall that $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ represents our system of equations

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \dots + p_{1n}(t)x_n + g_1(t)$$

$$x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \dots + p_{2n}(t)x_n + g_2(t)$$

\vdots

$$x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \dots + p_{nn}(t)x_n + g_n(t)$$

- Assuming \mathbf{P} and \mathbf{g} continuous on I , such a solution exists by Theorem 7.1.2.

Homogeneous Case; Vector Function Notation

- As in Chapters 3 and 4, we first examine the general homogeneous equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

- Also, the following notation for the vector functions

$\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)}, \dots$ will be used:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{pmatrix}, \quad \dots, \quad \mathbf{x}^{(k)}(t) = \begin{pmatrix} x_{1k}(t) \\ x_{2k}(t) \\ \vdots \\ x_{nk}(t) \end{pmatrix}, \quad \dots$$

Theorem 7.4.1

- If the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, then the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution for any constants c_1 and c_2 .

- Note: By repeatedly applying the result of this theorem, it can be seen that every finite linear combination

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \dots + c_k\mathbf{x}^{(k)}(t)$$

of solutions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)}$ is itself a solution to $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

Theorem 7.4.2

- If $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$ are linearly independent solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ for each point in $I: \alpha < t < \beta$, then each solution $\mathbf{x} = \phi(t)$ can be expressed uniquely in the form

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \dots + c_n\mathbf{x}^{(n)}(t)$$

- If solutions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are linearly independent for each point in $I: \alpha < t < \beta$, then they are **fundamental solutions on I** , and the **general solution** is given by

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \dots + c_n\mathbf{x}^{(n)}(t)$$

The Wronskian and Linear Independence

- The proof of Thm 7.4.2 uses the fact that if $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$ are linearly independent on I , then $\det \mathbf{X}(t) \neq 0$ on I , where

$$\mathbf{X}(t) = \begin{pmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & \ddots & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{pmatrix},$$

- The Wronskian of $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ is defined as

$$W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}](t) = \det \mathbf{X}(t).$$

- It follows that $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}](t) \neq 0$ on I iff $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are linearly independent for each point in I .

Theorem 7.4.3

- If $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$ are solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on $I: \alpha < t < \beta$, then the Wronskian $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}](t)$ is either identically zero on I or else is never zero on I .
- This result relies on Abel's formula for the Wronskian

$$\frac{dW}{dt} = (p_{11} + p_{22} + \dots + p_{nn}) \Rightarrow W(t) = ce^{\int (p_{11}(t) + p_{22}(t) + \dots + p_{nn}(t)) dt}$$
 where c is an arbitrary constant (Refer to Section 3.2)
- This result enables us to determine whether a given set of solutions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$ are fundamental solutions by evaluating $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}](t)$ at *any* point t in $\alpha < t < \beta$.

Theorem 7.4.4

- Let

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \mathbf{e}^{(n)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$
- Let $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$ be solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, $\alpha < t < \beta$, that satisfy the initial conditions

$$\mathbf{x}^{(1)}(t_0) = \mathbf{e}^{(1)}, \dots, \mathbf{x}^{(n)}(t_0) = \mathbf{e}^{(n)},$$
 respectively, where t_0 is any point in $\alpha < t < \beta$. Then $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$ are a fundamental set of solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

Theorem 7.4.5

- Consider the system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$$

where each element of \mathbf{P} is a real-valued continuous function. If $\mathbf{x} = \mathbf{u}(t) + i\mathbf{v}(t)$ is a complex-valued solution of Eq. (3), then its real part $\mathbf{u}(t)$ and its imaginary part $\mathbf{v}(t)$ are also solutions of this equation.

Boyce/DiPrima 10th ed, Ch 7.5: Homogeneous Linear Systems with Constant Coefficients

Elementary Differential Equations and Boundary Value Problems, 10th edition, by William E. Boyce and Richard C. DiPrima, ©2013 by John Wiley & Sons, Inc.

- We consider here a homogeneous system of n first order linear equations with constant, real coefficients:

$$x_1' = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$

$$x_2' = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$$

$$\vdots$$

$$x_n' = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n$$

- This system can be written as $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Equilibrium Solutions

- Note that if $n = 1$, then the system reduces to

$$x' = ax \Rightarrow x(t) = e^{at}$$
- Recall that $x = 0$ is the only equilibrium solution if $a \neq 0$.
- Further, $x = 0$ is an asymptotically stable solution if $a < 0$, since other solutions approach $x = 0$ in this case.
- Also, $x = 0$ is an unstable solution if $a > 0$, since other solutions depart from $x = 0$ in this case.
- For $n > 1$, equilibrium solutions are similarly found by solving $\mathbf{A}\mathbf{x} = \mathbf{0}$. We assume $\det \mathbf{A} \neq 0$, so that $\mathbf{x} = \mathbf{0}$ is the only solution. Determining whether $\mathbf{x} = \mathbf{0}$ is asymptotically stable or unstable is an important question here as well.

Phase Plane

- When $n = 2$, then the system reduces to

$$x_1' = a_{11}x_1 + a_{12}x_2$$

$$x_2' = a_{21}x_1 + a_{22}x_2$$
- This case can be visualized in the x_1x_2 -plane, which is called the **phase plane**.
- In the phase plane, a direction field can be obtained by evaluating $\mathbf{A}\mathbf{x}$ at many points and plotting the resulting vectors, which will be tangent to solution vectors.
- A plot that shows representative solution trajectories is called a **phase portrait**.
- Examples of phase planes, directions fields, and phase portraits will be given later in this section.

Solving Homogeneous System

- To construct a general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, assume a solution of the form $\mathbf{x} = \xi e^{rt}$, where the exponent r and the constant vector ξ are to be determined.
- Substituting $\mathbf{x} = \xi e^{rt}$ into $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we obtain

$$r\xi e^{rt} = \mathbf{A}\xi e^{rt} \Leftrightarrow r\xi = \mathbf{A}\xi \Leftrightarrow (\mathbf{A} - r\mathbf{I})\xi = \mathbf{0}$$
- Thus to solve the homogeneous system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we must find the eigenvalues and eigenvectors of \mathbf{A} .
- Therefore $\mathbf{x} = \xi e^{rt}$ is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ provided that r is an eigenvalue and ξ is an eigenvector of the coefficient matrix \mathbf{A} .

Example 1 (1 of 2)

- Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x}$$

- The most important feature of this system is that the coefficient matrix is a diagonal matrix. Thus, by writing the system in scalar form, we obtain

$$x_1' = 2x_1, \quad x_2' = -3x_2$$

- Each of these equations involves only one of the unknown variables, so we can solve the two equations separately. In this way we find that

$$x_1 = c_1 e^{2t}, \quad x_2 = c_2 e^{-3t}$$

where c_1 and c_2 are arbitrary constants.

Example 1 (2 of 2)

- Then, by writing the solution in vector form, we have

$$\mathbf{x} = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$$

- Now we define the two solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ so that

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$$

- The Wronskian of these solutions is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{vmatrix} = e^{-t}$$

which is never zero. Therefore, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions.

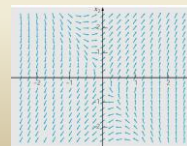
Example 2: Direction Field (1 of 9)

- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

- A direction field for this system is given below.
- Substituting $\mathbf{x} = \xi e^{rt}$ in for \mathbf{x} , and rewriting system as $(\mathbf{A} - r\mathbf{I})\xi = \mathbf{0}$, we obtain

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



Example 2: Eigenvalues (2 of 9)

- Our solution has the form $\mathbf{x} = \xi e^{rt}$, where r and ξ are found by solving

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Recalling that this is an eigenvalue problem, we determine r by solving $\det(\mathbf{A} - r\mathbf{I}) = 0$:

$$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 = r^2 - 2r - 3 = (r-3)(r+1)$$

- Thus $r_1 = 3$ and $r_2 = -1$.

Example 2: First Eigenvector (3 of 9)

- Eigenvector for $r_1 = 3$: Solve

$$(\mathbf{A} - r_1\mathbf{I})\xi = \mathbf{0} \Leftrightarrow \begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} -2 & 1 & 0 \\ 4 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1/2 & 0 \\ 4 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} 1\xi_1 - 1/2\xi_2 = 0 \\ 0\xi_2 = 0 \end{matrix}$$

$$\rightarrow \xi^{(1)} = \begin{pmatrix} 1/2\xi_2 \\ \xi_2 \end{pmatrix} = c \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \quad c \text{ arbitrary} \rightarrow \text{choose } \xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Example 2: Second Eigenvector (4 of 9)

- Eigenvector for $r_2 = -1$: Solve

$$(A - r_2 I)\xi = 0 \Leftrightarrow \begin{pmatrix} 1+1 & 1 \\ 4 & 1+1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1/2 & 0 \\ 4 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow 1\xi_1 + 1/2\xi_2 = 0 \\ \rightarrow \xi^{(2)} = \begin{pmatrix} -1/2\xi_2 \\ \xi_2 \end{pmatrix} = c \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}, c \text{ arbitrary} \rightarrow \text{choose } \xi^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Example 2: General Solution (5 of 9)

- The corresponding solutions $\mathbf{x} = \xi e^{rt}$ of $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

- The Wronskian of these two solutions is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{-2t} \neq 0$$

- Thus $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are fundamental solutions, and the general solution of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) \\ = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

Example 2: Phase Plane for $\mathbf{x}^{(1)}$ (6 of 9)

- To visualize solution, consider first $\mathbf{x} = c_1 \mathbf{x}^{(1)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} \Leftrightarrow x_1 = c_1 e^{3t}, \quad x_2 = 2c_1 e^{3t}$$

- Now

$$x_1 = c_1 e^{3t}, \quad x_2 = 2c_1 e^{3t} \Leftrightarrow e^{3t} = \frac{x_1}{c_1} = \frac{x_2}{2c_1} \Leftrightarrow x_2 = 2x_1$$

- Thus $\mathbf{x}^{(1)}$ lies along the straight line $x_2 = 2x_1$, which is the line through origin in direction of first eigenvector $\xi^{(1)}$
- If solution is trajectory of particle, with position given by (x_1, x_2) , then it is in Q1 when $c_1 > 0$, and in Q3 when $c_1 < 0$.
- In either case, particle moves away from origin as t increases.

Example 2: Phase Plane for $\mathbf{x}^{(2)}$ (7 of 9)

- Next, consider $\mathbf{x} = c_2 \mathbf{x}^{(2)}$:

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \Leftrightarrow x_1 = c_2 e^{-t}, \quad x_2 = -2c_2 e^{-t}$$

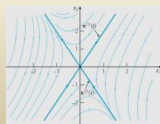
- Then $\mathbf{x}^{(2)}$ lies along the straight line $x_2 = -2x_1$, which is the line through origin in direction of 2nd eigenvector $\xi^{(2)}$
- If solution is trajectory of particle, with position given by (x_1, x_2) , then it is in Q4 when $c_2 > 0$, and in Q2 when $c_2 < 0$.
- In either case, particle moves towards origin as t increases.

Example 2: Phase Plane for General Solution (8 of 9)

- The general solution is $\mathbf{x} = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)}$:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

- As $t \rightarrow \infty$, $c_1 \mathbf{x}^{(1)}$ is dominant and $c_2 \mathbf{x}^{(2)}$ becomes negligible. Thus, for $c_1 \neq 0$, all solutions asymptotically approach the line $x_2 = 2x_1$ as $t \rightarrow \infty$.
- Similarly, for $c_2 \neq 0$, all solutions asymptotically approach the line $x_2 = -2x_1$ as $t \rightarrow -\infty$.
- The origin is a **saddle point**, and is unstable. See graph.

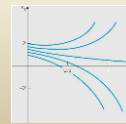


Example 2: Time Plots for General Solution (9 of 9)

- The general solution is $\mathbf{x} = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)}$:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \Leftrightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} + c_2 e^{-t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} \end{pmatrix}$$

- As an alternative to phase plane plots, we can graph x_1 or x_2 as a function of t . A few plots of x_1 are given below.
- Note that when $c_1 = 0$, $x_1(t) = c_2 e^{-t} \rightarrow 0$ as $t \rightarrow \infty$. Otherwise, $x_1(t) = c_1 e^{3t} + c_2 e^{-t}$ grows unbounded as $t \rightarrow \infty$.
- Graphs of x_2 are similarly obtained.



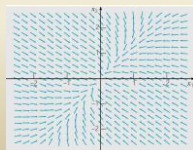
Example 3: Direction Field (1 of 9)

- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x}$$

- A direction field for this system is given below.
- Substituting $\mathbf{x} = \xi e^{rt}$ in for \mathbf{x} , and rewriting system as $(\mathbf{A} - r\mathbf{I})\xi = \mathbf{0}$, we obtain

$$\begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



Example 3: Eigenvalues (2 of 9)

- Our solution has the form $\mathbf{x} = \xi e^{rt}$, where r and ξ are found by solving

$$\begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Recalling that this is an eigenvalue problem, we determine r by solving $\det(\mathbf{A} - r\mathbf{I}) = 0$:

$$\begin{vmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{vmatrix} = (-3-r)(-2-r) - 2 = r^2 + 5r + 4 = (r+1)(r+4)$$

- Thus $r_1 = -1$ and $r_2 = -4$.

Example 3: First Eigenvector (3 of 9)

- Eigenvector for $r_1 = -1$: Solve

$$(\mathbf{A} - r_1\mathbf{I})\xi = \mathbf{0} \Leftrightarrow \begin{pmatrix} -3+1 & \sqrt{2} \\ \sqrt{2} & -2+1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ \sqrt{2} & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \xi^{(1)} = \begin{pmatrix} \sqrt{2}/2 \xi_2 \\ \xi_2 \end{pmatrix} \rightarrow \text{choose } \xi^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

Example 3: Second Eigenvector (4 of 9)

- Eigenvector for $r_2 = -4$: Solve

$$(\mathbf{A} - r_2\mathbf{I})\xi = \mathbf{0} \Leftrightarrow \begin{pmatrix} -3+4 & \sqrt{2} \\ \sqrt{2} & -2+4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \xi^{(2)} = \begin{pmatrix} -\sqrt{2}\xi_2 \\ \xi_2 \end{pmatrix}$$

$$\rightarrow \text{choose } \xi^{(2)} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

Example 3: General Solution (5 of 9)

- The corresponding solutions $\mathbf{x} = \xi e^{rt}$ of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

- The Wronskian of these two solutions is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix} e^{-t} & -\sqrt{2}e^{-4t} \\ \sqrt{2}e^{-t} & e^{-4t} \end{vmatrix} = 3e^{-5t} \neq 0$$

- Thus $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are fundamental solutions, and the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t)$$

$$= c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

Example 3: Phase Plane for $\mathbf{x}^{(1)}$ (6 of 9)

- To visualize solution, consider first $\mathbf{x} = c_1 \mathbf{x}^{(1)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} \Leftrightarrow x_1 = c_1 e^{-t}, \quad x_2 = \sqrt{2} c_1 e^{-t}$$

- Now

$$x_1 = c_1 e^{-t}, \quad x_2 = \sqrt{2} c_1 e^{-t} \Leftrightarrow e^{-t} = \frac{x_1}{c_1} = \frac{x_2}{\sqrt{2} c_1} \Leftrightarrow x_2 = \sqrt{2} x_1$$

- Thus $\mathbf{x}^{(1)}$ lies along the straight line $x_2 = \sqrt{2} x_1$, which is the line through origin in direction of first eigenvector $\xi^{(1)}$
- If solution is trajectory of particle, with position given by (x_1, x_2) , then it is in Q1 when $c_1 > 0$, and in Q3 when $c_1 < 0$.
- In either case, particle moves towards origin as t increases.

Example 3: Phase Plane for $\mathbf{x}^{(2)}$ (7 of 9)

- Next, consider $\mathbf{x} = c_2 \mathbf{x}^{(2)}$:

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} \Leftrightarrow x_1 = -\sqrt{2}c_2 e^{-4t}, \quad x_2 = c_2 e^{-4t}$$

- Then $\mathbf{x}^{(2)}$ lies along the straight line $x_2 = -2^{1/2}x_1$, which is the line through origin in direction of 2nd eigenvector $\xi^{(2)}$
- If solution is trajectory of particle, with position given by (x_1, x_2) , then it is in Q4 when $c_2 > 0$, and in Q2 when $c_2 < 0$.
- In either case, particle moves towards origin as t increases.

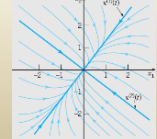
Example 3:

Phase Plane for General Solution (8 of 9)

- The general solution is $\mathbf{x} = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

- As $t \rightarrow \infty$, $c_1 \mathbf{x}^{(1)}$ is dominant and $c_2 \mathbf{x}^{(2)}$ becomes negligible. Thus, for $c_1 \neq 0$, all solutions asymptotically approach origin along the line $x_2 = 2^{1/2}x_1$ as $t \rightarrow \infty$.
- Similarly, all solutions are unbounded as $t \rightarrow -\infty$.
- The origin is a **node**, and is asymptotically stable.



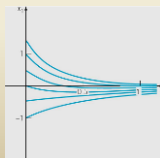
Example 3:

Time Plots for General Solution (9 of 9)

- The general solution is $\mathbf{x} = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)}$:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} \Leftrightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} - \sqrt{2}c_2 e^{-4t} \\ \sqrt{2}c_1 e^{-t} + c_2 e^{-4t} \end{pmatrix}$$

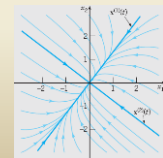
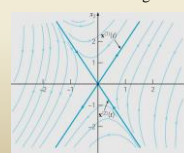
- As an alternative to phase plane plots, we can graph x_1 or x_2 as a function of t . A few plots of x_1 are given below.
- Graphs of x_2 are similarly obtained.



2 x 2 Case:

Real Eigenvalues, Saddle Points and Nodes

- The previous two examples demonstrate the two main cases for a 2 x 2 real system with real and different eigenvalues:
 - Both eigenvalues have opposite signs, in which case origin is a saddle point and is unstable.
 - Both eigenvalues have the same sign, in which case origin is a node, and is asymptotically stable if the eigenvalues are negative and unstable if the eigenvalues are positive.



Eigenvalues, Eigenvectors and Fundamental Solutions

- In general, for an $n \times n$ real linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$:
 - All eigenvalues are real and different from each other.
 - Some eigenvalues occur in complex conjugate pairs.
 - Some eigenvalues are repeated.
- If eigenvalues r_1, \dots, r_n are real & different, then there are n corresponding linearly independent eigenvectors $\xi^{(1)}, \dots, \xi^{(n)}$. The associated solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \xi^{(1)} e^{r_1 t}, \dots, \mathbf{x}^{(n)}(t) = \xi^{(n)} e^{r_n t}$$

- Using Wronskian, it can be shown that these solutions are linearly independent, and hence form a fundamental set of solutions. Thus general solution is

$$\mathbf{x} = c_1 \xi^{(1)} e^{r_1 t} + \dots + c_n \xi^{(n)} e^{r_n t}$$

Hermitian Case: Eigenvalues, Eigenvectors & Fundamental Solutions

- If \mathbf{A} is an $n \times n$ Hermitian matrix (real and symmetric), then all eigenvalues r_1, \dots, r_n are real, although some may repeat.
- In any case, there are n corresponding linearly independent and orthogonal eigenvectors $\xi^{(1)}, \dots, \xi^{(n)}$. The associated solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \xi^{(1)} e^{r_1 t}, \dots, \mathbf{x}^{(n)}(t) = \xi^{(n)} e^{r_n t}$$

and form a fundamental set of solutions.

Example 4: Hermitian Matrix (1 of 3)

- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$$

- The eigenvalues were found previously in Ch 7.3, and were:
 $r_1 = 2, r_2 = -1$ and $r_3 = -1$.
- Corresponding eigenvectors:

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \xi^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \xi^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Example 4: General Solution (2 of 3)

- The fundamental solutions are

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}, \mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

with general solution

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

Example 4: General Solution Behavior (3 of 3)

- The general solution is $\mathbf{x} = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} + c_3 \mathbf{x}^{(3)}$:

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

- As $t \rightarrow \infty$, $c_1 \mathbf{x}^{(1)}$ is dominant and $c_2 \mathbf{x}^{(2)}, c_3 \mathbf{x}^{(3)}$ become negligible.
- Thus, for $c_1 \neq 0$, all solns \mathbf{x} become unbounded as $t \rightarrow \infty$, while for $c_1 = 0$, all solns $\mathbf{x} \rightarrow \mathbf{0}$ as $t \rightarrow \infty$.
- The initial points that cause $c_1 = 0$ are those that lie in plane determined by $\xi^{(2)}$ and $\xi^{(3)}$. Thus solutions that start in this plane approach origin as $t \rightarrow \infty$.

Complex Eigenvalues and Fundamental Solns

- If some of the eigenvalues r_1, \dots, r_n occur in complex conjugate pairs, but otherwise are different, then there are still n corresponding linearly independent solutions

$$\mathbf{x}^{(1)}(t) = \xi^{(1)} e^{r_1 t}, \dots, \mathbf{x}^{(n)}(t) = \xi^{(n)} e^{r_n t},$$

which form a fundamental set of solutions. Some may be complex-valued, but real-valued solutions may be derived from them. This situation will be examined in Ch 7.6.

- If the coefficient matrix \mathbf{A} is complex, then complex eigenvalues need not occur in conjugate pairs, but solutions will still have the above form (if the eigenvalues are distinct) and these solutions may be complex-valued.

Repeated Eigenvalues and Fundamental Solns

- If some of the eigenvalues r_1, \dots, r_n are repeated, then there may not be n corresponding linearly independent solutions of the form

$$\mathbf{x}^{(1)}(t) = \xi^{(1)} e^{r_1 t}, \dots, \mathbf{x}^{(n)}(t) = \xi^{(n)} e^{r_n t}$$

- In order to obtain a fundamental set of solutions, it may be necessary to seek additional solutions of another form.
- This situation is analogous to that for an n th order linear equation with constant coefficients, in which case a repeated root gave rise solutions of the form

$$e^{rt}, t e^{rt}, t^2 e^{rt}, \dots$$

This case of repeated eigenvalues is examined in Section 7.8.

Boyce/DiPrima 10th ed, Ch 7.6: Complex Eigenvalues

Elementary Differential Equations and Boundary Value Problems, 10th edition, by William E. Boyce and Richard C. DiPrima, ©2013 by John Wiley & Sons, Inc.

- We consider again a homogeneous system of n first order linear equations with constant, real coefficients,

$$x_1' = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$

$$x_2' = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$$

$$\vdots$$

$$x_n' = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n,$$

and thus the system can be written as $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Conjugate Eigenvalues and Eigenvectors

- We know that $\mathbf{x} = \xi e^{rt}$ is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, provided r is an eigenvalue and ξ is an eigenvector of \mathbf{A} .
- The eigenvalues r_1, \dots, r_n are the roots of $\det(\mathbf{A} - r\mathbf{I}) = 0$, and the corresponding eigenvectors satisfy $(\mathbf{A} - r\mathbf{I})\xi = \mathbf{0}$.
- If \mathbf{A} is real, then the coefficients in the polynomial equation $\det(\mathbf{A} - r\mathbf{I}) = 0$ are real, and hence any complex eigenvalues must occur in conjugate pairs. Thus if $r_1 = \lambda + i\mu$ is an eigenvalue, then so is $r_2 = \lambda - i\mu$.
- The corresponding eigenvectors $\xi^{(1)}, \xi^{(2)}$ are conjugates also. To see this, recall \mathbf{A} and \mathbf{I} have real entries, and hence $(\mathbf{A} - r_1\mathbf{I})\xi^{(1)} = \mathbf{0} \Rightarrow (\mathbf{A} - \bar{r}_1\mathbf{I})\bar{\xi}^{(1)} = \mathbf{0} \Rightarrow (\mathbf{A} - r_2\mathbf{I})\bar{\xi}^{(1)} = \mathbf{0}$

Conjugate Solutions

- It follows from the previous slide that the solutions

$$\mathbf{x}^{(1)} = \xi^{(1)} e^{rt}, \quad \mathbf{x}^{(2)} = \xi^{(2)} e^{rt}$$

corresponding to these eigenvalues and eigenvectors are conjugates conjugates as well, since

$$\mathbf{x}^{(2)} = \xi^{(2)} e^{rt} = \bar{\xi}^{(1)} e^{\bar{r}t} = \bar{\mathbf{x}}^{(1)}$$

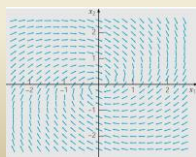
Example 1: Direction Field (1 of 7)

- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\mathbf{x}' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \mathbf{x}$$

- A direction field for this system is given below.
- Substituting $\mathbf{x} = \xi e^{rt}$ in for \mathbf{x} , and rewriting system as $(\mathbf{A} - r\mathbf{I})\xi = \mathbf{0}$, we obtain

$$\begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



Example 1: Complex Eigenvalues (2 of 7)

- We determine r by solving $\det(\mathbf{A} - r\mathbf{I}) = 0$. Now

$$\begin{vmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{vmatrix} = (r + 1/2)^2 + 1 = r^2 + r + \frac{5}{4}$$

- Thus

$$r = \frac{-1 \pm \sqrt{1^2 - 4(5/4)}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i$$

- Therefore the eigenvalues are $r_1 = -1/2 + i$ and $r_2 = -1/2 - i$.

Example 1: First Eigenvector (3 of 7)

- Eigenvector for $r_1 = -1/2 + i$: Solve

$$\begin{aligned} (\mathbf{A} - r_1\mathbf{I})\xi = \mathbf{0} &\Leftrightarrow \begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & i \\ -1 & -i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & i & 0 \\ -1 & -i & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \xi^{(1)} = \begin{pmatrix} -i\xi_2 \\ \xi_2 \end{pmatrix} \rightarrow \text{choose } \xi^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

- Thus $\xi^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example 1: Second Eigenvector (4 of 7)

- Eigenvector for $r_1 = -1/2 - i$: Solve

$$\begin{aligned} (\mathbf{A} - r_1\mathbf{I})\xi = \mathbf{0} &\Leftrightarrow \begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & -i \\ -1 & i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & -i & 0 \\ -1 & i & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \xi^{(2)} = \begin{pmatrix} i\xi_2 \\ \xi_2 \end{pmatrix} \rightarrow \text{choose } \xi^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

- Thus $\xi^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Example 1: General Solution (5 of 7)

- The corresponding solutions $\mathbf{x} = \xi e^{it}$ of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are

$$\mathbf{u}(t) = e^{-i/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t = e^{-i/2} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

$$\mathbf{v}(t) = e^{-i/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t = e^{-i/2} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

- The Wronskian of these two solutions is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix} e^{-i/2} \cos t & e^{-i/2} \sin t \\ -e^{-i/2} \sin t & e^{-i/2} \cos t \end{vmatrix} = e^{-i} \neq 0$$

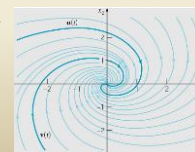
- Thus $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are real-valued fundamental solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with general solution $\mathbf{x} = c_1 \mathbf{u} + c_2 \mathbf{v}$.

Example 1: Phase Plane (6 of 7)

- Given below is the phase plane plot for solutions \mathbf{x} , with

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} e^{-i/2} \cos t \\ -e^{-i/2} \sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-i/2} \sin t \\ e^{-i/2} \cos t \end{pmatrix}$$

- Each solution trajectory approaches origin along a spiral path as $t \rightarrow \infty$, since coordinates are products of decaying exponential and sine or cosine factors.
- The graph of \mathbf{u} passes through $(1, 0)$, since $\mathbf{u}(0) = (1, 0)$. Similarly, the graph of \mathbf{v} passes through $(0, 1)$.
- The origin is a **spiral point**, and is asymptotically stable.

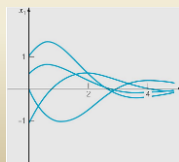


Example 1: Time Plots (7 of 7)

- The general solution is $\mathbf{x} = c_1 \mathbf{u} + c_2 \mathbf{v}$:

$$\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-i/2} \cos t + c_2 e^{-i/2} \sin t \\ -c_1 e^{-i/2} \sin t + c_2 e^{-i/2} \cos t \end{pmatrix}$$

- As an alternative to phase plane plots, we can graph x_1 or x_2 as a function of t . A few plots of x_1 are given below, each one a decaying oscillation as $t \rightarrow \infty$.



General Solution

- To summarize, suppose $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$, and that r_3, \dots, r_n are all real and distinct eigenvalues of \mathbf{A} . Let the corresponding eigenvectors be

$$\xi^{(1)} = \mathbf{a} + i\mathbf{b}, \xi^{(2)} = \mathbf{a} - i\mathbf{b}, \xi^{(3)}, \xi^{(4)}, \dots, \xi^{(n)}$$

- Then the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x} = c_1 \mathbf{u}(t) + c_2 \mathbf{v}(t) + c_3 \xi^{(3)} e^{r_3 t} + \dots + c_n \xi^{(n)} e^{r_n t}$$

where

$$\mathbf{u}(t) = e^{\lambda t} (\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t), \mathbf{v}(t) = e^{\lambda t} (\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t)$$

Real-Valued Solutions

- Thus for complex conjugate eigenvalues r_1 and r_2 , the corresponding solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are conjugates also.
- To obtain real-valued solutions, use real and imaginary parts of either $\mathbf{x}^{(1)}$ or $\mathbf{x}^{(2)}$. To see this, let $\xi^{(1)} = \mathbf{a} + i\mathbf{b}$. Then

$$\begin{aligned} \mathbf{x}^{(1)} &= \xi^{(1)} e^{(\lambda + i\mu)t} = (\mathbf{a} + i\mathbf{b}) e^{\lambda t} (\cos \mu t + i \sin \mu t) \\ &= e^{\lambda t} (\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t) + i e^{\lambda t} (\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t) \end{aligned}$$

where

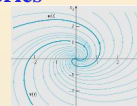
$$\mathbf{u}(t) = e^{\lambda t} (\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t), \mathbf{v}(t) = e^{\lambda t} (\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t),$$

are real valued solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and can be shown to be linearly independent.

Spiral Points, Centers, Eigenvalues, and Trajectories

- In previous example, general solution was

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} e^{-i/2} \cos t \\ -e^{-i/2} \sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-i/2} \sin t \\ e^{-i/2} \cos t \end{pmatrix}$$



- The origin was a **spiral point**, and was asymptotically stable.
- If real part of complex eigenvalues is positive, then trajectories spiral away, unbounded, from origin, and hence origin would be an unstable spiral point.
- If real part of complex eigenvalues is zero, then trajectories circle origin, neither approaching nor departing. Then origin is called a **center** and is stable, but not asymptotically stable. Trajectories periodic in time.
- The direction of trajectory motion depends on entries in \mathbf{A} .

Example 2: Second Order System with Parameter (1 of 2)

- The system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below contains a parameter α .

$$\mathbf{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \mathbf{x}$$

- Substituting $\mathbf{x} = \xi e^{rt}$ in for \mathbf{x} and rewriting system as $(\mathbf{A} - r\mathbf{I})\xi = \mathbf{0}$, we obtain

$$\begin{pmatrix} \alpha - r & 2 \\ -2 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Next, solve for r in terms of α :

$$\begin{vmatrix} \alpha - r & 2 \\ -2 & -r \end{vmatrix} = r(r - \alpha) + 4 = r^2 - \alpha r + 4 \Rightarrow r = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$$

Example 2: Eigenvalue Analysis (2 of 2)

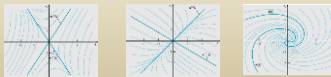
$$r = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$$

- The eigenvalues are given by the quadratic formula above.
- For $\alpha < -4$, both eigenvalues are real and negative, and hence origin is asymptotically stable node.
- For $\alpha > 4$, both eigenvalues are real and positive, and hence the origin is an unstable node.
- For $-4 < \alpha < 0$, eigenvalues are complex with a negative real part, and hence origin is asymptotically stable spiral point.
- For $0 < \alpha < 4$, eigenvalues are complex with a positive real part, and the origin is an unstable spiral point.
- For $\alpha = 0$, eigenvalues are purely imaginary, origin is a center. Trajectories closed curves about origin & periodic.
- For $\alpha = \pm 4$, eigenvalues real & equal, origin is a node (Ch 7.8)

Second Order Solution Behavior and Eigenvalues: Three Main Cases

- For second order systems, the three main cases are:
 - Eigenvalues are real and have opposite signs; $\mathbf{x} = \mathbf{0}$ is a saddle point.
 - Eigenvalues are real, distinct and have same sign; $\mathbf{x} = \mathbf{0}$ is a node.
 - Eigenvalues are complex with nonzero real part; $\mathbf{x} = \mathbf{0}$ a spiral point.
- Other possibilities exist and occur as transitions between two of the cases listed above:
 - A zero eigenvalue occurs during transition between saddle point and node. Real and equal eigenvalues occur during transition between nodes and spiral points. Purely imaginary eigenvalues occur during a transition between asymptotically stable and unstable spiral points.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



Example 3: Multiple Spring-Mass System (1 of 6)

- The equations for the system of two masses and three springs discussed in Section 7.1, assuming no external forces, can be expressed as:

$$m_1 \frac{d^2 x_1}{dt^2} = -(k_1 + k_2)x_1 + k_2 x_2 \quad \text{and} \quad m_2 \frac{d^2 x_2}{dt^2} = k_2 x_1 - (k_2 + k_3)x_2$$

$$\text{or } m_1 y_3' = -(k_1 + k_2)y_1 + k_2 y_2 \quad \text{and} \quad m_2 y_4' = k_2 y_1 - (k_2 + k_3)y_2$$

$$\text{where } y_1 = x_1, y_2 = x_2, y_3 = x_1', \text{ and } y_4 = x_2'$$

- Given $m_1 = 2$, $m_2 = 9/4$, $k_1 = 1$, $k_2 = 3$, and $k_3 = 15/4$, the equations become

$$y_1' = y_3, y_2' = y_4, y_3' = -2y_1 + 3/2 y_2, \text{ and } y_4' = 4/3 y_1 - 3y_2$$

Example 3: Multiple Spring-Mass System (2 of 6)

- Writing the system of equations in matrix form:

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 3/2 & 0 & 0 \\ 4/3 & -3 & 0 & 0 \end{pmatrix} \mathbf{y} = \mathbf{A}\mathbf{y}$$

- Assuming a solution of the form $\mathbf{y} = \xi e^{rt}$, where r must be an eigenvalue of the matrix \mathbf{A} and ξ is the corresponding eigenvector, the characteristic polynomial of \mathbf{A} is

$$r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4)$$

yielding the eigenvalues: $r_1 = i$, $r_2 = -i$, $r_3 = 2i$, and $r_4 = -2i$

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 3/2 & 0 & 0 \\ 4/3 & -3 & 0 & 0 \end{pmatrix} \mathbf{y} = \mathbf{A}\mathbf{y}$$

Example 3: Multiple Spring-Mass System (3 of 6)

- For the eigenvalues $r_1 = i$, $r_2 = -i$, $r_3 = 2i$, and $r_4 = -2i$ the corresponding eigenvectors are

$$\xi^{(1)} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix}, \xi^{(2)} = \begin{pmatrix} 3 \\ 2 \\ -3i \\ -2i \end{pmatrix}, \xi^{(3)} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix}, \text{ and } \xi^{(4)} = \begin{pmatrix} 3 \\ -4 \\ -6i \\ 8i \end{pmatrix}$$

- The products $\xi^{(1)} e^{it}$ and $\xi^{(3)} e^{2it}$ yield the complex-valued solutions:

$$\xi^{(1)} e^{it} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix} (\cos t + i \sin t) = \begin{pmatrix} 3 \cos t \\ 2 \cos t \\ -3 \sin t \\ -2 \sin t \end{pmatrix} + i \begin{pmatrix} 3 \sin t \\ 2 \sin t \\ 3 \cos t \\ 2 \cos t \end{pmatrix} = \mathbf{u}^{(1)}(t) + i \mathbf{v}^{(1)}(t)$$

$$\xi^{(3)} e^{2it} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix} (\cos 2t + i \sin 2t) = \begin{pmatrix} 3 \cos 2t \\ -4 \cos 2t \\ 6 \sin 2t \\ -8 \sin 2t \end{pmatrix} + i \begin{pmatrix} 3 \sin 2t \\ -4 \sin 2t \\ 6 \cos 2t \\ -8 \cos 2t \end{pmatrix} = \mathbf{u}^{(2)}(t) + i \mathbf{v}^{(2)}(t)$$

$$y_1' = y_3, y_2' = y_4, y_3' = -2y_1 + 3/2 y_2, \text{ and } y_4' = 4/3 y_1 - 3y_2$$

Example 3: Multiple Spring-Mass System (4 of 6)

- After validating that $\mathbf{u}^{(1)}(t)$, $\mathbf{v}^{(1)}(t)$, $\mathbf{u}^{(2)}(t)$, $\mathbf{v}^{(2)}(t)$ are linearly independent, the general solution of the system of equations can be written as

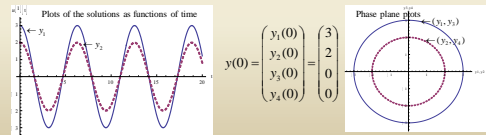
$$\mathbf{y} = c_1 \begin{pmatrix} 3 \cos t \\ 2 \cos t \\ -3 \sin t \\ -2 \sin t \end{pmatrix} + c_2 \begin{pmatrix} 3 \sin t \\ 2 \sin t \\ -4 \cos 2t \\ 2 \cos 2t \end{pmatrix} + c_3 \begin{pmatrix} 3 \cos 2t \\ -4 \cos 2t \\ -6 \sin 2t \\ 8 \sin 2t \end{pmatrix} + c_4 \begin{pmatrix} 3 \sin 2t \\ -4 \sin 2t \\ 6 \cos 2t \\ -8 \cos 2t \end{pmatrix}$$

- where c_1, c_2, c_3, c_4 are arbitrary constants.
- Each solution will be periodic with period 2π , so each trajectory is a closed curve. The first two terms of the solution describe motions with frequency 1 and period 2π while the second two terms describe motions with frequency 2 and period π . The motions of the two masses will be different relative to one another for solutions involving only the first two terms or the second two terms.

$$y_1 \text{ and } y_2 \text{ represent the motion of the masses and } y_3 = y_1', y_4 = y_2'$$

Example 3: Multiple Spring-Mass System (5 of 6)

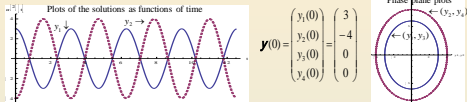
- To obtain the fundamental mode of vibration with frequency 1 $c_3 = c_4 = 0 \rightarrow$ occurs when $3y_2(0) = 2y_1(0)$ and $3y_4(0) = 2y_3(0)$
- To obtain the fundamental mode of vibration with frequency 2 $c_1 = c_2 = 0 \rightarrow$ occurs when $3y_2(0) = -4y_1(0)$ and $3y_4(0) = -4y_3(0)$
- Plots of y_1 and y_2 and parametric plots (y, y') are shown for a selected solution with frequency 1



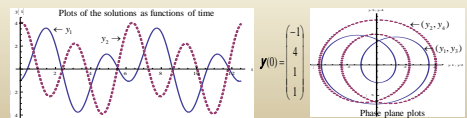
$$y_1 \text{ and } y_2 \text{ represent the motion of the masses and } y_3 = y_1', y_4 = y_2'$$

Example 3: Multiple Spring-Mass System (6 of 6)

- Plots of y_1 and y_2 and parametric plots (y, y') are shown for a selected solution with frequency 2



- Plots of y_1 and y_2 and parametric plots (y, y') are shown for a selected solution with mixed frequencies satisfying the initial condition stated



Boyce/DiPrima 10th ed, Ch 7.7: Fundamental Matrices

Elementary Differential Equations and Boundary Value Problems, 10th edition, by William E. Boyce and Richard C. DiPrima, ©2013 by John Wiley & Sons, Inc.

- Suppose that $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on $\alpha < t < \beta$.
- The matrix

$$\mathbf{\Psi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \dots & x_1^{(n)}(t) \\ \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & \dots & x_n^{(n)}(t) \end{pmatrix},$$

whose columns are $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(n)}(t)$, is a fundamental matrix for the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. This matrix is nonsingular since its columns are linearly independent, and hence $\det \mathbf{\Psi} \neq 0$.

- Note also that since $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(n)}(t)$ are solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, $\mathbf{\Psi}$ satisfies the matrix differential equation $\mathbf{\Psi}' = \mathbf{P}(t)\mathbf{\Psi}$.

Example 1:

- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

- In Example 2 of Chapter 7.5, we found the following fundamental solutions for this system:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

- Thus a fundamental matrix for this system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

Fundamental Matrices and General Solution

- The general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$

$$\mathbf{x} = c_1 \mathbf{x}^{(1)}(t) + \dots + c_n \mathbf{x}^{(n)}(t)$$

can be expressed $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c}$, where \mathbf{c} is a constant vector with components c_1, \dots, c_n :

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c} = \begin{pmatrix} x_1^{(1)}(t) & \dots & x_1^{(n)}(t) \\ \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & \dots & x_n^{(n)}(t) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Fundamental Matrix & Initial Value Problem

- Consider an initial value problem
 $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}^0$
 where $\alpha < t_0 < \beta$ and \mathbf{x}^0 is a given initial vector.
- Now the solution has the form $\mathbf{x} = \Psi(t)\mathbf{c}$, hence we choose \mathbf{c} so as to satisfy $\mathbf{x}(t_0) = \mathbf{x}^0$.
- Recalling $\Psi(t_0)$ is nonsingular, it follows that
 $\Psi(t_0)\mathbf{c} = \mathbf{x}^0 \Rightarrow \mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}^0$
- Thus our solution $\mathbf{x} = \Psi(t)\mathbf{c}$ can be expressed as

$$\mathbf{x} = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}^0$$

Recall: Theorem 7.4.4

- Let

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \mathbf{e}^{(n)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$
- Let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on $I: \alpha < t < \beta$ that satisfy the initial conditions

$$\mathbf{x}^{(1)}(t_0) = \mathbf{e}^{(1)}, \dots, \mathbf{x}^{(n)}(t_0) = \mathbf{e}^{(n)}, \quad \alpha < t_0 < \beta$$

 Then $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are fundamental solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

Fundamental Matrix & Theorem 7.4.4

- Suppose $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(n)}(t)$ form the fundamental solutions given by Thm 7.4.4. Denote the corresponding fundamental matrix by $\Phi(t)$. Then columns of $\Phi(t)$ are $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(n)}(t)$, and hence

$$\Phi(t_0) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = \mathbf{I}$$

- Thus $\Phi^{-1}(t_0) = \mathbf{I}$, and the hence general solution to the corresponding initial value problem is

$$\mathbf{x} = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}^0 = \Phi(t)\mathbf{x}^0$$

- It follows that for any fundamental matrix $\Psi(t)$,

$$\mathbf{x} = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}^0 = \Phi(t)\mathbf{x}^0 \Rightarrow \Phi(t) = \Psi(t)\Psi^{-1}(t_0)$$

The Fundamental Matrix Φ and Varying Initial Conditions

- Thus when using the fundamental matrix $\Phi(t)$, the general solution to an IVP is

$$\mathbf{x} = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}^0 = \Phi(t)\mathbf{x}^0$$
- This representation is useful if same system is to be solved for many different initial conditions, such as a physical system that can be started from many different initial states.
- Also, once $\Phi(t)$ has been determined, the solution to each set of initial conditions can be found by matrix multiplication, as indicated by the equation above.
- Thus $\Phi(t)$ represents a linear transformation of the initial conditions \mathbf{x}^0 into the solution $\mathbf{x}(t)$ at time t .

Example 2: Find $\Phi(t)$ for 2 x 2 System (1 of 5)

- Find $\Phi(t)$ such that $\Phi(0) = \mathbf{I}$ for the system below.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

- Solution: First, we must obtain $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ such that

$$\mathbf{x}^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- We know from previous results that the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

- Every solution can be expressed in terms of the general solution, and we use this fact to find $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$.

Example 2: Use General Solution (2 of 5)

- Thus, to find $\mathbf{x}^{(1)}(t)$, express it terms of the general solution

$$\mathbf{x}^{(1)}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

and then find the coefficients c_1 and c_2 .

- To do so, use the initial conditions to obtain

$$\mathbf{x}^{(1)}(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or equivalently,

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example 2: Solve for $\mathbf{x}^{(1)}(t)$ (3 of 5)

- To find $\mathbf{x}^{(1)}(t)$, we therefore solve

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -4 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{pmatrix}$$

$$\rightarrow \begin{matrix} c_1 & = & 1/2 \\ c_2 & = & 1/2 \end{matrix}$$

- Thus

$$\mathbf{x}^{(1)}(t) = \frac{1}{2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + \frac{1}{2} \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} = \begin{pmatrix} \frac{1}{2} e^{3t} + \frac{1}{2} e^{-t} \\ e^{3t} - e^{-t} \end{pmatrix}$$

Example 2: Solve for $\mathbf{x}^{(2)}(t)$ (4 of 5)

- To find $\mathbf{x}^{(2)}(t)$, we similarly solve

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1/4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -1/4 \end{pmatrix}$$

$$\rightarrow \begin{matrix} c_1 & = & 1/4 \\ c_2 & = & -1/4 \end{matrix}$$

- Thus

$$\mathbf{x}^{(2)}(t) = \frac{1}{4} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} - \frac{1}{4} \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} = \begin{pmatrix} \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t} \\ \frac{1}{2} e^{3t} + \frac{1}{2} e^{-t} \end{pmatrix}$$

Example 2: Obtain $\Phi(t)$ (5 of 5)

- The columns of $\Phi(t)$ are given by $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$, and thus from the previous slide we have

$$\Phi(t) = \begin{pmatrix} \frac{1}{2} e^{3t} + \frac{1}{2} e^{-t} & \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2} e^{3t} + \frac{1}{2} e^{-t} \end{pmatrix}$$

- Note $\Phi(t)$ is more complicated than $\Psi(t)$ found in Ex 1. However, now that we have $\Phi(t)$, it is much easier to determine the solution to any set of initial conditions.

$$\Psi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

Matrix Exponential Functions

- Consider the following two cases:
 - The solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$, is $\mathbf{x} = \mathbf{x}_0 e^{\mathbf{A}t}$, where $e^0 = \mathbf{I}$.
 - The solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$, is $\mathbf{x} = \Phi(t)\mathbf{x}^0$, where $\Phi(0) = \mathbf{I}$.
- Comparing the form and solution for both of these cases, we might expect $\Phi(t)$ to have an exponential character.
- Indeed, it can be shown that $\Phi(t) = e^{\mathbf{A}t}$, where

$$e^{\mathbf{A}t} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n t^n}{n!} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}$$

is a well defined matrix function that has all the usual properties of an exponential function. See text for details.

- Thus the solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$, is $\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}^0$.

Coupled Systems of Equations

- Recall that our constant coefficient homogeneous system

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n, \end{aligned}$$

written as $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix},$$

is a system of *coupled* equations that must be solved *simultaneously* to find all the unknown variables.

Uncoupled Systems & Diagonal Matrices

- In contrast, if each equation had only one variable, solved for independently of other equations, then task would be easier.
- In this case our system would have the form

$$\begin{aligned} x_1' &= d_{11}x_1 + 0x_2 + \dots + 0x_n \\ x_2' &= 0x_1 + d_{22}x_2 + \dots + 0x_n \\ &\vdots \\ x_n' &= 0x_1 + 0x_2 + \dots + d_{nn}x_n, \end{aligned}$$

or $\mathbf{x}' = \mathbf{D}\mathbf{x}$, where \mathbf{D} is a diagonal matrix:

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{pmatrix}$$

Uncoupling: Transform Matrix \mathbf{T}

- In order to explore transforming our given system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ of coupled equations into an uncoupled system $\mathbf{x}' = \mathbf{D}\mathbf{x}$, where \mathbf{D} is a diagonal matrix, we will use the eigenvectors of \mathbf{A} .
- Suppose \mathbf{A} is $n \times n$ with n linearly independent eigenvectors $\xi^{(1)}, \dots, \xi^{(n)}$, and corresponding eigenvalues $\lambda_1, \dots, \lambda_n$.
- Define $n \times n$ matrices \mathbf{T} and \mathbf{D} using the eigenvalues & eigenvectors of \mathbf{A} :

$$\mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \dots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \dots & \xi_n^{(n)} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

- Note that \mathbf{T} is nonsingular, and hence \mathbf{T}^{-1} exists.

Uncoupling: $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$

- Recall here the definitions of \mathbf{A} , \mathbf{T} and \mathbf{D} :

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \dots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \dots & \xi_n^{(n)} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

- Then the columns of $\mathbf{A}\mathbf{T}$ are $\mathbf{A}\xi^{(1)}, \dots, \mathbf{A}\xi^{(n)}$, and hence

$$\mathbf{A}\mathbf{T} = \begin{pmatrix} \lambda_1 \xi_1^{(1)} & \dots & \lambda_n \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \lambda_1 \xi_n^{(1)} & \dots & \lambda_n \xi_n^{(n)} \end{pmatrix} = \mathbf{T}\mathbf{D}$$

- It follows that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$.

Similarity Transformations

- Thus, if the eigenvalues and eigenvectors of \mathbf{A} are known, then \mathbf{A} can be transformed into a diagonal matrix \mathbf{D} , with $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$
- This process is known as a **similarity transformation**, and \mathbf{A} is said to be **similar** to \mathbf{D} . Alternatively, we could say that \mathbf{A} is **diagonalizable**.

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \dots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \dots & \xi_n^{(n)} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

Similarity Transformations: Hermitian Case

- Recall: Our similarity transformation of \mathbf{A} has the form

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$$

where \mathbf{D} is diagonal and columns of \mathbf{T} are eigenvectors of \mathbf{A} .

- If \mathbf{A} is Hermitian, then \mathbf{A} has n linearly independent orthogonal eigenvectors $\xi^{(1)}, \dots, \xi^{(n)}$, normalized so that $\langle \xi^{(i)}, \xi^{(j)} \rangle = 1$ for $i = 1, \dots, n$, and $\langle \xi^{(i)}, \xi^{(k)} \rangle = 0$ for $i \neq k$.
- With this selection of eigenvectors, it can be shown that $\mathbf{T}^{-1} = \mathbf{T}^*$. In this case we can write our similarity transform as

$$\mathbf{T}^*\mathbf{A}\mathbf{T} = \mathbf{D}$$

Nondiagonalizable \mathbf{A}

- Finally, if \mathbf{A} is $n \times n$ with fewer than n linearly independent eigenvectors, then there is no matrix \mathbf{T} such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$.
- In this case, \mathbf{A} is not similar to a diagonal matrix and \mathbf{A} is not diagonalizable.

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \dots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \dots & \xi_n^{(n)} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

Example 3:

Find Transformation Matrix \mathbf{T} (1 of 2)

- For the matrix \mathbf{A} below, find the similarity transformation matrix \mathbf{T} and show that \mathbf{A} can be diagonalized.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

- We already know that the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -1$ with corresponding eigenvectors

$$\xi^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \xi^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

- Thus

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Example 3: Similarity Transformation (2 of 2)

- To find T^{-1} , augment the identity to T and row reduce:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -4 & -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & -1/4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1/2 & 1/4 \\ 0 & 1 & 1/2 & -1/4 \end{pmatrix} \rightarrow T^{-1} = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix}$$

- Then

$$T^{-1}AT = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = D$$

- Thus A is similar to D , and hence A is diagonalizable.

Fundamental Matrices for Similar Systems (1 of 3)

- Recall our original system of differential equations $\mathbf{x}' = A\mathbf{x}$.
- If A is $n \times n$ with n linearly independent eigenvectors, then A is diagonalizable. The eigenvectors form the columns of the nonsingular transform matrix T , and the eigenvalues are the corresponding nonzero entries in the diagonal matrix D .
- Suppose \mathbf{x} satisfies $\mathbf{x}' = A\mathbf{x}$, let \mathbf{y} be the $n \times 1$ vector such that $\mathbf{x} = T\mathbf{y}$. That is, let \mathbf{y} be defined by $\mathbf{y} = T^{-1}\mathbf{x}$.
- Since $\mathbf{x}' = A\mathbf{x}$ and T is a constant matrix, we have $T\mathbf{y}' = AT\mathbf{y}$, and hence $\mathbf{y}' = T^{-1}AT\mathbf{y} = D\mathbf{y}$.
- Therefore \mathbf{y} satisfies $\mathbf{y}' = D\mathbf{y}$, the system similar to $\mathbf{x}' = A\mathbf{x}$.
- Both of these systems have fundamental matrices, which we examine next.

Fundamental Matrix for Diagonal System (2 of 3)

- A fundamental matrix for $\mathbf{y}' = D\mathbf{y}$ is given by $Q(t) = e^{Dt}$.
- Recalling the definition of e^{Dt} , we have

$$Q(t) = \sum_{n=0}^{\infty} \frac{D^n t^n}{n!} = \sum_{n=0}^{\infty} \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^n \end{pmatrix} \frac{t^n}{n!} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(\lambda_1 t)^n}{n!} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sum_{n=0}^{\infty} \frac{(\lambda_n t)^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n t} \end{pmatrix}$$

Fundamental Matrix for Original System (3 of 3)

- To obtain a fundamental matrix $\Psi(t)$ for $\mathbf{x}' = A\mathbf{x}$, recall that the columns of $\Psi(t)$ consist of fundamental solutions \mathbf{x} satisfying $\mathbf{x}' = A\mathbf{x}$. We also know $\mathbf{x} = T\mathbf{y}$, and hence it follows that

$$\Psi = TQ = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n t} \end{pmatrix} = \begin{pmatrix} \xi_1^{(1)} e^{\lambda_1 t} & \cdots & \xi_1^{(n)} e^{\lambda_n t} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} e^{\lambda_1 t} & \cdots & \xi_n^{(n)} e^{\lambda_n t} \end{pmatrix}$$

- The columns of $\Psi(t)$ given the expected fundamental solutions of $\mathbf{x}' = A\mathbf{x}$.

Example 4: Fundamental Matrices for Similar Systems

- We now use the analysis and results of the last few slides.
- Applying the transformation $\mathbf{x} = T\mathbf{y}$ to $\mathbf{x}' = A\mathbf{x}$ below, this system becomes $\mathbf{y}' = T^{-1}AT\mathbf{y} = D\mathbf{y}$:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x} \Rightarrow \mathbf{y}' = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y}$$

- A fundamental matrix for $\mathbf{y}' = D\mathbf{y}$ is given by $Q(t) = e^{Dt}$:

$$Q(t) = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}$$

- Thus a fundamental matrix $\Psi(t)$ for $\mathbf{x}' = A\mathbf{x}$ is

$$\Psi(t) = TQ = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

Boyce/DiPrima 10th ed, Ch 7.8: Repeated Eigenvalues

Elementary Differential Equations and Boundary Value Problems, 10th edition, by William E. Boyce and Richard C. DiPrima, ©2013 by John Wiley & Sons, Inc.

- We consider again a homogeneous system of n first order linear equations with constant real coefficients $\mathbf{x}' = A\mathbf{x}$.
- If the eigenvalues r_1, \dots, r_n of A are real and different, then there are n linearly independent eigenvectors $\xi_1^{(1)}, \dots, \xi_n^{(n)}$, and n linearly independent solutions of the form $\mathbf{x}^{(1)}(t) = \xi_1^{(1)} e^{r_1 t}, \dots, \mathbf{x}^{(n)}(t) = \xi_n^{(n)} e^{r_n t}$.
- If some of the eigenvalues r_1, \dots, r_n are repeated, then there may not be n corresponding linearly independent solutions of the above form.
- In this case, we will seek additional solutions that are products of polynomials and exponential functions.

Example 1: Eigenvalues (1 of 2)

- We need to find the eigenvectors for the matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

- The eigenvalues r and ξ eigenvectors satisfy the equation $(\mathbf{A} - r\mathbf{I})\xi = 0$ or

$$\begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- To determine r , solve $\det(\mathbf{A} - r\mathbf{I}) = 0$:

$$\begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = (r-1)(r-3) + 1 = r^2 - 4r + 4 = (r-2)^2$$

- Thus $r_1 = 2$ and $r_2 = 2$.

Example 1: Eigenvectors (2 of 2)

- To find the eigenvectors, we solve

$$(\mathbf{A} - r\mathbf{I})\xi = 0 \Leftrightarrow \begin{pmatrix} 1-2 & -1 \\ 1 & 3-2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow 1\xi_1 + 1\xi_2 = 0$$

$$\rightarrow \xi^{(1)} = \begin{pmatrix} -\xi_2 \\ \xi_2 \end{pmatrix} \rightarrow \text{choose } \xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- Thus there is only one linearly independent eigenvector for the repeated eigenvalue $r = 2$.

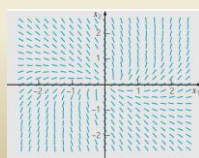
Example 2: Direction Field (1 of 10)

- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

- A direction field for this system is given below.
- Substituting $\mathbf{x} = \xi e^{rt}$ in for \mathbf{x} , where r is \mathbf{A} 's eigenvalue and ξ is its corresponding eigenvector, the previous example showed the existence of only one eigenvalue, $r = 2$, with one eigenvector:

$$\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



Example 2: First Solution; and Second Solution, First Attempt (2 of 10)

- The corresponding solution $\mathbf{x} = \xi e^{rt}$ of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$$

- Since there is no second solution of the form $\mathbf{x} = \xi e^{rt}$, we need to try a different form. Based on methods for second order linear equations in Ch 3.5, we first try $\mathbf{x} = \xi t e^{2t}$.
- Substituting $\mathbf{x} = \xi t e^{2t}$ into $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we obtain

$$\xi e^{2t} + 2\xi t e^{2t} = \mathbf{A}\xi t e^{2t}$$

or

$$2\xi t e^{2t} + \xi e^{2t} - \mathbf{A}\xi t e^{2t} = 0$$

Example 2: Second Solution, Second Attempt (3 of 10)

- From the previous slide, we have

$$2\xi t e^{2t} + \xi e^{2t} - \mathbf{A}\xi t e^{2t} = 0$$

- In order for this equation to be satisfied for all t , it is necessary for the coefficients of $t e^{2t}$ and e^{2t} to both be zero.
- From the e^{2t} term, we see that $\xi = 0$, and hence there is no nonzero solution of the form $\mathbf{x} = \xi t e^{2t}$.
- Since $t e^{2t}$ and e^{2t} appear in the above equation, we next consider a solution of the form

$$\mathbf{x} = \xi t e^{2t} + \eta e^{2t}$$

Example 2: Second Solution and its Defining Matrix Equations (4 of 10)

- Substituting $\mathbf{x} = \xi t e^{2t} + \eta e^{2t}$ into $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we obtain

$$\xi e^{2t} + 2\xi t e^{2t} + 2\eta e^{2t} = \mathbf{A}(\xi t e^{2t} + \eta e^{2t})$$

or

$$2\xi t e^{2t} + (\xi + 2\eta) e^{2t} = \mathbf{A}\xi t e^{2t} + \mathbf{A}\eta e^{2t}$$

- Equating coefficients yields $\mathbf{A}\xi = 2\xi$ and $\mathbf{A}\eta = \xi + 2\eta$, or

$$(\mathbf{A} - 2\mathbf{I})\xi = 0 \quad \text{and} \quad (\mathbf{A} - 2\mathbf{I})\eta = \xi$$

- The first equation is satisfied if ξ is an eigenvector of \mathbf{A} corresponding to the eigenvalue $r = 2$. Thus

$$\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example 2: Solving for Second Solution (5 of 10)

- Recall that

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}, \quad \boldsymbol{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- Thus to solve $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ for $\boldsymbol{\eta}$, we row reduce the corresponding augmented matrix:

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \eta_2 = -1 - \eta_1$$

$$\rightarrow \boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ -1 - \eta_1 \end{pmatrix} \rightarrow \boldsymbol{\eta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example 2: Second Solution (6 of 10)

- Our second solution $\mathbf{x} = \boldsymbol{\xi}te^{2t} + \boldsymbol{\eta}e^{2t}$ is now

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{2t} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t}$$

- Recalling that the first solution was

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t},$$

we see that our second solution is simply

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{2t},$$

since the last term of third term of \mathbf{x} is a multiple of $\mathbf{x}^{(1)}$.

Example 2: General Solution (7 of 10)

- The two solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{2t}$$

- The Wronskian of these two solutions is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix} e^{2t} & te^{2t} \\ -e^{2t} & te^{2t} - e^{2t} \end{vmatrix} = -e^{4t} \neq 0$$

- Thus $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are fundamental solutions, and the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

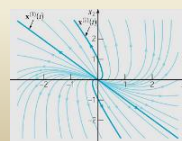
$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) \\ &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{2t} \right] \end{aligned}$$

Example 2: Phase Plane (8 of 10)

- The general solution is

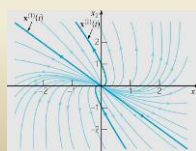
$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{2t} \right]$$

- Thus \mathbf{x} is unbounded as $t \rightarrow \infty$, and $\mathbf{x} \rightarrow \mathbf{0}$ as $t \rightarrow -\infty$.
- Further, it can be shown that as $t \rightarrow -\infty$, $\mathbf{x} \rightarrow \mathbf{0}$ asymptotic to the line $x_2 = -x_1$ determined by the first eigenvector.
- Similarly, as $t \rightarrow \infty$, \mathbf{x} is asymptotic to a line parallel to $x_2 = -x_1$.



Example 2: Phase Plane (9 of 10)

- The origin is an **improper node**, and is unstable. See graph.
- The pattern of trajectories is typical for two repeated eigenvalues with only one eigenvector.
- If the eigenvalues are negative, then the trajectories are similar but are traversed in the inward direction. In this case the origin is an asymptotically stable improper node.

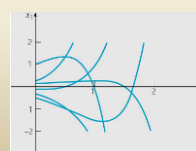


Example 2: Time Plots for General Solution (10 of 10)

- Time plots for $x_1(t)$ are given below, where we note that the general solution \mathbf{x} can be written as follows.

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{2t} \right]$$

$$\Leftrightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1e^{2t} + c_2te^{2t} \\ -(c_1 + c_2)e^{2t} - c_2te^{2t} \end{pmatrix}$$



General Case for Double Eigenvalues

- Suppose the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has a double eigenvalue $r = \rho$ and a single corresponding eigenvector ξ .
- The first solution is $\mathbf{x}^{(1)} = \xi e^{\rho t}$, where ξ satisfies $(\mathbf{A} - \rho \mathbf{I})\xi = \mathbf{0}$.
- As in Example 1, the second solution has the form $\mathbf{x}^{(2)} = \xi t e^{\rho t} + \eta e^{\rho t}$ where ξ is as above and η satisfies $(\mathbf{A} - \rho \mathbf{I})\eta = \xi$.
- Since ρ is an eigenvalue, $\det(\mathbf{A} - \rho \mathbf{I}) = 0$, and $(\mathbf{A} - \rho \mathbf{I})\eta = \mathbf{b}$ does not have a solution for all \mathbf{b} . However, it can be shown that $(\mathbf{A} - \rho \mathbf{I})\eta = \xi$ always has a solution.
- The vector η is called a **generalized eigenvector**.

Example 2 Extension: Fundamental Matrix Ψ (1 of 2)

- Recall that a fundamental matrix $\Psi(t)$ for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has linearly independent solution for its columns.
- In Example 1, our system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ was

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

and the two solutions we found were

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}$$

- Thus the corresponding fundamental matrix is

$$\Psi(t) = \begin{pmatrix} e^{2t} & t e^{2t} \\ -e^{2t} & -t e^{2t} - e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & t \\ -1 & -t-1 \end{pmatrix}$$

Example 2 Extension: Fundamental Matrix Φ (2 of 2)

- The fundamental matrix $\Phi(t)$ that satisfies $\Phi(0) = \mathbf{I}$ can be found using $\Phi(t) = \Psi(t)\Psi^{-1}(0)$, where

$$\Psi(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \Psi^{-1}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

where $\Psi^{-1}(0)$ is found as follows:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

- Thus

$$\Phi(t) = e^{2t} \begin{pmatrix} 1 & t \\ -1 & -t-1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = e^{2t} \begin{pmatrix} 1-t & -t \\ t & t+1 \end{pmatrix}$$

Jordan Forms

- If \mathbf{A} is $n \times n$ with n linearly independent eigenvectors, then \mathbf{A} can be diagonalized using a similarity transform $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$. The transform matrix \mathbf{T} consisted of eigenvectors of \mathbf{A} , and the diagonal entries of \mathbf{D} consisted of the eigenvalues of \mathbf{A} .
- In the case of repeated eigenvalues and fewer than n linearly independent eigenvectors, \mathbf{A} can be transformed into a nearly diagonal matrix \mathbf{J} , called the **Jordan form** of \mathbf{A} , with

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{J}.$$

Example 2 Extension: Transform Matrix (1 of 2)

- In Example 2, our system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ was

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

with eigenvalues $r_1 = 2$ and $r_2 = 2$ and eigenvectors

$$\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- Choosing $k = 0$, the transform matrix \mathbf{T} formed from the two eigenvectors ξ and η is

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

Example 2 Extension: Jordan Form (2 of 2)

- The Jordan form \mathbf{J} of \mathbf{A} is defined by $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{J}$.

- Now

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

and hence

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

- Note that the eigenvalues of \mathbf{A} , $r_1 = 2$ and $r_2 = 2$, are on the main diagonal of \mathbf{J} , and that there is a 1 directly above the second eigenvalue. This pattern is typical of Jordan forms.

Boyce/DiPrima 10th ed, Ch 7.9: Nonhomogeneous Linear Systems

Elementary Differential Equations and Boundary Value Problems, 10th edition, by William E. Boyce and Richard C. DiPrima, ©2013 by John Wiley & Sons, Inc.

- The general theory of a nonhomogeneous system of equations

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \dots + p_{1n}(t)x_n + g_1(t)$$

$$x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \dots + p_{2n}(t)x_n + g_2(t)$$

⋮

$$x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \dots + p_{nn}(t)x_n + g_n(t)$$

parallels that of a single n th order linear equation.

- This system can be written as $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$, where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}, \quad \mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{pmatrix}$$

General Solution

- The general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ on $I: \alpha < t < \beta$ has the form

$$\mathbf{x} = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) + \dots + c_n \mathbf{x}^{(n)}(t) + \mathbf{v}(t)$$

where

$$c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) + \dots + c_n \mathbf{x}^{(n)}(t)$$

is the general solution of the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ and $\mathbf{v}(t)$ is a particular solution of the nonhomogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$.

Diagonalization

- Suppose $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$, where \mathbf{A} is an $n \times n$ diagonalizable constant matrix.
- Let \mathbf{T} be the nonsingular transform matrix whose columns are the eigenvectors of \mathbf{A} , and \mathbf{D} the diagonal matrix whose diagonal entries are the corresponding eigenvalues of \mathbf{A} .
- Suppose \mathbf{x} satisfies $\mathbf{x}' = \mathbf{A}\mathbf{x}$, let \mathbf{y} be defined by $\mathbf{x} = \mathbf{T}\mathbf{y}$.
- Substituting $\mathbf{x} = \mathbf{T}\mathbf{y}$ into $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we obtain

$$\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g}(t).$$

$$\text{or} \quad \mathbf{y}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t)$$

$$\text{or} \quad \mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{h}(t), \text{ where } \mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t).$$

- Note that if we can solve diagonal system $\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{h}(t)$ for \mathbf{y} , then $\mathbf{x} = \mathbf{T}\mathbf{y}$ is a solution to the original system.

Solving Diagonal System

- Now $\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{h}(t)$ is a diagonal system of the form

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

where r_1, \dots, r_n are the eigenvalues of \mathbf{A} .

- Thus $\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{h}(t)$ is an uncoupled system of n linear first order equations in the unknowns $y_k(t)$, which can be isolated

$$y_k' = r_k y_k + h_k(t), \quad k = 1, \dots, n$$

and solved separately, using methods of Section 2.1:

$$y_k = e^{r_k t} \int_{t_0}^t e^{-r_k s} h_k(s) ds + c_k e^{r_k t}, \quad k = 1, \dots, n$$

Solving Original System

- The solution \mathbf{y} to $\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{h}(t)$ has components

$$y_k = e^{r_k t} \int_{t_0}^t e^{-r_k s} h_k(s) ds + c_k e^{r_k t}, \quad k = 1, \dots, n$$

- For this solution vector \mathbf{y} , the solution to the original system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$ is then $\mathbf{x} = \mathbf{T}\mathbf{y}$.
- Recall that \mathbf{T} is the nonsingular transform matrix whose columns are the eigenvectors of \mathbf{A} .
- Thus, when multiplied by \mathbf{T} , the second term on right side of y_k produces general solution of homogeneous equation, while the integral term of y_k produces a particular solution of nonhomogeneous system.

Example 1: General Solution of Homogeneous Case (1 of 5)

- Consider the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$ below.

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$$

- Note: \mathbf{A} is a Hermitian matrix, since it is real and symmetric.
- The eigenvalues of \mathbf{A} are $r_1 = -3$ and $r_2 = -1$, with corresponding eigenvectors

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- The general solution of the homogeneous system is then

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

Example 1: Transformation Matrix (2 of 5)

- Consider next the transformation matrix \mathbf{T} of eigenvectors. Using a Section 7.7 comment, and \mathbf{A} Hermitian, we have $\mathbf{T}^{-1} = \mathbf{T}^* = \mathbf{T}^T$, provided we normalize $\xi^{(1)}$ and $\xi^{(2)}$ so that $(\xi^{(1)}, \xi^{(1)}) = 1$ and $(\xi^{(2)}, \xi^{(2)}) = 1$. Thus normalize as follows:

$$\xi^{(1)} = \frac{1}{\sqrt{(1)(1) + (-1)(-1)}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\xi^{(2)} = \frac{1}{\sqrt{(1)(1) + (1)(1)}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- Then for this choice of eigenvectors,

$$\mathbf{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{T}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Example 1:

Diagonal System and its Solution (3 of 5)

- Under the transformation $\mathbf{x} = \mathbf{T}\mathbf{y}$, we obtain the diagonal system $\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t)$:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

$$= \begin{pmatrix} -3y_1 \\ -y_2 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{pmatrix}$$

- Then, using methods of Section 2.1,

$$y_1' + 3y_1 = \frac{\sqrt{2}e^{-t}}{2} - \frac{3}{\sqrt{2}}t \Rightarrow y_1 = \frac{\sqrt{2}}{2}e^{-t} - \frac{3}{\sqrt{2}}\left(\frac{t}{3} - \frac{1}{9}\right) + c_1e^{-3t}$$

$$y_2' + y_2 = \frac{\sqrt{2}e^{-t}}{\sqrt{2}} + \frac{3}{\sqrt{2}}t \Rightarrow y_2 = \sqrt{2}te^{-t} + \frac{3}{\sqrt{2}}(t-1) + c_2e^{-t}$$

Example 1:

Transform Back to Original System (4 of 5)

- We next use the transformation $\mathbf{x} = \mathbf{T}\mathbf{y}$ to obtain the solution to the original system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}e^{-t} - \left(\frac{t}{2} - \frac{1}{6}\right) + k_1e^{-3t} \\ te^{-t} + \frac{3}{2}(t-1) + k_2e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} k_1e^{-3t} + \left(k_2 + \frac{1}{2}\right)e^{-t} + t - \frac{4}{3} + te^{-t} \\ -k_1e^{-3t} + \left(k_2 - \frac{1}{2}\right)e^{-t} + 2t - \frac{5}{3} + te^{-t} \end{pmatrix}, \quad k_1 = \frac{c_1}{\sqrt{2}}, \quad k_2 = \frac{c_2}{\sqrt{2}}$$

Example 1:

Solution of Original System (5 of 5)

- Simplifying further, the solution \mathbf{x} can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1e^{-3t} + \left(k_2 + \frac{1}{2}\right)e^{-t} + t - \frac{4}{3} + te^{-t} \\ -k_1e^{-3t} + \left(k_2 - \frac{1}{2}\right)e^{-t} + 2t - \frac{5}{3} + te^{-t} \end{pmatrix}$$

$$= k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

- Note that the first two terms on right side form the general solution to homogeneous system, while the remaining terms are a particular solution to nonhomogeneous system.

Nondiagonal Case

- If \mathbf{A} cannot be diagonalized, (repeated eigenvalues and a shortage of eigenvectors), then it can be transformed to its Jordan form \mathbf{J} , which is nearly diagonal.
- In this case the differential equations are not totally uncoupled, because some rows of \mathbf{J} have two nonzero entries: an eigenvalue in diagonal position, and a 1 in adjacent position to the right of diagonal position.
- However, the equations for y_1, \dots, y_n can still be solved consecutively, starting with y_n . Then the solution \mathbf{x} to original system can be found using $\mathbf{x} = \mathbf{T}\mathbf{y}$.

Undetermined Coefficients

- A second way of solving $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is the method of undetermined coefficients. Assume \mathbf{P} is a constant matrix, and that the components of \mathbf{g} are polynomial, exponential or sinusoidal functions, or sums or products of these.
- The procedure for choosing the form of solution is usually directly analogous to that given in Section 3.6.
- The main difference arises when $\mathbf{g}(t)$ has the form $\mathbf{u}e^{\lambda t}$, where λ is a simple eigenvalue of \mathbf{P} . In this case, $\mathbf{g}(t)$ matches solution form of homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, and as a result, it is necessary to take nonhomogeneous solution to be of the form $\mathbf{a}te^{\lambda t} + \mathbf{b}e^{\lambda t}$. This form differs from the Section 3.6 analog, $\mathbf{a}te^{\lambda t}$.

Example 2: Undetermined Coefficients (1 of 5)

- Consider again the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$:

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t$$

- Assume a particular solution of the form

$$\mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}$$

where the vector coefficients \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} are to be determined.

- Since $r = -1$ is an eigenvalue of \mathbf{A} , it is necessary to include both $\mathbf{a}te^{-t}$ and $\mathbf{b}e^{-t}$, as mentioned on the previous slide.

Example 2: Matrix Equations for Coefficients (2 of 5)

- Substituting

$$\mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}$$

in for \mathbf{x} in our nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$,

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t,$$

we obtain

$$-\mathbf{a}te^{-t} + (\mathbf{a} - \mathbf{b})e^{-t} + \mathbf{c} = \mathbf{A}\mathbf{a}te^{-t} + \mathbf{A}\mathbf{b}e^{-t} + \mathbf{A}\mathbf{c}t + \mathbf{A}\mathbf{d} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t$$

- Equating coefficients, we conclude that

$$\mathbf{A}\mathbf{a} = -\mathbf{a}, \quad \mathbf{A}\mathbf{b} = \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \mathbf{A}\mathbf{c} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad \mathbf{A}\mathbf{d} = \mathbf{c}$$

Example 2: Solving Matrix Equation for (a) (3 of 5)

- Our matrix equations for the coefficients are:

$$\mathbf{A}\mathbf{a} = -\mathbf{a}, \quad \mathbf{A}\mathbf{b} = \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \mathbf{A}\mathbf{c} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad \mathbf{A}\mathbf{d} = \mathbf{c}$$

- From the first equation, we see that \mathbf{a} is an eigenvector of \mathbf{A} corresponding to eigenvalue $r = -1$, and hence has the form

$$\mathbf{a} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$$

- We will see on the next slide that $\alpha = 1$, and hence

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Example 2: Solving Matrix Equation for (b) (4 of 5)

- Our matrix equations for the coefficients are:

$$\mathbf{A}\mathbf{a} = -\mathbf{a}, \quad \mathbf{A}\mathbf{b} = \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \mathbf{A}\mathbf{c} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad \mathbf{A}\mathbf{d} = \mathbf{c}$$

- Substituting $\mathbf{a}^T = (\alpha, \alpha)$ into second equation,

$$\begin{aligned} \mathbf{A}\mathbf{b} &= \begin{pmatrix} \alpha - 2 \\ \alpha \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Leftrightarrow \left[\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha - 2 \\ \alpha \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha - 2 \\ \alpha \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha - 1 \end{pmatrix} \end{aligned}$$

- Thus $\alpha = 1$, and solving for \mathbf{b} , we obtain

$$\mathbf{b} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \text{choose } k = 0 \rightarrow \mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Example 2: Particular Solution (5 of 5)

- Our matrix equations for the coefficients are:

$$\mathbf{A}\mathbf{a} = -\mathbf{a}, \quad \mathbf{A}\mathbf{b} = \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \mathbf{A}\mathbf{c} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad \mathbf{A}\mathbf{d} = \mathbf{c}$$

- Solving third equation for \mathbf{c} , and then fourth equation for \mathbf{d} , it is straightforward to obtain $\mathbf{c}^T = (1, 2)$, $\mathbf{d}^T = (-4/3, -5/3)$.

- Thus our particular solution of $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$ is

$$\mathbf{v}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

- Comparing this to the result obtained in Example 1, we see that both particular solutions would be the same if we had chosen $k = 1/2$ for \mathbf{b} on previous slide, instead of $k = 0$.

Variation of Parameters: Preliminaries

- A more general way of solving $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is the method of variation of parameters.
- Assume $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are continuous on $\alpha < t < \beta$, and let $\mathbf{\Psi}(t)$ be a fundamental matrix for the homogeneous system.
- Recall that the columns of $\mathbf{\Psi}$ are linearly independent solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, and hence $\mathbf{\Psi}(t)$ is invertible on the interval $\alpha < t < \beta$, and also $\mathbf{\Psi}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)$.
- Next, recall that the solution of the homogeneous system can be expressed as $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c}$.
- Analogous to Section 3.7, assume the particular solution of the nonhomogeneous system has the form $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$, where $\mathbf{u}(t)$ is a vector to be found.

Variation of Parameters: Solution

- We assume a particular solution of the form $\mathbf{x} = \Psi(t)\mathbf{u}(t)$.
- Substituting this into $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$, we obtain

$$\Psi'(t)\mathbf{u}(t) + \Psi(t)\mathbf{u}'(t) = \mathbf{P}(t)\Psi(t)\mathbf{u}(t) + \mathbf{g}(t)$$
- Since $\Psi'(t) = \mathbf{P}(t)\Psi(t)$, the above equation simplifies to

$$\mathbf{u}'(t) = \Psi^{-1}(t)\mathbf{g}(t)$$
- Thus

$$\mathbf{u}(t) = \int \Psi^{-1}(t)\mathbf{g}(t)dt + \mathbf{c}$$

where the vector \mathbf{c} is an arbitrary constant of integration.
- The general solution to $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is therefore

$$\mathbf{x} = \Psi(t)\mathbf{c} + \Psi(t)\int_{t_1}^t \Psi^{-1}(s)\mathbf{g}(s)ds, \quad t_1 \in (\alpha, \beta) \text{ arbitrary}$$

Variation of Parameters: Initial Value Problem

- For an initial value problem

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \quad \mathbf{x}(t_0) = \mathbf{x}^{(0)},$$

the general solution to $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is

$$\mathbf{x} = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}^{(0)} + \Psi(t)\int_{t_0}^t \Psi^{-1}(s)\mathbf{g}(s)ds$$
- Alternatively, recall that the fundamental matrix $\Phi(t)$ satisfies $\Phi(t_0) = \mathbf{I}$, and hence the general solution is

$$\mathbf{x} = \Phi(t)\mathbf{x}^{(0)} + \Phi(t)\int_{t_0}^t \Psi^{-1}(s)\mathbf{g}(s)ds$$
- In practice, it may be easier to row reduce matrices and solve necessary equations than to compute $\Psi^{-1}(t)$ and substitute into equations. See next example.

Example 3: Variation of Parameters (1 of 3)

- Consider again the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$:

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t$$
- We have previously found general solution to homogeneous case, with corresponding fundamental matrix:

$$\Psi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$$
- Using variation of parameters method, our solution is given by $\mathbf{x} = \Psi(t)\mathbf{u}(t)$, where $\mathbf{u}(t)$ satisfies $\Psi(t)\mathbf{u}'(t) = \mathbf{g}(t)$, or

$$\begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

Example 3: Solving for $\mathbf{u}(t)$ (2 of 3)

- Solving $\Psi(t)\mathbf{u}'(t) = \mathbf{g}(t)$ by row reduction,

$$\begin{pmatrix} e^{-3t} & e^{-t} & 2e^{-t} \\ -e^{-3t} & e^{-t} & 3t \end{pmatrix} \rightarrow \begin{pmatrix} e^{-3t} & e^{-t} & 2e^{-t} \\ 0 & 2e^{-t} & 2e^{-t} + 3t \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} e^{-3t} & e^{-t} & 2e^{-t} \\ 0 & e^{-t} & e^{-t} + 3t/2 \end{pmatrix} \rightarrow \begin{pmatrix} e^{-3t} & 0 & e^{-t} - 3t/2 \\ 0 & e^{-t} & e^{-t} + 3t/2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & e^{2t} - 3te^{3t}/2 \\ 0 & 1 & 1 + 3te^{t}/2 \end{pmatrix} \rightarrow \begin{matrix} u_1' & = e^{2t} - 3te^{3t}/2 \\ u_2' & = 1 + 3te^{t}/2 \end{matrix}$$
- It follows that

$$\mathbf{u}(t) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} e^{2t}/2 - te^{3t}/2 + e^{3t}/6 + c_1 \\ t + 3te^{t}/2 - 3e^{t}/2 + c_2 \end{pmatrix}$$

Example 3: Solving for $\mathbf{x}(t)$ (3 of 3)

- Now $\mathbf{x}(t) = \Psi(t)\mathbf{u}(t)$, and hence we multiply

$$\mathbf{x} = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} e^{2t}/2 - te^{3t}/2 + e^{3t}/6 + c_1 \\ t + 3te^{t}/2 - 3e^{t}/2 + c_2 \end{pmatrix}$$

to obtain, after collecting terms and simplifying,

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$
- Note that this is the same solution as in Example 1.

Laplace Transforms

- The Laplace transform can be used to solve systems of equations. Here, the transform of a vector is the vector of component transforms, denoted by $\mathbf{X}(s)$:

$$L\{\mathbf{x}(t)\} = L\left\{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}\right\} = \begin{pmatrix} L\{x_1(t)\} \\ L\{x_2(t)\} \end{pmatrix} = \mathbf{X}(s)$$
- Then by extending Theorem 6.2.1, we obtain

$$L\{\mathbf{x}'(t)\} = s\mathbf{X}(s) - \mathbf{x}(0)$$

Example 4: Laplace Transform (1 of 5)

- Consider again the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$:

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

- Taking the Laplace transform of each term, we obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{G}(s)$$

where $\mathbf{G}(s)$ is the transform of $\mathbf{g}(t)$, and is given by

$$\mathbf{G}(s) = \begin{pmatrix} 2/(s+1) \\ 3/s^2 \end{pmatrix}$$

Example 4: Transfer Matrix (2 of 5)

- Our transformed equation is

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{G}(s)$$
- If we take $\mathbf{x}(0) = \mathbf{0}$, then the above equation becomes

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{G}(s)$$
 or

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{G}(s)$$
- Solving for $\mathbf{X}(s)$, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{G}(s)$$
- The matrix $(s\mathbf{I} - \mathbf{A})^{-1}$ is called the **transfer matrix**.

Example 4: Finding Transfer Matrix (3 of 5)

- Then

$$\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \Rightarrow (s\mathbf{I} - \mathbf{A}) = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}$$

- Solving for $(s\mathbf{I} - \mathbf{A})^{-1}$, we obtain

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+3)} \begin{pmatrix} s+2 & 1 \\ 1 & s+2 \end{pmatrix}$$

Example 4: Transfer Matrix (4 of 5)

- Next, $\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{G}(s)$, and hence

$$\mathbf{X}(s) = \frac{1}{(s+1)(s+3)} \begin{pmatrix} s+2 & 1 \\ 1 & s+2 \end{pmatrix} \begin{pmatrix} 2/(s+1) \\ 3/s^2 \end{pmatrix}$$

or

$$\mathbf{X}(s) = \begin{pmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} \end{pmatrix}$$

Example 4: Transfer Matrix (5 of 5)

- Thus

$$\mathbf{X}(s) = \begin{pmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} \end{pmatrix}$$

- To solve for $\mathbf{x}(t) = L^{-1}\{\mathbf{X}(s)\}$, use partial fraction expansions of both components of $\mathbf{X}(s)$, and then Table 6.2.1 to obtain:

$$\mathbf{x} = -\frac{2}{3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

- Since we assumed $\mathbf{x}(0) = \mathbf{0}$, this solution differs slightly from the previous particular solutions.

Summary (1 of 2)

- The method of undetermined coefficients requires no integration but is limited in scope and may involve several sets of algebraic equations.
- Diagonalization requires finding inverse of transformation matrix and solving uncoupled first order linear equations. When coefficient matrix is Hermitian, the inverse of transformation matrix can be found without calculation, which is very helpful for large systems.
- The Laplace transform method involves matrix inversion, matrix multiplication, and inverse transforms. This method is particularly useful for problems with discontinuous or impulsive forcing functions.

Summary (2 of 2)

- Variation of parameters is the most general method, but it involves solving linear algebraic equations with variable coefficients, integration, and matrix multiplication, and hence may be the most computationally complicated method.
- For many small systems with constant coefficients, all of these methods work well, and there may be little reason to select one over another.