Math 204: Final Exam
Spring 2018

- Write your full name and Student ID number in the space provided below and sign.

Last Name, First Name:	
Student ID Number:	
Signature:	

- Mark the section you are registered below.

  ☐ Section 1 (Mon. & Wed. 14:30-15:45, Instructor: Hasan İnci)

  ☐ Section 2 (Tue. & Thu. 16:00-17:15, Instructor: Tolga Etgü)

  ☐ Section 3 (Tue. & Thu. 13:00-14:15, Instructor: Tolga Etgü)

- You have 120 minutes.

- ***You must show all your work to receive full credit.***

To be filled by the grader:

PROBLEM	1	2	3	4	5	6	7	TOTAL
POINTS	12	16	16	14	12	14	16	100
SCORE								

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Problem 1. a) (8 pts) Given that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are solutions of $t^2y'' - 2y = 0, \quad t > 0$, find the general solution of the following equation.

$$t^2y'' - 2y = t^2 - 3, \quad t > 0$$

st. form: $y'' - \frac{2}{t^2} y = \frac{t^2-3}{t^2}$ ← $g(t)$

$W(y_1, y_2) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -3 \neq 0$

var. of par.: $y_p = u_1 t^2 + u_2 t^{-1}$

where $u_1 = -\int \frac{t^{-1}}{-3} \cdot \frac{t^2-3}{t^2} dt = \frac{\ln t}{3} + \frac{t^{-2}}{2} + c_1$

$u_2 = \int \frac{t^2}{-3} \cdot \frac{t^2-3}{t^2} dt = -\frac{t^3}{9} + t + c_2$

gen. sln:

$y = c_1 t^2 + c_2 t^{-1} + \frac{t^2 \ln t}{3} + \frac{1}{2} - \frac{t^2}{9} + 1$

$\left(= c_1 t^2 + c_2 t^{-1} + \frac{t^2 \ln t}{3} + \frac{3}{2} \right)$

b) (*no explanation required, 2 points*) True or false:

(i) $\{2t^2, t^2 + t^{-1}\}$ is a fundamental set of solutions of $t^2y'' - 2y = 0, \ t > 0$. **T** **F**

(ii) $\{3t^2 + 3t^{-1}, 4t^2 + 4t^{-1}\}$ is a fundamental set of solutions of $t^2y'' - 2y = 0, \ t > 0$. **T** **F**

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Problem 2. a) (12 pts) Find the general solution of the following equation.

$$D(D-2)(D+1)y = 3e^{2t}$$

$$(D-2)\, D(D-2)(D+1)y = 0$$

$$D(D-2)^2(D+1)\,y = 0$$

$$\to y = \boxed{C_1 + C_2 \cdot e^{2t}} + C_3 t \cdot e^{2t} \boxed{+ C_4 \cdot e^{-t}}$$ 3

sol. to ~~hom~~ eq.

3

$$\to y_p = A \cdot t e^{2t}$$

$$(D-2)\, Ate^{2t} = Ae^{2t} + 2Ate^{2t} - 2Ate^{2t}$$

$$= Ae^{2t}$$

$$(D+1)\, Ae^{2t} = 2Ae^{2t} + Ae^{2t} = 3Ae^{2t}$$

$$D(3Ae^{2t}) = 6Ae^{2t} \stackrel{!}{=} 3e^{2t}$$

$$\to A = \frac{1}{2}$$ 3

$\to$ general sol.

$$y = C_1 + C_2 \cdot e^{2t} + C_3 e^{-t} + \frac{1}{2} t e^{2t}$$ 3

b) (*no explanation required, 2 points each*) True or false:

(i) $y = 4 + \frac{e^{2t}}{2} + 3e^{-t} + te^{2t}$ is a solution of the equation above. T F 2

(ii) $y = e^{2t} + e^{-t} + \frac{te^{2t}}{2}$ is a solution of the equation above. T F

2

$\ast$

Problem 3. a) (12 pts) Solve the following initial value problem

$$y'' - y = \begin{cases} 1\,, & \text{if } 1 \leq t < 3 \\ 0\,, & \text{otherwise} \end{cases}$$

$$y(0) = 0, \quad y'(0) = 0$$

2 Let $g(t) = \begin{cases} 1, & \text{if } 1 \leq t < 3 \\ 0, & \text{otherwise} \end{cases}$ Then $g(t) = u_1(t) - u_3(t)$

2 $\begin{cases} \mathcal{L}(y''-y) = \mathcal{L}(g(t)) \\ \mathcal{L}(y''-y) = \mathcal{L}(y'') - \mathcal{L}(y) = s^2\mathcal{L}(y) - sy(0) - y'(0) - \mathcal{L}(y) = (s^2-1)\mathcal{L}(y) \end{cases}$

1 $\{ \mathcal{L}(g(t)) = \mathcal{L}(u_1(t)) - \mathcal{L}(u_3(t)) = \frac{e^{-s}}{s} - \frac{e^{-3s}}{s} \Rightarrow \mathcal{L}(y) = (e^{-s} - e^{-3s}) \frac{1}{s(s^2-1)}$

3 Let $F(s) = \frac{1}{s(s^2-1)} = \frac{a}{s} + \frac{b}{s-1} + \frac{c}{s+1} \Rightarrow a(s-1)(s+1) + b(s)(s+1) + c(s)(s-1) = 1$

Putting $s = 1 \Rightarrow 2b = 1 \Rightarrow b = \frac{1}{2}$

$s = -1 \Rightarrow 2c = 1 \Rightarrow c = 1/2$

$s = 0 \Rightarrow -a = 1 \Rightarrow a = -1$

$F(s) = -\frac{1}{s} + \left(\frac{1}{2}\right)\frac{1}{s-1} + \left(\frac{1}{2}\right)\frac{1}{s+1}$

4 $\mathcal{L}(y) = e^{-s}F(s) - e^{-3s}F(s) \Rightarrow y(t) = u_1(t)f(t-1) - u_3(t)f(t-3)$ where $F(s) = \mathcal{L}(f(t))$

$F(s) = -\frac{1}{s} + \left(\frac{1}{2}\right)\frac{1}{s-1} + \left(\frac{1}{2}\right)\frac{1}{s+1} \Rightarrow f(t) = -1 + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}$

$\Rightarrow y(t) = u_1(t)\left[-1 + \frac{1}{2}e^{t-1} + \frac{1}{2}e^{-t+1}\right] - u_3(t)\left[-1 + \frac{1}{2}e^{t-3} + \frac{1}{2}e^{-t+3}\right]$

b) (*no explanation required, 2 points each*) True or false:

(i) If the Laplace transforms $F(s) = \mathfrak{L}\{f(t)\}$ and $G(s) = \mathfrak{L}\{g(t)\}$ both exist for $s > a > 0$, then $2F(s) + 3G(s) = \mathfrak{L}\{2f(t) + 3g(t)\}$ for $s > a > 0$. **T** **F**

(ii) If the Laplace transforms $F(s) = \mathfrak{L}\{f(t)\}$ and $G(s) = \mathfrak{L}\{g(t)\}$ both exist for $s > a > 0$, then $5F(s)G(s) = \mathfrak{L}\{5f(t)g(t)\}$ for $s > a > 0$. **T** **F**

Problem 4. a) (12 pts) Find the solution of the following systems of equations that satisfies the given initial condition.

$$\mathbf{x}' = \underbrace{\begin{pmatrix} -3 & 6 \\ -1 & 2 \end{pmatrix}}_{A} \cdot \mathbf{x} \;, \quad \mathbf{x}(0) = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

Find eigenvalues and corresponding eigenvectors of A:

$$\begin{vmatrix} -3-\lambda & 6 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Leftrightarrow (-3-\lambda)(2-\lambda)+6=0 \Leftrightarrow -6+\lambda+\lambda^2+6=0$$
$$\Leftrightarrow \lambda(\lambda+1)=0$$

3

$$\Rightarrow \boxed{\text{Eigenvalues: } \lambda_1 = 0, \ \lambda_2 = -1}$$

$\lambda_1 = 0$: $\begin{pmatrix} -3-0 & 6 \\ -1 & 2-0 \end{pmatrix} v_1 = 0 \Rightarrow v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ 2

$\lambda_2 = -1$: $\begin{pmatrix} -3+1 & 6 \\ -1 & 2+1 \end{pmatrix} v_2 = 0 \Rightarrow v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ 2

General solution:

$x = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ 2

$x(0) = \begin{pmatrix} 9 \\ 4 \end{pmatrix} \Rightarrow c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

$\Rightarrow c_1 = 3$ 3

$c_2 = 1$

$x(t) = \begin{pmatrix} 6 \\ 3 \end{pmatrix} + e^{-t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

b) (*no explanation required, 2 points*) True or false:

The system $\left\{ \begin{array}{rcl} x_1' & = & -3x_1 + 6x_2 \\ x_2' & = & -x_1 + 2x_2 \end{array} \right\}$ is satisfied by the functions $x_1 = 2 - 3e^{-t}$ and $x_2 = 1 - e^{-t}$.

T F

Left

Problem 5. (12 pts) Find the general solution of the following system of equations.

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \mathbf{x}$$

1 $\quad \frac{1}{0}$

1/2

$\det \begin{pmatrix} r-1 & 2 \\ 0 & r-1 \end{pmatrix} = (r-1)^2 \Rightarrow r=1$ only eigenvalue

3 pts

$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x_2 = 0$

$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the only eigenvector of eigenvalue $r=1$.

3 pts.

$\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t$ is a solution of the system

6 pts

need to find a vector as follows

$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \xi \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + k\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2 pts

$\cancel{\Rightarrow \mathbf{x} = c_1\left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} t e^t\right)}$

$\mathbf{x} = c_1\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} t e^t + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} e^t\right) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t c_2$

4 pts

6 pts

2 pts – Wronskian.

$c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} t e^t + c_3\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} e^t$

w/o wronskian

8 pts

Problem 6. (14 pts) Given that $\phi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$ is a fundamental matrix for the system $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \cdot \mathbf{x}$, find the general solution of the following system of equations.

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \cdot \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$$

Solution:

$$x = \phi(t) \cdot u(t)$$

where $\phi(t) \cdot u'(t) = g(t)$

$$\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$$

$$e^t \cdot u_1' + e^{-t} u_2' = e^t \qquad u_2' = 0 \qquad u_2 = C_1$$

$$e^t \cdot u_1' + 3e^{-t} u_2' = e^t \qquad u_1' = 1 \qquad u_1 = t + C_2$$

$$X = \phi \cdot u = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \begin{pmatrix} t + C_2 \\ C_1 \end{pmatrix}$$

$$= \begin{pmatrix} C_2 e^t + t e^{+t} + C_1 e^{-t} \\ C_2 e^t + e^{+t} t + 3C_1 e^{-t} \end{pmatrix}$$

$$= \boxed{C_1 \begin{pmatrix} e^{-t} \\ 3e^{-t} \end{pmatrix} + C_2 \begin{pmatrix} e^{+t} \\ e^{+t} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{+t}}$$

✱

Problem 7. (16 pts) Find the general solution of the following system for $t > 0$.

$$\begin{cases} tx_1' &= 3x_1 - 2x_2 \\ tx_2' &= 2x_1 - 2x_2 \end{cases}$$

(Hint: Look for a solution of the form $x_1 = v_1 t^r$, $x_2 = v_2 t^r$, where v_1, v_2, and r are suitable constants.)

$x_1' = r v_1 t^{r-1}$, $x_2' = r v_2 t^{r-1}$

$$\begin{cases} t r v_1 t^{r-1} = 3v_1 t^r - 2v_2 t^r \\ t r v_2 t^{r-1} = 2v_1 t^r - 2v_2 t^r \end{cases}$$

$\xrightarrow{\text{divide by } t^r}$ $\begin{cases} r v_1 = 3v_1 - 2v_2 \\ r v_2 = 2v_1 - 2v_2 \end{cases} \iff \begin{cases} (r-3)v_1 + 2v_2 = 0 \\ -2v_1 + (r-2)v_2 = 0 \end{cases}$

$\Rightarrow$ nontrivial solutions for v_1, v_2 exist if and only if $\begin{vmatrix} r-3 & 2 \\ -2 & r-2 \end{vmatrix} = 0$

i.e. r is an eigenvalue and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a corresponding eigenvector of $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

eigenvalues: $r_1 = 2$, $r_2 = -1$

eigenvectors: $\vec{v}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\vec{v}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Hence $\vec{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2$ and $\vec{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1}$ are sols of the system. Since $W(\vec{x}^{(1)}, \vec{x}^{(2)}) = \begin{vmatrix} 2t^2 & t^{-1} \\ t^2 & 2t^{-1} \end{vmatrix} = 3t \neq 0$ $(t>0)$, the gen sol:

$$\boxed{\vec{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2 + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1}}$$