Math 204: Final Exam
Spring 2018

- Write your full name and Student ID number in the space provided below and sign.

Last Name, First Name:	
Student ID Number:	
Signature:	

- Mark the section you are registered below.

  ☐ Section 1 (Mon. & Wed. 14:30-15:45, Instructor: Hasan İnci)

  ☐ Section 2 (Tue. & Thu. 16:00-17:15, Instructor: Tolga Etgü)

  ☐ Section 3 (Tue. & Thu. 13:00-14:15, Instructor: Tolga Etgü)

- You have 120 minutes.
- ***You must show all your work to receive full credit.***

To be filled by the grader:

PROBLEM	1	2	3	4	5	6	7	TOTAL
POINTS	12	16	16	14	12	14	16	100
SCORE								

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Problem 1. a) (8 pts) Given that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are solutions of $t^2y'' - 2y = 0,\ \ t > 0$, find the general solution of the following equation.

$$t^2y'' - 2y = 3t^2 - 1, \quad t > 0$$

st. form: $y'' - \frac{2}{t^2} y = \frac{3t^2-1}{t^2}$

$W(y_1, y_2) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -3 \neq 0$

var. of par.: $y_p = u_1 t^2 + u_2 t^{-1}$

where $u_1 = -\int \frac{t^{-1}}{-3} \cdot \frac{3t^2-1}{t^2}\, dt = \ln t + \frac{t^{-2}}{6} + c_1$

$u_2 = \int \frac{t^2}{-3} \cdot \frac{3t^2-1}{t^2}\, dt = -\frac{t^3}{3} + \frac{t}{3} + c_2$

gen. sln.:

$y = c_1 t^2 + c_2 t^{-1} + t^2 \ln t + \frac{1}{6} - \frac{t^2}{3} + \frac{1}{3}$

$\left(= c_1 t^2 + c_2 t^{-1} + t^2 \ln t + \frac{1}{2} \right)$

b) (*no explanation required, 2 points*) True or false:

(i) $\{2t^2 + 2t^{-1}, 3t^2 + 3t^{-1}\}$ is a fundamental set of solutions of $t^2y'' - 2y = 0,\ t > 0$. **T F**

(ii) $\{2t^2 + t^{-1}, t^2\}$ is a fundamental set of solutions of $t^2y'' - 2y = 0,\ t > 0$. **T F**

$\times$

Problem 2. a) (12 pts) Find the general solution of the following equation.

$$D(D+2)(D-1)y = 2e^{-2t}$$

$\rightarrow$ $(D+2) \mid D(D+2)(D-1)y = 0$ 3

$D(D+2)^2(D-1)y = 0$

$\rightarrow$ $y = \boxed{C_1 + C_2 e^{-2t}} + C_3 t e^{-2t} \boxed{+ C_4 e^t}$ 3

sol. of hom. eq.

$\rightarrow$ Ansatz $y_p = At \cdot e^{-2t}$

$(D+2)y_p = Ae^{-2t} - 2Ate^{-2t} + 2Ate^{-2t} = Ae^{-2t}$

$(D-1)Ae^{-2t} = -2Ae^{-2t} - Ae^{-2t} = -3Ae^{-2t}$

$D(-3Ae^{-2t}) = 6Ae^{-2t} \overset{!}{=} 2e^{-2t}$

$\rightarrow A = \frac{1}{3}$ 3

$\rightarrow$ general sol

$y = C_1 + C_2 \cdot e^{-2t} + C_3 \cdot e^t + \frac{1}{3} t e^{-2t}$ 3

b) (*no explanation required, 2 points each*) True or false:

(i) $y = e^{-2t} + e^t + \frac{te^{-2t}}{3}$ is a solution of the equation above. T F 2

(ii) $y = 4 + \frac{e^{-2t}}{2} + 3e^{-t} + te^{-2t}$ is a solution of the equation above. T F 2

Problem 3. a) (12 pts) Solve the following initial value problem

$$y'' - y = \begin{cases} 1\,, & \text{if } 2 \le t < 4 \\ 0\,, & \text{otherwise} \end{cases}$$

$$y(0) = 0, \quad y'(0) = 0$$

2 Let $g(t) = \begin{cases} 1, & \text{if } 2 \le t < 4 \\ 0, & \text{otherwise} \end{cases}$. Then $g(t) = u_2(t) - u_4(t)$

2 $\mathcal{L}(y''-y) = \mathcal{L}(g(t))$

$\mathcal{L}(y''-y) = \mathcal{L}(y'') - \mathcal{L}(y) = s^2\mathcal{L}(y) - sy(0) - y'(0) - \mathcal{L}(y) = (s^2-1)\mathcal{L}(y)$

1 $\mathcal{L}(g(t)) = \mathcal{L}(u_2(t)) - \mathcal{L}(u_4(t)) = \dfrac{e^{-2s}}{s} - \dfrac{e^{-4s}}{s}$

$\Rightarrow \mathcal{L}(y) = (e^{-2s} - e^{-4s})\left(\dfrac{1}{s(s^2-1)}\right)$

Let $F(s) = \dfrac{1}{s(s^2-1)} = \dfrac{a}{s} + \dfrac{b}{s-1} + \dfrac{c}{s+1} \Rightarrow a(s-1)(s+1) + b(s)(s+1) + c(s)(s-1) = 1$

3 Putting $s = 1 \Rightarrow 2b = 1 \Rightarrow b = \frac{1}{2}$

$s = -1 \Rightarrow 2c = 1 \Rightarrow c = \frac{1}{2}$

$s = 0 \Rightarrow -a = 1 \Rightarrow a = -1$

$F(s) = \dfrac{1}{s(s^2-1)} = \dfrac{-1}{s} + \left(\dfrac{1}{2}\right)\dfrac{1}{s-1} + \left(\dfrac{1}{2}\right)\dfrac{1}{s+1}$

$\Rightarrow \mathcal{L}(y) = e^{-2s}F(s) - e^{-4s}F(s) \Rightarrow y(t) = u_2(t)f(t-2) - u_4(t)f(t-4)$ where $F(s) = \mathcal{L}(f(t))$.

4 $F(s) = -\frac{1}{s} + \left(\frac{1}{2}\right)\frac{1}{s-1} + \left(\frac{1}{2}\right)\frac{1}{s+1} \Rightarrow f(t) = -1 + \left(\frac{1}{2}\right)e^{t} + \left(\frac{1}{2}\right)e^{-t}$

$\Rightarrow y(t) = u_2(t)\left[-1 + \left(\frac{1}{2}\right)e^{t-2} + \frac{1}{2}e^{-t+2}\right] - u_4(t)\left[-1 + \left(\frac{1}{2}\right)e^{t-4} + \left(\frac{1}{2}\right)e^{-t+4}\right]$

b) (*no explanation required, 2 points each*) True or false:

(i) If the Laplace transforms $F(s) = \mathfrak{L}\{f(t)\}$ and $G(s) = \mathfrak{L}\{g(t)\}$ both exist for $s > a > 0$, then $5F(s)G(s) = \mathfrak{L}\{5f(t)g(t)\}$ for $s > a > 0$. **T** **F**

(ii) If the Laplace transforms $F(s) = \mathfrak{L}\{f(t)\}$ and $G(s) = \mathfrak{L}\{g(t)\}$ both exist for $s > a > 0$, then $2F(s) + 3G(s) = \mathfrak{L}\{2f(t) + 3g(t)\}$ for $s > a > 0$. **T** **F**

$\cancel{X}$

Problem 4. a) (12 pts) Find the solution of the following systems of equations that satisfies the given initial condition.

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}}_{A} \cdot \mathbf{x} \ , \quad \mathbf{x}(0) = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

Find eigenvalues and corresponding eigenvectors of A:

$$\begin{vmatrix} 3-\lambda & 6 \\ -1 & -2-\lambda \end{vmatrix} = 0 \Leftrightarrow (3-\lambda)(-2-\lambda)+6=0 \Leftrightarrow -6-\lambda+\lambda^2+6=0 \Leftrightarrow \lambda(\lambda-1)=0$$

3

$\Rightarrow$ Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 0$

$\underline{\lambda_1 = 1}$: $\begin{pmatrix} 3-1 & 6 \\ -1 & -2-1 \end{pmatrix} v_1 = 0 \Rightarrow v_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ 2 or $v_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = 0}$: $\begin{pmatrix} 3-0 & 6 \\ -1 & -2-0 \end{pmatrix} v_2 = 0 \Rightarrow v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ 2 or $v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\underline{\text{General Soln}}$:

$x(t) = c_1\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

2

$x(0) = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \Rightarrow c_1\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$

$\Rightarrow c_1 = -1$

$c_2 = 3$

3

$x(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + e^t\begin{pmatrix} 9 \\ -3 \end{pmatrix}$

b) (*no explanation required, 2 points*) True or false:

The system $\left\{ \begin{array}{rcl} x_1' & = & 3x_1 + 6x_2 \\ x_2' & = & -x_1 - 2x_2 \end{array} \right\}$ is satisfied by the functions $x_1 = 2 + 3e^t$ and $x_2 = 1 + e^t$. T F

Problem 5. (12 pts) Find the general solution of the following system of equations.

$$\mathbf{x}' = \underbrace{\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}}_{\text{this matrix } A.} \cdot \mathbf{x}$$

We need to find eigenvalues of this matrix A.

$$\begin{vmatrix} r+1 & -2 \\ 0 & r+1 \end{vmatrix} = (r+1)^2 = 0 \Rightarrow r=-1 \text{ is the repeated eigenvalue}$$

to find eigenvectors $A.v = -v \Leftrightarrow (A+I)v = 0 \Leftrightarrow \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow x_2 = 0$ $x_1 \in \mathbb{R}$, pick $x_1 = 1$

ie $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector

$x^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot e^{-t}$ is a solution.

another solution is given by

$x^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot t \cdot e^{-t} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} e^{-t}$ where $(A+I)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

ie $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Leftrightarrow y_2 = 1/2$ $y_1 \in \mathbb{R}$.

pick $y_1 = 0$.

$\Rightarrow$ general solution is $c_1 x^{(1)} + c_2 x^{(2)} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} te^{-t}$

$+ c_2 \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} e^{-t}$

$$W(x^{(1)}, x^{(2)}) = \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t}/2 \end{pmatrix} = \frac{e^{-2t}}{2} \neq 0 \ \forall t \in \mathbb{R}$$

Problem 6. (14 pts) Given that $\phi(t) = \begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix}$ is a fundamental matrix for the system $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \cdot \mathbf{x}$, find the general solution of the following system of equations.

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \cdot \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

$x = \varnothing(t) \cdot u(t)$

where $\varnothing(t)\, u'(t) = g(t) = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$

$$\begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$-2/ \quad 2e^{2t} u_1' + e^{-t} u_2' = e^t$

$e^{2t} u_1' + 2e^{-t} u_2' = -e^{+t}$

$+$

$-3e^{2t} u_1' = -3e^t$

$u_1' = e^{-t}$

$u_2' = -e^{2t}$

$$\boxed{\begin{aligned} u_1 &= -e^{-t} + c_1 \\ u_2 &= -\frac{e^{2t}}{2} + c_2 \end{aligned}}$$

Solution: $x = \varnothing(t) \cdot u(t)$

$$= \begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} -e^{-t} + c_1 \\ -\frac{e^{2t}}{2} + c_2 \end{pmatrix} = \begin{pmatrix} -2e^{t} + 2c_1 e^{2t} - \frac{e^t}{2} + c_2 e^{-t} \\ -e^t + c_1 e^{2t} - e^t + 2c_2 e^{-t} \end{pmatrix} = x(t)$$

$$= e^t \begin{pmatrix} -\frac{5}{2} \\ -2 \end{pmatrix} + c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Problem 7. (16 pts) Find the general solution of the following system for $t > 0$.

$$\begin{cases} tx_1' &= 2x_1 - x_2 \\ tx_2' &= 3x_1 - 2x_2 \end{cases}$$

(Hint: Look for a solution of the form $x_1 = v_1 t^r$, $x_2 = v_2 t^r$, where v_1, v_2, and r are suitable constants.)

$x_1' = r v_1 t^{r-1}, \quad x_2' = r v_2 t^{r-1}$

$$\begin{cases} t r v_1 t^{r-1} = 2v_1 t^r - v_2 t^r \\ t r v_2 t^{r-1} = 3v_1 t^r - 2v_2 t^r \end{cases}$$

(divide by t^r) $\Longrightarrow$ $\begin{cases} r v_1 = 2v_1 - v_2 \\ r v_2 = 3v_1 - 2v_2 \end{cases} \Longleftrightarrow \begin{cases} (r-2)v_1 + v_2 = 0 \\ -3v_1 + (r+2)v_2 = 0 \end{cases}$

$\Rightarrow$ Nontrivial solutions for v_1, v_2 exist if and only if $\begin{vmatrix} r-2 & 1 \\ -3 & r+2 \end{vmatrix} = 0$

i.e. r is an eigenvalue and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a corresponding eigenvector of $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$

eigenvalues: $r_1 = 1, \quad r_2 = -1$

eigenvectors: $\vec{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Hence $\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^1$ and $\vec{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^{-1}$ are sols of the system. $W(\vec{x}^{(1)}, \vec{x}^{(2)}) = \begin{vmatrix} t & t^{-1} \\ t & 3t^{-1} \end{vmatrix} = 2 \neq 0 \Rightarrow$ gen. sol.: $\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^{-1}$