## Boyce/DiPrima 10th ed, Ch 5.1: Review of Power Series

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- Finding the general solution of a linear differential equation depends on determining a fundamental set of solutions of the homogeneous equation.
- So far, we have a systematic procedure for constructing fundamental solutions if equation has constant coefficients.
- For a larger class of equations with variable coefficients, we must search for solutions beyond the familiar elementary functions of calculus.
- The principal tool we need is the representation of a given function by a power series.
- Then, similar to the undetermined coefficients method, we assume the solutions have power series representations, and then determine the coefficients so as to satisfy the equation.

## Convergent Power Series

- A **power series** about the point $x_0$ has the form
$$\sum_{n=1}^{\infty} a_n (x - x_0)^n$$
and is said to **converge** at a point $x$ if
$$\lim_{m\to\infty} \sum_{n=1}^{m} a_n (x - x_0)^n$$
exists for that $x$.
- Note that the series converges for $x = x_0$. It may converge for all $x$, or it may converge for some values of $x$ and not others.

## Absolute Convergence

- A power series about the point $x_0$
$$\sum_{n=1}^{\infty} a_n (x - x_0)^n$$
is said to **converge absolutely** at a point $x$ if the series
$$\sum_{n=1}^{\infty} \left| a_n (x - x_0)^n \right| = \sum_{n=1}^{\infty} |a_n| |x - x_0|^n$$
converges.
- If a series converges absolutely, then the series also converges. The converse, however, is not necessarily true.

## Ratio Test

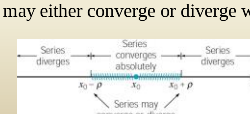
- One of the most useful tests for the absolute convergence of a power series
$$\sum_{n=1}^{\infty} a_n (x - x_0)^n$$
is the ratio test. If $a_n \neq 0$, and if, for a fixed value of $x$,
$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n (x - x_0)^n} \right| = |x - x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x - x_0| L,$$
then the power series converges absolutely at that value of $x$ if $|x - x_0| L < 1$ and diverges if $|x - x_0| L > 1$. The test is inconclusive if $|x - x_0| L = 1$.

## Example 1

- Find which values of $x$ does power series below converge.
$$\sum_{n=1}^{\infty} (-1)^{n+1} n (x - 2)^n$$
- Using the ratio test, we obtain
$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(n+1)(x-2)^{n+1}}{(-1)^{n+1} n (x-2)^n} \right| = |x - 2| \lim_{n\to\infty} \frac{n+1}{n} = |x - 2| < 1, \text{ for } 1 < x < 3$$
- At $x = 1$ and $x = 3$, the corresponding series are, respectively,
$$\sum_{n=1}^{\infty} (1-2)^n = \sum_{n=1}^{\infty} (-1)^n, \qquad \sum_{n=1}^{\infty} (3-2)^n = \sum_{n=1}^{\infty} (1)^n$$
- Both series diverge, since the $n$th terms do not approach zero.
- Therefore the interval of convergence is $(1, 3)$.

## Radius of Convergence

- There is a nonnegative number $\rho$, called the **radius of convergence**, such that $\Sigma a_n (x - x_0)^n$ converges absolutely for all $x$ satisfying $|x - x_0| < \rho$ and diverges for $|x - x_0| > \rho$.
- For a series that converges only at $x_0$, we define $\rho$ to be zero.
- For a series that converges for all $x$, we say that $\rho$ is infinite.
- If $\rho > 0$, then $|x - x_0| < \rho$ is called the **interval of convergence**.
- The series may either converge or diverge when $|x - x_0| = \rho$.



## Example 2

- Find the radius of convergence for the power series below.

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n\,2^n}$$

- Using the ratio test, we obtain

$$\lim_{n\to\infty}\left|\frac{(n)2^n(x+1)^{n+1}}{(n+1)2^{n+1}(x+1)^n}\right| = \frac{|x+1|}{2}\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \frac{|x+1|}{2} < 1, \text{ for } -3 < x < 1$$

- At $x$ = -3 and $x$ = 1, the corresponding series are, respectively,

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}, \qquad \sum_{n=1}^{\infty}\frac{(2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

- The alternating series on the left is convergent but not absolutely convergent. The series on the right, called the harmonic series is divergent. Therefore the interval of convergence is [-3, 1), and hence the radius of convergence is $\rho = 2$.

## Taylor Series

- Suppose that $\Sigma\, a_n(x - x_0)^n$ converges to $f(x)$ for $|x - x_0| < \rho$.
- Then the value of $a_n$ is given by

$$a_n = \frac{f^{(n)}(x_0)}{n!},$$

and the series is called the **Taylor series** for $f$ about $x = x_0$.

- Also, if

$$f(x) = \sum_{n=1}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

then $f$ is continuous and has derivatives of all orders on the interval of convergence. Further, the derivatives of $f$ can be computed by differentiating the relevant series term by term.

## Analytic Functions

- A function $f$ that has a Taylor series expansion about $x = x_0$

$$f(x) = \sum_{n=1}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

with a radius of convergence $\rho > 0$, is said to be **analytic** at $x_0$.

- All of the familiar functions of calculus are analytic.
- For example, $\sin x$ and $e^x$ are analytic everywhere, while $1/x$ is analytic except at $x = 0$, and $\tan x$ is analytic except at odd multiples of $\pi/2$.
- If $f$ and $g$ are analytic at $x_0$, then so are $f \pm g$, $fg$, and $f/g$; see text for details on these arithmetic combinations of series.

## Series Equality

- If two power series are equal, that is,

$$\sum_{n=1}^{\infty} a_n(x-x_0)^n = \sum_{n=1}^{\infty} b_n(x-x_0)^n$$

for each $x$ in some open interval with center $x_0$, then $a_n = b_n$ for $n = 0, 1, 2, 3,\ldots$

- In particular, if

$$\sum_{n=1}^{\infty} a_n(x-x_0)^n = 0$$

then $a_n = 0$ for $n = 0, 1, 2, 3,\ldots$

## Shifting Index of Summation

- The index of summation in an infinite series is a dummy parameter just as the integration variable in a definite integral is a dummy variable.
- Thus it is immaterial which letter is used for the index of summation:

$$\sum_{n=1}^{\infty} a_n(x-x_0)^n = \sum_{k=1}^{\infty} a_k(x-x_0)^k$$

- Just as we make changes in the variable of integration in a definite integral, we find it convenient to make changes of summation in calculating series solutions of differential equations.

## Example 3: Shifting Index of Summation

- We are asked to rewrite the series below as one starting with the index $n = 0$.

$$\sum_{n=2}^{\infty} a_n(x)^n$$

By letting $m = n - 2$ in this series. $n = 2$ corresponds to $m = 0$, and hence

$$\sum_{n=2}^{\infty} a_n(x)^n = \sum_{m=0}^{\infty} a_{m+2}(x)^{m+2}$$

- Replacing the dummy index $m$ with $n$, we obtain

$$\sum_{n=2}^{\infty} a_n(x)^n = \sum_{n=0}^{\infty} a_{n+2}(x)^{n+2}$$

as desired.

## Example 4: Rewriting Generic Term

- We can write the following series
$$\sum_{n=2}^{\infty}(n+2)(n+1)a_n(x-x_0)^{n-2}$$
as a sum whose generic term involves $(x-x_0)^n$ by letting $m = n-2$. Then $n = 2$ corresponds to $m = 0$
- It follows that
$$\sum_{n=2}^{\infty}(n+2)(n+1)a_n(x-x_0)^{n-2} = \sum_{m=0}^{\infty}(m+4)(m+3)a_{m+2}(x-x_0)^m$$
- Replacing the dummy index $m$ with $n$, we obtain
$$\sum_{n=0}^{\infty}(n+4)(n+3)a_{n+2}(x-x_0)^n$$
as desired.

## Example 5: Rewriting Generic Term

- We can write the following series
$$x^2\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1}$$
as a series whose generic term involves $x^{r+n}$
- Begin by taking $x^2$ inside the summation and letting $m = n+1$
$$x^2\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1} = \sum_{n=0}^{\infty}(r+n)a_n x^{r+n+1} = \sum_{m=1}^{\infty}(r+m-1)a_{m-1}x^{r+m}$$
- Replacing the dummy index $m$ with $n$, we obtain the desired result:
$$\sum_{n=1}^{\infty}(r+n-1)a_{n-1}x^{r+n}$$

## Example 6: Determining Coefficients (1 of 2)

- Assume that
$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$
- Determine what this implies about the coefficients.
- Begin by writing both series with the same powers of $x$. As before, for the series on the left, let $m = n-1$, then replace $m$ by as we have been doing. The above equality becomes:
$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty}a_n x^n \Rightarrow (n+1)a_{n+1} = a_n \Rightarrow a_{n+1} = \frac{a_n}{n+1}$$
for $n = 0, 1, 2, 3, \ldots$

## Example 6: Determining Coefficients (2 of 2)

- Using the recurrence relationship just derived:
$$a_{n+1} = \frac{a_n}{n+1}$$
- we can solve for the coefficients successively by letting $n = 0, 1, 2, \ldots n$:
$$a_1 = \frac{a_0}{2},\ a_2 = \frac{a_1}{3} = \frac{a_0}{6},\ a_3 = \frac{a_2}{4} = \frac{a_0}{24},\ \cdots,\ a_n = \frac{a_0}{n!}$$
- Using these coefficients in the original series, we get a recognizable Taylor series:
$$a_0\sum_{n=0}^{\infty}\frac{x^n}{n!} = a_0 e^x$$

## Boyce/DiPrima 10th ed, Ch 5.2: Series Solutions Near an Ordinary Point, Part I

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- In Chapter 3, we examined methods of solving second order linear differential equations with constant coefficients.
- We now consider the case where the coefficients are functions of the independent variable, which we will denote by $x$.
- It is sufficient to consider the homogeneous equation
$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0,$$
since the method for the nonhomogeneous case is similar.
- We primarily consider the case when $P$, $Q$, $R$ are polynomials, and hence also continuous.
- However, as we will see, the method of solution is also applicable when $P$, $Q$ and $R$ are general analytic functions.

## Ordinary Points

- Assume $P$, $Q$, $R$ are polynomials with no common factors, and that we want to solve the equation below in a neighborhood of a point of interest $x_0$:
$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$
- The point $x_0$ is called an **ordinary point** if $P(x_0) \neq 0$. Since $P$ is continuous, $P(x) \neq 0$ for all $x$ in some interval about $x_0$. For $x$ in this interval, divide the differential equation by $P$ to get
$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \text{ where } p(x) = \frac{Q(x)}{P(x)},\ \ q(x) = \frac{R(x)}{P(x)}$$
- Since $p$ and $q$ are continuous, Theorem 3.2.1 says there is a unique solution, given initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$

## Singular Points

- Suppose we want to solve the equation below in some neighborhood of a point of interest $x_0$:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \text{ where } p(x) = \frac{Q(x)}{P(x)}, \; q(x) = \frac{R(x)}{P(x)}$$

- The point $x_0$ is called an **singular point** if $P(x_0) = 0$.
- Since $P$, $Q$, $R$ are polynomials with no common factors, it follows that $Q(x_0) \neq 0$ or $R(x_0) \neq 0$, or both.
- Then at least one of $p$ or $q$ becomes unbounded as $x \to x_0$, and therefore Theorem 3.2.1 does not apply in this situation.
- Sections 5.4 through 5.8 deal with finding solutions in the neighborhood of a singular point.

## Series Solutions Near Ordinary Points

- In order to solve our equation near an ordinary point $x_0$,

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

we will assume a series representation of the unknown solution function $y$:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- As long as we are within the interval of convergence, this representation of $y$ is continuous and has derivatives of all orders.

## Example 1: Series Solution (1 of 8)

- Find a series solution of the equation

$$y'' + y = 0, \quad -\infty < x < \infty$$

- Here, $P(x) = 1$, $Q(x) = 0$, $R(x) = 1$. Thus every point $x$ is an ordinary point. We will take $x_0 = 0$.
- Assume a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

- Differentiate term by term to obtain

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

- Substituting these expressions into the equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

## Example 1: Combining Series (2 of 8)

- Our equation is

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

- Shifting indices, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n\right] x^n = 0$$

## Example 1: Recurrence Relation (3 of 8)

- Our equation is

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n\right] x^n = 0$$

- For this equation to be valid for all $x$, the coefficient of each power of $x$ must be zero, and hence

$$(n+2)(n+1) a_{n+2} + a_n = 0, \quad n = 0, 1, 2, \ldots$$

or

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, \ldots$$

- This type of equation is called a **recurrence relation**.
- Next, we find the individual coefficients $a_0$, $a_1$, $a_2$, …

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

## Example 1: Even Coefficients (4 of 8)

- To find $a_2$, $a_4$, $a_6$, …., we proceed as follows:

$$a_2 = -\frac{a_0}{2 \cdot 1},$$

$$a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1},$$

$$a_6 = -\frac{a_4}{6 \cdot 5} = -\frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1},$$

$$\vdots$$

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!}, \quad k = 1, 2, 3, \ldots$$

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

## Example: Odd Coefficients (5 of 8)

- To find $a_3$, $a_5$, $a_7$, …., we proceed as follows:

$$a_3 = -\frac{a_1}{3\cdot 2},$$
$$a_5 = -\frac{a_3}{5\cdot 4} = \frac{a_1}{5\cdot 4\cdot 3\cdot 2\cdot 1},$$
$$a_7 = -\frac{a_5}{7\cdot 6} = -\frac{a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1},$$
$$\vdots$$
$$a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}, \quad k = 1, 2, 3, \ldots$$

## Example 1: Solution (6 of 8)

- We now have the following information:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \text{ where } a_{2k} = \frac{(-1)^k}{(2k)!} a_0, \quad a_{2k+1} = \frac{(-1)^k}{(2k+1)!} a_1$$

- Thus

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

- Note: $a_0$ and $a_1$ are determined by the initial conditions. (Expand series a few terms to see this.)
- Also, by the ratio test it can be shown that these two series converge absolutely on (-∞, ∞), and hence the manipulations we performed on the series at each step are valid.

## Example 1: Functions Defined by IVP (7 of 8)

- Our solution is

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

- From Calculus, we know this solution is equivalent to

$$y(x) = a_0 \cos x + a_1 \sin x$$

- In hindsight, we see that cos $x$ and sin $x$ are indeed fundamental solutions to our original differential equation

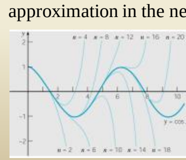
$$y'' + y = 0, \quad -\infty < x < \infty$$

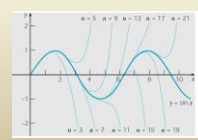
- While we are familiar with the properties of cos $x$ and sin $x$, many important functions are defined by the initial value problem that they solve.

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

## Example 1: Graphs (8 of 8)

- The graphs below show the partial sum approximations of $\cos x$ and $\sin x$.
- As the number of terms increases, the interval over which the approximation is satisfactory becomes longer, and for each $x$ in this interval the accuracy improves.
- However, the truncated power series provides only a local approximation in the neighborhood of $x$ = 0.





## Example 2: Airy's Equation (1 of 10)

- Find a series solution of Airy's equation about $x_0$ = 0:

$$y'' - xy = 0, \ -\infty < x < \infty$$

- Here, $P(x)$ = 1, $Q(x)$ = 0, $R(x)$ = - $x$. Thus every point $x$ is an ordinary point. We will take $x_0$ = 0.
- Assuming a series solution and differentiating, we obtain

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

- Substituting these expressions into the equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

## Example 2: Combine Series (2 of 10)

- Our equation is

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

- Shifting the indices, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

or

$$2\cdot 1\cdot a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1}\right] x^n = 0$$

## Example 2: Recurrence Relation (3 of 10)

- Our equation is

$$2\cdot 1\cdot a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_{n-1}\right]x^n = 0$$

- For this equation to be valid for all $x$, the coefficient of each power of $x$ must be zero; hence $a_2 = 0$ and

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}, \quad n = 1, 2, 3, \ldots$$

or

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)}, \quad n = 0, 1, 2, \ldots$$

## Example 2: Coefficients (4 of 10)

- We have $a_2 = 0$ and

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)}, \quad n = 0, 1, 2, \ldots$$

- For this recurrence relation, note that $a_2 = a_5 = a_8 = \ldots = 0$.
- Next, we find the coefficients $a_0, a_3, a_6, \ldots$.
- We do this by finding a formula $a_{3n}$, $n = 1, 2, 3, \ldots$
- After that, we find $a_1, a_4, a_7, \ldots$, by finding a formula for $a_{3n+1}$, $n = 1, 2, 3, \ldots$

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)}$$

## Example 2: Find $a_{3n}$ (5 of 10)

- Find $a_3, a_6, a_9, \ldots$.

$$a_3 = \frac{a_0}{2\cdot 3}, \quad a_6 = \frac{a_3}{5\cdot 6} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6}, \quad a_9 = \frac{a_6}{8\cdot 9} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}, \ldots$$

- The general formula for this sequence is

$$a_{3n} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdots(3n-4)(3n-3)(3n-1)(3n)}, \quad n = 1, 2, \ldots$$

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)}$$

## Example 2: Find $a_{3n+1}$ (6 of 10)

- Find $a_4, a_7, a_{10}, \ldots$

$$a_4 = \frac{a_1}{3\cdot 4}, \quad a_7 = \frac{a_4}{6\cdot 7} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7}, \quad a_{10} = \frac{a_7}{9\cdot 10} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}, \ldots$$

- The general formula for this sequence is

$$a_{3n+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdots(3n-3)(3n-2)(3n)(3n+1)}, \quad n = 1, 2, \ldots$$

## Example 2: Series and Coefficients (7 of 10)

- We now have the following information:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=3}^{\infty} a_n x^n$$

where $a_0$, $a_1$ are arbitrary, and

$$a_{3n} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdots(3n-4)(3n-3)(3n-1)(3n)}, \quad n = 1, 2, \ldots$$

$$a_{3n+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdots(3n-3)(3n-2)(3n)(3n+1)}, \quad n = 1, 2, \ldots$$

## Example 2: Solution (8 of 10)

- Thus our solution is

$$y(x) = a_0\left[1 + \sum_{n=1}^{\infty}\frac{x^{3n}}{2\cdot 3\cdots(3n-1)(3n)}\right] + a_1\left[x + \sum_{n=1}^{\infty}\frac{x^{3n+1}}{3\cdot 4\cdots(3n)(3n+1)}\right]$$

where $a_0$, $a_1$ are arbitrary (determined by initial conditions).
- Consider the two cases
  - (1) $a_0 = 1$, $a_1 = 0$ $\Leftrightarrow$ $y(0) = 1$, $y'(0) = 0$
  - (2) $a_0 = 0$, $a_1 = 1$ $\Leftrightarrow$ $y(0) = 0$, $y'(0) = 1$
- The corresponding solutions $y_1(x)$, $y_2(x)$ are linearly independent, since $W(y_1, y_2)(0) = 1 \neq 0$, where

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = y_1(0)y_2'(0) - y_1'(0)y_2(0)$$

## Example 2: Fundamental Solutions (9 of 10)

- Our solution:

$$y(x) = a_0\left[1+\sum_{n=1}^{\infty}\frac{x^{3n}}{2\cdot 3\cdots(3n-1)(3n)}\right] + a_1\left[x+\sum_{n=1}^{\infty}\frac{x^{3n+1}}{3\cdot 4\cdots(3n)(3n+1)}\right]$$

- For the cases

  (1) $a_0 = 1,\ a_1 = 0 \ \Leftrightarrow\ y(0) = 1, y'(0) = 0$

  (2) $a_0 = 0,\ a_1 = 1 \ \Leftrightarrow\ y(0) = 0, y'(0) = 1,$

  the corresponding solutions $y_1(x)$, $y_2(x)$ are linearly independent, and thus are fundamental solutions for Airy's equation, with general solution

$$y(x) = c_1 y_1(x) + c_1 y_2(x)$$

## Example 2: Graphs (10 of 10)

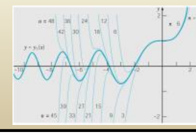
- Thus given the initial conditions

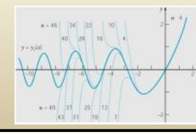
$$y(0) = 1, y'(0) = 0 \quad \text{and} \quad y(0) = 0, y'(0) = 1$$

  the solutions are, respectively,

$$y_1(x) = 1+\sum_{n=1}^{\infty}\frac{x^{3n}}{2\cdot 3\cdots(3n-1)(3n)},\quad y_2(x) = x+\sum_{n=1}^{\infty}\frac{x^{3n+1}}{3\cdot 4\cdots(3n)(3n+1)}$$

- The graphs of $y_1$ and $y_2$ are given below. Note the approximate intervals of accuracy for each partial sum





## Example 3: Airy's Equation (1 of 7)

- Find a series solution of Airy's equation about $x_0 = 1$:

$$y'' - xy = 0,\ -\infty < x < \infty$$

- Here, $P(x) = 1$, $Q(x) = 0$, $R(x) = -x$. Thus every point $x$ is an ordinary point. We will take $x_0 = 1$.
- Assuming a series solution and differentiating, we obtain

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n,\quad y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1},\quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

- Substituting these into ODE & shifting indices, we obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = x\sum_{n=0}^{\infty} a_n (x-1)^n$$

## Example 3: Rewriting Series Equation (2 of 7)

- Our equation is

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = x\sum_{n=0}^{\infty} a_n (x-1)^n$$

- The $x$ on right side can be written as $1 + (x - 1)$; and thus

$$\begin{aligned}\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n &= [1+(x-1)]\sum_{n=0}^{\infty} a_n (x-1)^n \\ &= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^{n+1} \\ &= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=1}^{\infty} a_{n-1} (x-1)^n\end{aligned}$$

## Example 3: Recurrence Relation (3 of 7)

- Thus our equation becomes

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = a_0 + \sum_{n=1}^{\infty} a_n (x-1)^n + \sum_{n=1}^{\infty} a_{n-1}(x-1)^n$$

- Thus the recurrence relation is

$$(n+2)(n+1)a_{n+2} = a_n + a_{n-1},\ (n \geq 1)$$

- Equating like powers of $x$ -1, we obtain

$$\begin{aligned} 2a_2 &= a_0 & &\Rightarrow a_2 = \frac{a_0}{2}, \\ 3\cdot 2 a_3 &= a_1 + a_0 & &\Rightarrow a_3 = \frac{a_0}{6} + \frac{a_1}{6}, \\ 4\cdot 3 a_4 &= a_2 + a_1 & &\Rightarrow a_4 = \frac{a_0}{24} + \frac{a_1}{12}, \\ &\vdots \end{aligned}$$

## Example 3: Solution (4 of 7)

- We now have the following information:

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

and

$$y(x) = a_0\left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \cdots\right] + a_1\left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \cdots\right]$$

$$\begin{aligned} a_0 &= \text{arbitrary} \\ a_1 &= \text{arbitrary} \\ a_2 &= \frac{a_0}{2}, \\ a_3 &= \frac{a_0}{6} + \frac{a_1}{6}, \\ a_4 &= \frac{a_0}{24} + \frac{a_1}{12}, \\ &\vdots \end{aligned}$$

## Example 3: Solution and Recursion (5 of 7)

- Our solution:

$$y(x) = a_0\left[1+\frac{(x-1)^2}{2}+\frac{(x-1)^3}{6}+\frac{(x-1)^4}{24}+\cdots\right]$$
$$+a_1\left[(x-1)+\frac{(x-1)^3}{6}+\frac{(x-1)^4}{12}+\cdots\right]$$

$$a_0 = \text{arbitrary}$$
$$a_1 = \text{arbitrary}$$
$$a_2 = \frac{a_0}{2},$$
$$a_3 = \frac{a_0}{6}+\frac{a_1}{6},$$
$$a_4 = \frac{a_0}{24}+\frac{a_1}{12},$$
$$\vdots$$

- The recursion has three terms,

$$(n+2)(n+1)a_{n+2} = a_n + a_{n-1},\ (n \ge 1)$$

and determining a general formula for the coefficients $a_n$ can be difficult or impossible.
- However, we can generate as many coefficients as we like, preferably with the help of a computer algebra system.

## Example 3: Solution and Convergence (6 of 7)

- Our solution:

$$y(x) = a_0\left[1+\frac{(x-1)^2}{2}+\frac{(x-1)^3}{6}+\frac{(x-1)^4}{24}+\cdots\right]$$
$$+a_1\left[(x-1)+\frac{(x-1)^3}{6}+\frac{(x-1)^4}{12}+\cdots\right]$$

- Since we don't have a general formula for the $a_n$, we cannot use a convergence test (i.e., ratio test) on our power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

- This means our manipulations of the power series to arrive at our solution are suspect. However, the results of Section 5.3 will confirm the convergence of our solution.

## Example 3: Fundamental Solutions (7 of 7)

- Our solution:

$$y(x) = a_0\left[1+\frac{(x-1)^2}{2}+\frac{(x-1)^3}{6}+\frac{(x-1)^4}{24}+\cdots\right]$$
$$+a_1\left[(x-1)+\frac{(x-1)^3}{6}+\frac{(x-1)^4}{12}+\cdots\right]$$

or

$$y(x) = a_0 y_3(x) + a_1 y_4(x)$$

- It can be shown that the solutions $y_3(x)$, $y_4(x)$ are linearly independent, and thus are fundamental solutions for Airy's equation, with general solution

$$y(x) = a_0 y_3(x) + a_1 y_4(x)$$

## Boyce/DiPrima 10th ed, Ch 5.3: Series Solutions Near an Ordinary Point, Part II

Elementary Differential Equations and Boundary Value Problems, 10th edition, by William E. Boyce and Richard C. DiPrima, ©2013 by John Wiley & Sons, Inc.

- A function $p$ is **analytic** at $x_0$ if it has a Taylor series expansion that converges to $p$ in some interval about $x_0$

$$p(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n$$

- The point $x_0$ is an **ordinary point** of the equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

if $p(x) = Q(x)/P(x)$ and $q(x)= R(x)/P(x)$ are analytic at $x_0$. Otherwise $x_0$ is a **singular point**.
- If $x_0$ is an ordinary point, then $p$ and $q$ are analytic and have derivatives of all orders at $x_0$, and this enables us to solve for $a_n$ in the solution expansion $y(x) = \Sigma\, a_n(x - x_0)^n$. See text.

## Theorem 5.3.1

- If $x_0$ is an ordinary point of the differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

then the general solution for this equation is

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1(x) + a_1 y_2(x)$$

where $a_0$ and $a_1$ are arbitrary, and $y_1$, $y_2$ are linearly independent series solutions that are analytic at $x_0$.
- Further, the radius of convergence for each of the series solutions $y_1$ and $y_2$ is at least as large as the minimum of the radii of convergence of the series for $p$ and $q$.

## Radius of Convergence

- Thus if $x_0$ is an ordinary point of the differential equation, then there exists a series solution $y(x) = \Sigma\, a_n(x - x_0)^n$.
- Further, the radius of convergence of the series solution is at least as large as the minimum of the radii of convergence of the series for $p$ and $q$.
- These radii of convergence can be found in two ways:
  1. Find the series for $p$ and $q$, and then determine their radii of convergence using a convergence test.
  2. If $P$, $Q$ and $R$ are polynomials with no common factors, then it can be shown that $Q/P$ and $R/P$ are analytic at $x_0$ if $P(x_0) \neq 0$, and the radius of convergence of the power series for $Q/P$ and $R/P$ about $x_0$ is the distance to the nearest zero of $P$ (including complex zeros).

## Example 1

- Let $f(x) = (1 + x^2)^{-1}$. Find the radius of convergence of the Taylor series of $f$ about $x_0 = 0$.
- The Taylor series of $f$ about $x_0 = 0$ is
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$
- Using the ratio test, we have
$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(-1)^n x^{2n}}\right| = \lim_{n\to\infty} x^2 < 1, \text{ for } |x| < 1$$
- Thus the radius of convergence is $\rho = 1$.
- Alternatively, note that the zeros of $1 + x^2$ are $x = \pm i$. Since the distance in the complex plane from 0 to $i$ or $-i$ is 1, we see again that $\rho = 1$.

## Example 2

- Find the radius of convergence of the Taylor series for $(x^2 - 2x + 1)^{-1}$ about $x_0 = 0$ and about $x_0 = 1$. First observe:
$$(x^2 - 2x + 1) = 0 \Rightarrow x = 1 \pm i$$
- Since the denominator cannot be zero, this establishes the bounds over which the function can be defined.
- In the complex plane, the distance from $x_0 = 0$ to $1 \pm i$ is $\sqrt{2}$, so the radius of convergence for the Taylor series expansion about $x_0 = 0$ is $\rho = \sqrt{2}$.
- In the complex plane, the distance from $x_0 = 1$ to $1 \pm i$ is 1, so the radius of convergence for the Taylor series expansion about $x_0 = 0$ is $\rho = 1$.

## Example 3: Legendre Equation (1 of 2)

- Determine a lower bound for the radius of convergence of the series solution about $x_0 = 0$ for the Legendre equation
$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \quad \alpha \text{ a constant.}$$
- Here, $P(x) = 1 - x^2$, $Q(x) = -2x$, $R(x) = \alpha(\alpha + 1)$.
- Thus $x_0 = 0$ is an ordinary point, since $p(x) = -2x/(1 - x^2)$ and $q(x) = \alpha(\alpha + 1)/(1 - x^2)$ are analytic at $x_0 = 0$.
- Also, $p$ and $q$ have singular points at $x = \pm 1$.
- Thus the radius of convergence for the Taylor series expansions of $p$ and $q$ about $x_0 = 0$ is $\rho = 1$.
- Therefore, by Theorem 5.3.1, the radius of convergence for the series solution about $x_0 = 0$ is at least $\rho = 1$.

## Example 3: Legendre Equation (2 of 2)

- Thus, for the Legendre equation
$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$
the radius of convergence for the series solution about $x_0 = 0$ is at least $\rho = 1$.
- It can be shown that if $\alpha$ is a positive integer, then one of the series solutions terminates after a finite number of terms, and hence converges for all $x$, not just for $|x| < 1$.

## Example 4: Radius of Convergence (1 of 2)

- Determine a lower bound for the radius of convergence of the series solution about $x_0 = 0$ for the equation
$$(1 + x^2)y'' + 2xy' + 4x^2 y = 0$$
- Here, $P(x) = 1 + x^2$, $Q(x) = 2x$, $R(x) = 4x^2$.
- Thus $x_0 = 0$ is an ordinary point, since $p(x) = 2x/(1 + x^2)$ and $q(x) = 4x^2/(1 + x^2)$ are analytic at $x_0 = 0$.
- Also, $p$ and $q$ have singular points at $x = \pm i$.
- Thus the radius of convergence for the Taylor series expansions of $p$ and $q$ about $x_0 = 0$ is $\rho = 1$.
- Therefore, by Theorem 5.3.1, the radius of convergence for the series solution about $x_0 = 0$ is at least $\rho = 1$.

## Example 4: Solution Theory (2 of 2)

- Thus for the equation
$$(1 + x^2)y'' + 2xy' + 4x^2 y = 0,$$
- the radius of convergence for the series solution about $x_0 = 0$ is at least $\rho = 1$, by Theorem 5.3.1.
- Suppose that initial conditions $y(0) = y_0$ and $y(0) = y_0'$ are given. Since $1 + x^2 \neq 0$ for all $x$, there exists a unique solution of the initial value problem on $-\infty < x < \infty$, by Theorem 3.2.1.
- On the other hand, Theorem 5.3.1 only guarantees a solution of the form $\Sigma a_n x^n$ for $-1 < x < 1$, where $a_0 = y_0$ and $a_1 = y_0'$.
- Thus the unique solution on $-\infty < x < \infty$ may not have a power series about $x_0 = 0$ that converges for all $x$.

## Example 5

- Determine a lower bound for the radius of convergence of the series solution about $x_0 = 0$ for the equation
$$y'' + (\sin x)y' + (1 + x^2)y = 0$$
- Here, $P(x) = 1$, $Q(x) = \sin x$, $R(x) = 1 + x^2$.
- Note that $p(x) = \sin x$ is not a polynomial, but recall that it does have a Taylor series about $x_0 = 0$ that converges for all $x$.
- Similarly, $q(x) = 1 + x^2$ has a Taylor series about $x_0 = 0$, namely$1 + x^2$, which converges for all $x$.
- Therefore, by Theorem 5.3.1, the radius of convergence for the series solution about $x_0 = 0$ is infinite.