6.2

6. Using partial fractions,

$$\frac{2s+1}{s^2-4} = \frac{1}{4}\left[\frac{5}{s-2} + \frac{3}{s+2}\right].$$

Hence $\mathcal{L}^{-1}[Y(s)] = (5e^{2t} + 3e^{-2t})/4$. Note that we can also write

$$\frac{2s+1}{s^2-4} = 2\,\frac{s}{s^2-4} + \frac{1}{2}\frac{2}{s^2-4}.$$

8. Using partial fractions,

$$\frac{8s^2-6s+12}{s(s^2+4)} = 3\,\frac{1}{s} + 5\,\frac{s}{s^2+4} - 3\,\frac{2}{s^2+4}.$$

Hence $\mathcal{L}^{-1}[Y(s)] = 3 + 5\,\cos 2t - 3\,\sin 2t$.

10. Note that the denominator $s^2+2s+10$ is irreducible over the reals. Completing the square, $s^2+2s+10 = (s+1)^2+9$. Now convert the function to a rational function of the variable $\xi = s+1$. That is,

$$\frac{2s-5}{s^2+2s+10} = \frac{2(s+1)-7}{(s+1)^2+9}.$$

We find that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+9} - \frac{7}{\xi^2+9}\right] = 2\,\cos 3t - \frac{7}{3}\,\sin 3t.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{2s-5}{s^2+2s+10}\right] = e^{-t}(2\,\cos 3t - \frac{7}{3}\,\sin 3t).$$

13. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2\,[s\,Y(s) - y(0)] + 2\,Y(s) = 0\,.$$

Applying the initial conditions,

$$s^2\,Y(s) - 2s\,Y(s) + 2\,Y(s) - s + 1 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s-1}{s^2 - 2s + 2}\,.$$

Since the denominator is irreducible, write the transform as a function of $\xi = s - 1$. That is,

$$\frac{s-1}{s^2 - 2s + 2} = \frac{s-1}{(s-1)^2 + 1}\,.$$

First note that

$$\mathcal{L}^{-1}\left[\frac{\xi}{\xi^2 + 1}\right] = \cos\,t\,.$$

Using the fact that $\mathcal{L}\left[e^{at} f(t)\right] = \mathcal{L}\left[f(t)\right]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{s-1}{s^2 - 2s + 2}\right] = e^t\,\cos\,t\,.$$

Hence $y(t) = e^t\,\cos\,t$.

18. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0) - \,Y(s) = 0\,.$$

Applying the initial conditions,

$$s^4 Y(s) - \,Y(s) - s^3 - s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2 - 1}\,.$$

By inspection, it follows that $y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = \cosh\,t$.

19. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0) - 4\,Y(s) = 0\,.$$

Applying the initial conditions,

$$s^4 Y(s) - \,4Y(s) - s^3 + 2s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2 + 2}\,.$$

It follows that $y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = \cos\,\sqrt{2}\,t$.

21. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) - 2\left[s\,Y(s) - y(0)\right] + 2\,Y(s) = \frac{s}{s^2+1}\,.$$

Applying the initial conditions,

$$s^2 Y(s) - 2s\,Y(s) + 2\,Y(s) - s + 1 = \frac{s}{s^2+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s}{(s^2-2s+2)(s^2+1)} + \frac{s-1}{s^2-2s+2}\,.$$

Using partial fractions on the first term,

$$\frac{s}{(s^2-2s+2)(s^2+1)} = \frac{1}{5}\left[\frac{s-2}{s^2+1} - \frac{s-4}{s^2-2s+2}\right].$$

Thus we can write

$$Y(s) = \frac{1}{5}\,\frac{s}{s^2+1} - \frac{2}{5}\,\frac{1}{s^2+1} + \frac{1}{5}\,\frac{4s-1}{s^2-2s+2}\,.$$

For the last term, we note that $s^2-2s+2 = (s-1)^2+1$. So that

$$\frac{4s-1}{s^2-2s+2} = \frac{4(s-1)+3}{(s-1)^2+1}\,.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{4\,\xi}{\xi^2+1} + \frac{3}{\xi^2+1}\right] = 4\,\cos t + 3\sin t\,.$$

Based on the translation property of the Laplace transform,

$$\mathcal{L}^{-1}\left[\frac{4s-1}{s^2-2s+2}\right] = e^t(4\,\cos t + 3\sin t).$$

Combining the above, the solution of the IVP is

$$y(t) = \frac{1}{5}\,\cos t - \frac{2}{5}\,\sin t + \frac{1}{5}\,e^t(4\,\cos t + 3\sin t)\,.$$

23. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) + 2\left[s\,Y(s) - y(0)\right] + Y(s) = \frac{4}{s+1}\,.$$

Applying the initial conditions,

$$s^2 Y(s) + 2s\,Y(s) + Y(s) - 2s - 3 = \frac{4}{s+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2}\,.$$

First write

$$\frac{2s+3}{(s+1)^2} = \frac{2(s+1)+1}{(s+1)^2} = \frac{2}{s+1} + \frac{1}{(s+1)^2}\,.$$

We note that

$$\mathcal{L}^{-1}\left[\frac{4}{\xi^3} + \frac{2}{\xi} + \frac{1}{\xi^2}\right] = 2\,t^2 + 2 + t\,.$$

So based on the translation property of the Laplace transform, the solution of the IVP is

$$y(t) = 2\,t^2 e^{-t} + t\,e^{-t} + 2\,e^{-t}.$$

(2) $F(s) = \dfrac{5}{(s-1)^3}$, We know from Table 6.21. that

$f(t) = t^n e^{at}$ then $F(s) = \dfrac{n!}{(s-a)^{n+1}}$. $s > a$.

then rewrite $F(s)$ as. $\dfrac{5}{2} \cdot \dfrac{2!}{(s-1)^{2+1}}$.

Then $a = 1$, $n = 2$. Thus $f(t) = \frac{5}{2} t^2 e^t$.

(4) $F(s) = \dfrac{2s}{s^2 - s - 6} = \dfrac{6}{5} \cdot \dfrac{1}{(s-3)} + \dfrac{4}{5} \cdot \dfrac{1}{s+2}$

Then from Table 6.21 $\Rightarrow f(t) = \frac{6}{5} e^{3t} + \frac{4}{5} e^{-2t}$.

(15) $y'' - 2y' + 4y = 0$, $y(0) = 3$, $y'(0) = 0$.

Taking the laplace transform of the ODE, we have:

$s^2 Y(s) - s\,y(0) - y'(0) - 2[s Y(s) - y(0)] + 4Y(s) = 0$.

Applying the initial conditions:

$s^2 Y(s) - 2s\,Y(s) + 4\,Y(s) - 3s + 6 = 0$.

$\Rightarrow Y(s) = \dfrac{3s - 6}{s^2 - 2s + 4} =$

$= \dfrac{3\,(s-1)}{(s-1)^2 + (\sqrt{3})^2} - \dfrac{3}{(s-1)^2 + (\sqrt{3})^2}$. from table 6.2

$\Rightarrow y(t) = 3e^t \cos\sqrt{3}t - e^t \sin\sqrt{3}t$ //

(11) $y'' - y' - 6y = 0 \quad , \quad y(0) = 2 \, , \quad y'(0) = -1 .$

$$\mathcal{L}(y'' - y' - 6y) = \mathcal{L}(0)$$

$$\left[s^2 \cdot Y(s) - s \cdot y(0) - y'(0)\right] - \left[s \cdot Y(s) - y(0)\right] - 6 \cdot Y(s) = 0.$$

$$\left[s^2 \cdot Y(s) - 2s + 1\right] - \left[s \cdot Y(s) - 2\right] - 6 \cdot Y(s) = 0.$$

$$Y(s) \cdot \left[s^2 - s - 6\right] - 2s + 3 = 0$$

$$Y(s) = \frac{2s-3}{s^2 - s - 6} = \frac{2s-3}{(s-3)(s+2)}$$

observe that, $\frac{1}{(s-3)(s+2)} = -\frac{1}{5}\left(\frac{1}{s+2} - \frac{1}{s-3}\right)$

OR,

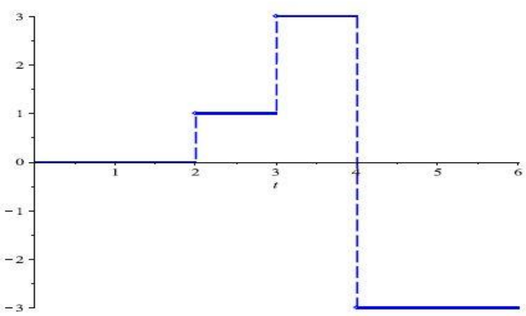
$$Y(s) = \frac{2s-3}{(s-3)(s+2)} = \frac{3}{5} \cdot \frac{1}{s-3} + \frac{7}{5} \cdot \frac{1}{s+2}$$

So that,

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{3}{5} \cdot e^{3t} + \frac{7}{5} \cdot e^{-2t} \quad //$$

6.3

1.



Problem 2: $g(t) = (t-3)u_2(t) - (t-4)u_3(t)$

$$u_2(t) = \begin{cases} 0, & t<2 \\ 1, & t \geq 2 \end{cases}$$

$$u_3(t) = \begin{cases} 0, & t<3 \\ 1, & t \geq 3 \end{cases}$$

So the critical points are $t=2$ and $t=3$.

if $t<2$: Then,

$$g(t) = (t-3)\cdot 0 - (t-4)\cdot 0 = 0$$

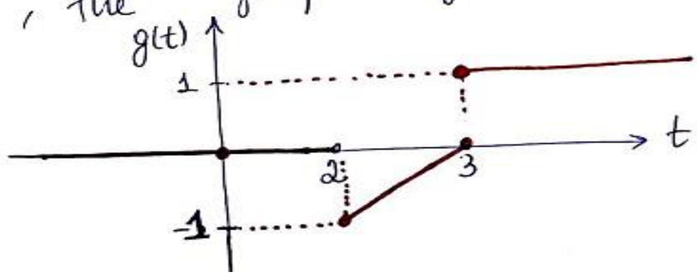
if $2 \leq t < 3$: Then,

$$g(t) = (t-3)\cdot 1 - (t-4)\cdot 0 = t-3$$

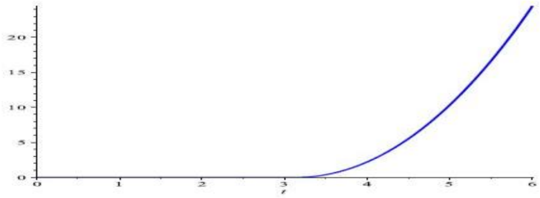
if $3 \leq t$: Then,

$$g(t) = (t-3)\cdot 1 - (t-4)\cdot 1 = 1$$

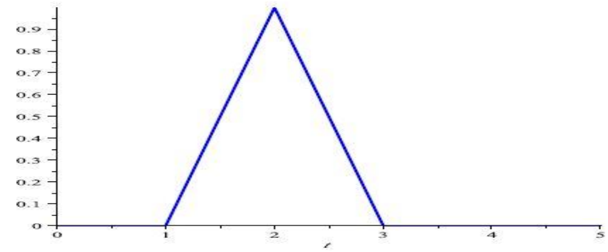
So, the graph of $g(t)$ for $t \geq 0$ is



3.



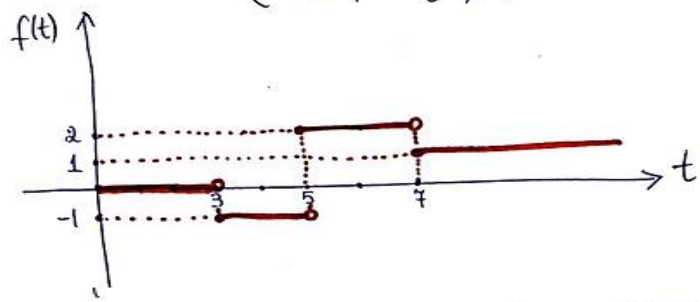
6.



Problem 7 :

$$f(t) = \begin{cases} 0 , & 0 \leq t < 3 \\ -1 , & 3 \leq t < 5 \\ 2 , & 5 \leq t < 7 \\ 1 , & t \geq 7 \end{cases}$$

(a)

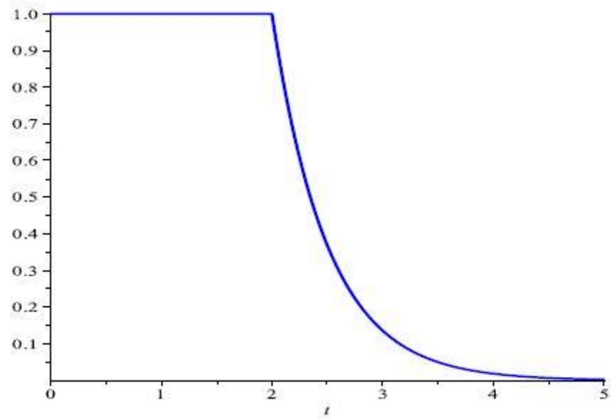


(b) we start with the function $f_1(t) = 0$, which agrees with $f(t)$ on $[0,3)$. To produce the negative jump of 1 units at $t=3$ corresponds to adding $-1 \cdot U_3(t)$, which gives

$f_2(t) = 0 - U_3(t)$, which agrees with $f(t)$ on $[0,5)$. Now, to produce the jump of 3 units at $t=5$, we add $3 \cdot U_5(t)$ to $f_2(t)$, obtaining $f_3(t) = -U_3(t) + 3 \cdot U_5(t)$ which agrees with $f(t)$ on $[0,7)$. Finally, produce the negative jump of 1 unit at $t=7$ is adding $-1 \cdot U_7(t)$ to $f_3(t)$ to obtain t,

$$\boxed{f(t) = -U_3(t) + 3 \cdot U_5(t) - U_7(t)}$$ //

9.(a)



(b) $f(t) = 1 + (e^{-2(t-2)} - 1)u_2(t)$.

13. Using the Heaviside function, we can write $f(t) = (t-2)^3\, u_2(t)$. The Laplace transform has the property that $\mathcal{L}\left[\, u_c(t) f(t-c)\right] = e^{-cs}\mathcal{L}\left[f(t)\right]$. Hence

$$\mathcal{L}\left[\,(t-2)^3\, u_2(t)\right] = \frac{6\, e^{-2s}}{s^4}.$$

15. The function can be expressed as $f(t) = (t-\pi)\,[u_\pi(t) - u_{2\pi}(t)]$. Before invoking the translation property of the transform, write the function as

$$f(t) = (t-\pi)\, u_\pi(t) - (t-2\pi)\, u_{2\pi}(t) - \pi\, u_{2\pi}(t)\,.$$

It follows that

$$\mathcal{L}\left[\, f(t)\right] = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-2\pi s}}{s}.$$

17. Before invoking the translation property of the transform, write the function as

$$f(t) = (t-2)\, u_2(t) - 2\, u_2(t) - (t-3)\, u_3(t) - u_3(t)\,.$$

It follows that

$$\mathcal{L}\left[\, f(t)\right] = \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}.$$

19. Using the fact that $\mathcal{L}\left[e^{at}f(t)\right] = \mathcal{L}\left[f(t)\right]_{s \to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{3!}{(s-5)^4}\right] = t^3 e^{5t}\,.$$

22. The inverse transform of the function $2/(s^2-4)$ is $f(t) = \sinh 2t$. Using the translation property of the transform,

$$\mathcal{L}^{-1}\left[\frac{2\,e^{-4s}}{s^2-4}\right] = \sinh\left(2(t-4)\right) \cdot u_4(t)\,.$$

24. Write the function as

$$F(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{e^{-4s}}{s}\,.$$

It follows from the translation property of the transform, that

$$\mathcal{L}^{-1}\left[\frac{e^{-s}+e^{-2s}-e^{-3s}+e^{-4s}}{s}\right] = u_1(t) + u_2(t) - u_3(t) + u_4(t)\,.$$

25.(a) By definition of the Laplace transform,

$$\mathcal{L}\left[\,f(ct)\right] = \int_0^\infty e^{-st} f(ct)dt\,.$$

Making a change of variable, $\tau = ct$, we have

$$\mathcal{L}\left[\,f(ct)\right] = \frac{1}{c}\int_0^\infty e^{-s(\tau/c)} f(\tau)d\tau = \frac{1}{c}\int_0^\infty e^{-(s/c)\tau} f(\tau)d\tau\,.$$

Hence $\mathcal{L}\left[\,f(ct)\right] = (1/c)\,F(s/c)$, where $s/c > a$.

(b) Using the result in part (a),

$$\mathcal{L}\left[f\left(\frac{t}{k}\right)\right] = k\,F(ks).$$

Hence

$$\mathcal{L}^{-1}\left[F(ks)\right] = \frac{1}{k} f\left(\frac{t}{k}\right).$$

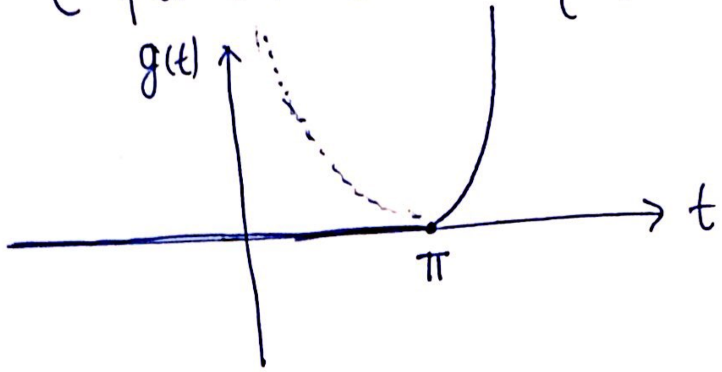
(c) From part (b), $\mathcal{L}^{-1}\left[F(as)\right] = (1/a)f(t/a)$ Note that $as+b = a(s+b/a)$. Using the fact that $\mathcal{L}\left[e^{ct}f(t)\right] = \mathcal{L}\left[f(t)\right]_{s \to s-c}$,

$$\mathcal{L}^{-1}\left[F(as+b)\right] = e^{-bt/a}\,\frac{1}{a} f\left(\frac{t}{a}\right).$$

③ $g(t) = f(t-\pi)\cdot u_\pi(t)$ where $f(t) = 3t^2$

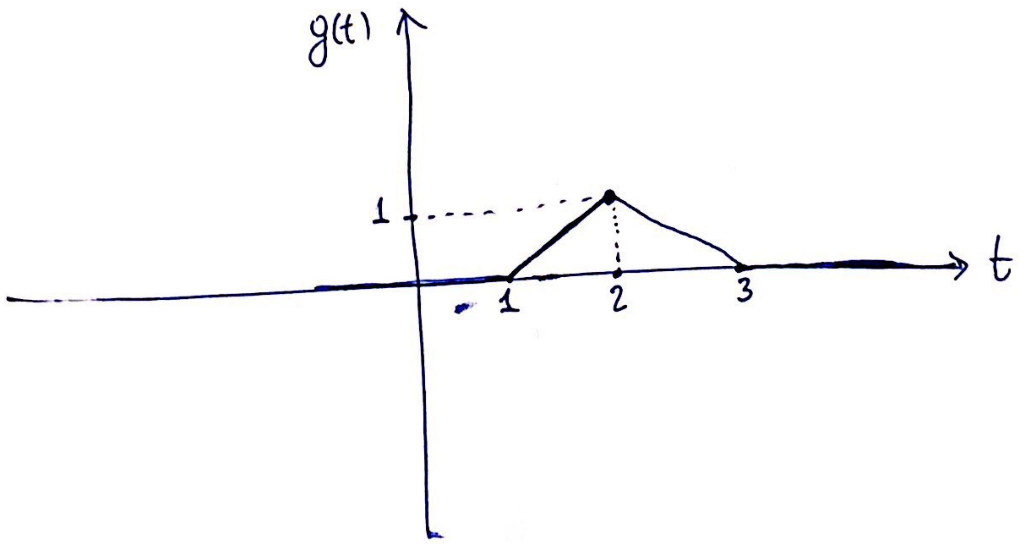
$$u_\pi(t) = \begin{cases} 0, & t<\pi \\ 1, & t\geq\pi \end{cases}$$

$$g(t) = \begin{cases} \cancel{f(t)}\cdot 0, & t<\pi \\ f(t-\pi), & t\geq\pi \end{cases} = \begin{cases} 0, & t<\pi \\ 3(t-\pi)^2, & t\geq\pi \end{cases}$$



⑥ $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

$$g(t) = \begin{cases} 0, & t<1 \\ t-1, & 1\leq t<2 \\ -t+3, & 2\leq t<3 \\ 0, & t\geq 3 \end{cases}$$



6.4

Problem 1 :

$$y'' + 9y = f(t)$$
$$y(0) = 0, \quad y'(0) = 1$$
$$f(t) = \begin{cases} 1, & 0 \le t < 3\pi \\ 0, & 3\pi \le t < \infty . \end{cases}$$

Taking the Laplace transform of both sides of the ODE, we obtain

$$\mathcal{L}(y'') = s^2 \cdot Y(s) - s\,y(0) - y'(0)$$
$$\mathcal{L}(y') = s \cdot Y(s) - y(0)$$
$$\mathcal{L}(y) = Y(s)$$

$$\mathcal{L}(y'') + 9 \cdot \mathcal{L}(y) = \mathcal{L}(f(t))$$

$$s^2 \cdot Y(s) - s \cdot \cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,1} + 9 \cdot Y(s) = \mathcal{L}(f(t))$$

$$(s^2+9) \cdot Y(s) - 1 = \frac{1}{s} - \frac{1}{s} \cdot e^{-3\pi s}$$

Since, $\mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st} dt = \int_0^{3\pi} 1 e^{-st} dt + \int_{3\pi}^{\infty} 0 \cdot e^{-st} dt$

$$= \int_0^{3\pi} e^{-st} dt = \frac{1}{s} - \frac{1}{s} e^{-3\pi s}.$$

Thus,

$$Y(s) = \frac{1}{s^2+9} + \frac{1}{s(s^2+9)} - \frac{1}{s\cdot(s^2+9)}\cdot e^{-3\pi s}$$

observe that,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right) = \frac{1}{3}\cdot \sin(3t)$$

and

$$\frac{1}{s\cdot(s^2+9)} = \frac{1}{9}\left(\frac{s}{s^2} - \frac{s}{s^2+9}\right) = \frac{1}{9}\cdot\frac{1}{s} - \frac{1}{9}\cdot\frac{s}{s^2+9}$$

So that,

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+9)}\right) = \mathcal{L}^{-1}\left(\frac{1}{9\cdot s}\right) - \mathcal{L}^{-1}\left(\frac{s}{9\cdot(s^2+9)}\right)$$

$$= \frac{1}{9}\cdot 1 - \frac{1}{9}\cdot\cos(3t)$$

And, by Theorem 6.3.1

$$\mathcal{L}^{-1}\left(\frac{e^{-3\pi s}}{s(s^2+9)}\right) = \mathcal{L}^{-1}\left(\frac{e^{-3\pi s}}{9s}\right) - \mathcal{L}^{-1}\left(\frac{s\cdot e^{-3\pi s}}{9(s^2+9)}\right)$$

$$= \left(\frac{1}{9} - \frac{1}{9}\cos(3(t-3\pi))\right)u_{3\pi} = \left(\frac{1}{9} + \frac{1}{9}\cos 3t\right)u_{3\pi}$$

Thus,

$$y(t) = \mathcal{L}^{-1}(Y(s))$$

$$= \frac{1}{3}\sin 3t + \left(\frac{1}{9} - \frac{1}{9}\cos 3t\right) + \left(\frac{1}{9} + \frac{1}{9}\cos 3t\right)u_{3\pi}$$

Problem 3 : $y'' + 4y = \sin t - u_{2\pi}(t) \cdot \sin(t-2\pi)$

$0 = y(0), \& \ y'(0) = 0$

Taking the Laplace transformation of both sides of the ODE, we obtain

$$\mathcal{L}(y'') + 4\,\mathcal{L}(y) = \mathcal{L}\left(\sin t - u_{2\pi}(t)\sin(t-2\pi)\right)$$

$$\left[s^2 \cdot Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0}\right] + 4 \cdot Y(s) = \frac{1}{s^2+1} - e^{2\pi s} \cdot \frac{1}{s^2+1}$$

$$Y(s) \cdot (s^2+4) = \frac{1}{s^2+1} - \frac{e^{2\pi s}}{s^2+1}$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)} - \frac{e^{2\pi s}}{(s^2+1)(s^2+4)}$$

observe that,

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

So that,

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)(s^2+4)}\right) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{1}{3} \cdot \sin t - \frac{1}{6} \cdot \sin 2t$$

and,

$$\mathcal{L}^{-1}\left(\frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}\right) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{e^{-2\pi s}}{s^2+1}\right) - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{e^{-2\pi s}}{s^2+4}\right)$$

$$= \frac{1}{3} u_{2\pi}(t)\cdot\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3} u_{2\pi}(t)\cdot\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{1}{3}\sin(t-2\pi)u_{2\pi}(t) - \frac{1}{3}u_{2\pi}(t)\frac{1}{2}\sin 2(t-2\pi)$$

$$= \left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right)u_{2\pi}(t)$$

Thus,

$$y(t) = \left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right) + \left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right)u_{2\pi}(t)$$

$$= \frac{1}{6}\left(2\sin t - \sin 2t\right) + \frac{1}{6}\left(2\sin t - \sin 2t\right)u_{2\pi}(t)$$

OR,

$$y(t) = \frac{1}{6}\left(1-u_{2\pi}(t)\right)\cdot\left(2\sin t - \sin 2t\right) \;//$$

5.(a) Let $f(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) + 3\,[s\,Y(s) - \,y(0)] + 2\,Y(s) = \mathcal{L}\,[f(t)]\,.$$

Applying the initial conditions,

$$s^2 Y(s) + 3s\,Y(s) + 2\,Y(s) - s - 3 = \mathcal{L}\,[f(t)]\,.$$

The transform of the forcing function is

$$\mathcal{L}\,[f(t)] = \frac{1}{s} - \frac{e^{-10s}}{s}\,.$$

Solving for the transform,

$$Y(s) = \frac{s+3}{s^2+3s+2} + \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right], \quad \frac{s+3}{s^2+3s+2} = \frac{2}{s+1} - \frac{1}{s+2}.$$

Hence

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+3s+2)}\right] = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}\,.$$

Based on Theorem 6.3.1,

$$\mathcal{L}^{-1}\left[\frac{e^{-10s}}{s(s^2+3s+2)}\right] = \frac{1}{2}\left[1 + e^{-2(t-10)} - 2e^{-(t-10)}\right] u_{10}(t)\,.$$

Hence the solution of the IVP is

$$y(t) = 2e^{-t} - e^{-2t} + \frac{1}{2}\,[1 - u_{10}(t)] + \frac{e^{-2t}}{2} - e^{-t} - \frac{1}{2}\left[e^{-(2t-20)} - 2e^{-(t-10)}\right] u_{10}(t)\,.$$

7.(a) Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2 Y(s) + Y(s) - 2s = \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{2s}{s^2+1} + \frac{e^{-3\pi s}}{s(s^2+1)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

Hence

$$Y(s) = \frac{2s}{s^2+1} + e^{-3\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]\,.$$

Taking the inverse transform, the solution of the IVP is

$$y(t) = 2\cos t + [1 - \cos(t - 3\pi)]\,u_{3\pi}(t) = 2\cos t + [1 + \cos t\,]\,u_{3\pi}(t)\,.$$

11.(a) Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) + 4Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2 Y(s) + 4Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-\pi s}}{s(s^2+4)} - \frac{e^{-3\pi s}}{s(s^2+4)}.$$

Using partial fractions,

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right].$$

Taking the inverse transform, and applying Theorem 6.3.1,

$$\begin{aligned} y(t) &= \frac{1}{4}\left[1-\cos(2t-2\pi)\right]u_\pi(t) - \frac{1}{4}\left[1-\cos(2t-6\pi)\right]u_{3\pi}(t) \\ &= \frac{1}{4}\left[u_\pi(t)-u_{3\pi}(t)\right] - \frac{1}{4}\cos 2t \cdot \left[u_\pi(t)-u_{3\pi}(t)\right]. \end{aligned}$$