REVIEW QUESTIONS for FINAL

6.3

15. The function can be expressed as

$$f(t) = (t - \pi)\left[u_\pi(t) - u_{2\pi}(t)\right].$$

Before invoking the translation property of the transform, write the function as

$$f(t) = (t - \pi)\, u_\pi(t) - (t - 2\pi)\, u_{2\pi}(t) - \pi\, u_{2\pi}(t).$$

It follows that

$$\mathcal{L}\left[\, f(t)\right] = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-2\pi s}}{s}.$$

21. First consider the function

$$G(s) = \frac{2(s-1)}{s^2 - 2s + 2}.$$

Completing the square in the denominator,

$$G(s) = \frac{2(s-1)}{(s-1)^2 + 1}.$$

It follows that

$$\mathcal{L}^{-1}\left[G(s)\right] = 2\, e^t \cos t\,.$$

Hence

$$\mathcal{L}^{-1}\left[e^{-2s} G(s)\right] = 2\, e^{t-2} \cos(t-2)\, u_2(t).$$

6.4.

2.(a) Let $h(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\, Y(s) - s\, y(0) - y'(0) + 2\left[s\, Y(s) - y(0)\right] + 2\, Y(s) = \mathcal{L}\left[h(t)\right].$$

Applying the initial conditions,

$$s^2\, Y(s) + 2s\, Y(s) + 2\, Y(s) - 1 = \mathcal{L}\left[h(t)\right].$$

The forcing function can be written as $h(t) = 2(u_\pi(t) - u_{2\pi}(t))$. Its transform is

$$\mathcal{L}\left[h(t)\right] = \frac{2(e^{-\pi s} - e^{-2\pi s})}{s}.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{s^2 + 2s + 2} + \frac{2(e^{-\pi s} - e^{-2\pi s})}{s(s^2 + 2s + 2)}.$$

First note that

$$\frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1}.$$

Using partial fractions,

$$\frac{2}{s(s^2+2s+2)} = \frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+1}\,.$$

Taking the inverse transform, term-by-term,

$$\mathcal{L}\left[\frac{1}{s^2+2s+2}\right] = \mathcal{L}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}\sin t\,.$$

Now let

$$G(s) = \frac{2}{s(s^2+2s+2)}\,.$$

Then

$$\mathcal{L}^{-1}\left[G(s)\right] = 1 - e^{-t}\cos t - e^{-t}\sin t\,.$$

Using Theorem 6.3.1,

$$\mathcal{L}^{-1}\left[e^{-cs}G(s)\right] = u_c(t) - e^{-(t-c)}\left[\cos(t-c) + \sin(t-c)\right]u_c(t)\,.$$

Hence the solution of the IVP is

$$\begin{aligned} y(t) &= e^{-t}\sin t + u_\pi(t) - e^{-(t-\pi)}\left[\cos(t-\pi) + \sin(t-\pi)\right]u_\pi(t) - \\ &\quad - u_{2\pi}(t) + e^{-(t-2\pi)}\left[\cos(t-2\pi) + \sin(t-2\pi)\right]u_{2\pi}(t)\,. \end{aligned}$$

That is,

$$\begin{aligned} y(t) &= e^{-t}\sin t + \left[u_\pi(t) - u_{2\pi}(t)\right] + e^{-(t-\pi)}\left[\cos t + \sin t\right]u_\pi(t) + \\ &\quad + e^{-(t-2\pi)}\left[\cos t + \sin t\right]u_{2\pi}(t)\,. \end{aligned}$$

12.(a) Taking the Laplace transform of the ODE, we obtain

$$s^4Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) +$$
$$+5\left[s^2Y(s) - s\,y(0) - y'(0)\right] + 4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

Applying the initial conditions,

$$s^4Y(s) + 5s^2Y(s) + 4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s(s^4 + 5s^2 + 4)} - \frac{e^{-\pi s}}{s(s^4 + 5s^2 + 4)}.$$

Using partial fractions,

$$\frac{1}{s(s^4 + 5s^2 + 4)} = \frac{1}{12}\left[\frac{3}{s} + \frac{s}{s^2+4} - \frac{4s}{s^2+1}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^4 + 5s^2 + 4)}\right] = \frac{1}{12}\left[3 + \cos 2t - 4\cos t\right].$$

Based on Theorem 6.3.1, the solution of the IVP is

$$y(t) = \frac{1}{4}\left[1 - u_\pi(t)\right] + \frac{1}{12}\left[\cos 2t - 4\cos t\right] -$$
$$-\frac{1}{12}\left[\cos 2(t-\pi) - 4\cos(t-\pi)\right]u_\pi(t).$$

That is,

$$y(t) = \frac{1}{4}\left[1 - u_\pi(t)\right] + \frac{1}{12}\left[\cos 2t - 4\cos t\right] -$$
$$-\frac{1}{12}\left[\cos 2t + 4\cos t\right]u_\pi(t).$$

6.6

6. We have $\mathcal{L}\left[e^{-t}\right] = 1/(s+1)$ and $\mathcal{L}\left[\sin t\right] = 1/(s^2+1)$. Based on Theorem 6.6.1,

$$\mathcal{L}\left[\int_0^t e^{-(t-\tau)}\sin(\tau)\,d\tau\right] = \frac{1}{s+1}\cdot\frac{1}{s^2+1} = \frac{1}{(s+1)(s^2+1)}.$$

13. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) - 1 + \omega^2 Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2 + \omega^2} + \frac{G(s)}{s^2 + \omega^2}.$$

As shown in a related situation, Problem 11,

$$\mathcal{L}^{-1}\left[\frac{G(s)}{s^2 + \omega^2}\right] = \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))\, g(\tau)\, d\tau\,.$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{\omega}\sin(\omega t) + \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))\, g(\tau)\, d\tau\,.$$

7.1

3. First divide both sides of the equation by t^2, and write

$$u'' = -\frac{1}{t}u' - (1 - \frac{1}{t^2})u\,.$$

Set $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$x_2' = u'' = -\frac{1}{t}u' - (1 - \frac{1}{t^2})u\,.$$

We obtain the system of equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -(1 - \frac{1}{t^2})x_1 - \frac{1}{t}x_2\,. \end{aligned}$$

7.2

11. First augment the given matrix by the identity matrix:

$$[\mathbf{A}\,|\,\mathbf{I}] = \begin{pmatrix} 3 & -1 & 1 & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.$$

Divide the first row by 3, to obtain

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.$$

Adding -6 times the first row to the second row results in

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 0 & 4 & -2 & 1 \end{pmatrix}.$$

Divide the second row by 4 , to obtain

$$\begin{pmatrix}1 & -1/3 & 1/3 & 0\\ 0 & 1 & -1/2 & 1/4\end{pmatrix}.$$

Finally, adding 1/3 times the second row to the first row results in

$$\begin{pmatrix}1 & 0 & 1/6 & 1/12\\ 0 & 1 & -1/2 & 1/4\end{pmatrix}.$$

Hence

$$\begin{pmatrix}3 & -1\\ 6 & 2\end{pmatrix}^{-1} = \frac{1}{12}\begin{pmatrix}2 & 1\\ -6 & 3\end{pmatrix}.$$

13. Elementary row operations yield

$$\begin{pmatrix}2 & 1 & 0 & 1 & 0 & 0\\ 0 & 2 & 1 & 0 & 1 & 0\\ 0 & 0 & 2 & 0 & 0 & 1\end{pmatrix} \to \begin{pmatrix}1 & 1/2 & 0 & 1/2 & 0 & 0\\ 0 & 1 & 1/2 & 0 & 1/2 & 0\\ 0 & 0 & 1 & 0 & 0 & 1/2\end{pmatrix} \to$$
$$\begin{pmatrix}1 & 0 & -1/4 & 1/2 & -1/4 & 0\\ 0 & 1 & 0 & 0 & 1/2 & -1/4\\ 0 & 0 & 1 & 0 & 0 & 1/2\end{pmatrix} \to \begin{pmatrix}1 & 0 & -1/4 & 1/2 & -1/4 & 0\\ 0 & 1 & 0 & 0 & 1/2 & -1/4\\ 0 & 0 & 1 & 0 & 0 & 1/2\end{pmatrix}$$

Finally, combining the first and third rows results in

$$\begin{pmatrix}1 & 0 & 0 & 1/2 & -1/4 & 1/8\\ 0 & 1 & 0 & 0 & 1/2 & -1/4\\ 0 & 0 & 1 & 0 & 0 & 1/2\end{pmatrix}, \text{ so } A^{-1} = \begin{pmatrix}1/2 & -1/4 & 1/8\\ 0 & 1/2 & -1/4\\ 0 & 0 & 1/2\end{pmatrix}.$$

7.3

4. The augmented matrix is

$$\left(\begin{array}{ccc|c}1 & 2 & -1 & 0\\ 2 & 1 & 1 & 0\\ 1 & -1 & 2 & 0\end{array}\right).$$

Adding -2 times the first row to the second row and subtracting the first row from the third row results in

$$\left(\begin{array}{ccc|c}1 & 2 & -1 & 0\\ 0 & -3 & 3 & 0\\ 0 & -3 & 3 & 0\end{array}\right).$$

Adding the negative of the second row to the third row results in

$$\left(\begin{array}{ccc|c}1 & 2 & -1 & 0\\ 0 & -3 & 3 & 0\\ 0 & 0 & 0 & 0\end{array}\right).$$

We evidently end up with an equivalent system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + x_3 &= 0\,. \end{aligned}$$

Since there is no unique solution, let $x_3 = \alpha$, where α is arbitrary. It follows that $x_2 = \alpha$, and $x_1 = -\alpha$. Hence all solutions have the form

$$x = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

8. Write the given vectors as columns of the matrix

$$\mathbf{X} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is evident that $\det(\mathbf{X}) = 0$. Hence the vectors are linearly dependent. In order to find a linear relationship between them, write $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$. The latter equation is equivalent to

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Performing elementary row operations,

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We obtain the system of equations

$$\begin{aligned} c_1 - 3c_3 &= 0 \\ c_2 + 5c_3 &= 0\,. \end{aligned}$$

Setting $c_3 = 1$, it follows that $c_1 = 3$ and $c_2 = -5$. Hence

$$3\mathbf{x}^{(1)} - 5\mathbf{x}^{(2)} + \mathbf{x}^{(3)} = \mathbf{0}\,.$$

7.5

3.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + r - 6 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = -3$. For $r = 2$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. Substitution of $r = -3$ results in the single equation $4\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1, -4)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}.$$

12. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^3 - 4r^2 - r + 4 = 0$. The roots of the characteristic equation are $r_1 = 4$, $r_2 = 1$ and $r_3 = -1$. Setting $r = 4$, we have

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 - \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (1\,,1\,,1)^T$. Setting $\lambda = 1$, the reduced system of equations is

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(2)} = (1\,,-2\,,1)^T$. Finally, setting $\lambda = -1$, the reduced system of equations is

$$\begin{aligned} \xi_1 + \xi_3 &= 0 \\ \xi_2 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(3)} = (1\,,0\,,-1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}.$$

15. Setting $\mathbf{x}=\boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A}-r\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 4 \\ c_1 + 3\,c_2 &= -2\,. \end{aligned}$$

Hence $c_1 = 7$ and $c_2 = -3$, and the solution of the IVP is

$$\mathbf{x} = 7\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - 3\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

7.6

2.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A}-r\mathbf{I}) = r^2 + 1 = 0$. The roots of the characteristic equation are $r = \pm i$. Setting $r = i$, the equations are equivalent to $\xi_1 - (2+i)\xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (2+i\,,1)^T$ and $\boldsymbol{\xi}^{(2)} = (2-i\,,1)^T$. Hence one of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}(\cos t + i \sin t) =$$

$$= \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}.$$

Therefore the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}.$$

The solution may also be written as

$$\mathbf{x} = c_1\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2\begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}.$$

10. Solution of the system of ODEs requires that

$$\begin{pmatrix} -3-r & 2 \\ -1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 4r + 5 = 0$, with roots $r = -2 \pm i$. Substituting $r = -2 + i$, the equations are equivalent to $\xi_1 - (1-i)\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1-i, 1)^T$. One of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{(-2+i)t} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{-2t}(\cos t + i \sin t) =$$

$$= e^{-2t}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + ie^{-2t}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.$$

Hence the general solution is

$$\mathbf{x} = c_1 e^{-2t}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 - c_2 &= 2 \\ c_1 &= -4\,. \end{aligned}$$

Solving for the coefficients, the solution of the initial value problem is

$$\begin{aligned} \mathbf{x} &= -4e^{-2t}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} - 6e^{-2t}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix} \\ &= 2e^{-2t}\begin{pmatrix} \cos t - 5 \sin t \\ -2\cos t - 3\sin t \end{pmatrix}. \end{aligned}$$

7.7

3.(a,b) The general solution, found in Problem 3, Section 7.6, is given by

$$\mathbf{x} = c_1\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2\begin{pmatrix} 5 \sin t \\ \cos t + 2\sin t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} 5c_1 &= 1 \\ 2c_1 - c_2 &= 0\,, \end{aligned}$$

resulting in $c_1 = 1/5$, $c_2 = 2/5$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$5c_1 = 0$$
$$2c_1 - c_2 = 1\,,$$

resulting in $c_1 = 0$, $c_2 = -1$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -5 \sin t \\ \cos t - 2 \sin t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\Phi(t) = \begin{pmatrix} \cos t + 2 \sin t & -5 \sin t \\ \sin t & \cos t - 2 \sin t \end{pmatrix}.$$

7.8

2 (c) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} -3 - r & \frac{5}{2} \\ -\frac{5}{2} & 2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + r + \frac{1}{4} = 0$. The only root is $r = -1/2$, which is an eigenvalue of multiplicity two. Setting $r = -1/2$ is the coefficient matrix reduces the system to the single equation $-\xi_1 + \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2}.$$

In order to obtain a second linearly independent solution, we find a solution of the system

$$\begin{pmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

There equations reduce to $-5\eta_1 + 5\eta_2 = 2$. Set $\eta_1 = k$, some arbitrary constant. Then $\eta_2 = k + 2/5$. A second solution is

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t/2} + \begin{pmatrix} k \\ k + 2/5 \end{pmatrix} e^{-t/2} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t/2} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} e^{-t/2} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2}. \end{aligned}$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t/2} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} e^{-t/2} \right].$$

7.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{5}{2}-r & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2}-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2\,r + 1 = 0$, with a single root $r = -1$. Setting $r = -1$, the two equations reduce to $-\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (1\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

A second linearly independent solution is obtained by solving the system

$$\begin{pmatrix} -3/2 & 3/2 \\ -3/2 & 3/2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The equations reduce to the single equation $-3\eta_1 + 3\eta_2 = 2$. Let $\eta_1 = k$. We obtain $\eta_2 = 2/3 + k$, and a second linearly independent solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} k \\ 2/3+k \end{pmatrix} e^{-t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t}\right].$$

Imposing the initial conditions, find that

$$c_1 = 3$$
$$c_1 + \frac{2}{3}c_2 = -1\,,$$

so that $c_1 = 3$ and $c_2 = -6$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-t} - \begin{pmatrix} 6 \\ 6 \end{pmatrix} te^{-t}.$$

7.9

10. Since the coefficient matrix is symmetric, the differential equations can be decoupled. The eigenvalues and eigenvectors are given by

$$r_1 = -4\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} \text{ and } r_2 = -1\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

Using the normalized eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix} \qquad \mathbf{T}^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix}.$$

Setting $\mathbf{x}$=$\mathbf{Ty}$, and $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$y_1' = -4y_1 + \frac{1}{\sqrt{3}}(1+\sqrt{2})e^{-t}$$
$$y_2' = -y_2 + \frac{1}{\sqrt{3}}(1-\sqrt{2})\, e^{-t}.$$

The solutions are easily obtained as

$$y_1(t) = k_1 e^{-4t} + \frac{1}{3\sqrt{3}}(1+\sqrt{2})e^{-t}, \qquad y_2(t) = k_2 e^{-t} + \frac{1}{\sqrt{3}}(1-\sqrt{2})te^{-t}.$$

Transforming back to the original variables, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}e^{-4t} + c_2\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}e^{-t} + \frac{1}{9}\begin{pmatrix} 2+\sqrt{2}+3\sqrt{3} \\ 3\sqrt{6}-\sqrt{2}-1 \end{pmatrix}e^{-t} + \frac{1}{3}\begin{pmatrix} 1-\sqrt{2} \\ \sqrt{2}-2 \end{pmatrix}te^{-t}.$$

Note that

$$\begin{pmatrix} 2+\sqrt{2}+3\sqrt{3} \\ 3\sqrt{6}-\sqrt{2}-1 \end{pmatrix} = \begin{pmatrix} 2+\sqrt{2} \\ -\sqrt{2}-1 \end{pmatrix} + 3\sqrt{3}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

The second vector is an eigenvector, hence the solution may be written as

$$\mathbf{x} = c_1\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}e^{-4t} + c_2\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}e^{-t} + \frac{1}{9}\begin{pmatrix} 2+\sqrt{2} \\ -\sqrt{2}-1 \end{pmatrix}e^{-t} + \frac{1}{3}\begin{pmatrix} 1-\sqrt{2} \\ \sqrt{2}-2 \end{pmatrix}te^{-t}.$$

11. Based on the solution of Problem 3 of Section 7.6, a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} \cos t & \sin t \\ -2\cos t - \sin t & \cos t - 2\sin t \end{pmatrix}.$$

The inverse of the fundamental matrix is easily determined as

$$\mathbf{\Psi}^{-1}(t) = \begin{pmatrix} \cos t - 2\sin t & -\sin t \\ 2\cos t + \sin t & \cos t \end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \begin{pmatrix} -\cos t \sin t \\ \cos^2 t \end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix} \frac{1}{2}\cos^2 t \\ \frac{1}{2}\cos t \sin t + \frac{1}{2}t \end{pmatrix}.$$

A particular solution is constructed as

$$\mathbf{v}(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt,$$

where

$$v_1(t) = \frac{1}{2}(\cos t + t\sin t), \qquad v_2(t) = -\cos t + \frac{1}{2}t\cos t - t\sin t.$$

Hence the general solution is

$$\begin{aligned}\mathbf{x} &= c_1\begin{pmatrix} \cos t \\ -2\cos t - \sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix} + \\ &+ t\sin t\begin{pmatrix} 1/2 \\ -1 \end{pmatrix} + t\cos t\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} + \cos t\begin{pmatrix} 1/2 \\ -1 \end{pmatrix}.\end{aligned}$$

13.(a) As shown in Problem 25 of Section 7.6, the solution of the homogeneous system is

$$\begin{pmatrix} x_1^{(c)} \\ x_2^{(c)} \end{pmatrix} = c_1 e^{-t/2}\begin{pmatrix} \cos(t/2) \\ 4\sin(t/2) \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} \sin(t/2) \\ -4\cos(t/2) \end{pmatrix}.$$

Therefore the associated fundamental matrix is given by

$$\mathbf{\Psi}(t) = e^{-t/2}\begin{pmatrix} \cos(t/2) & \sin(t/2) \\ 4\sin(t/2) & -4\cos(t/2) \end{pmatrix}.$$

(b) The inverse of the fundamental matrix is

$$\mathbf{\Psi}^{-1}(t) = \frac{e^{t/2}}{4}\begin{pmatrix} 4\cos(t/2) & \sin(t/2) \\ 4\sin(t/2) & -\cos(t/2) \end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \frac{1}{2}\begin{pmatrix} \cos(t/2) \\ \sin(t/2) \end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix} \sin(t/2) \\ -\cos(t/2) \end{pmatrix}.$$

A particular solution is constructed as

$$\mathbf{v}(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt,$$

where $v_1(t) = 0$, $v_2(t) = 4\,e^{-t/2}$. Hence the general solution is

$$\mathbf{x} = c_1 e^{-t/2}\begin{pmatrix}\cos(t/2)\\ 4\,\sin(t/2)\end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix}\sin(t/2)\\ -4\,\cos(t/2)\end{pmatrix} + 4\,e^{-t/2}\begin{pmatrix}0\\1\end{pmatrix}.$$

Imposing the initial conditions, we require that $c_1 = 0$, $-4c_2 + 4 = 0$, which results in $c_1 = 0$ and $c_2 = 1$. Therefore the solution of the IVP is

$$\mathbf{x} = e^{-t/2}\begin{pmatrix}\sin(t/2)\\ 4 - 4\,\cos(t/2)\end{pmatrix}.$$

16. Based on the hypotheses,

$$\phi'(t) = \mathbf{P}(t)\phi(t) + \mathbf{g}(t) \text{ and } \mathbf{v}'(t) = \mathbf{P}(t)\mathbf{v}(t) + \mathbf{g}(t)\,.$$

Subtracting the two equations results in

$$\phi'(t) - \mathbf{v}'(t) = \mathbf{P}(t)\phi(t) - \mathbf{P}(t)\mathbf{v}(t)\,,$$

that is,

$$[\phi(t) - \mathbf{v}(t)]' = \mathbf{P}(t)\,[\phi(t) - \mathbf{v}(t)]\,.$$

It follows that $\phi(t) - \mathbf{v}(t)$ is a solution of the homogeneous equation. According to Theorem 7.4.2,

$$\phi(t) - \mathbf{v}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t).$$

Hence

$$\phi(t) = \mathbf{u}(t) + \mathbf{v}(t),$$

in which $\mathbf{u}(t)$ is the general solution of the homogeneous problem.

17.(a) Setting $t_0 = 0$ in Equation (34),

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0 + \mathbf{\Phi}(t)\int_0^t \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds = \mathbf{\Phi}(t)\mathbf{x}^0 + \int_0^t \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds\,.$$

It was shown in Problem 15(c) in Section 7.7 that $\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s) = \mathbf{\Phi}(t-s)$. Therefore

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0 + \int_0^t \mathbf{\Phi}(t-s)\mathbf{g}(s)ds\,.$$

(b) The principal fundamental matrix is identified as $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$. Hence

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}^0 + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{g}(s)ds\,.$$

In Problem 27 of Section 3.6, the particular solution is given as

$$y(t) = \int_{t_0}^t K(t-s)g(s)ds\,,$$

in which the kernel $K(t)$ depends on the nature of the fundamental solutions.