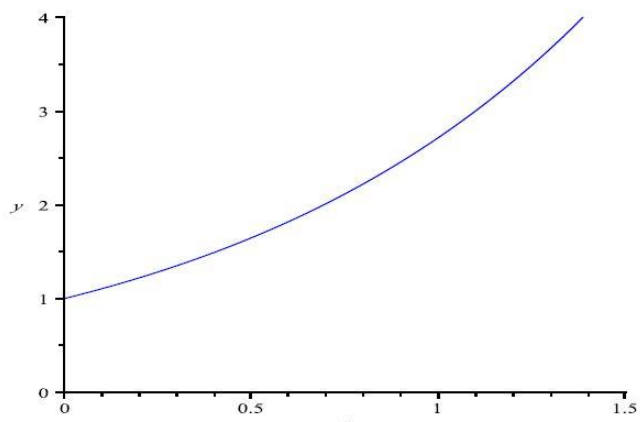
3.1

2. Let $y = e^{rt}$. Substitution of the assumed solution results in the characteristic equation $r^2 + 5r + 6 = 0$. The roots of the equation are $r = -3, -2$. Hence the general solution is $y = c_1e^{-2t} + c_2e^{-3t}$.

4. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $3r^2 - 4r + 1 = 0$. The roots of the equation are $r = 1/3, 1$. Hence the general solution is $y = c_1e^{t/3} + c_2e^{t}$.

6. The characteristic equation is $9r^2 - 16 = 0$, with roots $r = \pm 4/3$. Therefore the general solution is $y = c_1e^{-4t/3} + c_2e^{4t/3}$.

9. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + 2r - 3 = 0$. The roots of the equation are $r = -3, 1$. Hence the general solution is $y = c_1e^{-3t} + c_2e^{t}$. Its derivative is $y' = -3c_1e^{-3t} + c_2e^{t}$. Based on the first condition, $y(0) = 1$, we require that $c_1 + c_2 = 1$. In order to satisfy $y'(0) = 1$, we find that $-3c_1 + c_2 = 1$. Solving for the constants, $c_1 = 0$ and $c_2 = 1$. Hence the specific solution is $y(t) = e^{t}$. It clearly increases without bound as $t \to \infty$.



14. The characteristic equation is $2r^2 + r - 4 = 0$, with roots $r = (-1 \pm \sqrt{33})/4$. The general solution is $y = c_1 e^{(-1-\sqrt{33})t/4} + c_2 e^{(-1+\sqrt{33})t/4}$, with derivative

$$y' = \frac{-1-\sqrt{33}}{4}\, c_1 e^{(-1-\sqrt{33})t/4} + \frac{-1+\sqrt{33}}{4}\, c_2 e^{(-1+\sqrt{33})t/4}.$$

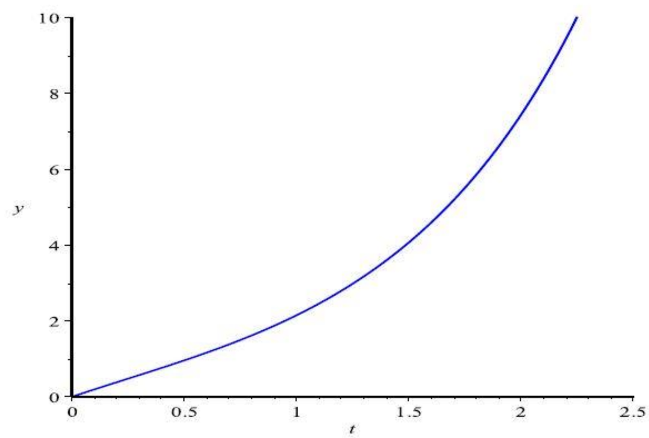
In order to satisfy the initial conditions, we require that

$$c_1 + c_2 = 0 \quad \text{and} \quad \frac{-1-\sqrt{33}}{4}\, c_1 + \frac{-1+\sqrt{33}}{4}\, c_2 = 2.$$

Solving for the coefficients, $c_1 = -4/\sqrt{33}$ and $c_2 = 4/\sqrt{33}$. The specific solution is

$$y(t) = -4\left[e^{(-1-\sqrt{33})t/4} - e^{(-1+\sqrt{33})t/4}\right]/\sqrt{33}.$$

It clearly increases without bound as $t \to \infty$.



22. The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Hence the general solution is $y = c_1 e^{-t/2} + c_2 e^{t/2}$ and $y' = -c_1 e^{-t/2}/2 + c_2 e^{t/2}/2$. Invoking the initial conditions, we require that $c_1 + c_2 = 2$ and $-c_1 + c_2 = 2\beta$. The specific solution is $y(t) = (1-\beta)e^{-t/2} + (1+\beta)e^{t/2}$. Based on the form of the solution, it is evident that as $t \to \infty$, $y(t) \to 0$ as long as $\beta = -1$.

3.2

3.

$$W(e^{-3t}, t\,e^{-3t}) = \begin{vmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & (1-3t)e^{-3t} \end{vmatrix} = e^{-6t}.$$

9. Write the equation as $y'' + (3/(t-4))y' + (5/t(t-4))y = 2/t(t-4)$. The coefficients are not continuous at $t = 0$ and $t = 4$. Since $t_0 \in (0, 4)$, the largest interval is $0 < t < 4$.

10. The coefficient $3\ln|t|$ is discontinuous at $t = 0$. Since $t_0 > 0$, the largest interval of existence is $0 < t < \infty$.

16. No. Substituting $y = \sin(t^2)$ into the differential equation,

$$-4t^2 \sin(t^2) + 2\cos(t^2) + 2t \cos(t^2)p(t) + \sin(t^2)q(t) = 0.$$

At $t = 0$, this equation becomes $2 = 0$ (if we suppose that $p(t)$ and $q(t)$ are continuous), which is impossible.

20. $W(f, g) = fg' - f'g = t\cos t - \sin t$, and $W(u, v) = -5fg' + 5f'g$. Hence $W(u, v) = -5t\cos t + 5\sin t$.

25. Clearly, $y_1 = e^{2t}$ is a solution. $y_2' = (1+2t)e^{2t}$, $y_2'' = (4+4t)e^{2t}$. Substitution into the ODE results in $(4+4t)e^{2t} - 4(1+2t)e^{2t} + 4t\,e^{2t} = 0$. Furthermore, $W(e^{2t}, te^{2t}) = e^{4t}$. Hence the solutions form a fundamental set of solutions.

35. The Wronskian associated with the solutions of the differential equation is given by $W(t) = c\,e^{-\int -2/t^2\,dt} = c\,e^{-2/t}$. Since $W(2) = 3$, it follows that for the hypothesized set of solutions, $c = 3\,e$. Hence $W(6) = 3e^{2/3}$.

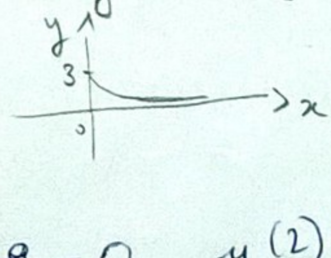
3.2.38. $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = 0$ at some point in I because y_1 and y_2 are zero at the same point in I. Hence, from Theorem 3.2.3 they cannot be a fundamental set of solutions on I.

Math 204 - HW #3.

Find the solution of the given initial value pb. Sketch the graph of the solution and describe its behaviour as t increases.

3.1, 10) $y'' + 4y' + 3y = 0, \quad y(0) = 3, \quad y'(0) = -1.$

Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + 4r + 3 = 0$. The roots of the equation $r = -3, -1$. Hence the general solution is $y = c_1 e^{-3t} + c_2 e^{-t}$. Its derivative is $y' = -3c_1 e^{-3t} - c_2 e^{-t}$.
Based on the first condition $y(0) = 3$, we require that $c_1 + c_2 = 3$. In order to satisfy $y'(0) = 1$, we find that $-3c_1 - c_2 = -1$. Solving for the constants, $c_1 = -1$ and $c_2 = 4$.
Hence the specific solution is $y(t) = -e^{-3t} + 4e^{-t}$.
The solution clearly converges to 0 as $t \to \infty$.



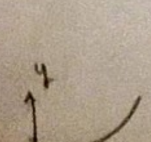
3.1.15) $y'' + 8y' - 9y = 0, \quad y(2) = 1, \quad y'(2) = 0.$

Assumed solution: $y(t) = e^{rt}$.
Characteristic eq: $r^2 + 8r - 9 = 0$, roots: $r = -9, 1$.
General solution: $y = c_1 e^{-9t} + c_2 e^{t}, \quad y' = -9c_1 e^{-9t} + c_2 e^{t}$.

$$y(2) = 1 \Rightarrow c_1 e^{-18} + c_2 e^{2} = 1$$
$$y'(2) = 0 \Rightarrow -9c_1 e^{-18} + c_2 e^{2} = 0$$
$$\Rightarrow c_1 = \frac{e^{18}}{10}, \quad c_2 = e^{-2} \cdot \frac{9}{10}.$$

Specific solution: $y(t) = \frac{e^{18}}{10} e^{-9t} + \frac{9}{10} e^{-2} e^{t}$

$$= \frac{1}{10} e^{18-9t} + \frac{9}{10} \cdot e^{-2+t} = \frac{1}{10} e^{-9(t-2)} + \frac{9}{10} e^{t-2}.$$



$y \to \infty$ as $t \to \infty$.

3.2, 2) Find the Wronskian of the given pair of functions.

cost, sint.

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t - (-\sin^2 t) = 1.$$

3.2, 8) Determine the longest interval in which the given initial value pb. is certain to have a unique twice-differentiable solution. Do not attempt to find the solut°.

$$(t-1)y'' - 3ty' + 5y = \sin t, \quad \begin{cases} y(-3) = 2 \\ y'(-3) = 1. \end{cases}$$

write the equation as

$$y'' - \frac{3t}{(t-1)} y' + \frac{5}{(t-1)} y = \frac{\sin t}{(t-1)}.$$

The coefficients are not continuous at $t = 1$.

Since $t_0 < 0$, the IVP has a unique solution for all t such that $-\infty < t < 1$.

3.2.22) Find the fundamental set of solutions specified by the theorem 3.25 for the given differential eq. and initial point.

$$y'' + 2y' - 3y = 0, \quad t_0 = 0.$$

$r^2 + 2r - 3 = 0$, $r = -3, 1 \Rightarrow$ The general solution: $y = c_1 e^{-3t} + c_2 e^{t}$.

$W(e^{-3t}, e^{t}) = 4e^{-2t}$, and hence the exponentials form a fundamental set of solutions. (which must satisfy the conditions $y_1(1) = 1$, $y_1'(1) = 0$, $y_2(1) = 0$, $y_2'(1) = 1$.

for y_1: $\left.\begin{matrix} c_1 + c_2 = 1 \\ -3c_1 + c_2 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} c_1 = 1/4 \\ c_2 = 3/4 \end{matrix}$ for y_2: $\left.\begin{matrix} c_1 + c_2 = 0 \\ -3c_1 + c_2 = 1 \end{matrix}\right\} \Rightarrow \begin{matrix} c_1 = -1/4 \\ c_2 = 1/4 \end{matrix}$

Hence the fundamental solutions are $y_1 = \frac{e^{-3t}}{4} + \frac{3}{4}e^{t}$, $y_2 = -\frac{e^{-3t}}{4} + \frac{e^{t}}{4}$.

3.2, 29) Find the Wronskian of two solutions of the given differential equation without solving the equation.

$$t^2 y'' - t(t+2)y' + (t+2)y = 0.$$

Writing the equation in standard form, we have

$P(x) = \dfrac{-t(t+2)}{t^2} = \dfrac{-(t+2)}{t}$ Hence the wronskian is

$$w(t) = c.\exp\left(-\int -\frac{(t+2)}{t}\,dt\right) = c.\exp\left(2\ln|x| + x + c_1\right)$$

$$= c\,x^2 e^x . e^{c_1}$$

$$= c_2 x^2 e^x$$