Math 204 - Differential Equations

Midterm 2 April 19, 2016

Duration: 90 minutes

Instructions: Calculators are not allowed. No books, no notes, no questions, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. If necessary, you can use the back of these pages, but make sure you have indicated doing so. **Print (i.e., use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: ANSWER KEY

Signature: ______________________

Section (Check One):

Section 1: E. Ceyhan (Tue-Thu 10:00) ______
Section 2: E. Ceyhan (Tue-Thu 08:30) ______
Section 3: H. Göral (Mon-Wed 16:00) ______

Question	Points	Score
1	17	
2	15	
3	15	
4	20	
5	18	
6	20	
Total	**105**	

1. (17 points) Find power series solutions of

$$y'' + x^2 y' + 3y = 0$$

around the ordinary point $x_0 = 0$. Find a fundamental set of solutions y_1 and y_2.

Let $y = \sum_{n=0}^{\infty} a_n \cdot x^n$ be a solution of the differential eq.

Note that $y' = \sum_{n=0}^{\infty} (n+1) \cdot a_{n+1} \cdot x^n$ and

$y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$, thus we have

$$0 = y'' + x^2 \cdot y' + 3y = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \underbrace{\sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+2}}_{\text{interchange}} + \sum_{n=0}^{\infty} 3a_n x^n$$

$$= \underbrace{\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n}_{\text{seperate } n=0,1 \text{ terms}} + \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n + \underbrace{\sum_{n=0}^{\infty} 3a_n x^n}_{\text{seperate } n=0,1 \text{ terms}}$$

$$= (2a_2 + \cancel{x}6a_3 + 3a_0 + 3a_1\cancel{x}) + \sum_{n=2}^{\infty} \left[(n+1)(n+2) a_{n+2} + (n-1) a_{n-1} + 3a_n \right] \cdot x^n$$

Thus, $2a_2 + \cancel{x}6a_3 + 3a_0 + 3a_1\cancel{x} = 0$ and

$(n+1)(n+2) a_{n+2} + (n-1) a_{n-1} + 3a_n = 0$ for $n \geq 2$.

So, $(2a_2 + 3a_0) + x(6a_3 + 3a_1) = 0$ implies,

$2a_2 + 3a_0 = 0$ and $6a_3 + 3a_1 = 0$

$\boxed{a_2 = -\frac{3}{2} a_0}$ and $\boxed{a_3 = -\frac{a_1}{2}}$

Here a_0, a_1 are free and our recurrence relation is

$$a_{n+2} = \frac{-3a_n - (n-1)a_{n-1}}{(n+1)(n+2)} \quad \text{for } n \geq 2.$$

If we let $a_0 = 1$ and $a_1 = 0$ then we get

$$y_1 = 1 - \frac{3}{2}x^2 + \dots$$

Similarly for $a_0 = 0$ and $a_1 = 1$ we get

$$y_2 = x - \frac{1}{2}x^3 + \dots$$

Next, observe that $W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

Hence, y_1 and y_2 form a fundamental set of solutions.

2. (15 points) Find the first 2 terms of each fundamental solution of

$$y'' + e^x y = 0$$

around $x_0 = 0$. (Hint: First find a set of fundamental solutions.)

let $y = \sum_{n=0}^{\infty} a_n x^n$ be a solution

As usual, $y'' = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, x^n$

Note that, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Plugging these into the equation,

$$0 = y'' + e^x y = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{6} + \dots\right) \sum_{n=0}^{\infty} a_n x^n$$

$$= 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \dots$$
$$+ a_0 + (a_0 + a_1) x + \left(\frac{a_0}{2} + a_1 + a_2\right) x^2 + \dots$$

Thus, $2a_2 + a_0 = 0$, $a_2 = -\frac{a_0}{2}$

Also, $6a_3 + a_0 + a_1 = 0$, $a_3 = -\frac{(a_0 + a_1)}{6}$

For $a_0 = 1$, $a_1 = 0$ we get $y_1 = 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots$

For $a_0 = 0$, $a_1 = 1$ we get $y_2 = x - \frac{1}{6}x^3 + \dots$

LAPLACE TRANSFORM TABLE:

$\mathcal{L}\{1\} = \frac{1}{s} \quad s > 0$ | $\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad s > a$ | $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} \quad s > 0$ | $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \quad s > 0$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad s > 0$ | $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2} \quad s > a$ | $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \quad s > a$

$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$

---

3. (a) (10 points) Suppose $r(t)$ is a continuous function. Show that for any constants a and b, the solution of the IVP $y' + ay = r(t)$, $y(0) = b$ is given by $y(t) = e^{-at} * r(t) + be^{-at}$. (Note that $*$ stands for the convolution of the two functions.)

Let $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{r(t)\} = R(s)$

then $\mathcal{L}\{y'+ay\} = \mathcal{L}\{r(t)\} \Rightarrow \mathcal{L}\{y'\} + a\mathcal{L}\{y\} = R(s)$

$\Rightarrow sY(s) - y(0) + aY(s) = R(s)$

$\Rightarrow (s+a)Y(s) - b = R(s)$

$\Rightarrow Y(s) = \frac{b}{s+a} + \frac{R(s)}{s+a}$

then $y(t) = be^{-at} + \mathcal{L}^{-1}\Big\{\underbrace{\frac{1}{s+a}}_{\mathcal{L}\{e^{-at}\}} \underbrace{R(s)}_{\mathcal{L}\{r(t)\}}\Big\}$

So $\underline{y(t) = be^{-at} + e^{-at} * r(t)}$

(b) (5 points) Find the solution of $y' + 2y = e^{-t}$, $y(0) = 2$ (make sure that your final answer does not involve any integrals).

For $y' + 2y = e^{-t}$, $y(0) = 2$, $a = 2$, $b = 2$ and $r(t) = e^{-t}$

So by part (a),

$y(t) = 2e^{-2t} + e^{-2t} * e^{-t}$

where $e^{-2t} * e^{-t} = \int_0^t e^{-2u} e^{-(t-u)}\, du = e^{-t}\int_0^t e^{-u} du$

$= -e^{-t}\left(e^{-u}\Big|_{u=0}^{t}\right) = e^{-t}(1 - e^{-t})$

So $y(t) = 2e^{-2t} + e^{-t}(1-e^{-t})$

$\underline{= e^{-t} + e^{-2t}}$

4. (a) (10 points) Solve the following IVP (initial value problem) using the Laplace transform

$$y'' + 9y = 3, \quad y(0) = 1, \quad y'(0) = 0.$$

Let $\mathcal{L}\{y(t)\} = Y(s)$, then

$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{3\}$

$\Rightarrow s^2 Y(s) - s y(0) - y'(0) + 9Y(s) = 3/s$

$\Rightarrow (s^2+9)Y(s) - s = \frac{3}{s} \Rightarrow Y(s) = \underbrace{\frac{s}{s^2+9}}_{\mathcal{L}\{\cos 3t\}} + \frac{3}{s(s^2+9)}$

For the second part using partial fractions

$\frac{3}{s(s^2+9)} = \frac{1}{3}\left(\frac{1}{s} - \frac{s}{s^2+9}\right)$ so $\mathcal{L}^{-1}\left\{\frac{3}{s(s^2+9)}\right\} = \frac{1}{3} - \frac{1}{3}\cos 3t$

so $y(t) = \cos 3t + \frac{1}{3} - \frac{1}{3}\cos 3t = \underline{\underline{\frac{1}{3} + \frac{2}{3}\cos 3t}}$

In parts (b) and (c), find the Laplace transform of the given functions.

(b) (5 points) $f(t) = te^{-t} - \cos^2 t$ (hint: $\cos 2t = \cos^2 t - \sin^2 t$)

Using the hint, $\cos^2 t = \frac{1+\cos 2t}{2} \Rightarrow f(t) = te^{-t} - \frac{(1+\cos 2t)}{2}$

so $\mathcal{L}\{f(t)\} = \mathcal{L}\{te^{-t}\} - \mathcal{L}\{\frac{1}{2}\} - \frac{1}{2}\mathcal{L}\{\cos 2t\}$

$= \frac{1}{(s+1)^2} - \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4}$

$= \underline{\underline{\frac{1}{(s+1)^2} - \frac{1}{2s} - \frac{1}{2(s^2+4)}}}$

(c) (5 points) $f(t) = \begin{cases} 7, & 0 \le t < 4 \\ t+9, & 4 \le t \end{cases}$

In terms of unit step fcns,

$f(t) = 7 + u_4(t)\left[(t+9) - 7\right] = 7 + u_4(t)(t+2)$

$= 7 + u_4(t)((t-4)+6)$

so $\mathcal{L}\{f(t)\} = \frac{7}{s} + \mathcal{L}\{u_4(t)(t-4)\} + \mathcal{L}\{6u_4(t)\}$

$= \frac{7}{s} + \frac{e^{-4s}}{s^2} + 6\frac{e^{-4s}}{s} = \underline{\underline{\frac{7}{s} + \frac{e^{-4s}}{s}\left(\frac{1}{s} + 6\right)}}$

5. Find the inverse Laplace transform of

(a) (5 points) $F(s) = \dfrac{s+3}{s^2-4s+10}$

$$F(s) = \frac{s+3}{(s-2)^2+6} = \frac{s-2}{(s-2)^2+6} + \frac{5}{(s-2)^2+6}$$

$$= \frac{s-2}{(s-2)^2+(\sqrt{6})^2} + \frac{5}{\sqrt{6}} \frac{\sqrt{6}}{(s-2)^2+(\sqrt{6})^2}$$

so $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+(\sqrt{6})^2}\right\} + \frac{5}{\sqrt{6}} \mathcal{L}^{-1}\left\{\frac{\sqrt{6}}{(s-2)^2+(\sqrt{6})^2}\right\}$

$$\Rightarrow f(t) = e^{2t}\cos(\sqrt{6}t) + \frac{5}{\sqrt{6}} e^{2t}\sin(\sqrt{6}t)$$

$$= \underline{e^{2t}\left(\cos(\sqrt{6}t) + \frac{5}{\sqrt{6}} \cdot \sin(\sqrt{6}t)\right)}$$

(b) (5 points) $F(s) = \dfrac{1}{s^2(s^2+1)}$

$$F(s) = \underbrace{\frac{1}{s^2}}_{\mathcal{L}\{t\}} \cdot \underbrace{\frac{1}{s^2+1}}_{\mathcal{L}\{\sin t\}} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = t * \sin t$$

$$= \int_0^t (\sin \tau)(t-\tau)\, d\tau$$

by integration by parts, we get

$$\underline{f(t) = t - \sin t}$$

OR by partial fractions,

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

so $\mathcal{L}^{-1}\{F(s)\} = t - \sin t$

(c) (8 points) $F(s) = \dfrac{1-e^{-3s}+se^{-4s}}{s^2}$

$$F(s) = \underbrace{\frac{1}{s^2}}_{\mathcal{L}\{t\}} - \underbrace{e^{-3s}\frac{1}{s^2}}_{e^{-3s}\mathcal{L}\{t\}} + \underbrace{\frac{e^{-4s}}{s}}_{e^{-4s}\mathcal{L}\{1\}}$$

so $\mathcal{L}^{-1}\{F(s)\} = f(t) = t - u_3(t)(t-3) + u_4(t)$

$$\Rightarrow f(t) = \begin{cases} t & 0 \le t < 3 \\ 3 & 3 \le t < 4 \\ 4 & t \ge 4 \end{cases}$$

6. (a) (5 points) For any positive c, define the function $f_c(t)$ by $f_c(t) = \left\{ \begin{array}{ll} 1/c, & 0 \le t < c \\ 0, & c \le t \end{array} \right\}$. Find $F_c(s) = \mathcal{L}\{f_c(t)\}$.

$$\mathcal{L}\{f_c(t)\} = \int_0^\infty e^{-st} f_c(t)\, dt = \int_0^c e^{-st} \frac{1}{c}\, dt$$

$$= \frac{-1}{sc}\left(e^{-st}\Big|_{t=0}^{c}\right) = \frac{1-e^{-sc}}{sc}$$

(b) (5 points) Define $\delta(t) := \lim_{c \to 0^+} f_c(t)$ (Admittedly, this is not an ordinary function, but such a function is a generalized function called Dirac's delta function). Then find $\mathcal{L}\{\delta(t)\}$. (Hint: the limit and the improper integral in the definition of the Laplace transform are interchangeable.)

$$\delta(t) = \lim_{c \to 0^+} f_c(t)$$

$$\to \mathcal{L}\{\delta(t)\} = \mathcal{L}\{\lim_{c\to 0^+} f_c(t)\} = \lim_{c \to 0^+} \mathcal{L}\{f_c(t)\}$$

$$= \lim_{c \to 0^+} \frac{1-e^{-sc}}{sc} = \lim_{c \to 0^+} \frac{s e^{-sc}}{s} = 1$$

by L'hospital rule

$$\to \mathcal{L}\{\delta(t)\} = 1$$

(c) (5 points) Find the Laplace transform of $f(t) = t^2 e^t - \int_0^t u \sin(t-u) du$.

$$f(t) = e^t t^2 - t * \sin t$$

$$\to \mathcal{L}\{f(t)\} = \mathcal{L}\{e^t t^2\} - \mathcal{L}\{t * \sin t\}$$

$$= \frac{2}{(s-1)^3} - \frac{1}{s^2(s^2+1)}$$

(d) (5 points) Find the Laplace transform of $f(t) = \tan t$ if exists. If not, show that it does not exist.

tant is not of exponential order, since $\tan t \le K e^{at}$ can not hold for any finite K and a;

so $\mathcal{L}\{\tan t\}$ does not exist.