7.1

1. Introduce the variables $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$x_2' = u'' = -2u - 3\,u'.$$

In terms of the new variables, we obtain the system of two first order ODEs

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 - 3\,x_2\,. \end{aligned}$$

3. First divide both sides of the equation by t^3, and write

$$u'' = -\frac{1}{t^2}\,u' - (\frac{1}{t} - \frac{1}{4t^3})u\,.$$

Set $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$x_2' = u'' = -\frac{1}{t^2}\,u' - (\frac{1}{t} - \frac{1}{4t^3})u\,.$$

We obtain the system of equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -(\frac{1}{t} - \frac{1}{4t^3})x_1 - \frac{1}{t^2}\,x_2\,. \end{aligned}$$

5. Let $x_1 = u$ and $x_2 = u'$; then $u'' = x_2'$. In terms of the new variables, we have

$$x_2' + 2x_2 + 4\,x_1 = 2\,\cos\,3t$$

with the initial conditions $x_1(0) = 1$ and $x_2(0) = -2$. The equivalent first order system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -4\,x_1 - 2\,x_2 + 2\,\cos\,3t \end{aligned}$$

with the above initial conditions.

7.(a) Solving the first equation for x_2, we have $x_2 = x_1' + 2x_1$. Substitution into the second equation results in $(x_1' + 2x_1)' = x_1 - 2(x_1' + 2x_1)$. That is, $x_1'' + 4\,x_1' + 3\,x_1 = 0$. The resulting equation is a second order differential equation with constant coefficients. The general solution is $x_1(t) = c_1 e^{-t} + c_2 e^{-3t}$. With x_2 given in terms of x_1, it follows that $x_2(t) = c_1 e^{-t} - c_2 e^{-3t}$.

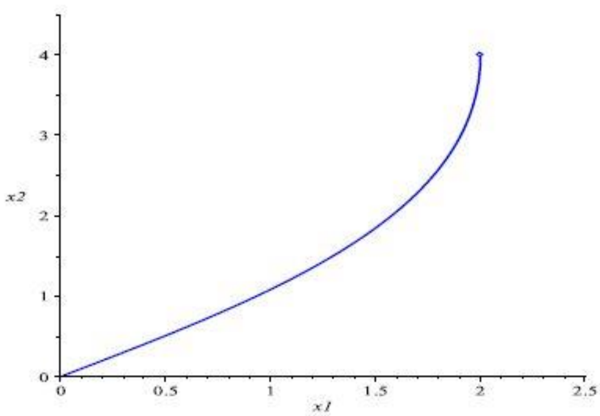
(b) Imposing the specified initial conditions, we obtain

$$c_1 + c_2 = 2, \qquad c_1 - c_2 = 4,$$

with solution $c_1 = 3$ and $c_2 = -1$. Hence

$$x_1(t) = 3e^{-t} - e^{-3t} \text{ and } x_2(t) = 3e^{-t} + e^{-3t}\,.$$

(c)



1. **(a)** $\begin{pmatrix} 6 & -6 & 3 \\ 5 & 9 & -2 \\ 2 & 3 & 8 \end{pmatrix}$ **(b)** $\begin{pmatrix} -15 & 6 & -12 \\ 7 & -18 & -1 \\ -26 & -3 & -5 \end{pmatrix}$

(c) $\begin{pmatrix} 6 & -12 & 3 \\ 4 & 3 & 7 \\ 9 & 12 & 0 \end{pmatrix}$ **(d)** $\begin{pmatrix} -8 & -9 & 11 \\ 14 & 12 & -5 \\ 5 & -8 & 5 \end{pmatrix}$

7.2

3.(c,d)

$$\begin{aligned} \mathbf{A}^T + \mathbf{B}^T &= \begin{pmatrix} -2 & 3 & 2 \\ 1 & 0 & -1 \\ 3 & -3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 2 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 4 & 0 \\ 3 & -1 & 0 \\ 5 & -4 & 1 \end{pmatrix} = (\mathbf{A}+\mathbf{B})^T. \end{aligned}$$

4.(b)

$$\overline{\mathbf{A}} = \begin{pmatrix} 3+2i & 1-2i \\ 2+i & -2-3i \end{pmatrix}.$$

(c) By definition,

$$\mathbf{A}^* = \overline{\mathbf{A}^T} = (\overline{\mathbf{A}})^T = \begin{pmatrix} 3+2i & 2+i \\ 1-2i & -2-3i \end{pmatrix}.$$

10. $\begin{pmatrix} \frac{3}{11} & -\frac{4}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$

12. $\begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$

8.(a) $\mathbf{x}^T\mathbf{y} = 2(-1+i) + 2(4i) + (1-i)(2+i) = 1 + 9i.$

(b) $\mathbf{y}^T\mathbf{y} = (-1+i)^2 + 2^2 + (2+i)^2 = 7 + 2i.$

(c) $(\mathbf{x}, \mathbf{y}) = 2(-1-i) + 2(4i) + (1-i)(2-i) = -1 + 3i.$

(d) $(\mathbf{y}, \mathbf{y}) = (-1+i)(-1-i) + 2^2 + (2+i)(2-i) = 11.$

7.3

4. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right).$$

Adding -2 times the first row to the second row and subtracting the first row from the third row results in

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right).$$

Adding the negative of the second row to the third row results in

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We evidently end up with an equivalent system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + x_3 &= 0\,. \end{aligned}$$

Since there is no unique solution, let $x_3 = \alpha$, where α is arbitrary. It follows that $x_2 = \alpha$, and $x_1 = -\alpha$. Hence all solutions have the form

$$x = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

13. By inspection, we find that

$$\mathbf{x}^{(1)}(t) - 2\mathbf{x}^{(2)}(t) = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix}.$$

Hence $4\,\mathbf{x}^{(1)}(t) - 8\,\mathbf{x}^{(2)}(t) + \mathbf{x}^{(3)}(t) = \mathbf{0}$, and the vectors are linearly dependent.

18. The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation

$$\begin{pmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(-3-\lambda)(-3-\lambda) - 1 = 0$, that is,

$$\lambda^2 + 6\lambda + 8 = 0\,.$$

The eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -2$. For $\lambda_1 = -4$, the system of equations becomes

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1\,,-1)^T$. Substituting $\lambda = \lambda_2 = -2$, we have

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equations reduce to $x_1 = x_2$. Hence a solution vector is given by $\mathbf{x}^{(2)} = (1\,,1)^T$.

7.5

15. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\,\xi_1 - \,\xi_2 = 0$. An

eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 3 \\ c_1 + 3\,c_2 &= -1\,. \end{aligned}$$

Hence $c_1 = 5$ and $c_2 = -2$, and the solution of the IVP is

$$\mathbf{x} = 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

7.1) 9a. $\begin{cases} x_1' = 1.25x_1 + 0.75x_2 & x_1(0) = -2 \\ x_2' = 0.75x_1 + 1.25x_2 & x_2(0) = 3 \end{cases}$

Solve the first equation for x_2 first: $x_2 = \dfrac{x_1'}{0.75} - \dfrac{5}{3}x_1$

Substitute into the second eqn: $0.75 = \frac{3}{4}$, $1.25 = \frac{5}{4}$

$$\Rightarrow \frac{4}{3}x_1'' - \frac{5}{3}x_1' = \frac{3}{4}x_1 + \frac{5}{4}\left(\frac{4}{3}x_1' - \frac{5}{3}x_1\right)$$

$$\Rightarrow \frac{4}{3}x_1'' - \frac{10}{3}x_1' + \frac{4}{3}x_1 = 0 \quad // \quad \equiv (2x_1'' - 5x_1 + 2x_1 = 0).$$

9.b. The general solution is $x_1(t) = c_1 e^{t/2} + c_2 e^{2t}$.

x_2 in terms of x_1:

$$x_2 = \frac{4}{3} \cdot \frac{c_1}{2} e^{t/2} + \frac{4}{3} \cdot 2c_2 e^{2t} - \frac{5}{3}c_1 e^{t/2} - \frac{5}{3}c_2 e^{2t}.$$

$$x_2 = -c_1 e^{t/2} + c_2 e^{2t}.$$

$$\begin{cases} x_1(0) = -2 \\ x_2(0) = 3 \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = -2 \\ -c_1 + c_2 = 3 \end{cases} \Rightarrow c_2 = 1/2, \; c_1 = -\frac{5}{2}.$$

Hence, $x_1 = -\frac{5}{2}e^{t/2} + \frac{1}{2}e^{2t}$, $x_2 = \frac{5}{2}e^{t/2} + \frac{1}{2}e^{2t}$.

7.3) 1. $\begin{cases} x_1 - x_3 = 0 \\ 3x_1 + x_2 + x_3 = 1 \\ -x_1 + x_2 + 2x_3 = -1 \end{cases}$

The augmented matrix is $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 3 & 1 & 1 & 1 \\ -1 & 1 & 2 & -1 \end{array}\right)$

Adding -3 times the first row to the second row and adding the first row to the third row: $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right)$

Substracting the second row from the third row: $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -3 & -2 \end{array}\right)$

We end up with an equivalent system of eqns:

$$\left.\begin{array}{r} x_1 - x_3 = 0 \\ x_2 + 4x_3 = 1 \\ -3x_3 = -2 \end{array}\right\} \Rightarrow \begin{array}{l} x_3 = 2/3 \\ x_2 = -5/3 \\ x_1 = 2/3 \, // \end{array}$$

2. The augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & -1 & 2 & 3 \end{array}\right) \xrightarrow[-1\times \text{---} + 3^{rd} \text{ row}]{-2\times 1^{st} \text{ row} + 2^{nd} \text{ row}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 2 \end{array}\right) \xrightarrow{2^{nd} + 3^{rd}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

$\Rightarrow$ From the third row, observe $0.x_1 + 0.x_2 + 0.x_3 = 2$!,

So no solution.

7.3) 7. $x^{(1)} = (1,1,1)$, $x^{(2)} = (0,1,1)$, $x^{(3)} = (1,0,1)$.

column vector

$$X = \left(x^{(1)} \quad x^{(2)} \quad x^{(3)}\right) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$\det X = 1(1-0) + 1(1-1) = 1 \neq 0$. So they are linearly independent.

9. $X = \begin{pmatrix} 1 & -1 & -2 & -3 \\ 2 & 0 & -1 & 0 \\ 2 & 3 & 1 & -1 \\ 3 & 1 & 0 & 3 \end{pmatrix} \Rightarrow \det X = 0.$

Hence the vectors are linearly dependent. In order to find a linear relationship between them, write $c_1 x^{(1)} + c_2 x^{(2)} + c_3 x^{(3)} + c_4 x^{(4)} = 0$. Let $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$

$\Rightarrow X c = 0$, $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Perform row operations on X

$\begin{cases} -2 \times 1^{st} \text{ row add to 2nd and } 3^{rd} \text{ row} \\ -3 \times \text{ —— last row} \end{cases}$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 0 & 2 & 3 & 6 & 0 \\ 0 & 5 & 5 & 5 & 0 \\ 0 & 4 & 6 & 12 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 0 & 2 & 3 & 6 & 0 \\ 0 & 0 & -5 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$-\frac{5}{2} \times 2^{nd}$ row $+ 3^{rd}$ row

$-2 \times$ —— + last row

$$\Rightarrow \begin{aligned} c_1 - c_2 - 2c_3 - 3c_4 &= 0 \\ 2c_2 + 3c_3 + 6c_4 &= 0 \\ -5c_3 - 10c_4 &= 0 \end{aligned}$$

Set $c_4 = 1$, $\Rightarrow c_3 = 4$, $c_2 = -3$, $c_1 = 2$,

Hence $2x^{(1)} - 3x^{(2)} + 4x^{(3)} + x^{(4)} = 0$.

7.3.) 16. The eigenvalues λ and eigenvectors x satisfy the equation $\begin{pmatrix} 5-\lambda & 3 \\ -1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

For a non-zero solution, we must have

$$(5-\lambda)(1-\lambda)+3=0$$

$$\lambda^2 - 6\lambda + 8 = 0 \Rightarrow \lambda_{1,2} = 2,4$$

For $\lambda_1 = 2$ we have:

$$\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad 3x_1 + 3x_2 = 0$$

$$\Rightarrow x_1 = -x_2$$

Hence $x^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda = 4$ we have:

$$\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = -3x_2$$

Hence $x^{(2)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

22. $\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$. For a non-zero sol. we must have:

$$\underbrace{(1-\lambda)}_{0} \text{ or } \underbrace{((1-\lambda)^2+4)}_{0} = 0$$

$$\Rightarrow \lambda_1 = 1, \quad \lambda^2 - 2\lambda + 5 = 0$$

$$\lambda_{2,3} = \frac{+2 \mp \sqrt{-16}}{2} = \frac{2 \mp 4i}{2}$$

$$\lambda_{2,3} = 1 \mp 2i$$

22. cont. For $\lambda_1 = 1$ we have:

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 - 2x_3 = 0 \Rightarrow \begin{cases} x_1 = x_3 \\ 2x_2 = -3x_1 \end{cases}$$
$$3x_1 + 2x_2 = 0 \Rightarrow$$

$$\begin{matrix} 2x_2 & = & -3x_1 \\ \downarrow & & \downarrow \\ -3 & & 2 \end{matrix}$$

Hence $x^{(1)} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$

$\lambda_2 = 1 - 2i$ we have:

$$\begin{pmatrix} 2i & 0 & 0 \\ 2 & 2i & -2 \\ 3 & 2 & 2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{cases} 2ix_1 = 0 \\ 2x_1 + 2i \cdot x_2 - 2x_3 = 0 \\ 3x_1 + 2x_2 + 2ix_3 = 0 \end{cases}$$

$$\Rightarrow \quad x_1 = 0 \quad \Rightarrow \quad 2ix_2 - 2x_3 = 0.$$

Hence $x^{(2)} = \begin{pmatrix} 0 \\ 2 \\ 2i \end{pmatrix}$. $x^{(3)}$ is $\overline{x^{(2)}} = \begin{pmatrix} 0 \\ 2 \\ -2i \end{pmatrix}$.

Section 7.4 : Problem 6.

$$x^{(1)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix} \quad , \quad x^{(2)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

(a) $W(x^{(1)}, x^{(2)})(t) = \det\begin{pmatrix} | & | \\ x^{(1)} & x^{(2)} \\ | & | \end{pmatrix} = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2$

(b) From (a), it follows that $x^{(1)}$ and $x^{(2)}$ are linearly independent at each point except $t=0$; they are linearly independent on every interval.

(c) At least one coefficient must be discontinuous at $t=0$.

(d) The general solution of the homogeneous system $x' = \mathbf{P} \cdot x$ is of the form

$$x = c_1 \cdot x^{(1)} + c_2 \cdot x^{(2)} \quad \text{so that,}$$

$$x(t) = c_1 \cdot \begin{pmatrix} t \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} t^2 \\ 2t \end{pmatrix} = \begin{pmatrix} c_1 \cdot t + c_2 \cdot t^2 \\ c_1 + c_2 \cdot 2t \end{pmatrix}$$

$$x'(t) = \begin{pmatrix} c_1 + 2tc_2 \\ c_1 \cdot 0 + 2 \cdot c_2 \end{pmatrix} \text{ and } P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

plugging the value of $x(t)$ in the equation, we find

$$P = \begin{pmatrix} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{pmatrix} \text{ so, } x' = \begin{pmatrix} 0 & 1 \\ -2t^{-2} & 2t^{-1} \end{pmatrix} x.$$

Section 7.5 - Q(1a). $x' = \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix} x.$

Finding eigenvalues of the matrix $\begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix}$,

$$0 = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ -2 & -2-\lambda \end{vmatrix}$$

$$0 = (3-\lambda)\cdot(-2-\lambda) - 2\cdot(-2)$$
$$0 = -6 - 3\lambda + 2\lambda + \lambda^2 + 4$$
$$0 = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$$

So $\lambda_1 = -1$ and $\lambda_2 = 2$.

To find eigenvectors,

$$0 = (A - \lambda_1 I)v = \begin{pmatrix} 4 & 2 \\ -2 & -1 \end{pmatrix} \cdot v \quad \text{so} \quad v_1 = \begin{pmatrix} -1 \\ +2 \end{pmatrix}$$

and,

$$0 = (A - \lambda_2 I)v = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} v \quad \text{so} \quad v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Thus, the general solution of the equation is,

$$x(t) = c_1 \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t}$$

Q5a: $x' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x$. $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

eigenvalues $0 = \det(A - \lambda I)$ i.e $\lambda_1 = -3$, $\lambda_2 = -1$
and corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Hence, $x(t) = c_1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$

$\boxed{7a}$: $x' = \begin{pmatrix} 4 & 8 \\ -3 & -6 \end{pmatrix} x$

$A = \begin{pmatrix} 4 & 8 \\ -3 & -6 \end{pmatrix}$

eigenvalues are λ_1, λ_2 such that $0 = \det(A - \lambda I)$ from which we obtain $\lambda_1 = 0$ and $\lambda_2 = -2$

And, corresponding eigenvectors $v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

Thus,

$$x(t) = c_1 \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot e^{0 \cdot t} + c_2 \cdot \begin{pmatrix} -4 \\ 3 \end{pmatrix} e^{-2t}$$

$\boxed{Q1b}$: $x' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

$A = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}$

eigenvalues are λ_1, λ_2 roots of $\det(A - \lambda I) = 0$ so that $\lambda_1 = -1$ and $\lambda_2 = 3$. And corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

So, $x(t) = c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$.

Imposing the initial value $x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$,

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{0} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{0} = \begin{pmatrix} c_1 + c_2 \\ c_1 + 5c_2 \end{pmatrix}$$

OR $c_1 = \frac{7}{4}$ and $c_2 = \frac{1}{4}$

Thus, $x(t) = \frac{7}{4} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{4} \cdot \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$. //