7.6

3.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 2-r & 1 \\ -5 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 1 = 0$. The roots of the characteristic equation are $r = \pm i$. Setting $r = i$, the equations are equivalent to $(2-i)\xi_1 + \xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (1, -2+i)^T$ and $\boldsymbol{\xi}^{(2)} = (1, -2-i)^T$. Hence one of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -2+i \end{pmatrix} e^{it} = \begin{pmatrix} 1 \\ -2+i \end{pmatrix} (\cos t + i \sin t) =$$

$$= \begin{pmatrix} \cos t \\ -2\cos t - \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix}.$$

Therefore the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} \cos t \\ -2\cos t - \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix}.$$

The solution may also be written as

$$\mathbf{x} = c_1 \begin{pmatrix} -2\cos t + \sin t \\ 5\cos t \end{pmatrix} + c_2 \begin{pmatrix} -2\sin t - \cos t \\ 5\sin t \end{pmatrix}.$$

5.(a) Setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 5 \\ -1 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2\,r + 2 = 0$, with roots $r = -1 \pm i$. Substituting $r = -1 - i$ reduces the system of equations to $(2+i)\xi_1 + 5\xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (-2+i\,,1)^T$ and $\boldsymbol{\xi}^{(2)} = (-2-i\,,1)^T$. Hence one of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} -2+i \\ 1 \end{pmatrix} e^{-(1+i)t} = \begin{pmatrix} -2+i \\ 1 \end{pmatrix} e^{-t}(\cos\, t - i\, \sin\, t) =$$

$$= e^{-t} \begin{pmatrix} -2\cos\, t + \sin t \\ \cos\, t \end{pmatrix} + i e^{-t} \begin{pmatrix} \cos t + 2\sin t \\ -\sin t \end{pmatrix}.$$

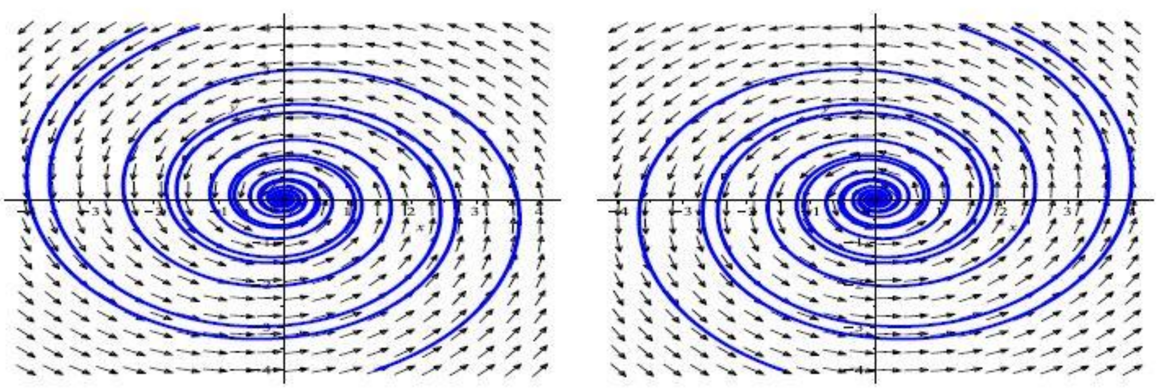
The general solution is

$$\mathbf{x} = c_1\, e^{-t} \begin{pmatrix} -2\cos\, t + \sin t \\ \cos\, t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \cos t + 2\sin t \\ -\sin t \end{pmatrix}.$$

13.(a) The characteristic equation is $r^2 - 2\alpha r + 1 + \alpha^2 = 0$, with roots $r = \alpha \pm i$.

(b) When $\alpha < 0$ and $\alpha > 0$, the equilibrium point $(0\,,0)$ is a stable spiral and an unstable spiral, respectively. The equilibrium point is a center when $\alpha = 0$.

(c)



(a) $\alpha = -1/8$ (b) $\alpha = 1/8$

7.7

1.(a) The eigenvalues and eigenvectors were found in Problem 1, Section 7.5.

$$r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \quad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -e^{-t} \\ 2\,e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} -2\,e^{2t} \\ e^{2t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} -e^{-t} & -2\,e^{2t} \\ 2\,e^{-t} & e^{2t} \end{pmatrix}.$$

(b) We now have

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix} \text{ and } \boldsymbol{\Psi}^{-1}(0) = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix},$$

So that

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) = \frac{1}{3}\begin{pmatrix} -e^{-t} + 4e^{2t} & -2e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & 4e^{-t} - e^{2t} \end{pmatrix}.$$

3.(a) The eigenvalues and eigenvectors were found in Problem 3, Section 7.5. The general solution of the system is

$$\mathbf{x} = c_1 \begin{pmatrix} -3e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{pmatrix}.$$

(b) Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} -3c_1 - c_2 &= 1 \\ c_1 + c_2 &= 0, \end{aligned}$$

to obtain $c_1 = -1/2$, $c_2 = 1/2$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \frac{3}{2}e^t - \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^t + \frac{1}{2}e^{-t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} -3c_1 - c_2 &= 0 \\ c_1 + c_2 &= 1, \end{aligned}$$

to obtain $c_1 = -1/2$, $c_2 = 3/2$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \frac{3}{2}e^t - \frac{3}{2}e^{-t} \\ -\frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \frac{1}{2} \begin{pmatrix} 3e^t - e^{-t} & 3e^t - 3e^{-t} \\ -e^t + e^{-t} & -e^t + 3e^{-t} \end{pmatrix}.$$

5.(a) The general solution, found in Problem 3, Section 7.6, is given by

$$\mathbf{x} = c_1 \begin{pmatrix} -2\cos t + \sin t \\ 5\cos t \end{pmatrix} + c_2 \begin{pmatrix} -2\sin t - \cos t \\ 5\sin t \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} -2\cos t + \sin t & -2\sin t - \cos t \\ 5\cos t & 5\sin t \end{pmatrix}.$$

(b) Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} -2c_1 - c_2 &= 1 \\ 5c_1 &= 0, \end{aligned}$$

resulting in $c_1 = 0$, $c_2 = -1$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \cos t + 2\sin t \\ -5\sin t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$-2c_1 - c_2 = 0$$
$$5c_1 = 1\,,$$

resulting in $c_1 = 1/5$, $c_2 = -2/5$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{pmatrix} \cos t + 2\sin t & \sin t \\ -5\sin t & \cos t - 2\sin t \end{pmatrix}.$$

7.(a) The general solution, found in Problem 15, Section 7.5, is given by

$$\mathbf{x} = c_1 \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} -3e^{4t} \\ e^{4t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} -e^{2t} & -3e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}.$$

(b) Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$-c_1 - 3c_2 = 1$$
$$c_1 + c_2 = 0\,,$$

resulting in $c_1 = 1/2$, $c_2 = -1/2$. The corresponding solution is

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} -e^{2t} + 3e^{4t} \\ e^{2t} - e^{4t} \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$ require that

$$-c_1 - 3c_2 = 0$$
$$c_1 + c_2 = 1\,,$$

resulting in $c_1 = 3/2$, $c_2 = -1/2$. The corresponding solution is

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} -3e^{2t} + 3e^{4t} \\ 3e^{2t} - e^{4t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \frac{1}{2}\begin{pmatrix} -e^{2t} + 3e^{4t} & -3e^{2t} + 3e^{4t} \\ e^{2t} - e^{4t} & 3e^{2t} - e^{4t} \end{pmatrix}.$$

7.8

7.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 6r + 9 = 0$. The only root is $r = -3$, which is an eigenvalue of multiplicity two. Substituting $r = -3$ into the coefficient matrix, the system reduces to the single equation $\xi_1 - \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

For a second linearly independent solution, we search for a generalized eigenvector. Its components satisfy

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

that is, $4\eta_1 - 4\eta_2 = 1$. Let $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = k + 1/4$. It follows that a second solution is given by

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} k + 1/4 \\ k \end{pmatrix} e^{-3t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} \right].$$

Imposing the initial conditions, we require that $c_1 + c_2/4 = 4$, $c_1 = 2$, which results in $c_1 = 2$ and $c_2 = 8$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 8 \\ 8 \end{pmatrix} te^{-3t}.$$

7.9

6. The eigenvalues of the coefficient matrix are $r_1 = 0$ and $r_2 = -5$. It follows that the solution of the homogeneous equation is

$$\mathbf{x}_c = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}-2e^{-5t}\\e^{-5t}\end{pmatrix}.$$

The coefficient matrix is symmetric. Hence the system is diagonalizable. Using the normalized eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix} \qquad \mathbf{T}^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}.$$

Setting $\mathbf{x}=\mathbf{T}\mathbf{y}$, and $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$y_1' = \frac{5+4t}{\sqrt{5}\,t}$$

$$y_2' = -5y_2 + \frac{2}{\sqrt{5}}.$$

The solutions are readily obtained as

$$y_1(t) = \sqrt{5}\,\ln t + \frac{4}{\sqrt{5}}t + c_1 \text{ and } y_2(t) = c_2\,e^{-5t} + \frac{2}{5\sqrt{5}}.$$

Transforming back to the original variables, we have $\mathbf{x}=\mathbf{T}\mathbf{y}$, with

$$\mathbf{x} = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\begin{pmatrix}y_1(t)\\y_2(t)\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}y_1(t) + \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}y_2(t).$$

Hence the general solution is

$$\mathbf{x} = k_1\begin{pmatrix}1\\2\end{pmatrix} + k_2\begin{pmatrix}-2e^{-5t}\\e^{-5t}\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix}\ln t + \frac{4}{5}\begin{pmatrix}1\\2\end{pmatrix}t + \frac{2}{25}\begin{pmatrix}-2\\1\end{pmatrix}.$$

9. Note that the coefficient matrix is symmetric. Hence the system is diagonalizable. The eigenvalues and eigenvectors are given by

$$r_1 = -\frac{1}{2}\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } r_2 = -2\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Using the normalized eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \mathbf{T}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Setting $\mathbf{x}$=$\mathbf{Ty}$, and $\mathbf{h}(t)$ =$\mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$y_1' = -\frac{1}{2}y_1 + 2\sqrt{2}\,t + \frac{1}{\sqrt{2}}e^t$$
$$y_2' = -2y_2 + 2\sqrt{2}\,t - \frac{1}{\sqrt{2}}e^t\,.$$

Using any elementary method for first order linear equations, the solutions are

$$y_1(t) = k_1 e^{-t/2} + \frac{\sqrt{2}}{3}e^t - 8\sqrt{2} + 4\sqrt{2}\,t$$
$$y_2(t) = k_2 e^{-2t} - \frac{1}{3\sqrt{2}}e^t - \frac{1}{\sqrt{2}} + \sqrt{2}\,t\,.$$

Transforming back to the original variables, $\mathbf{x}$=$\mathbf{Ty}$, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t/2} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-2t} - \frac{1}{2}\begin{pmatrix} 17 \\ 15 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \end{pmatrix}t + \frac{1}{6}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^t.$$

7.6.) 1.a Setting $x = \xi e^{rt}$ results:

$$\begin{pmatrix} 3-r & 4 \\ -2 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For a non-zero solution, we require that $\det(A - rI) = 0$

$$\Rightarrow (3-r)(-1-r) + 8 = 0$$

$$r_{1,2} = 1 \mp 2i \qquad r_1 = 1+2i, \quad r_2 = 1-2i$$

Substituting $r_{1,2}$, we have associated eigenvectors; respectively

$$\xi^{(1)} = \begin{pmatrix} -2 \\ 1-i \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} -2 \\ 1+i \end{pmatrix}$$

Hence one of the complex-valued solutions is given by:

$$x^{(1)} = \begin{pmatrix} -2 \\ 1-i \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} -2 \\ 1-i \end{pmatrix} e^{t} (\cos 2t + i \sin 2t)$$

$$= e^{t} \begin{pmatrix} -2\cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + i e^{t} \begin{pmatrix} -2\sin 2t \\ -\cos 2t + \sin 2t \end{pmatrix}$$

Based on the real and imaginary parts of this solution, the general solution is:

$$x = c_1 e^{t} \begin{pmatrix} -2\cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^{t} \begin{pmatrix} -2\sin 2t \\ -\cos 2t + \sin 2t \end{pmatrix}$$

7.6) 9. eigenvalues:

$$\begin{vmatrix} 1-r & -5 \\ 1 & -3-r \end{vmatrix} = 0 \Rightarrow \begin{array}{l} (1-r)(-3-r)+5=0 \\ r^2+2r+2=0 \\ \Rightarrow r_{1,2} = -1 \mp i \end{array}$$

$r_1 = -1+i \quad \xi^{(1)} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$

$$x^{(1)} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} e^{(-1+i)t} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} e^{-t} (\cos t + i \sin t)$$

$$= e^{-t} \begin{pmatrix} 5\cos t + i\, 5 \sin t \\ 2\cos t + \sin t + i(2\sin t - \cos t) \end{pmatrix}$$

$$= e^{-t} \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + e^{-t} \cdot i \begin{pmatrix} 5 \sin t \\ 2\sin t - \cos t \end{pmatrix}$$

Hence the general solution is:

$$x = c_1 e^{-t} \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$$

Subsitute the I.C. $X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\begin{cases} 5c_1 = 2 \\ 2c_1 - c_2 = 1 \end{cases} \Rightarrow \begin{array}{l} c_1 = 2/5 \\ c_2 = -\frac{1}{5} \end{array}$$

$$x = \frac{2}{5} e^{-t} \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} - \frac{1}{5} e^{-t} \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$$

$t \to \infty, \quad x \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

7.6.) 11.a. $\begin{vmatrix} \frac{3}{4}-\lambda & 1 \\ -2 & -\frac{5}{4}-\lambda \end{vmatrix} = 0$

$$\lambda^2 + \frac{1}{2}\lambda + \frac{17}{16} = 0$$

$$\Delta = -4$$

$$r_{1,2} = \frac{-\frac{1}{2} \mp \sqrt{-4}}{2} = -\frac{1}{4} \mp i$$

7.7.) 11. $x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

The solution of the initial problem is given by

$$x = \Phi(t)\, x(0) = \frac{1}{2}\begin{pmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

from ex. 3.

$$= \begin{pmatrix} 3e^t - e^{-t} - \frac{3}{2}e^t + \frac{3}{2}e^{-t} \\ 3e^t - 3e^{-t} - \frac{3}{2}e^t + \frac{9}{2}e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2}e^t + \frac{1}{2}e^{-t} \\ \frac{3}{2}e^t + \frac{3}{2}e^{-t} \end{pmatrix}$$

Section 7.8 : Problem (1c): $x' = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} x$

From, $\det(A - \lambda I) = 0$ we find eigenvalues

i.e $\det\begin{bmatrix} 3-\lambda & 1 \\ -4 & -1-\lambda \end{bmatrix} = 0$ i.e $(3-\lambda)(-1-\lambda) + 4 \cdot 1 = 0$

OR, $-3 - 3\lambda + \lambda + \lambda^2 + 4 = 0$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

So that $\lambda_1 = 1$ and $\lambda_2 = 1$.

corresponding eigenvector for $\lambda_1 = 1$ is v_1 such that $(A - \lambda_1 I)v_1 = 0$ i.e $v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

and hence $x^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t$.

and $x^{(2)} = v_1 \cdot t \cdot e^t + v_2 e^t$ where v_2 (gen. eigenvector) satisfies $(A - \lambda_2 I)v_2 = v_1$.

i.e $\begin{pmatrix} 3-\lambda_2 & 1 \\ -4 & -1-\lambda_2 \end{pmatrix} v_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} v_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$$

Thus, $x^{(2)} = t \cdot e^t \begin{pmatrix} -1 \\ 2 \end{pmatrix} + e^t \cdot \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$, And the general solution is $x(t) = c_1 \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t + c_2 \left[t \cdot e^t \begin{pmatrix} -1 \\ 2 \end{pmatrix} + e^t \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} \right]$.

Section 7.3 - Problem (3c): $X' = \begin{pmatrix} -3/2 & -1/4 \\ 1 & -1/2 \end{pmatrix} X$

$\det(A - \lambda I) = 0$ i.e $\det\begin{pmatrix} -3/2-\lambda & -1/4 \\ 1 & -\frac{1}{2}-\lambda \end{pmatrix} = 0$

$0 = \left(-\frac{3}{2}-\lambda\right)\left(-\frac{1}{2}-\lambda\right) - 1\cdot\left(-\frac{1}{4}\right)$

$0 = \frac{3}{4} + \frac{3}{2}\lambda + \frac{1}{2}\lambda + \lambda^2 + \frac{1}{4}$

$0 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$ so $\lambda_1 = \lambda_2 = -1$

From $(A - \lambda_1 I)v_1 = 0$ we get $v_1 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

and from $(A - \lambda_2 I)v_2 = v_1$ we get $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

So that,

$$X^{(1)}(t) = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} e^{-t} \text{ and,}$$

$$X^{(2)}(t) = te^{-t}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus, the general solution is of the form,

$$X(t) = c_1 \cdot X^{(1)} + c_2 X^{(2)}$$

$$= c_1 t e^{-t}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} + c_2 \cdot \left[te^{-t}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right].$$

Section 7.8 – Problem 9a :

$$X' = \begin{pmatrix} 2 & 3/2 \\ -3/2 & -1 \end{pmatrix} X$$

$$X(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

From $0 = \det(A - \lambda I)$

$$0 = (2-\lambda)\cdot(-1-\lambda) - \frac{3}{2}\cdot\left(-\frac{3}{2}\right)$$

$$0 = -2 - 2\lambda + \lambda + \lambda^2 + \frac{9}{4}$$

$$0 = \lambda^2 - \lambda + \frac{1}{4}$$

$$0 = \left(\lambda - \frac{1}{2}\right)^2 \qquad \lambda_1 = \lambda_2 = \frac{1}{2}$$

so that, from $(A - \lambda_1 I)\,v_1 = 0$ we obtain

$$\begin{pmatrix} 2-\frac{1}{2} & 3/2 \\ -\frac{3}{2} & -1-\frac{1}{2} \end{pmatrix} v_1 = 0 \quad \text{i.e} \quad v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

And from $(A - \lambda_2 I)\,v_2 = v_1$ we obtain

$$\begin{pmatrix} 2-1/2 & 3/2 \\ -3/2 & -1-\frac{1}{2} \end{pmatrix} v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{i.e} \quad v_2 = \begin{pmatrix} -2/3 \\ 0 \end{pmatrix}$$

So the general solution is

$$X^{(1)}(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t/2}, \quad X^{(2)} = t e^{t/2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{t/2} \cdot \begin{pmatrix} -4/3 \\ 0 \end{pmatrix}$$

$$X(t) = c_1 \cdot e^{t/2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left[t e^{t/2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{t/2} \begin{pmatrix} -2/3 \\ 0 \end{pmatrix} \right]$$

Imposing $X(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ we find $c_1 = -1$, $c_2 = -3$

$$X(t) = \begin{pmatrix} 3 + 3t \\ -1 - 3t \end{pmatrix} e^{t/2}.$$

Section 7.9 – Problem 1 :

$$x' = \underbrace{\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}}_{A} x + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

First of all, note that there are many ways to solve this problem. But, if A is diagonalizable then we'll use method of diagonalization provided in textbook, page 440.

eigenvalues: $0 = \det(A - \lambda I)$

$$0 = \det\begin{pmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{pmatrix} = (2-\lambda)(-2-\lambda) - 3\cdot(-1)$$

$$0 = -4 - 2\lambda + 2\lambda + \lambda^2 + 3$$

$$0 = \lambda^2 \cancel{-0\lambda} + 3 - 4 = \lambda^2 - 1$$

$$0 = (\lambda+1)(\lambda-1)$$

So $\lambda_1 = -1$ and $\lambda_2 = 1$ $(\lambda_1 \neq \lambda_2)$

Hence A is diagonalizable.

corresponding eigenvectors are

$(A - \lambda_1 I)v_1 = 0$ and $(A - \lambda_2 I)v_2 = 0$

So that $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

forming the matrix $T = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}$

from $x' = Ax + g(t)$, where $g(t) = \begin{pmatrix} e^t \\ t \end{pmatrix}$.

~~[illegible]~~ letting $x = Ty$, where y is a new variable, substitute in the equation to get

$$(Ty)' = A(Ty) + g(t)$$

$$Ty' = ATy + g(t)$$, now multiply both side by T^{-1}

$$\underbrace{T^{-1}T}_{I}y' = \underbrace{T^{-1}AT}_{D}y + T^{-1}g(t)$$

$$y' = Dy + T^{-1}(g(t))$$

where D is a diagonal matrix with diagonal entries λ_1 and λ_2

i.e $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

from $T = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix} \Rightarrow$, $T^{-1} = \frac{1}{2} \cdot \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix}$

so eqn,

$$y' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} y + \frac{1}{2}\begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} e^t \\ t \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} e^t + 3t \\ -e^t - t \end{pmatrix}$$

$$y_1' = -y_1 + \frac{1}{2}(e^t + 3t)$$

$$y_2' = y_2 + \frac{1}{2}(-e^{+t} - t)$$

But we already know how to solve this eqns,

$$y_1' + y_1 = \frac{1}{2}\left(e^t + 3t\right)$$

$$e^t\left(y_1' + y_1\right) = \frac{1}{2}\left(e^{2t} + 3te^t\right)$$

$$\left(y_1 \cdot e^t\right)' = \frac{1}{2}e^{2t} + \frac{3}{2}te^t$$

$$e^t \cdot y_1 = \int\left(\frac{1}{2}e^{2t} + \frac{3}{2}te^t\right)dt$$

$$= \frac{1}{2}\cdot\frac{1}{2}e^{2t} + \frac{3}{2}\left(te^t - e^t\right) + c_1$$

$$\boxed{y_1(t) = \frac{1}{4}\cdot e^t + \frac{3}{2}\cdot(t-1) + c_1 \cdot e^{-t}}$$

And, similarly

$$y_2' - y_2 = -\frac{1}{2}\cdot\left(e^t + t\right)$$

$$e^{-t}\left(y_2' - y_2\right) = -\frac{1}{2}\left(1 + te^{-t}\right)$$

$$\left(y_2 \cdot e^{-t}\right)' = -\frac{1}{2}\left(1 + te^{-t}\right)$$

$$y_2(t)\cdot e^{-t} = \int\left(-\frac{1}{2} - \frac{1}{2}te^{-t}\right)dt$$

$$y_2 \cdot e^{-t} = -\frac{1}{2}t - \frac{1}{2}\cdot\left(-te^{-t} - e^{-t}\right) + c_2$$

$$\boxed{y_2(t) = -\frac{1}{2}te^t + \frac{1}{2}\cdot(t-1) + c_2 \cdot e^t}$$

Finally, we write the solution in terms of the original variables

$$X = Ty$$
$$= \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} -y_1 - 3y_2 \\ y_1 + y_2 \end{pmatrix} = y_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$X(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix} te^{t}$$
$$+ \begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix} e^{t} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} //$$

Section 7.9 - Problem 3:

$$x' = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix} x + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

$$0 = \det(A - \lambda I) = (2-\lambda)\cdot(-2-\lambda) - 1\cdot(-5)$$

$$0 = -4 - 2\lambda + 2\lambda + \lambda^2 + 5$$

$$0 = \lambda^2 + 1$$

so $\lambda_1 = -i$ and $\lambda_2 = i$.

from $(A - \lambda_1 I)\vartheta_1 = 0$

$$\begin{pmatrix} 2+i & 1 \\ -5 & -2+i \end{pmatrix} \vartheta_1 = 0 \quad \text{gives} \quad \vartheta_1 = \begin{pmatrix} -1 \\ 2+i \end{pmatrix}$$

and $\vartheta_2 = \begin{pmatrix} -1 \\ 2-i \end{pmatrix}$ conjugate of ϑ_1.

So, $x^{(1)}(t) = \vartheta_1 \cdot e^{-it} = \begin{pmatrix} -1 \\ 2+i \end{pmatrix} e^{-it} = \begin{pmatrix} -1 \\ 2+i \end{pmatrix} (\cos t - i \sin t)$

$$= \begin{pmatrix} -\cos t + i\cdot\sin t \\ 2\cos t + \sin t + i\cdot(\cos t - 2\cdot\sin t) \end{pmatrix}$$

$$x^{(1)}(t) = \begin{pmatrix} -\cos t \\ 2\cos t + \sin t \end{pmatrix} + i \cdot \begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix}$$

Hence a set of real-valued solutions are

$$u(t) = \begin{pmatrix} -\cos t \\ 2\cos t + \sin t \end{pmatrix} \quad \text{and} \quad \vartheta(t) = \begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix}$$

$W(u, \vartheta)(t) = -1 \neq 0$ u, ϑ are linearly independent.

u and ϑ are hom. solutions of $x' = Ax$.

As for non-homogeneous part

let (guess) $x_p(t) = \sin t \cdot \alpha + \cos t \cdot \beta + t \cdot \sin t \cdot \gamma + t \cos t \cdot \theta$

where $\alpha, \beta, \gamma, \theta \in \mathbb{R}^2$.

plugging in the equation we obtain that

$\alpha = \begin{pmatrix} 1 \\ -5/2 \end{pmatrix}$, $\beta = \begin{pmatrix} -1/2 \\ 2 \end{pmatrix}$, $\gamma = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $\theta = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Hence,

$x(t) = c_1 \cdot u(t) + c_2 \cdot v(t) + x_p(t)$.

Section 7.9- Problem 4:

$$x' = \underbrace{\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}}_{A} x + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

$$0 = \det(A - \lambda I) = (1-\lambda)(-2-\lambda) - 4 \cdot 1$$

$$0 = -2 - \lambda + 2\lambda + \lambda^2 - 4$$

$$0 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

So $\lambda_1 = -3$ and $\lambda_2 = 2$

and corresponding eigenvalues are

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = v_1 \quad \text{and} \quad v_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

So, $x_h(t) = c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t}$

As for non-hom. part, guess particular solution

$x_p(t) = e^{-2t} \cdot \alpha + e^{t} \cdot \beta$ where $\alpha, \beta \in \mathbb{R}^2$.

plug in the eqn to obtain,

$$\alpha = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

thus, $x_p(t) = e^{-2t} \cdot \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} + e^{t} \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

finally, the general solution is,

$$x(t) = x_h(t) + x_p(t).$$