Math 204 - Differential Equations

Midterm 1 November 5, 2015

Duration: 90 minutes

Instructions: No calculators, no books, no notes, no questions, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. If necessary, you can use the back of these pages, but make sure you have indicated doing so. **Print (i.e., use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: ______________________

Signature: ______________________

Section (Check One):

Section 1: E. Ceyhan (Mon-Wed 10:00) ______

Section 2: E. Ceyhan (Mon-Wed 14:30) ______

Section 3: A. Erdoğan (Tue-Thu 16:00) ______

Question	Points	Score
1	22	
2	15	
3	18	
4	20	
5	20	
6	10	
Total	**105**	

1. (22 points) (a) Solve the differential equation

$$ty' + 2y = \sin t \quad (t > 0)$$

$y' + \frac{2}{t} y = \frac{\sin t}{t}, \quad t > 0$

$\Rightarrow \mu(t) = \exp\left(\int p(t)\,dt\right) = \exp\left(\int \frac{2}{t}\,dt\right) = \exp(2\ln t) = t^2$

$\Rightarrow t^2 y' + 2ty = t\sin t \Rightarrow (t^2 y)' = t \sin t$

so we need $\int t \sin t = -t\cos t + \int \cos t\,dt = -t\cos t + \sin t + c$

$u = t, \; dv = \sin t\,dt$

$du = dt, \; v = -\cos t$

$\Rightarrow t^2 y = -t\cos t + \sin t + c \Rightarrow \underline{y(t) = -t^{-1}\cos t + t^{-2}\sin t + ct^{-2}}$

(b) Solve the initial value problem (IVP) explicitly

$$y' = \frac{8y^3}{t^3} \quad y(1) = \frac{-1}{3}$$

$\frac{dy}{dt} = \frac{8y^3}{t^3} \Rightarrow \frac{dy}{8y^3} = \frac{dt}{t^3} \Rightarrow \frac{1}{8}\frac{y^{-2}}{-2} = \frac{t^{-2}}{-2} + c_1$

$\Rightarrow \frac{1}{8y^2} = \frac{1}{t^2} + c \Rightarrow \frac{1}{8y^2} = \frac{1 + ct^2}{t^2}$

$\Rightarrow y^2 = \frac{t^2}{8(1+ct^2)} \Rightarrow y = \mp\sqrt{\frac{t^2}{8(1+ct^2)}}$

with $y(1) = \frac{-1}{3} = -\sqrt{\frac{1}{8(1+c)}} \Rightarrow c = 1/8$

so $y(t) = -\sqrt{\frac{t^2}{8+t^2}}$

(c) Are the differential equations in parts (a) and (b) linear or nonlinear? (explain the reason briefly).

DE in (a) is linear, since coeff. of y & y' do not involve y and its derivatives

DE in (b) is nonlinear, since there is y^3 in it.

(d) Describe the behaviors of the solutions you found in parts (a) and (b) as $t \to \infty$.

In (a) $y(t) = -t^{-1}\cos t + t^{-2}\sin t + ct^{-2} \to 0$ as $t \to \infty$

In (b) $y(t) = -\sqrt{\frac{t^2}{8+t^2}} \to -1$ as $t \to \infty$.

2. (15 points) Write the general solution to each differential equation below:

(a) $y'' + 100y' = 0$

$r^2 + 100r = 0 \Rightarrow r(r+100) = 0$

$\Rightarrow r_1 = 0 \ , \ r_2 = -100$

so $y_1(t) = 1 \ , \ y_2 = e^{-100t}$

so $\underline{y(t) = c_1 + c_2 e^{-100t}}$

(b) $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0$

$r_{1,2} = \frac{-2 \mp \sqrt{4-8}}{2} = \frac{-2 \mp \sqrt{-4}}{2}$

$= -1 \mp i$

$\rightarrow \lambda = -1 \ , \ \mu = 1$

so $y_1(t) = e^{-t}\cos t \quad y_2(t) = e^{-t}\sin t$

$\Rightarrow \underline{y(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t}$

3. (18 points)

(a) Find the solution for the following IVP.

$$y'' + 24\,y' + 144\,y = 0, \qquad y(0) = 2 \text{ and } y'(0) = 0$$

$r^2 + 24r + 144 = 0$

$\Rightarrow (r+12)^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = -12$

$\Rightarrow y_1(t) = e^{-12t}, \quad y_2(t) = t\,e^{-12t}$

so $y(t) = c_1 e^{-12t} + c_2 t\,e^{-12t}$

with $y(0) = 2 \quad \Rightarrow \quad \underline{c_1 = 2}$

$y'(t) = -12\,c_1\,e^{-12t} + c_2\left(e^{-12t} - 12t\,e^{-12t}\right)$

$y'(0) = 0 \quad \Rightarrow \quad -12c_1 + c_2 = 0 \quad \Rightarrow \quad \underline{c_2 = 24}$

so $y(t) = 2e^{-12t} + 24\,t\,e^{-12t}$

(b) Consider the differential equation $t(t-4)y'' + 3ty' + 4y = 2$ with initial conditions $y(1) = 0$ and $y'(1) = 2$. Is this differential equation linear or nonlinear? (give your reasoning briefly).

It is linear, since coeff's of y and its derivatives involve only terms with t.

(c) Determine the largest open interval in which the IVP in part (b) is certain to have a unique solution.

$$y'' + \frac{3}{t-4}\,y' + \frac{4}{t(t-4)}\,y = \frac{2}{t(t-4)}$$

so, points of discontinuity are $t = 0, \; t = 4$

$\Rightarrow$ largest interval containing 1 is $\underline{(0,4)}$ is the answer.

4. (20 points) (a) The function $y_1(t) = t^{-1}$ is a solution of the differential equation

$$2t^2 y'' + ty' - 3y = 0$$

for $t > 0$. Find the other solution, $y_2(t)$.

$y(t) = v(t)t^{-1} \Rightarrow y' = v't^{-1} - vt^{-2}$

$y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}$

$\Rightarrow 2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + t(v't^{-1} - vt^{-2}) - 3(vt^{-1})$

$= 2tv'' - 3v' + \underbrace{(4t^{-1} - t^{-1} - 3t^{-1})}_{=0} v$

$\Rightarrow 2tv'' - 3v' = 0$ (letting $u = v'$)

$2tu' - 3u = 0 \Rightarrow 2t\frac{du}{dt} = 3u \Rightarrow \frac{du}{3u} = \frac{dt}{2t}$

$\Rightarrow \frac{1}{3}\ln u = \frac{1}{2}\ln t + c_2 \Rightarrow u^{1/3} = c_1 t^{1/2} \Rightarrow u = ct^{3/2}$

$\Rightarrow v'(t) = ct^{3/2} \Rightarrow v(t) = \frac{2c}{5}t^{5/2} + k$

$\Rightarrow y(t) = \frac{2}{5}ct^{3/2} + kt^{-1} \Rightarrow \underline{y_2(t) = t^{3/2}}$

(b) Show that the solutions $y_1(t)$ and $y_2(t)$ in part (a) above form a fundamental set of solutions.

$W = \begin{vmatrix} t^{-1} & t^{3/2} \\ -t^{-2} & \frac{3}{2}t^{1/2} \end{vmatrix} = \frac{3}{2}t^{-1/2} - t^{-1/2} = \frac{1}{2}t^{1/2} \neq 0$ for $t>0$

$\Rightarrow y_1$ & y_2 form a fund. set of sol'ns

(c) Write the general solution of the differential equation in part (a).

$y(t) = c_1 t^{-1} + c_2 t^{3/2}$

5. (a) (5 points) Show that the equation $2xy + (x^2 + y)y' = 0$ is exact.

Let $M(x, y) = 2xy$ and $N(x, y) = x^2 + y$. Then $M_y = 2x$ and $N_x = 2x$. Since $M_y = N_x$ the equation is exact.

(b) (10 points) Solve the equation given in part (a).

Since the given DE (differential equation) is exact, there exists a function $\Psi(x, y)$ such that $\Psi_x = M(x, y)$ and $\Psi_y = N(x, y)$. So

$$\Psi(x, y) = \int M(x, y)dx = \int 2xydx = x^2y + h(y).$$

But then we have that

$$N(x, y) = \Psi_y = x^2 + h'(y) \implies x^2 + y = x^2 + h'(y) \implies y = h'(y) \implies h(y) = y^2/2 + c$$

where $c \in \mathbb{R}$. So the solution of the DE is $\Psi(x, y) = x^2y + y^2/2 = c$.

(c) (5 points) Is the solution of the IVP $2xy + (x^2 + y)y' = 0$, $y(0) = 1$ unique in some interval containing $x = 0$? Explain.

Let

$$f(x, y) = \frac{-2xy}{x^2 + y}$$

so that the DE becomes $y' = f(x, y)$. Now

$$\frac{\partial f}{\partial y} = \frac{(-2x)(x^2 + y) + 2xy.1}{(x^2 + y)^2} = \frac{-2x^3}{x^2 + y)^2}.$$

We see that both $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on the rectangle $-\infty < x < \infty$, $1/2 < y < \infty$ which contains the initial point $(0, 1)$. So by existence and uniqueness theorem there exists an interval containing $x = 0$ on which the given IVP has a unique solution.

6. (10 points) Suppose that the Wronskian of any two solutions of $y'' + p(t)y' + q(t)y = 0$ is constant and that $y_1(t) = t \ln t$ is a solution of $y'' + p(t)y' + q(t)y = 0$. Find $p(t)$ and $q(t)$.

By Abel's theorem for any two solutions u_1 and u_2 of the DE we have that

$$W(u_1, u_2)(t) = ce^{\int -p(t)dt}$$

. Now there exists a pair of functions u_1 and u_2 which form a fundamental set of solutions of the given DE, i.e. $W(u_1, u_2)(t) \neq 0$. Since $W(u_1, u_2)(t)$ is constant, we have that $p(t) = 0$.

So the DE is of the form $y'' + q(t)y = 0$. It is given that $y_1 = t \ln t$ is a solution. Now $y_1' = \ln t + 1$ and $y_1'' = 1/t$, so

$$1/t + q(t)t \ln t = 0 \implies q(t) = -1/(t^2 \ln t).$$