Homework 7 Solutions

Section 5.1

3. Applying the ratio test,

$$\lim_{n\to\infty}\frac{|n!\,x^{4n+4}|}{|(n+1)!\,x^{4n}|}=\lim_{n\to\infty}\frac{x^4}{n+1}=0\,.$$

The series converges absolutely for all values of x. Thus the radius of convergence is $\rho=\infty$.

5) Applying the ratio test,

$$\lim_{n\to\infty}\frac{\left|\frac{(3x+1)^{n+1}}{(n+1)^2}\right|}{\left|\frac{(3x+1)^n}{n^2}\right|}=\lim_{n\to\infty}\underbrace{\frac{n^2}{(n+1)^2}}_{\to 1}|3x+1|=|3x+1|$$

the series converges abs. when $|3x+1|<1$

i.e., $-\frac{2}{3}<x<0$.

from $|3x+1|<1$ it follows that,

$|x+\frac{1}{3}|<\frac{1}{3}$ thus $\rho=\frac{1}{3}$

(the radius of convergence).

9) $f(x)=\sin 3x$, $x_0=0$.

a Taylor series expansion about $x_0=0$ is,

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot(x-0)^n$$

$f'(x)=3\cdot\cos 3x$, $f^{(2)}(x)=3^2\cdot(-1)\cdot\sin 3x$, $f^{(3)}(x)=3^3\cdot(-1)\cdot\cos 3x$

$f^{(4)}(x)=3^4\cdot(-1)^2\cdot\sin 3x$

So,
$f^{(4n)}(x)=3^{4n}\cdot(-1)^{2n}\cdot\sin 3x$

$f^{(4n+1)}(x)=3^{4n+1}\cdot(-1)^{2n}\cdot\cos 3x$

$f^{(4n+2)}(x)=3^{4n+2}\cdot(-1)^{2n+1}\cdot\sin 3x$

$f^{(4n+3)}(x)=3^{4n+3}\cdot(-1)^{2n+1}\cdot\cos 3x$

So that,

$$f^{(4n)}(0) = 0 \quad , \quad f^{(4n+1)}(0) = 3^{4n+1}$$

$$f^{(4n+2)}(0) = 0 \quad , \quad f^{(4n+3)}(0) = 3^{4n+3} \cdot (-1)^{2n+1}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n$$

$$f(x) = \frac{3^1 \cdot x^1}{1!} + \frac{3^3 \cdot x^3}{3!} \cdot (-1) + \frac{3^5 \cdot x^5}{5!} + \frac{3^7 \cdot x^7}{7!} \cdot (-1) + \dots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{3^n \cdot x^{2n+1}}{(2n+1)!} \cdot (-1)^n \quad //$$

From the ratio test,

$$\lim_{n\to\infty} \frac{|(2n+1)!| \cdot 3^{2n+1} \cdot |x^{2n+3}|}{|(2n+3)!| \cdot 3^{[illegible]} \, |x^{2n+1}|} \, |(-1)^n| = \lim_{n\to\infty} \frac{3x^2}{(2n+2)(2n+3)} = 0$$

the series converges absolutely for all values of x.. The radius of convergence is $\rho = \infty$.

13) $f(x) = \ln x, \quad x_0 = 1$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot (x - x_0)^n$$

$$f'(x) = (\ln x)' = \frac{1}{x}$$

$$f^{(2)}(x) = -\frac{1}{x^2} \quad , \quad f^{(3)}(x) = \frac{2}{x^3}$$

So by induction, $f^{(n)}(x) = (-1)^{n-1} \dfrac{(n-1)!}{x^n}$

and $f^{(n)}(x_0 = 1) = (-1)^{n-1} \cdot (n-1)!$

thus,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot (x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} \cdot (n-1)!}{n!} \cdot (x-1)^n$$

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{(x-1)^n}{n} \qquad \text{(for } n=0\text{, not def. Start from } n=1\text{)}$$

As for radius of convergence, observe ratio test,

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^n (x-1)^{n+1}}{n+1}}{\frac{(-1)^{n-1} \cdot (x-1)^n}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot |x-1| = \lim_{n\to\infty} |x-1| = |x-1|$$

is abs. convergent if $|x-1| < 1$

Hence $\rho = 1$ (Radius of Convergence)

17. Applying the ratio test,

$$\lim_{n \to \infty} \frac{\left|(n+1)^2 x^{n+1}\right|}{|n^2 x^n|} = \lim_{n \to \infty} \frac{n^2 + 2n + 1}{n^2} |x| = |x| .$$

The series converges absolutely for $|x| < 1$. Term-by-term differentiation results in

$$y' = \sum_{n=1}^{\infty} n^3 x^{n-1} = 1 + 8x + 27x^2 + 64x^3 + \ldots$$

$$y'' = \sum_{n=2}^{\infty} n^3 (n-1)\, x^{n-2} = 8 + 54x + 192x^2 + 500x^3 + \ldots$$

Shifting the indices, we can also write

$$y' = \sum_{n=0}^{\infty} (n+1)^3 x^n \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+2)^3 (n+1)\, x^n.$$

22. Shift the index down by 3, that is, set $m = n + 3$. It follows that

$$\sum_{n=0}^{\infty} a_n x^{n+3} = \sum_{m=3}^{\infty} a_{m-3}\, x^m = \sum_{n=3}^{\infty} a_{n-3}\, x^n \,.$$

26. Clearly,

$$\sum_{n=1}^{\infty} n a_n\, x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=1}^{\infty} n a_n\, x^{n-1} + \sum_{n=0}^{\infty} a_n\, x^{n+2}.$$

Shifting the index in the first series, that is, setting $k = n - 1$,

$$\sum_{n=1}^{\infty} n a_n\, x^{n-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k.$$

Shifting the index in the second series, that is, setting $k = n + 2$,

$$\sum_{n=0}^{\infty} a_n\, x^{n+2} = \sum_{k=2}^{\infty} a_{k-2} x^k.$$

Combining the series, and starting the summation at $n = 2$,

$$\sum_{n=1}^{\infty} n a_n\, x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n\, x^n = a_1 + 2a_2 x + \sum_{n=2}^{\infty} \left[(n+1) a_{n+1} + a_{n-2} \right] x^n.$$

SECTION 5.2

1.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right] x^n = 0 .$$

Equating all the coefficients to zero,

$$(n+2)(n+1)a_{n+2} - a_n = 0\,, \qquad n = 0, 1, 2, \ldots .$$

We obtain the recurrence relation

$$a_{n+2} = \frac{a_n}{(n+1)(n+2)}\,, \qquad n = 0, 1, 2, \ldots .$$

The subscripts differ by two, so for $k = 1, 2, \ldots$

$$a_{2k} = \frac{a_{2k-2}}{(2k-1)2k} = \frac{a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \ldots = \frac{a_0}{(2k)!}$$

and

$$a_{2k+1} = \frac{a_{2k-1}}{2k(2k+1)} = \frac{a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \ldots = \frac{a_1}{(2k+1)!} .$$

Hence

$$y = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} .$$

The linearly independent solutions are

$$y_1 = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots = \cosh x$$

$$y_2 = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots = \sinh x .$$

(c) The Wronskian at 0 is 1.

2) $y'' - xy' - y = 0, \quad x_0 = 0.$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n \cdot n(n-1) x^{n-2}$$

Then,

$$\sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) \cdot x^{n-2} - x \cdot \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} - \sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \cdot x^n - \sum_{n=0}^{\infty} n \cdot a_n \cdot x^n - \sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

$$\sum_{n=0}^{\infty} x^n \left[(n+2)(n+1) a_{n+2} - n a_n - a_n \right] = 0$$

$$\Rightarrow \quad 0 = (n+2)\underline{(n+1)} a_{n+2} - \underline{(n+1)} a_n$$

$$0 = (n+2) a_{n+2} - a_n$$

$$\boxed{a_{n+2} = \frac{1}{n+2} \cdot a_n}$$

$$a_2 = \frac{1}{2} a_0, \quad a_4 = \frac{1}{4} \cdot a_2 = \frac{1}{2 \cdot 4} a_0$$

So by induction,

$$a_{2n} = \frac{1}{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n)} \cdot a_0 = \frac{a_0}{2^n \cdot n!}$$

Similarly,

$$a_3 = \frac{1}{3} \cdot a_1, \qquad a_5 = \frac{1}{5} a_3 = \frac{1}{3 \cdot 5} a_1$$

$$a_{2n+1} = \frac{1}{3 \cdot 5 \cdots\cdots (2n+1)} \cdot a_1 = \frac{2^n \cdot n!}{(2n+1)!} a_1$$

5) $(1-x)y'' + y = 0, \qquad x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n \cdot x^n$$

$$y' = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n \cdot (n-1) \cdot a_n \cdot x^{n-2}$$

$$\Rightarrow \quad (1-x) \cdot \sum_{n=2}^{\infty} n(n-1) a_n \cdot x^{n-2} + \sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n \cdot x^{n-1} + \sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \cdot x^n - \sum_{n=1}^{\infty} (n+1) \cdot n \cdot a_{n+1} x^n + \sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

$$\underbrace{(2 \cdot a_2 + a_0)}_{0} + \sum_{n=1}^{\infty} x^n \Big(\underbrace{(n+2) a_{n+2} \cdot (n+1) - n(n+1) a_{n+1} + a_n}_{0} \Big) = 0$$

So, $2a_2 + a_0 = 0$ gives $a_2 = -\frac{1}{2} \cdot a_0$

And $\boxed{a_{n+2} \cdot (n+2)(n+1) - a_{n+1} \cdot (n+1)(n) + a_n = 0.}$

$n=0,\quad a_2\cdot 2 - a_1\cdot 0 + a_0 = 0 \quad$ i.e $\quad a_2 = -\frac{1}{2}\cdot a_0$

$n=1,\quad a_3\cdot 3\cdot 2 - a_2\cdot 2 + a_1 = 0$

$6\cdot a_3 = 2a_2 - a_1 = -a_0 - a_1 \rightarrow a_3 = \frac{-a_0}{6} - \frac{a_1}{6}$

$n=2,\quad a_4\cdot 12 - a_3\cdot 6 + a_2 = 0$

$a_4 = \frac{1}{12}\left(6a_3 - a_2\right) = \frac{1}{12}\left(-a_0 - a_1 - \overbrace{a_2}^{-a_0/2}\right)$

$= a_0\left(-\frac{1}{12} + \frac{1}{24}\right) - a_1\left(\frac{1}{12}\right)$

$= a_0\cdot\left(-\frac{1}{24}\right) - a_1\left(\frac{1}{12}\right)$

So,

$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$

$= a_0 + a_1\cdot x + \left(-\frac{1}{2}a_0\right)x^2 + \left(\frac{-a_0}{6} - \frac{a_1}{6}\right)x^3 + \left(\frac{-a_0}{24} - \frac{a_1}{12}\right)x^4 + \dots$

$= a_0\left(1 - \frac{x^2}{2} - \frac{1}{6}x^3 - \frac{x^4}{24} \dots\right) + a_1\left(x - \frac{x^3}{6} - \frac{x^4}{12} \dots\right)$

So $\quad y_1 = 1 - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} + \dots$ and

$y_2 = x - \frac{x^3}{6} - \frac{x^4}{12} - \frac{1}{24}x^5 + \dots$ → Compute yourself!

6.(a,b) Let $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$. Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n.$$

Substitution into the ODE results in

$$(2+x^2)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n - x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 4\sum_{n=0}^{\infty}a_n x^n = 0\,.$$

Before proceeding, write

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n = \sum_{n=2}^{\infty}n(n-1)a_n x^n$$

and

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=1}^{\infty}n\,a_n x^n.$$

It follows that

$$4a_0 + 4a_2 + (3a_1 + 12a_3)x +$$

$$+\sum_{n=2}^{\infty}\left[2(n+2)(n+1)a_{n+2} + n(n-1)a_n - n\,a_n + 4a_n\right]x^n = 0\,.$$

Equating the coefficients to zero, we find that $a_2 = -a_0$, $a_3 = -a_1/4$, and

$$a_{n+2} = -\frac{n^2-2n+4}{2(n+2)(n+1)}\,a_n\,, \quad n = 0, 1, 2, \ldots\,.$$

The indices differ by two, so for $k = 0, 1, 2, \ldots$

$$a_{2k+2} = -\frac{(2k)^2 - 4k + 4}{2(2k+2)(2k+1)}\,a_{2k}$$

and

$$a_{2k+3} = -\frac{(2k+1)^2 - 4k + 2}{2(2k+3)(2k+2)}\,a_{2k+1}.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \ldots$$

$$y_2(x) = x - \frac{x^3}{4} + \frac{7x^5}{160} - \frac{19x^7}{1920} + \ldots\,.$$

(c) The Wronskian at 0 is 1.

9.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$(1+x^2) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - 4x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 6 \sum_{n=0}^{\infty} a_n x^n = 0 .$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n .$$

It follows that

$$6a_0 + 2a_2 + (2a_1 + 6a_3)x +$$

$$+ \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + n(n-1) a_n - 4n\, a_n + 6a_n \right] x^n = 0 .$$

Setting the coefficients equal to zero, we obtain $a_2 = -3a_0$, $a_3 = -a_1/3$, and

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}\, a_n \,, \quad n = 0, 1, 2, \ldots .$$

Observe that for $n = 2$ and $n = 3$, we obtain $a_4 = a_5 = 0$. Since the indices differ by two, we also have $a_n = 0$ for $n \geq 4$. Therefore the general solution is a polynomial

$$y = a_0 + a_1 x - 3a_0 x^2 - a_1 x^3/3 .$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - 3x^2 \quad \text{and} \quad y_2(x) = x - x^3/3 .$$

(c) The Wronskian is $(x^2+1)^2$. At $x = 0$ it is 1.

10.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$(4-x^2)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1)a_n x^n .$$

It follows that

$$2a_0 + 8a_2 + (2a_1 + 24a_3)x +$$

$$+\sum_{n=2}^{\infty}\left[4(n+2)(n+1)a_{n+2} - n(n-1)a_n + 2a_n\right] x^n = 0\,.$$

We obtain $a_2 = -a_0/4$, $a_3 = -a_1/12$ and

$$4(n+2)a_{n+2} = (n-2)a_n\,, \quad n = 0, 1, 2, \ldots .$$

Note that for $n = 2$, $a_4 = 0$. Since the indices differ by two, we also have $a_{2k} = 0$ for $k = 2, 3, \ldots$. On the other hand, for $k = 1, 2, \ldots,$

$$a_{2k+1} = \frac{(2k-3)a_{2k-1}}{4(2k+1)} = \frac{(2k-5)(2k-3)a_{2k-3}}{4^2(2k-1)(2k+1)} = \ldots = \frac{-a_1}{4^k(2k-1)(2k+1)}\,.$$

Therefore the general solution is

$$y = a_0 + a_1 x - a_0\frac{x^2}{4} - a_1\sum_{n=1}^{\infty}\frac{x^{2n+1}}{4^n(2n-1)(2n+1)}\,.$$

Hence the linearly independent solutions are $y_1(x) = 1 - x^2/4$ and

$$y_2(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \ldots = x - \sum_{n=1}^{\infty}\frac{x^{2n+1}}{4^n(2n-1)(2n+1)}\,.$$

(c) The Wronskian at 0 is 1.

13.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$2\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 3\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

We then obtain

$$4a_2 + 3a_0 + \sum_{n=1}^{\infty} \left[2(n+2)(n+1) a_{n+2} + n\, a_n + 3a_n\right] x^n = 0\,.$$

It follows that $a_2 = -3a_0/4$ and

$$2(n+2)(n+1) a_{n+2} + (n+3) a_n = 0$$

for $n = 0, 1, 2, \ldots$. The indices differ by two, so for $k = 1, 2, \ldots$

$$\begin{aligned} a_{2k} &= -\frac{(2k+1) a_{2k-2}}{2(2k-1)(2k)} = \frac{(2k-1)(2k+1) a_{2k-4}}{2^2(2k-3)(2k-2)(2k-1)(2k)} = \ldots \\ &= \frac{(-1)^k 3 \cdot 5 \ldots (2k+1)}{2^k\,(2k)!}\, a_0\,. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= -\frac{(2k+2) a_{2k-1}}{2(2k)(2k+1)} = \frac{(2k)(2k+2) a_{2k-3}}{2^2(2k-2)(2k-1)(2k)(2k+1)} = \ldots \\ &= \frac{(-1)^k\, 4 \cdot 6 \ldots (2k)(2k+2)}{2^k\,(2k+1)!}\, a_1\,. \end{aligned}$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6 + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n 3 \cdot 5 \ldots (2n+1)}{2^n\,(2n)!} x^{2n}$$

$$y_2(x) = x - \frac{1}{3}x^3 + \frac{1}{20}x^5 - \frac{1}{210}x^7 + \ldots = x + \sum_{n=1}^{\infty} \frac{(-1)^n 4 \cdot 6 \ldots (2n+2)}{2^n\,(2n+1)!} x^{2n+1}.$$

(c) The Wronskian at 0 is 1.

14) $2y'' + (x+1)y' + 3y = 0, \quad x_0 = 2$

$$y(x) = \sum_{n=0}^{\infty} a_n \cdot (x-2)^n$$

$$y' = \sum_{n=1}^{\infty} a_n \cdot n \, (x-2)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n \cdot (n-1) \cdot a_n \cdot (x-2)^{n-2}$$

$$2 \sum_{n=2}^{\infty} n(n-1) a_n \cdot (x-2)^{n-2} + \underbrace{(x+1)}_{(x-2)+3} \cdot \sum_{n=1}^{\infty} n \cdot a_n (x-2)^{n-1} + 3 \sum_{n=0}^{\infty} a_n (x-2)^n$$

$$2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-2)^n + (x-2) \sum_{n=1} n a_n (x-2)^{n-1} + 3 \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} + 3 \sum_{n=0} a_n (x-2)^n = 0$$

$$\sum_{n=0}^{\infty} 2 \cdot a_{n+2} \cdot (n+2)(n+1)(x-2)^n + \sum_{n=0}^{\infty} n a_n (x-2)^n + \sum_{n=0} (3+3n) a_{n+1} (x-2)^n + 3 \sum_{n=0} a_n (x-2)^n = 0$$

$$\sum_{n=0}^{\infty} (x-2)^n \cdot \left[a_{n+2} \cdot (n+2)(n+1) \cdot 2 + 3(n+1) a_{n+1} + a_n (n+3) \right] = 0$$

So, $\boxed{2 \cdot (n+2)(n+1) \cdot a_{n+2} + 3 \cdot (n+1) \cdot a_{n+1} + (n+3) \cdot a_n = 0}$

$n = 0, \quad 4 \cdot a_2 + 3 a_1 + 3 a_0 = 0$

$$a_2 = -\frac{3}{4} a_1 - \frac{3}{4} a_0 \checkmark$$

$n = 1, \quad 12 \cdot a_3 + 6 a_2 + 4 a_1 = 0$

$$a_3 = -\frac{1}{2}a_2 - \frac{1}{3}a_1$$
$$= -\frac{1}{2}\left(-\frac{3}{4}a_1 - \frac{3}{4}a_0\right) - \frac{1}{3}a_1$$
$$= a_1\left(\frac{3}{8} - \frac{1}{3}\right) + a_0 \cdot \left(+\frac{3}{8}\right)$$
$$= a_1 \cdot \left(\frac{1}{24}\right) + a_0 \left(\frac{3}{8}\right)$$

$n=2, \quad 24 \cdot a_4 + 9a_3 + 5a_2 = 0$

$$a_4 = -\frac{9}{24}a_3 - \frac{5}{24}a_2$$
$$= -\frac{9}{24}\left(\frac{a_1}{24} + \frac{3a_0}{8}\right) - \frac{5}{24} \cdot \left(-\frac{3}{4}a_1 - \frac{3}{4}a_0\right)$$
$$= a_1 \cdot \left(\frac{-9}{576} + \frac{15}{96}\right) + a_0 \left(\frac{-27}{192} + \frac{15}{96}\right)$$
$$= a_1 \cdot \frac{81}{576} + a_0 \cdot \frac{3}{192} = a_1 \cdot \frac{9}{64} + a_0 \frac{1}{64}$$

$$y = \sum_{n=0}^{\infty} (x-2)^n \cdot a_n$$

$$= a_0 + a_1(x-2)^1 + a_2(x-2)^2 + a_3(x-2)^3 + a_4(x-2)^4 + \dots$$
$$= a_0 + a_1(x-2)^1 + (x-2)^2 \cdot \left[-\frac{3}{4}a_1 - \frac{3}{4}a_0\right] + (x-2)^3 \cdot \left[\frac{a_1}{24} + \frac{3a_0}{8}\right] + (x-2)^4 \left[\frac{9a_1}{64} + \frac{a_0}{64}\right] + \dots$$
$$= a_0\left[1 - \frac{3}{4}(x-2)^2 + \frac{3}{8}(x-2)^3 + \frac{1}{64}(x-2)^4 + \dots\right]$$
$$+ a_1\left[(x-2)^1 - \frac{3}{4}(x-2)^2 + \frac{1}{24}(x-2)^3 + \frac{9}{64}(x-2)^4 + \dots\right]$$

$$Y_1 = 1 - \frac{3}{4}(x-2)^2 + \frac{3}{8}(x-2)^3 + \frac{1}{64}(x-2)^4 + \dots$$
$$Y_2 = (x-2)^1 - \frac{3}{4}(x-2)^2 + \frac{1}{24}(x-2)^3 + \frac{9}{64}(x-2)^4 + \dots$$

Section 5.3

1) $y'' + xy' + y = 0$, $y(0) = 2$, $y'(0) = 0$

$y = \phi(x)$ is a sol. $x_0 = 0$.

given that $\phi(0) = 2$ and $\phi'(0) = 0$.

$y = \phi(x)$ is a sol. then plug in the eq.

$$\phi'' + x \cdot \phi' + \phi = 0$$

(1) — $\phi''(x) = -x \cdot \phi'(x) - \phi(x)$

Take derivative

(2) — $\phi'''(x) = (-1) \cdot \phi'(x) - x \cdot \phi''(x) - \phi'(x)$

Once more,

(3) — $\phi^{(4)}(x) = (-1) \cdot \phi''(x) - 1 \cdot \phi''(x) - x \cdot \phi'''(x) - \phi''(x)$

From (1) equation,

$\phi''(0) = -0 \cdot \phi'(0) - \phi(0) = -\phi(0) = -2$ ✓

From (2) equation;

$\phi'''(0) = -1 \cdot \underbrace{\phi'(0)}_{0} - 0 \cdot \phi''(0) - \underbrace{\phi'(0)}_{0} = 0$ ✓

From (3), $\phi^{(4)}(0) = -1 \cdot \phi''(0) - \phi''(0) - 0 \cdot \phi'''(0) - \phi''(0) = 2 + 2 + 0 + 2 = 6$ ✓

2. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -(\sin x)y' - (\cos x)y\,.$$

Differentiating twice,

$$\begin{aligned} y''' &= -(\sin x)y'' - 2(\cos x)y' + (\sin x)y \\ y^{(4)} &= -(\sin x)y''' - 3(\cos x)y'' + 3(\sin x)y' + (\cos x)y\,. \end{aligned}$$

Given that $\phi(0) = 0$ and $\phi'(0) = -1$, the first equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = 2$ and $\phi^{(4)}(0) = 0$.

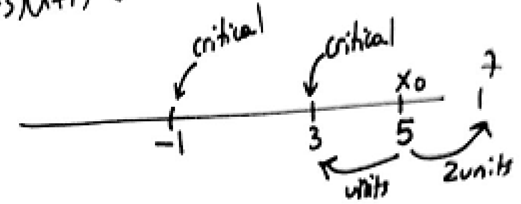
5. Clearly, $p(x) = 4$ and $q(x) = 6x$ are analytic for all x. Hence the series solutions converge everywhere.

Sciences

6) $(x^2-2x-3)\,y'' + xy' + 4y = 0.$ $x_0 = 5,\ x_0 = -5,\ x_0 = 0$

$$y'' + \frac{x}{(x-3)(x+1)}\cdot y' + \frac{4}{(x-3)(x+1)}\,y = 0.$$

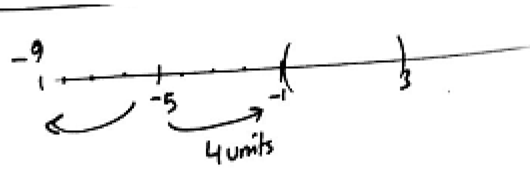
IF $x_0 = 5$ then



so, $\rho = 2$ units.

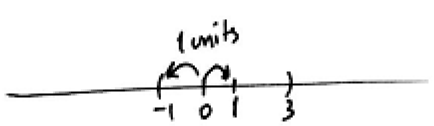
interval should not contain discontinuity points $\{3, -1\}$

IF $x_0 = -5$



$\rho = 4$

IF $x_0 = 0$



$\rho = 1.$

8. The only root of $P(x) = x$ is zero. Hence $\rho_{min} = 2$.

11) $y'' + (\sin x)y = 0$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n \cdot x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \Bigg(\underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}}\Bigg)\Bigg(\underbrace{\sum_{n=0}^{\infty} a_n x^n}_{a_0 + a_1x^1 + a_2x^2 + \dots}\Bigg) = 0$$

$$\left[2a_2x^0 + 6a_3x^1 + 12a_4x^2 + 20a_5x^3 + \dots\right] + \left[x \cdot a_0 + x^2 a_1 + x^3\left(-\frac{a_0}{6} + a_2\right) + x^4\left(-\frac{a_1}{6} + a_3\right) + \dots\dots\right. = 0$$

So, $2a_2 = 0$, $a_0 + 6a_3 = 0$, $12a_4 + a_1 = 0$, $20a_5 - \frac{a_0}{6} + a_2 = 0$

$\Rightarrow a_3 = -\frac{1}{6}a_0$, $a_2 = 0$, $a_4 = -\frac{1}{12}a_1$, $a_5 = \frac{a_0}{120}$, ...

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

$$= a_0 + a_1x + 0 + \frac{a_0}{-6}x^3 - \frac{1}{12}a_1x^4 + \frac{a_0}{120}x^5 + \dots$$

$$= a_0\left(1 - \frac{x^3}{6} + \frac{x^5}{120} + \dots\right)$$

$$+ a_1\left(x - \frac{x^4}{12} + \dots\dots\right)$$

So, $y_1 = 1 - \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^6}{180} + \dots$ → do yourself!

$y_2 = x - \frac{x^4}{12} + \frac{x^6}{180} + \frac{x^7}{504} + \dots$ → do yourself!

13. The Taylor series expansion of $\cos x$, about $x_0 = 0$, is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\,.$$

Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right]\left[\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n\right] + \sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

The coefficient of x^n in the product of the two series is

$$c_n = 2a_2 b_n + 6a_3 b_{n-1} + 12a_4 b_{n-2} + \ldots + (n+1)n a_{n+1} b_1 + (n+2)(n+1)a_{n+2} b_0,$$

in which $\cos x = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots$. It follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} c_n x^n + \sum_{n=1}^{\infty}(n-2)a_n x^n = 0\,.$$

Expanding the product of the series, it follows that

$$2a_2 - 2a_0 + 6a_3 x + (-a_2 + 12a_4)x^2 + (-3a_3 + 20a_5)x^3 + \ldots$$
$$\ldots - a_1\,x + a_3 x^3 + 2a_4 x^4 + \ldots = 0\,.$$

Setting the coefficients equal to zero, $a_2 - a_0 = 0$, $6a_3 - a_1 = 0$, $-a_2 + 12a_4 = 0$, $-3a_3 + 20a_5 + a_3 = 0$, $\ldots$. Hence the general solution is

$$y(x) = a_0 + a_1 x + a_0 x^2 + a_1\frac{x^3}{6} + a_0\frac{x^4}{12} + a_1\frac{x^5}{60} + a_0\frac{x^6}{120} + a_1\frac{x^7}{560} + \ldots\,.$$

We find that two linearly independent solutions ($W(y_1, y_2)(0) = 1$) are

$$y_1(x) = 1 + x^2 + \frac{x^4}{12} + \frac{x^6}{120} + \ldots$$

$$y_2(x) = x + \frac{x^3}{6} + \frac{x^5}{60} + \frac{x^7}{560} + \ldots$$

The nearest zero of $P(x) = \cos x$ is at $x = \pm\pi/2$. Hence $\rho_{min} = \pi/2$.