7.7

1.(a) The eigenvalues and eigenvectors were found in Problem 1, Section 7.5.

$$r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \quad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -e^{-t} \\ 2\,e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} -2\,e^{2t} \\ e^{2t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} -e^{-t} & -2\,e^{2t} \\ 2\,e^{-t} & e^{2t} \end{pmatrix}.$$

(b) We now have

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix} \text{ and } \boldsymbol{\Psi}^{-1}(0) = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix},$$

So that

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) = \frac{1}{3}\begin{pmatrix} -e^{-t} + 4e^{2t} & -2e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & 4e^{-t} - e^{2t} \end{pmatrix}.$$

3.(a) The eigenvalues and eigenvectors were found in Problem 3, Section 7.5. The general solution of the system is

$$\mathbf{x} = c_1 \begin{pmatrix} -3e^{t} \\ e^{t} \end{pmatrix} + c_2 \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} -3e^{t} & -e^{-t} \\ e^{t} & e^{-t} \end{pmatrix}.$$

(b) Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$-3c_1 - c_2 = 1$$
$$c_1 + c_2 = 0\,,$$

to obtain $c_1 = -1/2\,,\ c_2 = 1/2\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$-3c_1 - c_2 = 0$$
$$c_1 + c_2 = 1\,,$$

to obtain $c_1 = -1/2$, $c_2 = 3/2$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \frac{3}{2}e^t - \frac{3}{2}e^{-t} \\ -\frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \frac{1}{2}\begin{pmatrix} 3e^t - e^{-t} & 3e^t - 3e^{-t} \\ -e^t + e^{-t} & -e^t + 3e^{-t} \end{pmatrix}.$$

5.(a) The general solution, found in Problem 3, Section 7.6, is given by

$$\mathbf{x} = c_1 \begin{pmatrix} -2\cos t + \sin t \\ 5\cos t \end{pmatrix} + c_2 \begin{pmatrix} -2\sin t - \cos t \\ 5\sin t \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} -2\cos t + \sin t & -2\sin t - \cos t \\ 5\cos t & 5\sin t \end{pmatrix}.$$

(b) Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} -2c_1 - c_2 &= 1 \\ 5c_1 &= 0\,, \end{aligned}$$

resulting in $c_1 = 0$, $c_2 = -1$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \cos t + 2\sin t \\ -5\sin t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} -2c_1 - c_2 &= 0 \\ 5c_1 &= 1\,, \end{aligned}$$

resulting in $c_1 = 1/5$, $c_2 = -2/5$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{pmatrix} \cos t + 2\sin t & \sin t \\ -5\sin t & \cos t - 2\sin t \end{pmatrix}.$$

7.(a) The general solution, found in Problem 15, Section 7.5, is given by

$$\mathbf{x} = c_1 \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} -3e^{4t} \\ e^{4t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} -e^{2t} & -3e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}.$$

(b) Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$
\begin{aligned}
-c_1 - 3c_2 &= 1 \\
c_1 + c_2 &= 0\,,
\end{aligned}
$$

resulting in $c_1 = 1/2\,,\ c_2 = -1/2\,.$ The corresponding solution is

$$
\mathbf{x} = \frac{1}{2}\begin{pmatrix} -e^{2t} + 3e^{4t} \\ e^{2t} - e^{4t} \end{pmatrix}.
$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$ require that

$$
\begin{aligned}
-c_1 - 3c_2 &= 0 \\
c_1 + c_2 &= 1\,,
\end{aligned}
$$

resulting in $c_1 = 3/2\,,\ c_2 = -1/2\,.$ The corresponding solution is

$$
\mathbf{x} = \frac{1}{2}\begin{pmatrix} -3e^{2t} + 3e^{4t} \\ 3e^{2t} - e^{4t} \end{pmatrix}.
$$

Therefore the fundamental matrix is

$$
\mathbf{\Phi}(t) = \frac{1}{2}\begin{pmatrix} -e^{2t} + 3e^{4t} & -3e^{2t} + 3e^{4t} \\ e^{2t} - e^{4t} & 3e^{2t} - e^{4t} \end{pmatrix}.
$$

7.7) 11. $x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x\,, \quad x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

The solution of the initial problem is given by

$$
x = \Phi(t)\, x(0) = \frac{1}{2}\begin{pmatrix} 3e^{t} - e^{-t} & -e^{t} + e^{-t} \\ 3e^{t} - 3e^{-t} & -e^{t} + 3e^{-t} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}
$$

from ex. 3.

$$
= \begin{pmatrix} 3e^{t} - e^{-t} - \frac{3}{2}e^{t} + \frac{3}{2}e^{-t} \\ 3e^{t} - 3e^{-t} - \frac{3}{2}e^{t} + \frac{9}{2}e^{-t} \end{pmatrix}
$$

$$
= \begin{pmatrix} \frac{3}{2}e^{t} + \frac{1}{2}e^{-t} \\ \frac{3}{2}e^{t} + \frac{3}{2}e^{-t} \end{pmatrix}
$$

Section 7.8 : Problem (1c): $x' = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} x$

From, $\det(A - \lambda I) = 0$ we find eigenvalues

i.e $\det\begin{bmatrix} 3-\lambda & 1 \\ -4 & -1-\lambda \end{bmatrix} = 0$ i.e $(3-\lambda)(-1-\lambda) + 4\cdot 1 = 0$

OR, $-3 - 3\lambda + \lambda + \lambda^2 + 4 = 0$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

So that $\lambda_1 = 1$ and $\lambda_2 = 1$.

corresponding eigenvector for $\lambda_1 = 1$ is v_1 such that $(A - \lambda_1 I)v_1 = 0$ i.e $v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

and hence $x^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t$.

and $x^{(2)} = v_1 \cdot t \cdot e^t + v_2 e^t$ where v_2 (gen. eigenvector) satisfies $(A - \lambda_2 I)v_2 = v_1$.

i.e $\begin{pmatrix} 3-\lambda_2 & 1 \\ -4 & -1-\lambda_2 \end{pmatrix} v_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} v_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$$

Thus, $x^{(2)} = t\cdot e^t \begin{pmatrix} -1 \\ 2 \end{pmatrix} + e^t \cdot \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$, And the general solution is $x(t) = c_1 \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t + c_2 \left[t \cdot e^t \begin{pmatrix} -1 \\ 2 \end{pmatrix} + e^t \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} \right]$.

Section 7.3 – Problem (3c): $X' = \begin{pmatrix} -3/2 & -1/4 \\ 1 & -1/2 \end{pmatrix} X$

$\det(A - \lambda I) = 0$ i.e $\det\begin{pmatrix} -3/2-\lambda & -1/4 \\ 1 & -\frac{1}{2}-\lambda \end{pmatrix} = 0$

$$0 = \left(-\frac{3}{2}-\lambda\right)\left(-\frac{1}{2}-\lambda\right) - 1\cdot\left(-\frac{1}{4}\right)$$

$$0 = \frac{3}{4} + \frac{3}{2}\lambda + \frac{1}{2}\lambda + \lambda^2 + \frac{1}{4}$$

$0 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$ so $\lambda_1 = \lambda_2 = -1$

From $(A - \lambda_1 I)v_1 = 0$ we get $v_1 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

and from $(A - \lambda_2 I)v_2 = v_1$ we get $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

So that,

$$X^{(1)}(t) = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} e^{-t} \text{ and,}$$

$$X^{(2)}(t) = t e^{-t}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus, the general solution is of the form,

$$X(t) = c_1 \cdot X^{(1)} + c_2 X^{(2)}$$
$$= c_1 t e^{-t}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} + c_2 \cdot \left[t e^{-t}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right].$$

7.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 6r + 9 = 0$. The only root is $r = -3$, which is an eigenvalue of multiplicity two. Substituting $r = -3$ into the coefficient matrix, the system reduces to the single equation $\xi_1 - \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1\,, 1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

For a second linearly independent solution, we search for a generalized eigenvector. Its components satisfy

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

that is, $4\eta_1 - 4\eta_2 = 1$. Let $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = k + 1/4$. It follows that a second solution is given by

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} k + 1/4 \\ k \end{pmatrix} e^{-3t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} \right].$$

Imposing the initial conditions, we require that $c_1 + c_2/4 = 4$, $c_1 = 2$, which results in $c_1 = 2$ and $c_2 = 8$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 8 \\ 8 \end{pmatrix} te^{-3t}.$$

Section 7.8 – Problem 9a:

$$x' = \begin{pmatrix} 2 & 3/2 \\ -3/2 & -1 \end{pmatrix} x$$

$$x(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

From

$$0 = \det(A - \lambda I)$$

$$0 = (2-\lambda)\cdot(-1-\lambda) - \frac{3}{2}\cdot\left(-\frac{3}{2}\right)$$

$$0 = -2 - 2\lambda + \lambda + \lambda^2 + \frac{9}{4}$$

$$0 = \lambda^2 - \lambda + \frac{1}{4}$$

$$0 = \left(\lambda - \frac{1}{2}\right)^2 \qquad \lambda_1 = \lambda_2 = \frac{1}{2}$$

so that, from $(A - \lambda_1 I)\,v_1 = 0$ we obtain

$$\begin{pmatrix} 2-\frac{1}{2} & 3/2 \\ -\frac{3}{2} & -1-\frac{1}{2} \end{pmatrix} v_1 = 0 \quad \text{i.e} \quad v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

And from $(A - \lambda_2 I)\,v_2 = v_1$ we obtain

$$\begin{pmatrix} 2-1/2 & 3/2 \\ -3/2 & -1-\frac{1}{2} \end{pmatrix} v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{i.e} \quad v_2 = \begin{pmatrix} -2/3 \\ 0 \end{pmatrix}$$

so the general solution is

$$x^{(1)}(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t/2}, \quad x^{(2)} = te^{t/2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{t/2}\cdot\begin{pmatrix} -2/3 \\ 0 \end{pmatrix}$$

$$x(t) = c_1\cdot e^{t/2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\left[te^{t/2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{t/2}\begin{pmatrix} -2/3 \\ 0 \end{pmatrix}\right]$$

Imposing $x(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ we find $c_1 = -1$, $c_2 = -3$

$$x(t) = \begin{pmatrix} 3 + 3t \\ -1 - 3t \end{pmatrix} e^{t/2}.$$