Boyce/DiPrima 10th ed, Ch 2.1: Linear Equations; Method of Integrating Factors

- A linear first order ODE has the general form

$$\frac{dy}{dt} = f(t, y)$$

  where f is linear in y. Examples include equations with constant coefficients, such as those in Chapter 1,

$$y' = -ay + b$$

  or equations with variable coefficients:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Constant Coefficient Case

- For a first order linear equation with constant coefficients,

$$\frac{dy}{dt} = -ay + b,$$

  recall that we can use methods of calculus to solve:

$$\frac{dy/dt}{y - b/a} = -a$$

$$\int \frac{dy}{y - b/a} = -\int a\,dt$$

$$\ln\left|y - b/a\right| = -a\,t + C$$

$$y = b/a + ke^{at}, \quad k = \pm e^{C}$$

Variable Coefficient Case: Method of Integrating Factors

- We next consider linear first order ODEs with variable coefficients:

$$\frac{dy}{dt} + p(t)y = g(t)$$

- The method of integrating factors involves multiplying this equation by a function $\mu(t)$, chosen so that the resulting equation is easily integrated.

Example 2: Integrating Factor (1 of 2)

- Consider the following equation:

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$$

- Multiplying both sides by $\mu(t)$, we obtain

$$\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)y = \frac{1}{2}\mu(t)e^{t/3}$$

- We will choose $\mu(t)$ so that left side is derivative of known quantity. Consider the following, and recall product rule:

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y$$

- Choose $\mu(t)$ so that

$$\mu'(t) = \frac{1}{2}\mu(t) \;\Rightarrow\; \mu(t) = e^{t/2}$$

Example 2: General Solution (2 of 2)

- With $\mu(t) = e^{t/2}$, we solve the original equation as follows:

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$$

$$e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t/2}y = \frac{1}{2}e^{5t/6}$$

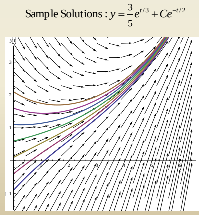
$$\frac{d}{dt}\left[e^{t/2}y\right] = \frac{1}{2}e^{5t/6}$$

$$e^{t/2}y = \frac{3}{5}e^{5t/6} + C$$

general solution:

$$y = \frac{3}{5}e^{t/3} + Ce^{-t/2}$$

Sample Solutions: $y = \frac{3}{5}e^{t/3} + Ce^{-t/2}$



Method of Integrating Factors: Variable Right Side

- In general, for variable right side $g(t)$, the solution can be found as follows:

$$\frac{dy}{dt} + ay = g(t)$$

$$\mu(t)\frac{dy}{dt} + a\mu(t)y = \mu(t)g(t)$$

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t)$$

$$\frac{d}{dt}\left[e^{at}y\right] = e^{at}g(t)$$

$$e^{at}y = \int e^{at}g(t)dt$$

$$y = e^{-at}\int e^{at}g(t)dt + Ce^{-at}$$

Example 3: General Solution (1 of 2)

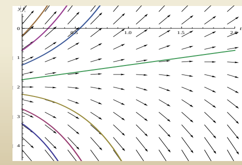
- We can solve the following equation
$$\frac{dy}{dt} - 2y = 4 - t$$
using the formula derived on the previous slide:
$$y = e^{-at}\int e^{at}g(t)dt + Ce^{-at} = e^{2t}\int e^{-2t}(4-t)dt + Ce^{2t}$$
- Integrating by parts, $\int e^{-2t}(4-t)dt = \int 4e^{-2t}dt - \int te^{-2t}dt$
$$= -2e^{t/5} - \left[-\frac{1}{2}te^{-2t} + \int \frac{1}{2}e^{-2t}dt\right]$$
$$= -\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t}$$
- Thus $y = e^{2t}\left(-\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t}\right) + Ce^{2t} = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}$

Example 3: Graphs of Solutions (2 of 2)

$$\frac{dy}{dt} - 2y = 4 - t$$

- The graph shows the direction field along with several integral curves. If we set $C = 0$, the exponential term drops out and you should notice how the solution in that case, through the point (0, -7/4), separates the solutions into those that grow exponentially in the positive direction from those that grow exponentially in the negative direction..

$$y = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}$$



Method of Integrating Factors for General First Order Linear Equation

- Next, we consider the general first order linear equation
$$\frac{dy}{dt} + p(t)y = g(t)$$
- Multiplying both sides by $\mu(t)$, we obtain
$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$
- Next, we want $\mu(t)$ such that $\mu'(t) = p(t)\mu(t)$, from which it will follow that
$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y$$

Integrating Factor for General First Order Linear Equation

- Thus we want to choose $\mu(t)$ such that $\mu'(t) = p(t)\mu(t)$.
- Assuming $\mu(t) > 0$, it follows that
$$\int \frac{d\mu(t)}{\mu(t)} = \int p(t)dt \;\Rightarrow\; \ln \mu(t) = \int p(t)dt + k$$
- Choosing $k = 0$, we then have
$$\mu(t) = e^{\int p(t)dt},$$
and note $\mu(t) > 0$ as desired.

Solution for General First Order Linear Equation

- Thus we have the following:
$$\frac{dy}{dt} + p(t)y = g(t)$$
$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t), \quad \text{where } \mu(t) = e^{\int p(t)dt}$$
- Then
$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)g(t)$$
$$\mu(t)y = \int \mu(t)g(t)dt + c$$
$$y = \frac{\int \mu(t)g(t)dt + c}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int p(t)dt}$$

Example 4: General Solution (1 of 2)

- To solve the initial value problem
$$ty' + 2y = 4t^2, \quad y(1) = 2,$$
first put into standard form:
$$y' + \frac{2}{t}y = 4t, \quad \text{for } t \neq 0$$
- Then
$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln(t^2)} = t^2$$
and hence
$$y = \frac{\int \mu(t)g(t)dt + C}{\mu(t)} = \frac{\int t^2(4t)dt + C}{t^2} = \frac{1}{t^2}\left[\int 4t^3dt + C\right] = t^2 + \frac{C}{t^2}$$

$ty' + 2y = 4t^2, \quad y(1) = 2,$

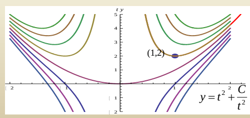
Example 4: Particular Solution (2 of 2)

- Using the initial condition $y(1) = 2$ and general solution

  it follows that $y = t^2 + \frac{C}{t^2}, y(1) = 1 + C = 2 \Rightarrow C = 1$
- The graphs below show solution curves for the differential equation, including a particular solution whose graph contains the initial point (1,2). Notice that when C=0, we get the parabolic solution (shown) and that solution separates the solutions into those that are asymptotic to the positive versus negative y-axis.

$$y = t^2 + \frac{1}{t^2}$$

$$y = t^2$$



Example 5: A Solution in Integral Form (1 of 2)

- To solve the initial value problem

$$2y' + ty = 2, \quad y(0) = 1,$$

  first put into standard form:

$$y' + \frac{t}{2}y = 1$$

- Then

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{t}{2}dt} = e^{\frac{t^2}{4}}$$

  and hence

$$y = e^{-t^2/4}\left(\int_0^t e^{s^2/4}ds + C\right) = e^{-t^2/4}\left(\int_0^t e^{s^2/4}ds\right) + Ce^{-t^2/4}$$

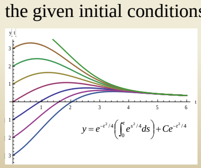
$2y' + ty = 2, \quad y(0) = 1,$

Example 5: A Solution in Integral Form (2 of 2)

- Notice that this solution must be left in the form of an integral, since there is no closed form for the integral.

$$y = e^{-t^2/4}\left(\int_0^t e^{s^2/4}ds\right) + Ce^{-t^2/4}$$

- Using software such as *Mathematica* or Maple, we can approximate the solution for the given initial conditions as well as for other initial conditions.
- Several solution curves are shown.



Boyce/DiPrima 10th ed, Ch 2.2: Separable Equations

- In this section we examine a subclass of linear and nonlinear first order equations. Consider the first order equation

$$\frac{dy}{dx} = f(x, y)$$

- We can rewrite this in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

- For example, let $M(x,y) = - f(x,y)$ and $N(x,y) = 1$. There may be other ways as well. In differential form,

$$M(x, y)dx + N(x, y)dy = 0$$

- If M is a function of x only and N is a function of y only, then

$$M(x)dx + N(y)dy = 0$$

- In this case, the equation is called **separable**.

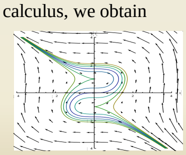
Example 1: Solving a Separable Equation

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2}$$

- Separating variables, and using calculus, we obtain

$$(1 - y^2)dy = (x^2)dx$$
$$\int (1 - y^2)dy = \int (x^2)dx$$
$$y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$
$$3y - y^3 = x^3 + C$$



- The equation above defines the solution y implicitly. A graph showing the direction field and implicit plots of several solution curves for the differential equation is given above.

Example 2: Implicit and Explicit Solutions (1 of 4)

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)}$$

- Separating variables and using calculus, we obtain

$$2(y - 1)dy = (3x^2 + 4x + 2)dx$$
$$2\int (y - 1)dy = \int (3x^2 + 4x + 2)dx$$
$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

- The equation above defines the solution y implicitly. An explicit expression for the solution can be found in this case:

$$y^2 - 2y - (x^3 + 2x^2 + 2x + C) = 0 \quad \Rightarrow \quad y = \frac{2 \pm \sqrt{4 + 4(x^3 + 2x^2 + 2x + C)}}{2}$$

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C}$$

$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$

Example 2: Initial Value Problem (2 of 4)

- Suppose we seek a solution satisfying $y(0) = -1$. Using the implicit expression of y, we obtain

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$
$$(-1)^2 - 2(-1) = C \;\Rightarrow\; C = 3$$

- Thus the implicit equation defining y is

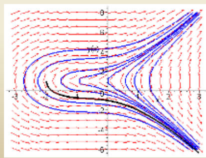
$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

- Using an explicit expression of y,

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C}$$
$$-1 = 1 \pm \sqrt{C} \;\Rightarrow\; C = 4$$

- It follows that

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

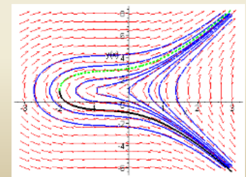


$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$

Example 2: Initial Condition $y(0) = 3$ (3 of 4)

- Note that if initial condition is $y(0) = 3$, then we choose the positive sign, instead of negative sign, on the square root term:

$$y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4}$$



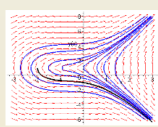
Example 2: Domain (4 of 4)

- Thus the solutions to the initial value problem

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \quad y(0) = -1$$

are given by

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3 \quad \text{(implicit)}$$
$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \quad \text{(explicit)}$$



- From explicit representation of y, it follows that

$$y = 1 - \sqrt{x^2(x+2) + 2(x+2)} = 1 - \sqrt{(x+2)(x^2+2)}$$

and hence the domain of y is $(-2, \infty)$. Note $x = -2$ yields $y = 1$, which makes the denominator of dy/dx zero (vertical tangent).
- Conversely, the domain of y can be estimated by locating vertical tangents on the graph (useful for implicitly defined solutions).

Example 3: Implicit Solution of an Initial Value Problem (1 of 2)

- Consider the following initial value problem:

$$y' = \frac{4x - x^3}{4 + y^3}, \quad y(0) = 1$$

- Separating variables and using calculus, we obtain

$$(4 + y^3)dy = (4x - x^3)dx$$
$$\int (4 + y^3)dy = \int (4x - x^3)dx$$
$$4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + c$$
$$y^4 + 16y + x^4 - 8x^2 = C \text{ where } C = 4c$$

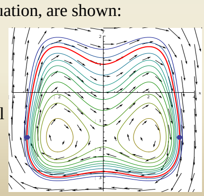
- Using the initial condition, $y(0)=1$, it follows that $C = 17$.

$$y^4 + 16y + x^4 - 8x^2 = 17$$

$y' = \frac{4x - x^3}{4 + y^3}, \quad y(0) = 1$

Example 3: Graph of Solutions (2 of 2)

- Thus the general solution is $y^4 + 16y + x^4 - 8x^2 = C$
and the solution through (0,2) is $y^4 + 16y + x^4 - 8x^2 = 17$
- The graph of this particular solution through (0, 2) is shown in red along with the graphs of the direction field and several other solution curves for this differential equation, are shown:
- The points identified with blue dots correspond to the points on the red curve where the tangent line is vertical: $y = \sqrt[3]{-4} \approx -1.5874$ $x \approx \pm 3.3488$ on the red curve, but at all points where the line connecting the blue points intersects solution curves the tangent line is vertical.



Parametric Equations

- The differential equation: $\frac{dy}{dx} = \frac{F(x,y)}{G(x,y)}$
is sometimes easier to solve if x and y are thought of as dependent variables of the independent variable t and rewriting the single differential equation as the system of differential equations:

$$\frac{dy}{dt} = F(x,y) \text{ and } \frac{dx}{dt} = G(x,y)$$

Chapter 9 is devoted to the solution of systems such as these.

Boyce/DiPrima 10th ed, Ch 2.6: Exact Equations and Integrating Factors

Elementary Differential Equations and Boundary Value Problems, 10th edition, by William E. Boyce and Richard C. DiPrima, ©2013 by John Wiley & Sons, Inc.

- Consider a first order ODE of the form
$$M(x,y)+N(x,y)y'=0$$
- Suppose there is a function ψ such that
$$\psi_x(x,y)=M(x,y),\ \psi_y(x,y)=N(x,y)$$
and such that $\psi(x,y) = c$ defines $y = \phi(x)$ implicitly. Then
$$M(x,y)+N(x,y)y'=\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\frac{dy}{dx}=\frac{d}{dx}\psi[x,\phi(x)]$$
and hence the original ODE becomes
$$\frac{d}{dx}\psi[x,\phi(x)]=0$$
- Thus $\psi(x,y) = c$ defines a solution implicitly.
- In this case, the ODE is said to be **exact**.

Example 1: Exact Equation

- Consider the equation:
$$2x+y^2+2xyy'=0$$
- It is neither linear nor separable, but there is a function φ such that
$$\frac{\partial\varphi}{\partial y}=2x+y^2 \text{ and } \frac{\partial\varphi}{\partial x}=2xy$$
- The function that works is $\varphi(x,y)=x^2+xy^2$
- Thinking of y as a function of x and calling upon the chain rule, the differential equation and its solution become
$$\frac{d\varphi}{dx}=\frac{d}{dx}(x^2+xy^2)=0 \Rightarrow \varphi(x,y)=x^2+xy^2=c$$

Theorem 2.6.1

- Suppose an ODE can be written in the form
$$M(x,y)+N(x,y)y'=0 \qquad (1)$$
where the functions M, N, M_y and N_x are all continuous in the rectangular region R: $(x, y) \in (\alpha, \beta) \times (\gamma, \delta)$. Then Eq. (1) is an **exact** differential equation iff
$$M_y(x,y)=N_x(x,y),\ \forall (x,y)\in R \qquad (2)$$
- That is, there exists a function ψ satisfying the conditions
$$\psi_x(x,y)=M(x,y),\ \psi_y(x,y)=N(x,y) \qquad (3)$$
if M and N satisfy Equation (2).

Example 2: Exact Equation (1 of 3)

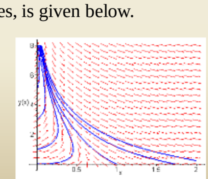
- Consider the following differential equation.
$$(y\cos x+2xe^y)+(\sin x+x^2e^y-1)y'=0$$
- Then
$$M(x,y)=y\cos x+2xe^y,\ N(x,y)=\sin x+x^2e^y-1$$
and hence
$$M_y(x,y)=\cos x+2xe^y=N_x(x,y) \Rightarrow \text{ODE is exact}$$
- From Theorem 2.6.1,
$$\psi_x(x,y)=M=y\cos x+2xe^y,\ \psi_y(x,y)=N=\sin x+x^2e^y-1$$
- Thus
$$\psi(x,y)=\int\psi_x(x,y)dx=\int\left(y\cos x+2xe^y\right)dx=y\sin x+x^2e^y+C(y)$$

Example 2: Solution (2 of 3)

- We have
$$\psi_x(x,y)=M=y\cos x+2xe^y,\ \psi_y(x,y)=N=\sin x+x^2e^y-1$$
and
$$\psi(x,y)=\int\psi_x(x,y)dx=\int\left(y\cos x+2xe^y\right)dx=y\sin x+x^2e^y+C(y)$$
- It follows that
$$\psi_y(x,y)=\sin x+x^2e^y-1=\sin x+x^2e^y+C'(y)$$
$$\Rightarrow C'(y)=-1 \Rightarrow C(y)=-y+k$$
- Thus
$$\psi(x,y)=y\sin x+x^2e^y-y+k$$
- By Theorem 2.6.1, the solution is given implicitly by
$$y\sin x+x^2e^y-y=c$$

Example 2: Direction Field and Solution Curves (3 of 3)

- Our differential equation and solutions are given by
$$(y\cos x+2xe^y)+(\sin x+x^2e^y-1)y'=0,$$
$$y\sin x+x^2e^y-y=c$$
- A graph of the direction field for this differential equation, along with several solution curves, is given below.



Example 3: Non-Exact Equation (1 of 2)

- Consider the following differential equation.

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

- Then

$$M(x,y) = 3xy + y^2,\ N(x,y) = x^2 + xy$$

and hence

$$M_y(x,y) = 3x + 2y \neq 2x + y = N_x(x,y) \Rightarrow \text{ODE is not exact}$$

- To show that our differential equation cannot be solved by this method, let us seek a function ψ such that

$$\psi_x(x,y) = M = 3xy + y^2,\ \psi_y(x,y) = N = x^2 + xy$$

- Thus

$$\psi(x,y) = \int \psi_x(x,y)dx = \int (3xy + y^2)dx = 3x^2y/2 + xy^2 + C(y)$$

Example 3: Non-Exact Equation (2 of 2)

- We seek ψ such that

$$\psi_x(x,y) = M = 3xy + y^2,\ \psi_y(x,y) = N = x^2 + xy$$

and

$$\psi(x,y) = \int \psi_x(x,y)dx = \int (3xy + y^2)dx = 3x^2y/2 + xy^2 + C(y)$$

- Then

$$\psi_y(x,y) = x^2 + xy = 3x^2/2 + 2xy + C'(y)$$

$$\Rightarrow C'(y) \overset{?}{=} -xy - x^2/2$$

- Because $C'(y)$ depends on x as well as y, there is no such function $\psi(x, y)$ such that

$$\frac{d\psi}{dx} = (3xy + y^2) + (x^2 + xy)y'$$

Integrating Factors

- It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor $\mu(x,y)$:

$$M(x,y) + N(x,y)y' = 0$$

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$$

- For this equation to be exact, we need

$$(\mu M)_y = (\mu N)_x \Leftrightarrow M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

- This partial differential equation may be difficult to solve. If μ is a function of x alone, then $\mu_y = 0$ and hence we solve

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu,$$

provided right side is a function of x only. Similarly if μ is a function of y alone. See text for more details.

Example 4: Non-Exact Equation

- Consider the following non-exact differential equation.

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

- Seeking an integrating factor, we solve the linear equation

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \Leftrightarrow \frac{d\mu}{dx} = \frac{\mu}{x} \Rightarrow \mu(x) = x$$

- Multiplying our differential equation by μ, we obtain the exact equation

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0,$$

which has its solutions given implicitly by

$$x^3y + \frac{1}{2}x^2y^2 = c$$