Answer Key

Math 204 - Differential Equations

Midterm 1 March 8, 2016

Duration: 90 minutes

Instructions: No calculators, no books, no notes, no questions, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. If necessary, you can use the back of these pages, but make sure you have indicated doing so. **Print (i.e., use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: ______________________

Signature: ______________________

Section (Check One):

Section 1: E. Ceyhan (Tue-Thu 10:00) ____

Section 2: E. Ceyhan (Tue-Thu 08:30) ____

Section 3: H. Göral (Mon-Wed 16:00) ____

Question	Points	Score
1	15	
2	20	
3	20	
4	20	
5	15	
6	15	
Total	**105**	

1. (15 points) Find the general solutions of the differential equations.

(a) $y'' - 4y' + 5y = 0$

$r^2 - 4r + 5 = 0 \Rightarrow r_{1,2} = \dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

$\Rightarrow \lambda = 2, \mu = 1$

so $y(t) = c_1 e^{2t} \cos t + c_2 e^{2t} \sin t$

(b) $y'' + 3y' - 4y = 0$

$r^2 + 3r - 4 = 0 \Rightarrow (r-1)(r+4) = 0$

$\begin{matrix} r & 4 \\ r & -1 \end{matrix}$ $\Rightarrow r_1 = 1, r_2 = -4$

so $y(t) = c_1 e^{t} + c_2 e^{-4t}$

(c) $4y'' + 4y' + y = 0$

$4r^2 + 4r + 1 = 0$

$\Rightarrow (2r+1)^2 = 0 \Rightarrow r_1 = r_2 = -\frac{1}{2}$

so $y(t) = c_1 e^{-t/2} + c_2 t e^{-t/2}$

2. (20 points) Find the particular solutions of the differential equations given below.

(a) $y'' - y' + 3y = \sin t$

Since $\sin t$ is not a sol'n to the hom. eq'n, we start with

$$\left.\begin{aligned} y(t) &= A\cos t + B\sin t \\ y'(t) &= -A\sin t + B\cos t \\ y''(t) &= -A\cos t - B\sin t \end{aligned}\right\} \Rightarrow \begin{aligned} &(-A\cos t - B\sin t) - (-A\sin t + B\cos t) \\ &+ 3(A\cos t + B\sin t) = \sin t \end{aligned}$$

$\Rightarrow \quad (-A - B + 3A)\cos t + (-B + A + 3B)\sin t = \sin t$

$\Rightarrow \quad (2A - B)\cos t + (A + 2B)\sin t = \sin t$

so $\quad 2A - B = 0 \Rightarrow B = 2A$

$\quad A + 2B = 1 \Rightarrow 5A = 1 \Rightarrow A = 1/5$ & $B = 2/5$

then $\underline{y_p(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t}$

(b) $y'' + y' - 6y = e^{2t}$

$r^2 + r - 6 = 0 \rightarrow (r+3)(r-2) = 0 \Rightarrow r_1 = 2,\ r_2 = -3$

So e^{2t} is a sol'n of the hom. eq'n.

so we try $\quad y(t) = Ate^{2t}$

$\quad y'(t) = A(e^{2t} + 2te^{2t})$

$\quad y''(t) = A(4e^{2t} + 4te^{2t})$

so $\quad 4Ae^{2t} + 4Ate^{2t} + Ae^{2t} + 2Ate^{2t} - 6Ate^{2t} = e^{2t}$

$\Rightarrow \quad (4A + A)e^{2t} + \underbrace{(4A + 2A - 6A)}_{0}te^{2t} = e^{2t}$

$\Rightarrow 5Ae^{2t} = e^{2t} \Rightarrow 5A = 1 \Rightarrow A = 1/5$

then $\underline{y_p(t) = \frac{1}{5}te^{2t}}$

3. (20 points)
(a) Solve the following initial value problem (IVP).

$$2xy' - y = x, \qquad y(1) = 1$$

In standard for $y' - \frac{1}{2x} y = \frac{1}{2}$

So $\mu(x) = \exp\left(\int -\frac{1}{2x} dx\right) = \exp\left(-\frac{1}{2}\int \frac{1}{x} dx\right) = \exp\left(-\frac{1}{2}\ln x\right) = x^{-1/2}$

$\Rightarrow x^{-1/2} \cdot y' - \frac{1}{2x^{3/2}} \cdot y = \frac{1}{2} x^{-1/2}$

$\Rightarrow \left(x^{-1/2} \cdot y\right)' = \frac{1}{2} \cdot x^{-1/2} \Rightarrow x^{-1/2} \cdot y = \frac{1}{2}\int x^{-1/2} dx = x^{1/2} + C$

So $y = x + C \cdot x^{1/2}$

Imposing the I.C $1 = 1 + C \Rightarrow C = 0$

So $\underline{y = x}$

(b) Determine the largest open x-interval in which the IVP in part (a) is certain to have a unique solution.

From (a), $p(x) = \frac{-1}{2x}$ & $g(x) = 1/2$

So $g(x)$ is continuous everywhere

& $p(x)$ is " on $(-\infty, 0) \cup (0, \infty)$

Since 1 is in $(0, \infty)$, the largest interval is $\underline{(0, \infty)}$

(c) Find the smallest positive integers a, b, and c so that $y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$ is the general solution of $ay'' + by' + cy = 0$.

So char. eqn has complex conjugate roots as

$\lambda \pm i \cdot \mu$ with $\lambda = -1$ & $\mu = 2$

So the roots are $r_{1,2} = -1 \pm 2i$

then the char. eqn is

$$(r + 1 - 2i) \cdot (r + 1 + 2i) = (r+1)^2 - (2i)^2$$

$$= r^2 + 2r + 1 + 4 = r^2 + 2r + 5$$

then the original diff. eqn is

$$y'' + 2y' + 5y = 0$$

so $a = 1$, $b = 2$, $c = 5$

4. (20 points) (a) Show that the following differential equation is not exact

$$(2xy - e^{-2x})dx + xdy = 0$$

$\Rightarrow (2xy - e^{-2x}) + xy' = 0$

$\Rightarrow M(x,y) = 2xy - e^{-2x}$ & $N(x,y) = x$

$\Rightarrow M_y = 2x \neq N_x = 1$

So DE is <u>not exact</u>.

(b) Find an integrating factor which makes it exact and verify that the new differential equation is exact. (DO NOT SOLVE THE DIFFERENTIAL EQUATION.)

$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu = \frac{2x-1}{x}\mu$ is a fcn of x only

so $\frac{\mu'}{\mu} = 2 - \frac{1}{x} \rightarrow \ln\mu = 2x - \ln x$

$\rightarrow \underline{\mu(x) = e^{2x}e^{-\ln x} = \frac{e^{2x}}{x}}$

So DE becomes $\left(2e^{2x}y - \frac{1}{x}\right) + e^{2x}y' = 0$ (*)

$\Rightarrow \frac{\partial}{\partial y}\left(2e^{2x}y - \frac{1}{x}\right) = 2e^{2x} = \frac{\partial}{\partial x}e^{2x}$.

$\Rightarrow$ <u>(*) is exact</u>

5. (15 points) Verify that $y = t^2$ is a solution of the differential equation

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0$$

for $t > 0$. Using Abel's theorem, find the general solution of this differential equation.

We first verify that t^2 is a solution:

$(t^2)'' - \frac{4}{t} \cdot (t^2)' + \frac{6}{t^2} \cdot t^2 = 2 - 8 + 6 = 0$. Let $y_1 = t^2$.

Note that $p(t) = \frac{-4}{t}$. By Abel's theorem we have that

$$W(y_1, y_2) = c \cdot e^{\int \frac{4}{t} dt} = c \cdot e^{\ln t^4} = ct^4$$

On the other hand,

$$W(t^2, y_2) = \begin{vmatrix} t^2 & y_2 \\ 2t & y_2' \end{vmatrix} = ct^4$$

Thus, $t^2 \cdot y_2' - 2t \cdot y_2 = ct^4$. Standard for is

$y_2' - \frac{2}{t} y_2 = ct^2$. This is a first order linear differential epuation with integrating factor

$\mu(t) = e^{\int -\frac{2}{t} dt} = \frac{1}{t^2}$.

Thus, $\left(\frac{y_2}{t^2}\right)' = c$ and $y_2 = ct^3$. We may take $c = 1$ and so $y_2 = t^3$.

The general solution is $\boxed{y = c_1 t^2 + c_2 t^3}$

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6. (15 points) If the differential equation

$$ty'' + 2y' + te^t y = 0$$

for $t > 0$ has y_1 and y_2 as fundamental set of solutions and if $W(y_1, y_2)(1) = 3$, then find the value of $W(y_1, y_2)(5)$. Justify your answer.

The standard form is $y'' + \frac{2}{t} y' + e^t \cdot y = 0$

Now we can use Abel's theorem, and we have

$$W(y_1, y_2) = C e^{\int -\frac{2}{t} dt} = \frac{C}{t^2}$$

for $t = 1$, we get $3 = W(y_1, y_2)(1) = C$

thus $C = 3$.

Now for $t = 5$, we get $W(y_1, y_2)(5) = \frac{3}{5^2} = \frac{3}{25}$