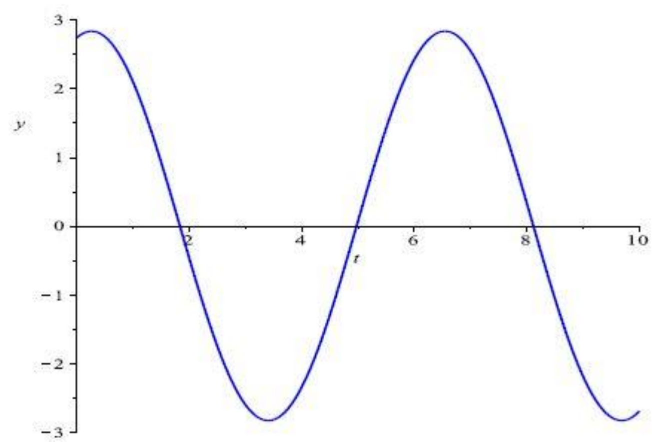
3.3

2. $e^{2-i} = e^2 e^{-i} = e^2(\cos 1 - i \sin 1)$.

3. $e^{3i\pi} = \cos 3\pi + i \sin 3\pi = -1$.

10. The characteristic equation is $r^2 + 4r + 5 = 0$, with roots $r = -2 \pm i$. Hence the general solution is $y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$.

20. The characteristic equation is $r^2 + 1 = 0$, with roots $r = \pm i$. Hence the general solution is $y = c_1 \cos t + c_2 \sin t$. Its derivative is $y' = -c_1 \sin t + c_2 \cos t$. Based on the first condition, $y(\pi/3) = 2$, we require that $c_1 + \sqrt{3}\, c_2 = 4$. In order to satisfy the condition $y'(\pi/3) = -2$, we find that $-\sqrt{3}\, c_1 + c_2 = -4$. Solving these for the constants, $c_1 = 1 + \sqrt{3}$ and $c_2 = \sqrt{3} - 1$. Hence the specific solution is a steady oscillation, given by $y(t) = (1 + \sqrt{3}) \cos t + (\sqrt{3} - 1) \sin t$.



Math 204 HW#4

3.3.5. $2^{2-i} = e^{(2-i)\ln 2} = e^{2\ln 2} e^{-i\ln 2}$

$= e^{2\ln 2}(\cos(\ln 2) - i\sin(\ln 2))$.

$= 4\cos(\ln 2) - 4i\sin(\ln 2)$

$\cong 3.0770 - 2.558i$ //

3.3.7. $y'' - 4y' + 5y = 0$.

$r^2 - 4r + 5 = 0$, $\quad r_{1,2} = \dfrac{4 \mp \sqrt{-4}}{2} = 2 \mp i$.

$\Delta = 16 - 20 = -4$

Hence the general solution:

$y(t) = c_1 e^{2t}\cos t + c_2 e^{2t}\sin t$ //

3.3.11. $y'' + 6y' + 10y = 0$.

$r^2 + 6r + 10 = 0 \Rightarrow r_{1,2} = \dfrac{-6 \mp \sqrt{-4}}{2} = -3 \mp i$

$\Delta = 36 - 40 = -4$

$\Rightarrow y(t) = c_1 e^{-3t}\cos t + c_2 e^{-3t}\sin t$ //

3.3.14. $9y'' + 3y' - 2y = 0$.

$y'' + \frac{1}{3}y' - \frac{2}{9}y = 0$

$r^2 + \frac{1}{3}r - \frac{2}{9} = 0$

$\left(r + \frac{2}{3}\right)\left(r - \frac{1}{3}\right) = 0 \Rightarrow r_1 = \frac{1}{3}, r_2 = \frac{-2}{3}$

$\Rightarrow y(t) = c_1 e^{t/3} + c_2 e^{-2/3 t}$ //

3.3.18. $y'' + 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = 0$

$r^2 + 4r + 5 = 0 \quad \Rightarrow r_{1,2} = \dfrac{-4 \mp \sqrt{-4}}{2} = -2 \mp i$

$\Delta = 16 - 20$
$= -4$

$\Rightarrow y(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$

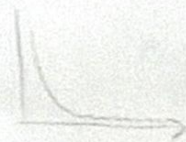
$y'(t) = -2c_1 e^{-2t} \cos t - c_1 e^{-2t} \sin t - 2c_2 e^{-2t} \sin t + \cos t \, c_2 e^{-2t}$.

$y(0) = 1 = c_1 . 1 . 1 + c_2 . 1 . 0 = c_1 \quad \Rightarrow \quad c_1 = 1$

$y'(0) = 0 = -2c_1 + c_2 \quad \Rightarrow c_2 = 2$

$\Rightarrow y(t) = e^{-2t} \cos t + 2e^{-2t} \sin t$.

$y$ decays oscillating as $t$ increases.



3.3.23. $3u'' - u' + 2u = 0$, $u(0) = 2$, $u'(0) = 0$.

a) $3r^2 - r + 2 = 0 \quad r_{1,2} = \dfrac{1 \mp \sqrt{23}}{2.3} = \dfrac{1}{6} \mp i\dfrac{\sqrt{23}}{6}$

$\Delta = 1 - 4.3.2$
$= -23$

$u(t) = c_1 e^{t/6} \cos(\sqrt{23}/6 \, t) + c_2 e^{t/6} \sin\left(\frac{\sqrt{23}}{6} t\right)$.

$u(0) = 2 = c_1 + 0 \quad \Rightarrow c_1 = 2$

$u'(t) = \frac{c_1}{6} e^{t/6} \cos\left(\frac{\sqrt{23}}{6} t\right) - c_1 e^{t/6} \frac{\sqrt{23}}{6} \sin\left(\frac{\sqrt{23}}{6} t\right) + \frac{c_2}{6} e^{t/6} \sin\left(\frac{\sqrt{23}}{6} t\right)$
$+ \cos\left(\frac{\sqrt{23}}{6} t\right) \frac{\sqrt{23}}{6} c_2 e^{t/6}$.

$u'(0) = 0 = \frac{c_1}{6} + \frac{\sqrt{23}}{6} c_2 \quad \Rightarrow c_2 = -\frac{2}{6} \cdot \frac{6}{\sqrt{23}} = \frac{-2}{\sqrt{23}}$

$\Rightarrow u(t) = 2e^{t/6} \cos\left(\frac{\sqrt{23}}{6} t\right) - \frac{2}{\sqrt{23}} e^{t/6} \sin\left(\frac{\sqrt{23}}{6} t\right)$.

3.3.23.

b) $|u(t)| = 10 = 2e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) - \frac{2}{2\sqrt{3}}e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right)$

$\rightarrow t = 10.7598$

3.3.25. $y'' + 2y' + 6y = 0$, $y(0) = 2$, $y'(0) = \alpha \geq 0$.

a) $r^2 + 2r + 6 = 0$ $\qquad r_{1,2} = \frac{-2 \mp \sqrt{-20}}{2} = -1 \mp \sqrt{5}$

$\Delta = 4 - 24 = -20$

$\Rightarrow y(t) = c_1 e^{-t}\cos(\sqrt{5}t) + c_2 e^{-t}\sin(\sqrt{5}t)$

$y(0) = 2 = c_1$

$y'(t) = -c_1 e^{-t}\cos(\sqrt{5}t) - c_1\sqrt{5}e^{-t}\sin(\sqrt{5}t) - c_2 e^{-t}\sin(\sqrt{5}t) + c_2\sqrt{5}e^{-t}\cos(\sqrt{5}t)$

$y'(0) = \alpha = -c_1 + c_2\sqrt{5} \Rightarrow c_2 = \frac{\alpha+2}{\sqrt{5}}$

$\Rightarrow y(t) = 2e^{-t}\cos\sqrt{5}t + \left(\frac{\alpha+2}{\sqrt{5}}\right)e^{-t}\sin\sqrt{5}t.$

b) $y(1) = \boxed{0 = 2e^{-1}\cos\sqrt{5} + \left(\frac{\alpha+2}{\sqrt{5}}\right)e^{-1}\sin\sqrt{5}}$

$\Rightarrow \frac{\alpha+2}{\sqrt{5}} = -\frac{2}{e}\cos\sqrt{5} \Big/ \frac{\sin\sqrt{5}}{e} = -2\cot\sqrt{5} = +1.5692$

$\Rightarrow \alpha = 1.5088$

c) $y = 0$, $t = ?$

$0 = 2e^{-t}\cos\sqrt{5}t + \left(\frac{\alpha+2}{\sqrt{5}}\right)e^{-t}\sin\sqrt{5}t. \Rightarrow 2 + ((\alpha+2)/\sqrt{5})\tan\sqrt{5}t = 0$

$\tan(\sqrt{5}t) = \frac{-2\sqrt{5}}{(\alpha+2)}$

$t = \{\pi - \arctan(2\sqrt{5}/\alpha+2)\}/\sqrt{5}$

d) $\lim_{\alpha\to\infty} \{\pi - \arctan[\underbrace{+2\sqrt{5}/(2+\alpha)}_{\to 0}]\}/\sqrt{5} = \pi/\sqrt{5} - 0 = \frac{\pi}{\sqrt{5}}$

# 3.4

3. The characteristic equation is $4r^2 - 8r - 5 = 0$, with roots $r = -1/2\,,\,5/2$. The general solution is $y(t) = c_1 e^{-t/2} + c_2 e^{5t/2}$.

6. The characteristic equation is $r^2 - 10r + 25 = 0$, with the double root $r = 5$. The general solution is $y(t) = c_1 e^{5t} + c_2 t\, e^{5t}$.

Problem 9 : The characteristic equation is

$25r^2 - 30r + 9 = 0$, with roots $r_1$ and $r_2$.

$\Delta = 30^2 - 4 \cdot 25 \cdot 9 = 0$

$r_1 = r_2 = \frac{30}{2 \cdot 25} = \frac{3}{5}$

The general solution is $y(t) = C_1 e^{+\frac{3}{5}t} + t \cdot C_2 \cdot e^{+3/5t}$.

Problem 12 : the characteristic equation is

$r^2 - 6r + 9 = 0$ with roots $r_1 = r_2 = 3$.

The general solution is,

$$\boxed{y(t) = c_1 e^{+3t} + t \cdot c_2 \cdot e^{+3t}}$$

From $y(0) = 0$ we get that,

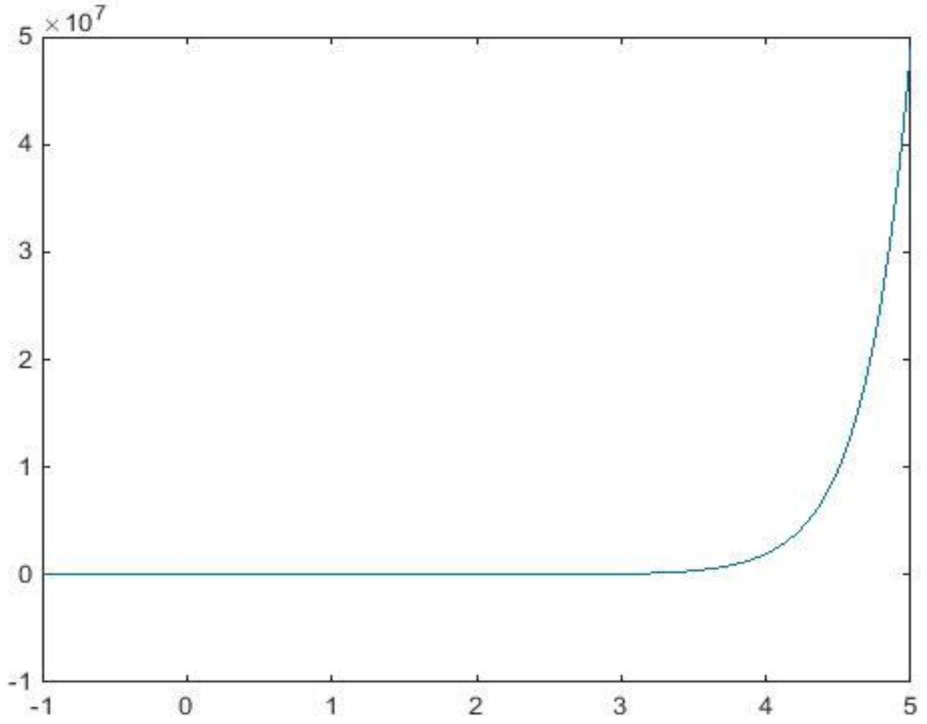
$\boxed{0 = c_1}$ So, $y(t) = c_2 t e^{+3t}$.

The derivative is $y'(t) = c_2 e^{+3t} + c_2 \cdot t \cdot 3 \cdot e^{+3t}$ and invoking the initial condition $y'(0) = 3$ we obtain,

$3 = c_2 e^{0} + 0 \rightarrow \boxed{c_2 = 3}$

The solution is $\boxed{y(t) = 3te^{+3t}}$

As $t \to \infty$, we have $y(t) \to \infty$.



Problem 14 : the characteristic equation is

$$r^2 + 4r + 4 = 0 \quad \text{with roots } r_1 = r_2 = -2$$

the general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$
$$y'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

From , $y(-1) = 2$ and $y'(-1) = 3$ we've

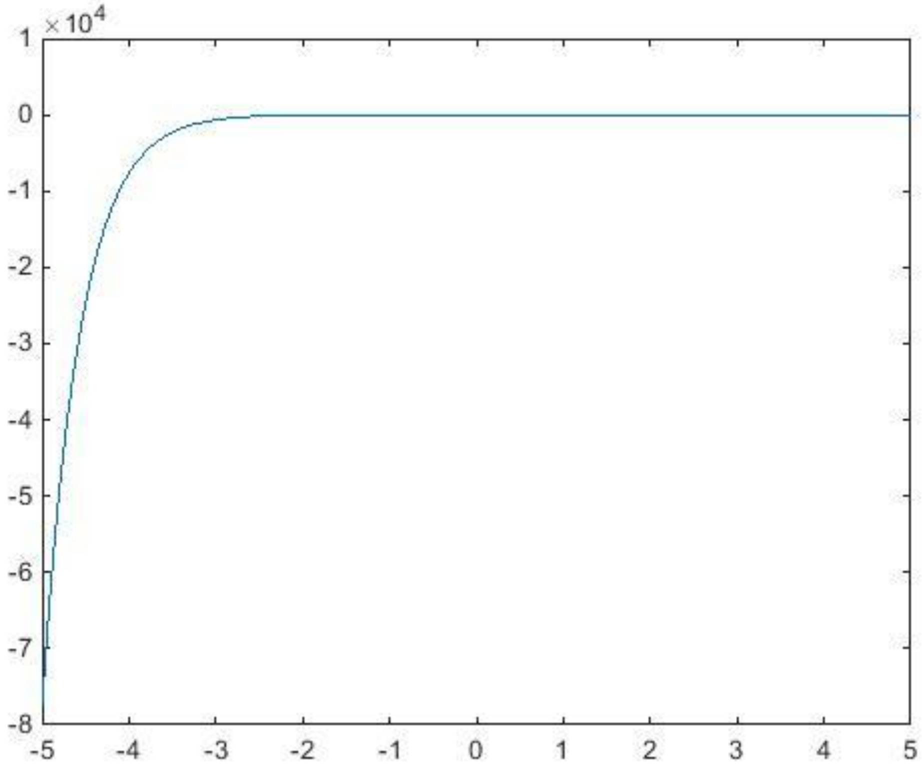
$$\begin{cases} 2 = c_1 e^2 - c_2 e^2 \\ 3 = -2c_1 e^2 + c_2 e^2 + 2c_2 e^2 \end{cases}$$

So that $c_2 = e^{-2} \cdot 7$ and $c_1 = 9e^{-2}$

Hence,

$$y(t) = 9e^{-2(t+1)} + 7t\, e^{-2(t+1)}$$

As $t \to \infty$ we have (clear) $y(t) \to 0$.



16. The characteristic roots are $r_1 = r_2 = 1/2$. Hence the general solution is given by $y(t) = c_1 e^{t/2} + c_2 t\, e^{t/2}$. Invoking the initial conditions, we require that $c_1 = 2$, and that $1 + c_2 = b$. The specific solution is $y(t) = 2e^{t/2} + (b-1)t\, e^{t/2}$. Since the second term dominates, the long-term solution depends on the sign of the coefficient $b - 1$. The critical value is $b = 1$.

23. Set $y_2(t) = t^3\, v(t)$. Substitution into the differential equation results in

$$t^2(t^3v'' + 6t^2v' + 6tv) - 4t(t^3v' + 3t^2v) + 6t^3v = 0\,.$$

After collecting terms, we end up with $t^5v'' + 2t^4v' = 0$. Hence $v(t) = c_1 + c_2/t$, and thus $y_2(t) = c_1t^3 + c_2t^2$. Setting $c_1 = 0$ and $c_2 = 1$, we obtain $y_2(t) = t^2$.

**25.** Set $y_2(t) = t^{-1}v(t)$. Substitution into the differential equation into the differential equation results in

$$t^2(2t^{-3}v - t^{-2}v' + v''t^{-1} - t^{-2}v') + 3t(v't^{-1} - t^{-2}v) + t^{-1}v = 0.$$

After collecting terms, we end up with $tv'' + v' = 0$. This equation is linear in variable $w = v'$. It follows that $v'(t) = t^{-1} + c_1$, and $v(t) = \ln(t) + c_1t + c_2$. Thus $y_2(t) = t^{-1}\ln(t) + c_1tt^{-1} + c_2t^{-1} = t^{-1}\ln(t) + c_1 + c_2t^{-1}$. Setting $c_1 = 0$ and $c_2 = 0$, we obtain $y_2(t) = t^{-1}\ln(t)$.

# 3.5

3. The characteristic equation for the homogeneous problem is $r^2 - r - 2 = 0$, with roots $r = -1, 2$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{2t}$. Set $Y = At^2 + Bt + C$. Substitution into the given differential equation, and comparing the coefficients, results in the system of equations $-2A = 4$, $-2A - 2B = 0$ and $2A - B - 2C = -3$. Hence $Y = -2t^2 + 2t - 3/2$. The general solution is $y(t) = y_c(t) + Y$.

5. The characteristic equation for the homogeneous problem is $r^2 - 2r - 3 = 0$, with roots $r = -1, 3$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{3t}$. Note that the assignment $Y = Ate^{-t}$ is not sufficient to match the coefficients. Try $Y = Ate^{-t} + Bt^2 e^{-t}$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $-4A + 2B = 0$ and $-8B = -6$. This implies that $Y = (3/8)te^{-t} + (3/4)t^2 e^{-t}$. The general solution is $y(t) = y_c(t) + Y$.

6. The characteristic equation for the homogeneous problem is $r^2 + 2r = 0$ with roots $r = 0, -2$. Hence $y_c = c_1 + c_2 e^{-2t}$. Note that the assignment $Y = A + B\sin 2t$ is not sufficient to match the coefficients. Try $Y = At + B\sin 2t + C\cos 2t$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $-4(B + C) = 4$ and $-4(C - B) = 0$. This implies that $Y = \frac{5}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$. The general solution is $y(t) = c_1 + c_2 e^{-2t} + \frac{5}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$.

9. The characteristic equation for the homogeneous problem is $2r^2 + 3r + 1 = 0$, with roots $r = -1, -1/2$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{-t/2}$. To simplify the analysis, set $g_1(t) = t^2$ and $g_2(t) = 3\sin t$. Based on the form of $g_1$, set $Y_1 = A + Bt + Ct^2$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $A + 3B + 4C = 0$, $B + 6C = 0$, and $C = 1$. Hence we obtain $Y_1 = 14 - 6t + t^2$. On the other hand, set $Y_2 = D\cos t + E\sin t$. After substitution into the ODE, we find that $D = -3/10$ and $E = 9/10$. The general solution is $y(t) = y_c(t) + Y_1 + Y_2$.

17. The characteristic equation for the homogeneous problem is $r^2 - 2r + 1 = 0$, with a double root $r = 1$. Hence $y_c(t) = c_1 e^t + c_2 t\, e^t$. Consider $g_1(t) = t\, e^t$. Note that $g_1$ is a solution of the homogeneous problem. Set $Y_1 = At^2 e^t + Bt^3 e^t$ (the first term is not sufficient for a match). Upon substitution, we obtain $Y_1 = t^3 e^t/6$. By inspection, $Y_2 = 4$. Hence the general solution is $y(t) = c_1 e^t + c_2 t\, e^t + t^3 e^t/6 + 4$. Invoking the initial conditions, we require that $c_1 + 4 = 2$ and $c_1 + c_2 = 1$. Hence $c_1 = -2$ and $c_2 = 3$.

13. The characteristic equation for the homogeneous problem is $r^2 + r = 0$ with roots $r = -\frac{1}{2} \pm \frac{\sqrt{15}}{2}i$. Hence $y = c_1 e^{-t/2} \cos \sqrt{15}2t + c_2 e^{-t/2} \sin \sqrt{15}2t$. Set $Y = Ae^t + Be^{-t}$. Substitution into the differential equation, and comparing the coefficients, we have $A = \frac{1}{3}$ and $B = -\frac{1}{2}$. The general solution is $y(t) = c_1 e^{-t/2} \cos \frac{\sqrt{15}}{2}t + c_2 e^{-t/2} \sin \frac{\sqrt{15}}{2}t + \frac{1}{3}e^t - \frac{1}{2}e^{-t}$.

21. a) Note that the assignment $Y = A_0t^4 + A_1t^3 + A_2t^2 + A_3t + A_4 + (Bt^2 + B_1t + B_2)e^{-3t} + D\sin 3t + E\cos 3t$ is not sufficient to match the coefficients. Try $Y = t(A_0t^4 + A_1t^3 + A_2t^2 + A_3t + A_4) + t(Bt^2 + B_1t + B_2)e^{-3t} + D\sin 3t + E\cos 3t$.