Section 5.3

1) $y'' + xy' + y = 0$, $y(0) = 2$, $y'(0) = 0$

$y = \phi(x)$ is a sol. $x_0 = 0$.

given that $\phi(0) = 2$ and $\phi'(0) = 0$.

$y = \phi(x)$ is a sol. then plug in the eq.

$$\phi'' + x\cdot\phi' + \phi = 0$$

(1) — $\phi''(x) = -x\cdot\phi'(x) - \phi(x)$

Take derivative

(2) — $\phi'''(x) = (-1)\cdot\phi'(x) - x\cdot\phi''(x) - \phi'(x)$

Once more,

(3) — $\phi^{(4)}(x) = (-1)\cdot\phi''(x) - 1\cdot\phi''(x) - x\cdot\phi'''(x) - \phi''(x)$

From (1) equation,

$\phi''(0) = -0\cdot\phi'(0) - \phi(0) = -\phi(0) = -2$ ✓

From (2) equation;

$\phi'''(0) = -1\cdot\underbrace{\phi'(0)}_{0} - 0\cdot\phi''(0) - \underbrace{\phi'(0)}_{0} = 0$ ✓

From (3), $\phi^{(4)}(0) = -1\cdot\phi''(0) - \phi''(0) - 0\cdot\phi'''(0) - \phi''(0) = 2+2+0+2 = 6$ ✓

2. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -(\sin x)y' - (\cos x)y\,.$$

Differentiating twice,

$$\begin{aligned} y''' &= -(\sin x)y'' - 2(\cos x)y' + (\sin x)y \\ y^{(4)} &= -(\sin x)y''' - 3(\cos x)y'' + 3(\sin x)y' + (\cos x)y\,. \end{aligned}$$

Given that $\phi(0) = 0$ and $\phi'(0) = -1$, the first equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = 2$ and $\phi^{(4)}(0) = 0$.

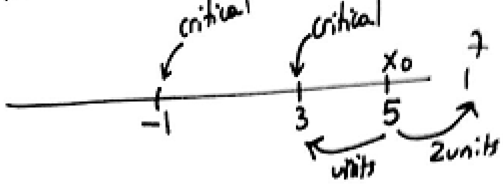
5. Clearly, $p(x) = 4$ and $q(x) = 6x$ are analytic for all x. Hence the series solutions converge everywhere.

Sciences

6) $(x^2-2x-3)\,y''+xy'+4y=0.$ $x_0=5,\ x_0=-5,\ x_0=0$

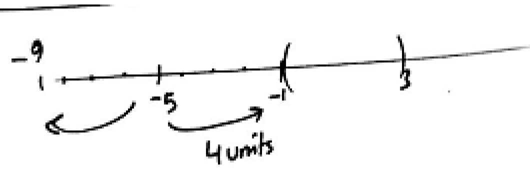
$$y''+\frac{x}{(x-3)(x+1)}\cdot y'+\frac{4}{(x-3)(x+1)}\,y=0.$$

IF $x_0=5$ then



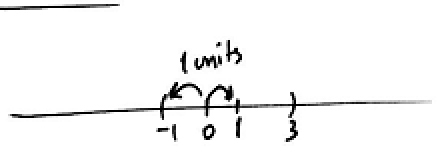
so, $\rho = 2$ units.
interval should not contain discontinuity points $\{3,-1\}$

IF $x_0=-5$



$\rho=4$

IF $x_0=0$



$\rho = 1.$

8. The only root of $P(x) = x$ is zero. Hence $\rho_{min} = 2$.

11) $y'' + (\sin x) y = 0$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n \cdot x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \underbrace{\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right)}_{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}} \underbrace{\left(\sum_{n=0}^{\infty} a_n x^n \right)}_{a_0 + a_1 x^1 + a_2 x^2 + \dots} = 0$$

$$\left[2a_2 x^0 + 6a_3 x^1 + 12a_4 x^2 + 20a_5 x^3 + \dots \right] + \left[x \cdot a_0 + x^2 a_1 + x^3 \cdot \left(-\frac{a_0}{6} + a_2 \right) + x^4 \cdot \left(-\frac{a_1}{6} + a_3 \right) + \dots \dots \right] = 0$$

So, $2a_2 = 0$, $a_0 + 6a_3 = 0$, $12a_4 + a_1 = 0$, $20a_5 - \frac{a_0}{6} + a_2 = 0$

$\Rightarrow a_3 = -\frac{1}{6} a_0$, $a_2 = 0$, $a_4 = -\frac{1}{12} a_1$, $a_5 = \frac{a_0}{120}$, ...

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

$$= a_0 + a_1 x + 0 + \frac{a_0}{-6} x^3 - \frac{1}{12} a_1 x^4 + \frac{a_0}{120} x^5 + \dots$$

$$= a_0 \left(1 - \frac{x^3}{6} + \frac{x^5}{120} + \dots \right)$$

$$+ a_1 \left(x - \frac{x^4}{12} + \dots \dots \right)$$

So, $y_1 = 1 - \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^6}{180} + \dots$ $\rightarrow$ do yourself!

$y_2 = x - \frac{x^4}{12} + \frac{x^6}{180} + \frac{x^7}{504} + \dots$ $\rightarrow$ do yourself!

13. The Taylor series expansion of $\cos x$, about $x_0 = 0$, is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \,.$$

Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right] \left[\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \, x^n \right] + \sum_{n=1}^{\infty} n a_n x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0 \,.$$

The coefficient of x^n in the product of the two series is

$$c_n = 2a_2 b_n + 6a_3 b_{n-1} + 12 a_4 b_{n-2} + \ldots + (n+1) n a_{n+1} b_1 + (n+2)(n+1) a_{n+2} b_0,$$

in which $\cos x = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots$. It follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} c_n x^n + \sum_{n=1}^{\infty} (n-2) a_n x^n = 0 \,.$$

Expanding the product of the series, it follows that

$$\begin{aligned} 2a_2 - 2a_0 + 6a_3 x + (-a_2 + 12a_4)x^2 + (-3a_3 + 20a_5)x^3 + \ldots \\ \ldots - a_1 x + a_3 x^3 + 2a_4 x^4 + \ldots = 0 \,. \end{aligned}$$

Setting the coefficients equal to zero, $a_2 - a_0 = 0$, $6a_3 - a_1 = 0$, $-a_2 + 12a_4 = 0$, $-3a_3 + 20a_5 + a_3 = 0$, $\ldots$. Hence the general solution is

$$y(x) = a_0 + a_1 x + a_0 x^2 + a_1 \frac{x^3}{6} + a_0 \frac{x^4}{12} + a_1 \frac{x^5}{60} + a_0 \frac{x^6}{120} + a_1 \frac{x^7}{560} + \ldots \,.$$

We find that two linearly independent solutions ($W(y_1, y_2)(0) = 1$) are

$$y_1(x) = 1 + x^2 + \frac{x^4}{12} + \frac{x^6}{120} + \ldots$$

$$y_2(x) = x + \frac{x^3}{6} + \frac{x^5}{60} + \frac{x^7}{560} + \ldots$$

The nearest zero of $P(x) = \cos x$ is at $x = \pm\pi/2$. Hence $\rho_{min} = \pi/2$.

15. Integrating by parts,

$$\int_0^A te^{at}\cdot e^{-st}dt = -\frac{t\,e^{(a-s)t}}{s-a}\bigg|_0^A + \int_0^A \frac{1}{s-a}e^{(a-s)t}dt =$$

$$= \frac{1-e^{A(a-s)}+A(a-s)e^{A(a-s)}}{(s-a)^2}\,.$$

Taking a limit, as $A\to\infty$,

$$\int_0^\infty te^{at}\cdot e^{-st}dt = \frac{1}{(s-a)^2}\,.$$

Note that the limit exists as long as $s>a$.

17. Observe that $t\,\sinh\,at = (t\,e^{at} - t\,e^{-at})/2$. For any value of c,

$$\int_0^A t\,e^{ct}\cdot e^{-st}dt = -\frac{t\,e^{(c-s)t}}{s-c}\bigg|_0^A + \int_0^A \frac{1}{s-c}e^{(c-s)t}dt =$$

$$= \frac{1-e^{A(c-s)}+A(c-s)e^{A(c-s)}}{(s-c)^2}\,.$$

Taking a limit, as $A\to\infty$,

$$\int_0^\infty te^{ct}\cdot e^{-st}dt = \frac{1}{(s-c)^2}\,.$$

Note that the limit exists as long as $s>|c|$. Therefore,

$$\int_0^\infty t\,\sinh\,at\cdot e^{-st}dt = \frac{1}{2}\left[\frac{1}{(s-a)^2}-\frac{1}{(s+a)^2}\right] = \frac{2as}{(s-a)^2(s+a)^2}\,.$$

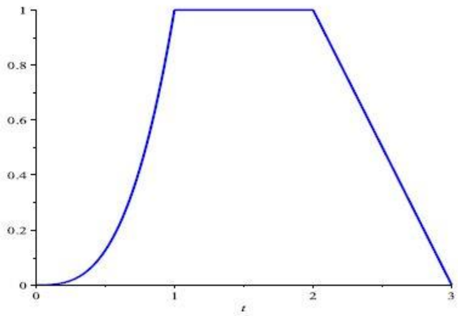
23. Using the definition of the Laplace transform and Problem 22, we get that

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st}f(t)\,dt = \int_0^3 e^{-st}t\,dt + \int_3^\infty e^{-st}\,dt =$$

$$= -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{e^{-3s}}{s} = -\frac{(2s+1)e^{-3s}}{s^2} + \frac{1}{s^2}.$$

6.1

3.



The function $f(t)$ is continuous.

7. Integration is a linear operation. It follows that

$$\int_0^A \cosh bt \cdot e^{-st} dt = \frac{1}{2}\int_0^A e^{bt} \cdot e^{-st} dt + \frac{1}{2}\int_0^A e^{-bt} \cdot e^{-st} dt =$$

$$= \frac{1}{2}\int_0^A e^{(b-s)t} dt + \frac{1}{2}\int_0^A e^{-(b+s)t} dt .$$

Hence

$$\int_0^A \cosh bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1-e^{(b-s)A}}{s-b}\right] + \frac{1}{2}\left[\frac{1-e^{-(b+s)A}}{s+b}\right].$$

Taking a limit, as $A \to \infty$,

$$\int_0^\infty \cosh bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1}{s-b}\right] + \frac{1}{2}\left[\frac{1}{s+b}\right] = \frac{s}{s^2-b^2} .$$

Note that the above is valid for $s > |b|$.

11. Using the linearity of the Laplace transform,

$$\mathcal{L}[\sin bt] = \frac{1}{2i}\mathcal{L}\left[e^{ibt}\right] - \frac{1}{2i}\mathcal{L}\left[e^{-ibt}\right].$$

Since

$$\int_0^\infty e^{(a+ib)t} e^{-st} dt = \frac{1}{s-a-ib} ,$$

we have

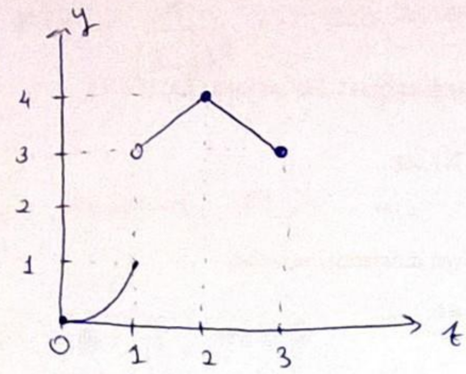
$$\int_0^\infty e^{\pm\, ibt} e^{-st} dt = \frac{1}{s \mp ib} .$$

Therefore

$$\mathcal{L}[\sin bt] = \frac{1}{2i}\left[\frac{1}{s-ib} - \frac{1}{s+ib}\right] = \frac{b}{s^2+b^2} .$$

The formula holds for $s > 0$.

1)



f is piecewise continuous on the interval $0 \le t \le 3$.

5) a) $f(t) = t$. $\quad F(s) = \int_0^\infty e^{-st} \cdot t \, dt$

integral by parts

$$= \lim_{A \to \infty} \left. -t \cdot \frac{e^{-st}}{s} \right]_0^A + \lim_{A \to \infty} \int_0^A \frac{e^{-st}}{s} dt$$

$$= 0 + \frac{1}{s^2} = \frac{1}{s^2}. \qquad s>0.$$

b) $f(t) = t^2$. $\quad F(s) = \int_0^\infty e^{-st} \, t^2 \, dt$ integral by parts.

$$= \frac{2}{s^3}.$$

c) $f(t) = t^n$, $\quad F(s) = \int_0^\infty e^{-st} t^n \, dt$ observe from previous results.

$$= \frac{n!}{s^{n+1}}, \quad s>0.$$

16) $f(t) = t \cdot \cos(at)$. we know; $\cos(at) = (e^{iat} + e^{-iat})/2$.

$$F(s) = \frac{1}{2} \left[\int_0^\infty t \, e^{iat-st} dt + \int_0^\infty t \, e^{-iat-st} dt \right]$$

$$= \frac{1}{2} \left[\int_0^\infty t \, e^{(ia-s)t} + \int_0^\infty t \cdot e^{-(ia+s)t} \right].$$

$$= \frac{1}{2}\left[\frac{1}{(ia-s)^2} + \frac{1}{(ia+s)^2}\right], \text{ from } (5.a).$$

$$= \frac{s^2 - a^2}{(a^2+s^2)^2}.$$

(21) $f(t) = \begin{cases} 1, & 0 \leq t < 2\pi \\ 0, & 2\pi \leq t < \infty. \end{cases}$

Using the fact that $f(t) = 0$ when $t \geq 2\pi$ and I by parts:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st}\, dt = \int_0^{2\pi} e^{-st}\, dt$$

$$= \left.\frac{-e^{-st}}{s}\right]_0^{2\pi} = \frac{1 - e^{-2\pi s}}{s}.$$