Math 204 - Differential Equations

Final Exam January 7, 2016

Duration: 150 minutes

Instructions: Calculators are not allowed. No books, no notes, no questions, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. If necessary, you can use the back of these pages, but make sure you have indicated doing so. **Print (i.e., use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: KEY

Signature: ____________________

Section (Check One):

Section 1: E. Ceyhan (Mon-Wed 10:00)

Section 2: E. Ceyhan (Mon-Wed 14:30)

Section 3: A. Erdoğan (Tue-Thu 16:00)

Question	Points	Score
1	20	
2	16	
3	10	
4	15	
5	12	
6	12	
7	20	
Total	**105**	

1. (20 points) (a) Solve the IVP $y' = \dfrac{x+1}{x^2(2y+1)}$, $y(1) = 0$.

$$\frac{dy}{dx} = \frac{x+1}{x^2(2y+1)} \Rightarrow (2y+1)\,dy = \left(\frac{x+1}{x^2}\right)dx \quad \text{so D.E. is separable}$$

$$\Rightarrow \int (2y+1)\,dy = \int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx \Rightarrow y^2 + y = \ln x - \frac{1}{x} + c$$

$$y(1) = 0 \Rightarrow \quad 0 = 0 - 1 + c \Rightarrow \underline{c = 1} \Rightarrow y^2 + y - (\ln x - \tfrac{1}{x} + 1) = 0$$

$$\Rightarrow y = \frac{-1 \mp \sqrt{5 + 4\ln x - 4/x}}{2} \qquad \text{Since } y(1) = 0 \text{ the sol'n is } \quad y = \frac{-1 + \sqrt{5 + 4\ln x - 4/x}}{2}$$

(b) How does the solution in part (a) behave as $x \to \infty$? How about as $x \to 1^+$?

As $x \to \infty$, $y(x) \to \infty$ since $\frac{4}{x} \to 0$ and $\ln x \to \infty$.

As $x \to 1^+$, $y(x) \to \dfrac{-1 + \sqrt{5 + 0 - 4}}{2} = 0$ (also notice that $y(1) = 0$)

(c) Solve the IVP $(t^2+1)y' + 2ty - te^t = 0$, $y(0) = 2$.

Standard form: $y' + \left(\dfrac{2t}{t^2+1}\right) y = \dfrac{te^t}{t^2+1}$

Integrating factor: $\mu(t) = \exp\left(\int \frac{2t}{t^2+1}\,dt\right)$, use substitution $u = t^2+1$, $du = 2t\,dt$

so $\mu(t) = \exp\left(\int \frac{du}{u}\right) = \exp(\ln u) = u = t^2 + 1$

so $((t^2+1)y)' = te^t \Rightarrow (t^2+1)y = \int te^t\,dt$ (using integration by parts with $u = t$, $dv = e^t dt$)

$= \dots = te^t - e^t + c$

$y(0) = 2 \Rightarrow \quad 2 = -1 + c \Rightarrow \underline{c = 3}$

$\Rightarrow (t^2+1)y = te^t - e^t + 3 \quad \Rightarrow \boxed{y = \dfrac{te^t - e^t + 3}{t^2+1}}$

(d) Find the largest interval in which a unique solution exists for the IVP in part (c).

In standard form $y' + p(t)y = g(t)$, $p(t) = \dfrac{2t}{t^2+1}$ and $g(t) = \dfrac{te^t}{t^2+1}$

so $p(t)$ & $g(t)$ are continuous for all t, hence largest interval on which a unique sol'n exists is $(-\infty, \infty)$

2. (16 points) (a) Find a particular solution of the following differential equation

$$(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t} \qquad 0 < t < 1$$

where $y_1(t) = e^t$ and $y_2(t) = t$ are the solutions for the corresponding homogeneous equation.

standard form. $y'' + \frac{t}{1-t} y' - \frac{1}{1-t} y = 2(1-t)e^{-t}, \quad 0<t<1$

Then $g(t) = 2(1-t)e^{-t}$, $y_1(t) = e^t$ and $y_2(t) = t$ are sol'ns of the hom. system.

The Wronskian of these sol'ns are

$W(y_1, y_2)(t) = \begin{vmatrix} e^t & t \\ e^t & 1 \end{vmatrix} = (1-t)e^t$, Using the method of variation of parameters,

the particular sol'n is $y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$

where $u_1(t) = -\int \frac{y_2(t) g(t)}{W(t)} dt = -\int 2t e^{-2t} dt = te^{-2t} + \frac{e^{-2t}}{2}$

and $u_2(t) = \int \frac{y_1(t) g(t)}{W(t)} dt = \int 2e^{-t} dt = -2e^{-t}$

Therefore, $y_p(t) = te^{-t} + \frac{e^{-t}}{2} - 2te^{-t} = \boxed{-te^{-t} + \frac{e^{-t}}{2}}$

(b) Find the general solution of $y'' + y' - 2y = e^t + \sin t$.

char. eqn: $r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1) = 0 \Rightarrow r = -2$ or $r = 1$

$\Rightarrow y_c(t) = c_1 e^t + c_2 e^{-2t}$, let $g(t) = \underbrace{e^t}_{g_1(t)} + \underbrace{\sin t}_{g_2(t)}$

For $g_1(t) = e^t$, particular sol'n has the form $Y_1 = Ate^t$ (since e^t is already a sol'n for hom. eq'n).

$Y_1' = A(te^t + e^t)$, $Y_1'' = A(te^t + 2e^t)$

$\Rightarrow A(te^t + 2e^t) + A(te^t + e^t) - 2Ate^t = e^t$

$\Rightarrow 3Ae^t = e^t \Rightarrow A = 1/3 \Rightarrow Y_1(t) = \frac{1}{3}e^t$

For $g_2(t) = \sin t$, $Y_2(t) = B\cos t + C\sin t$

$Y_2'(t) = -B\sin t + C\cos t$

$Y_2''(t) = -B\cos t - C\sin t$

$\Rightarrow (-B\cos t - C\sin t) + (-B\sin t + C\cos t) - 2(B\cos t + C\sin t) = \sin t$

$\Rightarrow (-B + C - 2B)\cos t + (-C - B - 2C)\sin t = \sin t$

$\Rightarrow C - 3B = 0, \quad -3C - B = 1 \Rightarrow B = -1/10, \quad C = \frac{-3}{10}$

$Y_2(t) = \frac{-1}{10}\cos t - \frac{3}{10}\sin t$

So $y_p(t) = Y_1(t) + Y_2(t)$

general sol'n: $\boxed{y(t) = c_1 e^t + c_2 e^{-2t} + \frac{1}{3}e^t - \frac{1}{10}\cos t - \frac{3}{10}\sin t}$

3. (10 points) Let $\{y_1, y_2\}$ be a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$ on $I = (-2, 2)$. Suppose that y_1 is nonzero on I and that $y_1(0) = 1$, $y_1(1) = 2$, $y_2(0) = 1$ and $y_2(1) = 4$. Show that the Wronskian of y_1 and y_2 is positive on I (Hint: What is the derivative of y_2/y_1?).

Since $\{y_1, y_2\}$ is a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$,

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) \neq 0$$

for any $t \in I$. So $W(y_1, y_2)$ is either positive or negative on I (by intermediate value theorem).

Now we compute the derivative of y_2/y_1;

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1 y_2' - y_1' y_2}{{y_1}^2} = \frac{W(y_1, y_2)}{{y_1}^2}$$

. Since ${y_1}^2 > 0$ on I, both $(y_2/y_1)'$ and $W(y_1, y_2)$ are either positive or negative on I. In particular y_2/y_1 is either strictly increasing or decreasing on I.

But

$$\frac{y_2(1)}{y_1(1)} = \frac{4}{2} = 2 > \frac{y_2(0)}{y_1(0)} = \frac{1}{1} = 1,$$

so y_2/y_1 must be increasing which implies that both $(y_2/y_1)'$ and $W(y_1, y_2)$ are positive on I.

LAPLACE TRANSFORM TABLE:

$\mathcal{L}\{1\} = \frac{1}{s}\ \ s>0$ | $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}\ \ s>a$ | $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}\ \ s>0$ | $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}\ \ s>0$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}\ \ s>0$ | $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}\ \ s>a$ | $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}\ \ s>a$

$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$

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4. (15 points) (a) Find $F(s) = \mathcal{L}(f(t))$ for the function $f(t) = \left\{\begin{array}{ll} t-3, & 0 \le t < 3 \\ t^2+1, & 3 \le t < 5 \\ 1, & 5 \le t \end{array}\right\}$.

Here $f(t) = (t-3) + u_3(t)\,(t^2-t+4) - u_5(t)\,t^2$

$= (t-3) + u_3(t)\left((t-3)^2 + 5(t-3) + 10\right) - u_5(t)\left((t-5)^2 + 10(t-5) + 25\right)$

$\Rightarrow F(s) = \frac{1}{s^2} - \frac{3}{s} + e^{-3s}\mathcal{L}(t^2+5t+10) - e^{-5s}\mathcal{L}(t^2+10t+25)$

$= \frac{1}{s^2} - \frac{3}{s} + e^{-3s}\left(\frac{2}{s^3} + \frac{5}{s^2} + \frac{10}{s}\right) - e^{-5s}\left(\frac{2}{s^3} + \frac{10}{s^2} + \frac{25}{s}\right)$

(b) Find $f(t) = \mathcal{L}^{-1}(F(s))$ for the function $F(s) = \dfrac{e^{-3s}(s+21)}{s^2+2s+5}$.

Note that $f(t) = u_3(t)\,g(t-3)$, $\mathcal{L}\{g(t)\} = G(s) = \frac{s+21}{s^2+2s+5}$

$\frac{s+21}{s^2+2s+5} = \frac{s+1}{(s+1)^2+4} + 10\,\frac{2}{(s+1)^2+4}$

So $g(t) = e^{-t}\cos(2t) + 10\,e^{-t}\sin(2t)$, Hence

$f(t) = u_3(t)\left(e^{-(t-3)}\cos(2(t-3)) + 10\,e^{-(t-3)}\sin 2(t-3)\right)$

(c) Compute the convolution $u_2(t) * \sin t$.

$f(t) = u_2(t) * \sin t$ then $F(s) = \mathcal{L}\{u_2(t)\}\,\mathcal{L}\{\sin t\}$

$= \frac{e^{-2s}}{s}\cdot\frac{1}{s^2+1} = \frac{e^{-2s}}{s} - \frac{s}{s^2+1}e^{-2s}$ since $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$

Applying $\mathcal{L}^{-1}$: $f(t) = u_2(t) - u_2(t)\cos(t-2)$

5. (12 points) (a) Find the solution $y(t)$ of the IVP $y'' - y = 1$, $y(0) = 0$ and $y'(0) = a$.

hom. eq'n: $y'' - y = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow r = \mp 1$

$\Rightarrow y_c(t) = c_1 e^t + c_2 e^{-t}$

$Y = A \Rightarrow Y' = Y'' = 0 \Rightarrow -A = 1 \Rightarrow A = -1 \Rightarrow y_p(t) = -1$

So general sol'n is $\underline{y(t) = c_1 e^t + c_2 e^{-t} - 1}$

Imposing the I.C.'s,

$y(0) = 0 \Rightarrow 0 = c_1 + c_2 - 1 \Rightarrow c_1 + c_2 = +1$

$y'(t) = c_1 e^t - c_2 e^{-t}$

so $y'(0) = a \Rightarrow c_1 - c_2 = a$

So $\left.\begin{array}{l} c_1 + c_2 = 1 \\ c_1 - c_2 = a \end{array}\right\} \Rightarrow c_1 = \frac{1+a}{2} \qquad c_2 = \frac{1-a}{2}$

So $\underline{y(t) = \left(\frac{1+a}{2}\right) e^t + \left(\frac{1-a}{2}\right) e^{-t} - 1}$

(b) For what value of a does $y(t)$ approach a constant finite limit as $t \to \infty$? What is the solution in this case?

As $t \to \infty$, $e^{-t} \to 0$ So second term vanishes

but $e^t \to \infty$, so to have constant limit, the coefficient of the first term must be 0;

$\Rightarrow \frac{1+a}{2} = 0 \Rightarrow \underline{a = -1}$

And with $a = -1$, sol'n is $\underline{y(t) = e^{-t} - 1}$

6. (12 points) Find the solution of the IVP

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\mathbf{x}, \;\; \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

First we compute the eigenvalues of A;

$$\det(A - rI) = \begin{vmatrix} 1-r & 1 \\ -1 & 3-r \end{vmatrix} = (r-2)^2 = 0.$$

So we have a repeated eigenvalue. Let $r = 2$. Now we find a corresponding eigenvector;

$$\begin{pmatrix} 1-r & 1 \\ -1 & 3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Longrightarrow \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So we find a solution as

$$\mathbf{x}^{(1)} = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In order to find a second independent solution we need to compute a generalized eigenvector of $r = 2$;

$$\begin{pmatrix} 1-r & 1 \\ -1 & 3-r \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \Longrightarrow \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \eta_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So a second independent solution is

$$\mathbf{x}^{(2)} = te^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the general solution is

$$\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\left[te^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

Now we put $t = 0$;

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Longrightarrow c_1 = 1, \; c_2 = 1.$$

Thus the solution of the IVP is

$$\mathbf{x} = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left[te^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + te^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

7. (20 points) (a) Find the general solution of the following system of equations.

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix} \mathbf{x}$$

Here $A = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix}$, $\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 3 \\ 1 & -4-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 5$
$= (\lambda+1)(\lambda+5)$

so eigenvalues of A are $\lambda = -1$ & $\lambda = -5$

For $\lambda = -1$, $\begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow v_1 = 3v_2 \Rightarrow v^{(1)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

For $\lambda = -5$, $\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow v_1 = -v_2 \Rightarrow v^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

so general sol'n is

$$y(t) = c_1 e^{-t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(b) Find the general solution of the following nonhomogeneous system of equations

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} + \mathbf{g}(t) = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4e^{-t} \\ 0 \end{pmatrix}$$

7. a) (10 points) Find the general solution of the following system of equations.

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix} \mathbf{x}$$

b) (10 points) Find the general solution of the following nonhomogeneous system of equations

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} + \mathbf{g}(t) = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4e^{-t} \\ 0 \end{pmatrix}$$

By part a) a fundamental matrix for $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$ is

$$\Psi(t) = \begin{pmatrix} e^{-5t} & 3e^{-t} \\ -e^{-5t} & e^{-t} \end{pmatrix}.$$

We use variation of parameters to solve $\mathbf{x}' = \mathbf{A}\,\mathbf{x} + \mathbf{g}(t)$; so that the solution is

$$\mathbf{x} = \Psi(t)\mathbf{u}(t) \quad \text{where} \quad \Psi(t)\mathbf{u}'(t) = \mathbf{g}(t).$$

Plug Ψ and $\mathbf{g}(t)$ into the last equation;

$$\begin{pmatrix} e^{-5t} & 3e^{-t} \\ -e^{-5t} & e^{-t} \end{pmatrix} \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 4e^{-t} \\ 0 \end{pmatrix}.$$

So we find

$$u_1 = e^{4t}, \; u_2' = 1 \implies u_1 = e^{4t}/4 + c_1, \; u_2 = t + c_2.$$

Hence the general solution is

$$\mathbf{x} = \Psi(t)\mathbf{u}(t) = \begin{pmatrix} e^{-5t} & 3e^{-t} \\ -e^{-5t} & e^{-t} \end{pmatrix} \begin{pmatrix} e^{4t}/4 + c_1 \\ t + c_2 \end{pmatrix} = \begin{pmatrix} e^{-5t} & 3e^{-t} \\ -e^{-5t} & e^{-t} \end{pmatrix} \left[\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^{4t}/4 \\ t \end{pmatrix} \right]$$

$$= c_1 \begin{pmatrix} e^{-5t} \\ -e^{-5t} \end{pmatrix} + c_2 \begin{pmatrix} 3e^{-t} \\ e^{-t} \end{pmatrix} + \begin{pmatrix} e^{-t}/4 \\ -e^{-t}/4 \end{pmatrix} + \begin{pmatrix} 3te^{-t} \\ te^{-t} \end{pmatrix}.$$