7.4

3. Equation (14) states that the Wronskian satisfies the first order linear ODE

$$\frac{dW}{dt} = (p_{11} + p_{22} + \cdots + p_{nn})W.$$

The general solution of this is given by Equation (15):

$$W(t) = C\, e^{\int (p_{11}+p_{22}+\cdots+p_{nn})dt},$$

in which C is an arbitrary constant. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be matrices representing two sets of fundamental solutions. It follows that

$$\begin{aligned} \det(\mathbf{X}_1) &= W_1(t) = C_1 e^{\int (p_{11}+p_{22}+\cdots+p_{nn})dt} \\ \det(\mathbf{X}_2) &= W_2(t) = C_2 e^{\int (p_{11}+p_{22}+\cdots+p_{nn})dt}\,. \end{aligned}$$

Hence $\det(\mathbf{X}_1)/\det(\mathbf{X}_2) = C_1/C_2$. Note that $C_2 \neq 0$.

4. First note that $p_{11} + p_{22} = -p(t)$. As shown in Problem 3,

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = c\, e^{-\int p(t)dt}.$$

For second order linear ODE, the Wronskian (as defined in Chapter 3) satisfies the first order differential equation $W' + p(t)W = 0$. It follows that

$$W\left[\mathbf{y}^{(1)}, \mathbf{y}^{(2)}\right] = c_1\, e^{-\int p(t)dt}.$$

Alternatively, based on the hypothesis,

$$\begin{aligned} \mathbf{y}^{(1)} &= \alpha_{11}\, x_{11} + \alpha_{12}\, x_{12} \\ \mathbf{y}^{(2)} &= \alpha_{21}\, x_{11} + \alpha_{22}\, x_{12}\,. \end{aligned}$$

Direct calculation shows that

$$\begin{aligned} W\left[\mathbf{y}^{(1)}, \mathbf{y}^{(2)}\right] &= \begin{vmatrix} \alpha_{11}\, x_{11} + \alpha_{12}\, x_{12} & \alpha_{21}\, x_{11} + \alpha_{22}\, x_{12} \\ \alpha_{11}\, x_{11}' + \alpha_{12}\, x_{12}' & \alpha_{21}\, x_{11}' + \alpha_{22}\, x_{12}' \end{vmatrix} \\ &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{11}x_{12}' - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{12}x_{11}' \\ &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{11}x_{22} - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{12}x_{21}\,. \end{aligned}$$

Here we used the fact that $\mathbf{x}_1' = \mathbf{x}_2$. Hence

$$W\left[\mathbf{y}^{(1)}, \mathbf{y}^{(2)}\right] = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right].$$

Section 7.4 : Problem 6.

$$X^{(1)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix} , \quad X^{(2)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

(a) $W(X^{(1)}, X^{(2)})(t) = \det\begin{pmatrix} | & | \\ X^{(1)} & X^{(2)} \\ | & | \end{pmatrix} = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2$

(b) From (a), it follows that $X^{(1)}$ and $X^{(2)}$ are linearly independent at each point except $t=0$; they are linearly independent on every interval.

(C) At least one coefficient must be discontinuous at $t=0$.

(d) The general solution of the homogeneous system $X' = P \cdot X$ is of the form

$X = C_1 \cdot X^{(1)} + C_2 \cdot X^{(2)}$ so that,

$$X(t) = C_1 \cdot \begin{pmatrix} t \\ 1 \end{pmatrix} + C_2 \cdot \begin{pmatrix} t^2 \\ 2t \end{pmatrix} = \begin{pmatrix} C_1 \cdot t + C_2 \cdot t^2 \\ C_1 + C_2 \cdot 2t \end{pmatrix}$$

$$X'(t) = \begin{pmatrix} C_1 + 2tC_2 \\ C_1 \cdot 0 + 2 \cdot C_2 \end{pmatrix} \text{ and } P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

plugging the value of $X(t)$ in the equation, we find

$$P = \begin{pmatrix} 0 & 1 \\ \frac{-2}{t^2} & \frac{2}{t} \end{pmatrix} \text{ so, } X' = \begin{pmatrix} 0 & 1 \\ -2t^{-2} & 2t^{-1} \end{pmatrix} X .$$

7.(a) By definition,

$$W\left[\mathbf{x}^{(1)},\mathbf{x}^{(2)}\right]=\begin{vmatrix} t & e^t \\ 1 & e^t \end{vmatrix}=(t-1)e^t.$$

(b) The Wronskian vanishes at $t_0=1$. Hence the vectors are linearly independent on $\mathcal{D}=(-\infty\,,1)\cup(1\,,\infty)$.

(c) It follows from Theorem 7.4.3 that one or more of the coefficients of the ODE must be discontinuous at $t_0=1$. If not, the Wronskian would not vanish.

(d) Let

$$\mathbf{x}=c_1\begin{pmatrix} t \\ 1 \end{pmatrix}+c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

Then

$$\mathbf{x}'=c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}+c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

On the other hand,

$$\begin{aligned}\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\mathbf{x} &= c_1\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix}+c_2\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} e^t \\ e^t \end{pmatrix} \\ &= \begin{pmatrix} c_1\left[p_{11}t+p_{12}\right]+c_2\left[p_{11}+p_{12}\right]e^t \\ c_1\left[p_{21}t+p_{22}\right]+c_2\left[p_{21}+p_{22}\right]e^t \end{pmatrix}.\end{aligned}$$

Comparing coefficients, we find that

$$\begin{aligned} p_{11}t+p_{12} &= 1 \\ p_{11}+p_{12} &= 1 \\ p_{21}t+p_{22} &= 0 \\ p_{21}+p_{22} &= 1\,. \end{aligned}$$

Solution of this system of equations results in

$$p_{11}(t)=0,\quad p_{12}(t)=1,\quad p_{21}(t)=\frac{1}{1-t},\quad p_{22}(t)=\frac{-t}{1-t}\,.$$

Hence the vectors are solutions of the ODE

$$\mathbf{x}'=\frac{1}{1-t}\begin{pmatrix} 0 & 1-t \\ 1 & -t \end{pmatrix}\mathbf{x}\,.$$