Math 204 - Differential Equations

Midterm 2 December 10, 2015

Duration: 90 minutes

Instructions: Calculators are not allowed. No books, no notes, no questions, and no talking allowed. You must always **explain your answers** and **show your work** to receive full **credit**. If necessary, you can use the back of these pages, but make sure you have indicated doing so. **Print (i.e., use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: KEY

Signature: ____________________

Section (Check One):

Section 1: E. Ceyhan (Mon-Wed 10:00)
Section 2: E. Ceyhan (Mon-Wed 14:30)
Section 3: A. Erdoğan (Tue-Thu 16:00)

Question	Points	Score
1	20	
2	15	
3	20	
4	10	
5	25	
6	15	
Total	**105**	

1. (20 points) Find the general solution of the differential equation

$$y'' + y = \sec(t)$$

on the interval $(-\pi/2, \pi/2)$. (Note that $\sec(t) = 1/\cos(t)$.)

First solve $y''+y=0$. (1)

The characteristic equation is $r^2+1=0 \Rightarrow r_{1,2} = \mp i$

Then the general solution of (1) is $y_c(t) = c_1 \cos t + c_2 \sin t$.

let $y_1 = \cos t$, $y_2 = \sin t$.

Check the Wronskian: $W(y_1, y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$.

Now we use variation of parameters;

let $y = u_1 y_1 + u_2 y_2$.

$u_1 = \int \frac{-y_2 \sec t}{W(y_1; y_2)} dt$, $\quad u_2 = \int \frac{y_1 \sec t}{W(y_1; y_2)} dt$.

$\Rightarrow \quad u_1 = \int -\frac{\sin t}{\cos t} dt = \ln|\cos t| + c_1$, $\quad u_2 = \int \frac{\cos t}{\cos t} dt = t + c_2$.

$u_1 = \ln(\cos t) + c_1$ since $t \in (-\pi/2, \pi/2)$.

Hence the general solution is

$y = u_1 y_1 + u_2 y_2 = c_1 \cos t + c_2 \sin t + \ln(\cos t)\cos t + t \sin t$

2. Find a particular solution for each the following differential equations.

(a) (5 points) $y'' + y' - 2y = t^2$

First solve $y''+y'-2y=0$. The characteristic equation is:
$r^2+r-2=0$. $\Rightarrow r_{1,2} = -2, 1$. $\Rightarrow y_c(t) = c_1 e^{-2t} + c_2 e^{t}$.

By the method of undetermined coefficients we try a particular solution as $Y(t) = At^2 + Bt + C$, so that $Y' = 2At + B$, $Y'' = 2A$.

$\Rightarrow Y'' + Y' - 2Y = 2A + 2At + B - 2At^2 - 2Bt - 2C = t^2$

$\Rightarrow A = -\frac{1}{2}$, $B = -\frac{1}{2}$, $C = \frac{-3}{4}$ $\Rightarrow Y(t) = -t^2/2 - t/2 - 3/4$ //.

(b) (5 points) $y'' + y' - 2y = e^t$

We know from (a) part that $y_c(t) = c_1 e^{-2t} + c_2 e^{t}$, so $Y(t) = Ae^t$ will not work!,

Try $Y(t) = Ate^t$, so that $Y' = Ae^t + Ate^t$
$Y'' = 2Ae^t + tAe^t$

$\Rightarrow Y'' + Y' - 2Y = 2Ae^t + Ate^t + Ae^t + Ate^t - 2Ate^t = e^t$

$\Rightarrow A = 1/3 \Rightarrow Y(t) = \frac{te^t}{3}$.

(c) (5 points) $y'' + y' - 2y = e^t + t^2$

$Y_1 = \frac{te^t}{3}$ for $y''+y'-2y = e^t$ and

$Y_2 = -\frac{t^2}{2} - \frac{t}{2} - \frac{3}{4}$ for $y''+y'-2y = t^2$

Thus $Y = Y_1 + Y_2 = \frac{te^t}{3} - \frac{t^2}{2} - \frac{t}{2} - \frac{3}{4}$ is a particular solution of $y''+y'-2y = e^t + t^2$

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3. (a) (15 points) Find the general solution of $(1+x^2)y'' + 3xy' + y = 0$ in terms of power series about 0. Determine the radius of convergence of the solution.

let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\Rightarrow (1+x^2)y'' + 3xy' + y = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

$n=0$ then $a_2 = -a_0/2$, $n=1$ then $a_3 = \frac{-2}{3} a_1$

$n \geq 2$ then $a_{n+2} = -a_n \frac{n+1}{n+2}$ $\Rightarrow \begin{cases} a_{2n} = (-1)^n \frac{1.3.\dots(2n-1)}{2.4.\dots(2n)} a_0 \\ a_{2n+1} = (-1)^n \frac{2.4.\dots(2n)}{1.3.\dots(2n+1)} a_1 \end{cases}$

let $\begin{cases} y_1 = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1.3.\dots(2n-1)}{2.6.\dots(2n)} x^{2n} \\ y_2 = x + \sum_{n=1}^{\infty} (-1)^n \frac{2.4.\dots(2n)}{1.3.\dots(2n+1)} x^{2n+1} \end{cases}$ $\Rightarrow$ general sol: $y = a_0 y_1 + a_1 y_2$.

Apply the ratio test to both series: $\begin{cases} \lim_{n\to\infty} x^2 \left(\frac{2n+1}{2n+2}\right) = x^2 < 1 \\ \lim_{n\to\infty} x^2 \left(\frac{2n+2}{2n+3}\right) = x^2 < 1 \end{cases}$

$\Rightarrow$ for both $|x| < 1 \Rightarrow R = 1$

Thus $R = 1$ for the general solution.

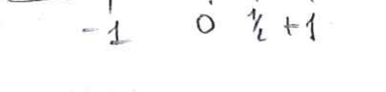
(b) (5 points) Find a lower bound for the radius of convergence of the power series solutions about 0 of $(1-x^2)(1-2x)y'' + x(1-2x)y' + (1-x^2)y = 0$.

Solve this as in Example 5.3.4 and use Theorem 5.3.1.

$p(x) = \frac{x}{1-x^2}$ and $q(x) = \frac{1}{1-2x}$.

$\Rightarrow$ $p(x)$ is analytic for all x except $x = \mp 1$.

$q(x)$ is analytic for all x except $x = \frac{1}{2}$.



So the minimum distance to $x_0 = 0$ is $1/2$.

$\Rightarrow$ Radius of convergence $\geq \frac{1}{2}$ //.

LAPLACE TRANSFORM TABLE:

$\mathcal{L}\{1\} = \frac{1}{s}$ $s > 0$ | $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ $s > a$ | $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$ $s > 0$ | $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$ $s > 0$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $s > 0$ | $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$ $s > a$ | $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$ $s > a$

$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$

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4. (10 points) Let $f(t)$ be a function whose Laplace transform is $F(s)$. Define a new function $g(t) = e^{-2t}f(3t)$. Determine the Laplace transform $G(s)$ of $g(t)$ in terms of F.

$$\mathcal{L}(g(t)) = \int_0^\infty e^{-st} e^{-2t} f(3t)\,dt$$

$$= \int_0^\infty e^{-(s+2)t} f(3t)\,dt \qquad , \begin{cases} u = 3t \\ du = 3dt \end{cases}$$

$$= \frac{1}{3}\int_0^\infty e^{-\left(\frac{s+2}{3}\right)u} f(u)\,du$$

Recall that $F(s) = \int_0^\infty e^{-su} f(u)\,du$

Thus $\mathcal{L}(g(t)) = \frac{1}{3} \cdot F\left(\frac{s+2}{3}\right)$ //

5. (a) (10 points) Find the inverse Laplace transform of $F(s)$, i.e., $f(t)$ for which $\mathcal{L}\{f(t)\} = F(s)$.

$$F(s) = \frac{2s - 3e^{-\pi s/2}}{s^2 + 2s + 10}$$

$$F(s) = \frac{2s - 3e^{-\pi s/2}}{(s+1)^2 + 9} = 2 \cdot \frac{s+1}{(s+1)^2 + 3^2} - \frac{2}{3} \cdot \frac{3}{(s+1)^2 + 3^2} - e^{-\pi s/2} \frac{3}{(s+1)^2 + 3^2}$$

$$= 2\mathcal{L}(e^{-t}\cos 3t) - \frac{2}{3}\mathcal{L}(e^{-t}\sin 3t) - \mathcal{L}\left(u_{\frac{\pi}{2}} e^{-(t-\frac{\pi}{2})} \cdot \sin\left(3(t-\tfrac{\pi}{2})\right)\right)$$

$$\Rightarrow f(t) = 2e^{-t}\cos 3t - \frac{2}{3}e^{-t}\sin 3t - u_{\frac{\pi}{2}}(t)\, e^{-(t-\pi/2)} \sin\left(3(t-\tfrac{\pi}{2})\right).$$

(b) (15 points) Let $g(t)$ be a forcing function defined as

$$g(t) = \left\{ \begin{array}{ll} 1, & 0 \le t < 1 \\ 0, & 1 < t \end{array} \right\}.$$

let $Y(s) = \mathcal{L}(y)$ &
$G(s) = \mathcal{L}(g)$.

Solve the following initial value problem.

$$y'' + y = g(t), \quad y(0) = 1, \quad y'(0) = 0$$

Here $g(t) = 1 - u_1(t)$. Take Laplace transform of both sides of IVP:

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = G(s).$$

$$Y(s)(s^2+1) - s = \frac{1}{s} - \frac{e^{-s}}{s} \Rightarrow Y(s) = \frac{s}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{e^{-s}}{s(s^2+1)}$$

$$= \frac{1}{s} - e^{-s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$Y(s) = \mathcal{L}(1) - e^{-s}\mathcal{L}(1 - \cos t)$$

$$\Rightarrow y(t) = 1 - u_1(t)\left(1 - \cos(t-1)\right).$$

6. (15 points) Let $g(t)$ be a function such that $\mathcal{L}\{g(t)\}$ exists. Find the solution of the initial value problem

$$2y'' + 3y' - 2y = g(t)\sin t, \quad y(0) = 0, \quad y'(0) = 0$$

in terms of convolution integrals.

Let $\mathcal{L}\{y(t)\} = Y(s)$ and $h(t) = g(t)\sin t$ and $\mathcal{L}(h(t)) = H(s)$.

Then take Laplace transformation of both sides of IVP:

$$2(s^2 Y(s) - s\,\cancel{y(0)} - \cancel{y'(0)}) + 3(s\,Y(s) - \cancel{y(0)}) - 2Y(s) = H(s)$$

$$Y(s) = \frac{H(s)}{2s^2+3s-2} = \frac{H(s)}{(2s-1)(s+2)}.$$

Note that $\frac{1}{(2s-1)(s+2)} = \frac{2}{5}\left(\frac{1}{2s-1}\right) - \frac{1}{5}\left(\frac{1}{s+2}\right)$.

So, $Y(s) = \left(\frac{1}{5}\cdot\left(\frac{1}{s-1/2}\right) - \frac{1}{5}\left(\frac{1}{s+2}\right)\right)H(s)$.

and $\mathcal{L}^{-1}\left\{\frac{1}{5}\left(\frac{1}{s-1/2}\right) - \frac{1}{5}\left(\frac{1}{s+2}\right)\right\} = \frac{1}{5}\left(e^{t/2} - e^{-2t}\right)$

and $\mathcal{L}^{-1}\{H(s)\} = h(t) = g(t)\sin t$.

Then, $y(t) = \frac{1}{5}\left(e^{t/2} - e^{-2t}\right) * h(t)$.

$$\Rightarrow y(t) = \frac{1}{5}\int_0^t g(t-\tau)\sin(t-\tau)\left(e^{\tau/2} - e^{-2\tau}\right)d\tau.$$

or $$y(t) = \frac{1}{5}\int_0^t \left(e^{(t-\tau)/2} - e^{-2(t-\tau)}\right) g(\tau)\sin\tau\, d\tau.$$