Math 204 - Differential Equations

Final Exam May 24, 2016

Duration: 150 minutes

Instructions: Calculators are not allowed. No books, no notes, no questions, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. If necessary, you can use the back of these pages, but make sure you have indicated doing so. **Print (i.e., use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: KEY

Signature: ______________________

Section (Check One):

Section 1: E. Ceyhan (Tue-Thu 10:00)

Section 2: E. Ceyhan (Tue-Thu 08:30)

Section 3: H. Göral (Mon-Wed 16:00)

Question	Points	Score
1	15	
2	15	
3	12	
4	15	
5	20	
6	15	
7	13	
Total	**105**	

1. (15 points)
(a) Find the solution of the initial value problem

$$ty' = \frac{1}{y+1}, \quad y(1) = 0$$

DE is seperable. Seperating the variables, we get

$(y+1)\,dy = \frac{1}{t}\,dt$; integrating

$\int (y+1)\,dy = \int \frac{1}{t}\,dt \Rightarrow \frac{y^2}{2} + y = \ln t + c$

Imposing the I.C $0 = 0 + c \Rightarrow c = 0$

So the solution is $y^2 + 2y = 2\ln t \Rightarrow y^2 + 2y + 1 = 2\ln t + 1$

$\Rightarrow (y+1)^2 = \ln t + 1 \Rightarrow y + 1 = \pm\sqrt{2\ln t + 1}$

$y(t) = \pm\sqrt{2\ln t + 1} - 1$ with I.C $y(1) = 0$ we have

$\boxed{y(t) = \sqrt{2\ln t + 1} - 1}$

(b) For what t-interval is the solution in part (a) defined?

The solution is defined and continuously differentiable provided that $2 \cdot \ln t + 1 > 0 \Rightarrow -\frac{1}{2} < \ln t < \infty$

$\Rightarrow \boxed{e^{-1/2} < t < \infty}$

(c) Find the solution of the differential equation $y' = \ln(2^y)$.

$y' = y \cdot \ln 2 \Rightarrow \frac{dy}{y} = \ln 2\,dx$

$\Rightarrow \int \frac{dy}{y} = \int \ln 2\,dx$

$\ln y = (\ln 2) \cdot x + c_1$

$\Rightarrow y = c \cdot (e^{\ln 2})^x = c \cdot 2^x$

2. (15 points) Suppose that a is a constant and consider the initial value problem

$$y' - y = e^{at}, \quad y(0) = 0$$

(a) Find the solution if $a \neq 1$.

An integrating factor is $\mu(t) = \exp\left(\int (-1)dt\right) = e^{-t}$

Multiply the D.E by e^{-t} and rearranging, we get

$$(e^{-t} \cdot y)' = e^{(a-1)t} \Rightarrow e^{-t} \cdot y = \int e^{(a-1)t} dt$$

$$\Rightarrow e^{-t} \cdot y = \frac{e^{(a-1)t}}{a-1} + c \Rightarrow y(t) = \frac{e^{at}}{a-1} + c \cdot e^{t}$$

with I.C $y(0) = 0 \Rightarrow 0 = \frac{1}{a-1} + c \Rightarrow c = -\frac{1}{a-1}$.

So, $\boxed{y(t) = \frac{e^{at} - e^{t}}{a-1}}$

(b) Find the solution if $a = 1$.

IF $a = 1$, $(e^{-t} \cdot y)' = 1$

$$\Rightarrow e^{-t} \cdot y = t + c \Rightarrow y = t \cdot e^{t} + c \cdot e^{t}$$

with I.C $y(0) = 0 \Rightarrow c = 0$.

So, $\boxed{y(t) = t \cdot e^{t}}$

(c) Show that the solution in part (b) is the limit of the solution in part (a) as $a \to 1$. (Hint: Use L'Hospital rule.)

$$\lim_{a \to 1} \frac{e^{at} - e^{t}}{a-1} = \frac{0}{0} \quad \text{by L'Hospital Rule,}$$

$$= \lim_{a \to 1} \frac{t \cdot e^{at}}{1} = t \cdot e^{t}$$

3. (12 points) Find the general solutions of the following differential equations
(a) $y'' - 4y' + 5y = 0$.

The characteristic equation is $r^2 - 4r + 5 = 0$

$$r_{1,2} = \frac{4 \mp \sqrt{16-20}}{2} = 2 \mp i$$

So the general solution is

$$\underline{\underline{y(t) = c_1 \cdot e^{2t} \cdot \cos t + c_2 \cdot e^{2t} \cdot \sin t}}$$

(b) $y'' + 3y' - 4y = 0$.

The characteristic equation is $r^2 + 3r - 4 = 0$

$$(r+4)(r-1) = 0$$

$$\Rightarrow r_1 = 1 \ , r_2 = -4$$

So the general solution is,

$$y(t) = c_1 \cdot e^{t} + c_2 \cdot e^{-4t}$$

4. (15 points) (a) Find the values of the constants m and n for which the differential equation

$$(xy^n + x^2)dx + (x^2y^m + y^3)dy = 0$$

is exact.

Here $M(x,y) = x \cdot y^n + x^2$ and $N(x,y) = x^2 \cdot y^m + y^3$

The theorem on exactness implies $M_y = N_x$

$\Rightarrow n \cdot x \cdot y^{n-1} = 2x \cdot y^m \Rightarrow \underline{\underline{n = 2, m = 1}}$

(b) Solve the differential equation in part (a) with the values you found for m and n.

with $m = 1$ & $n = 2$, the D.E is

$$(xy^2 + x^2)dx + (x^2y + y^3)dy = 0$$

So there exists a solution $\psi(x,y)$ such that

$\psi_x(x,y) = xy^2 + x^2$ and $\psi_y(x,y) = x^2y + y^3$

Integrating the first, we get

$$\psi(x,y) = \frac{x^2}{2} \cdot y^2 + \frac{x^3}{3} + h(y)$$

Then $\psi_y = x^2y + h'(y) = x^2y + y^3$

$\Rightarrow h'(y) = y^3 \Rightarrow h(y) = y^4/4.$

So $\underline{\underline{\psi(x,y) = \frac{x^2 \cdot y^2}{2} + \frac{x^3}{3} + \frac{y^4}{4} = C}}$

5. (20 points) (a) Find the general solution of

$$y'' - 2xy' + 2\lambda y = 0$$

in terms of a power series about 0 where λ is a constant.

Let $y = \sum_{n=0}^{\infty} a_n \cdot x^n$, so $y' = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n \cdot x^{n-2}$

Also $y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \cdot x^n$. Plug y, y' and y'' in D.E

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \cdot x^n - \sum_{n=1}^{\infty} 2n \cdot a_n \cdot x^n + \sum_{n=0}^{\infty} 2 \cdot \lambda \cdot a_n \cdot x^n = 0$$

$\underline{n=0}$: $2a_2 + 2\lambda a_0 = 0 \Rightarrow a_2 = -\lambda \cdot a_0$

$\underline{n \geq 1}$: $(n+2)(n+1) a_{n+2} = 2(n-\lambda) a_n \Rightarrow a_{n+2} = \dfrac{2(n-\lambda) a_n}{(n+2)(n+1)}$.

$$a_4 = \frac{2(2-\lambda)}{4 \cdot 3} = \frac{-2 \cdot (2-\lambda) \cdot \lambda}{4 \cdot 3} \cdot a_0 = \frac{-2^2 \cdot \lambda \cdot (2-\lambda)}{4!} a_0$$

$$a_6 = \frac{2(4-\lambda)}{6 \cdot 5} \cdot a_4 = -2^3 \cdot \frac{\lambda(2-\lambda)(4-\lambda)}{6!} a_0$$

$\vdots$

$$a_{2n} = \frac{-2^n \cdot \lambda(2-\lambda) \dots (2n-2-\lambda)}{(2n)!} a_0 \text{, similarly } a_{2n+1} = \frac{2^n (1-\lambda) \dots (2n-1-\lambda)}{(2n+1)!} a_1$$

so the general solution is

$$y = a_0 \cdot \sum_{n=0}^{\infty} \frac{-2^n \lambda \cdot (2-\lambda) \dots (2n-2-\lambda)}{(2n)!} x^{2n} + a_1 \cdot \sum_{n=0}^{\infty} \frac{2^n (1-\lambda)(3-\lambda) \dots (2n-1-\lambda)}{(2n+1)!} x^{2n+1}$$

(b) Determine the radius of convergence of the solution of the equation given in part (a).

From $y'' + q(x) \cdot y' + r(x) \cdot y = 0$, $q(x) = -2x$, $r(x) = 2\lambda$

$q(x)$ & $r(x)$ are analytic everywhere and the radius of convergence of power series expansion of both $q(x)$ and $r(x)$ is $R = \infty$. So the radius of convergence of the solution is $R = \infty$. //

LAPLACE TRANSFORM TABLE:

$\mathcal{L}\{1\} = \frac{1}{s}$ $s > 0$ | $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ $s > a$ | $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$ $s > 0$ | $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$ $s > 0$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $s > 0$ | $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$ $s > a$ | $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$ $s > a$

$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$

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6. (15 points) Let $\phi(t)$ be the solution of the initial value problem

$$y'' + 4y = g(t), \quad y(0) = a, \quad y'(0) = 0$$

where $a \in \mathbb{R}$ is constant and

$$g(t) = \left\{ \begin{array}{ll} \sin(t) & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi \leq t \end{array} \right\}.$$

Find $\phi(\pi/4)$.

Take Laplace transform of the D.E ;

$$s^2 \cdot Y(s) - s \cdot y(0) - y'(0) + 4 \cdot Y(s) = \mathcal{L}(g(t)) = \mathcal{L}\Big(\sin t \underbrace{- u_\pi(t)\sin t}_{= u_\pi(t)\cdot\sin(t-\pi)}\Big)$$

$$\Rightarrow (s^2+4)\cdot Y(s) - a \cdot s = \frac{1}{s^2+1} + e^{-\pi s} \cdot \frac{1}{s^2+1}$$

$$\Rightarrow Y(s) = a \cdot \frac{s}{s^2+4} + \frac{1}{(s^2+1)(s^2+4)} + e^{-\pi s} \cdot \frac{1}{(s^2+1)(s^2+4)}$$

$$\Rightarrow Y(s) = a \cdot \frac{s}{s^2+4} + \frac{1}{3} \cdot \left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] + \frac{1}{3} e^{-\pi s} \cdot \left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$$

$$\Rightarrow \phi(t) = a \cdot \cos 2t + \frac{1}{3} \cdot \left[\sin t - \frac{\sin 2t}{2}\right] + \frac{1}{3}\left[u_\pi(t)\left(\sin(t-\pi) - \frac{\sin(2t-2\pi)}{2}\right)\right]$$

$$\Rightarrow \phi(t) = a \cdot \cos 2t + \frac{1}{3}\left[\sin t - u_\pi(t) \cdot \sin t\right] - \frac{1}{6}\left[\sin 2t + u_\pi(t) \cdot \sin 2t\right]$$

$$\Rightarrow \phi(\pi/4) = \frac{1}{3} \cdot \left[\frac{\sqrt{2}}{2}\right] - \frac{1}{6} \cdot 1 = \frac{\sqrt{2}-1}{6} \; //$$

7. (13 points) Find the general solution of the following system of equations that satisfies the given initial condition.

$$\mathbf{y}' = \mathbf{A}\,\mathbf{y} = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\det(A - rI) = \begin{vmatrix} -1-r & -2 \\ 0 & -1-r \end{vmatrix} = (-1-r)^2 = 0$$

$\Rightarrow r_{1,2} = -1$ is the double root.

For $r = -1$, $\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} -2w_2 = 0 \\ w_2 = 0 \end{array}$

So $w^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector for $r = -1$.

So one solution is $y^{(1)}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$.

For the other solution, start with

$$y(t) = w \cdot t \cdot e^{-t} + \eta \cdot e^{-t}$$

$$\Rightarrow y'(t) = w \cdot e^{-t} - wt \cdot e^{-t} - \eta \cdot e^{-t}$$

So $(we^{-t} - wt\,e^{-t} - \eta e^{-t}) = A\,(wt\,e^{-t} + \eta \cdot e^{-t})$

$$w - \eta = A \cdot \eta \quad \ldots\ldots (1)$$

$$-w = Aw \quad \ldots\ldots\ldots (2)$$ this is just eigenvalue for $r = -1$.

In (1), $(A+I)\eta = w \Rightarrow \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \eta_2 = -1/2$ and $\eta_1 = c$

So $\eta = \begin{pmatrix} c \\ -1/2 \end{pmatrix} = c\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}$ and $y(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \cdot e^{-t} + \overbrace{c \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}}^{\text{a multiple of } y^{(1)}} + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} e^{-t}$

So, $y^{(2)}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} e^{-t} \Rightarrow$ general sol'n $\boxed{y(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} te^{-t} + c_2 \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} e^{-t}}$

Imposing the I.C, we get $c_1 = 1$ and $c_2 = 0$

thus, $y(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot e^{-t}$