Section 7.5 – Q(1a). $X' = \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix} X.$

Finding eigenvalues of the matrix $\begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix}$,

$$0 = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ -2 & -2-\lambda \end{vmatrix}$$

$$0 = (3-\lambda)\cdot(-2-\lambda) - 2\cdot(-2)$$
$$0 = -6 - 3\lambda + 2\lambda + \lambda^2 + 4$$
$$0 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

So $\lambda_1 = -1$ and $\lambda_2 = 2.$

To find eigenvectors,

$$0 = (A - \lambda_1 I)v = \begin{pmatrix} 4 & 2 \\ -2 & -1 \end{pmatrix} \cdot v \quad \text{so} \quad v_1 = \begin{pmatrix} -1 \\ +2 \end{pmatrix}$$

and,

$$0 = (A - \lambda_2 I)v = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} v \quad \text{so} \quad v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Thus, the general solution of the equation is,

$$X(t) = c_1 \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t}$$

$\boxed{Q5a}$: $X' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} X.$ $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

eigenvalues $0 = \det(A - \lambda I)$ i.e $\lambda_1 = -3$, $\lambda_2 = -1$

and corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Hence, $X(t) = c_1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$

3.(a) Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 2-r & 3 \\ -1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 1 = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = -1$. For $r = 1$, the system of equations reduces to $\xi_1 = -3\xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (-3, 1)^T$. Substitution of $r = -1$ results in the single equation $\xi_1 = -\xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (-1, 1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}.$$

15. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\,\xi_1 - \xi_2 = 0$. An

eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 3)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 3 \\ c_1 + 3\,c_2 &= -1\,. \end{aligned}$$

Hence $c_1 = 5$ and $c_2 = -2$, and the solution of the IVP is

$$\mathbf{x} = 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

$\boxed{7a}$ : $x' = \begin{pmatrix} 4 & 8 \\ -3 & -6 \end{pmatrix} x$

$A = \begin{pmatrix} 4 & 8 \\ -3 & -6 \end{pmatrix}$

eigenvalues are $\lambda_1, \lambda_2$ such that $0 = \det(A - \lambda I)$ from which we obtain $\lambda_1 = 0$ and $\lambda_2 = -2$

And, corresponding eigenvectors $v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

Thus,

$$x(t) = c_1 \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot e^{0 \cdot t} + c_2 \cdot \begin{pmatrix} -4 \\ 3 \end{pmatrix} e^{-2t}$$

$\boxed{Q1b}$: $x' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

$A = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}$

eigenvalues are $\lambda_1, \lambda_2$ roots of $\det(A - \lambda I) = 0$ so that $\lambda_1 = -1$ and $\lambda_2 = 3$. And corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

So, $x(t) = c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$.

Imposing the initial value $x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$,

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^0 + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^0 = \begin{pmatrix} c_1 + c_2 \\ c_1 + 5c_2 \end{pmatrix}$$

OR $c_1 = \frac{7}{4}$ and $c_2 = \frac{1}{4}$

Thus, $x(t) = \frac{7}{4} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{4} \cdot \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$. //

7.6) <u>1.a</u>. Setting $x = \xi e^{rt}$ results:

$$\begin{pmatrix} 3-r & 4 \\ -2 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For a non-zero solution, we require that $\det(A - rI) = 0$

$$\Rightarrow (3-r)(-1-r) + 8 = 0$$

$r_{1,2} = 1 \mp 2i$ $\qquad$ $r_1 = 1+2i$, $r_2 = 1-2i$

Substituting $r_{1,2}$, we have associated eigenvectors; respectively

$$\xi^{(1)} = \begin{pmatrix} -2 \\ 1-i \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} -2 \\ 1+i \end{pmatrix}$$

Hence one of the complex-valued solutions is given by:

$$x^{(1)} = \begin{pmatrix} -2 \\ 1-i \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} -2 \\ 1-i \end{pmatrix} e^{t}(\cos 2t + i \sin 2t)$$

$$= e^{t} \begin{pmatrix} -2\cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + i e^{t} \begin{pmatrix} -2\sin 2t \\ -\cos 2t + \sin 2t \end{pmatrix}$$

Based on the real and imaginary parts of this solution, the general solution is:

$$x = c_1 e^{t} \begin{pmatrix} -2\cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^{t} \begin{pmatrix} -2\sin 2t \\ -\cos 2t + \sin 2t \end{pmatrix}$$

3.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 2-r & 1 \\ -5 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 1 = 0$. The roots of the characteristic equation are $r = \pm i$. Setting $r = i$, the equations are equivalent to $(2-i)\xi_1 + \xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (1, -2+i)^T$ and $\boldsymbol{\xi}^{(2)} = (1, -2-i)^T$. Hence one of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -2+i \end{pmatrix} e^{it} = \begin{pmatrix} 1 \\ -2+i \end{pmatrix} (\cos t + i \sin t) =$$

$$= \begin{pmatrix} \cos t \\ -2\cos t - \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix}.$$

Therefore the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} \cos t \\ -2\cos t - \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix}.$$

The solution may also be written as

$$\mathbf{x} = c_1 \begin{pmatrix} -2\cos t + \sin t \\ 5\cos t \end{pmatrix} + c_2 \begin{pmatrix} -2\sin t - \cos t \\ 5\sin t \end{pmatrix}.$$

5.(a) Setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 5 \\ -1 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2r + 2 = 0$, with roots $r = -1 \pm i$. Substituting $r = -1 - i$ reduces the system of equations to $(2+i)\xi_1 + 5\xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (-2+i, 1)^T$ and $\boldsymbol{\xi}^{(2)} = (-2-i, 1)^T$. Hence one of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} -2+i \\ 1 \end{pmatrix} e^{-(1+i)t} = \begin{pmatrix} -2+i \\ 1 \end{pmatrix} e^{-t}(\cos t - i \sin t) =$$

$$= e^{-t} \begin{pmatrix} -2\cos t + \sin t \\ \cos t \end{pmatrix} + i e^{-t} \begin{pmatrix} \cos t + 2\sin t \\ -\sin t \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 e^{-t} \begin{pmatrix} -2\cos t + \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \cos t + 2\sin t \\ -\sin t \end{pmatrix}.$$

7.6) g. eigenvalues:

$$\begin{vmatrix} 1-r & -5 \\ 1 & -3-r \end{vmatrix} = 0 \Rightarrow \quad (1-r)(-3-r)+5=0$$

$$r^2+2r+2=0$$

$$\Rightarrow r_{1,2} = -1 \mp i$$

$r_1 = -1+i \qquad \xi^{(1)} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$

$$x^{(1)} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} e^{(-1+i)t} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} e^{-t}(\cos t + i \sin t)$$

$$= e^{-t}\begin{pmatrix} 5\cos t + i\, 5 \sin t \\ 2\cos t + \sin t + i(2\sin t - \cos t) \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + e^{-t} \cdot i \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$$

Hence the general solution is:

$$x = c_1 e^{-t}\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$$

Substitute the I.C. $X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\begin{cases} 5c_1 = 2 \\ 2c_1 - c_2 = 1 \end{cases} \Rightarrow \quad \begin{matrix} c_1 = 2/5 \\ c_2 = -\frac{1}{5} \end{matrix}$$

$$x = \frac{2}{5} e^{-t}\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} - \frac{1}{5} e^{-t}\begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$$

$t \to \infty, \quad x \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

7.6.) 11.a. $\begin{vmatrix} \frac{3}{4} - \lambda & 1 \\ -2 & -\frac{5}{4} - \lambda \end{vmatrix} = 0$

$\lambda^2 + \frac{1}{2}\lambda + \frac{17}{16} = 0$

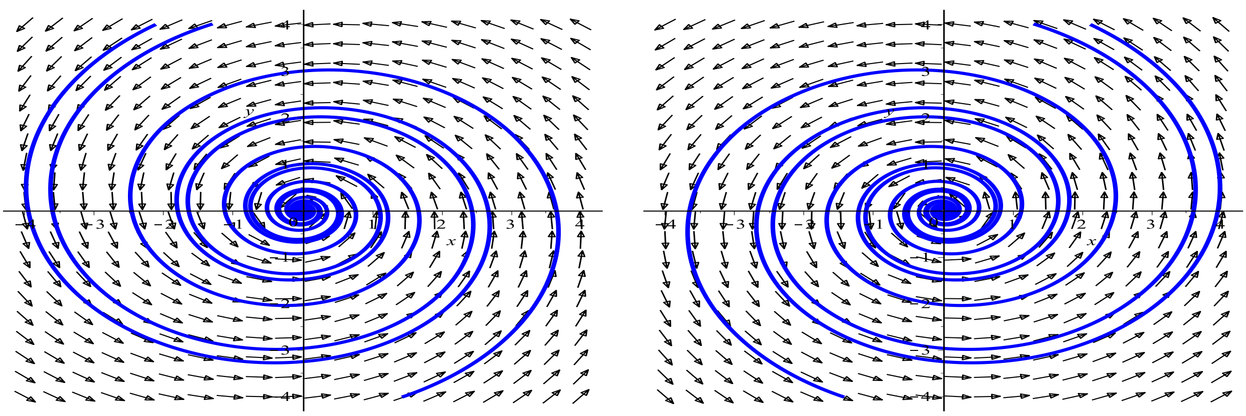
$\Delta = -4$

$r_{1,2} = \dfrac{-\frac{1}{2} \mp \sqrt{-4}}{2} = -\frac{1}{4} \mp i$.

13.(a) The characteristic equation is $r^2 - 2\alpha r + 1 + \alpha^2 = 0$, with roots $r = \alpha \pm i$.

(b) When $\alpha < 0$ and $\alpha > 0$, the equilibrium point $(0,0)$ is a stable spiral and an unstable spiral, respectively. The equilibrium point is a center when $\alpha = 0$.

(c)



(a) $\alpha = -1/8$ (b) $\alpha = 1/8$