#HW5

Section 4.1

3. Writing the equation in standard form, the coefficients are rational functions with singularities at $t = 0$ and $t = 1$. Hence the solutions are valid on the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$.

8 $f_1(t) = 2t - 3 \quad f_2(t) = 4t^2 + 2, \quad f_3(t) = 3t^2 + t$

$$W(f_1, f_2, f_3) = \begin{vmatrix} 2t-3 & 4t^2+2 & 3t^2+t \\ 2 & 8t & 6t+1 \\ 0 & 8 & 6 \end{vmatrix}$$

$$= (2t-3)(-1)^{1+1} \cdot \begin{vmatrix} 8t & 6t+1 \\ 8 & 6 \end{vmatrix} + 2(-1)^{2+1} \cdot \begin{vmatrix} 4t^2+2 & 3t^2+t \\ 8 & 6 \end{vmatrix}$$

$$= (2t-3)(48t - 48t - 8) - 2 \cdot (24t^2 + 12 - 24t^2 - 8t)$$

$$= (2t-3)(-8) - 2 \cdot (12 - 8t)$$

$$= -16t + 24 - 24 + 16t = 0$$

thus f_1, f_2, f_2 are linearly dependent

and, $2f_1(t) + 3f_2(t) - 4f_3(t) = 0$.

12. Substitution verifies that the functions are solutions of the differential equation. Furthermore, we have $W(1, t, \cos 3t, \sin 3t) = 243$.

16. Substitution verifies that the functions are solutions of the differential equation. Furthermore, we have $W(x, x^2, 1/x) = 6/x$.

Section 4.2

3. The magnitude of -4 is $R = 4$ and the polar angle is π. Hence $-4 = 4\,e^{i\pi}$.

8 $(i+1) = \frac{2}{\sqrt{2}} \left(\frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} \right)$

$$= \sqrt{2} \cdot (\cos 45^\circ + i \cdot \sin 45^\circ)$$

$$= 2^{1/2} \cdot e^{\frac{\pi}{4} i + 2\pi k i}$$

$$\Rightarrow (i+1)^{1/2} = 2^{1/4} \cdot e^{\frac{\pi}{8} i + \pi k i}, \text{ for } k = 0, 1$$

$k=0$ yields $2^{1/4} \cdot e^{\frac{\pi}{8} i}$ and

$k=1$ yields $2^{1/4} \cdot e^{\frac{9\pi}{8} i}$ as desired.

13. The characteristic equation is $2r^3 - r^2 - 2r + 1 = 0$, with roots $r = -1, 1, 1/2$. The roots are real and distinct, so the general solution is $y = c_1e^{-t} + c_2e^{t} + c_3e^{t/2}$.

16. The characteristic equation can be written as $(r^2 - 1)(r^2 - 9) = 0$. The roots are given by $r = \pm 1, \pm 3$. The roots are real and distinct, hence the general solution is $y = c_1e^{-t} + c_2e^{t} + c_3e^{-3t} + c_4e^{3t}$.

30 $\quad y^{(4)} + y = 0 \qquad y(0)=0,\ y'(0)=0$
$\qquad\qquad\qquad\qquad y''(0)=-2,\ y'''(0)=0$

from characteristic equation,

$r^4 + 1 = 0 \rightarrow r^4 = -1 = \cos\pi + i.\sin\pi$

so that $r^4 = e^{\pi i + 2\pi k i}$, $k = 0,1,2,3$

$r_1 = e^{\frac{1}{4}(\pi i + i2\pi 0)} = e^{\pi i/4} = \frac{\sqrt2}{2} + i.\frac{\sqrt2}{2}$

$r_2 = e^{\frac{1}{4}(\pi i + i2\pi)} = e^{\frac{3\pi i}{4}} = -\frac{\sqrt2}{2} + i.\frac{\sqrt2}{2}$

$r_3 = e^{\frac{1}{4}(\pi i + 4\pi i)} = e^{\frac{5\pi i}{4}} = -\frac{\sqrt2}{2} - i.\frac{\sqrt2}{2}$

$r_4 = e^{\frac{1}{4}(\pi i + 6\pi i)} = e^{\frac{7\pi i}{4}} = \frac{\sqrt2}{2} - i.\frac{\sqrt2}{2}$

so solution is $y(t) = c_1e^{-t/\sqrt2}.\cos(t/\sqrt2) + c_2e^{-t/\sqrt2}.\sin\left(\frac{t}{\sqrt2}\right) + c_3e^{t/\sqrt2}.\cos(t/\sqrt2) + c_4e^{t/\sqrt2}\sin\left(\frac{t}{\sqrt2}\right)$

Invoking initial conditions, we find $y(t) = e^{-t/\sqrt2}.\sin(t/\sqrt2) - e^{t/\sqrt2}.\sin(t/\sqrt2)$ //

$\boxed{4.3}$ (5) $y^{(4)} - 4y'' = t^2 + 4e^t$

char. eq $r^4 - 4r^2 = 0$

$r^2(r^2 - 4) = 0$

$r_1 = 0,\ r_2 = 0,\ r_3 = -2,\ r_4 = 2$

$y_h(t) = c_1 + c_2 t + c_3 e^{-2t} + c_4 e^{2t}$ ✓

for $g_1(t) = t^2$ guess is $y_{p_1}(t) = t^2(A + Bt + Ct^2)$

and for $g_2 = 4e^t$ guess is $y_{p_2} = D \cdot e^t$

putting all these into equation, we get,

$A = -\frac{1}{16},\ B = 0,\ C = -\frac{1}{48}$ and $D = -\frac{4}{3}$

so, $y(t) = y_h(t) + y_{p_1}(t) + y_{p_2}(t)$ //

(9) $y''' + 4y' = t,\quad y(0) = y'(0) = 0,\ y''(0) = 2.$

$r^3 + 4r = 0 \rightarrow r(r^2 + 4) = 0$

$r_1 = 0,\ r_2 = -2i$ and $r_3 = 2i$

$y_h(t) = c_1 + c_2 \cdot \cos 2t + c_3 \cdot \sin 2t$

and for particular solution, guess is $y_p(t) = t \cdot (A + Bt)$

putting into equation $\Rightarrow A = 0,\ B = \frac{1}{8}$

$y_p(t) = \frac{1}{8}t^2$ invoking initial values

we obtain, $\boxed{y(t) = \underbrace{\frac{7}{16}(1) - \frac{7}{16} \cdot \cos 2t}_{y_h} + \underbrace{\frac{1}{8}t^2}_{y_p}}$ //

$\boxed{4.3}$

(13) $y''' - 2y'' + y' = 3t^3 + 2e^t$

$r^3 - 2r^2 + r = 0$

$r(r^2 - 2r + 1) = 0$

$r_1 = 0 \;,\; r_2 = 1 \;,\; r_3 = 1$

$y_h(t) = C_1 + C_2e^t + C_3te^{t^2}$

for $g_1(t) = 3t^3$ a suitable form of particular solution is,

$y_{P1} = t(A_0t^3 + A_1t^2 + A_2t + A_3)$

and for $g_2(t) = 2e^t$ is,

$y_{P2} = t^2 \cdot (Be^t)$

(17) $y^{(4)} - y''' - y'' + y' = t^2 + 8 + t \cdot \sin t$

$r^4 - r^3 - r^2 + r = 0$

$r(r^3 - r^2 - r + 1) = 0$

$r(r-1)(r^2-1) = 0$

$r(r-1)(r-1)(r+1) = 0$

$r_1 = 0, r_2 = 1, r_3 = 1, r_4 = -1$

$y_h(t) = C_1 + C_2e^t + C_2te^t + C_3e^{-t}$ //

guess for $g_1(t) = t^2 + 8$ is

$y_{P1}(t) = t \cdot (A_0t^2 + A_1t + A_2)$ and

guess for $g_2(t) = t \cdot \sin t$ is,

$y_{P2}(t) = \cos t\,(B_0t + B_1) + \sin t\,(D_0t + D_1)$ //

Scanned by CamScanner

Section 4.4

3. From Problem 13 in Section 4.2, $y_c(t) = c_1 e^{-t} + c_2 e^{t} + c_3 e^{2t}$. The Wronskian is evaluated as $W(e^{-t}, e^{t}, e^{2t}) = 6\, e^{2t}$. Now compute the three determinants

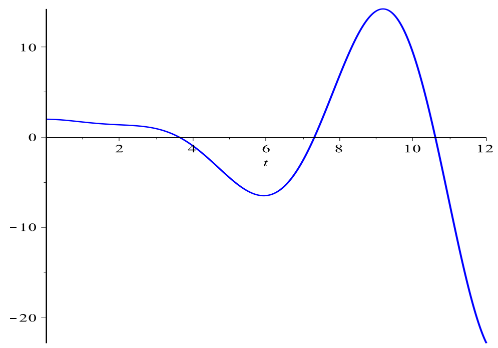
$$W_1(t) = \begin{vmatrix} 0 & e^t & e^{2t} \\ 0 & e^t & 2e^{2t} \\ 1 & e^t & 4e^{2t} \end{vmatrix} = e^{3t}, \quad W_2(t) = \begin{vmatrix} e^{-t} & 0 & e^{2t} \\ -e^{-t} & 0 & 2e^{2t} \\ e^{-t} & 1 & 4e^{2t} \end{vmatrix} = -3e^{t},$$

$$W_3(t) = \begin{vmatrix} e^{-t} & e^t & 0 \\ -e^{-t} & e^t & 0 \\ e^{-t} & e^t & 1 \end{vmatrix} = 2.$$

8. Based on the results in Problem 2, $y_c(t) = c_1 + c_2 e^{t} + c_3 e^{-t}$. It was also shown that $W(1, e^{t}, e^{-t}) = 2$, with $W_1(t) = -2$, $W_2(t) = e^{-t}$, $W_3(t) = e^{t}$. Therefore we have $u_1'(t) = -\csc t$, $u_2'(t) = e^{-t} \csc t/2$, $u_3'(t) = e^{t} \csc t/2$. The particular solution can be expressed as $Y(t) = [u_1(t)] + e^{-t}[u_2(t)] + e^{t}\, [u_3(t)]$. More specifically,

$$\begin{aligned} Y(t) &= \ln|\csc(t) + \cot(t)| + \frac{e^t}{2}\int_{t_0}^{t} e^{-s} \csc(s) ds + \frac{e^{-t}}{2}\int_{t_0}^{t} e^{s} \csc(s) ds \\ &= \ln|\csc(t) + \cot(t)| + \int_{t_0}^{t} \cosh(t-s) \csc(s) ds\,. \end{aligned}$$

10. From Problem 6, $y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t - t^2 \cos t/8$. In order to satisfy the initial conditions, we require that $c_1 = 2$, $c_2 + c_3 = 0$, $-c_1 + 2c_4 - 1/4 = -1$, $-c_2 - 3c_3 = 1$. Thus $y(t) = 2\cos t + \sin t/2 - t\cos t/2 + 5t \sin t/8 - t^2 \cos t/8$.



13. First write the equation as $y''' + x^{-1}y'' - 2x^{-2}y' + 2x^{-3}y = 2x$. The Wronskian is evaluated as $W(x, x^2, 1/x) = 6/x$. Now compute the three determinants

$$W_1(x) = \begin{vmatrix} 0 & x^2 & 1/x \\ 0 & 2x & -1/x^2 \\ 1 & 2 & 2/x^3 \end{vmatrix} = -3, \quad W_2(x) = \begin{vmatrix} x & 0 & 1/x \\ 1 & 0 & -1/x^2 \\ 0 & 1 & 2/x^3 \end{vmatrix} = 2/x,$$

$$W_3(x) = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2.$$

Hence $u_1'(x) = -x^2$, $u_2'(x) = 2x/3$, $u_3'(x) = x^4/3$. Therefore the particular solution can be expressed as

$$Y(x) = x\left[-x^3/3\right] + x^2\left[x^2/3\right] + \frac{1}{x}\left[x^5/15\right] = x^4/15\,.$$