Problem 1.

a) Find the general solution of the following equation (5 pts.)

$$D(D^2 + 4)(D^2 - 4)y = 0$$

$y = C_1 + C_2 \cdot \cos 2t + C_3 \cdot \sin 2t + C_4 \cdot e^{2t} + C_5 \cdot e^{-2t}$

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b) Determine a suitable form for a particular solution of the following equation (Do not calculate the coefficients) (10 pts.)

$$D(D^2 + 4)(D^2 - 4)y = 2t^2 + 5e^{2t} + \sin(2t) + 3e^{2t}\cos(3t)$$

Annihilator: $D^3 \cdot (D-2)(D^2+4)\underbrace{(D-(2+3i))(D-(2-3i))}_{(D^2-4D+13)}$ 4

$\rightarrow D^4(D-2)^2(D+2)(D^2+4)^2(D^2-4D+13)\,y = 0$

$\Rightarrow y = \boxed{C_1} + C_2 t + C_3 t^2 + C_4 t^3 \boxed{+ C_5 \cdot e^{2t}} + C_6 \cdot t \cdot e^{2t}$
$\boxed{+ C_7 \cdot e^{-2t}} \boxed{+ C_8 \cdot \cos 2t} \boxed{+ C_9 \cdot \sin 2t} + C_{10} \cdot t \cdot \cos 2t$
$+ C_{11} \cdot t \cdot \sin 2t + C_{12} \cdot e^{2t} \cdot \cos 3t + C_{13} \cdot e^{2t} \cdot \sin 3t$

$y_p = A \cdot t + B t^2 + C \cdot t^3 + D \cdot t \cdot e^{2t} + E\, t \cdot \cos 2t$
$+ F\, t \cdot \sin 3t + G \cdot e^{2t} \cos 3t + H \cdot e^{2t} \cdot \sin 3t$ 6

(−1) for missing terms

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Problem 2:

Determine the first 4 terms of the Power Series Solution about $x_0=0$ of the following Initial-value Problem

$$y'' - (x+1)y' - y = 0 \qquad y(0)=1, \ y'(0)=4$$

Find the radius of convergency and state the theorem you use to find this radius.

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-2) x^{n-2}$$

$$a_n = \frac{y^{(n)}(0)}{n!} \qquad y(0)=1 \qquad y'(0)=4$$

plug in $x=0$ to the equation:

$$y''(0) - (0+1)(y'(0)) - y(0) = 0$$

$$y''(0) = 5$$

Differentiate the diff. equation:

$$y''' - y' - (x+1)y'' - y' = 0$$

$$y'''(0) = y'(0) + y''(0) + y'(0) = 13$$

So, $a_2 = \dfrac{y''(0)}{2!} = \dfrac{5}{2} \qquad a_3 = \dfrac{13}{3!} = \dfrac{13}{6}$

$$Y = 1 + 4x + \frac{5}{2}x^2 + \frac{13}{6}x^3 + \cdots$$

If $P(x)y'' + Q(x)y' + R(x)y = 0$, then $p(x) = \dfrac{Q(x)}{P(x)}$, $q(x) = \dfrac{R(x)}{P(x)}$

If $p(x)$ and $q(x)$ are analytic at x_0, then the radius of convergency is as least as the minimum of the radii of convergency of p and q.

$p(x) = -x-1 \qquad q(x) = -1$, So, the radius of conv is ∞ for both p and q. $R = \infty$.

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Problem 3. Find the general terms of two linearly independent Power Series solutions of the following differential equation about the point $x_0 = 0$. (25 pts.)

$$y'' - 2x^2y' - 2xy = 0$$

$$2xy = \sum_{n=0}^{\infty} 2a_n x^{n+1}, \quad 2x^2y' = \sum_{n=1}^{\infty} 2n a_n x^{n+1},$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = y'' - 2x^2y' - 2xy = 2a_2 + (6a_3 - 2a_0)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - 2n a_{n-1}\right] x^n$$

So $a_2 = 0$
$a_3 = \frac{1}{3}a_0$ and for $n \geq 2$ $a_{n+2} = \frac{2n}{(n+2)(n+1)} a_{n-1}$

e.g. $a_4 = \frac{2 \cdot 2}{3 \cdot 4} a_1 = \frac{2 \cdot 2^2}{4!} a_1$

$a_5 = \frac{2 \cdot 3}{4 \cdot 5} a_2 = 0$

$a_6 = \frac{2 \cdot 4}{5 \cdot 6} a_3 = \frac{2 \cdot 4}{3 \cdot 5 \cdot 6} a_0 = \frac{2^2 \cdot 4^2}{6!} a_0$

$a_7 = \frac{2 \cdot 5}{6 \cdot 7} a_4 = \frac{2^2 \cdot 2^2 5^2}{7!} a_1$

$a_8 = 0$

$a_9 = \frac{2 \cdot 7}{8 \cdot 9} a_6 = \frac{2^3 \cdot 4^2 \cdot 7^2}{9!} a_0$

$a_{10} = \frac{2 \cdot 8}{9 \cdot 10} a_7 = \frac{2^3 \cdot 2^2 \cdot 5^2 \cdot 8^2}{10!} a_1$

...

Therefore

$$y = a_0 + a_1 x + \frac{2}{3!} a_0 x^3 + \frac{2 \cdot 2^2}{4!} a_1 x^4 + \frac{2^2 \cdot 4^2}{6!} a_0 x^6 + \frac{2^2 \cdot 2^2 \cdot 5^2}{7!} a_1 x^7 + \cdots$$

$$= a_0\left(1 + \frac{2}{3!}x^3 + \frac{2^2 4^2}{6!}x^6 + \frac{2^3 4^2 \cdot 7^2}{9!}x^9 + \cdots\right)$$

$$+ a_1\left(x + \frac{2 \cdot 2^2}{4!}x^4 + \frac{2^2 \cdot 2^2 5^2}{7!}x^7 + \frac{2^3 \cdot 2^2 5^2 8^2}{10!}x^{10} + \cdots\right)$$

Two lin. indep. P.S. solutions are

$$y_1 = 1 + \sum_{k=1}^{\infty} \frac{2^k 4^2 4^2 \cdots (3k-2)^2}{(3k)!} x^{3k},$$

$$y_2 = \sum_{k=0}^{\infty} \frac{2^k 2^2 5^2 \cdots (3k-1)^2}{(3k+1)!} x^{3k+1}$$

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Problem 4. Solve the following initial-value problem. (15 pts.)

$$x^2y'' + 3xy' + 5y = 0 \ , \ y(1) = 2 \ , \ y'(1) = 0$$

This is an Euler type differential equation. So plugging in $y = x^r$, we get ($y' = rx^{r-1}$; $y'' = r(r-1)x^{r-2}$)

$x^r(r(r-1) + 3r + 5) = 0 \Rightarrow r^2 + 2r + 5 = 0 \quad (x>0)$

$r = -1 \pm 2i$

$\Rightarrow$ homogeneous solutions are $y_1 = x^{-1}\cos(2\ln x)$

$y_2 = x^{-1}\sin(2\ln x)$

$y = c_1y_1 + c_2y_2 = c_1x^{-1}\cos(2\ln x) + c_2x^{-1}\sin(2\ln x)$

$y'(x) = \left(-x^{-2}\cos(2\ln x) - x^{-1}\sin(2\ln x)\cdot\frac{2}{x}\right)\cdot c_1$

$\left(-x^{-2}\sin(2\ln x) + x^{-1}\cos(2\ln x)\cdot\frac{2}{x}\right)\cdot c_2$

$y(1) = c_1\cdot 1\cdot\cos(2\cdot 0) + c_2\cdot 1\cdot\sin(2\cdot 0) = \boxed{c_1 = 2}$

$y'(1) = -1\cdot\cos(0)\cdot c_1 + 2\cdot 1\cdot\cos(0)\cdot c_2 = 0$

$\Rightarrow \quad 2c_2 - c_1 = 0 \Rightarrow \boxed{c_2 = 1}$

Right

Problem 5. Solve the following initial-value problem using the Laplace transform

(Hint: You may use the table on the last page of the exam) (25 pts.)

$$y'' - 6y' + 10y = 15e^t, \quad y(0) = 0,\ y'(0) = 1$$

$\mathcal{L}(y''-6y'+10y) = \mathcal{L}(15e^t)$

$\mathcal{L}(y''-6y'+10y) = \mathcal{L}(y'') - 6\mathcal{L}(y') + 10\mathcal{L}(y)$

$= [s^2\mathcal{L}(y) - sy(0) - y'(0)] - 6[s\mathcal{L}(y) - y(0)] + 10\mathcal{L}(y)$

Derivative Formula

$= (s^2-6s+10)\mathcal{L}(y) - 1$

$\mathcal{L}(15e^t) = 15\mathcal{L}(e^t) = \frac{15}{s-1}$

$\Rightarrow \mathcal{L}(y) = \frac{15}{(s-1)(s^2-6s+10)} + \frac{1}{s^2-6s+10}$

$\frac{15}{(s-1)(s^2-6s+10)} = \frac{a}{s-1} + \frac{bs+c}{s^2-6s+10} \Rightarrow a(s^2-6s+10) + (bs+c)(s-1) = 15$

$\Rightarrow (a+b)s^2 + (c-b-6a)s + (10a-c) = 15$

$\Rightarrow \left.\begin{array}{l} a+b=0 \\ c-b-6a=0 \\ 10a-c=15 \end{array}\right\} \begin{array}{l} a=3 \\ b=-3 \\ c=15 \end{array}$

$\Rightarrow \mathcal{L}(y) = \frac{3}{s-1} + \frac{-3s+15}{s^2-6s+10} + \frac{1}{s^2-6s+10} = \frac{3}{s-1} + \frac{-3s+16}{s^2-6s+10}$

$= \frac{3}{s-1} - \frac{3(s-3)}{s^2-6s+10} + \frac{7}{(s-3)^2+1}$

$= 3\mathcal{L}(e^t) - 3\mathcal{L}(e^{3t}\cos t) + 7\mathcal{L}(e^{3t}\sin t)$

$= \mathcal{L}(3e^t - 3e^{3t}\cos t + 7e^{3t}\sin t) \Rightarrow \boxed{y(t) = 3e^t - 3e^{3t}\cos t + 7e^{3t}\sin t}$