Math 204: Midterm Exam # 1
Spring 2018

- Write your name and Student ID number in the space provided below and sign.

Last Name, First Name:	
Student ID Number:	
Signature:	KEY

- Mark the section you are registered below.

  ☐ Section 1 (Mon. & Wed. 14:30-15:45, Instructor: Hasan İnci)

  ☐ Section 2 (Tue. & Thu. 16:00-17:15, Instructor: Tolga Etgü)

  ☐ Section 3 (Tue. & Thu. 13:00-14:15, Instructor: Tolga Etgü)

- You have 90 minutes.

- ***You must show all your work to receive full credit.***

To be filled by the grader:

Problem 1:	
Problem 2:	
Problem 3:	
Problem 4:	
Problem 5:	
Problem 6:	
Problem 7:	
Total Grade:	

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Problem 1. Solve the following initial-value problems.

a) (12 pts.) $t^2y' + ty = 3t^4 + t^2$, $y(2) = 9$

$y' + \frac{1}{t}y = 3t^2+1$ $\Big\}$ $\underbrace{ty' + y}_{(t\cdot y)'} = 3t^3+t$

$\mu = e^{\int \frac{1}{t}dt} \Rightarrow \mu = t$

$\Rightarrow t\cdot y = \int 3t^3+t\,dt = \frac{3t^4}{4} + \frac{t^2}{2} + C$

general solution: $y = \frac{3t^3}{4} + \frac{t}{2} + \frac{C}{t}$

$y(2)=9 \Rightarrow 9 = 6+1+\frac{C}{2} \Rightarrow C=4$

Unique solution of the IVP: $\boxed{y = \frac{3t^3}{4} + \frac{t}{2} + \frac{4}{t}}$

b) (10 pts.) $yy' = 3x^2(y^2+1)$, $y(0) = -2$

Separable equation: $\int \frac{y}{y^2+1}dy = \int 3x^2dx$

$\frac{1}{2}\ln(y^2+1) = x^3 + C \Rightarrow y^2+1 = A\cdot e^{2x^3}$

$y = \mp\sqrt{A\cdot e^{2x^3}-1}$

$y(0)=-2 \Rightarrow -2 = -\sqrt{A-1} \Rightarrow A=5$

Solution of the IVP: $\boxed{y = -\sqrt{5e^{2x^3}-1}}$

Problem 2. What is the largest interval on which the following initial-value problem has a unique solution? (10 pts.)

$$(t^2 + 2t - 8)y' + (\cos^3 t)y = \sin^2 t\ , \qquad y(1) = 2$$

$$y' + \frac{\cos^3 t}{(t+4)(t-2)}\, y = \frac{\sin^2 t}{(t+4)(t-2)}$$

The largest interval containing $t_0 = 1$ on which $\frac{\cos^3 t}{(t+4)(t-2)}$ and $\frac{\sin^2 t}{(t+4)(t-2)}$ are continuous is $(-4, 2)$. Therefore the IVP has a unique solution on $(-4, 2)$.

Problem 3. Find all the solutions of the following equation. (12 pts.)

$$(2xy + y^2 - 1)dx + (x + y)^2 dy = 0$$

$$\frac{\partial(2xy + y^2 - 1)}{\partial y} = 2x + 2y = \frac{\partial((x+y)^2)}{\partial x}$$

So, the equation is exact.

$$F(x,y) = \int 2xy + y^2 - 1\,dx = x^2 y + xy^2 - x + g(y)$$

$$\text{So, } (x+y)^2 = \frac{\partial F}{\partial y} = x^2 + 2xy + g'(y)$$

$$\Rightarrow g'(y) = y^2 \Rightarrow g(y) = \int y^2\,dy = \frac{y^3}{3} + c$$

Hence, the general solution is:

$$\boxed{x^2 y + xy^2 - x + \frac{y^3}{3} = C}$$

Problem 4. Solve the following initial-value problem. (10 pts.)

$$y'' + 25y = 0 \ , \ y(0) = 3 \ , \ y'(0) = -5$$

Characteristic Equation:

$r^2+25=0$

$r=\mp 5i$

General solution: $y = c_1 \cos 5t + c_2 \sin 5t$

$y' = -5c_1 \sin 5t + 5c_2 \cos 5t$

$y(0)=3$
$\Rightarrow c_1 = 3$

$y'(0)=-5$
$\Rightarrow c_2 = -1$

The solution of the IVP: $\boxed{y = 3\cos 5t - \sin 5t}$

Problem 5. Verify that $y(t) = t$ is a solution and solve the following equation. (12 pts.)
Hint: Look for a solution of the form $y(t) = v(t) \cdot t$

$$t^2 y'' - t(t+2)y' + (t+2)y = 0 \ , \ t > 0$$

$\left.\begin{array}{l} y=t \\ y'=1 \\ y''=0 \end{array}\right\}$ $t^2 \cdot 0 - t(t+2)\cdot 1 + (t+2)\cdot t = 0$ ✓

So, $y_1(t) = t$ is a solution.

$y_2 = v \cdot t \Rightarrow y_2' = v't + v, \quad y_2'' = v''t + 2v'$

Plugging into the equation:

$0 = (t^3 v'' + 2t^2 v') - (t^3 v' + t^2 v + 2t^2 v' + 2tv) + (t^2 v + 2tv)$

$0 = t^3 v'' - t^3 v' \Rightarrow v'' = v'$

Let $w = v'$, $w' = v''$ $\Rightarrow$ $\frac{dw}{w} = dt$ We can choose $w = e^t$

$v = \int e^t dt = e^t + C \Rightarrow v = e^t$

$\Rightarrow y_2 = t \cdot e^t$ is a solution

$W(y_1, y_2) = \begin{vmatrix} t & te^t \\ 1 & (t+1)e^t \end{vmatrix} = t^2 e^t \neq 0$ since $t > 0$

Therefore, the general solution is $\boxed{y = c_1 t + c_2 t e^t}$

Problem 6. a) Verify that $y_1(t) = t^3$ and $y_2(t) = t^{-2}$ are solutions, and also verify that they form a fundamental set of solutions of the following equation. (10 pts.)

$$y'' - 6t^{-2}y = 0 \ , \ t > 0$$

$$\left.\begin{aligned} y_1 &= t^3 \\ y_1' &= 3t^2 \\ y_1'' &= 6t \end{aligned}\right\} \Rightarrow 6t - 6t^{-2}t^3 = 0 \checkmark \Rightarrow y_1 \text{ is a solution}$$

$$W(y_1, y_2) = \begin{vmatrix} t^3 & t^{-2} \\ 3t^2 & -2t^{-3} \end{vmatrix} = -2 - 3 = -5 \neq 0$$

$$\left.\begin{aligned} y_2 &= t^{-2} \\ y_2' &= -2t^{-3} \\ y_2'' &= 6t^{-4} \end{aligned}\right\} \Rightarrow 6t^{-4} - 6t^{-2}t^{-2} = 0. \checkmark \Rightarrow y_2 \text{ is a solution.}$$

$\Rightarrow \{y_1, y_2\}$ is a fundamental set of solutions.

b) Find all the solutions of the following equation. (12 pts.)

$$y'' - 6t^{-2}y = 5 - t^{-1} \ , \ t > 0$$

Variation of parameters:

$y = u_1 t^3 + u_2 t^{-2}$ where

$$u_1 = \int \frac{-(5 - t^{-1})\, t^{-2}}{-5}\, dt = \frac{t^{-1}}{-1} + \frac{t^{-2}}{10} + c_1$$

$$u_2 = \int \frac{(5 - t^{-1})\, t^{3}}{-5}\, dt = \frac{-t^4}{4} + \frac{t^3}{15} + c_2$$

So, the general solution is:

$$\boxed{y = \left(-t^{-1} + \frac{t^{-2}}{10} + c_1\right) t^3 + \left(-\frac{t^4}{4} + \frac{t^3}{15} + c_2\right) t^{-2}}$$

OR $\boxed{y = c_1 t^3 + c_2 t^{-2} - \frac{5t^2}{4} + \frac{t}{6}}$

Problem 7. Suppose that $p(t)$ and $q(t)$ are continuous on an open interval I, (12 pts.) and y_1 and y_2 are solutions of

$$y'' + p(t)y' + q(t)y = 0$$

on I such that $y_1'(t_0) = y_2'(t_0) = 0$ for a point t_0 in I. Prove that the equation above has a solution on I which is not of the form $c_1y_1 + c_2y_2$, where c_1 and c_2 are constants. State the existence theorem you use in the proof.

Observe that $W(y_1,y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ 0 & 0 \end{vmatrix}$

$= 0$

So, $\{y_1, y_2\}$ is not a fondomental set of solutions.

On the other hand, let y_3 be the unique solution of the IVP:

$y'' + p(t)y' + q(t)y = 0, \; y(t_0) = 0, \; y'(t_0) = 1$

The existence and uniqueness theorem:

If p, q are continuous on an open interval I containing t_0, then the IVP

$y'' + py' + qy = 0, \; y(t_0) = A, \; y'(t_0) = B$

has a unique solution on I.

$y_3 \neq c_1y_1 + c_2y_2$ for any constants c_1 and c_2 since otherwise $y_3'(t_0) = 1$ would be $c_1y_1'(t_0) + c_2y_2'(t_0) = c_1 0 + c_2 0 = 0$.