

# HW # 6

## Section 5.1

3. Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|n! x^{4n+4}|}{|(n+1)! x^{4n}|} = \lim_{n \rightarrow \infty} \frac{x^4}{n+1} = 0.$$

The series converges absolutely for all values of  $x$ . Thus the radius of convergence is  $\rho = \infty$ .

5) Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{\left| \frac{(3x+1)^{n+1}}{(n+1)^2} \right|}{\left| \frac{(3x+1)^n}{n^2} \right|} = \lim_{n \rightarrow \infty} \left( \frac{n^2}{(n+1)^2} \right) |3x+1| = |3x+1|$$

the series converges abs. when  $|3x+1| < 1$

$$\text{i.e., } -\frac{2}{3} < x < 0.$$

from  $|3x+1| < 1$  it follows that,

$$\left| x + \frac{1}{3} \right| < \frac{1}{3} \quad \text{thus } \rho = \frac{1}{3} \quad (\text{the radius of convergence}).$$

9)  $f(x) = \sinh 3x$ ,  $x_0 = 0$ .

a Taylor series expansion about  $x_0 = 0$  is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot (x-0)^n$$

$$f'(x) = 3 \cdot \cosh 3x, \quad f^{(2)}(x) = 3^2 \cdot (-1) \cdot \sinh 3x, \quad f^{(3)}(x) = 3^3 \cdot (-1) \cdot \cosh 3x$$

$$f^{(4)}(x) = 3^4 \cdot (-1)^2 \cdot \sinh 3x$$

So, ~~the series is~~

$$f^{(4n)}(x) = 3^{4n} \cdot (-1)^{2n} \cdot \sinh 3x$$

$$f^{(4n+1)}(x) = 3^{4n+1} \cdot (-1)^{2n} \cdot \cosh 3x$$

$$f^{(4n+2)}(x) = 3^{4n+2} \cdot (-1)^{2n+1} \cdot \sinh 3x$$

$$f^{(4n+3)}(x) = 3^{4n+3} \cdot (-1)^{2n+1} \cdot \cosh 3x$$

So that,

$$f^{(4n)}(0) = 0, \quad f^{(4n+1)}(0) = 3^{4n+1}$$

$$f^{(4n+2)}(0) = 0, \quad f^{(4n+3)}(0) = 3^{4n+3} \cdot (-1)^{2n+1}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n$$

$$f(x) = \frac{3^1 \cdot x^1}{1!} + \frac{3^3 \cdot x^3}{3!} \cdot (-1) + \frac{3^5 \cdot x^5}{5!} + \frac{3^7 \cdot x^7}{7!} \cdot (-1) + \dots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{3^n \cdot x^{2n+1} \cdot (-1)^n}{(2n+1)!}$$

From the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|(2n+1)! \cdot 3^{2n+1} \cdot |x|^{2n+3}| \cdot |(-1)^n|}{|(2n+1)! \cdot 3^{2n} \cdot |x|^{2n+1}|} = \lim_{n \rightarrow \infty} \frac{3x^2}{(2n+1)(2n+3)} = 0$$

~~converges absolutely for all values of x~~

the series converges absolutely for all values of  $x$ . The radius of convergence is  $\rho = \infty$ .

13)  $f(x) = \ln x, \quad x_0 = 1$ :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot (x - x_0)^n$$

$$f'(x) = (\ln x)' = \frac{1}{x}$$

$$f^{(2)}(x) = -\frac{1}{x^2}, \quad f^{(3)}(x) = \frac{2}{x^3}$$

So by induction,  $f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{x^n}$

and  $f^{(n)}(x_0=1) = (-1)^{n-1} \cdot (n-1)!$

thus,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot (x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} \cdot (n-1)!}{n!} \cdot (x-1)^n$$

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{(x-1)^n}{n} \quad \left( \text{for } n=0, \text{ not def. start from } n=1 \right)$$

As for radius of convergence, observe ratio test,

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n \frac{(x-1)^{n+1}}{n+1}}{(-1)^{n-1} \frac{(x-1)^n}{n}} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot |x-1| = \lim_{n \rightarrow \infty} |x-1| = |x-1|$$

is abs. convergent if  $|x-1| < 1$   
 Hence  $\rho = 1$  (radius of convergence)

17. Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|(n+1)^2 x^{n+1}|}{|n^2 x^n|} = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{n^2} |x| = |x|.$$

The series converges absolutely for  $|x| < 1$ . Term-by-term differentiation results in

$$y' = \sum_{n=1}^{\infty} n^3 x^{n-1} = 1 + 8x + 27x^2 + 64x^3 + \dots$$

$$y'' = \sum_{n=2}^{\infty} n^3(n-1) x^{n-2} = 8 + 54x + 192x^2 + 500x^3 + \dots$$

Shifting the indices, we can also write

$$y' = \sum_{n=0}^{\infty} (n+1)^3 x^n \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+2)^3 (n+1) x^n.$$

21. Shifting the index by 2, that is, setting  $m = n - 2$ ,

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-2)a_n x^{n-2} &= \sum_{m=0}^{\infty} (m+2)ma_{m+2} x^m \\ &= \sum_{n=0}^{\infty} (n+2)na_{n+2} x^n. \end{aligned}$$

22. Shift the index down by 3, that is, set  $m = n + 3$ . It follows that

$$\sum_{n=0}^{\infty} a_n x^{n+3} = \sum_{m=3}^{\infty} a_{m-3} x^m = \sum_{n=3}^{\infty} a_{n-3} x^n.$$

26. Clearly,

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+2}.$$

Shifting the index in the first series, that is, setting  $k = n - 1$ ,

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k.$$

Shifting the index in the second series, that is, setting  $k = n + 2$ ,

$$\sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{k=2}^{\infty} a_{k-2} x^k.$$

Combining the series, and starting the summation at  $n = 2$ ,

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = a_1 + 2a_2 x + \sum_{n=2}^{\infty} [(n+1)a_{n+1} + a_{n-2}] x^n.$$

## SECTION 5.2

1.(a,b,d) Let  $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$ . Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_nx^n = 0$$

or

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - a_n]x^n = 0.$$

Equating all the coefficients to zero,

$$(n+2)(n+1)a_{n+2} - a_n = 0, \quad n = 0, 1, 2, \dots$$

We obtain the recurrence relation

$$a_{n+2} = \frac{a_n}{(n+1)(n+2)}, \quad n = 0, 1, 2, \dots$$

The subscripts differ by two, so for  $k = 1, 2, \dots$

$$a_{2k} = \frac{a_{2k-2}}{(2k-1)2k} = \frac{a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \dots = \frac{a_0}{(2k)!}$$

and

$$a_{2k+1} = \frac{a_{2k-1}}{2k(2k+1)} = \frac{a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \dots = \frac{a_1}{(2k+1)!}.$$

Hence

$$y = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

The linearly independent solutions are

$$y_1 = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots = \cosh x$$

$$y_2 = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots = \sinh x.$$

(c) The Wronskian at 0 is 1.

$$2) \quad y'' - xy' - y = 0, \quad x_0 = 0.$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n \cdot n(n-1) x^{n-2}$$

Then,

$$\sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) x^{n-2} - x \cdot \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n \cdot a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} x^n [(n+2)(n+1) a_{n+2} - n a_n - a_n] = 0$$

$$\Rightarrow 0 = (n+2)(n+1) a_{n+2} - (n+1) a_n$$

$$0 = (n+2) a_{n+2} - a_n$$

$$\boxed{a_{n+2} = \frac{1}{n+2} \cdot a_n}$$

$$a_2 = \frac{1}{2} a_0, \quad a_4 = \frac{1}{4} \cdot a_2 = \frac{1}{2 \cdot 4} a_0$$

So by induction,

$$a_{2n} = \frac{1}{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n)} \cdot a_0 = \frac{a_0}{2^n \cdot n!}$$

Similarly,

$$a_3 = \frac{1}{3} \cdot a_1, \quad a_5 = \frac{1}{5} a_3 = \frac{1}{3 \cdot 5} a_1$$

$$a_{2n+1} = \frac{1}{3 \cdot 5 \cdots (2n+1)} \cdot a_1 = \frac{2^n \cdot n!}{(2n+1)!} a_1$$

5)  $(1-x)y'' + y = 0, \quad x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n \cdot x^n$$

$$y' = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n \cdot (n-1) \cdot a_n \cdot x^{n-2}$$

$$\Rightarrow (1-x) \cdot \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \cdot x^n - \sum_{n=1}^{\infty} (n+1) \cdot n \cdot a_{n+1} x^n + \sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

$$\underbrace{(2 \cdot a_2 + a_0)}_0 + \sum_{n=1}^{\infty} x^n \cdot \underbrace{(n+2) a_{n+2} \cdot (n+1) - n(n+1) a_{n+1} + a_n}_0 = 0$$

So,  $2a_2 + a_0 = 0$  gives  $a_2 = -\frac{1}{2} \cdot a_0$

~~$a_{n+2} \cdot (n+2)(n+1) - a_{n+1} \cdot (n+1)(n) + a_n = 0$~~

And  $| a_{n+2} \cdot (n+2)(n+1) - a_{n+1} \cdot (n+1)(n) + a_n = 0. |$



$$n=0, \quad a_2 \cdot 2 - a_1 \cdot 0 + a_0 = 0 \quad \text{i.e.} \quad a_2 = -\frac{1}{2} \cdot a_0$$

$$n=1, \quad a_3 \cdot 3 \cdot 2 - a_2 \cdot 2 + a_1 = 0$$

$$6 \cdot a_3 = 2a_2 - a_1 = -a_0 - a_1 \rightarrow a_3 = -\frac{a_0}{6} - \frac{a_1}{6}$$

$$n=2, \quad a_4 \cdot 12 - a_3 \cdot 6 + a_2 = 0$$

$$a_4 = \frac{1}{12} (6a_3 - a_2) = \frac{1}{12} \left( -a_0 - a_1 - \sqrt{-a_0/2} \right)$$

$$= a_0 \left( -\frac{1}{12} + \frac{1}{24} \right) - a_1 \left( \frac{1}{12} \right)$$

$$= a_0 \left( -\frac{1}{24} \right) - a_1 \left( \frac{1}{12} \right)$$

So,

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$= a_0 + a_1 x + \left( -\frac{1}{2} a_0 \right) x^2 + \left( -\frac{a_0}{6} - \frac{a_1}{6} \right) x^3 + \left( -\frac{a_0}{24} - \frac{a_1}{12} \right) x^4 + \dots$$

$$= a_0 \left( 1 - \frac{x^2}{2} - \frac{1}{6} x^3 - \frac{x^4}{24} \dots \right) + a_1 \left( x - \frac{x^3}{6} - \frac{x^4}{12} \dots \right)$$

So  $y_1 = 1 - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} + \dots$  and

$$y_2 = x - \frac{x^3}{6} - \frac{x^4}{12} - \frac{1}{24} x^5 + \dots \rightarrow \text{compute yourself!}$$

6.(a,b) Let  $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$ . Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitution into the ODE results in

$$(2 + x^2) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - x \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n = \sum_{n=1}^{\infty} n a_n x^n.$$

It follows that

$$4a_0 + 4a_2 + (3a_1 + 12a_3)x + \sum_{n=2}^{\infty} [2(n+2)(n+1)a_{n+2} + n(n-1)a_n - n a_n + 4a_n] x^n = 0.$$

Equating the coefficients to zero, we find that  $a_2 = -a_0$ ,  $a_3 = -a_1/4$ , and

$$a_{n+2} = -\frac{n^2 - 2n + 4}{2(n+2)(n+1)} a_n, \quad n = 0, 1, 2, \dots$$

The indices differ by two, so for  $k = 0, 1, 2, \dots$

$$a_{2k+2} = -\frac{(2k)^2 - 4k + 4}{2(2k+2)(2k+1)} a_{2k}$$

and

$$a_{2k+3} = -\frac{(2k+1)^2 - 4k + 2}{2(2k+3)(2k+2)} a_{2k+1}.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \dots$$

$$y_2(x) = x - \frac{x^3}{4} + \frac{7x^5}{160} - \frac{19x^7}{1920} + \dots$$

(c) The Wronskian at 0 is 1.

9.(a,b,d) Let  $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$ . Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Substitution into the ODE results in

$$(1+x^2) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - 4x \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + 6 \sum_{n=0}^{\infty} a_nx^n = 0.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=2}^{\infty} n(n-1)a_nx^n$$

and

$$x \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n = \sum_{n=1}^{\infty} na_nx^n.$$

It follows that

$$6a_0 + 2a_2 + (2a_1 + 6a_3)x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + n(n-1)a_n - 4na_n + 6a_n]x^n = 0.$$

Setting the coefficients equal to zero, we obtain  $a_2 = -3a_0$ ,  $a_3 = -a_1/3$ , and

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}a_n, \quad n = 0, 1, 2, \dots$$

Observe that for  $n = 2$  and  $n = 3$ , we obtain  $a_4 = a_5 = 0$ . Since the indices differ by two, we also have  $a_n = 0$  for  $n \geq 4$ . Therefore the general solution is a polynomial

$$y = a_0 + a_1x - 3a_0x^2 - a_1x^3/3.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - 3x^2 \quad \text{and} \quad y_2(x) = x - x^3/3.$$

(c) The Wronskian is  $(x^2 + 1)^2$ . At  $x = 0$  it is 1.

10.(a,b,d) Let  $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$ . Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Substitution into the ODE results in

$$(4 - x^2) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + 2 \sum_{n=0}^{\infty} a_nx^n = 0.$$

First write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=2}^{\infty} n(n-1)a_nx^n.$$

It follows that

$$2a_0 + 8a_2 + (2a_1 + 24a_3)x + \sum_{n=2}^{\infty} [4(n+2)(n+1)a_{n+2} - n(n-1)a_n + 2a_n]x^n = 0.$$

We obtain  $a_2 = -a_0/4$ ,  $a_3 = -a_1/12$  and

$$4(n+2)a_{n+2} = (n-2)a_n, \quad n = 0, 1, 2, \dots$$

Note that for  $n = 2$ ,  $a_4 = 0$ . Since the indices differ by two, we also have  $a_{2k} = 0$  for  $k = 2, 3, \dots$ . On the other hand, for  $k = 1, 2, \dots$ ,

$$a_{2k+1} = \frac{(2k-3)a_{2k-1}}{4(2k+1)} = \frac{(2k-5)(2k-3)a_{2k-3}}{4^2(2k-1)(2k+1)} = \dots = \frac{-a_1}{4^k(2k-1)(2k+1)}.$$

Therefore the general solution is

$$y = a_0 + a_1x - a_0\frac{x^2}{4} - a_1 \sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n(2n-1)(2n+1)}.$$

Hence the linearly independent solutions are  $y_1(x) = 1 - x^2/4$  and

$$y_2(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \dots = x - \sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n(2n-1)(2n+1)}.$$

(c) The Wronskian at 0 is 1.

13.(a,b,d) Let  $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$ . Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitution into the ODE results in

$$2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 3 \sum_{n=0}^{\infty} a_n x^n = 0.$$

First write

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n a_n x^n.$$

We then obtain

$$4a_2 + 3a_0 + \sum_{n=1}^{\infty} [2(n+2)(n+1)a_{n+2} + n a_n + 3a_n] x^n = 0.$$

It follows that  $a_2 = -3a_0/4$  and

$$2(n+2)(n+1)a_{n+2} + (n+3)a_n = 0$$

for  $n = 0, 1, 2, \dots$ . The indices differ by two, so for  $k = 1, 2, \dots$

$$\begin{aligned} a_{2k} &= -\frac{(2k+1)a_{2k-2}}{2(2k-1)(2k)} = \frac{(2k-1)(2k+1)a_{2k-4}}{2^2(2k-3)(2k-2)(2k-1)(2k)} = \dots \\ &= \frac{(-1)^k 3 \cdot 5 \dots (2k+1)}{2^k (2k)!} a_0. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= -\frac{(2k+2)a_{2k-1}}{2(2k)(2k+1)} = \frac{(2k)(2k+2)a_{2k-3}}{2^2(2k-2)(2k-1)(2k)(2k+1)} = \dots \\ &= \frac{(-1)^k 4 \cdot 6 \dots (2k)(2k+2)}{2^k (2k+1)!} a_1. \end{aligned}$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6 + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n 3 \cdot 5 \dots (2n+1)}{2^n (2n)!} x^{2n}$$

$$y_2(x) = x - \frac{1}{3}x^3 + \frac{1}{20}x^5 - \frac{1}{210}x^7 + \dots = x + \sum_{n=1}^{\infty} \frac{(-1)^n 4 \cdot 6 \dots (2n+2)}{2^n (2n+1)!} x^{2n+1}.$$

(c) The Wronskian at 0 is 1.

$$14) \quad 2y'' + (x+1)y' + 3y = 0, \quad x_0 = 2$$

$$y(x) = \sum_{n=0}^{\infty} a_n \cdot (x-2)^n$$

$$y' = \sum_{n=1}^{\infty} a_n \cdot n \cdot (x-2)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n \cdot (n-1) \cdot a_n \cdot (x-2)^{n-2}$$

$$2 \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} + \underbrace{(x+1)}_{(x-2)+3} \cdot \sum_{n=1}^{\infty} n \cdot a_n (x-2)^{n-1} + 3 \sum_{n=0}^{\infty} a_n (x-2)^n$$

$$2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-2)^n + (x-2) \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} + 3 \sum_{n=1}^{\infty} n a_n (x-2)^n + 3 \sum_{n=0}^{\infty} a_n (x-2)^n$$

$$\sum_{n=0}^{\infty} a_{n+2} \cdot (n+2)(n+1) (x-2)^n + \sum_{n=0}^{\infty} n a_n (x-2)^n + \sum_{n=0}^{\infty} (n+3) a_{n+1} (x-2)^n + 3 \sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\textcircled{1} \quad \sum_{n=0}^{\infty} (x-2)^n \cdot [a_{n+2} \cdot (n+2)(n+1) + 3(n+1)a_{n+1} + a_n(n+3)] = 0$$

$$\text{So, } \underline{2 \cdot (n+2)(n+1) \cdot a_{n+2} + 3(n+1)a_{n+1} + (n+3) \cdot a_n = 0}$$

$$n=0, \quad 4 \cdot a_2 + 3a_1 + 3a_0 = 0$$

$$a_2 = -\frac{3}{4}a_1 - \frac{3}{4}a_0 \quad \checkmark$$

$$n=1, \quad 12 \cdot a_3 + 6a_2 + 4a_1 = 0$$

$$\begin{aligned}
 a_3 &= -\frac{1}{2} a_2 - \frac{1}{3} a_1 \\
 &= -\frac{1}{2} \left( -\frac{3}{4} a_1 - \frac{3}{4} a_0 \right) - \frac{1}{3} a_1 \\
 &= a_1 \left( \frac{3}{8} - \frac{1}{3} \right) + a_0 \left( +\frac{3}{8} \right) \\
 &= a_1 \left( \frac{1}{24} \right) + a_0 \left( \frac{3}{8} \right)
 \end{aligned}$$

$$n=2, \quad 24 \cdot a_4 + 9a_3 + 5a_2 = 0$$

$$\begin{aligned}
 a_4 &= -\frac{9}{24} a_3 - \frac{5}{24} a_2 \\
 &= -\frac{9}{24} \left( \frac{a_1}{24} + \frac{3a_0}{8} \right) - \frac{5}{24} \left( -\frac{3}{4} a_1 - \frac{3}{4} a_0 \right) \\
 &= a_1 \left( \frac{-9}{576} + \frac{15}{96} \right) + a_0 \left( \frac{-27}{192} + \frac{15}{96} \right) \\
 &= a_1 \cdot \frac{81}{576} + a_0 \cdot \frac{3}{192} = a_1 \cdot \frac{9}{64} + a_0 \cdot \frac{1}{64}
 \end{aligned}$$

$$y = \sum_{n=0}^{\infty} (x-2)^n \cdot a_n$$

$$\begin{aligned}
 &\cancel{= a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + a_4(x-2)^4 + \dots} \\
 &= a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + a_4(x-2)^4 + \dots \\
 &= a_0 + a_1(x-2) + (x-2)^2 \left[ -\frac{3}{4} a_1 - \frac{3}{4} a_0 \right] + (x-2)^3 \left[ \frac{a_1}{24} + \frac{3a_0}{8} \right] + (x-2)^4 \left[ \frac{9a_1}{64} + \frac{a_0}{64} \right] + \dots \\
 &= a_0 \left[ 1 - \frac{3}{4}(x-2)^2 + \frac{3}{8}(x-2)^3 + \frac{1}{64}(x-2)^4 + \dots \right] \\
 &\quad + a_1 \left[ (x-2) - \frac{3}{4}(x-2)^2 + \frac{1}{24}(x-2)^3 + \frac{9}{64}(x-2)^4 + \dots \right]
 \end{aligned}$$

$$Y_1 = 1 - \frac{3}{4}(x-2)^2 + \frac{3}{8}(x-2)^3 + \frac{1}{64}(x-2)^4 + \dots$$

$$Y_2 = (x-2) - \frac{3}{4}(x-2)^2 + \frac{1}{24}(x-2)^3 + \frac{9}{64}(x-2)^4 + \dots$$