Mathematical Model: Single-Day VRPSPD with Operational Constraints and Backlog Penalties

1 Problem Class

The project addresses a: *Capacitated Vehicle Routing Problem with Simultaneous Pickup and Delivery (VRPSPD)* with route-length, stop-count, vehicle-capacity, route-duration (shift) and neighborhood-adjacency constraints.

This classification follows directly from:

- simultaneous pickup and delivery at customer stops,
- dynamically changing vehicle load,
- capacity constraints,
- routes starting and ending at a single depot,
- no binding fleet size constraint (fleet chosen sufficiently large),
- operational constraints imposed by UPS (shift duration, stop limits, spatial compactness).

2 Sets and Indices

- $N = \{0, 1, \ldots, n\}$: set of nodes; node 0 is the depot (Istanbul Anadolu Aktarma Merkezi coordinates: (40.920153, 29.348245)), and nodes $1, \ldots, n$ are customer stops.
- $C = N \setminus \{0\}$: set of customer nodes.
- K: set of vehicles identified by unique IDs (vehicle dataset). Each $k \in K$ is a unique vehicle.
- L: set of neighborhoods (districts).
- V: set of vehicle types (stored as an attribute of each vehicle in the dataset).
- R: set of shipment types (e.g., $R = \{\text{Normal}, \text{Urgent}\}$).
- Indices: $i, j \in N$ (nodes), $i, j \in C$ (customers), $k \in K$ (vehicles), $r \in R$ (shipment type).

3 Parameters

- d_{ij}: distance between node i and node j.
- t_{ij}: travel time between node i and node j.
- s_i: service time at node i ($s_0 = 0$).
- Q: vehicle capacity (350 packages).
- $S_{\min}$: minimum stops per route (80).
- $S_{\max}$: maximum stops per route (110).

- $T_{\min}$: minimum route duration (420 minutes, 7 hours).
- $T_{\max}$: maximum route duration (510 minutes, 8.5 hours).
- $h_i \in L$: neighborhood of node i.
- $A_{lm} \in \{0,1\}$: 1 if neighborhoods l and m are adjacent, 0 otherwise.
- $\text{type}_k \in V$: type attribute of vehicle k (from the vehicle dataset).
- F_k: fixed cost of activating vehicle k.
- c_k^{dist}: distance cost coefficient for vehicle k.
- c_k^{time}: time cost coefficient for vehicle k.
- c^{stop}: per-stop cost (each visited customer stop).
- P_{ir}: pickup demand quantity at customer i of shipment type r.
- D_{ir}: delivery demand quantity at customer i of shipment type r.
- P_r^p: backlog penalty coefficient for pickup of type r.
- P_r^d: backlog penalty coefficient for delivery of type r.
- M: sufficiently large constant.

4 Decision Variables

- $x_{kij} \in \{0,1\}$: 1 if vehicle k travels from node i to node j.
- $y_{ki} \in \{0,1\}$: 1 if customer node i is visited/served by vehicle k.
- $z_k \in \{0,1\}$: 1 if vehicle k is used (activated).
- $q_{ki} \geq 0$: load of vehicle k after visiting node i.
- $a_{ki} \geq 0$: arrival time of vehicle k at node i.
- $T_k \geq 0$: total route duration of vehicle k (used for shift constraints, not in the objective).
- $\pi_{kir} \geq 0$: pickup amount of type r collected by vehicle k at customer i.
- $\delta_{kir} \geq 0$: delivery amount of type r delivered by vehicle k at customer i.
- $B_{ir}^p \geq 0$: pickup backlog amount for customer i and type r.
- $B_{ir}^d \geq 0$: delivery backlog amount for customer i and type r.
- $u_{ki} \geq 0$: MTZ ordering variable for subtour elimination.

5 Objective Function

Objective: Minimize total cost including distance/time costs (vehicle-dependent), fixed vehicle activation cost, per-stop cost, and type-based backlog penalties.

$$\min Z = \sum_{k\in K}\sum_{i\in N}\sum_{\substack{j\in N\\ j\neq i}} \left(c_k^{dist}\, d_{ij} + c_k^{time}\, t_{ij}\right) x_{kij} + \sum_{k\in K} F_k\, z_k + c^{stop}\sum_{k\in K}\sum_{i\in C} y_{ki} + \sum_{i\in C}\sum_{r\in R}\left(P_r^p\, B_{ir}^p + P_r^d\, B_{ir}^d\right) \tag{1}$$

EN: Total cost includes distance/time costs (vehicle-dependent), fixed vehicle usage cost, per-stop cost, and type-based backlog penalties (Urgent has higher penalty).
TR: Toplam maliyet; mesafe/süre maliyetleri (araç bazlı), araç kullanım sabit maliyeti, durak başı maliyet ve tür bazlı backlog cezasını içerir (Acil türün cezası daha yüksektir).

6 Constraints

6.1 1) Customer Assignment

Each customer is served exactly once:

$$\sum_{k \in K} y_{ki} = 1 \qquad \forall i \in C \tag{2}$$

6.2 2) Vehicle Activation

A vehicle is active if it serves at least one customer:

$$\sum_{i \in C} y_{ki} \leq M\, z_k \qquad \forall k \in K \tag{3}$$

6.3 3) Flow Conservation (Linking x and y)

For each customer node, if it is served by vehicle k, it must have exactly one incoming and one outgoing arc:

$$\sum_{\substack{j \in N \\ j \neq i}} x_{kij} = y_{ki} \qquad \forall i \in C,\ \forall k \in K \tag{4}$$

$$\sum_{\substack{j \in N \\ j \neq i}} x_{kji} = y_{ki} \qquad \forall i \in C,\ \forall k \in K \tag{5}$$

6.4 4) Depot Constraints

Each active vehicle starts and ends at the depot:

$$\sum_{j \in C} x_{k0j} = z_k \qquad \forall k \in K \tag{6}$$

$$\sum_{i \in C} x_{ki0} = z_k \qquad \forall k \in K \tag{7}$$

6.5 5) Stop Count Constraints

$$S_{\min}\, z_k \leq \sum_{i \in C} y_{ki} \leq S_{\max}\, z_k \qquad \forall k \in K \tag{8}$$

6.6 6) Neighborhood-Adjacency (Spatial Compactness)

Vehicles cannot directly travel between non-adjacent neighborhoods:

$$x_{kij} \leq A_{h_i h_j} \qquad \forall i \in C,\ \forall j \in C,\ i \neq j,\ \forall k \in K \tag{9}$$

6.7 7) Backlog Definition (Single-Day)

Unserved pickup/delivery demand becomes backlog (type-based):

$$\sum_{k\in K} \pi_{kir} + B^{p}_{ir} = P_{ir} \qquad \forall i \in C,\ \forall r \in R \tag{10}$$

$$\sum_{k\in K} \delta_{kir} + B^{d}_{ir} = D_{ir} \qquad \forall i \in C,\ \forall r \in R \tag{11}$$

Tie served quantities to whether customer is visited by vehicle k:

$$0 \le \pi_{kir} \le P_{ir}\, y_{ki} \qquad \forall i \in C,\ \forall k \in K,\ \forall r \in R \tag{12}$$

$$0 \le \delta_{kir} \le D_{ir}\, y_{ki} \qquad \forall i \in C,\ \forall k \in K,\ \forall r \in R \tag{13}$$

6.8 8) Dynamic Capacity Constraints (Simultaneous Pickup & Delivery)

Initial load at depot equals total deliveries assigned to the vehicle:

$$q_{k0} = \sum_{i\in C}\sum_{r\in R} \delta_{kir} \qquad \forall k \in K \tag{14}$$

Load propagation along arcs (two-sided Big-M to enforce equality when $x_{kij} = 1$):

$$q_{kj} \ge q_{ki} - \sum_{r\in R} \delta_{kjr} + \sum_{r\in R} \pi_{kjr} - M(1 - x_{kij}) \qquad \forall i \in N,\ \forall j \in C,\ i \neq j,\ \forall k \in K \tag{15}$$

$$q_{kj} \le q_{ki} - \sum_{r\in R} \delta_{kjr} + \sum_{r\in R} \pi_{kjr} + M(1 - x_{kij}) \qquad \forall i \in N,\ \forall j \in C,\ i \neq j,\ \forall k \in K \tag{16}$$

Capacity bounds:

$$0 \le q_{ki} \le Q \qquad \forall i \in N,\ \forall k \in K \tag{17}$$

6.9 9) Route Duration (Shift) Constraints

Time propagation:

$$a_{kj} \ge a_{ki} + s_i + t_{ij} - M(1 - x_{kij}) \qquad \forall i \in N,\ \forall j \in N,\ i \neq j,\ \forall k \in K \tag{18}$$

Fix start time at depot:

$$a_{k0} = 0 \qquad \forall k \in K \tag{19}$$

Route duration variable (arrival back to depot):

$$T_k \ge a_{ki} + s_i + t_{i0} - M(1 - x_{ki0}) \qquad \forall i \in C,\ \forall k \in K \tag{20}$$

Shift bounds:

$$T_{\min}\, z_k \le T_k \le T_{\max}\, z_k \qquad \forall k \in K \tag{21}$$

6.10 10) Subtour Elimination (MTZ)

MTZ constraints per vehicle:

$$u_{ki} - u_{kj} + |C|\, x_{kij} \le |C| - 1 \qquad \forall i \in C,\ \forall j \in C,\ i \neq j,\ \forall k \in K \tag{22}$$

Link ordering to visits:

$$y_{ki} \le u_{ki} \le |C|\, y_{ki} \qquad \forall i \in C,\ \forall k \in K \tag{23}$$

6.11 11) Variable Domains

$$x_{kij},\ y_{ki},\ z_k \in \{0, 1\} \tag{24}$$

$$q_{ki} \ge 0, \quad a_{ki} \ge 0, \quad T_k \ge 0, \quad \pi_{kir} \ge 0, \quad \delta_{kir} \ge 0, \quad B^p_{ir} \ge 0, \quad B^d_{ir} \ge 0, \quad u_{ki} \ge 0 \tag{25}$$