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Calculus of Variations

Concentration profile of endemic equilibrium of a reaction–diffusion–advection SIS epidemic model

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Abstract Cui and Lou (J Differ Equ 261:3305–3343, 2016) proposed a reaction–diffusion–advection SIS epidemic model in heterogeneous environments, and derived interesting results on the stability of the DFE (disease-free equilibrium) and the existence of EE (endemic equilibrium) under various conditions. In this paper, we are interested in the asymptotic profile of the EE (when it exists) in the three cases: (i) large advection; (ii) small diffusion of the susceptible population; (iii) small diffusion of the infected population. We prove that in case (i), the density of both the susceptible and infected populations concentrates only at the downstream behaving like a delta function; in case (ii), the density of the susceptible concentrates only at the downstream behaving like a delta function and the density of the infected vanishes on the entire habitat, and in case (iii), the density of the susceptible is positive while the density of the infected vanishes on the entire habitat. Our results show that in case (ii) and case (iii), the asymptotic profile is essentially different from that in the

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situation where no advection is present. As a consequence, we can conclude that the impact of advection on the spatial distribution of population densities is significant.

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1 Introduction

This paper is concerned with the following initial boundary value problem for a reaction–diffusion–advection system over the interval $[0, L]$:

$$\begin{cases} S_t = d_S S_{xx} + qS_x - \dfrac{\beta(x)SI}{S+I} + \gamma(x)I, & 0 < x < L,\ t > 0, \\ I_t \ = d_I I_{xx} \ + qI_x + \dfrac{\beta(x)SI}{S+I} - \gamma(x)I, & 0 < x < L,\ t > 0, \\ d_S S_x + qS = 0 = d_I I_x + qI, & x = 0, L,\ t > 0, \\ (S(x,0), I(x,0)) = (S_0(x), I_0(x)) \geq (0,0), \quad I_0(x) \not\equiv 0, & 0 < x < L. \end{cases} \tag{1.1}$$

System (1.1) is a susceptible-infected-susceptible (SIS) epidemic model which was proposed in [8] to describe the spatiotemporal behavior of the population densities of susceptible individuals and infected individuals in the one-dimensional habitat $[0, L]$.

In this model, the unknown functions $S(x, t)$ and $I(x, t)$ represent, respectively, the population density at position $x \in [0, L]$ and time $t > 0$ of the susceptible individuals and infected individuals, and the positive constants d_S and d_I are their respective diffusion coefficients; the functions $\beta(x)$ and $\gamma(x)$ are Hölder continuous positive functions in $[0, L]$ and stand for the rates of epidemic transmission and recovery at position x, respectively; the positive constant q represents the advection speed of a stream or wind which carries the susceptible and infected populations from the upstream $x = L$ to the downstream $x = 0$. The boundary value conditions suppose the no-flux situation at the downstream and the upstream. Given initial data $S_0,\ I_0$, thanks to the boundary value conditions, one can integrate the sum of the equations of (1.1) to yield the conservation law:

$$\int_0^L (S(x,t) + I(x,t))\,\mathrm{d}x = \int_0^L (S_0(x) + I_0(x))\,\mathrm{d}x =: N, \quad \forall t \geq 0. \tag{1.2}$$

The non-advection version ($q = 0$) of (1.1) was proposed by Allen et al. [2] in which they discussed the existence, uniqueness, and particularly the asymptotic profile of the equilibrium (steady state) as the diffusion rate d_S of the susceptible population tends to zero. Later, in [17–21], the third author and collaborators established further qualitative properties of solutions of (1.1) with $q = 0$. The advection version (1.1) was proposed by Cui and Lou [8]. They obtained many interesting results on the stability of the DFE (disease-free equilibrium) and the existence of the EE (endemic equilibrium) under various conditions; their main results will be listed in Sect. 2 below. Regarding other related studies and background on (1.1), one can refer to [1,3,4,6,7,9–12,14–16,22–25] and the references therein.

In the current paper, we will focus on the steady state (i.e., stationary) solutions corresponding to (1.1). The stationary problem consists of a pair of nonlinear ordinary differential equations:

$$d_S S_{xx} + q S_x - \frac{\beta(x) S I}{S + I} + \gamma(x) I = 0, \quad x \in (0, L), \tag{1.3a}$$

$$d_I I_{xx} + q I_x + \frac{\beta(x) S I}{S + I} - \gamma(x) I = 0, \quad x \in (0, L), \tag{1.3b}$$

subject to the no-flux boundary value conditions

$$d_s S_x + q S = 0 = d_I I_x + q I, \quad x = 0, L, \tag{1.3c}$$

and the conservation law associated with (1.2)

$$\int_0^L (S(x) + I(x))\, \mathrm{d}x = N. \tag{1.4}$$

From now on, we let N be a given positive constant.

From the ecological point of view, only nonnegative solutions of (1.3)–(1.4) are of our interest. Concerning (1.3)–(1.4), a solution (S, I) with $I = 0$ is called a *disease-free equilibrium* (DFE), whereas a componentwise positive solution is called an *endemic equilibrium* (EE). The disease-free equilibrium (DFE) $(\hat{S}, 0)$ can be easily obtained:

$$\hat{S}(x) = \frac{qN}{d_S\left(1 - e^{-\frac{q}{d_S} L}\right)} e^{-\frac{q}{d_S} x}$$

by solving (1.3)–(1.4) with $I = 0$.

It is well known that the *basic reproduction number* is a critical quantity in the study of the stability of DFE in the field of epidemic models. For the model (1.1), Cui and Lou [8] showed that the basic reproduction number $\mathcal{R}_0(d_I, q)$ is given as

$$\mathcal{R}_0(d_I, q) := \sup_{\varphi \in H^1(0,L), \varphi \neq 0} \left\{ \frac{\int_0^L \beta(x) e^{-\frac{q}{d_I} x} \varphi^2\, \mathrm{d}x}{d_I \int_0^L e^{-\frac{q}{d_I} x} \varphi_x^2\, \mathrm{d}x + \int_0^L \gamma(x) e^{-\frac{q}{d_I} x} \varphi^2\, \mathrm{d}x} \right\}, \tag{1.5}$$

and the stability of DFE is determined by the magnitude relationship between $\mathcal{R}_0(d_I, q)$ and 1: If $\mathcal{R}_0 < 1$, the DFE is globally asymptotically stable, whereas if $\mathcal{R}_0 > 1$, the DFE is unstable. Then it is important to derive the set $\{(d_I, q)\}$ satisfying $\mathcal{R}_0(d_I, q) = 1$. In [8], it was also proved that the above set forms a curve on (d_I, q)-plane under some assumptions on β and γ. Furthermore, they concluded that if (d_I, q) lies in in the region where $\mathcal{R}_0(d_I, q) > 1$, then (1.3)–(1.4) admits an EE. Unlike the case of no advection $q = 0$, the uniqueness and stability of EE of (1.3)–(1.4) remain open questions.

The purpose of the present paper is to establish the asymptotic behavior of EE (when it exists) in the three cases: (i) $q \to \infty$; (ii) $d_S \to 0$; (iii) $d_I \to 0$. We prove that in case (i), the density of both the susceptible and infected populations concentrates only at the downstream behaving like a delta function; in case (ii), the density of the susceptible concentrates only at the downstream behaving like a delta function and the density of the infected vanishes on the entire habitat, and in case (iii), on the entire habitat the density of the susceptible is positive while the density of the infected vanishes.

Our results show that in case (ii) and case (iii), the asymptotic profile is essentially different from that in the situation where no advection is present. As a consequence, we can conclude that the impact of advection on the spatial distribution of population densities is significant. The precise statement of our results will be given in Sect. 3, where more detailed discussions and comparisons with relevant results derived by [2,18] will be made.

We also like to mention that the paper [17] studied an SIS epidemic reaction–diffusion system with a linear source in a spatially heterogeneous environment. The main feature of such a model is that its total population number varies, compared to its counterpart treated by [1,17–21] and here. The theoretical results of [17] show that a varying total population can enhance persistence of infectious disease, and therefore the disease becomes more threatening and harder to control. A natural question arises: how could the advection effect the asymptotic behavior of the associated EE for large advection or small diffusion? We plan to explore it in future work.

Before ending the introduction, let us introduce the notion of high/low/moderate risk site used in [2,8,18,20], etc. We call $x \in [0, L]$ is a high-risk (respectively, moderate-risk, low-risk) site if $\beta(x) > \gamma(x)$ (respectively, $\beta(x) = \gamma(x)$, $\beta(x) < \gamma(x)$).

The rest of our paper is organized as follows. In Sect. 2, we will list the existence conditions of EE to (1.3)–(1.4) established in [8]. In Sect. 3, our main results will be presented. In Sect. 4, we prepare a priori estimates for EE of (1.3)–(1.4), which is fundamental in the analysis of the limiting behavior of EE. Sections 5, 6 and 7 are devoted to the proof of the three main results.

2 Known results

In this section, we collect some results on the existence of positive solutions of (1.3)–(1.4) obtained in [8]. Hereafter we assume that the positive functions $\beta(x)$ and $\gamma(x)$ on $[0, L]$ satisfy either

(C1) $\beta(0) - \gamma(0) > 0 > \beta(L) - \gamma(L)$

or

(C2) $\beta(0) - \gamma(0) < 0 < \beta(L) - \gamma(L)$.

From the epidemiological viewpoint, (C1) (respectively, (C2)) assumes the situation that the downstream $x = 0$ is a high (respectively, low) risk site, whereas the upstream $x = L$ is a low (respectively, high) risk site.

The following theorem depicts regions where the DFE is stable/unstable on (d_I, q)-plane when the entire habitat $(0, L)$ is of high risk.

Theorem 2.1 *([8]) Assume the interval $(0, L)$ is of high risk in the sense of $\int_0^L \beta\, \mathrm{d}x > \int_0^L \gamma\, \mathrm{d}x$. Then the following assertions hold.*

(i) *In case (C1), the DFE is unstable for any $d_I > 0$ and $q > 0$.*
(ii) *In case (C2), there exists a curve*

$$\Gamma_1 := \{\, (d_I, \rho_1(d_I)) \;:\; \mathcal{R}_0(d_I, \rho_1(d_I)) = 1, \quad d_I \in (0, \infty)\,\}$$

on (d_I, q)-plane such that, for each $d_I > 0$, the DFE is unstable for $0 < q < \rho_1(d_I)$ and the DFE is globally asymptotically stable for $q > \rho_1(d_I)$. Furthermore, the function $\rho_1 : (0, \infty) \to (0, \infty)$ satisfies

$$\lim_{d_I \to 0} \rho_1(d_I) = 0, \quad \lim_{d_I \to \infty} \frac{\rho_1(d_I)}{d_I} = \theta_1,$$

where θ_1 is a positive constant satisfying $\int_0^L (\beta(x) - \gamma(x))\, e^{-\theta_1 x}\, \mathrm{d}x = 0$.

The next theorem indicates regions where the DFE is stable/unstable on (d_I, q)-plane when $(0, L)$ is a low risk interval:

Theorem 2.2 *([8])*
Assume the interval $(0, L)$ is of low risk in the sense of $\int_0^L \beta\,\mathrm{d}x < \int_0^L \gamma\,\mathrm{d}x$. Then there exists a positive constant d_I^* such that the following assertions hold.

(i) *In case* (C1)*, then*

(i-1) *for $d_I \in (0, d_I^*]$, the DFE is unstable for any $q > 0$;*
(i-2) *for $d_I \in (d_I^*, \infty)$, there exists a curve*

$$\Gamma_2 := \{\, (d_I, \rho_2(d_I)) \;:\; \mathcal{R}_0(d_I, \rho_2(d_I)) = 1, \quad d_I \in (d_I^*, \infty)\,\}$$

on (d_I, q)-plane such that, for each $d_I \in (d_I^*, \infty)$, the DFE is globally asymptotically stable for $0 < q < \rho_2(d_I)$ and the DFE is unstable for $q > \rho_2(d_I)$. Furthermore, the function $\rho_2 : (d_1^*, \infty) \to (0, \infty)$ is monotone increasing with respect to d_I and satisfies

$$\lim_{d_I \downarrow d_I^*} \rho_2(d_I) = 0, \quad \lim_{d_I \to \infty} \frac{\rho_2(d_I)}{d_I} = \theta_2,$$

where θ_2 is a positive constant satisfying $\int_0^L (\beta(x) - \gamma(x))\, e^{-\theta_2 x}\,\mathrm{d}x = 0$.

(ii) *In case* (C2)*, then*

(ii-1) *for $d_I \in (0, d_I^*)$, there exists a curve*

$$\Gamma_3 := \{\, (d_I, \rho_2(d_I)) \;:\; \mathcal{R}_0(d_I, \rho_2(d_I)) = 1,\; d_I \in (0, d_I^*)\,\}$$

on (d_I, q)-plane such that, for each $d_I \in (0, d_I^*)$, the DFE is unstable for $0 < q < \rho_3(d_I)$ and the DFE is globally asymptotically stable for $q > \rho_3(d_I)$. Furthermore, the function $\rho_3 : (0, d_1^*) \to (0, \infty)$ satisfies

$$\lim_{d_I \to 0} \rho_3(d_I) = 0, \quad \lim_{d_I \uparrow d_I^*} \rho_3(d_I) = 0;$$

(ii-2) *for $d_I \in [d_I^*, \infty)$, the DFE is globally asymptotically stable for any $q > 0$.*

The following existence result on positive solutions of (1.3)–(1.4) is the starting point of our study in this paper.

Theorem 2.3 *([8]) Assume that $\beta(x)$ and $\gamma(x)$ satisfy* (C1) *or* (C2) *and there exists a unique zero of $\beta(x) - \gamma(x)$ on $(0, L)$. If the DFE is unstable, then* (1.3)–(1.4) *admits at least one EE.*

3 Main results

In what follows, when we discuss the asymptotic behavior of EE in the three cases: (i) $q \to \infty$; (ii) $d_S \to 0$; (iii) $d_I \to 0$, unless otherwise specified, we always assume that EE exists in that case. In particular, by Theorems 2.1–2.3, we know that if (C1) holds, $\beta(x) - \gamma(x)$ changes sign only once in $(0, L)$ and the entire habitat $(0, L)$ is of low risk or high risk, then (1.3)–(1.4) admits EE for all large q; nevertheless, if (C2) holds, (1.3)–(1.4) has no EE provided that q is large.

Our first main result concerns the asymptotic behavior of EE as $q \to \infty$, and reads as follows.

Theorem 3.1 *Let d_S, d_I be fixed. Assume that $\beta(x)$ and $\gamma(x)$ satisfy* (C1) *and there exists a unique zero of $\beta(x)-\gamma(x)$ in $(0, L)$. Let $(S(x,q), I(x,q)) := (S(x), I(x))$ be any EE of* (1.3)–(1.4)*. Then, as $q \to \infty$, we have*

- $(S(x,q), I(x,q)) \to (0,0)$ *locally uniformly in* $(0, L]$;
- *at the downstream,*

$$\lim_{q\to\infty}\frac{1}{q}S\left(\frac{y}{q},q\right)=C_Se^{-\frac{y}{d_S}},\qquad \lim_{q\to\infty}\frac{1}{q}I\left(\frac{y}{q},q\right)=C_Ie^{-\frac{y}{d_I}} \tag{3.1}$$

uniformly for y in any compact subset of $[0,\infty)$*, where (C_S, C_I) is a positive solution of*

$$\begin{cases} d_SC_S+d_IC_I=N,\\ \displaystyle\int_0^\infty \frac{C_Se^{-(\frac{1}{d_S}+\frac{1}{d_I})y}}{C_Se^{-\frac{y}{C_S}}+C_Ie^{-\frac{y}{C_I}}}\,\mathrm{d}y=\frac{\gamma(0)}{\beta(0)}d_I. \end{cases} \tag{3.2}$$

Theorem 3.1 tells us that, as the advection coefficient q becomes sufficiently large, the densities of the susceptible and infected populations concentrate only at the downstream $x = 0$ with the order $O(q)$, and both populations behave like delta functions with some masses at the downstream. These masses are determined by the conservation law and the integral in (3.2). We should mention that Cui et al. [7] have showed that (C_S, C_I) in our Theorem 3.1 is uniquely determined by (3.2); see Theorem 1.2 of [7]. Indeed, their results are sharper.

We next analyze the asymptotic behavior of EE as $d_S \to 0$, and can conclude the following result.

Theorem 3.2 *Let q, d_I be fixed. Assume that $\beta(x)$ and $\gamma(x)$ satisfy either* (C1) *or* (C2)*, and there exists a unique zero of $\beta(x)-\gamma(x)$ in $(0, L)$. Let $(S(x,d_S), I(x,d_S)) := (S(x), I(x))$ be any EE of* (1.3)–(1.4)*. Then, as $d_S \to 0$, we have*

- $S(x,d_S) \to 0$ *locally uniformly for $x \in (0, L]$, and* $\int_0^L S(x,d_S)\,\mathrm{d}x \to N$;
- $d_SS(d_Sy,d_S) \to Nqe^{-qy}$ *uniformly for y in any compact subset of* $[0,\infty)$;
- $I(x,d_S) \to 0$ *uniformly for $x \in [0, L]$.*

According to Theorem 3.2, when the mobility of the susceptible population is restricted to be small enough, the susceptible concentrates only at the downstream $x = 0$ as a delta function, and the density of the infected vanishes on the entire habitat. On the other hand, when no advection is imposed, [1] proved that, as the mobility of susceptible tends to zero, the density of the infected vanishes on the entire habitat, and the density of the susceptible is always positive on all low-risk and moderate-risk sites $\{x \in [0, L] : \beta(x) \le \gamma(x)\}$ and is also positive at some but not all high-risk sites. In particular, if no advection is present, the results of [1] showed that no concentration profile is formed for small mobility of the susceptible.

The last main result concerns the asymptotic behavior of EE as $d_I \to 0$. Indeed, we can state

Theorem 3.3 *Let q, d_S be fixed. Assume that $\beta(x)$ and $\gamma(x)$ satisfy either* (C1) *or* (C2)*, and there exists a unique zero of $\beta(x)-\gamma(x)$ in $(0, L)$. Let $(S(x,d_I), I(x,d_I)) := (S(x), I(x))$ be any EE of* (1.3)–(1.4)*. Then, as $d_I \to 0$, we have*

- $$S(x,\, d_I) \to \frac{qN}{d_S\left(1-e^{-\frac{qL}{d_S}}\right)}e^{-\frac{q}{d_S}x}\quad \textit{uniformly on } [0, L];$$

- $I(x, d_I) \to 0$ *locally uniformly in* $(0, L]$ *and* $\int_0^L I(x, d_I)\,\mathrm{d}x \to 0$.

We see from Theorem 3.3 that when the mobility of the infected population is small enough, the density of the infected vanishes on the entire habitat, while the density of the susceptible is positive but inhomogeneous everywhere. This asymptotic behavior of (S, I) as $d_I \to 0$ is totally different from that in the no advection case $q = 0$. If $q = 0$, [18] showed that, as $d_I \to 0$, the density of the infected vanishes on all low-risk and moderate-risk sites $\{x \in [0, L] : \beta(x) \le \gamma(x)\}$ and is positive and inhomogeneous at all high-risk sites, whereas the density of the susceptible becomes positive and homogeneous on the entire habitat. Therefore, Theorem 3.3 implies that the advection deprives a chance of the infected to remain positive somewhere in $(0, L]$ as $d_I \to 0$.

In light of the above discussion, we can draw the conclusion that advection has a qualitative impact of the spatial distribution of both the susceptible and infected populations.

From the proof in Sects. 6 and 7 below, it is not hard to see that both Theorem 3.2 and Theorem 3.3 remain valid as long as the EE exists (not only for the case that $\beta(x)$ and $\gamma(x)$ satisfy either (C1) or (C2)).

We would also like to mention that when the mobility of the susceptible tends to infinity, one can easily follow the argument of [18] to show that on the entire habitat, the density of the susceptible becomes positive and homogeneous and the density of the infected is also positive but inhomogeneous; on the other hand, when the mobility of the infected tends to infinity, on the entire habitat the density of the susceptible becomes positive and inhomogeneous, and the density of the infected is positive but homogeneous. In other words, advection has no essential effect on spatial distribution of population for large movement of either the susceptible or the infected, and no concentration phenomenon occurs.

4 A priori estimates

As a preliminary preparation, this section is devoted to a priori estimates for any EE of (1.3)–(1.4). First of all, we show the exponential decay of $I(x)/I(0)$ for any EE (S, I).

Lemma 4.1 *Let* (S, I) *be any EE of* (1.3)–(1.4). *Then the following assertions hold.*

(i) *For any* $q, d_S, d_I > 0$, *we have*

$$I(0)e^{-\frac{q}{d_I}\left(1+\|\gamma\|_\infty \frac{d_I}{q^2}\right)x} \le I(x), \quad \forall x \in [0, L].$$

(ii) *If* $q > d_I^{\frac{1}{2}}\|\beta\|_\infty/(\min_{x\in[0,L]}\gamma(x))^{\frac{1}{2}}$, *then*

$$I(x) \le I(0)e^{-\frac{q}{d_I}\left(1-\|\beta\|_\infty \frac{d_I}{q^2}\right)x}, \quad \forall x \in [0, L].$$

Proof Following Cui and Lou [8, Lemma 2.5], we employ the following change of variable

$$I(x) = e^{-\frac{q}{d_I}Ax}v(x), \tag{4.1}$$

where A is a positive constant which will be determined later. It follows from (4.1) that

$$\begin{aligned} I_x &= e^{-\frac{q}{d_I}Ax}\left(v_x - \frac{q}{d_I}Av\right), \\ I_{xx} &= e^{-\frac{q}{d_I}Ax}\left(v_{xx} - \frac{2q}{d_I}Av_x + \frac{q^2}{d_I^2}A^2v\right). \end{aligned} \tag{4.2}$$

Thus (1.3b) is equivalent to the equation

$$d_I v_{xx} + q(1-2A)v_x + \left\{\frac{q^2}{d_I}A(A-1) + \beta(x)\frac{S}{S+I} - \gamma(x)\right\}v = 0, \quad 0 < x < L. \tag{4.3}$$

By (4.1) and (4.2), the boundary condition (1.3c) is reduced to

$$d_I v_x = q(A-1)v, \quad x = 0, L. \tag{4.4}$$

(i) In (4.1), we set

$$A = 1 + \frac{Cd_I}{q^2},$$

where $C > 0$ is a positive constant which will be determined later. Then we know from (4.4) that

$$d_I v_x = \frac{Cd_I}{q}v > 0, \quad x = 0, L.$$

Let $v(x_*) = \min_{x\in[0,L]} v(x)$. We show $x_* = 0$ if C is suitably chosen. The above boundary condition implies $x_* \neq L$. Suppose for contradiction that $0 < x_* < L$. By virtue of $v_{xx}(x_*) \geq 0$ and $v_x(x_*) = 0$, substituting $x = x_*$ into (4.3) gives

$$C\left(1 + \frac{Cd_I}{q^2}\right) + \beta(x_*)\frac{S(x_*)}{S(x_*) + I(x_*)} - \gamma(x_*) \leq 0.$$

This implies

$$C\left(1 + \frac{Cd_I}{q^2}\right) < \gamma(x_*).$$

Therefore, we meet a contradiction by taking $C = \|\gamma\|_\infty$. Such a contradiction enables us to conclude that $x_* = 0$ by setting $C = \|\gamma\|_\infty$, that is,

$$\min_{x\in[0,L]} v(x) = v(0) \leq v(x), \quad \forall x \in [0, L].$$

By (4.1), we derive the following inequality

$$v(0) = I(0) \leq v(x) = e^{\frac{q}{d_I}\left(1+\|\gamma\|_\infty \frac{d_I}{q^2}\right)x} I(x), \quad \forall x \in [0, L],$$

which is exactly the inequality in (i).

(ii) In (4.1), we choose

$$A = 1 - \frac{Cd_I}{q^2},$$

where $C > 0$ is a positive constant to be determined later. It follows from (4.4) that

$$d_I v_x = -\frac{Cd_I}{q}v < 0, \quad x = 0, L.$$

Thus, letting $v(x^*) = \max_{x\in[0,L]} v(x)$, it is clear that $x^* \neq L$. Suppose for contradiction that $0 < x^* < L$. Since $v_{xx}(x^*) \leq 0$ and $v_x(x^*) = 0$, (4.3) with $x = x^*$ yields

$$-C\left(1 - \frac{Cd_I}{q^2}\right) + \beta(x^*)\frac{S(x^*)}{S(x^*) + I(x^*)} - \gamma(x^*) \geq 0.$$

Since $S/(S+I) < 1$, one can see

$$-C\left(1 - \frac{Cd_I}{q^2}\right) + \beta(x^*) - \min_{x\in[0,L]} \gamma(x) > 0.$$

Setting $C = \|\beta\|_\infty$ gives

$$\underbrace{\beta(x^*) - \|\beta\|_\infty}_{\leq 0} + d_I \left(\frac{\|\beta\|_\infty}{q}\right)^2 - \min_{x\in[0,L]} \gamma(x) > 0. \tag{4.5}$$

Therefore, (4.5) does not hold for any $q > 0$ satisfying

$$d_I \left(\frac{\|\beta\|_\infty}{q}\right)^2 < \min_{x\in[0,L]} \gamma(x), \quad \text{namely,} \quad q > \frac{d_I^{\frac{1}{2}} \|\beta\|_\infty}{(\min_{x\in[0,L]} \gamma(x))^{\frac{1}{2}}}. \tag{4.6}$$

Hence we can conclude $x^* = 0$ for any $q > 0$ satisfying (4.6). As $\max_{x\in[0,L]} v(x) = v(0) \geq v(x)$ $(0 \leq x \leq L)$, we know from (4.1) that

$$v(0) = I(0) \geq v(x) = e^{\frac{q}{d_I}\left(1 - \|\beta\|_\infty \frac{d_I}{q^2}\right)x} I(x), \quad forall x \in [0, L].$$

This is the desired inequality in (ii). □

Next we will discuss the decay estimate for the S-component of any EE (S, I) of (1.3)–(1.4).

Lemma 4.2 *Let (S, I) be any EE of (1.3)–(1.4). Then the following assertions hold.*

(i) *For any q, d_S, $d_I > 0$, then*

$$S(0)e^{-\frac{q}{d_S}\left(1 + \|\beta\|_\infty \frac{d_S}{q^2}\right)x} \leq S(x), \quad \forall x \in [0, L].$$

(ii) *For any q, d_S, $d_I > 0$, then*

$$S(x) \leq S(0)e^{-\frac{q}{d_S}x} + \frac{N}{q}\left(\|\beta\|_\infty - \min_{x\in[0,L]} \gamma(x)\right)\left(1 - e^{-\frac{q}{d_S}x}\right), \quad \forall x \in [0, L].$$

Proof (i) As in the proof of (i) of Lemma 4.1, we take

$$S(x) = e^{-\frac{q}{d_S}Ax} w(x), \tag{4.7}$$

where A is a positive constant to be determined. Then the desired estimate can be obtained by a similar procedure as in the proof of (i) of Lemma 4.1, using the equation satisfied by w. So we omit the details.

(ii) Integrating (1.3a) over $(0, x)$, we obtain

$$\begin{aligned} d_S S_x(x) + qS(x) &= \int_0^x \left(\beta(y)\frac{S(y)}{S(y) + I(y)} - \gamma(y)\right) I(y)\,\mathrm{d}y \\ &\leq \left(\|\beta\|_\infty - \min_{x\in[0,L]} \gamma\right) \int_0^L I(x)\,\mathrm{d}x. \end{aligned}$$

Thanks to $\int_0^L I\,\mathrm{d}x \le N$ by (1.4), one can see

$$S_x(x) + \frac{q}{d_S} S(x) \le \frac{1}{d_S}\left(\|\beta\|_\infty - \min_{x\in[0,L]} \gamma\right) N =: M, \quad \forall x \in [0, L].$$

Then the function

$$\widetilde{S}(x) := S(x) - \frac{d_S}{q} M$$

satisfies

$$\widetilde{S}_x + \frac{q}{d_S}\widetilde{S} \le 0, \quad \text{namely,} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\frac{q}{d_S}x}\widetilde{S}\right) \le 0, \quad \forall x \in [0, L].$$

Integrating the above inequality over $(0, x)$, we get

$$e^{\frac{q}{d_S}x}\widetilde{S}(x) - \widetilde{S}(0) \le 0,$$

that is,

$$S(x) - \frac{d_S}{q} M \le \left(S(0) - \frac{d_S}{q} M\right) e^{-\frac{q}{d_S}x}, \quad \forall x \in [0, L].$$

In light of the expression of M, the desired inequality is proved.

□

Combining Lemmas 4.1 and 4.2 with the conservation law (1.4), we can obtain the following upper estimates for any EE (S, I) of (1.3)–(1.4). These estimates will be useful to derive the asymptotic behaviors of (S, I) stated in Theorems 3.1–3.3.

Lemma 4.3 *Let (S, I) be any EE of (1.3)–(1.4). Then the following assertions hold:*

(i) *For any $q,\ d_S,\ d_I > 0$, we have*

$$S(x) \le \frac{qN\left(1 + \|\beta\|_\infty \frac{d_S}{q^2}\right)}{d_S\left[1 - e^{-\frac{q}{d_S}\left(1+\|\beta\|_\infty\frac{d_S}{q^2}\right)L}\right]} e^{-\frac{q}{d_S}x} + \frac{N}{q}\left(\|\beta\|_\infty - \min_{x\in[0,L]}\gamma(x)\right)\left(1 - e^{-\frac{q}{d_S}x}\right), \quad \forall x \in [0, L].$$

(ii) *For any $q,\ d_S,\ d_I > 0$, we have*

$$I(x) \le \frac{qN\left(1 + \|\gamma\|_\infty \frac{d_I}{q^2}\right)}{d_I\left[1 - e^{-\frac{q}{d_I}\left(1+\|\gamma\|_\infty\frac{d_I}{q^2}\right)L}\right]} e^{-\frac{q}{d_I}x} + \frac{N\|\gamma\|_\infty}{q}\left(1 - e^{-\frac{q}{d_I}x}\right), \quad \forall x \in [0, L].$$

(iii) *If $q > d_I^{\frac{1}{2}}\|\beta\|_\infty/(\min_{x\in[0,L]}\gamma(x))^{\frac{1}{2}}$, then,*

$$I(x) \le \frac{qN\left(1 + \|\gamma\|_\infty \frac{d_I}{q^2}\right)}{d_I\left[1 - e^{-\frac{q}{d_I}\left(1+\|\gamma\|_\infty\frac{d_I}{q^2}\right)L}\right]} e^{-\frac{q}{d_I}\left(1-\|\beta\|_\infty\frac{d_I}{q^2}\right)x}, \quad \forall x \in [0, L].$$

Proof It follows from (i) of Lemma 4.1 that

$$I(0)e^{-\frac{q}{d_I}\left(1+\|\gamma\|_\infty\frac{d_I}{q^2}\right)x} \le I(x), \quad \forall x \in [0, L].$$

Integrating this inequality over $(0, L)$, we have

$$I(0)\int_0^L e^{-\frac{q}{d_I}\left(1+\|\gamma\|_\infty\frac{d_I}{q^2}\right)x}\,\mathrm{d}x \le \int_0^L I(x)\,\mathrm{d}x. \tag{4.8}$$

The same procedure as applied to derive the inequality in (i) of Lemma 4.2 gives

$$S(0)\int_0^L e^{-\frac{q}{d_S}\left(1+\|\beta\|_\infty\frac{d_S}{q^2}\right)x}\,\mathrm{d}x \le \int_0^L S(x)\,\mathrm{d}x. \tag{4.9}$$

By adding (4.8)–(4.9), we obtain

$$\begin{aligned} &I(0)\int_0^L e^{-\frac{q}{d_I}\left(1+\|\gamma\|_\infty\frac{d_I}{q^2}\right)x}\,\mathrm{d}x + S(0)\int_0^L e^{-\frac{q}{d_S}\left(1+\|\beta\|_\infty\frac{d_S}{q^2}\right)x}\,\mathrm{d}x \\ &\le \int_0^L (S(x)+I(x))\,\mathrm{d}x = N. \end{aligned} \tag{4.10}$$

where the last equality comes from (1.4). Since (4.10) is reduced to

$$\begin{aligned} &I(0)\frac{d_I}{q\left(1+\|\gamma\|_\infty\frac{d_I}{q^2}\right)}\left(1-e^{-\frac{q}{d_I}\left(1+\|\gamma\|_\infty\frac{d_I}{q^2}\right)L}\right) \\ &+S(0)\frac{d_S}{q\left(1+\|\beta\|_\infty\frac{d_S}{q^2}\right)}\left(1-e^{-\frac{q}{d_S}\left(1+\|\beta\|_\infty\frac{d_S}{q^2}\right)L}\right) \le N, \end{aligned}$$

then for any $d_I,\ d_S,\ q > 0$, we find that

$$S(0) \le \frac{qN\left(1+\|\beta\|_\infty\frac{d_S}{q^2}\right)}{d_S\left[1-e^{-\frac{q}{d_S}\left(1+\|\beta\|_\infty\frac{d_S}{q^2}\right)L}\right]} \tag{4.11}$$

and

$$I(0) \le \frac{qN\left(1+\|\gamma\|_\infty\frac{d_I}{q^2}\right)}{d_I\left[1-e^{-\frac{q}{d_I}\left(1+\|\gamma\|_\infty\frac{d_I}{q^2}\right)L}\right]}. \tag{4.12}$$

Then by substituting (4.11) into the inequality obtained in (ii) of Lemma 4.2, we can derive the assertion (i).

In order to derive the assertion (ii), we apply a similar argument as in the proof of (ii) of Lemma 4.2 to (1.3b). By integrating (1.3b) over $(0, x)$, we obtain

$$d_I I_x(x) + qI(x) = \int_0^x \left(\gamma(y) - \beta(y)\frac{S(y)}{S(y)+I(y)}\right) I(y)\,\mathrm{d}y$$
$$\le \|\gamma\|_\infty \int_0^L I(x)\,\mathrm{d}x \le \|\gamma\|_\infty N, \quad \forall x \in [0, L],$$

where the last inequality follows from (1.4). Hence the same procedure as in the proof of (ii) of Lemma 4.2 leads us to

$$I(x) \le I(0)e^{-\frac{q}{d_I}x} + \frac{N\|\gamma\|_\infty}{q}\left(1 - e^{-\frac{q}{d_I}x}\right), \quad \forall x \in [0, L].$$

Then by substituting (4.12) into the above inequality, we obtain the assertion (ii).

In particular, if q and d_I satisfy

$$q > d_I^{\frac{1}{2}}\|\beta\|_\infty / (\min_{x\in[0,L]} \gamma(x))^{\frac{1}{2}},$$

(ii) of Lemma 4.1, combined with (4.12), yields the assertion (iii). □

The next lemma provides a lower bound of the integral of the S-component for any EE (S, I) to (1.3)–(1.4).

Lemma 4.4 *Let* (S, I) *be any EE of* (1.3)–(1.4). *Then we have*

$$\int_0^L S(x)\,\mathrm{d}x \ge \frac{N \min_{x\in[0,L]} \gamma}{\|\beta\|_\infty + \min_{x\in[0,L]} \gamma} > 0. \tag{4.13}$$

Proof Using the boundary condition (1.3c), we integrate (1.3b) over $(0, L)$ to find

$$\int_0^L \left(\beta(x)\frac{S(x)}{S(x)+I(x)} - \gamma(x)\right) I(x)\,\mathrm{d}x = 0.$$

This gives

$$\left(\min_{x\in[0,L]} \gamma\right)\int_0^L I\,\mathrm{d}x \le \int_0^L \gamma(x) I\,\mathrm{d}x = \int_0^L \beta(x)\frac{SI}{S+I}\,\mathrm{d}x \le \int_0^L \beta(x) S\,\mathrm{d}x \le \|\beta\|_\infty \int_0^L S\,\mathrm{d}x.$$

Substituting

$$\int_0^L I\,\mathrm{d}x = N - \int_0^L S\,\mathrm{d}x$$

into the left-hand side of the above inequality, we obtain

$$\left(\min_{x\in[0,L]} \gamma\right)\left(N - \int_0^L S\,\mathrm{d}x\right) \le \|\beta\|_\infty \int_0^L S\,\mathrm{d}x,$$

which is equivalent to (4.13). □

5 Profile of EE as $q \to \infty$: proof of Theorem 3.1

For any EE $(S(x,q), I(x,q))$ of (1.3)–(1.4), we introduce the following rescaled functions:

$$u(y,q) := \frac{1}{q} S\left(\frac{y}{q}, q\right), \quad v(y,q) := \frac{1}{q} I\left(\frac{y}{q}, q\right), \quad 0 \le y \le qL. \tag{5.1}$$

Corresponding to (1.3)–(1.4), (u, v) satisfies the following nonlinear ordinary differential equations over $(0, qL)$:

$$d_S u_{yy} + u_y - \frac{v(y,q)}{q^2}\left\{\beta\left(\frac{y}{q}\right)\frac{u(y,q)}{u(y,q)+v(y,q)} - \gamma\left(\frac{y}{q}\right)\right\} = 0, \quad y \in (0, qL), \tag{5.2a}$$

$$d_I v_{yy} + v_y + \frac{v(y,q)}{q^2}\left\{\beta\left(\frac{y}{q}\right)\frac{u(y,q)}{u(y,q)+v(y,q)} - \gamma\left(\frac{y}{q}\right)\right\} = 0, \quad y \in (0, qL) \tag{5.2b}$$

with the boundary conditions

$$d_s u_y + u = 0 = d_I v_y + v, \quad y = 0, \ qL. \tag{5.2c}$$

Furthermore, the conservation law (1.4) becomes

$$\int_0^{qL} (u(y,q) + v(y,q))\,\mathrm{d}y = N, \quad \forall q > 0. \tag{5.3}$$

Proof of Theorem 3.1 First of all, by (i) and (ii) (or (iii)) of Lemma 4.3, one can easily see that $S(x,q)$ and $I(x,q)$ converge to zero locally uniformly in $(0, L]$ as $q \to \infty$, respectively. It remains to show the limiting behavior of $S(x,q)$ and $I(x,q)$ at the downstream $x = 0$. The analysis below is quite long; for sake of clarity, we divide it into four steps.

Step 1 Convergence of $\{(u(y,q), v(y,q))\}$. It follows from (i) of Lemma 4.3 that when $q > 0$ is suitably large,

$$0 \le u(y,q) < \frac{2N}{d_S} e^{-\frac{y}{d_S}} + \frac{N}{q^2}\left(\|\beta\|_\infty - \min_{x\in[0,L]} \gamma(x)\right)(1 - e^{-\frac{y}{d_S}}), \quad \forall y \in [0, qL]. \tag{5.4}$$

Therefore, for any fixed $K > 0$, $\{u(y,q)\}$ is uniformly bounded in $[0, K]$ with respect to q satisfying $q > K/L$. From (iii) of Lemma 4.3, we obtain

$$v(y,q) \le \frac{2N}{d_I} e^{-\frac{y}{2d_I}}, \quad \forall y \in [0, qL]$$

for all sufficiently large $q > 0$. Clearly, $\{v(y,q)\}$ is also uniformly bounded in $[0, K]$ with respect to q with $q > K/L$. Thus, given a constant $K > 0$, the standard L^p-estimates (up to the boundary $y = 0$) for elliptic equations (see, for instance [13]) ensure that

$$\|u(y,q)\|_{W^{2,p}(0,K)} \le M, \quad \|v(y,q)\|_{W^{2,p}(0,K)} \le M$$

for some positive constant $M = M(K, p)$ and any $p > 1$. Hereafter, the constant M does not depend on all $q > K/L$ and is allowed to change from place to place. So from the embedding theorem (by taking p to be properly large), we have

$$\|u(y,q)\|_{C^{1,\alpha}([0,K])} \le M, \quad \|v(y,q)\|_{C^{1,\alpha}([0,K])} \le M$$

for some $\alpha \in (0, 1)$.

In view of such estimates, we can apply the Ascoli–Arzelà theorem, along with a diagonal argument, to find a positive sequence $\{q_n\}$ with $\lim_{n\to\infty} q_n = \infty$ and a nonnegative function (u^*, v^*) such that $u_n(y) := u(y, q_n)$ and $v_n(y) := v(y, q_n)$ satisfy

$$\lim_{n\to\infty}(u_n, v_n) = (u^*, v^*) \ \text{ in } \ C^1_{\mathrm{loc}}([0,\infty)) \times C^1_{\mathrm{loc}}([0,\infty)). \tag{5.5}$$

Since (u_n, v_n) satisfies (5.2)–(5.3) with $q = q_n$, we send $n \to \infty$ to know that (u^*, v^*) satisfies (in the weak sense and then in the classical sense) the limiting system

$$d_S u^*_{yy} + u^*_y = 0, \quad y \in (0,\infty), \tag{5.6a}$$

$$d_I v^*_{yy} + v^*_y = 0, \quad y \in (0,\infty) \tag{5.6b}$$

with the boundary conditions

$$d_s u^*_y(0) + u^*(0) = 0 = d_I v^*_y(0) + v^*(0). \tag{5.6c}$$

By virtue of (5.6a) one can see that $d_S u^*_y + u^*$ is a constant for all $y \geq 0$. Owing to (5.6c), clearly such a constant equals to zero: $d_S u^*_y + u^* = 0$ for any $y \geq 0$. Hence it follows that

$$u^*(y) = C_S e^{-\frac{y}{d_S}}$$

for some constant $C_S \geq 0$. Similarly,

$$v^*(y) = C_I e^{-\frac{y}{d_I}}$$

with some constant $C_I \geq 0$. So we get from (5.5) that

$$\lim_{n\to\infty}(u_n, v_n) = \left(C_S e^{-\frac{y}{d_S}}, C_I e^{-\frac{y}{d_I}}\right) \ \text{in } C^1_{\mathrm{loc}}([0,\infty)) \times C^1_{\mathrm{loc}}([0,\infty)). \tag{5.7}$$

Step 2 $C_S > 0$. By employing the change of variable $x = y/q_n$ in (4.13), we obtain

$$\int_0^{q_n L} u_n(y)\,\mathrm{d}y \geq \frac{N \min_{x\in[0,L]} \gamma}{\|\beta\|_\infty + \min_{x\in[0,L]} \gamma} > 0. \tag{5.8}$$

In view of

$$\lim_{n\to\infty} u_n(y) = C_S e^{-\frac{y}{d_S}} \ \text{ in } \ C^1_{\mathrm{loc}}([0,\infty))$$

from Step 1, we let $n \to \infty$ in the left-hand side of (5.8), and can claim

$$\int_0^{q_n L} u_n(y)\,\mathrm{d}y \to \int_0^{\infty} C_S e^{-\frac{y}{d_S}}\,\mathrm{d}y \tag{5.9}$$

and

$$d_S C_S = C_S \int_0^\infty e^{-\frac{y}{d_S}}\,\mathrm{d}y \geq \frac{N \min_{x\in[0,L]} \gamma}{\|\beta\|_\infty + \min_{x\in[0,L]} \gamma},$$

that is,

$$C_S \geq \frac{N \min_{x\in[0,L]} \gamma}{d_S(\|\beta\|_\infty + \min_{x\in[0,L]} \gamma)} > 0.$$

Then we obtain the positivity of C_S.

We now verify (5.9). Indeed, from (5.4) we have

$$0 \le u_n(y) \le \frac{2N}{d_S} e^{-\frac{y}{d_S}} + \frac{N}{q_n^2} \left(\|\beta\|_\infty - \min_{x \in [0,L]} \gamma(x) \right), \quad \forall y \in [0, q_n L].$$

Now we take $\varepsilon > 0$ arbitrarily. Then there exists a large $D > 0$ such that

$$2N e^{-\frac{D}{d_S}} < \frac{\varepsilon}{8}, \quad \int_D^\infty C_S e^{-\frac{y}{d_S}}\, \mathrm{d}y < \frac{\varepsilon}{4}.$$

and there exists $n_1 \in \mathbb{N}$ such that for $n \ge n_1$, $q_n L > D$ and

$$\frac{N(\|\beta\|_\infty - \min_{x \in [0,L]} \gamma(x))}{q_n} L < \frac{\varepsilon}{8}.$$

Hence for $n \ge n_1$ we obtain

$$\begin{aligned}
\int_D^{q_n L} u_n(y)\, \mathrm{d}y &\le \int_D^\infty \frac{2N}{d_S} e^{-\frac{y}{d_S}}\, \mathrm{d}y + \int_0^{q_n L} \frac{N}{q_n^2} (\|\beta\|_\infty - \min_{x \in [0,L]} \gamma(x))\, \mathrm{d}y \\
&\le 2N e^{-\frac{D}{d_S}} + \frac{N(\|\beta\|_\infty - \min_{x \in [0,L]} \gamma(x))}{q_n} L < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}.
\end{aligned}$$

From C^1_{loc} convergence of u_n to $C_S e^{-\frac{y}{d_S}}$, we can choose $n_2 \in \mathbb{N}$ with $n_2 \ge n_1$ such that for $n \ge n_2$

$$\int_0^D |u_n(y) - C_S e^{-\frac{y}{d_S}}|\, \mathrm{d}y < \frac{\varepsilon}{2}.$$

Therefore we have for $n \ge n_2$

$$\begin{aligned}
&\left| \int_0^\infty C_S e^{-\frac{y}{d_S}}\, \mathrm{d}y - \int_0^{q_n L} u_n(y)\, \mathrm{d}y \right| \\
&\le \left| \int_D^\infty C_S e^{-\frac{y}{d_S}}\, \mathrm{d}y - \int_D^{q_n L} u_n(y)\, \mathrm{d}y \right| + \left| \int_0^D \left(u_n(y) - C_S e^{-\frac{y}{d_S}} \right) \mathrm{d}y \right| \\
&\le \int_D^\infty C_S e^{-\frac{y}{d_S}}\, \mathrm{d}y + \int_D^{q_n L} u_n(y)\, \mathrm{d}y + \int_0^D |u_n(y) - C_S e^{-\frac{y}{d_S}}|\, \mathrm{d}y \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This means (5.9).

Step 3 $C_I > 0$. Suppose by way of contradiction that $C_I = 0$. Then it follows from (5.7) that

$$\lim_{n \to \infty} v_n(y) = 0 \ \text{ in } C^1_{\mathrm{loc}}([0, \infty)). \tag{5.10}$$

By Lemma 4.1(ii) and the definition of v_n, we observe that when

$$q_n > \max\left\{d_I^{\frac{1}{2}}\|\beta\|_\infty / \left(\min_{x\in[0,L]}\gamma(x)\right)^{\frac{1}{2}},\ 2d_I^{\frac{1}{2}}\|\beta\|_\infty^{\frac{1}{2}}\right\}, \tag{5.11}$$

v_n attains its maximum on $[0, q_n L]$ only at $y = 0$. Denote

$$\widetilde{v}_n(y) := \frac{v_n(y)}{v_n(0)}.$$

Thus, $\widetilde{v}_n(y) \le 1$ on $[0, q_n L]$ and $\widetilde{v}_n(0) = 1$ for all $n \ge 1$. By the similar argument as in Step 1, we can find a nonnegative limit $\widetilde{v}^*$ satisfying

$$\lim_{n\to\infty} \widetilde{v}_n = \widetilde{v}^* \ \text{ in } \ C^1_{\mathrm{loc}}([0,\infty)). \tag{5.12}$$

Dividing (5.2b) and (5.2c) by $v_n(0)$, one can see that $\widetilde{v}_n$ satisfies

$$\begin{cases} d_I(\widetilde{v}_n)_{yy} + (\widetilde{v}_n)_y + \dfrac{\widetilde{v}_n(y)}{q_n^2}\left\{\beta\left(\dfrac{y}{q_n}\right)\dfrac{u_n(y)}{u_n(y)+v_n(y)} - \gamma\left(\dfrac{y}{q_n}\right)\right\} = 0, & y \in (0, q_n L), \\ d_I(\widetilde{v}_n)_y + \widetilde{v}_n = 0, \quad y = 0,\ q_n L. \end{cases} \tag{5.13}$$

By letting $n \to \infty$ in (5.13), we derive the following limiting problem:

$$(d_I \widetilde{v}^*_y + \widetilde{v}^*)_y = 0, \quad y > 0, \qquad d_I \widetilde{v}^*_y(0) + \widetilde{v}^*(0) = 0.$$

Solving this problem yields

$$\widetilde{v}^*(y) = \widetilde{C}_I e^{-\frac{y}{d_I}}$$

for some nonnegative constant $\widetilde{C}_I$. Because of $\max_{y\in[0,q_nL]} \widetilde{v}_n(y) = 1$ for any $n \in \mathbb{N}$, the convergence (5.12) enables us to assert $\widetilde{C}_I = 1$, that is,

$$\lim_{n\to\infty} \widetilde{v}_n(y) = e^{-\frac{y}{d_I}} \ \text{ in } C^1_{\mathrm{loc}}([0,\infty)). \tag{5.14}$$

Integrating (5.13) over $(0, q_n L)$, we get

$$\int_0^{q_nL} \widetilde{v}_n(y)\left\{\beta\left(\frac{y}{q_n}\right)\frac{u_n(y)}{u_n(y)+v_n(y)} - \gamma\left(\frac{y}{q_n}\right)\right\} \mathrm{d}y = 0,\ quad\ \forall n \in \mathbb{N}. \tag{5.15}$$

With the help of (5.10) and (5.14), by sending $n \to \infty$ in (5.15), we can claim that

$$\int_0^\infty e^{-\frac{y}{d_I}}(\beta(0) - \gamma(0))\,\mathrm{d}y = 0, \ \text{ namely, } \ d_I(\beta(0) - \gamma(0)) = 0. \tag{5.16}$$

We now verify the above claim. Due to the decay property of the function $e^{-\frac{y}{d_I}}$ and the fact of

$$\left|\beta\left(\frac{y}{q_n}\right)\frac{u_n(y)}{u_n(y)+v_n(y)} - \gamma\left(\frac{y}{q_n}\right)\right| \le \|\beta\|_\infty + \|\gamma\|_\infty,$$

for any $\varepsilon > 0$, there exists a large integer n_0 such that for all $n \ge n_0$, there holds

$$\left|\int_{n_0L}^{q_nL} \tilde{v}_n(y)\left\{\beta\left(\frac{y}{q_n}\right)\frac{u_n(y)}{u_n(y)+v_n(y)} - \gamma\left(\frac{y}{q_n}\right)\right\}\mathrm{d}y\right| < \varepsilon \tag{5.17}$$

and

$$\left| \int_{n_0 L}^{\infty} e^{-\frac{y}{d_I}} (\beta(0) - \gamma(0))\,\mathrm{d}y \right| < \varepsilon. \tag{5.18}$$

Then it follows that

$$\begin{aligned}
&\left| \int_0^{\infty} e^{-\frac{y}{d_I}} (\beta(0) - \gamma(0))\,\mathrm{d}y - \int_0^{q_n L} \widetilde{v}_n(y) \left\{ \beta\left(\frac{y}{q_n}\right) \frac{u_n(y)}{u_n(y) + v_n(y)} - \gamma\left(\frac{y}{q_n}\right) \right\} \mathrm{d}y \right| \\
&\quad \le \int_0^{n_0 L} \left| e^{-\frac{y}{d_I}} (\beta(0) - \gamma(0)) - \widetilde{v}_n(y) \left\{ \beta\left(\frac{y}{q_n}\right) \frac{u_n(y)}{u_n(y) + v_n(y)} - \gamma\left(\frac{y}{q_n}\right) \right\} \right| \mathrm{d}y \\
&\qquad + \left| \int_{n_0 L}^{q_n L} \widetilde{v}_n(y) \left\{ \beta\left(\frac{y}{q_n}\right) \frac{u_n(y)}{u_n(y) + v_n(y)} - \gamma\left(\frac{y}{q_n}\right) \right\} \mathrm{d}y \right| \\
&\qquad + \left| \int_{n_0 L}^{\infty} e^{-\frac{y}{d_I}} (\beta(0) - \gamma(0))\,\mathrm{d}y \right|.
\end{aligned} \tag{5.19}$$

Thus, for all large n, the second and the third terms in the right-hand side of (5.19) can be estimated as (5.17) and (5.18). For the first term in the right-hand side of (5.19), the integrand converges to 0 pointwisely for $y \in [0, m]$ as $n \to \infty$ due to (5.10) and (5.14). Using the Lebesgue dominated convergence theorem, we know

$$\lim_{n \to \infty} \int_0^{n_0 L} \underbrace{\left| e^{-\frac{y}{d_I}} (\beta(0) - \gamma(0)) - \widetilde{v}_n(y) \left\{ \beta\left(\frac{y}{q_n}\right) \frac{u_n(y)}{u_n(y) + v_n(y)} - \gamma\left(\frac{y}{q_n}\right) \right\} \right|}_{\to 0\ (n \to \infty)} \mathrm{d}y = 0.$$

Since all terms of the right-hand side of (5.19) can be estimated, we conclude that for any $\varepsilon > 0$,

$$\left| \int_0^{\infty} e^{-\frac{y}{d_I}} (\beta(0) - \gamma(0))\,\mathrm{d}y - \int_0^{q_n L} \widetilde{v}_n(y) \left\{ \beta\left(\frac{y}{q_n}\right) \frac{u_n(y)}{u_n(y) + v_n(y)} - \gamma\left(\frac{y}{q_n}\right) \right\} \mathrm{d}y \right| < 3\varepsilon$$

if n is sufficiently large. This means that the integral in (5.15) converges to the integral in (5.16) as $n \to \infty$.

Consequently, we deduce $\beta(0) = \gamma(0)$. This contradicts our assumption (C1). Then it is necessary that $C_I > 0$.

Step 4 (C_S, C_I) satisfies (3.2). By letting $n \to \infty$ in (5.3): $\int_0^{q_n L} (u_n(y) + v_n(y))\,\mathrm{d}y = N$, arguing similarly as to verify (5.16), we know from (5.7) that

$$\int_0^{\infty} \left(C_S e^{-\frac{y}{d_S}} + C_I e^{-\frac{y}{d_I}} \right) \mathrm{d}y = N, \quad \text{that is,} \quad C_S d_S + C_I d_I = N.$$

Integrating (5.2a) or (5.2b) over $(0, q_n L)$ gives

$$\int_0^{q_n L} v_n(y)\left\{\beta\left(\frac{y}{q_n}\right)\frac{u_n(y)}{u_n(y)+v_n(y)} - \gamma\left(\frac{y}{q_n}\right)\right\} \mathrm{d}y = 0, \quad \forall n \in \mathbb{N}.$$

By sending $n \to \infty$ in the above equality, the similar argument as in Step 3 leads to

$$\int_0^{\infty} C_I e^{-\frac{y}{d_I}} \left\{\beta(0)\frac{C_S e^{-\frac{y}{d_S}}}{C_S e^{-\frac{y}{d_S}} + C_I e^{-\frac{y}{d_I}}} - \gamma(0)\right\} \mathrm{d}y = 0.$$

By virtue of $C_I > 0$, this is reduced to

$$\int_0^{\infty} \frac{C_S e^{-(\frac{1}{d_S}+\frac{1}{d_I})y}}{C_S e^{-\frac{y}{d_S}} + C_I e^{-\frac{y}{d_I}}} \mathrm{d}y = \frac{\gamma(0)}{\beta(0)} d_I.$$

So (C_S, C_I) is a positive solution of (3.2).

The proof of Theorem 3.1 is complete. □

6 Profile of EE as $d_S \to 0$: proof of Theorem 3.2

By the transformation

$$\widetilde{S} = e^{\frac{q}{d_S}x} S, \quad \widetilde{I} = e^{\frac{q}{d_I}x} I, \tag{6.1}$$

the system (1.3) is converted to the following one:

$$\begin{cases} d_S \tilde{S}_{xx} - q\tilde{S}_x - \beta(x)\dfrac{e^{-\frac{q}{d_I}x}\tilde{S}\tilde{I}}{e^{-\frac{q}{d_S}x}\tilde{S} + e^{-\frac{q}{d_I}x}\tilde{I}} + \gamma(x)e^{\left(\frac{q}{d_S}-\frac{q}{d_I}\right)x}\tilde{I} = 0, & 0 < x < L, \\ d_I \tilde{I}_{xx} - q\tilde{I}_x + \beta(x)\dfrac{e^{-\frac{q}{d_S}x}\tilde{S}\tilde{I}}{e^{-\frac{q}{d_S}x}\tilde{S} + e^{-\frac{q}{d_I}x}\tilde{I}} - \gamma(x)\tilde{I} = 0, & 0 < x < L, \\ \tilde{S}_x(0) = \tilde{S}_x(L) = 0, \ \tilde{I}_x(0) = \tilde{I}_x(L) = 0, \\ \displaystyle\int_0^L \left[e^{-\frac{q}{d_S}x}\tilde{S} + e^{-\frac{q}{d_I}x}\tilde{I}\right] \mathrm{d}x = N. \end{cases} \tag{6.2}$$

This equivalent system will be used below.

For any EE $(S(x, d_S), I(x, d_S))$ of (1.3)–(1.4), we also make the transformation [different from (5.1)]:

$$u(y, d_S) := d_S S(d_S y, d_S), \qquad v(y, d_S) := I(d_S y, d_S), \qquad 0 \le y \le d_S^{-1} L. \tag{6.3}$$

Thus, for $y \in (0, d_S^{-1} L)$, (1.3)–(1.4) is reduced to the following

$$u_{yy} + q u_y - d_S^2 v(y, d_S)\left\{\frac{\beta(d_S y)u(y, d_S)}{u(y, d_S) + d_S v(y, d_S)} - \gamma(d_S y)\right\} = 0, \tag{6.4a}$$

$$d_I v_{yy} + q d_S v_y + d_S^2 v(y, d_S)\left\{\frac{\beta(d_S y)u(y, d_S)}{u(y, d_S) + d_S v(y, d_S)} - \gamma(d_S y)\right\} = 0 \tag{6.4b}$$

with the boundary conditions

$$u_y + qu = 0 = d_I v_y + q d_S v, \quad y = 0,\ d_S^{-1} L \tag{6.4c}$$

and the conservation law

$$\int_0^{d_S^{-1}L} (u(y, d_S) + d_S v(y, d_S))\,\mathrm{d}y = N. \tag{6.5}$$

With the aid of the above two transformed systems, we are now ready to give the proof of Theorem 3.2.

Proof of Theorem 3.2 Our argument consists of four steps.

Step 1 Estimates of u and v for small d_S. Firstly, from (i) of Lemma 4.3, it immediately follows that $d_S S(x, d_S)$ is uniformly bounded on $[0, L]$ with respect to d_S for $0 < d_S \le 1$. Hence, taking into account the transformation (6.3), for any fixed constant $K > 0$, one can find a positive constant C such that

$$u(y, d_S) \le C, \quad \forall y \in [0, K]. \tag{6.6}$$

Here and in what follows, C can be different but is independent of all $0 < d_S < L/K$.

Given $K > 0$, we next aim to derive the estimate of $\|v(y, d_S)\|_{C([0,K])}$. By (6.1) and (1.4), one sees that

$$e^{-\frac{qL}{d_I}} \int_0^L \tilde{I}(x,\ d_S)\,\mathrm{d}x \le \int_0^L e^{-\frac{q}{d_I}x} \tilde{I}(x,\ d_S)\,\mathrm{d}x = \int_0^L I(x,\ d_S)\,\mathrm{d}x \le N.$$

This therefore implies

$$\min_{x\in[0,L]} \tilde{I}(x,\ d_S) \le \frac{1}{L}\int_0^L \tilde{I}(x,\ d_S)\,\mathrm{d}x \le \frac{N e^{\frac{qL}{d_I}}}{L}. \tag{6.7}$$

On the other hand, since the term

$$\frac{\beta(x) e^{-\frac{q}{d_S}x} \tilde{S}}{e^{-\frac{q}{d_S}x}\tilde{S} + e^{-\frac{q}{d_I}x}\tilde{I}} - \gamma(x)$$

is uniformly bounded on $[0, L]$, independent of all $d_S > 0$. The well-known Harnack inequality for elliptic equations, as applied to the equation of $\tilde{I}(x, d_S)$ in (6.2), tells us that

$$\max_{x\in[0,L]} \tilde{I}(x,\ d_S) \le C \min_{x\in[0,L]} \tilde{I}(x,\ d_S), \quad \forall d_S > 0, \tag{6.8}$$

which also implies that

$$\max_{x\in[0,L]} I(x,\ d_S) \le C \min_{x\in[0,L]} I(x,\ d_S), \quad \forall d_S > 0, \tag{6.9}$$

due to $\tilde{I}(x,\ d_S) = e^{\frac{q}{d_I}x} I(x,\ d_S)$. Thereby, (6.8), together with (6.7), shows that

$$\max_{x\in[0,L]} \tilde{I}(x,\ d_S) \le C \frac{N e^{\frac{qL}{d_I}}}{L}.$$

In light of (6.1), (6.3) and (6.9), there holds

$$\max_{y\in[0,d_S^{-1}L]} v(y,\,d_S)\le C,\quad \max_{x\in[0,L]} I(x,\,d_S)\le C,\quad \forall 0<d_S\le 1. \tag{6.10}$$

Reasoning as Step 1 in the proof of Theorem 3.1, using (6.6) and (6.10), we can conclude from (6.4a) and (6.4c) that for any fixed constant $K>0$, there is a positive constant $C=C(K)$, which does not depend on all $0<d_S<L/K$, such that

$$\|u(y\,,d_S)\|_{C^{1,\alpha}([0,K])}\le C,\quad \|v(y\,,d_S)\|_{C^{1,\alpha}([0,K])}\le C, \tag{6.11}$$

for some constant $\alpha\in(0,1)$.

Step 2 Convergence of $\{(u(y\,,d_S),\,v(y\,,d_S))\}$. Due to (6.11), there is a positive sequence of d_S, denoted by itself without loss of generality, and a nonnegative function $(\hat{u},\hat{v})$ such that $u_{d_S}(y):=u(y,d_S)$ and $v_{d_S}(y):=v(y,d_S)$ satisfy

$$\lim_{d_S\to 0}(u_{d_S},v_{d_S})=(\hat{u},\hat{v})\ \text{ in } C^1_{\text{loc}}([0,\infty))\times C^1_{\text{loc}}([0,\infty)). \tag{6.12}$$

Since (u_{d_S},v_{d_S}) solves (6.4)-(6.5), it is clear to see that $(\hat{u},\hat{v})$ satisfies the following

$$\hat{u}_{yy}+q\hat{u}_y=0,\quad y\in(0,\infty), \tag{6.13a}$$

$$\hat{v}_{yy}=0,\quad y\in(0,\infty) \tag{6.13b}$$

complemented by the boundary conditions

$$\hat{u}_y(0)+q\hat{u}(0)=0=\hat{v}_y(0). \tag{6.13c}$$

Hence, (6.13a), (6.13b) and (6.13c) give $\hat{u}(y)=\hat{C}_S e^{-qy}$ and $\hat{v}(y)=\hat{C}_I$, for $y\ge 0$, with some nonnegative constants $\hat{C}_S$ and $\hat{C}_I$. In addition, Step 2 in the proof of Theorem 3.1 can be easily adapted to show $\hat{C}_S>0$. We now have two cases to handle: $\hat{C}_I=0$ and $\hat{C}_I>0$.

When $\hat{C}_I=0$, it follows from (6.12) that

$$\lim_{d_S\to 0}(u_{d_S},v_{d_S})=(\hat{C}_S e^{-qy},0)\ \text{ in } C^1_{\text{loc}}([0,\infty))\times C^1_{\text{loc}}([0,\infty)). \tag{6.14}$$

Thus, by (6.3) and (6.12), we have $I(0,d_S)\to 0$ as $d_S\to 0$. In turn, by (6.9), we obtain

$$I(x,d_S)\to 0\quad \text{uniformly on } [0,L],\quad \text{as } d_S\to 0.$$

Furthermore, from this, (6.5) and (6.14), one can use the similar argument to that of proving (5.16) to conclude that

$$\begin{aligned} N&=\lim_{d_S\to 0}\int_0^L (S(x,d_S)+I(x,d_S))\,\mathrm{d}x=\lim_{d_S\to 0}\int_0^L S(x,d_S)\,\mathrm{d}x\\ &=\lim_{d_S\to 0}\int_0^{{d_S}^{-1}L} u_{d_S}(y)\,\mathrm{d}y=\int_0^\infty \hat{u}(y)\,\mathrm{d}y=\frac{\hat{C}_S}{q}. \end{aligned}$$

Hence, $\hat{C}_S=Nq$. So this implies

$$\lim_{d_S\to 0}(d_S S(d_S y,d_S))=Nqe^{-qy}\ \text{ in } C^1_{\text{loc}}([0,\infty)).$$

In particular, there holds

$$\lim_{d_S\to 0}(d_S S(0,d_S))=Nq. \tag{6.15}$$

On the other hand, following the proof of (ii) of Lemma 4.2, one can easily verify that the last N in (i) of Lemma 4.3 can be replaced by $\int_0^L I(x, d_S)\,\mathrm{d}x$, that is,

$$\begin{aligned} S(x, d_S) \leq & \frac{qN\left(1+\|\beta\|_\infty \frac{d_S}{q^2}\right)}{d_S\left[1-e^{-\frac{q}{d_S}\left(1+\|\beta\|_\infty \frac{d_S}{q^2}\right)L}\right]} e^{-\frac{q}{d_S}x} \\ & + \frac{\int_0^L I(x, d_S)\,\mathrm{d}x}{q}\left(\|\beta\|_\infty - \min_{x\in[0,L]}\gamma(x)\right)\left(1-e^{-\frac{q}{d_S}x}\right), \quad \forall x \in [0, L]. \end{aligned}$$

Because of $\int_0^L I(x, d_S)\,\mathrm{d}x \to 0$, it then follows that

$$S(x, d_S) \to 0 \text{ locally uniformly on } (0, L], \quad \text{as } d_S \to 0.$$

To complete the proof of Theorem 3.2, it is sufficient to show that $\hat{C}_I > 0$ is impossible. We proceed indirectly and suppose that $\hat{C}_I > 0$ in the rest of our analysis.

When $\hat{C}_I > 0$, from (6.3) and (6.12), it then follows that $I(0, d_S) \to \hat{C}_I > 0$ as $d_S \to 0$. Resorting to the equation satisfied by $I(x, d_S)$, together with (6.9) and the second estimate in (6.10), a standard compactness argument for elliptic equations ensures that, up to a further sequence of $\{d_S\}$, labelled by itself again,

$$I(x, d_S) \to I^*(x) \text{ in } C^1([0, L]), \quad \text{as } d_S \to 0, \tag{6.16}$$

where $I^* \in C^1([0, L])$ and is positive on $[0, L]$.

We now set $S(x^*_{d_S}, d_S) = \min_{x\in[0,L]} S(x, d_S)$. Then $x^*_{d_S} \in [0, L]$. Without loss of generality, we may assume that $x^*_{d_S} \to x^* \in [0, L]$ as $d_S \to 0$.

Step 3 $\lim_{d_S\to 0} S(x^*_{d_S}, d_S) = 0$ *and* $x^*_{d_S} = x^* = L$ *for all small* d_S. To prove the first statement, we shall employ an indirect argument again and suppose that $S(x, d_S) \geq \delta > 0$, $\forall x \in [0, L]$, for all small $d_S > 0$, where the positive constant δ is independent of $d_S > 0$.

With the help of the positive lower bound δ of $S(\cdot, d_S)$, we can use the similar technique used in the proof of Lemma 4.1 to establish a refined upper bound of $S(\cdot, d_S)$ when $d_S > 0$ is small. To reach such an aim, by setting

$$S(x, d_S) = e^{-\frac{q}{d_S}(1-\overline{C}d_S/q^2)x} w(x, d_S) \tag{6.17}$$

with the positive constant $\overline{C}$ to be chosen below, we find that w solves

$$d_S w_{xx} - q\left(1-\frac{2\overline{C}d_S}{q^2}\right)w_x + \left\{\overline{C}\left(\frac{\overline{C}d_S}{q^2}-1\right) - \frac{\beta(x)I}{S+I} + \frac{\gamma(x)I}{S}\right\}w = 0, \quad 0 < x < L, \tag{6.18}$$

coupled with the boundary condition

$$w_x = -\frac{\overline{C}}{q}w < 0, \quad x = 0, L.$$

Take $w(\hat{x}, d_S) = \max_{x\in[0,L]} w(x, d_S)$. Obviously, $\hat{x} \neq L$. Suppose that $0 < \hat{x} < L$. Due to $w_{xx}(\hat{x}, d_S) \leq 0$ and $w_x(\hat{x}, d_S) = 0$, from (6.18) it follows that

$$\overline{C}\left(\frac{\overline{C}d_S}{q^2}-1\right) - \frac{\beta(\hat{x})I(\hat{x}, d_S)}{S(\hat{x}, d_S)+I(\hat{x}, d_S)} + \frac{\gamma(\hat{x})I(\hat{x}, d_S)}{S(\hat{x}, d_S)} \geq 0.$$

Recall that (6.10) remains true and $S(x, d_S) \geq \delta > 0$, $\forall x \in [0, L]$, for all small $d_S > 0$. Thus, one gets

$$\overline{C}\Big(\frac{\overline{C}d_S}{q^2} - 1\Big) + \|\gamma\|_\infty \frac{C}{\delta} \geq 0. \tag{6.19}$$

where C is given as in (6.10). By choosing

$$\overline{C} = 2\|\gamma\|_\infty \frac{C}{\delta} \tag{6.20}$$

and requiring

$$0 < d_S < \frac{q^2}{2\overline{C}}, \tag{6.21}$$

we arrive at a contradiction against (6.19). This then implies that $\hat{x} = 0$ for any $d_S > 0$ satisfying (6.21) with $\overline{C}$ given by (6.20). As a result, for all such d_S, we have $w(0, d_S) = \max_{x\in[0,L]} w(x, d_S)$. In turn, it follows from (6.17) that

$$S(0, d_S) = w(0, d_S) \geq w(x, d_S) = e^{\frac{q}{d_S}(1-\overline{C}d_S/q^2)x} S(x, d_S), \quad \forall x \in [0, L].$$

Consequently, by virtue of this and (4.11), we obtain

$$S(x, d_S) \leq \frac{qN\left(1 + \|\beta\|_\infty \frac{d_S}{q^2}\right)}{d_S\left[1 - e^{-\frac{q}{d_S}\left(1+\|\beta\|_\infty \frac{d_S}{q^2}\right)L}\right]} e^{-\frac{q}{d_S}\left(1-\frac{\overline{C}d_S}{q^2}\right)x}, \quad \forall x \in [0, L] \tag{6.22}$$

for any d_S satisfying (6.21). Therefore, (6.22) shows that $S(x, d_S)$ converges to zero locally uniformly in $(0, L]$ as $d_S \to 0$, contradicting our previous hypothesis that $S(x, d_S) \geq \delta > 0$, $\forall x \in [0, L]$, for all small $d_S > 0$. So it is necessary that $\min_{x\in[0,L]} S(x, d_S) \to 0$ as $d_S \to 0$.

We next prove $x^* = L$. As $\lim_{d_S\to 0}(d_S S(0, d_S)) = \hat{C}_S > 0$ by what was proved in Step 2, clearly $x^* > 0$. Suppose for contradiction that $x^* \in (0, L)$. So $x^*_{d_S} \in (0, L)$ for all small $d_S > 0$. By the definition of $x^*_{d_S}$, $S_{xx}(x^*_{d_S}, d_S) \geq 0$ and $S_x(x^*_{d_S}, d_S) = 0$, which enables us to deduce from (1.3a) that

$$-\frac{\beta(x^*_{d_S})S(x^*_{d_S}, d_S)I(x^*_{d_S}, d_S)}{S(x^*_{d_S}, d_S) + I(x^*_{d_S}, d_S)} + \gamma(x^*_{d_S})I(x^*_{d_S}, d_S) \leq 0,$$

from which we further have

$$I(x^*_{d_S}, d_S) \min_{x\in[0,L]} \gamma(x) \leq \|\beta\|_\infty S(x^*_{d_S}, d_S) \to 0, \ \text{ as } d_S \to 0. \tag{6.23}$$

This is impossible because of (6.16). Thus, $x^* = L$. In fact, by (6.16) and (6.23), one can easily see that $x^*_{d_S} = x^* = L$ for all small $d_S > 0$.

Finally, we will exclude the possibility of $\hat{C}_I > 0$.

Step 4 $\hat{C}_I > 0$ is impossible. In view of (6.16), we may assume that

$$I^*(x) \geq \delta_1 > 0, \quad \forall x \in [0, L]$$

with some positive constant $\delta_1 > 0$, and

$$I(x, d_S) \geq \frac{\delta_1}{2}, \quad \forall x \in [0, L] \tag{6.24}$$

if d_S is sufficiently small.

We now set $w(x, d_S) := d_S S_x(x, d_S) + qS(x, d_S)$. It follows from (1.3a) that

$$w_x(x, d_S) = \frac{\beta(x)SI}{S+I} - \gamma(x)I, \quad x \in (0, L).$$

Since $\{I(x\ , d_S)\}$ is uniformly bounded in $C^1([0, L])$ with respect to $d_S \to 0$ by (6.16), then the above equation implies that $\{w_x(x\ , d_S)\}$ is uniformly bounded in $C([0, L])$ as $d_S \to 0$. By integrating this equation over $(0, x)$, we see that

$$w(x, d_S) = \int_0^x \left[\frac{\beta(y)SI}{S+I} - \gamma(y)I \right] \mathrm{d}y.$$

This ensures that $\{w(x\ , d_S)\}$ is also uniformly bounded in $C([0, L])$, thereby, in $C^1([0, L])$ as $d_S \to 0$. Then the Ascoli–Arzerà theorem enables us to conclude that there exists a sequence $\{d_{S,j}\}$ satisfying $d_{S,j} \to 0$ as $j \to \infty$, such that

$$\lim_{j\to\infty} w(x\ , d_{S,j}) = w^* \ \text{ uniformly in } [0, L] \tag{6.25}$$

for some $w^* \in C([0, L])$. It is noted that $w^*(0) = w^*(L) = 0$ from the boundary condition $w(0, d_S) = w(L, d_S) = 0$ for any $d_S > 0$.

In what follows, we are going to claim that there is a further sequence, say, $\{x_j\} \subset (0, L)$ such that $x_j \to 0$ as $j \to \infty$ and

$$\lim_{j\to\infty} S(x_j, d_{S,j}) = 0. \tag{6.26}$$

To the contrary, suppose that such a sequence $\{x_j\}$ does not exist. Then we can take small $\varepsilon_1 > 0$ and $\kappa_1 > 0$ (which are independent of $d_{S,j}$) such that

$$S(x, d_{S,j}) > \varepsilon_1, \quad \forall x \in [0, \kappa_1], \ \ \forall j \geq 1. \tag{6.27}$$

Therefore, it holds

$$\begin{aligned} d_{S,j} S_x(x, d_{S,j}) + \frac{q}{2} S(x, d_{S,j}) &= w(x, d_{S,j}) - \frac{q}{2} S(x, d_{S,j}) \\ &< w(x, d_{S,j}) - \frac{q\varepsilon_1}{2}, \quad \forall x \in [0, \kappa_1], \ \forall j \geq 1. \end{aligned} \tag{6.28}$$

We know from (6.25) that $w(x\ , d_{S,j}) - \frac{q\varepsilon_1}{2}$ uniformly converges to $w^* - \frac{q\varepsilon_1}{2}$ over $[0, L]$ as $j \to \infty$. Moreover, the boundary condition $w^*(0) = 0$ implies that $w^*(x) - \frac{q\varepsilon_1}{2} < 0$ for all sufficiently small $x \geq 0$. By virtue of (6.28), one can find $\kappa_2 > 0$ (which is independent of $j \geq 1$) such that

$$d_{S,j} S_x(x, d_{S,j}) + \frac{q}{2} S(x, d_{S,j}) < 0, \quad \forall x \in [0, \kappa_2]$$

for all sufficiently large j. This implies that

$$\left(e^{\frac{q}{2d_{S,j}} x} S \right)_x < 0, \quad \forall x \in [0, \kappa_2]$$

for all sufficiently large j.

Integrating the above inequality over $[0, x]$ gives

$$S(x, d_{S,j}) < S(0, d_{S,j}) e^{-\frac{q}{2d_{S,j}} x}, \quad \forall x \in [0, \kappa_2]$$

for all sufficiently large j. Together with (4.11), we get

$$S(x, d_{S,j}) < \frac{qN\left(1+\|\beta\|_\infty \frac{d_{S,j}}{q^2}\right)}{d_{S,j}\left[1-e^{-\frac{q}{d_{S,j}}\left(1+\|\beta\|_\infty \frac{d_{S,j}}{q^2}\right)L}\right]} e^{-\frac{q}{2d_{S,j}}x}, \quad \forall x \in [0, \kappa_2]$$

for all sufficiently large j. However, this contradicts (6.27) when j is sufficiently large. Therefore, we have obtained the desired sequence $\{x_j\} \subset (0, L)$ satisfying $x_j \to 0$ and (6.26).

Next we will show that

$$\lim_{j\to\infty} S(x, d_{S,j}) = 0 \text{ pointwise for each } x \in (0, L]. \tag{6.29}$$

By what was proved in Step 3, we may assume

$$\min_{x\in[0,L]} S(x, d_{S,j}) = S(L, S_{d,j}) \to 0. \tag{6.30}$$

In order to prove (6.29), we proceed indirectly again by supposing that there exist $x^* \in (0, L)$ and $\varepsilon_2 > 0$ (which are independent of j) such that

$$S(x^*, d_{S,j}) > \varepsilon_2, \quad \forall j \geq 1.$$

In addition, we recall the following local profiles of $S(x, d_{S,j})$:

- $S_x(0, d_{S,j}) = -\dfrac{q}{d_{S,j}} S(0, d_{S,j}) < 0, \quad \forall j \geq 1;$
- $S(x_j, d_{S,j}) \to 0$ for some sequence $\{x_j\} \subset (0, L)$ with $x_j \to 0$

and (6.30). As a consequence, one can find a sequence $\{y_j\} \subset (0, x^*)$ of local minimum points of $S(x, d_{S,j})$ satisfying

$$0 < S(y_j, d_{S,j}) \leq S(x_j, d_{S,j}) \to 0, \quad \text{as } j \to \infty.$$

Clearly, $S_x(y_j, d_{S,j}) = 0$ and $S_{xx}(y_j, d_{S,j}) \geq 0$. Thus, we deduce from (1.3a) that

$$I(y_j, d_{S,j})\left[\frac{\beta(y_j)S(y_j, d_{S,j})}{S(y_j, d_{S,j}) + I(y_j, d_{S,j})} - \gamma(y_j)\right] = d_{S,j}S_{xx}(y_j, d_{S,j}) + qS_x(y_j, d_{S,j}) \geq 0.$$

In turn, we have

$$\frac{\beta(y_j)S(y_j, d_{S,j})}{S(y_j, d_{S,j}) + I(y_j, d_{S,j})} - \gamma(y_j) \geq 0, \quad \forall j \geq 1. \tag{6.31}$$

One may also assume that $y_\infty := \lim_{j\to\infty} y_j \in [0, x^*]$. Then, due to (6.24), sending $n \to \infty$ in (6.31) leads to $\gamma(y_\infty) \leq 0$, which is a contradiction. Hence, (6.29) holds.

Now, multiplying (1.3a) by any nonnegative test function $\varphi \in C_0^\infty(L/2, L)$ and integrating by parts, we obtain

$$\int_{L/2}^{L} \left\{ d_{S,j}S(x, d_{S,j})\varphi_{xx}(x) - qS(x, d_{S,j})\varphi_x(x) \right.$$
$$\left. -\left[\frac{\beta(x)S(x, d_{S,j})I(x, d_{S,j})}{S(x, d_{S,j}) + I(x, d_{S,j})} - \gamma(x)I(x, d_{S,j})\right]\varphi(x) \right\} \mathrm{d}x = 0 \tag{6.32}$$

for any $j \geq 1$. By recalling (i) of Lemma 4.3, we notice that $\{S(\cdot, d_{S,j})\}$ is uniformly bounded in $C([L/2, L])$ as $j \to \infty$. Then, letting $j \to \infty$ in (6.32), one can apply the Lebesgue dominated convergence theorem to obtain

$$\int_{L/2}^{L} \gamma(x) I^*(x)\varphi(x)\,\mathrm{d}x = 0.$$

However, this is impossible by the assumption $I^*(x) \geq \delta_1 > 0$ over $[0, L]$ at the beginning of this step.

Consequently, it is necessary that $\hat{C}_I = 0$. The proof of Theorem 3.2 is thus complete. □

7 Profile of EE as $d_I \to 0$: proof of Theorem 3.3

In this section, we present the proof of Theorem 3.3.

Proof of Theorem 3.3 As $q > 0$ is given, for all small $d_I > 0$ satisfying

$$q > d_I^{\frac{1}{2}} \|\beta\|_\infty / \left(\min_{x\in[0,L]} \gamma(x)\right)^{\frac{1}{2}},$$

we have (iii) of Lemma 4.3, from which it is easily checked that $I(x, d_I) \to 0$ locally uniformly in $(0, L]$ as $d_I \to 0$.

We then claim that

$$\int_0^L I(x, d_I)\,\mathrm{d}x \to 0, \quad \text{as } d_I \to 0.$$

From (i) of Lemma 4.3, one first observes that $S(x, d_I) \leq C_0, \ \forall x \in [0, L]$ for some positive constant C_0 which is independent of all small d_I. On the other hand, there holds

$$\int_0^L \left(\frac{\beta(x)S(x, d_I)}{S(x, d_I) + I(x, d_I)} - \gamma(x)\right) I(x, d_I)\,\mathrm{d}x = 0, \quad \forall d_I > 0. \tag{7.1}$$

As

$$\left\| \frac{\beta(x)I(x, d_I)S(x, d_I)}{S(x, d_I) + I(x, d_I)} \right\|_\infty \leq C_0 \|\beta(x)\|_\infty \quad \text{for all small } d_I > 0,$$

and

$$0 < \frac{\beta(x)I(x, d_I)S(x, d_I)}{S(x, d_I) + I(x, d_I)} < \beta(x)I(x, d_I) \to 0$$

for any given $x \in (0, L]$ as $d_I \to 0$, an application of the Lebesgue dominated convergence theorem to (7.1) gives

$$\min_{x\in[0,L]} \gamma(x) \lim_{d_I\to 0} \int_0^L I(x, d_I)\,\mathrm{d}x \leq \lim_{d_I\to 0} \int_0^L \frac{\beta(x)S(x, d_I)}{S(x, d_I) + I(x, d_I)} I(x, d_I)\,\mathrm{d}x = 0,$$

as claimed.

According to (6.1) and (6.2), clearly $\tilde{S}(x, d_I) := \tilde{S}(x)$ solves

$$\begin{cases} -d_S \tilde{S}_{xx} + q\tilde{S}_x = -\dfrac{\beta(x)\tilde{S}(x, d_I)I(x, d_I)}{e^{-\frac{q}{d_S}x}\tilde{S}(x, d_I) + I(x, d_I)} + \gamma(x)e^{\frac{q}{d_S}x} I(x, d_I),\ 0 < x < L, \\ \tilde{S}_x(0, d_I) = \tilde{S}_x(L, d_I) = 0. \end{cases} \tag{7.2}$$

Notice that

$$\begin{aligned} \int_0^L \Big| -\frac{\beta(x)\tilde{S}I}{e^{-\frac{q}{d_S}x}\tilde{S} + I} + \gamma(x)e^{\frac{q}{d_S}x} \Big| \mathrm{d}x &\le (\|\beta\|_\infty + \|\gamma\|_\infty)e^{\frac{qL}{d_S}} \int_0^L I \,\mathrm{d}x \\ &\le N(\|\beta\|_\infty + \|\gamma\|_\infty)e^{\frac{qL}{d_S}}. \end{aligned}$$

Appealing to the L^1-estimate theory for elliptic equations (see [5]), for any given $p \in [1, \infty)$, we obtain from (7.2) that

$$\|\tilde{S}(x\,, d_I)\|_{W^{1,p}(0,L)} \le C_*$$

for some positive constant C_*, which does not depend on any d_I.

Since $W^{1,p}(0, L)$ is compactly embedded into $C([0, L])$ for any given $p > 1$ (see, for instance, [13]), passing to a subsequence of d_I, denoted by itself for convenience, we may assume that $\tilde{S}(\cdot\,, d_I) \to \tilde{w}$ uniformly on $[0, L]$ as $d_I \to 0$, where $\tilde{w} \in C([0, L])$ is nonnegative.

Integrating the equation in (7.2) from 0 to x, we find

$$\begin{aligned} -d_S \tilde{S}_x(x, d_I) &= q\tilde{S}(0, d_I) - q\tilde{S}(x, d_I) \\ &\quad - \int_0^x \Big[\frac{\beta(\tau)\tilde{S}(\tau, d_I)I(\tau, d_I)}{e^{-\frac{q}{d_S}\tau}\tilde{S}(\tau, d_I) + I(\tau, d_I)} - \gamma(\tau)e^{\frac{q}{d_S}\tau} I(\tau, d_I) \Big] \mathrm{d}\tau. \end{aligned} \tag{7.3}$$

Furthermore, it follows that

$$\begin{aligned} &\int_0^L \Big| \frac{\beta(x)\tilde{S}(x, d_I)I(x, d_I)}{e^{-\frac{q}{d_S}x}\tilde{S}(x, d_I) + I(x, d_I)} - \gamma(x)e^{\frac{q}{d_S}x} I(x, d_I) \Big| \mathrm{d}x \\ &\le (\|\beta\|_\infty + \|\gamma\|_\infty)e^{\frac{qL}{d_S}} \int_0^L I(x,\ d_I)\,\mathrm{d}x. \end{aligned}$$

We also recall that $\int_0^L I(x,\ d_I)\,\mathrm{d}x \to 0$ as $d_I \to 0$. Consequently, the integral on the right-hand side in (7.3) converges to zero uniformly on $[0, L]$. Thus, by sending $d_I \to 0$, it is easily seen that $\tilde{w}$ satisfies the following pointwise

$$\begin{cases} -d_S \tilde{w}_x + q\tilde{w} = q\tilde{w}(0), \quad 0 < x < L, \\ \tilde{w}_x(0) = 0. \end{cases} \tag{7.4}$$

Solving (7.4) yields that $\tilde{w}$ is a nonnegative constant. Due to (6.1), we have

$$S(x,\ d_I) \to \tilde{w}e^{-\frac{q}{d_S}x} \ \text{ uniformly on } [0, L], \ \text{ as } d_I \to 0.$$

In view of this fact, $\int_0^L I(x, d_I)\,\mathrm{d}x \to 0$ as $d_I \to 0$ and (1.4), one gets

$$\tilde{w} = \frac{qN}{d_S\left(1 - e^{-\frac{qL}{d_S}}\right)}.$$

This ends the proof of Theorem 3.3. □

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