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Remarks for the axisymmetric Navier–Stokes equations

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Abstract

We are concerned with 3-D incompressible Navier–Stokes equations when the initial data and the domain are cylindrically symmetric. We show that there exists a solution in a weighted space and certain weighted norms of vorticity of the solution remain finite if they are finite initially. Consequently, we can estimate the growth rate of the solution both spatially and temporally.
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1. Introduction

The 3-D incompressible Navier–Stokes equations have been casting many prominent problems in the area of partial differential equations. One of those problems, the regularity of the weak solution of the Navier–Stokes equations has not been solved yet. It is well known that if a initial velocity belongs to L^2 being divergence free there exists a corresponding global weak solution for the Navier–Stokes equations for various types of domains and that for almost every time $t > 0$, the solution is smooth [7]. It is also known that if the weak solution satisfies apriorily a certain integrability condition near a space–time point (x, t), then the solution become regular near the point [6]. However, we do not know yet whether such

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integrability conditions are satisfied in general. But, many efforts have been done in this direction till now and we know that the possible irregular set is small where the regularity of velocity fails [1,3] and that the solution become regular if the initial data and the domain have a certain symmetry. In particular, if the initial data and the domain are cylindrically symmetric and θ component of initial velocity is zero, the solution become regular [8].

In this paper, we consider the problem when the initial data and the domain is cylindrically symmetric in the presence of θ component of velocity. When the domain is distant from the axis of symmetry, a weak solution becomes indeed a strong solution, but it has not been known for a general cylindrically symmetric domain. As pointed out in [2], there is a possibility to obtain a uniform bound of a certain weighted L^p norm of vorticity for a general cylindrically symmetric domain. We shall show that such quantities indeed satisfy a simple differential inequality and thus that they remain finite for positive time if they are finite initially. Consequently, there exists a solution in a certain weighted space and it is possible to estimate the spatial and temporal growth of velocity and vorticity.

2. Axisymmetric Navier–Stokes equations

Let $x \in \mathbf{R}^3$, we denote the typical cylindrical coordinates of x by (r, θ, z). Namely, $r(x) = (x_1^2 + x_2^2)^{\frac{1}{2}}$, $\theta(x) = \tan^{-1}(\frac{x_2}{x_1})$, and $z(x) = x_3$. We also from time to time denote by r, θ, z functions defined as above. We consider several types of domains in this paper. That is, $\mathbf{R}^3$, an axisymmetric Lipschitz bounded domain, and an axisymmetric Lipschitz unbounded domain. From now on, we denote by Ω any of these domains and do not indicate exactly if it is not necessary and $L^p(\Omega)$, $H_0^p(\Omega)$, and $W_0^{k,p}(\Omega)$ will be denoted by L^p, H^p, $W^{k,p}$, respectively. Every integrals are also assumed to be done on Ω unless otherwise stated.

The incompressible Navier–Stokes equations on Ω are

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p + \nu \Delta v + f, \tag{1}$$

$$\nabla \cdot v = 0, \tag{2}$$

$$v = 0 \quad \text{on } \partial\Omega \tag{3}$$

with an initial condition

$$v(\cdot, 0) = v_0. \tag{4}$$

Here $v = (v_1(x,t), v_2(x,t), v_3(x,t))$ is the velocity of the fluid flows, $p = p(x,t)$ the scalar pressure, $f : \Omega \times \mathbf{R}^+ \to \mathbf{R}$ a given external force, and $\nabla \cdot v_0 = 0$. It is well known that there exists a weak solution v of (1)–(4) corresponding to $v_0 \in L^2$ and $f \in L^2_{\mathrm{loc}}[0, \infty; L^2)$ satisfying [4]

$$||v||^2_{L^2}(t) + \nu \int_0^t ||\nabla v||^2_{L^2} \leqslant C\left(||v_0||^2_{L^2} + \left(\int_0^t ||f||_{L^2}\right)^2\right) \tag{5}$$

and

$$\int_0^t ||v||_{L^\infty}(t) < C(v_0, f). \tag{6}$$

From now on, we assume $f = 0$ and $\nu = 1$ for simplicity. With some assumptions on the regularity of f, the arguments of this paper can be applied parallelly.

By an axisymmetric flow, we mean a flow of the form $v = v_r(r,z)e_r + v_\theta(r,z)e_\theta + v_z(r,z)e_z$. Here, e_r, e_θ, e_z are the basis vectors for the cylindrical coordinates. When v_0 is axisymmetric, it is natural to seek an axisymmetric solution for (1)–(4). Then, in terms of cylindrical components, (1) becomes

$$\frac{\partial v_r}{\partial t} + (\tilde{v} \cdot \nabla) v_r = -\partial_r p + \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial z^2}\right) v_r - \frac{1}{r^2} v_r + \frac{v_\theta^2}{r}, \tag{7}$$

$$\frac{\partial v_\theta}{\partial t} + (\tilde{v} \cdot \nabla) v_\theta = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial z^2}\right) v_\theta - \frac{1}{r^2} v_\theta - \frac{v_\theta v_r}{r}, \tag{8}$$

$$\frac{\partial v_z}{\partial t} + (\tilde{v} \cdot \nabla) v_z = -\partial_z p + \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial z^2}\right) v_z. \tag{9}$$

Here, $\tilde{v} = v_r e_r + v_z e_z$ and $p = p(r,z)$ since both sides of the θ component of (1) should be independent of θ. We note that $\nabla \cdot \tilde{v} = 0$. The vorticity of v is defined by $\omega = \nabla \times v$ and in the cylindrical coordinates

$$\omega_r = -\partial_z v_\theta, \quad \omega_\theta = \partial_z v_r - \partial_r v_z, \quad \omega_z = \frac{1}{r}\partial_r(r v_\theta).$$

Taking a curl to (1) we get the equations for ω;

$$\frac{\partial \omega_r}{\partial t} + (\tilde{v} \cdot \nabla)\omega_r = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial z^2}\right)\omega_r - \frac{1}{r^2}\omega_r + (\tilde{\omega} \cdot \nabla) v_r, \tag{10}$$

$$\frac{\partial \omega_\theta}{\partial t} + (\tilde{v} \cdot \nabla)\omega_\theta = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial z^2}\right)\omega_\theta - \frac{1}{r^2}\omega_\theta + \frac{1}{r}\partial_z v_\theta^2 + \frac{v_r \omega_\theta}{r}, \tag{11}$$

$$\frac{\partial \omega_z}{\partial t} + (\tilde{v} \cdot \nabla)\omega_z = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial z^2}\right)\omega_z + (\tilde{\omega} \cdot \nabla) v_z, \tag{12}$$

where $\tilde{\omega} = \omega_r e_r + \omega_z e_z$. We also note that $\nabla \cdot \tilde{\omega} = \nabla \cdot \omega = 0$ since ω is a curl of v. We know that if the domain Ω is bounded and apart from the axis of symmetry and the initial data is smooth, there exists unique global classical solution v for (7)–(9). From now on, we deal with this smooth solution and obtain some uniform estimates and extend it to the weak solution by an approximation argument in the successive sections. We first define $D = rv_\theta$, $E_\alpha = r^\alpha \omega_\theta$, $F_\alpha^1 = r^\alpha \omega_r$, and $F_\alpha^2 = r^\alpha \omega_z$ for $\alpha > 0$. From (8) and (11), D, E_α, and F_α^i, $i = 1, 2$ satisfy the following equations:

$$\frac{\partial D}{\partial t} + (\tilde{v} \cdot \nabla) D = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial z^2} \right) D - \frac{2}{r} \partial_r D, \tag{13}$$

$$\begin{aligned} \frac{\partial E_\alpha}{\partial t} + (\tilde{v} \cdot \nabla) E_\alpha &= \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial z^2} \right) E_\alpha + (\alpha^2 - 1) \frac{1}{r^2} E_\alpha \\ &\quad - \frac{2\alpha}{r} \partial_r E_\alpha + r^{\alpha - 1} \partial_z v_\theta^2 + (1 + \alpha) \frac{v_r}{r} E_\alpha, \end{aligned} \tag{14}$$

$$\begin{aligned} \frac{\partial F_\alpha^1}{\partial t} + (\tilde{v} \cdot \nabla) F_\alpha^1 &= \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial z^2} \right) F_\alpha^1 + (\alpha^2 - 1) \frac{1}{r^2} F_\alpha^1 \\ &\quad - \frac{2\alpha}{r} \partial_r F_\alpha^1 + \tilde{\omega} \cdot \nabla (r^\alpha v_r), \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{\partial F_\alpha^2}{\partial t} + (\tilde{v} \cdot \nabla) F_\alpha^2 &= \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial z^2} \right) F_\alpha^2 + \alpha^2 \frac{1}{r^2} F_\alpha^2 \\ &\quad - \frac{2\alpha}{r} \partial_r F_\alpha^2 + \alpha \left(\frac{v_r}{r} F_\alpha^2 - \frac{v_z}{r} F_\alpha^1 \right) + \tilde{\omega} \cdot \nabla (r^\alpha v_z). \end{aligned} \tag{16}$$

3. Basic estimates

We first denote $\tilde{\Omega} = \Omega \backslash \{r = 0\}$, $L_s = \{x | r(x) < s\}$, $H^{1,p}(\Omega) = \{u | u \in W^{1,p}(\Omega), \nabla \cdot u = 0\}$, and introduce $V_\alpha^p(\Omega)$, the closure of

$$\mathscr{D} = \{u | u \in C_c^\infty(\Omega), \nabla \cdot u = 0\}$$

under the corresponding norm,

$$||u||_{p,\alpha} = ||u||_{L^2} + ||\nabla \times u||_{L^p(r^{\alpha p} dx)}.$$

Due to the Calderon–Zygmund inequality, $V_\alpha^p \subset H_{\text{loc}}^{1,p}(\tilde{\Omega})$ and $u|_{\partial\Omega} = 0$ by the trace theorem. When Ω is bounded, $V_\alpha^p \supset H_0^{1,p}$ since the usual $W^{1,p}$ norm is stronger than the V_α^p norm.

Lemma 1. *For $p>1$ and $\alpha p>\max\{p-2,0\}$, we have $V_\alpha^p(\Omega)=V_\alpha^p(\tilde{\Omega})$.*

Proof. Clearly, $V_\alpha^p(\tilde{\Omega})$ is a closed subspace of $V_\alpha^p(\Omega)$. Let l be a continuous linear functional of $V_\alpha^p(\Omega)$ vanishing on $V_\alpha^p(\tilde{\Omega})$ and q be the Hölder conjugate of p. Then, for any $u\in V_\alpha^p$,

$$(l,u)=\int_\Omega l_1\cdot u\,dx+\int_\Omega l_2\cdot(\nabla\times u)r^{\alpha p}\,dx$$

for some $l_1\in L^2$, $l_2\in L^q(r^{\alpha p}\,dx)$. Indeed, We can identify $V_\alpha^p(\Omega)$ as a closed subspace of the product space $L^2\times L^p(r^{\alpha p}\,dx)$ and extend the linear functional on the whole product space by the Han–Banach theorem, then, applying the Riesz representation theorem, we get the above representation. Since the linear functional vanishes on $V_\alpha^p(\tilde{\Omega})$, there exists a distribution $\pi\in(C_c^\infty(\tilde{\Omega}))'$, $l=\nabla\pi$ by the Hodge decomposition theorem [7]. Since l is also a bounded linear functional on $L^2\times L^p(r^{\alpha p}\,dx)$(In fact, we extend l on $L^2\times L^p(r^{\alpha p}\,dx)$), $\pi\in L^q(r^{-\alpha q}\,dx,\tilde{\Omega})$ by the duality. Now, let $\phi\in\mathscr{D}(\Omega)$, $\xi\in C_c^\infty(\Omega)$, ξ be supported on the cylinder L_2, and $\xi_s(x)=\xi(r/s,\theta,z)$. Then,

$$\begin{aligned}(l,\phi\xi_s)&=\int_\Omega(l_1\phi\xi_s+l_2\nabla\times(\phi\xi_s)r^{\alpha p})\,dx\\&\leqslant C(\|\phi\|_{L^\infty}+\|\nabla\phi\|_{L^\infty})\int(|l_1|+|l_2|r^{\alpha p})|\xi_s|\\&\quad+\ C\|\phi\|_{L^\infty}\|r^\alpha\nabla\xi_s\|_{L^p}\|r^{\alpha p/q}l_2\|_{L^q}\\&\leqslant C(\phi)(s\|\xi\|_{L^2}\|l_1\|_{L^2}+s^{\alpha+2/p}\|r^\alpha\xi\|_{L^p}\|r^{\alpha p/q}l_2\|_{L^q})\\&\quad+\ C(\phi)s^{(\alpha p-p+2)/p}\|r^\alpha\nabla\xi\|_{L^p}\|r^{\alpha p/q}l_2\|_{L^q}.\end{aligned}$$

Thus, if $\alpha p>p-2$, $(l,\phi\xi_s)\to0$ as $s\to0$. Next, for any $\phi\in\mathscr{D}(\Omega)$, we take $\xi\in C_c^\infty$ supported on the cylinder L_2 such that $\xi=1$ on the set $L_1\cap\operatorname{supp}(\phi)$ and calculate

$$\begin{aligned}(l,\phi)&=(l,\phi\xi_s)+(l,\phi(1-\xi_s))=o(1)+\int\pi\phi\cdot\nabla\xi_s\,dx\\&=o(1)+Cs^{(\alpha p-p+2)/p}\|\nabla\xi\|_{L^p}\|\phi\|_{L^\infty}\|r^{-\alpha}\pi\|_{L^q}.\end{aligned}$$

Here, we use the property of l on $\tilde{\Omega}$ and the smallness of l near the origin. We then obtain that if $\alpha p>p-2$, $(l,\phi)=0$ for any $\phi\in V_\alpha^p(\Omega)$ sending s to zero, which proves the lemma. $\square$

Next, we present a version of Grönwall's inequality which we will use later for completeness. We adopt summation convention here.

Lemma 2. *Let* $y:[0,T]\to R^+$ *be continuous and satisfies*

$$\frac{d}{dx}y\leqslant h_i y^{a_i},\quad 1>a_1\geqslant\cdots\geqslant a_n\geqslant 0,\quad 0\leqslant h_i\in L^1(0,T),\quad i=1,\dots,n.$$

Then,

$$y(t)\leqslant\left(y(0)^{1-a_1}+1+C\int_0^t\sum_i h_i\right)^{1/(1-a_1)}. \tag{17}$$

Proof. Consider the region $S=\{y>1\}$. On S,

$$\frac{d}{dx}y\leqslant h_i y^{a_i}\leqslant\sum_i h_i y^{a_1}.$$

Thus, integrating the inequality, we have

$$y^{1-a_1}(t)\leqslant y^{1-a_1}(s)+C\int_s^t\sum_i h_i$$

for any s, t in the same open component of S. On the boundary of S, either $y=1$ or $y=y(0)$. Thus proved. $\square$

Lemma 3. *Let* v *and* ω *as before and* $v\in V^p_\alpha$, $\alpha\geqslant\frac{5p-6}{2p}$. *Denoting* $q=3p/(3-p)$ *if* $2\leqslant p<3$, $2\leqslant q<\infty$ *if* $p=3$, *and* $q=\infty$ *if* $p>3$, *we have*

$$\begin{aligned}&\|r^{\alpha-1}v\|_{L^p},\quad \|r^\alpha v\|_{L^q},\quad \|\nabla(r^\alpha v)\|_{L^p}\\ &\quad\leqslant C(1+\|v\|_{V^p_\alpha}+\|v\|_{L^2})(1+\|v\|_{L^2}+\|v\|_{L^\infty}^{1/2})\end{aligned} \tag{18}$$

if Ω *is bounded. The above inequality also holds for an unbounded* Ω *when* $\alpha=1$ *and* $p=2$.

Proof. We extend v on the whole of $\mathbf{R}^3$ setting $v(x)=0$ if $x\notin\Omega$. We note that the extended v belongs to $V^p_\alpha(\mathbf{R}^3)$. Now, $r^\alpha v$ satisfy the following equations on $\mathbf{R}^3$:

$$\nabla\cdot(r^\alpha v)=\alpha r^{\alpha-1}v_r,$$

$$\nabla\times(r^\alpha v)=r^\alpha\omega-\alpha r^{\alpha-1}e_r\times v.$$

Thus, we have

$$\|r^\alpha v\|_{L^q},\quad \|\nabla(r^\alpha v)\|_{L^p}\leqslant C\|r^\alpha\omega\|_{L^p}+C\|r^{\alpha-1}v\|_{L^p} \tag{19}$$

by the Calderon–Zygmund inequality when Ω is bounded. For an unbounded Ω and $p=2$, $\alpha=1$, (19) still holds since $rv \to 0$ as $|x| \to \infty$ due to the fact $v \in L^2$. Eq. (19) shows the lemma for $\alpha=1$, $p=2$. Now, for Ω bounded, we first consider the case, $p<3$. We note that $r^{\alpha-1}v = (r^\alpha v)^{(\alpha-1)/\alpha} v^{1/\alpha}$ and thus

$$\begin{aligned} ||r^{\alpha-1}v||_{L^p} &\leqslant C||r^\alpha v||_{L^q}^{(\alpha-1)/\alpha} ||v||_{L^a}^{1/\alpha} \\ &\leqslant \varepsilon ||r^\alpha v||_{L^q} + C_\varepsilon ||v||_{L^a} \end{aligned} \tag{20}$$

for any $\varepsilon>0$ and $a = pq/(\alpha q - \alpha p + p)$ by the Young and Hölder's inequality. Since $\alpha \geqslant (5p-6)/2p$, we have $a \leqslant 2$ and thus taking ε small enough and plugging the above inequality into (19), we arrive

$$||r^{\alpha-1}v||_{L^p}, \quad ||r^\alpha v||_{L^q}, \quad ||\nabla(r^\alpha v)||_{L^p} \leqslant C||v||_{V_\alpha^p} + C||v||_{L^2}. \tag{21}$$

Next, when $p \geqslant 3$, we replace q by s and take s large enough in (20) so that $a = ps/(\alpha s - p\alpha + p)$ be close enough to p/α. Since $V_\alpha^p \subset V_\alpha^b$ for any $b<3$,

$$||r^\alpha v||_{L^s} \leqslant C||v||_{V_\alpha^p} + C||v||_{L^2}$$

by (21). Meanwhile,

$$||v||_{L^a} \leqslant C||v||_{L^2}$$

if $a \leqslant 2$ and

$$||v||_{L^a} \leqslant ||v||_{L^2}^{2/a} ||v||_{L^\infty}^{(a-2)/a}$$

if $a>2$. However, $(a-2)/a$ can be made arbitrarily close to $(p-2\alpha)/p$ taking s large enough. Thus, if $a>2$,

$$||v||_{L^a}^{1/\alpha} \leqslant ||v||_{L^2} + ||v||_{L^\infty}^{(a-2)/(a\alpha-2)} \leqslant ||v||_{L^2} + ||v||_{L^\infty}^{1/2}.$$

Therefore, (20) is reduced

$$||r^{\alpha-1}v||_{L^p} \leqslant C(||v||_{V_\alpha^p} + ||v||_{L^2})^{(\alpha-1)/\alpha}(||v||_{L^2} + ||v||_{L^\infty}^{1/2}).$$

This shows (18) for $p>3$. $\square$

Lemma 4. *Let v and ω as before, E any of E_α, F_1, and F_2, $p \geqslant 2$, and $\alpha \geqslant \frac{5p-6}{p}$. For any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that*

$$\int \frac{1}{r^2}|E|^p \leqslant C_\varepsilon ||\omega||_{L^2}^{m_1} ||E||_{L^p}^{m_2} + \varepsilon ||\nabla |E|^{p/2}||_{L^2}^2, \tag{22}$$

where $m_1 = \frac{4p}{2p\alpha - 3p + 6} \leqslant 2$ *and* $m_2 = \frac{2p\alpha - 3p + 2}{2p\alpha - 3p + 6} p \leqslant p$ *and*

$$\int \frac{|v|}{r} |E|^p \leqslant C_\varepsilon ||\omega||_{L^2}^{m_3} ||E||_{L^p}^{m_4} + \varepsilon ||\nabla |E_\alpha|^{p/2}||_{L^2}^2 \tag{23}$$

with $m_3 = \frac{4(\alpha+1)p}{3\alpha p - 3p + 6} \leqslant 2$ *and* $m_4 = \frac{3\alpha p - 3p + 2}{3\alpha p - 3p + 6} p \leqslant p$.

Proof. Let $a = \frac{2}{\alpha} \leqslant 1$ and $0 \leqslant b \equiv \frac{4p - 3pa + 2a}{4} \leqslant p - a$, then

$$\begin{aligned}
\int \frac{1}{r^2} |E|^p &\leqslant C \int |\omega|^a |E|^b |E|^{p-a-b} \\
&\leqslant C ||\omega||_{L^2}^a ||E||_{L^p}^b ||E||_{L^{3p}}^{3(pa-2a)/4} \\
&\leqslant C ||\omega||_{L^2}^a ||E||_{L^p}^b ||\nabla |E|^{p/2}||_{L^2}^{3(pa-2a)/2p} \\
&\leqslant C_\varepsilon ||\omega||_{L^2}^{m_1} ||E||_{L^p}^{m_2} + \varepsilon ||\nabla |E|^{p/2}||_{L^2}^2
\end{aligned}$$

by the Hölder's, Sobolev, and Young's inequality. Here, the Hölder exponents $\frac{2}{a}$, $\frac{p}{b}$, and $\frac{3p}{p-a-b}$ are proper and thus proved. Next, we show (23). With $d = \frac{1}{\alpha} \leqslant 2$ and $e, s \geqslant 0$, we use the Hölder's, Sobolev, Young's inequality and get

$$\begin{aligned}
\int \frac{|v|}{r} |E|^p &= C \int |v| |\omega|^d |E|^{p-d-e} |E|^e \\
&\leqslant C ||v||_{L^s} ||\omega||_{L^2}^d ||E||_{L^p}^{p-d-e} ||E||_{L^{3p}}^e \\
&\leqslant C_\varepsilon (||v||_{L^s} ||\omega||_{L^2}^d ||E||_{L^p}^{p-d-e})^{p/(p-e)} + \varepsilon ||\nabla |E|^{p/2}||_{L^2}^2.
\end{aligned}$$

To make the above formal computation valid, we ask

$$0 \leqslant e \leqslant p - d, \quad 1 \leqslant s, \quad \frac{1}{s} + \frac{d}{2} + \frac{p-d-e}{p} + \frac{e}{3p} = 1. \tag{24}$$

We note $||v||_{L^s} \leqslant ||v||_{L^2}^{(6-s)/2s} ||v||_{L^6}^{(3s-6)/2s}$ when $2 \leqslant s \leqslant 6$ and the exponent of $||E_\alpha||_{L^p}$ is less than or equal to p. Thus, the coefficient of $||E_\alpha||_{L^p}$, $(||v||_{L^s} ||\omega||_{L^2}^d)^{p/(p-e)}$ is integrable if

$$2 \leqslant s \leqslant 6, \quad \left(\frac{3s-6}{2s} + d \right) \frac{p}{p-e} \leqslant 2 \tag{25}$$

by (5). Eqs. (24) and (25) are reduced to

$$\begin{aligned}
&\frac{p + 3pd - 6d}{4} \leqslant e \leqslant \min\left[p - d, \frac{3p + 3pd - 6d}{4} \right], \\
&s = \frac{6p}{4e + 6d - 3pd}, \quad \frac{5p - 6}{p} \leqslant \alpha.
\end{aligned}$$

Since there is always such e and s if $\frac{5p-6}{p} \leqslant \alpha$, we take simply $e = (p + 3pd - 6d)/4$ and $s = 6$ and obtain the exponents m_3 and m_4. $\square$

4. A priori estimates

Lemma 5. *Let D be as before, then*

$$\|D\|_{L^q}(t) \leqslant \|D\|_{L^q}(0) \tag{26}$$

for $1 < q < \infty$. *If we assume further that* $\|D\|_{L^q}(0) < \infty$ *for any* $q > 1$, (26) *holds for* $q = \infty$.

Proof. Applying the Hölder's inequality and divergence theorem to (13) after multiplying $D|D|^{q-2}$ to the equation and integrating it, we have

$$\frac{\partial}{\partial t}\|D\|^q_{L^q} \leqslant -C\|\nabla |D|^{q/2}\|^2_{L^2}.$$

Thus, we have (26) for $1 < q < \infty$ integrating the above equation. If $D(t) \in L^{q_0}$ for any fixed finite $q_0 > 1$, then $\|D\|_{L^q}(t) \to \|D\|_{L^\infty}(\mathrm{t})$ as $q \to \infty$. Since $D(t) \in L^{q_0}$ for all $t > 0$ if $D|_{t=0} \in L^{q_0}$ by (26), we have (26) for $q = \infty$ taking the limit of (26). $\square$

Now, we are ready to give two main lemmas, a priori estimation on weighted norms of vorticity. The basic ingredients are the simpler structure of Eqs. (10)–(12) due to axisymmetry, (5), and (6). We start with E_α.

Lemma 6. *Let* v, ω *as before. For any* $p \geqslant 2$ *and* $\alpha \geqslant \frac{5p-6}{p}$,

$$\|E_\alpha\|^p_{L^p}(t) \leqslant (\|E_\alpha\|^{p-m_4}_{L^p}(0) + 1 + C\int_0^t (G+H))^{p/(p-m_4)}. \tag{27}$$

Here, m_1, m_3, m_4 *are as in the Lemma* 4 *and*

$$G = \|\omega\|^{m_1}_{L^2} + \|\omega\|^{m_3}_{L^2},$$

$$H = \|rv_\theta\|^{4(\alpha-1)/p}_{L^q}\|\omega\|^{4(3-\alpha)/p}_{L^2}, \quad q = \frac{6p(\alpha-1)}{\alpha p - 3p + 6}$$

if $\alpha \leqslant 3$ or

$$H = \|rv_\theta\|^{4(p-1)/p}_{L^\infty}\|v_\theta\|^{2/p}_{L^2}\|r^{\alpha p-3p+2}v_\theta\|^{2/p}_{L^2}$$

if $\alpha > 3$.

Proof. We multiply by $E_\alpha|E_\alpha|^{p-2}$ and integrating (14) in the cylindrical coordinates. Due to the divergence theorem,

$$\int (\tilde{v}\cdot\nabla)E_\alpha E_\alpha|E_\alpha|^{p-2} = \frac{1}{p}\int \nabla\cdot(\tilde{v}|E_\alpha|^p) = 0,$$

$$\begin{aligned}\int \frac{2\alpha}{r}\partial_r E_\alpha E_\alpha|E_\alpha|^{p-2}\,dx &= \frac{2\alpha}{p}\int \partial_r|E_\alpha|^p\,dr\,dz\,d\theta\\ &= -\frac{2\alpha}{p}\int |E_\alpha|^p|_{r=0}\,dz\,d\theta = 0\end{aligned}$$

since E_a is smooth [2]. Applying the Hölder's inequality, we then have

$$\begin{aligned}\frac{\partial}{\partial t}||E_\alpha||_{L^p}^p + C_1||\nabla|E_\alpha|^{p/2}||_{L^2}^2 &\leqslant C_2\int\frac{1}{r^2}|E_\alpha|^p + C_2\int\frac{|v_r|}{r}|E_\alpha|^p\\ &+ C_2\int r^{\alpha-1}|v_\theta|^2|E_\alpha|^{(p-2)/2}\partial_z|E_\alpha|^{p/2} = \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}\tag{28}$$

Here, C_1 and C_2 are fixed constants. We use (22) and (23) with small enough ε to bound I and II and

$$\mathrm{III}\leqslant C\left(\int r^{(\alpha-1)p}|v_\theta|^{2p}\right)^{2/p}||E_\alpha||_{L^p}^{p-2} + \frac{C_1}{2}||\nabla|E_\alpha|^{p/2}||_{L^2}^2$$

by the Young's inequality. Then, we divide it into two cases, $\alpha\leqslant 3$ and $\alpha>3$. In the case of $\alpha\leqslant 3$,

$$\begin{aligned}\int r^{\alpha p-p}|v_\theta|^{2p} &\leqslant ||rv_\theta||_{L^q}^{2\alpha-2}||v_\theta||_{L^6}^{6-2\alpha}\\ &\leqslant C||rv_\theta||_{L^q}^{2\alpha-2}||\omega||_{L^2}^{6-2\alpha}.\end{aligned}$$

Note that all the Hölder exponents, $\frac{3p}{\alpha p-3p+6}$ and $\frac{3}{3-\alpha}$ are proper since $p\leqslant 3$ in this case. For the other case, by the Hölder's inequality

$$\int r^{\alpha p-p}|v_\theta|^{2p}\leqslant ||rv_\theta||_{L^\infty}^{2p-2}||v_\theta||_{L^2}||r^{\alpha p-3p+2}v_\theta||_{L^2}.$$

Note that $\alpha p-3p+2\geqslant 2$ so that we can apply (30). Plugging above results into (28), we have

$$\frac{\partial}{\partial t}||E_\alpha||_{L^p}^p\leqslant C(||\omega||_{L^2}^{m_1}||E_\alpha||_{L^p}^{m_2} + ||\omega||_{L^2}^{m_3}||E_\alpha||_{L^p}^{m_4} + H||E_\alpha||_{L^p}^{p-2}).$$

We note that $m_4 \geqslant m_2 \geqslant p-2$ and apply (17) to the above inequality to obtain (27). $\square$

Let us define $D_\beta = r^\beta v_\theta$ for $\beta > 0$. The equation for D_β is then

$$\frac{\partial D_\beta}{\partial t} + (\tilde{v} \cdot \nabla) D_\beta = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial z^2} \right) D_\beta - \frac{2\beta}{r} \partial_r D_\beta + (\beta^2 - 1) \frac{D_\beta}{r^2} + (\beta - 1) \frac{u_r}{r} D_\beta. \tag{29}$$

This equation is simpler than (14) and we can derive the following corollary by the parallel argument with the above lemma.

Corollary 1. *Provided that* $\beta \geqslant \frac{5p-6}{p}$, $p \geqslant 2$,

$$||D_\beta||^p_{L^p}(t) \leqslant \left(||D_\beta||^{p-m_4}_{L^p}(0) + 1 + C \int_0^t G \right)^{p/(p-m_4)}. \tag{30}$$

Here, G *is as in the Lemma* 6.

We now give a similar estimate for F^i_α, $i = 1, 2$.

Lemma 7. *Let* v, D, G, *and* F^i_α, $i = 1, 2$ *be as before. When* Ω *is bounded*,

$$||F_\alpha||^p_{L^p}(t) \leqslant (||F_\alpha||^{p-m_4}_{L^p}(0) + 1 + C \int_0^t (G + ||D||^2_{L^\infty} (||\tilde{v}||^2_{V^p_{\alpha-1}} + ||\omega||^2_{L^2})))^{p/(p-m_4)} \tag{31}$$

for $p \geqslant 2$ and $\alpha \geqslant \frac{5p-6}{p}$. *For a general* Ω, (31) *holds true for* $p = \alpha = 2$ *with* $||\tilde{v}||^2_{V^p_{\alpha-1}}$ *replaced with* $||\tilde{v}||^2_{V^2_2}$.

Proof. We repeat the similar argument with the proof of Lemma 6. Multiplying (15) and (16) by $|F^i_\alpha|^{p-1} F^i_\alpha$, $i = 1, 2$, respectively, we integrate it and use divergence theorem to get

$$\begin{aligned} \frac{\partial}{\partial t} ||F^i_\alpha||^p_{L^p} \leqslant & - C ||\nabla |F^i_\alpha|^{p/2}||^2_{L^2} + C \int \frac{1}{r^2} |F^i_\alpha|^p \\ & + C \int \frac{|v|}{r} |F_\alpha|^p + C \int \tilde{\omega} \cdot \nabla (r^\alpha \tilde{v}^i) |F^i_\alpha|^{p-2} F^i_\alpha \\ = & \, \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned} \tag{32}$$

Here, we denoted $F_\alpha^1 e_r + F_\alpha^2 e_z$ by F_α and $\tilde{v}^1 = v_r$, $\tilde{v}^2 = v_z$. The terms II and III are dominated as (22) and (23). We only need to show IV can be written as a integrable form to use (17). We use the special structure of $\tilde{\omega}$, that is, $\tilde{\omega} = 1/r(-\partial_z, \partial_r)(rv_\theta)$. Using this fact and applying the divergence theorem, the Hölder's inequality, and (18) to IV, we get

$$\begin{aligned}
\mathrm{IV} &= -C\int v_\theta \nabla(r^\alpha \tilde{v}^i)\cdot\nabla^\perp(|F_\alpha|^{p-2}F_\alpha^i)\\
&\leqslant C\int |rv_\theta|(|\nabla(r^{\alpha-1}\tilde{v}^i)| + |r^{\alpha-2}\tilde{v}^i|)|\nabla(|F_\alpha|^{p-2}F_\alpha^i)|\\
&\leqslant C\|D\|_{L^\infty}(\|\tilde{v}\|_{V^p_{\alpha-1}} + \|\omega\|_{L^2})\|F_\alpha\|_{L^p}^{(p-2)/2}\|\nabla|F_\alpha|^{p/2}\|_{L^2}\\
&\leqslant C_\varepsilon\|D\|^2_{L^\infty}(\|\tilde{v}\|^2_{V^p_{\alpha-1}} + \|\omega\|^2_{L^2})\|F_\alpha\|_{L^p}^{p-2} + \varepsilon\|\nabla|F_\alpha^i|^{p/2}\|^2_{L^2}.
\end{aligned}$$

We take ε small enough and plug the above inequality to (32) and apply (17) as in the Lemma 6 to finish the proof. We point it out that

$$\|r\omega_\theta\|^2_{L^2} \leqslant \|r^2\omega_\theta\|^2_{L^2} + \|\omega_\theta\|^2_{L^2}$$

for the remark for $p = \alpha = 2$. $\square$

5. Main theorem

Theorem 1. *Let $p \geqslant 2$, $\alpha \geqslant \frac{5p-6}{p}$, $v(0) \in L^2$ axisymmetric, $\tilde{v}(0) \in V^p_\alpha$, and $rv_\theta(0) \in L^1 \cap L^\infty$. Let further $r^{\alpha p - 3p + 2}v_\theta(0) \in L^2$ when $\alpha > 3$. Then for any $T > 0$, there exists a solution $v \in L^\infty(0, T; L^2) \cap L^2(0, T; W^{1,2})$ for (1)–(4) satisfying $rv_\theta \in L^q$ for all $1 < q \leqslant \infty$ and $\tilde{v} \in V^p_\alpha$ uniformly in t. Further,*

(A) *For $p = \alpha = 2$, $v \in V^2_2$ uniformly in t if further $v_\theta(0)e_\theta \in V^2_2$.*

(B) *For Ω bounded, $v \in V^p_\alpha$ uniformly in t if further $\alpha \geqslant \frac{5p-6}{p} + 1$, $\tilde{v}(0) \in V^p_{\alpha-1}$, and $v_\theta(0)e_\theta \in V^p_\alpha$.*

Proof. Since $\tilde{v}(0) \in V^p_\alpha$, we can approximate $\tilde{v}(0)$ by $\tilde{v}(0)^j \in C_c^\infty(\tilde{\Omega})$, $j = 1, \ldots, \infty$ due to the Lemma 1. Let also $v^j_\theta(0)(r, z) \in C_c^\infty(\tilde{\Omega})$ approximate $v_\theta(0)$ in L^2 with $\sup_j \|v^j_\theta(0)\|_{L^\infty} \leqslant \|v_\theta(0)\|_{L^\infty}$. We can assume that $\tilde{v}^j(0)$ is axisymmetric without loss of generality. Indeed, otherwise, we can take the average of $\tilde{v}^j_r(0)$, $\tilde{v}^j_z(0)$ on the circle instead of $\tilde{v}^j_r(0)$, $\tilde{v}^j_z(0)$. Due to the Minkowski inequality, the norms of $\tilde{v}^j(0)$ are not affected. For each $v(0)^j = \tilde{v}^j(0) + v_\theta(0)^j e_\theta$, we consider $\Omega^j \equiv \Omega \cap \{\varepsilon < r < R\}$ for a suitable ε, R so that the support of $v(0)^j$ is contained in Ω^j. We solve (1)–(3) for $v(0)^j$ on Ω^j and get the smooth solution v^j. For this solution, the apriori estimates

(26),(30), and (27) holds true and these estimates are independent of Ω. Since $||v^j(0)||_{L^2}$ bounded uniformly, (5), and (6), every terms in the righthand side of (26), (30), and (27) are bounded uniformly and thus the limit, v satisfies $\tilde{v}\in V^p_\alpha$, $v_\theta \in L^q$ uniformly in t. Since $v^j(0)$ converges to $v(0)$ strongly in L^2, v is a solution for $v(0)$. Due to the Lemma 7, (A) is true similarly. To show (B), we argue with (31). In this case, $r^s v_\theta(0) \in L^q$ for all $s \geqslant 1$, $1 < q \leqslant \infty$ since Ω is bounded. Then the condition for (27) replacing α with $\alpha - 1$ are satisfied and we finish the proof. $\square$

Remark. (1) When Ω is bounded, the integrability conditions in the above theorem is satisfied provided $v_\theta(0) \in L^\infty$ and $\omega(0) \in L^p$. We note that even if v is merely of L^2 initially, v satisfies these conditions for almost every positive time [7] and thus the solution is regularized outside the axis of symmetry.

(2) When $\Omega \cap \{r = 0\} = \emptyset$ such as $\Omega = \{x|\, |r - z| > 0\}$, the solution then become globally regular even if the distance between Ω and the axis of symmetry is zero and $v \sim r^{-\alpha} \times L^{3p/(3-p)}$ near the axis of symmetry and $v \sim r^{-1} L^2$ near at infinity if $p = \alpha = 2$ by (18).

(3) From (30), (27), and (31), $||r^\alpha v||_{L^p}$, $||\tilde{v}||_{V^p_\alpha}$, and $||v||_{V^p_\alpha}$ grow polynomially in t [5]. Using (5), we can calculate the rough growth powers. Since $m_1, m_3 \leqslant C$ and $p/(p - m_4) = O(\alpha p)$, $||r^\alpha v_\theta||_{L^p} = t^{O(\alpha)}$, $||\tilde{v}||_{V^p_\alpha} = t^{O(\alpha^2)}$, and $||r^\alpha v||_{L^q} \leqslant C||v||_{V^p_\alpha} = t^{O(\alpha^3)}$ for large p and α. It is interesting that the growth rates are uniformly bounded with respect to p. The author do not know whether the nonlinear dependence on α of the growth rates are due to the technical reasons.

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