Team Reference Document
Team #define true false, TU München
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IO

C++ Input/Output

```
#include <iostream>
#include <iomanip>

using namespace std;

int main()
{
    // Ouput a specific number of digits past the decimal point,
    // in this case 5
    cout.setf(ios::fixed); cout << setprecision(5);
    cout << 100.0/7.0 << endl;
    cout.unsetf(ios::fixed);

    // Output the decimal point and trailing zeros
    cout.setf(ios::showpoint);
    cout << 100.0 << endl;
    cout.unsetf(ios::showpoint);

    // Output a '+' before positive values
    cout.setf(ios::showpos);
    cout << 100 << " " << -100 << endl;
    cout.unsetf(ios::showpos);

    // Output numerical values in hexadecimal
    cout << hex << 100 << " " << 1000 << " " << 10000 << dec << endl;
}
```

Computations

Greates Common Divisor

```
long gcd(long a, long b)
{
if (b == 0)
return a;
else return gcd(b, a % b);
}
```

Binomial Coefficients

```
long binomial(long n, long k)
{
if (k > n - k)
return binomial(n, n - k);

long result = 1;
if (k > n)
return 0;

for (long next = 1; next <= k; ++next)
{
```

```
long cancelled = gcd(result, next);
result = (result / cancelled)*(n - next + 1);
result = result/(next/cancelled);
}

return result;
}
```

Data Structures

Union Find

```
initialize(): for all x, boss[x] = x, rank[x] = 0.

union(x, y)
   a = find(x); b = find(y);
   if (rank(a) < rank(b)) boss[a] = b;
   if (rank(a) > rank(b)) boss[b] = a;
   if (rank(a) == rank(b)) {boss[b] = a; rank[a] += 1;}

find(x)
   if (boss[x] == x] return x;
   boss[x] = find(boss[x]); // path compression
   return boss[x];
```

Math-Stuff

Euclid-Stuff

```
// This is a collection of useful code for solving problems that
// involve modular linear equations. Note that all of the
// algorithms described here work on nonnegative integers.

#include <iostream>
#include <vector>
#include <algorithm>

using namespace std;

typedef vector<int> VI;
typedef pair<int,int> PII;

// return a % b (positive value)
int mod(int a, int b) {
  return ((a%b)+b)%b;
}

// computes gcd(a,b)
int gcd(int a, int b) {
  int tmp;
  while(b){a%=b; tmp=a; a=b; b=tmp;}
  return a;
}

// computes lcm(a,b)
int lcm(int a, int b) {
  return a/gcd(a,b)*b;
}

// returns d = gcd(a,b); finds x,y such that d = ax + by
int extended_euclid(int a, int b, int &x, int &y) {
  int xx = y = 0;
  int yy = x = 1;
  while (b) {
    int q = a/b;
    int t = b; b = a%b; a = t;
    t = xx; xx = x-q*xx; x = t;
    t = yy; yy = y-q*yy; y = t;
  }
  return a;
}

// finds all solutions to ax = b (mod n)
VI modular_linear_equation_solver(int a, int b, int n) {
  int x, y;
  VI solutions;
  int d = extended_euclid(a, n, x, y);
  if (!(b%d)) {
    x = mod (x*(b/d), n);
    for (int i = 0; i < d; i++)
      solutions.push_back(mod(x + i*(n/d), n));
  }
  return solutions;
}

// computes b such that ab = 1 (mod n), returns -1 on failure
int mod_inverse(int a, int n) {
  int x, y;
  int d = extended_euclid(a, n, x, y);
  if (d > 1) return -1;
  return mod(x,n);
}

// Chinese remainder theorem (special case): find z such that
// z % x = a, z % y = b. Here, z is unique modulo M = lcm(x,y).
// Return (z,M). On failure, M = -1.
PII chinese_remainder_theorem(int x, int a, int y, int b) {
  int s, t;
  int d = extended_euclid(x, y, s, t);
  if (a%d != b%d) return make_pair(0, -1);
  return make_pair(mod(s*b*x+t*a*y,x*y)/d, x*y/d);
}

// Chinese remainder theorem: find z such that
// z % x[i] = a[i] for all i. Note that the solution is
// unique modulo M = lcm_i (x[i]). Return (z,M). On
// failure, M = -1. Note that we do not require the a[i]'s
// to be relatively prime.
PII chinese_remainder_theorem(const VI &x, const VI &a) {
  PII ret = make_pair(a[0], x[0]);
  for (int i = 1; i < x.size(); i++) {
    ret = chinese_remainder_theorem(ret.second, ret.first, x[i], a[i]);
    if (ret.second == -1) break;
  }
  return ret;
}

// computes x and y such that ax + by = c; on failure, x = y =-1
void linear_diophantine(int a, int b, int c, int &x, int &y) {
  int d = gcd(a,b);
  if (c%d) {
    x = y = -1;
  } else {
    x = c/d * mod_inverse(a/d, b/d);
    y = (c-a*x)/b;
  }
}

int main() {

  // expected: 2
  cout << gcd(14, 30) << endl;

  // expected: 2 -2 1
  int x, y;
  int d = extended_euclid(14, 30, x, y);
  cout << d << " " << x << " " << y << endl;

  // expected: 95 45
  VI sols = modular_linear_equation_solver(14, 30, 100);
  for (int i = 0; i < (int) sols.size(); i++) cout << sols[i] << " ";
  cout << endl;

  // expected: 8
  cout << mod_inverse(8, 9) << endl;

  // expected: 23 56
  //           11 12
  int xs[] = {3, 5, 7, 4, 6};
  int as[] = {2, 3, 2, 3, 5};
  PII ret = chinese_remainder_theorem(VI (xs, xs+3), VI(as, as+3));
  cout << ret.first << " " << ret.second << endl;
  ret = chinese_remainder_theorem (VI(xs+3, xs+5), VI(as+3, as+5));
  cout << ret.first << " " << ret.second << endl;

  // expected: 5 -15
  linear_diophantine(7, 2, 5, x, y);
  cout << x << " " << y << endl;

}
```

Gauss-Jordan

```
// Gauss-Jordan elimination with full pivoting.
//
// Uses:
//   (1) solving systems of linear equations (AX=B)
//   (2) inverting matrices (AX=I)
//   (3) computing determinants of square matrices
//
// Running time: O(n^3)
//
// INPUT:    a[][] = an nxn matrix
//           b[][] = an nxm matrix
//
// OUTPUT:   X      = an nxm matrix (stored in b[][])
//           A^{-1} = an nxn matrix (stored in a[][])
//           returns determinant of a[][]

#include <iostream>
#include <vector>
#include <cmath>

using namespace std;

const double EPS = 1e-10;

typedef vector<int> VI;
typedef double T;
typedef vector<T> VT;
typedef vector<VT> VVT;

T GaussJordan(VVT &a, VVT &b) {
  const int n = a.size();
  const int m = b[0].size();
  VI irow(n), icol(n), ipiv(n);
  T det = 1;

  for (int i = 0; i < n; i++) {
    int pj = -1, pk = -1;
    for (int j = 0; j < n; j++) if (!ipiv[j])
      for (int k = 0; k < n; k++) if (!ipiv[k])
   if (pj == -1 || fabs(a[j][k]) > fabs(a[pj][pk])) { pj = j; pk = k; }
    if (fabs(a[pj][pk]) < EPS) { cerr << "Matrix is singular." << endl; exit(0); }
    ipiv[pk]++;
    swap(a[pj], a[pk]);
    swap(b[pj], b[pk]);
    if (pj != pk) det *= -1;
    irow[i] = pj;
    icol[i] = pk;

    T c = 1.0 / a[pk][pk];
    det *= a[pk][pk];
    a[pk][pk] = 1.0;
    for (int p = 0; p < n; p++) a[pk][p] *= c;
    for (int p = 0; p < m; p++) b[pk][p] *= c;
    for (int p = 0; p < n; p++) if (p != pk) {
      c = a[p][pk];
      a[p][pk] = 0;
      for (int q = 0; q < n; q++) a[p][q] -= a[pk][q] * c;
      for (int q = 0; q < m; q++) b[p][q] -= b[pk][q] * c;
    }
  }

  for (int p = n-1; p >= 0; p--) if (irow[p] != icol[p]) {
    for (int k = 0; k < n; k++) swap(a[k][irow[p]], a[k][icol[p]]);
  }

  return det;
}

int main() {
  const int n = 4;
  const int m = 2;
  double A[n][n] = { {1,2,3,4},{1,0,1,0},{5,3,2,4},{6,1,4,6} };
  double B[n][m] = { {1,2},{4,3},{5,6},{8,7} };
  VVT a(n), b(n);
  for (int i = 0; i < n; i++) {
    a[i] = VT(A[i], A[i] + n);
    b[i] = VT(B[i], B[i] + m);
  }

  double det = GaussJordan(a, b);

  // expected: 60
  cout << "Determinant: " << det << endl;

  // expected: -0.233333 0.166667 0.133333 0.0666667
  //           0.166667 0.166667 0.333333 -0.333333
  //           0.233333 0.833333 -0.133333 -0.0666667
  //           0.05 -0.75 -0.1 0.2
  cout << "Inverse: " << endl;
  for (int i = 0; i < n; i++) {
    for (int j = 0; j < n; j++)
      cout << a[i][j] << ' ';
    cout << endl;
  }

  // expected: 1.63333 1.3
  //           -0.166667 0.5
  //           2.36667 1.7
  //           -1.85 -1.35
  cout << "Solution: " << endl;
  for (int i = 0; i < n; i++) {
    for (int j = 0; j < m; j++)
      cout << b[i][j] << ' ';
    cout << endl;
  }
}
```

Shortest Paths

Floyd-Warshall

Floyd-Warshall kommt mit negativen Gewichten zurecht. All sources, all targets.

```
procedure FloydWarshallWithPathReconstruction ()
    for k := 1 to n
       for i := 1 to n
          for j := 1 to n
             if (path[i][k] + path[k][j] < path[i][j]) {
               path[i][j] := path[i][k]+path[k][j];
               next[i][j] := next[i][k];
             }

function Path (i,j)
   if path[i][j] equals infinity then
       return "no path";
   int intermediate := next[i][j];
   if intermediate equals 'null' then
       return " ";
   else
       return Path(i,intermediate)
         + intermediate
         + Path(intermediate,j);
```

Dijkstra/Java

```
PriorityQueue<Item> q = new PriorityQueue<Item>();

Item[] index = new Item[n];
for(int i = 0; i < n; ++i)
{
index[i] = new Item(-1, oo);
}

index[start] = new Item(-1, 0);
q.add(new Item(start, 0));

while(!q.isEmpty())
{
Item curr = q.poll();
if(curr.value > index[curr.node].value)
{
continue;
}
/*if(curr.node == end)
{
// Ende
break;
}*/
ArrayList<Item> edges = v.get(curr.node);
for(int i = 0; i < edges.size(); ++i)
{
int nv = edges.get(i).value + curr.value;
int otherNode = edges.get(i).node;
Item oi = index[otherNode];
if(nv < oi.value)
{
oi.value = nv;
oi.node = curr.node;
q.add(new Item(otherNode, nv));
}
}
}
return index;
```

Bellman-Ford/Java

```
static class Item
{public int node;public double value;}

ArrayList<ArrayList<Item>> v = new ArrayList<ArrayList<Item>>(n);
for(int i = 0; i < n; ++i)
{
v.add(new ArrayList<Item>());
}
// Kanten einfuegen:
// v.get(a).add(new Item(b, c));
ArrayDeque<Integer> q = new ArrayDeque<Integer>();
Item[] index = new Item[n];
index[0] = new Item(-1, 0);
for(int i = 1; i < n; ++i)
{
index[i] = new Item(-1, oo);
}

boolean[] inQueue = new boolean[n];
inQueue[0] = true;
int phase = 0;
int nextPhaseStart = -1;
q.add(0);
boolean jackpot = false; // neg cycle
while(!q.isEmpty())
{
int i = q.poll();
inQueue[i] = false;
if(i == nextPhaseStart)
{
phase++;
nextPhaseStart = -1;
}
if(phase == n-1)
{
System.out.format("Case \#%d: Jackpot\n", numCase+1);
jackpot = true;
break;
}
Item it = index[i];
ArrayList<Item> e = v.get(i);
for(int x = 0; x < e.size(); ++x)
{
Item edge = e.get(x);
double nv = edge.value + it.value;
Item other = index[edge.node];
if(nv < other.value)
{
other.value = nv;
if(!inQueue[edge.node])
{
q.add(edge.node);
if(nextPhaseStart == -1)
{
nextPhaseStart = edge.node;
}
inQueue[edge.node] = true;
}
}
}
}
```

Flow

MaxFlow Push-Relabel

```
#include <cmath>
#include <vector>
#include <iostream>
#include <queue>
using namespace std;
typedef long long LL;

struct Edge {
  int from, to, cap, flow, index;
  Edge(int from, int to, int cap, int flow, int index) :
    from(from), to(to), cap(cap), flow(flow), index(index) {}
};

struct PushRelabel {
  int N;
  vector<vector<Edge> > G;
  vector<LL> excess;
  vector<int> dist, active, count;
  queue<int> Q;

  PushRelabel(int N) : N(N), G(N), excess(N), dist(N), active(N), count(2*N) {}

  void AddEdge(int from, int to, int cap) {
    G[from].push_back(Edge(from, to, cap, 0, G[to].size()));
    if (from == to) G[from].back().index++;
    G[to].push_back(Edge(to, from, 0, 0, G[from].size() - 1));
  }

  void Enqueue(int v) {
    if (!active[v] && excess[v] > 0) { active[v] = true; Q.push(v); }
  }

  void Push(Edge &e) {
    int amt = int(min(excess[e.from], LL(e.cap - e.flow)));
    if (dist[e.from] <= dist[e.to] || amt == 0) return;
    e.flow += amt;
    G[e.to][e.index].flow -= amt;
    excess[e.to] += amt;
    excess[e.from] -= amt;
    Enqueue(e.to);
  }

  void Gap(int k) {
    for (int v = 0; v < N; v++) {
      if (dist[v] < k) continue;
      count[dist[v]]--;
      dist[v] = max(dist[v], N+1);
      count[dist[v]]++;
      Enqueue(v);
    }
  }

  void Relabel(int v) {
    count[dist[v]]--;
    dist[v] = 2*N;
    for (int i = 0; i < G[v].size(); i++)
      if (G[v][i].cap - G[v][i].flow > 0)
 dist[v] = min(dist[v], dist[G[v][i].to] + 1);
    count[dist[v]]++;
    Enqueue(v);
  }

  void Discharge(int v) {
    for (int i = 0; excess[v] > 0 && i < G[v].size(); i++) Push(G[v][i]);
    if (excess[v] > 0) {
      if (count[dist[v]] == 1)
 Gap(dist[v]);
      else
 Relabel(v);
    }
  }

  LL GetMaxFlow(int s, int t) {
    count[0] = N-1;
    count[N] = 1;
    dist[s] = N;
    active[s] = active[t] = true;
    for (int i = 0; i < G[s].size(); i++) {
      excess[s] += G[s][i].cap;
      Push(G[s][i]);
    }

    while (!Q.empty()) {
      int v = Q.front();
      Q.pop();
      active[v] = false;
      Discharge(v);
    }

    LL totflow = 0;
    for (int i = 0; i < G[s].size(); i++) totflow += G[s][i].flow;
    return totflow;
  }
};
```

Matching

Max Bipartite Matching

```
#include <vector>

using namespace std;

typedef vector<int> VI;
typedef vector<VI> VVI;

bool FindMatch(int i, const VVI &w, VI &mr, VI &mc, VI &seen) {
  for (int j = 0; j < w[i].size(); j++) {
    if (w[i][j] && !seen[j]) {
      seen[j] = true;
      if (mc[j] < 0 || FindMatch(mc[j], w, mr, mc, seen)) {
        mr[i] = j;
        mc[j] = i;
        return true;
      }
    }
  }
  return false;
}

int BipartiteMatching(const VVI &w, VI &mr, VI &mc) {
  mr = VI(w.size(), -1);
  mc = VI(w[0].size(), -1);

  int ct = 0;
  for (int i = 0; i < w.size(); i++) {
    VI seen(w[0].size());
    if (FindMatch(i, w, mr, mc, seen)) ct++;
  }
  return ct;
}
```

Strings

Suffix Array

```
#include <vector>
#include <iostream>
#include <string>

using namespace std;

struct SuffixArray {
  const int L;
  string s;
  vector<vector<int> > P;
  vector<pair<pair<int,int>,int> > M;

  SuffixArray(const string &s) : L(s.length()), s(s), P(1, vector<int>(L, 0)), M(L) {
    for (int i = 0; i < L; i++) P[0][i] = int(s[i]);
    for (int skip = 1, level = 1; skip < L; skip *= 2, level++) {
      P.push_back(vector<int>(L, 0));
      for (int i = 0; i < L; i++)
   M[i] = make_pair(make_pair(P[level-1][i], i + skip < L ? P[level-1][i + skip] : -1000), i);
      sort(M.begin(), M.end());
      for (int i = 0; i < L; i++)
   P[level][M[i].second] = (i > 0 && M[i].first == M[i-1].first) ? P[level][M[i-1].second] : i;
    }
  }

  vector<int> GetSuffixArray() { return P.back(); }

  // returns the length of the longest common prefix of s[i...L-1] and s[j...L-1]
  int LongestCommonPrefix(int i, int j) {
    int len = 0;
    if (i == j) return L - i;
    for (int k = P.size() - 1; k >= 0 && i < L && j < L; k--) {
      if (P[k][i] == P[k][j]) {
   i += 1 << k;
   j += 1 << k;
   len += 1 << k;
      }
    }
    return len;
  }
};

int main() {

  // bobocel is the 0'th suffix
  //  obocel is the 5'th suffix
  //   bocel is the 1'st suffix
  //    ocel is the 6'th suffix
  //     cel is the 2'nd suffix
  //      el is the 3'rd suffix
  //       l is the 4'th suffix
  SuffixArray suffix("bobocel");
  vector<int> v = suffix.GetSuffixArray();

  // Expected output: 0 5 1 6 2 3 4
  //                  2
  for (int i = 0; i < v.size(); i++) cout << v[i] << " ";
  cout << endl;
  cout << suffix.LongestCommonPrefix(0, 2) << endl;
}
```

Knuth-Morris-Pratt Algorithm

```
/*
Searches for the string w in the string s (of length k). Returns the
0-based index of the first match (k if no match is found). Algorithm
runs in O(k) time.
*/
#include <iostream>
#include <string>
#include <vector>

using namespace std;

typedef vector<int> VI;
void buildTable(string& w, VI& t)
{
  t = VI(w.length());
  int i = 2, j = 0;
  t[0] = -1; t[1] = 0;
  while(i < w.length())
  {
    if(w[i-1] == w[j]) { t[i] = j+1; i++; j++; }
    else if(j > 0) j = t[j];
    else { t[i] = 0; i++; }
  }
}

int KMP(string& s, string& w)
{
  int m = 0, i = 0;
  VI t;

  buildTable(w, t);
  while(m+i < s.length())
  {
    if(w[i] == s[m+i])
    {
      i++;
      if(i == w.length()) return m;
    }
    else
    {
      m += i-t[i];
      if(i > 0) i = t[i];
    }
  }
  return s.length();
}

int main()
{
  string a = (string) "The example above illustrates the general technique for assembling "+
    "the table with a minimum of fuss. The principle is that of the overall search: "+
    "most of the work was already done in getting to the current position, so very "+
    "little needs to be done in leaving it. The only minor complication is that the "+
    "logic which is correct late in the string erroneously gives non-proper "+
    "substrings at the beginning. This necessitates some initialization code.";

  string b = "table";

  int p = KMP(a, b);
  cout << p << ": " << a.substr(p, b.length()) << " " << b << endl;
}
```

Geometry

Geometry/C++

```
// C++ routines for computational geometry.

#include <iostream>
#include <vector>
#include <cmath>
#include <cassert>

using namespace std;

double INF = 1e100;
double EPS = 1e-12;

struct PT {
  double x, y;
  PT() {}
  PT(double x, double y) : x(x), y(y) {}
  PT(const PT &p) : x(p.x), y(p.y)    {}
  PT operator + (const PT &p)  const { return PT(x+p.x, y+p.y); }
  PT operator - (const PT &p)  const { return PT(x-p.x, y-p.y); }
  PT operator * (double c)     const { return PT(x*c,   y*c  ); }
  PT operator / (double c)     const { return PT(x/c,   y/c  ); }
};

double dot(PT p, PT q)     { return p.x*q.x+p.y*q.y; }
double dist2(PT p, PT q)   { return dot(p-q,p-q); }
double cross(PT p, PT q)   { return p.x*q.y-p.y*q.x; }
ostream &operator<<(ostream &os, const PT &p) {
  os << "(" << p.x << "," << p.y << ")";
}

// rotate a point CCW or CW around the origin
PT RotateCCW90(PT p)   { return PT(-p.y,p.x); }
PT RotateCW90(PT p)    { return PT(p.y,-p.x); }
PT RotateCCW(PT p, double t) {
  return PT(p.x*cos(t)-p.y*sin(t), p.x*sin(t)+p.y*cos(t));
}

// project point c onto line through a and b
// assuming a != b
PT ProjectPointLine(PT a, PT b, PT c) {
  return a + (b-a)*dot(c-a, b-a)/dot(b-a, b-a);
}

// project point c onto line segment through a and b
PT ProjectPointSegment(PT a, PT b, PT c) {
  double r = dot(b-a,b-a);
  if (fabs(r) < EPS) return a;
  r = dot(c-a, b-a)/r;
  if (r < 0) return a;
  if (r > 1) return b;
  return a + (b-a)*r;
}

// compute distance from c to segment between a and b
double DistancePointSegment(PT a, PT b, PT c) {
  return sqrt(dist2(c, ProjectPointSegment(a, b, c)));
}

// compute distance between point (x,y,z) and plane ax+by+cz=d
double DistancePointPlane(double x, double y, double z,
                          double a, double b, double c, double d)
```

```
{
  return fabs(a*x+b*y+c*z-d)/sqrt(a*a+b*b+c*c);
}

// determine if lines from a to b and c to d are parallel or collinear
bool LinesParallel(PT a, PT b, PT c, PT d) {
  return fabs(cross(b-a, c-d)) < EPS;
}

bool LinesCollinear(PT a, PT b, PT c, PT d) {
  return LinesParallel(a, b, c, d)
      && fabs(cross(a-b, a-c)) < EPS
      && fabs(cross(c-d, c-a)) < EPS;
}

// determine if line segment from a to b intersects with
// line segment from c to d
bool SegmentsIntersect(PT a, PT b, PT c, PT d) {
  if (LinesCollinear(a, b, c, d)) {
    if (dist2(a, c) < EPS || dist2(a, d) < EPS ||
      dist2(b, c) < EPS || dist2(b, d) < EPS) return true;
    if (dot(c-a, c-b) > 0 && dot(d-a, d-b) > 0 && dot(c-b, d-b) > 0)
      return false;
    return true;
  }
  if (cross(d-a, b-a) * cross(c-a, b-a) > 0) return false;
  if (cross(a-c, d-c) * cross(b-c, d-c) > 0) return false;
  return true;
}

// compute intersection of line passing through a and b
// with line passing through c and d, assuming that unique
// intersection exists; for segment intersection, check if
// segments intersect first
PT ComputeLineIntersection(PT a, PT b, PT c, PT d) {
  b=b-a; d=c-d; c=c-a;
  assert(dot(b, b) > EPS && dot(d, d) > EPS);
  return a + b*cross(c, d)/cross(b, d);
}

// compute center of circle given three points
PT ComputeCircleCenter(PT a, PT b, PT c) {
  b=(a+b)/2;
  c=(a+c)/2;
  return ComputeLineIntersection(b, b+RotateCW90(a-b), c, c+RotateCW90(a-c));
}

// determine if point is in a possibly non-convex polygon (by William
// Randolph Franklin); returns 1 for strictly interior points, 0 for
// strictly exterior points, and 0 or 1 for the remaining points.
// Note that it is possible to convert this into an *exact* test using
// integer arithmetic by taking care of the division appropriately
// (making sure to deal with signs properly) and then by writing exact
// tests for checking point on polygon boundary
bool PointInPolygon(const vector<PT> &p, PT q) {
  bool c = 0;
  for (int i = 0; i < p.size(); i++){
    int j = (i+1)%p.size();
    if ((p[i].y <= q.y && q.y < p[j].y ||
      p[j].y <= q.y && q.y < p[i].y) &&
      q.x < p[i].x + (p[j].x - p[i].x) * (q.y - p[i].y) / (p[j].y - p[i].y))
      c = !c;
  }
  return c;
}

// determine if point is on the boundary of a polygon
bool PointOnPolygon(const vector<PT> &p, PT q) {
  for (int i = 0; i < p.size(); i++)
    if (dist2(ProjectPointSegment(p[i], p[(i+1)%p.size()], q), q) < EPS)
      return true;
    return false;
}

// compute intersection of line through points a and b with
// circle centered at c with radius r > 0
vector<PT> CircleLineIntersection(PT a, PT b, PT c, double r) {
  vector<PT> ret;
  b = b-a;
  a = a-c;
  double A = dot(b, b);
  double B = dot(a, b);
  double C = dot(a, a) - r*r;
  double D = B*B - A*C;
  if (D < -EPS) return ret;
  ret.push_back(c+a+b*(-B+sqrt(D+EPS))/A);
  if (D > EPS)
    ret.push_back(c+a+b*(-B-sqrt(D))/A);
  return ret;
}

// compute intersection of circle centered at a with radius r
// with circle centered at b with radius R
vector<PT> CircleCircleIntersection(PT a, PT b, double r, double R) {
  vector<PT> ret;
  double d = sqrt(dist2(a, b));
  if (d > r+R || d+min(r, R) < max(r, R)) return ret;
  double x = (d*d-R*R+r*r)/(2*d);
  double y = sqrt(r*r-x*x);
  PT v = (b-a)/d;
  ret.push_back(a+v*x + RotateCCW90(v)*y);
  if (y > 0)
    ret.push_back(a+v*x - RotateCCW90(v)*y);
  return ret;
}

// This code computes the area or centroid of a (possibly nonconvex)
// polygon, assuming that the coordinates are listed in a clockwise or
// counterclockwise fashion. Note that the centroid is often known as
// the "center of gravity" or "center of mass".
```

```
double ComputeSignedArea(const vector<PT> &p) {
  double area = 0;
  for(int i = 0; i < p.size(); i++) {
    int j = (i+1) % p.size();
    area += p[i].x*p[j].y - p[j].x*p[i].y;
  }
  return area / 2.0;
}

double ComputeArea(const vector<PT> &p) {
  return fabs(ComputeSignedArea(p));
}

PT ComputeCentroid(const vector<PT> &p) {
  PT c(0,0);
  double scale = 6.0 * ComputeSignedArea(p);
  for (int i = 0; i < p.size(); i++){
    int j = (i+1) % p.size();
    c = c + (p[i]+p[j])*(p[i].x*p[j].y - p[j].x*p[i].y);
  }
  return c / scale;
}

// tests whether or not a given polygon (in CW or CCW order) is simple
bool IsSimple(const vector<PT> &p) {
  for (int i = 0; i < p.size(); i++) {
    for (int k = i+1; k < p.size(); k++) {
      int j = (i+1) % p.size();
      int l = (k+1) % p.size();
      if (i == l || j == k) continue;
      if (SegmentsIntersect(p[i], p[j], p[k], p[l]))
        return false;
    }
  }
  return true;
}

int main() {

  // expected: (-5,2)
  cerr << RotateCCW90(PT(2,5)) << endl;

  // expected: (5,-2)
  cerr << RotateCW90(PT(2,5)) << endl;

  // expected: (-5,2)
  cerr << RotateCCW(PT(2,5),M_PI/2) << endl;

  // expected: (5,2)
  cerr << ProjectPointLine(PT(-5,-2), PT(10,4), PT(3,7)) << endl;

  // expected: (5,2) (7.5,3) (2.5,1)
  cerr << ProjectPointSegment(PT(-5,-2), PT(10,4), PT(3,7)) << " "
       << ProjectPointSegment(PT(7.5,3), PT(10,4), PT(3,7)) << " "
       << ProjectPointSegment(PT(-5,-2), PT(2.5,1), PT(3,7)) << endl;

// expected: 6.78903
cerr << DistancePointPlane(4,-4,3,2,-2,5,-8) << endl;

// expected: 1 0 1
cerr << LinesParallel(PT(1,1), PT(3,5), PT(2,1), PT(4,5)) << " "
     << LinesParallel(PT(1,1), PT(3,5), PT(2,0), PT(4,5)) << " "
     << LinesParallel(PT(1,1), PT(3,5), PT(5,9), PT(7,13)) << endl;

// expected: 0 0 1
cerr << LinesCollinear(PT(1,1), PT(3,5), PT(2,1), PT(4,5)) << " "
     << LinesCollinear(PT(1,1), PT(3,5), PT(2,0), PT(4,5)) << " "
     << LinesCollinear(PT(1,1), PT(3,5), PT(5,9), PT(7,13)) << endl;

// expected: 1 1 1 0
cerr << SegmentsIntersect(PT(0,0), PT(2,4), PT(3,1), PT(-1,3)) << " "
     << SegmentsIntersect(PT(0,0), PT(2,4), PT(4,3), PT(0,5)) << " "
     << SegmentsIntersect(PT(0,0), PT(2,4), PT(2,-1), PT(-2,1)) << " "
     << SegmentsIntersect(PT(0,0), PT(2,4), PT(5,5), PT(1,7)) << endl;

// expected: (1,2)
cerr << ComputeLineIntersection(PT(0,0), PT(2,4), PT(3,1), PT(-1,3)) << endl;

// expected: (1,1)
cerr << ComputeCircleCenter(PT(-3,4), PT(6,1), PT(4,5)) << endl;

vector<PT> v;
v.push_back(PT(0,0));
v.push_back(PT(5,0));
v.push_back(PT(5,5));
v.push_back(PT(0,5));

// expected: 1 1 1 0 0
cerr << PointInPolygon(v, PT(2,2)) << " "
     << PointInPolygon(v, PT(2,0)) << " "
     << PointInPolygon(v, PT(0,2)) << " "
     << PointInPolygon(v, PT(5,2)) << " "
     << PointInPolygon(v, PT(2,5)) << endl;

// expected: 0 1 1 1 1
cerr << PointOnPolygon(v, PT(2,2)) << " "
     << PointOnPolygon(v, PT(2,0)) << " "
     << PointOnPolygon(v, PT(0,2)) << " "
     << PointOnPolygon(v, PT(5,2)) << " "
     << PointOnPolygon(v, PT(2,5)) << endl;

// expected: (1,6)
//           (5,4) (4,5)
//           blank line
//           (4,5) (5,4)
//           blank line
//           (4,5) (5,4)
vector<PT> u = CircleLineIntersection(PT(0,6), PT(2,6), PT(1,1), 5);
for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
u = CircleLineIntersection(PT(0,9), PT(9,0), PT(1,1), 5);
```

```
  for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
  u = CircleCircleIntersection(PT(1,1), PT(10,10), 5, 5);
  for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
  u = CircleCircleIntersection(PT(1,1), PT(8,8), 5, 5);
  for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
  u = CircleCircleIntersection(PT(1,1), PT(4.5,4.5), 10, sqrt(2.0)/2.0);
  for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
  u = CircleCircleIntersection(PT(1,1), PT(4.5,4.5), 5, sqrt(2.0)/2.0);
  for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;

  // area should be 5.0
  // centroid should be (1.1666666, 1.166666)
  PT pa[] = { PT(0,0), PT(5,0), PT(1,1), PT(0,5) };
  vector<PT> p(pa, pa+4);
  PT c = ComputeCentroid(p);
  cerr << "Area: " << ComputeArea(p) << endl;
  cerr << "Centroid: " << c << endl;

  return 0;
}
```

Geometry/Java

```
P cross(P o)
{
return new P(y*o.z-z*o.y, z*o.x-x*o.z, x*o.y-y*o.x);
}

P scalar(P o)
{
return new P(x*o.x, y * o.y, z * o.z);
}

P r90()
{
return new P(-y, x, z);
}

P parallel(P p)
{
return cross(zeroOne).cross(p);
}

Point2D getPoint()
{
return new Point2D.Double(x / z, y / z);
}

static double computePolygonArea(ArrayList<Point2D.Double> points) {
Point2D.Double[] pts = points.toArray(new Point2D.Double[points.size()]);
double area = 0;
for (int i = 0; i < pts.length; i++){
int j = (i+1) % pts.length;
area += pts[i].x * pts[j].y - pts[j].x * pts[i].y;
}
return Math.abs(area)/2;
}
```

Graham Scan – Konvexe Huelle

1. Finde p_0 mit min y, Unentschieden: betrachte x
2. Sortiere $p_{1...n}$. $p_i < p_j = ccw(p_0, p_i, p_j)$
   (colinear → naechster zuerst)
3. Setze $p_{n+1} = p_0$
4. Push(p_0); Push(p_1); Push(p_2);
5. for $i = 3$ to $n + 1$
   (a) Solange Winkel der letzten zwei des Stacks und p_i rechtskurve: Pop()
   (b) Push(p_i)

```
int minPoint = 0;
for(int i = 1; i < n; ++i)
{
if(points[i].y < points[minPoint].y || (points[i].y == points[minPoint].y && points[i].
{
minPoint = i;
}
}
final int mx = points[minPoint].x;
final int my = points[minPoint].y;
Arrays.sort(points, new Comparator<Point>()
{
@Override
public int compare(Point a, Point b) {
int ccw = Line2D.relativeCCW(mx, my, a.x, a.y, b.x, b.y);
if(ccw == 0 || Line2D.relativeCCW(mx, my, b.x, b.y, a.x, a.y) == 0)
{
// gleich...
double d1 = a.distance(mx, my);
double d2 = b.distance(mx, my);
if((d2 < d1 && d2 != 0) || d1 == 0)
{
return 1;
}else
{
return -1;
}
}else if(ccw == 1)
{
// clockwise... -> zuerst b -> a > b
return 1;
}else if(ccw == -1)
{
return -1;
}else
{
System.out.println("shouldnt happen");
System.exit(1);
}
// return 0;
return 0;
}
});

ArrayList<Integer> stack = new ArrayList<Integer>();
stack.add(n-1);
for(int i = 0; i < n; ++i)
{
if(stack.size() < 2)
{
stack.add(i);
continue;
}
int last = stack.get(stack.size() - 1);
int l2 = stack.get(stack.size() - 2);
int ccw = Line2D.relativeCCW(points[l2].x, points[l2].y, points[last].x, points[last].y, points[i].x, points[i].y);
if(ccw != -1)
{
// clockwise oder gleiche Linie
stack.remove(stack.size() - 1);
i--;
}else
{
stack.add(i);
}
}
```

Trees

Binary Indexed Tree

```
//binary indexed tree
//verwaltet kumultative Summen in log(n)

int tree[1<<N];
int MaxVal = (1<<N)-1;

int readsum(int idx){//sum_{i in [1;idx]} f[i]
   int sum = 0;
   while (idx > 0){
      sum += tree[idx];
      idx -= (idx & -idx);
   }
   return sum;
}

int suminrange(int a, int b) { //sum_{i in [a;b[} f[i]
   return readsum(b-1)-readsum(a-1);
}

void update(int idx ,int val){ //updates f[idx]->val
   while (idx <= MaxVal){
      tree[idx] += val;
      idx += (idx & -idx);
   }
}
```

Segment Tree- TODO

TODO

KD-tree

```
// ----------------------------------------------------------------------
// A straightforward, but probably sub-optimal KD-tree implmentation that's
// probably good enough for most things (current it's a 2D-tree)
//
//  - constructs from n points in O(n lg^2 n) time
//  - handles nearest-neighbor query in O(lg n) if points are well distributed
//  - worst case for nearest-neighbor may be linear in pathological case
//
// Sonny Chan, Stanford University, April 2009
// ----------------------------------------------------------------------

#include <iostream>
#include <vector>
#include <limits>
#include <cstdlib>

using namespace std;

// number type for coordinates, and its maximum value
typedef long long ntype;
const ntype sentry = numeric_limits<ntype>::max();

// point structure for 2D-tree, can be extended to 3D
struct point {
    ntype x, y;
    point(ntype xx = 0, ntype yy = 0) : x(xx), y(yy) {}
};

bool operator==(const point &a, const point &b)
{
    return a.x == b.x && a.y == b.y;
}

// sorts points on x-coordinate
bool on_x(const point &a, const point &b)
{
    return a.x < b.x;
}

// sorts points on y-coordinate
bool on_y(const point &a, const point &b)
{
    return a.y < b.y;
}

// squared distance between points
ntype pdist2(const point &a, const point &b)
{
    ntype dx = a.x-b.x, dy = a.y-b.y;
    return dx*dx + dy*dy;
```

```
}

// bounding box for a set of points
struct bbox
{
    ntype x0, x1, y0, y1;

    bbox() : x0(sentry), x1(-sentry), y0(sentry), y1(-sentry) {}

    // computes bounding box from a bunch of points
    void compute(const vector<point> &v) {
        for (int i = 0; i < v.size(); ++i) {
            x0 = min(x0, v[i].x);   x1 = max(x1, v[i].x);
            y0 = min(y0, v[i].y);   y1 = max(y1, v[i].y);
        }
    }

    // squared distance between a point and this bbox, 0 if inside
    ntype distance(const point &p) {
        if (p.x < x0) {
            if (p.y < y0)       return pdist2(point(x0, y0), p);
            else if (p.y > y1)  return pdist2(point(x0, y1), p);
            else                return pdist2(point(x0, p.y), p);
        }
        else if (p.x > x1) {
            if (p.y < y0)       return pdist2(point(x1, y0), p);
            else if (p.y > y1)  return pdist2(point(x1, y1), p);
            else                return pdist2(point(x1, p.y), p);
        }
        else {
            if (p.y < y0)       return pdist2(point(p.x, y0), p);
            else if (p.y > y1)  return pdist2(point(p.x, y1), p);
            else                return 0;
        }
    }
};

// stores a single node of the kd-tree, either internal or leaf
struct kdnode
{
    bool leaf;      // true if this is a leaf node (has one point)
    point pt;       // the single point of this is a leaf
    bbox bound;     // bounding box for set of points in children

    kdnode *first, *second; // two children of this kd-node

    kdnode() : leaf(false), first(0), second(0) {}
    ~kdnode() { if (first) delete first; if (second) delete second; }

    // intersect a point with this node (returns squared distance)
    ntype intersect(const point &p) {
        return bound.distance(p);
    }

    // recursively builds a kd-tree from a given cloud of points
    void construct(vector<point> &vp)
    {
        // compute bounding box for points at this node
        bound.compute(vp);

        // if we're down to one point, then we're a leaf node
        if (vp.size() == 1) {
            leaf = true;
            pt = vp[0];
        }
        else {
            // split on x if the bbox is wider than high (not best heuristic...)
            if (bound.x1-bound.x0 >= bound.y1-bound.y0)
                sort(vp.begin(), vp.end(), on_x);
            // otherwise split on y-coordinate
            else
                sort(vp.begin(), vp.end(), on_y);

            // divide by taking half the array for each child
            // (not best performance if many duplicates in the middle)
            int half = vp.size()/2;
            vector<point> vl(vp.begin(), vp.begin()+half);
            vector<point> vr(vp.begin()+half, vp.end());
            first = new kdnode();   first->construct(vl);
            second = new kdnode();  second->construct(vr);
        }
    }
};

// simple kd-tree class to hold the tree and handle queries
struct kdtree
{
    kdnode *root;

    // constructs a kd-tree from a points (copied here, as it sorts them)
    kdtree(const vector<point> &vp) {
        vector<point> v(vp.begin(), vp.end());
        root = new kdnode();
        root->construct(v);
    }
    ~kdtree() { delete root; }

    // recursive search method returns squared distance to nearest point
    ntype search(kdnode *node, const point &p)
    {
        if (node->leaf) {
            // commented special case tells a point not to find itself
//            if (p == node->pt) return sentry;
//            else
                return pdist2(p, node->pt);
        }

        ntype bfirst = node->first->intersect(p);
        ntype bsecond = node->second->intersect(p);
        // choose the side with the closest bounding box to search first
        // (note that the other side is also searched if needed)
        if (bfirst < bsecond) {
            ntype best = search(node->first, p);
            if (bsecond < best)
                best = min(best, search(node->second, p));
            return best;
        }
        else {
            ntype best = search(node->second, p);
            if (bfirst < best)
                best = min(best, search(node->first, p));
            return best;
        }
    }

    // squared distance to the nearest
    ntype nearest(const point &p) {
        return search(root, p);
    }
};

// ------------------------------------------------------------------------
// some basic test code here

int main()
{
    // generate some random points for a kd-tree
    vector<point> vp;
    for (int i = 0; i < 100000; ++i) {
        vp.push_back(point(rand()%100000, rand()%100000));
    }
    kdtree tree(vp);

    // query some points
    for (int i = 0; i < 10; ++i) {
        point q(rand()%100000, rand()%100000);
        cout << "Closest squared distance to (" << q.x << ", " << q.y << ")"
             << " is " << tree.nearest(q) << endl;
    }

    return 0;
}

// ------------------------------------------------------------------------
```

Misc

Simulated Annealing

```
Random r = new Random();
int numChanges = 0;
double T = 10000;
double alpha = 0.99;
int decreaseAfter = 20;
int nChanges = 0;
for(int i = 0; i < 1000000; ++i)
{
// calculate newCost (apply 2-opt-step) (swap two things)
double delta = newCost - cost;
boolean accept = newCost <= cost;
if(!accept)
{
double R = r.nextDouble();
double calc = Math.exp(-delta / T);
double maxDiff = Math.exp(-10000/T);
if(calc < maxDiff && i < 1000000/2)
{
calc = maxDiff;
}
//System.out.println(calc);
if(calc > R)
{
accept = true;
}
}
if(i % 10000 == 0)
{
// System.out.println("after " + i + ": " + T);
}

if(nChanges >= decreaseAfter)
{
nChanges = 0;
T = alpha * T;
}
if(accept)
{
cost = newCost;
numChanges++;
nChanges++;
}else
{
// swap back
swap(trip, a, b);
}
}
```

Theoretical Computer Science Cheat Sheet

Definitions

$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)$ $\forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0$ $\forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim\limits_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in \mathbb{R}$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \le s$, $\forall s \in S$.
$\liminf\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\limsup\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[{n \atop k}\right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{{n \atop k}\right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle{n \atop k}\right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\ldots\pi_n$ on $\{1, 2, \ldots, n\}$ with k ascents.
$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \qquad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}, \qquad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \qquad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

1. $\displaystyle \binom{n}{k} = \frac{n!}{(n-k)!k!}$, **2.** $\displaystyle \sum_{k=0}^{n}\binom{n}{k} = 2^n$, **3.** $\displaystyle \binom{n}{k} = \binom{n}{n-k}$,

4. $\displaystyle \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, **5.** $\displaystyle \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

6. $\displaystyle \binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, **7.** $\displaystyle \sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

8. $\displaystyle \sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, **9.** $\displaystyle \sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

10. $\displaystyle \binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, **11.** $\displaystyle \left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$,

12. $\displaystyle \left\{{n \atop 2}\right\} = 2^{n-1} - 1$, **13.** $\displaystyle \left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$,

14. $\displaystyle \left[{n \atop 1}\right] = (n-1)!$, **15.** $\displaystyle \left[{n \atop 2}\right] = (n-1)!H_{n-1}$, **16.** $\displaystyle \left[{n \atop n}\right] = 1$, **17.** $\displaystyle \left[{n \atop k}\right] \ge \left\{{n \atop k}\right\}$,

18. $\displaystyle \left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, **19.** $\displaystyle \left\{{n \atop n-1}\right\} = \left[{n \atop n-1}\right] = \binom{n}{2}$, **20.** $\displaystyle \sum_{k=0}^{n}\left[{n \atop k}\right] = n!$, **21.** $\displaystyle C_n = \frac{1}{n+1}\binom{2n}{n}$,

22. $\displaystyle \left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$, **23.** $\displaystyle \left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-1-k}\right\rangle$, **24.** $\displaystyle \left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$,

25. $\displaystyle \left\langle{0 \atop k}\right\rangle = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases}$ **26.** $\displaystyle \left\langle{n \atop 1}\right\rangle = 2^n - n - 1$, **27.** $\displaystyle \left\langle{n \atop 2}\right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

28. $\displaystyle x^n = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{x+k}{n}$, **29.** $\displaystyle \left\langle{n \atop m}\right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, **30.** $\displaystyle m!\left\{{n \atop m}\right\} = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{k}{n-m}$,

31. $\displaystyle \left\langle{n \atop m}\right\rangle = \sum_{k=0}^{n}\left\{{n \atop k}\right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, **32.** $\displaystyle \left\langle\!\left\langle{n \atop 0}\right\rangle\!\right\rangle = 1$, **33.** $\displaystyle \left\langle\!\left\langle{n \atop n}\right\rangle\!\right\rangle = 0 \quad \text{for } n \ne 0$,

34. $\displaystyle \left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$, **35.** $\displaystyle \sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$,

36. $\displaystyle \left\{{x \atop x-n}\right\} = \sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle\binom{x+n-1-k}{2n}$, **37.** $\displaystyle \left\{{n+1 \atop m+1}\right\} = \sum_{k}\binom{n}{k}\left\{{k \atop m}\right\} = \sum_{k=0}^{n}\left\{{k \atop m}\right\}(m+1)^{n-k}$,

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Identities Cont.

38. $\begin{bmatrix} n+1\\ m+1\end{bmatrix} = \sum_k \begin{bmatrix} n\\ k\end{bmatrix}\binom{k}{m} = \sum_{k=0}^{n} \begin{bmatrix} k\\ m\end{bmatrix} n^{\underline{n-k}} = n!\sum_{k=0}^{n} \frac{1}{k!}\begin{bmatrix} k\\ m\end{bmatrix},$

39. $\begin{bmatrix} x\\ x-n\end{bmatrix} = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \binom{x+k}{2n},$

40. $\begin{Bmatrix} n\\ m\end{Bmatrix} = \sum_k \binom{n}{k}\begin{Bmatrix} k+1\\ m+1\end{Bmatrix}(-1)^{n-k},$

41. $\begin{bmatrix} n\\ m\end{bmatrix} = \sum_k \begin{bmatrix} n+1\\ k+1\end{bmatrix}\binom{k}{m}(-1)^{m-k},$

42. $\begin{Bmatrix} m+n+1\\ m\end{Bmatrix} = \sum_{k=0}^{m} k\begin{Bmatrix} n+k\\ k\end{Bmatrix},$

43. $\begin{bmatrix} m+n+1\\ m\end{bmatrix} = \sum_{k=0}^{m} k(n+k)\begin{bmatrix} n+k\\ k\end{bmatrix},$

44. $\binom{n}{m} = \sum_k \begin{Bmatrix} n+1\\ k+1\end{Bmatrix}\begin{bmatrix} k\\ m\end{bmatrix}(-1)^{m-k},$

45. $(n-m)!\binom{n}{m} = \sum_k \begin{bmatrix} n+1\\ k+1\end{bmatrix}\begin{Bmatrix} k\\ m\end{Bmatrix}(-1)^{m-k}, \quad \text{for } n \ge m,$

46. $\begin{Bmatrix} n\\ n-m\end{Bmatrix} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\begin{bmatrix} m+k\\ k\end{bmatrix},$

47. $\begin{bmatrix} n\\ n-m\end{bmatrix} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\begin{Bmatrix} m+k\\ k\end{Bmatrix},$

48. $\begin{Bmatrix} n\\ \ell+m\end{Bmatrix}\binom{\ell+m}{\ell} = \sum_k \begin{Bmatrix} k\\ \ell\end{Bmatrix}\begin{Bmatrix} n-k\\ m\end{Bmatrix}\binom{n}{k},$

49. $\begin{bmatrix} n\\ \ell+m\end{bmatrix}\binom{\ell+m}{\ell} = \sum_k \begin{bmatrix} k\\ \ell\end{bmatrix}\begin{bmatrix} n-k\\ m\end{bmatrix}\binom{n}{k}.$

Trees

Every tree with n vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \ldots, d_n$:

$$\sum_{i=1}^{n} 2^{-d_i} \le 1,$$

and equality holds only if every internal node has 2 sons.

Recurrences

Master method:

$$T(n) = aT(n/b) + f(n), \quad a \ge 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then

$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then

$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \le cf(n)$ for large n, then

$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence

$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have

$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get

$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find

$$u_{i+1} = \tfrac{1}{2} + u_i, \qquad u_1 = \tfrac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$. Summing factors (example): Consider the following recurrence

$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side

$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"

$$\begin{aligned} 1\big(T(n) - 3T(n/2) &= n\big)\\ 3\big(T(n/2) - 3T(n/4) &= n/2\big)\\ \vdots \quad \vdots \quad &\vdots\\ 3^{\log_2 n - 1}\big(T(2) - 3T(1) &= 2\big)\end{aligned}$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get

$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\tfrac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have

$$\begin{aligned} n\sum_{i=0}^{m-1} c^i &= n\left(\frac{c^m - 1}{c - 1}\right)\\ &= 2n(c^{\log_2 n} - 1)\\ &= 2n(c^{(k-1)\log_c n} - 1)\\ &= 2n^k - 2n,\end{aligned}$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider

$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that

$$T_{i+1} = 1 + \sum_{j=0}^{i} T_j.$$

Subtracting we find

$$\begin{aligned} T_{i+1} - T_i &= 1 + \sum_{j=0}^{i} T_j - 1 - \sum_{j=0}^{i-1} T_j\\ &= T_i.\end{aligned}$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i.
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i.

Example:

$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:

$$\sum_{i\ge0} g_{i+1}x^i = \sum_{i\ge0} 2g_i x^i + \sum_{i\ge0} x^i.$$

We choose $G(x) = \sum_{i\ge0} x^i g_i$. Rewrite in terms of $G(x)$:

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i\ge0} x^i.$$

Simplify:

$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:

$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:

$$\begin{aligned} G(x) &= x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right)\\ &= x\left(2\sum_{i\ge0} 2^i x^i - \sum_{i\ge0} x^i\right)\\ &= \sum_{i\ge0} (2^{i+1} - 1)x^{i+1}.\end{aligned}$$

So $g_i = 2^i - 1$.

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$\pi \approx 3.14159$, $e \approx 2.71828$, $\gamma \approx 0.57721$, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$, $\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$

i	2^i	p_i
1	2	2
2	4	3
3	8	5
4	16	7
5	32	11
6	64	13
7	128	17
8	256	19
9	512	23
10	1,024	29
11	2,048	31
12	4,096	37
13	8,192	41
14	16,384	43
15	32,768	47
16	65,536	53
17	131,072	59
18	262,144	61
19	524,288	67
20	1,048,576	71
21	2,097,152	73
22	4,194,304	79
23	8,388,608	83
24	16,777,216	89
25	33,554,432	97
26	67,108,864	101
27	134,217,728	103
28	268,435,456	107
29	536,870,912	109
30	1,073,741,824	113
31	2,147,483,648	127
32	4,294,967,296	131

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

General

Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},$$
$$B_6 = \tfrac{1}{42},\ B_8 = -\tfrac{1}{30},\ B_{10} = \tfrac{5}{66}.$$

Change of base, quadratic formula:

$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Euler's number e:

$$e = 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots$$
$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$
$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$
$$\left(1 + \tfrac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$$

Harmonic numbers:

$$1,\ \tfrac{3}{2},\ \tfrac{11}{6},\ \tfrac{25}{12},\ \tfrac{137}{60},\ \tfrac{49}{20},\ \tfrac{363}{140},\ \tfrac{761}{280},\ \tfrac{7129}{2520}, \ldots$$

$$\ln n < H_n < \ln n + 1,$$
$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Factorial, Stirling's approximation:

1, 2, 6, 24, 120, 720, 5040, 40320, 362880, ...

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

Ackermann's function and inverse:

$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$$
$$\alpha(i) = \min\{j \mid a(j,j) \geq i\}.$$

Binomial distribution:

$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \qquad q = 1 - p,$$
$$\mathrm{E}[X] = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} = np.$$

Poisson distribution:

$$\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \mathrm{E}[X] = \lambda.$$

Normal (Gaussian) distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad \mathrm{E}[X] = \mu.$$

The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is

$$nH_n.$$

Probability

Continuous distributions: If

$$\Pr[a < X < b] = \int_a^b p(x)\,dx,$$

then p is the probability density function of X. If

$$\Pr[X < a] = P(a),$$

then P is the distribution function of X. If P and p both exist then

$$P(a) = \int_{-\infty}^{a} p(x)\,dx.$$

Expectation: If X is discrete

$$\mathrm{E}[g(X)] = \sum_x g(x) \Pr[X = x].$$

If X continuous then

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dP(x).$$

Variance, standard deviation:

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2,$$
$$\sigma = \sqrt{\mathrm{VAR}[X]}.$$

For events A and B:

$$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$
$$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$$

iff A and B are independent.

$$\Pr[A|B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

For random variables X and Y:

$$\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y],$$

if X and Y are independent.

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y],$$
$$\mathrm{E}[cX] = c\,\mathrm{E}[X].$$

Bayes' theorem:

$$\Pr[A_i|B] = \frac{\Pr[B|A_i]\Pr[A_i]}{\sum_{j=1}^{n} \Pr[A_j]\Pr[B|A_j]}.$$

Inclusion-exclusion:

$$\Pr\left[\bigvee_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \Pr[X_i] + \sum_{k=2}^{n} (-1)^{k+1} \sum_{i_i < \cdots < i_k} \Pr\left[\bigwedge_{j=1}^{k} X_{i_j}\right].$$

Moment inequalities:

$$\Pr\left[|X| \geq \lambda\,\mathrm{E}[X]\right] \leq \frac{1}{\lambda},$$
$$\Pr\left[\left|X - \mathrm{E}[X]\right| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$$

Geometric distribution:

$$\Pr[X = k] = pq^{k-1}, \qquad q = 1 - p,$$
$$\mathrm{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$$

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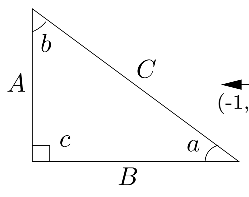
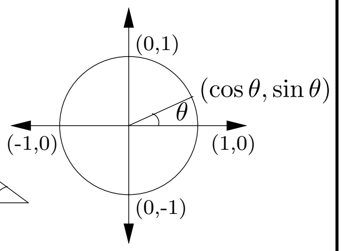
Trigonometry





Pythagorean theorem:

$$C^2 = A^2 + B^2.$$

Definitions:

$$\sin a = A/C, \quad \cos a = B/C,$$
$$\csc a = C/A, \quad \sec a = C/B,$$
$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:

$$\tfrac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:

$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x},$$
$$\tan x = \frac{1}{\cot x}, \qquad \sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$
$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$
$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$
$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot\tfrac{x}{2} - \cot x,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$
$$\sin 2x = 2 \sin x \cos x, \qquad \sin 2x = \frac{2\tan x}{1 + \tan^2 x},$$
$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$
$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2\cot x},$$
$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$
$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:

$$e^{ix} = \cos x + i \sin x, \qquad e^{i\pi} = -1.$$

Matrices

Multiplication:

$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.

$$\det A \cdot B = \det A \cdot \det B,$$
$$\det A = \sum_{\pi} \prod_{i=1}^{n} \operatorname{sign}(\pi) a_{i,\pi(i)}.$$

2×2 and 3×3 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g\begin{vmatrix} b & c \\ e & f \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix} + i\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= \begin{matrix} aei + bfg + cdh \\ -\, ceg - fha - ibd. \end{matrix}$$

Permanents:

$$\operatorname{perm} A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Hyperbolic Functions

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$
$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

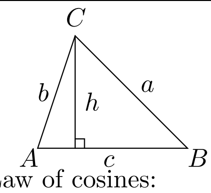
Identities:

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1,$$
$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$
$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh 2x = 2 \sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$
$$2\sinh^2 \tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2 \tfrac{x}{2} = \cosh x + 1.$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

… in mathematics you don't understand things, you just get used to them.
– J. von Neumann

More Trig.



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:

$$A = \tfrac{1}{2}hc,$$
$$= \tfrac{1}{2}ab\sin C,$$
$$= \frac{c^2 \sin A \sin B}{2\sin C}.$$

Heron's formula:

$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$
$$s = \tfrac{1}{2}(a+b+c),$$
$$s_a = s - a,$$
$$s_b = s - b,$$
$$s_c = s - c.$$

More identities:

$$\sin\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$
$$\cos\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$
$$\tan\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$
$$= \frac{1 - \cos x}{\sin x},$$
$$= \frac{\sin x}{1 + \cos x},$$
$$\cot\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$
$$= \frac{1 + \cos x}{\sin x},$$
$$= \frac{\sin x}{1 - \cos x},$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$
$$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$
$$= -i\frac{e^{2ix} - 1}{e^{2ix} + 1},$$
$$\sin x = \frac{\sinh ix}{i},$$
$$\cos x = \cosh ix,$$
$$\tan x = \frac{\tanh ix}{i}.$$

Theoretical Computer Science Cheat Sheet

Number Theory

The Chinese remainder theorem: There exists a number C such that:

$$\begin{gathered} C \equiv r_1 \bmod m_1 \\ \vdots \quad \vdots \quad \vdots \\ C \equiv r_n \bmod m_n \end{gathered}$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^{n} p_i^{e_i - 1}(p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^{n} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.
Wilson's theorem: n is a prime iff

$$(n-1)! \equiv -1 \bmod n.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\Big(\frac{a}{d}\Big).$$

Prime numbers:

$$p_n = n \ln n + n \ln\ln n - n + n \frac{\ln\ln n}{\ln n} + O\Big(\frac{n}{\ln n}\Big),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\Big(\frac{n}{(\ln n)^4}\Big).$$

Graph Theory

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \ldots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k-1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \lvert S\rvert$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embeded in the plane.
Plane graph	An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1,n_2}	Complete bipartite graph
$\mathrm{r}(k,\ell)$	Ramsey number

Geometry

Projective coordinates: triples (x, y, z), not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

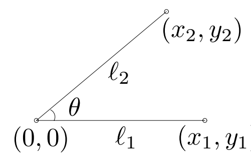
$$\big[|x_1 - x_0|^p + |y_1 - y_0|^p\big]^{1/p},$$

$$\lim_{p \to \infty} \big[|x_1 - x_0|^p + |y_1 - y_0|^p\big]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):

$$\tfrac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos\theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \tfrac{4}{3}\pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Issac Newton