$$
\begin{aligned}
V(k_0) &= \sum_{t=0}^{\infty}\Big[\beta^t \ln(1-\alpha\beta) + \beta^t \alpha \ln k_t\Big] \\
&= \ln(1-\alpha\beta)\sum_{t=0}^{\infty}\beta^t + \alpha\sum_{t=0}^{\infty}\beta^t\Big[\frac{1-(\alpha\beta)^t}{1-\alpha\beta}\ln\alpha\beta + \alpha^t \ln k_0\Big] \\
&= \frac{\alpha}{1-\alpha\beta}\ln k_0 + \frac{\ln(1-\alpha\beta)}{1-\beta} + \alpha\ln(\alpha\beta)\sum_{t=0}^{\infty}\Big[\frac{\beta^t}{1-\alpha} - \frac{(\alpha\beta)^t}{1-\alpha}\Big] \\
&= \frac{\alpha}{1-\alpha\beta}\ln k_0 + \frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta}{(1-\beta)(1-\alpha\beta)}\ln(\alpha\beta)
\end{aligned}
$$

$$
\begin{aligned}
\text{左边} = V(k) &= \frac{\alpha}{1-\alpha\beta}\ln k + \frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta}{(1-\beta)(1-\alpha\beta)}\ln(\alpha\beta) \\
&\triangleq \frac{\alpha}{1-\alpha\beta}\ln k + A
\end{aligned}
$$

$$
\text{右边} = \max\Big\{u\big(f(k)-y\big) + \beta V(y)\Big\}
$$

利用 FOC 和包络条件求解得 $y = g(k) = \alpha\beta k^\alpha$，代入，求右边。

$$
\begin{aligned}
\text{右边} &= \max\Big\{u\big(f(k)-y\big) + \beta V(y)\Big\} \\
&= u\big(f(k)-g(k)\big) + \beta\Big[\frac{\alpha}{1-\alpha\beta}\ln g(k) + A\Big] \\
&= \ln(k^\alpha - \alpha\beta k^\alpha) + \beta\Big[\frac{\alpha}{1-\alpha\beta}\ln\alpha\beta k^\alpha + A\Big] \\
&= \ln(1-\alpha\beta) + \alpha\ln k + \beta\Big[\frac{\alpha}{1-\alpha\beta}\big[\ln\alpha\beta + \alpha\ln k\big] + k\Big] \\
&= \alpha\ln k + \frac{\alpha\beta}{1-\alpha\beta}\alpha\ln k + \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln\alpha\beta + \beta A \\
&= \frac{\alpha}{1-\alpha\beta}\ln k + \underline{\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln\alpha\beta} + \beta A \\
&= \frac{\alpha}{1-\alpha\beta}\ln k + (1-\beta)A + \beta A \\
&= \frac{\alpha}{1-\alpha\beta}\ln k + A
\end{aligned}
$$

所以，左边 = 右边，证毕。