Proof #3

2022年12月2日 星期五 18:29

1. Proof that the amortized cost for a single operation applied to dynamic table is constant, expansion and contraction are allowed

Define the amortized cost $\hat{c}_i$ of the ith operation with respect to potential function ϕ → $\hat{c}_i = c_i + \phi(D_i) - \phi(D_{i-1})$

the actual cost of the ith operation

the data structure that results after applying the ith operation to data structure D_{i-1}

Define T.size : size of the table, T.num = number of items in the table

① For table expansion

- if T.size = T.num, we allocate a new table of size = 2T.size
- define a potential function ϕ that is 0 immediately after an expansion but builds to the table size by the time the table is full. → $\phi(T) = 2T.num - T.size$

→ If the ith operation does trigger an expansion, the we have $size_i = 2 size_{i-1}$, and $size_{i-1} = num_{i-1} = num_i - 1$, which implies that $size_i = 2\cdot(num_i - 1)$.

$\boxed{\alpha_{i-1} < \frac{1}{2} \quad \alpha_i > \frac{1}{2}}$

Thus, the amortized cost of the operation (Table-insert) is

$$\begin{aligned}
\hat{c}_i &= c_i + \phi_i - \phi_{i-1} \\
&= num_i + (2\cdot num_i - size_i) - (2num_{i-1} - size_{i-1}) \\
&= num_i + (2\cdot \cancel{num_i} - 2(\cancel{num_i} - 1)) - (2(num_i - 1) - (num_i - 1)) \\
&= num_i + 2 - (num_i - 1) \\
&= 3
\end{aligned}$$

→ If the ith operation cannot trigger an expansion, then we have $size_i = size_{i-1}$, the amortized cost of the ith operation is

$\boxed{\alpha_i < \frac{1}{2}}$

$$\begin{aligned}
\hat{c}_i &= c_i + \phi_i - \phi_{i-1} \\
&= 1 + (2\cdot num_i - size_i) - (2\cdot num_{i-1} - size_{i-1}) \\
&= 1 + (2\cdot num_i - size_i) - (2(num_i - 1) - size_i) \\
&= 3
\end{aligned}$$

∴ the amortized cost of a Table-insert operation is at most 3

② For table contraction

- If $\alpha_{i-1} \geq \frac{1}{2}$, Table-delete cannot contract, so $c_i = 1$ and $size_i = size_{i-1}$

case 1 : if $\alpha_{i-1} \geq \frac{1}{2}$

$$\begin{aligned}
\hat{c}_i &= c_i + \phi_i - \phi_{i-1} \\
&= 1 + (2num_i - size_i) - (2num_{i-1} - size_{i-1}) \\
&= 1 + (2(\cancel{num_{i-1}} - 1) - \cancel{size_i}) - (2\cancel{num_{i-1}} - \cancel{size_i}) \\
&= 1 + (-2) \\
&= -1
\end{aligned}$$

case 2 : if $\alpha_{i-1} < \frac{1}{2}$ → contraction

$$\begin{aligned}
\hat{c}_i &= c_i + \phi_i - \phi_{i-1} \\
&= (num_i + 1) + (size_i/2 - num_i) - (size_i/2 - num_{i-1}) \\
&= (\cancel{num_i} + 1) + ((\cancel{num_i} + 1) - \cancel{num_i}) - ((2\cdot \cancel{num_i} + 2) - (\cancel{num_i} + 1)) \\
&= 1
\end{aligned}$$

∴ the amortized cost of each operation is bounded by a constant

---

2. Compute $A_1(1)$, $A_2(1)$, $A_3(1)$, $A_4(1)$ Book 21.4

$$A_k(j) = \begin{cases} j+1 & \text{if } k=0 \\ A_{k-1}^{(j+1)}(j) & \text{if } k \geq 1 \end{cases}$$

$A_{k-1}^{(0)}(j) = j$

$A_{k-1}^{(i)}(j) = A_{k-1}(A_{k-1}^{(i-1)}(j))$ for $i \geq 1$

Basic case : $A_1^{(0)}(j) = j = 2^0(j+1) - 1$

$A_1^{(i)}(j) = 2^i(j+1) - 1$

$A_2(j) = A_1^{(j+1)}(j) = 2^{j+1}(j+1) - 1$

$A_0(1) = 1 + 1 = 2$

$A_1(1) = 2\cdot 1 + 1 = 3$

$A_2(1) = 2^{1+1}(1+1) - 1 = 4 \times 2 - 1 = 7$

$A_3(1) = A_2^{(2)}(1) = A_2(A_2(1)) = A_2(7) = 2^8(8) - 1 = 2047$

$A_4(1) = A_3^{(2)}(1) = A_3(A_3(1)) = A_3(2047) = A_2^{(2048)}(2047)$

$\gg A_2(2047) = 2^{2048}(2048) - 1 > 2^{2048} = (2^4)^{512} = 16^{512} \gg 10^{80}$