Proof of Master Theorem

2022年9月22日 星期四 22:24

Let $a \geq 1$, $b > 1$, $f(n)$ be a nonnegative function defined on exact power of b, then $g(n) = \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j)$ can be bounded for exact power of b

Define $T(n)$ on exact powers of b by the recurrence

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ aT(n/b) + f(n) & \text{if } n = b^i \end{cases}$$

where i is a positive integer. Then $T(n)$ has the following asymptotic bounds for exact powers of b:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$
3. If $af(n/b) \leq cf(n)$ for some constant $c < 1$ and for all sufficiently large n, then $T(n) = \Theta(f(n))$

Proof:

Case 1. $f(n) = O(n^{\log_b a - \epsilon})$ implies $f(n/b^j) = O((n/b^j)^{\log_b a - \epsilon})$

$$\Rightarrow g(n) = \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j) = O\left(\sum_{j=0}^{\log_b n - 1} a^j (n/b^j)^{\log_b a - \epsilon}\right)$$

$$= O\left(n^{\log_b a - \epsilon} \sum_{j=0}^{\log_b n - 1} a^j / (b^{\log_b a - \epsilon})^j\right)$$

$$= O\left(n^{\log_b a - \epsilon} \sum_{j=0}^{\log_b n - 1} a^j / (a^j (b^{-\epsilon})^j)\right)$$

$$= O\left(n^{\log_b a - \epsilon} \sum_{j=0}^{\log_b n - 1} (b^{\epsilon})^j\right)$$

$$= O\left(n^{\log_b a - \epsilon}\left(((b^{\epsilon})^{\log_b n} - 1)/(b^{\epsilon} - 1)\right)\right)$$

$$= O\left(n^{\log_b a - \epsilon}\left(((b^{\log_b n})^{\epsilon} - 1)/(b^{\epsilon} - 1)\right)\right)$$

$$= O\left(n^{\log_b a} n^{-\epsilon} \frac{n^{\epsilon} - 1}{b^{\epsilon} - 1}\right) \quad n^{\log_b a - \epsilon} O(n^{\epsilon}) = O(n^{\log_b a})$$

$$= O(n^{\log_b a})$$

$$T(n) = \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j) + \Theta(n^{\log_b a})$$

$$= O(n^{\log_b a}) + \Theta(n^{\log_b a}) = \Theta(n^{\log_b a}) \quad \#$$

Case 2. $f(n) = \Theta(n^{\log_b a})$ implies $f(n/b^j) = \Theta((n/b^j)^{\log_b a})$

$$\Rightarrow g(n) = \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j) = \Theta\left(\sum_{j=0}^{\log_b n - 1} a^j (n/b^j)^{\log_b a}\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{j=0}^{\log_b n - 1} a^j / (b^j)^{\log_b a}\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{j=0}^{\log_b n - 1} a^j / \underbrace{(b^{\log_b a})^j}_{= a^j}\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{j=0}^{\log_b n - 1} 1\right)$$

$$= \Theta(n^{\log_b a} \log_b n)$$

$$= \Theta(n^{\log_b a} \lg n)$$

$$T(n) = \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j) + \Theta(n^{\log_b a})$$

$$= \Theta(n^{\log_b a} \lg n) + \Theta(n^{\log_b a}) = \Theta(n^{\log_b a} \lg n) \quad \#$$

Case 3. $af(n/b) \leq cf(n) \Rightarrow a^j f(n/b^j) \leq c^j f(n)$

$$g(n) = \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j) \leq \sum_{j=0}^{\log_b n - 1} c^j f(n) \leq f(n) \sum_{j=0}^{\infty} c^j$$

$$= f(n) \frac{1}{1-c} = O(f(n))$$

because c is a constant, we can conclude that $g(n) = \Theta(f(n))$ for exact power of b

$$T(n) = \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j) + \Theta(n^{\log_b a})$$

$$= \underbrace{\Theta(f(n))}_{f(n) = \Omega(n^{\log_b a - \epsilon})} + \Theta(n^{\log_b a}) = \Theta(f(n)) \quad \#$$