Math 302: Homework Two

2026-01-26

Q1: Let $\boldsymbol{\Omega}$ be a sample space and $\mathbb{P}$ be a probability measure. Prove that there cannot exist events $\boldsymbol{E}, \boldsymbol{F}$ that satisfy

$$\mathbb{P}(E \setminus F) = \tfrac{2}{5}, \mathbb{P}(E \cup F) = \tfrac{1}{2}, \text{and } \mathbb{P}((E \cap F)^c) = \tfrac{3}{4}$$

Proof:
$\mathbb{P}((E \cap F)^c) + \mathbb{P}(E \cap F) = 1$ by def. of probability measure
$\mathbb{P}(E \cap F) = \frac{1}{4}$
$\mathbb{P}(E \setminus F) = (E \cap F^c)$
(1) show that $(E \cap F^c)$ and $(E \cap F)$ are disjoint
$(E \cap F^c) \cap (E \cap F)$
$(E \cap E) \cap (F^c \cap F)$ by associative and commutative property
$(E \cap \emptyset)$ by idempotent and complement
$= \emptyset$
$\therefore$ (1)
(2) show that $(E \cap F^c) \cup (E \cap F) \subseteq (E \cup F)$
$(E \cap F^c) \cup (E \cap F) = E \cap (F^c \cup F)$
$E \cup U = E$
$E \subseteq (E \cup F)$
$\therefore$ (2)

$\mathbb{P}(E \cap F^c) + \mathbb{P}(E \cap F) \leq P(E \cup F)$ by (1) and (2)
$\frac{2}{5} + \frac{1}{4} \nleq \frac{1}{2}$
$\blacksquare$ There cannot exist events E, F that satisfy the given conditions

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Q2: We roll a fair six-sided die until the first 1 comes up. What is the probabilty that the number of tosses is odd?
Success = {a 1 is rolled} Failure = {not 1}
trials are independent and probability of success is p for all trials
Therefore, $X \sim \text{Geom}(p)$
We are interested in odd tosses only, so:

$$P(\text{roll is odd}) = \sum_{n=0}^{\infty} P(2n+1) = \sum_{n=0}^{\infty} (1-p)^{(2n+1)-1} p = p \times \sum_{n=0}^{\infty} (1-p)^{2n}, \text{let } \alpha = (1-p)^2$$

$P(\text{roll is odd}) = p \times \sum_{n=0}^{\infty} \alpha^n = p \times \frac{1}{1-\alpha} = \frac{p}{1-(1-p)^2} = \frac{1}{2-p}$
$p = \frac{1}{6}$
$P(1 \text{ is rolled on an odd roll}) = \frac{6}{11}$

Q3: Given a sample space Ω and a probability measure $\mathbb{P}$, two events $A, B \subseteq \Omega$ are said to be independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Assume that the events E_1, E_2 are independent.

a) Prove that the events E_1^c, E_2^c are also independent.

$$P(E_1^c) = 1 - P(E_1) \text{ and } P(E_2^c) = 1 - P(E_2) \text{ by definition of } \mathbb{P}$$

$$P(E_1^c) \times P(E_2^c) = 1 - P(E_1) - P(E_2) + P(E_1)P(E_2) \tag{1}$$

$$P\big((E_1 \cup E_2)^c\big) = 1 - P(E_1 \cup E_2) \text{ by definition of } \mathbb{P}$$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \text{ by inclusion-exclusion}$$

$$P\big((E_1 \cup E_2)^c\big) = 1 - P(E_1) - P(E_2) + P(E_1 \cap E_2))$$

$$= P(E_1^c)P(E_2^c) - P(E_1)P(E_2) + P(E_1 \cap E_2) \text{ by substitution with (1)}$$

$$P\big((E_1 \cup E_2)^c\big) = P(E_1^c)P(E_2^c) - P(E_1 \cap E_2) + P(E_1 \cap E_2) \text{ since } E_1 \text{ and } E_2 \text{ are independent}$$

$$P(E_1^c \cap E_2^c) = P(E_1^c) \times P(E_2^c)$$

$\blacksquare E_1^c$ and E_2^c are independent

b) If, in addition, $\mathbb{P}(E_1) = \frac{1}{2}$ and $\mathbb{P}(E_2) = \frac{1}{3}$, Prove that

$$\mathbb{P}(E_1 \cup E_2) = \frac{2}{3}$$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= P(E_1) + P(E_2) - P(E_1)P(E_2) \qquad \text{since } E_1 \text{ and } E_2 \text{ are independent}$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{2} \times \frac{1}{3} = \frac{5}{6} - \frac{1}{6} = \frac{2}{3}$$

c) Let $\boldsymbol{E_3}$ be a third event where $\mathbb{P}(\boldsymbol{E_3}) = \frac{1}{4}$, such that all 3 events are independent (not jointly). Prove that:

$$\boldsymbol{\frac{17}{24} \leq \mathbb{P}(E_1 \cup E_2 \cup E_3) \leq \frac{19}{24}}$$

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3) = {} & P(E_1) + P(E_2) + P(E_3) \\ & -P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) \\ & +P(E_1 \cap E_2 \cap E_3) \end{aligned}$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{12} - \frac{1}{8} + P(E_1 \cap E_2 \cap E_3)$$

$$= \frac{17}{24} + P(E_1 \cap E_2 \cap E_3) \qquad (1)$$

We need to find the bounds on $P(E_1 \cap E_2 \cap E_3)$:

- The minimum value occurs when $E_1 \cap E_2 \cap E_3 = \emptyset$, so $P(E_1 \cap E_2 \cap E_3) = 0$
- To find the tightest upper bound on $P(E_1 \cap E_2 \cap E_3)$, we know that $(E_1 \cap E_2 \cap E_3) \subseteq (E_1 \cap E_2)$, *(note: I am picking $E_1 \cap E_2$ because the product gives me the tightest bound)*. This means that $P(E_1 \cap E_2 \cap E_3) \leq P(E_2 \cap E_3) = \frac{1}{3} \times \frac{1}{4} = \frac{2}{24}$.

Since $0 \leq P(E_1 \cap E_2 \cap E_3) \leq \frac{2}{24}$, by (1) we get,

$\blacksquare \frac{17}{24} \leq P(E_1 \cup E_2 \cup E_3) \leq \frac{19}{24}$

Q4: Eight rooks are placed randomly on a chess board (8x8). What is the probability that none of the rooks can capture any of the other rooks?

$P(A = 8 \text{ rooks are randomly placed on diagonals}) = \frac{|A|}{|\Omega|}$
Looking at the problem space, for the first rook, we place it on the 8x8 board, then we loose the row and column, then next valid space is the 7x7 square. Recursively we reduce

until we have 1 square left. Therefore, $|A| = 64 \times 49 \times 36 \times 25 \times 16 \times 9 \times 4 \times 1$
For $|\Omega|$, there are 64 squares on the chess board, and we have to pick eight, one for each rook. Since order matters we get $(64)_8$.
$P(A) = 64 \times 49 \times 36 \times 25 \times 16 \times 9 \times 4 \times \frac{1}{(64)_8}$

Q5: A fair six-sided die is rolled repeatedly.

(a) Given an expression for the probability that the first five rolls give a four at most two times.

We can roll a four 0, 1, or 2 times in 5 rolls. This means the probability of at most two fours in five rolls is denoted as the sum of these events.

$$\begin{gathered} P(A) = P(\{\text{No fours}\}) + P(\{\text{1 four}\}) + P(\{\text{2 four}\}) \\ = (1-p)^5 + (1-p)^4 p + (1-p)^3 p^2 \\ \text{where } p = \text{roll a four} = \frac{1}{6} \end{gathered}$$

(b) Calculate the probability that the first two does not appear before the fifth roll

The probability that the first two does not appear before the fifth roll is simply $\left(\frac{5}{6}\right)^4$, we are calculating the probability that we do not roll a two 4 times, after that the rest of the possible events are rolling a two or not rolling a two, of which we have a probability of 1.

(c) Calculate the probability that the first six appears before the twentieth roll, but not before the fifth roll

We first need to consider the case in which we roll a 6 before the 20th roll, this case includes the probability that we roll a six before the first 5 rolls. To get the probability that the first six appears between these bounds, we need to subtract the probability that we roll a six in the first 4 rolls from the probability that we roll a six in the first 19 rolls.
Let A = {first six appears before the twentieth roll, but not before the fifth roll}

$$P(A) = P(\{\text{roll a six before 20th roll}\}) - P(\{\text{roll a six before 5th roll}\})$$

$$P(\{\text{roll a six before 20th roll}\}) = 1 - P(\{\text{no 6 in first 20 rolls}\})$$

$$= 1 - \left(\frac{5}{6}\right)^{19}$$

$$P(\{\text{roll a six before 5th roll}\}) = 1 - \left(\frac{5}{6}\right)^4$$

$$P(A) = 1 - \left(\frac{5}{6}\right)^{19} - \left(1 - \left(\frac{5}{6}\right)^4\right) = \left(\frac{5}{6}\right)^4 - \left(\frac{5}{6}\right)^{19}$$

Q6: The statement "some days are snowy" has 16 letters (treating different appearances of the same letter as distinct). Pick one of them uniformly at random (i.e. each with equal probability). Let X be the length of the word to which the letter which was chosen belongs. Determine $\mathbb{P}[X = k]$ for $k \in \{3, 4, 5\}$

Letters are picked uniformly at random. Let $A_1 = \{\text{letters in 3 letter words}\}$, $A_2 = \{\text{letters in 4 letter words}\}$, $A_3 = \{\text{letters in 5 letter words}\}$, and $\Omega = \{\text{ways to randomly pick one letter from the sentence}\}$.

$$P(X = 3) = \frac{|A_1|}{|\Omega|} = \frac{3}{16}$$

$$P(X = 4) = \frac{|A_2|}{|\Omega|} = \frac{8}{16}$$

$$P(X = 4) = \frac{|A_3|}{|\Omega|} = \frac{5}{16}$$

Notice that the probability that the letter is in a word in the sentence (any of the words), we get the sum of these probabilities. Intuitively, the probability of doing so should be 1. And indeed, $\frac{3}{16} + \frac{8}{16} + \frac{5}{16} = 1$.