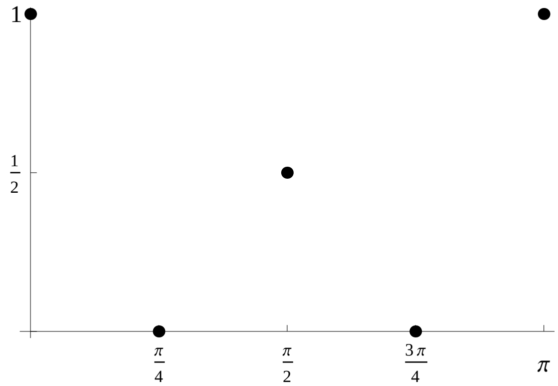
Dear Cera,

You want a function that is a simple combination of trig functions, and which takes certain values at certain places. Your data look like this:



Suppose you just want a simple sum of trig functions. You also want your festivity index to be positive. As trig functions tend to stray into the positive and the negative, you'll need a constant as well. Furthermore, note that your data repeats every π, not every 2π. Thus you want only terms of the form $\cos(nx)$ with n an *even* number.

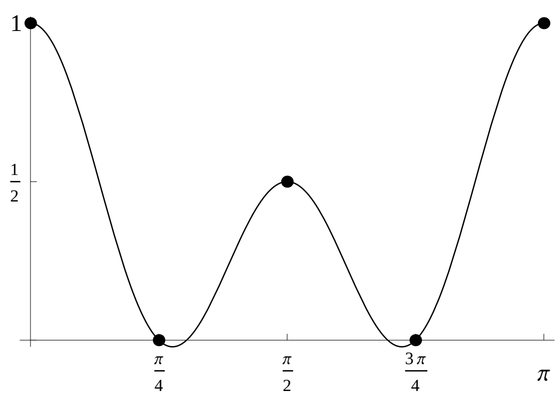
Let's try the simplest possible guess:

$$f(x) = A + B\cos(2x) + C\cos(4x).$$

If we plug in $x = 0$, $\pi/4$, $\pi/2$, $3\pi/2$ and π, we get

$$\begin{aligned}
f(0) &= A + B\cos(0) + C\cos(0) &&= A + B + C = 1 \\
f(\pi/4) &= A + B\cos(\pi/2) + C\cos(\pi) &&= A - C \qquad = 0 \\
f(\pi/2) &= A + B\cos(\pi) + C\cos(2\pi) &&= A - B + C = 1/2 \\
f(3\pi/4) &= A + B\cos(3\pi/2) + C\cos(3\pi) &&= A - C \qquad = 0 \\
f(\pi) &= A + B\cos(2\pi) + C\cos(4\pi) &&= A + B + C = 1.
\end{aligned}$$

Your five data points give us five equations for the three unknowns A, B and C. The good news is that two of these equations are redundant, so we can solve the first three equations in three unknowns to get $A = C = 3/8$ and $B = 1/4$. The bad news is that while the resulting function does match the data points, it is not always positive:

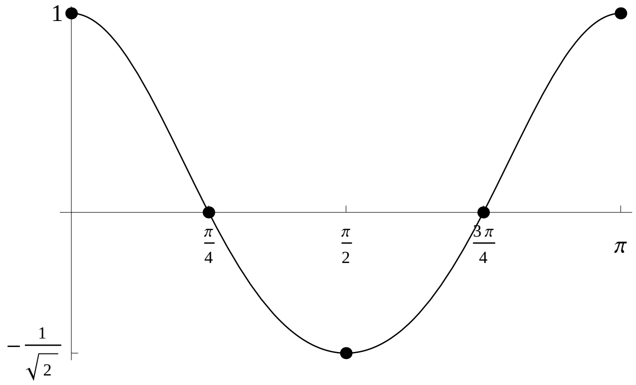


As you can see, there are tiny regions near $\pi/4$ and $3\pi/4$ where the function is negative.

Let's try another approach. To *guarantee* that we end up with something positive, let's let $f(x)$ be the square of a function. In fact, let's try

$$f(x) = \big[g(x)\big]^2 = \big[A + B\cos(2x) + C\cos(4x)\big]^2,$$

with $g(x) = A + B\cos(2x) + C\cos(4x)$. We want f to look like this:



Then $f = g^2$ will look like what we want. We do the same shenanigans as before:

$$\begin{aligned}
g(0) &= A + B\cos(0) + C\cos(0) &&= A + B + C = 1\\
g(\pi/4) &= A + B\cos(\pi/2) + C\cos(\pi) &&= A - C \qquad\quad = 0\\
g(\pi/2) &= A + B\cos(\pi) + C\cos(2\pi) &&= A - B + C = -1/\sqrt{2}\\
g(3\pi/4) &= A + B\cos(3\pi/2) + C\cos(3\pi) &&= A - C \qquad\quad = 0\\
g(\pi) &= A + B\cos(2\pi) + C\cos(4\pi) &&= A + B + C = 1.
\end{aligned}$$

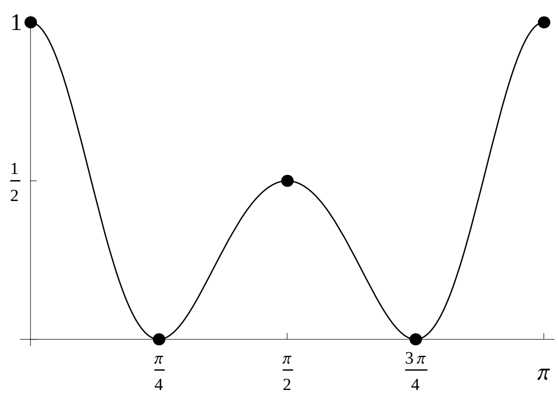
We can solve this to get

$$A = C = \frac{1}{8}(2 - \sqrt{2}\,), \qquad B = \frac{1}{4}(2 + \sqrt{2}\,).$$

This gives us the festivity index

$$f(x) = \left[\frac{1}{8}(2 - \sqrt{2}\,) + \frac{1}{4}(2 + \sqrt{2}\,)\cos(2x) + \frac{1}{8}(2 - \sqrt{2}\,)\cos(4x)\right]^2.$$

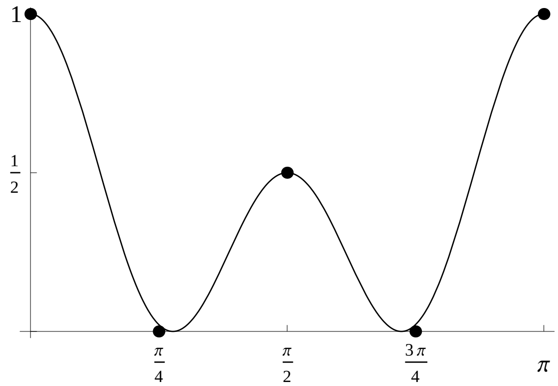
This matches the data and is always positive:



This is rather complicated, but then, you have an unusual set of conditions. A simpler function would be

$$g(x) = \left[\frac{1}{4}(2 - \sqrt{2}\,) + \frac{1}{4}(2 + \sqrt{2}\,)\cos(2x)\right]^2,$$

which has the graph



This doesn't *quite* vanish at $x = \pi/4$. Instead it vanishes at $x = \frac{1}{2}\arccos(2\sqrt{2} - 3) \approx 0.87$, which is close. The sole purpose of the $\cos(4x)$ term is just to nudge the zero of the function to the place where you want it.

An interesting puzzle. Thank you for it.

Yours,

Tim