PRODUCT ENGR.

BELLEVILLE SPRING CHARACTERISTICS

AND DESIGN GUIDELINES

20 August 1984

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FOREWORD

The following document is a completely revised edition of a former confidential Carleton Controls Corporation document covering Belleville Spring design. It is now intended to serve the same purpose for Moog Inc., Carleton Group engineers engaged in regulator design. This document does not present the theory of Belleville springs but rather thoroughly investigates the known equations and formulates from them simple relations very useful for design purposes.

Special recognition must go to Mr. George R. Ord for his encouragement, advice and guidance. Most of the material contained in this document comes from his personal notes and the long time experience of using Belleville spring characteristics in precision pressure regulators.

Hans G. Toews
20 August 1984

1.0 INTRODUCTION

Among the many theoretical theses and papers about Belleville springs that can be found in the literature, there are very little differences. With few exceptions they are basically all the same and derive the governing deflection formulas by assuming the small angle theory. It is not the intent of this document to provide a similar theoretical analysis of Belleville springs, that can be found in most college level textbooks, but rather use the appropriate equations in deriving certain important parameters directly useful for engineering applications. Using these parameters in design application over many years at Moog Inc., Carleton Group showed that calculated values agree well with actual test data.

This document is also intended to give examples of real engineering applications of Belleville springs and other useful design values not necessarily found any other place.

2.0 DEFINITION OF DIMENSIONAL & NONDIMENSIONAL PARAMETERS

OD or D = Outside diameter in inches

ID or d = Inside diameter in inches

$a = \frac{OD}{ID} = \frac{D}{d}$ Nondimensional diameter ratio

h = Inside heights in inches

t = Thickness in inches

R = h/t Nondimensional height to thickness ratio

$$\alpha = \tan^{-1} \frac{2h}{OD-ID} \quad \text{or} \quad A = \tan^{-1} \frac{2\,Rt}{D-d} \quad \text{Spring angle} \tag{1}$$

f or δ = Deflection from free position in inches

N = f/t Nondimensional deflection ratio

$r = \frac{D}{2}$ or $\frac{OD}{2}$ = Outside radius in inches

d_c = Diameter of hole circle for slotted springs in inches

R_s = Negative spring rate in lbf/in

M = $D/_2t$ Nondimensional radius to thickness ratio

d_h = Stress relief hole diameter in inches

d_t = Effective Belleville ID for slotted springs in inches

$d_t = d_c + .72\ d_h$ (assuming 12 or more slots are used)

P = Load in lbs.

P_f = Load at flat position in lbs.

S_a = Stress in PSI (upper inner edge)

S_b = Stress in PSI (lower inner edge)

S_c = Stress in PSI (lower ~~inner~~ edge) Should be "Lower Outer Edge"

Note: Positive sign denotes compressive stress.

ν = Poisson's ratio

$k = \dfrac{1}{(1 - \nu^2)}$, Poisson's ratio factor

E = Modulus of Elasticity

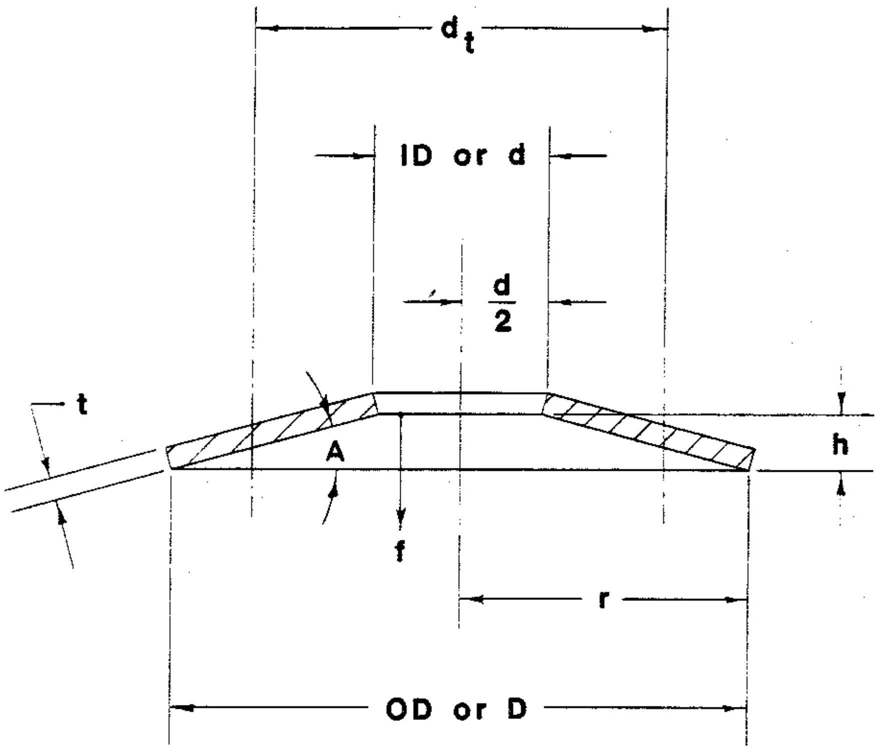
$K = \dfrac{E}{(1 - \nu^2)}$, Material factor

$Z = \dfrac{R^2 - 2}{3}$ Dimensionless Ord's ratio* (2)

"Lever Ratio" (ADV)

$$f_m = \frac{D - d}{D - d_t} \quad \text{Multiplication factor for slotted springs} \qquad (3)$$

* This ratio was determined by Mr. George Ord of Moog Inc., Carleton Group in 1956 and found to be very useful in solving the governing equations. It makes exact solutions of important values of the load factor C_1 possible.



Dimensional Parameters

Figure 1

2.1 Material Properties

Metal	E (PSI)	ν	K (PSI)	k
Steel	30×10^6	0.30	33.0×10^6	1.099
Phosphor Bronze	15×10^6	0.20	15.6×10^6	1.042
17-7PH Stainless	29×10^6	0.34	33.0×10^6	1.131
G.O. Exp.*	22.53×10^6			
302 Stainless	28×10^6	0.30	30.8×10^6	1.099
Beryllium Copper	18.5×10^6	0.33	20.8×10^6	1.122
Inconel	31×10^6	0.29	33.8×10^6	1.092
Inconel X	31×10^6	0.29	33.8×10^6	1.092

Material Constants

Table 1

*This value is based on experience. It relates actual test data to theory for closer agreement. This value may be used in force and rate design calculations. For stress calculation 29×10^6 should be used.

2.2 Belleville Characteristic Constants

The nondimensional constants listed below are found in nearly all Belleville reference papers. No effort is made to derive them here.

Dimension Load Factor:

$$C = \frac{\pi \ln a}{6}\left(\frac{a}{a-1}\right)^2 \tag{4}$$

Deflection Load Factor in Units of Inches to the Fourth Power:

$$C_1 = \frac{ft}{\boxed{t^4}}\left[(h-f)\left(h-\frac{f}{2}\right)+t^2\right] \tag{5}$$

See (11) on next page

NOTE: This factor depends on the h/t ratio R and gives the typical Belleville curve shape of load versus deflection.

Stress Factors for Upper and Lower ID Edges:

$$C_2 = \frac{6}{\pi \ln a}\left(\frac{a-1}{\ln a} - 1\right) \tag{6}$$

$$C_3 = \frac{3(a-1)}{\pi \ln a} \tag{7}$$

Stress Factors for Lower OD Edge:

$$C_4 = \frac{a \ln a - (a-1)}{\ln a} * \frac{a}{(a-1)^2} \tag{8}$$

$$C_5 = \frac{0.5a}{a-1} \tag{9}$$

3.0 BELLEVILLE SPRING FORCE EQUATIONS

As before, the governing force equation is listed with no derivation. The force generated by a Belleville spring is a function of the deflection load factor C_1. The unique nonlinear deflection characteristics are explored in the next paragraph.

The general load deflection relation found in most textbooks is given by:

$$F = \frac{CEft}{(1-\nu^2)r^2}\left[(h-f)\left(h-\frac{f}{2}\right)+t^2\right] \qquad (10)$$

This equation can be simplified by substituting C_1 as follows:

$$C_1 = \frac{f\left[(h-f)(h-\frac{1}{2}f) + t^2\right]}{t^3} \qquad (11)$$

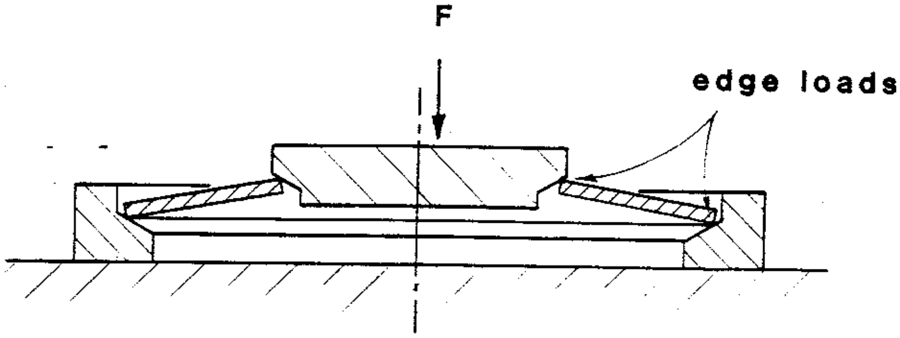
for $N = f/t$, $R = h/t$ the deflection load factor now is:

$$C_1 = N\left[(R-N)\left(R-\frac{1}{2}N\right) + 1\right] \qquad (12)$$

The force equation can therefore be simplified as follows:

$$F = \frac{CkEt^4}{r^2}\, C_1 \qquad (13)$$

Where C_1 is the deflection load factor from equation 5 or 12. The force F acts uniformly distributed around the inner upper edge balanced by the same force acting oppositely but also uniformly distributed around the lower outer edge. Hence a Belleville spring is properly tested by suitable machined plates or cups which uniformly load the edges as shown in Figure 2.



Force Testing of a Belleville Spring
Figure 2

When a Belleville spring is deflected to the flat position the force equation simplifies to a special case.

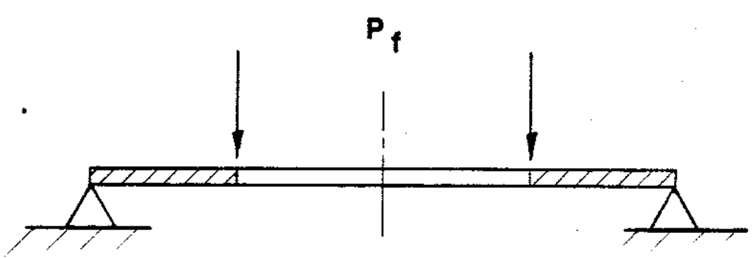
At flat N = R and C_1 reduces to R

$$P_f = \frac{CkEt^4}{r^2} R \tag{14}$$

or for R = h/t

$$P_f = \frac{CkEht^3}{r^2} \tag{15}$$

A flat deflected spring is shown in Figure 3.



Belleville Spring Deflected to Flat Position
Figure 3

3.1 Characteristics of Load Factor C_1

The nondimensional load or force factor C_1 is directly proportional to the Belleville force F as seen by the previous equations. Its behavior for different values of deflection gives the typical Belleville spring characteristic having high and low inflection points with a negative spring rate between them. The concept of "negative spring rate" is intuitively a little difficult to understand. It essentially says that the force required to produce further deflection becomes smaller. This is clearly opposite to the normal positive helical spring rate where the force to produce further deflection must always be increased.

The definition of C_1 (equation 5 or 12) exhibits certain characteristics which depend only on R and a unique function of R. The maximum and minimum values of C_1 for example which are most important in Belleville spring design can be expressed as simple functions of R, the h/t ratio. This holds true for all geometries and Belleville spring dimensions.

The values for C_{1H}, C_{1L} and C_{1M} are stated here without derivation. All calculations are carried out in nondimensional form for simplicity and generality and may be reviewed in the Appendix.

3.1.1 Expressions for C_{1H}, C_{1L} and C_{1M}

C_{1H} is the maximum value of C_1, C_{1L} is the minimum value of C_1 and C_{1M} the mean (middle) value of the function C_1 when the spring is deflected to a flat position.

The values are found by setting the derivative of C_1 with respect to N to zero. The result is a quadratic equation with the following solution:

$$N = R \pm \sqrt{\frac{R^2 - 2}{3}} \tag{16}$$

The value under the square root sign is defined to be Ord's ratio Z.

$$Z = \frac{R^2 - 2}{3} \tag{17}$$

or

$$N = R \pm \sqrt{Z} \tag{18}$$

For complete derivation see Appendix A. Since Z cannot be less than zero for a real solution, R must have a minimum value.

$$R_{min} = \sqrt{2} = 1.4142 \tag{19}$$

If the h/t ratio of a Belleville spring is less than or equal to 1.4142, the spring will have no maximum or minimum value and also no negative rate. In practice R values of 1.6 to 2.6 are used to provide a usable negative slope range.

The maximum value C_{1H} occurs at $N = R - \sqrt{Z}$ and minimum value C_{1L} at $N = R + \sqrt{Z}$: When $N = R$ the mean value C_{1M} which coincides with the flat position is obtained. All three values can be expressed as functions of R and Z.

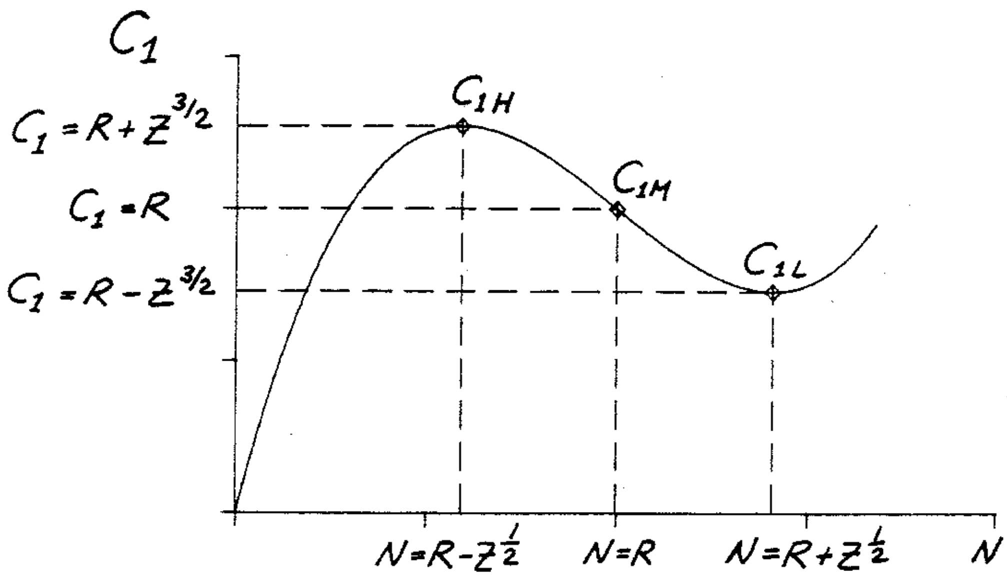
$$C_{1H} = R + Z^{3/2} \tag{20}$$

$$C_{1M} = R \tag{21}$$

$$C_{1L} = R - Z^{3/2} \tag{22}$$

Clearly for $R = \sqrt{Z}$ which means $Z = 0$, C_{1H}, C_{1L} and C_{1M} assume the same point. See Appendix for complete derivation.

These very important Belleville spring characteristic design values are related in a unique fashion by the Ord's ratio Z as shown graphically in Figure 4.



The maximum and minimum force factors are uniquely related to R and Z.

Figure 4

3.1.2 Determination of C'_{1H} and C'_{1L}

For Belleville spring design purposes for pressure regulators, it is useful to arrive at an approximate linearization of the negative spring rate. The idea is to define a straight line which best approximates the negative slope of the force deflection characteristic.

A straight line connecting the points C_{1H} and C_{1L} has large errors on either side of C_{1M} and therefore is not very useful. Another choice would be a straight line coinciding with the slope at C_{1M} which is a point of inflection. This line would be a better approximation with close agreement around the point C_{1M} but rather large diversions toward the high and low points C_{1H} and C_{1L}.

A very useful approximation is obtained when two more points on the curve are identified. The points are C'_{1H} and C'_{1L} half way between C_{1H}, C_{1L} and C_{1M} on the deflection scale N. A straight line negative slope approximation between these points nearly matches the exact curve. At the same time a maximum usable deflection range (regulator operating stroke) is defined.

Since the two new points C'_{1H} and C'_{1L} are exactly located between $N = R \pm \sqrt{Z}$ and $N = R$, it is clear that $N = R \pm 1/2 \sqrt{Z}$ gives these points. Substituting and solving for C_1 gives (see Appendix for derivation):

$$C'_1 = R \pm \frac{11}{16} Z^{3/2} \tag{23}$$

The two points which now define the negative linear usable spring rate are given by:

$$C'_{1H} = R + \frac{11}{16} Z^{3/2} \qquad @ \quad N = R - \frac{1}{2} Z^{1/2} \tag{24a,b}$$

$$C'_{1L} = R - \frac{11}{16} Z^{3/2} \qquad @ \quad N = R + \frac{1}{2} Z^{1/2} \tag{25a,b}$$

Two more points are now identified as unique functions of R and Z. The dimensionless ratio Z makes it possible to quickly find all five points C_{1H}, C'_{1H}, C_{1M}, C'_{1L} and C_{1L} for a selected value of R. This is a great advantage in Belleville spring design, because it makes the recalculation of the complete force-deflection equation unnecessary.

Further manipulation with these values will show that the values for C'_{1H} and C'_{1L} can also be expressed in terms of C_{1H} and C_{1L}.

$$C'_{1H} = \frac{32}{32}R + \frac{22}{32}Z^{3/2} = \frac{27}{32}\left(R+Z^{3/2}\right) + \frac{5}{32}\left(R-Z^{3/2}\right) \tag{26}$$

$$C'_{1L} = \frac{32}{32}R - \frac{22}{32}Z^{3/2} = \frac{27}{32}\left(R-Z^{3/2}\right) + \frac{5}{32}\left(R+Z^{3/2}\right) \tag{27}$$

$$C'_{1H} = \frac{27}{32}C_{1H} + \frac{5}{32}C_{1L} \tag{28}$$

$$C'_{1L} = \frac{27}{32}C_{1L} + \frac{5}{32}C_{1H} \tag{29}$$

In terms of high and low forces usually known from the force plots of a tested Belleville spring the relation are:

$$F'_H = \frac{27}{32}F_H + \frac{5}{32}F_L \tag{30}$$

$$F'_L = \frac{27}{32}F_L + \frac{5}{32}F_H \tag{31}$$

Also given are the reverse relationships:

$$F_L = \frac{F'_L - \frac{5}{32}F_H}{27/32} \tag{32}$$

$$F_H = \frac{\frac{27}{32}F'_H - \frac{5}{32}F'_L}{\left(\frac{27}{32}\right)^2 - \left(\frac{5}{32}\right)^2} \tag{33}$$

Figure 5 shows the five characteristic points of a typical Belleville spring curve:

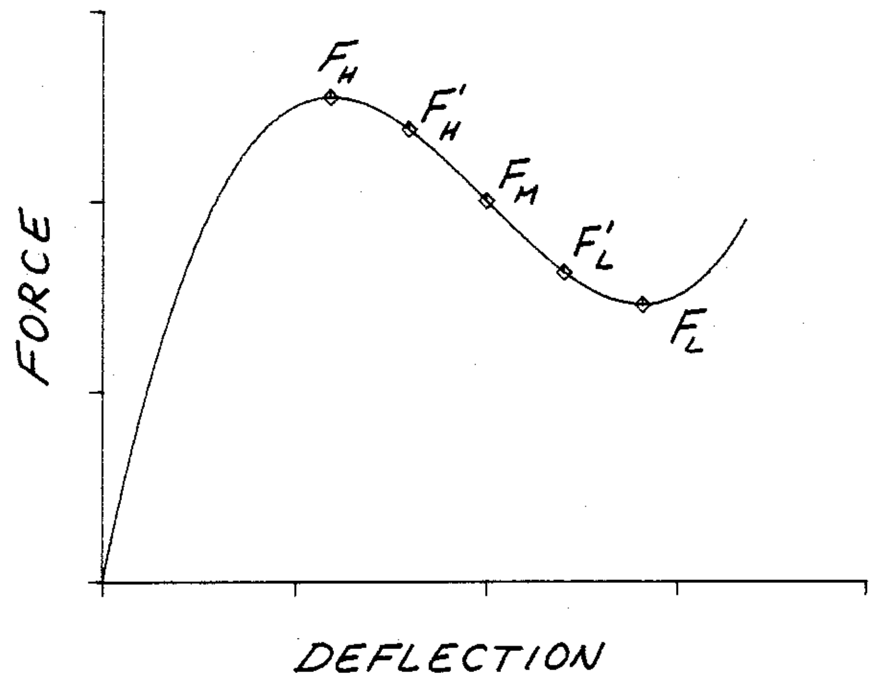


Figure 5

3.2 Negative Spring Rates

The points C'_{1H} and C'_{1L} define the usable stroke and negative spring rate. The slope S is given by:

$$S = \frac{\Delta C_1}{\Delta N} = \frac{(R-\frac{11}{16}Z^{3/2})-(R+\frac{11}{16}Z^{3/2})}{(R+\frac{1}{2}Z^{1/2})-(R-\frac{1}{2}Z^{1/2})}$$

$$S = -\frac{\frac{11}{8}Z^{3/2}}{Z^{1/2}} = -\frac{11}{8}Z$$

$$S = -1.375\ Z \qquad (34a,b,c)$$

The nondimensional slope S is again a very simple function of Ord's ratio Z. Noting that N=f/t the negative rate in dimensional form is derived as follows.

$$R_S = \frac{\Delta F}{\Delta f} = \frac{\frac{CkEt^4}{r^2}\ \Delta C_1}{t\ \ \Delta N} \qquad (35)$$

$$R_S = -1.375\ \frac{CkEt^3}{r^2}\ Z \qquad (36)$$

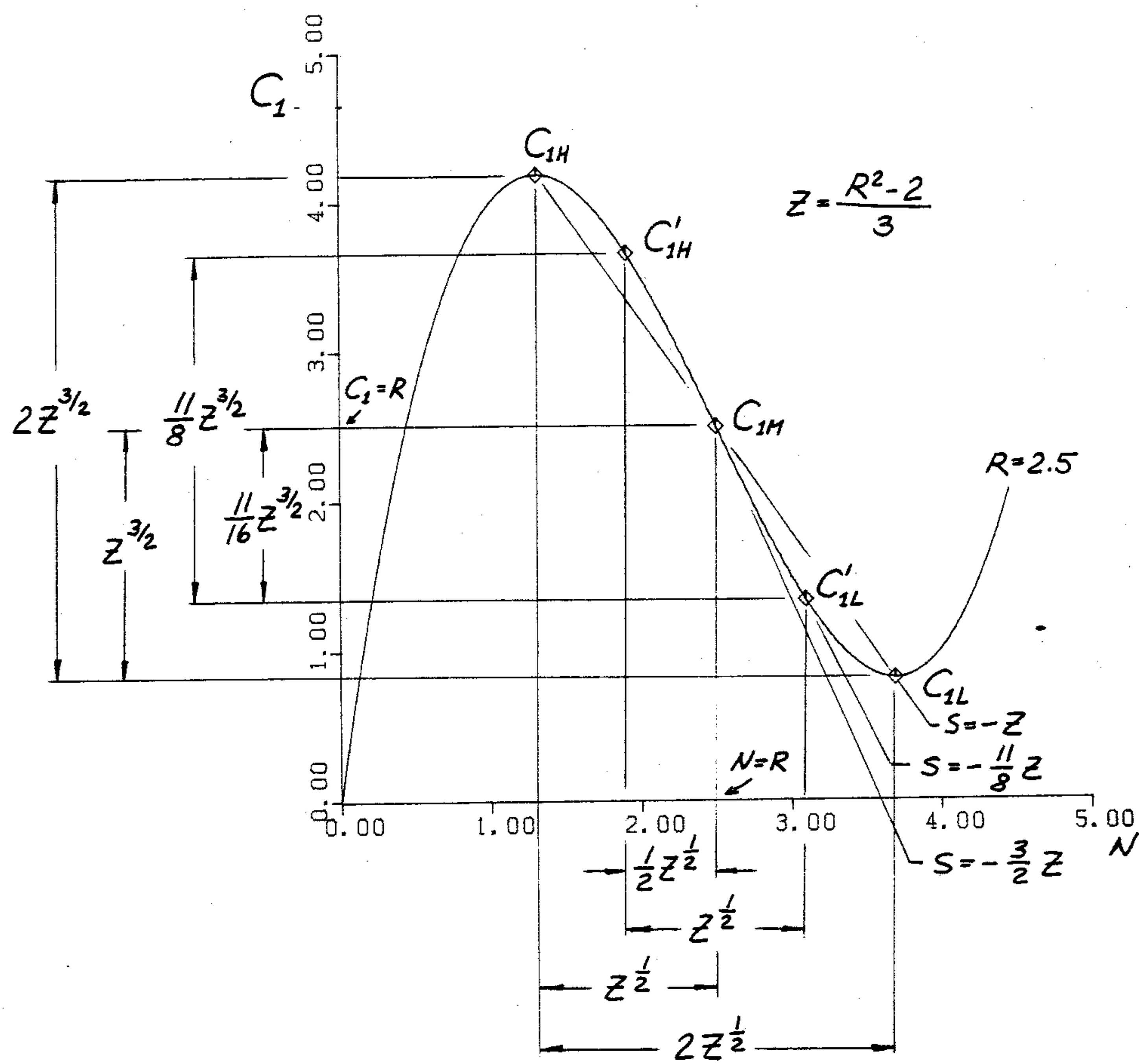
The two other slopes which may be of interest are given without derivation to complete the picture. The straight line slope between high and low is:

$$S_{H-L} = -Z \qquad (37)$$

and the tangent slope at C_{1M} is

$$S_M = -1.5\ Z \qquad (38)$$

All points of interest on a Belleville force deflection curve have now been defined in nondimensional terms of R and Z. The complete picture is shown in Figure 6. Note that once R has been selected for a spring design all points can be quickly calculated.



Force factor representation of Belleville spring characteristics in terms of nondimensional parameters of R and Z.

Figure 6

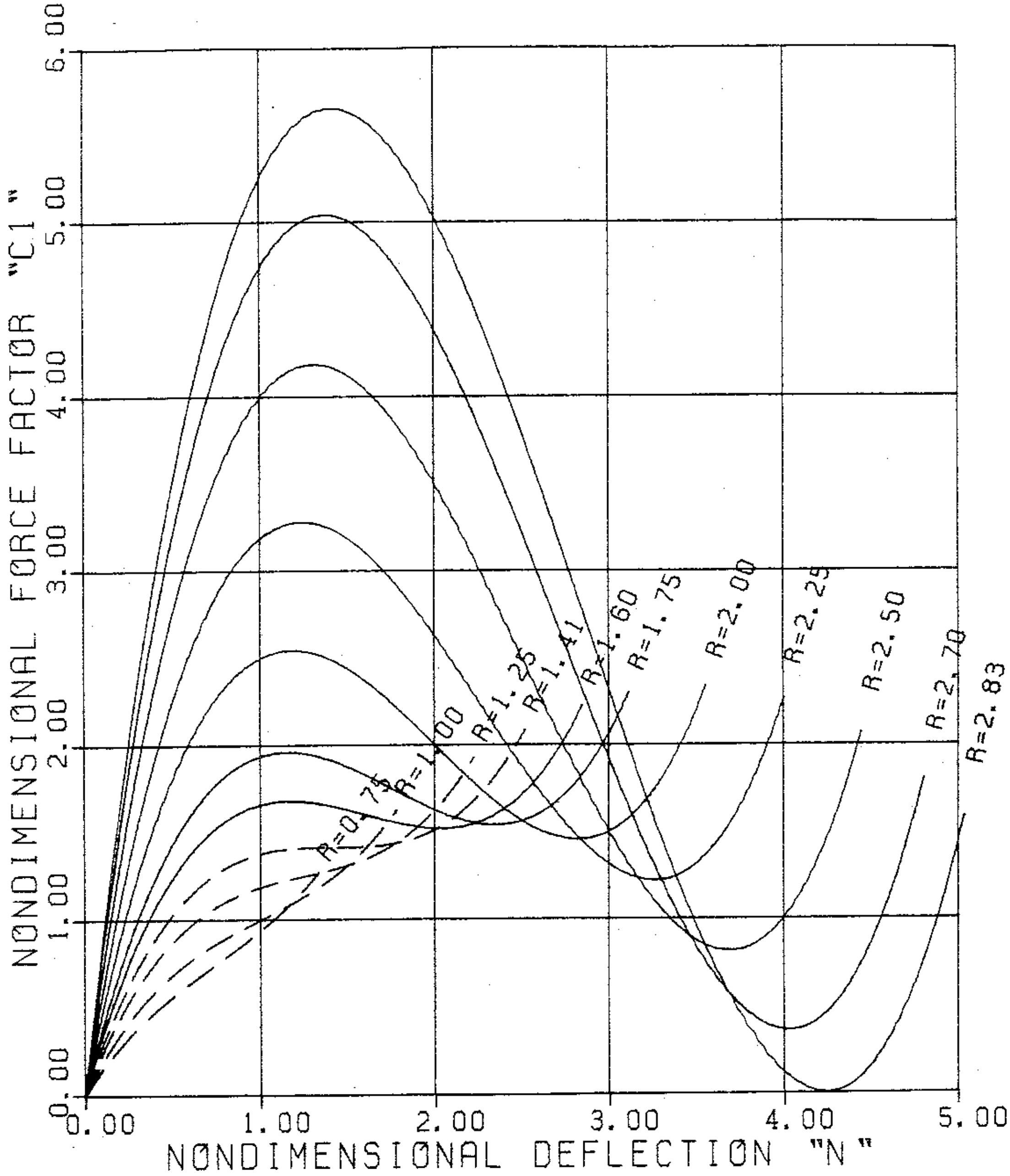
3.3 C_1 for Different Values of R

As shown in the previous paragraph, the curve of the load or force factor C_1 has a unique shape with identifiable points. A complete set of curves for different values of R has also unique characteristics on which depend the value of R. In Figure 7 the force factor C_1 is plotted versus deflection ratio for increasing values of R.

Inspection of these curves reveal that if $R < \sqrt{2}$ no negative slope exists. Springs of these proportions are not very useful for regulator design. They are however applied where very large forces with small working deflections are needed. Belleville springs of this type require very little installation volume when compared to an equivalent helical spring.

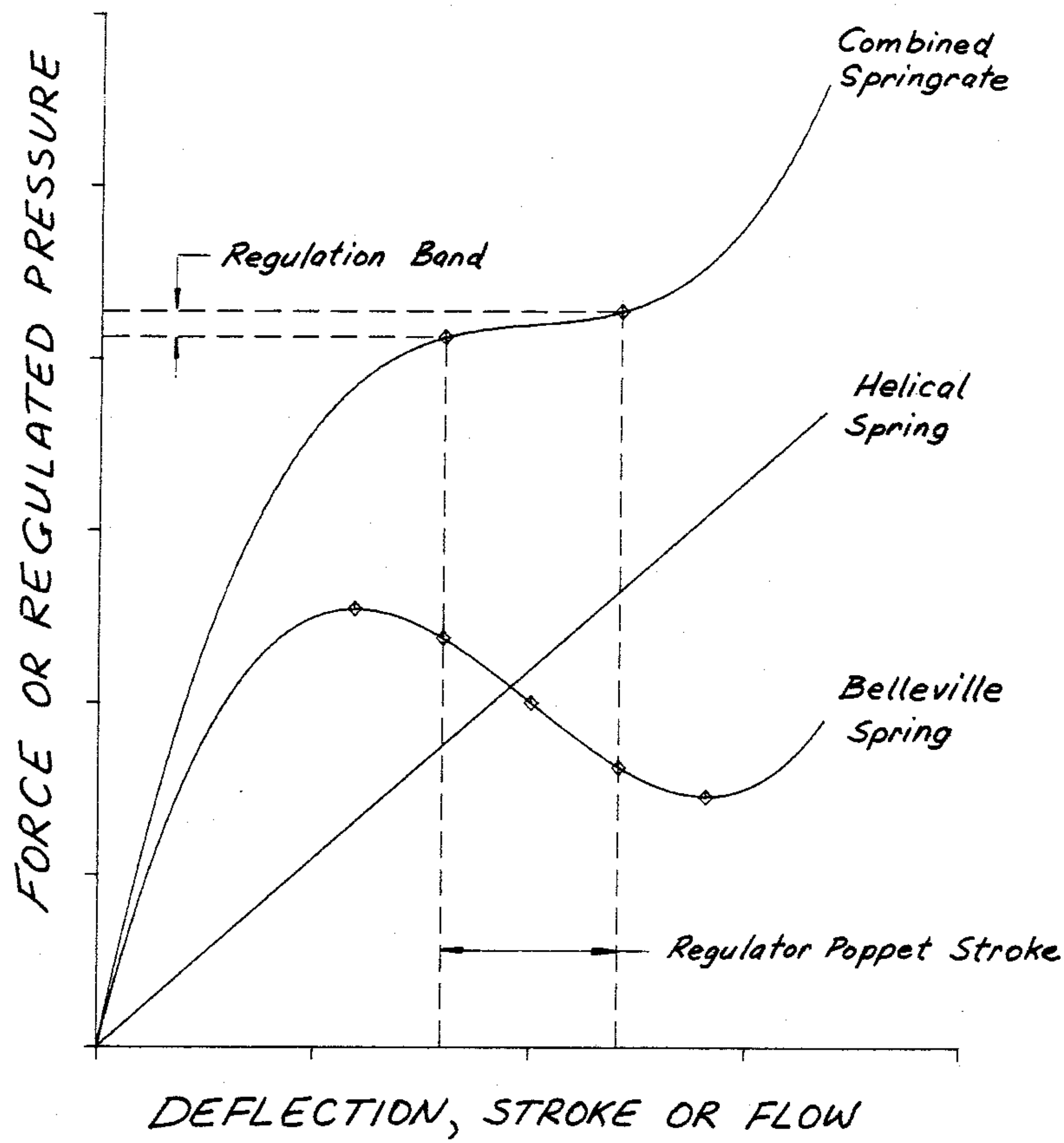
When $R = \sqrt{2}$ all five points (C_{1H}, C'_{1H}, C_{1M} , C'_{1L}, C_{1L}) are matched into a single point at $C_1 = \sqrt{2}$ and $N = \sqrt{2}$. The curve slope is zero at this point and starts to become negative for increasing values of R.

The extreme limits of the usable negative spring rates are between $\sqrt{2} < R < \sqrt{8}$ ($1.4142 < R < 2.8284$). In practice a range of $1.6 < R < 2.6$ is used for Belleville spring designs for regulators. In this range the typical negative spring rate behavior is balanced by a positive helical spring rate to provide a combined usable flat spring rate. Figure 8 shows why this combined spring rate gives good small tolerance regulation characteristics. Note that it is not necessary for the helical spring to start at $N = 0$. The point at which the helical spring should engage is usually determined by the total force of the system. In an actual regulator design the total load is shared between the Belleville and helical spring. In most designs the larger part of the load is carried by the helical spring. In other designs where space is at a premium most of the load is taken by one or several Belleville springs.



Force factor characteristics for various values of R.

Figure 7



For regulator design a Belleville spring is used with a helical spring to obtain a low but positive combined spring rate.

Figure 8

A NOTE OF CAUTION:

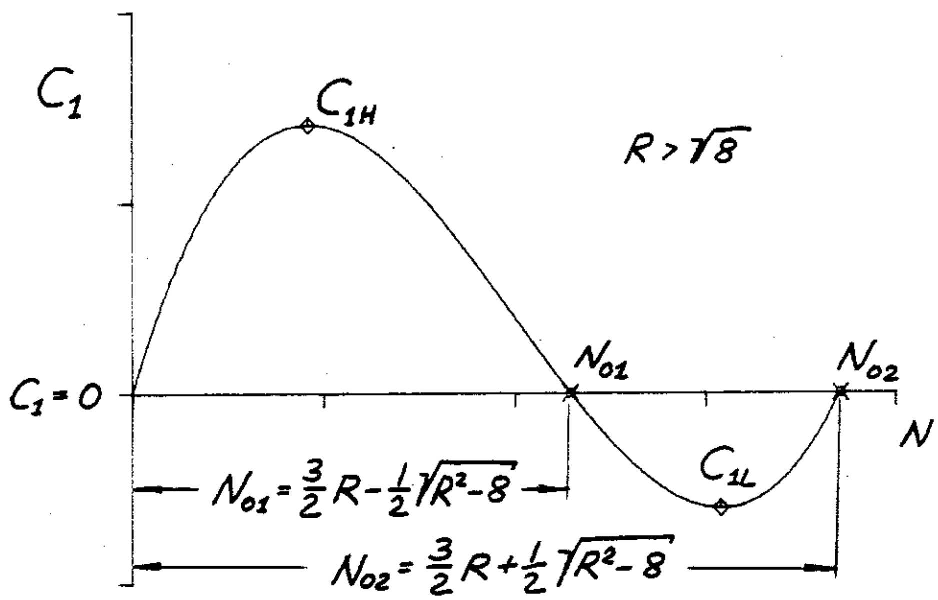
In an actual design task when a Belleville spring is calculated and the desired load and rate is established, one must always check whether the stresses are within safe limits. Since the stresses are also a function of deflection the stresses must be checked throughout the usable range. Stresses at high R rates grow very fast. In order to stay within safe stress limits the Belleville deflection must sometimes be restricted.

Spring designs beyond the value $R \geq 2.8284$ start to show a different behavior. The force factor curve goes through zero and comes back up again. When springs of this type are deflected beyond the minimum negative force value they will snap through and remain snapped through.

3.4 Bistable (Snap-Through) Springs

The characteristic points of snap-through Belleville springs are shown in Figure 9. Again, as before, the exact solutions for the points of interest are derived in the Appendix.

The maximum force factor, minimum force factor, and the three points in between with the associated slopes, obey the same equations as before. The two new points of interest are at N_{01} when the force factor becomes negative and at N_{02} when the second stable position is reached.



A snap-through Belleville has a new stable position at N_{02}.

Figure 9

The value of N_{01} and N_{02} are derived by setting the force factor equation to zero and solving for the roots of N.

The first root is clearly zero and the other two are:

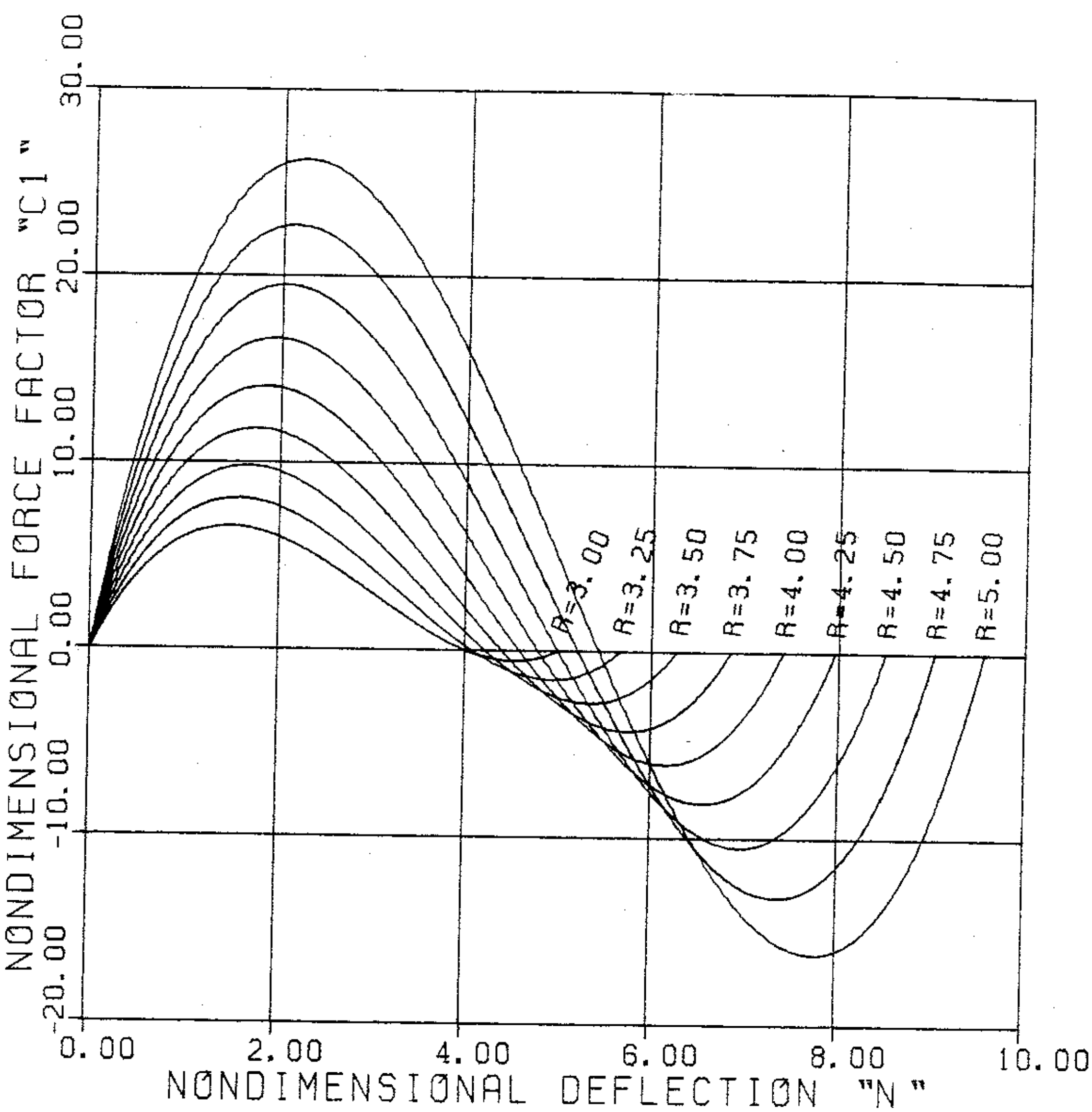
$$N_{01} = \frac{3}{2}R - \frac{1}{2}\sqrt{R^2 - 8} \tag{39}$$

$$N_{01} = \frac{3}{2}R + \frac{1}{2}\sqrt{R^2 - 8} \tag{40}$$

The characteristic curves of snap-through Belleville springs for R = 3 through R = 5 are shown in Figure 10. These springs have two stable points and are used in latching type valve designs. Latching mechanisms of this type must be designed very carefully, because the force and movement needed to affect latching changes drastically over the range of R as shown. For R = 3 for example, very little negative snap-through latching force is available; it is approximately only 9% of the maximum positive value. The percentage is increasing and reaches approximately 62% for R = 5. In any case the positive and negative latching forces will always be unequal, not necessarily a desirable characteristic. Sometimes however, like in the 2217 latching valve for example, this can be used to an advantage. In this case a higher closing force (sealing force) was required than the force needed to hold the valve open.

Snap-through springs are <u>not</u> used for regulator design. These are also inherently difficult to test and do not allow a smooth force versus deflection plot without elaborate fixturing.

The disadvantage of single snap-through springs however can be overcome by using two Belleville springs back to back.

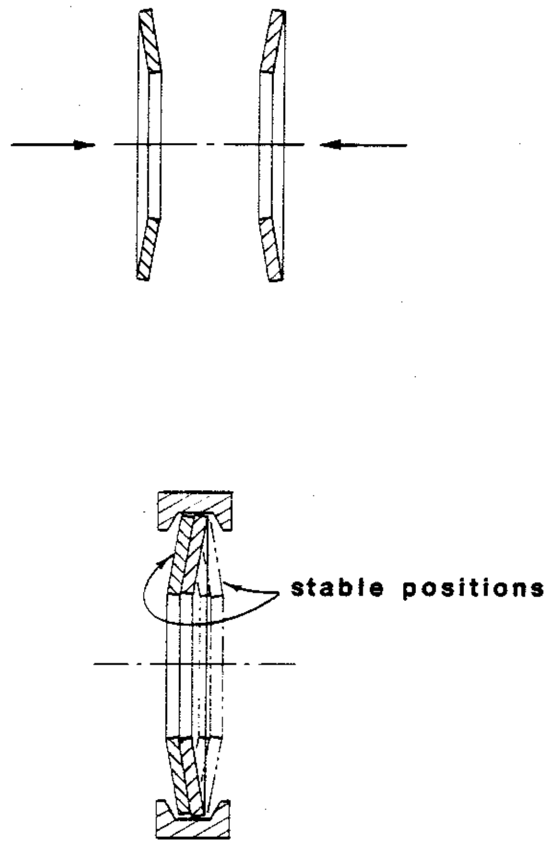


Snap-through Belleville spring characteristics for various R values.

Figure 10

3.5 Two Back-to-Back Springs

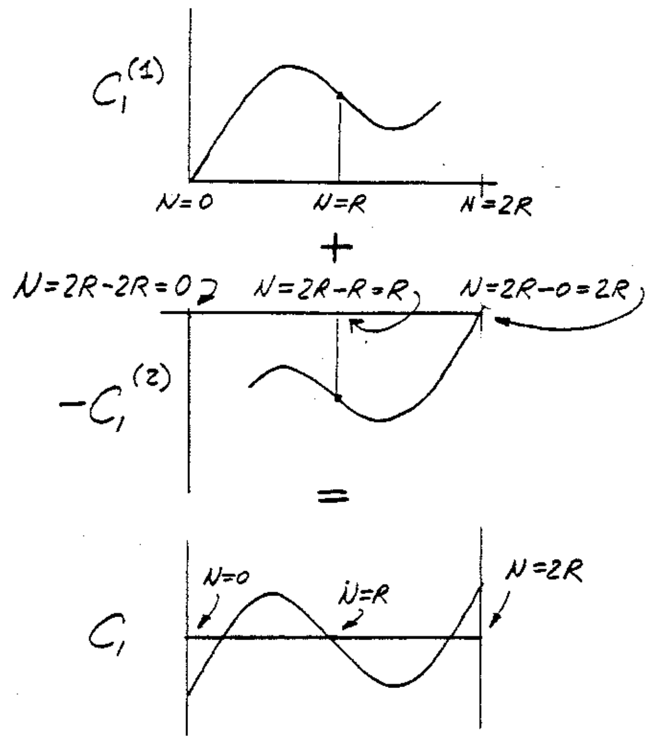
Two identical Belleville springs installed back to back as shown in Figure 11 will create a single spring with two stable endpoints with equal deflections and forces.



Two Belleville springs installed back to back create a single bistable spring.

Figure 11

In order to derive the combined force factor equation the two individual force factor curves must be added in a manner illustrated by Figure 12.



Graphic illustration of two identical back-to-back springs.

Figure 12

The first spring is evaluated at N. The second spring evaluated at N = 2R - N and its negative force factor subtracted from the first.

$$C_1^{(1)} = N\left[(R-N)\left(R-\tfrac{1}{2}N\right)+1\right] \tag{41}$$

$$-C_1^{(2)} = -(2R-N)\left[\left(R-(2R-N)\right)\left(R-\tfrac{1}{2}(2R-N)\right)+1\right] \tag{42}$$

The combined force factor is:

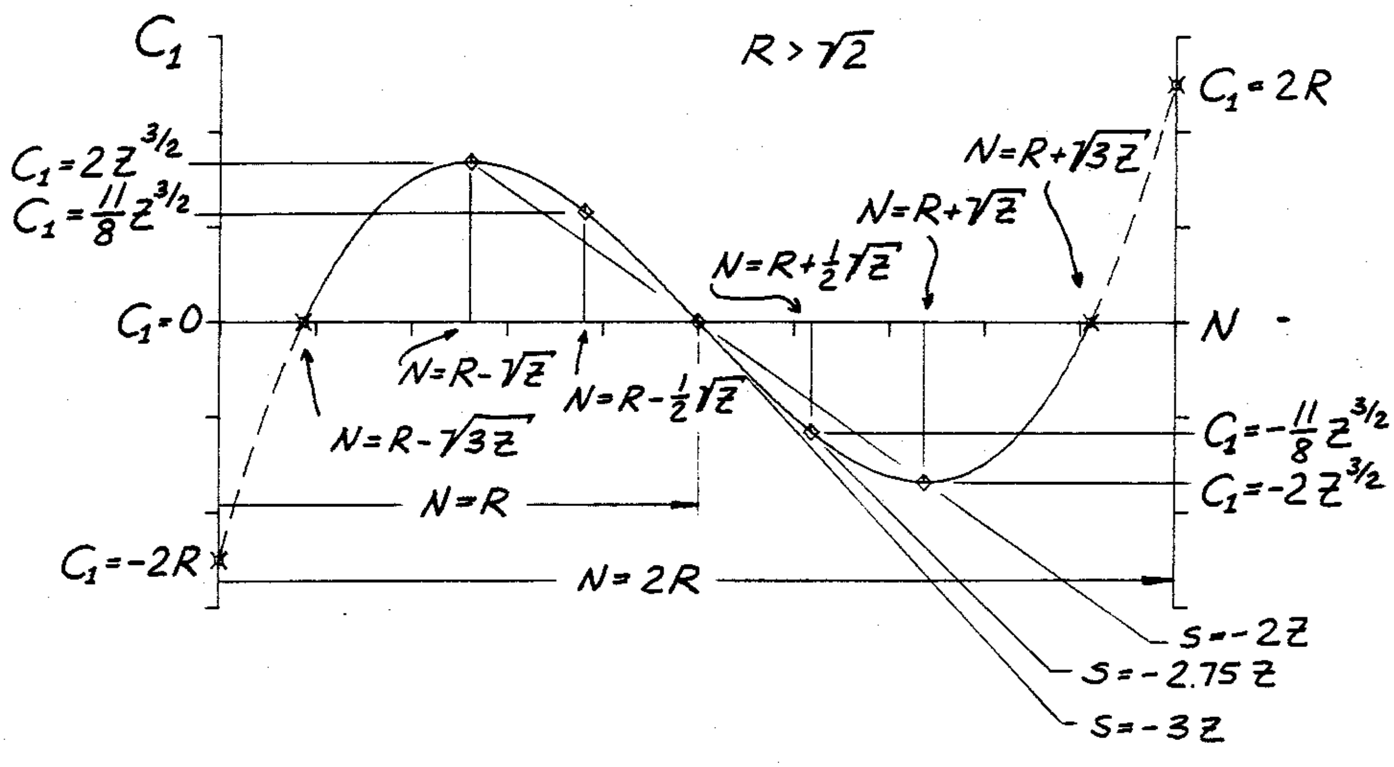
$$C_1^{B} = C_1^{(1)} + \left(-C_1^{(2)}\right) \tag{43}$$

After algebraic manipulation the following third order equation is obtained.

$$C_1^{B} = N^3 - 3RN^2 + 2(R^2+1)N - 2R \tag{44}$$

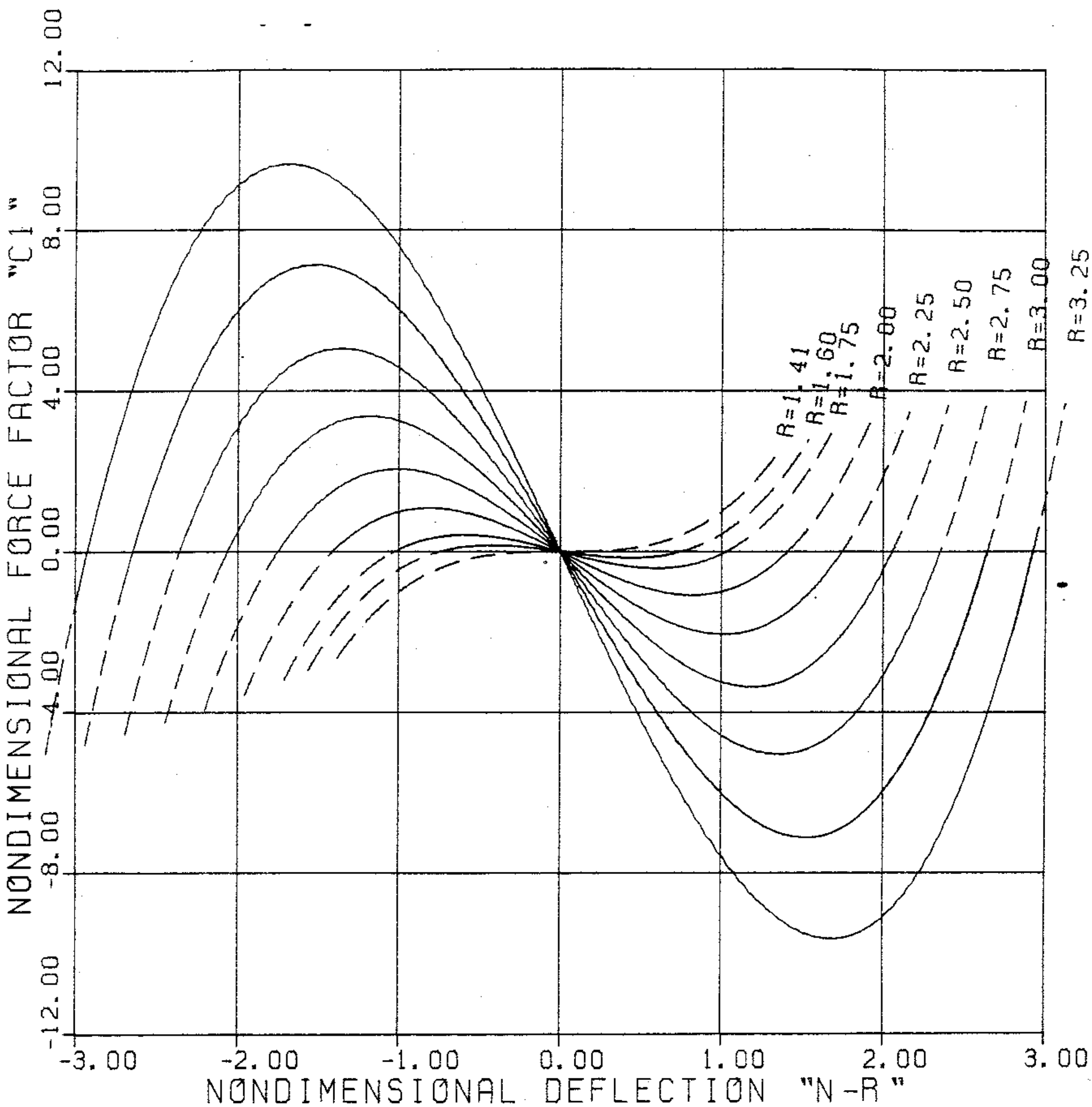
Investigating the characteristics of this equation it is again possible to define all points which are of interest for the design of a back-to-back snap-through spring system. Figure 13 identifies these points in terms of R and Z without derivation. The two bistable points are at $N = R \pm \sqrt{3Z}$ where the snap-through of one spring is exactly balanced by the normal deflection of the other spring. The dashed lines on either end are fictitious and do not exist in reality.

Unlike a single spring where snap-through action can only be achieved when $R > \sqrt{8}$, snap-through action for back to back Belleville springs can be achieved when each spring has at least a height over thickness ratio larger than $\sqrt{2}$. This behavior is illustrated by Figure 14 which shows snap-through characteristics from R = 1.41 to 3.25. Clearly a back-to-back spring design with low R ratios is useless because it exhibits no clearly defined force peaks. Note also that the latching force increases rapidly for larger ratios without a significant increase in latching stroke.



Force factor representation of two identical back-to-back Belleville springs in terms of nondimensional parameters R and Z.

Figure 13



Back-to-back spring characteristics for various R ratios.

Figure 14

4.0 SLOTTED BELLEVILLE SPRINGS

In general when designing Belleville springs, it is usually difficult to obtain low forces at relatively large deflections. This fact for nonslotted springs is illustrated by Figure 15 when the end points C_{1H} and C_{1L} of the usable deflection are connected by straight lines for various values of R. For low R ratios the usable stroke is very small, for high R ratios it gets larger but the force difference increases drastically. Good regulator design usually requires a large usable stroke with a low force differential. This is achieved with slotted springs as shown in Figure 16. The slotted spring overcomes the limit of a straight nonslotted Belleville by essentially amplifying by lever action the deflection of the inner edge to a new smaller inside diameter. The force at the new smaller inner diameter decreases proportionally. The large majority of Carleton designed regulators employ slotted springs because of this design flexibility. The lever action of the inner fingers (trapezoidal elements between slots) has the effect of increasing the usable deflection for a given force differential from high to low.

The analysis of a slotted Belleville spring is identical to all previous shown equations except that one new nondimensional factor f_m must be considered as explained in the following paragraph.

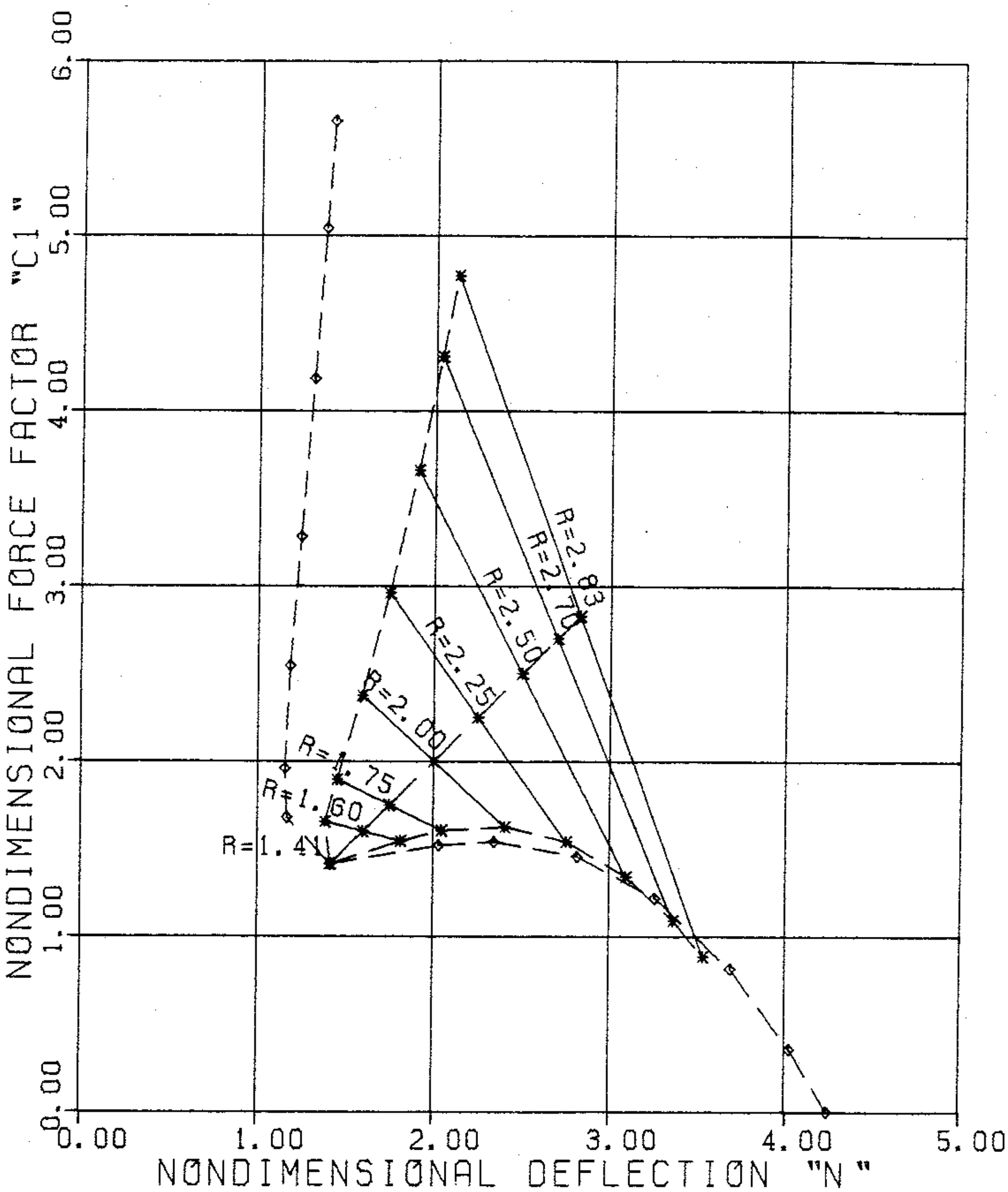
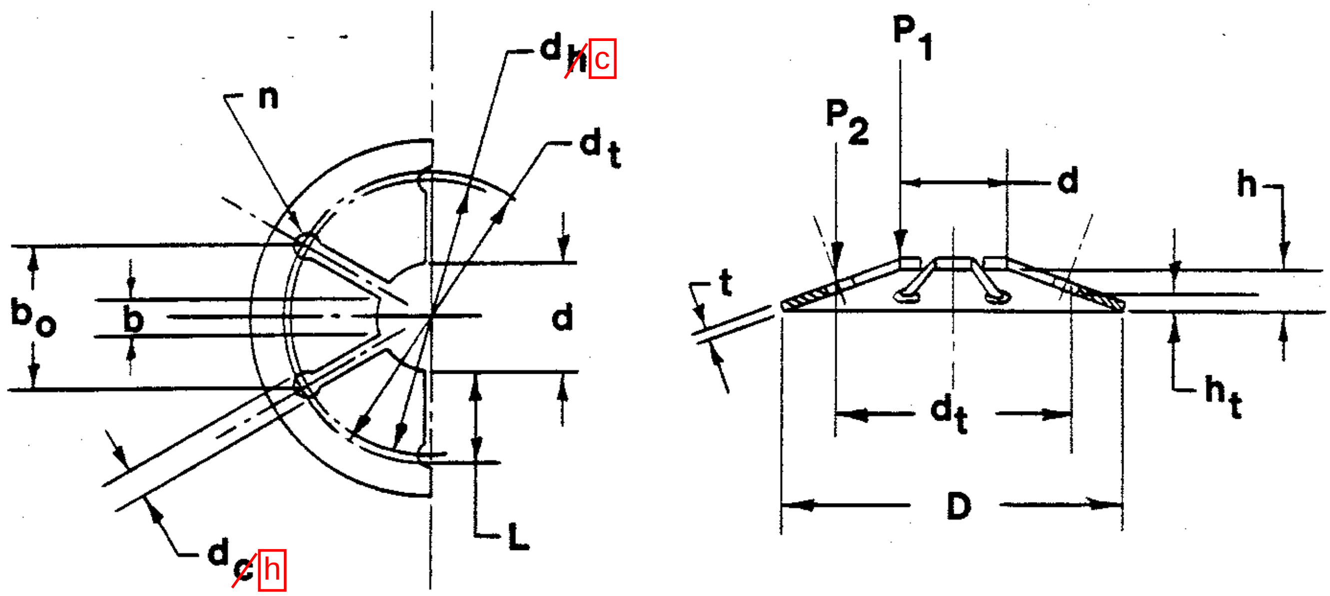


Illustration of negative rates versus usable stroke for nonslotted springs for various R ratios.

Figure 15



Slotted spring dimensional parameters

Figure 16

4.1 Rigid Finger Approach

For the slotted Belleville spring analysis one considers the inner finger to be perfectly rigid and connected to a new effective inside diameter d_t. The diameter d_t is now the inside diameter of the Belleville spring to be designed and has to be used in all previous equations. Most notably the nondimensional diameter ratio is now D/dt.

The new effective ID namely d_t is not identical to the hole circle diameter. An imperical relation was developed, which expresses d_t as function of the hole diameter and hole circle diameter.

$$d_t = 0.72\ d_h + d_c \tag{45}$$

This relation will be used henceforth for every application. In an actual design d_t is decided upon first, then the number of holes, the slot width, and the hole diameter. The hole circle diameter d_c is then calculated per above equation.

The slotted spring inside diameter d shown in Figure 16 is now referred to as <u>inner pivot</u> diameter in order to avoid any confusion with ID of nonslotted Belleville.

The rigid trapezoidal fingers are acting as a lever arm to reduce the force and increase the deflection at the pivot diameter d. Based on simple proportionality of movements with respect to the outer lower edge the following relation is developed.

$$F_1 * \frac{D-d}{2} = F_2 * \frac{D-d_t}{2} \tag{46}$$

$$F_2 = \frac{D-d}{D-d_t} F_1 \quad \text{where } f_m = \frac{D-d}{D-d_t} \tag{47a,b}$$

$$F2 = f_m F_1$$

F_2 can now be substituted for F in equation 13 and the Belleville spring force equation for a slotted spring is:

$$F = \frac{CkEt^4}{r^2} \frac{C_1}{f_m} \tag{49}$$

The force F acts on the inner pivot diameter. The slotted spring factor f_m is always greater than one. The constant C is a function of D and d_t.

The force deflection characteristic C_1 must be calculated with the non-dimensional deflection N at d_t. The deflection f at the inner pivot diameter must be divided by f_m.

$$N = \frac{f}{f_m t} \tag{50}$$

The h/t ratio R is usually an input for the calculation of C_1 and hence not affected. The slotted Belleville spring height however must be corrected for the f_m lever ratio.

$$h = R\ t\ f_m \tag{51}$$

The negative rate equation is established like equations 35 and 36.

$$R_S = \frac{\Delta F}{\Delta f} = \frac{\dfrac{C k E t^4}{r^2} \; \dfrac{\Delta C_1}{f_m}}{t \; f_m \; \Delta N} \tag{52}$$

$$R_S = -1.375 \frac{C k E\ t^3}{r^2 f_m^2} Z \tag{53}$$

This equation matches plotted data for F vs deflection. (ADV)

The spring rate on the inner pivot diameter decreases inversely proportional to the square of the lever ratio. If f_m is equal to 1, all equations reduce to the non-slotted Belleville configuration.

It is good design practice to select a large number of holes and slots. A slotted Belleville spring with many holes more closely matches the above equations. The number of holes is usually determined by the inner pivot diameter, if that diameter is small a limited number of slots and holes can be created. Care must be taken not to make the remaining trapezoidal element too narrow. In this case it will also bend under a force and the force deflection will not be as calculated. The reverse is also true; a rather large trapezoidal element will be very stiff and influence the Belleville spring behavior also adversely. In general however the above equations can be used reliably in Belleville spring design.

5.0 OPTIMIZATION FOR DEFLECTION

In Belleville spring design the greatest limitation is nearly always imposed by the deflection capability. A large deflection at the inside diameter always calls for a large outside diameter. A large O.D. means unrealistic size and weight. When a given deflection between high and low is required the following optimization process may be used.

The conditions for the following equations are:

a) Deflection between C_{1H} and C_{1L} is set or required by design.
b) A given force is required. Force at C_{1M} may be used.
c) R has been selected.
d) Effective "a" ratio (D/d_t) has been selected.
e) Allowable "M" has been determined.
f) A slotted type spring may be used.

By dividing top and bottom of the force deflection equation by t^2 it can be rewritten in terms of the nondimensional factor M.

$$F = \frac{CkEt^2}{M^2 f_m} C_1 \tag{54}$$

where

$$M = \frac{r}{t} = \frac{D}{2t} \tag{55}$$

The deflection at the inside diameter d is Δf between high and low. At d_t however we have to divide Δf by f_m. From Figure 6 it is known that ΔN at d_t equals to $2Z^{1/2}$. Hence:

$$\Delta N = \frac{\Delta f}{f_m t} = 2Z^{\frac{1}{2}} \tag{56}$$

or solving for t^2

$$t^2 = \frac{\Delta f^2}{4Z f_m^2} \tag{57}$$

Substituting t^2 back into the force equation gives:

$$F = \frac{CkE(\Delta f)^2}{M^2 4 Z f_m^3} C_1 \tag{58}$$

Solving now for f_m the best factor for a given deflection and force can be found:

$$f_m = \left[\frac{CkE(\Delta f)^2 C_1}{M^2 4 Z F}\right]^{\frac{1}{3}} \tag{59}$$

Since M has to be selected, it may be more advantageous to express M in terms of r or D/2.

$$f_m^3 = \frac{CkE(\Delta f)^2 C_1 4 t^2}{D^2 4 Z F} \tag{60}$$

and again substituting for t^2

$$f_m^5 = \frac{CkE(\Delta f)^4 C_1}{D^2 4 Z^2 F} \tag{61}$$

The last equation can now be used to find an optimized f_m for a given outside diameter force and deflection. Stresses however which are addressed in the following paragraph should also be checked.

In order to find the inside diameter d, the definition of f_m is solved for d.

$$d = D - f_m (D - d_t) \tag{62}$$

6.0 STRESSES IN BELLEVILLE SPRINGS

The detail derivation of Belleville stress equations can be found in college level textbooks and are not repeated here. A few general comments however are in order. Stress in Belleville springs is predominately shear stress; they can become very large with only minor changes in configuration, h/t or M ratios.

Non-Slotted Springs:

Stresses on the upper-inner edge are always in compression and overstressing may result in local edge yielding. Such yielding once accomplished is not considered to be very critical because it is in compression and the edge will work harden to higher stress level.

The more severe stresses in tension occur at the lower inner edge. With the onset of deflection the surface stress at this edge is in slight compression (see Figure 17b) but increases very rapidly in tension upon further deflection. The spring should be designed so that the tensile stress at this edge does not exceed the material yield point when deflected to its maximum operating range. Some safety factor should be considered. It is also important to have this edge well deburred and polished. Imperfections at this edge are likely to result in fatigue failure when cycle tested.

Stresses on the outer lower edge are also in tension, but the magnitude is usually less when compared to the other edges. However, this is not always true, particularly at small O.D. to I.D. ratios and large height to thickness ratios. It is necessary to examine the stresses at both lower edges to be certain that the allowable tensile stresses are not exceeded throughout the working deflection range.

Slotted Springs:

For slotted springs, the stress at the upper inner edge is concentrated around the relief hole. This type of spring should be designed with a lower stress at this edge than normal. A large number of holes are also useful in lowering the stress concentration around each hole.

Snap-through Springs:

Because of the high h/t ratio, snap-through springs are stressed higher than other springs. The maximum stress occurs at the snap-through point. The maximum stress on snap-through springs should be well below the material yield point in compression and tension. Excessive stresses on the inner edge could lead to fatigue problems, especially if the spring is also slotted.

Typical stress distribution on the upper and lower surfaces are shown in Figure 17.

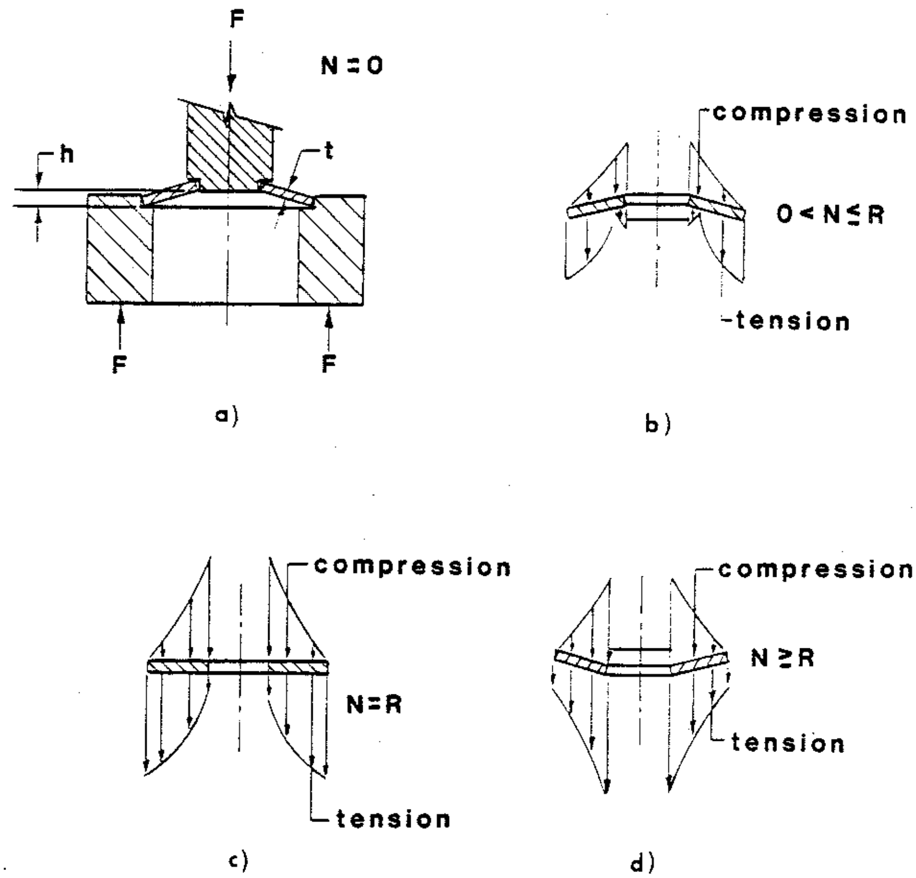


Illustration of surface stress distribution for various deflections. In (a) the supporting members for force application are shown. For clarity views b, c, & d are shown without supports.

Figure 17

6.1 Compressive Stress at Inner Upper Edge

The stress is given by:

$$S_a = \frac{CEf}{(1-\nu^2)r^2}\left[C_2\left(h - \frac{f}{2}\right) + C_3 t\right] \tag{63}$$

where C_2 and C_3 are constants given in Section 2.2. Rewriting this equation with the more familiar nondimensional factors and also using $M = r/t$ gives:

$$S_a = \frac{CkE}{M^2} N\left[C_2\left(R - \frac{1}{2}N\right) + C_3\right] \tag{64}$$

When this equation is differentiated the maximum stress can be found to occur at:

$$N = R + \frac{C_3}{C_2} \qquad \text{for } S_{amax} \tag{65}$$

This result reentered gives the following maximum stress S_{amax}.

$$S_{amax} = \frac{CkE}{M^2} \quad \frac{C_2}{2}\left(R - \frac{C_3}{C_2}\right)^2 \tag{66}$$

The result shows that the magnitude of the maximum compressive stress is independent of deflection. This stress is therefore taken as unity for Figure 18 and the Figures in Section 9. A multiplication factor is then given for the lower outer edge stress.

6.2 Tensile Stress at Inner Lower Edge

The stress at this edge is the same as before except for change in sign for C_3.

$$S_b = \frac{CEf}{(1-\nu^2)r^2}\left[C_2\left(h - \frac{f}{2}\right) - C_3 t\right] \tag{67}$$

Rewriting this equation again with nondimensional factors gives:

$$S_b = \frac{CkE}{M^2} N\left[C_2\left(R - \frac{1}{2}N\right) - C_3\right] \tag{68}$$

When this equation is differentiated a maximum compressive stress occurs at:

$$N = R - \frac{C_3}{C_2} \tag{69}$$

but no maximum tensile can be found. <u>The tensile stress at this edge keeps increasing with the square of the deflection.</u>

The maximum compressive stress occurs at low N values and is given by:

$$S_{bmax} = \frac{CkE}{M^2} \quad \frac{C_2}{2}\left(R - \frac{C_3}{C_2}\right)^2 \tag{70}$$

This stress is usually very small and of no design interest. The stress given by equation 68 is very important and must be checked for the maximum value of N.

6.3 <u>Tensile Stress at Outer Lower Edge</u>

Two new constants C_4 and C_5 given in section 2.2 determine this stress and the equation is:

$$S_c = \frac{Ef}{(1-\nu^2)r^2}\left[C_4\left(h - \frac{f}{2}\right) + C_5 t\right] \tag{71}$$

Note that the dimension constant C is not part of this equation. Again rewriting this equation in with nondimensional factors gives:

$$S_c = \frac{kE}{M^2} N\left[C_4\left(R - \frac{1}{2}N\right) + C_5\right] \tag{72}$$

When this equation is differentiated a maximum value at

$$N = R + \frac{C_5}{C_4} \tag{73}$$

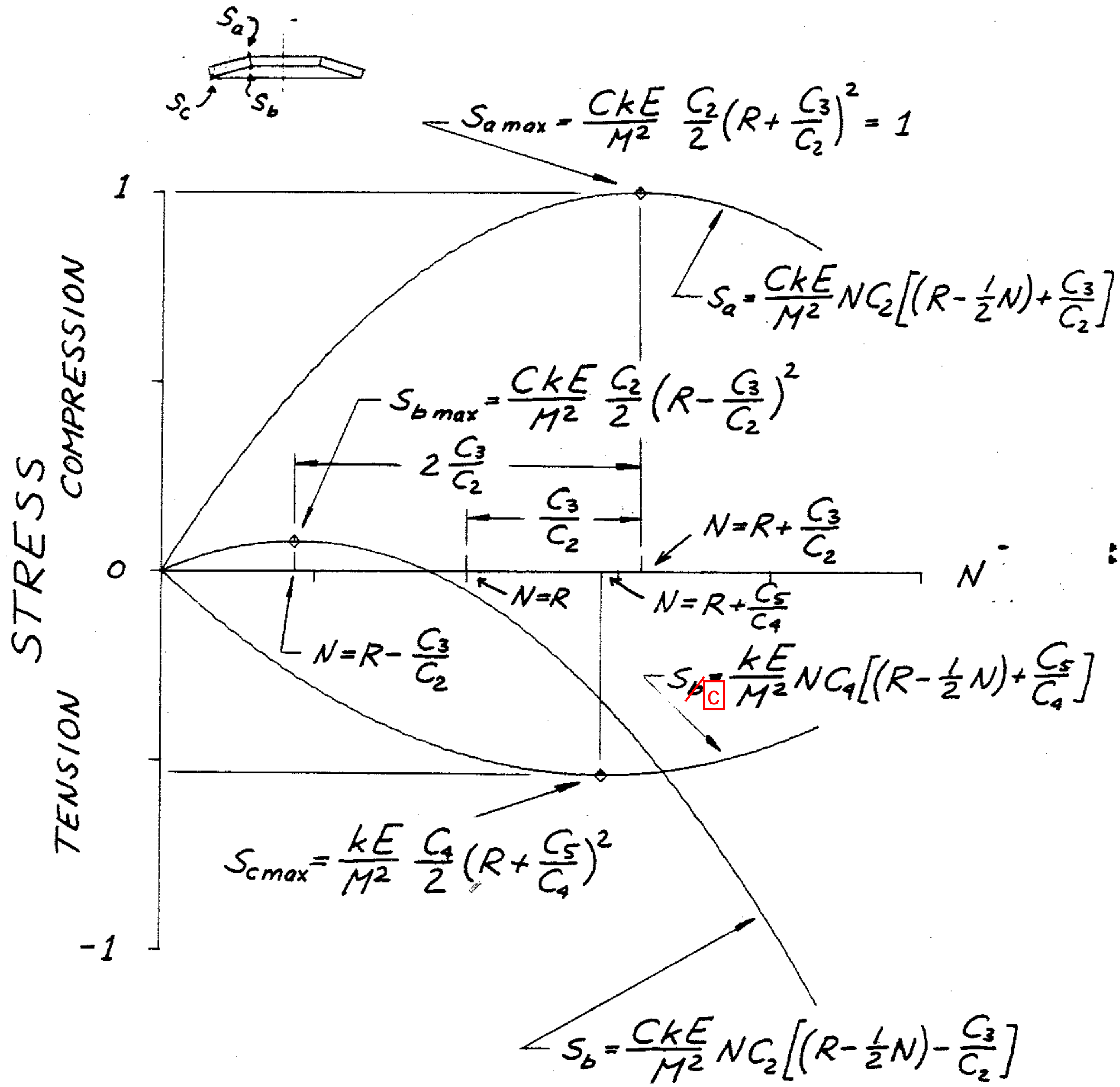
is found. Resubstituting gives the maximum tensile stress

$$S_{cmax} = \frac{kE}{M^2} \quad \frac{C_4}{2}\left(R + \frac{C_5}{C_4}\right)^2 \tag{74}$$

The magnitude of this maximum tensile is always less than the magnitude of the maximum compressive stress S_a. For large deflection in equation 72 the stress may actually become positive. This is not of practical importance because S_b has grown very large at this point already.

6.4 Stress Versus Deflection

All three sets of stress equations are plotted versus nondimensional deflection in Figure 18. The maximum stress S_a is used to nondimensional the stress axis. The resulting curves are typical for all Belleville springs. The curves shown in Figure 18 are based on R = 2 and a = 2. It is also seen from this plot that S_b increases rapidly and limits the spring deflection capability.



Typical stress versus deflection curves for Belleville springs (R = 2 and a = 2).

Figure 18

7.0 BELLEVILLE SPRING DESIGN USING THE "~~BELVPL~~ BELLVLE" COMPUTER PROGRAM

The computer program BELVPL is written in Fortran and solves all previous equations. It is designed to be interactive so very quickly new values can be recalculated. It provides printed or plotted output. If the plotted output is selected, the program will draw the force curve and all three stress curves and identify the proper points. It will also allow the user to make a scale drawing of the Belleville which was calculated.

A sample input and output listing is shown in Table 2a and 2b. The output plot is shown in Figure 19 and contains all design information which might be needed.

A complete listing of the computer program may be found in the Appendix.

```
B:BELVPL.

ENTER BELLEVILLE SPRING PARTNUMBER
?EXAMPLE

OUTSIDE DIA.
?1.75

IS THIS SPRING SLOTTED?  1 YES,  0 NO
?1

INNER PIVOT DIA
?.625

EFFECTIVE I.D.
?1.1

DO YOU WANT THE PROGRAM TO CALCULATE NO. OF HOLES ?
1  YES, 0  NO
?0

NUMBER OF HOLES ?
?18

HOLE DIAMETER ?
?.058

SLOT WIDTH ?
?.025

THICKNESS
?.022

H/T
?2.2

DATA VERIFICATION:
EXAMPLE
DO = 1.750000
DI = .6250000
DIT = 1.100000
T = .2200000E-01
R = 2.200000
FM = 1.730769
A = 1.590909
WGHT = .1534570E-01
C = 1.762187
C2 = 1.121603
C3 = 1.215313
C4 = 1.449953
C5 = 1.346154
NH = 18
DHC = 1.058240
DH = .5800000E-01
SLW = .2500000E-01
FOR PRINTED OUTPUT NTER 1, PLOTTED 2
?1
```

Table 2a

```
O.D. = 1.750 INCHES
I.D.(PIVOT DIA) = .625 INCHES
EFFECTIVE I.D. = 1.100 INCHES
THICKNESS = .0220 INCHES
CONE HEIGHT =.0838 INCHES
CONE ANGLE = 8.47 DEGREES
H/T = 2.20
MECHANICAL ADVANTAGE = 1.731


HIGH FORCE = 23.51 LBS AT .0467 INCHES
MID FORCE = 16.57 LBS AT .0838 INCHES
LOW FORCE =  9.63 LBS AT .1208 INCHES
RATE = -257.46 LBS/INCH
DELTA HIGH TO LOW= .0741 INCHES


THE MAXIMUM STRESSES ARE:

SAMAX   UPPER INNER EDGE = 214652. PSI AT .1250 INCHES
SBMAX   LOWER INNER EDGE =  24816. PSI AT .0425 INCHES
        LOWER INNER EDGE =-139255. PSI AT .1389 INCHES
SCMAX   LOWER OUTER EDGE =-142941. PSI AT .1191 INCHES


ENTER:  1 FOR NEW THICKNESS
        2 FOR NEW H/T
        3 FOR NEW T & H/T
        4 FOR ALL NEW DATA
        5 TO STOP
        6 TO PLOT
?6

DRAWING SCALE ?  ENTER 1, 2, 4, OR 10
?2

ZZZZZZZZZZ
```

Table 2b

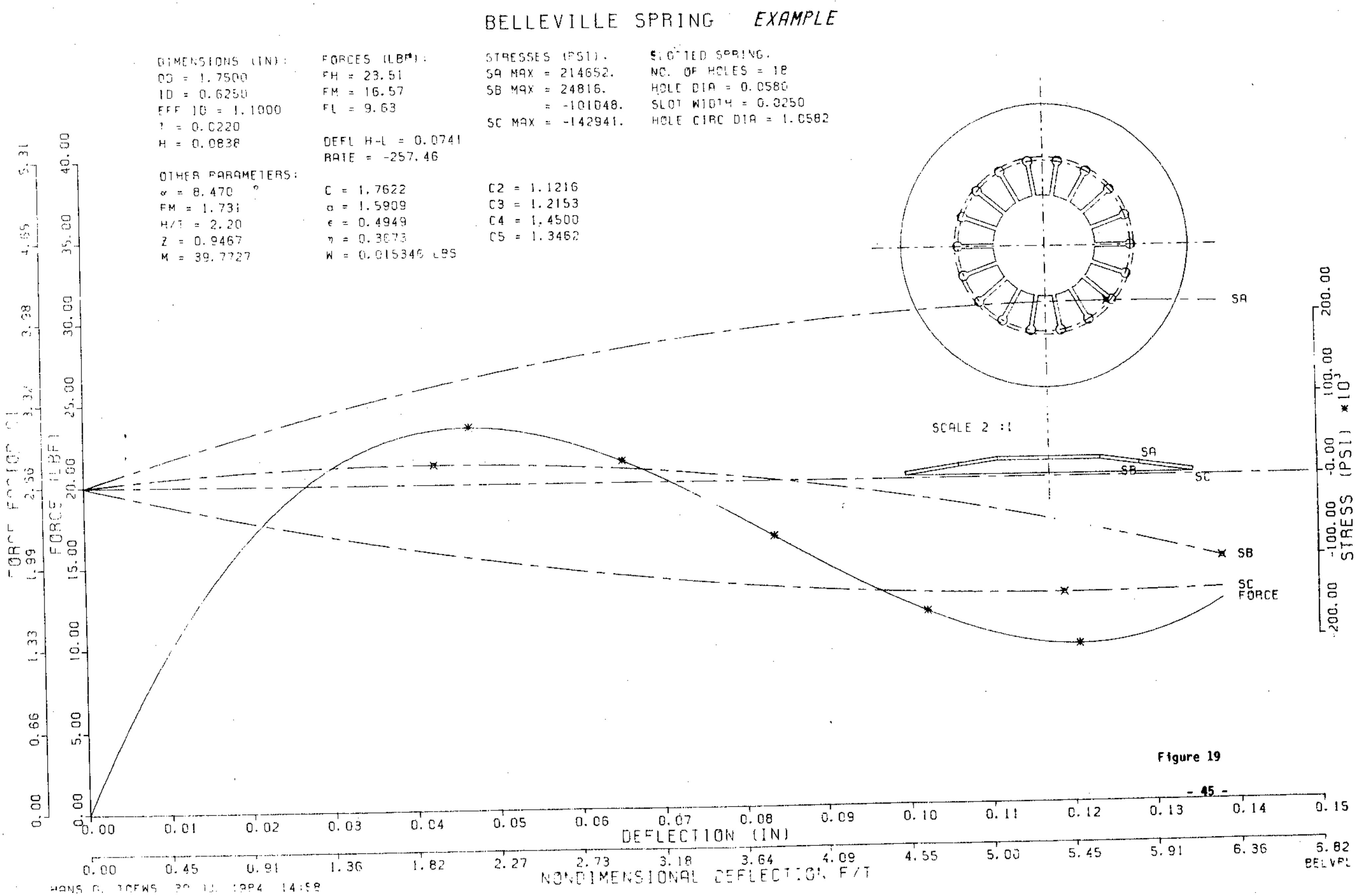
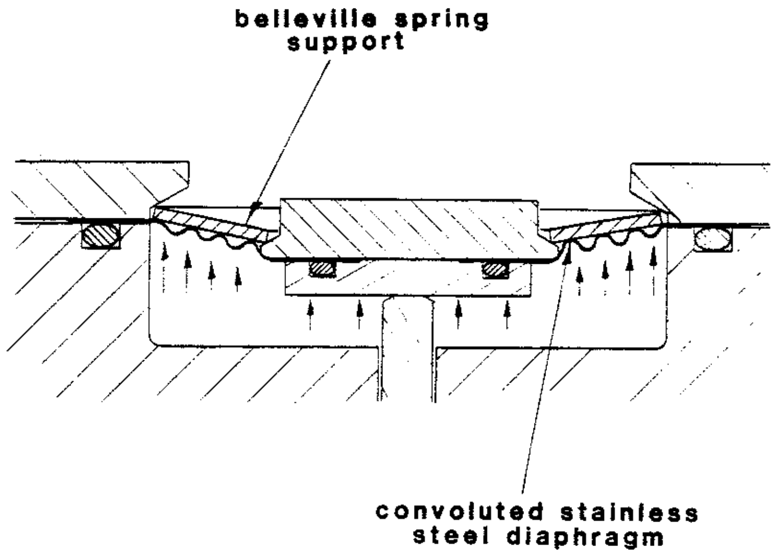

BELLEVILLE SPRING EXAMPLE
DIMENSIONS (IN):
OD = 1.7500
ID = 0.6250
EFF ID = 1.1000
T = 0.0220
H = 0.0838
FORCES (LBF):
FH = 23.51
FM = 16.57
FL = 9.63
DEFL H-L = 0.0741
RATE = -257.46
STRESSES (PSI).
SA MAX = 214652.
SB MAX = 24816.
= -101048.
SC MAX = -142941.
SLOTTED SPRING.
NO. OF HOLES = 18
HOLE DIA = 0.0580
SLOT WIDTH = 0.0250
HOLE CIRC DIA = 1.0582
OTHER PARAMETERS:
α = 8.470 °
FM = 1.731
H/T = 2.20
Z = 0.9467
M = 39.7727
C = 1.7622
α = 1.5909
ε = 0.4949
η = 0.3673
W = 0.015346 LBS
C2 = 1.1216
C3 = 1.2153
C4 = 1.4500
C5 = 1.3462
SCALE 2 :1
SA
SB
SC
FORCE
FORCE (LBF)
STRESS (PSI) $*10^3$
DEFLECTION (IN)
NONDIMENSIONAL DEFLECTION F/T
BELVPL

Figure 19

8.0 EFFECTIVE BELLEVILLE SPRING AREA AND WEIGHT

A typical pressure regulator design is shown in Figure 20. The total Belleville spring area, including the closed center hole, see pressure. The spring deflects with increasing pressure throughout its deflection range. Complete pressure sealing of the slots, the O.D. and I.D. is usually accomplished by a thin metal diaphragm. This thin diaphragm is sometimes made of rubber and of very negligible influence on the spring characteristics in most applications. The total force acting on the Belleville spring is now different and a new effective area factor must be considered.



There is a difference between pure inner edge loading and total area pressure loading. Note that in this case the spring is shown inverted.

Figure 20

So far in this discussion all formulas given are such that the force acts on the inside diameter edge and the opposing force on the outside diameter edge. In pressure regulator design, however this is not necessarily so, a distributed load, namely pressure, acts on the total Belleville area. An effective pressure area must now be determined which provides the correct force on the I.D. edge such that all previous equations are valid.

It is intuitively obvious that this effective pressure area must somehow be larger than the circular inside diameter area and smaller than the circular area formed by the outside diameter.

If ϵ is taken to be the coefficient, the effective area can be expressed as:

$$A_{eff} = \epsilon \frac{\pi D^2}{4} \tag{75}$$

Also, Aeff = pi/12*(D^2+d^2+Dd)
Derived from pi/4*Dh (vol of cylinder) = pi/3*h*(D^2+d^2+Dd)
(vol of frustrum of cone)

where ϵ is value less or equal to 1. When A_{eff} is multiplied by the pressure acting on the complete Belleville a force F results, which can be considered as acting on the ID edge.

8.1 The Effective Area Coefficient

The coefficient ϵ can be derived from moment consideration and uniform distributed loads. The derivation is rather lengthy and carried out in the Appendix. Only the result is given here as follows:

$$\epsilon = \frac{1-\left(\frac{d}{D}\right)^3}{3\left(1-\frac{d}{D}\right)} \tag{76}$$

where d is the Belleville inside diameter, for slotted or unslotted springs, and D the outside diameter.

In terms of "a" for unslotted springs:

$$\epsilon = \frac{1-\frac{1}{a^3}}{3\left(1-\frac{1}{a}\right)} = \frac{\frac{a^3-1}{a}}{3\frac{(a-1)}{a}} = \frac{a^3-1}{3a^2(a-1)} \tag{77}$$

The appropriate force F acting on d due to the pressure p is given now by:

$$F = \epsilon \frac{\pi D^2}{4} p \tag{78}$$

Figure 21 shows a plot of ϵ versus d/D. Clearly small values for d/D where d approaches zero are not practical.

8.2 The Equivalent Weight of a Belleville Spring

As in the previous paragraph, one cannot simply calculate the weight (or weigh a spring) and assume that this is the weight which acts dynamically. The outside edge is usually restrained and the inside edge is attached to other moving parts. It is of interest to know the equivalent force that appears on the inner edge due to its own weight. The complete derivation is shown in the Appendix. The result is given here. The factor η which modifies the weight is given by the following equation:

$$\eta = \frac{1 - 3\left(\frac{d}{D}\right)^2 + 2\left(\frac{d}{D}\right)^3}{3\left(1 - \frac{d}{D}\right)} \tag{79}$$

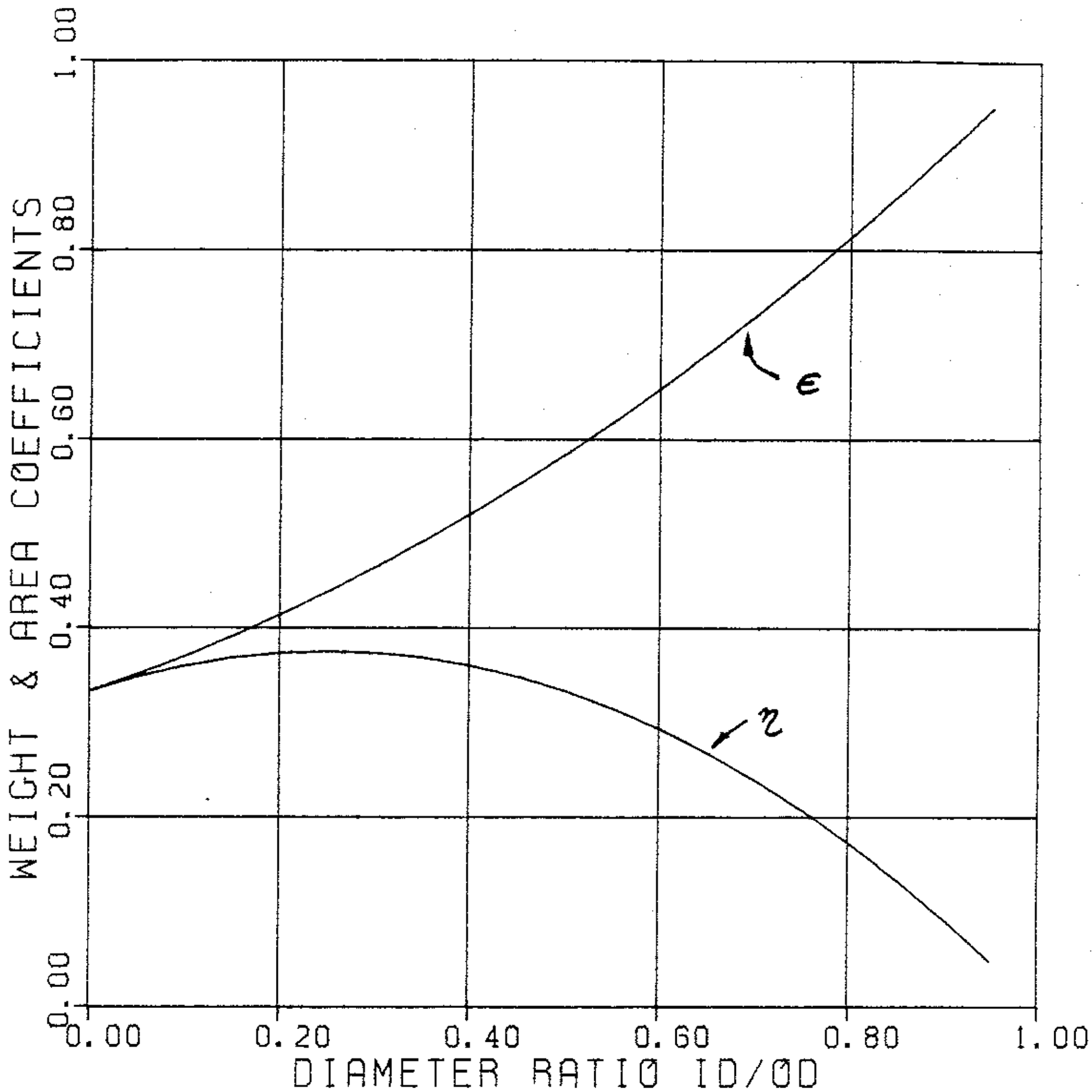
The equivalent weight acting on the inside edge is then given by

$$W_{equ} = \eta W = \eta \frac{\pi D^2 t}{4} \gamma \tag{80}$$

where γ is the density ~~specific gravity~~ of the material. Minor adjustments for slots and holes must be made for slotted Belleville springs.

8.3 Pressure Deflection Equation

Since in pressure regulators Belleville spring deflection is accomplished by pressure as shown in Figure 20, the Belleville force equation can be rewritten in terms of pressure involving a new constant C_p.



Effective area and equivalent weight coefficient for various diameter ratios d/D.

Figure 21

Recalling equation 13 from page 7 and substituting P A_{eff} for F gives:

$$P\,A_{eff} = \epsilon \frac{\pi D^2}{4} P = \frac{C k E t^4}{r^2} C_1 \qquad (81)$$

or

$$P\,A = P \frac{\pi D^2}{4} = C_p \frac{k E t^4}{r^2} C_1 \qquad (82)$$

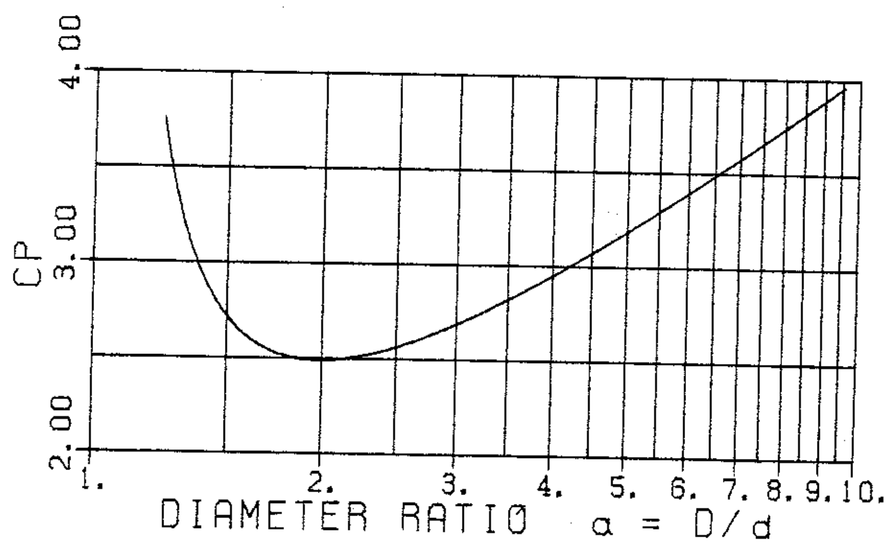
where the constant C_p is defined

$$C_p = \frac{C}{\epsilon} \qquad (83)$$

Both C and ϵ are functions of the O.D. and I.D. dimensions only and therefore, for unslotted springs, can be reduced to:

$$C_p = \frac{\pi}{2} \ln a \frac{a^4}{(a-1)(a^3-1)} \qquad (84)$$

C_p is plotted in Figure 22 and clearly shows a minimum value at a = 2. At this point C_p has the smallest value which means that for any given pressure acting on the total spring area the diameter ratio should be a = 2 to give the most efficient spring for such a design.



Pressure deflection coefficient of a Belleville spring with center hole closed.

Figure 22

9.0 INTERDEPENDENCE OF BELLEVILLE SPRING DESIGN PARAMETERS

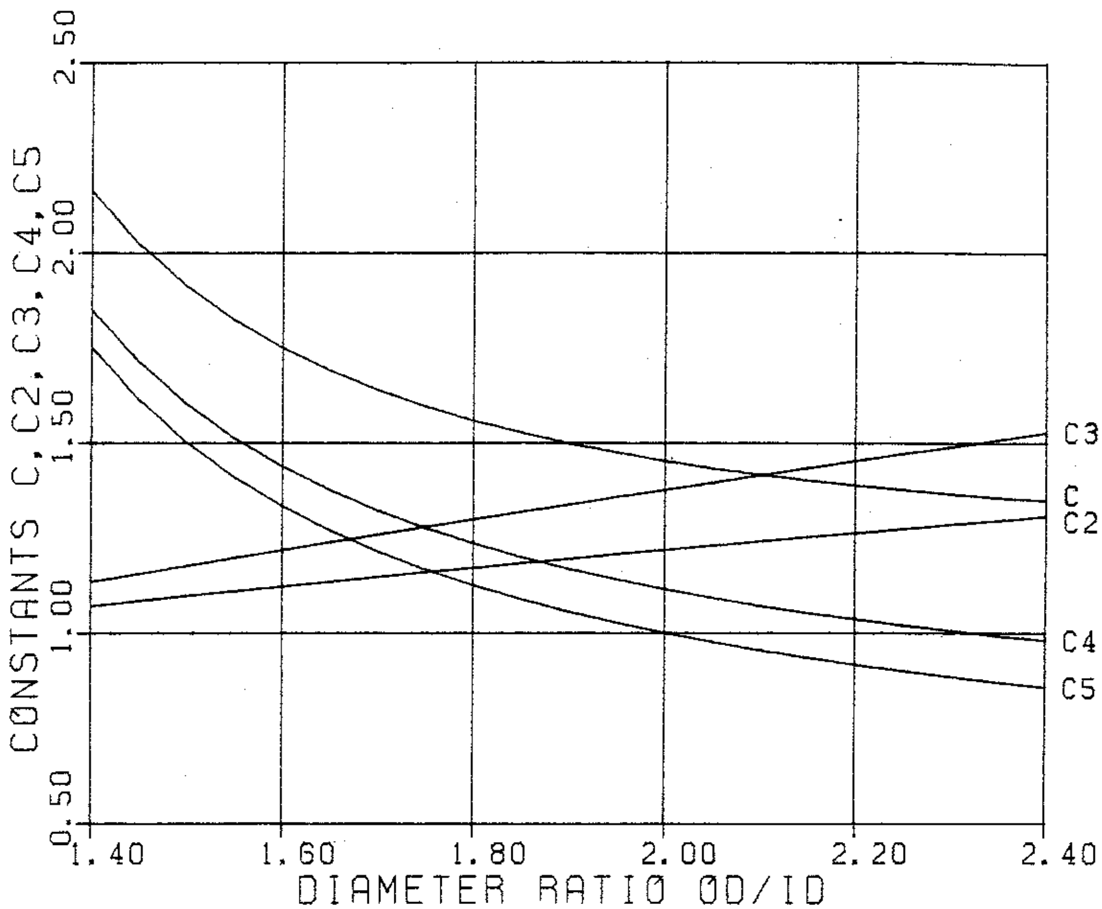
This section attempts to provide a "design feel" for Belleville springs by presenting several plots. The intent is to show the interdependence of height to thickness ratio $R = h/t$, the diameter ratio $a = D/d$ and the diameter to thickness ratio $M = D/2t$ as they relate to allowable or selected stress levels.

First Figure 23 shows all stress related constants versus diameter ratio as defined in Section 2.2. This graph may be used for quick scaling of all five C values.

Figures 24 through 26 are designed to show the effect on the three edge stresses when two out of the three parameters (R, a and M) are held constant. In each case the graphs are normalized so that 100% stress corresponds to S_{amax} = 200,000 PSI for R = 2 and a = 2. The value of M for this condition is fixed by the material properties and calculates to be 37.1705 for $\nu = .3$ and $E = 29 \times 10^6$.

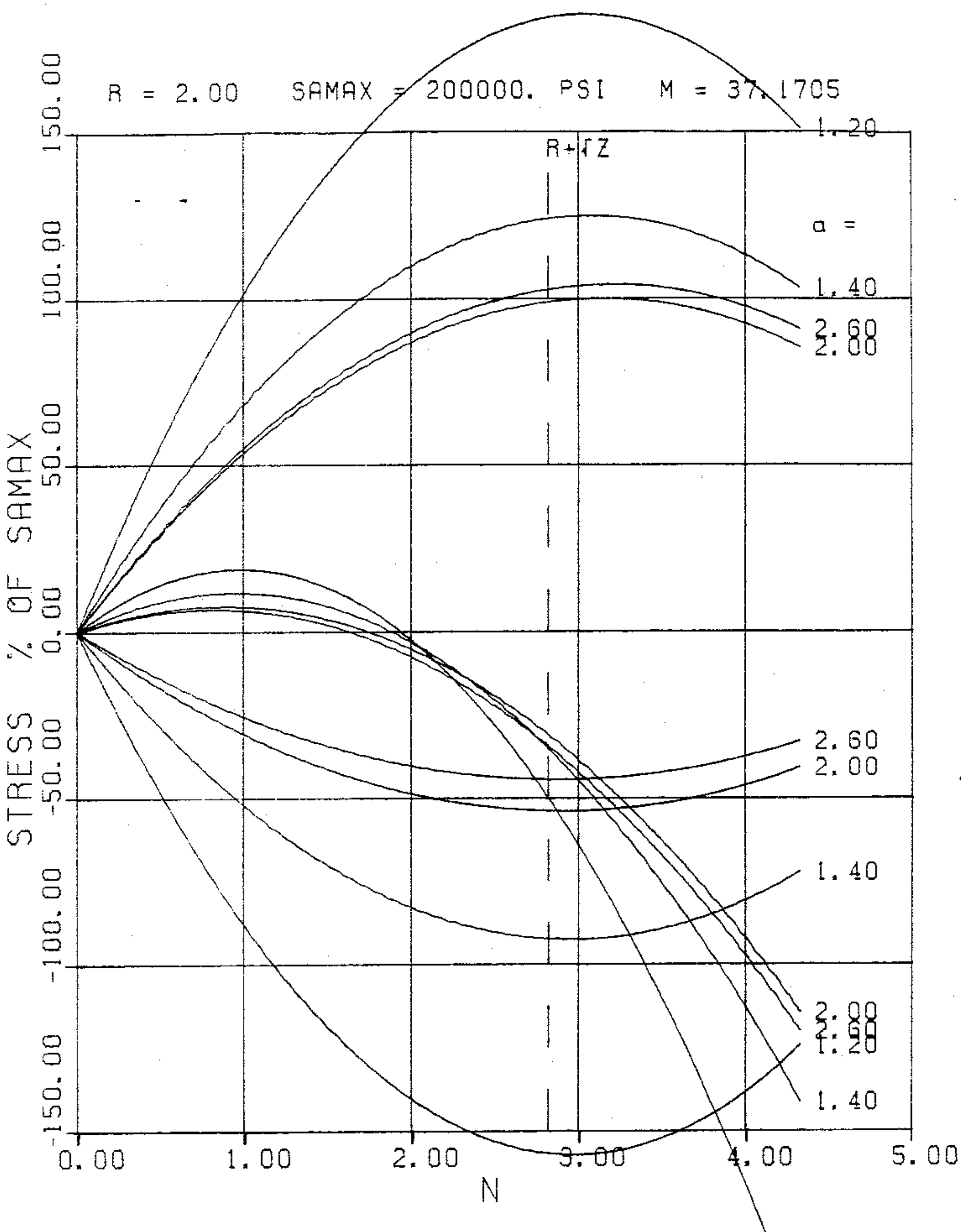
To complete the picture and obtain a feeling to what stress levels springs are usually deflected, a dashed vertical is shown. This line corresponds to the force at the low point at $N = R + \sqrt{Z}$. Deflection beyond this point is usually not required for regulator design.

Figure 26 shows that it is important to design Belleville springs with the correct M ratio to obtain allowable tensile stresses on the lower outer edge. This constraint relates directly to regulator design. In practice a regulated pressure (fixed force) and a given envelope (fixed O.D.) is usually specified. The force, proportionally to the fourth power of thickness, can in principle be achieved if there were no constraint on the outside diameter. The M ratio could simply be lowered until the allowable stresses are obtained. This however would make for a large regulator. The overriding constraint is frequently envelope, stipulating a maximum spring outside diameter. The next constraint is allowable stress and the force falls out. If the force is not enough to balance the required regulated pressure, multiple springs in parallel are often used.



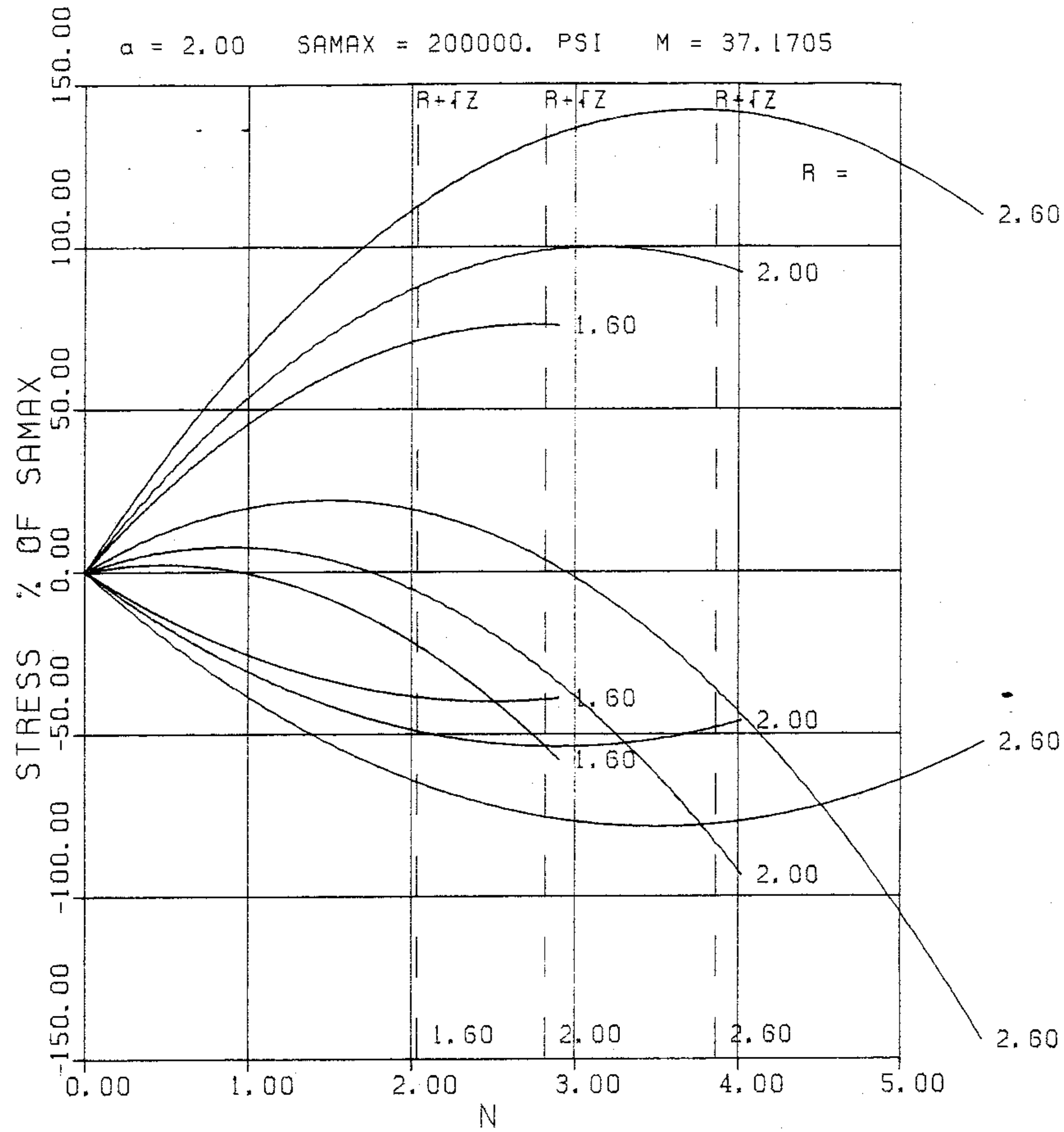
The stress constants C_2, C_3, C_4, C_5 and the dimension load constant C as function diameter ratio a = D/d.

Figure 23



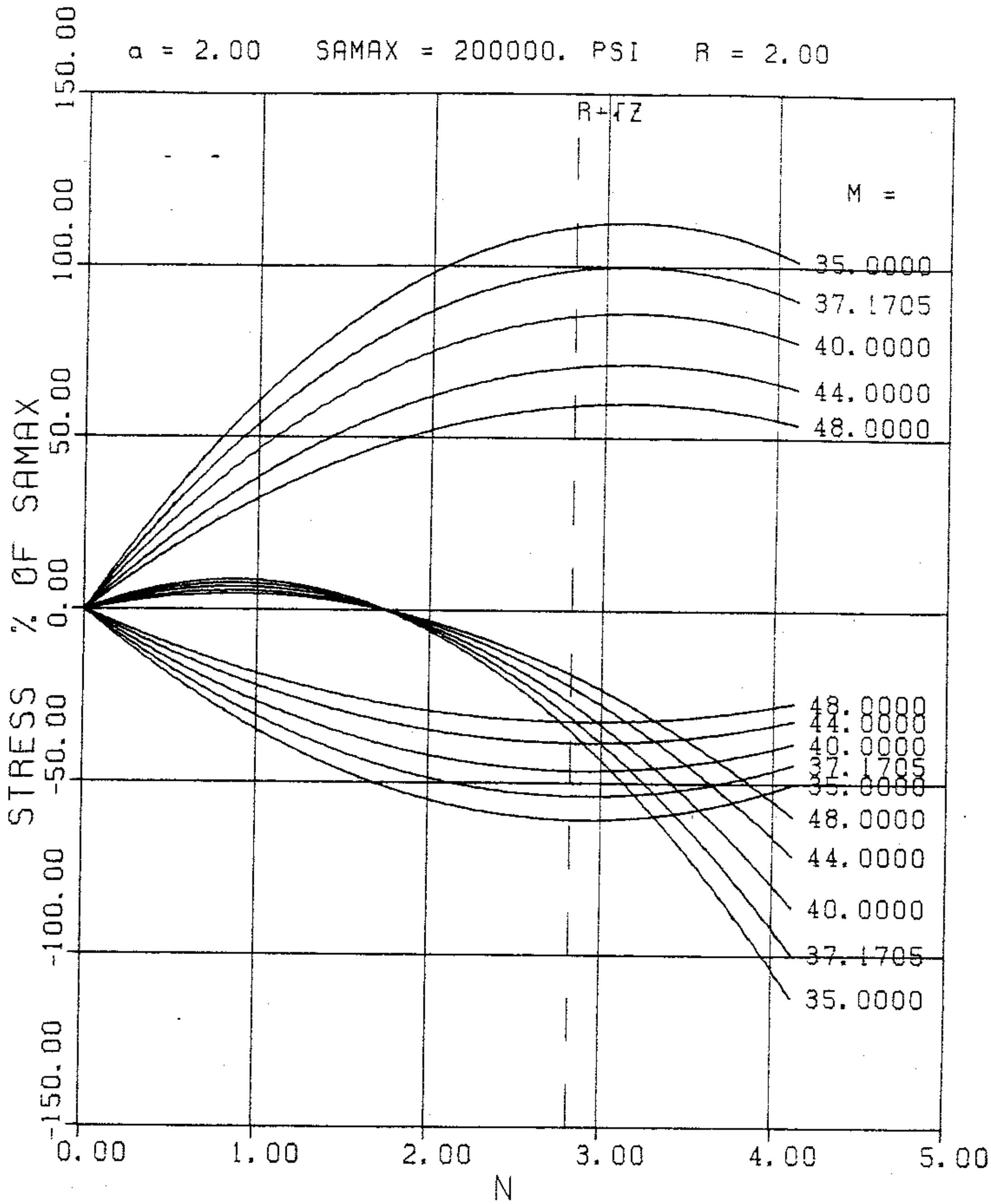
Variation in stress levels for various a's with R and M fixed. Note: Tensile stresses on the lower outer edge increases drastically with low diameter ratios.

Figure 24



Variation in stress levels for various R's with a and M fixed.

Figure 25



Variation in stress levels for various M's with a and R fixed. R = 2 and a = 2 are 100% when S_{amax} = 200,000 PSI.

Figure 26

9.1 STRESS DEFLECTION DESIGN ENVELOPE

To improve and extend the information given by Figures 24 through 26 the graph shown by Figure 27 was developed. This graph contains extremely usable and valuable design stress information on a single page. The behavior of Belleville edge stresses S_a, S_b and S_c versus nondimensional deflection for 36 realistic combinations of "R" and "a" are illustrated in Figure 27. This display of stress curves can be scaled directly for stress and deflection. The vertical scale is 200,000 PSI per 1/2 inch either in compression or tension and the horizontal scale relates N = 5 to 1 inch. The curves visually demonstrate the increase of stresses with decreasing "a" and increasing "R" ratios. Again for each "R" ratio the deflection to the low force point C_{1L} is shown by a vertical dashed line.

All curves as before are normalized to S_{amx} = 200,000 PSI (100%) for R = 2 and a = 2. M is then determined to be 37.1705 for ν = .3 and E = 29 x 10^6.

BELLEVILLE EDGE STRESSES VERSUS DEFLECTION

☆ 100% IS SAMAX = 200000. PSI AT R = 2.00 & α = 2.00 M = 37.1705

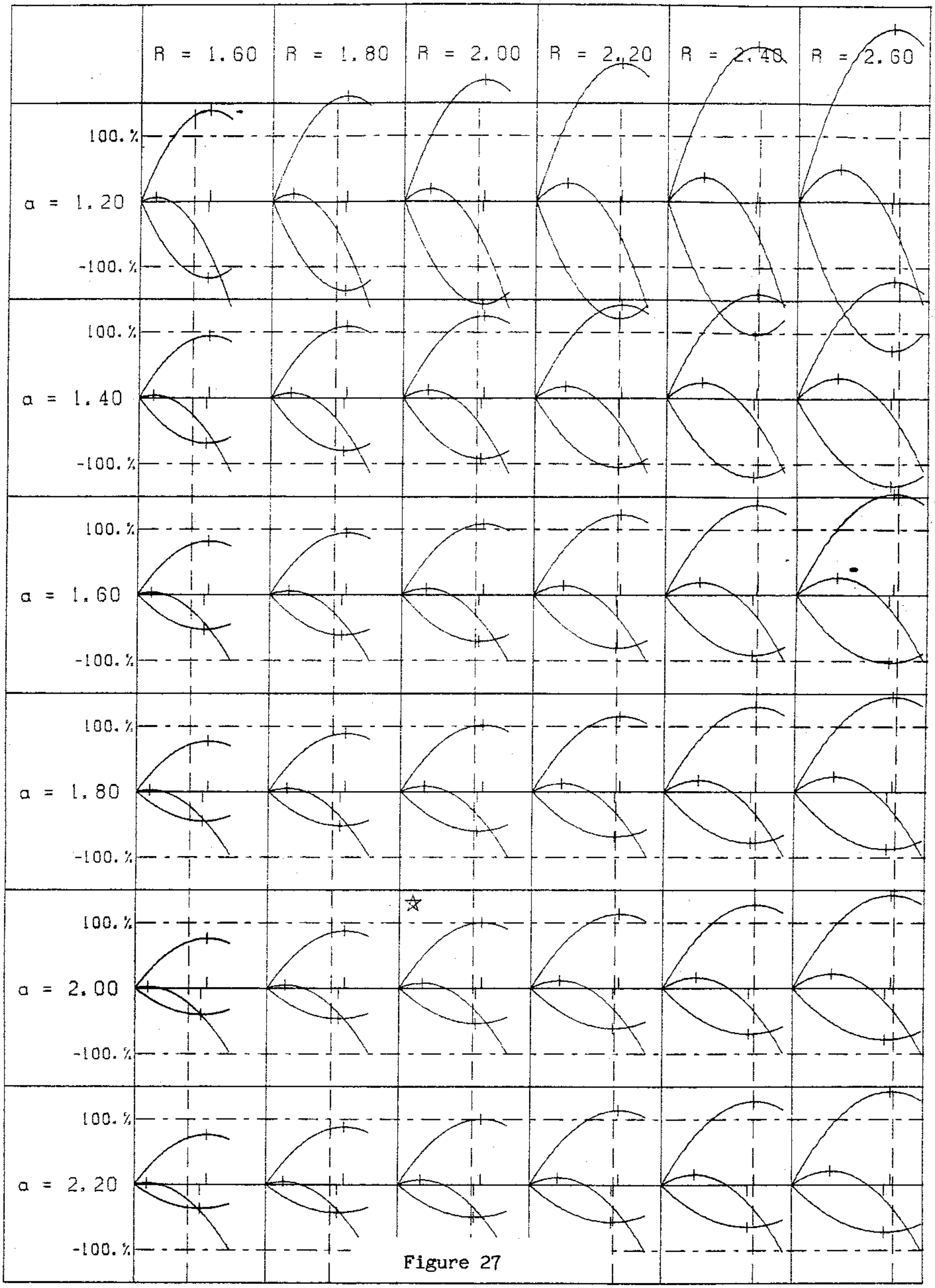


Figure 27

— — — — — — N = R + √Z SCALE N = 5 PER INCH

The same set of curves in Figure 27 can also be used for different values of M, realizing that M^2 is inversely proportional to stress:

$$S = \frac{CONST}{M^2}$$

For the identical ratios therefore:

$$S_1 M_1{}^2 = S_2 M_2{}^2 \tag{86}$$

or

$$S_2 = \frac{M_1^2}{M_2^2} S_1 \tag{87}$$

The stress multilplication factor f_s is defined to be:

$$f_s = \frac{M_1^2}{M_2^2} \tag{88}$$

For example, if $M_1 = 37.1705$ for R = 2 and a = 2 and the O.D. design constraint is such that only M = 34.5 can be obtained for given required force (no change in thickness t), the new stress calculates to be:

$$S = \frac{37.1705^2}{34.5^2} 200{,}000 = 232{,}160 \text{ PSI} \tag{89}$$

This is a 16% increase for all three edge stresses. For S_{cmax} (outer lower edge) this scales from -106,800 to -123,900 PSI. Minus sign denotes tensile stress. Stresses of this magnitude would still be allowable for 17-7PH material.

The same calculation may be made for all other "R" and "a" combinations. Figure 27 is therefore usable for all values of M by the definition of the stress multiplication factor f_s by equation 88. NOTE: Higher M values mean lower stresses or larger O.D.'s, lower M values mean higher stresses or smaller outside diameters for a given spring thickness.

The relationship between stresses and Belleville spring design ratios so graphically displayed in Figure 27 is shown in a more usable form in Figure 28. It contains actually more information because the height to thickness ratio "R" and the diameter ratio "a" vary continuously and are not limited to 36 combinations. Figure 28 shows lines of constant "M" for an infinite number of "R" and "a" combinations when S_{amax} is taken to be 200,000 PSI in all cases.

For example, a spring designed with a = 1.3 and R = 2.3 must have M = 49 for S_{amax} to be 200,000 PSI. The tensile stress on the lower outer edge is then approximately 80% (dashed line on graph) of S_{amax} namely 160,000 PSI. If now that same spring has a <u>different "M" value</u>, S_{amax} is <u>not</u> 200,000 PSI anymore. In this case the foregoing definition of the stress multiplication factor f_s may be used in conjunction with Figure 28. The previously used reference value M = 37.1705 is shown in its proper location for comparison. The 54% tension to compression stress ratio may be verified on Figure 27.

Figures 27 and 28 should always be used in Belleville spring design. They lead very quickly to the first estimate of allowable stress levels. In particular the graph of Figure 28 shows interaction of the three most important Belleville spring design parameters for a fixed stress value and material constants. If another stress value or other material constants were selected the family of curves shown would not change in shape but move with respect to the background grid of "a" and "R". The selected material constants of = 0.3 and E = 29 x 10^6 PSI however should be useful for a great many design applications.

Figure 29 has been added to show the same information as Figure 28 but based on the maximum tensile stress S_{cmax} on the lower outer edge. This stress is taken to be 100,000 PSI. The factors to calculate S_{amax} are now larger than one.

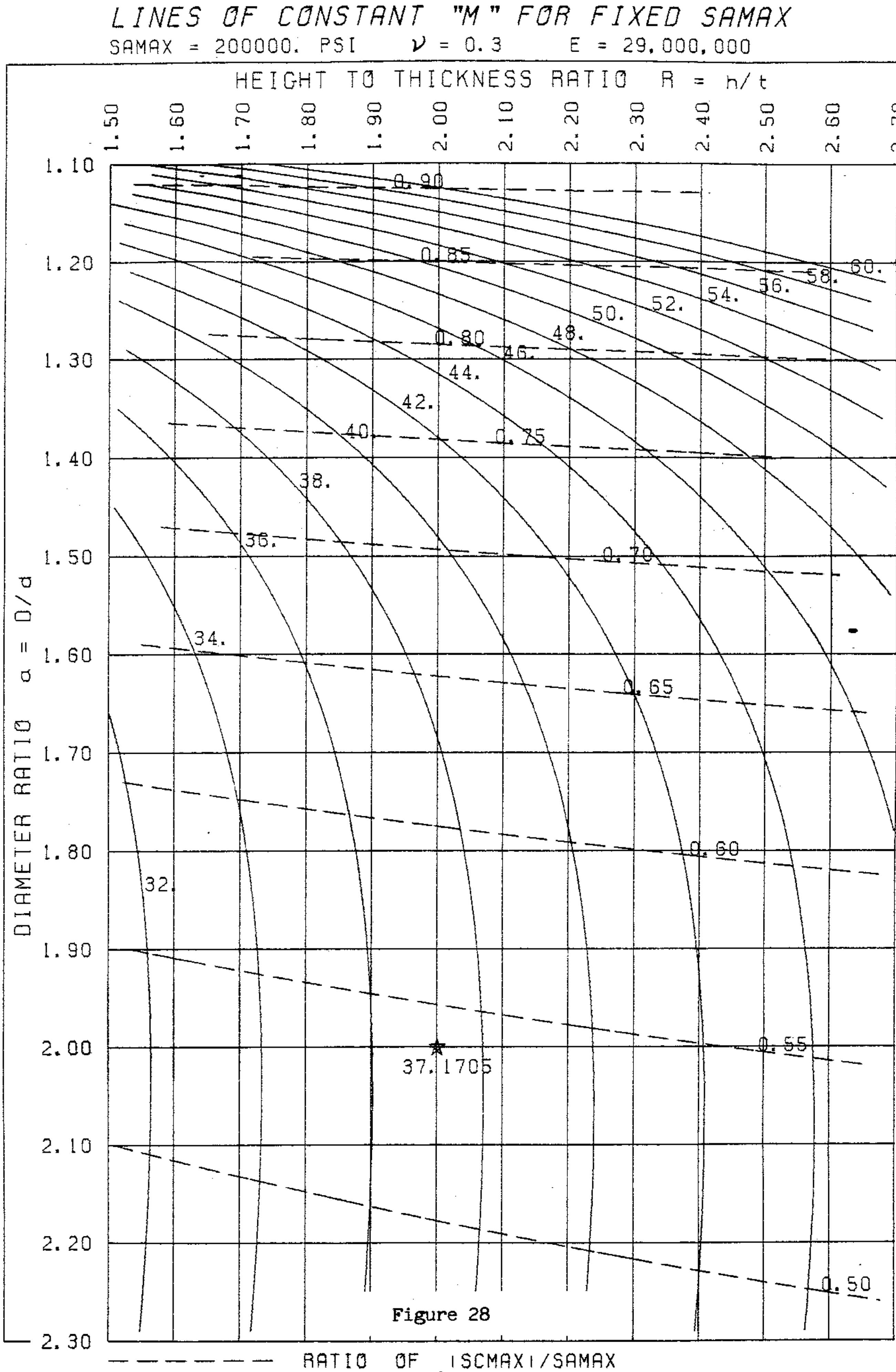

LINES OF CONSTANT "M" FOR FIXED SAMAX
SAMAX = 200000. PSI ν = 0.3 E = 29,000,000
HEIGHT TO THICKNESS RATIO R = h/t
1.50
1.60
1.70
1.80
1.90
2.00
2.10
2.20
2.30
2.40
2.50
2.60
2.70
DIAMETER RATIO α = D/d
1.10
1.20
1.30
1.40
1.50
1.60
1.70
1.80
1.90
2.00
2.10
2.20
2.30
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.55
0.50
32.
34.
36.
38.
40.
42.
44.
46.
48.
50.
52.
54.
56.
58.
60.
37.1705
RATIO OF |SCMAX|/SAMAX

Figure 28

LINES OF CONSTANT "M" FOR FIXED SCMAX

SCMAX = -100000. PSI ν = 0.3 E = 29,000,000

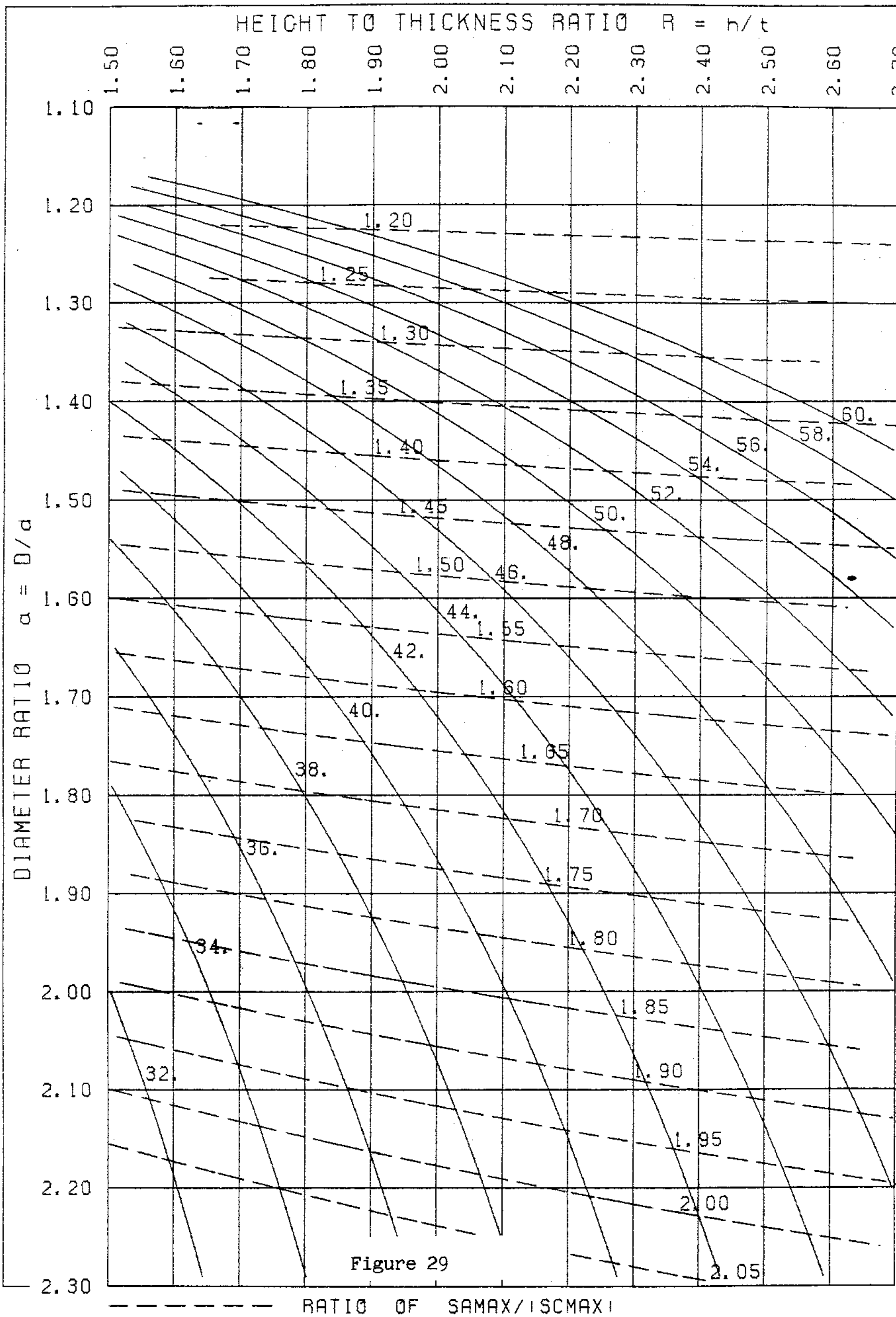


Figure 29

CONCLUDING REMARKS

In general it is possible to theoretically calculate a Belleville spring that will meet any force or stroke requirement. It may be a little unusual and have a large cone angle but calculations show that force and deflection requirements are met. If now the stresses are investigated, frequently surprising large numbers are the result; 300,000 to 400,000 or up to a million PSI is not unusual.

Any Belleville spring design should be checked against Figures 27 and 28. A Belleville spring design for a given application is really an iterative process going back and forth through formulas in this paper and of course Figures 27 and 28. This paper on Belleville spring design is intended to stand on its own and no other reference should be needed for an actual design application.

APPENDIX

Derivation of C_{1H}, C_{1L} and C_{1M}

$$C_1 = N\left[(R-N)(R-\tfrac{1}{2}N)+1\right]$$

$$C_1 = N\left(R^2-\tfrac{3}{2}RN+\tfrac{1}{2}N^2+1\right)$$

$$\frac{dC_1}{dN} = \left(R^2-\tfrac{3}{2}RN+\tfrac{1}{2}N^2+1\right)+N\left(N-\tfrac{3}{2}R\right)$$

$$\frac{dC_1}{dN} = R^2-3RN+\tfrac{3}{2}N^2+1$$

To find max & min values; $\frac{dC_1}{dN}=0$

$$R^2-3RN+\tfrac{3}{2}N^2+1=0$$

$$\tfrac{3}{2}N^2-3RN=-1-R^2$$

$$N^2-2RN=-\tfrac{2}{3}-\tfrac{2}{3}R^2$$

$$N^2-2RN+R^2=-\tfrac{2}{3}-\tfrac{2}{3}R^2+R^2$$

$$(N-R)^2=-\tfrac{2}{3}+\tfrac{1}{3}R^2$$

$$N=R\pm\sqrt{\frac{R^2-2}{3}}$$

let

$$Z=\frac{R^2-2}{3}\quad;\quad \underline{N=R\pm\sqrt{Z}}$$

$$Z = \frac{R^2 - 2}{3} = 0$$

$$R^2 = 2$$

$$R_{min} = 1.4142$$

Resubstitution to calculate C_{1H} & C_{1L}

$$C_1 = (R \pm Z^{\frac{1}{2}})\left[(R - R \mp Z^{\frac{1}{2}})(R - \tfrac{1}{2}R \mp \tfrac{1}{2}Z^{\frac{1}{2}}) + 1\right]$$

$$C_1 = (R \pm Z^{\frac{1}{2}})\left[\mp Z^{\frac{1}{2}}(\tfrac{1}{2}R \mp \tfrac{1}{2}Z^{\frac{1}{2}}) + 1\right]$$

$$C_1 = (R \pm Z^{\frac{1}{2}})(\mp Z^{\frac{1}{2}}\tfrac{1}{2}R + \tfrac{1}{2}Z + 1)$$

$$C_1 = \mp Z^{\frac{1}{2}}\tfrac{1}{2}R^2 + \cancel{\tfrac{1}{2}ZR} + R - \cancel{Z\tfrac{1}{2}R} \pm \tfrac{1}{2}Z^{3/2} \pm Z^{\frac{1}{2}}$$

$$C_1 = R \mp Z^{\frac{1}{2}}(\tfrac{1}{2}R^2 - 1) \pm \tfrac{1}{2}Z^{3/2}$$

$$C_1 = R \mp Z^{\frac{1}{2}}\left(\frac{R^2 - 2}{3} * \frac{3}{2}\right) \pm \tfrac{1}{2}Z^{3/2}$$

$$C_1 = R \mp Z^{\frac{1}{2}}(Z\tfrac{3}{2}) \pm \tfrac{1}{2}Z^{3/2}$$

$$C_1 = R \mp \tfrac{3}{2}Z^{3/2} \pm Z^{3/2}$$

$$\underline{C_1 = R \mp Z^{3/2}}$$

Hence

$$C_{1H} = R + Z^{3/2} \quad \text{when} \quad N = R - Z^{\frac{1}{2}}$$

$$C_{1L} = R - Z^{3/2} \quad \text{when} \quad N = R + Z^{\frac{1}{2}}$$

C_{1M} is defined as

$$C_{1M} = \frac{C_{1H} + C_{1L}}{2} = \frac{R + Z^{3/2} + R - Z^{3/2}}{2}$$

$$\underline{C_{1M} = R}$$

C_{1H}' & C_{1L}' are evaluated at $N = R \pm \frac{1}{2} \sqrt{Z}$

$$C_1' = \left(R \pm \tfrac{1}{2} Z^{\frac{1}{2}}\right)\left[\left(R - R \mp \tfrac{1}{2} Z^{\frac{1}{2}}\right)\left(R - \tfrac{1}{2}R \mp \tfrac{1}{4} Z^{\frac{1}{2}}\right) + 1\right]$$

$$C_1' = \left(R \pm \tfrac{1}{2} Z^{\frac{1}{2}}\right)\left[\mp \tfrac{1}{2} Z^{\frac{1}{2}}\left(\tfrac{1}{2}R \mp \tfrac{1}{4} Z^{\frac{1}{2}}\right) + 1\right]$$

$$C_1' = \left(R \pm \tfrac{1}{2} Z^{\frac{1}{2}}\right)\left(\mp \tfrac{1}{4} R Z^{\frac{1}{2}} + \tfrac{1}{8} Z + 1\right)$$

$$C_1' = R \mp \tfrac{1}{4} R^2 Z^{\frac{1}{2}} + \cancel{\tfrac{1}{8} ZR} - \cancel{\tfrac{1}{8} ZR} \pm \tfrac{1}{16} Z^{3/2} \pm \tfrac{1}{2} Z^{\frac{1}{2}}$$

but: $R^2 = 3Z - 2$

$$C_1' = R \mp \tfrac{1}{4}(3Z - 2) Z^{\frac{1}{2}} \pm \tfrac{1}{16} Z^{3/2} \pm \tfrac{1}{2} Z^{\frac{1}{2}}$$

$$C_1' = R \mp \tfrac{3}{4} Z^{3/2} \cancel{\mp \tfrac{1}{2} Z^{\frac{1}{2}}} \pm \tfrac{1}{16} Z^{3/2} \cancel{\pm \tfrac{1}{2} Z^{\frac{1}{2}}}$$

$$\underline{C_1' = R \mp \tfrac{11}{16} Z^{3/2}}$$

Hence:

$$C_{1H}' = R + \tfrac{11}{16} Z^{3/2} \quad @ \quad N = R - \tfrac{1}{2} Z^{\frac{1}{2}}$$

$$C_{1L}' = R - \tfrac{11}{16} Z^{3/2} \quad @ \quad N = R + \tfrac{1}{2} Z^{\frac{1}{2}}$$

Snap-Through Springs

C_1 becomes zero when:

$$C_1 = N\left[(R-N)(R-\tfrac{1}{2}N)+1\right] = 0$$

first root $N = 0 \quad \therefore C_1 = 0$

$$(R-N)(R-\tfrac{1}{2}N)+1 = 0$$

$$\tfrac{1}{2}N^2 - \tfrac{3}{2}RN + R^2 + 1 = 0$$

$$N^2 - 3RN = -2 - 2R^2$$

$$N^2 - 3RN + \tfrac{9}{4}R^2 = -2 - 2R^2 + \tfrac{9}{4}R^2$$

$$\left(N - \tfrac{3}{2}R\right)^2 = -2 + \tfrac{1}{4}R^2$$

$$N = \tfrac{3}{2}R \pm \sqrt{\tfrac{1}{4}R^2 - 2}$$

$$N = \tfrac{3}{2}R \pm \tfrac{1}{2}\sqrt{R^2 - 8} \qquad \text{(2nd and 3rd root)}$$

hence:

$C_1 = 0$ when $N_{01} = \tfrac{3}{2}R - \tfrac{1}{2}\sqrt{R^2-8}$

$C_1 = 0$ when $N_{02} = \tfrac{3}{2}R + \tfrac{1}{2}\sqrt{R^2-8}$

Back to Back Springs

$$C_1^{(1)} = N\left[(R-N)(R-\tfrac{1}{2}N) + 1\right]$$

for $C_1^{(2)}$ $\therefore$ $N = (2R-N)$

$$-C_1^{(2)} = -(2R-N)\left[(R-(2R-N))(R-\tfrac{1}{2}(2R-N))+1\right]$$

minus sign because $C_1^{(2)}$ is inverted with respect to $C_1^{(1)}$.

$$C_1^{(B)} = C_1^{(1)} + \left(-C_1^{(2)}\right)$$

$$\begin{aligned} C_1^{(1)} &= N\left[R^2 - \tfrac{3}{2}RN + \tfrac{1}{2}N^2 + 1\right] \\ &= \tfrac{1}{2}N^3 - \tfrac{3}{2}RN^2 + (R^2+1)N \end{aligned}$$

$$\begin{aligned} -C_1^{(2)} &= -(2R-N)\left[(N-R)\tfrac{1}{2}N + 1\right] \\ &= -(2R-N)\left(\tfrac{1}{2}N^2 - \tfrac{1}{2}RN + 1\right) \\ &= -\left(RN^2 - R^2N + 2R - \tfrac{1}{2}N^3 + \tfrac{1}{2}RN^2 - N\right) \\ &= -\left(-\tfrac{1}{2}N^3 + \tfrac{3}{2}RN^2 - (R^2+1)N + 2R\right) \\ &= \tfrac{1}{2}N^3 - \tfrac{3}{2}RN^2 + (R^2+1)N - 2R \end{aligned}$$

$$C_1^{(B)} = \tfrac{1}{2}N^3 - \tfrac{3}{2}RN^2 + (R^2+1)N + \left\{\tfrac{1}{2}N^3 - \tfrac{3}{2}RN^2 + (R^2+1)N - 2R\right\}$$

$$\underline{C_1^{(B)} = N^3 - 3RN^2 + 2(R^2+1)N - 2R}$$

Max and Min Values occur at:

$$C_1^B = N^3 - 3RN^2 + 2(R^2+1)N - 2R$$

$$\frac{dC_1^B}{dN} = 3N^2 - 6RN + 2(R^2+1) = 0$$

$$N^2 - 2RN + R^2 = -\frac{2}{3} R^2 + 1 + R^2$$

$$(N-R)^2 = -\frac{2}{3}(3Z+3) + (3Z+2)$$

$$= -2Z - 2 + 3Z + 2$$

$$= Z$$

$$\underline{N = R \pm \sqrt{Z}}$$ (same as for single spring)

Resubstitution to find Max and Min values:

$$N^3 = (R-\sqrt{Z})^3 = (R-\sqrt{Z})(R-\sqrt{Z})(R-\sqrt{Z})$$

$$= (R-\sqrt{Z})(R^2 - 2R\sqrt{Z} + Z)$$

$$= R^3 - 2R^2\sqrt{Z} + RZ - R^2\sqrt{Z} + 2RZ - Z\sqrt{Z}$$

$$= R^3 - 3R^2\sqrt{Z} + 3RZ - Z\sqrt{Z}$$

$$-3RN^2 = -3R(R^2 - 2R\sqrt{Z} + Z)$$

$$= -3R^3 + 6R^2\sqrt{Z} - 3RZ$$

$$2(R^2+1)N = 2(R^2+1)(R-\sqrt{Z})$$

$$\cdots = 2(R^3 - R^2\sqrt{Z} + R - \sqrt{Z})$$

$$= 2R^3 - 2R^2\sqrt{Z} + 2R - 2\sqrt{Z}$$

$$C_1 = N^3 - 3RN^2 + 2(R^2+1)N - 2R$$

$$\begin{array}{rrrrr}
\cancel{R^3} & -\,3R^2\sqrt{Z} & +\,\cancel{3RZ} & & -\,Z\sqrt{Z} \\
-\cancel{3R^3} & +\,6R^2\sqrt{Z} & -\,\cancel{3RZ} & & \\
\cancel{2R^3} & -\,2R^2\sqrt{Z} & +\,\cancel{2R} & & -\,2\sqrt{Z} \\
 & & -\,\cancel{2R} & & \\
\hline
\end{array}$$

$$R^2\sqrt{Z} - (Z+2)\sqrt{Z}$$

$$(3Z+2)\sqrt{Z} - (Z+2)\sqrt{Z}$$

$$3Z^{3/2} + 2\sqrt{Z} - Z^{3/2} - 2\sqrt{Z}$$

$$2Z^{3/2}$$

Hence:

$$C_{1MAX} = 2Z^{3/2} \quad @ \quad N = R - \sqrt{Z}$$

$$C_{1MIN} = -2Z^{3/} \quad @ \quad N = R + \sqrt{Z}$$

$@\ N = 0 \quad \therefore \quad \underline{C_1 = -2R}$

$@\ N = 2R$

$$C_1 = N^3 - 3RN^2 + 2(R^2+1)N - 2R$$

$$= 8R^3 - 3R(4R^2) + 2(R^2+1)2R - 2R$$

$$= 8R^3 - 12R^3 + 4R^3 + 4R - 2R$$

$$\underline{C_1 = 2R}$$

first root $\quad N=R$

$$C_1 = N^3 - 3RN^2 + 2(R^2+1)N - 2R$$
$$= R^3 - 3R^2 + 2(R^2+1)R - 2R$$
$$\underline{\underline{C_1 = 0}}$$

2nd and 3rd root @ $N = R \pm \sqrt{3Z}$

$$N^3 = (R-\sqrt{3Z})^3 = (R-\sqrt{3Z})(R-\sqrt{3Z})(R-\sqrt{3Z})$$
$$= (R-\sqrt{3Z})(R^2 - 2R\sqrt{3Z} + 3Z)$$
$$= R^3 - 2R^2\sqrt{3Z} + 3RZ - R^2\sqrt{3Z} + 2R3Z - 3Z\sqrt{3Z}$$
$$= R^3 - 3R^2\sqrt{3Z} + 9RZ - 3Z\sqrt{3Z}$$

$$-3RN^2 = -3R(R^2 - 2R\sqrt{3Z} + 3Z)$$
$$= -3R^3 + 6R^2\sqrt{3Z} - 9RZ$$

$$2(R^2+1)N = 2(R^2+1)(R-\sqrt{3Z})$$
$$= 2(R^3 - R^2\sqrt{3Z} + R - \sqrt{3Z})$$
$$= 2R^3 - 2R^2\sqrt{3Z} + 2R - 2\sqrt{3Z}$$

$$C_1^B = N^3 - 3RN^2 + 2(R^2+1)N - 2R$$

$$\cancel{R^3} - 3R^2\sqrt{3z} + 9Rz - 3z\sqrt{3z}$$
$$-3\cancel{R^3} + 6R^2\sqrt{3z} - 9\cancel{Rz}$$
$$2\cancel{R^3} - 2R^2\sqrt{3z} \qquad + 2\cancel{R} - 2\sqrt{3z}$$
$$-\cancel{2R}$$

$$R^2\sqrt{3z} - 3z\sqrt{3z} - 2\sqrt{3z}$$

$$(R^2-2)\sqrt{3z} - 3z\sqrt{3z}$$

$$\underline{3z\sqrt{3z} - 3z\sqrt{3z} = 0}$$

Root are @ $C_1^B = 0$

$$\begin{cases} N = R \\ N = R - \sqrt{3z} = R - \sqrt{R^2-2} \\ N = R + \sqrt{3z} = R + \sqrt{R^2-2} \end{cases}$$

---

let $N = R - \frac{1}{2}\sqrt{z}$

$$\begin{aligned} N^3 = \left(R - \tfrac{1}{2}\sqrt{z}\right)^3 &= \left(R-\tfrac{1}{2}\sqrt{z}\right)\left(R-\tfrac{1}{2}\sqrt{z}\right)\left(R-\tfrac{1}{2}\sqrt{z}\right) \\ &= \left(R-\tfrac{1}{2}\sqrt{z}\right)\left(R^2 - R\sqrt{z} + \tfrac{1}{4}\sqrt{z}\right) \\ &= R^3 - R^2\sqrt{z} + \tfrac{1}{4}Rz - \tfrac{1}{2}R^2\sqrt{z} + \tfrac{1}{2}Rz - \tfrac{1}{8}z^{3/2} \\ &= R^3 - \tfrac{3}{2}R^2\sqrt{z} + \tfrac{3}{4}Rz - \tfrac{1}{8}z^{3/2} \end{aligned}$$

$$-3RN^2 = -3R\left(R^2 + R\sqrt{z} + \tfrac{1}{4}z\right)$$
$$= -3R^3 + 3R^2\sqrt{z} - \tfrac{3}{4}Rz$$

$$2(R^2+1)N = 2(R^2+1)\left(R - \tfrac{1}{2}\sqrt{z}\right)$$
$$= 2\left(R^3 - \tfrac{1}{2}R^2\sqrt{z} + R - \tfrac{1}{2}\sqrt{z}\right)$$
$$= 2R^3 - 2R^2\sqrt{z} + 2R - \sqrt{z}$$

$$C_1^B = N^3 - 3RN^2 + 2(R^2+1)N - 2R$$

$$\begin{array}{l}
R^3 + \tfrac{3}{2}R^2\sqrt{z} + \tfrac{3}{4}\cancel{Rz} - \tfrac{1}{8}z^{3/2} \\
-3R^3 + 3R^2\sqrt{z} - \tfrac{3}{4}\cancel{Rz} \\
2R^3 - R^2\sqrt{z} + \cancel{2R} - \sqrt{z} \\
\quad -\cancel{2R} \\
\hline
\end{array}$$

$$\tfrac{1}{2}R^2\sqrt{z} - \tfrac{1}{8}z^{3/2} - \sqrt{z}$$

$$\tfrac{1}{2}(3z+2)\sqrt{z} - \tfrac{1}{8}z^{3/2} - \sqrt{z}$$

$$\tfrac{3}{2}z^{3/2} + \sqrt{z} - \tfrac{1}{8}z^{3/2} - \sqrt{z}$$

$$\left(\tfrac{12}{8} - \tfrac{1}{8}\right)z^{3/2} = \underline{\tfrac{11}{8}z^{3/2}}$$

Hence

$$C_{1H}^{1B} = \tfrac{11}{8}z^{3/2} \quad @ \quad N = R - \tfrac{1}{2}\sqrt{z}$$
$$C_{1L}^{1B} = -\tfrac{11}{8}z^{3/2} \quad @ \quad N = R + \tfrac{1}{2}\sqrt{z}$$

Slopes for Back to Back Springs:

Slope @ $N_0 = R$

$$\frac{dC^B}{dN} = 3N^2 - 6RN + 2(R^2+1)$$

$$= 3R^2 - 6R^2 + 2R^2 + 2$$

$$= -R^2 + 2 = -(3Z+2) + 2$$

$$\underline{S_0 = -3Z}$$

Slope between $+\frac{11}{8}Z^{3/2}$ and $-\frac{11}{8}Z^{3/2}$

$$S_1 = \frac{-\frac{11}{8}Z^{3/2} - \left(\frac{11}{8}Z^{3/2}\right)}{N + \frac{1}{2}Z^{\frac{1}{2}} - \left(N - \frac{1}{2}Z^{\frac{1}{2}}\right)} = \frac{-\frac{22}{8}Z^{3/2}}{Z^{\frac{1}{2}}}$$

$$= -\frac{11}{4}Z = \underline{-2.75Z}$$

Slope between $2Z^{3/2}$ and $-2Z^{3/2}$

$$S_2 = \frac{-2Z^{3/2} - (2Z^{3/2})}{N + Z^{\frac{1}{2}} - (N - Z^{\frac{1}{2}})} = \frac{-4Z^{3/2}}{2Z^{1/2}} = \underline{-2Z}$$

Maxima's of Stresses:

$$S_a = \frac{CKE}{M^2} C_2 N \left[\left(R - \tfrac{1}{2}N\right) + \frac{C_3}{C_2} \right]$$

$$= \frac{CKE}{M^2} C_2 \left(RN - \tfrac{1}{2}N^2 + \frac{C_3}{C_2} N \right)$$

$$\frac{dS_a}{dN} = \frac{CKE}{M^2} C_2 \left(R - N + \frac{C_3}{C_2} \right) = 0$$

$$@ \quad N = R + \frac{C_3}{C_2}$$

$$S_b = \frac{CKE}{M^2} C_2 \left(RN - \tfrac{1}{2}N^2 - \frac{C_3}{C_2} N \right)$$

$$\frac{dS_b}{dN} = \frac{CKE}{M^2} C_2 \left(R - N - \frac{C_3}{C_2} \right) = 0$$

$$@ \quad N = R - \frac{C_3}{C_2}$$

$$S_c = \frac{KE}{M^2} C_4 N \left[\left(R - \tfrac{1}{2}N\right) + \frac{C_5}{C_4} \right]$$

$$= \frac{KE}{M^2} C_4 \left(RN - \tfrac{1}{2}N^2 + \frac{C_5}{C_4} N \right)$$

$$\frac{dS_c}{dN} = \frac{KE}{M^2} C_4 \left(R - N + \frac{C_5}{C_4} \right) = 0$$

$$@ \quad N = R + \frac{C_5}{C_4}$$

Resubstitution:

$$S_{a\,max} = \frac{CKE}{M^2} C_2 \left(R + \frac{C_3}{C_2}\right)\left[R - \frac{1}{2}\left(R + \frac{C_3}{C_2}\right) + \frac{C_3}{C_2}\right]$$

$$= \frac{CKE}{M^2} C_2 \left(R + \frac{C_3}{C_2}\right)\left(\frac{1}{2}R + \frac{1}{2}\frac{C_3}{C_2}\right)$$

$$S_{a\,max} = \frac{CKE}{M^2} \frac{C_2}{2} \left(R + \frac{C_3}{C_2}\right)^2$$

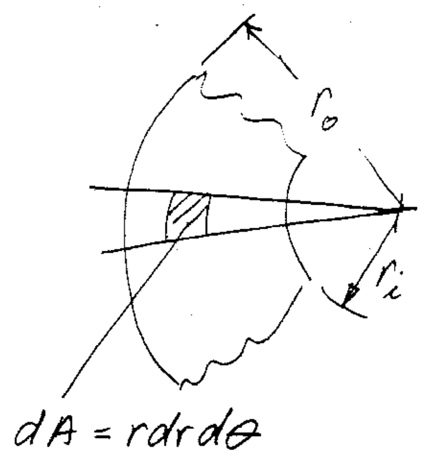
$$S_{b\,max} = \frac{CKE}{M^2} \frac{C_2}{2} \left(R - \frac{C_3}{C_2}\right)^2$$

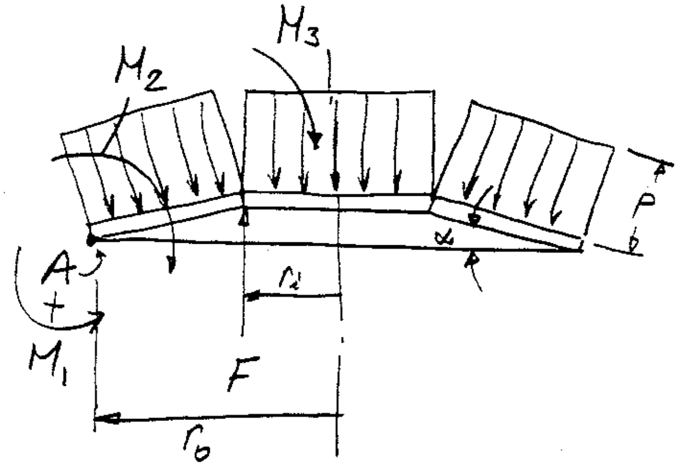
$$S_{c\,max} = \frac{KE}{M^2} C_4 \left(R + \frac{C_5}{C_4}\right)\left[R - \frac{1}{2}\left(R + \frac{C_5}{C_4}\right) + \frac{C_5}{C_4}\right]$$

$$= \frac{KE}{M^2} C_4 \left(R + \frac{C_5}{C_4}\right)\left(\frac{1}{2}R + \frac{1}{2}\frac{C_5}{C_4}\right)$$

$$S_{c\,max} = \frac{KE}{M^2} \frac{C_4}{2} \left(R + \frac{C_5}{C_4}\right)^2$$

Closed Center Belleville Spring Pressure Loading





$$M_1 = M_2 + M_3 \qquad (\text{moment balance about } A)$$

where $\quad M_1 = F(r_o - r_i)$

$$M_2 = -\int_0^{2\pi} d\theta \int_{r_o}^{r_i} \cos\alpha \, p\, r\, dr\, (r_o - r)$$

$$M_3 = -\int_0^{2\pi} d\theta \int_{r_o}^{0} p\, r\, dr\, (r_o - r_i)$$

$$\alpha = \text{small} \therefore \cos\alpha \cong 1$$

$$F(r_o - r_i) = -2\pi p \int_{r_o}^{r_i} (r r_o - r^2)\, dr - 2\pi p (r_o - r_i) \int_{r_i}^{0} r\, dr$$

$$= -2\pi p \left\{ \left(\frac{r^2 r_o}{2} - \frac{r^3}{3} \right) \Bigg|_{r_o}^{r_i} + (r_o - r_i) \frac{r^2}{2} \Bigg|_{r_i}^{0} \right\}$$

$$F(r_o - r_i) = -2\pi p \left\{ \frac{r_i^2 r_o}{2} - \frac{r_i^3}{3} - \left(\frac{r_o^3}{2} - \frac{r_o^3}{3} \right) + (r_o - r_i)\left(-\frac{r_i^2}{2}\right) \right\}$$

$$= -2\pi p \left(\cancel{\frac{r_i^2 r_o}{2}} - \frac{r_i^3}{3} - \frac{r_o^3}{2} + \frac{r_o^3}{3} - \cancel{\frac{r_i^2 r_o}{2}} + \frac{r_i^3}{2} \right)$$

$$= -2\pi p \frac{1}{6} (r_o^3 - r_i^3)$$

$$F = -\pi p \frac{(r_o^3 - r_i^3)}{3(r_o - r_i)} = -\pi p \frac{r_o^3 \left(1 - \frac{r_i^3}{r_o^3}\right)}{3 r_o \left(1 - \frac{r_i}{r_o}\right)}$$

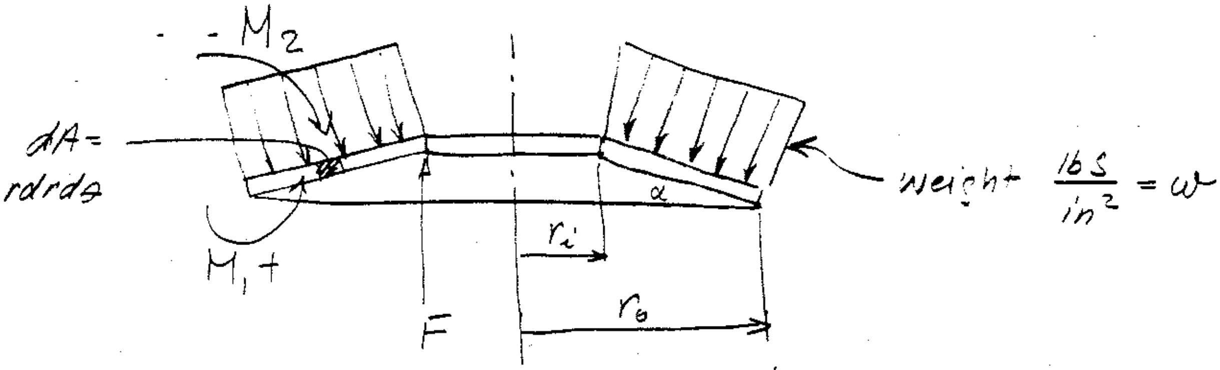
$$= -\frac{\pi D^2}{4} p \frac{\left(1 - \frac{d^3}{D^3}\right)}{3\left(1 - \frac{d}{D}\right)}$$

Hence

$$\epsilon = \frac{\left(1 - \left(\frac{d}{D}\right)^3\right)}{3\left(1 - \frac{d}{D}\right)}$$

Note $\frac{\pi D^2}{4} p$ = total pressure loading (closed center)

Equivalent Weight of a Belleville spring. (open center)



$$M_1 = F(r_o - r_i) \; ; \; M_2 = -\int_0^{2\pi} d\theta \int_{r_o}^{r_i} \cos\alpha \, w r\,dr\,(r_o - r)$$

$$F(r_o - r_i) = -2\pi w \int_{r_o}^{r_i} (r r_o - r^2)\,dr$$

$$= -2\pi w \left\{ \frac{r^2}{2} r_o - \frac{r^3}{3} \right\}_{r_o}^{r_i}$$

$$= -2\pi w \left\{ \frac{r_i^2 r_o}{2} - \frac{r_i^3}{3} - \left(\frac{r_o^3}{2} - \frac{r_o^3}{3} \right) \right\}$$

$$= -2\pi w \left\{ \frac{r_i^2 r_o}{2} - \frac{r_i^3}{3} - \frac{r_o^3}{6} \right\}$$

$$F r_o \left(1 - \frac{r_i}{r_o}\right) = -2\pi w r_o^{\cancel{3}2} \left\{ \frac{1}{2} \frac{r_i^2}{r_o^2} - \frac{1}{3} \frac{r_i^3}{r_o^3} - \frac{1}{6} \right\}$$

$$F = -\pi w r_o^2 \; \frac{1 - 3\left(\frac{r_i}{r_o}\right)^2 + 2\left(\frac{r_i}{r_o}\right)^3}{3\left(1 - \frac{r_i}{r_o}\right)}$$

but $w = \gamma t$

$$F_{equ} = \frac{\pi D^2}{4} \gamma t \left\{ \overbrace{\frac{1 - 3\left(\frac{d}{D}\right)^2 + 2\left(\frac{d}{D}\right)^3}{3\left(1 - \frac{d}{D}\right)}}^{\eta} \right\}$$

Nondimensional Stress Relations used for Figures:

$$S_{aN} = \frac{S_a}{S_{a\,max}} = \frac{\cancel{\frac{CKE}{M^2}}\, C_2 N\left[\left(R - \frac{1}{2}N\right) + \frac{C_3}{C_2}\right]}{\cancel{\frac{CKE}{M^2}}\, \frac{C_2}{2}\left(R + \frac{C_3}{C_2}\right)^2}$$

$$= \frac{2N\left[\left(R - \frac{1}{2}N\right) + C_3/C_2\right]}{\left(R + \frac{C_3}{C_2}\right)^2}$$

$$S_{bN} = \frac{S_b}{S_{a\,max}} = \frac{\frac{CKE}{M^2}\, C_2 N\left[\left(R - \frac{1}{2}N\right) - \frac{C_3}{C_2}\right]}{\frac{CKE}{M^2}\, \frac{C_2}{2}\left(R + \frac{C_3}{C_2}\right)^2}$$

$$= \frac{2N\left[\left(R - \frac{1}{2}N\right) - C_3/C_2\right]}{\left(R + \frac{C_3}{C_2}\right)^2}$$

$$S_{cN} = \frac{S_c}{S_{a\,max}} = \frac{\frac{KE}{M^2}\, C_4 N\left[\left(R - \frac{1}{2}N\right) + \frac{C_5}{C_4}\right]}{\frac{CKE}{M^2}\, \frac{C_2}{2}\left(R + \frac{C_3}{C_2}\right)^2}$$

$$= \frac{2NC_4\left[\left(R - \frac{1}{2}N\right) + \frac{C_5}{C_4}\right]}{C\,C_2\,\left(R + \frac{C_3}{C_2}\right)^2}$$

Nondimensional Maximum Stress Relation:

Max @ $N = R + \frac{C_3}{C_2}$

$$S_{a\,max\,N} = \frac{2\cancel{\left(R + \frac{C_3}{C_2}\right)}\left[R - \frac{1}{2}\left(R + \frac{C_3}{C_2}\right) + \frac{C_3}{C_2}\right]}{\left(R + \frac{C_3}{C_2}\right)^{\cancel{2}}}$$

$$= \frac{2\left(\frac{1}{2}R + \frac{1}{2}\frac{C_3}{C_2}\right)}{R + \frac{C_3}{C_2}} = 1 \quad \text{(unity)}$$

Max @ $N = R - \frac{C_3}{C_2}$

$$S_{b\,max\,N} = \frac{2\left(R - \frac{C_3}{C_2}\right)\left[\left(R - \frac{1}{2}\left(R - \frac{C_3}{C_2}\right) + \frac{C_3}{C_3}\right]}{\left(R + \frac{C_3}{C_2}\right)^2}$$

$$= \frac{\left(R \;\; \frac{C_3}{C_2}\right)^2}{\left(R + \frac{C_3}{C_2}\right)^2}$$

Max @ $N = R + \frac{C_5}{C_4}$

$$S_{c\,max\,N} = \frac{2\left(R + \frac{C_5}{C_4}\right) C_4 \left[\left(R - \frac{1}{2}\left(R + \frac{C_5}{C_4}\right) + \frac{C_5}{C_4}\right]}{C\,C_2\left(R + \frac{C_3}{C_2}\right)^2}$$

$$= \frac{C_4}{C\,C_2}\,\frac{\left(R + \frac{C_5}{C_4}\right)^2}{\left(R + \frac{C_3}{C_2}\right)^2}$$

Stress Relations as used for Figure 27

All values based on $R=2$ and $a=2$ denoted as unity or reference by subscript R

$$S_{aN} = \frac{C N C_2 \left[\left(R - \frac{1}{2} N \right) + C_3 / C_2 \right]}{\frac{C_R C_{2R}}{2} \left(R_R + \frac{C_{3R}}{C_{2R}} \right)^2}$$

$$S_{bN} = \frac{C N C_2 \left[\left(R - \frac{1}{2} N \right) - C_3 / C_2 \right]}{\frac{C_R C_{2R}}{2} \left(R_R + \frac{C_{3R}}{C_{2R}} \right)^2}$$

$$S_{cN} = \frac{N C_4 \left[\left(R - \frac{1}{2} N \right) + C_5 / C_4 \right]}{\frac{C_R C_{2R}}{2} \left(R_R + \frac{C_{3R}}{C_{2R}} \right)^2}$$

$C_R = C$ for $a = 2$

$C_{2R} = C_2$ for $a = 2$

$C_{3R} = C_3$ for $a = 2$

$R_R = R$ for $R = 2$

Lines of constant M's

For Figure 28:

$$S_{amax} = \frac{CkE}{M^2} \frac{C_2}{2}\left(R + \frac{C_3}{C_2}\right)^2 = 200{,}000$$

$$\left(R + \frac{C_3}{C_2}\right)^2 = \frac{S_{amax} M^2 2}{CkE\ C_2}$$

$$R = M\sqrt{\frac{2 S_{amax}}{CkEC_2}} - \frac{C_3}{C_2}$$

This equation solved for given values of M throughout the range of "a".

For Figure 29:

$$S_{cmax} = \frac{kE}{M^2} \frac{C_4}{2}\left(R + \frac{C_5}{C_4}\right)^2 = 100{,}000$$

$$\left(R + \frac{C_5}{C_4}\right)^2 = \frac{2 S_{cmax} M^2}{kE C_4}$$

$$R = M\sqrt{\frac{2 S_{cmax}}{kEC_4}} - \frac{C_5}{C_4}$$

This equation solved for given values of M throughout the range of "a".

Lines of constant stress ratios:

For Figure 28: $C_{32} = C_3/C_2$; $C_{54} = C_5/C_4$

$$S_R = \frac{S_{c\,max}}{S_{a\,max}} = \frac{\frac{C_4}{2}\left(R + C_{54}\right)^2}{C\,\frac{C_2}{2}\left(R + C_{32}\right)^2}$$

$$\left(R + C_{32}\right)^2 = \frac{C_4}{C\,C_2\,S_R}\left(R + C_{54}\right)^2$$

$$R + C_{32} = \sqrt{\frac{C_4}{C\,C_2\,S_R}}\left(R + C_{54}\right)$$

$$= R\sqrt{\frac{C_4}{C\,C_2\,S_R}} + C_{54}\sqrt{\frac{C_4}{C\,C_2\,S_R}}$$

$$R - R\sqrt{\frac{C_4}{C\,C_2\,S_R}} = C_{54}\sqrt{\frac{C_4}{C\,C_2\,S_R}} - C_{32}$$

$$R = \frac{C_{54}\sqrt{\frac{C_4}{C\,C_2\,S_R}} - C_{32}}{1 - \sqrt{\frac{C_4}{C\,C_2\,S_R}}}$$

This last equation solved for given values of S_R $(.5 < S_R < .9)$ for the total range of "a".

For Figure 29:

$$S_R = \frac{S_{a\,max}}{S_{c\,max}} = \frac{\frac{C C_2}{2}(R + C_{32})^2}{C_4 (R + C_{54})^2}$$

$$(R + C_{54})^2 = \frac{C C_2}{C_4 S_R}(R + C_{32})^2$$

$$R + C_{54} = \sqrt{\frac{C C_2}{C_4 S_R}}\,(R + C_{32})$$

$$R + C_{54} = R\sqrt{\frac{C C_2}{C_4 S_R}} + C_{32}\sqrt{\frac{C C_2}{C_4 S_R}}$$

$$R - R\sqrt{\frac{C C_2}{C_4 S_R}} = C_{32}\sqrt{\frac{C C_2}{C_4 S_R}} - C_{54}$$

$$R = \frac{C_{32}\sqrt{\frac{C C_2}{C_4 S_R}} - C_{54}}{1 - \sqrt{\frac{C C_2}{C_4 S_R}}}$$

This last equation solved for given values of S_R $(1.2 < S_R < 2.05)$ for the total range of "a".