Difference Equations

Weijie Chen
Department of Political and Economic Studies
University of Helsinki

16 Aug, 2011

Contents

Abstract

Difference equations are close cousin of differential equations, they have remarkable similarity as you will soon find out. So if you have learned differential equations, you will have a rather nice head start. Conventionally we study differential equations first, then difference equations, it is not simply because it is better to study them chronologically, it is mainly because diffence equations has a naturally stronger bond with computational science, sometimes we even need to study how to make a differential equation into its discrete version-difference equation-since which can be simulated by MATLAB. Difference equations are always the first lesson of any advanced time series analysis course, difference equation largely overshadows its econometric brother, lag operation, that is because difference equation can be expressed by matrix, which tremendously increase it power, you will see its shockingly powerful application of space-state model and Kalman filter.

1 First Order Difference Equation

Difference equations emerge because we need to deal with discrete time model, which is more realistic when we study econometrics, time series datasets are apparently discrete. First we will discuss about *iterative mathod*, which is almost the topic of first chapter of every time series textbook. In preface of Ender (2004)[5],'In my experience, this material (difference equation) and a knowledge of regression analysis is sufficient to bring students to the point where they are able to read the professional journals and embark on a serious applied study.' Although I do not full agree with his optimism, I do concur that the knowledge of difference equation is the key to all further study of time series and advanced macreconomic theory.

A simple difference equation is a dynamic model, which describes the time path of evolution, highly resembles the differential equations, so its solution should be a function of t, completed free of $y_{t+1} - y_t$, a time instant t can exactly locate the position of its variable.

1.1 Iterative Method

This method is also called *recursive substitution*, basically it means if you know the y_0, you know y_1, and rest of y_t can be expressed by a recursive relation. We start with a simple example which appears in very time series textbook,

$$y_t = ay_{t-1} + w_t \tag{1}$$

where

$$\boldsymbol{w} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_t \end{bmatrix}.$$

$\boldsymbol{w}$ is a determinstic vector, later we will drop this assumption when we study stochastic difference equations, but for the time being and simplicity, we assume $\boldsymbol{w}$ nonstochastic yet.

Notice that

$$\begin{aligned} y_0 &= ay_{-1} + w_0 \\ y_1 &= ay_0 + w_1 \\ y_2 &= ay_1 + w_2 \\ &\vdots \\ y_t &= ay_{t-1} + w_t \end{aligned}$$

Assume that we know y_{-1} and $\boldsymbol{w}$. Thus, substitute y_0 into y_1,

$$y_1 = a(ay_{-1} + w_0) + w_1 = a^2 y_{-1} + aw_0 + w_1$$

Now we have y_1, follow this procedure, substitute y_1 into y_2,

$$y_2 = a(a^2 y_{-1} + aw_0 + w_1) + w_2 = a^3 y_{-1} + a^2 w_0 + aw_1 + w_2$$

Again, substitute y_2 into y_3,

$$\begin{aligned} y_3 &= a(a^3 y_{-1} + a^2 w_0 + aw_1 + w_2) + w_3 \\ &= a^4 y_{-1} + a^3 w_0 + a^2 w_1 + aw_2 + w_3 \end{aligned}$$

I trust you are able to perceive the pattern of the dynamics,

$$y_t = a^{t+1} y_{-1} + \sum_{i=0}^{t} a^i w_{t-i}$$

Actually what we have done is not the simplest version of difference equation, to show your the outrageously simplest version, let us run through next example in a lightning fast speed. A homogeneous difference equation[1],

$$my_t - ny_{t-1} = 0$$

[1] If this is first time you hear about this, refer to my notes of *differential equation*[2].

To rearrange it in a familiar manner,

$$y_t = \left(\frac{n}{m}\right) y_{t-1}$$

As the recurisve methods implies, if we have an intial value ${y_0}^2$, then

$$\begin{aligned} y_1 &= \left(\frac{n}{m}\right) y_0 \\ y_2 &= \left(\frac{n}{m}\right) y_1 \\ \ldots \\ y_t &= \left(\frac{n}{m}\right) y_{t-1} \end{aligned}$$

Substitute y_1 into y_2,

$$y_2 = \left(\frac{n}{m}\right)\left(\frac{n}{m}\right) y_0 = \left(\frac{n}{m}\right)^2 y_0$$

Then substitute y_2 into y_3,

$$y_3 = \left(\frac{n}{m}\right)\left(\frac{n}{m}\right)^2 y_1 = \left(\frac{n}{m}\right)^3 y_0$$

Well the pattern is clear, and solution is

$$y_t = \left(\frac{n}{m}\right)^t y_0$$

If we denote $(n/m)^t$ as b^t and y_0 as A, it becomes Ab^t, this is the counterpart of solution of first order differential equation Ae^{rt}. They both play the fundamental role in solving differential or difference equations. Notice that the solution is a function of t, here only t is a varible, y_0 is an initial value which is a known constant.

1.2 General Method

As you no doubt have guess the general solution will difference equation will also be the counterpart of its differential version.

$$y = y_p + y_c$$

where y_p is the *particular solution* and y_c is the *comlementary solution*[3].

[2] For sake of mathematical convenience, we assume initial value y_0 rather than y_{-1} this time, but the essence is identical.

[3] All these terminology is full explained in my notes of *differential equations.*

The process will be clear with the help of an example,

$$y_{t+1} + ay_t = c$$

We follow the standard procedure, we will find the complementary solution first, which is the solution of the according homogeneous difference equation,

$$y_{t+1} + ay_t = 0$$

With the knowledge of last section, we can try a solution of Ab^t, when I say 'try' it does not mean we just randomly guess, because we don't need to perform the crude iterative method every time, we can make use of the solution of previously solved equation, so

$$y_{t+1} + ay_t = Ab^{t+1} + aAb^t = 0,$$

Cancel the common factor,

$$\begin{aligned} b + a &= 0 \\ b &= -a \end{aligned}$$

If $b = -a$, this solution will work, so the complementary solution is

$$y_c = Ab^t = A(-a)^t$$

Next step, find the particular solution of

$$y_{t+1} + ay_t = c.$$

The most intreguing thing comes, to make the equation above hold, we can choose any y_t to satisfy it. Might to your surprise, we can even choose $y_t = k$ for $-\infty < t < \infty$, which is just a constant time series, every period we have the same value k. So

$$\begin{aligned} k + ak &= c \\ k = y_p &= \frac{c}{1+a} \end{aligned}$$

You can easily notice that if we want to make this solution work, then $a \neq -1$. Then the question we ask will simply be, what if $a = -1$? Of course it is not define, we have to change the form of the solution, here we use $y_t = kt$ which is similar to the one we used in differential equation,

$$\begin{aligned} (k+1)t + akt &= c \\ k &= \frac{c}{t+1+at}, \end{aligned}$$

because $a = -1$,

$$k = c,$$

But this time $y_p = kt$, so $y_p = ct$, still it is a function of t.

Add y_p and y_c together,

$$y_t = A(-a)^t + \frac{c}{1+a} \qquad \text{if } a \neq -1 \tag{2}$$

$$y_t = A(-a)^t + ct \qquad \text{if } a = -1 \tag{3}$$

Last, of course you can solve A if you have an initial condition, say $y_t = y_0$, and $a \neq -1$,

$$y_0 = A + \frac{c}{1+a} \qquad A = y_0 - \frac{c}{1+a}$$

If $a = 1$,

$$y_0 = A(-a)^0 + 0 = A$$

Then just plug them back to the according solution (2) and (3).

1.2.1 One Example

Solve

$$y_{t+1} - 2y_t = 2$$

First solve the complementary equation,

$$y_{t+1} - 2y_t = 0$$

use Ab^t,

$$\begin{aligned} Ab^{t+1} - 2Ab^t &= 0 \\ b - 2 &= 0 \\ b &= 2 \end{aligned}$$

So, complementary solution,

$$y_c = A(2)^t$$

To find particular solution, let $y_t = k$ for $-\infty < t < \infty$,

$$\begin{aligned} k - 2k &= 2 \\ k = y_p &= -2 \end{aligned}$$

So the general solution,

$$y_t = y_p + y_c = -2 + 2^t A$$

If we are given an initial value, $y_0 = 4$,

$$4 = -2 + 2^0 A$$
$$A = 6$$

then the definite solution is

$$y_t = -2 + 6 \cdot 2^t$$

2 Second-Order Difference Equation

Second order difference equation is that an equation involves $\Delta^2 y_t$, which is

$$\begin{aligned}
\Delta(\Delta y_t) &= \Delta(y_{t+1} - y_t) \\
\Delta^2 y_t &= \Delta y_{t+1} - \Delta y_t \\
\Delta^2 y_t &= (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t) \\
\Delta^2 y_t &= y_{t+2} - 2y_{t+1} + y_t
\end{aligned}$$

We define a linear second-order difference equation as,

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$$

But we better not use iterative method to mess with it, trust me, it is more confusing then illuminating, so we come straightly to the general solution. As we we have studied by far, the general solution is the addition of the complementary solution and particular solution, we will talk about them next.

2.1 Complementary Solution

As interestingly as differential equation, we have several situations to discuss. So first we try to solve the complementary equation, some textbook might call this *reduced equation*, it is simply the homogeneous version of the orginal equation,

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$$

We learned from first-order difference equation that homogeneous difference equation has a solution form of $y_t = Ab^t$, we try it,

$$\begin{aligned}
Ab^{t+2} + a_1 Ab^{t+1} + a_2 Ab^t &= 0 \\
(b^2 + a_1 b + a_2) Ab^t &= 0
\end{aligned}$$

We assume that Ab^t is nonzero,

$$b^2 + a_1 b + a_2 = 0$$

This is our *characteristic equation*, same as we had seen in differential equation, high school math can sometime bring us a little fun,

$$b = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

I guess you are much familiar with it in textbook form, if we have a qudractic eqution $x^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ab}}{2a}$$

The same as we did in second-order differential equations, now we have three cases to discuss.

Case I $a_1^2 - 4a_2 > 0$ Then we have two distinct real roots, and the complementary solution will be

$$y_c = A_1 b_1^t + A_2 b_2^t$$

Note that, $A_1 b_1^t$ and $A_2 b_2^t$ are linearly independent, we can't only use one of them to represent complementary solution, because we need two constants A_1 and A_2.

Case II $a_1^2 - 4a_2 = 0$ Only one real root is available to us.

$$b = b_1 = b_2 = -\frac{a_1}{2}$$

Then the complementary solution collapses,

$$y_c = A_1 b^t + A_2 b^t = (A_1 + A_2) b^t$$

We just need another constant to fill the position, say A_4,

$$y_c = A_3 b^t + A_4 t b^t$$

where $A_3 = A_1 + A_2$, and tb^t is just old trick we have similiarly used in differential equation, which we used te^{rt}.

Case III $a_1^2 - 4a_2 < 0$ Complex numbers are our old friends, we need to make use of them again here.

$$b_1 = \alpha + i\beta$$
$$b_2 = \alpha - i\beta$$

where $\alpha = -\frac{a_1}{2}$, $\beta = \frac{\sqrt{4a_2 - a_1^2}}{2}$. Thus,

$$y_c = A_1(\alpha + i\beta)^t + A_2(\alpha - i\beta)^t$$

Here we simply need to make use of *De Moivre's theorem*,

$$y_c = A_1 \|b_1\|^t [\cos(t\theta) + i\sin(t\theta)] + A_2 \|b_2\|^t [\cos(t\theta) - i\sin(t\theta)]$$

where $\|b_1\| = \|b_2\|$, because

$$\|b_1\| = \|b_2\| = \sqrt{\alpha^2 + \beta^2}$$

Thus,

$$\begin{aligned}
y_c &= A_1 \|b\|^t [\cos(t\theta) + i\sin(t\theta)] + A_2 \|b\|^t [\cos(t\theta) - i\sin(t\theta)] \\
&= \|b\|^t \{A_1[\cos(t\theta) + i\sin(t\theta)] + A_2[\cos(t\theta) - i\sin(t\theta)]\} \\
&= \|b\|^t [(A_1 + A_2)\cos(t\theta) + (A_1 - A_2)i\sin(t\theta) \\
&= \|b\|^t [A_5 \cos(t\theta) + A_6 \sin(t\theta)]
\end{aligned}$$

2.2 Particular Solutions

We pick any y_p to satisfy

$$y_{t+2} + a_1 y_{y+1} + a_2 y_t = c$$

The simplest case is to choose $y_{t+2} = y_{t+1} = y_t = k$, thus

$$k + a_1 k + a_2 k = c, \text{ and } k = \frac{c}{1 + a_1 + a_2}$$

But we have to make sure that $a_1 + a_2 \neq -1$. If $a_1 + a_2 = -1$ happens we will use choose $y_p = kt$, and following this pattern there will be kt^2, kt^3, etc to choose depending on the situation we have.

2.3 One Example

Solve

$$y_{t+2} - 4y_{y+1} + 4y_t = 7$$

First, complementary solution, its characteristic equation is

$$\begin{aligned}
b^2 - 4b + 4 = 0 \\
(b+2)(b-2) = 0
\end{aligned}$$

The complementary solution is

$$y_c = A_1 b_1^2 + A_2 b_2^{-2}$$

For particular solution, we try $y_{t+2} = y_{t+1} = y_t = k$, then

$$k - 4k + 4k = 7, \text{ which is } k = 7$$

Then the general solution is

$$y = y_c + y_p = A_1 2^t - A_2(-2)^t + 7$$

And we have two initial conditions, $y_0 = 1$ and $y_1 = 3$,

$$y_0 = A_1 - A_2 + 7 = 1$$
$$y_1 = 2A_1 + 2A_2 + 7 = 3$$

Solve for

$$A_1 = -4 \quad A_2 = 2$$

Difinite solution is

$$y = y_c + y_p = -4 \cdot 2^t - 2(-2)^t + 7$$

3 p^{th}-Order Difference Equation

Previous sections are simply teaching your to solve the low order difference equations, no much of insight and difficulty. From this section on, difficulty increases significantly, for those of you who do not prepare linear algebra well enough, it might not be a good idea to study this section in a hurry. Every mathematics is built on another, it is not quite possible to progress in jumps. And one warning, as the mathematics is becoming deeper, the notation is also becoming complicated. However, don't get dismayed, this is not rocket science.

We generalize our difference equation into p^{th}-Order,

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_p y_{t-p} + \omega_t \tag{4}$$

This is actually a generalization of (1), we need to set ω here in order to prepare for stochastic difference equations. It is a conventional that we diffence backwards when deal with high orders. But here we still take it as constant. We can't handle this difference equation in this form since it doesn't leave us much of room to perform any useful algebraic operation. Even if you try to use old method, its characteristic equation will be

$$b^t - a_1 b^{t-1} - a_2 b^{t-2} \cdots + a_p b^{t-p} = 0,$$

factor out b^{t-p},

$$b^p - a_1 b^{p-1} - a_2 b^{p-2} \cdots a_p = 0$$

If the p is high enough make you have headache, it means this isn't the right way to advance.

Everything will be reshaped into its matrix form, define

$$\boldsymbol{\psi}_t = \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix}$$

which is a $p \times 1$ vector. And define

$$\boldsymbol{F} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{p-1} & a_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Finally, define

$$\boldsymbol{v}_t = \begin{bmatrix} \omega_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Put them together, we have new vector form first-order difference equation,

$$\boldsymbol{\psi}_t = \boldsymbol{F}\boldsymbol{\psi}_{t-1} + \boldsymbol{v}_t$$

you will see what is $\boldsymbol{\psi}_{t-1}$ in next explicit form,

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{p-1} & a_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} \omega_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The first equation is

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_p y_{t-p} + \omega_t$$

which is exactly (4). And it is quite obvious for you that from second to p^{th} equation is simple a $y_i = y_i$ form. The reason that we write it like this is not obvious till now, but one reason is that we reduce the system down to a first-order difference equation, although it is in matrix form, we can use old methods to analyse it.

3.1 Iterative Method

We list evolution of difference equations as follows,

$$\begin{aligned} \boldsymbol{\psi}_0 &= \boldsymbol{F}\boldsymbol{\psi}_{-1} + \boldsymbol{v}_0 \\ \boldsymbol{\psi}_1 &= \boldsymbol{F}\boldsymbol{\psi}_0 + \boldsymbol{v}_1 \\ \boldsymbol{\psi}_2 &= \boldsymbol{F}\boldsymbol{\psi}_1 + \boldsymbol{v}_2 \\ &\vdots \\ \boldsymbol{\psi}_t &= \boldsymbol{F}\boldsymbol{\psi}_{t-1} + \boldsymbol{v}_t \end{aligned}$$

And we need to assume that $\boldsymbol{\psi}_{-1}$ and $\boldsymbol{v}_0$ are known. Then old tricks of recursive substitution,

$$\begin{aligned}\boldsymbol{\psi}_1 &= \boldsymbol{F}(\boldsymbol{F}\boldsymbol{\psi}_{-1} + \boldsymbol{v}_0) + \boldsymbol{v}_1\\ &= \boldsymbol{F}^2\boldsymbol{\psi}_{-1} + \boldsymbol{F}\boldsymbol{v}_0 + \boldsymbol{v}_1\end{aligned}$$

Again,

$$\begin{aligned}\boldsymbol{\psi}_2 &= \boldsymbol{F}\boldsymbol{\psi}_1 + \boldsymbol{v}_2\\ &= \boldsymbol{F}(\boldsymbol{F}^2\boldsymbol{\psi}_{-1} + \boldsymbol{F}\boldsymbol{v}_0 + \boldsymbol{v}_1) + \boldsymbol{v}_2\\ &= \boldsymbol{F}^3\boldsymbol{\psi}_{-1} + \boldsymbol{F}^2\boldsymbol{v}_0 + \boldsymbol{F}\boldsymbol{v}_1 + \boldsymbol{v}_2\end{aligned}$$

Till step t,

$$\begin{aligned}\boldsymbol{\psi}_t &= \boldsymbol{F}^{t+1}\boldsymbol{\psi}_{-1} + \boldsymbol{F}^t\boldsymbol{v}_0 + \boldsymbol{F}^{t-1}\boldsymbol{v}_1 + \ldots + \boldsymbol{F}\boldsymbol{v}_{t-1} + \boldsymbol{v}_t\\ &= \boldsymbol{F}^{t+1}\boldsymbol{\psi}_{-1} + \sum_{i=0}^{t}\boldsymbol{F}^i\boldsymbol{v}_{t-i}\end{aligned}$$

which has explicit form,

$$\begin{bmatrix} y_t\\ y_{t-1}\\ y_{t-2}\\ \vdots\\ y_{t-p+1}\end{bmatrix} = \boldsymbol{F}^{t+1}\begin{bmatrix} y_{-1}\\ y_{-2}\\ y_{-3}\\ \vdots\\ y_{-p}\end{bmatrix} + \boldsymbol{F}^t\begin{bmatrix}\omega_0\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix} + \boldsymbol{F}^{t-1}\begin{bmatrix}\omega_1\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix} + \cdots + \boldsymbol{F}\begin{bmatrix}\omega_{t-1}\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix} + \begin{bmatrix}\omega_t\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix}$$

Note that how we use $\boldsymbol{v}_i$ here. Unfortunatly, more notations are needed. We denote the $(1,1)$ elelment of $\boldsymbol{F}^t$ as $f_{11}^{(t)}$, and so on so forth. We made these notation in order to extract the first equation from above unwieldy system, thus

$$\begin{aligned}y_t = f_{11}^{(t+1)}y_{-1} + f_{12}^{(t+1)}y_{-2} + f_{13}^{(t+1)}y_{-3} + \ldots + f_{1p}^{(t+1)}y_{-p}\\ + f_{11}^{(t)}\omega_0 + f_{11}^{(t-1)}\omega_1 + \ldots + f_{11}\omega_{t-1} + \omega_t\end{aligned}$$

I have to admit, most of time mathematics looks more difficult than it really is, notation is evil, and trust me, more evil stuff you haven't seen yet. However, keep on telling youself this is just the first equation of the system, nothing more. Because the idea is the same in matrix form, we are told that we know $\boldsymbol{\psi}_{-1}$ and $\boldsymbol{v}_0$ as initial value. Here we just write them in a explicit way, y_t is a function of initial values from y_{-1} to y_{-p}, and a sequence of ω_i. The scalar first-order difference equation need one initial value, and p^{th}-order need p initial values. But if we turn it into vector form, it need one initial value again, which is $\boldsymbol{\psi}_{-1}$.

If you want, you can even take partial derivative to find *dynamic multiplier*, say we find

$$\frac{\partial y_t}{\partial \omega_0} = f_{11}^{(t)}$$

this measure one-unit increase in ω_0, and is give by $f_{11}^{(t)}$.

3.2 Analytical Solution

We need to study further about matrix $\boldsymbol{F}$, because it is the core of the system, all information are hidden in this matrix. We'd better use a small scale $\boldsymbol{F}$ to study the general pattern. Let's set

$$\boldsymbol{F} = \begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix}$$

So the first equation of the system, we can write,

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + \omega_t, \text{ or you might like, } y_t - a_1 y_{t-1} - a_2 y_{t-2} = \omega_t$$

It is your familiar form, a nonhomogenenous second-order difference equation. And calculate its eigenvalue,

$$\begin{vmatrix} a_1 - \lambda & a_2 \\ 1 & -\lambda \end{vmatrix} = 0$$

Write determinant in algebraic form,

$$\lambda^2 - a_1\lambda - a_2 = 0$$

We finally reveal the myth why we always solve a characteristic equation first, because we are finding it eigenvalues. In general the characterstic equal of p^{th}-order difference equation will be,

$$\lambda^p - a_1\lambda^{p-1} - a_2\lambda^{p-2} - \cdots - a_p = 0 \tag{5}$$

we of course can prove it by using $|\boldsymbol{F} - \lambda \boldsymbol{I}| = 0$, but the process looks rather messy, the basic idea is to perform row operations to turn it into upper triangle matrix, then the determinant is just the product of diagonal entries. Not much insight we can get from it, so we omit the process here.

In second order differential or difference equation, we usually discuss three cases of solution, two distinct roots (now you know they are eigenvalues), one repeated root, complex roots. We have a root formula two catagorize them $b^2 - 4ac$, however, that is only used in second order, when we come to high orders, we don't have anything like that. But it actually makes high orders more interesting than low ones, because we can use linear algebra.

3.2.1 Distinct Real Eigenvalues

Distinct eigenvalue category here corresponds to two distinct real roots in second order. It is becoming hardcore in following content, be sure you are well-prepared in linear algebra.

If $\boldsymbol{F}$ has p distinct eigenvalues, you should be happy, because diagonalization is waving hands towards you. Recall that

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$$

where $\boldsymbol{P}$ is a nonsingular matrix, because distinct eigenvalues assure that we have linearly dependent eigenvectors. We need to make some cosmetic change to suit our need,

$$\boldsymbol{F} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$$

We use a capital Λ to indicate that all eigenvalues on the principle diagonal. You should naturally respond that

$$\boldsymbol{F}^t = \boldsymbol{P}\boldsymbol{\Lambda}^t\boldsymbol{P}^{-1},$$

simply because,

$$\begin{aligned}
\boldsymbol{F}^2 &= \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1} \\
&= \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{\Lambda}\boldsymbol{P}^{-1} \\
&= \boldsymbol{P}\boldsymbol{\Lambda^2}\boldsymbol{P}^{-1}
\end{aligned}$$

Diagonal matrix shows its convenience,

$$\Lambda^t = \begin{bmatrix} \lambda_1^t & 0 & \cdots & 0 \\ 0 & \lambda_2^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p^t \end{bmatrix}$$

And unfortunately again, we need more notation, we denote t_{ij} to be the i^{th} row, j^{th} column entry of $\boldsymbol{P}$, and t^{ij} to be the i^{th} row, j^{th} column entry of $\boldsymbol{P}^{-1}$.

We try to write $\boldsymbol{F}^t$ in explicit matrix form,

$$\begin{aligned}
\boldsymbol{F}^t &= \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1p} \\ t_{21} & t_{22} & \cdots & t_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ t_{p1} & t_{p2} & \cdots & t_{pp} \end{bmatrix} \begin{bmatrix} \lambda_1^t & 0 & \cdots & 0 \\ 0 & \lambda_2^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p^t \end{bmatrix} \begin{bmatrix} t^{11} & t^{12} & \cdots & t^{1p} \\ t^{21} & t^{22} & \cdots & t^{2p} \\ \vdots & \vdots & \ddots & \vdots \\ t^{p1} & t^{p2} & \cdots & t^{pp} \end{bmatrix} \\
&= \begin{bmatrix} t_{11}\lambda_1^t & t_{12}\lambda_2^t & \cdots & t_{1p}\lambda_p^t \\ t_{21}\lambda_1^t & t_{22}\lambda_2^t & \cdots & t_{2p}\lambda_p^t \\ \vdots & \vdots & \ddots & \vdots \\ t_{p1}\lambda_1^t & t_{p2}\lambda_2^t & \cdots & t_{pp}\lambda_p^t \end{bmatrix} \begin{bmatrix} t^{11} & t^{12} & \cdots & t^{1p} \\ t^{21} & t^{22} & \cdots & t^{2p} \\ \vdots & \vdots & \ddots & \vdots \\ t^{p1} & t^{p2} & \cdots & t^{pp} \end{bmatrix}
\end{aligned}$$

Then f_{11}^t is

$$f_{11}^t = t_{11}t^{11}\lambda_1^t + t_{12}t^{21}\lambda_2^t + \cdot + t_{1p}t^{p1}\lambda_p^t$$

if we denote $c_i = t_{1i}t^{i1}$, then

$$f_{11}^t = c_1\lambda_1^t + c_2\lambda_2^t + \cdot + c_p\lambda_p^t$$

Obviously if you pay attention to,

$$c_1 + c_2 + \ldots + c_p = t_{11}t^{11} + t_{12}t^{21} + \ldots + t_{1p}t^{p1},$$

you would realize it is a scalar product, it is from the first row of $\boldsymbol{P}$ and first column of $\boldsymbol{P}^{-1}$. In other words, it is the first element of $\boldsymbol{P}\boldsymbol{P}^{-1}$. Magically, $\boldsymbol{P}\boldsymbol{P}^{-1}$ is an identity matrix. Thus,

$$c_1 + c_2 + \ldots + c_p = 1$$

This time if you want to calculate dynamic multiplier,

$$\begin{aligned}\frac{\partial y_t}{\partial \omega_0} &= f_{11}^{(t)} \\ &= c_1\lambda_1^t + c_2\lambda_2^t + \cdot + c_p\lambda_p^t\end{aligned}$$

the dynamic multiplier is a weighted average of all t^{th} powered eigenvalues.

Here one problem must be solved before we move on, how can we find c_i? Or is it just some unclosed form expression, we can stop here?

What we do next might look very strange to you, but don't stop and finish it you will get a sense of what we are preparing here.

Set $\boldsymbol{t}_i$ to be the i^{th} eigenvalue,

$$\boldsymbol{t}_i = \begin{bmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ \lambda_i^1 \\ 1 \end{bmatrix}$$

Then,

$$\begin{aligned}\boldsymbol{F}\boldsymbol{t}_i &= \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{p-1} & a_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ \lambda_i^1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} a_1\lambda_i^{p-1} + a_2\lambda_i^{p-2} + \ldots + a_{p-1}\lambda_i^1 + a_p \\ \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ \lambda_i^1 \end{bmatrix}\end{aligned}$$

Recall you have seen the characteristic equation of p^{th}-order (5),

$$\lambda^p - a_1\lambda^{p-1} - a_2\lambda^{p-2} - \cdots - a_{p-1}\lambda - a_p = 0$$

Rearange,

$$\lambda^p = a_1\lambda^{p-1} + a_2\lambda^{p-2} + \cdots + a_{p-1}\lambda + a_p \tag{6}$$

Interestingly, we get what we want here, the right-hand side of last equation is just the first element of $\boldsymbol{F}\boldsymbol{t}_i$.

$$\boldsymbol{F}\boldsymbol{t}_i = \begin{bmatrix} \lambda_i^p \\ \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ \lambda_i^1 \end{bmatrix}$$

We factor out a λ_i,

$$\boldsymbol{F}\boldsymbol{t}_i = \lambda_i \begin{bmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ \lambda_i^1 \\ 1 \end{bmatrix} = \lambda_i \boldsymbol{t}_i$$

We have successfully showed that $\boldsymbol{t}_i$ is the eigenvalue of $\boldsymbol{F}$. Now we can set up a $\boldsymbol{P}$, its every column is eigenvalue,

$$\boldsymbol{P} = \begin{bmatrix} \lambda_1^{p-1} & \lambda_2^{p-1} & \cdots & \lambda_p^{p-1} \\ \lambda_1^{p-2} & \lambda_2^{p-2} & \cdots & \lambda_p^{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_p^1 \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

Actually this is a transposed *Vandermonde matrix*, which is mainly made use of in signal processing and polynomial interpolation. Since we do not know $\boldsymbol{P}^{-1}$[4]yet. We use the notation of t^{ij}, we postmultiply $\boldsymbol{P}$ by the first column of $\boldsymbol{P}^{-1}$,

$$\begin{bmatrix} \lambda_1^{p-1} & \lambda_2^{p-1} & \cdots & \lambda_p^{p-1} \\ \lambda_1^{p-2} & \lambda_2^{p-2} & \cdots & \lambda_p^{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_p^1 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} t^{11} \\ t^{21} \\ \vdots \\ t^{p1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

[4] Don't ever try to calculate its inverse matrix by ajugate matrix or Gauss-Jordon elimination by hands, it is very inaccurate and time consuming.

This is a linear equation system, solution will look like,

$$
\begin{aligned}
t^{11} &= \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\cdots(\lambda_1 - \lambda_p)} \\
t^{21} &= \frac{1}{(\lambda_2 - \lambda_2)(\lambda_2 - \lambda_3)\cdots(\lambda_2 - \lambda_p)} \\
&\vdots \\
t^{p1} &= \frac{1}{(\lambda_p - \lambda_2)(\lambda_p - \lambda_3)\cdots(\lambda_p - \lambda_p)}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
c_1 = t_{11}t^{11} &= \frac{\lambda_1^{p-1}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\cdots(\lambda_1 - \lambda_p)} \\
c_2 = t_{12}t^{21} &= \frac{\lambda_2^{p-1}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)\cdots(\lambda_2 - \lambda_p)} \\
\vdots \\
c_p = t_{1p}t^{p1} &= \frac{\lambda_p^{p-1}}{(\lambda_p - \lambda_2)(\lambda_p - \lambda_3)\cdots(\lambda_p - \lambda_{p-1})}
\end{aligned}
$$

I know all this must been quite uncomfortable to you if you don't fancy mathematics too much. We'd better go through a small example to get familiar with these artilleries.

Let's look at a second-order difference equation,

$$y_t = 0.4y_{t-1} + 0.7y_{t-2} + \omega_t.$$

But I guess you would prefer to it like this,

$$y_{t+2} - 0.4y_{t+1} - 0.7y_t = \omega_{t+2}.$$

Calculate its characteristic equation,

$$|\boldsymbol{F} - \lambda \boldsymbol{I}| = 0,$$

which is

$$\boldsymbol{F} - \lambda \boldsymbol{I} = \begin{bmatrix} 0.4 & 0.7 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0.4 - \lambda & 0.7 \\ 1 & -\lambda \end{bmatrix}$$

Now calculate its determinant,

$$\boldsymbol{F} - \lambda \boldsymbol{I} = \lambda^2 - 0.4\lambda - 0.7$$

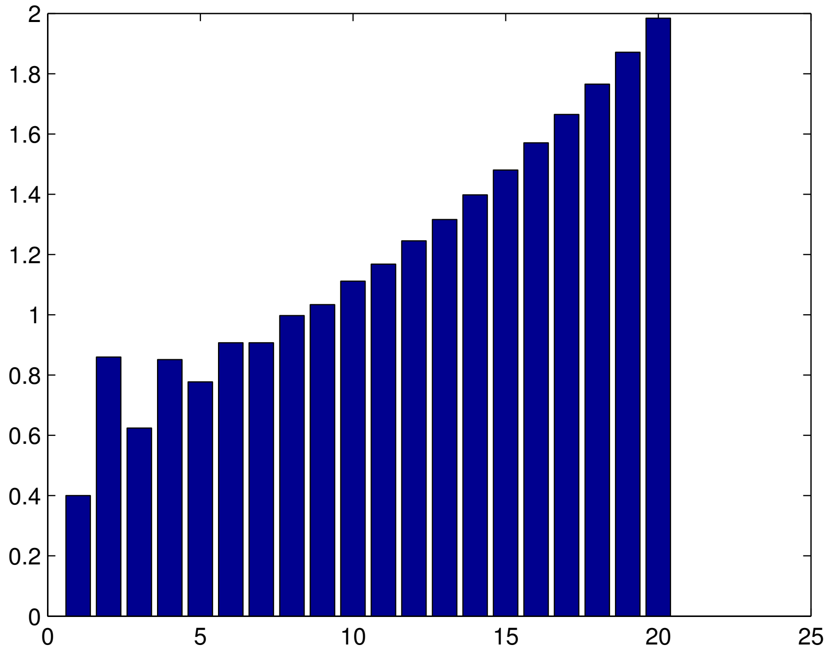

Figure 1: Dynamic multiplier as a function of t.

Use root formula,

$$\lambda_1 = \frac{0.4 + \sqrt{(-0.4)^2 - 4(-0.7)}}{2} = 1.0602$$
$$\lambda_2 = \frac{0.4 - \sqrt{(-0.4)^2 - 4(-0.7)}}{2} = -0.6602$$

For c_i,

$$c_1 = \frac{\lambda_1}{\lambda_1 - \lambda_2} = \frac{1.0602}{1.0602 + 0.6602} = 0.6163$$
$$c_2 = \frac{\lambda_2}{\lambda_2 - \lambda_1} = \frac{-0.6602}{-0.6602 - 1.0602} = 0.3837$$

And note that $0.6163 + 0.3837 = 1$. Dynamic multiplier is

$$\frac{\partial y_t}{\partial \omega_t} = c_1\lambda^t + c_2\lambda^t = 0.6163 \cdot 1.0602^t + 0.3837 \cdot (-0.6602)^t$$

The figure 1 shows the dynamic multiplier as a function of t, MATLAB code is at the appendix. The dynamic multiplier will explode as $t \to \infty$, because we have an eigenvalue $\lambda_1 > 1$.

3.2.2 Distinct Complex Eigenvalues

We also need to talk about distinct complex eigenvalues, but the example will still resort to second-order difference equation, because we have a handy root formula. The pace might be a little fast, but easily understandable. Suppose we have two complex eigenvalues,

$$\lambda_1 = \alpha + i\beta$$
$$\lambda_2 = \alpha - i\beta$$

And modulus of the conjugate pair is the same,

$$\|\lambda\| = \sqrt{\alpha^2 + \beta^2}$$

Then rewritten conjugate pair as,

$$\lambda_1 = \|\lambda\| (\cos\theta + i\sin\theta)$$
$$\lambda_2 = \|\lambda\| (\cos\theta - i\sin\theta)$$

According to *De Moivre's theorem*, to raise the power of complex number,

$$\lambda_1^t = \|\lambda\|^t (\cos t\theta + i\sin t\theta)$$
$$\lambda_2^t = \|\lambda\|^t (\cos t\theta - i\sin t\theta)$$

Back to dynamic multiplier,

$$\begin{aligned}\frac{\partial y_t}{\partial \omega_t} &= c_1\lambda_1^t + c_2\lambda_2^t \\ &= c_1 \|\lambda\|^t (\cos t\theta + i\sin t\theta) + c_2 \|\lambda\|^t (\cos t\theta - i\sin t\theta)\end{aligned}$$

rearrange, we have

$$= (c_1 + c_2) \|\lambda\|^t \cos t\theta + i(c_1 - c_2) \|\lambda\|^t \sin t\theta$$

However, c_i is calculated from λ_i, so since eigenvalues are complexe number, and so are c_i's. We can denote the conjugate pair as,

$$c_1 = \gamma + i\delta$$
$$c_2 = \gamma - i\delta$$

Thus,

$$\begin{aligned}c_1\lambda_1^t + c_2\lambda_2^t &= [(\gamma + i\delta) + (\gamma - i\delta)] \|\lambda\|^t \cos t\theta + i[((\gamma + i\delta) - (\gamma - i\delta)] \|\lambda\|^t \sin t\theta \\ &= 2\gamma \|\lambda\|^t \cos t\theta + i2i\delta \|\lambda\|^t \sin t\theta \\ &= 2\gamma \|\lambda\|^t \cos t\theta - 2\delta \|\lambda\|^t \sin t\theta\end{aligned}$$

it turns out real number again. And the time path is determined by modulus $\|\lambda\|$, if $\|\lambda\| > |1|$, the dynamic multiplier will explode, if $\|\lambda\| < |1|$ dynamic multiplier follow either a oscillating or nonoscillating decaying pattern.

3.2.3 Repeated Eigenvalues

If we have repeat eigenvalues, diagonalization might not be the choice, since we might encounter singular $\boldsymbol{P}$. But since we have enough mathematical artillery to use, we just switch to another kind of more general diagonalization, which is called *the Jordon decomposition.* Let's give a definition first, a $h \times h$ matrix which is called *Jordon block matrix*, is defined as

$$\boldsymbol{J}_h(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

This is a *Jordon canonical form*,[5]

The Jordon decomposition says, if we have a $m \times m$ matrix $\boldsymbol{A}$, then there must be a nonsingular matrix $\boldsymbol{B}$ such that,

$$\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B} = \boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_{h_1}(\lambda_1) & (0) & \cdots & (0) \\ (0) & \boldsymbol{J}_{h_2}(\lambda_2) & \cdots & (0) \\ \vdots & \vdots & \ddots & \vdots \\ (0) & (0) & \cdots & \boldsymbol{J}_{h_r}(\lambda_r) \end{bmatrix}$$

Cosmetically change to our notation,

$$\boldsymbol{F} = \boldsymbol{M}\boldsymbol{J}\boldsymbol{M}^{-1}$$

As you can imagine,

$$\boldsymbol{F}^t = \boldsymbol{M}\boldsymbol{J}^t\boldsymbol{M}^{-1}$$

And

$$\boldsymbol{J}^t = \begin{bmatrix} \boldsymbol{J}^t_{h_1}(\lambda_1) & (0) & \cdots & (0) \\ (0) & \boldsymbol{J}^t_{h_2}(\lambda_2) & \cdots & (0) \\ \vdots & \vdots & \ddots & \vdots \\ (0) & (0) & \cdots & \boldsymbol{J}^t_{h_r}(\lambda_r) \end{bmatrix}$$

What you need to do is just to calculate $\boldsymbol{J}^t_{h_i}(\lambda_i)$ one by one, of course we should leave this to computer. Then you can calculate dynamic multiplier as usual, once you figure out the $\boldsymbol{F}$, then pick the $(1,1)$ entry $f^{(t)}_{11}$ the rest will be done.

4 Lag Operator

In econometrics, the convention is to use *lag operator* to function as what we did with difference equation. Essentially, they are the identical. But there

[5] Study my notes of *Linear Algebra and Matrix Analysis II* [1].

is still some subtle conceptual discrepancy, difference equation is a kind of equation, a balanced expression on both sides, but lag operator represents a kind of operation, no different from addition, substraction or multiplication. Lag operator will turn a time series into a difference equation.

We conventionally use L to represent lag operator, for instance,

$$Lx_t = x_{t-1} \qquad L^2 x_t = x_{t-2}$$

4.1 p^{th}-order difference equation with lag operator

Basically this section is to reproduce the crucial results of what we did with difference equation. We are trying to show you that we can reach the same goal with the help of either difference equations or lag operators.

Turn this p^{th}-order difference equation into lag operator form,

$$y_t - a_1 y_{t-1} - a_2 y_{t-2} - \cdots - a_p y_{t-p} = \omega_t$$

which will be

$$(1 - a_1 L - a_2 L^2 - \cdots - a_p L^p) y_t = \omega_t$$

Because it is an operation in the equation above, some steps below might not be appropriate, but if we switch to

$$(1 - a_1 z - a_2 z^2 - \cdots - a_p z^p) y_t = \omega_t,$$

we can use some algebraic operation to analyse it. Multiply both sides by z^{-p},

$$\begin{aligned} z^{-p}(1 - a_1 z - a_2 z^2 - \cdots - a_p z^p) y_t &= z^{-p}\omega_t \\ (z^{-p} - a_1 z z^{-p} - a_2 z^2 z^{-p} - \cdots - a_p z^p z^{-p}) y_t &= z^{-p}\omega_t \\ (z^{-p} - a_1 z^{1-p} - a_2 z^{2-p} - \cdots - a_p) y_t &= z^{-p}\omega_t \end{aligned}$$

And define $\lambda = z^{-1}$, and set the right-hand side zero,

$$\lambda^p - a_1 \lambda^{p-1} - a_2 \lambda^{p-2} - \cdots - a_{p-1}\lambda - a_p = 0$$

This is the characteristic equation, reproduction of equation (6).

References

[1] Chen W. (2011): *Linear Algebra and Matrix Analysis II*, study notes

[2] Chen W. (2011): *Differential Equations*, study notes

[3] Chiang A.C. and Wainwright K.(2005): *Fundamental Methods of Mathematical Economics*, McGraw-Hill

[4] Simon C. and Blume L.(1994): *Mathematics For Economists*, W.W.Norton & Company

[5] Enders W. (2004): *Applied Econometric Time Series*, John Wiley & Sons