A Primer on Vector Autoregressions

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[DISCLAIMER]

These notes are meant to provide intuition on the basic mechanisms of VARs

As such, most of the material covered here is treated in a very informal way

If you crave a formal treatment of these topics, you should stop here and buy a copy of Hamilton's "Time Series Analysis"

The Matlab codes accompanying the notes are available at:
`https://github.com/ambropo/VAR-Toolbox`

The job of macro-econometricians

- In their 2001 Journal of Economic Perspectives' article "Vector Autoregressions" Stock and Watson describe the job of macroeconometricians as consisting of the following tasks
  - Describe and summarize macroeconomic time series
  - Make forecasts
  - Recover the structure of the macroeconomy from the data Main focus of these notes
  - Advise macroeconomic policy-makers
- Vector autoregressive models (VARs) are a statistical tool to perform these tasks

What can we do with VARs?

- Consider a bivariate VAR with the following variables:
  - Real GDP growth (y_t)
  - Policy rate (r_t)

- A VAR can help us answering the following questions
  [1] What is the dynamic behavior of these variables? How do these variables interact?
  [2] What is the most likely path of GDP in the next few quarters?
  [3] What is the effect of a monetary policy shock on GDP?
  [4] What has been the historical contribution of monetary policy shocks to GDP fluctuations?

VAR Basics

What is a Vector Autoregression (VAR)?

- Consider a (2×1) vector of zero-mean time series x_t, composed of t observations and an initial condition x_0

$$x_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1t} \\ x_{21} & x_{22} & \dots & x_{2t} \end{bmatrix} \quad \text{and} \quad x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

- Assume that the two time series in x_t are covariance stationary, which means (for $i = 1, 2$)
  - Constant mean $\mathbb{E}[x_{it}] = \mu_i$
  - Constant variance $Var[x_{it}] = \sigma_i^2$
  - Constant autocovariance $COV[x_{it}, x_{it+\tau}] = \gamma_i(\tau)$

- A **structural VAR** of order 1 is given by

$$x_t = \Phi x_{t-1} + B\varepsilon_t$$

where
  - Φ and B are (2×2) matrices of coefficients
  - ε_t is an (2×1) vector of unobservable zero-mean white noise processes

Three different ways of writing the same thing

- There are different ways to write the same structural VAR(1)

$$x_t = \Phi x_{t-1} + B\varepsilon_t$$

- For example, we can write it in matrix form

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

- Or as a system of linear equations

$$\begin{cases} x_{1t} = \phi_{11} x_{1,t-1} + \phi_{12} x_{1,t-1} + b_{11}\varepsilon_{1t} + b_{12}\varepsilon_{2t} \\ x_{2t} = \phi_{21} x_{2,t-1} + \phi_{22} x_{2,t-1} + b_{21}\varepsilon_{1t} + b_{22}\varepsilon_{2t} \end{cases}$$

The structural shocks

- We defined ε_t as a *vector of unobservable zero mean white noise processes*. **What does it mean?**

- The elements of ε_t are serially uncorrelated and independent of each other

- In other words we assumed

$$\varepsilon_t = (\varepsilon'_{1t}, \varepsilon'_{2t})' \sim \mathcal{N}(0, I_2)$$

where

$$Var(\varepsilon_t) = \Sigma_\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad CORR(\varepsilon_t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Why is it called 'structural' VAR?

- Go back to our bivariate structural VAR(1)

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

- The structural VAR can be thought of as a description of the true structure of the economy
  - E.g.: an approximation of the structure of a DSGE model
- The structural shocks are shocks with a well-defined economic interpretation
  - E.g.: TFP shocks or monetary policy shocks
  - As $\varepsilon_t \sim \mathcal{N}(0, I_2)$ we can move one shock keeping the other shocks fixed
  - That is: we can focus on the causal effect of one shock at the time

Structural VARs can answer many interesting questions...

- Go back to our bivariate structural VAR(1). To make a concrete example, assume that
  - x_{1t} and x_{2t} are output growth (y_t) and the policy rate (r_t), both demeaned
  - ε_{1t} and ε_{2t} are a demand shock (ε_t^{Demand}) and a monetary policy shock (ε_t^{MonPol})
  - B is known (we'll get back to this in a second)

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \underbrace{\boxed{\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}}}_{\text{Dynamic matrix}} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \underbrace{\boxed{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}}_{\text{Impact matrix}} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

- What is the effect of monetary policy shocks on output?
  - The coefficient b_{12} captures the 'impact effect' of a monetary policy shock on output growth
  - The Φ matrix allows us to trace the 'dynamic effect' of the monetary policy shock over time
  - (We can add additional variables and look at other shocks: aggregate supply, oil price,...)

... but the estimation of structural VARs is tricky

* **Problem** The structural shocks ε_t are unobserved. How can we estimate B?

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

* Best we can do is to 'bundle' the ε_t into a single object:

$$u_t = B\varepsilon_t \Rightarrow \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix} \Rightarrow \begin{cases} u_{yt} = b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{MonPol} \\ u_{rt} = b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{MonPol} \end{cases}$$

* Why is this useful? The VAR becomes

$$x_t = \Phi x_{t-1} + u_t$$

* Now we can estimate Φ and u_t with OLS (where u_t will be OLS residuals)

The reduced-form VAR

- This alternative formulation of the VAR is called the **reduced-form VAR** representation

$$x_t = \Phi x_{t-1} + u_t$$

- In matrix form

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} u_{yt} \\ u_{rt} \end{bmatrix}$$

- Or as a system of linear equations

$$\begin{cases} y_t = \phi_{11} y_{t-1} + \phi_{12} r_{t-1} + u_{yt} \\ r_t = \phi_{21} y_{t-1} + \phi_{22} r_{t-1} + u_{rt} \end{cases}$$

The reduced-form covariance matrix

- A key object of interest in VARs is the covariance matrix of the **reduced-form residuals**

$$\Sigma_u = \begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ \sigma_{yr}^2 & \sigma_r^2 \end{bmatrix}$$

- Differently from the structural shocks (which are orthogonal), the reduced-form residuals are correlated among each other

- This is because the elements of u_t inherit all the contemporaneous relations among the endogenous variables x_t
  - To see that, remember how the reduced form residuals are defined

$$\begin{cases} u_{yt} = b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{MonPol} \\ u_{rt} = b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{MonPol} \end{cases}$$

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- This is because the elements of u_t inherit all the contemporaneous relations among the endogenous variables x_t

- To make causal statements (e.g. the effects on y_t of a shock to ε_t^{MonPol}) we need to find a way to recover B

- This is the essence of **identification** in VARs

The Wold representation

- Before turning to identification, let's introduce another representation of the VAR that will be useful later

- Start from the structural VAR representation

$$x_t = \Phi x_{t-1} + B\varepsilon_t$$

- The **Wold representation** can be obtained by substituting recursively the elements on the right hand side of the equal sign

$$\begin{aligned} x_t &= \Phi x_{t-1} + B\varepsilon_t \\ &= \Phi \underbrace{\left(\Phi x_{t-2} + B\varepsilon_{t-1}\right)}_{x_{t-1}} + B\varepsilon_t = \Phi^2 x_{t-2} + \Phi B\varepsilon_{t-1} + B\varepsilon_t \\ &= \dots \\ &= \Phi^t x_0 + \sum_{j=0}^{t-1} \Phi^j B\varepsilon_{t-j} \end{aligned}$$

The Wold representation (cont'd)

- The Wold representation shows that each observation (x_t) can be re-written as a combination of two terms

$$x_t = \underbrace{\boxed{\Phi^t x_0}}_{\text{Initial condition}} + \underbrace{\boxed{\sum_{j=0}^{t-1} \Phi^j B \varepsilon_{t-j}}}_{\text{Current \& past shocks}}$$

  - The sum of current and past structural shocks
  - An initial condition

- Now let $t \to \infty$ to get

$$x_t = \Phi^{\infty} x_{t-\infty} + \sum_{j=0}^{\infty} \Phi^j B \varepsilon_{t-j}$$

- But: we assumed that x_t is covariance stationary. How do these infinite sums relate to that assumption?
  - Aren't the increasing powers of Φ exploding?

Stability of the VAR

- A VAR is stable if the effect of shocks progressively dissipate over time. For that to happen we need Φ^j to converge to zero

$$x_t = \Phi^{\infty} x_{t-\infty} + \sum_{j=0}^{\infty} \Phi^j B \varepsilon_{t-j}$$

- Why does this matter? If shocks have permanent effects
  - The mean and the variance of x_t will depend on the history of shocks
  - Violates covariance stationary assumption $\longrightarrow$ VAR displays unstable dynamics

- **Definition** A VAR is called stable iff all the eigenvalues of Φ are less than 1 in modulus. More formally:

$$\det\left(\Phi - \lambda I_2\right) = 0 \quad |\lambda| < 1$$

- **Implication** In the absence of shocks, the VAR will converge to its equilibrium (i.e. its unconditional mean)

The unconditional mean of the VAR

- First note that if the eigenvalues of Φ are less than 1 in modulus we have

$$\Phi^{\infty} = 0 \quad \text{and} \quad \sum_{j=0}^{\infty} \Phi^{j} = \boxed{(I_2 - \Phi)^{-1}}$$

Geometric series

- Because of white noise assumption of the ε_t, the unconditional mean is simply given by

$$\mathbb{E}\left[x_t\right] = \Phi^{\infty} x_{t-\infty} + \sum_{j=0}^{\infty} \Phi^{j} B \mathbb{E}\left[\varepsilon_{t-j}\right] = 0$$

- Note that if the VAR had a constant (α) an additional term would show up in the Wold representation

$$x_t = \Phi^{\infty} x_{t-\infty} + \sum_{j=0}^{\infty} \Phi^{j} \alpha + \sum_{j=0}^{\infty} \Phi^{j} B \varepsilon_{t-j}$$

- The unconditional mean in this case would be

$$\mathbb{E}\left[x_t\right] = (I_2 - \Phi)^{-1} \alpha$$

The general form of the stationary structural VAR(p) model

- The basic bivariate VAR(1) model used so far may be too parsimonious to sufficiently summarize the dynamic relations of the data

- Model can be enriched along the following dimensions
  - Increase the number of endogenous variables (k)
  - Increase the number of lags (p)
  - Add deterministic terms (e.g. time trend or seasonal dummy variables)
  - Add exogenous variables (e.g. price of oil from the point of view of a small country)

- The general form of the VAR(p) model with deterministic terms (Z_t) and exogenous variables (W_t) is given by

$$x_t = \Phi_1 x_{t-1} + \Phi_2 x_{t-2} + \ldots + \Phi_p x_{t-p} + \Lambda Z_t + \Psi W_t + B\varepsilon_t$$

The Identification Problem

Back to our reduced form VAR

* We have seen above that with OLS we can only estimate the reduced-form VAR (and not the structural VAR)

* Assume we already have an OLS estimate of $\hat{\Phi}$ and $\hat{u}_t$:

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} u_{yt} \\ u_{rt} \end{bmatrix}$$

* **Question** What is the effect of a monetary policy shock on GDP growth?

* Unfortunately, the reduced form innovations (u_{yt} or u_{rt}) are not going to help us in answering the question

Reduced-form VARs do not tell us anything about causality

* To see that, assume that the 'true' (and unobserved) model of the economy is given by

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

* It is obvious that the reduced form innovations are a linear combination of the two structural shocks

$$u_{yt} = b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{MonPol}$$
$$u_{rt} = b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{MonPol}$$

* An increase in u_{rt} is not a monetary policy shock!

Reduced-form VARs do not tell us anything about causality

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$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

- It is obvious that the reduced form innovations are a linear combination of the two structural shocks

$$\begin{aligned} u_{yt} &= b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{MonPol} \\ u_{rt} &= b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{MonPol} \end{aligned}$$

- An increase in u_{rt} could be due to
  [1] A positive demand shock that increases both output growth and the policy rate ($b_{21} > 0$)
  [2] Or a monetary policy shock that decreases output growth and increases the policy rate ($b_{22} > 0$)

- How to know whether is [1] or [2]? This is the very nature of the **identification problem**!

The identification problem

- The identification problem consists in finding a mapping from the reduced form VAR to its structural counterpart

$$u_t = B\varepsilon_t$$

- To do that, we can exploit the relation between reduced form and structural innovations to write

$$\Sigma_u = \mathbb{E}\left[u_t u_t'\right] = \mathbb{E}\left[B\varepsilon_t \left(B\varepsilon_t\right)'\right] = B\mathbb{E}(\varepsilon_t \varepsilon_t')B' = B\Sigma_\varepsilon B' = BB'$$

- The identification problem simply boils down to finding a B matrix that satisfies $\Sigma_u = BB'$

- Unfortunately this is not as easy as it sounds. Why?
  - Hint: There are infinite combinations of B that give you the same Σ_u

The identification problem (cont'd)

- Think of $\Sigma_u = BB'$ as a system of equations

$$\begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ - & \sigma_r^2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$$

- Can be rewritten as

$$\begin{cases} \sigma_y^2 = b_{11}^2 + b_{12}^2 \\ \sigma_{yr}^2 = b_{11}b_{21} + b_{12}b_{22} \\ \sigma_{yr}^2 = b_{11}b_{21} + b_{12}b_{22} \\ \sigma_r^2 = b_{21}^2 + b_{22}^2 \end{cases}$$

- **Problem** Because of the symmetry of the Σ_u matrix, the second and the third equation are identical

- We are left with 4 unknowns (the elements of B) but only 3 equations!

Identification Schemes

How to solve the identification problem?

- **Identification problem (recap)**
  - Identification → Find a B that satisfies $\Sigma_u = BB'$
  - There are infinite of such Bs

- In our simple example, we have to solve a system of 3 equations in 4 unknowns. How can we do it? Add a fourth equation :)

- Economic theory can help in providing the 'missing' equation
  - Make an assumption about the structure of the economy based on your beliefs (e.g. long-run monetary neutrality)
  - Try to map this assumption into an equation that involves the VAR parameters

- The additional equation is known as a **restriction**
  - That is: the additional equation restricts the set of infinite B matrices to a single one (or few ones) that are consistent with your assumption

Common identification schemes

- Zero (recursive) contemporaneous restrictions
- Zero (recursive) long-run restrictions
- Sign restrictions
- External instruments
- Combining sign restrictions and external instruments
- Other (narrative sign restrictions, maximization of forecast error variance,...)

Common Identification Schemes

Zero short-run restrictions

Zero contemporaneous restrictions

- **Intuition** Identification is achieved by assuming that some shocks have zero contemporaneous effect on some of the endogenous variables
- **References** Sims (1980), Christiano, Eichenbaum, Evans (1999)
- For example, assume that monetary policy works with a lag and has no contemporaneous effects on output
- But how can we impose restrictions on the effect of a structural shock?

Zero contemporaneous restrictions

- **Solution** Impose zero restrictions on the impact matrix B

- The b_{12} coefficient captures the contemporaneous effect of monetary policy on output growth

By assumption

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & \boxed{0} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

- **Implication** We now have 3 structural parameters to estimate (instead of 4) and 3 restrictions implied by Σ_u

Zero contemporaneous restrictions

How to achieve identification?

- The system of equations implied by $\Sigma_u = BB'$ now becomes

$$\begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ - & \sigma_r^2 \end{bmatrix} = \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ 0 & b_{22} \end{bmatrix}$$

- This yields

$$\begin{cases} \sigma_y^2 = b_{11}^2 \\ \sigma_{yr}^2 = b_{11} b_{21} \\ \sigma_r^2 = b_{21}^2 + b_{22}^2 \end{cases}$$

- And can be easily solved to get:

$$\begin{cases} b_{11} = \sigma_y \\ b_{21} = \sigma_{yr}^2 / \sigma_y \\ b_{22} = \sqrt{\sigma_r^2 - \frac{\left(\sigma_{yr}^2\right)^2}{\sigma_y^2}} \end{cases}$$

Zero contemporaneous restrictions

Impact effects

- We can now derive the impact effects of shocks by simply re-writing the structural VAR as

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma_y & 0 \\ \sigma_{yr}^2/\sigma_y & \sqrt{\sigma_r^2 - \frac{\left(\sigma_{yr}^2\right)^2}{\sigma_y^2}} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

- A one standard deviation shock to monetary policy ($\varepsilon_t^{MonPol} = 1$) in t leads to

$$\begin{cases} y_t = \boxed{0} \\ r_t = \sqrt{\sigma_r^2 - \frac{\left(\sigma_{yr}^2\right)^2}{\sigma_y^2}} \end{cases}$$

By assumption

- A one standard deviation shock to aggregate demand ($\varepsilon_t^{Demand} = 1$) in t leads to

$$\begin{cases} y_t = \sigma_y \\ r_t = \sigma_{yr}^2/\sigma_y \end{cases}$$

Zero contemporaneous restrictions

Aka Cholesky identification

* This identification scheme is normally implemented via a Cholesky decomposition of Σ_u

* A Cholesky decomposition allows us to decompose Σ_u into the product of a lower triangular matrix P times its transpose

$$\Sigma_u = PP'$$

* In matrix form we have

Lower Cholesky factor

$$\begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ \sigma_{yr}^2 & \sigma_r^2 \end{bmatrix} = \boxed{\begin{bmatrix} p_{11} & 0 \\ p_{21} & p_{22} \end{bmatrix}} \begin{bmatrix} p_{11} & p_{21} \\ 0 & p_{22} \end{bmatrix}$$

Cholesky decomposition of a matrix [Back to basics]

- The Cholesky decomposition is (roughly speaking) the square root of a matrix
  - As for a square root, you can't compute a Cholesky decomposition for a non positive-definite matrix

- A symmetric and positive-definite matrix A can be decomposed as:

$$A = PP'$$

where P is a lower triangular matrix (and therefore P' is upper triangular)

- The formula for the decomposition of a 2×2 matrix is

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad P = \begin{bmatrix} \sqrt{a} & 0 \\ \frac{b}{\sqrt{a}} & \sqrt{c - \frac{b^2}{a}} \end{bmatrix}$$

Zero contemporaneous restrictions

Aka Cholesky identification

* To see why the zero contemporaneous restrictions identification can be implemented with a Cholesky decomposition, first note that Σ_u is a positive semi-definite matrix

* Then we can use the Cholesky decomposition to write

$$\Sigma_u = PP'$$

* But remember that we assumed that B is also lower triangular ($b_{12} = 0$) and that

$$\Sigma_u = BB'$$

* As both P and B are lower triangular, it must follow that $P = B$

Common Identification Schemes

Zero long-run restrictions

Zero long-run restrictions

- **Intuition** Identification is achieved by assuming that some shocks have zero cumulative effect on some of the endogenous variables in the long run

- **References** Blanchard and Quah (1989), Gali (1999)

- For example, assume that monetary policy is neutral in the long-run and has no cumulative effect on the level of output

- But how can we impose restrictions on the long-run cumulative effect of a structural shock?

Zero long-run restrictions

How to compute the cumulative long-run effects of shocks?

- Re-write the VAR as

$$x_t = \Phi x_{t-1} + B\varepsilon_t$$

- If a shock ε_t hits in t, its **cumulative** impact on x_t in the long run is given by

$$x_{t,t+\infty} = \boxed{B\varepsilon_t} + \boxed{\Phi B\varepsilon_t} + \boxed{\Phi^2 B\varepsilon_t} + \ldots + \Phi^\infty B\varepsilon_t$$

Impact in t

Impact in t+1

etc...

- Note: for output growth, $y_{t,t+\infty}$ is the effect of ε_t on the level of output

Zero long-run restrictions

How to compute the cumulative long-run effects of shocks?

* Re-write the VAR as

$$x_t = \Phi x_{t-1} + B\varepsilon_t$$

* If a shock ε_t hits in t, its **cumulative** impact on x_t in the long run is given by

$$x_{t,t+\infty} = \underbrace{\boxed{B\varepsilon_t}}_{\text{Impact in } t} + \underbrace{\boxed{\Phi B\varepsilon_t}}_{\text{Impact in } t+1} + \underbrace{\boxed{\Phi^2 B\varepsilon_t}}_{\text{etc...}} + \ldots + \Phi^\infty B\varepsilon_t$$

* If the VAR is stable, we can rewrite

$$x_{t,t+\infty} = \sum_{j=0}^{\infty} \Phi^j B\varepsilon_t = (I - \Phi)^{-1} B\varepsilon_t = C\varepsilon_t$$

where $C \equiv (I - \Phi)^{-1}$ captures the cumulative effect of ε_t on x_t from t to ∞

Zero long-run restrictions

How to compute the cumulative long-run effects of shocks?

- What is the intuition for C?
- Go back to our output growth / policy rate example

$$\begin{bmatrix} y_{t,t+\infty} \\ r_{t,t+\infty} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

- Take the first equation: $y_{t,t+\infty} = c_{11}\varepsilon_t^{Demand} + c_{12}\varepsilon_t^{MonPol}$
  - The coefficient c_{12} represents the impact of a monetary policy shock (hitting in t) on the **level of GDP** in the long-run
  - If you believe in the long-run neutrality of monetary policy you would expect $c_{12} = 0$

Zero long-run restrictions

How to achieve identification?

- Remember that $C \equiv (I - \Phi)^{-1} B$ is **unobserved** as we don't know B. So, how does this help with the identification of B?

- To achieve identification define $\Omega \equiv CC'$ and note that

  1. Ω is known!
  $$\Omega = \left((I-\Phi)^{-1}\right) BB' \left((I-\Phi)^{-1}\right)' = \left((I-\Phi)^{-1}\right) \Sigma_u \left((I-\Phi)^{-1}\right)'$$

  2. Ω is a positive-definite symmetric matrix $\longrightarrow$ It admits a unique Cholesky decomposition
  $$\Omega = PP'$$

  3. Because of our assumption that C is lower triangular, it follows that $P = C$

- We achieved identification: $B = (I - \Phi)\, P$

Zero long-run restrictions

How to achieve identification?

- As before, we can rewrite the structural VAR with the B matrix implied by the zero long run restriction

$$B = (I_2 - \Phi)P = (I_2 - \Phi) \times \texttt{chol}\left(\left((I - \Phi)^{-1}\right)\Sigma_u\left((I - \Phi)^{-1}\right)'\right)$$

  where `chol` denotes the Cholesky factor

- Note that the B matrix **is not triangular**
  - This is different to what we had in the zero contemporaneous restrictions identification

- The impact effects are left unrestricted, the restrictions are on the C matrix
  - We'll check later that the restrictions is satisfied in a simple example with true data

Common Identification Schemes

Sign restrictions

Sign restrictions

- **Intuition** Exploit prior beliefs (typically informed by theoretical models) about the sign that certain shocks should have on certain endogenous variables

- **Intuition** Faust (1998), Canova and De Nicolo (2002), Uhlig (2005)

- For example
  - Demand shocks should lead to an increase in output and interest rates
  - Monetary policy shocks should lead to a fall in output and an increase in interest rates

	Demand (ε_t^{Demand})	Monetary Policy (ε_t^{MonPol})
Output growth (y_t)	$+$	$-$
Short-rate Int. Rate (r_t)	$+$	$+$

- But how can we impose restrictions on the signs of the effect of a structural shock?

Sign restrictions

How to achieve identification?

- The key intuition is based on the following three steps
  1. Consider a **random orthonormal matrix** Q such that

$$QQ' = I_2$$

  2. Consider the lower triangular B matrix corresponding to the Cholesky factor of Σ_u

$$\Sigma_u = PP'$$

  3. The following equality holds

$$\Sigma_u = PP' = PQQ'P' = \underbrace{(PQ)}_{B}\underbrace{(PQ)'}_{B'}$$

- The matrix $B = PQ$ is a valid 'candidate' impact matrix that solves the identification problem!
  - Differently from P, the matrix PQ is not lower triangular anymore

Orthonormal matrix [Back to basics]

* An orthonormal matrix Q is a real square matrix whose columns and rows are orthogonal unit vectors

* What does it mean? Take for example two 2×1 vectors q_1 and q_2, then the matrix $Q = (q_1, q_2)$ is orthonormal if
  * The vectors have unit norm: $\| q_i \| = 1$
  * The vectors are mutually orthogonal: $q_1^T q_2 = 0$

* It follows that

$$QQ' = I \quad \text{and} \quad Q' = Q^{-1}$$

* **Note** You can draw infinite matrices that satisfy the above conditions (we'll see how to do it in Matlab below)

Sign restrictions

How to achieve identification?

* But Q is a random matrix... How can we check that $B = PQ$ represents a plausible solution?
* **Solution** Check that the effects of shocks implied by $B = PQ$ satisfy a set of a priori sign restrictions. That is:

  [1] Consider the structural representation of our VAR

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

  [2] Then check that the elements of B satisfy

	Demand (ε_t^{Demand})	Monetary Policy (ε_t^{MonPol})
Output growth (y_t)	$b_{11} > 0$?	$b_{12} < 0$?
Short-rate Int. Rate (r_t)	$b_{21} > 0$?	$b_{22} > 0$?

Sign restriction in steps

* Perform N replications of the following steps
  [1] Draw a random orthonormal matrix Q
  [2] Compute $B = PQ$ where P is the Cholesky decomposition of the reduced form residuals Σ_u
  [3] Compute the impact effects of shocks associated with B
  [4] Are the sign restrictions satisfied?
    [4.1] Yes. Store B and go back to [1]
    [4.2] No. Discard B and go back to [1]
* All matrices in the set $B^{(i)}$ (for $i = 1, 2, \ldots, N$) represent admissible solutions to the identification problem
* In this sense, sign restricted VARs are only **set identified**

Common Identification Schemes

External Instruments (or Proxy SVARs)

External instruments

- **Intuition** Exploit information from a variable that is *external* to the VAR, but that is correlated with a particular shock of interest and uncorrelated with other shocks (the *instrument*)
- **References** Stock and Watson (2012), Mertens and Ravn (2013)
- For example, assume that you have some ‘narrative’ series of policy surprises (i.e. that are not just a response of the central bank to some development in the economy)
- But how can this help in finding the B matrix?

External instruments

* **Key element** Presence of an *instrument* that is correlated with a shock of interest and uncorrelated with all other shocks

* For example, assume that such an instrument exists (z_t) and satisfies the following properties:

$$\begin{aligned} \mathbb{E}[\varepsilon_t^{Demand} z_t'] &= 0, \\ \mathbb{E}[\varepsilon_t^{MonPol} z_t'] &= c, \end{aligned}$$

* Then, we can identify one column (in this example, the second one) of the B matrix:

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} - & b_{12} \\ - & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

External instruments identification: How does it work?

* Recall that the reduced form residuals are a linear combination of the two structural shocks

$$\begin{cases} u_{yt} = b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{MonPol} \\ u_{rt} = b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{MonPol} \end{cases}$$

* The OLS estimate of β in the following 'first stage' regression identifies b_{22} up to a scaling factor

$$u_{rt} = \beta z_t + \xi_t$$

* To see that, recall that the OLS β can be written as $\beta = Cov(u_{rt}, z_t)/Var(z_t)$
    * Focus on the Cov term and plug in the definition of u_{rt} to get

$$Cov(u_{rt}, z_t) = Cov(b_{21}\varepsilon_t^{Demand} + b_{22}\varepsilon_t^{MonPol}, z_t) = b_{22}Cov(\varepsilon_t^{MonPol}, z_t) = b_{22}c$$

    * It follows that $\beta = \frac{b_{22}c}{Var(z_t)}$

* As c is an unknown constant, $b_{22} = \beta Var(z_t)/c$ is only identified to a scaling factor

External instruments identification: How does it work?

- The OLS estimate of γ in the following 'second stage' regression identifies the ratio b_{12}/b_{22}

$$u_{yt} = \gamma \hat{u}_{rt} + \zeta_t = \gamma \left(\frac{b_{22}c}{Var(z_t)} \right) z_t + \zeta_t$$

- To see that, and recalling again that $\gamma = Cov(u_{yt}, \hat{u}_{rt})/Var(\hat{u}_{rt})$
  - Focus on the Cov term and plug in the definition of u_{yt} and $\hat{u}_{rt}$ to get

$$Cov(u_{rt}, \hat{u}_{rt}) = Cov\left(b_{11}\varepsilon_t^{Demand} + b_{12}\varepsilon_t^{MonPol}, \frac{b_{22}c}{Var(z_t)} z_t \right) = \frac{b_{12}b_{22}c^2}{Var(z_t)}$$

  - Then focus on the Var term to get

$$Var(\hat{u}_{rt}) = Var\left(\frac{b_{22}c}{Var(z_t)} z_t \right) = \frac{b_{22}^2 c^2}{Var\left(z_t\right)}$$

  - It follows that $\gamma = \frac{b_{12}}{b_{22}}$

External instruments: Partial identification

- In sum, we can normalize the effect of ε_t^{MonPol} on r_t to 1

$$b_{22} = 1$$

- And quantify the effect of ε_t^{MonPol} on y_t as

$$b_{12} = \gamma$$

- In other words, we have identified the column of the B matrix of the structural VAR representation up to a scaling factor

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} - & \gamma \\ - & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

  - **Note** It is actually possible to work out the true values of B. See footnote 4 of Gertler and Karadi (2015)

Common Identification Schemes

Combining Sign Restrictions & External Instruments

Combining Sign Restrictions & External Instruments

- **Intuition** Identifies one (or more) columns of B with external instruments and conditional on that the remaining columns with sign restrictions

- **References** Cesa-Bianchi and Sokol (2019), Cesa-Bianchi and Ferrero (2020)

- For example, assume that there are two shocks that imply similar signs (so that sign restrictions are not enough to identify the shocks), but you have an instrument for one of the two shocks

- How can we find the B matrix?

Combining Sign Restrictions & External Instruments

- Consider a k-variable version of our simple structural VAR(1)

$$\begin{bmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{kt} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1k} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k1} & \phi_{k2} & \cdots & \phi_{kk} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \\ \vdots \\ x_{kt-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kk} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \vdots \\ \varepsilon_{kt} \end{bmatrix}$$

- Assume that
  - The first structural shock (ε_{1t}) can be identified with an external instrument
  - The remaining structural shocks ($\varepsilon_{2t}, \ldots, \varepsilon_{kt}$) can be identified with sign restrictions

Combining Sign Restrictions & External Instruments

- Partition the structural matrix B as $[b\ \mathcal{B}]$

$$B = \left[\begin{array}{cccc} \underbrace{\begin{matrix} b_{11} \\ b_{21} \\ \vdots \\ b_{k1} \end{matrix}}_{b} & \underbrace{\begin{matrix} b_{12} & \cdots & b_{1k} \\ b_{22} & \cdots & b_{2k} \\ \vdots & \ddots & \vdots \\ b_{k2} & \cdots & b_{kk} \end{matrix}}_{\mathcal{B}} \end{array} \right]$$

- Column vector b captures the impact of the first shock, matrix $\mathcal{B}$ captures the impact of the remaining shocks

- We have seen above how to identify b with external instruments

- **Question** Once b is known, how can we find a $\mathcal{B}$ matrix that satisfies a set of sign restrictions?

Combining Sign Restrictions & External Instruments

- Let C be the Cholesky decomposition of Σ_u. Find a normal vector q of dimension $k \times 1$ that rotates the first column of C into the vector b, so that

$$Cq = b$$

- Given q, build a $(n \times n - 1)$ matrix $\mathcal{Q}$ such that $Q = [q\ \mathcal{Q}]$ is orthonormal

$$[q\ \mathcal{Q}][q\ \mathcal{Q}]' = QQ' = I_k$$

- As Q is an orthonormal matrix we have

$$\Sigma_u = CC' = CQQ'C' = (CQ)\,(CQ)'$$

- So $B = CQ$ is a valid candidate matrix that solves the identification problem as
  - $\Sigma_u = (CQ)\,(CQ)'$ holds
  - The first column of CQ is b

Combining Sign Restrictions & External Instruments: Steps

[1] Identify b, the first column of $B = [b \ \mathcal{B}]$, with the external instrument

[2] Compute the Cholesky decomposition C of the reduced form residuals' covariance matrix Σ_u

[3] Find a normal vector q that rotates the first column of C into the vector b, namely $Cq = b$

- [3.i] Given q, build the remaining $k-1$ columns of an orthonormal matrix $Q = [q \ \mathcal{Q}]$
- [3.ii] The matrix CQ then represents a candidate identification scheme because:

$$(CQ)(CQ)' = \Sigma_u \quad \text{and} \quad C[q \ \mathcal{Q}] = [b \ \mathcal{B}]$$

- [3.iii] If $\mathcal{B}$ satisfies the sign restrictions, retain it. Otherwise, go back to [4.i]

[4] Go back to [1] and repeat N times

Structural Dynamic Analysis

Structural Dynamic Analysis

* We have seen how to ‘identify’ structural shocks by imposing some restrictions on the data

* But what can we do with that?
  * Quantify the dynamic effect of a shock over time $\Rightarrow$ **Impulse responses**
  * Quantify how important a shock is in explaining the variation in the endogenous variables (on average) $\Rightarrow$ **Forecast error variance decomposition**
  * Quantify how important a shock was in driving the behavior of the endogenous variables in a specific time period in the past $\Rightarrow$ **Historical decompositions**

* We’ll turn to this structural dynamic analysis next

Structural Dynamic Analysis

Impulse responses

Impulse response functions

- Impulse response functions (*IR*) answer the following question:

  What is the response over time of the VAR's endogenous variables to an innovation in the structural shocks, assuming that the other structural shocks are kept to zero?

- *IR* allow to single out the effect of a shock (e.g. its impact and persistence) keeping all else equal

- **Example** What is the impact of a monetary policy shock to GDP?

How to compute impulse response functions

- Consider our simple bivariate VAR

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

- Define a 2×1 impulse selection vector (s) that takes value of one for the structural shock that we want to consider.

- For example, to compute the *IR* to the demand shock, define s as:

$$s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- The impulse responses to ε_t^{Demand} can be easily computed with the following equation

$$x_t = \Phi x_{t-1} + Bs$$

How to compute impulse response functions (cont'd)

- The IR can be computed recursively as follows

$$\begin{cases} IR_t = Bs & \text{for } t = 0 \\ IR_t = \Phi IR_{t-1} & \text{for } t = 1, \ldots, h \end{cases}$$

- Note that the impact response is simply given by the elements of the impact matrix B selected by s...

$$\begin{bmatrix} IR_0^y \\ IR_0^r \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$$

- ... while the responses at longer horizons are given the transition matrix

$$\begin{bmatrix} IR_t^y \\ IR_t^r \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} IR_{t-1}^y \\ IR_{t-1}^r \end{bmatrix}$$

The companion matrix [Back to basics]

* So far, we considered simple VAR(1) specifications. But what to do if the VAR has $p > 1$?
* Every VAR(p) can be written as a VAR(1) using the **companion representation**
  * For example, take a VAR(2)

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi^1_{11} & \phi^1_{12} \\ \phi^1_{21} & \phi^1_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} \phi^2_{11} & \phi^2_{12} \\ \phi^2_{21} & \phi^2_{22} \end{bmatrix} \begin{bmatrix} y_{t-2} \\ r_{t-2} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

  * Re-write the VAR(2) as

$$\begin{bmatrix} y_t \\ r_t \\ y_{t-1} \\ r_{t-1} \end{bmatrix} = \overbrace{\boxed{\begin{bmatrix} \phi^1_{11} & \phi^1_{12} & \phi^2_{11} & \phi^2_{12} \\ \phi^1_{21} & \phi^1_{22} & \phi^2_{21} & \phi^2_{22} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}}^{\text{Companion matrix}} \begin{bmatrix} y_{t-1} \\ r_{t-1} \\ y_{t-2} \\ r_{t-2} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Demand} \\ \varepsilon_t^{MonPol} \\ 0 \\ 0 \end{bmatrix}$$

  * Which is a VAR(1) where $\tilde{\Phi}$ is the **companion matrix**

$$\tilde{x}_t = \tilde{\Phi}\tilde{x}_{t-1} + \tilde{B}\varepsilon_t$$

Structural Dynamic Analysis

Forecast Error Variance Decompositions

Forecast error variance decompositions

- Forecast error variance decompositions (*VD*) answer the following question:

  What portion of the variance of the VAR's forecast errors (at a given horizon h) is due to each structural shock?

- *VD* provide information about the relative importance of each structural shock in affecting the forecast error variance of the VAR's endogenous variables

- **Example** What is the (average) importance of demand shocks in driving GDP forecast errors?

How to compute forecast error variance decompositions

- The forecast error of a variable at horizon $t+h$ is the change in the variable that couldn't have been forecast between $t-1$ and $t+h$ due to the realization of the structural shocks.
- For example, at $h=0$ we can compute the forecast error as

$$x_t - \mathbb{E}_{t-1}[x_t] = \Phi x_{t-1} + B\varepsilon_t - \Phi x_{t-1} = B\varepsilon_t$$

- At $h=1$, we have

$$\begin{aligned} x_{t+1} - \mathbb{E}_{t-1}[x_{t+1}] &= \Phi x_t + B\varepsilon_{t+1} - \Phi^2 x_{t-1} = \\ &= \Phi(\Phi x_{t-1} + B\varepsilon_t) + B\varepsilon_{t+1} - \Phi^2 x_{t-1} = \Phi B\varepsilon_t + B\varepsilon_{t+1} \end{aligned}$$

- So, in general we have

$$FE_{t+h} = x_{t+h} - E_{t-1}[x_{t+h}] = \sum_{i=0}^{h} \Phi^{h-i} B\varepsilon_{t+h}$$

- What is the variance of FE_{t+h}?

Basic properties of the variance [Back to basics]

- If X is a random variable x and a is a constant
  - $Var(x + a) = Var(x)$
  - $Var(ax) = a^2 Var(x)$

- If Y is a random variable and b is a constant
  - $Var(aX + bY) = a^2 Var(x) + b^2 Var(Y) + 2abCov(X, Y)$

- Since the structural errors are independent, it follows that $COV\left(\varepsilon_{t+1}^{Demand}, \varepsilon_{t+1}^{MonPol}\right) = 0$

How to compute forecast error variance decompositions (cont'd)

- For simplicity consider $h = 0$, namely

$$Var(FE_t) = Var(x_t - E_{t-1}[x_t]) = Var(B\varepsilon_t)$$

- Recalling that $Var(\varepsilon_t) = I_2$ and that the structural shocks are orthogonal to each other, the variance of the forecast error can be computed as

$$\begin{aligned} Var(y_t - E_{t-1}[y_t]) &= b_{11}^2 Var\left(\varepsilon_t^{Demand}\right) + b_{12}^2 Var\left(\varepsilon_t^{MonPol}\right) = b_{11}^2 + b_{12}^2 \\ Var(r_t - E_{t-1}[r_t]) &= b_{21}^2 Var\left(\varepsilon_t^{Demand}\right) + b_{22}^2 Var\left(\varepsilon_t^{MonPol}\right) = b_{21}^2 + b_{22}^2 \end{aligned}$$

- What portion of the variance of the forecast error at $h = 0$ is due to each structural shock?

$$\underbrace{\left\{ \begin{aligned} VD_{y_0}^{\varepsilon^{Demand}} &= \frac{b_{11}^2}{b_{11}^2 + b_{12}^2} \\ VD_{y_0}^{\varepsilon^{MonPol}} &= \frac{b_{12}^2}{b_{11}^2 + b_{12}^2} \end{aligned} \right.}_{\text{This sums up to 1}} \qquad \underbrace{\left\{ \begin{aligned} VD_{r_0}^{\varepsilon^{Demand}} &= \frac{b_{21}^2}{b_{21}^2 + b_{22}^2} \\ VD_{r_0}^{\varepsilon^{MonPol}} &= \frac{b_{22}^2}{b_{21}^2 + b_{22}^2} \end{aligned} \right.}_{\text{This sums up to 1}}$$

Structural Dynamic Analysis

Historical Decompositions

Historical decompositions

- Historical decompositions (*HD*) answer the following question:

  What is the historical contribution of each structural shock in driving deviations of the VAR's the endogenous variables away from their equilibrium?

- *HD* allow to track, at each point in time, the role of structural shocks in driving the VAR's endogenous variables away from their steady state

- **Example** What was the contribution of oil shocks in driving the fall in GDP growth in 1973:Q4?

How to compute historical decompositions

- As an example, let's compute the *HD* of of the endogenous variables for $t = 2$ in our simple bivariate VAR

- Historical decompositions can be easily understood from the Wold representation of the VAR

$$x_t = \Phi^t x_0 + \sum_{j=0}^{t-1} \Phi^j B \varepsilon_{t-j}$$

- Using the Wold representation, we can write x_2 as a function of present and past structural shocks (ε^{Demand} and ε^{MonPol}) plus the initial condition (x_0)

$$x_2 = \underbrace{\Phi^2 x_0}_{init} + \underbrace{\Phi B}_{\Theta_1} \varepsilon_1 + \underbrace{B}_{\Theta_0} \varepsilon_2$$

- Re-write x_2 in matrix form

$$\begin{bmatrix} y_2 \\ r_2 \end{bmatrix} = \begin{bmatrix} init_y \\ init_r \end{bmatrix} + \begin{bmatrix} \theta^1_{11} & \theta^1_{12} \\ \theta^1_{21} & \theta^1_{22} \end{bmatrix} \begin{bmatrix} \varepsilon^{Demand}_2 \\ \varepsilon^{MonPol}_2 \end{bmatrix} + \begin{bmatrix} \theta^0_{11} & \theta^0_{12} \\ \theta^0_{21} & \theta^0_{22} \end{bmatrix} \begin{bmatrix} \varepsilon^{Demand}_2 \\ \varepsilon^{MonPol}_2 \end{bmatrix}$$

How to compute historical decompositions (cont'd)

- Then x_2 can be expressed as

$$\begin{cases} y_2 = init_y + \theta^1_{11}\varepsilon^{Demand}_1 + \theta^1_{12}\varepsilon^{MonPol}_1 + \theta^0_{11}\varepsilon^{Demand}_2 + \theta^0_{12}\varepsilon^{MonPol}_2 \\ r_2 = init_r + \theta^1_{21}\varepsilon^{Demand}_1 + \theta^1_{22}\varepsilon^{MonPol}_1 + \theta^0_{21}\varepsilon^{Demand}_2 + \theta^0_{22}\varepsilon^{MonPol}_2 \end{cases}$$

- The historical decomposition is given by

$$\underbrace{\begin{cases} HD^{\varepsilon^{Demand}}_{y_2} = \theta^1_{11}\varepsilon^{Demand}_1 + \theta^2_{11}\varepsilon^{Demand}_2 \\ HD^{\varepsilon^{MonPol}}_{y_2} = \theta^1_{12}\varepsilon^{MonPol}_1 + \theta^2_{12}\varepsilon^{MonPol}_2 \\ HD^{init}_{y_2} = init_y \end{cases}}_{\text{This sums up to } y_2} \qquad \underbrace{\begin{cases} HD^{\varepsilon^{Demand}}_{r_2} = \theta^1_{21}\varepsilon^{Demand}_1 + \theta^0_{21}\varepsilon^{Demand}_2 \\ HD^{\varepsilon^{MonPol}}_{r_2} = \theta^1_{22}\varepsilon^{MonPol}_1 + \theta^0_{22}\varepsilon^{MonPol}_2 \\ HD^{init}_{r_2} = init_r \end{cases}}_{\text{This sums up to } r_2}$$

Practical Examples

The VAR Toolbox

- We'll see in practice how VARs work through a set of examples using the **VAR Toolbox 3.0**
- The VAR Toolbox is a collection of Matlab routines to perform VAR analysis
  - Codes are available at `https://github.com/ambropo/VAR-Toolbox`
  - No installation is required. Simply clone the folder from Github and add the folder (with subfolders) to your Matlab path
  - To save figures in high quality format, you need to download an install Ghostscript (freely available at `www.ghostscript.com`).
    - The first time you'll be saving a figure using the Toolbox, you'll be asked to locate Ghostscript on your local drive
- We'll start with a very simple example and then replicate the results from a few well-known papers

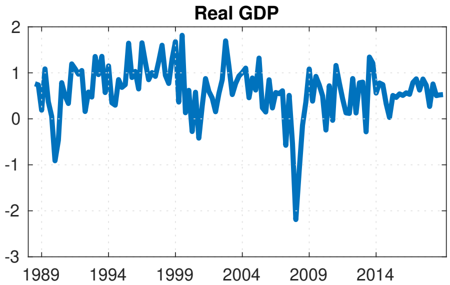
Adding the VAR Toolbox path to Matlab

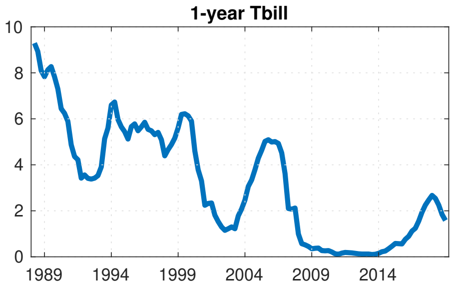
- To avoid clashes with functions from other toolboxes, it is recommendable to add and remove the Toolbox at beginning and end of your scripts
- If you download the toolbox to `/User/VAR-Toolbox/`, you can simply add the following lines at the beginning and end of your script

```
addpath(genpath('/User/VAR-Toolbox/v3dot0/'))
...
rmpath(genpath('/User/VAR-Toolbox/v3dot0'))
```

A simple bivariate VAR model

- Our first example will be a simple bivariate VAR as the one considered above
- US quarterly data from 1989:Q1 to 2019:Q4 on output growth (y_t) and the 1-year T-bill (r_t)





A simple bivariate VAR model

- As both GDP growth an the 1-year rate are non-zero means, we fit the data with a VAR(1) with a constant

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \alpha_y \\ \alpha_r \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} u_{yt} \\ u_{rt} \end{bmatrix}$$

- This means we will estimate the following parameters
  - $2+4$ coefficients, namely the elements of α and Φ
  - 2 variances of the reduced-form residuals, namely σ_y^2 and σ_r^2
  - 1 covariance of the reduced-form residuals, namely σ_{yr}^2
- We will store these coefficients in two Matlab matrices

$$\texttt{F} = \begin{bmatrix} \alpha_1 & \phi_{11} & \phi_{12} \\ \alpha_2 & \phi_{21} & \phi_{22} \end{bmatrix} \qquad \texttt{sigma} = \begin{bmatrix} \sigma_y^2 & \sigma_{yr}^2 \\ \sigma_{yr}^2 & \sigma_r^2 \end{bmatrix}$$

A simple bivariate VAR model

- In Matlab we store the data in the matrix `X`

$$\texttt{X} = \begin{bmatrix} y_1 & r_1 \\ y_2 & r_2 \\ \dots & \dots \\ y_T & r_T \end{bmatrix} = \left(y_t', r_t'\right) = x_t'$$

- The VAR can then be easily estimated with a few lines of code

```
% Set the deterministic variable in the VAR (1=constant, 2=trend)
det = 1;
% Set number of nlags
nlags = 1;
% Estimate VAR by OLS
[VAR, VARopt] = VARmodel(ENDO,nlags,det);
```

OLS estimation: Typical VAR output (cont'd)

- The off-diagonal elements of Σ capture the **average** contemporaneous relation between the endogenous variables

	GDP growth (u_y)	1-year T-Bill(u_r)
GDP growth (u_y)	0.2891	$\boxed{0.0782}$
1-year T-Bill (u_r)	0.0782	0.1473

$Cov(u_y, u_r) > 0$

- In our example output growth and interest rates are contemporaneously positively correlated
  - It means that, on average, when GDP growth increases interest rates increases, too

- Does it mean that a shock to interest rates always increase output growth?
  - No! Recall that reduced from residuals are not informative about structural shocks

Model checking & tuning

- These notes do not cover this aspect in detail but...
- ... before interpreting the VAR results you should check a number of assumptions
- Loosely speaking, you need to check that the reduced-form residuals are
  - Normally distributed
  - Not autocorrelated
  - Not heteroskedastic (i.e., have constant variance)
- ... and that the VAR is stable

Stability and equilibrium (cont'd)

- As our VAR is stable, its Wold representation will converge to the (finite) unconditional mean of the data

- To see that, first consider the Wold representation in the presence of a constant

$$x_t = \Phi^t x_0 + \sum_{j=0}^{t-1} \Phi^t \alpha + \sum_{j=0}^{t-1} \Phi^j B \varepsilon_{t-j}$$

- For t large enough and taking expectations we get

$$\mathbb{E}[x_t] = \sum_{j=0}^{t-1} \Phi^t \alpha = (I_2 - \Phi)^{-1} \alpha$$

- In absence of shocks, the VAR's variable will converge to its equilibrium $(I_2 - \Phi)^{-1} \alpha$ at a rate that depends on Φ

Examples of different identification schemes

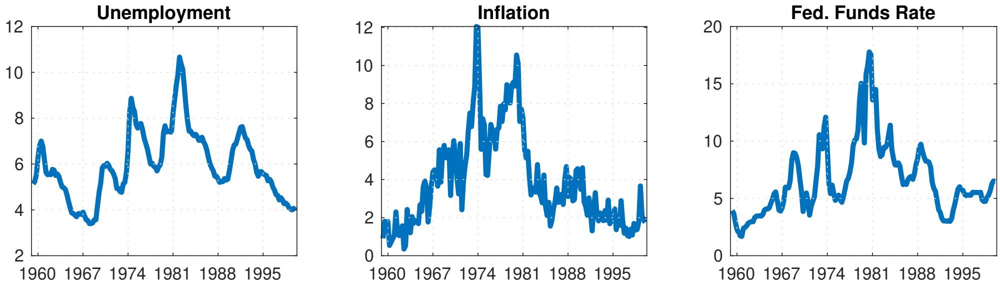
* Zero short-run restrictions
  * Stock and Watson (2001). "Vector Autoregressions," *Journal of Economic Perspectives*
* Zero long-run restrictions
  * Blanchard and Quah (1989). "The Dynamic Effects of Aggregate Demand and Supply Disturbances", *American Economic Review*
* Sign Restrictions
  * Uhlig (2005). "What are the effects of monetary policy on output? Results from an agnostic identification procedure," *Journal of Monetary Economics*
* External instruments
  * Gertler and Karadi (2015). "What are the effects of monetary policy on output? Results from an agnostic identification procedure," *American Economic Journal: Macroeconomics*

Practical Examples

Stock and Watson (2001, JEP)

Stock and Watson (2001): Zero short-run restrictions

- Stock and Watson (2001). “Vector Autoregressions,” *Journal of Economic Perspectives*
- US quarterly data from 1960:Q1 to 2000:Q4



Monetary policy shocks, inflation and unemployment

- **Objective** Infer the causal influence of monetary policy on unemployment and inflation
- Assume a VAR with $p = 4$ with inflation (π_t), unemployment (u_t), and the fed funds rate (r_t)
- **Key identifying assumptions**
  - MP (r_t) reacts contemporaneously to movements in inflation and in unemployment
  - MP shocks (ε_t^{MonPol}) do not affect inflation and unemployment within the quarter of the shock

$$\begin{bmatrix} \pi_t \\ u_t \\ r_t \end{bmatrix} = \sum_{p=1}^{4} \Phi_p x_{t-p} + \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \\ \varepsilon_t^{MonPol} \end{bmatrix}$$

Replicating Stock and Watson (2001) with the VAR Toolbox

- In Matlab, set lag length to 4 and estimate a VAR with a constant

```
% Set up and estimate VAR
det = 1;
nlags = 4;
[VAR, VARopt] = VARmodel(X,nlags,det);
```

- Then set the option for recursive identification `VARopt.ident ='short'` and compute the *IR* with the `VARir` function.
  - Note that the **ordering of the variables matter!**

```
% For zero contemporaneous restrictions set:
VARopt.ident = 'short';
% Compute IR
[IR, VAR] = VARir(VAR,VARopt);
```

- Note that the second output of the `VARir` function is `VAR` again
  - This is because the `VAR` structure is updated with the B matrix corresponding to the identification scheme chosen

Replicating Stock and Watson (2001) with the VAR Toolbox (cont'd)

* The `VARirband` function allows to compute confidence intervals

```
% Compute IR
[IR, VAR] = VARir(VAR,VARopt);
```

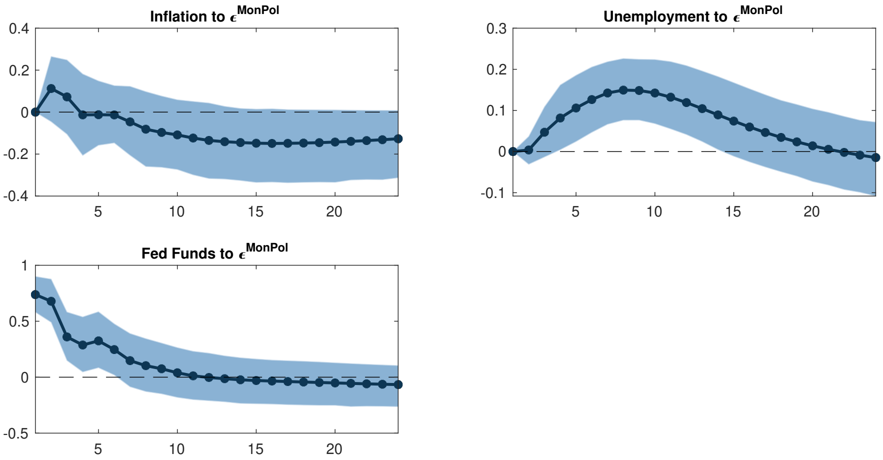
* You can control the options of the bootstrap procedure by modifying the `VARopt` structure (before running `VARir`)

* For example

```
% Some options for the bootstrap
VARopt.ndraws = 1000; % Number of draws
VARopt.pctg = 95; % Level for confidence intervals
VARopt.method = 'bs'; % 'bs' sampling with replacement; 'wild' wild bootstrap
```

The effect of a monetary policy shock

* Monetary policy shock raises inflation in the short run (price puzzle) and increases unemployment

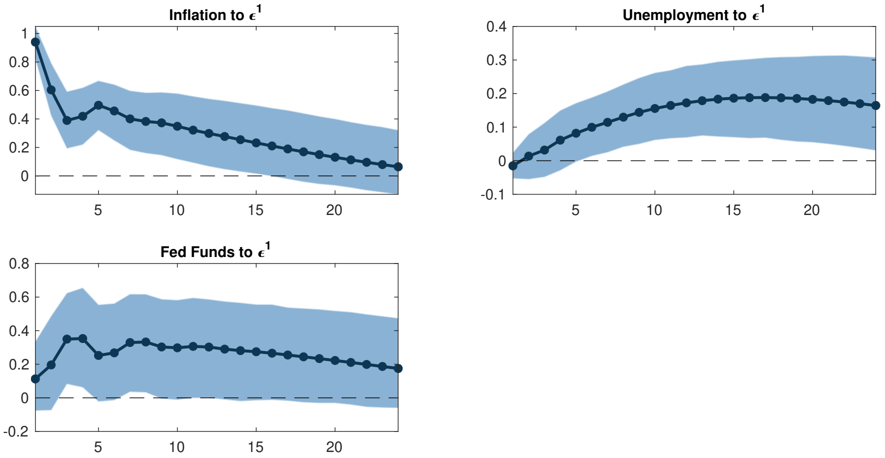


The other two shocks are identified by definition... but how can we interpret them?

- How about ε_t^1 and ε_t^2?
  - The shock ε_t^1 affects all variables contemporaneously
  - The shock ε_t^2 affects r_t contemporaneously but not π_t
- Can we interpret these shocks? Are the assumptions consistent with any theoretical mechanism?
- Some shocks may be better identified than others

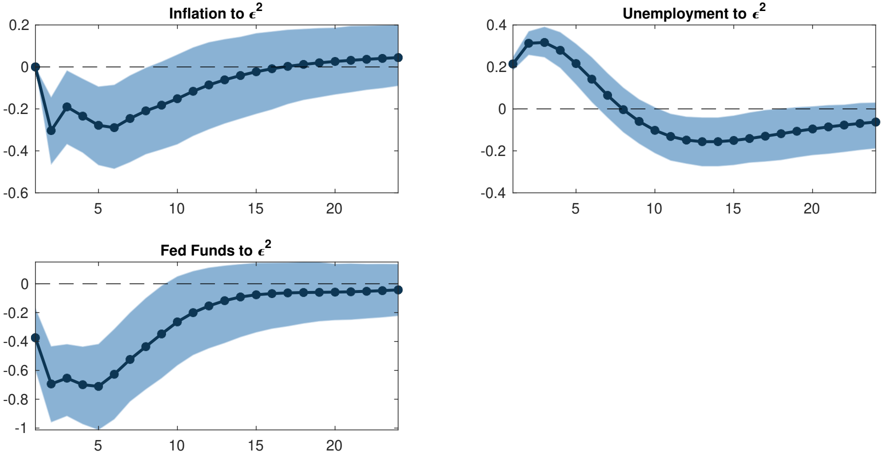
The other two shocks are identified by definition... but how can we interpret them?

* Shock to ε_t^1 behaves as a negative aggregate supply shock

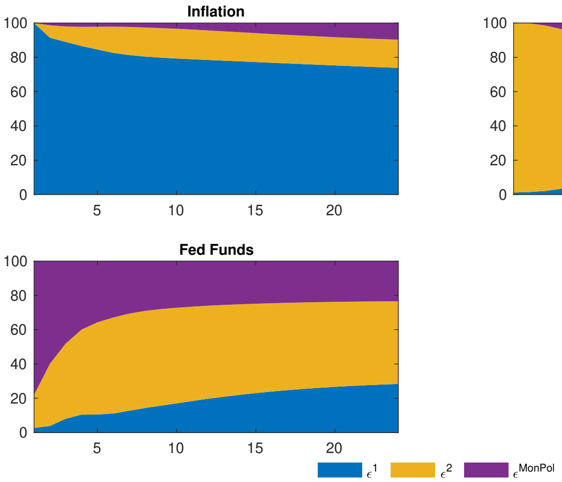


The other two shocks are identified by definition... but how can we interpret them?

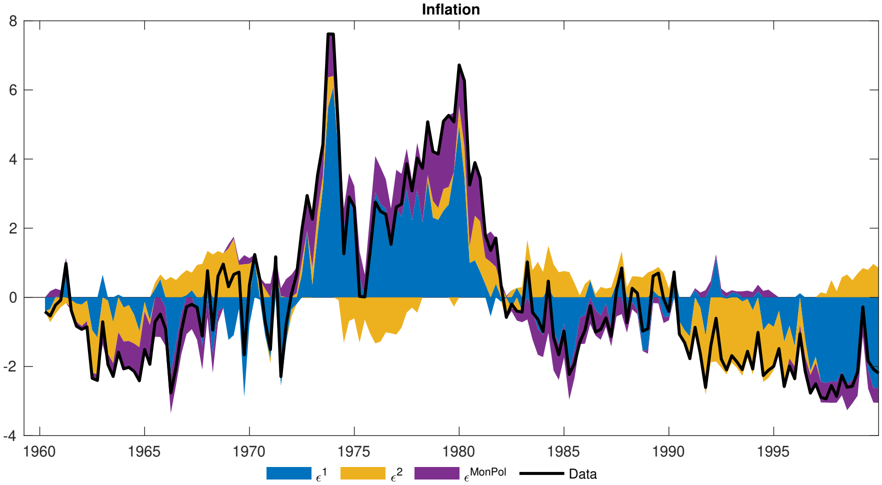
- Shock to ε_t^2 behaves as a negative aggregate demand shock



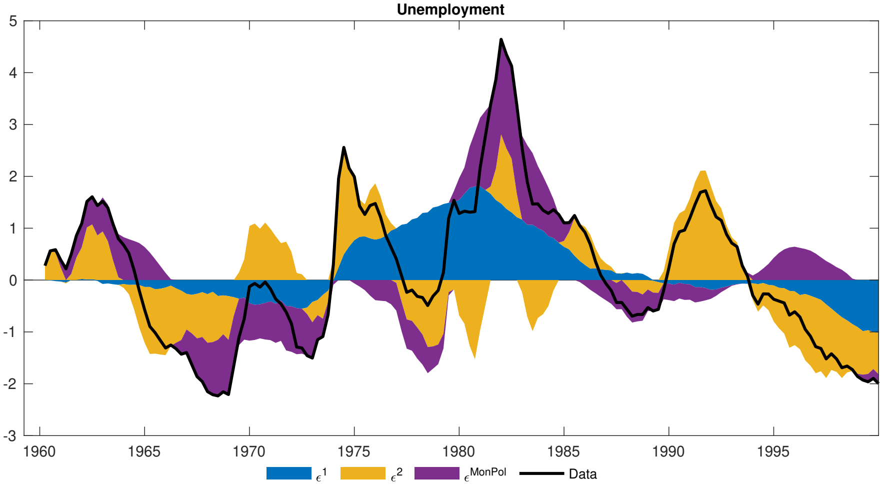
Forecast error variance decomposition



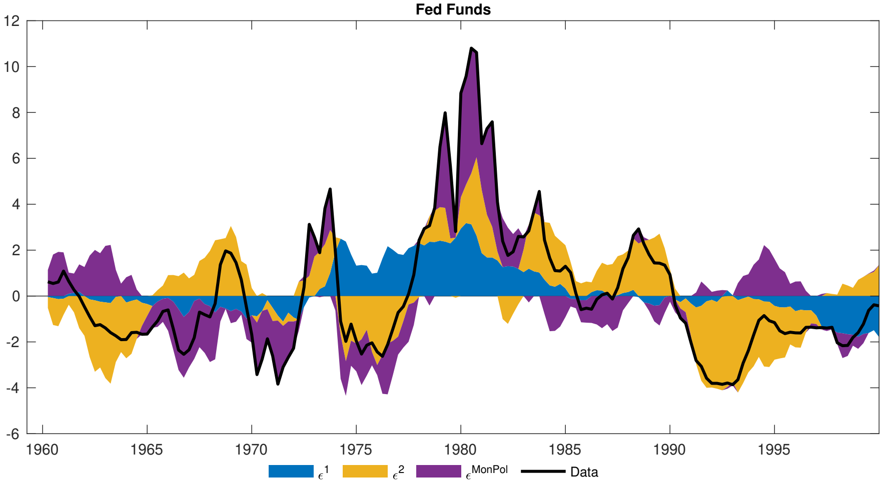
Historical decomposition



Historical decomposition



Historical decomposition



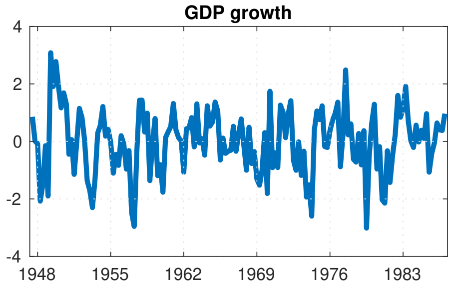
Practical Examples

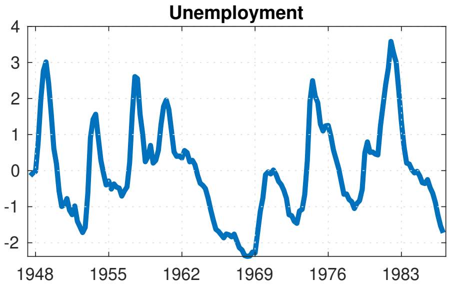
Blanchard and Quah (1989, AER)

Blanchard and Quah (1989): Zero long-run restrictions

- Blanchard and Quah (1989). “The Dynamic Effects of Aggregate Demand and Supply Disturbances”, *American Economic Review*

- US quarterly data from 1948:Q1 to 1987:Q4





What is the effect of demand and supply shocks?

- **Objective** Identify the effects of demand and supply shocks on output and unemployment

- Assume a bivariate VAR with $p = 8$ with output growth (y_t) and unemployment (u_t)

- **Key identifying assumption** Demand-side shocks have no long-run effect on the level of output, while supply-side shocks do

- Blanchard and Quah impose zero long-run restrictions on the cumulative effect of demand shocks on output growth (i.e. on output level) to identify the shocks

$$\begin{bmatrix} y_{t,t+\infty} \\ u_{t,t+\infty} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{Supply} \\ \varepsilon_t^{Demand} \end{bmatrix}$$

Monetary policy shocks, inflation and unemployment

- In Matlab, set lag length to 8 and estimate a VAR with a constant

```
% Set up and estimate VAR
det = 1;
nlags = 8;
[VAR, VARopt] = VARmodel(X,nlags,det);
```

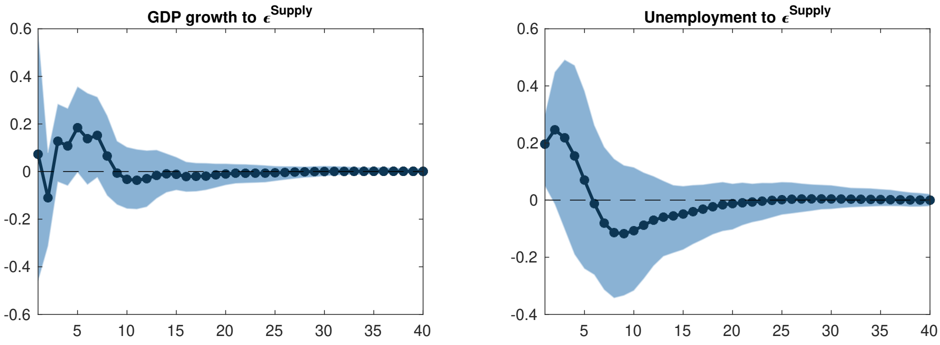
- Then set the option for zero long-run restrictions `VARopt.ident ='long'` and compute the *IR* with the `VARir` function.
  - Note that the **ordering of the variables matter!**

```
% For zero contemporaneous restrictions set:
VARopt.ident = 'long';
% Compute IR
[IR, VAR] = VARir(VAR,VARopt);
```

- The *B* matrix implied by the zero long-run restrictions is stored in `VAR.B`

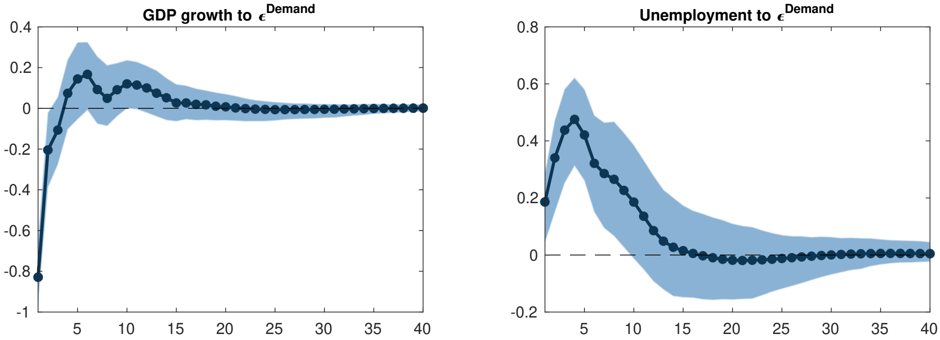
Aggregate supply shock

- Aggregate supply shock initially increases unemployment (puzzle of hours to productivity shocks)



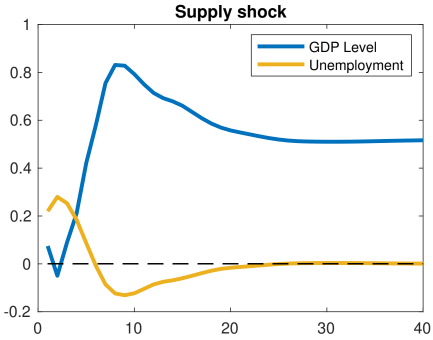
Aggregate demand shock

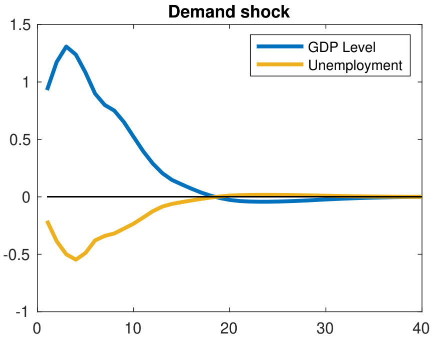
* Aggregate demand shocks have a hump-shaped effect on output and unemployment



What is the long run effect of demand and supply shocks on output level?

* Blanchard and Quah report (Figure 1) the cumulative sum of the impulse responses of output growth (i.e. the response of output level)
* By assumption, it should be zero for demand shocks ✔





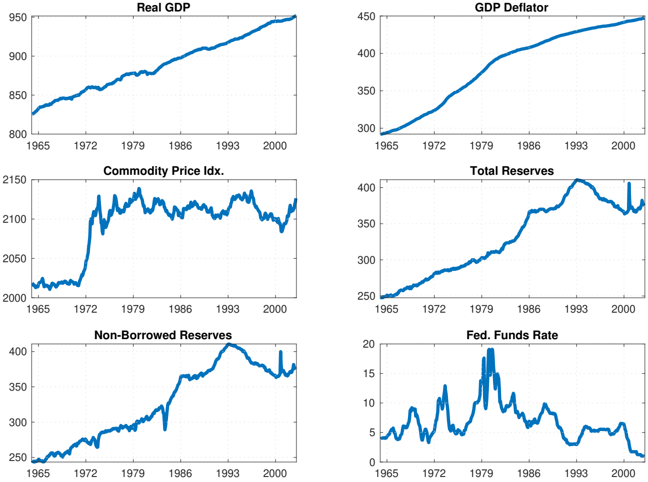
Practical Examples

Uhlig (2005, JME)

Uhlig (2005, JME): Sign restrictions

- Uhlig (2005). “What are the effects of monetary policy on output? Results from an agnostic identification procedure,” *Journal of Monetary Economics*

- US monthly data from 1965:M1 to 2003:M12



What are the effects of monetary policy on output?

- **Objective** Infer the causal effect of monetary policy on real GDP

- Assume a VAR with $p = 12$ with real GDP, real GDP deflator, a commodity price index, total reserves, non-borrowed reserves, and the fed. funds rate

- **Key identifying assumptions** According to conventional wisdom, monetary contractions should
    - Raise the federal funds rate
    - Lower prices
    - Decrease non-borrowed reserves

- Real GDP is left unrestricted

Monetary policy shock: The sign restrictions

* Uhlig imposes the following sign restrictions on the impulse responses of the VAR

	Monetary Policy Shock
Real GDP	?
Real GDP deflator	< 0
Commodity price index	?
Total reserves	?
Non-borrowed reserves	< 0
Fed. Funds Rate	> 0

* Restrictions are imposed for 6 periods

Monetary policy shock: The sign restrictions

- In Matlab, the sign restrictions can be set as follows

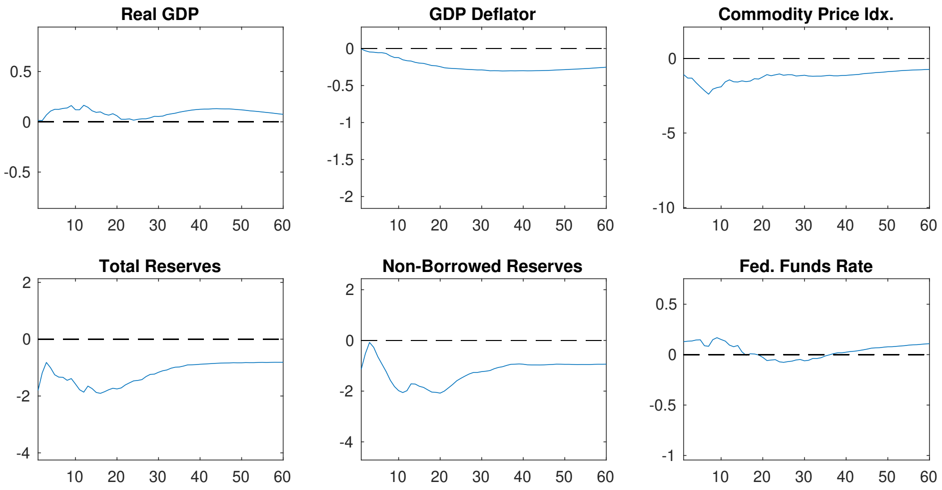
```
% Define the shock names
VARopt.snames = {'Mon. Policy Shock'};
% Define sign restrictions : positive 1, negative -1, unrestricted 0
SIGN = [ 0,0,0,0,0,0;  % Real GDP
        -1,0,0,0,0,0;  % Deflator
        -1,0,0,0,0,0;  % Commodity Price
         0,0,0,0,0,0;  % Total Reserves
        -1,0,0,0,0,0;  % NonBorr. Reserves
         1,0,0,0,0,0]; % Fed Funds
% Define the number of steps the restrictions are imposed for:
VARopt.sr_hor = 6;
```

- The sign restriction routine routine is then implemented with the `SR` function

```
% Function SR performs the sign restrictions identification and computes
% IRs, VDs, and HDs. All the results are stored in SRout
SRout = SR(VAR,SIGN,VARopt);
```

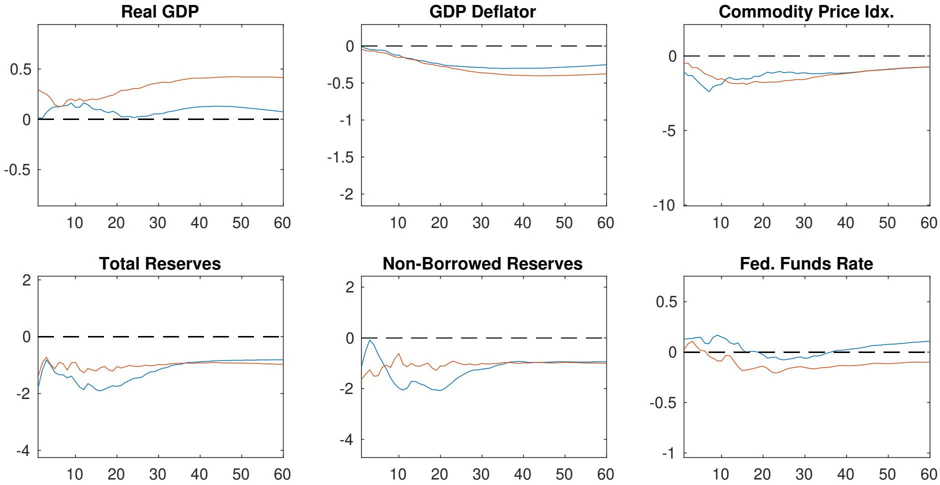
What happens when you do sign restrictions

- Start drawing orthonormal matrices Q until you find one that satisfies the restrictions...



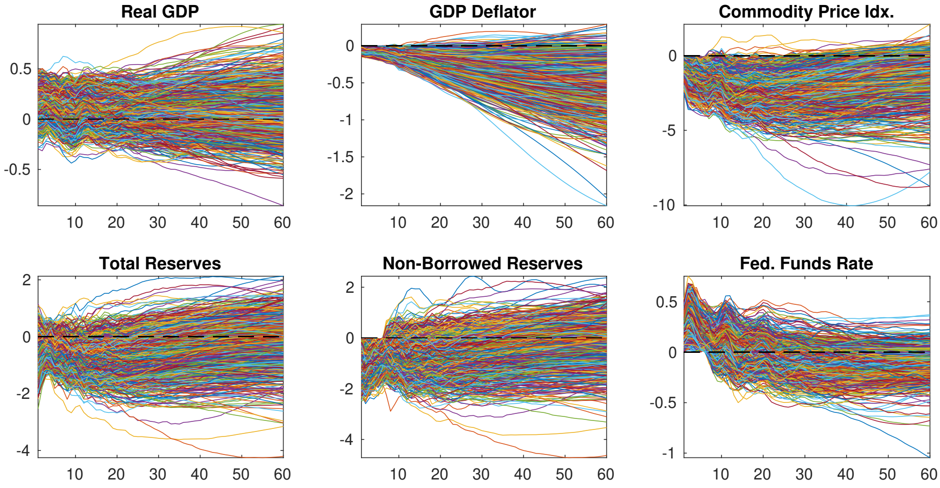
What happens when you do sign restrictions

- Keep on drawing Qs again until you find another one...



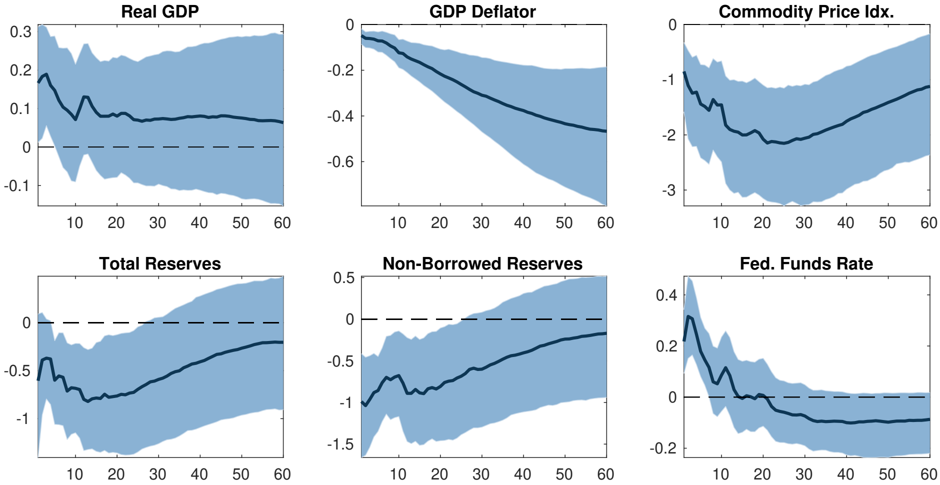
What happens when you do sign restrictions

- After a while...



What are the effects of monetary policy on output?

* Ambiguous effect on real GDP $\Longrightarrow$ Long-run monetary neutrality

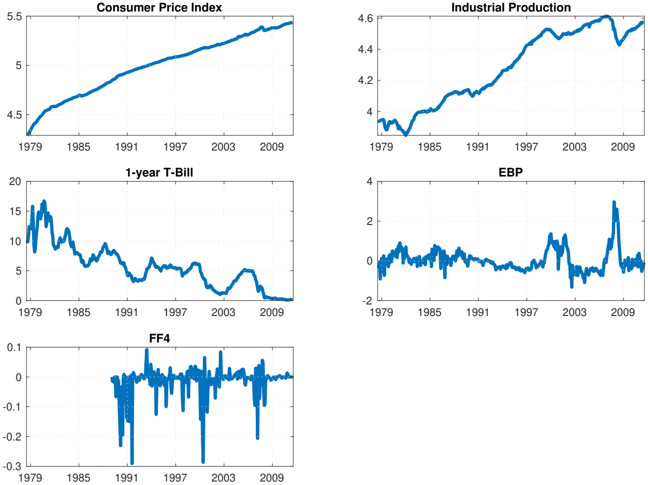


Practical Examples

Gertler and Karadi (2015, AEJ:M)

Gertler and Karadi (2015, AEJ:M): External instruments

- Gertler and Karadi (2015). "Monetary Policy Surprises, Credit Costs, and Economic Activity," *American Economic Journal: Macroeconomics*
- US monthly data from 1979:M7 to 2012:M6



What are the effects of monetary policy on output?

- **Objective** Infer the causal influence of monetary policy on real GDP
- Assume a VAR with $p = 12$ with industrial production, the consumer price index, the 1-year T-bill interest rate, and the Excess Bond Premium
- **Key identifying assumptions** There exists an external instrument (z_t) such that

$$
\begin{aligned}
\mathbb{E}\left[\varepsilon_t^i z_t'\right] &= 0 \quad \text{for } i \neq MonPol \\
\mathbb{E}[\varepsilon_t^{MonPol} z_t'] &= c
\end{aligned}
$$

- That is: z_t is correlated with the monetary policy shock and uncorrelated with all other structural shocks in the system

The instrument (z_t): High frequency monetary policy surprises

- Ingredients
  - Intra-daily data (τ denotes minutes)
  - A monetary policy announcement on day t at time τ (e.g., FOMC decision)
  - A policy indicator r (e.g., fed funds target)
  - Price of futures contract on r for j days ahead $P^j_{t,\tau} = 100 - \mathbb{E}_{t,\tau}\left(r^j\right)$

- Monetary policy surprise

$$s^j_{t,\tau} = -(P^j_{t,\tau+20} - P^j_{t,\tau-10}) = \mathbb{E}_{t,\tau+20}\left(r^j\right) - \mathbb{E}_{t,\tau-10}\left(r^j\right)$$

- **Intuition** Only monetary policy shocks affect the futures prices in this short 30-minute window

External instruments identification with the VAR Toolbox

- In Matlab, first add the instrument to the `VAR` structure

```
% Identification is achieved with the external instrument, which needs
% to be added to the VAR structure
VAR.IV = IV;
```

- Then update the options for identification and for computation of error bands

```
% Update the options in VARopt to be used in IR calculations and plots
VARopt.ident = 'iv';
VARopt.method = 'wild';
```

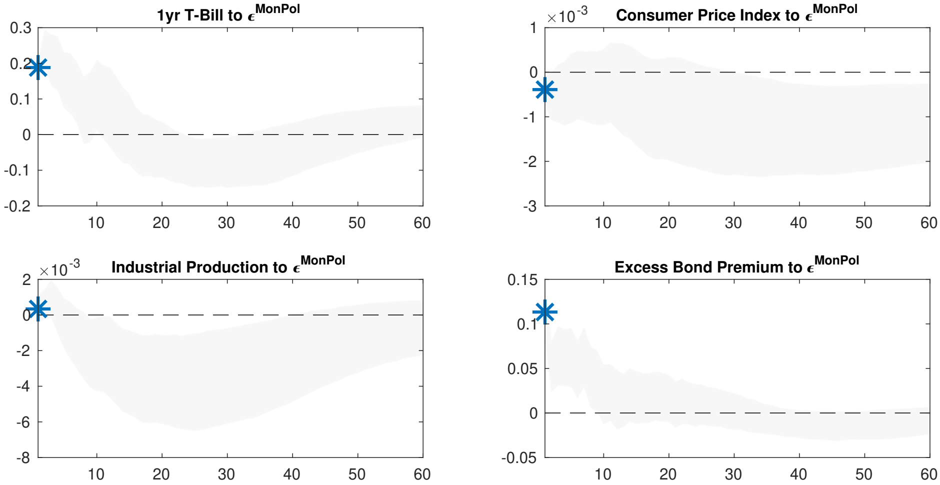
- Finally, compute the *IR* with the `VARir` function
  - The code instruments the residual of the first equation, so the **ordering of the variables matter!**

```
% Compute IR
[IR, VAR] = VARir(VAR,VARopt);
```

- The *b* matrix implied by the external instrument is stored in `VAR.b` and additional info on the first stage is stored in in `VAR.FirstStage`

Impulse response functions: Impact effect

- The impact effect (i.e. the *b* matrix) is given by the first and second stage regressions



Impulse response functions: Dynamic effect

- The dynamic effect is computed as usual with the Φ matrix