Using the Student's Guide

Each *Student's Guide* chapter at this site corresponds to a chapter in the book. (The one exception to this is a chapter corresponding to Appendix 6, which provides solutions to all the problems given but not solved in that Appendix.) Each *Student's Guide* chapter provides a brief summary of the chapter and solutions to the starred problems from the text. Note that many of the starred problems provide proofs or steps in proofs that have been left to the reader in the text; others provide detailed analysis of variations on results given in the text. In some cases, the *Student's Guide* chapter also provides commentary on how you might want to attack the chapter and/or suggestions about which problems to undertake. (Of course, my primary advice is to do *all* the problems, but that may be unrealistic.)

Students over the years have disagreed about how and when the chapter summaries are best consulted. Some students assert that it is best to read through the summaries before tackling the text, so you have a sense of where the chapter is going and how it will get there. This is particularly true for students who have seen this material from a different textbook; they assert that an overview of how I plan to attack the subject is helpful. But other students assert that the summaries are best as… summaries, read after consuming the chapter. You must make up your own mind as to what works best for you.

In addition, this site contains a link to a document, *Errata*, that provides a list of all errors (of the typo variety and worse) in the book of which I currently know. You should open that file immediately and make any necessary corrections to your copy of the book. Note that many errors found in the first year of publication have been corrected in later print runs; the file *Errata* explains how to discover if your copy has these corrections already or not.

If you find errors in the text not listed in this document, of the typo version or otherwise, please email them to me, and I will include them. If you discover any typos/thinkos in the solutions offered in this *Guide*, please email them to me. Finally, if you have feedback to offer on the book, good or bad, my email address is kreps@stanford.edu.

Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Appendix 6: Dynamic Programming

I present here the solutions to the problems left unsolved and/or given at the end of Appendix 6.

■ 3. ***The parking problem with uncertainty about the value of ρ.***

When there is uncertainty as to the value of ρ, the appropriate thing to do is to carry along as a state variable either (a) the posterior probability that $\rho = 0.7$ (or 0.9, one or the other), or (b) the number of spots so far observed that were occupied (or unoccupied). (The posterior probability is a sufficient statistic for everything you've seen so far, and the number of occupied spots so far observed [if you know where you are and when you began observing] allows you to compute the posterior.)

I'll work with the posterior: Let $C(s, p, n)$ be the expected cost (following the optimal policy) if you are at space $-n$, the spot is occupied ($s = 1$) or not ($s = 0$), and your posterior probability that $\rho = 0.7$ is p (this assessed having seen the "condition" of the current spot). If the spot is empty and you park, you get a payoff of n. If either it is empty and you choose to go on to the next spot, or if it is full (so you must go on to the next spot), then the chance the next spot is occupied is

$$0.7p + 0.9(1 - p) = 0.9 - 0.2p,$$

if the next spot is occupied, the next posterior probability (by Bayes' rule) is

$$\frac{0.7p}{0.9 - 0.2p},$$

and if the next spot is empty (with marginal probability $0.1 + 0.2p$), the next posterior probability is

$$\frac{0.3p}{0.1 + 0.2p.}$$

(Build a 2×2 probability table if you are unsure about the application of Bayes' Law here.) Hence Bellman's equation is

$$\begin{aligned} C(1, p, n) = \quad & (0.9 - 0.2p)\, C\left(1, \frac{0.7p}{0.9 - 0.2p}, n - 1\right) \\ + \; & (0.1 + 0.2p)\, C\left(0, \frac{0.3p}{0.1 + 0.2p}, n - 1\right), \text{and} \end{aligned}$$

$$\text{C(0,p,n)} = \min\{n, C(1, p, n)\}.$$

The boundary conditions are $C(0, p, 0) = 0$ (if you are at spot zero and it is unoccupied, park), and $C(1, p, 0) = p(10/3) + (1 - p)(10/1)$. (To see the latter, note that if we are at spot 0, we will park at the first unoccupied spot we meet, so we are interested in the expectation of a random variable X, where X is geometrically distributed (but is not 0) with unknown parameter $1 - d\rho$, and where $\rho = 0.7$ with probability p and $= 0.9$ with probability $1 - p$. To find the expectation of X, condition and then uncondition on the true value of ρ to get the formula I've given.)

I doubt this problem can be solved analytically, so we have to resort to a numerical solution.

One way to try to solve this numerically is to solve for a discrete grid of posteriors, using linear interpolation (or something fancier) for finding the value function at posteriors that lie between values in your grid. The alternative, which I think will be more precise, is to note that, for the given prior 0.5 that $\rho = 0.7$ (prior to seeing the occupancy status of spot -100), there are two possible posteriors you can hold at spot -100 (two, because you have to condition on what you see at spot -100); at $n = -99$, three (not four!); at $n = -98$, four (not eight!); and so forth. The reason is that at $n = 97$ (say), there are only four possible results in terms of the posterior: All three spots were occupied, all three were empty, two were occupied, or one was occupied. The order of occupied and empty spots doesn't matter. So if you specify the prior, you can (almost surely) work your way through to a full solution, for the order of n^2 "states" that might occur.

This is how I tackled the problem, numerically. The first step was to build a table of posteriors. Rows correspond to the spot at which the posterior is being computed or, equivalently, the number of spots visited so far: Row 1 is spot -100, row 2 is spot -99, and so forth. Columns are the number of those spots that were occupied—for row 1 the columns are 0 and 1; for row 2, the columns are 0, 1, and 2; and so forth. Note that we expect the posterior that $\rho = 0.7$ to fall the more occupied spots we've seen. I won't depict the full table of posteriors, but Table GA6.1 shows how it begins.

initial prior: 0.5

TABLE OF POSTERIORS

number of occupied spots / number of spots visited	0	1	2	3	4	5	6	7	8	9	10
1	0.750	0.438									
2	0.900	0.700	0.377								
3	0.964	0.875	0.645	0.320							
4	0.988	0.955	0.845	0.585	0.268						
5	0.996	0.984	0.942	0.809	0.523	0.222					
6	0.999	0.995	0.980	0.927	0.767	0.461	0.181				
7	1.000	0.998	0.993	0.974	0.908	0.719	0.399	0.147			
8	1.000	0.999	0.998	0.991	0.967	0.885	0.666	0.341	0.118		
9	1.000	1.000	0.999	0.997	0.989	0.958	0.857	0.608	0.287	0.094	
10	1.000	1.000	1.000	0.999	0.996	0.986	0.947	0.823	0.547	0.238	0.075

Table GA6.1. Posteriors for the Parking Problem

So, for instance, if you have visited 6 spots (have just seen spot -95 and 4 of those were occupied, you assess probability 0.767 that $\rho = 0.7$. But if 5 out of the 6 were occupied, your posterior assessment that $\rho = 0.7$ is 0.461.

With this table in place, we can proceed to compute the value functions. I found it easiest to have rows for the parking spot $(0, -1, -2, \ldots)$ and columns for the number of occupied spots observed up to but not including the current spot. Two tables were created of this sort, one for the case where the current spot is occupied, so the driver must drive on, and one for the case where the current spot is empty, so the driver has a choice to make. Since you never park before spot -2 when $\rho = 0.7$ or before spot -6 when $\rho = 0.9$, it is entirely intuitive that, in this case where ρ is either 0.7 or 0.9, you never park before spot -6. What do you do? Consult Table GA6.2, which is a part of the "value if the spot is unoccupied" table of values.

Where the value in the table is the absolute value of the spot number, the optimal strategy is to park immediately, so the earliest you park is at spot -6, if you have (prior to this spot) seen 79 or more occupied spots. If you reach spot -5, you park first chance you get if you have seen 79 or more occupied spots. The same is true for spots -4 and -3. At spot -2, park if you can and if you have seen 78 or more occupied spots. At spot -1, park if you can, no matter how many occupied spots you have seen.

■ 7. ***The simplest multi-armed bandit problem.***

I will give the optimal strategy in terms of p_t, the Bayesian assessment made by the decision maker at date t that the slot machine pays \$12 with probability 1/3. (Therefore, $p_0 = 0.8$.) Before doing this, let me say how p_t evolves: On each turn, if you

		number of occupied spots observed prior to this				
		77	78	79	80	81
	-10	6.521	6.554	6.563	6.565	6.566
	-9	6.426	6.531	6.557	6.564	6.565
	-8	6.137	6.462	6.539	6.559	6.564
spot	-7	5.388	6.248	6.488	6.545	6.561
number	-6	4.125	5.642	6	6	6
	-5	2.961	4.456	5	5	5
	-4	2.333	3.186	4	4	4
	-3	2.082	2.434	3	3	3
	-2	1.984	2	2	2	2
	-1	1	1	1	1	1
	0	0	0	0	0	0

Table GA6.2. Values at Unoccupied Spots as a Function of Occupied Spots Seen Prior to This. See the text for a fuller description, but this table implicitly identifies the optimal strategy.

choose not to play, you learn nothing, and so $p_{t+1} = p_t$. If you choose to play, the machine either pays \$0 or \$12. If it pays \$12, then you know that this is the sort of machine that will give \$12 with probability 1/3 each time; that is, $p_{t+1} = 1$, regardless of p_t (as long as $p_t > 0$.) If it pays \$0, then the probability that it is the "good" type of machine becomes (by Bayes' rule)

$$p_{t+1} = \frac{(2/3)p_t}{(2/3)p_t + (1-p_t)} = \frac{2p_t}{3-p_t}.$$

Note that this posterior probability is monotonically increasing in the prior, is 1 if $p_t = 1$, is 0 if $p_t = 0$, and is strictly less than p_t if p_t is strictly betweeen 0 and 1. Moreover, if $p_t < 1$, then the ratio of p_{t+1} to p_t is $2/(3-p_t)$, which is decreasing in p_t. Therefore, by a simple ratio test, we know that this sequence of posteriors decreases to zero, if the machine is played repeatedly and, each time, a \$0 reward is received.

With this as background, I assert the optimal strategy is to play the machine as long as $p_t > 1/13$, but to stop playing if and when $p_t \leq 1/13$.

I'll prove optimality of this strategy by proving that it is unimprovable. (Rewards are bounded above by \$11 per period and below by $\$-1$, and the discount factor is less than one.) I will do this in steps.

Step 1. The (discounted, expected net present) value of following the strategy given if p_t ever reaches 1 (because a \$12 prize is received) is \$30 (as a continuation value from that stage on).

To see this, note that if p_t reaches 1, it stays there forever. Therefore, the strategy says to play the game every time. The machine pays off an expected \$4 per round ((\$12)(1/3) + (\$0)(2/3)), netted against the \$1 it costs to play, so your expected net reward in each round is \$3. Discounted with discount rate 0.9, this has an discounted expected net present value of

$$\$3 + (0.9)\$3 + (0.9)^2\$3 + \ldots = \frac{\$3}{1 - 0.9} = \$30.$$

Step 2. The (discounted expected net present continuation) value of following the strategy given is \$0 if $p_t \leq 1/13$.

The strategy says not to play when $p_t \leq 1/13$. No information is received, so you will be choosing not to play next time and forevermore. So following this strategy nets \$0 each round, for a net present value of 0.

Step 3. The (discounted expected net present) value of following the strategy given is $\geq$ \$0 *if* p_t *is* $1/13$ *or more.*

This is the most complex step. Let me denote by q_0 the value $1/13$, and define inductively

$$q_{k+1} = \frac{3q_k}{2 + q_k}.$$

I assert that the sequence $\{q_k; k = 0, 1, \ldots\}$ has the properties that

(a) The sequence is increasing, with $q_k < 1$ for all k.

(b) The limit $\lim_k q_k = 1$.

(c) If $p_t = q_k$, the arm is pulled, and \$0 appears, then $p_{t+1} = q_{k-1}$.

In words, the q_k's "invert" the Bayesian inference process described above. To prove parts a and b, consider the function $f(x) = 3x/(2+x)$, for $x \geq 0$. By inspection, $f(0) = 0$ and $f(1) = 1$. Also, f' (the derivative of f) is $3/(2+x) - 3x/(2+x)^2 = 6/(2+x)^2 > 0$; f is a strictly increasing function. Since $f(q_k) = q_{k+1}$, we know that the sequence of q_k's is increasing, and since $f(1) = 1$, we know that if we start below 1, we do not exceed 1 in the sequence. That's part a. As for part b, suppose that the limit of the q_k's was something less than 1, call it q^*. Then by the continuity of f, $f(q^*) = q^*$. But for any $q < 1$, $f(q)/q = 3/(2+q) > 1$; there are no fixed points of f less than 1 (except for 0).

Part c is simple computation.

Now we proceed to prove step 3. Suppose $p_t > 1/13$. Suppose moreover that $p_t \in (q_0, q_1]$. Then the strategy calls for pulling the arm this turn and, if \$0 is the reward, since $p_t \leq q_0 = 1/13$, never playing again. The expected net (continuation) value is

$$(1/3)p_t(11 + (0.9)(30)) + (1 - (1/3)p_t)(-1 + (0.9)(0)), \qquad (*)$$

or the probability of getting a payback of \$12, which is $(1/3)p_t$, times the net immediate reward of \$11 and the discounted continuation value of \$30, plus the probability of getting back \$0, for a net of -1, times the discounted continuation value of \$0. Do the algebra, and you'll find that this is $13p_1 - 1$, which exceeds 0 as long as $p_t > 1/13$. Next suppose that $p_t \in (q_1, q_2]$. You do the same computations as above, except that in the equation ($*$), the continuation value of 0 on the right-hand side is replaced by something greater or equal to 0, since p_{t+1} will be in interval $(q_0, q_1]$. So this exceeds \$0 as long as $p_t > 1/13$. And so forth, "inductively," where the induction is on which interval $(q_k, q_{k+1}]$ contains p_t. (We know that p_t is in some such interval because of part b above.)

Step 4. The strategy is unimprovable (hence optimal).

There are two cases: $p_t \le 1/13$ and $p_t > 1/13$.

Taking the former first, the strategy says not to play, for a next present value of \$0. The alternative is to play for one round. If \$12 is paid back, $p_{t+1} = 1$, so reverting to the strategy, you get a continuation value of \$30; if \$0 is paid back, $p_{t+1} < p_t < 1/13$, so reverting to the strategy says not to play, for a continuation value of \$0. Therefore, if you play this round, you net (in expectation)

$$(1/3)p_t(11 + 0.9 \cdot 30) + (1 - p_t/3)(-1 + 0.9(0)) = 13p_t - 1,$$

which is less than \$0 for $p_t < 1/13$ and just equal for $p_t = 1/13$. So for $p_t \le 1/13$, the strategy of not playing is unimprovable (in a single step).

And if $p_t > 1/13$, the strategy says to play. This, we showed in step 3, generates an expected discounted net present value in excess of \$0. (This is true as well for $p_t = 1$, although the appeal here is to step 1, not step 3.) If you do the alternative, which is not to play this round, you neither win nor lose anything this round and, since your posterior doesn't change, precisely beginning next round you get the expected discounted net present value from following the strategy today, but discounted by an additional 0.9. Hence instead of the continuation value v from following the strategy, you get $0.9v$. Since $v \ge 0$, it is better (or, at least as good) to play today.

That does it.

■ 8. ***Another button-pushing, light-flashing machine.***

(a) Create two states, labeled x and y, where the state at date t is x if the decision maker pushed X at $t-1$, and y if she pushed Y. Transition probabilities on the states are thus deterministic, and expected within-period rewards are $r(x, X) = 0.75(10) + 0.25(0) = 7.5$, $r(x, Y) = 12.5$, $r(y, X) = 2$, and $r(y, Y) = 7$, where (for instance $r(x, Y)$ is the expected reward if the state is x and Y is pushed).

I assert that if $\delta \ge 10/11$, the optimal strategy is to push X regardless of state, and if $\delta \le 10/11$, the optimal strategy is to push Y regardless of state. There is no claim here that these are the only optimal strategies, and in fact at $\delta = 10/11$, there are many others. But the problem asks you to produce an optimal strategy for each value of δ, which is what I'm doing, once I verify optimality of these strategies. (I'm only interested in the cases where $\delta < 1$ here. If $\delta \geq 1$, any strategy gives you expected reward of $+\infty$.)

Regardless of δ, the strategy of pushing X every time gives expected total reward of $7.5+7.5\delta+7.5\delta^2+\ldots = 7.5/(1-\delta)$ starting in state x, and $2+7.5\delta+7.5\delta^2+\ldots = -5.5+7.5/(1-\delta)$ starting in state y. A one-step deviation in state x gives $12.5 + 2\delta + 7.5\delta^2 + 7.5\delta^3 + \ldots$ starting in state x and $7 + 2\delta + 7.5\delta^2 + 7.5\delta^3 + \ldots$ starting in state y, so this strategy is unimprovable if

$$7.5 + 7.5\delta \geq 12.5 + 2\delta \quad \text{and} \quad 2 + 7.5\delta \geq 7 + 2\delta,$$

both inequalities being true as long as $\delta \geq 5/5.5 = 10/11$.

(Now you fill in the details for $\delta \leq 10/11$ and the strategy of "always Y"; following the pattern set up in the preceding paragraph.)

(b) Now the states are called xa, if last time X was pushed and the machine flashed A, and z, if last time Y was pushed or the machine flashed B. Expected rewards are the same as in part (a), with xa replacing x and z replacing y. But transition probabilities are different: From xa, if the decision maker pushes Y, then transition is to z with certainty, and it is back to xa with probability 0.75 (and to z with probability 0.25) if she pushes X. From z, transition is to z with certainty if the decision maker pushes Y, and to xa with probability 0.2 if she pushes X.

I assert that "always push Y" is optimal, regardless of δ.

First to find the value of this strategy. Starting in state z, this strategy gives an expected value of 7 each time, for a total (discounted) expected value of $7/(1-\delta)$. Starting in state xa, this strategy gives 12.5 in the first period and 7 thereafter, or $5.5 + 7/(1-\delta)$ in total.

I'll check unimprovability of this strategy starting in xa: If you deviate for one step, you get 7.5 in the first round and transition to xa with probability 0.75 and to z with probability 0.25, for a net expected value of

$$\begin{aligned} 7.5+\delta[(0.75)(5.5 + 7/(1-\delta)) + (0.25)7/(1-\delta)] &= 7/(1-\delta) + 0.5 + (0.75)(5.5)\delta \\ &= 7/(1-\delta) + 0.5 + 4.125\delta \leq 7/(1-\delta) + 4.625, \end{aligned}$$

which is always less than $5.5 + 7/(1-\delta)$. You still need to check unimprovability starting in z, but that is even easier.

■ 9. The optimal strategy is to pick β whenever in state X. (There are no other choices to make.) Rewards in each period are bounded, and the discount factor (0.8) is strictly less than one, so the proof of optimality is a matter of checking that the strategy is unimprovable.

Step 1. Compute the value of following the strategy of choosing β whenever in state X. If we follow this strategy, then at any point we are either in state X or state Y. I will let v_X denote the expected value of all future rewards beginning with the current period if this period begins in state X, and I will let v_Y denote the expected value of being in state Y. Then v_X and v_Y satisfy the following two recursive equations:

$$v_X = 2 + 0.8 v_Y \quad \text{and} \quad v_Y = 0 + 0.8(0.1 v_Y + 0.9 v_X).$$

To explain, the value starting in state X is an immediate payment of 2, and then next period you are in Y for sure, so you get the discounted value of being in state Y. And if you start in state Y, you get an immediate 0, and then, discounted by 0.8, the value of being in Y with probability 0.1 and the value of being in X with probability 0.9.

Now, the second of these equations is $v_Y = 0.08 v_Y + 0.72 v_X$ or $v_Y = (0.72/0.92) v_X$. And substituting this into the first equation gives

$$v_X = 2 + 0.8(0.72/0.92) v_X, \quad \text{or} \quad v_X = \frac{2}{1 - 0.8(0.72/0.92)} = 5.34883721,$$

and, therefore,

$$v_Y = \frac{0.72}{0.92} 5.34883721 = 4.18604651.$$

Step 2. Check to ensure that this strategy is unimprovable (in a single step). You only have a choice in state X, where the only alternative would be to choose α. If, in state X, you choose α and then revert to the strategy I have hypothesized is optimal, you will net $1 + (0.8) v_X = 1 + (0.8)(5.34883721) = 5.27906977$, which is less than what you get by following the strategy. So the strategy in question is indeed unimprovable, and since rewards are bounded and discounted, its unimprovability implies that it is optimal.

■ 10. This is a discounted dynamic programming problem with an additive structure, where the discount factor 0.9 is strictly less than one, and where the per-period rewards are bounded: They are 0 if you haven't begun manufacturing, 2250 if you are manufacturing with a cost of 7, 9000 if you are manufacturing with a cost of 4, and 20250 if you are manufacturing with a cost of 1.

Therefore, all the tools of discounted dynamic programming are available. I use *unimprovable strategies are optimal*.

Moreover, the nature of the problem is very "transient." One you begin to manufacture the good, there are no further decisions to make. If you ever learn how to manufacture at a cost of 1, there is nothing more to learn, so it is evident that you should begin to manufacture immediately. If you ever learn how to manufacture at a cost of 4, then you never again need to be concerned about what happens if the cost is 7, since there are no circumstances in which that condition will ever recur.

I hypothesize that, if the cost of manufacture is 4 (and 1 is still possible), with p your probability assessment that theoretically best cost (TBC) is 4 and $1-p$ that the TBC is 1, your optimal strategy is to continue to explore if $p \leq 37/45$ and to begin manufacture if $p > 37/45$. Let me describe how I found this number 37/45 and then prove my assertion.

Given p, if the strategy is to hold off for a round and then start manufacturing, there is probability $(1-p)/2$ that you will reduce costs to 1 and $p + (1-p)/2 = (1+p)/2$ that you will not. So your expected reward of following this plan is

$$0.9\left[\frac{1-p}{2}(20250 \times 10) + \frac{1+p}{2}(9000 \times 10)\right].$$

The 0.9 is the discount factor, while 20250×10 is the net present value (next period) of manufacturing with a cost of 1, and 9000×10 is the NPV (next period) of manufacturing with a cost of 4. The alternative is to begin manufacturing immediately, with an NPV (this period) of 90,000. So I find the implications for p of the first quantity being greater or equal to the second, which is

$$.9\left[\frac{1-p}{2}(20250 \times 10) + \frac{1+p}{2}(9000 \times 10)\right] \geq 90000,$$

and this gives $p \leq 37/45$.

Note that this calculation verifies the unimprovability of my announced strategy for $p \geq 37/45$, since when I follow this strategy, my value function for $p \geq 37/45$ is 90,000 (the strategy says, start to manufacture immediately), and I've shown that for p in this range, beginning to manufacture is better than the one-step alternative of exploring for a round and then (since, if I don't move costs down to 1, my new assessment will be greater than my previous p, hence greater than 37/45) beginning to manufacture.

But it doesn't verify unimprovability for $p \leq 37/45$. To do that, I need the value of following this policy or, rather, I need to know that the value is greater than 90,000, which is the value if, instead of exploring, I move to the (irreversible) strategy of beginning to manufacture. I can show this by an argument of the sort used in the "simplest multi-armed bandit problem." But, in this case, I can revert to a more computational approach:

If ever costs fall from 7 to 4, at that point, $p = 0.5$; the chances of improving costs from 7 to 4 are independent of whether the TBC is 4 or 1, so their relative likelihoods stay the same (equal), and so the moment you learn that TBC is not 7, you are back to a 50-50 assessment on whether it is 4 or 1. Now if you explore for one round, either costs fall to 1 or they don't, and a simple application of Bayes' rule tells you that, if they don't, p rises to $2/3$. Another round of unsuccessful exploration leads to $p = 4/5$. And successive rounds give $p = 8/9$, $p = 16/17$, $p = 32/33$, and so forth. Why? The general rule is if $p = 2^n/(2^n + 1)$ in this round, then the chance of failure (no lowering of cost) this round is $2^n/(2^n + 1) + (1/2)(1/(2^n + 1)) = (2^{n+1} + 1)/(2^{n+1} + 2)$ and the posterior probability (given this event) that TBC is 4 is

$$\frac{2^n/(2^n+1)}{(2^{n+1}+1)/(2^{n+1}+2)} = \frac{2^{n+1}}{2^{n+1}+1}.$$

At $p = 8/9$, we are in the region where $p > 37/45 = 0.8222$, so we know that, in following my hypothesized strategy, we search for a lower cost (of 1) immediately, when $p = 2/3$, and when $p = 4/5$, but abandon the search and start manufacturing if we fail to lower cost on the third try (after costs are lowered to 4).

We can then use a simple finite-horizon recursion to compute the value of following this strategy. When $p = 4/5$, we get

$$.9\left[\frac{1}{10}202500 + \frac{9}{10}90000\right] = 91125.$$

This then gives us the value for when $p = 2/3$, as

$$.9\left[\frac{1}{6}202500 + \frac{5}{6}91125\right] = 98718.75$$

and, for when $p = 1/2$,

$$.9\left[\frac{1}{4}202500 + \frac{3}{4}98718.75\right] = 112197.656.$$

Now we can go back to the situation where best current cost is 7, and TBC could be 7, 4, or 1. Now, if a "breakthrough" happens and costs reduce to 4, the continuation value is 112197.656, so you can repeat the analysis given above to discover that you should keep trying to lower costs as long as q, the current posterior assessment that 7 is the TBC, exceeds 0.94425718. (That is, 0.94425718 is the solution to the equation $22500 = .9[22500(1+q^*)/2 + 112197.656(1-q^*)/2$.)

Given a starting value of $q = 1/3$ (with the assessments that TBC = 4 and that TBC = 1 always half of $1 - q$, as long as current cost is 7), the sequence of posteriors for q as long as no "breakthough" to a cost of 4 occur, is $1/2$, then $2/3$, then $4/5$, $8/9$, $16/17 = 0.94117697$, and $32/33 = 0.96969696\ldots$. So, by the same logic as before, you will keep trying to lower costs for six periods, giving up (and starting to manufacture) when q reaches $32/33$. To verify unimprovability of this, we need the values for each posterior, which are computed in the same fashion as before:

$$\text{for } q = \frac{16}{17}, \text{ the value is } .9\left[\left(1 - \frac{1}{34}\right)22500 + \frac{1}{34}112197.656\right] = 22624.3497,$$

$$\text{for } q = \frac{8}{9}, \text{ the value is } .9\left[\left(1 - \frac{1}{18}\right)22624.3497 + \frac{1}{18}112197.656\right] = 24849.5801,$$

and, similarly,

for $q = 4/5$, the value is 30218.6589,

for $q = 2/3$, the value is 39493,6426,

for $q = 1/2$, the value is 51902.6814, and

for $q = 1/3$, the value is 64800.9057.

Note that these computations carry with them the test of unimprovibility: For $q = \frac{1}{3}, \frac{1}{2} \cdot \frac{4}{5} \cdot \frac{8}{9}$, and $\frac{16}{17}$, the strategy is to explore, and we've computed the values from following this strategy (together with the "second half" of it, done before) as all being greater than 22500, the value of the alternative (manufacture immediately). While for any $q > 0.94425718$, which includes $q = \frac{32}{33}$, the inequality we implicitly computed previously shows that manufacturing is better than a one-step change of explore and then, if you fail to lower costs to 4, manufacture.

▪ 11. Forest Management.

(a) The optimal strategy is to cut the forest as soon as it reaches size 19 (or more).

If you following this strategy, the forest will be of size 19 ($= 1 + 2 + 4 \times 4$) each time it is cut, and you will have a profit of 18. The value of following this strategy can be written as v_n, the value of having a forest of size n going into a period (after the growth has taken place), for $n = 1, 3, 7, 11, \ldots$. Then

$$v_1 = .9^5(18 + .9^6 \times 18 + .9^{12} \times 18 + \ldots) = \frac{.9^5 \times 18}{1 - .9^6} = 22.684059,$$

$$v_3 = .9^4(18 + .9^6 \times 18 + \ldots) = 25.20451,$$

and similarly

$$v_n = \begin{cases} 28.005, & \text{for } n = 7, \\ 31.116679, & \text{for } n = 11, \\ 34.5740878, & \text{for } n = 15, \\ 38.415631, & \text{for } n = 19. \end{cases}$$

For larger n (of the form $n = 3 + 4m$ for some integer $m > 4$), this strategy says to cut immediately, for a value of

$$v_n = n - 1 + .9^6 \times 18 + .9^{12} \times 18 + \ldots = n + 19.4156531.$$

To show the optimality of this strategy, I must show that it is unimprovable. For a forest of size 1, 3, 11, or 15, the strategy calls for letting the forest grow, while a (one-step) variation would call for cutting the forest immediately. Cutting the forest immediately generates $n - 1$ immediately (where n is the size of the forest), and then (reverting back to the strategy) 18 in six periods, 18 again in twelve, etc., for a net present value of 19.4156531 + n (gotten the same way was we obtained v_n for $n > 19$). That is, the variant strategy nets

$$v'_n = \begin{cases} 19.4156531, & \text{for } n = 1, \\ 22.4156531, & \text{for } n = 3, \\ 26.4156531, & \text{for } n = 7, \\ 30.4156531, & \text{for } n = 11, \text{ and} \\ 34.4156531, & \text{for } n = 15. \end{cases}$$

Therefore, none of these variations is a one-step improvement on the allegedly optimal strategy. As for forests of size $n \geq 19$, the strategy says to cut them, so the one-step variation is to let them go for a year and then cut. This nets $.9(19.4156531 + n + 4) = 21.0740679 + .9n$, versus $19.4156531 + n$ which is obtained by following the allegedly optimal strategy. The former is less than the latter as long as $1.6584148 \leq .1n$ or $n \geq 16.584148$, which is true for these n.

This shows that the announced strategy is unimprovable, and therefore it is optimal.

(b) I assert that the optimal strategy is to wait until the forest reaches size 15 (or more), and then cut it on the first opportunity where $p_t = 1.5$. That is, you never cut the forest if $p_t = 1$, and you never cut it before it reaches size 15.

To show that this strategy is optimal, I first have to find out the value of following it. Begin with a forest of size $n \geq 15$. I'm going to evaluate the expected value of following the strategy, unconditional on the current price, calling this v_n. The strategy says to cut the forest on the first opportunity when $p_t = 1.5$, which means that you will cut the forest immediately with probability 1/2, in one period with probability 1/4, and so forth. Therefore, your expected net present value is

$$v_n = \frac{1}{2}\left[1.5(n-1)+.9^5 v_{15}\right]+\frac{.9}{4}\left[1.5(n+4-1)+.9^5 v_{15}\right]+\ldots+\frac{.9^k}{2^{k+1}}\left[1.5(n+4k-1)+.9^5 v_{15}\right]+\ldots.$$

To explain, with probability 1/2, the immediate price is 1.5 and you cut the forest down, netting $1.5(n-1)$ immediately and, beginning in five periods, v_{15}; with probability 1/4 the immediate price is 1 but the next price is 1.5, so you cut next time, netting an immediate $1.5(n+4-1)$ and, with a further five period delay, v_{15}, and so forth. Replacing n with 15 in the formula gives a recursive equation for v_{15}, namely

$$v_{15} = \frac{1}{2}\left[1.5(14)+.9^5 v_{15}\right]+\frac{.9}{4}\left[1.5(14+4)+.9^5 v_{15}\right]+\ldots+\frac{.9^k}{2^{k+1}}\left[1.5(14+4k)+.9^5 v_{15}\right]+\ldots,$$

which is

$$v_{15} = 21 \times \left(\frac{1}{2}+\frac{.9}{4}+\ldots\right)+\left(\frac{.9}{4}\times 6+\frac{.9^2}{8}\times 12+\ldots\right)+.9^5 v_{15}\left(\frac{1}{2}+\frac{.9}{4}+\ldots\right).$$

This is

$$v_{15} = \frac{10.5}{1-.45}+\frac{3\times .45}{.55^2}+\frac{.9^5 v_{15}}{2}\frac{1}{1-.45} = 19.090909+4.46281+.53680909 v_{15}.$$

Therefore,

$$v_{15} = \frac{19.090909+4.46281}{1-.53680909} = 50.85099588.$$

This then allows us to compute v_n for $n > 15$:

$$v_n = (1.5(n-1)+.9^5 v_{15})\left(\frac{1}{2}+\frac{.9}{4}+\ldots\right)+\left(\frac{.9}{4}\times 6+\frac{.9^2}{8}\times 12+\ldots\right) =$$

$$\frac{.75(n-1)+.9^5\times 50.85099588\times .5}{.55}+4.46281 = 1.3636n+30.39645,$$

which gives

$$v_{19} = 56.30485, v_{23} = 61.75925, v_{27} = 67.21365,$$

and so forth. For $n < 15$, the value of following the strategy is found by

$$v_n = \begin{cases} 0.9v_{15} = 45.7659, & \text{for } n = 11, \\ 0.9^2 v_{15} = 41.1893, & \text{for } n = 7, \\ 0.9^3 v_{15} = 37.07038, & \text{for } n = 3, \text{ and} \\ 0.9^4 v_{15} = 33.3633384, & \text{for } n = 1, \end{cases}$$

since in each case the strategy calls for waiting until the forest grows to size 15.

Now to verify unimprovability. First we'll check for sizes below 15. The strategy says to wait, regardless of the current price of wood, and so if cutting when the price is 1.5 is not a one step improvement, then neither is cutting when the price is 1. When the forest is of size 1, if the price is 1.5, cutting nets 0 immediately and v_{15} in five periods, which is clearly worse than getting v_{15} in four periods, which is what the strategy causes. So cutting when the forest is of size 1 is not a one-step improvement.

When the forest is of size 3 and the price is 1.5, cutting nets an immediate 3 plus v_{15} in five periods, or $3 + .9^5 v_{15} = 33.0270046$, which is worse than following the allegedly optimal strategy. Similar calculations verify unimprovability of the strategy when $n = 7$ and 11.

When the forest is of size 15, the strategy calls for cutting if the current price is 1.5 and leaving the forest if the current price is 1. So if the current price is 1.5, not cutting gives $.9v_{19}$ next period, which is $.9 \times 56.3055413 = 50.674365$, which is worse than carrying out the strategy, netting an immediate 21 plus $.9^5 v_{15}$, for a total of 51.027.

And when the forest is of size $n \geq 15$, the strategy calls for cutting if the current price is 1.5 and letting the forest grow for another period if the current price is 1. Suppose the current price is 1.5. Then following the strategy nets $1.5(n-1)$ immediately and a continuation value of $.9^5 v_{15}$, for a total $28.52700456 + 1.5n$. Letting the forest grow for another period nets an expected $.9v_{n+4} = .9(1.3636(n+4) + 30.39645) = 1.22724n + 32.265765$. So following the strategy is better. And if the price is 1, cutting the forest immediately nets $n - 1$ immediately and $.9^5 v_{15}$ as a continuation value, or a total of $n + 29.02668$, versus $1.22724n + 32.265765$ by following the strategy (of waiting until next period). Following the strategy is clearly better.

(c) I assert that the optimal strategy is to cut the forest either the first time it fails to grow or when it reaches size $n \geq 15$.

First step in verifying this is to compute the value function for following this strategy. Let v_n be the value beginning with a forest of size n that has just grown, let v'_n be the value of a forest of size n that has not grown, and let v be the continuation value following this strategy, immediately after cutting and replanting the forest.

Now consider the continuation value v obtained immediately after cutting the forest. The forest grows to sizes 1 and 3 in the next two periods. Then, in the third period, it doesn't grow at all with probability 0.2, and you harvest it and replant. You harvest and replant (at the size 7) in period 4 with probability $0.8 \times 0.2 = 0.16$; you harvest and replant at the size 11 in period 5 with probability $0.8^2 \times 0.2 = .128$; and you harvest and replant a forest of size 15 in period 5 with probability $0.8^3 = 0.512$. Therefore,

$$v = (0.9^3)(0.2)(2 + v) + (0.9^4)(0.16)(6 + v) + (0.9^5)(0.128)(10 + v) + (0.9^5)(0.512)(14 + v).$$

Collecting terms this is

$$v = 5.90991552 + 0.6286896v, \quad \text{or} \quad v = \frac{5.90991552}{1 - 0.6286896} = 15.9163749.$$

For $n \geq 15$, the strategy calls for cutting the forest whether it has just gown or not, so $v_n = v'_n = n - 1 + 15.9163749 = n + 14.9163749$. For any n, if the forest has not grown, the strategy says to cut immediately, so $v'_n = n + 14.9163749$. The harder values to compute are v_n for $n < 15$. In all such cases, the strategy says not to cut. Therefore:

- For $n = 11$, $v_{11} = 0.9(0.8v_{15}+0.2(10+v)) = 0.9(0.8\times(14+v)+0.2\times(10+v)) = 26.72306488$.
- For $n = 7$, $v_7 = 0.9(0.8v_{11} + 0.2(6 + v)) = 23.18555419$.
- For $n = 3$, the forest is certain to grow to size 7, so $v_3 = 0.9v_7 = 20.86699877$.
- For $n = 1$, the forest is certain to grow to size 3, so $v_1 = 0.9v_3 = 18.78029890$.

Now to verify unimprovability: For a forest of size 1, the alternative to the allegedly optimal strategy is to cut immediately, yielding 0 immediately and a continuation value of 15.9163749. Waiting is better.

For a forest of size 3, the alternative strategy, cutting immediately, nets 2 immediately and the continuation value of 15.9163749, for a total of 17.9163749, clearly worse than following the strategy.

For any forest of size $n > 3$ that has stopped growing, the strategy says to cut immediately. The alternative is not to cut, which simply delays by one period the immediate value and the continuation, since the forest is certain not to grow. So it is clearly the case that the alternative of not cutting for one stage is inferior to immediately cutting.

For a forest of size 7 that has just grown, cutting immediately nets $6 + 15.9163749$, versus 23.18... for following the strategy. Following the strategy is better.

For a forest of size 11 that has just grown, cutting immediately nets $10 + 15.9163749 = 25.9163749$, versus 26.73... for following the strategy. Following the strategy is better.

For a forest of size $n \geq 15$ that has just grown, the strategy says to cut immediately, netting $n+14.9163749$. The alternative one-step deviation is to wait a period (following which you will cut immediately, since you revert to the strategy): This nets

$$0.9\big(0.8(n + 3 + 15.9163749) + 0.2(n - 1 + 15.9163749)\big)$$
$$= 0.9(n + 18.1163749) = 0.9n + 16.3047374.$$

Following the strategy is at least as good as long as

$$0.1n \geq 16.3047374 - 14.9163749 = 1.38836251 \quad \text{or} \quad n \geq 13.8836251,$$

which is true.

(d) In this case, you have to keep track of how many times in a row the forest has experienced zero growth and the resulting posterior probability that it is in no-growth mode. Because a growth of 4 is incompatible with no-growth phase, every time you see growth of 4, you reassess that, for that period, the forest was indeed in growth mode, and so the probability it is in growth mode in the next period is 0.8, and the probability of a growth of 4 units is 0.72. But suppose the forest, in this next period, didn't grow (grew 0 units). The posterior probability that it was indeed in growth mode is

$$\frac{\text{Prob(0 units and growth mode)}}{\text{Prob(0 units)}} = \frac{0.8 \times 0.1}{0.8 \times 0.1 + 0.2 \times 1} = \frac{0.08}{0.28} = 0.2857,$$

and hence the probability that it is in growth mode next period is $.8 \times 0.2857 = 0.22857$, and the probability of 4-unit growth in that period is $0.22857 \times 0.9 = 0.2057$. If again it doesn't grow, the posterior probability that it was in growth mode, computed by the same Bayes' rule calculations, falls to 0.02877698, so the probability (following this) that it is in growth mode the next period is 0.02302158 and the probability that it grows 4 units is 0.020719. And so forth.

My initial hypothesis as to the optimal strategy is: As long as the forest keeps growing, let it grow to size 15 and then cut. If it ever fails to grow, cut immediately. Actually, my strategy needs to be specified a bit better than that: Precisely, it is to cut any forest that is of size 15 or larger, and cut any forest that has failed to grow in the last period, but let a forest of size 11 or less go if it has just grown. You can see this strategy depicted in Figure GA6.1, where I have drawn an "event tree" out to period 6 after a cutting. The open nodes represent states where the strategy says to let the forest grow, and the closed nodes are states where the strategy says to cut.

If I follow this strategy, I can evaluate the continuation value just after a cutting as the solution v to the following equation:

$$v = (.72)(.72)(.72)(.9^5)(14 + v) + (.72)(.72)(.28)(.9^5)(10 + v)$$
$$(.72)(.28)(.9^4)(6 + v) + (.28)(.9^3)(2 + v).$$

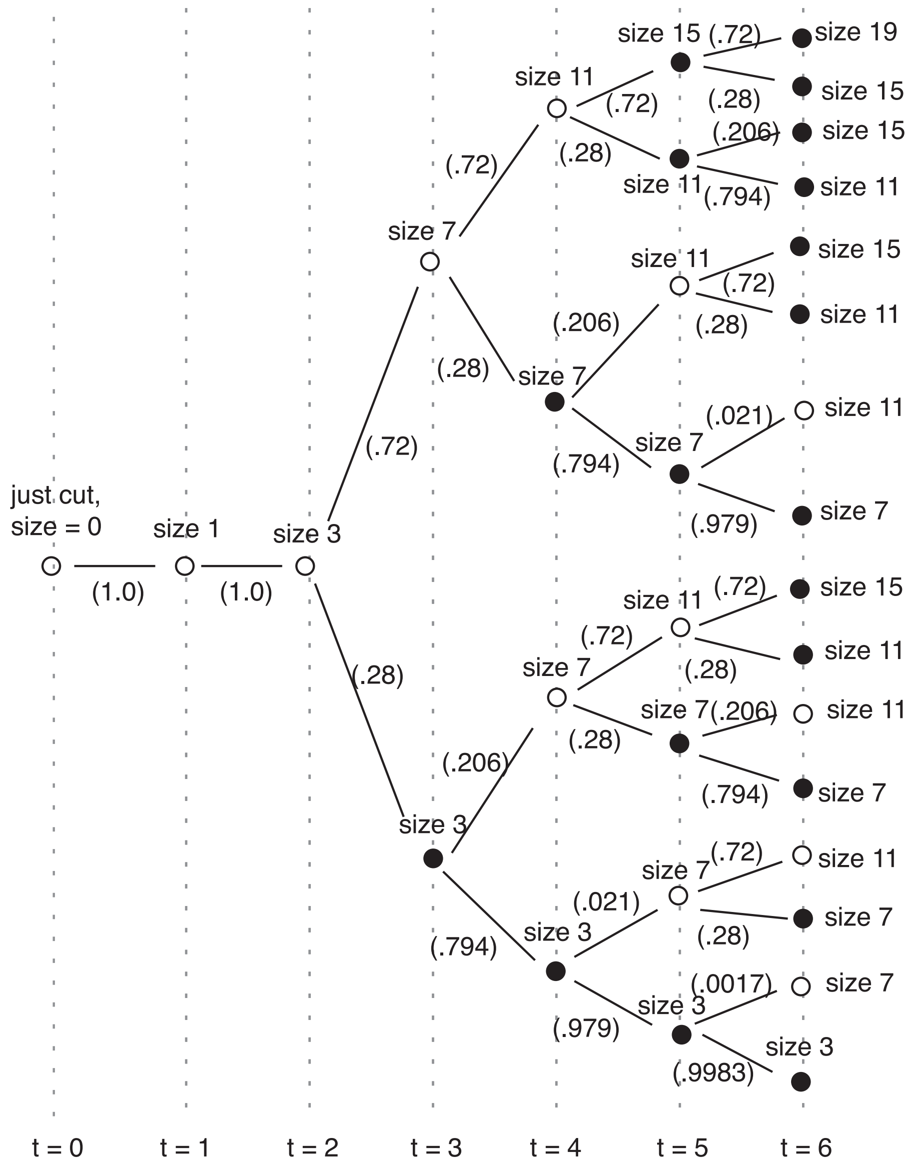

Figure GA6.1. The Optimal Strategy for Harvesting the Forest in Part d. An open node means to let the forest continue to grow, while a filled-in node means a position in which the forest is cut. Transition probabilities are indidcated on branches in parentheses.

This is just evaluating each branch that might occur, following this strategy. This equation, if you do the math, becomes

$$v = 5.14455557 + .64249978v \quad \text{which gives} \quad v = 14.3903562.$$

To evaluate one-step deviations from this strategy, we have to know the value of the hypothesized strategy from positions where you let the forest grow, according to the strategy. This includes situations where the forest has just grown and is of size 1, 3, 7, and 11.

- If the forest has just grown (so you know it is in growth mode) and is of size 11, you are supposed to leave it. It grows again with probability 0.72, and you get $14 + v$, and it fails to grow with probability .28, you cut it, and you net $10 + v$. So the value to you of being in this position is

$$.9\big[(.72)(14 + v) + (.28)(10 + v)\big] = 24.5433206.$$

- If the forest has just grown and is of size 7, you leave it. It grows again with probability 0.72, giving you a full continuation value of 24.5433206. It fails to grow with probability 0.28, so you cut it, netting $6 + v$. So the full value to you of being in this position is

$$.9\big[(.72)(24.5433206 + (.28)(6 + v)\big] = 21.0424415.$$

- If the forest is of size 3, you leave it. It grows to size 7 with probability .72, netting you the value just computed. It remains at size 3, in which case you are meant to cut it, netting $2 + v$. So your net value is

$$.9\big[(.72)(21.0424415) + (.28)(2 + v)\big] = 17.7658719.$$

- If the forest is of size 1, you leave it. It is sure to be size 3 next time, so your value in this position is $.9 \times 17.7658719 = 15.9892847$.

Now we can verify unimprovability. First we'll verify that, when the strategy says to let the forest go for another period, it isn't better to cut immediately. This involves the four situations just evaluated.

- If the forest is of size 11, having just grown, and you cut, you net $10+v = 24.3903562$, which is just slightly worse than the 24.5433206 you get by following the strategy.
- If the forest is of size 7, having just grown, and you cut, you net $6 + v = 20.3903562$, which is worse than following the strategy and getting 21.0424415.
- If the forest is of size 3, having just grown, cutting nets $2 + v = 16.3903562$. Following the strategy is better.
- If the forest is of size 1, cutting nets $v = 14.3903462$. Following the strategy is better.

Now for forests of all sizes 15 or more. The strategy says to cut immediately, which nets $n - 1 + v = 13.3903562 + n$, where n is the size of the forest. The alternative is to let the forest go a period, and at best it will grow by 4 units with probability 0.72 and not at all with probability 0.28, after which (reverting to the strategy) you cut. So the alternative at best yields

$$.9\big[(.72)(n + 17.3903562) + (.28)(n + 13.3903562)\big] = .9n + 14.6433206.$$

Following the strategy is at least as good as long as $.1n \geq 1.25296438$, or $n \geq 12.529$, which is certainly true.

For the smaller sized forests, the question is whether it is better to cut or to leave the forest after it has failed to grow. This only affects forests of size 11, 7, and 3.

- For a forest of size 11 that has just failed to grow, at best the odds of it growing next period is 0.2057. If you let it grow and then revert to strategy, you will cut it next period whether it grows or not. So letting it grow yields

  $$.9\big[(0.2057)(14 + v) + (0.7943)(10 + v)\big] = 22.6918406,$$

  while cutting it immediately nets $10 + 14.3903562$; cutting it immediately is better.

- For a forest of size 7 that has just failed to grow, the most optimistic probability of growth in the coming period is 0.2057, so at most the value of letting it grow is

  $$(.9)\big[(0.2057)(24.5433206) + (0.7943)(6 + v)\big] = 19.1201589,$$

  versus the 20.3903562 that you get from cutting it, so cutting it is better.

- And for a forest of size 3 that has just failed to grow, the most optimistic value of leaving it for a year (and reverting to the strategy) is

  $$(.9)\big[(0.2057)(21.0424415) + (0.7943)(2 + v)\big] = 15.6125611,$$

  versus the 16.3903562 that you get from cutting it, so cutting it is better.

Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Chapter 1: Choice, Preference, and Utility

This chapter discusses the basic microeconomic models of consumer choice, preference, and utility. It is very abstract, consisting primarily of proofs of mathematical (deductive) propositions. If you are rusty at reading (and constructing) mathematical proofs, it may be painful. If you are rusty, or if you haven't done anything like this before, please take it slow. Be sure to follow the details of the proofs one step at a time; it helps to have a pad and pen or pencil by your side, so you can follow along, make notes, finish arguments, and so forth. This recommendation extends to the entire book and, indeed, to any book or article you are reading that is mathematical in character. But it comes with a complementary recommendation: If you read carefully and slowly for details, you may lose the "plot line," which is just as important. So my suggestion is to read this sort of thing at least twice, with pad and pen or pencil each time: First, read to get the big picture. What is the framework? What are the results? How do the results tie together? And then go back and read for the details: How is each step done?

In a few places, I leave the proofs of propositions for you to complete; unless you are very confident in your ability to do this, you should write out proofs and have them checked by someone—a peer, a TA, your instructor—who is well versed in this skill. Constructing mathematical proofs is a skill you learn best—and perhaps only—by doing. (The proof of Proposition 1.19 will arrive in Chapter 2. Try it if you wish, but it takes considerable mathematical sophistication.)

On pedagogical grounds, it would be nice to begin with something more concrete. But this is the logical starting point for consumer theory, which in turn is the logical starting point of microeconomics. Persevere until Chapter 3, and you'll get to an application—the theory of the consumer—that isn't quite so abstract.

Summary of the Chapter

The chapter is about the standard economic model of consumer choice.

1. A set of *objects of choice*, X, is given.

2. A *choice function* c is given that, for each nonempty subset A of X, tells us the *set* of objects $c(A)$ the consumer would be content to have. We require that $c(A) \subseteq A$. We allow for the possibility that $c(A) = \emptyset$. The consumer gets only one element out of A; if $c(A)$ contains more than one element, the interpretation is that the consumer would be equally happy with any one. A more general formulation would have as domain of c some collection $\mathcal{A}$ of nonempty subsets of X, but for the balance of this chaper, we simplify by assuming that $c(A)$ is defined for *all* the nonempty subsets A of X, and we let $\mathcal{A}$ denote this domain.

3. The *preferences* of the consumer are specified by a binary relation $\succeq$, where $x \succeq y$ (for x and y from X) is read "x is as good as or better than y" or as "x is weakly preferred to y." Choice is generated by the preferences $\succeq$ if, for all A,

$$c(A) = \{x \in A : x \succeq y \text{ for all } y \in A\}. \tag{1.2}$$

   (Equation numbers are out of order here, so that they conform to the numbers in the text.)

4. A *utility function* for the consumer is a real-valued function $u : X \to R$, with the interpretation that the consumer regards items of higher utility as better. In accordance with this interpretation, we say that u represents the preference relation $\succeq$ if

$$x \succeq y \quad \text{if and only if} \quad u(x) \geq u(y). \tag{1.3}$$

   And the choice function c is generated by utility maximization with the utility function u if, for all A,

$$c(A) = \{x \in A : u(x) \geq u(y) \text{ for all } y \in A.\} \tag{1.1}$$

Most economic models have consumers who are utility maximizers or, at least, preference driven. The point of the chapter, then, is to say when choice behavior, given by a choice function c, is generated by preferences or by utility maximization for some utility function. The basic answers, in the context of a finite set X, are given by Definition 1.1 and Proposition 1.2:

Definition 1.1.

a. *A choice function c satisfies* ***finite nonemptiness*** *if $c(A)$ is nonempty for every finite $A \in \mathcal{A}$.*

b. *A choice function c satisfies **choice coherence** if, for every pair x and y from X and A and B from $\mathcal{A}$, if $x, y \in A \cap B$, $x \in c(A)$, and $y \notin c(A)$, then $y \notin c(B)$.*

c. *A preference relation on X is **complete** if for every pair x and y from X, either $x \succeq y$ or $y \succeq x$ (or both).*

d. *A preference relation on X is **transitive** if $x \succeq y$ and $y \succeq z$ implies that $x \succeq z$.*

Proposition 1.2. *Suppose that X is finite.*

a. *If a choice function c satisfies finite nonemptiness and choice coherence, then there exist both a utility function $u : X \to R$ and a complete and transitive preference relation $\succeq$ that produce choices according to c via the formulas (1.1) and (1.2), respectively.*

b. *If a preference relation $\succeq$ on X is complete and transitive, then the choice function it produces via formula (1.2) satisfies finite nonemptiness and choice coherence, and there exists a utility function $u : X \to R$ such that*

$$x \succeq y \text{ if and only if } u(x) \geq u(y). \tag{1.3}$$

c. *Given any utility function $u : X \to R$, the choice function it produces via formula (1.1) satisfies finite nonemptiness and choice coherence, the preference relation it produces via (1.3) is complete and transitive, and the choice function produced by that preference relation via (1.2) is precisely the choice function produced directly from u via (1.1).*

In words, choice behavior (for a finite X) that satisfies finite nonemptiness and choice coherence is equivalent to preference-maximization (that is, formula (1.2)) for complete and transitive preferences, both of which are equivalent to utility-maximization (via formulas (1.1) and (1.3)). Whether expressed in terms of choice, preference, or utility, this conglomerate (with the two pairs of assumptions) is the standard model of consumer choice in microeconomics.

The chapter goes on to prove and generalize Proposition 1.2, and to provide complements to it. This includes the following:

1. Those results that extend automatically to infinite sets X are extended.

2. For a complete and transitive preference relation $\succeq$, strict preference $\succ$ is defined by $x \succ y$ if $x \succeq y$ and not $y \succeq x$, and indifference $\sim$ is defined by $x \sim y$ if $x \succeq y$ and $y \succeq x$. Properties of these two relations are derived, and the derivation of $\succeq$ from $\succ$ is discussed.

3. Define the *no better than* x set NBT(x) := $\{y \in X : x \succeq y\}$. If $\succeq$ is complete and transitive, then $x \succeq y$ if and only if NBT(y) $\subseteq$ NBT(x), with strict set inclusion if and only if $x \succ y$. The no-better-than sets are used in many ways; in particular, they allow the construction of utility functions that represent $\succeq$.

4. Utility representations for infinite sets X are considered. Necessary and sufficient conditions on a complete and transitive binary relation on (infinite) X to have a

utility representation are provided; then the important special case of *continuous preferences* when X is a (nice) subset of finite dimensional Euclidean space (R^k) is discussed in detail.

5. The phenomenon of $c(A) = \emptyset$ for infinite sets A is discussed.
6. The relationship between two different utility functions for the same preferences is given, making in particular the point that utility numbers (in this chapter) have only ordinal and not cardinal significance.
7. A number of comments, extensions, variations, and criticisms of the standard model are provided.

Solutions to Starred Problems

■ 1.1. In case you had problems producing a counterexample, consider the four bottles x = California Red for \$20, x' = French white for \$20, x'' = California Red for \$25, and x''' =French red for \$30. With this list of bottles, you can produce a counterexample to the choice coherence axiom, using either two wine lists of three bottles apiece, or one having three bottles and another having two.

Here is one example:

From the wine list $L_1 = \{x, x', x''\}$, my friend's choice algorithm proceeds as follows: Two California bottles and one French, so take a California bottle. Two California reds, so take a red. Take the most expensive California Red, which is x''. Thus $c(\{x, x', x''\}) = \{x''\}$.

From the wine list $L_2 = \{x', x'', x'''\}$, he reasons: One California and two French, so take one of the French. Of the two bottles of French wine, one is white and one red. He must invoke his tie-breaking rule, which leads him to choose white. Since there is only one bottle of French white on the list, he chooses that: $c(\{x', x'', x'''\}) = \{x'\}$.

This constitutes a violation of choice coherence. Both x' and x'' are available on both wine lists, and x' but not x'' is chosen from the second list while x'' and not x' is chosen from the first.

■ 1.3. It is easy to see that $\succeq^*$ is complete: For any x and y, since $\succeq_{\text{Larry}}$ is complete, either $x \succeq_{\text{Larry}} y$ or $y \succeq_{\text{Larry}} x$, which immediately imply $x \succeq^* y$ and $y \succeq^* x$, respectively.

To see that $\succeq^*$ is not transitive, suppose that X has three elements, x, y, and z. Suppose Larry ranks the three $x \succ_{\text{Larry}} y \succ_{\text{Larry}} z$, while Moe ranks them $y \succ_{\text{Moe}} z \succ_{\text{Moe}} x$. Then $z \succeq^* x$, because Moe likes z at least as much as x. And $x \succeq^* y$, because Larry likes x at least as much as y. But it is not true that $z \succeq^* y$, because both Larry and Moe think that y is strictly better than z.

■ 1.6. (a) Suppose that $x \succeq y$ and $y \succeq z$ for $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$, and

$z = (z_1, \ldots, z_k)$. Then $x_i \geq y_i$ and $y_i \geq z_i$ for each i. Hence, by transitivity of $\geq$ for real numbers, $x_i \geq z_i$ for all i, and thus $x \succeq z$. This shows that $\succeq$ is transitive.

On the other hand, for $k = 2$, $x = (1,2)$, and $y = (2,1)$, neither $x \succeq y$ nor $y \succeq x$; $\succeq$ is not complete.

(b) $x \succ y$ if $x \succeq y$ and not $y \succeq x$, which is $x_i \geq y_i$ for all i, and not $y_i \geq x_i$ for all i, which is $x_i \geq y_i$ for all i, and $x_i > y_i$ for some i.

This is asymmetric: If $x \succ y$, then $x_i > y_i$ for some i. Thus neither $y \succeq x$ nor $y \succ x$ are possible.

But this is not negatively transitive: Take $x = (2,2)$, $z = (1,1)$, and $y = (3,0)$. We have $x \succ z$, but neither $x \succ y$ nor $y \succ z$ is true.

(c) $x \sim y$ if $x \succeq y$ and $y \succeq x$, which is $x_i \geq y_i$ and $y_i \geq x_i$ for all i, which is $x_i = y_i$ for all i. Thus we have

$$x \sim y \text{ if and only if } x = y.$$

This is clearly reflexive, symmetric, and (trivially) transitive.

■ 1.7. As suggested in the hint, the first thing to do is to characterize what *not* $y \succeq x$ means. For $y \succeq x$, two (or more) of y's three components must be at least as large as the corresponding components of x. For this to fail, two (or more) of those three components must be strictly less than the corresponding components of x. Thus

not $y \succeq x$ is equivalent to
two or more of x*'s components strictly exceed* y*'s corresponding components.*

(If you couldn't solve this problem because you didn't get this far, assume the characterization above and try parts (a) and (b) again.) If you aren't satisfied with this verbal argument, a very formal argument can be given, but it is gruesome. Here it is:

(i) $y \succeq x$ by definition is $[y_1 \geq x_1$ and $y_2 \geq x_2]$ or $[y_1 \geq x_1$ and $y_3 \geq x_3]$ or $[y_2 \geq x_2$ and $y_3 \geq x_3]$.

(ii) *Not [a or b or c]* is *[not a] and [not b] and [not c]*, so *not* $y \succeq x$ is

$$[\text{not } [y_1 \geq x_1 \text{ and } y_2 \geq x_2]] \text{ and } [\text{not } [y_1 \geq x_1 \text{ and } y_3]] \text{ and } [\text{not } [y_2 \geq x_2 \text{ and } y_3 \geq x_3]].$$

(iii) The negation of *a and b* is *not a or not b*, and the negation of $a \geq b$ is $b > a$, for real numbers a and b, so *not* $y \succeq x$ is

$$[x_1 > y_1 \text{ or } x_2 > y_2] \text{ and } [x_1 > y_1 \text{ or } x_3 > y_3] \text{ and } [x_2 > y_2 \text{ or } x_3 \geq y_3].$$

(iv) This, in turn, has the form $[\alpha$ or $\beta]$ and $[\alpha$ or $\gamma]$ and $[\beta$ or $\gamma]$. Either by considering the two cases α and not α or by constructing a Venn diagram, you can show that this is the same as [α and β] or [α and γ] or [β and γ]; i.e., two of the three must be true. Translating this back to the components of x and y, this is the desired conclusion.

As promised, this is rather gruesome, and you probably came to the correct conclusion without all these details, but if you are a freak for mathematical rigor, the mess just previous should make you happy.

(a) With this result, however obtained, the problem is easy. First, to show that $\succeq$ is complete, take any x and y. If it is not the case that $y \succeq x$, then x strictly exceeds y in at least two components, and so $x \succeq y$. Thus $\succeq$ is complete.

To show that $\succeq$ is not transitive, an example will do: $(2,2,1) \succeq (1,1,3)$, and $(1,1,3) \succeq (3,0,2)$, but it is not true that $(2,2,1) \succeq (3,0,2)$.

(b) Since not $y \succeq x$ implies $x \succeq y$ (see just previously), $x \succ y$ is equivalent to not $y \succeq x$, which we saw means that x strictly exceeds y in at least two components. This is clearly asymmetric: If x exceeds y in at least two components, then y can strictly exceed x in at most one. But negative transitivity fails, and the same example as we used before will work: $(2,2,1) \succ (1,1,3)$, but if the third bundle is $(3,0,2)$, then neither $(2,2,1) \succ (3,0,2)$ nor $(3,0,2) \succ (1,1,3)$.

■ 1.10. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. If $x_1 \neq y_1$, then either $x_1 > y_1$, in which case $x \succeq y$, or $y_1 > x_1$, in which case $y \succeq x$. And if $x_1 = y_1$, then either $x_2 \geq y_2$, implying $x \succeq y$, or $y_2 \geq x_2$, implying $y \succeq x$. Thus $\succeq$ is complete.

Suppose that $x = (x_1, x_2) \succeq y = (y_1, y_2)$, and $y \succeq z = (z_1, z_2)$. Since $x \succeq y$, either $x_1 > y_1$ or [$x_1 = y_1$ and $x_2 \geq y_2$]. Similarly, $y \succeq z$ implies either $y_1 > z_1$ or [$y_1 = z_1$ and $y_2 \geq z_2$]. It is boring, but the easiest way to proceed is to take all four = two-by-two cases seriatum:

Case 1: $x_1 > y_1$ and $y_1 > z_1$. In this case $x_1 > z_1$, so $x \succeq z$.

Case 2: $x_1 > y_1$ and [$y_1 = z_1$ and $y_2 \geq z_2$]. In this case $x_1 > z_1$, so $x \succeq z$.

Case 3: [$x_1 = y_1$ and $x_2 \geq y_2$] and $y_1 > z_1$. In this case $x_1 > z_1$, so $x \succeq z$.

Case 4: [$x_1 = y_1$ and $x_2 \geq y_2$] and [$y_1 = z_1$ and $y_2 \geq z_2$]. In this case $x_1 = z_1$ and $x_2 \geq z_2$, so $x \succeq z$.

In all four possible cases, we conclude $x \succeq z$, so $\succeq$ is transitive.

To show that there is no numerical representation, we could prove that no countable set X^* as is required for a numerical representation can be found. This is relatively easy to do: For every real number $r \in [0,1]$, consider the points $x_r = (r, 0.8)$ and $y_r = (r, 0.2)$. By the definition of the preference relation, $x_r \succ y_r$, so if a set X^* existed, it would have to contain a point x_r^* that lies between these two, perhaps tied with x_r. The set of candidates for x_r^* is $\{(r,q) : 0.8 \geq q > 0.2\}$. But this implies that for any two different real numbers r and r', $x_r^* \neq x_{r'}^*$, and since there are uncountably many r, the set X^* must have an uncountable number of elements.

An alternative proof is a bit more direct. Assume that u is a numerical representation for $\succeq$. Since for every $r \in [0,1]$, $x_r = (r, 0.8) \succ (r, 0.2) = y_r$, it follows that $u(x_r) > u(y_r)$. But then, since the rationals are dense in the real line, it follows that for every $r \in [0,1]$, there is a rational number q_r in the open interval $(u(y_r), u(x_r))$. This would constitute a one-to-one map from the the unit interval $[0,1]$ onto the rational numbers, which of course cannot be, since there are uncountably many elements of $[0,1]$ and only countably many rationals.

■ 1.11. The first step is to show that *c holds if and only if d holds.* Note that if $\succeq$ is complete and transitive, then for all pairs x and y, either $x \succeq y$ or $y \succ x$, but never both. Therefore, the sets NBT(x) = $\{y \in R^k_+ : x \succeq y\}$ and SBT(x) = $\{y \in R^k_+ : y \succ x\}$ are complements, and NWT(x) and SWT(x) are complements. Hence, the sets NBT(x) and NWT(x) are both closed if and only if their complements, SBT(x) and SWT(x) are (relatively) open.

Next I'll show that *c and d imply the original definition*: (This is by far the longest step.) Suppose $x \succ y$. I need to produce a w such that $x \succ w \succ y$. Here is one way to do it: Look at all convex combinations of x and y, $ax + (1-a)y$, for $a \in [0,1]$. Since c holds for $\succeq$, the set NWT(x) is closed, and therefore $NWT(x) \cap \{ax + (1-a)y : a \in [0,1]\}$ is a closed set. This intersection contains $a = 1$ and does not include $a = 0$ and, being closed, it contains its infimum (in terms of a); that is, if we let $a^* = \inf\{a \in [0,1] : ax + (1-a)y \succeq x\}$, we know that $a^*x + (1-a^*)y \succeq x$, $a^* > 0$, and, for all $a \in [0, a^*)$, $x \succ ax + (1-a)y$. Since the set NBT(x) is also closed, and $a^*x + (1-a^*)y$ can be approached by points all in NBT(x) (namely, $ax + (1-a)y$ for $a < a^*$), we know that $x \succeq a^*x + (1-a^*)y$; we conclude that $a^*x + (1-a^*)y \sim x \succ y$. Let z denote $a^*x+(1-a^*)y$. Now repeating the argument (but with inequalities reversed), we can find a $b^* < 1$ such that $b^*z + (1-b^*)y \sim y$ and $bz + (1-b)y \succ y$ for all $b \in (b^*, 1]$. Let $w = bz + (1-b)y$ for any $b \in (b^*, 1)$; note that w is a convex combination of x and y with weight less than a^* on x, so putting everything together, we know that $x \succ w \succ z$.

Now that I have w, the rest of this step is easy. I know that $x \in$ SBT(w) and this set is open, there is an open neighborhood of x, say all x' within $\epsilon_1 > 0$ of x, that is in SBT(w). And I know that $y \in$ SWT(w) and this set is open, so for some $\epsilon_2 > 0$, every y' within ϵ_2 of y is in SWT(w). But then letting $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, for all x' within ϵ of x and y' within ϵ of y, $x' \succ w \succ y'$, and by transitivity, $x' \succ y'$.

Now to show that *the definition implies b*: Suppose $\{x_n\}$ is a sequence with limit x and $x \succ y$. Per the definition, we can find $\epsilon > 0$ so that all x' within ϵ of x and all y' within ϵ of y, $x' \succ y'$. Taking $y' = y$, this tells us that all x' within ϵ of x satisfy $x' \succ y$. But since the sequence has limit x, for all sufficiently large n, x_n will be within ϵ of x. The other half is similar.

Next, I'll show that *b implies a*. Suppose that b holds, $\{x_n\}$ is a sequence with limit x, and $x_n \succeq y$ for all n If it is not true that $x \succeq y$, then $y \succ x$ must be true. But if b holds, then $y \succ x$ and $\lim_n x_n = x$, then it must be that $y \succ x_n$ for all sufficiently large n. To the contrary, the assumption is that $x_n \succeq y$ for all n. We've derived a contradiction to the assumption that $x \not\succeq y$. The argument for the other half is similar.

To conclude, I have to show that *a implies c and d*. It should be clear that showing *a implies c* is the way to go: To prove that NBT(x) is closed, I'll show that it contains all its limit points: Suppose $\{y_n\}$ is a sequence drawn from NBT(x) with limit y. Then $x \succeq y_n$ for each n. But then a tells us that $x \succeq y$, and $y \in$ NBT(x). The other half is similar.

■ 1.13. The proposition concerns the choice function c, so the first step is to note that since c satisfies nonemptiness, choice coherence, and Assumption 1.16, Proposition 1.17 tells us the $c \equiv c_{\succeq_c}$. So to show that $c(A) \neq \emptyset$ for every compact A, we must show that, for each compact A, $c_{\succeq_c}(A) = \{x \in A : x \succeq_c y \text{ for all } y \in A\}$ is nonempty. (We know, of course, that $\succeq_c$ is complete and transitive, and the problem tells us to assume that $\succeq_c$ is continuous.)

So suppose, by way of contradiction, that for some compact set A, $\{x \in A : x \succeq_c y$ for all $y \in A\}$ is empty. That is, for every $x \in A$, there is some $y \in A$ such that $x \not\succeq_c y$, or $y \succ_c x$. This means that for every $x \in A$, there is some $y \in A$ such that $x \in \text{SWT}(y) = \{z \in X : y \succ_c z\}$. So if we look at the union $\cup_{y \in A} \text{SWT}(y)$, this union contains all of A; the sets SWT(y) for $y \in A$ constitute a *cover* of A.

But Proposition 1.14 tells us that, if preferences $\succeq$ are continuous, then the strictly worse than y sets SWT(y) are all (relatively) open. Among the characterizations of compactness—in some sense, the basic characterization—is that for any compact set, every open cover of the set has a finite subcover. That is, for some finite collection $\{y_1, \ldots, y_n\}$ from A, A is a subset of the union of the SWT(y_j). We know that $c_{\succeq_c}(\{y_1, \ldots, y_n\})$ is nonempty from the finite nonemptiness property, so there is some $k = 1, \ldots, n$ such that $y_k \succeq_c y_j$ for $j = 1, \ldots, n$. But this $y_k \in A$, so it must be in some SWT(y_j) for some j. This is a contradiction; $y_k \in \text{SWT}(y_j)$ means that $y_j \succ_c y_k$. But y_k was chosen to satisfy $y_k \succeq_c y_j$ for all j. We have the desired contradition; there must be some $x \in A$ such that $x \succeq_c y$ for all $y \in A$, and $c(A) = c_{\succeq_c}(A)$ is nonempty. (Where in this proof did I use the very necessary assumption that A is nonempty?)

■ 1.16. (a) The idea here is simple, once you see the trick. For any set X, consider the weak preferences $\succeq^0$ given by $x \succeq^0 y$ for all x and y in X. That is, everything is weakly preferred to everything, thus the consumer is indifferent among all options. For these preferences, $c_{\succeq^0}(A) = A$ for all A. And as long as we see the consumer making a single choice from any set A, we have no evidence against the theory. If we can't infer something about strict preference from the observable data, the theory has no implications.

One way we learn about strict preferences from the data is if the data purport to show, for each A, the *entire* set $c_{\succeq}(A)$. Then if $x \notin c_{\succeq}(A)$, we infer that $y \succ x$ for every $y \in c_{\succeq}(A)$. This is the case we deal with in part (b) of this problem. Another way is coming in Chapter 4; as foreshadowing for this, I describe this alternative, although my description may not make sense to you until you get to Chapter 4: In Chapter 4, we will make inferences about strict preferences under a joint hypothesis that the consumer's observed choices are preference-driven *and* the consumer is locally insatiable. Therefore if there is a ball of positive diameter around the point x, all of which is contained in A, and if what is chosen from A is some distance from x, then we know that there is something in A that is strictly preferred to x (local insatiability) which was not chosen, hence what was chosen is at least as good as something that is strictly preferred to x, and thus the thing chosen must be strictly better than x.

(b) Suppose $X = \{x, y, z\}$, and we observe the following choices out of the three two-element sets:

$$c(\{x, y\}) = \{x\}, \quad c(\{y, z\}) = \{y\}, \quad \text{and} \quad c(\{x, z\}) = \{z\}.$$

(If you didn't get this far, see if you can finish the argument from here. You have to show (a) that there are no direct violations of choice coherence in these data, and (b) these data are inconsistent with preference–driven choice; i.e., there is no complete and transitive $\succeq$ that, if used to choose, would produce these data.)

The argument that there is no direct violation of choice coherence in these data is: We never have two distinct sets A and B and two distinct elements w and v such that both w and v are in both A and B. Since the *if* part of Houthakker's axiom is never satisfied by these data, the axiom has no content for these data.

Nonetheless, if these data could be explained by some complete and transitive preferences $\succeq$, $c(\{x, y\}) = \{x\}$ would mean that $x \succ y$ must be true, $c(\{y, z\}) = \{y\}$ would imply $y \succ z$, and $c(\{x, z\})$ would imply $z \succ x$. Since strict preference is transitive (Proposition 1.9), this would imply $x \succ x$, which violates the asymmetry of $\succ$. Hence these data are inconsistent with our standard model of choice driven by complete and transitive weak preferences.

(c) Fixing X and c, define (for $A \subseteq X$) $b(A) = A \setminus c(A)$. That is, $b(A)$ is the set of "bad" (really, less than best) elements out of A.

Suppose the data are consistent with choice according to some complete and transitive $\succeq$. Then $x \succeq^r y$ implies $x \succeq y$: if $x \succeq^r y$, then for some k, $x \in c(A_k)$ while $y \in A_k$. But if c is consistent with choice according to $\succeq$, $x \in c(A_k)$ implies that $x \succeq z$ for all $z \in A_k$, and this includes y.

Moreover, $x \succ^r y$ implies $x \succ y$: $x \succ^r y$ implies that, for some A_k containing both x and y, $x \in c(A_k)$ but $y \notin c(A_k)$. The former implies that $x \succeq z$ for all $z \in A_k$. Now if $x \not\succ y$, then $y \succeq x$, and by transitivity of $\succeq$, $y \succeq z$ for all $z \in A_k$, which would imply $y \in c(A_k)$, a contradiction.

So suppose the data violate SGARP. This means there is some set $\{x_1, \ldots, x_m\}$ that $x_i \succeq^r x_{i+1}$ for $i = 1, \ldots, m-1$ and $x_m \succeq^r x_1$. But then by the previous two paragraphs, $x_i \succeq x_{i+1}$ for $i = 1, \ldots, m-1$, and so by transitivity of $\succeq$, $x_1 \succeq x_m$, which contradicts $x_m \succ x_1$. Any violation of SGARP rules out the possibility that $c(\cdot)$ can be rationalized by a complete and transitive $\succeq$, which is the first half of Proposition 1.23.

The second half of the proposition, that no violations of SGARP means that the data are consistent with some $\succeq$, is a good deal harder. Because it is easy to get lost in the details of the proof, I will take it in steps, by first proving and then applying an abstract lemmas.

The lemma concerns a finite set of objects K, on which is defined a pair of binary relations, P and I. A binary relation (in case you don't know) is a mathematical object that concerns pairs of elements of a given set. For k and k' from K, we write kPk' if k stands in relation P to k', and we write *not* kPk' if not. Examples of binary relations are weak preference, strict preference, and indifference. But there are many others, such as: If K is the set of all students in a class, we might define a binary relation B by kBk' if k is the brother of k'. Note that order is important; it is certainly possible that kBk' and not $k'Bk$ (if, for example, k' is k's sister). Or, to take another example, in the binary relation $\succ$, order is crucial. In fact, $\succ$ is asymmetric, meaning that $x \succ y$ implies that *not* $y \succ x$.

The binary relations P and I on the finite set K have the following properties:

Property 1: *kPk' implies not kIk'.* (By contraposition, the reverse is true as well.)

Property 2: *I is reflexive. That is, for all $k \in K$, kIk.* (Note that this, together with property 1, implies that for no k is it true that kPk.)

Property 3: *I is symmetric. That is, for all k and $k' \in K$, kIk' implies $k'Ik$.*

Property 4: *(a) Both I and P are transitive; (b) kPk' and $k'Ik''$ implies kPk''; and (c) kIk' and $k'Pk''$ implies kPk''.*

(If you need a concrete example to think about, think of P as something like revealed strict preference and I as revealed indifference, or see below.)

Lemma G1.1. *Suppose that binary relations P and I on a finite set K satisfy properties 1 through 4. Then there exists a function $V : K \to R$ such that kPk' implies $V(k) > V(k')$ and kIk' implies $V(k) = V(k')$.*

This is like numerical representation of P and I, except that the implications run one way only.

Proof of the lemma. For each $k \in K$, let

$$\mathbf{W}(k) = \{k'' \in K : kPk''\},$$

and let $V(k)$ be the number of elements in $\mathbf{W}(k)$.

Suppose kIk'. Then $k'Ik$ (by property 3) and so if $k'' \in \mathbf{W}(k)$, kPk'' and hence $k'Pk''$ (by property 4(c)). Thus $k'' \in \mathbf{W}(k')$. By the symmetric argument, if kIk', then $\mathbf{W}(k) =$

$\mathbf{W}(k')$, and therefore $V(k) = V(k')$.

Suppose kPk'. Then $k' \in \mathbf{W}(k)$ by definition. We know (see the parenthetical remark in property 2) that not $k'Pk$, and thus $k \notin \mathbf{W}(k')$. Moreover, for all $k'' \in \mathbf{W}(k')$, $k'Pk''$ and thus by property 4(a), kPk''. So $\mathbf{W}(k')$ is a strict subset of $\mathbf{W}(k)$, and $V(k) > V(k')$. ∎

Now we return to the proof of the second half of Proposition 1.23. Recall where we are: We know $c(A_k)$ for a finite number of sets A_k, $k = 1, 2, \ldots, n$, and we know that these data admit no violations of SGARP.

To apply the lemma, we let $K = \{1, 2, \ldots, n\}$, and we define:

(1) kIk', if there is a finite sequence $k = k_1, \ldots, k_m = k'$ such that $c(A_{k_i}) \cap c(A_{k_{i+1}}) \neq \emptyset$, for $i = 1, \ldots, m-1$.

(2) kPk', if there is a finite sequence $k = k_1, \ldots, k_m = k'$ such that $c(A_{k_{i+1}}) \cap A_{k_i} \neq \emptyset$, for $i = 1, \ldots, m-1$, and $c(A_{k_{i+1}}) \cap b(A_{k_i}) \neq \emptyset$ for at least one i.

We must show that properties 1 through 4 hold in this case:

It is easiest to begin with reflexivity and symmetry of I; i.e., properties 2 and 3. The symmetry of I is clear, because the definition of I is symmetric in k and k'. As for reflexivity, take $m = 1$ (so there is a single element in the sequence) and apply the definition trivially.

For property 1, suppose kPk' and kIk'. If kPk', there is a finite sequence $k = k_1, \ldots, k_m = k'$ such that $c(A_{k_{i+1}}) \cap A_{k_i} \neq \emptyset$, for $i = 1, \ldots, m-1$, and $c(A_{k_{i+1}}) \cap b(A_{k_i}) \neq \emptyset$ for at least one i. Let x_{k_i} be the element of $c(A_{k_{i+1}}) \cap A_{k_i}$ for all i and the element of $c(A_{k_{i+1}}) \cap b(A_{k_i})$ for at least one i. Let x_{k_1} be any element of $c(A_{k_1})$. Then $x_{k_i} \succeq^r x_{k_{i+1}}$ for all i, and $x_{k_i} \succ^r x_{k_{i+1}}$ for the distinguished i. We can similarly use $k'Ik$ to construct a sequence of revealed weak preferences from x_{k_m} back to x_{k_1}. But putting the two sequences of revealed weak preferences, one with a revealed strict preference in at least one step, would be a violation of SGARP.

Property 4 holds by construction. All four forms of "transitivity" called for in the property involve stringing together pairs of sequences used to define I and P and noting that: Two sequences that define I can be strung together to give another that defines I, and any two, as long as one has a strict revealed preference (i.e., is used for P) when strung together gives a sequence that defines P.

Hence we can apply the lemma and know that there is a function $V : K \to R$ such that kPk' implies $V(k) > V(k')$ and kIK' implies $V(k) = V(k')$. Let L be any number strictly less than $V(k)$ for all k. Define $U : X \to R$ by

$$U(x) = \begin{cases} V(k), & \text{if } x \in c(A_k) \text{ for some } k \text{, and} \\ L, & \text{if } x \notin c(A_k) \text{ for all } k. \end{cases}$$

We must be sure that this is well-defined; i.e., if $x \in c(A_k) \cap c(A_{k'})$, then $V(k) = V(k')$. But if $x \in c(A_k) \cap c(A_{k'})$, then kIk', and $V(k) = V(k')$ follows.

Now define $\succeq$ as the weak preferences given by U. Obviously, $\succeq$ is complete and transitive. We are done if we show that for each k, $c(A_k) = c_{\succeq}(A_k)$.

To do this, fix A_k. The set $c_{\succeq}(A_k)$ contains all those elements of A_k that have the highest values according to U. By construction, all elements of $c(A_k)$ have the same value, namely $V(k)$. So we need only show that no $x \in b(A_k)$ has higher value than $V(k)$. But this is easy. If $x \notin c(A_{k'})$ for any other k', then $U(x) = L < V(k)$. Suppose $x \in c(A_{k'})$ Since $x \in b(A_k)$, we know that kPk'. Thus $U(x) = V(k') < V(k)$. Done. ∎

You are probably exhausted from all this work, but let me make a few remarks. First, the idea is not that hard: Because of SGARP, we are able to induce a revealed preference ordering among the $c(A_k)$. We select a utility function that reflects that ordering, giving everything that is never selected some utility less than anything that is ever selected (in the data). The hard part is in getting the right definition for the revealed preference ordering and showing that this is enough to produce a numerical representation. The structural properties needed to produce a numerical representation are properties 1 through 4, and the lemma and proposition establish the existence of the ordering. The definitions of I and P in this case, and the demonstration that SGARP implies property 1 for this definition, show that these structural properties hold.

Things are slightly easier if you assume that X is finite, as then you can work directly with X in the role of K. If you were able to do that much, you were doing quite well.

The adjective "simple" will be removed from GARP in Chapter 4, where we reconsider this result (and extend it) for the case of demand data.

Microeconomic Foundations I
Choice and Competitive Markets

Student's Guide

Chapter 2: Structural Properties of Preferences and Utility Functions

Summary of the Chapter

This chapter concerns structural properties that preferences and their utility functions might have. The setting, for the sake of definiteness, is $X = R^k_+$, but many of the results extend to more general spaces. The chapter provides three types of results:

1(a). *If preferences $\succeq$ have property A, then **every** utility function u that represents $\succeq$ has property Z.*

1(b). *If preferences $\succeq$ have property A, then **some** utility function u that represents $\succeq$ has property Z.*

2. *If preferences $\succeq$ are represented by a utility function u that has property Z, then $\succeq$ has property A.*

The categories for property A include three of the most important sets of structural assumptions in economic theory, namely *monotonicity*, *convexity*, and *continuity*. It also includes properties that are useful in the analysis of specific economic problems; the buzzwords here are *separability*, *quasi-linearity*, and *homotheticity*.

A summary of the results of this form is given in Table G2.1. (This is the solution to Problem 2.1 from the text; you might want to try that problem on your own before consulting this figure.) For example, the first line reads *Preferences are monotone implies that **all** utility function representations (of those preferences) are nondecreasing* and *Preferences are monotone is implied if **some** utility function representation is nondecreasing.* I give

Preferences are	implies	is implied if	utility function representation(s) is/are	
monotone	all	some	nondecreasing	Def. 2.1; Prop. 2.2
strictly monotone	all	some	strictly increasing	
convex	no implication	some	concave	Defs. 2.4 & 2.7; Prop. 2.8(a)
strictly convex	(see note 1)	some	strictly concave	
convex	all	some	quasi-concave	Defs. 2.4 & 2.7; Prop. 2.8(b)
semi-strictly convex	all	some	semi-strictly quasi-concave	
strictly convex	all	some	strictly quasi-concave	
continuous	some	some	continuous	Def. 1.13; Prop. 2.9 (Debreu's Theorem)
weakly separable	all	some	weakly separable form (see note 2)	Def. 2.11; Prop. 2.12
strongly separable	some (see n. 6)	some	additively separable (see note 3)	Def. 2.13; Prop. 2.14
such that they satisfy the 3 properties in Prop. 2.16	some	some	quasi-linear form (see note 4)	Def. 2.15; Prop. 2.16
continuous and homothetic (see note 5)	some	some	continuous and homogeneous	Defs. 2.17 & 2.18; Prop. 2.19

Table G2.1. The heart of Chapter 2. (See notes below.)

Notes: 1. Do not attempt to fill in the box marked ????. You were not given this information in the chapter. (In fact, I don’t know what is correct to put in this box.)
2. u takes the form $v(u_1(x_{J_1}), u_2(x_{J_2}), \ldots, u_N(x_{J_N}), x_{K\setminus(J_1\cup\ldots\cup J_N)})$ for v strictly increasing in its first N arguments.
3. u takes the form $\sum_{n=1}^{N} u_n(x_{J_n})$.
4. U takes the form $u(x) + m$.
5. In this row, a maintained hypothesis is that preferences are continuous.
6. Proposition 2.14, which is the basis for this “some,” also requires that preferences are continuous and that there are at least three nontrivial dimensions.

the results in the table and, in the right-hand margin, I refer you to the appropriate definitions and propositions from the text.

Note that in the direction of implication from the utility function to the representation—results of type 2—it is always the case that if *some* representation has the property, preferences have the corresponding property. But in the other direction, sometimes the implication is that *all* representations have the corresponding property, and sometimes it is that *some* representation does so. (Note that for convexity of preferences and concavity of the utility function, we don’t even have *some*.) In this regard, you should convince yourself that if there is an *all* in the second column, then the property in the fourth column is preserved under strictly increasing rescalings (and vice versa). That is, if $u : X \to R$ is a nondecreasing function, and if $f : R \to R$ is strictly increasing, then v defined by $v(x) = f(u(x))$ is also nondecreasing, because in the second column of row 1 we have an *all*. While in the second column of row 8, we have *some*, so we know that continuity of u is not necessarily preserved if we take a strictly increasing rescaling.

These are useful results primarily because the properties in the fourth column, properties of a representing utility function, are convenient for proving results. Take Proposition 1.19, for example, which states that if preferences are continuous, choice from a nonempty compact set A is nonempty. This can be proved from first principles based on the definition of continuous preferences, but it is easier to go by way of a continuous representation and the standard result from mathematics that a continuous function achieves its maximum on a nonempty compact set. The point is, if we want to assume that a utility representation of preferences has a property such as continuity, quasi-concavity, or quasi-linearity, we must answer the question, What does this entail about the underlying preferences? The results summarized in Figure G2.1 give us answers.

Why do we care? Continuity and convexity are important because they provide us with very useful characterizations of the set of solutions of (constrained) choice problems:

- If preferences $\succeq$ are continuous and A is a nonempty and compact set, then $c_{\succeq}(A)$ is nonempty (Proposition 1.19).
- If preferences $\succeq$ are convex and A is a convex set, then $c_{\succeq}(A)$ is a convex set (Proposition 2.6).
- If preferences $\succeq$ are strictly convex and A is a convex set, then $c_{\succeq}(A)$ contains either one element or is empty (also Proposition 2.6).

Monotonicity of preferences and the related properties of insatiability will play an important role in the theory of the consumer; anticipating developments in the next chapter, if preferences are locally insatiable, a consumer choosing a consumption bundle subject to a budget constraint will spend all her income. Separability, quasi-linearity, and homotheticity all play important simplifying roles in specific problems; this happens beginning next chapter and then throughout the book.

Solutions to Starred Problems

■ 2.1. (a) See Table G2.1.

(b) Lexicographic preferences (which I'll write $\succeq_L$) are strictly monotone, strictly convex, and weakly separable. None of these are hard to prove, so I won't spell out the details, but it is interesting to note, at least, that lexicographic preferences are *antisymmetric*, meaning that if $x \succeq_L y \succeq_L x$, then $x = y$. So if $x \succeq_L y$ and $x \neq y$, then $x \succ_L y$. This in turn implies (if $x = (x_1, x_2)$ and $y = (y_1, y_2)$) that either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$. In either case, for all $\alpha \in (0, 1)$, $x \succ_L \alpha x + (1 - \alpha)y \succ_L y$, which of course gives us strict convexity. Each of these three properties of preference relations imply *all* representing utility functions have the properties given in the right-hand column (and corresponding row) of Table G2.1, which is consistent with the fact that lexicographic preferences have *no* numerical (utility) representation.

We have *some* in the second-from-left-hand columns for continuity, strong separability, the 3 properties of Proposition 2.16, and continuous and homothetic preferences. So, since lexicographic preferences have no numerical representation, they must not have any of these properties. It is easy to see that lexicographic preferences are not continuous, which takes care of the first and fourth of these four. Concerning the three properties in Proposition 2.16, lexicographic preferences satisfy properties a and b. But they fail to satisfy c: If $x > x'$, then no amounts of the second component can compensate for this difference in the first component.

This leaves strong separability. As note 6 in Table G2.1 states, the *some* for that row requires that preferences are strongly separable *and continuous*, and that there are at least three nontrivial components. Lexicographic preferences as defined in Problem 1.10 were for a subset of R^2, so they fail on the three-nontrivial-component requirement. But it is easy to define lexicographic preferences for, say, R^n_+, where $n \geq 3$, and then there will be three or more nontrivial components. And, it is not hard to show, they satisfy Definition 2.13; that is, they are strongly separable (into individual components). So the "problem," so to speak—the reason that there is *some* in the second-from-left column, which implies the existence of a representation, while lexicographic preferences have no representation—is that lexicographic preferences are not continuous.

■ 2.3. Suppose that $\succeq$ is globally insatiable and semi-strictly convex. Let x be any point from X. Then by global insatiability, there is some $x' \in X$ such that $x' \succ x$. For any $\epsilon > 0$, there is some $a \in (0,1)$ (very close to zero) such that the distance from x to $ax' + (1-a)x$ is less than ϵ. But by semi-strict convexity, for every $a \in (0,1)$, $ax' + (1-a)x \succ x$. Thus preferences are locally insatiable.

Suppose $k = 1$. Define $\succeq$ on $X = R_+$ as the preferences represented by the following utility function:

$$u(x) = \begin{cases} x, & \text{if } x \leq 1, \\ 1, & \text{if } 1 \leq x \leq 2, \text{ and} \\ x - 1, & \text{if } x \geq 2. \end{cases}$$

Because these preferences give convex NWT sets, they are convex. And because u is unbounded above, preferences are globally insatiable. But preferences are not locally insatiable at, for example, $x = 3/2$. Being very pedantic, the problem is that convex preferences (without semi-strict convexity) can be "flat" for a while, which is incompatible with local insatiability.

■ 2.5. We give the same example for parts a and b. Let $k = 1$, and let $u(x) = x$. This is certainly concave and continuous. Now let f be given by

$$f(r) = \begin{cases} (r-1)^3, & \text{for } 0 \le r \le 2, \text{ and} \\ r, & \text{for } r > 2. \end{cases}$$

By inspection, f is strictly increasing. Also, $f(u(x)) = f(x)$. But f is neither concave nor continuous; it is convex over the region $x \in [1, 2]$, and it is discontinuous at $x = 2$. (It also has derivative zero at $x = 1$, which is something you may want to remember for problems upcoming in later chapters.)

■ 2.6. Suppose preferences are monotone and locally insatiable. Take any two points x and x' such that x' is strictly greater than x, component by component. Let $\epsilon = \min_{i=1,\ldots,k} x'_i - x_i$; then $\epsilon > 0$. Moreover, by construction, if x'' is within ϵ of x, then $x'' \le x'$. (If $x''_i > x'_i$ for some i, then $x''_i - x_i > \epsilon$, which implies that $\|x'' - x'\| > \epsilon$.) Use local insatiability to produce some x'' within ϵ of x such that $x'' \succ x$. Since $x'' \le x'$, by monotonicity, $x' \succeq x''$, and thus $x' \succ x$. This shows that preferences are strictly monotone for strict increases.

Conversely, suppose that preferences are strictly monotone for strict increases. Local insatiability is obvious: For any x and $\epsilon > 0$, let $x' = x + (\epsilon/k, \epsilon/k, \ldots, \epsilon/k)$. Then x' is strictly greater than x, then $x' \succ x$. By construction, $\|x' - x\| = \epsilon/\sqrt{k} \le \epsilon$. Monotonicity requires continuity. Suppose $x' \ge x$. Let $x^n = x' + (\frac{1}{n}, \ldots, \frac{1}{n})$; then x^n is strictly greater than x, hence $x^n \succ x$. By continuity, $\lim_{n\to\infty} x^n \succeq x$, but this limit is x', hence preferences are monotone.

Without continuity, preferences that are monotone and locally insatiable are strictly monotone for strict increases—the proof in the first paragraph works—and preferences that are strictly monotone for strict increases are locally insatiable, by virtue of the first part of the second paragraph. But preferences that are strictly monotone for strict increases need not be monotone. To see this, let $k = 2$ and define

$$u((x_1, x_2)) = \begin{cases} 0, & \text{if } x_1 = x_2 = 0, \\ -1, & \text{if } \max\{x_1, x_2\} > \min\{x_1, x_2\} = 0, \text{ and} \\ \min\{x_1, x_2\}, & \text{if } \min\{(x_1, x_2\} > 0. \end{cases}$$

In words, this utility function gives the "standard" preferences with right-angle indifference curves (along the 45° line) in R^2_+, except that all points along the two axes, except 0, are worse than 0. If you compare any two points x and x', one of which is strictly greater than the other, then the strictly greater point is off the axes, and it is simple to see that its utility exceeds that of the first. So these preferences are strictly monotone for strict increases. But clearly preferences are not monotone, since $(0, 0) \succ (0, 1)$.

■ 2.9. (a) For preferences given by the utility function $u((x_1, x_2)) = \alpha x_1 + \beta x_2$ $(\alpha, \beta > 0)$, the indifference curves are the loci of points $\alpha x_1 + \beta x_2 = c$, or $x_2 = (c - \alpha x_1)/\beta$, for all

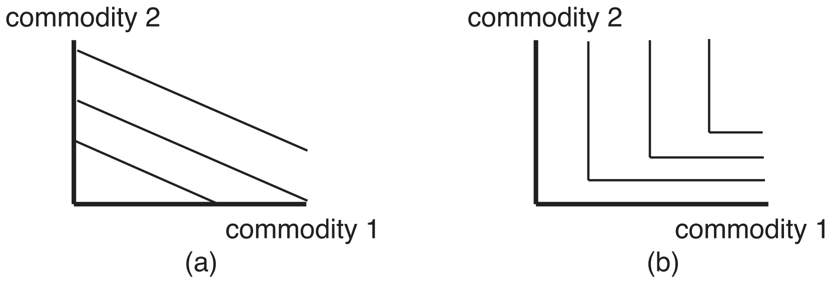

Figure G2.2. Problem 2.9: Two sets of indifference curves

constants c. These are parallel straight lines, with slope $-\alpha/\beta$; the picture is in Figure G2.2(a).

These preferences are: strictly monotone (since u is strictly increasing) and therefore increasing; locally insatiable (since they are strictly monotone); convex (since the NWT sets are convex); not strictly convex (indifference curves are linear); semi-strictly convex (if $x \succ y$, then $u(x) > u(y)$, and since u is linear, $u(ax + (1-a)y) = au(x) + (1-a)u(y) > u(y)$ for $a > 0$); and continuous (since u is continuous).

(b) For preferences given by the utility function $u((x_1, x_2)) = \min\{x_1/\alpha, x_2/\beta\}$, indifference curves are loci of points satisfying $\min\{x_1/\alpha, x_2/\beta\} = c$, or $[x_1 = \alpha c$ and $x_2 \geq \beta c]$ or $[x_1 \geq \alpha c$ and $x_2 = \beta c]$, for all constants c. These indifference curves are "right angles" whose angle comes at points along the line $x_1/\alpha = x_2/\beta$ or $x_2 = \beta x_1/\alpha$. The picture is in Figure G2.2(b).

These preferences are monotone (u is increasing) but not strictly monotone (for example, $(\alpha, \beta) \sim (\alpha + 1, \beta)$); locally insatiable (for any $x = (x_1, x_2)$, $y = (x_1 + \epsilon/2, x_2 + \epsilon/2)$ is within ϵ of x and is strictly better than x); convex (since the NWT sets are convex); not strictly convex (look at convex combinations of (α, β) and $(\alpha + 1, \beta)$); semi-strictly convex (proof given momentarily); and continuous (u is continuous).

The one slightly difficult thing to prove is that these preferences are semi-strictly convex. Suppose $x = (x_1, x_2) \succ y = (y_1, y_2)$. Then $x_1/\alpha \geq u(x) > u(y)$, $x_2/\beta \geq u(x) > u(y)$, $y_1/\alpha \geq u(y)$, and $y_2/\beta \geq u(y)$. Therefore, for any $a \in (0, 1)$,

$$\frac{ax_1 + (1-a)y_1}{\alpha} > u(y) \quad \text{and} \quad \frac{ax_2 + (1-a)y_2}{\beta} > u(y).$$

This implies that $u(ax + (1-a)y) > u(y)$, or $ax + (1-a)y \succ y$.

▪ 2.11. Proving that a quasi-linear representation implies properties a, b, and c is entirely straightforward, so I'll only give the proof that the three properties imply the representation.

Begin by observing that property a implies: *If* $m > m'$, *then* $(x, m) \succ (x, m')$ *for all* x. *And if* $(x, m) \succ (x, m')$ *for some* x, *then* $m > m'$. For the first part of this, suppose $m > m'$. The the property immediately implies that $(x, m) \succeq (x, m')$. So we only need to rule out $(x, m) \sim (x, m')$, but by the property, that would imply $m' \geq m$, which is false. Then for the second part, $(x, m) \succ (x, m')$ implies $(x, m) \succeq (x, m')$ implies $m \geq m'$, and $m = m'$ is precluded because if $m = m'$, then $(x, m) = (x, m')$ and hence $(x, m) \sim (x, m')$, which by hypothesis is not true.

Next, observe that property b implies: *For every* $x, x' \in R_+^{K-1}$ *and* m, m', *and* $m'' \in R_+^k$ *such that* $m - m'' \geq 0$ *and* $m' - m'' \geq 0, (x, m) \succeq (x', m')$ *if and only if* $(x, m - m'') \succeq (x', m' - m'')$. This simply involves letting $n = m - m''$ and $n' = m' - m''$ and invoking property b for n and n' in place of m and m'' with $m'', x,$ and x'.

Next observe the following: *If* $(x, m) \sim (x', m')$ *and* $(x, n) \sim (x', n')$, *then* $m - m' = n - n'$. To see this, assume the antecedent. Assume $n \geq m$. Let $\delta = n - m$; by property b from Proposition , $(x, m) \sim (x', m')$ implies $(x, m + \delta) \sim (x', m' + \delta)$. But $m + \delta = n$, and $(x, n) \sim (x', n')$, so that $(x', n') \sim (x', m' + \delta)$. By property a, $m' + \delta = n'$, or $n' - m' = \delta = n - m$. The argument if $m \geq n$ is entirely symmetrical.

Following the hint, fix some x^0. For each x, use c to find m_x and m'_x such that $(x, m_x) \sim (x^0, m'_x)$, and define $u(x) = m'_x - m_x$. The result in the previous paragraph implies that this definition is independent of the particular m_x and m'_x selected; that is, if $(x, n) \sim (x^0, n')$, then $u(x) := m'_x - m_x = n' - n$.

Take any x and $m \geq -u(x)$. Let $\delta = m - m_x$. If $\delta \geq 0$, use property b and $(x, m_x) \sim (x, m'_x)$ to conclude that $(x^0, m'_x + \delta) \sim (x, m_x + \delta)$. But $m_x + \delta = m$, while $m'_x + \delta = m'_x + m - m_x = m + u(x)$. Therefore, $(x^0, m + u(x)) \sim (x, m)$. And if $\delta \leq 0$, use the "extension" of property b given in the second paragraph of this solution to conclude that since $(x, m_x) \sim (x^0, m'_x)$, $(x, m) = (x, m_x + \delta) \sim (x^0, m'_x + \delta) = (x^0, m + u(x))$.

Let $U(x, m) = u(x) + m$. We are done if we show that U gives a representation of $\succeq$. To see this, fix any pair (x, m) and (x', m'). Let $K = \max\{-u(x), -u(x'), 0\}$. By property b, $(x, m) \succeq (x', m')$ if and only if $(x, m + K) \succeq (x, m' + K)$. But $(x, m + K) \sim (x^0, m + K + u(x))$ and $(x', m' + K) \sim (x^0, m' + K + u(x'))$, therefore $(x, m) \succeq (x', m')$ if and only if $(x^0, m + K - u(x)) \succeq (x^0, m' + K + u(x'))$ which, by property a and the first paragraph, is true if and only if $m + K + u(x) \geq m' + K + u(x')$ if and only if $m + u(x) \geq m' + u(x')$.

■ 2.14. We do not provide all the details of the proof, but following are some of the harder steps.

(a) Following the hint given in the text, let V be any numerical representation of $\succeq$. Let $\bar{v} = \sup\{V(x); x \in X\}$, and let $\underline{v} = \inf\{V(x); x \in X\}$.

If there is some $x \in X$ such that $V(x) = \bar{v}$, set z_1 to be any such x. Otherwise, let $\{z_1, z_2, \ldots\}$ be a sequence out of X such that $V(z_n)$ is strictly increasing in n and $\lim_n V(z_n) = \bar{v}$.

If there is some $x \in X$ such that $V(x) = \underline{v}$, set y_1 to be any such x. Otherwise, let $\{y_1, y_2, \ldots\}$ be a sequence out of X such that $V(y_n)$ is strictly decreasing in n and $\lim_n V(y_n) = \underline{v}$.

Let Z be the union of the following pieces: (1) In all cases, the line segment joining y_1 to z_1. (2) If $V(z_1) < \bar{v}$, the line segments joining z_1 to z_2, z_2 to z_3, z_3 to z_4, and so on. (3) If $V(y_1) > \underline{v}$, the line segments joining y_1 to y_2, y_2 to y_3, and so on.

Z is the union of line segments with joined endpoints, so it traces out a continuous one-dimensional "curve" in Z. And by construction, for all $x \in X$, there exist z_n and y_m with $V(z_n) \geq V(x) \geq V(y_m)$ for some n and m, thus $z_n \succeq x \succeq y_m$.

(b) The next step is to apply the lemma to conclude that for every $x \in X$ there is some $z(x) \in Z$ such that $x \sim z(x)$. There is nothing to prove here. The lemma applies directly.

(c) Next we are to prove that if U_Z is a continuous representation of $\succeq$ restricted to Z, we can extend U_Z to all of X to get a continuous representation of $\succeq$ on X.

To show this, note first that by part b, for each x there is some $z(x) \in Z$ such that $x \sim z(x)$. Define $U(x) = U_Z(z(x))$. We must show that the definition doesn't depend on which $z(x) \in Z$ such that $z(x) \sim x$ we pick. If $z \sim x \sim z'$,then $z \sim z'$ and $U_Z(z) = U_Z(z')$. Therefore, U is well defined on X.

Next, to show that U represents $\succeq$, we have: $x \succeq x'$ if and only if $z(x) \succeq z(x')$ (since $x \sim z(x)$ and $x' \sim z(x')$) if and only if $U_Z(z(x)) \geq U_Z(z(x'))$ (since U_Z represents $\succeq$ on Z) if and only if $U(x) \geq U(x')$ (by the definition of U).

To finish part b, we have to show that U is continuous. Suppose that $\{x_n\}$ has limit x in X, but that $U(x_n)$ does not have limit $U(x)$. By looking along a subsequence if necessary, we can assume that, for some $\delta > 0$, either $U(x_n) > U(x) + \delta$ for all sufficiently large n or $U(x_n) < U(x) - \delta$. Consider the former case. Since $U(x_n) = U_Z(z(x_n))$ and $U(x) = U_Z(z(x))$, by the continuity of U_Z and the intermediate-value theorem there is some $z' \in Z$ with $U_Z(z') = U(x) + \delta/2$. Of course, $z' \succ z(x)$ (since $U(z') > U(z(x))$). For all n sufficiently large, $U(x_n) \geq U(z')$, therefore $x_n \succeq z'$, and thus by continuity $x = \lim x_n \succeq z'$. But $x \succeq z' \succ z(x)$ contradicts $x \sim z(x)$. The case where $U(x_n) < U(x) - \delta$ for all sufficiently large n gives a similar contradiction, and U must be continuous.

So it remains is to produce a continuous representation on Z.

Recall what you were told to do next: Let Z' be a countable dense subset of Z. To be specific, let Z' be all the points on the various line segments that make up Z', that have the form $qx + (1 - q)x'$, where x and x' are endpoints of one of the line segments and q is a rational number. Note that this ensures that z^1 and y^1 are both in Z'. For each line segment, there are countably many such convex combinations of the endpoints, and there are countably many line segments; since the countable union of countably many sets is countable, Z' so defined is countable.

(d) This part asks you to show that this procedure produces a numerical representation of $\succeq$ on Z'. Details of this step are left to you, but the idea is: At each step, we produce a representation of $\succeq$ restricted to finite subsets of Z'. To show this use induction on the size of the finite subset. Then the extension to all of Z' gives a numerical representation because for any pair z and z' from Z', values of $U_{Z'}$ for z and z' were defined at some finite step in the procedure.

(e) Next: Let I be the smallest interval containing $U_{Z'}(z')$ for all $z' \in Z'$. We must show that the set of values $\{r = U_{Z'}(z')$ for some $z' \in Z'\}$ is dense in I. This is the trickiest part of the proof and is essential to what follows. It is shown as follows: Suppose it is not true. Then there is an interval J of length greater than zero inside I such that $U_{Z'}(z') \notin J$ for any z'. Let J^* be the largest open interval containing J for which this is so, and let δ be the width of J^*. There are then points z and z' from Z' such that $U_{Z'}(z)$ is above J^* but within $\delta/5$ of J^*'s right-hand endpoint, and $U_{Z'}(z')$ is below J^* but within $\delta/5$ of the left-hand endpoint of J^*. Let m be the larger of the indices of z and z' in the ennumeration of Z'. Consider $\{z'_1, \ldots, z'_m\}$, and let y' be the element of this set with largest $U_{Z'}$ value that is still to the left of J^*, and y be the element of this set with smallest $U_{Z'}$ value but still to the right of J^*. Of course, y and y' are within $\delta/5$ of the interval's nearer endpoint. Also, $y \succ y'$

By continuity of $\succeq$, there will be some subsequent z'_n such that $y \succ z'_n \succ y'$. Assume that n is the first index after m such that this happens. By construction, $U_{Z'}(z'_n) = (U_{Z'}(y) + U_{Z'}(y'))/2$. But again by construction, this average lies in the interval J^* (somewhere between two-fifths and three-fifths along the interval), which contradicts the supposition that J^* is not hit by $U_{Z'}$. Therefore, we have the desired denseness.

(f) The next step is to take any $z \in Z$, let $\{z'_n\}$ be a sequence out of Z' with limit z, and show that $\limsup U_{Z'}(z'_n) = \liminf U_{Z'}(z'_n)$. (The notation here is poor, because the subscript no longer refers to the ennumeration of the previous step.) Suppose this fails. Then we can produce subsequences n' and n'' such that $\lim_{n'} z'_{n'} = z = \lim_{n''} z'_{n''}$, but for all sufficiently large n' and n'', $U_{Z'}(z'_{n'}) > a + \delta$ and $U_{Z'}(z'_{n''}) < a - \delta$ for some a and $\delta > 0$. Use the denseness of $U_{Z'}(Z')$ to produce, out of Z', y' and y'' such that $U_{Z'}(y')$ and $U_{Z'}(y'')$ both come from the interval $(a - \delta/2, a + \delta/2)$, and $y' \succ y''$. Then for all sufficiently large n', $z'_{n'} \succ y'$, and by continuity, $z \succeq y'$. Similarly, for all sufficiently large n'', $y'' \succ z'_{n''}$, so by continuity $y'' \succeq z$. Therefore $z \succeq y' \succ y'' \succeq z$, a contradiction.

Now that we know that $\lim_n U_{Z'}(z'_n)$ exists (if $\lim z_n = z$), we have to show that this limit is finite. We will show that the limit is bounded above by considering two cases. (Showing that the limit is bounded below is symmetric.) (1) If there is a point $x \in X$ such that $V(x) = \bar{v}$, then since we were careful to put one such x in Z', once its $U_{Z'}$ value is assigned this puts an upper bound on $U_{Z'}$. (2) In the other case, where no x is $\succeq$-maximal in x, any $z \in Z$ is strictly worse than some other z^0 in Z, hence (by continuity) it is strictly worse than some z^* in Z'. By continuity again, this must be true for all sufficiently large z'_n, which would bound $U_{Z'}(z'_n)$ (by $U_{Z'}(z^*)$), again giving a finite limit.

(g) This allows us to define $U_Z : Z \to R$ by $U_Z(z) = \lim U_{Z'}(z'_n)$ for any sequence $\{z'_n\}$ out of Z' with limit z. This is clearly well-defined; the limit exists, is finite, and is independent of the sequence $\{z'_n\}$ used to define $U_Z(z)$. (Since Z' is dense in Z, such a sequence always exists.)

To show that U_Z is continuous on Z, let $\{z_n\}$ be a sequence in Z with limit z. By a standard diagonalization argument, we produce a sequence $\{z'_n\}$ from Z' with limit z such that $\lim U_{Z'}(z'_n) = \lim U_Z(z_n)$. (Find $z'_n \in Z'$ such that z'_n is within $1/n$ of z_n and $U_{Z'}(z'_n)$ is within $1/n$ of $U_Z(z_n)$.) Therefore, $\lim U_Z(z_n) = U_Z(z)$.

Finally, we have to show that U_Z represents $\succeq$ on Z. Suppose $z \succ y$ in Z. By continuity of $\succeq$ and denseness of Z' in Z, there are z' and y' in Z' such that $z \succ z' \succ y' \succ y$. We define $U_Z(z)$ by taking a sequence $\{z'_n\}$ out of Z' with limit z and setting $U_Z(z) = \lim U_{Z'}(z'_n)$; by continuity, $z'_n \succ z'$ for all sufficiently large n, thus $U_{Z'}(z'_n) > U_{Z'}(z')$, and hence $U_Z(z) \geq U_{Z'}(z')$. Similarly we can show that $U_{Z'}(y') \geq U_Z(y)$. But since $z' \succ y'$ and $U_{Z'}$ represents $\succeq$ on Z', this implies $U_Z(z) > U_Z(y)$.

Conversely, suppose $U_Z(z) > U_Z(y)$ for z and y from Z. Use the denseness of $U_{Z'}(Z')$ in its range to produce z' and y' such that $U_Z(z) > U_{Z'}(z') > U_{Z'}(y') > U_Z(y)$. Letting $\{z'_n\}$ be a sequence out of Z' with limit z, continuity implies that for all sufficiently large n, $U_{Z'}(z'_n) > U_{Z'}(z')$, thus $z'_n \succ z'$, and therefore, by continuity of $\succeq$, $z \succeq z'$. A similar argument shows that $y' \succeq y$, and hence $z \succ y$. ■

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Chapter 3: Basics of Consumer Demand

Summary of the Chapter

After the highly abstract Chapter 1 and the moderately abstract Chapter 2, the start of this chapter is likely to come as a relief; it is relatively simple and concrete, and it links back to things you are likely to have studied previously in economics. But by the time the discussion of correspondences and upper semi-continuity is reached, mathematics again takes over.

The chapter does, essentially, three things.

1. The classic consumer's problem of maximizing utility subject to a budget constraint is formulated, and basic properties (homogeneity of the problem in prices and income, existence of a solution, convexity of the solution set, exhaustion of the consumer's budget at the solution) are established.

2. Berge's Theorem, also known as the Theorem of the Maximum, is applied (for the first but far from the last time in this book): As prices and income vary parametrically, the set of solutions forms an upper semi-continuous and locally bounded correspondence, and the value of the solution—the *indirect utility function*—is continuous.

3. Calculus and the first-order/complementary-slackness conditions are used to solve the consumer's problem, and the optimality conditions are interpreted in (what I hope are) fairly intuitive terms.

Do not be misled by the relative shortness of this chapter. It contains three solid topics, and the second and third topics come with fairly tough appendices (Appendices

4 and 5, respectively). Expect this to be a hard chapter to master. In particular, if you are like most students, you will find the material in Appendix 4 to be new and challenging. But if you do master this chapter and its two appendices, a lot of the math that follows in later chapters will come relatively easily. I strongly recommend that you try Problem 3.14 (and, in the Instructor's Manual, I have strongly recommended to your instructor that he or she assign this problem). It isn't about the consumer's problem, *per se*, but if you want to fix the application of ideas from Appendix 4 in your mind, it is an excellent exercise.

Solutions to Starred Problems

■ 3.1. These preferences are locally insatiable everywhere but at the point 0. To see that they are not locally insatiable there, take $\epsilon = 1$. If preferences are locally insatiable at 0, it must be possible to find a consumption bundle $x^* \in R^k_+$ at distance 1 or less from 0 such that $x \succ 0$. But for strictly decreasing preferences, $0 \succeq x$ for all $x \in R^k_+$; indeed, $0 \succ x$ for all $x \in R^k_+$ except for $x = 0$ itself, so no such x^* can be found.

■ 3.4. I solve this using the language of bangs for the buck. The bang for the buck (henceforth, bfb) of goods 1, 2, and 3, respectively, are

$$\frac{1}{2(x_1+2)}, \quad \frac{2}{3(x_2+3)}, \quad \text{and} \quad \frac{4}{x_3+2}.$$

If these are to be equal (which is necessary if the solution has all three commodities strictly positive), we would have to have

$$2(x_1+2) = \frac{3}{2}(x_2+3) = \frac{1}{4}(x_3+2) \quad \text{or} \quad 8x_1+16 = 6x_2+18 = x_3+2.$$

(If a commodity level is zero, its bfb can be smaller than the largest, which means that the corresponding term in the previous display can be larger than the smallest. Since these preferences are locally insatiable (they are strictly monotone), we know that the consumer will spend all her income, or

$$2x_1 + 3x_2 + x_3 = y.$$

At this point, finding the solution for specific values of y is a matter of trial and error. Note that when x_1 is zero, the term corresponding to it in the last string of equalities, $8x_1 + 16$, takes on the value 16. Thus x_1 will be zero until $x_3 + 2 = 16$, or $x_3 = 14$. Similarly, x_2 must be zero until $x_3 + 2 = 18$, or $x_3 = 16$.

Thus when $y = 5$, even if the consumer spends all her wealth on good 3, she only gets $x_3 = 5$ units, and the bfb of good 3 still exceeds the initial bfbs (bangs for the buck) of goods 1 and 2. For $y = 5$, the solution is $x_1 = x_2 = 0$ and $x_3 = 5$.

When $y = 16.4$, the consumer cannot spend all her wealth on good 3: That would give $x_3 = 16.4$, which then requires positive levels of goods 1 and 2. How about spending money on goods 1 and 3 only? We need equal bfbs for these two commodities, or $x_3 + 2 = 8x_1 + 16$, and budget exhaustion, or $2x_1 + x_3 = 16.4$; rewrite the first of these as $x_3 = 8x_1 + 14$ and substitute into the second to get $10x_1 + 14 = 16.4$, or $x_1 = 0.24$, and thus $x_3 = 15.92$. This gives (equal) bfbs for goods 1 and 3 of 1/4.48, which is larger than 2/9, the bfb for good 2 at $x_2 = 0$. Therefore, we have our solution: $x_1 = .24, x_2 = 0,$ and $x_3 = 15.92$.

Finally, when $y = 100$, it is easy to guess that all three commodity levels will be strictly positive, so the equal bfbs condition must hold, or $8x_1 + 16 = 6x_2 + 18 = x_3 + 2$. Thus $x_3 = 6x_2 + 16$, and $x_1 = .75x_2 + .25$. Substituting in for x_1 and x_3 in the budget-exhaustion equation gives $2(.75x_2 + .25) + 3x_2 + 6x_2 + 16 = 100$, or $10.5x_2 = 83.5$, and thus $x_2 = 83.5/10.5 = 7.952380\ldots$; this gives $x_1 = 6.2142857\ldots$ and $x_3 = 63.7142857\ldots$.

Suppose we had misguessed in the case $y = 16.4$, guessing instead that the answer would involve all three commodities strictly positive. How would we have learned that this guess is wrong, algebraically? If you run the numbers, you'll find that setting the three bfbs equal, and combining these equations with the budget exhaustion equation, leads to a negative value of x_2, which is the tipoff that this was a bad guess.

■ 3.6. (a) Fixing p, J^* be the set of indices $j = 1, \ldots, k$ that achieve $\max_{j=1,\ldots,k} \alpha_j/p_j$. That is, $j^* \in J^*$ if $\alpha_{j^*}/p_{j^*} \geq \alpha_j/p_j$ for $j = 1, \ldots, k$. Then x^* is a solution of the CP if and only if $p \cdot x^* = y$ and $x_j^* > 0$ only if $j \in J^*$. In words, the consumer must spend all her income, and she must spend it only on commodities that have the highest ratio of α_j to p_j. This is easily derived from the optimality conditions, once you note that $\mathrm{MU}_j \equiv \alpha_j$.

(b) A point x that satisfies the budget constraint $p \cdot x \leq y$ and such that $\alpha_j x_j > \min_i \alpha_i x_i$ cannot be a solution of the CP, since from such a point utility will increase by decreasing x_j by $\epsilon = [\alpha_j x_j - \min_i \alpha_i x_i]/[2\alpha_j]$ (which lowers $\alpha_j x_j$, but not so much that $\alpha_j x_j$ becomes less or equal to $\min_i \alpha_i x_i$), and distributing the resulting income surplus equally over all the other commodities (which rasies the minimum over the other commodities by some strictly positive amount). Therefore, the solution x^* must involve $\alpha_j x_j^* = \alpha_i x_i^*$ for all i and j, or $x_j^* = \alpha_1 x_i^*/\alpha_j$. Thus once x_1^* is fixed, all other components of x^* are fixed, and $p \cdot x^* = \sum_i p^i[\alpha_1/\alpha_i]x_1^* = x_1^* \sum_i p_i[\alpha_1/\alpha_i]$. These preferences are locally insatiable and continuous, so we know that a solution exists and involves $p \cdot x^* = y$, hence

$$x_1^* = \frac{y}{\alpha_1 \sum_i [p_i/\alpha_i]} \quad ,$$

and hence

$$x_j^* = \frac{\alpha_1}{\alpha_j} x_1^* = \frac{y}{\alpha_j [\sum_i p_i/\alpha_i]} \quad .$$

■ 3.8. I will describe the procedure, but I am leaving it to you to create the pictures that go with it. I urge you to do so, at least for the example that is worked at as the method is described.

Step 1. For each $i = 1, \ldots, k$, define and draw the function $\lambda_i(y_i) = u_i'(y_i/p_i)/p_i$, where the prime denotes derivative. Note that $u_i'(0)$ is finite, and λ_i has strictly positive values, is continuous, and strictly decreasing. (Continuity follows from the implicit assumption that u_i is continuously differentiable. Strict decreasing follows because u_i is strictly concave. This also implies that the derivatives of the u_i are strictly positive.) The function $\lambda_i(y_i)$ is the *expenditure-driven* bang for the buck of commodity i function. That is, if y_i is spent on commodity i, the amount purchased is y_i/p_i, which has marginal utility $u_i'(y_i/p_i)$, hence bang for the buck equal to $u_i'(y_i/p_i)/p_i$.

Example: Suppose (as in Problem 3.4) that $u(x) = \ln(x_1 + 2) + 2\ln(x_2 + 3) + 4\ln(x_3 + 2)$, and $p = (2, 3, 1)$. Then $\lambda_1(y_1) = (1/(y_1/2 + 2))/2 = 1/(y_1 + 4)$, $\lambda_2(y_2) = (2/(y_2/3 + 3))/3 = 2/(y_2 + 9)$, and $\lambda_3(y_3) = 4/(y_3 + 2)$.

Step 2. Define the function $y_i^* : [0, \infty) \to [0, \infty)$ by letting $y_i^*(\lambda)$ be the unique value that solves the equation $u_i'(y_i^*(\lambda)/p_i)/p_i = \lambda$. If $\lambda > u_i'(0)/p_i$, then set $y_i^*(\lambda) = 0$. If $\lim_{y\to\infty} u_i'(y)/p_i \geq \lambda$, then set $y_i^*(\lambda) = \infty$. (This function y_i^* is the inverse of the function graphed in Step 1, so if we graph it with λ on the vertical axis, we get the "same" graph as in Step 1.)

Example: Since $\lambda_1(y_1) = 1/(y_1 + 4)$, we solve $y_1^*(\lambda) = 1/\lambda - 4$ (for $\lambda < 1/4$). Similarly, $y_2^*(\lambda) = 2/\lambda - 9$ (for $\lambda < 2/9$), and $y_3^*(\lambda) = 4/\lambda - 2$ (for $\lambda < 2$).

Step 3. Let $y^*(\lambda) = \sum_{i=1}^k y_i^*(\lambda)$. That is, $y^*(\lambda)$ is the horizontal sum of the y_I^* functions (horizontal because we put λ on the vertical axis). I assert that $y^*(\cdot)$ is strictly decreasing where it is finite and nonzero, continuous, and satisfies $\lim_{\lambda\to 0} y^*(\lambda) = \infty$. I leave the first two pieces of this assertion to your ingenuity and only indicate why the last is true: For any y, if $\lambda < u_i'(y/p_1)/p_1$, then $y^*(\lambda) > y$.

Example: We have

$$y^*(\lambda) = \begin{cases} 0, & \text{if } \lambda \geq 2, \\ 4/\lambda - 2, & \text{if } 2 > \lambda \geq 1/4, \\ 5/\lambda - 6, & \text{if } 1/4 > \lambda \geq 2/9, \text{ and} \\ 7/\lambda - 15, & \text{if } 2/9 > \lambda. \end{cases}$$

Step 4. Take the fixed income level $y > 0$ and find the unique value of λ such that $y^*(\lambda) = y$. A unique value of λ solves this equation because y^* is continuous and approaches ∞ as λ approaches zero (existence) and is strictly decreasing where it is positive (uniqueness). Then the solution to the CP for p and y is x^*, where $x_i^* = y_i^*(\lambda)/p_i$.

For the utility function of Problem 3.4, we have described the corresponding y^* function in the display in Step 3. Note the values of y at the critical values of λ: $y^*(2) = 0$; $y^*(1/4) = 14$; $y^*(2/9) = 33/2 = 16.5$. So for, say, $y = 100$, we know that the corresponding λ value must be in the range from 2/9 to 0, hence must be a solution of $7/\lambda - 15 = 100$, or $\lambda = 7/115$. This gives $y_1^*(7/115) = 115/7 - 4 = 12.4285\ldots$, hence x_1 (at the solution for $y = 100$) is $(12.4285\ldots)/2 = 6.2142\ldots$. (Compare this with the answer obtained in Problem 3.4.)

Why does this produce a solution? We know (because the u_i are strictly concave) that there is a unique solution to the problem, and the optimality conditions are necessary and sufficient for the solution. Then fixing p and y, simply check that this procedure generates a solution to the optimality conditions: The optimality conditions are $u_i'(x_i^*) = p_i\lambda$ for x_i^* that are strictly positive, and $u_i'(0) \leq p_i\lambda$ if $x_i^* = 0$, plus full expenditure, which is precisely the set of conditions generated by the procedure.

■ 3.9. Fix p and y. For each $i = 1, \ldots, k-1$, let $\hat{x}_i = 0$ if $u_i'(0) < p_i/p_k$ and otherwise satisfy $u_i'(\hat{x}_i) = p_i/p_k$. If $\sum_{i=1}^{k-1} p_i\hat{x}_i \leq y$, then the solution is $x_i^* = \hat{x}_i$ for $i = 1, \ldots, k-1$ with the remainder of y "spent" on the kth good, while if $\sum_{i=1}^{k-1} p_i\hat{x}_i > y$, then the solution is found by setting $x_k^* = 0$ and using the graphical procedure of Problem 3.8 for the first $k-1$ components.

I'll leave it to you to figure out why this works. (Of course, the reason is that this guarantees a solution to the optimality conditions, and your job is to verify that this is so.) But one comment is definitely in order: This formulation for utility — quasi-linear in a single commodity — is quite common in applications, where the commodity is "money left over," and the interpretation is that the $k-1$ goods are only a partial list of what the consumer will buy. Note that in our solution, hence in such applications, as long as the consumer has enough money in the sense that $y \geq \sum_{i=1}^{k-1} p_i\hat{x}_i$, the consumer's choice of the $k-1$ initial goods is independent of y; she buys good i to the point where its bang for the buck is $1/p_k$, which is the unchanging bang for the buck of the kth commodity.

■ 3.12. Let $k = 1$, $u(x_1) = (x_1 - 4)^3$, $p = (1)$, and $y = 4$. I should explain what $p = (1)$ means: there is a single commodity, so $p = (p_1)$, and $p_1 = 1$. Preferences are continuous (since u is), strictly increasing (hence, locally insatiable), and strictly convex (how do I know this?), so we know that at the solution to the CP, $p \cdot x_1^* = y = 4$, which means that $x_1^* = 4$. But $\text{MU}_1 = 3(x_1 - 4)^2$, which at $x_1^* = 4$ is zero, so the optimality conditions $\text{MU}_1(x_1^*)/p_1 = \lambda$ give $\lambda = 0$.

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Chapter 4: Revealed Preference and Afriat's Theorem

Summary of the Chapter

Chapter 4 concerns revealed preference. If we see a finite amount of data concerning the demand of a consumer, what can we tell about the consumer's preferences? In particular:

1. Are these data consistent with our model of a utility-maximizing consumer?
2. What patterns in the data can we expect to see? For instance, does demand for a commodity always rise when the price of that commodity falls (everything else held equal)?

Afriat's theorem, given in the text as Proposition 4.3, answers the first question, under the maintained hypothesis that the consumer in question has locally insatiable preferences. (Without some means of deducing from the data that the consumer strictly prefers one bundle to another, the answer to question 1 is always Yes: Any budget-feasible choices are consistent with utility maximization, namely by a consumer who is indifferent among all bundles. The maintained hypothesis of local insatiability is one way to "see" strict preferences in the data.) Since Afriat's theorem gives necessary and sufficient conditions for a finite set of demand data to be consistent with utility maximization, it can then be used to answer question 2; we can conceivably see any pattern in the data that is not precluded by Afriat's theorem. In particular, so-called *Giffen goods*, a good the demand for which rises with increases in the price of the good, are possible, at least in theory.

Two specific points made in the chapter merit special attention.

1. Concerning Afriat's theorem, the basic condition (no revealed preference cycle with at least one link a revelation of strict preference) is *necessary* if the data are generated by a locally insatiable, utility-maximizing consumer and is *sufficient* to guarantee that the data could come from a consumer with continuous, strictly increasing, and convex preferences. Hence the latter three conditions are empirically untestable from finite *market demand* data, except by throwing out the entire model of utility maximization with local insatiability. (And if you want to understand why I emphasized "market demand," see the text.)
2. Because of income effects, little in the way of comparative static conclusions can come from market (Marshallian) demand data with a fixed level of income. But comparative static conclusions can arise by dealing with compensated demand, where income levels are changed to compensate for changes in prices.

Chapter 4 is short and without much in the way of complications, except for the proof of Afriat's theorem. Your instructor will tell you how much attention to pay to the proof; it's an aesthetically beautiful proof, but not one whose techniques are used elsewhere in the book.

Solutions to Starred Problems

■ 4.1. For each bundle and each price vector, we compute the cost of the bundle at those prices:

consumption bundles	price vectors (1,1,1)	(3,1,1)	(1,2,2)	(1,1,2)
(10,5,5)	20	40	30	25
(3,5,6)	14	20	25	20
(13,3,3)	19	45	25	22
(15,3,1)	19	49	23	20

We verify first that, in each case, the chosen bundle exhausts the budget constraint. And we conclude from the first set of prices that (10,5,5) strictly dominates the other three. At the second set of prices (3,1,1), none of the other bundles are affordable, so this column generates no restrictions. The third set of prices reveals that $(13,3,3) \succ (15,3,1)$ and $(13,3,3) \succeq (3,5,6)$. And the fourth and final set of prices tells us (only) that $(15,3,1) \succeq (3,5,6)$. This gives no cycles that I can find and, indeed, the data are consistent with $(10,5,5) \succ (13,3,3) \succ (15,3,1) \succ (3,5,6)$. So Afriat's theorem tells us that these data could come from a locally insatiable, utility-maximizing consumer (with continuous, convex, and strictly increasing preferences).

■ 4.4. Since preferences are locally insatiable, $p \cdot x = y$ and $p' \cdot x' = p' \cdot x$. Since x' is chosen when prices are p' and income is $p' \cdot x$, $x' \succeq x$ must hold (since x is affordable). But then $p \cdot x' \geq y = p \cdot x$ must be true; otherwise, local insatiability would tell us that something strictly better than x' is affordable at prices p and income y, and such a bundle would be strictly preferred to x by transitivity.

Therefore, $p \cdot x' \geq p \cdot x$ and $p' \cdot x' = p' \cdot x$. But

$$(p' - p) \cdot (x' - x) = p' \cdot x' - p' \cdot x - p \cdot x' + p \cdot x \leq 0,$$

since the first two terms give 0 and the second two are less than or equal to 0.

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Chapter 5: Choice under Uncertainty

Summary of the Chapter

This chapter develops and discusses the basic models that economists use to model consumer or individual choice under uncertainty. At fifty pages, it is (with Chapter 16) one of the longest chapters in this volume, and only the most intrepid reader will want to do it all in one sitting. (If you make it all the way through Section 5.2 in one sitting, you are doing great. And see below about the last bits of Section 5.2.)

The plot of the chapter, roughly, is:

- Section 5.1 introduces two frameworks for discussing choice under uncertainty, one framework with *states of nature*, a second with *(objective) probability distributions over prizes*. It also introduces the two *expected utility* representations that are sought in the rest of the chapter: using the given probabilities in the second context; and employing probabilities over the states of nature that are derived from the decision maker's preferences in the first context. (Other names you will encounter in the literature are: the *von Neumann–Morgenstern expected utility model* or *representation* when the probabilities are given "objectively"; and the *subjective expected utility model* or the *Savage model* when the probabilities over states of nature are derived from the decision maker's preferences.)
- Section 5.2 does the heavy lifting for objective probabilities and expected utility, employing a more general result known as the *Mixture-Space Theorem*. At first everything is done for probability distributions that have finite support, so-called *simple probability distributions*, on a given space of prizes. And then the section discusses how one extends from simple to more complex probability distributions, and

why (in theory) this often comes at the price of bounded utility functions. This final discussion will be tough sledding for readers whose mathematical background doesn't include weak convergence of probability measures; if that is you, you can safely go on to Section 5.3 without a full understanding of this part of Section 5.2.

- Section 5.3 does the somewhat less-heavy lifting for subjective expected utility. While the classic derivation is due to L. J. Savage (1954), his treatment is too long and complex for this book, and we employ instead the very clever development of Anscombe and Aumann (1963). After all the hard work of Section 5.2, the mathematical derivations in this section seem quite simple. But this hides what is really going on, and an important part of the section is a discussion of how, in the Anscombe and Aumann treatment, the *formulation* of the problem contains a critical assumption. To appreciate fully this point, I urge you to do Problems 5.4 and 5.5 and, to the extent possible, discuss the solutions to those problems with your peers/in class.
- Having finished with the mathematical development, Section 5.4 discusses the point of view held by many economic theorists (but not by me) that differences in subjective probability assessment can only result from differences in information, the so-called *common-prior assumption.*
- The chapter concludes in Section 5.5 with a discussion of empirical and theoretical reasons to doubt the validity of the models developed in this chapter.

Solutions to Starred Problems

■ 5.1. (a) Gamble 1 has expected utility $(1.0)[\sqrt{10{,}000}] = 100$. Gamble 2 has expected utility $(1/3)[\sqrt{3600}] + (2/3)[\sqrt{14{,}400}] = 20 + 80 = 100$. Gamble 3 has expected utility $(1/5)[\sqrt{0}] + (1/5)[\sqrt{10{,}000}] + (2/5)[\sqrt{22{,}500}] = 0 + 20 + 90 = 110$. So the consumer will select gamble 3.

(b) The first act evaluates as $\sqrt{4} + 0.6\min\{3, 12\} + 0.4(7 + 7) = 2 + 1.8 + 5.6 = 9.4$. The second evaluates as $\sqrt{16} + 0.6\min\{2, 20\} + 0.4(10 + 0) = 4 + 1.2 + 4 = 9.2$. The third evaluates as $\sqrt{25} + 0.6\min\{5, 5\} + 0.4(5 + 5) = 5 + 3 + 4 = 12$. This consumer will choose the third act.

(c) The first act evaluates as $(0.5)(4)^{0.25} + (0.3)(36)^{0.25} + (0.2)(49)^{0.25} = 1.97$. The act evaluates as $(0.5)(16)^{0.25} + (0.3)(40)^{0.25} + (0.2)(0)^{0.25} = 1.754$. The act evaluates as $(0.5)(25)^{0.25} + (0.3)(25)^{0.25} + (0.2)(25)^{0.25} = 2.236$. So the third act is chosen.

(d) The first horse-race lottery evaluates as 13.92, and the second at 12. So the first is chosen.

(e) The first horse-race lottery evaluates at 2.214 and the second as 2.236, so the second is chosen.

■ 5.3. The problem is to prove that a through c in the Mixture-Space Theorem (Proposition 5.4) imply the representation d and e, without the extra assumption enlisted in the proof in the text.

Note that the lemmas in the proof do not use this extra assumption, so we have them all available.

So assume we have a mixture space Z and a preference relation $\succeq$ on Z that satisfies a through c. Either $z \sim z'$ for all z and z' from Z, or $z \succ z'$ for some pair z and z'. In the first case, any constant function u satisfies d and e, so we can proceed with the second case. Fix a pair, denoted $\overline{z}$ and $\underline{z}$, such that $\overline{z} \succ \underline{z}$.

Now choose any $z \in Z$ and define $u(z)$ as follows:

Case A: If $\overline{z} \succeq z \succeq \underline{z}$, let a be the unique number between zero and one such that $z \sim a\overline{z} + (1-a)\underline{z}$, and set $u(z) = a$.

Case B: If $z \succ \overline{z}$, let a be the unique number strictly between zero and one such that $\overline{z} \sim az + (1-a)\underline{z}$, and let $u(z) = 1/a$.

Case C: If $\underline{z} \succ z$, let a be the unique number strictly between zero and one such that $\underline{z} \sim a\overline{z} + (1-a)z$, and let $u(z) = a/(a-1)$.

Why are these scalars a all unique? In cases B and C, why are they strictly between 0 and 1? We note first that in Case A, $\overline{z} \succ \underline{z}$ by assumption, in case B, $z \succ \overline{z} \succ \underline{z}$, and in case C, $\overline{z} \succ \underline{z} \succ z$; then apply the lemmas and, in particular, Lemma 5.9.

We need to show that $u(z)$ so defined represents $\succ$ and is linear in convex combinations. In other words, if z and z' are both from Z and if $\alpha \in [0,1]$, then $z \succeq z'$ if and only if $u(z) \geq u(z')$, and $u(\alpha z + (1-\alpha)z') = \alpha u(z) + (1-\alpha)u(z')$.

So fix for the remainder of this solution some z and z' from Z and $\alpha \in [0,1]$.

Let z_1 be the $\succeq$-best of the threesome z, z', and $\overline{z}$ (if there is a tie for best, choose one of the best however you prefer) and let z_2 be the $\succeq$-worst of the threesome z, z' and $\underline{z}$ (again choosing one of the worst if there is a tie). Since $\overline{z} \succ \underline{z}$, $z_1 \succeq \overline{z}$, and $\underline{z} \succeq z_2$, this ensures that $z_1 \succ z_2$. Define

$$Z' = \{z \in Z : z_1 \succeq z \succeq z_2\}.$$

By the manner in which z_1 and z_2 where chosen, z, z', $\overline{z}$, and $\underline{z}$ are all in Z'. Moreover, I assert that: *If $\hat{z}$ and $\check{z}$ are both in Z' and if $a \in [0,1]$, then $a\hat{z} + (1-a)\check{z} \in Z'$.* To see why, take any $\hat{z}$ and $\check{z}$ from Z'. By completeness of $\succeq$, we know that either $\hat{z} \succeq \check{z}$ or $\check{z} \succeq \hat{z}$. Without loss of generality, suppose that $\hat{z} \succeq \check{z}$. Then Lemma 5.7 tells us that $\hat{z} \succeq a\hat{z} + (1-a)\check{z} \succeq \check{z}$ for all $a \in [0,1]$. But $z_1 \succeq \hat{z}$ and $\check{z} \succeq z_2$, so transitivity of $\succeq$ gives us $z_1 \succeq a\hat{z} + (1-a)\check{z} \succeq z_2$ for all a.

This implies that Z' is a mixture space: It is closed under the taking of convex combinations. And since a, b, and c from Proposition 5.3 hold on all of Z and are "for all" statements, they hold for all $z \in Z'$. So Z' is a mixture space satisfying a, b, and c. What is more, Z' contains a $\succeq$-best and a $\succeq$-worst element, namely z_1 and z_2, respectively. So the proof of Proposition 5.3 given in the text works perfectly well for Z': There exists a function $v : Z' \to R$ that represents $\succeq$ on Z' and that is linear in convex combinations.

Moreover, Proposition 5.3 tells us that this is true as well of any $v' : Z' \to R$ that has $v' = Av + B$, for any strictly positive scalar A and any scalar B.

We know that $\overline{z} \succ \underline{z}$, and of course both are elements of Z', so $v(\overline{z}) > v(\underline{z})$. So let

$$A = \frac{1}{v(\overline{z}) - v(\underline{z})} \quad \text{and} \quad B = \frac{-v(\underline{z})}{v(\overline{z}) - v(\underline{z})},$$

and let $v' = Av + B$ for this specific choice of $A > 0$ and B. Simple algebra gives $v'(\underline{z}) = Av(\underline{z}) + B = v(\underline{z})/(v(\overline{z}) - v(\underline{z})) - v(\underline{z})/(v(\overline{z}) - v(\underline{z})) = 0$, and $v'(\overline{z}) = Av(\overline{z}) + B = v(\overline{z})/(v(\overline{z}) - v(\underline{z})) - v(\underline{z})/(v(\overline{z}) - v(\underline{z})) = 1$.

Moreover, I asset that *for any* $\hat{z} \in Z'$, $v'(\hat{z}) = u(\hat{z})$. That is, v' coincides with the originally defined u on all of Z'. We go back to the three cases at the start of the problem, for $\hat{z}$ this time:

Case A: If $\overline{z} \succeq \hat{z} \succeq \underline{z}$, $u(a)$ is the unique number a such that $\hat{z} \sim a\overline{z} + (1-a)\underline{z}$. But if $\hat{z} \sim a\overline{z} + (1-a)\underline{z}$, $v'(\hat{z}) = v'(a\overline{z} + (1-a)\underline{z}) = av'(\overline{z}) + (1-a)v'(\underline{z}) = a$, since $v'(\overline{z}) = 1$ and $v'(\underline{z}) = 0$.

Case B: If $\hat{z} \succ \overline{z}$, $u(\hat{z})$ is $1/a$, where a is the unique number strictly between zero and one such that $\overline{z} \sim a\hat{z} + (1-a)\underline{z}$. But then $v'(\overline{z}) = v'(a\hat{z} + (1-a)\underline{z}) = av'(\hat{z}) + (1-a)v'(\underline{z})$, or $1 = av'(\hat{z})$, or $v'(\hat{z}) = 1/a$.

Case C: If $\underline{z} \succ \hat{z}$, $u(\hat{z}) = a/(a-1)$, where a is the unique number strictly between zero and one such that $\underline{z} \sim a\overline{z}+(1-a)\hat{z}$. But then $v'(\underline{z}) = v'(a\overline{z}+(1-a)\hat{z}) = av'(\overline{z})+(1-a)v'(\hat{z})$, giving us the equation $0 = a + (1-a)v'(\hat{z})$, which solves as $v'(\hat{z}) = a/(a-1)$.

Please note that in all three cases, I am repeatedly using the fact that for any two elements of Z', their convex combinations are also in Z'.

This (practically) finishes the proof. Since v' coincides with u on Z' and v' represents $\succeq$ and is linear in convex combinations on Z', we know that u represents $\succeq$ and is linear in convex combinations on Z'. And our originally (arbitrarily) chosen z and z' are both in Z', so we know that

$$z \succeq z' \text{ if and only if } u(z) \geq u(z'), \text{ and } u(\alpha z + (1-a)z') = \alpha u(z) + (1-\alpha)u(z').$$

That's what we needed to show.

■ 5.6. This solution will be completely unintelligible if you don't have the textbook open in front of you, as I'm not going to repeat the notation employed.

a. H is a mixture space, and the Mixture-Space Theorem applies. If $\succeq$ satisfies the three mixture-space axioms, there exists a function $F : H \to R$ that represents $\succeq$ and that is linear in convex combinations. (The other parts of the Mixture-Space Theorem, concerning the converse and the extent to which the representing function F is "unique," are also all true, of course.)

b. The key is to employ the "trick" of Anscombe and Aumann, namely

$$\frac{1}{n}[\ldots,\pi_{ij} \text{ on } A_{ij},\ldots] + \frac{n-1}{n}[\underline{x} \text{ on } S] = \\ \sum_{i=1}^{n} \frac{1}{n}[\pi_{i1} \text{ on } A_{i1},\ldots,\pi_{im} \text{ on } A_{im}, \underline{x} \text{ on } A_i^C].$$

Therefore,

$$\frac{1}{n}F([\ldots,\pi_{ij} \text{ on } A_{ij},\ldots]) + \frac{n-1}{n}F([\underline{x} \text{ on } S]) = \\ \sum_{i=1}^{n} \frac{1}{n}F([\pi_{i1} \text{ on } A_{i1},\ldots,\pi_{im} \text{ on } A_{im}, \underline{x} \text{ on } A_i^C]).$$

Since $F([\underline{x} \text{ on } S]) = 0$, that term disappears, and if you multiply through by n and recall the definition of F_A for any set A, we have part b.

c. In this context, Savage's Independence Axiom would say: Suppose A and B, both from $\mathcal{A}$, partition S. For $h_A, h'_A \in H_A$ and $h_B, h'_B \in H_B$,

$$[h_A \text{ on } A, h_B \text{ on } B] \succeq [h'_A \text{ on } A, h_B \text{ on } B] \text{ if and only if} \\ [h_A \text{ on } A, h'_B \text{ on } B] \succeq [h'_A \text{ on } A, h'_B \text{ on } B].$$

That is, if we compare two h's that agree on B, what is important to the comparison is how they disagree where they disagree. If we change how they agree where they agree, we don't change how they compare.

With part b, showing this is trivial:

$$
\begin{aligned}
&[h_A \text{ on } A, h_B \text{ on } B] \succeq [h'_A \text{ on } A, h_B \text{ on } B] \text{ if and only if} \\
&F([h_A \text{ on } A, h_B \text{ on } B]) \geq F([h'_A \text{ on } A, h_B \text{ on } B]) \text{ if and only if} \\
&F_A(h_A) + F_B(h_B) \geq F_A(h'_A) + F_B(h_B) \text{ if and only if} \\
&F_A(h_A) + F_B(h'_B) \geq F_A(h'_A) + F_B(h'_B) \text{ if and only if} \\
&F([h_A \text{ on } A, h'_B \text{ on } B]) \geq F([h'_A \text{ on } A, h'_B \text{ on } B]) \text{ if and only if} \\
&[h_A \text{ on } A, h'_B \text{ on } B] \succeq [h'_A \text{ on } A, h'_B \text{ on } B].
\end{aligned}
$$

d. Fix some $A \in \mathcal{A}$ and look at $\succeq$ restricted to h that take the form $[\pi \text{ on } A, \underline{x} \text{ on } A^C]$, for $\pi \in \Pi$. If $\succeq$ satisfies the three mixture-space axioms overall, it satisfies them on this restricted set as well, and so the von Neumann-Morgenstern Expected-Utility Theorem (Proposition 5.3) says that we have an expected utility representation, which also gives a mixture-space representation. Since F is a mixture-space representation, already, it can be used to provide the expected utility representation, where the utility of the prize x is the representing function evaluated at the lottery that gives x with certainty. Which is precisely how U_A is defined in this part of the problem. That is,

$$F_A([p \text{ on } A]) = F([p \text{ on } A, \underline{x} \text{ on } A^C]) = \sum_x U_A(x)p(x),$$

where the sum is over x in the support of p. Now apply part b.

e. For $\mathbf{p}$ as defined in the statement of the problem, $\mathbf{p}(S) = F([\overline{x} \text{ on } S]) = 1$. And if A and B are disjoint sets, then

$$\frac{1}{2}[\overline{x} \text{ on } A \cup B, \underline{x} \text{ on } (A \cup B)^C] + \frac{1}{2}\underline{x} = \frac{1}{2}[\overline{x} \text{ on } A, \underline{x} \text{ on } A^C] + \frac{1}{2}[\overline{x} \text{ on } B, \underline{x} \text{ on } B^C].$$

Apply the Mixture-Space Theorem to get

$$\frac{1}{2}F([\overline{x} \text{ on } A\cup B, \underline{x} \text{ on } (A\cup B)^C]) + \frac{1}{2}F(\underline{x}) = \frac{1}{2}F([\overline{x} \text{ on } A, \underline{x} \text{ on } A^C]) + \frac{1}{2}F([\overline{x} \text{ on } B, \underline{x} \text{ on } B^C]).$$

Recall that $F(\underline{x}) = 0$, so that term drops. Multiply on both sides by 2, and this is

$$\mathbf{p}(A \cup B) = \mathbf{p}(A) + \mathbf{p}(B).$$

So we have finite additivity.

We haven't used the assumption that $\overline{x} \succeq_A \underline{x}$ yet, so where does that come in? We need that to ensure that $\mathbf{p}(A) = F([\overline{x} \text{ on } A, \underline{x} \text{ on } A^C])$ is nonnegative (since we know

$F(\underline{x}) = 0$); otherwise, we produce not a finitely additive probability but a finitely additive signed measure on $(S, \mathcal{A})$.

Finally, we need to show that $\mathbf{p}(A) = 0$ if and only if A is null. But if A is null, then $\overline{x} \sim_A \underline{x}$, hence $F([\overline{x} \text{ on } A, \underline{x} \text{ on } A^C]) = F([\underline{x} \text{ on } A, \underline{x} \text{ on } A^C]) = F([\underline{x} \text{ on } S]) = 0$. And if A is not null, then $\overline{x} \succ_A \underline{x}$, and $\mathbf{p}(A) = F([\overline{x} \text{ on } A, \underline{x} \text{ on } A^C]) > F([\underline{x} \text{ on } A, \underline{x} \text{ on } A^C]) = F([\underline{x} \text{ on } S]) = 0$.

f. Given everything that has come before (and, in particular, part d), we are done once we show that U_A can be written $\mathbf{p}(A)U$ for some single function $U : X \to R$. For the function U, take $U(x) = F([x \text{ on } S])$. Then our normalization of F means that $U(\overline{x}) = 1$ and $U(\underline{x}) = 0$.

Take any $A \in \mathcal{A}$. If A is null, then $\pi \sim_A \pi'$ for all $\pi, \pi' \in \Pi$ and, in particular, $x \sim_A \underline{x}$ for all $x \in X$. Therefore, $U_A(x) = U_A(\underline{x}) = 0$ for all x. But then $\mathbf{p}(A)U(\cdot) = 0 = U_A(\cdot)$, since $\mathbf{p}(A) = 0$, and we are done.

And if A is not null, then $\succeq_A$ gives the same preference ordering on Π as does $\succeq$. Hence U_A must be a positive affine translate of U. We've fixed $U(\underline{x}) = U_A(\underline{x}) = 0$, hence $U_A(\cdot) = aU(\cdot)$ for some positive constant a. But we know that $U(\overline{x}) = 1$ and $U_A(\overline{x}) = \mathbf{p}(A)$, so this positive constant a must be $\mathbf{p}(A)$, completing the proof.

■ 5.7 or, **Concerning Proposition 5.10**:[1]

The topology of weak convergence (on the space P of Borel probability measures on $X = R^k$ or R^k_+) has the following "definition": A sequence $\{p_n\}$ out of P converges to $p \in P$ if

$$\lim_{n\to\infty} \int_X f(x)p_n(dx) = \int_X f(x)p(dx) \text{ for all bounded and continuous real-valued functions } f : X \to R.$$

The scare-quotes around *definition* are there because to define a topology, one ought to specify an appropriate collection of open sets or, at least, a neighborhood base for the topology; but given the definition of sequential convergence I've provided, readers who know about such things can probably produce for themselves a neighborhood base for this topology. We won't need to know this, but it turns out that this topology

[1] In early printings of the book, the discussion beginning on page 98 and leading to Proposition 5.10 referred to the *weak topology on* P. This is, at best, ambiguous; mathematically sophisticated readers are entitled to object that, in the most natural interpretation, it is wrong. What I wanted was the *the topology of weak convergence of probability measures*, which is a weak* topology and which I define in the text immediately following. Princeton University Press allowed me to correct typos in the book for later printings, and I fixed the language around Proposition 5.10 and added Problem 5.7, which asks the reader to prove the Proposition (now stated less ambiguously). If your copy of the book lacks Problem 5.7, you have an early printing, not corrected for typos, in which case the discussion to follow should help clarify the ambiguous text. If your copy of the book has Problem 5.7, then here is its solution.

on P is metrizable—look for references to the Prohorov metric in the literature. The facts we do need to know are:

- For every $p \in P$, we can construct a sequence $\{p_n\}$ consisting of *simple* probabilities on X that has the limit p.

- Suppose $p \in P$ has countable support $\{x_1, x_2, \ldots\}$, and we define a sequence of simple probabilites $\{p_m\}$, where p_m has support $\{x_0, \ldots, x_m\}$ (for arbitrary x_0) and $p_m(x_n) = p(x_n)$ for $n = 1, \ldots, m$ and $p_m(x_0) = \sum_{n=m=1}^{\infty} p(x_n)$. In words, p_m "truncates" the support of p at element #m, assigning any left-over probability to x_0. Then $\lim_m p_m = p$ in the topology of weak convergence. (I'm not going to give you a proof of this, but since I've characterized what it takes for a sequence to converge, you shouldn't have huge difficulty in proving this on your own.)

- Suppose $\{p_n\}$ is any sequence out of P with limit p in this topology; then for all $a \in [0,1]$ and $p' \in P$, $\lim_n ap_n + (1-a)p' = ap + (1-a)p'$. (This is even easier to prove from first principles, if your first principles are the definition of sequential convergence given above.)

Using these three facts,[2] here is a proof of the Proposition:

Step 1 is to invoke the mixture-space theorem: There exists some function $u : P \to R$ that represents $\succeq$ and that is linear in convex combinations. We now aim to show that this u takes the form of expected utility and where, moreover, the utility function U is just $U(x) := u(\delta_x)$. (Of course, if u takes the form of expected utility with U, then $U(x)$ must equal $u(\delta_x)$, since that is the expected utility of the lottery δ_x.) It should be noted that for all $p \in P_S$, the induction argument used in the chapter tells us already that $u(p) = \sum_{x \in \text{supp}(p)} u(\delta_x)p(x) = \sum_{x \in \text{supp}(p)} U(x)p(x)$.

Step 2 disposes of a trivial case. Suppose $\delta_x \sim \delta_y$ for all x, $y \in X$. Then (I assert) $p \sim q$ for all $p, q \in P$, so any constant function does the trick (and, of course, $U(x) = u(\delta_x)$ is one such constant function). To see why my assertion is true, suppose $\delta_x \sim \delta_y$ for all x and y. Then $U(x) = U(y)$ for all x and y; that is, U is a constant function. But then we know (from the representation on P_S or directly by induction) that $p \sim q$ for all simple probabilities p and q. Suppose $p \succ q$ for some (non-simple) p and q from P. Let $\{p_n\}$ be a sequence of simple probabilities with limit p, and let $\{q_n\}$ be a sequence of simple probabilities with limit q; continuity and $p \succ q$ would imply that for n large enough (for all large enough n), $p_n \succ q_n$. But we know that for simple probabilities, universal indifference must hold, a contradiction. So in this case, every Borel lottery p is indifferent to every other one, and we are finished.

Hence I can assume for the remainder of the proof that there exist $\overline{x}$ and $\underline{x}$ from X such that $\delta_{\overline{x}} \succ \delta_{\underline{x}}$.

[2] And, if your copy of the book says "the weak topology," changing this to "the topology of weak convergence,"

Step 3: The function U is continuous. Suppose not. Then there is some prize x and sequence of prizes $\{x_n\}$ with limit x such that $U(x) \neq \lim_n U(x_n)$. (By looking along subsequences, I can assume that $\lim_n U(x_n)$ exists, although it may be $+\infty$ or $-\infty$.) Again I need to look at cases, one case where the limit exceeds $U(x)$; the other where it is less. I'll assume $\lim_n U(x_n) > U(x)$ and let you handle the other case. Then we know that for some $\epsilon > 0$ and for all sufficiently large n, $U(x_n) \geq U(x) + \epsilon$. Choose any one of the x_n's for which this is true and call it x', so that $U(x') \geq U(x) + \epsilon$. This means that $\delta_{x'} \succ \delta_x$. We can find an $a \in (0,1)$ such that $aU(x)+(1-a)U(x') = U(x)+\epsilon/2$; this is a simple application of the intermediate value theorem. Of course, this implies that $a\delta_x + (1-a)\delta_{x'} \succ \delta_x$. But then for all sufficient large n,

$$U(x_n) \geq U(x) + \epsilon > U(x) + \epsilon/2 = aU(x) + (1-a)U(x') = u(a\delta_x + (1-a)\delta_{x'}),$$

so $\delta_{x_n} \succ a\delta_x + (1-a)\delta_{x'}$. By continuity, since δ_{x_n} has limit δ_x in the topology of weak convergence (since x_n has limit x), this implies that $\delta_x \succeq a\delta_x + (1-a)\delta_{x'}$, which contradicts $a\delta_x + (1-a)\delta_{x'} \succ \delta_x$.

Step 4: The function U is bounded. Again, suppose not. Then it is either unbounded above or below (or both). I'll do the case where U is unbounded above, and you can deal with unbounded below. Since U is unbounded above, we can find for $n = 1, 2, \ldots$ a prize x_n such that $U(x_n) > 1/2^n$. Let p be the (countable-support, non-simple) lottery that has prize x_n with probability $1/2^n$, and let p_n be the (finite support, hence simple) lottery that has prize x_m with probability $1/2^m$ for $m \leq n$ and some fixed prize x_0 with the remaining probability $1/2^n$. It isn't hard to see that $\lim_n p_n = p$ in the topology of weak convergence.

It is here that we employ $\delta_{\overline{x}}$ and $\delta_{\underline{x}}$. Since $\delta_{\overline{x}} \succ \delta_{\underline{x}}$, the third (continuity) mixture-space axiom tells us that for some $a > 0$, $\delta_{\overline{x}} \succ ap + (1-a)\delta_{\underline{x}}$. Again relying on properties of the topology of weak convergence, we observe that $\lim_n(ap_n+(1-a)\delta_{\underline{x}}) = ap+(1-a)\delta_{\underline{x}}$, hence by continuity of preference, for all sufficient large n, $\delta_{\overline{x}} \succ ap_n + (1-a)\delta_{\underline{x}}$. But $ap_n + (1-a)\delta_{\underline{x}}$ is a simple probability, and we (therefore) know that its utility is its expected utility under U. And for any $a > 0$, the way p and the p_n were constructed implies that the expected utility of $ap_n + (1-a)\delta_{\underline{x}}$ increases without bound, as long as $a > 0$. So for all sufficiently large n, we have $ap_n + (1-a)\delta_{\underline{x}} \succ \delta_{\overline{x}}$. This is a contradiction; U must be bounded.

Step 5. The punchline. Since U is bounded and continuous, $\int_X U(x)p(dx)$ is well defined for all Borel p. We merely need to confirm that $u(p)$ is this integral for all Borel p, and we're done. I'm going to give a slightly clumsy proof of this, but one that doesn't need any special knowledge of the theory of integration, besides what we've assumed so far.

Take any p. If $u(p) \neq \int_X U(x)p(dx)$, then either $u(p) >$ the integral or less. I'll deal with the case that it is greater; you can supply the details for the case where it is less. Let $u(p) - \int_X U(x)p(dx) = \epsilon > 0$.

Take some sequence of simple probabilities $\{p_n\}$ with limit p in the topology of weak convergence. Since U is bounded and continuous, we know that $\int_X U(x)p(dx) = \lim_n \int_X U(x)p_n(dx)$; that's implied by convergence in this topology. Hence, for some N and all $n > N$,

$$u(p_n) = \int_X U(x)p_n(dx) < \int_X U(x)p(x) + \epsilon/3.$$

Of course, this means that, for all $n > N$, $u(p_n) + 2\epsilon/3 < u(p)$; hence $p \succ p_n$ for all $n > N$. Re-lable p_{N+1} as p', and let $a \in (0,1)$ be such that $u(ap + (1-a)p') = au(p) + (1-a)u(p') = \int_X U(x)p(dx) + 2\epsilon/3$. Then for all $n > N$, $p_n \prec ap + (1-a)p'$. Since $\lim_n p_n = p$, this implies $p \preceq ap + (1-a)p'$, which contradicts $p \succ p'$. ∎

Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Chapter 6: Utility for Money

Summary of the Chapter

Chapter 5 discussed the basic models of choice under uncertainty in some generality. In this chapter, we specialize to the expected-utility model (where probabilities are given "objectively") and we further specialize to the case where the prizes are one dimensional and interpreted as "money." We always have a consumer who is an expected-utility maximizer for simple lotteries with money prizes and whose utility function is generally denoted by U. Within this context, three general tasks are undertaken:

- In the spirit of Chapter 2, we look in Section 6.1 for "natural" properties of the consumer's preferences over lotteries and for corresponding properties of the utility function U. Monotonicity and continuity of U are quickly disposed of (some discussion of the continuity of U requires some advanced mathematics; it helps if you know about the topology of weak convergence on probability measures). The heart of this section, though, involves risk aversion and the comparison of the levels of risk aversion of two different consumers or one consumer but at different levels of wealth. Risk aversion corresponds to concavity of U; measures of the level of risk aversion (for smooth enough utility functions) concern the so-called coefficient of risk aversion, $-U''/U'$. We conclude with a brief discussion (all proofs left as exercises) of two partial orders that are commonly applied to lotteries, first- and second-order stochastic dominance.

- Presumably, our consumer desires money because it will enable her to buy consumption goods. So preferences over lotteries with money prizes are *derived* preferences, derived from more primitive preferences over lotteries of consumption bundles. Section 6.2 discusses this and goes on to answer the question: What properties on preferences over lotteries of consumption bundles yield commonly assumed properties for preferences on lotteries over income? (In setting up this discussion, some commentary is provided about how economists model "derived" or "induced" preferences. This is the sort of thing that is easy to skim through; this is one place where you might want to slow down. In particular, if you think the points made about this are all obvious, try Problem 6.7.)

- The most basic common applications of these theories concern consumer demand for insurance and for risky assets. Section 6.3 presents the most basic result in the theory of insurance; you are asked to tackle the basics of demand for risky assets in the problems.

The material in this chapter is a bare introduction to aspects of *The Economics of Uncertainty*, about which whole courses are sometimes offered. (And this doesn't touch at all on *The Economics of Information*, which is a natural partner with the economics of uncertainty, but which requires tools and concepts that we only will reach in Volume II.) As for applications, applied choice under uncertainty is the basis of much of the theory of financial markets, about which other entire books have been written and courses are taught. So your instructor is likely to have his or her own ideas about which material in this chapter and what sorts of supplements are worth your while.

Solutions to Starred Problems

■ 6.1. (a) Suppose U is strictly increasing. Then $\delta_x \succ \delta_y$ if and only if $U(x) > U(y)$ if and only if $x > y$. Note in this chain of implications that the expected utility of δ_x is $U(x)$, and that of δ_y is $U(y)$. Conversely, suppose $\delta_x \succ \delta_y$ if and only if $x > y$. By the representation, we know that $\delta_x \succ \delta_y$ if and only if $U(x) > U(y)$, so we have $x > y$ if and only if $U(x) > U(y)$, and U is strictly increasing.

(b) Suppose U is concave. Then for all x_1, x_2, and $a \in [0,1]$, $U(ax_1 + (1-a)x_2) \geq aU(x_1) + (1-a)U(x_2)$. Consider the induction hypothesis, that for all $n \leq N$, lists of real numbers $x_1, \ldots, x_n$ in the domain of U, and lists of nonnegative numbers $a_1, \ldots a_n$ such that $\sum_{i=1}^{n} a_i = 1$, $U(a_1x_1 + \ldots + a_nx_n) \geq a_1U(x_1) + \ldots + a_nU(x_n)$. The induction hypothesis is trivially true for $N = 1$, and it is true for $N = 2$ by the definition of a concave function. So now assume it is true for some N; we will demonstrate it is true for $N+1$ and therefore show it is true for all positive integer N.

To show it is true for $N+1$, take a list of real numbers $x_1, \ldots, x_n$ and a list of nonnegative real numbers $a_1, \ldots, a_n$, where $n \leq N+1$, and such that $\sum a_i = 1$. If $n \leq N$, we know that $U(a_1x_1 + \ldots + a_nx_n) \geq a_1U(x_1) + \ldots + a_nU(x_n)$ by the induction hypothesis, so we need only consider the case where $n = N+1$. And if any of the a_i are zero, then we essentially are in the case $n \leq N$, so we can assume that $a_{N+1} > 0$, at least. For $i = 1, \ldots, N$, let $b_i = a_i/(1-a_{N+1})$. We know that $\sum_{i=1}^{N} b_i = [\sum_{i=1}^{N} a_i]/(1-a_{N+1}) = 1$, and so by the induction hypothesis,

$$U(b_1x_1 + \ldots + b_Nx_n) \geq \sum_{i=1}^{N} b_iU(x_i).$$

But by concavity of U,

$$\begin{aligned} U\big(a_1x_1 + \ldots + a_{N+1}x_{N+1}\big) &= U\big((1-a_{N+1})(b_1x_1 + \ldots + b_Nx_N) + a_{N+1}x_{N+1}\big) \\ &\geq (1-a_{N+1})U\big(b_1x_1 + \ldots b_Nx_N\big) + a_{N+1}U\big(x_{N+1}\big) \\ &\geq (1-a_{N+1})\left[\sum_{i=1}^{N} b_iU(x_i)\right] + a_{N+1}U(x_{N+1}) \\ &= a_1U(x_1) + \ldots + a_NU(x_N) + a_{N+1}U(x_{N+1}). \end{aligned}$$

This establishes the induction hypothesis. (Please note that nowhere in this proof did we use the fact that the x_is are real numbers. This works for any concave function on any convex domain.)

(c) We suppose that X is an interval of R and U is continuous. Take any simple lottery π. Since π has finite support, there are $\overline{x}$ and $\underline{x}$ in the support of π such that $U(\overline{x}) \geq U(x) \geq U(\underline{x})$ for all x in the support of π. Therefore, $U(\overline{x}) \geq \sum \pi(x)U(x) \geq U(\underline{x})$, where the sum is over the support of π. But then, by the intermediate value theorem, there is some x^* in the interval from $\overline{x}$ to $\underline{x}$ (we don't know which is greater as a real number, so I can't write the interval as $[\underline{x}, \overline{x}]$, for instance) such that $U(x^*) = \sum \pi(x)U(x)$, which makes x^* a certainty equivalent of π. (Note that this would work for arbitrary X, as long as it is simply connected.)

Suppose U is strictly increasing. Suppose by way of contradiction that some lottery π had more than one certainty equivalent; say, x_1 and x_2 are both certainty equivalents, with $x_1 \neq x_2$. Since the real line is simply ordered, if $x_1 \neq x_2$, then either $x_1 > x_2$ or $x_2 > x_1$. But then, in the first case, $U(x_1) > U(x_2)$, and in the second, $U(x_2) > U(x_1)$. In either case, they are unequal, so they both cannot equal expected utility under π, so they cannot both be certainty equivalents of π. (Note that this depends very much on the fact that the domain of X is the real line or, at least, is simply ordered with respect to the order under which U is strictly increasing.)

■ 6.3. The expected utility for a lottery with (net) prizes $x^0 + \epsilon$ and $x^0 - \epsilon$, each with probability 1/2, is $[U(x^0 + \epsilon) + U(x^0 - \epsilon)]/2$. Let δ be the risk premium for this lottery; in other words, the certainty equivalent of the lottery is its expected value, x^0, less δ. Therefore, δ is defined by the equation

$$U(x^0 - \delta) = \frac{U(x^0 + \epsilon) + U(x^0 - \epsilon)}{2}.$$

The Taylor's series expansion of the quantity on the right to the 2nd derivative is

$$\begin{aligned} & 0.5[U(x^0) + \epsilon U'(x^0) + (1/2)\epsilon^2\, U''(x^0) + o(\epsilon^2)] + \\ & \qquad 0.5[U(x^0) - \epsilon U'(x^0) + (1/2)(-\epsilon)^2\, U''(x^0) + o(\epsilon^2)] \\ = & \quad U(x^0) + (1/2)\epsilon^2 U''(x^0) + o(\epsilon^2). \end{aligned}$$

And the exact Taylor's series expansion of the term on the left, to the first derivative, is

$$U(x^0) - \delta U'(x^0) + \delta^2 U''(x^0 - \alpha\delta)/2,$$

where $\alpha \in [0,1]$. So we have

$$U(x^0) - \delta U'(x^0) + \delta^2 U''(x^0 - \alpha\delta)/2 = U(x^0) + \frac{1}{2}\epsilon^2 U''(x^0) + o(\epsilon^2), \text{ or}$$

$$\delta U'(x^0) + \delta^2 U''(x^0 - \alpha\delta)/2 = \frac{1}{2}\epsilon^2 U''(x^0) + o(\epsilon^2).$$

Divide both sides of this equation by ϵ. As $\epsilon \to 0$, the term on the right has limit zero. And on the left, since $\delta > -\epsilon$ by the monotonicity of U, the second term goes to zero. Therefore, $\delta/\epsilon \to 0$ as $\epsilon \to 0$; that is, δ is $o(\epsilon)$. But then the remainder term is $o(\epsilon^2)$, so the equation is

$$-\delta\, U'(x^0) = \frac{1}{2}\epsilon^2 U''(x^0) + o(\epsilon^2),$$

which, if you divide both sides by $-U'(x^0)$, is the conclusion of the proposition.

Concerning Problems 6.5 and 6.6. The solutions to Problems 6.5 and 6.6 that I give here require a bit of sophistication concerning the language of random variables, their expectations, and their conditional expectations. And in the solution to Problem 6.6, to avoid the use of algebra that hides some relatively simple ideas, I resort to "picture proofs" and "proof by example" when it is expositionally convenient. (It is worth noting that for the hard part in all this—part c of Problem 6.6—I essentially sketch the clever "direct construction" due to Machina and Pratt, 1997.)

■ 6.5. (a) We assume that π and ρ are simple lotteries such that for every nondecreasing utility function U, the expected utility under π is at least as large as under ρ. If this is so, it is so in particular for the utility function U_x defined by

$$U_r(x) = \begin{cases} 1, & \text{if } x > r, \text{ and} \\ 0, & \text{if } x \le r. \end{cases}$$

But for this utility function U_r, the expected utility under π is the probability that π gives a prize strictly greater than r, and similarly for ρ. Therefore,

$$\sum_{x>r} \pi(x) \quad \geq \quad \sum_{x>r} \rho(x), \tag{$\star$}$$

where the value of $\pi(x)$ is implicitly zero if x is not in the support of π, and similarly for ρ. Since the sum of all probabilities for any lottery is 1, this means that

$$F_\pi(r) = \sum_{x\leq r} \pi(x) \quad \leq \quad \sum_{x\leq r} \rho(x) = F_\rho(r),$$

and this is moreover true for all r. This is what we are asked to show.

You may be unhappy that the utility function U_r used here is discontinuous. If so, redo the analysis with $U_{r,\epsilon}$ defined by $U_{r,\epsilon}(x) = 0$ if $x \leq r$, $= 1$ if $x \geq r + \epsilon$, and $= (x - r)/\epsilon$ for $x \in (r, r + \epsilon)$. If you apply the fact that $\pi \geq^1 \rho$ for this utility function and then, for fixed r, take the limit of the resulting inequalities as $\epsilon \to 0$, you will get the inequality $\star$, from which the result follows.

(b) The random variable $X = x_i$ if $U \in (F_\pi(x_{i-1}), F_\pi(x_i)]$, which has probability $F_\pi(x_i) - F_\pi(x_{i-1}) = \sum_{x\leq x_i} \pi(x) - \sum_{x\leq x_{i-1}} \pi(x) = \pi(x_i)$. The only exception to this is for the case $i = 1$, and by the rule given, the probability that $X = x_1$ is the probability that $U \leq F_\pi(x_1)$, which is $\pi(x_1)$.

(c) On a probability space on which a uniformly distributed random variable U is defined, let $X_F = F^{-1}(U)$ and let $X_G = G^{-1}(U)$. By part b, X_F has distribution F and X_G has distribution G. And since $F(r) \leq G(r)$ for all r, $F^{-1}(u) \geq G^{-1}(u)$ for all $u \in [0, 1]$, hence $X_F \geq X_G$ for all values of U, and $X_F - X_G \geq 0$ with probability one.

(d) If X_π is a random variable with distribution π, then the expected utility under π is the expectation of the random variable $U(x_\pi)$, for which we write $E[U(X_\pi)]$. And if X_π and X_ρ are (joint) random variables on some probability space such that $X_\pi - X_\rho \geq 0$ with probability one, then $X_\pi \geq X_\rho$ with probability one. Therefore, if U is a nondecreasing function, $U(X_\pi) \geq U(X_\rho)$ with probability one, which implies that $E[U(X_\pi)] \geq E[U(X_\rho)]$. Since this is true for any nondecreasing function, we have satisfied the conditions for the definition of $\pi \geq^1 \rho$ to hold.

■ 6.6. Rather than write in full "cumulative distribution function," the abbreviation c.d.f. will be used.

(a) We suppose that $\pi \geq^2 \rho$, which means that expected utility computed for π is at least as large as expected utility computed for ρ, for any utility function U that is nondecreasing and concave. As suggested in the problem, we look at the expected utilities of π and ρ for the specific utility function $U_{r^0,r^1}(x)$ which equals $x - r^0$ for $x \leq r^1$ and equals $r^1 - r^0$ for $x \geq r^1$, where r^0 is any number less than all values in the supports of both π and ρ and r^1 is any number greater than r^0.

The expected utility of this utility function for π, which I'll denote by $E^\pi[U_{r^0,r^1}]$, is then

$$\sum_{x \in \text{supp}(\pi), x \leq r^1} (x - r^0)\pi(x) \quad + \quad (r^1 - r^0)(1 - F_\pi(r^1)),$$

and similarly for ρ. Because $\pi \geq^2 \rho$, we know that $E^\pi[U_{r^0,r^1}] \geq E^\rho[U_{r^0,r^1}]$, which implies that

$$(r^1 - r^0) - E^\pi[U_{r^0,r^1}] \leq (r^1 - r^0) - E^\rho[U_{r^0,r^1}].$$

But I assert that

$$(r^1 - r^0) - E^\pi[U_{r^0,r^1}] = \int_{r^0}^{r^1} F_\pi(r)dr, \qquad (\star\star)$$

and similarly for ρ, which (once I show this) gives part a. Why is $(\star\star)$ true? This is an application of integration by parts (which is how one would show this formally), most easily seen in a picture. See Figure G6.1. Panel a depicts (with dashed lines) the function F_π for a typical simple probability distribution π, writing the elements of the support of π as $\{x_1, x_2, x_3, x_4\}$ and with r^1 between x_2 and x_3. (You should mentally construct the picture for the cases where r^1 is in the support and where it lies above the elements in the support of π.) Panel b then shows, for this example, the terms that go into $E^\pi[U_{r^0,r^1}]$. There are two terms in the summation, the areas of the two more heavily shaded rectangles; the final term is the area of the more lightly shaded rectangle. Turning finally to panel c, you see a representation of $r^1 - r^0$, which is the area of the heavily bordered rectangle in both panels b and c less then shaded areas in panel b, or the shaded area in panel c. This is manifestly the integral on the right-hand side of $(\star\star)$.

(b) In this part of the problem, we are given two (joint) random variables X_π and X_ρ, where X_π has distribution π, X_ρ has distribution ρ, and $E[X_\rho - X_\pi | X_\pi] \leq 0$. The objective is to show that $\pi \geq^2 \rho$, which means that for all nondecreasing and concave utility functions U, the expected utility of U under π is greater or equal to the expected utility of U under ρ, or $E[U(X_\pi)] \geq E[U(X_\rho)]$. We verify this by showing that the conditional expectations, conditional on X_π bear this relation; that is

$$E[U(X_\rho)|X_\pi] \leq E[U(X_\pi)|X_\pi] = U(X_\pi);$$

once this is shown, the law of iterated conditional expectations gives the desired result. But since U is concave, we know that $E[U(X_\rho)|X_\pi] \leq U(E[X_\rho|X_\pi])$, and since U is

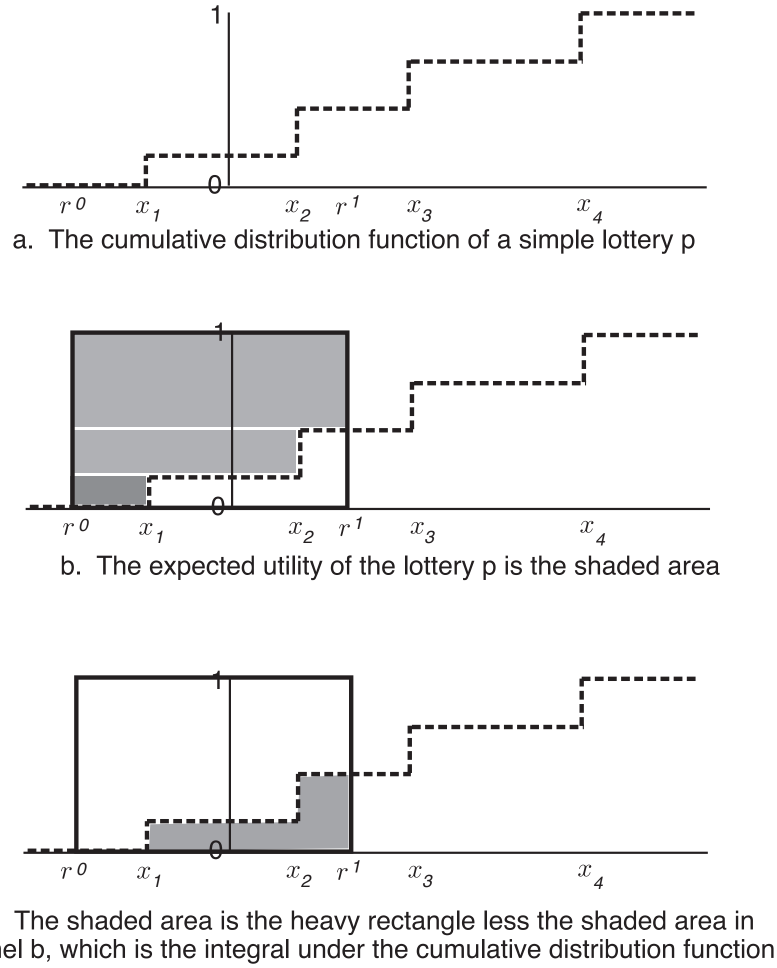


Figure G6.1. Proving Part a of Problem 6.6. The key step in the proof is an application of integration by parts, to show equation ($\star\star$). Panel a graphs the cumulative distribution function of a simple lottery with a dashed line. In panel b, $E^{\pi}[U_{r^0,r^1}]$ is computed "graphically"; for points in the support of π that are less than r^1, we get the probability of the point (the height of respective rectangle) times $x - r^0$ (the base); plus we get the term $(r^1 - r^0)(1 - F_{\pi}(r^1))$, the area of the third rectangle. We subtract this expected utility from $r^1 - r^0$, the area of the heavily outlined rectangle in both panels b and c, and we are left with the shaded area in panel c, which is clearly the integral of π's c.d.f. from r^0 up to r^1.

nondecreasing and $E[X_\rho|X_\pi] \leq X_\pi$, $U(E[X_\rho|X_\pi]) \leq U(X_\pi)$. Combining these two inequalities does it.

This "proof" is likely to seem like hand-waving to readers who are not familiar with conditional expectations at a somewhat advanced level (where a conditional expectation is itself a random variable). In fact, it is not hand-waving at all. But for readers who find it hard to follow, let me explain it in terms of the sort of compound probability trees you see in Figure 6.2 of the text. When computing the expected utility under ρ, you can convert each endpoint to utility and then compute expected utility in two rounds, first computing the expected utility for the "second-stage" uncertainty and then overall. For any concave utility function U, the expected utility computed for the second-stage will, for each first-stage result, be less than or equal to the utility if there is no second stage. Look, for instance, at Figure 6.2(a) and (b) and, in particular, at the middle branches: In panel a, we have the outcome 5 as an outcome under π. But for ρ, we have in the middle the outcomes $5-2=3$ with probability 0.75 and $5+3=8$ with probability 0.25. Because U is concave, the expected utility of 3 with probability 0.75 and 8 with probability 0.25 can be no greater than the utility of the expected value, which is (3)(0.75) + (8)(0.25) = 4.25. Since U is nondecreasing, this can be no greater than the utility of 5. In general, this always happens, because the "supplement" in the second stage to what happens in the first stage is always random (invoking the concavity of U) and has expected value less than zero (which invokes the assumption that U is nondecreasing.) That, in somewhat painful detail, is what the two inequalities with the conditional expectations are saying.

(c) This part of the problem is by far the hardest piece of these two problems. Because we are dealing with simple probability distributions, it can be done rather neatly and slickly—it takes a bit more math to do it in general. But it is not something that one would expect most students to come up with on their own. And even for simple probabilities, to write out all the details of a formal proof is a formidable task. So, instead, I will indicate how this is proved though an example.

We are trying to show that the "integral of the c.d.fs." condition implies the "$X_\rho = X_\pi +$ conditionally (on X_π) nonpositive-mean 'noise'" characterization. So the first thing to do is to understand the structure of of the integral-of-the-c.d.f. functions. Fix an r^0 less than every member of the support of π, and let

$$I_\pi(r) = \int_{r^0}^{r} F_\pi(r')dr', \quad \text{for } r \geq r^0.$$

This function is piece-wise linear and convex. Until the first (smallest) element of the support of π, it is zero (and in this respect, the choice of r^0 is irrelevant as long as it is smaller than this first element in the support); after the last (largest) element of the support of π, it has slope 1. Figure G6.2(a) provides an example; this is for the lottery π which gives prize 0 with probability 0.4, 5 with probability 0.2, and 10 with probability 0.4, the lottery in panel a of Figure 6.2 in the text. And in Figure G6.2(b) we superimpose (with dashed line segments) the function $I_\rho(r)$ for ρ that gives -3 with probability 0.2, 3 with probability 0.35, 8 with probability 0.25, and 12 with probability 0.2. (It is hard to see the kinks in I_ρ at the values 8 and 13, but they are there.) Note that $I_\pi \leq I_\rho$; we know this must be true because we know from parts a and b of this problem that ρ can be realized as π plus "noise" with nonpositive conditional means—that's what Figure 6.2 demonstrates—hence $\pi \geq^2 \rho$, and hence $I_\pi \leq I_\rho$.

Two facts about these functions should be noted:

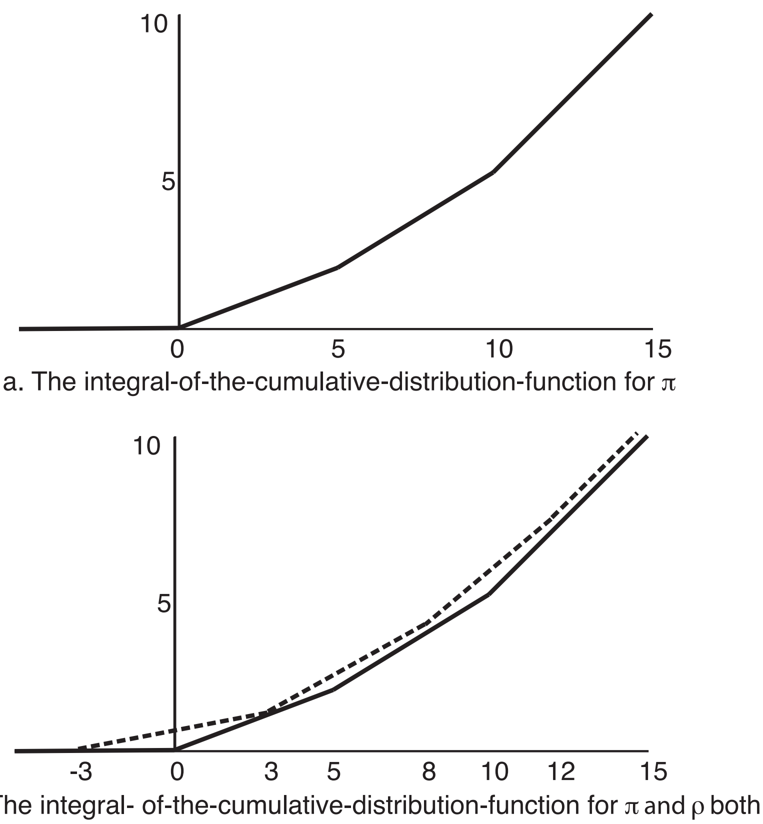


Figure G6.2. Graphing the integral-of-the-c.d.fs. In panel a, the function $I_\pi(r) = \int_{r^0}^{r} F_\pi(r')dr'$ is graphed, for π the simple lottery in panel a of Figure 6.2 of the text; π gives 0 with probability 0.4, 5 with probability 0.2, and 10 with probability 0.4. Note that this function is zero up to the first (smallest) value in the support of π, and then piecewise linear and convex, with "final" slope of 1. Panel b superimposes the same function for the simple lottery that is depicted in Figure 6.2(c) of the text. This second lottery is 2nd-order stochastically dominated by the first, so its integral-of-the-c.d.f. function is "higher", as guaranteed by parts a and b of this problem.

1. Any function of this sort—that is, piecewise-linear and convex, with value (and slope) zero below some value and with "terminal" slope 1 after some value—can be "unwound" into a corresponding simple lottery π: The location of the kinks constitute the support of π, and the size of the change in slope at the kink gives the probability of that value.

2. If I_π and I_ρ are two of these functions, the "higher" of the two once both hit slope 1 corresponds to the lottery with the greater mean. Therefore, π and ρ have the same mean if and only if I_π and I_ρ coincide after the last (greatest) value in the

union of their supports. To prove this, you need to enlist the integration-by-parts argument alluded to (and shown graphically) in part a of this problem. This is left for you to do.

Therefore, we know that if $\pi \geq^2 \rho$, then π has mean greater or equal to the mean of ρ. (We know this directly by using the utility function $U(x) = x$, of course.)

So much for preliminaries. Now for the heart of the argument. Supposing $I_\pi \leq I_\rho$, we "convert" I_π into I_ρ one step at a time. Each step involves adjusting one linear segment at a time, moving from the right (greater values of x) to the left. We are looking at a picture like panel b of Figure G6.2: In the first step, we take the "terminal segment" of I_ρ—the segment with slope 1—and extend it down and to the left, until it hits I_π. And we replace I_π with the integral-of-c.d.f. function that is I_π up to this point of intersection and is the extended "terminal segment" thereafter, creating a π^1 whose integral-of-the-c.d.f. function this is. (If π and ρ have the same mean, this first step is unnecessary, as their "terminal segments" are already coincident.) This in essence will "wipe out" one or more of the largest values in the support of π, replacing them with a lower value. If the extension (down and to the left) of I_ρ's final segment hits I_π right in a kink of I_π, it will mean increasing the probability of the value at that kink.

So, for instance, if we extend the final segment of I_ρ in Figure G6.2 down and to the right, we hit I_π at the value 9.625. We are, in essense, supplementing the prize 10 in π by -0.375, so the result is a lottery with prizes 0, 5, and 9.625, with probabilities 0.4, 0.2, and 0.4, respectively. Note that the value -0.375 is computed once we know the target value of 9.625, and we know that because that is where the extended final segment hits I_π. See panel a Figure G6.3.

In panel b of Figure G6.3, we have I_{π^1} and I_ρ, where π^1 is π modified as in the first stage; that is, the old prize of 10 is replaced by 9.625. Note that ρ and π^1 coincide past the value 12 (by design), so ρ and π^1 have the same mean. Hereafter, any changes we make will be zero conditional-mean changes.

And to make the first of these (which is our second change, from π^1 to π^2), we extend the next segment of I_ρ, the segment from 8 to 12, down and to the left, until it hits I_{π^1}. This, you can compute, happens at the value 7.25, and the extended segment "cuts out" the value 9.625 in π^1. In place of the 0.4 mass at 9.625, it puts mass 0.2 at 12 and 0.2 at 7.25. As shown in the right-hand portion of Figure 6.3b, this amounts to a "supplement" to the prize 9.625 of 2.375 with probability 0.5 and -2.375 with probability 0.5, giving π^2 which is 0 with probability 0.4, 5 with probability 0.2, 7.25 with probability 0.2, and 12 with probability 0.2. The fact that the supplement gives ± 2.375, that is, the same size prize on the upside as on the downside, is specific to this example; this does not happen in general. But the supplement will always be a zero-conditional mean supplement, because the means of the two lotteries start the same and end the same.

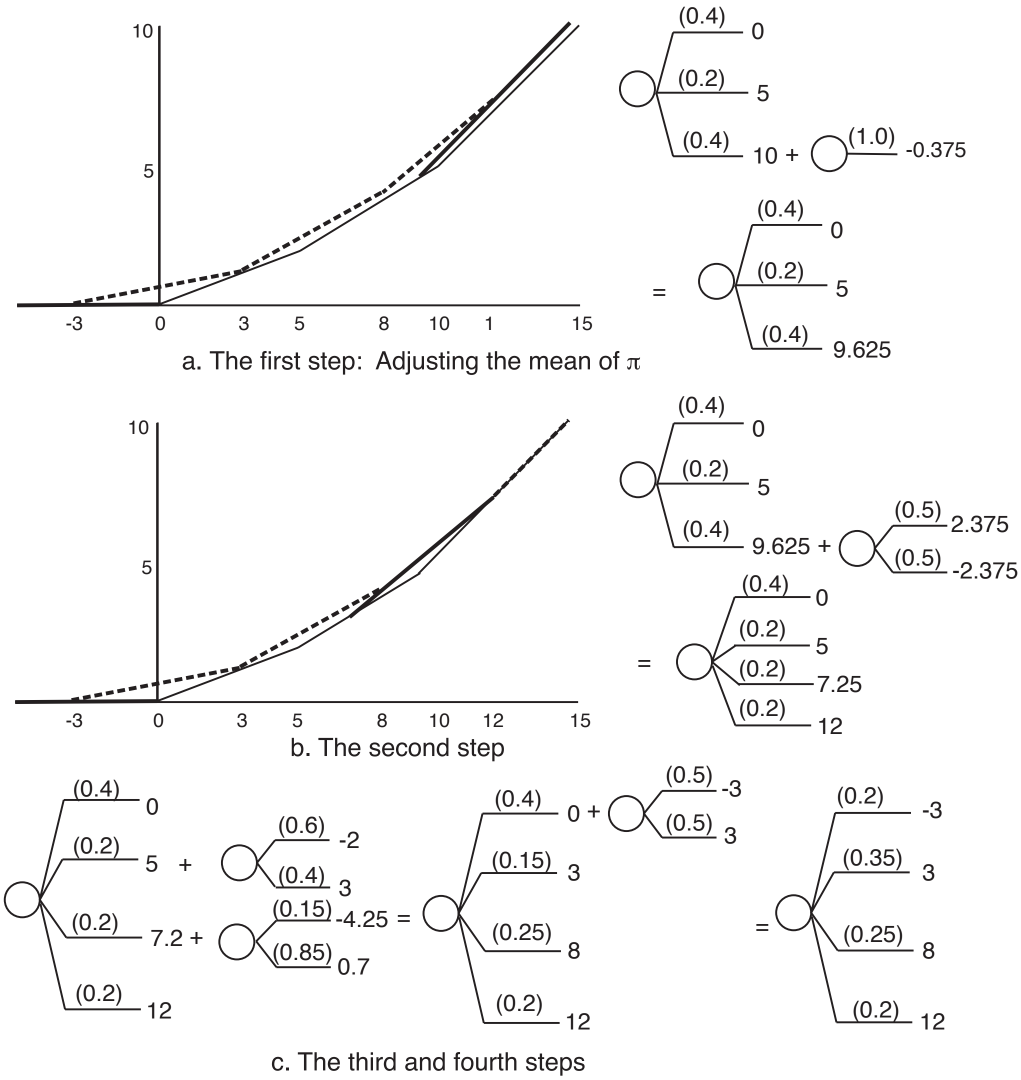


Figure G6.3. Converting π to ρ, one step at a time. See text for details.

In the next step (passing from π^2 to π^3), we "extend" the next segment of I_ρ down and to the left, until it hits I_{π^2}. In fact, since $I_\rho = I_\pi$ at the value 3, no extension is needed; we are going to replace in π^2 the *two* intermediate prizes 7.25 and 5 with prizes 8 and 3. For 7.25, this means a supplement of 0.75 on the upside and -4.25 on the downside which, to balance (to have zero conditional mean) requires probability 0.15 of -4.25 and 0.85 of 0.75. And for the supplement to 5, we need an addition of 3 and a subtraction of 2; to have zero conditional mean, this will take 0.4 probability for the supplement of 3 and 0.6 for the supplement -2. It takes a proof to show that, in general, these "balanced" or "zero conditional mean" supplements to the "passed over" kinks give the right probabilities in the end, but because the lotteries have to start and end with the same mean, this can be proved. In panel c of Figure 6.3, we show (without the graph) the transition from π^2 to π^3. And we add to panel c the final transition, from π^3 to ρ. This involves moving the 0.4 mass at 0 in π^3 to -3 and to $+3$.

Figure G6.4 summarizes the various steps in our transition from π to ρ and collapses (for each outcome of π) the changes made that led to ρ. This makes for an interesting comparison with Figure 6.2, which shows a different way to shift from π to ρ. Think in terms of the "supplementary random variable" Y that satisfies $X_\pi + Y = X_\rho$. In both Figure 6.2 in the text and here, the supplementary random variable has nonpositive conditional mean (conditional on X_π). But the supplement in Figure 6.2 is quite a different animal from the supplement here. In particular, they have different supports! The supplement in Figure 6.2 has support $\{-3, -2, 2, 3\}$, while the support of the supplement here includes -7 as well as the four other values. In general, there will be more than one way to move from a random-variable realization of one probability distribution π to the random-variable realization of a 2nd-degree stochastically dominated ρ, while respecting the rule that the supplement that accomplishes the move should have nonpositive conditional mean.

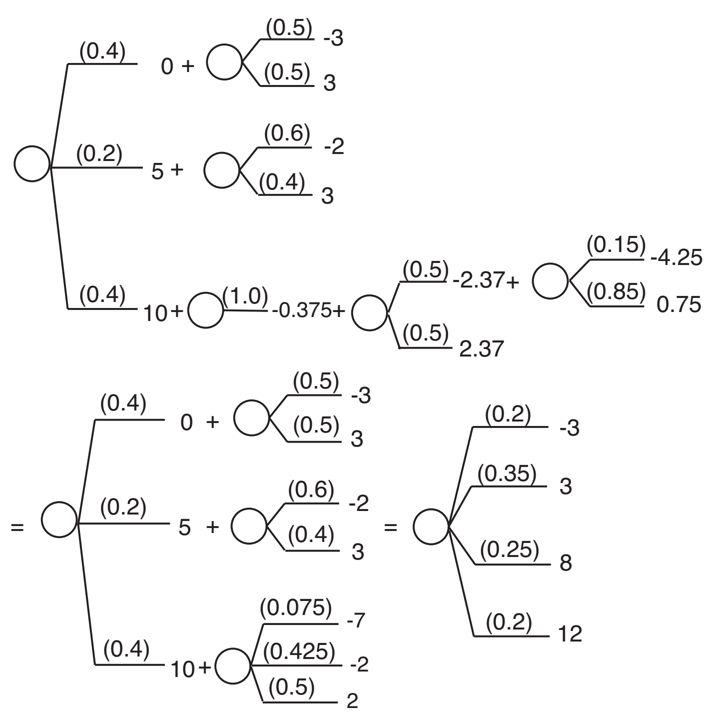


Figure G6.4. Summarizing the transition from π to ρ in Figure G6.3.

It hardly needs saying, I hope, that what just happened does not constitute a proof of what you were asked to prove in part c of Problem 6.6. The pictures we've drawn may convince you that this can be proved. But all we've done is to work through an illustrative example.

Writing out all the details of a proof for the case of simple probability distributions is not complex, just tedious. But since this result generalizes beyond simple probabilities, if you wish to pursue it further, you might want to see a proof for the more general case. If that is so, see Machina and Pratt (1997).

■ 6.8. (a) For these preferences, if $p = (1,3)$, the consumer will consume good one only, or the bundle $(y,0)$, while at the prices $p = (3,1)$, she'll choose $(0,y)$, so her utility (for certain) will be $f(y)$. But if we take the average prices $(2,2)$, she chooses (for instance) $(y/4, y/4)$ (or any bundle where $x_1 + x_2 = y/2$), for utility $f(y/2)$. Clearly she prefers the risky prices.

(b) At any prices (with this utility function), the consumer chooses to allocate her income to equalize the amounts of the goods she purchases. So at the prices $p = (\gamma,\gamma)$, she will consume $y/2\gamma$ of each good, for utility $f(y/(2\gamma))$, while at the prices $p = (1/\gamma, 1/\gamma)$, she'll get utility $f(\gamma y/2)$. Therefore, her average utility is

$$\frac{f(y/(2\gamma)) + f(y\gamma/2)}{2}.$$

But at the average prices, her utility is

$$f(y/(\gamma + 1/\gamma)).$$

Fix strictly increasing f. As γ increases toward infinity, the average utility is approximately $[f(0) + f(\gamma y/2)]/2$, while the utility at average prices approaches $f(0)$. So the former is greater. But fix γ. Since γ is greater than one,

$$\frac{y}{2\gamma} < \frac{y}{\gamma + 1/\gamma} < \frac{\gamma y}{2}.$$

No matter what are the values of $f(y/(2\gamma))$ and $f(y/(\gamma + 1/\gamma))$, by making f very concave, we can make $f(\gamma y/2)$ small enough so that the term in the middle is greater than the average of the two on the outside—the consumer prefers the sure prices.

■ 6.10. As in the proof of Proposition 6.17, the second half of Proposition A3.17(b) tells us that v as defined in (6.3) is concave. Moreover, the derivative of v is

$$v'(a) = \sum_{\theta \in \text{supp}(\pi)} V'(a(\theta - r) + rw)(\theta - r)\pi(\theta).$$

Therefore, $v'(0) = \sum(\theta - r)V'(rw)\pi(\theta) = V'(rw)(E\theta - r)$.

(a) If $E\theta < r$, then $v'(0) < 0$, and since v is concave, $v'(a) \leq v'(0) < 0$ for all $a \geq 0$. Therefore, the solution, and the only solution, is $a = 0$.

(b) If $E\theta = r$, then $v'(0) = 0$, and $a = 0$ is a solution.

(c) If $E\theta > r$, then $v'(0) > 0$, hence $a = 0$ is not a solution.

(d) If $V(y) = Ay + B$ for $A > 0$, the first-order condition is $\sum_{\theta \in \text{supp}(\pi)} (\theta - r) A \pi(\theta) = A(E\theta - r) \leq 0$.[1] Since a is entirely removed from v' in the first-order condition, there are three possibilities: If $E\theta < r$, the consumer chooses $a = 0$ (which we already knew). If $E\theta = r$, any a is a solution. And if $E\theta > r$, there is no solution.

(e) Since V is strictly increasing amd concave, $V' > 0$. If $\theta \geq r$ for all $\theta \in \text{supp}(\pi)$, then each term in $\sum_\theta (\theta - r) V'(a(\theta - r) + rw)\pi(\theta)$ is nonnegative. And if $\theta > r$ for at least one θ with positive probability, the corresponding term in the sum is always strictly positive. Hence the sum can never be nonpositive, and there is no solution to the problem.

(f) We show that v is strictly concave. (Once we know v is strictly concave, we know from Proposition A3.21 that (6.3) can have at most one solution over any convex set.) We do this by showing that v' is strictly decreasing. Note that v' is composed of a sum of terms. For those terms for which $\theta > r$, $\theta - r > 0$, and increasing a increases the argument of V', hence decreases V'. Hence increasing a decreases (strictly) all terms with $\theta > r$. While for terms for which $\theta < r$, increasing a decreases the argument of V', increasing V', but the coefficient $\theta - r$ is negative. So these terms get increasingly negative—that is, they decrease—as a is increased. (For $\theta = r$, changing a has no effect on the corresponding term in the sum.) Since the support of π contains at least two elements, one of them must be different from r, hence increasing a strictly decreases the sum, that is, $v'(a)$. ∎

■ 6.12. Suppose there are three assets available: A riskless asset, which returns $r = 5\%$ for sure, risky asset #1, which (say) returns 10% with probability 0.9 and 4% with probability 0.1, and:

- Risky asset #2, which returns 10% with probability 0.1 and 4% with probability 0.9, for an overall expected return of 4.6%. But this risky asset #2 returns 10% in precisely those states of the world where risky asset #1 returns 4%; the two are (perfectly) negatively correlated. Then consider any investor who, if given a portfolio choice of the safe asset and asset #1, would choose some of the safe asset. (Make the investor sufficiently risk averse, and this must happen.) If we add risky asset #2 to the possible investments, then this investor will necessarily choose some of risky asset #2: A portfolio of equal dollars invested in the two risky assets produces a riskless asset with overall (sure) return of 7%, dominating the riskless asset, so the optimal choice is necessarily a split between the two risky assets (only).

[1] Don't confuse A and a here!

- Risky asset #2, which returns 9% with probability 0.9 and 4% with probability 0.1, for an overall expected return of 8.5%. But this asset returns 9% precisely in those states of the world where asset #1 returns 10%. So this asset is dominated by risky asset #1, and no investor (who has an increasing utility function) would ever choose to invest in it.

The moral (well known in the theory of finance) is: When constructing an optimal portfolio out of various assets, how they correlate with one another is hugely important.

■ 6.14. Suppose that our consumer has constant absolute risk aversion; that is, if we denote her wealth by W, we have $u(W) = -e^{-\lambda W}$. Let $\tilde{X}_n$ be the (random) outcome of the nth gamble, so that if the consumer is given m gambles, her final wealth W will be

$$W = \sum_{i=1}^{m} \tilde{X}_i.$$

If all gambles are mutually independent, we have

$$E\left[u(W)\right] = E\left[-e^{-\lambda \sum_{i=1}^{m} \tilde{X}_i}\right] = E\left[-\prod_{i=1}^{m} e^{\lambda \tilde{X}_i}\right] = -\prod_{i=1}^{m} E\left[e^{-\lambda \tilde{X}_i}\right] = -\left(E\left[e^{-\lambda \tilde{X}_i}\right]\right)^m,$$

where the second-to-last equality follows by the independence assumption.

Therefore, independently of m, the consumer will take m copies of the gamble if and only if she is willing to take one, depending on whether $E[e^{-\lambda \tilde{X}_i}]$ is less than 1 or greater. This fits very well with the intuitive meaning of constant absolute risk aversion.

As for the challenge, (and using some results from the theories of random walks and martingales that you may not know): Whatever is the consumer's initial wealth (as long as it is not zero), as $N \to \infty$ the probability that the consumer will go bankrupt before the Nth gamble approaches some constant α (depending on the initial wealth) bounded away from one, and the probability that her wealth exceeds any finite level approaches $1 - \alpha$; that is, either she goes bankrupt eventually, or her wealth goes off to ∞. See, e.g., Chung (1974, Chapter 8).

Suppose u is unbounded above. Let Z be a wealth level sufficiently large so that $(1 - \alpha)u(Z) + \alpha u(0) > u(W_0)$, where W_0 is the consumer's initial level of wealth. (Since u is unbounded, such a Z exists.) Then for N sufficiently large, so that the probability of exceeding Z after N or more gambles is sufficiently close to $1 - \alpha$, the consumer will take the gambles. (Assume the consumer's initial wealth is a multiple of \$500, so we don't need to worry about her going bankrupt on a "part-loss." Also, throughout we are assuming that the consumer must take all the N gambles, limited by the bankruptcy constraint, or none at all, although you can work around that assumption for this part of the problem at least.)

When u is bounded above, things are a bit more delicate. Let $\hat{u} = \lim_{z\to\infty} u(z)$. Then as the number of gambles grows, and if the consumer chooses to gamble, her expected utility will approach $(1-\alpha)\hat{u} + \alpha u(0)$. To know whether this entices the consumer to take many gambles, we need to know whether $(1-\alpha)\hat{u} + \alpha u(0) > u(W_0)$. Assuming we know u, we know $u(0)$, $u(W_0)$, and $\hat{u}$. But what is α? What is the probability that the consumer's wealth will go off to infinity before she is bankrupt?

We can use martingale theory to find α.[2] Let W_n be the level of the consumer's wealth after n gambles (where we stop the process if the consumer is bankrupt, and we assume that the consumer takes the gambles), and let $X_n = .999610175^{W_n}$. The stochastic process $\{X_n\}$ is then a bounded martingale (or, rather, it will be if in place of the number .999610175 we have the root of the polynomial $.4\gamma^{1000} + .6\gamma^{-500} = 1$ that is near .999610175). This martingale either "absorbs" at the value of 1 (if the consumer goes bankrupt) or approaches 0 (if the consumer's wealth rises to infinity), and so by the martingale stopping theorem, we know that $1 \cdot \alpha + 0 \cdot (1-\alpha) = \alpha = .999610175^{W_0}$. This, then, gives us α; and—except for the knife-edge case where $(1-\alpha)\hat{u} + \alpha u(0) = u(W_0)$—given $\hat{u}$, $u(0)$, and W_0, we have everything we need to determine whether the consumer will take "many" gambles or not.

[2] Most advanced probability texts will cover this, as well as the random-walk theory we used previously. But for a definite reference, see Chung (1974), this time Chapter 9.

Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Chapter 7: Dynamic Choice

Summary of the Chapter

The theme of the first six chapters of the book has largely been choice by individuals, where the choices have been framed atemporally: The individual has a single choice to make; how do we model that? But most choices, and in particular most of the most important economic choices that individuals make, are part of a sequence of choices. So it is natural to ask (and try to answer) whether and how we adapt the models of choice we have developed to dynamic-choice contexts.

In most of the literature of economics, the answer is presumed to be so obvious that it is never explicitly discussed: An individual in a dynamic-choice situation thinks from the outset about all the choices she has to make and what will be the consequences for herself of the different *strategies* she might employ. She rates each strategy as being just as good as the outcome it engenders. And she chooses to employ, and proceeds flawlessly to implement, an *optimal* strategy; a strategy that leads to the best outcome available to her.

If this answer is accepted, then the issue becomes one of applying the model: Given a relatively complex dynamic-choice problem, how does one find an optimal strategy? The mathematical techniques of dynamic programming are generally employed in the economics literature; Appendix 6 provides you with a primer on these techniques.

That, for the most part, is the "story" about dynamic choice in the mainstream economic literature. The story is not subjected to critical examination. Chapter 7 relates the basics of the standard story and then suggests that critical examination may be in order. The standard story is predicated on (at least) three implicit assumptions:

1. The decision maker has (and acts on) fixed, atemporal preferences over outcomes. She will not change her mind about what she wants.

2. From the outset, the decision maker has full ”strategic awareness” of what options she has and will have and how her decisions will translate into outcomes. All this is allowed to depend on the resolution of uncertain events. But unforeseen contingencies, previously unappreciated options, and the general law of unintended consequences are not (relevant) parts of her vocabulary and, in particular, she makes no allowance today for any such possibilities affecting her in the future.
3. The decision maker has the cognitive and computational skills required to evaluate all the strategies she has available or, at least, to find one that is optimal.

Having made these assumptions explicit, the chapter suggests some possible alternatives to the standard story, if the assumptions are violated.

1. If the decision maker's tastes may change through time, she recognizes this, and she acts today in a way that tries to optimize the outcome from the perspective of today's tastes, then she may make choices today that eliminate from her later opportunity sets choices that, today, she would rather she didn't take later.
2. On the other hand, if she recognizes the possibility of unforeseen contingencies, and so forth, she may devote resources today to leaving herself with the flexibility later on to deal with them, even if today she doesn't foresee what they will be.
3. And faced with a truly complex dynamic decision problem, one that is intractable even with the technology of dynamic programming, she may resort to rules of thumb and decision heuristics, ranging from very simple forms of adaptive satisficing (take the first option that appears, that appears to be ”good enough”) to employing complex but mis-specified statistical models of her environment, to try to understand that environment.

While there are literatures on each of these alternatives, the literatures are small and under-developed. Following this chapter (certainly for the remainder of this volume), we employ the standard, strategic apporach to dynamic decision making. But as an informed consumer of economic methodology, you should (at least) know (1) that, notwithstanding the impression left in much of the literature that this approach is obvious, this approach is based on some pretty heroic assumptions, and (2) it is entirely possible and, I believe, desirable to explore alternatives.

Two sorts of problems can be connected to this chapter. The first sort provides you with practice solving dynamic choice problems of the standard sort, using the techniques of dynamic programming as described in Appendix 6. You should certainly do some problems of this sort; some are given in Appendix 6, with solutions provided in another “chapter” posted at this *Student's Guide*.

The other sort of problem explores alternatives to the standard model, and that's what you will find in the four problems at the end of Chapter 7: Problems 7.1, 7.2, and 7.3 concern the ”changing tastes” model, while 7.4 concerns ”preference for flexibility.” These problems do not build skills that you need for what follows in the book, and you shouldn't bother with them if you aren't curious about alternatives to the standard, strategic model. But if you are curious, here are solutions of the first three. (The text gives you a reference in which you can find the solution to the fourth problem).

Solutions to Starred Problems

■ 7.1. (a) The utility function is strictly concave, so there is a unique solution. The problem is completely stationary, so at the solution, $a_1 = a_2 = b_1 = b_2 = 25$. The utility obtained is 12.875...

(b) She will choose $a_1 = b_1 = 25$ and $s_1 = 50$ at the first stage, and $a_2 = b_2 = 25$ at the second stage.

(c) The problem here is to maximize $\ln(a_1) + (1 + a_1/25 \ln(a_2) + \ln(b_1) + \ln(b_2)$ subject to the constraint $a_1 + a_2 + b_1 + b_2 \leq 100$ and all variables nonnegative. I solved this numerically (using Excel and Solver) and got (to three decimal places) $a_1 = 61.876, a_2 = 24.179, b_1 = b_2 = 6.963$. Because her tastes are not changing (but are simply non-separable), it doesn't matter whether she chooses all at once or in two stages; if she has to save, she'll save (roughly) 13.962 at the first stage and wind up with the outcome given in the previous sentence.

(d) If the decision maker doesn't anticipate the shift in her tastes, she'll begin (naively) thinking she is in the setting of part b, consuming $a_1 = b_1 = 25$ and $s_1 = 50$. But then, when she gets to the second stage, she'll have her savings of 50 and want to maximize $\ln(b_2) + 2\ln(a_2)$, where the coefficient 2 comes from the formula $1 + a_1/25$ and the fact that $a_1 = 25$. We can work this out analytically, getting $b_2 = 33.333\ldots$ and $a_2 = 16.666\ldots$. Just for reference sake, let me evaluate the outome $(a_1, a_2, b_1, b_2) = (25, 33.333, 25, 16.666)$ with the first-stage utility function: it gives (to three decimal places) 12.758.

(e) A more sophisticated decision maker with tastes that change in this fashion reasons as follows: "If I consume a_1 and b_1 in the first period, saving $s_1 = 100 - a_1 - b_1$, then my second-period self will choose to maximize $(1 + a_1/25)\ln(a_2) + \ln(b_2)$ subject to the constraint that $a_2 + b_2 \leq s_1$. The solution she will find is

$$a_2 = \frac{25 + a_1}{50 + a_1} s_1 \quad \text{and} \quad b_2 = \frac{25}{50 + a_1} s_1.$$

So I want to choose a_1, b_1, and s_1 to maximize

$$\ln(a_1) + \ln(b_1) + \ln\left(\frac{25 + a_1}{50 + a_1} s_1\right) + \ln\left(\frac{25}{50 + a_1} s_1\right),$$

subject to the constraint that $s_1 + a_1 + b_1 \leq 100$. Using numerical methods, the solution (to three decimal places) is $a_1 = 22.189, b_1 = 25.937$, and $s_1 = 51.874$, with the second-stage choices then being $a_2 = 33.909$ and $b_2 = 17.965$, for a utility of 12.767. What is interesting here is that savings *increases*, which would seem to fly in the face of the discussion in the text, which is all about constraining future selves: Doesn't more savings give more freedom of choice to one's future self? The reason savings increases, though, is because it is clear that the second-stage self will choose a larger than desireable (by the first-period self) a_2 and a smaller than desireable b_2, and to diminish the deleterious effects of the second half of this, the first-period self saves more. But she also cuts back on her first-period consumption of asparagus, to reduce the impact of asparagus addiction.

■ 7.2(a) The problem is to maximize $c_0^{1/2} + 0.5c_1^{1/2} + 0.4c_2^{1/2}$ subject to $c_0 + c_1 + c_2 \leq 100$, for which the answer is $c_0 = 70.922, c_1 = 17.731, c_2 = 11.347$ (each of these approximately, to three decimal places), giving utility level 11.8743421.

(b) Being naive, the individual chooses the same c_0 as in part a, which means she saves $s_0 = 29.078$. But then, at time $t = 1$, she divides this between time $t = 1$ and $t = 2$ consumption in the way that maximizes $c_1^{1/2} + 0.5c_2^{1/2}$, which is, in a 4 to 1 ratio, or $c_1 = 23.262$ and $c_2 = 5.816$, for a $(t = 0)$ uility level 11.7976977.

(c) Being sophisticated, the individual knows that if she saves s_0 at time $t = 0$, at time $t = 1$ she'll consume 80% of that amount and leave 20% for time $t = 1$ consumption. So she picks c_0 (and $s_0 = 100 - c_0$) with that in mind. That is, she picks c_0 to maximize

$$c_0^{1/2} + 0.5[(0.8(100 - c_0))^{1/2}] + 0.4[(0.2)(100 - c_0))^{1/2}],$$

which gives $c_0 = 71.839$, $c_1 = 22.529$, and $c_2 = 5.632$ (as always, rounded to three decimal places), for a $(t = 0)$ utility level 11.798305. That's not much of an increase in her utility over part b, but it's there. Compared with problem 7.1, in this case sophistication leads her to consume *more* in the first period and to save less.

■ 7.3. Suppose $\dot{\succeq}$, defined on Z, is complete and transitive and satisfies

$$\text{For all } z \text{ and } z', \text{ either } z \dot{\sim} z \cup z' \text{ or } z' \dot{\sim} z \cup z', \qquad (G7.2.1)$$

and is antisymmetric when restricted to singleton sets. Define $\succeq_1$ and $\succeq_2$ on X as in the statement of the problem:

$$x \succeq_1 x' \text{ if } \{x\} \dot{\succeq} \{x'\} \quad \text{and} \quad x \succeq_2 x' \text{ if } \{x\} \dot{\sim} \{x, x'\}.$$

Since $\succeq_1$ is, essentially, $\dot{\succeq}$ restricted to singleton sets, it is evident that $\succeq_1$ is complete, transitive, and anti-symmetric.

As for $\succeq_2$, take any pair x and x' from X. By G7.2.1, either $\{x\} \dot{\sim} \{x, x'\} = \{x\} \cup \{x'\}$ or $\{x'\} \dot{\sim} \{x, x'\}$. Hence, by the definition of $\succeq_2$, $x \succeq_2 x'$ in the first case and $x' \succeq_2$ in the second case, therefore $\succeq_2$ is complete. Although it is redundant to say so, note that if $x = x'$, then $\{x\} = \{x, x'\}$, so $\{x\}\dot{\sim}\{x, x'\}$, so $x \succeq_2 x' = x$. On the other hand (and this part isn't redundant) if $x \neq x'$, then by then anti-symmetry of $\dot{\succeq}$, if $\{x\}\dot{\succeq}\{x'\}$, it cannot be the case that $\{x'\}\dot{\succeq}\{x\}$, and so $\{x\}\dot{\succ}\{x'\}$. We know that either $\{x\}\dot{\sim}\{x, x'\}$ or $\{x'\}\dot{\sim}\{x, x'\}$, but since (for $x \neq x'$) $\{x\}\dot{\sim}\{x'\}$ is not possible, we know that *exactly* one of these can be true, and $\succeq_2$ is anti-symmetric. (We'll show that $\succeq_2$ is transitive at the end.)

I assert that

$$\text{For every } z \in Z\text{, there exists some } x \in z \text{ such that } \{x\}\dot{\sim}z. \tag{G7.2.2}$$

This is shown using induction on the size of z. (The notation $\#z$ will be used for the cardinality of z.) This is clearly true for all singleton sets z, so suppose inductively that it is true for all sets of size $n-1$ and less. Suppose $\#z = n > 1$, and let x^* be any element of z and $z^0 = z \setminus \{x^*\}$. Then z^0 has cardinality $n-1$, and by the induction hypothesis, for some $x^0 \in z^0$, $\{x^0\}\dot{\sim}z^0$. But by G7.2.1, either

$$z^0\dot{\sim}z^0 \cup \{x^*\} = z \quad \text{or} \quad \{x^*\}\dot{\sim}z^0 \cup \{x^*\} = z.$$

In the second case, $z\dot{\sim}\{x^*\}$, while in the first case, $\{x^0\}\dot{\sim}z^0\dot{\sim}z$, hence $\{x^0\}\dot{\sim}z$. Since one of these two must hold, we know that z is indifferent to the singleton set consisting of one of its elements, completing the induction step and the proof by induction.

I assert that

$$\text{If } \{x\}\dot{\sim}z\text{, for } x \in z\text{, then } \{x\}\dot{\sim}z' \text{ for all sets } z' \text{ such that } x \in z' \subseteq z. \tag{G7.2.3}$$

The proof is again by induction on the size of z. This is vacuously true if z is a singleton set. Suppose it is true for all sets with $n-1$ or fewer elements, and let z be a set of n elements for $n > 1$. Suppose $\{x\}\dot{\sim}z$ for some $x \in z$. Let z' be any subset of z with $n-1$ elements, one of which is x, and let x' be the element of z left out of z'. Either $\{x'\}\dot{\sim}z$ or $z'\dot{\sim}z$ by G7.2.1. In the latter case, since $\{x\}\dot{\sim}z$, we conclude that $\{x\}\dot{\sim}z'$. In the former case, transitivity of $\dot{\sim}$ implies that $\{x\}\dot{\sim}\{x'\}$, contradicting anti-symmetry of $\dot{\succeq}$ restricted to singleton sets. So $\{x\}\dot{\sim}z'$. But by the induction hypothesis, this implies that $\{x\}\dot{\sim}z''$ for all subsets of z' that contain x. And since this is so for all subsets z' of z with $n-1$ elements, we have the induction step and, by induction, the asserted property.

Now we will show that $\succeq_2$ is transitive. Suppose $x \succeq_2 x'$ and $x' \succeq_2 x''$, which means that $\{x\}\dot{\sim}\{x, x'\}$ and $\{x'\}\dot{\sim}\{x'x''\}$. If $x = x' \neq x''$, then $x \succeq_2 x''$ is obvious, and similarly in the case $x \neq x' = x''$. If $x = x' = x''$, transitivity is clear. And the case $x \neq x'$ but $x = x''$ is impossible by anti-symmetry. So we are left with the case where these three elements are distinct from one another. I assert that $\{x\}\dot{\sim}\{x, x', x''\}$. We know that if this is not true, then either $\{x'\}\dot{\sim}\{x, x', x''\}$, or $\{x''\}\dot{\sim}\{x, x', x''\}$. In the first case, we would then know that $\{x'\}\dot{\sim}\{x,x'\}$, contradicting anti-symmetry. In the second case, we would know that $\{x''\}\dot{\sim}\{x',x''\}$, contradicting anti-symmetry. Hence it must be that $\{x\}\dot{\sim}\{x,x',x''\}$. But then $\{x\}\dot{\sim}\{x,x''\}$ by G7.2.3, and therefore $x \succeq_2 x''$.

G7.2.2 tells us that, for each $z \in Z$, some $x \in z$ satisfies $\{x\}\dot{\sim}z$. Since $\dot{\succeq}$ is antisymmetric on singleton sets, this x, which we henceforth label $x_2(z)$, is unique. Take any $x \neq x_2(z)$ from z. By G7.2.3, $x_2(z)\dot{\sim}\{x_2(z),x\}$ so, by definition, $x_2(z) \succeq_2 x$. Of course, $x_2(z) \succeq_2 x_2(z)$, and so

$$\text{For all } z,\ x_2(z) \succeq_2 x \text{ for all } x \in z. \qquad (G7.2.4)$$

Finally, by simple transitivity properties of $\dot{\succeq}$, for all z and z', $z\dot{\succeq}z'$ if and only if $\{x_2(z)\}\dot{\succeq}\{x_2(z')\}$ if and only if $x_2(z) \succeq_1 x_2(z')$. But this tells us that $\dot{\succeq}$ on Z is "explained" by the changing-tastes model, where first-stage preferences over meals are given by $\succeq_1$ and second-stage preferences over meals are given by $\succeq_2$.

As mentioned in the text, this proof, although perhaps a bit tedious, is greatly helped along by the assumption that $\dot{\succeq}$ is anti-symmetric when reduced to singleton sets (which, in words, is the assumption that the decision maker is not indifferent between any pair of distinct meals). If you wish to see the proof for the case where this assumption is dropped, go to Gul and Pesendorfer (2005).

Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Chapter 8: Social Choice and Efficiency

Summary of the Chapter

Social Choice Theory concerns the following problem. A set of individuals or households H and a set of social states X are given. Each individual has her own preferences concerning the various social states, and we want to *aggregate* those preferences, with a view towards choosing a social state from all of X, if all the social states are feasible, or perhaps from some subset of feasible states $A \subseteq X$. Moreover, we want to do this in a fashion that properly reflects the preferences of the individuals, where the definition of the term "properly" is part of the theory.

The chapter provides two answers to the question, How is this to be done? Sections 8.1 and 8.2 concern *Arrow's Theorem*, which says, in essence, that no completely satisfactory solution to the problem is possible. Assume that

1. The set of social states X is finite
2. The set of individuals H is finite
3. The preferences of individual h are given by a complete and transitive preference relation $\succeq_h$ defined on X

We look for a social preference function Φ, which maps every H-tuple of preferences $(\succeq_h)_{h \in H}$ into a complete and transitive preference relation $\succeq = \Phi[(\succeq_h)_{h \in H}]$. The assumption that Φ should work on every H-tuple of complete and transitive preferences is called the *universal domain* assumption. The assumption that the value of Φ at each argument is a complete and transitive preference relation on X is called the *coherence* assumption. These assumptions on the domain and range of Φ are stated formally in the text as Assumption 8.1.

We employ short-hand notation $\succeq$ for $\Phi[(\succeq_h)_{h\in H}]$ and $\succeq'$ for $\Phi[(\succeq'_h)_{h\in H}]$. The strict preference relation derived from $\succeq_h$ is denoted $\succ_h$; $\succ$ is strict preferences derived from $\succeq$, and so forth.

We don't want just any social preference function Φ, but one that has desirable properties and avoids undesirable properties. Definition 8.2 gives two properties that are deemed to be desirable and then a third, undesirable property:

Definition 8.2.

a. *The social preference function Φ satisfies **unanimity** if, for any profile of individual preferences $(\succeq_h)_{h\in H}$ and any pair of social states x and y, if $x \succ_h y$ for each $h \in H$, then $x \succ y$.*

b. *The social preference function Φ satisfies **independence of irrelevant alternatives (IIA)** if, for any two profiles of individual preferences $(\succeq_h)_{h\in H}$ and $(\succeq'_h)_{h\in H}$ and any two social states x and y such that $x \succeq_h y$ if and only if $x \succeq'_h y$ for all $h \in H$, $x \succeq y$ if and only if $x \succeq' y$.*

c. *The social preference function Φ is **dictatorial** if there is some $h^* \in H$ such that, for every profile of individual preferences $(\succeq_h)_{h\in H}$ and every pair of social states x and y, $x \succ_{h^*} y$ implies $x \succ y$.*

The text interprets these properties and gives arguments in favor of a and b and against c. Then:

Proposition 8.3. Arrow's Theorem. *Suppose that X contains three or more elements. If Φ satisfies Assumption 8.1 (the universal domain and coherence assumptions,) unanimity, and IIA, then Φ is dictatorial.*

This result tells us that standard social preference functions must be defective in one way or another. Two examples, majority rule and the *Borda Rule* (how athletic teams are often rank-ordered by the votes of sportswriters or coaches), are provided; majority rule can fail to provide transitive social preferences, while the Borda Rule will fail IIA.

Arrow's Theorem tells us that if we want to avoid dictatorial social preference functions, we must give up one (or more) of (a) universal domain, (b) coherence, (c) unanimity, or (d) IIA. Section 8.2 explores three of these options:

- No one seems very interested in giving up unanimity. If every member of society thinks x is strictly better than y, it is hard to imagine that society should conclude otherwise, for any social preference function that is "nice."
- A substantial literature concerns giving up on the universal domain assumption. Representative of and foremost in this literature is the so-called *assumption of single-peaked preferences*. This assumption and its consequences are developed.
- We might think of giving up on IIA. A primary rationalization for IIA is that we cannot make interpersonal comparisons of utility if our data consist only of the ordinal preferences of the individuals. But if we imagine that we have data that

allow us to calibrate intensity of preferences, for example via calibration with lotteries against benchmark social states, then we could use those data to make judgments based on interpersonal comparisons.

But none of these alternatives is viewed as altogether satisfactory, and we move in Section 8.3 to the "solution" adopted by mainstream economics and economists: Give up on producing a coherent (complete and transitive) social ordering. Instead, rank social states by efficiency or the so-called *Pareto ordering*, which is transitive but not complete:

Definition 8.6. *The outcome or social state x is* ***Pareto superior to*** *y (or Pareto dominates y), if $x \succeq_h y$ for every h in H and $x \succ_h y$ for at least one $h \in H$. The social state x is* ***strictly Pareto superior to*** *y (or strictly Pareto dominates y) if $x \succ_h y$ for all h. For a subset A of X and a point $x \in A$, x is* ***Pareto efficient*** *(or just efficient) within A if there is no $y \in A$ that Pareto dominates x. The set of Pareto efficient points within A is called the* ***Pareto frontier*** *of A.*

Discussion of the formal properties of Pareto efficiency follows this definition, together with analysis of the question, In applied settings, how would one "compute" the Pareto frontier? The answer to this question that is developed is, Under appropriate convexity assumptions, the Pareto frontier is, more or less, the set of solutions to the problems, Maximize weighted sums of the utilities of the various consumers, maximizing over the possible social states. After formalizing and proving this result, it is applied in Section 8.5 to the problem of *Syndicate Theory*: How should a finite collection of risk-averse expected-utility maximizers efficiently share in a collective set of risks that they face.

The analysis in this Section 8.4 is important for a reason beyond the results given: It is the first application in the text of the Separating-Hyperplane Theorem, which in the rest of the text is far and away the most useful and used hammer in our mathematical toolbox.

The chapter concludes (Section 8.6) with two cautions. Mainstream economics is, for the most part, passionate about the merits of efficiency. Perhaps because economists are generally unwilling to make value judgments among different efficient social states (because they are unwilling to make interpersonal utility tradeoffs), efficiency seemingly becomes the sole virtue.

- But no argument is offered in favor of the proposition that *any* efficient social state is better than any other state that is not efficient. A social state may be efficient and, at the same time, be highly inequitable, on which grounds it may be judged inferior as an outcome to some inefficient outcome. You should guard against choosing one process or policy or mechanism instead of a second merely because the first produces a Pareto-efficient outcome, while the second does not.

- Efficiency is based on the concept of consumer sovereignty. But, especially in dynamic contexts, where tastes change or the population of individuals changes with changes in the social state, applying consumer sovereignty can be problematic. Consumer sovereignty should command great deference and respect, but it should also be approached with at least a modicum of skepticism.

In terms of the problems at the end of this chapter, I urge every reader to try his or her hand at Problem 8.11.

Solutions to Starred Problems

▪ 8.3. (a) The argument that majority rule produces a complete binary relation (for every x and y, either $x \succeq y$ or $y \succeq x$), is established in the text: Since each $\succeq_h$ is complete, either $x \succeq_h y$ or $y \succeq_h x$ for each x. Therefore, at least half the h "vote" one way or the other (and more than half may "vote" each way, in which case $x \succeq y \succeq x$ would be the result of majority rule).

The hard part of the proof is to show that $\succeq$ is transitive. So suppose that $x \succeq y$ and $y \succeq z$, both according to majority rule. We must show that $x \succeq z$.

If $x = y$, then $y \succeq z$ tells us $x \succeq z$. If $y = z$, $x \succeq y$ tells us $x \succeq z$. And if $x = z$, then $x \succeq z$ follows because everyone votes for x being at least as good at itself (which is z). Therefore, we can proceed under the assumption that x, y, and z are distinct.

The proposition (to keep matters simple) assumes that each $\succeq_h$ is anti-symmetric. Since x, y, and z are distinct, for each h and every pair, $\succ_h$ holds between the two.

Now to enlist single-peakedness. Because the number of individuals is odd, we know that at least one individual must hold that $x \succ_h y \succ_h z$, since more than half the h have $x \succ_h y$ and more than half have $y \succ_h z$. By single-peakedness, this implies that z cannot lie between x and y (when these are viewed as numbers). Now ennumerate all the possible cases; each h must belong to one (and only one) of the following six sets:

$$A = \{h \in H : x \succ_h y \succ_h z\}, \qquad B = \{h \in H : x \succ_h z \succ_h y\},$$

$$C = \{h \in H : y \succ_h x \succ_h z\}, \qquad D = \{h \in H : y \succ_h z \succ_h x\},$$

$$E = \{h \in H : z \succ_h x \succ_h y\}, \text{ and } F = \{h \in H : z \succ_h y \succ_h x\}.$$

We know that half or more of the individuals belong to the union of A, B, and E, since these are the three sets with $x \succ_h y$. And since a majority have $y \succ_h z$, half or more must belong to the union of A, C, and D. To show that $x \succeq z$ by majority rule, we must show that half or more belong to the union of A, B, and C.

I assert that either D or E must be empty. We already know (from single-peakedness and the nonemptiness of A) that z cannot lie between x and y. So either x lies between y and z, which implies that D must be empty, or y must lie between x and z, which implies that E is empty. And if D is empty, then A and C are more than half of H, hence $x \succeq z$ by majority rule, while if E is empty, then A and B are more than half of H, and again $x \succeq z$ by majority rule.

(b) Suppose X consists of three (distinct) elements, $X = \{x, y, z\}$. Suppose there are seven individuals. Two have $x \succ_h y \succ_h z$, two have $z \succ_h x \succ_h y$, and three have $y \succ_h x \succ_h z$. Are these preferences consistent with single-peaked preferences? Yes: Suppose $y > x > z$. It is easy to verify that the preferences given are consistent with this geometric ordering and single-peakedness.

In a pairwise comparison of x and y, four out of the seven prefer x, so $x \succ y$. In a pairwise comparison of y with z, five vote for y, so $y \succ z$. (We know from part a, then, that a pairwise comparison of x with z must give a majority in favor of x, and in fact five vote this way.) But if each individual voted for his or her most preferred outcome, x would get two votes, z would get two, and y would get three. If this were a parliamentary election in Great Britain (and many other places), y would be the winner!

■ 8.5. This is an exercise in creative function construction. Let $v^0 = u(x^0)$, and let W be defined by

$$W(v) = \begin{cases} \sum_h (v_h - v_h^0), & \text{if } v \geq v^0 \text{, and} \\ \sum_h -e^{-(v_h - v_h^0)}, & \text{if } v \not\geq v^0. \end{cases}$$

Note that this function satisfies $W(v^0) = 0$ and $W(v) < 0$ for $v \not\geq v^0$. Therefore, it certainly will be maximized at v^0, if we are looking at the maximum over any set that contains v^0 but contains no other point that is $\geq v^0$. It is clearly strictly increasing on its two "pieces" (that is, on $\{v : v \geq v^0\}$ and on $\{v : v \not\geq v^0\}$ separately, as it is the component-by-component sum of strictly increasing functions on each component. And if $v \geq v'$, "across" the two pieces, then it must be that v is in the $\geq v^0$ piece: if $v \geq v' \geq v^0$, then $v \geq v^0$, and both v and v' are in the same piece. But if $v \geq v^0$ and $v' \not\geq v^0$, then $W(v) \geq 0$ while $W(v') < 0$, hence we have $W(v) > W(v')$. (Not only is W strictly increasing across the two pieces, but every $v \geq v^0$ has $W(v)$ strictly greater than every v' such that $v' \not\geq v^0$.)

Needless to say, this is a function that is absolutely tailor-made to pick out v^0, if it is at all feasible to do so!

■ 8.6. Suppose $X = \{x \in R^3 : x \geq 0, x_1 + x_2 + x_3 = 1\}$. That is, X is a two-dimensional unit simplex. H will have two members. The first member of H, denoted 1, has utility function $u_1(x) = x_1^{1/2} + 0.5x_2^{1/2}$. And individual 2 has utility function $u_2(x) = 0.5x_2^{1/2} + x_3^{1/2}$. To see what sort of set of utility imputations this produces, I plotted the utility imputations in Excel for values of (x_1, x_2, x_3) where each coordinate is a multiple of 0.05. The result is shown in Figure G8.1. The graininess of the plot is due, of course, to the graininess in the values of the coordinates of x. I have drawn in (freehand) what would be (approximately) the boundaries of the full set of utility imputations;

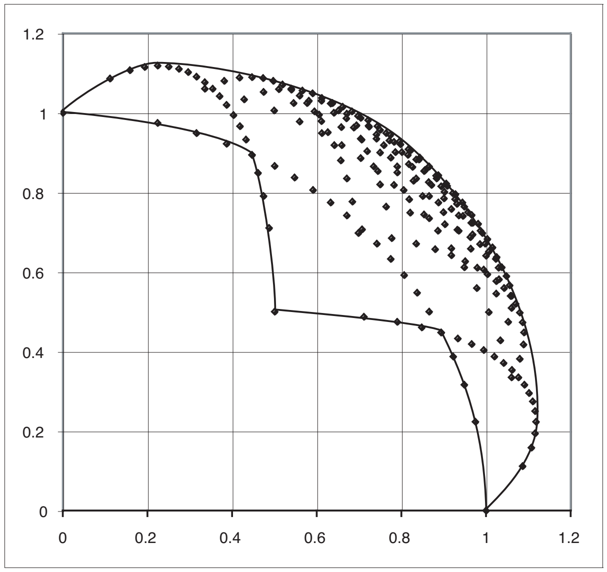


Figure G8.1. Problem 8.6: A Set of Utility Imputations

the set is convex to the north-east (as it must be per Proposition 8.10) but is not at all convex to the south-west.

■ 8.7. Suppose $x = (y_{hs})$ and $x' = (y'_{hs})$ are both feasible. We must show that for any $\alpha \in [0,1]$, $\alpha x + (1-\alpha)x'$ is feasible. Note that the sharing rule $\alpha x + (1-\alpha)x'$ involves sharing rules $(\alpha y_{hs} + (1-\alpha)y'_{hs})$. Feasibility is a matter of satisfying whichever constraints are imposed:

The adding-up constraint is always imposed: $\sum_h y_{hs} \leq \sum_j z_{js}$. If x and x' both satisfy this constraint, then $\sum_h y_{hs} \leq \sum_j z_{js}$, hence $\alpha \sum_h y_{hs} = \sum_h \alpha y_{hs} \leq \alpha \sum_j z_{js}$, and $\sum_h y'_{hs} \leq \sum_j z_{js}$, hence (1-$\alpha$) $\sum_h y'_{hs} = \sum_h (1-\alpha)y'_{hs} \leq (1-\alpha)\sum_j z_{js}$, therefore

$$\sum_h \alpha y_{hs} + \sum_h (1-\alpha)y'_{hs} = \sum_h [\alpha y_{hs} + (1-\alpha)y'_{hs}] \leq \alpha \sum_j z_{js} + (1-\alpha)\sum_j z_{js} = \sum_j z_{js},$$

and $\alpha x + (1-\alpha)x'$ satisfies the adding-up constraint.

If either the constraint $y_{hs} \geq 0$ or $y_{hs} + e_{hs} \geq 0$ is imposed, and if x and x' satisfy the constraint(s), then it is easy to see (by an argument similar to the one in the previous paragraph) that $\alpha x + (1-\alpha)x'$ satisfies the constraint(s).

There is finally the possible constraint $\sum_s \pi_h(s) U_h(y_{hs} + e_{hs}) \geq \sum_s \pi_h(s) U(e_{hs})$. Once we show that the functions

$$x \rightarrow u_h(x) = \sum_s \pi_h(s) U_h(y_{hs} + e_{hs})$$

are concave in $x = (y_{hs})$ for each h, satisfaction of this constraint by convex combinations of sharing rules that satisfy the constraint is immediate.

So it remains to show that the functions $x \rightarrow u_h(x)$ as defined above are concave. You can probably just cite Proposition A3.17(b) as justification for this and satisfy most graders/referees, but in case you want a bit of detail: Fixing x, x', and α, note first that

$$\begin{aligned} U_h([\alpha y_{hs} + (1-\alpha)y'_{hs}] + e_{hs}) = & U_h(\alpha[y_{hs} + e_{hs}] + (1-\alpha)[y'_{hs} + e_{hs}]) \geq \\ & \alpha U_h(y_{hs} + e_{hs}) + (1-\alpha)U_h(y'_{hs} + e_{hs}) \end{aligned},$$

by the concavity of each U_h, therefore

$$x \rightarrow U_h(y_{hs} + e_{hs})$$

is concave in x. But then so is $x \rightarrow \pi_h(s)U_h(y_{hs} + e_{hs})$, since a positive constant times a concave function is concave, and then so is $u_h(x)$, since the sum of concave functions is concave.

■ 8.10. To begin, let me resummarize what we know. Pareto-efficient sharing rules are precisely the solutions (if any) to problems of the form

$$\text{Maximize } \sum_h \left[\alpha_h \sum_s \pi_h(s)U_h(x_{hs})\right], \quad \text{subject to } \sum_h x_{hs} \leq W_s,$$

plus (perhaps) nonnegativity constraints on the x_{hs}. The maximization problems can be solved state-by-state, per Proposition 8.12, and assuming that the U_h are differentiable (which we are assuming), the FOCS conditions for a maximum are both necessary and sufficient. So Pareto efficient points correspond to the solutions to

$$\alpha_h \pi_h(s)U'_h(x_{hs}) = \lambda_s \quad \text{and} \sum_s x_{hs} = W_s$$

without the nonnegativity constraint, and

$$\alpha_h \pi_h(s)U'_h(x_{hs}) \leq \lambda_s, \text{ with equality if } x_{hs} > 0, \quad \text{and} \sum_s x_{hs} = W_s$$

if we do impose the nonnegativity constraints.

Now consider how we might generate a solution for a given weighting vector (α_h). Take first the case where the nonnegativity constraints are not imposed. Pick a state s. For each nonnegative real number λ_s and for each h, look for solutions x to the equation

$$U'_h(x) = \lambda_s/(\alpha_h \pi_h(s)).$$

Let $X_{hs}(\lambda_s)$ be the set of solutions (which, clearly, depends on α_h and the parameter $\pi_h(s)$). (If $U_h'(-\infty) < \lambda_s/(\alpha_h\pi_h(s))$, let $X_{hs}(\lambda) = \{-\infty\}$. If $U_h'(\infty) > \lambda_s/(\alpha_h\pi_h(s))$, let $X_{hs}(\lambda) = \{\infty\}$.) Because U_h is concave and continuously differentiable, the sets X_{hs} are intervals that move continuously downwards in λ_s, meaning, they never overlap, and as λ increases, the set of solutions decreases in the sense that every member of the set for a smaller λ is greater than every member for a larger λ. Moreover, if U_h' is strictly concave, then $X_{hs}(\lambda)$ is always a singleton set and, therefore, describes a continuous, decreasing function. (If this isn't pretty close to obvious to you, you might benefit by going back to review the solutions of Problems 3.8 and 3.9. The mathematics are practically identical.) A solution to the FOCS conditions is obtained for state s when we find some λ, which will be λ_s, for which W_s is in the set-by-set sum (over h of the sets $X_{hs}(\lambda)$ and where each of these sets must contribute a finite element to the sum—you don't have an answer if $X_{hs}(\lambda) = \{-\infty\}$ for some individuals and $= \{\infty\}$ for others).

If the nonnegativity constraints are imposed, you follow the same process, except that in constructing the sets $X_{hs}(\lambda)$, you only look at nonnegative values for x, and if $U'(0) \leq \lambda/(\alpha_h\pi_h(s))$, then set $X_{hs}(\lambda) = \{0\}$.

The first bullet point concerns the case where every individual except perhaps for one is strictly risk averse. This means that each $X_{hs}(\lambda)$ will be singleton for every value of λ, except perhaps for one individual. Since the X_{hs} sets decline with increasing λ, there can be at most one λ that solves the FOCS conditions for each state (excluding the trivial case where the nonnegativity constraints are applied and $W_s = 0$ for some state), and for that one λ, which will be λ_s, the shares of all but at most one of the individuals are fixed. The share of the last individual is then fixed by the adding-up constraint (which must hold with equality).

The second bullet point is obvious from the FOCS conditions. Solutions depend on the weighting vector, the individual's subjective probability assessments, the shape of their utility functions, and W_s. The division of W_s into "shared ventures" and "private endowments" plays no role, and so this division can have no impact on efficient sharing rules.

Suppose that all the subjective probability assessments are the same: $\pi_h(s) = \pi_{h'}(s)$ for all h, h', and s. Then, given weights (α_h), the state-s maximization problem

$$\text{Maximize } \sum_h \alpha_h\pi_h(s)U_h(x_{hs}), \quad \text{subject to } \sum_h x_{hs} \leq W_s$$

and, possibly, nonnegativity constraints, is the same as

$$\text{Maximize } \sum_h \alpha_h U_h(x_{hs}), \quad \text{subject to the same constraints.}$$

Solutions depend in no fashion on the probability assessments, but only on the total amount of wealth, W_s, that can be distributed, the weighting vector (α_h), and the individual utility functions. (In terms of the FOCS conditions, the multipliers λ_s get scaled by the common probability assessment, but the set of solutions in terms of the sharing rule is unchanged by those probabilities.)

Suppose that all the subjective probability assessments are the same, and one party, say h_0, is risk neutral. Let U' be the constant slope of h_0's utility function. Then given a weighting vector (α_h), the value of the multiplier in state s is forced: $\lambda_s = \alpha_{h_0}\pi(s)U'$ (where I am writing $\pi(s)$ for the now common probability assessment). This is true independent of W_s—if there is a solution to the FOCS conditions, the solution must have this value of λ_s. (We are assuming that h_0 is not subject to the nonnegativity constraint, so her weighted-by-α-and-probability marginal utility must equal the mulitplier in any solution of the FOCS conditions.)

But then, for all the other (strictly risk averse) members of the syndicate, their net-of-endowment amounts are fixed by the weighting vector: x_{hs} must satisfy the equation

$$\alpha_h \pi(s) U_h'(x_{hs}) = \lambda_s = \alpha_{h_0}\pi(s)U' \quad \text{or} \quad U_h'(x_{hs}) = \frac{\alpha_{h_0}}{\alpha_h}U'.$$

(If individual h must satisfy the nonnegativity constraint, this equation is replaced by $x_{hs} = 0$ if $U_h'(0) \leq (\alpha_{h_0}/\alpha_h)U'$.) This solution is unique (strict concavity of the U_h) and is independent of W_s; the one risk-neutral individual has $X_{h_0 s} = R$, so she is happy to soak up any residual risk after the rest of the syndicate gets their constant net-of-endowment shares. Note that the value of x_{hs} that h gets (net of endowment) varies in increasing fashion with h's weight relative to the weight on h_0; as α_{h_0}/α_h decreases, (as its reciprocal increases), the solution to $U_h'(x_{hs}) = (\alpha_{h_0}/\alpha_h)U'$ increases.

Finally, suppose $U_h(x) = -e^{-x/\tau_h}$ for each h. If nonnegativity constraints do not bind and there is a common probability assessment $\pi(s)$, the FOCS conditions are

$$\frac{\alpha_h}{\tau_h} e^{-x_{hs}/\tau_h} = \frac{\lambda_s}{\pi(s)} \quad \text{and} \quad \sum_s x_{hs} = W_s.$$

Write μ_s for $\lambda_s/\pi(s)$, and we can rewrite the first-order condition as

$$e^{-x_{hs}/\tau_h} = \frac{\tau_h \mu_s}{\alpha_h} \quad \text{or} \quad x_{hs} = -\tau_h \ln\left(\frac{\tau_h\mu_s}{\alpha_h}\right) = \tau_h \ln\left(\frac{\alpha_h}{\tau_h \mu_s}\right).$$

The adding-up constraint is then

$$\sum_h \tau_h \ln\left(\frac{\alpha_h}{\tau_h\mu_s}\right) = \sum_h \tau_h \ln\left(\frac{\alpha_h}{\tau_h}\right) - \ln(\mu_s)\sum_h \tau_h = W_s.$$

Let $K^* = \sum_h \tau_h \ln(\alpha_h/\tau_h)$, and let $T = \sum_h \tau_h$, and this becomes $K^* - T\ln(\mu_s) = W_s$, or $-\ln(\mu_s) = (W_s - K^*)/T$, where K^* is a constant (in s) that is parametrized by the vector of coefficients of risk tolerence (τ_h) and the weighting vector (α_h). And now we can go back and write

$$x_{hs} = \tau_h \ln\left(\frac{\alpha_h}{\tau_h}\right) - \tau_h \ln(\mu_s) = \tau_h \ln\left(\frac{\alpha_h}{\tau_h}\right) + \tau_h\left(\frac{W_s - K^*}{T}\right) =$$

$$\tau_h\left[\ln\left(\frac{\alpha_h}{\tau_h}\right) - \frac{K^*}{T}\right] \; + \; \frac{\tau_h}{T}W_s.$$

The term on the left-hand side of the plus sign is constant in s, depending (in a fairly complicated way) on the vectors (τ_h) and (α_h); moreover, these terms sum to zero.
And the term on the right-hand side of the plus sign, which says how h's net-of-endowment share varies with s, is as promised in the bullet point.

Microeconomic Foundations I
Choice and Competitive Markets

Student's Guide

Chapter 9: Competitive and Profit-Maximizing Firms

Summary of the Chapter

In the economics of this book, two types of economic entities interact: consumers and firms. Chapters 1 through 8 have concerned consumers, as will Chapters 10 and 11. In this chapter, firms are studied. (Subsequent chapters concern both consumers and firms.)

Firms are entities that have the capability of transforming a vector of commodities, the inputs to its production process, into another vector of commodities, its outputs. This is done, in this chapter at least, instantaneously. The model of a firm specifies two things:

1. Its technological capabilities: Which vectors of inputs can the firm transform, into which vectors of outputs?
2. What does the firm choose to do, given its various technological capabilities?

For almost all of this chapter, the firm's capabilities are modeled with a *production-possibility set*, discussed in Section 9.1. Its objective, which guides its choice of production plan given prices, is *profit maximization*, discussed in Section 9.2. Having put these two pieces of the model of the firm in place, the bulk of the chapter analyzes the firm's profit-maximization problem. Section 9.3 concerns the existence of solutions to the firm's profit-maximization problem—nontrivial because production-possibility sets are not compact in general—and basic properties of solutions. In particular, we show that, viewed as a correspondence whose domain is the space of strictly positive prices, the firm's optimal netput set is upper semi-continuous; viewed as a function with the same domain, the firm's profit function is convex, homogeneous of degree 1, and continuous. (Of course, certain assumptions about the production-possibility set are required to prove these results.) In Section 9.4, we derive results similar in character to Afriat's Theorem, for firms instead of for the market demand of a consumer.

Through Section 9.4, the analysis parallels the analysis of the consumer's utility-maximization problem in Chapters 3 and 4. Section 9.5 breaks new analytical ground. The focus is on the firm's *profit function*, which says how much profit the firm can (optimally) make as function of the prices it faces. In particular, we study in depth the question of *duality*: To what extent does the firm's profit function identify its production-possibility set?

1. Section 9.5 shows that if a function $\pi : R^k_{++} \to R$ is convex, homogeneous of degree 1, and continuous (which it must be if it is convex), then we can identify a production-possibility set that has π as its profit function. This set is convex, closed, and has free-disposal (meaning that with the same level or more inputs, the firm can always make the same level or less output).

2. More than one production-possibility set can produce the same profit function. Section 9.6 answers the questions, When do two production-possibility sets give the same profit function? When do they give different profit functions?

3. The results of Sections 9.5 and 9.6 establish a *duality* or one-to-one correspondence between profit functions (functions with domain R^k_{++} and range R) and production-possibility sets that are closed, convex, and have free disposal. Section 9.7 looks harder at the mathematics of this duality result, tieing it (in general) to the mathematical result that a convex set in R^k that is less than all of R^k is the intersection of the half-spaces that contain it, the so-called *Support-Function Theorem*.

4. In Section 9.8, we show that a profit function is differentiable at prices p^0 if and only if the firm's profit-maximization problem admits a single solution at p^0.

A different approach to modeling the firm involves *input-requirement sets* and the firm's cost-minimization problem. This approach is briefly outlined in Section 9.9.

What this chapter does not discuss

Throughout Chapter 9, the firm is *competitive* or a *price taker*, meaning that it acts in the belief that its choice of production plan will have no impact on the price of any commodity. While this may be a reasonable assumption for (most) consumers, it is less reasonable when it comes to firms and especially with regard to the outputs of firms. For instance, General Motors probably does not take the price of Cadillacs as given. (One reason to study the firm's cost-minimization theory is that this assumes that the firm is a price taker [only] for inputs to its production process, a generally more palatable assumption.) Theories of monopolistic firms, oligopolistic firms, and monopolistically competitive firms all address the "firm as entity with given production capabilities and with profit-maximizing objectives," but with some effect on prices. We do not study any of this here, leaving it for Volume II.

In some treatments of the firm in the literature, the firm has capabilities specified as here and may or may not have the ability to affect the prices it faces, but it does not seek maximal profits. Instead, it chooses its production plan in pursuit of other objectives, such as the maximization of its wage bill in so-called worker-managed firms or the maximization of its fixed asset base in some managerial theories of the firm. We do not discuss such models at all.

And, especially in the recent past, a lot of attention has been paid to thinking of firms not as entities with a given set of capabilities and a fixed objective but instead as an institution much akin to a market, within which various entities—workers, owners (shareholders), managers, customers, suppliers—interact. The study of firms as settings for the interaction of various parties, rather than as entities, is left in part to Volume II and especially to Volume III.

Solutions to Starred Problems

▪ 9.2. Suppose $0 \in Z$. Then it is clear that for every price $p \in R^k_{++}$, $\pi(p) \geq 0$. And if $\pi(p) \geq 0$ for all p, then 0 is obviously in the set

$$\{z \in R^k : p \cdot z \leq \pi(p) \text{ for all strictly positive } p\}.$$

So if π is nonnegative, then while 0 may not be in Z, 0 is in the closed free-disposal convex hull of Z. (Since you were told to assume that π is real-valued, Corollary 9.18 and Proposition 9.16 apply.)

▪ 9.4. Yes, $p_1 \to z_1(p_1)$ is continuous in this open domain. But to show this takes a bit of work. Corollary A4.10 says that if $p \Rightarrow Z^*(p)$ is singleton valued in an open domain, the function described is a continuous function, hence the projection of this function along any coordinate describes a continuous function. This is a corollary of Berge's Theorem. What is different here is that while $Z^*(p)$ may not be singleton valued, its projection along the first coordinate is singleton valued. But the proof mimics the proof from Berge's Theorem:

Take any sequence $\{p_1^n\}$ of prices for the first coordinate, lying within this open set and with limit p_1^0. We will show that $\lim_n z_1(p_1^n) = z_1(p_1^0)$, which demonstrates continuity of the $z_1(\cdot)$ function.

Let $p^n = (p_1^n, p_2, \ldots, p_k)$, and let z^n be any element of $Z^*(p^n)$. We know that the first coordinate of z^n, z_1^n, is $z_1(p_1^n)$. Because Z has the recession-cone property, we know from Lemma 9.8 that the sequence $\{z^n\}$ lives within a bounded set, so by looking at a subsequence if necessary, we know that $\{z^n\}$ has a limit z^0. By Berge's Theorem, $z^0 \in Z^*(p_1^0, p_2, \ldots, p_k)$ and, therefore, $z_1^0 = z_1(p_1^0)$. But this implies that $z_1(p_1^0)$ is indeed the limit of $\{z_1^n\} = \{z_1(p_1^n)\}$. That does it.

▪ 9.7. Suppose that $k = 2$, $Z = \{(x_1, x_2) : x_1 \leq 0\}$, $Z' = \{(x_1, x_2) : x_1 \leq -1\}$, and $Z'' = \{(x_1, x_2) : x_2 \leq -10\}$. In words, Z represents a technology which can get as much of the second good as desired with no inputs, Z' represents a technology which can get as much of the second good as desired if there is at least 1 unit of good 1 input, and Z'' represents a technology which can get as much of the first good as desired if there is at least 10 units of the second good input. Each of these is closed, convex, and has free disposal, and they are certainly distinct from one another. But at any strictly positive prices, each of them gives a profit of ∞, so they share the same ($\equiv \infty$) profit function (for strictly positive prices).

They have different profit functions if we can have zero prices. At the price vector $(1, 0)$, Z gives profit 0, Z' gives profit -1, and Z'' gives profit ∞, while at $(0, 1)$, Z and Z' both give profit ∞ while Z'' gives profit -10.

Now for the proof of the corollary: I'll show that first that, for any $Z \subseteq R^k$, Z and $\overline{\text{FDCH}}(Z)$ give the same profit for any nonnegative p. Therefore, if $\overline{\text{FDCH}}(Z) = \overline{\text{FDCH}}(Z')$, Z and Z' have the same extended profit function (which is the same as the extended profit function for their common $\overline{\text{FDCH}}$s.

Look at the proof of Proposition 9.16, starting with the second paragraph. No changes are necessary in the second paragraph. In the third paragraph, change "for some strictly positive price p" to "for some nonnegative price p" in the second sentence. Change "(since prices are strictly positive)" to "(since prices are nonnegative)" in the sixth sentence. With these changes, the argument goes through without a hitch.

For the converse, since we know that Z and $\overline{\text{FDCH}}(Z)$ have the same profit function and since Z' and $\overline{\text{FDCH}}(Z')$ have the same profit function, I have the result if I show that, *if Z and Z' are distinct closed, convex, and free-disposal subsets of R^k, then they have different extended profit functions.* This time, it is easier just to write down the argument.

Since Z and Z' are distinct, either there is some $z \in Z \setminus Z'$ or some $z' \in Z' \setminus Z$ (or both). Wlog, assume there is some $z \in Z \setminus Z'$. Then since Z' is convex and closed, we can find a hyperplane that strictly separates the point z from Z'; that is, for some $p \in R^k$ that is not identically 0, $p \cdot z > \sup_{z' \in Z'} p \cdot z'$. This implies, in particular, that $\sup_{z' \in Z'} p \cdot z' < \infty$, which implies that $p \geq 0$: If some component of p were strictly less than zero, then take any z^0 from Z'[1] and then look at $z^M = z^0 + (0, 0, \ldots, 0, -M, 0, \ldots, 0)$, where the $-M$ goes in the coordinate for which p is strictly negative. As M goes to ∞, $p \cdot z^M$ goes to ∞. But $z^M \in Z'$ by free disposal, which contradicts that $\sup_{z' \in Z'} p \cdot z' < \infty$.

But then $\overline{\pi}_Z(p) \geq p \cdot z > \sup_{z' \in Z'} p \cdot z' = \overline{\pi}_{Z'}(p)$, showing that Z and Z' have distinct extended profit functions.

■ 9.9. (a) First I'll show that if $Z' = \overline{\text{CH}}(Z)$, then $\tilde{\pi}_Z \equiv \tilde{\pi}_{Z'}$. This will immediately establish that if $\overline{\text{CH}}(Z) = \overline{\text{CH}}(Z')$, then $\tilde{\pi}_Z \equiv \tilde{\pi}_{Z'}$ in general.

So suppose that $Z' = \overline{\text{CH}}(Z)$. Since $Z \subseteq Z'$ and since the support functions are defined as suprema, it is clear that $\tilde{\pi}_{Z'} \geq \tilde{\pi}_Z$. Suppose by way of contradiction that $\tilde{\pi}_{Z'}(p) > \tilde{\pi}_Z(p)$ for some $p \in R^k$. Then for some $\epsilon > 0$, we can find $\hat{z} \in \overline{\text{CH}}(Z)$ such

[1] If $Z' = \emptyset$, then $\overline{\pi}_{Z'} \equiv -\infty$ which is different from $\overline{\pi}_Z$, since we know that Z is not empty.

that $p \cdot \hat{z} > \tilde{\pi}_Z(p) + \epsilon$. By continuity of the dot product, we can find some $z' \in \mathrm{CH}(Z)$ close enough to z so that $p \cdot z' > \tilde{\pi}_Z(p) + \epsilon/2$. Since $z' \in \mathrm{CH}(Z)$, z' is a convex combination of elements of Z; write $z' = \sum_{j=1}^n \alpha^j z^j$, where each $z^j \in Z$ and the scalars are nonnegative and sum to one. Since $p \cdot z' = \sum_{j=1}^n \alpha^j (p \cdot z^j) \geq \tilde{\pi}_Z(p) + \epsilon/2$, and since the α^j are nonnegative and sum to one, for some index j, $p \cdot z^j \geq \tilde{\pi}(p) + \epsilon/2$. Since this z^j is from Z, we have a contradiction.

For the converse, suppose $\overline{\mathrm{CH}}(Z) \neq \overline{\mathrm{CH}}(Z')$. Since the support functions of Z and of $\overline{\mathrm{CH}}(Z)$ are identical, as are those of Z' and $\overline{\mathrm{CH}}(Z')$, we need to show that the support functions of $\overline{\mathrm{CH}}(Z)$ and $\overline{\mathrm{CH}}(Z')$ are not the same. This amounts to showing that if Z and Z' are different closed and convex sets, they have different support functions, which is what I'll do.

So suppose Z and Z' are different closed and convex sets. Either $Z \setminus Z'$ or $Z' \setminus Z$ (or both) is nonempty. So, wlog, suppose that $z \in Z \setminus Z'$. Use the Strict-Separation Theorem to find a $p \in R^k$, not identically zero, such that $p \cdot z > \sup\{p \cdot z' : z' \in Z' = \tilde{\pi}_{Z'}(p)$. Since $\tilde{\pi}_Z(p) \geq p \cdot z$ for this z (since $z \in Z$), we're done.

(b) If Z is compact, it is obvious that, for every $p \in R^k$, $\tilde{\pi}_Z(p) = \sup\{p \cdot z : z \in Z\} < \infty$, since the dot product is a continuous function in z for every fixed p. Conversely, suppose Z is closed and convex and that, for every p, $\tilde{\pi}_Z(p) = \sup\{p \cdot z : z \in Z\} < \infty$. Suppose by way of contradiction that Z is not bounded. Then we can find a sequence $\{z^n\}$ from Z such that $\|z^n\| > n$. Let $\hat{z}^n = z^n/\|z^n\|$; $\{\hat{z}^n\}$ is a sequence that lies on the unique sphere, hence is has an accumulation point $\hat{z}^0$ lying on the unit sphere. $\hat{z}^0$ is not identically zero, so it has some nonzero component, say component i; let $e_i = 1$ or -1 depending on whether $\hat{z}_i^0$ is positive or negative, and let $p = (0, \ldots, 0, e_i, 0, \ldots, 0)$ where e_i is put in coordinate i. Then $p \cdot \hat{z}^0 = |z_i^0| > 0$. Therefore, by continuity, $p \cdot \hat{z}^n \to |z_i^0|$ as $n \to \infty$ along the subsequence for which $\hat{z}^n$ converges to $\hat{z}^0$. But then $p \cdot z^n = \|z^n\| p \cdot \hat{z}^n > np \cdot \hat{z}^n$ when n is large enough in the subsequence so that $p \cdot \hat{z}^n$ has become positive, and therefore $\lim_n p \cdot z^n$ along that subsequence is $+\infty$. This means that $\tilde{\pi}_Z(p) = \infty$ for this p, a contradiction.

Now we must show that $\tilde{\pi}_Z$ is convex and homogeneous of degree 1. If it is not convex, then there exist p, p', and $\alpha \in [0,1]$ such that $\alpha\tilde{\pi}_Z(p) + (1-\alpha)\tilde{\pi}_Z(p') < \tilde{\pi}_Z(\alpha p + (1-\alpha)p')$. For this to be true, there must exist $z \in Z$ such that $\alpha\tilde{\pi}_Z(p) + (1-\alpha)\tilde{\pi}_Z(p') < (\alpha p + (1-\alpha)p') \cdot z = \alpha(p \cdot z) + (1-\alpha)(p' \cdot z) \leq \alpha\tilde{\pi}_Z(p) + (1-\alpha)\tilde{\pi}_Z(p')$, which is a contradiction. (You should be careful that everything I said works, even if $\tilde{\pi}_Z$ can be infinite valued. But it does work; the key is the very first inequality: Convexity fails only if $\alpha\tilde{\pi}_Z(p) + (1-\alpha)\tilde{\pi}_Z(p') < \tilde{\pi}_Z(\alpha p + (1-\alpha)p')$ for some p, p', and α. This can't happen for $\alpha = 0$ or 1, since then the two sides are identical, therefore we can assume $\alpha \in (0,1)$. But then if the right-hand side is finite, so is the left-hand side (since it is less); and if the right-hand side is infinite, the left-hand side to be less must be finite. Once you see that, it all goes through very nicely.) And for homogeneity of degree one, this fails only if for some p and some $\alpha \geq 0$, $\tilde{\pi}_Z(\alpha p) \neq \alpha\tilde{\pi}_Z(p)$. If $\alpha = 0$, this can't happen by definition (since $\tilde{\pi}_Z(0) = 0$), so we can assume $\alpha > 0$. And then by perhaps replacing α with $1/\alpha$ and p with αp, we can assume wlog that we have $\alpha > 0$ and p such that $\tilde{\pi}_Z(\alpha p) > \alpha\tilde{\pi}_Z(p)$. But then there must exist $z \in Z$ such that

$\alpha\tilde{\pi}_Z(p) < (\alpha p) \cdot z = \alpha(p \cdot z) \leq \alpha\tilde{\pi}_Z(p)$, a contradiction.

If you compare these proofs of convexity and homogeneity with the proofs used for the profit function when we assumed it was everywhere finite valued, you will see that they are essentially identical.

(c) We are now assuming that we have a convex and homogeneous real-valued function $\tilde{\pi} : R^k \to R$, for which we define

$$Z = \{z \in R^k : p \cdot z \leq \tilde{\pi}(p) \text{ for all } p \in R^k\}.$$

If we then define

$$\tilde{\pi}_Z(p) = \sup\{p \cdot z : z \in Z\},$$

it is immediate that $\tilde{\pi}_Z(p) \leq \tilde{\pi}(p)$, since for each p, if $z \in Z$, then $p \cdot z \leq \tilde{\pi}(p)$, so taking the supremum of these values over all $z \in Z$ can't get us above this common upper bound.

The key, then, is to show that for each $p \in R^k$, there is some $z_p \in Z$ such that $p \cdot z_p = \tilde{\pi}(p)$. But this is easy: Fix p. Since $\tilde{\pi}$ is convex and since its domain is open (so that p is in the interior of the domain), there is a subgradient of $\tilde{\pi}$ at p. Because $\tilde{\pi}$ is homogeneous of degree 1, subgradients take the form of scalar multiplication by a non-zero vector (which I'll call) $z_p \in R^k$; that is, $p \cdot z_p = \tilde{\pi}(p)$ and $p' \cdot z_p \leq \tilde{\pi}(p')$ for all p'. Therefore, $z_p \in Z$. And $p \cdot z_p = \tilde{\pi}(p)$ implies $\tilde{\pi}_Z(p) \geq \tilde{\pi}(p)$. Combined with our earlier inequality (the other way), this gives us the result.

■ 9.11. (a) and (b) are trivial matters of comparing two ways to say the same things.

(c) If $x \in V(y)$, then $(y, -x, 0) \in Z$. If $x' \geq x$, then $-x' \leq -x$, so $(y, -x', 0) \leq (y, -x, 0)$, so $(y, -x', 0) \in Z$ by free disposal, and therefore $x' \in V(y)$.

See Figure G9.1(a) for an example where each $V(y)$ is comprehensive upwards, but Z does not have free disposal.

Suppose $\hat{Z}$ has free disposal. By the argument just given, this implies that each $V(y)$ is comprehensive upwards. And if $y \geq y'$ and $x \in V(y)$, then $(y, -x, 0) \in X$, hence (since $y \geq y'$ and $\hat{Z}$ has free disposal) $(y', -x, 0) \in \hat{Z}$, hence $x \in V(y')$, and $V(y) \subseteq V(y')$; the V sets nest.

In the other direction, suppose the $V(y)$ are comprehensive upwards and nest, and Y is "comprehensive downwards." If $(y, -x, 0) \in \hat{Z}$ and $(y', -x', 0) \leq (y, -x, 0)$, then $y \geq y' \geq 0$ and $x' \geq x$. Since $(y, -x, 0) \in \hat{Z}, x \in V(y)$. Since $y \geq y'$ and the V's nest, $y' \in Y$ and $x \in V(y) \subseteq V(y')$, hence $x \in V(y')$. Since the Vs are comprehensive upwards, $x' \in V(y')$. Therefore $(y', -x', 0) \in V(y')$; we have (modified for $\hat{Z}$) free disposal.

(d) Suppose $x, x' \in V(y)$. This implies $(y, -x, 0) \in Z$ and $(y, -x', 0) \in Z$. Therefore, for all $\alpha \in [0, 1]$, $(y, -(\alpha x + (1-\alpha)x'), 0) \in Z$, which implies $\alpha x + (1-\alpha)x' \in V(y)$.

Figure G9.1(a) is also a case where each $V(y)$ is convex, but $\hat{Z}$ (and Z) is not.

Proving the last part of d just amounts to rearranging definitions.

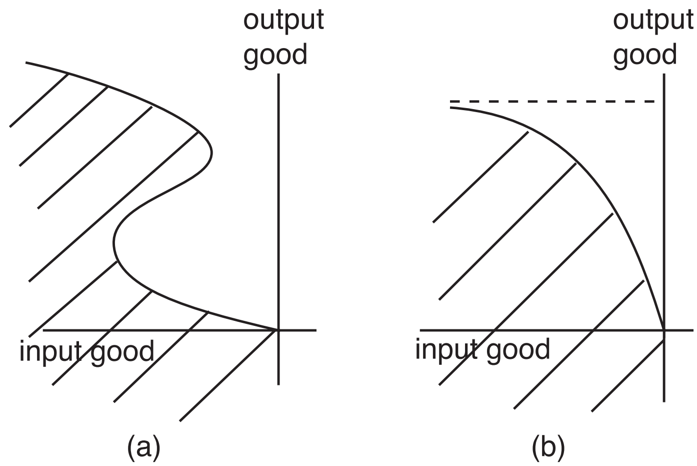


Fig. G9.1. Two Production-Possibility Sets. The two production-possibility sets depicted provide "counterexamples" for Problem 9.11. In each case, there is a single output good and a single input good, and the whole production possibility set Z is shown as the hatched area, with free disposal assumed. In panel a, each input requirement set $V(y)$ is convex and comprehensive upwards, but Z is neither convex not does it have free disposal. In panel b, the boundary of Z is meant to asymptote to the dashed line but never hit it; so Y includes all levels of output up to but not including the level of the dashed line; although Z is closed, Y is not.

(e) Suppose $y \Rightarrow V(y)$ is upper semi-continuous. Suppose that $(y^\ell, -x^\ell, 0)$ is a sequence from $\hat{Z}$ such that y^ℓ converges to y and $-x^\ell$ converges to some $-x$. Since we are assuming this sequence is drawn from $\hat{Z}$, this means that $x^\ell \in V(y^\ell)$ for each ℓ, and then upper semi-continuity of the V correspondence tells us that $x \in V(y)$, which implies that $(y, -x, 0) \in \hat{Z}$, and $\hat{Z}$ is closed.

Conversely, suppose that $\hat{Z}$ is closed and that we have sequences $\{y^\ell\}$ and $\{x^\ell\}$ with respective limits y and x and with $x^\ell \in V(y^\ell)$ for each ℓ. The last piece of this implies that $(y^\ell, -x^\ell, 0) \in \hat{Z}$ for all ℓ, and so the closedness of $\hat{Z}$ implies $(y, -x, 0)$ in $\hat{Z}$, which immediately yields $x \in V(y)$.

(Or you can subsume both paragraphs into the citation of Proposition A4.3, if you note the connection between $\hat{Z}$ and the graph of the V correspondence.)

The corollary statement, that if $\hat{Z}$ is closed, then each $V(y)$ is closed, follows immediately from Proposition A4.4.

But we cannot guarantee that Y is closed; see panel b of Figure G9.1.

■ 9.12. The proof of part a is obvious: $r \in R^n_{++}$ and $x \in R^n_+$, so the function over which we are minimizing, $r \cdot x$, is always nonnegative.

For part b, let x^0 be an arbitrary element of $V(y)$. Then any solution to this problem x^* must satisfy $r \cdot x^* \leq r \cdot x^0$. So we can restrict attention in the problem to the set

$$\{x \in V(y) : r \cdot x \leq r \cdot x^0\}.$$

But if $V(y)$ is closed, this set is compact: As the intersection of two closed sets, namely $V(y)$ and $\{x : r \cdot x \leq r \cdot x^0\}$, it is itself closed. And as r is strictly positive, the set in the display just above is strictly bounded, since $\{x : r \cdot x \leq r \cdot x^0\}$ is bounded: See the proof of Proposition 3.1(b).

Part c follows the usual lines: The objective function (being minimized) is linear, hence convex in x, and the constraint set is (by assumption) convex.

For part d, take any two vectors of factor prices r and r' and $\alpha \in (0,1)$. Let $r'' = \alpha r + (1-\alpha)r'$, and let $x^n \in V(y)$ be such that $r'' \cdot x^n \leq \mathrm{C}(r'') + 1/n$. Then

$$\alpha \mathrm{C}(r) + (1-\alpha)\mathrm{C}(r') \leq \alpha r \cdot x^n + (1-\alpha) r' \cdot x^n = (\alpha r + (1-\alpha r') \cdot x^n = r'' \cdot x^n \leq \mathrm{C}(r'') + 1/n.$$

Since n is arbitrary here, this implies $\alpha \mathrm{C}(r) + (1-\alpha)\mathrm{C}(r') \leq \mathrm{C}(r'')$, and C is concave.

Part e: Fix $r \in R^n_{++}$ and $\alpha > 0$. Suppose that $x^n \in V(y)$ is such that $r \cdot x^n \leq \mathrm{C}(r) + 1/n$. Then $\mathrm{C}(\alpha r) \leq (\alpha r) \cdot x^n = \alpha(r \cdot x^n) \leq \alpha \mathrm{C}(r) + \alpha/n$. Since n is arbitrary here, $\mathrm{C}(\alpha r) \leq \alpha \mathrm{C}(r)$. But if we replace α with $1/\alpha$ and r with αr in the inequality just derived, we get $\mathrm{C}(r) = \mathrm{C}((1/\alpha)\alpha r) \leq (1/\alpha)\mathrm{C}(\alpha r)$, and multiplying through both sides of this inequality by α gives $\alpha \mathrm{C}(r) \leq \mathrm{C}(\alpha r)$. Therefore, $\alpha \mathrm{C}(r) = \mathrm{C}(\alpha r)$.

Part f: If $x \in V^*(r)$, then $r \cdot x = \mathrm{C}(r)$ and $x \in V(y)$, hence $r' \cdot x \geq \mathrm{C}(r')$ for all r'. That makes x a supergradient of C at r.

Conversely, if x is in $V(y)$ and x is a supergradient of C at r, then for some scalar β, $r \cdot x + \beta = \mathrm{C}(r)$, and $r' \cdot x + \beta \geq \mathrm{C}(r')$ for all other r'. In particular, this is true for $r' = \alpha r$, for $\alpha > 0$, so that $\alpha r \cdot x + \beta \geq \mathrm{C}(\alpha r) = \alpha \mathrm{C}(r) = \alpha(r \cdot x + \beta) = \alpha r \cdot x + \alpha\beta$. Cancelling the term $\alpha r \cdot x$ from both sides of this inequality, we get $\beta \geq \alpha\beta$ for all $\alpha > 0$, which implies $\beta = 0$. Therefore, $r \cdot x = \mathrm{C}(r)$, which means that $x \in V^*(r)$.

That takes care of the easy parts of the proposition. Next comes part g, or *Berge's Theorem* (for fixed y and varying r):

Having fixed y, the problem is to minimize $r \cdot x$ subject to $x \in V(y)$. The objective function $r \cdot x$ is jointly continuous in r and x, because it is bilinear in the two arguments. The constraint correspondence $r \Rightarrow V(y)$ is constant in r, so it clearly is lower semi-continuous. So the only challenge here it to find the locally bounded, upper semi-continuous sub-correspondence within which all the solutions can be found. Let x^0 be any element of $V(y)$, and let $B(r) = \{x \in V(y) : r \cdot x \leq r \cdot x^0\}$. It is then clear minimizing $r \cdot x$ over $B(r)$ gives the same infimum as does minimizing over $V(y)$ and that any solution to the cost-minimization problem at prices r will be contained within $B(r)$ (for the fixed y), since any solution must have a total cost no greater than the cost of obtaining y using the (feasible) input set x^0. So we're done once we show that $r \Rightarrow B(r)$ is locally bounded and upper semi-continuous.

For local boundedness, fix $r^0 \in R^n_{++}$. Let $K = r^0 \cdot x^0$, $M = \max\{x^0_i; i = 1, \ldots, n\}$, and $\delta = \min\{r^0_i; i = 1, \ldots, n\}$. I assert that if $\|r - r^0\| \leq \delta/2$, then for all $x \in B(r)$, $\|x\| < 2n^2(K + M\delta/2)/\delta$. To see this, first note that if $\|r - r^0\| \leq \delta/2$, then by the triangle inequality, $|r_i - r^0_i| \leq \delta/2$ and therefore $r_i \geq \delta/2$. The first inequality tells us that $|r \cdot x^0 - r^0 \cdot x^0| \leq n \max\{|r_i - r^0_i| x^0_i; i = 1, \ldots, n\} \leq n(\delta/2)M$, therefore $r \cdot x^0 \leq$

$n(\delta/2)M + K \le n(K + M\delta/2)$. So for x to be in $B(r)$, we need that $r \cdot x \le n(K + M\delta/2)$. But since all the components of r are strictly positive and all the components of x are nonnegative, this implies that for each $i = 1, \ldots, n$, $r_i x_i \le n(K + M\delta/2)$. And since $r_i \ge \delta/2$, this in turn implies that $x_i \le n(K + M\delta/2)/(\delta/2) = 2n(K + M\delta/2)/\delta$. This then gives the upper bound on $\|x\|$ that I asserted.

And for upper semi-continuity of $r \Rightarrow B(r)$, suppose $r^n \to r$, $x^n \to x$, and $x^n \in B(r^n)$ for each n. The last part of this says that $x^n \in V(y)$ and $r^n \cdot x^n \le r^n \cdot x^0$ for each n. Since $V(y)$ is closed, $x \in V(y)$, and by continuity of the dot product, taking the limit on both sides of the inequalities $r^n \cdot x^n \le r^n \cdot x^0$ tells us that $r \cdot x \le r \cdot x^0$, which implies that $x \in B(r)$.

Part h:

Fix a set $X \subseteq R^n_+$. Let C be the "cost function" for X—that is, $\mathrm{C} : R^n_{++} \to R$ is defined by $\mathrm{C}(r) = \inf\{r \cdot x : x \in X\}$—and let C′ be the cost function for $\overline{\mathrm{CCH}}(X)$. Since $X \subseteq \overline{\mathrm{CCH}}(X)$ and the two functions are defined using infima, we know that $\mathrm{C}'(r) \le \mathrm{C}(r)$ for all r. Moreover, C and C′ are both nonnegative. So once we show that $\mathrm{C}'(r) < \mathrm{C}(r)$ is impossible, we'll know that $\mathrm{C} \equiv \mathrm{C}'$, proving the first half of the result.

Suppose $\mathrm{C}'(r) < \mathrm{C}(r)$ for some specific r. The value of $\mathrm{C}'(r)$ is gotten by taking the infimum of $r \cdot x$ for x in the closure of CCH(X), so (since the dot product is continuous) there must be some $x \in \mathrm{CCH}(X)$ such that $r \cdot x < \mathrm{C}(r)$. But for x to be in CCH(X), it must be that $x \ge \sum_{\ell}, \alpha_\ell x^\ell$ where the α's are nonnegative scalars that sum to one and each x^ℓ is in X. Since factor prices (components of r) are strictly positive, $r \cdot x \ge \sum_\ell \alpha_\ell (r \cdot x^\ell)$, therefore $\sum_\ell \alpha_\ell (r \cdot x^\ell) < \mathrm{C}(r)$. But since $x^\ell \in X$ for all ℓ, $r \cdot x^\ell \ge \mathrm{C}(r)$ for all ℓ, which gives us $\mathrm{C}(r) = \sum_\ell \alpha_\ell \mathrm{C}(r) \le \sum_\ell \alpha_\ell (r \cdot x^\ell) < \mathrm{C}(r)$, a contradiction.

For the converse, fix X and X' such that $\overline{\mathrm{CCH}}(X) \ne \overline{\mathrm{CCH}}(X')$. Without loss of generality, suppose $x^0 \in \overline{\mathrm{CCH}}(X) \setminus \overline{\mathrm{CCH}}(X')$. Since $\overline{\mathrm{CCH}}(X')$ is closed and convex, the Strict-Separation Theorem tells us that there exist $r^0 \in R^n$ (no sign restriction, yet) such that $r^0 \cdot x^0 < \inf\{r^0 \cdot x : x \in \overline{\mathrm{CCH}}(X')\}$. Since $\overline{\mathrm{CCH}}(X')$ is comprehensive upwards, it is easy to see that $r^0 \ge 0$. Now let e denote the vector $(1, 1, \ldots, 1) \in R^n$, and let $c = e \cdot x^0$. Let ϵ be small enough (but strictly positive) so that $r^0 \cdot x^0 + \epsilon c < \inf\{r^0 \cdot x : x \in \overline{\mathrm{CCH}}(X')\}$. Then if $r = r^0 + \epsilon e$, which is strictly positive, $r \cdot x^0 < \inf\{r \cdot x : x \in \overline{\mathrm{CCH}}(X')\}$ (since $r \cdot x \ge r^0 \cdot x$ for all nonnegative x), and therefore the cost function associated with X, evaluated at the strictly positive price vector r, will give a value less than the cost function associated with X'. The two have different cost functions.

Part i:

Rewrite X as $\cap_{r \in R^k_{++}} \{x \in R^k_+ : r \cdot x \ge \mathrm{C}(r)\}$, and you see that X is the intersection of closed and convex sets, hence it itself is closed and convex. And since the r's are strictly positive, it is clearly comprehensive upwards: If $x \in X$ and $x' \ge x$, then $r \cdot x' \ge r \cdot x$ for all r (the r are strictly positive) and $r \cdot x \ge \mathrm{C}(r)$ for all r ($x \in X$), therefore $r \cdot x' \ge \mathrm{C}(r)$ for all r, hence $x' \in X$.

Take any $r^0 \in R^n_{++}$, and let $t = \inf\{r^0 \cdot x : x \in X\}$. Since $x \in X$ only if $r \cdot x \ge \mathrm{C}(r)$

for all r, this must be true in particular for r^0, hence $t \geq \mathrm{C}(r^0)$. We must show that $t = \mathrm{C}(r^0)$ and, to justify the "min," that there is some specific $x^0 \in X$ such that $r^0 \cdot x^0 = \mathrm{C}(r^0)$. Here is where we use the concavity and homogeneity of C: r^0 is in the interior of the domain of C (every strictly positive price vector is), so there is a supergradient of C at r^0; that is, some $x^0 \in R^k$ and scalar β such that $r^0 \cdot x^0 + \beta = \mathrm{C}(r^0)$ and $r \cdot x^0 + \beta \geq \mathrm{C}(r)$ for all $r \in R^k_{++}$. The at-this-point usual argument employing homogeneity is enlisted to show that $\beta = 0$: We know that

$$\alpha r \cdot x^0 + \beta \geq \mathrm{C}(\alpha r) = \alpha \mathrm{C}(r) = \alpha r \cdot x^0 + \alpha\beta \quad \text{or} \quad \beta \geq \alpha\beta,$$

for all $\alpha > 0$, which implies that $\beta = 0$. So we are done once we show that $x^0 \geq 0$. This is where C being nonnegative-valued comes in: Suppose x^0 had a strictly negative component. Let r be the strictly positive price vector with coordinate values 1 everywhere except the one coordinate where x^0 is negative, in which coordinate have r take on the value B, for (very large) positive B. For B large enough, $r \cdot x^0$ will be negative. But $r \cdot x^0 \geq \mathrm{C}(r) \geq 0$, which is a contradiction.

(The problem didn't ask you to prove part j, so I will not include a proof here.)

■ 9.13. I'll deal first with the part about C being continuous, if we assume that $y \Rightarrow V(y)$ is continuous, and then go back to the part where we only assume that the V correspondence is upper semi-continuous.

We are going to apply Berge's Theorem, of course. Continuity of the objective function is obvious, and we've assumed that the constraint correspondence is lower semi-continuous. So the key is to produce the requisite locally bounded and upper semi-continuous sub-correspondence. Here is where the trick is enlisted. Continuity of C and upper semi-continuity of V^* are local properties; if they hold (say) in an open neighborhood of every (y^0, r^0), then they hold "globally." The r argument presents no problem; difficulties that arise come about because of the y argument. So fix any $y^0 \in Y^o$. By the definition of Y^o, some y' exists such that $y' \in Y$ and y' is strictly greater (in every component) than y^0. So we can put an open neighborhood around y^0 such that every y in that neighborhood is strictly less than y'. Now let x^0 be any point in $V(y')$. Because the nesting property holds for the V's, we know that $x^0 \in V(y)$ for all y in the neighborhood of y^0. So for each y in that neighborhood and $r \in R^n_{++}$, consider

$$V'(y, r) = \{x \in V(y') : r \cdot x \leq r \cdot x^0\}.$$

Clearly, since $x^0 \in V(y)$ for all these y, minimizing over $V'(y, r)$ (for a given r) gives the same infimum as does minimizing over $V(y)$, and all the solutions when minimizing over $V(y)$ are contained in $V'(y, r)$. And, following the details of the proof of part g of Proposition 9.24, the correspondence $(y, r) \Rightarrow V'(y, r)$ is upper semi-continuous and locally bounded. So Berge's Theorem can be applied for (y, r) where y is in this neighborhood of y^0 (and $r \in R^n_{++}$), and C is continuous and V^* is upper semi-continuous at y^0 and any r. Since y^0 was an arbitrarily picked point from Y^o, we have the result.

Now suppose that the V correspondence is only upper semi-continuous. Let $\{(y^\ell, r^\ell)\}$ be a sequence from $Y^o \times R^n_{++}$ converging to some $(y^0, r^0) \in Y^o \times R^n_{++}$. Let x^ℓ be any solution to the cost-minimization problem at (y^ℓ, r^ℓ). (We know a solution exists because $V(y^\ell)$ is closed.) The previous paragraph tells us that the x^ℓ live inside a compact set, so by looking along a subsequence if necessary, we can assume that $x^\ell \to x^0$. Upper semi-continuity of the V correspondence tells us that $x^0 \in V(y^0)$, and so C$(y^0, r^0) \leq r^0 \cdot x^0$.

But this tells us that for any sequence $\{(y^\ell, r^\ell)\}$ converging to some (y^0, r^0) in Y$^o \times R^n_{++}$, lim inf C$(y^\ell, r^\ell) \geq$ C(y^0, r^0). (Start with the sequence, and look along a subsequence along which the limit infimum is attained. Then plug in to the preceding paragraph.) This means that C is lower semi-continous (as a function).

Microeconomic Foundations I
Choice and Competitive Markets

Student's Guide

Chapter 10: The Expenditure-Minimization Problem

Summary of the Chapter

Chapter 9 began by providing results concerning the theory of the competitive and profit-maximizing firm that parallel developments for the theory of the utility-maximizing consumer from Chapters 3 and 4. But after that beginning, Chapter 9 went on to some results that have no parallel in Chapters 3 and 4. The most important of these were: (1) How to reconstruct, to the extent possible, a production-possibility set Z from its profit function. (2) Providing necessary and sufficient conditions for a function π to be a profit function for some production-possibility set Z.

In this chapter and the next, we go back to the theory of the utility-maximizing consumer. A variety of results will be provided, but our first and foremost objective is to replicate those two results from Chapter 9 in the context of the consumer's problem. Unhappily, the consumer's utility-maximization problem is more complex than the firm's profit-maximization problem on two grounds:

1. The firm's profit-maximization problem has the price vector p as parameter. In the consumer's utility-maximization problem, prices enter parametrically, but so does the consumer's income y.

2. In the firm's profit-maximization problem, prices enter the objective function, but the feasible set never changes. In the consumer's utility-maximization problem, prices (and income) shift the feasible set.

In this chapter, we study a problem related to the consumer's utility-maximization problem, the so-called *Expenditure-Minimization Problem*, which deals with the first of these complications while avoiding the second. In the Expenditure-Minimization Problem, the question asked is, Fixing a consumer (identified by her utility function u), for each (attainable) level of utility v and strictly positive price vector p, what is least expensive way at these prices for the consumer to attain utility level v or more. (This problem is sometimes called the *Dual Consumer's Problem*.) The chapter runs the following course:

1. The Expenditure-Minimization Problem, or the EMP, is defined in Section 10.1, and basic analysis of the problem (existence of solutions, convexity of the set of solutions, uniqueness of the solution, homogeneity properties) are provided in Section 10.2.

2. The correspondence that, for each pair (p, v), gives the set of solutions to the EMP at p and v, is named the *Hicksian-demand* correspondence, and the optimized value of the objective function is named the *expenditure function* in Section 10.3. Berge's Theorem is applied. In Section 10.4, basic properties of the expenditure function are derived; most siginficantly, the expenditure function is concave and homogeneous of degree 1 in p for fixed v. Differentiability of the expenditure function in p is shown to be equivalent to uniqueness of solutions to the EMP.

3. A variety of utility functions can give rise to the same expenditure function, and in Section 10.5, we ask and answer the question, When are two utility functions expenditure equivalent, in the sense that they give the same expenditure function? We show in particular that for any continuous utility function u, there is a unique continuous, quasi-concave, and nondecreasing utility function $\hat{u}$ that shares an expenditure function with u. And, in Section 10.6, we discuss how to take an expenditure function and "invert" it to find the unique continuous, quasi-concave, and nondecreasing utility function that generates it.

4. In Section 10.6, we assume that we are given a legitimate expenditure function to invert. In Section 10.7, we find necessary and sufficient conditions for an expenditure function to be "legitimate," in the sense that, when inverted, what results is a continuous, quasi-concave, and nondecreasing utility function. (Local insatiability is included in the list of properties for the utility function.)

5. The chapter concludes in Section 10.8, where we connect the EMP to the Consumer's Problem from Chapters 3 and 4.

Solutions to Starred Problems

▪ 10.1. (b) We prove part (b) first (since having the existence of a solution makes the proof for part (a) expositionally neater). The problem is to minimize $p \cdot x$ subject to $u(x) \geq v$, for $u(0) \leq v <$, where = sup $u(x)$. Since $v <$, there exists some x^0 such that $v \leq u(x^0) <$ (the supremum in the definition of is not attained, since u is globally insatiable). Fix such an x^0, and let $X = \{x \in R^k_+ : u(x) \geq v; p \cdot x \leq p \cdot x^0\} = \{x \in R^k_+ : u(x) \geq$

$v\} \cap \{x \in R^k_+ : p \cdot x \le p \cdot x^0\}$. Both sets in the intersection are closed, the first because u is a continuous function and the second because $p \cdot x$ is continuous in x. And the second set is bounded; p is strictly positive, so the arguments we gave back in Chapter 3 for the boundedness of budget sets work. Therefore, X is the intersection of two closed sets, one of which is bounded, and X is compact. Hence, the problem of minimizing $p \cdot x$ over X has a solution (minimizing a continuous function $x \to p \cdot x$ over a compact set). But any solution to this problem is a solution to the EMP for p and v: If x is feasible for the EMP, then $u(x) \ge v$, and if x is a solution, then $p \cdot x \le p \cdot x^0$, since x^0 is feasible. Therefore, if x solves the EMP, then $x \in X$. On the other hand, every point in X is feasible for the EMP; indeed, X is a subset of the set of feasible points for the EMP, so minimizing over X cannot improve matters. That is, solutions to the EMP are solutions to minimizing $p \cdot x$ over X (and vice versa), and there are solutions to the problem of minimizing $p \cdot x$ over X, therefore there are solutions to the EMP.

(a) Suppose x solves the EMP for prices p and target utility v but not for λp and v, for some $\lambda > 0$. Let x' be a solution to the EMP for λp and v (a solution exists by part b). Then $\lambda p \cdot x' < \lambda p \cdot x$, which (since $\lambda > 0$) implies that $p \cdot x' < p \cdot x$. But if x' is a solution to the EMP for λp and v, then $u(x') \ge v$, so x' is feasible for the EMP at p and v, in which cas $p \cdot x' < p \cdot x$ contradicts the original hypothesis that x solves the EMP for p and v. This contradiction proves that x must solve the EMP for λp and v (if it solves the EMP for p and v).

(c) Suppose x and x' solve the EMP at p and v. Then x and x' must both be feasible for this problem, which means that $u(x) \ge v$ and $u(x') \ge v$. For any $\alpha \in [0,1]$, $u(\alpha x + (1-\alpha)x') \ge \min\{u(x), u(x')\} \ge v$, by quasi-concavity of u, and therefore $\alpha x + (1-\alpha)x'$ is feasible for the EMP at p and v. But $p \cdot (\alpha x + (1-\alpha)x') = \alpha p \cdot x + (1-\alpha) p \cdot x' = p \cdot x = p \cdot x'$, since $p \cdot x = p \cdot x'$ (they are both solutions), which implies that $p \cdot (\alpha x + (1-\alpha)x')$ has the same (minimal) level of expenditure; it is also a solution. The set of solutions is convex.

And if u is strictly quasi-concave: If x and x' are distinct solutions of the EMP at p and v, one of them is not 0, and since p is strictly positive, $p \cdot x = p \cdot x' > 0$; that is, both are not 0. Let $x'' = 0.5x + 0.5x'$; by strict quasi-concavity, $u(x'') > \min\{u(x), u(x')\} = v$. But then by continuity, for $\beta < 1$ but sufficiently close to 1, $u(\beta x'') > v$, and $\beta x''$ is feasible. Now $p \cdot (\beta x'') = \beta p \cdot x'' < p \cdot x = p \cdot x'' = p \cdot x'$ (since $p \cdot x = p \cdot x' = p \cdot x'' > 0$), which (since $\beta x''$ is feasible) contradicts the alleged optimality of x and x'. There cannot be two distinct solutions if u is strictly quasi-concave.

(d) If x solves the EMP at p and v, then x must be feasible; that is, $u(x) \ge v$. So if $u(x) \ne v$, it must be that $u(x) > v$. Since $v \ge u(0)$, $x \ne 0$, and $p \cdot x > 0$. But then by continuity of u, for β strictly less than 1 but close to 1, $u(\beta x) > v$; that is, βx is feasible. And $p \cdot (\beta x) = \beta p \cdot x < p \cdot x$, which means that x isn't optimal for p and v, a contradiction. It must be that $u(x) = v$.

■ 10.3. The first step is to show that for all $p \in R^k_{++}$ and $v \in R_+$, then there exists some $x \in R^k_+$ that attains the infimum in the definition of e. Since, by assumption, sup $u(x) = \infty$, there exists some $x^0 \in R^k_+$ such that $u(x^0) \geq v$. By the same argument as was given in the proof of Proposition 10.2(b) (see the solution just given), if we let $X = \{x \in R^k_+ : u(x) \geq v \text{ and } p \cdot x \leq p \cdot x^0\}$, then the set of solutions (if any) to the problem of finding the infimum of $p \cdot x$ over all x such that $u(x) \geq v$ is the same as the set of solutions to the problem of finding the infimum over $x \in X$. And $X = \{x \in R^k_+ : u(x) \geq v\} \cap \{x \in R^k_+ : p \cdot x \leq p \cdot x^0\}$. The second set in the intersection is compact (again see the solution given just previously), while the first set is closed, since u is upper semi-continuous. Therefore we are minimizing a continuous function over a compact set, and a solution exists; the infimum is attained.

Moveover, $p \to e(p, v)$ is concave in p for each fixed v. The argument from the text (Proposition 10.4(d)) works like a charm: Fix p, p' and $p'' = \alpha p + (1-\alpha)p'$, for $\alpha \in [0, 1]$. Let x'' solve the minimization problem for p'' (a solution exists per the previous paragraph). Then $u(x'') \geq v$, so x'' is feasible for (p, v) and (p', v), and $e(p, v) \leq p \cdot x''$ and $e(p', v) \leq p' \cdot x''$. But then

$$\alpha e(p,v)+(1-\alpha)e(p',v) \leq \alpha(p\cdot x'')+(1-\alpha)p'\cdot x'' = (\alpha p+(1-\alpha)p')\cdot x'' = e(\alpha p+(1-\alpha)p'', v),$$

showing that e is concave in p. And, being concave in p (since p is drawn from an open domain), e is continuous in p.

It remains to show that e is lower semi-continuous in (p, v). Let $\{(p^n, v^n)\}$ be a sequence of price–target utility pairs with limit (p, v). Let x^n be a solution to the minimization problem at (p^n, v^n); that is, $u(x^n) \geq v^n$ and $p^n \cdot x^n = e(p^n, v^n)$. To show lower semi-continuity, we have to show that

$$e(p, v) \leq \liminf e(p^n, v^n).$$

Look along a subsequence, along which the limit infimum is attained. (I'll assume that the original sequence is that subsequence.) If the limit infimum is $+\infty$, then there is nothing more to show. So suppose the limit infimum is finite. This means that the set $\{e(p^n, v^n); n = 1, \ldots\}$ is bounded uniformly by some M. Moreover, since $p^n \to p$ and p is strictly positive, there is a lower bound m on all the components p^n_i for $i = 1, \ldots, k$ and $n = 1, \ldots$. And since $M \geq e(p^n, v^n) = p^n \cdot x^n$, this means there is an upper bound M/m on the components x^n_i for $i = 1, \ldots, k$, $n = 1, \ldots$. But then the vectors x^n live in a compact set, and by looking along a (further) subsequence, we can assume that x^n converges to some x. Since u is upper semi-continuous, $u(x) \geq \limsup u(x^n) \geq \lim v^n = v$, and therefore x is feasible at (p, v), so $e(p, v) \leq p \cdot x$. But by continuity of the dot product, $p \cdot x = \lim p^n \cdot x^n = \lim e(p^n, v^n)$, showing that e is lower semi-continuous.

■ 10.6. The problem asks how much of Proposition 10.18 survives if u is not necessarily locally insatiable and (only) upper semi-continuous. So to begin, we note that as long as u is upper semi-continuous, solutions to both the CP and the EMP are guaranteed to exist (for strictly positive prices, of course). The former is true because in the CP, we are maximizing an upper semi-continuous function on a compact set, which is enough for existence of a solution. As for the EMP, see the first paragraph in the solution to Problem 10.3 just given.

So fix p and y. Suppose $x \in (p, \nu(p, y))$. Let x' be any solution of the CP at p and y; so (of course) $p \cdot x' \leq y$ and $u(x') = \nu(p, y)$. The second of these conclusions tells us that x' is feasible for the EMP at p and $\nu(p, y)$, and (hence) x, being a solution of the EMP, must be no more expensive, or $p \cdot x \leq p \cdot x' \leq y$, which means that x is feasible for the CP. But $u(x) \geq \nu(p, y)$, which tells us that x must be a solution to the CP at p and y, or $(p, \nu(p, y)) \subseteq (p, y)$. Moreover, $e(p, \nu(p, y)) = p \cdot x \leq y$.

The following example shows that this set inclusion may be strict and that $e(p, \nu(p, y)) < y$ is possible. Take $k = 1$ and let

$$u(x) = \begin{cases} x, & \text{for } 0 \leq x \leq 5, \\ 10 - x, & \text{for } 5 < x \leq 6, \text{ and} \\ x - 2, & \text{for } x > 6. \end{cases}$$

(Draw the function u if you don't "see" it.) The key is that this function u has a local max at $x = 5$. Now for $p = 1$ and $y = 7$, there are two solutions to the CP, namely $x = 5$ and $x = 7$, giving utility level 5. But the cheapest way to achieve utility level 5 is with $x = 5$ only. This gives both things that we wanted to show.

Note that u in this example is continuous; it is the lack of local insatiability that causes the "one-way" implications. In particular, suppose that u is only upper semi-continuous but is also locally insatiable. And suppose that $x \in (p, y)$ but is not in $(p, \nu(p, y))$. This implies that there exists $x' \in (p, \nu(p, y))$ which is strictly cheaper than x at prices p; that is $p \cdot x' < p \cdot x$. By local insatiability, there exists x'' that is strictly preferred to x' but still is cheaper than x at prices p; that is, $u(x'') > u(x') \geq \nu(p, y)$ and $p \cdot x'' \leq p \cdot x \leq y$, which contradicts the supposed optimality (in the CP) of x. So if u is locally insatiable but (only) upper semi-continuous, then $(p, \nu(p, y)) = (p, y)$, and $e(p, \nu(p, y)) = y$.

Now fix p and v. Suppose $x \in (p, e(p, v))$. Then $p \cdot x \leq e(p, v)$. Suppose $x' \in (p, v)$. We know that $p \cdot x' = e(p, v)$, so x' is feasible for the CP at p and $e(p, v)$. Since x is optimal for the CP at those parameters, $u(x) \geq u(x') \geq v$, and x is feasbile for the EMP at p and v. But since x is feasible for the CP at p and $e(p, v)$, $p \cdot x \leq e(p, v)$, and (therefore) $x \in (p, v)$. This implies that $(p, e(p, v)) \subseteq (p, v)$. Moreover, $\nu(p, e(p, v)) = u(x) \geq v$.

If we know that u is continuous, the set inclusion and inequalities just given are equalities: First, if u is continuous and $x \in (p, v)$, we know that $u(x) = v$. But $u(x)$ here is $\nu(p, e(p, v))$. And, suppose $x' \in (p, v)$. Since $p \cdot x' = e(p, v)$ by definition, x' is feasible for the CP at p and $e(p, v)$. Let x be any arbitrary element of $(p, e(p, v))$. We proved that $x \in H(p, v)$ and so, since u is continuous, $u(x) = v$. Similarly, we know that $u(x') = v$. But then x' is feasible for the CP and provides as much utility as a solution to the CP, so $x' \in (p, e(p, v))$.

To finish off, we need an example where u is upper semi-continuous (only) and locally insatiable, and for which the set inclusion $(p, e(p, v)) \subseteq (p, v)$ and inequality $\nu(p, e(p, v)) \geq v$ are both strict. It takes two dimensions to do this, so the example is a bit complex: Let

$$u((x_1, x_2)) = \begin{cases} x_1 + x_2 + 1, & \text{for } x_1 + x_2 \geq 1 \text{ and } x_2 \geq x_1, \text{ and} \\ x_1 + x_2, & \text{otherwise.} \end{cases}$$

Set $p = (1, 2)$ and $v = 1.5$. To minimize expenditure and achieve this level of utility, there are two solutions: $(0.5, 0.5)$ and $(1.5, 0)$, each costing 1.5. But at these prices and a budget of 1.5, only $(0.5, 0.5)$ is utility maximizing, giving an indirect utility level of 2.

■ 10.8. If x^0 is a solution to the MEMP(p, x^0), then it is also a solution to the MCP(p, x^0), but not vice versa:

Suppose x^0 minimizes $p \cdot x$ subject to $u(x) \geq u(x^0)$. If $x^0 = 0$, then $p \cdot x^0 = 0$, and $x^0 = 0$ is the only feasible consumption bundle in the MCP(p, x^0), so is optimal. So we can assume that $x^0 \neq 0$ and $u(x^0) > u(0)$ (since if $x^0 \neq 0$ but $u(x^0) \leq u(0)$, then x^0 cannot be a solution to the MEMP(p, x^0), as 0 costs less at strictly positive prices and satisfies the utility constraint).

Now suppose in addition that x^0 does not maximize $u(x)$ subject to $p \cdot x \leq p \cdot x^0$. Then there exists some x^* such that $p \cdot x^* \leq p \cdot x^0$ but $u(x^*) > u(x^0) \geq u(0)$. Since $u(x^*) > u(0)$, $x^* \neq 0$. But then for β strictly less than one but close to one, continuity of u ensures that $u(\beta x^*) > u(x^0)$, and βx^* is feasible for the MEMP(p, x^0). This gives us a contradiction to the original assumption that x^0 is a solution to the MEMP(p, x^0), since βx^* costs strictly less than x^* (remember that $x^* \neq 0$), which in turn costs no more than x^0 at the prices p.

Now take the counterexample from the solution to Problem 10.7, namely $k = 1$ and let

$$u(x) = \begin{cases} x, & \text{for } 0 \leq x \leq 5, \\ 10 - x, & \text{for } 5 < x \leq 6, \text{ and} \\ x - 2, & \text{for } x > 6. \end{cases}$$

I assert that, for $p = 1$, 7 is a solution to the MCP$(p, 7)$, since with a budget of 7 and $p = 1$, the best bundles are 5 and 7, both giving utility 5. But with a target utility $u(7) = 5$, the cheapest bundle is 5; 7 is not a solution of MEMP$(p, 7)$.

Clearly, we need local insatiability of u. Suppose u is locally insatiable and x^0 is a solution of the MCP(p, x^0). Then we know that x^0 maximizes $u(x)$ on the set of x such that $p \cdot x \leq p \cdot x^0$. Suppose, however, x^0 does not minimize $p \cdot x$ on the set of x such that $u(x) \geq u(x^0)$. Then there is some x^* with $p \cdot x^* < p \cdot x^0$ and $u(x^*) \geq u(x^0)$. By local insatiability, we can find a close neighbor of x^*, call it x', close enough so that $p \cdot x' < p \cdot x^0$ and $u(x') > u(x^*) \geq u(x^0)$. But this contradicts the supposed optimality of x^0 in the MCP(p, x^0).

Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Chapter 11: Classic Demand Theory

Summary of the Chapter

This chapter completes the development of the theory of the consumer in parallel with the theory of the firm developed in Chapter 9. The key question, answered (more or less) by the Integrability Theorem (which is never formally stated, hence the "more or less") is, When is a (parametric) family of demand functions the Marshallian demand of a utility-maximizing consumer? But the path that takes us to this climax (and the denouement that follows) is long and winding:

1. We begin with *Roy's Identity* and the *Slutsky Equation*, two important "identities" from the theory of consumer demand. Simple derivations are provided, and intuition for the two results are given, relating them to the notion of compensated demand, wherein one tries to isolate the *substitution* and *income* effects of a change in a price to (a) the level of indirect utility (Roy's Identity) and (b) the quantities consumed (the Slutsky Equation).

2. Section 11.2 concerns differentiability of indirect utility, both in prices and in income. Included here is a more robust derivation of Roy's Identity.

3. Section 11.3 does for the indirect utility function what we did last chapter for the expenditure function: Which utility functions give rise to the same indirect utility? (How) Can you invert indirect utility to get the utility function (or, *a* utility function) that generates it? What are necessary and sufficient conditions on an indirect utility to be "legitimate," meaning, generated by a continuous (quasi-concave, nondecreasing, and locally insatiable) utility function?

4. In Section 11.4, the Implicit-Function Theorem is used to generate conditions under which Marshallian demand (if it is single-valued) is (locally) differentiable in prices and income.
5. Section 11.5 provides the climax, the Integrability Theorem. Or, rather, the section discusses the Integrability Theorem, describing what it says (roughly) and how it works (again, roughly), but without giving all the fine mathematical details.
6. The Slutsky Equation (and symmetry of the so-called Slutsky matrix) is used to motivate a discussion of complementary and substitute goods in Section 11.6.
7. Afriat's Theorem, based on revealed preference, gives one set of answers to the question, When do demand data arise from a utility-maximizing consumer? The Integrability Theorem gives a seemingly very different answer to what is, at its core, the same question. We conclude in Section 11.7 by asking, How are these connected? As with the Integrability Theorem, we discuss the connection instead of developing it in all its details.

Solutions to Starred Problems

■ 11.1. It is clear that at the optimal solution, x_1 and x_2 must both be strictly positive, since otherwise $U(x_1, x_2, x_3)$ will be $-\infty$. Their bangs for the buck are $1/x_1$ for commodity one and $3/2x_2$ for commodity 2. The bang for the buck of the third commodity is $1/3(x_3 + 10)$, which is $1/30$ at $x_3 = 0$. Thus consumption of the third commodity only begins when the bangs for the buck of the other two commodities fall to $1/30$, which is where

$$\frac{1}{x_1} = \frac{1}{30} \text{ or } x_1 = 30 \quad \text{and} \quad \frac{3}{2x_2} = \frac{1}{30} \text{ or } x_2 = 45,$$

at a total cost of 120. Hence:

- For $y \leq 120$, only commodities 1 and 2 are purchased, and equal bangs for the buck means

  $$x_1 = \frac{2x_2}{3},$$

  hence the budget equation is

  $$x_1 + 2x_2 = \frac{2x_2}{3} + 2x_2 = \frac{8x_2}{3} = y,$$

  hence $x_2 = 3y/8$ and $x_1 = y/4$. This gives indirect utility $\ln(y/4)+3\ln(3y/8)+\ln(10) = \ln(y)-\ln(4)+3\ln(y)+3\ln(3/8)+\ln(10) = 4\ln(y)+k$, for k some constant, and $\partial\nu/\partial y =$

$4/y$. The multiplier λ is, of course, the bang for the buck of the two commodities that are consumed, which is $1/(y/4) = 4/y$ for commodity 1 and $3/[2(3y/8)] = 4/y$ for commodity 2.

- For $y > 120$, all three commodities are purchased, and equal bangs for the buck gives

$$x_1 = \frac{2x_2}{3} = 3(x_3 + 10),$$

hence the budget equation is

$$x_1 + 2x_2 + 3x_3 = 3(x_3 + 10) + 9(x_3 + 10) + 3x_3 = 15x_3 + 120 = y,$$

or

$$x_3 = \frac{y - 120}{15}, x_1 = \frac{y + 30}{5}, \text{ and } x_2 = \frac{3(y + 30)}{10}.$$

Indirect utility is

$$\nu((1, 2, 3), y) = \ln(y + 30) + 3\ln(y + 30) + \ln(y + 30) + K,$$

where I've factored out a constant K, or $5\ln(y+30)+K$, so that $\partial\nu/\partial y = 5/(y+30)$. The multiplier is the bang for the buck of the three commodities: For commodity 1, this is $1/x_1 = 1/[(y + 30)/5] = 5/(y + 30)$. For commodity 2, it is $3/(2x_2) = 3/[2(3(y + 30)/10)] = 5/(y + 30)$. And for commodity 3, it is $1/[3(x_3 + 10) = 1/[3(y + 30)/15] = 5/(y + 30)$.

■ 11.4. Suppose that u is continuous and locally insatiable. By Corollary 11.6, the indirect utility function generated by u is identical to the indirect utility function generated by "its" nondecreasing and quasi-concave "equivalent," $\hat{u}$. Since this half of the proof involves demonstrating properties of ν, we can assume w.l.o.g. that u is also nondecreasing and quasi-concave. Fix $x^0 \in R^k_+$, and suppose $v \geq 0$ and $\epsilon > 0$ satisfy $\nu(p, p \cdot x^0) \geq v + \epsilon$ for all $p \in R^k_{++}$. Then according to Proposition 11.7, $u(x^0) \geq v + \epsilon$. Because u is continuous, there exists $\delta < 1$ (but close to 1) such that $u(\delta x^0) \geq v + \epsilon/2$. But then, for any strictly positive p, since δx^0 is feasible for the CP at prices p and income $y = p \cdot \delta x^0 = \delta p \cdot x^0$, we have $\nu(p, \delta p \cdot x^0) \geq u(\delta x^0) \geq v + \epsilon/2 > v$.

Conversely, suppose that ν has the property (11.7), and we construct u from ν via (11.5) and (11.6). We already know that u is upper semi-continuous, so to verify that u is continuous, we need only show that it is lower semi-continuous. Suppose, then, by way of contradiction that for some sequence $\{x^n\}$ with limit x^0, $\lim_n u(x^n) < u(x^0)$. Without loss of generality, we can assume that $u(x^n)$ is nondecreasing in n. Let $v' =$

$\lim_n u(x^n)$, and let $\epsilon = (u(x^0) - v')/2 > 0$ and $v = v' + \epsilon$. Then it is clear that $\nu(p, p \cdot x^0) \geq u(x^0) = v + \epsilon$ for all strictly positive p, so by property (11.7), for some $\delta < 1$, $\nu(p, \delta p \cdot x^0) = \nu(p, p \cdot \delta x^0) > v$ for all p. By the same argument as used in the proof of Proposition 10.17, for large-enough n, $x^n \geq \delta x^0$. And since ν is nondecreasing in its last argument, this implies that for all large-enough n, $\nu(p, p \cdot x^n) > v$ for all p. But then looking at (11.5) and (11.6), this implies that for all large n, $u(x^n) > v$, which is the desired contradiction.

■ 11.6. There are (at least) two ways to attack this problem and, despite what the problem statement says, in one of them you don't (necessarily) have to worry about first-order conditions for the EMP.

Of course, we must know first of all that Hicksian demand is a function and not a correspondence, in which case Hicksian demand will be the "same" as Marshallian demand. Writing $h(p, v)$ for Hicksian demand and $d(p, y)$ for Marshallian demand, this gives us the first line of attack: We have

$$h(p, v) = d(p, e(p, v)),$$

and so if d is continuously differentiable and e is continuously differentiable (in both p and v), then the chain-rule establishes the differentiability of h. Proposition 11.10 gives sufficient conditions for differentiability of d, and we (almost!) know what it takes for e to be differentiable; I add the parenthetical *almost!* because we only really proved differentiability in p. But following our proof of the differentiability of ν in y, it isn't hard to obtain differentiability of e in v. (If you did Problem 2, you will have done this.) So that's one line of attack.

Rather than finish that line of attack, I'll take a direct approach, mimicking the proof of Proposition 11.10. The first step is to discuss the solution of the EMP using calculus. We need to assume that u is (at least) continuously differentiable. The problem is to

$$\text{minimize } p \cdot x, \quad \text{subject to } u(x) \geq v, x \geq 0.$$

We will want to know that the first-order/complementary-slackness conditions are necessary at any optimal solution, so we need to know that, at any solution, the constraint qualification holds. We know that if u is differentiable, it is continuous, and we know that if u is continuous, then the constraint $u(x) \geq v$ must hold with equality. (See Proposition 10.2(d).) So that constraint is binding. Now if $v = u(0)$, then we are stuck: The solution is $x = 0$, all the nonnegativity constraints bind, and the constraint qualification fails miserably. So we must restrict attention to $v > u(0)$. (You may not recall, but back in Chapter 3, when we said that the constaint qualification held for the CP, it was only for the case $y > 0$.)

But if $v > u(0)$, then at any solution, at least one x_i must be strictly positive. (That is, for any x that meets the constraint $u(x) \geq v$, some $x_i > 0$.) This means that the constraint qualification will hold as long as $\partial u/\partial x_i > 0$ for any of the is that have $x_i > 0$. To ensure this is so, we need to add an assumption (it isn't true in general), and I'll simply assume that u is strictly increasing and, moreover, has strictly positive partial derivatives at all nonzero arguments. Then the constraint qualification holds (if you aren't sure about this, go to Appendix 5 to recall what this means and verify that it is so), and the first-order/complementary-slackness conditions are indeed necessary.

And what are they? They are

$$\begin{aligned} & p_i - \hat{\lambda}\frac{\partial u}{\partial x_i} - \hat{\mu}_i = 0, \ i = 1, \ldots, k \\ & \hat{\lambda} \geq 0, u(x) \geq v, \hat{\lambda}(u(x) - v) = 0, \\ & \hat{\mu}_i \geq 0, x_i \geq 0, \hat{\mu}_i x_i = 0, \ i = 1, \ldots, k \end{aligned} .$$

We can rewrite this by eliminating the multipliers on the nonnegativity constraints and noting that we know the constraint $u(x) \geq v$ must hold with equality; we have

$$\begin{aligned} & u(x) = v; \quad x_i \geq 0, i = 1, \ldots, k; \quad \hat{\lambda} \geq 0 \\ & p_i = \hat{\lambda}\frac{\partial u}{\partial x_i} \text{ if } x_i > 0; \quad p_i \geq \hat{\lambda}\frac{\partial u}{\partial x_i} \text{ if } x_i = 0. \end{aligned}$$

It is worth staring at these optimality conditions before moving on to the rest of the problem: We have the utility and feasibility constraints in the first line (and the constraint on the mulitplier; given our assumption on marginal utilities, this isn't much of a constraint given what must be true on the second line). As for the second line, let $\lambda = 1/\hat{\lambda}$, and rewrite the equality and inequality with λ on the left-hand side and $(\partial u/\partial x_i)/p_i$ on the right-hand side. (We know that $\hat{\lambda} > 0$ because some good is consumed at a positive level, and its price and marginal utility are both strictly positive.) This should look very familiar: The bangs-for-the-buck of goods that are strictly positive must all be equal, and they must (all, equally) exceed the bang-for-the-buck of any good that is up against the nonnegativity constraint. Where have we seen that before? In the first-order/complementary-slackness conditions of the CP, of course. This has the same first-order conditions as the CP, which in retrospect shouldn't come as a surprise: We know that solutions to the CP are solutions to the EMP (and vice versa) for the appropriate change of variables. So the marginal conditions on the commodities should be the same.

And, just to do a bit more on this, go back to the equalities and inequalities in line 2, and keep $\hat{\lambda}$ on the right-hand side, putting $p_i/(\partial u/\partial x_i)$ on the left-hand side. To give $p_i/(\partial u/\partial x_i)$ a name, it is the *buck-for-the-bang*, or how much expenditure can be saved per unit of utility lost, on the margin, by decreasing the consumption of good i. The first-order conditions are that, for goods that are at strictly positive levels, their bucks-for-the-bang must all be equal, and they must be no greater than the bucks-for-the-bang of goods that are up against the nonnegativity constraint. Logic similar to the logic of Chapter 3 applies: If the buck-for-the-bang of good i is less than that of good j, the thing to do is to increase the consumption of good i while decreasing that of good j in a ratio that keeps utility constant; this should lead to less expenditure. Which works, unless you can't decrease the consumption of good j, because it is at level 0, from which no decreases are feasible.

Now to finish the assigned problem. We're going to mimic the proof of Proposition 11.10 and apply the Implicit-Function Theorem to the equalities in the first-order, complementary slackness conditions, in a neighborhood in which the EMP has unique solutions and the set of binding constraints does not change. We know that the utility-level constraint will be binding, and some subset of the nonnegativity constraints will bind—for commodity indices i that are strictly positive, we have the equations

$$p_i = \hat{\lambda} \frac{\partial u}{\partial x_i}.$$

Renumber the variables so that these indices are 1 through n: We are looking at the function

$$G(p, v, \hat{\lambda}, x) = \begin{cases} u(x) - v \\ p_1 - \hat{\lambda} u_1(x) \\ \quad \dots \\ p_n - \hat{\lambda} u_n(x) \end{cases}$$

where u_i is short hand for $\partial u / \partial x_i$. Solutions of the first-order/complementary-slackness conditions (locally) are solutions to $G(p, v, \hat{\lambda}, x) = 0$, and we want to implicitly define functions $\hat{\lambda}(p, v)$ and $x(p, v)$ that satisfy

$$G(p, v, \hat{\lambda}(p, v), x(p, v)) = 0.$$

(In the proof of Proposition 11.10, I put hats on the implicitly defined functions, which is unfortunate given my use of a hat here in $\hat{\lambda}$. But I trust that if you are sophisticated enough to be following this discussion, this won't bother you.)

We'll need that u is twice-continuously differentiable to apply the Implicit-Function Theorem, of course, and we need to know that the matrix of partial derivatives in the eliminated variables in nonsingular. The partial derivative in $\hat{\lambda}$ is $(0, u_1, \dots, u_n)$ and in x_i is $u_i, \hat{\lambda} u_{1i}, \dots, \hat{\lambda} u_{ni}$. Multiply all the rows except the first by $1/\hat{\lambda}$ and then multiply the first column by $\hat{\lambda}$, and out pops the same bordered Hessian as in Section 11.4; non-singularity of the bordered Hessian is just what we need here, precisely as in that section.

To sum up, we get differentiability of Hicksian demand in neighborhoods of points in which the binding constraints in the EMP do not change, if u is twice-continuously differentiable, u is strictly increasing (and with strictly positive first derivatives), and the bordered Hessian from Proposition 11.10 is non-singular.

Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Chapter 12: Producer and Consumer Surplus

Summary of the Chapter

To this point, and with the exception of Chapter 8, this volume has concerned the behaviors of single consumers and single firms. The environment in which these consumers and firms act might contain other consumers and/or firms, but the interactions between diverse consumers and firms have not been an issue. That begins to change in this chapter.

A natural first step is to ask, Is the market behavior of a collection of firms or a collection of consumers similar to the behavior of single firms and consumers? That question, in microeconomics, goes by the name *aggregation*, and is the topic of next chapter. Anticipating a bit, we'll discover that aggregation works well for firms, but is problematic for consumers. Nonetheless, to make use of the models we've developed, it is helpful to have ways to "aggregate," and in this chapter we warm up to the harder developments of next chapter by asking, (How) Can we aggregate the impact on firms or consumers of a change in, say, the price of a single good?

Most readers will know the practical answer to this question: Economists measure the aggregate impact on firms of a change in the price of a good by the change in *producer surplus*, while the impact on consumers is measured by the change in *consumer surplus*. In each case, "impact" is measured on a scale of dollars (or whatever price numeraire is in use), which allows us to add up the impacts on individual firms and/or consumers, to get a measure of aggregate impact.

But what is the theoretical rationale for producer and consumer surplus? That is the specific question asked and answered in this chapter.

For firms, the answer is simple and straightforward. If (say) the price of commodity i changes, we can use the result that

$$\frac{\partial \pi}{\partial p_i} = z_i^*$$

(where π is the profit function of the firm and for prices where z_i^* is unique, which we show is "most" prices) to conclude that the change in producer surplus for a single firm is equal to the change in its profit owing to the price change. This aggregates very nicely.

But for consumers, the story is not so simple. Owing to income effects in Marshallian demand, there is no single dollar-denominated measure of "the impact on the consumer of a change in the price of a commodity." Two approximate measures, the so-called *compensating* and *equivalent variations*, are used by economists. These two measures can be expressed as changes in the value of the consumer's expenditure function (from Chapter 10) over the range of the changed prices. And we know from Chapter 10 that

$$\frac{\partial e}{\partial p_i} = h_i,$$

where e is the expenditure function for the consumer and h_i is Hicksian demand for commodity i and for prices where Hicksian demand is single valued (for commodity i), which we show is most of them. Hence the integral "under" the appropriate Hicksian demand function—consumer surplus based on Hicksian demand—can be used to compute the compensating and equivalent variations. And we conclude by giving conditions under which traditionally computed consumer surplus—the integral "under" Marshallian demand—must lie between these two variations.

Solutions to Starred Problems

■ 12.1. This is straightforward. We know that every full solution to the firm's profit-maximization problem at a given price vector p is a subgradient of the convex profit function π. Suppose z and z' were two solutions at p with $z_1 \neq z_1'$. Being a subgradient means that $p \cdot z = \pi(p)$ and $p' \cdot z \leq \pi(p')$ for all $p' \neq p$. Fixing all prices other than the first, this means that $p_1 z_1 + \sum_{i=2}^k p_i z_i = \pi(p)$ and $p_1' z_1 + \sum_{i=2}^k p_i z_i \leq \pi(p_1', p_2, \ldots, p_k)$ for all $p_1' \neq p_1$. That is, z_1 is a subgradient of the function $p_1' \to \pi(p_1', p_2, \ldots, p_k)$. And the same is true of z_1'.

But since π is convex overall, it is a convex function of p_1' alone; if it is differentiable in p_1, it has a unique subgradient at p_1, namely its derivative at p_1. So it is not possible that $z_1 \neq z_1'$; moreover, the sole element of $Z_1^{f*}(p_1^0)$ is indeed the derivative of π in its first argument at p_1.

■ 12.4. A picture (with its legend) suffices: See Figure G12.1.

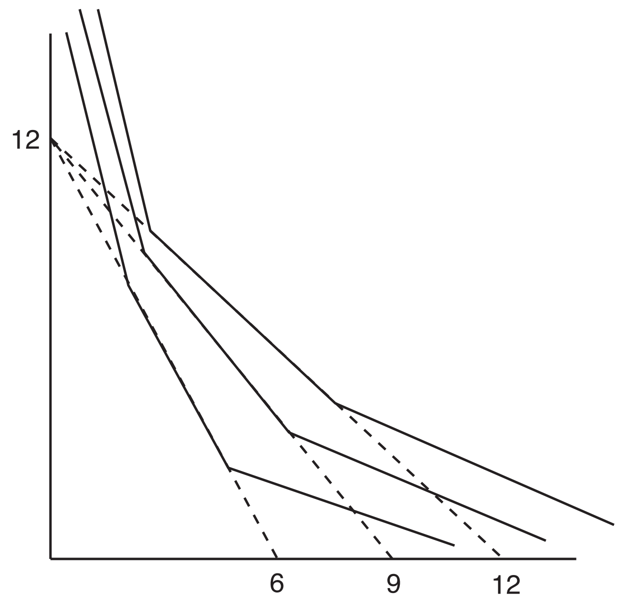


Figure G12.1. Solution to Problem 12.4. The figure shows indifference curves for a consumer in the case of two goods. To construct the figure, we first construct dashed lines that hit the x-axis at different values and the y-axis at $(0, 12)$. Then we construct indifference curves (the solid lines): They are parallel to one-another "above" the dashed lines, but run along the dashed lines (so are not parallel) each for an interval of values. Now imagine a consumer with these indifferent curves, with $y = 12$, with the price of the y good equal to 1, and for various prices of the x good. If the price of the x good is 1, the consumer chooses any of the points that lie along the highest indifference curve shown, where that curve lies along the dashed line; if the prices of the x good is 4/3, she chooses any of the points along the second indifference curve shown, where that curve hits "its" dashed line; if the price of the x good is 2, she chooses any of the points along the third indifference curve, where that curve hits "its" dashed line. The point is that for each price of the x good, the optimal level of Marshallian demand for that good consists of an interval of points. Of course, this is "exceptional": If the ratio of y to the price of the y good is anything other than 12 to 1, this behavior is not observed. But the point is that this sort of exceptional case is, in theory, possible.

Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Chapter 13: Aggregating Firms and Consumers

Summary of the Chapter

This chapter is primarily concerned with the following two questions:

- Suppose we have a finite number F of profit-maximizing firms and, more specifically, their optimal netput correspondences $p \Rightarrow Z^{f*}(p)$ and profit functions $p \rightarrow \pi^f(p)$, telling us what are the (possibly multiple) optimal production plans for the firms and their levels of profit as a function of the price vector p. We form the *aggregate optimal netput correspondence* $p \Rightarrow Z^*(p)$, defined by

$$Z^*(p) := \{z \in R^k : z = z^1 + \ldots + z^F \text{ for some selection of } z^f \in Z^{f*}(p), f = 1, \ldots, F\},$$

  and the aggregate profit function

$$\pi(p) := \sum_{f=1}^{F} \pi^f(p).$$

  Chapter 9 tells us a lot about $p \Rightarrow Z^{f*}(p)$ and $p \rightarrow \pi^f(p)$, if these come from a profit-maximizing firm. What can we say in this vein about the properties of the aggregate optimal netput correspondence and aggregate profit function?

- And suppose we have a finite number H of utility-maximizing consumers. Let $\mathbf{D}^h(p, y^h)$ be the Marshallian demand of consumer h, as a function of prices p and income y^h. Form the aggregate demand for these H consumers by summing up across consumers their Marshallian demands (sum a selection, one from each $\mathbf{D}^h(p, y^h)$, if $\mathbf{D}^h(p, y^h)$ is not singleton). Chapter 11 tells us a lot about the structure of individual consumer demand. What can we say in this vein about aggregate demand?

The answers obtained are

1. For firms, aggregation works like a charm. Aggregate optimal-netput from a (finite) number of competitive firms has precisely the properties of optimal-netput from a single competitive firm, where the "aggregate firm" is constructed as a simple merger of the production possibilities of the firms being aggregated.
2. But for consumers, things do not work very well. In some extraordinarily special cases, one can aggregate. But in general, the *Sonnenschein–Mantel–Debreu Theorem* tells us (more or less) that any continuous function from prices to excess demands that is continuous, homogeneous of degree zero, and that satisfies Walras' Law is the aggregate excess demand for a collection of consumers with continuous, nondecreasing, and convex preferences.
3. Finally, and in a different vein, aggregation can "smooth" out what is being aggregated. We illustrate with a result due to Shapley, Folkman, and Starr, which shows (for instance) that per capita aggregate demand correspondences become "asymptotically" convex-valued as the number of consumers increases, if the number of commodities stays fixed. (This result requires that some assumptions hold, of course.)

Solutions to Starred Problems

■ 13.1. If preferences are only identical and homothetic, the proposition as stated won't work. Suppose, for instance, $k = 2$, there are two consumers, and their preferences are given by $u((x_1, x_2)) = \max\{x_1, x_2\}$. Then for the price vector $p = (1, 1)$, $(p, y) = \{(y, 0), (0, y)\}$. Suppose1 we split y into y^1 and y^2 such that $y^1 + y^2 = y$; we get

$$(p, y^1) + (p, y^2) = \{(y, 0), (y^1, y^2), (y^2, y^1), (0, y)\}.$$

The Minkowski sum of the (two) individual demands will depend on how income is split up. One can show that the Minkowski sum of the individual demands is always a subset of Marshallian demand for the (homothetic) preferences that are given by the convexified preferences derived from the original preferences (see Proposition 10.13, although to plug directly into this, we should assume that the original preferences are nondecreasing, as well). But with identical and homothetic but nonconvex preferences, this is about the best you can do.

■ 13.4. (a) The scalars β^h just scale the objective function and can be ignored. If $y = 0$, then the only feasible point is 0 (the origin in R^H), so the result is true. Now suppose that $y > 0$. We know that, at any solution, $y^h > 0$ for all h, since otherwise you get 0 for the value of the objective function, and strictly positive values of the objective function are clearly feasible. And, having recognized this, we know that any solution will be interior. Now you can proceed to look at the first-order conditions for a maximum, although to make life even easier, subject the objective function to the monotonic transformation of taking its natural log. The problem (ignoring the β^h, remember) becomes

$$\max \sum_h \alpha^h \ln(y^h), \quad \text{subject to } \sum_h y^h \leq y.$$

This is a strictly concave objective function (so was the other, but it is easier to see it in this form), so there is a unique solution, and first-order, complementary-slackness conditions are $\alpha^h/y^h = \lambda$, where λ is the mulitplier on the constraint. This is $y^h = \alpha^h/\lambda$, and to satisfy the constraint, this is $\lambda = 1/y$ so that $y^h = \alpha^h y$.

(b) Since u^h is, by assumption, homogeneous of degree 1, we know that the solution for y is just the solution for $y = 1$ scaled up by y; that is, the solution is $y\hat{x}^h$ and the value of the objective function at the solution is $\beta^h y$. (If there are multiple solutions, they all scale, but we'll fix on one, namely $\hat{x}^h$.) It is worthwhile observing at this point that $\beta^h > 0$ by virtue of the assumption that there is some x such that $u^h(x) > u^h(0) = 0$, which also implies that the consumer spends all of her income at the solution.

(c) For each h, we know that $u^h(y\hat{x}^h) \geq u^h(x)$ for any x that satisfies $p \cdot x \leq y$. So if we set $\check{y}^h = p \cdot \check{x}^h$, we know that $u^h(\check{y}^h\hat{x}^h) \geq u^h(\check{x}^h)$ for each h. This implies that

$$U(x^0) = \prod_h (u^h(\check{x}^h))^{\alpha^h} \leq \prod_h (u^h(\check{y}^h\hat{x}^h))^{\alpha^h} = \prod_h (\check{y}^h\beta^h)^{\alpha^h}.$$

(d) But since $p \cdot \hat{x}^h = 1$, $p \cdot \left[\sum_h \alpha^h y\hat{x}^h\right] = \sum_h \alpha^h y = y$. So if we let $\hat{x} = \sum_h \alpha^h y\hat{x}^h$, we know that $p \cdot \hat{x} \leq y$, so $U(x^0) \geq U(\hat{x})$. Moreover,

$$\begin{aligned} U(x^0) \geq U(\hat{x}) = \max\left\{ \prod_h (u^h(\tilde{x}^h))^{\alpha^h} : \sum_h \tilde{x}^h \leq \hat{x} \right\} \\ \geq \prod_h (u^h(\alpha^h y\hat{x}^h))^{\alpha^h} = \prod_h (\alpha^h y\beta^h)^{\alpha^h}. \end{aligned}$$

(e) But by part a, since both $\sum_h \alpha^h y = y$ and $\sum_h \check{y}^h \leq y$, we know that

$$\prod_h (\alpha^h y\beta^h)^{\alpha^h} \geq \prod_h (\check{y}^h\beta^h)^{\alpha^h},$$

and moreover that the two will be equal only if $\check{y}^h = \alpha^h y$ for each h. But the string of inequalities we've produced say they must be equal. Therefore, $\check{y}^h = \alpha^h y$ for each y,

$\hat{x}$, which is the sum of individual demands, must be a solution the consumer's problem for the aggregate demand function, and each individual $\check{x}^h$ must solve each individual consumer's demand problem at prices p and income $\check{y}^h = \alpha^h y$, for if any fell short of the utility generated by $\alpha^h y \hat{x}^h$, then the string of equalities would fail. ■

■ 13.5. Here's an analogous result to Proposition 13.4, but stated in terms of excess demand:

Consider the problem

$$\text{Maximize } u^h(\zeta + e^h), \text{ subject to } p \cdot \zeta \leq 0,\ \zeta \in R^k,\ \zeta \geq -e^h$$

where u^h *is a continuous utility function defined on* R^k_+, $p \in R^k_{++}$, $e^h \in R^k_+$. *Let* $^h(p, e^h)$ *denote the set of solutions for this problem, and let* $\eta^h(p, e^h)$ *denote sup* $u^h(\zeta + e^h)$ *subject to the same constraints. Then:*

a. *A solution exists for each* p *and* e^h*; that is,* $^h(p, e^h)$ *is nonempty. Therefore the supremum that defines* $\eta^h(p, e^h)$ *is a maximum and* $\eta^h(p, e^h)$ *is finite for each* p *and* e^h.

b. *If* u^h *is quasi-concave, the* $^h(p, e^h)$ *is convex. If* u^h *is strictly quasi-concave, then* $^h(p, e^h)$ *is singleton.*

c. *The correspondence* $(p, e^h) \Rightarrow^h (p, e^h)$ *is upper semi-continuous and locally bounded, and the function* $(p, e^h) \to \eta^h(p, e^h)$ *is continuous.*

d. $^h(p, e^h) =^h (\lambda p, e^h)$ *for all* $\lambda > 0$*, and* $p \to \eta(p, e^h)$ *is homogeneous of degree 0.*

e. *If* u^h *is locally insatiable, then* $p \cdot \zeta = 0$ *for all* $\zeta \in^h (p, e^h)$.

It remains in this problem to prove Proposition 13.6:

Suppose $\zeta \in (p)$. Then $\zeta = \zeta^1 + \ldots + \zeta^H$ where each $\zeta^h \in^h (p)$. (The endowments are fixed throughout at e^h for consumer h.) For any $\lambda > 0$, $\zeta^h \in^h (\lambda p)$ (see d immediately above), so $\lambda\zeta = \lambda\zeta^1 + \ldots + \lambda\zeta^H \in (\lambda p)$. Reversing the argument for $\zeta \in (\lambda p)$ where we scale everything with $1/\lambda$ shows that $\zeta \in (p)$, and so $(p) = (\lambda p)$ for all $\lambda > 0$.

Suppose $\zeta \in (p)$. Then $\zeta = \zeta^1 + \ldots + \zeta^H$, where each $\zeta^h \in^h (p)$. But then $p \cdot \zeta^h = 0$ for each h (see e above, and recall that the proposition assumes that each consumer is locally insatiable), hence $0 = p \cdot \zeta^1 + \ldots p \cdot \zeta^H = p \cdot (\zeta^1 + \ldots + \zeta^H) = p \cdot \zeta$. Walras' Law holds.

Suppose each consumer has convex preferences. Fix $\zeta, \zeta' \in (p)$. Then $\zeta = \zeta^1 + \ldots + \zeta^H$ and $\zeta' = \zeta'^1 + \ldots + \zeta'^H$ where each ζ^h and $\zeta'^h \in^h (p)$. But since each $^h(p)$ is convex (see b above), for any $\lambda \in [0, 1]$, $\lambda\zeta^h + (1-\lambda)\zeta'^h \in^h (p)$, hence $\lambda\zeta + (1-\lambda)\zeta' = \sum_h \left(\lambda\zeta^h + (1-\lambda)\zeta'^h\right) \in (p)$. That is, (p) is convex.

Finally, to show that $p \Rightarrow (p)$ is upper semi-continuous: Suppose that $p^n \to p^0$ and $\zeta^n \to \zeta^0$ where each $\zeta^n \in (p^n)$. Write $\zeta^n = \zeta^{n1} + \ldots + \zeta^{nH}$, where $\zeta^{nh} \in^h (p^n)$. I assert that $\{\zeta^{nh}\}_{n=1,\ldots}$ lies inside a bounded set for each h. This follows from the fact that each $p \Rightarrow^h (p)$ is locally bounded, but let me give details, since they are simple: Since $p^n \to p$ and p is strictly positive, if we let $\epsilon = \min_i \left(p_i / \sum_{i'} p_{i'}\right)$, we know that for all sufficiently large n, $p_i^n / \left(\sum_{i'} p_{i'}^n\right) \geq \epsilon/2$. We know that $\zeta^{nh} \geq -e^h$, so if we let $M = \max_{i=1,\ldots,k; h=1,\ldots,H} e_i^h$, we know that $\zeta_i^{nh} \geq -M$ for all n, h, and i. But since $p^n \cdot \zeta^{nh} = 0$, this implies that for all n large enough so that $p_i^n / \left(\sum_{i'} p_{i'}^n\right) \geq \epsilon/2$, $\zeta_i^{nh} \leq M/(\epsilon/2) = 2M/\epsilon$. (If this were not so for some n, h, and i, then the contribution $p_i^n \zeta_i^{nh}$ to the dot product would be too large (and positive) to be counteracted by the contribution of all the other terms; the dot product would have to be strictly positive.) We already know that $-M$ is a (uniform) lower bound on the components of any $\zeta^h(p)$; this gives us a local upper bound.

Of course, once we know that each individual excess demand correspondence is locally bounded, we know that aggregate excess demand is also locally bounded.

And once we know that each $\{\zeta_n^h\}_{n=1,\ldots}$ lies within a bounded set, we can extract a subsequence along which $\{(\zeta_n^1, \ldots, \zeta_n^H)\}$ converges to some $(\zeta_0^1, \ldots, \zeta_0^H)$, and by the upper semi-continuity of $p \Rightarrow^h (p)$ for each h, we know that each ζ_0^h lies in ${}^h(p_0)$. But $\zeta_0 = \lim_n(\zeta_n^1 + \ldots + \zeta_n^H)$ and, along the subsequence, this is equals $\lim_n(\zeta_n^1 + \ldots + \zeta_n^H) = \zeta_0^1 + \ldots + \zeta_0^H$. Hence $\zeta_0 \in (p_0)$.

■ 13.7. For the set $X = \{1, 2, 3, 4\}$ in R^1, a point $x \in (X)$ is of greatest distance from points in X that (minimally) have x as a convex combination are points that are very close one of the points in X and so are very far from the "neighboring" point in X. That is, the worst case are points as $2 + \epsilon$ for very small $\epsilon > 0$. If you write this as a convex combination of points in the set, the most economical (in terms of distance from the further point) is to write it as a convex combination of 2 and 3, and the distance from 3 is $1 - \epsilon$. Hence the inner radius of this set is 1.

For the set $X = \{(z_1.z_2) \in R^2 : z_1 = 1, 2, 3, \text{ or } 4 \text{ and } z_2 = 1, 2, 3, \text{ or } 4\}$, the "worst case" $x \in (X)$ is where you are very close to but not on a lattice point in either direction. For instance, consider the point $(2 + \epsilon, 2 + \delta)$, $\delta \neq \epsilon$, $\delta, \epsilon > 0$. To make that a convex combination, we'll require $(2, 2)$ and $(3, 3)$ and either $(2, 3)$ or $(3, 2)$, depending on whether δ or ϵ is larger. (If $\delta = \epsilon$, we'd only need $(2, 2)$ and $(3, 3)$, although this won't reduce the calculation of "furthest needed point.") The furthest point is $(3, 3)$ (if both δ and ϵ are small), with a distance that approaches $\sqrt{2}$ as δ and ϵ go to zero. So $\sqrt{2}$ is the inner radius of the set.

Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Chapter 14: General Equilibrium

Summary of the Chapter

This chapter and the two to follow conclude this volume by looking at General Equilibrium Theory. We formulate economies populated by (finitely) many consumers and, perhaps, firms, and look at how the various economic actors interact via prices and markets, under the assumption that the consumers and firms are all price takers. We look for and at *Walrasian equilibria*, consisting of a price vector, consumption allocations for all the consumers, and production plans for all the firms, where each consumer's consumption allocation solves her utility-maximization problem at the given prices, each firm's production plan solves its profit-maximization problem at those prices, and markets clear, meaning that the demand for each and every good is no greater than its supply.

In this chapter, we are concerned with formulation, basic properties of Walrasian equilibria, the existence of at least one equilibrium, and (somewhat briefly) the mathematical nature of the set of equilibria for a given economy. Next chapter takes up issues of the efficiency of Walrasian equilibria, and Chapter 16 concludes by adapting the formalisms of this chapter to situations involving multiple periods and uncertainty.

More specifically for this chapter:

1. Section 14.1 provides basic definitions: the formal structure of an *economy*, both with firms and without (so-called *pure-trade* economies); and a precise definition of *Walrasian equilibrium*, the object of our attention for most of the remainder of the book.

2. In Section 14.2, basic properties of Walrasian equilibria are proved. In particular, we give conditions under which *Walras' Law* holds (the market value of what consumers consume equals the market value of all their resources) and under which equilibrium prices must be nonnegative and, then, strictly positive.

3. The *Edgeworth Box* is the topic of Section 14.3. This is a graphical depiction of a two-consumer, two-commodity pure-exchange economy, very useful for developing both intuition and illustrative examples.

4. Section 14.4 is the heart of Chapter 14, concerned with the question, Under what conditions on a given economy can we be sure that at least one Walrasian equilibrium exists. The approaches to existence taken in this chapter employ fixed-point theorems and, in particular, Kakutani's Fixed-Point Theorem, concerning which you should consult Appendix 8. Two general approaches are taken:

   a. First we take the approach of one of the two seminal papers in the subject, Arrow and Debreu (1954). In this approach, we define a so-called generalized game and prove an existence result for Nash equilibria of these games. This is then applied to the question of existence, where the economy (consisting of consumers and firms) is explicitly and formally specified.

   b. And then we give two existence results more in the spirit of the second seminal paper on the subject, McKenzie (1954). In this approach, consumers and firms are implicit; we begin formally with an aggregate excess-demand correspondence. We state and prove what is known as the Debreu-Gale-Kuhn-Nikaido Lemma, for aggregate excess-demand correspondences that are defined for all nonnegative price vectors. And we state and prove an existence result for aggregate excess-demand defined (only) for strictly positive prices due to Hildenbrand (1974).

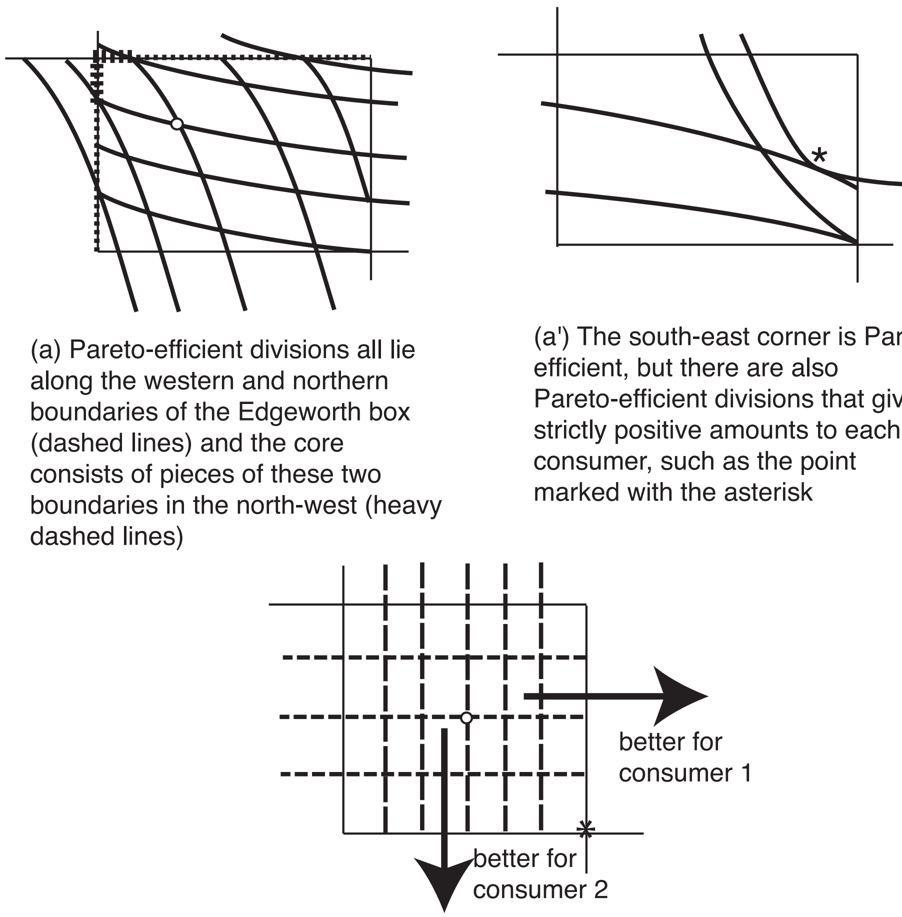
5. Having provided results that guarantee the existence of at least one equilibrium, we take the opposite tack in Section 14.5: What can be said to ensure that the number of equilibria isn't "too large?" Very little is proved in this section; this is more of a discussion than an explication of results. We argue that the set of equilibrium prices (normalized to lie in the unit simplex) for a given economy must be closed, but we then cite Mas-Collel's (1977) Theorem: For *any* closed set of prices in the unit simplex that does not intersect the boundary, there is a pure-exchange economy whose set of Walrasian equilibrium prices is that set.

6. Finally, we state (and leave for you to prove) a result along the lines of: the Walrasian equilibrium correspondence is upper semi-continuous in the parameters of the economy. *But this proposition (Proposition 14.14 on page 354) is not true as stated, on two grounds, one easily fixed, the other not. Immediately following this introductory discussion, I address this error in the text.*

The level of abstraction in this chapter is very high, making it somewhat easy for students to lose sight of what all the symbols mean. For this reason, I strongly recommend that, after you absorb sections 14.1 and 14.2, you take the time to compute the Walrasian equilibria of a few parametric examples of economies; solve Problems 14.2 through 14.5.

Solutions to Starred Problems

■ 14.1. Figure G14.1, panel a, shows indifference curves that give a Pareto-efficient set lying entirely along the west and north boundaries of the Edgeworth box. Both parties value both goods, and preferences are strictly convex, but consumer 1 is willing to give up a lot of good 1 for a little good 2, while consumer 2 has the opposite tradeoffs.



(a) Pareto-efficient divisions all lie along the western and northern boundaries of the Edgeworth box (dashed lines) and the core consists of pieces of these two boundaries in the north-west (heavy dashed lines)

(a') The south-east corner is Pareto efficient, but there are also Pareto-efficient divisions that give strictly positive amounts to each consumer, such as the point marked with the asterisk

(b) If consumer 1 only values good 1, and consumer 2 only values good 2, then it is Pareto efficient to give all of good 1 to consumer 1 and all of good 2 to consumer 2. The point marked with the asterisk is the only Pareto-efficient allocation, so it is the only core allocation

Figure G14.1. Solutions to Problem 14.1

Panel a′ provides the other example of part (a) of the problem: The south-east corner is Pareto efficient, because at that point, consumer 1's rate of trading off good 1 for good 2 (along the indifference curve) is much smaller than consumer 2's. But, at least as drawn, consumer 1's indifference curve at a higher level of utility changes shape radically, enough so that now an efficient point gives strictly positive amounts of each good to each consumer.

And panel b shows what things look like if consumer 1 only values good 1 and consumer 2 only values good 2. The only Pareo-efficient point is the south-east corner, and since any point in the core must be efficient, this point, marked with an asterisk in the picture, is the only core allocation. And, of course, the unique Walrasian equilibrium has $p_1 = p_2$, so that each consumer has wealth $20p_1 = 20p_2$, with consumer 1 demanding (and getting) $(20, 0)$ and consumer 2, $(0, 20)$.

■ 14.2. (a) Normalize prices to sum to 1, and let p be the price of the first good. Note that this means that Alice has initial wealth 10, while Bob's wealth is $5p + 10(1 - p) = 10 - 5p$. We solve the consumers' problems in the usual fashion, leading to the following demand functions:

$$x_1^A = \frac{4}{p}, \quad x_2^A = \frac{6}{1-p}, \quad x_1^B = \frac{10-5p}{2p}, \quad \text{and } x_2^B = \frac{10-5p}{2(1-p)}.$$

If you equate supply to total demand and solve for the equilibrium prices, you will find

$$p = \frac{18}{35}, \quad x_1^A = \frac{70}{9}, \quad x_2^A = \frac{210}{17}, \quad x_1^B = \frac{65}{9}, \quad \text{and } x_2^B = \frac{130}{17}.$$

These are the only solutions to supply equals demand: This economy has a single Walrasian equilibrium.

■ 14.3. I assert that at any Walrasian equilibrium of this economy, the price of all three goods must be strictly positive. First, Alice and Bob have strictly increasing utility functions in both consumption goods, so the first part of Proposition 14.5 applies, garanteeing that they have strictly positive prices. And then the second part of the proposition applies to the input good.

That being so, normalize prices so the price of good 3 is 1.

Alice and Bob will have strictly positive wealth. The value of their endowments will be 5, and firms can avoid taking losses (since $(0, 0, 0)$ is feasible for them), so each consumer's wealth will be at least 5. In fact, their wealth levels will be 5, because both firms have constant-returns-to-scale technologies: If prices allowed either firm to make positive profit, it would seek to make infinite profit, which is not possible in a Walrasian equilibrium.

Indeed, with p_3 normalized to be 1, we know that $p_1 \leq 1/3$ and $p_2 \leq 1/4$ are required (to avoid infinite profit opportunities). More than that, if $p_1 < 1/3$, then firm 1 will definitely choose not to produce good 1. That can't be part of an equilibrium: At any price for good 1, Alice and Bob (with wealth 5 each) both demand some good 1, and the only source for good 1 in this economy is production by firm 1. So we know that in any Walrasian equilibrium (with p_3 normalized to be 1), p_1 must be $1/3$. Similarly, p_2 must be $1/4$, to induce firm 2 to produce some of good 2, so that markets clear.

And now that we know the equilibrium price vector (namely, $(1/3, 1/4, 1)$), it is a simple matter to work out demands by Alice and Bob, therefore the amounts that the firms must produce of their respective goods, therefore their demands for good 3 input. The solution on the consumption side is $x_1^A = 6, x_2^A = 12, x_1^B = 7.5$, and $x_2^B = 10$. Therefore, the production plans must be $(13.5, 0, -4.5)$ for firm 1 and $(0, 22, -5.5)$ for firm 2.

It doesn't matter what are the shareholdings in this economy. The firms have constant-returns-to-scale technologies, so in any equilibrium, they must make zero profits, and the "dividends" paid to shareholders are zero, no matter how many or how few shares are held.

■ 14.6. Fix an argument $a = (a_1, \ldots, a_n)$. Recall that $a_{\neg\ell}$ is short-hand for $(a_m)_{m \neq \ell}$. For $\ell = 1, \ldots, n$, we know that $A_\ell^*(a_{\neg\ell})$ is nonempty, so some a_ℓ^* lies within $A_\ell^*(a_{\neg\ell})$. But then $(a_1^*, \ldots, a_n^*) \in A^*(a)$; this shows that the A^* correspondence is nonempty valued.

Continue to fix $a = (a_1, \ldots, a_n)$, and suppose $\hat{a}^* = (\hat{a}_1^*, \ldots, \hat{a}_n^*)$ and $\check{a}^* = (\check{a}_1^*, \ldots, \check{a}_n^*)$ are both in $A^*(a)$. This means that, for each ℓ, $\hat{a}_\ell^*$ and $\check{a}_\ell^*$ are in $A_\ell^*(a_{\neg\ell})$. But then for any $\beta \in [0, 1]$, since each A_ℓ^* is convex valued, $\beta\hat{a}_\ell^* + (1-\beta)\check{a}_\ell^* \in A_\ell^*(a_{\neg\ell})$, and hence $\left(\beta\hat{a}_1^* + (1-\beta)\check{a}_1^*, \ldots, \beta\hat{a}_n^* + (1-\beta)\check{a}_n^*\right) \in A^*(a)$. That is, A^* is convex valued.

Suppose $\{a^j\}$ is a sequence (where each a^j has the form $(a_1^j, \ldots, a_n^j)$) with limit $a = (a_1, \ldots, a_n)$, and for each j, $a^{*j} = (a_1^{*j}, \ldots, a_n^{*j}) \in A^*(a^j)$. And suppose $\lim_j a^{*j} = a^*$. For each ℓ from 1 to n, since $a^{*j} \in A^*(a^j)$, we know that $a_\ell^{*j} \in A_\ell^*(a_{\neg\ell}^j)$. And, of course, $\lim_j a_\ell^{*j} = a_\ell^*$. So since each A_ℓ^* is upper semi-continuous, $a_\ell^* \in A_\ell^*(a_{\neg\ell})$. But this implies that $a^* \in A^*(a)$, which demonstrates that A^* is upper semi-continuous.

■ 14.8. Since ζ is a continuous function, it is clear that $\xi : P \to R_+^k$ is continuous. Since $\xi(p) \geq p$, $\sum_{i=1}^k \xi_i(p) \geq 1$, and hence ϕ is continuous. Moreover, $\phi : P \to P$. So by Brouwer's Fixed-Point Theorem (Proposition A8.2), there exists $p \in P$ such that $\phi(p) = p$. I assert that $\phi(p) = p$ implies $\zeta(p) \leq 0$:

Suppose by way of contradiction that $\zeta_i(p) > 0$ for some i. Then $\xi_i(p) > p_i$. Note that this immediately implies that $\sum_\ell \xi_\ell(p) > 1$. If, for this i, $p_i = 0$, then $\phi_i(p) = \xi_i(p)/\sum_\ell \xi_\ell(p) > 0$, contradicting $\phi(p) = p$. So we can assume $p_i > 0$ and hence $p_i\zeta_i(p) > 0$. Since $p \cdot \zeta(p) \leq 0$, there must then be some j such that $p_j\zeta_j(p) < 0$ (to counteract the strictly positive $p_i\zeta_i(p)$ in the dot product), which implies $p_j > 0$ and $\zeta_j(p) < 0$. But then $\xi_j(p) = p_j$, and therefore $\phi_j(p) = \xi_j(p)/\sum_\ell \xi_\ell(p) = p_j/\sum_\ell \xi_\ell(p) < p_j$, which is the final contradiction.

At the risk of stating the obvious: The way this works is to increase the relative prices of goods for which there is positive excess demand. If there are any goods in positive excess demand, this lowers the relative price of goods for which excess demand is negative (as long as the price of the latter good is strictly positive). So at a fixed point of the function, no good can be in positive excess demand.

■ 14.9. The unit simplex of prices P is obviously nonempty, convex, and compact. So that part of the assumptions of Kakutani is clearly satisfied.

We need to show that the correspondence ϕ described in the problem is upper semi-continuous, nonempty-valued, and convex-valued. The construction of ϕ makes it obvious that it both nonempty- and convex-valued; we focus on upper semi-continuity.

Suppose $p^n \to p$ (in P) and $q^n \in \phi(p^n)$ for each n is such that $\lim_n q^n$ exists and equals q. We must show that $q \in \phi(p)$.

There are two cases to consider. The first case is if p is in the interior of P. Then if $q_i > 0$, $q^n \to q$ implies that for all sufficiently large n, $q_i^n > 0$. And because $p^n \to p$ and p is interior to p, it must be that for all sufficiently large n, p^n is interior to P. But if p^n is interior to P, $q^n \in \phi(p^n)$, and $q_i^n > 0$, it must be that $\zeta_i(p^n) \geq \zeta_j(p^n)$ for all j. And since ζ is continuous and $p^n \to p$, $\zeta_i(p^n) \geq \zeta_j(p^n)$ for all j and all sufficiently large n implies that $\zeta_i(p) \geq \zeta_j(p)$ for all j, which implies that $q \in \phi(p)$.

The second case is where p is on the boundary of P. Let $I \subset \{1, \ldots, k\}$ be the set of indices of p such that $p_i = 0$. We know, of course, that $I \neq \{1, \ldots, k\}$, since for $p \in P$, the sum of its components must be 1. So let $J = \{1, \ldots, k\} \setminus I$; then both I and J are nonempty.

If $q_i > 0$ implies $i \in I$, then $q \in \phi(p)$. So suppose by way of contradiction that $q_i > 0$ but $i \in J$, meaning $p_i > 0$. Because $q^n \to q$ and $p^n \to p$, we know that for all sufficiently large n, $q_i^n > 0$ and $p_i^n > 0$. More than that, we know that for large n, p_i^n is strictly bounded away from zero; that is, there exists some $\epsilon > 0$ such that $p_i^n \geq \epsilon$ for all large n. Now it cannot be true that (for large n) p^n is on the boundary of P, for if it were, $p_i^n > 0$ and $q^n \in \phi(p^n)$ would imply that $q_i^n = 0$. So we know that for all sufficiently large n, $q_i^n > 0$, $p_i^n \geq \epsilon$, and p^n lies in the interior of P.

But $p^n \cdot \zeta(p^n) = 0$, and there is a uniform lower bound on all components of $\zeta(p)$ for all p. If $-B$ is that lower bound (and wlog, we can assume $B > 0$), then $p_i^n \zeta_i(p^n) \leq B(k-1)$, since the other $k-1$ terms in the dot product $p^n \cdot \zeta(p^n)$ are bounded below by $1 \times (-B)$. Hence $\zeta_i(p^n) \leq B(k-1)/\epsilon$ for all sufficiently large n.

However, since $p^n \to p$ and some components of p are zero, we know that $\lim_n \sum_{j \in I} \zeta_j(p^n) = \infty$. This means that for all sufficiently large n, one of the components of $\zeta_j(p^n)$ for $j \in I$ must exceed $B(k-1)/\epsilon \geq \zeta_i(p^n)$, hence q_i^n must equal zero, a contradiction. Therefore, ϕ is indeed upper semi-continuous.

All the conditions of Kakutani's Fixed-Point Theorem hold, and we know there is a fixed point: For some $p^* \in P$, $p^* \in \phi(p^*)$. Obviously, this p^* cannot be on the boundary of P. For if $p_i^* > 0$ while $p_j^* = 0$ for some $j \neq i$, then the rules by which ϕ has been constructed tell us that $q_i = 0$ for all $q \in \phi(p^*)$. And if p^* is interior to P, then $p_i^* > 0$ for all i. But then by the rule by which ϕ has been constructed, $\zeta_i(p^*) \geq \zeta_j(p^*)$ for all i and j. That is, $\zeta_i(p^*) = \zeta_j(p^*)$ for all i and j. And since $p^* \cdot \zeta(p^*) = 0$ and all the components of p^* are strictly positive, this entails $\zeta_i(p^*) = 0$ for all components, or $\zeta(p^*) = 0$; p^* is an equilibrium price vector.

■ 14.11. If you possess an early printing of the book, the version of Proposition 14.14 in your book asserts that $e \Rightarrow \mathbf{W}(\mathcal{E}(e))$ is upper semi-continuous, as long as eqquilibrium prices are known to be nonnegative and nonzero. Later printings of the book have been corrected: This is not true, and we only (necessarily) get upper semi-continuity at limit points where prices are strictly positive. (Also, if you have an earlier printing, the statement of Problem 14.11 lacks an asterisk, so you might be surprised to find its solution here. Since I screwed up on the statement of the proposition in earlier printings, I decided it would be a good idea to move the discussion to the *Student's Guide*.)

Let me first prove the version of the proposition that is given in later printings. For those with an earlier printing, this is the assertion that, if $\{e_n\}$ is a sequence of endowments with limit e, if $(p_n, \boldsymbol{x}_n, \boldsymbol{z}_n)$ is a Walrasian equilibrium for the economy $\mathcal{E}(e_n)$, and if $\lim_n(p_n, \boldsymbol{x}_n, \boldsymbol{z}_n) = (p, \boldsymbol{x}, \boldsymbol{z})$ where p is strictly positive, then $(p, \boldsymbol{x}, \boldsymbol{z})$ is a Walrasian equilibrium for the economiy $\mathcal{E}(e)$.

There are four things to show. First is that markets clear in the limit economy. But market clearing in the nth economy is

$$\sum_h \boldsymbol{x}_n^h \leq \sum_h e_n^h + \sum_f \boldsymbol{z}_n^h,$$

and passing to the limit in n gives us market clearing for the limit economy. Second, for each firm f, $\boldsymbol{z}^f$ must maximize profit for f at prices p. Let $\hat{z}$ be any (other) production plan from Z^f. The profit-maximization condition for firm f in the nth economy implies that $p_n \cdot \boldsymbol{z}_n^f \geq p_n \cdot \hat{z}$, and passing to the limit in n gives $p \cdot \boldsymbol{z}^f \geq p \cdot \hat{z}$.

The third condition is that consumer h can afford $\boldsymbol{x}^h$ at prices p and given the production plans specified by $\boldsymbol{z}^f$. But this condition for the nth economy is that

$$p_n \cdot \boldsymbol{x}_n^h \leq p_n \cdot e_n^h + \sum_f s^{fh} p_n \cdot \boldsymbol{z}_n^f,$$

and passing to the limit in n once more gives us what we want.

And, finally, we must show that for each h, $\boldsymbol{x}^h$ is preference/utility maximizing for h subject to h's budget constraint in the limit economy. I'll do this in two steps.

Step 1. *If* $\hat{x} \succ^h \boldsymbol{x}^h$, *then* $p \cdot \hat{x} \geq p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f$. Or, in words, any bundle that is strictly preferred to $\boldsymbol{x}^h$ costs at least as much at the limit prices p as h has to spend. To see this, note that if $\hat{x} \succ^h \boldsymbol{x}^h$, then (since $\boldsymbol{x}^h = \lim_n \boldsymbol{x}_n^h$) there exists N such that, for all $n \geq N$, $\hat{x}^h \succ^h \boldsymbol{x}_n^h$. Hence for all large n,

$$p_n \cdot \hat{x} > p_n \cdot e_n^h + \sum_f s^{fh} p_n \cdot \boldsymbol{z}_n^f. \tag{SG14.1}$$

And (one more time!) passing to the limit in n gives the desired result. (For future reference, note that while (SG14.1) has a strict inequality, passing to the limit makes the limit inequality weak. However, if instead of the strict inequality in (SG14.1) we had only weak inequalities, we would still get a weak inequality in the limit.)

Step 2. *If $\hat{x} \succ^h \boldsymbol{x}^h$, then $p \cdot \hat{x} > p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f$.* That is, in the first step we showed that any bundle better than $\boldsymbol{x}^h$ would cost h at least as much as her full resources; now we will show that it costs strictly more. Suppose it does not cost more; then in view of the first step, we must be dealing with a case where $p \cdot \hat{x} = p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f$.

Since $\boldsymbol{x}^h \in R^k_+$ and p is nonnegative, $p \cdot \boldsymbol{x}^h \geq 0$. This implies that $p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f \geq 0$; that is, h has (in the limit) nonnegative wealth. Suppose that $p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f = 0$. Then in the limit economy, h has zero wealth, and the only feasible (nonnegative) bundle she can afford is the zero bundle. If $\hat{x} \succ^h \boldsymbol{x}^h = 0$, then $\hat{x}$ cannot be the zero bundle. But it must be nonnegative, so since prices p are strictly positive, $p \cdot \hat{x} > 0$, contradicting the hypothesis that $\hat{x}$ costs precisely h's financial resources.

Or it may be that $0 < p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f$ (which, $= p \cdot \hat{x}$). Consider bundles $\alpha \hat{x}^f$ for α less than but close to 1. By continuity of h's preferences, for some such α, $\alpha \hat{x} \succ^h \boldsymbol{x}^h$. Since $p \cdot \hat{x} = p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f > 0$, for $\alpha < 1$, $p \cdot (\alpha \hat{x}) < p \cdot \hat{x} = p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f$, contradicting what we showed in Step 1.

So we know that if $\hat{x} \succ^h \boldsymbol{x}^h$, $\hat{x}$ must cost more at prices p than h can afford, and $(p, \boldsymbol{x}, \boldsymbol{z})$ is a Walrasian equilibrium for the economy $\mathcal{E}(e)$. ∎

That finishes the proof of the Proposition (as stated, correctly, in later printings of the book.) But it is worth saying a few things more. Suppose we are in a setting where equilibrium prices are always nonnegative and nonzero. In this case, we can normalize any price vector p to lie in the unit simplex; for the remainder of this discussion, do so.

In the next chapter, the concept of a *Walrasian quasi-equilibriun* is defined. You can read the formal definition in the text in Definition 15.3; the short description is that this is a triple of (nonnegative, non-zero) prices p, consumption allocations $\boldsymbol{x}$, and production plans $\boldsymbol{z}$, such that (a) markets clear, (b) firms maximize their profit, (c) consumers can afford their part of the consumption allocation, and (d) any bundle $\hat{x}^h$ that is preferred by agent h to her allocated $\boldsymbol{x}^h$ costs (at prices p) at least as much as h's resources (given p, her endowment, her shareholdings, and the firms' production plans). This is the same as a Walrasian equilibrium, except that part d is a bit weaker; in a Walrasian equilibrium, if $\hat{x} \succ^h \boldsymbol{x}^h$, then $\hat{x}$ must cost strictly more than h's financial resources. Of course, this implies that every Walrasian equilibrium is also a Walrasian quasi-equilibrium.

Suppose that we let $\boldsymbol{Q}(\mathcal{E}(e))$ be the set of Walrasian quasi-equilibria for the economy $\mathcal{E}(e)$, where we look only at nonnegative, nonzero prices, that are normalized to be in the unit simplex of prices. Then I assert that the proof of Proposition 14.14 just given is easily adapted to show that $e \Rightarrow \boldsymbol{Q}(\mathcal{E}(e))$ is upper semi-continuous. Go through the steps, if this isn't clear to you; the key step is connected to the parenthetical remark made at the end of Step 1.

So why isn't $e \Rightarrow \mathbf{W}(\mathcal{E}(e))$ upper semi-continuous? What is wrong with the proposition as given in early printings of the book? There are two problems, one obvious and easily repaired, but the other requiring that we move from $\mathbf{W}$ to $\boldsymbol{Q}$.

The first, easily fixed problem concerns the normalization of prices. If $(p, \boldsymbol{x}, \boldsymbol{z})$ is a Walrasian equilibrium for $\mathcal{E}(e)$, then (of course) so is $(\lambda p, \boldsymbol{x}, \boldsymbol{z})$ for any strictly positive scalar λ. So if we don't normalize prices, we could have a sequence of equilibria $\{(p_n, \boldsymbol{x}_n, \boldsymbol{z}_n)\}$ where the price vectors converge to 0. Of course, those limit prices will not be Walrasian equilibrium prices (at least, as long as consumers are, say, globally insatiable). But the fix is easy: Be in a setting where the only prices considered are nonnegative and nonzero, and look (only) at prices normalized to be in the unit simplex. Then any limit price vector will be nonnegative, nonzero, and (in fact) in the unit simplex.

But the second, harder-to-fix problem, involves consumers whose wealth goes to zero. Let me provide a concrete example of a two-consumer, two-good, pure-exchange economy: The two consumers are Alice and Bob. Alice will have endowment $e^A = 0$ throughout. We can give Alice absolutely standard preferences: $u^A((x_1, x_2)) = x_1^{1/2} + x_2^{1/2}$, say. Bob's preferences will be quasi-linear in good 2, $u^B(x_1, x_2) = x_1^{1/2} + x_2$. And Bob's endowment in economy n will be

$$e_n^B = \left(\frac{1}{n}, 1 - \frac{1}{n^{1/2}}\right).$$

Note that this is set up so that Bob's endowment gives him utility 1 for all n; we are riding along his $u^B = 1$ indifference curve, decreasing to zero the amount of good 1 and compensating with more and more good 2.

I assert that, with this endowment, equilibrium prices in economy n (normalized to be in the unit simplex) are

$$p_n = \left(\frac{n^{1/2}}{2 + n^{1/2}}, \frac{2}{2 + n^{1/2}}\right).$$

At these prices, Alice (whose wealth is zero, because her endowment is 0), optimally chooses $\boldsymbol{x}_n^A = 0$, while Bob chooses his endowment $x_n^B = e_n^B$. There is nothing mysterious in this construction; since Alice has zero endowment, as long as prices are strictly positive, she will get 0, so Bob must be picking his endowment. And the equilibrium prices are chosen so that they support Bob's choice of his endowment.[1]

But note that in p_n, the price of the first good is getting close to 1, while the price of the second good is getting close to 0. So

$$\lim_{n\to\infty}(p_n, \boldsymbol{x}_n^A, \boldsymbol{x}_n^B) = (p, \boldsymbol{x}^A, \boldsymbol{x}^B) \quad \text{where} \quad p = (1,0),\ \boldsymbol{x}^A = (0,0),\ \text{and } \boldsymbol{x}^B = (0,1).$$

This clearly is not a Walrasian equilibrium for the limit economy (in which Bob's endowment is $e^B = (0,1)$); markets clear, but neither Alice nor Bob are satisfied with their assigned consumption bundles; since good 2 is free, both of them want infinite amounts of it.

But, of course, $(p, \boldsymbol{x}^A, \boldsymbol{x}^B)$ is a Walrasian quasi-equilibrium. The bundles that both Alice and Bob prefer and can afford cost the same as their equilibrium allocations, since these strictly preferred bundles involve large quantities of a good whose price is zero.

This is not to say that a triple $(p, \boldsymbol{x}, \boldsymbol{z})$ cannot be a (full-fledged) Walrasian equilibrium, if p has some zero components. But if we want to save the result that $e \Rightarrow \mathbf{W}(\mathcal{E}(e))$ is upper semi-continuous, we either must find ways to ensure that equilibrium prices are bounded uniformly away from zero (on the unit simplex), or we must make sure that the price of a good can go to zero only in situations where eveyone is satiated in the good.[2]

[1] Since both Alice and Bob have strictly increasing preferences, prices have to be strictly positive in equilibrium. As long as Alice has endowment 0, at strictly positive prices, she must choose 0, so Bob must consume his endowment, and so equilibrium prices must support this choice for him. That is to say, we have described the *only* Walrasian equilibrium for economy $\mathcal{E}(e_n)$. It may be worth noting a couple of things here: (1) If $e^A = (0,0)$ and $e^B = (0,1)$, there is no Walrasian equilibrium at all. There are a number of existence results in this chapter, and you might want to go through them all and figure out why none of them work in this case. (2) This could have been posed as a one-consumer, pure-exchange economy—Alice plays no real role in the counterexample—but I worried that readers might think that a one-person economy was somehow a "trick," so poor Alice was included.

[2] As an example of the first approach, Phil Reny suggests that we constrain endowments and production technologies so that there is a uniform upper bound on what the economy can allocate, and then assume that consumer preferences are such that the norm of aggregate consumption demand goes to infinity for any sequence of prices that approaches the boundary of P. Then Walrasian equilibrium prices must stay uniformly away from the boundary, for any sequence of endowments $\{e_n\}$. If, by way of contradiction, we had a sequence of endowment vectors $\{e_n\}$ and Walrasian equilibrium prices p_n for those endowment vectors, net consumer demand would have to diverge in norm. This would mean that, for large n, net consumer demand (in equilibrium) would have to exceed what the economy is (uniformly) capable of producing. That, of course, cannot happen.

Microeconomic Foundations I
Choice and Competitive Markets

Student's Guide

Chapter 15: General Equilibrium, Efficiency and the Core

Summary of the Chapter

Since Adam Smith's *Wealth of Nations*, the idea that market outcomes are socially "good" (at least, compared to other ways of organizing the production and distribution of goods) has been advanced by economists. There is a lot to this story, concerning innovation and growth, for which we lack important tools of analysis. But, in this chapter, we begin to understand this notion, with formal results that show:

- Walrasian-equilibrium allocations are Pareto efficient, the *First Theorem of Welfare Economics*.
- Conversely (and subject to some extra convexity assumptions and other qualifications), every Pareto-efficient allocation can be "decentralized" as a Walrasian-equilibrium allocation, if endowments and shareholdings redistributed, the *Second Theorem of Welfare Economics*.
- Moreover, every Walrasian-equilibrium allocation is in an appropriately defined core.
- With "enough" competition of the "right kind," core allocations are Walrasian-equilibrium allocations. This idea is formalized in the literature in several ways; we provide one formalization, the *Debreu-Scarf Theorem*.

It goes without saying that these results are not tautologies, and for them to hold, certain conditions must hold. For one thing, all economic entities—consumers and firms—must be price-takers. For another, there can be no externalities; the chapter closes with a brief discussion of externalities and *Lindahl equilibria*, an extension of Walrasian equilibria that is designed to get back the Pareto efficiency of equilibrium allocations in economies with externalities, by adding many more markets to the economy.

Solutions to Starred Problems

■ *2. (a) At the prices $(1,1)$ the symmetry and strict concavity of Alice's utility function implies that she will choose $x_1 = x_2$. Her preferences are strictly increasing, hence locally insatiable, and so she will satisfy her budget constraint. Hence her unique demand at these prices is $(1,1)$.

As for Bob: The function $x^3 - 9x^2 + 15x$ is increasing at $x = 0$, hits a local maximum at $x = 1$ where its value is 7, then declines until $x = 5$, and then increases to ∞. This means that, for Bob, $(1,1)$ is a local maximum. Moreover, for the price vector $(1,1)$, Bob's wealth is 4, so he cannot afford any bundle that does better than $(1,1)$. That is his (unique) demand.

So total demand is $(2,2)$, while the social endowment is $(3,3)$, and markets clear.

(b) But this allocation is not Pareto efficient. Since Alice has strictly increasing utility, it is optimal to give her anything Bob doesn't want. The equilibrium allocation is Pareto inferior to the allocation that leaves Bob at $(1,1)$ and gives Alice the remaining $(2,2)$.

(c) A key step in the proof of the First Theorem is to say that all Pareto-superior allocations correspond to bundles of goods that cost more than the consumer's wealth. While it is true, in this case, that allocations Pareto superior to the equilibrium allocation cost more than the equilibrium allocation, because Bob is not locally insatiable, his equilibrium allocation costs less than all his wealth. The proof breaks down.

(Can you produce an example where both consumers spend all their wealth at the equilibrium and yet the equilibrium allocation is not Pareto efficient? As a starting point, what is the value of x for $x > 5$ for which $x^3 - 9x^2 + 15x = 7$. Why is that significant?)

■ 4. See Figure G15.1. In this figure, consumer 1's indifference curves as $x_1 \to 0$ hit the axis, but do so at a slope that approaches infinity. Consumer 2, on the other hand, has no use for good 2; consumer 2 wants (only) as much good 1 as he can get. (In the figure, consumer 2's indifference curves are shown as dashed lines.) Hence Pareto-efficient points give all of good 2 to consumer 1. The social endowment of good 1 can be divided between them, but one Pareto-efficient point, and the one that is going to be problematic, is where consumer 1 gets all of good 2 while consumer 2 gets all of good 1. The prices that support this allocation are (multiples of) $p = (p_1, p_2) = (1, 0)$. But at these prices, consumer 1 wants an unlimited amount of good 2; bundles that give consumer 1 more than the social endowment of good 2 are strictly preferred to her (quasi-)equilibrium allocation, but cost just as much (namely, \$0).

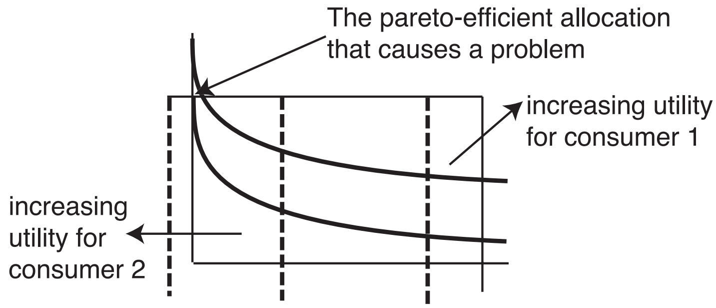


Figure G15.1. A quasi-equilibrium that is not a Walrasian equilibrium.

■ 5. If $\boldsymbol{x}$ is not in the core, then there must be some coalition J and some allocation $(\hat{x}_j)_{j\in J} \in \boldsymbol{X}^J$ such that $\hat{x}^j \succeq^j \boldsymbol{x}^j$ for all $j \in J$ and $\hat{x}^j \succ^j \boldsymbol{x}^j$ for some $j \in J$. But if $\hat{x}^j \succeq^j \boldsymbol{x}^j$, then $p \cdot \hat{x}^j \geq p \cdot e^j$; otherwise, consumer j could afford something close to and better than $\hat{x}^j$ at prices p, but then that bundle would be better than $\boldsymbol{x}^j$, which is not possible since $\boldsymbol{x}^j$ solves consumer j's utility-maximization problem at prices p. And for any j such that $\hat{x}^j \succ^j \boldsymbol{x}^j$, $p \cdot \hat{x}^j > p \cdot e^j$. Summing up over all the j, we have

$$p \cdot \left[\sum_{j\in J} \hat{x}^j\right] > p \cdot \left[\sum_{j\in J} e^j\right].$$

But if $(\hat{x}^j)_{j\in J} \in \boldsymbol{X}^J$, then $\sum_{j\in J} \hat{x}^j \leq \sum_{j\in J} e^j$. Prices p are nonnegative, so this implies that

$$p \cdot \left[\sum_{j\in J} \hat{x}^j\right] \leq p \cdot \left[\sum_{j\in J} e^j\right],$$

a direct contradiction to what we had before.

■ 7. The proof of Proposition 15.9 mimics that of Proposition 15.8 (see the solution to Problem 15.5 just above), except that you must take into account the profits of firms. Suppose first that $\boldsymbol{X}^J$ is defined using rule 5 for each J. We begin as in the solution to Problem 15.5:

If $\boldsymbol{x}$ is not in the core, then there must be some coalition J and some allocation $(\hat{x}_j)_{j\in J} \in \boldsymbol{X}^J$ such that $\hat{x}^j \succeq^j \boldsymbol{x}^j$ for all $j \in J$ and $\hat{x}^j \succ^j \boldsymbol{x}^j$ for some $j \in J$. But if $\hat{x}^j \succeq^j \boldsymbol{x}^j$, then $p \cdot \hat{x}^j \geq p \cdot e^j + \sum_f s^{fj} p \cdot \boldsymbol{z}^f$; otherwise, consumer j could afford something close to and better than $\hat{x}^j$ at prices p, but then that bundle would be better than $\boldsymbol{x}^j$, which is not possible since $\boldsymbol{x}^j$ solves consumer j's utility-maximization problem at prices p. And for any j such that $\hat{x}^j \succ^j \boldsymbol{x}^j$, $p \cdot \hat{x}^j > p \cdot e^j + \sum_f s^{fj} p \cdot \boldsymbol{z}^f$. Summing up over all the j, we have

$$p \cdot \left[\sum_{j\in J} \hat{x}^j\right] > p \cdot \left[\sum_{j\in J} e^j\right] + p \cdot \left[\sum_{j\in J}\sum_f s^{fj}\boldsymbol{z}^f\right] = p \cdot \left[\sum_{j\in J} e^j\right] + p \cdot \left[\sum_f s^{fJ}\boldsymbol{z}^f\right].$$

But if $(\hat{x}^j)_{j\in J} \in \boldsymbol{X}^J$, then $\sum_{j\in J} \hat{x}^j \le \sum_{j\in J} e^j + \sum_f s^{fJ}\hat{z}^f$, for some production plans $(\hat{z}^f)_{f\in F}$, since $\boldsymbol{X}^J$ is defined using rule 5. Prices p are nonnegative, so this implies that

$$p \cdot \left[\sum_{j\in J} \hat{x}^j\right] \le p \cdot \left[\sum_{j\in J} e^j\right] + p \cdot \left[\sum_f s^{fJ}\hat{z}^f\right].$$

And since $\boldsymbol{z}^f$ is profit maximizing for firm f at prices p, and $\hat{z}^f$ is a feasible production plan for firm f, we maintain the inequality just above if we replace each $\hat{z}^f$ with $\boldsymbol{z}^f$, giving us a direct contradiction to the inequality obtained two displays ago.

Since $\boldsymbol{X}^J$ under rules 1 and 3 is smaller than $\boldsymbol{X}^J$ under rule 5, this proof shows that every Walrasian equilibrium is in the core of the economy, if rules 1 or 3 are used. (The assumption that $0 \in Z^f$ for each f needs to be used here. Can you see why? If this is not assumed, why might rule 1 make it *easier* for a coalition to construct a valid objection to an equilibrium allocation?)

And if any of the rules is used, and all firms have constant-returns-to-scale technologies: Note first that, since $0 \in Z^f$ and all firms have c-r-s technologies, $p \cdot \boldsymbol{z}^f = 0$ for all firms f in any Walrasian equilibrium. (In fact, if I assume that all technologies are c-r-s, I don't need the assumption that $0 \in Z^f$ for each f. Why?) Rules 1, 3, and 5 are covered by the previous argument, so we only need to worry about rules 2 and 4. Rule 2 is the more liberal (gives more power to each coalition to form objections), so we can work with that specification of $\boldsymbol{X}^J$. Then at any Walrasian-equilibrium prices, $p \cdot \boldsymbol{z}^f = 0$ for the profit-maximizing plans $\boldsymbol{z}$, and $p \cdot \hat{z}^f \le 0$ for any other plans. Therefore, even if the most liberal rule 2 is used, feasibility of objection $(\hat{x}^j)$ implies that

$$\sum_{j\in J} \hat{x}^j \le \sum_{j\in J} e^j + \sum_{f\in F} \hat{z}^f,$$

for some production plan $(\hat{z}^f)_{f\in F}$. Evaluating both sides of the inequality with equilibrium prices p, and noting that $p \cdot z^f \le 0$ for any feasible production plan for firm f, gives

$$p \cdot \left[\sum_{j\in J} \hat{x}^j\right] \le p \cdot \left[\sum_{j\in J} e^j\right] + p \cdot \left[\sum_{f\in F} \hat{z}^f\right] \le p \cdot \left[\sum_{j\in J} e^j\right].$$

And, in the first part of argument, the consumer-by-consumer inequalities $p \cdot \hat{x}^j \ge p \cdot e^j + \sum_f s^{fj} p \cdot \boldsymbol{z}^f$ become $p \cdot \hat{x}^j \ge p \cdot e^j$, since the sum of the firm's profits is just a sum of zeros. Adding up over all consumers (and noting that some of the inequalities are strict) gives the reverse inequality, and the (by now) usual contradiction.

▪ 11. (a) This is a Chapter-8 style problem. Both utility functions are concave functions of the social state (an allocation of the social endowment), and the set of social states is convex, so we find all the Pareto-efficient points by maximizing weighted averages of the two utility functions (making due allowance for the ”endpoints” of such weighted averages).

If we give all the weight to Alice, the obvious Pareto-efficient allocation provides Alice with all the goods, or $(x^A, x^B) = ((4, 4,), (0, 0))$.

If we give all the weight to Bob, it is not so simple. It is clear that Bob should get all of good 2. But Bob gets utility from Alice consuming good 1, so we have to solve $\max \ln(x_1^B + 1) + 0.5 \ln(x_1^A + 1)$ subject to nonnegativity constraints and $x_1^A + x_1^B \leq 4$. Simple calculus reveals an answer of $x_1^B = 3$; the Pareto-efficient allocation corresponding to this weighting is $(x^A, x^B) = ((1, 0), (3, 4))$.

And if there is positive weight given to each: Normalize the weight on Bob to be 1, and let $\alpha \in (0, \infty)$ be the relative weight on Alice. The problem is to

$$\text{maximize}\, \alpha \big(\ln(x_1^A + 1) + \ln(x_2^A + 1) \big) + \big(\ln(x_1^B + 1) + \ln(x_2^B + 1) + 0.5 \ln(x_1^A + 1) \big),$$

subject to nonnegativity constraints and $x_1^A + x_1^B \leq 4$ and $x_2^A + x_2^B \leq 4$. Since the utility functions are strictly increasing in the variables, we know that the solution will have $x_1^B = 4 - x_1^A$ and $x_2^B = 4 - x_2^A$, so you can replace x_1^B and x_2^B in the objective function with $4 - x_1^A$ and $4 - x_2^A$, respectively. Also, the objective function is (strictly) concave in the variables, so we know that the first-order/complementary slackness conditions are necessary and sufficient for an optimum. Finally, we know that if the unconstrained maximum violates one of the constraints, the answer will be at the violated constraint. (This follows from the concavity of the objective function.) So we look at the simple first-order conditions on the objective function. We get

$$x_1^A = \frac{10\alpha + 3}{2\alpha + 3} \quad \text{and} \quad x_A^2 = \frac{5\alpha - 1}{\alpha + 1}.$$

Note that for some values of α, these equations violate the constraints, so the ”real” solution must be amended. Specifically, for large α (lots of weight on Alice), we get the simple first-order conditions of x_1^A and x_2^A both approaching 5. When these reach 4, there is no more to give Alice. And as α approaches zero (most of the weight on Bob), there is no problem with x_1^A, which approaches 1, but x_2^A approaches -1, and this has to stop at 0. Putting all this together, we get for the full Pareto frontier:

$$x^A, x^B = \begin{cases} \left(1, 0\right), \left(3, 4\right), & \text{for } \alpha = 0, \\ \left(\frac{10\alpha + 3}{2\alpha + 3}, 0\right), \left(\frac{9 - 2\alpha}{2\alpha + 3}, 4\right), & \text{for } 0 < \alpha \leq 1/5, \\ \left(\frac{10\alpha + 3}{2\alpha + 3}, \frac{5\alpha - 1}{\alpha + 1}\right), \left(\frac{9 - 2\alpha}{2\alpha + 3}, \frac{5 - \alpha}{\alpha + 1}\right), & \text{for } 1/5 \leq \alpha \leq 9/2, \\ \left(4, \frac{5\alpha - 1}{\alpha + 1}\right), \left(0, \frac{5 - \alpha}{\alpha + 1}\right), & \text{for } 9/2 \leq \alpha \leq 5, \text{ and} \\ \left(4, 4\right), \left(0, 0\right), & \text{for } \alpha \geq 5, \text{ including } \alpha = \infty. \end{cases}$$

(b) To find the Walrasian equilibria, we proceed as usual, simply ignoring the externality in Bob's utility function. Bob is going to take Alice's choices as something he cannot affect, and so that term enters his utility-maximization problem as a constant.

Given that this is so, all the symmetry and strict convexity (of preferences) ensures that the equilibrium will be symmetric, meaning that both Alice and Bob wind up with equilibrium allocations of $(2, 2)$. To support these allocations, normalized prices must be $(1, 1)$.

This allocation is (of course) *not* Pareto efficient. To allocate 2 units of good 1 to Alice along the Pareto frontier, we are looking at α in the range $\alpha \in (0, 9/2)$, for which $x_1^A = (10\alpha + 3)/(2\alpha + 3)$. Setting this equal to 2 gives $\alpha = 1/2$. Note well, $\alpha = 1/2$ is the only point along the Pareto frontier for which $x_1^A = 2$. But when $\alpha = 1/2$, the corresponding point on the Pareto frontier is $(x^A, x^B) = ((2, 1), (2, 3))$.

Now, this point is *not* a Pareto improvement on the equilibrium allocation, because it decreases Alice's utility. We've only demonstrated that the Walrasian equilibrium allocation is not on the Pareto-frontier that we computed in part a. A different way to see that this point is not Pareto efficient is to move from it (from $((2, 2), (2, 2))$) in Pareto-improving direction, and this means having Bob give up some of his 1-good to Alice while getting some 2-good in return. Suppose Bob gives up 0.1 units of the first good, and gets back 0.08 units of the second good in return. At the equilibrium allocation, Alice's utility is 2.19722458 and Bob's is 2.74653072, while at this alternative, Alice has utility 2.20298573 and Bob has 2.75534139. So this is a Pareto improvement. It isn't hard to see why. The terms of trade—.0.1 units of good 2 in exchange for 0.08 units of good 1—are good for Alice. And while they diminish those portions of Bob's utility which bear directly on his own compensation, the external effect of having Alice consume more of good 1 more than makes up for this.

(c) There is only one external effect here, so we need to take into account only one "extra" price in the Lindahl equilibrium, a transfer from Alice to Bob because of the external impact on Bob of Alice's consumption of good 1. Let this have a price r, and let the "regular prices" of goods 1 and 2 be normalized to be 1 and p, respectively. Alice's Lindahl problem is then

$$\max \ln(x_1^A + 1) + \ln(x_2^A + 1), \text{ subject to } (1 + r)x_1^A + px_2^A \leq 4.$$

Alice maximizes her utility, but in terms of her budget constraint, it "costs" her r per unit of the first good that she consumes.

As for Bob, his problem is

$$\max \ln(x_1^B + 1) + \ln(x_2^B + 1) + 0.5 \ln(x_1^A + 1), \text{ subject to } x_1^B + px_2^B \leq 4p + rx_1^A.$$

He also maximizes his utility, but for him, a change in x_1^A is a transfer to him, so we put this on the "income" side of his budget constraint.

Now look at the following prices: $p_1 = 1$, $p_2 = 0.82057582$, and $r = -0.27984671$. Note that $r < 0$; the activity x_1^A generates a positive externality for Bob, and so the transfer from Alice to Bob for this activity should be negative; that is, Bob should compensate (or, a better verb is, subsidize) Alice for her consumption of the first good.

If you run the math, you'll find that, at these prices, Alice chooses (2.84690988, 2.37612289) and Bob chooses $(1.15309012, 1.62387711)$, which clears markets. And, just as Proposition 15.13 tells us, this allocation is Pareto efficient, corresponding to $\alpha = 1.28669246$.

How did I find this equilibrium? One method you might try is to write down the optimal solutions to Alice's and Bob's problems as a function of the price vector $(1, p, r)$, and then see what it takes to get markets to clear. I tried that, and I have to admit that the algebra defeated me. It is certainly possible to do, but I kept making errors that led me astray.[1] But then it occurred to me to use Proposition 15.13. I know that the equilibrium allocation has to be Pareto efficient, so it must correspond to some value of α. I also know that, with only one positive externality in the problem, the ratios of the supporting prices have to match Bob's marginal utilities for the *three* variables in his utility function. That is, the full price vector (p_1, p_2, r) must be proportional to

$$\left(\frac{1}{x_1^B + 1}, \frac{1}{x_2^B + 1}, -\frac{0.5}{x_1^A + 1} \right).$$

(I knew that r had to be negative, because Bob wants to subsidize Alice's consumption of good 1.) So for a variety of values of α, I computed the allocations and then those relative prices, and then for each allocation and set of relative prices, I computed Bob's

[1] But see the next paragraph.

budget: the value of his endowment "less" the amount he subsidizes Alice, less the cost of his purchases. At the value of α (namely, 1.28...) where his budget just balanced, I knew I had my equilibrium. (Of course, I then checked that if Alice and Bob maximize at those relative prices, it all works. I did this numerically, and it all worked perfectly.)

Added after the fact: While I think that the method for finding the equilibrium in the previous paragraph is interesting enough to be retained, Alejandro Francetich (who, as a Ph.D. student, did a magnificent job of going carefully through the various pieces of the text and ancilliaries, finding numerous typos and think-os) was able to do the algebra needed for a direct solution of the equilibrium. Here is his solution:

▪ 15.12. The front-cover design gives a graphical depiction of the proof of the Debreu-Scarf Theorem for the case of two commodities and two consumers. On the left-hand side, an Edgeworth box is given, with endowment point e and a reallocation $\boldsymbol{x}$ of the endowment, together with indifference curves for the two consumers through $\boldsymbol{x}$. Since this is an Edgeworth box depiction, we know that the reallocation is non-wasteful; if we call the consumers Alice and Bob and we denote their endowments e^A and e^B and their portions of $\boldsymbol{x}$ by $\boldsymbol{x}^A$ and $\boldsymbol{x}^B$, then we know that $\boldsymbol{x}^A + \boldsymbol{x}^B = e^A + e^B$.

On the right-hand side of the cover, we depict the sets of net trades (from their endowments) for Alice and Bob that, together with those endowments, provide them with more utility than do $\boldsymbol{x}^A$ and $\boldsymbol{x}^B$, respectively. Note that $\boldsymbol{x}^A - e^A$, the red dot in the north-west quantrant, is on the boundary of this set for Alice, while $\boldsymbol{x}^B - e^B = e^A - \boldsymbol{x}^A$ (since the reallocation $(\boldsymbol{x}^A, \boldsymbol{x}^B)$ is non-wasteful) lies on the boundary of this set for Bob. The full boundary for Alice just takes her (blue) indifference curve from the Edgeworth box and translates it onto the axes on the right-hand side; for Bob, we take his (yellow) indifference curve from the Edgeworth box, rotate is 180 degrees, and translate it approppropriately. And then we form the convex hull—the shaded green area, bounded by the dashed green line—of these two regions.

Since this convex hull is disjoint from the strict negative orthant (the dark green area containing the author's name), we know (from the proof) that the reallocation $\boldsymbol{x}$ is a Walrasian-equilibirum allocation relative to e, with corresponding relative prices given by the dashed green line, which is the hyperplane that separates the convex hull from the strict negative orthant.

Note that, since $\boldsymbol{x}^A + \boldsymbol{x}^B = e^A + e^B$, or $\boldsymbol{x}^A - e^A = -(x^B - e^B)$, and since (in the depiction) preferences are clearly strictly increasing, we know that $\boldsymbol{x}^A - e^A$ is on the boundary of Alice's set of strictly preferred net trades, while $\boldsymbol{x}^B - e^B$ is on the boundary of Bob's set. Hence $(1/2)(\boldsymbol{x}^A - e^A + x^B - e^B) = (1/2)0 = 0$ is (at least) on the boundary of the convex hull. And the only way the origin will be on the boundary—that is, not in the interior, which would mean that there is intersection with the strict negative orthant—is if the line formed by joining $\boldsymbol{x}^A - e^A$ to $\boldsymbol{x}^B - e^B$ is tangent to the two sets of strictly preferred net trades. But this would mean (a) the indifference curves back in the Edgeworth box must be tangent to one another, and (b) the line of tangency must pass through the endowment point. Which is, of course, the condition for $\boldsymbol{x}$ to be a Walrasian-equilibrium allocation relative to e, framed in terms of the Edgeworth box picture.

Microeconomic Foundations I
Choice and Competitive Markets

Student's Guide

Chapter 16: General Equilibrium, Time, and Uncertainty

Summary of the Chapter

In this chapter, the model and results of general equilibrium, the stuff of Chapters 14 and 15, are adapted to account for time and uncertainty. The chapter begins with a discussion of how the all-markets-at-once framework of Chapters 14 and 15 can do this, if one considers as different commodities the same physical good or service consumed (or used as input or produced) at different times or in different contingencies. But then we look at economies with a sequence of markets, some meeting and clearing before others begin to function.

Models of this sort of dynamic economy can be divided roughly into two groups. In some, opportunties to move wealth across time and states of nature—which is accomplished by trading in various securities—are *complete*: Subject to one unifying budget constraint, any shifting of wealth is feasible. The equilibria of these complete-market economies provide outcomes identical to those provided by Walrasian equilibria of all-markets-at-once economies. The chapter provides a very complete theory of these.

But in other cases, opportunities to shift wealth across time and states of nature are insufficient to achieve every desired shift (again, subject to a budget constraint). These are *incomplete-market economies/equilibria*, and they admit many "nasty" phenomena. The chapter doesn't offer anything like a complete treatment of incomplete markets; in fact, all it really does is provide a couple of examples to show the sorts of nasties that occur.

This is, by page count, the longest chapter in the book, and it certainly has one of the more complex "plot lines." It also comes with one of the most complex (and sometimes confusing) systems of notation. *But, conceptually, it is not at all difficult.* If you read it carefully and slowly, keeping track as you go with the elaborate system of notation, you should find it reasonably straightforward. That said, you will probably need to take some breaks: Good places to take a break are between Sections 16.3 and 16.4, and then between 16.5 and 16.6.

The plot line is:

- Section 16.1 provides the setting, a time/uncertainty/information structure.
- In Section 16.2, the notion of a commodity is enriched to include the *contingency* (a time and state-of-information) in which the commodity is consumed (or otherwise used or produced by a firm), and the general equilibrium constructions of Chapters 14 and 15 are adapted to this enriched notion of a commodity. In this adaptation, we imagine that markets in all (contingent) commodities are conducted at a single point in time, before any production or consumption takes place.
- Section 16.3 gives a first cut at a more "dynamic" view of markets. Here, there are spot markets in basic commodities in each contingency and, in addition, a set of markets at the outset in so-called *Arrow-Debreu contingent claims*.
- Section 16.4 is the heart of the chapter. Here, more complex "securities" than simple Arrow-Debreu contingent claims are formulated and included in an equilibrium framework. Characterizations of equilibria are provided, and results are given concerning when the equilibrium consumption allocations in a dynamic economy are identical with the Walrasian-equilibrium allocations of the economy of Section 16.2
- In Section 16.4, securities pay *financial dividends*, dividends denominated in the numeraire (or in units of account). This gives rise to a massive indeterminancy in securities prices. So in Section 16.5, we "redo" the analysis of Section 16.4 but under the assumption that securities pay their dividends in contingent commodities. (We limit analysis to the case where all dividends are paid in one basic commodity, in each contingency.)
- The focus in Sections 16.3 through 16.5 is on "complete-markets" equilibria, although a number of results are proved that are more general. In Section 16.6, we briefly discuss what can happen when markets are "incomplete," illustrating with a very simple example.
- Firms are omitted from all the analysis of dynamic economies, from Section 16.3 through 16.6. Section 16.7 concludes the chapter by discussing how firms can be brought back into the story, including in that story the idea that amongst the securities being traded are shares of equity in the firms. When markets are competitive and complete, no difficulties arise. But when they are not complete, extreme pathologies can occur, which we illustrate with a final example. The nature of these pathologies suggests that the tools developed in this volume are inadequate to model important economic phenomenon, which is where this volume ends.

Solutions to Starred Problems

■ 16.2. (a) Assumption 16.1 implies 16.1′. If a consumer is globally insatiable in f_t wheat, then she is clearly globally insatiable in f_t consumption.

(b) Here is the modified Proposition 16.4.

Proposition 16.4′. *Assumption 16.1′ holds. Suppose $(p, \boldsymbol{x})$ is a Walrasian equilibrium for the all-at-once market structure. Then $(r, q, \boldsymbol{x}, \boldsymbol{y})$ where*

$$r_{if_t} = p_{if_t}, \quad q_{f_t} = 1, \quad \textit{and} \quad \boldsymbol{y}^h_{f_t} = p_{f_t} \cdot (\boldsymbol{x}^h_{f_t} - e^h_{f_t})$$

is an EPPPE for the economy with contingent financial markets. And if $(r, q, \boldsymbol{x}, \boldsymbol{y})$ is an EPPPE for the economy with contingent financial markets, then $(p, \boldsymbol{x})$ where

$$p_{f_0} = r_{f_0} \quad \textit{and} \quad p_{if_t} = q_{f_t} r_{if_t}, \textit{for } f_t \neq f_0$$

is a Walrasian equilibrium for the all-at-once market structure.

(c) At the risk of overdoing things, I reproduce the proof of Proposition 16.4, modified appropriately, below. Commentary and changes are provided in a sans serif font. But before launching into the proof, I repeat here the new budget constraint (for financial assets), giving them an "equation" number. The budget constraint for f_0 doesn't change; what are different (and simpler) are the constraints for $f_t \neq f_0$:

$$r_{f_0} \cdot x^h_{f_0} + q \cdot y^h \leq r_{f_0} \cdot e^h_{f_0}, \quad \text{and} \quad r_{f_t} \cdot x_{f_t} \leq r_{f_t} \cdot e^h_{f_t} + y_{f_t} \text{ for } f_t \neq f_0. \tag{16.1′}$$

Proof. *[The proof in the text begins with some fluff, which I omit.]* Suppose that $(p, \boldsymbol{x})$ is a Walrasian equilibrium. Define r, q, and $\boldsymbol{y}$ as shown *[which is a bit changed]*. To show that $(r, q, \boldsymbol{x}, \boldsymbol{y})$ is an EPPPE, we must verify that each consumer satisfies her budget constraints with $\boldsymbol{x}^h$ and $\boldsymbol{y}^h$ and is maximizing utility subject to those budget constraints, and that markets clear.

Concerning the budget constraints, since r is defined to be p, the definition of $\boldsymbol{y}$ can be rewritten as $\boldsymbol{y}^h_{f_t} = r_{f_t} \cdot (\boldsymbol{x}^h_{f_t} - e^h_{f_t})$ *[simpler than before]*. Therefore, by definition, the second "half" of (16.1′) *[change here]* holds with equality: $\boldsymbol{y}^h$ is defined to make this so. As for the first half, begin with the budget constraint in the Walrasian equilibrium, which holds with equality because all consumers are locally insatiable:

$$p \cdot \boldsymbol{x}^h = p \cdot e^h.$$

Break each sum into terms for contingency f_0 and for all others:

$$p_{f_0} \cdot \boldsymbol{x}^h_{f_0} + \sum_{f_t \neq f_0} p_{f_t} \cdot \boldsymbol{x}^h_{f_t} \quad = \quad p_{f_0} \cdot e^h_{f_0} + \sum_{f_t \neq f_0} p_{f_t} \cdot e^h_{f_t}.$$

Rearrange terms and substitute r for p (since they are identical), to get

$$r_{f_0} \cdot \boldsymbol{x}^h_{f_0} + \sum_{f_t \neq f_0} r_{f_t} \cdot (\boldsymbol{x}^h_{f_t} - e^h_{f_t}) \quad = \quad r_{f_0} \cdot e^h_{f_0}.$$

Substitute $\boldsymbol{y}^h_{f_t}$ for $r_{f_t} \cdot (\boldsymbol{x}^h_{f_t} - e^h_{f_t})$ in the summation and, recalling that $q \equiv 1$, we have the first half of (16.1').

Next is to show that the consumer is maximizing. Let $\hat{x}$ be any consumption bundle for h that, together with a corresponding $\hat{y}$, satisfies the budget constraints (16.1'). Sum up over all f_t the budget constraints for $\hat{x}$ and $\hat{y}$, and you get

$$r_{f_0} \cdot \hat{x}_{f_0} + q \cdot \hat{y} + \sum_{f_t \neq f_0} r_{f_t} \cdot \hat{x}_{f_t} \leq r_{f_0} \cdot e^h_{f_0} + \sum_{f_t \neq f_0} (r_{f_t} \cdot e^h_{f_t} + \hat{y}_{f_t}).$$

Note that $q \cdot \hat{y}$ on the left-hand side is the same as the summation of $\hat{y}$ terms on the right-hand side, since $q \equiv 1$, so these terms can be dropped. Then if you substitute ps for rs (they are identical), you get $p \cdot \hat{x} \leq p \cdot e^h$. Hence, $\hat{x}$ is a budget-feasible bundle for h in the economy with an all-at-once market structure and prices p. But since $\boldsymbol{x}^h$ is a Walrasian-equilibrium allocation at those prices, $u^h(\boldsymbol{x}^h) \geq u^h(\hat{x})$. That proves the maximization part of the definition of an EPPPE.

And to show that markets clear: The first half of (16.2) is the market-clearing condition for a Walrasian equilibrium (albeit expressed a bit less succinctly than usual), so it holds immediately. And, we know that $p \geq 0$ and $\sum_h \boldsymbol{x}^h \leq \sum_h e^h$, so clearly $p \cdot \sum_h (\boldsymbol{x}^h - e^h) \leq 0$. Moreover, this dot product consists only of nonpositive terms. And, because consumers are all locally insatiable, each satisfies Walras' Law with equality in any equilibrium, so this dot product is zero, and hence each term in it is zero. *That is, for each f_t,*

$$0 = p_{f_t} \cdot \sum_h \left(\boldsymbol{x}^h_{f_t} - e^h_{f_t}\right) = \sum_h p_{f_t} \cdot \left(\boldsymbol{x}^h_{f_t} - e^h_{f_t}\right) = \sum_h \boldsymbol{y}^h_{f_t}.$$

That is market clearing in the futures market; we know that $(r, q, \boldsymbol{x}, \boldsymbol{y})$ is an EPPPE.

For the other half, suppose that $(r, q, \boldsymbol{x}, \boldsymbol{y})$ is an EPPPE. To show that $(p, \boldsymbol{x})$ is a Walrasian equilibrium (for p defined as indicated from r and q), we have to show that each consumer is satisfying her (all-at-once) budget constraint, each is maximizing subject to that budget constraint, and that markets clear.

Market clearing is immediate from the first half of (16.2).

For the rest, we first argue that in any EPPPE, each q_{f_t} *[text eliminated here]* will be strictly positive. *Suppose to the contrary that, in an EPPPE, some $q_{f_t} \leq 0$. Temporarily denote the consumer whose preferences are globally insatiable in f_t by $\hat{h}$. We know then that* *there is some x' which is identical to $\boldsymbol{x}^{\hat{h}}$ (her equilibrium allocation) off of f_t and which satisfies $u^{\hat{h}}(x') > u^{\hat{h}}(\boldsymbol{x}^{\hat{h}})$. And she can afford x'; her budget constraints off of f_t are all satisfied with her original asset positions, since x' agrees with $\boldsymbol{x}^{\hat{h}}$ off of f_t. And to meet her f_t budget constraint, she simply buys enough financial claims for contingency-f_t numeraire as is required. Since $q_{f_t} \leq 0$, this at least maintains her f_0 budget constraint (if $q_{f_t} = 0$) and may even loosen it (if $q_{f_t} < 0$). In either case, x' becomes affordable, contradicting the optimality (for her) of $\boldsymbol{x}^{\hat{h}}$. This contradiction implies that q_{f_t} must be strictly positive for all f_t.*

Now take the EPPPE budget constraint for consumer h, given by (16.1′) *[change here, and...] Multiply the constraint for $f_t \neq f_0$ by q_{f_t}, and add all these constraints together (including the constraint for f_0), to get*

$$r_{f_0} \cdot \boldsymbol{x}^h_{f_0} + q \cdot \boldsymbol{y}^h + \sum_{f_t \neq f_0} q_{f_t} r_{f_t} \cdot \boldsymbol{x}^h_{f_t} \leq r_{f_0} \cdot e^h_{f_0} + \sum_{f_t \neq f_0} q_{f_t} \left(r_{f_t} \cdot e^h_{f_t} + \boldsymbol{y}^h_{f_t} \right).$$

Replace the (normalized) rs with ps, and cancel the common term $q \cdot \boldsymbol{y}^h$, and you get $p \cdot \boldsymbol{x}^h \leq p \cdot e^h$. So $\boldsymbol{x}^h$ is budget feasible for h at the prices p.

Finally, suppose that $\hat{x}$ is feasible for h at the prices p. Define $\hat{y} \in R^{N-1}$ by $\hat{y}_{f_t} = r_{f_t} \cdot (\hat{x}_{f_t} - e^h_{f_t})$ *[Change!]*. I assert that the pair $(\hat{x}, \hat{y})$ satisfies (16.1′), so that $\hat{x}$ is a feasible consumption bundle for h in the economy with futures markets and prices (r, q). *[Here and everywhere else that (16.1') appears, this is a change from before. I won't mention this again.]* Since $\boldsymbol{x}^h$ is an equilibrium bundle (in the EPPPE), once this is shown, we know that $u^h(\boldsymbol{x}^h) \geq u^h(\hat{x})$, proving that $\boldsymbol{x}^h$ solves h's utility maxmization problem in the all-markets-at-once economy, concluding the proof that $(p, \boldsymbol{x})$ is a Walrasian equilibrium for that economy.

Note that $\hat{y}$ is defined so that, for all f_t other than f_0, the second part of (16.1′) holds with equality. We only need to verify the first part of (16.1′). Since $\hat{x}$ is budget feasible in the all-markets-at-once economy at prices p, we know that $p \cdot \hat{x} \leq p \cdot e^h$. Write this as

$$p_{f_0} \cdot \hat{x}_{f_0} + \sum_{f_t \neq f_0} p_{f_t} \cdot \hat{x}_{f_t} \leq p_{f_0} \cdot e^h_{f_0} + \sum_{f_t \neq f_0} p_{f_t} \cdot e^h_{f_t},$$

hence

$$p_{f_0} \cdot \hat{x}_{f_0} + \sum_{f_t \neq f_0} p_{f_t} \cdot \left(\hat{x}_{f_t} - e^h_{f_t} \right) \leq p_{f_0} \cdot e^h_{f_0}.$$

In this inequality, replace each p with its definition in terms of q and r, and use the definition of $\hat{y}$, and you get *[the following equation is changed]*

$$r_{f_0} \cdot \hat{x}_{f_0} + \sum_{f_t \neq f_0} q_{f_t} r_{f_t} \cdot \left(\hat{x}_{f_t} - e^h_{f_t} \right) = r_{f_0} \cdot \hat{x}_{f_0} + \sum_{f_t \neq f_0} q_{f_t} \hat{y}_{f_t} = r_{f_0} \cdot \hat{x}_{f_0} + q \cdot \hat{y} \leq r_{f_0} \cdot e^h_{f_0}.$$

That is the first half of (16.1′), completing the proof. ∎

∎ 16.5. Recall the notation $\mathbf{S}(f_t)$ for the set of all successors to f_t.

(a) First, I'll give an arbitrage-based argument for the "qualitative" result that the price of a security is strictly positive at f_t if and only if it pays a strictly positive dividend in at least one successor to f_t:

Suppose that, for viable $\mathcal{S}$ and q, there is some $s \in \mathcal{S}$ and some $f_t \in \mathcal{F}(s)$ such that $q_{sf_t} > 0$, but $d_s(f'_{t'}) = 0$ for all $f'_{t'} \in \mathbf{S}(f_t)$. Then the plan of selling one unit of s in contingency f_t and holding to the end generates a wealth-transfer vector that is q_{sf_t} in contingency f_t and zero in every other contingency. This is an arbitrage opportunity, which is not allowed since $\mathcal{S}$ and q are viable. Hence, if $q_{sf_t} > 0$ for any $f_t \in \mathcal{F}(s)$, there must be some successor contingency to f_t in which s pays a strictly positive dividend.

On the other hand, suppose that we have $s \in \mathcal{S}$ and $f_t \in \mathcal{F}(s)$ such that $q_{sf_t} \leq 0$ and, in some successor contingency to f_t, s pays a strictly positive dividend. Then the plan of buying a unit of s in f_t and holding to the end generates $-q_{sf_t}$ in f_t, nonnegative dividends in all successors to f_t, and a strictly positive dividend in at least one successor to f_t. That's an arbitrage opportunity. But this can't be if $\mathcal{S}$ and q are viable: If they are viable, and if, for $f_t \in \mathcal{F}(s)$, there is some successor contingency in which s pays a strictly positive dividend, then q_{sf_t} must be strictly positive.

Now I'll prove the more exact result, that if $\mathcal{S}$ and q are viable, then for all s and for all $f^0_\tau \in \mathcal{F}(s)$,

$$q_{sf^0_\tau} = \frac{1}{\pi_{f^0_\tau}} \sum_{f_t \in \mathrm{S}(f^0_\tau)} \pi_{f_t} d_s(f_t), \tag{16.4}$$

for all $\pi \in (\mathcal{S}, q)$:

Suppose $\mathcal{S}$ and q are viable. Fix $s \in \mathcal{S}$ and some $f^0_\tau \in \mathcal{F}(s)$. Consider the following trading plan: In contingency f^0_τ, buy one unit of s; then hold it to the end. This trading plan generates the wealth-transfer vector

$$\xi(f_t) = \begin{cases} -q_{sf^0_\tau}, & \text{if } f_t = f^0_\tau, \\ d_s(f_t), & \text{if } f_t \in \mathbf{S}(f^0_\tau), \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

This ξ is clearly in $M(\mathcal{S}, q)$, and so for every $\pi \in (\mathcal{S}, q)$, $\pi \cdot \xi = 0$ is required. But this is

$$-\pi_{f^0_\tau} q_{sf^0_\tau} + \sum_{f_t \in \mathrm{S}(f^0_\tau)} \pi_{f_t} d_s(f_t) \quad = \quad 0.$$

Solve for $q_{sf^0_\tau}$, and you have the result.

(b) Let y be any legitimate trading plan for the securities in $\mathcal{S}$; the wealth transfer vector that is generated by y is ξ given by

$$\xi(f_t) = \sum_{s\in\mathcal{S}} \Big(q_{sf_t}\left[y(\hat{f}_t, s) - y(f_t, s)\right] + y(\hat{f}_t, s)d_s(f_t)] \Big).$$

Hence $\pi \cdot \xi$ is

$$\pi \cdot \xi = \sum_{f_t}\sum_{s\in\mathcal{S}} \Big(\pi_{f_t} q_{sf_t}\left[y(\hat{f}_t, s) - y(f_t, s)\right] + \pi_{f_t} y(\hat{f}_t, s)d_s(f_t)] \Big).$$

Once we show that this is equal to 0 for the q given in the statement of the proposition, we're done: This will show that $\pi \cdot \xi = 0$ for all $\xi \in M(\mathcal{S}, q)$, which (per Proposition 16.8(b)) is one test of viability of $\mathcal{S}$ and q.

To show that the double sum is 0, we will show that, for each s, the sum (only over f_t) is 0. (This shouldn't surprise you; if y is a legitimate trading plan, then so is the plan of executing only trades in a single s that are given by y.) Now, for a fixed security s,

$$\pi \cdot \xi = \sum_{f_t} \Big(\pi_{f_t} q_{sf_t}\left[y(\hat{f}_t, s) - y(f_t, s)\right] + \pi_{f_t} y(\hat{f}_t, s)d_s(f_t)] \Big).$$

Substitute the formula for q_{sf_t}, and this becomes

$$\sum_{f_t} \left\{ \pi_{f_t} \left(\frac{1}{\pi_{f_t}} \Big[\sum_{f'_{t'} \in \mathrm{S}(f_t)} \pi_{f'_{t'}} d_s(f'_{t'}) \Big] \right) \left[y(\hat{f}_t, s) - y(f_t, s)\right] + \pi_{f_t} y(\hat{f}_t, s) d_s(f_t) \right\},$$

which is

$$\sum_{f_t} \left\{ \Big[\sum_{f'_{t'} \in \mathrm{S}(f_t)} \pi_{f'_{t'}} d_s(f'_{t'}) \Big] \left[y(\hat{f}_t, s) - y(f_t, s)\right] + \pi_{f_t} y(\hat{f}_t, s) d_s(f_t) \right\}.$$

In these manipulations, you may be worried about the fact that some f_t may not be legitimate trading dates for t. But for those dates, the term $y(\hat{f}_t, s) - y(f_t, s)$ is zero, so it is irrelevant to the sum if we put these terms in, with what the formula says *would have been* the price of s, had it traded. Now for each f_t, recall that $\mathbf{P}(f_t)$ denotes the set of all predecessors of f_t, and recall that we have invented a predecessor for f_0 at which y is zero. I want to look at the term in the last display, running the sum over f_t in which s pays a nonzero dividend. This is

$$\sum_{f_t} \pi_{f_t} d_s(f_t) \left[y(\hat{f}_t, s) + \sum_{f'_{t'} \in \mathrm{P}(f_t)} \left(y(\hat{f}'_{t'}, s) - y(f'_{t'}, s) \right) \right].$$

Let me translate this: Two displays ago, we had the sum, for each f_t, of the trading plan's change in portfolio times the sum of normalized future dividends (normalized by π for the contingency in which the dividend is paid), plus the holdings of the previous contingency times any normalized dividend paid immediately. In the display immediately above, we look at all terms associated with a dividend paid in f_t, and sum over all the f_t. We have the normalized dividend multiplied by: The holdings from last period, and then the sum of all changes in holdings, going back to $\hat{f}_0$, in all predecessors of f_t. But, if you think about it just for a moment, you'll see that the term in square-brackets in the last display is a telescoping sum that, after cancelling out the same terms with different sign, leaves us with $y(\hat{f}_0, s)$, which is zero. So the last displayed term is zero, completing the proof.

For the nth time, I caution you not to be too impressed by this. It is (once again) simple bookkeeping and nothing more. If the price of a security is the properly "discounted" sum of its future dividends, then when you buy a unit of security, you are "paying" for those future dividends. And then, for as long as you hold the security, it pays some of those dividends; when and if you sell it, (by the rule that gives q) you get back the properly discounted sum of any dividends you didn't wait long enough to receive.

■ 16.6. Suppose that $\mathcal{S}$ is $\mathcal{S}^{\text{SFCC}}$, and prices q are given such that $\mathcal{S}$ and q are viable. Pick any f_t, $t > 0$, and some $\omega \in f_t$. (That is, ω is a state of nature that is still possible in contingency f_t.) If s^ω is the security in $\mathcal{S}^{\text{SFCC}}$ that goes with state ω, then we know that $q_{s^\omega f_t} > 0$ (since s^ω pays a strictly positive dividend in a successor to f_t, namely $\{\omega\}$ at time T). But then the trading plan of buying $1/q_{s^\omega f_t}$ units of s^ω at f_0 and then selling at time t (regardless of the contingency) generates the following wealth-transfer vector: At f_0, you pay $q_{s^\omega f_0}/q_{s^\omega f_t}$. At all other dates, except for t, you get nothing. And at date t, *if* the contingency is not f_t, you get nothing (for all other time-t contingencies, ω has been ruled out, the security is defunct, and so it is worthless), while in contingency f_t, you get 1 unit of numeraire. Hence, this trading plan transfers wealth from f_0 to f_t (and to no other contingency), satisfying the requirements of Lemma 16.10. The dimension of $M(\mathcal{S}^{\text{SFCC}}, q)$ must be $N-1$.

Supppose that $\mathcal{S} = \mathcal{S}^{\text{RFCC}}$ and q are viable. For each f_t, let s^{f_t} be the security that trades at $\hat{f}_t$ and pays (only) at f_t. As before, we create a trading plan that moves wealth from f_0 to f_t, for arbitrary f_t, and to no other contingency. Fixing f_t (for $t > 0$), let the path of predecessors of f_t from f_0 to f_t be denoted $f_0, f_1, \ldots, f_t$. The trading plan begins with the purchase of units of s^{f_1} in f_0. If the next contingency is f_1,

s^{f_1} pays a dividend, and the proceeds are used to buy units of s^{f_2}. (If the time 1 contingency is not f_1, no dividend is paid, and the plan involves no more trading.) Then if f_2 happens, s^{f_2} pays a dividend, used to buy units of s^{f_3}, and so forth. Clearly, as time moves from 0 to t, as long as f_t is still possible, the plan enables the purchase of units of the next element of $\mathcal{S}^{\text{RFCC}}$; if at time t, f_t is the contingency, the units of s^{f_t} that were purchased at time $t-1$, in contingency f_{t-1}, pay a dividend, which is "cashed out." Hence, this plan moves wealth from f_0 to f_t (and to no other contingency). Once we get the scale of this plan right (how many units of s^{f_1} are bought at f_0, to get the ball rolling, so that in the end we wind up with 1 unit of s^{f_t} and hence a dividend of 1 unit of the numeraiare), we have satisfied the requirements of Lemma 16.10, and we know that $M(\mathcal{S}^{\text{RFCC}}, q)$ has dimension $N-1$.

I'll leave the details of the argument here, except to say: To be precise about this, you first have to say that each s^{f_t} has a strictly positive price under q in the one contingency (namely, $\hat{f}_t$) in which it trades. But that is another application of first part of Lemma 16.11.

■ 16.8. Suppose we arbitrarily set $\pi = (1, 1, \ldots, 1)$ and use it and formula (16.4) to compute the prices of the L securities at the various time t-contingencies. The formula says that the price of security s for this π in contingency f_t will be

$$q_{sf_t} = \sum_{\omega \in f_t} d_s(\{\omega\}).$$

Since the f_t partition Ω, this tells us that, with probably one, all the square sub-matrices of the $L \times N_t$ matrix whose component corresponding to security s (which indexes the L dimension) and the time-t contingency f_t (which indexes the N_t dimension) is q_{sf_t} have full rank, with probability 1. Since there are only finitely many dates, this is true for all dates $t = 0, \ldots, T-1$ with probability one, as well as for the original matrix of dividends.

I assert that if this property holds (which it does, with probability 1), then for this q, $M(\mathcal{S}, q)$ has dimension $N-1$. The argument is: Take any f_t, $t < T$, and let f^i_{t+1} enumerate its $\ell(f_t)$ immediate successors. We know that $L \geq \ell(f_t)$, so choose any $\ell(f_t)$ securities and look at the $\ell(f_t) \times \ell(f_t)$ square matrix of the prices of these securities at date $t+1$ in the $\ell(f_t)$ contingencies f^i_{t+1}. (If $t = T-1$, look at the matrix of dividends paid by the $\ell(f_t)$ securities you select.) This has full rank, which means that, for each i, a portfolio of these $\ell(f_t)$ securities can be constructed in contingency f_t that is worth 1 in f^i_{t+1} and 0 otherwise. We know that these portfolios all have positive prices (in fact, given our choice of π, we know that each of these portfolios costs 1 unit exactly!), so these portfolios allow us to transfer wealth from any f_t to any one of its immediate successors, without generating gains or losses in any other contingency. This is enough to know that markets are complete. (This isn't quite the criterion given in Lemma 16.10, but if you look at the solution to the second half of 16.6, given previously, you'll see why it is true.)

But then, by exactly the argument given in the text, if $(p, \boldsymbol{x})$ is a Walrasian equilibrium, then for some $\boldsymbol{y}$ (which needs to be constructed, based on $\boldsymbol{x}$), this specific q, and for $r \equiv p$, $(r, q, \boldsymbol{x}, \boldsymbol{y})$ is an EPPPE.

Compare this with Proposition 16.13. That proposition says that, to get complete markets, we need *at least* $\ell(f_t)$ non-defunct securities in contingency f_t. This is almost a perfect converse; in that it says that, as long as securities trade in every non-terminal contingency and have positive dividends in every terminal contingency, any "random selection" of dividends for $\max_{f_t} \ell(f_t)$ securities will do.

But this trades rather heavily on the fact that we are allowed to choose π as we want. How does this work, if we instead have securities whose dividends are denominated in wheat? In fact, it works almost as well, in the following sense. Fix some Walrasian equilibrium $(p, \boldsymbol{x})$. We know that to get this equilibrium with an EPPPE where the price of wheat in every spot market is 1, we'll need $\pi_{f_t} = p_{1f_t}$. So now, when we look at the prices of the securities, instead of getting the simple sum of dividends, we get

$$q_{sf_t} = \frac{1}{p_{1f_t}} \sum_{\omega \in f_t} p_{1\{\omega\}} d_s(\{\omega\}).$$

The $1/p_{1f_t}$ is irrelevant but, when we take the sum of dividends, we are taking weighted sums. So we need to know: If we have a big matrix with each component randomly selected (in a fashion absolutely continuous with respect to Lebesgue measure), and we take weighted sums of the columns, the matrix that results (for specific weights) has full-rank square submatrices with probability one. This is true, if the weights are fixed in advance. But: the weights change with the Walrasian equilibrium we are looking at. *If* we know that the all-markets-at-once economy has only finitely many Walrasian equilibria, which is true for "most well-behaved" pure exchange economies, then we get the probability one statement for all of them at once. So if this *if* is met, this is almost as good as the result for economies with financial securities.

However: (1) In either case, the result is only true for "most" selections of dividends. It is a probability one statement. $L \geq \max_{f_t} \ell(f_t)$ securities might have their dividends chosen in a way that means markets are not complete. And (2) this says that, with probability 1, every Walrasian-equilibrium consumption allocation is an EPPPE consumption allocation. The converse is not claimed. For more on this, I urge you to try Problem 16.9.

■ 16.11. We know that $\mathcal{S}$ and q are viable if and only if there is some strictly positive π such that $\pi \cdot \xi = 0$ for all $\xi \in M(\mathcal{S}, q)$, and that the dimension of $M(\mathcal{S}, q)$ is $N - 1$ if and only if there is, up to normalization, a unique π of this character.

In the problem, it assumes the existence of one security s^0 that pays a dividend of 1 at every terminal (time-T) contingency and whose price, by choice of numeraire, is 1 in every contingency. In this context, imagine a strictly positive π such that $\pi \cdot \xi = 0$. In particular, look at any such π normalized so that $\pi_{f_0} = 1$.

One $\xi \in M(\mathcal{S}, q)$ is obtained by purchasing a share of the distinguished security in f_0 and holding it until time T, collecting dividends of 1. For this ξ, $\pi \cdot \xi = -\pi_{f_0} q_{s^0 f_0} + \sum_{\omega \in \Omega} \pi_{\{\omega\}} d_{s^0}(\{\omega\})$. Since $q_{s^0 f_0} = d_{s^0}(\{\omega\}) = 1$ for every ω, and since we have normalized π_{f_0} to be 1, this being equal to 0 is the same as

$$1 = \sum_{\omega \in \Omega} \pi_{\{\omega\}}.$$

So if we define, for each ω, $\mu(\omega) = \pi_{\{\omega\}}$, $\mu : \Omega \to R_{++}$ and sums to 1; μ constitutes a probability measure on Ω.

Moreover, another $\xi \in M(\mathcal{S}, q)$ is obtained by purchasing a share of the distinguished security in f_t and holding it until time T, collecting dividends of 1 in all terminal contingencies that are successors to f_t. If $\pi \cdot \xi = 0$ for this ξ, and if the price of s^0 is 1 in contingency f_t (which we've assumed), then by the same argument, we get

$$\pi_{f_t} = \sum_{\omega \in f_t} \pi\{\omega\} = \sum_{\omega \in f_t} \mu(\omega),$$

which is to say that π_{f_t} is the probability of contingency f_t under the probability distribution μ.

Now take any other security s (all securities trade in all contingencies, recall), and take any contingency f_t. Consider the trading rule of buying one unit of s in f_t and, if $t < T - 1$, selling it next period, while if $t = T - 1$, holding it for the dividends it will pay. This generates a ξ, and for $\pi \cdot \xi$ to equal 0 for this ξ, it is necessary that

$$\pi_{f_t} q_{s f_t} = \sum_{f_{t+1} \in \mathsf{S}(f_t)} \pi_{f_{t+1}} q_{s f_{t+1}},$$

where if $t = T - 1$, we interpret $q_{s f_{t+1}}$ as $d_s(f_{t+1})$. But if we replace the components of π with $\mu(f_t)$, the probability of contingency f_t under the probability measure μ, this is

$$q_{s f_t} = \sum_{f_{t+1} \in \mathsf{S}(f_t)} \frac{\mu(f_{t+1})}{\mu(f_t)} q_{s f_{t+1}},$$

with the same special treatment for $t = T - 1$, which says that, at contingency f_t, the conditional expected price next time of security s (or the conditional expected dividend it will pay, in the special case $t = T - 1$), is equal to its current price. That is, μ turns the price process of s into a martingale.

So, if there is some $\pi \in (\mathcal{S}, q)$, which is true if and only if $\mathcal{S}$ and q are viable, then there is an associated probability distribution μ under which every security is a martingale. And, if you run all the arguments we've given backwards (or use formula

(16.4) and, in particular, Lemma 16.11b, the converse emerges: If μ is a full-support probability on Ω that turns each security's price process (with the dividend at time T) into a martingale, then π_{f_t} equal to the probability of f_t (under that probability) is a strictly positive element of R^N that satisfies $\pi \cdot \xi = 0$ for every $\xi \in M(\mathcal{S}, q)$. So $\mathcal{S}$ *and* q *are viable (where* $\mathcal{S}$ *conforms to the assumptions of this problem) if and only if there is a (strictly positive) probability* μ *on* Ω *that makes each security price process into a martingale.*

And, moreover, for each $\pi \in (\mathcal{S}, q)$ normalized so that $\pi_{f_0} = 1$, there is a different "martingale measure" μ on Ω. So *the dimension of* $M(\mathcal{S}, q) = N - 1$, *which is to say that markets are complete, if and only if there exists a unique "martingale measure" for the given data.*

Although you weren't asked to show this, we can go even a step further: Suppose you are given a set of securities $\mathcal{S}$ with the properties of this problem and prices q for those securities, and you are asked: Suppose we add another security s^* to the set, which trades in every contingency and pays dividends only at date T. Can we put any bounds on what the equilibrium price of s^* might be, assuming that adding it leaves $\mathcal{S}$ and q unchanged? The answer is, The price of s^* in any contingency f_t must lie within the range of conditional expected values of its dividends, where we take conditional expectations with respect to all the "martingale measures" for $\mathcal{S}$ and q. In particular, if all those conditional expectations are the same, then s^* is "priced by arbitrage," meaning there is a trading plan involving the securities in $\mathcal{S}$ that precisely replicates what s^* does.

■ 16.13. To give a version of Proposition 16.15 that includes firms, we need to revise definitions a bit. We continue to suppose that there is a subspace $\mathcal{M}$ of R^{kN} of "traded bundles," and prices will be given by a linear functional $\pi : \mathcal{M} \to R$. Consumers will look at net trades from $\mathcal{M}$, and they will be assumed to be locally insatiable constrained to $\mathcal{M}$, as before. But we add: Firm f is characterized by a production-possibility set $Z^f \subseteq \mathcal{M}$ and by the desire, facing prices given by π, to maximize profit, given by $\pi(z)$ for $z \in Z^f$. Consumer h holds an s^{fh} share in firm f, where $\sum_h s^{fh} = 1$ for each f, and all the shares are nonnegative. An $\mathcal{M}$-constrained Walrasian equilibrium is a linear functional $\pi : \mathcal{M} \to R$, net trades ζ^h for each consumer, and production plans z^f for each firm, such that $\zeta^h \in \mathcal{M} \cap \mathcal{X}^h$ and ζ^h maximizes u^h over all $\hat{\zeta}^h$ that satisfy h's budget constraint $\pi(\hat{\zeta}^h) \leq \sum_f s^{fh}\pi(z^f)$, z^f maximizes $\pi(\hat{z}^f)$ over all $\hat{z}^f \in Z^f$, and markets clear: $\sum_h \zeta^h \leq \sum_f z^f$.

And then, in this expanded setting, we want to prove that if $(\pi, (\zeta^h)_h, (z^f)_f)$ is an $\mathcal{M}$-constrained Walrasian equilibrium for $\mathcal{M}$-constrained locally insatiable consumers and for π a nonnegative linear functional on $\mathcal{M}$, then no feasible $\mathcal{M}$-constrained consumption plan $(\hat{\zeta}^h)_h$ can Pareto-dominate (ζ^h), where feasibility means that $\hat{\zeta}^h \in \mathcal{M} \cup \mathcal{X}^h$ for each h, and there are feasible production plans $\hat{z}^f \in Z^f$, one for each firm, such that $\sum_h \hat{\zeta}^h \leq \sum_f \hat{z}^f$.

To prove this, suppose that $(\hat{\zeta}^h)$ is feasible, and the plans $(\hat{z}^f)$ are what make it so.

Then since each $\hat{\zeta}^h \in \mathcal{M}$ and each $\hat{z}^f \in \mathcal{M}$, so is $\sum_h \hat{\zeta}^h - \sum_f \hat{z}^f$. Feasibility ensures that $\sum_h \hat{\zeta}^h - \sum_f \hat{z}^f \leq 0$. So, since π is nonneative,

$$0 \geq \pi\left(\sum_h \hat{\zeta}^h - \sum_f \hat{z}^f\right) = \sum_h \pi(\hat{\zeta}^f) - \sum_f \pi(\hat{z}^f), \quad \text{or} \quad \sum_h \pi(\hat{\zeta}^h) \leq \sum_f \pi(\hat{z}^f).$$

Of course, since z^f maximizes firm z's profit at the prices π, $\pi(\hat{z}^f) \leq \pi(z^f)$ for each f, and so the last inequality in the display implies

$$\sum_h \pi(\hat{\zeta}^h) \leq \sum_f \pi(z^f). \tag{$\star$}$$

Now suppose that $(\zeta^h)_h$ is Pareto dominated by $(\hat{\zeta}^h)$. Since each consumer h is $\mathcal{M}$-constrained locally insatiable, we know that each h spends all her wealth in equilibrium. (If she didn't, she could find some other $\zeta' \in \mathcal{M} \cup \mathcal{X}$ arbitrarily close to ζ^h—close enough to still obey her budget constraint, which is strictly better for her than ζ^h.) So, for each h, $\pi(\zeta^h) = \sum_f s^{fh}\pi(z^f)$. And if $(\hat{\zeta}^h)_h$ Pareto dominates (ζ^h), then (1) by local insatiability again, we know that $\pi(\hat{\zeta}^h) \geq \pi(\zeta^h)$ for each h, and (2) for any h (and there must be one) such that $\hat{\zeta}^h$ is strictly preferred to ζ^h, $\pi(\hat{\zeta}^h) > \pi(\zeta^h)$. So summing over h,

$$\sum_h \pi(\hat{\zeta}^h) > \sum_h \pi(\zeta^h) = \sum_h \sum_f s^{fh}\pi(z^f) = \sum_f \pi(z^f)\left[\sum_h s^{fh}\right] = \sum_f \pi(z^f).$$

This directly contradicts ($\star$); a feasible $(\hat{\zeta}^h)_h$ cannot Pareto-dominate $(\zeta^h)_h$.

▪ 16.15. We are imagining an economy with consumers and firms. The description of consumers doesn't change from before, except that they will be endowed with initial shareholdings to be described later. There are also non-equity securities $\mathcal{S}$, just as before. For simplicity, I'll assume that these securities are financial; they pay dividends in units of account or numeraire, although it isn't hard to adapt this formulation to a case where all dividends (including those paid by firms) are paid in some single commodity whose equilibrium price is ensured to always be positive and that is chosen to be the numeraire in each contingency. The symbols x^h and $\boldsymbol{x}^h$ for consumption plans for consumer h, y^h and $\boldsymbol{y}^h$ for trading plans (in securities in h) for consumer h, r for prices in the spot markets, and q for prices for the securities in $\mathcal{S}$ continue to be used.

Firms are described by production-possibility sets $Z^f \subseteq R^{kN}$. (The double use of f for firms and for contingencies can't be avoided, unless we use a different letter for one or the other. I think it is less confusing to just go with the double use, so I do that.) If a firm chooses production plan z^f, it must *finance* this plan, which means it must issue dividends $d_f(f_t)$ and have a financial plan which involves holding a portfolio of the securities in $\mathcal{S}$ in each contingency f_t, where $w^f(f_t, s)$ will denote firm f's (post-) contingency f_t holdings of security s for $s \in \mathcal{S}$. Firm f's production, dividends, and financing plan (z^f, d_f, w^f) must satisfy two constraints: The firm's financial plan must respect any trading constraints on the securities $s \in \mathcal{S}$. And the firm must *balance its books* in every contigency, meaning that, given prices r and q, for each f_t,

$$r_{f_t} \cdot z^f_{f_t} + \sum_{s \in S} w^f(\hat{f}_t, s) d_s(f_t) = d_f(s_t) + \sum_{s \in S} q_{sf_t} \big[w^f(f_t, s) - w^f(\hat{f}_t, s) \big].$$

To translate: $r_{f_t} \cdot z^f_{f_t}$ is the net operating income of the firm, the market value of its outputs less the cost of inputs. Added to this are any dividends the firm might receive from the portfolio of securities it holds; of course, this is based on its holdings the contingency before, because we assume that securities trade *ex dividend*. On the right-hand side is the dividend this firm pays, plus the net cost to it of transactions it undertakes in the securities markets. Suppose, for instance, the firm is just starting out at f_0. It begins with no security holdings. And suppose that, at f_0, it must purchase inputs, so $r \cdot z < 0$. Then to achieve an equality in the "balance the books" equation for f_0, the firm must sell some securities, which, in real life, is called "borrowing some money" or "issuing debt," although as we noted in the text, without bankruptcy, debt is debt. A richer formulation could have it *float equity*, but I'll avoid that sort of thing in this answer. If you are courageous, add it in. Also, I am not allowing firms to trade in each other's equity or, even worse, to buy and sell its own equity or create securities not in $\mathcal{S}$. So long as markets are complete using $\mathcal{S}$ alone—which will be assumed—any security the firm created would not add trading possibilities, hence would be economically redundant. Of course, in real life, these things happen, and you can modify my formulation so that this is allowed. It is tedious but doable. But if $\mathcal{S}$ doesn't provide complete markets on its own, the difficulties you will encounter will be immensely greater.

We now add equity in each firm to the story. For each firm f, there is one more security added to the securities markets, equity in the firm. I'll use the symbol σ^f to represent a "share" of equity in the firm, where we normalize things so that there is only one share in the firm in existence. I'll assume that equity in a firm trades in every contingency f_t for $t < T$. Dividends are determined endogenously by firms, as part of their production, dividend, and financing plans. Only consumers (in this formulation) are allowed to hold equity in firms: A trading plan y^h has components $y^h(f_t, s)$ for $s \in S$ and $y^h(f_t, \sigma^f)$ for $f \in F$. Recall that $\hat{f}_0$ is used conventionally to describe the "predecessor" to f_0 for budget-equation purposes; when we write (say) $y^h(\hat{f}_0, s)$, we always mean 0, since this represents the endowment of consumer holdings of securities in $\mathcal{S}$. Now we continue to use $y^h(\hat{f}_0, \sigma^f)$, but these numbers are exogenously given, with the stipulation that they are always nonnegative and sum (over all consumers, for each firm) to 1. The consumer's budget constraints are changed to reflect these new securities, where $q_{\sigma^f f_t}$ is the equilibrium price of the (one) share of firm f's equity. (I reiterate, if you want to change my formulation to allow firms to float their own equity, go at it. But if you do this, since the value of a firm's equity will not always be zero, you will need to have "founders" or "entrepreneurs" in your model for each firm, who reap the benefits of the initial public offering. The nice thing is, the meta-theorem will still apply, as long as its rather strong assumptions are maintained.)

In this formulation, you must be careful about dividends paid by the firm. Since firms have shareholders from the start (at least, the way I've done this), you can allow the firm to pay dividends even at f_0, something we didn't allow for $s \in \mathcal{S}$. Also, the issue of negative dividends must be addressed. You can restrict the firm to pay only non-negative dividends. But if you do, you should probably assume that $0 \in Z^f$ for each firm, so you know (once you define profit, see below) that the firm can achieve zero profit. And this may make some level of participation in the securities markets a requirement, at least for some production plans. Imagine, for instance, a very productive z^f that, unhappily, requires significant inputs at time T to "clean up" (a legal requirement, say) after producing tons and tons of really valuable output at time $T-1$. To make this feasible with nonnegative dividends, the firm had better buy some securities at time $T-1$ that will allow it to finance the required operations at time T.

That completes the definition of the economy. (Whew!) The next step is to define an EPPPE. This, it turns out, is where things get murky. The problem, as described informally in the text, is: What is the profit of the firm? We need to have an objective function for the firm, but what is it?

To get to a formal statement of the meta-proposition, we duck this issue: We only define EPPPE with complete markets:

Definition G16.1. *A **complete-markets EPPPE** is an array* $(r, q, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{d}, \boldsymbol{w})$*, consisting of:*

a. consumption and trading plans $\boldsymbol{x}^h$ *and* $\boldsymbol{y}^h$ *for each consumer (where the trading plans include the possibility of trading in the equity of firms,*

b. production, dividend, and financing plans $\boldsymbol{z}^f$*,* $\boldsymbol{d}_f$*, and* $\boldsymbol{w}^f$ *for each firm,*

c. spot-market prices r_{f_t} *for the commodities in each* f_t*,*

d. prices q_{sf_t} *for the non-equity securities in* $\mathcal{S}$*, and*

e. prices $q_{\sigma^f f_t}$ *for the equity of each firm* f *in each contingency* f_t*,*

such that

f. the plans (consumption and trading) of each consumer respect the rules of trading in securities in $\mathcal{S}$ *and satisfy the consumer's budget constraints,*

g. *the plans (production, dividend, and financial) of each firm respect the rules of trading in securities in $\mathcal{S}$ and satisfy the firm's "balance the books" constraints in each f_t,*

h. *each consumer h maximizes her utility with $\boldsymbol{x}^h$, in comparison with any other $\hat{x}^h$ for which there is a trading plan $\hat{y}^h$ that meets all the constraints of part f,*

i. *the dimension of $M(\mathcal{S}, q)$ is $N-1$, (with $\pi \in R^N_{++}$ the unique vector in $(\mathcal{S}, q)$ normalized so that $\pi_{f_0} = 1$),*

j. *each firm f maximizes the value of its initial equity, $q_{\sigma^f f_0} + \boldsymbol{d}_f(f_0)$, as determined by its production, dividend, and financing plan $\boldsymbol{z}^f$, $\boldsymbol{d}_f$, and $\boldsymbol{w}^f$, in comparison with any other set of plans $\hat{z}^f$, $\hat{d}_f$, and $\hat{w}^f$ that together satisfy the constraints imposed in part g, where the firm asseses the value of its initial equity under the alternative, hat plan, as*

$$\sum_{f_t \in \mathcal{F}} \hat{d}_f(f_t)\pi_{f_t}, \quad \textit{and}$$

k. *all markets clear: with inequalities in the spot-commodity markets, with equality (for a net supply of zero) in the markets for securities in $\mathcal{S}$, and with equality (for a net supply of 1) in the markets for each firm's equity.*

The key to this definition is part j, which provides the objective function for the firm: it seeks to maximize the *initial value of its equity*, taking all prices, and in particular the implicit pricing vector π, as given. It may not be one-hundred percent clear that $q_{\sigma^f f_0} + d_f(f_0)$ is equal to $\sum_{f_t \in \mathcal{F}} \boldsymbol{d}_f(f_t)\pi_{f_t}$, so this needs to proved. (On the other hand, if you understood the proof of Lemma 16.11, this may be very clear.) It should be noted that the term *the value of [firm f's] initial equity* we mean the price the equity commands in the f_0 market plus any dividend the firm might choose to pay to its initial shareholders at f_0. (Securities are priced *ex dividend*, so the value of this dividend is not impounded in $q_{\sigma^f f_0}$.) Why did we choose this as the objective of the firm? We have the following result:

Proposition G16.2. *Suppose that $(r, q, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{d}, \boldsymbol{w})$ is a complete-markets EPPPE.*

a. *For each firm f,*

$$\boldsymbol{d}_f(f_0) + q_{\sigma^f f_t} = \sum_{f_t \in \mathcal{F}} \pi_{f_t} r_{f_t} \cdot \boldsymbol{z}^f_{f_t}.$$

b. For each f_t,

$$q_{\sigma^f f_t} = \frac{1}{\pi_{f_t}} \sum_{f'_{t'} \in S(f_t)} \pi_{f'_{t'}} d_f(f'_{t'}).$$

c. Suppose that any firm f *contemplates shifting its dividend and financing structure from* $\boldsymbol{d}_f$ *and* $\boldsymbol{w}^f$ *to some alternative plan* $\hat{d}_f$ *and* $\hat{w}^f$ *that, with* $\boldsymbol{z}^f$, *continues to satisfy the constraints in part j of the definition. Then* $(r, \hat{q}, \boldsymbol{x}, \hat{y}, \boldsymbol{z}, \hat{d}, \hat{w})$ *is a complete-markets EPPPE, where* $\hat{q}$ *is the same as the initial* q, *except for some possible repricing of the equity of this firm* f, $\hat{y}$ *involves adjusting the trading plans of consumers to accomodate the net dividend structure of firm* f *and its new financing plan,* $\hat{d}$ *is the same as* $\boldsymbol{d}$ *except for the one firm* f, *which shifts to* $\hat{d}^f$, *and* $\hat{w}$ *is the same as* $\boldsymbol{w}$, *except that firm* f *shifts to* $\hat{w}^f$.

To explain: Part a says that the value of initial equity is the discounted sum of contingency-by-contingency net operating profits of the firm. So, when we said that the objective of the firm is to maximize the value of its initial equity, we could just as well have defined its "profit" as the discounted sum of these net operating profits (where the discount factors π_{f_t} are driven by equilibrium prices for the securities in $\mathcal{S}$) and said that this is what firms maximize. Part b confirms that the formula used by the firm to evaluate alternatives to the plan it chooses is the formula that determines the value of its initial equity. As for part c, that is the formal statement of the famous Modgliani–Miller Theorem—-that the capital structure of a firm (what combination of debt and equity it uses to finance its operations) is irrelevant in perfect (or frictionless) markets, because investor/consumers can always use other markets to "undo" what the firm does in setting up its capital structure. Of course, the *capital structure* of a firm will affect the price of its equity (although *not* the value of its initial equity—that's what part a tells us); a firm that pays a big dividend in some contingency will have a lower value of equity (*ex dividend*) than a firm that takes the money that would have been used to pay that dividend and invests it for a while in securities from $\mathcal{S}$.

The proof of this proposition is one more exercise in accounting and arbitrage arguments. For part a, you have to construct an argument somewhat similar to the proof of Lemma 16.11b, where you show that the operating profit accrued in some contingency f_t has to reach shareholders (at a market-driven, fair rate of return) at some point in time. Part b follows the proof of Lemma 16.11a precisely. As for part c, the only difficult part is to show that, if one firm changes its dividend and financing plan, consumers can take the opposite side of its transactions without affecting the wealth they have in each contingency to purchase real commodities.

And, with those hints, I leave the details to you. Next comes

Proposition G16.3. *Suppose that* $(r, q, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{d}, \boldsymbol{w})$ *is a complete-markets EPPPE. Then there is a Walrasian equilibrium* $(p, \boldsymbol{x}, \boldsymbol{z})$ *for the all-markets-at-once economy (where shareholdings in the latter are the initial shareholdings in the dynamic economy). And, conversely, if*

$(p, \boldsymbol{x}, \boldsymbol{z})$ is a Walrasian equilibrium for the all-markets-at-once economy and if $\mathcal{S}$ is rich enough so that $M(\mathcal{S}, q)$ has dimension $N-1$, where q is defined for securities in S by formula (16.4), for some strictly positive $\pi \in R^N_{++}$, then there is a complete-markets EPPPE in which firms produce according to $\boldsymbol{z}$ and consumers consume according to $\boldsymbol{x}$.

This is another exercise in bookkeeping, which I leave for you.

There is one final point to address: In the text, I said something about how, in this sort of complete-markets EPPPE, shareholders unanimously prefer the production plan chosen by the firm over any other feasible plan (taking prices as given), and—subject to a mysterious caveat—this continues to be true of subsequent shareholders.

I can finally explain the mysterious caveat. In contingency f_t (say), the firm will be committed to having done certain things. It will have undertaken certain production activities. It will be holding a portfolio of securities and have paid dividends. None of this can be undone, and when I say that current shareholders prefer that the firm carry out its plans going forward, I mean, taking all those past actions as given; no time machines are available to undo what has been done. But if we allow any specification of $Z^f \subseteq R^{kN}$, the possibility exists that what the firm is capable of depends on what it would have done in contingencies now known to be impossible. Go back to the discussion in the text of the consumer whose marginal utility for consuming wheat in one contingency depends on what she would have consumed in another contingency that is now known not to be happening; this is just the "production" equivalent. For production technologies of this sort, shareholders at time t don't need a time-machine to say things like, "The firm planned to undertake $z^f_{f'_t}$ if today's contingency was f'_t instead of f_t, perhaps because this would have generated big operating profits and, hence, dividends. But we know f'_t isn't going to happen, and if the firm shifts what it would have done, Z^f is constructed so that it can do more (make more operating profits) today and going forward. So we like that shift."

This phenomenon, which is possible in theory, will prevent the "shareholders continue to unanimously prefer the original plan" from being true. What is needed is a restriction on feasible production technologies that rules this out; essentially, that says that things the firm might have done in a different state of nature doesn't constrain what it can do today and going forward. Only the past that was constrains the firm going forward; not the present and future that will never be. Formalized, this means some "state-by-state separability" in the construction of Z^f; if that separability is present, then the "shareholders continue to prefer…" result can be obtained. (You are invited to do so.)

Errata in the Text

Here is a list of errata found (so far) in the text. Please note: Princeton University Press is kindly allowing me to correct a number of these in later printings of the book. Assuming you have a physical copy of the book, look at page 122. If there is a Problem 5.7 on this page, you hold a copy of the book with a number of corrections made; if there is no Problem 5.7, you have an early printing, which does not have these corrections.[1] Below, I list corrections as follows: First are major flaws that were found after the corrections were made (and so are germane to all readers); then are minor flaws found after the corrections were made; then major flaws that appear (only) in early printings; and finally minor flaws found in early printings. (For the moment, there are no entries in the first two categories. But I'm relatively sure that some will appear.)

If you happen to find a typo (or something worse) that is not included in this list, I'd be grateful if you would email it to me, at kreps@stanford.edu.

References to line numbers mean lines of text, including any displays as a single line and any headers as a line. A reference, say, to line -10 means line 10 from the bottom.

I interpret the term "flaw" broadly, here, including errors of omission as well as of commission.

If you have an ebook version of the text (for instance, a Kindle edition): I know of at least one typo in the Kindle edition that is not in the paper version of the text; I assume there are others. I will not attempt to keep a record of all such "extra" typos here, but if you have an ebook version and you believe you have found a typo not on this list (or if you are just confused by what you see), you should check with a paper version of the book. Sorry for the added work, but as I write this, the publisher is as mystified as am I about how this could happen.

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Major flaws in all printings

(None so far)

Minor flaws in all printings

Page 44 first full paragraph: This is a flaw, but not easy to fix. In this praragraph, I use a lower case u for the utility function, where the argument is (x, m). The use of lower case for the utility function is pretty standard throughout the chapter, so that's why I did it. However, Definition 2.15 uses an upper case U for utility functions over bundles (x, m) and a lower case u for the sub-utility function over x. So, perhaps, in this paragraph, all those lower case u's would be better as upper case U's. (Best of

[1] If you have an ebook version of the book, the same test applies. But see following for a caveat.

all would be to have as the display in Definition $u(x, m) = v(x) + m$, for some sub-utility function v. But that change, made in all the places subsequently that would be required, would probably be more confusing than helpful. So, in this paragraph on page 44, use either u.c. or l.c. as you find more consistent.

Major flaws in early printings, corrected in later printings

There are seven of these, including some absolute howlers:

Page 35, as well as elsewhere in Chapter 2: There is nothing wrong with what the text says. But the text omits a subtle but substantial point. Line 13 proposes the conjecture that *if preferences $\succeq$ are convex, they admit* ***at least one*** *concave numerical representation.* The text correctly says that this conjecture is not true and, in fact, Problem 2.8 gives two examples (and challenges you to find an example involving preferences that are both convex and continuous, a much harder task). But there is a counterexample to the conjecture one can give that is a good deal simpler than the two counterexamples in Problem 2.8, namely lexicographic preferences, as defined in Problem 1.10. These preferences (extended, say, to all of R^2_+) are both strictly monotone—easy to see—and convex—maybe a bit harder to see—but they cannot be represented by *any* utility function, let alone one that is concave or strictly increasing.

This illustrates a larger issue of omission in the chapter. Contrast the two statements *If preferences $\succeq$ have property X, then* ***every*** *utility function u that represents $\succeq$ has property Y* and *If preferences $\succeq$ have property X, then* ***some*** *utility function u that represents $\succeq$ has property Y.* It is somewhat natural to feel—and the chapter may give you the feeling—that the first statement is stronger than the second; that is, whenever the first is true, the second is as well. But if you get that feeling, be careful! The first statement can be true because there is *no* utility function u that represents $\succeq$, hence every u has the property Y by default. But the second implicitly guarantees that $\succeq$ has at least one representation. I believe that, in this regard, the text never says anything that is wrong. But it sure doesn't do justice to this somewhat subtle point.

In later printings, to make this point, I add a part b to Problem 2.1: Consider lexicographic preferences as defined in Problem 1.10. Which of the properties listed in the left-hand column of Table 2.1 are satisfied by lexicographic preferences? How do your answers to this question square with (a) what you answer in the second-to-left column of the table and (b) the undisputed fact that lexicographic preferences admit no numerical (utility) representation? (This is discussed in the *Student's Guide* if you can't figure it out on your own.)

While I'm on this point, let me add: In Table 2.1, the third row from the bottom concerns strict separability. In filling in what strict separability implies in terms of numerical representations, note that the relevant proposition assumes strict separability *and more*. The answer given in the *Student's Guide* assumes the relevant *and more*.

Page 99 Proposition 5.10: The text reads "If preferences ... are continuous in the weak topology ...." Since I'm not defining the topology here, this is at best ambiguous; readers well-trained in mathematics will assert that it somewhere between an abuse of terminology and just wrong. It should say "If preferences ... are continuous in the topology of weak convergence [of probability measures]...," (which is in fact a weak* topology). I fix this in later printings, adding to this chapter Problem 5.7, which asks you to prove the proposition (and which supplies relevant facts about the topology of weak convergence). You can learn about this, including seeing (a sketch of) the proof of the proposition, by going to the final bits of the *Guide* Chapter 5.

Pages 248 to 249: The motivational paragraph that starts on the bottom of page 248 is, at best, misleading. It seems to say that for u to be locally insatiable, e should be strictly increasing. But according to Proposition 10.4 and its proof, e is strictly increasing whether u is locally insatiable or not. Both Proposition 10.3 and, more importantly, the proof of local insatiability inside the proof of Proposition 10.16 (page 251) make clear that local insatiability of u is related to *continuity* of e *jointly* in p and v. So, the paragraph at the bottom of page 248 makes a lot more sense if it reads "Besides this, e should be concave and homogenous of degree 1 in p. It should be nondecreasing in p, strictly increasing in v, and unbounded in v for each p. And, if we want u to be locally insatiable, it should be continuous jointly in p and v."

Page 354 Proposition 14.14: This proposition (as stated in early printings) is quite wrong. It has two problems, one easily fixed (a matter of normalizing prices) and one much less so. Because the issues raised are substantial, I have moved the solution of Problem 14.11 (which asks you to prove the proposition) to the *Student's Guide*. If you go to the solution of this problem in the *Guide* Chapter 14, a lengthy explanation is provided.

Page 468, Proposition A3.28: Pure "think-o." If the Hessian is negative definite, the function is strictly concave. But, as illustrated by the strictly concave function $f(x) = -x^4$, the Hessian of a strictly concave function can be (only) negative semi-definite in places. So please, in the proposition, change "strictly concave if and only if" to "strictly concave if," both times the expression appears.

Page 451 bottom: (A very grievous error!) This only works if the a_n are nonnegative, in which case the absolute value signs in the condition that $\sum_n |a_n| < \infty$ are superfluous.

Page 476, especially the statement of Berge's Theorem and footnote 4: I thought that I make enough assumptions to rule out $A(\theta) = \emptyset$, but I haven't. Suppose the domain of θ is R_+, $F(z, \theta) \equiv 0$, and $A(\theta) = \{1/\theta\}$ for $\theta > 0$ and $= \emptyset$ for $\theta = 0$. Of course, $Z(\theta)$ must be identically $A(\theta)$, and if $B(\theta) \equiv A(\theta)$, I believe all the stated assumptions hold. But the conclusions of the theorem do not hold. I see no way around adding the assumption that A is nonempty valued, in which case everything is fine. So that assumption is added in corrected versions, and you should add it here.

Once this correction is made, the proof offered is fine. But students have found the argument given in the proof to be hard to follow, and so I have rewritten the proof in a manner that, I hope, is clearer. Because Berge's theorem is so important to developments in the book, at the very end of this Errata document, I reproduce both the corrected statement of the Theorem and the new (and, I hope, clearer) proof.

Minor flaws in early printings

Page 8: line 15 should read "there are at least...; line 17 "abd" should be "and"; line 23 should not have a close parentheses after c before the full stop.

Page 20 line 8: "strictly preferred X to Y" should be "strictly preferred Y to X"

Page 35 line 2: "Proposition 1.19" should be "Proposition 1.20"

Page 40 first line of Proposition 2.12: Insert "are" so that it reads "Preferences $\succeq$ are weakly..."

Page 40 third line of Proposition 2.12: The fourth subscript + sign should subscript R and not the superscript of R

Page 40 display near the bottom of the page: $u(x^K)$ should be $u(x_K)$

Page 54 line 5: b should be p

Page 57 line 23 (the second line in ***Step 1*****:** The subscript i on μ should be a j.

Page 70 statement of Afriat's Theorem: I remind you (per footnote 2 on page 31) that as the book progresses, I become quite sloppy, using *increasing* for preferences instead of *monotone*. Here is an example (the first?).

Page 92, line -17: "not" should be "now"

Page 97, line 17: The end of the line should read "some function $U : X \to R$..." That is, the domain of U is X and not Z.

Page 101, line 4 after Section 5.3 header: Eliminate "we will"

Page 118 statement of Problem 5.4: It should read "With regard to the 'run the horses second' discussion beginning on the bottom of page 105..."

Page 142 Problem 6.7: You should assume that x_1 and x_2 are constrained to be nonnegative. And, in the middle of the second paragraph, c_1 should be x_1.

Page 182 third full paragraph: "social welfare functionals" should be "social utility functionals," twice.

Page 205 line 19 to 20: It should read "Let i be any component of z^* such that..."

Page 213 the display in Definition 9.15: The superscripts on the αs in the first line of the display should be subscripts

Page 220 line 5: Delete the word "is"

Page 226, lines 2 to 3: Given the concavity of V, I'm not certain I need this, but to be safe: Put the word "continuously" just before "differentiable," so it reads "...if this (vector-valued) function is continuously differentiable..."

Page 227 first line of Proposition 9.25: "Suppose y and y', both in Y, ..."

Page 250: (I seem to have been asleep when proof-reading this chapter. Lots of small stuff:) Line 9 should say "...Fix v' and v with $v' < v$ ..." Two lines further on, "...such the..." should be "...such that..." And, three lines from the bottom of the page, (10.5) should say (10.6). Skipping to page 253, in the third line after the italicized portion, (10.6) should be (10.7). And on page 254, in the statement of Proposition 10.18, y should be in R_+, not R_+^k.

Page 269 line -2 and line -1, and page 270, line 3: In all three cases, the suprema in y should be over $y \in R_+$ and not $y \in R_+^k$.

Page 270, proof of Proposition 11.5, line 6 of the proof: "...such that $v \geq u(x^i)$..." should be "...such that $v \geq u'(x^i)$..." And four lines further on, the v^0 should be v (no superscript 0).

Page 274 statement of Proposition 11.9: This isn't a typo, but to be clear, the (11.7) in the last line of the Proposition refers to Condition (11.7) and not Proposition 11.7.

Page 325 line 9: The u.c. T should be a l.c. t.

Page 328 line -15: "is it" should be "it is".

Page 334 line 17: u^i at the very start of the line should be u^h.

Page 339, Definition 14.7, 2nd line: it should read $\ell = 1, \ldots, n$ and not $\ldots, N$.

Page 340 line 3: "...then the only choice available to 1 is H."

Page 341, 2nd from last line in the proof: $\prod_{\ell=1}^n A_i$ should be $\prod_{\ell=1}^n A_\ell$.

Page 344 first few lines: all p_n should be p^n.

Page 349 3rd line before the endproof mark: change p_j to ζ_j.

Page 397 the second displayed equation: On the r.h.s., a p_{f_t} is mistakenly written p_{ft}

Page 405 second line after the display in part h: A q_{sf_t} is mistakenly written q_{sf_f}

Page 405 very last line: The last square bracket in this display is superfluous.

Page 422 very last line: near the end of the line, $p(\zeta^h)$ should be $\boldsymbol{p}(\zeta^h)$.

Page 454, statement of Carathédory's Theorem: Insert "be" between "can" and "written."

Page 454 display in the middle of the page: Change the ℓ in the summation to i

Page 475 line 6 in the proof: l.c. y should be an u.c. Y

Page 482 line 3: "pari pasu" should be "pari passu"

Page 489 line 10: techical should be technical

Page 493 line 2: "Kolmogov's" should be "Kolmogorov's."

Page 543 Arrow (1953): It should be the *Review of Economic Studies*

Correct statement of Berge's Theorem and the new proof:

Proposition A4.7 (Berge's Theorem, also known as the Theorem of the Maximum). *Consider the parametric constrained-maximization problem*

$$\textit{Maximize}\ \ F(z,\theta),\ \textit{subject to}\ z \in A(\theta).$$

Let $Z(\theta)$ *be the set of solutions of this problem for the parameter* θ*, and let* $f(\theta) = sup\ \{F(z,\theta); z \in A(\theta)\}$*. If*

a. F *is a continuous function in* (z,θ)*,*

b. $\theta \Rightarrow A(\theta)$ *is lower semi-continuous and nonempty valued (that is,* $A(\theta) \neq \emptyset$ *for all* θ*), and*

c. *there exists for each* θ *a set* $B(\theta) \subseteq A(\theta)$ *such that* $Z(\theta) \subseteq B(\theta)$*,* $sup\{F(z,\theta) : z \in B(\theta)\} = sup\{F(z,\theta) : z \in A(\theta)\}$*, and* $\theta \Rightarrow B(\theta)$ *is an upper semi-continuous and locally bounded correspondence.*

Then:

d. $Z(\theta)$ *is nonempty for all* θ*, and* $\theta \Rightarrow Z(\theta)$ *is an upper semi-continuous and locally bounded correspondence; and*

e. *the function* $\theta \to f(\theta)$ *is continuous.*

Identical conclusions hold if the optimization problem calls for minimizing F *rather than maximizing* F*.*

Before giving the proof of Berge's Theorem, we give a corollary that shows why we were interested last section in singleton-valued correspondences.

Corollary A4.8. *In the situation of Proposition A4.7, if in addition you know that* $Z(\theta)$ *is a singleton set* $\{z(\theta)\}$ *for all* θ *in some (relatively) open set of parameter values, then* $z(\theta)$ *is a continuous function over that set of parameter values.*

Proof of the corollary. Berge's Theorem establishes that the solution correspondence Z is upper semi-continuous and locally bounded. Apply Proposition A4.6. ∎

In standard statements of this theorem, $B(\theta)$ doesn't appear; it is assumed that $A(\theta)$ is continuous. But the more general version of the result given here permits smoother application of the result in some cases encountered in the text.

Proof of Berge's Theorem. Since B is upper semi-continuous and locally bounded, it is compact valued. Since $A(\theta)$ is nonempty valued, $\sup\{F(z,\theta) : z \in A(\theta)\}$ is either finite or $+\infty$; since $\sup\{F(z,\theta) : z \in A(\theta)\} = \sup\{F(z,\theta) : z \in B(\theta)\}$ (condition c), we conclude that $B(\theta)$ is nonempty for each θ. Nonemptiness of $Z(\theta)$ for all θ then follows from Proposition A2.21, because F is continuous in z and $B(\theta)$ is nonempty and compact for each θ. This also implies that $f(\theta) < \infty$ for all θ.[2]

To establish continuity of f, suppose $\{\theta_n\}$ is a sequence of parameter values with limit θ. Let z be any solution to the problem at θ, so that $f(\theta) = F(z,\theta)$. Since A is lower semi-continuous, we can find $z_n \in A(\theta_n)$ for each n such that $\lim_n z_n = z$. But then, by continuity of F, $\lim_n F(z_n,\theta_n) = F(z,\theta) = f(\theta)$. And since $f(\theta_n) \geq F(z_n,\theta_n)$ for each n, $\liminf_n f(\theta_n) \geq f(\theta)$; f is lower semi-continuous at θ. Hence, continuity of f at θ can fail only if, for some sequence $\{\theta_n\}$ converging to θ and some $\epsilon > 0$, $f(\theta_n) > f(\theta) + \epsilon$ for all n. Suppose such a sequence exists; choose $z_n \in Z(\theta_n)$ (hence $f(\theta_n) = F(z_n,\theta_n)$) for each n. $Z(\theta_n) \subseteq B(\theta_n)$ implies $z_n \in B(\theta_n)$ for each n. Since B is locally bounded and $\theta_n \to \theta$, we can extract from the sequence $\{z_n\}$ a convergent subsequence $\{z_{n'}\}$, and since B is upper semi-continuous, the limit z of this convergent subsequence is in $B(\theta) \subseteq A(\theta)$. But then $f(\theta) \geq F(z,\theta) = \lim_n F(z_n,\theta_n)$ (by continuity of F) $= \lim_n f(\theta_n)$, contradicting the alleged hypothesis. The function f is continuous.

By assumption, B is locally bounded and $Z(\theta) \subseteq B(\theta)$, hence Z is locally bounded. Suppose $\{\theta_n\}$ and $\{z_n\}$ are sequences of parameters and variables such that: $\lim_n \theta_n = \theta$; $\lim_n z_n = z$; and $z_n \in Z(\theta_n)$, so that $f(\theta_n) = F(z_n,\theta_n)$, for each n. Continuity of F (assumed) and f (just proved) then tells us that $F(z,\theta) = \lim_n F(z_n,\theta_n) = \lim_n f(\theta_n) = f(\theta)$. Since $z_n \in Z(\theta_n) \subseteq B(\theta_n)$ and $\theta \Rightarrow B(\theta)$ is upper semi-continuous, $z \in B(\theta) \subseteq A(\theta)$, hence $z \in Z(\theta)$. We conclude that Z is upper semi-continuous and locally bounded. ■

[2] Suppose $A(\theta) = \emptyset$ for some θ is permitted, with the convention that $\sup\{F(z,\theta) : z \in A(\theta)\} = -\infty$ when $A(\theta) = \emptyset$. In particular, suppose the domain of θ is R_+, $F(z,\theta) \equiv 0$, and $A(\theta) = \{1/\theta\}$ for all $\theta > 0$ and $= \emptyset$ for $\theta = 0$. For $A \equiv B$, do the rest of the conditions (besides assuming that A is nonempty valued) of Berge's Theorem hold? Does the conclusion hold?

About the "Envelope Theorem"

I have been taken to task by some colleagues because no where in the book do I give the general "Envelope Theorem." Applications of this "theorem" appear in several places, in Propositions 9.22, 10.6, and 11.1. But the general "theorem" is never discussed.

The scare quotes in the previous paragraph signal my excuse: As far as I can tell, there is no single, official result that can be called "The General Envelope Theorem," but instead a number of more or less general results that are given that title. In spirit, they are all of the following form: In the parametric optimization problem

$$\text{Maximize (or minimize) } F(x,\theta) \text{ over } x \in A(\theta),$$

if we write $f(\theta)$ as the maximized value, then under conditions $\ldots$, the function f is differentiable in θ and its derivative is $\ldots$ (involving the partial derivative of F in θ at the optimal value of x for θ.) As you might imagine from the three propositions listed, assumptions about the uniqueness of the solution to the problem at θ will be involved, and the precise connection between the derivative of f and the partial derivative of F will depend on whether and how the constraint set $A(\theta)$ changes with θ. Also, to deal with situations in which the solution to the problem is not unique for all θ but is unique for "most" of them, variations are given in "integrated" form.

Hence, there are a lot of variations.

If and when Volume II is done, I'll need to confront at least some of these variations, as the general result is employed in mechanism-design problems. But, in the meantime, in case you were feeling bereft of a general result of this type, here is a result that is a bit more complex than is the situation in Propositions 9.22 and 10.6 but still without confronting some of the difficulties in Proposition 11.1. (The objective function is pretty general, but the constraint set is constant.) You should be able to prove this on your own after you've consumed the proofs of the three propositions—it is good practice—but in case you need some help, I give the proof.

One variation on "The Envelope Theorem." *Consider the problem*

$$\textit{Maximize } F(x,\theta) \textit{ in } x, \textit{ for fixed } \theta \in \Theta, \textit{ subject to } x \in X.$$

Assume that $F : X \times \Theta \to R$, where $X \subseteq R^n$ is compact and $\Theta \subseteq R^m$ is open. Let $f(\theta)$ be the maximized value; that is,

$$f(\theta) := sup\,\{F(x,\theta); x \in X\}, \textit{ for each } \theta \in \Theta$$

Assume that F is jointly continuous in x and θ and that F is continuously differentiable in θ for each x; assume further that the gradient of F with respect to θ is jointly continuous in x and θ. Then if, for a given θ^0, there is a unique solution to the problem x^0, f is differentiable at θ^0, and

$$\left.\frac{\partial f}{\partial \theta_i}\right|_{(\theta^0)} = \left.\frac{\partial F}{\partial \theta_i}\right|_{(x^0,\theta^0)}.$$

Proof. The first step is to apply Berge's Theorem. This is a parametric optimization problem. Moreover, it is a very simple parametric optimization problem: The objective function is continuous, and the constraint set is compact and doesn't change with the parameter. Hence we know that Berge's Theorem applies; the optimal value function is continuous in θ, and the solution set correspondence is upper semi-continuous (and locally bounded, but since X is compact, that's superfluous).

Moreover, if at θ^0 there is a unique solution x^0, then if x^n is *any* solution at θ^n for a sequence $\{\theta^n\}$ that approaches θ^0, upper semi-continuity of the solution correpondence tells us that $x^n \to x^0$.

Now examine what we want to show: If $\{\theta^n\}$ is a sequence with limit θ^0, we wish to show that

$$\lim_{n\to\infty} \frac{f(\theta^n) - f(\theta^0) - \nabla_\theta F(x^0,\theta^0)\cdot(\theta^n - \theta^0)}{\|\theta^n - \theta^0\|} = 0,$$

where $\nabla_\theta F(x^0,\theta^0)$ is the gradient vector of F in θ evaluated at (x^0,θ^0).

Let x^n be any solution at θ^n. We know that

$$f(\theta^n) = F(x^n,\theta^n) \geq F(x^0,\theta^n) \quad \text{and} \quad f(\theta^0) = F(x^0,\theta^0) \geq F(x^n,\theta^0).$$

Therefore, for each n,

$$f(\theta^n) - f(\theta^0) - \nabla_\theta F(x^0,\theta^0)\cdot(\theta^n - \theta^0) = F(x^n,\theta^n) - F(x^0,\theta^0) - \nabla_\theta F(x^0,\theta^0)\cdot(\theta^n - \theta^0)$$

and therefore

$$\begin{aligned}
&F(x^n,\theta^n) - F(x^n,\theta^0) - \nabla_\theta F(x^0,\theta^0)\cdot(\theta^n - \theta^0) \geq \\
&f(\theta^n) - f(\theta^0) - \nabla_\theta F(x^0,\theta^0)\cdot(\theta^n - \theta^0) \geq \\
&F(x^0,\theta^n) - F(x^0,\theta^0) - \nabla_\theta F(x^0,\theta^0)\cdot(\theta^n - \theta^0).
\end{aligned}$$

So we have the desired result if we can show that the limit of the terms on the left- and right-hand side of the string of inequalities just given, when divided by $\|\theta^n - \theta^0\|$, are both zero.

Take the left-hand side first. By the exact form of Taylor's Theorem (which is just the mean-value theorem),

$$F(x^n, \theta^n) - F(x^n, \theta^0) = \nabla_\theta F(x^n, \hat{\theta}^n) \cdot (\theta^n - \theta^0),$$

where $\hat{\theta}^n$ is some convex combination of θ^n and θ^0. Hence,

$$F(x^n, \theta^n) - F(x^n, \theta^n) - \nabla_\theta F(x^0, \theta^0) \cdot (\theta^n - \theta^0) = \left[\nabla_\theta F(x^n, \hat{\theta}^n) - \nabla_\theta F(x^0, \theta^0)\right] \cdot (\theta^n - \theta^0).$$

Of course,

$$\left|\left[\nabla_\theta F(x^n, \hat{\theta}^n) - \nabla_\theta F(x^0, \theta^0)\right] \cdot (\theta^n - \theta^0)\right| \leq \left\|\nabla_\theta F(x^n, \hat{\theta}^n) - \nabla_\theta F(x^0, \theta^0)\right\| \times \|\theta^n - \theta^0\|.$$

Hence

$$\lim_{n\to\infty} \left| \frac{F(x^n, \theta^n) - F(x^n, \theta^0) - \nabla_\theta F(x^0, \theta^0) \cdot (\theta^n - \theta^0)}{\|\theta^n - \theta^0\|} \right| \leq$$

$$\lim_{n\to\infty} \left| \frac{\left\|\nabla_\theta F(x^n, \hat{\theta}^n) - \nabla_\theta F(x^0, \theta^0)\right\| \times \|\theta^n - \theta^0\|}{\|\theta^n - \theta^0\|} \right| = \lim_{n\to\infty} \left\|\nabla_\theta F(x^n, \hat{\theta}^n) - \nabla_\theta F(x^0, \theta^0)\right\|.$$

But the last limit is zero, since $(x^n, \hat{\theta}^n) \to (x^0, \theta^0)$, and the gradient of F in θ is (by assumption) jointly continuous in x and θ. The other side is handled in entirely similar fashion, completing the proof.