概率统计知识复习

#统计 #计量 #重要

样本方差矩阵表示

向量可理解为特殊的矩阵。$\vec{i}$ 是一个其元素都为 1 的 n 维列向量，即 $\vec{i}=(1,1,\cdots\cdots,1)$

显而易见，$\sum_{i=1}^{n} x_i=\vec{i}'\cdot\vec{x}$，$\sum_{i=1}^{n} x_i^2=\vec{x}'\cdot\vec{x}$，

事实上 $\vec{i}'\vec{i}=n$ 即 $\dfrac{\vec{i}'\vec{i}}{n}=1$，

$$\vec{i}\vec{i}'=\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \vdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$\overline{x}=\frac{\sum_{i=1}^{n}x_i}{n}=\frac{\vec{\boldsymbol{i}}'\cdot\vec{\boldsymbol{x}}}{n}$$

$$\begin{pmatrix}\bar{x}\\ \vdots\\ \bar{x}\end{pmatrix}=\vec{i}\bar{x}=\vec{i}\cdot\frac{\vec{i}'\cdot\vec{x}}{n}=\frac{\vec{i}\vec{i}'\vec{x}}{n}$$

$$\overset{x-\bar{x}}{=}(\vec{x}-\vec{i}\bar{x})=\left(\vec{x}-\frac{\vec{i}\vec{i}'\vec{x}}{n}\right)=\left(I-\frac{\vec{i}\cdot\vec{i}'}{n}\right)\vec{x}\triangleq\boldsymbol{M}^{\mathbf{0}}\vec{x}$$

$$\begin{aligned}S^2 &= \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2=\frac{1}{n}(\vec{x}-\bar{x})'(\vec{x}-\bar{x})\\ &= \frac{1}{n}\vec{x}\cdot\boldsymbol{M}^{\mathbf{0}\prime}\boldsymbol{M}^{\mathbf{0}}\vec{x}=\frac{1}{n}\vec{x}\boldsymbol{M}^{\mathbf{0}^2}\vec{x}=\frac{1}{n}\vec{x}'\boldsymbol{M}^{\mathbf{0}}\vec{x}\end{aligned}$$

$$P(B)=\sum_{i=1}^{\infty}P\left(A_iB\right)=\sum_{i=1}^{\infty}P\left(A_i\right)\cdot P\left(B\mid A_i\right)$$

$$P\left(A_i \mid B\right)=\frac{P\left(A_i\right) P\left(B \mid A_i\right)}{\sum_{i=1}^{\infty} P\left(A_i\right) P\left(B \mid A_i\right)}$$

协方差矩阵 (Covariance Matrix)

$$\begin{aligned}
\operatorname{cov}(\boldsymbol{X}) & =\mathbb{E}\left[(X-\mathbb{E}[\boldsymbol{X}])(X-\mathbb{E}[\boldsymbol{X}])^{\prime}\right] \\
& =\mathbb{E}\left[\begin{array}{cccc}
\left(X_1-\mu_1\right)^2 & \left(X_1-\mu_1\right)\left(X_2-\mu_2\right) & \cdots & \left(X_1-\mu_1\right)\left(X_n-\mu_n\right) \\
\left(X_2-\mu_2\right)\left(X_1-\mu_1\right) & \left(X_2-\mu_2\right)^2 & \cdots & \left(X_2-\mu_2\right)\left(X_n-\mu_n\right) \\
\cdots & \cdots & \cdots & \cdots \\
\left(X_n-\mu_n\right)\left(X_1-\mu_1\right) & \left(X_n-\mu_n\left(X_2-\mu_2\right)\right. & \cdots & \left(X_n-\mu_n\right)^2
\end{array}\right] \\
& =\left[\begin{array}{cccc}
\operatorname{var}\left(X_1\right) & \operatorname{cov}\left(X_1, X_2\right) & \cdots & \operatorname{cov}\left(X_1, X_n\right) \\
\operatorname{cov}\left(X_2, X_1\right) & \operatorname{var}\left(X_2\right) & \cdots & \operatorname{cov}\left(X_2, X_n\right) \\
\cdots & \cdots & \cdots & \cdots \\
\operatorname{cov}\left(X_n, X_1\right) & \operatorname{cov}\left(X_n, X_1\right) & \cdots & \operatorname{var}\left(X_n\right)
\end{array}\right] \\
& =\left[\begin{array}{cccc}
\sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\
\sigma_{21} & \sigma_{22}^2 & \cdots & \sigma_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
\sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2
\end{array}\right]
\end{aligned}$$

$\boldsymbol{X}$ 的协方差矩阵常常记为 $\boldsymbol{\Sigma}_{\boldsymbol{x}}$，它是一个正定矩阵

构造一个变量 $\boldsymbol{Y}=\boldsymbol{a}^{\prime} \boldsymbol{X}$ 那么 Y 的方差 $\operatorname{Var}(\boldsymbol{Y})=\operatorname{Var}\left(\boldsymbol{a}^{\prime} \boldsymbol{X}\right)=\boldsymbol{a}^{\prime} \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{a} \geq 0$

Var(Y)=Var [E(Y | X)] + E [Varx(Y | X)]

$$\operatorname{Var}(Y) \quad=\quad \operatorname{Var}[\mathbb{E}(Y \mid X)]+\mathbb{E}\left[\operatorname{Var}_x(Y \mid X)\right]$$

方差分解公式

$$\operatorname{Var}(Y) \quad = \quad \operatorname{Var}[\mathbb{E}(Y \mid X)] + \mathbb{E}[\operatorname{Var}_x(Y \mid X)]$$

$$\text{回归方差} = \operatorname{Var}_x[\mathbb{E}[y \mid x]]$$

$$\text{残差方差} = \operatorname{Var}_x[var[y \mid x]]$$

随机变量的收敛性 (Convergence of the Random Variable)

a) 依分布收敛 (Convergence in Distribution)

分布函数弱收敛的讨论启发我们引进如下定义。

Definition 2.7 (依分布收敛) 设随机变量 ξ_n, ξ 的分布函数分别为 $F_n(x)$ 及 $F(x)$ 如果 $F_n(x) \xrightarrow{w} F(x)$ 则称 ξ_n 依分布收敛于 ξ，并记为 $\xi_n \xrightarrow{L} \xi$

b) 依概率收敛 (Convergence in Probability)

Definition 2.8 (依概率收敛) 如果 $\lim_{n\to\infty} P\{|\xi_n - \xi| \geq \varepsilon\} = 0$ 对任意的 $\varepsilon > 0$ 成立，则称 ξ_n 依概率收敛于 ξ，并记为 $\xi_n \xrightarrow{P} \xi$