The LM statistic has a useful form. Let $\hat{\mathbf{g}}_{iR}$ denote the ith term in the gradient of the log-likelihood function. Then,

$$\hat{\mathbf{g}}_R = \sum_{i=1}^{n} \hat{\mathbf{g}}_{iR} = \hat{\mathbf{G}}_R'\mathbf{i},$$

where $\hat{\mathbf{G}}_R$ is the $n \times K$ matrix with ith row equal to $\mathbf{g}_{iR}'$ and $\mathbf{i}$ is a column of 1s. If we use the BHHH (outer product of gradients) estimator in (17-18) to estimate the Hessian, then

$$[\hat{\mathbf{I}}(\hat{\boldsymbol{\theta}})]^{-1} = [\hat{\mathbf{G}}_R'\hat{\mathbf{G}}_R]^{-1}$$

and

$$\text{LM} = \mathbf{i}'\hat{\mathbf{G}}_R[\hat{\mathbf{G}}_R'\hat{\mathbf{G}}_R]^{-1}\hat{\mathbf{G}}_R'\mathbf{i}.$$

Now, since $\mathbf{i}'\mathbf{i}$ equals n, $\text{LM} = n(\mathbf{i}'\hat{\mathbf{G}}_R[\hat{\mathbf{G}}_R'\hat{\mathbf{G}}_R]^{-1}\hat{\mathbf{G}}_R'\mathbf{i}/n) = nR_{\mathbf{i}}^2$, which is n times the uncentered squared multiple correlation coefficient in a linear regression of a column of 1s on the derivatives of the log-likelihood function computed at the restricted estimator. We will encounter this result in various forms at several points in the book.

17.5.4 AN APPLICATION OF THE LIKELIHOOD BASED TEST PROCEDURES

Consider, again, the data in Example C.1. In Example 17.4, the parameter β in the model

$$f(y_i \mid x_i, \beta) = \frac{1}{\beta + x_i} e^{-y_i/(\beta + x_i)} \tag{17-29}$$

was estimated by maximum likelihood. For convenience, let $\beta_i = 1/(\beta + x_i)$. This exponential density is a restricted form of a more general gamma distribution,

$$f(y_i \mid x_i, \beta, \rho) = \frac{\beta_i^{\rho}}{\Gamma(\rho)} y_i^{\rho-1} e^{-y_i \beta_i}. \tag{17-30}$$

The restriction is $\rho = 1$.[9] We consider testing the hypothesis

$$H_0\colon \rho = 1 \quad \text{versus} \quad H_1\colon \rho \neq 1$$

using the various procedures described previously. The log-likelihood and its derivatives are

$$\ln L(\beta, \rho) = \rho \sum_{i=1}^{n} \ln \beta_i - n \ln \Gamma(\rho) + (\rho - 1) \sum_{i=1}^{n} \ln y_i - \sum_{i=1}^{n} y_i \beta_i,$$

$$\frac{\partial \ln L}{\partial \beta} = -\rho \sum_{i=1}^{n} \beta_i + \sum_{i=1}^{n} y_i \beta_i^2, \qquad \frac{\partial \ln L}{\partial \rho} = \sum_{i=1}^{n} \ln \beta_i - n\Psi(\rho) + \sum_{i=1}^{n} \ln y_i, \tag{17-31}$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = \rho \sum_{i=1}^{n} \beta_i^2 - 2 \sum_{i=1}^{n} y_i \beta_i^3, \qquad \frac{\partial^2 \ln L}{\partial \rho^2} = -n\Psi'(\rho), \qquad \frac{\partial^2 \ln L}{\partial \beta \partial \rho} = -\sum_{i=1}^{n} \beta_i.$$

[9] The gamma function $\Gamma(\rho)$ and the gamma distribution are described in Sections B.4.5 and E.5.3.

TABLE 17.1 Maximum Likelihood Estimates

Quantity	*Unrestricted Estimate*[a]	*Restricted Estimate*
β	−4.7198 (2.344)	15.6052 (6.794)
ρ	3.1517 (0.7943)	1.0000 (0.000)
$\ln L$	−82.91444	−88.43771
$\partial \ln L/\partial\beta$	0.0000	0.0000
$\partial \ln L/\partial\rho$	0.0000	7.9162
$\partial^2 \ln L/\partial\beta^2$	−0.85628	−0.021659
$\partial^2 \ln L/\partial\rho^2$	−7.4569	−32.8987
$\partial^2 \ln L/\partial\beta\partial\rho$	−2.2423	−0.66885

[a]Estimated asymptotic standard errors based on **V** are given in parentheses.

[Recall that $\Psi(\rho) = d \ln \Gamma(\rho)/d\rho$ and $\Psi'(\rho) = d^2 \ln \Gamma(\rho)/d\rho^2$.] Unrestricted maximum likelihood estimates of β and ρ are obtained by equating the two first derivatives to zero. The restricted maximum likelihood estimate of β is obtained by equating $\partial \ln L/\partial\beta$ to zero while fixing ρ at one. The results are shown in Table 17.1. Three estimators are available for the asymptotic covariance matrix of the estimators of $\boldsymbol{\theta} = (\beta, \rho)'$. Using the actual Hessian as in (17-17), we compute $\mathbf{V} = [-\Sigma_i \partial^2 \ln L/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}']^{-1}$ at the maximum likelihood estimates. For this model, it is easy to show that $E[y_i \mid x_i] = \rho(\beta + x_i)$ (either by direct integration or, more simply, by using the result that $E[\partial \ln L/\partial\beta] = 0$ to deduce it). Therefore, we can also use the expected Hessian as in (17-16) to compute $\mathbf{V}_E = \{-\Sigma_i E[\partial^2 \ln L/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}']\}^{-1}$. Finally, by using the sums of squares and cross products of the first derivatives, we obtain the BHHH estimator in (17-18), $\mathbf{V}_B = [\Sigma_i(\partial \ln L/\partial\boldsymbol{\theta})(\partial \ln L/\partial\boldsymbol{\theta}')]^{-1}$. Results in Table 17.1 are based on $\mathbf{V}$.

The three estimators of the asymptotic covariance matrix produce notably different results:

$$\mathbf{V} = \begin{bmatrix} 5.495 & -1.652 \\ -1.652 & 0.6309 \end{bmatrix}, \qquad \mathbf{V}_E = \begin{bmatrix} 4.897 & -1.473 \\ -1.473 & 0.5770 \end{bmatrix}, \qquad \mathbf{V}_B = \begin{bmatrix} 13.35 & -4.314 \\ -4.314 & 1.535 \end{bmatrix}.$$

Given the small sample size, the differences are to be expected. Nonetheless, the striking difference of the BHHH estimator is typical of its erratic performance in small samples.

- **Confidence Interval Test:** A 95 percent confidence interval for ρ based on the unrestricted estimates is $3.1517 \pm 1.96\sqrt{0.6309} = [1.5942, 4.7085]$. This interval does not contain $\rho = 1$, so the hypothesis is rejected.
- **Likelihood Ratio Test:** The LR statistic is $\lambda = -2[-88.43771 - (-82.91444)] = 11.0465$. The table value for the test, with one degree of freedom, is 3.842. Since the computed value is larger than this critical value, the hypothesis is again rejected.
- **Wald Test:** The Wald test is based on the unrestricted estimates. For this restriction, $c(\boldsymbol{\theta}) - q = \rho - 1$, $dc(\hat{\rho})/d\hat{\rho} = 1$, Est.Asy. Var$[c(\hat{\rho}) - q] =$ Est.Asy. Var$[\hat{\rho}] = 0.6309$, so $W = (3.1517 - 1)^2/[0.6309] = 7.3384$.

The critical value is the same as the previous one. Hence, H_0 is once again rejected. Note that the Wald statistic is the square of the corresponding test statistic that would be used in the confidence interval test, $|3.1517 - 1|/\sqrt{0.6309} = 2.70895$.