8 Suppose $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j}$, and $\mathbf{c} = -4\mathbf{i}$. Find:

a $\mathbf{a} + \mathbf{b}$ **b** $3\mathbf{b} + \mathbf{c}$ **c** $\mathbf{a} - \mathbf{c}$

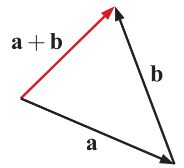
d $2\mathbf{b} - \mathbf{a}$ **e** $|\mathbf{c} + 2\mathbf{a}|$ **f** $|-2\mathbf{b}|$

9 Suppose $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Prove that:

a if $k\mathbf{a} = \mathbf{b}$, $k \neq 0$, then $\mathbf{a} = \frac{1}{k}\mathbf{b}$ **b** $|k\mathbf{a}| = |k|\,|\mathbf{a}|$

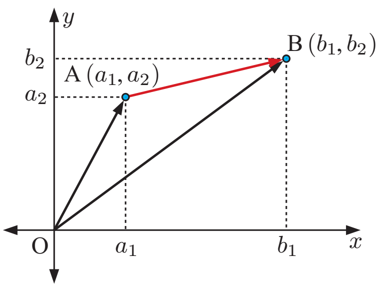
10 Prove that $|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$:

a using a geometric argument and the diagram



b by letting $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

D THE VECTOR BETWEEN TWO POINTS



In the diagram, point A has position vector $\overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$,

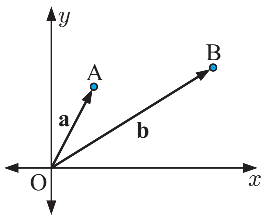
and point B has position vector $\overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

$$\begin{aligned} \therefore\ \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ &= \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} \end{aligned}$$

The **position vector of B relative to A** is $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$.

In general, for two points A and B with position vectors **a** and **b** respectively, we observe

$$\begin{aligned} \overrightarrow{AB} &= -\mathbf{a} + \mathbf{b} \\ &= \mathbf{b} - \mathbf{a} \\ &= \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \overrightarrow{BA} &= -\mathbf{b} + \mathbf{a} \\ &= \mathbf{a} - \mathbf{b} \\ &= \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix} \end{aligned}$$



Example 8 — Self Tutor

Given points A$(-1, 2)$, B$(3, 4)$, and C$(4, -5)$, find the position vector of:

a B from O **b** B from A **c** A from C

a The position vector of B relative to O is $\overrightarrow{OB} = \begin{pmatrix} 3-0 \\ 4-0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

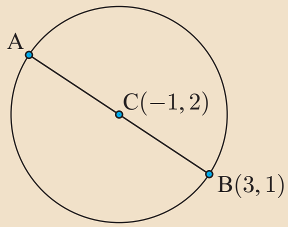
b The position vector of B relative to A is $\overrightarrow{AB} = \begin{pmatrix} 3--1 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

c The position vector of A relative to C is $\overrightarrow{CA} = \begin{pmatrix} -1-4 \\ 2--5 \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$.

Example 9 — Self Tutor

[AB] is the diameter of a circle with centre C$(-1, 2)$. If B is $(3, 1)$, find:

a $\overrightarrow{BC}$ **b** the coordinates of A.



a $\overrightarrow{BC} = \begin{pmatrix} -1-3 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

b If A has coordinates (a, b), then $\overrightarrow{CA} = \begin{pmatrix} a-(-1) \\ b-2 \end{pmatrix} = \begin{pmatrix} a+1 \\ b-2 \end{pmatrix}$

But $\overrightarrow{CA} = \overrightarrow{BC}$, so $\begin{pmatrix} a+1 \\ b-2 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$\therefore\ a+1=-4$ and $b-2=1$

$\therefore\ a=-5$ and $b=3$

$\therefore$ A is $(-5, 3)$.

EXERCISE 11D

1 Find $\overrightarrow{AB}$ given:

a A$(2, 3)$ and B$(4, 7)$ **b** A$(3, -1)$ and B$(1, 4)$ **c** A$(-2, 7)$ and B$(1, 4)$

d B$(3, 0)$ and A$(2, 5)$ **e** B$(6, -1)$ and A$(0, 4)$ **f** B$(0, 0)$ and A$(-1, -3)$

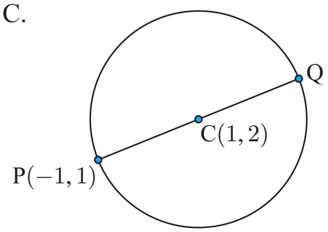
2 Consider the point A$(1, 4)$. Find the coordinates of:

a B given $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **b** C given $\overrightarrow{CA} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

3 [PQ] is the diameter of a circle with centre C.

a Find $\overrightarrow{PC}$.

b Hence find the coordinates of Q.



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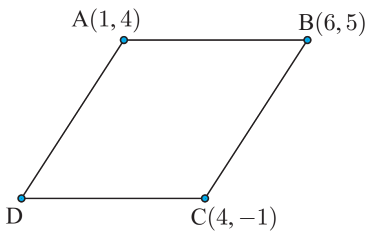


ABCD is a parallelogram.

a Find $\overrightarrow{AB}$.

b Find $\overrightarrow{CD}$.

c Hence find the coordinates of D.

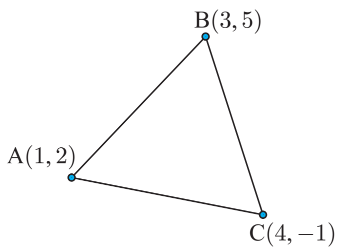
5 $A(-1,\ 3)$ and $B(3,\ k)$ are two points which are 5 units apart.

a Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

b Hence, find the two possible values of k.

c Show, by illustration, why k should have two possible values.

6



a Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b Explain why $\overrightarrow{BC} = -\overrightarrow{AB} + \overrightarrow{AC}$.

c Hence find $\overrightarrow{BC}$.

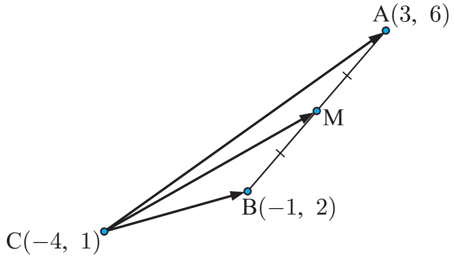
d Check your answer to **c** by direct evaluation.

7 **a** Given $\overrightarrow{BA} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, find $\overrightarrow{AC}$.

b Given $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\overrightarrow{CA} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, find $\overrightarrow{CB}$.

c Given $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $\overrightarrow{RS} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, find $\overrightarrow{SP}$.

8

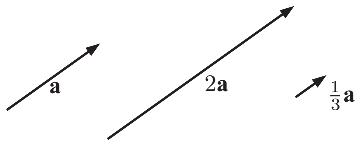


a Find the coordinates of M.

b Find vectors $\overrightarrow{CA}$, $\overrightarrow{CM}$, and $\overrightarrow{CB}$.

c Verify that $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB}$.

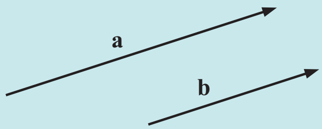
E PARALLELISM



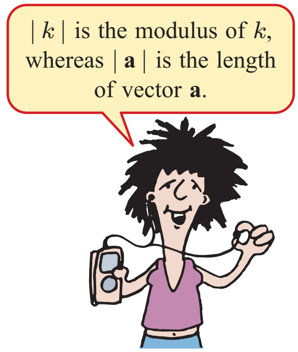
are parallel vectors of different length.

Two non-zero vectors are **parallel** if and only if one is a scalar multiple of the other.

Given any non-zero vector $\mathbf{a}$ and non-zero scalar k, the vector $k\mathbf{a}$ is parallel to $\mathbf{a}$.



- If $\mathbf{a}$ is parallel to $\mathbf{b}$, then there exists a scalar k such that $\mathbf{a} = k\mathbf{b}$.
- If $\mathbf{a} = k\mathbf{b}$ for some scalar k, then
  - $\mathbf{a}$ is parallel to $\mathbf{b}$, and
  - $|\mathbf{a}| = |k|\,|\mathbf{b}|$.



Example 10 Self Tutor

Find r given that $\mathbf{a} = \begin{pmatrix} -1 \\ r \end{pmatrix}$ is parallel to $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Since $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\mathbf{a} = k\mathbf{b}$ for some scalar k.

$$\therefore \begin{pmatrix} -1 \\ r \end{pmatrix} = k\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\therefore -1 = 2k \text{ and } r = -3k$$

$$\therefore k = -\tfrac{1}{2} \text{ and hence } r = -3(-\tfrac{1}{2}) = \tfrac{3}{2}$$

UNIT VECTORS

Given a non-zero vector $\mathbf{a}$, its magnitude $|\mathbf{a}|$ is a scalar quantity.

If we multiply $\mathbf{a}$ by the scalar $\dfrac{1}{|\mathbf{a}|}$, we obtain the parallel vector $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$ with length 1.

- A unit vector in the direction of $\mathbf{a}$ is $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$.
- A vector of length k in the same direction as $\mathbf{a}$ is $\dfrac{k}{|\mathbf{a}|}\mathbf{a}$.
- A vector of length k which is *parallel to* $\mathbf{a}$ could be $\pm\dfrac{k}{|\mathbf{a}|}\mathbf{a}$.

Example 11

Self Tutor

If $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$, find:

a a unit vector in the direction of **a**

b a vector of length 4 units in the direction of **a**

c vectors of length 4 units which are parallel to **a**.

a $|\,\mathbf{a}\,| = \sqrt{3^2 + (-1)^2}$
$= \sqrt{9+1}$
$= \sqrt{10}$ units

$\therefore$ the unit vector is $\frac{1}{\sqrt{10}}(3\mathbf{i} - \mathbf{j})$
$= \frac{3}{\sqrt{10}}\mathbf{i} - \frac{1}{\sqrt{10}}\mathbf{j}$

b This vector is $\frac{4}{\sqrt{10}}(3\mathbf{i} - \mathbf{j})$
$= \frac{12}{\sqrt{10}}\mathbf{i} - \frac{4}{\sqrt{10}}\mathbf{j}$

c The vectors are $\frac{12}{\sqrt{10}}\mathbf{i} - \frac{4}{\sqrt{10}}\mathbf{j}$ and $-\frac{12}{\sqrt{10}}\mathbf{i} + \frac{4}{\sqrt{10}}\mathbf{j}$.

EXERCISE 11E

1 Find r given that $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -6 \\ r \end{pmatrix}$ are parallel.

2 Find a given that $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} a \\ 2 \end{pmatrix}$ are parallel.

3 What can be deduced from the following?

a $\overrightarrow{AB} = 3\overrightarrow{CD}$ **b** $\overrightarrow{RS} = -\frac{1}{2}\overrightarrow{KL}$ **c** $\overrightarrow{AB} = 2\overrightarrow{BC}$

4 If $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, write down the vector:

a in the same direction as **a** and twice its length

b in the opposite direction to **a** and half its length.

5 Find the unit vector in the direction of:

a $\mathbf{i} + 2\mathbf{j}$ **b** $\mathbf{i} - 3\mathbf{j}$ **c** $2\mathbf{i} - \mathbf{j}$

6 Find a vector **v** which has:

a the same direction as $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and length 3 units

b the opposite direction to $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ and length 2 units.

7 A is $(3, 2)$ and point B is 4 units from A in the direction $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

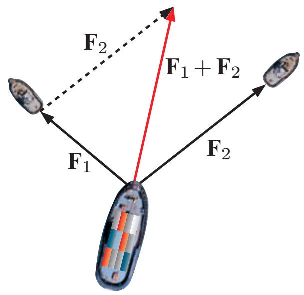
a Find $\overrightarrow{AB}$. **b** Find $\overrightarrow{OB}$ using $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$.

c Hence deduce the coordinates of B.

F PROBLEMS INVOLVING VECTOR OPERATIONS

When we apply vectors to problems in the real world, we often consider the combined effect when vectors are added together. This sum is called the **resultant vector**.

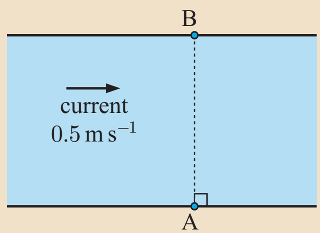
The diagram shows an example of vector addition. Two tug boats are being used to pull a ship into port. If the tugs tow with forces $\mathbf{F}_1$ and $\mathbf{F}_2$ then the resultant force is $\mathbf{F}_1 + \mathbf{F}_2$.



Example 12

Self Tutor

In still water, Jacques can swim at 1.5 m s^{-1}. Jacques is at point A on the edge of a canal, and considers point B directly opposite. A current is flowing from the left at a constant speed of 0.5 m s^{-1}.



a If Jacques dives in straight towards B, and swims without allowing for the current, what will his actual speed and direction be?

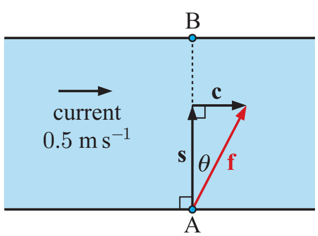
b Jacques wants to swim directly across the canal to point B.

 i At what angle should Jacques *aim* to swim in order that the current will correct his direction?

 ii What will Jacques' actual speed be?

Suppose $\mathbf{c}$ is the current's velocity vector,
$\mathbf{s}$ is the velocity vector Jacques would have if the water was still, and
$\mathbf{f} = \mathbf{c} + \mathbf{s}$ is Jacques' resultant velocity vector.

a Jacques aims directly across the river, but the current takes him downstream to the right.



$$|\mathbf{f}|^2 = |\mathbf{c}|^2 + |\mathbf{s}|^2$$
$$= 0.5^2 + 1.5^2$$
$$= 2.5$$
$$\therefore \ |\mathbf{f}| \approx 1.58$$

$$\tan\theta = \frac{0.5}{1.5}$$
$$\therefore \ \theta \approx 18.4°$$

Jacques has an actual speed of approximately 1.58 m s^{-1} and his direction of motion is approximately 18.4° to the right of his intended line.

b Jacques needs to aim to the left of B so the current will correct his direction.

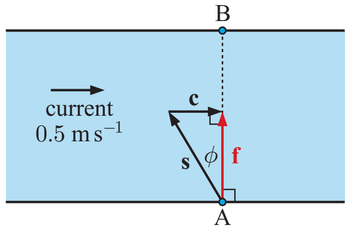
 i $\sin\phi = \dfrac{0.5}{1.5}$

 $\therefore \ \phi \approx 19.5°$

 Jacques needs to aim approximately 19.5° to the left of B.

ii $$|\mathbf{f}|^2 + |\mathbf{c}|^2 = |\mathbf{s}|^2$$
$$\therefore \quad |\mathbf{f}|^2 + 0.5^2 = 1.5^2$$
$$\therefore \quad |\mathbf{f}|^2 = 2$$
$$\therefore \quad |\mathbf{f}| \approx 1.41$$

In these conditions, Jacques' actual speed towards B is approximately 1.41 m s^{-1}.



Another example of vector addition is when an aircraft is affected by wind. A pilot needs to know how to compensate for the wind, especially during take-off and landing.

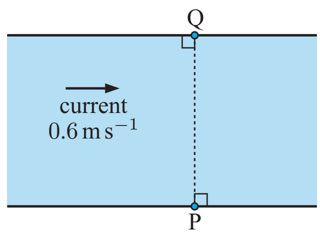
SIMULATION

EXERCISE 11F

1 A bird can normally fly with constant speed 6 m s^{-1}. Using a vector diagram to illustrate each situation, find the bird's speed if:

- **a** it is assisted by a wind of 1 m s^{-1} from directly behind it
- **b** it flies into a head wind of 1 m s^{-1}.

2 In still water, Mary can swim at 1.2 m s^{-1}. She is standing at point P on the edge of a canal, directly opposite point Q. The water is flowing to the right at a constant speed of 0.6 m s^{-1}.

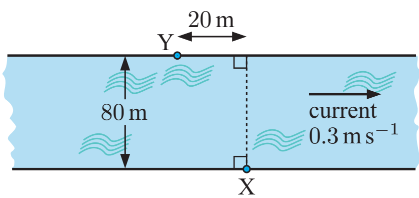


- **a** If Mary tries to swim directly from P to Q without allowing for the current, what will her actual velocity be?
- **b** Mary wants to swim directly across the canal to point Q.
  - **i** At what angle should she *aim* to swim in order that the current corrects her direction?
  - **ii** What will Mary's actual speed be?

3 A boat needs to travel south at a speed of 20 km h^{-1}. However, a constant current of 6 km h^{-1} is flowing from the south-east. Use vectors to find:

- **a** the equivalent speed in still water for the boat to achieve the actual speed of 20 km h^{-1}
- **b** the direction in which the boat must head to compensate for the current.

4 As part of an endurance race, Stephanie needs to swim from X to Y across a wide river.
Stephanie swims at 1.8 m s^{-1} in still water.
The river flows with a consistent current of 0.3 m s^{-1} as shown.



- **a** Find the distance from X to Y.
- **b** In which direction should Stephanie *aim* so that the current will push her onto a path directly towards Y?
- **c** Find the time Stephanie will take to cross the river.

5 An aeroplane needs to fly due east from one city to another at a speed of 400 km h^{-1}. However, a 50 km h^{-1} wind blows constantly from the north-east.

a How does the wind affect the speed of the aeroplane?

b In what direction must the aeroplane head to compensate for the wind?

G LINES

We have seen in Cartesian geometry that we can determine the **equation of a line** using its **direction** and any **fixed point** on the line. We can do the same using vectors.

DEMO

Suppose a line passes through a fixed point A with position vector **a**, and that the line is parallel to the vector **b**.

Consider a point R on the line so that $\overrightarrow{OR} = \mathbf{r}$.

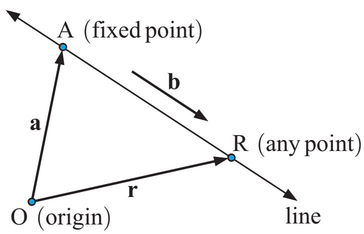
By vector addition, $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$

$\therefore \ \mathbf{r} = \mathbf{a} + \overrightarrow{AR}$.

Since $\overrightarrow{AR}$ is parallel to **b**,

$\overrightarrow{AR} = t\mathbf{b}$ for some scalar $t \in \mathbb{R}$

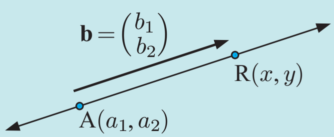
$\therefore \ \mathbf{r} = \mathbf{a} + t\mathbf{b}$



Suppose a line passes through a fixed point $A(a_1, a_2)$ with position vector **a**, and that the line is parallel to the vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. If $R(x, y)$ with position vector **r** is any point on the line, then:

- $\mathbf{r} = \mathbf{a} + t\mathbf{b}, \ t \in \mathbb{R}$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

  is the **vector equation** of the line.



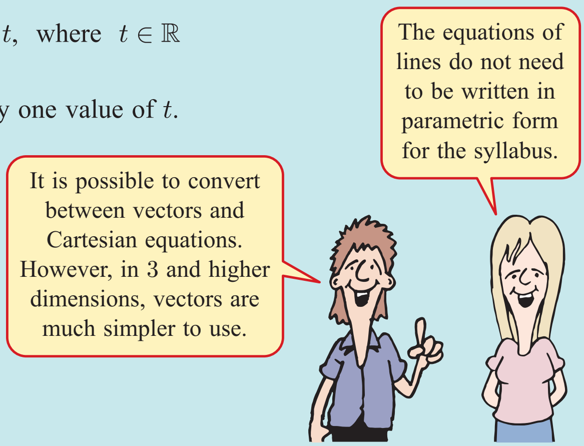
- The gradient of the line is $m = \dfrac{b_2}{b_1}$.

- Since $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix}$, the **parametric equations** of the line are $x = a_1 + b_1 t$ and $y = a_2 + b_2 t$, where $t \in \mathbb{R}$ is the **parameter**.

  Each point on the line corresponds to exactly one value of t.

- We can convert these equations into Cartesian form by equating t values.

  Using $t = \dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2}$ we obtain $b_2 x - b_1 y = b_2 a_1 - b_1 a_2$ which is the **Cartesian equation** of the line.



Example 13 **Self Tutor**

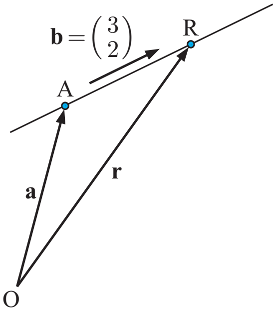
A line passes through the point A(1, 5) and has direction vector $\begin{pmatrix}3\\2\end{pmatrix}$. Describe the line using:

a a vector equation **b** parametric equations **c** a Cartesian equation.

a The vector equation is $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where

$\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix}1\\5\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}3\\2\end{pmatrix}$

$\therefore \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix} + t\begin{pmatrix}3\\2\end{pmatrix}, \quad t \in \mathbb{R}$



b $x = 1 + 3t$ and $y = 5 + 2t, \quad t \in \mathbb{R}$

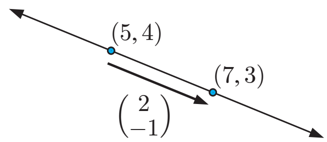
c Now $t = \dfrac{x-1}{3} = \dfrac{y-5}{2}$

$\therefore \ 2x - 2 = 3y - 15$

$\therefore \ 2x - 3y = -13$

NON-UNIQUENESS OF THE VECTOR EQUATION OF A LINE

Consider the line passing through (5, 4) and (7, 3). When writing the equation of the line, we could use either point to give the position vector **a**.



Similarly, we could use the direction vector $\begin{pmatrix}2\\-1\end{pmatrix}$, but we could also use $\begin{pmatrix}-2\\1\end{pmatrix}$ or indeed any non-zero scalar multiple of these vectors.

We could thus write the equation of the line as

$$\mathbf{x} = \begin{pmatrix}5\\4\end{pmatrix} + t\begin{pmatrix}2\\-1\end{pmatrix}, \quad t \in \mathbb{R} \quad \text{or} \quad \mathbf{x} = \begin{pmatrix}7\\3\end{pmatrix} + s\begin{pmatrix}-2\\1\end{pmatrix}, \quad s \in \mathbb{R} \quad \text{and so on.}$$

Notice how we use different parameters t and s when we write these equations. This is because the parameters are clearly not the same: when $t = 0$, we have the point (5, 4)
when $s = 0$, we have the point (7, 3).

In fact, the parameters are related by $s = 1 - t$.

EXERCISE 11G

1 Describe each of the following lines using:

i a vector equation **ii** parametric equations **iii** a Cartesian equation

a a line with direction $\begin{pmatrix}1\\4\end{pmatrix}$ which passes through $(3, -4)$

b a line parallel to $3\mathbf{i} + 7\mathbf{j}$ which cuts the x-axis at -6

c a line passing through $(-1, 11)$ and $(-3, 12)$.

2 A line passes through $(-1, 4)$ with direction vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

a Write parametric equations for the line using the parameter t.

b Find the points on the line for which $t = 0, 1, 3, -1$, and -4.

3 **a** Does $(3, -2)$ lie on the line with vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \end{pmatrix}$?

b $(k, 4)$ lies on the line with parametric equations $x = 1 - 2t$, $y = 1 + t$. Find k.

4 Line L has vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

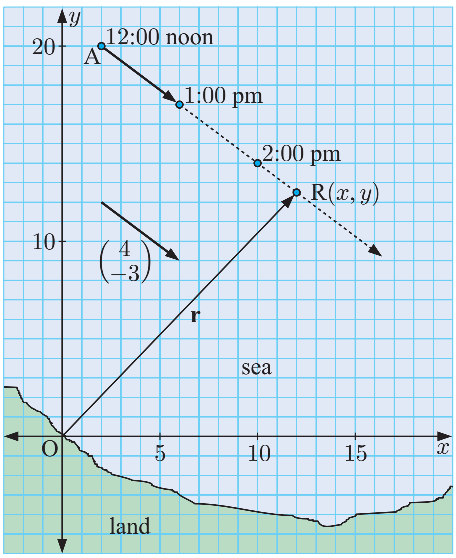
a Locate the point on the line corresponding to $t = 1$.

b Explain why the direction of the line could also be described by $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

c Use your answers to **a** and **b** to write an alternative vector equation for line L.

H CONSTANT VELOCITY PROBLEMS

A yacht club is situated at $(0, 0)$. At 12:00 noon a yacht is at point A$(2, 20)$. The yacht is moving with constant speed in the straight path shown in the diagram. The grid intervals are kilometres.



At 1:00 pm the yacht is at $(6, 17)$.
At 2:00 pm it is at $(10, 14)$.

In this case:

- the **initial position** of the yacht is given by the position vector $\mathbf{a} = \begin{pmatrix} 2 \\ 20 \end{pmatrix}$
- the direction of the yacht is given by the vector $\mathbf{b} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

Suppose that t hours after leaving A, the yacht is at R(x, y).

$$\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$$

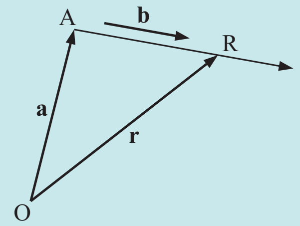
$$\therefore \quad \mathbf{r} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad \text{for } t \geqslant 0$$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix}$$ is the **vector equation** of the yacht's path.

If an object has initial position vector $\mathbf{a}$ and moves with constant velocity $\mathbf{b}$, its position at time t is given by

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \quad \text{for} \quad t \geqslant 0.$$

The **speed** of the object is $|\,\mathbf{b}\,|$.



Example 14

Self Tutor

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ is the vector equation of the path of an object.

The time t is in seconds, $t \geqslant 0$. The distance units are metres.

a Find the object's initial position.

b Plot the path of the object for $t = 0, 1, 2, 3$.

c Find the velocity vector of the object.

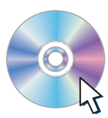
d Find the object's speed.

e If the object continues in the same direction but increases its speed to 30 m s^{-1}, state its new velocity vector.

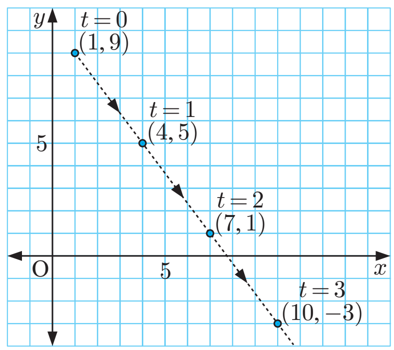
a At $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$

$\therefore$ the object is at $(1, 9)$.

DEMO



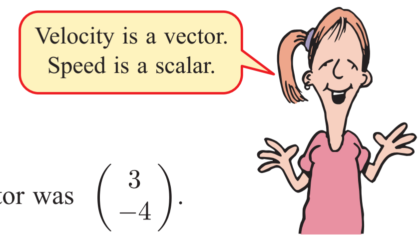
b



c The velocity vector is $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

d The speed is $\left|\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right| = \sqrt{3^2 + (-4)^2}$

$= 5$ m s^{-1}.



e Previously, the speed was 5 m s^{-1} and the velocity vector was $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

$\therefore$ the new velocity vector is $6\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 18 \\ -24 \end{pmatrix}$.

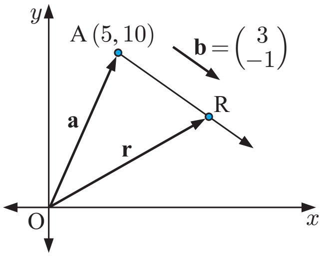
Example 15

Self Tutor

An object is initially at $(5, 10)$ and moves with velocity vector $3\mathbf{i} - \mathbf{j}$ metres per minute. Find:

- **a** the position of the object at time t minutes
- **b** the speed of the object
- **c** the position of the object at $t = 3$ minutes
- **d** the time when the object is due east of $(0, 0)$.

a



$$\mathbf{r} = \mathbf{a} + t\mathbf{b}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5+3t \\ 10-t \end{pmatrix}$$

After t minutes, the object is at $(5 + 3t, 10 - t)$.

b The speed of the object is $|\mathbf{b}| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$ metres per minute.

c At $t = 3$ minutes, $5 + 3t = 14$ and $10 - t = 7$. The object is at $(14, 7)$.

d When the object is due east of $(0, 0)$, y must be zero.

$\therefore \ 10 - t = 0$

$\therefore \ t = 10$

The object is due east of $(0, 0)$ after 10 minutes.

EXERCISE 11H

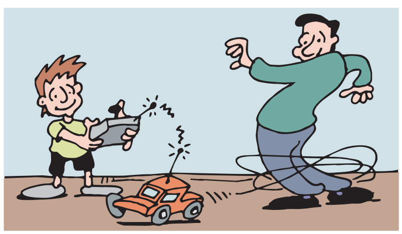
1 A particle at $P(x(t), y(t))$ moves such that $x(t) = 1 + 2t$ and $y(t) = 2 - 5t$, $t \geqslant 0$. The distances are in centimetres and t is in seconds.

- **a** Find the initial position of P.
- **b** Illustrate the initial part of the motion of P where $t = 0, 1, 2, 3$.
- **c** Find the velocity vector of P.
- **d** Find the speed of P.

2 **a** Find the vector equation of a boat initially at $(2, 3)$, which travels with velocity vector $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$. The grid units are kilometres and the time is in hours.

- **b** Locate the boat's position after 90 minutes.
- **c** How long will it take for the boat to reach the point $(5, -0.75)$?

3 A remote controlled toy car is initially at $(-3, -2)$. It moves with constant velocity $2\mathbf{i} + 4\mathbf{j}$. The distance units are centimetres, and the time is in seconds.



- **a** Write an expression for the position vector of the car at any time $t \geqslant 0$.
- **b** Find the position vector of the car at time $t = 2.5$.

c Find when the car is **i** due north **ii** due west of the observation point $(0,\ 0)$.

d Plot the car's positions at times $t = 0,\ \frac{1}{2},\ 1,\ 1\frac{1}{2},\ 2,\ 2\frac{1}{2},\ ....$

4 Each of the following vector equations represents the path of a moving object. t is measured in seconds, and $t \geqslant 0$. Distances are measured in metres. In each case, find:

i the initial position **ii** the velocity vector **iii** the speed of the object.

a $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 12 \\ 5 \end{pmatrix}$ **b** $x = 3 + 2t,\ \ y = -t$

5 Find the velocity vector of a speed boat moving parallel to:

a $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ with a speed of 150 km h^{-1} **b** $2\mathbf{i} + \mathbf{j}$ with a speed of 50 km h^{-1}.

6 Find the velocity vector of a swooping eagle moving in the direction $5\mathbf{i} - 12\mathbf{j}$ with a speed of 91 km h^{-1}.

7 Yacht A moves according to $x(t) = 4 + t,\ \ y(t) = 5 - 2t$ where the distance units are kilometres and the time units are hours. Yacht B moves according to $x(t) = 1 + 2t,\ \ y(t) = -8 + t,\ \ t \geqslant 0$.

a Find the initial position of each yacht.

b Find the velocity vector of each yacht.

c Show that the speed of each yacht is constant, and state these speeds.

d Find the Cartesian equation of the path of each yacht.

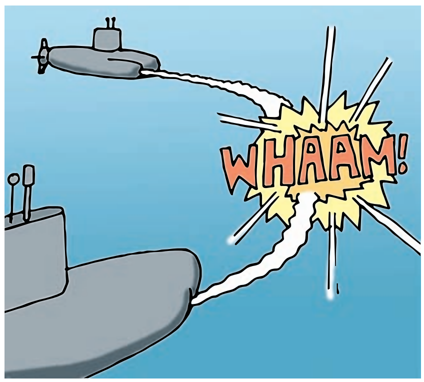
e *Hence* show that the paths of the yachts intersect at right angles.

f Will the yachts collide?

8 Submarine P is at $(-5,\ 4)$. It fires a torpedo with velocity vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ at 1:34 pm.

Submarine Q is at $(15,\ 7)$. a minutes after 1:34 pm, it fires a torpedo with velocity vector $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$.

Distances are measured in kilometres, and time is in minutes.



a Show that the position of P's torpedo can be written as $P(x_1(t),\ y_1(t))$ where $x_1(t) = -5 + 3t$ and $y_1(t) = 4 - t$.

b What is the speed of P's torpedo?

c Show that the position of Q's torpedo can be written as $Q(x_2(t),\ y_2(t))$ where $x_2(t) = 15 - 4(t - a)$ and $y_2(t) = 7 - 3(t - a)$.

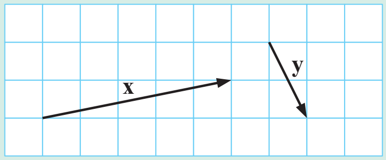
d Q's torpedo is successful in knocking out P's torpedo. At what time did Q fire its torpedo, and at what time did the explosion occur?

Review set 11A

1 **a** Write the given vectors in component form and in unit vector form.

b Find, in unit vector form:

i $\mathbf{x} + \mathbf{y}$ **ii** $\mathbf{y} - 2\mathbf{x}$



2 Consider the vector $3\mathbf{i} - \mathbf{j}$.

a Write the vector in component form.

b Illustrate the vector using a directed line segment.

c Write the negative of the vector.

d Find the length of the vector.

3 **a** Find k given that $\begin{pmatrix} k \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ is a unit vector.

b Find the vector which is 5 units long and has the opposite direction to $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

4 For $\mathbf{m} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and $\mathbf{p} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, find:

a $\mathbf{m} - \mathbf{n} + \mathbf{p}$ **b** $2\mathbf{n} - 3\mathbf{p}$ **c** $|\mathbf{m} + \mathbf{p}|$

5 Given points A(3, 1), B(5, −2), and C(8, 4), find:

a $\overrightarrow{AB}$ **b** $\overrightarrow{CB}$ **c** $|\overrightarrow{AC}|$

6 B(−3, −1) and C(k, 2) are 5 units apart.

a Find $\overrightarrow{BC}$ and $|\overrightarrow{BC}|$.

b Hence, find the two possible values of k.

c Show, by illustration, why k should have two possible values.

7 A small plane can fly at 350 km h^{-1} in still conditions. Its pilot needs to fly due north, but needs to deal with a 70 km h^{-1} wind from the east.

a In what direction should the pilot face the plane in order that its resultant velocity is due north?

b What will the speed of the plane be?

8 For the line that passes through (−6, 3) with direction $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, write down the corresponding:

a vector equation **b** parametric equations **c** Cartesian equation.

9 (−3, m) lies on the line with vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ -2 \end{pmatrix} + t\begin{pmatrix} -7 \\ 4 \end{pmatrix}$. Find m.

10 Find the velocity vector of an object moving in the direction $3\mathbf{i} - \mathbf{j}$ with speed 20 km h^{-1}.

11 Line L has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

a Locate the point on the line corresponding to $t = 1$.

b Explain why the direction of the line could also be described by $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$.

c Use your answers to **a** and **b** to write an alternative vector equation for line L.

12 A moving particle has coordinates $P(x(t), y(t))$ where $x(t) = -4 + 8t$ and $y(t) = 3 + 6t$. The distance units are metres, and $t \geqslant 0$ is the time in seconds. Find the:

a initial position of the particle

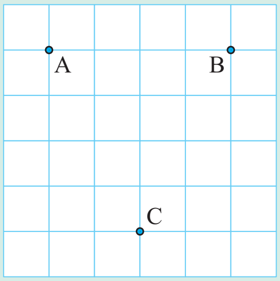
b position of the particle after 4 seconds

c particle's velocity vector

d speed of the particle.

Review set 11B

1



a Find in component form and in unit vector form:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CA}$

b Which two vectors in **a** have the same length? Explain your answer.

c Write the negative vector of $\overrightarrow{CA}$ in *three* different ways.

2 If $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ find:

a $|\mathbf{s}|$

b $|\mathbf{r}+\mathbf{s}|$

c $|2\mathbf{s}-\mathbf{r}|$

3 Find k if the following are unit vectors:

a $\begin{pmatrix} \frac{5}{13} \\ k \end{pmatrix}$

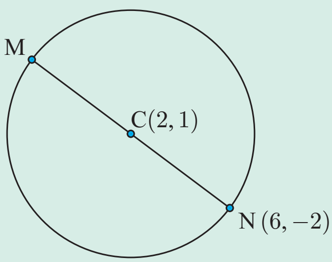
b $\begin{pmatrix} k \\ -k \end{pmatrix}$

4 If $\overrightarrow{PQ} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, and $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, find $\overrightarrow{SP}$.

5 [MN] is the diameter of a circle with centre C.

a Find the coordinates of M.

b Find the radius of the circle.



6 Find m if $\begin{pmatrix} 3 \\ m \end{pmatrix}$ and $\begin{pmatrix} -12 \\ -20 \end{pmatrix}$ are parallel vectors.

7 When an archer fires an arrow, he is suddenly aware of a breeze which pushes his shot off-target. The speed of the shot $|\mathbf{v}|$ is *not* affected by the wind, but the arrow's flight is $2°$ off-line.

a Draw a vector diagram to represent the situation.

b Hence explain why:

i the breeze must be $91°$ to the intended direction of the arrow

ii the speed of the breeze must be $2|\mathbf{v}|\sin 1°$.

8 Find the vector equation of the line which cuts the y-axis at $(0, 8)$ and has direction $5\mathbf{i} + 4\mathbf{j}$.

9 A yacht is sailing with constant speed $5\sqrt{10}$ km h^{-1} in the direction $-\mathbf{i} - 3\mathbf{j}$. Initially it is at point $(-6, 10)$. A beacon is at $(0, 0)$ at the centre of a tiny atoll. Distances are in kilometres.

a Find, in terms of $\mathbf{i}$ and $\mathbf{j}$:

i the initial position vector of the yacht

ii the velocity vector of the yacht

iii the position vector of the yacht at any time t hours, $t \geqslant 0$.

b Find the time when the yacht is due west of the beacon. How far away from the beacon is the yacht at this time?

10 Write down **i** a vector equation **ii** parametric equations for the line passing through:

a $(2, -3)$ with direction $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$

b $(-1, 6)$ and $(5, -2)$.

11 Submarine X23 is at $(2, 4)$. It fires a torpedo with velocity vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ at exactly 2:17 pm. Submarine Y18 is at $(11, 3)$. It fires a torpedo with velocity vector $\begin{pmatrix} -1 \\ a \end{pmatrix}$ at 2:19 pm to intercept the torpedo from X23. Distance units are kilometres. t is in minutes.

a Find $x_1(t)$ and $y_1(t)$ for the torpedo fired from submarine X23.

b Find $x_2(t)$ and $y_2(t)$ for the torpedo fired from submarine Y18.

c At what time does the interception occur?

d What was the direction and speed of the interception torpedo?

12

Matrices

Contents:

Opening problem

Aakriti owns a stationery shop. She sells two brands of pen, in three colours. Her sales for one week are shown in the table below.

	Brand	
Colour	*Pentex*	*Rollerball*
Blue	32	24
Black	25	16
Red	13	9

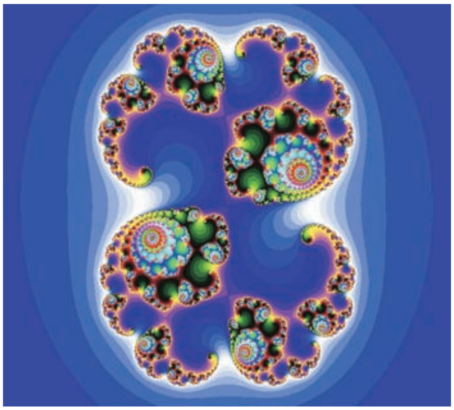
Pentex pens sell for \$1.19 each, and Rollerball pens sell for \$1.55 each.

Things to think about:

- **a** How can we convert the table into a 3×2 quantities matrix $\mathbf{Q}$?
- **b** How can we display the prices in a price matrix $\mathbf{P}$?
- **c** How can we multiply the matrices $\mathbf{Q}$ and $\mathbf{P}$? What does the matrix $\mathbf{QP}$ represent?
- **d** Can you find the total revenue for Aakriti in pen sales for the week?

Matrices are rectangular arrays of numbers which are used to organise numerical information. They are used in a wide range of fields, including:

- solving systems of equations in business, physics, and engineering
- linear programming where we may wish to optimise a linear expression subject to linear constraints
- business inventories involving stock control, cost, revenue, and profit calculations
- Markov chains for predicting long term probabilities such as in weather
- strategies in games where we wish to maximise our chance of winning
- economic modelling where the input from suppliers is needed to help a business be successful
- graph (network) theory used to determine routes for trucks and airlines to minimise distance travelled and therefore costs
- assignment problems to direct resources in the most cost-effective way
- forestry and fisheries management where we need to select an appropriate sustainable harvesting policy
- cubic spline interpolation used to construct curves and fonts
- computer graphics
- flight simulation
- Computer Aided Tomography (CAT scanning) and Magnetic Resonance Imaging (MRI)
- fractals and chaos
- genetics
- cryptography including coding, code breaking, and computer confidentiality.



The Julia set

A MATRIX STRUCTURE

A **matrix** is a rectangular array of numbers arranged in **rows** and **columns**.

Each number within a matrix has a particular meaning.

You have been using matrices for many years without realising it. For example, a football premiership table and a recipe can each be written as matrices.

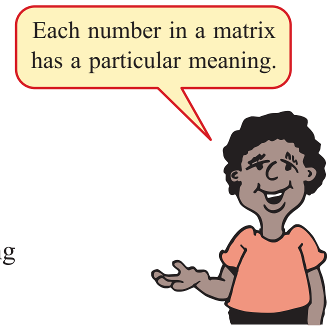
	Won	*Lost*	*Drew*	*Points*
Manchester United	28	5	5	89
Manchester City	23	9	6	78
Chelsea	22	9	7	75
Arsenal	21	10	7	73
⋮				

Ingredients	*Amount*
sugar	1 tspn
flour	1 cup
milk	200 mL
salt	1 pinch

Consider these two items of information:

Shopping list	
Bread	2 loaves
Juice	1 carton
Eggs	6
Cheese	1

Furniture inventory			
	chairs	tables	beds
Flat	6	1	2
Unit	9	2	3
House	10	3	4



We can write these tables as matrices by extracting the numbers and placing them in round brackets:

$$\begin{array}{c c} & \text{number} \\ \begin{matrix} B \\ J \\ E \\ C \end{matrix} & \begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c c} & \begin{matrix} C & \;T & \;B \end{matrix} \\ \begin{matrix} F \\ U \\ H \end{matrix} & \begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & 3 & 4 \end{pmatrix} \end{array} \quad \text{or simply} \quad \begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & 3 & 4 \end{pmatrix}$$

Notice how the organisation of the data is maintained in matrix form.

$\begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix}$ has 4 rows and 1 column, and we say that this is a 4×1 **column matrix** or **column vector**.

column 2

row 3

$$\begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & 3 & 4 \end{pmatrix}$$

this element is in row 3, column 2

has 3 rows and 3 columns, and is called a 3×3 **square matrix**.

$\begin{pmatrix} 3 & 0 & -1 & 2 \end{pmatrix}$ has 1 row and 4 columns, and is called a 1×4 **row matrix** or **row vector**.

An $m \times n$ matrix has m rows and n columns.

$m \times n$ specifies the **order** of a matrix.

Example 1

Self Tutor

Lisa goes shopping at store A to buy 2 loaves of bread at \$2.65 each, 3 litres of milk at \$1.55 per litre, and one 500 g tub of butter at \$2.35 .

a Represent the quantities purchased in a row matrix $\mathbf{Q}$, and the costs in a column matrix $\mathbf{A}$.

b When Lisa goes to a different supermarket (store B), she finds that the prices for the same items are \$2.25 for bread, \$1.50 for milk, and \$2.20 for butter.
Write the costs for both stores in a single costs matrix $\mathbf{C}$.

a The quantities matrix is

$$\mathbf{Q} = \begin{array}{c} \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \\ \text{bread} \quad \text{milk} \quad \text{butter} \end{array}$$

The costs matrix is

$$\mathbf{A} = \begin{array}{cc} \begin{pmatrix} 2.65 \\ 1.55 \\ 2.35 \end{pmatrix} & \begin{array}{l} \leftarrow \text{bread} \\ \leftarrow \text{milk} \\ \leftarrow \text{butter} \end{array} \end{array}$$

b We write the costs for each store in separate columns.

The new costs matrix is

$$\mathbf{C} = \begin{array}{cc} \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix} & \begin{array}{l} \leftarrow \text{bread} \\ \leftarrow \text{milk} \\ \leftarrow \text{butter} \end{array} \\ \text{store A} \quad \text{store B} & \end{array}$$

EXERCISE 12A

1 Write down the order of:

a $\begin{pmatrix} 5 & 1 & 0 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 5 & 1 & 0 \end{pmatrix}$

2 A grocery list consists of 2 loaves of bread, 1 kg of butter, 6 eggs, and 1 carton of cream. Each loaf of bread costs \$1.95, each kilogram of butter costs \$2.35, each egg costs \$0.45, and each carton of cream costs \$2.95 .

a Construct a row matrix showing quantities.

b Construct a column matrix showing prices.

c What is the significance of $(2 \times 1.95) + (1 \times 2.35) + (6 \times 0.45) + (1 \times 2.95)$?

3 A food processing factory produces cans of beans in three sizes: 200 g, 300 g, and 500 g. In February they produced respectively:

- 1000, 1500, and 1250 cans of each in week 1
- 1500, 1000, and 1000 cans of each in week 2
- 800, 2300, and 1300 cans of each in week 3
- 1200 cans of each in week 4.

Construct a matrix to show February's production levels.

4 Over a long weekend holiday, a baker produced the following food items: On Friday he baked 40 dozen pies, 50 dozen pasties, 55 dozen rolls, and 40 dozen buns. On Saturday he baked 25 dozen pies, 65 dozen pasties, 30 dozen buns, and 44 dozen rolls. On Sunday he baked 40 dozen pasties, 40 dozen rolls, and 35 dozen of each of pies and buns. On Monday he baked 40 dozen pasties, 50 dozen buns, and 35 dozen of each of pies and rolls. Represent this information as a matrix.

B MATRIX OPERATIONS AND DEFINITIONS

MATRIX NOTATION

Consider a matrix **A** which has order $m \times n$.

We can write

$$\mathbf{A} = (a_{ij}) \quad \text{where} \quad i = 1, 2, 3, ..., m$$
$$j = 1, 2, 3, ..., n$$
and a_{ij} is the element in the ith row, jth column.

By convention, the a_{ij} are labelled down then across.

$i \downarrow \begin{pmatrix} & \\ & \end{pmatrix}$ $\underset{j}{\longrightarrow}$

For example, a_{23} is the number in row 2 and column 3 of matrix **A**.

EQUALITY

Two matrices are **equal** if they have the **same order** *and* the elements in corresponding positions are equal.

$$\mathbf{A} = \mathbf{B} \quad \Leftrightarrow \quad a_{ij} = b_{ij} \quad \text{for all } i, j.$$

For example, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ then $a = w$, $b = x$, $c = y$, and $d = z$.

MATRIX ADDITION

Thao has three stores: A, B, and C. Her stock levels for dresses, skirts, and blouses are given by the matrix:

$$\begin{array}{c} \text{Store} \\ \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \begin{pmatrix} 23 & 41 & 68 \\ 28 & 39 & 79 \\ 46 & 17 & 62 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \text{dresses} \\ \text{skirts} \\ \text{blouses} \end{array}$$

Some newly ordered stock has just arrived. 20 dresses, 30 skirts, and 50 blouses must be added to the stock levels of each store. Her stock order is given by the matrix:

$$\begin{pmatrix} 20 & 20 & 20 \\ 30 & 30 & 30 \\ 50 & 50 & 50 \end{pmatrix}$$

Clearly the new levels are: $\begin{pmatrix} 23 & 41 & 68 \\ 28 & 39 & 79 \\ 46 & 17 & 62 \end{pmatrix} + \begin{pmatrix} 20 & 20 & 20 \\ 30 & 30 & 30 \\ 50 & 50 & 50 \end{pmatrix} = \begin{pmatrix} 43 & 61 & 88 \\ 58 & 69 & 109 \\ 96 & 67 & 112 \end{pmatrix}$

To **add** two matrices, they must be of the **same order**, and we **add corresponding elements**.

MATRIX SUBTRACTION

Suppose Thao's stock levels were $\begin{pmatrix} 29 & 51 & 19 \\ 31 & 28 & 32 \\ 40 & 17 & 29 \end{pmatrix}$ and her sales matrix for the week was $\begin{pmatrix} 15 & 12 & 6 \\ 20 & 16 & 19 \\ 19 & 8 & 14 \end{pmatrix}$.

Thao will be left with her original stock levels less what she has sold. Clearly, we need to subtract corresponding elements:

$$\begin{pmatrix} 29 & 51 & 19 \\ 31 & 28 & 32 \\ 40 & 17 & 29 \end{pmatrix} - \begin{pmatrix} 15 & 12 & 6 \\ 20 & 16 & 19 \\ 19 & 8 & 14 \end{pmatrix} = \begin{pmatrix} 14 & 39 & 13 \\ 11 & 12 & 13 \\ 21 & 9 & 15 \end{pmatrix}$$

To **subtract** matrices, they must be of the **same order**, and we **subtract corresponding elements**.

Summary:

- $\mathbf{A} \pm \mathbf{B} = (a_{ij}) \pm (b_{ij}) = (a_{ij} \pm b_{ij})$
- We can only add or subtract matrices of the same order.
- We add or subtract corresponding elements.
- The result of addition or subtraction is another matrix of the same order.

Example 2 Self Tutor

If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$

b $\mathbf{A} + \mathbf{C}$

a $$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1+2 & 2+1 & 3+6 \\ 6+0 & 5+3 & 4+5 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 3 & 9 \\ 6 & 8 & 9 \end{pmatrix} \end{aligned}$$

b $\mathbf{A} + \mathbf{C}$ cannot be found as the matrices do not have the same order.

Example 3 Self Tutor

If $\mathbf{A} = \begin{pmatrix} 3 & 4 & 8 \\ 2 & 1 & 0 \\ 1 & 4 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 0 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 3 \end{pmatrix}$,

find $\mathbf{A} - \mathbf{B}$.

$$\begin{aligned} \mathbf{A} - \mathbf{B} &= \begin{pmatrix} 3 & 4 & 8 \\ 2 & 1 & 0 \\ 1 & 4 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3-2 & 4-0 & 8-6 \\ 2-3 & 1-0 & 0-4 \\ 1-5 & 4-2 & 7-3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 4 & 2 \\ -1 & 1 & -4 \\ -4 & 2 & 4 \end{pmatrix} \end{aligned}$$

EXERCISE 12B.1

1 If $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -3 & 7 \\ -4 & -2 \end{pmatrix}$, find:

a $\mathbf{A}+\mathbf{B}$ **b** $\mathbf{A}+\mathbf{B}+\mathbf{C}$ **c** $\mathbf{B}+\mathbf{C}$ **d** $\mathbf{C}+\mathbf{B}-\mathbf{A}$

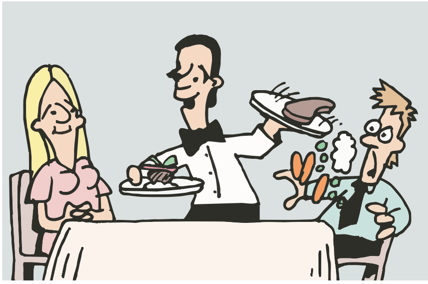
2 If $\mathbf{P} = \begin{pmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{pmatrix}$, find:

a $\mathbf{P}+\mathbf{Q}$ **b** $\mathbf{P}-\mathbf{Q}$ **c** $\mathbf{Q}-\mathbf{P}$

3 A restaurant served 85 men, 92 women, and 52 children on Friday night. On Saturday night they served 102 men, 137 women, and 49 children.

a Express this information in *two* column matrices.

b Use the matrices to find the totals of men, women, and children served over the two nights.



4 David bought shares in five companies on Monday, and he sold them on Friday. The details are shown in the table alongside.

a Write down David's column matrix for:

i cost price **ii** selling price.

b What matrix operation is needed to find David's profit or loss on each type of share?

c Find David's profit or loss matrix.

	Cost price per share	*Selling price per share*
A	\$1.72	\$1.79
B	\$27.85	\$28.75
C	\$0.92	\$1.33
D	\$2.53	\$2.25
E	\$3.56	\$3.51

5 In November, Lou E Gee sold 23 fridges, 17 stoves, and 31 microwave ovens. His partner Rose A Lee sold 19 fridges, 29 stoves, and 24 microwave ovens.

In December, Lou sold 18 fridges, 7 stoves, and 36 microwaves, and Rose sold 25 fridges, 13 stoves, and 19 microwaves.

a Write their sales for November as a 3×2 matrix.

b Write their sales for December as a 3×2 matrix.

c Write their total sales for November and December as a 3×2 matrix.

6 Find x and y if:

a $\begin{pmatrix} x & x^2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} y & 4 \\ 3 & y+1 \end{pmatrix}$ **b** $\begin{pmatrix} x & y \\ y & x \end{pmatrix} = \begin{pmatrix} -y & x \\ x & -y \end{pmatrix}$

7 **a** If $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix}$, find $\mathbf{A}+\mathbf{B}$ and $\mathbf{B}+\mathbf{A}$.

b Explain why $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

8 **a** For $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 1 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}$, find $(\mathbf{A}+\mathbf{B})+\mathbf{C}$ and $\mathbf{A}+(\mathbf{B}+\mathbf{C})$.

b Prove that, if **A**, **B**, and **C** are any 2×2 matrices, then $(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$.

Hint: Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

MULTIPLES OF MATRICES

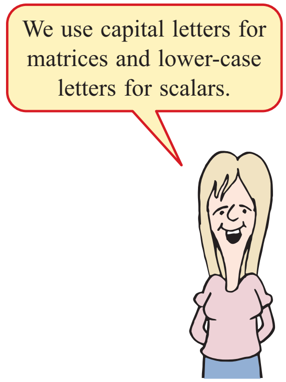
In the pantry there are 6 cans of peaches, 4 cans of apricots, and 8 cans of pears. We represent this by the column vector $\mathbf{C} = \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix}$.

If we doubled the cans in the pantry, we would have $\begin{pmatrix} 12 \\ 8 \\ 16 \end{pmatrix}$ which is $\mathbf{C}+\mathbf{C}$ or $2\mathbf{C}$.

Notice that to get $2\mathbf{C}$ from $\mathbf{C}$ we simply multiply all the matrix elements by 2.

Likewise, trebling the fruit cans in the pantry gives $3\mathbf{C} = \begin{pmatrix} 3\times 6 \\ 3\times 4 \\ 3\times 8 \end{pmatrix} = \begin{pmatrix} 18 \\ 12 \\ 24 \end{pmatrix}$

and halving them gives $\frac{1}{2}\mathbf{C} = \begin{pmatrix} \frac{1}{2}\times 6 \\ \frac{1}{2}\times 4 \\ \frac{1}{2}\times 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$.



If $\mathbf{A} = (a_{ij})$ has order $m \times n$, and k is a scalar, then $k\mathbf{A} = (ka_{ij})$.
So, to find $k\mathbf{A}$, we multiply each element in **A** by k.
The result is another matrix of order $m \times n$.

Example 4 — Self Tutor

If **A** is $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix}$, find:

a $3\mathbf{A}$ **b** $\frac{1}{2}\mathbf{A}$

a
$$3\mathbf{A} = 3\begin{pmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3\times 1 & 3\times 2 & 3\times 5 \\ 3\times 2 & 3\times 0 & 3\times 1 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 15 \\ 6 & 0 & 3 \end{pmatrix}$$

b
$$\tfrac{1}{2}\mathbf{A} = \tfrac{1}{2}\begin{pmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\times 1 & \frac{1}{2}\times 2 & \frac{1}{2}\times 5 \\ \frac{1}{2}\times 2 & \frac{1}{2}\times 0 & \frac{1}{2}\times 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & 2\frac{1}{2} \\ 1 & 0 & \frac{1}{2} \end{pmatrix}$$

EXERCISE 12B.2

1 If $\mathbf{B} = \begin{pmatrix} 6 & 12 \\ 24 & 6 \end{pmatrix}$, find:

a $2\mathbf{B}$ **b** $\frac{1}{3}\mathbf{B}$ **c** $\frac{1}{12}\mathbf{B}$ **d** $-\frac{1}{2}\mathbf{B}$

2 If $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} - \mathbf{B}$ **c** $2\mathbf{A} + \mathbf{B}$ **d** $3\mathbf{A} - \mathbf{B}$

3 A builder builds a block of 12 identical flats. Each flat is to contain 1 table, 4 chairs, 2 beds, and 1 wardrobe.

Let $\mathbf{F} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 1 \end{pmatrix}$ be the matrix representing the furniture in one flat.

In terms of $\mathbf{F}$, what is the matrix representing the furniture in **all** flats? Evaluate this matrix.

4 On weekdays, a video store finds that its average daily hirings are 75 DVD movies, 27 Blu-ray movies, and 102 games. On weekends, the average daily hirings are 43 Blu-ray movies, 136 DVD movies, and 129 games.

a Represent the data using *two* column matrices **A** and **B**.

b Find $5\mathbf{A} + 2\mathbf{B}$.

c What does the matrix in **b** represent?

$$\begin{pmatrix} \; \\ \; \\ \; \end{pmatrix} \begin{array}{l} \leftarrow \text{DVD movies} \\ \leftarrow \text{Blu-ray movies} \\ \leftarrow \text{games} \end{array}$$

5 Isabelle sells clothing made by four different companies which we will call A, B, C, and D.

Her usual monthly order is:

$$\begin{array}{r} \\ \text{skirt} \\ \text{dress} \\ \text{evening} \\ \text{suit} \end{array} \begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{pmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{pmatrix} \end{array}$$

Find her order, to the nearest whole number, if:

a she increases her total order by 15%

b she decreases her total order by 15%.

ZERO OR NULL MATRIX

A **zero matrix** is a matrix in which all the elements are zero.

For example, the 2×2 zero matrix is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and the 2×3 zero matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

If **A** is a matrix of any order and **O** is the corresponding **zero matrix**, then $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$.

For example: $\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$.

NEGATIVE MATRICES

The **negative** matrix **A**, denoted $-\mathbf{A}$, is actually $-1\mathbf{A}$.

$-\mathbf{A}$ is obtained from **A** by reversing the sign of each element of **A**.

For example, if $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$, then $-\mathbf{A} = \begin{pmatrix} -1\times 3 & -1\times -1 \\ -1\times 2 & -1\times 4 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & -4 \end{pmatrix}$

The addition of a matrix and its negative always produces a zero matrix.

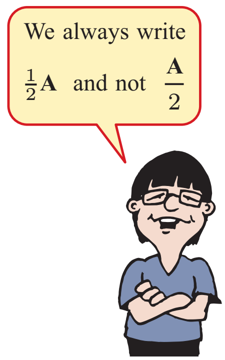
$$\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{O}$$

For example: $\begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 1 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

MATRIX ALGEBRA

We now compare our discoveries about matrices so far with ordinary algebra. We assume that **A** and **B** are matrices of the same order.

Ordinary algebra	Matrix algebra
• If a and b are real numbers then $a+b$ is also a real number.	• If **A** and **B** are matrices then $\mathbf{A}+\mathbf{B}$ is a matrix of the same order.
• $a+b=b+a$	• $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
• $(a+b)+c=a+(b+c)$	• $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$
• $a+0=0+a=a$	• $\mathbf{A}+\mathbf{O}=\mathbf{O}+\mathbf{A}=\mathbf{A}$
• $a+(-a)=(-a)+a=0$	• $\mathbf{A}+(-\mathbf{A})=(-\mathbf{A})+\mathbf{A}=\mathbf{O}$
• a half of a is $\frac{a}{2}$	• a half of **A** is $\frac{1}{2}\mathbf{A}$



Example 5 — Self Tutor

Show that:

a if $\mathbf{X}+\mathbf{A}=\mathbf{B}$ then $\mathbf{X}=\mathbf{B}-\mathbf{A}$ **b** if $3\mathbf{X}=\mathbf{A}$ then $\mathbf{X}=\frac{1}{3}\mathbf{A}$

a
$$\begin{aligned} \mathbf{X}+\mathbf{A} &= \mathbf{B} \\ \therefore\ \mathbf{X}+\mathbf{A}+(-\mathbf{A}) &= \mathbf{B}+(-\mathbf{A}) \\ \therefore\ \mathbf{X}+\mathbf{O} &= \mathbf{B}-\mathbf{A} \\ \therefore\ \mathbf{X} &= \mathbf{B}-\mathbf{A} \end{aligned}$$

b
$$\begin{aligned} 3\mathbf{X} &= \mathbf{A} \\ \therefore\ \tfrac{1}{3}(3\mathbf{X}) &= \tfrac{1}{3}\mathbf{A} \\ \therefore\ 1\mathbf{X} &= \tfrac{1}{3}\mathbf{A} \\ \therefore\ \mathbf{X} &= \tfrac{1}{3}\mathbf{A} \end{aligned}$$

EXERCISE 12B.3

1 Simplify:

a $\mathbf{A}+2\mathbf{A}$ **b** $3\mathbf{B}-3\mathbf{B}$ **c** $\mathbf{C}-2\mathbf{C}$

d $-\mathbf{B}+\mathbf{B}$ **e** $2(\mathbf{A}+\mathbf{B})$ **f** $-(\mathbf{A}+\mathbf{B})$

g $-(2\mathbf{A}-\mathbf{C})$ **h** $3\mathbf{A}-(\mathbf{B}-\mathbf{A})$ **i** $\mathbf{A}+2\mathbf{B}-(\mathbf{A}-\mathbf{B})$

2 Find **X** in terms of **A**, **B**, and **C** if:

a $\mathbf{X}+\mathbf{B}=\mathbf{A}$ **b** $\mathbf{B}+\mathbf{X}=\mathbf{C}$ **c** $4\mathbf{B}+\mathbf{X}=2\mathbf{C}$

d $2\mathbf{X}=\mathbf{A}$ **e** $3\mathbf{X}=\mathbf{B}$ **f** $\mathbf{A}-\mathbf{X}=\mathbf{B}$

g $\frac{1}{2}\mathbf{X}=\mathbf{C}$ **h** $2(\mathbf{X}+\mathbf{A})=\mathbf{B}$ **i** $\mathbf{A}-4\mathbf{X}=\mathbf{C}$

3 **a** Suppose $\mathbf{M}=\begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix}$ and $\frac{1}{3}\mathbf{X}=\mathbf{M}$. Find **X**.

b Suppose $\mathbf{N}=\begin{pmatrix}2 & -1\\ 3 & 5\end{pmatrix}$ and $4\mathbf{X}=\mathbf{N}$. Find **X**.

c Suppose $\mathbf{A}=\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 4\\ -1 & 1\end{pmatrix}$, and $\mathbf{A}-2\mathbf{X}=3\mathbf{B}$. Find **X**.

C MATRIX MULTIPLICATION

Suppose you go to a shop and purchase 3 cans of soft drink, 4 chocolate bars, and 2 ice creams.

The prices are:

soft drink cans	chocolate bars	ice creams
\$1.30	\$0.90	\$1.20

We can represent this by the quantities matrix $\mathbf{A}=\begin{pmatrix}3 & 4 & 2\end{pmatrix}$ and the costs matrix $\mathbf{B}=\begin{pmatrix}1.30\\ 0.90\\ 1.20\end{pmatrix}$.

We can find the total cost of the items by multiplying the number of each item by its respective cost, and then adding the results:

$$3\times \$1.30+4\times \$0.90+2\times \$1.20=\$9.90$$

We can also determine the total cost by the **matrix multiplication**:

$$\begin{aligned}\mathbf{AB}&=\begin{pmatrix}3 & 4 & 2\end{pmatrix}\begin{pmatrix}1.30\\ 0.90\\ 1.20\end{pmatrix}\\ &=(3\times 1.30)+(4\times 0.90)+(2\times 1.20)\\ &=9.90\end{aligned}$$

Notice that we write the **row matrix** first and the **column matrix** second.

In general, $$\begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}p\\ q\\ r\end{pmatrix}=ap+bq+cr.$$

EXERCISE 12C.1

1 Determine:

a $\begin{pmatrix}3 & -1\end{pmatrix}\begin{pmatrix}5\\ 4\end{pmatrix}$ **b** $\begin{pmatrix}1 & 3 & 2\end{pmatrix}\begin{pmatrix}5\\ 1\\ 7\end{pmatrix}$ **c** $\begin{pmatrix}6 & -1 & 2 & 3\end{pmatrix}\begin{pmatrix}1\\ 0\\ -1\\ 4\end{pmatrix}$