6 Solve $\tan x = \sqrt{3}$ for $0 \leqslant x \leqslant 2\pi$. Hence solve the following equations for $0 \leqslant x \leqslant 2\pi$:

a $\tan 4x = \sqrt{3}$ **b** $\tan^2 x = 3$

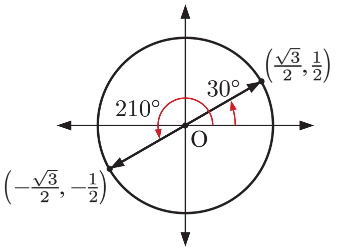
7 Solve $\sqrt{3}\tan 3x = 1$ for $0 \leqslant x \leqslant \pi$.

8 Solve for $-\pi \leqslant x \leqslant \pi$:

a $\sec x = -2$ **b** $\sqrt{3}\operatorname{cosec} 2x = 2$ **c** $\cot x = 0$

Example 10 — Self Tutor

Find the exact solutions of $\sqrt{3}\sin x = \cos x$ for $0° \leqslant x \leqslant 360°$.



$$\sqrt{3}\sin x = \cos x$$

$$\therefore \quad \frac{\sin x}{\cos x} = \frac{1}{\sqrt{3}} \quad \{\text{dividing both sides by } \sqrt{3}\cos x\}$$

$$\therefore \quad \tan x = \frac{1}{\sqrt{3}}$$

$$\therefore \quad x = 30° \text{ or } 210°$$

9 Solve for $0 \leqslant x \leqslant 2\pi$:

a $\sin x - \cos x = 0$ **b** $\sin x = -\cos x$

c $\sin 3x = \cos 3x$ **d** $\sin 2x = \sqrt{3}\cos 2x$

Check your answers using the graphing package.

GRAPHING PACKAGE

10 Solve for $0 \leqslant x \leqslant \pi$: $\sin x = \operatorname{cosec} x$

F TRIGONOMETRIC RELATIONSHIPS

There are a vast number of trigonometric relationships. However, we only need to remember a few because we can obtain the rest by rearrangement or substitution.

SIMPLIFYING TRIGONOMETRIC EXPRESSIONS

For any given angle θ, $\sin\theta$ and $\cos\theta$ are real numbers. $\tan\theta$ is also real whenever it is defined. The algebra of trigonometry is therefore identical to the algebra of real numbers.

An expression like $2\sin\theta + 3\sin\theta$ compares with $2x + 3x$, so $2\sin\theta + 3\sin\theta = 5\sin\theta$.

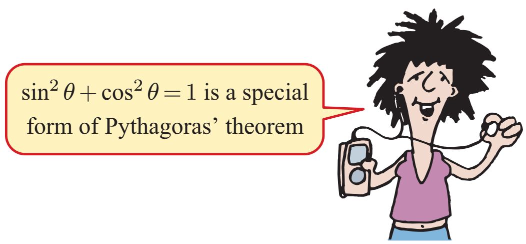
To simplify more complicated trigonometric expressions, we often use the identities:

$$\mathbf{\sin^2\theta + \cos^2\theta = 1}$$

$$\mathbf{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

$$\mathbf{\tan^2\theta + 1 = \sec^2\theta}$$

$$\mathbf{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$$



We can also use rearrangements of these formulae, such as:

$$\sin^2\theta = 1 - \cos^2\theta \qquad \tan^2\theta = \sec^2\theta - 1$$
$$\cos^2\theta = 1 - \sin^2\theta \qquad \cot^2\theta = \operatorname{cosec}^2\theta - 1$$

Example 11 — Self Tutor

Simplify:

a $3\cos\theta + 4\cos\theta$ **b** $\tan\alpha - 3\tan\alpha$

a $3\cos\theta + 4\cos\theta = 7\cos\theta$
{compare with $3x + 4x = 7x$}

b $\tan\alpha - 3\tan\alpha = -2\tan\alpha$
{compare with $x - 3x = -2x$}

Example 12 — Self Tutor

Simplify:

a $2 - 2\sin^2\theta$ **b** $\cos^2\theta\sin\theta + \sin^3\theta$

a
$$\begin{aligned}&2 - 2\sin^2\theta\\ &= 2(1-\sin^2\theta)\\ &= 2\cos^2\theta\end{aligned}$$
$\{\cos^2\theta + \sin^2\theta = 1\}$

b
$$\begin{aligned}&\cos^2\theta\sin\theta + \sin^3\theta\\ &= \sin\theta(\cos^2\theta + \sin^2\theta)\\ &= \sin\theta \times 1\\ &= \sin\theta\end{aligned}$$

EXERCISE 9F.1

1 Simplify:

a $\sin\theta + \sin\theta$ **b** $2\cos\theta + \cos\theta$ **c** $3\sin\theta - \sin\theta$

d $3\sin\theta - 2\sin\theta$ **e** $\tan\theta - 3\tan\theta$ **f** $2\cos^2\theta - 5\cos^2\theta$

Example 13 — Self Tutor

Expand and simplify: $(\cos\theta - \sin\theta)^2$

$$\begin{aligned}&(\cos\theta - \sin\theta)^2\\ &= \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta \qquad \{\text{using } (a-b)^2 = a^2 - 2ab + b^2\}\\ &= \cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta\\ &= 1 - 2\cos\theta\sin\theta\end{aligned}$$

2 Simplify:

a $3\sin^2\theta + 3\cos^2\theta$ **b** $-2\sin^2\theta - 2\cos^2\theta$ **c** $-\cos^2\theta - \sin^2\theta$

d $3 - 3\sin^2\theta$ **e** $4 - 4\cos^2\theta$ **f** $\cos^3\theta + \cos\theta\sin^2\theta$

g $\cos^2\theta - 1$ **h** $\sin^2\theta - 1$ **i** $2\cos^2\theta - 2$

j $\dfrac{1-\sin^2\theta}{\cos^2\theta}$ **k** $\dfrac{1-\cos^2\theta}{\sin\theta}$ **l** $\dfrac{\cos^2\theta - 1}{-\sin\theta}$

3 Simplify:

a $3\tan x - \dfrac{\sin x}{\cos x}$
b $\dfrac{\sin^2 x}{\cos^2 x}$
c $\tan x \cos x$
d $\dfrac{\sin x}{\tan x}$
e $3\sin x + 2\cos x \tan x$
f $\dfrac{2\tan x}{\sin x}$
g $\tan x \cot x$
h $\sin x \operatorname{cosec} x$
i $\sec x \cot x$
j $\sin x \cot x$
k $\dfrac{\cot x}{\operatorname{cosec} x}$
l $\dfrac{2\sin x \cot x + 3\cos x}{\cot x}$

4 Expand and simplify if possible:

a $(1+\sin\theta)^2$
b $(\sin\alpha - 2)^2$
c $(\tan\alpha - 1)^2$
d $(\sin\alpha + \cos\alpha)^2$
e $(\sin\beta - \cos\beta)^2$
f $-(2-\cos\alpha)^2$

5 Simplify:

a $1 - \sec^2\beta$
b $\dfrac{\tan^2\theta(\cot^2\theta + 1)}{\tan^2\theta + 1}$
c $\cos^2\alpha(\sec^2\alpha - 1)$
d $(\sin x + \tan x)(\sin x - \tan x)$
e $(2\sin\theta + 3\cos\theta)^2 + (3\sin\theta - 2\cos\theta)^2$
f $(1+\operatorname{cosec}\theta)(\sin\theta - \sin^2\theta)$
g $\sec A - \sin A \tan A - \cos A$

FACTORISING TRIGONOMETRIC EXPRESSIONS

Example 14 Self Tutor

Factorise:

a $\cos^2\alpha - \sin^2\alpha$
b $\tan^2\theta - 3\tan\theta + 2$

a $\cos^2\alpha - \sin^2\alpha$
$= (\cos\alpha + \sin\alpha)(\cos\alpha - \sin\alpha)$ {compare with $a^2 - b^2 = (a+b)(a-b)$}

b $\tan^2\theta - 3\tan\theta + 2$
$= (\tan\theta - 2)(\tan\theta - 1)$ {compare with $x^2 - 3x + 2 = (x-2)(x-1)$}

EXERCISE 9F.2

1 Factorise:

a $1 - \sin^2\theta$
b $\sin^2\alpha - \cos^2\alpha$
c $\tan^2\alpha - 1$
d $2\sin^2\beta - \sin\beta$
e $2\cos\phi + 3\cos^2\phi$
f $3\sin^2\theta - 6\sin\theta$
g $\tan^2\theta + 5\tan\theta + 6$
h $2\cos^2\theta + 7\cos\theta + 3$
i $6\cos^2\alpha - \cos\alpha - 1$
j $3\tan^2\alpha - 2\tan\alpha$
k $\sec^2\beta - \operatorname{cosec}^2\beta$
l $2\cot^2 x - 3\cot x + 1$
m $2\sin^2 x + 7\sin x\cos x + 3\cos^2 x$

Example 15 — Self Tutor

Simplify:

a $\dfrac{2-2\cos^2\theta}{1+\cos\theta}$ **b** $\dfrac{\cos\theta-\sin\theta}{\cos^2\theta-\sin^2\theta}$

a
$$\begin{aligned}
&\frac{2-2\cos^2\theta}{1+\cos\theta}\\
&=\frac{2(1-\cos^2\theta)}{1+\cos\theta}\\
&=\frac{2\cancelto{1}{(1+\cos\theta)}(1-\cos\theta)}{\cancelto{1}{(1+\cos\theta)}}\\
&=2(1-\cos\theta)
\end{aligned}$$

b
$$\begin{aligned}
&\frac{\cos\theta-\sin\theta}{\cos^2\theta-\sin^2\theta}\\
&=\frac{\cancelto{1}{(\cos\theta-\sin\theta)}}{(\cos\theta+\sin\theta)\cancelto{1}{(\cos\theta-\sin\theta)}}\\
&=\frac{1}{\cos\theta+\sin\theta}
\end{aligned}$$

2 Simplify:

a $\dfrac{1-\sin^2\alpha}{1-\sin\alpha}$ **b** $\dfrac{\tan^2\beta-1}{\tan\beta+1}$ **c** $\dfrac{\cos^2\phi-\sin^2\phi}{\cos\phi+\sin\phi}$

d $\dfrac{\cos^2\phi-\sin^2\phi}{\cos\phi-\sin\phi}$ **e** $\dfrac{\sin\alpha+\cos\alpha}{\sin^2\alpha-\cos^2\alpha}$ **f** $\dfrac{3-3\sin^2\theta}{6\cos\theta}$

g $1-\dfrac{\cos^2\theta}{1+\sin\theta}$ **h** $\dfrac{1+\cot\theta}{\operatorname{cosec}\theta}-\dfrac{\sec\theta}{\tan\theta+\cot\theta}$ **i** $\dfrac{\tan^2\theta}{\sec\theta-1}$

3 Show that:

a $(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2=2$ **b** $(2\sin\theta+3\cos\theta)^2+(3\sin\theta-2\cos\theta)^2=13$

c $(1-\cos\theta)\left(1+\dfrac{1}{\cos\theta}\right)=\tan\theta\sin\theta$ **d** $\left(1+\dfrac{1}{\sin\theta}\right)(\sin\theta-\sin^2\theta)=\cos^2\theta$

e $\sec A-\cos A=\tan A\sin A$ **f** $\dfrac{\cos\theta}{1-\sin\theta}=\sec\theta+\tan\theta$

g $\dfrac{\cos\alpha}{1-\tan\alpha}+\dfrac{\sin\alpha}{1-\cot\alpha}=\sin\alpha+\cos\alpha$ **h** $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}=2\operatorname{cosec}\theta$

i $\dfrac{\sin\theta}{1-\cos\theta}-\dfrac{\sin\theta}{1+\cos\theta}=2\cot\theta$ **j** $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}=2\sec^2\theta$

GRAPHING PACKAGE

Use a graphing package to check these simplifications by graphing each function on the same set of axes.

Discovery 5 — Double angle formulae

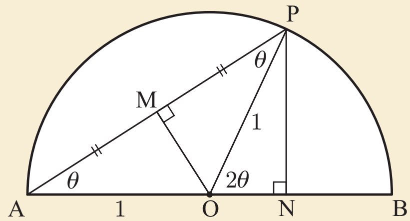
What to do:

1 Copy and complete, using angles of your choice as well:

θ	$\sin 2\theta$	$2\sin\theta$	$2\sin\theta\cos\theta$	$\cos 2\theta$	$2\cos\theta$	$\cos^2\theta-\sin^2\theta$
0.631						
57.81°						
-3.697						

2 Write down any discoveries from your table of values in **1**.

3 In the diagram alongside, the semi-circle has radius 1 unit, and $\widehat{PAB} = \theta$.



$\widehat{APO} = \theta$ {$\triangle$AOP is isosceles}

$\widehat{PON} = 2\theta$ {exterior angle of a triangle}

a Find in terms of θ, the lengths of:

i OM **ii** AM **iii** ON **iv** PN

b Use $\triangle$ANP and the lengths in **a** to show that:

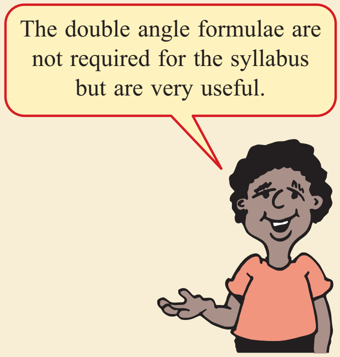
i $\cos\theta = \dfrac{\sin 2\theta}{2\sin\theta}$ **ii** $\cos\theta = \dfrac{1+\cos 2\theta}{2\cos\theta}$

c Hence deduce that:

i $\sin 2\theta = 2\sin\theta\cos\theta$ **ii** $\cos 2\theta = 2\cos^2\theta - 1$

4 Starting with $\cos 2\theta = 2\cos^2\theta - 1$, show that:

a $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ **b** $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$



G TRIGONOMETRIC EQUATIONS IN QUADRATIC FORM

Sometimes we may be given trigonometric equations in quadratic form.

For example, $2\sin^2 x + \sin x = 0$ and $2\cos^2 x + \cos x - 1 = 0$ are quadratic equations where the variables are $\sin x$ and $\cos x$ respectively.

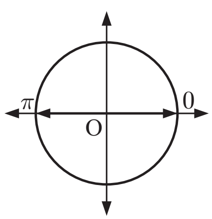
These equations can be factorised by quadratic factorisation and then solved for x.

Example 16 — Self Tutor

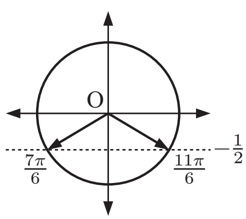
Solve for $0 \leqslant x \leqslant 2\pi$:

a $2\sin^2 x + \sin x = 0$ **b** $2\cos^2 x + \cos x - 1 = 0$

a
$$2\sin^2 x + \sin x = 0$$
$$\therefore \sin x(2\sin x + 1) = 0$$
$$\therefore \sin x = 0 \text{ or } -\tfrac{1}{2}$$



$\sin x = 0$ when
$x = 0, \pi, \text{ or } 2\pi$



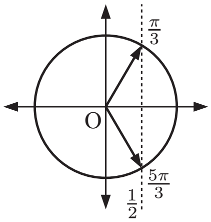
$\sin x = -\frac{1}{2}$ when
$x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

The solutions are: $x = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, \text{ or } 2\pi$.

b $\quad 2\cos^2 x + \cos x - 1 = 0$

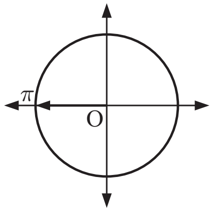
$\therefore \quad (2\cos x - 1)(\cos x + 1) = 0$

$\therefore \quad \cos x = \frac{1}{2}$ or -1



$\cos x = \frac{1}{2}$ when

$x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$



$\cos x = -1$ when

$x = \pi$

The solutions are: $\quad x = \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3}$.

EXERCISE 9G

1 Solve for $0 \leqslant x \leqslant 2\pi$:

a $2\sin^2 x + \sin x = 0$ **b** $2\cos^2 x = \cos x$ **c** $2\cos^2 x + \cos x - 1 = 0$

d $2\sin^2 x + 3\sin x + 1 = 0$ **e** $\sin^2 x = 2 - \cos x$ **f** $\cos x + \sec x = 2$

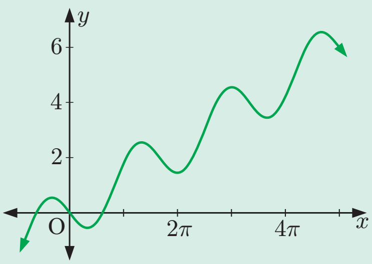
2 Solve for $0 \leqslant x \leqslant 2\pi$:

a $\sin^2 x + \cos x = -1$ **b** $2\cos^2 x = 3\sin x$

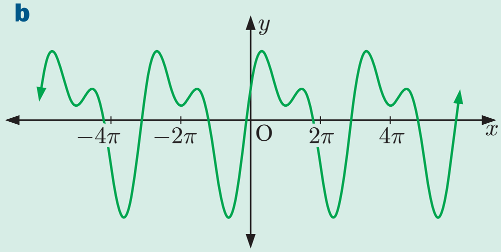
Review set 9A

1 Which of the following graphs displays periodic behaviour?

a



b



2 Draw each of the following graphs for $0 \leqslant x \leqslant 2\pi$:

a $y = 5\sin x$ **b** $y = \cos 3x - 1$ **c** $y = \tan 2x + 4$

3 State the minimum and maximum values of:

a $1 + \sin x$ **b** $2\cos 3x$ **c** $y = 3\sin 2x$ **d** $y = \cos 4x - 1$

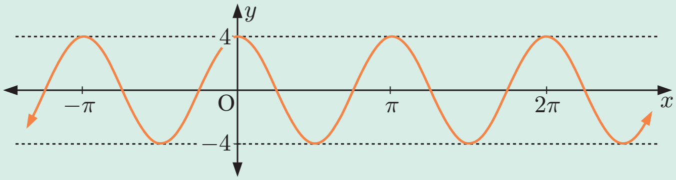
4 State the period of:

a $y = 4\sin x$ **b** $y = 2\cos 4x$ **c** $y = 4\cos 2x + 4$ **d** $y = 2\tan 3x$

5 Complete the table:

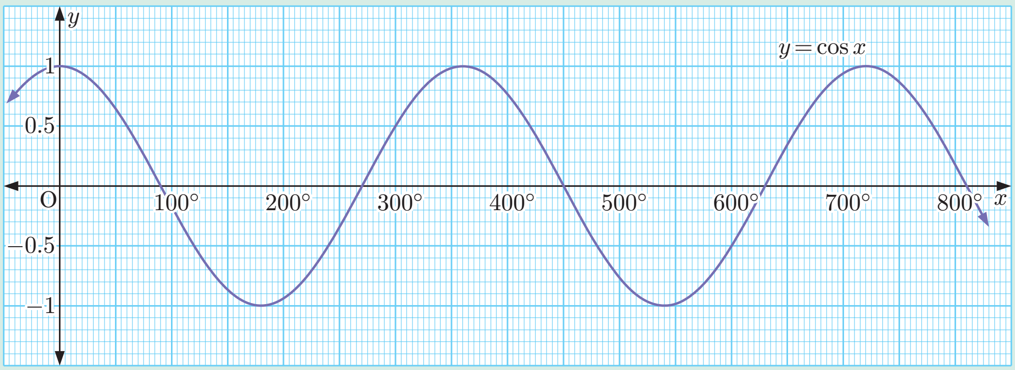
Function	*Period*	*Amplitude*	*Domain*	*Range*
$y = 3\sin 2x + 1$				
$y = \tan 2x$				
$y = 2\cos 3x - 3$				

6 Find the cosine function represented in the graph.



7 On the same set of axes, graph $y = 2\cos x$ and $y = |2\cos x|$ for $0 \leqslant x \leqslant 2\pi$.

8



Use the graph of $y = \cos x$ to find the solutions of:

a $\cos x = -0.4, \quad 0 \leqslant x \leqslant 800°$

b $\cos x = 0.9, \quad 0 \leqslant x \leqslant 600°$

9 Solve in terms of π:

a $2\sin x = -1$ for $0 \leqslant x \leqslant 4\pi$

b $\sqrt{2}\sin x - 1 = 0$ for $-2\pi \leqslant x \leqslant 2\pi$

c $2\sin 3x + \sqrt{3} = 0$ for $0 \leqslant x \leqslant 2\pi$

d $\sqrt{2}\cos x - 1 = 0$ for $0 \leqslant x \leqslant 4\pi$

10 Simplify:

a $\dfrac{1-\cos^2\theta}{1+\cos\theta}$

b $\dfrac{\sin\alpha - \cos\alpha}{\sin^2\alpha - \cos^2\alpha}$

c $\dfrac{4\sin^2\alpha - 4}{8\cos\alpha}$

d $\dfrac{\cot^2\theta}{\operatorname{cosec}\theta - 1}$

11 Show that $\dfrac{\cos\theta - \sec\theta}{\tan\theta}$ simplifies to $-\sin\theta$.

12 Find exact solutions for $-\pi \leqslant x \leqslant \pi$:

a $\tan 2x = -\sqrt{3}$

b $\tan^2 x - 3 = 0$

Review set 9B

1 Consider the graph alongside.

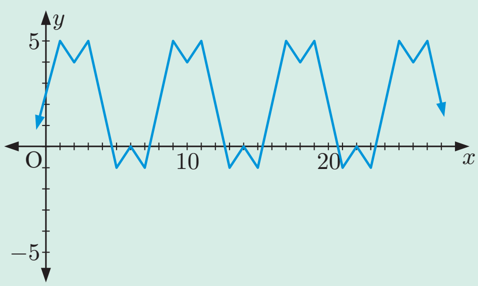
a Explain why this graph shows periodic behaviour.

b State:

i the period

ii the maximum value

iii the minimum value



2 Find b given that the function $y = \sin bx$, $b > 0$ has period:

a $\frac{\pi}{3}$ **b** $\frac{\pi}{12}$

3 State the minimum and maximum values of:

a $y = 5\sin x - 3$ **b** $y = 3\cos x + 1$ **c** $y = 4\cos 2x + 9$

4 On the same set of axes, for the domain $0 \leqslant x \leqslant 2\pi$, sketch:

a $y = \cos x$ and $y = \cos x - 3$ **b** $y = \tan x$ and $y = 2\tan x$

c $y = \cos x$ and $y = \cos 2x + 1$ **d** $y = \sin x$ and $y = 3\sin x + 1$

5 The function $y = a\sin bx + c$, $a > 0$, $b > 0$, has amplitude 2, period $\frac{\pi}{3}$, and principal axis $y = -2$.

a Find the values of a, b, and c. **b** Sketch the function for $0 \leqslant x \leqslant \pi$.

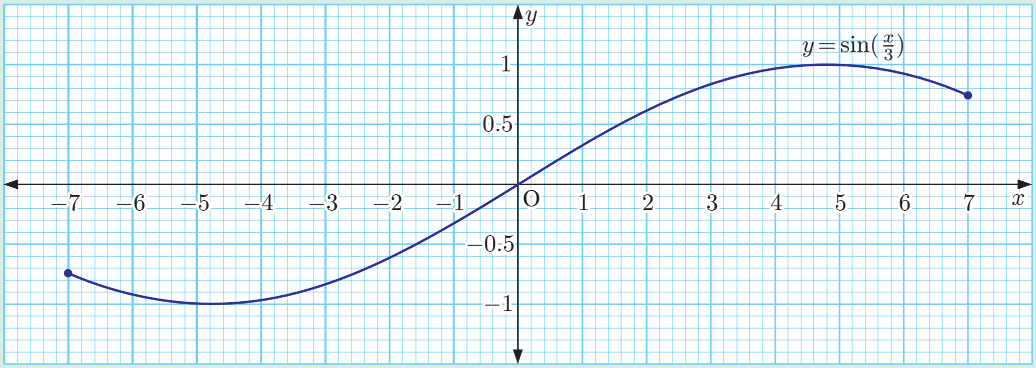
6 Consider the function $y = 2\tan x$.

a State a function which has the same shape, but has principal axis $y = 2$.

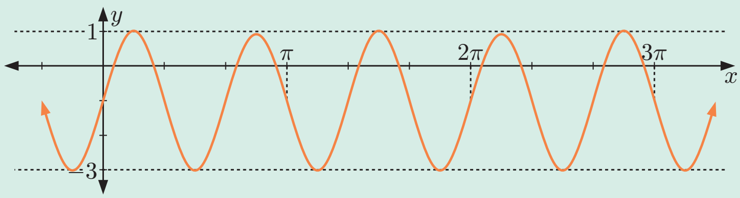
b Draw $y = 2\tan x$ and your function from **a** on the same set of axes, for $-2\pi \leqslant x \leqslant 2\pi$.

7 Consider $y = \sin(\frac{x}{3})$ on the domain $-7 \leqslant x \leqslant 7$. Use the graph to solve, correct to 1 decimal place:

a $\sin(\frac{x}{3}) = -0.9$ **b** $\sin(\frac{x}{3}) = \frac{1}{4}$



8 Find m and n given the following graph of the function $y = 2\sin mx + n$:



9 Solve for $0 \leqslant x \leqslant 2\pi$:

a $\sin^2 x - \sin x - 2 = 0$ **b** $4\sin^2 x = 1$

10 Simplify:

a $\cos^3\theta + \sin^2\theta\cos\theta$ **b** $\dfrac{\cos^2\theta - 1}{\sin\theta}$

c $5 - 5\sin^2\theta$ **d** $\dfrac{\sin^2\theta - 1}{\cos\theta}$

11 Expand and simplify if possible:

a $(2\sin\alpha - 1)^2$

b $(\cos\alpha - \sin\alpha)^2$

12 Show that:

a $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta} = 2\sec\theta$

b $(1+\sec\theta)\left(\cos\theta - \cos^2\theta\right) = \sin^2\theta$

10

Counting and the binomial expansion

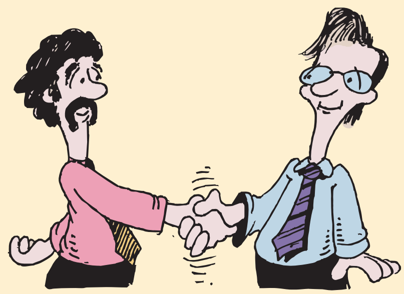
Contents:

Opening problem

At a mathematics teachers' conference there are 273 delegates present. The organising committee consists of 10 people.

Things to think about:

- **a** If each committee member shakes hands with every other committee member, how many handshakes take place?
  Can a 10-sided convex polygon be used to solve this problem?
- **b** If all 273 delegates shake hands with all other delegates, how many handshakes take place now?
- **c** If the organising committee lines up on stage to face the delegates in the audience, in how many different orders can they line up?



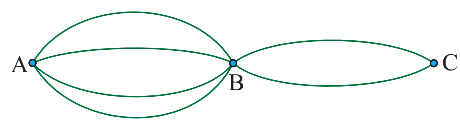
The **Opening Problem** is an example of a **counting** problem.

The following exercises will help us to solve counting problems without having to list and count the possibilities one by one. To do this we will examine:

- the product principle
- counting permutations
- counting combinations.

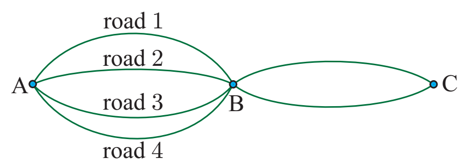
A THE PRODUCT PRINCIPLE

Suppose there are three towns A, B, and C. Four different roads could be taken from A to B, and two different roads from B to C.



How many different pathways are there from A to C going through B?

If we take road 1, there are two alternative roads to complete our trip.



Similarly, if we take road 2, there are two alternative roads to complete our trip.

The same is true for roads 3 and 4.

So, there are $2 + 2 + 2 + 2 = 4 \times 2$ different pathways from A to C going through B.

Notice that the 4 corresponds to the number of roads from A to B and the 2 corresponds to the number of roads from B to C.

THE PRODUCT PRINCIPLE

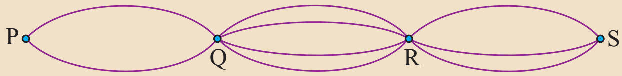
If there are m different ways of performing an operation, and for each of these there are n different ways of performing a second **independent** operation, then there are mn different ways of performing the two operations in succession.

The product principle can be extended to three or more successive independent operations.

Example 1

Self Tutor

P, Q, R, and S represent where Pauline, Quentin, Reiko, and Sam live. There are two different paths from P to Q, four different paths from Q to R, and 3 different paths from R to S.

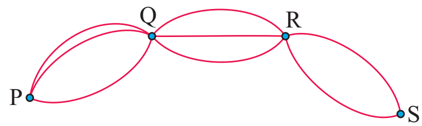


How many different pathways could Pauline take to visit Sam if she stops to see Quentin and then Reiko on the way?

The total number of different pathways $= 2 \times 4 \times 3 = 24$ {product principle}

EXERCISE 10A

1 The illustration shows the different map routes for a bus service which goes from P to S through both Q and R. How many different routes are possible?

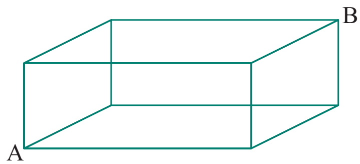


2 

In how many ways can the vertices of a rectangle be labelled with the letters A, B, C, and D:

- **a** in clockwise alphabetical order
- **b** in alphabetical order
- **c** in random order?

3 The wire frame shown forms the outline of a box. An ant crawls along the wire from A to B. How many different paths of shortest length lead from A to B?



4 A table tennis competition has 7 teams. In how many different ways can the top two positions be filled in order of premiership points obtained?

5 A football competition is organised between 8 teams. In how many ways can the top 4 places be filled in order of premiership points obtained?

6 How many 3-digit numbers can be formed using the digits 2, 3, 4, 5, and 6:

- **a** as often as desired
- **b** at most once each?

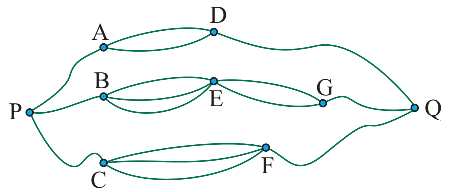
7 How many different alpha-numeric plates for motor car registration can be made if the first 3 places are English alphabet letters and the remaining places are 3 digits from 0 to 9?

8 In how many ways can:

- **a** 2 postcards be mailed into 2 mail boxes
- **b** 2 postcards be mailed into 3 mail boxes
- **c** 4 postcards be mailed into 3 mail boxes?

B COUNTING PATHS

Consider the road system illustrated which shows the roads from P to Q.



From A to Q there are 2 paths.
From B to Q there are $3 \times 2 = 6$ paths.
From C to Q there are 3 paths.

$\therefore$ from P to Q there are $2 + 6 + 3 = 11$ paths.

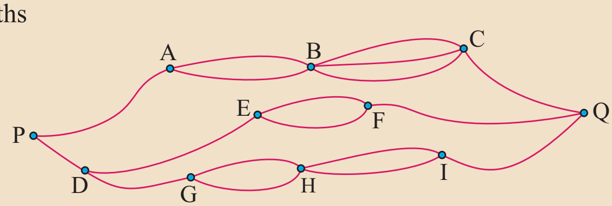
Notice that:

- When going from B to G, we go from B to E **and** then from E to G. We **multiply** the possibilities.
- When going from P to Q, we must first go from P to A **or** P to B **or** P to C. We **add** the possibilities from each of these first steps.

The word **and** suggests *multiplying* the possibilities.
The word **or** suggests *adding* the possibilities.

Example 2 — Self Tutor

How many different paths lead from P to Q?



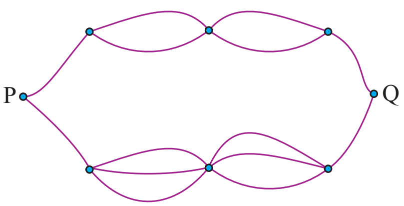
	From	P to A to B to C to Q	there are $2 \times 3 = 6$ paths
or	from	P to D to E to F to Q	there are 2 paths
or	from	P to D to G to H to I to Q	there are $2 \times 2 = 4$ paths.

In total there are $6 + 2 + 4 = 12$ different paths.

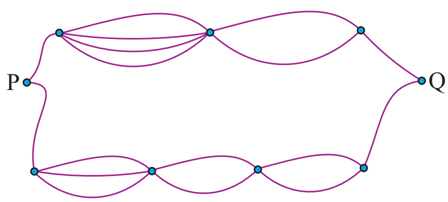
EXERCISE 10B

1 How many different paths lead from P to Q?

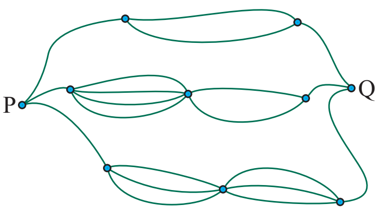
a



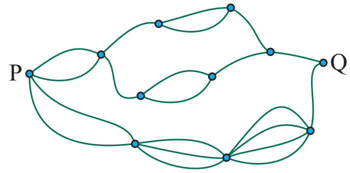
b



c

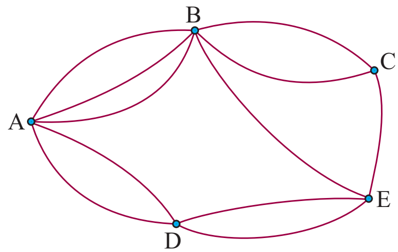


d



2 Katie is going on a long journey to visit her family. She lives in city A and is travelling to city E. Unfortunately there are no direct trains. However, she has the choice of several trains which stop in different cities along the way. These are illustrated in the diagram.

How many different train journeys does Katie have to choose from?



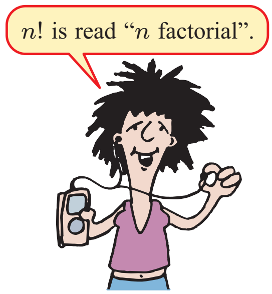
C FACTORIAL NOTATION

In problems involving counting, products of consecutive positive integers such as $8 \times 7 \times 6$ and $6 \times 5 \times 4 \times 3 \times 2 \times 1$ are common.

For convenience, we introduce **factorial numbers** to represent the products of consecutive positive integers.

For $n \geqslant 1$, $n!$ is the product of the first n positive integers.

$$\boldsymbol{n! = n(n-1)(n-2)(n-3) \ldots. \times 3 \times 2 \times 1}$$



For example, the product $6 \times 5 \times 4 \times 3 \times 2 \times 1$ can be written as $6!$.

Notice that $8 \times 7 \times 6$ can be written using factorial numbers only as

$$8 \times 7 \times 6 = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{8!}{5!}$$

An alternative **recursive definition** of factorial numbers is $\boldsymbol{n! = n \times (n-1)!}$ for $n \geqslant 1$

which can be extended to $n! = n(n-1)(n-2)!$ and so on.

Using the factorial rule with $n = 1$, we have $1! = 1 \times 0!$

Therefore, for completeness we define $\boldsymbol{0! = 1}$

Example 3 Self Tutor

Simplify: **a** $4!$ **b** $\dfrac{5!}{3!}$ **c** $\dfrac{7!}{4! \times 3!}$

a $4! = 4 \times 3 \times 2 \times 1 = 24$

b $\dfrac{5!}{3!} = \dfrac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 5 \times 4 = 20$

c $\dfrac{7!}{4! \times 3!} = \dfrac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1} = 35$

If your problem involves factorials of large numbers then it is important to cancel as many factors as possible before using a calculator to evaluate the rest.

For example, if you have $\dfrac{300!}{297!}$ in your problem, you will find you cannot calculate 300! on your calculator. However, we can see that

$$\begin{aligned}\frac{300!}{297!} = \frac{300 \times 299 \times 298 \times 297!}{297!} &= 300 \times 299 \times 298 \\ &= 26\,730\,600.\end{aligned}$$

EXERCISE 10C.1

1 Find $n!$ for $n = 0, 1, 2, 3, ..., 10$.

2 Simplify without using a calculator:

a $\dfrac{6!}{5!}$ **b** $\dfrac{6!}{4!}$ **c** $\dfrac{6!}{7!}$ **d** $\dfrac{4!}{6!}$ **e** $\dfrac{100!}{99!}$ **f** $\dfrac{7!}{5! \times 2!}$

3 Simplify:

a $\dfrac{n!}{(n-1)!}$ **b** $\dfrac{(n+2)!}{n!}$ **c** $\dfrac{(n+1)!}{(n-1)!}$

Example 4 — Self Tutor

Express in factorial form:

a $10 \times 9 \times 8 \times 7$ **b** $\dfrac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1}$

a $10 \times 9 \times 8 \times 7 = \dfrac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \dfrac{10!}{6!}$

b $\dfrac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = \dfrac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \dfrac{10!}{4! \times 6!}$

4 Express in factorial form:

a $7 \times 6 \times 5$ **b** 10×9 **c** $11 \times 10 \times 9 \times 8 \times 7$

d $\dfrac{13 \times 12 \times 11}{3 \times 2 \times 1}$ **e** $\dfrac{1}{6 \times 5 \times 4}$ **f** $\dfrac{4 \times 3 \times 2 \times 1}{20 \times 19 \times 18 \times 17}$

Example 5 — Self Tutor

Write as a product by factorising:

a $8! + 6!$ **b** $10! - 9! + 8!$

a
$$\begin{aligned}&8! + 6! \\ &= 8 \times 7 \times 6! + 6! \\ &= 6!(8 \times 7 + 1) \\ &= 6! \times 57\end{aligned}$$

b
$$\begin{aligned}&10! - 9! + 8! \\ &= 10 \times 9 \times 8! - 9 \times 8! + 8! \\ &= 8!(90 - 9 + 1) \\ &= 8! \times 82\end{aligned}$$

5 Write as a product by factorising:

a $5! + 4!$ **b** $11! - 10!$ **c** $7! + 9!$ **d** $12! - 10!$

e $9! + 8! + 7!$ **f** $7! - 6! + 8!$ **g** $12! - 2 \times 11!$ **h** $3 \times 9! + 5 \times 8!$

Example 6 Self Tutor

Simplify $\frac{7! - 6!}{6}$ by factorising.

$$\frac{7! - 6!}{6}$$
$$= \frac{7 \times 6! - 6!}{6}$$
$$= \frac{6!\cancel{(7-1)}^{1}}{\cancel{6}_{1}}$$
$$= 6!$$

6 Simplify by factorising:

a $\frac{12! - 11!}{11}$ **b** $\frac{10! + 9!}{11}$ **c** $\frac{10! - 8!}{89}$ **d** $\frac{10! - 9!}{9!}$

e $\frac{6! + 5! - 4!}{4!}$ **f** $\frac{n! + (n-1)!}{(n-1)!}$ **g** $\frac{n! - (n-1)!}{n-1}$ **h** $\frac{(n+2)! + (n+1)!}{n+3}$

THE BINOMIAL COEFFICIENT

The **binomial coefficient** is defined by

$$\binom{n}{r} = \underbrace{\frac{n(n-1)(n-2)\,....\,(n-r+2)(n-r+1)}{r(r-1)(r-2)\,....\,2 \times 1}}_{\text{factor form}} = \underbrace{\frac{n!}{r!(n-r)!}}_{\text{factorial form}}$$

The binomial coefficient is sometimes written ${}^nC_r$ or C_r^n.

Example 7 Self Tutor

Use the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to evaluate: **a** $\binom{5}{2}$ **b** $\binom{11}{7}$

a $$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2! \times 3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1} = 10$$

b $$\binom{11}{7} = \frac{11!}{7!(11-7)!} = \frac{11!}{7! \times 4!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1} = \frac{7920}{24} = 330$$

EXERCISE 10C.2

1 Use the formula $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ to evaluate:

a $\binom{3}{1}$ **b** $\binom{4}{2}$ **c** $\binom{7}{3}$ **d** $\binom{10}{4}$

Check your answers using technology.

2 **a** Use the formula $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ to evaluate:

i $\binom{8}{2}$ **ii** $\binom{8}{6}$

b Show that $\binom{n}{r} = \binom{n}{n-r}$ for all $n \in \mathbb{Z}^+$, $r = 0, 1, 2, ..., n$.

3 Find k if $\binom{9}{k} = 4\binom{7}{k-1}$.

D PERMUTATIONS

A **permutation** of a group of symbols is *any arrangement* of those symbols in a definite *order*.

For example, BAC is a permutation on the symbols A, B, and C in which all three of them are used. We say the symbols are "taken 3 at a time".

The set of all the different permutations on the symbols A, B, and C taken 3 at a time, is {ABC, ACB, BAC, BCA, CAB, CBA}.

Example 8 Self Tutor

List the set of all permutations on the symbols P, Q, and R taken:

a 1 at a time **b** 2 at a time **c** 3 at a time.

a {P, Q, R}

b {PQ, QP, RP, PR, QR, RQ}

c {PQR, PRQ, QPR, QRP, RPQ, RQP}

Example 9 Self Tutor

List all permutations on the symbols W, X, Y, and Z taken 4 at a time.

WXYZ	WXZY	WYXZ	WYZX	WZXY	WZYX
XWYZ	XWZY	XYWZ	XYZW	XZYW	XZWY
YWXZ	YWZX	YXWZ	YXZW	YZWX	YZXW
ZWXY	ZWYX	ZXWY	ZXYW	ZYWX	ZYXW

There are 24 of them.

For large numbers of symbols, listing the complete set of permutations is absurd. However, we can still count them by considering the number of options we have for filling each position.

Suppose we want to find the number of different permutations on the symbols A, B, C, D, E, F, and G, taken 3 at a time.

There are 3 positions to fill:

1st	2nd	3rd

In the 1st position, any of the 7 symbols could be used, so we have 7 options.

7		
1st	2nd	3rd

This leaves any of 6 symbols to go in the 2nd position, and this leaves any of 5 symbols to go in the 3rd position.

7	6	5
1st	2nd	3rd

So, the total number of permutations $= 7 \times 6 \times 5 \quad \{\text{product principle}\}$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}$$

$$= \frac{7!}{4!} \quad \text{or} \quad \frac{7!}{(7-3)!}$$

The number of **permutations** on n distinct symbols taken r at a time is:

$$\underbrace{\boldsymbol{n \times (n-1) \times (n-2) \times \dots \times (n-r+1)}}_{r \text{ of these}} = \boldsymbol{\frac{n!}{(n-r)!}}$$

If we are finding permutations on the complete set of n symbols, as in **Example 9**, then $r = n$, and the number of permutations is $n!$.

Example 10 — Self Tutor

A chess association runs a tournament with 16 teams. In how many different ways could the top 5 positions be filled on the competition ladder?

Any of the 16 teams could fill the 'top' position.
Any of the remaining 15 teams could fill the 2nd position.
Any of the remaining 14 teams could fill the 3rd position.
⋮
Any of the remaining 12 teams could fill the 5th position.

16	15	14	13	12
1st	2nd	3rd	4th	5th

The total number of permutations $= 16 \times 15 \times 14 \times 13 \times 12$

$$= \frac{16!}{11!}$$

$$= 524\,160$$

So the top 5 positions could be filled in 524 160 ways.

Example 11

Self Tutor

The alphabet blocks A, B, C, D, and E are placed in a row in front of you.

a How many different permutations could you have?

b How many permutations end in C?

c How many permutations have the form $\boxed{\dots}\boxed{A}\boxed{\dots}\boxed{B}\boxed{\dots}$?

d How many begin and end with a vowel (A or E)?

a There are 5 letters taken 5 at a time.

$\therefore$ the total number of permutations $= 5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$.

b $\underbrace{\boxed{4}\boxed{3}\boxed{2}\boxed{1}}_{\text{any others here}}\underset{\uparrow \text{ C here}}{\boxed{1}}$

C must be in the last position. The other 4 letters could go into the remaining 4 places in 4! ways.

$\therefore$ the number of permutations $= 1 \times 4! = 24$.

c $\boxed{3}\underset{\uparrow \text{ A}}{\boxed{1}}\boxed{2}\underset{\uparrow \text{ B}}{\boxed{1}}\boxed{1}$

A goes into 1 place. B goes into 1 place. The remaining 3 letters go into the remaining 3 places in 3! ways.

$\therefore$ the number of permutations $= 1 \times 1 \times 3! = 6$.

d $\underset{\uparrow \text{ A or E}}{\boxed{2}}\boxed{3}\boxed{2}\boxed{1}\underset{\uparrow \text{ remainder of A or E}}{\boxed{1}}$

A or E could go into the 1st position, so there are two options. The other one must go into the last position.

The remaining 3 letters could go into the 3 remaining places in 3! ways.

$\therefore$ the number of permutations $= 2 \times 1 \times 3! = 12$.

EXERCISE 10D

1 List the set of all permutations on the symbols W, X, Y, and Z taken:

a 1 at a time **b** two at a time **c** three at a time.

2 List the set of all permutations on the symbols A, B, C, D, and E taken:

a 2 at a time **b** 3 at a time.

3 In how many ways can:

a 5 different books be arranged on a shelf

b 3 different paintings be chosen from a collection of 8, and hung in a row

c a signal consisting of 4 coloured flags in a row be made if there are 10 different flags to choose from?

4 A captain and vice-captain are to be selected from a team of 11 cricketers. In how many ways can this be done?

5 Suppose you have 4 different coloured flags. How many different signals could you make using:

a 2 flags in a row **b** 3 flags in a row **c** 2 or 3 flags in a row?

6 Nine boxes are each labelled with a different whole number from 1 to 9. Five people are allowed to take one box each. In how many different ways can this be done if:

a there are no restrictions

b the first three people decide that they will take even numbered boxes?

7 **a** How many different permutations on the letters A, B, C, D, E, and F are there if each letter can be used once only?

b How many of these permutations:

i end in ED **ii** begin with F and end with A

iii begin and end with a vowel (A or E)?

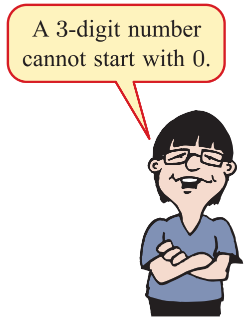
8 How many 3-digit numbers can be constructed from the digits 1, 2, 3, 4, 5, 6, and 7 if each digit may be used:

a as often as desired **b** only once **c** once only and the number must be odd?

9 3-digit numbers are constructed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 using each digit at most once. How many such numbers:

a can be constructed **b** end in 5

c end in 0 **d** are divisible by 5?



10 Arrangements containing 5 different letters from the word TRIANGLE are to be made. How many possible arrangements are there if:

a there are no restrictions

b the arrangement must start with R and end with A or E

c the arrangement must include the letter G?

Example 12 — Self Tutor

There are 6 different books arranged in a row on a shelf. In how many ways can two of the books, A and B, be together?

Method 1: We could have any of the following locations for A and B

$$\left.\begin{array}{cccccc} A & B & \times & \times & \times & \times \\ B & A & \times & \times & \times & \times \\ \times & A & B & \times & \times & \times \\ \times & B & A & \times & \times & \times \\ \times & \times & A & B & \times & \times \\ \times & \times & B & A & \times & \times \\ \times & \times & \times & A & B & \times \\ \times & \times & \times & B & A & \times \\ \times & \times & \times & \times & A & B \\ \times & \times & \times & \times & B & A \end{array}\right\}\text{10 of these}$$

If we consider any one of these, the remaining 4 books could be placed in 4! different orderings.

$\therefore$ total number of ways
$= 10 \times 4! = 240.$

Method 2: A and B can be put together in 2! ways (AB or BA).

Now consider this pairing as one book (effectively tying a string around them) which together with the other 4 books can be ordered in 5! different ways.

$\therefore$ the total number of ways $= 2! \times 5! = 240.$

11 In how many ways can 5 different books be arranged on a shelf if:

a there are no restrictions **b** books X and Y must be together

c books X and Y must not be together?

12 10 students sit in a row of 10 chairs. In how many ways can this be done if:

a there are no restrictions **b** students A, B, and C insist on sitting together?

13 3 boys and 3 girls are to sit in a row. How many ways can this be done if:

- **a** there are no restrictions
- **b** there is a girl at each end
- **c** boys and girls must alternate
- **d** all the boys sit together?

14 How many three-digit numbers can be made using the digits 0, 1, 3, 5, and 8 at most once each, if:

- **a** there are no restrictions
- **b** the numbers must be less than 500
- **c** the numbers must be even and greater than 300?

15 Consider the letters of the word MONDAY. How many permutations of four different letters can be chosen if:

- **a** there are no restrictions
- **b** at least one vowel (A or O) must be used
- **c** the two vowels are not together?

16 Alice has booked ten adjacent front-row seats for a basketball game for herself and nine friends.

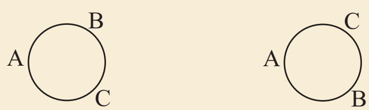
- **a** How many different arrangements are possible if there are no restrictions?
- **b** Due to a severe snowstorm, only five of Alice's friends are able to join her for the game. In how many different ways can they be seated in the 10 seats if:
  - **i** there are no restrictions
  - **ii** any two of Alice's friends are to sit next to her?

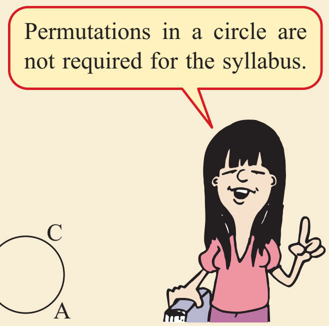
Discovery 1 — *Permutations in a circle*

There are 6 permutations on the symbols A, B, and C **in a line**. These are:

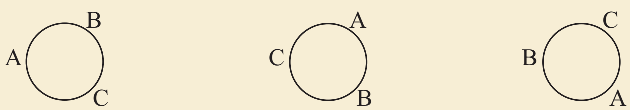
ABC ACB BAC BCA CAB CBA.

However **in a circle** there are only 2 different permutations on these 3 symbols. They are the only possibilities with different right-hand and left-hand neighbours.





In contrast, these three diagrams show the same cyclic permutation:



What to do:

1 Draw diagrams showing different cyclic permutations for:

- **a** one symbol: A
- **b** two symbols: A and B
- **c** three symbols: A, B, and C
- **d** four symbols: A, B, C, and D

2 Copy and complete:

Number of symbols	*Permutations in a line*	*Permutations in a circle*
1		
2		
3	$6 = 3!$	$2 = 2!$
4		

3 If there are n symbols to be arranged around a circle, how many different cyclic permutations are possible?

E COMBINATIONS

A **combination** is a selection of objects *without* regard to order.

For example, the possible teams of 3 people that can be selected from A, B, C, D, and E are:

ABC ABD ABE ACD ACE ADE
BCD BCE BDE
CDE

There are 10 combinations in total.

Now given the five people A, B, C, D, and E, we know that there are $5 \times 4 \times 3 = 60$ permutations for taking three of them at a time. So why is this 6 times larger than the number of combinations?

The answer is that for the combinations, order is not important. Selecting A, B, and C for the team is the same as selecting B, C, and A. For each of the 10 possible combinations, there are $3! = 6$ ways of ordering the members of the team.

In general, when choosing r objects from n objects,

$$\begin{aligned}\text{number of combinations } &= \text{number of permutations} \div r! \\ &= \frac{n!}{(n-r)!} \div r! \\ &= \frac{n!}{r!(n-r)!}\end{aligned}$$

This is the binomial coefficient we encountered in **Section C**.

The number of **combinations** on n distinct symbols taken r at a time is $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$.

Example 13 — Self Tutor

How many different teams of 4 can be selected from a squad of 7 if:

a there are no restrictions
b the teams must include the captain?

a There are 7 players up for selection and we want any 4 of them.
There are $\binom{7}{4} = 35$ possible combinations.

b The captain must be included *and* we need any 3 of the other 6.
There are $\binom{1}{1} \times \binom{6}{3} = 20$ possible combinations.

Example 14

Self Tutor

A committee of 4 is chosen from 7 men and 6 women. How many different committees can be chosen if:

a there are no restrictions

b there must be 2 of each sex

c there must be at least one of each sex?

a There are $7+6=13$ people up for selection and we want any 4 of them.

There are $\binom{13}{4}=715$ possible committees.

b The 2 men can be chosen out of 7 in $\binom{7}{2}$ ways.

The 2 women can be chosen out of 6 in $\binom{6}{2}$ ways.

$\therefore$ there are $\binom{7}{2}\times\binom{6}{2}=315$ possible committees.

c The total number of committees

= the number with 3 men and 1 woman + the number with 2 men and 2 women + the number with 1 man and 3 women

$=\binom{7}{3}\times\binom{6}{1}+\binom{7}{2}\times\binom{6}{2}+\binom{7}{1}\times\binom{6}{3}$

$=665$

or The total number of committees

$=\binom{13}{4}$ − the number with all men − the number with all women

$=\binom{13}{4}-\binom{7}{4}\times\binom{6}{0}-\binom{7}{0}\times\binom{6}{4}$

$=665$

EXERCISE 10E

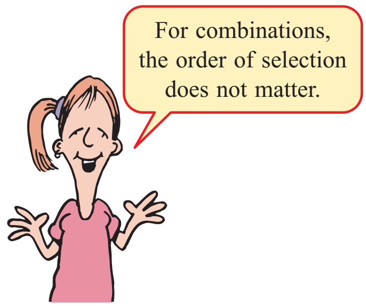
1 Determine whether the following are examples of combinations or permutations:

a making a 3-digit number using the digits 1, 2, 3, 4, and 5 at most once each

b selecting a committee of 3 people from a list of 5

c selecting the chairperson and treasurer from a committee of 8 people

d selecting 2 pieces of fruit to take to school from a bowl of 10 pieces.



2 List the different teams of 4 that can be chosen from a squad of 6 (named A, B, C, D, E, and F).

Check that the formula $\binom{n}{r}=\dfrac{n!}{r!(n-r)!}$ gives the total number of teams.

3 How many different teams of 11 can be chosen from a squad of 17?

4 Candidates for an examination are required to answer 5 questions out of 9.

a In how many ways can the questions be chosen if there are no restrictions?

b If question 1 was made compulsory, how many selections would be possible?

5 **a** How many different committees of 3 can be selected from 13 candidates?

b How many of these committees consist of the president and 2 others?

6 **a** How many different teams of 5 can be selected from a squad of 12?

b How many of these teams contain:

i the captain and vice-captain **ii** exactly one of the captain or the vice-captain?

7 A team of 9 is selected from a squad of 15. 3 particular players *must* be included, and another must be excluded because of injury. In how many ways can the team be chosen?

8 In how many ways can 4 people be selected from 10 if:

a one particular person *must* be selected

b two particular people are excluded from every selection

c one particular person is always included and two particular people are always excluded?

9 A committee of 5 is chosen from 10 men and 6 women. Determine the number of ways of selecting the committee if:

a there are no restrictions **b** it must contain 3 men and 2 women

c it must contain all men **d** it must contain at least 3 men

e it must contain at least one of each sex.

10 A committee of 8 is chosen from 9 boys and 6 girls. In how many ways can this be done if:

a there are no restrictions **b** there must be 5 boys and 3 girls

c all the girls are selected **d** there are more boys than girls?

11 A music class consists of 5 piano players, 7 guitarists, and 4 violinists. A band of 1 piano player, 3 guitarists, and 2 violinists must be chosen to play at a school concert. In how many different ways can the band be chosen?

12 A committee of 5 is chosen from 6 doctors, 3 dentists, and 7 others.
Determine the number of ways of selecting the committee if it is to contain:

a exactly 2 doctors and 1 dentist **b** exactly 2 doctors

c at least one person from either of the two given professions.

13 How many diagonals does a 20-sided convex polygon have?

14 There are 12 distinct points A, B, C, D, ..., L on a circle. Lines are drawn between each pair of points.

a How many lines: **i** are there in total **ii** pass through B?

b How many triangles: **i** are determined by the lines **ii** have one vertex B?

15 How many 4-digit numbers can be constructed for which the digits are in ascending order from left to right? You cannot start a number with 0.

16 **a** Give an example which demonstrates that:

$$\binom{5}{0}\times\binom{6}{4}+\binom{5}{1}\times\binom{6}{3}+\binom{5}{2}\times\binom{6}{2}+\binom{5}{3}\times\binom{6}{1}+\binom{5}{4}\times\binom{6}{0}=\binom{11}{4}.$$

b Copy and complete:

$$\binom{m}{0}\times\binom{n}{r}+\binom{m}{1}\times\binom{n}{r-1}+\binom{m}{2}\times\binom{n}{r-2}+\ldots.+\binom{m}{r-1}\times\binom{n}{1}+\binom{m}{r}\times\binom{n}{0}=\ldots.$$

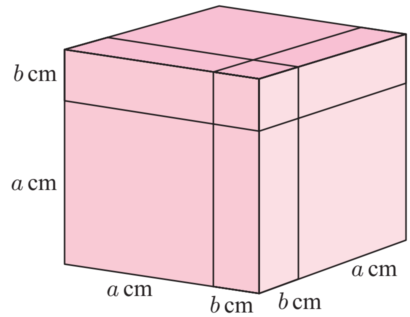
17 In how many ways can 12 people be divided into:

a two equal groups **b** three equal groups?

18 Answer the **Opening Problem** on page **256**.

F BINOMIAL EXPANSIONS

Consider the cube alongside, which has sides of length $(a+b)$ cm.

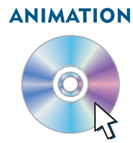


The cube has been subdivided into 8 blocks by making 3 cuts parallel to the cube's surfaces as shown.

We know that the total volume of the cube is $(a+b)^3$ cm^3. However, we can also find an expression for the cube's volume by adding the volumes of the 8 individual blocks.

We have:

1 block	$a \times a \times a$
3 blocks	$a \times a \times b$
3 blocks	$a \times b \times b$
1 block	$b \times b \times b$



$\therefore$ the cube's volume $= a^3 + 3a^2b + 3ab^2 + b^3$

$\therefore \quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

> The sum $\boldsymbol{a+b}$ is called a **binomial** as it contains two terms.
>
> Any expression of the form $\boldsymbol{(a+b)^n}$ is called a **power of a binomial**.

All binomials raised to a power can be expanded using the same general principles. In this chapter, therefore, we consider the expansion of the general expression $(a+b)^n$ where $n \in \mathbb{N}$.

Consider the following algebraic expansions:

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 \\ &= (a+b)(a^2+2ab+b^2) \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

> The **binomial expansion** of $\boldsymbol{(a+b)^2}$ is $\boldsymbol{a^2 + 2ab + b^2}$.
>
> The **binomial expansion** of $\boldsymbol{(a+b)^3}$ is $\boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3}$.

Discovery 2 — The binomial expansion

What to do:

1 Expand $(a+b)^4$ in the same way as for $(a+b)^3$ above.
Hence expand $(a+b)^5$ and $(a+b)^6$.

2 The cubic expansion $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ contains 4 terms. Observe that their coefficients are: 1 3 3 1

a What happens to the powers of a and b in each term of the expansion of $(a+b)^3$?

b Does the pattern in **a** continue for the expansions of $(a+b)^4$, $(a+b)^5$, and $(a+b)^6$?

c Write down the triangle of coefficients to row 6:

$$\begin{array}{lccccccc} n=1 & & & 1 & & 1 & & \\ n=2 & & 1 & & 2 & & 1 & \\ n=3 & 1 & & 3 & & 3 & & 1 \longleftarrow \text{row } 3 \\ & & & & \vdots & & & \end{array}$$

3 The triangle of coefficients in **c** above is called **Pascal's triangle**. Investigate:

a the predictability of each row from the previous one

b a formula for finding the sum of the numbers in the nth row of Pascal's triangle.

4 **a** Use your results from **3** to predict the elements of the 7th row of Pascal's triangle.

b Hence write down the binomial expansion of $(a+b)^7$.

c Check your result algebraically by using $(a+b)^7 = (a+b)(a+b)^6$ and your results from **1**.

From the **Discovery** we obtained
$$\begin{aligned}(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ &= a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + b^4\end{aligned}$$

Notice that:

- As we look from left to right across the expansion, the powers of a decrease by 1, while the powers of b increase by 1.
- The sum of the powers of a and b in each term of the expansion is 4.
- The number of terms in the expansion is $4+1=5$.
- The coefficients of the terms are row 4 of Pascal's triangle.

For the expansion of $(a+b)^n$ where $n \in \mathbb{N}$:

- As we look from left to right across the expansion, the powers of a *decrease* by 1, while the powers of b *increase* by 1.
- The sum of the powers of a and b in each term of the expansion is n.
- The number of terms in the expansion is $n+1$.
- The coefficients of the terms are row n of Pascal's triangle.

In the following examples we see how the general binomial expansion $(a+b)^n$ may be put to use.

Example 15 Self Tutor

Using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, find the binomial expansion of:

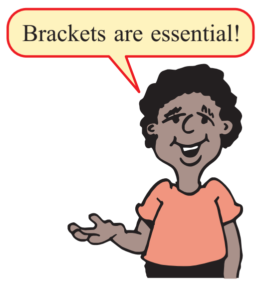
a $(2x+3)^3$ **b** $(x-5)^3$

a In the expansion of $(a+b)^3$ we substitute $a=(2x)$ and $b=(3)$.

$$\begin{aligned}\therefore\ (2x+3)^3 &= (2x)^3 + 3(2x)^2(3) + 3(2x)^1(3)^2 + (3)^3 \\ &= 8x^3 + 36x^2 + 54x + 27\end{aligned}$$

b We substitute $a=(x)$ and $b=(-5)$

$$\begin{aligned}\therefore\ (x-5)^3 &= (x)^3 + 3(x)^2(-5) + 3(x)(-5)^2 + (-5)^3 \\ &= x^3 - 15x^2 + 75x - 125\end{aligned}$$



Example 16

Self Tutor

Find the:

a 5th row of Pascal's triangle

b binomial expansion of $\left(x - \frac{2}{x}\right)^5$.

a

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \leftarrow \text{the 0th row, for } (a+b)^0 \\ &&&& 1 && 1 &&&& \leftarrow \text{the 1st row, for } (a+b)^1 \\ &&& 1 && 2 && 1 &&& \\ && 1 && 3 && 3 && 1 && \\ & 1 && 4 && 6 && 4 && 1 & \\ 1 && 5 && 10 && 10 && 5 && 1 \leftarrow \text{the 5th row, for } (a+b)^5 \end{array}$$

b Using the coefficients obtained in **a**, $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Letting $a = (x)$ and $b = \left(\frac{-2}{x}\right)$, we find

$$\left(x - \frac{2}{x}\right)^5 = (x)^5 + 5(x)^4\left(\frac{-2}{x}\right) + 10(x)^3\left(\frac{-2}{x}\right)^2 + 10(x)^2\left(\frac{-2}{x}\right)^3 + 5(x)\left(\frac{-2}{x}\right)^4 + \left(\frac{-2}{x}\right)^5$$

$$= x^5 - 10x^3 + 40x - \frac{80}{x} + \frac{80}{x^3} - \frac{32}{x^5}$$

EXERCISE 10F

1 Use the binomial expansion of $(a+b)^3$ to expand and simplify:

a $(p+q)^3$ **b** $(x+1)^3$ **c** $(x-3)^3$

d $(2+x)^3$ **e** $(3x-1)^3$ **f** $(2x+5)^3$

g $(2a-b)^3$ **h** $\left(3x - \frac{1}{3}\right)^3$ **i** $\left(2x + \frac{1}{x}\right)^3$

2 Use $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand and simplify:

a $(1+x)^4$ **b** $(p-q)^4$ **c** $(x-2)^4$

d $(3-x)^4$ **e** $(1+2x)^4$ **f** $(2x-3)^4$

g $(2x+b)^4$ **h** $\left(x + \frac{1}{x}\right)^4$ **i** $\left(2x - \frac{1}{x}\right)^4$

3 Expand and simplify:

a $(x+2)^5$ **b** $(x-2y)^5$ **c** $(1+2x)^5$ **d** $\left(x - \frac{1}{x}\right)^5$

4 Expand and simplify $(2+x)^5 + (2-x)^5$.

5 **a** Write down the 6th row of Pascal's triangle.

b Find the binomial expansion of:

i $(x+2)^6$ **ii** $(2x-1)^6$ **iii** $\left(x + \frac{1}{x}\right)^6$

6 Expand and simplify:

a $(1+\sqrt{2})^3$ **b** $(\sqrt{5}+2)^4$ **c** $(2-\sqrt{2})^5$

7 Find $\dfrac{(2+\sqrt{3})^3}{4+\sqrt{3}}$, giving your answer in the form $\dfrac{a+b\sqrt{3}}{c}$, where $a, b, c \in \mathbb{Z}$.

8 **a** Expand $(2+x)^6$. **b** Hence find the value of $(2.01)^6$.

9 The first two terms in a binomial expansion are: $(a+b)^3 = 8 + 12e^x + ...$

a Find a and b. **b** Hence determine the remaining two terms of the expansion.

10 Expand and simplify $(2x+3)(x+1)^4$.

11 Find the coefficient of:

a a^3b^2 in the expansion of $(3a+b)^5$ **b** a^3b^3 in the expansion of $(2a+3b)^6$.

G THE BINOMIAL THEOREM

Historical note — Binomial Theorem

The Binomial Theorem is one of the most important results in mathematics.

Multiplying out binomial terms is a basic process which dates back to the beginning of algebra. Mathematicians had noticed relationships between the coefficients for many centuries, and Pascal's triangle was certainly widely used long before Pascal.

Isaac Newton discovered the Binomial Theorem in 1665, but he did not publish his results until much later. Newton was the first person to give a formula for the binomial coefficients. He did this because he wanted to go further. Newton's ground-breaking result included a generalisation of the Binomial Theorem to the case of $(a+b)^n$ where n is a rational number, such as $\frac{1}{2}$. This results in a sum with an infinite number of terms, called an infinite series. In doing this, Newton was the first person to confidently use the exponential notation that we recognise today for both negative and fractional powers.

In the previous Section we saw how the coefficients of the binomial expansion $(a+b)^n$ can be found in the nth row of Pascal's triangle. These coefficients are in fact the **binomial coefficients** $\binom{n}{r}$ for $r = 0, 1, 2, ..., n$.

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array} \qquad \begin{array}{ccccccccc} & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \end{array}$$

The **Binomial Theorem** states that

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{r}a^{n-r}b^r + ... + b^n$$

where $\binom{n}{r}$ is the **binomial coefficient** of $a^{n-r}b^r$ and $r = 0, 1, 2, 3, ..., n$.

The **general term** or $(r+1)$th term in the binomial expansion is $T_{r+1} = \binom{n}{r}a^{n-r}b^r$.

Using sigma notation we write $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r}a^{n-r}b^r$.

Example 17 Self Tutor

Write down the first three and last two terms of the expansion of $\left(2x+\frac{1}{x}\right)^{12}$.

Do not simplify your answer.

$$\left(2x+\frac{1}{x}\right)^{12} = (2x)^{12} + \binom{12}{1}(2x)^{11}\left(\frac{1}{x}\right)^{1} + \binom{12}{2}(2x)^{10}\left(\frac{1}{x}\right)^{2} + \dots.$$

$$\dots. + \binom{12}{11}(2x)^{1}\left(\frac{1}{x}\right)^{11} + \left(\frac{1}{x}\right)^{12}$$

Example 18 Self Tutor

Find the 7th term of $\left(3x-\frac{4}{x^2}\right)^{14}$. Do not simplify your answer.

$a=(3x)$, $b=\left(\frac{-4}{x^2}\right)$, and $n=14$

Given the general term $T_{r+1}=\binom{n}{r}a^{n-r}b^r$, we let $r=6$

$$\therefore\ T_7=\binom{14}{6}(3x)^8\left(\frac{-4}{x^2}\right)^6$$

Example 19 Self Tutor

In the expansion of $\left(x^2+\frac{4}{x}\right)^{12}$, find:

a the coefficient of x^6

b the constant term.

$a=(x^2)$, $b=\left(\frac{4}{x}\right)$, and $n=12$

$$\begin{aligned}\therefore\ \text{the general term}\ T_{r+1} &= \binom{12}{r}(x^2)^{12-r}\left(\frac{4}{x}\right)^r\\ &= \binom{12}{r}x^{24-2r}\times\frac{4^r}{x^r}\\ &= \binom{12}{r}4^r x^{24-3r}\end{aligned}$$

a If $24-3r=6$
then $3r=18$
$\therefore\ r=6$

$\therefore\ T_7=\binom{12}{6}4^6x^6$

$\therefore$ the coefficient of x^6 is $\binom{12}{6}4^6$ or $3\,784\,704$.

b If $24-3r=0$
then $3r=24$
$\therefore\ r=8$

$\therefore\ T_9=\binom{12}{8}4^8x^0$

$\therefore$ the constant term is $\binom{12}{8}4^8$ or $32\,440\,320$.

EXERCISE 10G

1 Write down the first three and last two terms of the following binomial expansions. Do not simplify your answers.

a $(1+2x)^{11}$ **b** $\left(3x+\frac{2}{x}\right)^{15}$ **c** $\left(2x-\frac{3}{x}\right)^{20}$

2 Without simplifying, write down:

a the 6th term of $(2x+5)^{15}$ **b** the 4th term of $\left(x^2+y\right)^9$

c the 10th term of $\left(x-\frac{2}{x}\right)^{17}$ **d** the 9th term of $\left(2x^2-\frac{1}{x}\right)^{21}$.

3 In the expansion of $(2x+3)^{12}$, find:

a the coefficient of x^8 **b** the coefficient of x^5.

4 In the expansion of $(1-3x)^{10}$, find:

a the coefficient of x^3 **b** the coefficient of x^7.

5 In the expansion of $\left(x^2+\frac{2}{x}\right)^9$, find:

a the coefficient of x^{12} **b** the constant term **c** the coefficient of x^{-6}.

6 Consider the expansion of $(x+b)^7$.

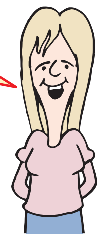
a Write down the general term of the expansion.

b Find b given that the coefficient of x^4 is -280.

7 Find the term independent of x in the expansion of:

a $\left(x+\frac{2}{x^2}\right)^{15}$ **b** $\left(x-\frac{3}{x^2}\right)^9$.

The "term independent of x" is the constant term.



8 Find the coefficient of:

a x^{10} in the expansion of $(3+2x^2)^{10}$ **b** x^3 in the expansion of $\left(2x^2-\frac{3}{x}\right)^6$

c x^6y^3 in the expansion of $\left(2x^2-3y\right)^6$ **d** x^{12} in the expansion of $\left(2x^2-\frac{1}{x}\right)^{12}$.

9 In the expansion of $(k+x)^8$, the coefficient of x^5 is 10 times the coefficient of x^6. Find the value of k.

10 The coefficient of x^5 in the expansion of $(ax-2)^7$ is twice the coefficient of x^5 in the expansion of $(a+x)^9$. Find the value of a.

11 In the expansion of $\left(ax+\frac{b}{x}\right)^6$, the constant term is 20 000, and the coefficient of x^4 is equal to the coefficient of x^2.

a Show that $ab=10$ and $b=\frac{2a}{5}$.

b Find a and b given that they are both positive.

Example 20 — Self Tutor

Find the coefficient of x^5 in the expansion of $(x+3)(2x-1)^6$.

$$\begin{aligned}
&(x+3)(2x-1)^6\\
&= (x+3)[(2x)^6 + \binom{6}{1}(2x)^5(-1) + \binom{6}{2}(2x)^4(-1)^2 + ...]\\
&= (x+3)(2^6x^6 - \binom{6}{1}2^5x^5 + \binom{6}{2}2^4x^4 - ....)
\end{aligned}$$

(2)

(1)

So, the terms containing x^5 are $\binom{6}{2}2^4x^5$ from (1)

and $-3\binom{6}{1}2^5x^5$ from (2)

$\therefore$ the coefficient of x^5 is $\binom{6}{2}2^4 - 3\binom{6}{1}2^5 = -336$

12 Find the coefficient of x^5 in the expansion of $(x+2)(x^2+1)^8$.

13 Find the term containing x^6 in the expansion of $(2-x)(3x+1)^9$.

14 Find the coefficient of x^4 in the expansion of:

a $(3-2x)^7$ **b** $(1+3x)(3-2x)^7$

15 Find:

a the coefficient of x^7 in the expansion of $(x^2-3)(2x-5)^8$

b the term independent of x in the expansion of $(1-x^2)\left(x+\frac{2}{x}\right)^6$.

16 When the expansion of $(a+bx)(1-x)^6$ is written in ascending powers of x, the first three terms are $3-20x+cx^2$. Find the values of a, b, and c.

Example 21 — Self Tutor

Consider the expansion of $(1+3x)^n$, where $n \in \mathbb{Z}^+$.

If the coefficient of x^2 is 90, find the value of n.

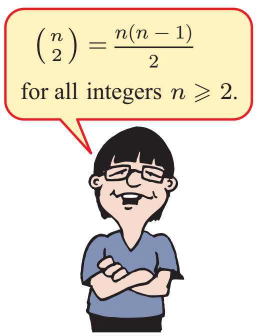
$(1+3x)^n$ has general term $T_{r+1} = \binom{n}{r}1^{n-r}(3x)^r$

$= \binom{n}{r}3^rx^r$

$\therefore$ $T_3 = \binom{n}{2}3^2x^2$ is the x^2 term.

Since the coefficient of x^2 is 90, $\binom{n}{2} \times 9 = 90$

$$\begin{aligned}
\therefore\ \frac{n(n-1)}{2} &= 10\\
\therefore\ n^2 - n &= 20\\
\therefore\ n^2 - n - 20 &= 0\\
\therefore\ (n-5)(n+4) &= 0\\
\therefore\ n &= 5 \qquad \{n > 0\}
\end{aligned}$$



17 The coefficient of x^2 in the expansion of $(1+2x)^n$ is 112. Find n.

18 The coefficient of x^2 in the expansion of $\left(1-\frac{x}{3}\right)^n$ is $\frac{5}{3}$. Find n.

19 The third term of $(1+x)^n$ is $36x^2$. Find the fourth term.

20 Suppose $(1+kx)^n = 1 - 12x + 60x^2 - ....$. Find the values of k and n.

Review set 10A

1 Simplify: **a** $\dfrac{n!}{(n-2)!}$ **b** $\dfrac{n!+(n+1)!}{n!}$

2 Eight people enter a room and each person shakes hands with every other person. How many hand shakes are made?

3 The letters P, Q, R, S, and T are to be arranged in a row. How many of the possible arrangements:

a end with T **b** begin with P and end with T?

4 **a** How many three digit numbers can be formed using the digits 0 to 9?

b How many of these numbers are divisible by 5?

5 The first two terms in a binomial expansion are: $(a+b)^4 = e^{4x} - 4e^{2x} + ...$

a Find a and b. **b** Copy and complete the expansion.

6 Expand and simplify $(\sqrt{3}+2)^5$, giving your answer in the form $a+b\sqrt{3}$, $a, b \in \mathbb{Z}$.

7 Find the constant term in the expansion of $\left(3x^2+\frac{1}{x}\right)^8$.

8 Find c given that the expansion $(1+cx)(1+x)^4$ includes the term $22x^3$.

9 Steven and nine of his classmates are in a school committee. The committee must select a president, vice-president, and secretary. In how many ways can this be done if:

a there are no restrictions **b** Steven must be the president

c Steven cannot hold any of the key positions?

10 Find the coefficient of x^3 in the expansion of $(x+5)^6$.

11 A team of five is chosen from six men and four women.

a How many different teams are possible with no restrictions?

b How many different teams contain at least one person of each sex?

12 Find the coefficient of x^{-6} in the expansion of $\left(2x-\frac{3}{x^2}\right)^{12}$.

13 Find the coefficient of x^5 in the expansion of $(2x+3)(x-2)^6$.

14 Find the possible values of a if the coefficient of x^3 in $\left(2x+\frac{1}{ax^2}\right)^9$ is 288.

15 In the expansion of $(kx-1)^6$, the coefficient of x^4 is equal to four times the coefficient of x^2. Find the possible values of k.

Review set 10B

1 Alpha-numeric number plates have two letters followed by four digits. How many plates are possible if:

a there are no restrictions

b the first letter must be a vowel

c no letter or digit may be repeated?

2 **a** How many committees of five can be selected from eight men and seven women?

b How many of the committees contain two men and three women?

c How many of the committees contain at least one man?

3 Use the binomial expansion to find:

a $(x-2y)^3$

b $(3x+2)^4$

4 Find the coefficient of x^3 in the expansion of $(2x+5)^6$.

5 Find the term independent of x in the expansion of $\left(2x^2-\frac{1}{x}\right)^6$.

6 Find $\dfrac{(3-\sqrt{2})^3}{\sqrt{2}+1}$, giving your answer in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Z}$.

7 Kristen's school offers 6 Group A subjects, 8 Group B subjects, and 5 Group C subjects. Kristen must select 2 Group A, 3 Group B, and 1 Group C subject to study. In how many ways can she make her selection?

8 Find the coefficient of x^4 in the expansion of $(x-3)(2x+1)^5$.

9 How many arrangements containing 4 different letters from the word DRAGONFLY are possible if:

a there are no restrictions

b the letters G and Y must not be included

c the arrangement must start with R and end with N?

10 Find the possible values of q if the constant terms in the expansions of $\left(x^3+\frac{q}{x^3}\right)^8$ and $\left(x^3+\frac{q}{x^3}\right)^4$ are equal.

11 Eight people enter a room and sit in a row of eight chairs. In how many ways can the sisters Cathy, Robyn, and Jane sit together in the row?

12 Find k in the expansion $(m-2n)^{10} = m^{10} - 20m^9n + km^8n^2 - .... + 1024n^{10}$.

13 A team of eight is chosen from 11 men and 7 women. How many different teams are possible if there:

a are no restrictions

b must be four of each sex on the team

c must be at least two women on the team

d must be more women than men?

14 The coefficient of x^2 in the expansion of $\left(1+\frac{x}{2}\right)^n$ is $\frac{21}{4}$. Find n.

15 The first three terms in the expansion of $(1+kx)^n$ are $1-4x+\frac{15}{2}x^2$. Find k and n.

11

Vectors

Contents:

Opening problem

An aeroplane in calm conditions is flying at 800 km h^{-1} due east. A cold wind suddenly blows from the south-west at 35 km h^{-1}, pushing the aeroplane slightly off course.

Things to think about:

a How can we use an array of numbers to represent the speed *and* direction of the plane?

b What operation do we need to perform to find the effect of the wind on the aeroplane?

c Can you determine the resulting speed and direction of the aeroplane?

A VECTORS AND SCALARS

In the **Opening Problem**, the effect of the wind on the aeroplane is determined by both its speed *and* its direction. The effect would be different if the wind was blowing against the aeroplane rather than from behind it.

> Quantities which have only magnitude are called **scalars**.
>
> Quantities which have both magnitude and direction are called **vectors**.

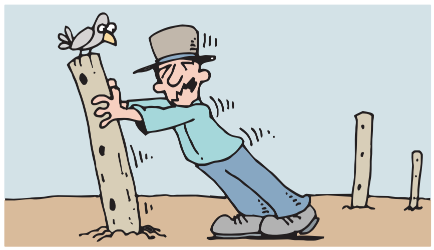
The *speed* of the plane is a scalar. It describes its size or strength.

The *velocity* of the plane is a vector. It includes both its speed and also its direction.

Other examples of vector quantities are:

- acceleration
- force
- displacement
- momentum

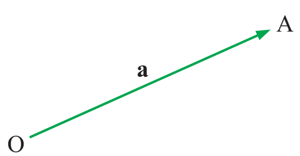
For example, farmer Giles needs to remove a fence post. He starts by pushing on the post sideways to loosen the ground. Giles has a choice of how hard to push the post, and in which direction. The force he applies is therefore a vector.



From previous courses, you should have seen how we can represent a vector quantity using a **directed line segment** or **arrow**. The **length of the arrow** represents the size or magnitude of the quantity, and the **arrowhead** shows its direction.

POSITION VECTORS

Consider the vector from the origin O to the point A. We call this the **position vector** of point A.



- This **position vector** could be represented by

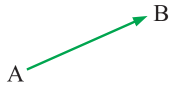
  $\overrightarrow{OA}$ or $\mathbf{a}$ or $\vec{a}$.

  $\mathbf{a}$: bold used in textbooks; $\vec{a}$: used by students

- The **magnitude** or **length** could be represented by

  $|\overrightarrow{OA}|$ or OA or $|\mathbf{a}|$ or $|\vec{a}|$.

Now consider the vector from point A to point B. We say that:

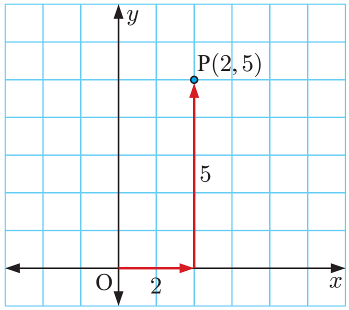


- $\overrightarrow{AB}$ is the vector which **originates** at A and **terminates** at B
- $\overrightarrow{AB}$ is the **position vector** of B relative to A.

When we plot points in the Cartesian plane, we move first in the x-direction and then in the y-direction.

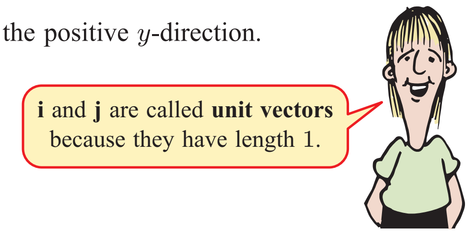
For example, to plot the point P(2, 5), we start at the origin, move 2 units in the x-direction, and then 5 units in the y-direction.

We therefore say that the vector from O to P is $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.



Suppose that $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a vector of length 1 unit in the positive x-direction

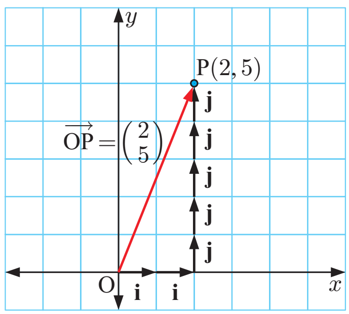
and that $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a vector of length 1 unit in the positive y-direction.



We can see that moving from O to P is equivalent to 2 lots of **i** plus 5 lots of **j**.

$$\overrightarrow{OP} = 2\mathbf{i} + 5\mathbf{j}$$

$$\therefore \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



The point P(x, y) has **position vector** $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = x\mathbf{i} + y\mathbf{j}$.

component form unit vector form

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the **base unit vector** in the x-direction.

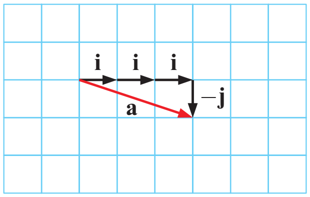
$\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the **base unit vector** in the y-direction.

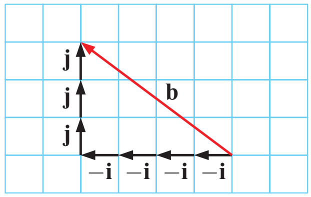
The set of vectors $\{\mathbf{i}, \mathbf{j}\}$ is the **standard basis** for the 2-dimensional (x, y) coordinate system.

All vectors in the plane can be described in terms of the base unit vectors **i** and **j**.

For example: $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$

$\mathbf{b} = -4\mathbf{i} + 3\mathbf{j}$



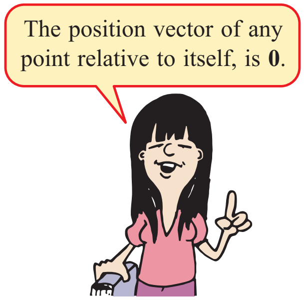


THE ZERO VECTOR

The **zero vector**, **0**, is a vector of length 0.

It is the only vector with no direction.

In component form, $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.



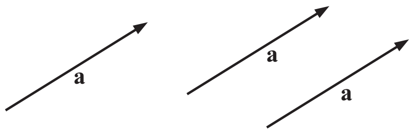
When we write the zero vector by hand, we usually write $\overrightarrow{0}$.

VECTOR EQUALITY

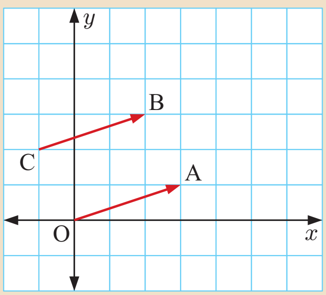
Two vectors are **equal** if they have the same magnitude and direction.

In component form, their x-components are equal *and* their y-components are equal.

Equal vectors are **parallel** and in the same direction, and are **equal in length**. The arrows that represent them are translations of one another.



Example 1 — Self Tutor



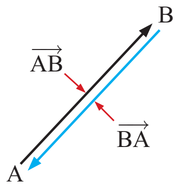
a Write $\overrightarrow{OA}$ and $\overrightarrow{CB}$ in component form and in unit vector form.

b Comment on your answers in **a**.

a $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3\mathbf{i} + \mathbf{j}$ $\qquad \overrightarrow{CB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3\mathbf{i} + \mathbf{j}$

b The vectors $\overrightarrow{OA}$ and $\overrightarrow{CB}$ are equal.

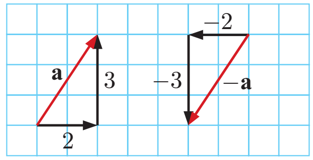
NEGATIVE VECTORS



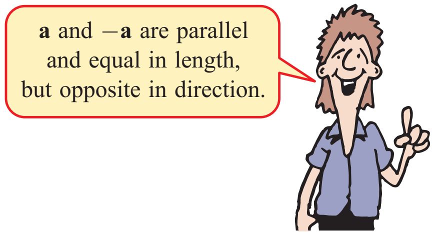
$\overrightarrow{AB}$ and $\overrightarrow{BA}$ have the same length, but they have opposite directions.

We say that $\overrightarrow{BA}$ is the **negative** of $\overrightarrow{AB}$, and write $\overrightarrow{BA} = -\overrightarrow{AB}$.

In the diagram we see the vector $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and its negative $-\mathbf{a} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

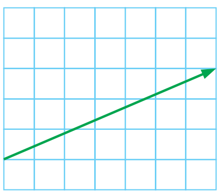


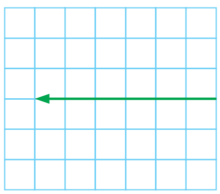
If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ then $-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$.

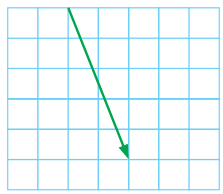


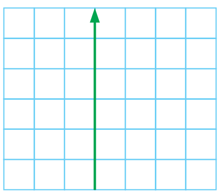
EXERCISE 11A

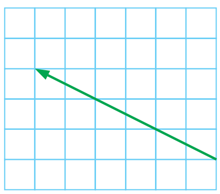
1 Write the illustrated vectors in component form and in unit vector form:

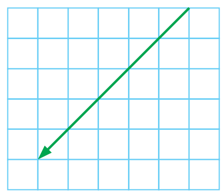
a 

b 

c 

d 

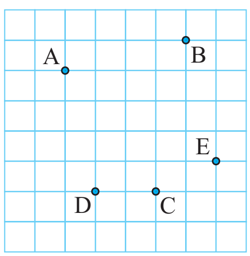
e 

f 

2 Write each vector in unit vector form, and illustrate it using an arrow diagram:

a $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ **d** $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$

3



a Find in component form and in unit vector form:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BA}$ **iii** $\overrightarrow{BC}$

iv $\overrightarrow{DC}$ **v** $\overrightarrow{AC}$ **vi** $\overrightarrow{DE}$

b Which two vectors in **a** are equal? Explain your answer.

c Which two vectors in **a** are negatives? Explain your answer.

4 Write in component form and illustrate using a directed line segment:

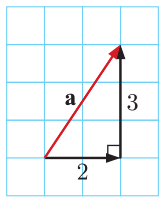
a $\mathbf{i} + 2\mathbf{j}$ **b** $-\mathbf{i} + 3\mathbf{j}$ **c** $-5\mathbf{j}$ **d** $4\mathbf{i} - 2\mathbf{j}$

5 Write down the negative of:

a $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ **d** $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$

B THE MAGNITUDE OF A VECTOR

Consider vector $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\mathbf{i} + 3\mathbf{j}$.



The **magnitude** or **length** of **a** is represented by $|\mathbf{a}|$.

By Pythagoras, $|\mathbf{a}|^2 = 2^2 + 3^2 = 4 + 9 = 13$

$\therefore |\mathbf{a}| = \sqrt{13}$ units {since $|\mathbf{a}| > 0$}

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1\mathbf{i} + a_2\mathbf{j}$, the **magnitude** or **length** of **a** is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$.

Example 2

Self Tutor

If $\mathbf{p} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\mathbf{q} = 2\mathbf{i} - 5\mathbf{j}$, find:

a $|\mathbf{p}|$ **b** $|\mathbf{q}|$

a $\mathbf{p} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

$\therefore |\mathbf{p}| = \sqrt{3^2 + (-5)^2}$

$= \sqrt{34}$ units

b $\mathbf{q} = 2\mathbf{i} - 5\mathbf{j} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

$\therefore |\mathbf{q}| = \sqrt{2^2 + (-5)^2}$

$= \sqrt{29}$ units

UNIT VECTORS

A **unit vector** is any vector which has a length of one unit.

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the base unit vectors in the positive x and y-directions respectively.

Example 3 **Self Tutor**

Find k given that $\begin{pmatrix} -\frac{1}{3} \\ k \end{pmatrix}$ is a unit vector.

Since $\begin{pmatrix} -\frac{1}{3} \\ k \end{pmatrix}$ is a unit vector,
$$\sqrt{(-\tfrac{1}{3})^2 + k^2} = 1$$
$$\therefore \sqrt{\tfrac{1}{9} + k^2} = 1$$
$$\therefore \tfrac{1}{9} + k^2 = 1 \quad \{\text{squaring both sides}\}$$
$$\therefore k^2 = \tfrac{8}{9}$$
$$\therefore k = \pm\tfrac{\sqrt{8}}{3}$$

EXERCISE 11B

1 Find the magnitude of:

a $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ **d** $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$ **e** $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$

2 Find the length of:

a $\mathbf{i} + \mathbf{j}$ **b** $5\mathbf{i} - 12\mathbf{j}$ **c** $-\mathbf{i} + 4\mathbf{j}$ **d** $3\mathbf{i}$ **e** $k\mathbf{j}$

3 Which of the following are unit vectors?

a $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ **c** $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ **d** $\begin{pmatrix} -\frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$ **e** $\begin{pmatrix} \frac{2}{7} \\ -\frac{5}{7} \end{pmatrix}$

4 Find k for the unit vectors:

a $\begin{pmatrix} 0 \\ k \end{pmatrix}$ **b** $\begin{pmatrix} k \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} k \\ 1 \end{pmatrix}$ **d** $\begin{pmatrix} k \\ k \end{pmatrix}$ **e** $\begin{pmatrix} \frac{1}{2} \\ k \end{pmatrix}$

5 Given $\mathbf{v} = \begin{pmatrix} 8 \\ p \end{pmatrix}$ and $|\mathbf{v}| = \sqrt{73}$ units, find the possible values of p.

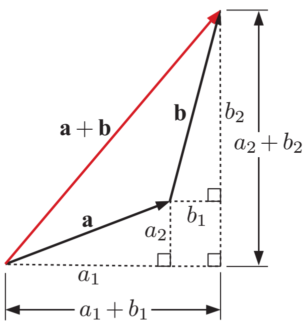
C OPERATIONS WITH PLANE VECTORS

VECTOR ADDITION

Consider adding vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Notice that:

- the horizontal step for $\mathbf{a} + \mathbf{b}$ is $a_1 + b_1$
- the vertical step for $\mathbf{a} + \mathbf{b}$ is $a_2 + b_2$.



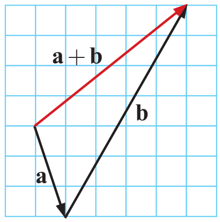
If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$.

Example 4 — Self Tutor

If $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, find $\mathbf{a} + \mathbf{b}$. Check your answer graphically.

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 1+4 \\ -3+7 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

Graphical check:



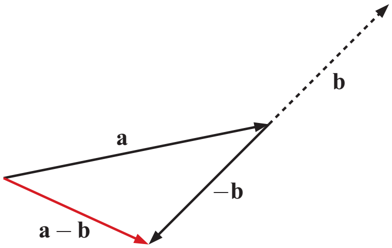
VECTOR SUBTRACTION

To subtract one vector from another, we simply **add its negative**.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

then $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$

$$= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$$



If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then $\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$.

Example 5 — Self Tutor

Given $\mathbf{p} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$, find:

a $\mathbf{q} - \mathbf{p}$

b $\mathbf{p} - \mathbf{q} - \mathbf{r}$

a $\mathbf{q} - \mathbf{p}$

$$= \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1-3 \\ 4+2 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

b $\mathbf{p} - \mathbf{q} - \mathbf{r}$

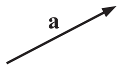
$$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ -5 \end{pmatrix} = \begin{pmatrix} 3-1+2 \\ -2-4+5 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

SCALAR MULTIPLICATION

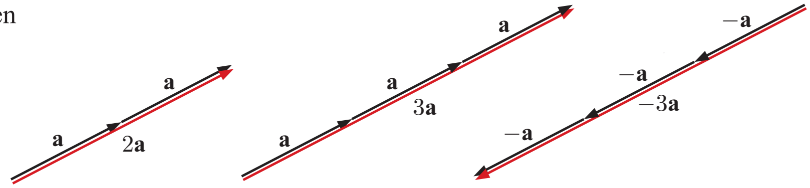
A **scalar** is a non-vector quantity. It has a size but no direction.

We can multiply vectors by scalars such as 2 and -3, or in fact any $k \in \mathbb{R}$.

If $\mathbf{a}$ is a vector, we define $2\mathbf{a} = \mathbf{a} + \mathbf{a}$ and $3\mathbf{a} = \mathbf{a} + \mathbf{a} + \mathbf{a}$
so $-3\mathbf{a} = 3(-\mathbf{a}) = (-\mathbf{a}) + (-\mathbf{a}) + (-\mathbf{a})$.

If $\mathbf{a}$ is 

then



So, $2\mathbf{a}$ is in the same direction as $\mathbf{a}$ but is twice as long as $\mathbf{a}$
$3\mathbf{a}$ is in the same direction as $\mathbf{a}$ but is three times longer than $\mathbf{a}$
$-3\mathbf{a}$ has the opposite direction to $\mathbf{a}$ and is three times longer than $\mathbf{a}$.

If $\mathbf{a}$ is a vector and k is a scalar, then $k\mathbf{a}$ is also a vector and we are performing **scalar multiplication**.

If $k > 0$, $k\mathbf{a}$ and $\mathbf{a}$ have the same direction.
If $k < 0$, $k\mathbf{a}$ and $\mathbf{a}$ have opposite directions.
If $k = 0$, $k\mathbf{a} = \mathbf{0}$, the zero vector.

If k is any scalar and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then $k\mathbf{v} = \begin{pmatrix} kv_1 \\ kv_2 \end{pmatrix}$.

VECTOR SCALAR MULTIPLICATION

Notice that:

- $(-1)\mathbf{v} = \begin{pmatrix} (-1)\,v_1 \\ (-1)\,v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} = -\mathbf{v}$
- $(0)\mathbf{v} = \begin{pmatrix} (0)\,v_1 \\ (0)\,v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$

Example 6 — Self Tutor

If $\mathbf{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, find: **a** $3\mathbf{q}$ **b** $\mathbf{p} + 2\mathbf{q}$ **c** $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$

a $3\mathbf{q}$
$$= 3\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} 6 \\ -9 \end{pmatrix}$$

b $\mathbf{p} + 2\mathbf{q}$
$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} 4 + 2(2) \\ 1 + 2(-3) \end{pmatrix}$$
$$= \begin{pmatrix} 8 \\ -5 \end{pmatrix}$$

c $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$
$$= \tfrac{1}{2}\begin{pmatrix} 4 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2}(4) - 3(2) \\ \frac{1}{2}(1) - 3(-3) \end{pmatrix}$$
$$= \begin{pmatrix} -4 \\ 9\frac{1}{2} \end{pmatrix}$$

Example 7 — Self Tutor

If $\mathbf{p} = 3\mathbf{i} - 5\mathbf{j}$ and $\mathbf{q} = -\mathbf{i} - 2\mathbf{j}$, find $|\mathbf{p} - 2\mathbf{q}|$.

$$\begin{aligned} \mathbf{p} - 2\mathbf{q} &= 3\mathbf{i} - 5\mathbf{j} - 2(-\mathbf{i} - 2\mathbf{j}) \\ &= 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{i} + 4\mathbf{j} \\ &= 5\mathbf{i} - \mathbf{j} \end{aligned}$$

$$\begin{aligned} \therefore\ |\mathbf{p} - 2\mathbf{q}| &= \sqrt{5^2 + (-1)^2} \\ &= \sqrt{26} \text{ units} \end{aligned}$$

EXERCISE 11C

VECTOR RACE GAME

1 If $\mathbf{a} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ find:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{b} + \mathbf{a}$ **c** $\mathbf{b} + \mathbf{c}$ **d** $\mathbf{c} + \mathbf{b}$
e $\mathbf{a} + \mathbf{c}$ **f** $\mathbf{c} + \mathbf{a}$ **g** $\mathbf{a} + \mathbf{a}$ **h** $\mathbf{b} + \mathbf{a} + \mathbf{c}$

2 Given $\mathbf{p} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ find:

a $\mathbf{p} - \mathbf{q}$ **b** $\mathbf{q} - \mathbf{r}$ **c** $\mathbf{p} + \mathbf{q} - \mathbf{r}$
d $\mathbf{p} - \mathbf{q} - \mathbf{r}$ **e** $\mathbf{q} - \mathbf{r} - \mathbf{p}$ **f** $\mathbf{r} + \mathbf{q} - \mathbf{p}$

3 Consider $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

a Use vector addition to show that $\mathbf{a} + \mathbf{0} = \mathbf{a}$.
b Use vector subtraction to show that $\mathbf{a} - \mathbf{a} = \mathbf{0}$.

4 For $\mathbf{p} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ find:

a $-3\mathbf{p}$ **b** $\frac{1}{2}\mathbf{q}$ **c** $2\mathbf{p} + \mathbf{q}$ **d** $\mathbf{p} - 2\mathbf{q}$
e $\mathbf{p} - \frac{1}{2}\mathbf{r}$ **f** $2\mathbf{p} + 3\mathbf{r}$ **g** $2\mathbf{q} - 3\mathbf{r}$ **h** $2\mathbf{p} - \mathbf{q} + \frac{1}{3}\mathbf{r}$

5 Consider $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Find geometrically and then comment on the results:

a $\mathbf{p} + \mathbf{p} + \mathbf{q} + \mathbf{q} + \mathbf{q}$ **b** $\mathbf{p} + \mathbf{q} + \mathbf{p} + \mathbf{q} + \mathbf{q}$ **c** $\mathbf{q} + \mathbf{p} + \mathbf{q} + \mathbf{p} + \mathbf{q}$

6 For $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ find:

a $|\mathbf{r}|$ **b** $|\mathbf{s}|$ **c** $|\mathbf{r} + \mathbf{s}|$ **d** $|\mathbf{r} - \mathbf{s}|$ **e** $|\mathbf{s} - 2\mathbf{r}|$

7 If $\mathbf{p} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ find:

a $|\mathbf{p}|$ **b** $|2\mathbf{p}|$ **c** $|-2\mathbf{p}|$ **d** $|3\mathbf{p}|$ **e** $|-3\mathbf{p}|$
f $|\mathbf{q}|$ **g** $|4\mathbf{q}|$ **h** $|-4\mathbf{q}|$ **i** $\left|\frac{1}{2}\mathbf{q}\right|$ **j** $\left|-\frac{1}{2}\mathbf{q}\right|$