Example 10 — Self Tutor

Write as powers of 2:

a 16 **b** $\frac{1}{16}$ **c** 1 **d** 4×2^n **e** $\dfrac{2^m}{8}$

a $16 = 2 \times 2 \times 2 \times 2 = 2^4$

b $\frac{1}{16} = \dfrac{1}{2^4} = 2^{-4}$

c $1 = 2^0$

d $4 \times 2^n = 2^2 \times 2^n = 2^{2+n}$

e $\dfrac{2^m}{8} = \dfrac{2^m}{2^3} = 2^{m-3}$

2 Write as powers of 2:

a 4 **b** $\frac{1}{4}$ **c** 8 **d** $\frac{1}{8}$ **e** 32 **f** $\frac{1}{32}$
g 2 **h** $\frac{1}{2}$ **i** 64 **j** $\frac{1}{64}$ **k** 128 **l** $\frac{1}{128}$

3 Write as powers of 3:

a 9 **b** $\frac{1}{9}$ **c** 27 **d** $\frac{1}{27}$ **e** 3 **f** $\frac{1}{3}$
g 81 **h** $\frac{1}{81}$ **i** 1 **j** 243 **k** $\frac{1}{243}$

4 Write as a single power of 2:

a 2×2^a **b** 4×2^b **c** 8×2^t **d** $(2^{x+1})^2$ **e** $(2^{1-n})^{-1}$
f $\dfrac{2^c}{4}$ **g** $\dfrac{2^m}{2^{-m}}$ **h** $\dfrac{4}{2^{1-n}}$ **i** $\dfrac{2^{x+1}}{2^x}$ **j** $\dfrac{4^x}{2^{1-x}}$

5 Write as a single power of 3:

a 9×3^p **b** 27^a **c** 3×9^n **d** 27×3^d **e** 9×27^t
f $\dfrac{3^y}{3}$ **g** $\dfrac{3}{3^y}$ **h** $\dfrac{9}{27^t}$ **i** $\dfrac{9^a}{3^{1-a}}$ **j** $\dfrac{9^{n+1}}{3^{2n-1}}$

Example 11 — Self Tutor

Write in simplest form, without brackets:

a $\left(-3a^2\right)^4$ **b** $\left(-\dfrac{2a^2}{b}\right)^3$

a
$$\left(-3a^2\right)^4 = (-3)^4 \times (a^2)^4 = 81 \times a^{2\times 4} = 81a^8$$

b
$$\left(-\frac{2a^2}{b}\right)^3 = \frac{(-2)^3 \times (a^2)^3}{b^3} = \frac{-8a^6}{b^3}$$

6 Write without brackets:

a $(2a)^2$ **b** $(3b)^3$ **c** $(ab)^4$ **d** $(pq)^3$ **e** $\left(\frac{m}{n}\right)^2$

f $\left(\frac{a}{3}\right)^3$ **g** $\left(\frac{b}{c}\right)^4$ **h** $\left(\frac{2a}{b}\right)^0$ **i** $\left(\frac{m}{3n}\right)^4$ **j** $\left(\frac{xy}{2}\right)^3$

7 Write the following in simplest form, without brackets:

a $(-2a)^2$ **b** $(-6b^2)^2$ **c** $(-2a)^3$ **d** $(-3m^2n^2)^3$

e $(-2ab^4)^4$ **f** $\left(\dfrac{-2a^2}{b^2}\right)^3$ **g** $\left(\dfrac{-4a^3}{b}\right)^2$ **h** $\left(\dfrac{-3p^2}{q^3}\right)^2$

i $\dfrac{(2x^2y)^2}{x}$ **j** $\dfrac{(4a^2b)^3}{2ab^2}$ **k** $\dfrac{(-5a^6b^3)^2}{5b^8}$ **l** $\dfrac{(-2x^7y^4)^3}{4x^3y^{15}}$

Example 12 — Self Tutor

Write without negative exponents: $\dfrac{a^{-3}b^2}{c^{-1}}$

$$a^{-3} = \frac{1}{a^3} \quad \text{and} \quad \frac{1}{c^{-1}} = c^1$$

$$\therefore \quad \frac{a^{-3}b^2}{c^{-1}} = \frac{b^2c}{a^3}$$

8 Write without negative exponents:

a ab^{-2} **b** $(ab)^{-2}$ **c** $(2ab^{-1})^2$ **d** $(3a^{-2}b)^2$ **e** $\dfrac{a^2b^{-1}}{c^2}$

f $\dfrac{a^2b^{-1}}{c^{-2}}$ **g** $\dfrac{1}{a^{-3}}$ **h** $\dfrac{a^{-2}}{b^{-3}}$ **i** $\dfrac{2a^{-1}}{d^2}$ **j** $\dfrac{12a}{m^{-3}}$

Example 13 — Self Tutor

Write $\dfrac{1}{2^{1-n}}$ in non-fractional form.

$$\begin{aligned}\frac{1}{2^{1-n}} &= 2^{-(1-n)} \\ &= 2^{-1+n} \\ &= 2^{n-1}\end{aligned}$$

9 Write in non-fractional form:

a $\dfrac{1}{a^n}$ **b** $\dfrac{1}{b^{-n}}$ **c** $\dfrac{1}{3^{2-n}}$ **d** $\dfrac{a^n}{b^{-m}}$ **e** $\dfrac{a^{-n}}{a^{2+n}}$

10 Simplify, giving your answers in simplest rational form:

a $\left(\frac{5}{3}\right)^0$ **b** $\left(\frac{7}{4}\right)^{-1}$ **c** $\left(\frac{1}{6}\right)^{-1}$ **d** $\dfrac{3^3}{3^0}$

e $\left(\frac{4}{3}\right)^{-2}$ **f** $2^1 + 2^{-1}$ **g** $\left(1\frac{2}{3}\right)^{-3}$ **h** $5^2 + 5^1 + 5^{-1}$

11 Write as powers of 2, 3 and/or 5:

a $\frac{1}{9}$ **b** $\frac{1}{16}$ **c** $\frac{1}{125}$ **d** $\frac{3}{5}$

e $\frac{4}{27}$ **f** $\dfrac{2^c}{8\times 9}$ **g** $\dfrac{9^k}{10}$ **h** $\dfrac{6^p}{75}$

12 Read about Nicomachus' pattern on page **108** and find the series of odd numbers for:

a 5^3 **b** 7^3 **c** 12^3

D RATIONAL INDICES

The index laws used previously can also be applied to **rational indices**, or indices which are written as a fraction.

The notation a^n is defined to mean "a multiplied together n times". Since we cannot multiply a together "half a time", the notation $a^{\frac{1}{2}}$ is an extension of the meaning of this notation. The goal is to extend the meaning of a^n so that the fundamental law

$$a^n a^m = a^{n+m}$$

remains true. If we assume that $a > 0$ then this law holds for rational indices.

Since $x^3 = -8$ has $x = -2$ as a solution, we would like to write

$$x = x^{\frac{3}{3}} = (x^3)^{\frac{1}{3}} = (-8)^{\frac{1}{3}} = (-2^3)^{\frac{1}{3}} = (-2)^{\frac{3}{3}} = -2.$$

Under some circumstances it is therefore possible to extend the meaning of a^n when n is rational and $a \leqslant 0$. However, this is not generally so easy, and so for this course we restrict ourselves to cases where $a > 0$.

For $a > 0$, notice that $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$ {index laws}
and $\sqrt{a} \times \sqrt{a} = a$ also.

So, $\boxed{a^{\frac{1}{2}} = \sqrt{a}}$ {by direct comparison}

Likewise $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1 = a$
and $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$

suggests $\boxed{a^{\frac{1}{3}} = \sqrt[3]{a}}$

In general, $a^{\frac{1}{n}} = \sqrt[n]{a}$ where $\sqrt[n]{a}$ reads 'the nth root of a', for $n \in \mathbb{Z}^+$.

We can now determine that $\sqrt[n]{a^m}$
$= (a^m)^{\frac{1}{n}}$
$= a^{\frac{m}{n}}$

$\therefore$ $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ for $a > 0$, $n \in \mathbb{Z}^+$, $m \in \mathbb{Z}$

Example 14 Self Tutor

Write as a single power of 2:

a $\sqrt[3]{2}$ **b** $\frac{1}{\sqrt{2}}$ **c** $\sqrt[5]{4}$

a $\sqrt[3]{2}$
$= 2^{\frac{1}{3}}$

b $\frac{1}{\sqrt{2}}$
$= \frac{1}{2^{\frac{1}{2}}}$
$= 2^{-\frac{1}{2}}$

c $\sqrt[5]{4}$
$= (2^2)^{\frac{1}{5}}$
$= 2^{2\times\frac{1}{5}}$
$= 2^{\frac{2}{5}}$

EXERCISE 4D

1 Write as a single power of 2:

a $\sqrt[5]{2}$ **b** $\frac{1}{\sqrt[5]{2}}$ **c** $2\sqrt{2}$ **d** $4\sqrt{2}$ **e** $\frac{1}{\sqrt[3]{2}}$

f $2\times\sqrt[3]{2}$ **g** $\frac{4}{\sqrt{2}}$ **h** $(\sqrt{2})^3$ **i** $\frac{1}{\sqrt[3]{16}}$ **j** $\frac{1}{\sqrt{8}}$

2 Write as a single power of 3:

a $\sqrt[3]{3}$ **b** $\frac{1}{\sqrt[3]{3}}$ **c** $\sqrt[4]{3}$ **d** $3\sqrt{3}$ **e** $\frac{1}{9\sqrt{3}}$

3 Write the following in the form a^x where a is a prime number and x is rational:

a $\sqrt[3]{7}$ **b** $\sqrt[4]{27}$ **c** $\sqrt[5]{16}$ **d** $\sqrt[3]{32}$ **e** $\sqrt[7]{49}$

f $\frac{1}{\sqrt[3]{7}}$ **g** $\frac{1}{\sqrt[4]{27}}$ **h** $\frac{1}{\sqrt[5]{16}}$ **i** $\frac{1}{\sqrt[3]{32}}$ **j** $\frac{1}{\sqrt[7]{49}}$

4 Use your calculator to evaluate:

a $3^{\frac{3}{4}}$ **b** $2^{\frac{7}{8}}$ **c** $2^{-\frac{1}{3}}$ **d** $4^{-\frac{3}{5}}$ **e** $\sqrt[4]{8}$

f $\sqrt[5]{27}$ **g** $\frac{1}{\sqrt[3]{7}}$

Example 15 Self Tutor

Without using a calculator, write in simplest rational form:

a $8^{\frac{4}{3}}$ **b** $27^{-\frac{2}{3}}$

a $8^{\frac{4}{3}}$
$= (2^3)^{\frac{4}{3}}$
$= 2^{3\times\frac{4}{3}}$ $\{(a^m)^n = a^{mn}\}$
$= 2^4$
$= 16$

b $27^{-\frac{2}{3}}$
$= (3^3)^{-\frac{2}{3}}$
$= 3^{3\times-\frac{2}{3}}$
$= 3^{-2}$
$= \frac{1}{9}$

5 Without using a calculator, write in simplest rational form:

a $4^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$ **c** $16^{\frac{3}{4}}$ **d** $25^{\frac{3}{2}}$ **e** $32^{\frac{2}{5}}$

f $4^{-\frac{1}{2}}$ **g** $9^{-\frac{3}{2}}$ **h** $8^{-\frac{4}{3}}$ **i** $27^{-\frac{4}{3}}$ **j** $125^{-\frac{2}{3}}$

E ALGEBRAIC EXPANSION AND FACTORISATION

EXPANSION

We can use the usual expansion laws to simplify expressions containing indices:

$$\begin{aligned}
a(b+c) &= ab+ac \\
(a+b)(c+d) &= ac+ad+bc+bd \\
(a+b)(a-b) &= a^2-b^2 \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a-b)^2 &= a^2-2ab+b^2
\end{aligned}$$

Example 16 Self Tutor

Expand and simplify: $x^{-\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}})$

$x^{-\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}})$

$= x^{-\frac{1}{2}} \times x^{\frac{3}{2}} + x^{-\frac{1}{2}} \times 2x^{\frac{1}{2}} - x^{-\frac{1}{2}} \times 3x^{-\frac{1}{2}}$ {each term is $\times$ by $x^{-\frac{1}{2}}$}

$= x^1 + 2x^0 - 3x^{-1}$ {adding exponents}

$= x + 2 - \dfrac{3}{x}$

Example 17 Self Tutor

Expand and simplify:

a $(2^x+3)(2^x+1)$ **b** $(7^x+7^{-x})^2$

a $(2^x+3)(2^x+1)$

$= 2^x \times 2^x + 2^x + 3 \times 2^x + 3$

$= 2^{2x} + 4 \times 2^x + 3$

$= 4^x + 2^{2+x} + 3$

b $(7^x+7^{-x})^2$

$= (7^x)^2 + 2 \times 7^x \times 7^{-x} + (7^{-x})^2$

$= 7^{2x} + 2 \times 7^0 + 7^{-2x}$

$= 7^{2x} + 2 + 7^{-2x}$

EXERCISE 4E.1

1 Expand and simplify:

a $x^2(x^3+2x^2+1)$ **b** $2^x(2^x+1)$ **c** $x^{\frac{1}{2}}(x^{\frac{1}{2}}+x^{-\frac{1}{2}})$

d $7^x(7^x+2)$ **e** $3^x(2-3^{-x})$ **f** $x^{\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}})$

g $2^{-x}(2^x+5)$ **h** $5^{-x}(5^{2x}+5^x)$ **i** $x^{-\frac{1}{2}}(x^2+x+x^{\frac{1}{2}})$

2 Expand and simplify:

a $(2^x - 1)(2^x + 3)$ **b** $(3^x + 2)(3^x + 5)$ **c** $(5^x - 2)(5^x - 4)$

d $(2^x + 3)^2$ **e** $(3^x - 1)^2$ **f** $(4^x + 7)^2$

3 Expand and simplify:

a $(x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} - 2)$ **b** $(2^x + 3)(2^x - 3)$ **c** $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$

d $(x + \frac{2}{x})^2$ **e** $(7^x - 7^{-x})^2$ **f** $(5 - 2^{-x})^2$

g $(x^{\frac{2}{3}} + x^{\frac{1}{3}})^2$ **h** $(x^{\frac{3}{2}} - x^{\frac{1}{2}})^2$ **i** $(2x^{\frac{1}{2}} - x^{-\frac{1}{2}})^2$

FACTORISATION AND SIMPLIFICATION

Example 18 — Self Tutor

Factorise: **a** $2^{n+3} + 2^n$ **b** $2^{n+3} + 8$ **c** $2^{3n} + 2^{2n}$

a
$$\begin{aligned} & 2^{n+3} + 2^n \\ &= 2^n 2^3 + 2^n \\ &= 2^n(2^3 + 1) \\ &= 2^n \times 9 \end{aligned}$$

b
$$\begin{aligned} & 2^{n+3} + 8 \\ &= 2^n 2^3 + 8 \\ &= 8(2^n) + 8 \\ &= 8(2^n + 1) \end{aligned}$$

c
$$\begin{aligned} & 2^{3n} + 2^{2n} \\ &= 2^{2n} 2^n + 2^{2n} \\ &= 2^{2n}(2^n + 1) \end{aligned}$$

Example 19 — Self Tutor

Factorise: **a** $4^x - 9$ **b** $9^x + 4(3^x) + 4$

a
$$\begin{aligned} & 4^x - 9 \\ &= (2^x)^2 - 3^2 && \{\text{compare } a^2 - b^2 = (a+b)(a-b)\} \\ &= (2^x + 3)(2^x - 3) \end{aligned}$$

b
$$\begin{aligned} & 9^x + 4(3^x) + 4 \\ &= (3^x)^2 + 4(3^x) + 4 && \{\text{compare } a^2 + 4a + 4\} \\ &= (3^x + 2)^2 && \{\text{as } a^2 + 4a + 4 = (a+2)^2\} \end{aligned}$$

EXERCISE 4E.2

1 Factorise:

a $5^{2x} + 5^x$ **b** $3^{n+2} + 3^n$ **c** $7^n + 7^{3n}$

d $5^{n+1} - 5$ **e** $6^{n+2} - 6$ **f** $4^{n+2} - 16$

2 Factorise:

a $9^x - 4$ **b** $4^x - 25$ **c** $16 - 9^x$

d $25 - 4^x$ **e** $9^x - 4^x$ **f** $4^x + 6(2^x) + 9$

g $9^x + 10(3^x) + 25$ **h** $4^x - 14(2^x) + 49$ **i** $25^x - 4(5^x) + 4$

3 Factorise:

a $4^x + 9(2^x) + 18$ **b** $4^x - 2^x - 20$ **c** $9^x + 9(3^x) + 14$

d $9^x + 4(3^x) - 5$ **e** $25^x + 5^x - 2$ **f** $49^x - 7^{x+1} + 12$

Example 20 — Self Tutor

Simplify:

a $\dfrac{6^n}{3^n}$ **b** $\dfrac{4^n}{6^n}$

a $\dfrac{6^n}{3^n}$ $= \dfrac{2^n \cancel{3^n}}{{}_1\cancel{3^n}}$ $= 2^n$ **or** $\dfrac{6^n}{3^n}$ $= \left(\frac{6}{3}\right)^n$ $= 2^n$

b $\dfrac{4^n}{6^n}$ $= \dfrac{\cancel{2^n}2^n}{\cancel{2^n}3^n}$ $= \dfrac{2^n}{3^n}$ **or** $\dfrac{4^n}{6^n}$ $= \left(\frac{4}{6}\right)^n$ $= \left(\frac{2}{3}\right)^n$

4 Simplify:

a $\dfrac{12^n}{6^n}$ **b** $\dfrac{20^a}{2^a}$ **c** $\dfrac{6^b}{2^b}$ **d** $\dfrac{4^n}{20^n}$

e $\dfrac{35^x}{7^x}$ **f** $\dfrac{6^a}{8^a}$ **g** $\dfrac{5^{n+1}}{5^n}$ **h** $\dfrac{5^{n+1}}{5}$

Example 21 — Self Tutor

Simplify:

a $\dfrac{3^n + 6^n}{3^n}$ **b** $\dfrac{2^{m+2} - 2^m}{2^m}$ **c** $\dfrac{2^{m+3} + 2^m}{9}$

a $\dfrac{3^n + 6^n}{3^n}$

$= \dfrac{3^n + 2^n 3^n}{3^n}$

$= \dfrac{\cancel{3^n}(1 + 2^n)}{{}_1\cancel{3^n}}$

$= 1 + 2^n$

b $\dfrac{2^{m+2} - 2^m}{2^m}$

$= \dfrac{2^m 2^2 - 2^m}{2^m}$

$= \dfrac{\cancel{2^m}(4-1)}{{}_1\cancel{2^m}}$

$= 3$

c $\dfrac{2^{m+3} + 2^m}{9}$

$= \dfrac{2^m 2^3 + 2^m}{9}$

$= \dfrac{2^m \cancel{(8+1)}}{{}_1\cancel{9}}$

$= 2^m$

5 Simplify:

a $\dfrac{6^m + 2^m}{2^m}$ **b** $\dfrac{2^n + 12^n}{2^n}$ **c** $\dfrac{8^n + 4^n}{2^n}$

d $\dfrac{12^x - 3^x}{3^x}$ **e** $\dfrac{6^n + 12^n}{1 + 2^n}$ **f** $\dfrac{5^{n+1} - 5^n}{4}$

g $\dfrac{5^{n+1} - 5^n}{5^n}$ **h** $\dfrac{4^n - 2^n}{2^n}$ **i** $\dfrac{2^n - 2^{n-1}}{2^n}$

6 Simplify:

a $2^n(n+1) + 2^n(n-1)$ **b** $3^n\left(\dfrac{n-1}{6}\right) - 3^n\left(\dfrac{n+1}{6}\right)$

F EXPONENTIAL EQUATIONS

An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent.

For example: $2^x = 8$ and $30 \times 3^x = 7$ are both exponential equations.

There are a number of methods we can use to solve exponential equations. These include graphing, using technology, and by using **logarithms**, which we will study in **Chapter 5**. However, in some cases we can solve algebraically.

If the base numbers are the same, we can **equate indices**.
If $\boldsymbol{a^x = a^k}$ then $\boldsymbol{x = k}$.

For example, if $2^x = 8$ then $2^x = 2^3$. Thus $x = 3$, and this is the only solution.

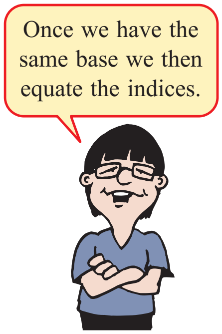
Example 22 Self Tutor

Solve for x:

a $2^x = 16$ **b** $3^{x+2} = \frac{1}{27}$

a
$$\begin{aligned} 2^x &= 16 \\ \therefore\ 2^x &= 2^4 \\ \therefore\ x &= 4 \end{aligned}$$

b
$$\begin{aligned} 3^{x+2} &= \tfrac{1}{27} \\ \therefore\ 3^{x+2} &= 3^{-3} \\ \therefore\ x+2 &= -3 \\ \therefore\ x &= -5 \end{aligned}$$



Example 23 Self Tutor

Solve for x:

a $4^x = 8$ **b** $9^{x-2} = \frac{1}{3}$

a
$$\begin{aligned} 4^x &= 8 \\ \therefore\ (2^2)^x &= 2^3 \\ \therefore\ 2^{2x} &= 2^3 \\ \therefore\ 2x &= 3 \\ \therefore\ x &= \tfrac{3}{2} \end{aligned}$$

b
$$\begin{aligned} 9^{x-2} &= \tfrac{1}{3} \\ \therefore\ (3^2)^{x-2} &= 3^{-1} \\ \therefore\ 3^{2(x-2)} &= 3^{-1} \\ \therefore\ 2(x-2) &= -1 \\ \therefore\ 2x-4 &= -1 \\ \therefore\ 2x &= 3 \\ \therefore\ x &= \tfrac{3}{2} \end{aligned}$$

EXERCISE 4F

1 Solve for x:

a $2^x = 8$ **b** $5^x = 25$ **c** $3^x = 81$ **d** $7^x = 1$

e $3^x = \frac{1}{3}$ **f** $2^x = \sqrt{2}$ **g** $5^x = \frac{1}{125}$ **h** $4^{x+1} = 64$

i $2^{x-2} = \frac{1}{32}$ **j** $3^{x+1} = \frac{1}{27}$ **k** $7^{x+1} = 343$ **l** $5^{1-2x} = \frac{1}{5}$

2 Solve for x:

a $8^x = 32$ **b** $4^x = \frac{1}{8}$ **c** $9^x = \frac{1}{27}$ **d** $25^x = \frac{1}{5}$

e $27^x = \frac{1}{9}$ **f** $16^x = \frac{1}{32}$ **g** $4^{x+2} = 128$ **h** $25^{1-x} = \frac{1}{125}$

i $4^{4x-1} = \frac{1}{2}$ **j** $9^{x-3} = 27$ **k** $(\frac{1}{2})^{x+1} = 8$ **l** $(\frac{1}{3})^{x+2} = 9$

m $81^x = 27^{-x}$ **n** $(\frac{1}{4})^{1-x} = 32$ **o** $(\frac{1}{7})^x = 49$ **p** $(\frac{1}{3})^{x+1} = 243$

3 Solve for x, if possible:

a $4^{2x+1} = 8^{1-x}$ **b** $9^{2-x} = (\frac{1}{3})^{2x+1}$ **c** $2^x \times 8^{1-x} = \frac{1}{4}$

4 Solve for x:

a $\dfrac{3^{2x+1}}{3^x} = 9^x$ **b** $\dfrac{25^x}{5^{x+4}} = 25^{1-x}$ **c** $\dfrac{4^x}{2^{x+2}} = \dfrac{2^{x+1}}{8^x}$

d $\dfrac{5^{2x-5}}{125^x} = \dfrac{25^{1-2x}}{5^{x+2}}$ **e** $\dfrac{4^x}{8^{2-x}} = 2^x \times 4^{x-1}$ **f** $\dfrac{9^{2x}}{27^{2-x}} = \dfrac{81^{3x+1}}{3^{1-2x}}$

5 Solve for x:

a $3 \times 2^x = 24$ **b** $7 \times 2^x = 28$ **c** $3 \times 2^{x+1} = 24$

d $12 \times 3^{-x} = \frac{4}{3}$ **e** $4 \times (\frac{1}{3})^x = 36$ **f** $5 \times (\frac{1}{2})^x = 20$

Example 24 — Self Tutor

Solve for x: $4^x + 2^x - 20 = 0$

$$\begin{aligned}
& 4^x + 2^x - 20 = 0 & \\
\therefore\ & (2^x)^2 + 2^x - 20 = 0 & \{\text{compare } a^2 + a - 20 = 0\} \\
\therefore\ & (2^x - 4)(2^x + 5) = 0 & \{a^2 + a - 20 = (a-4)(a+5)\} \\
\therefore\ & 2^x = 4 \text{ or } 2^x = -5 & \\
\therefore\ & 2^x = 2^2 & \{2^x \text{ cannot be negative}\} \\
\therefore\ & x = 2 &
\end{aligned}$$

6 Solve for x:

a $4^x - 6(2^x) + 8 = 0$ **b** $4^x - 2^x - 2 = 0$ **c** $9^x - 12(3^x) + 27 = 0$

d $9^x = 3^x + 6$ **e** $25^x - 23(5^x) - 50 = 0$ **f** $49^x + 1 = 2(7^x)$

G EXPONENTIAL FUNCTIONS

We have already seen how to evaluate b^n when $n \in \mathbb{Q}$, or in other words when n is a rational number.

But what about b^n when $n \in \mathbb{R}$, so n is real but not necessarily rational?

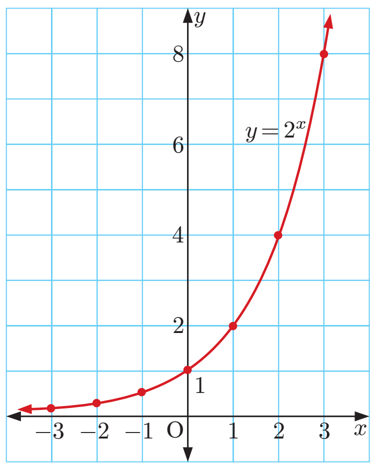
To answer this question, we can look at graphs of exponential functions.

The most simple general **exponential function** has the form $y = b^x$ where $b > 0$, $b \neq 1$.

For example, $y = 2^x$ is an exponential function.

We construct a table of values from which we graph the function:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



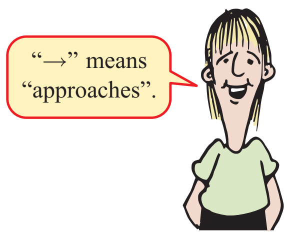
When $x = -10$, $y = 2^{-10} \approx 0.001$.
When $x = -50$, $y = 2^{-50} \approx 8.88 \times 10^{-16}$.

As x becomes large and negative, the graph of $y = 2^x$ approaches the x-axis from above but never touches it, since 2^x becomes very small but never zero.

So, as $x \to -\infty$, $y \to 0$ from above.

We say that $y = 2^x$ is '**asymptotic** to the x-axis' or '$y = 0$ is a **horizontal asymptote**'.

We now have a well-defined meaning for b^n where $b, n \in \mathbb{R}$ because simple exponential functions have smooth increasing or decreasing graphs.



Discovery 1 — *Graphs of exponential functions*

In this Discovery we examine the graphs of various families of exponential functions.

Click on the icon to run the **dynamic graphing package**, or else you could use your **graphics calculator**.

DYNAMIC GRAPHING PACKAGE

What to do:

1 Explore the family of curves of the form $y = b^x$ where $b > 0$.
For example, consider $y = 2^x$, $y = 3^x$, $y = 10^x$, and $y = (1.3)^x$.

- **a** What effect does changing b have on the shape of the graph?
- **b** What is the y-intercept of each graph?
- **c** What is the horizontal asymptote of each graph?

2 Explore the family of curves of the form $y = 2^x + d$ where d is a constant.
For example, consider $y = 2^x$, $y = 2^x + 1$, and $y = 2^x - 2$.

a What effect does changing d have on the position of the graph?
b What effect does changing d have on the shape of the graph?
c What is the horizontal asymptote of each graph?
d What is the horizontal asymptote of $y = 2^x + d$?
e To graph $y = 2^x + d$ from $y = 2^x$ what transformation is used?

3 Explore the family of curves of the form $y = 2^{x-c}$.
For example, consider $y = 2^x$, $y = 2^{x-1}$, $y = 2^{x+2}$, and $y = 2^{x-3}$.

a What effect does changing c have on the position of the graph?
b What effect does changing c have on the shape of the graph?
c What is the horizontal asymptote of each graph?
d To graph $y = 2^{x-c}$ from $y = 2^x$ what transformation is used?

4 Explore the relationship between $y = b^x$ and $y = b^{-x}$ where $b > 0$.
For example, consider $y = 2^x$ and $y = 2^{-x}$.

a What is the y-intercept of each graph?
b What is the horizontal asymptote of each graph?
c What transformation moves $y = 2^x$ to $y = 2^{-x}$?

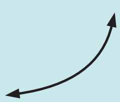
5 Explore the family of curves of the form $y = a \times 2^x$ where a is a constant.

a Consider functions where $a > 0$, such as $y = 2^x$, $y = 3 \times 2^x$, and $y = \frac{1}{2} \times 2^x$.
Comment on the effect on the graph.
b Consider functions where $a < 0$, such as $y = -2^x$, $y = -3 \times 2^x$, and $y = -\frac{1}{2} \times 2^x$.
Comment on the effect on the graph.
c What is the horizontal asymptote of each graph? Explain your answer.

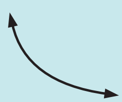
From **Discovery 1** you should have found that:

For the general exponential function $\boldsymbol{y = a \times b^{x-c} + d}$ where $b > 0$, $b \neq 1$, $a \neq 0$:

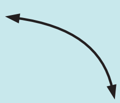
- b controls how steeply the graph increases or decreases
- c controls horizontal translation
- d controls vertical translation
- the equation of the horizontal asymptote is $y = d$
- if $a > 0$, $b > 1$ the function is increasing



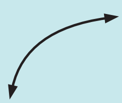
- if $a > 0$, $0 < b < 1$ the function is decreasing



- if $a < 0$, $b > 1$ the function is decreasing

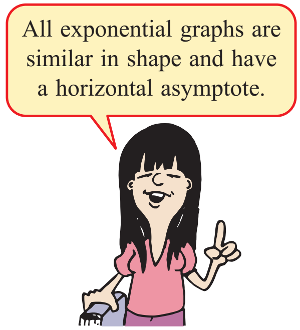


- if $a < 0$, $0 < b < 1$ the function is increasing.



We can sketch reasonably accurate graphs of exponential functions using:

- the horizontal asymptote
- the y-intercept
- two other points, for example, when $x = 2$, $x = -2$



Example 25

Self Tutor

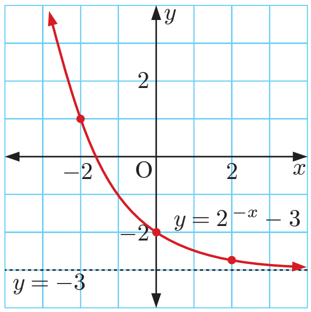
Sketch the graph of $y = 2^{-x} - 3$.
Hence state the domain and range of $f(x) = 2^{-x} - 3$.

For $y = 2^{-x} - 3$,
the horizontal asymptote is $y = -3$.

When $x = 0$, $y = 2^0 - 3$
$= 1 - 3$
$= -2$

$\therefore$ the y-intercept is -2.

When $x = 2$, $y = 2^{-2} - 3$
$= \frac{1}{4} - 3$
$= -2\frac{3}{4}$

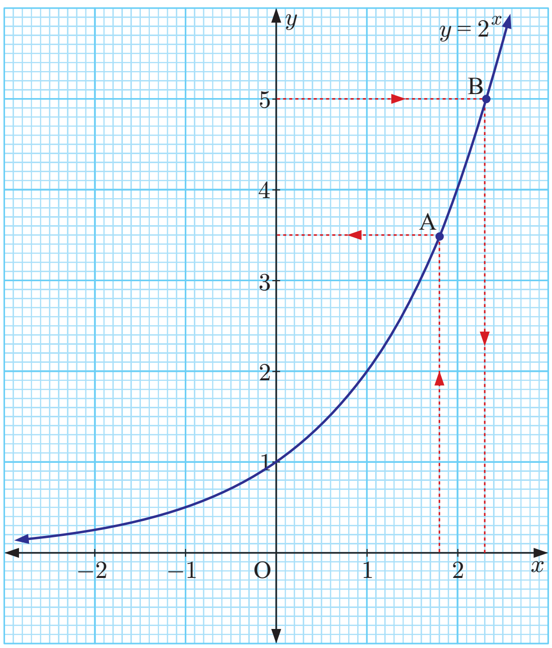


When $x = -2$, $y = 2^2 - 3 = 1$

The domain is $\{x : x \in \mathbb{R}\}$. The range is $\{y : y > -3\}$.

Consider the graph of $y = 2^x$ alongside. We can use the graph to estimate:

- the value of 2^x for a given value of x, for example $2^{1.8} \approx 3.5$ {point A}
- the solutions of the exponential equation $2^x = b$, for example if $2^x = 5$ then $x \approx 2.3$ {point B}.



EXERCISE 4G

1 Use the graph of $y = 2^x$ to estimate the value of:

a $2^{\frac{1}{2}}$ or $\sqrt{2}$ **b** $2^{0.8}$ **c** $2^{1.5}$ **d** $2^{-\sqrt{2}}$

2 Use the graph of $y = 2^x$ to estimate the solution to:

a $2^x = 3$ **b** $2^x = 0.6$ **c** $2^x = 4.3$ **d** $2^x = 0.3$

3 Use the graph of $y = 2^x$ to explain why $2^x = 0$ has no solutions.

4 Suppose $f(x) = 2 \times 3^x$. Find:

a $f(0)$ **b** $f(3)$ **c** $f(-2)$

5 Suppose $g(x) = 5^x + 2$.

a Find $g(0)$ and $g(-1)$. **b** Find a such that $g(a) = 27$.

6 Draw freehand sketches of the following pairs of graphs using your observations from the previous **Discovery**:

GRAPHING PACKAGE

a $y = 2^x$ and $y = 2^x - 2$ **b** $y = 2^x$ and $y = 2^{-x}$

c $y = 2^x$ and $y = 2^{x-2}$ **d** $y = 2^x$ and $y = 2(2^x)$

7 Draw freehand sketches of the following pairs of graphs:

a $y = 3^x$ and $y = 3^{-x}$ **b** $y = 3^x$ and $y = 3^x + 1$

c $y = 3^x$ and $y = -3^x$ **d** $y = 3^x$ and $y = 3^{x-1}$

8 For each of the functions below:

i Sketch the graph of the function.
ii State the domain and range.
iii Use your calculator to find the value of y when $x = \sqrt{2}$.
iv Discuss the behaviour of y as $x \to \pm\infty$.
v Determine the horizontal asymptotes.

a $y = 2^x + 1$ **b** $f(x) = 2 - 2^x$ **c** $y = 2^{-x} + 3$ **d** $f(x) = 3 - 2^{-x}$

Example 26 Self Tutor

An entomologist monitoring a grasshopper plague notices that the area affected by the grasshoppers is given by $A(n) = 1000 \times 2^{0.2n}$ hectares, where n is the number of weeks after the initial observation.

a Find the original affected area.

b Find the affected area after:

i 5 weeks **ii** 10 weeks **iii** 12 weeks.

c Draw the graph of A against n.

d How long will it take for the affected area to reach 8000 hectares?

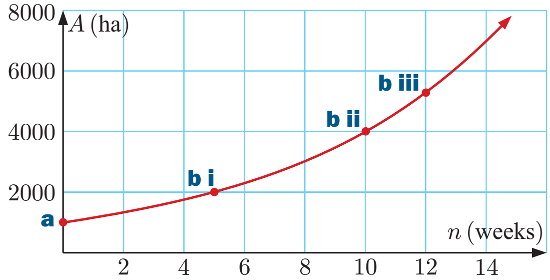
a $A(0) = 1000 \times 2^0$
$= 1000 \times 1$
$= 1000$ $\therefore$ the original affected area was 1000 ha.

b **i** $A(5) = 1000 \times 2^1$
$= 2000$
The affected area is 2000 ha.

ii $A(10) = 1000 \times 2^2$
$= 4000$
The affected area is 4000 ha.

iii $A(12) = 1000 \times 2^{0.2\times 12}$
$= 1000 \times 2^{2.4}$
≈ 5280
The affected area is about 5280 ha.

c



d We need to find n such that

$$A(n) = 8000$$
$$\therefore\ 1000 \times 2^{0.2n} = 8000$$
$$\therefore\ 2^{0.2n} = 8$$
$$\therefore\ 2^{0.2n} = 2^3$$
$$\therefore\ 0.2n = 3$$
$$\therefore\ n = 15$$

So, it will take 15 weeks.

9 A breeding program to ensure the survival of pygmy possums is established with an initial population of 50 (25 pairs). From a previous program, the expected population P in n years' time is given by $P(n) = P_0 \times 2^{0.5n}$.

a What is the value of P_0?

b What is the expected population after:
i 2 years **ii** 6 years **iii** 10 years?

c Sketch the graph of P against n using **a** and **b** only.

d How long will it take for the population to reach 800?


© Matt West, Healesville Sanctuary

10 The weight W of bacteria in a culture t hours after establishment is given by $W(t) = 100 \times 3^{0.1t}$ grams.

GRAPHING PACKAGE

a Find the initial weight.

b Find the weight after: **i** 5 hours **ii** 10 hours **iii** 24 hours.

c Sketch the graph of W against t using the results of **a** and **b** only.

d How long will it take for the weight to reach 900 g?

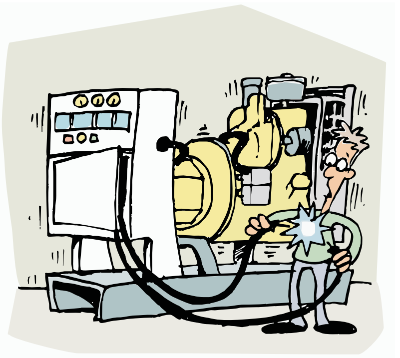
11 The current flowing in an electrical circuit t seconds after it is switched off is given by $I(t) = 32 \times 4^{-t}$ amps.

a What current was flowing at the instant when it was switched off?

b What current was still flowing after:
i 1 second **ii** 2 seconds?

c Plot I against t.

d How long will it take for the current to reach $\frac{1}{2}$ amp?

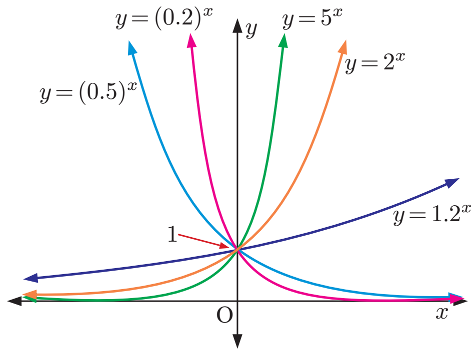


12 Answer the **Opening Problem** on page **102**.

H THE NATURAL EXPONENTIAL e^x

We have seen that the simplest exponential functions are of the form $f(x) = b^x$ where $b > 0$, $b \neq 1$.

Graphs of some of these functions are shown alongside.



We can see that for all positive values of the base b, the graph is always positive.

Hence $\boxed{b^x > 0 \text{ for all } b > 0.}$

There are an infinite number of possible choices for the base number.

However, where exponential data is examined in science, engineering, and finance, the base $e \approx 2.7183$ is commonly used.

e is a special number in mathematics. It is irrational like π, and just as π is the ratio of a circle's circumference to its diameter, e also has a physical meaning. We explore this meaning in the following **Discovery**.

Discovery 2 — Continuous compound interest

A formula for calculating the amount to which an investment grows is $u_n = u_0(1+i)^n$ where:

u_n is the final amount, u_0 is the initial amount,
i is the interest rate per compounding period,
n is the number of periods or number of times the interest is compounded.

We will investigate the final value of an investment for various values of n, and allow n to get extremely large.

What to do:

1 Suppose \$1000 is invested for one year at a fixed rate of 6% per annum. Use your calculator to find the final amount or *maturing value* if the interest is paid:

- **a** annually $(n = 1, \; i = 6\% = 0.06)$
- **b** quarterly $(n = 4, \; i = \frac{6\%}{4} = 0.015)$
- **c** monthly
- **d** daily
- **e** by the second
- **f** by the millisecond.

2 Comment on your answers from **1**.

3 If r is the percentage rate per year, t is the number of years, and N is the number of interest payments per year, then $i = \frac{r}{N}$ and $n = Nt$.

The growth formula becomes $u_n = u_0\left(1 + \frac{r}{N}\right)^{Nt}$.

If we let $a = \frac{N}{r}$, show that $u_n = u_0\left[\left(1 + \frac{1}{a}\right)^a\right]^{rt}$.

4 For continuous compound growth, the number of interest payments per year N gets very large.

a Explain why a gets very large as N gets very large.

b Copy and complete the table, giving your answers as accurately as technology permits.

a	$\left(1+\frac{1}{a}\right)^a$
10	
100	
1000	
10 000	
100 000	
1 000 000	
10 000 000	

5 You should have found that for very large values of a,

$$\left(1+\frac{1}{a}\right)^a \approx 2.718\,281\,828\,459....$$

Use the $\boxed{e^x}$ key of your calculator to find the value of e^1. What do you notice?

6 For continuous growth, $u_n = u_0 e^{rt}$ where u_0 is the initial amount
r is the annual percentage rate
t is the number of years

Use this formula to find the final value if \$1000 is invested for 4 years at a fixed rate of 6% per annum, where the interest is calculated continuously.

From **Discovery 2** we observe that:

If interest is paid *continuously* or *instantaneously* then the formula for calculating a compounding amount $u_n = u_0(1+i)^n$ can be replaced by $u_n = u_0 e^{rt}$, where r is the percentage rate per annum and t is the number of years.

Historical note

The natural exponential e was first described in 1683 by Swiss mathematician **Jacob Bernoulli**. He discovered the number while studying compound interest, just as we did in **Discovery 2**.

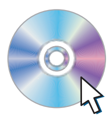
The natural exponential was first called e by Swiss mathematician and physicist **Leonhard Euler** in a letter to the German mathematician **Christian Goldbach** in 1731. The number was then published with this notation in 1736.

In 1748 Euler evaluated e correct to 18 decimal places.

One may think that e was chosen because it was the first letter of Euler's name or for the word exponential, but it is likely that it was just the next vowel available since he had already used a in his work.

Leonhard Euler

EXERCISE 4H



1 Sketch, on the same set of axes, the graphs of $y = 2^x$, $y = e^x$, and $y = 3^x$. Comment on any observations.

2 Sketch, on the same set of axes, the graphs of $y = e^x$ and $y = e^{-x}$. What is the geometric connection between these two graphs?

3 For the general exponential function $y = ae^{kx}$, what is the y-intercept?

4 Consider $y = 2e^x$.

a Explain why y can never be < 0. **b** Find y if: **i** $x = -20$ **ii** $x = 20$.

5 Find, to 3 significant figures, the value of:

a e^2 **b** e^3 **c** $e^{0.7}$ **d** $\sqrt{e}$ **e** e^{-1}

6 Write the following as powers of e:

a $\sqrt{e}$ **b** $\dfrac{1}{\sqrt{e}}$ **c** $\dfrac{1}{e^2}$ **d** $e\sqrt{e}$

7 On the same set of axes, sketch and clearly label the graphs of:

$f : x \mapsto e^x$, $g : x \mapsto e^{x-2}$, $h : x \mapsto e^x + 3$

State the domain and range of each function.

8 On the same set of axes, sketch and clearly label the graphs of:

$f : x \mapsto e^x$, $g : x \mapsto -e^x$, $h : x \mapsto 10 - e^x$

State the domain and range of each function.

9 Expand and simplify:

a $(e^x + 1)^2$ **b** $(1 + e^x)(1 - e^x)$ **c** $e^x(e^{-x} - 3)$

10 Solve for x:

a $e^x = \sqrt{e}$ **b** $e^{\frac{1}{2}x} = \dfrac{1}{e^2}$

11 Suppose $f : x \mapsto e^x$ and $g : x \mapsto 3x + 2$.

a Find $fg(x)$ and $gf(x)$. **b** Solve $fg(x) = \dfrac{1}{e}$.

12 Consider the function $f(x) = e^x$.

a On the same set of axes, sketch $y = f(x)$, $y = x$, and $y = f^{-1}(x)$.

b State the domain and range of f^{-1}.

Activity

Click on the icon to run a card game for exponential functions.

CARD GAME

Review set 4A

1 Simplify:

a $5\sqrt{3}(4 - \sqrt{3})$ **b** $(6 - 5\sqrt{2})^2$

2 Write with integer denominator:

a $\dfrac{2}{\sqrt{3}}$ **b** $\dfrac{\sqrt{7}}{\sqrt{5}}$ **c** $\dfrac{1}{4\sqrt{7}}$

3 Simplify using the laws of exponents:

a $a^4b^5 \times a^2b^2$ **b** $6xy^5 \div 9x^2y^5$ **c** $\dfrac{5(x^2y)^2}{(5x^2)^2}$

4 Let $f(x) = 3^x$.

a Write down the value of: **i** $f(4)$ **ii** $f(-1)$

b Find the value of k such that $f(x+2) = k\,f(x)$, $k \in \mathbb{Z}$.

5 Write without brackets or negative exponents:

a $x^{-2} \times x^{-3}$ **b** $2(ab)^{-2}$ **c** $2ab^{-2}$

6 Write as a single power of 3:

a $\dfrac{27}{9^a}$ **b** $(\sqrt{3})^{1-x} \times 9^{1-2x}$

7 Evaluate:

a $8^{\frac{2}{3}}$ **b** $27^{-\frac{2}{3}}$

8 Write without negative exponents:

a mn^{-2} **b** $(mn)^{-3}$ **c** $\dfrac{m^2n^{-1}}{p^{-2}}$ **d** $(4m^{-1}n)^2$

9 Expand and simplify:

a $(3 - e^x)^2$ **b** $(\sqrt{x}+2)(\sqrt{x}-2)$ **c** $2^{-x}(2^{2x} + 2^x)$

10 Find the positive solution of the equation $(8+\sqrt{13})x^2 + (2-\sqrt{13})x - 1 = 0$.
Give your solution in the form $x = a + b\sqrt{13}$, where $a, b \in \mathbb{Q}$.

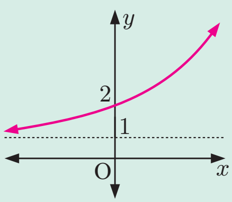
11 Solve for x:

a $2^{x-3} = \frac{1}{32}$ **b** $9^x = 27^{2-2x}$ **c** $e^{2x} = \dfrac{1}{\sqrt{e}}$

12 Match each equation to its corresponding graph:

a $y = -e^x$ **b** $y = 3 \times 2^x$ **c** $y = e^x + 1$ **d** $y = 3^{-x}$ **e** $y = -e^{-x}$

A



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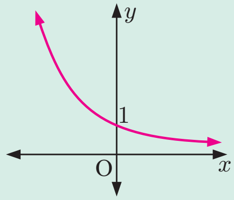

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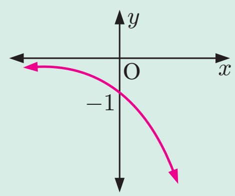

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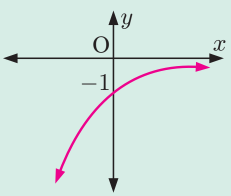
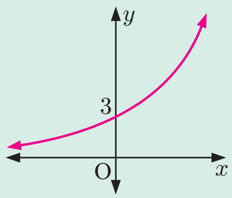


E



13 If $f(x) = 3 \times 2^x$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-2)$

14 Consider the function $f : x \mapsto e^{-x} - 3$.

a State the range of the function. **b** Find the value of $f(0)$.

c Solve $f(x) = \dfrac{\sqrt{e} - 3e}{e}$.

15 The temperature of a dish t minutes after it is removed from the microwave, is given by $T = 80 \times 2^{-0.1t}$ °C.

a Find the initial temperature of the dish.

b Find the temperature after: **i** 10 minutes **ii** 20 minutes.

c Draw the graph of T against t for $t \geqslant 0$.

d Find the time taken for the temperature of the dish to fall to 10°C.

Review set 4B

1 Simplify:

a $(7+2\sqrt{3})(5-3\sqrt{3})$ **b** $(6+2\sqrt{2})(6-2\sqrt{2})$

2 Rationalise the denominator:

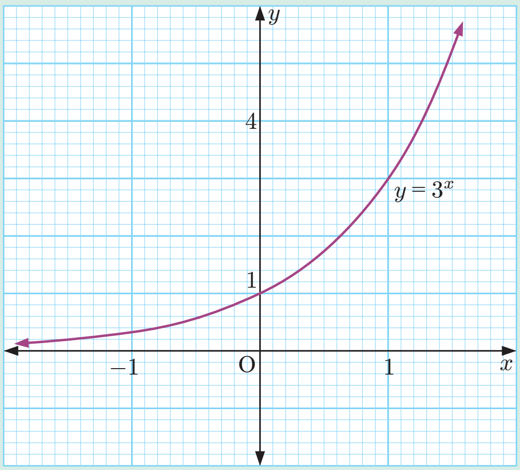
a $\dfrac{1}{5-\sqrt{3}}$ **b** $\dfrac{\sqrt{11}}{\sqrt{7}-2}$ **c** $\dfrac{8+\sqrt{2}}{3-\sqrt{2}}$ **d** $\dfrac{4+5\sqrt{5}}{6-3\sqrt{5}}$

3 Given the graph of $y = 3^x$ shown, estimate solutions to the exponential equations:

a $3^x = 5$

b $3^x = \frac{1}{2}$

c $6 \times 3^x = 20$



4 Write each of the following in the form $a \pm b\sqrt{2}$ where $a, b \in \mathbb{Z}^+$:

a $(\sqrt{2}-1)^2$ **b** $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$ **c** $\dfrac{1}{(\sqrt{2}+1)^2}$ **d** $\dfrac{1}{3+2\sqrt{2}}$

5 Simplify using the laws of exponents:

a $(a^7)^3$ **b** $pq^2 \times p^3q^4$ **c** $\dfrac{8ab^5}{2a^4b^4}$

6 Write the following as a power of 2:

a 2×2^{-4} **b** $16 \div 2^{-3}$ **c** 8^4

7 Write the following without brackets:

a $(2m^3)^2$ **b** $\left(\dfrac{-a^3}{b}\right)^3$ **c** $\dfrac{(3x^2y)^2}{3x}$ **d** $\dfrac{(2a^{\frac{1}{2}}b^{\frac{1}{5}})^4}{a}$

8 Simplify $\dfrac{2^{x+1}}{2^{1-x}}$.

9 Write as powers of 5 in simplest form:

a 1 **b** $5\sqrt{5}$ **c** $\dfrac{1}{\sqrt[4]{5}}$ **d** 25^{a+3}

10 Expand and simplify:

a $e^x(e^{-x}+e^x)$ **b** $(2^x+5)^2$ **c** $(x^{\frac{1}{2}}-7)(x^{\frac{1}{2}}+7)$

11 Solve for x:

a $6\times 2^x = 192$ **b** $4\times(\frac{1}{3})^x = 324$

12 Solve for x without using a calculator:

a $4^{x+1}=\left(\frac{1}{8}\right)^x$ **b** $\dfrac{25^x}{5^{x-3}}=\dfrac{5^x}{125^{x-2}}$ **c** $\dfrac{3^{x+2}}{9^{3-x}}=\dfrac{27^{1-2x}}{3^{2x}}$

13 Suppose $f(x)=2^{-x}+1$.

a Find $f(\frac{1}{2})$. **b** Find a such that $f(a)=3$.

14 On the same set of axes draw the graphs of $y=2^x$ and $y=2^x-4$. Include on your graph the y-intercept and the equation of the horizontal asymptote of each function.

15 Consider $y=3^x-5$.

a Find y when $x=0,\ \pm1,\ \pm2$. **b** Discuss y as $x\to\pm\infty$.

c Sketch the graph of $y=3^x-5$. **d** State the range of the function.

16 Consider $f: x\mapsto e^{2x-1}$ and $g: x\mapsto e^{\sqrt{2}x}$.

a State the range of f.

b Find the exact value of $g(\sqrt{2})$.

c Solve $f(x)=g(x)$, giving your answer in the form $x=a+b\sqrt{2}$ where $a,\ b\in\mathbb{Q}$.

5

Logarithms

Contents:

Opening problem

In a plentiful springtime, a population of 1000 mice will double every week.

The population after t weeks is given by the exponential function $P(t) = 1000 \times 2^t$ mice.

Things to think about:

- **a** What does the graph of the population over time look like?
- **b** How long will it take for the population to reach 20 000 mice?
- **c** Can we write a function for t in terms of P, which determines the time at which the population P is reached?
- **d** What does the graph of this function look like?

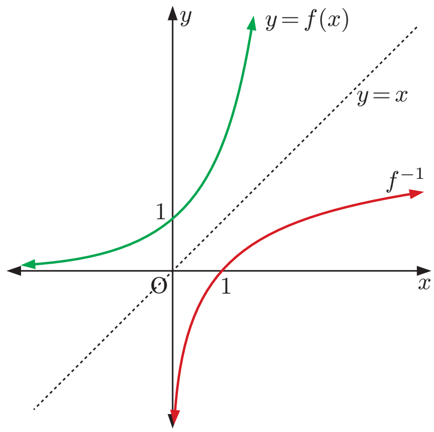
A LOGARITHMS IN BASE 10

Consider the exponential function $f : x \mapsto 10^x$ or $f(x) = 10^x$.

The graph of $y = f(x)$ is shown alongside, along with its inverse function f^{-1}.

Since f is defined by $y = 10^x$,
f^{-1} is defined by $x = 10^y$.
{interchanging x and y}

y is the exponent to which the base 10 is raised in order to get x.



We write this as $y = \log_{10} x$ or $\lg x$ and say that y is the **logarithm in base 10, of x**.

Logarithms are thus defined to be the inverse of exponential functions:

If $f(x) = 10^x$ then $f^{-1}(x) = \log_{10} x$ or $\lg x$.

WORKING WITH LOGARITHMS

Many positive numbers can be easily written in the form 10^x.

For example:

$$\begin{aligned} 10\,000 &= 10^4 \\ 1000 &= 10^3 \\ 100 &= 10^2 \\ 10 &= 10^1 \\ 1 &= 10^0 \\ 0.1 &= 10^{-1} \\ 0.01 &= 10^{-2} \\ 0.001 &= 10^{-3} \end{aligned}$$

Numbers like $\sqrt{10}$, $10\sqrt{10}$ and $\frac{1}{\sqrt[5]{10}}$ can also be written in the form 10^x as follows:

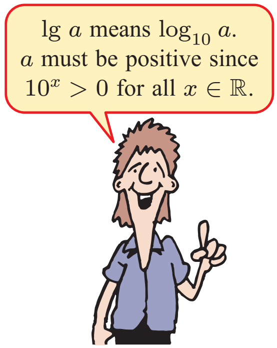
$$\sqrt{10} = 10^{\frac{1}{2}} = 10^{0.5} \qquad 10\sqrt{10} = 10^1 \times 10^{0.5} = 10^{1.5} \qquad \frac{1}{\sqrt[5]{10}} = 10^{-\frac{1}{5}} = 10^{-0.2}$$

In fact, all positive numbers can be written in the form 10^x.

> The **logarithm in base 10** of a positive number is the power that 10 must be raised to in order to obtain the number.

For example:

- Since $1000 = 10^3$, we write $\log_{10}(1000) = 3$
  or $\lg(1000) = 3$.
- Since $0.01 = 10^{-2}$, we write $\log_{10}(0.01) = -2$
  or $\lg(0.01) = -2$.



We hence conclude that:

$$\mathbf{lg}\, \mathbf{10}^{x} = x \quad \text{for any } x \in \mathbb{R}.$$
$$a = \mathbf{10}^{\lg a} \quad \text{for any } a > 0.$$

Example 1 — Self Tutor

Without using a calculator, find:

a $\log 100$ **b** $\log(\sqrt[4]{10})$

a $\log 100 = \log 10^2 = 2$

b $\log(\sqrt[4]{10}) = \log(10^{\frac{1}{4}}) = \frac{1}{4}$

EXERCISE 5A

1 Without using a calculator, find:

a $\lg 10\,000$ **b** $\lg 0.001$ **c** $\lg 10$ **d** $\lg 1$

e $\lg \sqrt{10}$ **f** $\lg(\sqrt[3]{10})$ **g** $\lg\left(\frac{1}{\sqrt[4]{10}}\right)$ **h** $\lg\left(10\sqrt{10}\right)$

i $\lg \sqrt[3]{100}$ **j** $\lg\left(\frac{100}{\sqrt{10}}\right)$ **k** $\lg\left(10 \times \sqrt[3]{10}\right)$ **l** $\lg\left(1000\sqrt{10}\right)$

Check your answers using your calculator.

2 Simplify:

a $\lg 10^n$ **b** $\lg\left(10^a \times 100\right)$ **c** $\lg\left(\frac{10}{10^m}\right)$ **d** $\lg\left(\frac{10^a}{10^b}\right)$

Example 2 — Self Tutor

Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

a 8 **b** 800 **c** 0.08

a $8 = 10^{\lg 8} \approx 10^{0.9031}$

b $800 = 10^{\lg 800} \approx 10^{2.9031}$

c $0.08 = 10^{\lg 0.08} \approx 10^{-1.0969}$

3 **a** Use your calculator to find $\lg 41$, giving your answer correct to 4 decimal places.
b Hence, write 41 as a power of 10.

4 Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006
f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** 0.000 15

5 Explain why you cannot find the logarithm of a negative number.

Example 3 — Self Tutor

a Use your calculator to find: **i** $\lg 2$ **ii** $\lg 20$
b Explain why $\lg 20 = \lg 2 + 1$.

a **i** $\lg 2 \approx 0.3010$
ii $\lg 20 \approx 1.3010$

b
$$\begin{aligned}\lg 20 &= \lg(2 \times 10)\\ &\approx \lg(10^{0.3010} \times 10^1)\\ &\approx \lg 10^{1.3010} \quad \{\text{adding exponents}\}\\ &\approx 1.3010\\ &\approx \lg 2 + 1\end{aligned}$$

6 **a** Use your calculator to find: **i** $\lg 3$ **ii** $\lg 300$
b Explain why $\lg 300 = \lg 3 + 2$.

7 **a** Use your calculator to find: **i** $\lg 5$ **ii** $\lg 0.05$
b Explain why $\lg 0.05 = \lg 5 - 2$.

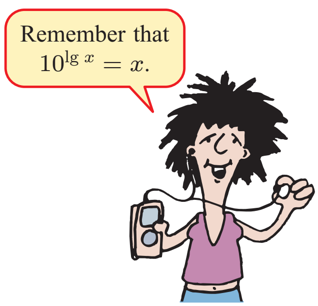
Example 4 — Self Tutor

Find x if:

a $\lg x = 3$ **b** $\lg x \approx -0.271$

a $\lg x = 3$
$\therefore\ 10^{\lg x} = 10^3$
$\therefore\ x = 1000$

b $\lg x \approx -0.271$
$\therefore\ 10^{\lg x} \approx 10^{-0.271}$
$\therefore\ x \approx 0.536$



8 Find x if:

a $\lg x = 2$	**b** $\lg x = 1$	**c** $\lg x = 0$
d $\lg x = -1$	**e** $\lg x = \frac{1}{2}$	**f** $\lg x = -\frac{1}{2}$
g $\lg x = 4$	**h** $\lg x = -5$	**i** $\lg x \approx 0.8351$
j $\lg x \approx 2.1457$	**k** $\lg x \approx -1.378$	**l** $\lg x \approx -3.1997$

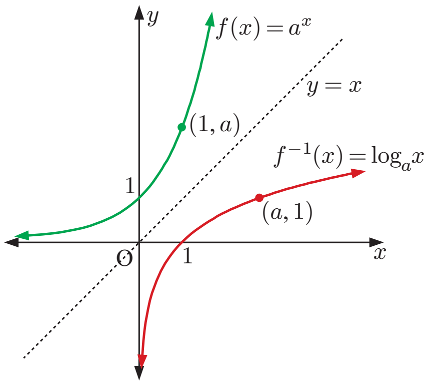
B LOGARITHMS IN BASE a

In the previous section we defined logarithms in base 10 as the inverse of the exponential function $f(x) = 10^x$.

If $f(x) = 10^x$ then $f^{-1}(x) = \log_{10} x$.

We can use the same principle to define logarithms in other bases:

If $f(x) = a^x$ then $f^{-1}(x) = \log_a x$.

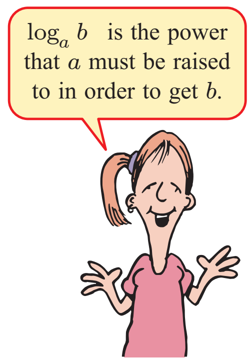


If $b = a^x$, $a \neq 1$, $a > 0$, we say that x is the **logarithm in base a, of b**, and that $b = a^x \Leftrightarrow x = \log_a b$, $b > 0$.

$b = a^x \Leftrightarrow x = \log_a b$ is read as "$b = a^x$ if and only if $x = \log_a b$".

It is a short way of writing:
"if $b = a^x$ then $x = \log_a b$, and if $x = \log_a b$ then $b = a^x$".

$b = a^x$ and $x = \log_a b$ are *equivalent* or *interchangeable* statements.



For example:

- $8 = 2^3$ means that $3 = \log_2 8$ and vice versa.
- $\log_5 25 = 2$ means that $25 = 5^2$ and vice versa.

If $y = a^x$ then $x = \log_a y$, and so $\boldsymbol{x = \log_a a^x}$.

If $x = a^y$ then $y = \log_a x$, and so $\boldsymbol{x = a^{\log_a x}}$ provided $x > 0$.

Example 5 — Self Tutor

a Write an equivalent exponential equation for $\log_{10} 1000 = 3$.

b Write an equivalent logarithmic equation for $3^4 = 81$.

a From $\log_{10} 1000 = 3$ we deduce that $10^3 = 1000$.

b From $3^4 = 81$ we deduce that $\log_3 81 = 4$.

EXERCISE 5B

1 Write an equivalent exponential equation for:

a $\log_{10} 100 = 2$ **b** $\log_{10} 10\,000 = 4$ **c** $\log_{10}(0.1) = -1$

d $\log_{10} \sqrt{10} = \frac{1}{2}$ **e** $\log_2 8 = 3$ **f** $\log_3 9 = 2$

g $\log_2(\frac{1}{4}) = -2$ **h** $\log_3 \sqrt{27} = 1.5$ **i** $\log_5 \left(\frac{1}{\sqrt{5}}\right) = -\frac{1}{2}$

2 Write an equivalent logarithmic equation for:

a $2^2 = 4$ **b** $4^3 = 64$ **c** $5^2 = 25$

d $7^2 = 49$ **e** $2^6 = 64$ **f** $2^{-3} = \frac{1}{8}$

g $10^{-2} = 0.01$ **h** $2^{-1} = \frac{1}{2}$ **i** $3^{-3} = \frac{1}{27}$

Example 6 — Self Tutor

Find:

a $\log_2 16$ **b** $\log_5 0.2$ **c** $\log_{10} \sqrt[5]{100}$ **d** $\log_2 \left(\frac{1}{\sqrt{2}}\right)$

a $\log_2 16$
$= \log_2 2^4$
$= 4$

b $\log_5 0.2$
$= \log_5(\frac{1}{5})$
$= \log_5 5^{-1}$
$= -1$

c $\log_{10} \sqrt[5]{100}$
$= \log_{10} \left(10^2\right)^{\frac{1}{5}}$
$= \log_{10} 10^{\frac{2}{5}}$
$= \frac{2}{5}$

d $\log_2 \left(\frac{1}{\sqrt{2}}\right)$
$= \log_2 2^{-\frac{1}{2}}$
$= -\frac{1}{2}$

3 Find:

a $\log_{10} 100\,000$ **b** $\log_{10}(0.01)$ **c** $\log_3 \sqrt{3}$ **d** $\log_2 8$

e $\log_2 64$ **f** $\log_2 128$ **g** $\log_5 25$ **h** $\log_5 125$

i $\log_2(0.125)$ **j** $\log_9 3$ **k** $\log_4 16$ **l** $\log_{36} 6$

m $\log_3 243$ **n** $\log_2 \sqrt[3]{2}$ **o** $\log_a a^n$ **p** $\log_8 2$

q $\log_t \left(\frac{1}{t}\right)$ **r** $\log_6 6\sqrt{6}$ **s** $\log_4 1$ **t** $\log_9 9$

4 Use your calculator to find:

a $\log_{10} 152$ **b** $\log_{10} 25$ **c** $\log_{10} 74$ **d** $\log_{10} 0.8$

5 Solve for x:

a $\log_2 x = 3$ **b** $\log_4 x = \frac{1}{2}$ **c** $\log_x 81 = 4$ **d** $\log_2(x-6) = 3$

6 Simplify:

a $\log_4 16$ **b** $\log_2 4$ **c** $\log_3 \left(\frac{1}{3}\right)$ **d** $\log_{10} \sqrt[4]{1000}$

e $\log_7 \left(\frac{1}{\sqrt{7}}\right)$ **f** $\log_5(25\sqrt{5})$ **g** $\log_3 \left(\frac{1}{\sqrt{27}}\right)$ **h** $\log_4 \left(\frac{1}{2\sqrt{2}}\right)$

i $\log_x x^2$ **j** $\log_x \sqrt{x}$ **k** $\log_m m^3$ **l** $\log_x(x\sqrt{x})$

m $\log_n \left(\frac{1}{n}\right)$ **n** $\log_a \left(\frac{1}{a^2}\right)$ **o** $\log_a \left(\frac{1}{\sqrt{a}}\right)$ **p** $\log_m \sqrt{m^5}$

Discussion

We have seen that $\sqrt{2}$ cannot be written in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$. We therefore say $\sqrt{2}$ is irrational.

More generally, $\sqrt{a}$ is only rational if a is a perfect square.

What about logarithms? The following is a proof that $\log_2 3$ is irrational.

Proof: If $\log_2 3$ is rational, then $\log_2 3 = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$

$$\therefore \quad 3 = 2^{\frac{p}{q}}$$
$$\therefore \quad 3^q = 2^p$$

The left hand side is always odd, and the right hand side is always even, so the statement is impossible.

Hence $\log_2 3$ must be irrational.

Under what circumstances will $\log_a b$ be rational?

C LAWS OF LOGARITHMS

Discovery — The laws of logarithms

What to do:

1 Use your calculator to find:

a $\lg 2 + \lg 3$ **b** $\lg 3 + \lg 7$ **c** $\lg 4 + \lg 20$

d $\lg 6$ **e** $\lg 21$ **f** $\lg 80$

From your answers, suggest a possible simplification for $\lg a + \lg b$.

2 Use your calculator to find:

a $\lg 6 - \lg 2$ **b** $\lg 12 - \lg 3$ **c** $\lg 3 - \lg 5$

d $\lg 3$ **e** $\lg 4$ **f** $\lg(0.6)$

From your answers, suggest a possible simplification for $\lg a - \lg b$.

3 Use your calculator to find:

a $3 \lg 2$ **b** $2 \lg 5$ **c** $-4 \lg 3$

d $\lg(2^3)$ **e** $\lg(5^2)$ **f** $\lg(3^{-4})$

From your answers, suggest a possible simplification for $n \lg a$.

From the **Discovery**, you should have found the three important **laws of logarithms**:

If A and B are both positive then:

- $\lg A + \lg B = \lg(AB)$
- $\lg A - \lg B = \lg\left(\frac{A}{B}\right)$
- $n \lg A = \lg(A^n)$

More generally, in any base c where $c \neq 1$, $c > 0$, we have these **laws of logarithms**:

If A and B are both positive then:

- $\log_c A + \log_c B = \log_c(AB)$
- $\log_c A - \log_c B = \log_c\left(\frac{A}{B}\right)$
- $n\log_c A = \log_c(A^n)$

Proof:

- $$\begin{aligned}&\log_c(AB)\\ &= \log_c(c^{\log_c A} \times c^{\log_c B})\\ &= \log_c(c^{\log_c A + \log_c B})\\ &= \log_c A + \log_c B\end{aligned}$$
- $$\begin{aligned}&\log_c\left(\frac{A}{B}\right)\\ &= \log_c\left(\frac{c^{\log_c A}}{c^{\log_c B}}\right)\\ &= \log_c(c^{\log_c A - \log_c B})\\ &= \log_c A - \log_c B\end{aligned}$$
- $$\begin{aligned}&\log_c(A^n)\\ &= \log_c((c^{\log_c A})^n)\\ &= \log_c(c^{n\log_c A})\\ &= n\log_c A\end{aligned}$$

Example 7 — Self Tutor

Use the laws of logarithms to write the following as a single logarithm or as an integer:

a $\lg 5 + \lg 3$ **b** $\log_3 24 - \log_3 8$ **c** $\log_2 5 - 1$

a
$$\begin{aligned}&\lg 5 + \lg 3\\ &= \lg(5 \times 3)\\ &= \lg 15\end{aligned}$$

b
$$\begin{aligned}&\log_3 24 - \log_3 8\\ &= \log_3\left(\tfrac{24}{8}\right)\\ &= \log_3 3\\ &= 1\end{aligned}$$

c
$$\begin{aligned}&\log_2 5 - 1\\ &= \log_2 5 - \log_2 2^1\\ &= \log_2\left(\tfrac{5}{2}\right)\end{aligned}$$

Example 8 — Self Tutor

Simplify by writing as a single logarithm or as a rational number:

a $2\lg 7 - 3\lg 2$ **b** $2\lg 3 + 3$ **c** $\dfrac{\lg 8}{\lg 4}$

a
$$\begin{aligned}&2\lg 7 - 3\lg 2\\ &= \lg(7^2) - \lg(2^3)\\ &= \lg 49 - \lg 8\\ &= \lg\left(\tfrac{49}{8}\right)\end{aligned}$$

b
$$\begin{aligned}&2\lg 3 + 3\\ &= \lg(3^2) + \lg(10^3)\\ &= \lg 9 + \lg 1000\\ &= \lg(9000)\end{aligned}$$

c
$$\begin{aligned}&\frac{\lg 8}{\lg 4}\\ &= \frac{\lg 2^3}{\lg 2^2}\\ &= \frac{3\lg 2}{2\lg 2}\\ &= \tfrac{3}{2}\end{aligned}$$

EXERCISE 5C

1 Write as a single logarithm or as an integer:

a $\lg 8+\lg 2$ **b** $\lg 4+\lg 5$ **c** $\lg 40-\lg 5$
d $\lg p-\lg m$ **e** $\log_4 8-\log_4 2$ **f** $\lg 5+\lg(0.4)$
g $\lg 2+\lg 3+\lg 4$ **h** $1+\log_2 3$ **i** $\lg 4-1$
j $\lg 5+\lg 4-\lg 2$ **k** $2+\lg 2$ **l** $t+\lg w$
m $\log_m 40-2$ **n** $\log_3 6-\log_3 2-\log_3 3$ **o** $\lg 50-4$
p $3-\log_5 50$ **q** $\log_5 100-\log_5 4$ **r** $\lg\left(\frac{4}{3}\right)+\lg 3+\lg 7$

2 Write as a single logarithm or integer:

a $5\lg 2+\lg 3$ **b** $2\lg 3+3\lg 2$ **c** $3\lg 4-\lg 8$
d $2\log_3 5-3\log_3 2$ **e** $\frac{1}{2}\log_6 4+\log_6 3$ **f** $\frac{1}{3}\lg\left(\frac{1}{8}\right)$
g $3-\lg 2-2\lg 5$ **h** $1-3\lg 2+\lg 20$ **i** $2-\frac{1}{2}\log_n 4-\log_n 5$

3 Simplify without using a calculator:

a $\dfrac{\lg 4}{\lg 2}$ **b** $\dfrac{\log_5 27}{\log_5 9}$ **c** $\dfrac{\lg 8}{\lg 2}$
d $\dfrac{\lg 3}{\lg 9}$ **e** $\dfrac{\log_3 25}{\log_3(0.2)}$ **f** $\dfrac{\log_4 8}{\log_4(0.25)}$

Check your answers using a calculator.

Example 9 — Self Tutor

Show that:

a $\lg\left(\frac{1}{9}\right)=-2\lg 3$ **b** $\lg 500=3-\lg 2$

a $\lg\left(\frac{1}{9}\right)$
$=\lg(3^{-2})$
$=-2\lg 3$

b $\lg 500$
$=\lg\left(\frac{1000}{2}\right)$
$=\lg 1000-\lg 2$
$=\lg 10^3-\lg 2$
$=3-\lg 2$

4 Show that:

a $\lg 9=2\lg 3$ **b** $\lg\sqrt{2}=\frac{1}{2}\lg 2$ **c** $\lg\left(\frac{1}{8}\right)=-3\lg 2$
d $\lg\left(\frac{1}{5}\right)=-\lg 5$ **e** $\lg 5=1-\lg 2$ **f** $\lg 5000=4-\lg 2$
g $\log_6 4+\log_6 9=2$ **h** $\log_{15} 3-\log_{15} 45=-1$ **i** $2\log_{12} 2+\frac{1}{2}\log_{12} 9=1$

5 Find the exact value of:

a $3\lg 2+2\lg 5-\frac{1}{2}\lg 4$ **b** $2\log_2 3-\log_2 6-\frac{1}{2}\log_2 9$
c $5\log_6 2+2\log_6 3-\frac{1}{2}\log_6 16-\log_6 12$

6 If $x = \log_2 P$, $y = \log_2 Q$, and $z = \log_2 R$, write in terms of x, y, and z:

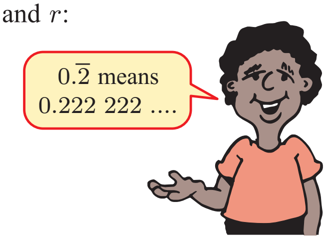
a $\log_2(PR)$ **b** $\log_2(RQ^2)$ **c** $\log_2\left(\dfrac{PR}{Q}\right)$

d $\log_2(P^2\sqrt{Q})$ **e** $\log_2\left(\dfrac{Q^3}{\sqrt{R}}\right)$ **f** $\log_2\left(\dfrac{R^2\sqrt{Q}}{P^3}\right)$

7 If $p = \log_b 2$, $q = \log_b 3$, and $r = \log_b 5$, write in terms of p, q, and r:

a $\log_b 6$ **b** $\log_b 45$ **c** $\log_b 108$

d $\log_b\left(\frac{5\sqrt{3}}{2}\right)$ **e** $\log_b\left(\frac{5}{32}\right)$ **f** $\log_b(0.\overline{2})$



8 If $\log_t M = 1.29$ and $\log_t N^2 = 1.72$, find:

a $\log_t N$ **b** $\log_t(MN)$ **c** $\log_t\left(\dfrac{N^2}{\sqrt{M}}\right)$

9 Suppose $\log_b P = 5$ and $\log_b(P^3Q^2) = 21$. Find $\log_b Q$.

10 Suppose that $\log_t(AB^3) = 15$ and $\log_t\left(\dfrac{A^2}{B}\right) = 9$.

a Write *two* equations connecting $\log_t A$ and $\log_t B$.

b Find the values of $\log_t A$ and $\log_t B$.

c Find $\log_t\left(B^5\sqrt{A}\right)$.

d Write B in terms of t.

D LOGARITHMIC EQUATIONS

We can use the laws of logarithms to write equations in a different form. This can be particularly useful if an unknown appears as an exponent.

For the logarithmic function, for every value of y, there is only one corresponding value of x. We can therefore take the logarithm of both sides of an equation without changing the solution. However, we can only do this if both sides are positive.

Example 10 Self Tutor

Write these as logarithmic equations (in base 10):

a $y = 5 \times 3^x$ **b** $P = \dfrac{20}{\sqrt{n}}$

a
$$y = 5 \times 3^x$$
$$\therefore \lg y = \lg(5 \times 3^x)$$
$$\therefore \lg y = \lg 5 + \lg 3^x$$
$$\therefore \lg y = \lg 5 + x \lg 3$$

b
$$P = \frac{20}{\sqrt{n}}$$
$$\therefore \lg P = \lg\left(\frac{20}{n^{\frac{1}{2}}}\right)$$
$$\therefore \lg P = \lg 20 - \lg n^{\frac{1}{2}}$$
$$\therefore \lg P = \lg 20 - \tfrac{1}{2}\lg n$$

Example 11 — Self Tutor

Write the following equations without logarithms:

a $\lg y = x\lg 4 + \lg 3$ **b** $\log_2 M = 3\log_2 a - 5$

a
$$\begin{aligned} \lg y &= x\lg 4 + \lg 3 \\ \therefore\ \lg y &= \lg 4^x + \lg 3 \\ \therefore\ \lg y &= \lg(3\times 4^x) \\ \therefore\ y &= 3\times 4^x \end{aligned}$$

b
$$\begin{aligned} \log_2 M &= 3\log_2 a - 5 \\ \therefore\ \log_2 M &= \log_2 a^3 - \log_2 2^5 \\ \therefore\ \log_2 M &= \log_2\left(\frac{a^3}{32}\right) \\ \therefore\ M &= \frac{a^3}{32} \end{aligned}$$

EXERCISE 5D.1

1 Write the following as logarithmic equations in base 10, assuming all terms are positive:

a $y = 2^x$ **b** $y = x^3$ **c** $M = d^4$
d $T = 5^x$ **e** $y = \sqrt{x}$ **f** $y = 7\times 3^x$
g $S = \dfrac{9}{t}$ **h** $M = 100\times 7^x$ **i** $T = 5\sqrt{d}$
j $F = \dfrac{1000}{\sqrt{n}}$ **k** $S = 200\times 2^t$ **l** $y = \sqrt{\dfrac{15}{x}}$

2 Write the following equations without logarithms:

a $\lg y = x\lg 7$ **b** $\lg D = \lg x + \lg 2$ **c** $\log_a F = \log_a 5 - \log_a t$
d $\lg y = x\lg 2 + \lg 6$ **e** $\lg P = \frac{1}{2}\lg x$ **f** $\lg N = -\frac{1}{3}\lg p$
g $\lg P = 3\lg x + 1$ **h** $\lg y = x - \lg 2$ **i** $\lg y = 2\lg x - 1$
j $\log_2 T = 5\log_2 k + 1$ **k** $\log_3 P = 4\log_3 n - 2$ **l** $\log_2 y = 4x + 3$

3 Suppose $\lg y = 3\lg x - \lg 2$.

a Write y in terms of x, without using logarithms.
b Find y when: **i** $x = 2$ **ii** $x = 4$

4 Suppose $\lg y = \frac{1}{3}x + 2$.

a Write y in the form $y = a(10^{bx})$ where $a, b \in \mathbb{Q}$.
b Find y when: **i** $x = 0$ **ii** $x = 3$

5 Copy and complete:

a If there is a *power* relationship between y and x, for example $y = 5x^3$, then there is a *linear* relationship between $\lg y$ and ......
b If there is an *exponential* relationship between y and x, for example $y = 4\times 2^x$, then there is a *linear* relationship between ...... and ......

SOLVING LOGARITHMIC EQUATIONS

Logarithmic equations can often be solved using the laws of logarithms. However, we must always check that our solutions satisfy the original equation, remembering that $\lg x$ is only defined for $x > 0$.

Example 12 Self Tutor

Solve for x:

a $\lg(x-6) + \lg 3 = 2\lg 6$

b $\lg x + \lg(x+5) = \lg 14$

a $\lg(x-6) + \lg 3 = 2\lg 6$

$\therefore \lg(x-6) = \lg 6^2 - \lg 3$

$\therefore \lg(x-6) = \lg\left(\frac{36}{3}\right)$

$\therefore x - 6 = 12$

$\therefore x = 18$

Check: $x - 6 > 0$, so $x > 6$ ✓

b $\lg x + \lg(x+5) = \lg 14$

$\therefore \lg(x(x+5)) = \lg 14$

$\therefore x(x+5) = 14$

$\therefore x^2 + 5x - 14 = 0$

$\therefore (x+7)(x-2) = 0$

$\therefore x = -7$ or 2

But $x > 0$ and $x + 5 > 0$

$\therefore x = 2$ is the only valid solution.

EXERCISE 5D.2

1 Solve for x:

a $\lg(x-4) = \lg 3 + \lg 7$

b $\lg(x+5) - \lg 8 = 2\lg 3$

c $\lg(2x) = 1 + \frac{1}{2}\lg 16$

d $\log_2 x = 3\log_2 5 - 6$

e $\lg x - \lg(x-4) = \lg 5$

f $\log_5(x-2) - \log_5(x+2) = \log_5 3$

g $\log_3 x - 2 = \log_3(x-1)$

h $\lg(x+2) - 1 = \lg(x-3) - \lg 12$

2 Solve for x:

a $\lg x + \lg(x+1) = \lg 30$

b $\log_5(x+9) + \log_5(x+2) = \log_5(20x)$

c $\log_7 x = \log_7 8 - \log_7(6-x)$

d $\log_6(x+4) + \log_6(x-1) = 1$

e $\lg x + \lg(2x+8) = 1$

f $\lg(x+2) + \lg(x+7) = \lg(2x+2)$

g $2\log_2 x - \log_2(8-3x) = 1$

h $\log_2 x + \log_2(2x-7) = 2$

Example 13 Self Tutor

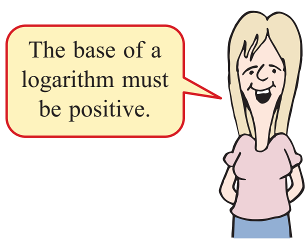
Solve for x: $\log_x 3 + \log_x 12 = 2$

$\log_x 3 + \log_x 12 = 2$

$\therefore \log_x(3 \times 12) = \log_x(x^2)$

$\therefore 36 = x^2$

$\therefore x = 6$ {since $x > 0$}



3 Solve for x:

a $\log_x 32 - \log_x 4 = 1$

b $\log_x 45 = 2 + \log_x 5$

c $\log_x 54 = 3 - \log_x 4$

d $2\log_x 2 - 3 = \log_x\left(\frac{1}{16}\right)$