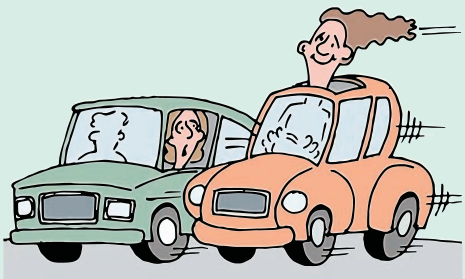
9 In a car club, 13 members drive a manual and 15 members have a sunroof on their car. 5 have manual cars with a sunroof, and 4 have neither.



- **a** Display this information on a Venn diagram.
- **b** How many members:
  - **i** are in the club
  - **ii** drive a manual car without a sunroof
  - **iii** do not drive a manual car?

10 All attendees of a camp left something at home. 11 forgot to bring their towel, and 23 forgot their hat. Of the 30 campers, how many had neither a hat nor a towel?

11 Consider the sets $\mathscr{E} = \{x \in \mathbb{Z}^+ : x \leqslant 40\}$, $A = \{\text{factors of } 40\}$, and $B = \{\text{factors of } 20\}$.

- **a** List the sets A and B.
- **b** What can be said about sets A and B?
- **c** Illustrate sets A and B on a Venn diagram.

12 At a conference, the 58 delegates speak many different languages. 28 speak Arabic, 27 speak Chinese, and 39 speak English. 12 speak Arabic and Chinese, 16 speak both Chinese and English, and 17 speak Arabic and English. 2 speak all three languages. How many delegates speak:

- **a** Chinese only
- **b** none of these languages
- **c** neither Arabic nor Chinese?

2

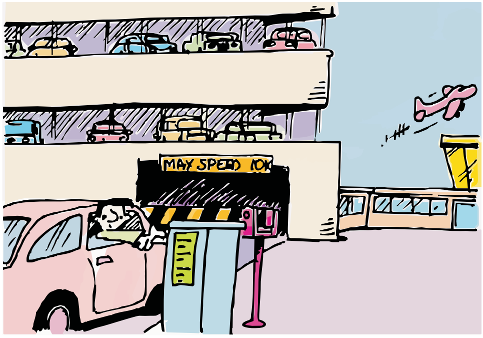
Functions

Contents:

A RELATIONS AND FUNCTIONS

The charges for parking a car in a short-term car park at an airport are shown in the table below. The total charge is *dependent* on the length of time t the car is parked.

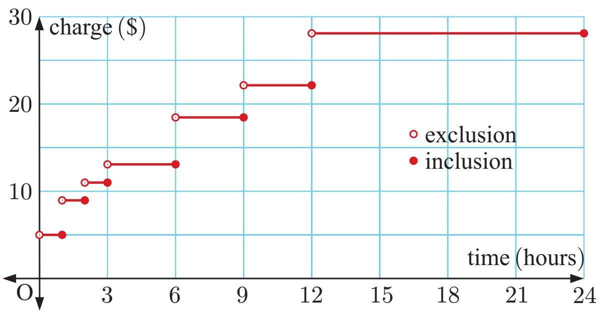
Car park charges	
Time t (hours)	*Charge*
0 - 1 hours	\$5.00
1 - 2 hours	\$9.00
2 - 3 hours	\$11.00
3 - 6 hours	\$13.00
6 - 9 hours	\$18.00
9 - 12 hours	\$22.00
12 - 24 hours	\$28.00



Looking at this table we might ask: How much would be charged for *exactly* one hour? Would it be \$5 or \$9?

To avoid confusion, we could adjust the table or draw a graph. We indicate that 2 - 3 hours really means a time over 2 hours up to and including 3 hours, by writing $2 < t \leqslant 3$ hours.

Car park charges	
Time t (hours)	*Charge*
$0 < t \leqslant 1$ hours	\$5.00
$1 < t \leqslant 2$ hours	\$9.00
$2 < t \leqslant 3$ hours	\$11.00
$3 < t \leqslant 6$ hours	\$13.00
$6 < t \leqslant 9$ hours	\$18.00
$9 < t \leqslant 12$ hours	\$22.00
$12 < t \leqslant 24$ hours	\$28.00



In mathematical terms, we have a relationship between two variables *time* and *charge*, so the schedule of charges is an example of a **relation**.

A relation may consist of a finite number of ordered pairs, such as $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$, or an infinite number of ordered pairs.

The parking charges example is clearly the latter, as every real value of time in the interval $0 < t \leqslant 24$ hours is represented.

The set of possible values of the variable on the horizontal axis is called the **domain** of the relation.

For example:
- the domain for the car park relation is $\{t \mid 0 < t \leqslant 24\}$
- the domain of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ is $\{-2, 1, 4\}$.

The set of possible values on the vertical axis is called the **range** of the relation.

For example:
- the range of the car park relation is $\{5, 9, 11, 13, 18, 22, 28\}$
- the range of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ is $\{3, 5, 6\}$.

We will now look at relations and functions more formally.

RELATIONS

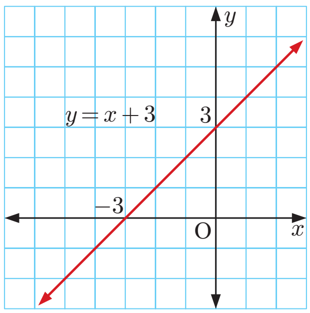
In **Chapter 1**, we saw that:

A **relation** is any set of points which connect two variables.

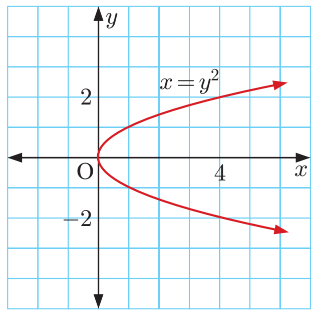
A relation is often expressed in the form of an **equation** connecting the **variables** x and y. The relation is a set of points (x, y) which can be viewed in the **Cartesian plane**.

For example, $y = x + 3$ and $x = y^2$ are the equations of two relations. Each equation generates a set of ordered pairs, which we can graph.

$y = x + 3$ is a set of points which lie in a straight line



$x = y^2$ is a set of points which lie in a smooth curve.



FUNCTIONS

A **function**, sometimes called a **mapping**, is a relation in which no two different ordered pairs have the same x-coordinate or first component.

We can see from the above definition that a function is a special type of relation.
Every function is a relation, but not every relation is a function.

TESTING FOR FUNCTIONS

Algebraic Test:

If a relation is given as an equation, and the substitution of any value for x results in one and only one value of y, then the relation is a function.

For example:

- $y = 3x - 1$ is a function, since for any value of x there is only one corresponding value of y
- $x = y^2$ is not a function, since if $x = 4$ then $y = \pm 2$.

Geometric Test or Vertical Line Test:

Suppose we draw all possible vertical lines on the graph of a relation.
- If each line cuts the graph at most once, then the relation is a function.
- If at least one line cuts the graph more than once, then the relation is *not* a function.

GRAPHICAL NOTE

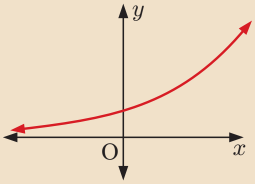
- If a graph contains a small **open circle** such as , this point is **not included**.
- If a graph contains a small **filled-in circle** such as , this point **is included**.
- If a graph contains an **arrow head** at an end such as , then the graph continues indefinitely in that general direction, or the shape may repeat as it has done previously.

Example 1

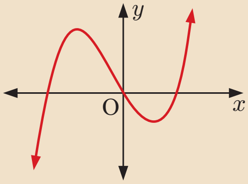
Self Tutor

Which of the following relations are functions?

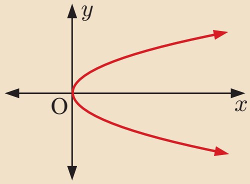
a



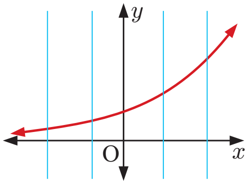
b



c

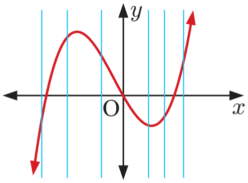


a



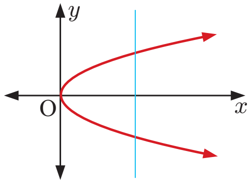
a function

b



a function

c



not a function

DEMO

EXERCISE 2A.1

1 Which of the following sets of ordered pairs are functions? Give reasons for your answers.

a $\{(1, 3), (2, 4), (3, 5), (4, 6)\}$

b $\{(1, 3), (3, 2), (1, 7), (-1, 4)\}$

c $\{(2, -1), (2, 0), (2, 3), (2, 11)\}$

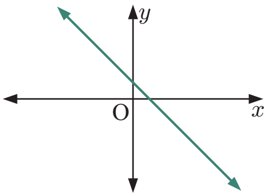
d $\{(7, 6), (5, 6), (3, 6), (-4, 6)\}$

e $\{(0, 0), (1, 0), (3, 0), (5, 0)\}$

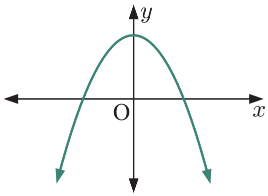
f $\{(0, 0), (0, -2), (0, 2), (0, 4)\}$

2 Use the vertical line test to determine which of the following relations are functions:

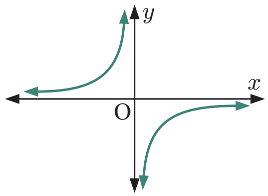
a



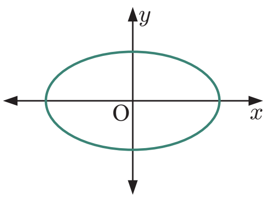
b



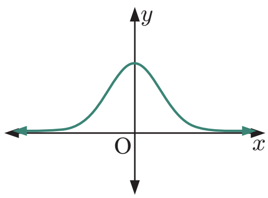
c



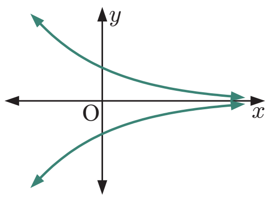
d



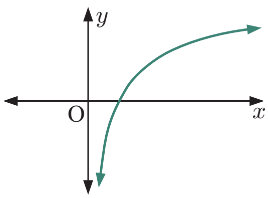
e



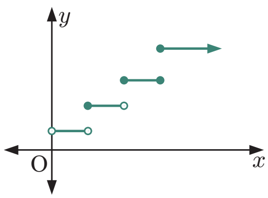
f



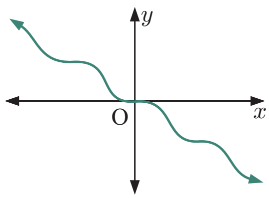
g



h



i

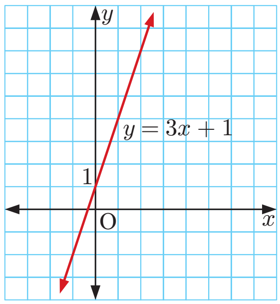


3 Give algebraic evidence to show that the relation $x^2 + y^2 = 16$ is not a function.

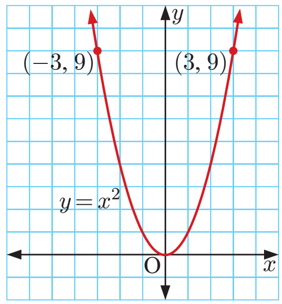
ONE-ONE FUNCTIONS

A **one-one function** is a function in which no two different ordered pairs have the same y-coordinate or second component.

For example:

- 

  The function $y = 3x + 1$ is one-one, since each distinct point on the graph has a different y-coordinate.

- 

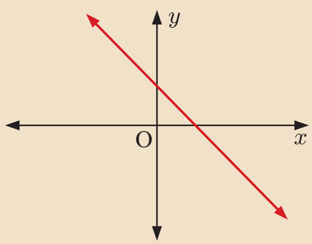
  The function $y = x^2$ is *not* one-one, since the graph contains distinct points $(-3,\ 9)$ and $(3,\ 9)$ which have the same y-coordinate.

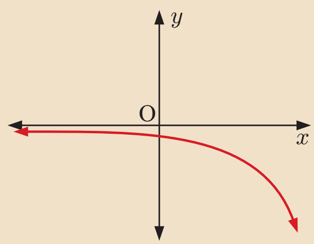
We can use the **horizontal line test** to determine whether a function is one-one:

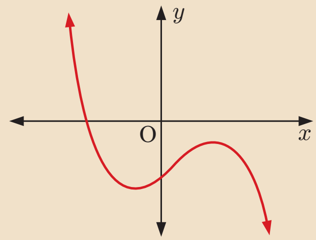
Suppose we draw all possible horizontal lines on the graph of a function.
- If each line cuts the graph at most once, then the function is one-one.
- If at least one line cuts the graph more than once, then the function is *not* one-one.

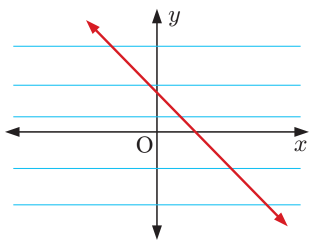
Example 2 — Self Tutor

Which of the following relations are one-one?

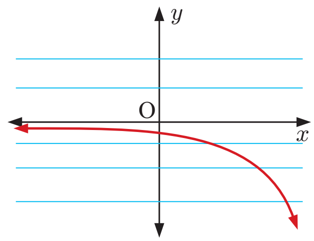
a


b


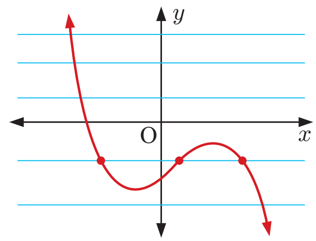
c


a


one-one

b


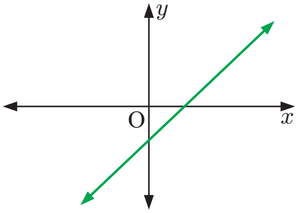
one-one

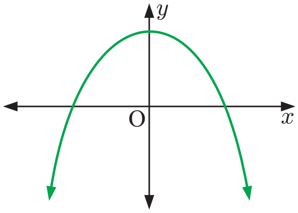
c


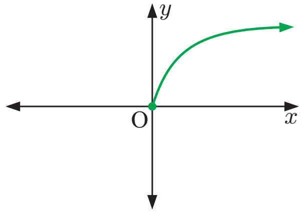
not one-one

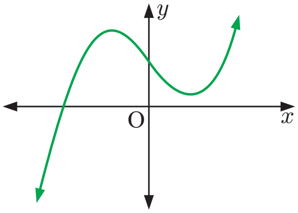
EXERCISE 2A.2

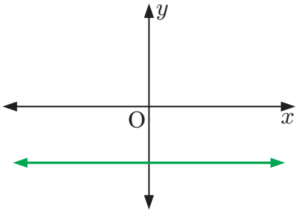
1 Which of the following functions are one-one?

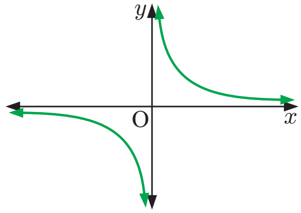
a


b


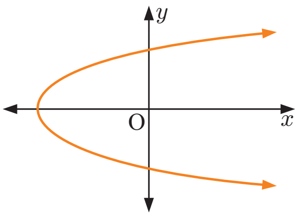
c


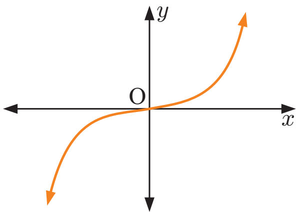
d


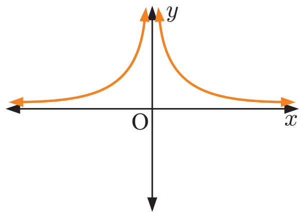
e


f


2 Determine whether the following relations are functions. If they are functions, determine whether they are one-one.

a


b


c


3 Consider the car park relation described on page **36**.

a **i** Richard parked his car for 4 hours. How much did he pay?

ii Suppose Susie parked her car for t hours. If you know the value of t, can you uniquely determine how much she paid?

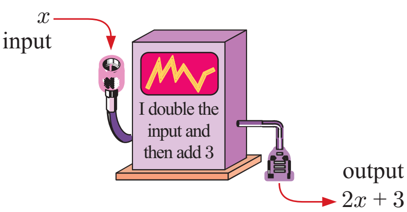
iii Is the car park relation a function?

b **i** Janette paid \$18 for parking. Can you uniquely determine how long she parked for?

ii Is the car park function one-one?

B FUNCTION NOTATION

Function machines are sometimes used to illustrate how functions behave.



If 4 is the input fed into the machine, the output is $2(4) + 3 = 11$.

The above 'machine' has been programmed to perform a particular function. If we use f to represent that particular function, we can write:

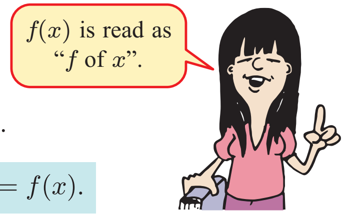
f is the function that will convert x into $2x + 3$.

So, f would convert 2 into $2(2)+3=7$ and
-4 into $2(-4)+3=-5$.

This function can be written as:

$$\boldsymbol{f : x \mapsto 2x+3}$$

function f such that x is converted into $2x+3$



Two other equivalent forms we use are $f(x)=2x+3$ and $y=2x+3$.

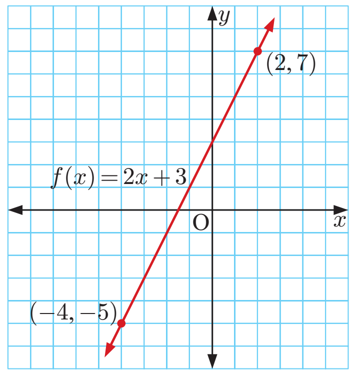
$f(x)$ is the value of y for a given value of x, so $y=f(x)$.

f is the function which converts x into $f(x)$, so we write $f : x \mapsto f(x)$.

$y=f(x)$ is sometimes called the **function value** or **image** of x.

For $f(x)=2x+3$:

- $f(2)=2(2)+3=7$.
  $\therefore$ the point (2, 7) lies on the graph of the function.
- $f(-4)=2(-4)+3=-5$.
  $\therefore$ the point (−4, −5) also lies on the graph.



A **linear function** is a function of the form $f(x)=ax+b$ where a, b are real constants.
The graph of a linear function is a straight line.

Example 3 — Self Tutor

If $f : x \mapsto 2x^2-3x$, find the value of: **a** $f(5)$ **b** $f(-4)$

$f(x)=2x^2-3x$

a $f(5)=2(5)^2-3(5)$ {replacing x with (5)}
$=2\times 25-15$
$=35$

b $f(-4)=2(-4)^2-3(-4)$ {replacing x with (−4)}
$=2(16)+12$
$=44$

EXERCISE 2B

1 If $f(x)=3x-x^2+2$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-3)$ **d** $f(-7)$ **e** $f(\frac{3}{2})$

2 If $g : x \mapsto x-\dfrac{4}{x}$, find the value of:

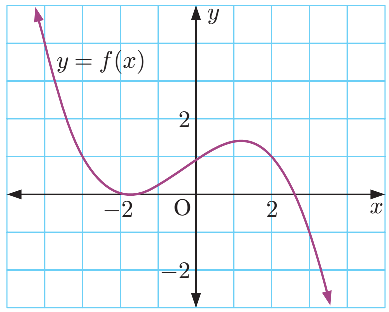
a $g(1)$ **b** $g(4)$ **c** $g(-1)$ **d** $g(-4)$ **e** $g(-\frac{1}{2})$

3 The graph of $y = f(x)$ is shown alongside.

a Find:

i $f(2)$ **ii** $f(3)$

b Find the value of x such that $f(x) = 4$.



Example 4 — Self Tutor

If $f(x) = 5 - x - x^2$, find in simplest form: **a** $f(-x)$ **b** $f(x+2)$

a $f(-x) = 5 - (-x) - (-x)^2$ {replacing x with $(-x)$}
$= 5 + x - x^2$

b $f(x+2) = 5 - (x+2) - (x+2)^2$ {replacing x with $(x+2)$}
$= 5 - x - 2 - [x^2 + 4x + 4]$
$= 3 - x - x^2 - 4x - 4$
$= -x^2 - 5x - 1$

4 If $f(x) = 7 - 3x$, find in simplest form:

a $f(a)$ **b** $f(-a)$ **c** $f(a+3)$ **d** $f(b-1)$ **e** $f(x+2)$ **f** $f(x+h)$

5 If $F(x) = 2x^2 + 3x - 1$, find in simplest form:

a $F(x+4)$ **b** $F(2-x)$ **c** $F(-x)$ **d** $F(x^2)$ **e** $F(x^2-1)$ **f** $F(x+h)$

6 Suppose $G(x) = \frac{2x+3}{x-4}$.

a Evaluate: **i** $G(2)$ **ii** $G(0)$ **iii** $G(-\frac{1}{2})$

b Find a value of x such that $G(x)$ does not exist.

c Find $G(x+2)$ in simplest form.

d Find x if $G(x) = -3$.

7 f represents a function. What is the difference in meaning between f and $f(x)$?

8 The value of a photocopier t years after purchase is given by $V(t) = 9650 - 860t$ dollars.

a Find $V(4)$ and state what $V(4)$ means.

b Find t when $V(t) = 5780$ and explain what this represents.

c Find the original purchase price of the photocopier.

9 On the same set of axes draw the graphs of three different functions $f(x)$ such that $f(2) = 1$ and $f(5) = 3$.

10 Find a linear function $f(x) = ax + b$ for which $f(2) = 1$ and $f(-3) = 11$.

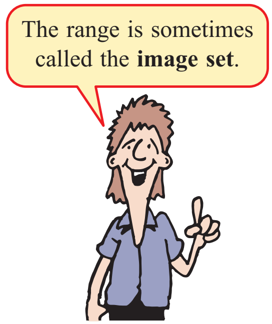
11 Given $f(x) = ax + \frac{b}{x}$, $f(1) = 1$, and $f(2) = 5$, find constants a and b.

12 Given $T(x) = ax^2 + bx + c$, $T(0) = -4$, $T(1) = -2$, and $T(2) = 6$, find a, b, and c.

C DOMAIN AND RANGE

The **domain** of a relation is the set of values of x in the relation.

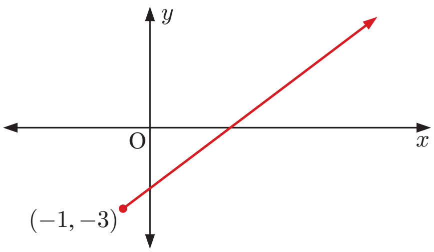
The **range** of a relation is the set of values of y in the relation.



The domain and range of a relation are often described using **interval notation**.

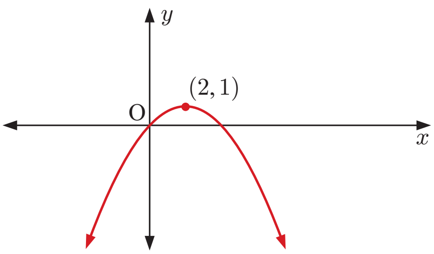
For example:

(1)



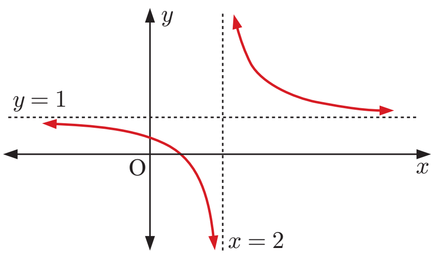
All values of $x \geqslant -1$ are included,
so the domain is $\{x : x \geqslant -1\}$.
All values of $y \geqslant -3$ are included,
so the range is $\{y : y \geqslant -3\}$.

(2)



x can take any value,
so the domain is $\{x : x \in \mathbb{R}\}$.
y cannot be > 1,
so the range is $\{y : y \leqslant 1\}$.

(3)



x can take all values except 2,
so the domain is $\{x : x \neq 2\}$.
y can take all values except 1,
so the range is $\{y : y \neq 1\}$.

DOMAIN AND RANGE OF FUNCTIONS

To fully describe a function, we need both a rule *and* a domain.

For example, we can specify $f(x) = x^2$ where $x \geqslant 0$.

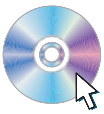
If a domain is not specified, we use the **natural domain**, which is the largest part of $\mathbb{R}$ for which $f(x)$ is defined.

For example, consider the domains in the table opposite:

$f(x)$	*Natural domain*
x^2	$x \in \mathbb{R}$
$\sqrt{x}$	$x \geqslant 0$
$\frac{1}{x}$	$x \neq 0$
$\frac{1}{\sqrt{x}}$	$x > 0$

Click on the icon to obtain software for finding the domain and range of different functions.

DOMAIN AND RANGE

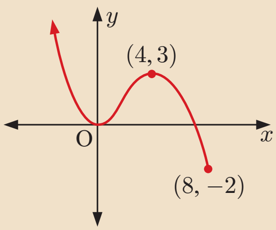


Example 5

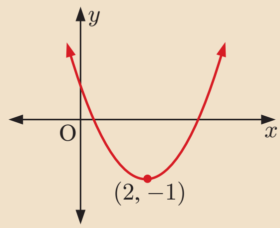
Self Tutor

For each of the following graphs, state the domain and range:

a



b



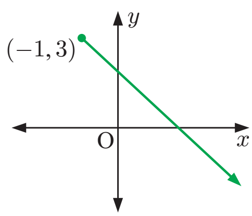
a Domain is $\{x : x \leqslant 8\}$
Range is $\{y : y \geqslant -2\}$

b Domain is $\{x : x \in \mathbb{R}\}$
Range is $\{y : y \geqslant -1\}$

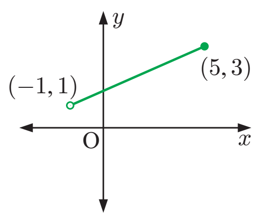
EXERCISE 2C

1 For each of the following graphs, state the domain and range:

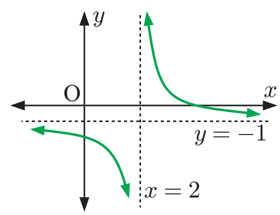
a



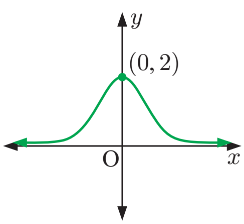
b



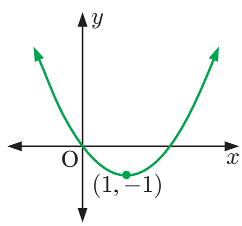
c



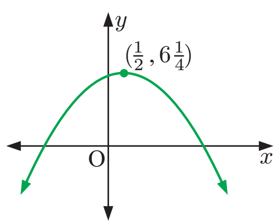
d



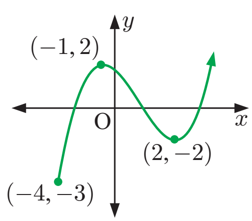
e



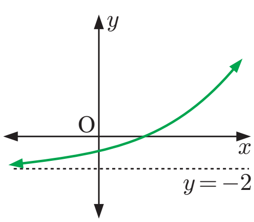
f



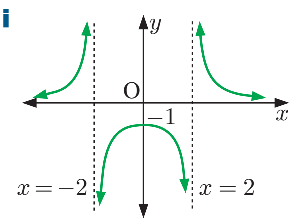
g



h



i



Example 6

Self Tutor

State the domain and range of each of the following functions:

a $f(x) = \sqrt{x-5}$ **b** $f(x) = \dfrac{1}{x-5}$ **c** $f(x) = \dfrac{1}{\sqrt{x-5}}$

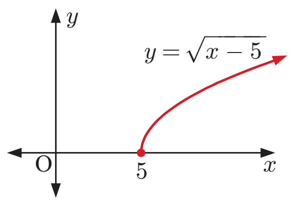
a $\sqrt{x-5}$ is defined when $x - 5 \geqslant 0$

$\therefore \ x \geqslant 5$

$\therefore$ the domain is $\{x : x \geqslant 5\}$.

A square root cannot be negative.

$\therefore$ the range is $\{y : y \geqslant 0\}$.



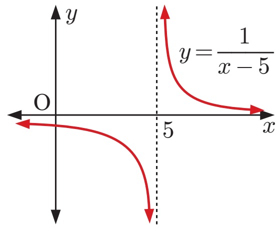
b $\dfrac{1}{x-5}$ is defined when $x - 5 \neq 0$

$\therefore \ x \neq 5$

$\therefore$ the domain is $\{x : x \neq 5\}$.

No matter how large or small x is, $y = f(x)$ is never zero.

$\therefore$ the range is $\{y : y \neq 0\}$.



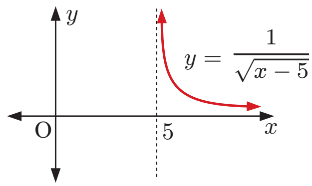
c $\dfrac{1}{\sqrt{x-5}}$ is defined when $x - 5 > 0$

$\therefore \ x > 5$

$\therefore$ the domain is $\{x : x > 5\}$.

$y = f(x)$ is always positive and never zero.

$\therefore$ the range is $\{y : y > 0\}$.



2 State the values of x for which $f(x)$ is defined, and hence state the domain of the function.

a $f(x) = \sqrt{x+6}$ **b** $f : x \mapsto \dfrac{1}{x^2}$ **c** $f(x) = \dfrac{-7}{\sqrt{3-2x}}$

3 Find the domain and range of each of the following functions:

a $f : x \mapsto 2x - 1$ **b** $f(x) = 3$ **c** $f : x \mapsto \sqrt{x}$

d $f(x) = \dfrac{1}{x+1}$ **e** $f(x) = -\dfrac{1}{\sqrt{x}}$ **f** $f : x \mapsto \dfrac{1}{3-x}$

4 Use technology to help sketch graphs of the following functions. Find the domain and range of each.

DOMAIN AND RANGE

a $f(x) = \sqrt{x-2}$ **b** $f : x \mapsto \dfrac{1}{x^2}$ **c** $f : x \mapsto \sqrt{4-x}$

d $y = x^2 - 7x + 10$ **e** $f(x) = \sqrt{x^2+4}$ **f** $f(x) = \sqrt{x^2-4}$

g $f : x \mapsto 5x - 3x^2$ **h** $f : x \mapsto x + \dfrac{1}{x}$ **i** $y = \dfrac{x+4}{x-2}$

j $y = x^3 - 3x^2 - 9x + 10$ **k** $f : x \mapsto \dfrac{3x-9}{x^2-x-2}$ **l** $y = x^2 + x^{-2}$

m $y = x^3 + \dfrac{1}{x^3}$ **n** $f : x \mapsto x^4 + 4x^3 - 16x + 3$

D THE MODULUS FUNCTION

The **modulus** or **absolute value** of a real number is its size, ignoring its sign.

We denote the absolute value of x by $|x|$.

For example, the modulus of 4 is 4, and the modulus of -9 is 9. We write $|4| = 4$ and $|-9| = 9$.

The absolute value of a number is always $\geqslant 0$.

Example 7 — Self Tutor

If $x = -3$, find the value of:

a $|x|$ **b** $x\,|x|$ **c** $|x^2 + x|$ **d** $\left|\dfrac{7x-1}{2}\right|$

a $|x|$
$= |-3|$
$= 3$

b $x\,|x|$
$= (-3)\,|-3|$
$= -3 \times 3$
$= -9$

c $|x^2 + x|$
$= \left|(-3)^2 + (-3)\right|$
$= |6|$
$= 6$

d $\left|\dfrac{7x-1}{2}\right|$
$= \left|\dfrac{7(-3)-1}{2}\right|$
$= |-11|$
$= 11$

EXERCISE 2D.1

1 Find the value of:

a $|5|$ **b** $|-5|$ **c** $|7-3|$ **d** $|3-7|$

e $|2^2 - 10|$ **f** $|15 - 3 \times 5|$ **g** $\left|\dfrac{3-1}{5+2}\right|$ **h** $\left|\dfrac{2^3}{(-3)^3}\right|$

2 If $x = 4$, find the value of:

a $|x-5|$ **b** $|10-x|$ **c** $|3x - x^2|$ **d** $\left|\dfrac{2x+1}{x-1}\right|$

3 If $x = -2$, find the value of:

a $|x|$ **b** $x\,|x|$ **c** $-\,|x - x^2|$ **d** $\dfrac{|1+3x|}{x+1}$

MODULUS EQUATIONS

The equation $|x| = 2$ has two solutions: $x = 2$ and $x = -2$.

Solving modulus equations is not needed for the syllabus.

If $|x| = a$ where $a > 0$, then $x = \pm a$.

If $|x| = |b|$ then $x = \pm b$.

We use these rules to solve equations involving the modulus function.

Example 8 — Self Tutor

Solve for x: **a** $|2x+3| = 7$ **b** $|3-2x| = -1$

a $|2x+3| = 7$

$\therefore\ 2x+3 = \pm 7$

$\therefore\ 2x+3 = 7$ **or** $2x+3 = -7$

$\therefore\ 2x = 4$ $\qquad\therefore\ 2x = -10$

$\therefore\ x = 2$ $\qquad\therefore\ x = -5$

So, $x = 2$ or -5

b $|3-2x| = -1$
has no solution as LHS is never negative.

Example 9 — Self Tutor

Solve for x: $|x+1| = |2x-3|$

If $|x+1| = |2x-3|$, then $x+1 = \pm(2x-3)$

$\therefore\ x+1 = 2x-3$ **or** $x+1 = -(2x-3)$

$\therefore\ 4 = x$ $\qquad\therefore\ x+1 = -2x+3$

$\therefore\ 3x = 2$

$\therefore\ x = \frac{2}{3}$

So, $x = \frac{2}{3}$ or 4.

EXERCISE 2D.2

1 Solve for x:

a $|x| = 3$ **b** $|x| = -5$ **c** $|x| = 0$

d $|x-1| = 3$ **e** $|3-x| = 4$ **f** $|x+5| = -1$

g $|3x-2| = 1$ **h** $|3-2x| = 3$ **i** $|2-5x| = 12$

2 Solve for x:

a $|3x-1| = |x+2|$ **b** $|2x+5| = |1-x|$ **c** $|x+1| = |2-x|$

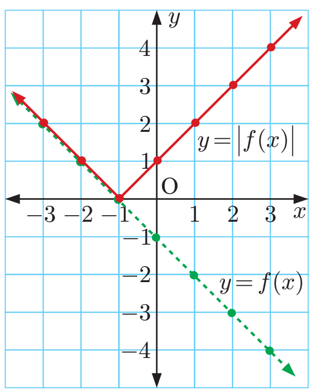
d $|x| = |5-x|$ **e** $|1-4x| = |x-1|$ **f** $|3x+2| = |2-x|$

THE GRAPH OF $y = |f(x)|$

Consider the function $f(x) = -x - 1$.

In the table below, we calculate the values of $f(x)$ and $|f(x)|$ for $x = -3, -2, -1, 0, 1, 2, 3$.

x	-3	-2	-1	0	1	2	3
$f(x)$	2	1	0	-1	-2	-3	-4
$\lvert f(x)\rvert$	2	1	0	1	2	3	4



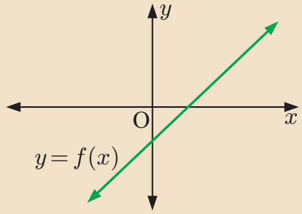
Using these values, we can plot $y = f(x)$ and $y = |f(x)|$ on the same set of axes.

To draw the graph of $y = |f(x)|$, any parts of $y = f(x)$ that are below the x-axis are reflected in the x-axis.

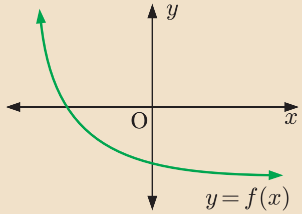
Example 10 — Self Tutor

For the following graphs, sketch the graph of $y = |f(x)|$:

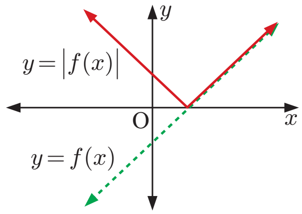
a



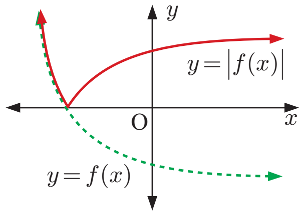
b



a



b

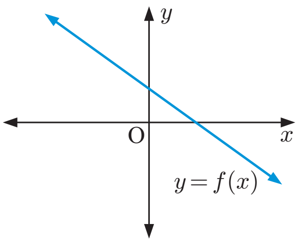


EXERCISE 2D.3

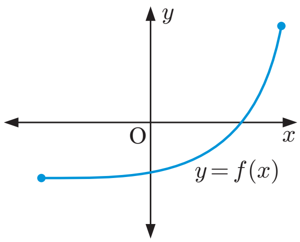
PRINTABLE DIAGRAMS

1 For the following graphs, sketch the graph of $y = |f(x)|$:

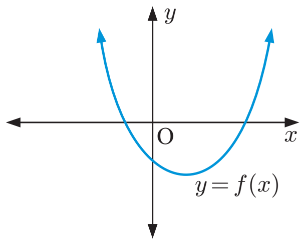
a



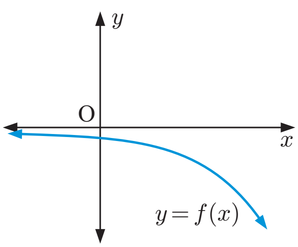
b



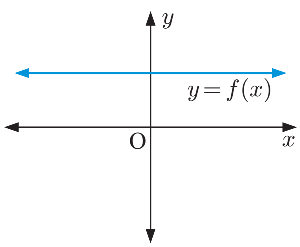
c



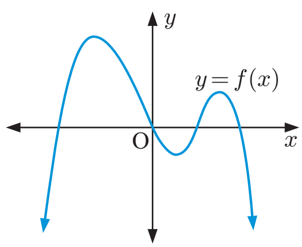
d



e



f



2 Which of the functions $y = |f(x)|$ in question **1** are one-one?

3 Suppose the range of $y = f(x)$ is $\{y : -6 \leqslant y \leqslant 2\}$. Write down the range of $y = |f(x)|$.

4 Determine whether the following statements are true or false:

a If $y = f(x)$ is one-one, then $y = |f(x)|$ is one-one.

b If $y = f(x)$ is not one-one, then $y = |f(x)|$ is not one-one.

c The graphs of $y = f(x)$ and $y = |f(x)|$ always meet the x-axis at the same point(s).

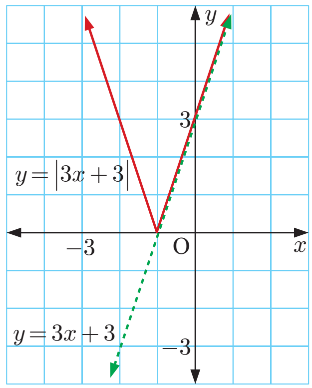
d The graphs of $y = f(x)$ and $y = |f(x)|$ always meet the y-axis at the same point.

Example 11 — Self Tutor

Draw the graph of $y = |3x + 3|$.

We first draw the graph of $y = 3x + 3$.

The part of the graph that is below the x-axis is then reflected in the x-axis to produce $y = |3x + 3|$.



5 Draw the graph of:

a $y = |x|$ **b** $y = |x + 3|$ **c** $y = |6 - 2x|$

d $y = |3x + 1|$ **e** $y = |10 - 4x|$ **f** $y = \left|\frac{1}{2}x + 2\right|$

E COMPOSITE FUNCTIONS

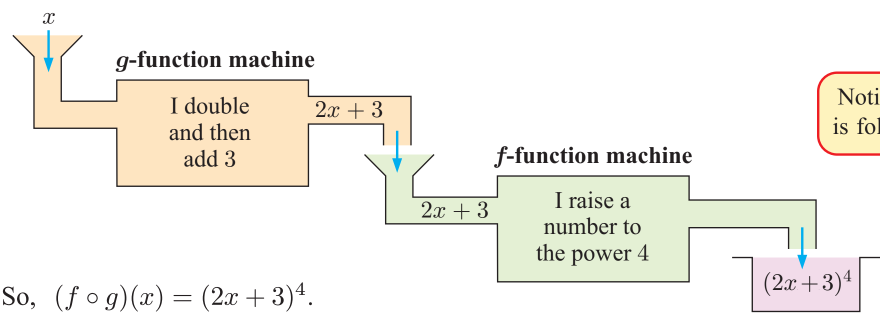
Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the **composite function** of f and g will convert x into $f(g(x))$.

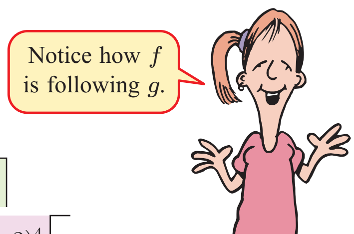
$f \circ g$ or fg is used to represent the composite function of f and g. It means "f following g".

$$(f \circ g)(x) \text{ or } fg(x) = f(g(x))$$

Consider $f : x \mapsto x^4$ and $g : x \mapsto 2x + 3$.

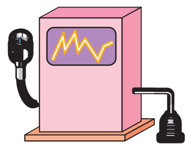
$f \circ g$ means that g converts x to $2x + 3$ and then f converts $(2x + 3)$ to $(2x + 3)^4$.





So, $(f \circ g)(x) = (2x + 3)^4$.

Algebraically, if $f(x) = x^4$ and $g(x) = 2x + 3$ then



$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x+3) && \{g \text{ operates on } x \text{ first}\} \\ &= (2x+3)^4 && \{f \text{ operates on } g(x) \text{ next}\}\end{aligned}$$

and
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^4) && \{f \text{ operates on } x \text{ first}\} \\ &= 2(x^4)+3 && \{g \text{ operates on } f(x) \text{ next}\} \\ &= 2x^4+3\end{aligned}$$

So, $f(g(x)) \neq g(f(x))$.

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

We can also compose a function f with itself. The resulting function is $(f \circ f)(x)$ or $f^2(x)$.

In general, $(f \circ f)(x) \neq (f(x))^2$.

Example 12 — Self Tutor

Given $f : x \mapsto 2x + 1$ and $g : x \mapsto 3 - 4x$, find in simplest form:

a $(f \circ g)(x)$ **b** $gf(x)$ **c** $f^2(x)$

$f(x) = 2x + 1$ and $g(x) = 3 - 4x$

a
$$\begin{aligned}&(f \circ g)(x) \\ &= f(g(x)) \\ &= f(3-4x) \\ &= 2(3-4x)+1 \\ &= 6-8x+1 \\ &= 7-8x\end{aligned}$$

b
$$\begin{aligned}&gf(x) \\ &= g(f(x)) \\ &= g(2x+1) \\ &= 3-4(2x+1) \\ &= 3-8x-4 \\ &= -8x-1\end{aligned}$$

c
$$\begin{aligned}&f^2(x) \\ &= f(f(x)) \\ &= f(2x+1) \\ &= 2(2x+1)+1 \\ &= 4x+2+1 \\ &= 4x+3\end{aligned}$$

In the previous **Example** you should have observed how we can substitute an expression into a function.

If $f(x) = 2x + 1$ then $f(\Delta) = 2(\Delta) + 1$
$\therefore\ f(3 - 4x) = 2(3 - 4x) + 1$.

Example 13 — Self Tutor

Given $f(x) = 6x - 5$ and $g(x) = x^2 + x$, determine:

a $(g \circ f)(-1)$ **b** $(f \circ f)(0)$

a $(g \circ f)(-1) = g(f(-1))$

Now $f(-1) = 6(-1) - 5$
$= -11$

$$\begin{aligned}\therefore\ (g \circ f)(-1) &= g(-11) \\ &= (-11)^2 + (-11) \\ &= 110\end{aligned}$$

b $(f \circ f)(0) = f(f(0))$

Now $f(0) = 6(0) - 5$
$= -5$

$$\begin{aligned}\therefore\ (f \circ f)(0) &= f(-5) \\ &= 6(-5) - 5 \\ &= -35\end{aligned}$$

The domain of the composite of two functions depends on the domain of the original functions.

For example, consider $f(x) = x^2$ with domain $x \in \mathbb{R}$ and $g(x) = \sqrt{x}$ with domain $x \geqslant 0$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= (\sqrt{x})^2 \\ &= x\end{aligned}$$

The domain of $(f \circ g)(x)$ is $x \geqslant 0$, not $\mathbb{R}$, since $(f \circ g)(x)$ is defined using function $g(x)$.

EXERCISE 2E

1 Given $f : x \mapsto 2x + 3$ and $g : x \mapsto 1 - x$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(f \circ g)(-3)$

2 Given $f(x) = 2 + x$ and $g(x) = 3 - x$, find:

a $fg(x)$ **b** $gf(x)$ **c** $f^2(x)$

3 Given $f(x) = \sqrt{6 - x}$ and $g(x) = 5x - 7$, find:

a $(g \circ g)(x)$ **b** $(f \circ g)(1)$ **c** $(g \circ f)(6)$

4 Given $f : x \mapsto x^2$ and $g : x \mapsto 2 - x$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.
Find also the domain and range of $f \circ g$ and $g \circ f$.

5 Suppose $f(x) = 3x + 5$ and $g(x) = 2x - 3$.

a Find $(f \circ g)(x)$. **b** Solve $(f \circ g)(x) = g(x - 2)$.

6 Suppose $f : x \mapsto x^2 + 1$ and $g : x \mapsto 3 - x$.

a Find in simplest form: **i** $fg(x)$ **ii** $gf(x)$

b Find the value(s) of x such that $gf(x) = f(x)$.

7 **a** If $ax + b = cx + d$ for all values of x, show that $a = c$ and $b = d$.
Hint: If it is true for all x, it is true for $x = 0$ and $x = 1$.

b Given $f(x) = 2x + 3$ and $g(x) = ax + b$ and that $(f \circ g)(x) = x$ for all values of x, deduce that $a = \frac{1}{2}$ and $b = -\frac{3}{2}$.

c Is the result in **b** true if $(g \circ f)(x) = x$ for all x?

8 Given $f(x) = \sqrt{1 - x}$ and $g(x) = x^2$, find:

a $(f \circ g)(x)$ **b** the domain and range of $(f \circ g)(x)$.

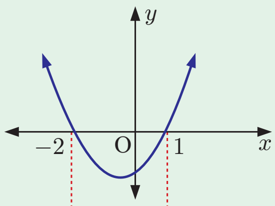
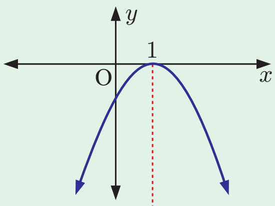
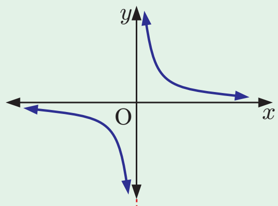
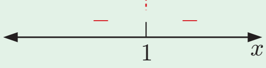
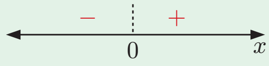
F SIGN DIAGRAMS

Sometimes we do not wish to draw a time-consuming graph of a function but wish to know when the function is positive, negative, zero, or undefined. A **sign diagram** enables us to do this and is relatively easy to construct.

For the function $f(x)$, the sign diagram consists of:

- a **horizontal line** which is really the x-axis
- **positive** $(+)$ and **negative** $(-)$ signs indicating that the graph is **above** and **below** the x-axis respectively
- the **zeros** of the function, which are the x-intercepts of the graph of $y = f(x)$, and the **roots** of the equation $f(x) = 0$
- values of x where the graph is undefined.

Consider the three functions given below.

Function	$y=(x+2)(x-1)$	$y=-2(x-1)^2$	$y=\dfrac{4}{x}$
Graph			
Sign diagram			

You should notice that:

- A sign change occurs about a zero of the function for single linear factors such as $(x+2)$ and $(x-1)$. This indicates **cutting** of the x-axis.
- No sign change occurs about a zero of the function for squared linear factors such as $(x-1)^2$. This indicates **touching** of the x-axis.

DEMO

-  indicates that a function is **undefined** at $x=0$.

In general:

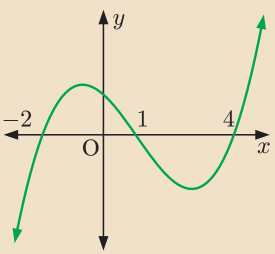
- when a linear factor has an **odd power** there is a change of sign about that zero
- when a linear factor has an **even power** there is no sign change about that zero.

Example 14

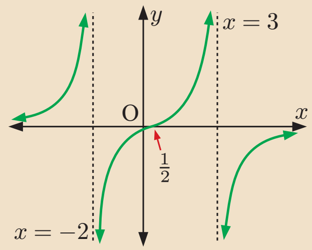
Self Tutor

Draw sign diagrams for:

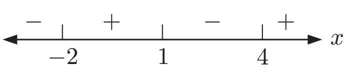
a



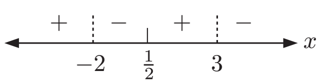
b



a

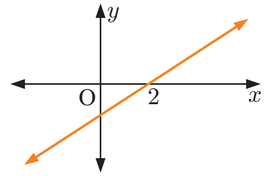


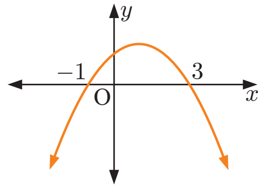
b

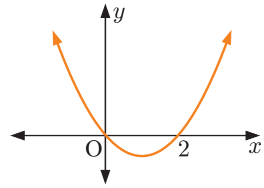


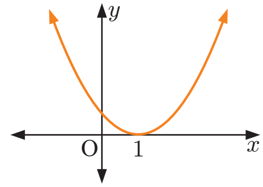
EXERCISE 2F

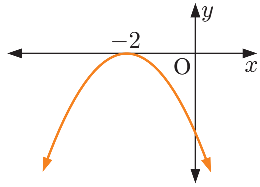
1 Draw sign diagrams for these graphs:

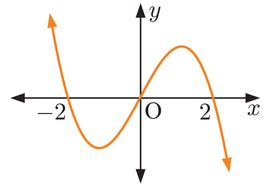
a


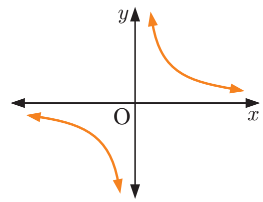
b


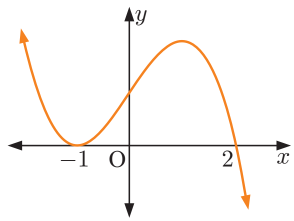
c


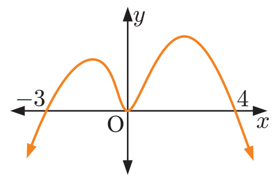
d


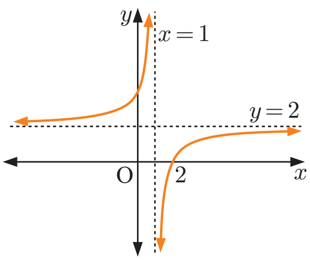
e


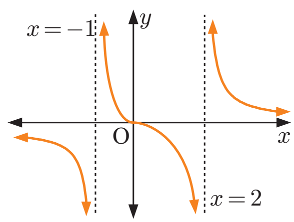
f


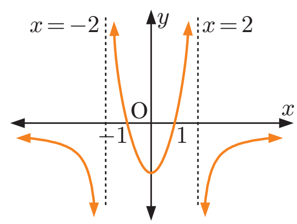
g


h


i


j


k


l


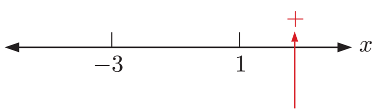
Example 15 — Self Tutor

Draw a sign diagram for:

a $(x+3)(x-1)$

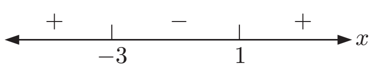
b $-4(x-3)^2$

a $(x+3)(x-1)$ has zeros -3 and 1.

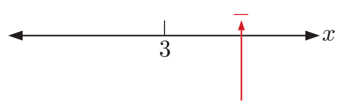


We substitute any number > 1.
When $x = 2$ we have $(5)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

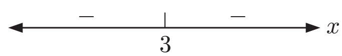


b $-4(x-3)^2$ has zero 3.



We substitute any number > 3.
When $x = 4$ we have $-4(1)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.



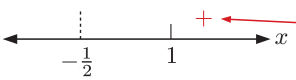
2 Draw sign diagrams for:

a $(x+4)(x-2)$ **b** $x(x-3)$ **c** $x(x+2)$
d $-(x+1)(x-3)$ **e** $(2x-1)(3-x)$ **f** $(5-x)(1-2x)$
g $(x+2)^2$ **h** $2(x-3)^2$ **i** $-3(x+4)^2$

Example 16 — Self Tutor

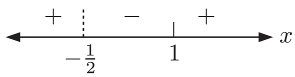
Draw a sign diagram for $\dfrac{x-1}{2x+1}$.

$\dfrac{x-1}{2x+1}$ is zero when $x=1$ and undefined when $x=-\frac{1}{2}$.



When $x=10$, $\dfrac{x-1}{2x+1}=\dfrac{9}{21}>0$

Since $(x-1)$ and $(2x+1)$ are single factors, the signs alternate.



3 Draw sign diagrams for:

a $\dfrac{x+2}{x-1}$ **b** $\dfrac{x}{x+3}$ **c** $\dfrac{2x+3}{4-x}$

d $\dfrac{4x-1}{2-x}$ **e** $\dfrac{3x}{x-2}$ **f** $\dfrac{-8x}{3-x}$

g $\dfrac{(x-1)^2}{x}$ **h** $\dfrac{4x}{(x+1)^2}$ **i** $\dfrac{(x+2)(x-1)}{3-x}$

j $\dfrac{x(x-1)}{2-x}$ **k** $\dfrac{(x+2)(x-2)}{-x}$ **l** $\dfrac{3-x}{(2x+3)(x-2)}$

4 Draw sign diagrams for:

a $1+\dfrac{3}{x+1}$ **b** $x-\dfrac{1}{x}$ **c** $x-\dfrac{1}{x^2}$

G INVERSE FUNCTIONS

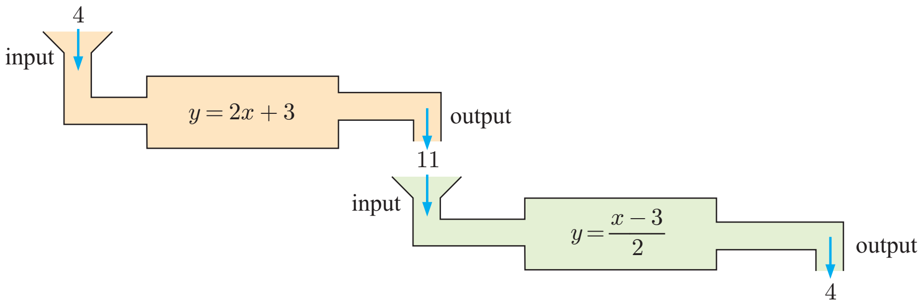
The operations of $+$ and $-$, $\times$ and $\div$, are **inverse operations** as one undoes what the other does.

For example, $x+3-3=x$ and $x\times 3\div 3=x$.

The function $y=2x+3$ can be "undone" by its **inverse function** $y=\dfrac{x-3}{2}$.

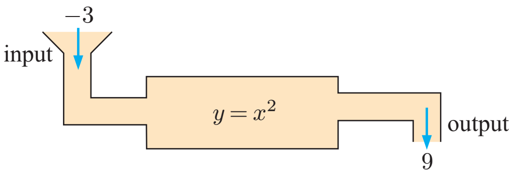
We can think of this as two machines. If the machines are inverses then the second machine *undoes* what the first machine does.

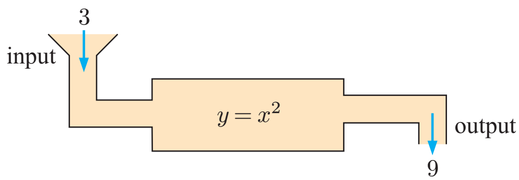
No matter what value of x enters the first machine, it is returned as the output from the second machine.



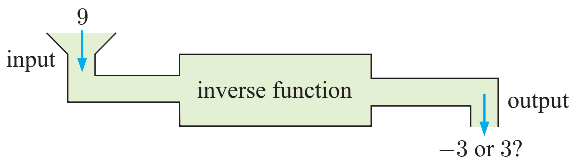
However, not all functions have an inverse function.

For example, consider the function $y = x^2$. The inputs -3 and 3 both produce an output of 9.





So, if we gave an inverse function the input 9, how would it know whether the output should be -3 or 3? It cannot answer both, since the inverse function would fail the vertical line test.



So, if a function has two inputs which produce the same output, then the function does not have an inverse function.

For a function to have an **inverse**, the function must be **one-one**. It must pass the horizontal line test.

If $y = f(x)$ has an **inverse function**, this new function:

- is denoted $f^{-1}(x)$
- is the reflection of $y = f(x)$ in the line $y = x$
- satisfies $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

The function $y = x$, defined as $f : x \mapsto x$, is the **identity function**.

f^{-1} is **not** the reciprocal of f.

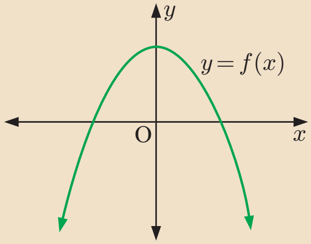
$$f^{-1}(x) \neq \frac{1}{f(x)}$$

Example 17

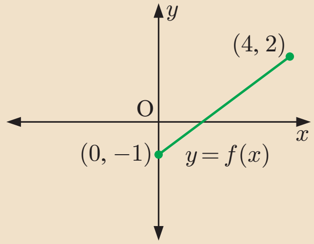
Self Tutor

If $y = f(x)$ has an inverse function, sketch $y = f^{-1}(x)$, and state the domain and range of $f(x)$ and $f^{-1}(x)$.

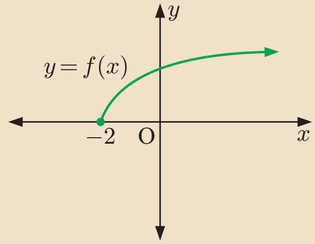
a



b

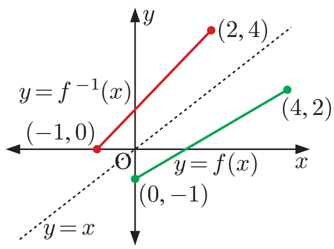


c



a The function fails the horizontal line test, so it is not one-one. The function does not have an inverse function.

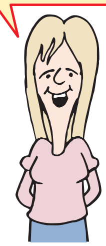
b



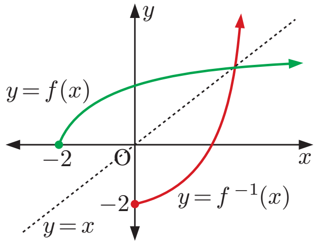
$f(x)$ has domain $\{x : 0 \leqslant x \leqslant 4\}$ and range $\{y : -1 \leqslant y \leqslant 2\}$.

$f^{-1}(x)$ has domain $\{x : -1 \leqslant x \leqslant 2\}$ and range $\{y : 0 \leqslant y \leqslant 4\}$.

$y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.



c



$f(x)$ has domain $\{x : x \geqslant -2\}$ and range $\{y : y \geqslant 0\}$.

$f^{-1}(x)$ has domain $\{x : x \geqslant 0\}$ and range $\{y : y \geqslant -2\}$.

From **Example 17**, we can see that:

The domain of f^{-1} is equal to the range of f.

The range of f^{-1} is equal to the domain of f.

If (x, y) lies on f, then (y, x) lies on f^{-1}. Reflecting the function in the line $y = x$ has the algebraic effect of interchanging x and y.

So, if the function is given as an equation, then we interchange the variables to find the equation of the inverse function.

For example, if f is given by $y = 5x + 2$ then f^{-1} is given by $x = 5y + 2$.

Example 18 — Self Tutor

Consider $f: x \mapsto 2x+3$.

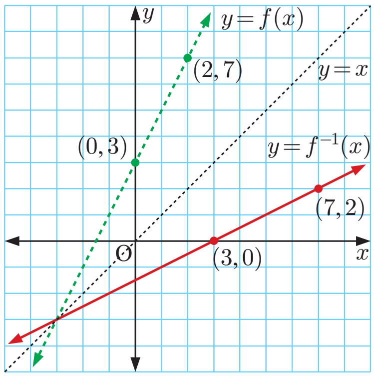
a On the same axes, graph f and its inverse function f^{-1}.

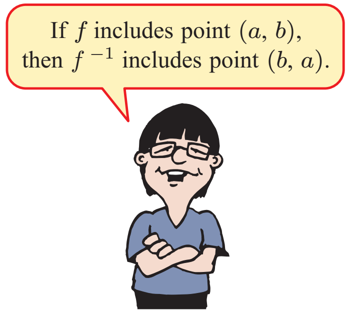
b Find $f^{-1}(x)$ using variable interchange.

c Check that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

a $f(x) = 2x+3$ passes through $(0, 3)$ and $(2, 7)$.

$\therefore$ $f^{-1}(x)$ passes through $(3, 0)$ and $(7, 2)$.





b f is $y = 2x+3$,

$\therefore$ f^{-1} is $x = 2y+3$

$\therefore$ $x-3 = 2y$

$\therefore$ $\dfrac{x-3}{2} = y$

$\therefore$ $f^{-1}(x) = \dfrac{x-3}{2}$

c $(f \circ f^{-1})(x)$

$= f(f^{-1}(x))$

$= f\left(\dfrac{x-3}{2}\right)$

$= 2\left(\dfrac{x-3}{2}\right)+3$

$= x$

and $(f^{-1} \circ f)(x)$

$= f^{-1}(f(x))$

$= f^{-1}(2x+3)$

$= \dfrac{(2x+3)-3}{2}$

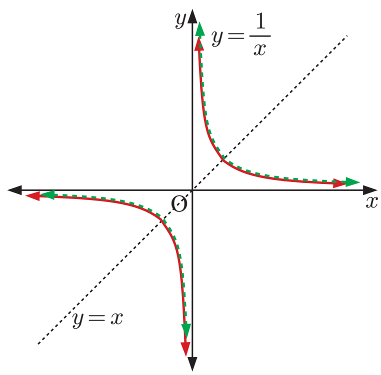
$= \dfrac{2x}{2}$

$= x$

Any function which has an inverse, and whose graph is symmetrical about the line $y = x$, is a **self-inverse function**.

If f is a self-inverse function then $f^{-1} = f$.

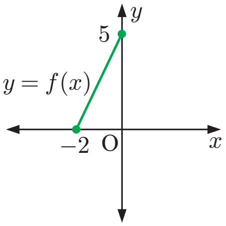
For example, the function $f(x) = \dfrac{1}{x}$, $x \neq 0$, is said to be self-inverse, as $f = f^{-1}$.

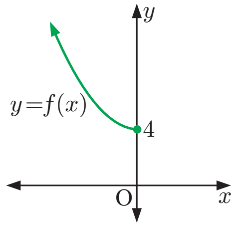


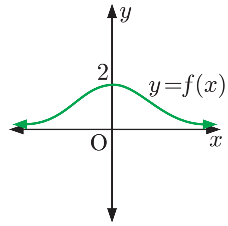
EXERCISE 2G

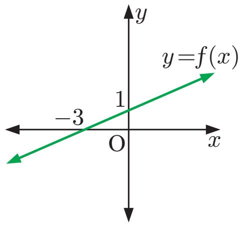
PRINTABLE GRAPHS

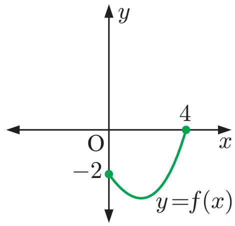
1 If $y = f(x)$ has an inverse function, sketch $y = f^{-1}(x)$, and state the domain and range of $f(x)$ and $f^{-1}(x)$.

a


b


c


d


e


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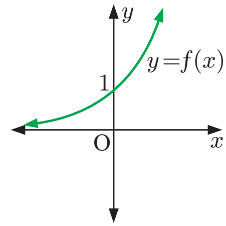

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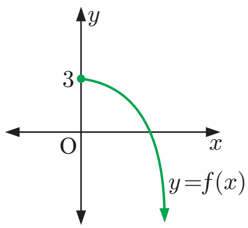

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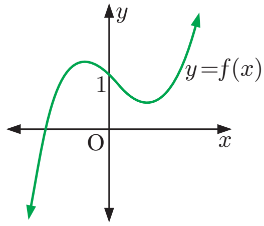
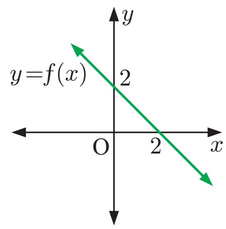


i


2 Which of the functions in **1** is a self-inverse function?

3 If the domain of $H(x)$ is $\{x : -2 \leqslant x < 3\}$, state the range of $H^{-1}(x)$.

4 For each of the following functions f:

- **i** On the same set of axes, sketch $y = x$, $y = f(x)$, and $y = f^{-1}(x)$.
- **ii** Find $f^{-1}(x)$ using variable interchange.

a $f : x \mapsto 3x + 1$ **b** $f : x \mapsto \dfrac{x+2}{4}$

5 For each of the following functions f:

- **i** Find $f^{-1}(x)$.
- **ii** Sketch $y = f(x)$, $y = f^{-1}(x)$, and $y = x$ on the same set of axes.
- **iii** Show that $(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$, the identity function.

a $f : x \mapsto 2x + 5$ **b** $f : x \mapsto x + 3$ **c** $f : x \mapsto \dfrac{x+6}{2}$

6 Given $f(x) = 2x - 5$, find $(f^{-1})^{-1}(x)$. What do you notice?

7 Sketch the graph of $f : x \mapsto x^3$ and its inverse function $f^{-1}(x)$.

8 Given $f : x \mapsto \dfrac{1}{x}$, $x \neq 0$, find f^{-1} algebraically and show that f is a self-inverse function.

9 Consider the function $f(x) = \frac{1}{2}x - 1$.

a Find $f^{-1}(x)$.

b Find: **i** $(f \circ f^{-1})(x)$ **ii** $(f^{-1} \circ f)(x)$.

10 Consider the functions $f : x \mapsto 2x + 5$ and $g : x \mapsto \dfrac{8-x}{2}$.

a Find $g^{-1}(-1)$.

b Show that $f^{-1}(-3) - g^{-1}(6) = 0$.

c Find x such that $(f \circ g^{-1})(x) = 9$.

11 Consider the functions $f : x \mapsto 5^x$ and $g : x \mapsto \sqrt{x}$.

a Find: **i** $f(2)$ **ii** $g^{-1}(4)$

b Solve the equation $(g^{-1} \circ f)(x) = 25$.

12 Which of these functions is a self-inverse function?

a $f(x) = 2x$

b $f(x) = x$

c $f(x) = -x$

d $f(x) = \dfrac{2}{x}$

e $f(x) = -\dfrac{6}{x}$

f $f(x) = \dfrac{x}{3}$

13 Given $f : x \mapsto 2x$ and $g : x \mapsto 4x - 3$, show that $(f^{-1} \circ g^{-1})(x) = (g \circ f)^{-1}(x)$.

Discovery — *Functions and form*

We already know that numbers have equivalent forms. For example, $\frac{1}{2}$, $\frac{3}{6}$, $\frac{5}{10}$, and 0.5 all represent the same number.

Similarly, a function might have different, but equivalent, algebraic representations.

Choosing a particular form for an expression helps us understand the behaviour of the function better. By anticipating what you are going to do with your function you can choose a form which will make the task easier.

For example, you will have seen in previous years that the equation of a straight line can be written in:

- gradient-intercept form $y = mx + c$ where m is the gradient and the y-intercept is c
- point-gradient form $y - b = m(x - a)$ where the line goes through (a, b) and has gradient m
- general-form $Ax + By = D$.

A given straight line can be converted between these forms easily, but each emphasises different features of the straight line.

What to do:

1 What different forms have you seen for a quadratic function $y = ax^2 + bx + c$?

2 Two expressions $f(x)$ and $g(x)$ are **equivalent** on the domain D if $f(x) = g(x)$ for all $x \in D$.

a Discuss whether: $f(x) = \dfrac{x^2-1}{x-1}$ and $g(x) = x + 1$ are equivalent on:

i $x \in \mathbb{R}$ **ii** $x \in \mathbb{R}^-$ **iii** $\{x : x > 1\}$ **iv** $\{x \in \mathbb{R} : x \neq 1\}$

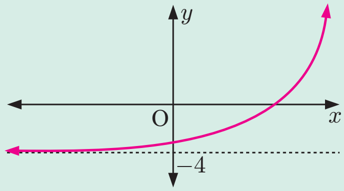
b When considering algebraically whether two functions are equivalent, what things do we need to be careful about?

Hint: $\dfrac{x^2-1}{x-1} = \dfrac{(x+1)(x-1)}{x-1} = x + 1$ only if $x \neq 1$.

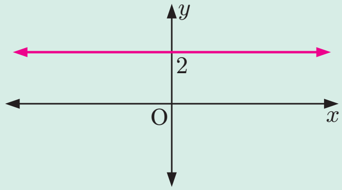
Review set 2A

1 Determine whether the following relations are functions:

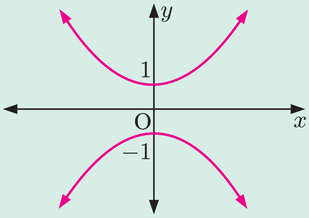
a



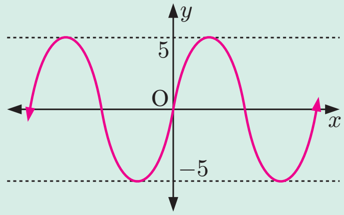
b



c



d



2 Suppose $f(x) = ax + b$ where a and b are constants.
If $f(1) = 7$ and $f(3) = -5$, find a and b.

3 Solve for x:

a $|x - 5| = 7$ **b** $|2x + 1| = |x - 4|$

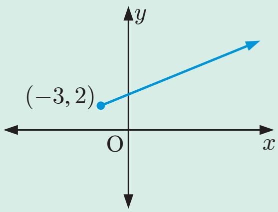
4 If $g(x) = x^2 - 3x$, find in simplest form:

a $g(-2)$ **b** $g(x + 1)$

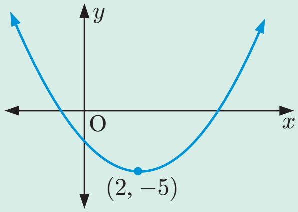
5 For each of the following functions:

i find the domain and range **ii** determine whether the function is one-one.

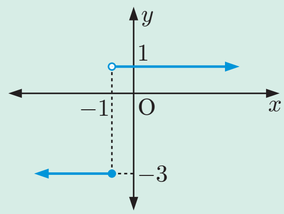
a



b



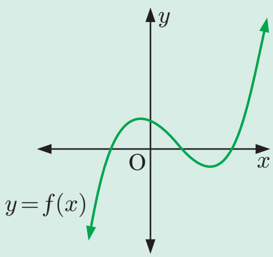
c



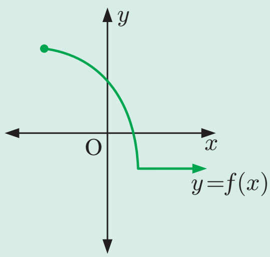
6 Draw the graph of $y = |f(x)|$ for:

PRINTABLE GRAPHS

a



b



7 Draw the graph of $y = |2x - 1|$.

8 Draw a sign diagram for:

a $(3x+2)(4-x)$ **b** $\dfrac{x-3}{(x+2)^2}$

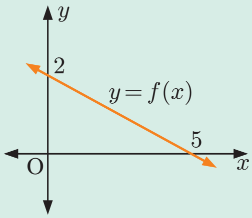
9 If $f(x) = 2x - 3$ and $g(x) = x^2 + 2$, find in simplest form:

a $fg(x)$ **b** $gf(x)$

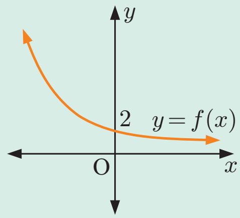
10 If $y = f(x)$ has an inverse, sketch the graph of $y = f^{-1}(x)$.

PRINTABLE GRAPHS

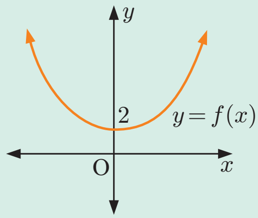
a



b



c



11 Find $f^{-1}(x)$ given that $f(x)$ is:

a $4x + 2$ **b** $\dfrac{3-5x}{4}$

12 Consider $f(x) = x^2$ and $g(x) = 1 - 6x$.

a Show that $f(-3) = g(-\frac{4}{3})$. **b** Find $(f \circ g)(-2)$.

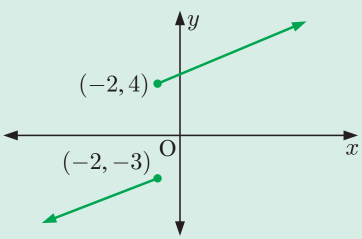
c Find x such that $g(x) = f(5)$.

13 Given $f : x \mapsto 3x + 6$ and $h : x \mapsto \dfrac{x}{3}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

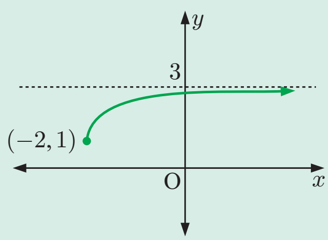
Review set 2B

1 Determine whether the following relations are functions. If they are functions, determine whether they are one-one.

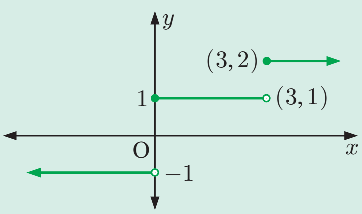
a



b



c



2 Given $f(x) = x^2 + 3$, find:

a $f(-3)$

b x such that $f(x) = 4$.

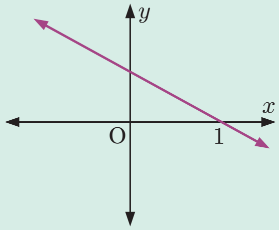
3 Solve for x:

a $|1 - 2x| = 11$

b $|5x - 1| = |9x - 13|$

4 Draw a sign diagram for each graph:

a



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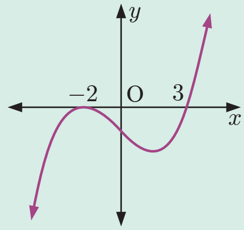


5 Given $h(x) = 7 - 3x$, find:

a $h(2x - 1)$

b $h^2(x)$

c $h^2(-1)$

6 Suppose the range of $y = f(x)$ is $\{y : -7 \leqslant y \leqslant -3\}$. Write down the range of $y = |f(x)|$.

7 Draw the graph of $y = \left|1 - \frac{1}{3}x\right|$.

8 Suppose $f(x) = 1 - 2x$ and $g(x) = 5x$.

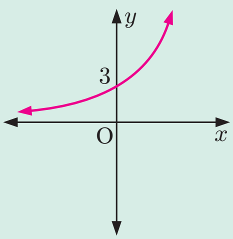
a Find in simplest form: **i** $fg(x)$ **ii** $gf(x)$.

b Solve $fg(x) = g(x + 2)$.

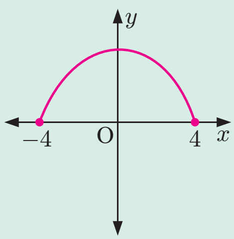
9 Suppose $f(x) = ax^2 + bx + c$, $f(0) = 5$, $f(-2) = 21$, and $f(3) = -4$. Find a, b, and c.

10 If $y = f(x)$ has an inverse, sketch the graph of $y = f^{-1}(x)$:

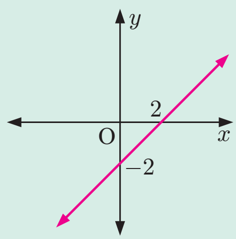
a



b



c



PRINTABLE DIAGRAMS

11 Find the inverse function $f^{-1}(x)$ for:

a $f(x) = 7 - 4x$

b $f(x) = \dfrac{3 + 2x}{5}$

12 Given $f : x \mapsto 5x - 2$ and $h : x \mapsto \dfrac{3x}{4}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

13 Given $f(x) = 2x + 11$ and $g(x) = x^2$, find $(g \circ f^{-1})(3)$.

3

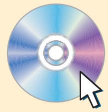
Quadratics

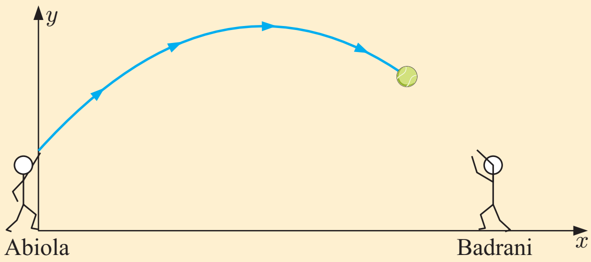
Contents:

Opening problem

Abiola and Badrani are standing 40 metres apart, throwing a ball between them. When Abiola throws the ball, it travels in a smooth arc. At the time when the ball has travelled x metres horizontally towards Badrani, its height is y metres.

SIMULATION





x (m)	0	5	10	15	20	25	30
y (m)	1.25	10	16.25	20	21.25	20	16.25

Things to think about:

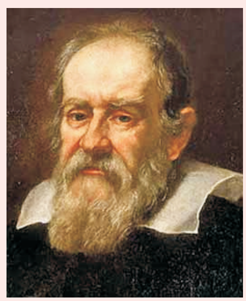
- **a** Use technology to plot these points.
- **b** What *shape* is the graph of y against x?
- **c** What is the maximum height reached by the ball?
- **d** What *formula* gives the height of the ball when it has travelled x metres horizontally towards Badrani?
- **e** Will the ball reach Badrani before it bounces?

Historical note

Galileo Galilei (1564 - 1642) was born in Pisa, Tuscany. He was a philosopher who played a significant role in the scientific revolution of that time.

Within his research he conducted a series of experiments on the paths of projectiles, attempting to find a mathematical description of falling bodies.

Two of Galileo's experiments consisted of rolling a ball down a grooved ramp that was placed at a fixed height above the floor and inclined at a fixed angle to the horizontal. In one experiment the ball left the end of the ramp and descended to the floor. In the second, a horizontal shelf was placed at the end of the ramp, and the ball travelled along this shelf before descending to the floor.



Galileo

In each experiment Galileo altered the release height h of the ball and measured the distance d the ball travelled before landing. The units of measurement were called 'punti' (points).

In both experiments Galileo found that once the ball left the ramp or shelf, its path was parabolic and could therefore be modelled by a quadratic function.

QUADRATICS

A **quadratic equation** is an equation of the form $ax^2 + bx + c = 0$ where a, b, and c are constants, $a \neq 0$.

A **quadratic function** is a function of the form $y = ax^2 + bx + c$, $a \neq 0$.

Quadratic functions are members of the family of **polynomials**. The first few members of this family are shown in the table.

Polynomial function	*Type*
$y = ax + b, \ \ a \neq 0$	linear
$y = ax^2 + bx + c, \ \ a \neq 0$	quadratic
$y = ax^3 + bx^2 + cx + d, \ \ a \neq 0$	cubic
$y = ax^4 + bx^3 + cx^2 + dx + e, \ \ a \neq 0$	quartic

A QUADRATIC EQUATIONS

Acme Leather Jacket Co. makes and sells x leather jackets each week. Their profit function is given by
$P = -12.5x^2 + 550x - 2125$ dollars.

How many jackets must be made and sold each week in order to obtain a weekly profit of $3000?

Clearly we need to solve the equation:

$$-12.5x^2 + 550x - 2125 = 3000$$

We can rearrange the equation to give

$$12.5x^2 - 550x + 5125 = 0,$$

which is of the form $ax^2 + bx + c = 0$ and is thus a quadratic equation.

SOLVING QUADRATIC EQUATIONS

To solve quadratic equations we have the following methods to choose from:

- **factorise** the quadratic and use the rule:

  If $ab = 0$ then $a = 0$ or $b = 0$.

- **complete the square**
- use the **quadratic formula**
- use **technology**.

The **roots** or **solutions** of $ax^2 + bx + c = 0$ are the values of x which satisfy the equation, or make it true.

For example: Consider $x^2 - 3x + 2 = 0$.

When $x = 2$, $x^2 - 3x + 2 = (2)^2 - 3(2) + 2$
$= 4 - 6 + 2$
$= 0$ ✓

So, $x = 2$ is a root of the equation $x^2 - 3x + 2 = 0$.

SOLVING BY FACTORISATION

Step 1: If necessary, rearrange the equation so one side is zero.
Step 2: Fully factorise the other side.
Step 3: Apply the rule: If $ab = 0$ then $a = 0$ or $b = 0$.
Step 4: Solve the resulting linear equations.

Example 1 — Self Tutor

Solve for x:

a $3x^2 + 5x = 0$ **b** $x^2 = 5x + 6$

a
$$\begin{aligned} 3x^2 + 5x &= 0 \\ \therefore\ x(3x+5) &= 0 \\ \therefore\ x = 0 \text{ or } 3x+5 &= 0 \\ \therefore\ x = 0 \text{ or } x &= -\tfrac{5}{3} \end{aligned}$$

b
$$\begin{aligned} x^2 &= 5x + 6 \\ \therefore\ x^2 - 5x - 6 &= 0 \\ \therefore\ (x-6)(x+1) &= 0 \\ \therefore\ x &= 6 \text{ or } -1 \end{aligned}$$

Example 2 — Self Tutor

Solve for x:

a $4x^2 + 1 = 4x$ **b** $6x^2 = 11x + 10$

a
$$\begin{aligned} 4x^2 + 1 &= 4x \\ \therefore\ 4x^2 - 4x + 1 &= 0 \\ \therefore\ (2x-1)^2 &= 0 \\ \therefore\ x &= \tfrac{1}{2} \end{aligned}$$

b
$$\begin{aligned} 6x^2 &= 11x + 10 \\ \therefore\ 6x^2 - 11x - 10 &= 0 \\ \therefore\ (2x-5)(3x+2) &= 0 \\ \therefore\ x &= \tfrac{5}{2} \text{ or } -\tfrac{2}{3} \end{aligned}$$

Caution:

- Do not be tempted to divide both sides by an expression involving x.
  If you do this then you may lose one of the solutions.
  For example, consider $x^2 = 5x$.

  Correct solution
  $$\begin{aligned} x^2 &= 5x \\ \therefore\ x^2 - 5x &= 0 \\ \therefore\ x(x-5) &= 0 \\ \therefore\ x &= 0 \text{ or } 5 \end{aligned}$$

  Incorrect solution
  $$\begin{aligned} x^2 &= 5x \\ \therefore\ \frac{x^2}{x} &= \frac{5x}{x} \\ \therefore\ x &= 5 \end{aligned}$$

  By dividing both sides by x, we lose the solution $x = 0$.

- Be careful when taking square roots of both sides of an equation. You may otherwise lose solutions.
  For example:
  - Consider $x^2 = 25$.

    Correct solution
    $$\begin{aligned} x^2 &= 25 \\ \therefore\ x &= \pm\sqrt{25} \\ \therefore\ x &= \pm 5 \end{aligned}$$

    Incorrect solution
    $$\begin{aligned} x^2 &= 25 \\ \therefore\ x &= \sqrt{25} \\ \therefore\ x &= 5 \end{aligned}$$

- Consider $(2x-7)^2 = (x+1)^2$.

Correct solution

$$(2x-7)^2 = (x+1)^2$$
$$\therefore \quad (2x-7)^2 - (x+1)^2 = 0$$
$$\therefore \quad (2x-7+x+1)(2x-7-x-1) = 0$$
$$\therefore \quad (3x-6)(x-8) = 0$$
$$\therefore \quad x = 2 \text{ or } 8$$

Incorrect solution

$$(2x-7)^2 = (x+1)^2$$
$$\therefore \quad 2x-7 = x+1$$
$$\therefore \quad x = 8$$

EXERCISE 3A.1

1 Solve the following by factorisation:

a $4x^2+7x=0$ **b** $6x^2+2x=0$ **c** $3x^2-7x=0$
d $2x^2-11x=0$ **e** $3x^2=8x$ **f** $9x=6x^2$
g $x^2-5x+6=0$ **h** $x^2=2x+8$ **i** $x^2+21=10x$
j $9+x^2=6x$ **k** $x^2+x=12$ **l** $x^2+8x=33$

2 Solve the following by factorisation:

a $9x^2-12x+4=0$ **b** $2x^2-13x-7=0$ **c** $3x^2=16x+12$
d $3x^2+5x=2$ **e** $2x^2+3=5x$ **f** $3x^2+8x+4=0$
g $3x^2=10x+8$ **h** $4x^2+4x=3$ **i** $4x^2=11x+3$
j $12x^2=11x+15$ **k** $7x^2+6x=1$ **l** $15x^2+2x=56$

Example 3 Self Tutor

Solve for x: $\quad 3x+\dfrac{2}{x}=-7$

$$3x+\frac{2}{x}=-7$$
$$\therefore \quad x\left(3x+\frac{2}{x}\right)=-7x \qquad \{\text{multiplying both sides by } x\}$$
$$\therefore \quad 3x^2+2=-7x \qquad \{\text{expanding the brackets}\}$$
$$\therefore \quad 3x^2+7x+2=0 \qquad \{\text{making the RHS } 0\}$$
$$\therefore \quad (x+2)(3x+1)=0 \qquad \{\text{factorising}\}$$
$$\therefore \quad x=-2 \text{ or } -\tfrac{1}{3}$$

3 Solve for x:

a $(x+1)^2=2x^2-5x+11$ **b** $(x+2)(1-x)=-4$
c $5-4x^2=3(2x+1)+2$ **d** $x+\dfrac{2}{x}=3$
e $2x-\dfrac{1}{x}=-1$ **f** $\dfrac{x+3}{1-x}=-\dfrac{9}{x}$

SOLVING BY 'COMPLETING THE SQUARE'

As you would be aware by now, not all quadratics factorise easily. For example, x^2+4x+1 cannot be factorised by simple factorisation. In other words, we cannot write x^2+4x+1 in the form $(x-a)(x-b)$ where a, b are rational.