

- 2 a** Show that the sum of w , x , y , and z is given by $(w \ x \ y \ z) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.
- b** Represent the *average* of w , x , y , and z in a similar way.
- 3** Lucy buys 4 shirts, 3 skirts, and 2 blouses costing \$27, \$35, and \$39 each respectively.
- a** Write down a quantities matrix \mathbf{Q} and a price matrix \mathbf{P} .
- b** Show how to use \mathbf{P} and \mathbf{Q} to determine the total cost of Lucy's clothes.
- 4** In the interschool public speaking competition, a first place is awarded 10 points, second place 6 points, third place 3 points, and fourth place 1 point. One school won 3 first places, 2 seconds, 4 thirds, and 2 fourths.
- a** Write down this information in terms of a points matrix \mathbf{P} and a numbers matrix \mathbf{N} .
- b** Show how to use \mathbf{P} and \mathbf{N} to find the total number of points awarded to the school.



MORE COMPLICATED MULTIPLICATIONS

Consider again **Example 1** on page 308 where Lisa needed 2 loaves of bread, 3 litres of milk, and 1 tub of butter.

We represented this by the quantities matrix $\mathbf{Q} = (2 \ 3 \ 1)$.

The prices for each store were summarised in the costs matrix $\mathbf{C} = \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix}$.

To find the *total cost* of the items in each store, Lisa needs to multiply the number of items by their respective cost.

In Store A, a loaf of bread is \$2.65, a litre of milk is \$1.55, and a tub of butter is \$2.35, so the total cost is $2 \times \$2.65 + 3 \times \$1.55 + 1 \times \$2.35 = \12.30 .

In Store B, a loaf of bread is \$2.25, a litre of milk is \$1.50, and a tub of butter is \$2.20, so the total cost is $2 \times \$2.25 + 3 \times \$1.50 + 1 \times \$2.20 = \11.20 .

To do this using matrices notice that:

$$\mathbf{QC} = (2 \ 3 \ 1) \times \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix} = \begin{pmatrix} 12.30 & 11.20 \end{pmatrix}$$

↑ row $\mathbf{Q} \times$ column 1 ↓ row $\mathbf{Q} \times$ column 2
 1 × 3 ← the same → 3 × 2 1 × 2
 ↑ resultant matrix

Now suppose Lisa's friend Olu needs 1 loaf of bread, 2 litres of milk, and 2 tubs of butter.

The quantities matrix for both Lisa and Olu would be $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

↑ Lisa ↑ Olu
 bread milk butter

Lisa's *total cost* at Store A is \$12.30, and at store B is \$11.20

$$\begin{aligned} \text{Olu's total cost at Store A is } & 1 \times \$2.65 + 2 \times \$1.55 + 2 \times \$2.35 = \$10.45, \\ \text{and at Store B is } & 1 \times \$2.25 + 2 \times \$1.50 + 2 \times \$2.20 = \$9.65. \end{aligned}$$

So, using matrices we require that

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \times \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix} = \begin{pmatrix} 12.30 & 11.20 \\ 10.45 & 9.65 \end{pmatrix}$$

row 1 × column 1 ↓
 row 1 × column 2 ↓
 ↑ row 2 × column 1 ↑ row 2 × column 2

2 × 3 ← the same → 3 × 2 2 × 2
 ↑ resultant matrix ←

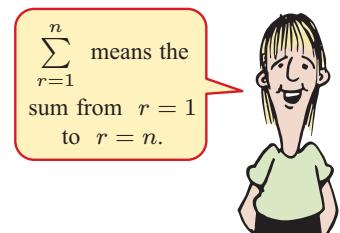
Having observed the usefulness of multiplying matrices in the contextual examples above, we now define matrix multiplication more formally.

The **product** of an $m \times n$ matrix **A** with an $n \times p$ matrix **B**, is the $m \times p$ matrix **AB** in which the element in the r th row and c th column is the sum of the products of the elements in the r th row of **A** with the corresponding elements in the c th column of **B**.

$$\text{If } \mathbf{C} = \mathbf{AB} \text{ then } c_{ij} = \sum_{r=1}^n a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

for each pair i and j with $1 \leq i \leq m$ and $1 \leq j \leq p$.

Note that the product \mathbf{AB} exists *only* if the number of columns of \mathbf{A} equals the number of rows of \mathbf{B} .



For example:

$$\text{If } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \text{ then } \mathbf{AB} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}.$$

$$\text{If } \mathbf{C} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{then} \quad \mathbf{CD} = \begin{pmatrix} ax + by + cz \\ dx + ey + fz \end{pmatrix}.$$

To get the matrix \mathbf{AB} you multiply **rows by columns**. To get the element in the 5th row and 3rd column of \mathbf{AB} (if it exists), multiply the 5th row of \mathbf{A} by the 3rd column of \mathbf{B} .

Example 6**Self Tutor**

For $\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{pmatrix}$, find:

- a** \mathbf{AC}
- b** \mathbf{BC}

a \mathbf{A} is 1×3 and \mathbf{C} is 3×2 $\therefore \mathbf{AC}$ is 1×2

$$\begin{aligned}\mathbf{AC} &= \begin{pmatrix} 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{pmatrix} \\ &= (1 \times 1 + 3 \times 2 + 5 \times 1 \quad 1 \times 0 + 3 \times 3 + 5 \times 4) \\ &= (12 \quad 29)\end{aligned}$$

b \mathbf{B} is 2×3 and \mathbf{C} is 3×2 $\therefore \mathbf{BC}$ is 2×2

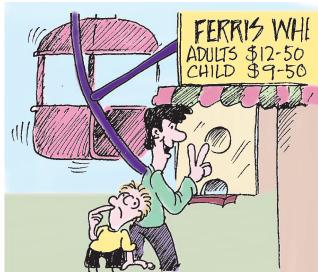
$$\begin{aligned}\mathbf{BC} &= \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{pmatrix} \\ &= (1 \times 1 + 3 \times 2 + 5 \times 1 \quad 1 \times 0 + 3 \times 3 + 5 \times 4) \\ &= (12 \quad 29) \\ &= (7 \quad 15)\end{aligned}$$

To get the element in the 2nd row and 1st column of \mathbf{BC} , multiply the 2nd row of \mathbf{B} by the 1st column of \mathbf{C} .

**EXERCISE 12C.2**

- 1** Explain why \mathbf{AB} cannot be found for $\mathbf{A} = \begin{pmatrix} 4 & 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.
- 2** Suppose \mathbf{A} is $2 \times n$ and \mathbf{B} is $m \times 3$.
 - a** When can we find \mathbf{AB} ?
 - b** If \mathbf{AB} can be found, what is its order?
 - c** Explain why \mathbf{BA} cannot be found.
- 3** For $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 6 \end{pmatrix}$, find:
 - a** \mathbf{AB}
 - b** \mathbf{BA}
- 4** For $\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$, find:
 - a** \mathbf{AB}
 - b** \mathbf{BA}
- 5** Find:
 - a** $(1 \ 2 \ 1) \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$
 - b** $\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$
- 6** Answer the **Opening Problem** on page 306.

7



At a fair, tickets for the Ferris wheel are \$12.50 per adult and \$9.50 per child. On the first day of the fair, 2375 adults and 5156 children ride this wheel. On the second day, 2502 adults and 3612 children ride the wheel.

- Write the costs as a 2×1 matrix \mathbf{C} , and the numbers as a 2×2 matrix \mathbf{N} .
- Find \mathbf{NC} and interpret the resulting matrix.
- Find the total income for the two days.

- 8 You and your friend each go to your local hardware stores A and B to price items you wish to purchase. You want to buy 1 hammer, 1 screwdriver, and 2 cans of white paint. Your friend wants 1 hammer, 2 screwdrivers, and 3 cans of white paint. The prices of these goods are:

	Hammer	Screwdriver	Can of paint
Store A	\$7	\$3	\$19
Store B	\$6	\$2	\$22

- Write the requirements matrix \mathbf{R} as a 3×2 matrix.
- Write the prices matrix \mathbf{P} as a 2×3 matrix.
- Find \mathbf{PR} .
- Find:
 - your costs at store A
 - your friend's costs at store B.
- Do any of the elements of \mathbf{PR} tell you and your friend the cheapest way to buy all your items? Explain your answer.



PROPERTIES OF MATRIX MULTIPLICATION

Discovery 1

Matrix multiplication

In this Discovery we find the properties of 2×2 matrix multiplication which are like those of ordinary number multiplication, and those which are not.

What to do:

- 1 For ordinary arithmetic $2 \times 3 = 3 \times 2$, and in algebra $ab = ba$.

For matrices, does \mathbf{AB} always equal \mathbf{BA} ?

Hint: Try $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix}$.

- 2 If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, find \mathbf{AO} and \mathbf{OA} .

- 3 Find \mathbf{AB} for:

a $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

b $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$

- 4 For all real numbers a , b , and c , we have the **distributive law** $a(b+c) = ab + ac$.

- a Use any three 2×2 matrices \mathbf{A} , \mathbf{B} and \mathbf{C} to verify that $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

b Now let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

Prove that in general, $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.

c Use the matrices you ‘made up’ in **a** to verify that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

d Prove that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

5 a If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $w = z = 1$ and $x = y = 0$ is a solution for any values of a, b, c , and d .

b For any real number a , we know that $a \times 1 = 1 \times a = a$.

Is there a matrix \mathbf{I} such that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices \mathbf{A} ?

6 Suppose $\mathbf{A}^2 = \mathbf{AA} = \mathbf{A} \times \mathbf{A}$ and that $\mathbf{A}^3 = \mathbf{AAA}$.

a Find \mathbf{A}^2 if $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$.

b Find \mathbf{A}^3 if $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix}$.

c If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ try to find \mathbf{A}^2 .

d Under what conditions can we square a matrix?

7 Show that if $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\mathbf{I}^2 = \mathbf{I}$ and $\mathbf{I}^3 = \mathbf{I}$.

$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
is called the
identity matrix.



In the **Discovery** you should have found that:

Ordinary algebra	Matrix algebra
<ul style="list-style-type: none"> If a and b are real numbers then so is ab. {closure} $ab = ba$ for all a, b {commutative} $a0 = 0a = 0$ for all a $ab = 0 \Leftrightarrow a = 0$ or $b = 0$ {Null Factor law} $a(b + c) = ab + ac$ {distributive law} $a \times 1 = 1 \times a = a$ {identity law} a^n exists for all $a \geq 0$ and $n \in \mathbb{R}$. 	<ul style="list-style-type: none"> If \mathbf{A} and \mathbf{B} are matrices that can be multiplied then \mathbf{AB} is also a matrix. {closure} In general $\mathbf{AB} \neq \mathbf{BA}$. {non-commutative} If \mathbf{O} is a zero matrix then $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$ for all \mathbf{A}. \mathbf{AB} may be \mathbf{O} without requiring $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ {distributive law} If \mathbf{I} is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices \mathbf{A}. {identity law} \mathbf{A}^n exists provided \mathbf{A} is square and $n \in \mathbb{Z}^+$.

Note that in general, $\mathbf{A}(k\mathbf{B}) = k(\mathbf{AB}) \neq k\mathbf{BA}$. We can change the order in which we multiply by a scalar, but we cannot reverse the order in which we multiply matrices.

Example 7**Self Tutor**

Expand and simplify where possible:

a $(A + 2I)^2$

$$\begin{aligned} &= (A + 2I)(A + 2I) \\ &= (A + 2I)A + (A + 2I)2I \\ &= A^2 + 2IA + 2AI + 4I^2 \\ &= A^2 + 2A + 2A + 4I \\ &= A^2 + 4A + 4I \end{aligned}$$

b $(A - B)^2$

$$\begin{aligned} &= (A - B)(A - B) \\ &= (A - B)A - (A - B)B \\ &= A^2 - BA - AB + B^2 \end{aligned}$$

$$\begin{aligned} &\{X^2 = XX \text{ by definition}\} \\ &\{B(C + D) = BC + BD\} \\ &\{(C + D)B = CB + DB\} \\ &\{AI = IA = A \text{ and } I^2 = I\} \end{aligned}$$

b cannot be simplified further since, in general, $AB \neq BA$.

**Example 8****Self Tutor**

If $A^2 = 2A + 3I$, find A^3 and A^4 in the linear form $kA + lI$ where k and l are scalars.

$$\begin{aligned} A^3 &= A \times A^2 \\ &= A(2A + 3I) \\ &= 2A^2 + 3AI \\ &= 2(2A + 3I) + 3AI \\ &= 7A + 6I \end{aligned}$$

$$\begin{aligned} A^4 &= A \times A^3 \\ &= A(7A + 6I) \\ &= 7A^2 + 6AI \\ &= 7(2A + 3I) + 6A \\ &= 20A + 21I \end{aligned}$$

EXERCISE 12C.3

- 1** Given that all matrices are 2×2 and I is the identity matrix, expand and simplify:

a $A(A + I)$

b $(B + 2I)B$

c $A(A^2 - 2A + I)$

d $A(A^2 + A - 2I)$

e $(A + B)(C + D)$

f $(A + B)^2$

g $(A + B)(A - B)$

h $(A + I)^2$

i $(3I - B)^2$

- 2** **a** If $A^2 = 2A - I$, find A^3 and A^4 in the linear form $kA + lI$ where k and l are scalars.

- b** If $B^2 = 2I - B$, find B^3 , B^4 , and B^5 in linear form.

- c** If $C^2 = 4C - 3I$, find C^3 and C^5 in linear form.

- 3** **a** If $A^2 = I$, simplify:

i $A(A + 2I)$

ii $(A - I)^2$

iii $A(A + 3I)^2$

b If $A^3 = I$, simplify $A^2(A + I)^2$.

c If $A^2 = O$, simplify:

i $A(2A - 3I)$

ii $A(A + 2I)(A - I)$

iii $A(A + I)^3$

- 4 a** If $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, determine \mathbf{A}^2 .

- b** Comment on the following argument for a 2×2 matrix \mathbf{A} such that $\mathbf{A}^2 = \mathbf{A}$:

$$\begin{aligned}\mathbf{A}^2 &= \mathbf{A} \\ \therefore \mathbf{A}^2 - \mathbf{A} &= \mathbf{0} \\ \therefore \mathbf{A}(\mathbf{A} - \mathbf{I}) &= \mathbf{0} \\ \therefore \mathbf{A} = \mathbf{O} \text{ or } \mathbf{A} - \mathbf{I} &= \mathbf{0} \\ \therefore \mathbf{A} = \mathbf{O} \text{ or } \mathbf{I} &\end{aligned}$$

- c** Find all 2×2 matrices \mathbf{A} for which $\mathbf{A}^2 = \mathbf{A}$. **Hint:** Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

- 5** Give one example which shows that “if $\mathbf{A}^2 = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$ ” is a false statement.

Example 9

Self Tutor

For $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$.

$$\begin{aligned}\text{Since } \mathbf{A}^2 &= a\mathbf{A} + b\mathbf{I}, \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = a \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + b \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \therefore \begin{pmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{pmatrix} &= \begin{pmatrix} a & 2a \\ 3a & 4a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \\ \therefore \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} &= \begin{pmatrix} a+b & 2a \\ 3a & 4a+b \end{pmatrix}\end{aligned}$$

Thus $a + b = 7$ and $2a = 10$

$$\therefore a = 5 \text{ and } b = 2$$

Checking for consistency:

$$3a = 3(5) = 15 \quad \checkmark \quad 4a + b = 4(5) + (2) = 22 \quad \checkmark$$

- 6** Find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$, given:

a $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix}$

- 7 a** For $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}$, find constants p and q such that $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$.

- b** Hence, write \mathbf{A}^3 in the linear form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

- c** Write \mathbf{A}^4 in linear form.

D**THE INVERSE OF A 2×2 MATRIX**

The real numbers 5 and $\frac{1}{5}$ are called **multiplicative inverses** because when they are multiplied together, the result is the multiplicative identity 1: $5 \times \frac{1}{5} = \frac{1}{5} \times 5 = 1$

For the matrices $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$, we notice that $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$
and $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$.

We say that $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ are *multiplicative inverses* of each other.

The **multiplicative inverse** of \mathbf{A} , denoted \mathbf{A}^{-1} , satisfies $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

To find the multiplicative inverse of a matrix \mathbf{A} , we need a matrix which, when multiplied by \mathbf{A} , gives the identity matrix \mathbf{I} .

We will now determine how to find the inverse of a matrix \mathbf{A} .

$$\text{Suppose } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{A}^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

$$\therefore \mathbf{AA}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \mathbf{I}$$

$$\therefore \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \begin{cases} aw + by = 1 & \dots (1) \\ cw + dy = 0 & \dots (2) \end{cases} \text{ and } \begin{cases} ax + bz = 0 & \dots (3) \\ cx + dz = 1 & \dots (4) \end{cases}$$

Solving (1) and (2) simultaneously for w and y gives: $w = \frac{d}{ad - bc}$ and $y = \frac{-c}{ad - bc}$.

Solving (3) and (4) simultaneously for x and z gives: $x = \frac{-b}{ad - bc}$ and $z = \frac{a}{ad - bc}$.

So, if $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc \neq 0$, then $\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$$\begin{aligned} \text{In this case } \mathbf{A}^{-1}\mathbf{A} &= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \frac{1}{ad - bc} \begin{pmatrix} ad - bc & bd - bd \\ ac - ac & -bc + ad \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \mathbf{I} \text{ also,} \end{aligned}$$

$$\text{so } \mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{I}$$

Just as the real number 0 does not have a multiplicative inverse, some matrices do not have a multiplicative inverse. This occurs when $\det \mathbf{A} = ad - bc = 0$.

For the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

- the value $ad - bc$ is called the **determinant** of matrix \mathbf{A} , denoted $\det \mathbf{A}$
- if $\det \mathbf{A} \neq 0$, then \mathbf{A} is **invertible** or **non-singular**, and $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- if $\det \mathbf{A} = 0$, then \mathbf{A} is **singular**, and \mathbf{A}^{-1} does not exist.

Example 10



Find, if it exists, the inverse matrix of:

a $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$

b $\mathbf{B} = \begin{pmatrix} 6 & 3 \\ -4 & -2 \end{pmatrix}$

a $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$

b $\mathbf{B} = \begin{pmatrix} 6 & 3 \\ -4 & -2 \end{pmatrix}$

$\therefore \det \mathbf{A} = 5(4) - 6(3) = 2$

$\therefore \det \mathbf{B} = 6(-2) - 3(-4)$

$$\begin{aligned} \therefore \mathbf{A}^{-1} &= \frac{1}{2} \begin{pmatrix} 4 & -6 \\ -3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &= -12 + 12 \\ &= 0 \end{aligned}$$

$\therefore \mathbf{B}^{-1}$ does not exist.

EXERCISE 12D.1

1 a Find $\begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -6 \\ -2 & 5 \end{pmatrix}$, and hence find the inverse of $\begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$.

b Find $\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$, and hence find the inverse of $\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}$.

2 Find $\det \mathbf{A}$ for \mathbf{A} equal to:

a $\begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix}$

b $\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$

c $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

d $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3 Find $\det \mathbf{B}$ for \mathbf{B} equal to:

a $\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}$

b $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

c $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

d $\begin{pmatrix} a & -a \\ 1 & a \end{pmatrix}$

4 For $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}$, find:

a $\det \mathbf{A}$

b $\det (-\mathbf{A})$

c $\det (2\mathbf{A})$

5 Prove that if \mathbf{A} is any 2×2 matrix and k is a constant, then $\det (k\mathbf{A}) = k^2 \times \det \mathbf{A}$.

6 Suppose $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

a Find:

i $\det \mathbf{A}$

ii $\det \mathbf{B}$

iii \mathbf{AB}

iv $\det(\mathbf{AB})$

b Hence show that $\det(\mathbf{AB}) = \det \mathbf{A} \times \det \mathbf{B}$ for all 2×2 matrices \mathbf{A} and \mathbf{B} .

7 Suppose $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$.

a Find $\det \mathbf{A}$ and $\det \mathbf{B}$.

b Find:

i $\det(2\mathbf{A})$

ii $\det(-\mathbf{A})$

iii $\det(-3\mathbf{B})$

iv $\det(\mathbf{AB})$

8 Find, if it exists, the inverse matrix of:

a $\begin{pmatrix} 2 & 4 \\ -1 & 5 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$

c $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

d $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

e $\begin{pmatrix} 5 & 0 \\ -1 & 2 \end{pmatrix}$

f $\begin{pmatrix} 3 & 5 \\ -6 & -10 \end{pmatrix}$

g $\begin{pmatrix} -1 & 2 \\ 4 & 7 \end{pmatrix}$

h $\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$

i $\begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}$

Example 11

Self Tutor

Suppose $\mathbf{A} = \begin{pmatrix} 4 & k \\ 2 & -1 \end{pmatrix}$.

Find \mathbf{A}^{-1} and state the values of k for which \mathbf{A}^{-1} exists.

$$\mathbf{A}^{-1} = \frac{1}{-4-2k} \begin{pmatrix} -1 & -k \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2k+4} & \frac{k}{2k+4} \\ \frac{2}{2k+4} & \frac{-4}{2k+4} \end{pmatrix}$$

$$\mathbf{A}^{-1} \text{ exists provided that } 2k+4 \neq 0 \\ \therefore k \neq -2$$

If $\det \mathbf{A} = 0$, the matrix \mathbf{A} is singular.



9 For each of the following matrices \mathbf{A} , find \mathbf{A}^{-1} and state the values of k for which \mathbf{A}^{-1} exists.

a $\mathbf{A} = \begin{pmatrix} k & 1 \\ -6 & 2 \end{pmatrix}$

b $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 0 & k \end{pmatrix}$

c $\mathbf{A} = \begin{pmatrix} k+1 & 2 \\ 1 & k \end{pmatrix}$

d $\mathbf{A} = \begin{pmatrix} k-2 & k \\ -3 & k \end{pmatrix}$

e $\mathbf{A} = \begin{pmatrix} k^2 & k-1 \\ 2k & 1 \end{pmatrix}$

f $\mathbf{A} = \begin{pmatrix} k+1 & 2 \\ k^2+2 & 3k \end{pmatrix}$

FURTHER MATRIX ALGEBRA

In this section we consider matrix algebra with inverse matrices. Be careful that you use multiplication correctly. In particular, remember that:

- We can only perform matrix multiplication if the orders of the matrices allow it.
- If we *premultiply* on one side then we must *premultiply* on the other. This is important because, in general, $\mathbf{AB} \neq \mathbf{BA}$. The same applies if we *postmultiply*.

Premultiply means multiply on the left of each side.
Postmultiply means multiply on the right of each side.



Discovery 2

Properties of inverse matrices

In this Discovery, we consider some properties of invertible 2×2 matrices.

What to do:

- 1** A matrix \mathbf{A} is **self-inverse** when $\mathbf{A} = \mathbf{A}^{-1}$.

For example, if $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ then $\mathbf{A}^{-1} = \frac{1}{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \mathbf{A}$.

- a** Show that if $\mathbf{A} = \mathbf{A}^{-1}$, then $\mathbf{A}^2 = \mathbf{I}$.

- b** Show that there are exactly 4 self-inverse matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

- 2 a** Given $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, find \mathbf{A}^{-1} and $(\mathbf{A}^{-1})^{-1}$.

- b** If \mathbf{A} is any invertible matrix, simplify $(\mathbf{A}^{-1})^{-1}(\mathbf{A}^{-1})$ and $(\mathbf{A}^{-1})(\mathbf{A}^{-1})^{-1}$ by replacing \mathbf{A}^{-1} by \mathbf{B} .

- c** What can be deduced from **b**?

- 3** Suppose k is a non-zero number and \mathbf{A} is an invertible matrix.

- a** Simplify $(k\mathbf{A})(\frac{1}{k}\mathbf{A}^{-1})$ and $(\frac{1}{k}\mathbf{A}^{-1})(k\mathbf{A})$.

- b** What can you conclude from your results?

- 4 a** If $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}$, find in simplest form:

i \mathbf{A}^{-1}

ii \mathbf{B}^{-1}

iii $(\mathbf{AB})^{-1}$

iv $(\mathbf{BA})^{-1}$

v $\mathbf{A}^{-1}\mathbf{B}^{-1}$

vi $\mathbf{B}^{-1}\mathbf{A}^{-1}$

- b** Choose any two invertible matrices and repeat **a**.

- c** What do the results of **a** and **b** suggest?

- d** Simplify $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1})$ and $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB})$ given that \mathbf{A}^{-1} and \mathbf{B}^{-1} exist.

What can you conclude from your results?

From the **Discovery** you should have found that if \mathbf{A} and \mathbf{B} are invertible, then:

$$\bullet \quad (\mathbf{A}^{-1})^{-1} = \mathbf{A} \qquad \bullet \quad (k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1} \qquad \bullet \quad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Example 12

Self Tutor

If $\mathbf{A}^2 = 2\mathbf{A} + 3\mathbf{I}$, find \mathbf{A}^{-1} in the linear form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

$$\begin{aligned} \mathbf{A}^2 &= 2\mathbf{A} + 3\mathbf{I} \\ \therefore \mathbf{A}^{-1}\mathbf{A}^2 &= \mathbf{A}^{-1}(2\mathbf{A} + 3\mathbf{I}) \quad \{\text{premultiplying both sides by } \mathbf{A}^{-1}\} \\ \therefore \mathbf{A}^{-1}\mathbf{AA} &= 2\mathbf{A}^{-1}\mathbf{A} + 3\mathbf{A}^{-1}\mathbf{I} \\ \therefore \mathbf{I}\mathbf{A} &= 2\mathbf{I} + 3\mathbf{A}^{-1} \\ \therefore \mathbf{A} - 2\mathbf{I} &= 3\mathbf{A}^{-1} \\ \therefore \mathbf{A}^{-1} &= \frac{1}{3}(\mathbf{A} - 2\mathbf{I}) \\ \therefore \mathbf{A}^{-1} &= \frac{1}{3}\mathbf{A} - \frac{2}{3}\mathbf{I} \end{aligned}$$

Premultiply means multiply on the left of each side.



EXERCISE 12D.2

- 1 Suppose $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$, and $\mathbf{AXB} = \mathbf{C}$. Find \mathbf{X} .
- 2 Suppose \mathbf{X} , \mathbf{Y} , and \mathbf{Z} are 2×1 matrices, and \mathbf{A} and \mathbf{B} are invertible 2×2 matrices. If $\mathbf{X} = \mathbf{AY}$ and $\mathbf{Y} = \mathbf{BZ}$, write:
 - a \mathbf{X} in terms of \mathbf{Z}
 - b \mathbf{Z} in terms of \mathbf{X} .
- 3 If $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$, write \mathbf{A}^2 in the linear form $p\mathbf{A} + q\mathbf{I}$ where p and q are scalars. Hence write \mathbf{A}^{-1} in the form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.
- 4 Write \mathbf{A}^{-1} in linear form given that:
 - a $\mathbf{A}^2 = 4\mathbf{A} - \mathbf{I}$
 - b $5\mathbf{A} = \mathbf{I} - \mathbf{A}^2$
 - c $2\mathbf{I} = 3\mathbf{A}^2 - 4\mathbf{A}$
- 5 It is known that $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$ where the matrices \mathbf{A} and \mathbf{B} are not necessarily invertible. Prove that $\mathbf{A}^2 = \mathbf{A}$.
Hint: From $\mathbf{AB} = \mathbf{A}$, you cannot deduce that $\mathbf{B} = \mathbf{I}$.
- 6 Under what condition is it true that “if $\mathbf{AB} = \mathbf{AC}$ then $\mathbf{B} = \mathbf{C}$ ”?
- 7 If $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{A}^3 = \mathbf{I}$, prove that $\mathbf{X}^3 = \mathbf{I}$.
- 8 If $a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{O}$ and $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$, prove that $a\mathbf{X}^2 + b\mathbf{X} + c\mathbf{I} = \mathbf{O}$.

E SIMULTANEOUS LINEAR EQUATIONS

We can solve $\begin{cases} 2x + 3y = 4 \\ 5x + 4y = 17 \end{cases}$ algebraically to get $x = 5, y = -2$.

Notice that this system can be written as a matrix equation $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix}$.

The solution $x = 5, y = -2$ is easily checked as

$$\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2(5) + 3(-2) \\ 5(5) + 4(-2) \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix} \quad \checkmark$$

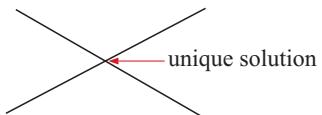
In general, a system of linear equations can be written in the form $\mathbf{AX} = \mathbf{B}$ where \mathbf{A} is the matrix of coefficients, \mathbf{X} is the unknown column matrix, and \mathbf{B} is a column matrix of constants.

We can use inverses to solve the matrix equation $\mathbf{AX} = \mathbf{B}$ for \mathbf{X} .

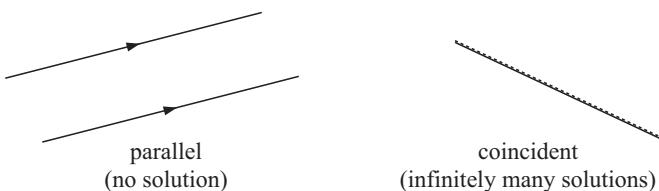
If we premultiply each side of $\mathbf{AX} = \mathbf{B}$ by \mathbf{A}^{-1} , we get

$$\begin{aligned} \mathbf{A}^{-1}(\mathbf{AX}) &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore \mathbf{IX} &= \mathbf{A}^{-1}\mathbf{B} \\ \text{and so } \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \end{aligned}$$

If the matrix of coefficients \mathbf{A} is invertible, then calculating $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ will give a unique solution to the pair of linear equations. This indicates that the lines intersect at a single point.



If the matrix of coefficients \mathbf{A} is singular, then we cannot calculate $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. This indicates that either the lines are parallel and there are no solutions, or that the lines are coincident and there are infinitely many solutions.



Example 13

Self Tutor

- a If $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$, find \mathbf{A}^{-1} .
- b Write the system $\begin{cases} 2x + 3y = 4 \\ 5x + 4y = 17 \end{cases}$ in matrix form.
- c Hence, solve the simultaneous linear equations.

a $\det \mathbf{A} = 2(4) - 3(5)$
 $= -7$

$$\therefore \mathbf{A}^{-1} = \frac{1}{-7} \begin{pmatrix} 4 & -3 \\ -5 & 2 \end{pmatrix}$$

- b** In matrix form, the system is $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix}$ which has the form $\mathbf{AX} = \mathbf{B}$.
- c** Premultiplying by \mathbf{A}^{-1} , $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$
 $\therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$
- $$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-7} \begin{pmatrix} 4 & -3 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 17 \end{pmatrix}$$
- $$= \frac{1}{-7} \begin{pmatrix} -35 \\ 14 \end{pmatrix}$$
- $$= \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$
- $$\therefore x = 5 \text{ and } y = -2.$$

EXERCISE 12E

- 1** Convert into matrix equations:

a $\begin{cases} 3x - y = 8 \\ 2x + 3y = 6 \end{cases}$ **b** $\begin{cases} 4x - 3y = 11 \\ 3x + 2y = -5 \end{cases}$ **c** $\begin{cases} 3a - b = 6 \\ 2a + 7b = -4 \end{cases}$

- 2** Use matrix algebra to solve the system:

a $\begin{cases} 2x - y = 6 \\ x + 3y = 14 \end{cases}$ **b** $\begin{cases} 5x - 4y = 5 \\ 2x + 3y = -13 \end{cases}$ **c** $\begin{cases} x - 2y = 7 \\ 5x + 3y = -2 \end{cases}$
d $\begin{cases} 3x + 5y = 4 \\ 2x - y = 11 \end{cases}$ **e** $\begin{cases} 4x - 7y = 8 \\ 3x - 5y = 0 \end{cases}$ **f** $\begin{cases} 7x + 11y = 18 \\ 11x - 7y = -11 \end{cases}$

- 3** **a** Show that if $\mathbf{AX} = \mathbf{B}$ then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$, whereas if $\mathbf{XA} = \mathbf{B}$ then $\mathbf{X} = \mathbf{BA}^{-1}$.

- b** Find \mathbf{X} if:

i $\begin{pmatrix} -6 & 5 \\ -3 & 4 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$ **ii** $\mathbf{X} \begin{pmatrix} 1 & 2 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 14 & -5 \\ 22 & 0 \end{pmatrix}$
iii $\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}$ **iv** $\mathbf{X} \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ -5 & 15 \end{pmatrix}$

- 4** **a** Consider the system $\begin{cases} 2x - 3y = 8 \\ 4x - y = 11 \end{cases}$.

- i** Write the equations in the form $\mathbf{AX} = \mathbf{B}$, and find $\det \mathbf{A}$.

- ii** Does the system have a unique solution? If so, find it.

- b** Consider the system $\begin{cases} 2x + ky = 8 \\ 4x - y = 11 \end{cases}$.

- i** Write the system in the form $\mathbf{AX} = \mathbf{B}$, and find $\det \mathbf{A}$.

- ii** For what value(s) of k does the system have a unique solution? Find the unique solution.

- iii** Find k when the system does not have a unique solution. How many solutions does the system have in this case?

Review set 12A

- 1** If $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$

b $3\mathbf{A}$

c $-2\mathbf{B}$

d $\mathbf{A} - \mathbf{B}$

e $\mathbf{B} - 2\mathbf{A}$

f $3\mathbf{A} - 2\mathbf{B}$

g \mathbf{AB}

h \mathbf{BA}

i \mathbf{A}^{-1}

j \mathbf{A}^2

k \mathbf{ABA}

l $(\mathbf{AB})^{-1}$

- 2** Find a , b , c , and d if:

a $\begin{pmatrix} a & b-2 \\ c & d \end{pmatrix} = \begin{pmatrix} -a & 3 \\ 2-c & -4 \end{pmatrix}$

b $\begin{pmatrix} 3 & 2a \\ b & -2 \end{pmatrix} + \begin{pmatrix} b & -a \\ c & d \end{pmatrix} = \begin{pmatrix} a & 2 \\ 2 & 6 \end{pmatrix}$

- 3** Write \mathbf{Y} in terms of \mathbf{A} , \mathbf{B} , and \mathbf{C} :

a $\mathbf{B} - \mathbf{Y} = \mathbf{A}$

b $2\mathbf{Y} + \mathbf{C} = \mathbf{A}$

c $\mathbf{AY} = \mathbf{B}$

d $\mathbf{YB} = \mathbf{C}$

e $\mathbf{C} - \mathbf{AY} = \mathbf{B}$

f $\mathbf{AY}^{-1} = \mathbf{B}$

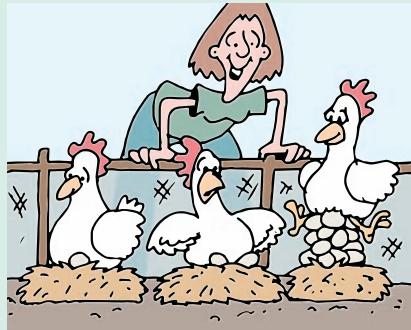
- 4** Susan keeps 3 hens in a pen. She calls them Anya, Betsy, and Charise. Each week the hens lay eggs according to the matrix

$$\mathbf{L} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Write, in terms of \mathbf{L} , a matrix to describe:

a the eggs laid by the hens over a 4 week period

b the eggs each hen loses each fortnight when Susan collects the eggs.



- 5** Suppose $\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 4 & -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -7 & 9 \\ 9 & -3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 2 & 1 \end{pmatrix}$.

Evaluate, if possible:

a $2\mathbf{A} - 2\mathbf{B}$

b \mathbf{AC}

c \mathbf{CB}

- 6** Given that all matrices are 2×2 and \mathbf{I} is the identity matrix, expand and simplify:

a $\mathbf{A}(\mathbf{I} - \mathbf{A})$

b $(\mathbf{A} - \mathbf{B})(\mathbf{B} + \mathbf{A})$

c $(2\mathbf{A} - \mathbf{I})^2$

- 7** If $\mathbf{A}^2 = 5\mathbf{A} + 2\mathbf{I}$, write \mathbf{A}^3 and \mathbf{A}^4 in the form $r\mathbf{A} + s\mathbf{I}$.

- 8** If $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$, find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$.

- 9** Find, if possible, the inverse matrix of:

a $\begin{pmatrix} 6 & 8 \\ 5 & 7 \end{pmatrix}$

b $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$

c $\begin{pmatrix} 11 & 5 \\ -6 & -3 \end{pmatrix}$

- 10** For what values of k does $\begin{cases} x + 4y = 2 \\ kx + 3y = -6 \end{cases}$ have a unique solution?

11 Solve using an inverse matrix:

a $\begin{cases} 3x - 4y = 2 \\ 5x + 2y = -1 \end{cases}$

b $\begin{cases} 4x - y = 5 \\ 2x + 3y = 9 \end{cases}$

12 Suppose $\mathbf{A} = 2\mathbf{A}^{-1}$.

a Show that $\mathbf{A}^2 = 2\mathbf{I}$.

b Simplify $(\mathbf{A} - \mathbf{I})(\mathbf{A} + 3\mathbf{I})$, giving your answer in the form $r\mathbf{A} + s\mathbf{I}$ where r and s are real numbers.

Review set 12B

1 For $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{pmatrix}$, find:

a $\mathbf{P} + \mathbf{Q}$

b $\mathbf{Q} - \mathbf{P}$

c $\frac{3}{2}\mathbf{P} - \mathbf{Q}$

2 A library owns several copies of a popular trilogy of novels, according to the matrix:

$$\begin{array}{cc} \text{paperback} & \text{hard cover} \\ \swarrow & \searrow \\ \mathbf{A} = \begin{pmatrix} 4 & 2 \\ 5 & 2 \\ 6 & 3 \end{pmatrix} & \leftarrow \begin{array}{l} \text{book 1} \\ \text{book 2} \\ \text{book 3} \end{array} \end{array}$$

a At present, the books on loan are described by the matrix $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 3 & 2 \end{pmatrix}$. Write a matrix to describe the books currently on the shelves.

b The values of the books (in dollars) are described by the matrix $\mathbf{C} = \begin{pmatrix} 7 & 7 & 8 \\ 15 & 16 & 20 \end{pmatrix}$.

- i** Which book has value \$16?
- ii** Find the total value of the books currently on loan.



3 Prove that for any square matrix \mathbf{A} , $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$.

4 Write \mathbf{X} in terms of \mathbf{A} and \mathbf{B} if:

a $2\mathbf{X} = \mathbf{B} - \mathbf{A}$

b $3(\mathbf{A} + \mathbf{X}) = 2\mathbf{B}$

c $\mathbf{B} - 4\mathbf{X} = \mathbf{A}$

5 Suppose $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix}$, and $\mathbf{A} + 2\mathbf{X} = -\mathbf{B}$. Find \mathbf{X} .

6 If \mathbf{A} is $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and \mathbf{B} is $\begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{pmatrix}$, find, if possible:

a $2\mathbf{B}$

b $\frac{1}{2}\mathbf{B}$

c \mathbf{AB}

d \mathbf{BA}

7 If \mathbf{A} and \mathbf{B} are square matrices, under what conditions are the following true?

a If $\mathbf{AB} = \mathbf{B}$ then $\mathbf{A} = \mathbf{I}$.

b $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$

8 For $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$, find:

a $\det \mathbf{A}$

b $\det(-2\mathbf{A})$

c $\det(\mathbf{A}^2)$

9 Solve using an inverse matrix:

a $\begin{cases} x + y = 5 \\ x - 2y = 4 \end{cases}$

b $\begin{cases} 3x + 2y = 3 \\ 5x + 3y = 4 \end{cases}$

10 If $\mathbf{M} = \begin{pmatrix} k & 2 \\ 2 & k \end{pmatrix} \begin{pmatrix} k-1 & -2 \\ -3 & k \end{pmatrix}$ has an inverse \mathbf{M}^{-1} , what values can k have?

11 For what values of k does the system $\begin{cases} kx + 3y = -6 \\ x + (k+2)y = 2 \end{cases}$ have a unique solution?

State the solution in this case.

12 Write $5\mathbf{A}^2 - 6\mathbf{A} = 3\mathbf{I}$ in the form $\mathbf{AB} = \mathbf{I}$. Hence write \mathbf{A}^{-1} in terms of \mathbf{A} and \mathbf{I} .

13 Prove that for any 2×2 matrix \mathbf{A} , \mathbf{A}^2 can be written in the linear form $a\mathbf{A} + b\mathbf{I}$.

13

Introduction to differential calculus

Contents:

- A** Limits
- B** Rates of change
- C** The derivative function
- D** Differentiation from first principles
- E** Simple rules of differentiation
- F** The chain rule
- G** The product rule
- H** The quotient rule
- I** Derivatives of exponential functions
- J** Derivatives of logarithmic functions
- K** Derivatives of trigonometric functions
- L** Second derivatives

Opening problem

In a BASE jumping competition from the Petronas Towers in Kuala Lumpur, the altitude of a professional jumper in the first 3 seconds is given by $f(t) = 452 - 4.8t^2$ metres, where $0 \leq t \leq 3$ seconds.

Things to think about:

- a** What will a graph of the altitude of the jumper in the first 3 seconds look like?
- b** Does the jumper travel with constant speed?
- c** Can you find the speed of the jumper when:
 - i** $t = 0$ seconds
 - ii** $t = 1$ second
 - iii** $t = 2$ seconds
 - iv** $t = 3$ seconds?



Calculus is a major branch of mathematics which builds on algebra, trigonometry, and analytic geometry. It has widespread applications in science, engineering, and financial mathematics.

The study of calculus is divided into two fields, **differential calculus** and **integral calculus**. These fields are linked by the **Fundamental Theorem of Calculus** which we will study later in the course.

Historical note

Calculus is a Latin word meaning ‘pebble’. Ancient Romans used stones for counting.

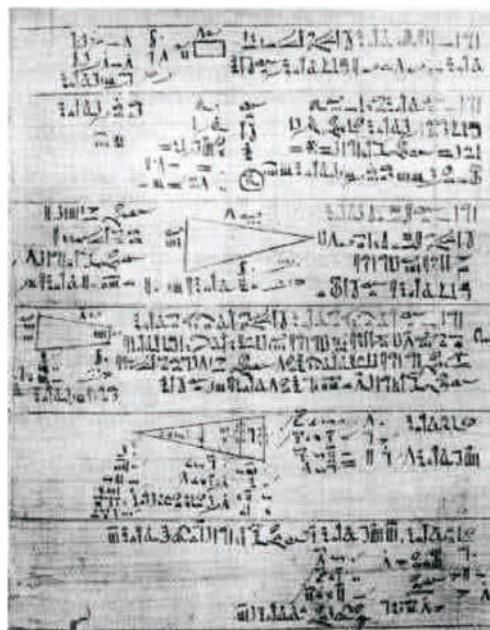
The history of calculus begins with the **Egyptian Moscow papyrus** from about 1850 BC.

The Greek mathematicians **Democritus**, **Zeno of Elea**, **Antiphon**, and **Eudoxes** studied **infinitesimals**, dividing objects into an infinite number of pieces in order to calculate the area of regions, and volume of solids.

Archimedes of Syracuse was the first to find the tangent to a curve other than a circle. His methods were the foundation of modern calculus developed almost 2000 years later.



Archimedes



Egyptian Moscow papyrus

A**LIMITS**

The concept of a **limit** is essential to differential calculus. We will see that calculating limits is necessary for finding the gradient of a tangent to a curve at any point on the curve.

The table alongside shows values for $f(x) = x^2$ where x is less than 2, but increasing and getting closer and closer to 2.

x	1	1.9	1.99	1.999	1.9999
$f(x)$	1	3.61	3.9601	3.996 00	3.999 60

We say that as x approaches 2 from the left, $f(x)$ approaches 4 from below.

We can construct a similar table of values where x is greater than 2, but decreasing and getting closer and closer to 2:

x	3	2.1	2.01	2.001	2.0001
$f(x)$	9	4.41	4.0401	4.004 00	4.000 40

We say that as x approaches 2 from the right, $f(x)$ approaches 4 from above.

So, as x approaches 2 from either direction, $f(x)$ approaches a limit of 4. We write this as $\lim_{x \rightarrow 2} x^2 = 4$.

INFORMAL DEFINITION OF A LIMIT

The following definition of a limit is informal but adequate for the purposes of this course:

If $f(x)$ can be made as close as we like to some real number A by making x sufficiently close to (but not equal to) a , then we say that $f(x)$ has a **limit** of A as x approaches a , and we write

$$\lim_{x \rightarrow a} f(x) = A.$$

In this case, $f(x)$ is said to **converge** to A as x approaches a .

Notice that the limit is defined for x close to but *not equal to* a . Whether the function f is defined or not at $x = a$ is not important to the definition of the limit of f as x approaches a . What *is* important is the behaviour of the function as x gets *very close to* a .

For example, if $f(x) = \frac{5x + x^2}{x}$ and we wish to find the limit as $x \rightarrow 0$, it is tempting for us to simply substitute $x = 0$ into $f(x)$. However, in doing this, not only do we get the meaningless value of $\frac{0}{0}$, but also we destroy the basic limit method.

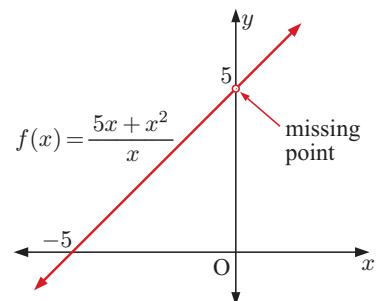
Observe that if $f(x) = \frac{5x + x^2}{x} = \frac{x(5+x)}{x}$ then $f(x) = \begin{cases} 5 + x & \text{if } x \neq 0 \\ \text{is undefined} & \text{if } x = 0. \end{cases}$

The graph of $y = f(x)$ is shown alongside. It is the straight line $y = x + 5$ with the point $(0, 5)$ missing, called a **point of discontinuity** of the function.

However, even though this point is missing, the *limit* of $f(x)$ as x approaches 0 does exist. In particular, as $x \rightarrow 0$ from either direction, $f(x) \rightarrow 5$.

We write $\lim_{x \rightarrow 0} \frac{5x + x^2}{x} = 5$ which reads:

“the limit as x approaches 0, of $f(x) = \frac{5x + x^2}{x}$, is 5”.



In practice we do not need to graph functions each time to determine limits, and most can be found algebraically.

Example 1**Self Tutor**

Evaluate: **a** $\lim_{x \rightarrow 2} x^2$

b $\lim_{x \rightarrow 0} \frac{x^2 + 3x}{x}$

c $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$

a x^2 can be made as close as we like to 4 by making x sufficiently close to 2.

$$\therefore \lim_{x \rightarrow 2} x^2 = 4.$$

b $\lim_{x \rightarrow 0} \frac{x^2 + 3x}{x}$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{\cancel{x}(x+3)}{\cancel{x}^1} \\ &= \lim_{x \rightarrow 0} (x+3) \quad \text{since } x \neq 0 \\ &= 3 \end{aligned}$$

c $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$

$$\begin{aligned} &= \lim_{x \rightarrow 3} \frac{(x+3)(x-3)}{x-3^1} \\ &= \lim_{x \rightarrow 3} (x+3) \quad \text{since } x \neq 3 \\ &= 6 \end{aligned}$$

EXERCISE 13A

1 Evaluate:

a $\lim_{x \rightarrow 3} (x + 4)$

b $\lim_{x \rightarrow -1} (5 - 2x)$

c $\lim_{x \rightarrow 4} (3x - 1)$

d $\lim_{x \rightarrow 2} (5x^2 - 3x + 2)$

e $\lim_{h \rightarrow 0} h^2(1 - h)$

f $\lim_{x \rightarrow 0} (x^2 + 5)$

2 Evaluate:

a $\lim_{x \rightarrow 0} 5$

b $\lim_{h \rightarrow 2} 7$

c $\lim_{x \rightarrow 0} c, \quad c \text{ a constant}$

3 Evaluate:

a $\lim_{x \rightarrow 1} \frac{x^2 - 3x}{x}$

b $\lim_{h \rightarrow 2} \frac{h^2 + 5h}{h}$

c $\lim_{x \rightarrow 0} \frac{x - 1}{x + 1}$

d $\lim_{x \rightarrow 0} \frac{x}{x}$

4 Evaluate the following limits:

a $\lim_{x \rightarrow 0} \frac{x^2 - 3x}{x}$

b $\lim_{x \rightarrow 0} \frac{x^2 + 5x}{x}$

c $\lim_{x \rightarrow 0} \frac{2x^2 - x}{x}$

d $\lim_{h \rightarrow 0} \frac{2h^2 + 6h}{h}$

e $\lim_{h \rightarrow 0} \frac{3h^2 - 4h}{h}$

f $\lim_{h \rightarrow 0} \frac{h^3 - 8h}{h}$

g $\lim_{x \rightarrow 1} \frac{x^2 - x}{x - 1}$

h $\lim_{x \rightarrow 2} \frac{x^2 - 2x}{x - 2}$

i $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3}$

B

RATES OF CHANGE

A **rate** is a comparison between two quantities with different units.

We often judge performances by rates. For example:

- Sir Donald Bradman's average batting rate at Test cricket level was 99.94 *runs per innings*.
- Michael Jordan's average basketball scoring rate was 20.0 *points per game*.
- Rangi's average typing rate is 63 *words per minute* with an error rate of 2.3 *errors per page*.

Speed is a commonly used rate. It is the rate of change in distance per unit of time.

We are familiar with the formula:

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

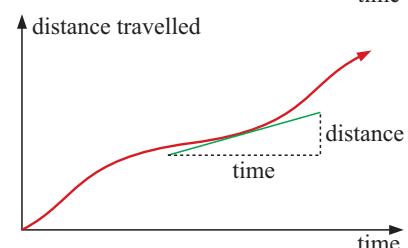
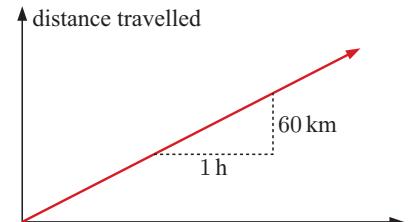
However, if a car has an average speed of 60 km h^{-1} for a journey, it does not mean that the car travels at exactly 60 km h^{-1} for the whole time.

In fact, the speed will probably vary continuously throughout the journey.

So, how can we calculate the car's speed at any particular time?

Suppose we are given a graph of the car's distance travelled against time taken. If this graph is a straight line, then we know the speed is constant and is given by the *gradient* of the line.

If the graph is a curve, then the car's instantaneous speed is given by the *gradient of the tangent* to the curve at that time.



Historical note

The modern study of **differential calculus** originated in the 17th century with the work of **Sir Isaac Newton** and **Gottfried Wilhelm Leibniz**. They developed the necessary theory while attempting to find algebraic methods for solving problems dealing with the **gradients of tangents** to curves, and finding the **rate of change** in one variable with respect to another.



Isaac Newton 1642 – 1727



Gottfried Leibniz 1646 – 1716

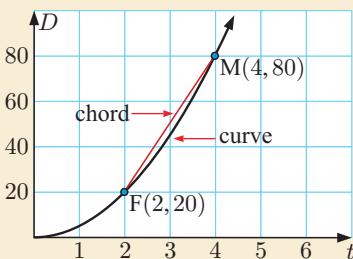
Discovery 1

A ball bearing is dropped from the top of a tall building. The distance D it has fallen after t seconds is recorded, and the following graph of distance against time obtained.

We choose a fixed point F on the curve when $t = 2$ seconds. We then choose another point M on the curve, and draw in the line segment or **chord** FM between the two points. To start with, we let M be the point when $t = 4$ seconds.

Instantaneous speed





The *average speed* in the time interval $2 \leq t \leq 4$

$$= \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{(80 - 20) \text{ m}}{(4 - 2) \text{ s}}$$

$$= \frac{60}{2} \text{ m s}^{-1}$$

$$= 30 \text{ m s}^{-1}$$

In this Discovery we will try to measure the *instantaneous speed* of the ball when $t = 2$ seconds.

What to do:

DEMO



- 1 Click on the icon to start the demonstration.

F is the point where $t = 2$ seconds, and M is another point on the curve.

To start with, M is at $t = 4$ seconds.

The number in the box marked *gradient* is the gradient of the chord FM. This is the *average speed* of the ball bearing in the interval from F to M. For M at $t = 4$ seconds, you should see the average speed is 30 m s^{-1} .

- 2 Click on M and drag it slowly towards F. Copy and complete the table alongside with the gradient of the chord FM for M being the points on the curve at the given varying times t .

- 3 Observe what happens as M reaches F. Explain why this is so.

- 4 Now move M to the origin, and then slide it towards F from the left. Copy and complete the table with the gradient of the chord FM for various times t .

- 5 a What can you say about the gradient of FM in the limit as $t \rightarrow 2$?
b What is the instantaneous speed of the ball bearing when $t = 2$ seconds? Explain your answer.

t	gradient of FM
3	
2.5	
2.1	
2.01	

t	gradient of FM
0	
1.5	
1.9	
1.99	

THE TANGENT TO A CURVE

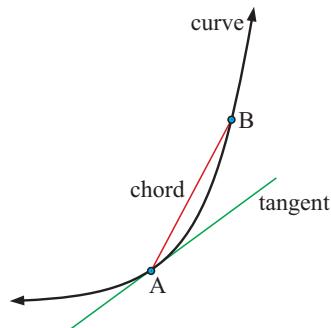
A **chord** of a curve is a straight line segment which joins any two points on the curve.

The gradient of the chord AB measures the average rate of change of the function values for the given change in x -values.

A **tangent** is a straight line which *touches* a curve at a single point. The tangent is the best approximating straight line to the curve through A.

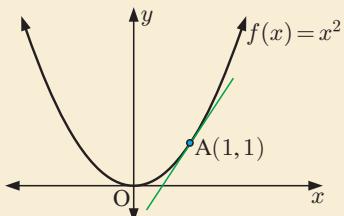
The gradient of the tangent at point A measures the instantaneous rate of change of the function at point A.

As B approaches A, the limit of the gradient of the chord AB will be the gradient of the tangent at A.



The **gradient of the tangent** to $y = f(x)$ at $x = a$ is the **instantaneous rate of change** in $f(x)$ with respect to x at that point.

Discovery 2



The gradient of a tangent

Given a curve $f(x)$, we wish to find the gradient of the tangent at the point $(a, f(a))$.

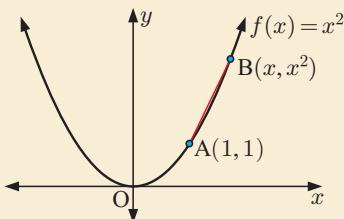
In this Discovery we find the gradient of the tangent to $f(x) = x^2$ at the point $A(1, 1)$.



What to do:

- 1 Suppose B lies on $f(x) = x^2$, and B has coordinates (x, x^2) .

- a Show that the chord AB has gradient $\frac{x^2 - 1}{x - 1}$.



x	Point B	gradient of AB
5	$(5, 25)$	6
3		
2		
1.5		
1.1		
1.01		
1.001		

- b Copy and complete the table shown.

- c Comment on the gradient of AB as x gets closer to 1.

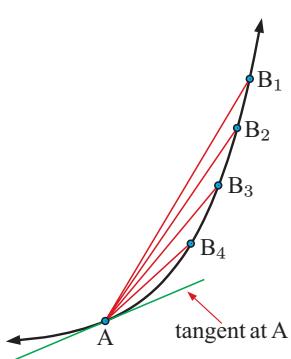
- 2 Repeat the process letting x get closer to 1, but from the left of A. Use the points where $x = 0, 0.8, 0.9, 0.99, \text{ and } 0.999$.

- 3 Click on the icon to view a demonstration of the process.

- 4 What do you suspect is the gradient of the tangent at A?

Fortunately we do not have to use a graph and table of values each time we wish to find the gradient of a tangent. Instead we can use an algebraic and geometric approach which involves **limits**.

From **Discovery 2**, the gradient of AB = $\frac{x^2 - 1}{x - 1}$.



As B approaches A, $x \rightarrow 1$ and the gradient of AB \rightarrow the gradient of the tangent at A.

So, the gradient of the tangent at the point A is

$$\begin{aligned} m_T &= \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{(x+1)(x-1)}{x-1} \\ &= \lim_{x \rightarrow 1} (x+1) \quad \text{since } x \neq 1 \\ &= 2 \end{aligned}$$

As B approaches A, the gradient of AB approaches or **converges** to 2.



EXERCISE 13B

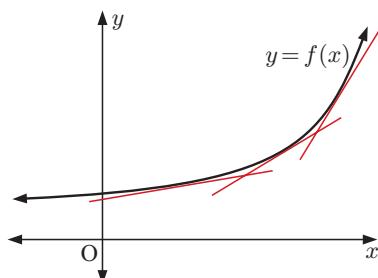
- 1 Use the method in **Discovery 1** to answer the **Opening Problem** on page 334.
- 2 **a** Use the method in **Discovery 2** to find the gradient of the tangent to $y = x^2$ at the point $(2, 4)$.
- b** Evaluate $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$, and provide a geometric interpretation of this result.

C

THE DERIVATIVE FUNCTION

For a non-linear function with equation $y = f(x)$, the gradients of the tangents at various points are different.

Our task is to determine a **gradient function** which gives the gradient of the tangent to $y = f(x)$ at $x = a$, for any point a in the domain of f .



The gradient function of $y = f(x)$ is called its **derivative function** and is labelled $f'(x)$.

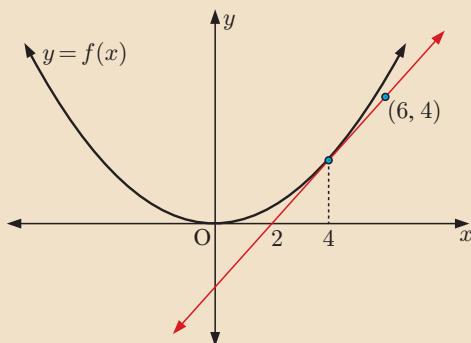
We read the derivative function as “eff dashed x ”.

The value of $f'(a)$ is the gradient of the tangent to $y = f(x)$ at the point where $x = a$.

Example 2

Self Tutor

For the given graph, find $f'(4)$ and $f(4)$.



The graph shows the tangent to the curve $y = f(x)$ at the point where $x = 4$.

The tangent passes through $(2, 0)$ and $(6, 4)$, so its gradient is $f'(4) = \frac{4-0}{6-2} = 1$.

The equation of the tangent is $y - 0 = 1(x - 2)$
 $\therefore y = x - 2$

When $x = 4$, $y = 2$, so the point of contact between the tangent and the curve is $(4, 2)$.

$$\therefore f(4) = 2$$