Example 13

Self Tutor

A particle moves in a straight line with displacement from O given by $s(t) = 3t - t^2$ metres at time t seconds. Find:

a the average velocity for the time interval from $t = 2$ to $t = 5$ seconds

b the average velocity for the time interval from $t = 2$ to $t = 2 + h$ seconds

c $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$ and comment on its significance.

a average velocity

$$= \frac{s(5) - s(2)}{5 - 2}$$
$$= \frac{(15 - 25) - (6 - 4)}{3}$$
$$= \frac{-10 - 2}{3}$$
$$= -4 \text{ m s}^{-1}$$

b average velocity

$$= \frac{s(2+h) - s(2)}{2 + h - 2}$$
$$= \frac{3(2+h) - (2+h)^2 - 2}{h}$$
$$= \frac{6 + 3h - 4 - 4h - h^2 - 2}{h}$$
$$= \frac{-h - h^2}{h}$$
$$= -1 - h \text{ m s}^{-1} \quad \text{provided } h \neq 0$$

c

$$\lim_{h \to 0} \frac{s(2+h) - s(2)}{h}$$
$$= \lim_{h \to 0} (-1 - h) \qquad \{\text{since } h \neq 0\}$$
$$= -1 \text{ m s}^{-1}$$

This is the instantaneous velocity of the particle at time $t = 2$ seconds.

EXERCISE 14C.1

1 A particle P moves in a straight line with displacement function $s(t) = t^2 + 3t - 2$ metres, where $t \geqslant 0$, t in seconds.

a Find the average velocity from $t = 1$ to $t = 3$ seconds.

b Find the average velocity from $t = 1$ to $t = 1 + h$ seconds.

c Find the value of $\lim\limits_{h \to 0} \dfrac{s(1+h) - s(1)}{h}$ and comment on its significance.

d Find the average velocity from time t to time $t + h$ seconds and interpret $\lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$.

2 A particle P moves in a straight line with displacement function $s(t) = 5 - 2t^2$ cm, where $t \geqslant 0$, t in seconds.

a Find the average velocity from $t = 2$ to $t = 5$ seconds.

b Find the average velocity from $t = 2$ to $t = 2 + h$ seconds.

c Find the value of $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$ and state the meaning of this value.

d Interpret $\lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$.

3 A particle moves in a straight line with velocity function $v(t) = 2\sqrt{t} + 3 \text{ cm s}^{-1}$, $t \geqslant 0$.

a Find the average acceleration from $t = 1$ to $t = 4$ seconds.

b Find the average acceleration from $t = 1$ to $t = 1 + h$ seconds.

c Find the value of $\lim\limits_{h \to 0} \dfrac{v(1+h) - v(1)}{h}$. Interpret this value.

d Interpret $\lim\limits_{h \to 0} \dfrac{v(t+h) - v(t)}{h}$.

4 An object moves in a straight line with displacement function $s(t)$ and velocity function $v(t)$, $t \geqslant 0$. State the meaning of:

a $\lim\limits_{h \to 0} \dfrac{s(4+h) - s(4)}{h}$

b $\lim\limits_{h \to 0} \dfrac{v(4+h) - v(4)}{h}$

VELOCITY AND ACCELERATION FUNCTIONS

If a particle P moves in a straight line and its position is given by the displacement function $s(t)$, $t \geqslant 0$, then:

- the **velocity** of P at time t is given by $\boldsymbol{v(t) = s'(t)}$
- the **acceleration** of P at time t is given by $\boldsymbol{a(t) = v'(t) = s''(t)}$
- $s(0)$, $v(0)$, and $a(0)$ give us the position, velocity, and acceleration of the particle at time $t = 0$, and these are called the **initial conditions**.

SIGN INTERPRETATION

Suppose a particle P moves in a straight line with displacement function $s(t)$ relative to an origin O. Its velocity function is $v(t)$ and its acceleration function is $a(t)$.

We can use **sign diagrams** to interpret:

- where the particle is located relative to O
- the direction of motion and where a change of direction occurs
- when the particle's velocity is increasing or decreasing.

SIGNS OF $s(t)$:

$s(t)$	*Interpretation*
$= 0$	P is at O
> 0	P is located to the right of O
< 0	P is located to the left of O

SIGNS OF $v(t)$:

$v(t)$	*Interpretation*
$= 0$	P is instantaneously at rest
> 0	P is moving to the right
< 0	P is moving to the left

SIGNS OF $a(t)$:

$a(t)$	*Interpretation*
> 0	velocity is increasing
< 0	velocity is decreasing
$= 0$	velocity may be a maximum or minimum or possibly constant

ZEROS:

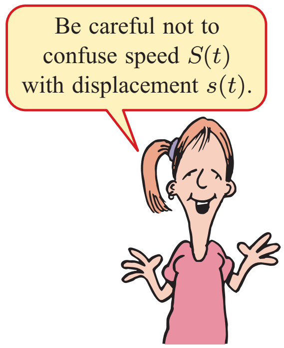
Phrase used in a question	t	s	v	a
initial conditions	0			
at the origin		0		
stationary			0	
reverses			0	
maximum or minimum displacement			0	
constant velocity				0
maximum or minimum velocity				0

When a particle reverses direction, its velocity must change sign.
This corresponds to a local maximum or local minimum distance from the origin O.

SPEED

As we have seen, velocities have size (magnitude) and sign (direction). In contrast, speed simply measures *how fast* something is travelling, regardless of the direction of travel. Speed is a *scalar* quantity which has size but no sign. Speed cannot be negative.

The **speed** at any instant is the magnitude of the object's velocity.
If $S(t)$ represents speed then $S = |v|$.



To determine when the speed $S(t)$ of an object P with displacement $s(t)$ is increasing or decreasing, we use a **sign test**.

- If the signs of $v(t)$ and $a(t)$ are the same (both positive or both negative), then the speed of P is increasing.
- If the signs of $v(t)$ and $a(t)$ are opposite, then the speed of P is decreasing.

Discovery — Displacement, velocity, and acceleration graphs

In this Discovery we examine the motion of a projectile which is fired in a vertical direction. The projectile is affected by gravity, which is responsible for the projectile's constant acceleration.

We then extend the Discovery to consider other cases of motion in a straight line.

What to do:

1 Click on the icon to examine vertical projectile motion.
Observe first the displacement along the line, then look at the velocity which is the rate of change in displacement. When is the velocity positive and when is it negative?

2 Examine the following graphs and comment on their shapes:

- *displacement* v *time*
- *velocity* v *time*
- *acceleration* v *time*

3 Pick from the menu or construct functions of your own choosing to investigate the relationship between displacement, velocity, and acceleration.

Example 14 — Self Tutor

A particle moves in a straight line with position relative to O given by $s(t) = t^3 - 3t + 1$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find expressions for the particle's velocity and acceleration, and draw sign diagrams for each of them.

b Find the initial conditions and hence describe the motion at this instant.

c Describe the motion of the particle at $t = 2$ seconds.

d Find the position of the particle when the changes in direction occur.

e Draw a motion diagram for the particle.

f For what time interval is the particle's speed increasing?

g What is the total distance travelled in the time from $t = 0$ to $t = 2$ seconds?

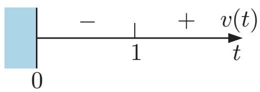
a $s(t) = t^3 - 3t + 1$ cm

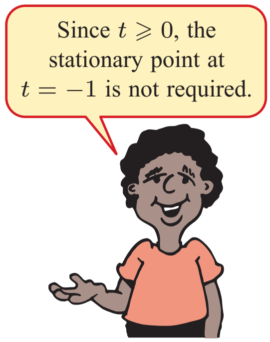
$\therefore\ v(t) = 3t^2 - 3$ {as $v(t) = s'(t)$}

$= 3(t^2 - 1)$

$= 3(t+1)(t-1)\ \text{cm s}^{-1}$

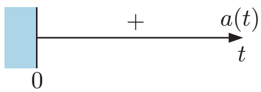
which has sign diagram:





and $a(t) = 6t\ \text{cm s}^{-2}$ {as $a(t) = v'(t)$}

which has sign diagram:



b When $t = 0$, $s(0) = 1$ cm

$v(0) = -3\ \text{cm s}^{-1}$

$a(0) = 0\ \text{cm s}^{-2}$

$\therefore$ the particle is 1 cm to the right of O, moving to the left at a speed of 3 cm s^{-1}.

c When $t = 2$, $s(2) = 8 - 6 + 1 = 3$ cm

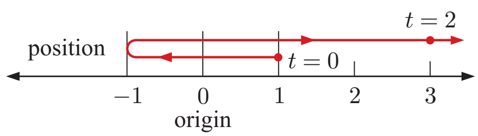
$v(2) = 12 - 3 = 9\ \text{cm s}^{-1}$

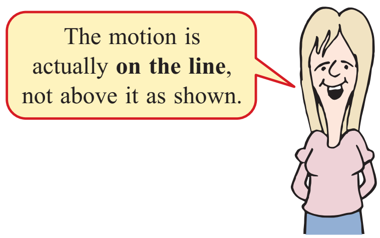
$a(2) = 12\ \text{cm s}^{-2}$

$\therefore$ the particle is 3 cm to the right of O, moving to the right at a speed of 9 cm s^{-1}.
Since a and v have the same sign, the speed of the particle is increasing.

d Since $v(t)$ changes sign when $t = 1$, a change of direction occurs at this instant.
$s(1) = 1 - 3 + 1 = -1$, so the particle changes direction when it is 1 cm to the left of O.

e





f Speed is increasing when $v(t)$ and $a(t)$ have the same sign. This is for $t \geqslant 1$.

g Total distance travelled $= 2 + 4 = 6$ cm.

In later chapters on integral calculus we will see another technique for finding the distances travelled and displacement over time.

EXERCISE 14C.2

1 An object moves in a straight line with position given by $s(t) = t^2 - 4t + 3$ cm from O, where t is in seconds, $t \geqslant 0$.

- **a** Find expressions for the object's velocity and acceleration, and draw sign diagrams for each function.
- **b** Find the initial conditions and explain what is happening to the object at that instant.
- **c** Describe the motion of the object at time $t = 2$ seconds.
- **d** At what time does the object reverse direction? Find the position of the object at this instant.
- **e** Draw a motion diagram for the object.
- **f** For what time intervals is the speed of the object decreasing?

2 A stone is projected vertically so that its position above ground level after t seconds is given by $s(t) = 98t - 4.9t^2$ metres, $t \geqslant 0$.

- **a** Find the velocity and acceleration functions for the stone, and draw sign diagrams for each function.
- **b** Find the initial position and velocity of the stone.
- **c** Describe the stone's motion at times $t = 5$ and $t = 12$ seconds.
- **d** Find the maximum height reached by the stone.
- **e** Find the time taken for the stone to hit the ground.

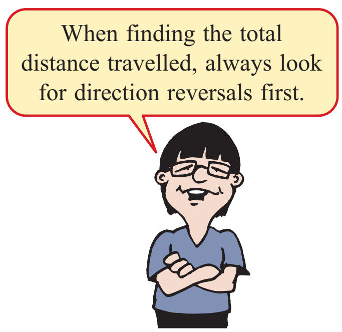
3 When a ball is thrown, its height above the ground is given by $s(t) = 1.2 + 28.1t - 4.9t^2$ metres where t is the time in seconds.

- **a** From what distance above the ground was the ball released?
- **b** Find $s'(t)$ and state what it represents.
- **c** Find t when $s'(t) = 0$. What is the significance of this result?
- **d** What is the maximum height reached by the ball?
- **e** Find the ball's speed:
  - **i** when released **ii** at $t = 2$ s **iii** at $t = 5$ s.

  State the significance of the sign of the derivative $s'(t)$.
- **f** How long will it take for the ball to hit the ground?
- **g** What is the significance of $s''(t)$?

4 The position of a particle moving along the x-axis is given by $x(t) = t^3 - 9t^2 + 24t$ metres where t is in seconds, $t \geqslant 0$.

- **a** Draw sign diagrams for the particle's velocity and acceleration functions.
- **b** Find the position of the particle at the times when it reverses direction, and hence draw a motion diagram for the particle.
- **c** At what times is the particle's:
  - **i** speed decreasing **ii** velocity decreasing?
- **d** Find the total distance travelled by the particle in the first 5 seconds of motion.



5 A particle P moves in a straight line with displacement function $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find the velocity and acceleration functions.

b Find the initial position, velocity, and acceleration of P.

c Sketch the graph of the velocity function.

d Find when the velocity of P is 80 cm per second.

6 A particle P moves along the x-axis with position given by $x(t) = 1 - 2\cos t$ cm where t is the time in seconds.

a State the initial position, velocity, and acceleration of P.

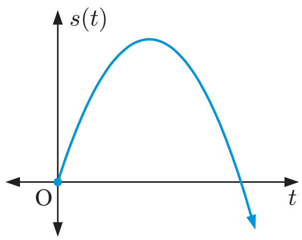
b Describe the motion when $t = \frac{\pi}{4}$ seconds.

c Find the times when the particle reverses direction on $0 < t < 2\pi$, and find the position of the particle at these instants.

d When is the particle's speed increasing on $0 \leqslant t \leqslant 2\pi$?

7 In an experiment, an object is fired vertically from the earth's surface. From the results, a two-dimensional graph of the position $s(t)$ metres above the earth's surface is plotted, where t is the time in seconds. It is noted that the graph is *parabolic*.

Assuming a constant gravitational acceleration g and an initial velocity of $v(0)$, show that:



a $v(t) = v(0) + gt$ **b** $s(t) = v(0) \times t + \frac{1}{2}gt^2$.

8 The table alongside shows data from a driving test in the United Kingdom.

A driver is travelling with constant speed. In response to a red light they must first react and press the brake. During this time the car travels a *thinking distance*. Once the brake is applied, the car travels a further *braking distance* before it comes to rest.

Speed (km h^{-1})	*Thinking distance* (m)	*Braking distance* (m)
32	6	6
48	9	14
64	12	24
80	15	38
96	18	55
112	21	75

a Using the data from the driving test, find the reaction time for the driver at 96 km h^{-1}.

b The distance $S(t)$ travelled by an object moving initially at speed u m s^{-1}, subject to constant acceleration a m s^{-2}, is $S(t) = ut + \frac{1}{2}at^2$ m.

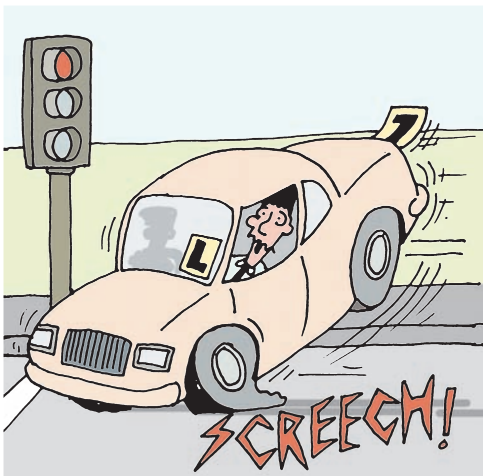
i Differentiate this formula with respect to time.

ii Hence calculate the time taken for the object to be at rest.

iii Using the data from the driving test, find the braking acceleration for the driver at 96 km h^{-1}.

iv Show that in general, an object starting at speed u comes to rest in a distance $-\frac{1}{2}\frac{u^2}{a}$ m.

v If a driver doubles their speed, what happens to their braking distance?



D RATES OF CHANGE

There are countless examples in the real world where quantities vary with time, or with respect to some other variable.

For example:
- temperature varies continuously
- the height of a tree varies as it grows
- the prices of stocks and shares vary with each day's trading.

We have already seen that if $y = f(x)$ then $f'(x)$ or $\frac{dy}{dx}$ is the gradient of the tangent to $y = f(x)$ at the given point.

$\frac{dy}{dx}$ gives the **rate of change in y with respect to x**.

We can therefore use the derivative of a function to tell us the **rate** at which something is happening.

For example:

- $\frac{dH}{dt}$ or $H'(t)$ could be the instantaneous rate of ascent of a person in a Ferris wheel.
  It might have units metres per second or m s^{-1}.
- $\frac{dC}{dt}$ or $C'(t)$ could be a person's instantaneous rate of change in lung capacity.
  It might have units litres per second or L s^{-1}.

Example 15

Self Tutor

According to a psychologist, the ability of a person to understand spatial concepts is given by $A = \frac{1}{3}\sqrt{t}$ where t is the age in years, $5 \leqslant t \leqslant 18$.

a Find the rate of improvement in ability to understand spatial concepts when a person is:
  i 9 years old **ii** 16 years old.

b Show that $\frac{dA}{dt} > 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

c Show that $\frac{d^2A}{dt^2} < 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

a $A = \frac{1}{3}\sqrt{t} = \frac{1}{3}t^{\frac{1}{2}}$

$\therefore \frac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}} = \frac{1}{6\sqrt{t}}$

i When $t = 9$, $\frac{dA}{dt} = \frac{1}{18}$

$\therefore$ the rate of improvement is $\frac{1}{18}$ units per year for a 9 year old.

ii When $t = 16$, $\frac{dA}{dt} = \frac{1}{24}$

$\therefore$ the rate of improvement is $\frac{1}{24}$ units per year for a 16 year old.

b Since $\sqrt{t}$ is never negative, $\frac{1}{6\sqrt{t}}$ is never negative

$\therefore \frac{dA}{dt} > 0$ for all $5 \leqslant t \leqslant 18$.

This means that the ability to understand spatial concepts increases with age.

c $\dfrac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}}$

$\therefore \dfrac{d^2A}{dt^2} = -\frac{1}{12}t^{-\frac{3}{2}} = -\dfrac{1}{12t\sqrt{t}}$

$\therefore \dfrac{d^2A}{dt^2} < 0$ for all $5 \leqslant t \leqslant 18$.

This means that while the ability to understand spatial concepts increases with age, the rate of increase slows down with age.

You are encouraged to use technology to graph each function you need to consider. This is often useful in interpreting results.

GRAPHING PACKAGE

EXERCISE 14D

1 The estimated future profits of a small business are given by $P(t) = 2t^2 - 12t + 118$ thousand dollars, where t is the time in years from now.

- **a** What is the current annual profit?
- **b** Find $\dfrac{dP}{dt}$ and state its units.
- **c** Explain the significance of $\dfrac{dP}{dt}$.
- **d** For what values of t will the profit:
  - **i** decrease **ii** increase on the previous year?
- **e** What is the minimum profit and when does it occur?
- **f** Find $\dfrac{dP}{dt}$ when $t = 4$, 10, and 25. What do these figures represent?

2 The quantity of a chemical in human skin which is responsible for its 'elasticity' is given by $Q = 100 - 10\sqrt{t}$ where t is the age of a person in years.

- **a** Find Q at:
  - **i** $t = 0$ **ii** $t = 25$ **iii** $t = 100$ years.
- **b** At what rate is the quantity of the chemical changing at the age of:
  - **i** 25 years **ii** 50 years?
- **c** Show that the quantity of the chemical is decreasing for all $t > 0$.

3 The height of *pinus radiata*, grown in ideal conditions, is given by $H = 20 - \dfrac{97.5}{t+5}$ metres, where t is the number of years after the tree was planted from an established seedling.

- **a** How high was the tree at the time of its planting?
- **b** Find the height of the tree after 4, 8, and 12 years.
- **c** Find the rate at which the tree is growing after 0, 5, and 10 years.
- **d** Show that $\dfrac{dH}{dt} > 0$ for all $t \geqslant 0$.

  What is the significance of this result?

Example 16

Self Tutor

The cost in dollars of producing x items in a factory each day is given by

$$C(x) = \underbrace{0.000\,13x^3 + 0.002x^2}_{\text{labour}} + \underbrace{5x}_{\text{raw materials}} + \underbrace{2200}_{\text{fixed costs}}$$

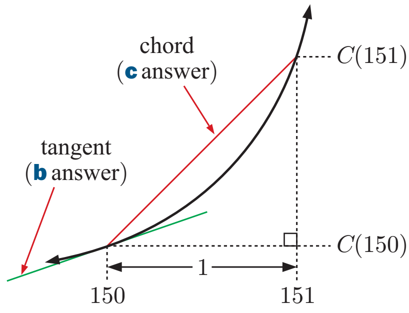
a Find $C'(x)$, which is called the marginal cost function.

b Find the marginal cost when 150 items are produced. Interpret this result.

c Find $C(151) - C(150)$. Compare this with the answer in **b**.

a The marginal cost function is
$C'(x) = 0.000\,39x^2 + 0.004x + 5$ dollars per item.

b $C'(150) = \$14.38$
This is the rate at which the costs are increasing with respect to the production level x when 150 items are made per day.
It gives an estimate of the cost of making the 151st item each day.



c $C(151) - C(150) \approx \$3448.19 - \$3433.75$
$\approx \$14.44$

This is the actual cost of making the 151st item each day, so the answer in **b** gives a good estimate.

4 Seablue make denim jeans. The cost model for making x pairs per day is

$$C(x) = 0.0003x^3 + 0.02x^2 + 4x + 2250 \text{ dollars.}$$

a Find the marginal cost function $C'(x)$.

b Find $C'(220)$. What does it estimate?

c Find $C(221) - C(220)$. What does this represent?

d Find $C''(x)$ and the value of x when $C''(x) = 0$. What is the significance of this point?

5

The total cost of running a train from Paris to Marseille is given by $C(v) = \frac{1}{5}v^2 + \frac{200\,000}{v}$ euros where v is the average speed of the train in km h^{-1}.

a Find the total cost of the journey if the average speed is:

i 50 km h^{-1} **ii** 100 km h^{-1}.

b Find the rate of change in the cost of running the train at speeds of:

i 30 km h^{-1} **ii** 90 km h^{-1}.

c At what speed will the cost be a minimum?

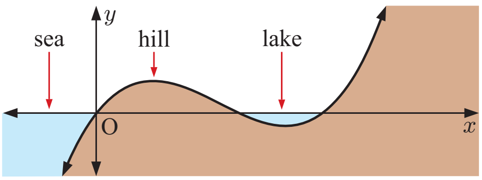
6 A tank contains 50 000 litres of water. The tap is left fully on and all the water drains from the tank in 80 minutes. The volume of water remaining in the tank after t minutes is given by $V = 50\,000\left(1 - \frac{t}{80}\right)^2$ litres where $0 \leqslant t \leqslant 80$.

a Find $\frac{dV}{dt}$ and draw the graph of $\frac{dV}{dt}$ against t.

b At what time was the outflow fastest?

c Show that $\frac{d^2V}{dt^2}$ is always constant and positive.
Interpret this result.

7



Alongside is a land and sea profile where the x-axis is sea level. The function $y = \frac{1}{10}x(x-2)(x-3)$ km gives the height of the land or sea bed relative to sea level at distance x km from the shore line.

a Find where the lake is located relative to the shore line of the sea.

b Find $\frac{dy}{dx}$ and interpret its value when $x = \frac{1}{2}$ km and when $x = 1\frac{1}{2}$ km.

c Find the deepest point of the lake, and the depth at this point.

8 A radioactive substance decays according to the formula $W = 20e^{-kt}$ grams where t is the time in hours.

a Find k given that after 50 hours the weight is 10 grams.

b Find the weight of radioactive substance present:
i initially **ii** after 24 hours **iii** after 1 week.

c How long will it take for the weight to reach 1 gram?

d Find the rate of radioactive decay at: **i** $t = 100$ hours **ii** $t = 1000$ hours.

e Show that $\frac{dW}{dt}$ is proportional to the weight of substance remaining.

9 The temperature of a liquid after being placed in a refrigerator is given by $T = 5 + 95e^{-kt}$ °C where k is a positive constant and t is the time in minutes.

a Find k if the temperature of the liquid is 20°C after 15 minutes.

b What was the temperature of the liquid when it was first placed in the refrigerator?

c Show that $\frac{dT}{dt} = c(T-5)$ for some constant c. Find the value of c.

d At what rate is the temperature changing at:
i $t = 0$ mins **ii** $t = 10$ mins **iii** $t = 20$ mins?

10 The height of a shrub t years after it is planted is given by $H(t) = 20\ln(3t+2) + 30$ cm, $t \geqslant 0$.

a How high was the shrub when it was planted?

b How long will it take for the shrub to reach a height of 1 m?

c At what rate is the shrub's height changing:
i 3 years after being planted **ii** 10 years after being planted?

11 In the conversion of sugar solution to alcohol, the chemical reaction obeys the law $A = s(1 - e^{-kt})$, $t \geqslant 0$ where t is the number of hours after the reaction commences, s is the original sugar concentration (%), and A is the alcohol produced, in litres.

a Find A when $t = 0$.

b Suppose $s = 10$ and $A = 5$ after 3 hours.

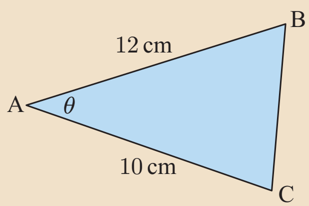
i Find k. **ii** Find the speed of the reaction at time 5 hours.

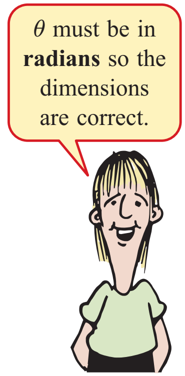
c Show that the speed of the reaction is proportional to $A - s$.

Example 17

Self Tutor

Find the rate of change in the area of triangle ABC as θ changes, at the time when $\theta = 60°$.





Area $A = \frac{1}{2} \times 10 \times 12 \times \sin\theta$ $\quad \{\text{Area} = \frac{1}{2}bc\sin A\}$

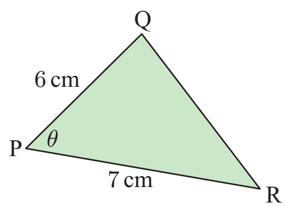
$\therefore\ A = 60\sin\theta$ cm^2

$\therefore\ \dfrac{dA}{d\theta} = 60\cos\theta$

When $\theta = \frac{\pi}{3}$, $\cos\theta = \frac{1}{2}$

$\therefore\ \dfrac{dA}{d\theta} = 30$ cm^2 per radian

12 Find the rate of change in the area of triangle PQR as θ changes, at the time when $\theta = 45°$.

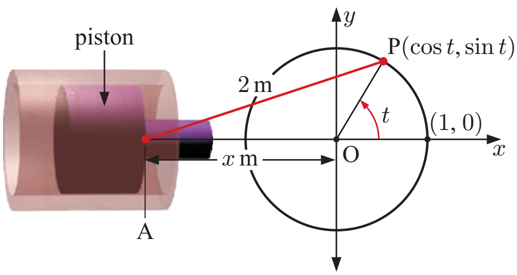


13 On the Indonesian coast, the depth of water at time t hours after midnight is given by $d = 9.3 + 6.8\cos(0.507t)$ metres.

a Find the rate of change in the depth of water at 8:00 am.

b Is the tide rising or falling at this time?

14 A piston is operated by rod [AP] attached to a flywheel of radius 1 m. AP = 2 m. P has coordinates $(\cos t, \sin t)$ and point A is $(-x, 0)$.



a Show that $x = \sqrt{4 - \sin^2 t} - \cos t$.

b Find the rate at which x is changing at the instant when:

i $t = 0$ **ii** $t = \frac{\pi}{2}$ **iii** $t = \frac{2\pi}{3}$

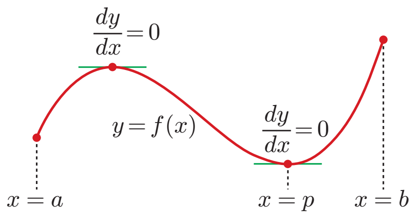
E OPTIMISATION

There are many problems for which we need to find the **maximum** or **minimum** value of a function. The solution is often referred to as the **optimum** solution and the process is called **optimisation**.

The maximum or minimum value does not always occur when the first derivative is zero.

It is essential to also examine the values of the function at the endpoint(s) of the interval under consideration for global maxima and minima.

For example:



The maximum value of y occurs at the endpoint $x=b$.

The minimum value of y occurs at the local minimum $x=p$.

OPTIMISATION PROBLEM SOLVING METHOD

Step 1: Draw a large, clear diagram of the situation.

Step 2: Construct a formula with the variable to be **optimised** as the subject. It should be written in terms of **one** convenient variable, for example x. You should write down what domain restrictions there are on x.

Step 3: Find the **first derivative** and find the values of x which make the first derivative **zero**.

Step 4: For each stationary point, use a sign diagram to determine if you have a local maximum or local minimum.

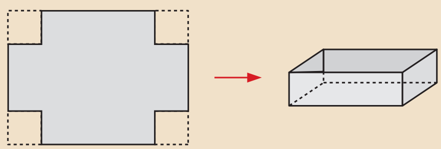
Step 5: Identify the optimum solution, also considering endpoints where appropriate.

Step 6: Write your answer in a sentence, making sure you specifically answer the question.

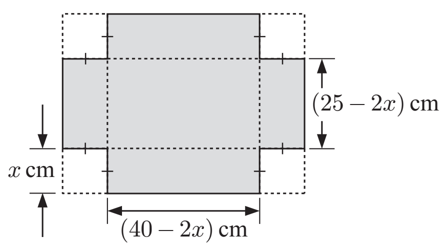
Example 18 — Self Tutor

A rectangular cake dish is made by cutting out squares from the corners of a 25 cm by 40 cm rectangle of tin-plate, and then folding the metal to form the container.

What size squares must be cut out to produce the cake dish of maximum volume?



Step 1: Let x cm be the side lengths of the squares that are cut out.



Step 2: Volume = length × width × depth

$$\begin{aligned}\text{Volume} &= (40-2x)(25-2x)x \\ &= (1000-80x-50x+4x^2)x \\ &= 1000x-130x^2+4x^3 \ \text{cm}^3\end{aligned}$$

Since the side lengths must be positive, $x>0$ and $25-2x>0$.

$\therefore$ $0<x<12.5$

Step 3: $\dfrac{dV}{dx} = 12x^2 - 260x + 1000$

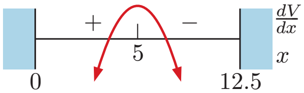
$= 4(3x^2 - 65x + 250)$

$= 4(3x - 50)(x - 5)$

$\therefore\ \dfrac{dV}{dx} = 0$ when $x = \frac{50}{3} = 16\frac{2}{3}$ or $x = 5$

DEMO

Step 4: $\dfrac{dV}{dx}$ has sign diagram:



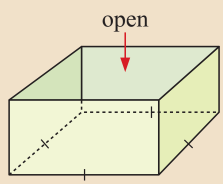
Step 5: There is a local maximum when $x = 5$. This is the global maximum for the given domain.

Step 6: The maximum volume is obtained when $x = 5$, which is when 5 cm squares are cut from the corners.

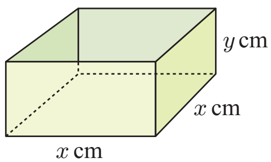
Example 19

Self Tutor

A 4 litre container must have a square base, vertical sides, and an open top. Find the most economical shape which minimises the surface area of material needed.



Step 1:



Let the base lengths be x cm and the depth be y cm.

The volume $V = \text{length} \times \text{width} \times \text{depth}$

$\therefore\ V = x^2y$

$\therefore\ 4000 = x^2y$ .... (1) {1 litre ≡ 1000 cm^3}

Step 2: The total surface area $A = \text{area of base} + 4(\text{area of one side})$

$= x^2 + 4xy$

$= x^2 + 4x\left(\dfrac{4000}{x^2}\right)$ {using (1)}

$\therefore\ A(x) = x^2 + 16\,000x^{-1}$ where $x > 0$

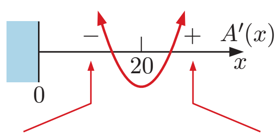
Step 3: $A'(x) = 2x - 16\,000x^{-2}$

$\therefore\ A'(x) = 0$ when $2x = \dfrac{16\,000}{x^2}$

$\therefore\ 2x^3 = 16\,000$

$\therefore\ x = \sqrt[3]{8000} = 20$

Step 4: $A'(x)$ has sign diagram:



If $x = 10$,

$A'(10) = 20 - \frac{16\,000}{100}$

$= 20 - 160$

$= -140$

If $x = 30$,

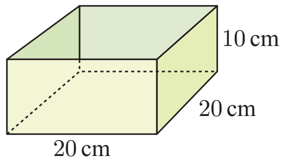
$A'(30) = 60 - \frac{16\,000}{900}$

$\approx 60 - 17.8$

≈ 42.2

Step 5: The minimum material is used to make the container when $x = 20$ and $y = \frac{4000}{20^2} = 10$.

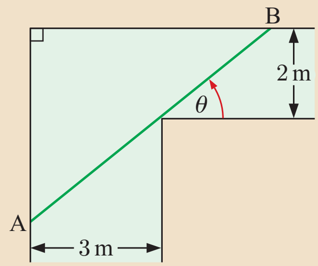
Step 6: The most economical shape has a square base $20 \text{ cm} \times 20 \text{ cm}$, and height 10 cm.



Example 20

Self Tutor

Two corridors meet at right angles and are 2 m and 3 m wide respectively. θ is the angle marked on the given figure. [AB] is a thin metal tube which must be kept horizontal and cannot be bent as it moves around the corner from one corridor to the other.



a Show that the length AB is given by $L = \dfrac{3}{\cos\theta} + \dfrac{2}{\sin\theta}$.

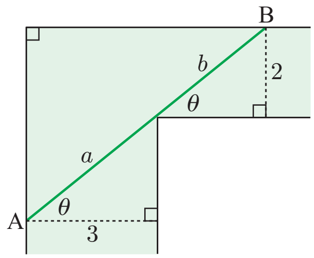
b Show that $\dfrac{dL}{d\theta} = 0$ when $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right) \approx 41.1°$.

c Find L when $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$ and comment on the significance of this value.

a $\cos\theta = \dfrac{3}{a}$ and $\sin\theta = \dfrac{2}{b}$

$\therefore\ a = \dfrac{3}{\cos\theta}$ and $b = \dfrac{2}{\sin\theta}$

$\therefore\ L = a + b = \dfrac{3}{\cos\theta} + \dfrac{2}{\sin\theta}$



DEMO

b $L = 3[\cos\theta]^{-1} + 2[\sin\theta]^{-1}$

$$\therefore\ \frac{dL}{d\theta} = -3[\cos\theta]^{-2}(-\sin\theta) - 2[\sin\theta]^{-2}\cos\theta$$
$$= \frac{3\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta}$$
$$= \frac{3\sin^3\theta - 2\cos^3\theta}{\cos^2\theta\sin^2\theta}$$

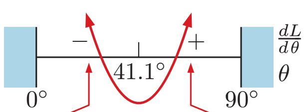
Thus $\dfrac{dL}{d\theta} = 0$ when $3\sin^3\theta = 2\cos^3\theta$

$\therefore\ \tan^3\theta = \frac{2}{3}$

$\therefore\ \tan\theta = \sqrt[3]{\frac{2}{3}}$

$\therefore\ \theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right) \approx 41.1°$

c Sign diagram of $\dfrac{dL}{d\theta}$:



When $\theta = 30°$, $\dfrac{dL}{d\theta} \approx -4.93 < 0$

When $\theta = 60°$, $\dfrac{dL}{d\theta} \approx 9.06 > 0$

Thus, AB is minimised when $\theta \approx 41.1°$. At this time $L \approx 7.02$ metres. Ignoring the width of the rod, the greatest length of rod able to be horizontally carried around the corner is 7.02 m.

Use **calculus techniques** to answer the following problems.

GRAPHING PACKAGE

In cases where finding the zeros of the derivatives is difficult you may use the **graphing package** to help you.

EXERCISE 14E

1 When a manufacturer makes x items per day, the cost function is $C(x) = 720 + 4x + 0.02x^2$ dollars, and the price function is $p(x) = 15 - 0.002x$ dollars per item. Find the production level that will maximise profits.

2 A duck farmer wishes to build a rectangular enclosure of area 100 m^2. The farmer must purchase wire netting for three of the sides, as the fourth side is an existing fence. Naturally, the farmer wishes to minimise the length (and therefore cost) of fencing required to complete the job.

a If the shorter sides have length x m, show that the required length of wire netting to be purchased is $L = 2x + \dfrac{100}{x}$.

b Find the minimum value of L and the corresponding value of x when this occurs.

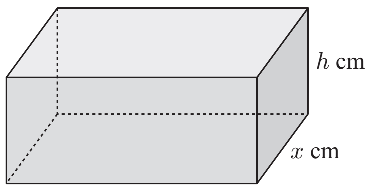
c Sketch the optimum situation, showing all dimensions.

3 The total cost of producing x blankets per day is $\frac{1}{4}x^2 + 8x + 20$ dollars, and for this production level each blanket may be sold for $(23 - \frac{1}{2}x)$ dollars.
How many blankets should be produced per day to maximise the total profit?

4 The cost of running a boat is $\left(\dfrac{v^2}{10} + 22\right)$ dollars per hour, where v km h^{-1} is the speed of the boat.
Find the speed which will minimise the total cost per kilometre.

5 A psychologist claims that the ability A to memorise simple facts during infancy years can be calculated using the formula $A(t) = t\ln t + 1$ where $0 < t \leqslant 5$, t being the age of the child in years. At what age is the child's memorising ability a minimum?

6 Radioactive waste is to be disposed of in fully enclosed lead boxes of inner volume 200 cm^3. The base of the box has dimensions in the ratio $2 : 1$.



a Show that $x^2 h = 100$.

b Show that the inner surface area of the box is given by $A(x) = 4x^2 + \dfrac{600}{x}$ cm^2.

c Find the minimum inner surface area of the box and the corresponding value of x.

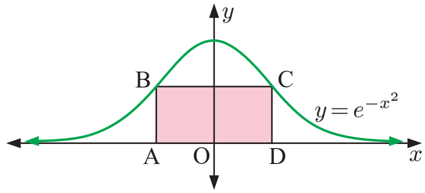
d Sketch the optimum box shape, showing all dimensions.

7 A manufacturer of electric kettles performs a cost control study. They discover that to produce x kettles per day, the cost per kettle is given by

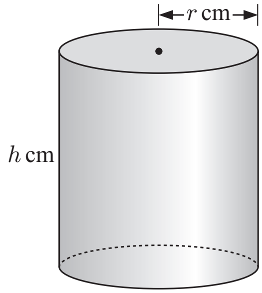
$$C(x) = 4\ln x + \left(\frac{30 - x}{10}\right)^2 \quad \text{dollars}$$

with a minimum production capacity of 10 kettles per day.
How many kettles should be manufactured to keep the cost per kettle to a minimum?

8 Infinitely many rectangles which sit on the x-axis can be inscribed under the curve $y = e^{-x^2}$.
Determine the coordinates of C such that rectangle ABCD has maximum area.



9 Consider the manufacture of cylindrical tin cans of 1 L capacity, where the cost of the metal used is to be minimised.

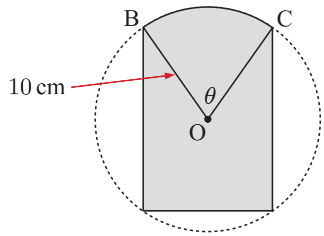


a Explain why the height h is given by $h = \dfrac{1000}{\pi r^2}$ cm.

b Show that the total surface area A is given by
$A = 2\pi r^2 + \dfrac{2000}{r}$ cm^2.

c Find the dimensions of the can which make A as small as possible.

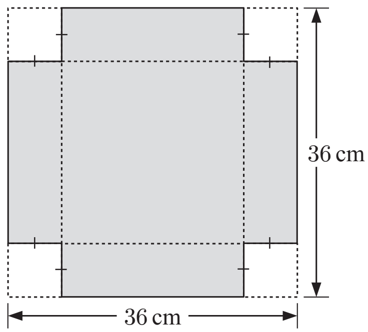
10 A circular piece of tinplate of radius 10 cm has 3 segments removed as illustrated. The angle θ is measured in radians.



a Show that the remaining area is given by
$A = 50(\theta + 3\sin\theta)$ cm^2.

b Find θ such that the area A is a maximum, and find the area A in this case.

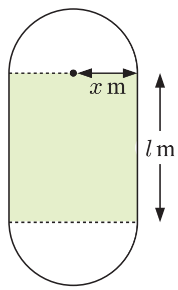
11 Sam has sheets of metal which are 36 cm by 36 cm square. He wants to cut out identical squares which are x cm by x cm from the corners of each sheet. He will then bend the sheets along the dashed lines to form an open container.



a Show that the volume of the container is given by
$V(x) = x(36 - 2x)^2$ cm^3.

b What sized squares should be cut out to produce the container of greatest capacity?

12 An athletics track has two 'straights' of length l m, and two semicircular ends of radius x m. The perimeter of the track is 400 m.



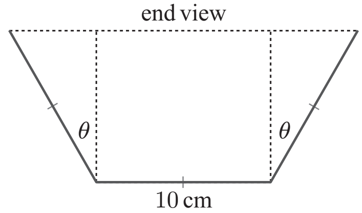
a Show that $l = 200 - \pi x$ and write down the possible values that x may have.

b What values of l and x maximise the shaded rectangle inside the track? What is this maximum area?

13 A small population of wasps is observed. After t weeks the population is modelled by $P(t) = \dfrac{50\,000}{1 + 1000e^{-0.5t}}$ wasps, where $0 \leqslant t \leqslant 25$.
Find when the wasp population is growing fastest.

14 When a new pain killing injection is administered, the effect is modelled by $E(t) = 750te^{-1.5t}$ units, where $t \geqslant 0$ is the time in hours after the injection.
At what time is the drug most effective?

15 A symmetrical gutter is made from a sheet of metal 30 cm wide by bending it twice as shown.

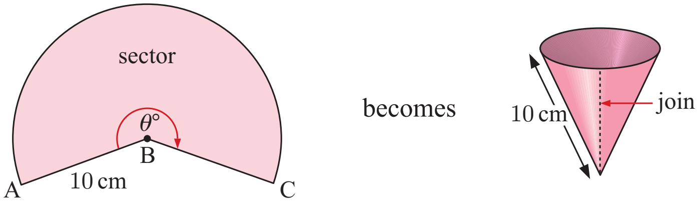


a Deduce that the cross-sectional area of the gutter is given by $A = 100\cos\theta(1 + \sin\theta)$.

b Show that $\dfrac{dA}{d\theta} = 0$ when $\sin\theta = \frac{1}{2}$ or -1.

c For what value of θ does the gutter have maximum carrying capacity? Find the cross-sectional area for this value of θ.

16 A sector of radius 10 cm and angle $\theta°$ is bent to form a conical cup as shown.



when edges [AB] and [CB] are joined with tape.

Suppose the resulting cone has base radius r cm and height h cm.

a Show using the sector that arc AC $= \dfrac{\theta\pi}{18}$.

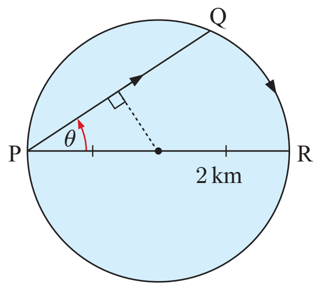
b Explain why $r = \dfrac{\theta}{36}$.

c Show that $h = \sqrt{100 - \left(\frac{\theta}{36}\right)^2}$.

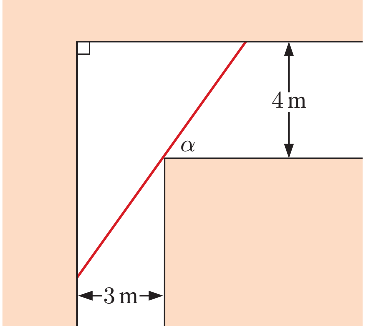
d Find the cone's capacity V in terms of θ only.

e Find θ when $V(\theta)$ is a maximum.

17 Hieu can row a boat across a circular lake of radius 2 km at 3 km h^{-1}. He can walk around the edge of the lake at 5 km h^{-1}.
What is the longest possible time Hieu could take to get from P to R by rowing from P to Q and then walking from Q to R?



18 In a hospital, two corridors 4 m wide and 3 m wide meet at right angles. What is the maximum possible length of an X-ray screen which can be carried upright around the corner?

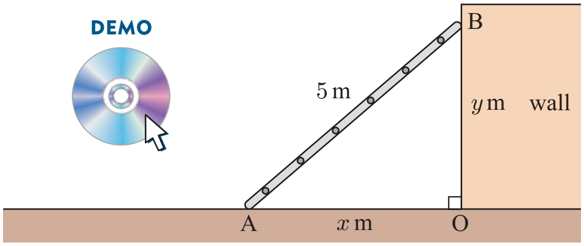


F RELATED RATES

A 5 m ladder rests against a vertical wall at point B. Its feet are at point A on horizontal ground.

The ladder slips and slides down the wall.

Click on the icon to view the motion of the sliding ladder.

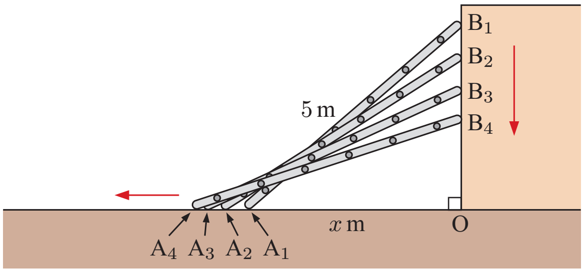


The following diagram shows the positions of the ladder at certain instances.

If AO $= x$ m and OB $= y$ m,
then $x^2 + y^2 = 5^2$. {Pythagoras}

Differentiating this equation with respect to time t gives $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$

or $x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = 0$.



This equation is called a **differential equation** and describes the motion of the ladder at any instant.

$\dfrac{dx}{dt}$ is the rate of change in x with respect to time t, and is the speed of A relative to point O.

$\dfrac{dx}{dt}$ is *positive* as x is increasing.

$\dfrac{dy}{dt}$ is the rate of change in y with respect to time t, and is the speed at which B moves downwards.

$\dfrac{dy}{dt}$ is *negative* as y is decreasing.

Problems involving differential equations where one of the variables is time t are called **related rates** problems.

The method for solving related rates problems is:

Step 1: Draw a large, clear **diagram** of the situation. Sometimes two or more diagrams are necessary.

Step 2: Write down the information, label the diagram(s), and make sure you distinguish between the **variables** and the **constants**.

Step 3: Write an **equation** connecting the variables. You will often need to use:
- Pythagoras' theorem
- similar triangles
- right angled triangle trigonometry
- sine and cosine rules.

Step 4: **Differentiate** the equation with respect to t to obtain a **differential equation**.

Step 5: Solve for the **particular case** which is some instant in time.

Warning:

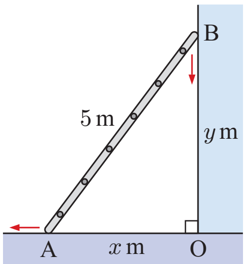
We **must not** substitute values for the particular case too early. Otherwise we will incorrectly treat variables as constants. The differential equation in fully generalised form must be established first.

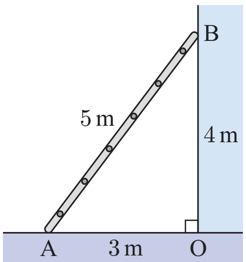
Example 21

Self Tutor

A 5 m long ladder rests against a vertical wall with its feet on horizontal ground. The feet on the ground slip, and at the instant when they are 3 m from the wall, they are moving at 10 m s^{-1}.

At what speed is the other end of the ladder moving at this instant?





Let OA $= x$ m and OB $= y$ m

$\therefore\quad x^2 + y^2 = 5^2 \qquad$ {Pythagoras}

Differentiating with respect to t gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\therefore\quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Particular case:

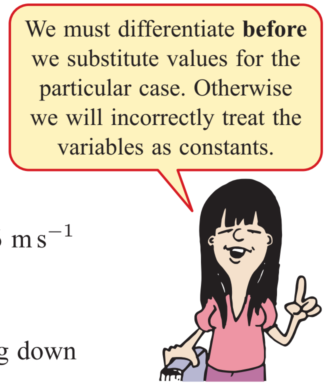
At the instant when $\dfrac{dx}{dt} = 10$ m s^{-1},

$$\therefore\quad 3(10) + 4\frac{dy}{dt} = 0$$

$$\therefore\quad \frac{dy}{dt} = -\tfrac{15}{2} = -7.5 \text{ m s}^{-1}$$

Thus OB is decreasing at 7.5 m s^{-1}.

$\therefore$ the other end of the ladder is moving down the wall at 7.5 m s^{-1} at that instant.



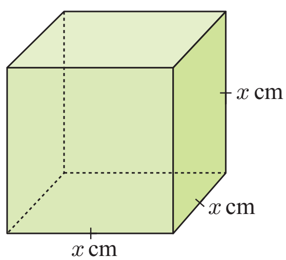
Example 22

Self Tutor

A cube is expanding so its volume increases at a constant rate of 10 cm^3 s^{-1}. Find the rate of change in its total surface area, at the instant when its sides are 20 cm long.

Let x cm be the lengths of the sides of the cube, so the surface area $A = 6x^2$ cm^2 and the volume $V = x^3$ cm^3.

$$\therefore\quad \frac{dA}{dt} = 12x\frac{dx}{dt} \quad\text{and}\quad \frac{dV}{dt} = 3x^2\frac{dx}{dt}$$



Particular case:

At the instant when $x = 20$, $\dfrac{dV}{dt} = 10$

$$\therefore\quad 10 = 3 \times 20^2 \times \frac{dx}{dt}$$

$$\therefore\quad \frac{dx}{dt} = \frac{10}{1200} = \frac{1}{120} \text{ cm s}^{-1}$$

Thus $\dfrac{dA}{dt} = 12 \times 20 \times \frac{1}{120}$ cm^2 s^{-1}

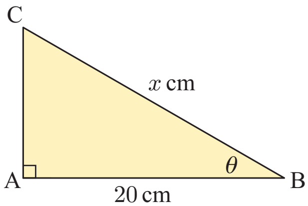
$= 2$ cm^2 s^{-1}

$\therefore$ the surface area is increasing at 2 cm^2 s^{-1}.

Example 23

Self Tutor

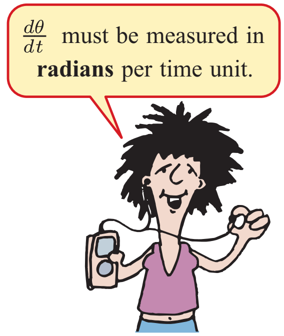
Triangle ABC is right angled at A, and $AB = 20$ cm. $A\widehat{B}C$ increases at a constant rate of $1°$ per minute. At what rate is BC changing at the instant when $A\widehat{B}C$ measures $30°$?



Let $A\widehat{B}C = \theta$ and $BC = x$ cm

Now $\cos\theta = \frac{20}{x} = 20x^{-1}$

$\therefore\ -\sin\theta\,\frac{d\theta}{dt} = -20x^{-2}\,\frac{dx}{dt}$

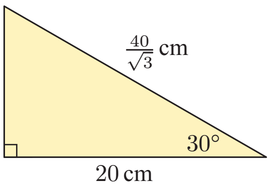


Particular case:

When $\theta = 30°$, $\cos 30° = \frac{20}{x}$

$\therefore\ \frac{\sqrt{3}}{2} = \frac{20}{x}$

$\therefore\ x = \frac{40}{\sqrt{3}}$



Also, $\frac{d\theta}{dt} = 1°$ per min

$= \frac{\pi}{180}$ radians per min

Thus $-\sin 30° \times \frac{\pi}{180} = -20 \times \frac{3}{1600} \times \frac{dx}{dt}$

$\therefore\ -\frac{1}{2} \times \frac{\pi}{180} = -\frac{3}{80}\,\frac{dx}{dt}$

$\therefore\ \frac{dx}{dt} = \frac{\pi}{360} \times \frac{80}{3}$ cm per min

≈ 0.2327 cm per min

$\therefore$ BC is increasing at approximately 0.233 cm per min.

EXERCISE 14F

1 a and b are variables related by the equation $ab^3 = 40$. At the instant when $a = 5$, b is increasing at 1 unit per second. What is happening to a at this instant?

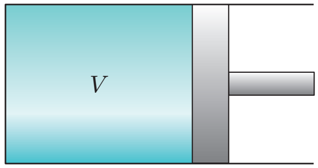
2 The length of a rectangle is decreasing at 1 cm per minute. However, the area of the rectangle remains constant at 100 cm^2. At what rate is the breadth increasing at the instant when the rectangle is a square?

3 A stone is thrown into a lake and a circular ripple moves out at a constant speed of 1 m s^{-1}. Find the rate at which the circle's area is increasing at the instant when:

a $t = 2$ seconds **b** $t = 4$ seconds.

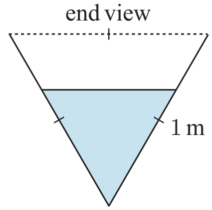
4 Air is pumped into a spherical weather balloon at a constant rate of 6π m^3 per minute. Find the rate of change in its surface area at the instant when the radius of the balloon is 2 m.

5 For a given mass of gas in a piston, $pV^{1.5} = 400$ where p is the pressure in N m^{-2}, and V is the volume in m^3.
Suppose the pressure increases at a constant rate of 3 N m^{-2} per minute. Find the rate at which the volume is changing at the instant when the pressure is 50 N m^{-2}.



6 Wheat runs from a hole in a silo at a constant rate and forms a conical heap whose base radius is treble its height. After 1 minute, the height of the heap is 20 cm. Find the rate at which the height is rising at this instant.

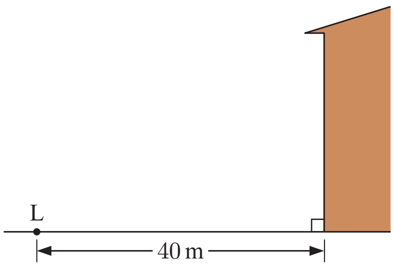
7 A trough of length 6 m has a uniform cross-section which is an equilateral triangle with sides of length 1 m. Water leaks from the bottom of the trough at a constant rate of 0.1 m^3 per min.
Find the rate at which the water level is falling at the instant when the water is 20 cm deep.



8 Two jet aeroplanes fly on parallel courses which are 12 km apart. Their air speeds are 200 m s^{-1} and 250 m s^{-1} respectively. How fast is the distance between them changing at the instant when the slower jet is 5 km ahead of the faster one?

9 A ground-level floodlight located 40 m from the foot of a building shines in the direction of the building.
A 2 m tall person walks directly from the floodlight towards the building at 1 m s^{-1}. How fast is the person's shadow on the building shortening at the instant when the person is:

a 20 m from the building
b 10 m from the building?



10 A right angled triangle ABC has a fixed hypotenuse [AC] of length 10 cm, and side [AB] increases in length at 0.1 cm s^{-1}. At what rate is $\widehat{CAB}$ decreasing at the instant when the triangle is isosceles?

11 Triangle PQR is right angled at Q, and [PQ] is 6 cm long. [QR] increases in length at 2 cm per minute. Find the rate of change in $\widehat{QPR}$ at the instant when [QR] is 8 cm long.

Review set 14A

1 Find the equation of the tangent to:

a $y = -2x^2$ at the point where $x = -1$

b $f(x) = 4\ln(2x)$ at the point $(1, 4\ln 2)$

c $f(x) = \dfrac{e^x}{x-1}$ at the point where $x = 2$.

2 The tangent to $y = \dfrac{ax+b}{\sqrt{x}}$ at $x = 1$ is $2x - y = 1$. Find a and b.

3 Suppose $f(x) = x^3 + ax$, $a < 0$ has a turning point when $x = \sqrt{2}$.

a Find a.
b Find the position and nature of all stationary points of $y = f(x)$.
c Sketch the graph of $y = f(x)$.

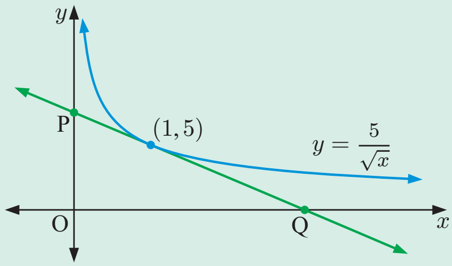
4 Find the equation of the normal to:

a $y = \dfrac{x+1}{x^2-2}$ at the point where $x = 1$ **b** $\sqrt{x+1}$ at the point where $x = 3$.

5 The tangent to $y = x^2\sqrt{1-x}$ at $x = -3$ cuts the axes at points A and B.
Determine the area of triangle OAB.

6 The line through A(2, 4) and B(0, 8) is a tangent to $y = \dfrac{a}{(x+2)^2}$. Find a.

7 Find the coordinates of P and Q if PQ is the tangent to $y = \dfrac{5}{\sqrt{x}}$ at (1, 5).



8 Show that $y = 2 - \dfrac{7}{1+2x}$ has no horizontal tangents.

9 Show that the curves whose equations are $y = \sqrt{3x+1}$ and $y = \sqrt{5x - x^2}$ have a common tangent at their point of intersection. Find the equation of this common tangent.

10 Consider the function $f(x) = x + \ln x$.

a Find the values of x for which $f(x)$ is defined.

b Find the sign of $f'(x)$ and comment on its geometrical significance.

c Sketch the graph of $y = f(x)$.

d Find the equation of the normal at the point where $x = 1$.

11 **a** Sketch the graph of $x \mapsto \dfrac{4}{x}$ for $x > 0$.

b Find the equation of the tangent to the function at the point where $x = k$, $k > 0$.

c If the tangent in **b** cuts the x-axis at A and the y-axis at B, find the coordinates of A and B.

d What can be deduced about the area of triangle OAB?

e Find k if the normal to the curve at $x = k$ passes through the point (1, 1).

12 A particle P moves in a straight line with position relative to the origin O given by $s(t) = 2t^3 - 9t^2 + 12t - 5$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find expressions for the particle's velocity and acceleration and draw sign diagrams for each of them.

b Find the initial conditions.

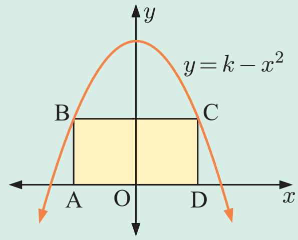
c Describe the motion of the particle at time $t = 2$ seconds.

d Find the times and positions where the particle changes direction.

e Draw a diagram to illustrate the motion of P.

f Determine the time intervals when the particle's speed is increasing.

13 Rectangle ABCD is inscribed within the parabola $y = k - x^2$ and the x-axis, as shown.



a If $\text{OD} = x$, show that the rectangle ABCD has area function $A(x) = 2kx - 2x^3$.

b If the area of ABCD is a maximum when $\text{AD} = 2\sqrt{3}$, find k.

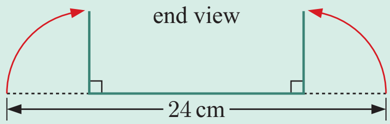
14 A particle moves in a straight line along the x-axis with position given by $x(t) = 3 + \sin(2t)$ cm after t seconds.

a Find the initial position, velocity, and acceleration of the particle.

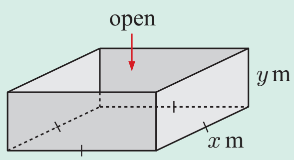
b Find the times when the particle changes direction during $0 \leqslant t \leqslant \pi$ seconds.

c Find the total distance travelled by the particle in the first π seconds.

15 A rectangular gutter is formed by bending a 24 cm wide sheet of metal as shown.
Where must the bends be made in order to maximise the capacity of the gutter?



16 A manufacturer of open steel boxes has to make one with a square base and a capacity of 1 m^3. The steel costs \$2 per square metre.



a If the base measures x m by x m and the height is y m, find y in terms of x.

b Hence, show that the total cost of the steel is $C(x) = 2x^2 + \dfrac{8}{x}$ dollars.

c Find the dimensions of the box which would cost the least in steel to make.

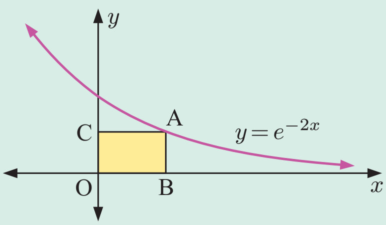
17 A particle P moves in a straight line with position from O given by $s(t) = 15t - \dfrac{60}{(t+1)^2}$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find velocity and acceleration functions for P's motion.

b Describe the motion of P at $t = 3$ seconds.

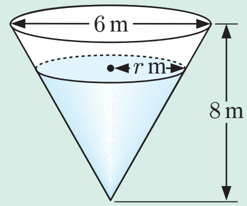
c For what values of t is the particle's speed increasing?

18 Infinitely many rectangles can be inscribed under the curve $y = e^{-2x}$ as shown.
Determine the coordinates of A such that the rectangle OBAC has maximum area.



19 A man on a jetty pulls a boat directly towards him so the rope is coming in at a rate of 20 metres per minute. The rope is attached to the boat 1 m above water level, and the man's hands are 6 m above the water level. How fast is the boat approaching the jetty at the instant when it is 15 m from the jetty?

20 Water exits a conical tank at a constant rate of 0.2 m^3 per minute. Suppose the surface of the water has radius r.



a Find $V(r)$, the volume of the water remaining in the tank.

b Find the rate at which the surface radius is changing at the instant when the water is 5 m deep.

Review set 14B

1 Find the equation of the normal to:

a $y = \dfrac{1-2x}{x^2}$ at the point where $x = 1$

b $y = e^{-x^2}$ at the point where $x = 1$

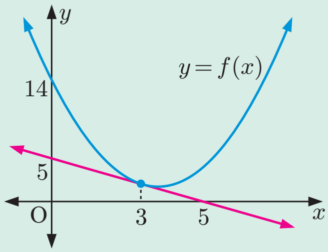
c $y = \dfrac{1}{\sqrt{x}}$ at the point where $x = 4$.

2 The curve $y = 2x^3 + ax + b$ has a tangent with gradient 10 at the point $(-2, 33)$. Find the values of a and b.

3 $y = f(x)$ is the parabola shown.

a Find $f(3)$ and $f'(3)$.

b Hence find $f(x)$ in the form $f(x) = ax^2 + bx + c$.



4 Find the equation of:

a the tangent to $y = \dfrac{1}{\sin x}$ at the point where $x = \frac{\pi}{3}$

b the normal to $y = \cos(\frac{x}{2})$ at the point where $x = \frac{\pi}{2}$.

5 At the point where $x = 0$, the tangent to $f(x) = e^{4x} + px + q$ has equation $y = 5x - 7$. Find p and q.

6 Find where the tangent to $y = 2x^3 + 4x - 1$ at $(1, 5)$ cuts the curve again.

7 Find a given that the tangent to $y = \dfrac{4}{(ax+1)^2}$ at $x = 0$ passes through $(1, 0)$.

8 Consider the function $f(x) = e^x - x$.

a Find and classify any stationary points of $y = f(x)$.

b Show that $e^x \geqslant x + 1$ for all x.

c Find $f''(x)$.

9 Find where the tangent to $y = \ln(x^4 + 3)$ at $x = 1$ cuts the y-axis.

10 Consider the function $f(x) = 2x^3 - 19x^2 + 52x - 35$.

a Find the y-intercept of the graph $y = f(x)$.

b Show that $x = 1$ is a root of the function, and hence find all roots.

c Find and classify all stationary points.

d Sketch the graph of $y = f(x)$, showing all important features.

11 If the normal to $f(x) = \dfrac{3x}{1+x}$ at $(2, 2)$ cuts the axes at B and C, determine the length BC.

12 The height of a tree t years after it was planted is given by $H(t) = 60 + 40\ln(2t+1)$ cm, $t \geqslant 0$.

a How high was the tree when it was planted?

b How long does it take for the tree to reach:

i 150 cm **ii** 300 cm?

c At what rate is the tree's height increasing after:

i 2 years **ii** 20 years?

13 A particle P moves in a straight line with position given by $s(t) = 80e^{-\frac{t}{10}} - 40t$ m where t is the time in seconds, $t \geqslant 0$.

a Find the velocity and acceleration functions.

b Find the initial position, velocity, and acceleration of P.

c Sketch the graph of the velocity function.

d Find the exact time when the velocity is -44 m s^{-1}.

14 The cost per hour of running a freight train is given by $C(v) = \dfrac{v^2}{30} + \dfrac{9000}{v}$ dollars where v is the average speed of the train in km h^{-1}.

a Find the cost of running the train for:

i two hours at 45 km h^{-1} **ii** 5 hours at 64 km h^{-1}.

b Find the rate of change in the hourly cost of running the train at speeds of:

i 50 km h^{-1} **ii** 66 km h^{-1}.

c At what speed will the cost per hour be a minimum?

15 A particle moves along the x-axis with position relative to origin O given by $x(t) = 3t - \sqrt{t+1}$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find expressions for the particle's velocity and acceleration at any time t, and draw sign diagrams for each function.

b Find the initial conditions, and hence describe the motion at that instant.

c Describe the motion of the particle at $t = 8$ seconds.

d Find the time and position when the particle reverses direction.

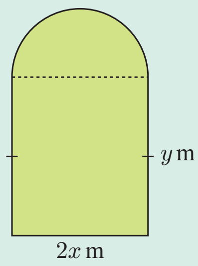
e Determine the time interval when the particle's speed is decreasing.

16 A 200 m fence is placed around a lawn which has the shape of a rectangle with a semi-circle on one of its sides.

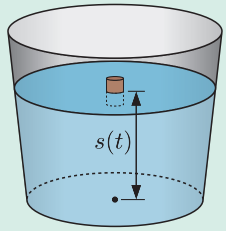
a Using the dimensions shown on the figure, show that $y = 100 - x - \frac{\pi}{2}x$.

b Find the area of the lawn A in terms of x only.

c Find the dimensions of the lawn if it has the maximum possible area.



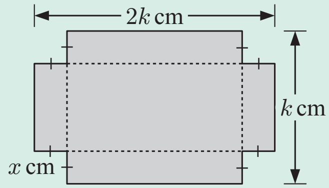
17 A cork bobs up and down in a bucket of water such that the distance from the centre of the cork to the bottom of the bucket is given by $s(t) = 30 + \cos(\pi t)$ cm, $t \geqslant 0$ seconds.



a Find the cork's velocity at times $t = 0, \frac{1}{2}, 1, 1\frac{1}{2}$, and 2 s.

b Find the time intervals when the cork is falling.

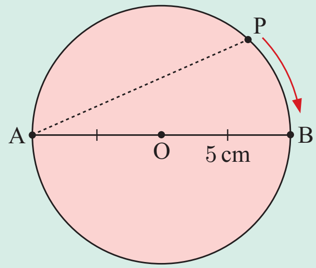
18 A rectangular sheet of tin-plate is $2k$ cm by k cm. Four squares, each with sides x cm, are cut from its corners. The remainder is bent into the shape of an open rectangular container. Find the value of x which will maximise the capacity of the container.



19 Two runners run in different directions, 60° apart. A runs at 5 m s^{-1} and B runs at 4 m s^{-1}. B passes through X 3 seconds after A passes through X.
At what rate is the distance between them increasing at the time when A is 20 metres past X?

20 Rectangle PQRS has PQ of fixed length 20 cm, and [QR] increases in length at a constant rate of 2 cm s^{-1}. At what rate is the acute angle between the diagonals of the rectangle changing at the instant when QR is 15 cm long?

21 AOB is a fixed diameter of a circle of radius 5 cm. Point P moves around the circle at a constant rate of 1 revolution in 10 seconds. Find the rate at which the distance AP is changing at the instant when:



a AP = 5 cm and increasing

b P is at B.

22 Consider the parabola $f(x) = \dfrac{1}{4b}x^2$ where $b > 0$.

a **i** Find the equation of the tangent to $y = f(x)$ at the point $P(a, f(a))$.

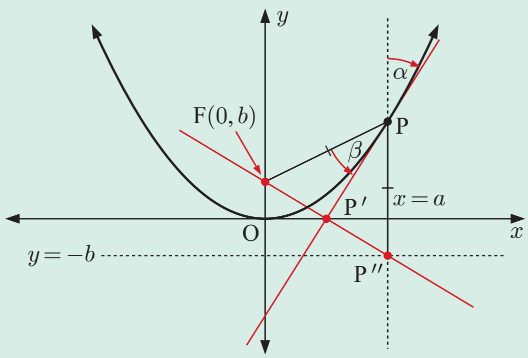
ii Show that this meets the x-axis at the point $P'(\frac{a}{2}, 0)$.

b **i** Find the equation of the line perpendicular to this tangent line, and which passes through P′.

ii Show that this line has y-intercept $F(0, b)$.

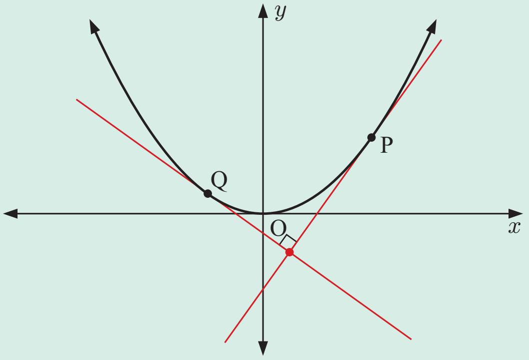
iii Show that the distance FP equals the distance from P to the line $y = -b$.

c The point $F(0, b)$ is *invariant* since it is independent of our choice of a. F is called the *focus* of the parabola. The line $y = -b$ is called the *directrix*.



i Prove the *reflective* property of the parabola, that any vertical ray will be reflected off the parabola into the focus.
Hint: Show that $\alpha = \beta$.

ii Suppose $a \neq 0$ and that $\mathrm{Q}(c,\ f(c))$ is another point on the parabola such that the tangents from P and Q are perpendicular. Show that the intersection of the tangents occurs on the directrix.



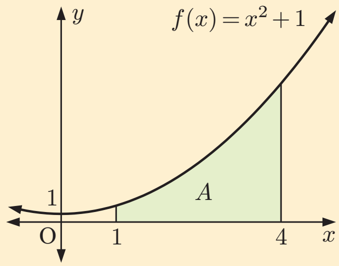
15

Integration

Contents:

Opening problem

The function $f(x) = x^2 + 1$ lies above the x-axis for all $x \in \mathbb{R}$.



Things to think about:

a How can we calculate the shaded area A, which is the area under the curve for $1 \leqslant x \leqslant 4$?

b What function has $x^2 + 1$ as its derivative?

In the previous chapters we used differential calculus to find the derivatives of many types of functions. We also used it in problem solving, in particular to find the gradients of graphs and rates of changes, and to solve optimisation problems.

In this chapter we consider **integral calculus**. This involves **antidifferentiation** which is the reverse process of differentiation. Integral calculus also has many useful applications, including:

- finding areas of shapes with curved boundaries
- finding volumes of revolution
- finding distances travelled from velocity functions
- solving problems in economics, biology, and statistics
- solving differential equations.

A THE AREA UNDER A CURVE

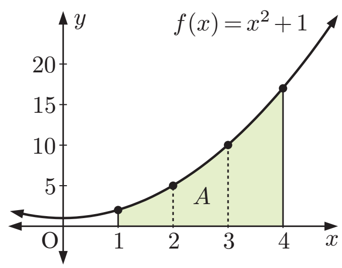
The task of finding the area under a curve has been important to mathematicians for thousands of years. In the history of mathematics it was fundamental to the development of integral calculus. We will therefore begin our study by calculating the area under a curve using the same methods as the ancient mathematicians.

UPPER AND LOWER RECTANGLES

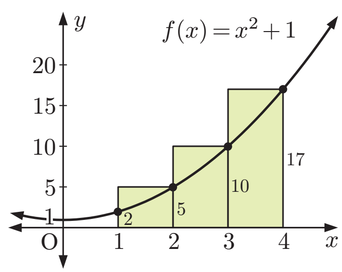
Consider the function $f(x) = x^2 + 1$.

We wish to estimate the area A enclosed by $y = f(x)$, the x-axis, and the vertical lines $x = 1$ and $x = 4$.

Suppose we divide the interval $1 \leqslant x \leqslant 4$ into three strips of width 1 unit as shown. We obtain three subintervals of equal width.



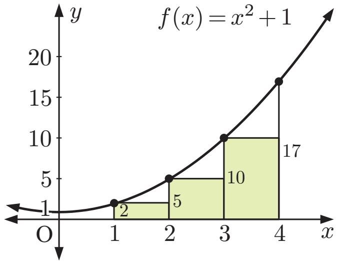
The diagram alongside shows **upper rectangles**, which are rectangles with top edges at the maximum value of the curve on that subinterval.



The area of the upper rectangles,

$A_U = 1 \times f(2) + 1 \times f(3) + 1 \times f(4)$
$\quad = 5 + 10 + 17$
$\quad = 32 \text{ units}^2$

The next diagram shows **lower rectangles**, which are rectangles with top edges at the minimum value of the curve on that subinterval.



The area of the lower rectangles,

$$\begin{aligned} A_L &= 1 \times f(1) + 1 \times f(2) + 1 \times f(3) \\ &= 2 + 5 + 10 \\ &= 17 \text{ units}^2 \end{aligned}$$

Now clearly $A_L < A < A_U$, so the area A lies between 17 units^2 and 32 units^2.

If the interval $1 \leqslant x \leqslant 4$ was divided into 6 subintervals, each of width $\frac{1}{2}$, then

$$\begin{aligned} A_U &= \tfrac{1}{2}f(1\tfrac{1}{2}) + \tfrac{1}{2}f(2) + \tfrac{1}{2}f(2\tfrac{1}{2}) + \tfrac{1}{2}f(3) + \tfrac{1}{2}f(3\tfrac{1}{2}) + \tfrac{1}{2}f(4) \\ &= \tfrac{1}{2}(\tfrac{13}{4} + 5 + \tfrac{29}{4} + 10 + \tfrac{53}{4} + 17) \\ &= 27.875 \text{ units}^2 \end{aligned}$$

$$\begin{aligned} \text{and} \quad A_L &= \tfrac{1}{2}f(1) + \tfrac{1}{2}f(1\tfrac{1}{2}) + \tfrac{1}{2}f(2) + \tfrac{1}{2}f(2\tfrac{1}{2}) + \tfrac{1}{2}f(3) + \tfrac{1}{2}f(3\tfrac{1}{2}) \\ &= \tfrac{1}{2}(2 + \tfrac{13}{4} + 5 + \tfrac{29}{4} + 10 + \tfrac{53}{4}) \\ &= 20.375 \text{ units}^2 \end{aligned}$$

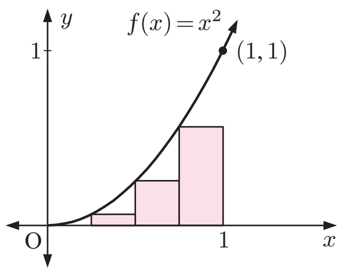
From this refinement we conclude that the area A lies between 20.375 and 27.875 units^2.

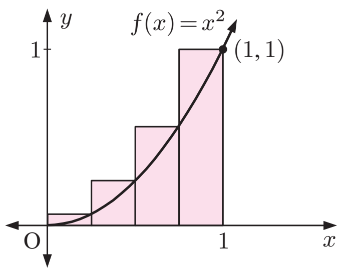
As we create more subintervals, the estimates A_L and A_U will become more and more accurate. In fact, as the subinterval width is reduced further and further, both A_L and A_U will **converge** to A.

We illustrate this process by estimating the area A between the graph of $y = x^2$ and the x-axis for $0 \leqslant x \leqslant 1$.

This example is of historical interest. **Archimedes** (287 - 212 BC) found the exact area. In an article that contains 24 propositions, he developed the essential theory for what is now known as integral calculus.

Consider $f(x) = x^2$ and divide the interval $0 \leqslant x \leqslant 1$ into 4 subintervals of equal width.





$$\begin{aligned} A_L &= \tfrac{1}{4}(0)^2 + \tfrac{1}{4}(\tfrac{1}{4})^2 + \tfrac{1}{4}(\tfrac{1}{2})^2 + \tfrac{1}{4}(\tfrac{3}{4})^2 \\ &\approx 0.219 \end{aligned}$$

and

$$\begin{aligned} A_U &= \tfrac{1}{4}(\tfrac{1}{4})^2 + \tfrac{1}{4}(\tfrac{1}{2})^2 + \tfrac{1}{4}(\tfrac{3}{4})^2 + \tfrac{1}{4}(1)^2 \\ &\approx 0.469 \end{aligned}$$

Now suppose there are n subintervals between $x = 0$ and $x = 1$, each of width $\dfrac{1}{n}$.

AREA FINDER

We can use the **area finder** software to help calculate A_L and A_U for large values of n.

The table alongside summarises the results you should obtain for $n = 4$, 10, 25, and 50.

The exact value of A is in fact $\frac{1}{3}$, as we will find later in the chapter. Notice how both A_L and A_U are converging to this value as n increases.

n	A_L	A_U	*Average*
4	0.218 75	0.468 75	0.343 75
10	0.285 00	0.385 00	0.335 00
25	0.313 60	0.353 60	0.333 60
50	0.323 40	0.343 40	0.333 40

EXERCISE 15A.1

1 Consider the area between $y = x$ and the x-axis from $x = 0$ to $x = 1$.

a Divide the interval into 5 strips of equal width, then estimate the area using:

i upper rectangles **ii** lower rectangles.

b Calculate the actual area and compare it with your answers in **a**.

2 Consider the area between $y = \dfrac{1}{x}$ and the x-axis from $x = 2$ to $x = 4$. Divide the interval into 6 strips of equal width, then estimate the area using:

a upper rectangles **b** lower rectangles.

3 Use rectangles to find lower and upper sums for the area between the graph of $y = x^2$ and the x-axis for $1 \leqslant x \leqslant 2$. Use $n = 10$, 25, 50, 100, and 500. Give your answers to 4 decimal places.
As n gets larger, both A_L and A_U converge to the same number which is a simple fraction. What is it?

AREA FINDER

4 **a** Use lower and upper sums to estimate the area between each of the following functions and the x-axis for $0 \leqslant x \leqslant 1$. Use values of $n = 5$, 10, 50, 100, 500, 1000, and 10 000. Give your answer to 5 decimal places in each case.

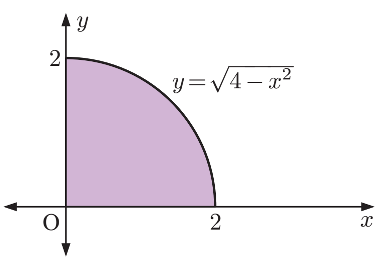
i $y = x^3$ **ii** $y = x$ **iii** $y = x^{\frac{1}{2}}$ **iv** $y = x^{\frac{1}{3}}$

b For each case in **a**, A_L and A_U converge to the same number which is a simple fraction. What fractions are they?

c Using your answer to **b**, predict the area between the graph of $y = x^a$ and the x-axis for $0 \leqslant x \leqslant 1$ and any number $a > 0$.

5 Consider the quarter circle of centre $(0, 0)$ and radius 2 units illustrated.

Its area is $\frac{1}{4}$(full circle of radius 2)
$= \frac{1}{4} \times \pi \times 2^2$
$= \pi$ units^2



a Estimate the area using lower and upper rectangles for $n = 10$, 50, 100, 200, 1000, and 10 000. Hence, find rational bounds for π.

b Archimedes found the famous approximation $3\frac{10}{71} < \pi < 3\frac{1}{7}$.
For what value of n is your estimate for π better than that of Archimedes?