

**Example 19****Self Tutor**

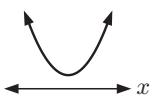
Use the discriminant to determine the relationship between the graph of each function and the  $x$ -axis:

**a**  $y = x^2 + 3x + 4$

$$\begin{aligned} \mathbf{a} \quad a &= 1, \quad b = 3, \quad c = 4 \\ \therefore \Delta &= b^2 - 4ac \\ &= 9 - 4(1)(4) \\ &= -7 \end{aligned}$$

Since  $\Delta < 0$ , the graph does not cut the  $x$ -axis.

Since  $a > 0$ , the graph is concave up.



The graph is positive definite, which means that it lies entirely above the  $x$ -axis.

**b**  $y = -2x^2 + 5x + 1$

$$\begin{aligned} \mathbf{b} \quad a &= -2, \quad b = 5, \quad c = 1 \\ \therefore \Delta &= b^2 - 4ac \\ &= 25 - 4(-2)(1) \\ &= 33 \end{aligned}$$

Since  $\Delta > 0$ , the graph cuts the  $x$ -axis twice.

Since  $a < 0$ , the graph is concave down.

**EXERCISE 3D.4**

- 1** Use the discriminant to determine the relationship between the graph and  $x$ -axis for:

**a**  $y = x^2 + x - 2$

**b**  $y = x^2 - 4x + 1$

**c**  $f(x) = -x^2 - 3$

**d**  $f(x) = x^2 + 7x - 2$

**e**  $y = x^2 + 8x + 16$

**f**  $f(x) = -2x^2 + 3x + 1$

**g**  $y = 6x^2 + 5x - 4$

**h**  $f(x) = -x^2 + x + 6$

**i**  $y = 9x^2 + 6x + 1$

- 2** Consider the graph of  $y = 2x^2 - 5x + 1$ .

- a** Describe the shape of the graph.

- b** Use the discriminant to show that the graph cuts the  $x$ -axis twice.

- c** Find the  $x$ -intercepts, rounding your answers to 2 decimal places.

- d** State the  $y$ -intercept.

- e** Hence, sketch the function.

- 3** Consider the graph of  $f(x) = -x^2 + 4x - 7$ .

- a** Use the discriminant to show that the graph does not cut the  $x$ -axis.

- b** Is the graph positive definite or negative definite?

- c** Find the vertex and  $y$ -intercept.

- d** Hence, sketch the function.

- 4** Show that:

**a**  $x^2 - 3x + 6 > 0$  for all  $x$

**b**  $4x - x^2 - 6 < 0$  for all  $x$

**c**  $2x^2 - 4x + 7$  is positive definite

**d**  $-2x^2 + 3x - 4$  is negative definite.

- 5** Explain why  $3x^2 + kx - 1$  is never positive definite for any value of  $k$ .

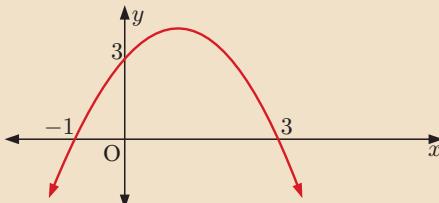
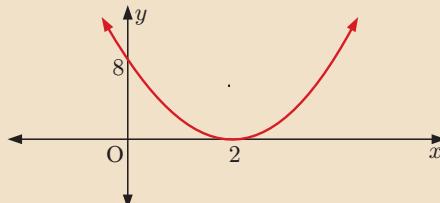
- 6** Under what conditions is  $2x^2 + kx + 2$  positive definite?

**E****FINDING A QUADRATIC FROM ITS GRAPH**

If we are given sufficient information on or about a graph, we can determine the quadratic function in whatever form is required.

**Example 20****Self Tutor**

Find the equation of the quadratic function with graph:

**a****b**

- a** Since the  $x$ -intercepts are  $-1$  and  $3$ ,

$$y = a(x + 1)(x - 3).$$

The graph is concave down, so  $a < 0$ .

When  $x = 0$ ,  $y = 3$

$$\therefore 3 = a(1)(-3)$$

$$\therefore a = -1$$

The quadratic function is

$$y = -(x + 1)(x - 3).$$

- b** The graph touches the  $x$ -axis at  $x = 2$ ,

$$\text{so } y = a(x - 2)^2.$$

The graph is concave up, so  $a > 0$ .

When  $x = 0$ ,  $y = 8$

$$\therefore 8 = a(-2)^2$$

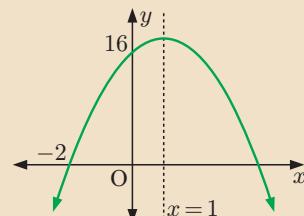
$$\therefore a = 2$$

The quadratic function is

$$y = 2(x - 2)^2.$$

**Example 21****Self Tutor**

Find the equation of the quadratic function with graph:



The axis of symmetry  $x = 1$  lies midway between the  $x$ -intercepts.

$\therefore$  the other  $x$ -intercept is 4.

$\therefore$  the quadratic has the form

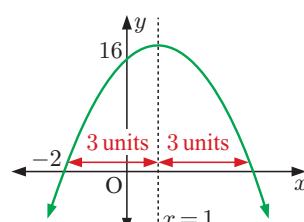
$$y = a(x + 2)(x - 4) \text{ where } a < 0$$

But when  $x = 0$ ,  $y = 16$

$$\therefore 16 = a(2)(-4)$$

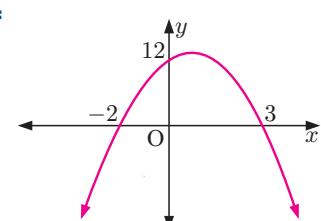
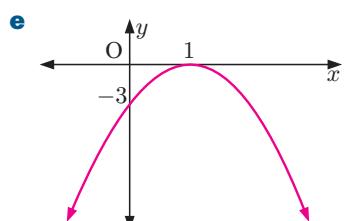
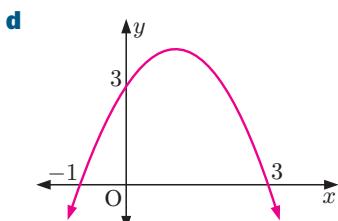
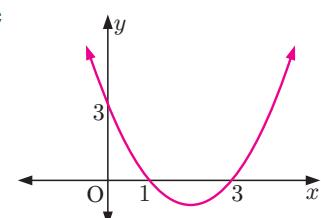
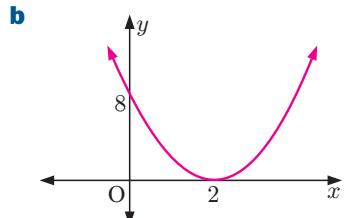
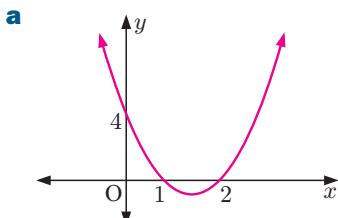
$$\therefore a = -2$$

The quadratic is  $y = -2(x + 2)(x - 4)$ .

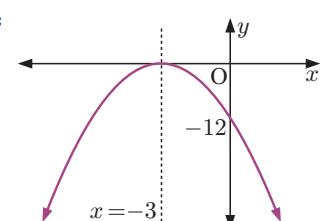
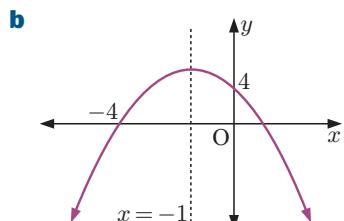
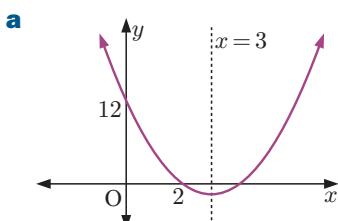


**EXERCISE 3E**

- 1** Find the equation of the quadratic with graph:



- 2** Find the quadratic with graph:

**Example 22****Self Tutor**

Find the equation of the quadratic whose graph cuts the  $x$ -axis at 4 and  $-3$ , and which passes through the point  $(2, -20)$ . Give your answer in the form  $y = ax^2 + bx + c$ .

Since the  $x$ -intercepts are 4 and  $-3$ , the quadratic has the form  $y = a(x - 4)(x + 3)$  where  $a \neq 0$ .

When  $x = 2$ ,  $y = -20$

$$\therefore -20 = a(2 - 4)(2 + 3)$$

$$\therefore -20 = a(-2)(5)$$

$$\therefore a = 2$$

$$\begin{aligned} \text{The quadratic is } y &= 2(x - 4)(x + 3) \\ &= 2(x^2 - x - 12) \\ &= 2x^2 - 2x - 24 \end{aligned}$$

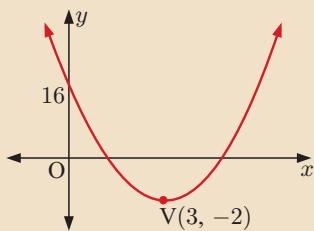
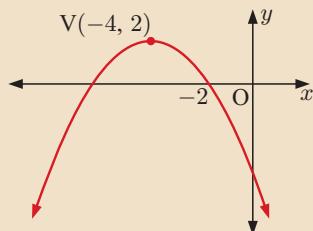
- 3** Find, in the form  $f(x) = ax^2 + bx + c$ , the equation of the quadratic whose graph:

- a** cuts the  $x$ -axis at 5 and 1, and passes through  $(2, -9)$
- b** cuts the  $x$ -axis at 2 and  $-\frac{1}{2}$ , and passes through  $(3, -14)$
- c** touches the  $x$ -axis at 3 and passes through  $(-2, -25)$
- d** touches the  $x$ -axis at  $-2$  and passes through  $(-1, 4)$

- 4** Find, in the form  $f(x) = ax^2 + bx + c$ , the equation of the quadratic whose graph:
- cuts the  $x$ -axis at 3, passes through  $(5, 12)$ , and has axis of symmetry  $x = 2$
  - cuts the  $x$ -axis at 5, passes through  $(2, 5)$ , and has axis of symmetry  $x = 1$ .

**Example 23****Self Tutor**

Find the equation of each quadratic function given its graph:

**a****b**

- a** Since the vertex is  $(3, -2)$ , the quadratic has the form

$$y = a(x - 3)^2 - 2 \quad \text{where } a > 0.$$

When  $x = 0$ ,  $y = 16$

$$\begin{aligned} \therefore 16 &= a(-3)^2 - 2 \\ \therefore 16 &= 9a - 2 \\ \therefore 18 &= 9a \\ \therefore a &= 2 \end{aligned}$$

The quadratic is  $y = 2(x - 3)^2 - 2$ .

- b** Since the vertex is  $(-4, 2)$ , the quadratic has the form

$$y = a(x + 4)^2 + 2 \quad \text{where } a < 0.$$

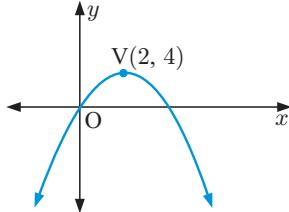
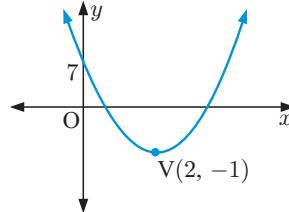
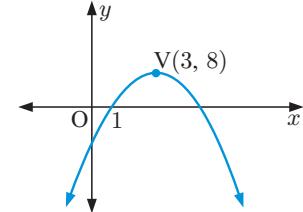
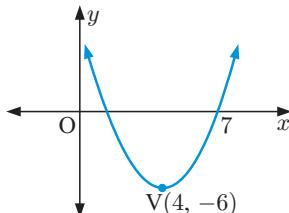
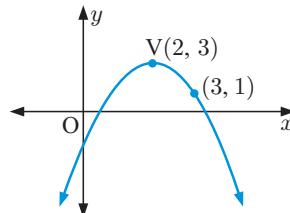
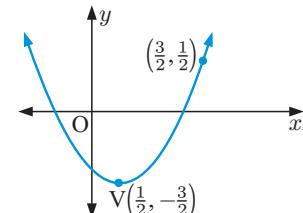
When  $x = -2$ ,  $y = 0$

$$\begin{aligned} \therefore 0 &= a(2)^2 + 2 \\ \therefore 4a &= -2 \\ \therefore a &= -\frac{1}{2} \end{aligned}$$

The quadratic is

$$y = -\frac{1}{2}(x + 4)^2 + 2.$$

- 5** If  $V$  is the vertex, find the equation of the quadratic function with graph:

**a****b****c****d****e****f**

**Discovery 3**

For the quadratic function  $y = 2x^2 + 3x + 7$  we can find a table of values for  $x = 0, 1, 2, 3, 4, 5$ .

We turn this table into a **difference table** by adding two further rows:

- the row  $\Delta_1$  gives the differences between successive  $y$ -values
- the row  $\Delta_2$  gives the differences between successive  $\Delta_1$ -values.

**What to do:**

- 1 Construct difference tables for  $x = 0, 1, 2, 3, 4, 5$  for each of the following quadratic functions:
  - $y = x^2 + 4x + 3$
  - $y = 3x^2 - 4x$
  - $y = 5x - x^2$
  - $y = 4x^2 - 5x + 2$
- 2 What do you notice about the  $\Delta_2$  row for each of the quadratic functions in 1?
- 3 Consider the general quadratic  $y = ax^2 + bx + c$ ,  $a \neq 0$ .
  - Copy and complete the following difference table:

$x$	0	1	2	3	4	5
$y$	(C)	$a + b + c$	$4a + 2b + c$	.....	.....	.....
$\Delta_1$	○	.....	.....	.....	.....	.....
$\Delta_2$	○	.....	.....	.....	.....	.....

- b Comment on the  $\Delta_2$  row.
- c What can the encircled numbers be used for?
- 4 Use your observations in 3 to determine, if possible, the quadratic functions with the following tables of values:

a	$x$	0	1	2	3	4
	$y$	6	5	8	15	26

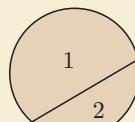
c	$x$	0	1	2	3	4
	$y$	1	2	-1	-8	-19

b	$x$	0	1	2	3	4
	$y$	8	10	18	32	52

d	$x$	0	1	2	3	4
	$y$	5	3	-1	-7	-15

- 5 We wish to determine the **maximum** number of pieces into which a pizza can be cut using  $n$  cuts across it.

For example, for  $n = 1$  we have



which has 2 pieces

for  $n = 3$  we have



which has 7 pieces.

- a Copy and complete:

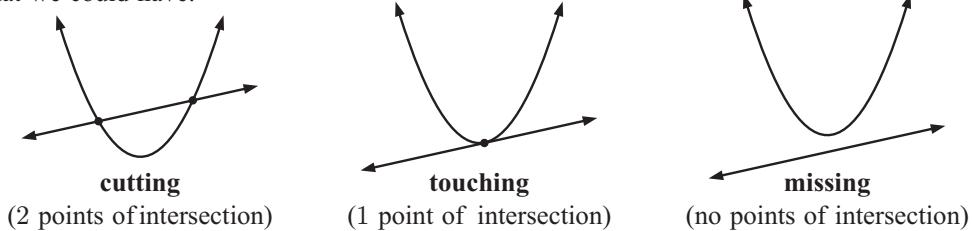
Number of cuts, $n$	0	1	2	3	4	5
Maximum number of pieces, $P_n$						

- b Complete the  $\Delta_1$  and  $\Delta_2$  rows. Hence determine a quadratic formula for  $P_n$ .  
 c For a huge pizza with 12 cuts across it, find the maximum number of pieces which can result.

**F****WHERE FUNCTIONS MEET**

Consider the graphs of a quadratic function and a linear function on the same set of axes.

Notice that we could have:



If the graphs meet, the coordinates of the points of intersection of the graphs can be found by *solving the two equations simultaneously*.

**Example 24****► Self Tutor**

Find the coordinates of the points of intersection of the graphs with equations  $y = x^2 - x - 18$  and  $y = x - 3$ .

$y = x^2 - x - 18$  meets  $y = x - 3$  where

$$\begin{aligned}x^2 - x - 18 &= x - 3 \\ \therefore x^2 - 2x - 15 &= 0 && \{\text{RHS} = 0\} \\ \therefore (x - 5)(x + 3) &= 0 && \{\text{factorising}\} \\ \therefore x &= 5 \text{ or } -3\end{aligned}$$

Substituting into  $y = x - 3$ , when  $x = 5$ ,  $y = 2$  and when  $x = -3$ ,  $y = -6$ .

$\therefore$  the graphs meet at  $(5, 2)$  and  $(-3, -6)$ .

**EXERCISE 3F**

- 1 Find the coordinates of the point(s) of intersection of:

- |  |   |
|--|---|
| <b>a</b> $y = x^2 - 2x + 8$ and $y = x + 6$  | <b>b</b> $f(x) = -x^2 + 3x + 9$ and $g(x) = 2x - 3$ |
| <b>c</b> $y = x^2 - 4x + 3$ and $y = 2x - 6$ | <b>d</b> $f(x) = -x^2 + 4x - 7$ and $g(x) = 5x - 4$ |

**Example 25****Self Tutor**

$y = 2x + k$  is a tangent to  $y = 2x^2 - 3x + 4$ . Find  $k$ .

$y = 2x + k$  meets  $y = 2x^2 - 3x + 4$  where

$$2x^2 - 3x + 4 = 2x + k$$

$$\therefore 2x^2 - 5x + (4 - k) = 0$$

Since the graphs touch, this quadratic has  $\Delta = 0$

$$\therefore (-5)^2 - 4(2)(4 - k) = 0$$

$$\therefore 25 - 8(4 - k) = 0$$

$$\therefore 25 - 32 + 8k = 0$$

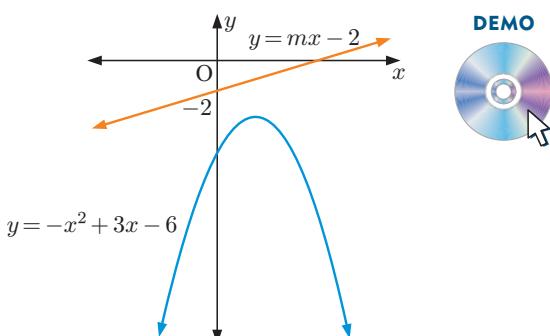
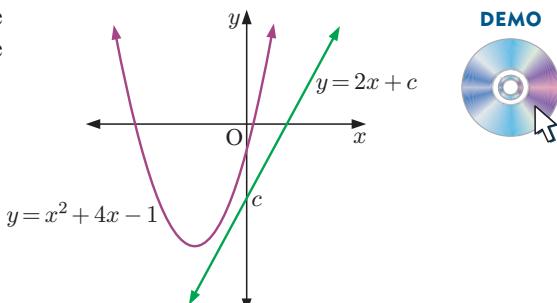
$$\therefore 8k = 7$$

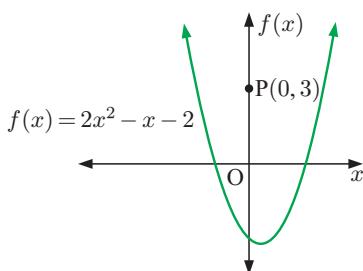
$$\therefore k = \frac{7}{8}$$

A line which is a tangent to a quadratic will **touch** the curve.



- 2 For which value of  $c$  is the line  $y = 3x + c$  a tangent to the parabola with equation  $y = x^2 - 5x + 7$ ?
- 3 Find the values of  $m$  for which the lines  $y = mx - 2$  are tangents to the curve with equation  $y = x^2 - 4x + 2$ .
- 4 Find the gradients of the lines with  $y$ -intercept 1 that are tangents to the curve  $f(x) = 3x^2 + 5x + 4$ .
- 5 **a** For what values of  $c$  do the lines  $y = x + c$  never meet the parabola with equation  $y = 2x^2 - 3x - 7$ ?  
**b** Choose one of the values of  $c$  found in part **a** above. Illustrate with a sketch that these graphs never meet.
- 6 Consider the curve  $y = x^2 + 4x - 1$  and the line  $y = 2x + c$ . Find the value(s) of  $c$  for which the line:
  - a** meets the curve twice
  - b** is a tangent to the curve
  - c** does not meet the curve.
- 7 Consider the curve  $f(x) = -x^2 + 3x - 6$  and the line  $g(x) = mx - 2$ . Find the values of  $m$  for which the line:
  - a** meets the curve twice
  - b** is a tangent to the curve
  - c** does not meet the curve.



**8**

Show that any linear function passing through  $P(0, 3)$  will meet the curve  $f(x) = 2x^2 - x - 2$  twice.

**G****PROBLEM SOLVING WITH QUADRATICS**

Some real world problems can be solved using a quadratic equation. We are generally only interested in any **real solutions** which result.

Any answer we obtain must be checked to see if it is reasonable. For example:

- if we are finding a length then it must be positive and we reject any negative solutions
- if we are finding ‘how many people are present’ then clearly the answer must be a positive integer.

We employ the following general problem solving method:

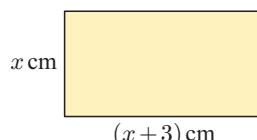
- Step 1:* If the information is given in words, translate it into algebra using a variable such as  $x$  for the unknown. Write down the resulting equation. Be sure to define what the variable  $x$  represents, and include units if appropriate.
- Step 2:* Solve the equation by a suitable method.
- Step 3:* Examine the solutions carefully to see if they are acceptable.
- Step 4:* Give your answer in a sentence.

**Example 26**

A rectangle has length 3 cm longer than its width. Its area is  $42 \text{ cm}^2$ . Find its width.

If the width is  $x$  cm then the length is  $(x + 3)$  cm.

$$\begin{aligned}\therefore x(x+3) &= 42 \quad \{\text{equating areas}\} \\ \therefore x^2 + 3x - 42 &= 0 \\ \therefore x &= \frac{-3 \pm \sqrt{3^2 - 4(1)(-42)}}{2} \\ \therefore x &= \frac{-3 \pm \sqrt{177}}{2} \\ \therefore x &\approx -8.15 \text{ or } 5.15\end{aligned}$$



We reject the negative solution as lengths are positive.

The width is about 5.15 cm.

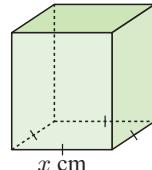
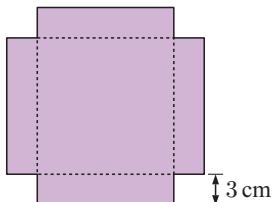
**EXERCISE 3G**

- Two integers differ by 12 and the sum of their squares is 74. Find the integers.
- The sum of a number and its reciprocal is  $\frac{26}{5}$ . Find the number.

- 3** The sum of a natural number and its square is 210. Find the number.
- 4** The product of two consecutive even numbers is 360. Find the numbers.
- 5** The number of diagonals of an  $n$ -sided polygon is given by the formula  $D = \frac{n}{2}(n - 3)$ .  
A polygon has 90 diagonals. How many sides does it have?
- 6** The length of a rectangle is 4 cm longer than its width. The rectangle has area  $26 \text{ cm}^2$ . Find its width.

- 7** A rectangular box has a square base with sides of length  $x$  cm. Its height is 1 cm longer than its base side length. The total surface area of the box is  $240 \text{ cm}^2$ .

- a Show that the total surface area is given by  $A = 6x^2 + 4x \text{ cm}^2$ .
- b Find the dimensions of the box.

**8**

An open box can hold  $80 \text{ cm}^3$ . It is made from a square piece of tinplate with  $3 \text{ cm}$  squares cut from each of its 4 corners. Find the dimensions of the original piece of tinplate.

### Example 27

### Self Tutor

Is it possible to bend a 12 cm length of wire to form the perpendicular sides of a right angled triangle with area  $20 \text{ cm}^2$ ?

Suppose the wire is bent  $x$  cm from one end.

$$\text{The area } A = \frac{1}{2}x(12 - x)$$

$$\therefore \frac{1}{2}x(12 - x) = 20$$

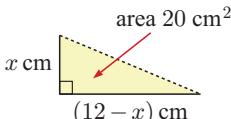
$$\therefore x(12 - x) = 40$$

$$\therefore 12x - x^2 - 40 = 0$$

$$\therefore x^2 - 12x + 40 = 0$$

$$\begin{array}{c} x \text{ cm} \quad (12 - x) \text{ cm} \\ \hline \longleftarrow 12 \text{ cm} \longrightarrow \end{array}$$

becomes



$$\text{Now } \Delta = (-12)^2 - 4(1)(40)$$

$$= -16 \text{ which is } < 0$$

There are no real solutions, indicating this situation is **impossible**.

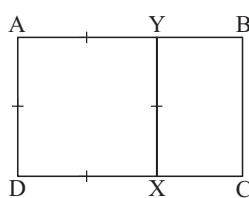
- 9** Is it possible to bend a 20 cm length of wire into the shape of a rectangle which has an area of  $30 \text{ cm}^2$ ?
- 10** The rectangle ABCD is divided into a square and a smaller rectangle by [XY] which is parallel to its shorter sides.

The smaller rectangle BCXY is *similar* to the original rectangle, so rectangle ABCD is a **golden rectangle**.

The ratio  $\frac{AB}{AD}$  is called the **golden ratio**.

Show that the golden ratio is  $\frac{1 + \sqrt{5}}{2}$ .

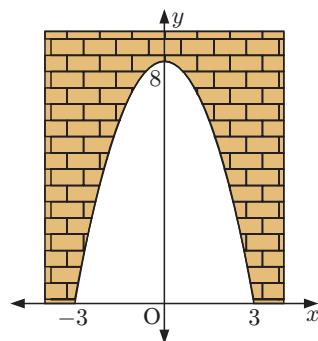
**Hint:** Let  $AB = x$  units and  $AD = 1$  unit.



- 11** A truck carrying a wide load needs to pass through the parabolic tunnel shown. The units are metres.

The truck is 5 m high and 4 m wide.

- Find the quadratic function which describes the shape of the tunnel.
- Determine whether the truck will fit.



- 12** Answer the **Opening Problem** on page 64.

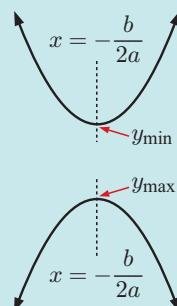
## H

## QUADRATIC OPTIMISATION

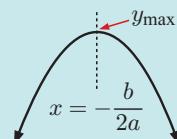
The process of finding the maximum or minimum value of a function is called **optimisation**.

For the quadratic function  $y = ax^2 + bx + c$ , we have already seen that the vertex has  $x$ -coordinate  $-\frac{b}{2a}$ .

- If  $a > 0$ , the **minimum** value of  $y$  occurs at  $x = -\frac{b}{2a}$ .



- If  $a < 0$ , the **maximum** value of  $y$  occurs at  $x = -\frac{b}{2a}$ .



### Example 28

### Self Tutor

Find the maximum or minimum value of the following quadratic functions, and the corresponding value of  $x$ :

**a**  $y = x^2 + x - 3$

**b**  $y = 3 + 3x - 2x^2$

**a**  $y = x^2 + x - 3$  has  
 $a = 1$ ,  $b = 1$ , and  $c = -3$ .

Since  $a > 0$ , the shape is

The minimum value occurs

when  $x = \frac{-b}{2a} = \frac{-1}{2}$

and  $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 3 = -3\frac{1}{4}$

So, the minimum value of  $y$  is  $-3\frac{1}{4}$ , occurring when  $x = -\frac{1}{2}$ .

**b**  $y = -2x^2 + 3x + 3$  has  
 $a = -2$ ,  $b = 3$ , and  $c = 3$ .

Since  $a < 0$ , the shape is

The maximum value occurs

when  $x = \frac{-b}{2a} = \frac{-3}{-4} = \frac{3}{4}$

and  $y = -2(\frac{3}{4})^2 + 3(\frac{3}{4}) + 3 = 4\frac{1}{8}$

So, the maximum value of  $y$  is  $4\frac{1}{8}$ , occurring when  $x = \frac{3}{4}$ .

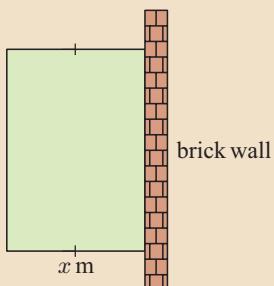
**EXERCISE 3H**

- 1** Find the maximum or minimum values of the following quadratic functions, and the corresponding values of  $x$ :
- a**  $y = x^2 - 2x$       **b**  $f(x) = 7 - 2x - x^2$       **c**  $y = 8 + 2x - 3x^2$   
**d**  $f(x) = 2x^2 + x - 1$       **e**  $y = 4x^2 - x + 5$       **f**  $f(x) = 7x - 2x^2$
- 2** The profit in manufacturing  $x$  refrigerators per day, is given by the profit relation  $P = -3x^2 + 240x - 800$  dollars.
- a** How many refrigerators should be made each day to maximise the total profit?
- b** What is the maximum profit?

**Example 29****Self Tutor**

A gardener has 40 m of fencing to enclose a rectangular garden plot, where one side is an existing brick wall. Suppose the two new equal sides are  $x$  m long.

- a** Show that the area enclosed is given by  $A = x(40 - 2x)$  m<sup>2</sup>.
- b** Find the dimensions of the garden of maximum area.



- a** Side [XY] has length  $(40 - 2x)$  m.

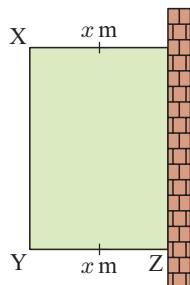
Now, area = length  $\times$  width

$$\therefore A = x(40 - 2x) \text{ m}^2$$

- b**  $A = 0$  when  $x = 0$  or 20.

The vertex of the function lies midway between these values, so  $x = 10$ .

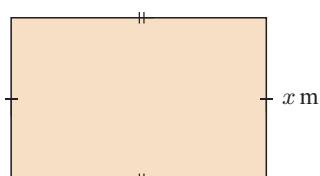
Since  $a < 0$ , the shape is



$\therefore$  the area is maximised when  $YZ = 10$  m and  $XY = 20$  m.

- 3** A rectangular plot is enclosed by 200 m of fencing and has an area of  $A$  square metres. Show that:

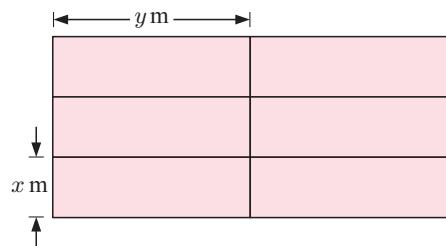
- a**  $A = 100x - x^2$  where  $x$  m is the length of one of its sides  
**b** the area is maximised if the rectangle is a square.



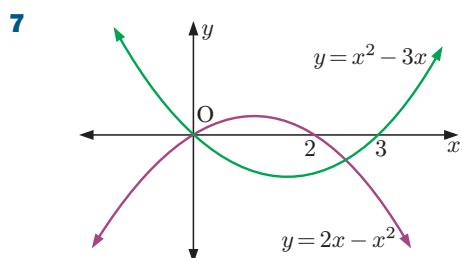
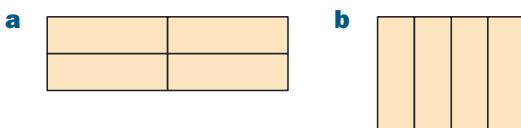
- 4** Three sides of a rectangular paddock are to be fenced, the fourth side being an existing straight water drain. If 1000 m of fencing is available, what dimensions should be used for the paddock so that it encloses the maximum possible area?

- 5 1800 m of fencing is available to fence six identical pens as shown in the diagram.

- a Explain why  $9x + 8y = 1800$ .
- b Show that the area of each pen is given by  $A = -\frac{9}{8}x^2 + 225x \text{ m}^2$ .
- c If the area enclosed is to be maximised, what are the dimensions of each pen?



- 6 500 m of fencing is available to make 4 rectangular pens of identical shape. Find the dimensions that maximise the area of each pen if the plan is:

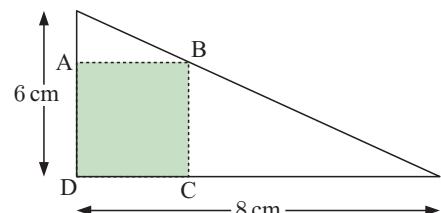


The graphs of  $y = x^2 - 3x$  and  $y = 2x - x^2$  are illustrated.

- a Show that the graphs meet where  $x = 0$  and  $x = 2\frac{1}{2}$ .
- b Find the maximum vertical separation between the curves for  $0 \leq x \leq 2\frac{1}{2}$ .

- 8 Infinitely many rectangles may be inscribed within the right angled triangle shown alongside. One of them is illustrated.

- a Let  $AB = x \text{ cm}$  and  $BC = y \text{ cm}$ . Use similar triangles to find  $y$  in terms of  $x$ .
- b Find the dimensions of rectangle ABCD of maximum area.



### Discovery 4

### Sum and product of roots

#### What to do:

- 1 Suppose  $ax^2 + bx + c = 0$  has roots  $p$  and  $q$ .

Prove that  $p + q = -\frac{b}{a}$  and  $pq = \frac{c}{a}$ .

- 2 Suppose  $2x^2 - 5x + 1 = 0$  has roots  $p$  and  $q$ . Without finding the values of  $p$  and  $q$ , find:

a  $p + q$

b  $pq$

c  $p^2 + q^2$

d  $\frac{1}{p} + \frac{1}{q}$

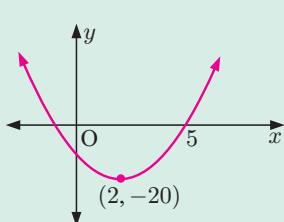
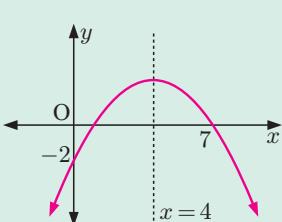
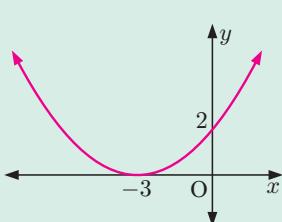
- 3 Find all quadratic equations with roots which are:

a one more than the roots of  $2x^2 - 5x + 1 = 0$

b the squares of the roots of  $2x^2 - 5x + 1 = 0$

c the reciprocals of the roots of  $2x^2 - 5x + 1 = 0$ .

**Review set 3A**

- 1** Consider the quadratic function  $f(x) = -2(x + 2)(x - 1)$ .
- State the  $x$ -intercepts.
  - State the equation of the axis of symmetry.
  - Find the  $y$ -intercept.
  - Find the coordinates of the vertex.
  - Sketch the function.
  - State the range of the function.
- 2** Solve the following equations, giving exact answers:
- $3x^2 - 12x = 0$
  - $3x^2 - x - 10 = 0$
  - $x^2 - 11x = 60$
- 3** Solve using the quadratic formula:
- $x^2 + 5x + 3 = 0$
  - $3x^2 + 11x - 2 = 0$
- 4** Solve for  $x$ :
- $x^2 - 4x - 21 < 0$
  - $3x^2 - 2 \geq 5x$
- 5** Use the vertex, axis of symmetry, and  $y$ -intercept to graph:
- $y = (x - 2)^2 - 4$
  - $y = -\frac{1}{2}(x + 4)^2 + 6$
- 6** Find, in the form  $y = ax^2 + bx + c$ , the equation of the quadratic whose graph:
- touches the  $x$ -axis at 4 and passes through (2, 12)
  - has vertex  $(-4, 1)$  and passes through (1, 11).
- 7** Find the maximum or minimum value of the relation  $f(x) = -2x^2 + 4x + 3$  and the value of  $x$  at which this occurs.
- 8** Find the points of intersection of  $y = x^2 - 3x$  and  $y = 3x^2 - 5x - 24$ .
- 9** For what values of  $k$  does the graph of  $y = -2x^2 + 5x + k$  not cut the  $x$ -axis?
- 10** Find the values of  $m$  for which  $2x^2 - 3x + m = 0$  has:
  - a repeated root
  - two distinct real roots
  - no real roots.
- 11** The sum of a number and its reciprocal is  $2\frac{1}{30}$ . Find the number.
- 12** Show that no line with a  $y$ -intercept of  $(0, 10)$  will ever be tangential to the curve with equation  $y = 3x^2 + 7x - 2$ .
- 13** **a** Write the quadratic  $y = 2x^2 + 4x - 3$  in the form  $y = a(x - h)^2 + k$ .  
**b** Hence, sketch the graph of the quadratic.
- 14** Find the equation of the quadratic function with graph:
- a** 
- b** 
- c** 
- 15** Find the range of  $y = x^2 - 6x - 4$  on the domain  $-1 \leq x \leq 8$ .

- 16** When Annie hits a softball, the height of the ball above the ground after  $t$  seconds is given by  $f(t) = -4.9t^2 + 19.6t + 1.4$  metres. Find the maximum height reached by the ball.



### Review set 3B

- 1** Consider the quadratic function  $y = \frac{1}{2}(x - 2)^2 - 4$ .
  - a** State the equation of the axis of symmetry.
  - b** Find the coordinates of the vertex.
  - c** Find the  $y$ -intercept.
  - d** Sketch the function.
  - e** State the range of the function.
- 2** Solve the following equations:
 

<b>a</b> $x^2 - 5x - 3 = 0$	<b>b</b> $2x^2 - 7x - 3 = 0$
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- 3** Solve for  $x$ :
 

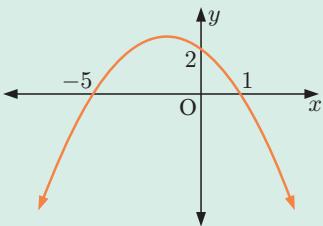
<b>a</b> $x^2 + 5x \leq 14$	<b>b</b> $2x^2 + 7x > 2(x + 6)$
-----------------------------	---------------------------------
- 4** Consider the quadratic function  $f(x) = -3x^2 + 8x + 7$ . Find the equation of the axis of symmetry, and the coordinates of the vertex.
- 5** Use the discriminant only to find the relationship between the graph and the  $x$ -axis for:
 

<b>a</b> $y = 2x^2 + 3x - 7$	<b>b</b> $y = -3x^2 - 7x + 4$
------------------------------	-------------------------------
- 6** Determine whether each quadratic function is positive definite, negative definite, or neither:
 

<b>a</b> $y = -2x^2 + 3x + 2$	<b>b</b> $f(x) = 3x^2 + x + 11$
-------------------------------	---------------------------------
- 7** Find the equation of the quadratic function with vertex  $(2, 25)$  and  $y$ -intercept 1.
- 8** For what values of  $m$  does the line  $y = mx - 10$  meet the curve  $y = 3x^2 + 7x + 2$  twice?
- 9** Consider the quadratic function  $y = 2x^2 + 4x - 1$ .
  - a** State the axis of symmetry.
  - b** Find the coordinates of the vertex.
  - c** Find the axes intercepts.
  - d** Hence sketch the function.
- 10** Find the range of  $y = -2x^2 + 6x + 1$  on the domain  $-4 \leq x \leq 5$ .
- 11** Find the values of  $k$  for which  $qx^2 + kx - 2$  has:
 

<b>a</b> a repeated root	<b>b</b> two distinct real roots	<b>c</b> no real roots.
--------------------------	----------------------------------	-------------------------
- 12** **a** For what values of  $c$  do the lines with equations  $y = 3x + c$  intersect the parabola  $y = x^2 + x - 5$  in two distinct points?  
**b** Choose one such value of  $c$  from part **a** and find the points of intersection in this case.

- 13** **a** Find the equation of the quadratic function illustrated.  
**b** Find the vertex of the quadratic.



- 14** Find the maximum or minimum value of the quadratic, and the corresponding value of  $x$ :

**a**  $y = 3x^2 + 4x + 7$

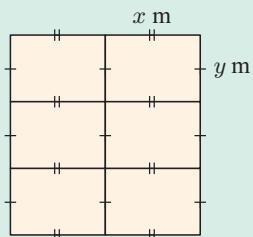
**b**  $y = -2x^2 - 5x + 2$

- 15** 600 m of fencing is used to construct 6 rectangular animal pens as shown.

**a** Show that the area  $A$  of each pen is  $A = x \left( \frac{600 - 8x}{9} \right) \text{ m}^2$ .

- b** Find the dimensions of each pen so that it has the maximum possible area.

- c** What is the area of each pen in this case?



- 16** Sketch the graph of  $f(x) = |x^2 + x - 20|$ .

# 4

# Surds, indices, and exponentials

## Contents:

- A** Surds
- B** Indices
- C** Index laws
- D** Rational indices
- E** Algebraic expansion and factorisation
- F** Exponential equations
- G** Exponential functions
- H** The natural exponential  $e^x$

## Opening problem

The interior of a freezer has temperature  $-10^{\circ}\text{C}$ . When a packet of peas is placed in the freezer, its temperature after  $t$  minutes is given by  $T(t) = -10 + 32 \times 2^{-0.2t}^{\circ}\text{C}$ .

### Things to think about:

- a What was the temperature of the packet of peas?
  - i when it was first placed in the freezer
  - ii after 5 minutes
  - iii after 10 minutes
  - iv after 15 minutes?
- b What does the graph of temperature over time look like?
- c According to this model, will the temperature of the packet of peas ever reach  $-10^{\circ}\text{C}$ ? Explain your answer.

We often deal with numbers that are repeatedly multiplied together. Mathematicians use **indices**, also called **powers** or **exponents**, to construct such expressions.

Indices have many applications in the areas of finance, engineering, physics, electronics, biology, and computer science.

## A SURDS

A **radical** is any number which is written with the **radical sign**  $\sqrt{\phantom{x}}$ .

A **surd** is a real, irrational radical such as  $\sqrt{2}$ ,  $\sqrt{3}$ ,  $\sqrt{5}$ , or  $\sqrt{6}$ . Surds are present in solutions to some quadratic equations.  $\sqrt{4}$  is a radical, but is not a surd as it simplifies to 2.

$\sqrt{a}$  is the non-negative number such that  $\sqrt{a} \times \sqrt{a} = a$ .

Important properties of surds are:

- $\sqrt{a}$  is never negative, so  $\sqrt{a} \geq 0$ .
- $\sqrt{a}$  is only real if  $a \geq 0$ .
- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$  for  $a \geq 0$  and  $b \geq 0$ .
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$  for  $a \geq 0$  and  $b > 0$ .

### Example 1

### Self Tutor

Write as a single surd:

a  $\sqrt{2} \times \sqrt{3}$       b  $\frac{\sqrt{18}}{\sqrt{6}}$

$$\begin{aligned} \mathbf{a} \quad & \sqrt{2} \times \sqrt{3} \\ &= \sqrt{2 \times 3} \\ &= \sqrt{6} \end{aligned} \quad \begin{aligned} \mathbf{b} \quad & \frac{\sqrt{18}}{\sqrt{6}} \\ &= \sqrt{\frac{18}{6}} \\ &= \sqrt{3} \end{aligned}$$

**EXERCISE 4A.1**

**1** Write as a single surd or rational number:

**a**  $\sqrt{11} \times \sqrt{11}$

**b**  $\sqrt{3} \times \sqrt{5}$

**c**  $(\sqrt{3})^2$

**d**  $\sqrt{5} \times \sqrt{6}$

**e**  $2\sqrt{2} \times \sqrt{2}$

**f**  $3\sqrt{2} \times 2\sqrt{2}$

**g**  $3\sqrt{7} \times 2\sqrt{7}$

**h**  $(3\sqrt{5})^2$

**i**  $\frac{\sqrt{12}}{\sqrt{2}}$

**j**  $\frac{\sqrt{18}}{\sqrt{3}}$

**k**  $\frac{\sqrt{20}}{\sqrt{5}}$

**l**  $\frac{\sqrt{6} \times \sqrt{10}}{\sqrt{12}}$

**Example 2****Self Tutor**

Write  $\sqrt{18}$  in the form  $a\sqrt{b}$  where  $a$  and  $b$  are integers and  $a$  is as large as possible.

$$\begin{aligned}\sqrt{18} &= \sqrt{9 \times 2} && \{9 \text{ is the largest perfect square factor of } 18\} \\ &= \sqrt{9} \times \sqrt{2} \\ &= 3\sqrt{2}\end{aligned}$$

**2** Write in the form  $a\sqrt{b}$  where  $a$  and  $b$  are integers and  $a$  is as large as possible:

**a**  $\sqrt{8}$

**b**  $\sqrt{12}$

**c**  $\sqrt{20}$

**d**  $\sqrt{32}$

**e**  $\sqrt{27}$

**f**  $\sqrt{45}$

**g**  $\sqrt{48}$

**h**  $\sqrt{54}$

**i**  $\sqrt{50}$

**j**  $\sqrt{80}$

**k**  $\sqrt{96}$

**l**  $\sqrt{108}$

**OPERATING WITH SURDS**

The rules for adding, subtracting, and multiplying by surds are the same as those for ordinary algebra.

**Example 3****Self Tutor**

Simplify:

**a**  $3\sqrt{3} + 5\sqrt{3}$

**b**  $2\sqrt{2} - 5\sqrt{2}$

$$\begin{aligned}\mathbf{a} \quad 3\sqrt{3} + 5\sqrt{3} &= 8\sqrt{3} \\ &= 8\sqrt{3}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad 2\sqrt{2} - 5\sqrt{2} &= -3\sqrt{2}\end{aligned}$$

In **b**, compare with  
 $2x - 5x = -3x$

**Example 4****Self Tutor**

Simplify:

**a**  $\sqrt{5}(6 - \sqrt{5})$

**b**  $(6 + \sqrt{3})(1 + 2\sqrt{3})$

$$\begin{aligned}\mathbf{a} \quad \sqrt{5}(6 - \sqrt{5}) &= \sqrt{5} \times 6 + \sqrt{5} \times -\sqrt{5} \\ &= 6\sqrt{5} - 5\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad (6 + \sqrt{3})(1 + 2\sqrt{3}) &= 6 + 6(2\sqrt{3}) + \sqrt{3}(1) + \sqrt{3}(2\sqrt{3}) \\ &= 6 + 12\sqrt{3} + \sqrt{3} + 6 \\ &= 12 + 13\sqrt{3}\end{aligned}$$

**EXERCISE 4A.2****1** Simplify:

**a**  $2\sqrt{2} + 3\sqrt{2}$

**e**  $3\sqrt{5} - 5\sqrt{5}$

**b**  $2\sqrt{2} - 3\sqrt{2}$

**f**  $7\sqrt{3} + 2\sqrt{3}$

**c**  $5\sqrt{5} - 3\sqrt{5}$

**g**  $9\sqrt{6} - 12\sqrt{6}$

**d**  $5\sqrt{5} + 3\sqrt{5}$

**h**  $\sqrt{2} + \sqrt{2} + \sqrt{2}$

**2** Simplify:

**a**  $\sqrt{2}(3 - \sqrt{2})$

**e**  $-\sqrt{3}(5 + \sqrt{3})$

**b**  $\sqrt{5}(\sqrt{5} + 1)$

**f**  $2\sqrt{6}(\sqrt{6} - 7)$

**c**  $\sqrt{10}(3 + 2\sqrt{10})$

**g**  $-\sqrt{8}(\sqrt{8} - 5)$

**d**  $\sqrt{7}(3\sqrt{7} - 4)$

**h**  $-3\sqrt{2}(4 - 6\sqrt{2})$

**3** Simplify:

**a**  $(5 + \sqrt{2})(4 + \sqrt{2})$

**d**  $(\sqrt{3} + 1)(2 - 3\sqrt{3})$

**b**  $(7 + 2\sqrt{3})(4 + \sqrt{3})$

**e**  $(\sqrt{8} - 6)(2\sqrt{8} - 3)$

**c**  $(9 - \sqrt{7})(4 + 2\sqrt{7})$

**f**  $(2\sqrt{5} - 7)(1 - 4\sqrt{5})$

**Example 5** **Self Tutor**

Simplify:

**a**  $(5 - \sqrt{2})^2$

**b**  $(7 + 2\sqrt{5})(7 - 2\sqrt{5})$

**a**  $(5 - \sqrt{2})^2$

$$\begin{aligned} &= 5^2 + 2(5)(-\sqrt{2}) + (\sqrt{2})^2 \\ &= 25 - 10\sqrt{2} + 2 \\ &= 27 - 10\sqrt{2} \end{aligned}$$

**b**  $(7 + 2\sqrt{5})(7 - 2\sqrt{5})$

$$\begin{aligned} &= 7^2 - (2\sqrt{5})^2 \\ &= 49 - (4 \times 5) \\ &= 29 \end{aligned}$$

**4** Simplify:

**a**  $(3 + \sqrt{2})^2$

**e**  $(4 + 2\sqrt{3})^2$

**b**  $(6 - \sqrt{3})^2$

**f**  $(3\sqrt{5} + 1)^2$

**c**  $(\sqrt{5} + 1)^2$

**g**  $(7 - 2\sqrt{10})^2$

**d**  $(\sqrt{8} - 3)^2$

**h**  $(5\sqrt{6} - 4)^2$

**5** Simplify:

**a**  $(3 + \sqrt{7})(3 - \sqrt{7})$

**d**  $(2\sqrt{2} + 1)(2\sqrt{2} - 1)$

**b**  $(\sqrt{2} + 5)(\sqrt{2} - 5)$

**e**  $(4 + 3\sqrt{8})(4 - 3\sqrt{8})$

**c**  $(4 - \sqrt{3})(4 + \sqrt{3})$

**f**  $(9\sqrt{3} - 5)(9\sqrt{3} + 5)$

**DIVISION BY SURDS**

Numbers like  $\frac{6}{\sqrt{2}}$  and  $\frac{9}{5 + \sqrt{2}}$  involve dividing by a surd.

It is customary to ‘simplify’ these numbers by rewriting them without the surd in the denominator.

For any fraction of the form  $\frac{b}{\sqrt{a}}$ , we can remove the surd from the denominator by multiplying by  $\frac{\sqrt{a}}{\sqrt{a}}$ .

Since  $\frac{\sqrt{a}}{\sqrt{a}} = 1$ , this does not change the value of the fraction.

**Example 6****Self Tutor**

Write with an integer denominator:

**a**  $\frac{6}{\sqrt{5}}$

**b**  $\frac{35}{\sqrt{7}}$

**a**  $\frac{6}{\sqrt{5}}$

$$= \frac{6}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{6\sqrt{5}}{5}$$

**b**  $\frac{35}{\sqrt{7}}$

$$= \frac{35}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$$

$$= \frac{35\sqrt{7}}{7}$$

$$= 5\sqrt{7}$$

Multiplying the original number by  $\frac{\sqrt{5}}{\sqrt{5}}$  or  $\frac{\sqrt{7}}{\sqrt{7}}$  does not change its value.



For any fraction of the form  $\frac{c}{a + \sqrt{b}}$ , we can remove the surd from the denominator by multiplying by  $\frac{a - \sqrt{b}}{a - \sqrt{b}}$ .

Expressions such as  $a + \sqrt{b}$  and  $a - \sqrt{b}$  are known as **radical conjugates**. They are identical except for the sign in the middle.

The product of radical conjugates is rational, since we have the difference between two squares. Multiplying by  $\frac{a - \sqrt{b}}{a - \sqrt{b}}$  therefore produces a rational denominator, so it is sometimes called **rationalising the denominator**.

**Example 7****Self Tutor**

Write  $\frac{5}{3 - \sqrt{2}}$  with an integer denominator.

$$\begin{aligned} \frac{5}{3 - \sqrt{2}} &= \left( \frac{5}{3 - \sqrt{2}} \right) \left( \frac{3 + \sqrt{2}}{3 + \sqrt{2}} \right) \\ &= \frac{5(3 + \sqrt{2})}{3^2 - (\sqrt{2})^2} \\ &= \frac{15 + 5\sqrt{2}}{7} \end{aligned}$$

The radical conjugate of  $3 - \sqrt{2}$  is  $3 + \sqrt{2}$ .

**EXERCISE 4A.3**

- 1** Write with integer denominator:

**a**  $\frac{1}{\sqrt{3}}$

**b**  $\frac{3}{\sqrt{3}}$

**c**  $\frac{9}{\sqrt{3}}$

**d**  $\frac{11}{\sqrt{3}}$

**e**  $\frac{\sqrt{2}}{3\sqrt{3}}$

**f**  $\frac{2}{\sqrt{2}}$

**g**  $\frac{6}{\sqrt{2}}$

**h**  $\frac{12}{\sqrt{2}}$

**i**  $\frac{\sqrt{3}}{\sqrt{2}}$

**j**  $\frac{1}{4\sqrt{2}}$

**2** Write with integer denominator:

a  $\frac{5}{\sqrt{5}}$

b  $\frac{15}{\sqrt{5}}$

c  $\frac{-3}{\sqrt{5}}$

d  $\frac{200}{\sqrt{5}}$

e  $\frac{1}{3\sqrt{5}}$

f  $\frac{7}{\sqrt{7}}$

g  $\frac{21}{\sqrt{7}}$

h  $\frac{2}{\sqrt{11}}$

i  $\frac{26}{\sqrt{13}}$

j  $\frac{1}{(\sqrt{3})^3}$

**3** Rationalise the denominator:

a  $\frac{1}{3 + \sqrt{2}}$

b  $\frac{2}{3 - \sqrt{2}}$

c  $\frac{1}{2 + \sqrt{5}}$

d  $\frac{\sqrt{2}}{2 - \sqrt{2}}$

e  $\frac{10}{\sqrt{6} - 1}$

f  $\frac{\sqrt{3}}{\sqrt{7} + 2}$

g  $\frac{1 + \sqrt{2}}{1 - \sqrt{2}}$

h  $\frac{\sqrt{3}}{4 - \sqrt{3}}$

i  $\frac{-2\sqrt{2}}{1 - \sqrt{2}}$

j  $\frac{1 + \sqrt{5}}{2 - \sqrt{5}}$

k  $\frac{\sqrt{3} + 2}{\sqrt{3} - 1}$

l  $\frac{\sqrt{10} - 7}{\sqrt{10} + 4}$

### Example 8

### Self Tutor

Write  $\frac{1}{5 + \sqrt{2}}$  in the form  $a + b\sqrt{2}$  where  $a, b \in \mathbb{Q}$ .

$$\begin{aligned}\frac{1}{5 + \sqrt{2}} &= \left( \frac{1}{5 + \sqrt{2}} \right) \times \left( \frac{5 - \sqrt{2}}{5 - \sqrt{2}} \right) \\ &= \frac{5 - \sqrt{2}}{25 - 2} \\ &= \frac{5 - \sqrt{2}}{23} \\ &= \frac{5}{23} - \frac{1}{23}\sqrt{2}\end{aligned}$$

**4** Write in the form  $a + b\sqrt{2}$  where  $a, b \in \mathbb{Q}$ :

a  $\frac{3}{\sqrt{2} - 3}$

b  $\frac{4}{2 + \sqrt{2}}$

c  $\frac{\sqrt{2}}{\sqrt{2} - 5}$

d  $\frac{-2\sqrt{2}}{\sqrt{2} + 1}$

**5** Write in the form  $a + b\sqrt{3}$  where  $a, b \in \mathbb{Q}$ :

a  $\frac{4}{1 - \sqrt{3}}$

b  $\frac{6}{\sqrt{3} + 2}$

c  $\frac{\sqrt{3}}{2 - \sqrt{3}}$

d  $\frac{1 + 2\sqrt{3}}{3 + \sqrt{3}}$

**6** a Suppose  $a, b$ , and  $c$  are integers,  $c > 0$ . Show that  $(a + b\sqrt{c})(a - b\sqrt{c})$  is also an integer.

b Write with an integer denominator:

i  $\frac{1}{1 + 2\sqrt{3}}$

ii  $\frac{\sqrt{2}}{3\sqrt{2} - 5}$

iii  $\frac{\sqrt{2} - 1}{3 - 2\sqrt{2}}$

**7** a Suppose  $a$  and  $b$  are positive integers. Show that  $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$  is also an integer.

b Write with an integer denominator:

i  $\frac{1}{\sqrt{2} + \sqrt{3}}$

ii  $\frac{\sqrt{3}}{\sqrt{3} - \sqrt{5}}$

iii  $\frac{\sqrt{11} - \sqrt{14}}{\sqrt{11} + \sqrt{14}}$

**8** Solve the equation  $2x - 3\sqrt{3} = 1 - x\sqrt{3}$ . Give your solution in the form  $x = a + b\sqrt{3}$ , where  $a$  and  $b$  are integers.

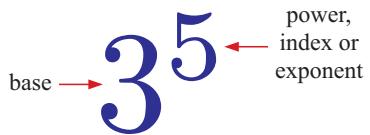
**9** Find the positive solution of the equation  $(9 + \sqrt{5})x^2 + (5 - 2\sqrt{5})x - 5 = 0$ . Give your answer in the form  $a + b\sqrt{5}$ , where  $a, b \in \mathbb{Q}$ .

**B****INDICES**

If  $n$  is a positive integer, then  $a^n$  is the product of  $n$  factors of  $a$ .

$$a^n = \underbrace{a \times a \times a \times a \times \dots \times a}_{n \text{ factors}}$$

We say that  $a$  is the **base**, and  $n$  is the **index or exponent**.

**NEGATIVE BASES**

$$(-1)^1 = -1$$

$$(-1)^2 = -1 \times -1 = 1$$

$$(-1)^3 = -1 \times -1 \times -1 = -1$$

$$(-1)^4 = -1 \times -1 \times -1 \times -1 = 1$$

$$(-2)^1 = -2$$

$$(-2)^2 = -2 \times -2 = 4$$

$$(-2)^3 = -2 \times -2 \times -2 = -8$$

$$(-2)^4 = -2 \times -2 \times -2 \times -2 = 16$$

From the patterns above we can see that:

A negative base raised to an **odd** index is **negative**.

A negative base raised to an **even** index is **positive**.

**EXERCISE 4B**

- 1** List the first six powers of:

**a** 2      **b** 3      **c** 4

- 2** Copy and complete the values of these common powers:

**a**  $5^1 = \dots$ ,  $5^2 = \dots$ ,  $5^3 = \dots$ ,  $5^4 = \dots$

**b**  $6^1 = \dots$ ,  $6^2 = \dots$ ,  $6^3 = \dots$ ,  $6^4 = \dots$

**c**  $7^1 = \dots$ ,  $7^2 = \dots$ ,  $7^3 = \dots$ ,  $7^4 = \dots$

- 3** Simplify, then use a calculator to check your answer:

<b>a</b> $(-1)^5$	<b>b</b> $(-1)^6$	<b>c</b> $(-1)^{14}$	<b>d</b> $(-1)^{19}$	<b>e</b> $(-1)^8$	<b>f</b> $-1^8$
<b>g</b> $-(-1)^8$	<b>h</b> $(-2)^5$	<b>i</b> $-2^5$	<b>j</b> $-(-2)^6$	<b>k</b> $(-5)^4$	<b>l</b> $-(5)^4$

- 4** Use your calculator to find the value of:

<b>a</b> $4^7$	<b>b</b> $7^4$	<b>c</b> $-5^5$	<b>d</b> $(-5)^5$	<b>e</b> $8^6$	<b>f</b> $(-8)^6$
<b>g</b> $-8^6$	<b>h</b> $2.13^9$	<b>i</b> $-2.13^9$	<b>j</b> $(-2.13)^9$		

- 5** Use your calculator to find the values of:

<b>a</b> $9^{-1}$	<b>b</b> $\frac{1}{9^1}$	<b>c</b> $6^{-2}$	<b>d</b> $\frac{1}{6^2}$	<b>e</b> $3^{-4}$	<b>f</b> $\frac{1}{3^4}$
<b>g</b> $17^0$	<b>h</b> $(0.366)^0$				

What do you notice?

- 6** Consider  $3^1, 3^2, 3^3, 3^4, 3^5, \dots$ . Look for a pattern and hence find the last digit of  $3^{101}$ .

- 7** What is the last digit of  $7^{217}$ ?

### Historical note

Nicomachus discovered an interesting number pattern involving cubes and sums of odd numbers.

Nicomachus was born in Roman Syria (now Jerash, Jordan) around 100 AD. He wrote in Greek, and was a Pythagorean, which means he followed the teaching of **Pythagoras**.

$$1 = 1^3$$

$$3 + 5 = 8 = 2^3$$

$$7 + 9 + 11 = 27 = 3^3$$

⋮

## C INDEX LAWS

The **index laws** for  $m, n \in \mathbb{Z}$  are:

$$a^m \times a^n = a^{m+n}$$

To **multiply** numbers with the **same base**, keep the base and **add** the indices.

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

To **divide** numbers with the same base, keep the base and **subtract** the indices.

$$(a^m)^n = a^{m \times n}$$

When **raising a power** to a **power**, keep the base and **multiply** the indices.

$$(ab)^n = a^n b^n$$

The power of a product is the product of the powers.

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \quad b \neq 0$$

The power of a quotient is the quotient of the powers.

$$a^0 = 1, \quad a \neq 0$$

Any non-zero number raised to the power of zero is 1.

$$a^{-n} = \frac{1}{a^n} \quad \text{and} \quad \frac{1}{a^{-n}} = a^n \quad \text{and in particular} \quad a^{-1} = \frac{1}{a}, \quad a \neq 0.$$

### Example 9

### Self Tutor

Simplify using the index laws:

a  $3^5 \times 3^4$

b  $\frac{5^3}{5^5}$

c  $(m^4)^3$

a  $3^5 \times 3^4$

$= 3^{5+4}$

$= 3^9$

b  $\frac{5^3}{5^5}$

$= 5^{3-5}$

$= 5^{-2}$

$= \frac{1}{25}$

c  $(m^4)^3$

$= m^{4 \times 3}$

$= m^{12}$

### EXERCISE 4C

1 Simplify using the index laws:

a  $5^4 \times 5^7$

b  $d^2 \times d^6$

c  $\frac{k^8}{k^3}$

d  $\frac{7^5}{7^6}$

e  $(x^2)^5$

f  $(3^4)^4$

g  $\frac{p^3}{p^7}$

h  $n^3 \times n^9$

i  $(5^t)^3$

j  $7^x \times 7^2$

k  $\frac{10^3}{10^q}$

l  $(c^4)^m$

**Example 10** **Self Tutor**

Write as powers of 2:

**a** 16

**b**  $\frac{1}{16}$

**c** 1

**d**  $4 \times 2^n$

**e**  $\frac{2^m}{8}$

$$\begin{aligned}\mathbf{a} \quad & 16 \\ & = 2 \times 2 \times 2 \times 2 \\ & = 2^4\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & \frac{1}{16} \\ & = \frac{1}{2^4} \\ & = 2^{-4}\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad & 1 \\ & = 2^0\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad & 4 \times 2^n \\ & = 2^2 \times 2^n \\ & = 2^{2+n}\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad & \frac{2^m}{8} \\ & = \frac{2^m}{2^3} \\ & = 2^{m-3}\end{aligned}$$

**2** Write as powers of 2:

**a** 4

**b**  $\frac{1}{4}$

**c** 8

**d**  $\frac{1}{8}$

**e** 32

**f**  $\frac{1}{32}$

**g** 2

**h**  $\frac{1}{2}$

**i** 64

**j**  $\frac{1}{64}$

**k** 128

**l**  $\frac{1}{128}$

**3** Write as powers of 3:

**a** 9

**b**  $\frac{1}{9}$

**c** 27

**d**  $\frac{1}{27}$

**e** 3

**f**  $\frac{1}{3}$

**g** 81

**h**  $\frac{1}{81}$

**i** 1

**j** 243

**k**  $\frac{1}{243}$

**4** Write as a single power of 2:

**a**  $2 \times 2^a$

**b**  $4 \times 2^b$

**c**  $8 \times 2^t$

**d**  $(2^{x+1})^2$

**e**  $(2^{1-n})^{-1}$

**f**  $\frac{2^c}{4}$

**g**  $\frac{2^m}{2^{-m}}$

**h**  $\frac{4}{2^{1-n}}$

**i**  $\frac{2^{x+1}}{2^x}$

**j**  $\frac{4^x}{2^{1-x}}$

**5** Write as a single power of 3:

**a**  $9 \times 3^p$

**b**  $27^a$

**c**  $3 \times 9^n$

**d**  $27 \times 3^d$

**e**  $9 \times 27^t$

**f**  $\frac{3^y}{3}$

**g**  $\frac{3}{3^y}$

**h**  $\frac{9}{27^t}$

**i**  $\frac{9^a}{3^{1-a}}$

**j**  $\frac{9^{n+1}}{3^{2n-1}}$

**Example 11** **Self Tutor**

Write in simplest form, without brackets:

**a**  $(-3a^2)^4$

**b**  $\left(-\frac{2a^2}{b}\right)^3$

$$\begin{aligned}\mathbf{a} \quad & (-3a^2)^4 \\ & = (-3)^4 \times (a^2)^4 \\ & = 81 \times a^{2 \times 4} \\ & = 81a^8\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & \left(-\frac{2a^2}{b}\right)^3 \\ & = \frac{(-2)^3 \times (a^2)^3}{b^3} \\ & = \frac{-8a^6}{b^3}\end{aligned}$$

**6** Write without brackets:

<b>a</b> $(2a)^2$	<b>b</b> $(3b)^3$	<b>c</b> $(ab)^4$	<b>d</b> $(pq)^3$	<b>e</b> $\left(\frac{m}{n}\right)^2$
<b>f</b> $\left(\frac{a}{3}\right)^3$	<b>g</b> $\left(\frac{b}{c}\right)^4$	<b>h</b> $\left(\frac{2a}{b}\right)^0$	<b>i</b> $\left(\frac{m}{3n}\right)^4$	<b>j</b> $\left(\frac{xy}{2}\right)^3$

**7** Write the following in simplest form, without brackets:

<b>a</b> $(-2a)^2$	<b>b</b> $(-6b^2)^2$	<b>c</b> $(-2a)^3$	<b>d</b> $(-3m^2n^2)^3$
<b>e</b> $(-2ab^4)^4$	<b>f</b> $\left(\frac{-2a^2}{b^2}\right)^3$	<b>g</b> $\left(\frac{-4a^3}{b}\right)^2$	<b>h</b> $\left(\frac{-3p^2}{q^3}\right)^2$
<b>i</b> $\frac{(2x^2y)^2}{x}$	<b>j</b> $\frac{(4a^2b)^3}{2ab^2}$	<b>k</b> $\frac{(-5a^6b^3)^2}{5b^8}$	<b>l</b> $\frac{(-2x^7y^4)^3}{4x^3y^{15}}$

### Example 12

### Self Tutor

Write without negative exponents:  $\frac{a^{-3}b^2}{c^{-1}}$

$$\begin{aligned} a^{-3} &= \frac{1}{a^3} \quad \text{and} \quad \frac{1}{c^{-1}} = c^1 \\ \therefore \frac{a^{-3}b^2}{c^{-1}} &= \frac{b^2c}{a^3} \end{aligned}$$

**8** Write without negative exponents:

<b>a</b> $ab^{-2}$	<b>b</b> $(ab)^{-2}$	<b>c</b> $(2ab^{-1})^2$	<b>d</b> $(3a^{-2}b)^2$	<b>e</b> $\frac{a^2b^{-1}}{c^2}$
<b>f</b> $\frac{a^2b^{-1}}{c^{-2}}$	<b>g</b> $\frac{1}{a^{-3}}$	<b>h</b> $\frac{a^{-2}}{b^{-3}}$	<b>i</b> $\frac{2a^{-1}}{d^2}$	<b>j</b> $\frac{12a}{m^{-3}}$

### Example 13

### Self Tutor

Write  $\frac{1}{2^{1-n}}$  in non-fractional form.

$$\begin{aligned} \frac{1}{2^{1-n}} &= 2^{-(1-n)} \\ &= 2^{-1+n} \\ &= 2^{n-1} \end{aligned}$$

**9** Write in non-fractional form:

<b>a</b> $\frac{1}{a^n}$	<b>b</b> $\frac{1}{b^{-n}}$	<b>c</b> $\frac{1}{3^{2-n}}$	<b>d</b> $\frac{a^n}{b^{-m}}$	<b>e</b> $\frac{a^{-n}}{a^{2+n}}$
--------------------------	-----------------------------	------------------------------	-------------------------------	-----------------------------------

**10** Simplify, giving your answers in simplest rational form:

<b>a</b> $\left(\frac{5}{3}\right)^0$	<b>b</b> $\left(\frac{7}{4}\right)^{-1}$	<b>c</b> $\left(\frac{1}{6}\right)^{-1}$	<b>d</b> $\frac{3^3}{3^0}$
<b>e</b> $\left(\frac{4}{3}\right)^{-2}$	<b>f</b> $2^1 + 2^{-1}$	<b>g</b> $\left(1\frac{2}{3}\right)^{-3}$	<b>h</b> $5^2 + 5^1 + 5^{-1}$

**11** Write as powers of 2, 3 and/or 5:

**a**  $\frac{1}{9}$

**b**  $\frac{1}{16}$

**c**  $\frac{1}{125}$

**d**  $\frac{3}{5}$

**e**  $\frac{4}{27}$

**f**  $\frac{2^c}{8 \times 9}$

**g**  $\frac{9^k}{10}$

**h**  $\frac{6^p}{75}$

**12** Read about Nicomachus' pattern on page 108 and find the series of odd numbers for:

**a**  $5^3$

**b**  $7^3$

**c**  $12^3$

## D

## RATIONAL INDICES

The index laws used previously can also be applied to **rational indices**, or indices which are written as a fraction.

The notation  $a^n$  is defined to mean “ $a$  multiplied together  $n$  times”. Since we cannot multiply  $a$  together “half a time”, the notation  $a^{\frac{1}{2}}$  is an extension of the meaning of this notation. The goal is to extend the meaning of  $a^n$  so that the fundamental law

$$a^n a^m = a^{n+m}$$

remains true. If we assume that  $a > 0$  then this law holds for rational indices.

Since  $x^3 = -8$  has  $x = -2$  as a solution, we would like to write

$$x = x^{\frac{3}{3}} = (x^3)^{\frac{1}{3}} = (-8)^{\frac{1}{3}} = (-2^3)^{\frac{1}{3}} = (-2)^{\frac{3}{3}} = -2.$$

Under some circumstances it is therefore possible to extend the meaning of  $a^n$  when  $n$  is rational and  $a \leq 0$ . However, this is not generally so easy, and so for this course we restrict ourselves to cases where  $a > 0$ .

For  $a > 0$ , notice that  $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2} + \frac{1}{2}} = a^1 = a$  {index laws}  
and  $\sqrt{a} \times \sqrt{a} = a$  also.

So,  $a^{\frac{1}{2}} = \sqrt{a}$  {by direct comparison}

Likewise  $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1 = a$   
and  $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$

suggests  $a^{\frac{1}{3}} = \sqrt[3]{a}$

In general,  $a^{\frac{1}{n}} = \sqrt[n]{a}$  where  $\sqrt[n]{a}$  reads ‘the  $n$ th root of  $a$ ’, for  $n \in \mathbb{Z}^+$ .

We can now determine that  $\sqrt[n]{a^m}$

$$\begin{aligned} &= (a^m)^{\frac{1}{n}} \\ &= a^{\frac{m}{n}} \end{aligned}$$

$\therefore a^{\frac{m}{n}} = \sqrt[n]{a^m}$  for  $a > 0$ ,  $n \in \mathbb{Z}^+$ ,  $m \in \mathbb{Z}$

**Example 14** **Self Tutor**

Write as a single power of 2:

**a**  $\sqrt[3]{2}$

**b**  $\frac{1}{\sqrt{2}}$

**c**  $\sqrt[5]{4}$

**a**  $\sqrt[3]{2}$

$$= 2^{\frac{1}{3}}$$

**b**  $\frac{1}{\sqrt{2}}$

$$= \frac{1}{2^{\frac{1}{2}}} \\ = 2^{-\frac{1}{2}}$$

**c**  $\sqrt[5]{4}$

$$= (2^2)^{\frac{1}{5}} \\ = 2^{2 \times \frac{1}{5}} \\ = 2^{\frac{2}{5}}$$

**EXERCISE 4D**

**1** Write as a single power of 2:

**a**  $\sqrt[5]{2}$

**b**  $\frac{1}{\sqrt[5]{2}}$

**c**  $2\sqrt{2}$

**d**  $4\sqrt{2}$

**e**  $\frac{1}{\sqrt[3]{2}}$

**f**  $2 \times \sqrt[3]{2}$

**g**  $\frac{4}{\sqrt{2}}$

**h**  $(\sqrt{2})^3$

**i**  $\frac{1}{\sqrt[3]{16}}$

**j**  $\frac{1}{\sqrt{8}}$

**2** Write as a single power of 3:

**a**  $\sqrt[3]{3}$

**b**  $\frac{1}{\sqrt[3]{3}}$

**c**  $\sqrt[4]{3}$

**d**  $3\sqrt{3}$

**e**  $\frac{1}{\sqrt[9]{3}}$

**3** Write the following in the form  $a^x$  where  $a$  is a prime number and  $x$  is rational:

**a**  $\sqrt[3]{7}$

**b**  $\sqrt[4]{27}$

**c**  $\sqrt[5]{16}$

**d**  $\sqrt[3]{32}$

**e**  $\sqrt[7]{49}$

**f**  $\frac{1}{\sqrt[3]{7}}$

**g**  $\frac{1}{\sqrt[4]{27}}$

**h**  $\frac{1}{\sqrt[5]{16}}$

**i**  $\frac{1}{\sqrt[3]{32}}$

**j**  $\frac{1}{\sqrt[7]{49}}$

**4** Use your calculator to evaluate:

**a**  $3^{\frac{3}{4}}$

**b**  $2^{\frac{7}{8}}$

**c**  $2^{-\frac{1}{3}}$

**d**  $4^{-\frac{3}{5}}$

**e**  $\sqrt[4]{8}$

**f**  $\sqrt[5]{27}$

**g**  $\frac{1}{\sqrt[3]{7}}$

**Example 15** **Self Tutor**

Without using a calculator, write in simplest rational form:

**a**  $8^{\frac{4}{3}}$

**b**  $27^{-\frac{2}{3}}$

**a**  $8^{\frac{4}{3}}$

**b**  $27^{-\frac{2}{3}}$

$$= (2^3)^{\frac{4}{3}}$$

$$= (3^3)^{-\frac{2}{3}}$$

$$= 2^{3 \times \frac{4}{3}}$$

$$= 3^{3 \times -\frac{2}{3}}$$

$$= 2^4$$

$$= 3^{-2}$$

$$= 16$$

$$= \frac{1}{9}$$

5 Without using a calculator, write in simplest rational form:

a  $4^{\frac{3}{2}}$

b  $8^{\frac{5}{3}}$

c  $16^{\frac{3}{4}}$

d  $25^{\frac{3}{2}}$

e  $32^{\frac{2}{5}}$

f  $4^{-\frac{1}{2}}$

g  $9^{-\frac{3}{2}}$

h  $8^{-\frac{4}{3}}$

i  $27^{-\frac{4}{3}}$

j  $125^{-\frac{2}{3}}$

## E

# ALGEBRAIC EXPANSION AND FACTORISATION

## EXPANSION

We can use the usual expansion laws to simplify expressions containing indices:

$$\begin{aligned} a(b+c) &= ab + ac \\ (a+b)(c+d) &= ac + ad + bc + bd \\ (a+b)(a-b) &= a^2 - b^2 \\ (a+b)^2 &= a^2 + 2ab + b^2 \\ (a-b)^2 &= a^2 - 2ab + b^2 \end{aligned}$$

### Example 16



Expand and simplify:  $x^{-\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}})$

$$\begin{aligned} &x^{-\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}) \\ &= x^{-\frac{1}{2}} \times x^{\frac{3}{2}} + x^{-\frac{1}{2}} \times 2x^{\frac{1}{2}} - x^{-\frac{1}{2}} \times 3x^{-\frac{1}{2}} \quad \{\text{each term is } \times \text{ by } x^{-\frac{1}{2}}\} \\ &= x^1 + 2x^0 - 3x^{-1} \quad \{\text{adding exponents}\} \\ &= x + 2 - \frac{3}{x} \end{aligned}$$

### Example 17



Expand and simplify:

a  $(2^x + 3)(2^x + 1)$

b  $(7^x + 7^{-x})^2$

$$\begin{aligned} \mathbf{a} \quad &(2^x + 3)(2^x + 1) \\ &= 2^x \times 2^x + 2^x + 3 \times 2^x + 3 \\ &= 2^{2x} + 4 \times 2^x + 3 \\ &= 4^x + 2^{2+x} + 3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad &(7^x + 7^{-x})^2 \\ &= (7^x)^2 + 2 \times 7^x \times 7^{-x} + (7^{-x})^2 \\ &= 7^{2x} + 2 \times 7^0 + 7^{-2x} \\ &= 7^{2x} + 2 + 7^{-2x} \end{aligned}$$

## EXERCISE 4E.1

1 Expand and simplify:

a  $x^2(x^3 + 2x^2 + 1)$

b  $2^x(2^x + 1)$

c  $x^{\frac{1}{2}}(x^{\frac{1}{2}} + x^{-\frac{1}{2}})$

d  $7^x(7^x + 2)$

e  $3^x(2 - 3^{-x})$

f  $x^{\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}})$

g  $2^{-x}(2^x + 5)$

h  $5^{-x}(5^{2x} + 5^x)$

i  $x^{-\frac{1}{2}}(x^2 + x + x^{\frac{1}{2}})$

**2** Expand and simplify:

**a**  $(2^x - 1)(2^x + 3)$

**b**  $(3^x + 2)(3^x + 5)$

**c**  $(5^x - 2)(5^x - 4)$

**d**  $(2^x + 3)^2$

**e**  $(3^x - 1)^2$

**f**  $(4^x + 7)^2$

**3** Expand and simplify:

**a**  $(x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} - 2)$

**b**  $(2^x + 3)(2^x - 3)$

**c**  $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$

**d**  $(x + \frac{2}{x})^2$

**e**  $(7^x - 7^{-x})^2$

**f**  $(5 - 2^{-x})^2$

**g**  $(x^{\frac{2}{3}} + x^{\frac{1}{3}})^2$

**h**  $(x^{\frac{3}{2}} - x^{\frac{1}{2}})^2$

**i**  $(2x^{\frac{1}{2}} - x^{-\frac{1}{2}})^2$

## FACTORISATION AND SIMPLIFICATION

### Example 18



Factorise: **a**  $2^{n+3} + 2^n$

$$\begin{aligned}\mathbf{a} \quad & 2^{n+3} + 2^n \\ &= 2^n 2^3 + 2^n \\ &= 2^n(2^3 + 1) \\ &= 2^n \times 9\end{aligned}$$

**b**  $2^{n+3} + 8$

$$\begin{aligned}\mathbf{b} \quad & 2^{n+3} + 8 \\ &= 2^n 2^3 + 8 \\ &= 8(2^n) + 8 \\ &= 8(2^n + 1)\end{aligned}$$

**c**  $2^{3n} + 2^{2n}$

$$\begin{aligned}\mathbf{c} \quad & 2^{3n} + 2^{2n} \\ &= 2^{2n} 2^n + 2^{2n} \\ &= 2^{2n}(2^n + 1)\end{aligned}$$

### Example 19



Factorise: **a**  $4^x - 9$

$$\begin{aligned}\mathbf{a} \quad & 4^x - 9 \\ &= (2^x)^2 - 3^2 \\ &= (2^x + 3)(2^x - 3)\end{aligned}$$

**b**  $9^x + 4(3^x) + 4$

$$\begin{aligned}\mathbf{b} \quad & 9^x + 4(3^x) + 4 \\ &= (3^x)^2 + 4(3^x) + 4 \\ &= (3^x + 2)^2\end{aligned}$$

{compare  $a^2 - b^2 = (a + b)(a - b)$ }

{compare  $a^2 + 4a + 4$ }

{as  $a^2 + 4a + 4 = (a + 2)^2$ }

## EXERCISE 4E.2

**1** Factorise:

**a**  $5^{2x} + 5^x$

**b**  $3^{n+2} + 3^n$

**c**  $7^n + 7^{3n}$

**d**  $5^{n+1} - 5$

**e**  $6^{n+2} - 6$

**f**  $4^{n+2} - 16$

**2** Factorise:

**a**  $9^x - 4$

**b**  $4^x - 25$

**c**  $16 - 9^x$

**d**  $25 - 4^x$

**e**  $9^x - 4^x$

**f**  $4^x + 6(2^x) + 9$

**g**  $9^x + 10(3^x) + 25$

**h**  $4^x - 14(2^x) + 49$

**i**  $25^x - 4(5^x) + 4$

**3** Factorise:

a  $4^x + 9(2^x) + 18$

d  $9^x + 4(3^x) - 5$

b  $4^x - 2^x - 20$

e  $25^x + 5^x - 2$

c  $9^x + 9(3^x) + 14$

f  $49^x - 7^{x+1} + 12$

### Example 20

### Self Tutor

Simplify:

a  $\frac{6^n}{3^n}$

b  $\frac{4^n}{6^n}$

a  $\frac{6^n}{3^n}$  or  $\frac{6^n}{3^n}$   
 $= \frac{2^n 3^n}{1 3^n}$   
 $= 2^n$

b  $\frac{4^n}{6^n}$  or  $\frac{4^n}{6^n}$   
 $= \frac{2^n 2^n}{2^n 3^n}$   
 $= \frac{2^n}{3^n}$

$= \left(\frac{6}{3}\right)^n$   
 $= \left(\frac{4}{6}\right)^n$   
 $= \left(\frac{2}{3}\right)^n$

**4** Simplify:

a  $\frac{12^n}{6^n}$

b  $\frac{20^a}{2^a}$

c  $\frac{6^b}{2^b}$

d  $\frac{4^n}{20^n}$

e  $\frac{35^x}{7^x}$

f  $\frac{6^a}{8^a}$

g  $\frac{5^{n+1}}{5^n}$

h  $\frac{5^{n+1}}{5}$

### Example 21

### Self Tutor

Simplify:

a  $\frac{3^n + 6^n}{3^n}$

b  $\frac{2^{m+2} - 2^m}{2^m}$

c  $\frac{2^{m+3} + 2^m}{9}$

a  $\frac{3^n + 6^n}{3^n}$   
 $= \frac{3^n + 2^n 3^n}{3^n}$   
 $= \frac{3^n (1 + 2^n)}{1 3^n}$   
 $= 1 + 2^n$

b  $\frac{2^{m+2} - 2^m}{2^m}$   
 $= \frac{2^m 2^2 - 2^m}{2^m}$   
 $= \frac{2^m (4 - 1)}{1 2^m}$   
 $= 3$

c  $\frac{2^{m+3} + 2^m}{9}$   
 $= \frac{2^m 2^3 + 2^m}{9}$   
 $= \frac{2^m (8 + 1)}{1 9}$   
 $= 2^m$

**5** Simplify:

a  $\frac{6^m + 2^m}{2^m}$

b  $\frac{2^n + 12^n}{2^n}$

c  $\frac{8^n + 4^n}{2^n}$

d  $\frac{12^x - 3^x}{3^x}$

e  $\frac{6^n + 12^n}{1 + 2^n}$

f  $\frac{5^{n+1} - 5^n}{4}$

g  $\frac{5^{n+1} - 5^n}{5^n}$

h  $\frac{4^n - 2^n}{2^n}$

i  $\frac{2^n - 2^{n-1}}{2^n}$

**6** Simplify:

a  $2^n(n+1) + 2^n(n-1)$

b  $3^n \left(\frac{n-1}{6}\right) - 3^n \left(\frac{n+1}{6}\right)$

## F

**EXPONENTIAL EQUATIONS**

An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent.

For example:  $2^x = 8$  and  $30 \times 3^x = 7$  are both exponential equations.

There are a number of methods we can use to solve exponential equations. These include graphing, using technology, and by using **logarithms**, which we will study in **Chapter 5**. However, in some cases we can solve algebraically.

If the base numbers are the same, we can **equate indices**.

If  $a^x = a^k$  then  $x = k$ .

For example, if  $2^x = 8$  then  $2^x = 2^3$ . Thus  $x = 3$ , and this is the only solution.

Remember that  
 $a > 0$ .

**Example 22****Self Tutor**

Solve for  $x$ :

**a**  $2^x = 16$

**b**  $3^{x+2} = \frac{1}{27}$

**a**  $2^x = 16$

$\therefore 2^x = 2^4$

$\therefore x = 4$

**b**  $3^{x+2} = \frac{1}{27}$

$\therefore 3^{x+2} = 3^{-3}$

$\therefore x + 2 = -3$

$\therefore x = -5$

Once we have the  
same base we then  
equate the indices.

**Example 23****Self Tutor**

Solve for  $x$ :

**a**  $4^x = 8$

**b**  $9^{x-2} = \frac{1}{3}$

**a**  $4^x = 8$

$\therefore (2^2)^x = 2^3$

$\therefore 2^{2x} = 2^3$

$\therefore 2x = 3$

$\therefore x = \frac{3}{2}$

**b**  $9^{x-2} = \frac{1}{3}$

$\therefore (3^2)^{x-2} = 3^{-1}$

$\therefore 3^{2(x-2)} = 3^{-1}$

$\therefore 2(x-2) = -1$

$\therefore 2x - 4 = -1$

$\therefore 2x = 3$

$\therefore x = \frac{3}{2}$

**EXERCISE 4F****1** Solve for  $x$ :

**a**  $2^x = 8$

**e**  $3^x = \frac{1}{3}$

**i**  $2^{x-2} = \frac{1}{32}$

**b**  $5^x = 25$

**f**  $2^x = \sqrt{2}$

**j**  $3^{x+1} = \frac{1}{27}$

**c**  $3^x = 81$

**g**  $5^x = \frac{1}{125}$

**k**  $7^{x+1} = 343$

**d**  $7^x = 1$

**h**  $4^{x+1} = 64$

**l**  $5^{1-2x} = \frac{1}{5}$

**2** Solve for  $x$ :

**a**  $8^x = 32$

**e**  $27^x = \frac{1}{9}$

**i**  $4^{4x-1} = \frac{1}{2}$

**m**  $81^x = 27^{-x}$

**b**  $4^x = \frac{1}{8}$

**f**  $16^x = \frac{1}{32}$

**j**  $9^{x-3} = 27$

**n**  $(\frac{1}{4})^{1-x} = 32$

**c**  $9^x = \frac{1}{27}$

**g**  $4^{x+2} = 128$

**k**  $(\frac{1}{2})^{x+1} = 8$

**o**  $(\frac{1}{7})^x = 49$

**d**  $25^x = \frac{1}{5}$

**h**  $25^{1-x} = \frac{1}{125}$

**l**  $(\frac{1}{3})^{x+2} = 9$

**p**  $(\frac{1}{3})^{x+1} = 243$

**3** Solve for  $x$ , if possible:

**a**  $4^{2x+1} = 8^{1-x}$

**b**  $9^{2-x} = (\frac{1}{3})^{2x+1}$

**c**  $2^x \times 8^{1-x} = \frac{1}{4}$

**4** Solve for  $x$ :

**a**  $\frac{3^{2x+1}}{3^x} = 9^x$

**b**  $\frac{25^x}{5^{x+4}} = 25^{1-x}$

**c**  $\frac{4^x}{2^{x+2}} = \frac{2^{x+1}}{8^x}$

**d**  $\frac{5^{2x-5}}{125^x} = \frac{25^{1-2x}}{5^{x+2}}$

**e**  $\frac{4^x}{8^{2-x}} = 2^x \times 4^{x-1}$

**f**  $\frac{9^{2x}}{27^{2-x}} = \frac{81^{3x+1}}{3^{1-2x}}$

**5** Solve for  $x$ :

**a**  $3 \times 2^x = 24$

**b**  $7 \times 2^x = 28$

**c**  $3 \times 2^{x+1} = 24$

**d**  $12 \times 3^{-x} = \frac{4}{3}$

**e**  $4 \times (\frac{1}{3})^x = 36$

**f**  $5 \times (\frac{1}{2})^x = 20$

**Example 24****Self Tutor**Solve for  $x$ :  $4^x + 2^x - 20 = 0$ 

$$4^x + 2^x - 20 = 0$$

$$\therefore (2^x)^2 + 2^x - 20 = 0$$

$$\therefore (2^x - 4)(2^x + 5) = 0$$

$$\therefore 2^x = 4 \text{ or } 2^x = -5$$

$$\therefore 2^x = 2^2$$

$$\therefore x = 2 \quad \{2^x \text{ cannot be negative}\}$$

{compare  $a^2 + a - 20 = 0$ }{ $a^2 + a - 20 = (a - 4)(a + 5)$ }**6** Solve for  $x$ :

**a**  $4^x - 6(2^x) + 8 = 0$

**b**  $4^x - 2^x - 2 = 0$

**c**  $9^x - 12(3^x) + 27 = 0$

**d**  $9^x = 3^x + 6$

**e**  $25^x - 23(5^x) - 50 = 0$

**f**  $49^x + 1 = 2(7^x)$

**G****EXPONENTIAL FUNCTIONS**

We have already seen how to evaluate  $b^n$  when  $n \in \mathbb{Q}$ , or in other words when  $n$  is a rational number.

But what about  $b^n$  when  $n \in \mathbb{R}$ , so  $n$  is real but not necessarily rational?

To answer this question, we can look at graphs of exponential functions.

The most simple general **exponential function** has the form  $y = b^x$  where  $b > 0$ ,  $b \neq 1$ .

For example,  $y = 2^x$  is an exponential function.

We construct a table of values from which we graph the function:

$x$	-3	-2	-1	0	1	2	3
$y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

When  $x = -10$ ,  $y = 2^{-10} \approx 0.001$ .

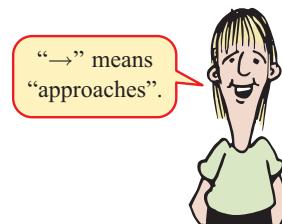
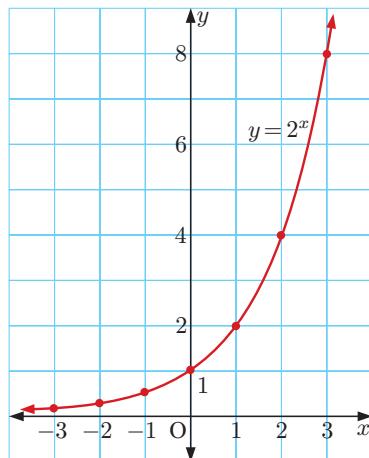
When  $x = -50$ ,  $y = 2^{-50} \approx 8.88 \times 10^{-16}$ .

As  $x$  becomes large and negative, the graph of  $y = 2^x$  approaches the  $x$ -axis from above but never touches it, since  $2^x$  becomes very small but never zero.

So, as  $x \rightarrow -\infty$ ,  $y \rightarrow 0$  from above.

We say that  $y = 2^x$  is ‘**asymptotic** to the  $x$ -axis’ or ‘ $y = 0$  is a **horizontal asymptote**’.

We now have a well-defined meaning for  $b^n$  where  $b, n \in \mathbb{R}$  because simple exponential functions have smooth increasing or decreasing graphs.

**Discovery 1****Graphs of exponential functions**

In this Discovery we examine the graphs of various families of exponential functions.

Click on the icon to run the **dynamic graphing package**, or else you could use your **graphics calculator**.

**What to do:**

- 1 Explore the family of curves of the form  $y = b^x$  where  $b > 0$ .

For example, consider  $y = 2^x$ ,  $y = 3^x$ ,  $y = 10^x$ , and  $y = (1.3)^x$ .

- a What effect does changing  $b$  have on the shape of the graph?
- b What is the  $y$ -intercept of each graph?
- c What is the horizontal asymptote of each graph?

**DYNAMIC  
GRAPHING  
PACKAGE**



- 2** Explore the family of curves of the form  $y = 2^x + d$  where  $d$  is a constant.  
 For example, consider  $y = 2^x$ ,  $y = 2^x + 1$ , and  $y = 2^x - 2$ .
- What effect does changing  $d$  have on the position of the graph?
  - What effect does changing  $d$  have on the shape of the graph?
  - What is the horizontal asymptote of each graph?
  - What is the horizontal asymptote of  $y = 2^x + d$ ?
  - To graph  $y = 2^x + d$  from  $y = 2^x$  what transformation is used?
- 3** Explore the family of curves of the form  $y = 2^{x-c}$ .  
 For example, consider  $y = 2^x$ ,  $y = 2^{x-1}$ ,  $y = 2^{x+2}$ , and  $y = 2^{x-3}$ .
- What effect does changing  $c$  have on the position of the graph?
  - What effect does changing  $c$  have on the shape of the graph?
  - What is the horizontal asymptote of each graph?
  - To graph  $y = 2^{x-c}$  from  $y = 2^x$  what transformation is used?
- 4** Explore the relationship between  $y = b^x$  and  $y = b^{-x}$  where  $b > 0$ .  
 For example, consider  $y = 2^x$  and  $y = 2^{-x}$ .
- What is the  $y$ -intercept of each graph?
  - What is the horizontal asymptote of each graph?
  - What transformation moves  $y = 2^x$  to  $y = 2^{-x}$ ?
- 5** Explore the family of curves of the form  $y = a \times 2^x$  where  $a$  is a constant.
- Consider functions where  $a > 0$ , such as  $y = 2^x$ ,  $y = 3 \times 2^x$ , and  $y = \frac{1}{2} \times 2^x$ .  
 Comment on the effect on the graph.
  - Consider functions where  $a < 0$ , such as  $y = -2^x$ ,  $y = -3 \times 2^x$ , and  $y = -\frac{1}{2} \times 2^x$ .  
 Comment on the effect on the graph.
  - What is the horizontal asymptote of each graph? Explain your answer.

From **Discovery 1** you should have found that:

For the general exponential function  $y = a \times b^{x-c} + d$  where  $b > 0$ ,  $b \neq 1$ ,  $a \neq 0$ :

- $b$  controls how steeply the graph increases or decreases
  - $c$  controls horizontal translation
  - $d$  controls vertical translation
  - the equation of the horizontal asymptote is  $y = d$
  - if  $a > 0$ ,  $b > 1$   
 the function is increasing
  - if  $a > 0$ ,  $0 < b < 1$   
 the function is decreasing
  - if  $a < 0$ ,  $b > 1$   
 the function is decreasing
  - if  $a < 0$ ,  $0 < b < 1$   
 the function is increasing.
-

We can sketch reasonably accurate graphs of exponential functions using:

- the horizontal asymptote
- the  $y$ -intercept
- two other points, for example, when  $x = 2$ ,  $x = -2$

All exponential graphs are similar in shape and have a horizontal asymptote.



### Example 25

### Self Tutor

Sketch the graph of  $y = 2^{-x} - 3$ .

Hence state the domain and range of  $f(x) = 2^{-x} - 3$ .

For  $y = 2^{-x} - 3$ ,  
the horizontal asymptote is  $y = -3$ .

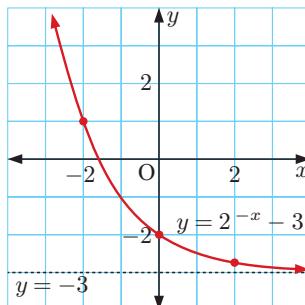
$$\begin{aligned}\text{When } x = 0, \quad y &= 2^0 - 3 \\ &= 1 - 3 \\ &= -2\end{aligned}$$

$\therefore$  the  $y$ -intercept is  $-2$ .

$$\begin{aligned}\text{When } x = 2, \quad y &= 2^{-2} - 3 \\ &= \frac{1}{4} - 3 \\ &= -2\frac{3}{4}\end{aligned}$$

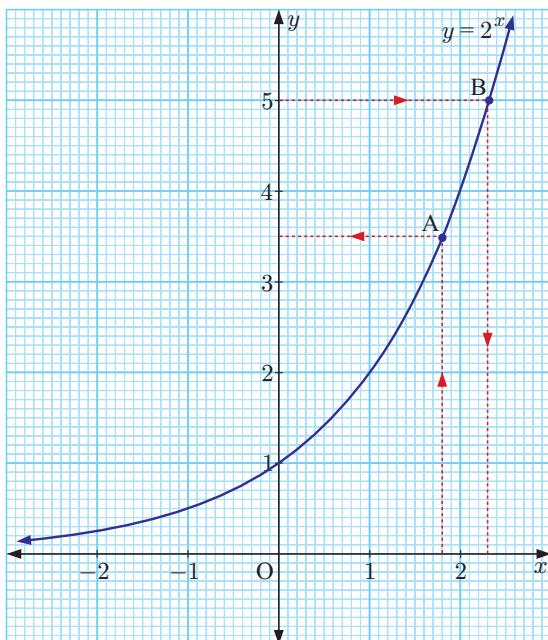
$$\text{When } x = -2, \quad y = 2^2 - 3 = 1$$

The domain is  $\{x : x \in \mathbb{R}\}$ . The range is  $\{y : y > -3\}$ .



Consider the graph of  $y = 2^x$  alongside. We can use the graph to estimate:

- the value of  $2^x$  for a given value of  $x$ , for example  $2^{1.8} \approx 3.5$  {point A}
- the solutions of the exponential equation  $2^x = b$ , for example if  $2^x = 5$  then  $x \approx 2.3$  {point B}.



**EXERCISE 4G**

**1** Use the graph of  $y = 2^x$  to estimate the value of:

**a**  $2^{\frac{1}{2}}$  or  $\sqrt{2}$

**b**  $2^{0.8}$

**c**  $2^{1.5}$

**d**  $2^{-\sqrt{2}}$

**2** Use the graph of  $y = 2^x$  to estimate the solution to:

**a**  $2^x = 3$

**b**  $2^x = 0.6$

**c**  $2^x = 4.3$

**d**  $2^x = 0.3$

**3** Use the graph of  $y = 2^x$  to explain why  $2^x = 0$  has no solutions.

**4** Suppose  $f(x) = 2 \times 3^x$ . Find:

**a**  $f(0)$

**b**  $f(3)$

**c**  $f(-2)$

**5** Suppose  $g(x) = 5^x + 2$ .

**a** Find  $g(0)$  and  $g(-1)$ .

**b** Find  $a$  such that  $g(a) = 27$ .

**6** Draw freehand sketches of the following pairs of graphs using your observations from the previous **Discovery**:

**a**  $y = 2^x$  and  $y = 2^x - 2$

**b**  $y = 2^x$  and  $y = 2^{-x}$

**c**  $y = 2^x$  and  $y = 2^{x-2}$

**d**  $y = 2^x$  and  $y = 2(2^x)$

**GRAPHING PACKAGE**



**7** Draw freehand sketches of the following pairs of graphs:

**a**  $y = 3^x$  and  $y = 3^{-x}$

**b**  $y = 3^x$  and  $y = 3^x + 1$

**c**  $y = 3^x$  and  $y = -3^x$

**d**  $y = 3^x$  and  $y = 3^{x-1}$

**8** For each of the functions below:

**i** Sketch the graph of the function.

**ii** State the domain and range.

**iii** Use your calculator to find the value of  $y$  when  $x = \sqrt{2}$ .

**iv** Discuss the behaviour of  $y$  as  $x \rightarrow \pm\infty$ .

**v** Determine the horizontal asymptotes.

**a**  $y = 2^x + 1$

**b**  $f(x) = 2 - 2^x$

**c**  $y = 2^{-x} + 3$

**d**  $f(x) = 3 - 2^{-x}$

**Example 26****Self Tutor**

An entomologist monitoring a grasshopper plague notices that the area affected by the grasshoppers is given by  $A(n) = 1000 \times 2^{0.2n}$  hectares, where  $n$  is the number of weeks after the initial observation.

**a** Find the original affected area.

**b** Find the affected area after:

**i** 5 weeks    **ii** 10 weeks    **iii** 12 weeks.

**c** Draw the graph of  $A$  against  $n$ .

**d** How long will it take for the affected area to reach 8000 hectares?



**a**  $A(0) = 1000 \times 2^0$

$$= 1000 \times 1$$

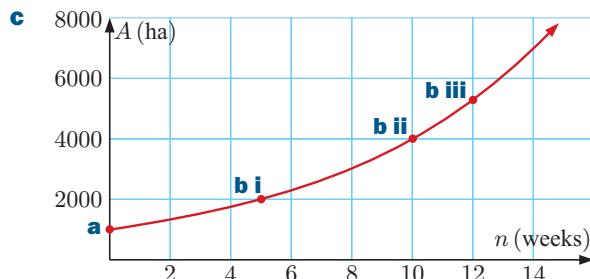
$$= 1000 \quad \therefore \text{the original affected area was } 1000 \text{ ha.}$$

**b** **i**  $A(5) = 1000 \times 2^1$   
 $= 2000$

The affected area is 2000 ha.

**iii**  $A(12) = 1000 \times 2^{0.2 \times 12}$   
 $= 1000 \times 2^{2.4}$   
 $\approx 5280$

The affected area is about 5280 ha.



**ii**  $A(10) = 1000 \times 2^2$   
 $= 4000$

The affected area is 4000 ha.

**d** We need to find  $n$  such that

$$\begin{aligned} A(n) &= 8000 \\ \therefore 1000 \times 2^{0.2n} &= 8000 \\ \therefore 2^{0.2n} &= 8 \\ \therefore 2^{0.2n} &= 2^3 \\ \therefore 0.2n &= 3 \\ \therefore n &= 15 \end{aligned}$$

So, it will take 15 weeks.

- 9** A breeding program to ensure the survival of pygmy possums is established with an initial population of 50 (25 pairs). From a previous program, the expected population  $P$  in  $n$  years' time is given by  $P(n) = P_0 \times 2^{0.5n}$ .

- a** What is the value of  $P_0$ ?  
**b** What is the expected population after:  
**i** 2 years      **ii** 6 years      **iii** 10 years?  
**c** Sketch the graph of  $P$  against  $n$  using **a** and **b** only.  
**d** How long will it take for the population to reach 800?

- 10** The weight  $W$  of bacteria in a culture  $t$  hours after establishment is given by  $W(t) = 100 \times 3^{0.1t}$  grams.

- a** Find the initial weight.  
**b** Find the weight after:    **i** 5 hours    **ii** 10 hours    **iii** 24 hours.  
**c** Sketch the graph of  $W$  against  $t$  using the results of **a** and **b** only.  
**d** How long will it take for the weight to reach 900 g?

- 11** The current flowing in an electrical circuit  $t$  seconds after it is switched off is given by  $I(t) = 32 \times 4^{-t}$  amps.

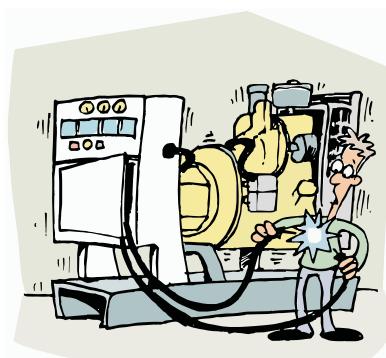
- a** What current was flowing at the instant when it was switched off?  
**b** What current was still flowing after:  
**i** 1 second      **ii** 2 seconds?  
**c** Plot  $I$  against  $t$ .  
**d** How long will it take for the current to reach  $\frac{1}{2}$  amp?

- 12** Answer the **Opening Problem** on page 102.



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#### GRAPHING PACKAGE



**H****THE NATURAL EXPONENTIAL  $e^x$** 

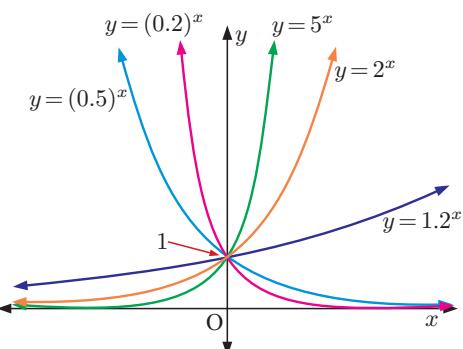
We have seen that the simplest exponential functions are of the form  $f(x) = b^x$  where  $b > 0$ ,  $b \neq 1$ .

Graphs of some of these functions are shown alongside.

We can see that for all positive values of the base  $b$ , the graph is always positive.

Hence  $b^x > 0$  for all  $b > 0$ .

There are an infinite number of possible choices for the base number.



However, where exponential data is examined in science, engineering, and finance, the base  $e \approx 2.7183$  is commonly used.

$e$  is a special number in mathematics. It is irrational like  $\pi$ , and just as  $\pi$  is the ratio of a circle's circumference to its diameter,  $e$  also has a physical meaning. We explore this meaning in the following **Discovery**.

**Discovery 2****Continuous compound interest**

A formula for calculating the amount to which an investment grows is  $u_n = u_0(1 + i)^n$  where:

$u_n$  is the final amount,  $u_0$  is the initial amount,

$i$  is the interest rate per compounding period,

$n$  is the number of periods or number of times the interest is compounded.

We will investigate the final value of an investment for various values of  $n$ , and allow  $n$  to get extremely large.

**What to do:**

- 1** Suppose \$1000 is invested for one year at a fixed rate of 6% per annum. Use your calculator to find the final amount or *maturing value* if the interest is paid:

- |  |  |
|--|--|
| <b>a</b> annually ( $n = 1$ , $i = 6\% = 0.06$ ) | <b>b</b> quarterly ( $n = 4$ , $i = \frac{6\%}{4} = 0.015$ ) |
| <b>c</b> monthly                                 | <b>d</b> daily   |
| <b>e</b> by the second                           | <b>f</b> by the millisecond.                                 |

- 2** Comment on your answers from **1**.

- 3** If  $r$  is the percentage rate per year,  $t$  is the number of years, and  $N$  is the number of interest payments per year, then  $i = \frac{r}{N}$  and  $n = Nt$ .

The growth formula becomes  $u_n = u_0 \left(1 + \frac{r}{N}\right)^{Nt}$ .

If we let  $a = \frac{N}{r}$ , show that  $u_n = u_0 \left[\left(1 + \frac{1}{a}\right)^a\right]^{rt}$ .

- 4** For continuous compound growth, the number of interest payments per year  $N$  gets very large.

- a Explain why  $a$  gets very large as  $N$  gets very large.  
 b Copy and complete the table, giving your answers as accurately as technology permits.

- 5** You should have found that for very large values of  $a$ ,

$$\left(1 + \frac{1}{a}\right)^a \approx 2.718\,281\,828\,459\dots$$

Use the  $e^x$  key of your calculator to find the value of  $e^1$ . What do you notice?

- 6** For continuous growth,  $u_n = u_0 e^{rt}$  where  $u_0$  is the initial amount  
 $r$  is the annual percentage rate  
 $t$  is the number of years

Use this formula to find the final value if \$1000 is invested for 4 years at a fixed rate of 6% per annum, where the interest is calculated continuously.

$a$	$\left(1 + \frac{1}{a}\right)^a$
10	
100	
1000	
10 000	
100 000	
1 000 000	
10 000 000	

From **Discovery 2** we observe that:

If interest is paid *continuously* or *instantaneously* then the formula for calculating a compounding amount  $u_n = u_0(1+i)^n$  can be replaced by  $u_n = u_0 e^{rt}$ , where  $r$  is the percentage rate per annum and  $t$  is the number of years.

### Historical note

The natural exponential  $e$  was first described in 1683 by Swiss mathematician **Jacob Bernoulli**. He discovered the number while studying compound interest, just as we did in **Discovery 2**.

The natural exponential was first called  $e$  by Swiss mathematician and physicist **Leonhard Euler** in a letter to the German mathematician **Christian Goldbach** in 1731. The number was then published with this notation in 1736.

In 1748 Euler evaluated  $e$  correct to 18 decimal places.

One may think that  $e$  was chosen because it was the first letter of Euler's name or for the word exponential, but it is likely that it was just the next vowel available since he had already used  $a$  in his work.



Leonhard Euler

## EXERCISE 4H

- 1** Sketch, on the same set of axes, the graphs of  $y = 2^x$ ,  $y = e^x$ , and  $y = 3^x$ . Comment on any observations.
- 2** Sketch, on the same set of axes, the graphs of  $y = e^x$  and  $y = e^{-x}$ . What is the geometric connection between these two graphs?
- 3** For the general exponential function  $y = ae^{kx}$ , what is the  $y$ -intercept?

GRAPHING PACKAGE



- 4** Consider  $y = 2e^x$ .
- a** Explain why  $y$  can never be  $< 0$ .  
**b** Find  $y$  if: **i**  $x = -20$  **ii**  $x = 20$ .
- 5** Find, to 3 significant figures, the value of:
- a**  $e^2$       **b**  $e^3$       **c**  $e^{0.7}$       **d**  $\sqrt{e}$       **e**  $e^{-1}$
- 6** Write the following as powers of  $e$ :
- a**  $\sqrt{e}$       **b**  $\frac{1}{\sqrt{e}}$       **c**  $\frac{1}{e^2}$       **d**  $e\sqrt{e}$
- 7** On the same set of axes, sketch and clearly label the graphs of:  
 $f : x \mapsto e^x$ ,     $g : x \mapsto e^{x-2}$ ,     $h : x \mapsto e^x + 3$   
State the domain and range of each function.
- 8** On the same set of axes, sketch and clearly label the graphs of:  
 $f : x \mapsto e^x$ ,     $g : x \mapsto -e^x$ ,     $h : x \mapsto 10 - e^x$   
State the domain and range of each function.
- 9** Expand and simplify:
- a**  $(e^x + 1)^2$       **b**  $(1 + e^x)(1 - e^x)$       **c**  $e^x(e^{-x} - 3)$
- 10** Solve for  $x$ :
- a**  $e^x = \sqrt{e}$       **b**  $e^{\frac{1}{2}x} = \frac{1}{e^2}$
- 11** Suppose  $f : x \mapsto e^x$  and  $g : x \mapsto 3x + 2$ .
- a** Find  $fg(x)$  and  $gf(x)$ .  
**b** Solve  $fg(x) = \frac{1}{e}$ .
- 12** Consider the function  $f(x) = e^x$ .
- a** On the same set of axes, sketch  $y = f(x)$ ,  $y = x$ , and  $y = f^{-1}(x)$ .  
**b** State the domain and range of  $f^{-1}$ .

### Activity

Click on the icon to run a card game for exponential functions.

CARD GAME



### Review set 4A

- 1** Simplify:
- a**  $5\sqrt{3}(4 - \sqrt{3})$       **b**  $(6 - 5\sqrt{2})^2$
- 2** Write with integer denominator:
- a**  $\frac{2}{\sqrt{3}}$       **b**  $\frac{\sqrt{7}}{\sqrt{5}}$       **c**  $\frac{1}{4\sqrt{7}}$
- 3** Simplify using the laws of exponents:
- a**  $a^4b^5 \times a^2b^2$       **b**  $6xy^5 \div 9x^2y^5$       **c**  $\frac{5(x^2y)^2}{(5x^2)^2}$

**4** Let  $f(x) = 3^x$ .

**a** Write down the value of: **i**  $f(4)$     **ii**  $f(-1)$

**b** Find the value of  $k$  such that  $f(x+2) = k f(x)$ ,  $k \in \mathbb{Z}$ .

**5** Write without brackets or negative exponents:

**a**  $x^{-2} \times x^{-3}$

**b**  $2(ab)^{-2}$

**c**  $2ab^{-2}$

**6** Write as a single power of 3:

**a**  $\frac{27}{9a}$

**b**  $(\sqrt{3})^{1-x} \times 9^{1-2x}$

**7** Evaluate:

**a**  $8^{\frac{2}{3}}$

**b**  $27^{-\frac{2}{3}}$

**8** Write without negative exponents:

**a**  $mn^{-2}$

**b**  $(mn)^{-3}$

**c**  $\frac{m^2 n^{-1}}{p^{-2}}$

**d**  $(4m^{-1}n)^2$

**9** Expand and simplify:

**a**  $(3 - e^x)^2$

**b**  $(\sqrt{x} + 2)(\sqrt{x} - 2)$

**c**  $2^{-x}(2^{2x} + 2^x)$

**10** Find the positive solution of the equation  $(8 + \sqrt{13})x^2 + (2 - \sqrt{13})x - 1 = 0$ .

Give your solution in the form  $x = a + b\sqrt{13}$ , where  $a, b \in \mathbb{Q}$ .

**11** Solve for  $x$ :

**a**  $2^{x-3} = \frac{1}{32}$

**b**  $9^x = 27^{2-2x}$

**c**  $e^{2x} = \frac{1}{\sqrt{e}}$

**12** Match each equation to its corresponding graph:

**a**  $y = -e^x$

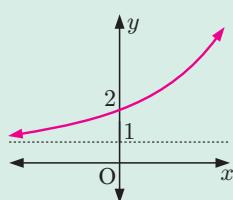
**b**  $y = 3 \times 2^x$

**c**  $y = e^x + 1$

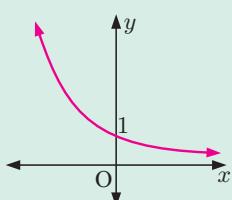
**d**  $y = 3^{-x}$

**e**  $y = -e^{-x}$

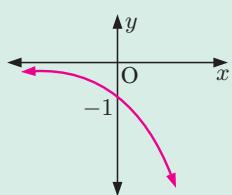
**A**



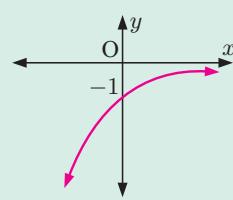
**B**



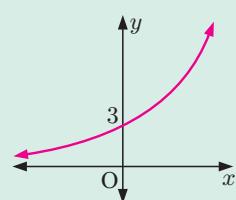
**C**



**D**



**E**



**13** If  $f(x) = 3 \times 2^x$ , find the value of:

**a**  $f(0)$

**b**  $f(3)$

**c**  $f(-2)$

**14** Consider the function  $f : x \mapsto e^{-x} - 3$ .

**a** State the range of the function.

**b** Find the value of  $f(0)$ .

**c** Solve  $f(x) = \frac{\sqrt{e} - 3e}{e}$ .

- 15** The temperature of a dish  $t$  minutes after it is removed from the microwave, is given by  $T = 80 \times 2^{-0.1t}$  °C.
- Find the initial temperature of the dish.
  - Find the temperature after:
    - 10 minutes
    - 20 minutes.
  - Draw the graph of  $T$  against  $t$  for  $t \geq 0$ .
  - Find the time taken for the temperature of the dish to fall to 10°C.

### Review set 4B

- 1** Simplify:
- $(7 + 2\sqrt{3})(5 - 3\sqrt{3})$
  - $(6 + 2\sqrt{2})(6 - 2\sqrt{2})$
- 2** Rationalise the denominator:
- $\frac{1}{5 - \sqrt{3}}$
  - $\frac{\sqrt{11}}{\sqrt{7} - 2}$
  - $\frac{8 + \sqrt{2}}{3 - \sqrt{2}}$
  - $\frac{4 + 5\sqrt{5}}{6 - 3\sqrt{5}}$
- 3** Given the graph of  $y = 3^x$  shown, estimate solutions to the exponential equations:
- $3^x = 5$
  - $3^x = \frac{1}{2}$
  - $6 \times 3^x = 20$
- 
- 4** Write each of the following in the form  $a \pm b\sqrt{2}$  where  $a, b \in \mathbb{Z}^+$ :
- $(\sqrt{2} - 1)^2$
  - $\frac{\sqrt{2} - 1}{\sqrt{2} + 1}$
  - $\frac{1}{(\sqrt{2} + 1)^2}$
  - $\frac{1}{3 + 2\sqrt{2}}$
- 5** Simplify using the laws of exponents:
- $(a^7)^3$
  - $pq^2 \times p^3q^4$
  - $\frac{8ab^5}{2a^4b^4}$
- 6** Write the following as a power of 2:
- $2 \times 2^{-4}$
  - $16 \div 2^{-3}$
  - $8^4$
- 7** Write the following without brackets:
- $(2m^3)^2$
  - $\left(\frac{-a^3}{b}\right)^3$
  - $\frac{(3x^2y)^2}{3x}$
  - $\frac{(2a^{\frac{1}{2}}b^{\frac{1}{5}})^4}{a}$
- 8** Simplify  $\frac{2^{x+1}}{2^{1-x}}$ .
- 9** Write as powers of 5 in simplest form:
- 1
  - $5\sqrt{5}$
  - $\frac{1}{\sqrt[4]{5}}$
  - $25^{a+3}$

**10** Expand and simplify:

**a**  $e^x(e^{-x} + e^x)$

**b**  $(2^x + 5)^2$

**c**  $(x^{\frac{1}{2}} - 7)(x^{\frac{1}{2}} + 7)$

**11** Solve for  $x$ :

**a**  $6 \times 2^x = 192$

**b**  $4 \times (\frac{1}{3})^x = 324$

**12** Solve for  $x$  without using a calculator:

**a**  $4^{x+1} = (\frac{1}{8})^x$

**b**  $\frac{25^x}{5^{x-3}} = \frac{5^x}{125^{x-2}}$

**c**  $\frac{3^{x+2}}{9^{3-x}} = \frac{27^{1-2x}}{3^{2x}}$

**13** Suppose  $f(x) = 2^{-x} + 1$ .

**a** Find  $f(\frac{1}{2})$ .

**b** Find  $a$  such that  $f(a) = 3$ .

**14** On the same set of axes draw the graphs of  $y = 2^x$  and  $y = 2^x - 4$ . Include on your graph the  $y$ -intercept and the equation of the horizontal asymptote of each function.

**15** Consider  $y = 3^x - 5$ .

**a** Find  $y$  when  $x = 0, \pm 1, \pm 2$ .

**b** Discuss  $y$  as  $x \rightarrow \pm\infty$ .

**c** Sketch the graph of  $y = 3^x - 5$ .

**d** State the range of the function.

**16** Consider  $f : x \mapsto e^{2x-1}$  and  $g : x \mapsto e^{\sqrt{2}x}$ .

**a** State the range of  $f$ .

**b** Find the exact value of  $g(\sqrt{2})$ .

**c** Solve  $f(x) = g(x)$ , giving your answer in the form  $x = a + b\sqrt{2}$  where  $a, b \in \mathbb{Q}$ .

# 5

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# Logarithms

## Contents:

- A** Logarithms in base 10
- B** Logarithms in base  $a$
- C** Laws of logarithms
- D** Logarithmic equations
- E** Natural logarithms
- F** Solving exponential equations using logarithms
- G** The change of base rule
- H** Graphs of logarithmic functions

## Opening problem

In a plentiful springtime, a population of 1000 mice will double every week.

The population after  $t$  weeks is given by the exponential function  $P(t) = 1000 \times 2^t$  mice.

### Things to think about:

- What does the graph of the population over time look like?
- How long will it take for the population to reach 20 000 mice?
- Can we write a function for  $t$  in terms of  $P$ , which determines the time at which the population  $P$  is reached?
- What does the graph of this function look like?



## A

## LOGARITHMS IN BASE 10

Consider the exponential function  $f : x \mapsto 10^x$  or  $f(x) = 10^x$ .

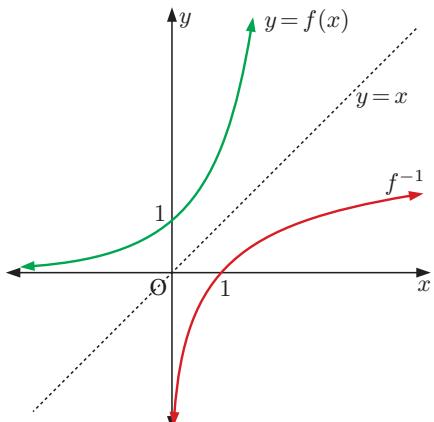
The graph of  $y = f(x)$  is shown alongside, along with its inverse function  $f^{-1}$ .

Since  $f$  is defined by  $y = 10^x$ ,

$f^{-1}$  is defined by  $x = 10^y$ .

{interchanging  $x$  and  $y$ }

$y$  is the exponent to which the base 10 is raised in order to get  $x$ .



We write this as  $y = \log_{10} x$  or  $\lg x$  and say that  $y$  is the **logarithm in base 10, of  $x$** .

Logarithms are thus defined to be the inverse of exponential functions:

$$\text{If } f(x) = 10^x \text{ then } f^{-1}(x) = \log_{10} x \text{ or } \lg x.$$

## WORKING WITH LOGARITHMS

Many positive numbers can be easily written in the form  $10^x$ .

For example:

$$10\ 000 = 10^4$$

$$1000 = 10^3$$

$$100 = 10^2$$

$$10 = 10^1$$

$$1 = 10^0$$

$$0.1 = 10^{-1}$$

$$0.01 = 10^{-2}$$

$$0.001 = 10^{-3}$$

Numbers like  $\sqrt{10}$ ,  $10\sqrt{10}$  and  $\frac{1}{\sqrt[5]{10}}$  can also be written in the form  $10^x$  as follows:

$$\begin{aligned}\sqrt{10} &= 10^{\frac{1}{2}} & 10\sqrt{10} &= 10^1 \times 10^{0.5} & \frac{1}{\sqrt[5]{10}} &= 10^{-\frac{1}{5}} \\ &= 10^{0.5} & &= 10^{1.5} & &= 10^{-0.2}\end{aligned}$$

In fact, all positive numbers can be written in the form  $10^x$ .

The **logarithm in base 10** of a positive number is the power that 10 must be raised to in order to obtain the number.

For example:

- Since  $1000 = 10^3$ , we write  $\log_{10}(1000) = 3$   
or  $\lg(1000) = 3$ .
- Since  $0.01 = 10^{-2}$ , we write  $\log_{10}(0.01) = -2$   
or  $\lg(0.01) = -2$ .

$\lg a$  means  $\log_{10} a$ .  
 $a$  must be positive since  
 $10^x > 0$  for all  $x \in \mathbb{R}$ .



We hence conclude that:

$$\begin{aligned}\lg 10^x &= x \quad \text{for any } x \in \mathbb{R}. \\ a &= 10^{\lg a} \quad \text{for any } a > 0.\end{aligned}$$

### Example 1

### Self Tutor

Without using a calculator, find:

a  $\log 100$

b  $\log(\sqrt[4]{10})$

a  $\log 100 = \log 10^2 = 2$

b  $\log(\sqrt[4]{10}) = \log(10^{\frac{1}{4}}) = \frac{1}{4}$

## EXERCISE 5A

- 1 Without using a calculator, find:

a  $\lg 10\,000$

b  $\lg 0.001$

c  $\lg 10$

d  $\lg 1$

e  $\lg \sqrt{10}$

f  $\lg(\sqrt[3]{10})$

g  $\lg\left(\frac{1}{\sqrt[4]{10}}\right)$

h  $\lg(10\sqrt{10})$

i  $\lg \sqrt[3]{100}$

j  $\lg\left(\frac{100}{\sqrt[5]{10}}\right)$

k  $\lg(10 \times \sqrt[3]{10})$

l  $\lg(1000\sqrt{10})$

Check your answers using your calculator.

- 2 Simplify:

a  $\lg 10^n$

b  $\lg(10^a \times 100)$

c  $\lg\left(\frac{10}{10^m}\right)$

d  $\lg\left(\frac{10^a}{10^b}\right)$

**Example 2****Self Tutor**

Use your calculator to write the following in the form  $10^x$  where  $x$  is correct to 4 decimal places:

a 8

a 8

$$\begin{aligned} &= 10^{\lg 8} \\ &\approx 10^{0.9031} \end{aligned}$$

b 800

b 800

$$\begin{aligned} &= 10^{\lg 800} \\ &\approx 10^{2.9031} \end{aligned}$$

c 0.08

c 0.08

$$\begin{aligned} &= 10^{\lg 0.08} \\ &\approx 10^{-1.0969} \end{aligned}$$

- 3** a Use your calculator to find  $\lg 41$ , giving your answer correct to 4 decimal places.  
 b Hence, write 41 as a power of 10.
- 4** Use your calculator to write the following in the form  $10^x$  where  $x$  is correct to 4 decimal places:  
 a 6                    b 60                    c 6000                    d 0.6                    e 0.006  
 f 15                    g 1500                    h 1.5                    i 0.15                    j 0.00015
- 5** Explain why you cannot find the logarithm of a negative number.

**Example 3****Self Tutor**

a Use your calculator to find:

i  $\lg 2$

ii  $\lg 20$

b Explain why  $\lg 20 = \lg 2 + 1$ .

a i  $\lg 2 \approx 0.3010$

ii  $\lg 20 \approx 1.3010$

b  $\lg 20 = \lg(2 \times 10)$

$\approx \lg(10^{0.3010} \times 10^1)$

$\approx \lg 10^{1.3010}$  {adding exponents}

$\approx 1.3010$

$\approx \lg 2 + 1$

- 6** a Use your calculator to find: i  $\lg 3$                     ii  $\lg 300$   
 b Explain why  $\lg 300 = \lg 3 + 2$ .
- 7** a Use your calculator to find: i  $\lg 5$                     ii  $\lg 0.05$   
 b Explain why  $\lg 0.05 = \lg 5 - 2$ .

**Example 4****Self Tutor**Find  $x$  if:

a  $\lg x = 3$

b  $\lg x \approx -0.271$

$$\begin{aligned} \text{a} \quad &\lg x = 3 \\ \therefore &10^{\lg x} = 10^3 \\ \therefore &x = 1000 \end{aligned}$$

$$\begin{aligned} \text{b} \quad &\lg x \approx -0.271 \\ \therefore &10^{\lg x} \approx 10^{-0.271} \\ \therefore &x \approx 0.536 \end{aligned}$$

Remember that  
 $10^{\lg x} = x$ .



8 Find  $x$  if:

- a  $\lg x = 2$
- d  $\lg x = -1$
- g  $\lg x = 4$
- j  $\lg x \approx 2.1457$

- b  $\lg x = 1$
- e  $\lg x = \frac{1}{2}$
- h  $\lg x = -5$
- k  $\lg x \approx -1.378$

- c  $\lg x = 0$
- f  $\lg x = -\frac{1}{2}$
- i  $\lg x \approx 0.8351$
- l  $\lg x \approx -3.1997$

## B

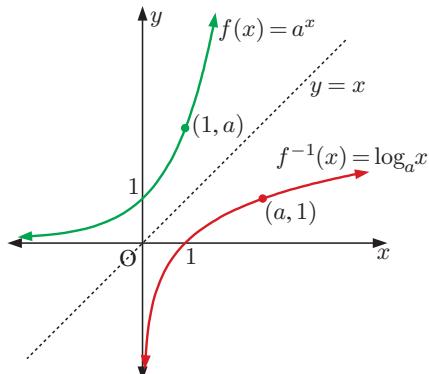
## LOGARITHMS IN BASE $a$

In the previous section we defined logarithms in base 10 as the inverse of the exponential function  $f(x) = 10^x$ .

If  $f(x) = 10^x$  then  $f^{-1}(x) = \log_{10} x$ .

We can use the same principle to define logarithms in other bases:

If  $f(x) = a^x$  then  $f^{-1}(x) = \log_a x$ .



If  $b = a^x$ ,  $a \neq 1$ ,  $a > 0$ , we say that  $x$  is the **logarithm in base  $a$ , of  $b$** , and that  $b = a^x \Leftrightarrow x = \log_a b$ ,  $b > 0$ .

$b = a^x \Leftrightarrow x = \log_a b$  is read as “ $b = a^x$  if and only if  $x = \log_a b$ ”.

It is a short way of writing:

“if  $b = a^x$  then  $x = \log_a b$ , and if  $x = \log_a b$  then  $b = a^x$ ”.

$b = a^x$  and  $x = \log_a b$  are *equivalent* or *interchangeable* statements.

For example:

- $8 = 2^3$  means that  $3 = \log_2 8$  and vice versa.
- $\log_5 25 = 2$  means that  $25 = 5^2$  and vice versa.

If  $y = a^x$  then  $x = \log_a y$ , and so

$$x = \log_a a^x.$$

If  $x = a^y$  then  $y = \log_a x$ , and so

$$x = a^{\log_a x} \text{ provided } x > 0.$$

$\log_a b$  is the power that  $a$  must be raised to in order to get  $b$ .



### Example 5

### Self Tutor

- a Write an equivalent exponential equation for  $\log_{10} 1000 = 3$ .
- b Write an equivalent logarithmic equation for  $3^4 = 81$ .

a From  $\log_{10} 1000 = 3$  we deduce that  $10^3 = 1000$ .

b From  $3^4 = 81$  we deduce that  $\log_3 81 = 4$ .

**EXERCISE 5B**

**1** Write an equivalent exponential equation for:

**a**  $\log_{10} 100 = 2$

**b**  $\log_{10} 10\,000 = 4$

**c**  $\log_{10}(0.1) = -1$

**d**  $\log_{10} \sqrt{10} = \frac{1}{2}$

**e**  $\log_2 8 = 3$

**f**  $\log_3 9 = 2$

**g**  $\log_2(\frac{1}{4}) = -2$

**h**  $\log_3 \sqrt{27} = 1.5$

**i**  $\log_5 \left( \frac{1}{\sqrt{5}} \right) = -\frac{1}{2}$

**2** Write an equivalent logarithmic equation for:

**a**  $2^2 = 4$

**b**  $4^3 = 64$

**c**  $5^2 = 25$

**d**  $7^2 = 49$

**e**  $2^6 = 64$

**f**  $2^{-3} = \frac{1}{8}$

**g**  $10^{-2} = 0.01$

**h**  $2^{-1} = \frac{1}{2}$

**i**  $3^{-3} = \frac{1}{27}$

**Example 6****Self Tutor**

Find:

**a**  $\log_2 16$

**b**  $\log_5 0.2$

**c**  $\log_{10} \sqrt[5]{100}$

**d**  $\log_2 \left( \frac{1}{\sqrt{2}} \right)$

**a**  $\log_2 16$

$$\begin{aligned} &= \log_2 2^4 \\ &= 4 \end{aligned}$$

**b**  $\log_5 0.2$

$$\begin{aligned} &= \log_5 \left( \frac{1}{5} \right) \\ &= \log_5 5^{-1} \\ &= -1 \end{aligned}$$

**c**  $\log_{10} \sqrt[5]{100}$

$$\begin{aligned} &= \log_{10} \left( 10^2 \right)^{\frac{1}{5}} \\ &= \log_{10} 10^{\frac{2}{5}} \\ &= \frac{2}{5} \end{aligned}$$

**d**  $\log_2 \left( \frac{1}{\sqrt{2}} \right)$

$$\begin{aligned} &= \log_2 2^{-\frac{1}{2}} \\ &= -\frac{1}{2} \end{aligned}$$

**3** Find:

**a**  $\log_{10} 100\,000$

**b**  $\log_{10}(0.01)$

**c**  $\log_3 \sqrt{3}$

**d**  $\log_2 8$

**e**  $\log_2 64$

**f**  $\log_2 128$

**g**  $\log_5 25$

**h**  $\log_5 125$

**i**  $\log_2(0.125)$

**j**  $\log_9 3$

**k**  $\log_4 16$

**l**  $\log_{36} 6$

**m**  $\log_3 243$

**n**  $\log_2 \sqrt[3]{2}$

**o**  $\log_a a^n$

**p**  $\log_8 2$

**q**  $\log_t \left( \frac{1}{t} \right)$

**r**  $\log_6 6\sqrt{6}$

**s**  $\log_4 1$

**t**  $\log_9 9$

**4** Use your calculator to find:

**a**  $\log_{10} 152$

**b**  $\log_{10} 25$

**c**  $\log_{10} 74$

**d**  $\log_{10} 0.8$

**5** Solve for  $x$ :

**a**  $\log_2 x = 3$

**b**  $\log_4 x = \frac{1}{2}$

**c**  $\log_x 81 = 4$

**d**  $\log_2(x - 6) = 3$

**6** Simplify:

**a**  $\log_4 16$

**b**  $\log_2 4$

**c**  $\log_3 \left( \frac{1}{3} \right)$

**d**  $\log_{10} \sqrt[4]{1000}$

**e**  $\log_7 \left( \frac{1}{\sqrt{7}} \right)$

**f**  $\log_5(25\sqrt{5})$

**g**  $\log_3 \left( \frac{1}{\sqrt{27}} \right)$

**h**  $\log_4 \left( \frac{1}{2\sqrt{2}} \right)$

**i**  $\log_x x^2$

**j**  $\log_x \sqrt{x}$

**k**  $\log_m m^3$

**l**  $\log_x(x\sqrt{x})$

**m**  $\log_n \left( \frac{1}{n} \right)$

**n**  $\log_a \left( \frac{1}{a^2} \right)$

**o**  $\log_a \left( \frac{1}{\sqrt{a}} \right)$

**p**  $\log_m \sqrt{m^5}$

## Discussion

We have seen that  $\sqrt{2}$  cannot be written in the form  $\frac{p}{q}$  where  $p, q \in \mathbb{Z}$ ,  $q \neq 0$ . We therefore say  $\sqrt{2}$  is irrational.

More generally,  $\sqrt{a}$  is only rational if  $a$  is a perfect square.

What about logarithms? The following is a proof that  $\log_2 3$  is irrational.

**Proof:** If  $\log_2 3$  is rational, then  $\log_2 3 = \frac{p}{q}$  where  $p, q \in \mathbb{Z}$ ,  $q \neq 0$

$$\therefore 3 = 2^{\frac{p}{q}}$$

$$\therefore 3^q = 2^p$$

The left hand side is always odd, and the right hand side is always even, so the statement is impossible.

Hence  $\log_2 3$  must be irrational.

Under what circumstances will  $\log_a b$  be rational?

## C LAWS OF LOGARITHMS

### Discovery

### The laws of logarithms

#### What to do:

1 Use your calculator to find:

a  $\lg 2 + \lg 3$

b  $\lg 3 + \lg 7$

c  $\lg 4 + \lg 20$

d  $\lg 6$

e  $\lg 21$

f  $\lg 80$

From your answers, suggest a possible simplification for  $\lg a + \lg b$ .

2 Use your calculator to find:

a  $\lg 6 - \lg 2$

b  $\lg 12 - \lg 3$

c  $\lg 3 - \lg 5$

d  $\lg 3$

e  $\lg 4$

f  $\lg(0.6)$

From your answers, suggest a possible simplification for  $\lg a - \lg b$ .

3 Use your calculator to find:

a  $3 \lg 2$

b  $2 \lg 5$

c  $-4 \lg 3$

d  $\lg(2^3)$

e  $\lg(5^2)$

f  $\lg(3^{-4})$

From your answers, suggest a possible simplification for  $n \lg a$ .

From the **Discovery**, you should have found the three important **laws of logarithms**:

If  $A$  and  $B$  are both positive then:

- $\lg A + \lg B = \lg(AB)$

- $\lg A - \lg B = \lg\left(\frac{A}{B}\right)$

- $n \lg A = \lg(A^n)$

More generally, in any base  $c$  where  $c \neq 1, c > 0$ , we have these **laws of logarithms**:

If  $A$  and  $B$  are both positive then:

- $\log_c A + \log_c B = \log_c(AB)$
- $\log_c A - \log_c B = \log_c\left(\frac{A}{B}\right)$
- $n \log_c A = \log_c(A^n)$

**Proof:**

<ul style="list-style-type: none"> <li>• <math>\log_c(AB)</math></li> </ul> $\begin{aligned} &= \log_c(c^{\log_c A} \times c^{\log_c B}) \\ &= \log_c(c^{\log_c A + \log_c B}) \\ &= \log_c A + \log_c B \end{aligned}$	<ul style="list-style-type: none"> <li>• <math>\log_c\left(\frac{A}{B}\right)</math></li> </ul> $\begin{aligned} &= \log_c\left(\frac{c^{\log_c A}}{c^{\log_c B}}\right) \\ &= \log_c(c^{\log_c A - \log_c B}) \\ &= \log_c A - \log_c B \end{aligned}$	<ul style="list-style-type: none"> <li>• <math>\log_c(A^n)</math></li> </ul> $\begin{aligned} &= \log_c((c^{\log_c A})^n) \\ &= \log_c(c^{n \log_c A}) \\ &= n \log_c A \end{aligned}$
---	---	--

### Example 7



Use the laws of logarithms to write the following as a single logarithm or as an integer:

a  $\lg 5 + \lg 3$

b  $\log_3 24 - \log_3 8$

c  $\log_2 5 - 1$

a  $\lg 5 + \lg 3$

$$\begin{aligned} &= \lg(5 \times 3) \\ &= \lg 15 \end{aligned}$$

b  $\log_3 24 - \log_3 8$

$$\begin{aligned} &= \log_3\left(\frac{24}{8}\right) \\ &= \log_3 3 \\ &= 1 \end{aligned}$$

c  $\log_2 5 - 1$

$$\begin{aligned} &= \log_2 5 - \log_2 2^1 \\ &= \log_2\left(\frac{5}{2}\right) \end{aligned}$$

### Example 8



Simplify by writing as a single logarithm or as a rational number:

a  $2 \lg 7 - 3 \lg 2$

b  $2 \lg 3 + 3$

c  $\frac{\lg 8}{\lg 4}$

a  $2 \lg 7 - 3 \lg 2$

$$\begin{aligned} &= \lg(7^2) - \lg(2^3) \\ &= \lg 49 - \lg 8 \\ &= \lg\left(\frac{49}{8}\right) \end{aligned}$$

b  $2 \lg 3 + 3$

$$\begin{aligned} &= \lg(3^2) + \lg(10^3) \\ &= \lg 9 + \lg 1000 \\ &= \lg(9000) \end{aligned}$$

c  $\frac{\lg 8}{\lg 4}$

$$\begin{aligned} &= \frac{\lg 2^3}{\lg 2^2} \\ &= \frac{3 \lg 2}{2 \lg 2} \\ &= \frac{3}{2} \end{aligned}$$

**EXERCISE 5C**

**1** Write as a single logarithm or as an integer:

**a**  $\lg 8 + \lg 2$

**d**  $\lg p - \lg m$

**g**  $\lg 2 + \lg 3 + \lg 4$

**j**  $\lg 5 + \lg 4 - \lg 2$

**m**  $\log_m 40 - 2$

**p**  $3 - \log_5 50$

**b**  $\lg 4 + \lg 5$

**e**  $\log_4 8 - \log_4 2$

**h**  $1 + \log_2 3$

**k**  $2 + \lg 2$

**n**  $\log_3 6 - \log_3 2 - \log_3 3$

**q**  $\log_5 100 - \log_5 4$

**c**  $\lg 40 - \lg 5$

**f**  $\lg 5 + \lg(0.4)$

**i**  $\lg 4 - 1$

**l**  $t + \lg w$

**o**  $\lg 50 - 4$

**r**  $\lg\left(\frac{4}{3}\right) + \lg 3 + \lg 7$

**2** Write as a single logarithm or integer:

**a**  $5\lg 2 + \lg 3$

**d**  $2\log_3 5 - 3\log_3 2$

**g**  $3 - \lg 2 - 2\lg 5$

**b**  $2\lg 3 + 3\lg 2$

**e**  $\frac{1}{2}\log_6 4 + \log_6 3$

**h**  $1 - 3\lg 2 + \lg 20$

**c**  $3\lg 4 - \lg 8$

**f**  $\frac{1}{3}\lg\left(\frac{1}{8}\right)$

**i**  $2 - \frac{1}{2}\log_n 4 - \log_n 5$

**3** Simplify without using a calculator:

**a**  $\frac{\lg 4}{\lg 2}$

**d**  $\frac{\lg 3}{\lg 9}$

**b**  $\frac{\log_5 27}{\log_5 9}$

**e**  $\frac{\log_3 25}{\log_3 (0.2)}$

**c**  $\frac{\lg 8}{\lg 2}$

**f**  $\frac{\log_4 8}{\log_4 (0.25)}$

Check your answers using a calculator.

**Example 9** **Self Tutor**

Show that:

**a**  $\lg\left(\frac{1}{9}\right) = -2\lg 3$

**b**  $\lg 500 = 3 - \lg 2$

**a** 
$$\begin{aligned} &\lg\left(\frac{1}{9}\right) \\ &= \lg(3^{-2}) \\ &= -2\lg 3 \end{aligned}$$

**b** 
$$\begin{aligned} &\lg 500 \\ &= \lg\left(\frac{1000}{2}\right) \\ &= \lg 1000 - \lg 2 \\ &= \lg 10^3 - \lg 2 \\ &= 3 - \lg 2 \end{aligned}$$

**4** Show that:

**a**  $\lg 9 = 2\lg 3$

**d**  $\lg\left(\frac{1}{5}\right) = -\lg 5$

**g**  $\log_6 4 + \log_6 9 = 2$

**b**  $\lg\sqrt{2} = \frac{1}{2}\lg 2$

**e**  $\lg 5 = 1 - \lg 2$

**h**  $\log_{15} 3 - \log_{15} 45 = -1$

**c**  $\lg\left(\frac{1}{8}\right) = -3\lg 2$

**f**  $\lg 5000 = 4 - \lg 2$

**i**  $2\log_{12} 2 + \frac{1}{2}\log_{12} 9 = 1$

**5** Find the exact value of:

**a**  $3\lg 2 + 2\lg 5 - \frac{1}{2}\lg 4$

**c**  $5\log_6 2 + 2\log_6 3 - \frac{1}{2}\log_6 16 - \log_6 12$

**b**  $2\log_2 3 - \log_2 6 - \frac{1}{2}\log_2 9$

6 If  $x = \log_2 P$ ,  $y = \log_2 Q$ , and  $z = \log_2 R$ , write in terms of  $x$ ,  $y$ , and  $z$ :

a  $\log_2(PR)$

b  $\log_2(RQ^2)$

c  $\log_2\left(\frac{PR}{Q}\right)$

d  $\log_2(P^2\sqrt{Q})$

e  $\log_2\left(\frac{Q^3}{\sqrt{R}}\right)$

f  $\log_2\left(\frac{R^2\sqrt{Q}}{P^3}\right)$

7 If  $p = \log_b 2$ ,  $q = \log_b 3$ , and  $r = \log_b 5$ , write in terms of  $p$ ,  $q$ , and  $r$ :

a  $\log_b 6$

b  $\log_b 45$

c  $\log_b 108$

d  $\log_b\left(\frac{5\sqrt{3}}{2}\right)$

e  $\log_b\left(\frac{5}{32}\right)$

f  $\log_b(0.\bar{2})$

$0.\bar{2}$  means  
0.222 222 ....



8 If  $\log_t M = 1.29$  and  $\log_t N^2 = 1.72$ , find:

a  $\log_t N$

b  $\log_t(MN)$

c  $\log_t\left(\frac{N^2}{\sqrt{M}}\right)$

9 Suppose  $\log_b P = 5$  and  $\log_b(P^3Q^2) = 21$ . Find  $\log_b Q$ .

10 Suppose that  $\log_t(AB^3) = 15$  and  $\log_t\left(\frac{A^2}{B}\right) = 9$ .

a Write two equations connecting  $\log_t A$  and  $\log_t B$ .

b Find the values of  $\log_t A$  and  $\log_t B$ .

c Find  $\log_t(B^5\sqrt{A})$ .

d Write  $B$  in terms of  $t$ .

## D

## LOGARITHMIC EQUATIONS

We can use the laws of logarithms to write equations in a different form. This can be particularly useful if an unknown appears as an exponent.

For the logarithmic function, for every value of  $y$ , there is only one corresponding value of  $x$ . We can therefore take the logarithm of both sides of an equation without changing the solution. However, we can only do this if both sides are positive.

### Example 10

### Self Tutor

Write these as logarithmic equations (in base 10):

a  $y = 5 \times 3^x$

b  $P = \frac{20}{\sqrt{n}}$

a  $y = 5 \times 3^x$

$\therefore \lg y = \lg(5 \times 3^x)$

$\therefore \lg y = \lg 5 + \lg 3^x$

$\therefore \lg y = \lg 5 + x \lg 3$

b  $P = \frac{20}{\sqrt{n}}$

$\therefore \lg P = \lg\left(\frac{20}{\sqrt{n}}\right)$

$\therefore \lg P = \lg 20 - \lg n^{\frac{1}{2}}$

$\therefore \lg P = \lg 20 - \frac{1}{2} \lg n$

**Example 11** **Self Tutor**

Write the following equations without logarithms:

**a**  $\lg y = x \lg 4 + \lg 3$

**b**  $\log_2 M = 3 \log_2 a - 5$

**a**  $\lg y = x \lg 4 + \lg 3$

$\therefore \lg y = \lg 4^x + \lg 3$

$\therefore \lg y = \lg(3 \times 4^x)$

$\therefore y = 3 \times 4^x$

**b**  $\log_2 M = 3 \log_2 a - 5$

$\therefore \log_2 M = \log_2 a^3 - \log_2 2^5$

$\therefore \log_2 M = \log_2 \left( \frac{a^3}{32} \right)$

$\therefore M = \frac{a^3}{32}$

**EXERCISE 5D.1**

- 1** Write the following as logarithmic equations in base 10, assuming all terms are positive:

**a**  $y = 2^x$

**b**  $y = x^3$

**c**  $M = d^4$

**d**  $T = 5^x$

**e**  $y = \sqrt{x}$

**f**  $y = 7 \times 3^x$

**g**  $S = \frac{9}{t}$

**h**  $M = 100 \times 7^x$

**i**  $T = 5\sqrt{d}$

**j**  $F = \frac{1000}{\sqrt{n}}$

**k**  $S = 200 \times 2^t$

**l**  $y = \sqrt{\frac{15}{x}}$

- 2** Write the following equations without logarithms:

**a**  $\lg y = x \lg 7$

**b**  $\lg D = \lg x + \lg 2$

**c**  $\log_a F = \log_a 5 - \log_a t$

**d**  $\lg y = x \lg 2 + \lg 6$

**e**  $\lg P = \frac{1}{2} \lg x$

**f**  $\lg N = -\frac{1}{3} \lg p$

**g**  $\lg P = 3 \lg x + 1$

**h**  $\lg y = x - \lg 2$

**i**  $\lg y = 2 \lg x - 1$

**j**  $\log_2 T = 5 \log_2 k + 1$

**k**  $\log_3 P = 4 \log_3 n - 2$

**l**  $\log_2 y = 4x + 3$

- 3** Suppose  $\lg y = 3 \lg x - \lg 2$ .

- a** Write  $y$  in terms of  $x$ , without using logarithms.

- b** Find  $y$  when:      **i**  $x = 2$       **ii**  $x = 4$

- 4** Suppose  $\lg y = \frac{1}{3}x + 2$ .

- a** Write  $y$  in the form  $y = a(10^{bx})$  where  $a, b \in \mathbb{Q}$ .

- b** Find  $y$  when:      **i**  $x = 0$       **ii**  $x = 3$

- 5** Copy and complete:

- a** If there is a *power* relationship between  $y$  and  $x$ , for example  $y = 5x^3$ , then there is a *linear* relationship between  $\lg y$  and ..... .

- b** If there is an *exponential* relationship between  $y$  and  $x$ , for example  $y = 4 \times 2^x$ , then there is a *linear* relationship between ..... and .....

## SOLVING LOGARITHMIC EQUATIONS

Logarithmic equations can often be solved using the laws of logarithms. However, we must always check that our solutions satisfy the original equation, remembering that  $\lg x$  is only defined for  $x > 0$ .

### Example 12

### Self Tutor

Solve for  $x$ :

a  $\lg(x - 6) + \lg 3 = 2 \lg 6$

b  $\lg x + \lg(x + 5) = \lg 14$

a  $\lg(x - 6) + \lg 3 = 2 \lg 6$

b  $\lg x + \lg(x + 5) = \lg 14$

$$\therefore \lg(x - 6) = \lg 6^2 - \lg 3$$

$$\therefore \lg(x - 6) = \lg\left(\frac{36}{3}\right)$$

$$\therefore x - 6 = 12$$

$$\therefore x^2 + 5x - 14 = 0$$

$$\therefore x = 18$$

$$\therefore (x + 7)(x - 2) = 0$$

Check:  $x - 6 > 0$ , so  $x > 6$  ✓

$$\therefore x = -7 \text{ or } 2$$

But  $x > 0$  and  $x + 5 > 0$

$\therefore x = 2$  is the only valid solution.

## EXERCISE 5D.2

1 Solve for  $x$ :

a  $\lg(x - 4) = \lg 3 + \lg 7$

b  $\lg(x + 5) - \lg 8 = 2 \lg 3$

c  $\lg(2x) = 1 + \frac{1}{2} \lg 16$

d  $\log_2 x = 3 \log_2 5 - 6$

e  $\lg x - \lg(x - 4) = \lg 5$

f  $\log_5(x - 2) - \log_5(x + 2) = \log_5 3$

g  $\log_3 x - 2 = \log_3(x - 1)$

h  $\lg(x + 2) - 1 = \lg(x - 3) - \lg 12$

2 Solve for  $x$ :

a  $\lg x + \lg(x + 1) = \lg 30$

b  $\log_5(x + 9) + \log_5(x + 2) = \log_5(20x)$

c  $\log_7 x = \log_7 8 - \log_7(6 - x)$

d  $\log_6(x + 4) + \log_6(x - 1) = 1$

e  $\lg x + \lg(2x + 8) = 1$

f  $\lg(x + 2) + \lg(x + 7) = \lg(2x + 2)$

g  $2 \log_2 x - \log_2(8 - 3x) = 1$

h  $\log_2 x + \log_2(2x - 7) = 2$

### Example 13

### Self Tutor

Solve for  $x$ :  $\log_x 3 + \log_x 12 = 2$

$$\log_x 3 + \log_x 12 = 2$$

$$\therefore \log_x(3 \times 12) = \log_x(x^2)$$

$$\therefore 36 = x^2$$

$$\therefore x = 6 \quad \{ \text{since } x > 0 \}$$

The base of a logarithm must be positive.



3 Solve for  $x$ :

a  $\log_x 32 - \log_x 4 = 1$

b  $\log_x 45 = 2 + \log_x 5$

c  $\log_x 54 = 3 - \log_x 4$

d  $2 \log_x 2 - 3 = \log_x \left(\frac{1}{16}\right)$

## Historical note

## The invention of logarithm

It is easy to take modern technology, such as the electronic calculator, for granted. Until electronic computers became affordable in the 1980s, a “calculator” was a *profession*, literally someone who would spend their time performing calculations by hand. They used mechanical calculators and technology such as logarithms. They often worked in banks, but sometimes for astronomers and other scientists.

The logarithm was invented by **John Napier** (1550 - 1617) and first published in 1614 in a Latin book which translates as a *Description of the Wonderful Canon of Logarithms*. John Napier was the 8th Lord of Merchiston, which is now part of Edinburgh, Scotland. Napier wrote a number of other books on many subjects including religion and mathematics. One of his other inventions was a device for performing long multiplication which is now called “Napier’s Bones”. Other calculators, such as slide rules, used logarithms as part of their design. He also popularised the use of the decimal point in mathematical notation.



*John Napier*

In Napier’s time, mathematicians did not use the same notation  $a^b$  for indices, nor did they make use of the general concept of a function as described in this course. It was therefore impossible for Napier to explain logarithms as we have done. Instead, Napier’s definition was based on the continuous movement of two points.

Gr.	9		+			
min	Sinus	Logarithmi	Differente	logarithmi	Sinus	
0	1564345	18451174	18427203	12381	9876883	60
1	1567218	18532820	18408484	124342	9876427	59
2	1570091	18414511	18389707	124804	9875971	58
3	1572064	18496231	18370968	125267	9875114	57
4	1575837	18477984	18352253	125731	9875056	56
5	1578709	18459772	18333576	126196	9874597	55
6	1581581	18441594	18314933	126661	9874137	54
7	1584453	18423451	18296324	127127	9873677	53
8	1587325	18405341	18277747	127594	9873216	52
9	1590197	18387265	18259203	128062	9872754	51
10	1593069	18362223	18240692	128531	9872291	50
11	1595941	18351214	18222213	129001	9871827	49
12	1598812	18333237	18203765	129472	9871362	48
13	1601684	18315204	18185351	129943	9870897	47
14	1604555	18297384	18166969	130415	9870431	46
15	1607426	18279507	18148619	130888	9866964	45

Logarithms were an extremely important development and they had an immediate effect on the seventeenth century scientific community. **Johannes Kepler** used Napier’s tables to assist with his calculations. This helped him develop his laws of planetary motion. Without logarithms these calculations would have taken many years. Kepler published a letter congratulating and acknowledging Napier. Kepler’s laws gave **Sir Isaac Newton** important evidence to support his theory of universal gravitation. 200 years later, **Laplace** said that logarithms “by shortening the labours, doubled the life of the astronomer”.

To enable people to actually use logarithms, he calculated tables of numbers by hand to seven places of decimals. This took him many years of work. To find the logarithm of a particular number, you would look it up in the table. Although this seems awkward to us, it is much quicker to use tables than calculate multiplication, division, and square roots by hand.



*Johannes Kepler*

**E****NATURAL LOGARITHMS**

In **Chapter 4** we came across the **natural exponential**  $e \approx 2.71828$ .

Given the exponential function  $f(x) = e^x$ , the inverse function  $f^{-1} = \log_e x$  is the logarithm in base  $e$ .

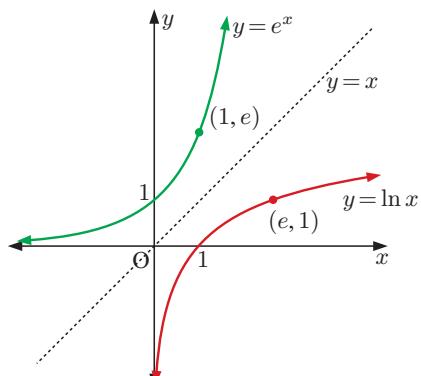
We use  $\ln x$  to represent  $\log_e x$ , and call  $\ln x$  the **natural logarithm** of  $x$ .

$y = \ln x$  is the reflection of  $y = e^x$  in the mirror line  $y = x$ .

Notice that:

- $\ln 1 = \ln e^0 = 0$
- $\ln e = \ln e^1 = 1$
- $\ln e^2 = 2$
- $\ln \sqrt{e} = \ln e^{\frac{1}{2}} = \frac{1}{2}$
- $\ln \left(\frac{1}{e}\right) = \ln e^{-1} = -1$

$$\ln e^x = x \quad \text{and} \quad e^{\ln x} = x.$$



$$\text{Since } a^x = (e^{\ln a})^x = e^{x \ln a}, \quad a^x = e^{x \ln a}, \quad a > 0.$$

### EXERCISE 5E.1

**1** Without using a calculator find:

- |   |                            |   |  |
|---|----------------------------|---|--|
| <b>a</b> $\ln e^2$                      | <b>b</b> $\ln e^3$         | <b>c</b> $\ln \sqrt{e}$                   | <b>d</b> $\ln 1$                               |
| <b>e</b> $\ln \left(\frac{1}{e}\right)$ | <b>f</b> $\ln \sqrt[3]{e}$ | <b>g</b> $\ln \left(\frac{1}{e^2}\right)$ | <b>h</b> $\ln \left(\frac{1}{\sqrt{e}}\right)$ |

Check your answers using a calculator.

**2** Simplify:

- |                      |                        |                       |                         |
|----------------------|------------------------|-----------------------|-------------------------|
| <b>a</b> $e^{\ln 3}$ | <b>b</b> $e^{2 \ln 3}$ | <b>c</b> $e^{-\ln 5}$ | <b>d</b> $e^{-2 \ln 2}$ |
|----------------------|------------------------|-----------------------|-------------------------|

**3** Explain why  $\ln(-2)$  and  $\ln 0$  cannot be found.

**4** Simplify:

- |                    |                              |                                |                       |  |
|--------------------|------------------------------|--------------------------------|-----------------------|--|
| <b>a</b> $\ln e^a$ | <b>b</b> $\ln(e \times e^a)$ | <b>c</b> $\ln(e^a \times e^b)$ | <b>d</b> $\ln(e^a)^b$ | <b>e</b> $\ln\left(\frac{e^a}{e^b}\right)$ |
|--------------------|------------------------------|--------------------------------|-----------------------|--|

#### Example 14

#### Self Tutor

Use your calculator to write the following in the form  $e^k$  where  $k$  is correct to 4 decimal places:

- |             |                |
|-------------|----------------|
| <b>a</b> 50 | <b>b</b> 0.005 |
|-------------|----------------|

<b>a</b> 50	<b>b</b> 0.005
$= e^{\ln 50}$ {using $x = e^{\ln x}$ }	$= e^{\ln 0.005}$
$\approx e^{3.9120}$	$\approx e^{-5.2983}$

5 Use your calculator to write the following in the form  $e^k$  where  $k$  is correct to 4 decimal places:

a 6

b 60

c 6000

d 0.6

e 0.006

f 15

g 1500

h 1.5

i 0.15

j 0.00015

**Example 15****Self Tutor**

Find  $x$  if:

a  $\ln x = 2.17$

b  $\ln x = -0.384$

a  $\ln x = 2.17$

b  $\ln x = -0.384$

∴  $x = e^{2.17}$

∴  $x = e^{-0.384}$

∴  $x \approx 8.76$

∴  $x \approx 0.681$

If  $\ln x = a$   
then  $x = e^a$ .



6 Find  $x$  if:

a  $\ln x = 3$

b  $\ln x = 1$

c  $\ln x = 0$

d  $\ln x = -1$

e  $\ln x = -5$

f  $\ln x \approx 0.835$

g  $\ln x \approx 2.145$

h  $\ln x \approx -3.2971$

**LAWS OF NATURAL LOGARITHMS**

The laws for natural logarithms are the laws for logarithms written in base  $e$ :

For positive  $A$  and  $B$ :

- $\ln A + \ln B = \ln(AB)$
- $\ln A - \ln B = \ln\left(\frac{A}{B}\right)$
- $n \ln A = \ln(A^n)$

**Example 16****Self Tutor**

Use the laws of logarithms to write the following as a single logarithm:

a  $\ln 5 + \ln 3$

b  $\ln 24 - \ln 8$

c  $\ln 5 - 1$

$$\begin{aligned} \mathbf{a} \quad & \ln 5 + \ln 3 \\ &= \ln(5 \times 3) \\ &= \ln 15 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \ln 24 - \ln 8 \\ &= \ln\left(\frac{24}{8}\right) \\ &= \ln 3 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \ln 5 - 1 \\ &= \ln 5 - \ln e^1 \\ &= \ln\left(\frac{5}{e}\right) \end{aligned}$$

**Example 17****Self Tutor**

Use the laws of logarithms to simplify:

a  $2 \ln 7 - 3 \ln 2$

b  $2 \ln 3 + 3$

$$\begin{aligned} \mathbf{a} \quad & 2 \ln 7 - 3 \ln 2 \\ &= \ln(7^2) - \ln(2^3) \\ &= \ln 49 - \ln 8 \\ &= \ln\left(\frac{49}{8}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & 2 \ln 3 + 3 \\ &= \ln(3^2) + \ln e^3 \\ &= \ln 9 + \ln e^3 \\ &= \ln(9e^3) \end{aligned}$$

**EXERCISE 5E.2**

**1** Write as a single logarithm or integer:

**a**  $\ln 15 + \ln 3$

**b**  $\ln 15 - \ln 3$

**c**  $\ln 20 - \ln 5$

**d**  $\ln 4 + \ln 6$

**e**  $\ln 5 + \ln(0.2)$

**f**  $\ln 2 + \ln 3 + \ln 5$

**g**  $1 + \ln 4$

**h**  $\ln 6 - 1$

**i**  $\ln 5 + \ln 8 - \ln 2$

**j**  $2 + \ln 4$

**k**  $\ln 20 - 2$

**l**  $\ln 12 - \ln 4 - \ln 3$

**2** Write in the form  $\ln a$ ,  $a \in \mathbb{Q}$ :

**a**  $5 \ln 3 + \ln 4$

**b**  $3 \ln 2 + 2 \ln 5$

**c**  $3 \ln 2 - \ln 8$

**d**  $3 \ln 4 - 2 \ln 2$

**e**  $\frac{1}{3} \ln 8 + \ln 3$

**f**  $\frac{1}{3} \ln(\frac{1}{27})$

**g**  $-\ln 2$

**h**  $-\ln(\frac{1}{2})$

**i**  $-2 \ln(\frac{1}{4})$

**Example 18****Self Tutor**

Show that:

**a**  $\ln(\frac{1}{9}) = -2 \ln 3$

**b**  $\ln(\frac{e}{4}) = 1 - 2 \ln 2$

$$\begin{aligned}\mathbf{a} \quad & \ln(\frac{1}{9}) \\ &= \ln(3^{-2}) \\ &= -2 \ln 3\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & \ln(\frac{e}{4}) = \ln e - \ln 4 \\ &= \ln e^1 - \ln 2^2 \\ &= 1 - 2 \ln 2\end{aligned}$$

**3** Show that:

**a**  $\ln 27 = 3 \ln 3$

**b**  $\ln \sqrt{3} = \frac{1}{2} \ln 3$

**c**  $\ln(\frac{1}{16}) = -4 \ln 2$

**d**  $\ln(\frac{1}{6}) = -\ln 6$

**e**  $\ln(\frac{1}{\sqrt{2}}) = -\frac{1}{2} \ln 2$

**f**  $\ln(\frac{e}{5}) = 1 - \ln 5$

**4** Show that:

**a**  $\ln \sqrt[3]{5} = \frac{1}{3} \ln 5$

**b**  $\ln(\frac{1}{32}) = -5 \ln 2$

**c**  $\ln(\frac{1}{\sqrt[5]{2}}) = -\frac{1}{5} \ln 2$

**d**  $\ln(\frac{e^2}{8}) = 2 - 3 \ln 2$

**Example 19****Self Tutor**

Write the following equations without logarithms:

**a**  $\ln A = 2 \ln c + 3$

**b**  $\ln M = 3a - \ln 2$

**a**  $\ln A = 2 \ln c + 3$

**b**  $\ln M = 3a - \ln 2$

$\therefore \ln A = \ln c^2 + \ln e^3$

$\therefore \ln M = \ln e^{3a} - \ln 2$

$\therefore \ln A = \ln(c^2 e^3)$

$\therefore \ln M = \ln\left(\frac{e^{3a}}{2}\right)$

$\therefore A = c^2 e^3$

$\therefore M = \frac{1}{2} e^{3a}$

**5** Write the following equations without logarithms, assuming all terms are positive:

**a**  $\ln D = \ln x + 1$

**b**  $\ln F = -\ln p + 2$

**c**  $\ln P = 2x + \ln 5$

**d**  $\ln M = 2 \ln y + 3$

**e**  $\ln B = 3t - \ln 4$

**f**  $\ln N = -\frac{1}{3} \ln g$

**g**  $\ln Q \approx 3 \ln x + 2.159$

**h**  $\ln D \approx 0.4 \ln n - 0.6582$

**i**  $\ln T \approx -x + 1.578$

## F

# SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS

In **Chapter 4** we found solutions to simple exponential equations where we could make equal bases and then equate exponents. However, it is not always easy to make the bases the same. In these situations we use **logarithms** to find the solution.

### Example 20



Solve for  $x$ , giving your answers correct to 3 significant figures:

**a**  $2^x = 7$

**b**  $5^{3x-1} = 90$

**a**  $2^x = 7$

$\therefore \lg 2^x = \lg 7$

$\therefore x \lg 2 = \lg 7 \quad \{\lg(a^n) = n \lg a\}$

$$\therefore x = \frac{\lg 7}{\lg 2}$$

$\therefore x \approx 2.81$

**b**  $5^{3x-1} = 90$

$\therefore \lg 5^{3x-1} = \lg 90$

$\therefore (3x-1) \lg 5 = \lg 90 \quad \{\lg(a^n) = n \lg a\}$

$$\therefore 3x-1 = \frac{\lg 90}{\lg 5}$$

$$\therefore x = \frac{1}{3} \left( 1 + \frac{\lg 90}{\lg 5} \right)$$

$\therefore x \approx 1.27$

### Example 21



Find  $x$  exactly:

**a**  $e^x = 30$

**b**  $3e^{\frac{x}{2}} = 21$

**a**  $e^x = 30$

$\therefore x = \ln 30$

**b**  $3e^{\frac{x}{2}} = 21$

$\therefore e^{\frac{x}{2}} = 7$

$\therefore \frac{x}{2} = \ln 7$

$\therefore x = 2 \ln 7$

## EXERCISE 5F

**1** Solve for  $x$ , giving your answer correct to 3 significant figures:

**a**  $2^x = 10$

**b**  $3^x = 20$

**c**  $4^x = 100$

**d**  $\left(\frac{1}{2}\right)^x = 0.0625$

**e**  $\left(\frac{3}{4}\right)^x = 0.1$

**f**  $10^x = 0.00001$

- 2** Solve for  $x$ , giving your answer correct to 3 significant figures:

a  $5^{2x} = 100$

b  $2^{4x} = 75$

c  $(0.8)^{3x} = 0.1$

d  $3^{x-1} = 200$

e  $4^{x+2} = 2.5$

f  $6^{2x-1} = 800$

g  $7^{2x+3} = 1000$

h  $(3^{x+1})^2 = 480$

i  $(2^{x-3})^{\frac{1}{2}} = 10$

- 3** Solve for  $x$ , giving an exact answer:

a  $e^x = 10$

b  $e^x = 1000$

c  $2e^x = 0.3$

d  $e^{\frac{x}{2}} = 5$

e  $e^{2x} = 18$

f  $e^{-\frac{x}{2}} = 1$

- 4 a** Solve  $e^{2x} = 300$  exactly.

**b** Use your calculator to find the solution correct to 2 decimal places.

### Example 22

### Self Tutor

A farmer monitoring an insect plague notices that the area affected by the insects is given by  $A = 1000 \times 2^{0.7n}$  hectares, where  $n$  is the number of weeks after the initial observation. How long will it take for the affected area to reach 5000 hectares?

When  $A = 5000$ ,

$$1000 \times 2^{0.7n} = 5000$$

$$\therefore 2^{0.7n} = 5$$

$$\therefore \lg 2^{0.7n} = \lg 5$$

$$\therefore 0.7n \lg 2 = \lg 5$$

$$\therefore n = \frac{\lg 5}{0.7 \times \lg 2}$$

$$\therefore n \approx 3.32$$

$\therefore$  it takes about 3 weeks and 2 days.

Logarithms allow us to solve exponential equations even if we cannot write both sides with the same base.



- 5** Solve for  $x$ , giving an exact answer:

a  $4 \times 2^{-x} = 0.12$

b  $300 \times 5^{0.1x} = 1000$

c  $32 \times 3^{-0.25x} = 4$

- 6** The weight  $W$  of bacteria in a culture  $t$  hours after establishment is given by  $W = 20 \times 2^{0.15t}$  grams. Find, using logarithms, the time for the weight of the culture to reach:

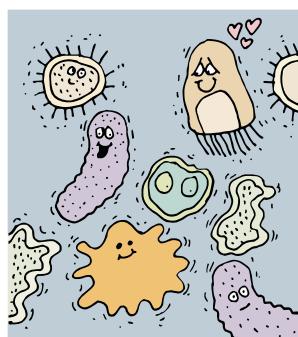
a 30 grams

b 100 grams.

- 7** The mass  $M$  of bacteria in a culture  $t$  hours after establishment is given by  $M = 25 \times e^{0.1t}$  grams.

a Show that the time required for the mass of the culture to reach 50 grams is  $10 \ln 2$  hours.

b Find the time required correct to 2 decimal places.



- 8** The weight of radioactive uranium remaining after  $t$  years is given by the formula  $W(t) = 50 \times 2^{-0.0002t}$  grams,  $t \geq 0$ .

a Find the initial weight of the uranium.

b Find the time required for the weight to fall to 8 grams.

**Example 23****Self Tutor**

Find algebraically the exact points of intersection of  $y = e^x - 3$  and  $y = 1 - 3e^{-x}$ .

The functions meet where

$$\begin{aligned} e^x - 3 &= 1 - 3e^{-x} \\ \therefore e^x - 4 + 3e^{-x} &= 0 \\ \therefore e^{2x} - 4e^x + 3 &= 0 \quad \{\text{multiplying each term by } e^x\} \\ \therefore (e^x - 1)(e^x - 3) &= 0 \\ \therefore e^x &= 1 \text{ or } 3 \\ \therefore x &= \ln 1 \text{ or } \ln 3 \\ \therefore x &= 0 \text{ or } \ln 3 \end{aligned}$$

**GRAPHING PACKAGE**



When  $x = 0$ ,  $y = e^0 - 3 = -2$

When  $x = \ln 3$ ,  $y = e^{\ln 3} - 3 = 0$

$\therefore$  the functions meet at  $(0, -2)$  and at  $(\ln 3, 0)$ .

**9** Solve for  $x$ :

**a**  $e^{2x} = 2e^x$

**b**  $e^x = e^{-x}$

**c**  $e^{2x} - 5e^x + 6 = 0$

**d**  $e^x + 2 = 3e^{-x}$

**e**  $1 + 12e^{-x} = e^x$

**f**  $e^x + e^{-x} = 3$

**10** Find algebraically the point(s) of intersection of:

**a**  $y = e^x$  and  $y = e^{2x} - 6$

**b**  $y = 2e^x + 1$  and  $y = 7 - e^x$

**c**  $y = 3 - e^x$  and  $y = 5e^{-x} - 3$

**G****THE CHANGE OF BASE RULE**

A logarithm in base  $b$  can be written with a different base  $c$  using the **change of base rule**:

$$\log_b a = \frac{\log_c a}{\log_c b} \quad \text{for } a, b, c > 0 \text{ and } b, c \neq 1.$$

**Proof:**

If  $\log_b a = x$ , then  $b^x = a$

$$\therefore \log_c b^x = \log_c a \quad \{\text{taking logarithms in base } c\}$$

$$\therefore x \log_c b = \log_c a \quad \{\text{power law of logarithms}\}$$

$$\therefore x = \frac{\log_c a}{\log_c b}$$

$$\therefore \log_b a = \frac{\log_c a}{\log_c b}$$

We can use this rule to write logarithms in base 10 or base  $e$ . This is useful in helping us evaluate them on our calculator.

**Example 24****Self Tutor**

Evaluate  $\log_2 9$  by:

- a** changing to base 10      **b** changing to base  $e$ .

$$\mathbf{a} \quad \log_2 9 = \frac{\log_{10} 9}{\log_{10} 2} \approx 3.17 \quad \mathbf{b} \quad \log_2 9 = \frac{\ln 9}{\ln 2} \approx 3.17$$

The rule can also be used to solve equations involving logarithms with different bases.

**Example 25****Self Tutor**

Solve for  $x$ :  $\log_2 x = \log_8 15$

$$\begin{aligned} \log_2 x &= \log_8 15 \\ \therefore \log_2 x &= \frac{\log_2 15}{\log_2 8} \quad \{ \text{writing RHS with base 2} \} \\ \therefore \log_2 x &= \frac{\log_2 15}{3} \\ \therefore \log_2 x &= \log_2 15^{\frac{1}{3}} \\ \therefore x &= \sqrt[3]{15} \end{aligned}$$

**EXERCISE 5G**

- 1** Use the rule  $\log_b a = \frac{\log_{10} a}{\log_{10} b}$  to evaluate, correct to 3 significant figures:

**a**  $\log_3 12$       **b**  $\log_{\frac{1}{2}} 1250$       **c**  $\log_3(0.067)$       **d**  $\log_{0.4}(0.006984)$

- 2** Use the rule  $\log_b a = \frac{\ln a}{\ln b}$  to solve, correct to 3 significant figures:

**a**  $2^x = 0.051$       **b**  $4^x = 213.8$       **c**  $3^{2x+1} = 4.069$

- 3** Write:

**a**  $\log_9 26$  in the form  $a \log_3 b$ , where  $a, b \in \mathbb{Q}$   
**b**  $\log_2 11$  in the form  $a \log_4 b$ , where  $a, b \in \mathbb{Z}$   
**c**  $\frac{6}{\log_7 25}$  in the form  $a \log_5 b$ , where  $a, b \in \mathbb{Z}$ .

If  $2^x = a$ ,  
then  $x = \log_2 a$ .



- 4** Solve for  $x$ :

**a**  $\log_3 x = \log_{27} 50$       **b**  $\log_2 x = \log_4 13$       **c**  $\log_{25} x = \log_5 7$   
**d**  $\log_3 \sqrt{x} + \log_9 x = \log_3 5$       **e**  $\log_8 x^2 - \log_2 \sqrt[3]{x} = 1$       **f**  $\log_4 x^3 + \log_2 \sqrt{x} = 8$

- 5** **a** Show that  $\log_a b = \frac{1}{\log_b a}$ .

**b** Solve for  $x$ :

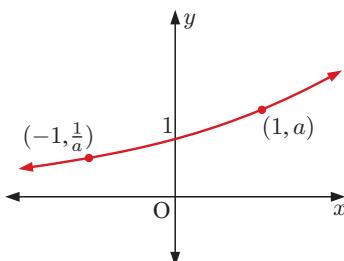
**i**  $\log_3 x = 4 \log_x 3$       **ii**  $\log_2 x - 4 = 5 \log_x 2$       **iii**  $2 \log_4 x + 3 \log_x 4 = 7$

**H****GRAPHS OF LOGARITHMIC FUNCTIONS**

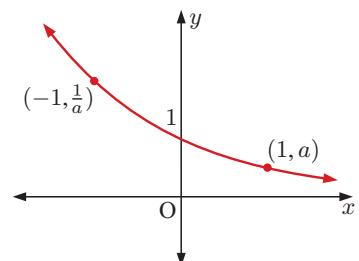
Consider the general exponential function  $f(x) = a^x$ ,  $a > 0$ ,  $a \neq 1$ .

The graph of  $y = a^x$  is:

For  $a > 1$ :



For  $0 < a < 1$ :



The **horizontal asymptote** for all of these functions is the  $x$ -axis  $y = 0$ .

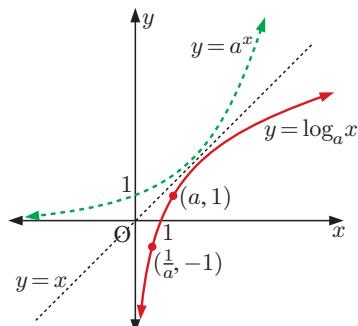
The inverse function  $f^{-1}$  is given by  $x = a^y$ , so  $y = \log_a x$ .

$$\text{If } f(x) = a^x \text{ where } a > 0, a \neq 1, \text{ then } f^{-1}(x) = \log_a x.$$

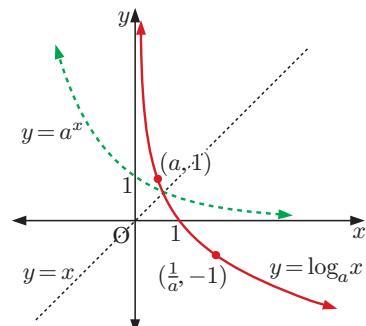
Since  $f^{-1}(x) = \log_a x$  is an inverse function, it is a reflection of  $f(x) = a^x$  in the line  $y = x$ . We may therefore deduce the following properties:

Function	$f(x) = a^x$	$f^{-1}(x) = \log_a x$
Domain	$\{x : x \in \mathbb{R}\}$	$\{x : x > 0\}$
Range	$\{y : y > 0\}$	$\{y : y \in \mathbb{R}\}$
Asymptote	horizontal $y = 0$	vertical $x = 0$

The graph of  $y = \log_a x$  for  $a > 1$ :



The graph of  $y = \log_a x$  for  $0 < a < 1$ :



The **vertical asymptote** of  $y = \log_a x$  is the  $y$ -axis  $x = 0$ .

Since we can only find logarithms of positive numbers, the domain of  $f^{-1}(x) = \log_a x$  is  $\{x | x > 0\}$ .

In general,  $y = \log_a(g(x))$  is defined when  $g(x) > 0$ .

**Example 26****Self Tutor**

Consider the function  $f(x) = \log_2(x - 1) + 1$ .

- Find the domain and range of  $f$ .
- Find any asymptotes and axes intercepts.
- Sketch the graph of  $f$  showing all important features.
- Find  $f^{-1}$ .

**a**  $x - 1 > 0$  when  $x > 1$

So, the domain is  $\{x : x > 1\}$  and the range is  $y \in \mathbb{R}$ .

- b** As  $x \rightarrow 1$  from the right,  $y \rightarrow -\infty$ , so the vertical asymptote is  $x = 1$ .  
As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ .

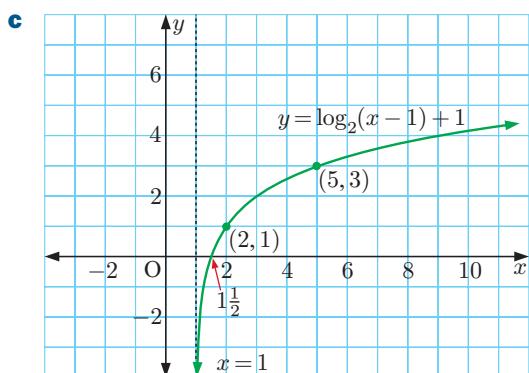
When  $x = 0$ ,  $y$  is undefined, so there is no  $y$ -intercept.

When  $y = 0$ ,  $\log_2(x - 1) = -1$

$$\therefore x - 1 = 2^{-1}$$

$$\therefore x = 1\frac{1}{2}$$

So, the  $x$ -intercept is  $1\frac{1}{2}$ .



**d**  $f$  is defined by  $y = \log_2(x - 1) + 1$   
 $\therefore f^{-1}$  is defined by  $x = \log_2(y - 1) + 1$   
 $\therefore x - 1 = \log_2(y - 1)$   
 $\therefore y - 1 = 2^{x-1}$   
 $\therefore y = 2^{x-1} + 1$   
 $\therefore f^{-1}(x) = 2^{x-1} + 1$

which has the horizontal asymptote  $y = 1$  ✓  
 Its domain is  $\{x : x \in \mathbb{R}\}$ , and  
 its range is  $\{y : y > 1\}$ .

**EXERCISE 5H**

- 1** For the following functions  $f$ :

- Find the domain and range.
- Find any asymptotes and axes intercepts.
- Sketch the graph of  $y = f(x)$  showing all important features.
- Solve  $f(x) = -1$  algebraically and check the solution on your graph.
- Find  $f^{-1}$ .

**a**  $f : x \mapsto \log_3(x + 1)$ ,  $x > -1$

**c**  $f : x \mapsto \log_5(x - 2) - 2$ ,  $x > 2$

**e**  $f : x \mapsto 1 - 2 \log_2 x$ ,  $x > 0$

**b**  $f : x \mapsto 1 - \log_3(x + 1)$ ,  $x > -1$

**d**  $f : x \mapsto 1 - \log_5(x - 2)$ ,  $x > 2$

**Example 27****Self Tutor**

Consider the function  $f : x \mapsto e^{x-3}$ .

- Find the equation defining  $f^{-1}$ .
- Sketch the graphs of  $f$  and  $f^{-1}$  on the same set of axes.
- State the domain and range of  $f$  and  $f^{-1}$ .
- Find any asymptotes and intercepts of  $f$  and  $f^{-1}$ .

**a**  $f(x) = e^{x-3}$

$\therefore f^{-1}$  is  $x = e^{y-3}$

$\therefore y - 3 = \ln x$

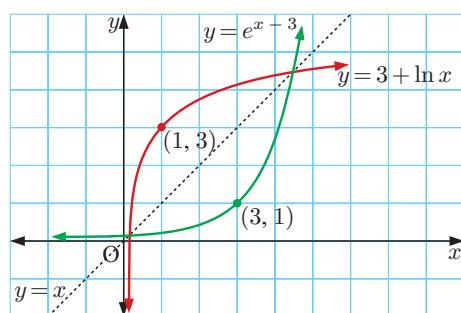
$\therefore y = 3 + \ln x$

So,  $f^{-1}(x) = 3 + \ln x$

**c**

Function	$f$	$f^{-1}$
Domain	$x \in \mathbb{R}$	$x > 0$
Range	$y > 0$	$y \in \mathbb{R}$

**b**



- d** For  $f$ , the horizontal asymptote is  $y = 0$ , and the  $y$ -intercept is  $e^{-3}$ .

For  $f^{-1}$ , the vertical asymptote is  $x = 0$ , and the  $x$ -intercept is  $e^{-3}$ .

- 2** For the following functions  $f$ :

- Find the equation of  $f^{-1}$ .
- Sketch the graphs of  $f$  and  $f^{-1}$  on the same set of axes.
- State the domain and range of  $f$  and  $f^{-1}$ .
- Find any asymptotes and intercepts of  $f$  and  $f^{-1}$ .

**a**  $f(x) = e^x + 5$

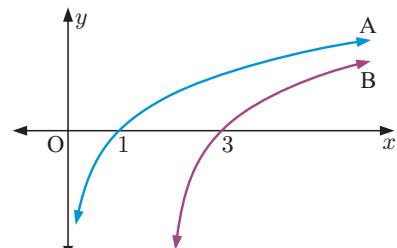
**b**  $f(x) = e^{x+1} - 3$

**c**  $f(x) = \ln x - 4, x > 0$

**d**  $f(x) = \ln(x-1) + 2, x > 1$

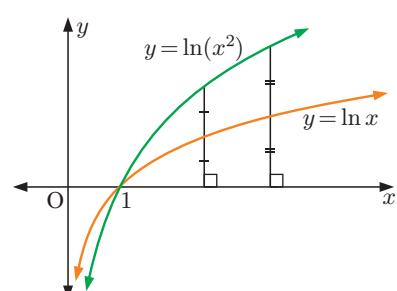
- 3** Consider the graphs A and B. One of them is the graph of  $y = \ln x$  and the other is the graph of  $y = \ln(x-2)$ .

- Identify which is which. Give evidence for your answer.
- Copy the graphs onto a new set of axes and add to them the graph of  $y = \ln(x+2)$ .
- Find the equation of the vertical asymptote for each graph.



- 4** Kelly said that in order to graph  $y = \ln(x^2), x > 0$ , you could first graph  $y = \ln x$  and then double the distance of each point on the graph from the  $x$ -axis.

Is Kelly correct? Explain your answer.



**5** Consider the function  $f : x \mapsto e^{x+3} + 2$ .

a Find the defining equation for  $f^{-1}$ .

b Find the values of  $x$  for which:

i  $f(x) < 2.1$       ii  $f(x) < 2.01$       iii  $f(x) < 2.001$       iv  $f(x) < 2.0001$

Hence conjecture the horizontal asymptote for the graph of  $f$ .

c Determine the equation of the horizontal asymptote of  $f(x)$  by discussing the behaviour of  $f(x)$  as  $x \rightarrow \pm\infty$ .

d Hence, determine the vertical asymptote and the domain of  $f^{-1}$ .

**6** Consider  $f(x) = \log_2(x+3)$ .

a Find: i  $f(5)$       ii  $f(x^2)$       iii  $f(2x-1)$

b State the domain of  $f(x)$ .

c Solve  $f(x^2+4) = 5$ .

**7** Suppose  $f(x) = e^{3x} + 1$ .

a State the range of  $f(x)$ .

b Find  $f^{-1}(x)$ .

c Find  $f^{-1}(10)$ .

d State the domain of  $f^{-1}(x)$ .

e Find  $(f \circ f^{-1})(x)$  and  $(f^{-1} \circ f)(x)$ .

**8** Suppose  $f : x \mapsto e^{2x}$  and  $g : x \mapsto 2x - 1$ .

a Find: i  $(f^{-1} \circ g)(x)$       ii  $(g \circ f)^{-1}(x)$

b Solve  $(f^{-1} \circ g)(x) = \ln 5$ .

**9** Consider  $f : x \mapsto 10e^{-x}$  and  $g : x \mapsto \ln(x-3)$ .

a Find  $f(1)$  and  $g(6)$ .

b Find the  $x$ -intercept of  $g(x)$ .

c Find  $fg(x)$ .

d Solve  $f(x) = g^{-1}(x)$ .

**10** Let  $f(x) = \ln(x+6)$  and  $g(x) = x - \ln 3$ .

a State the domain of  $f(x)$ .

b Find  $f^{-1}(x)$ .

c Find the axes intercepts of  $f(x)$ .

d Solve  $gf(x) = f(x^2 - 12)$ .

## Activity

Click on the icon to obtain a card game for logarithmic functions.

CARD GAME



## Review set 5A

**1** Find the following, showing all working:

a  $\log_4 64$

b  $\log_2 256$

c  $\log_2(0.25)$

d  $\log_{25} 5$

e  $\log_8 1$

f  $\log_{81} 3$

g  $\log_9(0.\bar{1})$

h  $\log_k \sqrt{k}$

**2** Find:

**a**  $\lg \sqrt{10}$

**b**  $\lg \frac{1}{\sqrt[3]{10}}$

**c**  $\lg(10^a \times 10^{b+1})$

**3** Simplify:

**a**  $4 \ln 2 + 2 \ln 3$

**b**  $\frac{1}{2} \ln 9 - \ln 2$

**c**  $2 \ln 5 - 1$

**d**  $\frac{1}{4} \ln 81$

**4** Find:

**a**  $\ln(e\sqrt{e})$

**b**  $\ln\left(\frac{1}{e^3}\right)$

**c**  $\ln(e^{2x})$

**d**  $\ln\left(\frac{e}{e^x}\right)$

**5** Write as a single logarithm:

**a**  $\lg 16 + 2 \lg 3$

**b**  $\log_2 16 - 2 \log_2 3$

**c**  $2 + \log_4 5$

**6** Write as logarithmic equations:

**a**  $P = 3 \times 7^x$

**b**  $m = \frac{n^3}{5}$

**7** Solve for  $x$ :

**a**  $\log_2(x+5) - \log_2(x-2) = 3$

**b**  $\lg x + \lg(x+15) = 2$

**8** Show that  $\log_3 7 \times 2 \log_7 x = 2 \log_3 x$ .

**9** Write the following equations without logarithms:

**a**  $\lg T = 2 \lg x - \lg 5$

**b**  $\log_2 K = x + \log_2 3$

**10** Write in the form  $a \ln k$  where  $a$  and  $k$  are positive whole numbers and  $k$  is prime:

**a**  $\ln 32$

**b**  $\ln 125$

**c**  $\ln 729$

**11** Copy and complete:

Function	$y = \log_2 x$	$y = \ln(x+5)$
Domain		
Range		

**12** If  $A = \log_5 2$  and  $B = \log_5 3$ , write in terms of  $A$  and  $B$ :

**a**  $\log_5 36$

**b**  $\log_5 54$

**c**  $\log_5(8\sqrt{3})$

**d**  $\log_5(20.25)$

**e**  $\log_5(0.8)$

**13** Solve for  $x$ :

**a**  $3e^x - 5 = -2e^{-x}$

**b**  $2 \ln x - 3 \ln\left(\frac{1}{x}\right) = 10$

**14** Solve for  $x$ , giving your answer to 2 decimal places:

**a**  $7^x = 120$

**b**  $6 \times 2^{3x} = 300$

**15** A population of seals is given by  $P = 20 \times 2^{\frac{t}{3}}$  where  $t$  is the time in years,  $t \geq 0$ . Find the time required for the population to reach 100.

**16** Consider  $f : x \mapsto 5e^{-x} + 1$ .

**a** State the range of  $f$ .

**b** Find: **i**  $f^{-1}(x)$     **ii**  $f^{-1}(2)$

**c** State the domain of  $f^{-1}$ .

**d** Solve  $f^{-1}(x) = 0$ .

**e** Sketch the graphs of  $f$ ,  $f^{-1}$ , and  $y = x$  on the same set of axes.

**Review set 5B**

- 1** Without using a calculator, find the base 10 logarithms of:
- a**  $\sqrt{1000}$       **b**  $\frac{10}{\sqrt[3]{10}}$       **c**  $\frac{10^a}{10^{-b}}$
- 2** Write in the form  $10^x$  giving  $x$  correct to 4 decimal places:
- a** 32      **b** 0.0013      **c**  $8.963 \times 10^{-5}$
- 3** Find  $x$  if:
- a**  $\log_2 x = -3$       **b**  $\log_5 x \approx 2.743$       **c**  $\log_3 x \approx -3.145$
- 4** Write the following equations without logarithms:
- a**  $\log_2 k \approx 1.699 + x$       **b**  $\log_a Q = 3 \log_a P + \log_a 5$       **c**  $\lg A = x \lg 2 + \lg 6$
- 5** Solve for  $x$ , giving exact answers:
- a**  $5^x = 7$       **b**  $20 \times 2^{2x+1} = 640$
- 6** Find the exact value of  $\log_{12} 3 - 2 \log_{12} 6$ .
- 7** Write  $\log_8 30$  in the form  $a \log_2 b$ , where  $a, b \in \mathbb{Q}$ .
- 8** Solve for  $x$ :
- a**  $\log_4 x + \log_4(2x - 8) = 3$       **b**  $\log_x 135 = 3 + \log_x 5$
- 9** Consider  $f(x) = e^x$  and  $g(x) = \ln(x + 4)$ ,  $x > -4$ . Find:
- a**  $(f \circ g)(5)$       **b**  $(g \circ f)(0)$
- 10** Write as a single logarithm:
- a**  $\ln 60 - \ln 20$       **b**  $\ln 4 + \ln 1$       **c**  $\ln 200 - \ln 8 + \ln 5$
- 11** Write as logarithmic equations:
- a**  $M = 5 \times 6^x$       **b**  $T = \frac{5}{\sqrt{l}}$       **c**  $G = \frac{4}{c}$
- 12** Solve exactly for  $x$ :
- a**  $e^{2x} = 3e^x$       **b**  $e^{2x} - 7e^x + 12 = 0$
- 13** Consider the function  $g : x \mapsto \log_3(x + 2) - 2$ .
- a** Find the domain and range.
- b** Find any asymptotes and axes intercepts for the graph of the function.
- c** Find the defining equation for  $g^{-1}$ .
- d** Sketch the graphs of  $g$ ,  $g^{-1}$ , and  $y = x$  on the same axes.
- 14** The weight of a radioactive isotope remaining after  $t$  weeks is given by  $W = 8000 \times e^{-\frac{t}{20}}$  grams. Find the time for the weight to halve.
- 15** Solve for  $x$ :
- a**  $\log_2 x + \log_4 x^4 = \log_2 125$       **b**  $\log_2 x = 25 \log_x 2$       **c**  $\log_3 x + 8 \log_x 3 = 6$
- 16** Consider  $f(x) = 5e^{2x}$  and  $g(x) = \ln(x - 4)$ .
- a** State the domain and range of  $g$ .
- b** Find the axes intercepts of  $g$ .
- c** Find the exact solution to  $fg(x) = 30$ .
- d** Solve  $f(x) = g^{-1}(x)$ .

# 6

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# Polynomials

## Contents:

- A** Real polynomials
- B** Zeros, roots, and factors
- C** The Remainder theorem
- D** The Factor theorem
- E** Cubic equations

## Opening problem

To determine whether 7 is a **factor** of 56, we divide 56 by 7. The result is exactly 8. Since there is no remainder, 7 is a factor of 56.

### Things to think about:

- Can we perform a similar test for *algebraic* factors? For example, how can we determine whether  $x - 3$  is a factor of  $x^3 - 4x^2 + 2x + 3$ ?
- Given that  $x - 3$  is a factor of  $x^3 - 4x^2 + 2x + 3$ , what does this tell us about the graph of  $f(x) = x^3 - 4x^2 + 2x + 3$ ?

Up to this point we have studied linear and quadratic functions at some depth, with perhaps occasional reference to cubic functions. These are part of a larger family of functions called the **polynomials**.

## A REAL POLYNOMIALS

A **polynomial function** is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0, \quad a_1, \dots, a_n \text{ constant}, \quad a_n \neq 0.$$

We say that:

$x$  is the **variable**

$a_0$  is the **constant term**

$a_n$  is the **leading coefficient** and is non-zero

$a_r$  is the **coefficient of  $x^r$**  for  $r = 0, 1, 2, \dots, n$

$n$  is the **degree** of the polynomial, being the highest power of the variable.

In **summation notation**, we write  $P(x) = \sum_{r=0}^n a_r x^r$ ,

which reads: “the sum from  $r = 0$  to  $n$ , of  $a_r x^r$ ”.

A **real polynomial**  $P(x)$  is a polynomial for which  $a_r \in \mathbb{R}$ ,  $r = 0, 1, 2, \dots, n$ .

The low degree members of the polynomial family have special names, some of which you are already familiar with. For these polynomials, we commonly write their coefficients as  $a, b, c, \dots$

Polynomial function	Degree	Name
$ax + b, \quad a \neq 0$	1	linear
$ax^2 + bx + c, \quad a \neq 0$	2	quadratic
$ax^3 + bx^2 + cx + d, \quad a \neq 0$	3	cubic
$ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0$	4	quartic

## ADDITION AND SUBTRACTION

To **add** or **subtract** two polynomials, we collect ‘like’ terms.

**Example 1****Self Tutor**

If  $P(x) = x^3 - 2x^2 + 3x - 5$  and  $Q(x) = 2x^3 + x^2 - 11$ , find:

**a**  $P(x) + Q(x)$

**b**  $P(x) - Q(x)$

$$\begin{aligned}\mathbf{a} \quad & P(x) + Q(x) \\ &= x^3 - 2x^2 + 3x - 5 \\ &\quad + 2x^3 + x^2 - 11 \\ &= \underline{3x^3 - x^2 + 3x - 16}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & P(x) - Q(x) \\ &= x^3 - 2x^2 + 3x - 5 - (2x^3 + x^2 - 11) \\ &= x^3 - 2x^2 + 3x - 5 \\ &\quad - 2x^3 - x^2 + 11 \\ &= \underline{-x^3 - 3x^2 + 3x + 6}\end{aligned}$$

Collecting ‘like’ terms is made easier by writing them one above the other.

It is a good idea to place brackets around expressions which are subtracted.

**SCALAR MULTIPLICATION**

To **multiply** a polynomial by a **scalar** (constant) we multiply each term by the scalar.

**Example 2****Self Tutor**

If  $P(x) = x^4 - 2x^3 + 4x + 7$ , find:

**a**  $3P(x)$

**b**  $-2P(x)$

$$\begin{aligned}\mathbf{a} \quad & 3P(x) \\ &= 3(x^4 - 2x^3 + 4x + 7) \\ &= 3x^4 - 6x^3 + 12x + 21\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & -2P(x) \\ &= -2(x^4 - 2x^3 + 4x + 7) \\ &= -2x^4 + 4x^3 - 8x - 14\end{aligned}$$

**POLYNOMIAL MULTIPLICATION**

To **multiply** two polynomials, we multiply each term of the first polynomial by each term of the second polynomial, and then collect like terms.

**Example 3****Self Tutor**

If  $P(x) = x^3 - 2x + 4$  and  $Q(x) = 2x^2 + 3x - 5$ , find  $P(x)Q(x)$ .

$$\begin{aligned}P(x)Q(x) &= (x^3 - 2x + 4)(2x^2 + 3x - 5) \\ &= x^3(2x^2 + 3x - 5) - 2x(2x^2 + 3x - 5) + 4(2x^2 + 3x - 5) \\ &= 2x^5 + 3x^4 - 5x^3 \\ &\quad - 4x^3 - 6x^2 + 10x \\ &\quad + 8x^2 + 12x - 20 \\ &= \underline{2x^5 + 3x^4 - 9x^3 + 2x^2 + 22x - 20}\end{aligned}$$

**EXERCISE 6A.1**

**1** If  $P(x) = x^2 + 2x + 3$  and  $Q(x) = 4x^2 + 5x + 6$ , find in simplest form:

- a**  $3P(x)$       **b**  $P(x) + Q(x)$       **c**  $P(x) - 2Q(x)$       **d**  $P(x)Q(x)$

**2** If  $f(x) = x^2 - x + 2$  and  $g(x) = x^3 - 3x + 5$ , find in simplest form:

- a**  $f(x) + g(x)$       **b**  $g(x) - f(x)$       **c**  $2f(x) + 3g(x)$   
**d**  $g(x) + xf(x)$       **e**  $f(x)g(x)$       **f**  $[f(x)]^2$

**3** Expand and simplify:

- a**  $(x^2 - 2x + 3)(2x + 1)$       **b**  $(x - 1)^2(x^2 + 3x - 2)$       **c**  $(x + 2)^3$   
**d**  $(2x^2 - x + 3)^2$       **e**  $(2x - 1)^4$       **f**  $(3x - 2)^2(2x + 1)(x - 4)$

**4** Find the following products:

- a**  $(2x^2 - 3x + 5)(3x - 1)$       **b**  $(4x^2 - x + 2)(2x + 5)$   
**c**  $(2x^2 + 3x + 2)(5 - x)$       **d**  $(x - 2)^2(2x + 1)$   
**e**  $(x^2 - 3x + 2)(2x^2 + 4x - 1)$       **f**  $(3x^2 - x + 2)(5x^2 + 2x - 3)$   
**g**  $(x^2 - x + 3)^2$       **h**  $(2x^2 + x - 4)^2$   
**i**  $(2x + 5)^3$       **j**  $(x^3 + x^2 - 2)^2$

**Discussion**

Suppose  $f(x)$  is a polynomial of degree  $m$ , and  $g(x)$  is a polynomial of degree  $n$ .

What is the degree of:

- $f(x) + g(x)$
- $5f(x)$
- $[f(x)]^2$
- $f(x)g(x)?$

**DIVISION OF POLYNOMIALS**

The division of polynomials is only useful if we divide a polynomial of degree  $n$  by another of degree  $n$  or less.

**Division by linears**

Consider  $(2x^2 + 3x + 4)(x + 2) + 7$ .

If we expand this expression we obtain  $(2x^2 + 3x + 4)(x + 2) + 7 = 2x^3 + 7x^2 + 10x + 15$ .

Dividing both sides by  $(x + 2)$ , we obtain

$$\begin{aligned} \frac{2x^3 + 7x^2 + 10x + 15}{x + 2} &= \frac{(2x^2 + 3x + 4)(x + 2) + 7}{x + 2} \\ &= \frac{(2x^2 + 3x + 4)(x + 2)}{x + 2} + \frac{7}{x + 2} \\ &= 2x^2 + 3x + 4 + \frac{7}{x + 2} \end{aligned}$$

The division of polynomials is not required for the syllabus, but is useful for understanding the Remainder and Factor theorems.

where  $x + 2$  is the divisor,  
 $2x^2 + 3x + 4$  is the quotient,  
and  $7$  is the remainder.



If  $P(x)$  is divided by  $ax + b$  until a constant remainder  $R$  is obtained, then

$$\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b} \quad \text{where } ax + b \text{ is the divisor, } D(x), \\ Q(x) \text{ is the quotient,} \\ \text{and } R \text{ is the remainder.}$$

Notice that  $P(x) = Q(x) \times (ax + b) + R$ .

### Division algorithm

We can divide a polynomial by another polynomial using an algorithm similar to that used for division of whole numbers:

*Step 1:* What do we multiply  $x$  by to get  $2x^3$ ?

The answer is  $2x^2$ ,

and  $2x^2(x + 2) = \underline{\underline{2x^3 + 4x^2}}$ .

*Step 2:* Subtract  $2x^3 + 4x^2$  from  $2x^3 + 7x^2$ .

The answer is  $3x^2$ .

*Step 3:* Bring down the  $10x$  to obtain  $3x^2 + 10x$ .

Return to *Step 1* with the question:

“What must we multiply  $x$  by to get  $3x^2$ ? ”

The answer is  $3x$ , and  $3x(x + 2) = 3x^2 + 6x$  ....

We continue the process until we are left with a constant.

$$\begin{array}{r} 2x^2 + 3x + 4 \\ x+2 \overline{)2x^3 + 7x^2 + 10x + 15} \\ - (2x^3 + 4x^2) \\ \hline 3x^2 + 10x \\ - (3x^2 + 6x) \\ \hline 4x + 15 \\ - (4x + 8) \\ \hline 7 \end{array}$$

$$\text{So, } \frac{2x^3 + 7x^2 + 10x + 15}{x + 2} = 2x^2 + 3x + 4 + \frac{7}{x + 2}$$

### Example 4

### Self Tutor

Find the quotient and remainder for  $\frac{x^3 - x^2 - 3x - 5}{x - 3}$ .

Hence write  $x^3 - x^2 - 3x - 5$  in the form  $Q(x) \times (x - 3) + R$ .

$$\begin{array}{r} x^2 + 2x + 3 \\ x-3 \overline{)x^3 - x^2 - 3x - 5} \\ - (x^3 - 3x^2) \\ \hline 2x^2 - 3x \\ - (2x^2 - 6x) \\ \hline 3x - 5 \\ - (3x - 9) \\ \hline 4 \end{array}$$

The quotient is  $x^2 + 2x + 3$  and the remainder is 4.

$$\therefore \frac{x^3 - x^2 - 3x - 5}{x - 3} = x^2 + 2x + 3 + \frac{4}{x - 3}$$

$$\therefore x^3 - x^2 - 3x - 5 = (x^2 + 2x + 3)(x - 3) + 4.$$

Check your answer by expanding the RHS.



**Example 5****Self Tutor**

Perform the division  $\frac{x^4 + 2x^2 - 1}{x + 3}$ .

Hence write  $x^4 + 2x^2 - 1$  in the form  $Q(x) \times (x + 3) + R$ .

$$\begin{array}{r} x^3 - 3x^2 + 11x - 33 \\ x+3 \overline{)x^4 + 0x^3 + 2x^2 + 0x - 1} \\ \underline{- (x^4 + 3x^3)} \quad \downarrow \quad \downarrow \\ \quad - 3x^3 + 2x^2 \\ \underline{- (-3x^3 - 9x^2)} \quad \downarrow \\ \quad 11x^2 + 0x \\ \underline{- (11x^2 + 33x)} \quad \downarrow \\ \quad - 33x - 1 \\ \underline{- (-33x - 99)} \\ \qquad \qquad \qquad 98 \end{array}$$

Notice the insertion of  $0x^3$  and  $0x$ .



$$\therefore \frac{x^4 + 2x^2 - 1}{x + 3} = x^3 - 3x^2 + 11x - 33 + \frac{98}{x + 3}$$

$$\therefore x^4 + 2x^2 - 1 = (x^3 - 3x^2 + 11x - 33)(x + 3) + 98$$

**EXERCISE 6A.2**

- 1** Find the quotient and remainder for the following, and hence write the division in the form  $P(x) = Q(x)D(x) + R$ , where  $D(x)$  is the divisor.

a  $\frac{x^2 + 2x - 3}{x + 2}$

b  $\frac{x^2 - 5x + 1}{x - 1}$

c  $\frac{2x^3 + 6x^2 - 4x + 3}{x - 2}$

- 2** Perform the following divisions, and hence write the division in the form  $P(x) = Q(x)D(x) + R$ .

a  $\frac{x^2 - 3x + 6}{x - 4}$

b  $\frac{x^2 + 4x - 11}{x + 3}$

c  $\frac{2x^2 - 7x + 2}{x - 2}$

d  $\frac{2x^3 + 3x^2 - 3x - 2}{2x + 1}$

e  $\frac{3x^3 + 11x^2 + 8x + 7}{3x - 1}$

f  $\frac{2x^4 - x^3 - x^2 + 7x + 4}{2x + 3}$

- 3** Perform the divisions:

a  $\frac{x^2 + 5}{x - 2}$

b  $\frac{2x^2 + 3x}{x + 1}$

c  $\frac{3x^2 + 2x - 5}{x + 2}$

d  $\frac{x^3 + 2x^2 - 5x + 2}{x - 1}$

e  $\frac{2x^3 - x}{x + 4}$

f  $\frac{x^3 + x^2 - 5}{x - 2}$

## DIVISION BY QUADRATICS

As with division by linears, we can use the **division algorithm** to divide polynomials by quadratics. The division process stops when the remainder has degree less than that of the divisor, so

If  $P(x)$  is divided by  $ax^2 + bx + c$  then

$$\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{ex + f}{ax^2 + bx + c} \quad \text{where } ax^2 + bx + c \text{ is the divisor,}$$

$Q(x)$  is the quotient,  
and  $ex + f$  is the remainder.

The remainder will be linear if  $e \neq 0$ , and constant if  $e = 0$ .

### Example 6



Find the quotient and remainder for  $\frac{x^4 + 4x^3 - x + 1}{x^2 - x + 1}$ .

Hence write  $x^4 + 4x^3 - x + 1$  in the form  $Q(x) \times (x^2 - x + 1) + R(x)$ .

$$\begin{array}{r} & x^2 + 5x + 4 \\ x^2 - x + 1 & \overline{\bigg|} \begin{array}{r} x^4 + 4x^3 + 0x^2 - x + 1 \\ - (x^4 - x^3 + x^2) \\ \hline 5x^3 - x^2 - x \\ - (5x^3 - 5x^2 + 5x) \\ \hline 4x^2 - 6x + 1 \\ - (4x^2 - 4x + 4) \\ \hline - 2x - 3 \end{array} \end{array}$$

The quotient is  $x^2 + 5x + 4$   
and the remainder is  $-2x - 3$ .  
 $\therefore x^4 + 4x^3 - x + 1$   
 $= (x^2 + 5x + 4)(x^2 - x + 1) - 2x - 3$

### EXERCISE 6A.3

- 1 Find the quotient and remainder for:

a  $\frac{x^3 + 2x^2 + x - 3}{x^2 + x + 1}$

b  $\frac{3x^2 - x}{x^2 - 1}$

c  $\frac{3x^3 + x - 1}{x^2 + 1}$

d  $\frac{x - 4}{x^2 + 2x - 1}$

- 2 Carry out the following divisions and also write each in the form  $P(x) = Q(x)D(x) + R(x)$ :

a  $\frac{x^2 - x + 1}{x^2 + x + 1}$

b  $\frac{x^3}{x^2 + 2}$

c  $\frac{x^4 + 3x^2 + x - 1}{x^2 - x + 1}$

d  $\frac{2x^3 - x + 6}{(x - 1)^2}$

e  $\frac{x^4}{(x + 1)^2}$

f  $\frac{x^4 - 2x^3 + x + 5}{(x - 1)(x + 2)}$

- 3 Suppose  $P(x) = (x - 2)(x^2 + 2x + 3) + 7$ . Find the quotient and remainder when  $P(x)$  is divided by  $x - 2$ .

- 4 Suppose  $f(x) = (x - 1)(x + 2)(x^2 - 3x + 5) + 15 - 10x$ . Find the quotient and remainder when  $f(x)$  is divided by  $x^2 + x - 2$ .

## B ZEROS, ROOTS, AND FACTORS

A **zero** of a polynomial is a value of the variable which makes the polynomial equal to zero.

$\alpha$  is a **zero** of polynomial  $P(x) \Leftrightarrow P(\alpha) = 0$ .

The **roots** of a polynomial **equation** are the solutions to the equation.

$\alpha$  is a **root** (or **solution**) of  $P(x) = 0 \Leftrightarrow P(\alpha) = 0$ .

The **roots** of  $P(x) = 0$  are the **zeros** of  $P(x)$  and the  $x$ -intercepts of the graph of  $y = P(x)$ .

Consider  $P(x) = x^3 + 2x^2 - 3x - 10$

$$\begin{aligned}\therefore P(2) &= 2^3 + 2(2)^2 - 3(2) - 10 \\ &= 8 + 8 - 6 - 10 \\ &= 0\end{aligned}$$

An equation has **roots**.  
A polynomial has **zeros**.



This tells us:

- 2 is a zero of  $x^3 + 2x^2 - 3x - 10$
- 2 is a root of  $x^3 + 2x^2 - 3x - 10 = 0$
- the graph of  $y = x^3 + 2x^2 - 3x - 10$  has the  $x$ -intercept 2.

If  $P(x) = (x+1)(2x-1)(x+2)$ , then  $(x+1)$ ,  $(2x-1)$ , and  $(x+2)$  are its **linear factors**.

Likewise  $P(x) = (x+3)^2(2x+3)$  has been factorised into 3 linear factors, one of which is repeated.

$x - \alpha$  is a **factor** of the polynomial  $P(x) \Leftrightarrow$  there exists a polynomial  $Q(x)$  such that  $P(x) = (x - \alpha)Q(x)$ .

### Example 7

### Self Tutor

Find the zeros of:

a  $x^2 - 6x + 2$

b  $x^3 - 5x$

- a We wish to find  $x$  such that

$$x^2 - 6x + 2 = 0$$

$$\therefore x = \frac{6 \pm \sqrt{36 - 4(1)(2)}}{2}$$

$$\therefore x = \frac{6 \pm \sqrt{28}}{2}$$

$$\therefore x = \frac{6 \pm 2\sqrt{7}}{2}$$

$$\therefore x = 3 \pm \sqrt{7}$$

The zeros are  $3 - \sqrt{7}$  and  $3 + \sqrt{7}$ .

- b We wish to find  $x$  such that

$$x^3 - 5x = 0$$

$$\therefore x(x^2 - 5) = 0$$

$$\therefore x(x + \sqrt{5})(x - \sqrt{5}) = 0$$

$$\therefore x = 0 \text{ or } \pm \sqrt{5}$$

The zeros are  $-\sqrt{5}$ , 0, and  $\sqrt{5}$ .

**EXERCISE 6B.1**

**1** Find the zeros of:

**a**  $2x^2 - 5x - 12$

**b**  $x^2 + 6x - 1$

**c**  $x^2 - 10x + 6$

**d**  $x^3 - 4x$

**e**  $x^3 - 11x$

**f**  $x^4 - 6x^2 + 8$

**2** Find the roots of:

**a**  $5x^2 = 3x + 2$

**b**  $(2x + 1)(x^2 - 3) = 0$

**c**  $(3x - 1)(x^2 + x - 6) = 0$

**d**  $-2x(x^2 - 2x - 2) = 0$

**e**  $x^3 = 7x$

**f**  $x^4 = 7x^2 - 10$

**Example 8**

Factorise:

**a**  $2x^3 + 5x^2 - 3x$

**b**  $x^2 + 4x - 1$

**a**  $2x^3 + 5x^2 - 3x$   
 $= x(2x^2 + 5x - 3)$   
 $= x(2x - 1)(x + 3)$

**b**  $x^2 + 4x - 1$  is zero when  $x = \frac{-4 \pm \sqrt{16 - 4(1)(-1)}}{2}$   
 $\therefore x = \frac{-4 \pm \sqrt{20}}{2}$   
 $\therefore x = \frac{-4 \pm 2\sqrt{5}}{2}$   
 $\therefore x = -2 \pm \sqrt{5}$   
 $\therefore x^2 + 4x - 1 = (x - [-2 + \sqrt{5}])(x - [-2 - \sqrt{5}])$   
 $= (x + 2 - \sqrt{5})(x + 2 + \sqrt{5})$

**3** Find the linear factors of:

**a**  $2x^2 - 7x - 15$

**b**  $x^3 - 11x^2 + 28x$

**c**  $x^2 - 6x + 3$

**d**  $x^3 + 2x^2 - 4x$

**e**  $6x^3 - x^2 - 2x$

**f**  $x^4 - 6x^2 + 5$

**4** If  $P(x) = a(x - \alpha)(x - \beta)(x - \gamma)$  then  $\alpha, \beta$ , and  $\gamma$  are its zeros.

Verify this statement by finding  $P(\alpha)$ ,  $P(\beta)$ , and  $P(\gamma)$ .

**Example 9**

Find all cubic polynomials with zeros  $\frac{1}{2}$  and  $-3 \pm \sqrt{2}$ .

The zeros  $-3 \pm \sqrt{2}$  have sum  $= -3 + \sqrt{2} - 3 - \sqrt{2} = -6$  and product  $= (-3 + \sqrt{2})(-3 - \sqrt{2}) = 7$

$\therefore$  they come from the quadratic factor  $x^2 + 6x + 7$

$\frac{1}{2}$  comes from the linear factor  $2x - 1$ .

$\therefore P(x) = a(2x - 1)(x^2 + 6x + 7)$ ,  $a \neq 0$ .

**5** Find all cubic polynomials with zeros:

**a**  $-3, 4, 5$

**b**  $\pm 2, 3$

**c**  $3, 1 \pm \sqrt{5}$

**d**  $-1, -2 \pm \sqrt{2}$

**Example 10****Self Tutor**

Find all quartic polynomials with zeros  $2$ ,  $-\frac{1}{3}$ , and  $-1 \pm \sqrt{5}$ .

The zeros  $-1 \pm \sqrt{5}$  have sum  $= -1 + \sqrt{5} - 1 - \sqrt{5} = -2$  and product  $= (-1 + \sqrt{5})(-1 - \sqrt{5}) = -4$

$\therefore$  they come from the quadratic factor  $x^2 + 2x - 4$ .

The zeros  $2$  and  $-\frac{1}{3}$  come from the linear factors  $x - 2$  and  $3x + 1$ .

$\therefore P(x) = a(x - 2)(3x + 1)(x^2 + 2x - 4)$ ,  $a \neq 0$ .

- 6** Find all quartic polynomials with zeros of:

**a**  $\pm 1, \pm \sqrt{2}$       **b**  $2, -\frac{1}{5}, \pm \sqrt{3}$       **c**  $-3, \frac{1}{4}, 1 \pm \sqrt{2}$       **d**  $2 \pm \sqrt{5}, -2 \pm \sqrt{7}$

**POLYNOMIAL EQUALITY**

Two polynomials are **equal** if and only if they have the **same degree** (order), and corresponding terms have equal coefficients.

If we know that two polynomials are **equal** then we can **equate coefficients** to find unknown coefficients.

For example, if  $2x^3 + 3x^2 - 4x + 6 = ax^3 + bx^2 + cx + d$ , where  $a, b, c, d \in \mathbb{R}$ , then  $a = 2$ ,  $b = 3$ ,  $c = -4$ , and  $d = 6$ .

**Example 11****Self Tutor**

Find constants  $a$ ,  $b$ , and  $c$  given that:

$$6x^3 + 7x^2 - 19x + 7 = (2x - 1)(ax^2 + bx + c) \text{ for all } x.$$

$$6x^3 + 7x^2 - 19x + 7 = (2x - 1)(ax^2 + bx + c)$$

$$\therefore 6x^3 + 7x^2 - 19x + 7 = 2ax^3 + 2bx^2 + 2cx - ax^2 - bx - c$$

$$\therefore 6x^3 + 7x^2 - 19x + 7 = 2ax^3 + (2b - a)x^2 + (2c - b)x - c$$

Since this is true for all  $x$ , we equate coefficients:

$$\therefore \underbrace{2a}_{x^3 \text{ s}} = 6 \quad \underbrace{2b - a}_{x^2 \text{ s}} = 7 \quad \underbrace{2c - b}_{x \text{ s}} = -19 \quad \text{and} \quad \underbrace{7}_{\text{constants}} = -c$$

$$\therefore a = 3 \quad \text{and} \quad c = -7 \quad \text{and consequently} \quad \underbrace{2b - 3 = 7}_{\therefore b = 5} \quad \text{and} \quad \underbrace{-14 - b = -19}_{\text{in both equations}}$$

So,  $a = 3$ ,  $b = 5$ , and  $c = -7$ .

**Example 12****Self Tutor**

Find constants  $a$  and  $b$  if  $z^4 + 9 = (z^2 + az + 3)(z^2 + bz + 3)$  for all  $z$ .

$$\begin{aligned} z^4 + 9 &= (z^2 + az + 3)(z^2 + bz + 3) \quad \text{for all } z \\ \therefore z^4 + 9 &= z^4 + bz^3 + 3z^2 \\ &\quad + az^3 + abz^2 + 3az \\ &\quad + 3z^2 + 3bz + 9 \\ \therefore z^4 + 9 &= z^4 + (a+b)z^3 + (ab+6)z^2 + (3a+3b)z + 9 \quad \text{for all } z \end{aligned}$$

Equating coefficients gives

$$\left\{ \begin{array}{l} a+b=0 \quad \dots (1) \quad \{z^3 \text{ s}\} \\ ab+6=0 \quad \dots (2) \quad \{z^2 \text{ s}\} \\ 3a+3b=0 \quad \dots (3) \quad \{z \text{ s}\} \end{array} \right.$$

When simultaneously solving more equations than there are unknowns, we must check that any solutions fit **all** equations. If they do not, there are **no solutions**.



From (1) and (3) we see that  $b = -a$

$$\therefore \text{in (2), } a(-a) + 6 = 0$$

$$\therefore a^2 = 6$$

$$\therefore a = \pm\sqrt{6} \quad \text{and so} \quad b = \mp\sqrt{6}$$

$$\therefore a = \sqrt{6}, \quad b = -\sqrt{6} \quad \text{or} \quad a = -\sqrt{6}, \quad b = \sqrt{6}$$

**EXERCISE 6B.2**

**1** Find constants  $a$ ,  $b$ , and  $c$  given that:

- a**  $2x^2 + 4x + 5 = ax^2 + [2b - 6]x + c$  for all  $x$
- b**  $2x^3 - x^2 + 6 = (x - 1)^2(2x + a) + bx + c$  for all  $x$
- c**  $6x^3 - 13x^2 + 7x + 4 = (3x + 1)(ax^2 + bx + c)$  for all  $x$ .

**2** Find constants  $a$  and  $b$  if:

- a**  $z^4 + 4 = (z^2 + az + 2)(z^2 + bz + 2)$  for all  $z$
- b**  $2z^4 + 5z^3 + 4z^2 + 7z + 6 = (z^2 + az + 2)(2z^2 + bz + 3)$  for all  $z$ .

**3** **a** Given that  $x^3 + 9x^2 + 11x - 21 = (x + 3)(ax^2 + bx + c)$ , find the values of  $a$ ,  $b$ , and  $c$ .

**b** Hence, fully factorise  $x^3 + 9x^2 + 11x - 21$ .

**4** **a** Given that  $4x^3 + 12x^2 + 3x - 5 = (2x - 1)(px^2 + qx + r)$ , find the values of  $p$ ,  $q$ , and  $r$ .

**b** Hence, find the solutions to  $4x^3 + 12x^2 + 3x - 5 = 0$ .

**5** **a** Given that  $3x^3 + 10x^2 - 7x + 4 = (x + 4)(ax^2 + bx + c)$ , find the values of  $a$ ,  $b$ , and  $c$ .

**b** Hence, show that  $3x^3 + 10x^2 - 7x + 4$  has only one real zero.

**6** Suppose  $3x^3 + kx^2 - 7x - 2 = (3x + 2)(ax^2 + bx + c)$ .

**a** Find the values of  $a$ ,  $b$ ,  $c$ , and  $k$ .

**b** Hence, find the roots of  $3x^3 + kx^2 - 7x - 2 = 0$ .

**7** **a** Find real numbers  $a$  and  $b$  such that  $x^4 - 4x^2 + 8x - 4 = (x^2 + ax + 2)(x^2 + bx - 2)$ .

**b** Hence, find the real roots of  $x^4 + 8x = 4x^2 + 4$ .

**Example 13****Self Tutor**

$x + 3$  is a factor of  $P(x) = x^3 + ax^2 - 7x + 6$ . Find  $a \in \mathbb{R}$  and the other factors.

Since  $x + 3$  is a factor,

The coefficient of

$x^3$  is  $1 \times 1 = 1$

This must be 2 so the

constant term is  $3 \times 2 = 6$

$$\begin{aligned} x^3 + ax^2 - 7x + 6 &= (x+3)(x^2 + bx + 2) \quad \text{for some constant } b \\ &= x^3 + bx^2 + 2x \\ &\quad + 3x^2 + 3bx + 6 \\ &= x^3 + (b+3)x^2 + (3b+2)x + 6 \end{aligned}$$

Equating coefficients gives  $3b + 2 = -7$  and  $a = b + 3$

$$\therefore b = -3 \quad \text{and} \quad a = 0$$

$$\begin{aligned} \therefore P(x) &= (x+3)(x^2 - 3x + 2) \\ &= (x+3)(x-1)(x-2) \end{aligned}$$

The other factors are  $(x-1)$  and  $(x-2)$ .

**8**  $2x - 3$  is a factor of  $2x^3 + 3x^2 + ax + 3$ . Find  $a \in \mathbb{R}$  and all zeros of the cubic.

**Example 14****Self Tutor**

$2x + 3$  and  $x - 1$  are factors of  $2x^4 + ax^3 - 3x^2 + bx + 3$ .

Find constants  $a$  and  $b$  and all zeros of the polynomial.

Since  $2x + 3$  and  $x - 1$  are factors,

The coefficient of  $x^4$

is  $2 \times 1 \times 1 = 2$

This must be  $-1$  so the constant

term is  $3 \times -1 \times -1 = 3$

$$\begin{aligned} 2x^4 + ax^3 - 3x^2 + bx + 3 &= (2x+3)(x-1)(x^2 + cx - 1) \quad \text{for some } c \\ &= (2x^2 + x - 3)(x^2 + cx - 1) \\ &= 2x^4 + 2cx^3 - 2x^2 \\ &\quad + x^3 + cx^2 - x \\ &\quad - 3x^2 - 3cx + 3 \\ &= 2x^4 + (2c+1)x^3 + (c-5)x^2 + (-1-3c)x + 3 \end{aligned}$$

Equating coefficients gives  $2c + 1 = a$ ,  $c - 5 = -3$ , and  $-1 - 3c = b$

$$\therefore c = 2$$

$$\therefore a = 5 \quad \text{and} \quad b = -7$$

$$\therefore P(x) = (2x+3)(x-1)(x^2 + 2x - 1)$$

$$\text{Now } x^2 + 2x - 1 \text{ has zeros } \frac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

$$\therefore P(x) \text{ has zeros } -\frac{3}{2}, 1, \text{ and } -1 \pm \sqrt{2}.$$

**9**  $2x + 1$  and  $x - 2$  are factors of  $P(x) = 2x^4 + ax^3 + bx^2 + 18x + 8$ .

**a** Find  $a$  and  $b$ .

**b** Hence, solve  $P(x) = 0$ .

**10**  $x^3 + 3x^2 - 9x + c$ ,  $c \in \mathbb{R}$ , has two identical linear factors. Prove that  $c$  is either 5 or  $-27$ , and factorise the cubic into linear factors in each case.

## C

## THE REMAINDER THEOREM

Consider the cubic polynomial  $P(x) = x^3 + 5x^2 - 11x + 3$ .

If we divide  $P(x)$  by  $x - 2$ , we find that

$$\frac{x^3 + 5x^2 - 11x + 3}{x - 2} = x^2 + 7x + 3 + \frac{9}{x - 2} \text{ remainder}$$

So, when  $P(x)$  is divided by  $x - 2$ , the remainder is 9.

Notice also that  $P(2) = 8 + 20 - 22 + 3 = 9$ , which is the remainder.

By considering other examples like the one above, we formulate the **Remainder theorem**.

### The Remainder Theorem

When a polynomial  $P(x)$  is divided by  $x - k$  until a constant remainder  $R$  is obtained, then  $R = P(k)$ .

#### Proof:

By the division algorithm,  $P(x) = Q(x)(x - k) + R$   
 Letting  $x = k$ ,  $P(k) = Q(k) \times 0 + R$   
 $\therefore P(k) = R$

When using the Remainder theorem, it is important to realise that the following statements are equivalent:

- $P(x) = (x - k)Q(x) + R$
- $P(k) = R$
- $P(x)$  divided by  $x - k$  leaves a remainder of  $R$ .

### Example 15

### Self Tutor

Use the Remainder theorem to find the remainder when  $x^4 - 3x^3 + x - 4$  is divided by  $x + 2$ .

If  $P(x) = x^4 - 3x^3 + x - 4$ , then

$$\begin{aligned} P(-2) &= (-2)^4 - 3(-2)^3 + (-2) - 4 \\ &= 16 + 24 - 2 - 4 \\ &= 34 \end{aligned}$$

$\therefore$  when  $x^4 - 3x^3 + x - 4$  is divided by  $x + 2$ , the remainder is 34. {Remainder theorem}

The Remainder theorem allows us to find a remainder without having to perform the division.



**Example 16** **Self Tutor**

When  $2x^3 + 2x^2 + ax + b$  is divided by  $x + 3$ , the remainder is  $-11$ .

When the same polynomial is divided by  $x - 2$ , the remainder is  $9$ .

Find  $a$  and  $b$ .

$$\text{Let } P(x) = 2x^3 + 2x^2 + ax + b$$

$$\text{Now } P(-3) = -11 \text{ and } P(2) = 9 \quad \{\text{Remainder theorem}\}$$

$$\text{So, } 2(-3)^3 + 2(-3)^2 + a(-3) + b = -11$$

$$\therefore -54 + 18 - 3a + b = -11$$

$$\therefore -3a + b = 25 \quad \dots (1)$$

$$\text{and } 2(2)^3 + 2(2)^2 + a(2) + b = 9$$

$$\therefore 16 + 8 + 2a + b = 9$$

$$\therefore 2a + b = -15 \quad \dots (2)$$

$$\text{Solving simultaneously: } \begin{aligned} 3a - b &= -25 && \{-1 \times (1)\} \\ 2a + b &= -15 && \{(2)\} \end{aligned}$$

$$\text{Adding, } \begin{array}{r} 5a \\ = -40 \end{array}$$

$$\therefore a = -8$$

$$\text{Substituting } a = -8 \text{ in (2) gives } 2(-8) + b = -15$$

$$\therefore b = 1$$

### EXERCISE 6C

- 1** For  $P(x)$  a real polynomial, write two equivalent statements for each of:
  - a** If  $P(2) = 7$ , then .....
  - b** If  $P(x) = Q(x)(x + 3) - 8$ , then .....
  - c** If  $P(x)$  divided by  $x - 5$  has a remainder of  $11$  then .....
- 2** Without performing division, find the remainder when:
  - a**  $x^3 + 2x^2 - 7x + 8$  is divided by  $x - 1$
  - b**  $2x^3 + x^2 - 5x + 11$  is divided by  $x + 3$
  - c**  $x^4 - 2x^2 + 3x - 1$  is divided by  $x + 2$ .
- 3** Use the Remainder theorem to find the remainder when  $x^3 - x^2 - 3x - 5$  is divided by  $x - 3$ . Check that your answer is the same as when this long division was performed on page 159.
- 4** Find  $a \in \mathbb{R}$  such that:
  - a** when  $x^3 - 2x + a$  is divided by  $x - 2$ , the remainder is  $7$
  - b** when  $2x^3 + x^2 + ax - 5$  is divided by  $x + 1$ , the remainder is  $-8$ .
- 5** When  $x^3 + 2x^2 + ax + b$  is divided by  $x - 1$  the remainder is  $4$ , and when divided by  $x + 2$  the remainder is  $16$ . Find  $a$  and  $b$ .
- 6** When  $x^3 + 4x^2 + ax + b$  is divided by  $x - 2$  the remainder is  $20$ , and when divided by  $x + 5$  the remainder is  $6$ . Find  $a$  and  $b$ .

- 7** Consider  $f(x) = 2x^3 + ax^2 - 3x + b$ . When  $f(x)$  is divided by  $x + 1$ , the remainder is 7. When  $f(x)$  is divided by  $x - 2$ , the remainder is 28. Find the remainder when  $f(x)$  is divided by  $x + 3$ .
- 8 a** Suppose a polynomial  $P(x)$  is divided by  $2x - 1$  until a constant remainder  $R$  is obtained. Show that  $R = P(\frac{1}{2})$ .
- Hint:**  $P(x) = Q(x)(2x - 1) + R$ .
- b** Find the remainder when:
- i**  $4x^2 - 10x + 1$  is divided by  $2x - 1$
  - ii**  $2x^3 - 5x^2 + 8$  is divided by  $2x - 1$
  - iii**  $4x^3 + 7x - 3$  is divided by  $2x + 1$ .
- 9** When  $2x^3 + ax^2 + bx + 4$  is divided by  $x + 1$  the remainder is  $-5$ , and when divided by  $2x - 1$  the remainder is  $10$ . Find  $a$  and  $b$ .
- 10** When  $P(z)$  is divided by  $z^3 - 3z + 2$  the remainder is  $4z - 7$ .  
Find the remainder when  $P(z)$  is divided by:  
**a**  $z - 1$       **b**  $z - 2$ .

## D THE FACTOR THEOREM

For any polynomial  $P(x)$ ,  $k$  is a zero of  $P(x) \Leftrightarrow x - k$  is a factor of  $P(x)$ .

**Proof:**

$k$ is a zero of $P(x) \Leftrightarrow P(k) = 0$	{definition of a zero}
$\Leftrightarrow R = 0$	{Remainder theorem}
$\Leftrightarrow P(x) = Q(x)(x - k)$	{division algorithm}
$\Leftrightarrow x - k$ is a factor of $P(x)$	{definition of a factor}

The **Factor theorem** says that if  $2$  is a zero of  $P(x)$  then  $x - 2$  is a factor of  $P(x)$ , and vice versa.

We can use the Factor theorem to determine whether  $x - k$  is a factor of a polynomial, without having to perform the long division.

### Example 17



Determine whether:

- a**  $x - 2$  is a factor of  $x^3 + 3x^2 - 13x + 6$       **b**  $x + 3$  is a factor of  $x^3 - 8x + 7$ .

**a** Let  $P(x) = x^3 + 3x^2 - 13x + 6$   
 $\therefore P(2) = (2)^3 + 3(2)^2 - 13(2) + 6$   
 $= 8 + 12 - 26 + 6$   
 $= 0$

Since  $P(2) = 0$ ,  $x - 2$  is a factor of  $x^3 + 3x^2 - 13x + 6$ . {Factor theorem}

**b** Let  $P(x) = x^3 - 8x + 7$   
 $\therefore P(-3) = (-3)^3 - 8(-3) + 7$   
 $= -27 + 24 + 7$   
 $= 4$

Since  $P(-3) \neq 0$ ,  $x + 3$  is *not* a factor of  $x^3 - 8x + 7$ . {Factor theorem}

When  $x^3 - 8x + 7$  is divided by  $x + 3$ , a remainder of 4 is left over.



**Example 18****Self Tutor**

$x - 2$  is a factor of  $P(x) = x^3 + kx^2 - 3x + 6$ .

Find  $k$ , and write  $P(x)$  as a product of linear factors.

Since  $x - 2$  is a factor,  $P(2) = 0$  {Factor theorem}

$$\therefore (2)^3 + k(2)^2 - 3(2) + 6 = 0$$

$$\therefore 8 + 4k = 0$$

$$\therefore k = -2$$

The coefficient of  
 $x^3$  is  $1 \times 1 = 1$       The constant term  
is  $-2 \times -3 = 6$

$$\begin{aligned} \text{So, } P(x) &= x^3 - 2x^2 - 3x + 6 = (x - 2)(x^2 + bx - 3) \\ &= x^3 + (b - 2)x^2 + (-2b - 3)x + 6 \end{aligned}$$

Equating  $x^2$ 's:  $b - 2 = -2$

$$\therefore b = 0$$

$$\begin{aligned} \text{So, } P(x) &= (x - 2)(x^2 - 3) \\ &= (x - 2)(x + \sqrt{3})(x - \sqrt{3}) \end{aligned}$$

**Example 19****Self Tutor**

$2x - 1$  is a factor of  $f(x) = 4x^3 - 4x^2 + ax + b$ , and the remainder when  $f(x)$  is divided by  $x - 1$  is  $-1$ . Find the values of  $a$  and  $b$ .

$2x - 1$  is a factor of  $f(x)$ , so  $f(\frac{1}{2}) = 0$

$$\therefore 4(\frac{1}{2})^3 - 4(\frac{1}{2})^2 + a(\frac{1}{2}) + b = 0$$

$$\therefore \frac{1}{2}a + b = \frac{1}{2} \quad \dots (1)$$

If  $2x - 1$  is a factor of  $f(x)$ ,  
then  $f(\frac{1}{2}) = 0$ .

Also,  $f(1) = -1$  {Remainder theorem}

$$\therefore 4(1)^3 - 4(1)^2 + a(1) + b = -1$$

$$\therefore a + b = -1 \quad \dots (2)$$

Solving simultaneously:  $\begin{array}{r} -a - 2b = -1 \quad \{-2 \times (1)\} \\ a + b = -1 \quad \{(2)\} \\ \hline -b = -2 \end{array}$

Adding,  $\underline{-b = -2}$

$$\therefore b = 2 \quad \text{and} \quad a = -3$$

**EXERCISE 6D**

- 1** Use the Factor theorem to determine whether:

- a**  $x - 1$  is a factor of  $4x^3 - 7x^2 + 5x - 2$       **b**  $x - 3$  is a factor of  $x^4 - x^3 - 4x^2 - 15$   
**c**  $x + 2$  is a factor of  $3x^3 + 5x^2 - 6x - 8$       **d**  $x + 4$  is a factor of  $2x^3 + 6x^2 + 4x + 16$ .

- 2** **a** Find  $c$  given that  $x + 1$  is a factor of  $5x^3 - 3x^2 + cx + 10$ .
- b** Find  $c$  given that  $x - 3$  is a factor of  $x^4 - 2x^3 + cx^2 - 4x + 3$ .
- c** Find  $b$  given that  $x + 2$  is a factor of  $x^6 + bx^5 - 2x^3 - 5x + 6$ .
- 3**  $x + 2$  is a factor of  $P(x) = 2x^3 + x^2 + kx - 4$ .  
Find  $k$ , and hence write  $P(x)$  as a product of linear factors.
- 4**  $x - 3$  is a factor of  $P(x) = 3x^3 + kx^2 - 5x + 6$ .
- a** Find  $k$ . **b** Write  $P(x)$  in the form  $P(x) = (x - 3)(ax^2 + bx + c)$ .
- c** Find all solutions to  $P(x) = 0$ .
- 5**  $2x^3 + ax^2 + bx + 5$  has factors  $x - 1$  and  $x + 5$ . Find  $a$  and  $b$ .
- 6**  $x - 2$  is a factor of  $f(x) = x^3 + ax^2 - 11x + b$ . The remainder when  $f(x)$  is divided by  $x + 1$  is 15. Find  $a$  and  $b$ .
- 7**  $x + 3$  is a factor of  $P(x) = 2x^3 + 9x^2 + ax + b$ . When  $P(x)$  is divided by  $x + 4$ , the remainder is  $-18$ .
- a** Find  $a$  and  $b$ .
- b** Find the remainder when  $P(x)$  is divided by  $x - 2$ .
- c** Write  $P(x)$  in the form  $P(x) = (x + 3)(px^2 + qx + r)$ .
- d** Find the zeros of  $P(x)$ .
- 8**  $2x - 1$  is a factor of  $P(x) = 2x^3 + ax^2 - 8x + b$ . When  $P(x)$  is divided by  $x - 1$ , the remainder is 3.
- a** Find  $a$  and  $b$ .
- b** Find the irrational roots of  $P(x) = 0$ , giving your answer in the form  $x = p \pm \sqrt{q}$  where  $p, q \in \mathbb{Z}$ .
- 9** **a** Consider  $P(x) = x^3 - a^3$  where  $a$  is real.
- i** Find  $P(a)$ . What is the significance of this result?
- ii** Factorise  $x^3 - a^3$  as the product of a real linear and a quadratic factor.
- b** Now consider  $P(x) = x^3 + a^3$ , where  $a$  is real.
- i** Find  $P(-a)$ . What is the significance of this result?
- ii** Factorise  $x^3 + a^3$  as the product of a real linear and a quadratic factor.
- 10** Find the real number  $a$  such that  $(x - 1 - a)$  is a factor of  $P(x) = x^3 - 3ax - 9$ .

**E****CUBIC EQUATIONS**

In **Discovery 4** in **Chapter 3** on page 97, we considered the sum and products of roots of a quadratic. In particular, we saw that

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta.$$

If we perform a similar expansion for a cubic, we find that

$$(x - \alpha)(x - \beta)(x - \gamma) = x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma.$$

In both cases, the *product of the roots* has the same size as the constant term in the expanded polynomial.

If the leading coefficient of the polynomial  $\neq 1$ , then we need to multiply by this as well:

$$a(x - \alpha)(x - \beta)(x - \gamma) = ax^3 - a(\alpha + \beta + \gamma)x^2 + a(\alpha\beta + \beta\gamma + \gamma\alpha)x - a\alpha\beta\gamma.$$

If you think a cubic equation has integer roots, try to find them by factorising the constant term.

### Example 20

### Self Tutor

Solve for  $x$ :  $x^3 - 31x - 30 = 0$ .

Let  $P(x) = x^3 - 31x - 30$ .

The constant term is  $-30$ , so the product of the roots is  $30$ .

Since  $30 = 5 \times 3 \times 2 \times 1$ , likely integer roots are  $\pm 1, \pm 2, \pm 3, \pm 5$ . They could also be  $\pm 6$  since  $2 \times 3 = 6$ , and so on.

Now  $P(1) = -60$ , so  $1$  is not a root.

But  $P(-1) = 0$ , so  $-1$  is a root, and  $(x + 1)$  is a factor of  $P(x)$ .

The coefficient of $x^3$ is $1 \times 1 = 1$	The constant term is $1 \times -30 = -30$
---	--

$$\begin{aligned} \text{So, } P(x) &= x^3 + 0x^2 - 31x - 30 = (x + 1)(x^2 + bx - 30) \\ &= x^3 + (b + 1)x^2 + (b - 30)x - 30 \end{aligned}$$

Equating  $x^2$ s:  $b + 1 = 0$

$$\therefore b = -1$$

$$\begin{aligned} \text{Hence } P(x) &= (x + 1)(x^2 - x - 30) \\ &= (x + 1)(x + 5)(x - 6) \end{aligned}$$

$\therefore$  the solutions are  $-1, -5$ , and  $6$ .

Note that this method only works for those cubics with all integer roots.

### EXERCISE 6E

**1** Solve for  $x$ :

- |                                     |                                      |                                       |
|-------------------------------------|--------------------------------------|---------------------------------------|
| <b>a</b> $x^3 - 6x^2 + 11x - 6 = 0$ | <b>b</b> $x^3 - 3x^2 + 4 = 0$        | <b>c</b> $x^3 + 2x^2 - x - 2 = 0$     |
| <b>d</b> $x^3 - 6x^2 + 5x + 12 = 0$ | <b>e</b> $x^3 + 5x^2 - 16x - 80 = 0$ | <b>f</b> $x^3 + 13x^2 + 55x + 75 = 0$ |

**2** Solve for  $x$ :

- |                                      |                                       |   |
|--------------------------------------|---------------------------------------|---|
| <b>a</b> $2x^3 - 6x^2 - 8x + 24 = 0$ | <b>b</b> $2x^3 - 2x^2 - 48x - 72 = 0$ | <b>c</b> $3x^3 - 24x^2 - 15x + 252 = 0$ |
|--------------------------------------|---------------------------------------|---|

Take out a common factor first!



## Discussion

Consider the general cubic  $p(x) = ax^3 + bx^2 + cx + d$ ,  $a, b, c, d \in \mathbb{R}$ .

What happens to  $p(x)$  if  $x$  gets:

- very large and positive
- very large and negative?

What does this tell you about the number of solutions that  $p(x) = 0$  may have?

## Review set 6A

**1** Given  $p(x) = 5x^2 - x + 4$  and  $q(x) = 3x^2 + 7x - 1$ , find:

**a**  $p(x) + q(x)$

**b**  $2p(x) - q(x)$

**c**  $p(x)q(x)$

**2** Find the quotient and remainder of:

**a** 
$$\frac{2x^2 + 11x + 18}{x + 3}$$

**b** 
$$\frac{x^3 - 6x^2 + 10x - 9}{x - 2}$$

**3** Find the zeros of:

**a**  $3x^2 + 2x - 8$

**b**  $x^2 + 8x + 11$

**4** **a** Given that  $x^3 + x^2 - 3x + 9 = (x + 3)(ax^2 + bx + c)$ , find the values of  $a$ ,  $b$ , and  $c$ .

**b** Show that  $x^3 + x^2 - 3x + 9$  has only one real zero.

**5** Use the Remainder theorem to find the remainder when:

**a**  $x^3 - 4x^2 + 5x - 1$  is divided by  $x - 2$

**b**  $2x^3 + 6x^2 - 7x + 12$  is divided by  $x + 5$ .

**6** Use the Factor theorem to determine whether:

**a**  $x + 1$  is a factor of  $2x^4 - 9x^2 - 6x - 1$

**b**  $x - 3$  is a factor of  $x^4 - 2x^3 - 4x^2 + 5x - 6$ .

**7**  $2x^2 + kx - 5$  has remainder 3 when divided by  $x + 4$ . Find  $k$ .

**8**  $ax^3 + 5x^2 - x + b$  has remainder 7 when divided by  $x - 1$ , and remainder  $-11$  when divided by  $x + 2$ . Find  $a$  and  $b$ .

**9** Find  $c$  given that  $x - 2$  is a factor of  $x^5 - 2x^4 + cx^3 - 7x^2 + 5x - 6$ .

**10**  $x - 4$  is a factor of  $f(x) = x^3 + 2x^2 + ax + b$ , and when  $f(x)$  is divided by  $x + 2$  the remainder is 18.

**a** Find  $a$  and  $b$ .

**b** Find all zeros of  $f(x)$ .

**11** Solve for  $x$ :  $x^3 - x^2 - 17x - 15 = 0$

## Review set 6B

**1** Expand and simplify:

**a**  $(3x^3 + 2x - 5)(4x - 3)$

**b**  $(2x^2 - x + 3)^2$

- 2** Carry out the following divisions:

**a**  $\frac{x^3}{x+2}$

**b**  $\frac{x^3}{(x+2)(x+3)}$

- 3** Find all cubic polynomials with zeros  $\frac{1}{4}, 1 \pm \sqrt{5}$ .

- 4** If  $f(x) = x^3 - 3x^2 - 9x + b$  has  $(x - k)^2$  as a factor, show that there are two possible values of  $k$ . For each of these two values of  $k$ , find the corresponding value for  $b$ , and hence solve  $f(x) = 0$ .

- 5** Find the remainder when:

**a**  $x^3 - 5x^2 + 9$  is divided by  $x - 2$       **b**  $4x^3 + 7x - 11$  is divided by  $2x - 1$ .

- 6** When  $f(x) = 2x^3 - x^2 + ax - 4$  is divided by  $x - 3$ , the remainder is 56.

**a** Find  $a$ .      **b** Find the remainder when  $f(x)$  is divided by  $x + 1$ .

- 7** **a** Use the Factor theorem to show that  $x - 2$  is a factor of  $x^3 - 13x + 18$ .

**b** Write  $x^3 - 13x + 18$  in the form  $(x - 2)(ax^2 + bx + c)$ , where  $a, b, c \in \mathbb{Z}$ .

**c** Find the real roots of  $x^3 + 18 = 13x$ .

- 8**  $x - 2$  and  $x + 3$  are factors of  $ax^3 - 3x^2 - 11x + b$ . Find  $a$  and  $b$ .

- 9**  $x + 1$  is a factor of  $f(x) = x^3 + 5x^2 + kx + 4$ . Find  $k$ , and the zeros of  $f(x)$ .

- 10**  $2x - 1$  is a factor of  $f(x) = 2x^3 - 9x^2 + ax + b$ , and when  $f(x)$  is divided by  $x - 1$  the remainder is  $-15$ .

**a** Find  $a$  and  $b$ .      **b** Write  $f(x)$  as a product of linear factors.

- 11** Solve for  $x$ :  $2x^3 - 2x^2 - 28x + 48 = 0$

# 7

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# Straight line graphs

## Contents:

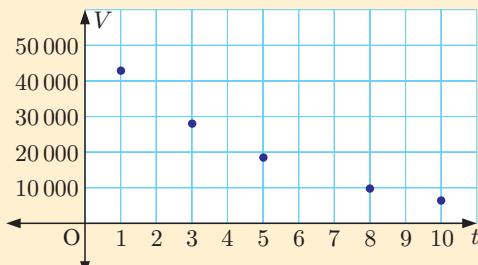
- A Equations of straight lines**
- B Intersection of straight lines**
- C Intersection of a straight line and a curve**
- D Transforming relationships to straight line form**
- E Finding relationships from data**

### Opening problem

This table shows the value  $V$  of Doug's father's car  $t$  years after purchase.

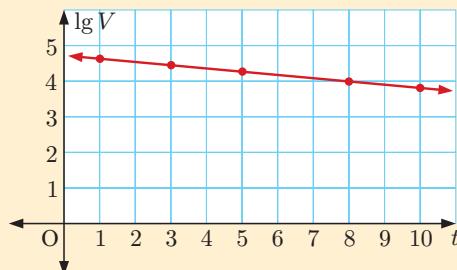
$t$ (years)	1	3	5	8	10
$V$ (dollars)	42 900	28 000	18 500	9 800	6 400

Doug is trying to work out the equation connecting  $V$  and  $t$ . When he plots the values on a graph, the result is a curve:



Doug's father suggests that he plots  $\lg V$  against  $t$ . When Doug does this, the result is a straight line:

$t$	1	3	5	8	10
$\lg V$	4.63	4.45	4.27	3.99	3.81



### Things to think about:

- a Is it easier to find the equation of a curve or a straight line?
- b How can Doug use the equation of the straight line to determine the relationship between  $V$  and  $t$ ?

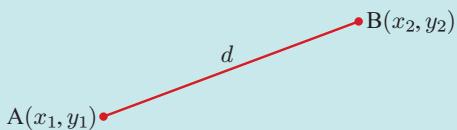
## BACKGROUND KNOWLEDGE

You should be familiar with the following facts involving points and lines on the coordinate plane:

### Distance between two points

The **distance** between  $A(x_1, y_1)$  and  $B(x_2, y_2)$  is

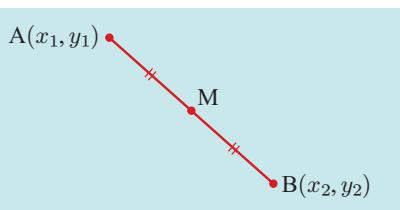
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



### Midpoint

If A is  $(x_1, y_1)$  and B is  $(x_2, y_2)$ , then the midpoint of AB is

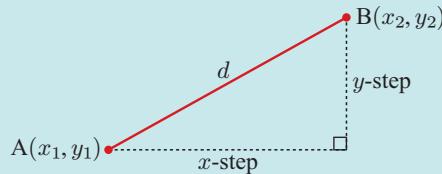
$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



## Gradient

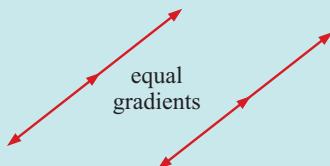
The **gradient** of a line passing through  $A(x_1, y_1)$  and  $B(x_2, y_2)$  is

$$\frac{y\text{-step}}{x\text{-step}} = \frac{y_2 - y_1}{x_2 - x_1}.$$



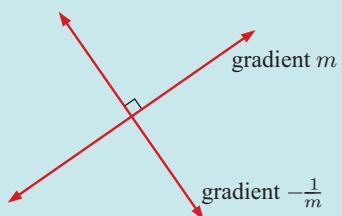
## Gradients of parallel and perpendicular lines

- If two lines are **parallel**, then their gradients are **equal**.



- If two lines are **perpendicular**, then their gradients are **negative reciprocals**.

If the gradient of one line is  $m$ , then the gradient of the other line is  $-\frac{1}{m}$ .



## A

# EQUATIONS OF STRAIGHT LINES

The **equation of a line** is an equation which connects the  $x$  and  $y$  values for every point on the line.

The equation of a straight line can be written in:

- gradient-intercept form**  $y = mx + c$ , or
- general form**  $Ax + By = D$ .

A line with equation  
 $y = mx + c$   
has gradient  $m$   
and  $y$ -intercept  $c$ .



## FINDING THE EQUATION OF A LINE

In order to find the equation of a line, we need to know some information.

Suppose we know the gradient of the line is 2, and that the line passes through  $(4, 1)$ .

We suppose  $(x, y)$  is any point on the line.

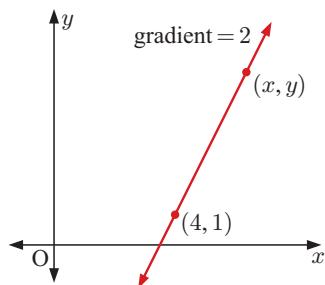
The gradient between  $(4, 1)$  and  $(x, y)$  is  $\frac{y-1}{x-4}$ , and this gradient must equal 2.

$$\text{So, } \frac{y-1}{x-4} = 2$$

$$\therefore y - 1 = 2(x - 4) \quad \{\text{multiplying both sides by } (x - 4)\}$$

$$\therefore y - 1 = 2x - 8 \quad \{\text{expanding the brackets}\}$$

$$\therefore y = 2x - 7 \quad \{\text{adding 1 to both sides}\}$$



This is the equation of the line in gradient-intercept form.

We can find the equation of a line if we know:

- its **gradient** and the **coordinates of any point** on the line, or
- the **coordinates of two distinct points** on the line.

If a straight line has gradient  $m$  and passes through the point  $(x_1, y_1)$

then its equation is  $\frac{y - y_1}{x - x_1} = m$  or  $y - y_1 = m(x - x_1)$ .

We can rearrange this equation into either gradient-intercept or general form.

### Example 1

### Self Tutor

Find, in *gradient-intercept form*, the equation of the line through  $(-1, 3)$  with a gradient of 5.

The equation of the line is  $y - 3 = 5(x - (-1))$

$$\therefore y - 3 = 5x + 5$$

$$\therefore y = 5x + 8$$

### Example 2

### Self Tutor

Find, in *general form*, the equation of the line with gradient  $\frac{3}{4}$  which passes through  $(5, -2)$ .

The equation of the line is  $y - (-2) = \frac{3}{4}(x - 5)$

$$\therefore 4(y + 2) = 3(x - 5)$$

$$\therefore 4y + 8 = 3x - 15$$

$$\therefore 3x - 4y = 23$$

**EXERCISE 7A.1**

**1** Find the gradient and  $y$ -intercept of the line with equation:

**a**  $y = 3x + 5$

**b**  $y = 4x - 2$

**c**  $y = \frac{1}{5}x + \frac{3}{5}$

**d**  $y = -7x - 3$

**e**  $y = \frac{x+2}{6}$

**f**  $y = \frac{8-5x}{3}$

**2** Find the equation of the line with:

**a** gradient 1 and  $y$ -intercept  $-2$

**b** gradient  $-1$  and  $y$ -intercept  $4$

**c** gradient 2 and  $y$ -intercept  $0$

**d** gradient  $-\frac{1}{2}$  and  $y$ -intercept  $3$ .

**3** Find, in *gradient-intercept form*, the equation of the line through:

**a**  $(2, -5)$  with gradient 4

**b**  $(-1, -2)$  with gradient  $-3$

**c**  $(7, -3)$  with gradient  $-5$

**d**  $(1, 4)$  with gradient  $\frac{1}{2}$

**e**  $(-1, 3)$  with gradient  $-\frac{1}{3}$

**f**  $(2, 6)$  with gradient  $0$ .

**4** Find, in *general form*, the equation of the line through:

**a**  $(2, 5)$  having gradient  $\frac{2}{3}$

**b**  $(-1, 4)$  having gradient  $\frac{3}{5}$

**c**  $(5, 0)$  having gradient  $-\frac{1}{3}$

**d**  $(6, -2)$  having gradient  $-\frac{2}{7}$

**e**  $(-3, -1)$  having gradient 4

**f**  $(5, -3)$  having gradient  $-2$

**g**  $(4, -5)$  having gradient  $-3\frac{1}{2}$

**h**  $(-7, -2)$  having gradient 6.

**Example 3****Self Tutor**

Find the equation of the line which passes through the points  $A(-1, 5)$  and  $B(2, 3)$ .

The gradient of the line is  $\frac{3-5}{2-(-1)} = -\frac{2}{3}$ .

Using point A, the equation is

$$y - 5 = -\frac{2}{3}(x - (-1))$$

$$\therefore 3(y - 5) = -2(x + 1)$$

$$\therefore 3y - 15 = -2x - 2$$

$$\therefore 2x + 3y = 13$$

We would get the same equation using point B.  
Try it for yourself.



**5** Find, in *gradient-intercept form*, the equation of the line which passes through the points:

**a**  $A(2, 3)$  and  $B(4, 8)$

**b**  $A(0, 3)$  and  $B(-1, 5)$

**c**  $A(-1, -2)$  and  $B(4, -2)$

**d**  $C(-3, 1)$  and  $D(2, 0)$

**e**  $P(5, -1)$  and  $Q(-1, -2)$

**f**  $R(-1, -3)$  and  $S(-4, -1)$ .

**6** Find, in *general form*, the equation of the line which passes through:

**a**  $(0, 1)$  and  $(3, 2)$

**b**  $(1, 4)$  and  $(0, -1)$

**c**  $(2, -1)$  and  $(-1, -4)$

**d**  $(0, -2)$  and  $(5, 2)$

**e**  $(3, 2)$  and  $(-1, 0)$

**f**  $(-1, -1)$  and  $(2, -3)$ .

**7** Consider the points A(2, 5) and B(-4, 2). Find:

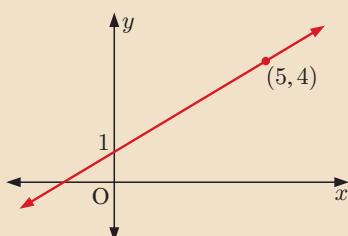
- the distance between A and B
- the midpoint of AB
- the gradient of the line which passes through A and B
- the equation of the line which passes through A and B.

### Example 4

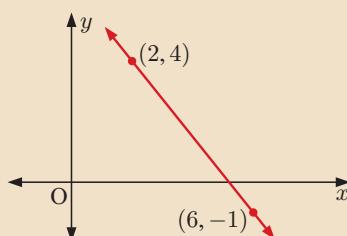
### Self Tutor

Find the equation of the line with graph:

a



b



- a Two points on the line are (0, 1) and (5, 4).

$$\therefore \text{the gradient } m = \frac{4-1}{5-0} = \frac{3}{5}$$

and the  $y$ -intercept  $c = 1$ .

The equation is  $y = \frac{3}{5}x + 1$   
{gradient-intercept form}

- b Two points on the line are (2, 4) and (6, -1).

$$\therefore \text{the gradient } m = \frac{-1-4}{6-2} = -\frac{5}{4}$$

Since we do not know the  $y$ -intercept we use the general form.

The equation is  $y - 4 = -\frac{5}{4}(x - 2)$

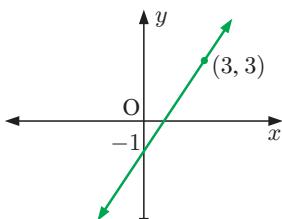
$$\therefore 4(y - 4) = -5(x - 2)$$

$$\therefore 4y - 16 = -5x + 10$$

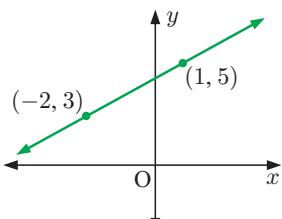
$$\therefore 5x + 4y = 26$$

**8** Find the equations of the illustrated lines:

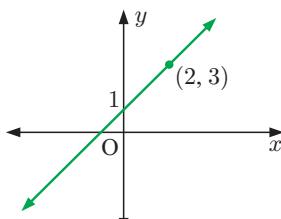
a



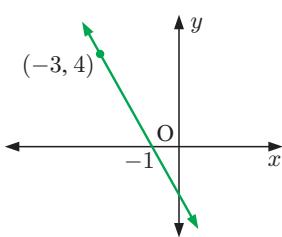
b



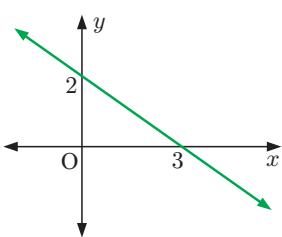
c



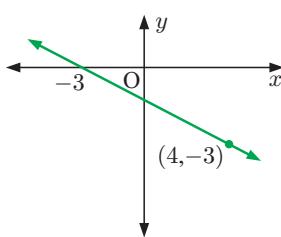
d



e

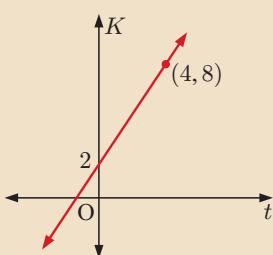


f



**Example 5****Self Tutor**

Find the equation connecting the variables.



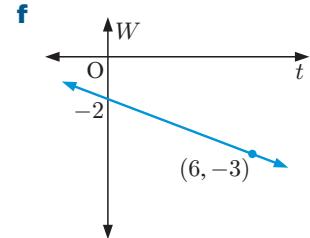
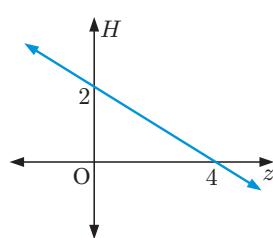
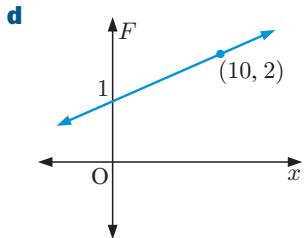
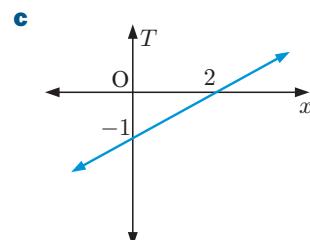
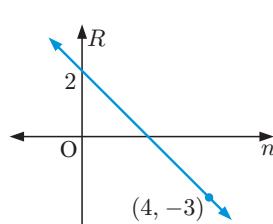
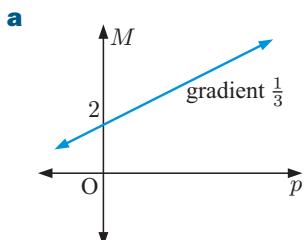
(0, 2) and (4, 8) lie on the straight line.

$\therefore$  the gradient  $m = \frac{8-2}{4-0} = \frac{6}{4} = \frac{3}{2}$ , and the  $y$ -intercept  $c = 2$ .

In this case  $K$  is on the vertical axis and  $t$  is on the horizontal axis.

$\therefore$  the equation is  $K = \frac{3}{2}t + 2$ .

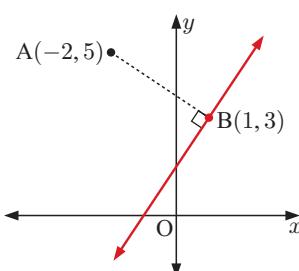
- 9 Find the equation connecting the variables:

**Example 6****Self Tutor**

Consider the points  $A(-2, 5)$  and  $B(1, 3)$ . A line perpendicular to AB, passes through B.

- a Find the equation of the line.  
b Find the coordinates of the point where the line cuts the  $x$ -axis.

a The gradient of AB =  $\frac{3-5}{1-(-2)} = -\frac{2}{3}$   
 $\therefore$  the perpendicular line has gradient  $\frac{3}{2}$ , and passes through B(1, 3).  
 $\therefore$  its equation is  $y - 3 = \frac{3}{2}(x - 1)$   
 $\therefore 2(y - 3) = 3(x - 1)$   
 $\therefore 2y - 6 = 3x - 3$   
 $\therefore 3x - 2y = -3$



- b** The line cuts the  $x$ -axis when  $y = 0$   
 $\therefore 3x - 2(0) = -3$   
 $\therefore x = -1$   
 $\therefore$  the line cuts the  $x$ -axis at  $(-1, 0)$ .

**10** Consider the points  $P(-3, -2)$  and  $Q(1, 6)$ . A line perpendicular to  $PQ$ , passes through  $Q$ .

- a** Find the equation of the line.  
**b** Find the coordinates of the point where the line cuts the  $x$ -axis.

**11** Suppose  $A$  has coordinates  $(-7, 4)$  and  $B$  has coordinates  $(3, -2)$ . A line parallel to  $AB$ , passes through  $C(5, -1)$ .

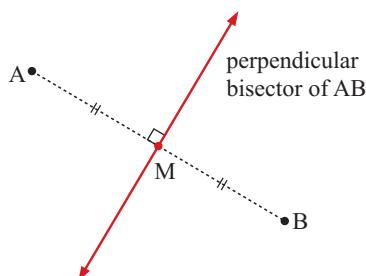
- a** Find the equation of the line.  
**b** Find the coordinates of the point where the line cuts the  $y$ -axis.

**12** Suppose  $P$  has coordinates  $(3, 8)$  and  $Q$  has coordinates  $(-5, 2)$ . The line perpendicular to  $PQ$  and passing through  $P$ , cuts the  $x$ -axis at  $R$  and the  $y$ -axis at  $S$ . Find the area of triangle  $ORS$ , where  $O$  is the origin.

## PERPENDICULAR BISECTORS

We have already seen that the **midpoint**  $M$  of the line segment  $AB$  is the point on the line segment that is halfway between  $A$  and  $B$ .

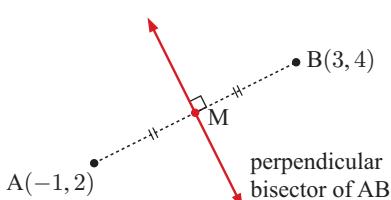
The **perpendicular bisector** of  $AB$  is the line which is perpendicular to  $AB$ , and which passes through its midpoint  $M$ .



### Example 7

### Self Tutor

Find the equation of the perpendicular bisector of  $AB$  given  $A(-1, 2)$  and  $B(3, 4)$ .



The midpoint  $M$  of  $AB$  is  $\left(\frac{-1+3}{2}, \frac{2+4}{2}\right)$   
or  $M(1, 3)$ .

The gradient of  $AB$  is  $\frac{4-2}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$

$\therefore$  the gradient of the perpendicular bisector is  $-\frac{2}{1}$   
{the negative reciprocal of  $\frac{1}{2}$ }

The equation of the perpendicular bisector is  $y - 3 = -2(x - 1)$  {using  $M(1, 3)$ }

$$\therefore y - 3 = -2x + 2$$

$$\therefore y = -2x + 5$$

**EXERCISE 7A.2**

- 1** Consider the points  $P(-3, 7)$  and  $Q(1, -5)$ . Find:
- the distance between  $P$  and  $Q$
  - the midpoint of  $PQ$
  - the gradient of  $PQ$
  - the equation of the perpendicular bisector of  $PQ$ .
- 2** Find the equation of the perpendicular bisector of  $AB$  given:
- |                                    |                                     |
|------------------------------------|-------------------------------------|
| <b>a</b> $A(3, -3)$ and $B(1, -1)$ | <b>b</b> $A(1, 3)$ and $B(-3, 5)$   |
| <b>c</b> $A(3, 1)$ and $B(-3, 6)$  | <b>d</b> $A(4, -2)$ and $B(4, 4)$ . |
- 3** Consider the points  $P(-1, 5)$  and  $Q(3, 7)$ . The perpendicular bisector of  $PQ$  cuts the  $x$ -axis at  $R$ . Find the area of triangle  $PQR$ .

**B****INTERSECTION OF STRAIGHT LINES**

To find where straight lines meet, we need to solve the equations of the lines simultaneously.

**Example 8**

Find where the line:

- $y = 2x - 5$  meets the line  $4x + 3y = 15$
- $x + 3y = 5$  meets the line  $2x - 5y = -12$ .

- a** Substituting  $y = 2x - 5$  into  $4x + 3y = 15$  gives

$$4x + 3(2x - 5) = 15$$

$$\therefore 4x + 6x - 15 = 15$$

$$\therefore 10x = 30$$

$$\therefore x = 3 \quad \text{and} \quad y = 2(3) - 5 = 1$$

The lines meet at  $(3, 1)$ .

- b**  $x + 3y = 5$ , so  $x = 5 - 3y$ .

Substituting  $x = 5 - 3y$  into  $2x - 5y = -12$  gives

$$2(5 - 3y) - 5y = -12$$

$$\therefore 10 - 6y - 5y = -12$$

$$\therefore -11y = -22$$

$$\therefore y = 2 \quad \text{and} \quad x = 5 - 3(2) = -1$$

The lines meet at  $(-1, 2)$ .

**EXERCISE 7B**

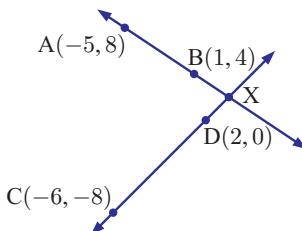
- 1** Find the intersection point of each pair of lines:

- |   |   |
|---|---|
| <b>a</b> $y = 4x - 1$ and $2x + y = 5$    | <b>b</b> $y = 9 - 2x$ and $4x + 3y = 15$    |
| <b>c</b> $x + 4y = 7$ and $5x - 2y = -31$ | <b>d</b> $3x + y = -5$ and $4x - 7y = 10$ . |

- 2** Line  $l_1$  has equation  $y = 2x + 7$ . Line  $l_2$  passes through  $(-7, 6)$  and  $(3, 0)$ .

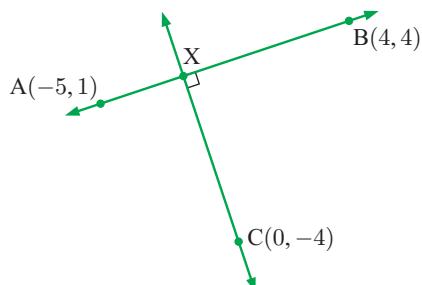
- a** Find the equation of  $l_2$ .
- b** Find the intersection point of  $l_1$  and  $l_2$ .

- 3** Find the coordinates of X.



- 4** In the diagram alongside, a line has been drawn through C, perpendicular to the line AB. The point of intersection of the lines is X. Find:

- a** the equation of AB
- b** the equation of CX
- c** the coordinates of X.

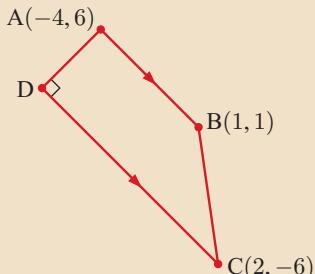


### Example 9

### Self Tutor

ABCD is a trapezium in which AB is parallel to DC, and  $\widehat{ADC} = 90^\circ$ . Find:

- a** the coordinates of D
- b** the area of the trapezium.



- a** Point D is the intersection of AD and DC.

$$\text{The gradient of AB is } \frac{1-6}{1-(-4)} = \frac{-5}{5} = -1$$

$\therefore$  DC also has gradient  $-1$ , and has equation  $y - (-6) = -1(x - 2)$

$$\therefore y + 6 = -x + 2$$

$$\therefore y = -x - 4 \quad \dots (1)$$

AD is perpendicular to DC, so its gradient is  $1$ , and its equation is  $y - 6 = 1(x - (-4))$

$$\therefore y - 6 = x + 4$$

$$\therefore y = x + 10 \quad \dots (2)$$

Substituting (1) into (2) gives  $-x - 4 = x + 10$

$$\therefore -2x = 14$$

$$\therefore x = -7 \quad \text{and} \quad y = -(-7) - 4 = 3$$

$\therefore$  D is  $(-7, 3)$ .

**b** The length of AB =  $\sqrt{(1 - -4)^2 + (1 - 6)^2} = \sqrt{50} = 5\sqrt{2}$  units

The length of DC =  $\sqrt{(2 - -7)^2 + (-6 - 3)^2} = \sqrt{162} = 9\sqrt{2}$  units

The length of AD =  $\sqrt{(-7 - -4)^2 + (3 - 6)^2} = \sqrt{18} = 3\sqrt{2}$  units

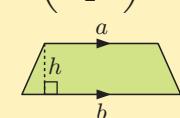
$$\therefore \text{the area of the trapezium} = \left( \frac{5\sqrt{2} + 9\sqrt{2}}{2} \right) \times 3\sqrt{2}$$

$$= 7\sqrt{2} \times 3\sqrt{2}$$

$$= 42 \text{ units}^2$$

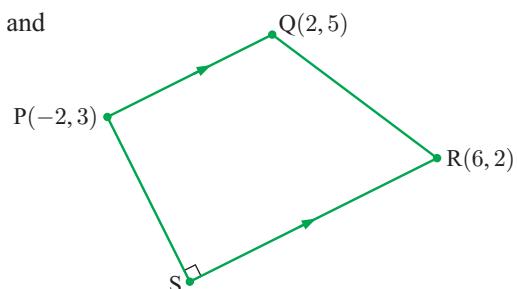


$$\text{Area of trapezium} = \left( \frac{a+b}{2} \right) \times h$$

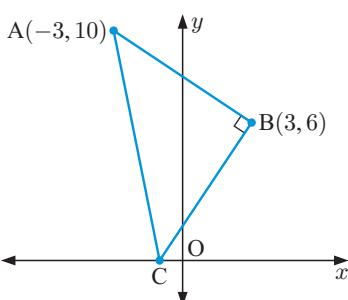


- 5** PQRS is a trapezium in which PQ is parallel to SR, and  $\widehat{PSR} = 90^\circ$ . Find:

- a** the coordinates of S
- b** the area of the trapezium.



**6**



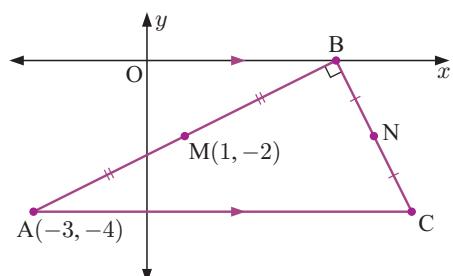
ABC is a triangle in which  $\widehat{ABC} = 90^\circ$ , and C lies on the x-axis. Find:

- a** the coordinates of C
- b** the area of the triangle.

- 7** A trapezium ABCD has vertices A(3, 0), B(-2, -5), C(-4, 1), and D. The side AD is parallel to BC, and the side CD is perpendicular to BC. Find the area of the trapezium.

- 8** ABC is a triangle in which  $\widehat{ABC} = 90^\circ$ , AC is parallel to the x-axis, M is the midpoint of AB, and N is the midpoint of BC.

- a** Find the coordinates of:
  - i** B
  - ii** C
  - iii** N
- b** Show that MN is parallel to AC.
- c** Find the area of:
  - i** trapezium AMNC
  - ii** triangle ABC.



## C

## INTERSECTION OF A STRAIGHT LINE AND A CURVE

To find where a straight line intersects a curve, we first rearrange the equation of the line so that  $x$  or  $y$  is the subject. We then substitute this expression for  $x$  or  $y$  into the equation of the curve.

While a straight line meets another straight line at most once, a straight line may meet a curve more than once.

### Example 10

### Self Tutor

Find the points where the line  $x - 3y = 4$  intersects the curve  $x^2 + y^2 = 34$ .

Substituting  $x = 3y + 4$  into  $x^2 + y^2 = 34$  gives

$$(3y + 4)^2 + y^2 = 34$$

$$\therefore 9y^2 + 24y + 16 + y^2 = 34$$

$$\therefore 10y^2 + 24y - 18 = 0$$

$$\therefore 2(5y^2 + 12y - 9) = 0$$

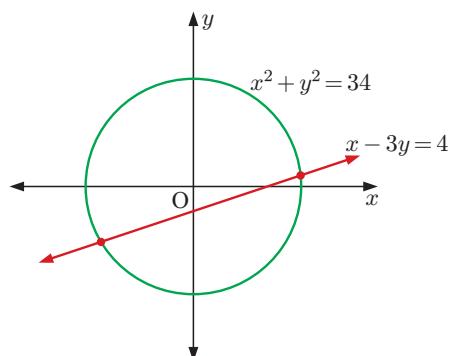
$$\therefore 2(5y - 3)(y + 3) = 0$$

$$\therefore y = \frac{3}{5} \text{ or } -3$$

$$\text{When } y = \frac{3}{5}, \quad x = 3\left(\frac{3}{5}\right) + 4 = \frac{29}{5}$$

$$\text{When } y = -3, \quad x = 3(-3) + 4 = -5$$

$\therefore$  the line intersects the curve at  $(\frac{29}{5}, \frac{3}{5})$  and  $(-5, -3)$ .



### Example 11

### Self Tutor

Find the points where the line  $2x + 3y = 5$  intersects the curve  $\frac{1}{x} - \frac{3}{y} = 2$ .

If  $2x + 3y = 5$ , then  $y = \frac{5 - 2x}{3}$ .

Substituting into  $\frac{1}{x} - \frac{3}{y} = 2$  gives  $\frac{1}{x} - \frac{3}{\frac{5-2x}{3}} = 2$

$$\therefore \frac{1}{x} - \frac{9}{5-2x} = 2$$

$$\therefore (5-2x) - 9x = 2x(5-2x) \quad \{ \times \text{ both sides by } x(5-2x) \}$$

$$\therefore 5 - 11x = 10x - 4x^2$$

$$\therefore 4x^2 - 21x + 5 = 0$$

$$\therefore (4x - 1)(x - 5) = 0$$

$$\therefore x = \frac{1}{4} \text{ or } 5$$

$$\text{When } x = \frac{1}{4}, \quad y = \frac{5 - 2(\frac{1}{4})}{3} = \frac{3}{2}, \quad \text{and when } x = 5, \quad y = \frac{5 - 2(5)}{3} = -\frac{5}{3}.$$

$\therefore$  the line intersects the curve at  $(\frac{1}{4}, \frac{3}{2})$  and  $(5, -\frac{5}{3})$ .

**EXERCISE 7C**

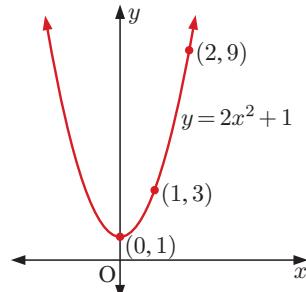
- 1** Find the points where the line  $x - 2y = 3$  intersects the curve  $x^2 + y^2 = 5$ .
- 2** The line  $x + y = 7$  meets the curve  $x^2 + y^2 = 29$  at A and B. Find the distance between A and B.
- 3** The line  $2x + y = 5$  meets the curve  $x^2 + y^2 = 10$  at P and Q. Find the equation of the perpendicular bisector of PQ.
- 4** Find the points where the line  $x - 2y = 4$  intersects the curve  $3x^2 + y^2 + xy + 3y = 8$ .
- 5** The line  $y = 2x + 1$  meets the curve  $x^2 + y^2 + xy + 16x = 29$  at P and Q. Find the distance between P and Q.
- 6** The line  $3x + y = 1$  intersects the curve  $2x^2 + y^2 + 5xy - 7x = -31$  at A and B. Find the equation of the perpendicular bisector of AB.
- 7** Find the points where the line  $x - 2y = 6$  intersects the curve  $\frac{4}{x} - \frac{1}{y} = 2$ .
- 8** The line  $3x + 2y = 12$  intersects the curve  $\frac{4}{x} + \frac{3}{y} = 3$  at P and Q. Find the midpoint of PQ.

**D****TRANSFORMING RELATIONSHIPS  
TO STRAIGHT LINE FORM**

Even if  $x$  and  $y$  are not linearly related, it is sometimes still possible to use a straight line graph to display the relationship. We do this by changing the variables on the axes.

For example, consider the relationship  $y = 2x^2 + 1$ .

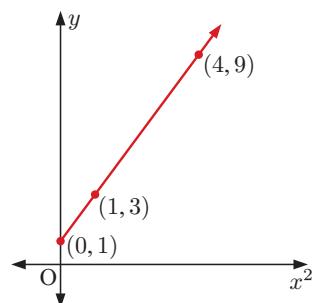
$x$  and  $y$  are not linearly related, but  $x^2$  and  $y$  are linearly related since  $y = 2(x^2) + 1$ .



We can use a table of values to plot  $y$  against  $x^2$ :

$x$	0	1	2
$x^2$	0	1	4
$y$	1	3	9

The graph of  $y$  against  $x^2$  is a straight line with gradient 2 and  $y$ -intercept 1.



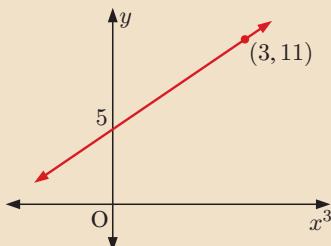
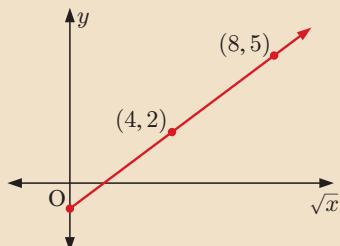
Click on the icon to view a demonstration of how the two graphs are related.

Observe that for the graph of  $y$  against  $x^2$ , the line terminates at  $(0, 1)$ , since  $x^2 \geq 0$  for all  $x$ . We need to be careful with the domain and range when we transform relationships.



**Example 12****Self Tutor**

Find  $y$  in terms of  $x$ :

**a****b**

- a** The graph of  $y$  against  $x^3$  is linear.

The gradient is  $\frac{11 - 5}{3 - 0} = 2$ , and  
the  $y$ -intercept is 5.

$\therefore$  the equation is  $y = 2x^3 + 5$ .

- b** The graph of  $y$  against  $\sqrt{x}$  is linear.

The gradient is  $\frac{5 - 2}{8 - 4} = \frac{3}{4}$ .

$\therefore$  the equation is

$$y - 2 = \frac{3}{4}(\sqrt{x} - 4)$$

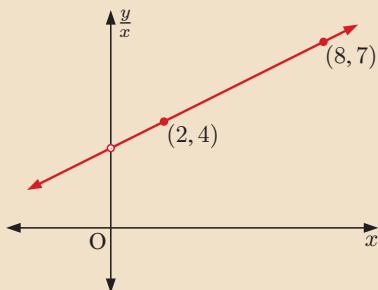
$$\therefore y - 2 = \frac{3}{4}\sqrt{x} - 3$$

$$\therefore y = \frac{3}{4}\sqrt{x} - 1, \quad x \geq 0$$

**Example 13****Self Tutor**

- a** Find  $y$  in terms of  $x$ .

- b** Find  $y$  when  $x = 4$ .



- a** The graph of  $\frac{y}{x}$  against  $x$  is linear.

The gradient is  $\frac{7 - 4}{8 - 2} = \frac{1}{2}$ .

$\therefore$  the equation is  $\frac{y}{x} - 4 = \frac{1}{2}(x - 2)$

$$\therefore \frac{y}{x} - 4 = \frac{1}{2}x - 1$$

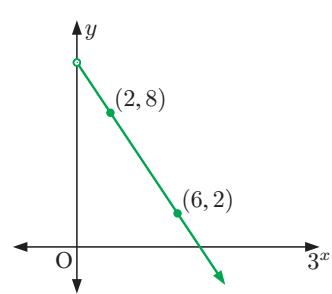
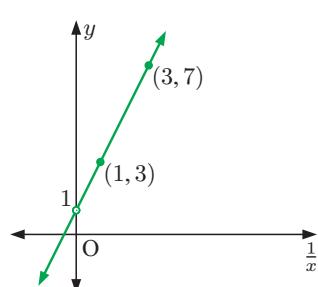
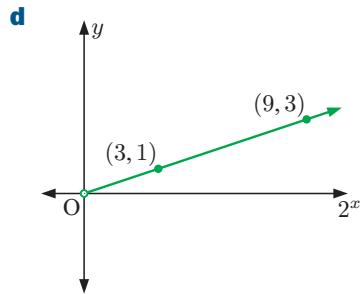
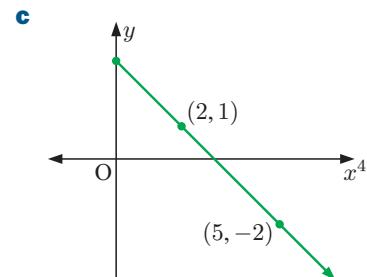
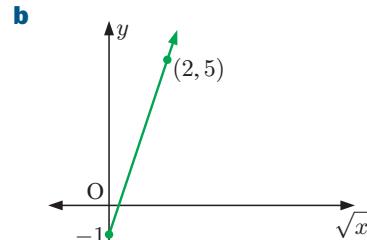
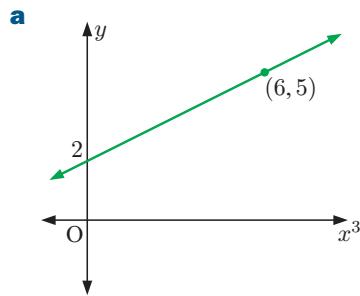
$$\therefore \frac{y}{x} = \frac{1}{2}x + 3$$

$$\therefore y = \frac{1}{2}x^2 + 3x$$

- b** When  $x = 4$ ,  $y = \frac{1}{2}(4)^2 + 3(4)$   
 $= 20$

**EXERCISE 7D**

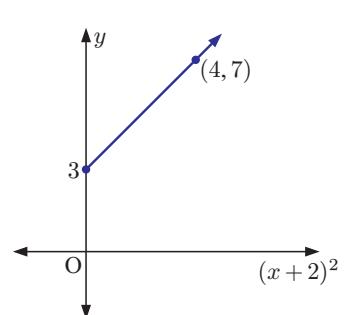
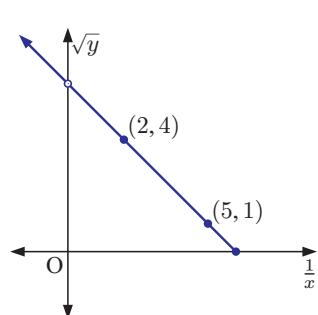
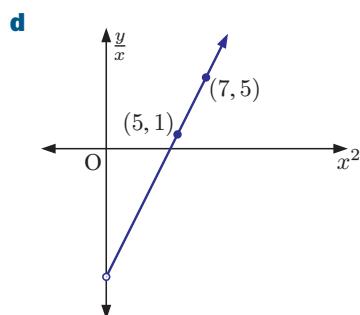
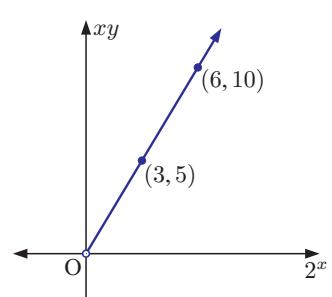
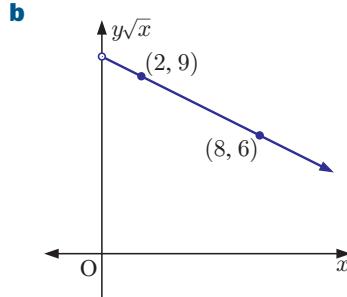
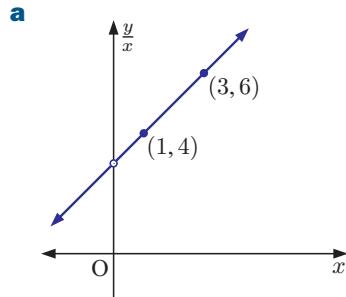
- 1** Find  $y$  in terms of  $x$ :



- 2** For each of the following relations:

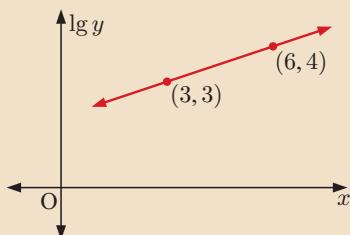
i find  $y$  in terms of  $x$

ii find the value of  $y$  when  $x = 3$ .



**Example 14****Self Tutor**

Write  $y$  in terms of  $x$ , giving your answer in the form  $y = a \times 10^{bx}$ , where  $a, b \in \mathbb{Q}$ .



The graph of  $\lg y$  against  $x$  is linear.

$$\text{The gradient is } \frac{4 - 3}{6 - 3} = \frac{1}{3}.$$

$$\therefore \text{the equation is } \lg y - 3 = \frac{1}{3}(x - 3)$$

$$\lg y - 3 = \frac{1}{3}x - 1$$

$$\therefore \lg y = \frac{1}{3}x + 2$$

$$\therefore y = 10^{\frac{1}{3}x+2} \quad \{ \text{if } \lg p = q \text{ then } p = 10^q \}$$

$$\therefore y = 10^{\frac{1}{3}x} \times 10^2$$

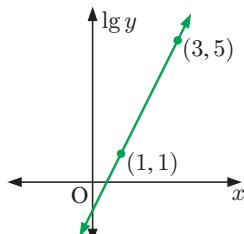
$$\therefore y = 100 \times 10^{\frac{1}{3}x}$$

In Chapter 5, we saw that a linear relationship between  $\lg y$  and  $x$  indicates an exponential relationship between  $y$  and  $x$ .



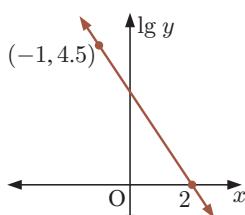
- 3 a** Find  $\lg y$  in terms of  $x$ .

- b** Write  $y$  in terms of  $x$ , giving your answer in the form  $y = a \times 10^{bx}$ , where  $a, b \in \mathbb{Q}$ .

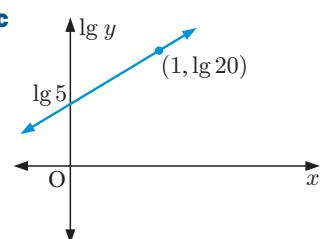
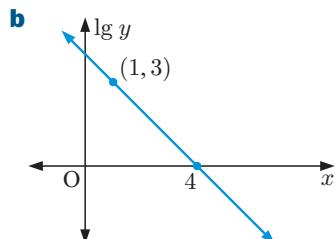
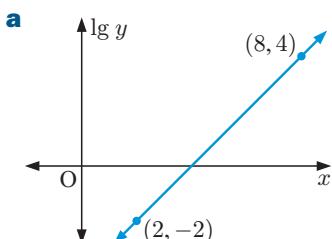


- 4** Write  $y$  in terms of  $x$ , giving your answer in the form

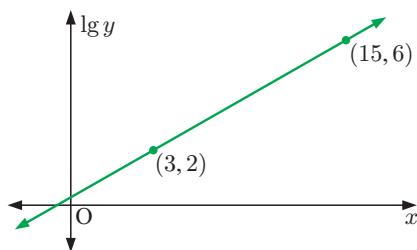
$$y = a \times 10^{bx}, \text{ where } a, b \in \mathbb{Q}.$$



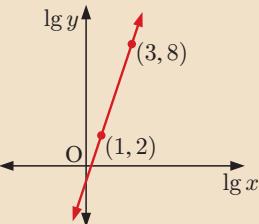
- 5** Write  $y$  in terms of  $x$ , giving your answer in the form  $y = a \times b^x$ , where  $a, b \in \mathbb{Q}$ .



- 6 a** Write  $y$  in terms of  $x$ , giving your answer in the form  $y = a \times 10^{bx}$ , where  $a, b \in \mathbb{Q}$ .  
**b** Find  $y$  when  $x = 6$ .

**Example 15**

Write  $y$  in terms of  $x$ , giving your answer in the form  $y = a \times x^b$ , where  $a, b \in \mathbb{Q}$ .



The graph of  $\lg y$  against  $\lg x$  is linear.

The gradient is  $\frac{8 - 2}{3 - 1} = 3$ .

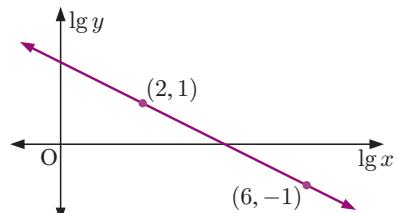
$$\begin{aligned}\therefore \text{the equation is } \lg y - 2 &= 3(\lg x - 1) \\ \therefore \lg y - 2 &= 3 \lg x - 3 \\ \therefore \lg y &= 3 \lg x - 1 \\ \therefore \lg y &= \lg x^3 - \lg 10 \\ \therefore \lg y &= \lg \left( \frac{x^3}{10} \right) \\ \therefore y &= \frac{1}{10} \times x^3\end{aligned}$$

A linear relationship between  $\lg y$  and  $\lg x$  indicates a power relationship between  $y$  and  $x$ .



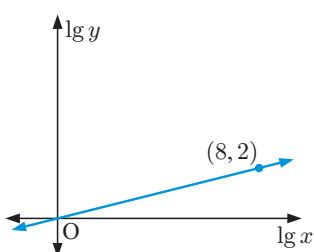
- 7** Consider the graph alongside.

- a** Write an equation for the line in the form  $\lg y = m \lg x + c$ .  
**b** Hence write  $y$  in terms of  $x$ .

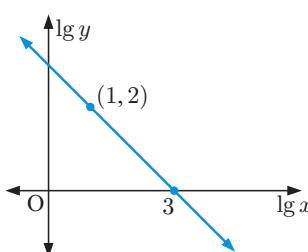


- 8** Write  $y$  in terms of  $x$ :

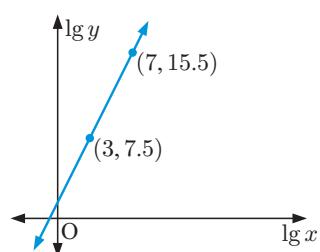
**a**



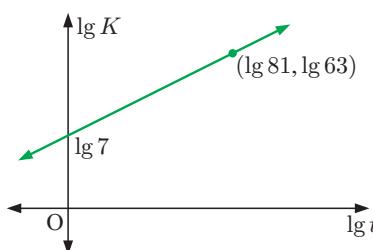
**b**



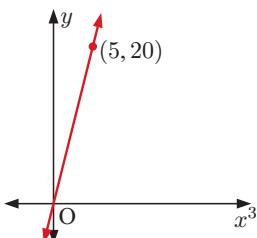
**c**



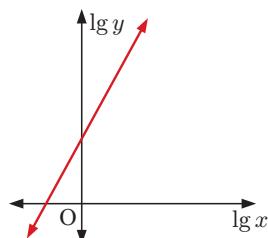
- 9** **a** Write  $K$  in terms of  $t$ .  
**b** Hence find  $K$  when  $t = 9$ .



- 10** *Graph A*



- Graph B*



The relationship between  $x$  and  $y$  in *Graph A* can also be plotted as a straight line in *Graph B*. For the straight line in *Graph B*, find the:

- a** gradient      **b** intercept on the vertical axis.

## E

## FINDING RELATIONSHIPS FROM DATA

We have seen how the transformation of variables may allow us to display a non-linear relationship using a straight line graph. It is particularly useful to do this if we are trying to use a function to model data.



Exponential, power, and logarithmic models can be transformed to straight line graphs.

### Case study

### Exponential growth and decay

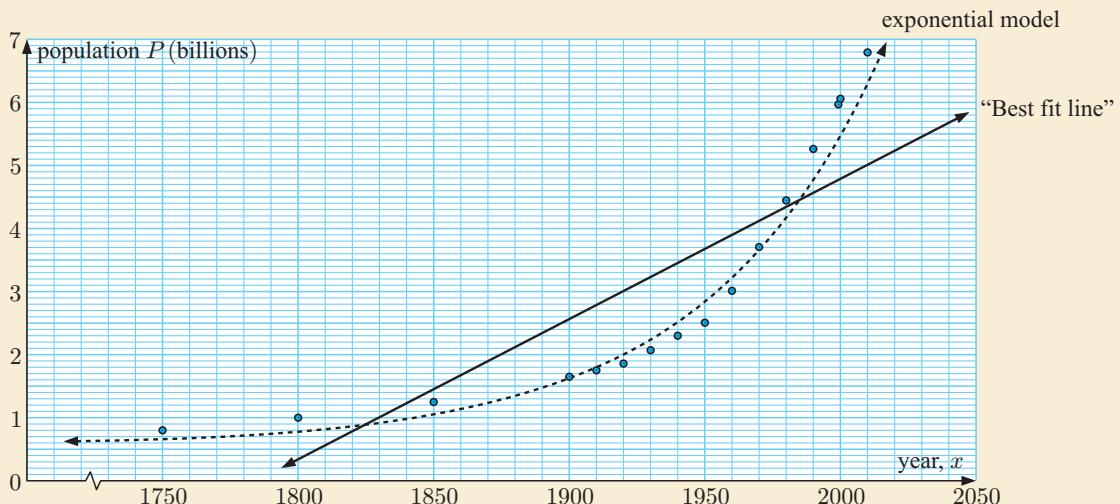
Logarithms are particularly important in science. Many physical processes are modelled accurately by exponential laws.

For example, the United Nations published the following data on world population:

Year	Population $P$ (in billions)	$\lg P$
1750	0.79	-0.236
1800	0.98	-0.0202
1850	1.26	0.231
1900	1.65	0.501
1910	1.75	0.560
1920	1.86	0.621
1930	2.07	0.728
1940	2.30	0.833

Year	Population $P$ (in billions)	$\lg P$
1950	2.52	0.924
1960	3.02	1.11
1970	3.70	1.31
1980	4.44	1.49
1990	5.27	1.66
1999	5.98	1.79
2000	6.06	1.80
2010	6.79	1.92

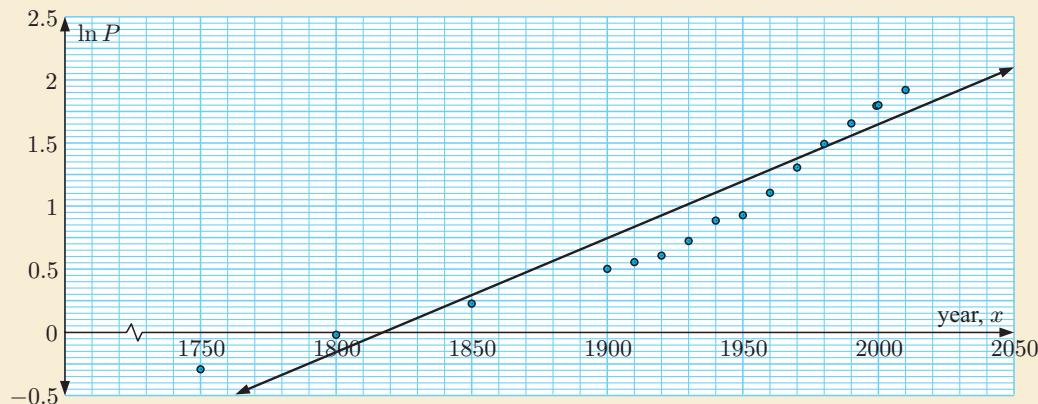
The population data is presented on the graph below:



The “best fit line”,  $P = 0.0222x - 39.6$ , does not fit the data very well. Instead, we try to fit an exponential curve of the form  $P = ae^{mx}$ .

Taking the natural logarithm of both sides, we have  $\ln P = mx + \ln a$ , which is the equation of a straight line.

We now plot  $\ln P$  against  $x$ :



The equation of this “best fit line” is  $\ln P = -15.5 + 0.00855x$ .

Using our model this means that  $\ln P = -15.5 + 0.00855x$ ,  
 $\therefore$  the data can be modelled by  $P = e^{-15.5+0.00855x}$ .

The “best fit line” is not a perfect fit because we are using real data.

This is shown as a dashed line on the original graph. This is not a perfect fit either, but is a considerable improvement on the original straight line graph.



**Example 16****Self Tutor**

Consider this table of data connecting  $x$  and  $y$ :

$x$	1	2	3	4
$y$	3.5	10	22.5	44

- a Copy and complete the following table:

$x^2$				
$\frac{y}{x}$				

- b Plot  $\frac{y}{x}$  against  $x^2$ , and draw a straight line through the points.

- c Find  $y$  in terms of  $x$ .

a	$x^2$	1	4	9	16
	$\frac{y}{x}$	3.5	5	7.5	11

- c The graph of  $\frac{y}{x}$  against  $x^2$  is linear.

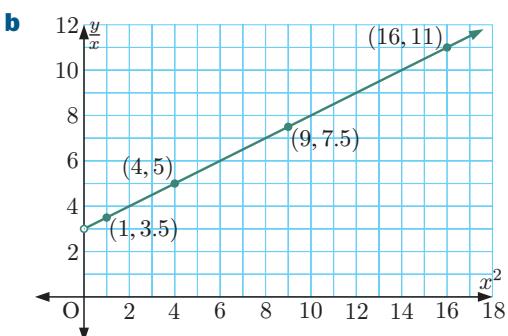
Using the points  $(4, 5)$  and  $(16, 11)$ ,

$$\text{the gradient is } \frac{11 - 5}{16 - 4} = \frac{1}{2}$$

$$\therefore \text{the equation is } \frac{y}{x} - 5 = \frac{1}{2}(x^2 - 4)$$

$$\therefore \frac{y}{x} = \frac{1}{2}x^2 + 3$$

$$\therefore y = \frac{1}{2}x^3 + 3x$$



$x^2 \geq 0$  for all  $x$ .  
 $\frac{y}{x}$  is undefined when  $x = 0$ .  
 $\therefore$  the point on the vertical axis is not included.

**EXERCISE 7E**

- 1 Consider this table of data connecting  $x$  and  $y$ :

$x$	1	2	3	4
$y$	2	11	26	47

$x^2 \geq 0$  for all  $x$ .

- a Copy and complete the following table:

$x^2$				
$y$				



- b Plot  $y$  against  $x^2$ , and draw a straight line through the points.

- c Find  $y$  in terms of  $x$ .

- 2** This table shows experimental data values for  $x$  and  $y$ :

$x$	1	2	3	4
$y$	9	9.90	10.97	12

$\sqrt{x}$  is only defined for  $x \geq 0$ .

- a** Copy and complete the following table:

$x$				
$y\sqrt{x}$				



- b** Plot  $y\sqrt{x}$  against  $x$ , and draw a straight line through the points.

- c** Find  $y$  in terms of  $x$ .

- d** Find  $y$  when  $x = 16$ .

- 3** This table shows experimental values for  $x$  and  $y$ .

$x$	1	2	3	4
$y$	-1	0	0.11	0.12

It is known that  $x$  and  $y$  are related by the equation  $y = \frac{a}{x} + \frac{b}{x^2}$ , where  $a$  and  $b$  are constants.

- a** Copy and complete the following table:

$\frac{1}{x}$				
$xy$				

$\frac{1}{x}$  is not defined when  $x = 0$ .



- b** Plot  $xy$  against  $\frac{1}{x}$ , and draw a straight line through the points.

- c** Hence find  $a$  and  $b$ .

- d** Find  $y$  when  $x = 10$ .

- 4** This table shows values of  $x$  and  $y$ :

$x$	2	4	6	8
$y$	5.24	5	5.45	6.12

- a** Copy and complete the following table:

$x\sqrt{x}$				
$y\sqrt{x}$				

- b** Plot  $y\sqrt{x}$  against  $x\sqrt{x}$ , and draw a straight line through the points.

- c** Find  $y$  in terms of  $x$ .

- d** Find  $y$  when  $x = 9$ .

- 5** The mass of bacteria in a culture is measured each day for 5 days.

$t$ (days)	1	2	3	4	5
$M$ (grams)	3.98	6.31	10	15.85	25.12

This experiment starts at  $t = 0$  days.

- a** Copy and complete the following table:

$t$				
$\lg M$				



- b** Plot  $\lg M$  against  $t$ , and draw a straight line through the points.

- c** Find  $M$  in terms of  $t$ .

- d** Find the original mass of the bacteria.

**Example 17****Self Tutor**

This table shows experimental data values for  $x$  and  $y$ :

By plotting a suitable straight line graph, show that  $y$  and  $x$  are related by the equation  $y = ax + \frac{b}{x}$ .

$x$	1	2	3	4
$y$	14	10	10	11

If  $y = ax + \frac{b}{x}$ , then

$$xy = ax^2 + b$$

$\therefore$  if  $y$  and  $x$  are related in this way, then we should observe a linear relationship between  $xy$  and  $x^2$ .

There may be more than one way to transform the variables.



$x^2$	1	4	9	16
$xy$	14	20	30	44

The graph of  $xy$  against  $x^2$  is linear.

Using points  $(1, 14)$  and  $(4, 20)$ ,

the gradient is  $\frac{20 - 14}{4 - 1} = 2$ .

$\therefore$  the equation is  $xy - 14 = 2(x^2 - 1)$

$$\therefore xy - 14 = 2x^2 - 2$$

$$\therefore xy = 2x^2 + 12$$

$$\therefore y = 2x + \frac{12}{x} \quad \{a = 2, b = 12\}$$



- 6** This table shows experimental values of  $x$  and  $y$ :

$x$	1	2	3	4
$y$	1	26	99	244

It is known that  $x$  and  $y$  are related by the equation  $y = ax^3 + bx$ , where  $a$  and  $b$  are constants.

- a** A straight line graph is to be drawn to represent this information. If  $\frac{y}{x}$  is plotted on the vertical axis, which variable should be plotted on the horizontal axis?
- b** Draw the straight line graph.
- c** Find the values of  $a$  and  $b$ .
- d** Find  $y$  when  $x = 5$ .

- 7** This table shows experimental values of  $x$  and  $y$ :

$x$	1	2	3	4
$y$	4	1.17	0.36	0

By plotting a suitable straight line graph, show that  $x$  and  $y$  are related by the equation  $y = \frac{a}{x} + \frac{b}{\sqrt{x}}$ .

- 8** A stone is dropped from the top of an 80 m high cliff. This table shows the distance the stone has fallen at various times.

Time ( $t$ s)	1	1.7	2	2.7
Distance ( $D$ m)	4.9	14.16	19.6	35.72

- a** By plotting a suitable straight line graph, show that  $t$  and  $D$  are related by the equation  $D = a \times t^b$ , where  $a$  and  $b$  are constants.
- b** How far had the stone fallen after 3 seconds?
- c** How long did the stone take to hit the water?



### Research

### Logarithmic scales in science

If your data ranges over many orders of magnitude, it can be difficult to compare or represent on a graph.

For example, the Richter scale for earthquake measurement uses logarithms in base 10. An earthquake measuring 6.0 on the Richter scale has a shaking amplitude  $10^{6-4} = 100$  times larger than one that measures 4.0.

Research some other scientific scales that use logarithms to compress very large ranges into manageable values. You may like to consider:

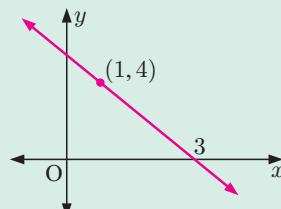
- the decibel scale for the loudness of sound
- the stellar magnitude scale for brightness of stars
- the pH scale for acidity and alkalinity
- counting f-stops for ratios of photographic exposure.

### Review set 7A

- 1** Consider the points  $A(-1, 6)$  and  $B(5, 4)$ . Find:

- a** the distance between  $A$  and  $B$
- b** the midpoint of  $AB$
- c** the equation of the line through  $A$  and  $B$ .

- 2** Determine the equation of the illustrated line:



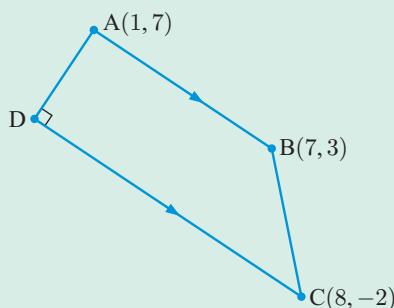
- 3** Explain why the vertical straight line in the plane cannot be written in gradient-intercept form  $y = mx + c$ .

- 4** Suppose  $P$  has coordinates  $(-2, -3)$ , and  $Q$  has coordinates  $(1, 3)$ . A line perpendicular to  $PQ$ , passes through  $Q$ .
- a** Find the equation of the line.
  - b** Find the coordinates of the point where the line cuts the  $x$ -axis.

- 5 Find the point of intersection of the lines  $x - 2y = 5$  and  $4x + 3y = 9$ .

- 6 ABCD is a trapezium in which AB is parallel to DC, and  $\hat{ADC} = 90^\circ$ . Find:

- a the coordinates of D  
b the area of the trapezium.



- 7 Find the points where the line  $3x + y = 1$  intersects the curve  $x^2 + y^2 = 29$ .

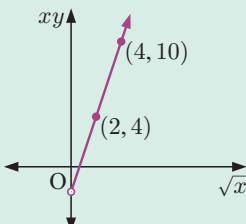
- 8 The line  $x + y = 5$  meets the curve  $x^2 + y^2 + 3xy + 5x = 1$  at P and Q. Find the equation of the perpendicular bisector of PQ.

- 9 Consider two distinct points in the plane  $(a_1, b_1)$  and  $(a_2, b_2)$  where  $a_1 \neq a_2$ . Show that the straight line passing through them has equation:

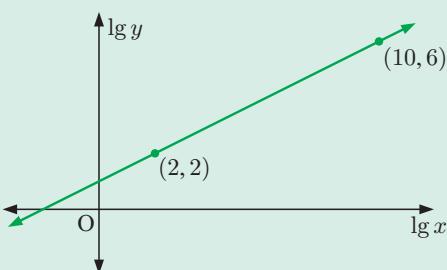
- a  $y = \frac{b_1 - b_2}{a_1 - a_2} x + \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}$  in gradient-intercept form  
b  $(b_1 - b_2)x + (a_2 - a_1)y = a_2 b_1 - a_1 b_2$  in general form.

- 10 a Write  $y$  in terms of  $x$ .

- b Hence find  $y$  when  $x = 4$ .



11



Consider the graph alongside.

- a Write an equation for the line in the form  $\lg y = m \lg x + c$ .  
b Hence write  $y$  in terms of  $x$ .

- 12 This table shows experimental values of  $x$  and  $y$ :

$x$	1	2	3	4
$y$	8	7.5	11.33	17.75

- a Copy and complete the following table:

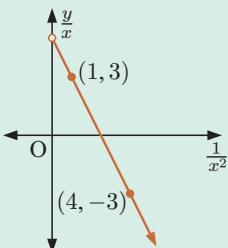
$x^3$				
$xy$				

- b Plot  $xy$  against  $x^3$ , and draw a straight line through the points.  
c Find  $y$  in terms of  $x$ .  
d Hence find  $y$  when  $x = 7$ .

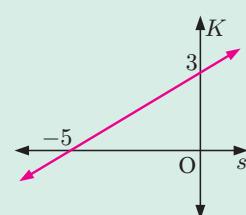
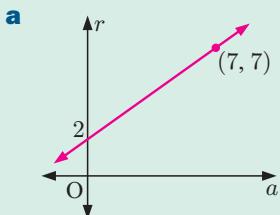
**Review set 7B**

- 1** Find the equation of the perpendicular bisector of AB given A(-2, 3) and B(4, 5).
- 2** The line  $x - 2y = 3$  meets the curve  $x^2 + 2y^2 - 2xy + 3x = 8$  at P and Q. Find the distance between P and Q.

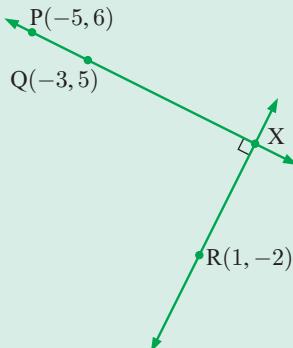
- 3** **a** Write  $y$  in terms of  $x$ .  
**b** Hence find  $y$  when  $x = 8$ .



- 4** Find the equation linking the variables in each graph:



- 5** Find the coordinates of X.



- 6** Find, in general form, the equation of the line passing through  $(-5, -7)$  and  $(3, -2)$ .

- 7** Consider this table of data connecting  $x$  and  $y$ :

$x$	1	2	3	4
$y$	2.5	5.29	8.13	11

- a** Copy and complete the following table:

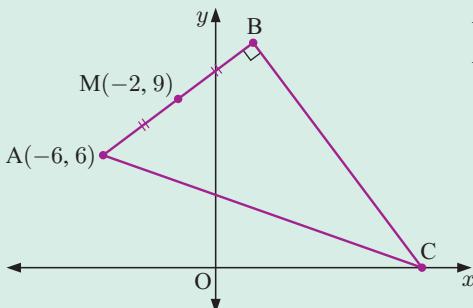
$\sqrt{x}$				
$\frac{y}{\sqrt{x}}$				

- b** Plot  $\frac{y}{\sqrt{x}}$  against  $\sqrt{x}$ , and draw a straight line through the points.

- c** Hence write  $y$  in terms of  $x$ .

- 8** Consider the points  $(a, 0)$  and  $(0, b)$ .

- a Find the equation of the straight line through these points, in general form.  
 b Let  $\theta$  be the angle between the line and the  $x$ -axis. Show that the general form of the equation of the line is  $(\sin \theta)x + (\cos \theta)y = d$  where  $d = \frac{ab}{\sqrt{a^2 + b^2}}$  is the shortest distance from the line to the origin.

**9**

ABC is a triangle in which M is the midpoint of AB,  $\widehat{ABC} = 90^\circ$ , and C lies on the  $x$ -axis.

- a Find the coordinates of:  
 i B      ii C  
 b Find the area of the triangle.

- 10** The line  $4x - 3y = 2$  intersects the curve  $\frac{3}{y} - \frac{1}{x} = 1$  at A and B. Find the midpoint of AB.

- 11** This table shows experimental values of  $x$  and  $y$ :

$x$	2	4	6	8
$y$	21.54	4.64	1	0.21

- a By plotting a suitable straight line graph, show that  $x$  and  $y$  are related by the equation  $y = a \times b^x$ , where  $a$  and  $b$  are constants.  
 b Hence find  $y$  when  $x = 1$ .

# 8

# The unit circle and radian measure

## Contents:

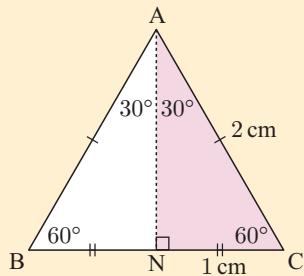
- A** Radian measure
- B** Arc length and sector area
- C** The unit circle and the trigonometric ratios
- D** Applications of the unit circle
- E** Multiples of  $\frac{\pi}{6}$  and  $\frac{\pi}{4}$
- F** Reciprocal trigonometric ratios

## Opening problem

Consider an equilateral triangle with sides 2 cm long. Altitude AN bisects side BC and the vertical angle BAC.

### Things to think about:

- a Can you use this figure to explain why  $\sin 30^\circ = \frac{1}{2}$ ?
- b Use your calculator to find the value of:
  - i  $\sin 150^\circ$
  - ii  $\sin 390^\circ$
  - iii  $\sin(-330^\circ)$
- c Can you explain each result in b, even though the angles are not between  $0^\circ$  and  $90^\circ$ ?



## A RADIANT MEASURE

### DEGREE MEASUREMENT OF ANGLES

We have seen previously that one full revolution makes an angle of  $360^\circ$ , and the angle on a straight line is  $180^\circ$ .

One degree,  $1^\circ$ , is  $\frac{1}{360}$ th of one full revolution.

This measure of angle is commonly used by surveyors and architects.

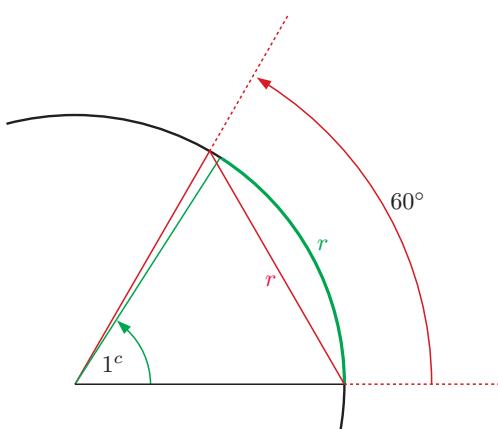
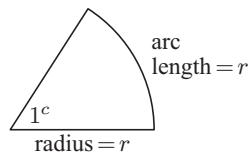
### RADIANT MEASUREMENT OF ANGLES

An angle is said to have a measure of one **radian**,  $1^c$ , if it is subtended at the centre of a circle by an arc equal in length to the radius.

The symbol ' $c$ ' is used for radian measure but is usually omitted. By contrast, the degree symbol is *always* used when the measure of an angle is given in degrees.

From the diagram below, it can be seen that  $1^c$  is slightly smaller than  $60^\circ$ . In fact,  $1^c \approx 57.3^\circ$ .

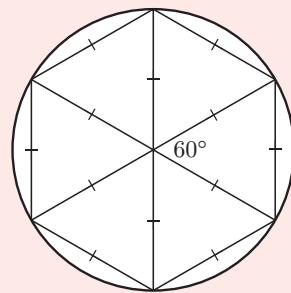
The word 'radian' is an abbreviation of 'radial angle'.



### Historical note

There are several theories for why one complete turn was divided into 360 degrees:

- 360 is approximately the number of days in a year.
- The Babylonians used a counting system in base 60. If they drew 6 equilateral triangles within a circle as shown, and divided each angle into 60 subdivisions, then there were 360 subdivisions in one turn. The division of an hour into 60 minutes, and a minute into 60 seconds, is from this base 60 counting system.
- 360 has 24 divisors, including every integer from 1 to 10 except 7.

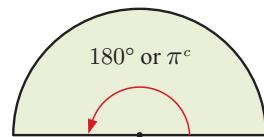


The idea of measuring an angle by the length of an arc dates to around 1400 and the Persian mathematician **Al-Kashi**. The concept of a radian is generally credited to **Roger Cotes**, however, who described it as we know it today.

### DEGREE-RADIAN CONVERSIONS

If the radius of a circle is  $r$ , then an arc of length  $\pi r$ , or half the circumference, will subtend an angle of  $\pi$  radians.

Therefore,  $\pi$  radians =  $180^\circ$ .

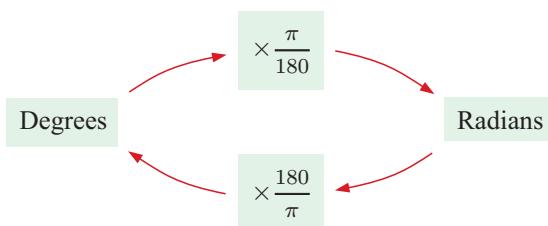


So,  $1^\circ = \left(\frac{180}{\pi}\right)^\circ \approx 57.3^\circ$  and  $1^\circ = \left(\frac{\pi}{180}\right)^c \approx 0.0175^c$ .

To convert from degrees to radians, we multiply by  $\frac{\pi}{180}$ .

To convert from radians to degrees, we multiply by  $\frac{180}{\pi}$ .

We indicate degrees with a small  $^\circ$ .  
To indicate radians we use a small  $^c$   
or else use no symbol at all.



#### Example 1

#### Self Tutor

Convert  $45^\circ$  to radians in terms of  $\pi$ .

$$\begin{aligned} 45^\circ &= (45 \times \frac{\pi}{180}) \text{ radians} & \text{or} & \quad 180^\circ = \pi \text{ radians} \\ &= \frac{\pi}{4} \text{ radians} & \therefore & \quad \left(\frac{180}{4}\right)^\circ = \frac{\pi}{4} \text{ radians} \\ && \therefore & \quad 45^\circ = \frac{\pi}{4} \text{ radians} \end{aligned}$$



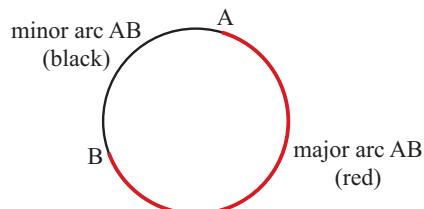
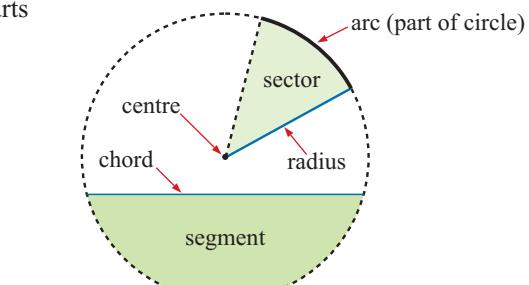
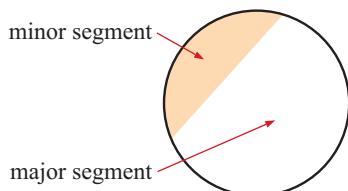
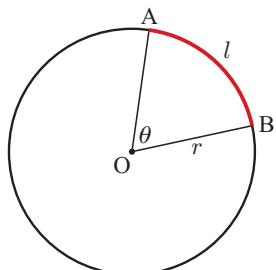
**B****ARC LENGTH AND SECTOR AREA**

The diagram alongside illustrates terms relating to the parts of a circle.

An arc, sector, or segment is described as:

- **minor** if it involves less than half the circle
- **major** if it involves more than half the circle.

For example:

**ARC LENGTH**

In the diagram, the **arc length** AB is  $l$ .

Angle  $\theta$  is measured in **radians**.

We use a ratio to obtain:

$$\frac{\text{arc length}}{\text{circumference}} = \frac{\theta}{2\pi}$$

$$\therefore \frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

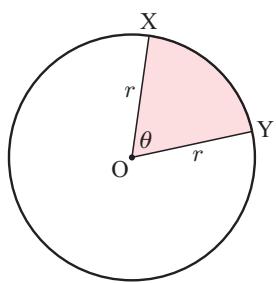
$$\therefore l = \theta r$$

Radians are used in pure mathematics because they make formulae simpler.



For  $\theta$  in **radians**, arc length  $l = \theta r$ .

For  $\theta$  in **degrees**, arc length  $l = \frac{\theta}{360} \times 2\pi r$ .

**AREA OF SECTOR**

In the diagram, the area of minor sector XZY is shaded.

$\theta$  is measured in **radians**.

We use a ratio to obtain:

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi}$$

$$\therefore \frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\therefore A = \frac{1}{2}\theta r^2$$

For  $\theta$  in **radians**, area of sector  $A = \frac{1}{2}\theta r^2$ .

For  $\theta$  in **degrees**, area of sector  $A = \frac{\theta}{360} \times \pi r^2$ .

**Example 4****Self Tutor**

A sector has radius 12 cm and angle 3 radians. Find its:

**a** arc length

**b** area

$$\begin{aligned}\mathbf{a} \quad \text{arc length} &= \theta r \\ &= 3 \times 12 \\ &= 36 \text{ cm}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \text{area} &= \frac{1}{2}\theta r^2 \\ &= \frac{1}{2} \times 3 \times 12^2 \\ &= 216 \text{ cm}^2\end{aligned}$$

**EXERCISE 8B**

- 1** Use radians to find the arc length and area of a sector of a circle of:

**a** radius 9 cm and angle  $\frac{7\pi}{4}$

**b** radius 4.93 cm and angle 4.67 radians.

- 2** A sector has an angle of  $107.9^\circ$  and an arc length of 5.92 m. Find its:

**a** radius

**b** area.

- 3** A sector has an angle of 1.19 radians and an area of  $20.8 \text{ cm}^2$ . Find its:

**a** radius

**b** perimeter.

**Example 5****Self Tutor**

Find the area of a sector with radius 8.2 cm and arc length 13.3 cm.

For  $\theta$  in radians,  $l = \theta r$

$$\therefore \theta = \frac{l}{r} = \frac{13.3}{8.2}$$

$$\begin{aligned}\therefore \text{area} &= \frac{1}{2}\theta r^2 \\ &= \frac{1}{2} \times \frac{13.3}{8.2} \times 8.2^2 \\ &\approx 54.5 \text{ cm}^2\end{aligned}$$

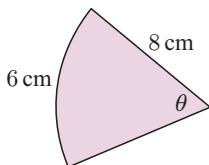
- 4** Find, in radians, the angle of a sector of:

**a** radius 4.3 m and arc length 2.95 m

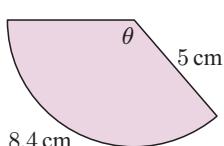
**b** radius 10 cm and area  $30 \text{ cm}^2$ .

- 5** Find  $\theta$  (in radians) for each of the following, and hence find the area of each figure:

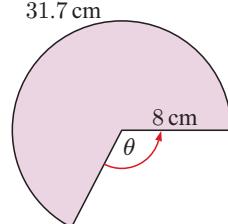
**a**



**b**



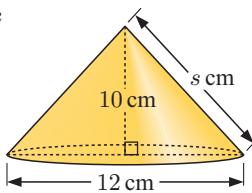
**c**



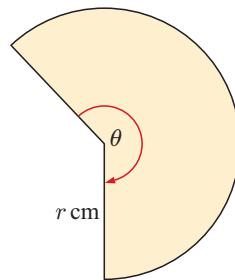
- 6** Find the arc length and area of a sector of radius 5 cm and angle 2 radians.

- 7** If a sector has radius  $2x$  cm and arc length  $x$  cm, show that its area is  $x^2 \text{ cm}^2$ .

**8** The cone



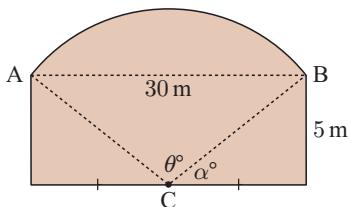
is made from this sector:



Find, correct to 3 significant figures:

- a** the slant length  $s$  cm
- b** the value of  $r$
- c** the arc length of the sector
- d** the sector angle  $\theta$  in radians.

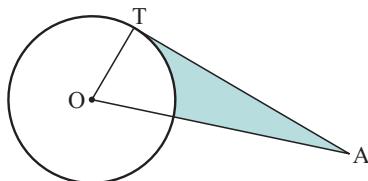
**9**



The end wall of a building has the shape illustrated, where the centre of arc AB is C. Find:

- a**  $\alpha$  to 4 significant figures
- b**  $\theta$  to 4 significant figures
- c** the area of the wall.

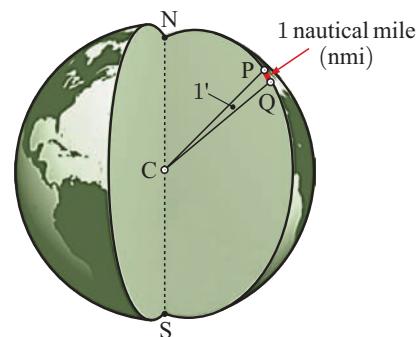
**10**



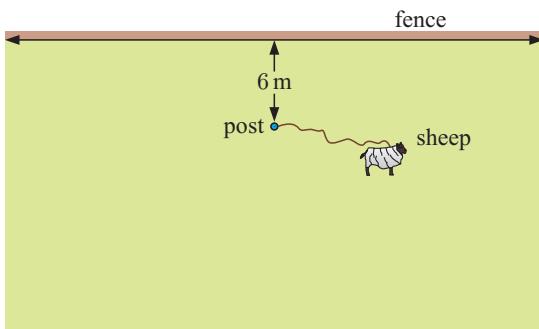
[AT] is a tangent to the given circle.  $OA = 13$  cm and the circle has radius 5 cm. Find the perimeter of the shaded region.

- 11** A **nautical mile** (nmi) is the distance on the Earth's surface that subtends an angle of 1 minute (or  $\frac{1}{60}$  of a degree) of the Great Circle arc measured from the centre of the Earth.  
A **knot** is a speed of 1 nautical mile per hour.

- a** Given that the radius of the Earth is 6370 km, show that  $1 \text{ nmi} \approx 1.853 \text{ km}$ .
- b** Calculate how long it would take a plane to fly from London to Moscow (a distance of 2508 km) if the plane can fly at 480 knots.



**12**

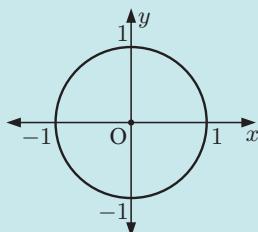


A sheep is tethered to a post which is 6 m from a long fence. The length of rope is 9 m. Find the area which the sheep can feed on.

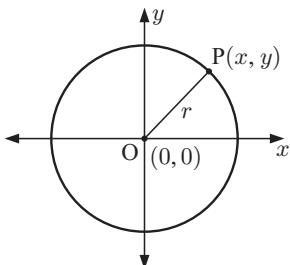
## C

# THE UNIT CIRCLE AND THE TRIGONOMETRIC RATIOS

The **unit circle** is the circle with centre  $(0, 0)$  and radius 1 unit.



## CIRCLES WITH CENTRE $(0, 0)$



Consider a circle with centre  $(0, 0)$  and radius  $r$  units.  
Suppose  $P(x, y)$  is any point on this circle.

$$\begin{aligned} \text{Since } OP &= r, \\ \sqrt{(x-0)^2 + (y-0)^2} &= r \quad \{\text{distance formula}\} \\ \therefore x^2 + y^2 &= r^2 \end{aligned}$$

$x^2 + y^2 = r^2$  is the equation of a circle with centre  $(0, 0)$  and radius  $r$ .

The equation of the **unit circle** is  $x^2 + y^2 = 1$ .

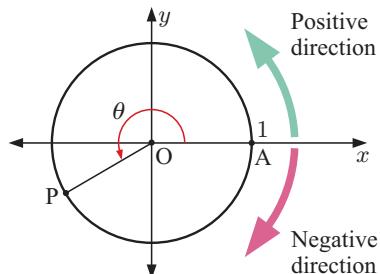
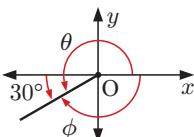
## ANGLE MEASUREMENT

Suppose  $P$  lies anywhere on the unit circle, and  $A$  is  $(1, 0)$ .  
Let  $\theta$  be the angle measured from  $[OA]$  on the positive  $x$ -axis.

$\theta$  is **positive** for anticlockwise rotations and **negative** for clockwise rotations.

For example:  $\theta = 210^\circ = \frac{7\pi}{6}$

$$\phi = -150^\circ = -\frac{5\pi}{6}$$



## DEFINITION OF SINE AND COSINE

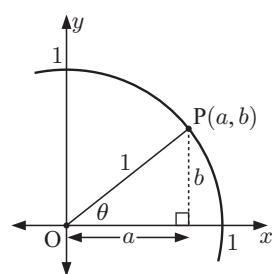
Consider a point  $P(a, b)$  which lies on the unit circle in the first quadrant.  $[OP]$  makes an angle  $\theta$  with the  $x$ -axis as shown.

Using right angled triangle trigonometry:

$$\cos \theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{a}{1} = a$$

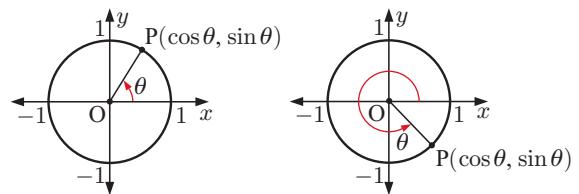
$$\sin \theta = \frac{\text{OPP}}{\text{HYP}} = \frac{b}{1} = b$$

$$\tan \theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{b}{a} = \frac{\sin \theta}{\cos \theta}$$



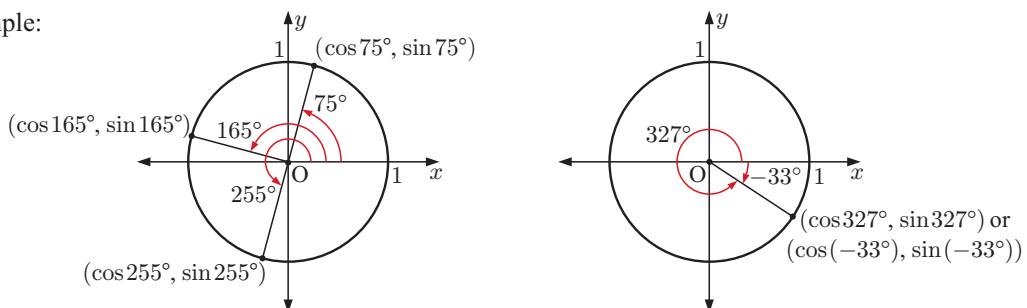
In general, for a point P anywhere on the unit circle:

- $\cos \theta$  is the  $x$ -coordinate of P
- $\sin \theta$  is the  $y$ -coordinate of P



We can hence find the coordinates of any point on the unit circle with given angle  $\theta$  measured from the positive  $x$ -axis.

For example:



Since the unit circle has equation  $x^2 + y^2 = 1$ ,  $(\cos \theta)^2 + (\sin \theta)^2 = 1$  for all  $\theta$ .

We commonly write this as

$$\cos^2 \theta + \sin^2 \theta = 1.$$

For all points on the unit circle,  $-1 \leq x \leq 1$  and  $-1 \leq y \leq 1$ .

So,

$$-1 \leq \cos \theta \leq 1 \text{ and } -1 \leq \sin \theta \leq 1 \text{ for all } \theta.$$

## DEFINITION OF TANGENT

Suppose we extend [OP] to meet the tangent from A(1, 0).

We let the intersection between these lines be point Q.

Note that as P moves, so does Q.

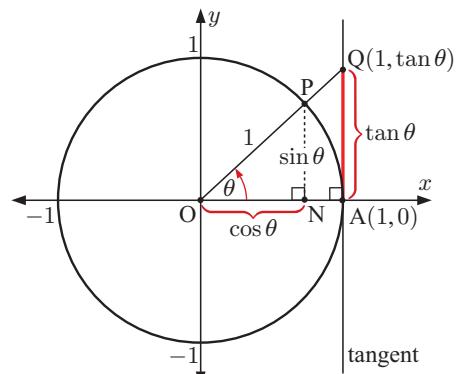
The position of Q relative to A is defined as the **tangent function**.

Notice that  $\triangle$ s ONP and OAQ are equiangular and therefore similar.

Consequently  $\frac{AQ}{OA} = \frac{NP}{ON}$  and hence  $\frac{AQ}{1} = \frac{\sin \theta}{\cos \theta}$ .

Under the definition that  $AQ = \tan \theta$ ,

$$\tan \theta = \frac{\sin \theta}{\cos \theta}.$$



**Discovery 1****The trigonometric ratios**

In this Discovery we explore the signs of the trigonometric ratios in each quadrant of the unit circle.

**What to do:**

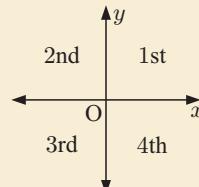
- Click on the icon to run the Unit Circle software.

Drag the point P slowly around the circle.

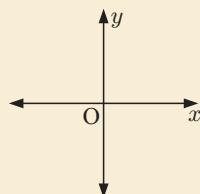
Note the *sign* of each trigonometric ratio in each quadrant.

Quadrant	$\cos \theta$	$\sin \theta$	$\tan \theta$
1	positive		
2			
3			
4			

THE UNIT CIRCLE



- Hence note down the trigonometric ratios which are *positive* in each quadrant.



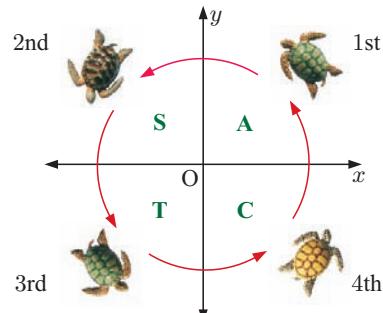
From the **Discovery** you should have found that:

- $\sin \theta$ ,  $\cos \theta$ , and  $\tan \theta$  are positive in quadrant 1
- only  $\sin \theta$  is positive in quadrant 2
- only  $\tan \theta$  is positive in quadrant 3
- only  $\cos \theta$  is positive in quadrant 4.

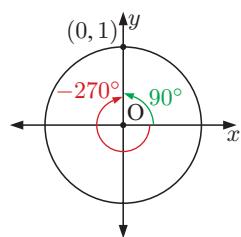
We can use a letter to show which trigonometric ratios are positive in each quadrant. The A stands for *all* of the ratios.

You might like to remember them using

All Silly Turtles Crawl.

**Example 6****Self Tutor**

Use a unit circle diagram to find the values of  $\cos(-270^\circ)$  and  $\sin(-270^\circ)$ .



$$\begin{aligned}\cos(-270^\circ) &= 0 && \{\text{the } x\text{-coordinate}\} \\ \sin(-270^\circ) &= 1 && \{\text{the } y\text{-coordinate}\}\end{aligned}$$

## PERIODICITY OF TRIGONOMETRIC RATIOS

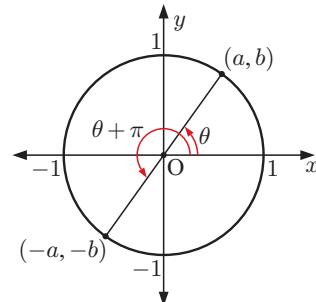
Since there are  $2\pi$  radians in a full revolution, if we add any integer multiple of  $2\pi$  to  $\theta$  (in radians) then the position of P on the unit circle is unchanged.

For  $\theta$  in radians and  $k \in \mathbb{Z}$ ,

$$\cos(\theta + 2k\pi) = \cos \theta \text{ and } \sin(\theta + 2k\pi) = \sin \theta.$$

We notice that for any point  $(\cos \theta, \sin \theta)$  on the unit circle, the point directly opposite is  $(-\cos \theta, -\sin \theta)$

$$\begin{aligned} \therefore \cos(\theta + \pi) &= -\cos \theta \\ \sin(\theta + \pi) &= -\sin \theta \\ \text{and } \tan(\theta + \pi) &= \frac{-\sin \theta}{-\cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta \end{aligned}$$



For  $\theta$  in radians and  $k \in \mathbb{Z}$ ,  $\tan(\theta + k\pi) = \tan \theta$ .

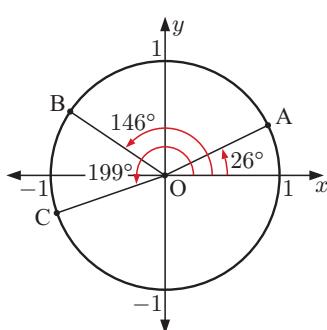
This **periodic** feature is an important property of the trigonometric functions.

### EXERCISE 8C

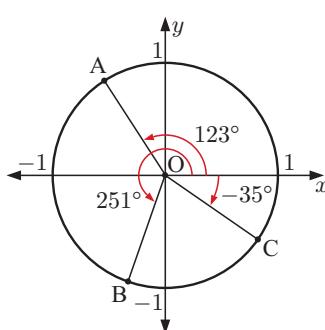
- 1** For each unit circle illustrated:

- i state the exact coordinates of points A, B, and C in terms of sine and cosine
- ii use your calculator to give the coordinates of A, B, and C correct to 3 significant figures.

a



b



- 2** With the aid of a unit circle, complete the following table:

$\theta$ (degrees)	$0^\circ$	$90^\circ$	$180^\circ$	$270^\circ$	$360^\circ$	$450^\circ$
$\theta$ (radians)						
sine						
cosine						
tangent						

- 3 a** Use your calculator to evaluate: **i**  $\frac{1}{\sqrt{2}}$     **ii**  $\frac{\sqrt{3}}{2}$
- b** Copy and complete the following table. If necessary, use your calculator to evaluate the trigonometric ratios, then **a** to write them exactly.

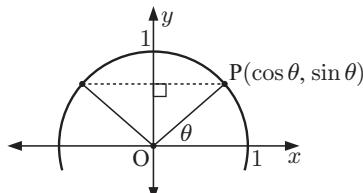
$\theta$ (degrees)	30°	45°	60°	135°	150°	240°	315°
$\theta$ (radians)							
sine							
cosine							
tangent							

- 4 a** Use your calculator to evaluate:
- |                           |                           |                             |                              |
|---------------------------|---------------------------|-----------------------------|------------------------------|
| <b>i</b> $\sin 100^\circ$ | <b>ii</b> $\sin 80^\circ$ | <b>iii</b> $\sin 120^\circ$ | <b>iv</b> $\sin 60^\circ$    |
| <b>v</b> $\sin 150^\circ$ | <b>vi</b> $\sin 30^\circ$ | <b>vii</b> $\sin 45^\circ$  | <b>viii</b> $\sin 135^\circ$ |

- b** Use the results from **a** to copy and complete:

$$\sin(180^\circ - \theta) = \dots$$

- c** Justify your answer using the diagram alongside:



- d** Find the obtuse angle with the same sine as:

<b>i</b> $45^\circ$	<b>ii</b> $51^\circ$	<b>iii</b> $\frac{\pi}{3}$	<b>iv</b> $\frac{\pi}{6}$
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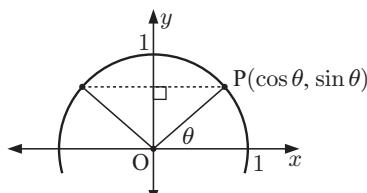
- 5 a** Use your calculator to evaluate:

<b>i</b> $\cos 70^\circ$	<b>ii</b> $\cos 110^\circ$	<b>iii</b> $\cos 60^\circ$	<b>iv</b> $\cos 120^\circ$
<b>v</b> $\cos 25^\circ$	<b>vi</b> $\cos 155^\circ$	<b>vii</b> $\cos 80^\circ$	<b>viii</b> $\cos 100^\circ$

- b** Use the results from **a** to copy and complete:

$$\cos(180^\circ - \theta) = \dots$$

- c** Justify your answer using the diagram alongside:



- d** Find the obtuse angle which has the negative cosine of:

<b>i</b> $40^\circ$	<b>ii</b> $19^\circ$	<b>iii</b> $\frac{\pi}{5}$	<b>iv</b> $\frac{2\pi}{5}$
---------------------	----------------------	----------------------------	----------------------------

- 6** Without using your calculator, find:

- |   |  |
|---|--|
| <b>a</b> $\sin 137^\circ$ if $\sin 43^\circ \approx 0.6820$ | <b>b</b> $\sin 59^\circ$ if $\sin 121^\circ \approx 0.8572$  |
| <b>c</b> $\cos 143^\circ$ if $\cos 37^\circ \approx 0.7986$ | <b>d</b> $\cos 24^\circ$ if $\cos 156^\circ \approx -0.9135$ |
| <b>e</b> $\sin 115^\circ$ if $\sin 65^\circ \approx 0.9063$ | <b>f</b> $\cos 132^\circ$ if $\cos 48^\circ \approx 0.6691$  |

- 7 a** Copy and complete:

Quadrant	Degree measure	Radian measure	$\cos \theta$	$\sin \theta$	$\tan \theta$
1	$0^\circ < \theta < 90^\circ$	$0 < \theta < \frac{\pi}{2}$	positive	positive	
2					
3					
4					

**b** In which quadrants are the following true?

**i**  $\cos \theta$  is positive.

**ii**  $\cos \theta$  is negative.

**iii**  $\cos \theta$  and  $\sin \theta$  are both negative.

**iv**  $\cos \theta$  is negative and  $\sin \theta$  is positive.

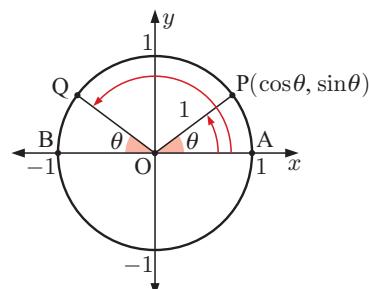
**8 a** If  $\widehat{AOP} = \widehat{BOQ} = \theta$ , what is the measure of  $\widehat{AOQ}$ ?

**b** Copy and complete:

[OQ] is a reflection of [OP] in the .....

and so Q has coordinates .....

**c** What trigonometric formulae can be deduced from **a** and **b**?



**9 a** Copy and complete:

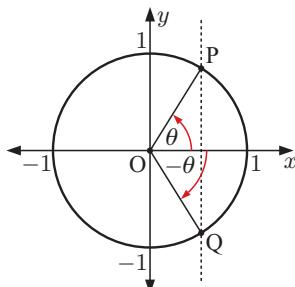
$\theta^c$	$\sin \theta$	$\sin(-\theta)$	$\cos \theta$	$\cos(-\theta)$
0.75				
1.772				
3.414				
6.25				
-1.17				

**b** What trigonometric formulae can be deduced from your results in **a**?

**c**

The coordinates of P in the figure are  $(\cos \theta, \sin \theta)$ .

- i** By finding the coordinates of Q in terms of  $\theta$  in *two different ways*, prove your formulae in **b**.
- ii** Hence explain why  $\cos(2\pi - \theta) = \cos \theta$ .



## D

## APPLICATIONS OF THE UNIT CIRCLE

The identity  $\cos^2 \theta + \sin^2 \theta = 1$  is essential for finding trigonometric ratios.

### Example 7

### Self Tutor

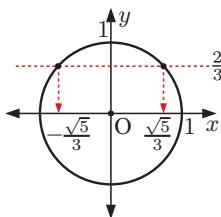
Find the possible values of  $\cos \theta$  for  $\sin \theta = \frac{2}{3}$ . Illustrate your answers.

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \cos^2 \theta + \left(\frac{2}{3}\right)^2 = 1$$

$$\therefore \cos^2 \theta = \frac{5}{9}$$

$$\therefore \cos \theta = \pm \frac{\sqrt{5}}{3}$$



**EXERCISE 8D.1**

**1** Find the possible values of  $\cos \theta$  for:

**a**  $\sin \theta = \frac{1}{2}$

**b**  $\sin \theta = -\frac{1}{3}$

**c**  $\sin \theta = 0$

**d**  $\sin \theta = -1$

**2** Find the possible values of  $\sin \theta$  for:

**a**  $\cos \theta = \frac{4}{5}$

**b**  $\cos \theta = -\frac{3}{4}$

**c**  $\cos \theta = 1$

**d**  $\cos \theta = 0$

**Example 8****Self Tutor**

If  $\sin \theta = -\frac{3}{4}$  and  $\pi < \theta < \frac{3\pi}{2}$ , find  $\cos \theta$  and  $\tan \theta$ . Give exact answers.

Now  $\cos^2 \theta + \sin^2 \theta = 1$

$\therefore \cos^2 \theta + \frac{9}{16} = 1$

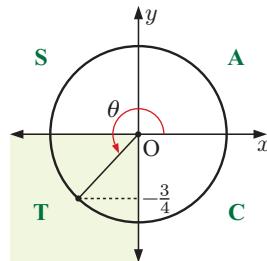
$\therefore \cos^2 \theta = \frac{7}{16}$

$\therefore \cos \theta = \pm \frac{\sqrt{7}}{4}$

But  $\pi < \theta < \frac{3\pi}{2}$ , so  $\theta$  is a quadrant 3 angle.

$\therefore \cos \theta$  is negative.

$$\therefore \cos \theta = -\frac{\sqrt{7}}{4} \text{ and } \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{-\frac{3}{4}}{-\frac{\sqrt{7}}{4}} = \frac{3}{\sqrt{7}}$$



**3** Find the exact value of:

**a**  $\sin \theta$  if  $\cos \theta = \frac{2}{3}$  and  $0 < \theta < \frac{\pi}{2}$

**b**  $\cos \theta$  if  $\sin \theta = \frac{2}{5}$  and  $\frac{\pi}{2} < \theta < \pi$

**c**  $\cos \theta$  if  $\sin \theta = -\frac{3}{5}$  and  $\frac{3\pi}{2} < \theta < 2\pi$

**d**  $\sin \theta$  if  $\cos \theta = -\frac{5}{13}$  and  $\pi < \theta < \frac{3\pi}{2}$ .

**4** Find the exact value of  $\tan \theta$  given that:

**a**  $\sin \theta = \frac{1}{3}$  and  $\frac{\pi}{2} < \theta < \pi$

**b**  $\cos \theta = \frac{1}{5}$  and  $\frac{3\pi}{2} < \theta < 2\pi$

**c**  $\sin \theta = -\frac{1}{\sqrt{3}}$  and  $\pi < \theta < \frac{3\pi}{2}$

**d**  $\cos \theta = -\frac{3}{4}$  and  $\frac{\pi}{2} < \theta < \pi$ .

**Example 9****Self Tutor**

If  $\tan \theta = -2$  and  $\frac{3\pi}{2} < \theta < 2\pi$ , find  $\sin \theta$  and  $\cos \theta$ . Give exact answers.

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = -2$$

$$\therefore \sin \theta = -2 \cos \theta$$

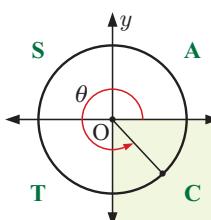
Now  $\sin^2 \theta + \cos^2 \theta = 1$

$$\therefore (-2 \cos \theta)^2 + \cos^2 \theta = 1$$

$$\therefore 4 \cos^2 \theta + \cos^2 \theta = 1$$

$$\therefore 5 \cos^2 \theta = 1$$

$$\therefore \cos \theta = \pm \frac{1}{\sqrt{5}}$$



But  $\frac{3\pi}{2} < \theta < 2\pi$ , so  $\theta$  is a quadrant 4 angle.

$\therefore \cos \theta$  is positive and  $\sin \theta$  is negative.

$$\therefore \cos \theta = \frac{1}{\sqrt{5}} \text{ and } \sin \theta = -\frac{2}{\sqrt{5}}$$

**5** Find exact values for  $\sin x$  and  $\cos x$  given that:

a  $\tan x = \frac{2}{3}$  and  $0 < x < \frac{\pi}{2}$

b  $\tan x = -\frac{4}{3}$  and  $\frac{\pi}{2} < x < \pi$

c  $\tan x = \frac{\sqrt{5}}{3}$  and  $\pi < x < \frac{3\pi}{2}$

d  $\tan x = -\frac{12}{5}$  and  $\frac{3\pi}{2} < x < 2\pi$

**6** Suppose  $\tan \theta = k$  where  $k$  is a constant and  $\pi < \theta < \frac{3\pi}{2}$ . Write expressions for  $\sin \theta$  and  $\cos \theta$  in terms of  $k$ .

## FINDING ANGLES WITH PARTICULAR TRIGONOMETRIC RATIOS

From Exercise 8C you should have discovered that:

For  $\theta$  in radians:

- $\sin(\pi - \theta) = \sin \theta$
- $\cos(\pi - \theta) = -\cos \theta$
- $\cos(2\pi - \theta) = \cos \theta$

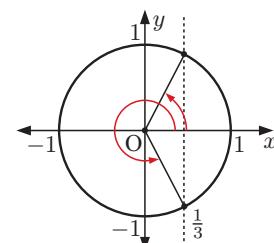
We need results such as these, and also the periodicity of the trigonometric ratios, to find angles which have a particular sine, cosine, or tangent.

### Example 10



Find the two angles  $\theta$  on the unit circle, with  $0 \leq \theta \leq 2\pi$ , such that:

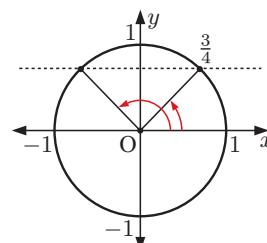
a  $\cos \theta = \frac{1}{3}$



$$\therefore \theta \approx 1.23 \text{ or } 2\pi - 1.23$$

$$\therefore \theta \approx 1.23 \text{ or } 5.05$$

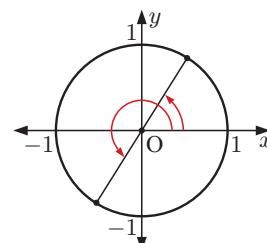
b  $\sin \theta = \frac{3}{4}$



$$\therefore \theta \approx 0.848 \text{ or } \pi - 0.848$$

$$\therefore \theta \approx 0.848 \text{ or } 2.29$$

c  $\tan \theta = 2$



$$\therefore \theta \approx 1.11 \text{ or } \pi + 1.11$$

$$\therefore \theta \approx 1.11 \text{ or } 4.25$$

If  $\cos \theta$ ,  $\sin \theta$ , or  $\tan \theta$  is positive, your calculator will give  $\theta$  in the domain  $0 < \theta < \frac{\pi}{2}$ .



## EXERCISE 8D.2

**1** Find two angles  $\theta$  on the unit circle, with  $0 \leq \theta \leq 2\pi$ , such that:

a  $\tan \theta = 4$

b  $\cos \theta = 0.83$

c  $\sin \theta = \frac{3}{5}$

d  $\cos \theta = 0$

e  $\tan \theta = 1.2$

f  $\cos \theta = 0.7816$

g  $\sin \theta = \frac{1}{11}$

h  $\tan \theta = 20.2$

i  $\sin \theta = \frac{39}{40}$

**Example 11****Self Tutor**

Find two angles  $\theta$  on the unit circle, with  $0 \leq \theta \leq 2\pi$ , such that:

a  $\sin \theta = -0.4$

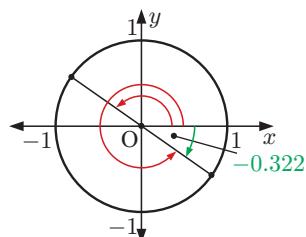
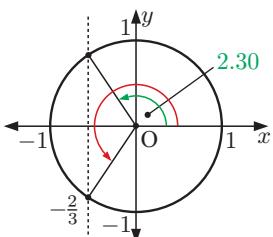
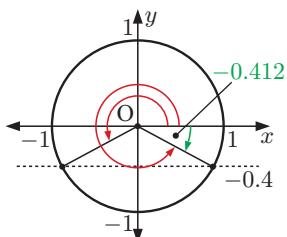
b  $\cos \theta = -\frac{2}{3}$

c  $\tan \theta = -\frac{1}{3}$

a  $\sin^{-1}(-0.4) \approx -0.412$

b  $\cos^{-1}(-\frac{2}{3}) \approx 2.30$

c  $\tan^{-1}(-\frac{1}{3}) \approx -0.322$



But  $0 \leq \theta \leq 2\pi$

$$\therefore \theta \approx \pi + 0.412 \text{ or}$$

$$2\pi - 0.412$$

$$\therefore \theta \approx 3.55 \text{ or } 5.87$$

But  $0 \leq \theta \leq 2\pi$

$$\therefore \theta \approx 2.30 \text{ or}$$

$$2\pi - 2.30$$

$$\therefore \theta \approx 2.30 \text{ or } 3.98$$

But  $0 \leq \theta \leq 2\pi$

$$\therefore \theta \approx \pi - 0.322 \text{ or}$$

$$2\pi - 0.322$$

$$\therefore \theta \approx 2.82 \text{ or } 5.96$$

If  $\sin \theta$  or  $\tan \theta$  is negative, your calculator will give  $\theta$  in the domain  $-\frac{\pi}{2} < \theta < 0$ .



The green arrow shows the angle that your calculator gives.



**2** Find two angles  $\theta$  on the unit circle, with  $0 \leq \theta \leq 2\pi$ , such that:

a  $\cos \theta = -\frac{1}{4}$

b  $\sin \theta = 0$

c  $\tan \theta = -3.1$

d  $\sin \theta = -0.421$

e  $\tan \theta = -6.67$

f  $\cos \theta = -\frac{2}{17}$

g  $\tan \theta = -\sqrt{5}$

h  $\cos \theta = -\frac{1}{\sqrt{3}}$

i  $\sin \theta = -\frac{\sqrt{2}}{\sqrt{5}}$

**Discovery 2****Parametric equations**

Usually we write functions in the form  $y = f(x)$ .

For example:  $y = 3x + 7$ ,  $y = x^2 - 6x + 8$ ,  $y = \sin x$

However, sometimes it is useful to express **both**  $x$  and  $y$  in terms of another variable  $t$ , called the **parameter**. In this case we say we have **parametric equations**.

**What to do:**

- 1 a Use the graphing package to plot  $\{(x, y) : x = \cos t, y = \sin t, 0^\circ \leq t \leq 360^\circ\}$ .  
Use the same scale on both axes.

PARAMETRIC PLOTTER



The use of parametric equations is not required for the syllabus.



- b** Describe the resulting graph. Is it the graph of a function?  
**c** Evaluate  $x^2 + y^2$ . Hence determine the equation of this graph in terms of  $x$  and  $y$  only.

**2** Use the graphing package to plot:

- a**  $\{(x, y) : x = 2 \cos t, y = \sin(2t), 0^\circ \leq t \leq 360^\circ\}$   
**b**  $\{(x, y) : x = 2 \cos t, y = 2 \sin(3t), 0^\circ \leq t \leq 360^\circ\}$   
**c**  $\{(x, y) : x = 2 \cos t, y = \cos t - \sin t, 0^\circ \leq t \leq 360^\circ\}$   
**d**  $\{(x, y) : x = \cos^2 t + \sin 2t, y = \cos t, 0^\circ \leq t \leq 360^\circ\}$   
**e**  $\{(x, y) : x = \cos^3 t, y = \sin t, 0^\circ \leq t \leq 360^\circ\}$

**E****MULTIPLES OF  $\frac{\pi}{6}$  AND  $\frac{\pi}{4}$** 

Angles which are multiples of  $\frac{\pi}{6}$  and  $\frac{\pi}{4}$  occur frequently, so it is important for us to write their trigonometric ratios exactly.

**MULTIPLES OF  $\frac{\pi}{4}$  OR  $45^\circ$** 

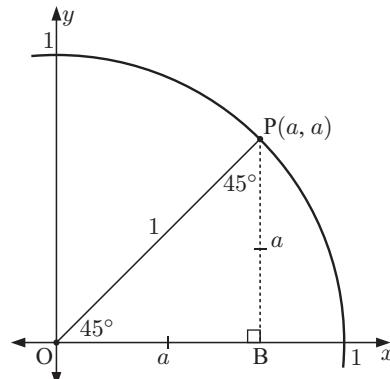
Triangle OBP is isosceles as angle OPB also measures  $45^\circ$ .

Letting  $OB = BP = a$ ,

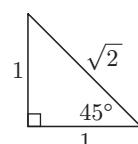
$$\begin{aligned} a^2 + a^2 &= 1^2 \quad \{\text{Pythagoras}\} \\ \therefore 2a^2 &= 1 \\ \therefore a^2 &= \frac{1}{2} \\ \therefore a &= \frac{1}{\sqrt{2}} \quad \{\text{as } a > 0\} \end{aligned}$$

So, P is  $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$  where  $\frac{1}{\sqrt{2}} \approx 0.707$ .

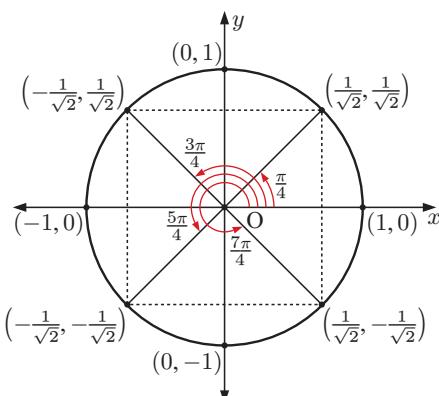
$$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$



You should remember these values. If you forget, draw a right angled isosceles triangle with equal sides of length 1.



For multiples of  $\frac{\pi}{4}$ , we have:



## MULTIPLES OF $\frac{\pi}{6}$ OR $30^\circ$

Since  $OA = OP$ , triangle OAP is isosceles.

The remaining angles are therefore also  $60^\circ$ , and so triangle AOP is equilateral.

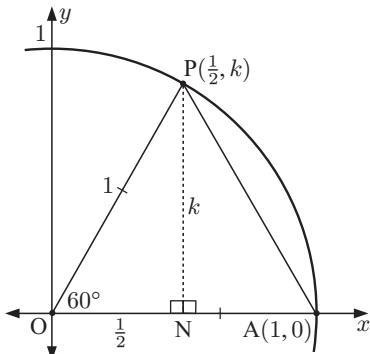
The altitude [PN] bisects base [OA], so  $ON = \frac{1}{2}$ .

$$\text{If } P \text{ is } (\frac{1}{2}, k), \text{ then } (\frac{1}{2})^2 + k^2 = 1 \quad \{\text{Pythagoras}\}$$

$$\therefore k^2 = \frac{3}{4}$$

$$\therefore k = \frac{\sqrt{3}}{2} \quad \{\text{as } k > 0\}$$

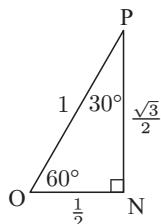
So, P is  $(\frac{1}{2}, \frac{\sqrt{3}}{2})$  where  $\frac{\sqrt{3}}{2} \approx 0.866$ .



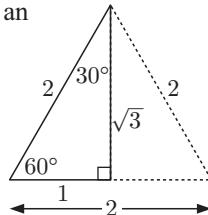
$$\cos \frac{\pi}{3} = \frac{1}{2} \quad \text{and} \quad \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Now  $\widehat{NPO} = \frac{\pi}{6} = 30^\circ$ .

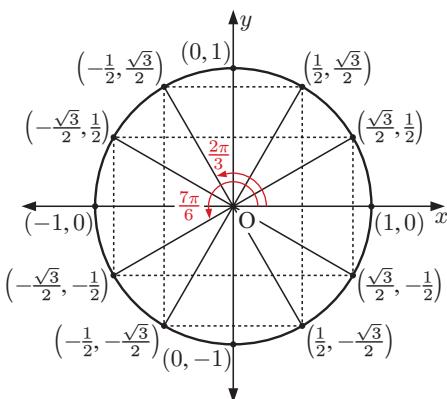
$$\text{Hence } \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin \frac{\pi}{6} = \frac{1}{2}$$



You should remember these values. If you forget, divide in two an equilateral triangle with side length 2.



For multiples of  $\frac{\pi}{6}$ , we have:



### Summary

- For **multiples of  $\frac{\pi}{2}$** , the coordinates of the points on the unit circle involve 0 and  $\pm 1$ .
- For **other multiples of  $\frac{\pi}{4}$** , the coordinates involve  $\pm \frac{1}{\sqrt{2}}$ .
- For **other multiples of  $\frac{\pi}{6}$** , the coordinates involve  $\pm \frac{1}{2}$  and  $\pm \frac{\sqrt{3}}{2}$ .
- The signs of the coordinates are determined by which quadrant the angle is in.

You should be able to use this summary to find the trigonometric ratios for angles which are multiples of  $\frac{\pi}{6}$  and  $\frac{\pi}{4}$ .

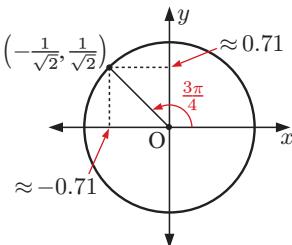
**Example 12****Self Tutor**

Find the exact values of  $\sin \alpha$ ,  $\cos \alpha$ , and  $\tan \alpha$  for:

**a**  $\alpha = \frac{3\pi}{4}$

**a**  $\frac{3\pi}{4}$  is a multiple of  $\frac{\pi}{4}$ .

The angle lies in quadrant 2, so only  $\sin \frac{3\pi}{4}$  is positive.



$$\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

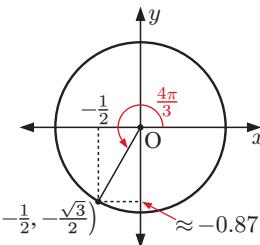
$$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\tan\left(\frac{3\pi}{4}\right) = -1$$

**b**  $\alpha = \frac{4\pi}{3}$

**b**  $\frac{4\pi}{3}$  is a multiple of  $\frac{\pi}{6}$ .

The angle lies in quadrant 3, so only  $\tan \frac{4\pi}{3}$  is positive.



$$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$$

$$\tan\left(\frac{4\pi}{3}\right) = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

**EXERCISE 8E**

1 Use a unit circle diagram to find exact values for  $\sin \theta$ ,  $\cos \theta$ , and  $\tan \theta$ , for  $\theta$  equal to:

**a**  $\frac{\pi}{4}$

**b**  $\frac{\pi}{2}$

**c**  $\frac{7\pi}{4}$

**d**  $\pi$

**e**  $-\frac{3\pi}{4}$

2 Use a unit circle diagram to find exact values for  $\sin \beta$ ,  $\cos \beta$ , and  $\tan \beta$ , for  $\beta$  equal to:

**a**  $\frac{\pi}{6}$

**b**  $\frac{2\pi}{3}$

**c**  $\frac{7\pi}{6}$

**d**  $\frac{5\pi}{3}$

**e**  $\frac{11\pi}{6}$

3 Find the exact values of:

**a**  $\cos 120^\circ$ ,  $\sin 120^\circ$ , and  $\tan 120^\circ$

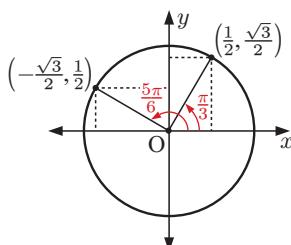
**b**  $\cos(-45^\circ)$ ,  $\sin(-45^\circ)$ , and  $\tan(-45^\circ)$

4 **a** Find the exact values of  $\cos 270^\circ$  and  $\sin 270^\circ$ .

**b** What can you say about  $\tan 270^\circ$ ?

**Example 13****Self Tutor**

Without using a calculator, show that  $8 \sin\left(\frac{\pi}{3}\right) \cos\left(\frac{5\pi}{6}\right) = -6$ .



$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\begin{aligned} \therefore 8 \sin\left(\frac{\pi}{3}\right) \cos\left(\frac{5\pi}{6}\right) &= 8\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) \\ &= 2(-3) \\ &= -6 \end{aligned}$$

- 5** Without using a calculator, evaluate:

**a**  $\sin^2 60^\circ$

**d**  $1 - \cos^2(\frac{\pi}{6})$

**g**  $\sin(\frac{3\pi}{4}) - \cos(\frac{5\pi}{4})$

**j**  $\tan^2(\frac{\pi}{3}) - 2 \sin^2(\frac{\pi}{4})$

**b**  $\sin 30^\circ \cos 60^\circ$

**e**  $\sin^2(\frac{2\pi}{3}) - 1$

**h**  $1 - 2 \sin^2(\frac{7\pi}{6})$

**k**  $2 \tan(-\frac{5\pi}{4}) - \sin(\frac{3\pi}{2})$

**c**  $4 \sin 60^\circ \cos 30^\circ$

**f**  $\cos^2(\frac{\pi}{4}) - \sin(\frac{7\pi}{6})$

**i**  $\cos^2(\frac{5\pi}{6}) - \sin^2(\frac{5\pi}{6})$

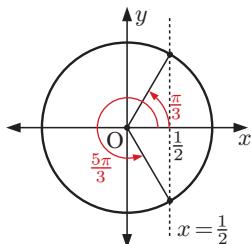
**l**  $\frac{2 \tan 150^\circ}{1 - \tan^2 150^\circ}$

Check your answers using your calculator.

### Example 14

### Self Tutor

Find all angles  $0 \leq \theta \leq 2\pi$  with a cosine of  $\frac{1}{2}$ .



Since the cosine is  $\frac{1}{2}$ , we draw the vertical line  $x = \frac{1}{2}$ .

Because  $\frac{1}{2}$  is involved, we know the required angles are multiples of  $\frac{\pi}{6}$ .

They are  $\frac{\pi}{3}$  and  $\frac{5\pi}{3}$ .

- 6** Find all angles between  $0^\circ$  and  $360^\circ$  with:

**a** a sine of  $\frac{1}{2}$

**d** a cosine of  $-\frac{1}{2}$

**b** a sine of  $\frac{\sqrt{3}}{2}$

**e** a cosine of  $-\frac{1}{\sqrt{2}}$

**c** a cosine of  $\frac{1}{\sqrt{2}}$

**f** a sine of  $-\frac{\sqrt{3}}{2}$

- 7** Find all angles between  $0$  and  $2\pi$  (inclusive) which have:

**a** a tangent of 1

**d** a tangent of 0

**b** a tangent of  $-1$

**e** a tangent of  $\frac{1}{\sqrt{3}}$

**c** a tangent of  $\sqrt{3}$

**f** a tangent of  $-\sqrt{3}$

- 8** Find all angles between  $0$  and  $4\pi$  with:

**a** a cosine of  $\frac{\sqrt{3}}{2}$

**b** a sine of  $-\frac{1}{2}$

**c** a sine of  $-1$

- 9** Find  $\theta$  if  $0 \leq \theta \leq 2\pi$  and:

**a**  $\cos \theta = \frac{1}{2}$

**e**  $\cos \theta = -\frac{1}{\sqrt{2}}$

**i**  $\tan \theta = -\frac{1}{\sqrt{3}}$

**b**  $\sin \theta = \frac{\sqrt{3}}{2}$

**f**  $\sin^2 \theta = 1$

**j**  $\tan^2 \theta = 3$

**c**  $\cos \theta = -1$

**g**  $\cos^2 \theta = 1$

**h**  $\cos^2 \theta = \frac{1}{2}$

- 10** Find *all* values of  $\theta$  for which  $\tan \theta$  is:      **a** zero      **b** undefined.

**F****RECIPROCAL TRIGONOMETRIC RATIOS**

We define the reciprocal trigonometric functions  $\text{cosec } \theta$ ,  $\sec \theta$ , and  $\cot \theta$  as:

$$\text{cosec } \theta = \frac{1}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \text{and} \quad \cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

Using these definitions we can derive the identities:

$$\tan^2 \theta + 1 = \sec^2 \theta \quad \text{and} \quad 1 + \cot^2 \theta = \text{cosec}^2 \theta$$

**Proof:** Using  $\sin^2 \theta + \cos^2 \theta = 1$ ,

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \quad \{ \text{dividing each term by } \cos^2 \theta \}$$

$$\therefore \tan^2 \theta + 1 = \sec^2 \theta$$

Also using  $\sin^2 \theta + \cos^2 \theta = 1$ ,

$$\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} \quad \{ \text{dividing each term by } \sin^2 \theta \}$$

$$\therefore 1 + \cot^2 \theta = \text{cosec}^2 \theta$$

**EXERCISE 8F**

**1** Without using a calculator, find:

- |   |   |   |                       |
|---|---|---|-----------------------|
| <b>a</b> $\text{cosec} \left( \frac{\pi}{3} \right)$  | <b>b</b> $\cot \left( \frac{2\pi}{3} \right)$ | <b>c</b> $\sec \left( \frac{5\pi}{6} \right)$ | <b>d</b> $\cot (\pi)$ |
| <b>e</b> $\text{cosec} \left( \frac{4\pi}{3} \right)$ | <b>f</b> $\sec \left( \frac{7\pi}{4} \right)$ |   |                       |

**2** Without using a calculator, find  $\text{cosec } x$ ,  $\sec x$ , and  $\cot x$  for:

**a**  $\sin x = \frac{3}{5}$ ,  $0 \leq x \leq \frac{\pi}{2}$       **b**  $\cos x = \frac{2}{3}$ ,  $\frac{3\pi}{2} < x < 2\pi$

**3** Find the other five trigonometric ratios if:

- |   |  |
|---|--|
| <b>a</b> $\cos \theta = \frac{3}{4}$ and $\frac{3\pi}{2} < \theta < 2\pi$ | <b>b</b> $\sin x = -\frac{2}{3}$ and $\pi < x < \frac{3\pi}{2}$          |
| <b>c</b> $\sec x = 2\frac{1}{2}$ and $0 < x < \frac{\pi}{2}$              | <b>d</b> $\text{cosec} \theta = 2$ and $\frac{\pi}{2} < \theta < \pi$    |
| <b>e</b> $\tan \beta = \frac{1}{2}$ and $\pi < \beta < \frac{3\pi}{2}$    | <b>f</b> $\cot \theta = \frac{4}{3}$ and $\pi < \theta < \frac{3\pi}{2}$ |

**4** Find all values of  $\theta$  for which:

- |  |                                      |
|--|--------------------------------------|
| <b>a</b> $\text{cosec } \theta$ is undefined | <b>b</b> $\sec \theta$ is undefined  |
| <b>c</b> $\cot \theta$ is zero               | <b>d</b> $\cot \theta$ is undefined. |

**Review set 8A**

**1** Convert these to radians in terms of  $\pi$ :

**a**  $120^\circ$       **b**  $225^\circ$       **c**  $150^\circ$       **d**  $540^\circ$

**2** Find the acute angles that would have the same:

**a** sine as  $\frac{2\pi}{3}$       **b** sine as  $165^\circ$       **c** cosine as  $276^\circ$ .

**3** Find:

**a**  $\sin 159^\circ$  if  $\sin 21^\circ \approx 0.358$

**c**  $\cos 75^\circ$  if  $\cos 105^\circ \approx -0.259$

**b**  $\cos 92^\circ$  if  $\cos 88^\circ \approx 0.035$

**d**  $\sin(-133^\circ)$  if  $\sin 47^\circ \approx 0.731$

**4** Determine the area of a sector of angle  $\frac{5\pi}{12}$  and radius 13 cm.

**5** Use the unit circle to find  $\theta$  such that  $\cos \theta = -\sin \theta$ ,  $0 \leq \theta \leq 2\pi$ .

**6** Find exact values for  $\sin \theta$ ,  $\cos \theta$ , and  $\tan \theta$  for  $\theta$  equal to:

**a**  $360^\circ$

**b**  $\frac{2\pi}{3}$

**c**  $-\pi$

**d**  $\frac{8\pi}{3}$

**7** If  $\cos \theta = \frac{3}{4}$  find the possible values of  $\sin \theta$ .

**8** Evaluate:

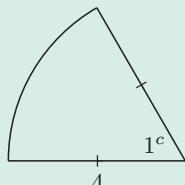
**a**  $2 \sin\left(\frac{\pi}{3}\right) \cos\left(\frac{\pi}{3}\right)$

**b**  $\tan^2\left(\frac{\pi}{4}\right) - 1$

**c**  $\cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right)$

**9** Given  $\tan x = -\frac{3}{2}$  and  $\frac{3\pi}{2} < x < 2\pi$ , find: **a**  $\cos x$     **b**  $\sin x$ .

**10**



Find the perimeter and area of the sector.

**11** Suppose  $\cos \theta = \frac{\sqrt{11}}{\sqrt{17}}$  and  $\theta$  is acute. Find the exact value of  $\tan \theta$ .

**12** Find all angles between  $0^\circ$  and  $360^\circ$  which have:

**a** a cosine of  $-\frac{\sqrt{3}}{2}$

**b** a secant of  $\sqrt{2}$

**c** a cotangent of  $-\frac{1}{\sqrt{3}}$

**13** Find  $\theta$  for  $0 \leq \theta \leq 2\pi$  if:

**a**  $\cos \theta = -1$

**b**  $\sin^2 \theta = \frac{3}{4}$

**14** If  $\sin x = -\frac{1}{4}$  and  $\pi < x < \frac{3\pi}{2}$ , find the other five trigonometric ratios exactly.

## Review set 8B

**1** Convert these radian measurements to degrees:

**a**  $\frac{2\pi}{5}$

**b**  $\frac{5\pi}{4}$

**c**  $\frac{7\pi}{9}$

**d**  $\frac{11\pi}{6}$

**2** Illustrate the regions where  $\sin \theta$  and  $\cos \theta$  have the same sign.

**3** Use a unit circle diagram to find:

**a**  $\cos\left(\frac{3\pi}{2}\right)$  and  $\sin\left(\frac{3\pi}{2}\right)$

**b**  $\cos\left(-\frac{\pi}{2}\right)$  and  $\sin\left(-\frac{\pi}{2}\right)$

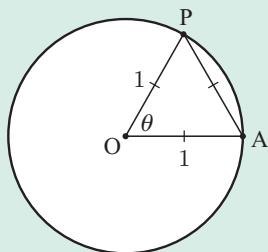
**4** Suppose  $m = \sin p$ , where  $p$  is acute. Write an expression in terms of  $m$  for:

**a**  $\sin(\pi - p)$

**b**  $\sin(p + 2\pi)$

**c**  $\cos p$

**d**  $\tan p$

**5****a** State the value of  $\theta$  in:**i** degrees**ii** radians.**b** State the arc length AP.**c** State the area of the minor sector OAP.

**6** Show that  $\cos\left(\frac{3\pi}{4}\right) - \sin\left(\frac{3\pi}{4}\right) = -\sqrt{2}$ .

**7** If  $\cos\theta = -\frac{2}{5}$ ,  $\frac{\pi}{2} < \theta < \pi$  find the other *five* trigonometric ratios exactly.

**8** Without using a calculator, evaluate:

**a**  $\tan^2 60^\circ - \sin^2 45^\circ$       **b**  $\cos^2\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right)$       **c**  $\cos\left(\frac{5\pi}{3}\right) - \tan\left(\frac{5\pi}{4}\right)$

**9** Find two angles on the unit circle with  $0 \leq \theta \leq 2\pi$ , such that:

**a**  $\cos\theta = \frac{2}{3}$       **b**  $\sin\theta = -\frac{1}{4}$       **c**  $\tan\theta = 3$

**10** Find the perimeter and area of a sector of radius 11 cm and angle  $63^\circ$ .

**11** Find the radius and area of a sector of perimeter 36 cm with an angle of  $\frac{2\pi}{3}$ .

**12** Simplify:

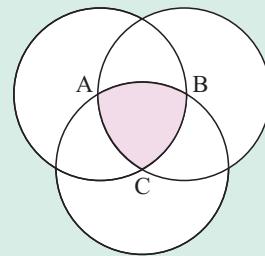
**a**  $\sin(\pi - \theta) - \sin\theta$       **b**  $\cos\theta \tan\theta$

**13** If  $\sec\alpha = -3\frac{1}{3}$  and  $0 < \alpha < \pi$ , find the other *five* trigonometric ratios exactly.

**14** Three circles with radius  $r$  are drawn as shown, each with its centre on the circumference of the other two circles. A, B, and C are the centres of the three circles.

Prove that an expression for the area of the shaded

region is  $A = \frac{r^2}{2}(\pi - \sqrt{3})$ .





# 9

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# Trigonometric functions

## Contents:

- A** Periodic behaviour
- B** The sine function
- C** The cosine function
- D** The tangent function
- E** Trigonometric equations
- F** Trigonometric relationships
- G** Trigonometric equations in quadratic form

## Opening problem

A Ferris wheel rotates at a constant speed. The wheel's radius is 10 m and the bottom of the wheel is 2 m above ground level. From a point in front of the wheel, Andrew is watching a green light on the perimeter of the wheel. Andrew notices that the green light moves in a circle. He estimates how high the light is above ground level at two second intervals, and draws a scatter diagram of his results.

### Things to think about:

- a** What will Andrew's scatter diagram look like?
- b** What function can be used to model the data?
- c** How could this function be used to find:
  - i** the light's position at any point in time
  - ii** the times when the light is at its maximum and minimum heights?
- d** What part of the function indicates the time for one full revolution of the wheel?

Click on the icon to visit a simulation of the Ferris wheel. You will be able to view the light from:

- in front of the wheel
- a side-on position
- above the wheel.



You can then observe the graph of the green light's position as the wheel rotates at a constant rate.

## A PERIODIC BEHAVIOUR

**Periodic phenomena** occur all the time in the physical world. Their behaviour repeats again and again over time.

We see periodic behaviour in:

- seasonal variations in our climate
- variations in average maximum and minimum monthly temperatures
- the number of daylight hours at a particular location
- tidal variations in the depth of water in a harbour
- the phases of the moon
- animal populations.

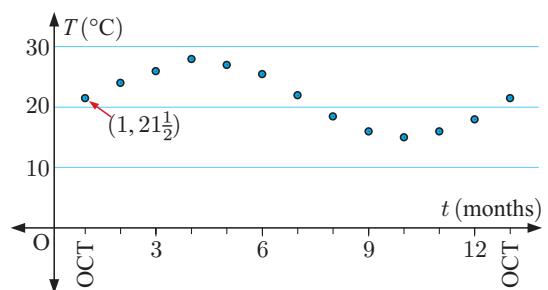
In this chapter we will see how trigonometric functions can be used to model periodic phenomena.

### OBSERVING PERIODIC BEHAVIOUR

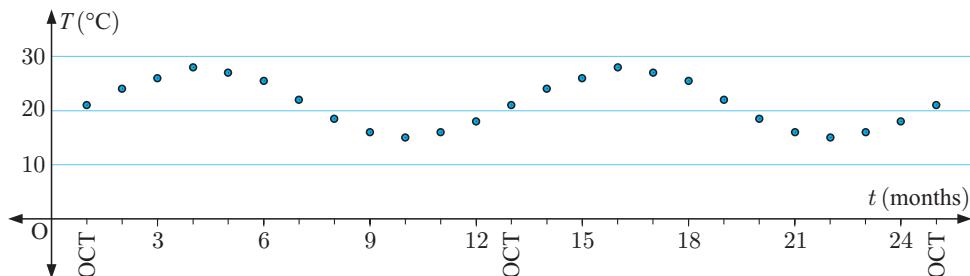
The table below shows the mean monthly maximum temperature for Cape Town, South Africa.

Month	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep
Temperature $T$ ( $^{\circ}\text{C}$ )	$21\frac{1}{2}$	24	26	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18

On the scatter diagram alongside we plot the temperature  $T$  on the vertical axis. We assign October as  $t = 1$  month, November as  $t = 2$  months, and so on for the rest of the year.

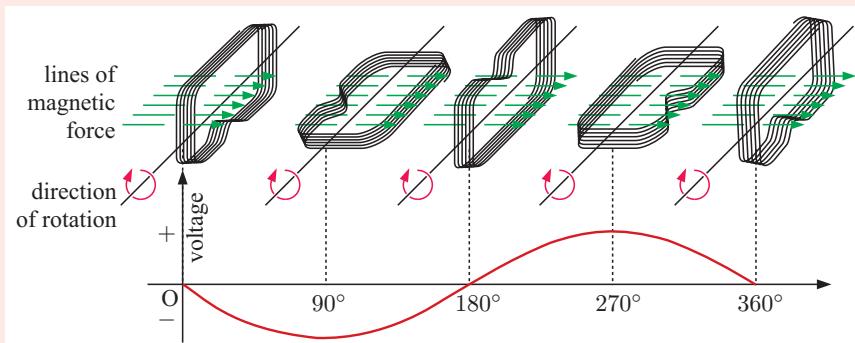


The cycle will approximately repeat itself for each subsequent 12 month period. By the end of the chapter we will be able to establish a **periodic function** which approximately fits this set of points.



Graphs with this basic shape, where the cycle is repeated over and over, are called **sine waves**.

### Historical note



In 1831 **Michael Faraday** discovered that an electric current was generated by rotating a coil of wire in a magnetic field. The electric current produced showed a voltage which varied between positive and negative values as the coil rotated through  $360^\circ$ .

## GATHERING PERIODIC DATA

Data on a number of periodic phenomena can be found online or in other publications. For example:

- Maximum and minimum monthly temperatures can be found at [www.weatherbase.com](http://www.weatherbase.com)
- Tidal details can be obtained from daily newspapers or internet sites such as <http://tidesandcurrents.noaa.gov> or <http://www.bom.gov.au/oceanography>

## TERMINOLOGY USED TO DESCRIBE PERIODICITY

A **periodic function** is one which repeats itself over and over in a horizontal direction, in intervals of the same length.

The **period** of a periodic function is the length of one repetition or cycle.

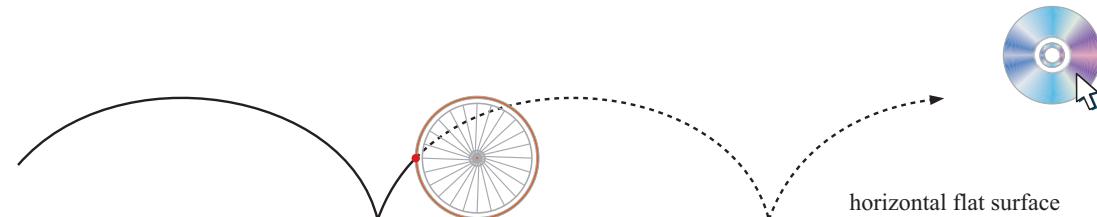
$f(x)$  is a periodic function with period  $p \Leftrightarrow f(x + p) = f(x)$  for all  $x$ , and  $p$  is the smallest positive value for this to be true.

$\Leftrightarrow$  means  
“if and only if”.



A **cycloid** is an example of a periodic function. It is the curve traced out by a point on a circle as the circle rolls across a flat surface in a straight line.

DEMO



Use a **graphing package** to examine the function  $f(x) = x - [x]$  where  $[x]$  is “the largest integer less than or equal to  $x$ ”.

Is  $f(x)$  periodic? What is its period?

GRAPHING PACKAGE



## WAVES

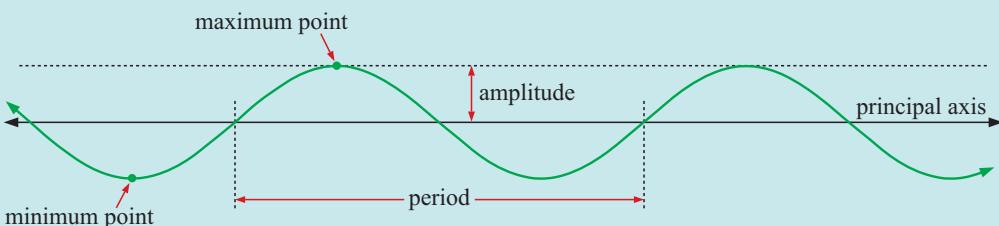
In this course we are mainly concerned with periodic phenomena which show a wave pattern.

The wave oscillates about a horizontal line called the **principal axis** or **mean line** which has equation  $y = \frac{\max + \min}{2}$ .

A **maximum point** occurs at the top of a crest, and a **minimum point** at the bottom of a trough.

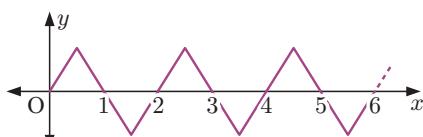
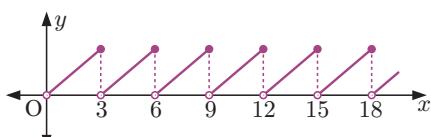
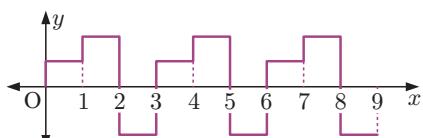
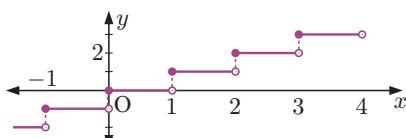
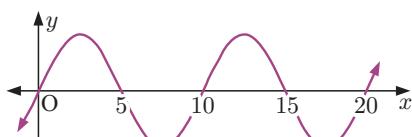
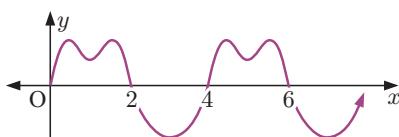
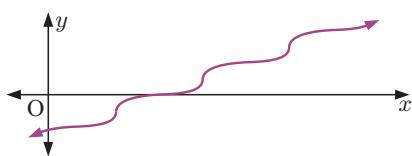
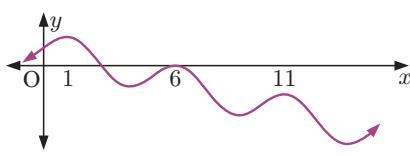
The **amplitude** is the distance between a maximum (or minimum) point and the principal axis.

$$\text{amplitude} = \frac{\max - \min}{2}$$



**EXERCISE 9A**

- 1** Which of these graphs show periodic behaviour?

**a****b****c****d****e****f****g****h**

- 2** The table below shows the height above the ground of a point on a bicycle wheel as it is rolled along a flat surface.

Distance travelled (cm)	0	20	40	60	80	100	120	140	160	180	200
Height above ground (cm)	0	6	23	42	57	64	59	43	23	7	1

Distance travelled (cm)	220	240	260	280	300	320	340	360	380	400
Height above ground (cm)	5	27	40	55	63	60	44	24	9	3

- a** Plot the graph of height against distance.

- b** Is it reasonable to fit a curve to this data, or should we leave it as discrete points?

- c** Is the data periodic? If so, estimate:

**i** the equation of the principal axis

**ii** the maximum value

**iii** the period

**iv** the amplitude.

- 3** Draw a scatter diagram for each set of data below. Is there evidence to suggest the data is periodic?

**a**

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	1	1.4	1	0	-1	-1.4	-1	0	1	1.4	1	0

**b**

x	0	2	3	4	5	6	7	8	9	10	12
y	0	4.7	3.4	1.7	2.1	5.2	8.9	10.9	10.2	8.4	10.4

## B THE SINE FUNCTION

In previous studies of trigonometry we have only considered static situations where an angle is fixed. However, when an object moves around a circle, the situation is dynamic. The angle  $\theta$  between the radius [OP] and the positive  $x$ -axis continually changes with time.

Consider again the **Opening Problem** in which a Ferris wheel of radius 10 m revolves at constant speed. We let P represent the green light on the wheel.

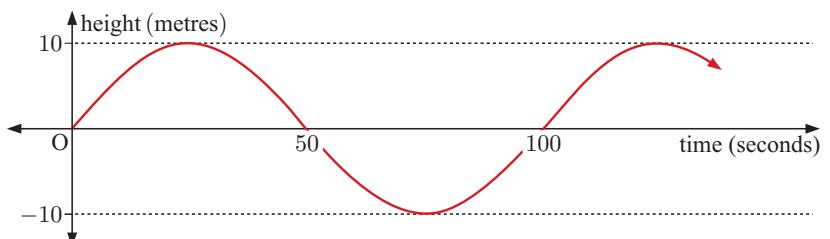
The height of P relative to the  $x$ -axis can be determined using right angled triangle trigonometry:

$$\sin \theta = \frac{h}{10}, \text{ so } h = 10 \sin \theta.$$

As time goes by,  $\theta$  changes and so does  $h$ .

So, we can write  $h$  as a function of  $\theta$ , or alternatively we can write  $h$  as a function of time  $t$ .

For example, suppose the Ferris wheel observed by Andrew takes 100 seconds for a full revolution. The graph below shows the height of the light above or below the principal axis against the time in seconds.



We observe that the amplitude is 10 metres and the period is 100 seconds.

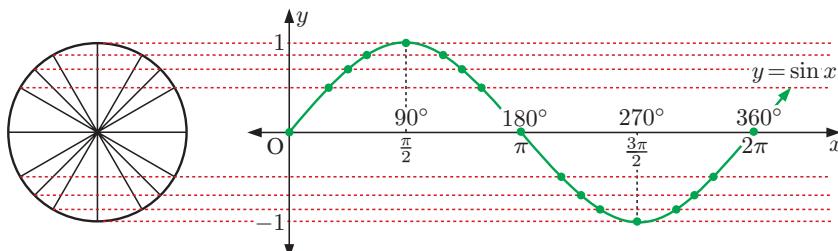
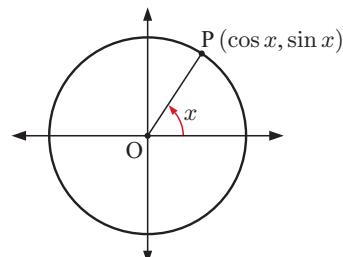
### THE BASIC SINE CURVE $y = \sin x$

Suppose point P moves around the unit circle so the angle [OP] makes with the positive horizontal axis is  $x$ . In this case P has coordinates  $(\cos x, \sin x)$ .

If we project the values of  $\sin x$  from the unit circle to a set of axes alongside, we can obtain the graph of  $y = \sin x$ .

Note carefully that  $x$  on the unit circle diagram is an *angle*, and becomes the horizontal coordinate of the sine function.

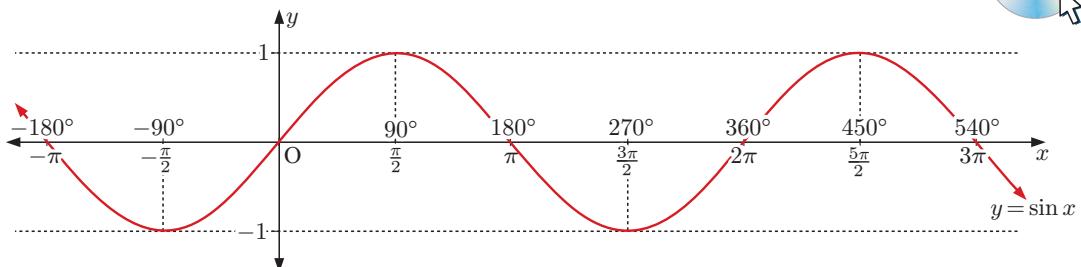
Unless indicated otherwise, you should assume that  $x$  is measured in radians. Degrees are only included on this graph for the sake of completeness.



Click on the icon to generate the sine function for yourself.

You should observe that the sine function can be continued beyond  $0 \leq x \leq 2\pi$  in either direction.

### SINE FUNCTION



The unit circle repeats itself after one full revolution, so the **period** of  $y = \sin x$  is  $2\pi$ .

The *maximum* value is 1 and the *minimum* is -1, as  $-1 \leq y \leq 1$  on the unit circle.

The **amplitude** of  $y = \sin x$  is 1.

## TRANSFORMATIONS OF THE SINE CURVE

In the **Discoveries** that follow, we will consider different transformations of the sine curve  $y = \sin x$ . We will hence be able to generate the curve for the general sine function  $y = a \sin bx + c$ ,  $a > 0$ ,  $b > 0$ .

### Discovery 1

### The family $y = a \sin x$ , $a > 0$

Click on the icon to explore the family  $y = a \sin x$ ,  $a > 0$ .

#### What to do:

- 1 Use the slider to vary the value of  $a$ . Observe the changes to the graph of the function.
- 2 Use the software to help complete the table:

$a$	Function	Maximum	Minimum	Period	Amplitude
1	$y = \sin x$	1	-1	$2\pi$	1
2	$y = 2 \sin x$				
3	$y = 3 \sin x$				
0.5	$y = 0.5 \sin x$				
$a$	$y = a \sin x$				

### DYNAMIC SINE FUNCTION



*x* is measured in radians.



- 3 How does  $a$  affect the function  $y = a \sin x$ ?

### Discovery 2

### The family $y = \sin bx$ , $b > 0$

Click on the icon to explore the family  $y = \sin bx$ ,  $b > 0$ .

#### What to do:

- 1 Use the slider to vary the value of  $b$ . Observe the changes to the graph of the function.

### DYNAMIC SINE FUNCTION



- 2 Use the software to help complete the table:

$b$	Function	Maximum	Minimum	Period	Amplitude
1	$y = \sin x$	1	-1	$2\pi$	1
2	$y = \sin 2x$				
3	$y = \sin 3x$				
$\frac{1}{2}$	$y = \sin(\frac{1}{2}x)$				
$b$	$y = \sin bx$				

- 3 How does  $b$  affect the function  $y = \sin bx$ ?

### Discovery 3

### The family $y = \sin x + c$

Click on the icon to explore the family  $y = \sin x + c$ .

DYNAMIC  
SINE FUNCTION



#### What to do:

- 1 Use the slider to vary the value of  $c$ . Observe the changes to the graph of the function.
- 2 Use the software to help complete the table:

$d$	Function	Maximum	Minimum	Period	Amplitude
0	$y = \sin x$	1	-1	$2\pi$	1
3	$y = \sin x + 3$				
-2	$y = \sin x - 2$				
$d$	$y = \sin x + d$				

- 3 How does  $c$  affect the function  $y = \sin x + c$ ?

## THE GENERAL SINE FUNCTION

The **general sine function** is

$$y = a \sin bx + c \quad \text{where } a > 0, b > 0.$$

affects amplitude      affects period      affects vertical translation

The **principal axis** of the general sine function is  $y = c$ .

The **period** of the general sine function is  $\frac{2\pi}{b}$ .

The **amplitude** of the general sine function is  $a$ .

**Example 1****Self Tutor**

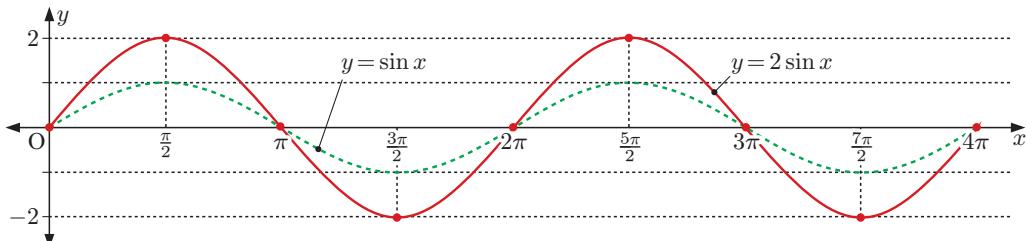
Without using technology, sketch the following graphs for  $0 \leq x \leq 4\pi$ :

**a**  $y = 2 \sin x$

**b**  $y = \sin 2x$

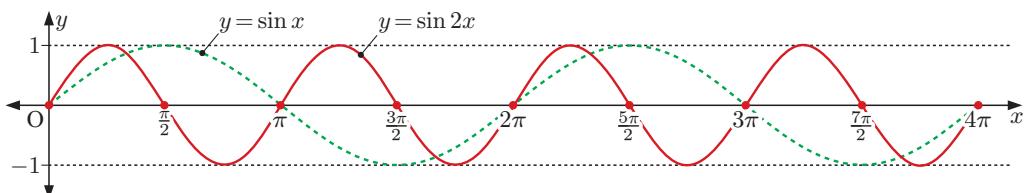
**c**  $y = \sin x - 1$

- a** The amplitude is 2 and the period is  $2\pi$ .



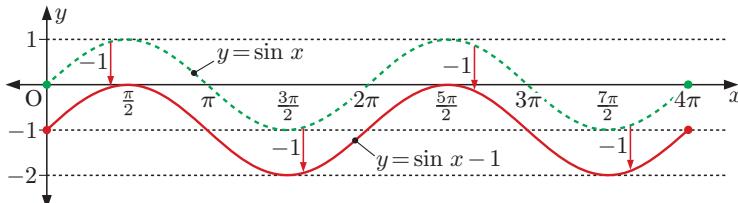
- b** The period is  $\frac{2\pi}{2} = \pi$ .

$\therefore$  the maximum values are  $\pi$  units apart.



- c** This is a vertical translation of  $y = \sin x$  downwards by 1 unit.  
The principal axis is now  $y = -1$ .

Since  $\sin 2x$  has half the period of  $\sin x$ , the first maximum is at  $\frac{\pi}{4}$  not  $\frac{\pi}{2}$ .

**EXERCISE 9B**

- 1** Without using technology, sketch the following graphs for  $0 \leq x \leq 4\pi$ :

**a**  $y = 3 \sin x$

**b**  $y = 4 \sin x$

**c**  $y = \sin 3x$

**d**  $y = \sin 4x$

**e**  $y = \sin x + 2$

**f**  $y = \sin x - 3$

Check your answers using technology.

- 2** Find the value of  $a$  given that the function  $y = a \sin x$ ,  $a > 0$ , has amplitude:

**a** 2

**b** 5

**c** 11

- 3** Find the value of  $b$  given that the function  $y = \sin bx$ ,  $b > 0$ , has period:

**a**  $\frac{2\pi}{3}$

**b**  $\frac{2\pi}{5}$

**c**  $\frac{\pi}{3}$

**d**  $\frac{\pi}{2}$

GRAPHING PACKAGE



- 4 Find the value of  $c$  given that the function  $y = \sin x + c$  has principal axis:

- a**  $y = 3$       **b**  $y = -1$       **c**  $y = 5$

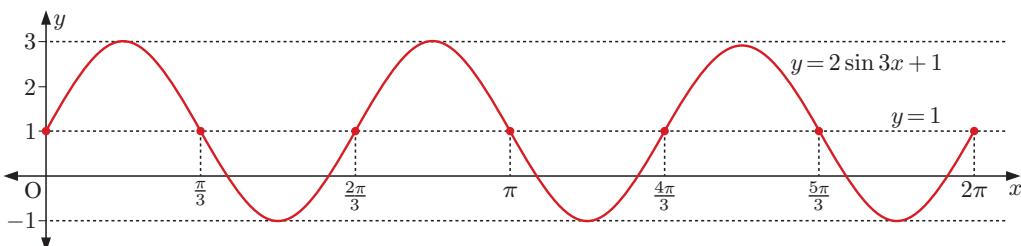
### Example 2

### Self Tutor

Without using technology, sketch  $y = 2 \sin 3x + 1$  for  $0 \leq x \leq 2\pi$ .

We start with  $y = \sin x$ . We then:

- double the amplitude to produce  $y = 2 \sin x$ , then
- divide the period by 3 to produce  $y = 2 \sin 3x$ , then
- translate the graph 1 unit upwards to produce  $y = 2 \sin 3x + 1$ , so the principal axis is now  $y = 1$ .



- 5 Without using technology, sketch the following graphs for  $0 \leq x \leq 2\pi$ :

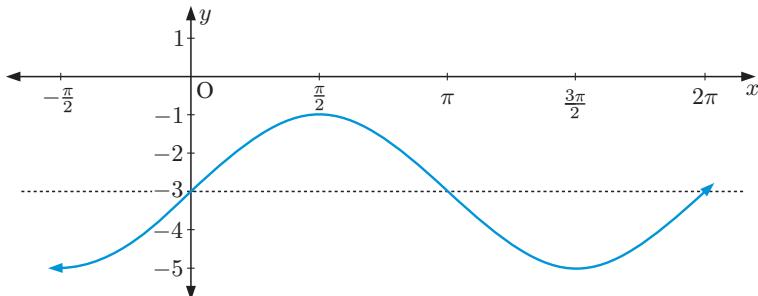
- a**  $y = 3 \sin x - 1$       **b**  $y = 2 \sin 3x$       **c**  $y = \sin 2x + 3$   
**d**  $y = 3 \sin 2x - 1$       **e**  $y = 5 \sin 2x + 3$       **f**  $y = 4 \sin 3x - 2$

Check your answers using technology.

- 6 Find  $a$ ,  $b$ , and  $c$  given that the function  $y = a \sin bx + c$ ,  $a > 0$ ,  $b > 0$ , has:

- a** amplitude 3, period  $2\pi$ , and principal axis  $y = 0$   
**b** amplitude 2, period  $\frac{2\pi}{5}$ , and principal axis  $y = 6$   
**c** amplitude 5, period  $\frac{2\pi}{3}$ , and principal axis  $y = -2$ .

- 7 Find  $m$  and  $n$  given the following graph of the function  $y = m \sin x + n$ .



- 8 On the same set of axes, sketch for  $0 \leq x \leq 2\pi$ :

- a**  $y = \sin x$  and  $y = |\sin x|$       **b**  $y = 3 \sin 2x$  and  $y = |3 \sin 2x|$

**Discovery 4****Modelling using sine functions**

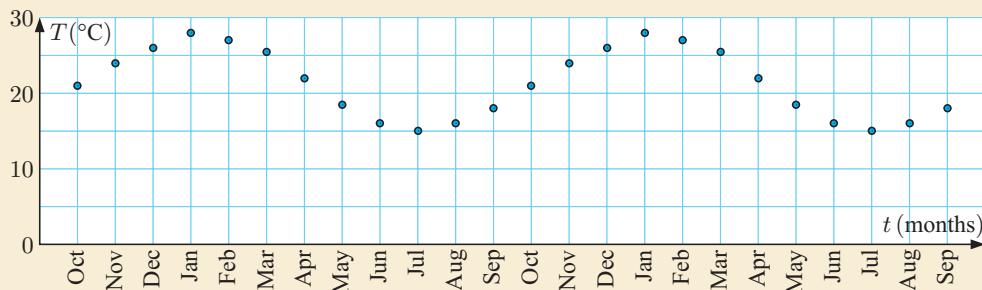
When patterns of variation can be identified and quantified using a formula or equation, predictions may be made about behaviour in the future. Examples of this include tidal movement which can be predicted many months ahead, and the date of a future full moon.

**What to do:**

- 1** Consider again the mean monthly maximum temperature for Cape Town:

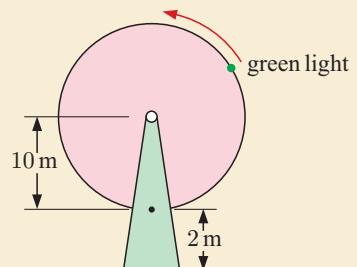
Month	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep
Temperature $T$ ( $^{\circ}\text{C}$ )	$21\frac{1}{2}$	24	26	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18

The graph over a two year period is shown below:



We attempt to model this data using the general sine function  $y = a \sin bx + c$ ,  
or in this case  $T = a \sin bt + c$ .

- a** State the period of the function. Hence show that  $b = \frac{\pi}{6}$ .
  - b** Use the amplitude to show that  $a \approx 6.5$ .
  - c** Use the principal axis to show that  $c \approx 21.5$ .
  - d** Superimpose the model  $T \approx 6.5 \sin(\frac{\pi}{6}t) + 21.5$  on the original data to confirm its accuracy.
- 2** Some of the largest tides in the world are observed in Canada's Bay of Fundy. The difference between high and low tides is 14 metres, and the average time difference between high tides is about 12.4 hours.
- Suppose the mean tide occurs at midnight.
- a** Find a sine model for the height of the tide  $H$  in terms of the time  $t$ .
  - b** Sketch the graph of the model over one period.
- 3** Revisit the **Opening Problem** on page 226.
- The wheel takes 100 seconds to complete one revolution. Find the sine model which gives the height of the light above the ground at any point in time. Assume that at time  $t = 0$ , the light is at its mean position.

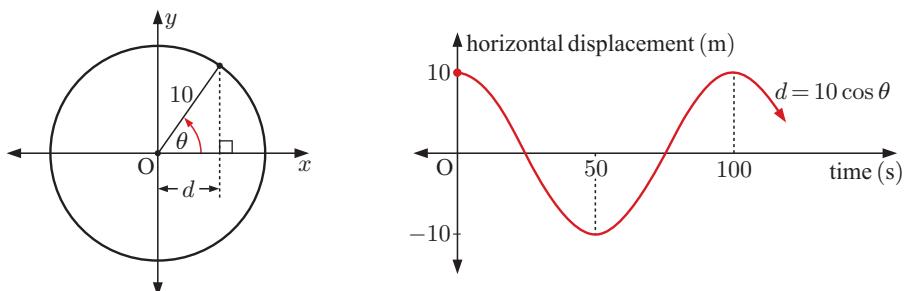


## C THE COSINE FUNCTION

We return to the Ferris wheel and now view the movement of the green light from above.

$$\text{Now } \cos \theta = \frac{d}{10} \text{ so } d = 10 \cos \theta.$$

The graph being generated over time is therefore a **cosine function**.



Use the graphing package to graph  $y = \cos x$  and  $y = \sin x$  on the same set of axes.

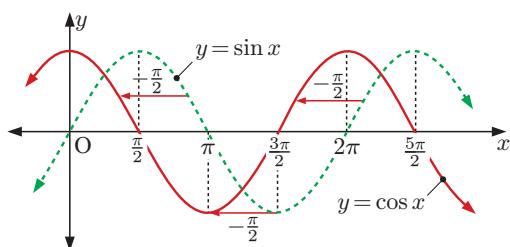
Like the sine curve  $y = \sin x$ , the cosine curve  $y = \cos x$  has a **period** of  $2\pi$ , an **amplitude** of 1, and its **range** is  $-1 \leq y \leq 1$ .

You should observe that  $y = \cos x$  and  $y = \sin x$  are identical in shape, but the cosine function is  $\frac{\pi}{2}$  units left of the sine function.

Use the graphing package to graph  $y = \cos x$  and  $y = \sin(x + \frac{\pi}{2})$  on the same set of axes.

You should observe that  $\cos x = \sin(x + \frac{\pi}{2})$ .

**GRAPHING PACKAGE**



### THE GENERAL COSINE FUNCTION

The **general cosine function** is  $y = a \cos bx + c$  where  $a > 0$ ,  $b > 0$ .

Since the cosine function is a horizontal translation of the sine function, the constants  $a$ ,  $b$ , and  $c$  have the same effects as for the general sine function. Click on the icon to check this.

**DYNAMIC COSINE FUNCTION**



The **principal axis** of the general cosine function is  $y = c$ .

The **period** of the general cosine function is  $\frac{2\pi}{b}$ .

The **amplitude** of the general cosine function is  $a$ .

$y = a \cos bx + c$   
has a maximum  
when  $x = 0$ .

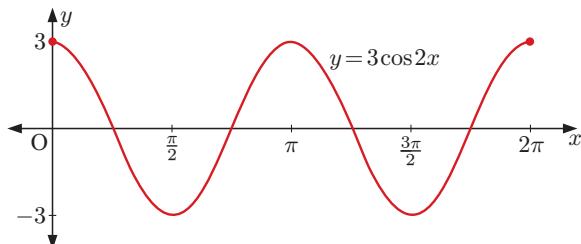


**Example 3****Self Tutor**

Without using technology, sketch the graph of  $y = 3 \cos 2x$  for  $0 \leq x \leq 2\pi$ .

$a = 3$ , so the amplitude is 3.

$b = 2$ , so the period is  $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$ .

**EXERCISE 9C**

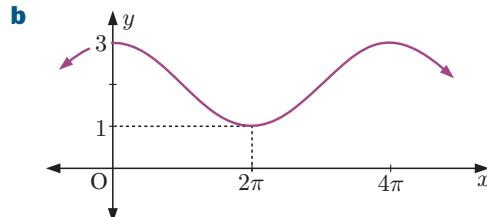
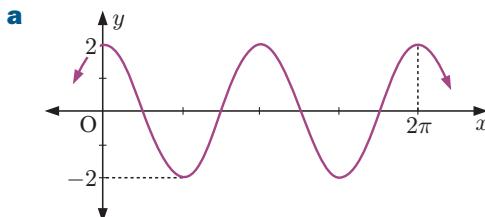
- 1** Without using technology, sketch the following graphs for  $0 \leq x \leq 2\pi$ :

- |                              |                              |                              |
|------------------------------|------------------------------|------------------------------|
| <b>a</b> $y = 3 \cos x$      | <b>b</b> $y = 5 \cos x$      | <b>c</b> $y = \cos 2x$       |
| <b>d</b> $y = \cos 3x$       | <b>e</b> $y = \cos x + 2$    | <b>f</b> $y = \cos x - 1$    |
| <b>g</b> $y = 2 \cos 2x$     | <b>h</b> $y = \cos 3x + 1$   | <b>i</b> $y = 4 \cos x + 10$ |
| <b>j</b> $y = 2 \cos 3x + 4$ | <b>k</b> $y = 4 \cos 2x - 2$ | <b>l</b> $y = 3 \cos 2x + 5$ |

- 2** Find  $a$ ,  $b$ , and  $c$  given that the function  $y = a \cos bx + c$ ,  $a > 0$ ,  $b > 0$ , has:

- a** amplitude 4, period  $\frac{2\pi}{3}$ , and principal axis  $y = -1$
- b** amplitude 3, period  $\frac{2\pi}{5}$ , and principal axis  $y = 3$ .

- 3** Find the cosine function shown in the graph:

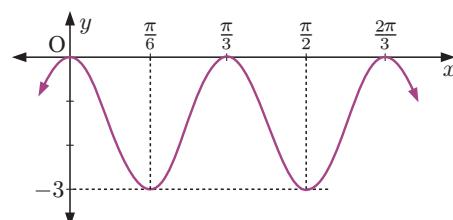


- 4** The function  $y = a \cos bx + c$ ,  $a > 0$ ,  $b > 0$ , has amplitude 5, period  $2\pi$ , and principal axis  $y = 1$ .

- a** Find the values of  $a$ ,  $b$ , and  $c$ .
- b** Sketch the function for  $0 \leq x \leq 2\pi$ .

- 5** The graph shown has the form  $y = a \cos bx + c$  where  $a > 0$ ,  $b > 0$ .

- a** Find the values of  $a$ ,  $b$ , and  $c$ .
- b** Sketch the reflection of the function in the  $x$ -axis.
- c** Write down the equation of the reflection in **b**.

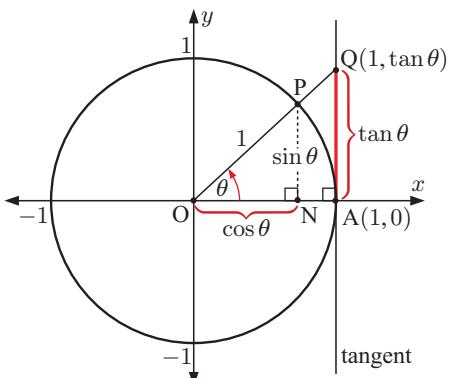


## D THE TANGENT FUNCTION

We have seen that if  $P(\cos \theta, \sin \theta)$  is a point which is free to move around the unit circle, and if  $[OP]$  is extended to meet the tangent at  $A(1, 0)$ , the intersection between these lines occurs at  $Q(1, \tan \theta)$ .

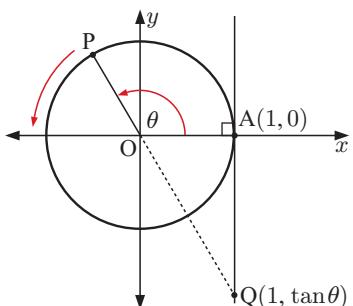
This enables us to define the **tangent function**

$$\tan \theta = \frac{\sin \theta}{\cos \theta}.$$

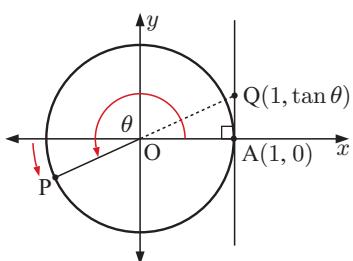


For  $\theta$  in quadrant 2,  $\sin \theta$  is positive and  $\cos \theta$  is negative and so  $\tan \theta = \frac{\sin \theta}{\cos \theta}$  is negative.

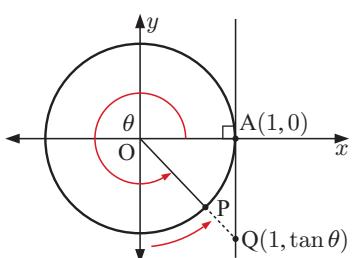
As before,  $[OP]$  is extended to meet the tangent at  $A$  at  $Q(1, \tan \theta)$ . We see that  $Q$  is below the  $x$ -axis.



For  $\theta$  in quadrant 3,  $\sin \theta$  and  $\cos \theta$  are both negative and so  $\tan \theta$  is positive. This is clearly demonstrated as  $Q$  is back above the  $x$ -axis.



For  $\theta$  in quadrant 4,  $\sin \theta$  is negative and  $\cos \theta$  is positive.  $\tan \theta$  is again negative. We see that  $Q$  is below the  $x$ -axis.



### Discussion

What happens to  $\tan \theta$  when  $P$  is at  $(0, 1)$  and  $(0, -1)$ ?