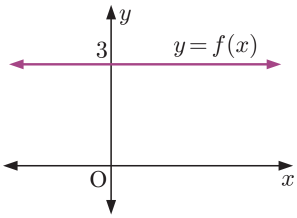
EXERCISE 13C

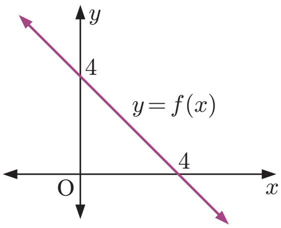
1 Using the graph below, find:

a $f(2)$ **b** $f'(2)$

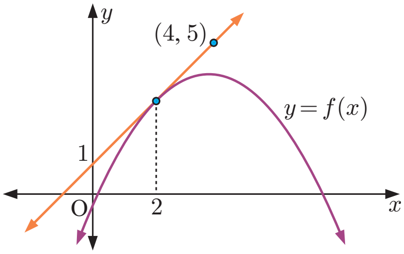


2 Using the graph below, find:

a $f(0)$ **b** $f'(0)$



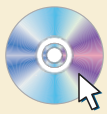
3 Consider the graph alongside.
Find $f(2)$ and $f'(2)$.



Discovery 3 — Gradient functions

The software on the CD can be used to find the gradient of the tangent to a function $f(x)$ at any point. By sliding the point along the graph we can observe the changing gradient of the tangent. We can hence generate the gradient function $f'(x)$.

GRADIENT FUNCTIONS



What to do:

1 Consider the functions $f(x) = 0$, $f(x) = 2$, and $f(x) = 4$.

a For each of these functions, what is the gradient?

b Is the gradient constant for all values of x?

2 Consider the function $f(x) = mx + c$.

a State the gradient of the function. **b** Is the gradient constant for all values of x?

c Use the CD software to graph the following functions and observe the gradient function $f'(x)$. Hence verify that your answer in **b** is correct.

i $f(x) = x - 1$ **ii** $f(x) = 3x + 2$ **iii** $f(x) = -2x + 1$

3 **a** Observe the function $f(x) = x^2$ using the CD software. What *type* of function is the gradient function $f'(x)$?

b Observe the following quadratic functions using the CD software:

i $f(x) = x^2 + x - 2$ **ii** $f(x) = 2x^2 - 3$

iii $f(x) = -x^2 + 2x - 1$ **iv** $f(x) = -3x^2 - 3x + 6$

c What *type* of function is each of the gradient functions $f'(x)$ in **b**?

4 **a** Observe the function $f(x) = \ln x$ using the CD software.

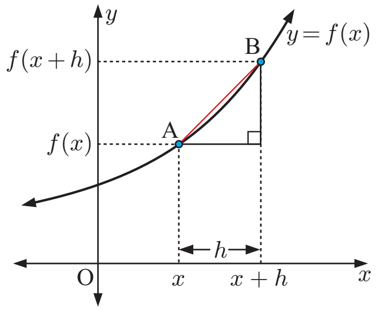
b What *type* of function is the gradient function $f'(x)$?

c What is the *domain* of the gradient function $f'(x)$?

5 **a** Observe the function $f(x) = e^x$ using the CD software.
b What is the gradient function $f'(x)$?

D DIFFERENTIATION FROM FIRST PRINCIPLES

Consider a general function $y = f(x)$ where A is the point $(x, f(x))$ and B is the point $(x + h, f(x + h))$.



The chord AB has gradient $= \dfrac{f(x+h) - f(x)}{x + h - x}$

$= \dfrac{f(x+h) - f(x)}{h}$

If we let B approach A, then the gradient of AB approaches the gradient of the tangent at A.

So, the gradient of the tangent at the variable point $(x, f(x))$ is $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

This formula gives the gradient of the tangent to the curve $y = f(x)$ at the point $(x, f(x))$ for any value of x for which this limit exists. Since there is at most one value of the gradient for each value of x, the formula is actually a function.

The **derivative function** or simply **derivative** of $y = f(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

When we evaluate this limit to find a derivative function, we say we are **differentiating from first principles**.

Example 3 — Self Tutor

Use the definition of $f'(x)$ to find the gradient function of $f(x) = x^2$.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(2x + h)}{\cancel{h}_1} \\
&= \lim_{h \to 0} (2x + h) \qquad \{\text{as } h \neq 0\} \\
&= 2x
\end{aligned}$$

ALTERNATIVE NOTATION

If we are given a function $f(x)$ then $f'(x)$ represents the derivative function.

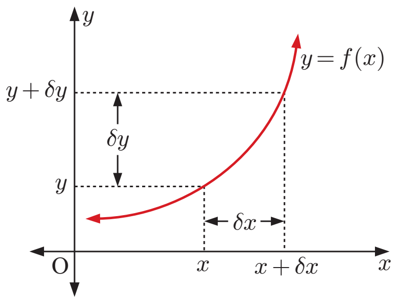
If we are given y in terms of x then y' or $\dfrac{dy}{dx}$ are commonly used to represent the derivative.

$\dfrac{dy}{dx}$ reads "dee y by dee x" or "the derivative of y with respect to x".

$\dfrac{dy}{dx}$ is **not a fraction**. However, the notation $\dfrac{dy}{dx}$ is a result of taking the limit of a fraction. If we replace h by δx and $f(x+h) - f(x)$ by δy, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{becomes}$$

$$\begin{aligned} f'(x) &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \\ &= \frac{dy}{dx}. \end{aligned}$$



THE DERIVATIVE WHEN $x = a$

The gradient of the tangent to $y = f(x)$ at the point where $x = a$ is denoted $f'(a)$, where

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 4 — Self Tutor

Use the first principles formula $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ to find the instantaneous rate of change in $f(x) = x^2 + 2x$ at the point where $x = 5$.

$$f(5) = 5^2 + 2(5) = 35$$

$$\begin{aligned} \therefore \quad f'(5) &= \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} \\ &= \lim_{h \to 0} \frac{(5+h)^2 + 2(5+h) - 35}{h} \\ \therefore \quad f'(5) &= \lim_{h \to 0} \frac{\cancel{25} + 10h + h^2 + \cancel{10} + 2h - \cancel{35}}{h} \\ &= \lim_{h \to 0} \frac{h^2 + 12h}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(h+12)}{\cancel{h}_1} \qquad \{\text{as } h \neq 0\} \\ &= 12 \end{aligned}$$

$\therefore$ the instantaneous rate of change in $f(x)$ at $x = 5$ is 12.

EXERCISE 13D

1 Find, from first principles, the gradient function of:

a $f(x) = x$ **b** $f(x) = 5$ **c** $f(x) = 2x + 5$

2 Find $\frac{dy}{dx}$ from first principles given:

a $y = 4 - x$ **b** $y = x^2 - 3x$ **c** $y = 2x^2 + x - 1$

3 Use the first principles formula $f'(a) = \lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$ to find the gradient of the tangent to:

a $f(x) = 3x + 5$ at $x = -2$ **b** $f(x) = 5 - 2x^2$ at $x = 3$

c $f(x) = x^2 + 3x - 4$ at $x = 3$ **d** $f(x) = 5 - 2x - 3x^2$ at $x = -2$

E SIMPLE RULES OF DIFFERENTIATION

Differentiation is the process of finding a derivative or gradient function.

Given a function $f(x)$, we obtain $f'(x)$ by **differentiating with respect to** the variable x.

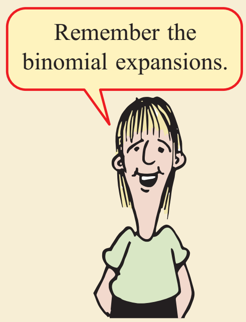
There are a number of rules associated with differentiation. These rules can be used to differentiate more complicated functions without having to use first principles.

Discovery 4 — *Simple rules of differentiation*

In this Discovery we attempt to differentiate functions of the form x^n, cx^n where c is a constant, and functions which are a sum or difference of polynomial terms of the form cx^n.

What to do:

1 Differentiate from first principles: **a** x^2 **b** x^3 **c** x^4



2 Consider the binomial expansion:

$$(x+h)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}h + \binom{n}{2} x^{n-2}h^2 + ... + \binom{n}{n} h^n$$
$$= x^n + nx^{n-1}h + \binom{n}{2} x^{n-2}h^2 + ... + h^n$$

Use the first principles formula $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

to find the derivative of $f(x) = x^n$ for $x \in \mathbb{Z}^+$.

3 **a** Find, from first principles, the derivatives of: **i** $4x^2$ **ii** $2x^3$

b By comparison with **1**, copy and complete: "If $f(x) = cx^n$, then $f'(x) = ...$"

4 **a** Use first principles to find $f'(x)$ for:

i $f(x) = x^2 + 3x$ **ii** $f(x) = x^3 - 2x^2$

b Copy and complete: "If $f(x) = u(x) + v(x)$ then $f'(x) = ...$"

The rules you found in the **Discovery** are much more general than the cases you just considered.

For example, if $f(x) = x^n$ then $f'(x) = nx^{n-1}$ is true not just for all $n \in \mathbb{Z}^+$, but actually for all $n \in \mathbb{R}$.

We can summarise the following rules:

$f(x)$	$f'(x)$	*Name of rule*
c (a constant)	0	**differentiating a constant**
x^n	nx^{n-1}	**differentiating x^n**
$c\,u(x)$	$c\,u'(x)$	**constant times a function**
$u(x)+v(x)$	$u'(x)+v'(x)$	**addition rule**

The last two rules can be proved using the first principles definition of $f'(x)$.

- If $f(x) = c\,u(x)$ where c is a constant, then $f'(x) = c\,u'(x)$.

Proof:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h \to 0} \frac{c\,u(x+h)-c\,u(x)}{h} \\ &= \lim_{h \to 0} c\left[\frac{u(x+h)-u(x)}{h}\right] \\ &= c \lim_{h \to 0} \frac{u(x+h)-u(x)}{h} \\ &= c\,u'(x) \end{aligned}$$

- If $f(x) = u(x) + v(x)$ then $f'(x) = u'(x) + v'(x)$

Proof:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h \to 0} \left(\frac{u(x+h)+v(x+h)-[u(x)+v(x)]}{h}\right) \\ &= \lim_{h \to 0} \left(\frac{u(x+h)-u(x)+v(x+h)-v(x)}{h}\right) \\ &= \lim_{h \to 0} \frac{u(x+h)-u(x)}{h} + \lim_{h \to 0} \frac{v(x+h)-v(x)}{h} \\ &= u'(x) + v'(x) \end{aligned}$$

Using the rules we have now developed we can differentiate sums of powers of x.

For example, if $f(x) = 3x^4 + 2x^3 - 5x^2 + 7x + 6$ then

$$\begin{aligned} f'(x) &= 3(4x^3) + 2(3x^2) - 5(2x) + 7(1) + 0 \\ &= 12x^3 + 6x^2 - 10x + 7 \end{aligned}$$

Example 5

Self Tutor

If $y = 3x^2 - 4x$, find $\dfrac{dy}{dx}$ and interpret its meaning.

As $y = 3x^2 - 4x$, $\dfrac{dy}{dx} = 6x - 4$.

$\dfrac{dy}{dx}$ is:
- the gradient function or derivative of $y = 3x^2 - 4x$ from which the gradient of the tangent at any point on the curve can be found
- the instantaneous rate of change of y with respect to x.

Example 6

Self Tutor

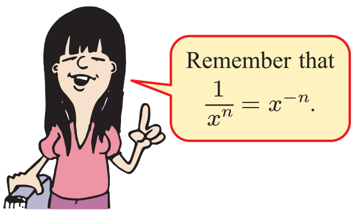
Find $f'(x)$ for $f(x)$ equal to:

a $5x^3 + 6x^2 - 3x + 2$

b $7x - \dfrac{4}{x} + \dfrac{3}{x^3}$

a
$$\begin{aligned} f(x) &= 5x^3 + 6x^2 - 3x + 2 \\ \therefore\ f'(x) &= 5(3x^2) + 6(2x) - 3(1) \\ &= 15x^2 + 12x - 3 \end{aligned}$$

b
$$\begin{aligned} f(x) &= 7x - \frac{4}{x} + \frac{3}{x^3} \\ &= 7x - 4x^{-1} + 3x^{-3} \\ \therefore\ f'(x) &= 7(1) - 4(-1x^{-2}) + 3(-3x^{-4}) \\ &= 7 + 4x^{-2} - 9x^{-4} \\ &= 7 + \frac{4}{x^2} - \frac{9}{x^4} \end{aligned}$$



Example 7

Self Tutor

Find the gradient function of $y = x^2 - \dfrac{4}{x}$ and hence find the gradient of the tangent to the function at the point where $x = 2$.

$$\begin{aligned} y &= x^2 - \frac{4}{x} \\ &= x^2 - 4x^{-1} \end{aligned}$$

$$\begin{aligned} \therefore\ \frac{dy}{dx} &= 2x - 4(-1x^{-2}) \\ &= 2x + 4x^{-2} \\ &= 2x + \frac{4}{x^2} \end{aligned}$$

When $x = 2$, $\dfrac{dy}{dx} = 4 + 1 = 5$.

So, the tangent has gradient 5.

Example 8

Self Tutor

Find the gradient function for each of the following:

a $f(x) = 3\sqrt{x} + \dfrac{2}{x}$ **b** $g(x) = x^2 - \dfrac{4}{\sqrt{x}}$

a
$$\begin{aligned} f(x) &= 3\sqrt{x} + \frac{2}{x} \\ &= 3x^{\frac{1}{2}} + 2x^{-1} \\ \therefore\ f'(x) &= 3(\tfrac{1}{2}x^{-\frac{1}{2}}) + 2(-1x^{-2}) \\ &= \tfrac{3}{2}x^{-\frac{1}{2}} - 2x^{-2} \\ &= \frac{3}{2\sqrt{x}} - \frac{2}{x^2} \end{aligned}$$

b
$$\begin{aligned} g(x) &= x^2 - \frac{4}{\sqrt{x}} \\ &= x^2 - 4x^{-\frac{1}{2}} \\ \therefore\ g'(x) &= 2x - 4(-\tfrac{1}{2}x^{-\frac{3}{2}}) \\ &= 2x + 2x^{-\frac{3}{2}} \\ &= 2x + \frac{2}{x\sqrt{x}} \end{aligned}$$

EXERCISE 13E

1 Find $f'(x)$ given that $f(x)$ is:

a x^3 **b** $2x^3$ **c** $7x^2$ **d** $6\sqrt{x}$

e $3\sqrt[3]{x}$ **f** $x^2 + x$ **g** $4 - 2x^2$ **h** $x^2 + 3x - 5$

i $\frac{1}{2}x^4 - 6x^2$ **j** $\dfrac{3x-6}{x}$ **k** $\dfrac{2x-3}{x^2}$ **l** $\dfrac{x^3+5}{x}$

m $\dfrac{x^3+x-3}{x}$ **n** $\dfrac{1}{\sqrt{x}}$ **o** $(2x-1)^2$ **p** $(x+2)^3$

2 Find $\dfrac{dy}{dx}$ for:

a $y = 2.5x^3 - 1.4x^2 - 1.3$ **b** $y = \pi x^2$ **c** $y = \dfrac{1}{5x^2}$

d $y = 100x$ **e** $y = 10(x+1)$ **f** $y = 4\pi x^3$

3 Differentiate with respect to x:

a $6x + 2$ **b** $x\sqrt{x}$ **c** $(5-x)^2$ **d** $\dfrac{6x^2 - 9x^4}{3x}$

e $(x+1)(x-2)$ **f** $\dfrac{1}{x^2} + 6\sqrt{x}$ **g** $4x - \dfrac{1}{4x}$ **h** $x(x+1)(2x-5)$

4 Find the gradient of the tangent to:

a $y = x^2$ at $x = 2$ **b** $y = \dfrac{8}{x^2}$ at the point $(9, \frac{8}{81})$

c $y = 2x^2 - 3x + 7$ at $x = -1$ **d** $y = \dfrac{2x^2 - 5}{x}$ at the point $(2, \frac{3}{2})$

e $y = \dfrac{x^2 - 4}{x^2}$ at the point $(4, \frac{3}{4})$ **f** $y = \dfrac{x^3 - 4x - 8}{x^2}$ at $x = -1$.

5 Suppose $f(x) = x^2 + (b+1)x + 2c$, $f(2) = 4$, and $f'(-1) = 2$.
Find the constants b and c.

6 Find the gradient function of:

a $f(x) = 4\sqrt{x} + x$ **b** $f(x) = \sqrt[3]{x}$ **c** $f(x) = -\dfrac{2}{\sqrt{x}}$

d $f(x) = 2x - \sqrt{x}$ **e** $f(x) = \dfrac{4}{\sqrt{x}} - 5$ **f** $f(x) = 3x^2 - x\sqrt{x}$

g $f(x) = \dfrac{5}{x^2\sqrt{x}}$ **h** $f(x) = 2x - \dfrac{3}{x\sqrt{x}}$

7 **a** If $y = 4x - \dfrac{3}{x}$, find $\dfrac{dy}{dx}$ and interpret its meaning.

b The position of a car moving along a straight road is given by $S = 2t^2 + 4t$ metres where t is the time in seconds. Find $\dfrac{dS}{dt}$ and interpret its meaning.

c The cost of producing x toasters each week is given by $C = 1785 + 3x + 0.002x^2$ dollars. Find $\dfrac{dC}{dx}$ and interpret its meaning.

F THE CHAIN RULE

In **Chapter 2** we defined the **composite** of two functions g and f as $(g \circ f)(x)$ or $gf(x)$.

We can often write complicated functions as the composite of two or more simpler functions.

For example $y = (x^2 + 3x)^4$ could be rewritten as $y = u^4$ where $u = x^2 + 3x$, or as $y = gf(x)$ where $g(x) = x^4$ and $f(x) = x^2 + 3x$.

Example 9 Self Tutor

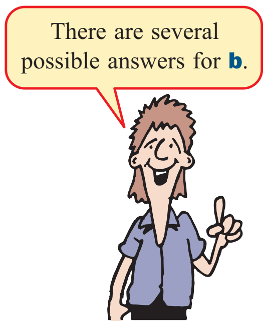
Find: **a** $gf(x)$ if $g(x) = \sqrt{x}$ and $f(x) = 2 - 3x$

b $g(x)$ and $f(x)$ such that $gf(x) = \dfrac{1}{x - x^2}$.

a $gf(x)$
$= g(2 - 3x)$
$= \sqrt{2 - 3x}$

b $gf(x) = \dfrac{1}{x - x^2} = \dfrac{1}{f(x)}$

$\therefore$ $g(x) = \dfrac{1}{x}$ and $f(x) = x - x^2$



EXERCISE 13F.1

1 Find $gf(x)$ if:

a $g(x) = x^2$ and $f(x) = 2x + 7$ **b** $g(x) = 2x + 7$ and $f(x) = x^2$

c $g(x) = \sqrt{x}$ and $f(x) = 3 - 4x$ **d** $g(x) = 3 - 4x$ and $f(x) = \sqrt{x}$

e $g(x) = \dfrac{2}{x}$ and $f(x) = x^2 + 3$ **f** $g(x) = x^2 + 3$ and $f(x) = \dfrac{2}{x}$

2 Find $g(x)$ and $f(x)$ such that $gf(x)$ is:

a $(3x + 10)^3$ **b** $\dfrac{1}{2x + 4}$ **c** $\sqrt{x^2 - 3x}$ **d** $\dfrac{10}{(3x - x^2)^3}$

DERIVATIVES OF COMPOSITE FUNCTIONS

The reason we are interested in writing complicated functions as composite functions is to make finding derivatives easier.

Discovery 5 — Differentiating composite functions

The purpose of this Discovery is to learn how to differentiate composite functions.

Based on the rule "if $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$", we might suspect that if $y = (2x+1)^2$ then $\frac{dy}{dx} = 2(2x+1)^1$. But is this so?

What to do:

1 Expand $y = (2x+1)^2$ and hence find $\frac{dy}{dx}$. How does this compare with $2(2x+1)^1$?

2 Expand $y = (3x+1)^2$ and hence find $\frac{dy}{dx}$. How does this compare with $2(3x+1)^1$?

3 Expand $y = (ax+1)^2$ where a is a constant, and hence find $\frac{dy}{dx}$. How does this compare with $2(ax+1)^1$?

4 Suppose $y = u^2$.

a Find $\frac{dy}{du}$.

b Now suppose $u = ax+1$, so $y = (ax+1)^2$.

i Find $\frac{du}{dx}$.

ii Write $\frac{dy}{du}$ from **a** in terms of x.

iii Hence find $\frac{dy}{du} \times \frac{du}{dx}$.

iv Compare your answer to the result in **3**.

c If $y = u^2$ where u is a function of x, what do you suspect $\frac{dy}{dx}$ will be equal to?

5 Expand $y = (x^2+3x)^2$ and hence find $\frac{dy}{dx}$.

Does your answer agree with the rule you suggested in **4 c**?

6 Consider $y = (2x+1)^3$.

a Expand the brackets and hence find $\frac{dy}{dx}$.

b If we let $u = 2x+1$, then $y = u^3$.

i Find $\frac{du}{dx}$.

ii Find $\frac{dy}{du}$, and write it in terms of x.

iii Hence find $\frac{dy}{du} \times \frac{du}{dx}$.

iv Compare your answer to the result in **a**.

7 Copy and complete: "If y is a function of u, and u is a function of x, then $\frac{dy}{dx} = ...$"

THE CHAIN RULE

If $\boldsymbol{y = g(u)}$ where $\boldsymbol{u = f(x)}$ then $\boldsymbol{\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}}$.

This rule is extremely important and enables us to differentiate complicated functions much faster.

For example, for any function $f(x)$:

If $\boldsymbol{y = [f(x)]^n}$ then $\boldsymbol{\dfrac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)}$.

Example 10 — Self Tutor

Find $\dfrac{dy}{dx}$ if:

a $y = (x^2 - 2x)^4$

b $y = \dfrac{4}{\sqrt{1-2x}}$

a
$$y = (x^2 - 2x)^4$$
$$\therefore\ y = u^4 \quad \text{where} \quad u = x^2 - 2x$$
$$\text{Now}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}$$
$$= 4u^3(2x-2)$$
$$= 4(x^2-2x)^3(2x-2)$$

b
$$y = \frac{4}{\sqrt{1-2x}}$$
$$\therefore\ y = 4u^{-\frac{1}{2}} \quad \text{where} \quad u = 1 - 2x$$
$$\text{Now}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}$$
$$= 4 \times (-\tfrac{1}{2}u^{-\frac{3}{2}}) \times (-2)$$
$$= 4u^{-\frac{3}{2}}$$
$$= 4(1-2x)^{-\frac{3}{2}}$$

EXERCISE 13F.2

1 Write in the form au^n, clearly stating what u is:

a $\dfrac{1}{(2x-1)^2}$ **b** $\sqrt{x^2-3x}$ **c** $\dfrac{2}{\sqrt{2-x^2}}$

d $\sqrt[3]{x^3-x^2}$ **e** $\dfrac{4}{(3-x)^3}$ **f** $\dfrac{10}{x^2-3}$

2 Find the gradient function $\dfrac{dy}{dx}$ for:

a $y = (4x-5)^2$ **b** $y = \dfrac{1}{5-2x}$ **c** $y = \sqrt{3x-x^2}$

d $y = (1-3x)^4$ **e** $y = 6(5-x)^3$ **f** $y = \sqrt[3]{2x^3-x^2}$

g $y = \dfrac{6}{(5x-4)^2}$ **h** $y = \dfrac{4}{3x-x^2}$ **i** $y = 2\left(x^2 - \dfrac{2}{x}\right)^3$

3 Find the gradient of the tangent to:

a $y = \sqrt{1-x^2}$ at $x = \frac{1}{2}$

b $y = (3x+2)^6$ at $x = -1$

c $y = \dfrac{1}{(2x-1)^4}$ at $x = 1$

d $y = 6 \times \sqrt[3]{1-2x}$ at $x = 0$

e $y = \dfrac{4}{x + 2\sqrt{x}}$ at $x = 4$

f $y = \left(x + \dfrac{1}{x}\right)^3$ at $x = 1$.

4 The gradient function of $f(x) = (2x-b)^a$ is $f'(x) = 24x^2 - 24x + 6$.
Find the constants a and b.

5 Suppose $y = \dfrac{a}{\sqrt{1+bx}}$ where a and b are constants. When $x = 3$, $y = 1$ and $\dfrac{dy}{dx} = -\frac{1}{8}$.
Find a and b.

6 If $y = x^3$ then $x = y^{\frac{1}{3}}$.

a Find $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$, and hence show that $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$.

b Explain why $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$ whenever these derivatives exist for any general function $y = f(x)$.

G THE PRODUCT RULE

We have seen the addition rule:

$$\text{If} \quad f(x) = u(x) + v(x) \quad \text{then} \quad f'(x) = u'(x) + v'(x).$$

We now consider the case $f(x) = u(x)\,v(x)$. Is $f'(x) = u'(x)\,v'(x)$?

In other words, does the derivative of a product of two functions equal the product of the derivatives of the two functions?

Discovery 6 — The product rule

Suppose $u(x)$ and $v(x)$ are two functions of x, and that $f(x) = u(x)\,v(x)$ is the product of these functions.

The purpose of this Discovery is to find a rule for determining $f'(x)$.

What to do:

1 Suppose $u(x) = x$ and $v(x) = x$, so $f(x) = x^2$.

a Find $f'(x)$ by direct differentiation.

b Find $u'(x)$ and $v'(x)$.

c Does $f'(x) = u'(x)\,v'(x)$?

2 Suppose $u(x) = x$ and $v(x) = \sqrt{x}$, so $f(x) = x\sqrt{x} = x^{\frac{3}{2}}$.

a Find $f'(x)$ by direct differentiation.

b Find $u'(x)$ and $v'(x)$.

c Does $f'(x) = u'(x)\,v'(x)$?

3 Copy and complete the following table, finding $f'(x)$ by direct differentiation.

$f(x)$	$f'(x)$	$u(x)$	$v(x)$	$u'(x)$	$v'(x)$	$u'(x)\,v(x)+u(x)\,v'(x)$
x^2		x	x			
$x^{\frac{3}{2}}$		x	$\sqrt{x}$			
$x(x+1)$		x	$x+1$			
$(x-1)(2-x^2)$		$x-1$	$2-x^2$			

4 Copy and complete: "If $f(x)=u(x)\,v(x)$ then $f'(x)=$ ......"

THE PRODUCT RULE

If $\boldsymbol{f(x)=u(x)\,v(x)}$ then $\boldsymbol{f'(x)=u'(x)\,v(x)+u(x)\,v'(x)}$.

Alternatively, if $\boldsymbol{y=uv}$ where u and v are functions of x, then $\boldsymbol{\dfrac{dy}{dx}=\dfrac{du}{dx}v+u\dfrac{dv}{dx}}$.

Example 11 — Self Tutor

Find $\dfrac{dy}{dx}$ if:

a $y=\sqrt{x}(2x+1)^3$

b $y=x^2(x^2-2x)^4$

a $y=\sqrt{x}(2x+1)^3$ is the product of $u=x^{\frac{1}{2}}$ and $v=(2x+1)^3$

$\therefore\ u'=\frac{1}{2}x^{-\frac{1}{2}}$ and $v'=3(2x+1)^2\times 2$ {chain rule}

$\qquad\qquad\qquad\qquad\ \ =6(2x+1)^2$

Now $\dfrac{dy}{dx}=u'v+uv'$ {product rule}

$$=\tfrac{1}{2}x^{-\frac{1}{2}}(2x+1)^3+x^{\frac{1}{2}}\times 6(2x+1)^2$$

$$=\tfrac{1}{2}x^{-\frac{1}{2}}(2x+1)^3+6x^{\frac{1}{2}}(2x+1)^2$$

b $y=x^2(x^2-2x)^4$ is the product of $u=x^2$ and $v=(x^2-2x)^4$

$\therefore\ u'=2x$ and $v'=4(x^2-2x)^3(2x-2)$ {chain rule}

Now $\dfrac{dy}{dx}=u'v+uv'$ {product rule}

$$=2x(x^2-2x)^4+x^2\times 4(x^2-2x)^3(2x-2)$$

$$=2x(x^2-2x)^4+4x^2(x^2-2x)^3(2x-2)$$

EXERCISE 13G

1 Use the product rule to differentiate:

a $f(x)=x(x-1)$

b $f(x)=2x(x+1)$

c $f(x)=x^2\sqrt{x+1}$

2 Find $\frac{dy}{dx}$ using the product rule:

a $y = x^2(2x-1)$ **b** $y = 4x(2x+1)^3$ **c** $y = x^2\sqrt{3-x}$

d $y = \sqrt{x}(x-3)^2$ **e** $y = 5x^2(3x^2-1)^2$ **f** $y = \sqrt{x}(x-x^2)^3$

3 Find the gradient of the tangent to:

a $y = x^4(1-2x)^2$ at $x = -1$ **b** $y = \sqrt{x}(x^2-x+1)^2$ at $x = 4$

c $y = x\sqrt{1-2x}$ at $x = -4$ **d** $y = x^3\sqrt{5-x^2}$ at $x = 1$.

4 Consider $y = \sqrt{x}(3-x)^2$.

a Show that $\frac{dy}{dx} = \frac{(3-x)(3-5x)}{2\sqrt{x}}$.

b Find the x-coordinates of all points on $y = \sqrt{x}(3-x)^2$ where the tangent is horizontal.

c For what values of x is $\frac{dy}{dx}$ undefined?

5 Suppose $y = -2x^2(x+4)$. For what values of x does $\frac{dy}{dx} = 10$?

H THE QUOTIENT RULE

Expressions like $\frac{x^2+1}{2x-5}$, $\frac{\sqrt{x}}{1-3x}$, and $\frac{x^3}{(x-x^2)^4}$ are called **quotients** because they represent the division of one function by another.

Quotient functions have the form $Q(x) = \frac{u(x)}{v(x)}$.

Notice that $u(x) = Q(x)\,v(x)$

$$\therefore \quad u'(x) = Q'(x)\,v(x) + Q(x)\,v'(x) \qquad \{\text{product rule}\}$$

$$\therefore \quad u'(x) - Q(x)\,v'(x) = Q'(x)\,v(x)$$

$$\therefore \quad Q'(x)\,v(x) = u'(x) - \frac{u(x)}{v(x)}\,v'(x)$$

$$\therefore \quad Q'(x)\,v(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{v(x)}$$

$$\therefore \quad Q'(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{[v(x)]^2} \qquad \text{when this exists.}$$

THE QUOTIENT RULE

If $\boldsymbol{Q(x) = \frac{u(x)}{v(x)}}$ then $\boldsymbol{Q'(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{[v(x)]^2}}$.

Alternatively, if $\boldsymbol{y = \frac{u}{v}}$ where u and v are functions of x, then $\boldsymbol{\frac{dy}{dx} = \frac{u'v - uv'}{v^2}}$.

Example 12

Self Tutor

Use the quotient rule to find $\dfrac{dy}{dx}$ if:

a $y = \dfrac{1+3x}{x^2+1}$

b $y = \dfrac{\sqrt{x}}{(1-2x)^2}$

a $y = \dfrac{1+3x}{x^2+1}$ is a quotient with $u = 1+3x$ and $v = x^2+1$

$\therefore$ $u' = 3$ and $v' = 2x$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$$

$$= \frac{3(x^2+1) - (1+3x)2x}{(x^2+1)^2}$$

$$= \frac{3x^2+3-2x-6x^2}{(x^2+1)^2}$$

$$= \frac{3-2x-3x^2}{(x^2+1)^2}$$

b $y = \dfrac{\sqrt{x}}{(1-2x)^2}$ is a quotient with $u = x^{\frac{1}{2}}$ and $v = (1-2x)^2$

$\therefore$ $u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(1-2x)^1 \times (-2)$ $\{\text{chain rule}\}$

$= -4(1-2x)$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 - x^{\frac{1}{2}} \times (-4(1-2x))}{(1-2x)^4}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 + 4x^{\frac{1}{2}}(1-2x)}{(1-2x)^4}$$

$$= \frac{\cancel{(1-2x)}\left[\frac{1-2x}{2\sqrt{x}} + 4\sqrt{x}\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)\right]}{(1-2x)^{\cancel{4}3}} \quad \{\text{look for common factors}\}$$

$$= \frac{1-2x+8x}{2\sqrt{x}(1-2x)^3}$$

$$= \frac{6x+1}{2\sqrt{x}(1-2x)^3}$$

Simplification of $\frac{dy}{dx}$ is often unnecessary, especially if you simply want the gradient of a tangent at a given point. In such cases, substitute a value for x without simplifying the derivative function first.

EXERCISE 13H

1 Use the quotient rule to find $\frac{dy}{dx}$ if:

a $y = \frac{1+3x}{2-x}$ **b** $y = \frac{x^2}{2x+1}$ **c** $y = \frac{x}{x^2-3}$

d $y = \frac{\sqrt{x}}{1-2x}$ **e** $y = \frac{x^2-3}{3x-x^2}$ **f** $y = \frac{x}{\sqrt{1-3x}}$

2 Find the gradient of the tangent to:

a $y = \frac{x}{1-2x}$ at $x = 1$ **b** $y = \frac{x^3}{x^2+1}$ at $x = -1$

c $y = \frac{\sqrt{x}}{2x+1}$ at $x = 4$ **d** $y = \frac{x^2}{\sqrt{x^2+5}}$ at $x = -2$.

3 **a** If $y = \frac{2\sqrt{x}}{1-x}$, show that $\frac{dy}{dx} = \frac{x+1}{\sqrt{x}(1-x)^2}$.

b For what values of x is $\frac{dy}{dx}$ **i** zero **ii** undefined?

4 **a** If $y = \frac{x^2-3x+1}{x+2}$, show that $\frac{dy}{dx} = \frac{x^2+4x-7}{(x+2)^2}$.

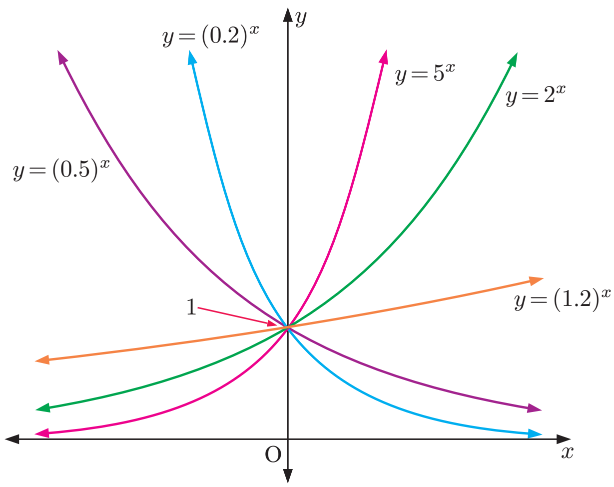
b For what values of x is $\frac{dy}{dx}$ **i** zero **ii** undefined?

I DERIVATIVES OF EXPONENTIAL FUNCTIONS

In **Chapter 4** we saw that the simplest **exponential functions** have the form $f(x) = b^x$ where b is any positive constant, $b \neq 1$.

The graphs of all members of the exponential family $f(x) = b^x$ have the following properties:

- pass through the point $(0, 1)$
- asymptotic to the x-axis at one end
- lie above the x-axis for all x.



Discovery 7 — The derivative of $y = b^x$

The purpose of this Discovery is to observe the nature of the derivatives of $f(x) = b^x$ for various values of b.

What to do:

CALCULUS DEMO

1 Use the software provided to help fill in the table for $y = 2^x$:

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0			
0.5			
1			
1.5			
2			

2 Repeat **1** for the following functions:

a $y = 3^x$ **b** $y = 5^x$ **c** $y = (0.5)^x$

3 Use your observations from **1** and **2** to write a statement about the derivative of the general exponential $y = b^x$ for $b > 0$, $b \neq 1$.

From the **Discovery** you should have found that:

$$\text{If } f(x) = b^x \text{ then } f'(x) = f'(0) \times b^x.$$

Proof:

If $f(x) = b^x$,

then $f'(x) = \lim\limits_{h \to 0} \dfrac{b^{x+h} - b^x}{h}$ {first principles definition of the derivative}

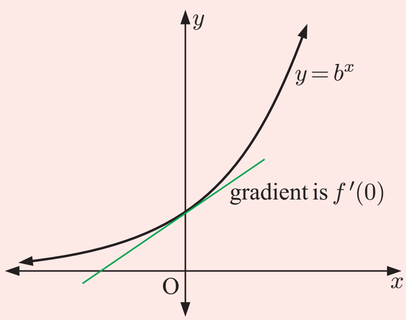
$= \lim\limits_{h \to 0} \dfrac{b^x(b^h - 1)}{h}$

$= b^x \times \left(\lim\limits_{h \to 0} \dfrac{b^h - 1}{h} \right)$ {as b^x is independent of h}

But $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$= \lim\limits_{h \to 0} \dfrac{b^h - 1}{h}$

$\therefore f'(x) = b^x \times f'(0)$



Given this result, if we can find a value of b such that $f'(0) = 1$, then we will have found *a function which is its own derivative*!

We have already shown that if $f(x)=b^x$ then $f'(x)=b^x\left(\lim\limits_{h\to 0}\dfrac{b^h-1}{h}\right)$.

So if $f'(x)=b^x$ then we require $\lim\limits_{h\to 0}\dfrac{b^h-1}{h}=1$.

$$\therefore\ \lim_{h\to 0} b^h=\lim_{h\to 0}(1+h)$$

Letting $h=\dfrac{1}{n}$, we notice that $\dfrac{1}{n}\to 0$ if $n\to\infty$

$$\therefore\ \lim_{n\to\infty} b^{\frac{1}{n}}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

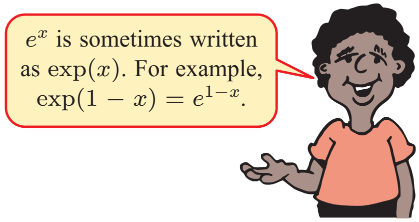
$$\therefore\ b=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad \text{if this limit exists}$$

We have in fact already seen this limit in **Chapter 4 Discovery 2** on page **123**.

We found that as $n\to\infty$,

$$\left(1+\frac{1}{n}\right)^n\to 2.718\,281\,828\,459\,045\,235\ldots.$$

and this irrational number is the natural exponential e.



We now have: $\boxed{\text{If } f(x)=e^x \text{ then } f'(x)=e^x.}$

THE DERIVATIVE OF $e^{f(x)}$

The functions e^{-x}, e^{2x+3}, and e^{-x^2} all have the form $e^{f(x)}$.

Since $e^x>0$ for all x, $e^{f(x)}>0$ for all x, no matter what the function $f(x)$.

Suppose $y=e^{f(x)}=e^u$ where $u=f(x)$.

$$\begin{aligned}\text{Now}\quad \frac{dy}{dx}&=\frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}\\ &=e^u\frac{du}{dx}\\ &=e^{f(x)}\times f'(x)\end{aligned}$$

Function	*Derivative*
e^x	e^x
$e^{f(x)}$	$e^{f(x)}\times f'(x)$

Example 13 — Self Tutor

Find the gradient function for y equal to:

a $2e^x+e^{-3x}$ **b** x^2e^{-x} **c** $\dfrac{e^{2x}}{x}$

a If $y=2e^x+e^{-3x}$ then
$$\begin{aligned}\frac{dy}{dx}&=2e^x+e^{-3x}(-3)\\ &=2e^x-3e^{-3x}\end{aligned}$$

b If $y=x^2e^{-x}$ then
$$\begin{aligned}\frac{dy}{dx}&=2xe^{-x}+x^2e^{-x}(-1) \qquad \{\text{product rule}\}\\ &=2xe^{-x}-x^2e^{-x}\end{aligned}$$

c If $y = \dfrac{e^{2x}}{x}$ then $\dfrac{dy}{dx} = \dfrac{e^{2x}(2)x - e^{2x}(1)}{x^2}$ {quotient rule}

$= \dfrac{e^{2x}(2x-1)}{x^2}$

Example 14

Self Tutor

Find the gradient function for y equal to: **a** $(e^x - 1)^3$ **b** $\dfrac{1}{\sqrt{2e^{-x}+1}}$

a $y = (e^x - 1)^3$

$= u^3$ where $u = e^x - 1$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 3u^2 \dfrac{du}{dx}$

$= 3(e^x - 1)^2 \times e^x$

$= 3e^x(e^x - 1)^2$

b $y = (2e^{-x} + 1)^{-\frac{1}{2}}$

$= u^{-\frac{1}{2}}$ where $u = 2e^{-x} + 1$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= -\frac{1}{2}u^{-\frac{3}{2}}\dfrac{du}{dx}$

$= -\frac{1}{2}(2e^{-x}+1)^{-\frac{3}{2}} \times 2e^{-x}(-1)$

$= e^{-x}(2e^{-x}+1)^{-\frac{3}{2}}$

EXERCISE 13I

1 Find the gradient function for $f(x)$ equal to:

a e^{4x} **b** $e^x + 3$ **c** $\exp(-2x)$ **d** $e^{\frac{x}{2}}$

e $2e^{-\frac{x}{2}}$ **f** $1 - 2e^{-x}$ **g** $4e^{\frac{x}{2}} - 3e^{-x}$ **h** $\dfrac{e^x + e^{-x}}{2}$

i e^{-x^2} **j** $e^{\frac{1}{x}}$ **k** $10(1 + e^{2x})$ **l** $20(1 - e^{-2x})$

m e^{2x+1} **n** $e^{\frac{x}{4}}$ **o** e^{1-2x^2} **p** $e^{-0.02x}$

2 Find the derivative of:

a xe^x **b** x^3e^{-x} **c** $\dfrac{e^x}{x}$ **d** $\dfrac{x}{e^x}$

e x^2e^{3x} **f** $\dfrac{e^x}{\sqrt{x}}$ **g** $\sqrt{x}e^{-x}$ **h** $\dfrac{e^x + 2}{e^{-x} + 1}$

3 Find the gradient of the tangent to:

a $y = (e^x + 2)^4$ at $x = 0$ **b** $y = \dfrac{1}{2 - e^{-x}}$ at $x = 0$

c $y = \sqrt{e^{2x} + 10}$ at $x = \ln 3$.

4 Given $f(x) = e^{kx} + x$ and $f'(0) = -8$, find k.

5 **a** By substituting $e^{\ln 2}$ for 2 in $y = 2^x$, find $\dfrac{dy}{dx}$.

b Show that if $y = b^x$ where $b > 0$, $b \neq 1$, then $\dfrac{dy}{dx} = b^x \times \ln b$.

6 The tangent to $f(x) = x^2e^{-x}$ at point P is horizontal. Find the possible coordinates of P.

J DERIVATIVES OF LOGARITHMIC FUNCTIONS

Discovery 8 — The derivative of $\ln x$

If $y = \ln x$, what is the gradient function?

CALCULUS DEMO

What to do:

1. Click on the icon to see the graph of $y = \ln x$. Observe the gradient function being drawn as the point moves from left to right along the graph.
2. Predict a formula for the gradient function of $y = \ln x$.
3. Find the gradient of the tangent to $y = \ln x$ for $x = 0.25, 0.5, 1, 2, 3, 4$, and 5. Do your results confirm your prediction in **2**?

From the **Discovery** you should have observed: If $y = \ln x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

The proof of this result is beyond the scope of this course.

THE DERIVATIVE OF $\ln f(x)$

Suppose $y = \ln f(x)$

$\therefore$ $y = \ln u$ where $u = f(x)$.

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$$\therefore \frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{f'(x)}{f(x)}$$

Function	*Derivative*
$\ln x$	$\dfrac{1}{x}$
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$

Example 15 — Self Tutor

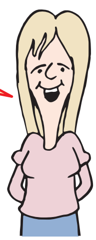
Find the gradient function of:

a $y = \ln(kx)$, k a constant **b** $y = \ln(1-3x)$ **c** $y = x^3 \ln x$

a $y = \ln(kx)$

$$\therefore \frac{dy}{dx} = \frac{k}{kx} = \frac{1}{x}$$

$\ln(kx) = \ln k + \ln x$
$= \ln x + \text{constant}$
so $\ln(kx)$ and $\ln x$ both have derivative $\frac{1}{x}$.



b $y = \ln(1-3x)$

$$\therefore \frac{dy}{dx} = \frac{-3}{1-3x} = \frac{3}{3x-1}$$

c $y = x^3 \ln x$

$$\therefore \frac{dy}{dx} = 3x^2 \ln x + x^3\left(\frac{1}{x}\right) \quad \text{\{product rule\}}$$

$$= 3x^2 \ln x + x^2$$

$$= x^2(3\ln x + 1)$$

The laws of logarithms can help us to differentiate some logarithmic functions more easily.

For $a > 0$, $b > 0$, $n \in \mathbb{R}$: $\ln(ab) = \ln a + \ln b$

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$\ln(a^n) = n \ln a$$

Example 16

Self Tutor

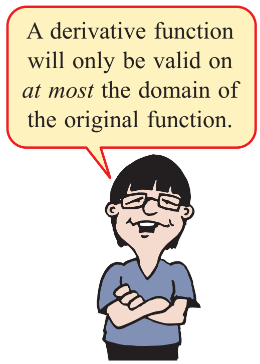
Differentiate with respect to x:

a $y = \ln(xe^{-x})$

b $y = \ln\left[\dfrac{x^2}{(x+2)(x-3)}\right]$

a
$$\begin{aligned} y &= \ln(xe^{-x}) \\ &= \ln x + \ln e^{-x} \quad \{\ln(ab) = \ln a + \ln b\} \\ &= \ln x - x \quad \{\ln e^a = a\} \end{aligned}$$
$$\therefore \frac{dy}{dx} = \frac{1}{x} - 1$$

b
$$\begin{aligned} y &= \ln\left[\frac{x^2}{(x+2)(x-3)}\right] \\ &= \ln x^2 - \ln[(x+2)(x-3)] \quad \left\{\ln\left(\frac{a}{b}\right) = \ln a - \ln b\right\} \\ &= 2\ln x - [\ln(x+2) + \ln(x-3)] \\ &= 2\ln x - \ln(x+2) - \ln(x-3) \end{aligned}$$
$$\therefore \frac{dy}{dx} = \frac{2}{x} - \frac{1}{x+2} - \frac{1}{x-3}$$



EXERCISE 13J

1 Find the gradient function of:

a $y = \ln(7x)$ **b** $y = \ln(2x+1)$ **c** $y = \ln(x - x^2)$

d $y = 3 - 2\ln x$ **e** $y = x^2 \ln x$ **f** $y = \dfrac{\ln x}{2x}$

g $y = e^x \ln x$ **h** $y = (\ln x)^2$ **i** $y = \sqrt{\ln x}$

j $y = e^{-x} \ln x$ **k** $y = \sqrt{x}\ln(2x)$ **l** $y = \dfrac{2\sqrt{x}}{\ln x}$

m $y = 3 - 4\ln(1-x)$ **n** $y = x\ln(x^2+1)$

2 Find $\dfrac{dy}{dx}$ for:

a $y = x \ln 5$ **b** $y = \ln(x^3)$ **c** $y = \ln(x^4 + x)$

d $y = \ln(10 - 5x)$ **e** $y = [\ln(2x+1)]^3$ **f** $y = \dfrac{\ln(4x)}{x}$

g $y = \ln\left(\dfrac{1}{x}\right)$ **h** $y = \ln(\ln x)$ **i** $y = \dfrac{1}{\ln x}$

3 Use the laws of logarithms to help differentiate with respect to x:

a $y = \ln\sqrt{1-2x}$

b $y = \ln\left(\dfrac{1}{2x+3}\right)$

c $y = \ln\left(e^x\sqrt{x}\right)$

d $y = \ln\left(x\sqrt{2-x}\right)$

e $y = \ln\left(\dfrac{x+3}{x-1}\right)$

f $y = \ln\left(\dfrac{x^2}{3-x}\right)$

g $f(x) = \ln\left((3x-4)^3\right)$

h $f(x) = \ln\left(x(x^2+1)\right)$

i $f(x) = \ln\left(\dfrac{x^2+2x}{x-5}\right)$

4 Find the gradient of the tangent to:

a $y = x\ln x$ at the point where $x = e$

b $y = \ln\left(\dfrac{x+2}{x^2}\right)$ at the point where $x = 1$.

5 Suppose $f(x) = a\ln(2x+b)$ where $f(e) = 3$ and $f'(e) = \dfrac{6}{e}$. Find the constants a and b.

K DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

In **Chapter 9** we saw that sine and cosine curves arise naturally from motion in a circle.

DEMO

Click on the icon to observe the motion of point P around the unit circle. Observe the graphs of P's height relative to the x-axis, and then P's horizontal displacement from the y-axis. The resulting graphs are those of $y = \sin t$ and $y = \cos t$.

Discovery 9 — Derivatives of $\sin x$ and $\cos x$

Our aim is to use a computer demonstration to investigate the derivatives of $\sin x$ and $\cos x$.

What to do:

DERIVATIVES DEMO

1 Click on the icon to observe the graph of $y = \sin x$. A tangent with x-step of length 1 unit moves across the curve, and its y-step is translated onto the gradient graph. Predict the derivative of the function $y = \sin x$.

2 Repeat the process in **1** for the graph of $y = \cos x$. Hence predict the derivative of the function $y = \cos x$.

From the **Discovery** you should have deduced that:

For x in radians:	If	$f(x) = \sin x$	then	$f'(x) = \cos x$.
	If	$f(x) = \cos x$	then	$f'(x) = -\sin x$.

THE DERIVATIVE OF $\tan x$

DERIVATIVE DEMO

Consider $y = \tan x = \dfrac{\sin x}{\cos x}$

We let $u = \sin x$ and $v = \cos x$

$$\therefore \quad \frac{du}{dx} = \cos x \quad \text{and} \quad \frac{dv}{dx} = -\sin x$$

$$\begin{aligned}
\therefore \quad \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \qquad \{\text{quotient rule}\} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{[\cos x]^2} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \qquad \{\text{since} \quad \sin^2 x + \cos^2 x = 1\} \\
&= \sec^2 x
\end{aligned}$$

Function	*Derivative*
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

THE DERIVATIVES OF $\sin[f(x)]$, $\cos[f(x)]$, AND $\tan[f(x)]$

Suppose $y = \sin[f(x)]$

If we let $u = f(x)$, then $y = \sin u$.

But $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ $\qquad \{\text{chain rule}\}$

$$\begin{aligned}
\therefore \quad \frac{dy}{dx} &= \cos u \times f'(x) \\
&= \cos[f(x)] \times f'(x)
\end{aligned}$$

We can perform the same procedure for $\cos[f(x)]$ and $\tan[f(x)]$, giving the following results:

Function	*Derivative*
$\sin[f(x)]$	$\cos[f(x)]\, f'(x)$
$\cos[f(x)]$	$-\sin[f(x)]\, f'(x)$
$\tan[f(x)]$	$\sec^2[f(x)] f'(x)$

Example 17 — Self Tutor

Differentiate with respect to x:

a $x \sin x$ **b** $4\tan^2(3x)$

a If $y = x \sin x$

then by the product rule

$$\begin{aligned}
\frac{dy}{dx} &= (1)\sin x + (x)\cos x \\
&= \sin x + x \cos x
\end{aligned}$$

b If

$$\begin{aligned}
y &= 4\tan^2(3x) \\
&= 4[\tan(3x)]^2 \\
&= 4u^2 \quad \text{where} \quad u = \tan(3x)
\end{aligned}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}$$

$$\begin{aligned}
\therefore \quad \frac{dy}{dx} &= 8u \times \frac{du}{dx} \\
&= 8\tan(3x) \times 3\sec^2(3x) \\
&= 24\sin(3x)\sec^3(3x)
\end{aligned}$$

EXERCISE 13K

1 Find $\frac{dy}{dx}$ for:

a $y = \sin(2x)$ **b** $y = \sin x + \cos x$ **c** $y = \cos(3x) - \sin x$
d $y = \sin(x+1)$ **e** $y = \cos(3 - 2x)$ **f** $y = \tan(5x)$
g $y = \sin(\frac{x}{2}) - 3\cos x$ **h** $y = 3\tan(\pi x)$ **i** $y = 4\sin x - \cos(2x)$

2 Differentiate with respect to x:

a $x^2 + \cos x$ **b** $\tan x - 3\sin x$ **c** $e^x \cos x$ **d** $e^{-x}\sin x$
e $\ln(\sin x)$ **f** $e^{2x}\tan x$ **g** $\sin(3x)$ **h** $\cos(\frac{x}{2})$
i $3\tan(2x)$ **j** $x\cos x$ **k** $\frac{\sin x}{x}$ **l** $x\tan x$

3 Differentiate with respect to x:

a $\sin(x^2)$ **b** $\cos(\sqrt{x})$ **c** $\sqrt{\cos x}$ **d** $\sin^2 x$
e $\cos^3 x$ **f** $\cos x \sin(2x)$ **g** $\cos(\cos x)$ **h** $\cos^3(4x)$
i $\frac{1}{\sin x}$ **j** $\frac{1}{\cos(2x)}$ **k** $\frac{2}{\sin^2(2x)}$ **l** $\frac{8}{\tan^3(\frac{x}{2})}$

4 Find the gradient of the tangent to:

a $f(x) = \sin^3 x$ at the point where $x = \frac{2\pi}{3}$
b $f(x) = \cos x \sin x$ at the point where $x = \frac{\pi}{4}$.

L SECOND DERIVATIVES

Given a function $f(x)$, the derivative $f'(x)$ is known as the **first derivative**.

The **second derivative** of $f(x)$ is the derivative of $f'(x)$, or **the derivative of the first derivative**.

We use $f''(x)$ or y'' or $\frac{d^2y}{dx^2}$ to represent the second derivative.

$f''(x)$ reads "*f double dashed x*".

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ reads "*dee two y by dee x squared*".

Example 18 — Self Tutor

Find $f''(x)$ given that $f(x) = x^3 - \frac{3}{x}$.

$$\begin{aligned} \text{Now} \quad f(x) &= x^3 - 3x^{-1} \\ \therefore \quad f'(x) &= 3x^2 + 3x^{-2} \\ \therefore \quad f''(x) &= 6x - 6x^{-3} \\ &= 6x - \frac{6}{x^3} \end{aligned}$$

EXERCISE 13L

1 Find $f''(x)$ given that:

a $f(x)=3x^2-6x+2$ **b** $f(x)=\frac{2}{\sqrt{x}}-1$ **c** $f(x)=2x^3-3x^2-x+5$

d $f(x)=\frac{2-3x}{x^2}$ **e** $f(x)=(1-2x)^3$ **f** $f(x)=\frac{x+2}{2x-1}$

2 Find $\frac{d^2y}{dx^2}$ given that:

a $y=x-x^3$ **b** $y=x^2-\frac{5}{x^2}$ **c** $y=2-\frac{3}{\sqrt{x}}$

d $y=\frac{4-x}{x}$ **e** $y=(x^2-3x)^3$ **f** $y=x^2-x+\frac{1}{1-x}$

3 Given $f(x)=x^3-2x+5$, find:

a $f(2)$ **b** $f'(2)$ **c** $f''(2)$

4 Suppose $y=Ae^{kx}$ where A and k are constants. Show that:

a $\frac{dy}{dx}=ky$ **b** $\frac{d^2y}{dx^2}=k^2y$

5 Find the value(s) of x such that $f''(x)=0$, given:

a $f(x)=2x^3-6x^2+5x+1$ **b** $f(x)=\frac{x}{x^2+2}$

6 Consider the function $f(x)=2x^3-x$.
Complete the following table by indicating whether $f(x)$, $f'(x)$, and $f''(x)$ are positive (+), negative (–), or zero (0) at the given values of x.

x	-1	0	1
$f(x)$	$-$		
$f'(x)$			
$f''(x)$			

7 Suppose $f(x)=2\sin^3 x-3\sin x$.

a Show that $f'(x)=-3\cos x\cos 2x$. **b** Find $f''(x)$.

8 Find $\frac{d^2y}{dx^2}$ given:

a $y=-\ln x$ **b** $y=x\ln x$ **c** $y=(\ln x)^2$

9 Given $f(x)=x^2-\frac{1}{x}$, find:

a $f(1)$ **b** $f'(1)$ **c** $f''(1)$

10 If $y=2e^{3x}+5e^{4x}$, show that $\frac{d^2y}{dx^2}-7\frac{dy}{dx}+12y=0$.

11 If $y=\sin(2x+3)$, show that $\frac{d^2y}{dx^2}+4y=0$.

12 If $y=2\sin x+3\cos x$, show that $y''+y=0$ where y'' represents $\frac{d^2y}{dx^2}$.

Review set 13A

1 Evaluate:

a $\lim\limits_{x \to 1} (6x - 7)$ **b** $\lim\limits_{h \to 0} \dfrac{2h^2 - h}{h}$ **c** $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x - 4}$

2 Find, from first principles, the derivative of:

a $f(x) = x^2 + 2x$ **b** $y = 4 - 3x^2$

3 In the **Opening Problem** on page **334**, the altitude of the jumper is given by $f(t) = 452 - 4.8t^2$ metres, where $0 \leqslant t \leqslant 3$ seconds.

a Find $f'(t) = \lim\limits_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$.

b *Hence* find the speed of the jumper when $t = 2$ seconds.

4 If $f(x) = 7 + x - 3x^2$, find: **a** $f(3)$ **b** $f'(3)$ **c** $f''(3)$.

5 Find $\dfrac{dy}{dx}$ for: **a** $y = 3x^2 - x^4$ **b** $y = \dfrac{x^3 - x}{x^2}$

6 At what point on the curve $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ does the tangent have gradient 1?

7 Find $\dfrac{dy}{dx}$ if: **a** $y = e^{x^3+2}$ **b** $y = \ln\left(\dfrac{x+3}{x^2}\right)$

8 Given $y = 3e^x - e^{-x}$, show that $\dfrac{d^2y}{dx^2} = y$.

9 Differentiate with respect to x:

a $5x - 3x^{-1}$ **b** $(3x^2 + x)^4$ **c** $(x^2+1)(1-x^2)^3$

10 Find all points on the curve $y = 2x^3 + 3x^2 - 10x + 3$ where the gradient of the tangent is 2.

11 If $y = \sqrt{5 - 4x}$, find: **a** $\dfrac{dy}{dx}$ **b** $\dfrac{d^2y}{dx^2}$

12 Differentiate with respect to x:

a $\sin(5x)\ln(x)$ **b** $\sin(x)\cos(2x)$ **c** $e^{-2x}\tan x$

13 Find the gradient of the tangent to $y = \sin^2 x$ at the point where $x = \frac{\pi}{3}$.

14 Find the derivative with respect to x of:

a $f(x) = (x^2+3)^4$ **b** $g(x) = \dfrac{\sqrt{x+5}}{x^2}$

15 Find $f''(2)$ for:

a $f(x) = 3x^2 - \dfrac{1}{x}$ **b** $f(x) = \sqrt{x}$

16 Differentiate with respect to x:

a $10x - \sin(10x)$ **b** $\ln\left(\dfrac{1}{\cos x}\right)$ **c** $\sin(5x)\ln(2x)$

Review set 13B

1 Evaluate the limits:

a $\lim\limits_{h \to 0} \dfrac{h^3 - 3h}{h}$ **b** $\lim\limits_{x \to 1} \dfrac{3x^2 - 3x}{x - 1}$ **c** $\lim\limits_{x \to 2} \dfrac{x^2 - 3x + 2}{2 - x}$

2 Given $f(x) = 5x - x^2$, find $f'(1)$ from first principles.

3 **a** Given $y = 2x^2 - 1$, find $\dfrac{dy}{dx}$ from first principles.

b Hence state the gradient of the tangent to $y = 2x^2 - 1$ at the point where $x = 4$.

c For what value of x is the gradient of the tangent to $y = 2x^2 - 1$ equal to -12?

4 Differentiate with respect to x: **a** $y = x^3\sqrt{1 - x^2}$ **b** $y = \dfrac{x^2 - 3x}{\sqrt{x + 1}}$

5 Find $\dfrac{d^2y}{dx^2}$ for: **a** $y = 3x^4 - \dfrac{2}{x}$ **b** $y = x^3 - x + \dfrac{1}{\sqrt{x}}$

6 Find all points on the curve $y = xe^x$ where the gradient of the tangent is $2e$.

7 Differentiate with respect to x: **a** $f(x) = \ln(e^x + 3)$ **b** $f(x) = \ln\left[\dfrac{(x+2)^3}{x}\right]$

8 Suppose $y = \left(x - \dfrac{1}{x}\right)^4$. Find $\dfrac{dy}{dx}$ when $x = 1$.

9 Find $\dfrac{dy}{dx}$ if: **a** $y = \ln(x^3 - 3x)$ **b** $y = \dfrac{e^x}{x^2}$

10 Find x if $f''(x) = 0$ and $f(x) = 2x^4 - 4x^3 - 9x^2 + 4x + 7$.

11 If $f(x) = x - \cos x$, find

a $f(\pi)$ **b** $f'(\frac{\pi}{2})$ **c** $f''(\frac{3\pi}{4})$

12 **a** Find $f'(x)$ and $f''(x)$ for $f(x) = \sqrt{x}\cos(4x)$.

b Hence find $f'(\frac{\pi}{16})$ and $f''(\frac{\pi}{8})$.

13 Suppose $y = 3\sin 2x + 2\cos 2x$. Show that $4y + \dfrac{d^2y}{dx^2} = 0$.

14 Consider $f(x) = \dfrac{6x}{3 + x^2}$. Find the value(s) of x when:

a $f(x) = -\frac{1}{2}$ **b** $f'(x) = 0$ **c** $f''(x) = 0$

15 The function f is defined by $f : x \mapsto -10\sin 2x \cos 2x$, $0 \leqslant x \leqslant \pi$.

a Write down an expression for $f(x)$ in the form $k\sin 4x$.

b Solve $f'(x) = 0$, giving exact answers.

16 Given that a and b are constants, differentiate $y = 3\sin bx - a\cos 2x$ with respect to x.

Find a and b if $y + \dfrac{d^2y}{dx^2} = 6\cos 2x$.

14

Applications of differential calculus

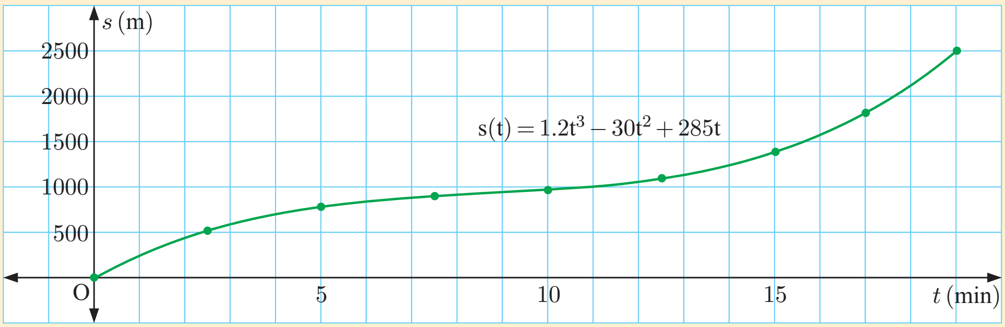
Contents:

Opening problem

Michael rides up a hill and down the other side to his friend's house. The dots on the graph show Michael's position at various times t.

DEMO

The distance Michael has travelled at various times is given by the function
$s(t) = 1.2t^3 - 30t^2 + 285t$ metres for $0 \leqslant t \leqslant 19$ minutes.



Things to think about:

- **a** Can you find a function for Michael's *speed* at any time t?
- **b** Michael's *acceleration* is the rate at which his speed is changing with respect to time. How can we interpret $s''(t)$?
- **c** Can you find Michael's speed and acceleration at the time $t = 15$ minutes?
- **d** At what point do you think the hill was steepest? How far had Michael travelled to this point?

In the previous chapter we saw how to differentiate many types of functions.

In this chapter we will use derivatives to find:

- tangents and normals to curves
- turning points which are local minima and maxima.

We will then look at applying these techniques to real world problems including:

- kinematics (motion problems of displacement, velocity, and acceleration)
- rates of change
- optimisation (maxima and minima).

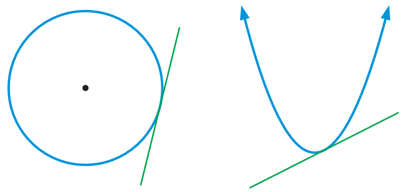
A TANGENTS AND NORMALS

TANGENTS

The **tangent** to a curve at point A is the best approximating straight line to the curve at A.

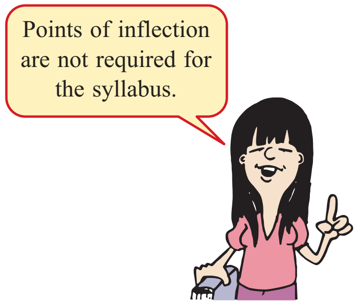
In cases we have seen already, the tangent *touches* the curve.

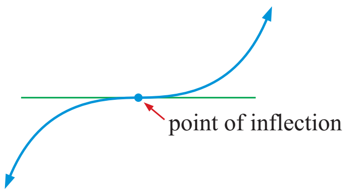
For example, consider tangents to a circle or a quadratic.



However, we note that for some functions:

- The tangent may intersect the curve again somewhere else.
- It is possible for the tangent to pass through the curve at the point of tangency. If this happens, we call it a **point of inflection**.





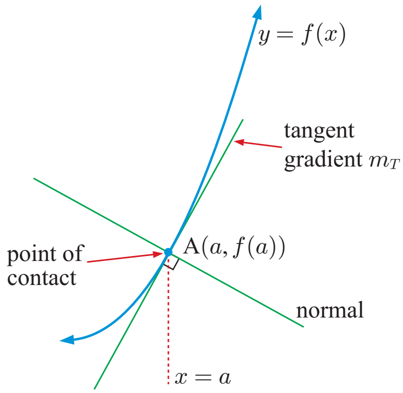
Consider a curve $y = f(x)$.

If A is the point with x-coordinate a, then the gradient of the tangent to the curve at this point is $f'(a) = m_T$.

The equation of the tangent is

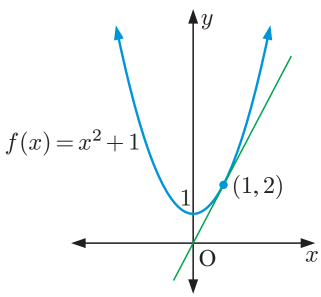
$$y - f(a) = f'(a)(x - a)$$

or $\boldsymbol{y = f(a) + f'(a)(x - a)}$.



Example 1 — Self Tutor

Find the equation of the tangent to $f(x) = x^2 + 1$ at the point where $x = 1$.



Since $f(1) = 1 + 1 = 2$, the point of contact is $(1, 2)$.

Now $f'(x) = 2x$, so $m_T = f'(1) = 2$

$\therefore$ the tangent has equation $y = 2 + 2(x - 1)$

which is $y = 2x$.

NORMALS

A **normal** to a curve is a line which is perpendicular to the tangent at the point of contact.

The gradients of perpendicular lines are negative reciprocals of each other, so:

> The gradient of the normal to the curve at $x = a$ is $m_N = -\dfrac{1}{f'(a)}$.
>
> The equation of the normal to the curve at $x = a$ is $\boldsymbol{y = f(a) - \dfrac{1}{f'(a)}(x - a)}$.

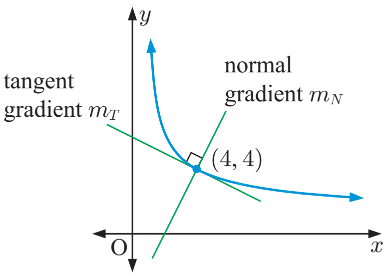
Reminder: If a line has gradient $\frac{4}{5}$ and passes through $(2, -3)$, another quick way to write down its equation is $4x - 5y = 4(2) - 5(-3)$ or $4x - 5y = 23$.

If the gradient was $-\frac{4}{5}$, we would have:
$4x + 5y = 4(2) + 5(-3)$ or $4x + 5y = -7$.

Example 2 — Self Tutor

Find the equation of the normal to $y = \dfrac{8}{\sqrt{x}}$ at the point where $x = 4$.

When $x = 4$, $y = \dfrac{8}{\sqrt{4}} = \dfrac{8}{2} = 4$. So, the point of contact is $(4, 4)$.



Now as $y = 8x^{-\frac{1}{2}}$, $\dfrac{dy}{dx} = -4x^{-\frac{3}{2}}$

$\therefore$ when $x = 4$, $m_T = -4 \times 4^{-\frac{3}{2}} = -\frac{1}{2}$

$\therefore$ the normal at $(4, 4)$ has gradient $m_N = \frac{2}{1}$.

$\therefore$ the equation of the normal is
$2x - 1y = 2(4) - 1(4)$
or $2x - y = 4$

EXERCISE 14A

1 Find the equation of the tangent to:

a $y = x - 2x^2 + 3$ at $x = 2$

b $y = \sqrt{x} + 1$ at $x = 4$

c $y = x^3 - 5x$ at $x = 1$

d $y = \dfrac{4}{\sqrt{x}}$ at $(1, 4)$

e $y = \dfrac{3}{x} - \dfrac{1}{x^2}$ at $(-1, -4)$

f $y = 3x^2 - \dfrac{1}{x}$ at $x = -1$.

2 Find the equation of the normal to:

a $y = x^2$ at the point $(3, 9)$

b $y = x^3 - 5x + 2$ at $x = -2$

c $y = \dfrac{5}{\sqrt{x}} - \sqrt{x}$ at the point $(1, 4)$

d $y = 8\sqrt{x} - \dfrac{1}{x^2}$ at $x = 1$.

Example 3 — Self Tutor

Find the equations of any horizontal tangents to $y = x^3 - 12x + 2$.

Since $y = x^3 - 12x + 2$, $\frac{dy}{dx} = 3x^2 - 12$

Horizontal tangents have gradient 0, so $3x^2 - 12 = 0$

$$\therefore\ 3(x^2 - 4) = 0$$
$$\therefore\ 3(x+2)(x-2) = 0$$
$$\therefore\ x = -2 \text{ or } 2$$

When $x = 2$, $y = 8 - 24 + 2 = -14$

When $x = -2$, $y = -8 + 24 + 2 = 18$

$\therefore$ the points of contact are $(2, -14)$ and $(-2, 18)$

$\therefore$ the tangents are $y = -14$ and $y = 18$.

3 Find the equations of any horizontal tangents to $y = 2x^3 + 3x^2 - 12x + 1$.

4 Find the points of contact where horizontal tangents meet the curve $y = 2\sqrt{x} + \frac{1}{\sqrt{x}}$.

5 Find k if the tangent to $y = 2x^3 + kx^2 - 3$ at the point where $x = 2$ has gradient 4.

6 Find the equation of another tangent to $y = 1 - 3x + 12x^2 - 8x^3$ which is parallel to the tangent at $(1, 2)$.

7 Consider the curve $y = x^2 + ax + b$ where a and b are constants. The tangent to this curve at the point where $x = 1$ is $2x + y = 6$. Find the values of a and b.

8 Consider the curve $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$ where a and b are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of a and b.

9 Show that the equation of the tangent to $y = 2x^2 - 1$ at the point where $x = a$, is $4ax - y = 2a^2 + 1$.

10 Find the equation of the tangent to:

a $y = \sqrt{2x+1}$ at $x = 4$

b $y = \frac{1}{2-x}$ at $x = -1$

c $f(x) = \frac{x}{1-3x}$ at $(-1, -\frac{1}{4})$

d $f(x) = \frac{x^2}{1-x}$ at $(2, -4)$.

11 Find the equation of the normal to:

a $y = \frac{1}{(x^2+1)^2}$ at $(1, \frac{1}{4})$

b $y = \frac{1}{\sqrt{3-2x}}$ at $x = -3$

c $f(x) = \sqrt{x}(1-x)^2$ at $x = 4$

d $f(x) = \frac{x^2-1}{2x+3}$ at $x = -1$.

12 Consider the curve $y = a\sqrt{1-bx}$ where a and b are constants. The tangent to this curve at the point where $x = -1$ is $3x + y = 5$. Find the values of a and b.

Example 4

Self Tutor

Show that the equation of the tangent to $y = \ln x$ at the point where $y = -1$ is $y = ex - 2$.

When $y = -1$, $\ln x = -1$

$\therefore\ x = e^{-1} = \frac{1}{e}$

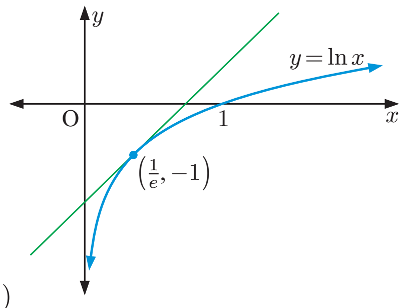
$\therefore$ the point of contact is $\left(\frac{1}{e}, -1\right)$.

Now $f(x) = \ln x$ has derivative $f'(x) = \dfrac{1}{x}$

$\therefore$ the tangent at $\left(\frac{1}{e}, -1\right)$ has gradient $\dfrac{1}{\frac{1}{e}} = e$

$\therefore$ the tangent has equation $y = -1 + e(x - \frac{1}{e})$

which is $y = ex - 2$



13 Find the equation of:

a the tangent to $f : x \mapsto e^{-x}$ at the point where $x = 1$

b the tangent to $y = \ln(2 - x)$ at the point where $x = -1$

c the normal to $y = \ln\sqrt{x}$ at the point where $y = -1$.

Example 5

Self Tutor

Find the equation of the tangent to $y = \tan x$ at the point where $x = \frac{\pi}{4}$.

When $x = \frac{\pi}{4}$, $y = \tan\frac{\pi}{4} = 1$

$\therefore$ the point of contact is $(\frac{\pi}{4}, 1)$.

Now $f(x) = \tan x$ has derivative $f'(x) = \sec^2 x$

$\therefore$ the tangent at $(\frac{\pi}{4}, 1)$ has gradient $\sec^2\frac{\pi}{4} = (\sqrt{2})^2 = 2$

$\therefore$ the tangent has equation $y = 1 + 2(x - \frac{\pi}{4})$

which is $y = 2x + (1 - \frac{\pi}{2})$

14 Show that the curve with equation $y = \dfrac{\cos x}{1 + \sin x}$ does not have any horizontal tangents.

15 Find the equation of:

a the tangent to $y = \sin x$ at the origin

b the tangent to $y = \tan x$ at the origin

c the normal to $y = \cos x$ at the point where $x = \frac{\pi}{6}$

d the normal to $y = \dfrac{1}{\sin(2x)}$ at the point where $x = \frac{\pi}{4}$.

Example 6

Self Tutor

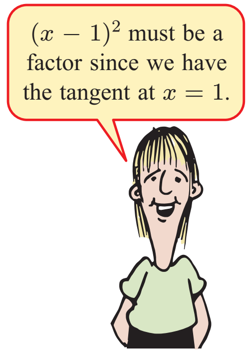
Find where the tangent to $y = x^3 + x + 2$ at $(1, 4)$ meets the curve again.

Let $f(x) = x^3 + x + 2$

$\therefore\ f'(x) = 3x^2 + 1$ and $\therefore\ f'(1) = 3 + 1 = 4$

$\therefore$ the equation of the tangent at $(1, 4)$ is $4x - y = 4(1) - 4$

or $y = 4x$.



The curve meets the tangent again when $x^3 + x + 2 = 4x$

$$\therefore\ x^3 - 3x + 2 = 0$$

$$\therefore\ (x-1)^2(x+2) = 0$$

When $x = -2$, $y = (-2)^3 + (-2) + 2 = -8$

$\therefore$ the tangent meets the curve again at $(-2, -8)$.

16 **a** Find where the tangent to the curve $y = x^3$ at the point where $x = 2$, meets the curve again.

b Find where the tangent to the curve $y = -x^3 + 2x^2 + 1$ at the point where $x = -1$, meets the curve again.

17 Consider the function $f(x) = x^2 + \dfrac{4}{x^2}$.

a Find $f'(x)$.

b Find the values of x at which the tangent to the curve is horizontal.

c Show that the tangents at these points are the same line.

18 The tangent to $y = x^2e^x$ at $x = 1$ cuts the x and y-axes at A and B respectively. Find the coordinates of A and B.

Example 7

Self Tutor

Find the equations of the tangents to $y = x^2$ from the external point $(2, 3)$.

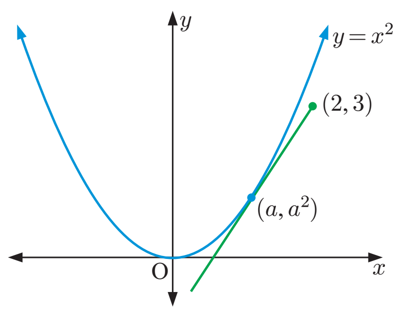
Let (a, a^2) be a general point on $f(x) = x^2$.

Now $f'(x) = 2x$, so $f'(a) = 2a$

$\therefore$ the equation of the tangent at (a, a^2) is

$$y = a^2 + 2a(x - a)$$

which is $y = 2ax - a^2$



Thus the tangents which pass through $(2, 3)$ satisfy

$$3 = 2a(2) - a^2$$

$$\therefore\ a^2 - 4a + 3 = 0$$

$$\therefore\ (a-1)(a-3) = 0$$

$$\therefore\ a = 1 \text{ or } 3$$

$\therefore$ exactly two tangents pass through the external point $(2, 3)$.

If $a = 1$, the tangent has equation $y = 2x - 1$ with point of contact $(1, 1)$.

If $a = 3$, the tangent has equation $y = 6x - 9$ with point of contact $(3, 9)$.

19 **a** Find the equation of the tangent to $y = x^2 - x + 9$ at the point where $x = a$.

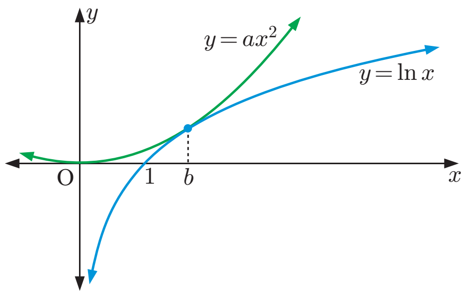
b Hence, find the equations of the two tangents from $(0, 0)$ to the curve. State the coordinates of the points of contact.

20 Find the equations of the tangents to $y = x^3$ from the external point $(-2, 0)$.

21 Find the equation of the normal to $y = \sqrt{x}$ from the external point $(4, 0)$.
Hint: There is no normal at the point where $x = 0$, as this is the endpoint of the function.

22 Find the equation of the tangent to $y = e^x$ at the point where $x = a$.
Hence, find the equation of the tangent to $y = e^x$ which passes through the origin.

23 A quadratic of the form $y = ax^2$, $a > 0$, touches the logarithmic function $y = \ln x$ as shown.

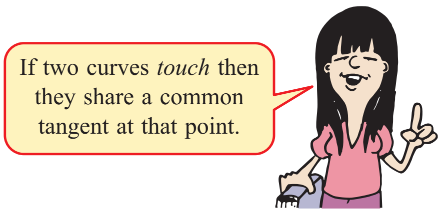


a If the x-coordinate of the point of contact is b, explain why $ab^2 = \ln b$ and $2ab = \frac{1}{b}$.

b Deduce that the point of contact is $(\sqrt{e}, \frac{1}{2})$.

c Find the value of a.

d Find the equation of the common tangent.



24 Find, correct to 2 decimal places, the angle between the tangents to $y = 3e^{-x}$ and $y = 2 + e^x$ at their point of intersection.

25 Consider the cubic function $f(x) = 2x^3 + 5x^2 - 4x - 3$.

a Show that the equation of the tangent to the curve at the point where $x = -1$ can be written in the form $y = 4 - 8(x + 1)$.

b Show that $f(x)$ can be written in the form $f(x) = 4 - 8(x + 1) - (x + 1)^2 + 2(x + 1)^3$.

c Hence explain why the tangent is the best approximating straight line to the curve at the point where $x = -1$.

26 A cubic has three real roots. Prove that the tangent line at the average of any two roots of the cubic, passes through the third root.
Hint: Let $f(x) = a(x - \alpha)(x - \beta)(x - \gamma)$.

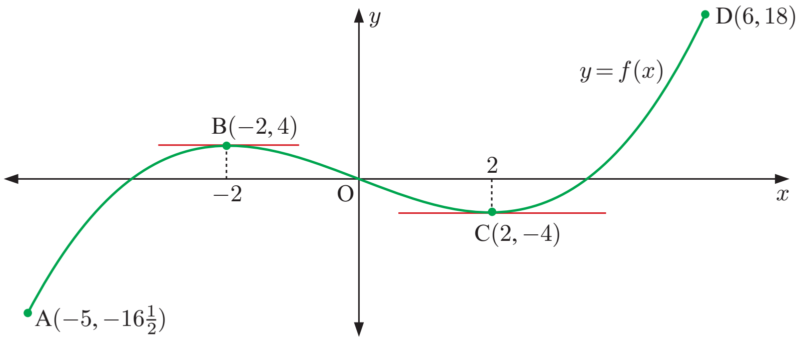
B STATIONARY POINTS

A **stationary point** of a function is a point where $f'(x) = 0$.

It could be a local maximum, local minimum, or stationary inflection.

TURNING POINTS (MAXIMA AND MINIMA)

Consider the following graph which has a restricted domain of $-5 \leqslant x \leqslant 6$.

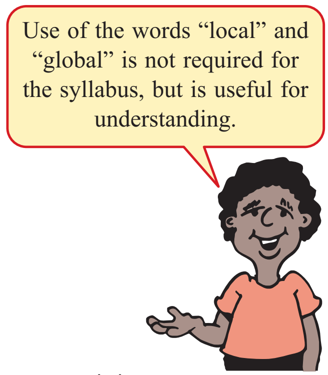


A is a **global minimum** as it has the minimum value of y on the entire domain.

B is a **local maximum** as it is a turning point where $f'(x) = 0$ and the curve has shape  .

C is a **local minimum** as it is a turning point where $f'(x) = 0$ and the curve has shape  .

D is a **global maximum** as it is the maximum value of y on the entire domain.



For many functions, a local maximum or minimum is also the global maximum or minimum.

For example, for $y = x^2$ the point $(0, 0)$ is a local minimum and is also the global minimum.

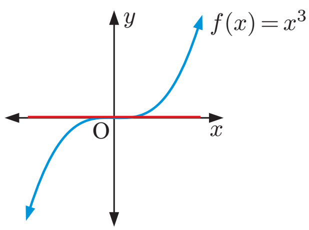
STATIONARY POINTS OF INFLECTION

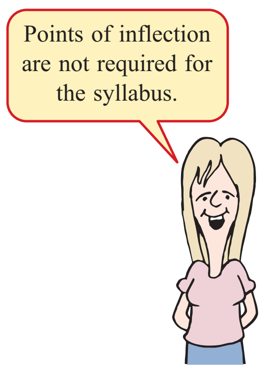
It is not always true that whenever we find a value of x where $f'(x) = 0$, we have a local maximum or minimum.

For example,

$f(x) = x^3$ has $f'(x) = 3x^2$,

so $f'(x) = 0$ when $x = 0$.





The x-axis is a tangent to the curve which actually crosses over the curve at O$(0, 0)$. This tangent is horizontal, but O$(0, 0)$ is neither a local maximum nor a local minimum. It is called a **stationary inflection** as the curve changes its curvature or shape.

SIGN DIAGRAMS

A **sign diagram** is used to display the intervals on which a function is positive and negative.

In calculus we commonly use sign diagrams of the *derivative function* $f'(x)$ so we can determine the nature of a stationary point.

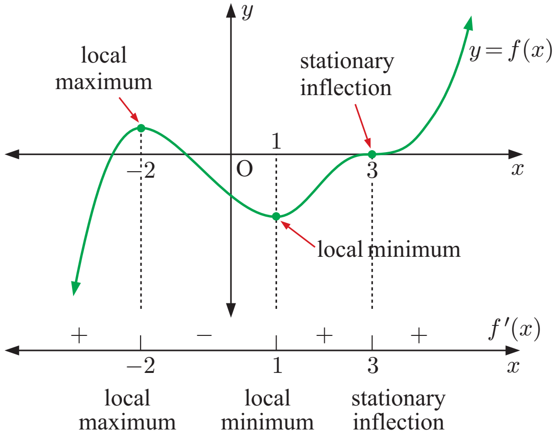
Consider the graph alongside.

The sign diagram of its gradient function is shown directly beneath it.

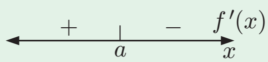
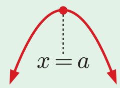
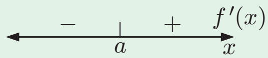
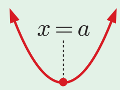
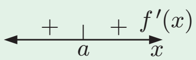
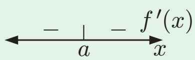
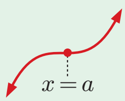
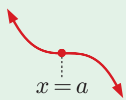
We can use the sign diagram to describe the stationary points of the function.

The signs on the sign diagram of $f'(x)$ indicate whether the gradient of $y = f(x)$ is positive or negative in that interval.

DEMO



We observe the following properties:

Stationary point where $f'(a) = 0$	*Sign diagram of* $f'(x)$ *near* $x = a$	*Shape of curve near* $x = a$
local maximum		
local minimum		
stationary inflection	 or 	 or 

Example 8 — Self Tutor

Consider the function $f(x) = x^3 - 3x^2 - 9x + 5$.

a Find the y-intercept.

b Find and classify all stationary points.

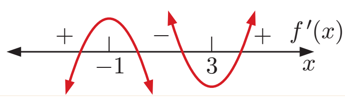
c Hence sketch the curve $y = f(x)$.

a $f(0) = 5$, so the y-intercept is 5.

b $f(x) = x^3 - 3x^2 - 9x + 5$

$$\begin{aligned} \therefore \quad f'(x) &= 3x^2 - 6x - 9 \\ &= 3(x^2 - 2x - 3) \\ &= 3(x-3)(x+1) \end{aligned}$$

which has sign diagram:

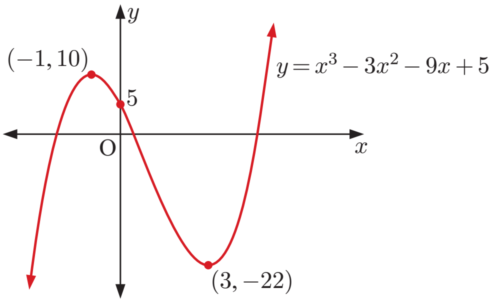


So, we have a local maximum at $x=-1$ and a local minimum at $x=3$.

$$f(-1)=(-1)^3-3(-1)^2-9(-1)+5=10$$
$$f(3)=3^3-3\times 3^2-9\times 3+5=-22$$

$\therefore$ there is a local maximum at $(-1,\ 10)$ and a local minimum at $(3,\ -22)$.

c



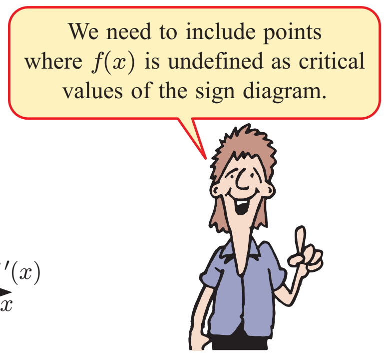
Example 9

Self Tutor

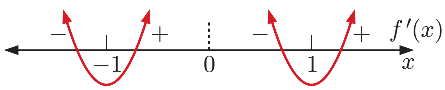
Find and classify all stationary points of $f(x)=\dfrac{x^2+1}{x}$.

$$=\frac{x^2+1}{x}$$

$$\therefore\ f'(x)=\frac{2x(x)-(x^2+1)}{x^2}$$
$$=\frac{x^2-1}{x^2}$$
$$=\frac{(x+1)(x-1)}{x}$$



$f'(x)$ has sign diagram:



So, we have local minima when $x=\pm 1$.

$$f(-1)=\frac{(-1)^2+1}{(-1)}=-2 \quad\text{and}\quad f(1)=\frac{1^2+1}{1}=2$$

$\therefore$ there are local minima at $(-1,\ -2)$ and $(1,\ 2)$.

SECOND DERIVATIVES AND STATIONARY POINTS

The second derivative of a function can be used to determine the nature of its stationary points.

For a function $f(x)$ with a stationary point at $x=a$:

- If $f''(a)>0$, then it is a **local minimum**.
- If $f''(a)<0$, then it is a **local maximum**.
- If $f''(a)=0$, then it could be a **local maximum**, a **local minimum**, or a **stationary inflection point**.

Example 10 — Self Tutor

Find and classify all stationary points of $f(x) = 2x^3 + 3x^2 - 12$.

$$f(x) = 2x^3 + 3x^2 - 12$$
$$\therefore\ f'(x) = 6x^2 + 6x = 6x(x+1)$$
$$\therefore\ f'(x) = 0 \text{ when } 6x = 0 \text{ or } x+1=0$$
$$\therefore\ x = 0 \text{ or } x = -1$$

Also, $f''(x) = 12x + 6$

$\therefore\ f''(0) = 12(0) + 6 = 6$ which is > 0

and $f''(-1) = 12(-1) + 6 = -6$ which is < 0

So, we have a local minimum at $x = 0$ and a local maximum at $x = -1$.

Now $f(0) = 2(0)^3 + 3(0)^2 - 12 = -12$

$f(-1) = 2(-1)^3 + 3(-1)^2 - 12 = -11$

$\therefore$ there is a local minimum at $(0, -12)$ and a local maximum at $(-1, -11)$.

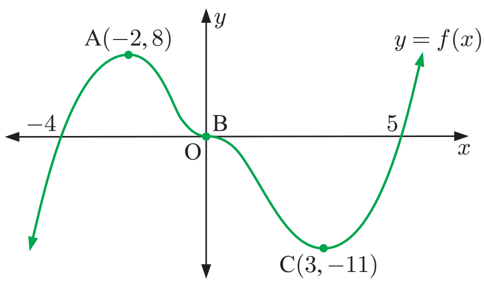
EXERCISE 14B

1 The tangents at points A, B, and C are horizontal.

a Classify points A, B, and C.

b Draw a sign diagram for:

i $f(x)$ **ii** $f'(x)$



2 For each of the following functions, find and classify any stationary points. Sketch the function, showing all important features.

a $f(x) = x^2 - 2$

b $f(x) = x^3 + 1$

c $f(x) = x^3 - 3x + 2$

d $f(x) = x^4 - 2x^2$

e $f(x) = x^3 - 6x^2 + 12x - 7$

f $f(x) = \sqrt{x} + 2$

g $f(x) = x - \sqrt{x}$

h $f(x) = x^4 - 6x^2 + 8x - 3$

i $f(x) = 1 - x\sqrt{x}$

j $f(x) = x^4 - 2x^2 - 8$

GRAPHING PACKAGE

3 At what value of x does the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, have a stationary point? Under what conditions is the stationary point a local maximum or a local minimum?

4 $f(x) = 2x^3 + ax^2 - 24x + 1$ has a local maximum at $x = -4$. Find a.

5 $f(x) = x^3 + ax + b$ has a stationary point at $(-2, 3)$.

a Find the values of a and b.

b Find the position and nature of all stationary points.

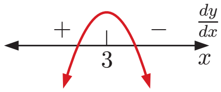
Example 11 — Self Tutor

Find the exact position and nature of the stationary point of $y = (x-2)e^{-x}$.

$$\frac{dy}{dx} = (1)e^{-x} + (x-2)e^{-x}(-1) \qquad \{\text{product rule}\}$$
$$= e^{-x}(1-(x-2))$$
$$= \frac{3-x}{e^x} \quad \text{where } e^x \text{ is positive for all } x$$

So, $\frac{dy}{dx} = 0$ when $x = 3$.

The sign diagram of $\frac{dy}{dx}$ is:



$\therefore$ at $x = 3$ we have a local maximum.

But when $x = 3$, $y = (1)e^{-3} = \frac{1}{e^3}$

$\therefore$ the local maximum is at $(3, \frac{1}{e^3})$.

To determine the nature of a stationary point, we can use a sign diagram or the second derivative.

6 Find the position and nature of the stationary point(s) of:

a $y = xe^{-x}$ **b** $y = x^2e^x$ **c** $y = \frac{e^x}{x}$ **d** $y = e^{-x}(x+2)$

7 Consider $f(x) = x\ln x$.

a For what values of x is $f(x)$ defined? **b** Show that the global minimum value of $f(x)$ is $-\frac{1}{e}$.

8 Find the greatest and least value of:

a $x^3 - 12x - 2$ for $-3 \leqslant x \leqslant 5$ **b** $4 - 3x^2 + x^3$ for $-2 \leqslant x \leqslant 3$

9 The cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ touches the line with equation $y = 9x + 2$ at the point $(0, 2)$, and has a stationary point at $(-1, -7)$. Find $P(x)$.

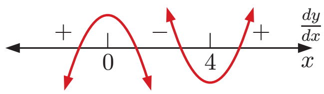
Example 12 — Self Tutor

Find the greatest and least value of $y = x^3 - 6x^2 + 5$ on the interval $-2 \leqslant x \leqslant 5$.

Now $\frac{dy}{dx} = 3x^2 - 12x$
$= 3x(x-4)$

$\therefore \frac{dy}{dx} = 0$ when $x = 0$ or 4

The sign diagram of $\frac{dy}{dx}$ is:



If the domain is restricted, we need to check the value of the function at the endpoints of the domain.

$\therefore$ there is a local maximum at $x = 0$, and a local minimum at $x = 4$.

Critical value (x)	$f(x)$
-2 (endpoint)	-27
0 (local max)	5
4 (local min)	-27
5 (endpoint)	-20

The greatest of these values is 5 when $x = 0$.

The least of these values is -27 when $x = -2$ and when $x = 4$.

10 For each of the following, determine the position and nature of the stationary points on the interval $0 \leqslant x \leqslant 2\pi$, then show them on a graph of the function.

GRAPHING PACKAGE

a $f(x) = \sin x$ **b** $f(x) = \cos(2x)$ **c** $f(x) = \sin^2 x$

d $f(x) = e^{\sin x}$ **e** $f(x) = \sin(2x) + 2\cos x$

11 Show that $y = 4e^{-x}\sin x$ has a local maximum when $x = \frac{\pi}{4}$.

12 Prove that $\dfrac{\ln x}{x} \leqslant \dfrac{1}{e}$ for all $x > 0$. **Hint:** Let $f(x) = \dfrac{\ln x}{x}$ and find its greatest value.

13 Consider the function $f(x) = x - \ln x$.

a Show that the graph of $y = f(x)$ has a local minimum and that this is the only turning point.

b Hence prove that $\ln x \leqslant x - 1$ for all $x > 0$.

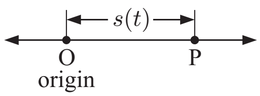
C KINEMATICS

In the **Opening Problem** we are dealing with the movement of Michael riding his bicycle. We do not know the direction Michael is travelling, so we talk simply about the *distance* he has travelled and his *speed*.

For problems of **motion in a straight line**, we can include the direction the object is travelling along the line. We therefore can talk about *displacement* and *velocity*.

DISPLACEMENT

Suppose an object P moves along a straight line so that its position s from an origin O is given as some function of time t. We write $s = s(t)$ where $t \geqslant 0$.



$s(t)$ is a **displacement function** and for any value of t it gives the displacement from O.

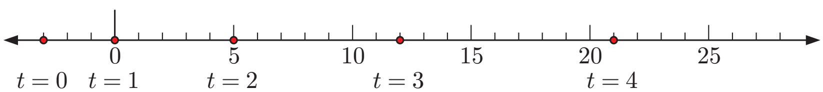
$s(t)$ is a vector quantity. Its magnitude is the distance from O, and its sign indicates the direction from O.

For example, consider $s(t) = t^2 + 2t - 3$ cm.

$s(0) = -3$ cm, $s(1) = 0$ cm, $s(2) = 5$ cm, $s(3) = 12$ cm, $s(4) = 21$ cm.

To appreciate the motion of P we draw a **motion graph**. You can also view the motion by clicking on the icon.

DEMO



VELOCITY

The **average velocity** of an object moving in a straight line in the time interval from $t = t_1$ to $t = t_2$ is the ratio of the change in displacement to the time taken.

If $s(t)$ is the displacement function then **average velocity** $= \dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$.

On a graph of $s(t)$ against t for the time interval from $t = t_1$ to $t = t_2$, the average velocity is the gradient of a chord through the points $(t_1, s(t_1))$ and $(t_2, s(t_2))$.

In **Chapter 13** we established that the instantaneous rate of change of a quantity is given by its derivative.

If $s(t)$ is the displacement function of an object moving in a straight line, then $v(t) = s'(t) = \lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$ is the **instantaneous velocity** or **velocity function** of the object at time t.

On a graph of $s(t)$ against t, the instantaneous velocity at a particular time is the gradient of the tangent to the graph at that point.

ACCELERATION

If an object moves in a straight line with velocity function $v(t)$ then:

- the **average acceleration** for the time interval from $t = t_1$ to $t = t_2$ is the ratio of the change in velocity to the time taken

$$\textbf{average acceleration} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

- the **instantaneous acceleration** at time t is $a(t) = v'(t) = \lim\limits_{h \to 0} \dfrac{v(t+h) - v(t)}{h}$.

UNITS

Each time we differentiate with respect to time t, we calculate a rate per unit of time. So, for a displacement in metres and time in seconds:

- the units of velocity are m s^{-1}
- the units of acceleration are m s^{-2}.

Discussion

- What is the relationship between the displacement function $s(t)$ and the acceleration function $a(t)$?
- How are the units of velocity and acceleration related to their formulae? You may wish to research "dimensional analysis".