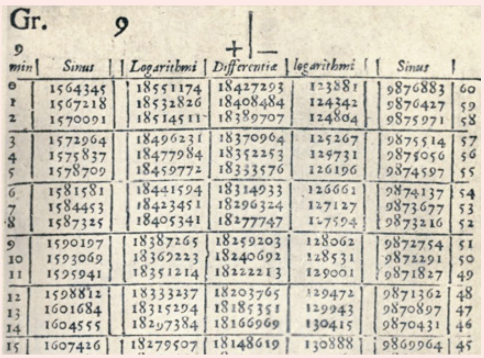
Historical note — *The invention of logarithm*

It is easy to take modern technology, such as the electronic calculator, for granted. Until electronic computers became affordable in the 1980s, a "calculator" was a *profession*, literally someone who would spend their time performing calculations by hand. They used mechanical calculators and technology such as logarithms. They often worked in banks, but sometimes for astronomers and other scientists.

The logarithm was invented by **John Napier** (1550 - 1617) and first published in 1614 in a Latin book which translates as a *Description of the Wonderful Canon of Logarithms.* John Napier was the 8th Lord of Merchiston, which is now part of Edinburgh, Scotland. Napier wrote a number of other books on many subjects including religion and mathematics. One of his other inventions was a device for performing long multiplication which is now called "Napier's Bones". Other calculators, such as slide rules, used logarithms as part of their design. He also popularised the use of the decimal point in mathematical notation.

John Napier

In Napier's time, mathematicians did not use the same notation a^b for indices, nor did they make use of the general concept of a function as described in this course. It was therefore impossible for Napier to explain logarithms as we have done. Instead, Napier's definition was based on the continuous movement of two points.



Gr. 9

9

min	Sinus	Logarithmi	+ Differentiæ −	logarithmi	Sinus	
0	1564345	18551174	18427293	123881	9876883	60
1	1567218	18532826	18408484	124342	9876427	59
2	1570091	18514511	18389707	124804	9875971	58
3	1572964	18496231	18370964	125267	9875514	57
4	1575837	18477984	18352253	125731	9875056	56
5	1578709	18459772	18333576	126196	9874597	55
6	1581581	18441594	18314933	126661	9874137	54
7	1584453	18423451	18296324	127127	9873677	53
8	1587325	18405341	18277747	127594	9873216	52
9	1590197	18387265	18259203	128062	9872754	51
10	1593069	18369223	18240692	128531	9872291	50
11	1595941	18351214	18222213	129001	9871827	49
12	1598812	18333237	18203765	129472	9871362	48
13	1601684	18315294	18185351	129943	9870897	47
14	1604555	18297384	18166969	130415	9870431	46
15	1607426	18279507	18148619	130888	9869964	45

To enable people to actually use logarithms, he calculated tables of numbers by hand to seven places of decimals. This took him many years of work. To find the logarithm of a particular number, you would look it up in the table. Although this seems awkward to us, it is much quicker to use tables than calculate multiplication, division, and square roots by hand.

Logarithms were an extremely important development and they had an immediate effect on the seventeenth century scientific community. **Johannes Kepler** used Napier's tables to assist with his calculations. This helped him develop his laws of planetary motion. Without logarithms these calculations would have taken many years. Kepler published a letter congratulating and acknowledging Napier. Kepler's laws gave **Sir Isaac Newton** important evidence to support his theory of universal gravitation. 200 years later, **Laplace** said that logarithms "by shortening the labours, doubled the life of the astronomer".

Johannes Kepler

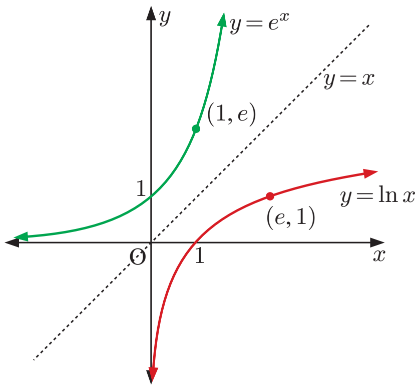
E NATURAL LOGARITHMS

In **Chapter 4** we came across the **natural exponential** $e \approx 2.718\,28$.

Given the exponential function $f(x) = e^x$, the inverse function $f^{-1} = \log_e x$ is the logarithm in base e.

We use $\ln x$ to represent $\log_e x$, and call $\ln x$ the **natural logarithm** of x.

$y = \ln x$ is the reflection of $y = e^x$ in the mirror line $y = x$.



Notice that:

- $\ln 1 = \ln e^0 = 0$
- $\ln e = \ln e^1 = 1$
- $\ln e^2 = 2$
- $\ln \sqrt{e} = \ln e^{\frac{1}{2}} = \frac{1}{2}$
- $\ln\left(\frac{1}{e}\right) = \ln e^{-1} = -1$

$$\boldsymbol{\ln e^x = x \quad \text{and} \quad e^{\ln x} = x.}$$

Since $a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$, $\boldsymbol{a^x = e^{x \ln a}}$, $a > 0$.

EXERCISE 5E.1

1 Without using a calculator find:

a $\ln e^2$ **b** $\ln e^3$ **c** $\ln \sqrt{e}$ **d** $\ln 1$

e $\ln\left(\frac{1}{e}\right)$ **f** $\ln \sqrt[3]{e}$ **g** $\ln\left(\frac{1}{e^2}\right)$ **h** $\ln\left(\frac{1}{\sqrt{e}}\right)$

Check your answers using a calculator.

2 Simplify:

a $e^{\ln 3}$ **b** $e^{2\ln 3}$ **c** $e^{-\ln 5}$ **d** $e^{-2\ln 2}$

3 Explain why $\ln(-2)$ and $\ln 0$ cannot be found.

4 Simplify:

a $\ln e^a$ **b** $\ln(e \times e^a)$ **c** $\ln\left(e^a \times e^b\right)$ **d** $\ln(e^a)^b$ **e** $\ln\left(\frac{e^a}{e^b}\right)$

Example 14 — Self Tutor

Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 50 **b** 0.005

a 50
$= e^{\ln 50}$ $\{$using $x = e^{\ln x}\}$
$\approx e^{3.9120}$

b 0.005
$= e^{\ln 0.005}$
$\approx e^{-5.2983}$

5 Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006

f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** $0.000\,15$

Example 15 — Self Tutor

Find x if:

a $\ln x = 2.17$ **b** $\ln x = -0.384$

a $\ln x = 2.17$

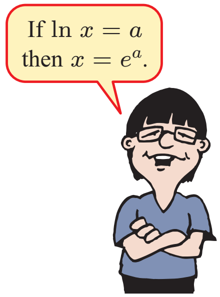
$\therefore\ x = e^{2.17}$

$\therefore\ x \approx 8.76$

b $\ln x = -0.384$

$\therefore\ x = e^{-0.384}$

$\therefore\ x \approx 0.681$



6 Find x if:

a $\ln x = 3$ **b** $\ln x = 1$ **c** $\ln x = 0$ **d** $\ln x = -1$

e $\ln x = -5$ **f** $\ln x \approx 0.835$ **g** $\ln x \approx 2.145$ **h** $\ln x \approx -3.2971$

LAWS OF NATURAL LOGARITHMS

The laws for natural logarithms are the laws for logarithms written in base e:

For positive A and B:

- $\ln A + \ln B = \ln(AB)$
- $\ln A - \ln B = \ln\left(\dfrac{A}{B}\right)$
- $n \ln A = \ln(A^n)$

Example 16 — Self Tutor

Use the laws of logarithms to write the following as a single logarithm:

a $\ln 5 + \ln 3$ **b** $\ln 24 - \ln 8$ **c** $\ln 5 - 1$

a $\ln 5 + \ln 3$

$= \ln(5 \times 3)$

$= \ln 15$

b $\ln 24 - \ln 8$

$= \ln\left(\frac{24}{8}\right)$

$= \ln 3$

c $\ln 5 - 1$

$= \ln 5 - \ln\ e^1$

$= \ln\left(\frac{5}{e}\right)$

Example 17 — Self Tutor

Use the laws of logarithms to simplify:

a $2\ln 7 - 3\ln 2$ **b** $2\ln 3 + 3$

a $2\ln 7 - 3\ln 2$

$= \ln(7^2) - \ln(2^3)$

$= \ln 49 - \ln 8$

$= \ln\left(\frac{49}{8}\right)$

b $2\ln 3 + 3$

$= \ln(3^2) + \ln e^3$

$= \ln 9 + \ln e^3$

$= \ln(9e^3)$

EXERCISE 5E.2

1 Write as a single logarithm or integer:

a $\ln 15 + \ln 3$ **b** $\ln 15 - \ln 3$ **c** $\ln 20 - \ln 5$

d $\ln 4 + \ln 6$ **e** $\ln 5 + \ln(0.2)$ **f** $\ln 2 + \ln 3 + \ln 5$

g $1 + \ln 4$ **h** $\ln 6 - 1$ **i** $\ln 5 + \ln 8 - \ln 2$

j $2 + \ln 4$ **k** $\ln 20 - 2$ **l** $\ln 12 - \ln 4 - \ln 3$

2 Write in the form $\ln a,\ \ a \in \mathbb{Q}$:

a $5\ln 3 + \ln 4$ **b** $3\ln 2 + 2\ln 5$ **c** $3\ln 2 - \ln 8$

d $3\ln 4 - 2\ln 2$ **e** $\frac{1}{3}\ln 8 + \ln 3$ **f** $\frac{1}{3}\ln(\frac{1}{27})$

g $-\ln 2$ **h** $-\ln(\frac{1}{2})$ **i** $-2\ln(\frac{1}{4})$

Example 18 Self Tutor

Show that:

a $\ln\left(\frac{1}{9}\right) = -2\ln 3$ **b** $\ln\left(\frac{e}{4}\right) = 1 - 2\ln 2$

a $\ln\left(\frac{1}{9}\right)$
$= \ln(3^{-2})$
$= -2\ln 3$

b $\ln\left(\frac{e}{4}\right) = \ln e - \ln 4$
$= \ln e^1 - \ln 2^2$
$= 1 - 2\ln 2$

3 Show that:

a $\ln 27 = 3\ln 3$ **b** $\ln\sqrt{3} = \frac{1}{2}\ln 3$ **c** $\ln(\frac{1}{16}) = -4\ln 2$

d $\ln(\frac{1}{6}) = -\ln 6$ **e** $\ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln 2$ **f** $\ln\left(\frac{e}{5}\right) = 1 - \ln 5$

4 Show that:

a $\ln\sqrt[3]{5} = \frac{1}{3}\ln 5$ **b** $\ln(\frac{1}{32}) = -5\ln 2$

c $\ln\left(\frac{1}{\sqrt[5]{2}}\right) = -\frac{1}{5}\ln 2$ **d** $\ln\left(\frac{e^2}{8}\right) = 2 - 3\ln 2$

Example 19 Self Tutor

Write the following equations without logarithms:

a $\ln A = 2\ln c + 3$ **b** $\ln M = 3a - \ln 2$

a $\ln A = 2\ln c + 3$
$\therefore\ \ln A = \ln c^2 + \ln e^3$
$\therefore\ \ln A = \ln(c^2 e^3)$
$\therefore\ A = c^2 e^3$

b $\ln M = 3a - \ln 2$
$\therefore\ \ln M = \ln e^{3a} - \ln 2$
$\therefore\ \ln M = \ln\left(\frac{e^{3a}}{2}\right)$
$\therefore\ M = \frac{1}{2}e^{3a}$

5 Write the following equations without logarithms, assuming all terms are positive:

a $\ln D = \ln x + 1$
b $\ln F = -\ln p + 2$
c $\ln P = 2x + \ln 5$
d $\ln M = 2\ln y + 3$
e $\ln B = 3t - \ln 4$
f $\ln N = -\frac{1}{3}\ln g$
g $\ln Q \approx 3\ln x + 2.159$
h $\ln D \approx 0.4\ln n - 0.6582$
i $\ln T \approx -x + 1.578$

F SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS

In **Chapter 4** we found solutions to simple exponential equations where we could make equal bases and then equate exponents. However, it is not always easy to make the bases the same. In these situations we use **logarithms** to find the solution.

Example 20 — Self Tutor

Solve for x, giving your answers correct to 3 significant figures:

a $2^x = 7$
b $5^{3x-1} = 90$

a
$$\begin{aligned} 2^x &= 7 \\ \therefore\ \lg 2^x &= \lg 7 \\ \therefore\ x\lg 2 &= \lg 7 \qquad \{\lg(a^n) = n\lg a\} \\ \therefore\ x &= \frac{\lg 7}{\lg 2} \\ \therefore\ x &\approx 2.81 \end{aligned}$$

b
$$\begin{aligned} 5^{3x-1} &= 90 \\ \therefore\ \lg 5^{3x-1} &= \lg 90 \\ \therefore\ (3x-1)\lg 5 &= \lg 90 \qquad \{\lg(a^n) = n\lg a\} \\ \therefore\ 3x-1 &= \frac{\lg 90}{\lg 5} \\ \therefore\ x &= \frac{1}{3}\left(1 + \frac{\lg 90}{\lg 5}\right) \\ \therefore\ x &\approx 1.27 \end{aligned}$$

Example 21 — Self Tutor

Find x exactly:

a $e^x = 30$
b $3e^{\frac{x}{2}} = 21$

a
$$\begin{aligned} e^x &= 30 \\ \therefore\ x &= \ln 30 \end{aligned}$$

b
$$\begin{aligned} 3e^{\frac{x}{2}} &= 21 \\ \therefore\ e^{\frac{x}{2}} &= 7 \\ \therefore\ \frac{x}{2} &= \ln 7 \\ \therefore\ x &= 2\ln 7 \end{aligned}$$

EXERCISE 5F

1 Solve for x, giving your answer correct to 3 significant figures:

a $2^x = 10$
b $3^x = 20$
c $4^x = 100$
d $\left(\frac{1}{2}\right)^x = 0.0625$
e $\left(\frac{3}{4}\right)^x = 0.1$
f $10^x = 0.000\,01$

2 Solve for x, giving your answer correct to 3 significant figures:

a $5^{2x} = 100$ **b** $2^{4x} = 75$ **c** $(0.8)^{3x} = 0.1$

d $3^{x-1} = 200$ **e** $4^{x+2} = 2.5$ **f** $6^{2x-1} = 800$

g $7^{2x+3} = 1000$ **h** $\left(3^{x+1}\right)^2 = 480$ **i** $\left(2^{x-3}\right)^{\frac{1}{2}} = 10$

3 Solve for x, giving an exact answer:

a $e^x = 10$ **b** $e^x = 1000$ **c** $2e^x = 0.3$

d $e^{\frac{x}{2}} = 5$ **e** $e^{2x} = 18$ **f** $e^{-\frac{x}{2}} = 1$

4 **a** Solve $e^{2x} = 300$ exactly.

b Use your calculator to find the solution correct to 2 decimal places.

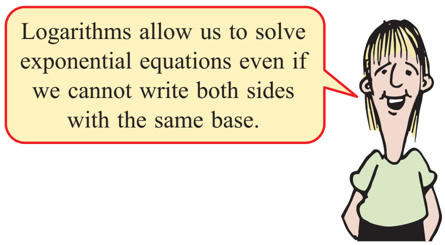
Example 22 Self Tutor

A farmer monitoring an insect plague notices that the area affected by the insects is given by $A = 1000 \times 2^{0.7n}$ hectares, where n is the number of weeks after the initial observation. How long will it take for the affected area to reach 5000 hectares?

When $A = 5000$,

$$1000 \times 2^{0.7n} = 5000$$
$$\therefore\ 2^{0.7n} = 5$$
$$\therefore\ \lg 2^{0.7n} = \lg 5$$
$$\therefore\ 0.7n \lg 2 = \lg 5$$
$$\therefore\ n = \frac{\lg 5}{0.7 \times \lg 2}$$
$$\therefore\ n \approx 3.32$$

$\therefore$ it takes about 3 weeks and 2 days.

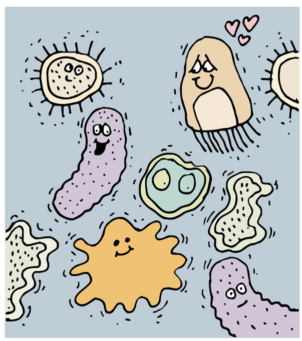


5 Solve for x, giving an exact answer:

a $4 \times 2^{-x} = 0.12$ **b** $300 \times 5^{0.1x} = 1000$ **c** $32 \times 3^{-0.25x} = 4$

6 The weight W of bacteria in a culture t hours after establishment is given by $W = 20 \times 2^{0.15t}$ grams. Find, using logarithms, the time for the weight of the culture to reach:

a 30 grams **b** 100 grams.



7 The mass M of bacteria in a culture t hours after establishment is given by $M = 25 \times e^{0.1t}$ grams.

a Show that the time required for the mass of the culture to reach 50 grams is $10 \ln 2$ hours.

b Find the time required correct to 2 decimal places.

8 The weight of radioactive uranium remaining after t years is given by the formula $W(t) = 50 \times 2^{-0.0002t}$ grams, $t \geqslant 0$.

a Find the initial weight of the uranium.

b Find the time required for the weight to fall to 8 grams.

Example 23

Self Tutor

Find algebraically the exact points of intersection of $y = e^x - 3$ and $y = 1 - 3e^{-x}$.

The functions meet where

$$\begin{aligned} e^x - 3 &= 1 - 3e^{-x} \\ \therefore\ e^x - 4 + 3e^{-x} &= 0 \\ \therefore\ e^{2x} - 4e^x + 3 &= 0 \quad \{\text{multiplying each term by } e^x\} \\ \therefore\ (e^x - 1)(e^x - 3) &= 0 \\ \therefore\ e^x &= 1 \text{ or } 3 \\ \therefore\ x &= \ln 1 \text{ or } \ln 3 \\ \therefore\ x &= 0 \text{ or } \ln 3 \end{aligned}$$

GRAPHING PACKAGE

When $x = 0$, $y = e^0 - 3 = -2$
When $x = \ln 3$, $y = e^{\ln 3} - 3 = 0$

$\therefore$ the functions meet at $(0, -2)$ and at $(\ln 3, 0)$.

9 Solve for x:

a $e^{2x} = 2e^x$ **b** $e^x = e^{-x}$ **c** $e^{2x} - 5e^x + 6 = 0$

d $e^x + 2 = 3e^{-x}$ **e** $1 + 12e^{-x} = e^x$ **f** $e^x + e^{-x} = 3$

10 Find algebraically the point(s) of intersection of:

a $y = e^x$ and $y = e^{2x} - 6$ **b** $y = 2e^x + 1$ and $y = 7 - e^x$

c $y = 3 - e^x$ and $y = 5e^{-x} - 3$

G THE CHANGE OF BASE RULE

A logarithm in base b can be written with a different base c using the **change of base rule**:

$$\log_b a = \frac{\log_c a}{\log_c b} \quad \text{for } a, b, c > 0 \text{ and } b, c \neq 1.$$

Proof: If $\log_b a = x$, then $b^x = a$

$$\begin{aligned} \therefore\ \log_c b^x &= \log_c a \quad \{\text{taking logarithms in base } c\} \\ \therefore\ x \log_c b &= \log_c a \quad \{\text{power law of logarithms}\} \\ \therefore\ x &= \frac{\log_c a}{\log_c b} \\ \therefore\ \log_b a &= \frac{\log_c a}{\log_c b} \end{aligned}$$

We can use this rule to write logarithms in base 10 or base e. This is useful in helping us evaluate them on our calculator.

Example 24 — Self Tutor

Evaluate $\log_2 9$ by:

a changing to base 10 **b** changing to base e.

a $\log_2 9 = \dfrac{\log_{10} 9}{\log_{10} 2} \approx 3.17$ **b** $\log_2 9 = \dfrac{\ln 9}{\ln 2} \approx 3.17$

The rule can also be used to solve equations involving logarithms with different bases.

Example 25 — Self Tutor

Solve for x: $\log_2 x = \log_8 15$

$$\begin{aligned} \log_2 x &= \log_8 15 \\ \therefore\ \log_2 x &= \frac{\log_2 15}{\log_2 8} \quad \{\text{writing RHS with base 2}\} \\ \therefore\ \log_2 x &= \frac{\log_2 15}{3} \\ \therefore\ \log_2 x &= \log_2 15^{\frac{1}{3}} \\ \therefore\ x &= \sqrt[3]{15} \end{aligned}$$

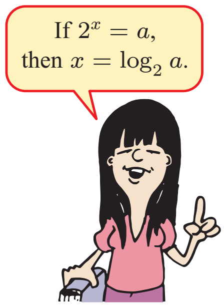
EXERCISE 5G

1 Use the rule $\log_b a = \dfrac{\log_{10} a}{\log_{10} b}$ to evaluate, correct to 3 significant figures:

a $\log_3 12$ **b** $\log_{\frac{1}{2}} 1250$ **c** $\log_3(0.067)$ **d** $\log_{0.4}(0.006\,984)$

2 Use the rule $\log_b a = \dfrac{\ln a}{\ln b}$ to solve, correct to 3 significant figures:

a $2^x = 0.051$ **b** $4^x = 213.8$ **c** $3^{2x+1} = 4.069$



3 Write:

a $\log_9 26$ in the form $a \log_3 b$, where $a, b \in \mathbb{Q}$

b $\log_2 11$ in the form $a \log_4 b$, where $a, b \in \mathbb{Z}$

c $\dfrac{6}{\log_7 25}$ in the form $a \log_5 b$, where $a, b \in \mathbb{Z}$.

4 Solve for x:

a $\log_3 x = \log_{27} 50$ **b** $\log_2 x = \log_4 13$ **c** $\log_{25} x = \log_5 7$

d $\log_3 \sqrt{x} + \log_9 x = \log_3 5$ **e** $\log_8 x^2 - \log_2 \sqrt[3]{x} = 1$ **f** $\log_4 x^3 + \log_2 \sqrt{x} = 8$

5 **a** Show that $\log_a b = \dfrac{1}{\log_b a}$.

b Solve for x:

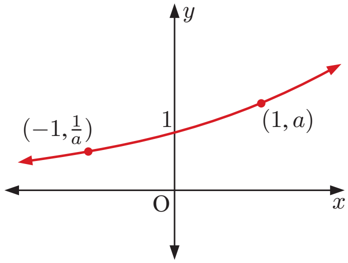
i $\log_3 x = 4 \log_x 3$ **ii** $\log_2 x - 4 = 5 \log_x 2$ **iii** $2 \log_4 x + 3 \log_x 4 = 7$

H GRAPHS OF LOGARITHMIC FUNCTIONS

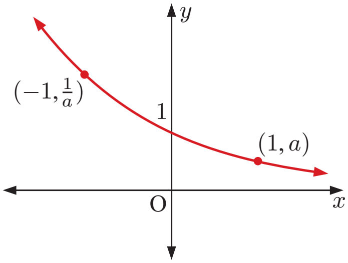
Consider the general exponential function $f(x) = a^x$, $a > 0$, $a \neq 1$.

The graph of $y = a^x$ is:

For $a > 1$:



For $0 < a < 1$:



The **horizontal asymptote** for all of these functions is the x-axis $y = 0$.

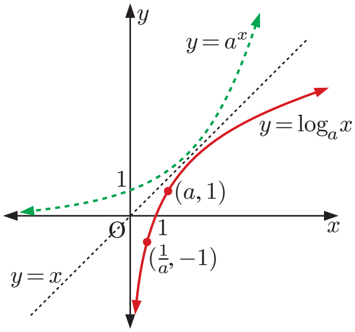
The inverse function f^{-1} is given by $x = a^y$, so $y = \log_a x$.

If $\boldsymbol{f(x) = a^x}$ where $\boldsymbol{a > 0,\ a \neq 1}$, then $\boldsymbol{f^{-1}(x) = \log_a x}$.

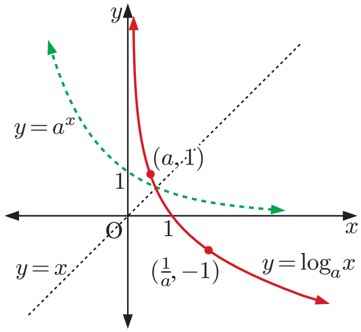
Since $f^{-1}(x) = \log_a x$ is an inverse function, it is a reflection of $f(x) = a^x$ in the line $y = x$. We may therefore deduce the following properties:

Function	$f(x) = a^x$	$f^{-1}(x) = \log_a x$
Domain	$\{x : x \in \mathbb{R}\}$	$\{x : x > 0\}$
Range	$\{y : y > 0\}$	$\{y : y \in \mathbb{R}\}$
Asymptote	horizontal $y = 0$	vertical $x = 0$

The graph of $y = \log_a x$ for $a > 1$:



The graph of $y = \log_a x$ for $0 < a < 1$:



The **vertical asymptote** of $y = \log_a x$ is the y-axis $x = 0$.

Since we can only find logarithms of positive numbers, the domain of $f^{-1}(x) = \log_a x$ is $\{x \mid x > 0\}$.

In general, $y = \log_a(g(x))$ is defined when $g(x) > 0$.

Example 26

Self Tutor

Consider the function $f(x) = \log_2(x-1)+1$.

a Find the domain and range of f.

b Find any asymptotes and axes intercepts.

c Sketch the graph of f showing all important features.

d Find f^{-1}.

a $x - 1 > 0$ when $x > 1$

So, the domain is $\{x : x > 1\}$ and the range is $y \in \mathbb{R}$.

b As $x \to 1$ from the right, $y \to -\infty$, so the vertical asymptote is $x = 1$.

As $x \to \infty$, $y \to \infty$.

When $x = 0$, y is undefined, so there is no y-intercept.

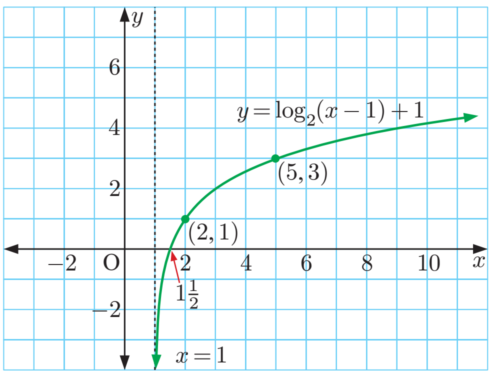
When $y = 0$, $\log_2(x-1) = -1$

$\therefore \quad x - 1 = 2^{-1}$

$\therefore \quad x = 1\frac{1}{2}$

So, the x-intercept is $1\frac{1}{2}$.

c



d f is defined by $y = \log_2(x-1)+1$

$\therefore$ f^{-1} is defined by $x = \log_2(y-1)+1$

$\therefore \quad x - 1 = \log_2(y-1)$

$\therefore \quad y - 1 = 2^{x-1}$

$\therefore \quad y = 2^{x-1}+1$

$\therefore \quad f^{-1}(x) = 2^{x-1}+1$

which has the horizontal asymptote $y = 1$ ✓

Its domain is $\{x : x \in \mathbb{R}\}$, and

its range is $\{y : y > 1\}$.

EXERCISE 5H

1 For the following functions f:

i Find the domain and range.

ii Find any asymptotes and axes intercepts.

iii Sketch the graph of $y = f(x)$ showing all important features.

iv Solve $f(x) = -1$ algebraically and check the solution on your graph.

v Find f^{-1}.

a $f : x \mapsto \log_3(x+1), \quad x > -1$

b $f : x \mapsto 1 - \log_3(x+1), \quad x > -1$

c $f : x \mapsto \log_5(x-2) - 2, \quad x > 2$

d $f : x \mapsto 1 - \log_5(x-2), \quad x > 2$

e $f : x \mapsto 1 - 2\log_2 x, \quad x > 0$

Example 27

Self Tutor

Consider the function $f : x \mapsto e^{x-3}$.

a Find the equation defining f^{-1}.

b Sketch the graphs of f and f^{-1} on the same set of axes.

c State the domain and range of f and f^{-1}.

d Find any asymptotes and intercepts of f and f^{-1}.

a $f(x) = e^{x-3}$

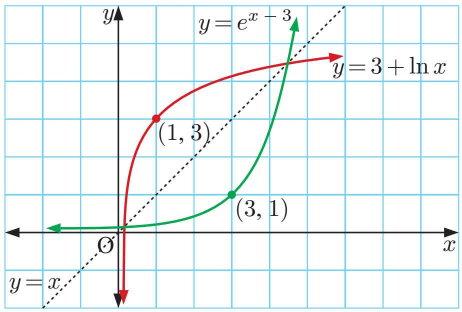
$\therefore$ f^{-1} is $x = e^{y-3}$

$\therefore$ $y - 3 = \ln x$

$\therefore$ $y = 3 + \ln x$

So, $f^{-1}(x) = 3 + \ln x$

b



c

Function	f	f^{-1}
Domain	$x \in \mathbb{R}$	$x > 0$
Range	$y > 0$	$y \in \mathbb{R}$

d For f, the horizontal asymptote is $y = 0$, and the y-intercept is e^{-3}.

For f^{-1}, the vertical asymptote is $x = 0$, and the x-intercept is e^{-3}.

2 For the following functions f:

i Find the equation of f^{-1}.

ii Sketch the graphs of f and f^{-1} on the same set of axes.

iii State the domain and range of f and f^{-1}.

iv Find any asymptotes and intercepts of f and f^{-1}.

a $f(x) = e^x + 5$

b $f(x) = e^{x+1} - 3$

c $f(x) = \ln x - 4, \quad x > 0$

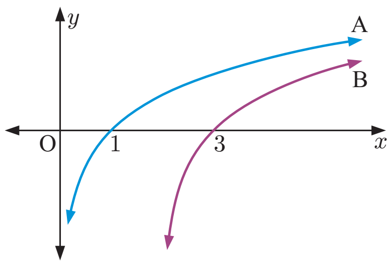
d $f(x) = \ln(x-1) + 2, \quad x > 1$

3 Consider the graphs A and B. One of them is the graph of $y = \ln x$ and the other is the graph of $y = \ln(x-2)$.

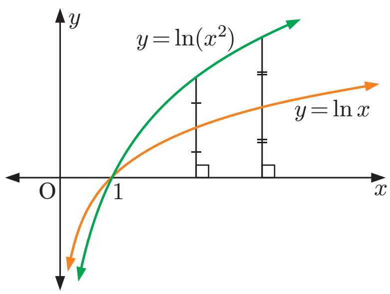
a Identify which is which. Give evidence for your answer.

b Copy the graphs onto a new set of axes and add to them the graph of $y = \ln(x+2)$.

c Find the equation of the vertical asymptote for each graph.



4 Kelly said that in order to graph $y = \ln(x^2)$, $x > 0$, you could first graph $y = \ln x$ and then double the distance of each point on the graph from the x-axis.
Is Kelly correct? Explain your answer.



5 Consider the function $f : x \mapsto e^{x+3} + 2$.

a Find the defining equation for f^{-1}.

b Find the values of x for which:

i $f(x) < 2.1$ **ii** $f(x) < 2.01$ **iii** $f(x) < 2.001$ **iv** $f(x) < 2.0001$

Hence conjecture the horizontal asymptote for the graph of f.

c Determine the equation of the horizontal asymptote of $f(x)$ by discussing the behaviour of $f(x)$ as $x \to \pm\infty$.

d Hence, determine the vertical asymptote and the domain of f^{-1}.

6 Consider $f(x) = \log_2(x+3)$.

a Find: **i** $f(5)$ **ii** $f(x^2)$ **iii** $f(2x-1)$

b State the domain of $f(x)$.

c Solve $f(x^2+4) = 5$.

7 Suppose $f(x) = e^{3x} + 1$.

a State the range of $f(x)$.

b Find $f^{-1}(x)$.

c Find $f^{-1}(10)$.

d State the domain of $f^{-1}(x)$.

e Find $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$.

8 Suppose $f : x \mapsto e^{2x}$ and $g : x \mapsto 2x - 1$.

a Find: **i** $(f^{-1} \circ g)(x)$ **ii** $(g \circ f)^{-1}(x)$

b Solve $(f^{-1} \circ g)(x) = \ln 5$.

9 Consider $f : x \mapsto 10e^{-x}$ and $g : x \mapsto \ln(x-3)$.

a Find $f(1)$ and $g(6)$.

b Find the x-intercept of $g(x)$.

c Find $fg(x)$.

d Solve $f(x) = g^{-1}(x)$.

10 Let $f(x) = \ln(x+6)$ and $g(x) = x - \ln 3$.

a State the domain of $f(x)$.

b Find $f^{-1}(x)$.

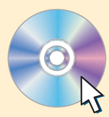
c Find the axes intercepts of $f(x)$.

d Solve $gf(x) = f(x^2 - 12)$.

Activity

Click on the icon to obtain a card game for logarithmic functions.

CARD GAME



Review set 5A

1 Find the following, showing all working:

a $\log_4 64$ **b** $\log_2 256$ **c** $\log_2(0.25)$ **d** $\log_{25} 5$

e $\log_8 1$ **f** $\log_{81} 3$ **g** $\log_9(0.\overline{1})$ **h** $\log_k \sqrt{k}$

2 Find:

a $\lg \sqrt{10}$ **b** $\lg \dfrac{1}{\sqrt[3]{10}}$ **c** $\lg(10^a \times 10^{b+1})$

3 Simplify:

a $4\ln 2 + 2\ln 3$ **b** $\frac{1}{2}\ln 9 - \ln 2$ **c** $2\ln 5 - 1$ **d** $\frac{1}{4}\ln 81$

4 Find:

a $\ln(e\sqrt{e})$ **b** $\ln\left(\dfrac{1}{e^3}\right)$ **c** $\ln(e^{2x})$ **d** $\ln\left(\dfrac{e}{e^x}\right)$

5 Write as a single logarithm:

a $\lg 16 + 2\lg 3$ **b** $\log_2 16 - 2\log_2 3$ **c** $2 + \log_4 5$

6 Write as logarithmic equations:

a $P = 3 \times 7^x$ **b** $m = \dfrac{n^3}{5}$

7 Solve for x:

a $\log_2(x+5) - \log_2(x-2) = 3$ **b** $\lg x + \lg(x+15) = 2$

8 Show that $\log_3 7 \times 2\log_7 x = 2\log_3 x$.

9 Write the following equations without logarithms:

a $\lg T = 2\lg x - \lg 5$ **b** $\log_2 K = x + \log_2 3$

10 Write in the form $a\ln k$ where a and k are positive whole numbers and k is prime:

a $\ln 32$ **b** $\ln 125$ **c** $\ln 729$

11 Copy and complete:

Function	$y = \log_2 x$	$y = \ln(x+5)$
Domain		
Range		

12 If $A = \log_5 2$ and $B = \log_5 3$, write in terms of A and B:

a $\log_5 36$ **b** $\log_5 54$ **c** $\log_5(8\sqrt{3})$ **d** $\log_5(20.25)$ **e** $\log_5(0.\overline{8})$

13 Solve for x:

a $3e^x - 5 = -2e^{-x}$ **b** $2\ln x - 3\ln\left(\dfrac{1}{x}\right) = 10$

14 Solve for x, giving your answer to 2 decimal places:

a $7^x = 120$ **b** $6 \times 2^{3x} = 300$

15 A population of seals is given by $P = 20 \times 2^{\frac{t}{3}}$ where t is the time in years, $t \geqslant 0$. Find the time required for the population to reach 100.

16 Consider $f : x \mapsto 5e^{-x} + 1$.

a State the range of f.

b Find: **i** $f^{-1}(x)$ **ii** $f^{-1}(2)$

c State the domain of f^{-1}.

d Solve $f^{-1}(x) = 0$.

e Sketch the graphs of f, f^{-1}, and $y = x$ on the same set of axes.

Review set 5B

1 Without using a calculator, find the base 10 logarithms of:

a $\sqrt{1000}$ **b** $\dfrac{10}{\sqrt[3]{10}}$ **c** $\dfrac{10^a}{10^{-b}}$

2 Write in the form 10^x giving x correct to 4 decimal places:

a 32 **b** 0.0013 **c** 8.963×10^{-5}

3 Find x if:

a $\log_2 x = -3$ **b** $\log_5 x \approx 2.743$ **c** $\log_3 x \approx -3.145$

4 Write the following equations without logarithms:

a $\log_2 k \approx 1.699 + x$ **b** $\log_a Q = 3\log_a P + \log_a 5$ **c** $\lg A = x \lg 2 + \lg 6$

5 Solve for x, giving exact answers:

a $5^x = 7$ **b** $20 \times 2^{2x+1} = 640$

6 Find the exact value of $\log_{12} 3 - 2\log_{12} 6$.

7 Write $\log_8 30$ in the form $a\log_2 b$, where $a, b \in \mathbb{Q}$.

8 Solve for x:

a $\log_4 x + \log_4(2x-8) = 3$ **b** $\log_x 135 = 3 + \log_x 5$

9 Consider $f(x) = e^x$ and $g(x) = \ln(x+4)$, $x > -4$. Find:

a $(f \circ g)(5)$ **b** $(g \circ f)(0)$

10 Write as a single logarithm:

a $\ln 60 - \ln 20$ **b** $\ln 4 + \ln 1$ **c** $\ln 200 - \ln 8 + \ln 5$

11 Write as logarithmic equations:

a $M = 5 \times 6^x$ **b** $T = \dfrac{5}{\sqrt{l}}$ **c** $G = \dfrac{4}{c}$

12 Solve exactly for x:

a $e^{2x} = 3e^x$ **b** $e^{2x} - 7e^x + 12 = 0$

13 Consider the function $g : x \mapsto \log_3(x+2) - 2$.

a Find the domain and range.

b Find any asymptotes and axes intercepts for the graph of the function.

c Find the defining equation for g^{-1}.

d Sketch the graphs of g, g^{-1}, and $y = x$ on the same axes.

14 The weight of a radioactive isotope remaining after t weeks is given by $W = 8000 \times e^{-\frac{t}{20}}$ grams. Find the time for the weight to halve.

15 Solve for x:

a $\log_2 x + \log_4 x^4 = \log_2 125$ **b** $\log_2 x = 25\log_x 2$ **c** $\log_3 x + 8\log_x 3 = 6$

16 Consider $f(x) = 5e^{2x}$ and $g(x) = \ln(x-4)$.

a State the domain and range of g. **b** Find the axes intercepts of g.

c Find the exact solution to $fg(x) = 30$. **d** Solve $f(x) = g^{-1}(x)$.

6

Polynomials

Contents:

Opening problem

To determine whether 7 is a **factor** of 56, we divide 56 by 7. The result is exactly 8. Since there is no remainder, 7 is a factor of 56.

Things to think about:

a Can we perform a similar test for *algebraic* factors? For example, how can we determine whether $x-3$ is a factor of x^3-4x^2+2x+3?

b Given that $x-3$ is a factor of x^3-4x^2+2x+3, what does this tell us about the graph of $f(x)=x^3-4x^2+2x+3$?

Up to this point we have studied linear and quadratic functions at some depth, with perhaps occasional reference to cubic functions. These are part of a larger family of functions called the **polynomials**.

A REAL POLYNOMIALS

A **polynomial function** is a function of the form

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+....+a_2x^2+a_1x+a_0, \quad a_1, ..., a_n \text{ constant}, \ a_n \neq 0.$$

We say that: x is the **variable**
a_0 is the **constant term**
a_n is the **leading coefficient** and is non-zero
a_r is the **coefficient of** $\boldsymbol{x^r}$ for $r=0, 1, 2, ..., n$
n is the **degree** of the polynomial, being the highest power of the variable.

In **summation notation**, we write $P(x)=\sum_{r=0}^{n} a_rx^r$,

which reads: "the sum from $r=0$ to n, of a_rx^r".

A **real polynomial** $P(x)$ is a polynomial for which $a_r \in \mathbb{R}$, $r=0, 1, 2, ..., n$.

The low degree members of the polynomial family have special names, some of which you are already familiar with. For these polynomials, we commonly write their coefficients as $a, b, c, ...$

Polynomial function	*Degree*	*Name*
$ax+b, \quad a\neq 0$	1	linear
$ax^2+bx+c, \quad a\neq 0$	2	quadratic
$ax^3+bx^2+cx+d, \quad a\neq 0$	3	cubic
$ax^4+bx^3+cx^2+dx+e, \quad a\neq 0$	4	quartic

ADDITION AND SUBTRACTION

To **add** or **subtract** two polynomials, we collect 'like' terms.

Example 1 — Self Tutor

If $P(x) = x^3 - 2x^2 + 3x - 5$ and $Q(x) = 2x^3 + x^2 - 11$, find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$

a
$$\begin{aligned} & P(x) + Q(x) \\ = \; & x^3 - 2x^2 + 3x - 5 \\ & +2x^3 + x^2 - 11 \\ \hline = \; & 3x^3 - x^2 + 3x - 16 \end{aligned}$$

b
$$\begin{aligned} & P(x) - Q(x) \\ = \; & x^3 - 2x^2 + 3x - 5 - (2x^3 + x^2 - 11) \\ = \; & x^3 - 2x^2 + 3x - 5 \\ & -2x^3 - x^2 + 11 \\ \hline = \; & -x^3 - 3x^2 + 3x + 6 \end{aligned}$$

SCALAR MULTIPLICATION

To **multiply** a polynomial by a **scalar** (constant) we multiply each term by the scalar.

Example 2 — Self Tutor

If $P(x) = x^4 - 2x^3 + 4x + 7$, find:

a $3P(x)$ **b** $-2P(x)$

a
$$\begin{aligned} & 3P(x) \\ & = 3(x^4 - 2x^3 + 4x + 7) \\ & = 3x^4 - 6x^3 + 12x + 21 \end{aligned}$$

b
$$\begin{aligned} & -2P(x) \\ & = -2(x^4 - 2x^3 + 4x + 7) \\ & = -2x^4 + 4x^3 - 8x - 14 \end{aligned}$$

POLYNOMIAL MULTIPLICATION

To **multiply** two polynomials, we multiply each term of the first polynomial by each term of the second polynomial, and then collect like terms.

Example 3 — Self Tutor

If $P(x) = x^3 - 2x + 4$ and $Q(x) = 2x^2 + 3x - 5$, find $P(x)Q(x)$.

$$\begin{aligned} P(x)Q(x) & = (x^3 - 2x + 4)(2x^2 + 3x - 5) \\ & = x^3(2x^2 + 3x - 5) - 2x(2x^2 + 3x - 5) + 4(2x^2 + 3x - 5) \\ & = 2x^5 + 3x^4 - 5x^3 \\ & \qquad\qquad - 4x^3 - 6x^2 + 10x \\ & \qquad\qquad\qquad + 8x^2 + 12x - 20 \\ \hline & = 2x^5 + 3x^4 - 9x^3 + 2x^2 + 22x - 20 \end{aligned}$$

EXERCISE 6A.1

1 If $P(x) = x^2 + 2x + 3$ and $Q(x) = 4x^2 + 5x + 6$, find in simplest form:

a $3P(x)$ **b** $P(x) + Q(x)$ **c** $P(x) - 2Q(x)$ **d** $P(x)Q(x)$

2 If $f(x) = x^2 - x + 2$ and $g(x) = x^3 - 3x + 5$, find in simplest form:

a $f(x) + g(x)$ **b** $g(x) - f(x)$ **c** $2f(x) + 3g(x)$
d $g(x) + xf(x)$ **e** $f(x)\,g(x)$ **f** $[f(x)]^2$

3 Expand and simplify:

a $(x^2 - 2x + 3)(2x + 1)$ **b** $(x - 1)^2(x^2 + 3x - 2)$ **c** $(x + 2)^3$
d $(2x^2 - x + 3)^2$ **e** $(2x - 1)^4$ **f** $(3x - 2)^2(2x + 1)(x - 4)$

4 Find the following products:

a $(2x^2 - 3x + 5)(3x - 1)$ **b** $(4x^2 - x + 2)(2x + 5)$
c $(2x^2 + 3x + 2)(5 - x)$ **d** $(x - 2)^2(2x + 1)$
e $(x^2 - 3x + 2)(2x^2 + 4x - 1)$ **f** $(3x^2 - x + 2)(5x^2 + 2x - 3)$
g $(x^2 - x + 3)^2$ **h** $(2x^2 + x - 4)^2$
i $(2x + 5)^3$ **j** $(x^3 + x^2 - 2)^2$

Discussion

Suppose $f(x)$ is a polynomial of degree m, and $g(x)$ is a polynomial of degree n.

What is the degree of:

- $f(x) + g(x)$
- $5f(x)$
- $[f(x)]^2$
- $f(x)g(x)$?

DIVISION OF POLYNOMIALS

The division of polynomials is only useful if we divide a polynomial of degree n by another of degree n or less.

Division by linears

Consider $(2x^2 + 3x + 4)(x + 2) + 7$.

If we expand this expression we obtain $(2x^2 + 3x + 4)(x + 2) + 7 = 2x^3 + 7x^2 + 10x + 15$.

Dividing both sides by $(x + 2)$, we obtain

The division of polynomials is not required for the syllabus, but is useful for understanding the Remainder and Factor theorems.

$$\begin{aligned}\frac{2x^3 + 7x^2 + 10x + 15}{x + 2} &= \frac{(2x^2 + 3x + 4)(x + 2) + 7}{x + 2} \\ &= \frac{(2x^2 + 3x + 4)(x + 2)}{x + 2} + \frac{7}{x + 2} \\ &= 2x^2 + 3x + 4 + \frac{7}{x + 2}\end{aligned}$$

where $x + 2$ is the divisor,
$2x^2 + 3x + 4$ is the quotient,
and 7 is the remainder.

If $P(x)$ is divided by $ax+b$ until a constant remainder R is obtained, then

$$\frac{P(x)}{ax+b} = Q(x) + \frac{R}{ax+b} \quad \text{where} \quad \begin{array}{rl} ax+b & \text{is the **divisor**, } D(x), \\ Q(x) & \text{is the **quotient**,} \\ \text{and} \quad R & \text{is the **remainder**.} \end{array}$$

Notice that $P(x) = Q(x) \times (ax+b) + R$.

Division algorithm

We can divide a polynomial by another polynomial using an algorithm similar to that used for division of whole numbers:

Step 1: What do we multiply x by to get $2x^3$?
The answer is $2x^2$,
and $2x^2(x+2) = \underbrace{2x^3+4x^2}$.

Step 2: Subtract $2x^3+4x^2$ from $2x^3+7x^2$.
The answer is $3x^2$.

Step 3: Bring down the $10x$ to obtain $3x^2+10x$.

Return to *Step 1* with the question:
"What must we multiply x by to get $3x^2$?"
The answer is $3x$, and $3x(x+2) = 3x^2+6x$ ....

We continue the process until we are left with a constant.

$$\begin{array}{r|l}
 & \quad 2x^2 + \ \ 3x + 4 \\ \hline
x+2 & 2x^3 + 7x^2 + 10x + 15 \\
 & -(2x^3 + 4x^2) \\ \hline
 & \qquad 3x^2 + 10x \\
 & \quad\ -(3x^2 + \ \ 6x) \\ \hline
 & \qquad\qquad 4x + 15 \\
 & \qquad\quad -(4x + 8) \\ \hline
 & \qquad\qquad\qquad 7
\end{array}$$

So, $\dfrac{2x^3+7x^2+10x+15}{x+2} = 2x^2+3x+4+\dfrac{7}{x+2}$

Example 4 — Self Tutor

Find the quotient and remainder for $\dfrac{x^3-x^2-3x-5}{x-3}$.

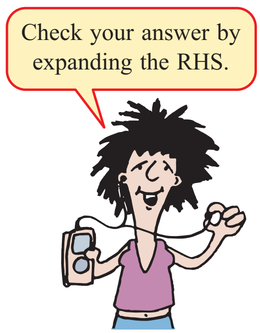
Hence write x^3-x^2-3x-5 in the form $Q(x) \times (x-3) + R$.

$$\begin{array}{r|l}
 & \qquad x^2 + 2x + 3 \\ \hline
x-3 & x^3 - \ \ x^2 - 3x - 5 \\
 & -(x^3 - 3x^2) \\ \hline
 & \qquad 2x^2 - 3x \\
 & \quad -(2x^2 - 6x) \\ \hline
 & \qquad\qquad 3x - 5 \\
 & \qquad\quad -(3x - 9) \\ \hline
 & \qquad\qquad\qquad 4
\end{array}$$

The quotient is x^2+2x+3 and the remainder is 4.

$\therefore \dfrac{x^3-x^2-3x-5}{x-3} = x^2+2x+3+\dfrac{4}{x-3}$

$\therefore x^3-x^2-3x-5 = (x^2+2x+3)(x-3)+4$.



Example 5 — Self Tutor

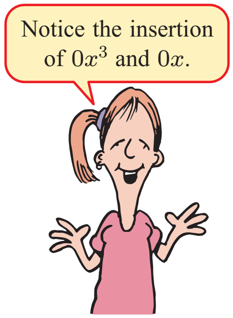
Perform the division $\dfrac{x^4+2x^2-1}{x+3}$.

Hence write x^4+2x^2-1 in the form $Q(x)\times(x+3)+R$.

$$
\begin{array}{r|l}
 & \phantom{x^4+{}} x^3-3x^2+11x-33 \\
\hline
x+3 & x^4+0x^3+2x^2+\ 0x-\ 1 \\
 & -(x^4+3x^3) \\
\cline{2-2}
 & -3x^3+2x^2 \\
 & -(-3x^3-9x^2) \\
\cline{2-2}
 & 11x^2+\ 0x \\
 & -(11x^2+33x) \\
\cline{2-2}
 & -33x-\ 1 \\
 & -(-33x-99) \\
\cline{2-2}
 & 98
\end{array}
$$

$\therefore \quad \dfrac{x^4+2x^2-1}{x+3}=x^3-3x^2+11x-33+\dfrac{98}{x+3}$

$\therefore \quad x^4+2x^2-1=(x^3-3x^2+11x-33)(x+3)+98$



EXERCISE 6A.2

1 Find the quotient and remainder for the following, and hence write the division in the form $P(x)=Q(x)\,D(x)+R$, where $D(x)$ is the divisor.

a $\dfrac{x^2+2x-3}{x+2}$ **b** $\dfrac{x^2-5x+1}{x-1}$ **c** $\dfrac{2x^3+6x^2-4x+3}{x-2}$

2 Perform the following divisions, and hence write the division in the form $P(x)=Q(x)\,D(x)+R$.

a $\dfrac{x^2-3x+6}{x-4}$ **b** $\dfrac{x^2+4x-11}{x+3}$ **c** $\dfrac{2x^2-7x+2}{x-2}$

d $\dfrac{2x^3+3x^2-3x-2}{2x+1}$ **e** $\dfrac{3x^3+11x^2+8x+7}{3x-1}$ **f** $\dfrac{2x^4-x^3-x^2+7x+4}{2x+3}$

3 Perform the divisions:

a $\dfrac{x^2+5}{x-2}$ **b** $\dfrac{2x^2+3x}{x+1}$ **c** $\dfrac{3x^2+2x-5}{x+2}$

d $\dfrac{x^3+2x^2-5x+2}{x-1}$ **e** $\dfrac{2x^3-x}{x+4}$ **f** $\dfrac{x^3+x^2-5}{x-2}$

DIVISION BY QUADRATICS

As with division by linears, we can use the **division algorithm** to divide polynomials by quadratics. The division process stops when the remainder has degree less than that of the divisor, so

> If $P(x)$ is divided by ax^2+bx+c then
>
> $$\frac{P(x)}{ax^2+bx+c} = Q(x) + \frac{ex+f}{ax^2+bx+c}$$
>
> where ax^2+bx+c is the **divisor**,
> $Q(x)$ is the **quotient**,
> and $ex+f$ is the **remainder**.
>
> The remainder will be linear if $e \neq 0$, and constant if $e = 0$.

Example 6 Self Tutor

Find the quotient and remainder for $\dfrac{x^4+4x^3-x+1}{x^2-x+1}$.

Hence write x^4+4x^3-x+1 in the form $Q(x)\times(x^2-x+1)+R(x)$.

$$\begin{array}{r|l}
 & \quad\ \ x^2+5x+4 \\ \hline
x^2-x+1 & x^4+4x^3+0x^2-x+1 \\
 & -(x^4-x^3+x^2) \\ \hline
 & \qquad 5x^3-x^2-x \\
 & \quad -(5x^3-5x^2+5x) \\ \hline
 & \qquad\qquad 4x^2-6x+1 \\
 & \qquad\quad -(4x^2-4x+4) \\ \hline
 & \qquad\qquad\quad -2x-3
\end{array}$$

The quotient is x^2+5x+4
and the remainder is $-2x-3$.

$\therefore\ x^4+4x^3-x+1$
$= (x^2+5x+4)(x^2-x+1)-2x-3$

EXERCISE 6A.3

1 Find the quotient and remainder for:

a $\dfrac{x^3+2x^2+x-3}{x^2+x+1}$ **b** $\dfrac{3x^2-x}{x^2-1}$ **c** $\dfrac{3x^3+x-1}{x^2+1}$ **d** $\dfrac{x-4}{x^2+2x-1}$

2 Carry out the following divisions and also write each in the form $P(x)=Q(x)\,D(x)+R(x)$:

a $\dfrac{x^2-x+1}{x^2+x+1}$ **b** $\dfrac{x^3}{x^2+2}$ **c** $\dfrac{x^4+3x^2+x-1}{x^2-x+1}$

d $\dfrac{2x^3-x+6}{(x-1)^2}$ **e** $\dfrac{x^4}{(x+1)^2}$ **f** $\dfrac{x^4-2x^3+x+5}{(x-1)(x+2)}$

3 Suppose $P(x)=(x-2)(x^2+2x+3)+7$. Find the quotient and remainder when $P(x)$ is divided by $x-2$.

4 Suppose $f(x)=(x-1)(x+2)(x^2-3x+5)+15-10x$. Find the quotient and remainder when $f(x)$ is divided by x^2+x-2.

B ZEROS, ROOTS, AND FACTORS

A **zero** of a polynomial is a value of the variable which makes the polynomial equal to zero.

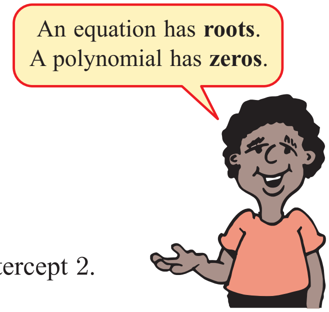
α is a **zero** of polynomial $P(x) \Leftrightarrow P(\alpha)=0$.

The **roots** of a polynomial **equation** are the solutions to the equation.

α is a **root** (or **solution**) of $P(x)=0 \Leftrightarrow P(\alpha)=0$.

The **roots** of $P(x)=0$ are the **zeros** of $P(x)$ and the x-intercepts of the graph of $y=P(x)$.

Consider
$$\begin{aligned} P(x) &= x^3+2x^2-3x-10 \\ \therefore\ P(2) &= 2^3+2(2)^2-3(2)-10 \\ &= 8+8-6-10 \\ &= 0 \end{aligned}$$



This tells us:
- 2 is a zero of $x^3+2x^2-3x-10$
- 2 is a root of $x^3+2x^2-3x-10=0$
- the graph of $y=x^3+2x^2-3x-10$ has the x-intercept 2.

If $P(x)=(x+1)(2x-1)(x+2)$, then $(x+1)$, $(2x-1)$, and $(x+2)$ are its **linear factors**.

Likewise $P(x)=(x+3)^2(2x+3)$ has been factorised into 3 linear factors, one of which is repeated.

$x-\alpha$ is a **factor** of the polynomial $P(x) \Leftrightarrow$ there exists a polynomial $Q(x)$ such that $P(x)=(x-\alpha)Q(x)$.

Example 7 — Self Tutor

Find the zeros of:

a x^2-6x+2

b x^3-5x

a We wish to find x such that

$x^2-6x+2=0$

$$\therefore\ x=\frac{6\pm\sqrt{36-4(1)(2)}}{2}$$
$$\therefore\ x=\frac{6\pm\sqrt{28}}{2}$$
$$\therefore\ x=\frac{6\pm 2\sqrt{7}}{2}$$
$$\therefore\ x=3\pm\sqrt{7}$$

The zeros are $3-\sqrt{7}$ and $3+\sqrt{7}$.

b We wish to find x such that

$$x^3-5x=0$$
$$\therefore\ x(x^2-5)=0$$
$$\therefore\ x(x+\sqrt{5})(x-\sqrt{5})=0$$
$$\therefore\ x=0 \text{ or } \pm\sqrt{5}$$

The zeros are $-\sqrt{5}$, 0, and $\sqrt{5}$.

EXERCISE 6B.1

1 Find the zeros of:

a $2x^2 - 5x - 12$ **b** $x^2 + 6x - 1$ **c** $x^2 - 10x + 6$

d $x^3 - 4x$ **e** $x^3 - 11x$ **f** $x^4 - 6x^2 + 8$

2 Find the roots of:

a $5x^2 = 3x + 2$ **b** $(2x+1)(x^2-3) = 0$ **c** $(3x-1)(x^2+x-6) = 0$

d $-2x(x^2 - 2x - 2) = 0$ **e** $x^3 = 7x$ **f** $x^4 = 7x^2 - 10$

Example 8 Self Tutor

Factorise:

a $2x^3 + 5x^2 - 3x$ **b** $x^2 + 4x - 1$

a
$$\begin{aligned} & 2x^3 + 5x^2 - 3x \\ &= x(2x^2 + 5x - 3) \\ &= x(2x-1)(x+3) \end{aligned}$$

b $x^2 + 4x - 1$ is zero when $x = \dfrac{-4 \pm \sqrt{16 - 4(1)(-1)}}{2}$

$$\therefore \quad x = \frac{-4 \pm \sqrt{20}}{2}$$

$$\therefore \quad x = \frac{-4 \pm 2\sqrt{5}}{2}$$

$$\therefore \quad x = -2 \pm \sqrt{5}$$

$$\begin{aligned} \therefore \quad x^2 + 4x - 1 &= (x - [-2+\sqrt{5}])(x - [-2-\sqrt{5}]) \\ &= (x + 2 - \sqrt{5})(x + 2 + \sqrt{5}) \end{aligned}$$

3 Find the linear factors of:

a $2x^2 - 7x - 15$ **b** $x^3 - 11x^2 + 28x$ **c** $x^2 - 6x + 3$

d $x^3 + 2x^2 - 4x$ **e** $6x^3 - x^2 - 2x$ **f** $x^4 - 6x^2 + 5$

4 If $P(x) = a(x-\alpha)(x-\beta)(x-\gamma)$ then α, β, and γ are its zeros.
Verify this statement by finding $P(\alpha)$, $P(\beta)$, and $P(\gamma)$.

Example 9 Self Tutor

Find *all* cubic polynomials with zeros $\frac{1}{2}$ and $-3 \pm \sqrt{2}$.

The zeros $-3 \pm \sqrt{2}$ have sum $= -3 + \sqrt{2} - 3 - \sqrt{2} = -6$ and
product $= (-3+\sqrt{2})(-3-\sqrt{2}) = 7$

$\therefore$ they come from the quadratic factor $x^2 + 6x + 7$

$\frac{1}{2}$ comes from the linear factor $2x - 1$.

$\therefore$ $P(x) = a(2x-1)(x^2+6x+7)$, $a \neq 0$.

5 Find *all* cubic polynomials with zeros:

a -3, 4, 5 **b** ± 2, 3 **c** 3, $1 \pm \sqrt{5}$ **d** -1, $-2 \pm \sqrt{2}$

Example 10 — Self Tutor

Find *all* quartic polynomials with zeros 2, $-\frac{1}{3}$, and $-1 \pm \sqrt{5}$.

The zeros $-1 \pm \sqrt{5}$ have sum $= -1 + \sqrt{5} - 1 - \sqrt{5} = -2$ and product $= (-1 + \sqrt{5})(-1 - \sqrt{5}) = -4$

$\therefore$ they come from the quadratic factor $x^2 + 2x - 4$.

The zeros 2 and $-\frac{1}{3}$ come from the linear factors $x - 2$ and $3x + 1$.

$\therefore$ $P(x) = a(x-2)(3x+1)(x^2+2x-4)$, $a \neq 0$.

6 Find *all* quartic polynomials with zeros of:

a ± 1, $\pm\sqrt{2}$ **b** 2, $-\frac{1}{5}$, $\pm\sqrt{3}$ **c** -3, $\frac{1}{4}$, $1 \pm \sqrt{2}$ **d** $2 \pm \sqrt{5}$, $-2 \pm \sqrt{7}$

POLYNOMIAL EQUALITY

Two polynomials are **equal** if and only if they have the **same degree** (order), and corresponding terms have equal coefficients.

If we know that two polynomials are **equal** then we can **equate coefficients** to find unknown coefficients.

For example, if $2x^3 + 3x^2 - 4x + 6 = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$, then $a = 2$, $b = 3$, $c = -4$, and $d = 6$.

Example 11 — Self Tutor

Find constants a, b, and c given that:

$$6x^3 + 7x^2 - 19x + 7 = (2x-1)(ax^2 + bx + c) \quad \text{for all } x.$$

$$6x^3 + 7x^2 - 19x + 7 = (2x-1)(ax^2+bx+c)$$
$$\therefore\ 6x^3 + 7x^2 - 19x + 7 = 2ax^3 + 2bx^2 + 2cx - ax^2 - bx - c$$
$$\therefore\ 6x^3 + 7x^2 - 19x + 7 = 2ax^3 + (2b-a)x^2 + (2c-b)x - c$$

Since this is true for all x, we equate coefficients:

$$\therefore\ \underbrace{2a = 6}_{x^3\text{ s}} \quad \underbrace{2b - a = 7}_{x^2\text{ s}} \quad \underbrace{2c - b = -19}_{x\text{ s}} \quad \text{and} \quad \underbrace{7 = -c}_{\text{constants}}$$

$$\therefore\ a = 3 \quad \text{and} \quad c = -7 \quad \text{and consequently} \quad \underbrace{2b - 3 = 7 \quad \text{and} \quad -14 - b = -19}_{\substack{\therefore\ b = 5 \\ \text{in both equations}}}$$

So, $a = 3$, $b = 5$, and $c = -7$.

Example 12

Self Tutor

Find constants a and b if $z^4+9=(z^2+az+3)(z^2+bz+3)$ for all z.

$$z^4+9=(z^2+az+3)(z^2+bz+3) \quad \text{for all } z$$

$$\begin{aligned}\therefore\ z^4+9 = z^4 &+ bz^3 + 3z^2\\ &+ az^3 + abz^2 + 3az\\ &\qquad\quad + 3z^2 + 3bz + 9\end{aligned}$$

$$\therefore\ z^4+9=z^4+(a+b)z^3+(ab+6)z^2+(3a+3b)z+9 \quad \text{for all } z$$

Equating coefficients gives $\begin{cases} a+b=0 & \text{.... (1)} & \{z^3\text{ s}\}\\ ab+6=0 & \text{.... (2)} & \{z^2\text{ s}\}\\ 3a+3b=0 & \text{.... (3)} & \{z\text{ s}\}\end{cases}$

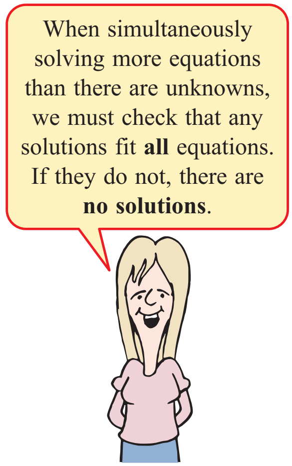
From (1) and (3) we see that $b=-a$

$\therefore$ in (2), $a(-a)+6=0$

$\therefore\ a^2=6$

$\therefore\ a=\pm\sqrt{6}$ and so $b=\mp\sqrt{6}$

$\therefore\ a=\sqrt{6},\ b=-\sqrt{6}$ *or* $a=-\sqrt{6},\ b=\sqrt{6}$



EXERCISE 6B.2

1 Find constants a, b, and c given that:

- **a** $2x^2+4x+5=ax^2+[2b-6]x+c$ for all x
- **b** $2x^3-x^2+6=(x-1)^2(2x+a)+bx+c$ for all x
- **c** $6x^3-13x^2+7x+4=(3x+1)(ax^2+bx+c)$ for all x.

2 Find constants a and b if:

- **a** $z^4+4=(z^2+az+2)(z^2+bz+2)$ for all z
- **b** $2z^4+5z^3+4z^2+7z+6=(z^2+az+2)(2z^2+bz+3)$ for all z.

3 **a** Given that $x^3+9x^2+11x-21=(x+3)(ax^2+bx+c)$, find the values of a, b, and c.

- **b** Hence, fully factorise $x^3+9x^2+11x-21$.

4 **a** Given that $4x^3+12x^2+3x-5=(2x-1)(px^2+qx+r)$, find the values of p, q, and r.

- **b** Hence, find the solutions to $4x^3+12x^2+3x-5=0$.

5 **a** Given that $3x^3+10x^2-7x+4=(x+4)(ax^2+bx+c)$, find the values of a, b, and c.

- **b** Hence, show that $3x^3+10x^2-7x+4$ has only one real zero.

6 Suppose $3x^3+kx^2-7x-2=(3x+2)(ax^2+bx+c)$.

- **a** Find the values of a, b, c, and k.
- **b** Hence, find the roots of $3x^3+kx^2-7x-2=0$.

7 **a** Find real numbers a and b such that $x^4-4x^2+8x-4=(x^2+ax+2)(x^2+bx-2)$.

- **b** Hence, find the real roots of $x^4+8x=4x^2+4$.