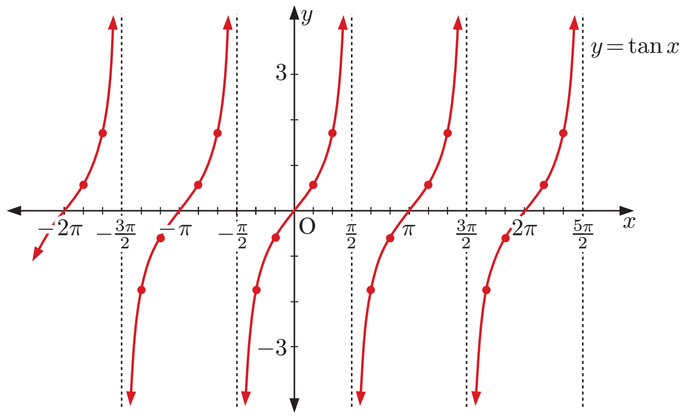
THE GRAPH OF $y = \tan x$

Since $\tan x = \dfrac{\sin x}{\cos x}$, $\tan x$ will be undefined whenever $\cos x = 0$.

The zeros of the function $y = \cos x$ correspond to vertical asymptotes of the function $y = \tan x$.



DEMO

We observe that $y = \tan x$ has:

- **period** π
- **range** $y \in \mathbb{R}$
- **vertical asymptotes** $x = \frac{\pi}{2} + k\pi$ for all $k \in \mathbb{Z}$.

TANGENT FUNCTION

Click on the icon to explore how the tangent function is produced from the unit circle.

THE GENERAL TANGENT FUNCTION

The **general tangent function** is $y = a \tan bx + c$, $a > 0$, $b > 0$.

- The **principal axis** is $y = c$.
- The **period** of this function is $\dfrac{\pi}{b}$.
- The **amplitude** of this function is undefined.

DYNAMIC TANGENT FUNCTION

Click on the icon to explore the properties of this function.

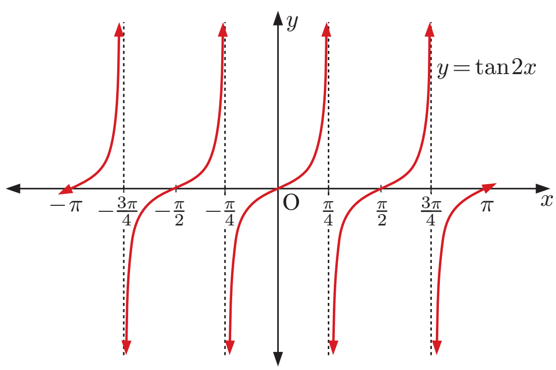
Example 4

Self Tutor

Without using technology, sketch the graph of $y = \tan 2x$ for $-\pi \leqslant x \leqslant \pi$.

Since $b = 2$, the period is $\frac{\pi}{2}$.

The vertical asymptotes are $x = \pm\frac{\pi}{4}$, $x = \pm\frac{3\pi}{4}$, and the x-axis intercepts are at $0, \pm\frac{\pi}{2}, \pm\pi$.



Discussion

- Discuss how to find the x-intercepts of $y = \tan x$.
- How can we simplify $\tan(x - \pi)$?
- How many solutions does the equation $\tan x = 2$ have?

EXERCISE 9D

GRAPHING PACKAGE

1 *Sketch* the following functions for $-\pi \leqslant x \leqslant \pi$:

a $y = 2\tan x$ **b** $y = \tan 3x$ **c** $y = \tan x + 2$

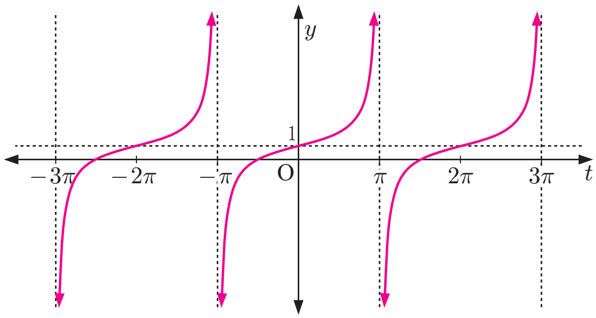
d $y = 3\tan 2x$ **e** $y = 2\tan x - 1$ **f** $y = 2\tan 3x + 2$

Use technology to check your answers.

2 Find b and c given that the function $y = \tan bx + c$, $b > 0$, has:

a period $\frac{2\pi}{3}$ and principal axis $y = 2$ **b** period $\frac{\pi}{2}$ and principal axis $y = -3$.

3 Find p and q given the following graph of the function $y = \tan pt + q$.



Activity

Click on the icon to run a card game for trigonometric functions.

CARD GAME

E TRIGONOMETRIC EQUATIONS

Linear equations such as $2x + 3 = 11$ have exactly one solution. Quadratic equations of the form $ax^2 + bx + c = 0$, $a \neq 0$ have at most two real solutions.

Trigonometric equations generally have infinitely many solutions unless a restricted domain such as $0 \leqslant x \leqslant 3\pi$ is given.

For example, suppose that Andrew in the **Opening Problem** wants to know when the green light will be 16 metres above the ground. To find out, he will need to solve a trigonometric equation. If the wheel keeps rotating, the equation would have infinitely many solutions. Andrew may therefore specify that he is interested in the *first* time the green light is 16 metres above the ground.

If a periodic function $f(x)$ has period p then the domain $0 \leqslant x < p$ is called the **principal domain**. By solving an equation on the principal domain, all the other solutions can be found using the periodic behaviour.

If $x = a$ is a solution, then $x = a + kp$ will also be a solution for all $k \in \mathbb{Z}$.

For example, $\sin x$ has period 2π, so it is normal to consider the domain $0 \leqslant x < 2\pi$.

Discussion

What would you choose as the principal domain for:

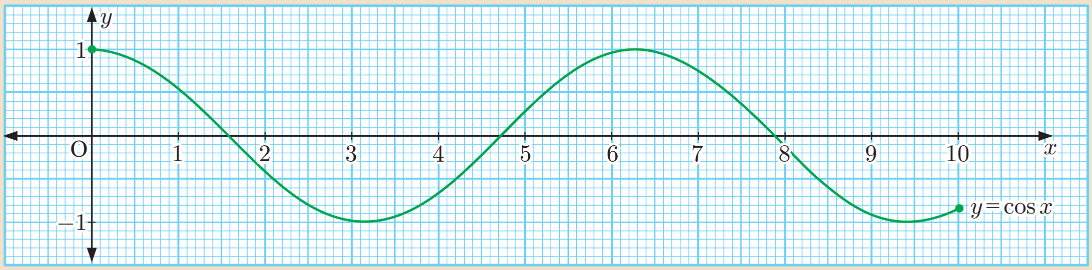
- $y = \cos x$
- $y = \sin(2x)$
- $y = \tan x$?

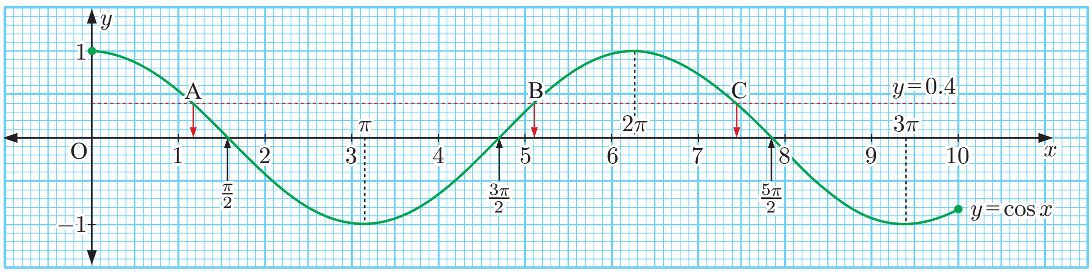
GRAPHICAL SOLUTION OF TRIGONOMETRIC EQUATIONS

Sometimes simple trigonometric graphs are available on grid paper. In such cases we can estimate solutions straight from the graph.

Example 5 Self Tutor

Solve $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$ radians using the graph of $y = \cos x$.



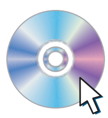


$y = 0.4$ meets $y = \cos x$ at A, B, and C. Hence $x \approx 1.2$, 5.1, or 7.4 .

The solutions of $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$ radians are 1.2, 5.1, and 7.4 .

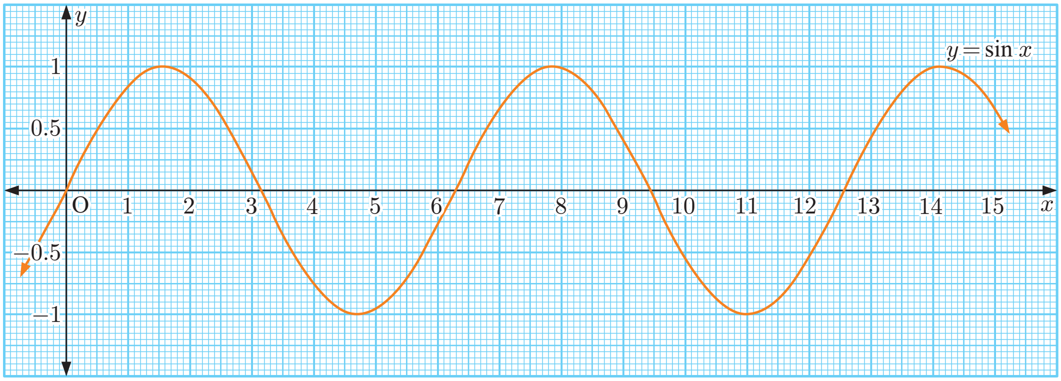
Trigonometric equations may also be solved using the graphing package.

GRAPHING PACKAGE



EXERCISE 9E.1

1

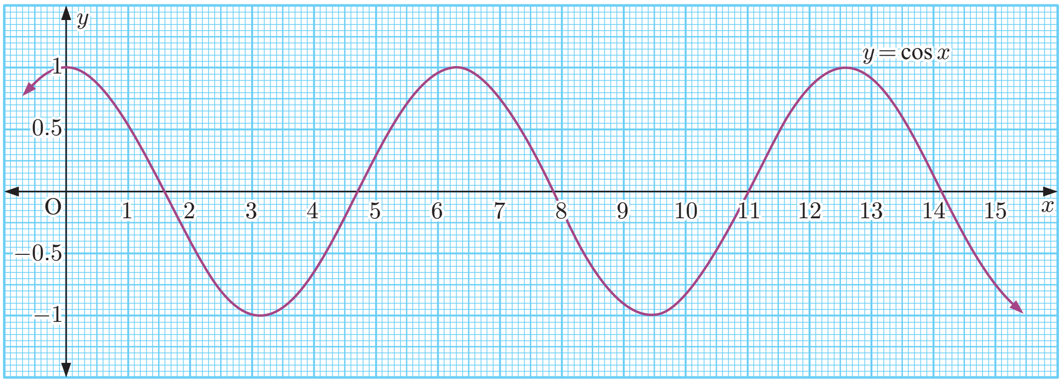


Use the graph of $y = \sin x$ to find, correct to 1 decimal place, the solutions of:

a $\sin x = 0.3$ for $0 \leqslant x \leqslant 15$

b $\sin x = -0.4$ for $5 \leqslant x \leqslant 15$.

2

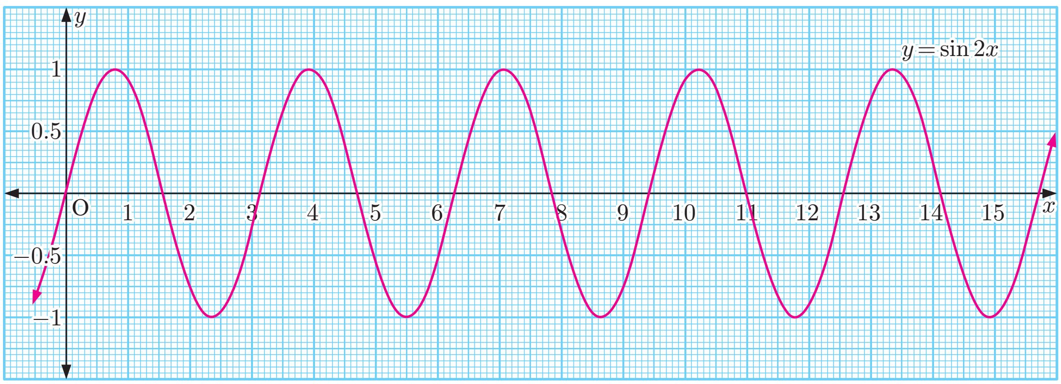


Use the graph of $y = \cos x$ to find, correct to 1 decimal place, the solutions of:

a $\cos x = 0.6$ for $0 \leqslant x \leqslant 10$

b $\cos x = -0.3$ for $4 \leqslant x \leqslant 12$.

3

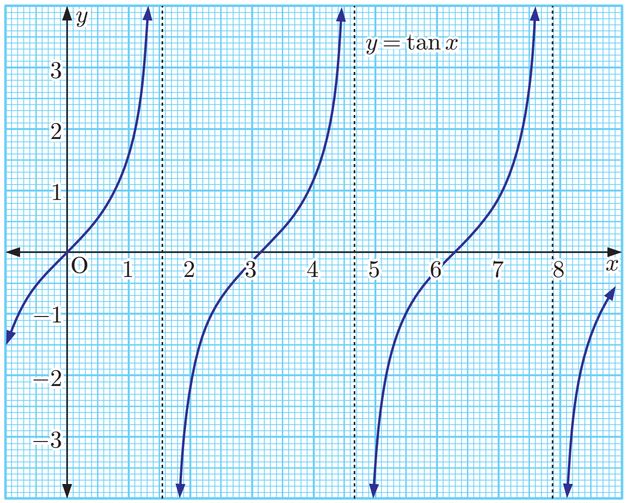


Use the graph of $y = \sin 2x$ to find, correct to 1 decimal place, the solutions of:

a $\sin 2x = 0.7$ for $0 \leqslant x \leqslant 16$

b $\sin 2x = -0.3$ for $0 \leqslant x \leqslant 16$.

4



The graph of $y = \tan x$ is illustrated.

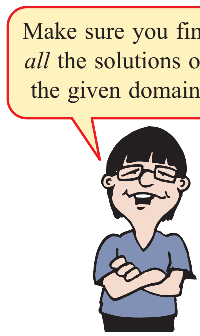
a Use the graph to estimate: **i** $\tan 1$ **ii** $\tan 2.3$
Check your answers using a calculator.

b Find, correct to 1 decimal place, the solutions of:
i $\tan x = 2$ for $0 \leqslant x \leqslant 8$ **ii** $\tan x = -1.4$ for $2 \leqslant x \leqslant 7$.

5 Use the graphing package to solve for x on the domain $0 < x < 4\pi$:
a $\sin x = 0.431$ **b** $\cos x = -0.814$ **c** $3\tan x - 2 = 0$

6 Use the graphing package to solve for x on the domain $-5 \leqslant x \leqslant 5$:
a $5\cos x - 4 = 0$ **b** $2\tan x + 13 = 0$ **c** $8\sin x + 3 = 0$

7 **a** Use the graphing package to solve $\sin^2 x + \sin x - 2 = 0$ for $0 \leqslant x \leqslant 2\pi$.
b Solve for m: $m^2 + m - 2 = 0$.
c Hence explain your answer in **a**.



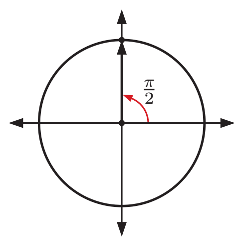
SOLVING TRIGONOMETRIC EQUATIONS USING ALGEBRA

Using a graph we get approximate decimal or **numerical** solutions to trigonometric equations.

Sometimes exact solutions are needed in terms of π, and these arise when the solutions are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$. Exact solutions obtained using algebra are called **analytical** solutions.

We use the periodicity of the trigonometric functions to give us all solutions in the required domain.

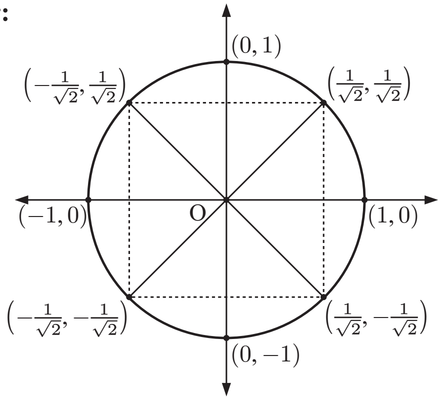
For example, consider $\sin x = 1$. We know from the unit circle that a solution is $x = \frac{\pi}{2}$. However, since the period of $\sin x$ is 2π, there are infinitely many solutions spaced 2π apart.

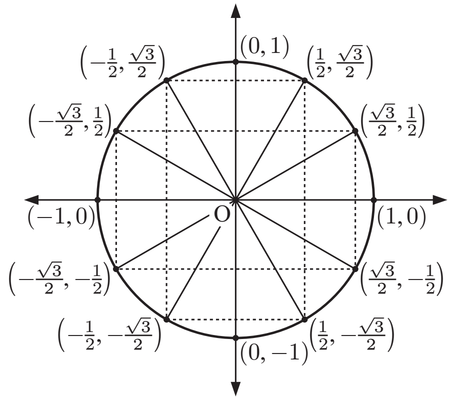


Hence $x = \frac{\pi}{2} + k2\pi$ is a solution for any $k \in \mathbb{Z}$.

In this course we will be solving equations on a fixed domain. This means there will be a finite number of solutions.

Reminder:





Example 6 — Self Tutor

Solve for x: $2\sin x - 1 = 0, \quad 0 \leqslant x \leqslant \pi$

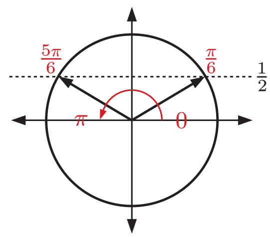
$2\sin x - 1 = 0$

$\therefore \quad \sin x = \frac{1}{2}$

There are two points on the unit circle with sine $\frac{1}{2}$.

They correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

These are the only solutions on the domain $0 \leqslant x \leqslant \pi$, so $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.



Since the tangent function is periodic with period π we see that $\tan(x+\pi) = \tan x$ for all values of x. This means that equal $\tan$ values are π units apart.

Example 7 — Self Tutor

Solve $\tan x + \sqrt{3} = 0$ for $0 < x < 4\pi$.

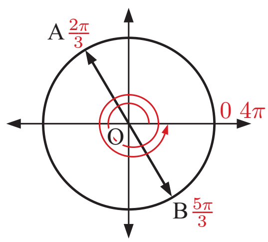
$\tan x + \sqrt{3} = 0$

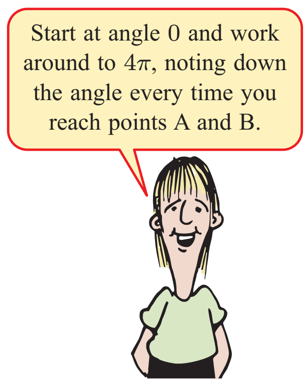
$\therefore \quad \tan x = -\sqrt{3}$

There are two points on the unit circle with tangent $-\sqrt{3}$.

They correspond to angles $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

For the domain $0 < x < 4\pi$ we have 4 solutions: $x = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}$, or $\frac{11\pi}{3}$.





EXERCISE 9E.2

1 Solve for x on the domain $0 \leqslant x \leqslant 4\pi$:

a $2\cos x - 1 = 0$ **b** $\sqrt{2}\sin x = 1$ **c** $\tan x = 1$

2 Solve for x on the domain $-2\pi \leqslant x \leqslant 2\pi$:

a $2\sin x - \sqrt{3} = 0$ **b** $\sqrt{2}\cos x + 1 = 0$ **c** $\tan x = -1$

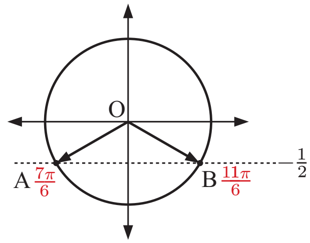
Example 8 — Self Tutor

Solve exactly for $0 \leqslant x \leqslant 3\pi$: **a** $\sin x = -\frac{1}{2}$ **b** $\sin 2x = -\frac{1}{2}$

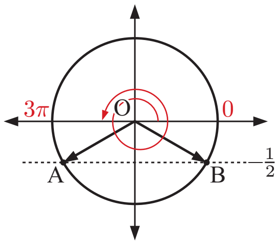
The equations both have the form $\sin\theta = -\frac{1}{2}$.

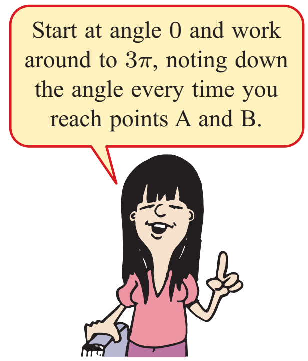
There are two points on the unit circle with sine $-\frac{1}{2}$.

They correspond to angles $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.



a In this case θ is simply x, so we have the domain $0 \leqslant x \leqslant 3\pi$.
The only solutions for this domain are $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.





b In this case θ is $2x$.
If $0 \leqslant x \leqslant 3\pi$ then $0 \leqslant 2x \leqslant 6\pi$.
$\therefore\ 2x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{31\pi}{6}$, or $\frac{35\pi}{6}$
$\therefore\ x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \frac{31\pi}{12}$, or $\frac{35\pi}{12}$

3 Solve exactly for $0 \leqslant x \leqslant 3\pi$: **a** $\cos x = \frac{1}{2}$ **b** $\cos 2x = \frac{1}{2}$

4 Solve exactly for $0 \leqslant x \leqslant 2\pi$: **a** $\sin x = -\frac{1}{\sqrt{2}}$ **b** $\sin 3x = -\frac{1}{\sqrt{2}}$

5 Find the exact solutions of:

a $\cos x = -\frac{1}{2},\ 0 \leqslant x \leqslant 5\pi$

b $2\sin x - 1 = 0,\ -360° \leqslant x \leqslant 360°$

c $2\cos x + \sqrt{3} = 0,\ 0 \leqslant x \leqslant 3\pi$

d $3\cos 2x + 3 = 0,\ 0 \leqslant x \leqslant 3\pi$

e $4\cos 3x + 2 = 0,\ -\pi \leqslant x \leqslant \pi$

Example 9 — Self Tutor

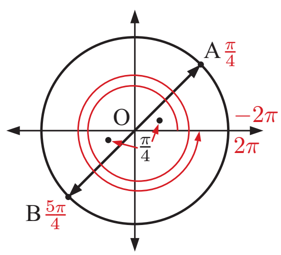
Solve $\tan 2x + 1 = 2$ for $-\pi \leqslant x \leqslant \pi$.

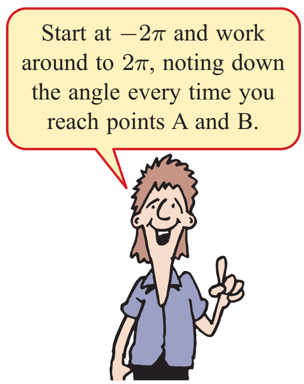
$\tan 2x = 1$

There are two points on the unit circle which have tangent 1.

Since $-\pi \leqslant x \leqslant \pi$,
$-2\pi \leqslant 2x \leqslant 2\pi$

So, $2x = -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}$, or $\frac{5\pi}{4}$
$\therefore\ x = -\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}$, or $\frac{5\pi}{8}$





6 Solve $\tan x = \sqrt{3}$ for $0 \leqslant x \leqslant 2\pi$. Hence solve the following equations for $0 \leqslant x \leqslant 2\pi$:

a $\tan 4x = \sqrt{3}$ **b** $\tan^2 x = 3$

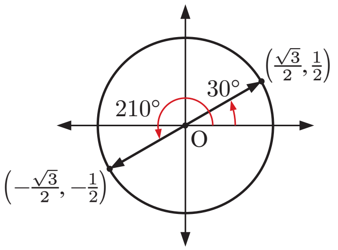
7 Solve $\sqrt{3}\tan 3x = 1$ for $0 \leqslant x \leqslant \pi$.

8 Solve for $-\pi \leqslant x \leqslant \pi$:

a $\sec x = -2$ **b** $\sqrt{3}\,\mathrm{cosec}\, 2x = 2$ **c** $\cot x = 0$

Example 10 — Self Tutor

Find the exact solutions of $\sqrt{3}\sin x = \cos x$ for $0° \leqslant x \leqslant 360°$.



$$\sqrt{3}\sin x = \cos x$$

$$\therefore \quad \frac{\sin x}{\cos x} = \frac{1}{\sqrt{3}} \quad \{\text{dividing both sides by } \sqrt{3}\cos x\}$$

$$\therefore \quad \tan x = \frac{1}{\sqrt{3}}$$

$$\therefore \quad x = 30° \text{ or } 210°$$

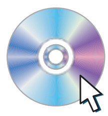
9 Solve for $0 \leqslant x \leqslant 2\pi$:

a $\sin x - \cos x = 0$ **b** $\sin x = -\cos x$

c $\sin 3x = \cos 3x$ **d** $\sin 2x = \sqrt{3}\cos 2x$

Check your answers using the graphing package.

GRAPHING PACKAGE



10 Solve for $0 \leqslant x \leqslant \pi$: $\sin x = \mathrm{cosec}\, x$

F TRIGONOMETRIC RELATIONSHIPS

There are a vast number of trigonometric relationships. However, we only need to remember a few because we can obtain the rest by rearrangement or substitution.

SIMPLIFYING TRIGONOMETRIC EXPRESSIONS

For any given angle θ, $\sin\theta$ and $\cos\theta$ are real numbers. $\tan\theta$ is also real whenever it is defined. The algebra of trigonometry is therefore identical to the algebra of real numbers.

An expression like $2\sin\theta + 3\sin\theta$ compares with $2x + 3x$, so $2\sin\theta + 3\sin\theta = 5\sin\theta$.

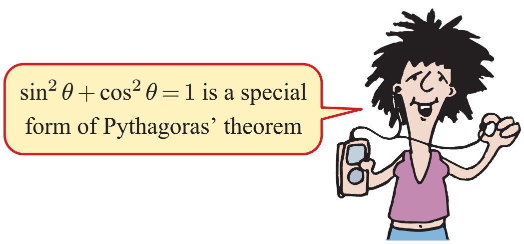
To simplify more complicated trigonometric expressions, we often use the identities:

$$\boldsymbol{\sin^2\theta + \cos^2\theta = 1}$$

$$\boldsymbol{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

$$\boldsymbol{\tan^2\theta + 1 = \sec^2\theta}$$

$$\boldsymbol{1 + \cot^2\theta = \mathrm{cosec}^2\theta}$$



We can also use rearrangements of these formulae, such as:

$$\sin^2\theta = 1 - \cos^2\theta \qquad \tan^2\theta = \sec^2\theta - 1$$
$$\cos^2\theta = 1 - \sin^2\theta \qquad \cot^2\theta = \csc^2\theta - 1$$

Example 11 — Self Tutor

Simplify:

a $3\cos\theta + 4\cos\theta$ **b** $\tan\alpha - 3\tan\alpha$

a $3\cos\theta + 4\cos\theta = 7\cos\theta$
{compare with $3x + 4x = 7x$}

b $\tan\alpha - 3\tan\alpha = -2\tan\alpha$
{compare with $x - 3x = -2x$}

Example 12 — Self Tutor

Simplify:

a $2 - 2\sin^2\theta$ **b** $\cos^2\theta\sin\theta + \sin^3\theta$

a
$$\begin{aligned} &2 - 2\sin^2\theta \\ &= 2(1 - \sin^2\theta) \\ &= 2\cos^2\theta \\ &\{\cos^2\theta + \sin^2\theta = 1\} \end{aligned}$$

b
$$\begin{aligned} &\cos^2\theta\sin\theta + \sin^3\theta \\ &= \sin\theta(\cos^2\theta + \sin^2\theta) \\ &= \sin\theta \times 1 \\ &= \sin\theta \end{aligned}$$

EXERCISE 9F.1

1 Simplify:

a $\sin\theta + \sin\theta$ **b** $2\cos\theta + \cos\theta$ **c** $3\sin\theta - \sin\theta$

d $3\sin\theta - 2\sin\theta$ **e** $\tan\theta - 3\tan\theta$ **f** $2\cos^2\theta - 5\cos^2\theta$

Example 13 — Self Tutor

Expand and simplify: $(\cos\theta - \sin\theta)^2$

$$\begin{aligned} &(\cos\theta - \sin\theta)^2 \\ &= \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta \qquad \{\text{using } (a-b)^2 = a^2 - 2ab + b^2\} \\ &= \cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta \\ &= 1 - 2\cos\theta\sin\theta \end{aligned}$$

2 Simplify:

a $3\sin^2\theta + 3\cos^2\theta$ **b** $-2\sin^2\theta - 2\cos^2\theta$ **c** $-\cos^2\theta - \sin^2\theta$

d $3 - 3\sin^2\theta$ **e** $4 - 4\cos^2\theta$ **f** $\cos^3\theta + \cos\theta\sin^2\theta$

g $\cos^2\theta - 1$ **h** $\sin^2\theta - 1$ **i** $2\cos^2\theta - 2$

j $\dfrac{1-\sin^2\theta}{\cos^2\theta}$ **k** $\dfrac{1-\cos^2\theta}{\sin\theta}$ **l** $\dfrac{\cos^2\theta - 1}{-\sin\theta}$

3 Simplify:

a $3\tan x - \dfrac{\sin x}{\cos x}$ **b** $\dfrac{\sin^2 x}{\cos^2 x}$ **c** $\tan x \cos x$

d $\dfrac{\sin x}{\tan x}$ **e** $3\sin x + 2\cos x \tan x$ **f** $\dfrac{2\tan x}{\sin x}$

g $\tan x \cot x$ **h** $\sin x \operatorname{cosec} x$ **i** $\sec x \cot x$

j $\sin x \cot x$ **k** $\dfrac{\cot x}{\operatorname{cosec} x}$ **l** $\dfrac{2\sin x \cot x + 3\cos x}{\cot x}$

4 Expand and simplify if possible:

a $(1+\sin\theta)^2$ **b** $(\sin\alpha - 2)^2$ **c** $(\tan\alpha - 1)^2$

d $(\sin\alpha + \cos\alpha)^2$ **e** $(\sin\beta - \cos\beta)^2$ **f** $-(2-\cos\alpha)^2$

5 Simplify:

a $1-\sec^2\beta$ **b** $\dfrac{\tan^2\theta(\cot^2\theta+1)}{\tan^2\theta+1}$

c $\cos^2\alpha(\sec^2\alpha - 1)$ **d** $(\sin x + \tan x)(\sin x - \tan x)$

e $(2\sin\theta + 3\cos\theta)^2 + (3\sin\theta - 2\cos\theta)^2$ **f** $(1+\operatorname{cosec}\theta)(\sin\theta - \sin^2\theta)$

g $\sec A - \sin A \tan A - \cos A$

FACTORISING TRIGONOMETRIC EXPRESSIONS

Example 14 — Self Tutor

Factorise:

a $\cos^2\alpha - \sin^2\alpha$ **b** $\tan^2\theta - 3\tan\theta + 2$

a $\cos^2\alpha - \sin^2\alpha$
$= (\cos\alpha + \sin\alpha)(\cos\alpha - \sin\alpha)$ {compare with $a^2 - b^2 = (a+b)(a-b)$}

b $\tan^2\theta - 3\tan\theta + 2$
$= (\tan\theta - 2)(\tan\theta - 1)$ {compare with $x^2 - 3x + 2 = (x-2)(x-1)$}

EXERCISE 9F.2

1 Factorise:

a $1-\sin^2\theta$ **b** $\sin^2\alpha - \cos^2\alpha$

c $\tan^2\alpha - 1$ **d** $2\sin^2\beta - \sin\beta$

e $2\cos\phi + 3\cos^2\phi$ **f** $3\sin^2\theta - 6\sin\theta$

g $\tan^2\theta + 5\tan\theta + 6$ **h** $2\cos^2\theta + 7\cos\theta + 3$

i $6\cos^2\alpha - \cos\alpha - 1$ **j** $3\tan^2\alpha - 2\tan\alpha$

k $\sec^2\beta - \operatorname{cosec}^2\beta$ **l** $2\cot^2 x - 3\cot x + 1$

m $2\sin^2 x + 7\sin x\cos x + 3\cos^2 x$

Example 15 — Self Tutor

Simplify:

a $\dfrac{2-2\cos^2\theta}{1+\cos\theta}$ **b** $\dfrac{\cos\theta-\sin\theta}{\cos^2\theta-\sin^2\theta}$

a
$$\begin{aligned}&\frac{2-2\cos^2\theta}{1+\cos\theta}\\ &=\frac{2(1-\cos^2\theta)}{1+\cos\theta}\\ &=\frac{2\cancel{(1+\cos\theta)}^{1}(1-\cos\theta)}{\cancel{(1+\cos\theta)}_{1}}\\ &=2(1-\cos\theta)\end{aligned}$$

b
$$\begin{aligned}&\frac{\cos\theta-\sin\theta}{\cos^2\theta-\sin^2\theta}\\ &=\frac{\cancel{(\cos\theta-\sin\theta)}^{1}}{(\cos\theta+\sin\theta)\cancel{(\cos\theta-\sin\theta)}_{1}}\\ &=\frac{1}{\cos\theta+\sin\theta}\end{aligned}$$

2 Simplify:

a $\dfrac{1-\sin^2\alpha}{1-\sin\alpha}$ **b** $\dfrac{\tan^2\beta-1}{\tan\beta+1}$ **c** $\dfrac{\cos^2\phi-\sin^2\phi}{\cos\phi+\sin\phi}$

d $\dfrac{\cos^2\phi-\sin^2\phi}{\cos\phi-\sin\phi}$ **e** $\dfrac{\sin\alpha+\cos\alpha}{\sin^2\alpha-\cos^2\alpha}$ **f** $\dfrac{3-3\sin^2\theta}{6\cos\theta}$

g $1-\dfrac{\cos^2\theta}{1+\sin\theta}$ **h** $\dfrac{1+\cot\theta}{\operatorname{cosec}\theta}-\dfrac{\sec\theta}{\tan\theta+\cot\theta}$ **i** $\dfrac{\tan^2\theta}{\sec\theta-1}$

3 Show that:

a $(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2=2$ **b** $(2\sin\theta+3\cos\theta)^2+(3\sin\theta-2\cos\theta)^2=13$

c $(1-\cos\theta)\left(1+\dfrac{1}{\cos\theta}\right)=\tan\theta\sin\theta$ **d** $\left(1+\dfrac{1}{\sin\theta}\right)(\sin\theta-\sin^2\theta)=\cos^2\theta$

e $\sec A-\cos A=\tan A\sin A$ **f** $\dfrac{\cos\theta}{1-\sin\theta}=\sec\theta+\tan\theta$

g $\dfrac{\cos\alpha}{1-\tan\alpha}+\dfrac{\sin\alpha}{1-\cot\alpha}=\sin\alpha+\cos\alpha$ **h** $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}=2\operatorname{cosec}\theta$

i $\dfrac{\sin\theta}{1-\cos\theta}-\dfrac{\sin\theta}{1+\cos\theta}=2\cot\theta$ **j** $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}=2\sec^2\theta$

GRAPHING PACKAGE

Use a graphing package to check these simplifications by graphing each function on the same set of axes.

Discovery 5 — Double angle formulae

What to do:

1 Copy and complete, using angles of your choice as well:

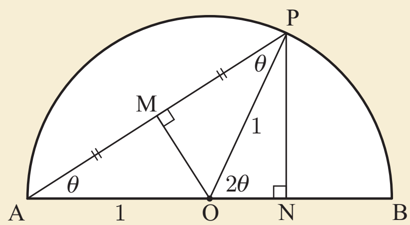
θ	$\sin 2\theta$	$2\sin\theta$	$2\sin\theta\cos\theta$	$\cos 2\theta$	$2\cos\theta$	$\cos^2\theta-\sin^2\theta$
0.631						
57.81°						
-3.697						

2 Write down any discoveries from your table of values in **1**.

3 In the diagram alongside, the semi-circle has radius 1 unit, and $\widehat{PAB} = \theta$.

$\widehat{APO} = \theta$ {$\triangle$AOP is isosceles}

$\widehat{PON} = 2\theta$ {exterior angle of a triangle}



a Find in terms of θ, the lengths of:

i OM **ii** AM **iii** ON **iv** PN

b Use $\triangle$ANP and the lengths in **a** to show that:

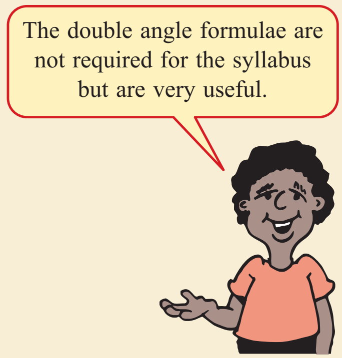
i $\cos\theta = \dfrac{\sin 2\theta}{2\sin\theta}$ **ii** $\cos\theta = \dfrac{1+\cos 2\theta}{2\cos\theta}$

c Hence deduce that:

i $\sin 2\theta = 2\sin\theta\cos\theta$ **ii** $\cos 2\theta = 2\cos^2\theta - 1$

4 Starting with $\cos 2\theta = 2\cos^2\theta - 1$, show that:

a $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ **b** $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$



G TRIGONOMETRIC EQUATIONS IN QUADRATIC FORM

Sometimes we may be given trigonometric equations in quadratic form.

For example, $2\sin^2 x + \sin x = 0$ and $2\cos^2 x + \cos x - 1 = 0$ are quadratic equations where the variables are $\sin x$ and $\cos x$ respectively.

These equations can be factorised by quadratic factorisation and then solved for x.

Example 16 — Self Tutor

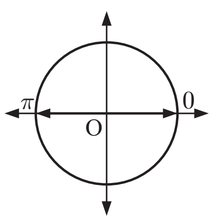
Solve for $0 \leqslant x \leqslant 2\pi$:

a $2\sin^2 x + \sin x = 0$ **b** $2\cos^2 x + \cos x - 1 = 0$

a $2\sin^2 x + \sin x = 0$

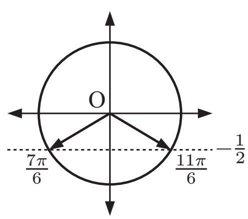
$\therefore \sin x(2\sin x + 1) = 0$

$\therefore \sin x = 0$ or $-\frac{1}{2}$



$\sin x = 0$ when

$x = 0, \pi$, or 2π



$\sin x = -\frac{1}{2}$ when

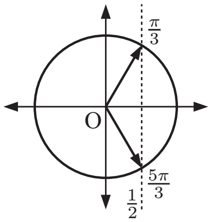
$x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

The solutions are: $x = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$, or 2π.

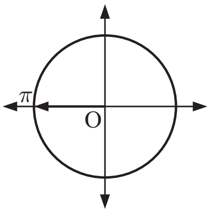
b $2\cos^2 x + \cos x - 1 = 0$

$\therefore\ (2\cos x - 1)(\cos x + 1) = 0$

$\therefore\ \cos x = \frac{1}{2}$ or -1



$\cos x = \frac{1}{2}$ when $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$



$\cos x = -1$ when $x = \pi$

The solutions are: $x = \frac{\pi}{3}$, π, or $\frac{5\pi}{3}$.

EXERCISE 9G

1 Solve for $0 \leqslant x \leqslant 2\pi$:

a $2\sin^2 x + \sin x = 0$ **b** $2\cos^2 x = \cos x$ **c** $2\cos^2 x + \cos x - 1 = 0$

d $2\sin^2 x + 3\sin x + 1 = 0$ **e** $\sin^2 x = 2 - \cos x$ **f** $\cos x + \sec x = 2$

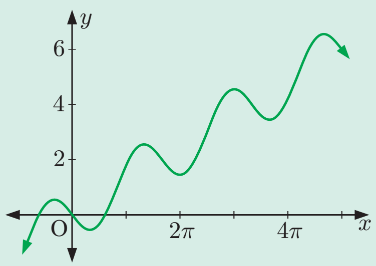
2 Solve for $0 \leqslant x \leqslant 2\pi$:

a $\sin^2 x + \cos x = -1$ **b** $2\cos^2 x = 3\sin x$

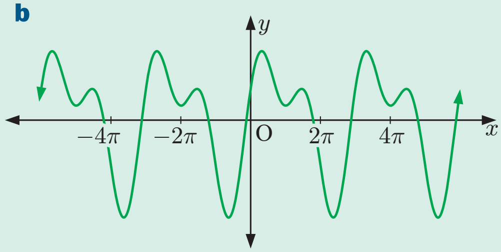
Review set 9A

1 Which of the following graphs displays periodic behaviour?

a



b



2 Draw each of the following graphs for $0 \leqslant x \leqslant 2\pi$:

a $y = 5\sin x$ **b** $y = \cos 3x - 1$ **c** $y = \tan 2x + 4$

3 State the minimum and maximum values of:

a $1 + \sin x$ **b** $2\cos 3x$ **c** $y = 3\sin 2x$ **d** $y = \cos 4x - 1$

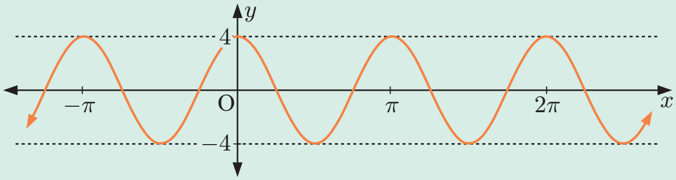
4 State the period of:

a $y = 4\sin x$ **b** $y = 2\cos 4x$ **c** $y = 4\cos 2x + 4$ **d** $y = 2\tan 3x$

5 Complete the table:

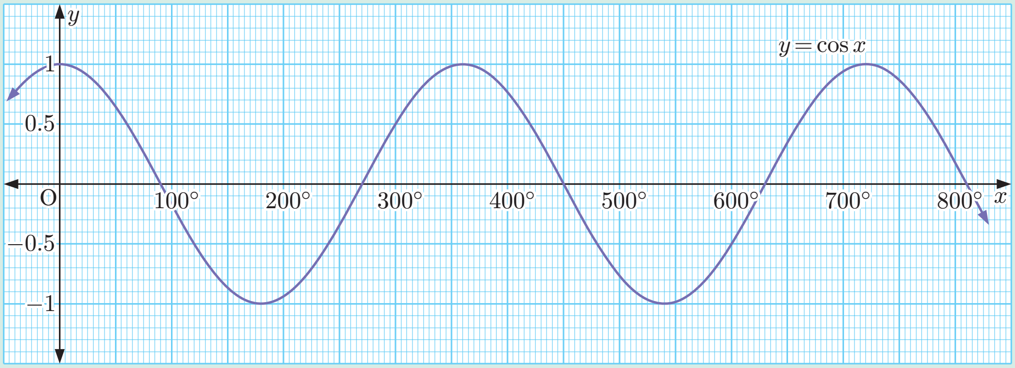
Function	*Period*	*Amplitude*	*Domain*	*Range*
$y = 3\sin 2x + 1$				
$y = \tan 2x$				
$y = 2\cos 3x - 3$				

6 Find the cosine function represented in the graph.



7 On the same set of axes, graph $y = 2\cos x$ and $y = |2\cos x|$ for $0 \leqslant x \leqslant 2\pi$.

8



Use the graph of $y = \cos x$ to find the solutions of:

a $\cos x = -0.4, \quad 0 \leqslant x \leqslant 800°$

b $\cos x = 0.9, \quad 0 \leqslant x \leqslant 600°$

9 Solve in terms of π:

a $2\sin x = -1$ for $0 \leqslant x \leqslant 4\pi$

b $\sqrt{2}\sin x - 1 = 0$ for $-2\pi \leqslant x \leqslant 2\pi$

c $2\sin 3x + \sqrt{3} = 0$ for $0 \leqslant x \leqslant 2\pi$

d $\sqrt{2}\cos x - 1 = 0$ for $0 \leqslant x \leqslant 4\pi$

10 Simplify:

a $\dfrac{1-\cos^2\theta}{1+\cos\theta}$

b $\dfrac{\sin\alpha - \cos\alpha}{\sin^2\alpha - \cos^2\alpha}$

c $\dfrac{4\sin^2\alpha - 4}{8\cos\alpha}$

d $\dfrac{\cot^2\theta}{\operatorname{cosec}\theta - 1}$

11 Show that $\dfrac{\cos\theta - \sec\theta}{\tan\theta}$ simplifies to $-\sin\theta$.

12 Find exact solutions for $-\pi \leqslant x \leqslant \pi$:

a $\tan 2x = -\sqrt{3}$

b $\tan^2 x - 3 = 0$

Review set 9B

1 Consider the graph alongside.

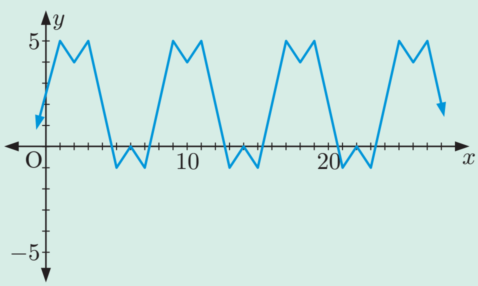
a Explain why this graph shows periodic behaviour.

b State:

i the period

ii the maximum value

iii the minimum value



2 Find b given that the function $y = \sin bx$, $b > 0$ has period:

a $\frac{\pi}{3}$ **b** $\frac{\pi}{12}$

3 State the minimum and maximum values of:

a $y = 5\sin x - 3$ **b** $y = 3\cos x + 1$ **c** $y = 4\cos 2x + 9$

4 On the same set of axes, for the domain $0 \leqslant x \leqslant 2\pi$, sketch:

a $y = \cos x$ and $y = \cos x - 3$ **b** $y = \tan x$ and $y = 2\tan x$

c $y = \cos x$ and $y = \cos 2x + 1$ **d** $y = \sin x$ and $y = 3\sin x + 1$

5 The function $y = a\sin bx + c$, $a > 0$, $b > 0$, has amplitude 2, period $\frac{\pi}{3}$, and principal axis $y = -2$.

a Find the values of a, b, and c. **b** Sketch the function for $0 \leqslant x \leqslant \pi$.

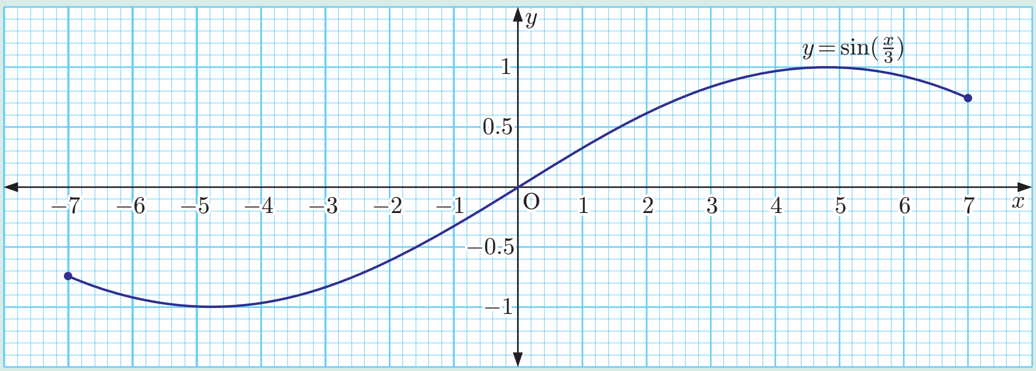
6 Consider the function $y = 2\tan x$.

a State a function which has the same shape, but has principal axis $y = 2$.

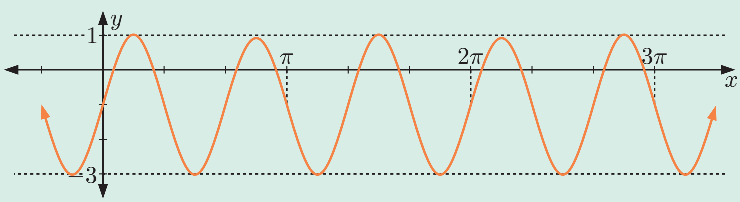
b Draw $y = 2\tan x$ and your function from **a** on the same set of axes, for $-2\pi \leqslant x \leqslant 2\pi$.

7 Consider $y = \sin(\frac{x}{3})$ on the domain $-7 \leqslant x \leqslant 7$. Use the graph to solve, correct to 1 decimal place:

a $\sin(\frac{x}{3}) = -0.9$ **b** $\sin(\frac{x}{3}) = \frac{1}{4}$



8 Find m and n given the following graph of the function $y = 2\sin mx + n$:



9 Solve for $0 \leqslant x \leqslant 2\pi$:

a $\sin^2 x - \sin x - 2 = 0$ **b** $4\sin^2 x = 1$

10 Simplify:

a $\cos^3\theta + \sin^2\theta\cos\theta$ **b** $\dfrac{\cos^2\theta - 1}{\sin\theta}$

c $5 - 5\sin^2\theta$ **d** $\dfrac{\sin^2\theta - 1}{\cos\theta}$

11 Expand and simplify if possible:

a $(2\sin\alpha - 1)^2$

b $(\cos\alpha - \sin\alpha)^2$

12 Show that:

a $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta} = 2\sec\theta$

b $(1+\sec\theta)\left(\cos\theta - \cos^2\theta\right) = \sin^2\theta$

10

Counting and the binomial expansion

Contents:

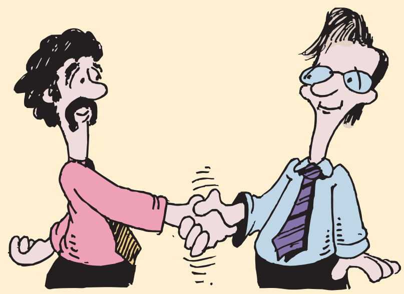
A The product principle
B Counting paths
C Factorial notation
D Permutations
E Combinations
F Binomial expansions
G The Binomial Theorem

Opening problem

At a mathematics teachers' conference there are 273 delegates present. The organising committee consists of 10 people.

Things to think about:

- **a** If each committee member shakes hands with every other committee member, how many handshakes take place?
  Can a 10-sided convex polygon be used to solve this problem?
- **b** If all 273 delegates shake hands with all other delegates, how many handshakes take place now?
- **c** If the organising committee lines up on stage to face the delegates in the audience, in how many different orders can they line up?



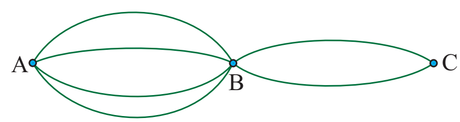
The **Opening Problem** is an example of a **counting** problem.

The following exercises will help us to solve counting problems without having to list and count the possibilities one by one. To do this we will examine:

- the product principle
- counting permutations
- counting combinations.

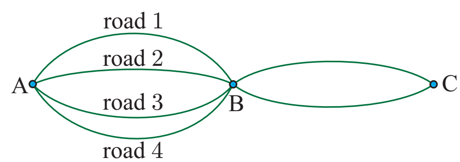
A THE PRODUCT PRINCIPLE

Suppose there are three towns A, B, and C. Four different roads could be taken from A to B, and two different roads from B to C.



How many different pathways are there from A to C going through B?

If we take road 1, there are two alternative roads to complete our trip.



Similarly, if we take road 2, there are two alternative roads to complete our trip.

The same is true for roads 3 and 4.

So, there are $2+2+2+2=4\times 2$ different pathways from A to C going through B.

Notice that the 4 corresponds to the number of roads from A to B and the 2 corresponds to the number of roads from B to C.

THE PRODUCT PRINCIPLE

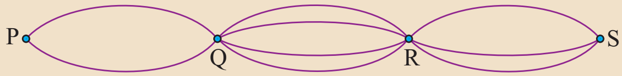
If there are m different ways of performing an operation, and for each of these there are n different ways of performing a second **independent** operation, then there are mn different ways of performing the two operations in succession.

The product principle can be extended to three or more successive independent operations.

Example 1

Self Tutor

P, Q, R, and S represent where Pauline, Quentin, Reiko, and Sam live. There are two different paths from P to Q, four different paths from Q to R, and 3 different paths from R to S.

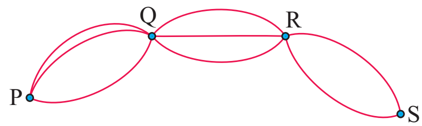


How many different pathways could Pauline take to visit Sam if she stops to see Quentin and then Reiko on the way?

The total number of different pathways $= 2 \times 4 \times 3 = 24$ {product principle}

EXERCISE 10A

1 The illustration shows the different map routes for a bus service which goes from P to S through both Q and R. How many different routes are possible?

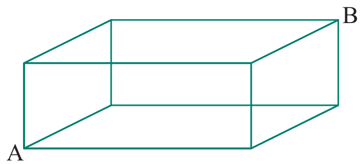


2 

In how many ways can the vertices of a rectangle be labelled with the letters A, B, C, and D:

a in clockwise alphabetical order
b in alphabetical order
c in random order?

3 The wire frame shown forms the outline of a box. An ant crawls along the wire from A to B. How many different paths of shortest length lead from A to B?



4 A table tennis competition has 7 teams. In how many different ways can the top two positions be filled in order of premiership points obtained?

5 A football competition is organised between 8 teams. In how many ways can the top 4 places be filled in order of premiership points obtained?

6 How many 3-digit numbers can be formed using the digits 2, 3, 4, 5, and 6:

a as often as desired
b at most once each?

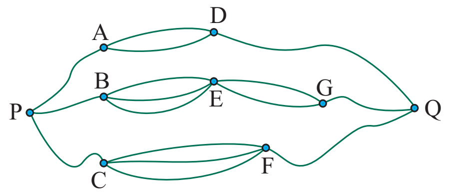
7 How many different alpha-numeric plates for motor car registration can be made if the first 3 places are English alphabet letters and the remaining places are 3 digits from 0 to 9?

8 In how many ways can:

a 2 postcards be mailed into 2 mail boxes
b 2 postcards be mailed into 3 mail boxes
c 4 postcards be mailed into 3 mail boxes?

B COUNTING PATHS

Consider the road system illustrated which shows the roads from P to Q.



From A to Q there are 2 paths.
From B to Q there are $3 \times 2 = 6$ paths.
From C to Q there are 3 paths.

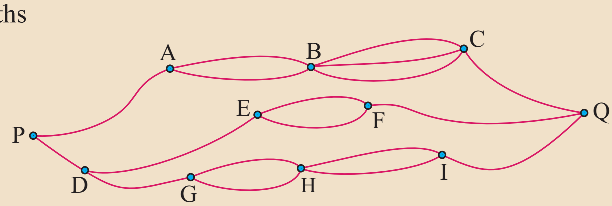
$\therefore$ from P to Q there are $2 + 6 + 3 = 11$ paths.

Notice that:
- When going from B to G, we go from B to E **and** then from E to G. We **multiply** the possibilities.
- When going from P to Q, we must first go from P to A **or** P to B **or** P to C. We **add** the possibilities from each of these first steps.

The word **and** suggests *multiplying* the possibilities.
The word **or** suggests *adding* the possibilities.

Example 2 — Self Tutor

How many different paths lead from P to Q?



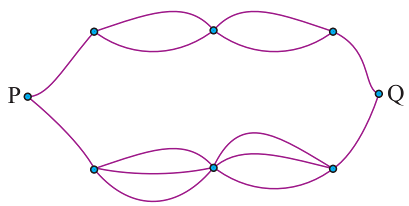
	From	P to A to B to C to Q	there are $2 \times 3 = 6$ paths
or	from	P to D to E to F to Q	there are 2 paths
or	from	P to D to G to H to I to Q	there are $2 \times 2 = 4$ paths.

In total there are $6 + 2 + 4 = 12$ different paths.

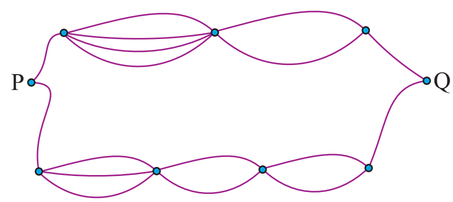
EXERCISE 10B

1 How many different paths lead from P to Q?

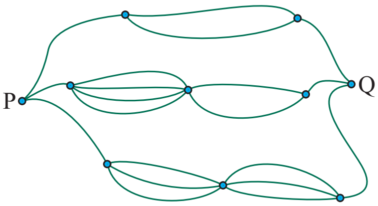
a



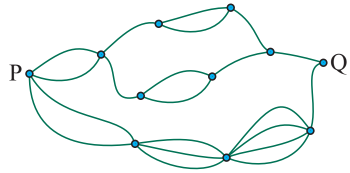
b



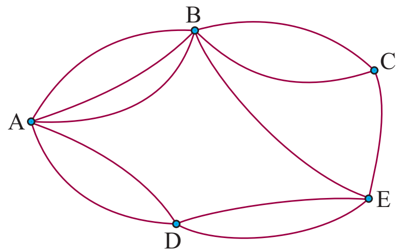
c



d



2 Katie is going on a long journey to visit her family. She lives in city A and is travelling to city E. Unfortunately there are no direct trains. However, she has the choice of several trains which stop in different cities along the way. These are illustrated in the diagram.
How many different train journeys does Katie have to choose from?



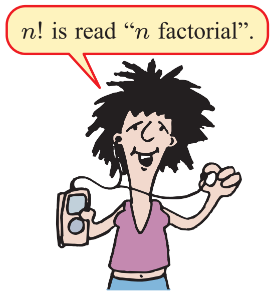
C FACTORIAL NOTATION

In problems involving counting, products of consecutive positive integers such as $8 \times 7 \times 6$ and $6 \times 5 \times 4 \times 3 \times 2 \times 1$ are common.

For convenience, we introduce **factorial numbers** to represent the products of consecutive positive integers.

For $n \geqslant 1$, $n!$ is the product of the first n positive integers.

$$\boldsymbol{n! = n(n-1)(n-2)(n-3)\ldots. \times 3 \times 2 \times 1}$$



For example, the product $6 \times 5 \times 4 \times 3 \times 2 \times 1$ can be written as $6!$.

Notice that $8 \times 7 \times 6$ can be written using factorial numbers only as

$$8 \times 7 \times 6 = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{8!}{5!}$$

An alternative **recursive definition** of factorial numbers is $\boldsymbol{n! = n \times (n-1)!}$ for $n \geqslant 1$

which can be extended to $n! = n(n-1)(n-2)!$ and so on.

Using the factorial rule with $n = 1$, we have $1! = 1 \times 0!$

Therefore, for completeness we define $\boldsymbol{0! = 1}$

Example 3 Self Tutor

Simplify: **a** $4!$ **b** $\frac{5!}{3!}$ **c** $\frac{7!}{4! \times 3!}$

a $4! = 4 \times 3 \times 2 \times 1 = 24$

b $\frac{5!}{3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 5 \times 4 = 20$

c $\frac{7!}{4! \times 3!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1} = 35$

If your problem involves factorials of large numbers then it is important to cancel as many factors as possible before using a calculator to evaluate the rest.

For example, if you have $\frac{300!}{297!}$ in your problem, you will find you cannot calculate 300! on your calculator. However, we can see that

$$\frac{300!}{297!} = \frac{300 \times 299 \times 298 \times 297!}{297!} = 300 \times 299 \times 298$$
$$= 26\,730\,600.$$

EXERCISE 10C.1

1 Find $n!$ for $n = 0, 1, 2, 3, ..., 10$.

2 Simplify without using a calculator:

a $\frac{6!}{5!}$ **b** $\frac{6!}{4!}$ **c** $\frac{6!}{7!}$ **d** $\frac{4!}{6!}$ **e** $\frac{100!}{99!}$ **f** $\frac{7!}{5! \times 2!}$

3 Simplify:

a $\frac{n!}{(n-1)!}$ **b** $\frac{(n+2)!}{n!}$ **c** $\frac{(n+1)!}{(n-1)!}$

Example 4 — Self Tutor

Express in factorial form:

a $10 \times 9 \times 8 \times 7$ **b** $\frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1}$

a $10 \times 9 \times 8 \times 7 = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{10!}{6!}$

b $\frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{10!}{4! \times 6!}$

4 Express in factorial form:

a $7 \times 6 \times 5$ **b** 10×9 **c** $11 \times 10 \times 9 \times 8 \times 7$

d $\frac{13 \times 12 \times 11}{3 \times 2 \times 1}$ **e** $\frac{1}{6 \times 5 \times 4}$ **f** $\frac{4 \times 3 \times 2 \times 1}{20 \times 19 \times 18 \times 17}$

Example 5 — Self Tutor

Write as a product by factorising:

a $8! + 6!$ **b** $10! - 9! + 8!$

a
$$\begin{aligned} & 8! + 6! \\ &= 8 \times 7 \times 6! + 6! \\ &= 6!(8 \times 7 + 1) \\ &= 6! \times 57 \end{aligned}$$

b
$$\begin{aligned} & 10! - 9! + 8! \\ &= 10 \times 9 \times 8! - 9 \times 8! + 8! \\ &= 8!(90 - 9 + 1) \\ &= 8! \times 82 \end{aligned}$$

5 Write as a product by factorising:

a $5! + 4!$ **b** $11! - 10!$ **c** $7! + 9!$ **d** $12! - 10!$

e $9! + 8! + 7!$ **f** $7! - 6! + 8!$ **g** $12! - 2 \times 11!$ **h** $3 \times 9! + 5 \times 8!$

Example 6 — Self Tutor

Simplify $\dfrac{7! - 6!}{6}$ by factorising.

$$\frac{7! - 6!}{6}$$
$$= \frac{7 \times 6! - 6!}{6}$$
$$= \frac{6!\cancel{(7-1)}^{1}}{\cancel{6}_{1}}$$
$$= 6!$$

6 Simplify by factorising:

a $\dfrac{12! - 11!}{11}$ **b** $\dfrac{10! + 9!}{11}$ **c** $\dfrac{10! - 8!}{89}$ **d** $\dfrac{10! - 9!}{9!}$

e $\dfrac{6! + 5! - 4!}{4!}$ **f** $\dfrac{n! + (n-1)!}{(n-1)!}$ **g** $\dfrac{n! - (n-1)!}{n-1}$ **h** $\dfrac{(n+2)! + (n+1)!}{n+3}$

THE BINOMIAL COEFFICIENT

The **binomial coefficient** is defined by

$$\binom{n}{r} = \underbrace{\frac{n(n-1)(n-2)\ldots.(n-r+2)(n-r+1)}{r(r-1)(r-2)\ldots.2 \times 1}}_{\text{factor form}} = \underbrace{\frac{n!}{r!(n-r)!}}_{\text{factorial form}}$$

The binomial coefficient is sometimes written ${}^nC_r$ or C_r^n.

Example 7 — Self Tutor

Use the formula $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ to evaluate: **a** $\binom{5}{2}$ **b** $\binom{11}{7}$

a $\binom{5}{2} = \dfrac{5!}{2!(5-2)!}$

$= \dfrac{5!}{2! \times 3!}$

$= \dfrac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1}$

$= 10$

b $\binom{11}{7} = \dfrac{11!}{7!(11-7)!}$

$= \dfrac{11!}{7! \times 4!}$

$= \dfrac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1}$

$= \dfrac{7920}{24}$

$= 330$

EXERCISE 10C.2

1 Use the formula $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ to evaluate:

a $\binom{3}{1}$ **b** $\binom{4}{2}$ **c** $\binom{7}{3}$ **d** $\binom{10}{4}$

Check your answers using technology.

2 **a** Use the formula $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ to evaluate:

i $\binom{8}{2}$ **ii** $\binom{8}{6}$

b Show that $\binom{n}{r} = \binom{n}{n-r}$ for all $n \in \mathbb{Z}^+$, $r = 0, 1, 2, ..., n$.

3 Find k if $\binom{9}{k} = 4\binom{7}{k-1}$.

D PERMUTATIONS

A **permutation** of a group of symbols is *any arrangement* of those symbols in a definite *order*.

For example, BAC is a permutation on the symbols A, B, and C in which all three of them are used. We say the symbols are "taken 3 at a time".

The set of all the different permutations on the symbols A, B, and C taken 3 at a time, is {ABC, ACB, BAC, BCA, CAB, CBA}.

Example 8 Self Tutor

List the set of all permutations on the symbols P, Q, and R taken:

a 1 at a time **b** 2 at a time **c** 3 at a time.

a {P, Q, R}

b {PQ, QP, RP, PR, QR, RQ}

c {PQR, PRQ, QPR, QRP, RPQ, RQP}

Example 9 Self Tutor

List all permutations on the symbols W, X, Y, and Z taken 4 at a time.

WXYZ	WXZY	WYXZ	WYZX	WZXY	WZYX
XWYZ	XWZY	XYWZ	XYZW	XZYW	XZWY
YWXZ	YWZX	YXWZ	YXZW	YZWX	YZXW
ZWXY	ZWYX	ZXWY	ZXYW	ZYWX	ZYXW

There are 24 of them.

For large numbers of symbols, listing the complete set of permutations is absurd. However, we can still count them by considering the number of options we have for filling each position.

Suppose we want to find the number of different permutations on the symbols A, B, C, D, E, F, and G, taken 3 at a time.

There are 3 positions to fill:

1st	2nd	3rd

In the 1st position, any of the 7 symbols could be used, so we have 7 options.

7		
1st	2nd	3rd

This leaves any of 6 symbols to go in the 2nd position, and this leaves any of 5 symbols to go in the 3rd position.

7	6	5
1st	2nd	3rd

$$\begin{aligned}\text{So, the total number of permutations} &= 7 \times 6 \times 5 \quad \{\text{product principle}\}\\ &= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}\\ &= \frac{7!}{4!} \quad \text{or} \quad \frac{7!}{(7-3)!}\end{aligned}$$

The number of **permutations** on n distinct symbols taken r at a time is:

$$\underbrace{\boldsymbol{n \times (n-1) \times (n-2) \times \ldots\ldots \times (n-r+1)}}_{r \text{ of these}} = \boldsymbol{\frac{n!}{(n-r)!}}$$

If we are finding permutations on the complete set of n symbols, as in **Example 9**, then $r = n$, and the number of permutations is $n!$.

Example 10 — Self Tutor

A chess association runs a tournament with 16 teams. In how many different ways could the top 5 positions be filled on the competition ladder?

Any of the 16 teams could fill the 'top' position.
Any of the remaining 15 teams could fill the 2nd position.
Any of the remaining 14 teams could fill the 3rd position.
⋮
Any of the remaining 12 teams could fill the 5th position.

16	15	14	13	12
1st	2nd	3rd	4th	5th

$$\begin{aligned}\text{The total number of permutations} &= 16 \times 15 \times 14 \times 13 \times 12\\ &= \frac{16!}{11!}\\ &= 524\,160\end{aligned}$$

So the top 5 positions could be filled in 524 160 ways.

Example 11

Self Tutor

The alphabet blocks A, B, C, D, and E are placed in a row in front of you.

a How many different permutations could you have?

b How many permutations end in C?

c How many permutations have the form $\boxed{\ldots}\boxed{A}\boxed{\ldots}\boxed{B}\boxed{\ldots}$?

d How many begin and end with a vowel (A or E)?

a There are 5 letters taken 5 at a time.

$\therefore$ the total number of permutations $= 5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$.

b $\underbrace{\boxed{4}\boxed{3}\boxed{2}\boxed{1}}_{\text{any others here}}\boxed{1}$

C here

C must be in the last position. The other 4 letters could go into the remaining 4 places in 4! ways.

$\therefore$ the number of permutations $= 1 \times 4! = 24$.

c $\boxed{3}\boxed{1}\boxed{2}\boxed{1}\boxed{1}$

A B

A goes into 1 place. B goes into 1 place. The remaining 3 letters go into the remaining 3 places in 3! ways.

$\therefore$ the number of permutations $= 1 \times 1 \times 3! = 6$.

d $\boxed{2}\boxed{3}\boxed{2}\boxed{1}\boxed{1}$

A or E remainder of A or E

A or E could go into the 1st position, so there are two options. The other one must go into the last position.

The remaining 3 letters could go into the 3 remaining places in 3! ways.

$\therefore$ the number of permutations $= 2 \times 1 \times 3! = 12$.

EXERCISE 10D

1 List the set of all permutations on the symbols W, X, Y, and Z taken:

a 1 at a time **b** two at a time **c** three at a time.

2 List the set of all permutations on the symbols A, B, C, D, and E taken:

a 2 at a time **b** 3 at a time.

3 In how many ways can:

a 5 different books be arranged on a shelf

b 3 different paintings be chosen from a collection of 8, and hung in a row

c a signal consisting of 4 coloured flags in a row be made if there are 10 different flags to choose from?

4 A captain and vice-captain are to be selected from a team of 11 cricketers. In how many ways can this be done?

5 Suppose you have 4 different coloured flags. How many different signals could you make using:

a 2 flags in a row **b** 3 flags in a row **c** 2 or 3 flags in a row?

6 Nine boxes are each labelled with a different whole number from 1 to 9. Five people are allowed to take one box each. In how many different ways can this be done if:

a there are no restrictions

b the first three people decide that they will take even numbered boxes?

7 **a** How many different permutations on the letters A, B, C, D, E, and F are there if each letter can be used once only?

b How many of these permutations:

i end in ED **ii** begin with F and end with A

iii begin and end with a vowel (A or E)?

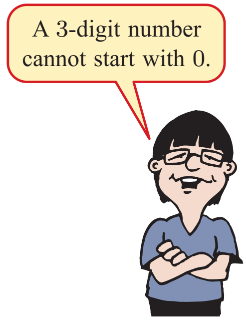
8 How many 3-digit numbers can be constructed from the digits 1, 2, 3, 4, 5, 6, and 7 if each digit may be used:

a as often as desired **b** only once **c** once only and the number must be odd?

9 3-digit numbers are constructed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 using each digit at most once. How many such numbers:

a can be constructed **b** end in 5

c end in 0 **d** are divisible by 5?



10 Arrangements containing 5 different letters from the word TRIANGLE are to be made. How many possible arrangements are there if:

a there are no restrictions

b the arrangement must start with R and end with A or E

c the arrangement must include the letter G?

Example 12 — Self Tutor

There are 6 different books arranged in a row on a shelf. In how many ways can two of the books, A and B, be together?

Method 1: We could have any of the following locations for A and B

$$\left.\begin{array}{cccccc} \text{A} & \text{B} & \times & \times & \times & \times \\ \text{B} & \text{A} & \times & \times & \times & \times \\ \times & \text{A} & \text{B} & \times & \times & \times \\ \times & \text{B} & \text{A} & \times & \times & \times \\ \times & \times & \text{A} & \text{B} & \times & \times \\ \times & \times & \text{B} & \text{A} & \times & \times \\ \times & \times & \times & \text{A} & \text{B} & \times \\ \times & \times & \times & \text{B} & \text{A} & \times \\ \times & \times & \times & \times & \text{A} & \text{B} \\ \times & \times & \times & \times & \text{B} & \text{A} \end{array}\right\} \text{10 of these}$$

If we consider any one of these, the remaining 4 books could be placed in 4! different orderings.

$\therefore$ total number of ways
$= 10 \times 4! = 240.$

Method 2: A and B can be put together in 2! ways (AB or BA).

Now consider this pairing as one book (effectively tying a string around them) which together with the other 4 books can be ordered in 5! different ways.

$\therefore$ the total number of ways $= 2! \times 5! = 240.$

11 In how many ways can 5 different books be arranged on a shelf if:

a there are no restrictions **b** books X and Y must be together

c books X and Y must not be together?

12 10 students sit in a row of 10 chairs. In how many ways can this be done if:

a there are no restrictions **b** students A, B, and C insist on sitting together?

13 3 boys and 3 girls are to sit in a row. How many ways can this be done if:

a there are no restrictions
b there is a girl at each end
c boys and girls must alternate
d all the boys sit together?

14 How many three-digit numbers can be made using the digits 0, 1, 3, 5, and 8 at most once each, if:

a there are no restrictions
b the numbers must be less than 500
c the numbers must be even and greater than 300?

15 Consider the letters of the word MONDAY. How many permutations of four different letters can be chosen if:

a there are no restrictions
b at least one vowel (A or O) must be used
c the two vowels are not together?

16 Alice has booked ten adjacent front-row seats for a basketball game for herself and nine friends.

a How many different arrangements are possible if there are no restrictions?

b Due to a severe snowstorm, only five of Alice's friends are able to join her for the game. In how many different ways can they be seated in the 10 seats if:

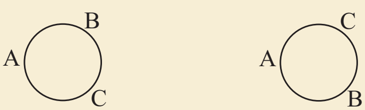
i there are no restrictions
ii any two of Alice's friends are to sit next to her?

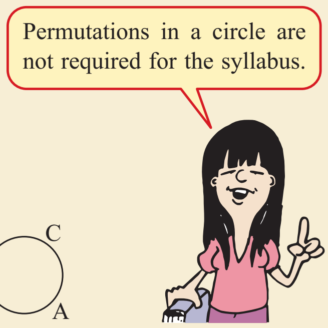
Discovery 1 — *Permutations in a circle*

There are 6 permutations on the symbols A, B, and C **in a line**. These are:

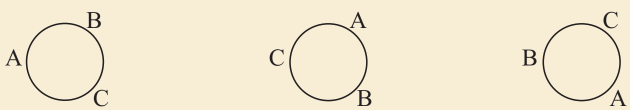
ABC ACB BAC BCA CAB CBA.

However **in a circle** there are only 2 different permutations on these 3 symbols. They are the only possibilities with different right-hand and left-hand neighbours.





In contrast, these three diagrams show the same cyclic permutation:



What to do:

1 Draw diagrams showing different cyclic permutations for:

a one symbol: A
b two symbols: A and B
c three symbols: A, B, and C
d four symbols: A, B, C, and D

2 Copy and complete:

Number of symbols	*Permutations in a line*	*Permutations in a circle*
1		
2		
3	$6 = 3!$	$2 = 2!$
4		

3 If there are n symbols to be arranged around a circle, how many different cyclic permutations are possible?

E COMBINATIONS

A **combination** is a selection of objects *without* regard to order.

For example, the possible teams of 3 people that can be selected from A, B, C, D, and E are:

ABC ABD ABE ACD ACE ADE
BCD BCE BDE
CDE

There are 10 combinations in total.

Now given the five people A, B, C, D, and E, we know that there are $5 \times 4 \times 3 = 60$ permutations for taking three of them at a time. So why is this 6 times larger than the number of combinations?

The answer is that for the combinations, order is not important. Selecting A, B, and C for the team is the same as selecting B, C, and A. For each of the 10 possible combinations, there are $3! = 6$ ways of ordering the members of the team.

In general, when choosing r objects from n objects,

$$\begin{aligned}\text{number of combinations} &= \text{number of permutations} \div r! \\ &= \frac{n!}{(n-r)!} \div r! \\ &= \frac{n!}{r!(n-r)!}\end{aligned}$$

This is the binomial coefficient we encountered in **Section C**.

The number of **combinations** on n distinct symbols taken r at a time is $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$.

Example 13 — Self Tutor

How many different teams of 4 can be selected from a squad of 7 if:

a there are no restrictions **b** the teams must include the captain?

a There are 7 players up for selection and we want any 4 of them.
There are $\binom{7}{4} = 35$ possible combinations.

b The captain must be included *and* we need any 3 of the other 6.
There are $\binom{1}{1} \times \binom{6}{3} = 20$ possible combinations.

Example 14

Self Tutor

A committee of 4 is chosen from 7 men and 6 women. How many different committees can be chosen if:

a there are no restrictions

b there must be 2 of each sex

c there must be at least one of each sex?

a There are $7 + 6 = 13$ people up for selection and we want any 4 of them.
There are $\binom{13}{4} = 715$ possible committees.

b The 2 men can be chosen out of 7 in $\binom{7}{2}$ ways.
The 2 women can be chosen out of 6 in $\binom{6}{2}$ ways.
$\therefore$ there are $\binom{7}{2} \times \binom{6}{2} = 315$ possible committees.

c The total number of committees
= the number with 3 men and 1 woman + the number with 2 men and 2 women + the number with 1 man and 3 women
$= \binom{7}{3} \times \binom{6}{1} + \binom{7}{2} \times \binom{6}{2} + \binom{7}{1} \times \binom{6}{3}$
$= 665$

or The total number of committees
$= \binom{13}{4}$ − the number with all men − the number with all women
$= \binom{13}{4} - \binom{7}{4} \times \binom{6}{0} - \binom{7}{0} \times \binom{6}{4}$
$= 665$

EXERCISE 10E

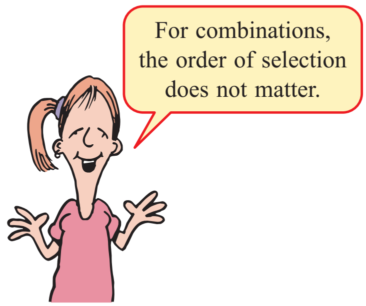
1 Determine whether the following are examples of combinations or permutations:

a making a 3-digit number using the digits 1, 2, 3, 4, and 5 at most once each

b selecting a committee of 3 people from a list of 5

c selecting the chairperson and treasurer from a committee of 8 people

d selecting 2 pieces of fruit to take to school from a bowl of 10 pieces.



2 List the different teams of 4 that can be chosen from a squad of 6 (named A, B, C, D, E, and F).
Check that the formula $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ gives the total number of teams.

3 How many different teams of 11 can be chosen from a squad of 17?

4 Candidates for an examination are required to answer 5 questions out of 9.

a In how many ways can the questions be chosen if there are no restrictions?

b If question 1 was made compulsory, how many selections would be possible?

5 **a** How many different committees of 3 can be selected from 13 candidates?

b How many of these committees consist of the president and 2 others?

6 **a** How many different teams of 5 can be selected from a squad of 12?

b How many of these teams contain:

i the captain and vice-captain **ii** exactly one of the captain or the vice-captain?

7 A team of 9 is selected from a squad of 15. 3 particular players *must* be included, and another must be excluded because of injury. In how many ways can the team be chosen?

8 In how many ways can 4 people be selected from 10 if:

a one particular person *must* be selected

b two particular people are excluded from every selection

c one particular person is always included and two particular people are always excluded?

9 A committee of 5 is chosen from 10 men and 6 women. Determine the number of ways of selecting the committee if:

a there are no restrictions **b** it must contain 3 men and 2 women

c it must contain all men **d** it must contain at least 3 men

e it must contain at least one of each sex.

10 A committee of 8 is chosen from 9 boys and 6 girls. In how many ways can this be done if:

a there are no restrictions **b** there must be 5 boys and 3 girls

c all the girls are selected **d** there are more boys than girls?

11 A music class consists of 5 piano players, 7 guitarists, and 4 violinists. A band of 1 piano player, 3 guitarists, and 2 violinists must be chosen to play at a school concert. In how many different ways can the band be chosen?

12 A committee of 5 is chosen from 6 doctors, 3 dentists, and 7 others.
Determine the number of ways of selecting the committee if it is to contain:

a exactly 2 doctors and 1 dentist **b** exactly 2 doctors

c at least one person from either of the two given professions.

13 How many diagonals does a 20-sided convex polygon have?

14 There are 12 distinct points A, B, C, D, ..., L on a circle. Lines are drawn between each pair of points.

a How many lines: **i** are there in total **ii** pass through B?

b How many triangles: **i** are determined by the lines **ii** have one vertex B?

15 How many 4-digit numbers can be constructed for which the digits are in ascending order from left to right? You cannot start a number with 0.

16 **a** Give an example which demonstrates that:

$$\binom{5}{0}\times\binom{6}{4}+\binom{5}{1}\times\binom{6}{3}+\binom{5}{2}\times\binom{6}{2}+\binom{5}{3}\times\binom{6}{1}+\binom{5}{4}\times\binom{6}{0}=\binom{11}{4}.$$

b Copy and complete:

$$\binom{m}{0}\times\binom{n}{r}+\binom{m}{1}\times\binom{n}{r-1}+\binom{m}{2}\times\binom{n}{r-2}+\ldots.+\binom{m}{r-1}\times\binom{n}{1}+\binom{m}{r}\times\binom{n}{0}=\ldots.$$

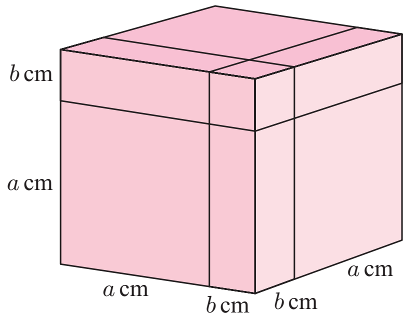
17 In how many ways can 12 people be divided into:

a two equal groups **b** three equal groups?

18 Answer the **Opening Problem** on page **256**.

F BINOMIAL EXPANSIONS

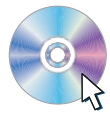
Consider the cube alongside, which has sides of length $(a+b)$ cm.



The cube has been subdivided into 8 blocks by making 3 cuts parallel to the cube's surfaces as shown.

We know that the total volume of the cube is $(a+b)^3$ cm^3. However, we can also find an expression for the cube's volume by adding the volumes of the 8 individual blocks.

We have:
1 block $a \times a \times a$
3 blocks $a \times a \times b$
3 blocks $a \times b \times b$
1 block $b \times b \times b$



$\therefore$ the cube's volume $= a^3 + 3a^2b + 3ab^2 + b^3$

$\therefore \quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

> The sum $\boldsymbol{a+b}$ is called a **binomial** as it contains two terms.
>
> Any expression of the form $\boldsymbol{(a+b)^n}$ is called a **power of a binomial**.

All binomials raised to a power can be expanded using the same general principles. In this chapter, therefore, we consider the expansion of the general expression $(a+b)^n$ where $n \in \mathbb{N}$.

Consider the following algebraic expansions:

$(a+b)^1 = a+b$
$(a+b)^2 = a^2 + 2ab + b^2$

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 \\ &= (a+b)(a^2+2ab+b^2) \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

> The **binomial expansion** of $\boldsymbol{(a+b)^2}$ is $\boldsymbol{a^2+2ab+b^2}$.
> The **binomial expansion** of $\boldsymbol{(a+b)^3}$ is $\boldsymbol{a^3+3a^2b+3ab^2+b^3}$.

Discovery 2 — The binomial expansion

What to do:

1 Expand $(a+b)^4$ in the same way as for $(a+b)^3$ above.
Hence expand $(a+b)^5$ and $(a+b)^6$.

2 The cubic expansion $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ contains 4 terms. Observe that their coefficients are: 1 3 3 1

- **a** What happens to the powers of a and b in each term of the expansion of $(a+b)^3$?
- **b** Does the pattern in **a** continue for the expansions of $(a+b)^4$, $(a+b)^5$, and $(a+b)^6$?

c Write down the triangle of coefficients to row 6:

$$\begin{array}{lccccccc} n=1 & & & 1 & & 1 & & \\ n=2 & & 1 & & 2 & & 1 & \\ n=3 & 1 & & 3 & & 3 & & 1 \longleftarrow \text{row } 3 \\ & & & & \vdots & & & \end{array}$$

3 The triangle of coefficients in **c** above is called **Pascal's triangle**. Investigate:

a the predictability of each row from the previous one

b a formula for finding the sum of the numbers in the nth row of Pascal's triangle.

4 **a** Use your results from **3** to predict the elements of the 7th row of Pascal's triangle.

b Hence write down the binomial expansion of $(a+b)^7$.

c Check your result algebraically by using $(a+b)^7 = (a+b)(a+b)^6$ and your results from **1**.

From the **Discovery** we obtained
$$\begin{aligned}(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ &= a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + b^4\end{aligned}$$

Notice that:

- As we look from left to right across the expansion, the powers of a decrease by 1, while the powers of b increase by 1.
- The sum of the powers of a and b in each term of the expansion is 4.
- The number of terms in the expansion is $4+1=5$.
- The coefficients of the terms are row 4 of Pascal's triangle.

For the expansion of $(a+b)^n$ where $n \in \mathbb{N}$:

- As we look from left to right across the expansion, the powers of a *decrease* by 1, while the powers of b *increase* by 1.
- The sum of the powers of a and b in each term of the expansion is n.
- The number of terms in the expansion is $n+1$.
- The coefficients of the terms are row n of Pascal's triangle.

In the following examples we see how the general binomial expansion $(a+b)^n$ may be put to use.

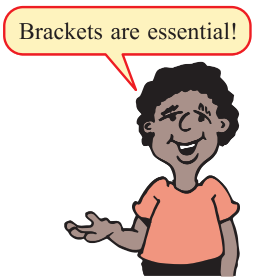
Example 15 — Self Tutor

Using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, find the binomial expansion of:

a $(2x+3)^3$ **b** $(x-5)^3$

a In the expansion of $(a+b)^3$ we substitute $a=(2x)$ and $b=(3)$.

$$\begin{aligned}\therefore\ (2x+3)^3 &= (2x)^3 + 3(2x)^2(3) + 3(2x)^1(3)^2 + (3)^3 \\ &= 8x^3 + 36x^2 + 54x + 27\end{aligned}$$



b We substitute $a=(x)$ and $b=(-5)$

$$\begin{aligned}\therefore\ (x-5)^3 &= (x)^3 + 3(x)^2(-5) + 3(x)(-5)^2 + (-5)^3 \\ &= x^3 - 15x^2 + 75x - 125\end{aligned}$$

Example 16 Self Tutor

Find the:

a 5th row of Pascal's triangle **b** binomial expansion of $\left(x - \frac{2}{x}\right)^5$.

a

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \leftarrow \text{the 0th row, for } (a+b)^0 \\
 & & & & 1 & & 1 & & & & \leftarrow \text{the 1st row, for } (a+b)^1 \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \leftarrow \text{the 5th row, for } (a+b)^5
\end{array}
$$

b Using the coefficients obtained in **a**, $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Letting $a = (x)$ and $b = \left(\frac{-2}{x}\right)$, we find

$$\left(x - \frac{2}{x}\right)^5 = (x)^5 + 5(x)^4\left(\frac{-2}{x}\right) + 10(x)^3\left(\frac{-2}{x}\right)^2 + 10(x)^2\left(\frac{-2}{x}\right)^3 + 5(x)\left(\frac{-2}{x}\right)^4 + \left(\frac{-2}{x}\right)^5$$

$$= x^5 - 10x^3 + 40x - \frac{80}{x} + \frac{80}{x^3} - \frac{32}{x^5}$$

EXERCISE 10F

1 Use the binomial expansion of $(a+b)^3$ to expand and simplify:

a $(p+q)^3$ **b** $(x+1)^3$ **c** $(x-3)^3$

d $(2+x)^3$ **e** $(3x-1)^3$ **f** $(2x+5)^3$

g $(2a-b)^3$ **h** $\left(3x - \frac{1}{3}\right)^3$ **i** $\left(2x + \frac{1}{x}\right)^3$

2 Use $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand and simplify:

a $(1+x)^4$ **b** $(p-q)^4$ **c** $(x-2)^4$

d $(3-x)^4$ **e** $(1+2x)^4$ **f** $(2x-3)^4$

g $(2x+b)^4$ **h** $\left(x + \frac{1}{x}\right)^4$ **i** $\left(2x - \frac{1}{x}\right)^4$

3 Expand and simplify:

a $(x+2)^5$ **b** $(x-2y)^5$ **c** $(1+2x)^5$ **d** $\left(x - \frac{1}{x}\right)^5$

4 Expand and simplify $(2+x)^5 + (2-x)^5$.

5 **a** Write down the 6th row of Pascal's triangle.

b Find the binomial expansion of:

i $(x+2)^6$ **ii** $(2x-1)^6$ **iii** $\left(x + \frac{1}{x}\right)^6$

6 Expand and simplify:

a $(1+\sqrt{2})^3$ **b** $(\sqrt{5}+2)^4$ **c** $(2-\sqrt{2})^5$

7 Find $\dfrac{(2+\sqrt{3})^3}{4+\sqrt{3}}$, giving your answer in the form $\dfrac{a+b\sqrt{3}}{c}$, where $a, b, c \in \mathbb{Z}$.

8 **a** Expand $(2+x)^6$. **b** Hence find the value of $(2.01)^6$.

9 The first two terms in a binomial expansion are: $(a+b)^3 = 8 + 12e^x + ...$

a Find a and b. **b** Hence determine the remaining two terms of the expansion.

10 Expand and simplify $(2x+3)(x+1)^4$.

11 Find the coefficient of:

a a^3b^2 in the expansion of $(3a+b)^5$ **b** a^3b^3 in the expansion of $(2a+3b)^6$.

G THE BINOMIAL THEOREM

Historical note *Binomial Theorem*

The Binomial Theorem is one of the most important results in mathematics.

Multiplying out binomial terms is a basic process which dates back to the beginning of algebra. Mathematicians had noticed relationships between the coefficients for many centuries, and Pascal's triangle was certainly widely used long before Pascal.

Isaac Newton discovered the Binomial Theorem in 1665, but he did not publish his results until much later. Newton was the first person to give a formula for the binomial coefficients. He did this because he wanted to go further. Newton's ground-breaking result included a generalisation of the Binomial Theorem to the case of $(a+b)^n$ where n is a rational number, such as $\frac{1}{2}$. This results in a sum with an infinite number of terms, called an infinite series. In doing this, Newton was the first person to confidently use the exponential notation that we recognise today for both negative and fractional powers.

In the previous Section we saw how the coefficients of the binomial expansion $(a+b)^n$ can be found in the nth row of Pascal's triangle. These coefficients are in fact the **binomial coefficients** $\binom{n}{r}$ for $r = 0, 1, 2, ..., n$.

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array} \qquad \begin{array}{ccccccccc} & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \end{array}$$

The **Binomial Theorem** states that

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{r}a^{n-r}b^r + ... + b^n$$

where $\binom{n}{r}$ is the **binomial coefficient** of $a^{n-r}b^r$ and $r = 0, 1, 2, 3, ..., n$.

The **general term** or $(r+1)$th term in the binomial expansion is $T_{r+1} = \binom{n}{r}a^{n-r}b^r$.

Using sigma notation we write $(a+b)^n = \displaystyle\sum_{r=0}^{n} \binom{n}{r}a^{n-r}b^r$.

Example 17

Self Tutor

Write down the first three and last two terms of the expansion of $\left(2x+\frac{1}{x}\right)^{12}$.

Do not simplify your answer.

$$\left(2x+\frac{1}{x}\right)^{12} = (2x)^{12} + \binom{12}{1}(2x)^{11}\left(\frac{1}{x}\right)^{1} + \binom{12}{2}(2x)^{10}\left(\frac{1}{x}\right)^{2} + \dots.$$

$$\dots. + \binom{12}{11}(2x)^{1}\left(\frac{1}{x}\right)^{11} + \left(\frac{1}{x}\right)^{12}$$

Example 18

Self Tutor

Find the 7th term of $\left(3x-\frac{4}{x^2}\right)^{14}$. Do not simplify your answer.

$a = (3x)$, $b = \left(\frac{-4}{x^2}\right)$, and $n = 14$

Given the general term $T_{r+1} = \binom{n}{r} a^{n-r} b^r$, we let $r = 6$

$$\therefore \quad T_7 = \binom{14}{6}(3x)^8\left(\frac{-4}{x^2}\right)^6$$

Example 19

Self Tutor

In the expansion of $\left(x^2+\frac{4}{x}\right)^{12}$, find:

a the coefficient of x^6

b the constant term.

$a = (x^2)$, $b = \left(\frac{4}{x}\right)$, and $n = 12$

$$\begin{aligned}\therefore \text{ the general term } T_{r+1} &= \binom{12}{r}(x^2)^{12-r}\left(\frac{4}{x}\right)^r \\ &= \binom{12}{r} x^{24-2r} \times \frac{4^r}{x^r} \\ &= \binom{12}{r} 4^r x^{24-3r}\end{aligned}$$

a If $24 - 3r = 6$

then $3r = 18$

$\therefore \quad r = 6$

$\therefore \quad T_7 = \binom{12}{6} 4^6 x^6$

$\therefore$ the coefficient of x^6 is $\binom{12}{6} 4^6$ or $3\,784\,704$.

b If $24 - 3r = 0$

then $3r = 24$

$\therefore \quad r = 8$

$\therefore \quad T_9 = \binom{12}{8} 4^8 x^0$

$\therefore$ the constant term is $\binom{12}{8} 4^8$ or $32\,440\,320$.

EXERCISE 10G

1 Write down the first three and last two terms of the following binomial expansions. Do not simplify your answers.

a $(1+2x)^{11}$ **b** $\left(3x+\frac{2}{x}\right)^{15}$ **c** $\left(2x-\frac{3}{x}\right)^{20}$

2 Without simplifying, write down:

a the 6th term of $(2x+5)^{15}$ **b** the 4th term of $(x^2+y)^9$

c the 10th term of $\left(x-\frac{2}{x}\right)^{17}$ **d** the 9th term of $\left(2x^2-\frac{1}{x}\right)^{21}$.

3 In the expansion of $(2x+3)^{12}$, find:

a the coefficient of x^8 **b** the coefficient of x^5.

4 In the expansion of $(1-3x)^{10}$, find:

a the coefficient of x^3 **b** the coefficient of x^7.

5 In the expansion of $\left(x^2+\frac{2}{x}\right)^9$, find:

a the coefficient of x^{12} **b** the constant term **c** the coefficient of x^{-6}.

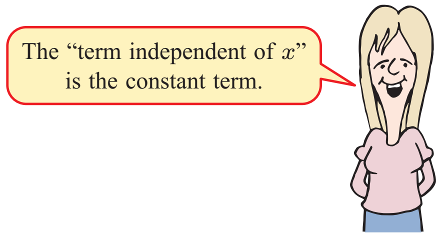
6 Consider the expansion of $(x+b)^7$.

a Write down the general term of the expansion.

b Find b given that the coefficient of x^4 is -280.

7 Find the term independent of x in the expansion of:

a $\left(x+\frac{2}{x^2}\right)^{15}$ **b** $\left(x-\frac{3}{x^2}\right)^9$.



8 Find the coefficient of:

a x^{10} in the expansion of $(3+2x^2)^{10}$ **b** x^3 in the expansion of $\left(2x^2-\frac{3}{x}\right)^6$

c x^6y^3 in the expansion of $(2x^2-3y)^6$ **d** x^{12} in the expansion of $\left(2x^2-\frac{1}{x}\right)^{12}$.

9 In the expansion of $(k+x)^8$, the coefficient of x^5 is 10 times the coefficient of x^6. Find the value of k.

10 The coefficient of x^5 in the expansion of $(ax-2)^7$ is twice the coefficient of x^5 in the expansion of $(a+x)^9$. Find the value of a.

11 In the expansion of $\left(ax+\frac{b}{x}\right)^6$, the constant term is 20 000, and the coefficient of x^4 is equal to the coefficient of x^2.

a Show that $ab=10$ and $b=\frac{2a}{5}$.

b Find a and b given that they are both positive.

Example 20 — Self Tutor

Find the coefficient of x^5 in the expansion of $(x+3)(2x-1)^6$.

$$\begin{aligned} &(x+3)(2x-1)^6 \\ &= (x+3)[(2x)^6 + \binom{6}{1}(2x)^5(-1) + \binom{6}{2}(2x)^4(-1)^2 + ...] \\ &= (x+3)(2^6x^6 - \binom{6}{1}2^5x^5 + \binom{6}{2}2^4x^4 - ....) \end{aligned}$$

So, the terms containing x^5 are $\binom{6}{2}2^4x^5$ from (1)

and $-3\binom{6}{1}2^5x^5$ from (2)

$\therefore$ the coefficient of x^5 is $\binom{6}{2}2^4 - 3\binom{6}{1}2^5 = -336$

12 Find the coefficient of x^5 in the expansion of $(x+2)(x^2+1)^8$.

13 Find the term containing x^6 in the expansion of $(2-x)(3x+1)^9$.

14 Find the coefficient of x^4 in the expansion of:

a $(3-2x)^7$ **b** $(1+3x)(3-2x)^7$

15 Find:

a the coefficient of x^7 in the expansion of $(x^2-3)(2x-5)^8$

b the term independent of x in the expansion of $(1-x^2)\left(x+\dfrac{2}{x}\right)^6$.

16 When the expansion of $(a+bx)(1-x)^6$ is written in ascending powers of x, the first three terms are $3-20x+cx^2$. Find the values of a, b, and c.

Example 21 — Self Tutor

Consider the expansion of $(1+3x)^n$, where $n \in \mathbb{Z}^+$.

If the coefficient of x^2 is 90, find the value of n.

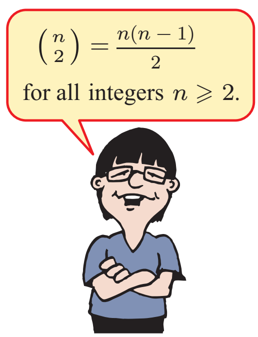
$(1+3x)^n$ has general term $T_{r+1} = \binom{n}{r}1^{n-r}(3x)^r$

$= \binom{n}{r}3^rx^r$

$\therefore$ $T_3 = \binom{n}{2}3^2x^2$ is the x^2 term.

Since the coefficient of x^2 is 90, $\binom{n}{2} \times 9 = 90$

$$\begin{aligned} \therefore \quad \frac{n(n-1)}{2} &= 10 \\ \therefore \quad n^2 - n &= 20 \\ \therefore \quad n^2 - n - 20 &= 0 \\ \therefore \quad (n-5)(n+4) &= 0 \\ \therefore \quad n &= 5 \qquad \{n > 0\} \end{aligned}$$



17 The coefficient of x^2 in the expansion of $(1+2x)^n$ is 112. Find n.

18 The coefficient of x^2 in the expansion of $\left(1-\frac{x}{3}\right)^n$ is $\frac{5}{3}$. Find n.

19 The third term of $(1+x)^n$ is $36x^2$. Find the fourth term.

20 Suppose $(1+kx)^n = 1 - 12x + 60x^2 - ....$. Find the values of k and n.

Review set 10A

1 Simplify: **a** $\dfrac{n!}{(n-2)!}$ **b** $\dfrac{n!+(n+1)!}{n!}$

2 Eight people enter a room and each person shakes hands with every other person. How many hand shakes are made?

3 The letters P, Q, R, S, and T are to be arranged in a row. How many of the possible arrangements:

a end with T **b** begin with P and end with T?

4 **a** How many three digit numbers can be formed using the digits 0 to 9?

b How many of these numbers are divisible by 5?

5 The first two terms in a binomial expansion are: $(a+b)^4 = e^{4x} - 4e^{2x} + ...$

a Find a and b. **b** Copy and complete the expansion.

6 Expand and simplify $(\sqrt{3}+2)^5$, giving your answer in the form $a+b\sqrt{3}$, $a, b \in \mathbb{Z}$.

7 Find the constant term in the expansion of $\left(3x^2+\frac{1}{x}\right)^8$.

8 Find c given that the expansion $(1+cx)(1+x)^4$ includes the term $22x^3$.

9 Steven and nine of his classmates are in a school committee. The committee must select a president, vice-president, and secretary. In how many ways can this be done if:

a there are no restrictions **b** Steven must be the president

c Steven cannot hold any of the key positions?

10 Find the coefficient of x^3 in the expansion of $(x+5)^6$.

11 A team of five is chosen from six men and four women.

a How many different teams are possible with no restrictions?

b How many different teams contain at least one person of each sex?

12 Find the coefficient of x^{-6} in the expansion of $\left(2x-\frac{3}{x^2}\right)^{12}$.

13 Find the coefficient of x^5 in the expansion of $(2x+3)(x-2)^6$.

14 Find the possible values of a if the coefficient of x^3 in $\left(2x+\frac{1}{ax^2}\right)^9$ is 288.

15 In the expansion of $(kx-1)^6$, the coefficient of x^4 is equal to four times the coefficient of x^2. Find the possible values of k.

Review set 10B

1 Alpha-numeric number plates have two letters followed by four digits. How many plates are possible if:

a there are no restrictions

b the first letter must be a vowel

c no letter or digit may be repeated?

2 **a** How many committees of five can be selected from eight men and seven women?

b How many of the committees contain two men and three women?

c How many of the committees contain at least one man?

3 Use the binomial expansion to find:

a $(x-2y)^3$

b $(3x+2)^4$

4 Find the coefficient of x^3 in the expansion of $(2x+5)^6$.

5 Find the term independent of x in the expansion of $\left(2x^2-\frac{1}{x}\right)^6$.

6 Find $\frac{(3-\sqrt{2})^3}{\sqrt{2}+1}$, giving your answer in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Z}$.

7 Kristen's school offers 6 Group A subjects, 8 Group B subjects, and 5 Group C subjects. Kristen must select 2 Group A, 3 Group B, and 1 Group C subject to study. In how many ways can she make her selection?

8 Find the coefficient of x^4 in the expansion of $(x-3)(2x+1)^5$.

9 How many arrangements containing 4 different letters from the word DRAGONFLY are possible if:

a there are no restrictions

b the letters G and Y must not be included

c the arrangement must start with R and end with N?

10 Find the possible values of q if the constant terms in the expansions of $\left(x^3+\frac{q}{x^3}\right)^8$ and $\left(x^3+\frac{q}{x^3}\right)^4$ are equal.

11 Eight people enter a room and sit in a row of eight chairs. In how many ways can the sisters Cathy, Robyn, and Jane sit together in the row?

12 Find k in the expansion $(m-2n)^{10} = m^{10} - 20m^9n + km^8n^2 - .... + 1024n^{10}$.

13 A team of eight is chosen from 11 men and 7 women. How many different teams are possible if there:

a are no restrictions

b must be four of each sex on the team

c must be at least two women on the team

d must be more women than men?

14 The coefficient of x^2 in the expansion of $\left(1+\frac{x}{2}\right)^n$ is $\frac{21}{4}$. Find n.

15 The first three terms in the expansion of $(1+kx)^n$ are $1-4x+\frac{15}{2}x^2$. Find k and n.

11

Vectors

Contents:

A Vectors and scalars
B The magnitude of a vector
C Operations with plane vectors
D The vector between two points
E Parallelism
F Problems involving vector operations
G Lines
H Constant velocity problems

Opening problem

An aeroplane in calm conditions is flying at 800 km h^{-1} due east. A cold wind suddenly blows from the south-west at 35 km h^{-1}, pushing the aeroplane slightly off course.

Things to think about:

a How can we use an array of numbers to represent the speed *and* direction of the plane?

b What operation do we need to perform to find the effect of the wind on the aeroplane?

c Can you determine the resulting speed and direction of the aeroplane?

A VECTORS AND SCALARS

In the **Opening Problem**, the effect of the wind on the aeroplane is determined by both its speed *and* its direction. The effect would be different if the wind was blowing against the aeroplane rather than from behind it.

> Quantities which have only magnitude are called **scalars**.
>
> Quantities which have both magnitude and direction are called **vectors**.

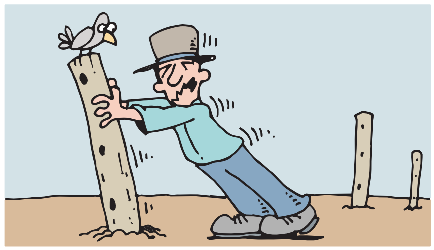
The *speed* of the plane is a scalar. It describes its size or strength.

The *velocity* of the plane is a vector. It includes both its speed and also its direction.

Other examples of vector quantities are:

- acceleration
- force
- displacement
- momentum

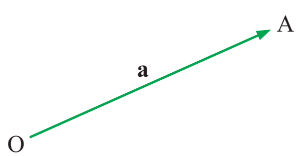
For example, farmer Giles needs to remove a fence post. He starts by pushing on the post sideways to loosen the ground. Giles has a choice of how hard to push the post, and in which direction. The force he applies is therefore a vector.



From previous courses, you should have seen how we can represent a vector quantity using a **directed line segment** or **arrow**. The **length of the arrow** represents the size or magnitude of the quantity, and the **arrowhead** shows its direction.

POSITION VECTORS

Consider the vector from the origin O to the point A. We call this the **position vector** of point A.



- This **position vector** could be represented by

  $\overrightarrow{OA}$ or $\mathbf{a}$ or $\vec{a}$.

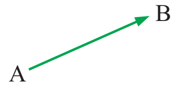
  $\mathbf{a}$: bold used in textbooks

  $\vec{a}$: used by students

- The **magnitude** or **length** could be represented by

  $|\overrightarrow{OA}|$ or OA or $|\mathbf{a}|$ or $|\vec{a}|$.

Now consider the vector from point A to point B. We say that:

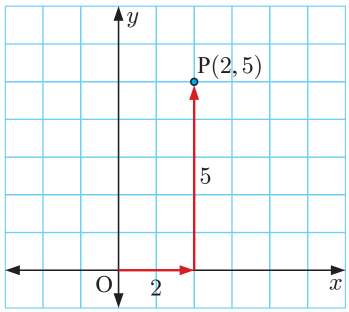


- $\overrightarrow{AB}$ is the vector which **originates** at A and **terminates** at B
- $\overrightarrow{AB}$ is the **position vector** of B relative to A.

When we plot points in the Cartesian plane, we move first in the x-direction and then in the y-direction.

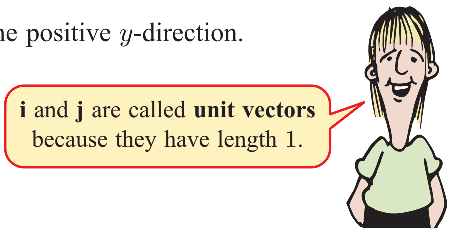
For example, to plot the point P(2, 5), we start at the origin, move 2 units in the x-direction, and then 5 units in the y-direction.

We therefore say that the vector from O to P is $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.



Suppose that $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a vector of length 1 unit in the positive x-direction

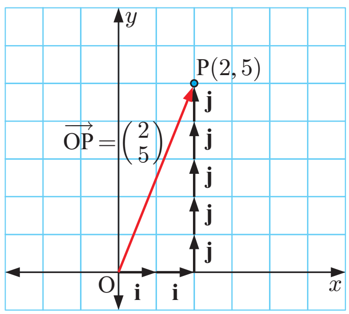
and that $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a vector of length 1 unit in the positive y-direction.



We can see that moving from O to P is equivalent to 2 lots of **i** plus 5 lots of **j**.

$$\overrightarrow{OP} = 2\mathbf{i} + 5\mathbf{j}$$

$$\therefore \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



The point P(x, y) has **position vector** $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = x\mathbf{i} + y\mathbf{j}$.

component form unit vector form

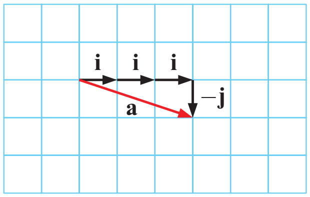
$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the **base unit vector** in the x-direction.

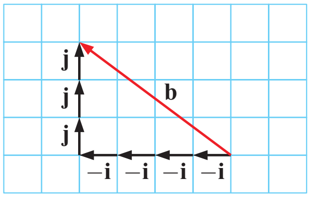
$\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the **base unit vector** in the y-direction.

The set of vectors $\{\mathbf{i}, \mathbf{j}\}$ is the **standard basis** for the 2-dimensional (x, y) coordinate system.

All vectors in the plane can be described in terms of the base unit vectors **i** and **j**.

For example: $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$
$\mathbf{b} = -4\mathbf{i} + 3\mathbf{j}$



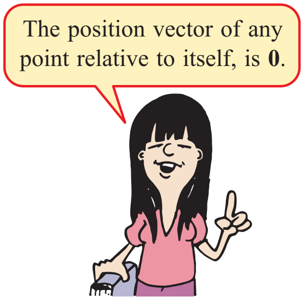


THE ZERO VECTOR

The **zero vector**, **0**, is a vector of length 0.

It is the only vector with no direction.

In component form, $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.



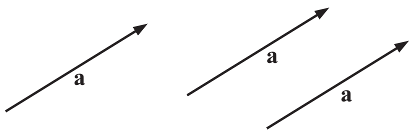
When we write the zero vector by hand, we usually write $\vec{0}$.

VECTOR EQUALITY

Two vectors are **equal** if they have the same magnitude and direction.

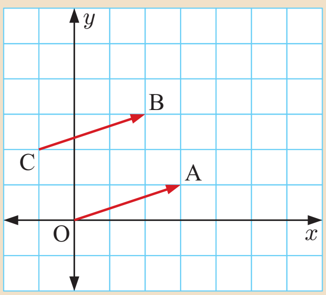
In component form, their x-components are equal *and* their y-components are equal.

Equal vectors are **parallel** and in the same direction, and are **equal in length**. The arrows that represent them are translations of one another.



Example 1

Self Tutor



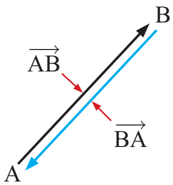
a Write $\overrightarrow{OA}$ and $\overrightarrow{CB}$ in component form and in unit vector form.

b Comment on your answers in **a**.

a $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3\mathbf{i} + \mathbf{j}$ $\quad$ $\overrightarrow{CB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3\mathbf{i} + \mathbf{j}$

b The vectors $\overrightarrow{OA}$ and $\overrightarrow{CB}$ are equal.

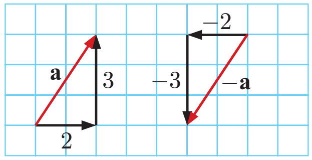
NEGATIVE VECTORS



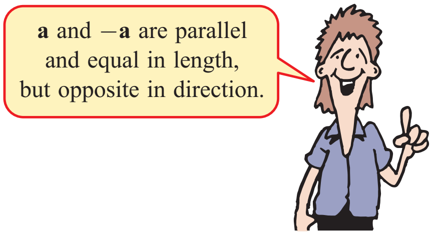
$\overrightarrow{AB}$ and $\overrightarrow{BA}$ have the same length, but they have opposite directions.

We say that $\overrightarrow{BA}$ is the **negative** of $\overrightarrow{AB}$, and write $\overrightarrow{BA} = -\overrightarrow{AB}$.

In the diagram we see the vector $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and its negative $-\mathbf{a} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

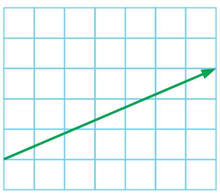


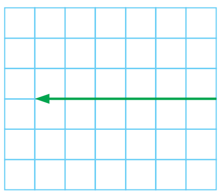
If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ then $-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$.

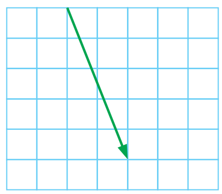


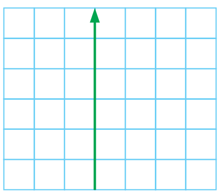
EXERCISE 11A

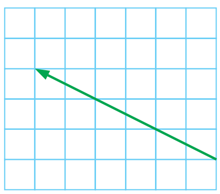
1 Write the illustrated vectors in component form and in unit vector form:

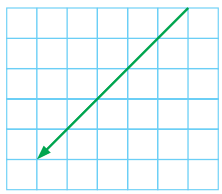
a 

b 

c 

d 

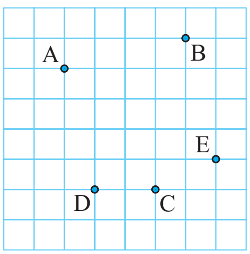
e 

f 

2 Write each vector in unit vector form, and illustrate it using an arrow diagram:

a $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ **d** $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$

3



a Find in component form and in unit vector form:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BA}$ **iii** $\overrightarrow{BC}$

iv $\overrightarrow{DC}$ **v** $\overrightarrow{AC}$ **vi** $\overrightarrow{DE}$

b Which two vectors in **a** are equal? Explain your answer.

c Which two vectors in **a** are negatives? Explain your answer.

4 Write in component form and illustrate using a directed line segment:

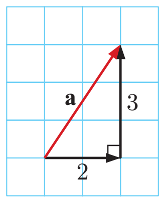
a $\mathbf{i} + 2\mathbf{j}$ **b** $-\mathbf{i} + 3\mathbf{j}$ **c** $-5\mathbf{j}$ **d** $4\mathbf{i} - 2\mathbf{j}$

5 Write down the negative of:

a $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ **d** $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$

B THE MAGNITUDE OF A VECTOR

Consider vector $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\mathbf{i} + 3\mathbf{j}$.



The **magnitude** or **length** of **a** is represented by $|\mathbf{a}|$.

By Pythagoras, $|\mathbf{a}|^2 = 2^2 + 3^2 = 4 + 9 = 13$

$\therefore\ |\mathbf{a}| = \sqrt{13}$ units {since $|\mathbf{a}| > 0$}

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1\mathbf{i} + a_2\mathbf{j}$, the **magnitude** or **length** of **a** is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$.

Example 2 Self Tutor

If $\mathbf{p} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\mathbf{q} = 2\mathbf{i} - 5\mathbf{j}$, find:

a $|\mathbf{p}|$ **b** $|\mathbf{q}|$

a $\mathbf{p} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

$\therefore\ |\mathbf{p}| = \sqrt{3^2 + (-5)^2}$

$= \sqrt{34}$ units

b $\mathbf{q} = 2\mathbf{i} - 5\mathbf{j} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

$\therefore\ |\mathbf{q}| = \sqrt{2^2 + (-5)^2}$

$= \sqrt{29}$ units

UNIT VECTORS

A **unit vector** is any vector which has a length of one unit.

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the base unit vectors in the positive x and y-directions respectively.

Example 3 — Self Tutor

Find k given that $\begin{pmatrix} -\frac{1}{3} \\ k \end{pmatrix}$ is a unit vector.

Since $\begin{pmatrix} -\frac{1}{3} \\ k \end{pmatrix}$ is a unit vector,

$$\begin{aligned} \sqrt{(-\tfrac{1}{3})^2 + k^2} &= 1 \\ \therefore \sqrt{\tfrac{1}{9} + k^2} &= 1 \\ \therefore \tfrac{1}{9} + k^2 &= 1 \quad \{\text{squaring both sides}\} \\ \therefore k^2 &= \tfrac{8}{9} \\ \therefore k &= \pm\tfrac{\sqrt{8}}{3} \end{aligned}$$

EXERCISE 11B

1 Find the magnitude of:

a $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ **d** $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$ **e** $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$

2 Find the length of:

a $\mathbf{i} + \mathbf{j}$ **b** $5\mathbf{i} - 12\mathbf{j}$ **c** $-\mathbf{i} + 4\mathbf{j}$ **d** $3\mathbf{i}$ **e** $k\mathbf{j}$

3 Which of the following are unit vectors?

a $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ **c** $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ **d** $\begin{pmatrix} -\frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$ **e** $\begin{pmatrix} \frac{2}{7} \\ -\frac{5}{7} \end{pmatrix}$

4 Find k for the unit vectors:

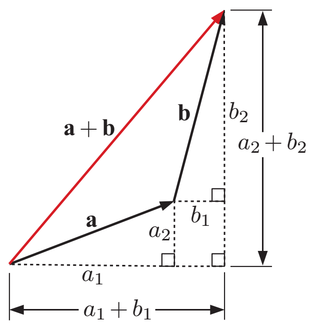
a $\begin{pmatrix} 0 \\ k \end{pmatrix}$ **b** $\begin{pmatrix} k \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} k \\ 1 \end{pmatrix}$ **d** $\begin{pmatrix} k \\ k \end{pmatrix}$ **e** $\begin{pmatrix} \frac{1}{2} \\ k \end{pmatrix}$

5 Given $\mathbf{v} = \begin{pmatrix} 8 \\ p \end{pmatrix}$ and $|\mathbf{v}| = \sqrt{73}$ units, find the possible values of p.

C OPERATIONS WITH PLANE VECTORS

VECTOR ADDITION

Consider adding vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.



Notice that:

- the horizontal step for $\mathbf{a} + \mathbf{b}$ is $a_1 + b_1$
- the vertical step for $\mathbf{a} + \mathbf{b}$ is $a_2 + b_2$.

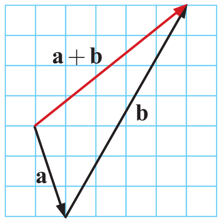
If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$.

Example 4 Self Tutor

If $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, find $\mathbf{a}+\mathbf{b}$. Check your answer graphically.

$$\begin{aligned}\mathbf{a}+\mathbf{b} &= \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 7 \end{pmatrix} \\ &= \begin{pmatrix} 1+4 \\ -3+7 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 4 \end{pmatrix}\end{aligned}$$

Graphical check:

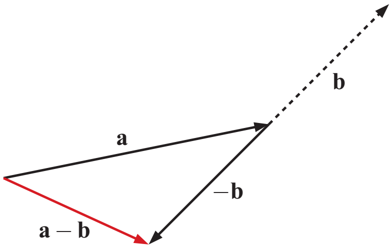


VECTOR SUBTRACTION

To subtract one vector from another, we simply **add its negative**.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

then $$\begin{aligned}\mathbf{a}-\mathbf{b} &= \mathbf{a}+(-\mathbf{b}) \\ &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1-b_1 \\ a_2-b_2 \end{pmatrix}\end{aligned}$$



If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then $\mathbf{a}-\mathbf{b} = \begin{pmatrix} a_1-b_1 \\ a_2-b_2 \end{pmatrix}$.

Example 5 Self Tutor

Given $\mathbf{p} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$, find:

a $\mathbf{q}-\mathbf{p}$

b $\mathbf{p}-\mathbf{q}-\mathbf{r}$

a $$\begin{aligned}&\mathbf{q}-\mathbf{p} \\ &= \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 1-3 \\ 4+2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 6 \end{pmatrix}\end{aligned}$$

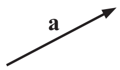
b $$\begin{aligned}&\mathbf{p}-\mathbf{q}-\mathbf{r} \\ &= \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} 3-1+2 \\ -2-4+5 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -1 \end{pmatrix}\end{aligned}$$

SCALAR MULTIPLICATION

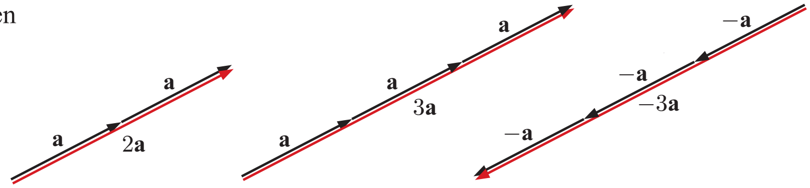
A **scalar** is a non-vector quantity. It has a size but no direction.

We can multiply vectors by scalars such as 2 and -3, or in fact any $k \in \mathbb{R}$.

If **a** is a vector, we define $2\mathbf{a} = \mathbf{a} + \mathbf{a}$ and $3\mathbf{a} = \mathbf{a} + \mathbf{a} + \mathbf{a}$
so $-3\mathbf{a} = 3(-\mathbf{a}) = (-\mathbf{a}) + (-\mathbf{a}) + (-\mathbf{a})$.

If **a** is 

then



So, $2\mathbf{a}$ is in the same direction as **a** but is twice as long as **a**
$3\mathbf{a}$ is in the same direction as **a** but is three times longer than **a**
$-3\mathbf{a}$ has the opposite direction to **a** and is three times longer than **a**.

If **a** is a vector and k is a scalar, then $k\mathbf{a}$ is also a vector and we are performing **scalar multiplication**.

If $k > 0$, $k\mathbf{a}$ and **a** have the same direction.
If $k < 0$, $k\mathbf{a}$ and **a** have opposite directions.
If $k = 0$, $k\mathbf{a} = \mathbf{0}$, the zero vector.

If k is any scalar and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then $k\mathbf{v} = \begin{pmatrix} kv_1 \\ kv_2 \end{pmatrix}$.

VECTOR SCALAR MULTIPLICATION

Notice that:

- $(-1)\mathbf{v} = \begin{pmatrix} (-1)\,v_1 \\ (-1)\,v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} = -\mathbf{v}$
- $(0)\mathbf{v} = \begin{pmatrix} (0)\,v_1 \\ (0)\,v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$

Example 6 — Self Tutor

If $\mathbf{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, find: **a** $3\mathbf{q}$ **b** $\mathbf{p} + 2\mathbf{q}$ **c** $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$

a $3\mathbf{q}$

$$= 3\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} 6 \\ -9 \end{pmatrix}$$

b $\mathbf{p} + 2\mathbf{q}$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} 4 + 2(2) \\ 1 + 2(-3) \end{pmatrix}$$
$$= \begin{pmatrix} 8 \\ -5 \end{pmatrix}$$

c $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$

$$= \tfrac{1}{2}\begin{pmatrix} 4 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2}(4) - 3(2) \\ \frac{1}{2}(1) - 3(-3) \end{pmatrix}$$
$$= \begin{pmatrix} -4 \\ 9\frac{1}{2} \end{pmatrix}$$

Example 7 — Self Tutor

If $\mathbf{p} = 3\mathbf{i} - 5\mathbf{j}$ and $\mathbf{q} = -\mathbf{i} - 2\mathbf{j}$, find $|\mathbf{p} - 2\mathbf{q}|$.

$$\begin{aligned} \mathbf{p} - 2\mathbf{q} &= 3\mathbf{i} - 5\mathbf{j} - 2(-\mathbf{i} - 2\mathbf{j}) \\ &= 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{i} + 4\mathbf{j} \\ &= 5\mathbf{i} - \mathbf{j} \end{aligned}$$

$$\begin{aligned} \therefore\ |\mathbf{p} - 2\mathbf{q}| &= \sqrt{5^2 + (-1)^2} \\ &= \sqrt{26} \text{ units} \end{aligned}$$

EXERCISE 11C

VECTOR RACE GAME

1 If $\mathbf{a} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ find:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{b} + \mathbf{a}$ **c** $\mathbf{b} + \mathbf{c}$ **d** $\mathbf{c} + \mathbf{b}$
e $\mathbf{a} + \mathbf{c}$ **f** $\mathbf{c} + \mathbf{a}$ **g** $\mathbf{a} + \mathbf{a}$ **h** $\mathbf{b} + \mathbf{a} + \mathbf{c}$

2 Given $\mathbf{p} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ find:

a $\mathbf{p} - \mathbf{q}$ **b** $\mathbf{q} - \mathbf{r}$ **c** $\mathbf{p} + \mathbf{q} - \mathbf{r}$
d $\mathbf{p} - \mathbf{q} - \mathbf{r}$ **e** $\mathbf{q} - \mathbf{r} - \mathbf{p}$ **f** $\mathbf{r} + \mathbf{q} - \mathbf{p}$

3 Consider $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

a Use vector addition to show that $\mathbf{a} + \mathbf{0} = \mathbf{a}$.
b Use vector subtraction to show that $\mathbf{a} - \mathbf{a} = \mathbf{0}$.

4 For $\mathbf{p} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ find:

a $-3\mathbf{p}$ **b** $\frac{1}{2}\mathbf{q}$ **c** $2\mathbf{p} + \mathbf{q}$ **d** $\mathbf{p} - 2\mathbf{q}$
e $\mathbf{p} - \frac{1}{2}\mathbf{r}$ **f** $2\mathbf{p} + 3\mathbf{r}$ **g** $2\mathbf{q} - 3\mathbf{r}$ **h** $2\mathbf{p} - \mathbf{q} + \frac{1}{3}\mathbf{r}$

5 Consider $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Find geometrically and then comment on the results:

a $\mathbf{p} + \mathbf{p} + \mathbf{q} + \mathbf{q} + \mathbf{q}$ **b** $\mathbf{p} + \mathbf{q} + \mathbf{p} + \mathbf{q} + \mathbf{q}$ **c** $\mathbf{q} + \mathbf{p} + \mathbf{q} + \mathbf{p} + \mathbf{q}$

6 For $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ find:

a $|\mathbf{r}|$ **b** $|\mathbf{s}|$ **c** $|\mathbf{r} + \mathbf{s}|$ **d** $|\mathbf{r} - \mathbf{s}|$ **e** $|\mathbf{s} - 2\mathbf{r}|$

7 If $\mathbf{p} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ find:

a $|\mathbf{p}|$ **b** $|2\mathbf{p}|$ **c** $|-2\mathbf{p}|$ **d** $|3\mathbf{p}|$ **e** $|-3\mathbf{p}|$
f $|\mathbf{q}|$ **g** $|4\mathbf{q}|$ **h** $|-4\mathbf{q}|$ **i** $\left|\frac{1}{2}\mathbf{q}\right|$ **j** $\left|-\frac{1}{2}\mathbf{q}\right|$

8 Suppose $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j}$, and $\mathbf{c} = -4\mathbf{i}$. Find:

a $\mathbf{a} + \mathbf{b}$ **b** $3\mathbf{b} + \mathbf{c}$ **c** $\mathbf{a} - \mathbf{c}$

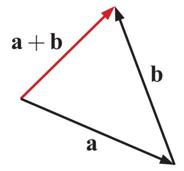
d $2\mathbf{b} - \mathbf{a}$ **e** $|\mathbf{c} + 2\mathbf{a}|$ **f** $|-2\mathbf{b}|$

9 Suppose $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Prove that:

a if $k\mathbf{a} = \mathbf{b}$, $k \neq 0$, then $\mathbf{a} = \frac{1}{k}\mathbf{b}$ **b** $|k\mathbf{a}| = |k||\mathbf{a}|$

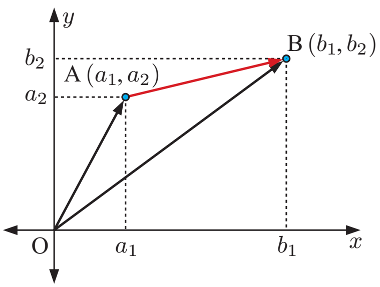
10 Prove that $|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$:

a using a geometric argument and the diagram



b by letting $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

D THE VECTOR BETWEEN TWO POINTS



In the diagram, point A has position vector $\overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$,

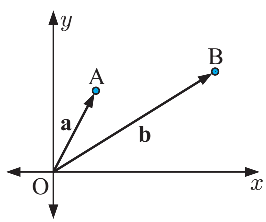
and point B has position vector $\overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

$$\begin{aligned} \therefore \quad \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ &= \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} \end{aligned}$$

The **position vector of B relative to A** is $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$.

In general, for two points A and B with position vectors **a** and **b** respectively, we observe

$$\begin{aligned} \overrightarrow{AB} &= -\mathbf{a} + \mathbf{b} \\ &= \mathbf{b} - \mathbf{a} \\ &= \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \overrightarrow{BA} &= -\mathbf{b} + \mathbf{a} \\ &= \mathbf{a} - \mathbf{b} \\ &= \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix} \end{aligned}$$



Example 8 — Self Tutor

Given points A$(-1, 2)$, B$(3, 4)$, and C$(4, -5)$, find the position vector of:

a B from O **b** B from A **c** A from C

a The position vector of B relative to O is $\overrightarrow{OB} = \begin{pmatrix} 3-0 \\ 4-0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

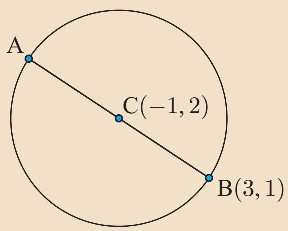
b The position vector of B relative to A is $\overrightarrow{AB} = \begin{pmatrix} 3--1 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

c The position vector of A relative to C is $\overrightarrow{CA} = \begin{pmatrix} -1-4 \\ 2--5 \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$.

Example 9 — Self Tutor

[AB] is the diameter of a circle with centre C$(-1, 2)$. If B is $(3, 1)$, find:

a $\overrightarrow{BC}$ **b** the coordinates of A.



a $\overrightarrow{BC} = \begin{pmatrix} -1-3 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

b If A has coordinates (a, b), then $\overrightarrow{CA} = \begin{pmatrix} a-(-1) \\ b-2 \end{pmatrix} = \begin{pmatrix} a+1 \\ b-2 \end{pmatrix}$

But $\overrightarrow{CA} = \overrightarrow{BC}$, so $\begin{pmatrix} a+1 \\ b-2 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$\therefore\ a+1=-4$ and $b-2=1$

$\therefore\ a=-5$ and $b=3$

$\therefore$ A is $(-5, 3)$.

EXERCISE 11D

1 Find $\overrightarrow{AB}$ given:

a A$(2, 3)$ and B$(4, 7)$ **b** A$(3, -1)$ and B$(1, 4)$ **c** A$(-2, 7)$ and B$(1, 4)$

d B$(3, 0)$ and A$(2, 5)$ **e** B$(6, -1)$ and A$(0, 4)$ **f** B$(0, 0)$ and A$(-1, -3)$

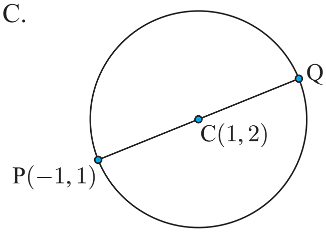
2 Consider the point A$(1, 4)$. Find the coordinates of:

a B given $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **b** C given $\overrightarrow{CA} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

3 [PQ] is the diameter of a circle with centre C.

a Find $\overrightarrow{PC}$.

b Hence find the coordinates of Q.



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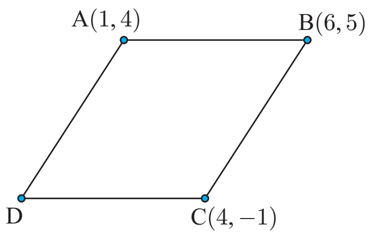


ABCD is a parallelogram.

a Find $\overrightarrow{AB}$.

b Find $\overrightarrow{CD}$.

c Hence find the coordinates of D.

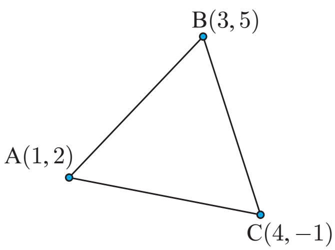
5 $A(-1,\ 3)$ and $B(3,\ k)$ are two points which are 5 units apart.

a Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

b Hence, find the two possible values of k.

c Show, by illustration, why k should have two possible values.

6



a Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b Explain why $\overrightarrow{BC} = -\overrightarrow{AB} + \overrightarrow{AC}$.

c Hence find $\overrightarrow{BC}$.

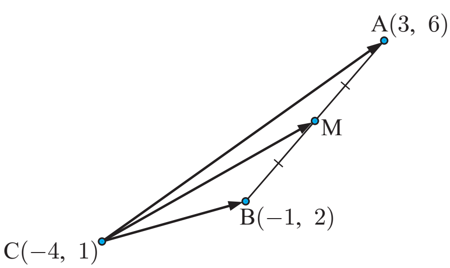
d Check your answer to **c** by direct evaluation.

7 **a** Given $\overrightarrow{BA} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, find $\overrightarrow{AC}$.

b Given $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\overrightarrow{CA} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, find $\overrightarrow{CB}$.

c Given $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $\overrightarrow{RS} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, find $\overrightarrow{SP}$.

8

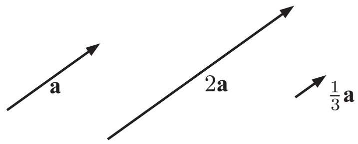


a Find the coordinates of M.

b Find vectors $\overrightarrow{CA}$, $\overrightarrow{CM}$, and $\overrightarrow{CB}$.

c Verify that $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB}$.

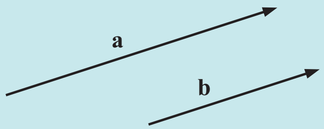
E PARALLELISM



are parallel vectors of different length.

Two non-zero vectors are **parallel** if and only if one is a scalar multiple of the other.

Given any non-zero vector $\mathbf{a}$ and non-zero scalar k, the vector $k\mathbf{a}$ is parallel to $\mathbf{a}$.



- If $\mathbf{a}$ is parallel to $\mathbf{b}$, then there exists a scalar k such that $\mathbf{a} = k\mathbf{b}$.
- If $\mathbf{a} = k\mathbf{b}$ for some scalar k, then
  - $\mathbf{a}$ is parallel to $\mathbf{b}$, and
  - $|\mathbf{a}| = |k|\,|\mathbf{b}|$.

$|k|$ is the modulus of k, whereas $|\mathbf{a}|$ is the length of vector $\mathbf{a}$.

Example 10 **Self Tutor**

Find r given that $\mathbf{a} = \begin{pmatrix} -1 \\ r \end{pmatrix}$ is parallel to $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Since $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\mathbf{a} = k\mathbf{b}$ for some scalar k.

$$\therefore \begin{pmatrix} -1 \\ r \end{pmatrix} = k\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\therefore -1 = 2k \text{ and } r = -3k$$

$$\therefore k = -\tfrac{1}{2} \text{ and hence } r = -3(-\tfrac{1}{2}) = \tfrac{3}{2}$$

UNIT VECTORS

Given a non-zero vector $\mathbf{a}$, its magnitude $|\mathbf{a}|$ is a scalar quantity.

If we multiply $\mathbf{a}$ by the scalar $\dfrac{1}{|\mathbf{a}|}$, we obtain the parallel vector $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$ with length 1.

- A unit vector in the direction of $\mathbf{a}$ is $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$.
- A vector of length k in the same direction as $\mathbf{a}$ is $\dfrac{k}{|\mathbf{a}|}\mathbf{a}$.
- A vector of length k which is *parallel to* $\mathbf{a}$ could be $\pm\dfrac{k}{|\mathbf{a}|}\mathbf{a}$.

Example 11 — Self Tutor

If $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$, find:

a a unit vector in the direction of $\mathbf{a}$

b a vector of length 4 units in the direction of $\mathbf{a}$

c vectors of length 4 units which are parallel to $\mathbf{a}$.

a $|\mathbf{a}| = \sqrt{3^2 + (-1)^2}$
$= \sqrt{9+1}$
$= \sqrt{10}$ units

$\therefore$ the unit vector is $\frac{1}{\sqrt{10}}(3\mathbf{i} - \mathbf{j})$
$= \frac{3}{\sqrt{10}}\mathbf{i} - \frac{1}{\sqrt{10}}\mathbf{j}$

b This vector is $\frac{4}{\sqrt{10}}(3\mathbf{i} - \mathbf{j})$
$= \frac{12}{\sqrt{10}}\mathbf{i} - \frac{4}{\sqrt{10}}\mathbf{j}$

c The vectors are $\frac{12}{\sqrt{10}}\mathbf{i} - \frac{4}{\sqrt{10}}\mathbf{j}$ and $-\frac{12}{\sqrt{10}}\mathbf{i} + \frac{4}{\sqrt{10}}\mathbf{j}$.

EXERCISE 11E

1 Find r given that $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -6 \\ r \end{pmatrix}$ are parallel.

2 Find a given that $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} a \\ 2 \end{pmatrix}$ are parallel.

3 What can be deduced from the following?

a $\overrightarrow{AB} = 3\overrightarrow{CD}$ **b** $\overrightarrow{RS} = -\frac{1}{2}\overrightarrow{KL}$ **c** $\overrightarrow{AB} = 2\overrightarrow{BC}$

4 If $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, write down the vector:

a in the same direction as $\mathbf{a}$ and twice its length

b in the opposite direction to $\mathbf{a}$ and half its length.

5 Find the unit vector in the direction of:

a $\mathbf{i} + 2\mathbf{j}$ **b** $\mathbf{i} - 3\mathbf{j}$ **c** $2\mathbf{i} - \mathbf{j}$

6 Find a vector $\mathbf{v}$ which has:

a the same direction as $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and length 3 units

b the opposite direction to $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ and length 2 units.

7 A is $(3, 2)$ and point B is 4 units from A in the direction $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

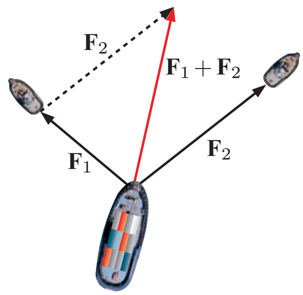
a Find $\overrightarrow{AB}$. **b** Find $\overrightarrow{OB}$ using $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$.

c Hence deduce the coordinates of B.

F PROBLEMS INVOLVING VECTOR OPERATIONS

When we apply vectors to problems in the real world, we often consider the combined effect when vectors are added together. This sum is called the **resultant vector**.

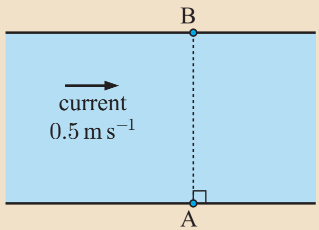
The diagram shows an example of vector addition. Two tug boats are being used to pull a ship into port. If the tugs tow with forces $\mathbf{F}_1$ and $\mathbf{F}_2$ then the resultant force is $\mathbf{F}_1 + \mathbf{F}_2$.



Example 12

Self Tutor

In still water, Jacques can swim at 1.5 m s^{-1}. Jacques is at point A on the edge of a canal, and considers point B directly opposite. A current is flowing from the left at a constant speed of 0.5 m s^{-1}.



a If Jacques dives in straight towards B, and swims without allowing for the current, what will his actual speed and direction be?

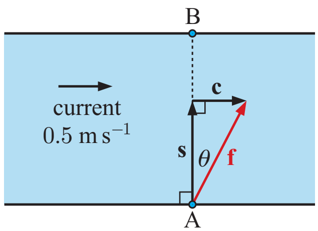
b Jacques wants to swim directly across the canal to point B.

i At what angle should Jacques *aim* to swim in order that the current will correct his direction?

ii What will Jacques' actual speed be?

Suppose $\mathbf{c}$ is the current's velocity vector,
$\mathbf{s}$ is the velocity vector Jacques would have if the water was still, and
$\mathbf{f} = \mathbf{c} + \mathbf{s}$ is Jacques' resultant velocity vector.

a Jacques aims directly across the river, but the current takes him downstream to the right.



$$|\mathbf{f}|^2 = |\mathbf{c}|^2 + |\mathbf{s}|^2$$
$$= 0.5^2 + 1.5^2$$
$$= 2.5$$
$$\therefore \ |\mathbf{f}| \approx 1.58$$

$$\tan\theta = \frac{0.5}{1.5}$$
$$\therefore \ \theta \approx 18.4^\circ$$

Jacques has an actual speed of approximately 1.58 m s^{-1} and his direction of motion is approximately 18.4° to the right of his intended line.

b Jacques needs to aim to the left of B so the current will correct his direction.

i $\sin\phi = \dfrac{0.5}{1.5}$

$\therefore \ \phi \approx 19.5^\circ$

Jacques needs to aim approximately 19.5° to the left of B.

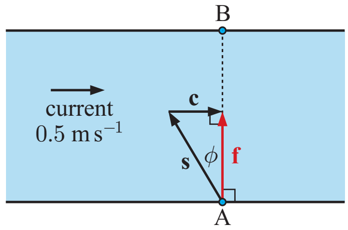
ii $|\mathbf{f}|^2 + |\mathbf{c}|^2 = |\mathbf{s}|^2$

$\therefore\ |\mathbf{f}|^2 + 0.5^2 = 1.5^2$

$\therefore\ |\mathbf{f}|^2 = 2$

$\therefore\ |\mathbf{f}| \approx 1.41$

In these conditions, Jacques' actual speed towards B is approximately 1.41 m s^{-1}.



Another example of vector addition is when an aircraft is affected by wind. A pilot needs to know how to compensate for the wind, especially during take-off and landing.

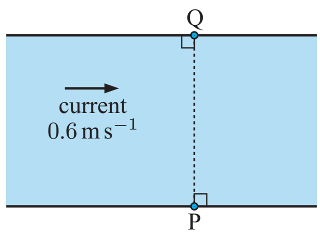
SIMULATION

EXERCISE 11F

1 A bird can normally fly with constant speed 6 m s^{-1}. Using a vector diagram to illustrate each situation, find the bird's speed if:

- **a** it is assisted by a wind of 1 m s^{-1} from directly behind it
- **b** it flies into a head wind of 1 m s^{-1}.

2 In still water, Mary can swim at 1.2 m s^{-1}. She is standing at point P on the edge of a canal, directly opposite point Q. The water is flowing to the right at a constant speed of 0.6 m s^{-1}.

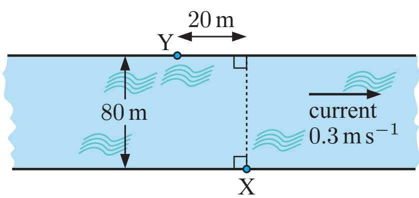


- **a** If Mary tries to swim directly from P to Q without allowing for the current, what will her actual velocity be?
- **b** Mary wants to swim directly across the canal to point Q.
  - **i** At what angle should she *aim* to swim in order that the current corrects her direction?
  - **ii** What will Mary's actual speed be?

3 A boat needs to travel south at a speed of 20 km h^{-1}. However, a constant current of 6 km h^{-1} is flowing from the south-east. Use vectors to find:

- **a** the equivalent speed in still water for the boat to achieve the actual speed of 20 km h^{-1}
- **b** the direction in which the boat must head to compensate for the current.

4 As part of an endurance race, Stephanie needs to swim from X to Y across a wide river.
Stephanie swims at 1.8 m s^{-1} in still water.
The river flows with a consistent current of 0.3 m s^{-1} as shown.



- **a** Find the distance from X to Y.
- **b** In which direction should Stephanie *aim* so that the current will push her onto a path directly towards Y?
- **c** Find the time Stephanie will take to cross the river.

5 An aeroplane needs to fly due east from one city to another at a speed of 400 km h^{-1}. However, a 50 km h^{-1} wind blows constantly from the north-east.

a How does the wind affect the speed of the aeroplane?

b In what direction must the aeroplane head to compensate for the wind?

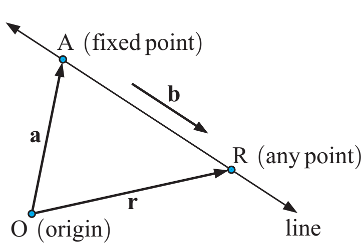
G LINES

We have seen in Cartesian geometry that we can determine the **equation of a line** using its **direction** and any **fixed point** on the line. We can do the same using vectors.

Suppose a line passes through a fixed point A with position vector **a**, and that the line is parallel to the vector **b**.

DEMO

Consider a point R on the line so that $\overrightarrow{OR} = \mathbf{r}$.



By vector addition, $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$

$\therefore \quad \mathbf{r} = \mathbf{a} + \overrightarrow{AR}.$

Since $\overrightarrow{AR}$ is parallel to **b**,

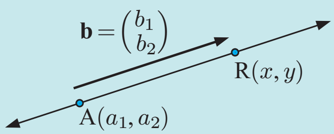
$\overrightarrow{AR} = t\mathbf{b}$ for some scalar $t \in \mathbb{R}$

$\therefore \quad \mathbf{r} = \mathbf{a} + t\mathbf{b}$

Suppose a line passes through a fixed point $A(a_1, a_2)$ with position vector **a**, and that the line is parallel to the vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. If $R(x, y)$ with position vector **r** is any point on the line, then:

- $\mathbf{r} = \mathbf{a} + t\mathbf{b}, \ t \in \mathbb{R}$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

  is the **vector equation** of the line.



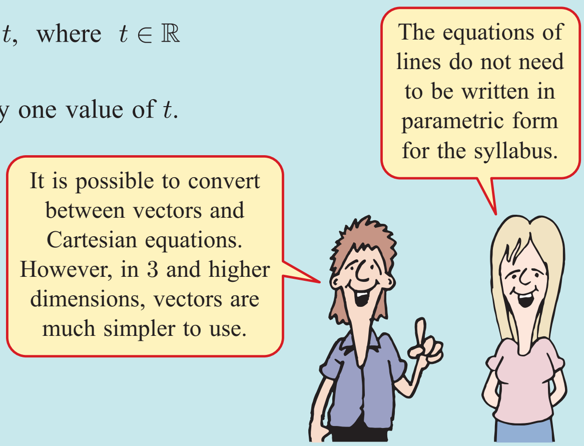
- The gradient of the line is $m = \dfrac{b_2}{b_1}$.

- Since $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix}$, the **parametric equations** of the line are $x = a_1 + b_1 t$ and $y = a_2 + b_2 t$, where $t \in \mathbb{R}$ is the **parameter**.

  Each point on the line corresponds to exactly one value of t.

- We can convert these equations into Cartesian form by equating t values.

  Using $t = \dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2}$ we obtain $b_2 x - b_1 y = b_2 a_1 - b_1 a_2$ which is the **Cartesian equation** of the line.



Example 13

Self Tutor

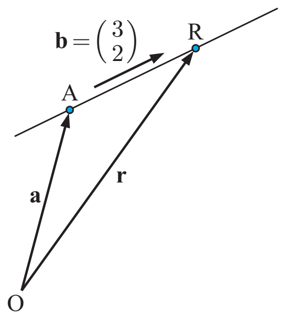
A line passes through the point A(1, 5) and has direction vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Describe the line using:

a a vector equation **b** parametric equations **c** a Cartesian equation.

a The vector equation is $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where

$\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}$



b $x = 1 + 3t$ and $y = 5 + 2t$, $t \in \mathbb{R}$

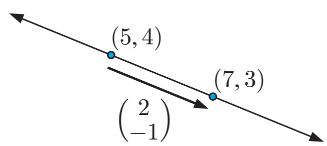
c Now $t = \dfrac{x-1}{3} = \dfrac{y-5}{2}$

$\therefore \; 2x - 2 = 3y - 15$

$\therefore \; 2x - 3y = -13$

NON-UNIQUENESS OF THE VECTOR EQUATION OF A LINE

Consider the line passing through (5, 4) and (7, 3). When writing the equation of the line, we could use either point to give the position vector **a**.



Similarly, we could use the direction vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, but we could also use $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ or indeed any non-zero scalar multiple of these vectors.

We could thus write the equation of the line as

$$\mathbf{x} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R} \quad \text{or} \quad \mathbf{x} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R} \quad \text{and so on.}$$

Notice how we use different parameters t and s when we write these equations. This is because the parameters are clearly not the same: when $t = 0$, we have the point (5, 4)
when $s = 0$, we have the point (7, 3).

In fact, the parameters are related by $s = 1 - t$.

EXERCISE 11G

1 Describe each of the following lines using:

i a vector equation **ii** parametric equations **iii** a Cartesian equation

a a line with direction $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ which passes through $(3, -4)$

b a line parallel to $3\mathbf{i} + 7\mathbf{j}$ which cuts the x-axis at -6

c a line passing through $(-1, 11)$ and $(-3, 12)$.

2 A line passes through $(-1,\ 4)$ with direction vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

a Write parametric equations for the line using the parameter t.

b Find the points on the line for which $t = 0,\ 1,\ 3,\ -1$, and -4.

3 **a** Does $(3,\ -2)$ lie on the line with vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \end{pmatrix}$?

b $(k,\ 4)$ lies on the line with parametric equations $x = 1 - 2t$, $y = 1 + t$. Find k.

4 Line L has vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

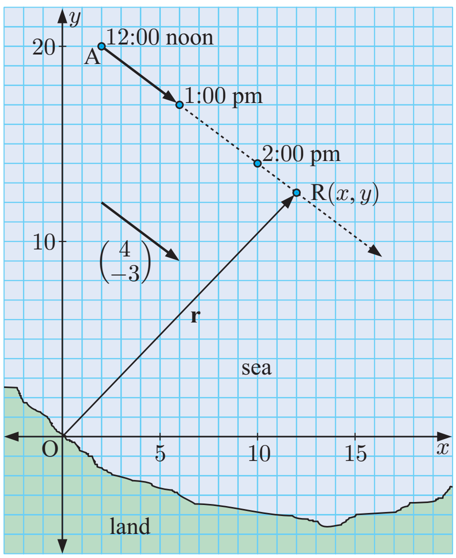
a Locate the point on the line corresponding to $t = 1$.

b Explain why the direction of the line could also be described by $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

c Use your answers to **a** and **b** to write an alternative vector equation for line L.

H CONSTANT VELOCITY PROBLEMS

A yacht club is situated at $(0,\ 0)$. At 12:00 noon a yacht is at point $\text{A}(2,\ 20)$. The yacht is moving with constant speed in the straight path shown in the diagram. The grid intervals are kilometres.



At 1:00 pm the yacht is at $(6,\ 17)$.
At 2:00 pm it is at $(10,\ 14)$.

In this case:

- the **initial position** of the yacht is given by the position vector $\mathbf{a} = \begin{pmatrix} 2 \\ 20 \end{pmatrix}$
- the direction of the yacht is given by the vector $\mathbf{b} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

  Suppose that t hours after leaving A, the yacht is at $\text{R}(x,\ y)$.

$$\overrightarrow{\text{OR}} = \overrightarrow{\text{OA}} + \overrightarrow{\text{AR}}$$

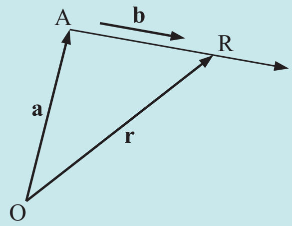
$$\therefore\ \mathbf{r} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad \text{for } t \geqslant 0$$

$$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix}$$ is the **vector equation** of the yacht's path.

If an object has initial position vector **a** and moves with constant velocity **b**, its position at time t is given by

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \quad \text{for} \quad t \geqslant 0.$$

The **speed** of the object is $|\mathbf{b}|$.



Example 14

Self Tutor

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ is the vector equation of the path of an object.

The time t is in seconds, $t \geqslant 0$. The distance units are metres.

a Find the object's initial position.

b Plot the path of the object for $t = 0, 1, 2, 3$.

c Find the velocity vector of the object.

d Find the object's speed.

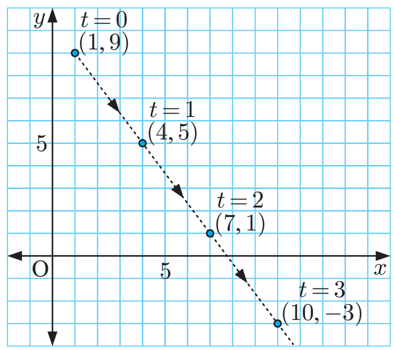
e If the object continues in the same direction but increases its speed to 30 m s^{-1}, state its new velocity vector.

a At $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$

$\therefore$ the object is at $(1, 9)$.

DEMO

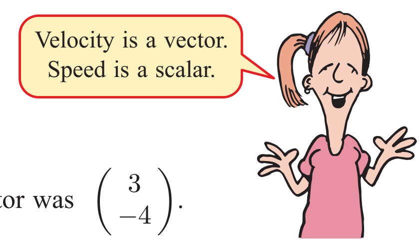
b



c The velocity vector is $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

d The speed is $\left|\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right| = \sqrt{3^2 + (-4)^2}$

$= 5 \text{ m s}^{-1}$.



e Previously, the speed was 5 m s^{-1} and the velocity vector was $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

$\therefore$ the new velocity vector is $6\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 18 \\ -24 \end{pmatrix}$.

Example 15

Self Tutor

An object is initially at $(5, 10)$ and moves with velocity vector $3\mathbf{i} - \mathbf{j}$ metres per minute. Find:

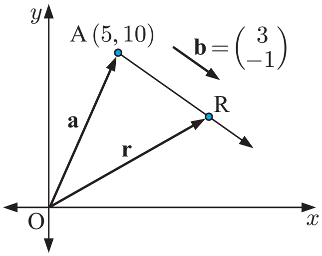
a the position of the object at time t minutes

b the speed of the object

c the position of the object at $t = 3$ minutes

d the time when the object is due east of $(0, 0)$.

a



$\mathbf{r} = \mathbf{a} + t\mathbf{b}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5+3t \\ 10-t \end{pmatrix}$

After t minutes, the object is at $(5 + 3t, 10 - t)$.

b The speed of the object is $|\mathbf{b}| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$ metres per minute.

c At $t = 3$ minutes, $5 + 3t = 14$ and $10 - t = 7$. The object is at $(14, 7)$.

d When the object is due east of $(0, 0)$, y must be zero.

$\therefore \ 10 - t = 0$

$\therefore \ t = 10$

The object is due east of $(0, 0)$ after 10 minutes.

EXERCISE 11H

1 A particle at $P(x(t), y(t))$ moves such that $x(t) = 1 + 2t$ and $y(t) = 2 - 5t$, $t \geqslant 0$. The distances are in centimetres and t is in seconds.

a Find the initial position of P.

b Illustrate the initial part of the motion of P where $t = 0, 1, 2, 3$.

c Find the velocity vector of P.

d Find the speed of P.

2 **a** Find the vector equation of a boat initially at $(2, 3)$, which travels with velocity vector $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$. The grid units are kilometres and the time is in hours.

b Locate the boat's position after 90 minutes.

c How long will it take for the boat to reach the point $(5, -0.75)$?

3 A remote controlled toy car is initially at $(-3, -2)$. It moves with constant velocity $2\mathbf{i} + 4\mathbf{j}$. The distance units are centimetres, and the time is in seconds.

a Write an expression for the position vector of the car at any time $t \geqslant 0$.

b Find the position vector of the car at time $t = 2.5$.

c Find when the car is **i** due north **ii** due west of the observation point $(0, 0)$.

d Plot the car's positions at times $t = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, ...$

4 Each of the following vector equations represents the path of a moving object. t is measured in seconds, and $t \geqslant 0$. Distances are measured in metres. In each case, find:

i the initial position **ii** the velocity vector **iii** the speed of the object.

a $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 12 \\ 5 \end{pmatrix}$ **b** $x = 3 + 2t, \ y = -t$

5 Find the velocity vector of a speed boat moving parallel to:

a $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ with a speed of 150 km h^{-1} **b** $2\mathbf{i} + \mathbf{j}$ with a speed of 50 km h^{-1}.

6 Find the velocity vector of a swooping eagle moving in the direction $5\mathbf{i} - 12\mathbf{j}$ with a speed of 91 km h^{-1}.

7 Yacht A moves according to $x(t) = 4 + t, \ y(t) = 5 - 2t$ where the distance units are kilometres and the time units are hours. Yacht B moves according to $x(t) = 1 + 2t, \ y(t) = -8 + t, \ t \geqslant 0$.

a Find the initial position of each yacht.

b Find the velocity vector of each yacht.

c Show that the speed of each yacht is constant, and state these speeds.

d Find the Cartesian equation of the path of each yacht.

e *Hence* show that the paths of the yachts intersect at right angles.

f Will the yachts collide?

8 Submarine P is at $(-5, 4)$. It fires a torpedo with velocity vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ at 1:34 pm.

Submarine Q is at $(15, 7)$. a minutes after 1:34 pm, it fires a torpedo with velocity vector $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$.

Distances are measured in kilometres, and time is in minutes.

a Show that the position of P's torpedo can be written as $P(x_1(t), \ y_1(t))$ where $x_1(t) = -5 + 3t$ and $y_1(t) = 4 - t$.

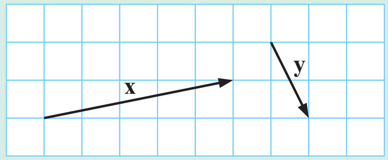
b What is the speed of P's torpedo?

c Show that the position of Q's torpedo can be written as $Q(x_2(t), \ y_2(t))$ where $x_2(t) = 15 - 4(t - a)$ and $y_2(t) = 7 - 3(t - a)$.

d Q's torpedo is successful in knocking out P's torpedo. At what time did Q fire its torpedo, and at what time did the explosion occur?

Review set 11A

1 **a** Write the given vectors in component form and in unit vector form.



b Find, in unit vector form:

i $\mathbf{x} + \mathbf{y}$ **ii** $\mathbf{y} - 2\mathbf{x}$

2 Consider the vector $3\mathbf{i} - \mathbf{j}$.

a Write the vector in component form.

b Illustrate the vector using a directed line segment.

c Write the negative of the vector.

d Find the length of the vector.

3 **a** Find k given that $\begin{pmatrix} k \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ is a unit vector.

b Find the vector which is 5 units long and has the opposite direction to $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

4 For $\mathbf{m} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and $\mathbf{p} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, find:

a $\mathbf{m} - \mathbf{n} + \mathbf{p}$ **b** $2\mathbf{n} - 3\mathbf{p}$ **c** $|\mathbf{m} + \mathbf{p}|$

5 Given points $A(3, 1)$, $B(5, -2)$, and $C(8, 4)$, find:

a $\overrightarrow{AB}$ **b** $\overrightarrow{CB}$ **c** $|\overrightarrow{AC}|$

6 $B(-3, -1)$ and $C(k, 2)$ are 5 units apart.

a Find $\overrightarrow{BC}$ and $|\overrightarrow{BC}|$.

b Hence, find the two possible values of k.

c Show, by illustration, why k should have two possible values.

7 A small plane can fly at 350 km h^{-1} in still conditions. Its pilot needs to fly due north, but needs to deal with a 70 km h^{-1} wind from the east.

a In what direction should the pilot face the plane in order that its resultant velocity is due north?

b What will the speed of the plane be?

8 For the line that passes through $(-6, 3)$ with direction $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, write down the corresponding:

a vector equation **b** parametric equations **c** Cartesian equation.

9 $(-3, m)$ lies on the line with vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ -2 \end{pmatrix} + t\begin{pmatrix} -7 \\ 4 \end{pmatrix}$. Find m.

10 Find the velocity vector of an object moving in the direction $3\mathbf{i} - \mathbf{j}$ with speed 20 km h^{-1}.

11 Line L has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

a Locate the point on the line corresponding to $t = 1$.

b Explain why the direction of the line could also be described by $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$.

c Use your answers to **a** and **b** to write an alternative vector equation for line L.

12 A moving particle has coordinates $P(x(t),\ y(t))$ where $x(t) = -4 + 8t$ and $y(t) = 3 + 6t$. The distance units are metres, and $t \geqslant 0$ is the time in seconds. Find the:

a initial position of the particle

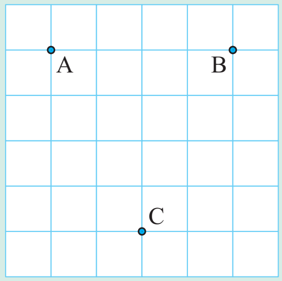
b position of the particle after 4 seconds

c particle's velocity vector

d speed of the particle.

Review set 11B

1



a Find in component form and in unit vector form:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CA}$

b Which two vectors in **a** have the same length? Explain your answer.

c Write the negative vector of $\overrightarrow{CA}$ in *three* different ways.

2 If $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ find:

a $|\mathbf{s}|$

b $|\mathbf{r}+\mathbf{s}|$

c $|2\mathbf{s}-\mathbf{r}|$

3 Find k if the following are unit vectors:

a $\begin{pmatrix} \frac{5}{13} \\ k \end{pmatrix}$

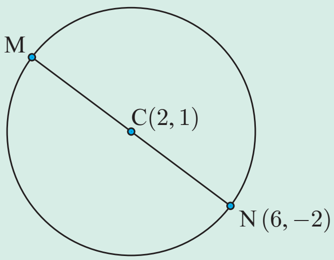
b $\begin{pmatrix} k \\ -k \end{pmatrix}$

4 If $\overrightarrow{PQ} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, and $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, find $\overrightarrow{SP}$.

5 [MN] is the diameter of a circle with centre C.

a Find the coordinates of M.

b Find the radius of the circle.



6 Find m if $\begin{pmatrix} 3 \\ m \end{pmatrix}$ and $\begin{pmatrix} -12 \\ -20 \end{pmatrix}$ are parallel vectors.

7 When an archer fires an arrow, he is suddenly aware of a breeze which pushes his shot off-target. The speed of the shot $|\mathbf{v}|$ is *not* affected by the wind, but the arrow's flight is $2°$ off-line.

a Draw a vector diagram to represent the situation.

b Hence explain why:

i the breeze must be $91°$ to the intended direction of the arrow

ii the speed of the breeze must be $2|\mathbf{v}|\sin 1°$.

8 Find the vector equation of the line which cuts the y-axis at $(0, 8)$ and has direction $5\mathbf{i} + 4\mathbf{j}$.

9 A yacht is sailing with constant speed $5\sqrt{10}$ km h^{-1} in the direction $-\mathbf{i} - 3\mathbf{j}$. Initially it is at point $(-6, 10)$. A beacon is at $(0, 0)$ at the centre of a tiny atoll. Distances are in kilometres.

a Find, in terms of $\mathbf{i}$ and $\mathbf{j}$:

i the initial position vector of the yacht

ii the velocity vector of the yacht

iii the position vector of the yacht at any time t hours, $t \geqslant 0$.

b Find the time when the yacht is due west of the beacon. How far away from the beacon is the yacht at this time?

10 Write down **i** a vector equation **ii** parametric equations for the line passing through:

a $(2, -3)$ with direction $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$

b $(-1, 6)$ and $(5, -2)$.

11 Submarine X23 is at $(2, 4)$. It fires a torpedo with velocity vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ at exactly 2:17 pm. Submarine Y18 is at $(11, 3)$. It fires a torpedo with velocity vector $\begin{pmatrix} -1 \\ a \end{pmatrix}$ at 2:19 pm to intercept the torpedo from X23. Distance units are kilometres. t is in minutes.

a Find $x_1(t)$ and $y_1(t)$ for the torpedo fired from submarine X23.

b Find $x_2(t)$ and $y_2(t)$ for the torpedo fired from submarine Y18.

c At what time does the interception occur?

d What was the direction and speed of the interception torpedo?

12

Matrices

Contents:

A Matrix structure
B Matrix operations and definitions
C Matrix multiplication
D The inverse of a 2×2 matrix
E Simultaneous linear equations

Opening problem

Aakriti owns a stationery shop. She sells two brands of pen, in three colours. Her sales for one week are shown in the table below.

	Brand	
Colour	*Pentex*	*Rollerball*
Blue	32	24
Black	25	16
Red	13	9

Pentex pens sell for \$1.19 each, and Rollerball pens sell for \$1.55 each.

Things to think about:

a How can we convert the table into a 3×2 quantities matrix $\mathbf{Q}$?

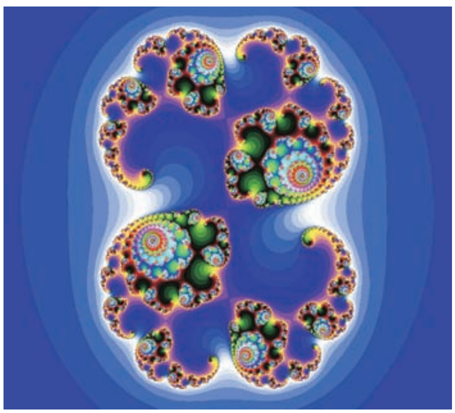
b How can we display the prices in a price matrix $\mathbf{P}$?

c How can we multiply the matrices $\mathbf{Q}$ and $\mathbf{P}$? What does the matrix $\mathbf{QP}$ represent?

d Can you find the total revenue for Aakriti in pen sales for the week?

Matrices are rectangular arrays of numbers which are used to organise numerical information. They are used in a wide range of fields, including:

- solving systems of equations in business, physics, and engineering
- linear programming where we may wish to optimise a linear expression subject to linear constraints
- business inventories involving stock control, cost, revenue, and profit calculations
- Markov chains for predicting long term probabilities such as in weather
- strategies in games where we wish to maximise our chance of winning
- economic modelling where the input from suppliers is needed to help a business be successful
- graph (network) theory used to determine routes for trucks and airlines to minimise distance travelled and therefore costs
- assignment problems to direct resources in the most cost-effective way
- forestry and fisheries management where we need to select an appropriate sustainable harvesting policy
- cubic spline interpolation used to construct curves and fonts
- computer graphics
- flight simulation
- Computer Aided Tomography (CAT scanning) and Magnetic Resonance Imaging (MRI)
- fractals and chaos
- genetics
- cryptography including coding, code breaking, and computer confidentiality.



The Julia set

A MATRIX STRUCTURE

A **matrix** is a rectangular array of numbers arranged in **rows** and **columns**.

Each number within a matrix has a particular meaning.

You have been using matrices for many years without realising it. For example, a football premiership table and a recipe can each be written as matrices.

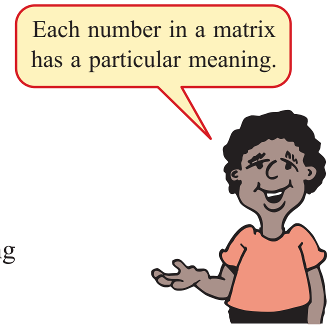
	Won	*Lost*	*Drew*	*Points*
Manchester United	28	5	5	89
Manchester City	23	9	6	78
Chelsea	22	9	7	75
Arsenal	21	10	7	73
⋮				

Ingredients	*Amount*
sugar	1 tspn
flour	1 cup
milk	200 mL
salt	1 pinch

Consider these two items of information:

Shopping list	
Bread	2 loaves
Juice	1 carton
Eggs	6
Cheese	1

Furniture inventory			
	chairs	tables	beds
Flat	6	1	2
Unit	9	2	3
House	10	3	4



We can write these tables as matrices by extracting the numbers and placing them in round brackets:

$$\begin{array}{c c} & \text{number} \\ \begin{array}{c} B \\ J \\ E \\ C \end{array} & \begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c c} & \begin{array}{ccc} C & T & B \end{array} \\ \begin{array}{c} F \\ U \\ H \end{array} & \begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & 3 & 4 \end{pmatrix} \end{array} \quad \text{or simply} \quad \begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & 3 & 4 \end{pmatrix}$$

Notice how the organisation of the data is maintained in matrix form.

$\begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix}$ has 4 rows and 1 column, and we say that this is a 4×1 **column matrix** or **column vector**.

column 2

row 3 $\longrightarrow$ $\begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & 3 & 4 \end{pmatrix}$ has 3 rows and 3 columns, and is called a 3×3 **square matrix**.

this element is in row 3, column 2

$\begin{pmatrix} 3 & 0 & -1 & 2 \end{pmatrix}$ has 1 row and 4 columns, and is called a 1×4 **row matrix** or **row vector**.

An $m \times n$ matrix has m rows and n columns.

$m \times n$ specifies the **order** of a matrix.

Example 1

Self Tutor

Lisa goes shopping at store A to buy 2 loaves of bread at \$2.65 each, 3 litres of milk at \$1.55 per litre, and one 500 g tub of butter at \$2.35.

a Represent the quantities purchased in a row matrix $\mathbf{Q}$, and the costs in a column matrix $\mathbf{A}$.

b When Lisa goes to a different supermarket (store B), she finds that the prices for the same items are \$2.25 for bread, \$1.50 for milk, and \$2.20 for butter.
Write the costs for both stores in a single costs matrix $\mathbf{C}$.

a The quantities matrix is $\mathbf{Q} = \begin{array}{c} \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{bread} & \text{milk} & \text{butter} \end{array} \end{array}$

The costs matrix is $\mathbf{A} = \begin{pmatrix} 2.65 \\ 1.55 \\ 2.35 \end{pmatrix} \begin{array}{l} \leftarrow \text{bread} \\ \leftarrow \text{milk} \\ \leftarrow \text{butter} \end{array}$

b We write the costs for each store in separate columns.

The new costs matrix is $\mathbf{C} = \begin{array}{c} \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{store A} & \text{store B} \end{array} \end{array} \begin{array}{l} \leftarrow \text{bread} \\ \leftarrow \text{milk} \\ \leftarrow \text{butter} \\ \\ \\ \end{array}$

EXERCISE 12A

1 Write down the order of:

a $\begin{pmatrix} 5 & 1 & 0 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 5 & 1 & 0 \end{pmatrix}$

2 A grocery list consists of 2 loaves of bread, 1 kg of butter, 6 eggs, and 1 carton of cream. Each loaf of bread costs \$1.95, each kilogram of butter costs \$2.35, each egg costs \$0.45, and each carton of cream costs \$2.95.

a Construct a row matrix showing quantities.

b Construct a column matrix showing prices.

c What is the significance of $(2 \times 1.95) + (1 \times 2.35) + (6 \times 0.45) + (1 \times 2.95)$?

3 A food processing factory produces cans of beans in three sizes: 200 g, 300 g, and 500 g. In February they produced respectively:

- 1000, 1500, and 1250 cans of each in week 1
- 1500, 1000, and 1000 cans of each in week 2
- 800, 2300, and 1300 cans of each in week 3
- 1200 cans of each in week 4.

Construct a matrix to show February's production levels.

4 Over a long weekend holiday, a baker produced the following food items: On Friday he baked 40 dozen pies, 50 dozen pasties, 55 dozen rolls, and 40 dozen buns. On Saturday he baked 25 dozen pies, 65 dozen pasties, 30 dozen buns, and 44 dozen rolls. On Sunday he baked 40 dozen pasties, 40 dozen rolls, and 35 dozen of each of pies and buns. On Monday he baked 40 dozen pasties, 50 dozen buns, and 35 dozen of each of pies and rolls. Represent this information as a matrix.

B MATRIX OPERATIONS AND DEFINITIONS

MATRIX NOTATION

Consider a matrix **A** which has order $m \times n$.

We can write

$$\mathbf{A} = (a_{ij}) \quad \text{where} \quad i = 1, 2, 3, ..., m$$
$$j = 1, 2, 3, ..., n$$
and a_{ij} is the element in the ith row, jth column.

By convention, the a_{ij} are labelled down then across.

i j

For example, a_{23} is the number in row 2 and column 3 of matrix **A**.

EQUALITY

Two matrices are **equal** if they have the **same order** *and* the elements in corresponding positions are equal.

$$\mathbf{A} = \mathbf{B} \quad \Leftrightarrow \quad a_{ij} = b_{ij} \quad \text{for all } i, j.$$

For example, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ then $a = w$, $b = x$, $c = y$, and $d = z$.

MATRIX ADDITION

Thao has three stores: A, B, and C. Her stock levels for dresses, skirts, and blouses are given by the matrix:

$$\begin{array}{cc} & \text{Store} \\ & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ & \begin{pmatrix} 23 & 41 & 68 \\ 28 & 39 & 79 \\ 46 & 17 & 62 \end{pmatrix} \begin{array}{l} \text{dresses} \\ \text{skirts} \\ \text{blouses} \end{array} \end{array}$$

Some newly ordered stock has just arrived. 20 dresses, 30 skirts, and 50 blouses must be added to the stock levels of each store. Her stock order is given by the matrix:

$$\begin{pmatrix} 20 & 20 & 20 \\ 30 & 30 & 30 \\ 50 & 50 & 50 \end{pmatrix}$$

Clearly the new levels are: $\begin{pmatrix} 23 & 41 & 68 \\ 28 & 39 & 79 \\ 46 & 17 & 62 \end{pmatrix} + \begin{pmatrix} 20 & 20 & 20 \\ 30 & 30 & 30 \\ 50 & 50 & 50 \end{pmatrix} = \begin{pmatrix} 43 & 61 & 88 \\ 58 & 69 & 109 \\ 96 & 67 & 112 \end{pmatrix}$

To **add** two matrices, they must be of the **same order**, and we **add corresponding elements**.

MATRIX SUBTRACTION

Suppose Thao's stock levels were $\begin{pmatrix} 29 & 51 & 19 \\ 31 & 28 & 32 \\ 40 & 17 & 29 \end{pmatrix}$ and her sales matrix for the week was $\begin{pmatrix} 15 & 12 & 6 \\ 20 & 16 & 19 \\ 19 & 8 & 14 \end{pmatrix}$.

Thao will be left with her original stock levels less what she has sold. Clearly, we need to subtract corresponding elements:

$$\begin{pmatrix} 29 & 51 & 19 \\ 31 & 28 & 32 \\ 40 & 17 & 29 \end{pmatrix} - \begin{pmatrix} 15 & 12 & 6 \\ 20 & 16 & 19 \\ 19 & 8 & 14 \end{pmatrix} = \begin{pmatrix} 14 & 39 & 13 \\ 11 & 12 & 13 \\ 21 & 9 & 15 \end{pmatrix}$$

To **subtract** matrices, they must be of the **same order**, and we **subtract corresponding elements**.

Summary:

- $\mathbf{A} \pm \mathbf{B} = (a_{ij}) \pm (b_{ij}) = (a_{ij} \pm b_{ij})$
- We can only add or subtract matrices of the same order.
- We add or subtract corresponding elements.
- The result of addition or subtraction is another matrix of the same order.

Example 2 — Self Tutor

If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} + \mathbf{C}$

a
$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1+2 & 2+1 & 3+6 \\ 6+0 & 5+3 & 4+5 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 3 & 9 \\ 6 & 8 & 9 \end{pmatrix} \end{aligned}$$

b $\mathbf{A} + \mathbf{C}$ cannot be found as the matrices do not have the same order.

Example 3 — Self Tutor

If $\mathbf{A} = \begin{pmatrix} 3 & 4 & 8 \\ 2 & 1 & 0 \\ 1 & 4 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 0 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 3 \end{pmatrix}$, find $\mathbf{A} - \mathbf{B}$.

$$\begin{aligned} \mathbf{A} - \mathbf{B} &= \begin{pmatrix} 3 & 4 & 8 \\ 2 & 1 & 0 \\ 1 & 4 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3-2 & 4-0 & 8-6 \\ 2-3 & 1-0 & 0-4 \\ 1-5 & 4-2 & 7-3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 4 & 2 \\ -1 & 1 & -4 \\ -4 & 2 & 4 \end{pmatrix} \end{aligned}$$

EXERCISE 12B.1

1 If $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -3 & 7 \\ -4 & -2 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} + \mathbf{B} + \mathbf{C}$ **c** $\mathbf{B} + \mathbf{C}$ **d** $\mathbf{C} + \mathbf{B} - \mathbf{A}$

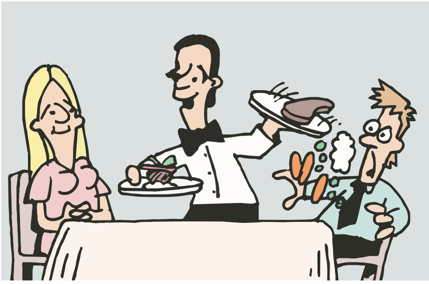
2 If $\mathbf{P} = \begin{pmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{pmatrix}$, find:

a $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{P} - \mathbf{Q}$ **c** $\mathbf{Q} - \mathbf{P}$

3 A restaurant served 85 men, 92 women, and 52 children on Friday night. On Saturday night they served 102 men, 137 women, and 49 children.

a Express this information in *two* column matrices.

b Use the matrices to find the totals of men, women, and children served over the two nights.



4 David bought shares in five companies on Monday, and he sold them on Friday. The details are shown in the table alongside.

a Write down David's column matrix for:

i cost price **ii** selling price.

b What matrix operation is needed to find David's profit or loss on each type of share?

c Find David's profit or loss matrix.

	Cost price per share	*Selling price per share*
A	\$1.72	\$1.79
B	\$27.85	\$28.75
C	\$0.92	\$1.33
D	\$2.53	\$2.25
E	\$3.56	\$3.51

5 In November, Lou E Gee sold 23 fridges, 17 stoves, and 31 microwave ovens. His partner Rose A Lee sold 19 fridges, 29 stoves, and 24 microwave ovens.

In December, Lou sold 18 fridges, 7 stoves, and 36 microwaves, and Rose sold 25 fridges, 13 stoves, and 19 microwaves.

a Write their sales for November as a 3×2 matrix.

b Write their sales for December as a 3×2 matrix.

c Write their total sales for November and December as a 3×2 matrix.

6 Find x and y if:

a $\begin{pmatrix} x & x^2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} y & 4 \\ 3 & y+1 \end{pmatrix}$ **b** $\begin{pmatrix} x & y \\ y & x \end{pmatrix} = \begin{pmatrix} -y & x \\ x & -y \end{pmatrix}$

7 **a** If $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix}$, find $\mathbf{A} + \mathbf{B}$ and $\mathbf{B} + \mathbf{A}$.

b Explain why $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

8 **a** For $\mathbf{A}=\begin{pmatrix}-1 & 0\\ 1 & 5\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}3 & 4\\ -1 & -2\end{pmatrix}$, and $\mathbf{C}=\begin{pmatrix}4 & -1\\ -1 & 3\end{pmatrix}$, find $(\mathbf{A}+\mathbf{B})+\mathbf{C}$ and $\mathbf{A}+(\mathbf{B}+\mathbf{C})$.

b Prove that, if **A**, **B**, and **C** are any 2×2 matrices, then $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$.

Hint: Let $\mathbf{A}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}p & q\\ r & s\end{pmatrix}$, and $\mathbf{C}=\begin{pmatrix}w & x\\ y & z\end{pmatrix}$.

MULTIPLES OF MATRICES

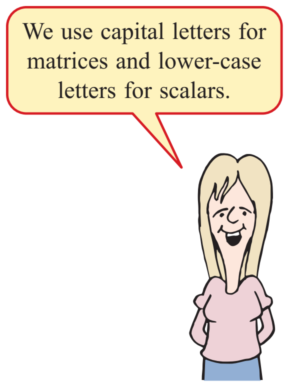
In the pantry there are 6 cans of peaches, 4 cans of apricots, and 8 cans of pears. We represent this by the column vector $\mathbf{C}=\begin{pmatrix}6\\ 4\\ 8\end{pmatrix}$.

If we doubled the cans in the pantry, we would have $\begin{pmatrix}12\\ 8\\ 16\end{pmatrix}$ which is $\mathbf{C}+\mathbf{C}$ or $2\mathbf{C}$.

Notice that to get $2\mathbf{C}$ from $\mathbf{C}$ we simply multiply all the matrix elements by 2.

Likewise, trebling the fruit cans in the pantry gives $3\mathbf{C}=\begin{pmatrix}3\times 6\\ 3\times 4\\ 3\times 8\end{pmatrix}=\begin{pmatrix}18\\ 12\\ 24\end{pmatrix}$

and halving them gives $\frac{1}{2}\mathbf{C}=\begin{pmatrix}\frac{1}{2}\times 6\\ \frac{1}{2}\times 4\\ \frac{1}{2}\times 8\end{pmatrix}=\begin{pmatrix}3\\ 2\\ 4\end{pmatrix}$.



If $\mathbf{A}=(a_{ij})$ has order $m\times n$, and k is a scalar, then $k\mathbf{A}=(ka_{ij})$.

So, to find $k\mathbf{A}$, we multiply each element in **A** by k.

The result is another matrix of order $m\times n$.

Example 4

Self Tutor

If **A** is $\begin{pmatrix}1 & 2 & 5\\ 2 & 0 & 1\end{pmatrix}$, find:

a $3\mathbf{A}$

b $\frac{1}{2}\mathbf{A}$

a $$3\mathbf{A}=3\begin{pmatrix}1 & 2 & 5\\ 2 & 0 & 1\end{pmatrix}=\begin{pmatrix}3\times 1 & 3\times 2 & 3\times 5\\ 3\times 2 & 3\times 0 & 3\times 1\end{pmatrix}=\begin{pmatrix}3 & 6 & 15\\ 6 & 0 & 3\end{pmatrix}$$

b $$\tfrac{1}{2}\mathbf{A}=\tfrac{1}{2}\begin{pmatrix}1 & 2 & 5\\ 2 & 0 & 1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\times 1 & \frac{1}{2}\times 2 & \frac{1}{2}\times 5\\ \frac{1}{2}\times 2 & \frac{1}{2}\times 0 & \frac{1}{2}\times 1\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & 1 & 2\frac{1}{2}\\ 1 & 0 & \frac{1}{2}\end{pmatrix}$$

EXERCISE 12B.2

1 If $\mathbf{B} = \begin{pmatrix} 6 & 12 \\ 24 & 6 \end{pmatrix}$, find:

a $2\mathbf{B}$ **b** $\frac{1}{3}\mathbf{B}$ **c** $\frac{1}{12}\mathbf{B}$ **d** $-\frac{1}{2}\mathbf{B}$

2 If $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} - \mathbf{B}$ **c** $2\mathbf{A} + \mathbf{B}$ **d** $3\mathbf{A} - \mathbf{B}$

3 A builder builds a block of 12 identical flats. Each flat is to contain 1 table, 4 chairs, 2 beds, and 1 wardrobe.

Let $\mathbf{F} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 1 \end{pmatrix}$ be the matrix representing the furniture in one flat.

In terms of $\mathbf{F}$, what is the matrix representing the furniture in **all** flats? Evaluate this matrix.

4 On weekdays, a video store finds that its average daily hirings are 75 DVD movies, 27 Blu-ray movies, and 102 games. On weekends, the average daily hirings are 43 Blu-ray movies, 136 DVD movies, and 129 games.

a Represent the data using *two* column matrices **A** and **B**.

b Find $5\mathbf{A} + 2\mathbf{B}$.

c What does the matrix in **b** represent?

$$\begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix} \begin{array}{l} \leftarrow \text{DVD movies} \\ \leftarrow \text{Blu-ray movies} \\ \leftarrow \text{games} \end{array}$$

5 Isabelle sells clothing made by four different companies which we will call A, B, C, and D.
Her usual monthly order is:

$$\begin{array}{r c} & \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{array}{r} \text{skirt} \\ \text{dress} \\ \text{evening} \\ \text{suit} \end{array} & \begin{pmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{pmatrix} \end{array}$$

Find her order, to the nearest whole number, if:

a she increases her total order by 15%

b she decreases her total order by 15%.

ZERO OR NULL MATRIX

A **zero matrix** is a matrix in which all the elements are zero.

For example, the 2×2 zero matrix is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and the 2×3 zero matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

If **A** is a matrix of any order and **O** is the corresponding **zero matrix**, then
$\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$.

For example: $\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$.

NEGATIVE MATRICES

The **negative** matrix $\mathbf{A}$, denoted $-\mathbf{A}$, is actually $-1\mathbf{A}$.

$-\mathbf{A}$ is obtained from $\mathbf{A}$ by reversing the sign of each element of $\mathbf{A}$.

For example, if $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$, then $-\mathbf{A} = \begin{pmatrix} -1 \times 3 & -1 \times -1 \\ -1 \times 2 & -1 \times 4 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & -4 \end{pmatrix}$

The addition of a matrix and its negative always produces a zero matrix.

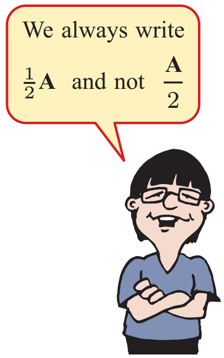
$$\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{O}$$

For example: $\begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 1 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

MATRIX ALGEBRA

We now compare our discoveries about matrices so far with ordinary algebra. We assume that $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same order.

Ordinary algebra	Matrix algebra
• If a and b are real numbers then $a+b$ is also a real number.	• If $\mathbf{A}$ and $\mathbf{B}$ are matrices then $\mathbf{A}+\mathbf{B}$ is a matrix of the same order.
• $a+b=b+a$	• $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
• $(a+b)+c=a+(b+c)$	• $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$
• $a+0=0+a=a$	• $\mathbf{A}+\mathbf{O}=\mathbf{O}+\mathbf{A}=\mathbf{A}$
• $a+(-a)=(-a)+a=0$	• $\mathbf{A}+(-\mathbf{A})=(-\mathbf{A})+\mathbf{A}=\mathbf{O}$
• a half of a is $\frac{a}{2}$	• a half of $\mathbf{A}$ is $\frac{1}{2}\mathbf{A}$



Example 5 — Self Tutor

Show that:

a if $\mathbf{X}+\mathbf{A}=\mathbf{B}$ then $\mathbf{X}=\mathbf{B}-\mathbf{A}$ **b** if $3\mathbf{X}=\mathbf{A}$ then $\mathbf{X}=\frac{1}{3}\mathbf{A}$

a
$$\begin{aligned} \mathbf{X}+\mathbf{A} &= \mathbf{B} \\ \therefore\ \mathbf{X}+\mathbf{A}+(-\mathbf{A}) &= \mathbf{B}+(-\mathbf{A}) \\ \therefore\ \mathbf{X}+\mathbf{O} &= \mathbf{B}-\mathbf{A} \\ \therefore\ \mathbf{X} &= \mathbf{B}-\mathbf{A} \end{aligned}$$

b
$$\begin{aligned} 3\mathbf{X} &= \mathbf{A} \\ \therefore\ \tfrac{1}{3}(3\mathbf{X}) &= \tfrac{1}{3}\mathbf{A} \\ \therefore\ 1\mathbf{X} &= \tfrac{1}{3}\mathbf{A} \\ \therefore\ \mathbf{X} &= \tfrac{1}{3}\mathbf{A} \end{aligned}$$

EXERCISE 12B.3

1 Simplify:

a $\mathbf{A}+2\mathbf{A}$ **b** $3\mathbf{B}-3\mathbf{B}$ **c** $\mathbf{C}-2\mathbf{C}$

d $-\mathbf{B}+\mathbf{B}$ **e** $2(\mathbf{A}+\mathbf{B})$ **f** $-(\mathbf{A}+\mathbf{B})$

g $-(2\mathbf{A}-\mathbf{C})$ **h** $3\mathbf{A}-(\mathbf{B}-\mathbf{A})$ **i** $\mathbf{A}+2\mathbf{B}-(\mathbf{A}-\mathbf{B})$

2 Find **X** in terms of **A**, **B**, and **C** if:

a $\mathbf{X}+\mathbf{B}=\mathbf{A}$ **b** $\mathbf{B}+\mathbf{X}=\mathbf{C}$ **c** $4\mathbf{B}+\mathbf{X}=2\mathbf{C}$

d $2\mathbf{X}=\mathbf{A}$ **e** $3\mathbf{X}=\mathbf{B}$ **f** $\mathbf{A}-\mathbf{X}=\mathbf{B}$

g $\frac{1}{2}\mathbf{X}=\mathbf{C}$ **h** $2(\mathbf{X}+\mathbf{A})=\mathbf{B}$ **i** $\mathbf{A}-4\mathbf{X}=\mathbf{C}$

3 **a** Suppose $\mathbf{M}=\begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix}$ and $\frac{1}{3}\mathbf{X}=\mathbf{M}$. Find **X**.

b Suppose $\mathbf{N}=\begin{pmatrix}2 & -1\\ 3 & 5\end{pmatrix}$ and $4\mathbf{X}=\mathbf{N}$. Find **X**.

c Suppose $\mathbf{A}=\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 4\\ -1 & 1\end{pmatrix}$, and $\mathbf{A}-2\mathbf{X}=3\mathbf{B}$. Find **X**.

C MATRIX MULTIPLICATION

Suppose you go to a shop and purchase 3 cans of soft drink, 4 chocolate bars, and 2 ice creams.

The prices are:

soft drink cans	chocolate bars	ice creams
\$1.30	\$0.90	\$1.20

We can represent this by the quantities matrix $\mathbf{A}=\begin{pmatrix}3 & 4 & 2\end{pmatrix}$ and the costs matrix $\mathbf{B}=\begin{pmatrix}1.30\\ 0.90\\ 1.20\end{pmatrix}$.

We can find the total cost of the items by multiplying the number of each item by its respective cost, and then adding the results:

$$3\times \$1.30+4\times \$0.90+2\times \$1.20=\$9.90$$

We can also determine the total cost by the **matrix multiplication**:

$$\begin{aligned}\mathbf{AB}&=\begin{pmatrix}3 & 4 & 2\end{pmatrix}\begin{pmatrix}1.30\\ 0.90\\ 1.20\end{pmatrix}\\ &=(3\times 1.30)+(4\times 0.90)+(2\times 1.20)\\ &=9.90\end{aligned}$$

Notice that we write the **row matrix** first and the **column matrix** second.

In general,

$$\begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}p\\ q\\ r\end{pmatrix}=ap+bq+cr.$$

EXERCISE 12C.1

1 Determine:

a $\begin{pmatrix}3 & -1\end{pmatrix}\begin{pmatrix}5\\ 4\end{pmatrix}$ **b** $\begin{pmatrix}1 & 3 & 2\end{pmatrix}\begin{pmatrix}5\\ 1\\ 7\end{pmatrix}$ **c** $\begin{pmatrix}6 & -1 & 2 & 3\end{pmatrix}\begin{pmatrix}1\\ 0\\ -1\\ 4\end{pmatrix}$

2 **a** Show that the sum of w, x, y, and z is given by $\begin{pmatrix} w & x & y & z \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

b Represent the *average* of w, x, y, and z in a similar way.

3 Lucy buys 4 shirts, 3 skirts, and 2 blouses costing \$27, \$35, and \$39 each respectively.

a Write down a quantities matrix **Q** and a price matrix **P**.

b Show how to use **P** and **Q** to determine the total cost of Lucy's clothes.

4 In the interschool public speaking competition, a first place is awarded 10 points, second place 6 points, third place 3 points, and fourth place 1 point. One school won 3 first places, 2 seconds, 4 thirds, and 2 fourths.

a Write down this information in terms of a points matrix **P** and a numbers matrix **N**.

b Show how to use **P** and **N** to find the total number of points awarded to the school.

MORE COMPLICATED MULTIPLICATIONS

Consider again **Example 1** on page **308** where Lisa needed 2 loaves of bread, 3 litres of milk, and 1 tub of butter.

We represented this by the quantities matrix $\mathbf{Q} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$.

The prices for each store were summarised in the costs matrix $\mathbf{C} = \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix}$.

To find the *total cost* of the items in each store, Lisa needs to multiply the number of items by their respective cost.

In Store A, a loaf of bread is \$2.65, a litre of milk is \$1.55, and a tub of butter is \$2.35, so the total cost is $2 \times \$2.65 + 3 \times \$1.55 + 1 \times \$2.35 = \$12.30$.

In Store B, a loaf of bread is \$2.25, a litre of milk is \$1.50, and a tub of butter is \$2.20, so the total cost is $2 \times \$2.25 + 3 \times \$1.50 + 1 \times \$2.20 = \$11.20$.

To do this using matrices notice that:

row **Q** × column 1

row **Q** × column 2

$$\mathbf{QC} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \times \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix} = \begin{pmatrix} 12.30 & 11.20 \end{pmatrix}$$

1×3 ← the same → 3×2 $\qquad$ 1×2

resultant matrix

Now suppose Lisa's friend Olu needs 1 loaf of bread, 2 litres of milk, and 2 tubs of butter.

The quantities matrix for both Lisa and Olu would be $\begin{array}{c} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \begin{array}{l} \leftarrow \text{Lisa} \\ \leftarrow \text{Olu} \end{array} \\ \begin{array}{ccc} \text{bread} & \text{milk} & \text{butter} \end{array} \end{array}$

Lisa's *total cost* at Store A is \$12.30, and at store B is \$11.20 .

Olu's *total cost* at Store A is $1 \times \$2.65 + 2 \times \$1.55 + 2 \times \$2.35 = \$10.45$,
and at Store B is $1 \times \$2.25 + 2 \times \$1.50 + 2 \times \$2.20 = \$9.65$.

So, using matrices we require that

row 1 × column 1 row 1 × column 2

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \times \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix} = \begin{pmatrix} 12.30 & 11.20 \\ 10.45 & 9.65 \end{pmatrix}$$

row 2 × column 1 row 2 × column 2

2×3 ← the same → 3×2 2×2

resultant matrix

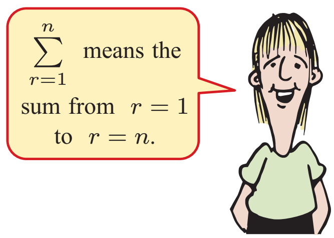
Having observed the usefulness of multiplying matrices in the contextual examples above, we now define matrix multiplication more formally.

The **product** of an $m \times n$ matrix **A** with an $n \times p$ matrix **B**, is the $m \times p$ matrix **AB** in which the element in the rth row and cth column is the sum of the products of the elements in the rth row of **A** with the corresponding elements in the cth column of **B**.

If $\mathbf{C} = \mathbf{AB}$ then $c_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}$

for each pair i and j with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant p$.

Note that the product **AB** exists *only* if the number of columns of **A** equals the number of rows of **B**.



For example:

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then $\mathbf{AB} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$.

If $\mathbf{C} = \underbrace{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}}_{2 \times 3}$ and $\mathbf{D} = \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{3 \times 1}$, then $\mathbf{CD} = \underbrace{\begin{pmatrix} ax + by + cz \\ dx + ey + fz \end{pmatrix}}_{2 \times 1}$.

To get the matrix **AB** you multiply **rows by columns**. To get the element in the 5th row and 3rd column of **AB** (if it exists), multiply the 5th row of **A** by the 3rd column of **B**.

Example 6 — Self Tutor

For $\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{pmatrix}$, find: **a** $\mathbf{AC}$ **b** $\mathbf{BC}$

a $\mathbf{A}$ is 1×3 and $\mathbf{C}$ is 3×2 $\therefore$ $\mathbf{AC}$ is 1×2

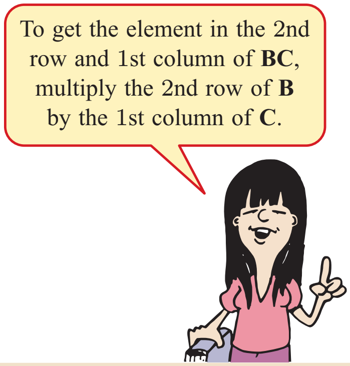
✓

$$\begin{aligned} \mathbf{AC} &= \begin{pmatrix} 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 1 + 3 \times 2 + 5 \times 1 & 1 \times 0 + 3 \times 3 + 5 \times 4 \end{pmatrix} \\ &= \begin{pmatrix} 12 & 29 \end{pmatrix} \end{aligned}$$

b $\mathbf{B}$ is 2×3 and $\mathbf{C}$ is 3×2 $\therefore$ $\mathbf{BC}$ is 2×2

✓

$$\begin{aligned} \mathbf{BC} &= \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 1 + 3 \times 2 + 5 \times 1 & 1 \times 0 + 3 \times 3 + 5 \times 4 \\ 2 \times 1 + 1 \times 2 + 3 \times 1 & 2 \times 0 + 1 \times 3 + 3 \times 4 \end{pmatrix} \\ &= \begin{pmatrix} 12 & 29 \\ 7 & 15 \end{pmatrix} \end{aligned}$$



EXERCISE 12C.2

1 Explain why $\mathbf{AB}$ cannot be found for $\mathbf{A} = \begin{pmatrix} 4 & 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

2 Suppose $\mathbf{A}$ is $2 \times n$ and $\mathbf{B}$ is $m \times 3$.

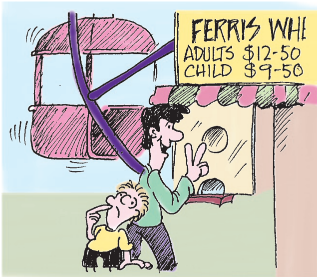
- **a** When can we find $\mathbf{AB}$?
- **b** If $\mathbf{AB}$ can be found, what is its order?
- **c** Explain why $\mathbf{BA}$ cannot be found.

3 For $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 6 \end{pmatrix}$, find: **a** $\mathbf{AB}$ **b** $\mathbf{BA}$

4 For $\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$, find: **a** $\mathbf{AB}$ **b** $\mathbf{BA}$

5 Find: **a** $\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

6 Answer the **Opening Problem** on page **306**.

7 

At a fair, tickets for the Ferris wheel are \$12.50 per adult and \$9.50 per child. On the first day of the fair, 2375 adults and 5156 children ride this wheel. On the second day, 2502 adults and 3612 children ride the wheel.

a Write the costs as a 2×1 matrix $\mathbf{C}$, and the numbers as a 2×2 matrix $\mathbf{N}$.

b Find $\mathbf{NC}$ and interpret the resulting matrix.

c Find the total income for the two days.

8 You and your friend each go to your local hardware stores A and B to price items you wish to purchase. You want to buy 1 hammer, 1 screwdriver, and 2 cans of white paint. Your friend wants 1 hammer, 2 screwdrivers, and 3 cans of white paint. The prices of these goods are:

	Hammer	*Screwdriver*	*Can of paint*
Store A	\$7	\$3	\$19
Store B	\$6	\$2	\$22

a Write the requirements matrix $\mathbf{R}$ as a 3×2 matrix.

b Write the prices matrix $\mathbf{P}$ as a 2×3 matrix.

c Find $\mathbf{PR}$.

d Find:

i your costs at store A **ii** your friend's costs at store B.

e Do any of the elements of $\mathbf{PR}$ tell you and your friend the cheapest way to buy all your items? Explain your answer.

PROPERTIES OF MATRIX MULTIPLICATION

Discovery 1 — Matrix multiplication

In this Discovery we find the properties of 2×2 matrix multiplication which are like those of ordinary number multiplication, and those which are not.

What to do:

1 For ordinary arithmetic $2 \times 3 = 3 \times 2$, and in algebra $ab = ba$.
For matrices, does $\mathbf{AB}$ always equal $\mathbf{BA}$?

Hint: Try $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix}$.

2 If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, find $\mathbf{AO}$ and $\mathbf{OA}$.

3 Find $\mathbf{AB}$ for:

a $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$

4 For all real numbers a, b, and c, we have the **distributive law** $a(b+c) = ab + ac$.

a Use any three 2×2 matrices $\mathbf{A}$ $\mathbf{B}$ and $\mathbf{C}$ to verify that $\mathbf{A(B+C)} = \mathbf{AB} + \mathbf{AC}$

b Now let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

Prove that in general, $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$.

c Use the matrices you 'made up' in **a** to verify that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

d Prove that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

5 **a** If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $w = z = 1$ and $x = y = 0$ is a solution for any values of a, b, c, and d.

b For any real number a, we know that $a \times 1 = 1 \times a = a$.
Is there a matrix $\mathbf{I}$ such that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$?

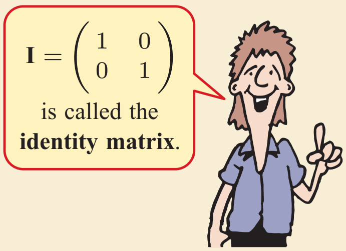
6 Suppose $\mathbf{A}^2 = \mathbf{AA} = \mathbf{A} \times \mathbf{A}$ and that $\mathbf{A}^3 = \mathbf{AAA}$.

a Find $\mathbf{A}^2$ if $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$.

b Find $\mathbf{A}^3$ if $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix}$.

c If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ try to find $\mathbf{A}^2$.

d Under what conditions can we square a matrix?



7 Show that if $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\mathbf{I}^2 = \mathbf{I}$ and $\mathbf{I}^3 = \mathbf{I}$.

In the **Discovery** you should have found that:

Ordinary algebra	**Matrix algebra**
• If a and b are real numbers then so is ab. {closure}	• If $\mathbf{A}$ and $\mathbf{B}$ are matrices that can be multiplied then $\mathbf{AB}$ is also a matrix. {closure}
• $ab = ba$ for all a, b {commutative}	• In general $\mathbf{AB} \neq \mathbf{BA}$. {non-commutative}
• $a0 = 0a = 0$ for all a	• If $\mathbf{O}$ is a zero matrix then $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$ for all $\mathbf{A}$.
• $ab = 0 \Leftrightarrow a = 0$ or $b = 0$ {Null Factor law}	• $\mathbf{AB}$ may be $\mathbf{O}$ without requiring $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$.
• $a(b+c) = ab+ac$ {distributive law}	• $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$ {distributive law}
• $a \times 1 = 1 \times a = a$ {identity law}	• If $\mathbf{I}$ is the **identity matrix** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$. {identity law}
• a^n exists for all $a \geqslant 0$ and $n \in \mathbb{R}$.	• $\mathbf{A}^n$ exists provided $\mathbf{A}$ is square and $n \in \mathbb{Z}^+$.

Note that in general, $\mathbf{A}(k\mathbf{B}) = k(\mathbf{AB}) \neq k\mathbf{BA}$. We can change the order in which we multiply by a scalar, but we cannot reverse the order in which we multiply matrices.

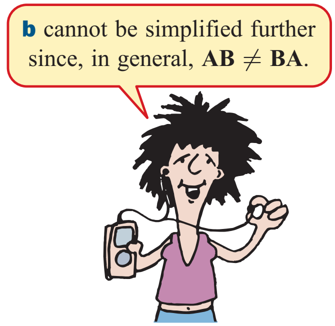
Example 7 — Self Tutor

Expand and simplify where possible:

a $(\mathbf{A}+2\mathbf{I})^2$ **b** $(\mathbf{A}-\mathbf{B})^2$

a
$$\begin{aligned}
&(\mathbf{A}+2\mathbf{I})^2 \\
&= (\mathbf{A}+2\mathbf{I})(\mathbf{A}+2\mathbf{I}) && \{\mathbf{X}^2=\mathbf{XX} \text{ by definition}\} \\
&= (\mathbf{A}+2\mathbf{I})\mathbf{A}+(\mathbf{A}+2\mathbf{I})2\mathbf{I} && \{\mathbf{B}(\mathbf{C}+\mathbf{D})=\mathbf{BC}+\mathbf{BD}\} \\
&= \mathbf{A}^2+2\mathbf{IA}+2\mathbf{AI}+4\mathbf{I}^2 && \{(\mathbf{C}+\mathbf{D})\mathbf{B}=\mathbf{CB}+\mathbf{DB}\} \\
&= \mathbf{A}^2+2\mathbf{A}+2\mathbf{A}+4\mathbf{I} && \{\mathbf{AI}=\mathbf{IA}=\mathbf{A} \text{ and } \mathbf{I}^2=\mathbf{I}\} \\
&= \mathbf{A}^2+4\mathbf{A}+4\mathbf{I}
\end{aligned}$$

b
$$\begin{aligned}
&(\mathbf{A}-\mathbf{B})^2 \\
&= (\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B}) && \{\mathbf{X}^2=\mathbf{XX} \text{ by definition}\} \\
&= (\mathbf{A}-\mathbf{B})\mathbf{A}-(\mathbf{A}-\mathbf{B})\mathbf{B} && \{\mathbf{C}(\mathbf{D}-\mathbf{E})=\mathbf{CD}-\mathbf{CE}\} \\
&= \mathbf{A}^2-\mathbf{BA}-\mathbf{AB}+\mathbf{B}^2 && \{(\mathbf{D}-\mathbf{E})\mathbf{C}=\mathbf{DC}-\mathbf{EC}\}
\end{aligned}$$



Example 8 — Self Tutor

If $\mathbf{A}^2=2\mathbf{A}+3\mathbf{I}$, find $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A}+l\mathbf{I}$ where k and l are scalars.

$$\begin{aligned}
\mathbf{A}^3 &= \mathbf{A}\times\mathbf{A}^2 \\
&= \mathbf{A}(2\mathbf{A}+3\mathbf{I}) \\
&= 2\mathbf{A}^2+3\mathbf{AI} \\
&= 2(2\mathbf{A}+3\mathbf{I})+3\mathbf{AI} \\
&= 7\mathbf{A}+6\mathbf{I}
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}^4 &= \mathbf{A}\times\mathbf{A}^3 \\
&= \mathbf{A}(7\mathbf{A}+6\mathbf{I}) \\
&= 7\mathbf{A}^2+6\mathbf{AI} \\
&= 7(2\mathbf{A}+3\mathbf{I})+6\mathbf{A} \\
&= 20\mathbf{A}+21\mathbf{I}
\end{aligned}$$

EXERCISE 12C.3

1 Given that all matrices are 2×2 and $\mathbf{I}$ is the identity matrix, expand and simplify:

a $\mathbf{A}(\mathbf{A}+\mathbf{I})$ **b** $(\mathbf{B}+2\mathbf{I})\mathbf{B}$ **c** $\mathbf{A}(\mathbf{A}^2-2\mathbf{A}+\mathbf{I})$

d $\mathbf{A}(\mathbf{A}^2+\mathbf{A}-2\mathbf{I})$ **e** $(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D})$ **f** $(\mathbf{A}+\mathbf{B})^2$

g $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})$ **h** $(\mathbf{A}+\mathbf{I})^2$ **i** $(3\mathbf{I}-\mathbf{B})^2$

2 **a** If $\mathbf{A}^2=2\mathbf{A}-\mathbf{I}$, find $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A}+l\mathbf{I}$ where k and l are scalars.

b If $\mathbf{B}^2=2\mathbf{I}-\mathbf{B}$, find $\mathbf{B}^3$, $\mathbf{B}^4$, and $\mathbf{B}^5$ in linear form.

c If $\mathbf{C}^2=4\mathbf{C}-3\mathbf{I}$, find $\mathbf{C}^3$ and $\mathbf{C}^5$ in linear form.

3 **a** If $\mathbf{A}^2=\mathbf{I}$, simplify:

i $\mathbf{A}(\mathbf{A}+2\mathbf{I})$ **ii** $(\mathbf{A}-\mathbf{I})^2$ **iii** $\mathbf{A}(\mathbf{A}+3\mathbf{I})^2$

b If $\mathbf{A}^3=\mathbf{I}$, simplify $\mathbf{A}^2(\mathbf{A}+\mathbf{I})^2$.

c If $\mathbf{A}^2=\mathbf{O}$, simplify:

i $\mathbf{A}(2\mathbf{A}-3\mathbf{I})$ **ii** $\mathbf{A}(\mathbf{A}+2\mathbf{I})(\mathbf{A}-\mathbf{I})$ **iii** $\mathbf{A}(\mathbf{A}+\mathbf{I})^3$

4 **a** If $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, determine $\mathbf{A}^2$.

b Comment on the following argument for a 2×2 matrix $\mathbf{A}$ such that $\mathbf{A}^2 = \mathbf{A}$:

$$\begin{aligned} \mathbf{A}^2 &= \mathbf{A} \\ \therefore\ \mathbf{A}^2 - \mathbf{A} &= \mathbf{O} \\ \therefore\ \mathbf{A}(\mathbf{A} - \mathbf{I}) &= \mathbf{O} \\ \therefore\ \mathbf{A} = \mathbf{O} &\text{ or } \mathbf{A} - \mathbf{I} = \mathbf{O} \\ \therefore\ \mathbf{A} = \mathbf{O} &\text{ or } \mathbf{I} \end{aligned}$$

c Find *all* 2×2 matrices $\mathbf{A}$ for which $\mathbf{A}^2 = \mathbf{A}$. **Hint:** Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

5 Give *one* example which shows that "if $\mathbf{A}^2 = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$" is a *false* statement.

Example 9 — Self Tutor

For $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$.

Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = a\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{pmatrix} = \begin{pmatrix} a & 2a \\ 3a & 4a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$$

$$\therefore \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} = \begin{pmatrix} a+b & 2a \\ 3a & 4a+b \end{pmatrix}$$

Thus $a + b = 7$ and $2a = 10$

$\therefore$ $a = 5$ and $b = 2$

Checking for consistency:

$3a = 3(5) = 15$ ✓ $\quad$ $4a + b = 4(5) + (2) = 22$ ✓

6 Find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$, given:

a $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$ $\qquad$ **b** $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix}$

7 **a** For $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}$, find constants p and q such that $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$.

b Hence, write $\mathbf{A}^3$ in the linear form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

c Write $\mathbf{A}^4$ in linear form.

D THE INVERSE OF A 2×2 MATRIX

The real numbers 5 and $\frac{1}{5}$ are called **multiplicative inverses** because when they are multiplied together, the result is the multiplicative identity 1: $5\times\frac{1}{5}=\frac{1}{5}\times 5=1$

For the matrices $\begin{pmatrix}2&5\\1&3\end{pmatrix}$ and $\begin{pmatrix}3&-5\\-1&2\end{pmatrix}$, we notice that $\begin{pmatrix}2&5\\1&3\end{pmatrix}\begin{pmatrix}3&-5\\-1&2\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\mathbf{I}$

and $\begin{pmatrix}3&-5\\-1&2\end{pmatrix}\begin{pmatrix}2&5\\1&3\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\mathbf{I}$.

We say that $\begin{pmatrix}2&5\\1&3\end{pmatrix}$ and $\begin{pmatrix}3&-5\\-1&2\end{pmatrix}$ are *multiplicative inverses* of each other.

The **multiplicative inverse** of $\mathbf{A}$, denoted $\mathbf{A}^{-1}$, satisfies $\mathbf{A}\mathbf{A}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$.

To find the multiplicative inverse of a matrix $\mathbf{A}$, we need a matrix which, when multiplied by $\mathbf{A}$, gives the identity matrix $\mathbf{I}$.

We will now determine how to find the inverse of a matrix $\mathbf{A}$.

Suppose $\mathbf{A}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and $\mathbf{A}^{-1}=\begin{pmatrix}w&x\\y&z\end{pmatrix}$

$$\therefore\ \mathbf{A}\mathbf{A}^{-1}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}w&x\\y&z\end{pmatrix}=\mathbf{I}$$

$$\therefore\ \begin{pmatrix}aw+by&ax+bz\\cw+dy&cx+dz\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\therefore\ \begin{cases}aw+by=1 & \text{.... (1)}\\ cw+dy=0 & \text{.... (2)}\end{cases}\quad\text{and}\quad\begin{cases}ax+bz=0 & \text{.... (3)}\\ cx+dz=1 & \text{.... (4)}\end{cases}$$

Solving (1) and (2) simultaneously for w and y gives: $w=\dfrac{d}{ad-bc}$ and $y=\dfrac{-c}{ad-bc}$.

Solving (3) and (4) simultaneously for x and z gives: $x=\dfrac{-b}{ad-bc}$ and $z=\dfrac{a}{ad-bc}$.

So, if $\mathbf{A}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ where $ad-bc\neq 0$, then $\mathbf{A}^{-1}=\dfrac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$.

In this case
$$\begin{aligned}\mathbf{A}^{-1}\mathbf{A}&=\frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\\&=\frac{1}{ad-bc}\begin{pmatrix}ad-bc&bd-bd\\ac-ac&-bc+ad\end{pmatrix}\\&=\begin{pmatrix}1&0\\0&1\end{pmatrix}\\&=\mathbf{I}\quad\text{also,}\end{aligned}$$

so $\mathbf{A}^{-1}\mathbf{A}=\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}$

Just as the real number 0 does not have a multiplicative inverse, some matrices do not have a multiplicative inverse. This occurs when $\det \mathbf{A} = ad - bc = 0$.

For the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

- the value $ad - bc$ is called the **determinant** of matrix $\mathbf{A}$, denoted $\det \mathbf{A}$
- if $\det \mathbf{A} \neq 0$, then $\mathbf{A}$ is **invertible** or **non-singular**, and $\mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- if $\det \mathbf{A} = 0$, then $\mathbf{A}$ is **singular**, and $\mathbf{A}^{-1}$ does not exist.

Example 10

Self Tutor

Find, if it exists, the inverse matrix of:

a $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$

b $\mathbf{B} = \begin{pmatrix} 6 & 3 \\ -4 & -2 \end{pmatrix}$

a $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$

$\therefore \det \mathbf{A} = 5(4) - 6(3) = 2$

$\therefore \mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -6 \\ -3 & 5 \end{pmatrix}$

$= \begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix}$

b $\mathbf{B} = \begin{pmatrix} 6 & 3 \\ -4 & -2 \end{pmatrix}$

$\therefore \det \mathbf{B} = 6(-2) - 3(-4)$

$= -12 + 12$

$= 0$

$\therefore \mathbf{B}^{-1}$ does not exist.

EXERCISE 12D.1

1 **a** Find $\begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -6 \\ -2 & 5 \end{pmatrix}$, and hence find the inverse of $\begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$.

b Find $\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$, and hence find the inverse of $\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}$.

2 Find $\det \mathbf{A}$ for $\mathbf{A}$ equal to:

a $\begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$ **c** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3 Find $\det \mathbf{B}$ for $\mathbf{B}$ equal to:

a $\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ **c** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **d** $\begin{pmatrix} a & -a \\ 1 & a \end{pmatrix}$

4 For $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}$, find:

a $\det \mathbf{A}$ **b** $\det(-\mathbf{A})$ **c** $\det(2\mathbf{A})$

5 Prove that if $\mathbf{A}$ is any 2×2 matrix and k is a constant, then $\det(k\mathbf{A}) = k^2 \times \det \mathbf{A}$.

6 Suppose $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

a Find:

i $\det \mathbf{A}$ **ii** $\det \mathbf{B}$ **iii** $\mathbf{AB}$ **iv** $\det(\mathbf{AB})$

b Hence show that $\det(\mathbf{AB}) = \det \mathbf{A} \times \det \mathbf{B}$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

7 Suppose $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$.

a Find $\det \mathbf{A}$ and $\det \mathbf{B}$.

b Find:

i $\det(2\mathbf{A})$ **ii** $\det(-\mathbf{A})$ **iii** $\det(-3\mathbf{B})$ **iv** $\det(\mathbf{AB})$

8 Find, if it exists, the inverse matrix of:

a $\begin{pmatrix} 2 & 4 \\ -1 & 5 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

d $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **e** $\begin{pmatrix} 5 & 0 \\ -1 & 2 \end{pmatrix}$ **f** $\begin{pmatrix} 3 & 5 \\ -6 & -10 \end{pmatrix}$

g $\begin{pmatrix} -1 & 2 \\ 4 & 7 \end{pmatrix}$ **h** $\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$ **i** $\begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}$

Example 11 Self Tutor

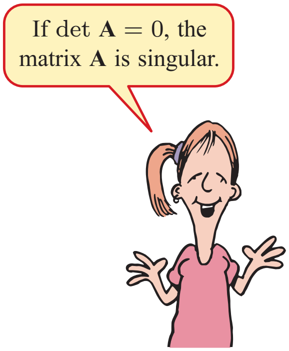
Suppose $\mathbf{A} = \begin{pmatrix} 4 & k \\ 2 & -1 \end{pmatrix}$.

Find $\mathbf{A}^{-1}$ and state the values of k for which $\mathbf{A}^{-1}$ exists.

$$\mathbf{A}^{-1} = \frac{1}{-4-2k}\begin{pmatrix} -1 & -k \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2k+4} & \frac{k}{2k+4} \\ \frac{2}{2k+4} & \frac{-4}{2k+4} \end{pmatrix}$$

$\mathbf{A}^{-1}$ exists provided that $2k + 4 \neq 0$

$\therefore \ k \neq -2$



9 For each of the following matrices $\mathbf{A}$, find $\mathbf{A}^{-1}$ and state the values of k for which $\mathbf{A}^{-1}$ exists.

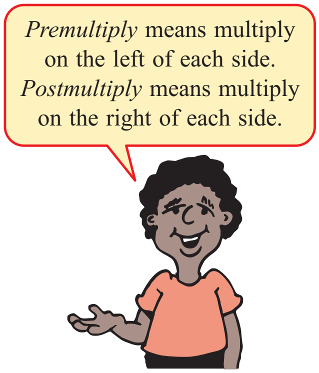
a $\mathbf{A} = \begin{pmatrix} k & 1 \\ -6 & 2 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 0 & k \end{pmatrix}$ **c** $\mathbf{A} = \begin{pmatrix} k+1 & 2 \\ 1 & k \end{pmatrix}$

d $\mathbf{A} = \begin{pmatrix} k-2 & k \\ -3 & k \end{pmatrix}$ **e** $\mathbf{A} = \begin{pmatrix} k^2 & k-1 \\ 2k & 1 \end{pmatrix}$ **f** $\mathbf{A} = \begin{pmatrix} k+1 & 2 \\ k^2+2 & 3k \end{pmatrix}$

FURTHER MATRIX ALGEBRA

In this section we consider matrix algebra with inverse matrices. Be careful that you use multiplication correctly. In particular, remember that:

- We can only perform matrix multiplication if the orders of the matrices allow it.
- If we *pre*multiply on one side then we must *pre*multiply on the other. This is important because, in general, $\mathbf{AB} \neq \mathbf{BA}$. The same applies if we *post*multiply.



Discovery 2 — Properties of inverse matrices

In this Discovery, we consider some properties of invertible 2×2 matrices.

What to do:

1 A matrix $\mathbf{A}$ is **self-inverse** when $\mathbf{A} = \mathbf{A}^{-1}$.

For example, if $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ then $\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \mathbf{A}$.

a Show that if $\mathbf{A} = \mathbf{A}^{-1}$, then $\mathbf{A}^2 = \mathbf{I}$.

b Show that there are exactly 4 self-inverse matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

2 **a** Given $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, find $\mathbf{A}^{-1}$ and $(\mathbf{A}^{-1})^{-1}$.

b If $\mathbf{A}$ is any invertible matrix, simplify $(\mathbf{A}^{-1})^{-1}(\mathbf{A}^{-1})$ and $(\mathbf{A}^{-1})(\mathbf{A}^{-1})^{-1}$ by replacing $\mathbf{A}^{-1}$ by $\mathbf{B}$.

c What can be deduced from **b**?

3 Suppose k is a non-zero number and $\mathbf{A}$ is an invertible matrix.

a Simplify $(k\mathbf{A})(\frac{1}{k}\mathbf{A}^{-1})$ and $(\frac{1}{k}\mathbf{A}^{-1})(k\mathbf{A})$.

b What can you conclude from your results?

4 **a** If $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}$, find in simplest form:

i $\mathbf{A}^{-1}$ **ii** $\mathbf{B}^{-1}$ **iii** $(\mathbf{AB})^{-1}$
iv $(\mathbf{BA})^{-1}$ **v** $\mathbf{A}^{-1}\mathbf{B}^{-1}$ **vi** $\mathbf{B}^{-1}\mathbf{A}^{-1}$

b Choose any two invertible matrices and repeat **a**.

c What do the results of **a** and **b** suggest?

d Simplify $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1})$ and $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB})$ given that $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist. What can you conclude from your results?

From the **Discovery** you should have found that if $\mathbf{A}$ and $\mathbf{B}$ are invertible, then:

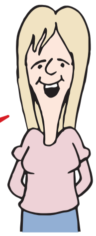
- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1}$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Example 12

Self Tutor

If $\mathbf{A}^2 = 2\mathbf{A} + 3\mathbf{I}$, find $\mathbf{A}^{-1}$ in the linear form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

$$\begin{aligned}
\mathbf{A}^2 &= 2\mathbf{A} + 3\mathbf{I} \\
\therefore\ \mathbf{A}^{-1}\mathbf{A}^2 &= \mathbf{A}^{-1}(2\mathbf{A} + 3\mathbf{I}) \qquad \{\text{premultiplying both sides by } \mathbf{A}^{-1}\} \\
\therefore\ \mathbf{A}^{-1}\mathbf{A}\mathbf{A} &= 2\mathbf{A}^{-1}\mathbf{A} + 3\mathbf{A}^{-1}\mathbf{I} \\
\therefore\ \mathbf{IA} &= 2\mathbf{I} + 3\mathbf{A}^{-1} \\
\therefore\ \mathbf{A} - 2\mathbf{I} &= 3\mathbf{A}^{-1} \\
\therefore\ \mathbf{A}^{-1} &= \tfrac{1}{3}(\mathbf{A} - 2\mathbf{I}) \\
\therefore\ \mathbf{A}^{-1} &= \tfrac{1}{3}\mathbf{A} - \tfrac{2}{3}\mathbf{I}
\end{aligned}$$



EXERCISE 12D.2

1 Suppose $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$, and $\mathbf{AXB} = \mathbf{C}$. Find $\mathbf{X}$.

2 Suppose $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are 2×1 matrices, and $\mathbf{A}$ and $\mathbf{B}$ are invertible 2×2 matrices.
If $\mathbf{X} = \mathbf{AY}$ and $\mathbf{Y} = \mathbf{BZ}$, write:

a $\mathbf{X}$ in terms of $\mathbf{Z}$

b $\mathbf{Z}$ in terms of $\mathbf{X}$.

3 If $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$, write $\mathbf{A}^2$ in the linear form $p\mathbf{A} + q\mathbf{I}$ where p and q are scalars.

Hence write $\mathbf{A}^{-1}$ in the form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

4 Write $\mathbf{A}^{-1}$ in linear form given that:

a $\mathbf{A}^2 = 4\mathbf{A} - \mathbf{I}$

b $5\mathbf{A} = \mathbf{I} - \mathbf{A}^2$

c $2\mathbf{I} = 3\mathbf{A}^2 - 4\mathbf{A}$

5 It is known that $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$ where the matrices $\mathbf{A}$ and $\mathbf{B}$ are not necessarily invertible.
Prove that $\mathbf{A}^2 = \mathbf{A}$.
Hint: From $\mathbf{AB} = \mathbf{A}$, you cannot deduce that $\mathbf{B} = \mathbf{I}$.

6 Under what condition is it true that "if $\mathbf{AB} = \mathbf{AC}$ then $\mathbf{B} = \mathbf{C}$"?

7 If $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{A}^3 = \mathbf{I}$, prove that $\mathbf{X}^3 = \mathbf{I}$.

8 If $a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{O}$ and $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$, prove that $a\mathbf{X}^2 + b\mathbf{X} + c\mathbf{I} = \mathbf{O}$.

E SIMULTANEOUS LINEAR EQUATIONS

We can solve $\begin{cases} 2x+3y=4 \\ 5x+4y=17 \end{cases}$ algebraically to get $x=5$, $y=-2$.

Notice that this system can be written as a matrix equation $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix}$.

The solution $x=5$, $y=-2$ is easily checked as

$$\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2(5)+3(-2) \\ 5(5)+4(-2) \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix} \quad \checkmark$$

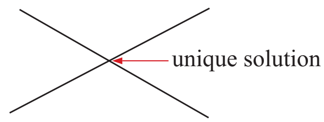
In general, a system of linear equations can be written in the form $\mathbf{AX}=\mathbf{B}$ where $\mathbf{A}$ is the matrix of coefficients, $\mathbf{X}$ is the unknown column matrix, and $\mathbf{B}$ is a column matrix of constants.

We can use inverses to solve the matrix equation $\mathbf{AX}=\mathbf{B}$ for $\mathbf{X}$.

If we premultiply each side of $\mathbf{AX}=\mathbf{B}$ by $\mathbf{A}^{-1}$, we get

$$\begin{aligned} \mathbf{A}^{-1}(\mathbf{AX}) &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\ (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\ \mathbf{IX} &= \mathbf{A}^{-1}\mathbf{B} \\ \text{and so}\quad \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \end{aligned}$$

If the matrix of coefficients $\mathbf{A}$ is invertible, then calculating $\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$ will give a unique solution to the pair of linear equations. This indicates that the lines intersect at a single point.



If the matrix of coefficients $\mathbf{A}$ is singular, then we cannot calculate $\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$. This indicates that either the lines are parallel and there are no solutions, or that the lines are coincident and there are infinitely many solutions.

parallel
(no solution)

coincident
(infinitely many solutions)

Example 13 — Self Tutor

a If $\mathbf{A}=\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$, find $\mathbf{A}^{-1}$.

b Write the system $\begin{cases} 2x+3y=4 \\ 5x+4y=17 \end{cases}$ in matrix form.

c Hence, solve the simultaneous linear equations.

a $\det \mathbf{A} = 2(4)-3(5)$
$= -7$

$\therefore\ \mathbf{A}^{-1} = \frac{1}{-7}\begin{pmatrix} 4 & -3 \\ -5 & 2 \end{pmatrix}$

b In matrix form, the system is $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix}$ which has the form $\mathbf{AX} = \mathbf{B}$.

c Premultiplying by $\mathbf{A}^{-1}$, $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$

$$\therefore\ \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

$$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{-7}\begin{pmatrix} 4 & -3 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 17 \end{pmatrix}$$

$$= \tfrac{1}{-7}\begin{pmatrix} -35 \\ 14 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$\therefore$ $x = 5$ and $y = -2$.

EXERCISE 12E

1 Convert into matrix equations:

a $\begin{cases} 3x - y = 8 \\ 2x + 3y = 6 \end{cases}$ **b** $\begin{cases} 4x - 3y = 11 \\ 3x + 2y = -5 \end{cases}$ **c** $\begin{cases} 3a - b = 6 \\ 2a + 7b = -4 \end{cases}$

2 Use matrix algebra to solve the system:

a $\begin{cases} 2x - y = 6 \\ x + 3y = 14 \end{cases}$ **b** $\begin{cases} 5x - 4y = 5 \\ 2x + 3y = -13 \end{cases}$ **c** $\begin{cases} x - 2y = 7 \\ 5x + 3y = -2 \end{cases}$

d $\begin{cases} 3x + 5y = 4 \\ 2x - y = 11 \end{cases}$ **e** $\begin{cases} 4x - 7y = 8 \\ 3x - 5y = 0 \end{cases}$ **f** $\begin{cases} 7x + 11y = 18 \\ 11x - 7y = -11 \end{cases}$

3 **a** Show that if $\mathbf{AX} = \mathbf{B}$ then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$, whereas if $\mathbf{XA} = \mathbf{B}$ then $\mathbf{X} = \mathbf{BA}^{-1}$.

b Find $\mathbf{X}$ if:

i $\begin{pmatrix} -6 & 5 \\ -3 & 4 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$ **ii** $\mathbf{X}\begin{pmatrix} 1 & 2 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 14 & -5 \\ 22 & 0 \end{pmatrix}$

iii $\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}$ **iv** $\mathbf{X}\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ -5 & 15 \end{pmatrix}$

4 **a** Consider the system $\begin{cases} 2x - 3y = 8 \\ 4x - y = 11 \end{cases}$.

i Write the equations in the form $\mathbf{AX} = \mathbf{B}$, and find $\det \mathbf{A}$.

ii Does the system have a unique solution? If so, find it.

b Consider the system $\begin{cases} 2x + ky = 8 \\ 4x - y = 11 \end{cases}$.

i Write the system in the form $\mathbf{AX} = \mathbf{B}$, and find $\det \mathbf{A}$.

ii For what value(s) of k does the system have a unique solution? Find the unique solution.

iii Find k when the system does not have a unique solution. How many solutions does the system have in this case?

Review set 12A

1 If $\mathbf{A}=\begin{pmatrix}3 & 2\\ 0 & -1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & 0\\ -2 & 4\end{pmatrix}$, find:

a $\mathbf{A}+\mathbf{B}$ **b** $3\mathbf{A}$ **c** $-2\mathbf{B}$ **d** $\mathbf{A}-\mathbf{B}$
e $\mathbf{B}-2\mathbf{A}$ **f** $3\mathbf{A}-2\mathbf{B}$ **g** $\mathbf{AB}$ **h** $\mathbf{BA}$
i $\mathbf{A}^{-1}$ **j** $\mathbf{A}^2$ **k** $\mathbf{ABA}$ **l** $(\mathbf{AB})^{-1}$

2 Find a, b, c, and d if:

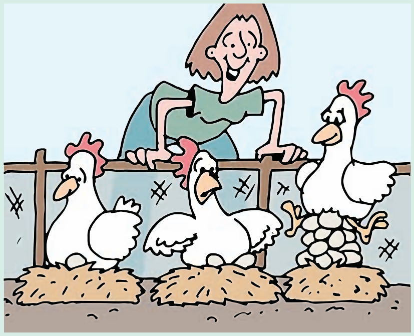
a $\begin{pmatrix}a & b-2\\ c & d\end{pmatrix}=\begin{pmatrix}-a & 3\\ 2-c & -4\end{pmatrix}$ **b** $\begin{pmatrix}3 & 2a\\ b & -2\end{pmatrix}+\begin{pmatrix}b & -a\\ c & d\end{pmatrix}=\begin{pmatrix}a & 2\\ 2 & 6\end{pmatrix}$

3 Write $\mathbf{Y}$ in terms of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$:

a $\mathbf{B}-\mathbf{Y}=\mathbf{A}$ **b** $2\mathbf{Y}+\mathbf{C}=\mathbf{A}$ **c** $\mathbf{AY}=\mathbf{B}$
d $\mathbf{YB}=\mathbf{C}$ **e** $\mathbf{C}-\mathbf{AY}=\mathbf{B}$ **f** $\mathbf{AY}^{-1}=\mathbf{B}$

4 Susan keeps 3 hens in a pen. She calls them Anya, Betsy, and Charise. Each week the hens lay eggs according to the matrix

$$\mathbf{L}=\begin{pmatrix}a\\ b\\ c\end{pmatrix}.$$



Write, in terms of $\mathbf{L}$, a matrix to describe:

a the eggs laid by the hens over a 4 week period
b the eggs each hen loses each fortnight when Susan collects the eggs.

5 Suppose $\mathbf{A}=\begin{pmatrix}-2 & 3\\ 4 & -1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-7 & 9\\ 9 & -3\end{pmatrix}$, and $\mathbf{C}=\begin{pmatrix}-1 & 0 & 3\\ 0 & 2 & 1\end{pmatrix}$.

Evaluate, if possible:

a $2\mathbf{A}-2\mathbf{B}$ **b** $\mathbf{AC}$ **c** $\mathbf{CB}$

6 Given that all matrices are 2×2 and $\mathbf{I}$ is the identity matrix, expand and simplify:

a $\mathbf{A}(\mathbf{I}-\mathbf{A})$ **b** $(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{A})$ **c** $(2\mathbf{A}-\mathbf{I})^2$

7 If $\mathbf{A}^2=5\mathbf{A}+2\mathbf{I}$, write $\mathbf{A}^3$ and $\mathbf{A}^4$ in the form $r\mathbf{A}+s\mathbf{I}$.

8 If $\mathbf{A}=\begin{pmatrix}2 & -1\\ 3 & 2\end{pmatrix}$, find constants a and b such that $\mathbf{A}^2=a\mathbf{A}+b\mathbf{I}$.

9 Find, if possible, the inverse matrix of:

a $\begin{pmatrix}6 & 8\\ 5 & 7\end{pmatrix}$ **b** $\begin{pmatrix}4 & -3\\ 8 & -6\end{pmatrix}$ **c** $\begin{pmatrix}11 & 5\\ -6 & -3\end{pmatrix}$

10 For what values of k does $\begin{cases} x+4y=2\\ kx+3y=-6\end{cases}$ have a unique solution?

11 Solve using an inverse matrix:

a $\begin{cases} 3x - 4y = 2 \\ 5x + 2y = -1 \end{cases}$ **b** $\begin{cases} 4x - y = 5 \\ 2x + 3y = 9 \end{cases}$

12 Suppose $\mathbf{A} = 2\mathbf{A}^{-1}$.

a Show that $\mathbf{A}^2 = 2\mathbf{I}$.

b Simplify $(\mathbf{A} - \mathbf{I})(\mathbf{A} + 3\mathbf{I})$, giving your answer in the form $r\mathbf{A} + s\mathbf{I}$ where r and s are real numbers.

Review set 12B

1 For $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{pmatrix}$, find:

a $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{Q} - \mathbf{P}$ **c** $\frac{3}{2}\mathbf{P} - \mathbf{Q}$

2 A library owns several copies of a popular trilogy of novels, according to the matrix:

$$\begin{array}{cc} & \begin{array}{cc} \text{paperback} & \text{hard cover} \end{array} \\ \mathbf{A} = & \begin{pmatrix} 4 & 2 \\ 5 & 2 \\ 6 & 3 \end{pmatrix} \begin{array}{l} \leftarrow \text{book 1} \\ \leftarrow \text{book 2} \\ \leftarrow \text{book 3} \end{array} \end{array}$$

a At present, the books on loan are described by the matrix $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 3 & 2 \end{pmatrix}$. Write a matrix to describe the books currently on the shelves.

b The values of the books (in dollars) are described by the matrix $\mathbf{C} = \begin{pmatrix} 7 & 7 & 8 \\ 15 & 16 & 20 \end{pmatrix}$.

i Which book has value \$16?

ii Find the total value of the books currently on loan.

3 Prove that for any square matrix $\mathbf{A}$, $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$.

4 Write $\mathbf{X}$ in terms of $\mathbf{A}$ and $\mathbf{B}$ if:

a $2\mathbf{X} = \mathbf{B} - \mathbf{A}$ **b** $3(\mathbf{A} + \mathbf{X}) = 2\mathbf{B}$ **c** $\mathbf{B} - 4\mathbf{X} = \mathbf{A}$

5 Suppose $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix}$, and $\mathbf{A} + 2\mathbf{X} = -\mathbf{B}$. Find $\mathbf{X}$.

6 If $\mathbf{A}$ is $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and $\mathbf{B}$ is $\begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{pmatrix}$, find, if possible:

a $2\mathbf{B}$ **b** $\frac{1}{2}\mathbf{B}$ **c** $\mathbf{AB}$ **d** $\mathbf{BA}$

7 If $\mathbf{A}$ and $\mathbf{B}$ are square matrices, under what conditions are the following true?

a If $\mathbf{AB} = \mathbf{B}$ then $\mathbf{A} = \mathbf{I}$. **b** $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$

8 For $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$, find:

a $\det \mathbf{A}$ **b** $\det(-2\mathbf{A})$ **c** $\det(\mathbf{A}^2)$

9 Solve using an inverse matrix:

a $\begin{cases} x + y = 5 \\ x - 2y = 4 \end{cases}$ **b** $\begin{cases} 3x + 2y = 3 \\ 5x + 3y = 4 \end{cases}$

10 If $\mathbf{M} = \begin{pmatrix} k & 2 \\ 2 & k \end{pmatrix}\begin{pmatrix} k-1 & -2 \\ -3 & k \end{pmatrix}$ has an inverse $\mathbf{M}^{-1}$, what values can k have?

11 For what values of k does the system $\begin{cases} kx + 3y = -6 \\ x + (k+2)y = 2 \end{cases}$ have a unique solution?

State the solution in this case.

12 Write $5\mathbf{A}^2 - 6\mathbf{A} = 3\mathbf{I}$ in the form $\mathbf{AB} = \mathbf{I}$. Hence write $\mathbf{A}^{-1}$ in terms of $\mathbf{A}$ and $\mathbf{I}$.

13 Prove that for any 2×2 matrix $\mathbf{A}$, $\mathbf{A}^2$ can be written in the linear form $a\mathbf{A} + b\mathbf{I}$.

13

Introduction to differential calculus

Contents:

A Limits
B Rates of change
C The derivative function
D Differentiation from first principles
E Simple rules of differentiation
F The chain rule
G The product rule
H The quotient rule
I Derivatives of exponential functions
J Derivatives of logarithmic functions
K Derivatives of trigonometric functions
L Second derivatives

Opening problem

In a BASE jumping competition from the Petronas Towers in Kuala Lumpur, the altitude of a professional jumper in the first 3 seconds is given by $f(t) = 452 - 4.8t^2$ metres, where $0 \leqslant t \leqslant 3$ seconds.

Things to think about:

a What will a graph of the altitude of the jumper in the first 3 seconds look like?

b Does the jumper travel with constant speed?

c Can you find the speed of the jumper when:

i $t = 0$ seconds **ii** $t = 1$ second
iii $t = 2$ seconds **iv** $t = 3$ seconds?

Calculus is a major branch of mathematics which builds on algebra, trigonometry, and analytic geometry. It has widespread applications in science, engineering, and financial mathematics.

The study of calculus is divided into two fields, **differential calculus** and **integral calculus**. These fields are linked by the **Fundamental Theorem of Calculus** which we will study later in the course.

Historical note

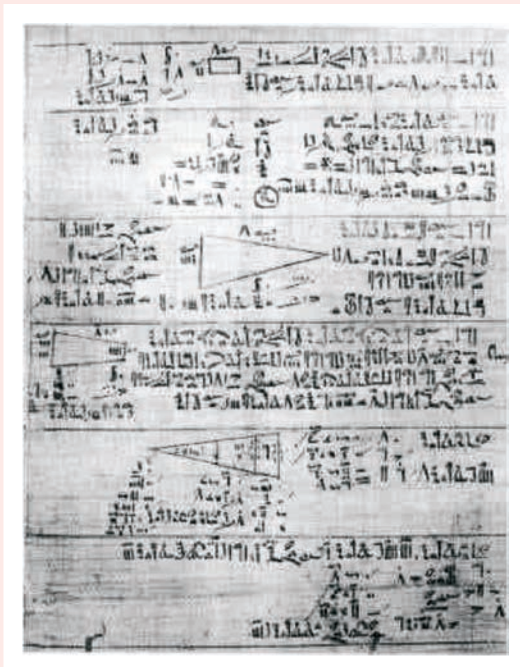
Calculus is a Latin word meaning 'pebble'. Ancient Romans used stones for counting.

The history of calculus begins with the **Egyptian Moscow papyrus** from about 1850 BC.

The Greek mathematicians **Democritus**, **Zeno of Elea**, **Antiphon**, and **Eudoxes** studied **infinitesimals**, dividing objects into an infinite number of pieces in order to calculate the area of regions, and volume of solids.

Archimedes of Syracuse was the first to find the tangent to a curve other than a circle. His methods were the foundation of modern calculus developed almost 2000 years later.

Archimedes



Egyptian Moscow papyrus

A LIMITS

The concept of a **limit** is essential to differential calculus. We will see that calculating limits is necessary for finding the gradient of a tangent to a curve at any point on the curve.

The table alongside shows values for $f(x) = x^2$ where x is less than 2, but increasing and getting closer and closer to 2.

x	1	1.9	1.99	1.999	1.9999
$f(x)$	1	3.61	3.9601	3.996 00	3.999 60

We say that as x approaches 2 from the left, $f(x)$ approaches 4 from below.

We can construct a similar table of values where x is greater than 2, but decreasing and getting closer and closer to 2:

x	3	2.1	2.01	2.001	2.0001
$f(x)$	9	4.41	4.0401	4.004 00	4.000 40

We say that as x approaches 2 from the right, $f(x)$ approaches 4 from above.

So, as x approaches 2 from either direction, $f(x)$ approaches a limit of 4. We write this as $\lim\limits_{x\to 2} x^2 = 4$.

INFORMAL DEFINITION OF A LIMIT

The following definition of a limit is informal but adequate for the purposes of this course:

If $f(x)$ can be made as close as we like to some real number A by making x sufficiently close to (but not equal to) a, then we say that $f(x)$ has a **limit** of A as x approaches a, and we write

$$\lim_{x\to a} f(x) = A.$$

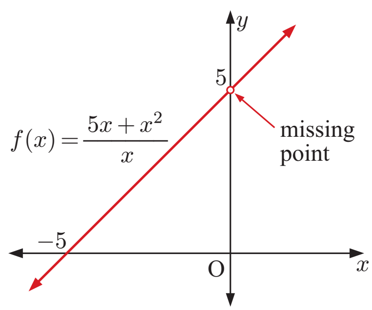
In this case, $f(x)$ is said to **converge** to A as x approaches a.

Notice that the limit is defined for x close to but *not equal to* a. Whether the function f is defined or not at $x = a$ is not important to the definition of the limit of f as x approaches a. What *is* important is the behaviour of the function as x gets *very close to* a.

For example, if $f(x) = \dfrac{5x + x^2}{x}$ and we wish to find the limit as $x \to 0$, it is tempting for us to simply substitute $x = 0$ into $f(x)$. However, in doing this, not only do we get the meaningless value of $\frac{0}{0}$, but also we destroy the basic limit method.

Observe that if $f(x) = \dfrac{5x + x^2}{x} = \dfrac{x(5+x)}{x}$ then $f(x) = \begin{cases} 5 + x & \text{if } x \neq 0 \\ \text{is undefined if } x = 0. \end{cases}$

The graph of $y = f(x)$ is shown alongside. It is the straight line $y = x + 5$ with the point $(0, 5)$ missing, called a **point of discontinuity** of the function.



However, even though this point is missing, the *limit* of $f(x)$ as x approaches 0 does exist. In particular, as $x \to 0$ from either direction, $f(x) \to 5$.

We write $\lim\limits_{x\to 0} \dfrac{5x + x^2}{x} = 5$ which reads:

"the limit as x approaches 0, of $f(x) = \dfrac{5x + x^2}{x}$, is 5".

In practice we do not need to graph functions each time to determine limits, and most can be found algebraically.

Example 1

Self Tutor

Evaluate: **a** $\lim\limits_{x \to 2} x^2$ **b** $\lim\limits_{x \to 0} \dfrac{x^2+3x}{x}$ **c** $\lim\limits_{x \to 3} \dfrac{x^2-9}{x-3}$

a x^2 can be made as close as we like to 4 by making x sufficiently close to 2.

$\therefore \lim\limits_{x \to 2} x^2 = 4$.

b
$$\begin{aligned}&\lim_{x \to 0} \frac{x^2+3x}{x}\\ &= \lim_{x \to 0} \frac{{}^{1}\cancel{x}(x+3)}{\cancel{x}_{1}}\\ &= \lim_{x \to 0} (x+3) \quad \text{since } x \neq 0\\ &= 3\end{aligned}$$

c
$$\begin{aligned}&\lim_{x \to 3} \frac{x^2-9}{x-3}\\ &= \lim_{x \to 3} \frac{(x+3)\cancel{(x-3)}^{1}}{\cancel{x-3}_{1}}\\ &= \lim_{x \to 3} (x+3) \quad \text{since } x \neq 3\\ &= 6\end{aligned}$$

EXERCISE 13A

1 Evaluate:

a $\lim\limits_{x \to 3} (x+4)$ **b** $\lim\limits_{x \to -1} (5-2x)$ **c** $\lim\limits_{x \to 4} (3x-1)$

d $\lim\limits_{x \to 2} (5x^2-3x+2)$ **e** $\lim\limits_{h \to 0} h^2(1-h)$ **f** $\lim\limits_{x \to 0} (x^2+5)$

2 Evaluate:

a $\lim\limits_{x \to 0} 5$ **b** $\lim\limits_{h \to 2} 7$ **c** $\lim\limits_{x \to 0} c$, c a constant

3 Evaluate:

a $\lim\limits_{x \to 1} \dfrac{x^2-3x}{x}$ **b** $\lim\limits_{h \to 2} \dfrac{h^2+5h}{h}$ **c** $\lim\limits_{x \to 0} \dfrac{x-1}{x+1}$ **d** $\lim\limits_{x \to 0} \dfrac{x}{x}$

4 Evaluate the following limits:

a $\lim\limits_{x \to 0} \dfrac{x^2-3x}{x}$ **b** $\lim\limits_{x \to 0} \dfrac{x^2+5x}{x}$ **c** $\lim\limits_{x \to 0} \dfrac{2x^2-x}{x}$

d $\lim\limits_{h \to 0} \dfrac{2h^2+6h}{h}$ **e** $\lim\limits_{h \to 0} \dfrac{3h^2-4h}{h}$ **f** $\lim\limits_{h \to 0} \dfrac{h^3-8h}{h}$

g $\lim\limits_{x \to 1} \dfrac{x^2-x}{x-1}$ **h** $\lim\limits_{x \to 2} \dfrac{x^2-2x}{x-2}$ **i** $\lim\limits_{x \to 3} \dfrac{x^2-x-6}{x-3}$

B RATES OF CHANGE

A **rate** is a comparison between two quantities with different units.

We often judge performances by rates. For example:

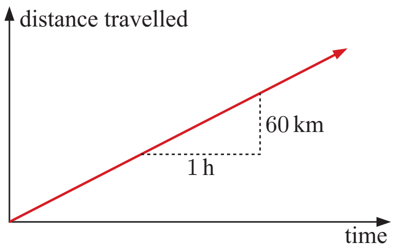
- Sir Donald Bradman's average batting rate at Test cricket level was 99.94 *runs per innings*.
- Michael Jordan's average basketball scoring rate was 20.0 *points per game*.
- Rangi's average typing rate is 63 *words per minute* with an error rate of 2.3 *errors per page*.

Speed is a commonly used rate. It is the rate of change in distance per unit of time.

We are familiar with the formula:

$$\textbf{average speed} = \frac{\textbf{distance travelled}}{\textbf{time taken}}$$

However, if a car has an average speed of 60 km h^{-1} for a journey, it does not mean that the car travels at exactly 60 km h^{-1} for the whole time.

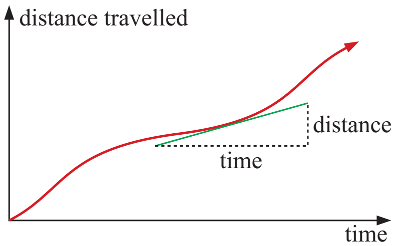


In fact, the speed will probably vary continuously throughout the journey.

So, how can we calculate the car's speed at any particular time?

Suppose we are given a graph of the car's distance travelled against time taken. If this graph is a straight line, then we know the speed is constant and is given by the *gradient* of the line.

If the graph is a curve, then the car's instantaneous speed is given by the *gradient of the tangent* to the curve at that time.



Historical note

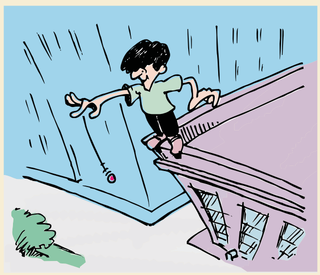
The modern study of **differential calculus** originated in the 17th century with the work of **Sir Isaac Newton** and **Gottfried Wilhelm Leibniz**. They developed the necessary theory while attempting to find algebraic methods for solving problems dealing with the **gradients of tangents** to curves, and finding the **rate of change** in one variable with respect to another.

Isaac Newton 1642 – 1727

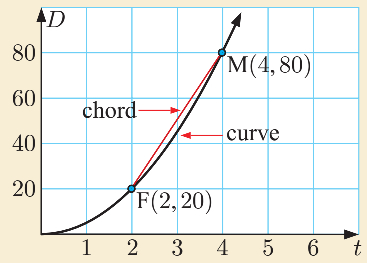
Gottfried Leibniz 1646 – 1716

Discovery 1 — *Instantaneous speed*

A ball bearing is dropped from the top of a tall building. The distance D it has fallen after t seconds is recorded, and the following graph of distance against time obtained.



We choose a fixed point F on the curve when $t = 2$ seconds. We then choose another point M on the curve, and draw in the line segment or **chord** FM between the two points. To start with, we let M be the point when $t = 4$ seconds.



The *average* speed in the time interval $2 \leqslant t \leqslant 4$

$$= \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{(80-20)\text{ m}}{(4-2)\text{ s}}$$

$$= \tfrac{60}{2}\text{ m s}^{-1}$$

$$= 30\text{ m s}^{-1}$$

In this Discovery we will try to measure the *instantaneous* speed of the ball when $t = 2$ seconds.

What to do:

DEMO

1 Click on the icon to start the demonstration.
F is the point where $t = 2$ seconds, and M is another point on the curve.
To start with, M is at $t = 4$ seconds.
The number in the box marked *gradient* is the gradient of the chord FM. This is the *average speed* of the ball bearing in the interval from F to M. For M at $t = 4$ seconds, you should see the average speed is 30 m s^{-1}.

2 Click on M and drag it slowly towards F. Copy and complete the table alongside with the gradient of the chord FM for M being the points on the curve at the given varying times t.

t	gradient of FM
3	
2.5	
2.1	
2.01	

3 Observe what happens as M reaches F. Explain why this is so.

4 Now move M to the origin, and then slide it towards F from the left. Copy and complete the table with the gradient of the chord FM for various times t.

t	gradient of FM
0	
1.5	
1.9	
1.99	

5 **a** What can you say about the gradient of FM in the limit as $t \to 2$?

b What is the instantaneous speed of the ball bearing when $t = 2$ seconds? Explain your answer.

THE TANGENT TO A CURVE

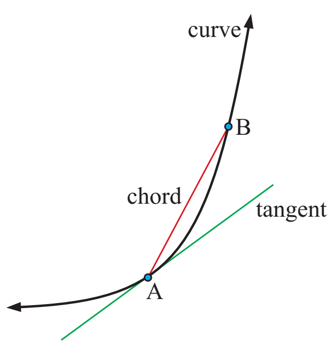
A **chord** of a curve is a straight line segment which joins any two points on the curve.

The gradient of the chord AB measures the average rate of change of the function values for the given change in x-values.

A **tangent** is a straight line which *touches* a curve at a single point. The tangent is the best approximating straight line to the curve through A.

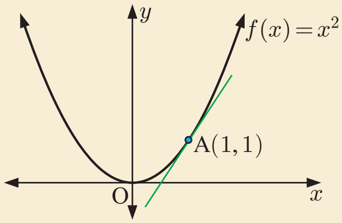
The gradient of the tangent at point A measures the instantaneous rate of change of the function at point A.

As B approaches A, the limit of the gradient of the chord AB will be the gradient of the tangent at A.



The **gradient of the tangent** to $y = f(x)$ at $x = a$ is the **instantaneous rate of change** in $f(x)$ with respect to x at that point.

Discovery 2 — The gradient of a tangent



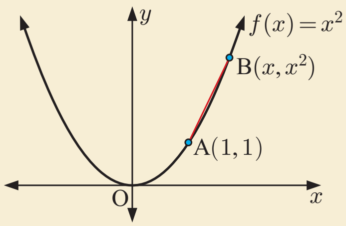
Given a curve $f(x)$, we wish to find the gradient of the tangent at the point $(a, f(a))$.

In this Discovery we find the gradient of the tangent to $f(x) = x^2$ at the point A(1, 1).

What to do:

1 Suppose B lies on $f(x) = x^2$, and B has coordinates (x, x^2).

a Show that the chord AB has gradient $\dfrac{x^2 - 1}{x - 1}$.



x	Point B	gradient of AB
5	(5, 25)	6
3		
2		
1.5		
1.1		
1.01		
1.001		

b Copy and complete the table shown.

c Comment on the gradient of AB as x gets closer to 1.

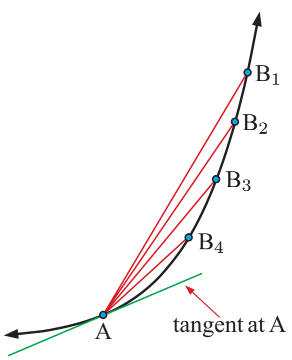
2 Repeat the process letting x get closer to 1, but from the left of A. Use the points where $x = 0$, 0.8, 0.9, 0.99, and 0.999 .

3 Click on the icon to view a demonstration of the process.

4 What do you suspect is the gradient of the tangent at A?

Fortunately we do not have to use a graph and table of values each time we wish to find the gradient of a tangent. Instead we can use an algebraic and geometric approach which involves **limits**.

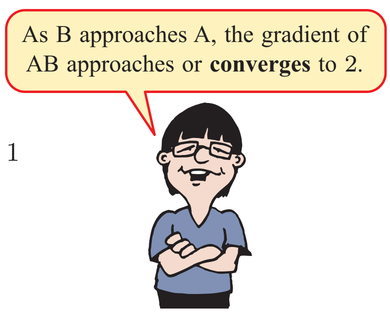
From **Discovery 2**, the gradient of AB $= \dfrac{x^2 - 1}{x - 1}$.



As B approaches A, $x \to 1$ and
the gradient of AB $\to$ the gradient of the tangent at A.

So, the gradient of the tangent at the point A is

$$\begin{aligned} m_T &= \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \to 1} \frac{(x+1)(x-1)}{x - 1} \\ &= \lim_{x \to 1} (x + 1) \quad \text{since } x \neq 1 \\ &= 2 \end{aligned}$$



EXERCISE 13B

1 Use the method in **Discovery 1** to answer the **Opening Problem** on page **334**.

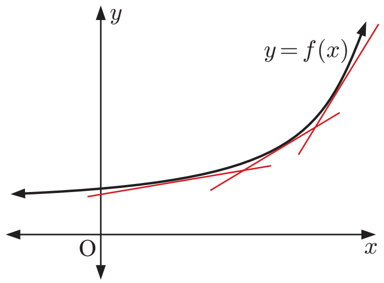
2 **a** Use the method in **Discovery 2** to find the gradient of the tangent to $y = x^2$ at the point $(2, 4)$.

b Evaluate $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$, and provide a geometric interpretation of this result.

C THE DERIVATIVE FUNCTION

For a non-linear function with equation $y = f(x)$, the gradients of the tangents at various points are different.

Our task is to determine a **gradient function** which gives the gradient of the tangent to $y = f(x)$ at $x = a$, for any point a in the domain of f.



The gradient function of $y = f(x)$ is called its **derivative function** and is labelled $f'(x)$.

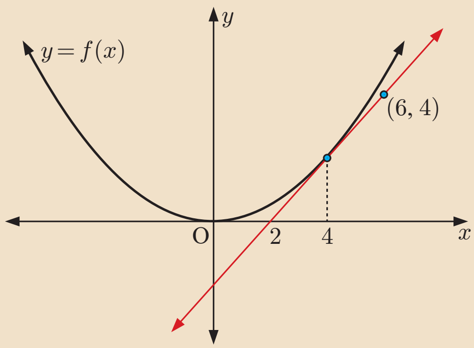
We read the derivative function as "eff dashed x".

The value of $f'(a)$ is the gradient of the tangent to $y = f(x)$ at the point where $x = a$.

Example 2

Self Tutor

For the given graph, find $f'(4)$ and $f(4)$.



The graph shows the tangent to the curve $y = f(x)$ at the point where $x = 4$.

The tangent passes through $(2, 0)$ and $(6, 4)$, so its gradient is $f'(4) = \dfrac{4 - 0}{6 - 2} = 1$.

The equation of the tangent is $y - 0 = 1(x - 2)$

$\therefore \quad y = x - 2$

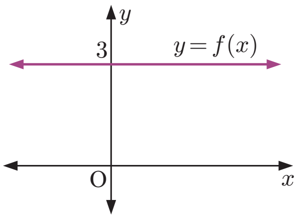
When $x = 4$, $y = 2$, so the point of contact between the tangent and the curve is $(4, 2)$.

$\therefore \quad f(4) = 2$

EXERCISE 13C

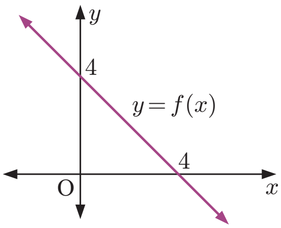
1 Using the graph below, find:

a $f(2)$ **b** $f'(2)$

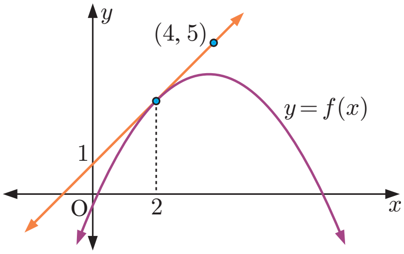


2 Using the graph below, find:

a $f(0)$ **b** $f'(0)$



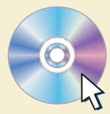
3 Consider the graph alongside. Find $f(2)$ and $f'(2)$.



Discovery 3 — Gradient functions

The software on the CD can be used to find the gradient of the tangent to a function $f(x)$ at any point. By sliding the point along the graph we can observe the changing gradient of the tangent. We can hence generate the gradient function $f'(x)$.

GRADIENT FUNCTIONS



What to do:

1 Consider the functions $f(x) = 0$, $f(x) = 2$, and $f(x) = 4$.

a For each of these functions, what is the gradient?

b Is the gradient constant for all values of x?

2 Consider the function $f(x) = mx + c$.

a State the gradient of the function. **b** Is the gradient constant for all values of x?

c Use the CD software to graph the following functions and observe the gradient function $f'(x)$. Hence verify that your answer in **b** is correct.

i $f(x) = x - 1$ **ii** $f(x) = 3x + 2$ **iii** $f(x) = -2x + 1$

3 **a** Observe the function $f(x) = x^2$ using the CD software. What *type* of function is the gradient function $f'(x)$?

b Observe the following quadratic functions using the CD software:

i $f(x) = x^2 + x - 2$ **ii** $f(x) = 2x^2 - 3$

iii $f(x) = -x^2 + 2x - 1$ **iv** $f(x) = -3x^2 - 3x + 6$

c What *type* of function is each of the gradient functions $f'(x)$ in **b**?

4 **a** Observe the function $f(x) = \ln x$ using the CD software.

b What *type* of function is the gradient function $f'(x)$?

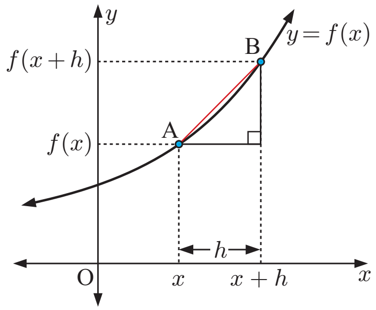
c What is the *domain* of the gradient function $f'(x)$?

5 **a** Observe the function $f(x) = e^x$ using the CD software.

b What is the gradient function $f'(x)$?

D DIFFERENTIATION FROM FIRST PRINCIPLES

Consider a general function $y = f(x)$ where A is the point $(x, f(x))$ and B is the point $(x+h, f(x+h))$.



The chord AB has gradient $= \dfrac{f(x+h) - f(x)}{x+h-x}$

$= \dfrac{f(x+h) - f(x)}{h}$

If we let B approach A, then the gradient of AB approaches the gradient of the tangent at A.

So, the gradient of the tangent at the variable point $(x, f(x))$ is $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

This formula gives the gradient of the tangent to the curve $y = f(x)$ at the point $(x, f(x))$ for any value of x for which this limit exists. Since there is at most one value of the gradient for each value of x, the formula is actually a function.

The **derivative function** or simply **derivative** of $y = f(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

When we evaluate this limit to find a derivative function, we say we are **differentiating from first principles**.

Example 3 — Self Tutor

Use the definition of $f'(x)$ to find the gradient function of $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}_1} \\ &= \lim_{h \to 0} (2x+h) \qquad \{\text{as } h \neq 0\} \\ &= 2x \end{aligned}$$

ALTERNATIVE NOTATION

If we are given a function $f(x)$ then $f'(x)$ represents the derivative function.

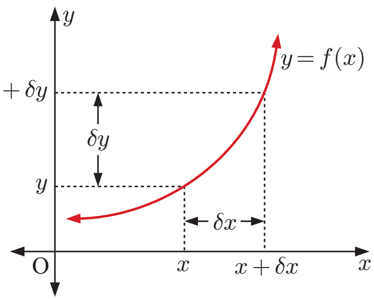
If we are given y in terms of x then y' or $\dfrac{dy}{dx}$ are commonly used to represent the derivative.

$\dfrac{dy}{dx}$ reads "dee y by dee x" or "the derivative of y with respect to x".

$\dfrac{dy}{dx}$ is **not a fraction**. However, the notation $\dfrac{dy}{dx}$ is a result of taking the limit of a fraction. If we replace h by δx and $f(x+h)-f(x)$ by δy, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{becomes}$$

$$\begin{aligned} f'(x) &= \lim_{\delta x\to 0} \frac{\delta y}{\delta x} \\ &= \frac{dy}{dx}. \end{aligned}$$



THE DERIVATIVE WHEN $x = a$

The gradient of the tangent to $y = f(x)$ at the point where $x = a$ is denoted $f'(a)$, where

$$\boldsymbol{f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}$$

Example 4 — Self Tutor

Use the first principles formula $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ to find the instantaneous rate of change in $f(x) = x^2 + 2x$ at the point where $x = 5$.

$$\begin{aligned} & f(5) = 5^2 + 2(5) = 35 \\ \therefore\ & f'(5) = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h} \\ & \qquad = \lim_{h\to 0} \frac{(5+h)^2 + 2(5+h) - 35}{h} \\ \therefore\ & f'(5) = \lim_{h\to 0} \frac{\cancel{25} + 10h + h^2 + \cancel{10} + 2h - \cancel{35}}{h} \\ & \qquad = \lim_{h\to 0} \frac{h^2 + 12h}{h} \\ & \qquad = \lim_{h\to 0} \frac{\cancel{h}(h+12)}{\cancel{h}_1} \qquad \{\text{as } h \neq 0\} \\ & \qquad = 12 \end{aligned}$$

$\therefore$ the instantaneous rate of change in $f(x)$ at $x = 5$ is 12.

EXERCISE 13D

1 Find, from first principles, the gradient function of:

a $f(x) = x$ **b** $f(x) = 5$ **c** $f(x) = 2x + 5$

2 Find $\frac{dy}{dx}$ from first principles given:

a $y = 4 - x$ **b** $y = x^2 - 3x$ **c** $y = 2x^2 + x - 1$

3 Use the first principles formula $f'(a) = \lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$ to find the gradient of the tangent to:

a $f(x) = 3x + 5$ at $x = -2$ **b** $f(x) = 5 - 2x^2$ at $x = 3$

c $f(x) = x^2 + 3x - 4$ at $x = 3$ **d** $f(x) = 5 - 2x - 3x^2$ at $x = -2$

E SIMPLE RULES OF DIFFERENTIATION

Differentiation is the process of finding a derivative or gradient function.

Given a function $f(x)$, we obtain $f'(x)$ by **differentiating with respect to** the variable x.

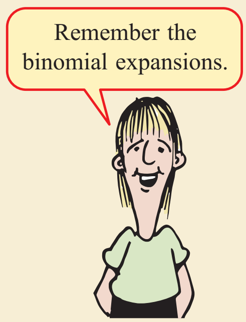
There are a number of rules associated with differentiation. These rules can be used to differentiate more complicated functions without having to use first principles.

Discovery 4 — Simple rules of differentiation

In this Discovery we attempt to differentiate functions of the form x^n, cx^n where c is a constant, and functions which are a sum or difference of polynomial terms of the form cx^n.

What to do:

1 Differentiate from first principles: **a** x^2 **b** x^3 **c** x^4



2 Consider the binomial expansion:

$$(x+h)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}h + \binom{n}{2} x^{n-2}h^2 + ... + \binom{n}{n} h^n$$
$$= x^n + nx^{n-1}h + \binom{n}{2} x^{n-2}h^2 + ... + h^n$$

Use the first principles formula $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

to find the derivative of $f(x) = x^n$ for $x \in \mathbb{Z}^+$.

3 **a** Find, from first principles, the derivatives of: **i** $4x^2$ **ii** $2x^3$

b By comparison with **1**, copy and complete: "If $f(x) = cx^n$, then $f'(x) = ...$"

4 **a** Use first principles to find $f'(x)$ for:

i $f(x) = x^2 + 3x$ **ii** $f(x) = x^3 - 2x^2$

b Copy and complete: "If $f(x) = u(x) + v(x)$ then $f'(x) = ...$"

The rules you found in the **Discovery** are much more general than the cases you just considered.

For example, if $f(x) = x^n$ then $f'(x) = nx^{n-1}$ is true not just for all $n \in \mathbb{Z}^+$, but actually for all $n \in \mathbb{R}$.

We can summarise the following rules:

$f(x)$	$f'(x)$	*Name of rule*
c (a constant)	0	**differentiating a constant**
x^n	nx^{n-1}	**differentiating x^n**
$c\,u(x)$	$c\,u'(x)$	**constant times a function**
$u(x)+v(x)$	$u'(x)+v'(x)$	**addition rule**

The last two rules can be proved using the first principles definition of $f'(x)$.

- If $f(x) = c\,u(x)$ where c is a constant, then $f'(x) = c\,u'(x)$.

Proof:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{c\,u(x+h)-c\,u(x)}{h} \\ &= \lim_{h\to 0} c\left[\frac{u(x+h)-u(x)}{h}\right] \\ &= c \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} \\ &= c\,u'(x) \end{aligned}$$

- If $f(x) = u(x) + v(x)$ then $f'(x) = u'(x) + v'(x)$

Proof:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{u(x+h)+v(x+h)-[u(x)+v(x)]}{h}\right) \\ &= \lim_{h\to 0} \left(\frac{u(x+h)-u(x)+v(x+h)-v(x)}{h}\right) \\ &= \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} + \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} \\ &= u'(x)+v'(x) \end{aligned}$$

Using the rules we have now developed we can differentiate sums of powers of x.

For example, if $f(x) = 3x^4 + 2x^3 - 5x^2 + 7x + 6$ then

$$\begin{aligned} f'(x) &= 3(4x^3) + 2(3x^2) - 5(2x) + 7(1) + 0 \\ &= 12x^3 + 6x^2 - 10x + 7 \end{aligned}$$

Example 5 — Self Tutor

If $y = 3x^2 - 4x$, find $\dfrac{dy}{dx}$ and interpret its meaning.

As $y = 3x^2 - 4x$, $\dfrac{dy}{dx} = 6x - 4$.

$\dfrac{dy}{dx}$ is:
- the gradient function or derivative of $y = 3x^2 - 4x$ from which the gradient of the tangent at any point on the curve can be found
- the instantaneous rate of change of y with respect to x.

Example 6 — Self Tutor

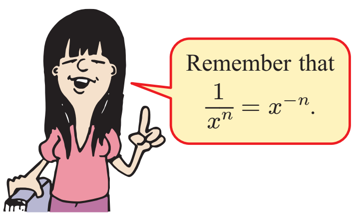
Find $f'(x)$ for $f(x)$ equal to:

a $5x^3 + 6x^2 - 3x + 2$

b $7x - \dfrac{4}{x} + \dfrac{3}{x^3}$

a
$$\begin{aligned} f(x) &= 5x^3 + 6x^2 - 3x + 2 \\ \therefore\ f'(x) &= 5(3x^2) + 6(2x) - 3(1) \\ &= 15x^2 + 12x - 3 \end{aligned}$$

b
$$\begin{aligned} f(x) &= 7x - \frac{4}{x} + \frac{3}{x^3} \\ &= 7x - 4x^{-1} + 3x^{-3} \\ \therefore\ f'(x) &= 7(1) - 4(-1x^{-2}) + 3(-3x^{-4}) \\ &= 7 + 4x^{-2} - 9x^{-4} \\ &= 7 + \frac{4}{x^2} - \frac{9}{x^4} \end{aligned}$$



Example 7 — Self Tutor

Find the gradient function of $y = x^2 - \dfrac{4}{x}$ and hence find the gradient of the tangent to the function at the point where $x = 2$.

$$\begin{aligned} y &= x^2 - \frac{4}{x} \\ &= x^2 - 4x^{-1} \end{aligned}$$

$$\begin{aligned} \therefore\ \frac{dy}{dx} &= 2x - 4(-1x^{-2}) \\ &= 2x + 4x^{-2} \\ &= 2x + \frac{4}{x^2} \end{aligned}$$

When $x = 2$, $\dfrac{dy}{dx} = 4 + 1 = 5$.

So, the tangent has gradient 5.

Example 8 **Self Tutor**

Find the gradient function for each of the following:

a $f(x) = 3\sqrt{x} + \frac{2}{x}$ **b** $g(x) = x^2 - \frac{4}{\sqrt{x}}$

a
$$\begin{aligned} f(x) &= 3\sqrt{x} + \frac{2}{x} \\ &= 3x^{\frac{1}{2}} + 2x^{-1} \\ \therefore \quad f'(x) &= 3(\tfrac{1}{2}x^{-\frac{1}{2}}) + 2(-1x^{-2}) \\ &= \tfrac{3}{2}x^{-\frac{1}{2}} - 2x^{-2} \\ &= \frac{3}{2\sqrt{x}} - \frac{2}{x^2} \end{aligned}$$

b
$$\begin{aligned} g(x) &= x^2 - \frac{4}{\sqrt{x}} \\ &= x^2 - 4x^{-\frac{1}{2}} \\ \therefore \quad g'(x) &= 2x - 4(-\tfrac{1}{2}x^{-\frac{3}{2}}) \\ &= 2x + 2x^{-\frac{3}{2}} \\ &= 2x + \frac{2}{x\sqrt{x}} \end{aligned}$$

EXERCISE 13E

1 Find $f'(x)$ given that $f(x)$ is:

a x^3 **b** $2x^3$ **c** $7x^2$ **d** $6\sqrt{x}$
e $3\sqrt[3]{x}$ **f** $x^2 + x$ **g** $4 - 2x^2$ **h** $x^2 + 3x - 5$
i $\frac{1}{2}x^4 - 6x^2$ **j** $\frac{3x-6}{x}$ **k** $\frac{2x-3}{x^2}$ **l** $\frac{x^3+5}{x}$
m $\frac{x^3+x-3}{x}$ **n** $\frac{1}{\sqrt{x}}$ **o** $(2x-1)^2$ **p** $(x+2)^3$

2 Find $\frac{dy}{dx}$ for:

a $y = 2.5x^3 - 1.4x^2 - 1.3$ **b** $y = \pi x^2$ **c** $y = \frac{1}{5x^2}$
d $y = 100x$ **e** $y = 10(x+1)$ **f** $y = 4\pi x^3$

3 Differentiate with respect to x:

a $6x + 2$ **b** $x\sqrt{x}$ **c** $(5-x)^2$ **d** $\frac{6x^2 - 9x^4}{3x}$
e $(x+1)(x-2)$ **f** $\frac{1}{x^2} + 6\sqrt{x}$ **g** $4x - \frac{1}{4x}$ **h** $x(x+1)(2x-5)$

4 Find the gradient of the tangent to:

a $y = x^2$ at $x = 2$ **b** $y = \frac{8}{x^2}$ at the point $(9, \frac{8}{81})$
c $y = 2x^2 - 3x + 7$ at $x = -1$ **d** $y = \frac{2x^2-5}{x}$ at the point $(2, \frac{3}{2})$
e $y = \frac{x^2-4}{x^2}$ at the point $(4, \frac{3}{4})$ **f** $y = \frac{x^3 - 4x - 8}{x^2}$ at $x = -1$.

5 Suppose $f(x) = x^2 + (b+1)x + 2c$, $f(2) = 4$, and $f'(-1) = 2$.
Find the constants b and c.

6 Find the gradient function of:

a $f(x) = 4\sqrt{x} + x$ **b** $f(x) = \sqrt[3]{x}$ **c** $f(x) = -\dfrac{2}{\sqrt{x}}$

d $f(x) = 2x - \sqrt{x}$ **e** $f(x) = \dfrac{4}{\sqrt{x}} - 5$ **f** $f(x) = 3x^2 - x\sqrt{x}$

g $f(x) = \dfrac{5}{x^2\sqrt{x}}$ **h** $f(x) = 2x - \dfrac{3}{x\sqrt{x}}$

7 **a** If $y = 4x - \dfrac{3}{x}$, find $\dfrac{dy}{dx}$ and interpret its meaning.

b The position of a car moving along a straight road is given by $S = 2t^2 + 4t$ metres where t is the time in seconds. Find $\dfrac{dS}{dt}$ and interpret its meaning.

c The cost of producing x toasters each week is given by $C = 1785 + 3x + 0.002x^2$ dollars. Find $\dfrac{dC}{dx}$ and interpret its meaning.

F THE CHAIN RULE

In **Chapter 2** we defined the **composite** of two functions g and f as $(g \circ f)(x)$ or $gf(x)$.

We can often write complicated functions as the composite of two or more simpler functions.

For example $y = (x^2 + 3x)^4$ could be rewritten as $y = u^4$ where $u = x^2 + 3x$, or as $y = gf(x)$ where $g(x) = x^4$ and $f(x) = x^2 + 3x$.

Example 9 Self Tutor

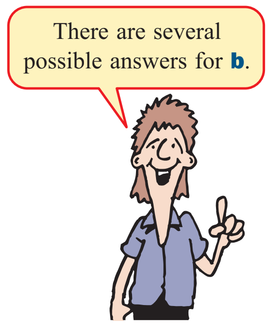
Find: **a** $gf(x)$ if $g(x) = \sqrt{x}$ and $f(x) = 2 - 3x$

b $g(x)$ and $f(x)$ such that $gf(x) = \dfrac{1}{x - x^2}$.

a $gf(x)$
$= g(2 - 3x)$
$= \sqrt{2 - 3x}$

b $gf(x) = \dfrac{1}{x - x^2} = \dfrac{1}{f(x)}$

$\therefore$ $g(x) = \dfrac{1}{x}$ and $f(x) = x - x^2$



EXERCISE 13F.1

1 Find $gf(x)$ if:

a $g(x) = x^2$ and $f(x) = 2x + 7$ **b** $g(x) = 2x + 7$ and $f(x) = x^2$

c $g(x) = \sqrt{x}$ and $f(x) = 3 - 4x$ **d** $g(x) = 3 - 4x$ and $f(x) = \sqrt{x}$

e $g(x) = \dfrac{2}{x}$ and $f(x) = x^2 + 3$ **f** $g(x) = x^2 + 3$ and $f(x) = \dfrac{2}{x}$

2 Find $g(x)$ and $f(x)$ such that $gf(x)$ is:

a $(3x + 10)^3$ **b** $\dfrac{1}{2x + 4}$ **c** $\sqrt{x^2 - 3x}$ **d** $\dfrac{10}{(3x - x^2)^3}$

DERIVATIVES OF COMPOSITE FUNCTIONS

The reason we are interested in writing complicated functions as composite functions is to make finding derivatives easier.

Discovery 5 — Differentiating composite functions

The purpose of this Discovery is to learn how to differentiate composite functions.

Based on the rule "if $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$", we might suspect that if $y = (2x+1)^2$ then $\frac{dy}{dx} = 2(2x+1)^1$. But is this so?

What to do:

1 Expand $y = (2x+1)^2$ and hence find $\frac{dy}{dx}$. How does this compare with $2(2x+1)^1$?

2 Expand $y = (3x+1)^2$ and hence find $\frac{dy}{dx}$. How does this compare with $2(3x+1)^1$?

3 Expand $y = (ax+1)^2$ where a is a constant, and hence find $\frac{dy}{dx}$. How does this compare with $2(ax+1)^1$?

4 Suppose $y = u^2$.

- **a** Find $\frac{dy}{du}$.
- **b** Now suppose $u = ax + 1$, so $y = (ax+1)^2$.
  - **i** Find $\frac{du}{dx}$.
  - **ii** Write $\frac{dy}{du}$ from **a** in terms of x.
  - **iii** Hence find $\frac{dy}{du} \times \frac{du}{dx}$.
  - **iv** Compare your answer to the result in **3**.
- **c** If $y = u^2$ where u is a function of x, what do you suspect $\frac{dy}{dx}$ will be equal to?

5 Expand $y = (x^2 + 3x)^2$ and hence find $\frac{dy}{dx}$.

Does your answer agree with the rule you suggested in **4 c**?

6 Consider $y = (2x+1)^3$.

- **a** Expand the brackets and hence find $\frac{dy}{dx}$.
- **b** If we let $u = 2x + 1$, then $y = u^3$.
  - **i** Find $\frac{du}{dx}$.
  - **ii** Find $\frac{dy}{du}$, and write it in terms of x.
  - **iii** Hence find $\frac{dy}{du} \times \frac{du}{dx}$.
  - **iv** Compare your answer to the result in **a**.

7 Copy and complete: "If y is a function of u, and u is a function of x, then $\frac{dy}{dx} = ...$"

THE CHAIN RULE

If $y = g(u)$ where $u = f(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

This rule is extremely important and enables us to differentiate complicated functions much faster.

For example, for any function $f(x)$:

If $y = [f(x)]^n$ then $\dfrac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$.

Example 10

Self Tutor

Find $\dfrac{dy}{dx}$ if:

a $y = (x^2 - 2x)^4$

b $y = \dfrac{4}{\sqrt{1-2x}}$

a
$$y = (x^2 - 2x)^4$$
$$\therefore\ y = u^4 \quad \text{where} \quad u = x^2 - 2x$$
$$\text{Now } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\}$$
$$= 4u^3(2x-2)$$
$$= 4(x^2-2x)^3(2x-2)$$

b
$$y = \frac{4}{\sqrt{1-2x}}$$
$$\therefore\ y = 4u^{-\frac{1}{2}} \quad \text{where} \quad u = 1 - 2x$$
$$\text{Now } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\}$$
$$= 4 \times (-\tfrac{1}{2}u^{-\frac{3}{2}}) \times (-2)$$
$$= 4u^{-\frac{3}{2}}$$
$$= 4(1-2x)^{-\frac{3}{2}}$$

EXERCISE 13F.2

1 Write in the form au^n, clearly stating what u is:

a $\dfrac{1}{(2x-1)^2}$ **b** $\sqrt{x^2-3x}$ **c** $\dfrac{2}{\sqrt{2-x^2}}$

d $\sqrt[3]{x^3-x^2}$ **e** $\dfrac{4}{(3-x)^3}$ **f** $\dfrac{10}{x^2-3}$

2 Find the gradient function $\dfrac{dy}{dx}$ for:

a $y = (4x-5)^2$ **b** $y = \dfrac{1}{5-2x}$ **c** $y = \sqrt{3x-x^2}$

d $y = (1-3x)^4$ **e** $y = 6(5-x)^3$ **f** $y = \sqrt[3]{2x^3-x^2}$

g $y = \dfrac{6}{(5x-4)^2}$ **h** $y = \dfrac{4}{3x-x^2}$ **i** $y = 2\left(x^2 - \dfrac{2}{x}\right)^3$

3 Find the gradient of the tangent to:

a $y = \sqrt{1-x^2}$ at $x = \frac{1}{2}$

b $y = (3x+2)^6$ at $x = -1$

c $y = \dfrac{1}{(2x-1)^4}$ at $x = 1$

d $y = 6 \times \sqrt[3]{1-2x}$ at $x = 0$

e $y = \dfrac{4}{x+2\sqrt{x}}$ at $x = 4$

f $y = \left(x + \dfrac{1}{x}\right)^3$ at $x = 1$.

4 The gradient function of $f(x) = (2x-b)^a$ is $f'(x) = 24x^2 - 24x + 6$.
Find the constants a and b.

5 Suppose $y = \dfrac{a}{\sqrt{1+bx}}$ where a and b are constants. When $x = 3$, $y = 1$ and $\dfrac{dy}{dx} = -\frac{1}{8}$.
Find a and b.

6 If $y = x^3$ then $x = y^{\frac{1}{3}}$.

a Find $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$, and hence show that $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$.

b Explain why $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$ whenever these derivatives exist for any general function $y = f(x)$.

G THE PRODUCT RULE

We have seen the addition rule:

$$\text{If} \quad f(x) = u(x) + v(x) \quad \text{then} \quad f'(x) = u'(x) + v'(x).$$

We now consider the case $f(x) = u(x)\,v(x)$. Is $f'(x) = u'(x)\,v'(x)$?

In other words, does the derivative of a product of two functions equal the product of the derivatives of the two functions?

Discovery 6 — The product rule

Suppose $u(x)$ and $v(x)$ are two functions of x, and that $f(x) = u(x)\,v(x)$ is the product of these functions.

The purpose of this Discovery is to find a rule for determining $f'(x)$.

What to do:

1 Suppose $u(x) = x$ and $v(x) = x$, so $f(x) = x^2$.

a Find $f'(x)$ by direct differentiation.

b Find $u'(x)$ and $v'(x)$.

c Does $f'(x) = u'(x)\,v'(x)$?

2 Suppose $u(x) = x$ and $v(x) = \sqrt{x}$, so $f(x) = x\sqrt{x} = x^{\frac{3}{2}}$.

a Find $f'(x)$ by direct differentiation.

b Find $u'(x)$ and $v'(x)$.

c Does $f'(x) = u'(x)\,v'(x)$?

3 Copy and complete the following table, finding $f'(x)$ by direct differentiation.

$f(x)$	$f'(x)$	$u(x)$	$v(x)$	$u'(x)$	$v'(x)$	$u'(x)\,v(x)+u(x)\,v'(x)$
x^2		x	x			
$x^{\frac{3}{2}}$		x	$\sqrt{x}$			
$x(x+1)$		x	$x+1$			
$(x-1)(2-x^2)$		$x-1$	$2-x^2$			

4 Copy and complete: "If $f(x)=u(x)\,v(x)$ then $f'(x)=$ ......"

THE PRODUCT RULE

If $\boldsymbol{f(x)=u(x)\,v(x)}$ then $\boldsymbol{f'(x)=u'(x)\,v(x)+u(x)\,v'(x)}$.

Alternatively, if $\boldsymbol{y=uv}$ where u and v are functions of x, then $\boldsymbol{\dfrac{dy}{dx}=\dfrac{du}{dx}v+u\dfrac{dv}{dx}}$.

Example 11 — Self Tutor

Find $\dfrac{dy}{dx}$ if:

a $y=\sqrt{x}(2x+1)^3$ **b** $y=x^2(x^2-2x)^4$

a $y=\sqrt{x}(2x+1)^3$ is the product of $u=x^{\frac{1}{2}}$ and $v=(2x+1)^3$

$\therefore\quad u'=\frac{1}{2}x^{-\frac{1}{2}}$ and $v'=3(2x+1)^2\times 2$ {chain rule}

$\qquad\qquad\qquad\qquad\qquad\quad =6(2x+1)^2$

Now $\dfrac{dy}{dx}=u'v+uv'$ {product rule}

$$=\tfrac{1}{2}x^{-\frac{1}{2}}(2x+1)^3+x^{\frac{1}{2}}\times 6(2x+1)^2$$

$$=\tfrac{1}{2}x^{-\frac{1}{2}}(2x+1)^3+6x^{\frac{1}{2}}(2x+1)^2$$

b $y=x^2(x^2-2x)^4$ is the product of $u=x^2$ and $v=(x^2-2x)^4$

$\therefore\quad u'=2x$ and $v'=4(x^2-2x)^3(2x-2)$ {chain rule}

Now $\dfrac{dy}{dx}=u'v+uv'$ {product rule}

$$=2x(x^2-2x)^4+x^2\times 4(x^2-2x)^3(2x-2)$$

$$=2x(x^2-2x)^4+4x^2(x^2-2x)^3(2x-2)$$

EXERCISE 13G

1 Use the product rule to differentiate:

a $f(x)=x(x-1)$ **b** $f(x)=2x(x+1)$ **c** $f(x)=x^2\sqrt{x+1}$

2 Find $\frac{dy}{dx}$ using the product rule:

a $y = x^2(2x-1)$ **b** $y = 4x(2x+1)^3$ **c** $y = x^2\sqrt{3-x}$

d $y = \sqrt{x}(x-3)^2$ **e** $y = 5x^2(3x^2-1)^2$ **f** $y = \sqrt{x}(x-x^2)^3$

3 Find the gradient of the tangent to:

a $y = x^4(1-2x)^2$ at $x=-1$ **b** $y = \sqrt{x}(x^2-x+1)^2$ at $x=4$

c $y = x\sqrt{1-2x}$ at $x=-4$ **d** $y = x^3\sqrt{5-x^2}$ at $x=1$.

4 Consider $y = \sqrt{x}(3-x)^2$.

a Show that $\frac{dy}{dx} = \frac{(3-x)(3-5x)}{2\sqrt{x}}$.

b Find the x-coordinates of all points on $y = \sqrt{x}(3-x)^2$ where the tangent is horizontal.

c For what values of x is $\frac{dy}{dx}$ undefined?

5 Suppose $y = -2x^2(x+4)$. For what values of x does $\frac{dy}{dx} = 10$?

H THE QUOTIENT RULE

Expressions like $\frac{x^2+1}{2x-5}$, $\frac{\sqrt{x}}{1-3x}$, and $\frac{x^3}{(x-x^2)^4}$ are called **quotients** because they represent the division of one function by another.

Quotient functions have the form $Q(x) = \frac{u(x)}{v(x)}$.

$$\begin{aligned}
\text{Notice that}\quad u(x) &= Q(x)\,v(x)\\
\therefore\quad u'(x) &= Q'(x)\,v(x) + Q(x)\,v'(x) \qquad \{\text{product rule}\}\\
\therefore\quad u'(x) - Q(x)\,v'(x) &= Q'(x)\,v(x)\\
\therefore\quad Q'(x)\,v(x) &= u'(x) - \frac{u(x)}{v(x)}\,v'(x)\\
\therefore\quad Q'(x)\,v(x) &= \frac{u'(x)\,v(x) - u(x)\,v'(x)}{v(x)}\\
\therefore\quad Q'(x) &= \frac{u'(x)\,v(x) - u(x)\,v'(x)}{[v(x)]^2} \qquad \text{when this exists.}
\end{aligned}$$

THE QUOTIENT RULE

If $Q(x) = \frac{u(x)}{v(x)}$ then $Q'(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{[v(x)]^2}$.

Alternatively, if $y = \frac{u}{v}$ where u and v are functions of x, then $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$.

Example 12

Self Tutor

Use the quotient rule to find $\dfrac{dy}{dx}$ if:

a $y = \dfrac{1+3x}{x^2+1}$

b $y = \dfrac{\sqrt{x}}{(1-2x)^2}$

a $y = \dfrac{1+3x}{x^2+1}$ is a quotient with $u = 1+3x$ and $v = x^2+1$

$\therefore$ $u' = 3$ and $v' = 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{3(x^2+1) - (1+3x)2x}{(x^2+1)^2}$$

$$= \frac{3x^2 + 3 - 2x - 6x^2}{(x^2+1)^2}$$

$$= \frac{3 - 2x - 3x^2}{(x^2+1)^2}$$

b $y = \dfrac{\sqrt{x}}{(1-2x)^2}$ is a quotient with $u = x^{\frac{1}{2}}$ and $v = (1-2x)^2$

$\therefore$ $u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(1-2x)^1 \times (-2)$ {chain rule}

$= -4(1-2x)$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 - x^{\frac{1}{2}} \times (-4(1-2x))}{(1-2x)^4}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 + 4x^{\frac{1}{2}}(1-2x)}{(1-2x)^4}$$

$$= \frac{\cancel{(1-2x)}\left[\frac{1-2x}{2\sqrt{x}} + 4\sqrt{x}\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)\right]}{(1-2x)^{\cancel{4}3}} \quad \text{\{look for common factors\}}$$

$$= \frac{1 - 2x + 8x}{2\sqrt{x}(1-2x)^3}$$

$$= \frac{6x+1}{2\sqrt{x}(1-2x)^3}$$

Simplification of $\frac{dy}{dx}$ is often unnecessary, especially if you simply want the gradient of a tangent at a given point. In such cases, substitute a value for x without simplifying the derivative function first.

EXERCISE 13H

1 Use the quotient rule to find $\dfrac{dy}{dx}$ if:

a $y = \dfrac{1+3x}{2-x}$ **b** $y = \dfrac{x^2}{2x+1}$ **c** $y = \dfrac{x}{x^2-3}$

d $y = \dfrac{\sqrt{x}}{1-2x}$ **e** $y = \dfrac{x^2-3}{3x-x^2}$ **f** $y = \dfrac{x}{\sqrt{1-3x}}$

2 Find the gradient of the tangent to:

a $y = \dfrac{x}{1-2x}$ at $x = 1$ **b** $y = \dfrac{x^3}{x^2+1}$ at $x = -1$

c $y = \dfrac{\sqrt{x}}{2x+1}$ at $x = 4$ **d** $y = \dfrac{x^2}{\sqrt{x^2+5}}$ at $x = -2$.

3 **a** If $y = \dfrac{2\sqrt{x}}{1-x}$, show that $\dfrac{dy}{dx} = \dfrac{x+1}{\sqrt{x}(1-x)^2}$.

b For what values of x is $\dfrac{dy}{dx}$ **i** zero **ii** undefined?

4 **a** If $y = \dfrac{x^2-3x+1}{x+2}$, show that $\dfrac{dy}{dx} = \dfrac{x^2+4x-7}{(x+2)^2}$.

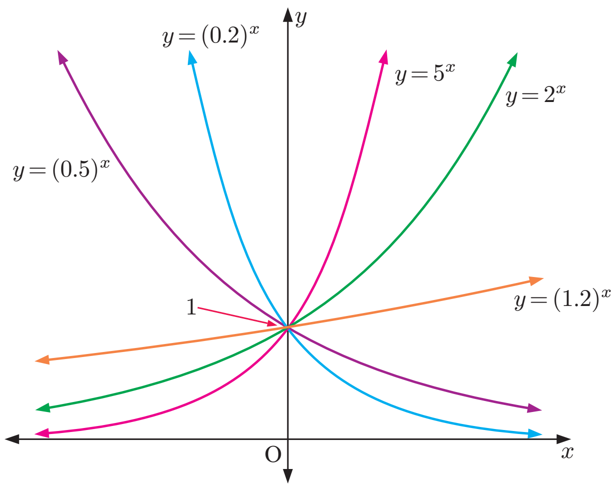
b For what values of x is $\dfrac{dy}{dx}$ **i** zero **ii** undefined?

I DERIVATIVES OF EXPONENTIAL FUNCTIONS

In **Chapter 4** we saw that the simplest **exponential functions** have the form $f(x) = b^x$ where b is any positive constant, $b \neq 1$.

The graphs of all members of the exponential family $f(x) = b^x$ have the following properties:

- pass through the point $(0, 1)$
- asymptotic to the x-axis at one end
- lie above the x-axis for all x.



Discovery 7 — The derivative of $y = b^x$

The purpose of this Discovery is to observe the nature of the derivatives of $f(x) = b^x$ for various values of b.

What to do:

CALCULUS DEMO

1 Use the software provided to help fill in the table for $y = 2^x$:

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0			
0.5			
1			
1.5			
2			

2 Repeat **1** for the following functions:

a $y = 3^x$ **b** $y = 5^x$ **c** $y = (0.5)^x$

3 Use your observations from **1** and **2** to write a statement about the derivative of the general exponential $y = b^x$ for $b > 0$, $b \neq 1$.

From the **Discovery** you should have found that:

If $f(x) = b^x$ then $f'(x) = f'(0) \times b^x$.

Proof:

If $f(x) = b^x$,

then $f'(x) = \lim_{h \to 0} \dfrac{b^{x+h} - b^x}{h}$ {first principles definition of the derivative}

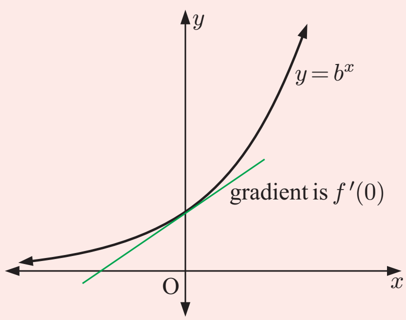
$= \lim_{h \to 0} \dfrac{b^x(b^h - 1)}{h}$

$= b^x \times \left(\lim_{h \to 0} \dfrac{b^h - 1}{h} \right)$ {as b^x is independent of h}

But $f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0} \dfrac{b^h - 1}{h}$

$\therefore f'(x) = b^x \times f'(0)$



Given this result, if we can find a value of b such that $f'(0) = 1$, then we will have found *a function which is its own derivative*!

We have already shown that if $f(x)=b^x$ then $f'(x)=b^x\left(\lim\limits_{h\to 0}\dfrac{b^h-1}{h}\right)$.

So if $f'(x)=b^x$ then we require $\lim\limits_{h\to 0}\dfrac{b^h-1}{h}=1$.

$$\therefore \lim_{h\to 0} b^h = \lim_{h\to 0}(1+h)$$

Letting $h=\dfrac{1}{n}$, we notice that $\dfrac{1}{n}\to 0$ if $n\to\infty$

$$\therefore \lim_{n\to\infty} b^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

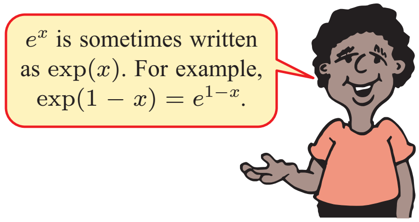
$$\therefore b = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad \text{if this limit exists}$$

We have in fact already seen this limit in **Chapter 4 Discovery 2** on page **123**.

We found that as $n\to\infty$,

$$\left(1+\frac{1}{n}\right)^n \to 2.718\,281\,828\,459\,045\,235\,....$$

and this irrational number is the natural exponential e.



We now have: **If $f(x)=e^x$ then $f'(x)=e^x$.**

THE DERIVATIVE OF $e^{f(x)}$

The functions e^{-x}, e^{2x+3}, and e^{-x^2} all have the form $e^{f(x)}$.

Since $e^x>0$ for all x, $e^{f(x)}>0$ for all x, no matter what the function $f(x)$.

Suppose $y=e^{f(x)}=e^u$ where $u=f(x)$.

$$\text{Now } \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}$$
$$=e^u\frac{du}{dx}$$
$$=e^{f(x)}\times f'(x)$$

Function	*Derivative*
e^x	e^x
$e^{f(x)}$	$e^{f(x)}\times f'(x)$

Example 13 — Self Tutor

Find the gradient function for y equal to:

a $2e^x+e^{-3x}$ **b** x^2e^{-x} **c** $\dfrac{e^{2x}}{x}$

a If $y=2e^x+e^{-3x}$ then $\dfrac{dy}{dx}=2e^x+e^{-3x}(-3)$

$$=2e^x-3e^{-3x}$$

b If $y=x^2e^{-x}$ then $\dfrac{dy}{dx}=2xe^{-x}+x^2e^{-x}(-1)$ {product rule}

$$=2xe^{-x}-x^2e^{-x}$$

c If $y = \dfrac{e^{2x}}{x}$ then $\dfrac{dy}{dx} = \dfrac{e^{2x}(2)x - e^{2x}(1)}{x^2}$ {quotient rule}

$= \dfrac{e^{2x}(2x-1)}{x^2}$

Example 14 — Self Tutor

Find the gradient function for y equal to: **a** $(e^x - 1)^3$ **b** $\dfrac{1}{\sqrt{2e^{-x}+1}}$

a
$$\begin{aligned} y &= (e^x-1)^3 \\ &= u^3 \quad \text{where} \quad u = e^x - 1 \\ \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\} \\ &= 3u^2\frac{du}{dx} \\ &= 3(e^x-1)^2 \times e^x \\ &= 3e^x(e^x-1)^2 \end{aligned}$$

b
$$\begin{aligned} y &= (2e^{-x}+1)^{-\frac{1}{2}} \\ &= u^{-\frac{1}{2}} \quad \text{where} \quad u = 2e^{-x}+1 \\ \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\} \\ &= -\tfrac{1}{2}u^{-\frac{3}{2}}\frac{du}{dx} \\ &= -\tfrac{1}{2}(2e^{-x}+1)^{-\frac{3}{2}} \times 2e^{-x}(-1) \\ &= e^{-x}(2e^{-x}+1)^{-\frac{3}{2}} \end{aligned}$$

EXERCISE 13I

1 Find the gradient function for $f(x)$ equal to:

a e^{4x} **b** $e^x + 3$ **c** $\exp(-2x)$ **d** $e^{\frac{x}{2}}$

e $2e^{-\frac{x}{2}}$ **f** $1 - 2e^{-x}$ **g** $4e^{\frac{x}{2}} - 3e^{-x}$ **h** $\dfrac{e^x + e^{-x}}{2}$

i e^{-x^2} **j** $e^{\frac{1}{x}}$ **k** $10(1+e^{2x})$ **l** $20(1-e^{-2x})$

m e^{2x+1} **n** $e^{\frac{x}{4}}$ **o** e^{1-2x^2} **p** $e^{-0.02x}$

2 Find the derivative of:

a xe^x **b** x^3e^{-x} **c** $\dfrac{e^x}{x}$ **d** $\dfrac{x}{e^x}$

e x^2e^{3x} **f** $\dfrac{e^x}{\sqrt{x}}$ **g** $\sqrt{x}e^{-x}$ **h** $\dfrac{e^x+2}{e^{-x}+1}$

3 Find the gradient of the tangent to:

a $y = (e^x+2)^4$ at $x = 0$ **b** $y = \dfrac{1}{2-e^{-x}}$ at $x = 0$

c $y = \sqrt{e^{2x}+10}$ at $x = \ln 3$.

4 Given $f(x) = e^{kx} + x$ and $f'(0) = -8$, find k.

5 **a** By substituting $e^{\ln 2}$ for 2 in $y = 2^x$, find $\dfrac{dy}{dx}$.

b Show that if $y = b^x$ where $b > 0$, $b \neq 1$, then $\dfrac{dy}{dx} = b^x \times \ln b$.

6 The tangent to $f(x) = x^2e^{-x}$ at point P is horizontal. Find the possible coordinates of P.

J DERIVATIVES OF LOGARITHMIC FUNCTIONS

Discovery 8 — The derivative of $\ln x$

If $y = \ln x$, what is the gradient function?

CALCULUS DEMO

What to do:

1. Click on the icon to see the graph of $y = \ln x$. Observe the gradient function being drawn as the point moves from left to right along the graph.
2. Predict a formula for the gradient function of $y = \ln x$.
3. Find the gradient of the tangent to $y = \ln x$ for $x = 0.25$, 0.5, 1, 2, 3, 4, and 5. Do your results confirm your prediction in **2**?

From the **Discovery** you should have observed: If $y = \ln x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

The proof of this result is beyond the scope of this course.

THE DERIVATIVE OF $\ln f(x)$

Suppose $y = \ln f(x)$

$\therefore$ $y = \ln u$ where $u = f(x)$.

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\}$$

$$\therefore \frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}$$

$$= \frac{f'(x)}{f(x)}$$

Function	*Derivative*
$\ln x$	$\dfrac{1}{x}$
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$

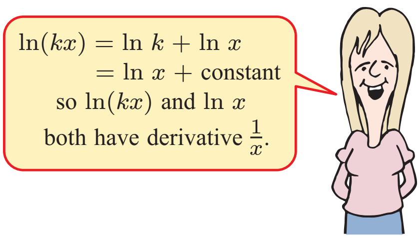
Example 15 — Self Tutor

Find the gradient function of:

a $y = \ln(kx)$, k a constant **b** $y = \ln(1-3x)$ **c** $y = x^3 \ln x$

a $y = \ln(kx)$

$$\therefore \frac{dy}{dx} = \frac{k}{kx} = \frac{1}{x}$$



b $y = \ln(1-3x)$

$$\therefore \frac{dy}{dx} = \frac{-3}{1-3x} = \frac{3}{3x-1}$$

c $y = x^3 \ln x$

$$\therefore \frac{dy}{dx} = 3x^2 \ln x + x^3\left(\frac{1}{x}\right) \quad \{\text{product rule}\}$$

$$= 3x^2 \ln x + x^2$$

$$= x^2(3\ln x + 1)$$

The laws of logarithms can help us to differentiate some logarithmic functions more easily.

For $a > 0,\ b > 0,\ n \in \mathbb{R}$: $\quad \ln(ab) = \ln a + \ln b$

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$\ln(a^n) = n \ln a$$

Example 16

Self Tutor

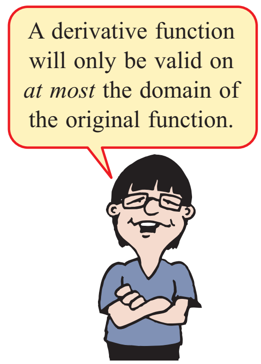
Differentiate with respect to x:

a $y = \ln(xe^{-x})$

b $y = \ln\left[\dfrac{x^2}{(x+2)(x-3)}\right]$

a
$$\begin{aligned} y &= \ln(xe^{-x}) \\ &= \ln x + \ln e^{-x} && \{\ln(ab) = \ln a + \ln b\} \\ &= \ln x - x && \{\ln e^a = a\} \\ \therefore\ \frac{dy}{dx} &= \frac{1}{x} - 1 \end{aligned}$$

b
$$\begin{aligned} y &= \ln\left[\frac{x^2}{(x+2)(x-3)}\right] \\ &= \ln x^2 - \ln[(x+2)(x-3)] && \left\{\ln\left(\frac{a}{b}\right) = \ln a - \ln b\right\} \\ &= 2\ln x - [\ln(x+2) + \ln(x-3)] \\ &= 2\ln x - \ln(x+2) - \ln(x-3) \\ \therefore\ \frac{dy}{dx} &= \frac{2}{x} - \frac{1}{x+2} - \frac{1}{x-3} \end{aligned}$$



EXERCISE 13J

1 Find the gradient function of:

a $y = \ln(7x)$
b $y = \ln(2x+1)$
c $y = \ln(x - x^2)$
d $y = 3 - 2\ln x$
e $y = x^2 \ln x$
f $y = \dfrac{\ln x}{2x}$
g $y = e^x \ln x$
h $y = (\ln x)^2$
i $y = \sqrt{\ln x}$
j $y = e^{-x} \ln x$
k $y = \sqrt{x}\ln(2x)$
l $y = \dfrac{2\sqrt{x}}{\ln x}$
m $y = 3 - 4\ln(1-x)$
n $y = x\ln(x^2+1)$

2 Find $\dfrac{dy}{dx}$ for:

a $y = x\ln 5$
b $y = \ln(x^3)$
c $y = \ln(x^4 + x)$
d $y = \ln(10 - 5x)$
e $y = [\ln(2x+1)]^3$
f $y = \dfrac{\ln(4x)}{x}$
g $y = \ln\left(\dfrac{1}{x}\right)$
h $y = \ln(\ln x)$
i $y = \dfrac{1}{\ln x}$

3 Use the laws of logarithms to help differentiate with respect to x:

a $y = \ln\sqrt{1-2x}$

b $y = \ln\left(\dfrac{1}{2x+3}\right)$

c $y = \ln\left(e^x\sqrt{x}\right)$

d $y = \ln\left(x\sqrt{2-x}\right)$

e $y = \ln\left(\dfrac{x+3}{x-1}\right)$

f $y = \ln\left(\dfrac{x^2}{3-x}\right)$

g $f(x) = \ln\left((3x-4)^3\right)$

h $f(x) = \ln\left(x(x^2+1)\right)$

i $f(x) = \ln\left(\dfrac{x^2+2x}{x-5}\right)$

4 Find the gradient of the tangent to:

a $y = x\ln x$ at the point where $x = e$

b $y = \ln\left(\dfrac{x+2}{x^2}\right)$ at the point where $x = 1$.

5 Suppose $f(x) = a\ln(2x+b)$ where $f(e) = 3$ and $f'(e) = \dfrac{6}{e}$. Find the constants a and b.

K DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

In **Chapter 9** we saw that sine and cosine curves arise naturally from motion in a circle.

DEMO

Click on the icon to observe the motion of point P around the unit circle. Observe the graphs of P's height relative to the x-axis, and then P's horizontal displacement from the y-axis. The resulting graphs are those of $y = \sin t$ and $y = \cos t$.

Discovery 9 — *Derivatives of* $\sin x$ *and* $\cos x$

Our aim is to use a computer demonstration to investigate the derivatives of $\sin x$ and $\cos x$.

What to do:

DERIVATIVES DEMO

1 Click on the icon to observe the graph of $y = \sin x$. A tangent with x-step of length 1 unit moves across the curve, and its y-step is translated onto the gradient graph. Predict the derivative of the function $y = \sin x$.

2 Repeat the process in **1** for the graph of $y = \cos x$. Hence predict the derivative of the function $y = \cos x$.

From the **Discovery** you should have deduced that:

For x in radians:	If	$f(x) = \sin x$	then	$f'(x) = \cos x$.
	If	$f(x) = \cos x$	then	$f'(x) = -\sin x$.

THE DERIVATIVE OF $\tan x$

DERIVATIVE DEMO

Consider $y = \tan x = \dfrac{\sin x}{\cos x}$

We let $u = \sin x$ and $v = \cos x$

$\therefore \dfrac{du}{dx} = \cos x$ and $\dfrac{dv}{dx} = -\sin x$

$$\begin{aligned}
\therefore \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \qquad \{\text{quotient rule}\}\\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{[\cos x]^2}\\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\
&= \frac{1}{\cos^2 x} \qquad \{\text{since} \quad \sin^2 x + \cos^2 x = 1\}\\
&= \sec^2 x
\end{aligned}$$

Function	*Derivative*
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

THE DERIVATIVES OF $\sin[f(x)]$, $\cos[f(x)]$, AND $\tan[f(x)]$

Suppose $y = \sin[f(x)]$

If we let $u = f(x)$, then $y = \sin u$.

But $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ $\qquad$ {chain rule}

$$\begin{aligned}
\therefore \frac{dy}{dx} &= \cos u \times f'(x)\\
&= \cos[f(x)] \times f'(x)
\end{aligned}$$

We can perform the same procedure for $\cos[f(x)]$ and $\tan[f(x)]$, giving the following results:

Function	*Derivative*
$\sin[f(x)]$	$\cos[f(x)]\, f'(x)$
$\cos[f(x)]$	$-\sin[f(x)]\, f'(x)$
$\tan[f(x)]$	$\sec^2[f(x)] f'(x)$

Example 17

Self Tutor

Differentiate with respect to x:

a $x \sin x$ $\qquad$ **b** $4\tan^2(3x)$

a If $y = x \sin x$

then by the product rule

$$\begin{aligned}
\frac{dy}{dx} &= (1)\sin x + (x)\cos x\\
&= \sin x + x\cos x
\end{aligned}$$

b If

$$\begin{aligned}
y &= 4\tan^2(3x)\\
&= 4[\tan(3x)]^2\\
&= 4u^2 \quad \text{where} \quad u = \tan(3x)\\
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}\\
\therefore \frac{dy}{dx} &= 8u \times \frac{du}{dx}\\
&= 8\tan(3x) \times 3\sec^2(3x)\\
&= 24\sin(3x)\sec^3(3x)
\end{aligned}$$

EXERCISE 13K

1 Find $\dfrac{dy}{dx}$ for:

a $y = \sin(2x)$ **b** $y = \sin x + \cos x$ **c** $y = \cos(3x) - \sin x$
d $y = \sin(x+1)$ **e** $y = \cos(3-2x)$ **f** $y = \tan(5x)$
g $y = \sin(\frac{x}{2}) - 3\cos x$ **h** $y = 3\tan(\pi x)$ **i** $y = 4\sin x - \cos(2x)$

2 Differentiate with respect to x:

a $x^2 + \cos x$ **b** $\tan x - 3\sin x$ **c** $e^x \cos x$ **d** $e^{-x}\sin x$
e $\ln(\sin x)$ **f** $e^{2x}\tan x$ **g** $\sin(3x)$ **h** $\cos(\frac{x}{2})$
i $3\tan(2x)$ **j** $x\cos x$ **k** $\dfrac{\sin x}{x}$ **l** $x\tan x$

3 Differentiate with respect to x:

a $\sin(x^2)$ **b** $\cos(\sqrt{x})$ **c** $\sqrt{\cos x}$ **d** $\sin^2 x$
e $\cos^3 x$ **f** $\cos x \sin(2x)$ **g** $\cos(\cos x)$ **h** $\cos^3(4x)$
i $\dfrac{1}{\sin x}$ **j** $\dfrac{1}{\cos(2x)}$ **k** $\dfrac{2}{\sin^2(2x)}$ **l** $\dfrac{8}{\tan^3(\frac{x}{2})}$

4 Find the gradient of the tangent to:

a $f(x) = \sin^3 x$ at the point where $x = \frac{2\pi}{3}$

b $f(x) = \cos x \sin x$ at the point where $x = \frac{\pi}{4}$.

L SECOND DERIVATIVES

Given a function $f(x)$, the derivative $f'(x)$ is known as the **first derivative**.

The **second derivative** of $f(x)$ is the derivative of $f'(x)$, or **the derivative of the first derivative**.

We use $f''(x)$ or y'' or $\dfrac{d^2y}{dx^2}$ to represent the second derivative.

$f''(x)$ reads "*f double dashed x*".

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ reads "*dee two y by dee x squared*".

Example 18 Self Tutor

Find $f''(x)$ given that $f(x) = x^3 - \dfrac{3}{x}$.

Now $f(x) = x^3 - 3x^{-1}$

$\therefore\ f'(x) = 3x^2 + 3x^{-2}$

$\therefore\ f''(x) = 6x - 6x^{-3}$

$= 6x - \dfrac{6}{x^3}$

EXERCISE 13L

1 Find $f''(x)$ given that:

a $f(x) = 3x^2 - 6x + 2$ **b** $f(x) = \frac{2}{\sqrt{x}} - 1$ **c** $f(x) = 2x^3 - 3x^2 - x + 5$

d $f(x) = \frac{2-3x}{x^2}$ **e** $f(x) = (1-2x)^3$ **f** $f(x) = \frac{x+2}{2x-1}$

2 Find $\frac{d^2y}{dx^2}$ given that:

a $y = x - x^3$ **b** $y = x^2 - \frac{5}{x^2}$ **c** $y = 2 - \frac{3}{\sqrt{x}}$

d $y = \frac{4-x}{x}$ **e** $y = (x^2 - 3x)^3$ **f** $y = x^2 - x + \frac{1}{1-x}$

3 Given $f(x) = x^3 - 2x + 5$, find:

a $f(2)$ **b** $f'(2)$ **c** $f''(2)$

4 Suppose $y = Ae^{kx}$ where A and k are constants. Show that:

a $\frac{dy}{dx} = ky$ **b** $\frac{d^2y}{dx^2} = k^2y$

5 Find the value(s) of x such that $f''(x) = 0$, given:

a $f(x) = 2x^3 - 6x^2 + 5x + 1$ **b** $f(x) = \frac{x}{x^2+2}$

6 Consider the function $f(x) = 2x^3 - x$.
Complete the following table by indicating whether $f(x)$, $f'(x)$, and $f''(x)$ are positive (+), negative (–), or zero (0) at the given values of x.

x	-1	0	1
$f(x)$	–		
$f'(x)$			
$f''(x)$			

7 Suppose $f(x) = 2\sin^3 x - 3\sin x$.

a Show that $f'(x) = -3\cos x \cos 2x$. **b** Find $f''(x)$.

8 Find $\frac{d^2y}{dx^2}$ given:

a $y = -\ln x$ **b** $y = x\ln x$ **c** $y = (\ln x)^2$

9 Given $f(x) = x^2 - \frac{1}{x}$, find:

a $f(1)$ **b** $f'(1)$ **c** $f''(1)$

10 If $y = 2e^{3x} + 5e^{4x}$, show that $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$.

11 If $y = \sin(2x+3)$, show that $\frac{d^2y}{dx^2} + 4y = 0$.

12 If $y = 2\sin x + 3\cos x$, show that $y'' + y = 0$ where y'' represents $\frac{d^2y}{dx^2}$.

Review set 13A

1 Evaluate:

a $\lim\limits_{x \to 1} (6x - 7)$ **b** $\lim\limits_{h \to 0} \dfrac{2h^2 - h}{h}$ **c** $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x - 4}$

2 Find, from first principles, the derivative of:

a $f(x) = x^2 + 2x$ **b** $y = 4 - 3x^2$

3 In the **Opening Problem** on page **334**, the altitude of the jumper is given by $f(t) = 452 - 4.8t^2$ metres, where $0 \leqslant t \leqslant 3$ seconds.

a Find $f'(t) = \lim\limits_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$.

b *Hence* find the speed of the jumper when $t = 2$ seconds.

4 If $f(x) = 7 + x - 3x^2$, find: **a** $f(3)$ **b** $f'(3)$ **c** $f''(3)$.

5 Find $\dfrac{dy}{dx}$ for: **a** $y = 3x^2 - x^4$ **b** $y = \dfrac{x^3 - x}{x^2}$

6 At what point on the curve $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ does the tangent have gradient 1?

7 Find $\dfrac{dy}{dx}$ if: **a** $y = e^{x^3+2}$ **b** $y = \ln\left(\dfrac{x+3}{x^2}\right)$

8 Given $y = 3e^x - e^{-x}$, show that $\dfrac{d^2y}{dx^2} = y$.

9 Differentiate with respect to x:

a $5x - 3x^{-1}$ **b** $(3x^2 + x)^4$ **c** $(x^2+1)(1-x^2)^3$

10 Find all points on the curve $y = 2x^3 + 3x^2 - 10x + 3$ where the gradient of the tangent is 2.

11 If $y = \sqrt{5 - 4x}$, find: **a** $\dfrac{dy}{dx}$ **b** $\dfrac{d^2y}{dx^2}$

12 Differentiate with respect to x:

a $\sin(5x)\ln(x)$ **b** $\sin(x)\cos(2x)$ **c** $e^{-2x}\tan x$

13 Find the gradient of the tangent to $y = \sin^2 x$ at the point where $x = \frac{\pi}{3}$.

14 Find the derivative with respect to x of:

a $f(x) = (x^2+3)^4$ **b** $g(x) = \dfrac{\sqrt{x+5}}{x^2}$

15 Find $f''(2)$ for:

a $f(x) = 3x^2 - \dfrac{1}{x}$ **b** $f(x) = \sqrt{x}$

16 Differentiate with respect to x:

a $10x - \sin(10x)$ **b** $\ln\left(\dfrac{1}{\cos x}\right)$ **c** $\sin(5x)\ln(2x)$

Review set 13B

1 Evaluate the limits:

a $\lim\limits_{h\to 0} \dfrac{h^3 - 3h}{h}$ **b** $\lim\limits_{x\to 1} \dfrac{3x^2 - 3x}{x - 1}$ **c** $\lim\limits_{x\to 2} \dfrac{x^2 - 3x + 2}{2 - x}$

2 Given $f(x) = 5x - x^2$, find $f'(1)$ from first principles.

3 **a** Given $y = 2x^2 - 1$, find $\dfrac{dy}{dx}$ from first principles.

b Hence state the gradient of the tangent to $y = 2x^2 - 1$ at the point where $x = 4$.

c For what value of x is the gradient of the tangent to $y = 2x^2 - 1$ equal to -12?

4 Differentiate with respect to x: **a** $y = x^3\sqrt{1 - x^2}$ **b** $y = \dfrac{x^2 - 3x}{\sqrt{x + 1}}$

5 Find $\dfrac{d^2y}{dx^2}$ for: **a** $y = 3x^4 - \dfrac{2}{x}$ **b** $y = x^3 - x + \dfrac{1}{\sqrt{x}}$

6 Find all points on the curve $y = xe^x$ where the gradient of the tangent is $2e$.

7 Differentiate with respect to x: **a** $f(x) = \ln(e^x + 3)$ **b** $f(x) = \ln\left[\dfrac{(x + 2)^3}{x}\right]$

8 Suppose $y = \left(x - \dfrac{1}{x}\right)^4$. Find $\dfrac{dy}{dx}$ when $x = 1$.

9 Find $\dfrac{dy}{dx}$ if: **a** $y = \ln(x^3 - 3x)$ **b** $y = \dfrac{e^x}{x^2}$

10 Find x if $f''(x) = 0$ and $f(x) = 2x^4 - 4x^3 - 9x^2 + 4x + 7$.

11 If $f(x) = x - \cos x$, find

a $f(\pi)$ **b** $f'(\frac{\pi}{2})$ **c** $f''(\frac{3\pi}{4})$

12 **a** Find $f'(x)$ and $f''(x)$ for $f(x) = \sqrt{x}\cos(4x)$.

b Hence find $f'(\frac{\pi}{16})$ and $f''(\frac{\pi}{8})$.

13 Suppose $y = 3\sin 2x + 2\cos 2x$. Show that $4y + \dfrac{d^2y}{dx^2} = 0$.

14 Consider $f(x) = \dfrac{6x}{3 + x^2}$. Find the value(s) of x when:

a $f(x) = -\frac{1}{2}$ **b** $f'(x) = 0$ **c** $f''(x) = 0$

15 The function f is defined by $f : x \mapsto -10\sin 2x\cos 2x$, $0 \leqslant x \leqslant \pi$.

a Write down an expression for $f(x)$ in the form $k\sin 4x$.

b Solve $f'(x) = 0$, giving exact answers.

16 Given that a and b are constants, differentiate $y = 3\sin bx - a\cos 2x$ with respect to x.

Find a and b if $y + \dfrac{d^2y}{dx^2} = 6\cos 2x$.

14

Applications of differential calculus

Contents:

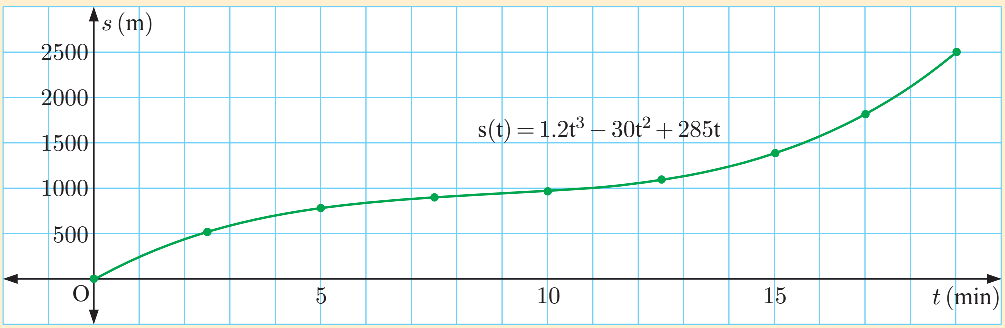
A Tangents and normals
B Stationary points
C Kinematics
D Rates of change
E Optimisation
F Related rates

Opening problem

Michael rides up a hill and down the other side to his friend's house. The dots on the graph show Michael's position at various times t.

DEMO

The distance Michael has travelled at various times is given by the function
$s(t) = 1.2t^3 - 30t^2 + 285t$ metres for $0 \leqslant t \leqslant 19$ minutes.



Things to think about:

a Can you find a function for Michael's *speed* at any time t?

b Michael's *acceleration* is the rate at which his speed is changing with respect to time. How can we interpret $s''(t)$?

c Can you find Michael's speed and acceleration at the time $t = 15$ minutes?

d At what point do you think the hill was steepest? How far had Michael travelled to this point?

In the previous chapter we saw how to differentiate many types of functions.

In this chapter we will use derivatives to find:

- tangents and normals to curves
- turning points which are local minima and maxima.

We will then look at applying these techniques to real world problems including:

- kinematics (motion problems of displacement, velocity, and acceleration)
- rates of change
- optimisation (maxima and minima).

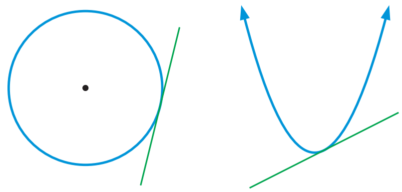
A TANGENTS AND NORMALS

TANGENTS

The **tangent** to a curve at point A is the best approximating straight line to the curve at A.

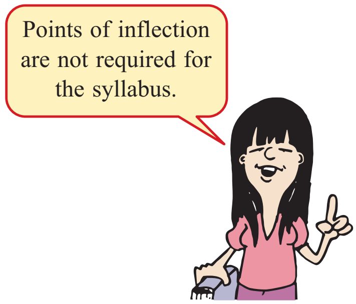
In cases we have seen already, the tangent *touches* the curve.

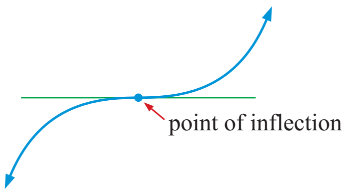
For example, consider tangents to a circle or a quadratic.



However, we note that for some functions:

- The tangent may intersect the curve again somewhere else.
- It is possible for the tangent to pass through the curve at the point of tangency. If this happens, we call it a **point of inflection**.





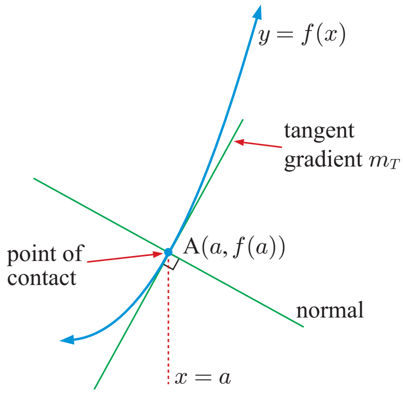
Consider a curve $y = f(x)$.

If A is the point with x-coordinate a, then the gradient of the tangent to the curve at this point is $f'(a) = m_T$.

The equation of the tangent is

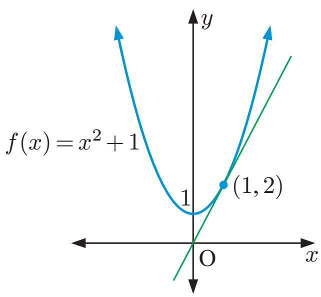
$$y - f(a) = f'(a)(x - a)$$

or $\boldsymbol{y = f(a) + f'(a)(x - a)}$.



Example 1 — Self Tutor

Find the equation of the tangent to $f(x) = x^2 + 1$ at the point where $x = 1$.



Since $f(1) = 1 + 1 = 2$, the point of contact is $(1,\ 2)$.

Now $f'(x) = 2x$, so $m_T = f'(1) = 2$

$\therefore$ the tangent has equation $y = 2 + 2(x - 1)$

which is $y = 2x$.

NORMALS

A **normal** to a curve is a line which is perpendicular to the tangent at the point of contact.

The gradients of perpendicular lines are negative reciprocals of each other, so:

> The gradient of the normal to the curve at $x = a$ is $m_N = -\dfrac{1}{f'(a)}$.
>
> The equation of the normal to the curve at $x = a$ is $\boldsymbol{y = f(a) - \dfrac{1}{f'(a)}(x - a)}$.

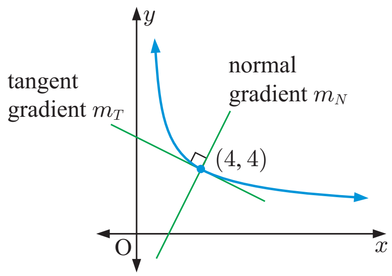
Reminder: If a line has gradient $\frac{4}{5}$ and passes through $(2, -3)$, another quick way to write down its equation is $4x - 5y = 4(2) - 5(-3)$ or $4x - 5y = 23$.

If the gradient was $-\frac{4}{5}$, we would have:
$4x + 5y = 4(2) + 5(-3)$ or $4x + 5y = -7$.

Example 2 — Self Tutor

Find the equation of the normal to $y = \dfrac{8}{\sqrt{x}}$ at the point where $x = 4$.

When $x = 4$, $y = \dfrac{8}{\sqrt{4}} = \dfrac{8}{2} = 4$. So, the point of contact is $(4, 4)$.



Now as $y = 8x^{-\frac{1}{2}}$, $\dfrac{dy}{dx} = -4x^{-\frac{3}{2}}$

$\therefore$ when $x = 4$, $m_T = -4 \times 4^{-\frac{3}{2}} = -\frac{1}{2}$

$\therefore$ the normal at $(4, 4)$ has gradient $m_N = \frac{2}{1}$.

$\therefore$ the equation of the normal is
$2x - 1y = 2(4) - 1(4)$
or $2x - y = 4$

EXERCISE 14A

1 Find the equation of the tangent to:

a $y = x - 2x^2 + 3$ at $x = 2$

b $y = \sqrt{x} + 1$ at $x = 4$

c $y = x^3 - 5x$ at $x = 1$

d $y = \dfrac{4}{\sqrt{x}}$ at $(1, 4)$

e $y = \dfrac{3}{x} - \dfrac{1}{x^2}$ at $(-1, -4)$

f $y = 3x^2 - \dfrac{1}{x}$ at $x = -1$.

2 Find the equation of the normal to:

a $y = x^2$ at the point $(3, 9)$

b $y = x^3 - 5x + 2$ at $x = -2$

c $y = \dfrac{5}{\sqrt{x}} - \sqrt{x}$ at the point $(1, 4)$

d $y = 8\sqrt{x} - \dfrac{1}{x^2}$ at $x = 1$.

Example 3

Self Tutor

Find the equations of any horizontal tangents to $y = x^3 - 12x + 2$.

Since $y = x^3 - 12x + 2$, $\dfrac{dy}{dx} = 3x^2 - 12$

Horizontal tangents have gradient 0, so $3x^2 - 12 = 0$

$\therefore\ 3(x^2 - 4) = 0$

$\therefore\ 3(x+2)(x-2) = 0$

$\therefore\ x = -2$ or 2

When $x = 2$, $y = 8 - 24 + 2 = -14$

When $x = -2$, $y = -8 + 24 + 2 = 18$

$\therefore$ the points of contact are $(2, -14)$ and $(-2, 18)$

$\therefore$ the tangents are $y = -14$ and $y = 18$.

3 Find the equations of any horizontal tangents to $y = 2x^3 + 3x^2 - 12x + 1$.

4 Find the points of contact where horizontal tangents meet the curve $y = 2\sqrt{x} + \dfrac{1}{\sqrt{x}}$.

5 Find k if the tangent to $y = 2x^3 + kx^2 - 3$ at the point where $x = 2$ has gradient 4.

6 Find the equation of another tangent to $y = 1 - 3x + 12x^2 - 8x^3$ which is parallel to the tangent at $(1, 2)$.

7 Consider the curve $y = x^2 + ax + b$ where a and b are constants. The tangent to this curve at the point where $x = 1$ is $2x + y = 6$. Find the values of a and b.

8 Consider the curve $y = a\sqrt{x} + \dfrac{b}{\sqrt{x}}$ where a and b are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of a and b.

9 Show that the equation of the tangent to $y = 2x^2 - 1$ at the point where $x = a$, is $4ax - y = 2a^2 + 1$.

10 Find the equation of the tangent to:

a $y = \sqrt{2x+1}$ at $x = 4$

b $y = \dfrac{1}{2-x}$ at $x = -1$

c $f(x) = \dfrac{x}{1-3x}$ at $(-1, -\frac{1}{4})$

d $f(x) = \dfrac{x^2}{1-x}$ at $(2, -4)$.

11 Find the equation of the normal to:

a $y = \dfrac{1}{(x^2+1)^2}$ at $(1, \frac{1}{4})$

b $y = \dfrac{1}{\sqrt{3-2x}}$ at $x = -3$

c $f(x) = \sqrt{x}(1-x)^2$ at $x = 4$

d $f(x) = \dfrac{x^2-1}{2x+3}$ at $x = -1$.

12 Consider the curve $y = a\sqrt{1-bx}$ where a and b are constants. The tangent to this curve at the point where $x = -1$ is $3x + y = 5$. Find the values of a and b.

Example 4

Self Tutor

Show that the equation of the tangent to $y = \ln x$ at the point where $y = -1$ is $y = ex - 2$.

When $y = -1$, $\ln x = -1$

$\therefore$ $x = e^{-1} = \frac{1}{e}$

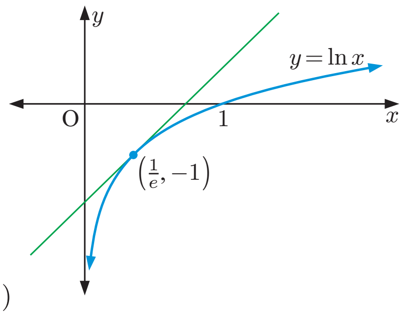
$\therefore$ the point of contact is $\left(\frac{1}{e}, -1\right)$.

Now $f(x) = \ln x$ has derivative $f'(x) = \dfrac{1}{x}$

$\therefore$ the tangent at $\left(\frac{1}{e}, -1\right)$ has gradient $\dfrac{1}{\frac{1}{e}} = e$

$\therefore$ the tangent has equation $y = -1 + e(x - \frac{1}{e})$

which is $y = ex - 2$



13 Find the equation of:

- **a** the tangent to $f : x \mapsto e^{-x}$ at the point where $x = 1$
- **b** the tangent to $y = \ln(2 - x)$ at the point where $x = -1$
- **c** the normal to $y = \ln\sqrt{x}$ at the point where $y = -1$.

Example 5

Self Tutor

Find the equation of the tangent to $y = \tan x$ at the point where $x = \frac{\pi}{4}$.

When $x = \frac{\pi}{4}$, $y = \tan\frac{\pi}{4} = 1$

$\therefore$ the point of contact is $(\frac{\pi}{4}, 1)$.

Now $f(x) = \tan x$ has derivative $f'(x) = \sec^2 x$

$\therefore$ the tangent at $(\frac{\pi}{4}, 1)$ has gradient $\sec^2\frac{\pi}{4} = (\sqrt{2})^2 = 2$

$\therefore$ the tangent has equation $y = 1 + 2(x - \frac{\pi}{4})$

which is $y = 2x + (1 - \frac{\pi}{2})$

14 Show that the curve with equation $y = \dfrac{\cos x}{1 + \sin x}$ does not have any horizontal tangents.

15 Find the equation of:

- **a** the tangent to $y = \sin x$ at the origin
- **b** the tangent to $y = \tan x$ at the origin
- **c** the normal to $y = \cos x$ at the point where $x = \frac{\pi}{6}$
- **d** the normal to $y = \dfrac{1}{\sin(2x)}$ at the point where $x = \frac{\pi}{4}$.

Example 6

Self Tutor

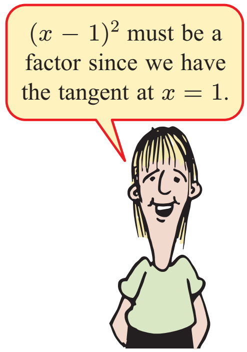
Find where the tangent to $y = x^3 + x + 2$ at $(1, 4)$ meets the curve again.

Let $f(x) = x^3 + x + 2$

$\therefore\ f'(x) = 3x^2 + 1$ and $\therefore\ f'(1) = 3 + 1 = 4$

$\therefore$ the equation of the tangent at $(1, 4)$ is $4x - y = 4(1) - 4$

or $y = 4x$.



The curve meets the tangent again when $x^3 + x + 2 = 4x$

$$\therefore\ x^3 - 3x + 2 = 0$$
$$\therefore\ (x-1)^2(x+2) = 0$$

When $x = -2$, $y = (-2)^3 + (-2) + 2 = -8$

$\therefore$ the tangent meets the curve again at $(-2, -8)$.

16 **a** Find where the tangent to the curve $y = x^3$ at the point where $x = 2$, meets the curve again.

b Find where the tangent to the curve $y = -x^3 + 2x^2 + 1$ at the point where $x = -1$, meets the curve again.

17 Consider the function $f(x) = x^2 + \dfrac{4}{x^2}$.

a Find $f'(x)$.

b Find the values of x at which the tangent to the curve is horizontal.

c Show that the tangents at these points are the same line.

18 The tangent to $y = x^2e^x$ at $x = 1$ cuts the x and y-axes at A and B respectively. Find the coordinates of A and B.

Example 7

Self Tutor

Find the equations of the tangents to $y = x^2$ from the external point $(2, 3)$.

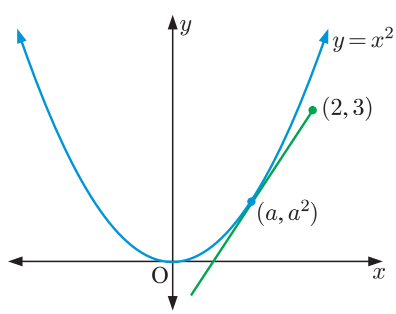
Let (a, a^2) be a general point on $f(x) = x^2$.

Now $f'(x) = 2x$, so $f'(a) = 2a$

$\therefore$ the equation of the tangent at (a, a^2) is

$$y = a^2 + 2a(x - a)$$

which is $y = 2ax - a^2$



Thus the tangents which pass through $(2, 3)$ satisfy

$$3 = 2a(2) - a^2$$
$$\therefore\ a^2 - 4a + 3 = 0$$
$$\therefore\ (a-1)(a-3) = 0$$
$$\therefore\ a = 1 \text{ or } 3$$

$\therefore$ exactly two tangents pass through the external point $(2, 3)$.

If $a = 1$, the tangent has equation $y = 2x - 1$ with point of contact $(1, 1)$.

If $a = 3$, the tangent has equation $y = 6x - 9$ with point of contact $(3, 9)$.

19 **a** Find the equation of the tangent to $y = x^2 - x + 9$ at the point where $x = a$.

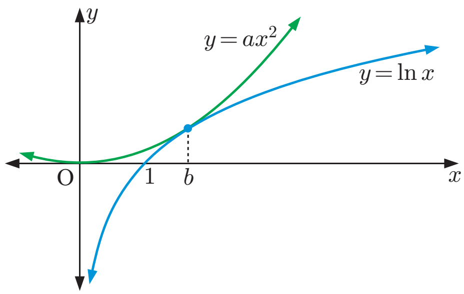
b Hence, find the equations of the two tangents from $(0, 0)$ to the curve. State the coordinates of the points of contact.

20 Find the equations of the tangents to $y = x^3$ from the external point $(-2, 0)$.

21 Find the equation of the normal to $y = \sqrt{x}$ from the external point $(4, 0)$.
Hint: There is no normal at the point where $x = 0$, as this is the endpoint of the function.

22 Find the equation of the tangent to $y = e^x$ at the point where $x = a$.
Hence, find the equation of the tangent to $y = e^x$ which passes through the origin.

23 A quadratic of the form $y = ax^2$, $a > 0$, touches the logarithmic function $y = \ln x$ as shown.

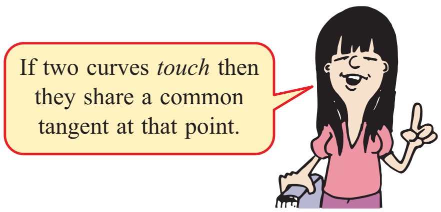


a If the x-coordinate of the point of contact is b, explain why $ab^2 = \ln b$ and $2ab = \frac{1}{b}$.

b Deduce that the point of contact is $(\sqrt{e}, \frac{1}{2})$.

c Find the value of a.

d Find the equation of the common tangent.



24 Find, correct to 2 decimal places, the angle between the tangents to $y = 3e^{-x}$ and $y = 2 + e^x$ at their point of intersection.

25 Consider the cubic function $f(x) = 2x^3 + 5x^2 - 4x - 3$.

a Show that the equation of the tangent to the curve at the point where $x = -1$ can be written in the form $y = 4 - 8(x + 1)$.

b Show that $f(x)$ can be written in the form $f(x) = 4 - 8(x + 1) - (x + 1)^2 + 2(x + 1)^3$.

c Hence explain why the tangent is the best approximating straight line to the curve at the point where $x = -1$.

26 A cubic has three real roots. Prove that the tangent line at the average of any two roots of the cubic, passes through the third root.
Hint: Let $f(x) = a(x - \alpha)(x - \beta)(x - \gamma)$.

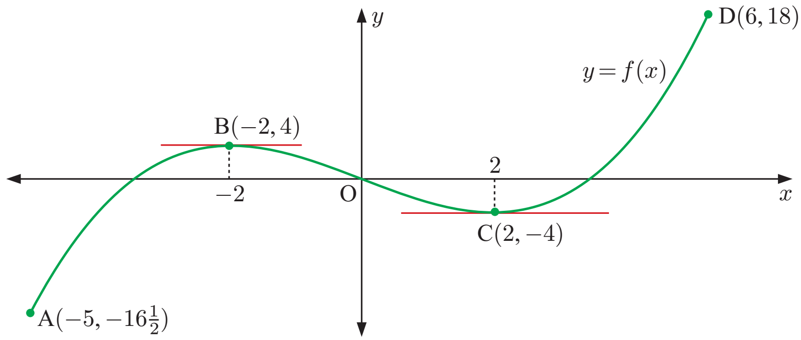
B STATIONARY POINTS

A **stationary point** of a function is a point where $f'(x) = 0$.

It could be a local maximum, local minimum, or stationary inflection.

TURNING POINTS (MAXIMA AND MINIMA)

Consider the following graph which has a restricted domain of $-5 \leqslant x \leqslant 6$.

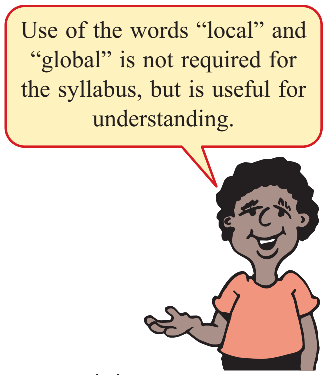


A is a **global minimum** as it has the minimum value of y on the entire domain.

B is a **local maximum** as it is a turning point where $f'(x) = 0$ and the curve has shape  .

C is a **local minimum** as it is a turning point where $f'(x) = 0$ and the curve has shape  .

D is a **global maximum** as it is the maximum value of y on the entire domain.



For many functions, a local maximum or minimum is also the global maximum or minimum.

For example, for $y = x^2$ the point $(0, 0)$ is a local minimum and is also the global minimum.

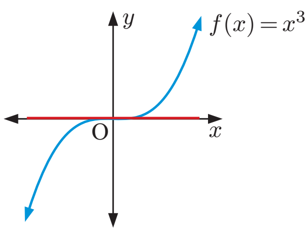
STATIONARY POINTS OF INFLECTION

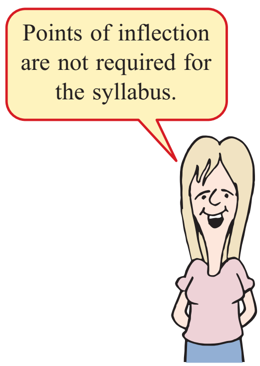
It is not always true that whenever we find a value of x where $f'(x) = 0$, we have a local maximum or minimum.

For example,

$f(x) = x^3$ has $f'(x) = 3x^2$,

so $f'(x) = 0$ when $x = 0$.





The x-axis is a tangent to the curve which actually crosses over the curve at $O(0, 0)$. This tangent is horizontal, but $O(0, 0)$ is neither a local maximum nor a local minimum. It is called a **stationary inflection** as the curve changes its curvature or shape.

SIGN DIAGRAMS

A **sign diagram** is used to display the intervals on which a function is positive and negative.

In calculus we commonly use sign diagrams of the *derivative function* $f'(x)$ so we can determine the nature of a stationary point.

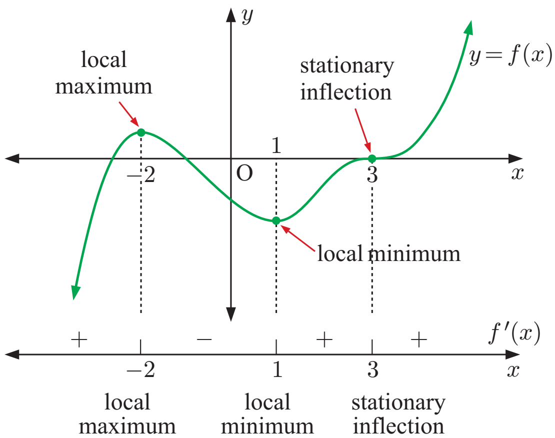
Consider the graph alongside.

The sign diagram of its gradient function is shown directly beneath it.

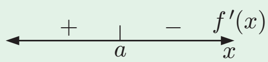
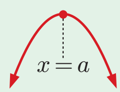
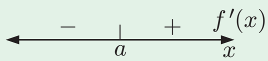
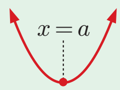
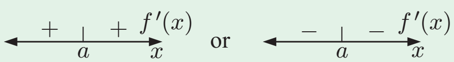
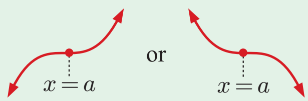
We can use the sign diagram to describe the stationary points of the function.

The signs on the sign diagram of $f'(x)$ indicate whether the gradient of $y = f(x)$ is positive or negative in that interval.

DEMO



We observe the following properties:

Stationary point where $f'(a) = 0$	*Sign diagram of* $f'(x)$ *near* $x = a$	*Shape of curve near* $x = a$
local maximum		
local minimum		
stationary inflection		

Example 8

Self Tutor

Consider the function $f(x) = x^3 - 3x^2 - 9x + 5$.

a Find the y-intercept.

b Find and classify all stationary points.

c Hence sketch the curve $y = f(x)$.

a $f(0) = 5$, so the y-intercept is 5.

b
$$\begin{aligned} f(x) &= x^3 - 3x^2 - 9x + 5 \\ \therefore \quad f'(x) &= 3x^2 - 6x - 9 \\ &= 3(x^2 - 2x - 3) \\ &= 3(x-3)(x+1) \end{aligned}$$
which has sign diagram: