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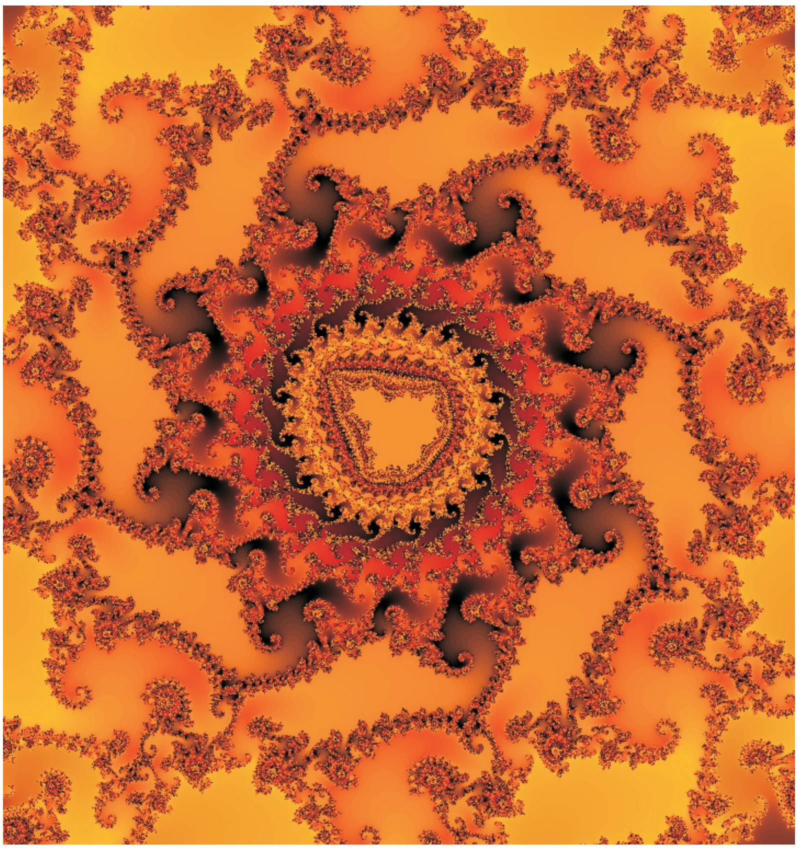
# Cambridge Additional Mathematics



*IGCSE® (0606)*

*O Level (4037)*

*Michael Haese*

*Sandra Haese*

*Mark Humphries*

*Chris Sangwin*

**CAMBRIDGE ADDITIONAL MATHEMATICS (0606) (4037)**


| | |
|---|---|
| Michael Haese | B.Sc.(Hons.), Ph.D. |
| Sandra Haese | B.Sc. |
| Mark Humphries | B.Sc.(Hons.) |
| Chris Sangwin | M.A., M.Sc., Ph.D. |

Haese Mathematics
152 Richmond Road, Marleston, SA 5033, AUSTRALIA
Telephone: +61 8 8210 4666, Fax: +61 8 8354 1238
Email: info@haesemathematics.com.au
Web: www.haesemathematics.com.au



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## FOREWORD

This book has been written to cover the '*Cambridge O Level Additional Mathematics (4037)*' and the '*Cambridge IGCSE® Additional Mathematics (0606)*' courses over a one-year period.

These syllabuses enable learners to extend the mathematics skills, knowledge, and understanding developed in the Cambridge IGCSE or O Level Mathematics courses, and use skills in the context of more advanced techniques.

The syllabuses have a Pure Mathematics only content which enables learners to acquire a suitable foundation in mathematics for further study in the subject. Knowledge of the content of the Cambridge IGCSE or O Level Mathematics syllabus (or an equivalent syllabus) is assumed.

Learners who successfully complete these courses gain lifelong skills, including:

- the further development of mathematical concepts and principles
- an ability to solve problems, present solutions logically, and interpret results.

This book is an attempt to cover, in one volume, the content outlined in the Cambridge O Level Additional Mathematics (4037) and Cambridge IGCSE Additional Mathematics (0606) syllabuses. The book can be used as a preparation for GCE Advanced Level Mathematics. The book has been endorsed by Cambridge.

To reflect the principles on which the course is based, we have attempted to produce a book and CD package that embraces understanding and problem solving in order to give students different learning experiences. Review exercises appear at the end of each chapter. Answers are given at the end of the book, followed by an index.

The interactive CD contains **Self Tutor** software (see p. 5), geometry and graphics software, demonstrations and simulations. The CD also contains the text of the book so that students can load it on a home computer and keep the textbook at school.

The examinations for Cambridge Additional Mathematics are in the form of two papers. Many of the problems in this textbook have been written to reflect the style of the examination questions. The questions, worked solutions and comments that appear in the book and CD were written by the authors.

The book can be used as a scheme of work but it is expected that the teacher will choose the order of topics. Exercises in the book range from routine practice and consolidation of basic skills, to problem solving exercises that are quite demanding.

In this changing world of mathematics education, we believe that the contextual approach shown in this book will enhance the students' understanding, knowledge and appreciation of mathematics, and its universal application.

We welcome your feedback.

Email: info@haesemathematics.com.au
Web: www.haesemathematics.com.au

*PMH, SHH, MH, CS*

## ABOUT THE AUTHORS

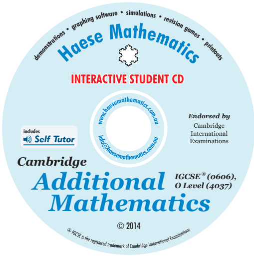
**Michael Haese** completed a BSc at the University of Adelaide, majoring in Infection and Immunity, and Applied Mathematics. He then completed Honours in Applied Mathematics, and a PhD in high speed fluid flows. He has a keen interest in education and a desire to see mathematics come alive in the classroom through its history and relationship with other subject areas. Michael has been the principal editor for Haese Mathematics since 2008.

**Sandra Haese** completed a BSc at the University of Adelaide, majoring in Pure Mathematics and Statistics. She taught mathematics at Underdale High School and later at Westminster School in Adelaide. In 1979, Sandra's husband Bob Haese began to write textbooks for mathematics students at high schools, and Sandra assumed the role of proof reader. She continues to work for Haese Mathematics as an editor and proof reader, and she produces much of the audio work for the Self Tutor software. In 2007 she was awarded Life Membership of the Mathematics Association of South Australia.

**Mark Humphries** completed an honours degree in Pure Mathematics and an Economics degree at the University of Adelaide. His mathematical interests include public key cryptography and number theory. He has been working at Haese Mathematics since 2006.

**Chris Sangwin** completed a BA in Mathematics at the University of Oxford, and an MSc and PhD in Mathematics at the University of Bath. He spent thirteen years in the Mathematics Department at the University of Birmingham, and from 2000 - 2011 was seconded half time to the UK Higher Education Academy "Maths Stats and OR Network" to promote learning and teaching of university mathematics. He was awarded a National Teaching Fellowship in 2006, and is now a Senior Lecturer in Mathematics Education in the Mathematics Education Centre at Loughborough University. His learning and teaching interests include automatic assessment of mathematics using computer algebra, and problem solving using the Moore method and similar student-centred approaches.

## USING THE INTERACTIVE CD

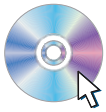



The interactive Student CD that comes with this book is designed for those who want to utilise technology in teaching and learning Mathematics.

The CD icon that appears throughout the book denotes an active link on the CD. Simply click on the icon when running the CD to access a large range of interactive features that includes:

- printable worksheets
- graphing packages
- demonstrations
- simulations
- revision games
- SELF TUTOR

INTERACTIVE LINK



**SELF TUTOR** is an exciting feature of this book.

The ***Self Tutor*** icon on each worked example denotes an active link on the CD.

Simply 'click' on the ***Self Tutor*** (or anywhere in the example box) to access the worked example, with a teacher's voice explaining each step necessary to reach the answer.

Play any line as often as you like. See how the basic processes come alive using movement and colour on the screen.

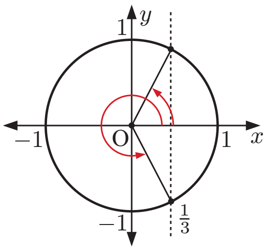
Ideal for students who have missed lessons or need extra help.

### Example 10 — *Self Tutor*

Find the two angles $\theta$ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

**a** $\cos\theta = \frac{1}{3}$ **b** $\sin\theta = \frac{3}{4}$ **c** $\tan\theta = 2$

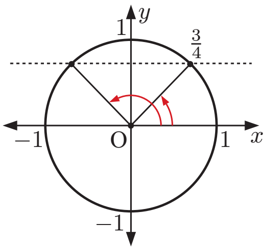
**a** $\cos^{-1}(\frac{1}{3}) \approx 1.23$




$\therefore\ \theta \approx 1.23$ or $2\pi - 1.23$

$\therefore\ \theta \approx 1.23$ or $5.05$

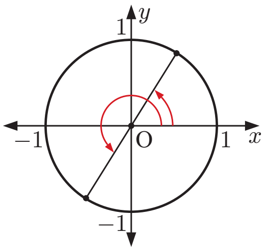
**b** $\sin^{-1}(\frac{3}{4}) \approx 0.848$




$\therefore\ \theta \approx 0.848$ or $\pi - 0.848$

$\therefore\ \theta \approx 0.848$ or $2.29$

**c** $\tan^{-1}(2) \approx 1.11$




$\therefore\ \theta \approx 1.11$ or $\pi + 1.11$

$\therefore\ \theta \approx 1.11$ or $4.25$

See **Chapter 8**, **The unit circle and radian measure**, page 209

## SYMBOLS AND NOTATION USED IN THIS BOOK

| | |
|---|---|
| $\mathbb{N}$ | the set of natural numbers, $\{1, 2, 3, ....\}$ |
| $\mathbb{Z}$ | the set of integers, $\{0, \pm1, \pm2, \pm3, ....\}$ |
| $\mathbb{Z}^+$ | the set of positive integers, $\{1, 2, 3, ....\}$ |
| $\mathbb{Q}$ | the set of rational numbers |
| $\mathbb{Q}^+$ | the set of positive rational numbers, $\{x \in \mathbb{Q},\ \ x > 0\}$ |
| $\mathbb{R}$ | the set of real numbers |
| $\mathbb{R}^+$ | the set of positive real numbers, $\{x \in \mathbb{R},\ \ x > 0\}$ |
| $[a, b]$ | the closed interval $\{x \in \mathbb{R} : a \leqslant x \leqslant b\}$ |
| $[a, b)$ | the interval $\{x \in \mathbb{R} : a \leqslant x < b\}$ |
| $(a, b]$ | the interval $\{x \in \mathbb{R} : a < x \leqslant b\}$ |
| $(a, b)$ | the open interval $\{x \in \mathbb{R} : a < x < b\}$ |
| $\{x_1, x_2, ....\}$ | the set with elements $x_1, x_2, ....$ |
| $n(A)$ | the number of elements in the finite set $A$ |
| $\{x : ....$ | the set of all $x$ such that |
| $\in$ | is an element of |
| $\notin$ | is not an element of |
| $\varnothing$ or $\{\ \}$ | the empty set |
| $\mathscr{E}$ | the universal set |
| $\cup$ | union |
| $\cap$ | intersection |
| $\subseteq$ | is a subset of |
| $\subset$ | is a proper subset of |
| $\nsubseteq$ | is not a subset of |
| $\not\subset$ | is not a proper subset of |
| $A'$ | the complement of the set $A$ |
| $a^{\frac{1}{n}},\ \sqrt[n]{a}$ | $a$ to the power of $\dfrac{1}{n}$, $n$th root of $a$ (if $a \geqslant 0$ then $\sqrt[n]{a} \geqslant 0$) |
| $a^{\frac{1}{2}},\ \sqrt{a}$ | $a$ to the power $\frac{1}{2}$, square root of $a$ (if $a \geqslant 0$ then $\sqrt{a} \geqslant 0$) |
| $\lvert x \rvert$ | the modulus or absolute value of $x$, that is $\begin{cases} x \text{ for } x \geqslant 0, & x \in \mathbb{R} \\ -x \text{ for } x < 0, & x \in \mathbb{R} \end{cases}$ |
| $\equiv$ | identity or is equivalent to |
| $\approx$ | is approximately equal to |
| $n!$ | $n$ factorial for $n \in \mathbb{N}$ $(0! = 1)$ |
| $\binom{n}{r}$ | the binomial coefficient $\dfrac{n!}{r!(n-r)!}$ for $n, r \in \mathbb{N},\ 0 \leqslant r \leqslant n$ |
| $>$ | is greater than |
| $\geq$ or $\geqslant$ | is greater than or equal to |
| $<$ | is less than |
| $\leq$ or $\leqslant$ | is less than or equal to |

| | |
|---|---|
| $\sum_{i=1}^{n} a_i$ | $a_1 + a_2 + .... + a_n$ |
| $f$ | function $f$ |
| $f : x \mapsto y$ | $f$ is a function under which $x$ is mapped to $y$ |
| $f(x)$ | the image of $x$ under the function $f$ |
| $f^{-1}$ | the inverse function of the function $f$ |
| $g \circ f$, $gf$ | the composite function of $f$ and $g$ |
| $\lim_{x \to a} f(x)$ | the limit of $f(x)$ as $x$ tends to $a$ |
| $\frac{dy}{dx}$ | the derivative of $y$ with respect to $x$ |
| $\frac{d^2y}{dx^2}$ | the second derivative of $y$ with respect to $x$ |
| $f'(x)$ | the derivative of $f(x)$ with respect to $x$ |
| $f''(x)$ | the second derivative of $f(x)$ with respect to $x$ |
| $\int y \, dx$ | the indefinite integral of $y$ with respect to $x$ |
| $\int_a^b y \, dx$ | the definite integral of $y$ with respect to $x$ for values of $x$ between $a$ and $b$ |
| $e$ | base of natural logarithms |
| $e^x$ | exponential function of $x$ |
| $\lg x$ | logarithm of $x$ to base 10 |
| $\ln x$ | natural logarithm of $x$ |
| $\log_a x$ | logarithm to the base $a$ of $x$ |
| sin, cos, tan, cosec, sec, cot | the circular functions |
| $\mathrm{A}(x, y)$ | the point A in the plane with Cartesian coordinates $x$ and $y$ |
| AB | the line segment with endpoints A and B<br>the distance from A to B<br>the line containing points A and B |
| $\widehat{\mathrm{A}}$ | the angle at A |
| $\mathrm{C\widehat{A}B}$ | the angle between CA and AB |
| $\triangle\mathrm{ABC}$ | the triangle whose vertices are A, B, and C |
| $\mathbf{a}$ | the vector $\mathbf{a}$ |
| $\overrightarrow{\mathrm{AB}}$ | the vector represented in magnitude and direction by the directed line segment from A to B |
| $\lvert\mathbf{a}\rvert$ | the magnitude of vector $\mathbf{a}$ |
| $\lvert\overrightarrow{\mathrm{AB}}\rvert$ | the magnitude of $\overrightarrow{\mathrm{AB}}$ |
| $\mathbf{i}$, $\mathbf{j}$ | unit vectors in the directions of the Cartesian coordinate axes |
| $\mathbf{M}$ | a matrix $\mathbf{M}$ |
| $\mathbf{M}^{-1}$ | the inverse of the square matrix $\mathbf{M}$ |
| $\det \mathbf{M}$ | the determinant of the square matrix $\mathbf{M}$ |

# TABLE OF CONTENTS

# 1

# Sets and Venn diagrams

**Contents:**

## Opening problem

A city has three football teams in the national league: $A$, $B$, and $C$.

In the last season, 20% of the city's population saw team $A$ play, 24% saw team $B$, and 28% saw team $C$. Of these, 4% saw both $A$ and $B$, 5% saw both $A$ and $C$, and 6% saw both $B$ and $C$. 1% saw all three teams play.

**Things to think about:**

**a** Writing out all of this information in sentences is very complicated. How can we represent this information more simply on a diagram?

**b** What percentage of the population:

**i** saw only team $A$ play

**ii** saw team $A$ *or* team $B$ play but not team $C$

**iii** did not see any of the teams play?

# A SETS

## SET NOTATION

A **set** is a collection of numbers or objects.

For example:

- the set of digits which we use to write numbers is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- if $V$ is the set of all vowels, then $V = \{a, e, i, o, u\}$.

The numbers or objects in a set are called the **elements** or **members** of the set.

We use the symbol $\in$ to mean *is an element of* and $\notin$ to mean *is not an element of*.

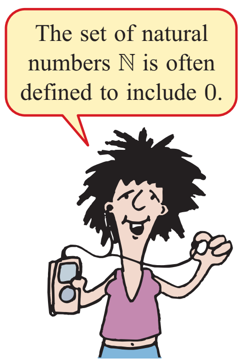
So, for the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ we can say $4 \in A$ but $9 \notin A$.

The set $\{\ \}$ or $\varnothing$ is called the **empty set** and contains no elements.

## SPECIAL NUMBER SETS

The following is a list of some special number sets you should be familiar with:

- $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, ....\}$ is the set of all **natural** or **counting numbers**.
- $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, ....\}$ is the set of all **integers**.
- $\mathbb{Z}^+ = \{1, 2, 3, 4, 5, 6, 7, ....\}$ is the set of all **positive integers**.
- $\mathbb{Z}^- = \{-1, -2, -3, -4, -5, ....\}$ is the set of all **negative integers**.
- $\mathbb{Q}$ is the set of all **rational numbers**, or numbers which can be written in the form $\frac{p}{q}$ where $p$ and $q$ are integers and $q \neq 0$.
- $\mathbb{R}$ is the set of all **real numbers**, which are all numbers which can be placed on the number line.



## COUNTING ELEMENTS OF SETS

The number of elements in set $A$ is written $n(A)$.

For example, the set $A = \{2, 3, 5, 8, 13, 21\}$ has 6 elements, so we write $n(A) = 6$.

A set which has a finite number of elements is called a **finite set**.

For example: $A = \{2, 3, 5, 8, 13, 21\}$ is a finite set.
$\varnothing$ is also a finite set, since $n(\varnothing) = 0$.

**Infinite sets** are sets which have infinitely many elements.

For example, the set of positive integers $\{1, 2, 3, 4, ....\}$ does not have a largest element, but rather keeps on going forever. It is therefore an infinite set.

In fact, the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$, $\mathbb{Q}$, and $\mathbb{R}$ are all infinite sets.

## SUBSETS

Suppose $A$ and $B$ are two sets. $A$ is a **subset** of $B$ if every element of $A$ is also an element of $B$. We write $A \subseteq B$.

For example, $\{2, 3, 5\} \subseteq \{1, 2, 3, 4, 5, 6\}$ as every element in the first set is also in the second set.

$A$ is a **proper subset** of $B$ if $A$ is a subset of $B$ but is *not equal to* $B$.
We write $A \subset B$.

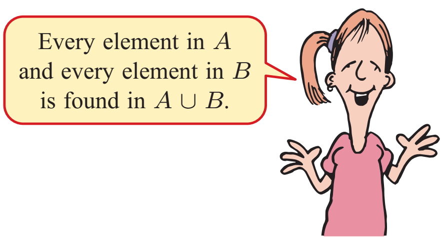
For example, $\mathbb{Z} \subset \mathbb{Q}$ since any integer $n = \frac{n}{1} \in \mathbb{Q}$. However, $\frac{1}{2} \in \mathbb{Q}$ but $\frac{1}{2} \notin \mathbb{Z}$.

We use $A \nsubseteq B$ to indicate that $A$ is *not* a subset of $B$
and $A \not\subset B$ to indicate that $A$ is *not* a proper subset of $B$.

## UNION AND INTERSECTION

If $A$ and $B$ are two sets, then

- $A \cap B$ is the **intersection** of $A$ and $B$, and consists of all elements which are in **both** $A$ **and** $B$
- $A \cup B$ is the **union** of $A$ and $B$, and consists of all elements which are in $A$ **or** $B$.




For example:

- If $A = \{1, 3, 4\}$ and $B = \{2, 3, 5\}$ then $A \cap B = \{3\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$.
- The set of integers is made up of the set of negative integers, zero, and the set of positive integers: $\mathbb{Z} = (\mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+)$

DEMO

## DISJOINT SETS

Two sets are **disjoint** or **mutually exclusive** if they have no elements in common.

If $A$ and $B$ are disjoint then $A \cap B = \varnothing$.

### Example 1

Self Tutor

$M = \{2, 3, 5, 7, 8, 9\}$ and $N = \{3, 4, 6, 9, 10\}$

**a** True or false? **i** $4 \in M$ **ii** $6 \notin M$

**b** List the sets: **i** $M \cap N$ **ii** $M \cup N$

**c** Is **i** $M \subseteq N$ **ii** $\{9, 6, 3\} \subseteq N$?

**a** **i** 4 is not an element of $M$, so $4 \in M$ is false.

**ii** 6 is not an element of $M$, so $6 \notin M$ is true.

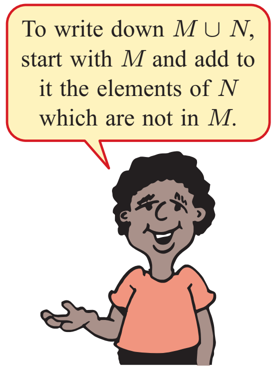
**b** **i** $M \cap N = \{3, 9\}$ since 3 and 9 are elements of both sets.

**ii** Every element which is in either $M$ or $N$ is in the union of $M$ and $N$.

$\therefore$ $M \cup N = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$

**c** **i** No. Not every element of $M$ is an element of $N$.

**ii** Yes, as 9, 6, and 3 are also in $N$.




## EXERCISE 1A

**1** Write using set notation:

**a** 5 is an element of set $D$

**b** 6 is not an element of set $G$

**c** $d$ is not an element of the set of all English vowels

**d** $\{2, 5\}$ is a subset of $\{1, 2, 3, 4, 5, 6\}$

**e** $\{3, 8, 6\}$ is not a subset of $\{1, 2, 3, 4, 5, 6\}$.

**2** Find **i** $A \cap B$ **ii** $A \cup B$ for:

**a** $A = \{6, 7, 9, 11, 12\}$ and $B = \{5, 8, 10, 13, 9\}$

**b** $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$

**c** $A = \{1, 3, 5, 7\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

**3** Suppose $A = \{0, 3, 5, 8, 14\}$ and $B = \{1, 4, 5, 8, 11, 13\}$. Write down the number of elements in:

**a** $A$ **b** $B$ **c** $A \cap B$ **d** $A \cup B$

**4** True or false?

**a** $\mathbb{Z}^+ \subseteq \mathbb{N}$ **b** $\mathbb{N} \subset \mathbb{Z}$ **c** $\mathbb{N} = \mathbb{Z}^+$ **d** $\mathbb{Z}^- \subseteq \mathbb{Z}$

**e** $\mathbb{Q} \subset \mathbb{Z}$ **f** $\{0\} \subseteq \mathbb{Z}$ **g** $\mathbb{Z} \subseteq \mathbb{Q}$ **h** $\mathbb{Z}^+ \cup \mathbb{Z}^- = \mathbb{Z}$

**5** Describe the following sets as either finite or infinite:

**a** the set of counting numbers between 10 and 20

**b** the set of counting numbers greater than 5

**c** the set of all rational numbers $\mathbb{Q}$

**d** the set of all rational numbers between 0 and 1.

**6** True or false?

**a** $127 \in \mathbb{N}$ **b** $\frac{138}{279} \in \mathbb{Q}$ **c** $3\frac{1}{7} \notin \mathbb{Q}$ **d** $-\frac{4}{11} \in \mathbb{Q}$

**7** Which of these pairs of sets are disjoint?

**a** $A = \{3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8\}$ **b** $P = \{3, 5, 6, 7, 8, 10\}$ and $Q = \{4, 9, 10\}$

**8** True or false? If $R$ and $S$ are two non-empty sets and $R \cap S = \varnothing$, then $R$ and $S$ are disjoint.

**9** **a** How many proper subsets does the set $\{a, b, c, d\}$ have?

**b** Copy and complete: "If a set has $n$ elements then it has ...... proper subsets."

# B INTERVAL NOTATION

To avoid having to list all members of a set, we often use a general description of its members. We often describe a set of all values of $x$ with a particular property.

The notation $\{x : ......\}$ or $\{x \mid ......\}$ is used to describe "the set of all $x$ such that ......".

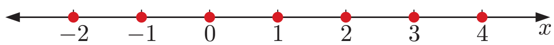
For example:

- $A = \{x \in \mathbb{Z} : -2 \leqslant x \leqslant 4\}$ reads "the set of all integers $x$ such that $x$ is between $-2$ and 4, including $-2$ and 4."

  such that

  the set of all

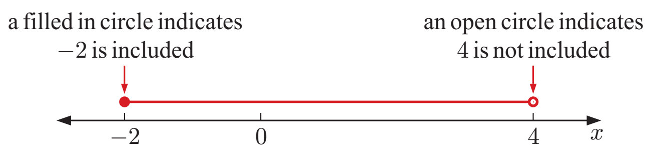
  We can represent $A$ on a number line as:




  $A$ is a finite set, and $n(A) = 7$.

- $B = \{x \in \mathbb{R} : -2 \leqslant x < 4\}$ reads "the set of all real $x$ such that $x$ is greater than or equal to $-2$ and less than 4."

  We represent $B$ on a number line as:




  $B$ is an infinite set, and $n(B) = \infty$.

  We could also write $B = \{x : -2 \leqslant x < 4\}$, in which case we would *assume* that $x \in \mathbb{R}$.

## Example 2 Self Tutor

Suppose $A = \{x \in \mathbb{Z} : 3 < x \leqslant 10\}$.

**a** Write down the meaning of the interval notation.

**b** List the elements of set $A$. **c** Find $n(A)$.

**a** The set of all integers $x$ such that $x$ is between 3 and 10, including 10.

**b** $A = \{4, 5, 6, 7, 8, 9, 10\}$ **c** There are 7 elements, so $n(A) = 7$.

## CLOSED AND OPEN INTERVALS

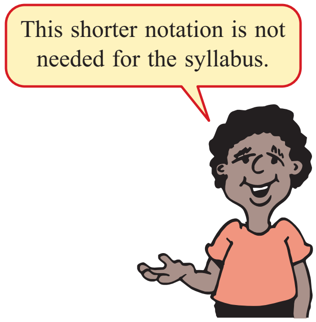
An **interval** is a connected subset of the number line $\mathbb{R}$.

An interval is **closed** if *both* of its endpoints are included.

An interval is **open** if *both* of its endpoints are *not* included.

For $x \in \mathbb{R}$, we commonly use the following notation to concisely write intervals:

$[a, b]$ represents the closed interval $\{x \in \mathbb{R} : a \leqslant x \leqslant b\}$
$[a, b)$ represents the interval $\{x \in \mathbb{R} : a \leqslant x < b\}$
$(a, b]$ represents the interval $\{x \in \mathbb{R} : a < x \leqslant b\}$
$(a, b)$ represents the open interval $\{x \in \mathbb{R} : a < x < b\}$

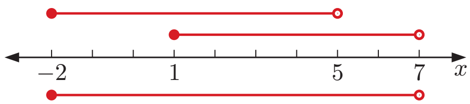



An interval which extends to infinity has no defined endpoint.
So, for $\{x \in \mathbb{R} : x \geqslant a\}$ we write $[a, \infty)$.

## INTERVALS WHICH OVERLAP

When two intervals overlap, we consolidate them into a single interval.

For example: $[-2, 5) \cup [1, 7) = [-2, 7)$




## EXERCISE 1B

**1** Explain whether the following sets are finite or infinite:

**a** $\{x \in \mathbb{Z} : -2 \leqslant x \leqslant 1\}$ **b** $\{x \in \mathbb{R} : -2 \leqslant x \leqslant 1\}$ **c** $\{x \in \mathbb{Z} : x \geqslant 5\}$
**d** $\{x \in \mathbb{Q} : 0 \leqslant x \leqslant 1\}$ **e** $(2, 4)$ **f** $[-3, 7]$
**g** $(-\infty, 0)$

**2** For the following sets:

**i** Write down the meaning of the interval notation.
**ii** If possible, list the elements of $A$.
**iii** Find $n(A)$.
**iv** If possible, sketch $A$ on a number line.


**a** $A = \{x \in \mathbb{Z} : -1 \leqslant x \leqslant 7\}$ **b** $A = \{x \in \mathbb{N} : -2 < x < 8\}$
**c** $A = \{x \in \mathbb{R} : 0 \leqslant x \leqslant 1\}$ **d** $A = \{x \in \mathbb{Q} : 5 \leqslant x \leqslant 6\}$
**e** $A = [-1, 5)$ **f** $A = \{x \in \mathbb{R} : 3 < x \leqslant 5 \ \cup \ x > 7\}$
**g** $A = (-\infty, 1] \cup (2, \infty)$ **h** $A = (-\infty, 2) \cup [1, \infty)$

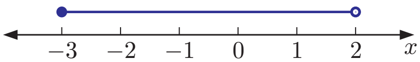
**3** Write in interval notation:

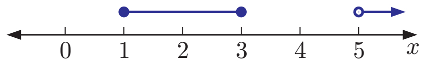
**a** the set of all integers between $-100$ and $100$
**b** the set of all real numbers greater than $1000$
**c** the set of all rational numbers between 2 and 3, including 2 and 3.

**4** Write using interval notation:

**a** $\{-2, -1, 0, 1, 2, 3\}$

**b** $\{...., -6, -5, -4, -3\}$

**c** 


**d** 


**5** State whether $A \subseteq B$:

**a** $A = \varnothing, \quad B = \{2, 5, 7, 9\}$

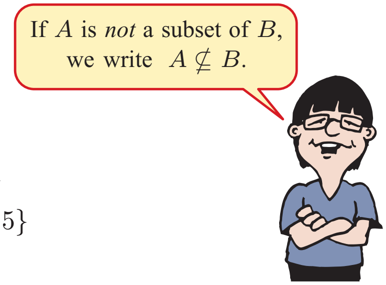
**b** $A = \{2, 5, 8, 9\}, \quad B = \{8, 9\}$

**c** $A = \{x \in \mathbb{R} : 2 \leqslant x \leqslant 3\}, \quad B = \{x \in \mathbb{R}\}$

**d** $A = \{x \in \mathbb{Q} : 3 \leqslant x \leqslant 9\}, \quad B = \{x \in \mathbb{R} : 0 \leqslant x \leqslant 10\}$

**e** $A = \{x \in \mathbb{Z} : -10 \leqslant x \leqslant 10\}, \quad B = \{z \in \mathbb{Z} : 0 \leqslant z \leqslant 5\}$

**f** $A = \{x \in \mathbb{Q} : 0 \leqslant x \leqslant 1\}, \quad B = \{y \in \mathbb{Q} : 0 < y \leqslant 2\}$




**6** For each of the following sets, determine whether the interval described is closed, open, or neither:

**a** $[2, 5)$ **b** $(-1, 3)$ **c** $(-\infty, -4]$ **d** $(4, \infty)$ **e** $[-2, 2]$ **f** $[0, 11)$

**7** Given that $\mathbb{Q}$ is the set of rational numbers, we can define
$\mathbb{Q}^+$ as the set of positive rational numbers, $\{x \in \mathbb{Q} : x > 0\}$
and $\mathbb{Q}_0^+$ as the set of positive rational numbers and zero, $\{x \in \mathbb{Q} : x \geqslant 0\}$.

**a** Explain why the set $\mathbb{Q}$ cannot be illustrated on a number line.

**b** Describe in words, in interval notation, and using a number line, what would be meant by the set:

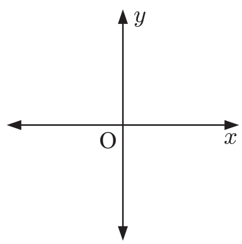
**i** $\mathbb{R}^+$ **ii** $\mathbb{R}_0^+$

## C RELATIONS

A **relation** is any set of points which connect two variables.

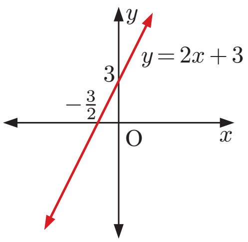
You should be familiar with points $(x, y)$ in the Cartesian plane.

Any set of these points is a relation.

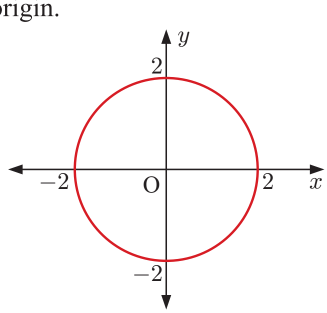



For example:

- $\{(x, y) : y = 2x + 3\}$ is the set of points which form a straight line with gradient 2 and $y$-intercept 3.




- $\{(x, y) : x^2 + y^2 = 4\}$ is the set of points which form a circle with radius 2 units centred at the origin.



## EXERCISE 1C

GRAPHING PACKAGE

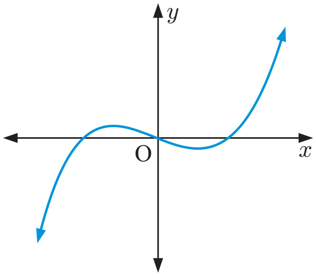
**1** Illustrate the following sets in the Cartesian plane. In each case state whether the set is finite or infinite.

**a** $\{(x, y) : y = x\}$ **b** $\{(x, y) : x + y = 1\}$

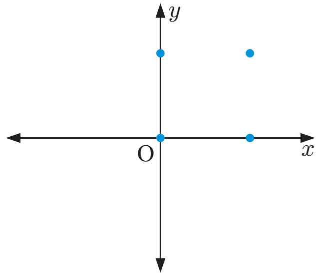
**c** $\{(x, y) : x > 0,\ y > 0\}$ **d** $\{(x, y) : x + y > 1\}$

**2** Let $A$ be the set of points in each graph below. State whether $A$ is finite or infinite.

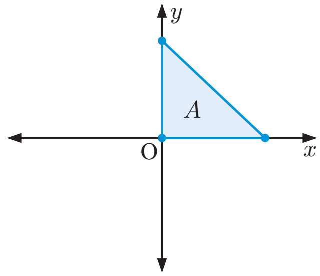
**a**




**b**




**c**




**3** Suppose $A$ is the set of points which define a straight line and $B$ is the set of points which define a circle.

**a** Describe in words the meaning of: **i** $A \cap B$ **ii** $A \cup B$

**b** Describe, with illustration, what it means if $n(A \cap B)$ equals: **i** 2 **ii** 1 **iii** 0

# D COMPLEMENTS OF SETS

## UNIVERSAL SETS

Suppose we are only interested in the natural numbers from 1 to 20, and we want to consider subsets of this set. We say the set $\mathcal{E} = \{x \in \mathbb{N} : 1 \leqslant x \leqslant 20\}$ is the *universal set* in this situation.

The symbol $\mathcal{E}$ is used to represent the **universal set** under consideration.

## COMPLEMENTARY SETS

The **complement** of $A$, denoted $A'$, is the set of all elements of $\mathcal{E}$ which are *not* in $A$.

$$A' = \{x \in \mathcal{E} : x \notin A\}$$

For example, if the universal set $\mathcal{E} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and the set $A = \{1, 3, 5, 7, 8\}$, then the complement of $A$ is $A' = \{2, 4, 6\}$.

Three obvious relationships are observed connecting $A$ and $A'$. These are:

- $A \cap A' = \varnothing$ as $A'$ and $A$ have no common members.
- $A \cup A' = \mathcal{E}$ as all elements of $A$ and $A'$ combined make up $\mathcal{E}$.
- $n(A) + n(A') = n(\mathcal{E})$

For example, $\mathbb{Q} \cap \mathbb{Q}' = \varnothing$ and $\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$.

### Example 3 — Self Tutor

Find $C'$ given that:

**a** $\mathscr{E} = \{\text{all positive integers}\}$ and $C = \{\text{all even integers}\}$

**b** $C = \{x \in \mathbb{Z} : x \geqslant 2\}$ and $\mathscr{E} = \mathbb{Z}$

**a** $C' = \{\text{all odd integers}\}$ **b** $C' = \{x \in \mathbb{Z} : x \leqslant 1\}$

### Example 4 — Self Tutor

Suppose $\mathscr{E} = \{x \in \mathbb{Z} : -5 \leqslant x \leqslant 5\}$, $A = \{x \in \mathbb{Z} : 1 \leqslant x \leqslant 4\}$, and $B = \{x \in \mathbb{Z} : -3 \leqslant x < 2\}$. List the elements of:

**a** $A$ **b** $B$ **c** $A'$ **d** $B'$
**e** $A \cap B$ **f** $A \cup B$ **g** $A' \cap B$ **h** $A' \cup B'$

**a** $A = \{1, 2, 3, 4\}$
**b** $B = \{-3, -2, -1, 0, 1\}$
**c** $A' = \{-5, -4, -3, -2, -1, 0, 5\}$
**d** $B' = \{-5, -4, 2, 3, 4, 5\}$
**e** $A \cap B = \{1\}$
**f** $A \cup B = \{-3, -2, -1, 0, 1, 2, 3, 4\}$
**g** $A' \cap B = \{-3, -2, -1, 0\}$
**h** $A' \cup B' = \{-5, -4, -3, -2, -1, 0, 2, 3, 4, 5\}$

## EXERCISE 1D

**1** Find the complement of $C$ given that:

**a** $\mathscr{E} = \{\text{letters of the English alphabet}\}$ and $C = \{\text{vowels}\}$

**b** $\mathscr{E} = \{\text{integers}\}$ and $C = \{\text{negative integers}\}$

**c** $\mathscr{E} = \mathbb{Z}$ and $C = \{x \in \mathbb{Z} : x \leqslant -5\}$

**d** $\mathscr{E} = \mathbb{Q}$ and $C = \{x \in \mathbb{Q} : x \leqslant 2 \cup x \geqslant 8\}$

**2** Suppose $\mathscr{E} = \{x \in \mathbb{Z} : 0 \leqslant x \leqslant 8\}$, $A = \{x \in \mathbb{Z} : 2 \leqslant x \leqslant 7\}$, and $B = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 8\}$. List the elements of:

**a** $A$ **b** $A'$ **c** $B$ **d** $B'$
**e** $A \cap B$ **f** $A \cup B$ **g** $A \cap B'$ **h** $A' \cup B'$

**3** Suppose $P$ and $Q'$ are subsets of $\mathscr{E}$. $n(\mathscr{E}) = 15$, $n(P) = 6$, and $n(Q') = 4$. Find:

**a** $n(P')$ **b** $n(Q)$

**4** True or false?

**a** If $n(\mathscr{E}) = a$ and $n(A) = b$ where $A \subseteq \mathscr{E}$, then $n(A') = b - a$.

**b** If $Q$ is a subset of $\mathscr{E}$ then $Q' = \{x \in \mathscr{E} : x \notin Q\}$.

**5** Suppose $\mathscr{E} = \{x \in \mathbb{Z} : 0 < x \leqslant 12\}$, $A = \{x \in \mathbb{Z} : 2 \leqslant x \leqslant 7\}$, $B = \{x \in \mathbb{Z} : 3 \leqslant x \leqslant 9\}$, and $C = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 11\}$.

List the elements of:

**a** $B'$ **b** $C'$ **c** $A'$ **d** $A \cap B$
**e** $(A \cap B)'$ **f** $A' \cap C$ **g** $B' \cup C$ **h** $(A \cup C) \cap B'$

**6** Consider the set of real numbers $\mathbb{R}$. Write down the complement of:

**a** $(-\infty, 0)$ **b** $[1, \infty)$ **c** $[-3, 2)$ **d** $(-5, 7]$
**e** $(-\infty, 1) \cup [3, \infty)$ **f** $[-5, 0) \cup (1, \infty)$

## E PROPERTIES OF UNION AND INTERSECTION

In this section we will explore the number of elements in unions and intersections of sets.

### Example 5 — Self Tutor

Suppose $\mathscr{E} = \{\text{positive integers}\}$, $P = \{\text{multiples of 4 less than 50}\}$, and $Q = \{\text{multiples of 6 less than 50}\}$.

**a** List $P$ and $Q$. **b** Find $P \cap Q$. **c** Find $P \cup Q$.
**d** Verify that $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$.

**a** $P = \{4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48\}$
$Q = \{6, 12, 18, 24, 30, 36, 42, 48\}$

**b** $P \cap Q = \{12, 24, 36, 48\}$

**c** $P \cup Q = \{4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 44, 48\}$

**d** $n(P \cup Q) = 16$ and $n(P) + n(Q) - n(P \cap Q) = 12 + 8 - 4 = 16$
So, $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$ is verified.

### EXERCISE 1E

**1** Suppose $\mathscr{E} = \mathbb{Z}^+$, $P = \{\text{prime numbers} < 25\}$, and $Q = \{2, 4, 5, 11, 12, 15\}$.

**a** List $P$. **b** Find $P \cap Q$. **c** Find $P \cup Q$.
**d** Verify that $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$.

**2** Suppose $\mathscr{E} = \mathbb{Z}^+$, $P = \{\text{factors of 28}\}$, and $Q = \{\text{factors of 40}\}$.

**a** List $P$ and $Q$. **b** Find $P \cap Q$. **c** Find $P \cup Q$.
**d** Verify that $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$.

**3** Suppose $\mathscr{E} = \mathbb{Z}^+$, $M = \{\text{multiples of 4 between 30 and 60}\}$, and $N = \{\text{multiples of 6 between 30 and 60}\}$.

**a** List $M$ and $N$. **b** Find $M \cap N$. **c** Find $M \cup N$.
**d** Verify that $n(M \cup N) = n(M) + n(N) - n(M \cap N)$.

**4** Suppose $\mathscr{E} = \mathbb{Z}$, $R = \{x \in \mathbb{Z} : -2 \leqslant x \leqslant 4\}$, and $S = \{x \in \mathbb{Z} : 0 \leqslant x < 7\}$.

**a** List $R$ and $S$. **b** Find $R \cap S$. **c** Find $R \cup S$.
**d** Verify that $n(R \cup S) = n(R) + n(S) - n(R \cap S)$.

**5** Suppose $\mathscr{E} = \mathbb{Z}$, $C = \{y \in \mathbb{Z} : -4 \leqslant y \leqslant -1\}$, and $D = \{y \in \mathbb{Z} : -7 \leqslant y < 0\}$.

**a** List $C$ and $D$. **b** Find $C \cap D$. **c** Find $C \cup D$.
**d** Verify that $n(C \cup D) = n(C) + n(D) - n(C \cap D)$.

**6** Suppose $\mathscr{E} = \mathbb{Z}^+$, $P = \{\text{factors of } 12\}$, $Q = \{\text{factors of } 18\}$, and $R = \{\text{factors of } 27\}$.

**a** List the sets $P$, $Q$, and $R$.

**b** Find: **i** $P \cap Q$ **ii** $P \cap R$ **iii** $Q \cap R$ **iv** $P \cup Q$ **v** $P \cup R$ **vi** $Q \cup R$

**c** Find: **i** $P \cap Q \cap R$ **ii** $P \cup Q \cup R$

**7** Suppose $\mathscr{E} = \mathbb{Z}^+$, $A = \{\text{multiples of 4 less than } 40\}$, $B = \{\text{multiples of 6 less than } 40\}$, and $C = \{\text{multiples of 12 less than } 40\}$.

**a** List the sets $A$, $B$, and $C$.

**b** Find: **i** $A \cap B$ **ii** $B \cap C$ **iii** $A \cap C$ **iv** $A \cap B \cap C$ **v** $A \cup B \cup C$

**c** Verify that
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.

**8** Suppose $\mathscr{E} = \mathbb{Z}^+$, $A = \{\text{multiples of 6 less than } 31\}$, $B = \{\text{factors of } 30\}$, and $C = \{\text{primes } < 30\}$.

**a** List the sets $A$, $B$, and $C$.

**b** Find: **i** $A \cap B$ **ii** $B \cap C$ **iii** $A \cap C$ **iv** $A \cap B \cap C$ **v** $A \cup B \cup C$

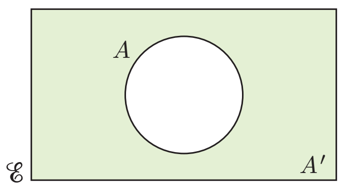
**c** Verify that
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.

## F VENN DIAGRAMS

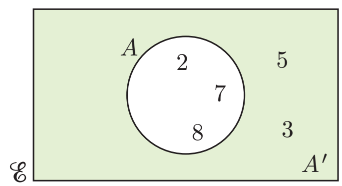
A **Venn diagram** consists of a universal set $\mathscr{E}$ represented by a rectangle. Sets within the universal set are usually represented by circles.

For example:

- This Venn diagram shows set $A$ within the universal set $\mathscr{E}$. $A'$, the complement of $A$, is the shaded region outside the circle.



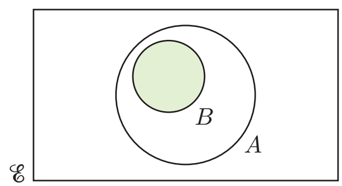

- The sets $\mathscr{E} = \{2, 3, 5, 7, 8\}$, $A = \{2, 7, 8\}$, and $A' = \{3, 5\}$ are represented by:




### SUBSETS

If $B \subseteq A$ then every element of $B$ is also in $A$.

The circle representing $B$ is placed within the circle representing $A$.

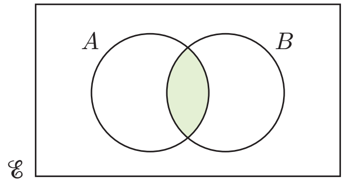



$B \subseteq A$

## INTERSECTION

$A \cap B$ consists of all elements common to both $A$ and $B$.

It is the shaded region where the circles representing $A$ and $B$ overlap.

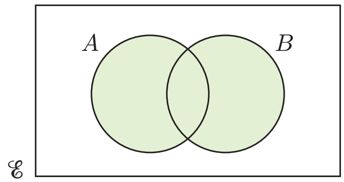


$A \cap B$

## UNION

$A \cup B$ consists of all elements in $A$ or $B$ or both.

It is the shaded region which includes both circles.



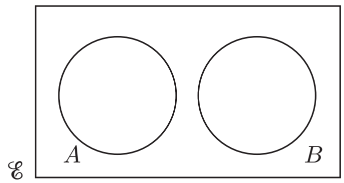
$A \cup B$

## DISJOINT OR MUTUALLY EXCLUSIVE SETS

Disjoint sets do not have common elements.

They are represented by non-overlapping circles.

For example, if $A = \{2, 3, 8\}$ and $B = \{4, 5, 9\}$
then $A \cap B = \varnothing$.



$A \cap B = \varnothing$

### Example 6 — Self Tutor

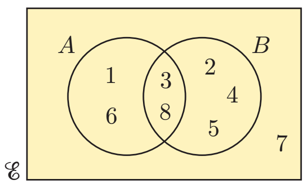
Suppose $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Illustrate on a Venn diagram the sets:

**a** $A = \{1, 3, 6, 8\}$ and $B = \{2, 3, 4, 5, 8\}$

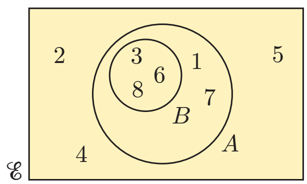
**b** $A = \{1, 3, 6, 7, 8\}$ and $B = \{3, 6, 8\}$

**c** $A = \{2, 4, 8\}$ and $B = \{3, 6, 7\}$.

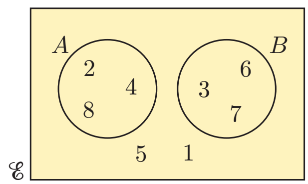
**a** $A \cap B = \{3, 8\}$




**b** $A \cap B = \{3, 6, 8\}$,
$B \subseteq A$




**c** $A \cap B = \varnothing$



## EXERCISE 1F.1

**1** Represent sets $A$ and $B$ on a Venn diagram, given:

**a** $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{2, 4, 5, 6\}$, and $B = \{1, 4, 6, 7\}$

**b** $\mathscr{E} = \{2, 3, 4, 5, 6, 7\}$, $A = \{2, 4, 6\}$, and $B = \{5, 7\}$

**c** $\mathscr{E} = \{2, 3, 4, 5, 6, 7\}$, $A = \{2, 4, 6\}$, and $B = \{3, 5, 7\}$

**d** $\mathscr{E} = \{3, 4, 5, 7\}$, $A = \{3, 4, 5, 7\}$, and $B = \{3, 5\}$

**2** Suppose $\mathscr{E} = \{x \in \mathbb{Z} : 1 \leqslant x \leqslant 10\}$, $A = \{\text{odd numbers} < 10\}$, and $B = \{\text{primes} < 10\}$.

**a** List sets $A$ and $B$.

**b** Find $A \cap B$ and $A \cup B$.

**c** Represent the sets $A$ and $B$ on a Venn diagram.

**3** Suppose $\mathscr{E} = \{x \in \mathbb{Z} : 1 \leqslant x \leqslant 9\}$, $A = \{\text{factors of } 6\}$, and $B = \{\text{factors of } 9\}$.

**a** List sets $A$ and $B$.

**b** Find $A \cap B$ and $A \cup B$.

**c** Represent the sets $A$ and $B$ on a Venn diagram.

**4** Suppose $\mathscr{E} = \{\text{even numbers between 0 and 30}\}$,
$P = \{\text{multiples of 4 less than 30}\}$, and
$Q = \{\text{multiples of 6 less than 30}\}$.

**a** List sets $P$ and $Q$.

**b** Find $P \cap Q$ and $P \cup Q$.

**c** Represent the sets $P$ and $Q$ on a Venn diagram.

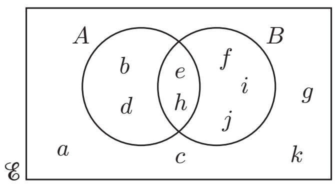
**5** Suppose $\mathscr{E} = \{x \in \mathbb{Z}^+ : x \leqslant 30\}$, $R = \{\text{primes less than 30}\}$, and
$S = \{\text{composites less than 30}\}$.

**a** List sets $R$ and $S$.

**b** Find $R \cap S$ and $R \cup S$.

**c** Represent the sets $R$ and $S$ on a Venn diagram.

**6**




List the elements of set:

**a** $A$

**b** $B$

**c** $A'$

**d** $B'$

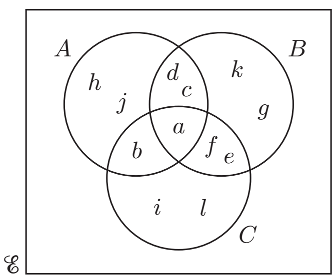
**e** $A \cap B$

**f** $A \cup B$

**g** $(A \cup B)'$

**h** $A' \cup B'$

**7**




This Venn diagram consists of three overlapping circles $A$, $B$, and $C$.

**a** List the letters in set:

**i** $A$

**ii** $B$

**iii** $C$

**iv** $A \cap B$

**v** $A \cup B$

**vi** $B \cap C$

**vii** $A \cap B \cap C$

**viii** $A \cup B \cup C$

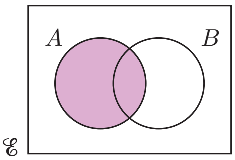
**b** Find: **i** $n(A \cup B \cup C)$

**ii** $n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

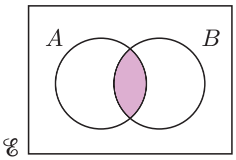
**c** What do you notice about your answers in **b**?

## USING VENN DIAGRAMS TO ILLUSTRATE REGIONS

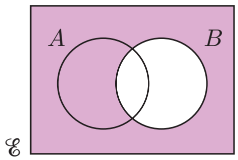
We can use shading to show various sets on a Venn diagram.
For example, for two intersecting sets $A$ and $B$:



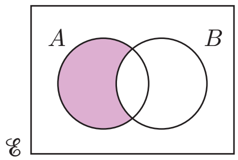
$A$ is shaded



$A \cap B$ is shaded



$B'$ is shaded



$A \cap B'$ is shaded

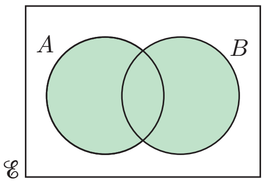
### Example 7

Self Tutor

Shade the following regions for two intersecting sets $A$ and $B$:

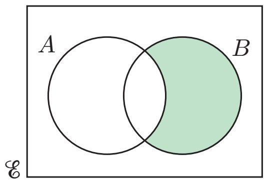
**a** $A \cup B$ **b** $A' \cap B$ **c** $(A \cap B)'$

**a**



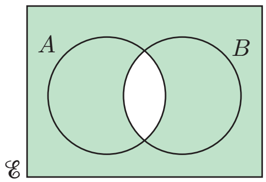
(in $A$, $B$, or both)

**b**



(outside $A$, intersected with $B$)

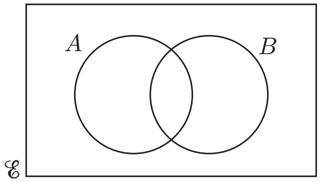
**c**



(outside $A \cap B$)

## EXERCISE 1F.2

**1**

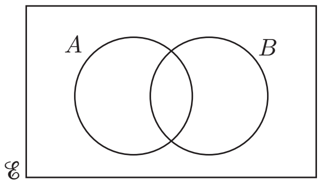



On separate Venn diagrams, shade regions for:

**a** $A \cap B$ **b** $A \cap B'$
**c** $A' \cup B$ **d** $A \cup B'$
**e** $A' \cap B$ **f** $A' \cap B'$

PRINTABLE VENN DIAGRAMS (OVERLAPPING)

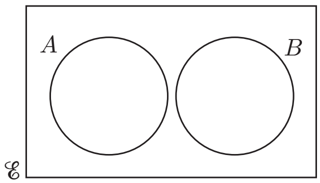
**2**




On separate Venn diagrams, shade regions for:

**a** $A \cup B$ **b** $(A \cup B)'$
**c** $(A \cap B)'$ **d** $A' \cup B'$
**e** $(A' \cup B')'$ **f** $(A \cup B')'$

**3**

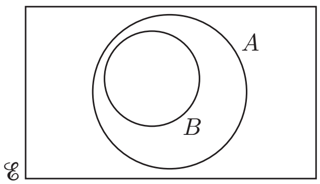



Suppose $A$ and $B$ are two disjoint sets. Shade on separate Venn diagrams:

**a** $A$ **b** $B$
**c** $A'$ **d** $B'$
**e** $A \cap B$ **f** $A \cup B$
**g** $A' \cap B$ **h** $A \cup B'$
**i** $(A \cap B)'$

PRINTABLE VENN DIAGRAMS (DISJOINT)

**4**

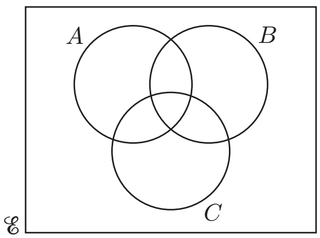



Suppose $B \subseteq A$, as shown in the given Venn diagram. Shade on separate Venn diagrams:

**a** $A$ **b** $B$
**c** $A'$ **d** $B'$
**e** $A \cap B$ **f** $A \cup B$
**g** $A' \cap B$ **h** $A \cup B'$
**i** $(A \cap B)'$

**PRINTABLE VENN DIAGRAMS (SUBSET)**

**5**




This Venn diagram consists of three intersecting sets. Shade on separate Venn diagrams:

**a** $A$ **b** $B'$
**c** $B \cap C$ **d** $A \cup B$
**e** $A \cap B \cap C$ **f** $A \cup B \cup C$
**g** $(A \cap B \cap C)'$ **h** $(B \cap C) \cup A$
**i** $(A \cup B) \cap C$ **j** $(A \cap C) \cup (B \cap C)$
**k** $(A \cap B) \cup C$ **l** $(A \cup C) \cap (B \cup C)$

**PRINTABLE VENN DIAGRAMS (3 SETS)**

Click on the icon to practise shading regions representing various subsets. You can practise with both two and three intersecting sets.

**VENN DIAGRAMS**

## *Discovery* — *The algebra of sets*

For the set of real numbers $\mathbb{R}$, we can write laws for the operations $+$ and $\times$:

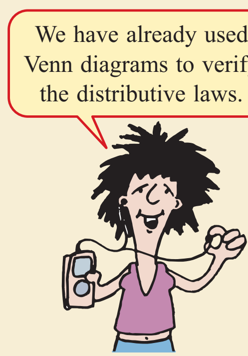
For any real numbers $a$, $b$, and $c$:

- **commutative** $a + b = b + a$ and $ab = ba$
- **identity** Identity elements 0 and 1 exist such that $a + 0 = 0 + a = a$ and $a \times 1 = 1 \times a = a$.
- **associativity** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$
- **distributive** $a(b + c) = ab + ac$

The following are the **laws for the algebra of sets** under the operations $\cup$ and $\cap$:

For any subsets $A$, $B$, and $C$ of the universal set $\mathscr{E}$:

- **commutative** $A \cap B = B \cap A$ and $A \cup B = B \cup A$
- **associativity** $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$
- **distributive** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **identity** $A \cup \varnothing = A$ and $A \cap \mathscr{E} = A$
- **complement** $A \cup A' = \mathscr{E}$ and $A \cap A' = \varnothing$
- **domination** $A \cup \mathscr{E} = \mathscr{E}$ and $A \cap \varnothing = \varnothing$
- **idempotent** $A \cap A = A$ and $A \cup A = A$
- **DeMorgan's** $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$
- **involution** $(A')' = A$



**What to do:**

**1** With the aid of Venn diagrams, explain why the following laws are valid:

**a** the *complement* law $(A')' = A$

**b** the *commutative* laws $A \cap B = B \cap A$ and $A \cup B = B \cup A$

**c** the *idempotent* laws $A \cap A = A$ and $A \cup A = A$

**d** the *associative* laws $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$

**e** the *distributive* laws $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

**2** Use the laws for the algebra of sets to show that:

**a** $A \cup (B \cup A') = \mathscr{E}$

**b** $A \cap (B \cap A') = \varnothing$

**c** $A \cup (B \cap A') = A \cup B$

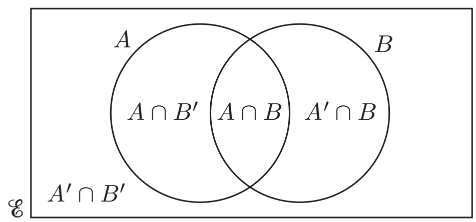
**d** $(A' \cup B')' = A \cap B$

**e** $(A \cup B) \cap (A' \cap B') = \varnothing$

**f** $(A \cup B) \cap (C \cup D) = (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$.

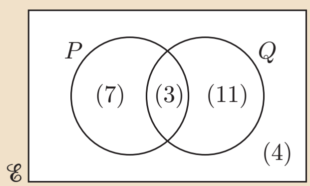
# G NUMBERS IN REGIONS

We have seen that there are four regions on a Venn diagram which contains two overlapping sets $A$ and $B$.




There are many situations where we are only interested in the **number of elements** of $\mathscr{E}$ that are in each region. We do not need to show all the elements on the diagram, so instead we write the number of elements in each region in brackets.

## Example 8 Self Tutor




In the Venn diagram given, (3) means that there are 3 elements in the set $P \cap Q$.

How many elements are there in:

**a** $P$

**b** $Q'$

**c** $P \cup Q$

**d** $P$, but not $Q$

**e** $Q$, but not $P$

**f** neither $P$ nor $Q$?

**a** $n(P) = 7 + 3 = 10$

**b** $n(Q') = 7 + 4 = 11$

**c** $n(P \cup Q) = 7 + 3 + 11 = 21$

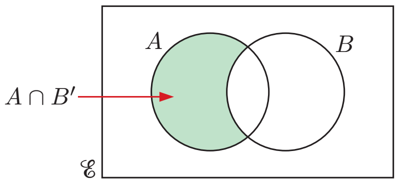
**d** $n(P, \text{but not } Q) = 7$

**e** $n(Q, \text{but not } P) = 11$

**f** $n(\text{neither } P \text{ nor } Q) = 4$

Venn diagrams allow us to easily visualise identities such as

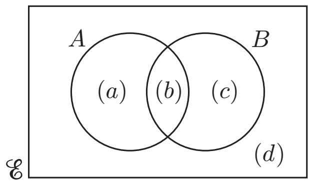
$$n(A \cap B') = n(A) - n(A \cap B)$$
$$n(A' \cap B) = n(B) - n(A \cap B)$$




## Example 9 — Self Tutor

Given $n(\mathscr{E}) = 30$, $n(A) = 14$, $n(B) = 17$, and $n(A \cap B) = 6$, find:

**a** $n(A \cup B)$ **b** $n(A, \text{ but not } B)$

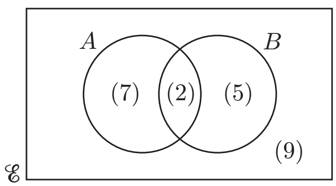



We see that $b = 6$ {as $n(A \cap B) = 6$}
$a + b = 14$ {as $n(A) = 14$}
$b + c = 17$ {as $n(B) = 17$}
$a + b + c + d = 30$ {as $n(\mathscr{E}) = 30$}
$\therefore$ $b = 6$, $a = 8$, and $c = 11$
$\therefore$ $8 + 6 + 11 + d = 30$
$\therefore$ $d = 5$

**a** $n(A \cup B) = a + b + c = 25$ **b** $n(A, \text{ but not } B) = a = 8$

## EXERCISE 1G

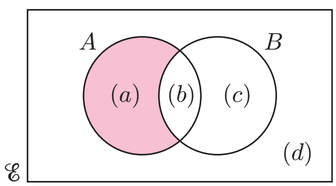
**1**




In the Venn diagram given, (2) means that there are 2 elements in the set $A \cap B$.
How many elements are there in:

**a** $B$ **b** $A'$
**c** $A \cup B$ **d** $A$, but not $B$
**e** $B$, but not $A$ **f** neither $A$ nor $B$?

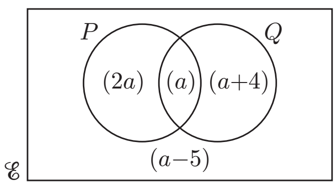
**2**




In the Venn diagram given, $(a)$ means that there are $a$ elements in the shaded region.
Notice that $n(A) = a + b$. Find:

**a** $n(B)$ **b** $n(A')$
**c** $n(A \cap B)$ **d** $n(A \cup B)$
**e** $n((A \cap B)')$ **f** $n((A \cup B)')$

**3**

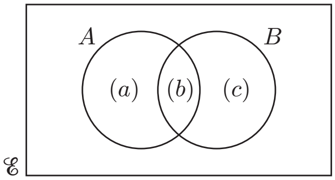



The Venn diagram shows that $n(P \cap Q) = a$ and $n(P) = 3a$.

**a** Find:
**i** $n(Q)$ **ii** $n(P \cup Q)$ **iii** $n(Q')$ **iv** $n(\mathscr{E})$

**b** Find $a$ if:
**i** $n(\mathscr{E}) = 29$ **ii** $n(\mathscr{E}) = 31$

Comment on your results.

**4** 


Use the Venn diagram to show that:
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

**5** Given $n(\mathscr{E}) = 26$, $n(A) = 11$, $n(B) = 12$, and $n(A \cap B) = 8$, find:

**a** $n(A \cup B)$ **b** $n(B, \text{ but not } A)$

**6** Given $n(\mathscr{E}) = 32$, $n(M) = 13$, $n(M \cap N) = 5$, and $n(M \cup N) = 26$, find:

**a** $n(N)$ **b** $n((M \cup N)')$

**7** Given $n(\mathscr{E}) = 50$, $n(S) = 30$, $n(R) = 25$, and $n(R \cup S) = 48$, find:

**a** $n(R \cap S)$ **b** $n(S, \text{ but not } R)$

## H PROBLEM SOLVING WITH VENN DIAGRAMS

In this section we use Venn diagrams to illustrate real world situations. We can solve problems by considering the number of elements in each region.

### Example 10 Self Tutor

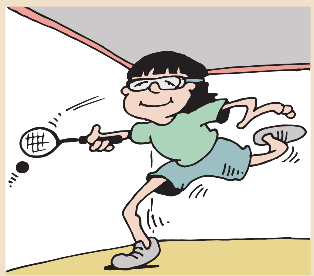
A squash club has 27 members. 19 have black hair, 14 have brown eyes, and 11 have both black hair and brown eyes.

**a** Place this information on a Venn diagram.

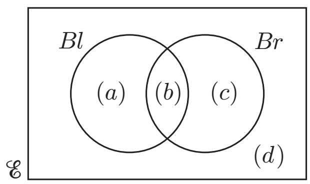
**b** Hence find the number of members with:

**i** black hair or brown eyes

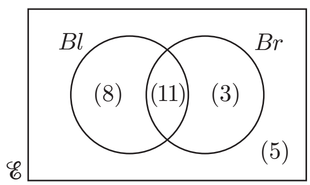
**ii** black hair, but not brown eyes.



**a** Let $Bl$ represent the black hair set and $Br$ represent the brown eyes set.




$a + b + c + d = 27$ {total members}
$a + b = 19$ {black hair}
$b + c = 14$ {brown eyes}
$b = 11$ {black hair and brown eyes}
$\therefore \; a = 8, \; c = 3, \; d = 5$




**b** **i** $n(Bl \cup Br) = 8 + 11 + 3 = 22$

22 members have black hair or brown eyes.

**ii** $n(Bl \cap Br') = 8$

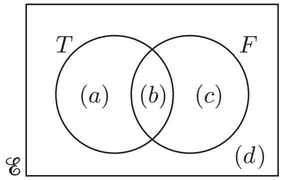
8 members have black hair, but not brown eyes.

## Example 11

Self Tutor

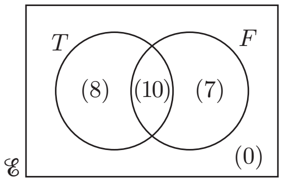
A platform diving squad of 25 has 18 members who dive from 10 m and 17 who dive from 5 m. How many dive from both platforms?

Let $T$ represent those who dive from 10 m and $F$ represent those who dive from 5 m.




$d = 0$ {as all divers in the squad must dive from at least one of the platforms}

$$a + b = 18$$
$$b + c = 17 \qquad \therefore\ a = 8,\ b = 10,\ c = 7$$
$$a + b + c = 25$$

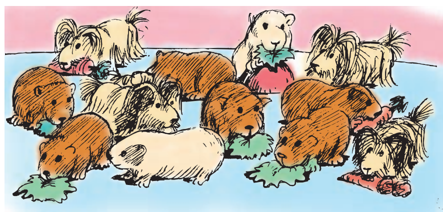



$$n(\text{both } T \text{ and } F) = n(T \cap F)$$
$$= 10$$

10 members dive from both platforms.

## EXERCISE 1H

**1** Pelé has 14 cavies as pets. Five have long hair and 8 are brown. Two are both brown and have long hair.



- **a** Place this information on a Venn diagram.
- **b** Hence find the number of cavies that:
  - **i** do not have long hair
  - **ii** have long hair and are not brown
  - **iii** are neither long-haired nor brown.

**2**

During a 2 week period, Murielle took her umbrella with her on 8 days. It rained on 9 days, and Murielle took her umbrella on five of the days when it rained.

- **a** Display this information on a Venn diagram.
- **b** Hence find the number of days that:
  - **i** Murielle did not take her umbrella and it rained
  - **ii** Murielle did not take her umbrella and it did not rain.

**3** A badminton club has 31 playing members. 28 play singles and 16 play doubles. How many play both singles and doubles?

**4** In a factory, 56 people work on the assembly line. 47 work day shifts and 29 work night shifts. How many work both day shifts and night shifts?

### Example 12

**Self Tutor**

Consider the **Opening Problem** on page **12**:

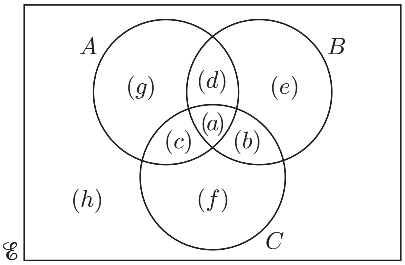
A city has three football teams in the national league: $A$, $B$, and $C$.

In the last season, 20% of the city's population saw team $A$ play, 24% saw team $B$, and 28% saw team $C$. Of these, 4% saw both $A$ and $B$, 5% saw both $A$ and $C$, and 6% saw both $B$ and $C$. 1% saw all three teams play.

Using a Venn diagram, find the percentage of the city's population which:

**a** saw only team $A$ play
**b** saw team $A$ or team $B$ play but not team $C$
**c** did not see any of the teams play.

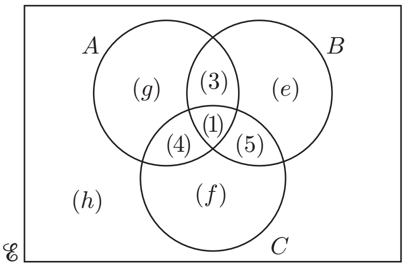
We construct the Venn diagram in terms of percentages.




Using the given information,

$$\begin{aligned} a &= 1 && \{1\% \text{ saw all three teams play}\} \\ a+d &= 4 && \{4\% \text{ saw } A \text{ and } B\} \\ a+b &= 6 && \{6\% \text{ saw } B \text{ and } C\} \\ a+c &= 5 && \{5\% \text{ saw } A \text{ and } C\} \end{aligned}$$

$\therefore\ d = 3,\ b = 5,$ and $c = 4$

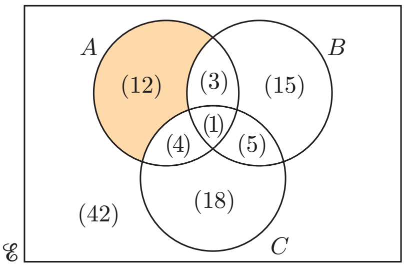



In total, 20% saw team $A$ play,
so $g + 1 + 4 + 3 = 20 \quad \therefore\ g = 12$

24% saw team $B$ play,
so $e + 1 + 5 + 3 = 24 \quad \therefore\ e = 15$

28% saw team $C$ play,
so $f + 1 + 5 + 4 = 28 \quad \therefore\ f = 18$

In total we cover 100% of the population, so $h = 42$.




**a** $n(\text{saw } A \text{ only}) = 12\%$ {shaded}

**b** $n(A \text{ or } B, \text{ but not } C)$
$= 12\% + 3\% + 15\%$
$= 30\%$

**c** $n(\text{saw none of the teams}) = 42\%$

**5** In a year group of 63 students, 22 study Biology, 26 study Chemistry, and 25 study Physics. 18 study both Physics and Chemistry, 4 study both Biology and Chemistry, and 3 study both Physics and Biology. 1 studies all three subjects.

**a** Display this information on a Venn diagram.

**b** How many students study:

**i** Biology only
**ii** Physics or Chemistry
**iii** none of Biology, Physics, or Chemistry
**iv** Physics but not Chemistry?

**6** 36 students participated in the mid-year adventure trip. 19 students went paragliding, 21 went abseiling, and 16 went white water rafting. 7 went abseiling and rafting, 8 went paragliding and rafting, and 11 went paragliding and abseiling. 5 students did all three activities.
Find the number of students who:

**a** went paragliding or abseiling
**b** only went white water rafting
**c** did not participate in any of the activities mentioned
**d** did exactly two of the activities mentioned.

**7**

There are 32 students in the woodwind section of the school orchestra. 11 students can play the flute, 15 can play the clarinet, and 12 can play the saxophone. 2 can play the flute and the saxophone, 2 can play the flute and the clarinet, and 6 can play the clarinet and the saxophone. 1 student can play all three instruments.
Find the number of students who can play:

**a** none of the instruments mentioned
**b** only the saxophone
**c** the saxophone and the clarinet, but not the flute
**d** only one of the clarinet, saxophone, or flute.

**8** In a particular region, most farms have livestock and crops. A survey of 21 farms showed that 15 grow crops, 9 have cattle, and 11 have sheep. 4 have sheep and cattle, 7 have cattle and crops, and 8 have sheep and crops. 3 have cattle, sheep, and crops. Find the number of farms with:

**a** only crops **b** only animals **c** exactly one type of animal, and crops.

## Review set 1A

**1** Suppose $S = \{x \in \mathbb{Z} : 2 < x \leqslant 7\}$.

**a** List the elements of $S$.
**b** Find $n(S)$.
**c** How many proper subsets does $S$ have?

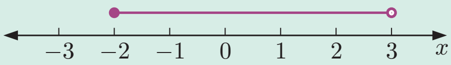
**2** Determine whether $A \subseteq B$ for the following sets:

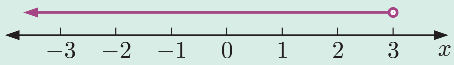
**a** $A = \{2, 4, 6, 8\}$ and $B = \{x \in \mathbb{Z} : 0 < x < 10\}$
**b** $A = \varnothing$ and $B = \{x \in (2, 3)\}$
**c** $A = \{x \in \mathbb{Q} : 2 < x \leqslant 4\}$ and $B = \{x \in \mathbb{R} : 0 \leqslant x < 4\}$
**d** $A = \{x \in (-\infty, 3)\}$ and $B = \{x \in (-\infty, 4]\}$

**3** Find the complement of $X$ given that:

**a** $\mathscr{E} = \{\text{the 7 colours of the rainbow}\}$ and $X = \{\text{red, indigo, violet}\}$
**b** $\mathscr{E} = \{x \in \mathbb{Z} : -5 \leqslant x \leqslant 5\}$ and $X = \{-4, -1, 3, 4\}$
**c** $\mathscr{E} = \{x \in \mathbb{Q}\}$ and $X = \{x \in \mathbb{Q} : x < -8\}$
**d** $\mathscr{E} = \{x \in \mathbb{R}\}$ and $X = \{x \in [-3, 1) \cup (4, \infty)\}$

**4** Write using interval notation, and state whether the interval is closed, open, or neither:

**a** 

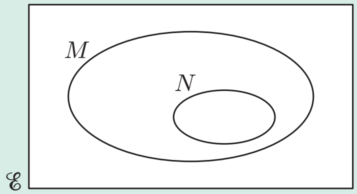

**b** 


**5** Illustrate in the Cartesian plane:

**a** $\{(x, y) : y = -2x\}$ **b** $\{(x, y) : x \leqslant y\}$

**6** On separate Venn diagrams like the one alongside, shade:

**a** $N'$ **b** $M \cap N$ **c** $M \cap N'$




**7** Let $\mathscr{E} = \{$the letters in the English alphabet$\}$, $A = \{$the letters in "springbok"$\}$, and $B = \{$the letters in "waterbuck"$\}$.

**a** Find:

**i** $A \cup B$ **ii** $A \cap B$ **iii** $A \cap B'$

**b** Write a description for each of the sets in **a**.

**c** Show $\mathscr{E}$, $A$, and $B$ on a Venn diagram.

**8** Let $\mathscr{E} = \{x \in \mathbb{Z}^+ : x \leqslant 30\}$, $P = \{$factors of 24$\}$, and $Q = \{$factors of 30$\}$.

**a** List the elements of:

**i** $P$ **ii** $Q$ **iii** $P \cap Q$ **iv** $P \cup Q$

**b** Illustrate the sets $P$ and $Q$ on a Venn diagram.

**9** A school has 564 students. During Term 1, 229 of them were absent for at least one day due to sickness, and 111 students missed some school because of family holidays. 296 students attended every day of Term 1.

**a** Display this information on a Venn diagram.

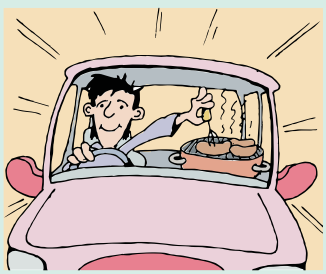
**b** Find the number of students who were away:

**i** for both sickness and holidays **ii** for holidays but not sickness

**iii** during Term 1 for any reason.

**10** The main courses at a restaurant all contain rice or onion. Of the 23 choices, 17 contain onion and 14 contain rice. How many dishes contain both rice and onion?

**11** 38 students were asked what life skills they had. 15 could swim, 12 could drive, and 23 could cook. 9 could cook and swim, 5 could swim and drive, and 6 could drive and cook. There was 1 student who could do all three. Find the number of students who:



**a** could only cook

**b** could not do any of these things

**c** had exactly two life skills.

**12** Consider the sets $\mathscr{E} = \{x \in \mathbb{Z}^+ : x \leqslant 10\}$, $P = \{\text{odd numbers less than } 10\}$, and $Q = \{\text{even numbers less than } 11\}$.

**a** List the sets $P$ and $Q$.

**b** What can be said about sets $P$ and $Q$?

**c** Illustrate sets $P$ and $Q$ on a Venn diagram.

## *Review set 1B*

**1** True or false?

**a** $\mathbb{N} \subset \mathbb{Q}$

**b** $0 \in \mathbb{Z}^+$

**c** $0 \in \mathbb{Q}$

**d** $\mathbb{R} \subseteq \mathbb{Q}$

**e** $\mathbb{Z}^+ \cap \mathbb{Z}^- = \{0\}$

**2** **a** Write using interval notation:

**i** the real numbers between 5 and 12

**ii** the integers between $-4$ and 7, including $-4$

**iii** the natural numbers greater than 45.

**b** Which sets in **a** are finite and which are infinite?

**3** List the subsets of $\{1, 3, 5\}$.

**4** Let $\mathscr{E} = \{x \in \mathbb{Z} : 0 < x < 10\}$, $A = \{\text{the even integers between 0 and 9}\}$, and $B = \{\text{the factors of } 8\}$.

**a** List the elements of:

**i** $A$

**ii** $A \cap B$

**iii** $(A \cup B)'$

**b** Represent this information on a Venn diagram.

**5** $S$ and $T$ are disjoint sets. $n(S) = s$ and $n(T) = t$. Find:

**a** $S \cap T$

**b** $n(S \cup T)$

**6** For each of the following sets, determine whether the interval described is closed, open, or neither:

**a** $x \in (-4, 3]$

**b** $x \in [-2, 2]$

**c** $x \in \mathbb{R}$

**7** Suppose $A$ and $B$ are each sets of points which define straight lines.

**a** Describe in words the meaning of:

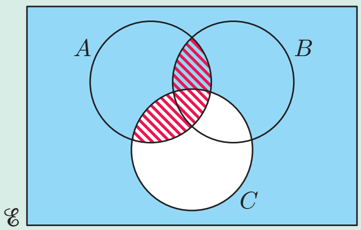
**i** $A \cap B$

**ii** $A \cup B$

**b** Is $A \cap B$ necessarily finite? Explain your answer.

**c** If $A \cap B$ is finite, what possible values can $n(A \cap B)$ have?

**8**




Give an expression for the region shaded in:

**a** blue

**b** red.

**9** In a car club, 13 members drive a manual and 15 members have a sunroof on their car. 5 have manual cars with a sunroof, and 4 have neither.

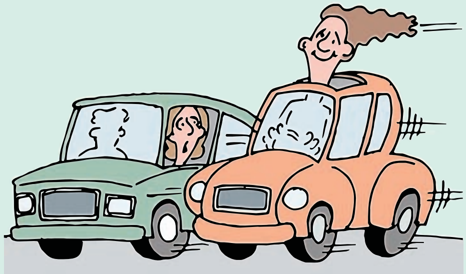
**a** Display this information on a Venn diagram.

**b** How many members:

**i** are in the club

**ii** drive a manual car without a sunroof

**iii** do not drive a manual car?



**10** All attendees of a camp left something at home. 11 forgot to bring their towel, and 23 forgot their hat. Of the 30 campers, how many had neither a hat nor a towel?

**11** Consider the sets $\mathscr{E} = \{x \in \mathbb{Z}^+ : x \leqslant 40\}$, $A = \{\text{factors of } 40\}$, and $B = \{\text{factors of } 20\}$.

**a** List the sets $A$ and $B$.

**b** What can be said about sets $A$ and $B$?

**c** Illustrate sets $A$ and $B$ on a Venn diagram.

**12** At a conference, the 58 delegates speak many different languages. 28 speak Arabic, 27 speak Chinese, and 39 speak English. 12 speak Arabic and Chinese, 16 speak both Chinese and English, and 17 speak Arabic and English. 2 speak all three languages. How many delegates speak:

**a** Chinese only

**b** none of these languages

**c** neither Arabic nor Chinese?

# 2

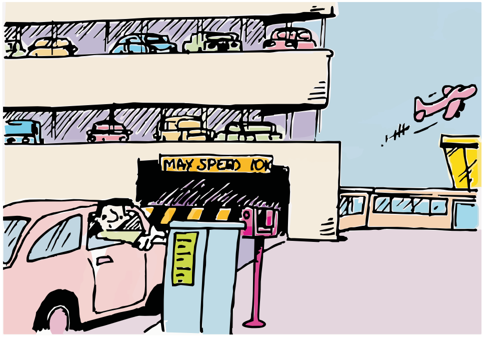
# Functions

**Contents:**

# A RELATIONS AND FUNCTIONS

The charges for parking a car in a short-term car park at an airport are shown in the table below. The total charge is *dependent* on the length of time $t$ the car is parked.

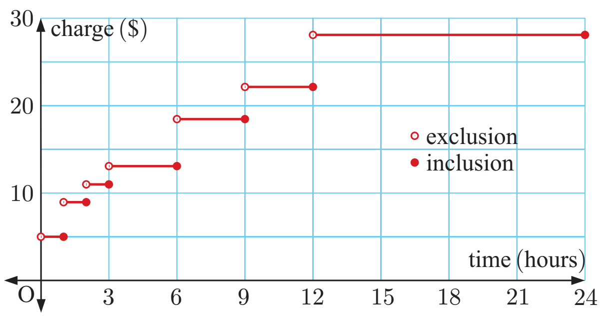
| Car park charges | |
|---|---|
| *Time t* (hours) | *Charge* |
| 0 - 1 hours | \$5.00 |
| 1 - 2 hours | \$9.00 |
| 2 - 3 hours | \$11.00 |
| 3 - 6 hours | \$13.00 |
| 6 - 9 hours | \$18.00 |
| 9 - 12 hours | \$22.00 |
| 12 - 24 hours | \$28.00 |



Looking at this table we might ask: How much would be charged for *exactly* one hour? Would it be \$5 or \$9?

To avoid confusion, we could adjust the table or draw a graph. We indicate that 2 - 3 hours really means a time over 2 hours up to and including 3 hours, by writing $2 < t \leqslant 3$ hours.

| Car park charges | |
|---|---|
| *Time t* (hours) | *Charge* |
| $0 < t \leqslant 1$ hours | \$5.00 |
| $1 < t \leqslant 2$ hours | \$9.00 |
| $2 < t \leqslant 3$ hours | \$11.00 |
| $3 < t \leqslant 6$ hours | \$13.00 |
| $6 < t \leqslant 9$ hours | \$18.00 |
| $9 < t \leqslant 12$ hours | \$22.00 |
| $12 < t \leqslant 24$ hours | \$28.00 |




In mathematical terms, we have a relationship between two variables *time* and *charge*, so the schedule of charges is an example of a **relation**.

A relation may consist of a finite number of ordered pairs, such as $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$, or an infinite number of ordered pairs.

The parking charges example is clearly the latter, as every real value of time in the interval $0 < t \leqslant 24$ hours is represented.

The set of possible values of the variable on the horizontal axis is called the **domain** of the relation.

For example:
- the domain for the car park relation is $\{t \mid 0 < t \leqslant 24\}$
- the domain of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ is $\{-2, 1, 4\}$.

The set of possible values on the vertical axis is called the **range** of the relation.

For example:
- the range of the car park relation is $\{5, 9, 11, 13, 18, 22, 28\}$
- the range of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ is $\{3, 5, 6\}$.

We will now look at relations and functions more formally.

## RELATIONS

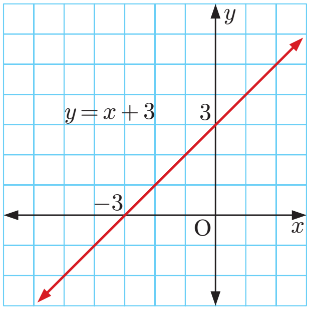
In **Chapter 1**, we saw that:

A **relation** is any set of points which connect two variables.

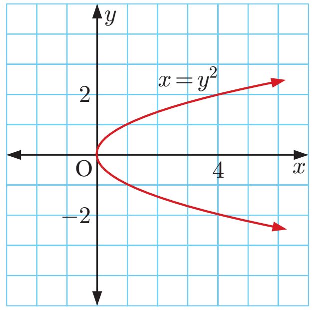
A relation is often expressed in the form of an **equation** connecting the **variables** $x$ and $y$. The relation is a set of points $(x, y)$ which can be viewed in the **Cartesian plane**.

For example, $y = x + 3$ and $x = y^2$ are the equations of two relations. Each equation generates a set of ordered pairs, which we can graph.

$y = x + 3$ is a set of points which lie in a straight line




$x = y^2$ is a set of points which lie in a smooth curve.




## FUNCTIONS

A **function**, sometimes called a **mapping**, is a relation in which no two different ordered pairs have the same $x$-coordinate or first component.

We can see from the above definition that a function is a special type of relation.

Every function is a relation, but not every relation is a function.

## TESTING FOR FUNCTIONS

**Algebraic Test:**

If a relation is given as an equation, and the substitution of any value for $x$ results in one and only one value of $y$, then the relation is a function.

For example:

- $y = 3x - 1$ is a function, since for any value of $x$ there is only one corresponding value of $y$
- $x = y^2$ is not a function, since if $x = 4$ then $y = \pm 2$.

**Geometric Test or Vertical Line Test:**

Suppose we draw all possible vertical lines on the graph of a relation.

- If each line cuts the graph at most once, then the relation is a function.
- If at least one line cuts the graph more than once, then the relation is *not* a function.

## GRAPHICAL NOTE

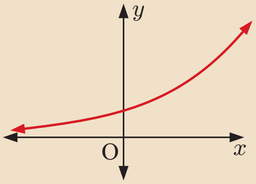
- If a graph contains a small **open circle** such as  , this point is **not included**.
- If a graph contains a small **filled-in circle** such as  , this point **is included**.
- If a graph contains an **arrow head** at an end such as  , then the graph continues indefinitely in that general direction, or the shape may repeat as it has done previously.

## Example 1

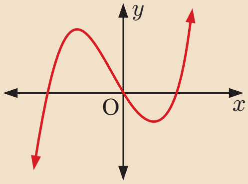
Self Tutor

Which of the following relations are functions?

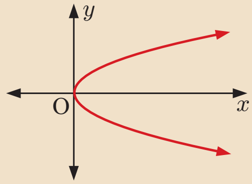
**a**



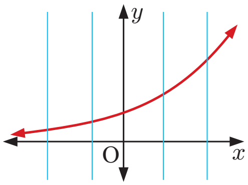

**b**




**c**

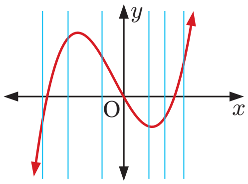



**a**



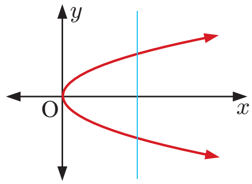

a function

**b**




a function

**c**




not a function

DEMO

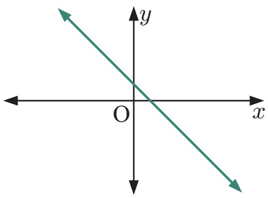
## EXERCISE 2A.1

**1** Which of the following sets of ordered pairs are functions? Give reasons for your answers.

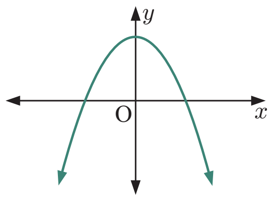
**a** $\{(1, 3), (2, 4), (3, 5), (4, 6)\}$
**b** $\{(1, 3), (3, 2), (1, 7), (-1, 4)\}$
**c** $\{(2, -1), (2, 0), (2, 3), (2, 11)\}$
**d** $\{(7, 6), (5, 6), (3, 6), (-4, 6)\}$
**e** $\{(0, 0), (1, 0), (3, 0), (5, 0)\}$
**f** $\{(0, 0), (0, -2), (0, 2), (0, 4)\}$

**2** Use the vertical line test to determine which of the following relations are functions:

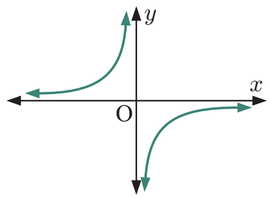
**a**



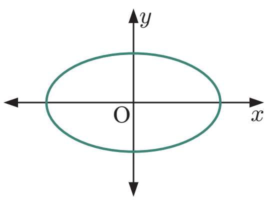

**b**



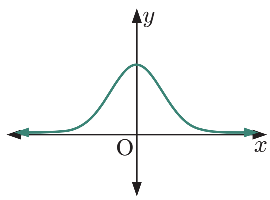

**c**



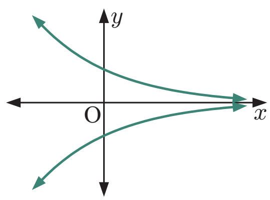

**d**



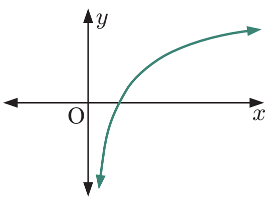

**e**



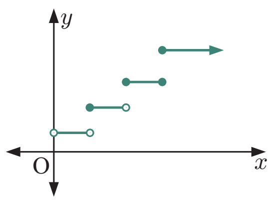

**f**



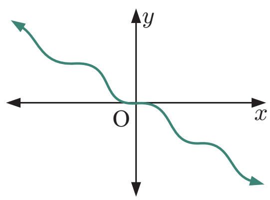

**g**




**h**




**i**

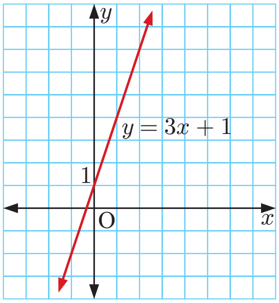



**3** Give algebraic evidence to show that the relation $x^2 + y^2 = 16$ is not a function.

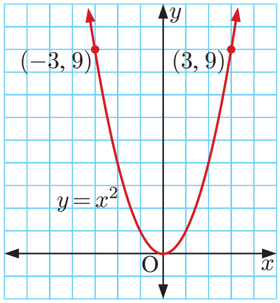
## ONE-ONE FUNCTIONS

A **one-one function** is a function in which no two different ordered pairs have the same $y$-coordinate or second component.

For example:

- 


The function $y = 3x + 1$ is one-one, since each distinct point on the graph has a different $y$-coordinate.

- 


The function $y = x^2$ is *not* one-one, since the graph contains distinct points $(-3, 9)$ and $(3, 9)$ which have the same $y$-coordinate.

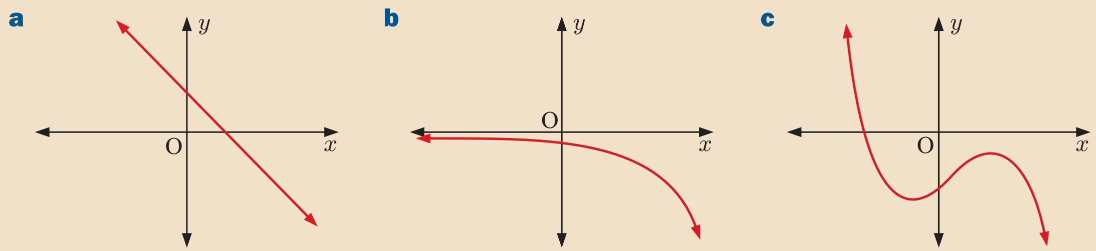

We can use the **horizontal line test** to determine whether a function is one-one:

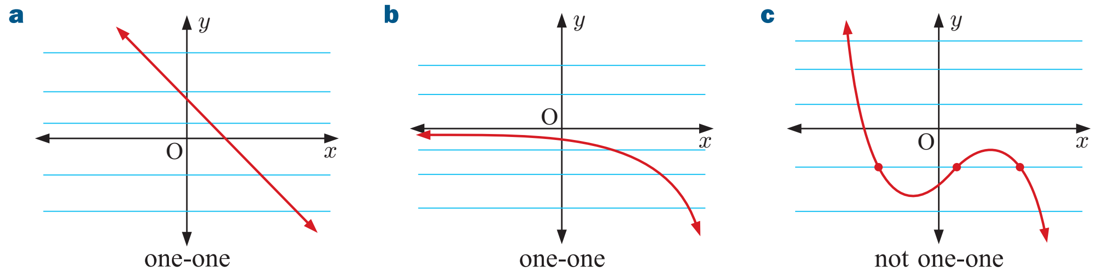
Suppose we draw all possible horizontal lines on the graph of a function.

- If each line cuts the graph at most once, then the function is one-one.
- If at least one line cuts the graph more than once, then the function is *not* one-one.

### Example 2 — Self Tutor

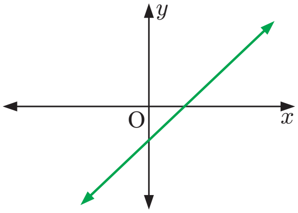
Which of the following relations are one-one?

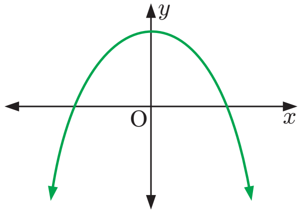


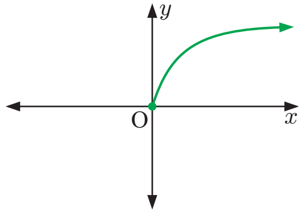



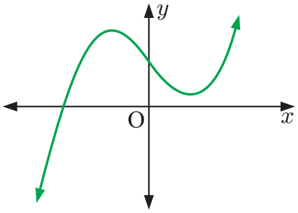
### EXERCISE 2A.2

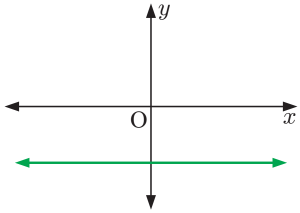
**1** Which of the following functions are one-one?

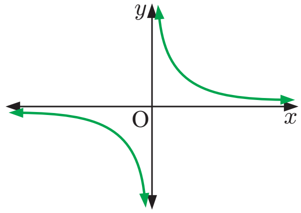
**a**



**b**


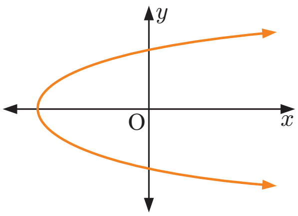

**c**


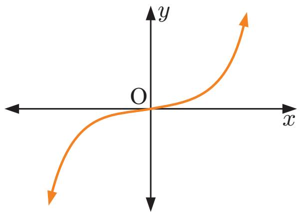

**d**


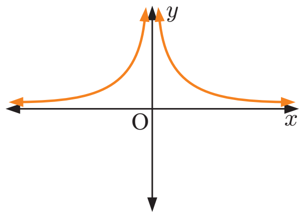

**e**



**f**



**2** Determine whether the following relations are functions. If they are functions, determine whether they are one-one.

**a**



**b**



**c**



**3** Consider the car park relation described on page **36**.

**a** **i** Richard parked his car for 4 hours. How much did he pay?

**ii** Suppose Susie parked her car for $t$ hours. If you know the value of $t$, can you uniquely determine how much she paid?

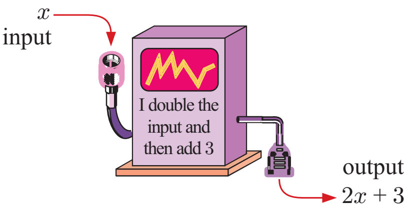
**iii** Is the car park relation a function?

**b** **i** Janette paid \$18 for parking. Can you uniquely determine how long she parked for?

**ii** Is the car park function one-one?

## B FUNCTION NOTATION

**Function machines** are sometimes used to illustrate how functions behave.




If 4 is the input fed into the machine, the output is $2(4)+3=11$.

The above 'machine' has been programmed to perform a particular function. If we use $f$ to represent that particular function, we can write:

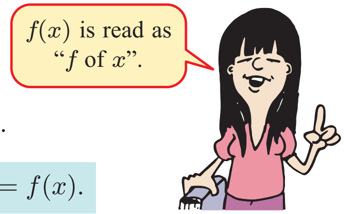
$f$ is the function that will convert $x$ into $2x+3$.

So, $f$ would convert $2$ into $2(2)+3=7$ and
$-4$ into $2(-4)+3=-5$.

This function can be written as:

$$\boldsymbol{f : x \mapsto \underbrace{2x+3}}$$

function $f$ such that $x$ is converted into $2x+3$




Two other equivalent forms we use are $f(x)=2x+3$ and $y=2x+3$.

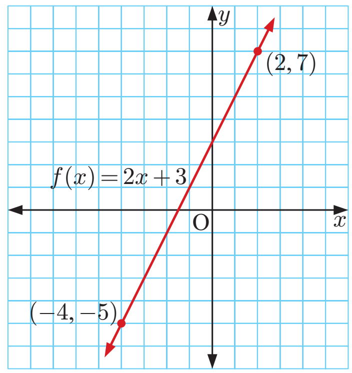
> $f(x)$ is the value of $y$ for a given value of $x$, so $y=f(x)$.

$f$ is the function which converts $x$ into $f(x)$, so we write $f : x \mapsto f(x)$.

$y=f(x)$ is sometimes called the **function value** or **image** of $x$.

For $f(x)=2x+3$:

- $f(2)=2(2)+3=7$.
  $\therefore$ the point (2, 7) lies on the graph of the function.
- $f(-4)=2(-4)+3=-5$.
  $\therefore$ the point $(-4, -5)$ also lies on the graph.




> A **linear function** is a function of the form $f(x)=ax+b$ where $a$, $b$ are real constants.
> The graph of a linear function is a straight line.

### Example 3 — Self Tutor

If $f : x \mapsto 2x^2-3x$, find the value of: **a** $f(5)$ **b** $f(-4)$

$f(x)=2x^2-3x$

**a** $f(5)=2(5)^2-3(5)$ {replacing $x$ with (5)}
$=2\times 25-15$
$=35$

**b** $f(-4)=2(-4)^2-3(-4)$ {replacing $x$ with $(-4)$}
$=2(16)+12$
$=44$

## EXERCISE 2B

**1** If $f(x)=3x-x^2+2$, find the value of:

**a** $f(0)$ **b** $f(3)$ **c** $f(-3)$ **d** $f(-7)$ **e** $f(\frac{3}{2})$

**2** If $g : x \mapsto x-\dfrac{4}{x}$, find the value of:

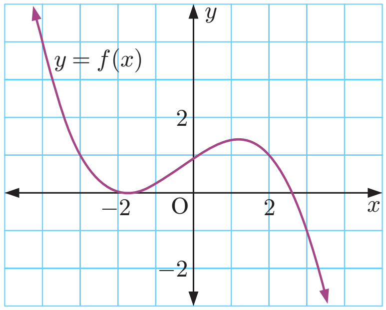
**a** $g(1)$ **b** $g(4)$ **c** $g(-1)$ **d** $g(-4)$ **e** $g(-\frac{1}{2})$

**3** The graph of $y = f(x)$ is shown alongside.

**a** Find:

**i** $f(2)$ **ii** $f(3)$

**b** Find the value of $x$ such that $f(x) = 4$.




### Example 4 Self Tutor

If $f(x) = 5 - x - x^2$, find in simplest form: **a** $f(-x)$ **b** $f(x+2)$

**a** $f(-x) = 5 - (-x) - (-x)^2$ {replacing $x$ with $(-x)$}
$= 5 + x - x^2$

**b** $f(x+2) = 5 - (x+2) - (x+2)^2$ {replacing $x$ with $(x+2)$}
$= 5 - x - 2 - [x^2 + 4x + 4]$
$= 3 - x - x^2 - 4x - 4$
$= -x^2 - 5x - 1$

**4** If $f(x) = 7 - 3x$, find in simplest form:

**a** $f(a)$ **b** $f(-a)$ **c** $f(a+3)$ **d** $f(b-1)$ **e** $f(x+2)$ **f** $f(x+h)$

**5** If $F(x) = 2x^2 + 3x - 1$, find in simplest form:

**a** $F(x+4)$ **b** $F(2-x)$ **c** $F(-x)$ **d** $F(x^2)$ **e** $F(x^2-1)$ **f** $F(x+h)$

**6** Suppose $G(x) = \dfrac{2x+3}{x-4}$.

**a** Evaluate: **i** $G(2)$ **ii** $G(0)$ **iii** $G(-\frac{1}{2})$

**b** Find a value of $x$ such that $G(x)$ does not exist.

**c** Find $G(x+2)$ in simplest form.

**d** Find $x$ if $G(x) = -3$.

**7** $f$ represents a function. What is the difference in meaning between $f$ and $f(x)$?

**8** The value of a photocopier $t$ years after purchase is given by $V(t) = 9650 - 860t$ dollars.

**a** Find $V(4)$ and state what $V(4)$ means.

**b** Find $t$ when $V(t) = 5780$ and explain what this represents.

**c** Find the original purchase price of the photocopier.

**9** On the same set of axes draw the graphs of three different functions $f(x)$ such that $f(2) = 1$ and $f(5) = 3$.

**10** Find a linear function $f(x) = ax + b$ for which $f(2) = 1$ and $f(-3) = 11$.

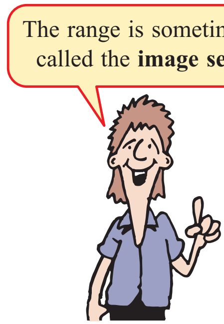
**11** Given $f(x) = ax + \dfrac{b}{x}$, $f(1) = 1$, and $f(2) = 5$, find constants $a$ and $b$.

**12** Given $T(x) = ax^2 + bx + c$, $T(0) = -4$, $T(1) = -2$, and $T(2) = 6$, find $a$, $b$, and $c$.

# C DOMAIN AND RANGE

The **domain** of a relation is the set of values of $x$ in the relation.

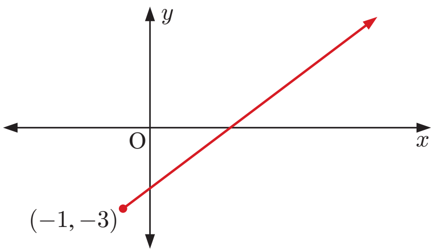
The **range** of a relation is the set of values of $y$ in the relation.




The domain and range of a relation are often described using **interval notation**.

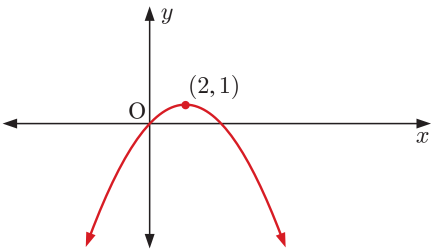
For example:

**(1)**



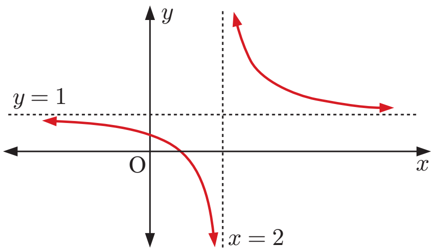

All values of $x \geqslant -1$ are included,
so the domain is $\{x : x \geqslant -1\}$.
All values of $y \geqslant -3$ are included,
so the range is $\{y : y \geqslant -3\}$.

**(2)**




$x$ can take any value,
so the domain is $\{x : x \in \mathbb{R}\}$.
$y$ cannot be $> 1$,
so the range is $\{y : y \leqslant 1\}$.

**(3)**




$x$ can take all values except 2,
so the domain is $\{x : x \neq 2\}$.
$y$ can take all values except 1,
so the range is $\{y : y \neq 1\}$.

## DOMAIN AND RANGE OF FUNCTIONS

To fully describe a function, we need both a rule *and* a domain.

For example, we can specify $f(x) = x^2$ where $x \geqslant 0$.

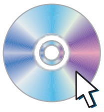
If a domain is not specified, we use the **natural domain**, which is the largest part of $\mathbb{R}$ for which $f(x)$ is defined.

For example, consider the domains in the table opposite:

| $f(x)$ | *Natural domain* |
|---|---|
| $x^2$ | $x \in \mathbb{R}$ |
| $\sqrt{x}$ | $x \geqslant 0$ |
| $\frac{1}{x}$ | $x \neq 0$ |
| $\frac{1}{\sqrt{x}}$ | $x > 0$ |

Click on the icon to obtain software for finding the domain and range of different functions.

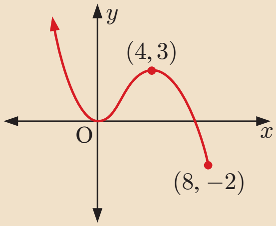
**DOMAIN AND RANGE**



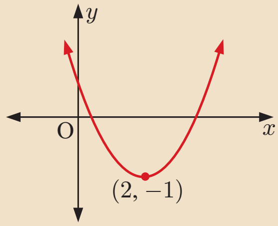
## *Example 5* — Self Tutor

For each of the following graphs, state the domain and range:

**a**




**b**



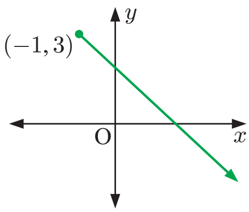

**a** Domain is $\{x : x \leqslant 8\}$
Range is $\{y : y \geqslant -2\}$

**b** Domain is $\{x : x \in \mathbb{R}\}$
Range is $\{y : y \geqslant -1\}$

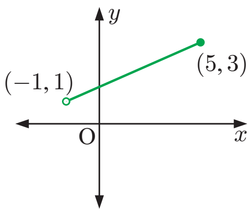
## *EXERCISE 2C*

**1** For each of the following graphs, state the domain and range:

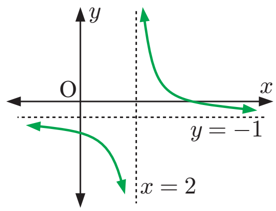
**a**



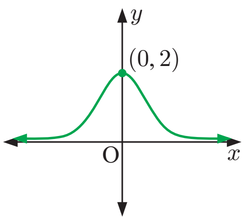

**b**



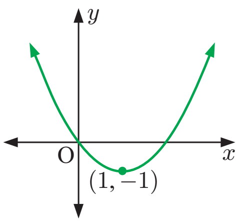

**c**



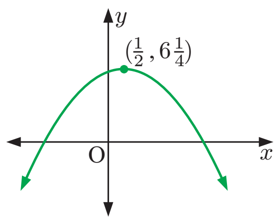

**d**



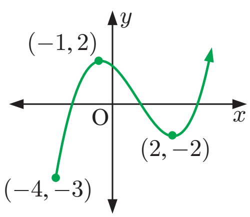

**e**



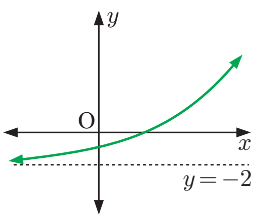

**f**



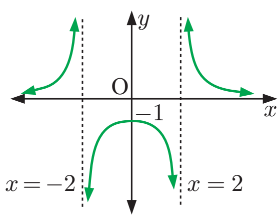

**g**




**h**




**i**



### Example 6

Self Tutor

State the domain and range of each of the following functions:

**a** $f(x)=\sqrt{x-5}$ **b** $f(x)=\dfrac{1}{x-5}$ **c** $f(x)=\dfrac{1}{\sqrt{x-5}}$

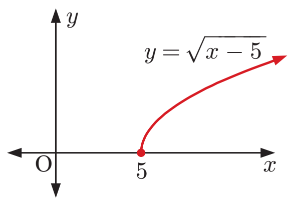
**a** $\sqrt{x-5}$ is defined when $x-5\geqslant 0$

$\therefore\ x\geqslant 5$

$\therefore$ the domain is $\{x : x\geqslant 5\}$.

A square root cannot be negative.

$\therefore$ the range is $\{y : y\geqslant 0\}$.



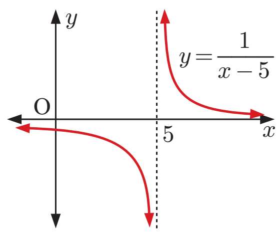

**b** $\dfrac{1}{x-5}$ is defined when $x-5\neq 0$

$\therefore\ x\neq 5$

$\therefore$ the domain is $\{x : x\neq 5\}$.

No matter how large or small $x$ is, $y=f(x)$ is never zero.

$\therefore$ the range is $\{y : y\neq 0\}$.



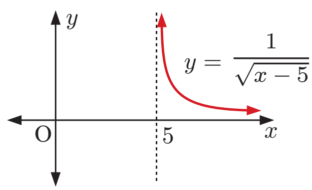

**c** $\dfrac{1}{\sqrt{x-5}}$ is defined when $x-5>0$

$\therefore\ x>5$

$\therefore$ the domain is $\{x : x>5\}$.

$y=f(x)$ is always positive and never zero.

$\therefore$ the range is $\{y : y>0\}$.




**2** State the values of $x$ for which $f(x)$ is defined, and hence state the domain of the function.

**a** $f(x)=\sqrt{x+6}$ **b** $f : x\mapsto \dfrac{1}{x^2}$ **c** $f(x)=\dfrac{-7}{\sqrt{3-2x}}$

**3** Find the domain and range of each of the following functions:

**a** $f : x\mapsto 2x-1$ **b** $f(x)=3$ **c** $f : x\mapsto\sqrt{x}$

**d** $f(x)=\dfrac{1}{x+1}$ **e** $f(x)=-\dfrac{1}{\sqrt{x}}$ **f** $f : x\mapsto\dfrac{1}{3-x}$

**4** Use technology to help sketch graphs of the following functions. Find the domain and range of each.

**a** $f(x)=\sqrt{x-2}$ **b** $f : x\mapsto\dfrac{1}{x^2}$ **c** $f : x\mapsto\sqrt{4-x}$

**d** $y=x^2-7x+10$ **e** $f(x)=\sqrt{x^2+4}$ **f** $f(x)=\sqrt{x^2-4}$

**g** $f : x\mapsto 5x-3x^2$ **h** $f : x\mapsto x+\dfrac{1}{x}$ **i** $y=\dfrac{x+4}{x-2}$

**j** $y=x^3-3x^2-9x+10$ **k** $f : x\mapsto\dfrac{3x-9}{x^2-x-2}$ **l** $y=x^2+x^{-2}$

**m** $y=x^3+\dfrac{1}{x^3}$ **n** $f : x\mapsto x^4+4x^3-16x+3$

DOMAIN AND RANGE

# D THE MODULUS FUNCTION

The **modulus** or **absolute value** of a real number is its size, ignoring its sign.

We denote the absolute value of $x$ by $|x|$.

The absolute value of a number is always $\geqslant 0$.

For example, the modulus of 4 is 4, and the modulus of $-9$ is 9. We write $|4| = 4$ and $|-9| = 9$.

### Example 7 — Self Tutor

If $x = -3$, find the value of:

**a** $|x|$ **b** $x\,|x|$ **c** $|x^2 + x|$ **d** $\left|\frac{7x-1}{2}\right|$

**a** $|x|$
$= |-3|$
$= 3$

**b** $x\,|x|$
$= (-3)\,|-3|$
$= -3 \times 3$
$= -9$

**c** $|x^2 + x|$
$= |(-3)^2 + (-3)|$
$= |6|$
$= 6$

**d** $\left|\frac{7x-1}{2}\right|$
$= \left|\frac{7(-3)-1}{2}\right|$
$= |-11|$
$= 11$

## EXERCISE 2D.1

**1** Find the value of:

**a** $|5|$ **b** $|-5|$ **c** $|7-3|$ **d** $|3-7|$

**e** $|2^2 - 10|$ **f** $|15 - 3 \times 5|$ **g** $\left|\frac{3-1}{5+2}\right|$ **h** $\left|\frac{2^3}{(-3)^3}\right|$

**2** If $x = 4$, find the value of:

**a** $|x-5|$ **b** $|10-x|$ **c** $|3x - x^2|$ **d** $\left|\frac{2x+1}{x-1}\right|$

**3** If $x = -2$, find the value of:

**a** $|x|$ **b** $x\,|x|$ **c** $-\,|x - x^2|$ **d** $\frac{|1+3x|}{x+1}$

## MODULUS EQUATIONS

Solving modulus equations is not needed for the syllabus.

The equation $|x| = 2$ has two solutions: $x = 2$ and $x = -2$.

If $|x| = a$ where $a > 0$, then $x = \pm a$.

If $|x| = |b|$ then $x = \pm b$.

We use these rules to solve equations involving the modulus function.

### Example 8 — Self Tutor

Solve for $x$: **a** $|2x+3|=7$ **b** $|3-2x|=-1$

**a** $|2x+3|=7$

$\therefore\ 2x+3=\pm 7$

$\therefore\ 2x+3=7$ **or** $2x+3=-7$

$\therefore\ 2x=4$ $\qquad\therefore\ 2x=-10$

$\therefore\ x=2$ $\qquad\therefore\ x=-5$

So, $x=2$ or $-5$

**b** $|3-2x|=-1$ has no solution as LHS is never negative.

### Example 9 — Self Tutor

Solve for $x$: $|x+1|=|2x-3|$

If $|x+1|=|2x-3|$, then $x+1=\pm(2x-3)$

$\therefore\ x+1=2x-3$ **or** $x+1=-(2x-3)$

$\therefore\ 4=x$ $\qquad\therefore\ x+1=-2x+3$

$\therefore\ 3x=2$

$\therefore\ x=\frac{2}{3}$

So, $x=\frac{2}{3}$ or 4.

## EXERCISE 2D.2

**1** Solve for $x$:

**a** $|x|=3$ **b** $|x|=-5$ **c** $|x|=0$

**d** $|x-1|=3$ **e** $|3-x|=4$ **f** $|x+5|=-1$

**g** $|3x-2|=1$ **h** $|3-2x|=3$ **i** $|2-5x|=12$

**2** Solve for $x$:

**a** $|3x-1|=|x+2|$ **b** $|2x+5|=|1-x|$ **c** $|x+1|=|2-x|$

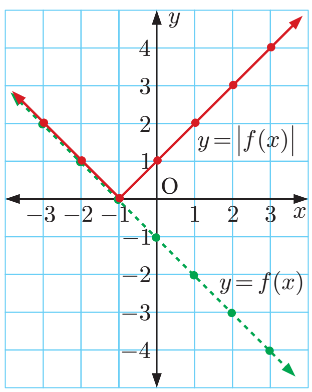
**d** $|x|=|5-x|$ **e** $|1-4x|=|x-1|$ **f** $|3x+2|=|2-x|$

## THE GRAPH OF $y=|f(x)|$

Consider the function $f(x)=-x-1$.

In the table below, we calculate the values of $f(x)$ and $|f(x)|$ for $x=-3, -2, -1, 0, 1, 2, 3$.

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 2 | 1 | 0 | $-1$ | $-2$ | $-3$ | $-4$ |
| $\|f(x)\|$ | 2 | 1 | 0 | 1 | 2 | 3 | 4 |




Using these values, we can plot $y=f(x)$ and $y=|f(x)|$ on the same set of axes.

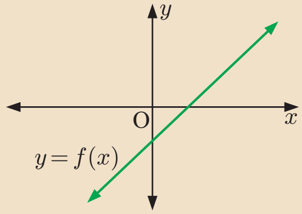
To draw the graph of $y = |f(x)|$, any parts of $y = f(x)$ that are below the $x$-axis are reflected in the $x$-axis.

### *Example 10*

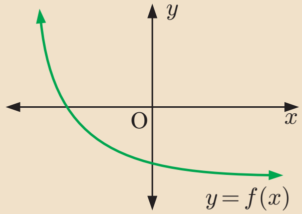
**Self Tutor**

For the following graphs, sketch the graph of $y = |f(x)|$:

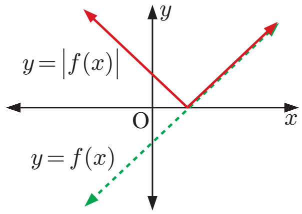
**a**



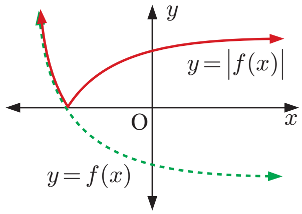

**b**




**a**




**b**

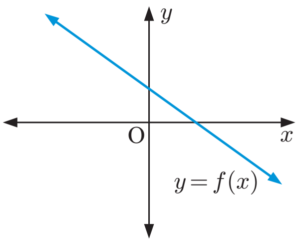



## *EXERCISE 2D.3*

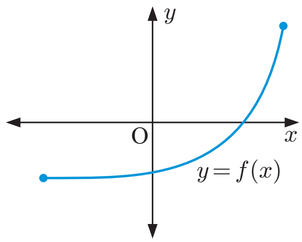
**PRINTABLE DIAGRAMS**

**1** For the following graphs, sketch the graph of $y = |f(x)|$:

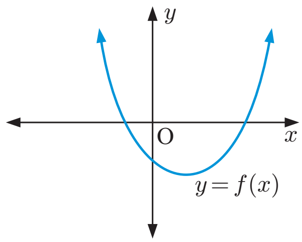
**a**



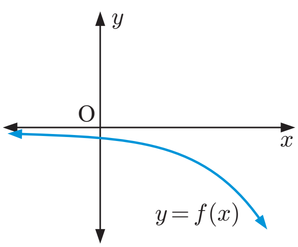

**b**



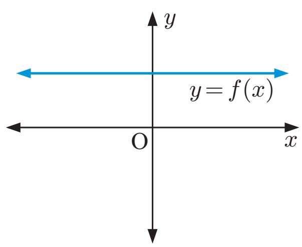

**c**



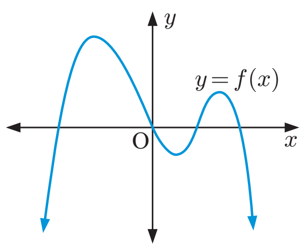

**d**




**e**




**f**




**2** Which of the functions $y = |f(x)|$ in question **1** are one-one?

**3** Suppose the range of $y = f(x)$ is $\{y : -6 \leqslant y \leqslant 2\}$. Write down the range of $y = |f(x)|$.

**4** Determine whether the following statements are true or false:

**a** If $y = f(x)$ is one-one, then $y = |f(x)|$ is one-one.

**b** If $y = f(x)$ is not one-one, then $y = |f(x)|$ is not one-one.

**c** The graphs of $y = f(x)$ and $y = |f(x)|$ always meet the $x$-axis at the same point(s).

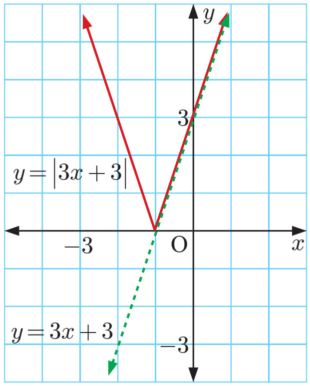
**d** The graphs of $y = f(x)$ and $y = |f(x)|$ always meet the $y$-axis at the same point.

### Example 11 — Self Tutor

Draw the graph of $y = |3x + 3|$.

We first draw the graph of $y = 3x + 3$.

The part of the graph that is below the $x$-axis is then reflected in the $x$-axis to produce $y = |3x + 3|$.




**5** Draw the graph of:

**a** $y = |x|$ **b** $y = |x + 3|$ **c** $y = |6 - 2x|$

**d** $y = |3x + 1|$ **e** $y = |10 - 4x|$ **f** $y = \left|\frac{1}{2}x + 2\right|$

## E COMPOSITE FUNCTIONS

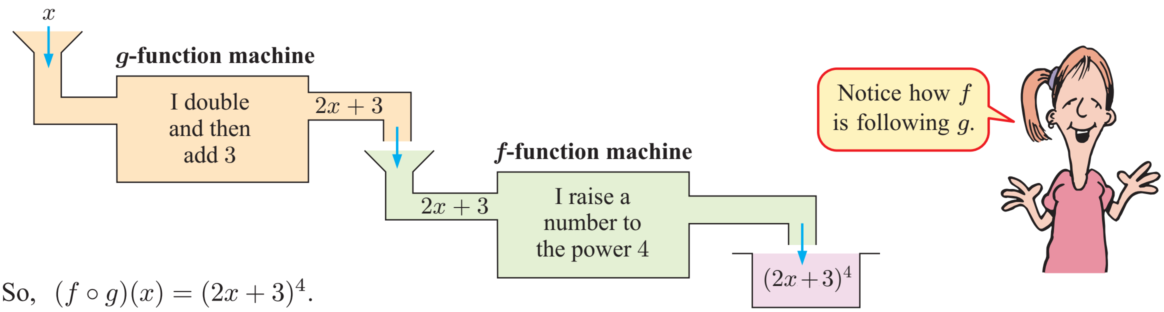
Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the **composite function** of $f$ and $g$ will convert $x$ into $f(g(x))$.

$f \circ g$ or $fg$ is used to represent the composite function of $f$ and $g$. It means "$f$ following $g$".

$$(\boldsymbol{f \circ g})(\boldsymbol{x}) \text{ or } \boldsymbol{fg(x) = f(g(x))}$$

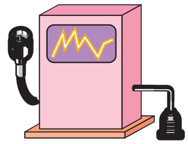
Consider $f : x \mapsto x^4$ and $g : x \mapsto 2x + 3$.

$f \circ g$ means that $g$ converts $x$ to $2x + 3$ and then $f$ converts $(2x + 3)$ to $(2x + 3)^4$.




So, $(f \circ g)(x) = (2x + 3)^4$.

Algebraically, if $f(x) = x^4$ and $g(x) = 2x + 3$ then



$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x+3) && \{g \text{ operates on } x \text{ first}\} \\ &= (2x+3)^4 && \{f \text{ operates on } g(x) \text{ next}\}\end{aligned}$$

and
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^4) && \{f \text{ operates on } x \text{ first}\} \\ &= 2(x^4) + 3 && \{g \text{ operates on } f(x) \text{ next}\} \\ &= 2x^4 + 3\end{aligned}$$

So, $f(g(x)) \neq g(f(x))$.

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

We can also compose a function $f$ with itself. The resulting function is $(f \circ f)(x)$ or $f^2(x)$.

In general, $(f \circ f)(x) \neq (f(x))^2$.

### Example 12 Self Tutor

Given $f : x \mapsto 2x + 1$ and $g : x \mapsto 3 - 4x$, find in simplest form:

**a** $(f \circ g)(x)$ **b** $gf(x)$ **c** $f^2(x)$

$f(x) = 2x + 1$ and $g(x) = 3 - 4x$

**a**
$$\begin{aligned}&(f \circ g)(x) \\ &= f(g(x)) \\ &= f(3-4x) \\ &= 2(3-4x)+1 \\ &= 6-8x+1 \\ &= 7-8x\end{aligned}$$

**b**
$$\begin{aligned}&gf(x) \\ &= g(f(x)) \\ &= g(2x+1) \\ &= 3-4(2x+1) \\ &= 3-8x-4 \\ &= -8x-1\end{aligned}$$

**c**
$$\begin{aligned}&f^2(x) \\ &= f(f(x)) \\ &= f(2x+1) \\ &= 2(2x+1)+1 \\ &= 4x+2+1 \\ &= 4x+3\end{aligned}$$

In the previous **Example** you should have observed how we can substitute an expression into a function.

If $f(x) = 2x + 1$ then
$$\begin{aligned} f(\Delta) &= 2(\Delta) + 1 \\ \therefore\ f(3-4x) &= 2(3-4x) + 1.\end{aligned}$$

### Example 13 Self Tutor

Given $f(x) = 6x - 5$ and $g(x) = x^2 + x$, determine:

**a** $(g \circ f)(-1)$ **b** $(f \circ f)(0)$

**a** $(g \circ f)(-1) = g(f(-1))$

Now $f(-1) = 6(-1) - 5$
$= -11$

$$\begin{aligned}\therefore\ (g \circ f)(-1) &= g(-11) \\ &= (-11)^2 + (-11) \\ &= 110\end{aligned}$$

**b** $(f \circ f)(0) = f(f(0))$

Now $f(0) = 6(0) - 5$
$= -5$

$$\begin{aligned}\therefore\ (f \circ f)(0) &= f(-5) \\ &= 6(-5) - 5 \\ &= -35\end{aligned}$$

The domain of the composite of two functions depends on the domain of the original functions.

For example, consider $f(x) = x^2$ with domain $x \in \mathbb{R}$ and $g(x) = \sqrt{x}$ with domain $x \geqslant 0$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= (\sqrt{x})^2 \\ &= x\end{aligned}$$

The domain of $(f \circ g)(x)$ is $x \geqslant 0$, not $\mathbb{R}$, since $(f \circ g)(x)$ is defined using function $g(x)$.

## EXERCISE 2E

**1** Given $f : x \mapsto 2x + 3$ and $g : x \mapsto 1 - x$, find in simplest form:

**a** $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(f \circ g)(-3)$

**2** Given $f(x) = 2 + x$ and $g(x) = 3 - x$, find:

**a** $fg(x)$ **b** $gf(x)$ **c** $f^2(x)$

**3** Given $f(x) = \sqrt{6 - x}$ and $g(x) = 5x - 7$, find:

**a** $(g \circ g)(x)$ **b** $(f \circ g)(1)$ **c** $(g \circ f)(6)$

**4** Given $f : x \mapsto x^2$ and $g : x \mapsto 2 - x$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.
Find also the domain and range of $f \circ g$ and $g \circ f$.

**5** Suppose $f(x) = 3x + 5$ and $g(x) = 2x - 3$.

**a** Find $(f \circ g)(x)$. **b** Solve $(f \circ g)(x) = g(x - 2)$.

**6** Suppose $f : x \mapsto x^2 + 1$ and $g : x \mapsto 3 - x$.

**a** Find in simplest form: **i** $fg(x)$ **ii** $gf(x)$

**b** Find the value(s) of $x$ such that $gf(x) = f(x)$.

**7** **a** If $ax + b = cx + d$ for all values of $x$, show that $a = c$ and $b = d$.
**Hint:** If it is true for all $x$, it is true for $x = 0$ and $x = 1$.

**b** Given $f(x) = 2x + 3$ and $g(x) = ax + b$ and that $(f \circ g)(x) = x$ for all values of $x$, deduce that $a = \frac{1}{2}$ and $b = -\frac{3}{2}$.

**c** Is the result in **b** true if $(g \circ f)(x) = x$ for all $x$?

**8** Given $f(x) = \sqrt{1 - x}$ and $g(x) = x^2$, find:

**a** $(f \circ g)(x)$ **b** the domain and range of $(f \circ g)(x)$.

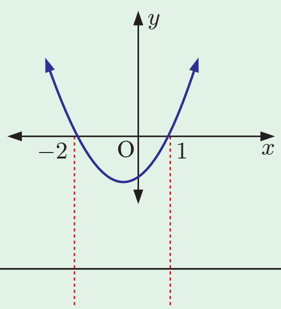
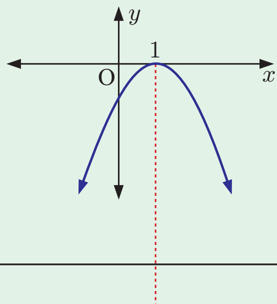
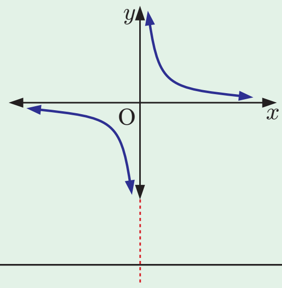
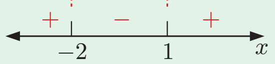
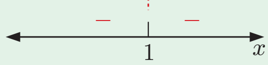
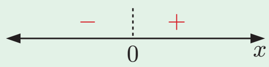
# F SIGN DIAGRAMS

Sometimes we do not wish to draw a time-consuming graph of a function but wish to know when the function is positive, negative, zero, or undefined. A **sign diagram** enables us to do this and is relatively easy to construct.

For the function $f(x)$, the sign diagram consists of:

- a **horizontal line** which is really the $x$-axis
- **positive** $(+)$ and **negative** $(-)$ signs indicating that the graph is **above** and **below** the $x$-axis respectively
- the **zeros** of the function, which are the $x$-intercepts of the graph of $y = f(x)$, and the **roots** of the equation $f(x) = 0$
- values of $x$ where the graph is undefined.

Consider the three functions given below.

| **Function** | $y = (x+2)(x-1)$ | $y = -2(x-1)^2$ | $y = \dfrac{4}{x}$ |
|---|---|---|---|
| **Graph** |  |  |  |
| **Sign diagram** |  |  |  |

You should notice that:

DEMO

- A sign change occurs about a zero of the function for single linear factors such as $(x+2)$ and $(x-1)$. This indicates **cutting** of the $x$-axis.
- No sign change occurs about a zero of the function for squared linear factors such as $(x-1)^2$. This indicates **touching** of the $x$-axis.
-  indicates that a function is **undefined** at $x = 0$.

In general:

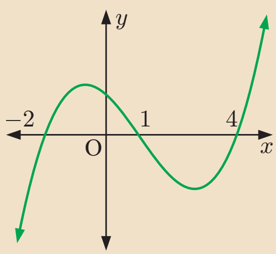
- when a linear factor has an **odd power** there is a change of sign about that zero
- when a linear factor has an **even power** there is no sign change about that zero.

## *Example 14*

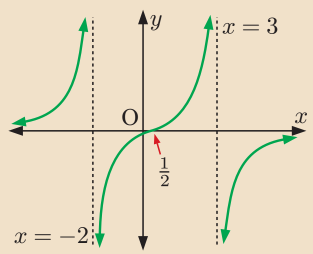
*Self Tutor*

Draw sign diagrams for:

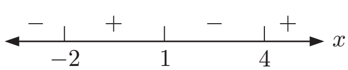
**a**



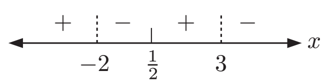

**b**




**a**

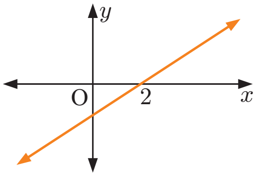


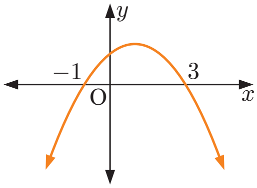

**b**

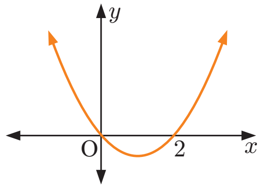


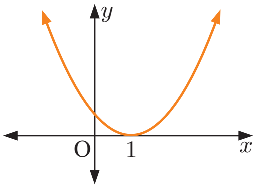
## EXERCISE 2F

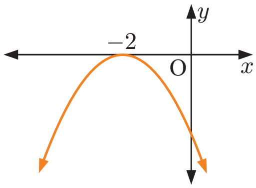
**1** Draw sign diagrams for these graphs:

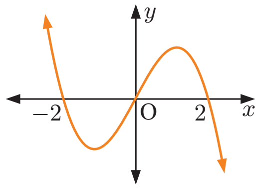
**a**


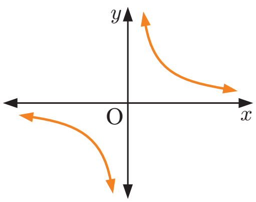

**b**


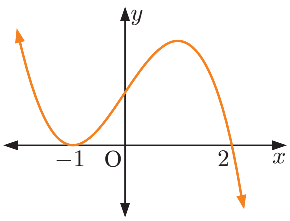

**c**


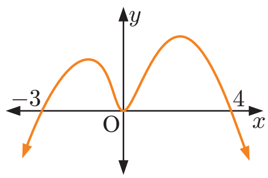

**d**


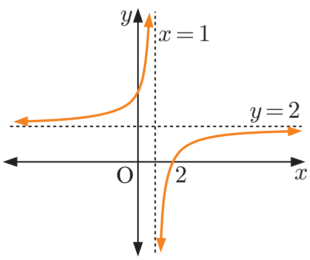

**e**


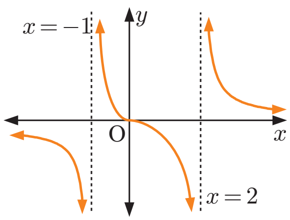

**f**


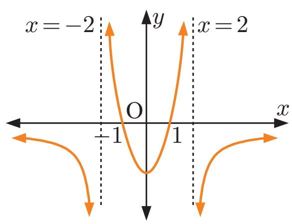

**g**



**h**



**i**



**j**



**k**



**l**


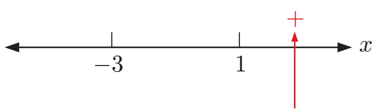

### Example 15 — Self Tutor

Draw a sign diagram for:

**a** $(x+3)(x-1)$

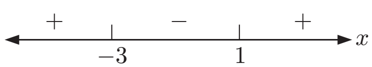
**b** $-4(x-3)^2$

**a** $(x+3)(x-1)$ has zeros $-3$ and $1$.

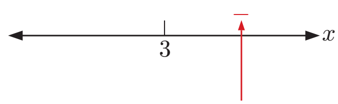



We substitute any number $> 1$.
When $x = 2$ we have $(5)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

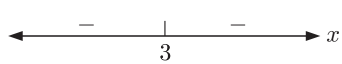



**b** $-4(x-3)^2$ has zero $3$.




We substitute any number $> 3$.
When $x = 4$ we have $-4(1)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.



**2** Draw sign diagrams for:

**a** $(x+4)(x-2)$ **b** $x(x-3)$ **c** $x(x+2)$

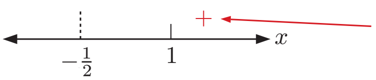
**d** $-(x+1)(x-3)$ **e** $(2x-1)(3-x)$ **f** $(5-x)(1-2x)$

**g** $(x+2)^2$ **h** $2(x-3)^2$ **i** $-3(x+4)^2$

### Example 16 Self Tutor

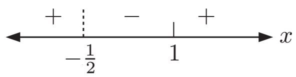
Draw a sign diagram for $\dfrac{x-1}{2x+1}$.

$\dfrac{x-1}{2x+1}$ is zero when $x=1$ and undefined when $x=-\frac{1}{2}$.



When $x=10$, $\dfrac{x-1}{2x+1}=\dfrac{9}{21}>0$

Since $(x-1)$ and $(2x+1)$ are single factors, the signs alternate.



**3** Draw sign diagrams for:

**a** $\dfrac{x+2}{x-1}$ **b** $\dfrac{x}{x+3}$ **c** $\dfrac{2x+3}{4-x}$

**d** $\dfrac{4x-1}{2-x}$ **e** $\dfrac{3x}{x-2}$ **f** $\dfrac{-8x}{3-x}$

**g** $\dfrac{(x-1)^2}{x}$ **h** $\dfrac{4x}{(x+1)^2}$ **i** $\dfrac{(x+2)(x-1)}{3-x}$

**j** $\dfrac{x(x-1)}{2-x}$ **k** $\dfrac{(x+2)(x-2)}{-x}$ **l** $\dfrac{3-x}{(2x+3)(x-2)}$

**4** Draw sign diagrams for:

**a** $1+\dfrac{3}{x+1}$ **b** $x-\dfrac{1}{x}$ **c** $x-\dfrac{1}{x^2}$

## G INVERSE FUNCTIONS

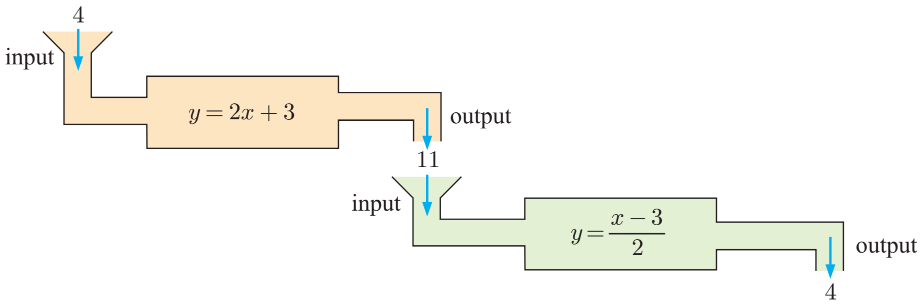
The operations of $+$ and $-$, $\times$ and $\div$, are **inverse operations** as one undoes what the other does.

For example, $x+3-3=x$ and $x\times 3\div 3=x$.

The function $y=2x+3$ can be "undone" by its **inverse function** $y=\dfrac{x-3}{2}$.

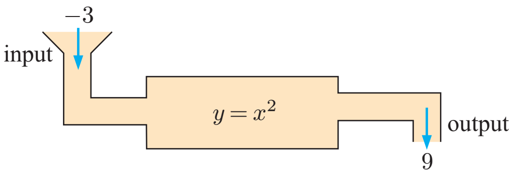
We can think of this as two machines. If the machines are inverses then the second machine *undoes* what the first machine does.

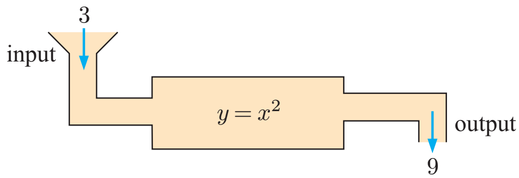
No matter what value of $x$ enters the first machine, it is returned as the output from the second machine.



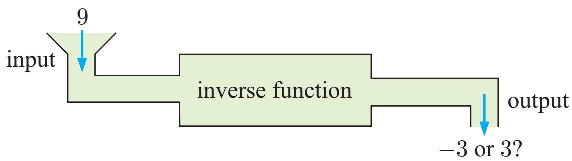

However, not all functions have an inverse function.

For example, consider the function $y = x^2$. The inputs $-3$ and $3$ both produce an output of $9$.




So, if we gave an inverse function the input $9$, how would it know whether the output should be $-3$ or $3$? It cannot answer both, since the inverse function would fail the vertical line test.




So, if a function has two inputs which produce the same output, then the function does not have an inverse function.

For a function to have an **inverse**, the function must be **one-one**. It must pass the horizontal line test.

If $y = f(x)$ has an **inverse function**, this new function:

- is denoted $f^{-1}(x)$
- is the reflection of $y = f(x)$ in the line $y = x$
- satisfies $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

The function $y = x$, defined as $f : x \mapsto x$, is the **identity function**.

$f^{-1}$ is **not** the reciprocal of $f$.

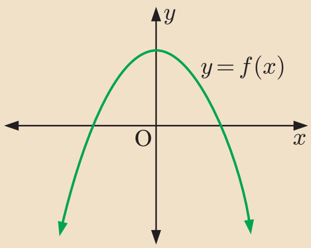
$$f^{-1}(x) \neq \frac{1}{f(x)}$$

## Example 17

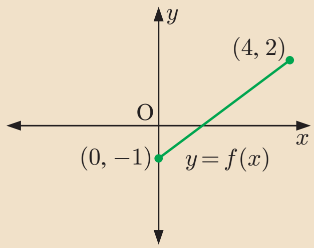
**Self Tutor**

If $y = f(x)$ has an inverse function, sketch $y = f^{-1}(x)$, and state the domain and range of $f(x)$ and $f^{-1}(x)$.

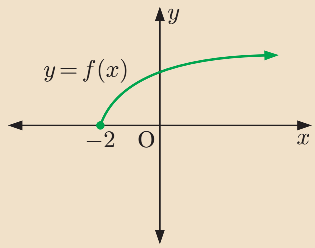
**a**




**b**

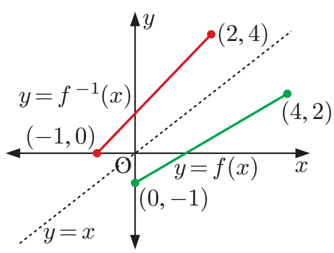



**c**



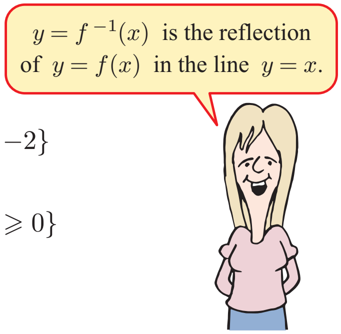

**a** The function fails the horizontal line test, so it is not one-one. The function does not have an inverse function.

**b**

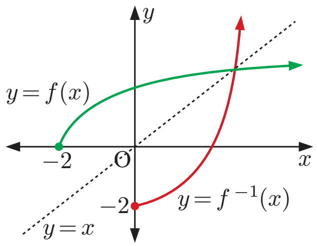



$f(x)$ has domain $\{x : 0 \leqslant x \leqslant 4\}$ and range $\{y : -1 \leqslant y \leqslant 2\}$.

$f^{-1}(x)$ has domain $\{x : -1 \leqslant x \leqslant 2\}$ and range $\{y : 0 \leqslant y \leqslant 4\}$.




**c**




$f(x)$ has domain $\{x : x \geqslant -2\}$ and range $\{y : y \geqslant 0\}$.

$f^{-1}(x)$ has domain $\{x : x \geqslant 0\}$ and range $\{y : y \geqslant -2\}$.

From **Example 17**, we can see that:

The domain of $f^{-1}$ is equal to the range of $f$.

The range of $f^{-1}$ is equal to the domain of $f$.

If $(x, y)$ lies on $f$, then $(y, x)$ lies on $f^{-1}$. Reflecting the function in the line $y = x$ has the algebraic effect of interchanging $x$ and $y$.

So, if the function is given as an equation, then we interchange the variables to find the equation of the inverse function.

For example, if $f$ is given by $y = 5x + 2$ then $f^{-1}$ is given by $x = 5y + 2$.

### Example 18

Self Tutor

Consider $f : x \mapsto 2x + 3$.

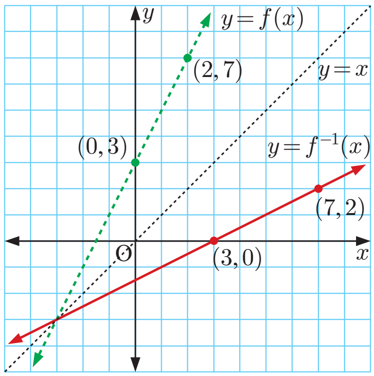
**a** On the same axes, graph $f$ and its inverse function $f^{-1}$.

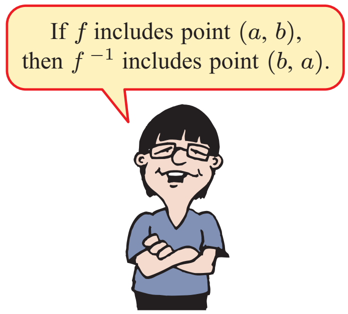
**b** Find $f^{-1}(x)$ using variable interchange.

**c** Check that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

**a** $f(x) = 2x + 3$ passes through $(0, 3)$ and $(2, 7)$.

$\therefore$ $f^{-1}(x)$ passes through $(3, 0)$ and $(7, 2)$.




**b**

$f$ is $y = 2x + 3$,

$\therefore$ $f^{-1}$ is $x = 2y + 3$

$\therefore$ $x - 3 = 2y$

$\therefore$ $\dfrac{x-3}{2} = y$

$\therefore$ $f^{-1}(x) = \dfrac{x-3}{2}$

**c**

$$\begin{aligned}
&(f \circ f^{-1})(x) \\
&= f(f^{-1}(x)) \\
&= f\left(\frac{x-3}{2}\right) \\
&= 2\left(\frac{x-3}{2}\right) + 3 \\
&= x
\end{aligned}$$

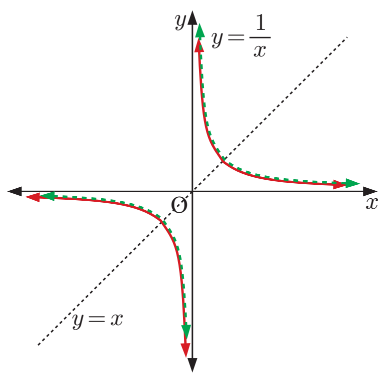
and

$$\begin{aligned}
&(f^{-1} \circ f)(x) \\
&= f^{-1}(f(x)) \\
&= f^{-1}(2x + 3) \\
&= \frac{(2x + 3) - 3}{2} \\
&= \frac{2x}{2} \\
&= x
\end{aligned}$$

Any function which has an inverse, and whose graph is symmetrical about the line $y = x$, is a **self-inverse function**.

If $f$ is a self-inverse function then $f^{-1} = f$.

For example, the function $f(x) = \dfrac{1}{x}$, $x \neq 0$, is said to be self-inverse, as $f = f^{-1}$.

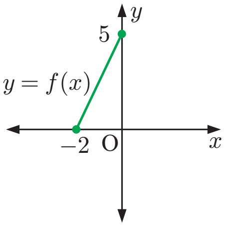


## EXERCISE 2G

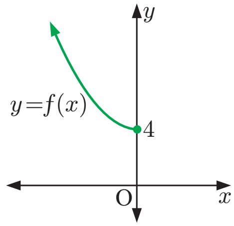
PRINTABLE GRAPHS

**1** If $y = f(x)$ has an inverse function, sketch $y = f^{-1}(x)$, and state the domain and range of $f(x)$ and $f^{-1}(x)$.

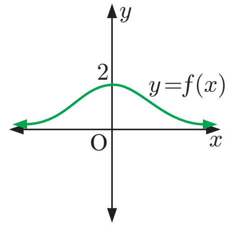
**a**



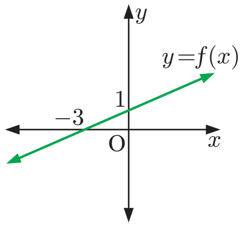

**b**



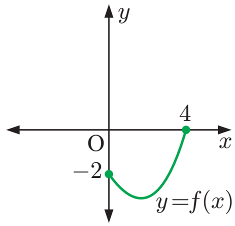

**c**



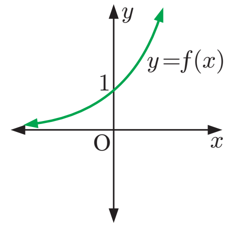

**d**



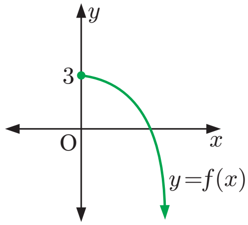

**e**



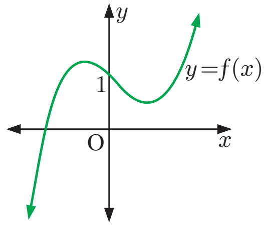

**f**



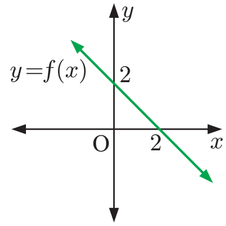

**g**




**h**




**i**




**2** Which of the functions in **1** is a self-inverse function?

**3** If the domain of $H(x)$ is $\{x : -2 \leqslant x < 3\}$, state the range of $H^{-1}(x)$.

**4** For each of the following functions $f$:

- **i** On the same set of axes, sketch $y = x$, $y = f(x)$, and $y = f^{-1}(x)$.
- **ii** Find $f^{-1}(x)$ using variable interchange.

**a** $f : x \mapsto 3x + 1$ **b** $f : x \mapsto \dfrac{x+2}{4}$

**5** For each of the following functions $f$:

- **i** Find $f^{-1}(x)$.
- **ii** Sketch $y = f(x)$, $y = f^{-1}(x)$, and $y = x$ on the same set of axes.
- **iii** Show that $(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$, the identity function.

**a** $f : x \mapsto 2x + 5$ **b** $f : x \mapsto x + 3$ **c** $f : x \mapsto \dfrac{x+6}{2}$

**6** Given $f(x) = 2x - 5$, find $(f^{-1})^{-1}(x)$. What do you notice?

**7** Sketch the graph of $f : x \mapsto x^3$ and its inverse function $f^{-1}(x)$.

**8** Given $f : x \mapsto \dfrac{1}{x}$, $x \neq 0$, find $f^{-1}$ algebraically and show that $f$ is a self-inverse function.

**9** Consider the function $f(x)=\frac{1}{2}x-1$.

**a** Find $f^{-1}(x)$.

**b** Find: **i** $(f\circ f^{-1})(x)$ **ii** $(f^{-1}\circ f)(x)$.

**10** Consider the functions $f: x\mapsto 2x+5$ and $g: x\mapsto \dfrac{8-x}{2}$.

**a** Find $g^{-1}(-1)$.

**b** Show that $f^{-1}(-3)-g^{-1}(6)=0$.

**c** Find $x$ such that $(f\circ g^{-1})(x)=9$.

**11** Consider the functions $f: x\mapsto 5^x$ and $g: x\mapsto \sqrt{x}$.

**a** Find: **i** $f(2)$ **ii** $g^{-1}(4)$

**b** Solve the equation $(g^{-1}\circ f)(x)=25$.

**12** Which of these functions is a self-inverse function?

**a** $f(x)=2x$

**b** $f(x)=x$

**c** $f(x)=-x$

**d** $f(x)=\dfrac{2}{x}$

**e** $f(x)=-\dfrac{6}{x}$

**f** $f(x)=\dfrac{x}{3}$

**13** Given $f: x\mapsto 2x$ and $g: x\mapsto 4x-3$, show that $(f^{-1}\circ g^{-1})(x)=(g\circ f)^{-1}(x)$.

## *Discovery* — *Functions and form*

We already know that numbers have equivalent forms. For example, $\frac{1}{2}$, $\frac{3}{6}$, $\frac{5}{10}$, and 0.5 all represent the same number.

Similarly, a function might have different, but equivalent, algebraic representations.

Choosing a particular form for an expression helps us understand the behaviour of the function better. By anticipating what you are going to do with your function you can choose a form which will make the task easier.

For example, you will have seen in previous years that the equation of a straight line can be written in:

- gradient-intercept form $y=mx+c$ where $m$ is the gradient and the $y$-intercept is $c$
- point-gradient form $y-b=m(x-a)$ where the line goes through $(a,b)$ and has gradient $m$
- general-form $Ax+By=D$.

A given straight line can be converted between these forms easily, but each emphasises different features of the straight line.

**What to do:**

**1** What different forms have you seen for a quadratic function $y=ax^2+bx+c$?

**2** Two expressions $f(x)$ and $g(x)$ are **equivalent** on the domain $D$ if $f(x)=g(x)$ for all $x\in D$.

**a** Discuss whether: $f(x)=\dfrac{x^2-1}{x-1}$ and $g(x)=x+1$ are equivalent on:

**i** $x\in\mathbb{R}$ **ii** $x\in\mathbb{R}^-$ **iii** $\{x: x>1\}$ **iv** $\{x\in\mathbb{R}: x\neq 1\}$

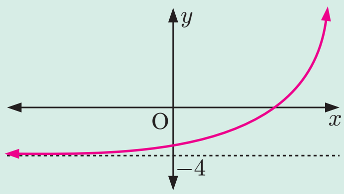
**b** When considering algebraically whether two functions are equivalent, what things do we need to be careful about?

**Hint:** $\dfrac{x^2-1}{x-1}=\dfrac{(x+1)(x-1)}{x-1}=x+1$ only if $x\neq 1$.

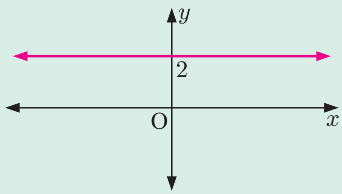
## Review set 2A

**1** Determine whether the following relations are functions:

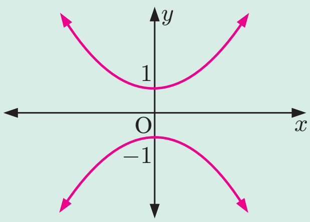
**a**



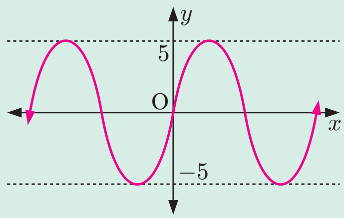

**b**




**c**




**d**




**2** Suppose $f(x) = ax + b$ where $a$ and $b$ are constants.
If $f(1) = 7$ and $f(3) = -5$, find $a$ and $b$.

**3** Solve for $x$:

**a** $|x - 5| = 7$ **b** $|2x + 1| = |x - 4|$

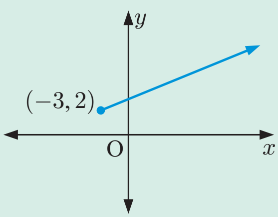
**4** If $g(x) = x^2 - 3x$, find in simplest form:

**a** $g(-2)$ **b** $g(x + 1)$

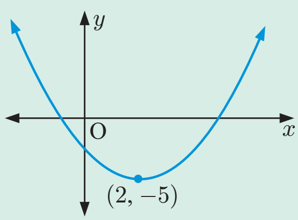
**5** For each of the following functions:

**i** find the domain and range **ii** determine whether the function is one-one.

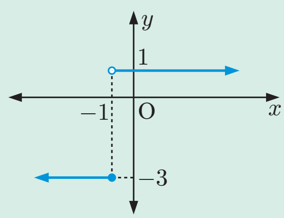
**a**




**b**



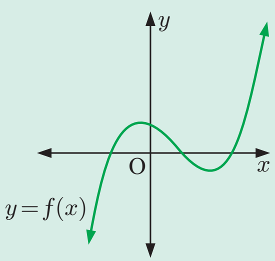

**c**



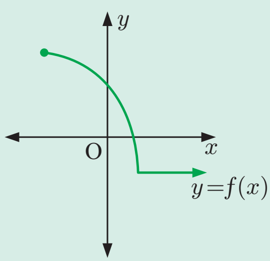

**6** Draw the graph of $y = |f(x)|$ for:

PRINTABLE GRAPHS

**a**




**b**



**7** Draw the graph of $y = |2x - 1|$.

**8** Draw a sign diagram for:

**a** $(3x+2)(4-x)$ **b** $\dfrac{x-3}{(x+2)^2}$

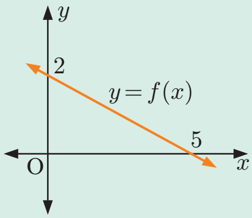
**9** If $f(x) = 2x - 3$ and $g(x) = x^2 + 2$, find in simplest form:

**a** $fg(x)$ **b** $gf(x)$

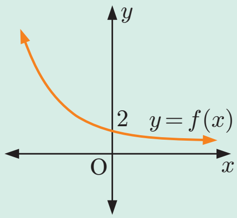
**10** If $y = f(x)$ has an inverse, sketch the graph of $y = f^{-1}(x)$.

PRINTABLE GRAPHS

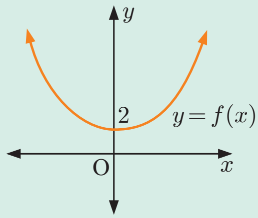
**a**




**b**




**c**




**11** Find $f^{-1}(x)$ given that $f(x)$ is:

**a** $4x + 2$ **b** $\dfrac{3-5x}{4}$

**12** Consider $f(x) = x^2$ and $g(x) = 1 - 6x$.

**a** Show that $f(-3) = g(-\frac{4}{3})$. **b** Find $(f \circ g)(-2)$.

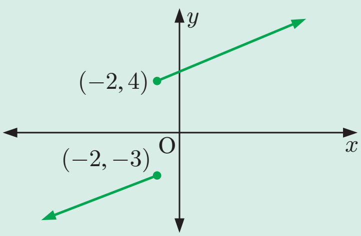
**c** Find $x$ such that $g(x) = f(5)$.

**13** Given $f : x \mapsto 3x + 6$ and $h : x \mapsto \dfrac{x}{3}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

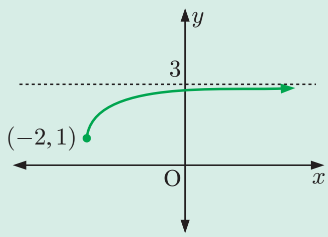
## *Review set 2B*

**1** Determine whether the following relations are functions. If they are functions, determine whether they are one-one.

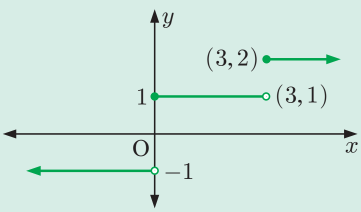
**a**




**b**




**c**



**2** Given $f(x) = x^2 + 3$, find:

**a** $f(-3)$

**b** $x$ such that $f(x) = 4$.

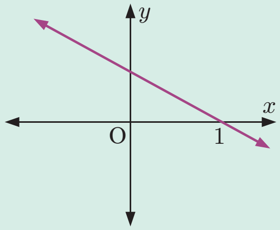
**3** Solve for $x$:

**a** $|1 - 2x| = 11$

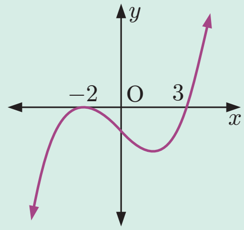
**b** $|5x - 1| = |9x - 13|$

**4** Draw a sign diagram for each graph:

**a**




**b**




**5** Given $h(x) = 7 - 3x$, find:

**a** $h(2x - 1)$

**b** $h^2(x)$

**c** $h^2(-1)$

**6** Suppose the range of $y = f(x)$ is $\{y : -7 \leqslant y \leqslant -3\}$. Write down the range of $y = |f(x)|$.

**7** Draw the graph of $y = \left|1 - \frac{1}{3}x\right|$.

**8** Suppose $f(x) = 1 - 2x$ and $g(x) = 5x$.

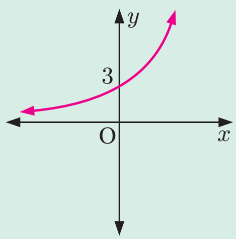
**a** Find in simplest form: **i** $fg(x)$ **ii** $gf(x)$.

**b** Solve $fg(x) = g(x + 2)$.

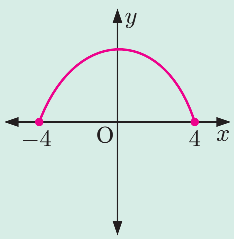
**9** Suppose $f(x) = ax^2 + bx + c$, $f(0) = 5$, $f(-2) = 21$, and $f(3) = -4$. Find $a$, $b$, and $c$.

**10** If $y = f(x)$ has an inverse, sketch the graph of $y = f^{-1}(x)$:

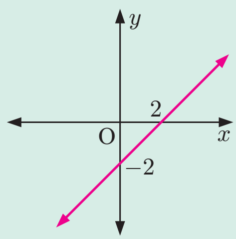
**a**




**b**




**c**




**PRINTABLE DIAGRAMS**

**11** Find the inverse function $f^{-1}(x)$ for:

**a** $f(x) = 7 - 4x$

**b** $f(x) = \dfrac{3 + 2x}{5}$

**12** Given $f : x \mapsto 5x - 2$ and $h : x \mapsto \dfrac{3x}{4}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

**13** Given $f(x) = 2x + 11$ and $g(x) = x^2$, find $(g \circ f^{-1})(3)$.

3

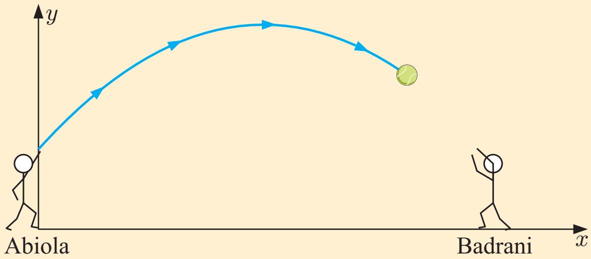
# Quadratics

Contents:

## Opening problem

Abiola and Badrani are standing 40 metres apart, throwing a ball between them. When Abiola throws the ball, it travels in a smooth arc. At the time when the ball has travelled $x$ metres horizontally towards Badrani, its height is $y$ metres.

SIMULATION




| $x$ (m) | 0 | 5 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|---|
| $y$ (m) | 1.25 | 10 | 16.25 | 20 | 21.25 | 20 | 16.25 |

**Things to think about:**

**a** Use technology to plot these points.

**b** What *shape* is the graph of $y$ against $x$?

**c** What is the maximum height reached by the ball?

**d** What *formula* gives the height of the ball when it has travelled $x$ metres horizontally towards Badrani?

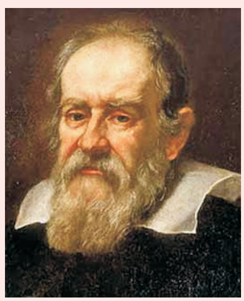
**e** Will the ball reach Badrani before it bounces?

## Historical note

**Galileo Galilei** (1564 - 1642) was born in Pisa, Tuscany. He was a philosopher who played a significant role in the scientific revolution of that time.

Within his research he conducted a series of experiments on the paths of projectiles, attempting to find a mathematical description of falling bodies.

Two of Galileo's experiments consisted of rolling a ball down a grooved ramp that was placed at a fixed height above the floor and inclined at a fixed angle to the horizontal. In one experiment the ball left the end of the ramp and descended to the floor. In the second, a horizontal shelf was placed at the end of the ramp, and the ball travelled along this shelf before descending to the floor.



***Galileo***

In each experiment Galileo altered the release height $h$ of the ball and measured the distance $d$ the ball travelled before landing. The units of measurement were called 'punti' (points).

In both experiments Galileo found that once the ball left the ramp or shelf, its path was parabolic and could therefore be modelled by a quadratic function.

## QUADRATICS

A **quadratic equation** is an equation of the form $ax^2 + bx + c = 0$ where $a$, $b$, and $c$ are constants, $a \neq 0$.

A **quadratic function** is a function of the form $y = ax^2 + bx + c$, $a \neq 0$.

Quadratic functions are members of the family of **polynomials**. The first few members of this family are shown in the table.

| *Polynomial function* | *Type* |
|---|---|
| $y = ax + b, \ a \neq 0$ | linear |
| $y = ax^2 + bx + c, \ a \neq 0$ | quadratic |
| $y = ax^3 + bx^2 + cx + d, \ a \neq 0$ | cubic |
| $y = ax^4 + bx^3 + cx^2 + dx + e, \ a \neq 0$ | quartic |

# A QUADRATIC EQUATIONS

Acme Leather Jacket Co. makes and sells $x$ leather jackets each week. Their profit function is given by
$P = -12.5x^2 + 550x - 2125$ dollars.


How many jackets must be made and sold each week in order to obtain a weekly profit of \$3000?

Clearly we need to solve the equation:

$$-12.5x^2 + 550x - 2125 = 3000$$

We can rearrange the equation to give

$$12.5x^2 - 550x + 5125 = 0,$$

which is of the form $ax^2 + bx + c = 0$ and is thus a quadratic equation.

## SOLVING QUADRATIC EQUATIONS

To solve quadratic equations we have the following methods to choose from:

- **factorise** the quadratic and use the rule:

  If $ab = 0$ then $a = 0$ or $b = 0$.

- **complete the square**
- use the **quadratic formula**
- use **technology**.

The **roots** or **solutions** of $ax^2 + bx + c = 0$ are the values of $x$ which satisfy the equation, or make it true.

For example: Consider $x^2 - 3x + 2 = 0$.

When $x = 2$, $x^2 - 3x + 2 = (2)^2 - 3(2) + 2$
$= 4 - 6 + 2$
$= 0$ ✓

So, $x = 2$ is a root of the equation $x^2 - 3x + 2 = 0$.

## SOLVING BY FACTORISATION

*Step 1*: If necessary, rearrange the equation so one side is zero.
*Step 2*: Fully factorise the other side.
*Step 3*: Apply the rule: If $ab = 0$ then $a = 0$ or $b = 0$.
*Step 4*: Solve the resulting linear equations.

### Example 1 — Self Tutor

Solve for $x$:

**a** $3x^2 + 5x = 0$ **b** $x^2 = 5x + 6$

**a**
$$\begin{aligned} 3x^2 + 5x &= 0 \\ \therefore\ x(3x+5) &= 0 \\ \therefore\ x = 0 \ \text{ or } \ 3x+5 &= 0 \\ \therefore\ x = 0 \ \text{ or } \ x &= -\tfrac{5}{3} \end{aligned}$$

**b**
$$\begin{aligned} x^2 &= 5x + 6 \\ \therefore\ x^2 - 5x - 6 &= 0 \\ \therefore\ (x-6)(x+1) &= 0 \\ \therefore\ x &= 6 \ \text{ or } \ -1 \end{aligned}$$

### Example 2 — Self Tutor

Solve for $x$:

**a** $4x^2 + 1 = 4x$ **b** $6x^2 = 11x + 10$

**a**
$$\begin{aligned} 4x^2 + 1 &= 4x \\ \therefore\ 4x^2 - 4x + 1 &= 0 \\ \therefore\ (2x-1)^2 &= 0 \\ \therefore\ x &= \tfrac{1}{2} \end{aligned}$$

**b**
$$\begin{aligned} 6x^2 &= 11x + 10 \\ \therefore\ 6x^2 - 11x - 10 &= 0 \\ \therefore\ (2x-5)(3x+2) &= 0 \\ \therefore\ x &= \tfrac{5}{2} \ \text{ or } \ -\tfrac{2}{3} \end{aligned}$$

**Caution:**

- Do not be tempted to divide both sides by an expression involving $x$.
  If you do this then you may lose one of the solutions.
  For example, consider $x^2 = 5x$.

  *Correct solution*
  $$\begin{aligned} x^2 &= 5x \\ \therefore\ x^2 - 5x &= 0 \\ \therefore\ x(x-5) &= 0 \\ \therefore\ x &= 0 \text{ or } 5 \end{aligned}$$

  *Incorrect solution*
  $$\begin{aligned} x^2 &= 5x \\ \therefore\ \frac{x^2}{x} &= \frac{5x}{x} \\ \therefore\ x &= 5 \end{aligned}$$

  By dividing both sides by $x$, we lose the solution $x = 0$.

- Be careful when taking square roots of both sides of an equation. You may otherwise lose solutions. For example:
  - Consider $x^2 = 25$.

    *Correct solution*
    $$\begin{aligned} x^2 &= 25 \\ \therefore\ x &= \pm\sqrt{25} \\ \therefore\ x &= \pm 5 \end{aligned}$$

    *Incorrect solution*
    $$\begin{aligned} x^2 &= 25 \\ \therefore\ x &= \sqrt{25} \\ \therefore\ x &= 5 \end{aligned}$$

- Consider $(2x-7)^2 = (x+1)^2$.

*Correct solution*

$$\begin{aligned} (2x-7)^2 &= (x+1)^2 \\ \therefore\ (2x-7)^2 - (x+1)^2 &= 0 \\ \therefore\ (2x-7+x+1)(2x-7-x-1) &= 0 \\ \therefore\ (3x-6)(x-8) &= 0 \\ \therefore\ x &= 2 \text{ or } 8 \end{aligned}$$

*Incorrect solution*

$$\begin{aligned} (2x-7)^2 &= (x+1)^2 \\ \therefore\ 2x-7 &= x+1 \\ \therefore\ x &= 8 \end{aligned}$$

## EXERCISE 3A.1

**1** Solve the following by factorisation:

**a** $4x^2+7x=0$ **b** $6x^2+2x=0$ **c** $3x^2-7x=0$
**d** $2x^2-11x=0$ **e** $3x^2=8x$ **f** $9x=6x^2$
**g** $x^2-5x+6=0$ **h** $x^2=2x+8$ **i** $x^2+21=10x$
**j** $9+x^2=6x$ **k** $x^2+x=12$ **l** $x^2+8x=33$

**2** Solve the following by factorisation:

**a** $9x^2-12x+4=0$ **b** $2x^2-13x-7=0$ **c** $3x^2=16x+12$
**d** $3x^2+5x=2$ **e** $2x^2+3=5x$ **f** $3x^2+8x+4=0$
**g** $3x^2=10x+8$ **h** $4x^2+4x=3$ **i** $4x^2=11x+3$
**j** $12x^2=11x+15$ **k** $7x^2+6x=1$ **l** $15x^2+2x=56$

### Example 3 — Self Tutor

Solve for $x$: $3x+\frac{2}{x}=-7$

$$\begin{aligned} 3x+\frac{2}{x} &= -7 \\ \therefore\ x\left(3x+\frac{2}{x}\right) &= -7x && \{\text{multiplying both sides by } x\} \\ \therefore\ 3x^2+2 &= -7x && \{\text{expanding the brackets}\} \\ \therefore\ 3x^2+7x+2 &= 0 && \{\text{making the RHS } 0\} \\ \therefore\ (x+2)(3x+1) &= 0 && \{\text{factorising}\} \\ \therefore\ x &= -2 \text{ or } -\tfrac{1}{3} \end{aligned}$$

RHS is short for Right Hand Side.

**3** Solve for $x$:

**a** $(x+1)^2=2x^2-5x+11$ **b** $(x+2)(1-x)=-4$
**c** $5-4x^2=3(2x+1)+2$ **d** $x+\frac{2}{x}=3$
**e** $2x-\frac{1}{x}=-1$ **f** $\frac{x+3}{1-x}=-\frac{9}{x}$

## SOLVING BY 'COMPLETING THE SQUARE'

As you would be aware by now, not all quadratics factorise easily. For example, $x^2+4x+1$ cannot be factorised by simple factorisation. In other words, we cannot write $x^2+4x+1$ in the form $(x-a)(x-b)$ where $a$, $b$ are rational.

An alternative way to solve equations like $x^2+4x+1=0$ is by 'completing the square'.

Equations of the form $ax^2+bx+c=0$ can be converted to the form $(x+p)^2=q$, from which the solutions are easy to obtain.

### Example 4 — Self Tutor

Solve exactly for $x$:

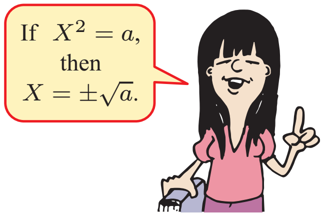
**a** $(x+2)^2=7$ **b** $(x-1)^2=-5$

**a** $(x+2)^2=7$

$\therefore\ x+2=\pm\sqrt{7}$

$\therefore\ x=-2\pm\sqrt{7}$

**b** $(x-1)^2=-5$ has no real solutions since the square $(x-1)^2$ cannot be negative.




The completed square form of an equation is $(x+p)^2=q$.

If we expand this out, $x^2+2px+p^2=q$.

Notice that the *coefficient of* $x$ equals $2p$. Therefore, $p$ is half the coefficient of $x$ in the expanded form.

If we have $x^2+2px=q$, then we "complete the square" by adding in $p^2$ to both sides of the equation.

### Example 5 — Self Tutor

Solve for exact values of $x$: $x^2+4x+1=0$

$x^2+4x+1=0$

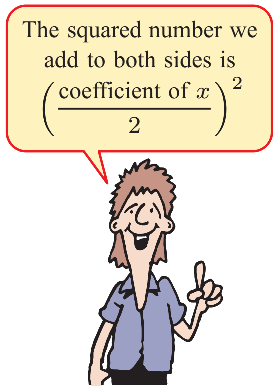
$\therefore\ x^2+4x=-1$ {put the constant on the RHS}

$\therefore\ x^2+4x+2^2=-1+2^2$ {completing the square}

$\therefore\ (x+2)^2=3$ {factorising LHS}

$\therefore\ x+2=\pm\sqrt{3}$

$\therefore\ x=-2\pm\sqrt{3}$




### Example 6 — Self Tutor

Solve exactly for $x$: $-3x^2+12x+5=0$

$-3x^2+12x+5=0$

$\therefore\ x^2-4x-\frac{5}{3}=0$ {dividing both sides by $-3$}

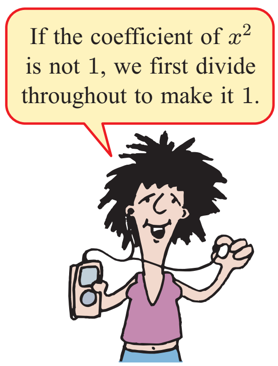
$\therefore\ x^2-4x=\frac{5}{3}$ {putting the constant on the RHS}

$\therefore\ x^2-4x+2^2=\frac{5}{3}+2^2$ {completing the square}

$\therefore\ (x-2)^2=\frac{17}{3}$ {factorising LHS}

$\therefore\ x-2=\pm\sqrt{\frac{17}{3}}$

$\therefore\ x=2\pm\sqrt{\frac{17}{3}}$



## EXERCISE 3A.2

**1** Solve exactly for $x$:

**a** $(x+5)^2 = 2$ **b** $(x+6)^2 = -11$ **c** $(x-4)^2 = 8$

**d** $(x-8)^2 = 7$ **e** $2(x+3)^2 = 10$ **f** $3(x-2)^2 = 18$

**g** $(x+1)^2 + 1 = 11$ **h** $(2x+1)^2 = 3$ **i** $(1-3x)^2 - 7 = 0$

**2** Solve exactly by completing the square:

**a** $x^2 - 4x + 1 = 0$ **b** $x^2 + 6x + 2 = 0$ **c** $x^2 - 14x + 46 = 0$

**d** $x^2 = 4x + 3$ **e** $x^2 + 6x + 7 = 0$ **f** $x^2 = 2x + 6$

**g** $x^2 + 6x = 2$ **h** $x^2 + 10 = 8x$ **i** $x^2 + 6x = -11$

**3** Solve exactly by completing the square:

**a** $2x^2 + 4x + 1 = 0$ **b** $2x^2 - 10x + 3 = 0$ **c** $3x^2 + 12x + 5 = 0$

**d** $3x^2 = 6x + 4$ **e** $5x^2 - 15x + 2 = 0$ **f** $4x^2 + 4x = 5$

**4** Solve for $x$:

**a** $3x - \dfrac{2}{x} = 4$ **b** $1 - \dfrac{1}{x} = -5x$ **c** $3 + \dfrac{1}{x^2} = -\dfrac{5}{x}$

**5** Suppose $ax^2 + bx + c = 0$ where $a$, $b$, and $c$ are constants, $a \neq 0$.
Solve for $x$ by completing the square.

## THE QUADRATIC FORMULA

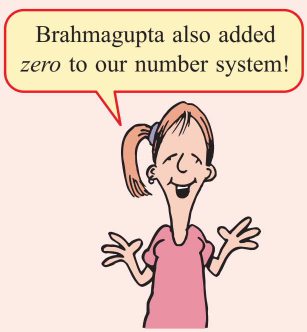
### Historical note — The quadratic formula

Thousands of years ago, people knew how to calculate the area of a shape given its side lengths. When they wanted to find the side lengths necessary to give a certain area, however, they ended up with a quadratic equation which they needed to solve.

The first known solution of a quadratic equation is written on the Berlin Papyrus from the Middle Kingdom (2160 - 1700 BC) in Egypt. By 400 BC, the Babylonians were using the method of 'completing the square'.

**Pythagoras** and **Euclid** both used geometric methods to explore the problem. Pythagoras noted that the square root was not always an integer, but he refused to accept that irrational solutions existed. Euclid also discovered that the square root was not always rational, but concluded that irrational numbers *did* exist.

A major jump forward was made in India around 700 AD, when Hindu mathematician **Brahmagupta** devised a general (but incomplete) solution for the quadratic equation $ax^2 + bx = c$ which was equivalent to $x = \dfrac{\sqrt{4ac+b^2}-b}{2a}$. Taking into account the sign of $c$, this is one of the two solutions we know today.




The final, complete solution as we know it today first came around 1100 AD, by another Hindu mathematician called **Baskhara**. He was the first to recognise that any positive number has two square roots, which could be negative or irrational. In fact, the quadratic formula is known in some countries today as 'Baskhara's Formula'.

While the Indians had knowledge of the quadratic formula even at this early stage, it took somewhat longer for the quadratic formula to arrive in Europe.

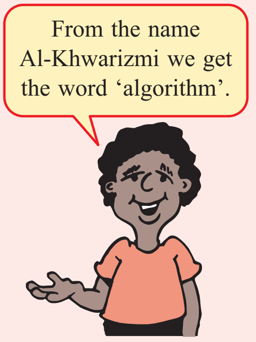
*Muhammad Al-Khwarizmi*

Around 820 AD, the Islamic mathematician **Muhammad bin Musa Al-Khwarizmi**, who was familiar with the work of Brahmagupta, recognised that for a quadratic equation to have real solutions, the value $b^2 - 4ac$ could not be negative. Al-Khwarizmi's work was brought to Europe by the Jewish mathematician and astronomer **Abraham bar Hiyya** (also known as Savasorda) who lived in Barcelona around 1100.

By 1545, **Girolamo Cardano** had blended the algebra of Al-Khwarizmi with the Euclidean geometry. His work allowed for the existence of complex or imaginary roots, as well as negative and irrational roots.

At the end of the 16th Century the mathematical notation and symbolism was introduced by **François Viète** in France.

In 1637, when **René Descartes** published *La Géométrie*, the quadratic formula adopted the form we see today.




If $\boldsymbol{ax^2 + bx + c = 0}$, $\boldsymbol{a \neq 0}$, then $\boldsymbol{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$.

**Proof:**

If $ax^2 + bx + c = 0$, $a \neq 0$

then $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ {dividing each term by $a$, as $a \neq 0$}

$\therefore \; x^2 + \frac{b}{a}x = -\frac{c}{a}$

$\therefore \; x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$ {completing the square on LHS}

$\therefore \; \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ {factorising}

$\therefore \; x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

$\therefore \; x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

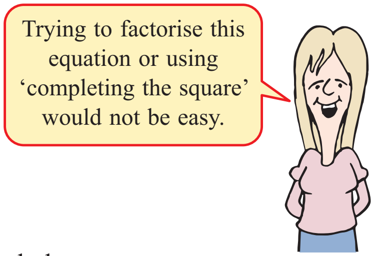
For example, consider the Acme Leather Jacket Co. equation from page **65**.

We need to solve: $12.5x^2 - 550x + 5125 = 0$
so in this case $a = 12.5$, $b = -550$, $c = 5125$

$$\therefore \quad x = \frac{550 \pm \sqrt{(-550)^2 - 4(12.5)(5125)}}{2(12.5)}$$

$$= \frac{550 \pm \sqrt{46\,250}}{25}$$

$$\approx 30.60 \text{ or } 13.40$$




However, for this application the number of jackets $x$ needs to be a whole number, so $x = 13$ or $31$ would produce a profit of around \$3000 each week.

### Example 7 — Self Tutor

Solve for $x$:

**a** $x^2 - 2x - 6 = 0$ **b** $2x^2 + 3x - 6 = 0$

**a** $x^2 - 2x - 6 = 0$ has
$a = 1$, $b = -2$, $c = -6$

$$\therefore \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-6)}}{2(1)}$$

$$\therefore \quad x = \frac{2 \pm \sqrt{4 + 24}}{2}$$

$$\therefore \quad x = \frac{2 \pm \sqrt{28}}{2}$$

$$\therefore \quad x = \frac{2 \pm 2\sqrt{7}}{2}$$

$$\therefore \quad x = 1 \pm \sqrt{7}$$

**b** $2x^2 + 3x - 6 = 0$ has
$a = 2$, $b = 3$, $c = -6$

$$\therefore \quad x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-6)}}{2(2)}$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{9 + 48}}{4}$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{57}}{4}$$

## EXERCISE 3A.3

**1** Use the quadratic formula to solve exactly for $x$:

**a** $x^2 - 4x - 3 = 0$ **b** $x^2 + 6x + 7 = 0$ **c** $x^2 + 1 = 4x$
**d** $x^2 + 4x = 1$ **e** $x^2 - 4x + 2 = 0$ **f** $2x^2 - 2x - 3 = 0$
**g** $3x^2 - 5x - 1 = 0$ **h** $-x^2 + 4x + 6 = 0$ **i** $-2x^2 + 7x - 2 = 0$

**2** Rearrange the following equations so they are written in the form $ax^2 + bx + c = 0$, then use the quadratic formula to solve exactly for $x$.

**a** $(x + 2)(x - 1) = 2 - 3x$ **b** $(2x + 1)^2 = 3 - x$ **c** $(x - 2)^2 = 1 + x$
**d** $(3x + 1)^2 = -2x$ **e** $(x + 3)(2x + 1) = 9$ **f** $(2x + 3)(2x - 3) = x$
**g** $\dfrac{x - 1}{2 - x} = 2x + 1$ **h** $x - \dfrac{1}{x} = 1$ **i** $2x - \dfrac{1}{x} = 3$

# B QUADRATIC INEQUALITIES

An **equation** is a mathematical statement that two expressions are equal.

Sometimes we have a statement that one expression is *greater than*, or else *greater than or equal to*, another. We call this an **inequality**.

$x^2 + 7x > 18$ is an example of a quadratic inequality.

While quadratic equations have 0, 1, or 2 solutions, quadratic inequalities may have 0, 1, or infinitely many solutions. We use interval notation to describe the set of solutions.

To solve quadratic inequalities we use these steps:

- Make the RHS zero by shifting all terms to the LHS.
- Fully factorise the LHS.
- Draw a sign diagram for the LHS.
- Determine the values required from the sign diagram.

### Example 8 — Self Tutor

Solve for $x$:

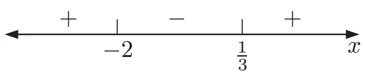
**a** $3x^2 + 5x \geqslant 2$  **b** $x^2 + 9 < 6x$

**a** $3x^2 + 5x \geqslant 2$

$\therefore\ 3x^2 + 5x - 2 \geqslant 0$ {make RHS zero}

$\therefore\ (3x-1)(x+2) \geqslant 0$ {factorising LHS}

Sign diagram of LHS is



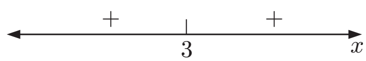

$\therefore\ x \leqslant -2$ or $x \geqslant \frac{1}{3}$.

**b** $x^2 + 9 < 6x$

$\therefore\ x^2 - 6x + 9 < 0$ {make RHS zero}

$\therefore\ (x-3)^2 < 0$ {factorising LHS}

Sign diagram of LHS is




So, the inequality is not true for any real $x$.

## EXERCISE 3B

**1** Solve for $x$:

**a** $(x-2)(x+3) \geqslant 0$ **b** $(x+1)(x-4) < 0$ **c** $(2x+1)(x-3) > 0$

**d** $x^2 - x \geqslant 0$ **e** $x^2 \geqslant 3x$ **f** $3x^2 + 2x < 0$

**g** $x^2 < 4$ **h** $2x^2 \geqslant 18$ **i** $x^2 + 4x + 4 > 0$

**j** $x^2 + 2x - 15 > 0$ **k** $x^2 - 11x + 28 \leqslant 0$ **l** $x(x+10) < -24$

**m** $x^2 - 30 \geqslant 13x$ **n** $2x^2 - x - 3 \geqslant 0$ **o** $4x^2 - 4x + 1 < 0$

**p** $6x^2 + 7x < 3$ **q** $3x^2 > 8(x+2)$ **r** $2x^2 - 4x + 2 < 0$

**s** $6x^2 + 1 \leqslant 5x$ **t** $(4x+1)(3x+2) \geqslant 16x - 4$ **u** $(2x+3)^2 < x + 6$

**2** In $3x^2 + 12 \ \Box\ 12x$, replace $\Box$ with $>$, $\geqslant$, $<$, or $\leqslant$ so that the resulting inequality has:

**a** no solutions **b** one solution **c** infinitely many solutions.

# C THE DISCRIMINANT OF A QUADRATIC

In the quadratic formula, the quantity $b^2 - 4ac$ under the square root sign is called the **discriminant**.

The symbol **delta** $\Delta$ is used to represent the discriminant, so $\Delta = b^2 - 4ac$.

The quadratic formula becomes $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta$ replaces $b^2 - 4ac$.

- If $\Delta = 0$, $x = \dfrac{-b}{2a}$ is the **only solution** (a **repeated** or **double root**)
- If $\Delta > 0$, $\sqrt{\Delta}$ is a positive real number, so there are **two distinct real roots**

$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{\Delta}}{2a}$$

- If $\Delta < 0$, $\sqrt{\Delta}$ is not a real number and so there are **no real roots**.
- If $a$, $b$, and $c$ are rational and $\Delta$ is a **square** then the equation has two rational roots which can be found by factorisation.

### Example 9 — Self Tutor

Use the discriminant to determine the nature of the roots of:

**a** $2x^2 - 2x + 3 = 0$

**b** $3x^2 - 4x - 2 = 0$

**a**
$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-2)^2 - 4(2)(3)\\ &= -20\end{aligned}$$
Since $\Delta < 0$, there are no real roots.

**b**
$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-4)^2 - 4(3)(-2)\\ &= 40\end{aligned}$$
Since $\Delta > 0$, but 40 is not a square, there are 2 distinct irrational roots.

### Example 10 — Self Tutor

Consider $x^2 - 2x + m = 0$. Find the discriminant $\Delta$, and hence find the values of $m$ for which the equation has:

**a** a repeated root **b** 2 distinct real roots **c** no real roots.

$x^2 - 2x + m = 0$ has $a = 1$, $b = -2$, and $c = m$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac\\ &= (-2)^2 - 4(1)(m)\\ &= 4 - 4m\end{aligned}$$

**a** For a repeated root
$$\Delta = 0$$
$$\therefore\ 4 - 4m = 0$$
$$\therefore\ 4 = 4m$$
$$\therefore\ m = 1$$

**b** For 2 distinct real roots
$$\Delta > 0$$
$$\therefore\ 4 - 4m > 0$$
$$\therefore\ -4m > -4$$
$$\therefore\ m < 1$$

**c** For no real roots
$$\Delta < 0$$
$$\therefore\ 4 - 4m < 0$$
$$\therefore\ -4m < -4$$
$$\therefore\ m > 1$$

### Example 11 — Self Tutor

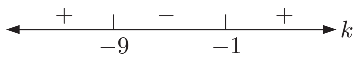
Consider the equation $kx^2 + (k+3)x = 1$. Find the discriminant $\Delta$ and draw its sign diagram. Hence, find the value of $k$ for which the equation has:

**a** two distinct real roots
**b** two real roots
**c** a repeated root
**d** no real roots.

$kx^2 + (k+3)x - 1 = 0$ has $a = k$, $b = (k+3)$, and $c = -1$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac \\ &= (k+3)^2 - 4(k)(-1) \\ &= k^2 + 6k + 9 + 4k \\ &= k^2 + 10k + 9 \\ &= (k+9)(k+1)\end{aligned}$$

So, $\Delta$ has sign diagram:




**a** For two distinct real roots, $\Delta > 0$ $\therefore$ $k < -9$ or $k > -1$, $k \neq 0$.
**b** For two real roots, $\Delta \geqslant 0$ $\therefore$ $k \leqslant -9$ or $k \geqslant -1$, $k \neq 0$.
**c** For a repeated root, $\Delta = 0$ $\therefore$ $k = -9$ or $k = -1$.
**d** For no real roots, $\Delta < 0$ $\therefore$ $-9 < k < -1$.

**Summary:**

| *Factorisation of quadratic* | *Roots of quadratic* | *Discriminant value* |
|---|---|---|
| two distinct linear factors | two real distinct roots | $\Delta > 0$ |
| two identical linear factors | two identical real roots (repeated) | $\Delta = 0$ |
| unable to factorise | no real roots | $\Delta < 0$ |

## EXERCISE 3C

**1** By using the discriminant only, state the nature of the solutions of:

**a** $x^2 + 7x - 3 = 0$ **b** $x^2 - 3x + 2 = 0$ **c** $3x^2 + 2x - 1 = 0$
**d** $5x^2 + 4x - 3 = 0$ **e** $x^2 + x + 5 = 0$ **f** $16x^2 - 8x + 1 = 0$

**2** By using the discriminant only, determine which of the following quadratic equations have rational roots which can be found by factorisation.

**a** $6x^2 - 5x - 6 = 0$ **b** $2x^2 - 7x - 5 = 0$ **c** $3x^2 + 4x + 1 = 0$
**d** $6x^2 - 47x - 8 = 0$ **e** $4x^2 - 3x + 2 = 0$ **f** $8x^2 + 2x - 3 = 0$

**3** For each of the following quadratic equations, determine the discriminant $\Delta$ in simplest form and draw its sign diagram. Hence find the value(s) of $m$ for which the equation has:

**i** a repeated root **ii** two distinct real roots **iii** no real roots.

**a** $x^2 + 4x + m = 0$ **b** $mx^2 + 3x + 2 = 0$ **c** $mx^2 - 3x + 1 = 0$

**4** For each of the following quadratic equations, find the discriminant $\Delta$ and hence draw its sign diagram. Find all values of $k$ for which the equation has:

**i** two distinct real roots **ii** two real roots **iii** a repeated root **iv** no real roots.

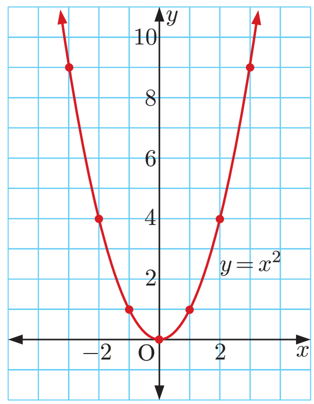
**a** $2x^2 + kx - k = 0$
**b** $kx^2 - 2x + k = 0$
**c** $x^2 + (k+2)x + 4 = 0$
**d** $2x^2 + (k-2)x + 2 = 0$
**e** $x^2 + (3k-1)x + (2k+10) = 0$
**f** $(k+1)x^2 + kx + k = 0$

# D QUADRATIC FUNCTIONS

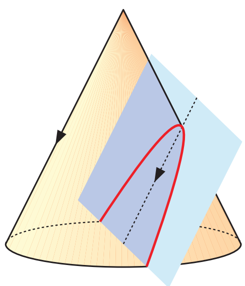
A **quadratic function** has the form $y = ax^2 + bx + c$ where $a \neq 0$.

The simplest quadratic function is $y = x^2$. Its graph can be drawn from a table of values.

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y$ | $9$ | $4$ | $1$ | $0$ | $1$ | $4$ | $9$ |




The graph of a quadratic function is called a **parabola**.



The parabola is one of the **conic sections**, the others being circles, hyperbolae, and ellipses. They are called conic sections because they can be obtained by cutting a cone with a plane. A parabola is produced by cutting the cone with a plane parallel to its slant side.

There are many examples of parabolas in everyday life, including water fountains, suspension bridges, and radio telescopes.



## TERMINOLOGY

The graph of a quadratic function $y = ax^2 + bx + c$, $a \neq 0$ is called a **parabola**.

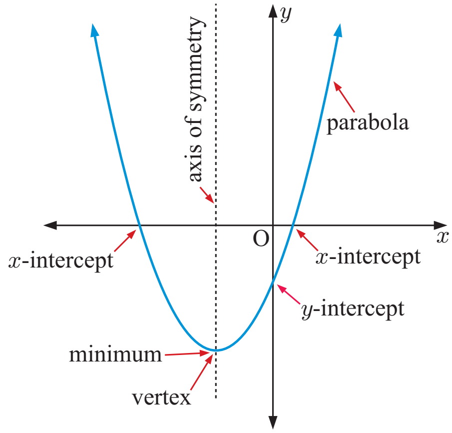
The point where the graph 'turns' is called the **vertex**.

If the graph opens upwards, the vertex is the **minimum** or **minimum turning point**, and the graph is **concave upwards**.

If the graph opens downwards, the vertex is the **maximum** or **maximum turning point**, and the graph is **concave downwards**.

The vertical line that passes through the vertex is called the **axis of symmetry**. Every parabola is symmetrical about its axis of symmetry.

The point where the graph crosses the $y$-axis is the **$y$-intercept**.




The points (if they exist) where the graph crosses the $x$-axis are called the **$x$-intercepts**. They correspond to the **roots** of the equation $y = 0$.

### Discovery 1 — Graphing $y = a(x-p)(x-q)$

This Discovery is best done using a **graphing package** or **graphics calculator**.

GRAPHING PACKAGE

**What to do:**

**1** **a** Use technology to help you to sketch:
$y = (x-1)(x-3)$, $y = 2(x-1)(x-3)$, $y = -(x-1)(x-3)$,
$y = -3(x-1)(x-3)$, and $y = -\frac{1}{2}(x-1)(x-3)$

**b** Find the $x$-intercepts for each function in **a**.

**c** What is the geometrical significance of $a$ in $y = a(x-1)(x-3)$?

**2** **a** Use technology to help you to sketch:
$y = 2(x-1)(x-4)$, $y = 2(x-3)(x-5)$, $y = 2(x+1)(x-2)$,
$y = 2x(x+5)$, and $y = 2(x+2)(x+4)$

**b** Find the $x$-intercepts for each function in **a**.

**c** What is the geometrical significance of $p$ and $q$ in $y = 2(x-p)(x-q)$?

**3** **a** Use technology to help you to sketch:
$y = 2(x-1)^2$, $y = 2(x-3)^2$, $y = 2(x+2)^2$, $y = 2x^2$

**b** Find the $x$-intercepts for each function in **a**.

**c** What is the geometrical significance of $p$ in $y = 2(x-p)^2$?

**4** Copy and complete:

- If a quadratic has the form $y = a(x-p)(x-q)$ then it ...... the $x$-axis at ......
- If a quadratic has the form $y = a(x-p)^2$ then it ...... the $x$-axis at ......

## Discovery 2 — Graphing $y = a(x-h)^2 + k$

This Discovery is also best done using technology.

GRAPHING PACKAGE

**What to do:**

**1** **a** Use technology to help you to sketch:
$y = (x-3)^2 + 2$, $y = 2(x-3)^2 + 2$, $y = -2(x-3)^2 + 2$,
$y = -(x-3)^2 + 2$, and $y = -\frac{1}{3}(x-3)^2 + 2$

**b** Find the coordinates of the vertex for each function in **a**.

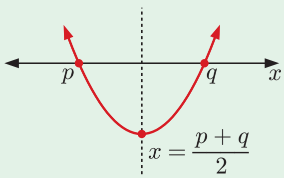
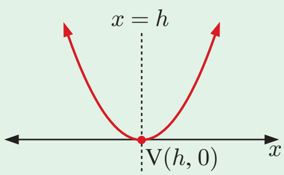
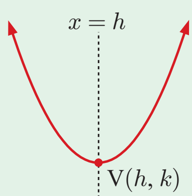
**c** What is the geometrical significance of $a$ in $y = a(x-3)^2 + 2$?

**2** **a** Use technology to help you to sketch:
$y = 2(x-1)^2 + 3$, $y = 2(x-2)^2 + 4$, $y = 2(x-3)^2 + 1$,
$y = 2(x+1)^2 + 4$, $y = 2(x+2)^2 - 5$, and $y = 2(x+3)^2 - 2$

**b** Find the coordinates of the vertex for each function in **a**.

**c** What is the geometrical significance of $h$ and $k$ in $y = 2(x-h)^2 + k$?

**3** Copy and complete:
If a quadratic has the form $y = a(x-h)^2 + k$ then its vertex has coordinates ......
The graph of $y = a(x-h)^2 + k$ is a ...... of the graph of $y = ax^2$ with vector ......

| *Quadratic form,* $a \neq 0$ | *Graph* | *Facts* |
|---|---|---|
| • $y = a(x-p)(x-q)$<br>$p$, $q$ are real |  | $x$-intercepts are $p$ and $q$<br>axis of symmetry is $x = \frac{p+q}{2}$<br>vertex is $\left(\frac{p+q}{2}, f\left(\frac{p+q}{2}\right)\right)$ |
| • $y = a(x-h)^2$<br>$h$ is real |  | touches $x$-axis at $h$<br>axis of symmetry is $x = h$<br>vertex is $(h, 0)$ |
| • $y = a(x-h)^2 + k$ |  | axis of symmetry is $x = h$<br>vertex is $(h, k)$ |

You should have found that $a$, the coefficient of $x^2$, controls the width of the graph and whether it opens upwards or downwards.

For a quadratic function $y = ax^2 + bx + c$, $a \neq 0$:

- $a > 0$ produces the shape  called concave up.

  $a < 0$ produces the shape  called concave down.

- If $-1 < a < 1$, $a \neq 0$ the graph is wider than $y = x^2$.
  If $a < -1$ or $a > 1$ the graph is narrower than $y = x^2$.

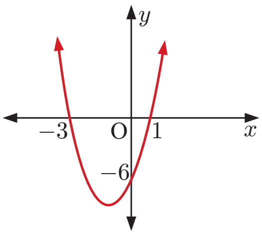
## Example 12

Self Tutor

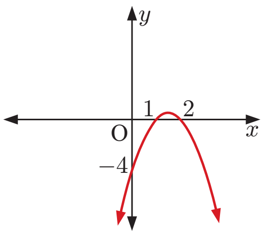
Using axes intercepts only, sketch the graphs of:

**a** $y = 2(x+3)(x-1)$ **b** $y = -2(x-1)(x-2)$ **c** $y = \frac{1}{2}(x+2)^2$

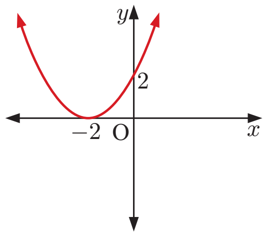
**a** $y = 2(x+3)(x-1)$
has $x$-intercepts $-3$, $1$
When $x = 0$,
$y = 2(3)(-1)$
$= -6$
$\therefore$ $y$-intercept is $-6$




**b** $y = -2(x-1)(x-2)$
has $x$-intercepts $1$, $2$
When $x = 0$,
$y = -2(-1)(-2)$
$= -4$
$\therefore$ $y$-intercept is $-4$



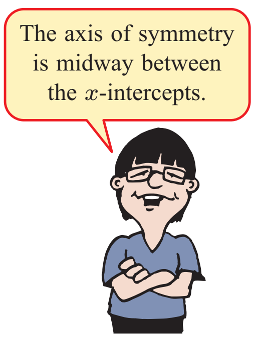

**c** $y = \frac{1}{2}(x+2)^2$
touches $x$-axis at $-2$
When $x = 0$,
$y = \frac{1}{2}(2)^2$
$= 2$
$\therefore$ $y$-intercept is $2$




## EXERCISE 3D.1

**1** Using axes intercepts only, sketch the graphs of:

**a** $y = (x-4)(x+2)$ **b** $f(x) = -(x-4)(x+2)$
**c** $y = 2(x+3)(x+5)$ **d** $f(x) = -3(x+1)(x+5)$
**e** $f(x) = 2(x+3)^2$ **f** $y = -\frac{1}{4}(x+2)^2$




**2** State the equation of the axis of symmetry for each graph in question **1**.

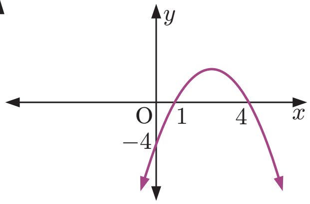
**3** Match each quadratic function with its corresponding graph.

**a** $y = 2(x-1)(x-4)$ **b** $y = -(x+1)(x-4)$ **c** $y = (x-1)(x-4)$

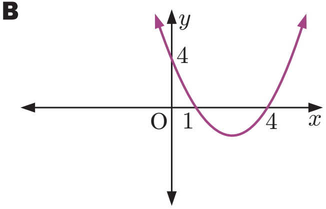
**d** $y = (x+1)(x-4)$ **e** $y = 2(x+4)(x-1)$ **f** $y = -3(x+4)(x-1)$

**g** $y = -(x-1)(x-4)$ **h** $y = -3(x-1)(x-4)$

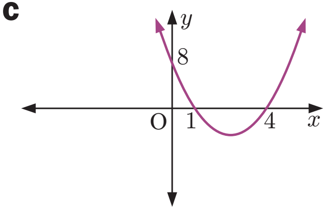
**A**



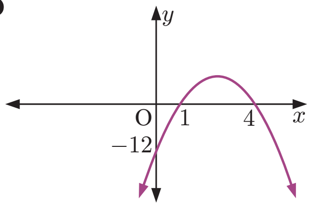

**B**



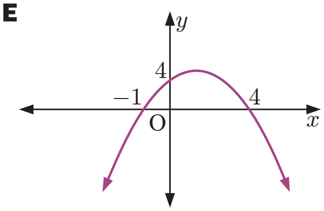

**C**



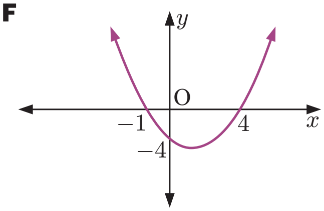

**D**



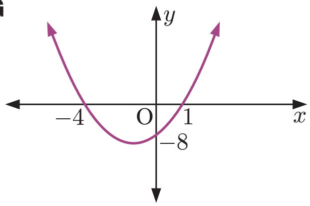

**E**



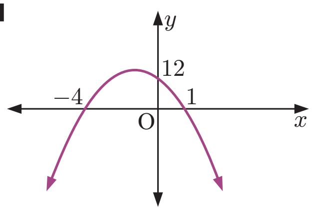

**F**




**G**




**H**



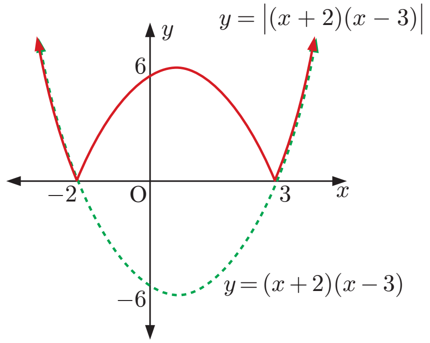

### *Example 13* **Self Tutor**

Sketch the graph of $y = |(x+2)(x-3)|$.

We first sketch $y = (x+2)(x-3)$.

$y = (x+2)(x-3)$ has $x$-intercepts $-2$ and $3$, and $y$-intercept $2(-3) = -6$.

The part of the graph that is below the $x$-axis is then reflected in the $x$-axis to produce the graph of $y = |(x+2)(x-3)|$.




**4** Sketch the graph of:

**a** $y = |(x+4)(x-5)|$ **b** $f(x) = |-(x-1)(x-6)|$

**c** $y = |2(x-2)(x+2)|$ **d** $f(x) = \left|-3(x+3)^2\right|$

### Example 14 — Self Tutor

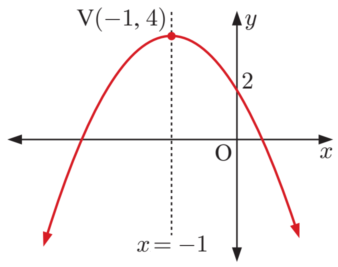
Use the vertex, axis of symmetry, and $y$-intercept to graph $y = -2(x+1)^2 + 4$.

The vertex is $(-1, 4)$.

The axis of symmetry is $x = -1$.

When $x = 0$, $y = -2(1)^2 + 4$
$= 2$

$a < 0$ so the shape is 




**5** Use the vertex, axis of symmetry, and $y$-intercept to graph:

**a** $y = (x-1)^2 + 3$ **b** $f(x) = 2(x+2)^2 + 1$ **c** $y = -2(x-1)^2 - 3$

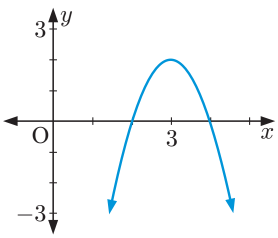
**d** $f(x) = \frac{1}{2}(x-3)^2 + 2$ **e** $y = -\frac{1}{3}(x-1)^2 + 4$ **f** $f(x) = -\frac{1}{10}(x+2)^2 - 3$

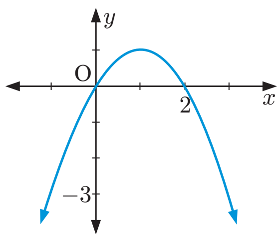
**6** Match each quadratic function with its corresponding graph:

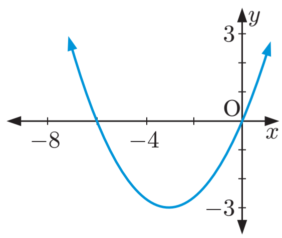
**a** $y = -(x+1)^2 + 3$ **b** $y = -2(x-3)^2 + 2$ **c** $y = x^2 + 2$

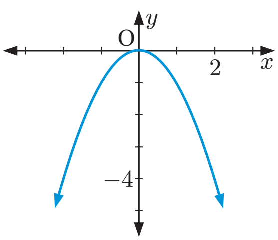
**d** $y = -(x-1)^2 + 1$ **e** $y = (x-2)^2 - 2$ **f** $y = \frac{1}{3}(x+3)^2 - 3$

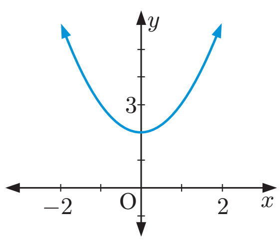
**g** $y = -x^2$ **h** $y = -\frac{1}{2}(x-1)^2 + 1$ **i** $y = 2(x+2)^2 - 1$

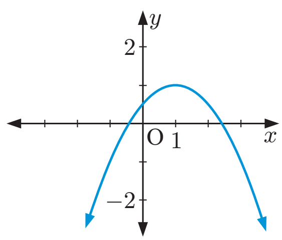
**A**


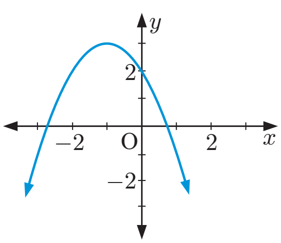

**B**


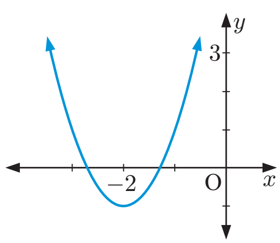

**C**


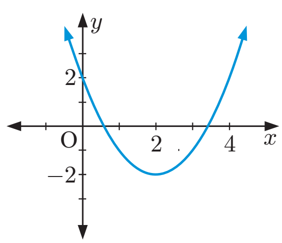

**D**



**E**



**F**



**G**



**H**



**I**


## SKETCHING GRAPHS BY 'COMPLETING THE SQUARE'

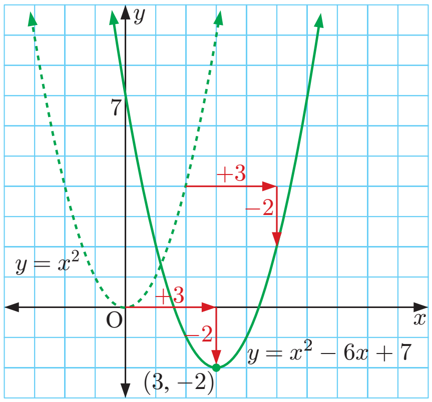
If we wish to graph a quadratic given in general form $y = ax^2 + bx + c$, one approach is to convert it to the form $y = a(x-h)^2 + k$ where we can read off the coordinates of the vertex $(h, k)$. To do this, we '**complete the square**'.

Consider the simple case $y = x^2 - 6x + 7$, for which $a = 1$.

$$\begin{aligned} y &= x^2 - 6x + 7 \\ \therefore\ y &= \underbrace{x^2 - 6x + 3^2}_{(x-3)^2}\ \underbrace{+7 - 3^2}_{-2} \\ \therefore\ y &= (x-3)^2 - 2 \end{aligned}$$

So, the vertex is $(3, -2)$.

To obtain the graph of $y = x^2 - 6x + 7$ from the graph of $y = x^2$, we shift it 3 units to the right and 2 units down.




### Example 15 Self Tutor

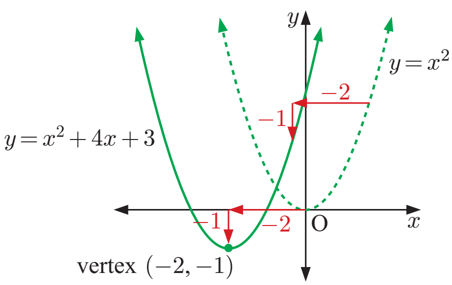
Write $y = x^2 + 4x + 3$ in the form $y = (x-h)^2 + k$ by 'completing the square'.

Hence sketch $y = x^2 + 4x + 3$, stating the coordinates of the vertex.

$$\begin{aligned} y &= x^2 + 4x + 3 \\ \therefore\ y &= x^2 + 4x + 2^2 + 3 - 2^2 \\ \therefore\ y &= (x+2)^2 - 1 \end{aligned}$$

shift 2 units left    shift 1 unit down

The vertex is $(-2, -1)$ and the $y$-intercept is 3.




### Example 16 Self Tutor

**a** Convert $y = 3x^2 - 4x + 1$ to the form $y = a(x-h)^2 + k$.

**b** Hence, write down the coordinates of its vertex and sketch the quadratic.

**a** $y = 3x^2 - 4x + 1$

$= 3[x^2 - \frac{4}{3}x + \frac{1}{3}]$ {taking out a factor of 3}

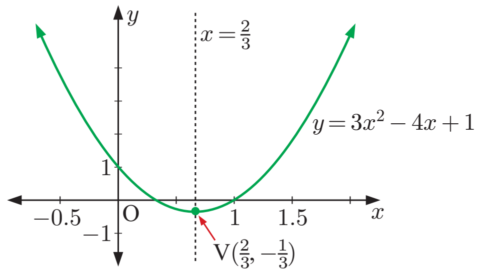
$= 3[x^2 - 2(\frac{2}{3})x + (\frac{2}{3})^2 - (\frac{2}{3})^2 + \frac{1}{3}]$ {completing the square}

$= 3[(x - \frac{2}{3})^2 - \frac{4}{9} + \frac{3}{9}]$ {writing as a perfect square}

$= 3[(x - \frac{2}{3})^2 - \frac{1}{9}]$

$= 3(x - \frac{2}{3})^2 - \frac{1}{3}$

**b** The vertex is $\left(\frac{2}{3}, -\frac{1}{3}\right)$ and the $y$-intercept is 1.




## EXERCISE 3D.2

**1** Write the following quadratics in the form $y = (x - h)^2 + k$ by 'completing the square'. Hence sketch each function, stating the coordinates of the vertex.

**a** $y = x^2 - 2x + 3$ **b** $y = x^2 + 4x - 2$ **c** $y = x^2 - 4x$

**d** $y = x^2 + 3x$ **e** $y = x^2 + 5x - 2$ **f** $y = x^2 - 3x + 2$

**g** $y = x^2 - 6x + 5$ **h** $y = x^2 + 8x - 2$ **i** $y = x^2 - 5x + 1$

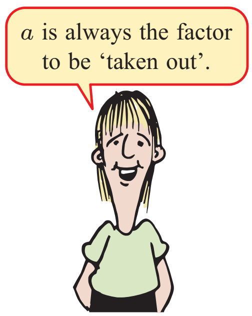
**2** For each of the following quadratics:

**i** Write the quadratic in the form $y = a(x - h)^2 + k$.
**ii** State the coordinates of the vertex.
**iii** Find the $y$-intercept.
**iv** Sketch the graph of the quadratic.

**a** $y = 2x^2 + 4x + 5$ **b** $y = 2x^2 - 8x + 3$

**c** $y = 2x^2 - 6x + 1$ **d** $y = 3x^2 - 6x + 5$

**e** $y = -x^2 + 4x + 2$ **f** $y = -2x^2 - 5x + 3$




## QUADRATIC FUNCTIONS OF THE FORM $y = ax^2 + bx + c$

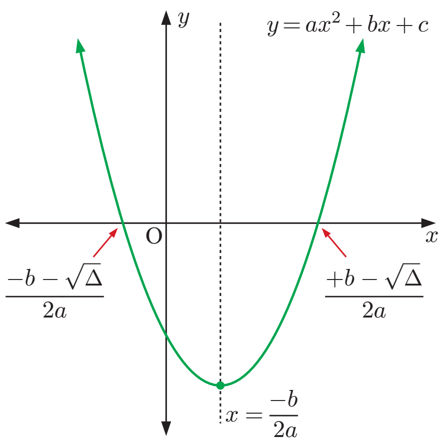
We now consider a method of graphing quadratics of the form $y = ax^2 + bx + c$ directly, without having to first convert them to a different form.

We know that the quadratic equation $ax^2 + bx + c = 0$ has solutions $\dfrac{-b - \sqrt{\Delta}}{2a}$ and $\dfrac{-b + \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$.

If $\Delta \geqslant 0$, these are the $x$-intercepts of the quadratic function $y = ax^2 + bx + c$.

The average of the values is $\dfrac{-b}{2a}$, so we conclude that:

- the axis of symmetry is $x = \dfrac{-b}{2a}$
- the vertex of the quadratic has $x$-coordinate $\dfrac{-b}{2a}$.



To graph a quadratic of the form $y = ax^2 + bx + c$, we:

- find the axis of symmetry $x = \dfrac{-b}{2a}$
- substitute to find the $y$-coordinate of the vertex
- state the $y$-intercept $c$
- find the $x$-intercepts by solving $ax^2 + bx + c = 0$, either by factorisation or using the quadratic formula.

### Example 17

Self Tutor

Consider the quadratic $f(x) = 2x^2 + 8x - 10$.

**a** Find the axis of symmetry.
**b** Find the coordinates of the vertex.
**c** Find the axes intercepts.
**d** Hence, sketch the function.
**e** State the range of the function.

$f(x) = 2x^2 + 8x - 10$ has $a = 2$, $b = 8$, and $c = -10$.

$a > 0$, so the shape is 

**a** $\dfrac{-b}{2a} = \dfrac{-8}{2(2)} = -2$

The axis of symmetry is $x = -2$.

**b** $f(-2) = 2(-2)^2 + 8(-2) - 10$
$= -18$

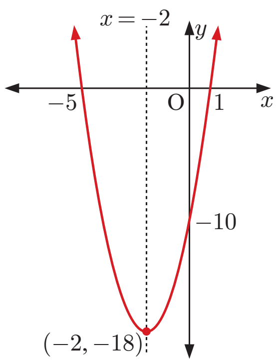
The vertex is $(-2, -18)$.

**c** The $y$-intercept is $-10$.

When $y = 0$, $2x^2 + 8x - 10 = 0$
$\therefore\ 2(x^2 + 4x - 5) = 0$
$\therefore\ 2(x+5)(x-1) = 0$
$\therefore\ x = -5$ or $1$

$\therefore$ the $x$-intercepts are $-5$ and $1$.

**d**

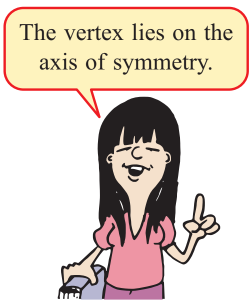



**e** The range is $\{y : y \geqslant -18\}$.

## EXERCISE 3D.3

**1** Locate the turning point or vertex for each of the following quadratic functions:

**a** $f(x) = x^2 - 4x + 2$
**b** $y = x^2 + 2x - 3$
**c** $y = 2x^2 + 4$
**d** $f(x) = -3x^2 + 1$
**e** $y = 2x^2 + 8x - 7$
**f** $f(x) = -x^2 - 4x - 9$
**g** $y = 2x^2 + 6x - 1$
**h** $f(x) = 2x^2 - 10x + 3$
**i** $y = -\frac{1}{2}x^2 + x - 5$



**2** For each of the following quadratics:

- **i** state the axis of symmetry
- **ii** find the coordinates of the vertex
- **iii** find the axes intercepts
- **iv** sketch the quadratic
- **v** state the range.

**a** $y = x^2 - 8x + 7$ **b** $y = -x^2 - 6x - 8$ **c** $f(x) = 6x - x^2$
**d** $y = -x^2 + 3x - 2$ **e** $y = 2x^2 + 4x - 24$ **f** $f(x) = -3x^2 + 4x - 1$
**g** $f(x) = 2x^2 - 5x + 2$ **h** $y = 4x^2 - 8x - 5$ **i** $y = -\frac{1}{4}x^2 + 2x - 3$

**3** For each of the following quadratics:

- **i** write the quadratic in factored form and hence find the roots
- **ii** write the quadratic in completed square form and hence find the coordinates of the vertex
- **iii** sketch the quadratic, showing the details you have found.

**a** $y = x^2 - 10x + 16$ **b** $y = x^2 + 10x + 9$ **c** $y = x^2 - 14x + 45$

**4** Sketch the graph of:

**a** $y = \left|x^2 + 4x - 12\right|$ **b** $f(x) = \left|-x^2 - 3x + 10\right|$ **c** $y = \left|4x^2 - 12x + 5\right|$

## Example 18 — Self Tutor

Find the range of $y = x^2 - 6x - 2$ on the domain $-2 \leqslant x \leqslant 7$.

$y = x^2 - 6x - 2$ has $a = 1$, $b = -6$, and $c = -2$.

$a > 0$, so the shape is 

$\dfrac{-b}{2a} = \dfrac{-(-6)}{2(1)} = 3$

When $x = 3$, $y = 3^2 - 6(3) - 2$
$= -11$

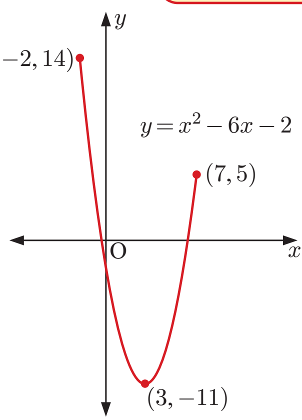
$\therefore$ the vertex is $(3, -11)$.

When $x = -2$, $y = (-2)^2 - 6(-2) - 2$
$= 14$

When $x = 7$, $y = 7^2 - 6(7) - 2$
$= 5$

So, on the domain $\{x : -2 \leqslant x \leqslant 7\}$,
the range is $\{y : -11 \leqslant y \leqslant 14\}$.

To find the range of a function on a given domain, you must evaluate the function at the endpoints of the domain.



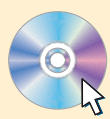

**5** Find the range of:

**a** $f(x) = x^2 + 4x - 6$ on $-6 \leqslant x \leqslant 3$ **b** $y = -x^2 + 8x + 3$ on $0 \leqslant x \leqslant 7$
**c** $y = 2x^2 - 12x + 5$ on $-2 \leqslant x \leqslant 6$ **d** $f(x) = 7x - x^2$ on $-1 \leqslant x \leqslant 5$

## Activity

Click on the icon to run a card game for quadratic functions.

CARD GAME



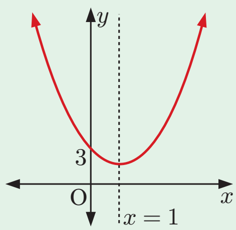
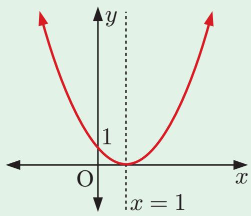
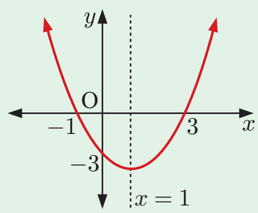
## THE DISCRIMINANT AND THE QUADRATIC GRAPH

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $\Delta = b^2 - 4ac$.

We used $\Delta$ to determine the number of real roots of the equation. If they exist, these roots correspond to zeros of the quadratic $y = ax^2 + bx + c$. $\Delta$ therefore tells us about the relationship between a quadratic function and the $x$-axis.

The graphs of $y = x^2 - 2x + 3$, $y = x^2 - 2x + 1$, and $y = x^2 - 2x - 3$ all have the same axis of symmetry, $x = 1$.

Consider the following table:

| $y = x^2 - 2x + 3$ | $y = x^2 - 2x + 1$ | $y = x^2 - 2x - 3$ |
|---|---|---|
|   |   |   |
| $\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(3)$ $= -8$ | $\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(1)$ $= 0$ | $\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(-3)$ $= 16$ |
| $\Delta < 0$ | $\Delta = 0$ | $\Delta > 0$ |
| does not cut the $x$-axis | touches the $x$-axis | cuts the $x$-axis twice |

For a quadratic function $y = ax^2 + bx + c$, we consider the discriminant $\Delta = b^2 - 4ac$.

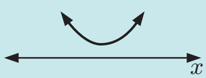
If $\Delta < 0$, the graph does not cut the $x$-axis.

If $\Delta = 0$, the graph *touches* the $x$-axis.

If $\Delta > 0$, the graph cuts the $x$-axis twice.

## POSITIVE DEFINITE AND NEGATIVE DEFINITE QUADRATICS

**Positive definite quadratics** are quadratics which are positive for all values of $x$. So, $ax^2 + bx + c > 0$ for all $x \in \mathbb{R}$.




**Test:** A quadratic is **positive definite** if and only if $a > 0$ and $\Delta < 0$.

**Negative definite quadratics** are quadratics which are negative for all values of $x$. So, $ax^2 + bx + c < 0$ for all $x \in \mathbb{R}$.


**Test:** A quadratic is **negative definite** if and only if $a < 0$ and $\Delta < 0$.