Example 13 Self Tutor

$x+3$ is a factor of $P(x) = x^3 + ax^2 - 7x + 6$. Find $a \in \mathbb{R}$ and the other factors.

Since $x+3$ is a factor,

The coefficient of x^3 is $1 \times 1 = 1$

This must be 2 so the constant term is $3 \times 2 = 6$

$$\begin{aligned} x^3 + ax^2 - 7x + 6 &= (x+3)(x^2 + bx + 2) \quad \text{for some constant } b \\ &= x^3 + bx^2 + 2x \\ &\qquad + 3x^2 + 3bx + 6 \\ &= x^3 + (b+3)x^2 + (3b+2)x + 6 \end{aligned}$$

Equating coefficients gives $3b + 2 = -7$ and $a = b + 3$

$\therefore$ $b = -3$ and $a = 0$

$$\begin{aligned} \therefore \quad P(x) &= (x+3)(x^2 - 3x + 2) \\ &= (x+3)(x-1)(x-2) \end{aligned}$$

The other factors are $(x-1)$ and $(x-2)$.

8 $2x - 3$ is a factor of $2x^3 + 3x^2 + ax + 3$. Find $a \in \mathbb{R}$ and all zeros of the cubic.

Example 14 Self Tutor

$2x+3$ and $x-1$ are factors of $2x^4 + ax^3 - 3x^2 + bx + 3$.

Find constants a and b and all zeros of the polynomial.

Since $2x+3$ and $x-1$ are factors,

The coefficient of x^4 is $2 \times 1 \times 1 = 2$

This must be -1 so the constant term is $3 \times -1 \times -1 = 3$

$$\begin{aligned} 2x^4 + ax^3 - 3x^2 + bx + 3 &= (2x+3)(x-1)(x^2 + cx - 1) \quad \text{for some } c \\ &= (2x^2 + x - 3)(x^2 + cx - 1) \\ &= 2x^4 + 2cx^3 - 2x^2 \\ &\qquad + x^3 + cx^2 - x \\ &\qquad\qquad - 3x^2 - 3cx + 3 \\ &= 2x^4 + (2c+1)x^3 + (c-5)x^2 + (-1-3c)x + 3 \end{aligned}$$

Equating coefficients gives $2c + 1 = a$, $c - 5 = -3$, and $-1 - 3c = b$

$\therefore$ $c = 2$

$\therefore$ $a = 5$ and $b = -7$

$\therefore$ $P(x) = (2x+3)(x-1)(x^2 + 2x - 1)$

Now $x^2 + 2x - 1$ has zeros $\dfrac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2} = \dfrac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

$\therefore$ $P(x)$ has zeros $-\frac{3}{2}$, 1, and $-1 \pm \sqrt{2}$.

9 $2x+1$ and $x-2$ are factors of $P(x) = 2x^4 + ax^3 + bx^2 + 18x + 8$.

a Find a and b. **b** Hence, solve $P(x) = 0$.

10 $x^3 + 3x^2 - 9x + c$, $c \in \mathbb{R}$, has two identical linear factors. Prove that c is either 5 or -27, and factorise the cubic into linear factors in each case.

C THE REMAINDER THEOREM

Consider the cubic polynomial $P(x) = x^3 + 5x^2 - 11x + 3$.

If we divide $P(x)$ by $x-2$, we find that

$$\frac{x^3+5x^2-11x+3}{x-2} = x^2 + 7x + 3 + \frac{\overset{\text{remainder}}{9}}{x-2}$$

So, when $P(x)$ is divided by $x-2$, the remainder is 9.

Notice also that $P(2) = 8 + 20 - 22 + 3$
$= 9$, which is the remainder.

By considering other examples like the one above, we formulate the **Remainder theorem**.

The Remainder Theorem

When a polynomial $P(x)$ is divided by $x-k$ until a constant remainder R is obtained, then $R = P(k)$.

Proof: By the division algorithm, $P(x) = Q(x)(x-k) + R$
Letting $x = k$, $P(k) = Q(k) \times 0 + R$
$\therefore$ $P(k) = R$

When using the Remainder theorem, it is important to realise that the following statements are equivalent:

- $P(x) = (x-k)Q(x) + R$
- $P(k) = R$
- $P(x)$ divided by $x-k$ leaves a remainder of R.

Example 15 Self Tutor

Use the Remainder theorem to find the remainder when $x^4 - 3x^3 + x - 4$ is divided by $x+2$.

If $P(x) = x^4 - 3x^3 + x - 4$, then

$$\begin{aligned} P(-2) &= (-2)^4 - 3(-2)^3 + (-2) - 4 \\ &= 16 + 24 - 2 - 4 \\ &= 34 \end{aligned}$$

$\therefore$ when $x^4 - 3x^3 + x - 4$ is divided by $x+2$, the remainder is 34. {Remainder theorem}

The Remainder theorem allows us to find a remainder without having to perform the division.

Example 16 — Self Tutor

When $2x^3+2x^2+ax+b$ is divided by $x+3$, the remainder is -11.

When the same polynomial is divided by $x-2$, the remainder is 9.

Find a and b.

Let $P(x)=2x^3+2x^2+ax+b$

Now $P(-3)=-11$ and $P(2)=9$ {Remainder theorem}

So, $2(-3)^3+2(-3)^2+a(-3)+b=-11$

$\therefore\ -54+18-3a+b=-11$

$\therefore\ -3a+b=25$ .... (1)

and $2(2)^3+2(2)^2+a(2)+b=9$

$\therefore\ 16+8+2a+b=9$

$\therefore\ 2a+b=-15$ .... (2)

Solving simultaneously: $3a-b=-25$ $\{-1\times(1)\}$

$2a+b=-15$ $\{(2)\}$

Adding, $5a\quad=-40$

$\therefore\ a=-8$

Substituting $a=-8$ in (2) gives $2(-8)+b=-15$

$\therefore\ b=1$

EXERCISE 6C

1 For $P(x)$ a real polynomial, write two equivalent statements for each of:

- **a** If $P(2)=7$, then ......
- **b** If $P(x)=Q(x)(x+3)-8$, then ......
- **c** If $P(x)$ divided by $x-5$ has a remainder of 11 then ......

2 Without performing division, find the remainder when:

- **a** x^3+2x^2-7x+8 is divided by $x-1$
- **b** $2x^3+x^2-5x+11$ is divided by $x+3$
- **c** x^4-2x^2+3x-1 is divided by $x+2$.

3 Use the Remainder theorem to find the remainder when x^3-x^2-3x-5 is divided by $x-3$. Check that your answer is the same as when this long division was performed on page **159**.

4 Find $a\in\mathbb{R}$ such that:

- **a** when x^3-2x+a is divided by $x-2$, the remainder is 7
- **b** when $2x^3+x^2+ax-5$ is divided by $x+1$, the remainder is -8.

5 When x^3+2x^2+ax+b is divided by $x-1$ the remainder is 4, and when divided by $x+2$ the remainder is 16. Find a and b.

6 When x^3+4x^2+ax+b is divided by $x-2$ the remainder is 20, and when divided by $x+5$ the remainder is 6. Find a and b.

7 Consider $f(x) = 2x^3 + ax^2 - 3x + b$. When $f(x)$ is divided by $x + 1$, the remainder is 7. When $f(x)$ is divided by $x - 2$, the remainder is 28. Find the remainder when $f(x)$ is divided by $x + 3$.

8 **a** Suppose a polynomial $P(x)$ is divided by $2x - 1$ until a constant remainder R is obtained. Show that $R = P(\frac{1}{2})$.

Hint: $P(x) = Q(x)(2x - 1) + R$.

b Find the remainder when:

i $4x^2 - 10x + 1$ is divided by $2x - 1$ **ii** $2x^3 - 5x^2 + 8$ is divided by $2x - 1$

iii $4x^3 + 7x - 3$ is divided by $2x + 1$.

9 When $2x^3 + ax^2 + bx + 4$ is divided by $x + 1$ the remainder is -5, and when divided by $2x - 1$ the remainder is 10. Find a and b.

10 When $P(z)$ is divided by $z^3 - 3z + 2$ the remainder is $4z - 7$.
Find the remainder when $P(z)$ is divided by: **a** $z - 1$ **b** $z - 2$.

D THE FACTOR THEOREM

For any polynomial $P(x)$, k is a zero of $P(x)$ $\Leftrightarrow$ $x - k$ is a factor of $P(x)$.

Proof:

$$\begin{aligned} k \text{ is a zero of } P(x) &\Leftrightarrow P(k) = 0 && \{\text{definition of a zero}\} \\ &\Leftrightarrow R = 0 && \{\text{Remainder theorem}\} \\ &\Leftrightarrow P(x) = Q(x)(x - k) && \{\text{division algorithm}\} \\ &\Leftrightarrow x - k \text{ is a factor of } P(x) && \{\text{definition of a factor}\} \end{aligned}$$

The **Factor theorem** says that if 2 is a zero of $P(x)$ then $x - 2$ is a factor of $P(x)$, and vice versa.

We can use the Factor theorem to determine whether $x - k$ is a factor of a polynomial, without having to perform the long division.

Example 17 Self Tutor

Determine whether:

a $x - 2$ is a factor of $x^3 + 3x^2 - 13x + 6$ **b** $x + 3$ is a factor of $x^3 - 8x + 7$.

a Let $P(x) = x^3 + 3x^2 - 13x + 6$

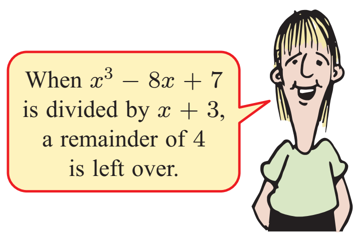
$$\begin{aligned} \therefore\ P(2) &= (2)^3 + 3(2)^2 - 13(2) + 6 \\ &= 8 + 12 - 26 + 6 \\ &= 0 \end{aligned}$$

Since $P(2) = 0$, $x - 2$ is a factor of $x^3 + 3x^2 - 13x + 6$. {Factor theorem}

b Let $P(x) = x^3 - 8x + 7$

$$\begin{aligned} \therefore\ P(-3) &= (-3)^3 - 8(-3) + 7 \\ &= -27 + 24 + 7 \\ &= 4 \end{aligned}$$

Since $P(-3) \neq 0$, $x + 3$ is *not* a factor of $x^3 - 8x + 7$. {Factor theorem}



Example 18 — Self Tutor

$x-2$ is a factor of $P(x) = x^3 + kx^2 - 3x + 6$.

Find k, and write $P(x)$ as a product of linear factors.

Since $x-2$ is a factor, $P(2) = 0$ {Factor theorem}

$$\therefore\ (2)^3 + k(2)^2 - 3(2) + 6 = 0$$
$$\therefore\ 8 + 4k = 0$$
$$\therefore\ k = -2$$

The coefficient of x^3 is $1 \times 1 = 1$

The constant term is $-2 \times -3 = 6$

So, $P(x) = x^3 - 2x^2 - 3x + 6 = (x-2)(x^2 + bx - 3)$
$$= x^3 + (b-2)x^2 + (-2b-3)x + 6$$

Equating x^2 s: $b - 2 = -2$
$$\therefore\ b = 0$$

So, $P(x) = (x-2)(x^2-3)$
$$= (x-2)(x+\sqrt{3})(x-\sqrt{3})$$

Example 19 — Self Tutor

$2x-1$ is a factor of $f(x) = 4x^3 - 4x^2 + ax + b$, and the remainder when $f(x)$ is divided by $x-1$ is -1. Find the values of a and b.

$2x-1$ is a factor of $f(x)$, so $f(\frac{1}{2}) = 0$

$$\therefore\ 4(\tfrac{1}{2})^3 - 4(\tfrac{1}{2})^2 + a(\tfrac{1}{2}) + b = 0$$
$$\therefore\ \tfrac{1}{2}a + b = \tfrac{1}{2} \quad .... (1)$$

Also, $f(1) = -1$ {Remainder theorem}

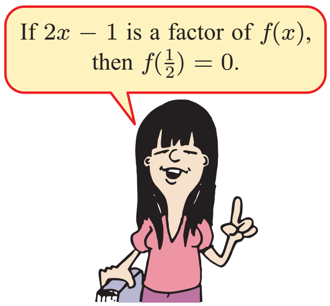
$$\therefore\ 4(1)^3 - 4(1)^2 + a(1) + b = -1$$
$$\therefore\ a + b = -1 \quad .... (2)$$

Solving simultaneously:

$$-a - 2b = -1 \quad \{-2 \times (1)\}$$
$$a + b = -1 \quad \{(2)\}$$

Adding, $-b = -2$

$$\therefore\ b = 2 \quad \text{and} \quad a = -3$$



EXERCISE 6D

1 Use the Factor theorem to determine whether:

a $x-1$ is a factor of $4x^3 - 7x^2 + 5x - 2$

b $x-3$ is a factor of $x^4 - x^3 - 4x^2 - 15$

c $x+2$ is a factor of $3x^3 + 5x^2 - 6x - 8$

d $x+4$ is a factor of $2x^3 + 6x^2 + 4x + 16$.

2 **a** Find c given that $x+1$ is a factor of $5x^3 - 3x^2 + cx + 10$.

b Find c given that $x-3$ is a factor of $x^4 - 2x^3 + cx^2 - 4x + 3$.

c Find b given that $x+2$ is a factor of $x^6 + bx^5 - 2x^3 - 5x + 6$.

3 $x+2$ is a factor of $P(x) = 2x^3 + x^2 + kx - 4$.
Find k, and hence write $P(x)$ as a product of linear factors.

4 $x-3$ is a factor of $P(x) = 3x^3 + kx^2 - 5x + 6$.

a Find k. **b** Write $P(x)$ in the form $P(x) = (x-3)(ax^2 + bx + c)$.

c Find all solutions to $P(x) = 0$.

5 $2x^3 + ax^2 + bx + 5$ has factors $x-1$ and $x+5$. Find a and b.

6 $x-2$ is a factor of $f(x) = x^3 + ax^2 - 11x + b$. The remainder when $f(x)$ is divided by $x+1$ is 15. Find a and b.

7 $x+3$ is a factor of $P(x) = 2x^3 + 9x^2 + ax + b$. When $P(x)$ is divided by $x+4$, the remainder is -18.

a Find a and b.

b Find the remainder when $P(x)$ is divided by $x-2$.

c Write $P(x)$ in the form $P(x) = (x+3)(px^2 + qx + r)$.

d Find the zeros of $P(x)$.

8 $2x-1$ is a factor of $P(x) = 2x^3 + ax^2 - 8x + b$. When $P(x)$ is divided by $x-1$, the remainder is 3.

a Find a and b.

b Find the irrational roots of $P(x) = 0$, giving your answer in the form $x = p \pm \sqrt{q}$ where $p, q \in \mathbb{Z}$.

9 **a** Consider $P(x) = x^3 - a^3$ where a is real.

i Find $P(a)$. What is the significance of this result?

ii Factorise $x^3 - a^3$ as the product of a real linear and a quadratic factor.

b Now consider $P(x) = x^3 + a^3$, where a is real.

i Find $P(-a)$. What is the significance of this result?

ii Factorise $x^3 + a^3$ as the product of a real linear and a quadratic factor.

10 Find the real number a such that $(x - 1 - a)$ is a factor of $P(x) = x^3 - 3ax - 9$.

E CUBIC EQUATIONS

In **Discovery 4** in **Chapter 3** on page **97**, we considered the sum and products of roots of a quadratic.
In particular, we saw that

$$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta.$$

If we perform a similar expansion for a cubic, we find that

$$(x-\alpha)(x-\beta)(x-\gamma) = x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma.$$

In both cases, the *product of the roots* has the same size as the constant term in the expanded polynomial.

If the leading coefficient of the polynomial $\neq 1$, then we need to multiply by this as well:

$$a(x-\alpha)(x-\beta)(x-\gamma) = ax^3 - a(\alpha+\beta+\gamma)x^2 + a(\alpha\beta+\beta\gamma+\gamma\alpha)x - a\alpha\beta\gamma.$$

If you think a cubic equation has integer roots, try to find them by factorising the constant term.

Example 20 — Self Tutor

Solve for x: $\quad x^3 - 31x - 30 = 0.$

Let $P(x) = x^3 - 31x - 30.$

The constant term is -30, so the product of the roots is 30.

Since $30 = 5 \times 3 \times 2 \times 1$, likely integer roots are $\pm1, \pm2, \pm3, \pm5$. They could also be ±6 since $2 \times 3 = 6$, and so on.

Now $P(1) = -60$, so 1 is not a root.

But $P(-1) = 0$, so -1 is a root, and $(x+1)$ is a factor of $P(x)$.

The coefficient of x^3 is $1 \times 1 = 1$ — The constant term is $1 \times -30 = -30$

$$\begin{aligned} \text{So,}\quad P(x) = x^3 + 0x^2 - 31x - 30 &= (x+1)(x^2+bx-30) \\ &= x^3 + (b+1)x^2 + (b-30)x - 30 \end{aligned}$$

Equating x^2s: $\quad b + 1 = 0$

$\therefore \quad b = -1$

$$\begin{aligned} \text{Hence}\quad P(x) &= (x+1)(x^2-x-30) \\ &= (x+1)(x+5)(x-6) \end{aligned}$$

$\therefore$ the solutions are -1, -5, and 6.

Note that this method only works for those cubics with all integer roots.

EXERCISE 6E

1 Solve for x:

a $x^3 - 6x^2 + 11x - 6 = 0$ **b** $x^3 - 3x^2 + 4 = 0$ **c** $x^3 + 2x^2 - x - 2 = 0$

d $x^3 - 6x^2 + 5x + 12 = 0$ **e** $x^3 + 5x^2 - 16x - 80 = 0$ **f** $x^3 + 13x^2 + 55x + 75 = 0$

2 Solve for x:

a $2x^3 - 6x^2 - 8x + 24 = 0$ **b** $2x^3 - 2x^2 - 48x - 72 = 0$ **c** $3x^3 - 24x^2 - 15x + 252 = 0$

Discussion

Consider the general cubic $p(x) = ax^3 + bx^2 + cx + d$, $a, b, c, d \in \mathbb{R}$.

What happens to $p(x)$ if x gets:

- very large and positive
- very large and negative?

What does this tell you about the number of solutions that $p(x) = 0$ may have?

Review set 6A

1 Given $p(x) = 5x^2 - x + 4$ and $q(x) = 3x^2 + 7x - 1$, find:

a $p(x) + q(x)$ **b** $2p(x) - q(x)$ **c** $p(x)q(x)$

2 Find the quotient and remainder of:

a $\dfrac{2x^2 + 11x + 18}{x + 3}$ **b** $\dfrac{x^3 - 6x^2 + 10x - 9}{x - 2}$

3 Find the zeros of:

a $3x^2 + 2x - 8$ **b** $x^2 + 8x + 11$

4 **a** Given that $x^3 + x^2 - 3x + 9 = (x + 3)(ax^2 + bx + c)$, find the values of a, b, and c.

b Show that $x^3 + x^2 - 3x + 9$ has only one real zero.

5 Use the Remainder theorem to find the remainder when:

a $x^3 - 4x^2 + 5x - 1$ is divided by $x - 2$

b $2x^3 + 6x^2 - 7x + 12$ is divided by $x + 5$.

6 Use the Factor theorem to determine whether:

a $x + 1$ is a factor of $2x^4 - 9x^2 - 6x - 1$

b $x - 3$ is a factor of $x^4 - 2x^3 - 4x^2 + 5x - 6$.

7 $2x^2 + kx - 5$ has remainder 3 when divided by $x + 4$. Find k.

8 $ax^3 + 5x^2 - x + b$ has remainder 7 when divided by $x - 1$, and remainder -11 when divided by $x + 2$. Find a and b.

9 Find c given that $x - 2$ is a factor of $x^5 - 2x^4 + cx^3 - 7x^2 + 5x - 6$.

10 $x - 4$ is a factor of $f(x) = x^3 + 2x^2 + ax + b$, and when $f(x)$ is divided by $x + 2$ the remainder is 18.

a Find a and b. **b** Find all zeros of $f(x)$.

11 Solve for x: $x^3 - x^2 - 17x - 15 = 0$

Review set 6B

1 Expand and simplify:

a $(3x^3 + 2x - 5)(4x - 3)$ **b** $(2x^2 - x + 3)^2$

2 Carry out the following divisions:

a $\dfrac{x^3}{x+2}$ **b** $\dfrac{x^3}{(x+2)(x+3)}$

3 Find *all* cubic polynomials with zeros $\frac{1}{4}$, $1 \pm \sqrt{5}$.

4 If $f(x) = x^3 - 3x^2 - 9x + b$ has $(x-k)^2$ as a factor, show that there are two possible values of k. For each of these two values of k, find the corresponding value for b, and hence solve $f(x) = 0$.

5 Find the remainder when:

a $x^3 - 5x^2 + 9$ is divided by $x - 2$ **b** $4x^3 + 7x - 11$ is divided by $2x - 1$.

6 When $f(x) = 2x^3 - x^2 + ax - 4$ is divided by $x - 3$, the remainder is 56.

a Find a. **b** Find the remainder when $f(x)$ is divided by $x + 1$.

7 **a** Use the Factor theorem to show that $x - 2$ is a factor of $x^3 - 13x + 18$.

b Write $x^3 - 13x + 18$ in the form $(x-2)(ax^2 + bx + c)$, where $a, b, c \in \mathbb{Z}$.

c Find the real roots of $x^3 + 18 = 13x$.

8 $x - 2$ and $x + 3$ are factors of $ax^3 - 3x^2 - 11x + b$. Find a and b.

9 $x + 1$ is a factor of $f(x) = x^3 + 5x^2 + kx + 4$. Find k, and the zeros of $f(x)$.

10 $2x - 1$ is a factor of $f(x) = 2x^3 - 9x^2 + ax + b$, and when $f(x)$ is divided by $x - 1$ the remainder is -15.

a Find a and b. **b** Write $f(x)$ as a product of linear factors.

11 Solve for x: $2x^3 - 2x^2 - 28x + 48 = 0$

7

Straight line graphs

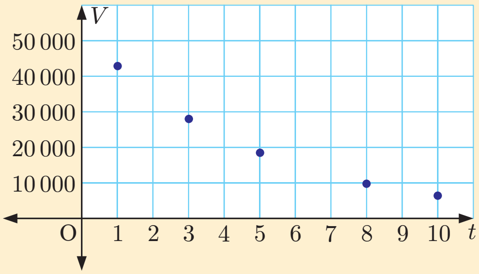
Contents:

Opening problem

This table shows the value $\$V$ of Doug's father's car t years after purchase.

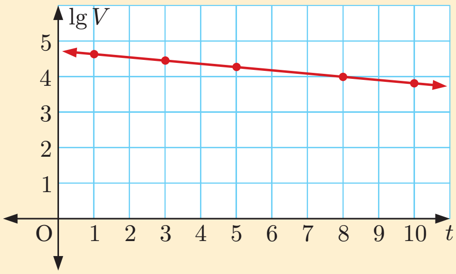
t (years)	1	3	5	8	10
V (dollars)	42 900	28 000	18 500	9800	6400

Doug is trying to work out the equation connecting V and t. When he plots the values on a graph, the result is a curve:



Doug's father suggests that he plots $\lg V$ against t. When Doug does this, the result is a straight line:

t	1	3	5	8	10
$\lg V$	4.63	4.45	4.27	3.99	3.81



Things to think about:

a Is it easier to find the equation of a curve or a straight line?

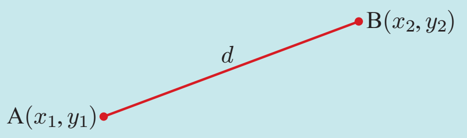
b How can Doug use the equation of the straight line to determine the relationship between V and t?

BACKGROUND KNOWLEDGE

You should be familiar with the following facts involving points and lines on the coordinate plane:

Distance between two points

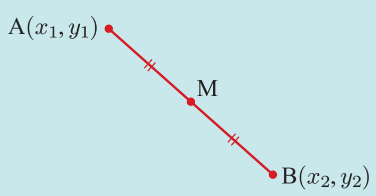
The **distance** between $A(x_1, y_1)$ and $B(x_2, y_2)$ is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.



Midpoint

If A is (x_1, y_1) and B is (x_2, y_2), then the midpoint of AB is

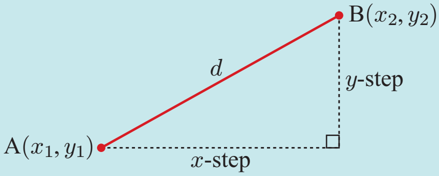
$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



Gradient

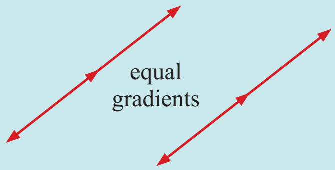
The **gradient** of a line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\frac{y\text{-step}}{x\text{-step}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

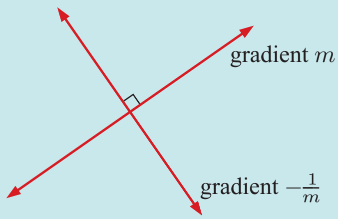


Gradients of parallel and perpendicular lines

- If two lines are **parallel**, then their gradients are **equal**.



- If two lines are **perpendicular**, then their gradients are **negative reciprocals**.
  If the gradient of one line is m, then the gradient of the other line is $-\frac{1}{m}$.

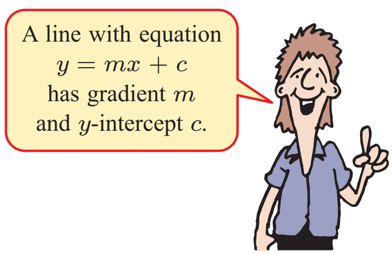


A EQUATIONS OF STRAIGHT LINES

The **equation of a line** is an equation which connects the x and y values for every point on the line.

The equation of a straight line can be written in:

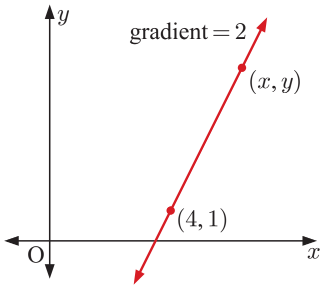
- **gradient-intercept form** $y = mx + c$, or
- **general form** $Ax + By = D$.



FINDING THE EQUATION OF A LINE

In order to find the equation of a line, we need to know some information.

Suppose we know the gradient of the line is 2, and that the line passes through $(4, 1)$.



We suppose (x, y) is any point on the line.

The gradient between $(4, 1)$ and (x, y) is $\dfrac{y-1}{x-4}$, and this gradient must equal 2.

So, $\dfrac{y-1}{x-4} = 2$

$\therefore\ y - 1 = 2(x-4)$ {multiplying both sides by $(x-4)$}

$\therefore\ y - 1 = 2x - 8$ {expanding the brackets}

$\therefore\ y = 2x - 7$ {adding 1 to both sides}

This is the equation of the line in gradient-intercept form.

We can find the equation of a line if we know:
- its **gradient** and the **coordinates of any point** on the line, or
- the **coordinates of two distinct points** on the line.

If a straight line has gradient m and passes through the point (x_1, y_1) then its equation is $\boldsymbol{\dfrac{y-y_1}{x-x_1} = m}$ or $\boldsymbol{y - y_1 = m(x - x_1)}$.

We can rearrange this equation into either gradient-intercept or general form.

Example 1 Self Tutor

Find, in *gradient-intercept form*, the equation of the line through $(-1, 3)$ with a gradient of 5.

The equation of the line is $y - 3 = 5(x - (-1))$

$\therefore\ y - 3 = 5x + 5$

$\therefore\ y = 5x + 8$

Example 2 Self Tutor

Find, in *general form*, the equation of the line with gradient $\frac{3}{4}$ which passes through $(5, -2)$.

The equation of the line is $y - (-2) = \frac{3}{4}(x-5)$

$\therefore\ 4(y+2) = 3(x-5)$

$\therefore\ 4y + 8 = 3x - 15$

$\therefore\ 3x - 4y = 23$

EXERCISE 7A.1

1 Find the gradient and y-intercept of the line with equation:

a $y = 3x + 5$ **b** $y = 4x - 2$ **c** $y = \frac{1}{5}x + \frac{3}{5}$

d $y = -7x - 3$ **e** $y = \dfrac{x+2}{6}$ **f** $y = \dfrac{8-5x}{3}$

2 Find the equation of the line with:

a gradient 1 and y-intercept -2 **b** gradient -1 and y-intercept 4

c gradient 2 and y-intercept 0 **d** gradient $-\frac{1}{2}$ and y-intercept 3.

3 Find, in *gradient-intercept form*, the equation of the line through:

a $(2, -5)$ with gradient 4 **b** $(-1, -2)$ with gradient -3

c $(7, -3)$ with gradient -5 **d** $(1, 4)$ with gradient $\frac{1}{2}$

e $(-1, 3)$ with gradient $-\frac{1}{3}$ **f** $(2, 6)$ with gradient 0.

4 Find, in *general form*, the equation of the line through:

a $(2, 5)$ having gradient $\frac{2}{3}$ **b** $(-1, 4)$ having gradient $\frac{3}{5}$

c $(5, 0)$ having gradient $-\frac{1}{3}$ **d** $(6, -2)$ having gradient $-\frac{2}{7}$

e $(-3, -1)$ having gradient 4 **f** $(5, -3)$ having gradient -2

g $(4, -5)$ having gradient $-3\frac{1}{2}$ **h** $(-7, -2)$ having gradient 6.

Example 3 Self Tutor

Find the equation of the line which passes through the points $A(-1, 5)$ and $B(2, 3)$.

The gradient of the line is $\dfrac{3-5}{2-(-1)} = -\dfrac{2}{3}$.

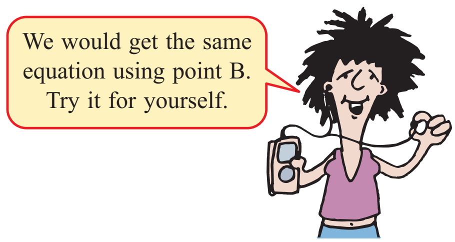
Using point A, the equation is

$$y - 5 = -\tfrac{2}{3}(x - (-1))$$

$\therefore\ 3(y-5) = -2(x+1)$

$\therefore\ 3y - 15 = -2x - 2$

$\therefore\ 2x + 3y = 13$



5 Find, in *gradient-intercept form*, the equation of the line which passes through the points:

a $A(2, 3)$ and $B(4, 8)$ **b** $A(0, 3)$ and $B(-1, 5)$

c $A(-1, -2)$ and $B(4, -2)$ **d** $C(-3, 1)$ and $D(2, 0)$

e $P(5, -1)$ and $Q(-1, -2)$ **f** $R(-1, -3)$ and $S(-4, -1)$.

6 Find, in *general form*, the equation of the line which passes through:

a $(0, 1)$ and $(3, 2)$ **b** $(1, 4)$ and $(0, -1)$ **c** $(2, -1)$ and $(-1, -4)$

d $(0, -2)$ and $(5, 2)$ **e** $(3, 2)$ and $(-1, 0)$ **f** $(-1, -1)$ and $(2, -3)$.

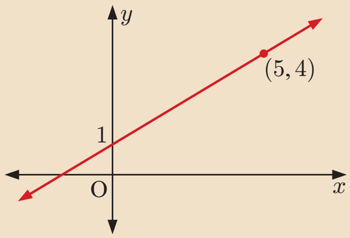
7 Consider the points A(2, 5) and B(−4, 2). Find:

- **a** the distance between A and B
- **b** the midpoint of AB
- **c** the gradient of the line which passes through A and B
- **d** the equation of the line which passes through A and B.

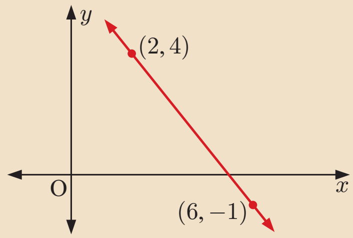
Example 4 — Self Tutor

Find the equation of the line with graph:

a



b



a Two points on the line are $(0, 1)$ and $(5, 4)$.

$\therefore$ the gradient $m = \dfrac{4-1}{5-0} = \frac{3}{5}$

and the y-intercept $c = 1$.

The equation is $y = \frac{3}{5}x + 1$ {gradient-intercept form}

b Two points on the line are $(2, 4)$ and $(6, -1)$.

$\therefore$ the gradient $m = \dfrac{-1-4}{6-2} = -\frac{5}{4}$

Since we do not know the y-intercept we use the general form.

The equation is $y - 4 = -\frac{5}{4}(x-2)$

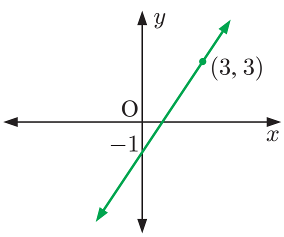
$\therefore \; 4(y-4) = -5(x-2)$

$\therefore \; 4y - 16 = -5x + 10$

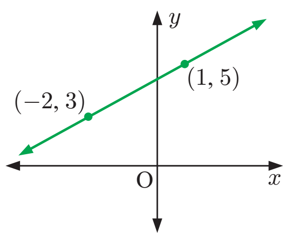
$\therefore \; 5x + 4y = 26$

8 Find the equations of the illustrated lines:

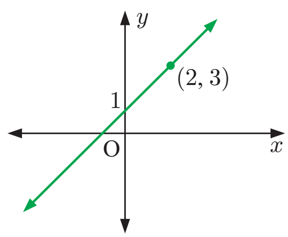
a



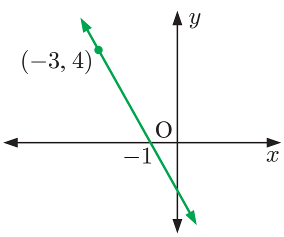
b



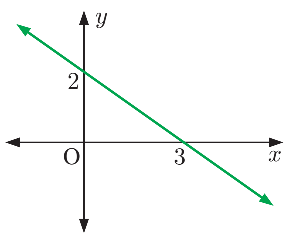
c



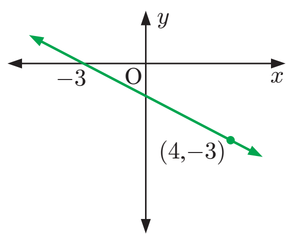
d



e



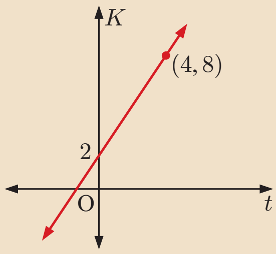
f



Example 5

Self Tutor

Find the equation connecting the variables.



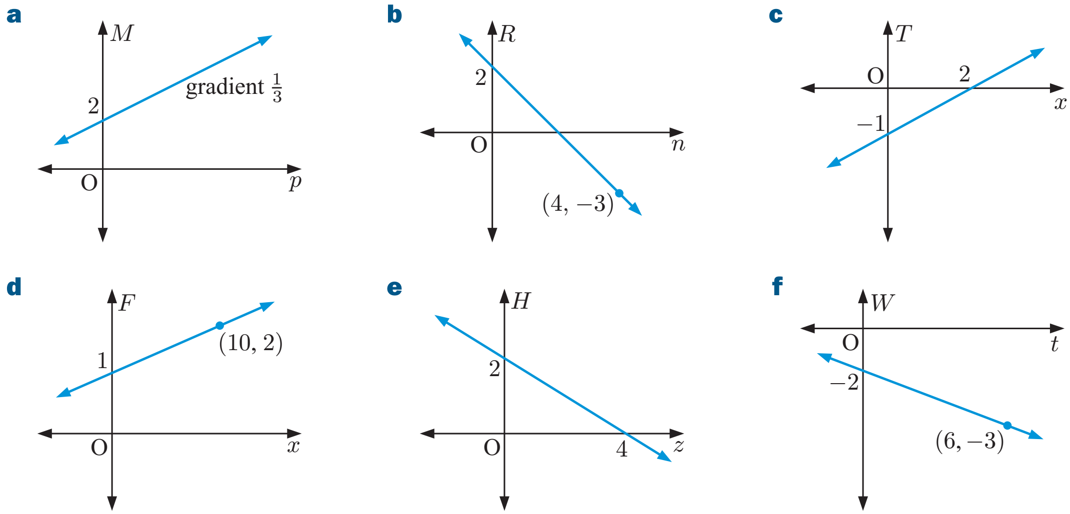
$(0, 2)$ and $(4, 8)$ lie on the straight line.

$\therefore$ the gradient $m = \frac{8-2}{4-0} = \frac{6}{4} = \frac{3}{2}$, and the y-intercept $c = 2$.

In this case K is on the vertical axis and t is on the horizontal axis.

$\therefore$ the equation is $K = \frac{3}{2}t + 2$.

9 Find the equation connecting the variables:



Example 6

Self Tutor

Consider the points $A(-2, 5)$ and $B(1, 3)$. A line perpendicular to AB, passes through B.

a Find the equation of the line.

b Find the coordinates of the point where the line cuts the x-axis.

a The gradient of AB $= \frac{3-5}{1-(-2)} = -\frac{2}{3}$

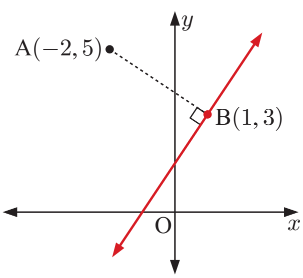
$\therefore$ the perpendicular line has gradient $\frac{3}{2}$, and passes through $B(1, 3)$.

$\therefore$ its equation is $y - 3 = \frac{3}{2}(x-1)$

$\therefore$ $2(y-3) = 3(x-1)$

$\therefore$ $2y - 6 = 3x - 3$

$\therefore$ $3x - 2y = -3$



b The line cuts the x-axis when $y = 0$

$\therefore\ 3x - 2(0) = -3$

$\therefore\ x = -1$

$\therefore$ the line cuts the x-axis at $(-1, 0)$.

10 Consider the points $P(-3, -2)$ and $Q(1, 6)$. A line perpendicular to PQ, passes through Q.

a Find the equation of the line.

b Find the coordinates of the point where the line cuts the x-axis.

11 Suppose A has coordinates $(-7, 4)$ and B has coordinates $(3, -2)$. A line parallel to AB, passes through $C(5, -1)$.

a Find the equation of the line.

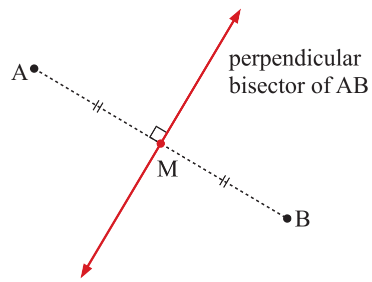
b Find the coordinates of the point where the line cuts the y-axis.

12 Suppose P has coordinates $(3, 8)$ and Q has coordinates $(-5, 2)$. The line perpendicular to PQ and passing through P, cuts the x-axis at R and the y-axis at S. Find the area of triangle ORS, where O is the origin.

PERPENDICULAR BISECTORS

We have already seen that the **midpoint** M of the line segment AB is the point on the line segment that is halfway between A and B.

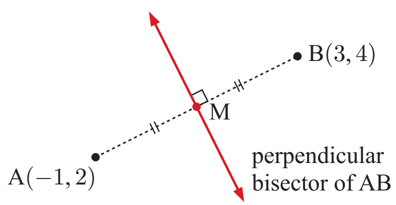
The **perpendicular bisector** of AB is the line which is perpendicular to AB, and which passes through its midpoint M.



Example 7

Self Tutor

Find the equation of the perpendicular bisector of AB given $A(-1, 2)$ and $B(3, 4)$.



The midpoint M of AB is $\left(\dfrac{-1+3}{2}, \dfrac{2+4}{2}\right)$ or $M(1, 3)$.

The gradient of AB is $\dfrac{4-2}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$

$\therefore$ the gradient of the perpendicular bisector is $-\frac{2}{1}$ {the negative reciprocal of $\frac{1}{2}$}

The equation of the perpendicular bisector is $y - 3 = -2(x - 1)$ {using $M(1, 3)$}

$\therefore\ y - 3 = -2x + 2$

$\therefore\ y = -2x + 5$

EXERCISE 7A.2

1 Consider the points P$(-3,\ 7)$ and Q$(1,\ -5)$. Find:

- **a** the distance between P and Q
- **b** the midpoint of PQ
- **c** the gradient of PQ
- **d** the equation of the perpendicular bisector of PQ.

2 Find the equation of the perpendicular bisector of AB given:

- **a** A$(3,\ -3)$ and B$(1,\ -1)$
- **b** A$(1,\ 3)$ and B$(-3,\ 5)$
- **c** A$(3,\ 1)$ and B$(-3,\ 6)$
- **d** A$(4,\ -2)$ and B$(4,\ 4)$.

3 Consider the points P$(-1,\ 5)$ and Q$(3,\ 7)$. The perpendicular bisector of PQ cuts the x-axis at R. Find the area of triangle PQR.

B INTERSECTION OF STRAIGHT LINES

To find where straight lines meet, we need to solve the equations of the lines simultaneously.

Example 8 — Self Tutor

Find where the line:

- **a** $y = 2x - 5$ meets the line $4x + 3y = 15$
- **b** $x + 3y = 5$ meets the line $2x - 5y = -12$.

a Substituting $y = 2x - 5$ into $4x + 3y = 15$ gives

$$\begin{aligned} 4x + 3(2x-5) &= 15 \\ \therefore\ 4x + 6x - 15 &= 15 \\ \therefore\ 10x &= 30 \\ \therefore\ x &= 3 \quad \text{and} \quad y = 2(3) - 5 = 1 \end{aligned}$$

The lines meet at $(3,\ 1)$.

b $x + 3y = 5$, so $x = 5 - 3y$.

Substituting $x = 5 - 3y$ into $2x - 5y = -12$ gives

$$\begin{aligned} 2(5-3y) - 5y &= -12 \\ \therefore\ 10 - 6y - 5y &= -12 \\ \therefore\ -11y &= -22 \\ \therefore\ y &= 2 \quad \text{and} \quad x = 5 - 3(2) = -1 \end{aligned}$$

The lines meet at $(-1,\ 2)$.

EXERCISE 7B

1 Find the intersection point of each pair of lines:

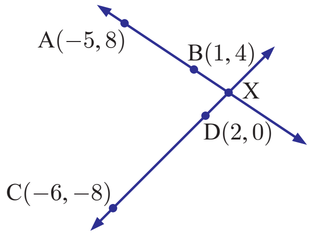
- **a** $y = 4x - 1$ and $2x + y = 5$
- **b** $y = 9 - 2x$ and $4x + 3y = 15$
- **c** $x + 4y = 7$ and $5x - 2y = -31$
- **d** $3x + y = -5$ and $4x - 7y = 10$.

2 Line l_1 has equation $y = 2x + 7$. Line l_2 passes through $(-7, 6)$ and $(3, 0)$.

a Find the equation of l_2.

b Find the intersection point of l_1 and l_2.

3 Find the coordinates of X.

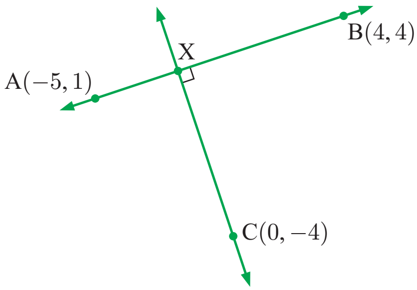


4 In the diagram alongside, a line has been drawn through C, perpendicular to the line AB. The point of intersection of the lines is X. Find:

a the equation of AB

b the equation of CX

c the coordinates of X.

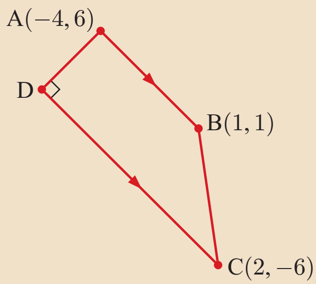


Example 9 — Self Tutor

ABCD is a trapezium in which AB is parallel to DC, and $\widehat{ADC} = 90°$. Find:

a the coordinates of D

b the area of the trapezium.



a Point D is the intersection of AD and DC.

The gradient of AB is $\dfrac{1-6}{1-(-4)} = \dfrac{-5}{5} = -1$

$\therefore$ DC also has gradient -1, and has equation $y - (-6) = -1(x - 2)$

$$\therefore\ y + 6 = -x + 2$$

$$\therefore\ y = -x - 4 \quad .... (1)$$

AD is perpendicular to DC, so its gradient is 1, and its equation is $y - 6 = 1(x - (-4))$

$$\therefore\ y - 6 = x + 4$$

$$\therefore\ y = x + 10 \quad .... (2)$$

Substituting (1) into (2) gives $-x - 4 = x + 10$

$$\therefore\ -2x = 14$$

$$\therefore\ x = -7 \quad \text{and} \quad y = -(-7) - 4 = 3$$

$\therefore$ D is $(-7, 3)$.

b The length of AB $= \sqrt{(1 - -4)^2 + (1 - 6)^2} = \sqrt{50} = 5\sqrt{2}$ units

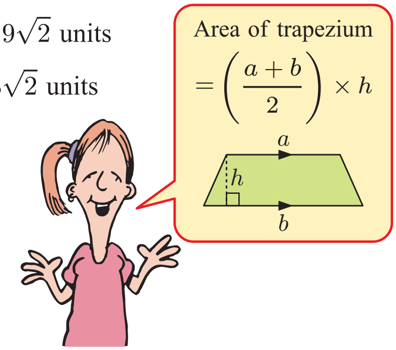
The length of DC $= \sqrt{(2 - -7)^2 + (-6 - 3)^2} = \sqrt{162} = 9\sqrt{2}$ units

The length of AD $= \sqrt{(-7 - -4)^2 + (3 - 6)^2} = \sqrt{18} = 3\sqrt{2}$ units

$\therefore$ the area of the trapezium $= \left(\dfrac{5\sqrt{2} + 9\sqrt{2}}{2}\right) \times 3\sqrt{2}$

$= 7\sqrt{2} \times 3\sqrt{2}$

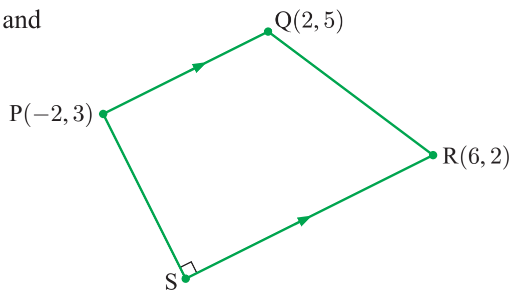
$= 42$ units^2



5 PQRS is a trapezium in which PQ is parallel to SR, and $\widehat{PSR} = 90°$. Find:

a the coordinates of S

b the area of the trapezium.



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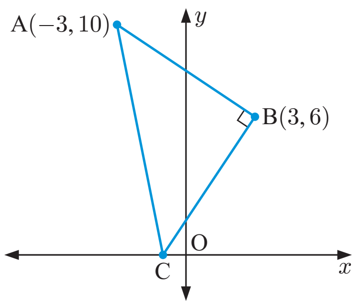


ABC is a triangle in which $\widehat{ABC} = 90°$, and C lies on the x-axis. Find:

a the coordinates of C

b the area of the triangle.

7 A trapezium ABCD has vertices A(3, 0), B(−2, −5), C(−4, 1), and D. The side AD is parallel to BC, and the side CD is perpendicular to BC. Find the area of the trapezium.

8 ABC is a triangle in which $\widehat{ABC} = 90°$, AC is parallel to the x-axis, M is the midpoint of AB, and N is the midpoint of BC.

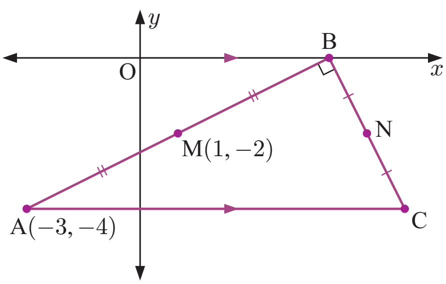
a Find the coordinates of:

i B **ii** C **iii** N

b Show that MN is parallel to AC.

c Find the area of:

i trapezium AMNC **ii** triangle ABC.



C INTERSECTION OF A STRAIGHT LINE AND A CURVE

To find where a straight line intersects a curve, we first rearrange the equation of the line so that x or y is the subject. We then substitute this expression for x or y into the equation of the curve.

While a straight line meets another straight line at most once, a straight line may meet a curve more than once.

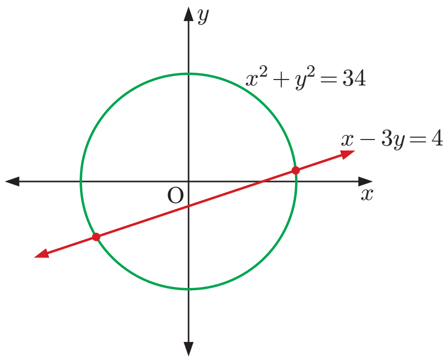
Example 10

Self Tutor

Find the points where the line $x - 3y = 4$ intersects the curve $x^2 + y^2 = 34$.

Substituting $x = 3y + 4$ into $x^2 + y^2 = 34$ gives

$$\begin{aligned}
(3y+4)^2 + y^2 &= 34 \\
\therefore\ 9y^2 + 24y + 16 + y^2 &= 34 \\
\therefore\ 10y^2 + 24y - 18 &= 0 \\
\therefore\ 2(5y^2 + 12y - 9) &= 0 \\
\therefore\ 2(5y-3)(y+3) &= 0 \\
\therefore\ y &= \tfrac{3}{5} \text{ or } -3
\end{aligned}$$



When $y = \frac{3}{5}$, $x = 3(\frac{3}{5}) + 4 = \frac{29}{5}$

When $y = -3$, $x = 3(-3) + 4 = -5$

$\therefore$ the line intersects the curve at $(\frac{29}{5}, \frac{3}{5})$ and $(-5, -3)$.

Example 11

Self Tutor

Find the points where the line $2x + 3y = 5$ intersects the curve $\dfrac{1}{x} - \dfrac{3}{y} = 2$.

If $2x + 3y = 5$, then $y = \dfrac{5-2x}{3}$.

Substituting into $\dfrac{1}{x} - \dfrac{3}{y} = 2$ gives $\dfrac{1}{x} - \dfrac{3}{\frac{5-2x}{3}} = 2$

$$\begin{aligned}
\therefore\ \frac{1}{x} - \frac{9}{5-2x} &= 2 \\
\therefore\ (5-2x) - 9x &= 2x(5-2x) \qquad \{\times \text{ both sides by } x(5-2x)\} \\
\therefore\ 5 - 11x &= 10x - 4x^2 \\
\therefore\ 4x^2 - 21x + 5 &= 0 \\
\therefore\ (4x-1)(x-5) &= 0 \\
\therefore\ x &= \tfrac{1}{4} \text{ or } 5
\end{aligned}$$

When $x = \frac{1}{4}$, $y = \dfrac{5 - 2(\frac{1}{4})}{3} = \frac{3}{2}$, and when $x = 5$, $y = \dfrac{5-2(5)}{3} = -\frac{5}{3}$.

$\therefore$ the line intersects the curve at $(\frac{1}{4}, \frac{3}{2})$ and $(5, -\frac{5}{3})$.

EXERCISE 7C

1 Find the points where the line $x - 2y = 3$ intersects the curve $x^2 + y^2 = 5$.

2 The line $x + y = 7$ meets the curve $x^2 + y^2 = 29$ at A and B. Find the distance between A and B.

3 The line $2x + y = 5$ meets the curve $x^2 + y^2 = 10$ at P and Q. Find the equation of the perpendicular bisector of PQ.

4 Find the points where the line $x - 2y = 4$ intersects the curve $3x^2 + y^2 + xy + 3y = 8$.

5 The line $y = 2x + 1$ meets the curve $x^2 + y^2 + xy + 16x = 29$ at P and Q. Find the distance between P and Q.

6 The line $3x + y = 1$ intersects the curve $2x^2 + y^2 + 5xy - 7x = -31$ at A and B. Find the equation of the perpendicular bisector of AB.

7 Find the points where the line $x - 2y = 6$ intersects the curve $\frac{4}{x} - \frac{1}{y} = 2$.

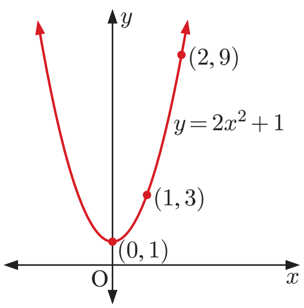
8 The line $3x + 2y = 12$ intersects the curve $\frac{4}{x} + \frac{3}{y} = 3$ at P and Q. Find the midpoint of PQ.

D TRANSFORMING RELATIONSHIPS TO STRAIGHT LINE FORM

Even if x and y are not linearly related, it is sometimes still possible to use a straight line graph to display the relationship. We do this by changing the variables on the axes.

For example, consider the relationship $y = 2x^2 + 1$.

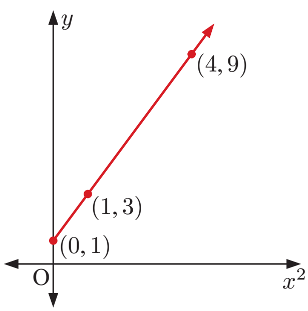
x and y are not linearly related, but x^2 and y *are* linearly related since $y = 2(x^2) + 1$.



We can use a table of values to plot y against x^2:

x	0	1	2
x^2	0	1	4
y	1	3	9

The graph of y against x^2 is a straight line with gradient 2 and y-intercept 1.



Click on the icon to view a demonstration of how the two graphs are related.

DEMO

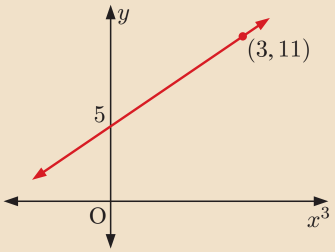
Observe that for the graph of y against x^2, the line terminates at $(0, 1)$, since $x^2 \geqslant 0$ for all x. We need to be careful with the domain and range when we transform relationships.

Example 12

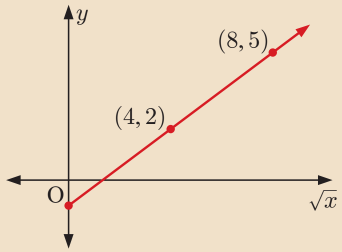
Self Tutor

Find y in terms of x:

a



b



a The graph of y against x^3 is linear.

The gradient is $\dfrac{11-5}{3-0} = 2$, and

the y-intercept is 5.

$\therefore$ the equation is $y = 2x^3 + 5$.

b The graph of y against $\sqrt{x}$ is linear.

The gradient is $\dfrac{5-2}{8-4} = \dfrac{3}{4}$.

$\therefore$ the equation is

$$y - 2 = \tfrac{3}{4}(\sqrt{x} - 4)$$

$$\therefore\ y - 2 = \tfrac{3}{4}\sqrt{x} - 3$$

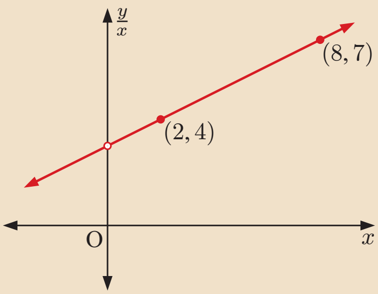
$$\therefore\ y = \tfrac{3}{4}\sqrt{x} - 1,\quad x \geqslant 0$$

Example 13

Self Tutor

a Find y in terms of x.

b Find y when $x = 4$.



a The graph of $\dfrac{y}{x}$ against x is linear.

The gradient is $\dfrac{7-4}{8-2} = \dfrac{1}{2}$.

$\therefore$ the equation is $\dfrac{y}{x} - 4 = \dfrac{1}{2}(x - 2)$

$$\therefore\ \frac{y}{x} - 4 = \frac{1}{2}x - 1$$

$$\therefore\ \frac{y}{x} = \frac{1}{2}x + 3$$

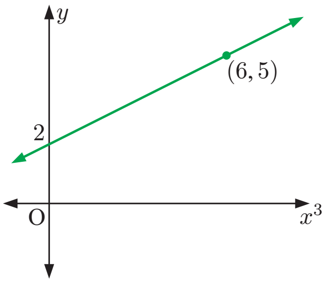
$$\therefore\ y = \tfrac{1}{2}x^2 + 3x$$

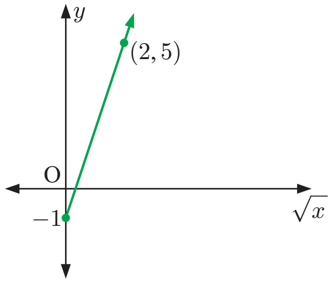
b When $x = 4$, $y = \tfrac{1}{2}(4)^2 + 3(4)$

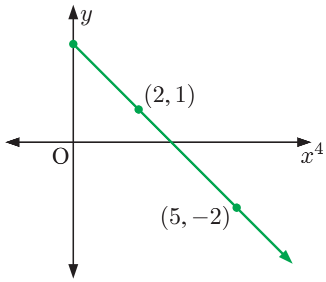
$= 20$

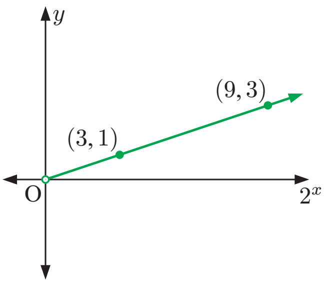
EXERCISE 7D

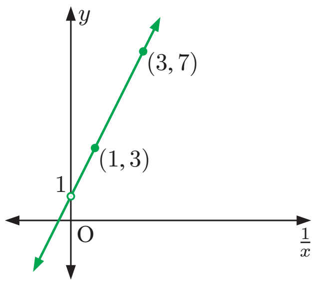
1 Find y in terms of x:

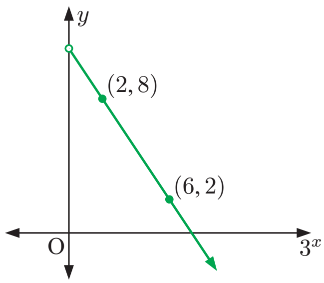
a


b


c


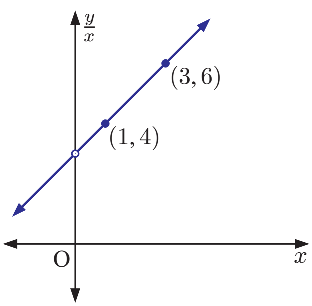
d


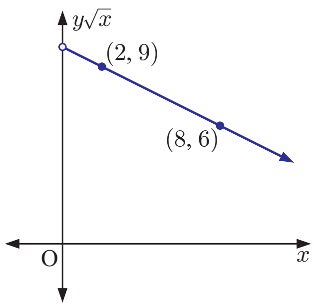
e


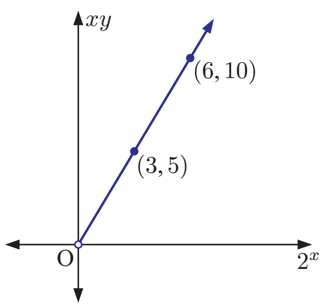
f


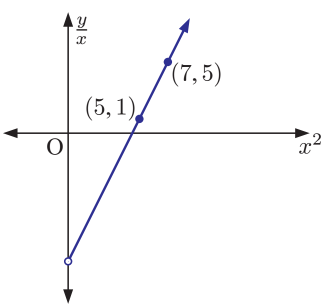
2 For each of the following relations:

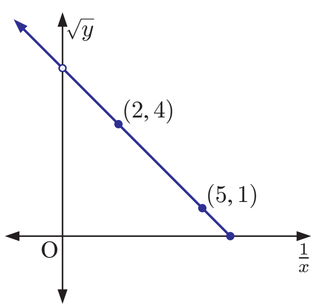
i find y in terms of x **ii** find the value of y when $x = 3$.

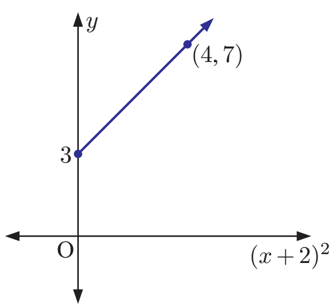
a


b


c


d


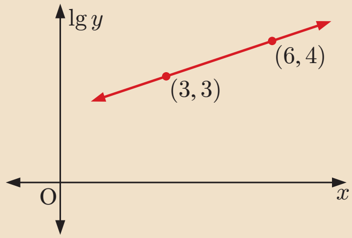
e


f


Example 14

Self Tutor

Write y in terms of x, giving your answer in the form $y = a \times 10^{bx}$, where $a, b \in \mathbb{Q}$.



The graph of $\lg y$ against x is linear.

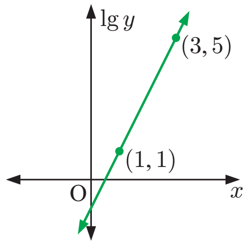
The gradient is $\dfrac{4-3}{6-3} = \frac{1}{3}$.

$\therefore$ the equation is

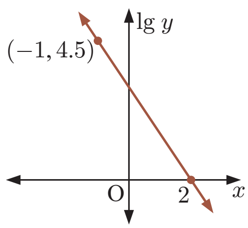
$$\begin{aligned}
\lg y - 3 &= \tfrac{1}{3}(x-3) \\
\lg y - 3 &= \tfrac{1}{3}x - 1 \\
\therefore \lg y &= \tfrac{1}{3}x + 2 \\
\therefore y &= 10^{\frac{1}{3}x+2} \qquad \{\text{if } \lg p = q \text{ then } p = 10^q\} \\
\therefore y &= 10^{\frac{1}{3}x} \times 10^2 \\
\therefore y &= 100 \times 10^{\frac{1}{3}x}
\end{aligned}$$

3 **a** Find $\lg y$ in terms of x.

b Write y in terms of x, giving your answer in the form $y = a \times 10^{bx}$, where $a, b \in \mathbb{Q}$.

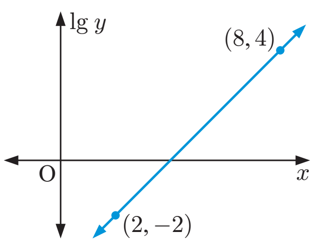


4 Write y in terms of x, giving your answer in the form $y = a \times 10^{bx}$, where $a, b \in \mathbb{Q}$.

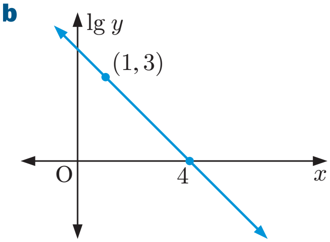


5 Write y in terms of x, giving your answer in the form $y = a \times b^x$, where $a, b \in \mathbb{Q}$.

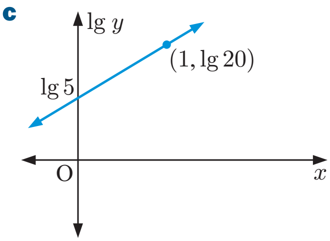
a



b

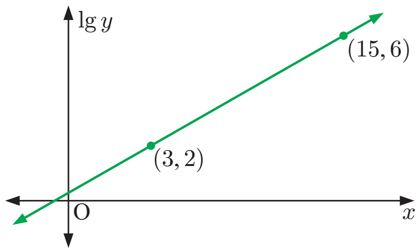


c



6 **a** Write y in terms of x, giving your answer in the form $y = a \times 10^{bx}$, where $a, b \in \mathbb{Q}$.

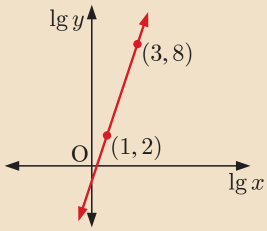
b Find y when $x = 6$.



Example 15

Self Tutor

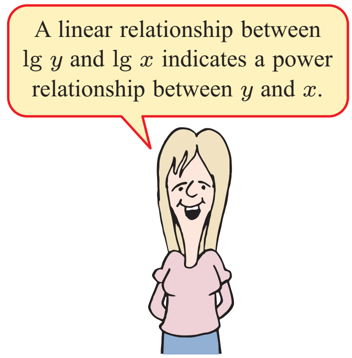
Write y in terms of x, giving your answer in the form $y = a \times x^b$, where $a, b \in \mathbb{Q}$.



The graph of $\lg y$ against $\lg x$ is linear.

The gradient is $\dfrac{8-2}{3-1} = 3$.

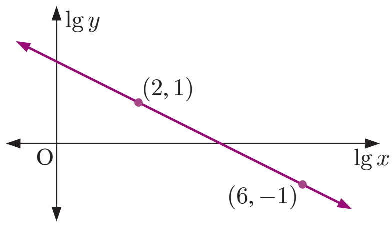
$\therefore$ the equation is
$$\begin{aligned} \lg y - 2 &= 3(\lg x - 1) \\ \therefore \lg y - 2 &= 3\lg x - 3 \\ \therefore \lg y &= 3\lg x - 1 \\ \therefore \lg y &= \lg x^3 - \lg 10 \\ \therefore \lg y &= \lg\left(\frac{x^3}{10}\right) \\ \therefore y &= \tfrac{1}{10} \times x^3 \end{aligned}$$



7 Consider the graph alongside.

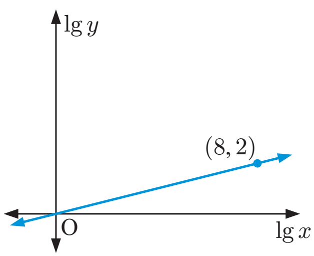
a Write an equation for the line in the form $\lg y = m \lg x + c$.

b Hence write y in terms of x.

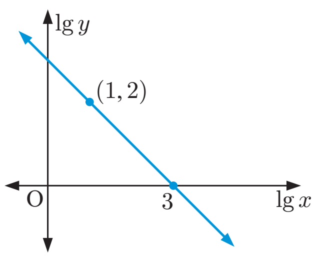


8 Write y in terms of x:

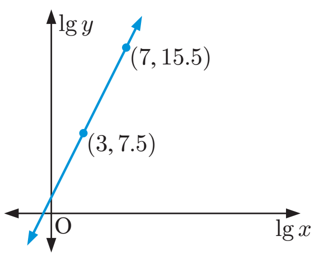
a



b

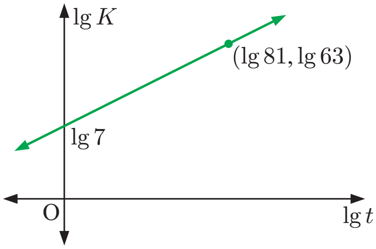


c



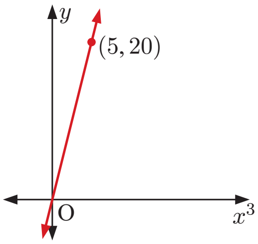
9 **a** Write K in terms of t.

b Hence find K when $t = 9$.

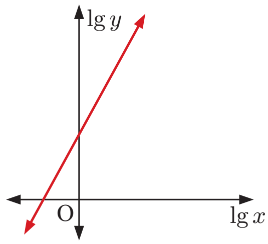


10

Graph A



Graph B

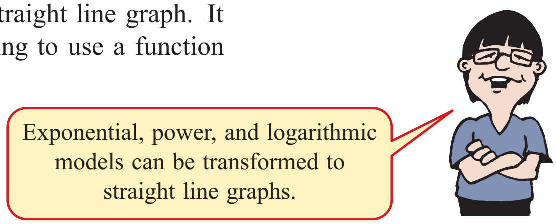


The relationship between x and y in *Graph A* can also be plotted as a straight line in *Graph B*. For the straight line in *Graph B*, find the:

a gradient **b** intercept on the vertical axis.

E FINDING RELATIONSHIPS FROM DATA

We have seen how the transformation of variables may allow us to display a non-linear relationship using a straight line graph. It is particularly useful to do this if we are trying to use a function to model data.



Case study *Exponential growth and decay*

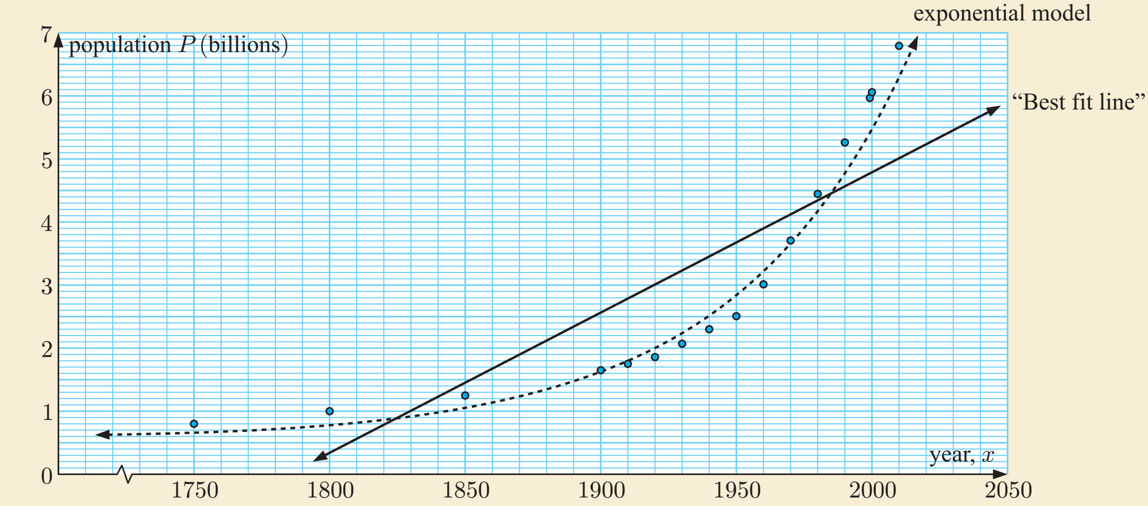
Logarithms are particularly important in science. Many physical processes are modelled accurately by exponential laws.

For example, the United Nations published the following data on world population:

Year	*Population P* (in billions)	lg P
1750	0.79	-0.236
1800	0.98	-0.0202
1850	1.26	0.231
1900	1.65	0.501
1910	1.75	0.560
1920	1.86	0.621
1930	2.07	0.728
1940	2 30	0 833

Year	*Population P* (in billions)	lg P
1950	2.52	0.924
1960	3.02	1.11
1970	3.70	1.31
1980	4.44	1.49
1990	5.27	1.66
1999	5.98	1.79
2000	6.06	1.80
2010	6 79	1 92

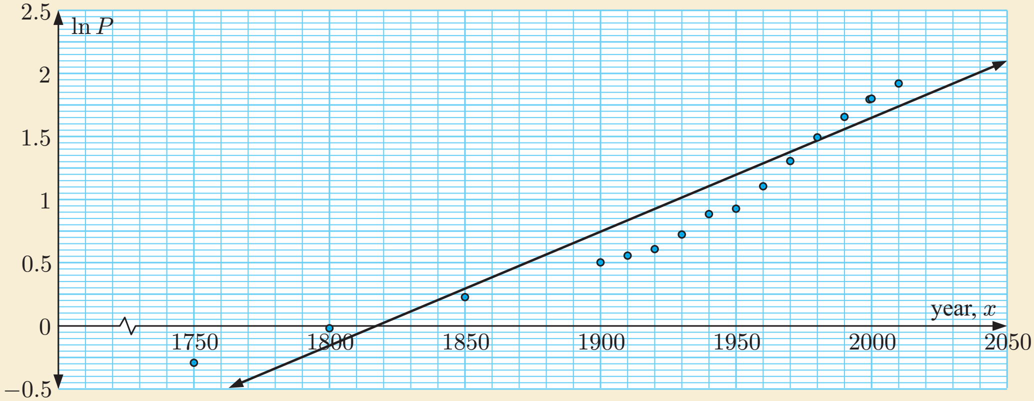
The population data is presented on the graph below:



The "best fit line", $P = 0.0222x - 39.6$, does not fit the data very well. Instead, we try to fit an exponential curve of the form $P = ae^{mx}$.

Taking the natural logarithm of both sides, we have $\ln P = mx + \ln a$, which is the equation of a straight line.

We now plot $\ln P$ against x:



The equation of this "best fit line" is $\ln P = -15.5 + 0.008\,55x$.

Using our model this means that $\ln y = -15.5 + 0.008\,55x$,

$\therefore$ the data can be modelled by $P = e^{-15.5+0.008\,55x}$.

This is shown as a dashed line on the original graph. This is not a perfect fit either, but is a considerable improvement on the original straight line graph.

Example 16

Self Tutor

Consider this table of data connecting x and y:

x	1	2	3	4
y	3.5	10	22.5	44

a Copy and complete the following table:

x^2				
$\frac{y}{x}$				

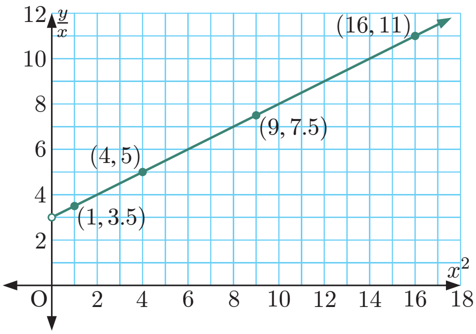
b Plot $\frac{y}{x}$ against x^2, and draw a straight line through the points.

c Find y in terms of x.

a

x^2	1	4	9	16
$\frac{y}{x}$	3.5	5	7.5	11

b



c The graph of $\frac{y}{x}$ against x^2 is linear.

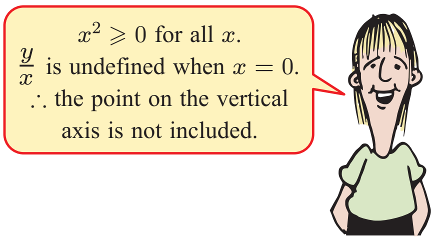
Using the points $(4, 5)$ and $(16, 11)$,

the gradient is $\frac{11-5}{16-4} = \frac{1}{2}$

$\therefore$ the equation is $\frac{y}{x} - 5 = \frac{1}{2}(x^2 - 4)$

$\therefore \frac{y}{x} = \frac{1}{2}x^2 + 3$

$\therefore y = \frac{1}{2}x^3 + 3x$



EXERCISE 7E

1 Consider this table of data connecting x and y:

x	1	2	3	4
y	2	11	26	47

a Copy and complete the following table:

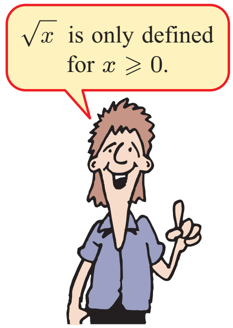
x^2				
y				

b Plot y against x^2, and draw a straight line through the points.

c Find y in terms of x.

2 This table shows experimental data values for x and y:

x	1	2	3	4
y	9	9.90	10.97	12



a Copy and complete the following table:

x				
$y\sqrt{x}$				

b Plot $y\sqrt{x}$ against x, and draw a straight line through the points.

c Find y in terms of x.

d Find y when $x = 16$.

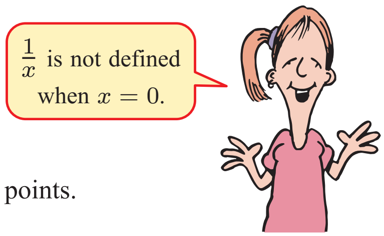
3 This table shows experimental values for x and y.

x	1	2	3	4
y	-1	0	0.11	0.12

It is known that x and y are related by the equation $y = \frac{a}{x} + \frac{b}{x^2}$, where a and b are constants.

a Copy and complete the following table:

$\frac{1}{x}$				
xy				



b Plot xy against $\frac{1}{x}$, and draw a straight line through the points.

c Hence find a and b.

d Find y when $x = 10$.

4 This table shows values of x and y:

x	2	4	6	8
y	5.24	5	5.45	6.12

a Copy and complete the following table:

$x\sqrt{x}$				
$y\sqrt{x}$				

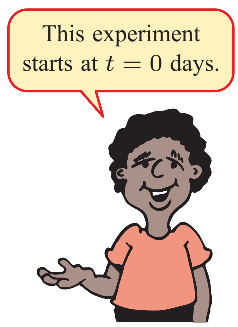
b Plot $y\sqrt{x}$ against $x\sqrt{x}$, and draw a straight line through the points.

c Find y in terms of x.

d Find y when $x = 9$.

5 The mass of bacteria in a culture is measured each day for 5 days.

t (days)	1	2	3	4	5
M (grams)	3.98	6.31	10	15.85	25.12



a Copy and complete the following table:

t					
$\lg M$					

b Plot $\lg M$ against t, and draw a straight line through the points.

c Find M in terms of t.

d Find the original mass of the bacteria.

Example 17

Self Tutor

This table shows experimental data values for x and y:

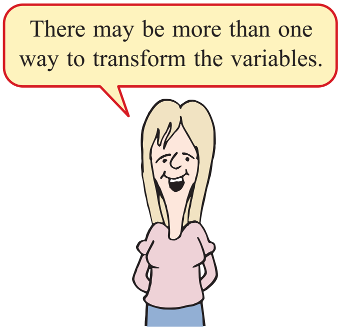
x	1	2	3	4
y	14	10	10	11

By plotting a suitable straight line graph, show that y and x are related by the equation $y = ax + \frac{b}{x}$.

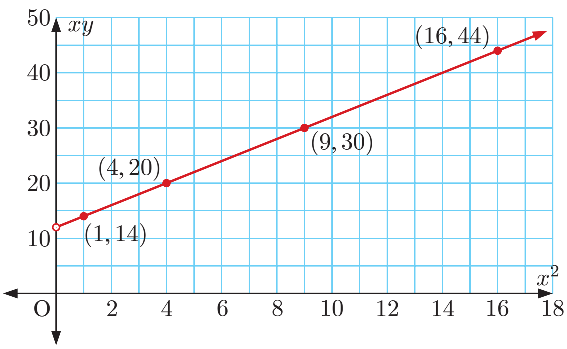
If $y = ax + \frac{b}{x}$, then

$xy = ax^2 + b$

$\therefore$ if y and x are related in this way, then we should observe a linear relationship between xy and x^2.



x^2	1	4	9	16
xy	14	20	30	44



The graph of xy against x^2 is linear.

Using points $(1, 14)$ and $(4, 20)$,

the gradient is $\frac{20 - 14}{4 - 1} = 2$.

$\therefore$ the equation is $xy - 14 = 2(x^2 - 1)$

$\therefore xy - 14 = 2x^2 - 2$

$\therefore xy = 2x^2 + 12$

$\therefore y = 2x + \frac{12}{x}$ $\{a = 2, \; b = 12\}$

6 This table shows experimental values of x and y:

x	1	2	3	4
y	1	26	99	244

It is known that x and y are related by the equation $y = ax^3 + bx$, where a and b are constants.

- **a** A straight line graph is to be drawn to represent this information. If $\frac{y}{x}$ is plotted on the vertical axis, which variable should be plotted on the horizontal axis?
- **b** Draw the straight line graph.
- **c** Find the values of a and b.
- **d** Find y when $x = 5$.

7 This table shows experimental values of x and y:

x	1	2	3	4
y	4	1.17	0.36	0

By plotting a suitable straight line graph, show that x and y are related by the equation $y = \frac{a}{x} + \frac{b}{\sqrt{x}}$.

8 A stone is dropped from the top of an 80 m high cliff. This table shows the distance the stone has fallen at various times.

Time (t s)	1	1.7	2	2.7
Distance (D m)	4.9	14.16	19.6	35.72

a By plotting a suitable straight line graph, show that t and D are related by the equation $D = a \times t^b$, where a and b are constants.

b How far had the stone fallen after 3 seconds?

c How long did the stone take to hit the water?

Research — Logarithmic scales in science

If your data ranges over many orders of magnitude, it can be difficult to compare or represent on a graph.

For example, the Richter scale for earthquake measurement uses logarithms in base 10. An earthquake measuring 6.0 on the Richter scale has a shaking amplitude $10^{6-4} = 100$ times larger than one that measures 4.0.

Research some other scientific scales that use logarithms to compress very large ranges into manageable values. You may like to consider:

- the decibel scale for the loudness of sound
- the stellar magnitude scale for brightness of stars
- the pH scale for acidity and alkalinity
- counting f-stops for ratios of photographic exposure.

Review set 7A

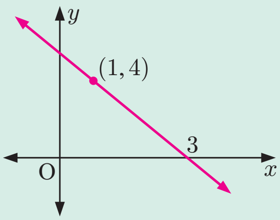
1 Consider the points A$(-1, 6)$ and B$(5, 4)$. Find:

a the distance between A and B

b the midpoint of AB

c the equation of the line through A and B.

2 Determine the equation of the illustrated line:



3 Explain why the vertical straight line in the plane cannot be written in gradient-intercept form $y = mx + c$.

4 Suppose P has coordinates $(-2, -3)$, and Q has coordinates $(1, 3)$. A line perpendicular to PQ, passes through Q.

a Find the equation of the line.

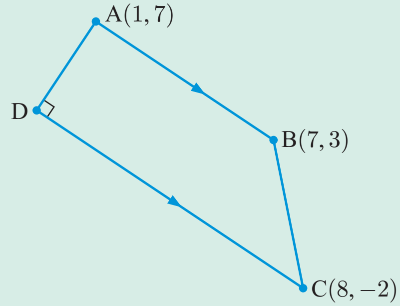
b Find the coordinates of the point where the line cuts the x-axis.

5 Find the point of intersection of the lines $x - 2y = 5$ and $4x + 3y = 9$.

6 ABCD is a trapezium in which AB is parallel to DC, and $\widehat{ADC} = 90°$. Find:

a the coordinates of D

b the area of the trapezium.



7 Find the points where the line $3x + y = 1$ intersects the curve $x^2 + y^2 = 29$.

8 The line $x + y = 5$ meets the curve $x^2 + y^2 + 3xy + 5x = 1$ at P and Q. Find the equation of the perpendicular bisector of PQ.

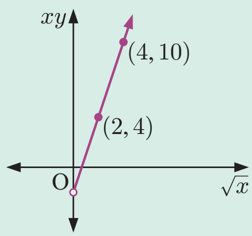
9 Consider two distinct points in the plane (a_1, b_1) and (a_2, b_2) where $a_1 \neq a_2$. Show that the straight line passing through them has equation:

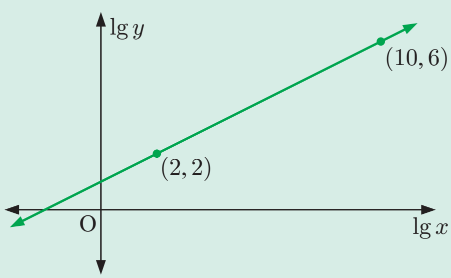
a $y = \dfrac{b_1 - b_2}{a_1 - a_2}x + \dfrac{a_1b_2 - a_2b_1}{a_1 - a_2}$ in gradient-intercept form

b $(b_1 - b_2)x + (a_2 - a_1)y = a_2b_1 - a_1b_2$ in general form.

10 **a** Write y in terms of x.

b Hence find y when $x = 4$.



11 

Consider the graph alongside.

a Write an equation for the line in the form $\lg y = m \lg x + c$.

b Hence write y in terms of x.

12 This table shows experimental values of x and y:

x	1	2	3	4
y	8	7.5	11.33	17.75

a Copy and complete the following table:

x^3				
xy				

b Plot xy against x^3, and draw a straight line through the points.

c Find y in terms of x.

d Hence find y when $x = 7$.

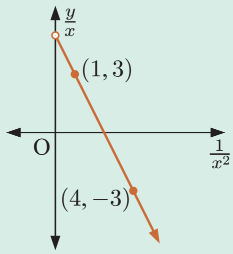
Review set 7B

1 Find the equation of the perpendicular bisector of AB given A$(-2, 3)$ and B$(4, 5)$.

2 The line $x - 2y = 3$ meets the curve $x^2 + 2y^2 - 2xy + 3x = 8$ at P and Q. Find the distance between P and Q.

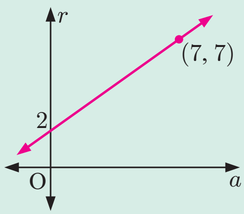
3 **a** Write y in terms of x.

b Hence find y when $x = 8$.

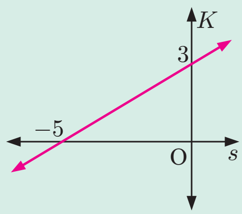


4 Find the equation linking the variables in each graph:

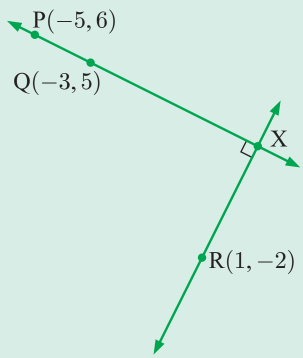
a



b



5 Find the coordinates of X.



6 Find, in general form, the equation of the line passing through $(-5, -7)$ and $(3, -2)$.

7 Consider this table of data connecting x and y:

x	1	2	3	4
y	2.5	5.29	8.13	11

a Copy and complete the following table:

$\sqrt{x}$				
$\frac{y}{\sqrt{x}}$				

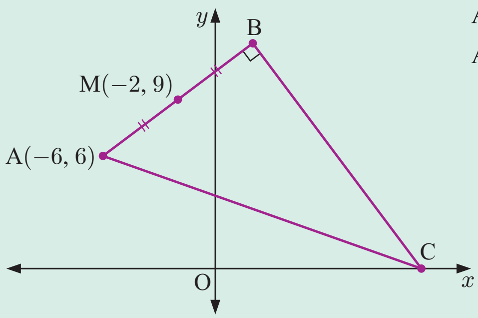
b Plot $\frac{y}{\sqrt{x}}$ against $\sqrt{x}$, and draw a straight line through the points.

c Hence write y in terms of x.

8 Consider the points $(a, 0)$ and $(0, b)$.

a Find the equation of the straight line through these points, in general form.

b Let θ be the angle between the line and the x-axis. Show that the general form of the equation of the line is $(\sin\theta)x + (\cos\theta)y = d$ where $d = \dfrac{ab}{\sqrt{a^2+b^2}}$ is the shortest distance from the line to the origin.

9 

ABC is a triangle in which M is the midpoint of AB, $A\widehat{B}C = 90°$, and C lies on the x-axis.

a Find the coordinates of:

i B **ii** C

b Find the area of the triangle.

10 The line $4x - 3y = 2$ intersects the curve $\dfrac{3}{y} - \dfrac{1}{x} = 1$ at A and B. Find the midpoint of AB.

11 This table shows experimental values of x and y:

x	2	4	6	8
y	21.54	4.64	1	0.21

a By plotting a suitable straight line graph, show that x and y are related by the equation $y = a \times b^x$, where a and b are constants.

b Hence find y when $x = 1$.

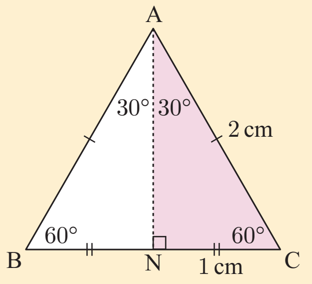
8

The unit circle and radian measure

Contents:

Opening problem

Consider an equilateral triangle with sides 2 cm long. Altitude AN bisects side BC and the vertical angle BAC.



Things to think about:

a Can you use this figure to explain why $\sin 30° = \frac{1}{2}$?

b Use your calculator to find the value of:

i $\sin 150°$ **ii** $\sin 390°$ **iii** $\sin(-330°)$

c Can you explain each result in **b**, even though the angles are not between 0° and 90°?

A RADIAN MEASURE

DEGREE MEASUREMENT OF ANGLES

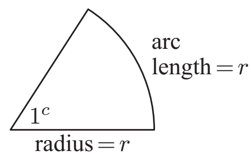
We have seen previously that one full revolution makes an angle of 360°, and the angle on a straight line is 180°.

One **degree**, 1°, is $\frac{1}{360}$th of one full revolution.

This measure of angle is commonly used by surveyors and architects.

RADIAN MEASUREMENT OF ANGLES

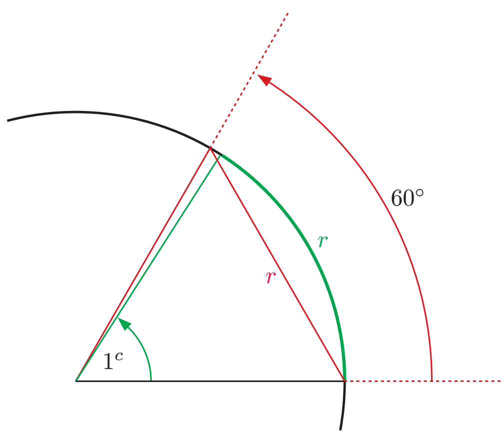
An angle is said to have a measure of one **radian**, 1^c, if it is subtended at the centre of a circle by an arc equal in length to the radius.



The symbol 'c' is used for radian measure but is usually omitted. By contrast, the degree symbol is *always* used when the measure of an angle is given in degrees.

From the diagram below, it can be seen that 1^c is slightly smaller than 60°. In fact, $1^c \approx 57.3°$.

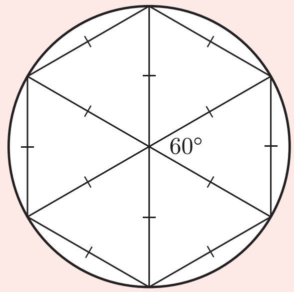
The word 'radian' is an abbreviation of 'radial angle'.



Historical note

There are several theories for why one complete turn was divided into 360 degrees:

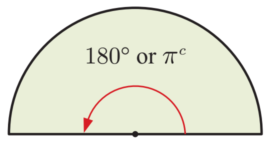
- 360 is approximately the number of days in a year.
- The Babylonians used a counting system in base 60. If they drew 6 equilateral triangles within a circle as shown, and divided each angle into 60 subdivisions, then there were 360 subdivisions in one turn. The division of an hour into 60 minutes, and a minute into 60 seconds, is from this base 60 counting system.
- 360 has 24 divisors, including every integer from 1 to 10 except 7.



The idea of measuring an angle by the length of an arc dates to around 1400 and the Persian mathematician **Al-Kashi**. The concept of a radian is generally credited to **Roger Cotes**, however, who described it as we know it today.

DEGREE-RADIAN CONVERSIONS

If the radius of a circle is r, then an arc of length πr, or half the circumference, will subtend an angle of π radians.

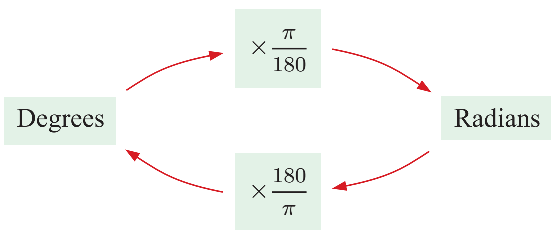


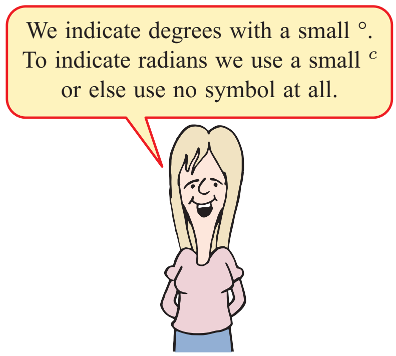
Therefore, $\boxed{\pi \text{ radians} = 180°.}$

So, $1^c = \left(\frac{180}{\pi}\right)^\circ \approx 57.3°$ and $1° = \left(\frac{\pi}{180}\right)^c \approx 0.0175^c$.

To convert from degrees to radians, we multiply by $\frac{\pi}{180}$.

To convert from radians to degrees, we multiply by $\frac{180}{\pi}$.





Example 1 — Self Tutor

Convert 45° to radians in terms of π.

$$45° = \left(45 \times \tfrac{\pi}{180}\right) \text{ radians}$$
$$= \tfrac{\pi}{4} \text{ radians}$$

or

$$180° = \pi \text{ radians}$$
$$\therefore \left(\tfrac{180}{4}\right)^\circ = \tfrac{\pi}{4} \text{ radians}$$
$$\therefore 45° = \tfrac{\pi}{4} \text{ radians}$$

Example 2 — Self Tutor

Convert 126.5° to radians.

$$\begin{aligned} & 126.5^\circ \\ = & \left(126.5 \times \tfrac{\pi}{180}\right) \text{ radians} \\ \approx & \ 2.21 \text{ radians} \end{aligned}$$

Angles in radians may be expressed either in terms of π or as decimals.

EXERCISE 8A

1 Convert to radians, in terms of π:

a 90° **b** 60° **c** 30° **d** 18° **e** 9°
f 135° **g** 225° **h** 270° **i** 360° **j** 720°
k 315° **l** 540° **m** 36° **n** 80° **o** 230°

2 Convert to radians, correct to 3 significant figures:

a 36.7° **b** 137.2° **c** 317.9° **d** 219.6° **e** 396.7°

Example 3 — Self Tutor

Convert the following radian measures to degrees:

a $\frac{5\pi}{6}$ **b** 0.638

a
$$\begin{aligned} & \tfrac{5\pi}{6} \\ = & \left(\tfrac{5\pi}{6} \times \tfrac{180}{\pi}\right)^\circ \\ = & \ 150^\circ \end{aligned}$$

b
$$\begin{aligned} & 0.638 \text{ radians} \\ = & \left(0.638 \times \tfrac{180}{\pi}\right)^\circ \\ \approx & \ 36.6^\circ \end{aligned}$$

3 Convert the following radian measures to degrees:

a $\frac{\pi}{5}$ **b** $\frac{3\pi}{5}$ **c** $\frac{3\pi}{4}$ **d** $\frac{\pi}{18}$ **e** $\frac{\pi}{9}$
f $\frac{7\pi}{9}$ **g** $\frac{\pi}{10}$ **h** $\frac{3\pi}{20}$ **i** $\frac{7\pi}{6}$ **j** $\frac{\pi}{8}$

4 Convert the following radian measures to degrees. Give your answers correct to 2 decimal places.

a 2 **b** 1.53 **c** 0.867 **d** 3.179 **e** 5.267

5 Copy and complete, giving your answers in terms of π:

a

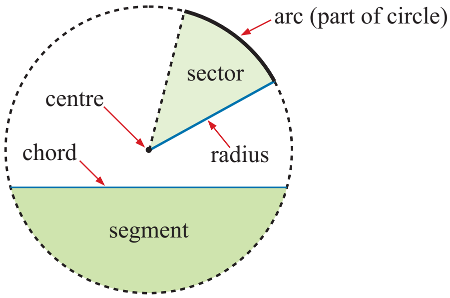
Degrees	0	45	90	135	180	225	270	315	360
Radians									

b

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians													

B ARC LENGTH AND SECTOR AREA

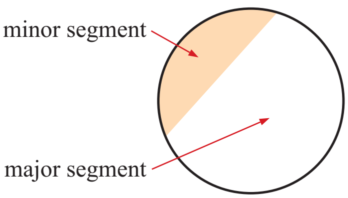
The diagram alongside illustrates terms relating to the parts of a circle.

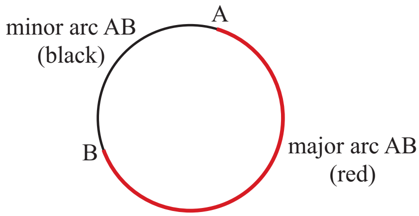


An arc, sector, or segment is described as:

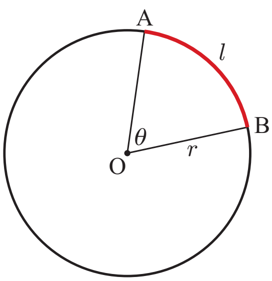
- **minor** if it involves less than half the circle
- **major** if it involves more than half the circle.

For example:





ARC LENGTH



In the diagram, the **arc length** AB is l.

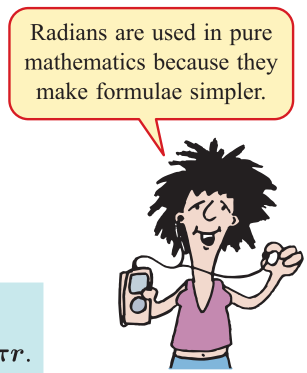
Angle θ is measured in **radians**.

We use a ratio to obtain:

$$\frac{\text{arc length}}{\text{circumference}} = \frac{\theta}{2\pi}$$

$$\therefore \quad \frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

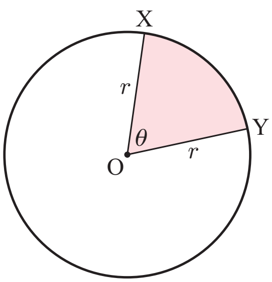
$$\therefore \quad l = \theta r$$



For θ in **radians**, arc length $\boldsymbol{l = \theta r}$.

For θ in **degrees**, arc length $\boldsymbol{l = \frac{\theta}{360} \times 2\pi r}$.

AREA OF SECTOR



In the diagram, the area of minor sector XOY is shaded.

θ is measured in **radians**.

We use a ratio to obtain:

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi}$$

$$\therefore \quad \frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\therefore \quad A = \tfrac{1}{2}\theta r^2$$

For θ in **radians**, area of sector $\boldsymbol{A = \frac{1}{2}\theta r^2}$.

For θ in **degrees**, area of sector $\boldsymbol{A = \frac{\theta}{360} \times \pi r^2}$.

Example 4 — Self Tutor

A sector has radius 12 cm and angle 3 radians. Find its:

a arc length **b** area

a arc length $= \theta r$
$= 3 \times 12$
$= 36$ cm

b area $= \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 3 \times 12^2$
$= 216$ cm^2

EXERCISE 8B

1 Use radians to find the arc length and area of a sector of a circle of:

a radius 9 cm and angle $\frac{7\pi}{4}$ **b** radius 4.93 cm and angle 4.67 radians.

2 A sector has an angle of $107.9°$ and an arc length of 5.92 m. Find its:

a radius **b** area.

3 A sector has an angle of 1.19 radians and an area of 20.8 cm^2. Find its:

a radius **b** perimeter.

Example 5 — Self Tutor

Find the area of a sector with radius 8.2 cm and arc length 13.3 cm.

For θ in radians, $l = \theta r$

$$\therefore \theta = \frac{l}{r} = \frac{13.3}{8.2}$$

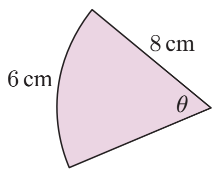
$$\therefore \text{area} = \frac{1}{2}\theta r^2 = \frac{1}{2} \times \frac{13.3}{8.2} \times 8.2^2 \approx 54.5 \text{ cm}^2$$

4 Find, in radians, the angle of a sector of:

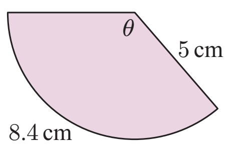
a radius 4.3 m and arc length 2.95 m **b** radius 10 cm and area 30 cm^2.

5 Find θ (in radians) for each of the following, and hence find the area of each figure:

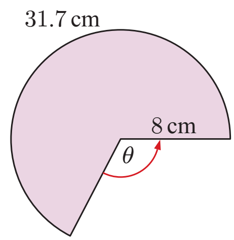
a



b



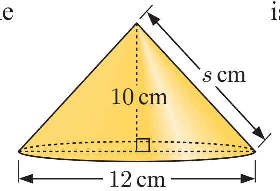
c



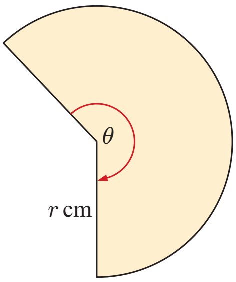
6 Find the arc length and area of a sector of radius 5 cm and angle 2 radians.

7 If a sector has radius $2x$ cm and arc length x cm, show that its area is x^2 cm^2.

8 The cone



is made from this sector:



Find, correct to 3 significant figures:

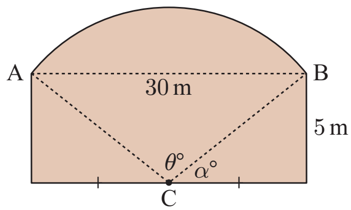
a the slant length s cm

b the value of r

c the arc length of the sector

d the sector angle θ in radians.

9



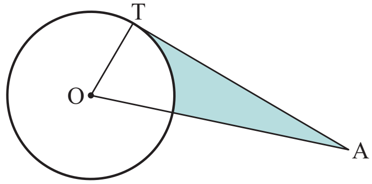
The end wall of a building has the shape illustrated, where the centre of arc AB is C. Find:

a α to 4 significant figures

b θ to 4 significant figures

c the area of the wall.

10

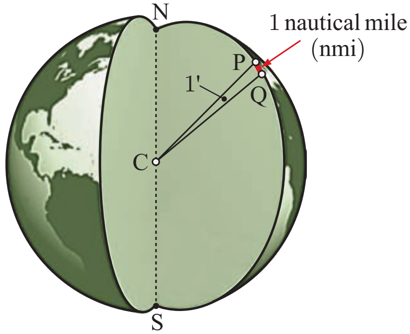


[AT] is a tangent to the given circle. $\text{OA} = 13$ cm and the circle has radius 5 cm. Find the perimeter of the shaded region.

11 A **nautical mile** (nmi) is the distance on the Earth's surface that subtends an angle of 1 minute (or $\frac{1}{60}$ of a degree) of the Great Circle arc measured from the centre of the Earth.
A **knot** is a speed of 1 nautical mile per hour.

a Given that the radius of the Earth is 6370 km, show that 1 nmi $\approx$ 1.853 km.

b Calculate how long it would take a plane to fly from London to Moscow (a distance of 2508 km) if the plane can fly at 480 knots.



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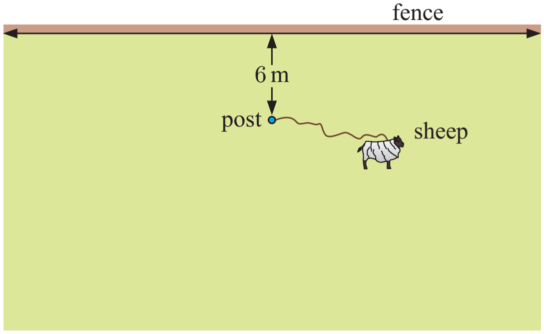
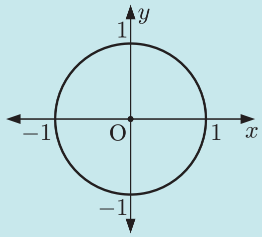


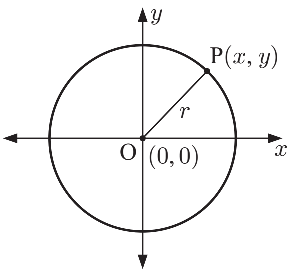
A sheep is tethered to a post which is 6 m from a long fence. The length of rope is 9 m. Find the area which the sheep can feed on.

C THE UNIT CIRCLE AND THE TRIGONOMETRIC RATIOS

The **unit circle** is the circle with centre $(0,\ 0)$ and radius 1 unit.



CIRCLES WITH CENTRE $(0,\ 0)$



Consider a circle with centre $(0,\ 0)$ and radius r units.
Suppose $\mathrm{P}(x,\ y)$ is any point on this circle.

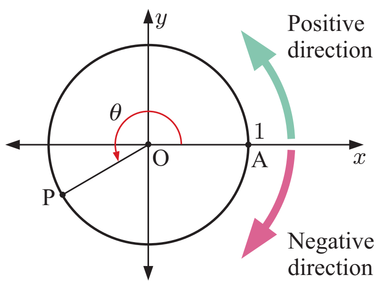
$$\begin{aligned} \text{Since}\quad \mathrm{OP} &= r, \\ \sqrt{(x-0)^2+(y-0)^2} &= r \qquad \{\text{distance formula}\} \\ \therefore\quad x^2+y^2 &= r^2 \end{aligned}$$

$\boldsymbol{x^2+y^2=r^2}$ is the equation of a circle with centre $(0,\ 0)$ and radius r.

The equation of the **unit circle** is $x^2+y^2=1$.

ANGLE MEASUREMENT

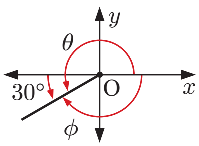
Suppose P lies anywhere on the unit circle, and A is $(1,\ 0)$.
Let θ be the angle measured from [OA] on the positive x-axis.



θ is **positive** for anticlockwise rotations and **negative** for clockwise rotations.

For example: $\theta = 210° = \frac{7\pi}{6}$

$\phi = -150° = -\frac{5\pi}{6}$



DEFINITION OF SINE AND COSINE

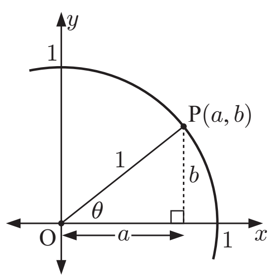
Consider a point $\mathrm{P}(a,\ b)$ which lies on the unit circle in the first quadrant. [OP] makes an angle θ with the x-axis as shown.

Using right angled triangle trigonometry:

$$\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{a}{1} = a$$

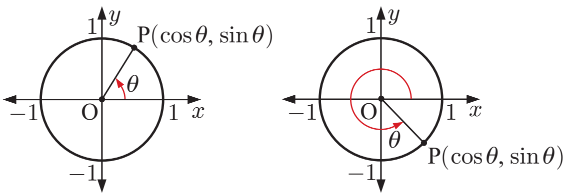
$$\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{b}{1} = b$$

$$\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{b}{a} = \frac{\sin\theta}{\cos\theta}$$



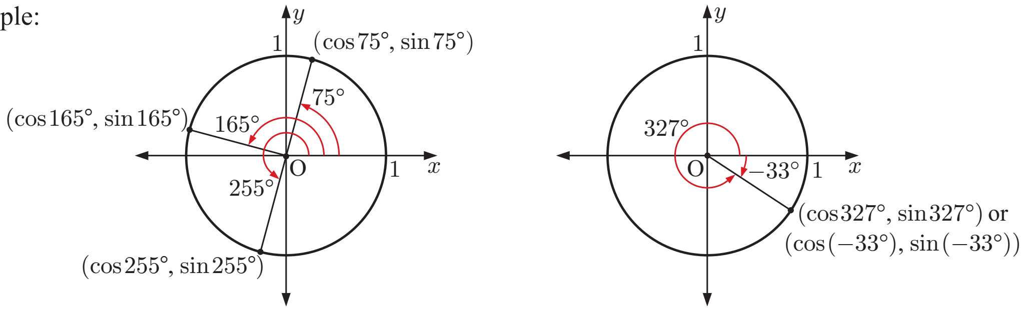
In general, for a point P anywhere on the unit circle:

- **cos θ** is the x-coordinate of P
- **sin θ** is the y-coordinate of P



We can hence find the coordinates of any point on the unit circle with given angle θ measured from the positive x-axis.

For example:



Since the unit circle has equation $x^2 + y^2 = 1$, $(\cos\theta)^2 + (\sin\theta)^2 = 1$ for all θ.

We commonly write this as $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$.

For all points on the unit circle, $-1 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant 1$.

So, $\boldsymbol{-1 \leqslant \cos\theta \leqslant 1}$ and $\boldsymbol{-1 \leqslant \sin\theta \leqslant 1}$ for all θ.

DEFINITION OF TANGENT

Suppose we extend [OP] to meet the tangent from A(1, 0).

We let the intersection between these lines be point Q.

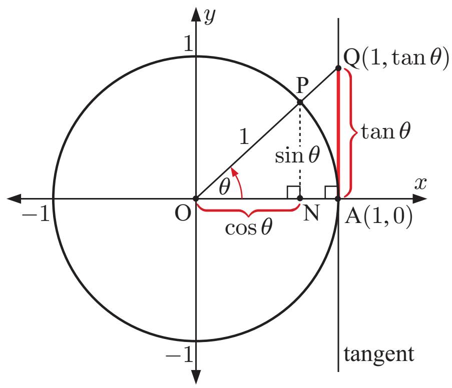
Note that as P moves, so does Q.

The position of Q relative to A is defined as the **tangent function**.

Notice that △s ONP and OAQ are equiangular and therefore similar.

Consequently $\dfrac{AQ}{OA} = \dfrac{NP}{ON}$ and hence $\dfrac{AQ}{1} = \dfrac{\sin\theta}{\cos\theta}$.

Under the definition that $AQ = \tan\theta$, $\boldsymbol{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$.

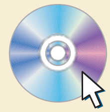


Discovery 1 — *The trigonometric ratios*

In this Discovery we explore the signs of the trigonometric ratios in each quadrant of the unit circle.

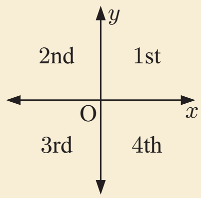
What to do:

THE UNIT CIRCLE

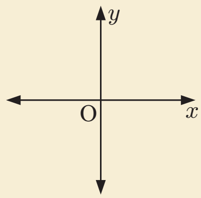


1 Click on the icon to run the Unit Circle software.
Drag the point P slowly around the circle.
Note the *sign* of each trigonometric ratio in each quadrant.

Quadrant	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	positive		
2			
3			
4			

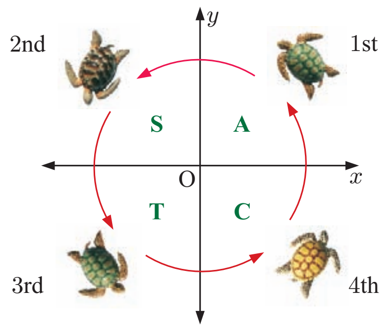


2 Hence note down the trigonometric ratios which are *positive* in each quadrant.



From the **Discovery** you should have found that:

- $\sin\theta$, $\cos\theta$, and $\tan\theta$ are positive in quadrant 1
- only $\sin\theta$ is positive in quadrant 2
- only $\tan\theta$ is positive in quadrant 3
- only $\cos\theta$ is positive in quadrant 4.



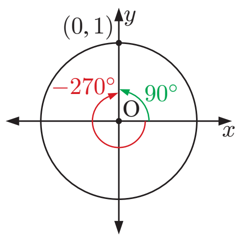
We can use a letter to show which trigonometric ratios are positive in each quadrant. The A stands for *all* of the ratios.

You might like to remember them using

All **S**illy **T**urtles **C**rawl.

Example 6 — Self Tutor

Use a unit circle diagram to find the values of $\cos(-270°)$ and $\sin(-270°)$.



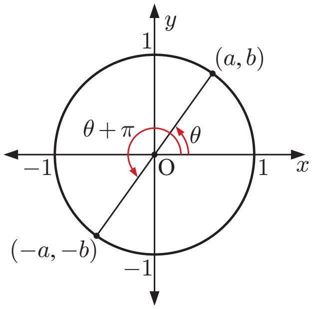
$\cos(-270°) = 0$ {the x-coordinate}
$\sin(-270°) = 1$ {the y-coordinate}

PERIODICITY OF TRIGONOMETRIC RATIOS

Since there are 2π radians in a full revolution, if we add any integer multiple of 2π to θ (in radians) then the position of P on the unit circle is unchanged.

> For θ in radians and $k \in \mathbb{Z}$,
> $\boldsymbol{\cos(\theta + 2k\pi) = \cos\theta}$ and $\boldsymbol{\sin(\theta + 2k\pi) = \sin\theta}$.

We notice that for any point $(\cos\theta, \sin\theta)$ on the unit circle, the point directly opposite is $(-\cos\theta, -\sin\theta)$



$$\therefore \cos(\theta + \pi) = -\cos\theta$$
$$\sin(\theta + \pi) = -\sin\theta$$
$$\text{and} \quad \tan(\theta + \pi) = \frac{-\sin\theta}{-\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

> For θ in radians and $k \in \mathbb{Z}$, $\boldsymbol{\tan(\theta + k\pi) = \tan\theta}$.

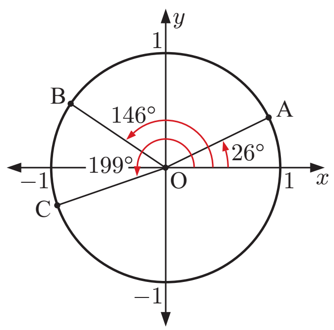
This **periodic** feature is an important property of the trigonometric functions.

EXERCISE 8C

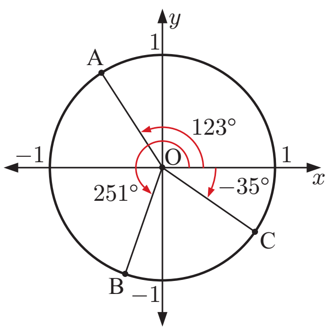
1 For each unit circle illustrated:

- **i** state the exact coordinates of points A, B, and C in terms of sine and cosine
- **ii** use your calculator to give the coordinates of A, B, and C correct to 3 significant figures.

a



b



2 With the aid of a unit circle, complete the following table:

θ (degrees)	0°	90°	180°	270°	360°	450°
θ (radians)						
sine						
cosine						
tangent						

3 **a** Use your calculator to evaluate: **i** $\frac{1}{\sqrt{2}}$ **ii** $\frac{\sqrt{3}}{2}$

b Copy and complete the following table. If necessary, use your calculator to evaluate the trigonometric ratios, then **a** to write them exactly.

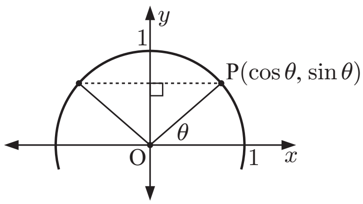
θ (degrees)	$30°$	$45°$	$60°$	$135°$	$150°$	$240°$	$315°$
θ (radians)							
sine							
cosine							
tangent							

4 **a** Use your calculator to evaluate:

i $\sin 100°$ **ii** $\sin 80°$ **iii** $\sin 120°$ **iv** $\sin 60°$
v $\sin 150°$ **vi** $\sin 30°$ **vii** $\sin 45°$ **viii** $\sin 135°$

b Use the results from **a** to copy and complete:
$\sin(180° - \theta) = ...$

c Justify your answer using the diagram alongside:



d Find the obtuse angle with the same sine as:

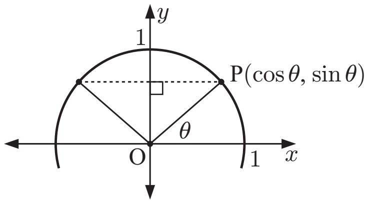
i $45°$ **ii** $51°$ **iii** $\frac{\pi}{3}$ **iv** $\frac{\pi}{6}$

5 **a** Use your calculator to evaluate:

i $\cos 70°$ **ii** $\cos 110°$ **iii** $\cos 60°$ **iv** $\cos 120°$
v $\cos 25°$ **vi** $\cos 155°$ **vii** $\cos 80°$ **viii** $\cos 100°$

b Use the results from **a** to copy and complete:
$\cos(180° - \theta) = ...$

c Justify your answer using the diagram alongside:



d Find the obtuse angle which has the negative cosine of:

i $40°$ **ii** $19°$ **iii** $\frac{\pi}{5}$ **iv** $\frac{2\pi}{5}$

6 Without using your calculator, find:

a $\sin 137°$ if $\sin 43° \approx 0.6820$
b $\sin 59°$ if $\sin 121° \approx 0.8572$
c $\cos 143°$ if $\cos 37° \approx 0.7986$
d $\cos 24°$ if $\cos 156° \approx -0.9135$
e $\sin 115°$ if $\sin 65° \approx 0.9063$
f $\cos 132°$ if $\cos 48° \approx 0.6691$

7 **a** Copy and complete:

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0° < \theta < 90°$	$0 < \theta < \frac{\pi}{2}$	positive	positive	
2					
3					
4					

b In which quadrants are the following true?

i $\cos\theta$ is positive.

ii $\cos\theta$ is negative.

iii $\cos\theta$ and $\sin\theta$ are both negative.

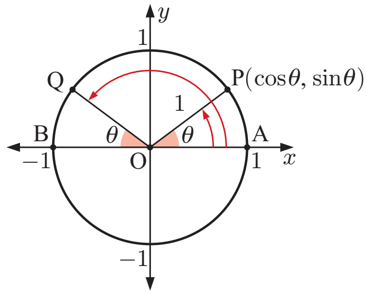
iv $\cos\theta$ is negative and $\sin\theta$ is positive.

8 **a** If $\mathrm{A\widehat{O}P} = \mathrm{B\widehat{O}Q} = \theta$, what is the measure of $\mathrm{A\widehat{O}Q}$?

b Copy and complete:

[OQ] is a reflection of [OP] in the ......
and so Q has coordinates ......

c What trigonometric formulae can be deduced from **a** and **b**?

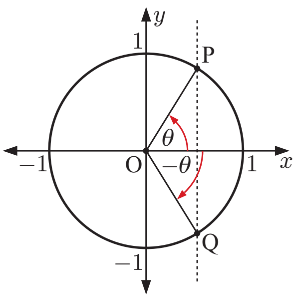


9 **a** Copy and complete:

θ^c	$\sin\theta$	$\sin(-\theta)$	$\cos\theta$	$\cos(-\theta)$
0.75				
1.772				
3.414				
6.25				
-1.17				

b What trigonometric formulae can be deduced from your results in **a**?

c



The coordinates of P in the figure are $(\cos\theta, \sin\theta)$.

i By finding the coordinates of Q in terms of θ in *two different* ways, prove your formulae in **b**.

ii Hence explain why $\cos(2\pi - \theta) = \cos\theta$.

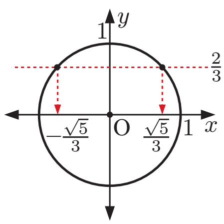
D APPLICATIONS OF THE UNIT CIRCLE

The identity $\cos^2\theta + \sin^2\theta = 1$ is essential for finding trigonometric ratios.

Example 7 Self Tutor

Find the possible values of $\cos\theta$ for $\sin\theta = \frac{2}{3}$. Illustrate your answers.

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore \cos^2\theta + \left(\tfrac{2}{3}\right)^2 = 1$$
$$\therefore \cos^2\theta = \tfrac{5}{9}$$
$$\therefore \cos\theta = \pm\tfrac{\sqrt{5}}{3}$$



EXERCISE 8D.1

1 Find the possible values of $\cos\theta$ for:

a $\sin\theta = \frac{1}{2}$ **b** $\sin\theta = -\frac{1}{3}$ **c** $\sin\theta = 0$ **d** $\sin\theta = -1$

2 Find the possible values of $\sin\theta$ for:

a $\cos\theta = \frac{4}{5}$ **b** $\cos\theta = -\frac{3}{4}$ **c** $\cos\theta = 1$ **d** $\cos\theta = 0$

Example 8 Self Tutor

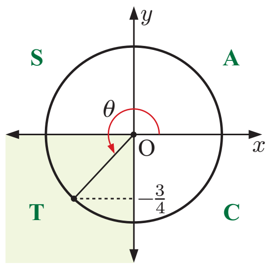
If $\sin\theta = -\frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\cos\theta$ and $\tan\theta$. Give exact answers.

Now $\cos^2\theta + \sin^2\theta = 1$

$\therefore\ \cos^2\theta + \frac{9}{16} = 1$

$\therefore\ \cos^2\theta = \frac{7}{16}$

$\therefore\ \cos\theta = \pm\frac{\sqrt{7}}{4}$



But $\pi < \theta < \frac{3\pi}{2}$, so θ is a quadrant 3 angle.

$\therefore$ $\cos\theta$ is negative.

$\therefore\ \cos\theta = -\frac{\sqrt{7}}{4}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{3}{4}}{-\frac{\sqrt{7}}{4}} = \frac{3}{\sqrt{7}}$

3 Find the exact value of:

a $\sin\theta$ if $\cos\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$

b $\cos\theta$ if $\sin\theta = \frac{2}{5}$ and $\frac{\pi}{2} < \theta < \pi$

c $\cos\theta$ if $\sin\theta = -\frac{3}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$

d $\sin\theta$ if $\cos\theta = -\frac{5}{13}$ and $\pi < \theta < \frac{3\pi}{2}$.

4 Find the exact value of $\tan\theta$ given that:

a $\sin\theta = \frac{1}{3}$ and $\frac{\pi}{2} < \theta < \pi$

b $\cos\theta = \frac{1}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$

c $\sin\theta = -\frac{1}{\sqrt{3}}$ and $\pi < \theta < \frac{3\pi}{2}$

d $\cos\theta = -\frac{3}{4}$ and $\frac{\pi}{2} < \theta < \pi$.

Example 9 Self Tutor

If $\tan\theta = -2$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\sin\theta$ and $\cos\theta$. Give exact answers.

$\tan\theta = \frac{\sin\theta}{\cos\theta} = -2$

$\therefore\ \sin\theta = -2\cos\theta$

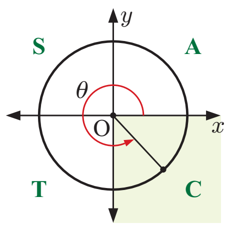
Now $\sin^2\theta + \cos^2\theta = 1$

$\therefore\ (-2\cos\theta)^2 + \cos^2\theta = 1$

$\therefore\ 4\cos^2\theta + \cos^2\theta = 1$

$\therefore\ 5\cos^2\theta = 1$

$\therefore\ \cos\theta = \pm\frac{1}{\sqrt{5}}$



But $\frac{3\pi}{2} < \theta < 2\pi$, so θ is a quadrant 4 angle.

$\therefore$ $\cos\theta$ is positive and $\sin\theta$ is negative.

$\therefore\ \cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = -\frac{2}{\sqrt{5}}$.