An alternative way to solve equations like $x^2+4x+1=0$ is by 'completing the square'.

Equations of the form $ax^2+bx+c=0$ can be converted to the form $(x+p)^2=q$, from which the solutions are easy to obtain.

Example 4 — Self Tutor

Solve exactly for x:

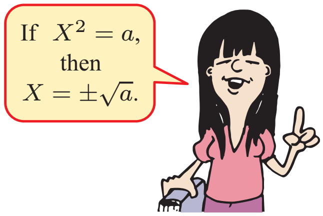
a $(x+2)^2=7$ **b** $(x-1)^2=-5$

a $(x+2)^2=7$

$\therefore\ x+2=\pm\sqrt{7}$

$\therefore\ x=-2\pm\sqrt{7}$

b $(x-1)^2=-5$ has no real solutions since the square $(x-1)^2$ cannot be negative.



The completed square form of an equation is $(x+p)^2=q$.

If we expand this out, $x^2+2px+p^2=q$.

Notice that the *coefficient of* x equals $2p$. Therefore, p is half the coefficient of x in the expanded form.

If we have $x^2+2px=q$, then we "complete the square" by adding in p^2 to both sides of the equation.

Example 5 — Self Tutor

Solve for exact values of x: $x^2+4x+1=0$

$x^2+4x+1=0$

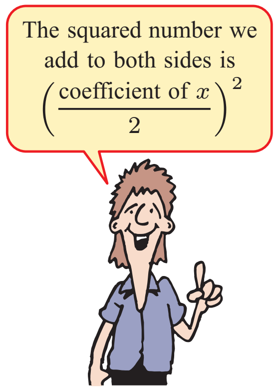
$\therefore\ x^2+4x=-1$ {put the constant on the RHS}

$\therefore\ x^2+4x+2^2=-1+2^2$ {completing the square}

$\therefore\ (x+2)^2=3$ {factorising LHS}

$\therefore\ x+2=\pm\sqrt{3}$

$\therefore\ x=-2\pm\sqrt{3}$



Example 6 — Self Tutor

Solve exactly for x: $-3x^2+12x+5=0$

$-3x^2+12x+5=0$

$\therefore\ x^2-4x-\frac{5}{3}=0$ {dividing both sides by -3}

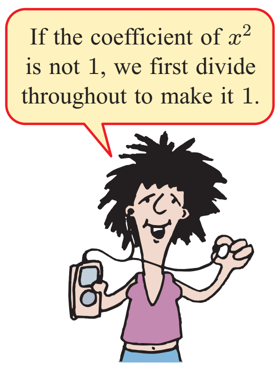
$\therefore\ x^2-4x=\frac{5}{3}$ {putting the constant on the RHS}

$\therefore\ x^2-4x+2^2=\frac{5}{3}+2^2$ {completing the square}

$\therefore\ (x-2)^2=\frac{17}{3}$ {factorising LHS}

$\therefore\ x-2=\pm\sqrt{\frac{17}{3}}$

$\therefore\ x=2\pm\sqrt{\frac{17}{3}}$



EXERCISE 3A.2

1 Solve exactly for x:

a $(x+5)^2 = 2$ **b** $(x+6)^2 = -11$ **c** $(x-4)^2 = 8$

d $(x-8)^2 = 7$ **e** $2(x+3)^2 = 10$ **f** $3(x-2)^2 = 18$

g $(x+1)^2 + 1 = 11$ **h** $(2x+1)^2 = 3$ **i** $(1-3x)^2 - 7 = 0$

2 Solve exactly by completing the square:

a $x^2 - 4x + 1 = 0$ **b** $x^2 + 6x + 2 = 0$ **c** $x^2 - 14x + 46 = 0$

d $x^2 = 4x + 3$ **e** $x^2 + 6x + 7 = 0$ **f** $x^2 = 2x + 6$

g $x^2 + 6x = 2$ **h** $x^2 + 10 = 8x$ **i** $x^2 + 6x = -11$

3 Solve exactly by completing the square:

a $2x^2 + 4x + 1 = 0$ **b** $2x^2 - 10x + 3 = 0$ **c** $3x^2 + 12x + 5 = 0$

d $3x^2 = 6x + 4$ **e** $5x^2 - 15x + 2 = 0$ **f** $4x^2 + 4x = 5$

4 Solve for x:

a $3x - \frac{2}{x} = 4$ **b** $1 - \frac{1}{x} = -5x$ **c** $3 + \frac{1}{x^2} = -\frac{5}{x}$

5 Suppose $ax^2 + bx + c = 0$ where a, b, and c are constants, $a \neq 0$.
Solve for x by completing the square.

THE QUADRATIC FORMULA

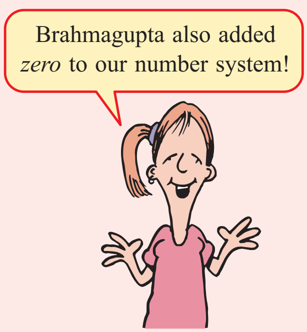
Historical note — The quadratic formula

Thousands of years ago, people knew how to calculate the area of a shape given its side lengths. When they wanted to find the side lengths necessary to give a certain area, however, they ended up with a quadratic equation which they needed to solve.

The first known solution of a quadratic equation is written on the Berlin Papyrus from the Middle Kingdom (2160 - 1700 BC) in Egypt. By 400 BC, the Babylonians were using the method of 'completing the square'.

Pythagoras and **Euclid** both used geometric methods to explore the problem. Pythagoras noted that the square root was not always an integer, but he refused to accept that irrational solutions existed. Euclid also discovered that the square root was not always rational, but concluded that irrational numbers *did* exist.

A major jump forward was made in India around 700 AD, when Hindu mathematician **Brahmagupta** devised a general (but incomplete) solution for the quadratic equation $ax^2 + bx = c$ which was equivalent to $x = \dfrac{\sqrt{4ac+b^2} - b}{2a}$. Taking into account the sign of c, this is one of the two solutions we know today.



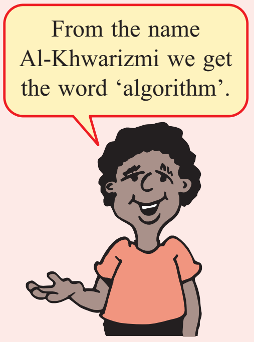
The final, complete solution as we know it today first came around 1100 AD, by another Hindu mathematician called **Baskhara**. He was the first to recognise that any positive number has two square roots, which could be negative or irrational. In fact, the quadratic formula is known in some countries today as 'Baskhara's Formula'.

While the Indians had knowledge of the quadratic formula even at this early stage, it took somewhat longer for the quadratic formula to arrive in Europe.

Muhammad Al-Khwarizmi

Around 820 AD, the Islamic mathematician **Muhammad bin Musa Al-Khwarizmi**, who was familiar with the work of Brahmagupta, recognised that for a quadratic equation to have real solutions, the value $b^2 - 4ac$ could not be negative. Al-Khwarizmi's work was brought to Europe by the Jewish mathematician and astronomer **Abraham bar Hiyya** (also known as Savasorda) who lived in Barcelona around 1100.

By 1545, **Girolamo Cardano** had blended the algebra of Al-Khwarizmi with the Euclidean geometry. His work allowed for the existence of complex or imaginary roots, as well as negative and irrational roots.



At the end of the 16th Century the mathematical notation and symbolism was introduced by **François Viète** in France.

In 1637, when **René Descartes** published *La Géométrie*, the quadratic formula adopted the form we see today.

If $\boldsymbol{ax^2 + bx + c = 0}$, $\boldsymbol{a \neq 0}$, then $\boldsymbol{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$.

Proof:

If $ax^2 + bx + c = 0$, $a \neq 0$

then $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ {dividing each term by a, as $a \neq 0$}

$\therefore \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$

$\therefore \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$ {completing the square on LHS}

$\therefore \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ {factorising}

$\therefore \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

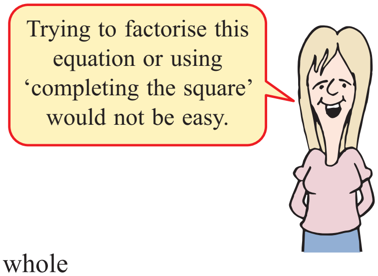
For example, consider the Acme Leather Jacket Co. equation from page **65**.

We need to solve: $12.5x^2 - 550x + 5125 = 0$
so in this case $a = 12.5$, $b = -550$, $c = 5125$

$$\therefore \quad x = \frac{550 \pm \sqrt{(-550)^2 - 4(12.5)(5125)}}{2(12.5)}$$

$$= \frac{550 \pm \sqrt{46\,250}}{25}$$

$$\approx 30.60 \text{ or } 13.40$$



However, for this application the number of jackets x needs to be a whole number, so $x = 13$ or 31 would produce a profit of around \$3000 each week.

Example 7

Self Tutor

Solve for x:

a $x^2 - 2x - 6 = 0$

b $2x^2 + 3x - 6 = 0$

a $x^2 - 2x - 6 = 0$ has
$a = 1$, $b = -2$, $c = -6$

$$\therefore \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-6)}}{2(1)}$$

$$\therefore \quad x = \frac{2 \pm \sqrt{4 + 24}}{2}$$

$$\therefore \quad x = \frac{2 \pm \sqrt{28}}{2}$$

$$\therefore \quad x = \frac{2 \pm 2\sqrt{7}}{2}$$

$$\therefore \quad x = 1 \pm \sqrt{7}$$

b $2x^2 + 3x - 6 = 0$ has
$a = 2$, $b = 3$, $c = -6$

$$\therefore \quad x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-6)}}{2(2)}$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{9 + 48}}{4}$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{57}}{4}$$

EXERCISE 3A.3

1 Use the quadratic formula to solve exactly for x:

a $x^2 - 4x - 3 = 0$
b $x^2 + 6x + 7 = 0$
c $x^2 + 1 = 4x$
d $x^2 + 4x = 1$
e $x^2 - 4x + 2 = 0$
f $2x^2 - 2x - 3 = 0$
g $3x^2 - 5x - 1 = 0$
h $-x^2 + 4x + 6 = 0$
i $-2x^2 + 7x - 2 = 0$

2 Rearrange the following equations so they are written in the form $ax^2 + bx + c = 0$, then use the quadratic formula to solve exactly for x.

a $(x + 2)(x - 1) = 2 - 3x$
b $(2x + 1)^2 = 3 - x$
c $(x - 2)^2 = 1 + x$
d $(3x + 1)^2 = -2x$
e $(x + 3)(2x + 1) = 9$
f $(2x + 3)(2x - 3) = x$
g $\dfrac{x - 1}{2 - x} = 2x + 1$
h $x - \dfrac{1}{x} = 1$
i $2x - \dfrac{1}{x} = 3$

B QUADRATIC INEQUALITIES

An **equation** is a mathematical statement that two expressions are equal.

Sometimes we have a statement that one expression is *greater than*, or else *greater than or equal to*, another. We call this an **inequality**.

$x^2 + 7x > 18$ is an example of a quadratic inequality.

While quadratic equations have 0, 1, or 2 solutions, quadratic inequalities may have 0, 1, or infinitely many solutions. We use interval notation to describe the set of solutions.

To solve quadratic inequalities we use these steps:

- Make the RHS zero by shifting all terms to the LHS.
- Fully factorise the LHS.
- Draw a sign diagram for the LHS.
- Determine the values required from the sign diagram.

Example 8 — Self Tutor

Solve for x:

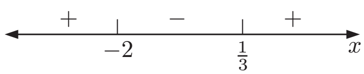
a $3x^2 + 5x \geqslant 2$ **b** $x^2 + 9 < 6x$

a $3x^2 + 5x \geqslant 2$

$\therefore\ 3x^2 + 5x - 2 \geqslant 0$ {make RHS zero}

$\therefore\ (3x - 1)(x + 2) \geqslant 0$ {factorising LHS}

Sign diagram of LHS is



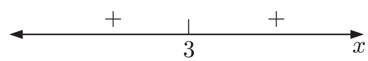
$\therefore\ x \leqslant -2$ or $x \geqslant \frac{1}{3}$.

b $x^2 + 9 < 6x$

$\therefore\ x^2 - 6x + 9 < 0$ {make RHS zero}

$\therefore\ (x - 3)^2 < 0$ {factorising LHS}

Sign diagram of LHS is



So, the inequality is not true for any real x.

EXERCISE 3B

1 Solve for x:

a $(x - 2)(x + 3) \geqslant 0$ **b** $(x + 1)(x - 4) < 0$ **c** $(2x + 1)(x - 3) > 0$

d $x^2 - x \geqslant 0$ **e** $x^2 \geqslant 3x$ **f** $3x^2 + 2x < 0$

g $x^2 < 4$ **h** $2x^2 \geqslant 18$ **i** $x^2 + 4x + 4 > 0$

j $x^2 + 2x - 15 > 0$ **k** $x^2 - 11x + 28 \leqslant 0$ **l** $x(x + 10) < -24$

m $x^2 - 30 \geqslant 13x$ **n** $2x^2 - x - 3 \geqslant 0$ **o** $4x^2 - 4x + 1 < 0$

p $6x^2 + 7x < 3$ **q** $3x^2 > 8(x + 2)$ **r** $2x^2 - 4x + 2 < 0$

s $6x^2 + 1 \leqslant 5x$ **t** $(4x + 1)(3x + 2) \geqslant 16x - 4$ **u** $(2x + 3)^2 < x + 6$

2 In $3x^2 + 12 \,\square\, 12x$, replace $\square$ with $>$, $\geqslant$, $<$, or $\leqslant$ so that the resulting inequality has:

a no solutions **b** one solution **c** infinitely many solutions.

C THE DISCRIMINANT OF A QUADRATIC

In the quadratic formula, the quantity $b^2 - 4ac$ under the square root sign is called the **discriminant**.

The symbol **delta** Δ is used to represent the discriminant, so $\Delta = b^2 - 4ac$.

The quadratic formula becomes $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ where Δ replaces $b^2 - 4ac$.

- If $\Delta = 0$, $x = \dfrac{-b}{2a}$ is the **only solution** (a **repeated** or **double root**)
- If $\Delta > 0$, $\sqrt{\Delta}$ is a positive real number, so there are **two distinct real roots**
$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{\Delta}}{2a}$$
- If $\Delta < 0$, $\sqrt{\Delta}$ is not a real number and so there are **no real roots**.
- If a, b, and c are rational and Δ is a **square** then the equation has two rational roots which can be found by factorisation.

Example 9 — Self Tutor

Use the discriminant to determine the nature of the roots of:

a $2x^2 - 2x + 3 = 0$

b $3x^2 - 4x - 2 = 0$

a $\Delta = b^2 - 4ac$
$= (-2)^2 - 4(2)(3)$
$= -20$

Since $\Delta < 0$, there are no real roots.

b $\Delta = b^2 - 4ac$
$= (-4)^2 - 4(3)(-2)$
$= 40$

Since $\Delta > 0$, but 40 is not a square, there are 2 distinct irrational roots.

Example 10 — Self Tutor

Consider $x^2 - 2x + m = 0$. Find the discriminant Δ, and hence find the values of m for which the equation has:

a a repeated root

b 2 distinct real roots

c no real roots.

$x^2 - 2x + m = 0$ has $a = 1$, $b = -2$, and $c = m$

$\therefore \Delta = b^2 - 4ac$
$= (-2)^2 - 4(1)(m)$
$= 4 - 4m$

a For a repeated root
$\Delta = 0$
$\therefore 4 - 4m = 0$
$\therefore 4 = 4m$
$\therefore m = 1$

b For 2 distinct real roots
$\Delta > 0$
$\therefore 4 - 4m > 0$
$\therefore -4m > -4$
$\therefore m < 1$

c For no real roots
$\Delta < 0$
$\therefore 4 - 4m < 0$
$\therefore -4m < -4$
$\therefore m > 1$

Example 11

Self Tutor

Consider the equation $kx^2 + (k+3)x = 1$. Find the discriminant Δ and draw its sign diagram. Hence, find the value of k for which the equation has:

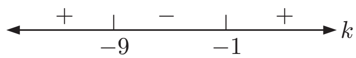
a two distinct real roots **b** two real roots

c a repeated root **d** no real roots.

$kx^2 + (k+3)x - 1 = 0$ has $a = k$, $b = (k+3)$, and $c = -1$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac \\ &= (k+3)^2 - 4(k)(-1) \\ &= k^2 + 6k + 9 + 4k \\ &= k^2 + 10k + 9 \\ &= (k+9)(k+1)\end{aligned}$$

So, Δ has sign diagram:



a For two distinct real roots, $\Delta > 0$ $\therefore$ $k < -9$ or $k > -1$, $k \neq 0$.

b For two real roots, $\Delta \geqslant 0$ $\therefore$ $k \leqslant -9$ or $k \geqslant -1$, $k \neq 0$.

c For a repeated root, $\Delta = 0$ $\therefore$ $k = -9$ or $k = -1$.

d For no real roots, $\Delta < 0$ $\therefore$ $-9 < k < -1$.

Summary:

Factorisation of quadratic	*Roots of quadratic*	*Discriminant value*
two distinct linear factors	two real distinct roots	$\Delta > 0$
two identical linear factors	two identical real roots (repeated)	$\Delta = 0$
unable to factorise	no real roots	$\Delta < 0$

EXERCISE 3C

1 By using the discriminant only, state the nature of the solutions of:

a $x^2 + 7x - 3 = 0$ **b** $x^2 - 3x + 2 = 0$ **c** $3x^2 + 2x - 1 = 0$

d $5x^2 + 4x - 3 = 0$ **e** $x^2 + x + 5 = 0$ **f** $16x^2 - 8x + 1 = 0$

2 By using the discriminant only, determine which of the following quadratic equations have rational roots which can be found by factorisation.

a $6x^2 - 5x - 6 = 0$ **b** $2x^2 - 7x - 5 = 0$ **c** $3x^2 + 4x + 1 = 0$

d $6x^2 - 47x - 8 = 0$ **e** $4x^2 - 3x + 2 = 0$ **f** $8x^2 + 2x - 3 = 0$

3 For each of the following quadratic equations, determine the discriminant Δ in simplest form and draw its sign diagram. Hence find the value(s) of m for which the equation has:

i a repeated root **ii** two distinct real roots **iii** no real roots.

a $x^2 + 4x + m = 0$ **b** $mx^2 + 3x + 2 = 0$ **c** $mx^2 - 3x + 1 = 0$

4 For each of the following quadratic equations, find the discriminant Δ and hence draw its sign diagram. Find all values of k for which the equation has:

i two distinct real roots **ii** two real roots **iii** a repeated root **iv** no real roots.

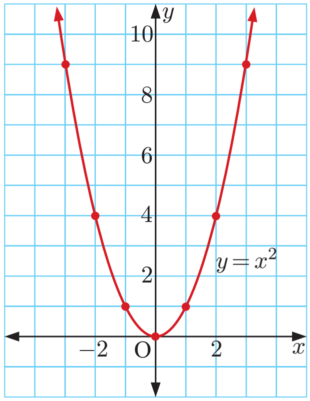
a $2x^2 + kx - k = 0$
b $kx^2 - 2x + k = 0$
c $x^2 + (k+2)x + 4 = 0$
d $2x^2 + (k-2)x + 2 = 0$
e $x^2 + (3k-1)x + (2k+10) = 0$
f $(k+1)x^2 + kx + k = 0$

D QUADRATIC FUNCTIONS

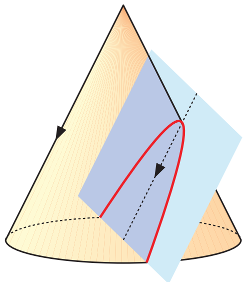
A **quadratic function** has the form $y = ax^2 + bx + c$ where $a \neq 0$.

The simplest quadratic function is $y = x^2$. Its graph can be drawn from a table of values.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9



The graph of a quadratic function is called a **parabola**.



The parabola is one of the **conic sections**, the others being circles, hyperbolae, and ellipses. They are called conic sections because they can be obtained by cutting a cone with a plane. A parabola is produced by cutting the cone with a plane parallel to its slant side.

There are many examples of parabolas in everyday life, including water fountains, suspension bridges, and radio telescopes.

TERMINOLOGY

The graph of a quadratic function $y = ax^2 + bx + c$, $a \neq 0$ is called a **parabola**.

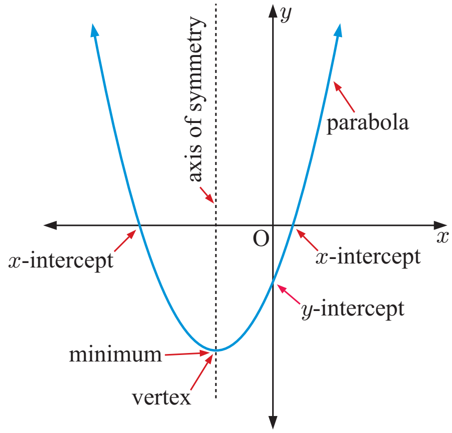
The point where the graph 'turns' is called the **vertex**.

If the graph opens upwards, the vertex is the **minimum** or **minimum turning point**, and the graph is **concave upwards**.

If the graph opens downwards, the vertex is the **maximum** or **maximum turning point**, and the graph is **concave downwards**.

The vertical line that passes through the vertex is called the **axis of symmetry**. Every parabola is symmetrical about its axis of symmetry.

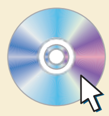
The point where the graph crosses the y-axis is the **y-intercept**.



The points (if they exist) where the graph crosses the x-axis are called the **x-intercepts**. They correspond to the **roots** of the equation $y = 0$.

Discovery 1 — Graphing $y = a(x-p)(x-q)$

This Discovery is best done using a **graphing package** or **graphics calculator**.



What to do:

1 **a** Use technology to help you to sketch:
$y = (x-1)(x-3)$, $y = 2(x-1)(x-3)$, $y = -(x-1)(x-3)$,
$y = -3(x-1)(x-3)$, and $y = -\frac{1}{2}(x-1)(x-3)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of a in $y = a(x-1)(x-3)$?

2 **a** Use technology to help you to sketch:
$y = 2(x-1)(x-4)$, $y = 2(x-3)(x-5)$, $y = 2(x+1)(x-2)$,
$y = 2x(x+5)$, and $y = 2(x+2)(x+4)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of p and q in $y = 2(x-p)(x-q)$?

3 **a** Use technology to help you to sketch:
$y = 2(x-1)^2$, $y = 2(x-3)^2$, $y = 2(x+2)^2$, $y = 2x^2$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of p in $y = 2(x-p)^2$?

4 Copy and complete:

- If a quadratic has the form $y = a(x-p)(x-q)$ then it ...... the x-axis at ......
- If a quadratic has the form $y = a(x-p)^2$ then it ...... the x-axis at ......

Discovery 2 — Graphing $y = a(x-h)^2 + k$

This Discovery is also best done using technology.

GRAPHING PACKAGE

What to do:

1 **a** Use technology to help you to sketch:
$y = (x-3)^2 + 2$, $y = 2(x-3)^2 + 2$, $y = -2(x-3)^2 + 2$,
$y = -(x-3)^2 + 2$, and $y = -\frac{1}{3}(x-3)^2 + 2$

b Find the coordinates of the vertex for each function in **a**.

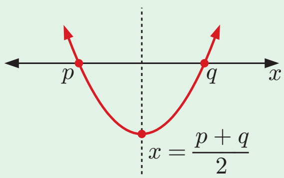
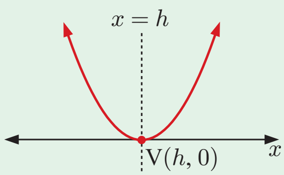
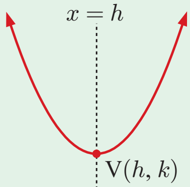
c What is the geometrical significance of a in $y = a(x-3)^2 + 2$?

2 **a** Use technology to help you to sketch:
$y = 2(x-1)^2 + 3$, $y = 2(x-2)^2 + 4$, $y = 2(x-3)^2 + 1$,
$y = 2(x+1)^2 + 4$, $y = 2(x+2)^2 - 5$, and $y = 2(x+3)^2 - 2$

b Find the coordinates of the vertex for each function in **a**.

c What is the geometrical significance of h and k in $y = 2(x-h)^2 + k$?

3 Copy and complete:
If a quadratic has the form $y = a(x-h)^2 + k$ then its vertex has coordinates ......
The graph of $y = a(x-h)^2 + k$ is a ...... of the graph of $y = ax^2$ with vector ......

Quadratic form, $a \neq 0$	*Graph*	*Facts*
• $y = a(x-p)(x-q)$ p, q are real		x-intercepts are p and q axis of symmetry is $x = \frac{p+q}{2}$ vertex is $\left(\frac{p+q}{2}, f\left(\frac{p+q}{2}\right)\right)$
• $y = a(x-h)^2$ h is real		touches x-axis at h axis of symmetry is $x = h$ vertex is $(h, 0)$
• $y = a(x-h)^2 + k$		axis of symmetry is $x = h$ vertex is (h, k)

You should have found that a, the coefficient of x^2, controls the width of the graph and whether it opens upwards or downwards.

For a quadratic function $y = ax^2 + bx + c$, $a \neq 0$:

- $a > 0$ produces the shape  called concave up.

  $a < 0$ produces the shape  called concave down.
- If $-1 < a < 1$, $a \neq 0$ the graph is wider than $y = x^2$.
  If $a < -1$ or $a > 1$ the graph is narrower than $y = x^2$.

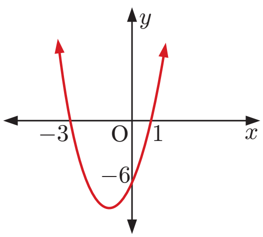
Example 12

Self Tutor

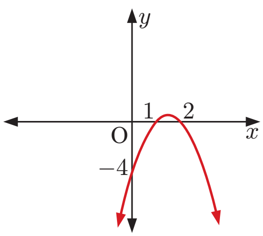
Using axes intercepts only, sketch the graphs of:

a $y = 2(x+3)(x-1)$ **b** $y = -2(x-1)(x-2)$ **c** $y = \frac{1}{2}(x+2)^2$

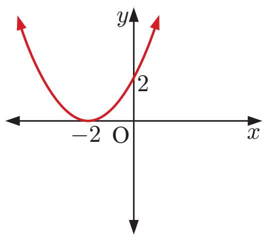
a $y = 2(x+3)(x-1)$
has x-intercepts -3, 1
When $x = 0$,
$y = 2(3)(-1)$
$= -6$
$\therefore$ y-intercept is -6



b $y = -2(x-1)(x-2)$
has x-intercepts 1, 2
When $x = 0$,
$y = -2(-1)(-2)$
$= -4$
$\therefore$ y-intercept is -4



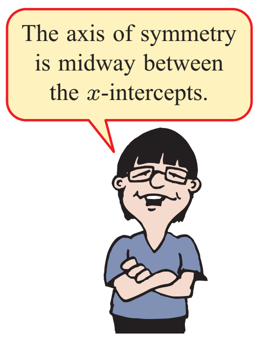
c $y = \frac{1}{2}(x+2)^2$
touches x-axis at -2
When $x = 0$,
$y = \frac{1}{2}(2)^2$
$= 2$
$\therefore$ y-intercept is 2



EXERCISE 3D.1

1 Using axes intercepts only, sketch the graphs of:

a $y = (x-4)(x+2)$ **b** $f(x) = -(x-4)(x+2)$
c $y = 2(x+3)(x+5)$ **d** $f(x) = -3(x+1)(x+5)$
e $f(x) = 2(x+3)^2$ **f** $y = -\frac{1}{4}(x+2)^2$



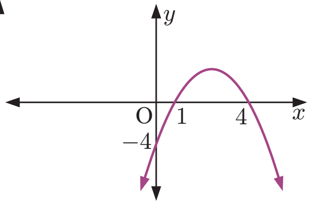
2 State the equation of the axis of symmetry for each graph in question **1**.

3 Match each quadratic function with its corresponding graph.

a $y = 2(x-1)(x-4)$ **b** $y = -(x+1)(x-4)$ **c** $y = (x-1)(x-4)$

d $y = (x+1)(x-4)$ **e** $y = 2(x+4)(x-1)$ **f** $y = -3(x+4)(x-1)$

g $y = -(x-1)(x-4)$ **h** $y = -3(x-1)(x-4)$

A


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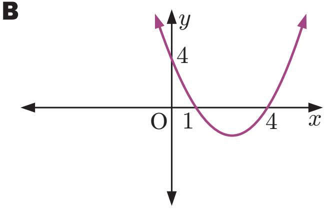
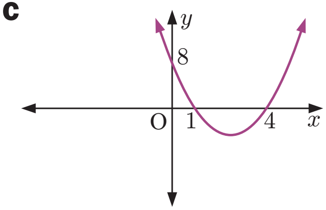
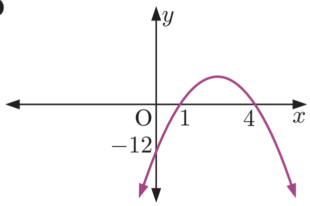
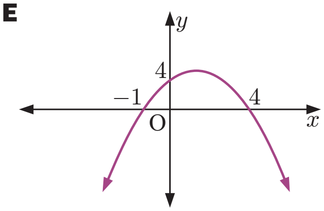
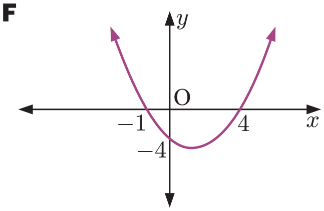
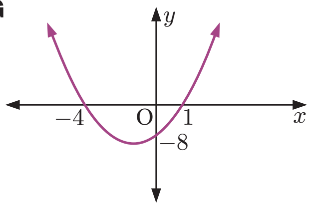


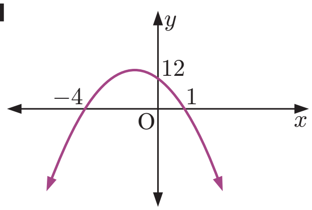
C


D


E


F


G


H


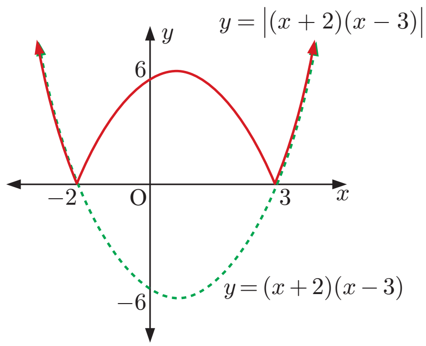
Example 13 Self Tutor

Sketch the graph of $y = |(x+2)(x-3)|$.

We first sketch $y = (x+2)(x-3)$.

$y = (x+2)(x-3)$ has x-intercepts -2 and 3, and y-intercept $2(-3) = -6$.

The part of the graph that is below the x-axis is then reflected in the x-axis to produce the graph of $y = |(x+2)(x-3)|$.



4 Sketch the graph of:

a $y = |(x+4)(x-5)|$ **b** $f(x) = |-(x-1)(x-6)|$

c $y = |2(x-2)(x+2)|$ **d** $f(x) = \left|-3(x+3)^2\right|$

Example 14

Self Tutor

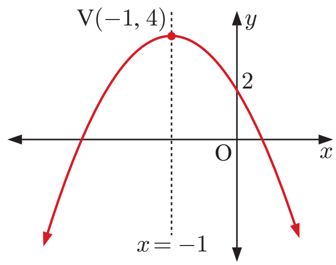
Use the vertex, axis of symmetry, and y-intercept to graph $y = -2(x+1)^2 + 4$.

The vertex is $(-1, 4)$.

The axis of symmetry is $x = -1$.

When $x = 0$, $y = -2(1)^2 + 4$
$= 2$

$a < 0$ so the shape is 



5 Use the vertex, axis of symmetry, and y-intercept to graph:

a $y = (x-1)^2 + 3$ **b** $f(x) = 2(x+2)^2 + 1$ **c** $y = -2(x-1)^2 - 3$

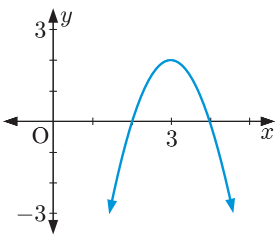
d $f(x) = \frac{1}{2}(x-3)^2 + 2$ **e** $y = -\frac{1}{3}(x-1)^2 + 4$ **f** $f(x) = -\frac{1}{10}(x+2)^2 - 3$

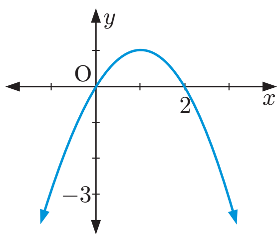
6 Match each quadratic function with its corresponding graph:

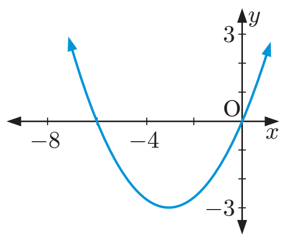
a $y = -(x+1)^2 + 3$ **b** $y = -2(x-3)^2 + 2$ **c** $y = x^2 + 2$

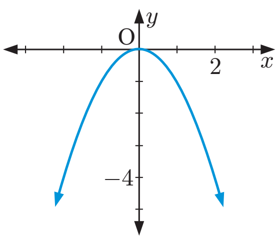
d $y = -(x-1)^2 + 1$ **e** $y = (x-2)^2 - 2$ **f** $y = \frac{1}{3}(x+3)^2 - 3$

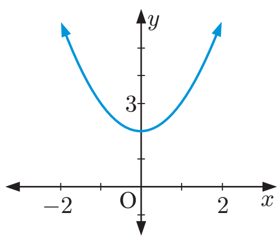
g $y = -x^2$ **h** $y = -\frac{1}{2}(x-1)^2 + 1$ **i** $y = 2(x+2)^2 - 1$

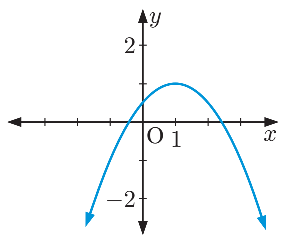
A


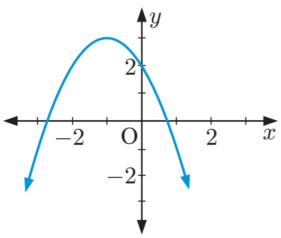
B


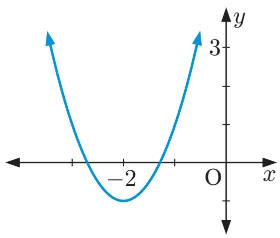
C


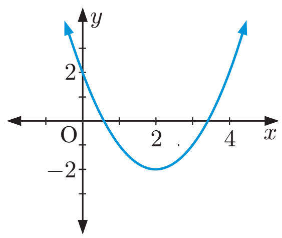
D


E


F


G


H


I


SKETCHING GRAPHS BY 'COMPLETING THE SQUARE'

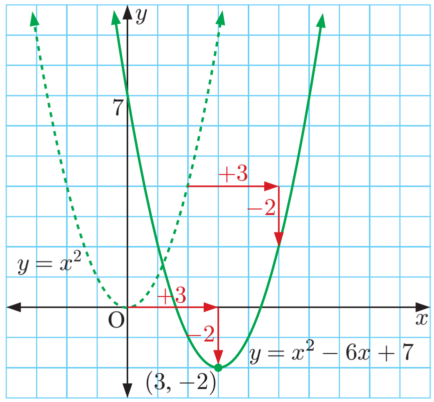
If we wish to graph a quadratic given in general form $y = ax^2 + bx + c$, one approach is to convert it to the form $y = a(x-h)^2 + k$ where we can read off the coordinates of the vertex (h, k). To do this, we '**complete the square**'.

Consider the simple case $y = x^2 - 6x + 7$, for which $a = 1$.

$$\begin{aligned} y &= x^2 - 6x + 7 \\ \therefore\ y &= \underbrace{x^2 - 6x + 3^2}\ \underbrace{+7 - 3^2} \\ \therefore\ y &= \quad (x-3)^2 \qquad -2 \end{aligned}$$

So, the vertex is $(3, -2)$.

To obtain the graph of $y = x^2 - 6x + 7$ from the graph of $y = x^2$, we shift it 3 units to the right and 2 units down.



Example 15 Self Tutor

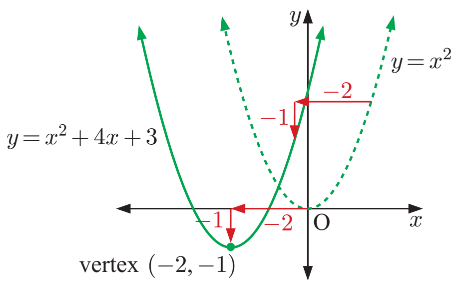
Write $y = x^2 + 4x + 3$ in the form $y = (x-h)^2 + k$ by 'completing the square'.

Hence sketch $y = x^2 + 4x + 3$, stating the coordinates of the vertex.

$$\begin{aligned} y &= x^2 + 4x + 3 \\ \therefore\ y &= x^2 + 4x + 2^2 + 3 - 2^2 \\ \therefore\ y &= (x+2)^2 - 1 \end{aligned}$$

shift 2 units left   shift 1 unit down

The vertex is $(-2, -1)$ and the y-intercept is 3.



Example 16 Self Tutor

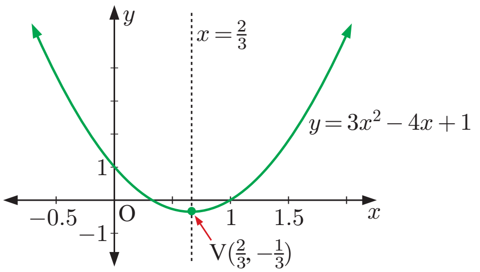
a Convert $y = 3x^2 - 4x + 1$ to the form $y = a(x-h)^2 + k$.

b Hence, write down the coordinates of its vertex and sketch the quadratic.

a $y = 3x^2 - 4x + 1$

$$\begin{aligned} &= 3[x^2 - \tfrac{4}{3}x + \tfrac{1}{3}] && \{\text{taking out a factor of 3}\} \\ &= 3[x^2 - 2(\tfrac{2}{3})x + (\tfrac{2}{3})^2 - (\tfrac{2}{3})^2 + \tfrac{1}{3}] && \{\text{completing the square}\} \\ &= 3[(x - \tfrac{2}{3})^2 - \tfrac{4}{9} + \tfrac{3}{9}] && \{\text{writing as a perfect square}\} \\ &= 3[(x - \tfrac{2}{3})^2 - \tfrac{1}{9}] \\ &= 3(x - \tfrac{2}{3})^2 - \tfrac{1}{3} \end{aligned}$$

b The vertex is $\left(\frac{2}{3}, -\frac{1}{3}\right)$ and the y-intercept is 1.



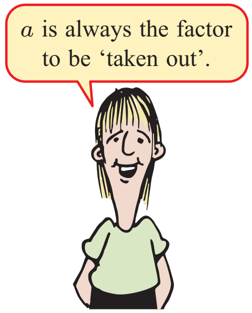
EXERCISE 3D.2

1 Write the following quadratics in the form $y = (x-h)^2 + k$ by 'completing the square'. Hence sketch each function, stating the coordinates of the vertex.

a $y = x^2 - 2x + 3$ **b** $y = x^2 + 4x - 2$ **c** $y = x^2 - 4x$
d $y = x^2 + 3x$ **e** $y = x^2 + 5x - 2$ **f** $y = x^2 - 3x + 2$
g $y = x^2 - 6x + 5$ **h** $y = x^2 + 8x - 2$ **i** $y = x^2 - 5x + 1$

2 For each of the following quadratics:

i Write the quadratic in the form $y = a(x-h)^2 + k$.
ii State the coordinates of the vertex.
iii Find the y-intercept.
iv Sketch the graph of the quadratic.



a $y = 2x^2 + 4x + 5$ **b** $y = 2x^2 - 8x + 3$
c $y = 2x^2 - 6x + 1$ **d** $y = 3x^2 - 6x + 5$
e $y = -x^2 + 4x + 2$ **f** $y = -2x^2 - 5x + 3$

QUADRATIC FUNCTIONS OF THE FORM $y = ax^2 + bx + c$

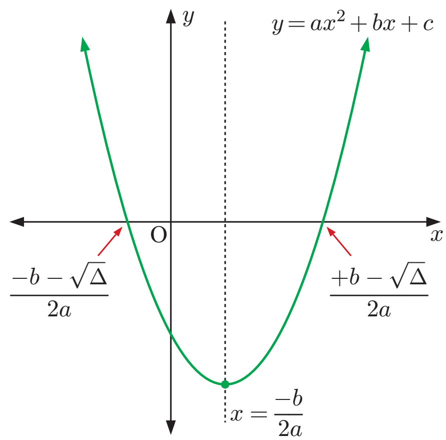
We now consider a method of graphing quadratics of the form $y = ax^2 + bx + c$ directly, without having to first convert them to a different form.

We know that the quadratic equation $ax^2 + bx + c = 0$ has solutions $\dfrac{-b-\sqrt{\Delta}}{2a}$ and $\dfrac{-b+\sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$.

If $\Delta \geqslant 0$, these are the x-intercepts of the quadratic function $y = ax^2 + bx + c$.

The average of the values is $\dfrac{-b}{2a}$, so we conclude that:

- the axis of symmetry is $x = \dfrac{-b}{2a}$
- the vertex of the quadratic has x-coordinate $\dfrac{-b}{2a}$.



To graph a quadratic of the form $y = ax^2 + bx + c$, we:

- find the axis of symmetry $x = \dfrac{-b}{2a}$
- substitute to find the y-coordinate of the vertex
- state the y-intercept c
- find the x-intercepts by solving $ax^2 + bx + c = 0$, either by factorisation or using the quadratic formula.

Example 17 — Self Tutor

Consider the quadratic $f(x) = 2x^2 + 8x - 10$.

a Find the axis of symmetry.
b Find the coordinates of the vertex.
c Find the axes intercepts.
d Hence, sketch the function.
e State the range of the function.

$f(x) = 2x^2 + 8x - 10$ has $a = 2$, $b = 8$, and $c = -10$.

$a > 0$, so the shape is 

a $\dfrac{-b}{2a} = \dfrac{-8}{2(2)} = -2$

The axis of symmetry is $x = -2$.

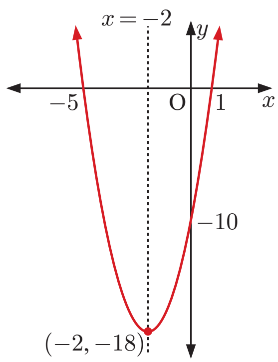
b $f(-2) = 2(-2)^2 + 8(-2) - 10$
$= -18$

The vertex is $(-2, -18)$.

c The y-intercept is -10.

When $y = 0$, $2x^2 + 8x - 10 = 0$
$\therefore\ 2(x^2 + 4x - 5) = 0$
$\therefore\ 2(x + 5)(x - 1) = 0$
$\therefore\ x = -5$ or 1

$\therefore$ the x-intercepts are -5 and 1.

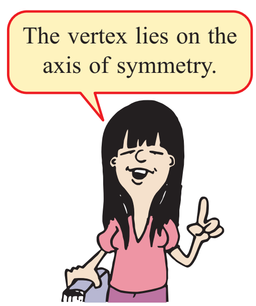
d 

e The range is $\{y : y \geqslant -18\}$.

EXERCISE 3D.3

1 Locate the turning point or vertex for each of the following quadratic functions:

a $f(x) = x^2 - 4x + 2$
b $y = x^2 + 2x - 3$
c $y = 2x^2 + 4$
d $f(x) = -3x^2 + 1$
e $y = 2x^2 + 8x - 7$
f $f(x) = -x^2 - 4x - 9$
g $y = 2x^2 + 6x - 1$
h $f(x) = 2x^2 - 10x + 3$
i $y = -\frac{1}{2}x^2 + x - 5$



2 For each of the following quadratics:

i state the axis of symmetry **ii** find the coordinates of the vertex
iii find the axes intercepts **iv** sketch the quadratic **v** state the range.

a $y = x^2 - 8x + 7$ **b** $y = -x^2 - 6x - 8$ **c** $f(x) = 6x - x^2$
d $y = -x^2 + 3x - 2$ **e** $y = 2x^2 + 4x - 24$ **f** $f(x) = -3x^2 + 4x - 1$
g $f(x) = 2x^2 - 5x + 2$ **h** $y = 4x^2 - 8x - 5$ **i** $y = -\frac{1}{4}x^2 + 2x - 3$

3 For each of the following quadratics:

i write the quadratic in factored form and hence find the roots
ii write the quadratic in completed square form and hence find the coordinates of the vertex
iii sketch the quadratic, showing the details you have found.

a $y = x^2 - 10x + 16$ **b** $y = x^2 + 10x + 9$ **c** $y = x^2 - 14x + 45$

4 Sketch the graph of:

a $y = |x^2 + 4x - 12|$ **b** $f(x) = |-x^2 - 3x + 10|$ **c** $y = |4x^2 - 12x + 5|$

Example 18 — Self Tutor

Find the range of $y = x^2 - 6x - 2$ on the domain $-2 \leqslant x \leqslant 7$.

$y = x^2 - 6x - 2$ has $a = 1$, $b = -6$, and $c = -2$.

$a > 0$, so the shape is 

$\frac{-b}{2a} = \frac{-(-6)}{2(1)} = 3$

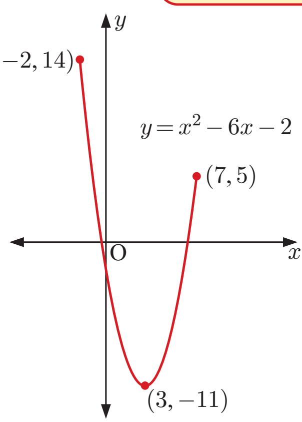
When $x = 3$, $y = 3^2 - 6(3) - 2$
$= -11$

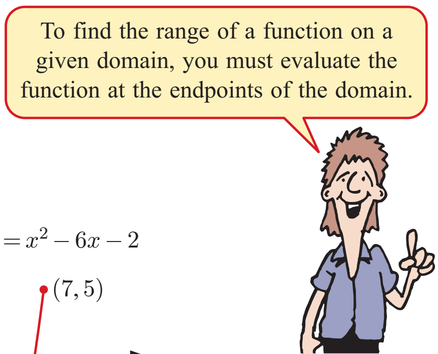
$\therefore$ the vertex is $(3, -11)$.

When $x = -2$, $y = (-2)^2 - 6(-2) - 2$
$= 14$

When $x = 7$, $y = 7^2 - 6(7) - 2$
$= 5$

So, on the domain $\{x : -2 \leqslant x \leqslant 7\}$,
the range is $\{y : -11 \leqslant y \leqslant 14\}$.





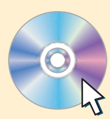
5 Find the range of:

a $f(x) = x^2 + 4x - 6$ on $-6 \leqslant x \leqslant 3$ **b** $y = -x^2 + 8x + 3$ on $0 \leqslant x \leqslant 7$
c $y = 2x^2 - 12x + 5$ on $-2 \leqslant x \leqslant 6$ **d** $f(x) = 7x - x^2$ on $-1 \leqslant x \leqslant 5$

Activity

Click on the icon to run a card game for quadratic functions.

CARD GAME



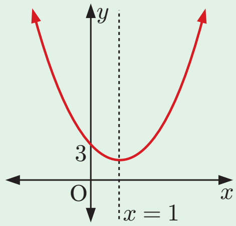
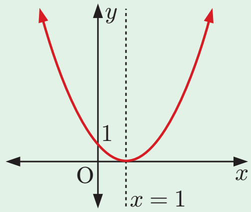
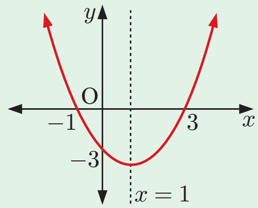
THE DISCRIMINANT AND THE QUADRATIC GRAPH

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $\Delta = b^2 - 4ac$.

We used Δ to determine the number of real roots of the equation. If they exist, these roots correspond to zeros of the quadratic $y = ax^2 + bx + c$. Δ therefore tells us about the relationship between a quadratic function and the x-axis.

The graphs of $y = x^2 - 2x + 3$, $y = x^2 - 2x + 1$, and $y = x^2 - 2x - 3$ all have the same axis of symmetry, $x = 1$.

Consider the following table:

$y = x^2 - 2x + 3$	$y = x^2 - 2x + 1$	$y = x^2 - 2x - 3$
		
$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(3)$ $= -8$	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(1)$ $= 0$	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(-3)$ $= 16$
$\Delta < 0$	$\Delta = 0$	$\Delta > 0$
does not cut the x-axis	touches the x-axis	cuts the x-axis twice

For a quadratic function $y = ax^2 + bx + c$, we consider the discriminant $\Delta = b^2 - 4ac$.

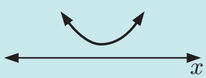
If $\Delta < 0$, the graph does not cut the x-axis.

If $\Delta = 0$, the graph *touches* the x-axis.

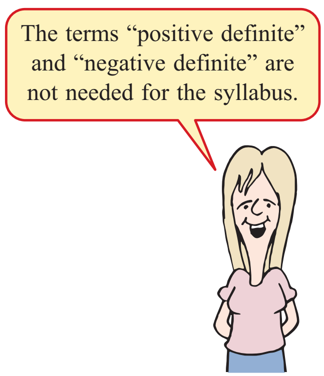
If $\Delta > 0$, the graph cuts the x-axis twice.

POSITIVE DEFINITE AND NEGATIVE DEFINITE QUADRATICS

Positive definite quadratics are quadratics which are positive for all values of x. So, $ax^2 + bx + c > 0$ for all $x \in \mathbb{R}$.



Test: A quadratic is **positive definite** if and only if $a > 0$ and $\Delta < 0$.



Negative definite quadratics are quadratics which are negative for all values of x. So, $ax^2 + bx + c < 0$ for all $x \in \mathbb{R}$.

Test: A quadratic is **negative definite** if and only if $a < 0$ and $\Delta < 0$.

Example 19

Self Tutor

Use the discriminant to determine the relationship between the graph of each function and the x-axis:

a $y = x^2 + 3x + 4$

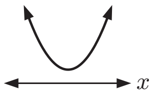
b $y = -2x^2 + 5x + 1$

a $a = 1, \quad b = 3, \quad c = 4$

$$\therefore \quad \Delta = b^2 - 4ac = 9 - 4(1)(4) = -7$$

Since $\Delta < 0$, the graph does not cut the x-axis.

Since $a > 0$, the graph is concave up.



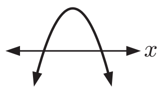
The graph is positive definite, which means that it lies entirely above the x-axis.

b $a = -2, \quad b = 5, \quad c = 1$

$$\therefore \quad \Delta = b^2 - 4ac = 25 - 4(-2)(1) = 33$$

Since $\Delta > 0$, the graph cuts the x-axis twice.

Since $a < 0$, the graph is concave down.



EXERCISE 3D.4

1 Use the discriminant to determine the relationship between the graph and x-axis for:

a $y = x^2 + x - 2$ **b** $y = x^2 - 4x + 1$ **c** $f(x) = -x^2 - 3$

d $f(x) = x^2 + 7x - 2$ **e** $y = x^2 + 8x + 16$ **f** $f(x) = -2x^2 + 3x + 1$

g $y = 6x^2 + 5x - 4$ **h** $f(x) = -x^2 + x + 6$ **i** $y = 9x^2 + 6x + 1$

2 Consider the graph of $y = 2x^2 - 5x + 1$.

a Describe the shape of the graph.

b Use the discriminant to show that the graph cuts the x-axis twice.

c Find the x-intercepts, rounding your answers to 2 decimal places.

d State the y-intercept.

e Hence, sketch the function.

3 Consider the graph of $f(x) = -x^2 + 4x - 7$.

a Use the discriminant to show that the graph does not cut the x-axis.

b Is the graph positive definite or negative definite?

c Find the vertex and y-intercept.

d Hence, sketch the function.

4 Show that:

a $x^2 - 3x + 6 > 0$ for all x **b** $4x - x^2 - 6 < 0$ for all x

c $2x^2 - 4x + 7$ is positive definite **d** $-2x^2 + 3x - 4$ is negative definite.

5 Explain why $3x^2 + kx - 1$ is never positive definite for any value of k.

6 Under what conditions is $2x^2 + kx + 2$ positive definite?

E FINDING A QUADRATIC FROM ITS GRAPH

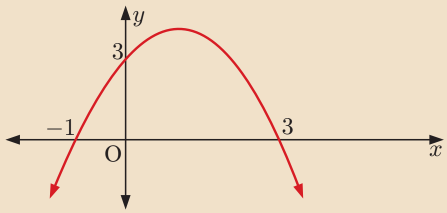
If we are given sufficient information on or about a graph, we can determine the quadratic function in whatever form is required.

Example 20

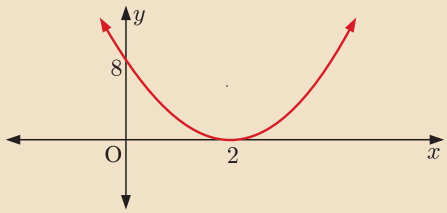
Self Tutor

Find the equation of the quadratic function with graph:

a



b



a Since the x-intercepts are -1 and 3,
$y = a(x+1)(x-3)$.
The graph is concave down, so $a < 0$.
When $x = 0$, $y = 3$

$\therefore\ 3 = a(1)(-3)$

$\therefore\ a = -1$

The quadratic function is
$y = -(x+1)(x-3)$.

b The graph touches the x-axis at $x = 2$,
so $y = a(x-2)^2$.
The graph is concave up, so $a > 0$.
When $x = 0$, $y = 8$

$\therefore\ 8 = a(-2)^2$

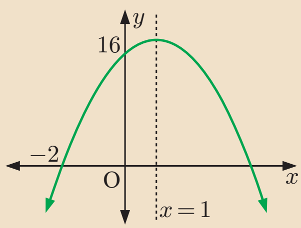
$\therefore\ a = 2$

The quadratic function is
$y = 2(x-2)^2$.

Example 21

Self Tutor

Find the equation of the quadratic function with graph:



The axis of symmetry $x = 1$ lies midway between the x-intercepts.

$\therefore$ the other x-intercept is 4.

$\therefore$ the quadratic has the form

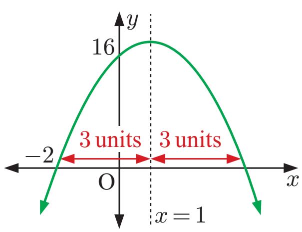
$$y = a(x+2)(x-4) \quad \text{where} \quad a < 0$$

But when $x = 0$, $y = 16$

$\therefore\ 16 = a(2)(-4)$

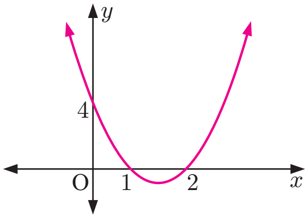
$\therefore\ a = -2$

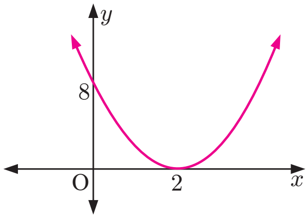
The quadratic is $y = -2(x+2)(x-4)$.

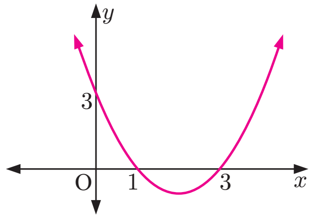


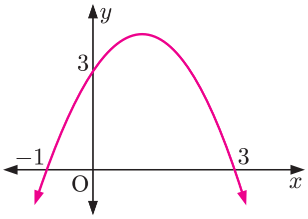
EXERCISE 3E

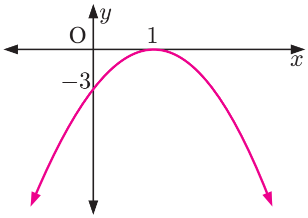
1 Find the equation of the quadratic with graph:

a


b


c


d


e


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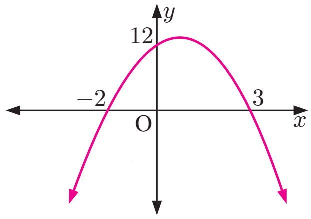
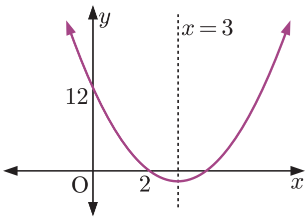
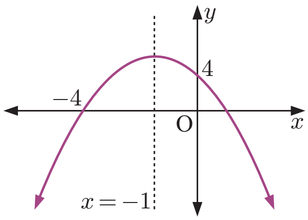
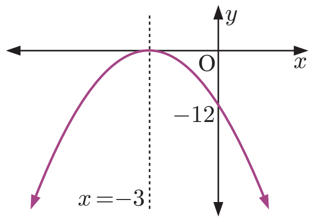


2 Find the quadratic with graph:

a


b


c


Example 22 Self Tutor

Find the equation of the quadratic whose graph cuts the x-axis at 4 and -3, and which passes through the point $(2, -20)$. Give your answer in the form $y = ax^2 + bx + c$.

Since the x-intercepts are 4 and -3, the quadratic has the form $y = a(x-4)(x+3)$ where $a \neq 0$.

When $x = 2$, $y = -20$

$$\therefore \ -20 = a(2-4)(2+3)$$
$$\therefore \ -20 = a(-2)(5)$$
$$\therefore \ a = 2$$

The quadratic is
$$\begin{aligned} y &= 2(x-4)(x+3) \\ &= 2(x^2 - x - 12) \\ &= 2x^2 - 2x - 24 \end{aligned}$$

3 Find, in the form $f(x) = ax^2 + bx + c$, the equation of the quadratic whose graph:

a cuts the x-axis at 5 and 1, and passes through $(2, -9)$

b cuts the x-axis at 2 and $-\frac{1}{2}$, and passes through $(3, -14)$

c touches the x-axis at 3 and passes through $(-2, -25)$

d touches the x-axis at -2 and passes through $(-1, 4)$

4 Find, in the form $f(x) = ax^2 + bx + c$, the equation of the quadratic whose graph:

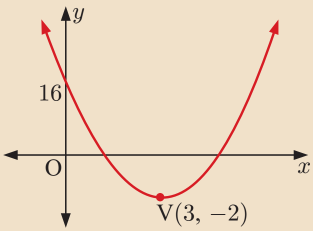
a cuts the x-axis at 3, passes through $(5, 12)$, and has axis of symmetry $x = 2$

b cuts the x-axis at 5, passes through $(2, 5)$, and has axis of symmetry $x = 1$.

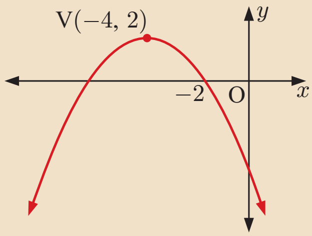
Example 23 — Self Tutor

Find the equation of each quadratic function given its graph:

a



b

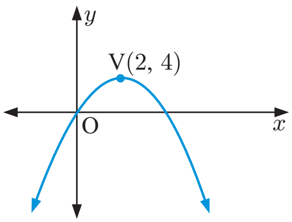


a Since the vertex is $(3, -2)$, the quadratic has the form
$y = a(x-3)^2 - 2$ where $a > 0$.
When $x = 0$, $y = 16$
$\therefore\ 16 = a(-3)^2 - 2$
$\therefore\ 16 = 9a - 2$
$\therefore\ 18 = 9a$
$\therefore\ a = 2$
The quadratic is $y = 2(x-3)^2 - 2$.

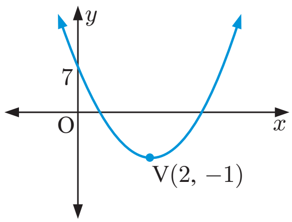
b Since the vertex is $(-4, 2)$, the quadratic has the form
$y = a(x+4)^2 + 2$ where $a < 0$.
When $x = -2$, $y = 0$
$\therefore\ 0 = a(2)^2 + 2$
$\therefore\ 4a = -2$
$\therefore\ a = -\frac{1}{2}$
The quadratic is
$y = -\frac{1}{2}(x+4)^2 + 2$.

5 If V is the vertex, find the equation of the quadratic function with graph:

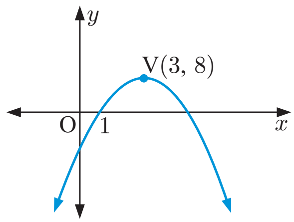
a



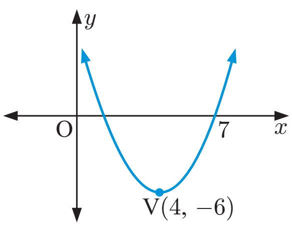
b



c



d



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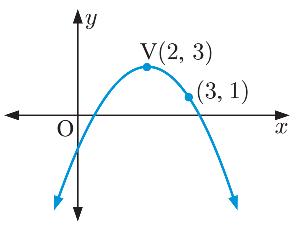

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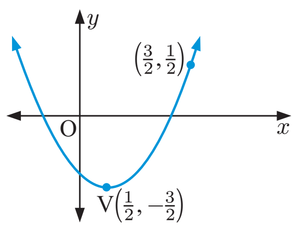


Discovery 3 — Finding quadratic functions

For the quadratic function $y = 2x^2 + 3x + 7$ we can find a table of values for $x = 0, 1, 2, 3, 4, 5$.

x	0	1	2	3	4	5
y	7	12	21	34	51	72

We turn this table into a **difference table** by adding two further rows:

- the row Δ_1 gives the differences between successive y-values
- the row Δ_2 gives the differences between successive Δ_1-values.

x	0	1	2	3	4	5
y	7	12	21	34	51	72
Δ_1	5	9	$\boxed{13}$	17	$\boxed{21}$	
Δ_2		$\boxed{4}$	4	4	4	

$9 - 5$ $\quad$ $34 - 21$ $\quad$ $72 - 51$

What to do:

1 Construct difference tables for $x = 0, 1, 2, 3, 4, 5$ for each of the following quadratic functions:

a $y = x^2 + 4x + 3$ **b** $y = 3x^2 - 4x$

c $y = 5x - x^2$ **d** $y = 4x^2 - 5x + 2$

2 What do you notice about the Δ_2 row for each of the quadratic functions in **1**?

3 Consider the general quadratic $y = ax^2 + bx + c, \quad a \neq 0$.

a Copy and complete the following difference table:

x	0	1	2	3	4	5
y	c	$a + b + c$	$4a + 2b + c$	......	......	......
Δ_1	○	......	......	......	......	
Δ_2		○	......	......	......	

b Comment on the Δ_2 row.

c What can the encircled numbers be used for?

4 Use your observations in **3** to determine, if possible, the quadratic functions with the following tables of values:

a

x	0	1	2	3	4
y	6	5	8	15	26

b

x	0	1	2	3	4
y	8	10	18	32	52

c

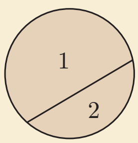
x	0	1	2	3	4
y	1	2	-1	-8	-19

d

x	0	1	2	3	4
y	5	3	-1	-7	-15

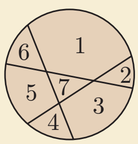
5 We wish to determine the **maximum** number of pieces into which a pizza can be cut using n cuts across it.

For example, for $n = 1$ we have



which has 2 pieces

for $n = 3$ we have



which has 7 pieces.

a Copy and complete:

Number of cuts, n	0	1	2	3	4	5
Maximum number of pieces, P_n						

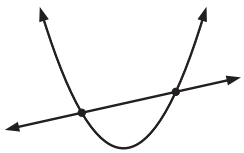
b Complete the Δ_1 and Δ_2 rows. Hence determine a quadratic formula for P_n.

c For a huge pizza with 12 cuts across it, find the maximum number of pieces which can result.

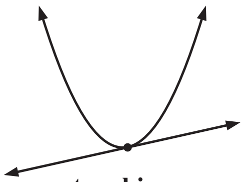
F WHERE FUNCTIONS MEET

Consider the graphs of a quadratic function and a linear function on the same set of axes.

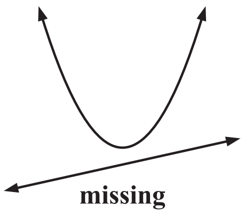
Notice that we could have:



cutting
(2 points of intersection)



touching
(1 point of intersection)



missing
(no points of intersection)

If the graphs meet, the coordinates of the points of intersection of the graphs can be found by *solving the two equations simultaneously.*

Example 24 — Self Tutor

Find the coordinates of the points of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$.

$y = x^2 - x - 18$ meets $y = x - 3$ where

$$x^2 - x - 18 = x - 3$$
$$\therefore\ x^2 - 2x - 15 = 0 \qquad \{\text{RHS} = 0\}$$
$$\therefore\ (x-5)(x+3) = 0 \qquad \{\text{factorising}\}$$
$$\therefore\ x = 5 \text{ or } -3$$

Substituting into $y = x - 3$, when $x = 5$, $y = 2$ and when $x = -3$, $y = -6$.

$\therefore$ the graphs meet at $(5, 2)$ and $(-3, -6)$.

EXERCISE 3F

1 Find the coordinates of the point(s) of intersection of:

a $y = x^2 - 2x + 8$ and $y = x + 6$

b $f(x) = -x^2 + 3x + 9$ and $g(x) = 2x - 3$

c $y = x^2 - 4x + 3$ and $y = 2x - 6$

d $f(x) = -x^2 + 4x - 7$ and $g(x) = 5x - 4$

Example 25 — Self Tutor

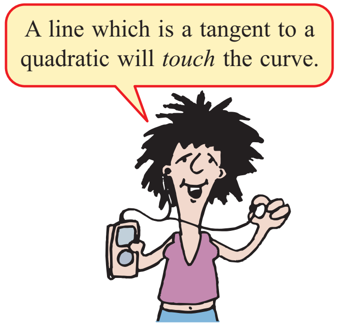
$y = 2x + k$ is a tangent to $y = 2x^2 - 3x + 4$. Find k.

$y = 2x + k$ meets $y = 2x^2 - 3x + 4$ where

$$2x^2 - 3x + 4 = 2x + k$$
$$\therefore\ 2x^2 - 5x + (4 - k) = 0$$

Since the graphs touch, this quadratic has $\Delta = 0$

$$\therefore\ (-5)^2 - 4(2)(4 - k) = 0$$
$$\therefore\ 25 - 8(4 - k) = 0$$
$$\therefore\ 25 - 32 + 8k = 0$$
$$\therefore\ 8k = 7$$
$$\therefore\ k = \tfrac{7}{8}$$



2 For which value of c is the line $y = 3x + c$ a tangent to the parabola with equation $y = x^2 - 5x + 7$?

3 Find the values of m for which the lines $y = mx - 2$ are tangents to the curve with equation $y = x^2 - 4x + 2$.

4 Find the gradients of the lines with y-intercept 1 that are tangents to the curve $f(x) = 3x^2 + 5x + 4$.

5 **a** For what values of c do the lines $y = x + c$ never meet the parabola with equation $y = 2x^2 - 3x - 7$?

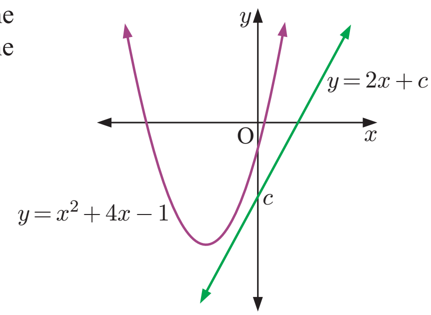
b Choose one of the values of c found in part **a** above. Illustrate with a sketch that these graphs never meet.

6 Consider the curve $y = x^2 + 4x - 1$ and the line $y = 2x + c$. Find the value(s) of c for which the line:

a meets the curve twice

b is a tangent to the curve

c does not meet the curve.



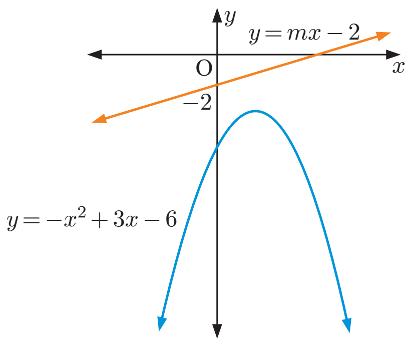
DEMO

7 Consider the curve $f(x) = -x^2 + 3x - 6$ and the line $g(x) = mx - 2$. Find the values of m for which the line:

a meets the curve twice

b is a tangent to the curve

c does not meet the curve.



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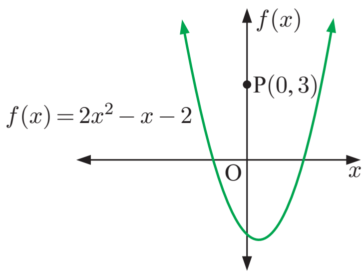
8 

Show that any linear function passing through P(0, 3) will meet the curve $f(x) = 2x^2 - x - 2$ twice.

G PROBLEM SOLVING WITH QUADRATICS

Some real world problems can be solved using a quadratic equation. We are generally only interested in any **real solutions** which result.

Any answer we obtain must be checked to see if it is reasonable. For example:

- if we are finding a length then it must be positive and we reject any negative solutions
- if we are finding 'how many people are present' then clearly the answer must be a positive integer.

We employ the following general problem solving method:

Step 1: If the information is given in words, translate it into algebra using a variable such as x for the unknown. Write down the resulting equation. Be sure to define what the variable x represents, and include units if appropriate.

Step 2: Solve the equation by a suitable method.

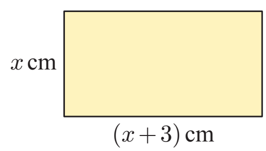
Step 3: Examine the solutions carefully to see if they are acceptable.

Step 4: Give your answer in a sentence.

Example 26 — Self Tutor

A rectangle has length 3 cm longer than its width. Its area is 42 cm^2. Find its width.

If the width is x cm then the length is $(x+3)$ cm.



$\therefore\ x(x+3) = 42$ {equating areas}

$\therefore\ x^2 + 3x - 42 = 0$

$\therefore\ x = \dfrac{-3 \pm \sqrt{3^2 - 4(1)(-42)}}{2}$

$\therefore\ x = \dfrac{-3 \pm \sqrt{177}}{2}$

$\therefore\ x \approx -8.15$ or 5.15

We reject the negative solution as lengths are positive.

The width is about 5.15 cm.

EXERCISE 3G

1 Two integers differ by 12 and the sum of their squares is 74. Find the integers.

2 The sum of a number and its reciprocal is $\frac{26}{5}$. Find the number.

3 The sum of a natural number and its square is 210. Find the number.

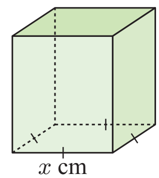
4 The product of two consecutive even numbers is 360. Find the numbers.

5 The number of diagonals of an n-sided polygon is given by the formula $D = \frac{n}{2}(n-3)$.

A polygon has 90 diagonals. How many sides does it have?

6 The length of a rectangle is 4 cm longer than its width. The rectangle has area 26 cm^2. Find its width.

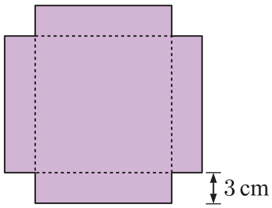
7 A rectangular box has a square base with sides of length x cm. Its height is 1 cm longer than its base side length. The total surface area of the box is 240 cm^2.



a Show that the total surface area is given by $A = 6x^2 + 4x$ cm^2.

b Find the dimensions of the box.

8



An open box can hold 80 cm^3. It is made from a square piece of tinplate with 3 cm squares cut from each of its 4 corners. Find the dimensions of the original piece of tinplate.

Example 27

Self Tutor

Is it possible to bend a 12 cm length of wire to form the perpendicular sides of a right angled triangle with area 20 cm^2?

Suppose the wire is bent x cm from one end.

The area $A = \frac{1}{2}x(12-x)$

$\therefore \frac{1}{2}x(12-x) = 20$

$\therefore x(12-x) = 40$

$\therefore 12x - x^2 - 40 = 0$

$\therefore x^2 - 12x + 40 = 0$

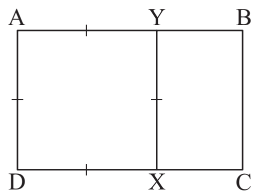
Now $\Delta = (-12)^2 - 4(1)(40)$

$= -16$ which is < 0

There are no real solutions, indicating this situation is **impossible**.

9 Is it possible to bend a 20 cm length of wire into the shape of a rectangle which has an area of 30 cm^2?

10 The rectangle ABCD is divided into a square and a smaller rectangle by [XY] which is parallel to its shorter sides.
The smaller rectangle BCXY is *similar* to the original rectangle, so rectangle ABCD is a **golden rectangle**.



The ratio $\frac{\text{AB}}{\text{AD}}$ is called the **golden ratio**.

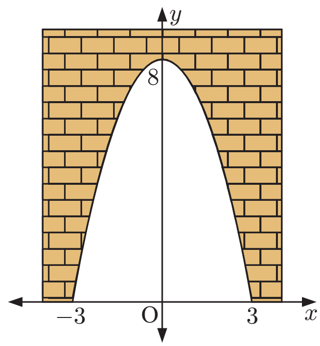
Show that the golden ratio is $\frac{1+\sqrt{5}}{2}$.

Hint: Let AB $= x$ units and AD $= 1$ unit.

11 A truck carrying a wide load needs to pass through the parabolic tunnel shown. The units are metres.
The truck is 5 m high and 4 m wide.

a Find the quadratic function which describes the shape of the tunnel.

b Determine whether the truck will fit.



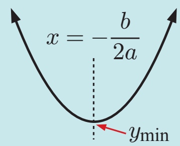
12 Answer the **Opening Problem** on page **64**.

H QUADRATIC OPTIMISATION

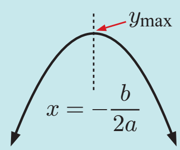
The process of finding the maximum or minimum value of a function is called **optimisation**.

For the quadratic function $y = ax^2 + bx + c$, we have already seen that the vertex has x-coordinate $-\frac{b}{2a}$.

- If $a > 0$, the **minimum** value of y occurs at $x = -\frac{b}{2a}$.



- If $a < 0$, the **maximum** value of y occurs at $x = -\frac{b}{2a}$.



Example 28

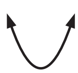
Self Tutor

Find the maximum or minimum value of the following quadratic functions, and the corresponding value of x:

a $y = x^2 + x - 3$

b $y = 3 + 3x - 2x^2$

a $y = x^2 + x - 3$ has
$a = 1$, $b = 1$, and $c = -3$.

Since $a > 0$, the shape is 

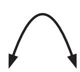
The minimum value occurs

when $x = \frac{-b}{2a} = -\frac{1}{2}$

and $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 3 = -3\frac{1}{4}$

So, the minimum value of y is $-3\frac{1}{4}$, occurring when $x = -\frac{1}{2}$.

b $y = -2x^2 + 3x + 3$ has
$a = -2$, $b = 3$, and $c = 3$.

Since $a < 0$, the shape is 

The maximum value occurs

when $x = \frac{-b}{2a} = \frac{-3}{-4} = \frac{3}{4}$

and $y = -2(\frac{3}{4})^2 + 3(\frac{3}{4}) + 3 = 4\frac{1}{8}$

So, the maximum value of y is $4\frac{1}{8}$, occurring when $x = \frac{3}{4}$.

EXERCISE 3H

1 Find the maximum or minimum values of the following quadratic functions, and the corresponding values of x:

a $y = x^2 - 2x$ **b** $f(x) = 7 - 2x - x^2$ **c** $y = 8 + 2x - 3x^2$

d $f(x) = 2x^2 + x - 1$ **e** $y = 4x^2 - x + 5$ **f** $f(x) = 7x - 2x^2$

2 The profit in manufacturing x refrigerators per day, is given by the profit relation
$P = -3x^2 + 240x - 800$ dollars.

a How many refrigerators should be made each day to maximise the total profit?

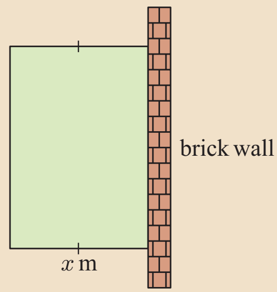
b What is the maximum profit?

Example 29 Self Tutor

A gardener has 40 m of fencing to enclose a rectangular garden plot, where one side is an existing brick wall. Suppose the two new equal sides are x m long.

a Show that the area enclosed is given by $A = x(40 - 2x)$ m^2.

b Find the dimensions of the garden of maximum area.



a Side [XY] has length $(40 - 2x)$ m.

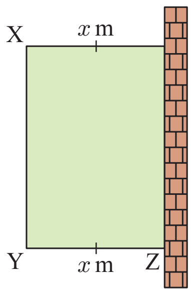
Now, area = length × width

$\therefore$ $A = x(40 - 2x)$ m^2

b $A = 0$ when $x = 0$ or 20.

The vertex of the function lies midway between these values, so $x = 10$.

Since $a < 0$, the shape is 

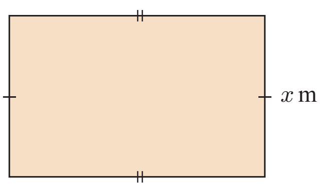


$\therefore$ the area is maximised when YZ = 10 m and XY = 20 m.

3 A rectangular plot is enclosed by 200 m of fencing and has an area of A square metres. Show that:

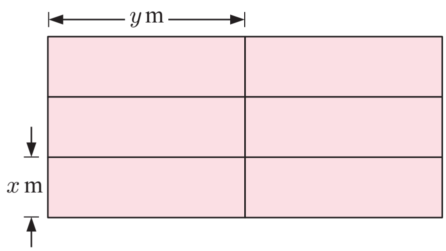
a $A = 100x - x^2$ where x m is the length of one of its sides

b the area is maximised if the rectangle is a square.



4 Three sides of a rectangular paddock are to be fenced, the fourth side being an existing straight water drain. If 1000 m of fencing is available, what dimensions should be used for the paddock so that it encloses the maximum possible area?

5 1800 m of fencing is available to fence six identical pens as shown in the diagram.

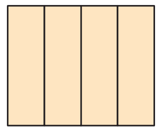


- **a** Explain why $9x + 8y = 1800$.
- **b** Show that the area of each pen is given by $A = -\frac{9}{8}x^2 + 225x$ m^2.
- **c** If the area enclosed is to be maximised, what are the dimensions of each pen?

6 500 m of fencing is available to make 4 rectangular pens of identical shape. Find the dimensions that maximise the area of each pen if the plan is:

a

b



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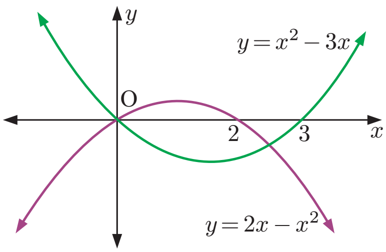
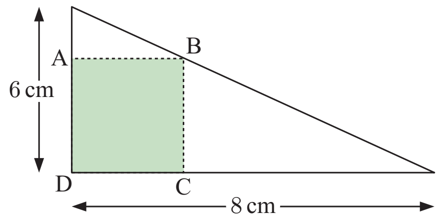


The graphs of $y = x^2 - 3x$ and $y = 2x - x^2$ are illustrated.

- **a** Show that the graphs meet where $x = 0$ and $x = 2\frac{1}{2}$.
- **b** Find the maximum vertical separation between the curves for $0 \leqslant x \leqslant 2\frac{1}{2}$.

8 Infinitely many rectangles may be inscribed within the right angled triangle shown alongside. One of them is illustrated.



- **a** Let AB = x cm and BC = y cm.
  Use similar triangles to find y in terms of x.
- **b** Find the dimensions of rectangle ABCD of maximum area.

Discovery 4 — *Sum and product of roots*

What to do:

1 Suppose $ax^2 + bx + c = 0$ has roots p and q.

Prove that $p + q = \dfrac{-b}{a}$ and $pq = \dfrac{c}{a}$.

2 Suppose $2x^2 - 5x + 1 = 0$ has roots p and q.
Without finding the values of p and q, find:

- **a** $p + q$
- **b** pq
- **c** $p^2 + q^2$
- **d** $\dfrac{1}{p} + \dfrac{1}{q}$

3 Find *all* quadratic equations with roots which are:

- **a** one more than the roots of $2x^2 - 5x + 1 = 0$
- **b** the squares of the roots of $2x^2 - 5x + 1 = 0$
- **c** the reciprocals of the roots of $2x^2 - 5x + 1 = 0$.

Review set 3A

1 Consider the quadratic function $f(x) = -2(x+2)(x-1)$.

a State the x-intercepts.
b State the equation of the axis of symmetry.
c Find the y-intercept.
d Find the coordinates of the vertex.
e Sketch the function.
f State the range of the function.

2 Solve the following equations, giving exact answers:

a $3x^2 - 12x = 0$
b $3x^2 - x - 10 = 0$
c $x^2 - 11x = 60$

3 Solve using the quadratic formula:

a $x^2 + 5x + 3 = 0$
b $3x^2 + 11x - 2 = 0$

4 Solve for x:

a $x^2 - 4x - 21 < 0$
b $3x^2 - 2 \geqslant 5x$

5 Use the vertex, axis of symmetry, and y-intercept to graph:

a $y = (x-2)^2 - 4$
b $y = -\frac{1}{2}(x+4)^2 + 6$

6 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph:

a touches the x-axis at 4 and passes through $(2, 12)$
b has vertex $(-4, 1)$ and passes through $(1, 11)$.

7 Find the maximum or minimum value of the relation $f(x) = -2x^2 + 4x + 3$ and the value of x at which this occurs.

8 Find the points of intersection of $y = x^2 - 3x$ and $y = 3x^2 - 5x - 24$.

9 For what values of k does the graph of $y = -2x^2 + 5x + k$ not cut the x-axis?

10 Find the values of m for which $2x^2 - 3x + m = 0$ has:

a a repeated root
b two distinct real roots
c no real roots.

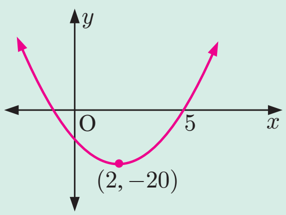
11 The sum of a number and its reciprocal is $2\frac{1}{30}$. Find the number.

12 Show that no line with a y-intercept of $(0, 10)$ will ever be tangential to the curve with equation $y = 3x^2 + 7x - 2$.

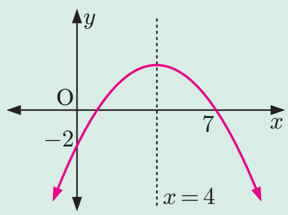
13 **a** Write the quadratic $y = 2x^2 + 4x - 3$ in the form $y = a(x-h)^2 + k$.
b Hence, sketch the graph of the quadratic.

14 Find the equation of the quadratic function with graph:

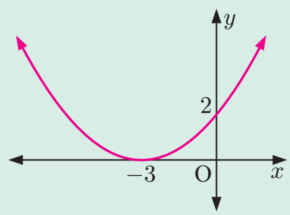
a



b



c



15 Find the range of $y = x^2 - 6x - 4$ on the domain $-1 \leqslant x \leqslant 8$.

16 When Annie hits a softball, the height of the ball above the ground after t seconds is given by $f(t) = -4.9t^2 + 19.6t + 1.4$ metres. Find the maximum height reached by the ball.

Review set 3B

1 Consider the quadratic function $y = \frac{1}{2}(x-2)^2 - 4$.

a State the equation of the axis of symmetry.
b Find the coordinates of the vertex.
c Find the y-intercept.
d Sketch the function.
e State the range of the function.

2 Solve the following equations:

a $x^2 - 5x - 3 = 0$
b $2x^2 - 7x - 3 = 0$

3 Solve for x:

a $x^2 + 5x \leqslant 14$
b $2x^2 + 7x > 2(x+6)$

4 Consider the quadratic function $f(x) = -3x^2 + 8x + 7$. Find the equation of the axis of symmetry, and the coordinates of the vertex.

5 Use the discriminant only to find the relationship between the graph and the x-axis for:

a $y = 2x^2 + 3x - 7$
b $y = -3x^2 - 7x + 4$

6 Determine whether each quadratic function is positive definite, negative definite, or neither:

a $y = -2x^2 + 3x + 2$
b $f(x) = 3x^2 + x + 11$

7 Find the equation of the quadratic function with vertex (2, 25) and y-intercept 1.

8 For what values of m does the line $y = mx - 10$ meet the curve $y = 3x^2 + 7x + 2$ twice?

9 Consider the quadratic function $y = 2x^2 + 4x - 1$.

a State the axis of symmetry.
b Find the coordinates of the vertex.
c Find the axes intercepts.
d Hence sketch the function.

10 Find the range of $y = -2x^2 + 6x + 1$ on the domain $-4 \leqslant x \leqslant 5$.

11 Find the values of k for which $kx^2 + kx - 2$ has:

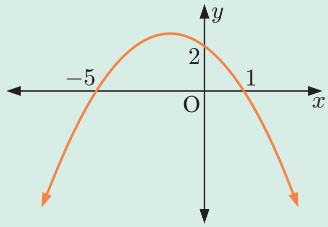
a a repeated root
b two distinct real roots
c no real roots.

12 **a** For what values of c do the lines with equations $y = 3x + c$ intersect the parabola $y = x^2 + x - 5$ in two distinct points?

b Choose one such value of c from part **a** and find the points of intersection in this case.

13 **a** Find the equation of the quadratic function illustrated.

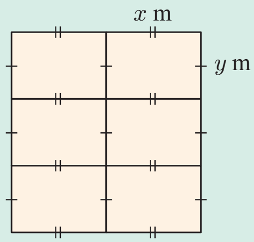
b Find the vertex of the quadratic.



14 Find the maximum or minimum value of the quadratic, and the corresponding value of x:

a $y = 3x^2 + 4x + 7$ **b** $y = -2x^2 - 5x + 2$

15 600 m of fencing is used to construct 6 rectangular animal pens as shown.



a Show that the area A of each pen is $A = x\left(\dfrac{600-8x}{9}\right)$ m^2.

b Find the dimensions of each pen so that it has the maximum possible area.

c What is the area of each pen in this case?

16 Sketch the graph of $f(x) = \left|x^2 + x - 20\right|$.

4

Surds, indices, and exponentials

Contents:

Opening problem

The interior of a freezer has temperature $-10°C$. When a packet of peas is placed in the freezer, its temperature after t minutes is given by $T(t) = -10 + 32 \times 2^{-0.2t}$ °C.

Things to think about:

a What was the temperature of the packet of peas:
  i when it was first placed in the freezer **ii** after 5 minutes
  iii after 10 minutes **iv** after 15 minutes?

b What does the graph of temperature over time look like?

c According to this model, will the temperature of the packet of peas ever reach $-10°C$? Explain your answer.

We often deal with numbers that are repeatedly multiplied together. Mathematicians use **indices**, also called **powers** or **exponents**, to construct such expressions.

Indices have many applications in the areas of finance, engineering, physics, electronics, biology, and computer science.

A SURDS

A **radical** is any number which is written with the **radical sign** $\sqrt{}$.

A **surd** is a real, irrational radical such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, or $\sqrt{6}$. Surds are present in solutions to some quadratic equations. $\sqrt{4}$ is a radical, but is not a surd as it simplifies to 2.

> $\sqrt{a}$ is the non-negative number such that $\sqrt{a} \times \sqrt{a} = a$.

Important properties of surds are:

- $\sqrt{a}$ is never negative, so $\sqrt{a} \geqslant 0$.
- $\sqrt{a}$ is only real if $a \geqslant 0$.
- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ for $a \geqslant 0$ and $b \geqslant 0$.
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ for $a \geqslant 0$ and $b > 0$.

Example 1 — Self Tutor

Write as a single surd: **a** $\sqrt{2} \times \sqrt{3}$ **b** $\frac{\sqrt{18}}{\sqrt{6}}$

a $\sqrt{2} \times \sqrt{3}$
$= \sqrt{2 \times 3}$
$= \sqrt{6}$

b $\frac{\sqrt{18}}{\sqrt{6}}$
$= \sqrt{\frac{18}{6}}$
$= \sqrt{3}$

EXERCISE 4A.1

1 Write as a single surd or rational number:

a $\sqrt{11}\times\sqrt{11}$ **b** $\sqrt{3}\times\sqrt{5}$ **c** $(\sqrt{3})^2$ **d** $\sqrt{5}\times\sqrt{6}$

e $2\sqrt{2}\times\sqrt{2}$ **f** $3\sqrt{2}\times 2\sqrt{2}$ **g** $3\sqrt{7}\times 2\sqrt{7}$ **h** $(3\sqrt{5})^2$

i $\dfrac{\sqrt{12}}{\sqrt{2}}$ **j** $\dfrac{\sqrt{18}}{\sqrt{3}}$ **k** $\dfrac{\sqrt{20}}{\sqrt{5}}$ **l** $\dfrac{\sqrt{6}\times\sqrt{10}}{\sqrt{12}}$

Example 2 — Self Tutor

Write $\sqrt{18}$ in the form $a\sqrt{b}$ where a and b are integers and a is as large as possible.

$\sqrt{18}$
$= \sqrt{9\times 2}$ {9 is the largest perfect square factor of 18}
$= \sqrt{9}\times\sqrt{2}$
$= 3\sqrt{2}$

2 Write in the form $a\sqrt{b}$ where a and b are integers and a is as large as possible:

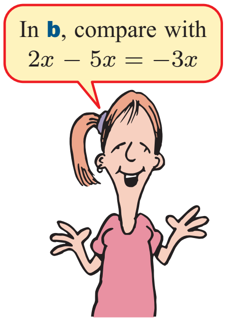
a $\sqrt{8}$ **b** $\sqrt{12}$ **c** $\sqrt{20}$ **d** $\sqrt{32}$

e $\sqrt{27}$ **f** $\sqrt{45}$ **g** $\sqrt{48}$ **h** $\sqrt{54}$

i $\sqrt{50}$ **j** $\sqrt{80}$ **k** $\sqrt{96}$ **l** $\sqrt{108}$

OPERATING WITH SURDS

The rules for adding, subtracting, and multiplying by surds are the same as those for ordinary algebra.



Example 3 — Self Tutor

Simplify: **a** $3\sqrt{3}+5\sqrt{3}$ **b** $2\sqrt{2}-5\sqrt{2}$

a $3\sqrt{3}+5\sqrt{3}$
$= 8\sqrt{3}$

b $2\sqrt{2}-5\sqrt{2}$
$= -3\sqrt{2}$

Example 4 — Self Tutor

Simplify:

a $\sqrt{5}(6-\sqrt{5})$ **b** $(6+\sqrt{3})(1+2\sqrt{3})$

a $\sqrt{5}(6-\sqrt{5})$
$= \sqrt{5}\times 6 + \sqrt{5}\times -\sqrt{5}$
$= 6\sqrt{5} - 5$

b $(6+\sqrt{3})(1+2\sqrt{3})$
$= 6 + 6(2\sqrt{3}) + \sqrt{3}(1) + \sqrt{3}(2\sqrt{3})$
$= 6 + 12\sqrt{3} + \sqrt{3} + 6$
$= 12 + 13\sqrt{3}$

EXERCISE 4A.2

1 Simplify:

a $2\sqrt{2}+3\sqrt{2}$ **b** $2\sqrt{2}-3\sqrt{2}$ **c** $5\sqrt{5}-3\sqrt{5}$ **d** $5\sqrt{5}+3\sqrt{5}$

e $3\sqrt{5}-5\sqrt{5}$ **f** $7\sqrt{3}+2\sqrt{3}$ **g** $9\sqrt{6}-12\sqrt{6}$ **h** $\sqrt{2}+\sqrt{2}+\sqrt{2}$

2 Simplify:

a $\sqrt{2}(3-\sqrt{2})$ **b** $\sqrt{5}(\sqrt{5}+1)$ **c** $\sqrt{10}(3+2\sqrt{10})$ **d** $\sqrt{7}(3\sqrt{7}-4)$

e $-\sqrt{3}(5+\sqrt{3})$ **f** $2\sqrt{6}(\sqrt{6}-7)$ **g** $-\sqrt{8}(\sqrt{8}-5)$ **h** $-3\sqrt{2}(4-6\sqrt{2})$

3 Simplify:

a $(5+\sqrt{2})(4+\sqrt{2})$ **b** $(7+2\sqrt{3})(4+\sqrt{3})$ **c** $(9-\sqrt{7})(4+2\sqrt{7})$

d $(\sqrt{3}+1)(2-3\sqrt{3})$ **e** $(\sqrt{8}-6)(2\sqrt{8}-3)$ **f** $(2\sqrt{5}-7)(1-4\sqrt{5})$

Example 5 — Self Tutor

Simplify:

a $(5-\sqrt{2})^2$ **b** $(7+2\sqrt{5})(7-2\sqrt{5})$

a
$$\begin{aligned}&(5-\sqrt{2})^2\\ &=5^2+2(5)(-\sqrt{2})+(\sqrt{2})^2\\ &=25-10\sqrt{2}+2\\ &=27-10\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}&(7+2\sqrt{5})(7-2\sqrt{5})\\ &=7^2-(2\sqrt{5})^2\\ &=49-(4\times 5)\\ &=29\end{aligned}$$

4 Simplify:

a $(3+\sqrt{2})^2$ **b** $(6-\sqrt{3})^2$ **c** $(\sqrt{5}+1)^2$ **d** $(\sqrt{8}-3)^2$

e $(4+2\sqrt{3})^2$ **f** $(3\sqrt{5}+1)^2$ **g** $(7-2\sqrt{10})^2$ **h** $(5\sqrt{6}-4)^2$

5 Simplify:

a $(3+\sqrt{7})(3-\sqrt{7})$ **b** $(\sqrt{2}+5)(\sqrt{2}-5)$ **c** $(4-\sqrt{3})(4+\sqrt{3})$

d $(2\sqrt{2}+1)(2\sqrt{2}-1)$ **e** $(4+3\sqrt{8})(4-3\sqrt{8})$ **f** $(9\sqrt{3}-5)(9\sqrt{3}+5)$

DIVISION BY SURDS

Numbers like $\frac{6}{\sqrt{2}}$ and $\frac{9}{5+\sqrt{2}}$ involve dividing by a surd.

It is customary to 'simplify' these numbers by rewriting them without the surd in the denominator.

For any fraction of the form $\frac{b}{\sqrt{a}}$, we can remove the surd from the denominator by multiplying by $\frac{\sqrt{a}}{\sqrt{a}}$.

Since $\frac{\sqrt{a}}{\sqrt{a}}=1$, this does not change the value of the fraction.

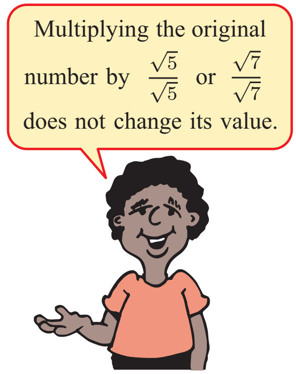
Example 6 — Self Tutor

Write with an integer denominator:

a $\dfrac{6}{\sqrt{5}}$ **b** $\dfrac{35}{\sqrt{7}}$

a
$$\frac{6}{\sqrt{5}}$$
$$= \frac{6}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{6\sqrt{5}}{5}$$

b
$$\frac{35}{\sqrt{7}}$$
$$= \frac{35}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$$
$$= \frac{\cancelto{5}{35}\sqrt{7}}{\cancelto{1}{7}}$$
$$= 5\sqrt{7}$$



For any fraction of the form $\dfrac{c}{a+\sqrt{b}}$, we can remove the surd from the denominator by multiplying by $\dfrac{a-\sqrt{b}}{a-\sqrt{b}}$.

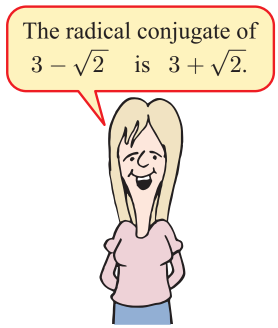
Expressions such as $a+\sqrt{b}$ and $a-\sqrt{b}$ are known as **radical conjugates**. They are identical except for the sign in the middle.

The product of radical conjugates is rational, since we have the difference between two squares. Multiplying by $\dfrac{a-\sqrt{b}}{a-\sqrt{b}}$ therefore produces a rational denominator, so it is sometimes called **rationalising the denominator**.

Example 7 — Self Tutor

Write $\dfrac{5}{3-\sqrt{2}}$ with an integer denominator.

$$\frac{5}{3-\sqrt{2}} = \left(\frac{5}{3-\sqrt{2}}\right)\left(\frac{3+\sqrt{2}}{3+\sqrt{2}}\right)$$
$$= \frac{5(3+\sqrt{2})}{3^2-(\sqrt{2})^2}$$
$$= \frac{15+5\sqrt{2}}{7}$$



EXERCISE 4A.3

1 Write with integer denominator:

a $\dfrac{1}{\sqrt{3}}$ **b** $\dfrac{3}{\sqrt{3}}$ **c** $\dfrac{9}{\sqrt{3}}$ **d** $\dfrac{11}{\sqrt{3}}$ **e** $\dfrac{\sqrt{2}}{3\sqrt{3}}$

f $\dfrac{2}{\sqrt{2}}$ **g** $\dfrac{6}{\sqrt{2}}$ **h** $\dfrac{12}{\sqrt{2}}$ **i** $\dfrac{\sqrt{3}}{\sqrt{2}}$ **j** $\dfrac{1}{4\sqrt{2}}$

2 Write with integer denominator:

a $\frac{5}{\sqrt{5}}$ **b** $\frac{15}{\sqrt{5}}$ **c** $\frac{-3}{\sqrt{5}}$ **d** $\frac{200}{\sqrt{5}}$ **e** $\frac{1}{3\sqrt{5}}$

f $\frac{7}{\sqrt{7}}$ **g** $\frac{21}{\sqrt{7}}$ **h** $\frac{2}{\sqrt{11}}$ **i** $\frac{26}{\sqrt{13}}$ **j** $\frac{1}{(\sqrt{3})^3}$

3 Rationalise the denominator:

a $\frac{1}{3+\sqrt{2}}$ **b** $\frac{2}{3-\sqrt{2}}$ **c** $\frac{1}{2+\sqrt{5}}$ **d** $\frac{\sqrt{2}}{2-\sqrt{2}}$

e $\frac{10}{\sqrt{6}-1}$ **f** $\frac{\sqrt{3}}{\sqrt{7}+2}$ **g** $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ **h** $\frac{\sqrt{3}}{4-\sqrt{3}}$

i $\frac{-2\sqrt{2}}{1-\sqrt{2}}$ **j** $\frac{1+\sqrt{5}}{2-\sqrt{5}}$ **k** $\frac{\sqrt{3}+2}{\sqrt{3}-1}$ **l** $\frac{\sqrt{10}-7}{\sqrt{10}+4}$

Example 8 Self Tutor

Write $\frac{1}{5+\sqrt{2}}$ in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Q}$.

$$\begin{aligned}\frac{1}{5+\sqrt{2}} &= \left(\frac{1}{5+\sqrt{2}}\right) \times \left(\frac{5-\sqrt{2}}{5-\sqrt{2}}\right) \\ &= \frac{5-\sqrt{2}}{25-2} \\ &= \frac{5-\sqrt{2}}{23} \\ &= \tfrac{5}{23} - \tfrac{1}{23}\sqrt{2}\end{aligned}$$

4 Write in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Q}$:

a $\frac{3}{\sqrt{2}-3}$ **b** $\frac{4}{2+\sqrt{2}}$ **c** $\frac{\sqrt{2}}{\sqrt{2}-5}$ **d** $\frac{-2\sqrt{2}}{\sqrt{2}+1}$

5 Write in the form $a+b\sqrt{3}$ where $a, b \in \mathbb{Q}$:

a $\frac{4}{1-\sqrt{3}}$ **b** $\frac{6}{\sqrt{3}+2}$ **c** $\frac{\sqrt{3}}{2-\sqrt{3}}$ **d** $\frac{1+2\sqrt{3}}{3+\sqrt{3}}$

6 **a** Suppose a, b, and c are integers, $c > 0$. Show that $(a+b\sqrt{c})(a-b\sqrt{c})$ is also an integer.

b Write with an integer denominator:

i $\frac{1}{1+2\sqrt{3}}$ **ii** $\frac{\sqrt{2}}{3\sqrt{2}-5}$ **iii** $\frac{\sqrt{2}-1}{3-2\sqrt{2}}$

7 **a** Suppose a and b are positive integers. Show that $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$ is also an integer.

b Write with an integer denominator:

i $\frac{1}{\sqrt{2}+\sqrt{3}}$ **ii** $\frac{\sqrt{3}}{\sqrt{3}-\sqrt{5}}$ **iii** $\frac{\sqrt{11}-\sqrt{14}}{\sqrt{11}+\sqrt{14}}$

8 Solve the equation $2x - 3\sqrt{3} = 1 - x\sqrt{3}$. Give your solution in the form $x = a + b\sqrt{3}$, where a and b are integers.

9 Find the positive solution of the equation $(9+\sqrt{5})x^2 + (5-2\sqrt{5})x - 5 = 0$. Give your answer in the form $a+b\sqrt{5}$, where $a, b \in \mathbb{Q}$.

B INDICES

If n is a positive integer, then a^n is the product of n factors of a.

$$a^n = \underbrace{a \times a \times a \times a \times \ ...... \ \times a}_{n \text{ factors}}$$

We say that a is the **base**, and n is the **index** or **exponent**.

base → 3^5 ← power, index or exponent

NEGATIVE BASES

$(-1)^1 = -1$	$(-2)^1 = -2$
$(-1)^2 = -1 \times -1 = 1$	$(-2)^2 = -2 \times -2 = 4$
$(-1)^3 = -1 \times -1 \times -1 = -1$	$(-2)^3 = -2 \times -2 \times -2 = -8$
$(-1)^4 = -1 \times -1 \times -1 \times -1 = 1$	$(-2)^4 = -2 \times -2 \times -2 \times -2 = 16$

From the patterns above we can see that:

A **negative** base raised to an **odd** index is **negative**.
A **negative** base raised to an **even** index is **positive**.

EXERCISE 4B

1 List the first six powers of:

a 2 **b** 3 **c** 4

2 Copy and complete the values of these common powers:

a $5^1 = ...$, $5^2 = ...$, $5^3 = ...$, $5^4 = ...$

b $6^1 = ...$, $6^2 = ...$, $6^3 = ...$, $6^4 = ...$

c $7^1 = ...$, $7^2 = ...$, $7^3 = ...$, $7^4 = ...$

3 Simplify, then use a calculator to check your answer:

a $(-1)^5$ **b** $(-1)^6$ **c** $(-1)^{14}$ **d** $(-1)^{19}$ **e** $(-1)^8$ **f** -1^8

g $-(-1)^8$ **h** $(-2)^5$ **i** -2^5 **j** $-(-2)^6$ **k** $(-5)^4$ **l** $-(-5)^4$

4 Use your calculator to find the value of:

a 4^7 **b** 7^4 **c** -5^5 **d** $(-5)^5$ **e** 8^6 **f** $(-8)^6$

g -8^6 **h** 2.13^9 **i** -2.13^9 **j** $(-2.13)^9$

5 Use your calculator to find the values of:

a 9^{-1} **b** $\frac{1}{9^1}$ **c** 6^{-2} **d** $\frac{1}{6^2}$ **e** 3^{-4} **f** $\frac{1}{3^4}$

g 17^0 **h** $(0.366)^0$

What do you notice?

6 Consider 3^1, 3^2, 3^3, 3^4, 3^5 .... Look for a pattern and hence find the last digit of 3^{101}.

7 What is the last digit of 7^{217}?

Historical note

Nicomachus discovered an interesting number pattern involving cubes and sums of odd numbers.

Nicomachus was born in Roman Syria (now Jerash, Jordan) around 100 AD. He wrote in Greek, and was a Pythagorean, which means he followed the teaching of **Pythagoras**.

$$\begin{aligned} 1 &= 1^3 \\ 3+5 = 8 &= 2^3 \\ 7+9+11 = 27 &= 3^3 \\ &\vdots \end{aligned}$$

C INDEX LAWS

The **index laws** for $m, n \in \mathbb{Z}$ are:

$a^m \times a^n = a^{m+n}$	To **multiply** numbers with the **same base**, keep the base and **add** the indices.
$\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$	To **divide** numbers with the same base, keep the base and **subtract** the indices.
$(a^m)^n = a^{m\times n}$	When **raising** a **power** to a **power**, keep the base and **multiply** the indices.
$(ab)^n = a^n b^n$	The power of a product is the product of the powers.
$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0$	The power of a quotient is the quotient of the powers.
$a^0 = 1, \quad a \neq 0$	Any non-zero number raised to the power of zero is 1.

$a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$ and in particular $a^{-1} = \dfrac{1}{a}, \quad a \neq 0.$

Example 9 — Self Tutor

Simplify using the index laws:

a $3^5 \times 3^4$ **b** $\dfrac{5^3}{5^5}$ **c** $(m^4)^3$

a $3^5 \times 3^4$
$= 3^{5+4}$
$= 3^9$

b $\dfrac{5^3}{5^5}$
$= 5^{3-5}$
$= 5^{-2}$
$= \frac{1}{25}$

c $(m^4)^3$
$= m^{4\times 3}$
$= m^{12}$

EXERCISE 4C

1 Simplify using the index laws:

a $5^4 \times 5^7$ **b** $d^2 \times d^6$ **c** $\dfrac{k^8}{k^3}$ **d** $\dfrac{7^5}{7^6}$ **e** $(x^2)^5$ **f** $(3^4)^4$

g $\dfrac{p^3}{p^7}$ **h** $n^3 \times n^9$ **i** $(5^t)^3$ **j** $7^x \times 7^2$ **k** $\dfrac{10^3}{10^q}$ **l** $(c^4)^m$