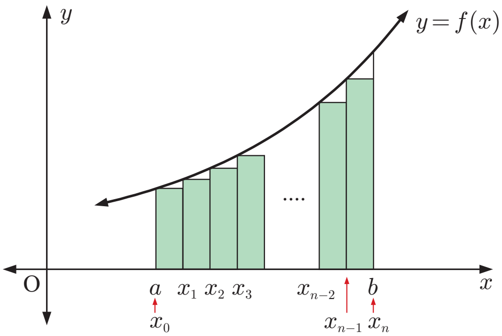
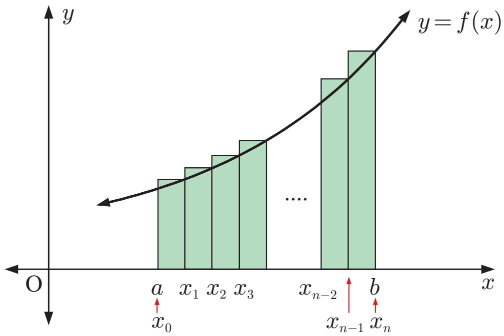
THE DEFINITE INTEGRAL

Consider the lower and upper rectangle sums for a function which is positive and increasing on the interval $a \leqslant x \leqslant b$.

We divide the interval into n subintervals, each of width $w = \dfrac{b-a}{n}$.

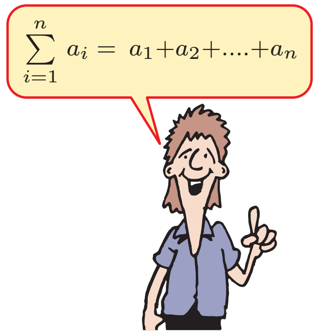




Since the function is increasing:

$$A_L = w\,f(x_0) + w\,f(x_1) + ... + w\,f(x_{n-2}) + w\,f(x_{n-1}) = w\sum_{i=0}^{n-1} f(x_i)$$

$$A_U = w\,f(x_1) + w\,f(x_2) + ... + w\,f(x_{n-1}) + w\,f(x_n) = w\sum_{i=1}^{n} f(x_i)$$

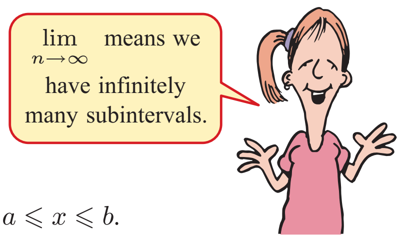


Notice that $\quad A_U - A_L = w\,(f(x_n) - f(x_0))$

$$= \frac{1}{n}(b-a)\,(f(b) - f(a))$$

$\therefore \quad \lim_{n\to\infty} (A_U - A_L) = 0 \qquad \{\text{since } \lim_{n\to\infty} \frac{1}{n} = 0\}$

$\therefore \quad \lim_{n\to\infty} A_L = \lim_{n\to\infty} A_U \quad \{\text{when both limits exist}\}$



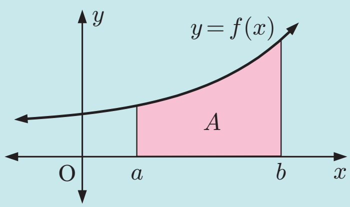
$\therefore$ since $A_L < A < A_U$ for all values of n, it follows that
$\lim_{n\to\infty} A_L = A = \lim_{n\to\infty} A_U$

This fact is true for all positive continuous functions on an interval $a \leqslant x \leqslant b$.

The value A is known as the "**definite integral** of $f(x)$ from a to b", written $A = \displaystyle\int_a^b f(x)\,dx$.

If $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$, then

$\displaystyle\int_a^b f(x)\,dx$ is equal to the shaded area.



Historical note

The word **integration** means "*to put together into a whole*". An **integral** is the "whole" produced from integration, since the areas $f(x_i) \times w$ of the thin rectangular strips are put together into one whole area.

The symbol $\int$ is called an **integral sign**. In the time of **Newton** and **Leibniz** it was the stretched out letter s, but it is no longer part of the alphabet.

Example 1

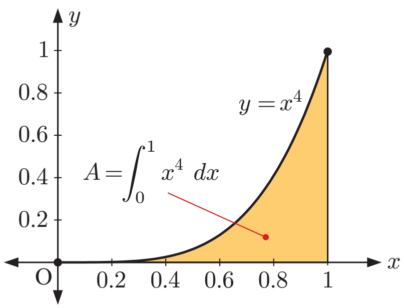
Self Tutor

a Sketch the graph of $y = x^4$ for $0 \leqslant x \leqslant 1$. Shade the area described by $\int_0^1 x^4\, dx$.

b Use technology to calculate the lower and upper rectangle sums for n equal subintervals where $n = 5$, 10, 50, 100, and 500.

c Hence find $\int_0^1 x^4\, dx$ to 2 significant figures.

a



b

n	A_L	A_U
5	0.1133	0.3133
10	0.1533	0.2533
50	0.1901	0.2101
100	0.1950	0.2050
500	0.1990	0.2010

c When $n = 500$, $A_L \approx A_U \approx 0.20$, to 2 significant figures.

$\therefore$ since $A_L < \int_0^1 x^4\, dx < A_U$, $\int_0^1 x^4\, dx \approx 0.20$

EXERCISE 15A.2

1 **a** Sketch the graph of $y = \sqrt{x}$ for $0 \leqslant x \leqslant 1$.

Shade the area described by $\int_0^1 \sqrt{x}\, dx$.

b Find the lower and upper rectangle sums for $n = 5$, 10, 50, 100, and 500.

c Hence find $\int_0^1 \sqrt{x}\, dx$ to 2 significant figures.

AREA FINDER

2 Consider the region enclosed by $y = \sqrt{1 + x^3}$ and the x-axis for $0 \leqslant x \leqslant 2$.

a Write expressions for the lower and upper rectangle sums using n subintervals, $n \in \mathbb{N}$.

b Find the lower and upper rectangle sums for $n = 50$, 100, and 500.

c Hence estimate $\int_0^2 \sqrt{1 + x^3}\, dx$.

GRAPHING PACKAGE

3 The integral $\int_{-3}^{3} e^{-\frac{x^2}{2}}\,dx$ is of considerable interest to statisticians.

a Use the graphing package to help sketch $y = e^{-\frac{x^2}{2}}$ for $-3 \leqslant x \leqslant 3$.

b Calculate the upper and lower rectangular sums for the interval $0 \leqslant x \leqslant 3$ using $n = 2250$.

c Use the symmetry of $y = e^{-\frac{x^2}{2}}$ to find upper and lower rectangular sums for $-3 \leqslant x \leqslant 0$ for $n = 2250$.

d Hence estimate $\int_{-3}^{3} e^{-\frac{x^2}{2}}\,dx$.

How accurate is your estimate compared with $\sqrt{2\pi}$?

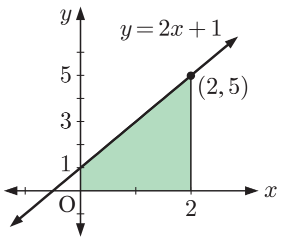
AREA FINDER

Example 2

Self Tutor

Use graphical evidence and known area facts to find: **a** $\int_0^2 (2x+1)\,dx$ **b** $\int_0^1 \sqrt{1-x^2}\,dx$

a



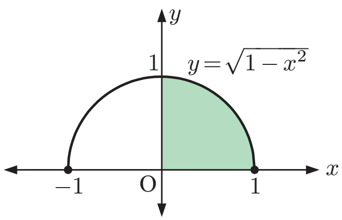
$$\int_0^2 (2x+1)\,dx$$

$=$ shaded area

$= \left(\frac{1+5}{2}\right) \times 2$

$= 6$

b If $y = \sqrt{1-x^2}$ then $y^2 = 1 - x^2$ and so $x^2 + y^2 = 1$ which is the equation of the unit circle. $y = \sqrt{1-x^2}$ is the upper half.



$$\int_0^1 \sqrt{1-x^2}\,dx$$

$=$ shaded area

$= \frac{1}{4}(\pi r^2)$ where $r = 1$

$= \frac{\pi}{4}$

4 Use graphical evidence and known area facts to find:

a $\int_1^3 (1+4x)\,dx$ **b** $\int_{-1}^{2} (2-x)\,dx$ **c** $\int_{-2}^{2} \sqrt{4-x^2}\,dx$

B ANTIDIFFERENTIATION

In many problems in calculus we know the rate of change of one variable with respect to another, but we do not have a formula which relates the variables. In other words, we know $\frac{dy}{dx}$, but we need to know y in terms of x.

The process of finding y from $\dfrac{dy}{dx}$, or $f(x)$ from $f'(x)$, is the reverse process of differentiation. We call it **antidifferentiation**.

$$\begin{tikzcd} y \text{ or } f(x) \arrow[r, bend left, "\text{differentiation}"] & \dfrac{dy}{dx} \text{ or } f'(x) \arrow[l, bend left, "\text{antidifferentiation}"] \end{tikzcd}$$

Consider $\dfrac{dy}{dx} = x^2$.

From our work on differentiation, we know that when we differentiate power functions the index reduces by 1. We hence know that y must involve x^3.

Now if $y = x^3$ then $\dfrac{dy}{dx} = 3x^2$, so if we start with $y = \frac{1}{3}x^3$ then $\dfrac{dy}{dx} = x^2$.

However, for all of the cases $y = \frac{1}{3}x^3 + 2$, $y = \frac{1}{3}x^3 + 100$, and $y = \frac{1}{3}x^3 - 7$, we find that $\dfrac{dy}{dx} = x^2$.

In fact, there are infinitely many functions of the form $y = \frac{1}{3}x^3 + c$ where c is an arbitrary constant, which will give $\dfrac{dy}{dx} = x^2$. Ignoring the arbitrary constant, we say that $\frac{1}{3}x^3$ is the **antiderivative** of x^2. It is the simplest function which, when differentiated, gives x^2.

If $F(x)$ is a function where $F'(x) = f(x)$ we say that:

- the **derivative** of $F(x)$ is $f(x)$ and
- the **antiderivative** of $f(x)$ is $F(x)$.

Example 3 — Self Tutor

Find the antiderivative of: **a** x^3 **b** e^{2x} **c** $\dfrac{1}{\sqrt{x}}$

a $\dfrac{d}{dx}\left(x^4\right) = 4x^3$

$\therefore \dfrac{d}{dx}\left(\frac{1}{4}x^4\right) = x^3$

$\therefore$ the antiderivative of x^3 is $\frac{1}{4}x^4$.

b $\dfrac{d}{dx}\left(e^{2x}\right) = e^{2x} \times 2$

$\therefore \dfrac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = \frac{1}{2} \times e^{2x} \times 2 = e^{2x}$

$\therefore$ the antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$.

c $\dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

Now $\dfrac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore \dfrac{d}{dx}(2x^{\frac{1}{2}}) = 2(\frac{1}{2})x^{-\frac{1}{2}} = x^{-\frac{1}{2}}$

$\therefore$ the antiderivative of $\dfrac{1}{\sqrt{x}}$ is $2\sqrt{x}$.

EXERCISE 15B

1 **a** Find the antiderivative of:

i x **ii** x^2 **iii** x^5 **iv** x^{-2} **v** x^{-4} **vi** $x^{\frac{1}{3}}$ **vii** $x^{-\frac{1}{2}}$

b Predict a general rule for the antiderivative of x^n, for $n \neq -1$.

2 **a** Find the antiderivative of:

i e^{2x} **ii** e^{5x} **iii** $e^{\frac{1}{2}x}$ **iv** $e^{0.01x}$ **v** $e^{\pi x}$ **vi** $e^{\frac{x}{3}}$

b Predict a general rule for the antiderivative of e^{kx} where k is a constant, $k \neq 0$.

3 Find the antiderivative of:

a $6x^2 + 4x$ by first differentiating $x^3 + x^2$

b e^{3x+1} by first differentiating e^{3x+1}

c $\sqrt{x}$ by first differentiating $x\sqrt{x}$

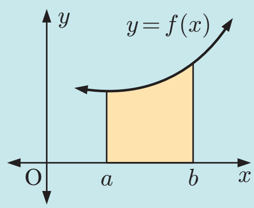
d $(2x+1)^3$ by first differentiating $(2x+1)^4$.

C THE FUNDAMENTAL THEOREM OF CALCULUS

Sir Isaac Newton and **Gottfried Wilhelm Leibniz** showed the link between differential calculus and the definite integral or limit of an area sum we saw in **Section A**. This link is called the **fundamental theorem of calculus**. The beauty of this theorem is that it enables us to evaluate complicated summations.

We have already observed that:

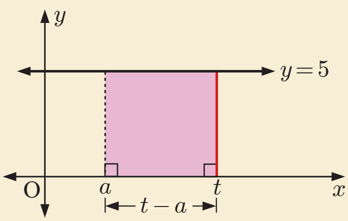
If $f(x)$ is a continuous positive function on an interval $a \leqslant x \leqslant b$ then the area under the curve between $x = a$ and $x = b$ is $\displaystyle\int_a^b \boldsymbol{f(x)\, dx}$.



Discovery — The area function

Consider the constant function $f(x) = 5$.

We wish to find an **area function** which will give the area under the function between $x = a$ and some other value of x which we will call t.



The area function is $A(t) = \displaystyle\int_a^t 5\, dx$

$= \text{shaded area in graph}$

$= (t - a)5$

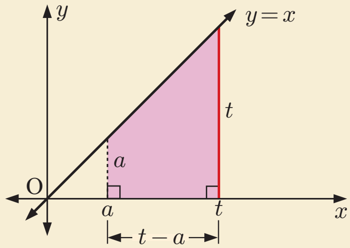
$= 5t - 5a$

$\therefore$ we can write $A(t)$ in the form $F(t) - F(a)$ where $F(t) = 5t$ or equivalently $F(x) = 5x$

What to do:

1 What is the derivative $F'(x)$ of the function $F(x)=5x$? How does this relate to the function $f(x)$?

2 Consider the simplest linear function $f(x)=x$.
The corresponding area function is

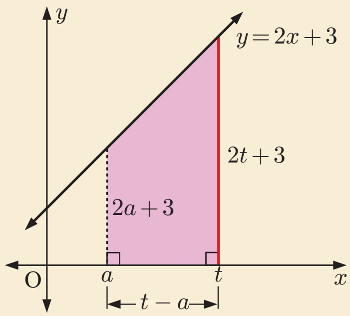


$$\begin{aligned} A(t) &= \int_a^t x\,dx \\ &= \text{shaded area in graph} \\ &= \left(\frac{t+a}{2}\right)(t-a) \end{aligned}$$

a Write $A(t)$ in the form $F(t)-F(a)$.

b What is the derivative $F'(x)$? How does it relate to the function $f(x)$?

3 Consider $f(x)=2x+3$. The corresponding area function is



$$\begin{aligned} A(t) &= \int_a^t (2x+3)\,dx \\ &= \text{shaded area in graph} \\ &= \left(\frac{2t+3+2a+3}{2}\right)(t-a) \end{aligned}$$

a Write $A(t)$ in the form $F(t)-F(a)$.

b What is the derivative $F'(x)$?
How does it relate to the function $f(x)$?

4 Repeat the procedure in **2** and **3** to find area functions for:

a $f(x)=\frac{1}{2}x+3$ **b** $f(x)=5-2x$

Do your results fit with your earlier observations?

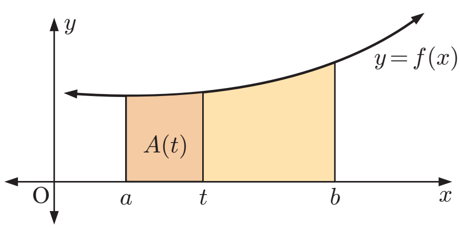
5 If $f(x)=3x^2+4x+5$, predict what $F(x)$ would be without performing the algebraic procedure.

From the **Discovery** you should have found that, for $f(x)\geqslant 0$,

$$\int_a^t f(x)\,dx = F(t)-F(a) \quad \text{where} \quad F'(x)=f(x).$$ $F(x)$ is the **antiderivative** of $f(x)$.

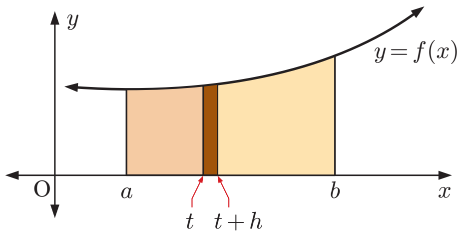
The following argument shows why this is true for all functions $f(x)\geqslant 0$.

Consider a function $y=f(x)$ which has antiderivative $F(x)$ and an area function $A(t)=\int_a^t f(x)\,dx$ which is the area from $x=a$ to $x=t$.



$A(t)$ is clearly an increasing function and
$A(a)=0$ .... (1)

Consider the narrow strip between $x = t$ and $x = t + h$. The area of this strip is $A(t+h) - A(t)$, but we also know it must lie between a lower and upper rectangle on the interval $t \leqslant x \leqslant t + h$ of width h.



area of smaller rectangle $\leqslant A(t+h) - A(t) \leqslant$ area of larger rectangle

If $f(x)$ is increasing on this interval then

$$hf(t) \leqslant A(t+h) - A(t) \leqslant hf(t+h)$$

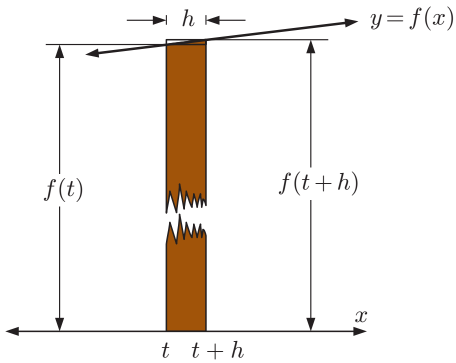
$$\therefore \quad f(t) \leqslant \frac{A(t+h) - A(t)}{h} \leqslant f(t+h)$$

Equivalently, if $f(x)$ is decreasing on this interval then $f(t+h) \leqslant \dfrac{A(t+h) - A(t)}{h} \leqslant f(t)$.

Taking the limit as $h \to 0$ gives $f(t) \leqslant A'(t) \leqslant f(t)$

$$\therefore \quad A'(t) = f(t)$$

So, the area function $A(t)$ must only differ from the antiderivative of $f(t)$ by a constant.



$\therefore \quad A(t) = F(t) + c$

Letting $t = a$, $A(a) = F(a) + c$

But from (1), $A(a) = 0$

$\therefore \quad c = -F(a)$

$\therefore \quad A(t) = F(t) - F(a)$

Letting $t = b$, $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

This result is in fact true for all continuous functions $f(x)$.

THE FUNDAMENTAL THEOREM OF CALCULUS

From the geometric argument above, the Fundamental Theorem of Calculus can be stated in two forms:

For a continuous function $f(x)$, if we define the area function from $x = a$ as $A(t) = \displaystyle\int_a^t f(x)\,dx$, then $A'(x) = f(x)$.

or more commonly:

For a continuous function $f(x)$ with antiderivative $F(x)$, $\displaystyle\int_a^b \boldsymbol{f(x)\,dx = F(b) - F(a)}$.

PROPERTIES OF DEFINITE INTEGRALS

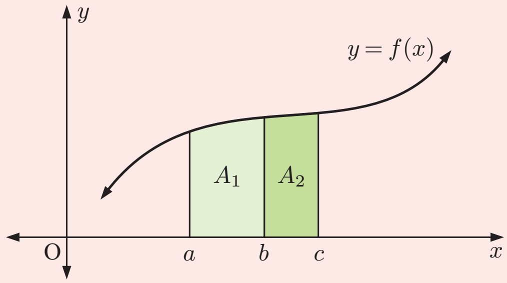
The following properties of definite integrals can all be deduced from the fundamental theorem of calculus:

- $\displaystyle\int_a^a f(x)\,dx = 0$
- $\displaystyle\int_a^b c\,dx = c(b-a)$ $\{c$ is a constant$\}$
- $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\displaystyle\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

- $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\displaystyle\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Example proof:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$
$$= F(b) - F(a) + F(c) - F(b)$$
$$= F(c) - F(a)$$
$$= \int_a^c f(x)\,dx$$



In particular, for the case where $a \leqslant b \leqslant c$ and $f(x) \geqslant 0$ for $a \leqslant x \leqslant c$, we observe that

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = A_1 + A_2 = \int_a^c f(x)\,dx$$

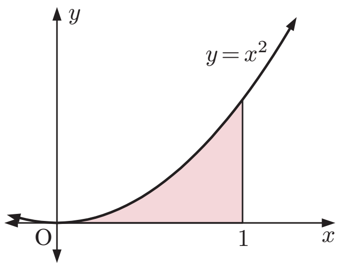
Example 4 — *Self Tutor*

Use the fundamental theorem of calculus to find the area between:

a the x-axis and $y = x^2$ from $x = 0$ to $x = 1$

b the x-axis and $y = \sqrt{x}$ from $x = 1$ to $x = 9$.

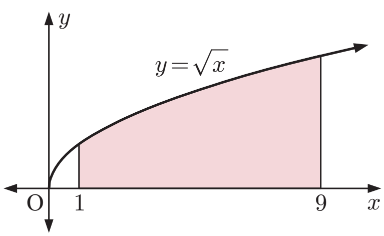
a



$f(x) = x^2$ has antiderivative $F(x) = \dfrac{x^3}{3}$

$$\therefore \text{ the area } = \int_0^1 x^2\,dx$$
$$= F(1) - F(0)$$
$$= \tfrac{1}{3} - 0$$
$$= \tfrac{1}{3} \text{ units}^2$$

b



$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$$F(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \tfrac{2}{3}x\sqrt{x}$$

$$\therefore \text{ the area } = \int_1^9 x^{\frac{1}{2}}\,dx$$
$$= F(9) - F(1)$$
$$= \tfrac{2}{3} \times 27 - \tfrac{2}{3} \times 1$$
$$= 17\tfrac{1}{3} \text{ units}^2$$

EXERCISE 15C

1 Use the fundamental theorem of calculus to find the area between:

a the x-axis and $y = x^3$ from $x = 0$ to $x = 1$

b the x-axis and $y = x^2$ from $x = 1$ to $x = 2$

c the x-axis and $y = \sqrt{x}$ from $x = 0$ to $x = 1$.

2 Use the fundamental theorem of calculus to show that:

a $\int_a^a f(x)\,dx = 0$ and explain the result graphically

b $\int_a^b c\,dx = c(b-a)$ where c is a constant

c $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

d $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$ where c is a constant

e $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3 Use the fundamental theorem of calculus to find the area between the x-axis and:

a $y = x^3$ from $x = 1$ to $x = 2$

b $y = x^2 + 3x + 2$ from $x = 1$ to $x = 3$

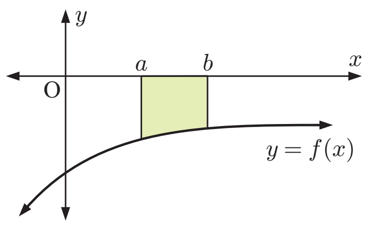
c $y = \sqrt{x}$ from $x = 1$ to $x = 2$

d $y = e^x$ from $x = 0$ to $x = 1.5$

e $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = 4$

4 **a** Use the fundamental theorem of calculus to show that

$$\int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx$$



b Hence show that if $f(x) \leqslant 0$ for all x on $a \leqslant x \leqslant b$ then the shaded area $= -\int_a^b f(x)\,dx$.

c Calculate the following integrals, and give graphical interpretations of your answers:

i $\int_0^1 (-x^2)\,dx$ **ii** $\int_0^1 (x^2 - x)\,dx$ **iii** $\int_{-2}^0 3x\,dx$

d Use graphical evidence and known area facts to find $\int_0^2 \left(-\sqrt{4-x^2}\right)\,dx$.

D INTEGRATION

Earlier, we showed that the **antiderivative** of x^2 is $\frac{1}{3}x^3$, and that any function of the form $\frac{1}{3}x^3 + c$ where c is a constant, has derivative x^2.

We say that the **indefinite integral** or **integral** of x^2 is $\frac{1}{3}x^3 + c$, and write $\int x^2\,dx = \frac{1}{3}x^3 + c$.

We read this as "the integral of x^2 with respect to x is $\frac{1}{3}x^3 + c$, where c is a constant".

$$\text{If} \quad F'(x) = f(x) \quad \text{then} \quad \int f(x)\,dx = F(x) + c.$$

This process is known as **indefinite integration**. It is indefinite because it is not being applied to a particular interval.

DISCOVERING INTEGRALS

Since integration is the reverse process of differentiation we can sometimes discover integrals by differentiation. For example:

- if $F(x) = x^4$ then $F'(x) = 4x^3$
  $$\therefore \int 4x^3\,dx = x^4 + c$$
- if $F(x) = \sqrt{x} = x^{\frac{1}{2}}$ then $F'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$
  $$\therefore \int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x} + c$$

The following rules may prove useful:

- Any constant may be written in front of the integral sign.

$$\int k\,f(x)\,dx = k\int f(x)\,dx, \quad k \text{ is a constant}$$

Proof: Consider differentiating $kF(x)$ where $F'(x) = f(x)$.

$$\frac{d}{dx}(k\,F(x)) = k\,F'(x) = k\,f(x)$$
$$\therefore \int k\,f(x)\,dx = k\,F(x)$$
$$= k\int f(x)\,dx$$

- The integral of a sum is the sum of the separate integrals. This rule enables us to integrate term by term.

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Example 5

Self Tutor

If $y = x^4 + 2x^3$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int (2x^3 + 3x^2)\,dx$.

If $y = x^4 + 2x^3$ then $\dfrac{dy}{dx} = 4x^3 + 6x^2$

$$\therefore \int (4x^3 + 6x^2)\,dx = x^4 + 2x^3 + c$$

$$\therefore \int 2(2x^3 + 3x^2)\,dx = x^4 + 2x^3 + c$$

$$\therefore 2\int (2x^3 + 3x^2)\,dx = x^4 + 2x^3 + c$$

$$\therefore \int (2x^3 + 3x^2)\,dx = \tfrac{1}{2}x^4 + x^3 + c$$

c represents a general constant, so is simply any value $c \in \mathbb{R}$. Instead of writing $\frac{c}{2}$, we can therefore still write just c.

EXERCISE 15D

1 If $y = x^7$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int x^6\,dx$.

2 If $y = x^3 + x^2$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int (3x^2 + 2x)\,dx$.

3 If $y = e^{2x+1}$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int e^{2x+1}\,dx$.

4 If $y = (2x+1)^4$ find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int (2x+1)^3\,dx$.

Example 6

Self Tutor

Suppose $y = \sqrt{5x-1}$.

a Find $\dfrac{dy}{dx}$.

b Hence find $\displaystyle\int \frac{1}{\sqrt{5x-1}}\,dx$.

a
$$y = \sqrt{5x-1}$$
$$= (5x-1)^{\frac{1}{2}}$$
$$\therefore \frac{dy}{dx} = \tfrac{1}{2}(5x-1)^{-\frac{1}{2}}(5)\quad \{\text{chain rule}\}$$
$$= \frac{5}{2\sqrt{5x-1}}$$

b Using **a**, $\displaystyle\int \frac{5}{2\sqrt{5x-1}}\,dx = \sqrt{5x-1} + c$

$$\therefore \frac{5}{2}\int \frac{1}{\sqrt{5x-1}}\,dx = \sqrt{5x-1} + c$$

$$\therefore \int \frac{1}{\sqrt{5x-1}}\,dx = \tfrac{2}{5}\sqrt{5x-1} + c$$

5 If $y = x\sqrt{x}$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int \sqrt{x}\,dx$.

6 If $y = \dfrac{1}{\sqrt{x}}$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int \frac{1}{x\sqrt{x}}\,dx$.

7 If $y = \cos 2x$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int \sin 2x\, dx$.

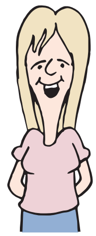
8 If $y = \sin(1-5x)$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int \cos(1-5x)\, dx$.

9 By considering $\dfrac{d}{dx}(x^2-x)^3$, find $\displaystyle\int (2x-1)(x^2-x)^2\, dx$.

10 Prove the rule $\displaystyle\int [f(x)+g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$.

11 Find $\dfrac{dy}{dx}$ if $y = \sqrt{1-4x}$. Hence find $\displaystyle\int \frac{1}{\sqrt{1-4x}}\, dx$.

We can check that an integral is correct by differentiating the answer. It should give us the **integrand**, the function we originally integrated.



E RULES FOR INTEGRATION

In **Chapter 13** we developed a set of rules to help us differentiate functions more efficiently:

Function	*Derivative*	*Name*
c, a constant	0	
$mx+c$, m and c are constants	m	
x^n	nx^{n-1}	**power rule**
$cu(x)$	$cu'(x)$	
$u(x)+v(x)$	$u'(x)+v'(x)$	**addition rule**
$u(x)v(x)$	$u'(x)v(x)+u(x)v'(x)$	**product rule**
$\dfrac{u(x)}{v(x)}$	$\dfrac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}$	**quotient rule**
$y=f(u)$ where $u=u(x)$	$\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dx}$	**chain rule**
e^x	e^x	
$e^{f(x)}$	$e^{f(x)}f'(x)$	
$\ln x$	$\dfrac{1}{x}$	
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$	
$[f(x)]^n$	$n[f(x)]^{n-1}f'(x)$	
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	

These rules or combinations of them can be used to differentiate all of the functions we consider in this course. Given an algebraic formula, we can repeatedly apply these rules until we get to basic functions such as x^n or $\sin x$, which we know how to differentiate.

However, the task of finding **antiderivatives** is not so easy. Given an algebraic formula there is no simple list of rules to find the antiderivative.

The problem was finally solved in 1968 by Robert Henry Risch. He devised a method for deciding if a function has an elementary antiderivative, and if it does, finding it. The original summary of his method took over 100 pages. Later developments from this are now used in all computer algebra systems.

Fortunately, our course is restricted to a few special cases.

RULES FOR INTEGRATION

For k a constant, $\dfrac{d}{dx}(kx+c)=k$ $\quad\therefore\quad$ $\boldsymbol{\int k\,dx = kx + c}$

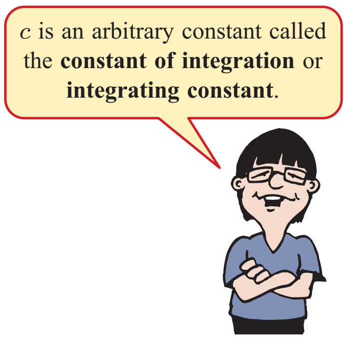
If $n \neq -1$, $\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}+c\right)=\dfrac{(n+1)x^n}{n+1}=x^n$ $\quad\therefore\quad$ $\boldsymbol{\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c,\ \ n \neq -1}$

$\dfrac{d}{dx}(e^x+c)=e^x$ $\quad\therefore\quad$ $\boldsymbol{\int e^x\,dx = e^x + c}$

$\dfrac{d}{dx}(\sin x+c)=\cos x$ $\quad\therefore\quad$ $\boldsymbol{\int \cos x\,dx = \sin x + c}$

$\dfrac{d}{dx}(-\cos x+c)=\sin x$ $\quad\therefore\quad$ $\boldsymbol{\int \sin x\,dx = -\cos x + c}$

Function	*Integral*
k, a constant	$kx+c$
x^n, $n \neq -1$	$\dfrac{x^{n+1}}{n+1}+c$
e^x	e^x+c
$\cos x$	$\sin x+c$
$\sin x$	$-\cos x+c$



Remember that you can always check your integration by differentiating the resulting function.

Example 7

Self Tutor

Find:

a $\int (x^3 - 2x^2 + 5)\, dx$ **b** $\int \left(\frac{1}{x^3} - \sqrt{x}\right) dx$ **c** $\int (2\sin x - \cos x)\, dx$

a
$$\begin{aligned}&\int (x^3 - 2x^2 + 5)\, dx\\ &= \frac{x^4}{4} - \frac{2x^3}{3} + 5x + c\end{aligned}$$

b
$$\begin{aligned}&\int \left(\frac{1}{x^3} - \sqrt{x}\right) dx\\ &= \int (x^{-3} - x^{\frac{1}{2}})\, dx\\ &= \frac{x^{-2}}{-2} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c\\ &= -\frac{1}{2x^2} - \tfrac{2}{3}x^{\frac{3}{2}} + c\end{aligned}$$

c
$$\begin{aligned}&\int (2\sin x - \cos x)\, dx\\ &= 2(-\cos x) - \sin x + c\\ &= -2\cos x - \sin x + c\end{aligned}$$

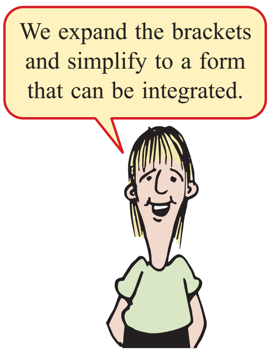
There is no product or quotient rule for integration. Consequently we often have to carry out multiplication or division before we integrate.

Example 8

Self Tutor

Find: **a** $\int \left(3x + \frac{2}{x}\right)^2 dx$ **b** $\int \left(\frac{x^2 - 2}{\sqrt{x}}\right) dx$

a
$$\begin{aligned}&\int \left(3x + \frac{2}{x}\right)^2 dx\\ &= \int \left(9x^2 + 12 + \frac{4}{x^2}\right) dx\\ &= \int (9x^2 + 12 + 4x^{-2})\, dx\\ &= \frac{9x^3}{3} + 12x + \frac{4x^{-1}}{-1} + c\\ &= 3x^3 + 12x - \frac{4}{x} + c\end{aligned}$$



b
$$\begin{aligned}&\int \left(\frac{x^2 - 2}{\sqrt{x}}\right) dx\\ &= \int \left(\frac{x^2}{\sqrt{x}} - \frac{2}{\sqrt{x}}\right) dx\\ &= \int (x^{\frac{3}{2}} - 2x^{-\frac{1}{2}})\, dx\\ &= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c\\ &= \tfrac{2}{5}x^2\sqrt{x} - 4\sqrt{x} + c\end{aligned}$$

EXERCISE 15E.1

1 Find:

a $\int (x^4 - x^2 - x + 2)\, dx$ **b** $\int (5x^4 - 4x^3 - 6x^2 - 7)\, dx$ **c** $\int (\sqrt{x} + e^x)\, dx$

d $\int (3e^x + x^2)\, dx$ **e** $\int (x\sqrt{x} - 2)\, dx$ **f** $\int \left(\frac{1}{x\sqrt{x}} + 4x\right) dx$

g $\int \left(\tfrac{1}{2}x^3 - x^4 + x^{\frac{1}{3}}\right) dx$ **h** $\int \left(\frac{x}{2} + x^2 - e^x\right) dx$ **i** $\int (5e^x + \tfrac{1}{3}x^3 - \sqrt{x})\, dx$

2 Integrate with respect to x:

a $3\sin x - 2$ **b** $4x - 2\cos x$ **c** $\sin x - 2\cos x + e^x$

d $x^2\sqrt{x} - 10\sin x$ **e** $\dfrac{x(x-1)}{3} + \cos x$ **f** $-\sin x + 2\sqrt{x}$

3 Find:

a $\displaystyle\int (x^2 + 3x - 2)\, dx$ **b** $\displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$ **c** $\displaystyle\int \left(2e^x - \frac{1}{x^2}\right) dx$

d $\displaystyle\int \frac{1-4x}{x\sqrt{x}}\, dx$ **e** $\displaystyle\int (2x+1)^2\, dx$ **f** $\displaystyle\int \left(x + \frac{1}{x}\right)^2 dx$

g $\displaystyle\int \frac{2x-1}{\sqrt{x}}\, dx$ **h** $\displaystyle\int \frac{x^2 - 4x + 10}{x^2\sqrt{x}}\, dx$ **i** $\displaystyle\int (x+1)^3\, dx$

4 Find:

a $\displaystyle\int \left(\sqrt{x} + \tfrac{1}{2}\cos x\right) dx$ **b** $\displaystyle\int (2e^t - 4\sin t)\, dt$ **c** $\displaystyle\int (3\cos t - \sin t)\, dt$

5 Find y if:

a $\dfrac{dy}{dx} = 6$ **b** $\dfrac{dy}{dx} = 4x^2$ **c** $\dfrac{dy}{dx} = 5\sqrt{x} - x^2$

d $\dfrac{dy}{dx} = \dfrac{1}{x^2}$ **e** $\dfrac{dy}{dx} = 2e^x - 5$ **f** $\dfrac{dy}{dx} = 4x^3 + 3x^2$

6 Find $f(x)$ if:

a $f'(x) = (1-2x)^2$ **b** $f'(x) = \sqrt{x} - \dfrac{2}{\sqrt{x}}$ **c** $f'(x) = \dfrac{x^2-5}{x^2}$

PARTICULAR VALUES

We can find the constant of integration c if we are given a particular value of the function.

Example 9 — Self Tutor

Find $f(x)$ given that:

a $f'(x) = x^3 - 2x^2 + 3$ and $f(0) = 2$ **b** $f'(x) = 2\sin x - \sqrt{x}$ and $f(0) = 4$.

a Since $f'(x) = x^3 - 2x^2 + 3$,

$$f(x) = \int (x^3 - 2x^2 + 3)\, dx$$

$$\therefore\ f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + c$$

But $f(0) = 2$, so $c = 2$

Thus $f(x) = \dfrac{x^4}{4} - \dfrac{2x^3}{3} + 3x + 2$

b $$f(x) = \int \left(2\sin x - x^{\frac{1}{2}}\right) dx$$

$$\therefore\ f(x) = 2 \times (-\cos x) - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\therefore\ f(x) = -2\cos x - \tfrac{2}{3}x^{\frac{3}{2}} + c$$

But $f(0) = 4$,

so $-2\cos 0 - 0 + c = 4$

$\therefore\ c = 6$

Thus $f(x) = -2\cos x - \tfrac{2}{3}x^{\frac{3}{2}} + 6$.

If we are given the second derivative we need to integrate twice to find the function. This creates two integrating constants, so we need two other facts about the curve in order to determine these constants.

Example 10 — Self Tutor

Find $f(x)$ given that $f''(x) = 12x^2 - 4$, $f'(0) = -1$, and $f(1) = 4$.

If $f''(x) = 12x^2 - 4$

then $f'(x) = \frac{12x^3}{3} - 4x + c$ {integrating with respect to x}

$\therefore$ $f'(x) = 4x^3 - 4x + c$

But $f'(0) = -1$, so $c = -1$

Thus $f'(x) = 4x^3 - 4x - 1$

$\therefore$ $f(x) = \frac{4x^4}{4} - \frac{4x^2}{2} - x + d$ {integrating again}

$\therefore$ $f(x) = x^4 - 2x^2 - x + d$

But $f(1) = 4$, so $1 - 2 - 1 + d = 4$ and hence $d = 6$

Thus $f(x) = x^4 - 2x^2 - x + 6$

EXERCISE 15E.2

1 Find $f(x)$ given that:

a $f'(x) = 2x - 1$ and $f(0) = 3$

b $f'(x) = 3x^2 + 2x$ and $f(2) = 5$

c $f'(x) = e^x + \frac{1}{\sqrt{x}}$ and $f(1) = 1$

d $f'(x) = x - \frac{2}{\sqrt{x}}$ and $f(1) = 2$

2 Find $f(x)$ given that:

a $f'(x) = x^2 - 4\cos x$ and $f(0) = 3$

b $f'(x) = 2\cos x - 3\sin x$ and $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

3 Find $f(x)$ given that:

a $f''(x) = 2x + 1$, $f'(1) = 3$, and $f(2) = 7$

b $f''(x) = 15\sqrt{x} + \frac{3}{\sqrt{x}}$, $f'(1) = 12$, and $f(0) = 5$

c $f''(x) = \cos x$, $f'(\frac{\pi}{2}) = 0$, and $f(0) = 3$

d $f''(x) = 2x$ and the points $(1, 0)$ and $(0, 5)$ lie on the curve.

F INTEGRATING $f(ax + b)$

In this section we deal with integrals of functions which are composite with the linear function $ax + b$.

Notice that $\frac{d}{dx}\left(\frac{1}{a}e^{ax+b}\right) = \frac{1}{a}e^{ax+b} \times a = e^{ax+b}$

$$\therefore \quad \boxed{\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c \quad \text{for } a \neq 0}$$

Likewise if $n \neq -1$,

$$\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1}\right) = \frac{1}{a(n+1)}(n+1)(ax+b)^n \times a,$$
$$= (ax+b)^n$$

$$\therefore \quad \boxed{\int (ax+b)^n \, dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{(n+1)} + c \quad \text{for } n \neq -1}$$

We can perform the same process for the circular functions:

$$\frac{d}{dx}(\sin(ax+b)) = a\cos(ax+b)$$
$$\therefore \quad \int a\cos(ax+b)\,dx = \sin(ax+b)+c$$
$$\therefore \quad a\int \cos(ax+b)\,dx = \sin(ax+b)+c$$

So, $$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c \quad \text{for } a \neq 0.$$

Likewise we can show $$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c \quad \text{for } a \neq 0.$$

For a, b constants with $a \neq 0$, we have:

Function	*Integral*
e^{ax+b}	$\frac{1}{a}e^{ax+b}+c$
$(ax+b)^n, \ n \neq -1$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b)+c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b)+c$

Example 11

Self Tutor

Find: **a** $\int (2x+3)^4\,dx$ **b** $\int \frac{1}{\sqrt{1-2x}}\,dx$

a
$$\int (2x+3)^4\,dx$$
$$= \tfrac{1}{2} \times \frac{(2x+3)^5}{5} + c$$
$$= \tfrac{1}{10}(2x+3)^5 + c$$

b
$$\int \frac{1}{\sqrt{1-2x}}\,dx$$
$$= \int (1-2x)^{-\frac{1}{2}}\,dx$$
$$= \tfrac{1}{-2} \times \frac{(1-2x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= -\sqrt{1-2x} + c$$

Example 12

Self Tutor

Integrate with respect to x:

a $2e^{2x} - e^{-3x}$

b $2\sin(3x) + \cos(4x + \pi)$

a $\int (2e^{2x} - e^{-3x})\,dx$

$= 2(\frac{1}{2})e^{2x} - (\frac{1}{-3})e^{-3x} + c$

$= e^{2x} + \frac{1}{3}e^{-3x} + c$

b $\int (2\sin(3x) + \cos(4x+\pi))\,dx$

$= 2 \times -\frac{1}{3}\cos(3x) + \frac{1}{4}\sin(4x+\pi) + c$

$= -\frac{2}{3}\cos(3x) + \frac{1}{4}\sin(4x+\pi) + c$

EXERCISE 15F

1 Find:

a $\int (2x+5)^3\,dx$ **b** $\int \frac{1}{(3-2x)^2}\,dx$ **c** $\int \frac{4}{(2x-1)^4}\,dx$

d $\int (4x-3)^7\,dx$ **e** $\int \sqrt{3x-4}\,dx$ **f** $\int \frac{10}{\sqrt{1-5x}}\,dx$

g $\int 3(1-x)^4\,dx$ **h** $\int \frac{4}{\sqrt{3-4x}}\,dx$

2 Integrate with respect to x:

a $\sin(3x)$ **b** $2\cos(-4x)+1$ **c** $3\cos\left(\frac{x}{2}\right)$

d $3\sin(2x) - e^{-x}$ **e** $2\sin\left(2x+\frac{\pi}{6}\right)$ **f** $-3\cos\left(\frac{\pi}{4}-x\right)$

g $\cos(2x)+\sin(2x)$ **h** $2\sin(3x)+5\cos(4x)$ **i** $\frac{1}{2}\cos(8x) - 3\sin x$

3 Find $y = f(x)$ given $\frac{dy}{dx} = \sqrt{2x-7}$ and that $y = 11$ when $x = 8$.

4 The function $f(x)$ has gradient function $f'(x) = \frac{4}{\sqrt{1-x}}$, and the curve $y = f(x)$ passes through the point $(-3, -11)$.
Find the point on the graph of $y = f(x)$ with x-coordinate -8.

5 Find:

a $\int 3(2x-1)^2\,dx$ **b** $\int (x^2-x)^2\,dx$ **c** $\int (1-3x)^3\,dx$

d $\int (1-x^2)^2\,dx$ **e** $\int 4\sqrt{5-x}\,dx$ **f** $\int (x^2+1)^3\,dx$

6 Find:

a $\int \left(2e^x + 5e^{2x}\right)\,dx$ **b** $\int \left(3e^{5x-2}\right)\,dx$ **c** $\int \left(e^{7-3x}\right)\,dx$

d $\int (e^x + e^{-x})^2\,dx$ **e** $\int (e^{-x}+2)^2\,dx$ **f** $\int \left(x - \frac{5}{(1-x)^2}\right)\,dx$

7 Find an expression for y given that $\frac{dy}{dx} = (1-e^x)^2$, and that the graph has y-intercept 4.

8 Suppose $f'(x) = p\sin\left(\frac{1}{2}x\right)$ where p is a constant. $f(0) = 1$ and $f(2\pi) = 0$. Find p and hence $f(x)$.

9 Consider a function g such that $g''(x) = -\sin 2x$.
Show that the gradients of the tangents to $y = g(x)$ when $x = \pi$ and $x = -\pi$ are equal.

10 Find $f(x)$ given $f'(x) = 2e^{-2x}$ and $f(0) = 3$.

11 A curve has gradient function $\sqrt{x} + \frac{1}{2}e^{-4x}$ and passes through $(1, 0)$. Find the equation of the function.

G DEFINITE INTEGRALS

Earlier we saw the **fundamental theorem of calculus**:

If $F(x)$ is the antiderivative of $f(x)$ where $f(x)$ is continuous on the interval $a \leqslant x \leqslant b$, then the **definite integral** of $f(x)$ on this interval is $\displaystyle\int_a^b \boldsymbol{f(x)\,dx = F(b) - F(a)}$.

$\displaystyle\int_a^b f(x)\,dx$ reads "the integral from $x = a$ to $x = b$ of $f(x)$ with respect to x"
or "the integral from a to b of $f(x)$ with respect to x".

It is called a **definite** integral because there are lower and upper limits for the integration, and it therefore results in a numerical answer.

When calculating definite integrals we can omit the constant of integration c as this will always cancel out in the subtraction process.

It is common to write $F(b) - F(a)$ as $[F(x)]_a^b$, and so $\displaystyle\int_a^b \boldsymbol{f(x)\,dx = [F(x)]_a^b = F(b) - F(a)}$

Earlier in the chapter we proved the following properties of definite integrals using the fundamental theorem of calculus:

- $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c is any constant
- $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Example 13

Self Tutor

Find:

a $\displaystyle\int_1^3 (x^2+2)\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{3}} \sin x\,dx$

a
$$\int_1^3 (x^2+2)\,dx$$
$$= \left[\frac{x^3}{3}+2x\right]_1^3$$
$$= \left(\tfrac{3^3}{3}+2(3)\right)-\left(\tfrac{1^3}{3}+2(1)\right)$$
$$= (9+6)-(\tfrac{1}{3}+2)$$
$$= 12\tfrac{2}{3}$$

b
$$\int_0^{\frac{\pi}{3}} \sin x\,dx$$
$$= [-\cos x]_0^{\frac{\pi}{3}}$$
$$= (-\cos\tfrac{\pi}{3})-(-\cos 0)$$
$$= -\tfrac{1}{2}+1$$
$$= \tfrac{1}{2}$$

EXERCISE 15G

Use questions **1** to **4** to check the properties of definite integrals.

1 Find: **a** $\displaystyle\int_1^4 \sqrt{x}\,dx$ and $\displaystyle\int_1^4 (-\sqrt{x})\,dx$ **b** $\displaystyle\int_0^1 x^7\,dx$ and $\displaystyle\int_0^1 (-x^7)\,dx$

2 Find: **a** $\displaystyle\int_0^1 x^2\,dx$ **b** $\displaystyle\int_1^2 x^2\,dx$ **c** $\displaystyle\int_0^2 x^2\,dx$ **d** $\displaystyle\int_0^1 3x^2\,dx$

3 Find: **a** $\displaystyle\int_0^2 (x^3-4x)\,dx$ **b** $\displaystyle\int_2^3 (x^3-4x)\,dx$ **c** $\displaystyle\int_0^3 (x^3-4x)\,dx$

4 Find: **a** $\displaystyle\int_0^1 x^2\,dx$ **b** $\displaystyle\int_0^1 \sqrt{x}\,dx$ **c** $\displaystyle\int_0^1 (x^2+\sqrt{x})\,dx$

5 Evaluate:

a $\displaystyle\int_0^1 x^3\,dx$ **b** $\displaystyle\int_0^2 (x^2-x)\,dx$ **c** $\displaystyle\int_0^1 e^x\,dx$

d $\displaystyle\int_0^{\frac{\pi}{6}} \cos x\,dx$ **e** $\displaystyle\int_1^4 \left(x-\frac{3}{\sqrt{x}}\right)dx$ **f** $\displaystyle\int_4^9 \frac{x-3}{\sqrt{x}}\,dx$

g $\displaystyle\int_1^3 \frac{1}{x}\,dx$ **h** $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\,dx$ **i** $\displaystyle\int_1^2 (e^{-x}+1)^2\,dx$

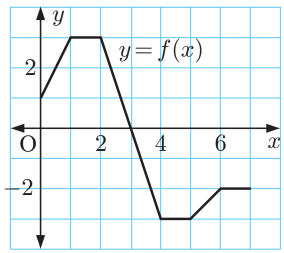
j $\displaystyle\int_2^6 \frac{1}{\sqrt{2x-3}}\,dx$ **k** $\displaystyle\int_0^1 e^{1-x}\,dx$ **l** $\displaystyle\int_0^{\frac{\pi}{6}} \sin(3x)\,dx$

6 Find m such that $\displaystyle\int_m^{2m} (2x-1)\,dx = 4$.

7 **a** Use the identity $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$ to help evaluate $\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$.

b Use the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$ to help evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$.

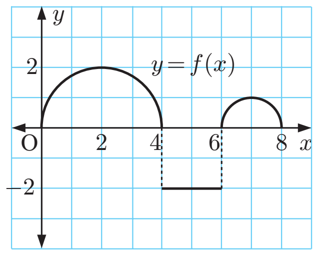
8 Evaluate the following integrals using area interpretation:



a $\int_0^3 f(x)\, dx$ **b** $\int_3^7 f(x)\, dx$

c $\int_2^4 f(x)\, dx$ **d** $\int_0^7 f(x)\, dx$

9 Evaluate the following integrals using area interpretation:



a $\int_0^4 f(x)\, dx$ **b** $\int_4^6 f(x)\, dx$

c $\int_6^8 f(x)\, dx$ **d** $\int_0^8 f(x)\, dx$

10 Write as a single integral:

a $\int_2^4 f(x)\, dx + \int_4^7 f(x)\, dx$ **b** $\int_1^3 g(x)\, dx + \int_3^8 g(x)\, dx + \int_8^9 g(x)\, dx$

11 **a** If $\int_1^3 f(x)\, dx = 2$ and $\int_1^6 f(x)\, dx = -3$, find $\int_3^6 f(x)\, dx$.

b If $\int_0^2 f(x)\, dx = 5$, $\int_4^6 f(x)\, dx = -2$, and $\int_0^6 f(x)\, dx = 7$, find $\int_2^4 f(x)\, dx$.

12 Given that $\int_{-1}^1 f(x)\, dx = -4$, determine the value of:

a $\int_1^{-1} f(x)\, dx$ **b** $\int_{-1}^1 (2 + f(x))\, dx$ **c** $\int_{-1}^1 2f(x)\, dx$

d k such that $\int_{-1}^1 kf(x)\, dx = 7$

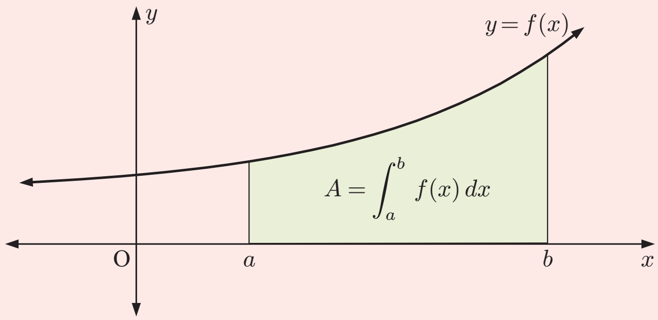
13 If $g(2) = 4$ and $g(3) = 5$, calculate $\int_2^3 (g'(x) - 1)\, dx$.

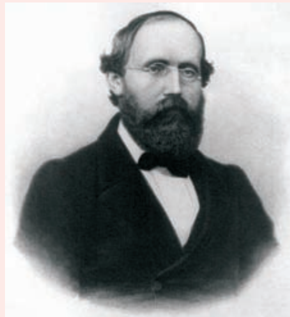
Historical note

Following the work of Newton and Leibniz, integration was rigorously formalised using limits by the German mathematician **Bernhard Riemann** (1826 - 1866).

If $f(x) \geqslant 0$ on the interval $a \leqslant x \leqslant b$, we have seen that the area under the curve is $A = \int_a^b f(x)\, dx$.

This is known as the **Riemann integral**.





Bernhard Riemann

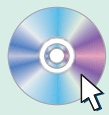
Review set 15A

1 **a** Sketch the region between the curve $y = \dfrac{4}{1+x^2}$ and the x-axis for $0 \leqslant x \leqslant 1$.

Divide the interval into 5 equal parts and display the 5 upper and lower rectangles.

b Use the **area finder** software to find the lower and upper rectangle sums for $n = 5$, 50, 100, and 500.

AREA FINDER

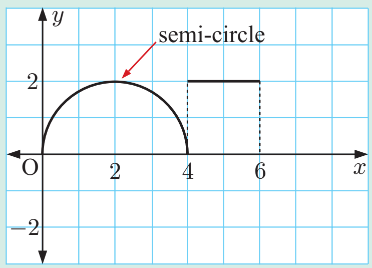


c Give your best estimate for $\displaystyle\int_0^1 \frac{4}{1+x^2}\, dx$ and compare this answer with π.

2 The graph of $y = f(x)$ is illustrated:

Evaluate the following using area interpretation:

a $\displaystyle\int_0^4 f(x)\, dx$ **b** $\displaystyle\int_4^6 f(x)\, dx$



3 Integrate with respect to x:

a $\dfrac{4}{\sqrt{x}}$ **b** $\sin(4x-5)$ **c** e^{4-3x}

4 Find the exact value of:

a $\displaystyle\int_{-5}^{-1} \sqrt{1-3x}\, dx$ **b** $\displaystyle\int_0^{\frac{\pi}{2}} \cos\left(\tfrac{x}{2}\right) dx$

5 By differentiating $y = \sqrt{x^2-4}$, find $\displaystyle\int \frac{x}{\sqrt{x^2-4}}\, dx$.

6 Find the values of b such that $\displaystyle\int_0^b \cos x \, dx = \frac{1}{\sqrt{2}}, \quad 0 < b < \pi$.

7 Find y if:

a $\dfrac{dy}{dx} = (x^2 - 1)^2$ **b** $\dfrac{dy}{dx} = 400 - 20e^{-\frac{x}{2}}$

8 A curve $y = f(x)$ has $f''(x) = 18x + 10$. Find $f(x)$ if $f(0) = -1$ and $f(1) = 13$.

9 If $\displaystyle\int_0^a e^{1-2x} \, dx = \frac{e}{4}$, find a in the form $\ln k$.

10 Suppose $f''(x) = 3x^2 + 2x$ and $f(0) = f(2) = 3$. Find:

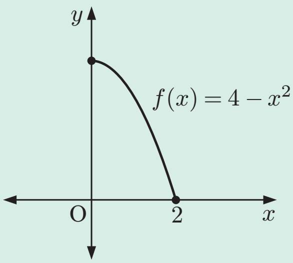
a $f(x)$ **b** the equation of the normal to $y = f(x)$ at $x = 2$.

11 **a** Find $(e^x + 2)^3$ using the binomial expansion.

b Hence find the exact value of $\displaystyle\int_0^1 (e^x + 2)^3 \, dx$.

Review set 15B

1



a Use *four* upper and lower rectangles to find rational numbers A and B such that:

$$A < \int_0^2 (4 - x^2) \, dx < B.$$

b Hence, find a good estimate for

$$\int_0^2 (4 - x^2) \, dx.$$

2 Find:

a $\displaystyle\int (2e^{-x} + 3) \, dx$ **b** $\displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$ **c** $\displaystyle\int \left(3 + e^{2x-1}\right)^2 dx$

3 Given that $f'(x) = x^2 - 3x + 2$ and $f(1) = 3$, find $f(x)$.

4 Find the exact value of $\displaystyle\int_2^3 \frac{1}{\sqrt{3x - 4}} \, dx$.

5 By differentiating $(3x^2 + x)^3$, find $\displaystyle\int (3x^2 + x)^2 (6x + 1) \, dx$.

6 If $\displaystyle\int_1^4 f(x) \, dx = 3$, determine:

a $\displaystyle\int_1^4 (f(x) + 1) \, dx$ **b** $\displaystyle\int_1^2 f(x) \, dx - \int_4^2 f(x) \, dx$

7 Given that $f''(x) = 2\sin(2x)$, $f'(\frac{\pi}{2}) = 0$, and $f(0) = 3$, find the exact value of $f(\frac{\pi}{2})$.

8 Find $\dfrac{d}{dx}(e^{-2x} \sin x)$ and hence find $\displaystyle\int_0^{\frac{\pi}{2}} \left[e^{-2x}(\cos x - 2\sin x)\right] dx$

9 Find $\int (2x+3)^n \, dx$ for all integers $n \neq -1$.

10 A function has gradient function $2\sqrt{x} + \frac{a}{\sqrt{x}}$ and passes through the points $(0, 2)$ and $(1, 4)$. Find a and hence explain why the function $y = f(x)$ has no stationary points.

11 $\int_a^{2a} (x^2 + ax + 2) \, dx = \frac{73a}{2}$. Find a.

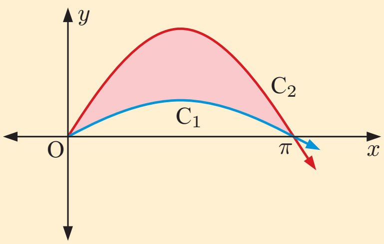
16

Applications of integration

Contents:

Opening problem

The illustrated curves are those of $y = \sin x$ and $y = 3\sin x$.



Things to think about:

a Can you identify each curve?

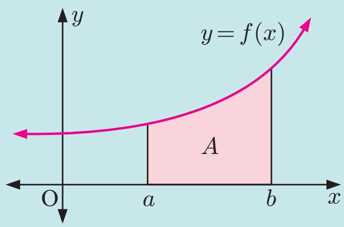
b Can you find the shaded area enclosed by C_1 and C_2 for $0 \leqslant x \leqslant \pi$?

We have already seen how definite integrals can be related to the areas between functions and the x-axis. In this chapter we explore this relationship further, and consider other applications of integral calculus including kinematics.

A THE AREA UNDER A CURVE

We have already established in **Chapter 15** that:

If $f(x)$ is positive and continuous on the interval $a \leqslant x \leqslant b$, then the area bounded by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by $A = \int_a^b f(x)\,dx$ or $\int_a^b y\,dx$.



Example 1

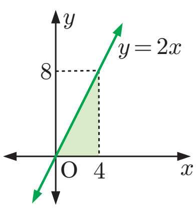
Self Tutor

Find the area of the region enclosed by $y = 2x$, the x-axis, $x = 0$, and $x = 4$ by using:

a a geometric argument

b integration.

a



Area $= \frac{1}{2} \times 4 \times 8$

$= 16$ units^2

b Area $= \int_0^4 2x\,dx$

$= \left[x^2\right]_0^4$

$= 4^2 - 0^2$

$= 16$ units^2

EXERCISE 16A

1 Find the area of each of the regions described below by using:

i a geometric argument

ii integration

a $y = 5$, the x-axis, $x = -6$, and $x = 0$

b $y = x$, the x-axis, $x = 4$, and $x = 5$

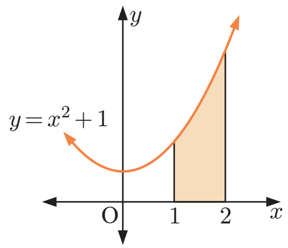
c $y = -3x$, the x-axis, $x = -3$, and $x = 0$

d $y = -x$, the x-axis, $x = 0$, and $x = 2$

Example 2

Self Tutor

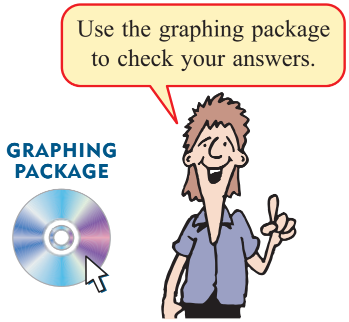
Find the area of the region enclosed by $y = x^2 + 1$, the x-axis, $x = 1$, and $x = 2$.



$$\begin{aligned}\text{Area} &= \int_1^2 (x^2+1)\,dx \\ &= \left[\frac{x^3}{3} + x\right]_1^2 \\ &= \left(\tfrac{8}{3} + 2\right) - \left(\tfrac{1}{3} + 1\right) \\ &= 3\tfrac{1}{3} \text{ units}^2\end{aligned}$$

2 Find the area of the region bounded by:

- **a** $y = x^2$, the x-axis, and $x = 1$
- **b** $y = \sin x$, the x-axis, $x = 0$, and $x = \pi$
- **c** $y = x^3$, the x-axis, $x = 1$, and $x = 4$
- **d** $y = e^x$, the x-axis, the y-axis, and $x = 1$
- **e** the x-axis and the part of $y = 6 + x - x^2$ above the x-axis
- **f** the axes and $y = \sqrt{9 - x}$
- **g** $y = \dfrac{1}{x^2}$, the x-axis, $x = 1$, and $x = 2$
- **h** $y = 2 - \dfrac{1}{\sqrt{x}}$, the x-axis, and $x = 4$
- **i** $y = e^x + e^{-x}$, the x-axis, $x = -1$, and $x = 1$

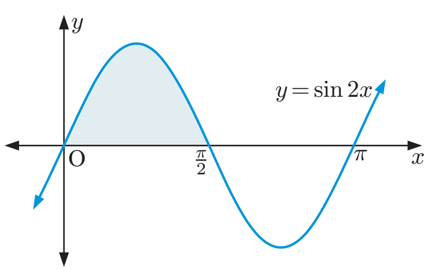


Example 3

Self Tutor

Find the area enclosed by one arch of the curve $y = \sin 2x$ and the x-axis.

The period of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$, so the first positive x-intercept is $\frac{\pi}{2}$.



$$\begin{aligned}\text{The required area} &= \int_0^{\frac{\pi}{2}} \sin 2x\,dx \\ &= \left[\tfrac{1}{2}(-\cos 2x)\right]_0^{\frac{\pi}{2}} \\ &= -\tfrac{1}{2}\left[\cos 2x\right]_0^{\frac{\pi}{2}} \\ &= -\tfrac{1}{2}(\cos\pi - \cos 0) \\ &= 1 \text{ unit}^2\end{aligned}$$

3 Find the area enclosed by one arch of the curve $y = \cos 3x$ and the x-axis.

Discovery — $\int_a^b f(x)\,dx$ and areas

Does $\int_a^b f(x)\,dx$ always give us an area?

What to do:

1 Find $\int_0^1 x^3\,dx$ and $\int_{-1}^1 x^3\,dx$.

2 Using a graph, explain why the first integral in **1** gives an area, whereas the second integral does not.

3 Find $\int_{-1}^0 x^3\,dx$ and explain why the answer is negative.

4 Show that $\int_{-1}^0 x^3\,dx + \int_0^1 x^3\,dx = \int_{-1}^1 x^3\,dx$.

5 Find $\int_0^{-1} x^3\,dx$ and interpret its meaning.

6 Suppose $f(x)$ is a function such that $f(x) \leqslant 0$ for all $a \leqslant x \leqslant b$. Suggest an expression for the area between the curve and the function for $a \leqslant x \leqslant b$.

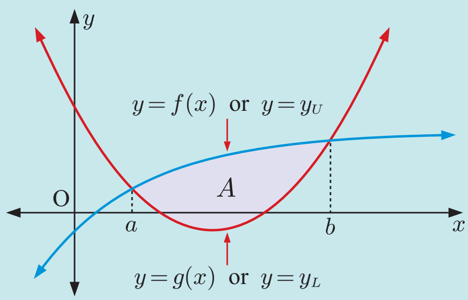
B THE AREA BETWEEN TWO FUNCTIONS

If two functions $f(x)$ and $g(x)$ intersect at $x = a$ and $x = b$, and $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$, then the area of the shaded region between their points of intersection is given by

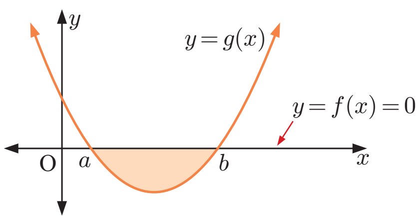
$$A = \int_a^b [f(x) - g(x)]\,dx.$$

Alternatively, if the upper and lower functions are $y = y_U$ and $y = y_L$ respectively, then the area is

$$A = \int_a^b [y_U - y_L]\,dx.$$



We can see immediately that if $f(x)$ is the x-axis $f(x) = 0$, then the enclosed area is $\int_a^b [-g(x)]\,dx$ or $-\int_a^b g(x)\,dx$.



Example 4 Self Tutor

Use $\int_a^b [y_U - y_L]\,dx$ to find the area bounded by the x-axis and $y = x^2 - 2x$.

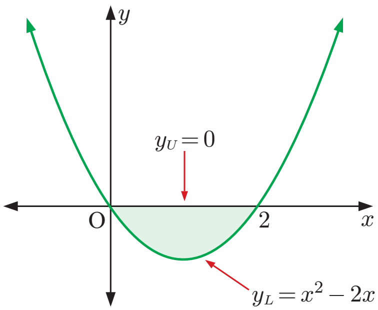
The curve cuts the x-axis when $y = 0$

$\therefore\ x^2 - 2x = 0$

$\therefore\ x(x-2) = 0$

$\therefore\ x = 0$ or 2

$\therefore$ the x-intercepts are 0 and 2.



$$\begin{aligned}\text{Area} &= \int_0^2 [y_U - y_L]\,dx \\ &= \int_0^2 [0 - (x^2 - 2x)]\,dx \\ &= \int_0^2 (2x - x^2)\,dx \\ &= \left[x^2 - \frac{x^3}{3}\right]_0^2 \\ &= \left(4 - \tfrac{8}{3}\right) - (0)\end{aligned}$$

$\therefore$ the area is $\frac{4}{3}$ units^2.

Example 5 Self Tutor

Find the area of the region enclosed by $y = x + 2$ and $y = x^2 + x - 2$.

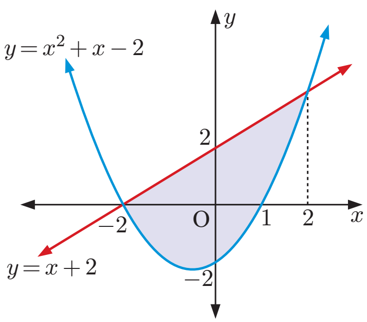
$y = x + 2$ meets $y = x^2 + x - 2$

where $x^2 + x - 2 = x + 2$

$\therefore\ x^2 - 4 = 0$

$\therefore\ (x+2)(x-2) = 0$

$\therefore\ x = \pm 2$



$$\begin{aligned}\text{Area} &= \int_{-2}^2 [y_U - y_L]\,dx \\ &= \int_{-2}^2 [(x+2) - (x^2 + x - 2)]\,dx \\ &= \int_{-2}^2 (4 - x^2)\,dx \\ &= \left[4x - \frac{x^3}{3}\right]_{-2}^2 \\ &= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right) \\ &= 10\tfrac{2}{3} \text{ units}^2\end{aligned}$$

$\therefore$ the area is $10\frac{2}{3}$ units^2.

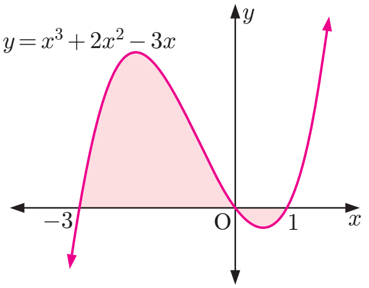
Example 6

Self Tutor

Find the total area of the regions contained by $y = f(x)$ and the x-axis for $f(x) = x^3 + 2x^2 - 3x$.

$f(x) = x^3 + 2x^2 - 3x$
$= x(x^2 + 2x - 3)$
$= x(x-1)(x+3)$

$\therefore\ y = f(x)$ cuts the x-axis at 0, 1, and -3.



Total area

$$= \int_{-3}^{0} (x^3 + 2x^2 - 3x)\,dx - \int_{0}^{1} (x^3 + 2x^2 - 3x)\,dx$$
$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{-3}^{0} - \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{0}^{1}$$
$$= \left(0 - -11\tfrac{1}{4}\right) - \left(-\tfrac{7}{12} - 0\right)$$
$$= 11\tfrac{5}{6} \text{ units}^2$$

EXERCISE 16B

1 Find the exact value of the area bounded by:

- **a** the x-axis and $y = x^2 + x - 2$
- **b** the x-axis, $y = e^{-x} - 1$, and $x = 2$
- **c** the x-axis and the part of $y = 3x^2 - 8x + 4$ below the x-axis
- **d** $y = \cos x$, the x-axis, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$
- **e** $y = x^3 - 4x$, the x-axis, $x = 1$, and $x = 2$
- **f** $y = \sin x - 1$, the x-axis, $x = 0$, and $x = \frac{\pi}{2}$

2 Find the area of the region enclosed by $y = x^2 - 2x$ and $y = 3$.

3 Consider the graphs of $y = x - 3$ and $y = x^2 - 3x$.

- **a** Sketch the graphs on the same set of axes.
- **b** Find the coordinates of the points where the graphs meet.
- **c** Find the area of the region enclosed by the two graphs.

4 Determine the area of the region enclosed by $y = \sqrt{x}$ and $y = x^2$.

5
- **a** On the same set of axes, graph $y = e^x - 1$ and $y = 2 - 2e^{-x}$, showing axes intercepts and asymptotes.
- **b** Find algebraically the points of intersection of $y = e^x - 1$ and $y = 2 - 2e^{-x}$.
- **c** Find the area of the region enclosed by the two curves.

6 Find the area of the region bounded by $y = 2e^x$, $y = e^{2x}$, and $x = 0$.

7 On the same set of axes, sketch $y = 2x$ and $y = 4x^2$.
Find the area of the region enclosed by these functions.

8 Sketch the circle with equation $x^2 + y^2 = 9$.

a Explain why the upper half of the circle has equation $y = \sqrt{9 - x^2}$.

b Hence, determine $\int_0^3 \sqrt{9 - x^2}\, dx$ without actually integrating the function.

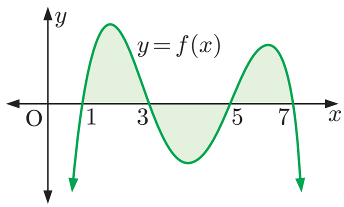
9 Find the area enclosed by the function $y = f(x)$ and the x-axis for:

a $f(x) = x^3 - 9x$ **b** $f(x) = -x(x-2)(x-4)$ **c** $f(x) = x^4 - 5x^2 + 4$.

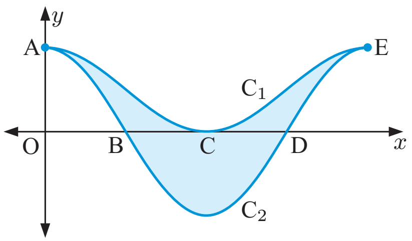
10 Answer the **Opening Problem** on page **438**.

11 **a** Explain why the total area shaded is *not* equal to $\int_1^7 f(x)\, dx$.

b Write an expression for the total shaded area in terms of integrals.



12



The illustrated curves are $y = \cos(2x)$ and $y = \frac{1}{2} + \frac{1}{2}\cos(2x)$.

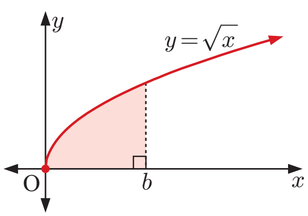
a Identify each curve as C_1 or C_2.

b Determine the coordinates of A, B, C, D, and E.

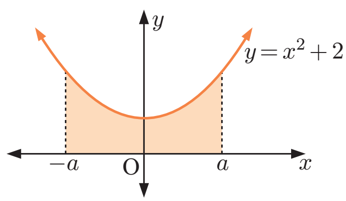
c Show that the area of the shaded region is $\frac{\pi}{2}$ units^2.

13 Explain why the area between two functions $f(x)$ and $g(x)$ on the interval $a \leqslant x \leqslant b$ is given by $A = \int_a^b |f(x) - g(x)|\, dx$.

14 The shaded area is 1 unit^2.
Find b, correct to 4 decimal places.



15 The shaded area is $6a$ units^2.
Find the exact value of a.



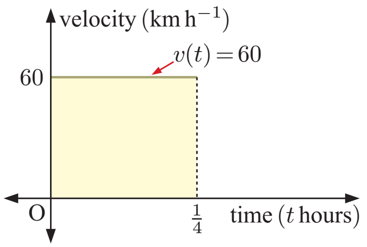
C KINEMATICS

DISTANCES FROM VELOCITY GRAPHS

Suppose a car travels at a constant positive velocity of 60 km h^{-1} for 15 minutes.

We know the distance travelled $=$ speed $\times$ time

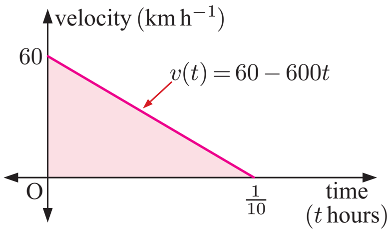
$$= 60 \text{ km h}^{-1} \times \tfrac{1}{4} \text{ h}$$
$$= 15 \text{ km.}$$



When we graph *velocity* against *time*, the graph is a horizontal line, and we can see that the distance travelled is the area shaded.

So, the distance travelled can also be found by the definite integral $\displaystyle\int_0^{\frac{1}{4}} 60\, dt = 15$ km.

Now suppose the velocity decreases at a constant rate, so that the car, initially travelling at 60 km h^{-1}, stops in 6 minutes or $\frac{1}{10}$ hour.



In this case the *average* speed is 30 km h^{-1}, so the distance travelled $= 30 \text{ km h}^{-1} \times \frac{1}{10}$ h

$= 3$ km

But the triangle has area $= \frac{1}{2} \times$ base $\times$ altitude

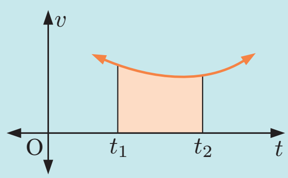
$$= \tfrac{1}{2} \times \tfrac{1}{10} \times 60 = 3$$

So, once again the shaded area gives us the distance travelled, and we can find it using the definite integral $\displaystyle\int_0^{\frac{1}{10}} (60 - 600t)\, dt = 3$.

These results suggest that:

For a velocity-time function $v(t)$ where $v(t) \geqslant 0$ on the interval $t_1 \leqslant t \leqslant t_2$,

$$\textbf{distance travelled} = \int_{t_1}^{t_2} v(t)\, dt.$$

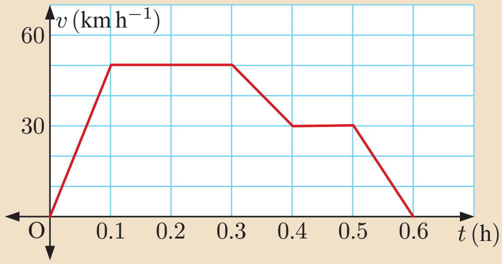


If we have a change of direction within the time interval then the velocity will change sign. We therefore need to add the components of area above and below the t-axis to find the total distance travelled.

Example 7

Self Tutor

The velocity-time graph for a train journey is illustrated in the graph alongside. Find the total distance travelled by the train.



Total distance travelled

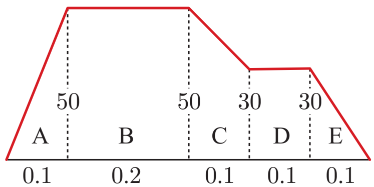
$= $ total area under the graph

$= $ area A + area B + area C + area D + area E

$= \frac{1}{2}(0.1)50 + (0.2)50 + \left(\frac{50+30}{2}\right)(0.1) + (0.1)30 + \frac{1}{2}(0.1)30$

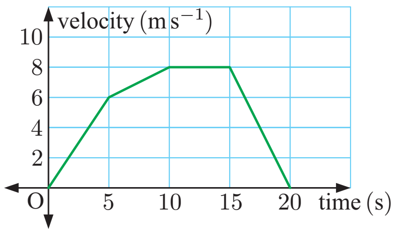
$= 2.5 + 10 + 4 + 3 + 1.5$

$= 21$ km

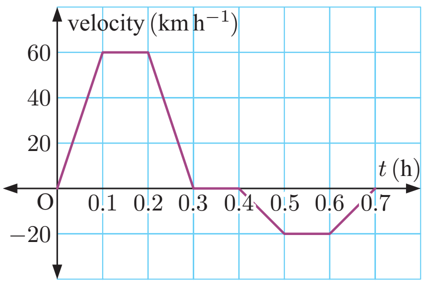


EXERCISE 16C.1

1 A runner has the velocity-time graph shown. Find the total distance travelled by the runner.



2



A car travels along a straight road with the velocity-time function illustrated.

a What is the significance of the graph:

  i above the t-axis

  ii below the t-axis?

b Find the total *distance* travelled by the car.

c Find the final *displacement* of the car from its starting point.

3 A cyclist rides off from rest, accelerating at a constant rate for 3 minutes until she reaches 40 km h^{-1}. She then maintains a constant speed for 4 minutes until reaching a hill. She slows down at a constant rate over one minute to 30 km h^{-1}, then continues at this rate for 10 minutes. At the top of the hill she reduces her speed uniformly and is stationary 2 minutes later.

a Draw a graph to show the cyclist's motion.

b How far has the cyclist travelled?

DISPLACEMENT AND VELOCITY FUNCTIONS

In this section we are concerned with **motion in a straight line**.

For some displacement function $s(t)$, the velocity function is $v(t) = s'(t)$.

So, given a velocity function we can determine the displacement function by the integral

$$\boldsymbol{s(t) = \int v(t)\, dt}$$

The constant of integration determines the **initial position** on the line where the object begins.

Using the displacement function we can determine the change in displacement in a time interval $t_1 \leqslant t \leqslant t_2$ using the integral:

$$\textbf{Displacement} = \boldsymbol{s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\, dt}$$

TOTAL DISTANCE TRAVELLED

To determine the total distance travelled in a time interval $t_1 \leqslant t \leqslant t_2$, we need to account for any changes of direction in the motion.

To find the total distance travelled given a velocity function $v(t) = s'(t)$ on $t_1 \leqslant t \leqslant t_2$:

- Draw a sign diagram for $v(t)$ so we can determine any changes of direction.
- Determine $s(t)$ by integration, including a constant of integration.
- Find $s(t_1)$ and $s(t_2)$. Also find $s(t)$ at each time the direction changes.
- Draw a motion diagram.
- Determine the total distance travelled from the motion diagram.

VELOCITY AND ACCELERATION FUNCTIONS

We know that the acceleration function is the derivative of the velocity function, so $a(t) = v'(t)$.

So, given an acceleration function, we can determine the velocity function by integration:

$$\boldsymbol{v(t) = \int a(t)\, dt}$$

Summary

$$\begin{tikzcd}[column sep=huge]
\begin{array}{c} s(t) \\ \textbf{displacement} \end{array} \arrow[r, bend left, "\text{differentiate}"] & \begin{array}{c} v(t) = \dfrac{ds}{dt} \\ \textbf{velocity} \end{array} \arrow[l, bend left, "\text{integrate}"] \arrow[r, bend left, "\text{differentiate}"] & \begin{array}{c} a(t) = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} \\ \textbf{acceleration} \end{array} \arrow[l, bend left, "\text{integrate}"]
\end{tikzcd}$$

Example 8

Self Tutor

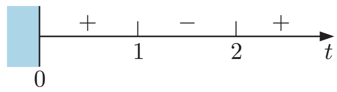
A particle P moves in a straight line with velocity function $v(t) = t^2 - 3t + 2$ m s^{-1}.

a How far does P travel in the first 4 seconds of motion?

b Find the displacement of P after 4 seconds.

a $v(t) = s'(t) = t^2 - 3t + 2$
$\qquad\qquad\qquad = (t-1)(t-2)$

$\therefore$ the sign diagram of v is:



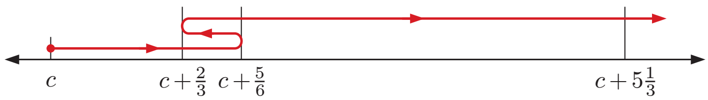
Since the signs change, P reverses direction at $t = 1$ and $t = 2$ seconds.

Now $s(t) = \int (t^2 - 3t + 2)\, dt = \dfrac{t^3}{3} - \dfrac{3t^2}{2} + 2t + c$

Hence $s(0) = c$ $\qquad s(1) = \frac{1}{3} - \frac{3}{2} + 2 + c = c + \frac{5}{6}$

$\qquad s(2) = \frac{8}{3} - 6 + 4 + c = c + \frac{2}{3}$ $\qquad s(4) = \frac{64}{3} - 24 + 8 + c = c + 5\frac{1}{3}$

Motion diagram:



$\therefore$ total distance travelled $= (c + \frac{5}{6} - c) + (c + \frac{5}{6} - [c + \frac{2}{3}]) + (c + 5\frac{1}{3} - [c + \frac{2}{3}])$
$= \frac{5}{6} + \frac{5}{6} - \frac{2}{3} + 5\frac{1}{3} - \frac{2}{3}$
$= 5\frac{2}{3}$ m

b Displacement = final position − original position
$= s(4) - s(0)$
$= c + 5\frac{1}{3} - c$
$= 5\frac{1}{3}$ m

So, the displacement is $5\frac{1}{3}$ m to the right.

EXERCISE 16C.2

1 A particle has velocity function $v(t) = 1 - 2t$ cm s^{-1} as it moves in a straight line. The particle is initially 2 cm to the right of O.

- **a** Write a formula for the displacement function $s(t)$.
- **b** Find the total distance travelled in the first second of motion.
- **c** Find the displacement of the particle at the end of one second.

2 Particle P is initially at the origin O. It moves with the velocity function $v(t) = t^2 - t - 2$ cm s^{-1}.

- **a** Write a formula for the displacement function $s(t)$.
- **b** Find the total distance travelled in the first 3 seconds of motion.
- **c** Find the displacement of the particle at the end of three seconds.

3 An object has velocity function $v(t) = \cos(2t)$ m s^{-1}. If $s(\frac{\pi}{4}) = 1$ m, determine $s(\frac{\pi}{3})$ exactly.

4 The velocity of a moving object is given by $v(t) = 32 + 4t$ m s^{-1}.

a If $s = 16$ m when $t = 0$ seconds, find the displacement function.

b Explain why the displacement of the object and its total distance travelled in the interval $0 \leqslant t \leqslant t_1$, can both be represented by the definite integral $\int_0^{t_1} (32 + 4t)\,dt$.

c Show that the object is travelling with constant acceleration.

5 A particle moves along the x-axis with velocity function $s'(t) = 16t - 4t^3$ units per second. Find the total distance travelled in the time interval:

a $0 \leqslant t \leqslant 3$ seconds **b** $1 \leqslant t \leqslant 3$ seconds.

6 A particle moves in a straight line with velocity function $v(t) = \cos t$ m s^{-1}.

a Show that the particle oscillates between two points.

b Find the distance between the two points in **a**.

7 The velocity of a particle travelling in a straight line is given by $v(t) = 50 - 10e^{-0.5t}$ m s^{-1}, where $t \geqslant 0$, t in seconds.

a State the initial velocity of the particle.

b Find the velocity of the particle after 3 seconds.

c How long will it take for the particle's velocity to increase to 45 m s^{-1}?

d Discuss $v(t)$ as $t \to \infty$.

e Show that the particle's acceleration is always positive.

f Draw the graph of $v(t)$ against t.

g Find the total distance travelled by the particle in the first 3 seconds of motion.

Example 9

Self Tutor

A particle is initially at the origin and moving to the right at 5 cm s^{-1}. It accelerates with time according to $a(t) = 4 - 2t$ cm s^{-2}.

a Find the velocity function of the particle, and sketch its graph for $0 \leqslant t \leqslant 6$ s.

b For the first 6 seconds of motion, determine the:

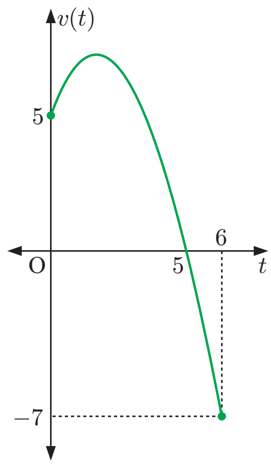
i displacement of the particle **ii** total distance travelled.

a $v(t) = \int a(t)\,dt = \int (4 - 2t)\,dt$

$= 4t - t^2 + c$

But $v(0) = 5$, so $c = 5$

$\therefore$ $v(t) = -t^2 + 4t + 5$ cm s^{-1}



b $s(t) = \int v(t)\,dt = \int (-t^2 + 4t + 5)\,dt$

$= -\frac{1}{3}t^3 + 2t^2 + 5t + c$ cm

But $s(0) = 0$, so $c = 0$

$\therefore$ $s(t) = -\frac{1}{3}t^3 + 2t^2 + 5t$ cm

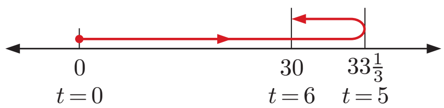
i Displacement $= s(6) - s(0)$

$= -\frac{1}{3}(6)^3 + 2(6)^2 + 5(6)$

$= 30$ cm

ii The particle changes direction when $t = 5$ s.
Now $s(5) = -\frac{1}{3}(5)^3 + 2(5)^2 + 5(5) = 33\frac{1}{3}$ cm

Motion diagram:



$\therefore$ the total distance travelled $= 33\frac{1}{3} + 3\frac{1}{3}$
$= 36\frac{2}{3}$ cm

8 A particle is initially stationary at the origin. It accelerates according to the function $a(t) = \dfrac{2}{(t+1)^3}$ m s^{-2}.

a Find the velocity function $v(t)$ for the particle.
b Find the displacement function $s(t)$ for the particle.
c Describe the motion of the particle at the time $t = 2$ seconds.

9 A train moves along a straight track with acceleration $\dfrac{t}{10} - 3$ m s^{-2}. The initial velocity of the train is 45 m s^{-1}.

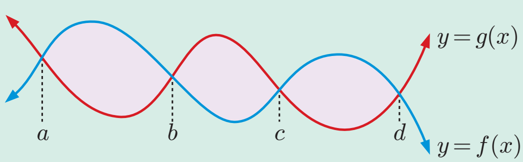
a Determine the velocity function $v(t)$.
b Evaluate $\displaystyle\int_0^{60} v(t)\,dt$ and explain what this value represents.

10 An object has initial velocity 20 m s^{-1} as it moves in a straight line with acceleration function $4e^{-\frac{t}{20}}$ m s^{-2}.

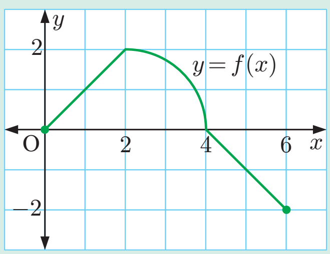
a Show that as t increases the object approaches a limiting velocity.
b Find the total distance travelled in the first 10 seconds of motion.

Review set 16A

1 Write an expression for the total shaded area.

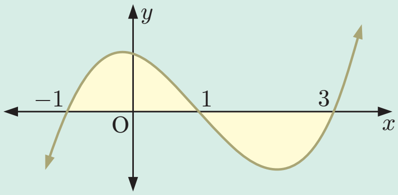


2

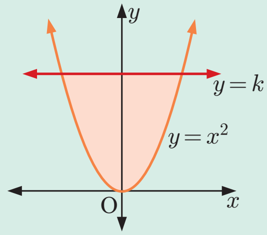


Find:
a $\displaystyle\int_0^4 f(x)\,dx$
b $\displaystyle\int_4^6 f(x)\,dx$
c $\displaystyle\int_0^6 f(x)\,dx$

3 Does $\displaystyle\int_{-1}^{3} f(x)\,dx$ represent the area of the shaded region?
Explain your answer briefly.



4 Determine k if the enclosed region has area $5\frac{1}{3}$ units^2.



5 Find the area of the region enclosed by $y = x^2 + 4x + 1$ and $y = 3x + 3$.

6 A particle moves in a straight line with velocity $v(t) = t^2 - 6t + 8$ m s^{-1}, for $t \geqslant 0$ seconds.

- **a** Draw a sign diagram for $v(t)$.
- **b** Describe what happens to the particle in the first 5 seconds of motion.
- **c** After 5 seconds, how far is the particle from its original position?
- **d** Find the total distance travelled in the first 5 seconds of motion.

7 Determine the area enclosed by the axes and $y = 4e^x - 1$.

8 A particle moves in a straight line with velocity given by $v(t) = \sin t$ m s^{-1}, where $t \geqslant 0$ seconds. Find the total distance travelled by the particle in the first 4 seconds of motion.

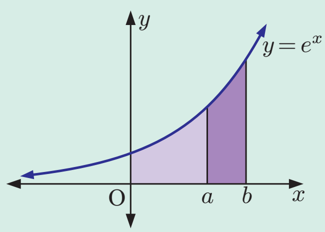
Review set 16B

1 At time $t = 0$ a particle passes through the origin with velocity 27 cm s^{-1}. Its acceleration t seconds later is $6t - 30$ cm s^{-2}.

- **a** Write an expression for the particle's velocity.
- **b** Calculate the displacement from the origin after 6 seconds.

2
- **a** Sketch the graphs of $y = \frac{1}{2} - \frac{1}{2}\cos 2x$ and $y = \sin x$ on the same set of axes for $0 \leqslant x \leqslant \pi$.
- **b** Verify that both graphs pass through the points $(0, 0)$ and $(\frac{\pi}{2}, 1)$.
- **c** Find the area enclosed by these curves for $0 \leqslant x \leqslant \frac{\pi}{2}$.

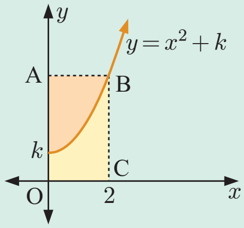
3 Find a given that the area of the region between $y = e^x$ and the x-axis from $x = 0$ to $x = a$ is 2 units^2.
Hence determine b such that the area of the region from $x = a$ to $x = b$ is also 2 units^2.



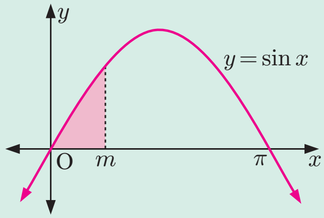
4 A particle moves in a straight line with velocity $v(t) = 2t - 3t^2$ m s^{-1}, for $t \geqslant 0$ seconds.

- **a** Find a formula for the acceleration function $a(t)$.
- **b** Find a formula for the displacement function $s(t)$.
- **c** Find the change in displacement after two seconds.

5 OABC is a rectangle and the two shaded regions are equal in area. Find k.



6 The shaded region has area $\frac{1}{2}$ unit^2. Find the value of m.



7 Find the area of the region enclosed by $y = x$ and $y = \sin\left(\frac{\pi x}{2}\right)$.

8 A boat travelling in a straight line has its engine turned off at time $t = 0$. Its velocity at time t seconds thereafter is given by $v(t) = \frac{100}{(t+2)^2}$ m s^{-1}.

- **a** Find the initial velocity of the boat, and its velocity after 3 seconds.
- **b** Discuss $v(t)$ as $t \to \infty$.
- **c** Sketch the graph of $v(t)$ against t.
- **d** Find how long it takes for the boat to travel 30 metres from when the engine is turned off.
- **e** Find the acceleration of the boat at any time t.
- **f** Show that $\frac{dv}{dt} = -kv^{\frac{3}{2}}$, and find the value of the constant k.

ANSWERS

EXERCISE 1A

1 **a** $5 \in D$ **b** $6 \notin G$ **c** $d \notin \{a, e, i, o, u\}$
d $\{2, 5\} \subseteq \{1, 2, 3, 4, 5, 6\}$
e $\{3, 8, 6\} \nsubseteq \{1, 2, 3, 4, 5, 6\}$

2 **a** **i** $\{9\}$ **ii** $\{5, 6, 7, 8, 9, 10, 11, 12, 13\}$
b **i** $\varnothing$ **ii** $\{1, 2, 3, 4, 5, 6, 7, 8\}$
c **i** $\{1, 3, 5, 7\} = A$ **ii** $\{1, 2, 3, 4, 5, 6, 7, 8, 9\} = B$

3 **a** 5 **b** 6 **c** 2 **d** 9

4 **a** true **b** true **c** true **d** true
e false **f** true **g** true **h** false

5 **a** finite **b** infinite **c** infinite **d** infinite

6 **a** true **b** true **c** false **d** true

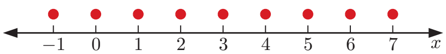
7 **a** disjoint **b** not disjoint **8** true

9 **a** 15 subsets **b** $2^n - 1$, $n \in \mathbb{Z}^+$

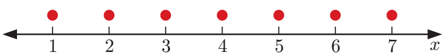
EXERCISE 1B

1 **a** finite **b** infinite **c** infinite **d** infinite
e infinite **f** infinite **g** infinite

2 **a** **i** The set of all integers x such that x is between -1 and 7, including -1 and 7.
ii $\{-1, 0, 1, 2, 3, 4, 5, 6, 7\}$ **iii** 9
iv



b **i** The set of all natural numbers x such that x is between -2 and 8.
ii $\{1, 2, 3, 4, 5, 6, 7\}$ **iii** 7
iv

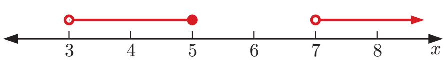


c **i** The set of all real numbers x such that x is between 0 and 1, including 0 and 1.
ii not possible **iii** infinite
iv

d **i** The set of all rational numbers x such that x is between 5 and 6, including 5 and 6.
ii not possible **iii** infinite
iv cannot be illustrated

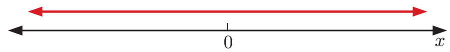
e **i** The set of all real numbers x such that x is between -1 and 5, including -1.
ii not possible **iii** infinite
iv

f **i** The set of all real numbers x such that x is between 3 and 5 (including 5), or greater than 7.
ii not possible **iii** infinite
iv



g **i** The set of all real numbers x such that x is less than or equal to 1, or greater than 2.
ii not possible **iii** infinite
iv

h **i** The set of all real numbers x such that x is less than 2, or greater than or equal to 1. (So, A is the set of all real numbers.)
ii not possible **iii** infinite
iv



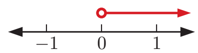
3 **a** $A = \{x \in \mathbb{Z} : -100 < x < 100\}$
b $A = \{x \in \mathbb{R} : x > 1000\}$
c $A = \{x \in \mathbb{Q} : 2 \leqslant x \leqslant 3\}$

4 **a** $A = \{x \in \mathbb{Z} : -2 \leqslant x \leqslant 3\}$
b $A = \{x \in \mathbb{Z} : x \leqslant -3\}$
c $A = \{x \in \mathbb{R} : -3 \leqslant x < 2\}$
d $A = \{x \in \mathbb{R} : 1 \leqslant x \leqslant 3 \cup x > 5\}$

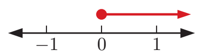
5 **a** $A \subseteq B$ **b** $A \nsubseteq B$ **c** $A \subseteq B$ **d** $A \subseteq B$
e $A \nsubseteq B$ **f** $A \nsubseteq B$

6 **a** neither **b** open **c** neither **d** open
e closed **f** neither

7 **a** There are infinitely many rational numbers within any given interval, so we cannot represent $\mathbb{Q}$ as a series of dots like we can with $\mathbb{Z}$. We cannot represent $\mathbb{Q}$ with a continuous line either (like we do with $\mathbb{R}$), as this would imply that irrational numbers are part of $\mathbb{Q}$.
b **i** the set of positive real numbers, $\{x \in \mathbb{R} : x > 0\}$

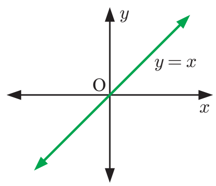


ii the set of positive real numbers and zero, $\{x \in \mathbb{R} : x \geqslant 0\}$

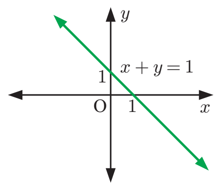


EXERCISE 1C

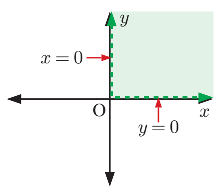
1 **a** infinite



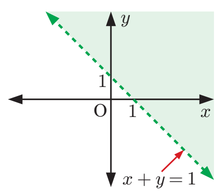
b infinite



c infinite



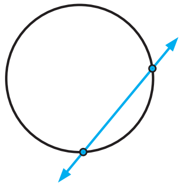
d infinite



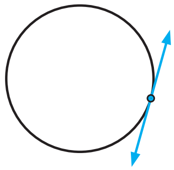
2 **a** infinite **b** finite **c** infinite

3 **a** **i** The set of all points of intersection between the line and the circle.
ii The set of all points that lie on either the straight line or the circle.

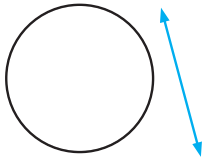
b **i** There are two points of intersection between the straight line and the circle.



ii There is one point of intersection between the straight line and the circle (that is, the straight line is a tangent to the circle).



iii The straight line and the circle do not intersect.



EXERCISE 1D

1 **a** $C' = \{\text{consonants}\}$ **b** $C' = \{x \in \mathbb{Z} : x \geqslant 0\}$
c $C' = \{x \in \mathbb{Z} : x \geqslant -4\}$ **d** $C' = \{x \in \mathbb{Q} : 2 < x < 8\}$

2 **a** $\{2, 3, 4, 5, 6, 7\}$ **b** $\{0, 1, 8\}$ **c** $\{5, 6, 7, 8\}$
d $\{0, 1, 2, 3, 4\}$ **e** $\{5, 6, 7\}$ **f** $\{2, 3, 4, 5, 6, 7, 8\}$
g $\{2, 3, 4\}$ **h** $\{0, 1, 2, 3, 4, 8\}$

3 **a** 9 **b** 11 **4** **a** false **b** true

5 **a** $\{1, 2, 10, 11, 12\}$ **b** $\{1, 2, 3, 4, 12\}$
c $\{1, 8, 9, 10, 11, 12\}$ **d** $\{3, 4, 5, 6, 7\}$
e $\{1, 2, 8, 9, 10, 11, 12\}$ **f** $\{8, 9, 10, 11\}$
g $\{1, 2, 5, 6, 7, 8, 9, 10, 11, 12\}$ **h** $\{2, 10, 11\}$

6 **a** $[0, \infty)$ **b** $(-\infty, 1)$ **c** $(-\infty, 3) \cup [2, \infty)$
d $(-\infty, -5] \cup (7, \infty)$ **e** $[1, 3)$
f $(-\infty, -5) \cup [0, 1]$

EXERCISE 1E

1 **a** $P = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$ **b** $\{2, 5, 11\}$
c $\{2, 3, 4, 5, 7, 11, 12, 13, 15, 17, 19, 23\}$
d $12 = 9 + 6 - 3$ ✓

2 **a** $P = \{1, 2, 4, 7, 14, 28\}$, $Q = \{1, 2, 4, 5, 8, 10, 20, 40\}$
b $\{1, 2, 4\}$ **c** $\{1, 2, 4, 5, 7, 8, 10, 14, 20, 28, 40\}$
d $11 = 6 + 8 - 3$ ✓

3 **a** $M = \{32, 36, 40, 44, 48, 52, 56\}$, $N = \{36, 42, 48, 54\}$
b $\{36, 48\}$ **c** $\{32, 36, 40, 42, 44, 48, 52, 54, 56\}$
d $9 = 7 + 4 - 2$ ✓

4 **a** $R = \{-2, -1, 0, 1, 2, 3, 4\}$, $S = \{0, 1, 2, 3, 4, 5, 6\}$
b $\{0, 1, 2, 3, 4\}$ **c** $\{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$
d $9 = 7 + 7 - 5$ ✓

5 **a** $C = \{-4, -3, -2, -1\}$
$D = \{-7, -6, -5, -4, -3, -2, -1\}$
b $\{-4, -3, -2, -1\}$ **c** $\{-7, -6, -5, -4, -3, -2, -1\}$
d $7 = 4 + 7 - 4$ ✓

6 **a** $P = \{1, 2, 3, 4, 6, 12\}$, $Q = \{1, 2, 3, 6, 9, 18\}$,
$R = \{1, 3, 9, 27\}$
b **i** $\{1, 2, 3, 6\}$ **ii** $\{1, 3\}$ **iii** $\{1, 3, 9\}$
iv $\{1, 2, 3, 4, 6, 9, 12, 18\}$ **v** $\{1, 2, 3, 4, 6, 9, 12, 27\}$
vi $\{1, 2, 3, 6, 9, 18, 27\}$

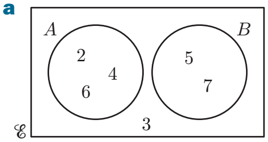
c **i** $\{1, 3\}$ **ii** $\{1, 2, 3, 4, 6, 9, 12, 18, 27\}$

7 **a** $A = \{4, 8, 12, 16, 20, 24, 28, 32, 36\}$
$B = \{6, 12, 18, 24, 30, 36\}$, $C = \{12, 24, 36\}$
b **i** $\{12, 24, 36\}$ **ii** $\{12, 24, 36\}$
iii $\{12, 24, 36\}$ **iv** $\{12, 24, 36\}$
v $\{4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36\}$
c $12 = 9 + 6 + 3 - 3 - 3 - 3 + 3$ ✓

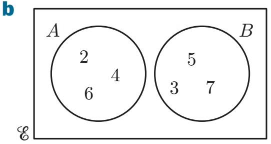
8 **a** $A = \{6, 12, 18, 24, 30\}$, $B = \{1, 2, 3, 5, 6, 10, 15, 30\}$
$C = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$
b **i** $\{6, 30\}$ **ii** $\{2, 3, 5\}$ **iii** $\varnothing$ **iv** $\varnothing$
v $\{1, 2, 3, 5, 6, 7, 10, 11, 12, 13, 15, 17, 18, 19, 23, 24, 29, 30\}$
c $18 = 5 + 8 + 10 - 2 - 3 - 0 + 0$ ✓

EXERCISE 1F.1

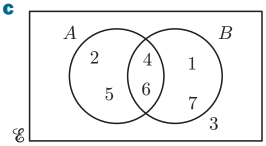
1 **a**



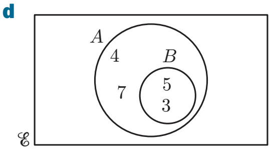
b



c

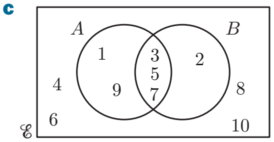


d



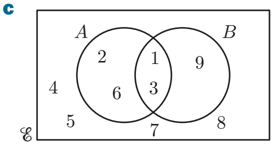
2 **a** $A = \{1, 3, 5, 7, 9\}$
$B = \{2, 3, 5, 7\}$
b $A \cap B = \{3, 5, 7\}$
$A \cup B = \{1, 2, 3, 5, 7, 9\}$

c



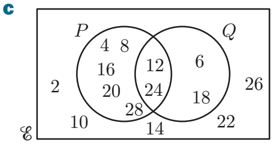
3 **a** $A = \{1, 2, 3, 6\}$
$B = \{1, 3, 9\}$
b $A \cap B = \{1, 3\}$
$A \cup B = \{1, 2, 3, 6, 9\}$

c



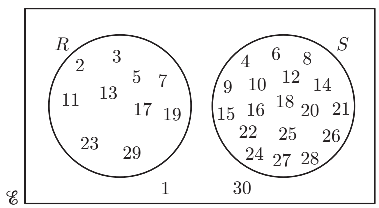
4 **a** $P = \{4, 8, 12, 16, 20, 24, 28\}$
$Q = \{6, 12, 18, 24\}$
b $P \cap Q = \{12, 24\}$
$P \cup Q = \{4, 6, 8, 12, 16, 18, 20, 24, 28\}$

c



5 **a** $R = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$
$S = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28\}$
b $R \cap S = \varnothing$
$R \cup S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\}$

c



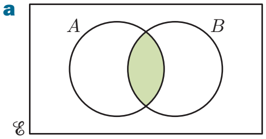
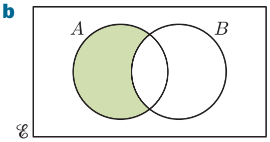
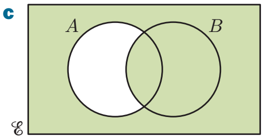
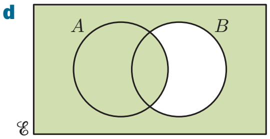
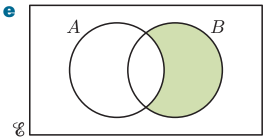
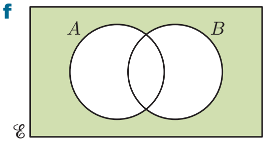
6 **a** $\{b, d, e, h\}$ **b** $\{e, f, h, i, j\}$ **c** $\{a, c, f, g, i, j, k\}$
d $\{a, b, c, d, g, k\}$ **e** $\{e, h\}$ **f** $\{b, d, e, f, h, i, j\}$
g $\{a, c, g, k\}$ **h** $\{a, b, c, d, f, g, i, j, k\}$

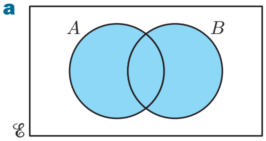
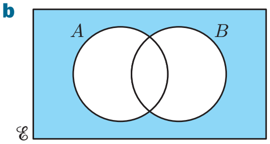
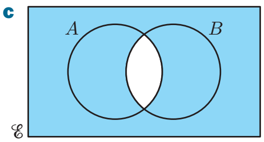
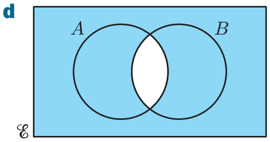
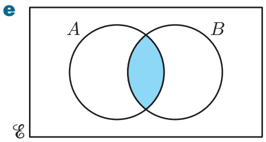
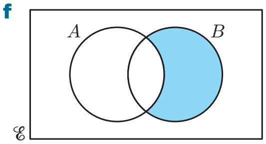
7 **a** **i** $\{a, b, c, d, h, j\}$ **ii** $\{a, c, d, e, f, g, k\}$
iii $\{a, b, e, f, i, l\}$ **iv** $\{a, c, d\}$
v $\{a, b, c, d, e, f, g, h, j, k\}$ **vi** $\{a, e, f\}$
vii $\{a\}$ **viii** $\{a, b, c, d, e, f, g, h, i, j, k, l\}$

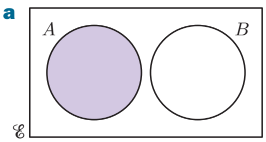
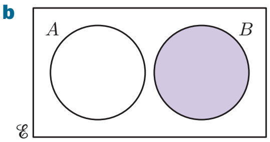
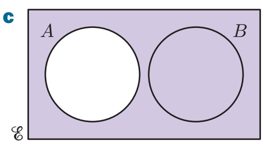
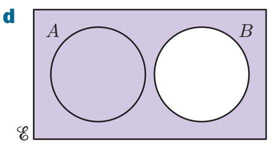
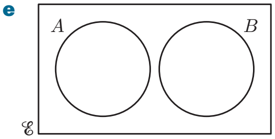
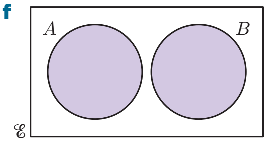
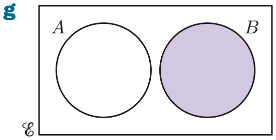
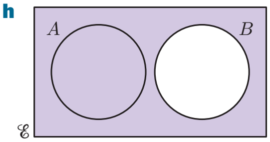
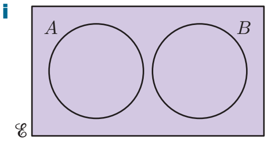
b **i** 12 **ii** 12

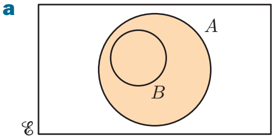
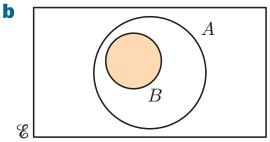
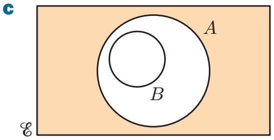
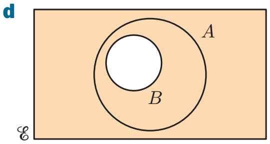
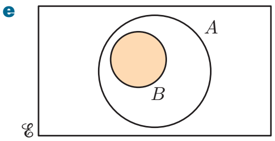
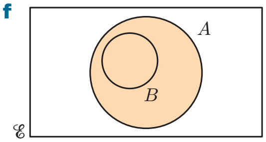
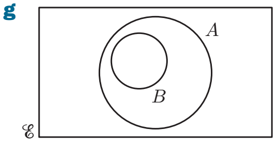
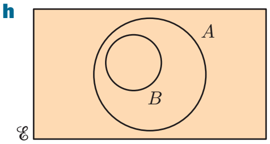
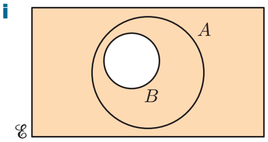
c $n(A \cup B \cup C)$
$= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C)$
$- n(B \cap C) + n(A \cap B \cap C)$

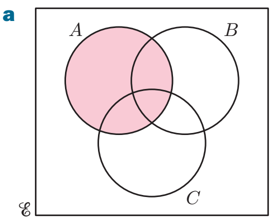
EXERCISE 1F.2

1 **a**

b

c

d

e

f


2 **a**

b

c

d

e

f


3 **a**

b

c

d

e

f

g

h

i


4 **a**

b

c

d

e

f

g

h

i


5 **a**

b