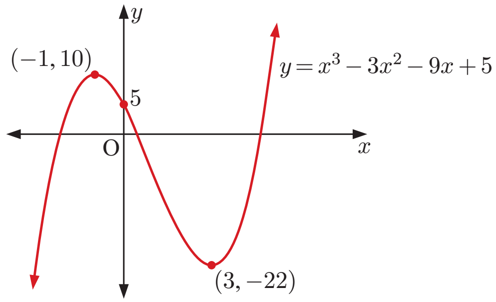
So, we have a local maximum at $x = -1$ and a local minimum at $x = 3$.

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = 10$$

$$f(3) = 3^3 - 3 \times 3^2 - 9 \times 3 + 5 = -22$$

$\therefore$ there is a local maximum at $(-1,\ 10)$ and a local minimum at $(3,\ -22)$.

**c**




## Example 9

**Self Tutor**

Find and classify all stationary points of $f(x) = \dfrac{x^2+1}{x}$.

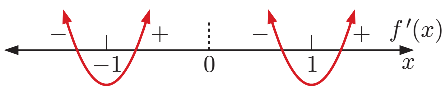
$$= \frac{x^2+1}{x}$$

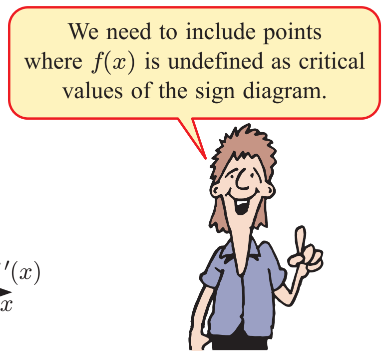
$$\therefore \quad f'(x) = \frac{2x(x) - (x^2+1)}{x^2}$$

$$= \frac{x^2-1}{x^2}$$

$$= \frac{(x+1)(x-1)}{x}$$

$f'(x)$ has sign diagram:




So, we have local minima when $x = \pm 1$.

$$f(-1) = \frac{(-1)^2+1}{(-1)} = -2 \quad \text{and} \quad f(1) = \frac{1^2+1}{1} = 2$$

$\therefore$ there are local minima at $(-1,\ -2)$ and $(1,\ 2)$.

## SECOND DERIVATIVES AND STATIONARY POINTS

The second derivative of a function can be used to determine the nature of its stationary points.

For a function $f(x)$ with a stationary point at $x = a$:

- If $f''(a) > 0$, then it is a **local minimum**.
- If $f''(a) < 0$, then it is a **local maximum**.
- If $f''(a) = 0$, then it could be a **local maximum**, a **local minimum**, or a **stationary inflection point**.

## Example 10

Self Tutor

Find and classify all stationary points of $f(x) = 2x^3 + 3x^2 - 12$.

$$f(x) = 2x^3 + 3x^2 - 12$$
$$\therefore\ f'(x) = 6x^2 + 6x$$
$$= 6x(x+1)$$
$$\therefore\ f'(x) = 0 \text{ when } 6x = 0 \text{ or } x + 1 = 0$$
$$\therefore\ x = 0 \text{ or } x = -1$$

Also, $f''(x) = 12x + 6$

$\therefore\ f''(0) = 12(0) + 6 = 6$ which is $> 0$

and $f''(-1) = 12(-1) + 6 = -6$ which is $< 0$

So, we have a local minimum at $x = 0$ and a local maximum at $x = -1$.

Now $f(0) = 2(0)^3 + 3(0)^2 - 12 = -12$

$f(-1) = 2(-1)^3 + 3(-1)^2 - 12 = -11$

$\therefore$ there is a local minimum at $(0, -12)$ and a local maximum at $(-1, -11)$.

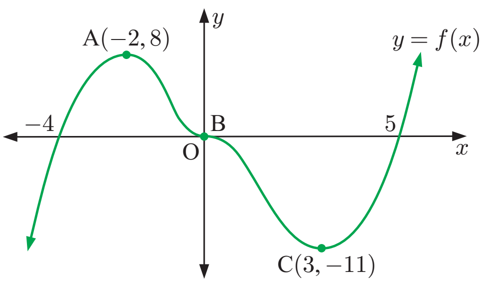
## EXERCISE 14B

**1** The tangents at points A, B, and C are horizontal.

**a** Classify points A, B, and C.

**b** Draw a sign diagram for:

**i** $f(x)$ **ii** $f'(x)$




**2** For each of the following functions, find and classify any stationary points. Sketch the function, showing all important features.

**a** $f(x) = x^2 - 2$

**b** $f(x) = x^3 + 1$

**c** $f(x) = x^3 - 3x + 2$

**d** $f(x) = x^4 - 2x^2$

**e** $f(x) = x^3 - 6x^2 + 12x - 7$

**f** $f(x) = \sqrt{x} + 2$

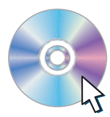
**g** $f(x) = x - \sqrt{x}$

**h** $f(x) = x^4 - 6x^2 + 8x - 3$

**i** $f(x) = 1 - x\sqrt{x}$

**j** $f(x) = x^4 - 2x^2 - 8$

GRAPHING PACKAGE



**3** At what value of $x$ does the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, have a stationary point? Under what conditions is the stationary point a local maximum or a local minimum?

**4** $f(x) = 2x^3 + ax^2 - 24x + 1$ has a local maximum at $x = -4$. Find $a$.

**5** $f(x) = x^3 + ax + b$ has a stationary point at $(-2, 3)$.

**a** Find the values of $a$ and $b$.

**b** Find the position and nature of all stationary points.

### Example 11 — Self Tutor

Find the exact position and nature of the stationary point of $y = (x-2)e^{-x}$.

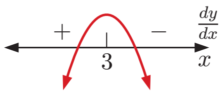
$\dfrac{dy}{dx} = (1)e^{-x} + (x-2)e^{-x}(-1)$ {product rule}

$= e^{-x}(1-(x-2))$

$= \dfrac{3-x}{e^x}$ where $e^x$ is positive for all $x$

So, $\dfrac{dy}{dx} = 0$ when $x = 3$.

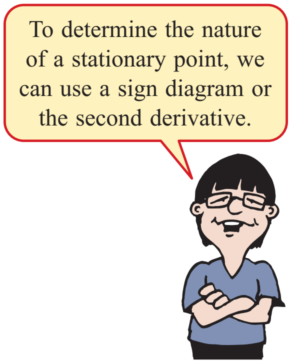
The sign diagram of $\dfrac{dy}{dx}$ is:




$\therefore$ at $x = 3$ we have a local maximum.

But when $x = 3$, $y = (1)e^{-3} = \dfrac{1}{e^3}$

$\therefore$ the local maximum is at $(3, \frac{1}{e^3})$.




**6** Find the position and nature of the stationary point(s) of:

**a** $y = xe^{-x}$ **b** $y = x^2e^x$ **c** $y = \dfrac{e^x}{x}$ **d** $y = e^{-x}(x+2)$

**7** Consider $f(x) = x\ln x$.

**a** For what values of $x$ is $f(x)$ defined? **b** Show that the global minimum value of $f(x)$ is $-\frac{1}{e}$.

**8** Find the greatest and least value of:

**a** $x^3 - 12x - 2$ for $-3 \leqslant x \leqslant 5$ **b** $4 - 3x^2 + x^3$ for $-2 \leqslant x \leqslant 3$

**9** The cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ touches the line with equation $y = 9x + 2$ at the point $(0, 2)$, and has a stationary point at $(-1, -7)$. Find $P(x)$.

### Example 12 — Self Tutor

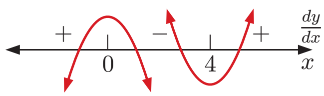
Find the greatest and least value of $y = x^3 - 6x^2 + 5$ on the interval $-2 \leqslant x \leqslant 5$.

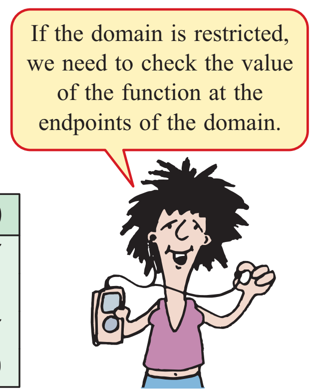
Now $\dfrac{dy}{dx} = 3x^2 - 12x$

$= 3x(x-4)$

$\therefore \dfrac{dy}{dx} = 0$ when $x = 0$ or $4$

The sign diagram of $\dfrac{dy}{dx}$ is:




$\therefore$ there is a local maximum at $x = 0$, and a local minimum at $x = 4$.

| Critical value ($x$) | $f(x)$ |
|---|---|
| $-2$ (endpoint) | $-27$ |
| $0$ (local max) | $5$ |
| $4$ (local min) | $-27$ |
| $5$ (endpoint) | $-20$ |

The greatest of these values is $5$ when $x = 0$.

The least of these values is $-27$ when $x = -2$ and when $x = 4$.

**10** For each of the following, determine the position and nature of the stationary points on the interval $0 \leqslant x \leqslant 2\pi$, then show them on a graph of the function.

GRAPHING PACKAGE

- **a** $f(x) = \sin x$
- **b** $f(x) = \cos(2x)$
- **c** $f(x) = \sin^2 x$
- **d** $f(x) = e^{\sin x}$
- **e** $f(x) = \sin(2x) + 2\cos x$

**11** Show that $y = 4e^{-x}\sin x$ has a local maximum when $x = \frac{\pi}{4}$.

**12** Prove that $\dfrac{\ln x}{x} \leqslant \dfrac{1}{e}$ for all $x > 0$. **Hint:** Let $f(x) = \dfrac{\ln x}{x}$ and find its greatest value.

**13** Consider the function $f(x) = x - \ln x$.

- **a** Show that the graph of $y = f(x)$ has a local minimum and that this is the only turning point.
- **b** Hence prove that $\ln x \leqslant x - 1$ for all $x > 0$.

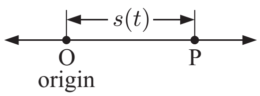
# C KINEMATICS

In the **Opening Problem** we are dealing with the movement of Michael riding his bicycle. We do not know the direction Michael is travelling, so we talk simply about the *distance* he has travelled and his *speed*.

For problems of **motion in a straight line**, we can include the direction the object is travelling along the line. We therefore can talk about *displacement* and *velocity*.

## DISPLACEMENT

Suppose an object P moves along a straight line so that its position $s$ from an origin O is given as some function of time $t$. We write $s = s(t)$ where $t \geqslant 0$.




$s(t)$ is a **displacement function** and for any value of $t$ it gives the displacement from O.

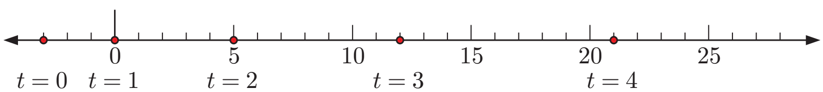
$s(t)$ is a vector quantity. Its magnitude is the distance from O, and its sign indicates the direction from O.

For example, consider $s(t) = t^2 + 2t - 3$ cm.

$s(0) = -3$ cm, $s(1) = 0$ cm, $s(2) = 5$ cm, $s(3) = 12$ cm, $s(4) = 21$ cm.

To appreciate the motion of P we draw a **motion graph**. You can also view the motion by clicking on the icon.

DEMO



## VELOCITY

The **average velocity** of an object moving in a straight line in the time interval from $t = t_1$ to $t = t_2$ is the ratio of the change in displacement to the time taken.

If $s(t)$ is the displacement function then **average velocity** $= \dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$.

On a graph of $s(t)$ against $t$ for the time interval from $t = t_1$ to $t = t_2$, the average velocity is the gradient of a chord through the points $(t_1, s(t_1))$ and $(t_2, s(t_2))$.

In **Chapter 13** we established that the instantaneous rate of change of a quantity is given by its derivative.

If $s(t)$ is the displacement function of an object moving in a straight line, then $v(t) = s'(t) = \lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$ is the **instantaneous velocity** or **velocity function** of the object at time $t$.

On a graph of $s(t)$ against $t$, the instantaneous velocity at a particular time is the gradient of the tangent to the graph at that point.

## ACCELERATION

If an object moves in a straight line with velocity function $v(t)$ then:

- the **average acceleration** for the time interval from $t = t_1$ to $t = t_2$ is the ratio of the change in velocity to the time taken

$$\textbf{average acceleration} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

- the **instantaneous acceleration** at time $t$ is $a(t) = v'(t) = \lim\limits_{h \to 0} \dfrac{v(t+h) - v(t)}{h}$.

## UNITS

Each time we differentiate with respect to time $t$, we calculate a rate per unit of time. So, for a displacement in metres and time in seconds:

- the units of velocity are $\text{m}\,\text{s}^{-1}$
- the units of acceleration are $\text{m}\,\text{s}^{-2}$.

### Discussion

- What is the relationship between the displacement function $s(t)$ and the acceleration function $a(t)$?
- How are the units of velocity and acceleration related to their formulae? You may wish to research "dimensional analysis".

### Example 13

Self Tutor

A particle moves in a straight line with displacement from O given by $s(t) = 3t - t^2$ metres at time $t$ seconds. Find:

**a** the average velocity for the time interval from $t = 2$ to $t = 5$ seconds

**b** the average velocity for the time interval from $t = 2$ to $t = 2 + h$ seconds

**c** $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$ and comment on its significance.

**a** average velocity

$$= \frac{s(5) - s(2)}{5 - 2}$$
$$= \frac{(15 - 25) - (6 - 4)}{3}$$
$$= \frac{-10 - 2}{3}$$
$$= -4 \text{ m s}^{-1}$$

**b** average velocity

$$= \frac{s(2+h) - s(2)}{2 + h - 2}$$
$$= \frac{3(2+h) - (2+h)^2 - 2}{h}$$
$$= \frac{6 + 3h - 4 - 4h - h^2 - 2}{h}$$
$$= \frac{-h - h^2}{h}$$
$$= -1 - h \text{ m s}^{-1} \quad \text{provided } h \neq 0$$

**c** $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$

$$= \lim_{h \to 0} (-1 - h) \qquad \{\text{since } h \neq 0\}$$
$$= -1 \text{ m s}^{-1}$$

This is the instantaneous velocity of the particle at time $t = 2$ seconds.

## EXERCISE 14C.1

**1** A particle P moves in a straight line with displacement function $s(t) = t^2 + 3t - 2$ metres, where $t \geqslant 0$, $t$ in seconds.

**a** Find the average velocity from $t = 1$ to $t = 3$ seconds.

**b** Find the average velocity from $t = 1$ to $t = 1 + h$ seconds.

**c** Find the value of $\lim\limits_{h \to 0} \dfrac{s(1+h) - s(1)}{h}$ and comment on its significance.

**d** Find the average velocity from time $t$ to time $t + h$ seconds and interpret $\lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$.

**2** A particle P moves in a straight line with displacement function $s(t) = 5 - 2t^2$ cm, where $t \geqslant 0$, $t$ in seconds.

**a** Find the average velocity from $t = 2$ to $t = 5$ seconds.

**b** Find the average velocity from $t = 2$ to $t = 2 + h$ seconds.

**c** Find the value of $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$ and state the meaning of this value.

**d** Interpret $\lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$.

**3** A particle moves in a straight line with velocity function $v(t) = 2\sqrt{t} + 3$ $\text{cm s}^{-1}$, $t \geqslant 0$.

**a** Find the average acceleration from $t = 1$ to $t = 4$ seconds.

**b** Find the average acceleration from $t = 1$ to $t = 1 + h$ seconds.

**c** Find the value of $\lim\limits_{h \to 0} \dfrac{v(1+h) - v(1)}{h}$. Interpret this value.

**d** Interpret $\lim\limits_{h \to 0} \dfrac{v(t+h) - v(t)}{h}$.

**4** An object moves in a straight line with displacement function $s(t)$ and velocity function $v(t)$, $t \geqslant 0$. State the meaning of:

**a** $\lim\limits_{h \to 0} \dfrac{s(4+h) - s(4)}{h}$

**b** $\lim\limits_{h \to 0} \dfrac{v(4+h) - v(4)}{h}$

## VELOCITY AND ACCELERATION FUNCTIONS

If a particle P moves in a straight line and its position is given by the displacement function $s(t)$, $t \geqslant 0$, then:

- the **velocity** of P at time $t$ is given by $\boldsymbol{v(t) = s'(t)}$
- the **acceleration** of P at time $t$ is given by $\boldsymbol{a(t) = v'(t) = s''(t)}$
- $s(0)$, $v(0)$, and $a(0)$ give us the position, velocity, and acceleration of the particle at time $t = 0$, and these are called the **initial conditions**.

## SIGN INTERPRETATION

Suppose a particle P moves in a straight line with displacement function $s(t)$ relative to an origin O. Its velocity function is $v(t)$ and its acceleration function is $a(t)$.

We can use **sign diagrams** to interpret:

- where the particle is located relative to O
- the direction of motion and where a change of direction occurs
- when the particle's velocity is increasing or decreasing.

**SIGNS OF $s(t)$:**

| $s(t)$ | *Interpretation* |
|---|---|
| $= 0$ | P is at O |
| $> 0$ | P is located to the right of O |
| $< 0$ | P is located to the left of O |

**SIGNS OF $v(t)$:**

| $v(t)$ | *Interpretation* |
|---|---|
| $= 0$ | P is instantaneously at rest |
| $> 0$ | P is moving to the right |
| $< 0$ | P is moving to the left |

**SIGNS OF $a(t)$:**

| $a(t)$ | *Interpretation* |
|---|---|
| $> 0$ | velocity is increasing |
| $< 0$ | velocity is decreasing |
| $= 0$ | velocity may be a maximum or minimum or possibly constant |

### ZEROS:

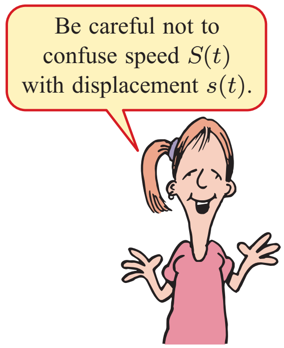
| *Phrase used in a question* | $t$ | $s$ | $v$ | $a$ |
|---|---|---|---|---|
| initial conditions | 0 | | | |
| at the origin | | 0 | | |
| stationary | | | 0 | |
| reverses | | | 0 | |
| maximum or minimum displacement | | | 0 | |
| constant velocity | | | | 0 |
| maximum or minimum velocity | | | | 0 |

When a particle reverses direction, its velocity must change sign.
This corresponds to a local maximum or local minimum distance from the origin O.

## SPEED

As we have seen, velocities have size (magnitude) and sign (direction). In contrast, speed simply measures *how fast* something is travelling, regardless of the direction of travel. Speed is a *scalar* quantity which has size but no sign. Speed cannot be negative.

The **speed** at any instant is the magnitude of the object's velocity.
If $S(t)$ represents speed then $S = |v|$.




To determine when the speed $S(t)$ of an object P with displacement $s(t)$ is increasing or decreasing, we use a **sign test**.

- If the signs of $v(t)$ and $a(t)$ are the same (both positive or both negative), then the speed of P is increasing.
- If the signs of $v(t)$ and $a(t)$ are opposite, then the speed of P is decreasing.

### Discovery — Displacement, velocity, and acceleration graphs

In this Discovery we examine the motion of a projectile which is fired in a vertical direction. The projectile is affected by gravity, which is responsible for the projectile's constant acceleration.


We then extend the Discovery to consider other cases of motion in a straight line.

**What to do:**

**1** Click on the icon to examine vertical projectile motion.
Observe first the displacement along the line, then look at the velocity which is the rate of change in displacement. When is the velocity positive and when is it negative?

**2** Examine the following graphs and comment on their shapes:
- *displacement* v *time*
- *velocity* v *time*
- *acceleration* v *time*

**3** Pick from the menu or construct functions of your own choosing to investigate the relationship between displacement, velocity, and acceleration.

## Example 14

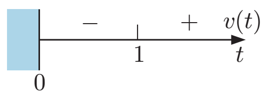
**Self Tutor**

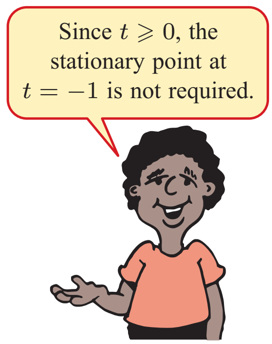
A particle moves in a straight line with position relative to O given by $s(t) = t^3 - 3t + 1$ cm, where $t$ is the time in seconds, $t \geqslant 0$.

- **a** Find expressions for the particle's velocity and acceleration, and draw sign diagrams for each of them.
- **b** Find the initial conditions and hence describe the motion at this instant.
- **c** Describe the motion of the particle at $t = 2$ seconds.
- **d** Find the position of the particle when the changes in direction occur.
- **e** Draw a motion diagram for the particle.
- **f** For what time interval is the particle's speed increasing?
- **g** What is the total distance travelled in the time from $t = 0$ to $t = 2$ seconds?

**a**
$$s(t) = t^3 - 3t + 1 \text{ cm}$$
$$\therefore\ v(t) = 3t^2 - 3 \qquad \{\text{as } v(t) = s'(t)\}$$
$$= 3(t^2 - 1)$$
$$= 3(t+1)(t-1) \text{ cm s}^{-1}$$

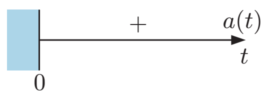
which has sign diagram:




and $a(t) = 6t \text{ cm s}^{-2}$ $\qquad \{\text{as } a(t) = v'(t)\}$

which has sign diagram:




**b** When $t = 0$, $s(0) = 1$ cm
$v(0) = -3 \text{ cm s}^{-1}$
$a(0) = 0 \text{ cm s}^{-2}$

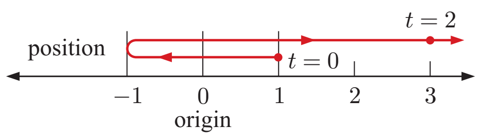
$\therefore$ the particle is 1 cm to the right of O, moving to the left at a speed of 3 $\text{cm s}^{-1}$.

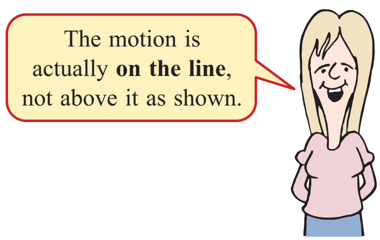
**c** When $t = 2$, $s(2) = 8 - 6 + 1 = 3$ cm
$v(2) = 12 - 3 = 9 \text{ cm s}^{-1}$
$a(2) = 12 \text{ cm s}^{-2}$

$\therefore$ the particle is 3 cm to the right of O, moving to the right at a speed of 9 $\text{cm s}^{-1}$.
Since $a$ and $v$ have the same sign, the speed of the particle is increasing.

**d** Since $v(t)$ changes sign when $t = 1$, a change of direction occurs at this instant.
$s(1) = 1 - 3 + 1 = -1$, so the particle changes direction when it is 1 cm to the left of O.

**e**




**f** Speed is increasing when $v(t)$ and $a(t)$ have the same sign. This is for $t \geqslant 1$.

**g** Total distance travelled $= 2 + 4 = 6$ cm.

In later chapters on integral calculus we will see another technique for finding the distances travelled and displacement over time.

## EXERCISE 14C.2

**1** An object moves in a straight line with position given by $s(t) = t^2 - 4t + 3$ cm from O, where $t$ is in seconds, $t \geqslant 0$.

- **a** Find expressions for the object's velocity and acceleration, and draw sign diagrams for each function.
- **b** Find the initial conditions and explain what is happening to the object at that instant.
- **c** Describe the motion of the object at time $t = 2$ seconds.
- **d** At what time does the object reverse direction? Find the position of the object at this instant.
- **e** Draw a motion diagram for the object.
- **f** For what time intervals is the speed of the object decreasing?

**2** A stone is projected vertically so that its position above ground level after $t$ seconds is given by $s(t) = 98t - 4.9t^2$ metres, $t \geqslant 0$.

- **a** Find the velocity and acceleration functions for the stone, and draw sign diagrams for each function.
- **b** Find the initial position and velocity of the stone.
- **c** Describe the stone's motion at times $t = 5$ and $t = 12$ seconds.
- **d** Find the maximum height reached by the stone.
- **e** Find the time taken for the stone to hit the ground.

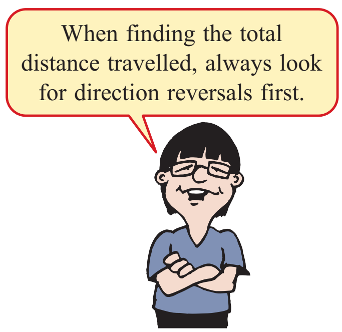
**3** When a ball is thrown, its height above the ground is given by $s(t) = 1.2 + 28.1t - 4.9t^2$ metres where $t$ is the time in seconds.

- **a** From what distance above the ground was the ball released?
- **b** Find $s'(t)$ and state what it represents.
- **c** Find $t$ when $s'(t) = 0$. What is the significance of this result?
- **d** What is the maximum height reached by the ball?
- **e** Find the ball's speed:
  - **i** when released **ii** at $t = 2$ s **iii** at $t = 5$ s.

  State the significance of the sign of the derivative $s'(t)$.
- **f** How long will it take for the ball to hit the ground?
- **g** What is the significance of $s''(t)$?

**4** The position of a particle moving along the $x$-axis is given by $x(t) = t^3 - 9t^2 + 24t$ metres where $t$ is in seconds, $t \geqslant 0$.

- **a** Draw sign diagrams for the particle's velocity and acceleration functions.
- **b** Find the position of the particle at the times when it reverses direction, and hence draw a motion diagram for the particle.
- **c** At what times is the particle's:
  - **i** speed decreasing **ii** velocity decreasing?
- **d** Find the total distance travelled by the particle in the first 5 seconds of motion.



**5** A particle P moves in a straight line with displacement function $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm, where $t$ is the time in seconds, $t \geqslant 0$.

**a** Find the velocity and acceleration functions.

**b** Find the initial position, velocity, and acceleration of P.

**c** Sketch the graph of the velocity function.

**d** Find when the velocity of P is 80 cm per second.

**6** A particle P moves along the $x$-axis with position given by $x(t) = 1 - 2\cos t$ cm where $t$ is the time in seconds.

**a** State the initial position, velocity, and acceleration of P.

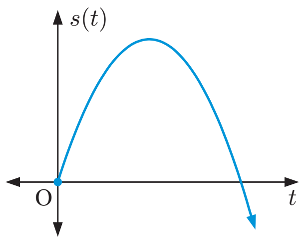
**b** Describe the motion when $t = \frac{\pi}{4}$ seconds.

**c** Find the times when the particle reverses direction on $0 < t < 2\pi$, and find the position of the particle at these instants.

**d** When is the particle's speed increasing on $0 \leqslant t \leqslant 2\pi$?

**7** In an experiment, an object is fired vertically from the earth's surface. From the results, a two-dimensional graph of the position $s(t)$ metres above the earth's surface is plotted, where $t$ is the time in seconds. It is noted that the graph is *parabolic*.

Assuming a constant gravitational acceleration $g$ and an initial velocity of $v(0)$, show that:




**a** $v(t) = v(0) + gt$ **b** $s(t) = v(0) \times t + \frac{1}{2}gt^2$.

**8** The table alongside shows data from a driving test in the United Kingdom.

A driver is travelling with constant speed. In response to a red light they must first react and press the brake. During this time the car travels a *thinking distance*. Once the brake is applied, the car travels a further *braking distance* before it comes to rest.

| *Speed* (km h$^{-1}$) | *Thinking distance* (m) | *Braking distance* (m) |
|---|---|---|
| 32 | 6 | 6 |
| 48 | 9 | 14 |
| 64 | 12 | 24 |
| 80 | 15 | 38 |
| 96 | 18 | 55 |
| 112 | 21 | 75 |

**a** Using the data from the driving test, find the reaction time for the driver at 96 km h$^{-1}$.

**b** The distance $S(t)$ travelled by an object moving initially at speed $u$ m s$^{-1}$, subject to constant acceleration $a$ m s$^{-2}$, is $S(t) = ut + \frac{1}{2}at^2$ m.

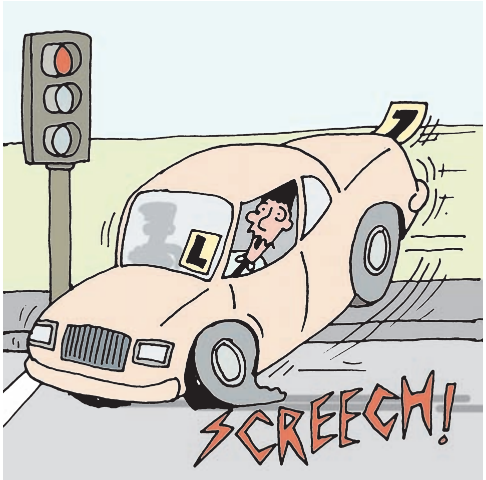
**i** Differentiate this formula with respect to time.

**ii** Hence calculate the time taken for the object to be at rest.

**iii** Using the data from the driving test, find the braking acceleration for the driver at 96 km h$^{-1}$.

**iv** Show that in general, an object starting at speed $u$ comes to rest in a distance $-\frac{1}{2}\frac{u^2}{a}$ m.

**v** If a driver doubles their speed, what happens to their braking distance?



# D RATES OF CHANGE

There are countless examples in the real world where quantities vary with time, or with respect to some other variable.

For example:
- temperature varies continuously
- the height of a tree varies as it grows
- the prices of stocks and shares vary with each day's trading.

We have already seen that if $y = f(x)$ then $f'(x)$ or $\dfrac{dy}{dx}$ is the gradient of the tangent to $y = f(x)$ at the given point.

> $\dfrac{dy}{dx}$ gives the **rate of change in $y$ with respect to $x$**.

We can therefore use the derivative of a function to tell us the **rate** at which something is happening.

For example:

- $\dfrac{dH}{dt}$ or $H'(t)$ could be the instantaneous rate of ascent of a person in a Ferris wheel.
  It might have units metres per second or $\text{m s}^{-1}$.
- $\dfrac{dC}{dt}$ or $C'(t)$ could be a person's instantaneous rate of change in lung capacity.
  It might have units litres per second or $\text{L s}^{-1}$.

### Example 15 — Self Tutor

According to a psychologist, the ability of a person to understand spatial concepts is given by $A = \frac{1}{3}\sqrt{t}$ where $t$ is the age in years, $5 \leqslant t \leqslant 18$.

**a** Find the rate of improvement in ability to understand spatial concepts when a person is:
  **i** 9 years old  **ii** 16 years old.

**b** Show that $\dfrac{dA}{dt} > 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

**c** Show that $\dfrac{d^2A}{dt^2} < 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

**a** $A = \frac{1}{3}\sqrt{t} = \frac{1}{3}t^{\frac{1}{2}}$

$\therefore \dfrac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}} = \dfrac{1}{6\sqrt{t}}$

**i** When $t = 9$, $\dfrac{dA}{dt} = \frac{1}{18}$

$\therefore$ the rate of improvement is $\frac{1}{18}$ units per year for a 9 year old.

**ii** When $t = 16$, $\dfrac{dA}{dt} = \frac{1}{24}$

$\therefore$ the rate of improvement is $\frac{1}{24}$ units per year for a 16 year old.

**b** Since $\sqrt{t}$ is never negative, $\dfrac{1}{6\sqrt{t}}$ is never negative

$\therefore \dfrac{dA}{dt} > 0$ for all $5 \leqslant t \leqslant 18$.

This means that the ability to understand spatial concepts increases with age.

**c** $\frac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}}$

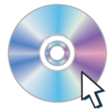
$\therefore \frac{d^2A}{dt^2} = -\frac{1}{12}t^{-\frac{3}{2}} = -\frac{1}{12t\sqrt{t}}$

$\therefore \frac{d^2A}{dt^2} < 0$ for all $5 \leqslant t \leqslant 18$.

This means that while the ability to understand spatial concepts increases with age, the rate of increase slows down with age.

You are encouraged to use technology to graph each function you need to consider. This is often useful in interpreting results.

GRAPHING PACKAGE



## EXERCISE 14D

**1** The estimated future profits of a small business are given by $P(t) = 2t^2 - 12t + 118$ thousand dollars, where $t$ is the time in years from now.

**a** What is the current annual profit?

**b** Find $\frac{dP}{dt}$ and state its units.

**c** Explain the significance of $\frac{dP}{dt}$.

**d** For what values of $t$ will the profit:

**i** decrease **ii** increase on the previous year?

**e** What is the minimum profit and when does it occur?

**f** Find $\frac{dP}{dt}$ when $t = 4$, 10, and 25. What do these figures represent?

**2** The quantity of a chemical in human skin which is responsible for its 'elasticity' is given by $Q = 100 - 10\sqrt{t}$ where $t$ is the age of a person in years.

**a** Find $Q$ at:

**i** $t = 0$ **ii** $t = 25$ **iii** $t = 100$ years.

**b** At what rate is the quantity of the chemical changing at the age of:

**i** 25 years **ii** 50 years?

**c** Show that the quantity of the chemical is decreasing for all $t > 0$.

**3** The height of *pinus radiata*, grown in ideal conditions, is given by $H = 20 - \frac{97.5}{t+5}$ metres, where $t$ is the number of years after the tree was planted from an established seedling.

**a** How high was the tree at the time of its planting?

**b** Find the height of the tree after 4, 8, and 12 years.

**c** Find the rate at which the tree is growing after 0, 5, and 10 years.

**d** Show that $\frac{dH}{dt} > 0$ for all $t \geqslant 0$.

What is the significance of this result?

## Example 16

Self Tutor

The cost in dollars of producing $x$ items in a factory each day is given by

$$C(x) = \underbrace{0.000\,13x^3 + 0.002x^2}_{\text{labour}} + \underbrace{5x}_{\text{raw materials}} + \underbrace{2200}_{\text{fixed costs}}$$

**a** Find $C'(x)$, which is called the marginal cost function.

**b** Find the marginal cost when 150 items are produced. Interpret this result.

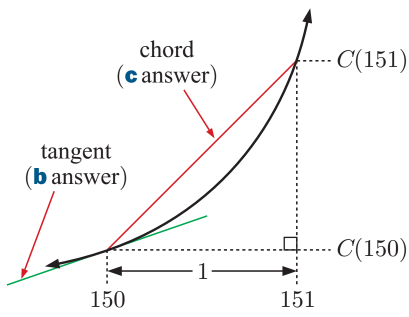
**c** Find $C(151) - C(150)$. Compare this with the answer in **b**.

**a** The marginal cost function is
$C'(x) = 0.000\,39x^2 + 0.004x + 5$ dollars per item.

**b** $C'(150) = \$14.38$

This is the rate at which the costs are increasing with respect to the production level $x$ when 150 items are made per day.

It gives an estimate of the cost of making the 151st item each day.




**c** $C(151) - C(150) \approx \$3448.19 - \$3433.75$
$\approx \$14.44$

This is the actual cost of making the 151st item each day, so the answer in **b** gives a good estimate.

**4** Seablue make denim jeans. The cost model for making $x$ pairs per day is

$$C(x) = 0.0003x^3 + 0.02x^2 + 4x + 2250 \text{ dollars.}$$

**a** Find the marginal cost function $C'(x)$.

**b** Find $C'(220)$. What does it estimate?

**c** Find $C(221) - C(220)$. What does this represent?

**d** Find $C''(x)$ and the value of $x$ when $C''(x) = 0$. What is the significance of this point?

**5**

The total cost of running a train from Paris to Marseille is given by $C(v) = \frac{1}{5}v^2 + \frac{200\,000}{v}$ euros where $v$ is the average speed of the train in $\text{km h}^{-1}$.

**a** Find the total cost of the journey if the average speed is:

**i** $50\ \text{km h}^{-1}$ **ii** $100\ \text{km h}^{-1}$.

**b** Find the rate of change in the cost of running the train at speeds of:

**i** $30\ \text{km h}^{-1}$ **ii** $90\ \text{km h}^{-1}$.

**c** At what speed will the cost be a minimum?

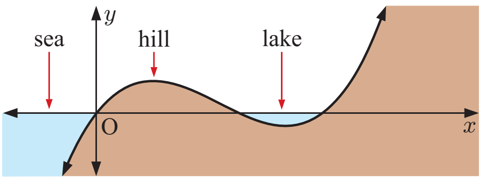
**6** A tank contains 50 000 litres of water. The tap is left fully on and all the water drains from the tank in 80 minutes. The volume of water remaining in the tank after $t$ minutes is given by $V = 50\,000\left(1 - \frac{t}{80}\right)^2$ litres where $0 \leqslant t \leqslant 80$.

**a** Find $\frac{dV}{dt}$ and draw the graph of $\frac{dV}{dt}$ against $t$.

**b** At what time was the outflow fastest?

**c** Show that $\frac{d^2V}{dt^2}$ is always constant and positive.
Interpret this result.

**7**




Alongside is a land and sea profile where the $x$-axis is sea level. The function $y = \frac{1}{10}x(x-2)(x-3)$ km gives the height of the land or sea bed relative to sea level at distance $x$ km from the shore line.

**a** Find where the lake is located relative to the shore line of the sea.

**b** Find $\frac{dy}{dx}$ and interpret its value when $x = \frac{1}{2}$ km and when $x = 1\frac{1}{2}$ km.

**c** Find the deepest point of the lake, and the depth at this point.

**8** A radioactive substance decays according to the formula $W = 20e^{-kt}$ grams where $t$ is the time in hours.

**a** Find $k$ given that after 50 hours the weight is 10 grams.

**b** Find the weight of radioactive substance present:
  **i** initially **ii** after 24 hours **iii** after 1 week.

**c** How long will it take for the weight to reach 1 gram?

**d** Find the rate of radioactive decay at: **i** $t = 100$ hours **ii** $t = 1000$ hours.

**e** Show that $\frac{dW}{dt}$ is proportional to the weight of substance remaining.

**9** The temperature of a liquid after being placed in a refrigerator is given by $T = 5 + 95e^{-kt}$ °C where $k$ is a positive constant and $t$ is the time in minutes.

**a** Find $k$ if the temperature of the liquid is 20°C after 15 minutes.

**b** What was the temperature of the liquid when it was first placed in the refrigerator?

**c** Show that $\frac{dT}{dt} = c(T-5)$ for some constant $c$. Find the value of $c$.

**d** At what rate is the temperature changing at:
  **i** $t = 0$ mins **ii** $t = 10$ mins **iii** $t = 20$ mins?

**10** The height of a shrub $t$ years after it is planted is given by $H(t) = 20\ln(3t+2) + 30$ cm, $t \geqslant 0$.

**a** How high was the shrub when it was planted?

**b** How long will it take for the shrub to reach a height of 1 m?

**c** At what rate is the shrub's height changing:
  **i** 3 years after being planted **ii** 10 years after being planted?

**11** In the conversion of sugar solution to alcohol, the chemical reaction obeys the law $A = s(1 - e^{-kt})$, $t \geqslant 0$ where $t$ is the number of hours after the reaction commences, $s$ is the original sugar concentration (%), and $A$ is the alcohol produced, in litres.

**a** Find $A$ when $t = 0$.

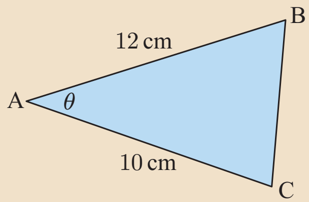
**b** Suppose $s = 10$ and $A = 5$ after 3 hours.

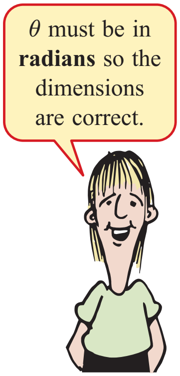
**i** Find $k$. **ii** Find the speed of the reaction at time 5 hours.

**c** Show that the speed of the reaction is proportional to $A - s$.

### Example 17 — Self Tutor

Find the rate of change in the area of triangle ABC as $\theta$ changes, at the time when $\theta = 60°$.




Area $A = \frac{1}{2} \times 10 \times 12 \times \sin\theta$ $\{\text{Area} = \frac{1}{2}bc\sin A\}$

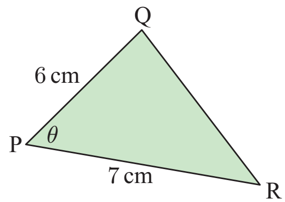
$\therefore$ $A = 60\sin\theta$ cm$^2$

$\therefore$ $\dfrac{dA}{d\theta} = 60\cos\theta$

When $\theta = \frac{\pi}{3}$, $\cos\theta = \frac{1}{2}$

$\therefore$ $\dfrac{dA}{d\theta} = 30$ cm$^2$ per radian

**12** Find the rate of change in the area of triangle PQR as $\theta$ changes, at the time when $\theta = 45°$.

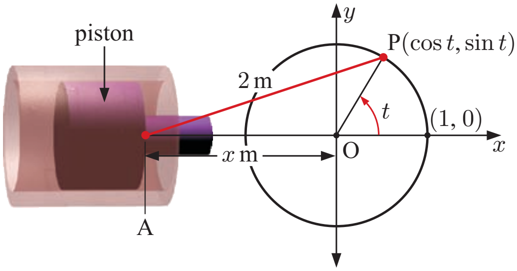



**13** On the Indonesian coast, the depth of water at time $t$ hours after midnight is given by $d = 9.3 + 6.8\cos(0.507t)$ metres.

**a** Find the rate of change in the depth of water at 8:00 am.

**b** Is the tide rising or falling at this time?

**14** A piston is operated by rod [AP] attached to a flywheel of radius 1 m. AP = 2 m. P has coordinates $(\cos t, \sin t)$ and point A is $(-x, 0)$.




**a** Show that $x = \sqrt{4 - \sin^2 t} - \cos t$.

**b** Find the rate at which $x$ is changing at the instant when:

**i** $t = 0$ **ii** $t = \frac{\pi}{2}$ **iii** $t = \frac{2\pi}{3}$

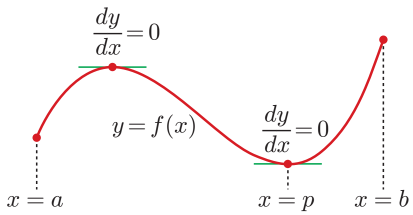
# E OPTIMISATION

There are many problems for which we need to find the **maximum** or **minimum** value of a function. The solution is often referred to as the **optimum** solution and the process is called **optimisation**.

The maximum or minimum value does not always occur when the first derivative is zero.

It is essential to also examine the values of the function at the endpoint(s) of the interval under consideration for global maxima and minima.

For example:




The maximum value of $y$ occurs at the endpoint $x = b$.

The minimum value of $y$ occurs at the local minimum $x = p$.

## OPTIMISATION PROBLEM SOLVING METHOD

*Step 1*: Draw a large, clear diagram of the situation.

*Step 2*: Construct a formula with the variable to be **optimised** as the subject. It should be written in terms of **one** convenient variable, for example $x$. You should write down what domain restrictions there are on $x$.

*Step 3*: Find the **first derivative** and find the values of $x$ which make the first derivative **zero**.

*Step 4*: For each stationary point, use a sign diagram to determine if you have a local maximum or local minimum.

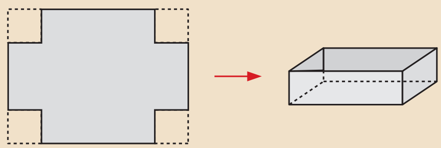
*Step 5*: Identify the optimum solution, also considering endpoints where appropriate.

*Step 6*: Write your answer in a sentence, making sure you specifically answer the question.

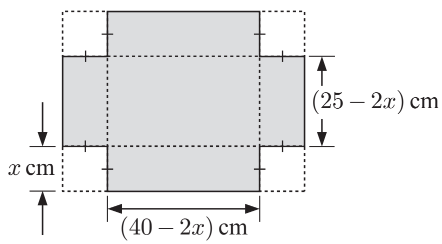
### Example 18 — Self Tutor

A rectangular cake dish is made by cutting out squares from the corners of a 25 cm by 40 cm rectangle of tin-plate, and then folding the metal to form the container.

What size squares must be cut out to produce the cake dish of maximum volume?



*Step 1*: Let $x$ cm be the side lengths of the squares that are cut out.




*Step 2*: Volume = length × width × depth

$$\begin{aligned} &= (40-2x)(25-2x)x \\ &= (1000-80x-50x+4x^2)x \\ &= 1000x-130x^2+4x^3 \quad \text{cm}^3 \end{aligned}$$

Since the side lengths must be positive, $x > 0$ and $25 - 2x > 0$.

$\therefore \quad 0 < x < 12.5$

*Step 3*: $\dfrac{dV}{dx} = 12x^2 - 260x + 1000$

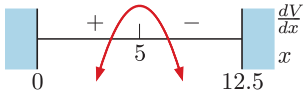
$= 4(3x^2 - 65x + 250)$

$= 4(3x - 50)(x - 5)$

$\therefore \dfrac{dV}{dx} = 0$ when $x = \frac{50}{3} = 16\frac{2}{3}$ or $x = 5$

DEMO

*Step 4*: $\dfrac{dV}{dx}$ has sign diagram:



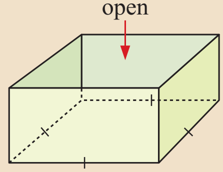

*Step 5*: There is a local maximum when $x = 5$. This is the global maximum for the given domain.

*Step 6*: The maximum volume is obtained when $x = 5$, which is when 5 cm squares are cut from the corners.

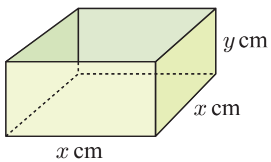
## Example 19

Self Tutor

A 4 litre container must have a square base, vertical sides, and an open top. Find the most economical shape which minimises the surface area of material needed.




*Step 1*:




Let the base lengths be $x$ cm and the depth be $y$ cm.

The volume $V = \text{length} \times \text{width} \times \text{depth}$

$\therefore V = x^2y$

$\therefore 4000 = x^2y$ .... (1) {1 litre ≡ 1000 cm$^3$}

*Step 2*: The total surface area $A = \text{area of base} + 4(\text{area of one side})$

$= x^2 + 4xy$

$= x^2 + 4x\left(\dfrac{4000}{x^2}\right)$ {using (1)}

$\therefore A(x) = x^2 + 16\,000x^{-1}$ where $x > 0$

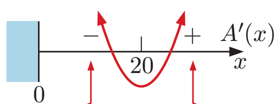
*Step 3*: $A'(x) = 2x - 16\,000x^{-2}$

$\therefore A'(x) = 0$ when $2x = \dfrac{16\,000}{x^2}$

$\therefore 2x^3 = 16\,000$

$\therefore x = \sqrt[3]{8000} = 20$

*Step 4*: $A'(x)$ has sign diagram:

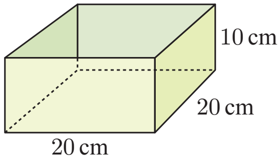



If $x = 10$,
$A'(10) = 20 - \frac{16\,000}{100}$
$= 20 - 160$
$= -140$

If $x = 30$,
$A'(30) = 60 - \frac{16\,000}{900}$
$\approx 60 - 17.8$
$\approx 42.2$

*Step 5*: The minimum material is used to make the container when $x = 20$ and $y = \dfrac{4000}{20^2} = 10$.

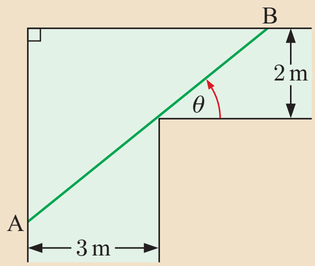
*Step 6*: The most economical shape has a square base $20 \text{ cm} \times 20 \text{ cm}$, and height 10 cm.




## Example 20

**Self Tutor**

Two corridors meet at right angles and are 2 m and 3 m wide respectively. $\theta$ is the angle marked on the given figure. [AB] is a thin metal tube which must be kept horizontal and cannot be bent as it moves around the corner from one corridor to the other.




**a** Show that the length AB is given by $L = \dfrac{3}{\cos\theta} + \dfrac{2}{\sin\theta}$.

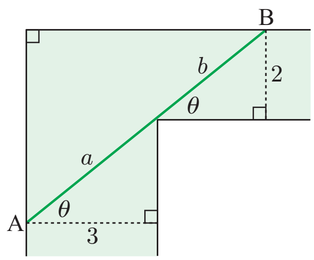
**b** Show that $\dfrac{dL}{d\theta} = 0$ when $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right) \approx 41.1°$.

**c** Find $L$ when $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$ and comment on the significance of this value.

**a** $\cos\theta = \dfrac{3}{a}$ and $\sin\theta = \dfrac{2}{b}$

$\therefore \quad a = \dfrac{3}{\cos\theta}$ and $b = \dfrac{2}{\sin\theta}$

$\therefore \quad L = a + b = \dfrac{3}{\cos\theta} + \dfrac{2}{\sin\theta}$





**b** $L = 3[\cos\theta]^{-1} + 2[\sin\theta]^{-1}$

$$\therefore \quad \frac{dL}{d\theta} = -3[\cos\theta]^{-2}(-\sin\theta) - 2[\sin\theta]^{-2}\cos\theta$$

$$= \frac{3\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta}$$

$$= \frac{3\sin^3\theta - 2\cos^3\theta}{\cos^2\theta\sin^2\theta}$$

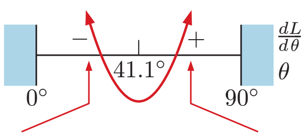
Thus $\dfrac{dL}{d\theta} = 0$ when $3\sin^3\theta = 2\cos^3\theta$

$\therefore \quad \tan^3\theta = \frac{2}{3}$

$\therefore \quad \tan\theta = \sqrt[3]{\frac{2}{3}}$

$\therefore \quad \theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right) \approx 41.1°$

**c** Sign diagram of $\dfrac{dL}{d\theta}$:




When $\theta = 30°$, $\dfrac{dL}{d\theta} \approx -4.93 < 0$

When $\theta = 60°$, $\dfrac{dL}{d\theta} \approx 9.06 > 0$

Thus, AB is minimised when $\theta \approx 41.1°$. At this time $L \approx 7.02$ metres. Ignoring the width of the rod, the greatest length of rod able to be horizontally carried around the corner is 7.02 m.

Use **calculus techniques** to answer the following problems.

GRAPHING PACKAGE

In cases where finding the zeros of the derivatives is difficult you may use the **graphing package** to help you.

## EXERCISE 14E

**1** When a manufacturer makes $x$ items per day, the cost function is $C(x) = 720 + 4x + 0.02x^2$ dollars, and the price function is $p(x) = 15 - 0.002x$ dollars per item. Find the production level that will maximise profits.

**2** A duck farmer wishes to build a rectangular enclosure of area $100\text{ m}^2$. The farmer must purchase wire netting for three of the sides, as the fourth side is an existing fence. Naturally, the farmer wishes to minimise the length (and therefore cost) of fencing required to complete the job.


**a** If the shorter sides have length $x$ m, show that the required length of wire netting to be purchased is $L = 2x + \dfrac{100}{x}$.

**b** Find the minimum value of $L$ and the corresponding value of $x$ when this occurs.

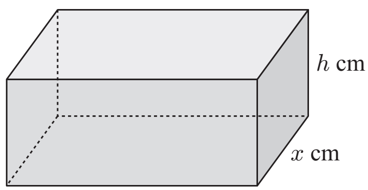
**c** Sketch the optimum situation, showing all dimensions.

**3** The total cost of producing $x$ blankets per day is $\frac{1}{4}x^2 + 8x + 20$ dollars, and for this production level each blanket may be sold for $(23 - \frac{1}{2}x)$ dollars.
How many blankets should be produced per day to maximise the total profit?

**4** The cost of running a boat is $\left(\dfrac{v^2}{10} + 22\right)$ dollars per hour, where $v\text{ km h}^{-1}$ is the speed of the boat.
Find the speed which will minimise the total cost per kilometre.

**5** A psychologist claims that the ability $A$ to memorise simple facts during infancy years can be calculated using the formula $A(t) = t\ln t + 1$ where $0 < t \leqslant 5$, $t$ being the age of the child in years. At what age is the child's memorising ability a minimum?

**6** Radioactive waste is to be disposed of in fully enclosed lead boxes of inner volume $200\text{ cm}^3$. The base of the box has dimensions in the ratio $2 : 1$.




**a** Show that $x^2h = 100$.

**b** Show that the inner surface area of the box is given by $A(x) = 4x^2 + \dfrac{600}{x}\text{ cm}^2$.

**c** Find the minimum inner surface area of the box and the corresponding value of $x$.

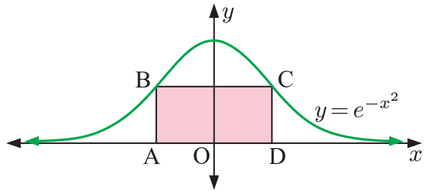
**d** Sketch the optimum box shape, showing all dimensions.

**7** A manufacturer of electric kettles performs a cost control study. They discover that to produce $x$ kettles per day, the cost per kettle is given by

$$C(x) = 4\ln x + \left(\frac{30 - x}{10}\right)^2 \quad \text{dollars}$$

with a minimum production capacity of 10 kettles per day.
How many kettles should be manufactured to keep the cost per kettle to a minimum?

**8** Infinitely many rectangles which sit on the $x$-axis can be inscribed under the curve $y = e^{-x^2}$.
Determine the coordinates of C such that rectangle ABCD has maximum area.

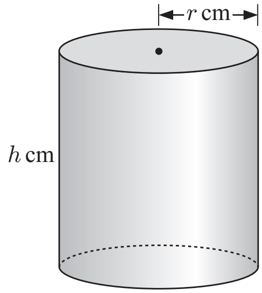



**9** Consider the manufacture of cylindrical tin cans of 1 L capacity, where the cost of the metal used is to be minimised.

**a** Explain why the height $h$ is given by $h = \dfrac{1000}{\pi r^2}$ cm.

**b** Show that the total surface area $A$ is given by
$A = 2\pi r^2 + \dfrac{2000}{r}$ cm$^2$.

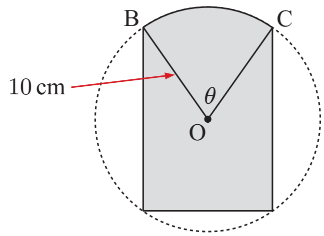
**c** Find the dimensions of the can which make $A$ as small as possible.




**10** A circular piece of tinplate of radius 10 cm has 3 segments removed as illustrated. The angle $\theta$ is measured in radians.

**a** Show that the remaining area is given by
$A = 50(\theta + 3\sin\theta)$ cm$^2$.

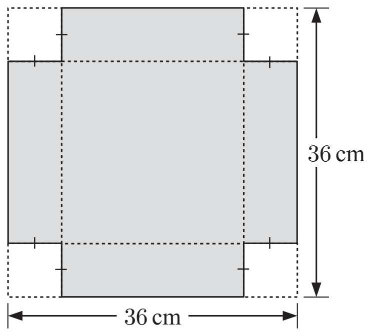
**b** Find $\theta$ such that the area $A$ is a maximum, and find the area $A$ in this case.




**11** Sam has sheets of metal which are 36 cm by 36 cm square. He wants to cut out identical squares which are $x$ cm by $x$ cm from the corners of each sheet. He will then bend the sheets along the dashed lines to form an open container.

**a** Show that the volume of the container is given by
$V(x) = x(36 - 2x)^2$ cm$^3$.

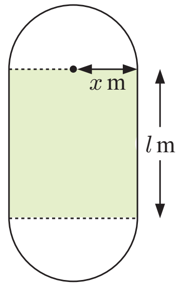
**b** What sized squares should be cut out to produce the container of greatest capacity?




**12** An athletics track has two 'straights' of length $l$ m, and two semicircular ends of radius $x$ m. The perimeter of the track is 400 m.

**a** Show that $l = 200 - \pi x$ and write down the possible values that $x$ may have.

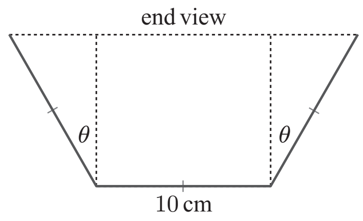
**b** What values of $l$ and $x$ maximise the shaded rectangle inside the track? What is this maximum area?




**13** A small population of wasps is observed. After $t$ weeks the population is modelled by $P(t) = \dfrac{50\,000}{1 + 1000e^{-0.5t}}$ wasps, where $0 \leqslant t \leqslant 25$.
Find when the wasp population is growing fastest.

**14** When a new pain killing injection is administered, the effect is modelled by $E(t) = 750te^{-1.5t}$ units, where $t \geqslant 0$ is the time in hours after the injection.
At what time is the drug most effective?

**15** A symmetrical gutter is made from a sheet of metal 30 cm wide by bending it twice as shown.

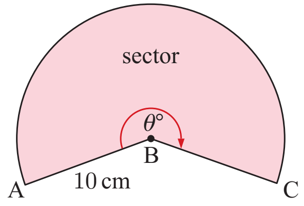



**a** Deduce that the cross-sectional area of the gutter is given by $A = 100\cos\theta(1+\sin\theta)$.

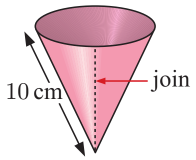
**b** Show that $\dfrac{dA}{d\theta} = 0$ when $\sin\theta = \frac{1}{2}$ or $-1$.

**c** For what value of $\theta$ does the gutter have maximum carrying capacity? Find the cross-sectional area for this value of $\theta$.

**16** A sector of radius 10 cm and angle $\theta°$ is bent to form a conical cup as shown.




becomes




when edges [AB] and [CB] are joined with tape.

Suppose the resulting cone has base radius $r$ cm and height $h$ cm.

**a** Show using the sector that arc AC $= \dfrac{\theta\pi}{18}$.

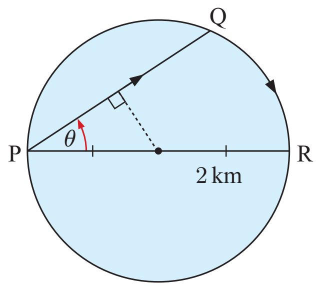
**b** Explain why $r = \dfrac{\theta}{36}$.

**c** Show that $h = \sqrt{100 - \left(\frac{\theta}{36}\right)^2}$.

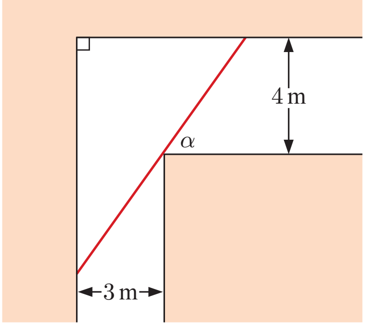
**d** Find the cone's capacity $V$ in terms of $\theta$ only.

**e** Find $\theta$ when $V(\theta)$ is a maximum.

**17** Hieu can row a boat across a circular lake of radius 2 km at 3 km h$^{-1}$. He can walk around the edge of the lake at 5 km h$^{-1}$.
What is the longest possible time Hieu could take to get from P to R by rowing from P to Q and then walking from Q to R?




**18** In a hospital, two corridors 4 m wide and 3 m wide meet at right angles. What is the maximum possible length of an X-ray screen which can be carried upright around the corner?

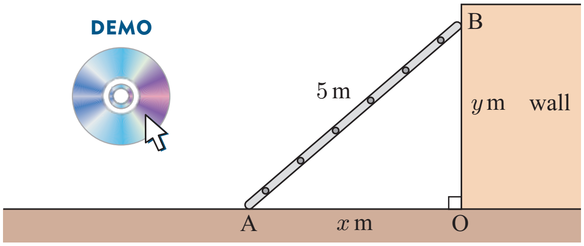



# F RELATED RATES

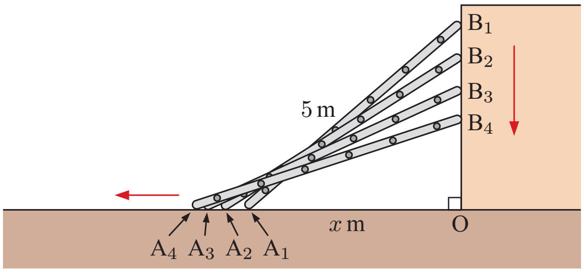
A 5 m ladder rests against a vertical wall at point B. Its feet are at point A on horizontal ground.

The ladder slips and slides down the wall.

Click on the icon to view the motion of the sliding ladder.




The following diagram shows the positions of the ladder at certain instances.




If AO $= x$ m and OB $= y$ m,
then $x^2 + y^2 = 5^2$. {Pythagoras}

Differentiating this equation with respect to time $t$ gives $\quad 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$

or $\quad x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = 0.$

This equation is called a **differential equation** and describes the motion of the ladder at any instant.

$\dfrac{dx}{dt}$ is the rate of change in $x$ with respect to time $t$, and is the speed of A relative to point O.

$\dfrac{dx}{dt}$ is *positive* as $x$ is increasing.

$\dfrac{dy}{dt}$ is the rate of change in $y$ with respect to time $t$, and is the speed at which B moves downwards.

$\dfrac{dy}{dt}$ is *negative* as $y$ is decreasing.

Problems involving differential equations where one of the variables is time $t$ are called **related rates** problems.

The method for solving related rates problems is:

*Step 1*: Draw a large, clear **diagram** of the situation. Sometimes two or more diagrams are necessary.

*Step 2*: Write down the information, label the diagram(s), and make sure you distinguish between the **variables** and the **constants**.

*Step 3*: Write an **equation** connecting the variables. You will often need to use:
- Pythagoras' theorem
- similar triangles
- right angled triangle trigonometry
- sine and cosine rules.

*Step 4*: **Differentiate** the equation with respect to $t$ to obtain a **differential equation**.

*Step 5*: Solve for the **particular case** which is some instant in time.

**Warning:**

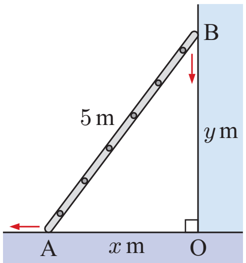
We **must not** substitute values for the particular case too early. Otherwise we will incorrectly treat variables as constants. The differential equation in fully generalised form must be established first.

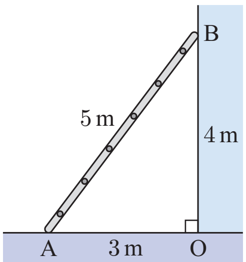
## Example 21

Self Tutor

A 5 m long ladder rests against a vertical wall with its feet on horizontal ground. The feet on the ground slip, and at the instant when they are 3 m from the wall, they are moving at 10 m s$^{-1}$.

At what speed is the other end of the ladder moving at this instant?




Let OA $= x$ m and OB $= y$ m

$\therefore \quad x^2 + y^2 = 5^2 \qquad$ {Pythagoras}

Differentiating with respect to $t$ gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\therefore \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

*Particular case*:

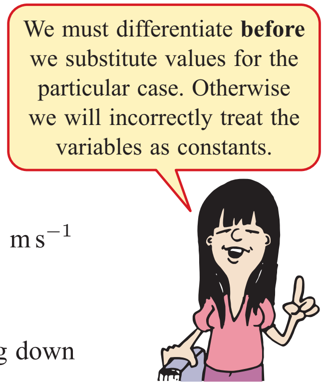
At the instant when $\dfrac{dx}{dt} = 10$ m s$^{-1}$,

$$\therefore \quad 3(10) + 4\frac{dy}{dt} = 0$$

$$\therefore \quad \frac{dy}{dt} = -\tfrac{15}{2} = -7.5 \text{ m s}^{-1}$$

Thus OB is decreasing at 7.5 m s$^{-1}$.

$\therefore$ the other end of the ladder is moving down the wall at 7.5 m s$^{-1}$ at that instant.




## Example 22

Self Tutor

A cube is expanding so its volume increases at a constant rate of 10 cm$^3$ s$^{-1}$. Find the rate of change in its total surface area, at the instant when its sides are 20 cm long.

Let $x$ cm be the lengths of the sides of the cube, so the surface area $A = 6x^2$ cm$^2$ and the volume $V = x^3$ cm$^3$.

$$\therefore \quad \frac{dA}{dt} = 12x\frac{dx}{dt} \quad \text{and} \quad \frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

*Particular case*:

At the instant when $x = 20$, $\dfrac{dV}{dt} = 10$

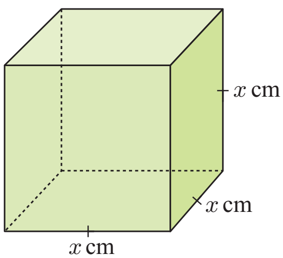
$$\therefore \quad 10 = 3 \times 20^2 \times \frac{dx}{dt}$$

$$\therefore \quad \frac{dx}{dt} = \tfrac{10}{1200} = \tfrac{1}{120} \text{ cm s}^{-1}$$

Thus $\dfrac{dA}{dt} = 12 \times 20 \times \tfrac{1}{120}$ cm$^2$ s$^{-1}$

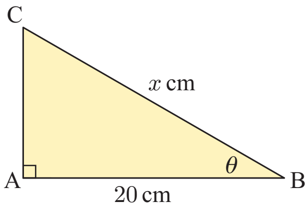
$= 2$ cm$^2$ s$^{-1}$

$\therefore$ the surface area is increasing at 2 cm$^2$ s$^{-1}$.




### Example 23 — Self Tutor

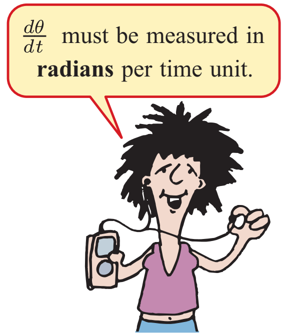
Triangle ABC is right angled at A, and AB $= 20$ cm. $A\widehat{B}C$ increases at a constant rate of $1°$ per minute. At what rate is BC changing at the instant when $A\widehat{B}C$ measures $30°$?




Let $A\widehat{B}C = \theta$ and BC $= x$ cm

Now $\cos\theta = \dfrac{20}{x} = 20x^{-1}$

$\therefore \ -\sin\theta \dfrac{d\theta}{dt} = -20x^{-2}\dfrac{dx}{dt}$

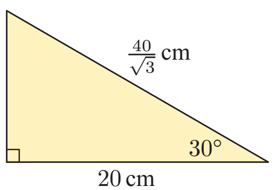



*Particular case*:

When $\theta = 30°$, $\cos 30° = \dfrac{20}{x}$

$\therefore \ \dfrac{\sqrt{3}}{2} = \dfrac{20}{x}$

$\therefore \ x = \dfrac{40}{\sqrt{3}}$




Also, $\dfrac{d\theta}{dt} = 1°$ per min

$= \frac{\pi}{180}$ radians per min

Thus $-\sin 30° \times \frac{\pi}{180} = -20 \times \dfrac{3}{1600} \times \dfrac{dx}{dt}$

$\therefore \ -\frac{1}{2} \times \frac{\pi}{180} = -\frac{3}{80}\dfrac{dx}{dt}$

$\therefore \ \dfrac{dx}{dt} = \frac{\pi}{360} \times \frac{80}{3}$ cm per min

$\approx 0.2327$ cm per min

$\therefore$ BC is increasing at approximately 0.233 cm per min.

## EXERCISE 14F

**1** $a$ and $b$ are variables related by the equation $ab^3 = 40$. At the instant when $a = 5$, $b$ is increasing at 1 unit per second. What is happening to $a$ at this instant?

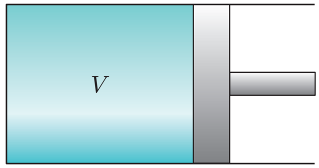
**2** The length of a rectangle is decreasing at 1 cm per minute. However, the area of the rectangle remains constant at 100 cm$^2$. At what rate is the breadth increasing at the instant when the rectangle is a square?

**3** A stone is thrown into a lake and a circular ripple moves out at a constant speed of 1 m s$^{-1}$. Find the rate at which the circle's area is increasing at the instant when:

**a** $t = 2$ seconds **b** $t = 4$ seconds.

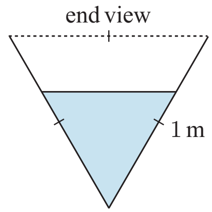
**4** Air is pumped into a spherical weather balloon at a constant rate of $6\pi$ m$^3$ per minute. Find the rate of change in its surface area at the instant when the radius of the balloon is 2 m.

**5** For a given mass of gas in a piston, $pV^{1.5} = 400$ where $p$ is the pressure in $\text{N m}^{-2}$, and $V$ is the volume in $\text{m}^3$.
Suppose the pressure increases at a constant rate of 3 $\text{N m}^{-2}$ per minute. Find the rate at which the volume is changing at the instant when the pressure is 50 $\text{N m}^{-2}$.




**6** Wheat runs from a hole in a silo at a constant rate and forms a conical heap whose base radius is treble its height. After 1 minute, the height of the heap is 20 cm. Find the rate at which the height is rising at this instant.

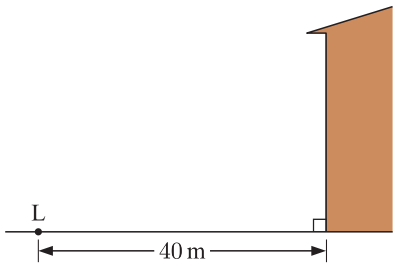
**7** A trough of length 6 m has a uniform cross-section which is an equilateral triangle with sides of length 1 m. Water leaks from the bottom of the trough at a constant rate of 0.1 $\text{m}^3$ per min.
Find the rate at which the water level is falling at the instant when the water is 20 cm deep.




**8** Two jet aeroplanes fly on parallel courses which are 12 km apart. Their air speeds are 200 $\text{m s}^{-1}$ and 250 $\text{m s}^{-1}$ respectively. How fast is the distance between them changing at the instant when the slower jet is 5 km ahead of the faster one?

**9** A ground-level floodlight located 40 m from the foot of a building shines in the direction of the building.
A 2 m tall person walks directly from the floodlight towards the building at 1 $\text{m s}^{-1}$. How fast is the person's shadow on the building shortening at the instant when the person is:

**a** 20 m from the building
**b** 10 m from the building?




**10** A right angled triangle ABC has a fixed hypotenuse [AC] of length 10 cm, and side [AB] increases in length at 0.1 $\text{cm s}^{-1}$. At what rate is $\widehat{CAB}$ decreasing at the instant when the triangle is isosceles?

**11** Triangle PQR is right angled at Q, and [PQ] is 6 cm long. [QR] increases in length at 2 cm per minute. Find the rate of change in $\widehat{QPR}$ at the instant when [QR] is 8 cm long.

## Review set 14A

**1** Find the equation of the tangent to:

**a** $y = -2x^2$ at the point where $x = -1$

**b** $f(x) = 4\ln(2x)$ at the point $(1, 4\ln 2)$

**c** $f(x) = \dfrac{e^x}{x-1}$ at the point where $x = 2$.

**2** The tangent to $y = \dfrac{ax+b}{\sqrt{x}}$ at $x = 1$ is $2x - y = 1$. Find $a$ and $b$.

**3** Suppose $f(x) = x^3 + ax$, $a < 0$ has a turning point when $x = \sqrt{2}$.

**a** Find $a$.
**b** Find the position and nature of all stationary points of $y = f(x)$.
**c** Sketch the graph of $y = f(x)$.

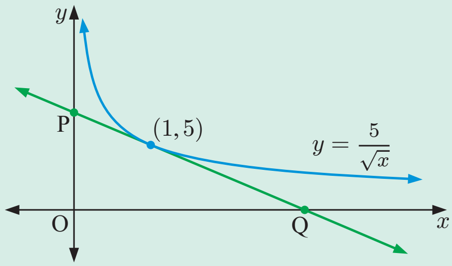
**4** Find the equation of the normal to:

**a** $y = \dfrac{x+1}{x^2-2}$ at the point where $x = 1$ **b** $\sqrt{x+1}$ at the point where $x = 3$.

**5** The tangent to $y = x^2\sqrt{1-x}$ at $x = -3$ cuts the axes at points A and B. Determine the area of triangle OAB.

**6** The line through A(2, 4) and B(0, 8) is a tangent to $y = \dfrac{a}{(x+2)^2}$. Find $a$.

**7** Find the coordinates of P and Q if PQ is the tangent to $y = \dfrac{5}{\sqrt{x}}$ at (1, 5).




**8** Show that $y = 2 - \dfrac{7}{1+2x}$ has no horizontal tangents.

**9** Show that the curves whose equations are $y = \sqrt{3x+1}$ and $y = \sqrt{5x - x^2}$ have a common tangent at their point of intersection. Find the equation of this common tangent.

**10** Consider the function $f(x) = x + \ln x$.

**a** Find the values of $x$ for which $f(x)$ is defined.

**b** Find the sign of $f'(x)$ and comment on its geometrical significance.

**c** Sketch the graph of $y = f(x)$.

**d** Find the equation of the normal at the point where $x = 1$.

**11** **a** Sketch the graph of $x \mapsto \dfrac{4}{x}$ for $x > 0$.

**b** Find the equation of the tangent to the function at the point where $x = k$, $k > 0$.

**c** If the tangent in **b** cuts the $x$-axis at A and the $y$-axis at B, find the coordinates of A and B.

**d** What can be deduced about the area of triangle OAB?

**e** Find $k$ if the normal to the curve at $x = k$ passes through the point (1, 1).

**12** A particle P moves in a straight line with position relative to the origin O given by $s(t) = 2t^3 - 9t^2 + 12t - 5$ cm, where $t$ is the time in seconds, $t \geqslant 0$.

**a** Find expressions for the particle's velocity and acceleration and draw sign diagrams for each of them.

**b** Find the initial conditions.

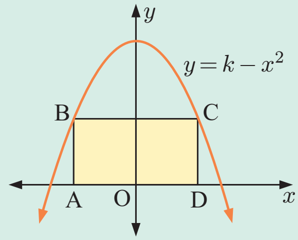
**c** Describe the motion of the particle at time $t = 2$ seconds.

**d** Find the times and positions where the particle changes direction.

**e** Draw a diagram to illustrate the motion of P.

**f** Determine the time intervals when the particle's speed is increasing.

**13** Rectangle ABCD is inscribed within the parabola $y = k - x^2$ and the $x$-axis, as shown.

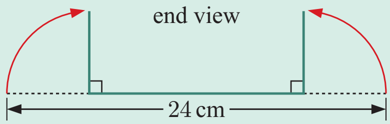



- **a** If $\text{OD} = x$, show that the rectangle ABCD has area function $A(x) = 2kx - 2x^3$.
- **b** If the area of ABCD is a maximum when $\text{AD} = 2\sqrt{3}$, find $k$.

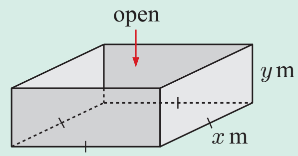
**14** A particle moves in a straight line along the $x$-axis with position given by $x(t) = 3 + \sin(2t)$ cm after $t$ seconds.

- **a** Find the initial position, velocity, and acceleration of the particle.
- **b** Find the times when the particle changes direction during $0 \leqslant t \leqslant \pi$ seconds.
- **c** Find the total distance travelled by the particle in the first $\pi$ seconds.

**15** A rectangular gutter is formed by bending a 24 cm wide sheet of metal as shown.
Where must the bends be made in order to maximise the capacity of the gutter?




**16** A manufacturer of open steel boxes has to make one with a square base and a capacity of 1 m$^3$. The steel costs \$2 per square metre.

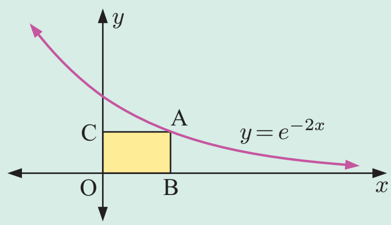



- **a** If the base measures $x$ m by $x$ m and the height is $y$ m, find $y$ in terms of $x$.
- **b** Hence, show that the total cost of the steel is $C(x) = 2x^2 + \dfrac{8}{x}$ dollars.
- **c** Find the dimensions of the box which would cost the least in steel to make.

**17** A particle P moves in a straight line with position from O given by $s(t) = 15t - \dfrac{60}{(t+1)^2}$ cm, where $t$ is the time in seconds, $t \geqslant 0$.

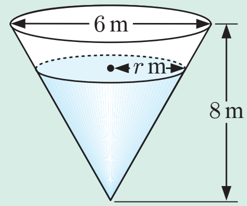
- **a** Find velocity and acceleration functions for P's motion.
- **b** Describe the motion of P at $t = 3$ seconds.
- **c** For what values of $t$ is the particle's speed increasing?

**18** Infinitely many rectangles can be inscribed under the curve $y = e^{-2x}$ as shown.
Determine the coordinates of A such that the rectangle OBAC has maximum area.




**19** A man on a jetty pulls a boat directly towards him so the rope is coming in at a rate of 20 metres per minute. The rope is attached to the boat 1 m above water level, and the man's hands are 6 m above the water level. How fast is the boat approaching the jetty at the instant when it is 15 m from the jetty?

**20** Water exits a conical tank at a constant rate of $0.2\ \text{m}^3$ per minute. Suppose the surface of the water has radius $r$.




**a** Find $V(r)$, the volume of the water remaining in the tank.

**b** Find the rate at which the surface radius is changing at the instant when the water is 5 m deep.

## Review set 14B

**1** Find the equation of the normal to:

**a** $y = \dfrac{1-2x}{x^2}$ at the point where $x = 1$

**b** $y = e^{-x^2}$ at the point where $x = 1$

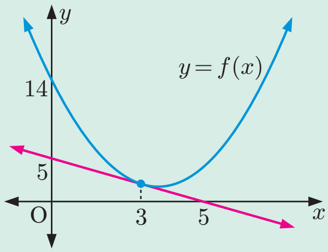
**c** $y = \dfrac{1}{\sqrt{x}}$ at the point where $x = 4$.

**2** The curve $y = 2x^3 + ax + b$ has a tangent with gradient 10 at the point $(-2, 33)$. Find the values of $a$ and $b$.

**3** $y = f(x)$ is the parabola shown.

**a** Find $f(3)$ and $f'(3)$.

**b** Hence find $f(x)$ in the form $f(x) = ax^2 + bx + c$.




**4** Find the equation of:

**a** the tangent to $y = \dfrac{1}{\sin x}$ at the point where $x = \frac{\pi}{3}$

**b** the normal to $y = \cos(\frac{x}{2})$ at the point where $x = \frac{\pi}{2}$.

**5** At the point where $x = 0$, the tangent to $f(x) = e^{4x} + px + q$ has equation $y = 5x - 7$. Find $p$ and $q$.

**6** Find where the tangent to $y = 2x^3 + 4x - 1$ at $(1, 5)$ cuts the curve again.

**7** Find $a$ given that the tangent to $y = \dfrac{4}{(ax+1)^2}$ at $x = 0$ passes through $(1, 0)$.

**8** Consider the function $f(x) = e^x - x$.

**a** Find and classify any stationary points of $y = f(x)$.

**b** Show that $e^x \geqslant x + 1$ for all $x$.

**c** Find $f''(x)$.

**9** Find where the tangent to $y = \ln(x^4 + 3)$ at $x = 1$ cuts the $y$-axis.

**10** Consider the function $f(x) = 2x^3 - 19x^2 + 52x - 35$.

**a** Find the $y$-intercept of the graph $y = f(x)$.

**b** Show that $x = 1$ is a root of the function, and hence find all roots.

**c** Find and classify all stationary points.

**d** Sketch the graph of $y = f(x)$, showing all important features.

**11** If the normal to $f(x) = \dfrac{3x}{1+x}$ at $(2, 2)$ cuts the axes at B and C, determine the length BC.

**12** The height of a tree $t$ years after it was planted is given by $H(t) = 60 + 40\ln(2t+1)$ cm, $t \geqslant 0$.

**a** How high was the tree when it was planted?

**b** How long does it take for the tree to reach:

**i** 150 cm **ii** 300 cm?

**c** At what rate is the tree's height increasing after:

**i** 2 years **ii** 20 years?

**13** A particle P moves in a straight line with position given by $s(t) = 80e^{-\frac{t}{10}} - 40t$ m where $t$ is the time in seconds, $t \geqslant 0$.

**a** Find the velocity and acceleration functions.

**b** Find the initial position, velocity, and acceleration of P.

**c** Sketch the graph of the velocity function.

**d** Find the exact time when the velocity is $-44$ m s$^{-1}$.

**14** The cost per hour of running a freight train is given by $C(v) = \dfrac{v^2}{30} + \dfrac{9000}{v}$ dollars where $v$ is the average speed of the train in km h$^{-1}$.

**a** Find the cost of running the train for:

**i** two hours at 45 km h$^{-1}$ **ii** 5 hours at 64 km h$^{-1}$.

**b** Find the rate of change in the hourly cost of running the train at speeds of:

**i** 50 km h$^{-1}$ **ii** 66 km h$^{-1}$.

**c** At what speed will the cost per hour be a minimum?

**15** A particle moves along the $x$-axis with position relative to origin O given by $x(t) = 3t - \sqrt{t+1}$ cm, where $t$ is the time in seconds, $t \geqslant 0$.

**a** Find expressions for the particle's velocity and acceleration at any time $t$, and draw sign diagrams for each function.

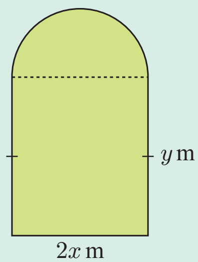
**b** Find the initial conditions, and hence describe the motion at that instant.

**c** Describe the motion of the particle at $t = 8$ seconds.

**d** Find the time and position when the particle reverses direction.

**e** Determine the time interval when the particle's speed is decreasing.

**16** A 200 m fence is placed around a lawn which has the shape of a rectangle with a semi-circle on one of its sides.

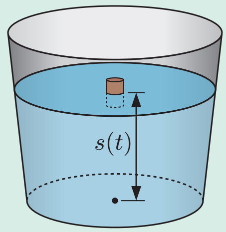



**a** Using the dimensions shown on the figure, show that $y = 100 - x - \frac{\pi}{2}x$.

**b** Find the area of the lawn $A$ in terms of $x$ only.

**c** Find the dimensions of the lawn if it has the maximum possible area.

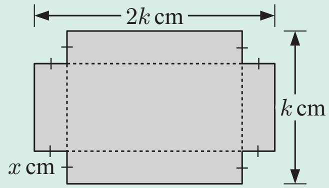
**17** A cork bobs up and down in a bucket of water such that the distance from the centre of the cork to the bottom of the bucket is given by $s(t) = 30 + \cos(\pi t)$ cm, $t \geqslant 0$ seconds.




**a** Find the cork's velocity at times $t = 0,\ \frac{1}{2},\ 1,\ 1\frac{1}{2}$, and 2 s.

**b** Find the time intervals when the cork is falling.

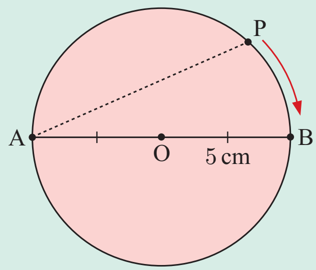
**18** A rectangular sheet of tin-plate is $2k$ cm by $k$ cm. Four squares, each with sides $x$ cm, are cut from its corners. The remainder is bent into the shape of an open rectangular container. Find the value of $x$ which will maximise the capacity of the container.




**19** Two runners run in different directions, $60°$ apart. A runs at 5 m s$^{-1}$ and B runs at 4 m s$^{-1}$. B passes through X 3 seconds after A passes through X.
At what rate is the distance between them increasing at the time when A is 20 metres past X?

**20** Rectangle PQRS has PQ of fixed length 20 cm, and [QR] increases in length at a constant rate of 2 cm s$^{-1}$. At what rate is the acute angle between the diagonals of the rectangle changing at the instant when QR is 15 cm long?

**21** AOB is a fixed diameter of a circle of radius 5 cm. Point P moves around the circle at a constant rate of 1 revolution in 10 seconds. Find the rate at which the distance AP is changing at the instant when:




**a** AP = 5 cm and increasing

**b** P is at B.

**22** Consider the parabola $f(x) = \dfrac{1}{4b}x^2$ where $b > 0$.

**a** **i** Find the equation of the tangent to $y = f(x)$ at the point $P(a,\ f(a))$.

**ii** Show that this meets the $x$-axis at the point $P'(\frac{a}{2},\ 0)$.

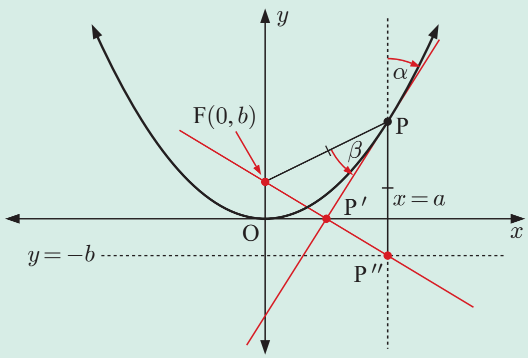
**b** **i** Find the equation of the line perpendicular to this tangent line, and which passes through P$'$.

**ii** Show that this line has $y$-intercept $F(0,\ b)$.

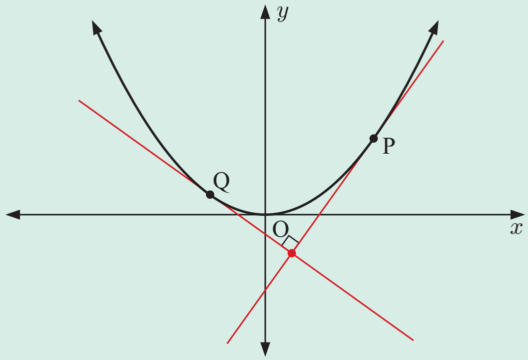
**iii** Show that the distance FP equals the distance from P to the line $y = -b$.

**c** The point $F(0,\ b)$ is *invariant* since it is independent of our choice of $a$. F is called the *focus* of the parabola. The line $y = -b$ is called the *directrix*.

**i** Prove the *reflective* property of the parabola, that any vertical ray will be reflected off the parabola into the focus.
**Hint:** Show that $\alpha = \beta$.



**ii** Suppose $a \neq 0$ and that $Q(c,\ f(c))$ is another point on the parabola such that the tangents from P and Q are perpendicular. Show that the intersection of the tangents occurs on the directrix.



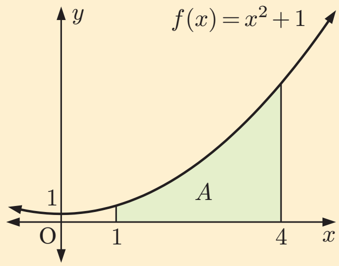
# 15

# Integration

Contents:

## Opening problem

The function $f(x) = x^2 + 1$ lies above the $x$-axis for all $x \in \mathbb{R}$.




**Things to think about:**

**a** How can we calculate the shaded area $A$, which is the area under the curve for $1 \leqslant x \leqslant 4$?

**b** What function has $x^2 + 1$ as its derivative?

In the previous chapters we used differential calculus to find the derivatives of many types of functions. We also used it in problem solving, in particular to find the gradients of graphs and rates of changes, and to solve optimisation problems.

In this chapter we consider **integral calculus**. This involves **antidifferentiation** which is the reverse process of differentiation. Integral calculus also has many useful applications, including:

- finding areas of shapes with curved boundaries
- finding volumes of revolution
- finding distances travelled from velocity functions
- solving problems in economics, biology, and statistics
- solving differential equations.

# A THE AREA UNDER A CURVE

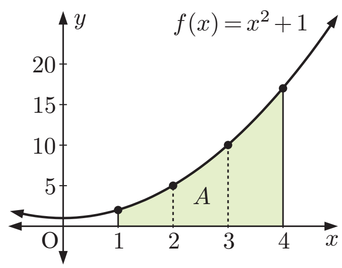
The task of finding the area under a curve has been important to mathematicians for thousands of years. In the history of mathematics it was fundamental to the development of integral calculus. We will therefore begin our study by calculating the area under a curve using the same methods as the ancient mathematicians.

## UPPER AND LOWER RECTANGLES

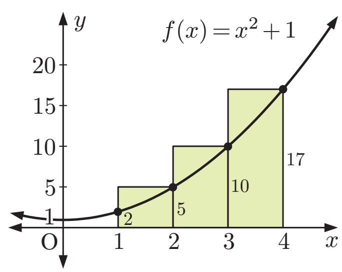
Consider the function $f(x) = x^2 + 1$.

We wish to estimate the area $A$ enclosed by $y = f(x)$, the $x$-axis, and the vertical lines $x = 1$ and $x = 4$.

Suppose we divide the interval $1 \leqslant x \leqslant 4$ into three strips of width 1 unit as shown. We obtain three subintervals of equal width.



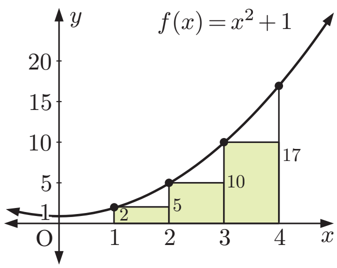

The diagram alongside shows **upper rectangles**, which are rectangles with top edges at the maximum value of the curve on that subinterval.




The area of the upper rectangles,

$$\begin{aligned} A_U &= 1 \times f(2) + 1 \times f(3) + 1 \times f(4) \\ &= 5 + 10 + 17 \\ &= 32 \text{ units}^2 \end{aligned}$$

The next diagram shows **lower rectangles**, which are rectangles with top edges at the minimum value of the curve on that subinterval.




The area of the lower rectangles,

$$\begin{aligned} A_L &= 1 \times f(1) + 1 \times f(2) + 1 \times f(3) \\ &= 2 + 5 + 10 \\ &= 17 \text{ units}^2 \end{aligned}$$

Now clearly $A_L < A < A_U$, so the area $A$ lies between 17 units$^2$ and 32 units$^2$.

If the interval $1 \leqslant x \leqslant 4$ was divided into 6 subintervals, each of width $\frac{1}{2}$, then

$$\begin{aligned} A_U &= \tfrac{1}{2}f(1\tfrac{1}{2}) + \tfrac{1}{2}f(2) + \tfrac{1}{2}f(2\tfrac{1}{2}) + \tfrac{1}{2}f(3) + \tfrac{1}{2}f(3\tfrac{1}{2}) + \tfrac{1}{2}f(4) \\ &= \tfrac{1}{2}(\tfrac{13}{4} + 5 + \tfrac{29}{4} + 10 + \tfrac{53}{4} + 17) \\ &= 27.875 \text{ units}^2 \end{aligned}$$

$$\begin{aligned} \text{and} \quad A_L &= \tfrac{1}{2}f(1) + \tfrac{1}{2}f(1\tfrac{1}{2}) + \tfrac{1}{2}f(2) + \tfrac{1}{2}f(2\tfrac{1}{2}) + \tfrac{1}{2}f(3) + \tfrac{1}{2}f(3\tfrac{1}{2}) \\ &= \tfrac{1}{2}(2 + \tfrac{13}{4} + 5 + \tfrac{29}{4} + 10 + \tfrac{53}{4}) \\ &= 20.375 \text{ units}^2 \end{aligned}$$

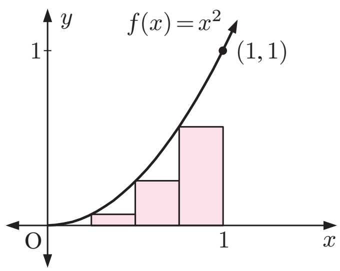
From this refinement we conclude that the area $A$ lies between 20.375 and 27.875 units$^2$.

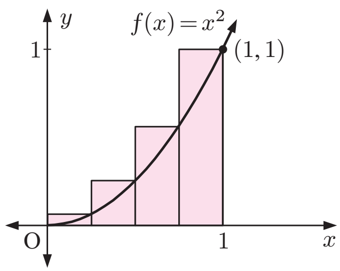
As we create more subintervals, the estimates $A_L$ and $A_U$ will become more and more accurate. In fact, as the subinterval width is reduced further and further, both $A_L$ and $A_U$ will **converge** to $A$.

We illustrate this process by estimating the area $A$ between the graph of $y = x^2$ and the $x$-axis for $0 \leqslant x \leqslant 1$.

This example is of historical interest. **Archimedes** (287 - 212 BC) found the exact area. In an article that contains 24 propositions, he developed the essential theory for what is now known as integral calculus.

Consider $f(x) = x^2$ and divide the interval $0 \leqslant x \leqslant 1$ into 4 subintervals of equal width.




$$\begin{aligned} A_L &= \tfrac{1}{4}(0)^2 + \tfrac{1}{4}(\tfrac{1}{4})^2 + \tfrac{1}{4}(\tfrac{1}{2})^2 + \tfrac{1}{4}(\tfrac{3}{4})^2 \\ &\approx 0.219 \end{aligned}$$

and

$$\begin{aligned} A_U &= \tfrac{1}{4}(\tfrac{1}{4})^2 + \tfrac{1}{4}(\tfrac{1}{2})^2 + \tfrac{1}{4}(\tfrac{3}{4})^2 + \tfrac{1}{4}(1)^2 \\ &\approx 0.469 \end{aligned}$$

Now suppose there are $n$ subintervals between $x = 0$ and $x = 1$, each of width $\dfrac{1}{n}$.

**AREA FINDER**

We can use the **area finder** software to help calculate $A_L$ and $A_U$ for large values of $n$.

The table alongside summarises the results you should obtain for $n = 4$, 10, 25, and 50.

The exact value of $A$ is in fact $\frac{1}{3}$, as we will find later in the chapter. Notice how both $A_L$ and $A_U$ are converging to this value as $n$ increases.

| $n$ | $A_L$ | $A_U$ | *Average* |
|---|---|---|---|
| 4 | 0.218 75 | 0.468 75 | 0.343 75 |
| 10 | 0.285 00 | 0.385 00 | 0.335 00 |
| 25 | 0.313 60 | 0.353 60 | 0.333 60 |
| 50 | 0.323 40 | 0.343 40 | 0.333 40 |

## EXERCISE 15A.1

**1** Consider the area between $y = x$ and the $x$-axis from $x = 0$ to $x = 1$.

**a** Divide the interval into 5 strips of equal width, then estimate the area using:

**i** upper rectangles **ii** lower rectangles.

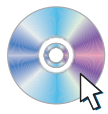
**b** Calculate the actual area and compare it with your answers in **a**.

**2** Consider the area between $y = \dfrac{1}{x}$ and the $x$-axis from $x = 2$ to $x = 4$. Divide the interval into 6 strips of equal width, then estimate the area using:

**a** upper rectangles **b** lower rectangles.

**3** Use rectangles to find lower and upper sums for the area between the graph of $y = x^2$ and the $x$-axis for $1 \leqslant x \leqslant 2$. Use $n = 10$, 25, 50, 100, and 500. Give your answers to 4 decimal places.
As $n$ gets larger, both $A_L$ and $A_U$ converge to the same number which is a simple fraction. What is it?

AREA FINDER



**4** **a** Use lower and upper sums to estimate the area between each of the following functions and the $x$-axis for $0 \leqslant x \leqslant 1$. Use values of $n = 5$, 10, 50, 100, 500, 1000, and 10 000. Give your answer to 5 decimal places in each case.

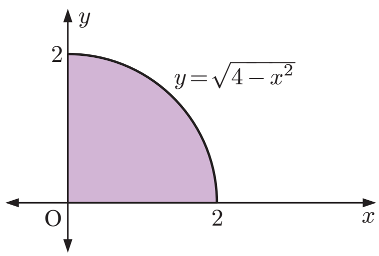
**i** $y = x^3$ **ii** $y = x$ **iii** $y = x^{\frac{1}{2}}$ **iv** $y = x^{\frac{1}{3}}$

**b** For each case in **a**, $A_L$ and $A_U$ converge to the same number which is a simple fraction. What fractions are they?

**c** Using your answer to **b**, predict the area between the graph of $y = x^a$ and the $x$-axis for $0 \leqslant x \leqslant 1$ and any number $a > 0$.

**5** Consider the quarter circle of centre $(0, 0)$ and radius 2 units illustrated.

Its area is $\frac{1}{4}$(full circle of radius 2)
$= \frac{1}{4} \times \pi \times 2^2$
$= \pi$ units$^2$



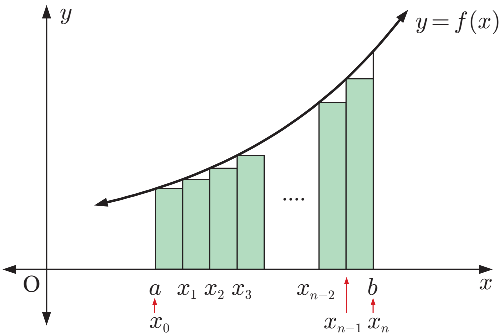

**a** Estimate the area using lower and upper rectangles for $n = 10$, 50, 100, 200, 1000, and 10 000. Hence, find rational bounds for $\pi$.

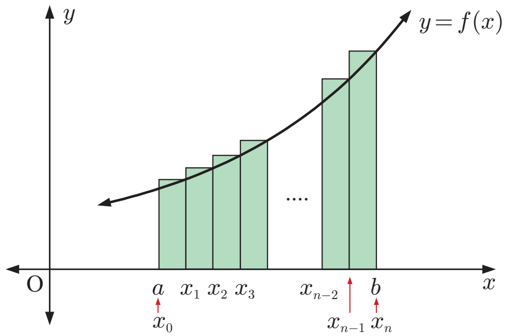
**b** Archimedes found the famous approximation $3\frac{10}{71} < \pi < 3\frac{1}{7}$.
For what value of $n$ is your estimate for $\pi$ better than that of Archimedes?

## THE DEFINITE INTEGRAL

Consider the lower and upper rectangle sums for a function which is positive and increasing on the interval $a \leqslant x \leqslant b$.

We divide the interval into $n$ subintervals, each of width $w = \dfrac{b-a}{n}$.




Since the function is increasing:

$$A_L = w\,f(x_0) + w\,f(x_1) + .... + w\,f(x_{n-2}) + w\,f(x_{n-1}) = w\sum_{i=0}^{n-1} f(x_i)$$

$$A_U = w\,f(x_1) + w\,f(x_2) + .... + w\,f(x_{n-1}) + w\,f(x_n) = w\sum_{i=1}^{n} f(x_i)$$

$\sum_{i=1}^{n} a_i = a_1 + a_2 + .... + a_n$

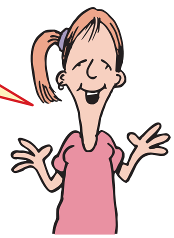
Notice that $\quad A_U - A_L = w\,(f(x_n) - f(x_0))$

$$= \frac{1}{n}(b-a)\,(f(b) - f(a))$$

$$\therefore \quad \lim_{n\to\infty} (A_U - A_L) = 0 \qquad \{\text{since } \lim_{n\to\infty} \frac{1}{n} = 0\}$$

$$\therefore \quad \lim_{n\to\infty} A_L = \lim_{n\to\infty} A_U \quad \{\text{when both limits exist}\}$$

$\lim_{n\to\infty}$ means we have infinitely many subintervals.



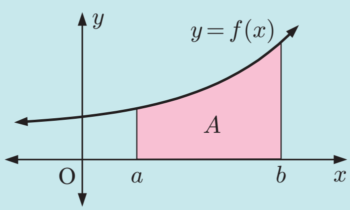
$\therefore$ since $A_L < A < A_U$ for all values of $n$, it follows that $\lim_{n\to\infty} A_L = A = \lim_{n\to\infty} A_U$

This fact is true for all positive continuous functions on an interval $a \leqslant x \leqslant b$.

The value $A$ is known as the "**definite integral** of $f(x)$ from $a$ to $b$", written $A = \displaystyle\int_a^b f(x)\,dx$.

If $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$, then

$\displaystyle\int_a^b f(x)\,dx$ is equal to the shaded area.



## Historical note

The word **integration** means "*to put together into a whole*". An **integral** is the "whole" produced from integration, since the areas $f(x_i) \times w$ of the thin rectangular strips are put together into one whole area.

The symbol $\int$ is called an **integral sign**. In the time of **Newton** and **Leibniz** it was the stretched out letter s, but it is no longer part of the alphabet.

## Example 1

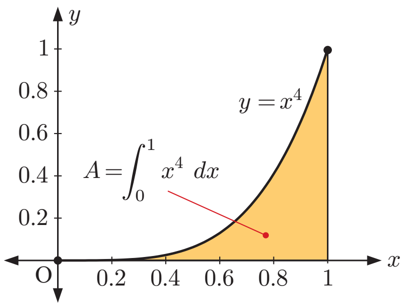
Self Tutor

**a** Sketch the graph of $y = x^4$ for $0 \leqslant x \leqslant 1$. Shade the area described by $\int_0^1 x^4\, dx$.

**b** Use technology to calculate the lower and upper rectangle sums for $n$ equal subintervals where $n = 5$, 10, 50, 100, and 500.

**c** Hence find $\int_0^1 x^4\, dx$ to 2 significant figures.

**a**




**b**

| $n$ | $A_L$ | $A_U$ |
|---|---|---|
| 5 | 0.1133 | 0.3133 |
| 10 | 0.1533 | 0.2533 |
| 50 | 0.1901 | 0.2101 |
| 100 | 0.1950 | 0.2050 |
| 500 | 0.1990 | 0.2010 |

**c** When $n = 500$, $A_L \approx A_U \approx 0.20$, to 2 significant figures.

$\therefore$ since $A_L < \int_0^1 x^4\, dx < A_U$, $\int_0^1 x^4\, dx \approx 0.20$

## EXERCISE 15A.2

AREA FINDER

**1** **a** Sketch the graph of $y = \sqrt{x}$ for $0 \leqslant x \leqslant 1$.

Shade the area described by $\int_0^1 \sqrt{x}\, dx$.

**b** Find the lower and upper rectangle sums for $n = 5$, 10, 50, 100, and 500.

**c** Hence find $\int_0^1 \sqrt{x}\, dx$ to 2 significant figures.

GRAPHING PACKAGE

**2** Consider the region enclosed by $y = \sqrt{1 + x^3}$ and the $x$-axis for $0 \leqslant x \leqslant 2$.

**a** Write expressions for the lower and upper rectangle sums using $n$ subintervals, $n \in \mathbb{N}$.

**b** Find the lower and upper rectangle sums for $n = 50$, 100, and 500.

**c** Hence estimate $\int_0^2 \sqrt{1 + x^3}\, dx$.

**3** The integral $\displaystyle\int_{-3}^{3} e^{-\frac{x^2}{2}}\,dx$ is of considerable interest to statisticians.

**a** Use the graphing package to help sketch $y = e^{-\frac{x^2}{2}}$ for $-3 \leqslant x \leqslant 3$.

**b** Calculate the upper and lower rectangular sums for the interval $0 \leqslant x \leqslant 3$ using $n = 2250$.

**c** Use the symmetry of $y = e^{-\frac{x^2}{2}}$ to find upper and lower rectangular sums for $-3 \leqslant x \leqslant 0$ for $n = 2250$.

**d** Hence estimate $\displaystyle\int_{-3}^{3} e^{-\frac{x^2}{2}}\,dx$.

How accurate is your estimate compared with $\sqrt{2\pi}$?

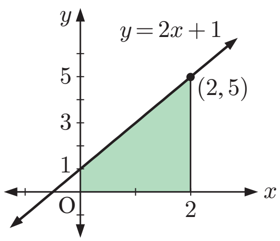
**AREA FINDER**

## *Example 2*

***Self Tutor***

Use graphical evidence and known area facts to find: **a** $\displaystyle\int_0^2 (2x+1)\,dx$ **b** $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

**a**



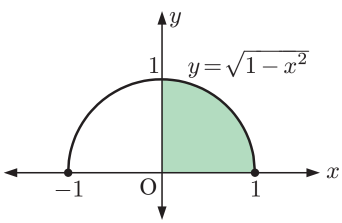

$\displaystyle\int_0^2 (2x+1)\,dx$

$=$ shaded area

$= \left(\frac{1+5}{2}\right) \times 2$

$= 6$

**b** If $y = \sqrt{1-x^2}$ then $y^2 = 1 - x^2$ and so $x^2 + y^2 = 1$ which is the equation of the unit circle. $y = \sqrt{1-x^2}$ is the upper half.




$\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

$=$ shaded area

$= \frac{1}{4}(\pi r^2)$ where $r = 1$

$= \frac{\pi}{4}$

**4** Use graphical evidence and known area facts to find:

**a** $\displaystyle\int_1^3 (1+4x)\,dx$ **b** $\displaystyle\int_{-1}^2 (2-x)\,dx$ **c** $\displaystyle\int_{-2}^2 \sqrt{4-x^2}\,dx$

# B ANTIDIFFERENTIATION

In many problems in calculus we know the rate of change of one variable with respect to another, but we do not have a formula which relates the variables. In other words, we know $\dfrac{dy}{dx}$, but we need to know $y$ in terms of $x$.

The process of finding $y$ from $\frac{dy}{dx}$, or $f(x)$ from $f'(x)$, is the reverse process of differentiation. We call it **antidifferentiation**.

$$\begin{tikzcd} y \text{ or } f(x) \arrow[rr, bend left, "\text{differentiation}"] & & \frac{dy}{dx} \text{ or } f'(x) \arrow[ll, bend left, "\text{antidifferentiation}"] \end{tikzcd}$$

Consider $\frac{dy}{dx} = x^2$.

From our work on differentiation, we know that when we differentiate power functions the index reduces by 1. We hence know that $y$ must involve $x^3$.

Now if $y = x^3$ then $\frac{dy}{dx} = 3x^2$, so if we start with $y = \frac{1}{3}x^3$ then $\frac{dy}{dx} = x^2$.

However, for all of the cases $y = \frac{1}{3}x^3 + 2$, $y = \frac{1}{3}x^3 + 100$, and $y = \frac{1}{3}x^3 - 7$, we find that $\frac{dy}{dx} = x^2$.

In fact, there are infinitely many functions of the form $y = \frac{1}{3}x^3 + c$ where $c$ is an arbitrary constant, which will give $\frac{dy}{dx} = x^2$. Ignoring the arbitrary constant, we say that $\frac{1}{3}x^3$ is the **antiderivative** of $x^2$. It is the simplest function which, when differentiated, gives $x^2$.

If $F(x)$ is a function where $F'(x) = f(x)$ we say that:

- the **derivative** of $F(x)$ is $f(x)$ and
- the **antiderivative** of $f(x)$ is $F(x)$.

## Example 3

Self Tutor

Find the antiderivative of: **a** $x^3$ **b** $e^{2x}$ **c** $\frac{1}{\sqrt{x}}$

**a** $\frac{d}{dx}\left(x^4\right) = 4x^3$

$\therefore$ $\frac{d}{dx}\left(\frac{1}{4}x^4\right) = x^3$

$\therefore$ the antiderivative of $x^3$ is $\frac{1}{4}x^4$.

**b** $\frac{d}{dx}\left(e^{2x}\right) = e^{2x} \times 2$

$\therefore$ $\frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = \frac{1}{2} \times e^{2x} \times 2 = e^{2x}$

$\therefore$ the antiderivative of $e^{2x}$ is $\frac{1}{2}e^{2x}$.

**c** $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

Now $\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore$ $\frac{d}{dx}(2x^{\frac{1}{2}}) = 2(\frac{1}{2})x^{-\frac{1}{2}} = x^{-\frac{1}{2}}$

$\therefore$ the antiderivative of $\frac{1}{\sqrt{x}}$ is $2\sqrt{x}$.

## EXERCISE 15B

**1** **a** Find the antiderivative of:

**i** $x$ **ii** $x^2$ **iii** $x^5$ **iv** $x^{-2}$ **v** $x^{-4}$ **vi** $x^{\frac{1}{3}}$ **vii** $x^{-\frac{1}{2}}$

**b** Predict a general rule for the antiderivative of $x^n$, for $n \neq -1$.

**2** **a** Find the antiderivative of:

**i** $e^{2x}$ **ii** $e^{5x}$ **iii** $e^{\frac{1}{2}x}$ **iv** $e^{0.01x}$ **v** $e^{\pi x}$ **vi** $e^{\frac{x}{3}}$

**b** Predict a general rule for the antiderivative of $e^{kx}$ where $k$ is a constant, $k \neq 0$.

**3** Find the antiderivative of:

**a** $6x^2 + 4x$ by first differentiating $x^3 + x^2$

**b** $e^{3x+1}$ by first differentiating $e^{3x+1}$

**c** $\sqrt{x}$ by first differentiating $x\sqrt{x}$

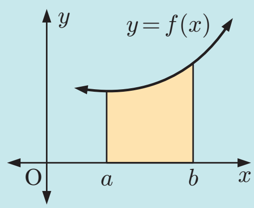
**d** $(2x+1)^3$ by first differentiating $(2x+1)^4$.

# C THE FUNDAMENTAL THEOREM OF CALCULUS

**Sir Isaac Newton** and **Gottfried Wilhelm Leibniz** showed the link between differential calculus and the definite integral or limit of an area sum we saw in **Section A**. This link is called the **fundamental theorem of calculus**. The beauty of this theorem is that it enables us to evaluate complicated summations.

We have already observed that:

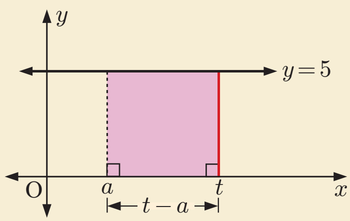
If $f(x)$ is a continuous positive function on an interval $a \leqslant x \leqslant b$ then the area under the curve between $x = a$ and $x = b$ is $\displaystyle\int_a^b \boldsymbol{f(x)\,dx}$.




## Discovery — The area function

Consider the constant function $f(x) = 5$.

We wish to find an **area function** which will give the area under the function between $x = a$ and some other value of $x$ which we will call $t$.




$$\text{The area function is}\quad A(t) = \int_a^t 5\,dx$$
$$= \text{shaded area in graph}$$
$$= (t-a)5$$
$$= 5t - 5a$$

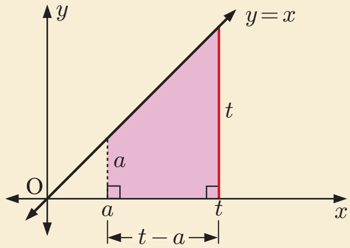
$\therefore$ we can write $A(t)$ in the form $F(t) - F(a)$ where $F(t) = 5t$ or equivalently $F(x) = 5x$

**What to do:**

**1** What is the derivative $F'(x)$ of the function $F(x) = 5x$? How does this relate to the function $f(x)$?

**2** Consider the simplest linear function $f(x) = x$.
The corresponding area function is

$$\begin{aligned} A(t) &= \int_a^t x\, dx \\ &= \text{shaded area in graph} \\ &= \left(\frac{t+a}{2}\right)(t-a) \end{aligned}$$

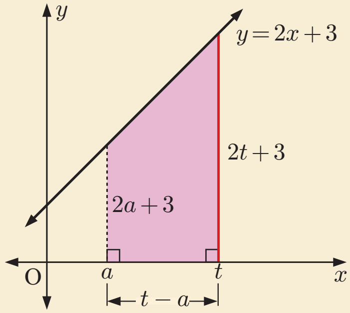



**a** Write $A(t)$ in the form $F(t) - F(a)$.

**b** What is the derivative $F'(x)$? How does it relate to the function $f(x)$?

**3** Consider $f(x) = 2x + 3$. The corresponding area function is

$$\begin{aligned} A(t) &= \int_a^t (2x+3)\, dx \\ &= \text{shaded area in graph} \\ &= \left(\frac{2t+3+2a+3}{2}\right)(t-a) \end{aligned}$$




**a** Write $A(t)$ in the form $F(t) - F(a)$.

**b** What is the derivative $F'(x)$?
How does it relate to the function $f(x)$?

**4** Repeat the procedure in **2** and **3** to find area functions for:

**a** $f(x) = \frac{1}{2}x + 3$ **b** $f(x) = 5 - 2x$

Do your results fit with your earlier observations?

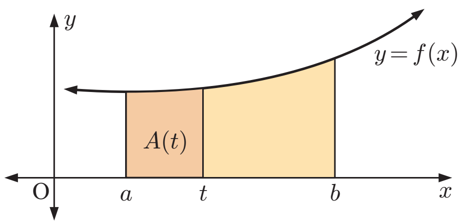
**5** If $f(x) = 3x^2 + 4x + 5$, predict what $F(x)$ would be without performing the algebraic procedure.

From the **Discovery** you should have found that, for $f(x) \geqslant 0$,

$$\int_a^t f(x)\, dx = F(t) - F(a) \quad \text{where} \quad F'(x) = f(x).$$ $F(x)$ is the **antiderivative** of $f(x)$.

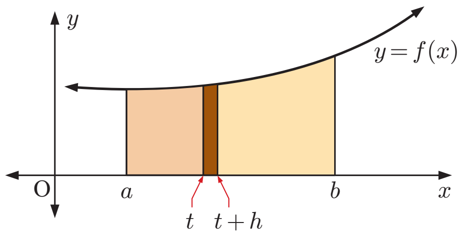
The following argument shows why this is true for all functions $f(x) \geqslant 0$.

Consider a function $y = f(x)$ which has antiderivative $F(x)$ and an area function $A(t) = \int_a^t f(x)\, dx$ which is the area from $x = a$ to $x = t$.




$A(t)$ is clearly an increasing function and
$A(a) = 0$ .... (1)

Consider the narrow strip between $x = t$ and $x = t + h$. The area of this strip is $A(t+h) - A(t)$, but we also know it must lie between a lower and upper rectangle on the interval $t \leqslant x \leqslant t+h$ of width $h$.




$$\text{area of smaller rectangle} \leqslant A(t+h) - A(t) \leqslant \text{area of larger rectangle}$$

If $f(x)$ is increasing on this interval then

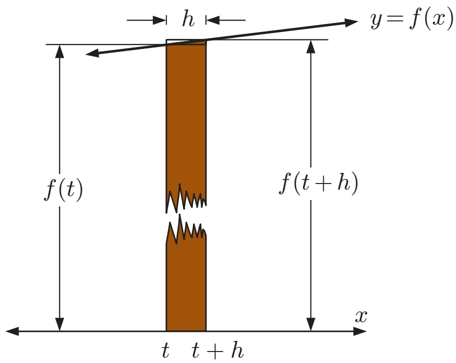
$$hf(t) \leqslant A(t+h) - A(t) \leqslant hf(t+h)$$
$$\therefore \quad f(t) \leqslant \frac{A(t+h) - A(t)}{h} \leqslant f(t+h)$$

Equivalently, if $f(x)$ is decreasing on this interval then $f(t+h) \leqslant \dfrac{A(t+h) - A(t)}{h} \leqslant f(t)$.

Taking the limit as $h \to 0$ gives $f(t) \leqslant A'(t) \leqslant f(t)$

$$\therefore \quad A'(t) = f(t)$$

So, the area function $A(t)$ must only differ from the antiderivative of $f(t)$ by a constant.



enlarged strip

$\therefore \quad A(t) = F(t) + c$

Letting $t = a$, $A(a) = F(a) + c$

But from (1), $A(a) = 0$

$\therefore \quad c = -F(a)$

$\therefore \quad A(t) = F(t) - F(a)$

Letting $t = b$, $\displaystyle\int_a^b f(x)\, dx = F(b) - F(a)$

This result is in fact true for all continuous functions $f(x)$.

## THE FUNDAMENTAL THEOREM OF CALCULUS

From the geometric argument above, the Fundamental Theorem of Calculus can be stated in two forms:

For a continuous function $f(x)$, if we define the area function from $x = a$ as $A(t) = \displaystyle\int_a^t f(x)\, dx$, then $A'(x) = f(x)$.

or more commonly:

For a continuous function $f(x)$ with antiderivative $F(x)$, $\displaystyle\int_a^b \boldsymbol{f(x)\, dx = F(b) - F(a)}$.

## PROPERTIES OF DEFINITE INTEGRALS

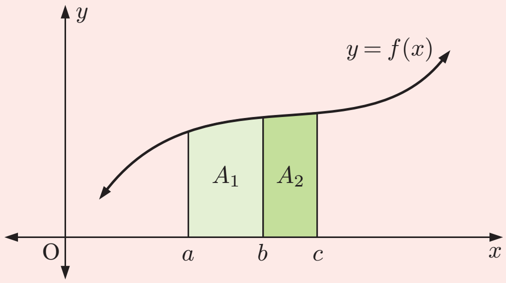
The following properties of definite integrals can all be deduced from the fundamental theorem of calculus:

- $\displaystyle\int_a^a f(x)\, dx = 0$
- $\displaystyle\int_a^b c\, dx = c(b-a)$ $\{c$ is a constant$\}$
- $\displaystyle\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$
- $\displaystyle\int_a^b c\, f(x)\, dx = c\int_a^b f(x)\, dx$

- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

**Example proof:**

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$
$$= F(b) - F(a) + F(c) - F(b)$$
$$= F(c) - F(a)$$
$$= \int_a^c f(x)\,dx$$




In particular, for the case where $a \leqslant b \leqslant c$ and $f(x) \geqslant 0$ for $a \leqslant x \leqslant c$, we observe that

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = A_1 + A_2 = \int_a^c f(x)\,dx$$

## Example 4

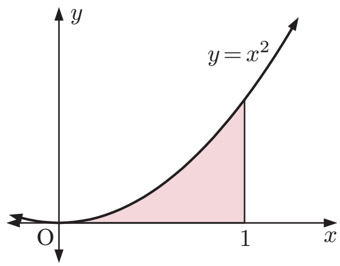
Self Tutor

Use the fundamental theorem of calculus to find the area between:

**a** the $x$-axis and $y = x^2$ from $x = 0$ to $x = 1$

**b** the $x$-axis and $y = \sqrt{x}$ from $x = 1$ to $x = 9$.

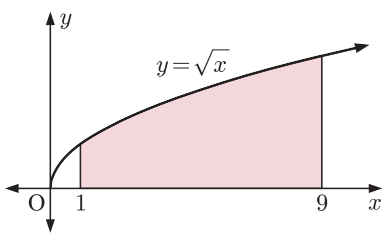
**a**




$f(x) = x^2$ has antiderivative $F(x) = \frac{x^3}{3}$

$$\therefore \text{ the area } = \int_0^1 x^2\,dx$$
$$= F(1) - F(0)$$
$$= \tfrac{1}{3} - 0$$
$$= \tfrac{1}{3} \text{ units}^2$$

**b**




$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$$F(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \tfrac{2}{3}x\sqrt{x}$$

$$\therefore \text{ the area } = \int_1^9 x^{\frac{1}{2}}\,dx$$
$$= F(9) - F(1)$$
$$= \tfrac{2}{3} \times 27 - \tfrac{2}{3} \times 1$$
$$= 17\tfrac{1}{3} \text{ units}^2$$

## EXERCISE 15C

**1** Use the fundamental theorem of calculus to find the area between:

**a** the $x$-axis and $y = x^3$ from $x = 0$ to $x = 1$

**b** the $x$-axis and $y = x^2$ from $x = 1$ to $x = 2$

**c** the $x$-axis and $y = \sqrt{x}$ from $x = 0$ to $x = 1$.

**2** Use the fundamental theorem of calculus to show that:

**a** $\displaystyle\int_a^a f(x)\,dx = 0$ and explain the result graphically

**b** $\displaystyle\int_a^b c\,dx = c(b-a)$ where $c$ is a constant

**c** $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

**d** $\displaystyle\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$ where $c$ is a constant

**e** $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

**3** Use the fundamental theorem of calculus to find the area between the $x$-axis and:

**a** $y = x^3$ from $x = 1$ to $x = 2$

**b** $y = x^2 + 3x + 2$ from $x = 1$ to $x = 3$

**c** $y = \sqrt{x}$ from $x = 1$ to $x = 2$

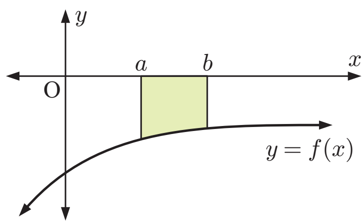
**d** $y = e^x$ from $x = 0$ to $x = 1.5$

**e** $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = 4$

**4** **a** Use the fundamental theorem of calculus to show that

$$\int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx$$

**b** Hence show that if $f(x) \leqslant 0$ for all $x$ on $a \leqslant x \leqslant b$ then the shaded area $= -\displaystyle\int_a^b f(x)\,dx$.




**c** Calculate the following integrals, and give graphical interpretations of your answers:

**i** $\displaystyle\int_0^1 (-x^2)\,dx$ **ii** $\displaystyle\int_0^1 (x^2-x)\,dx$ **iii** $\displaystyle\int_{-2}^0 3x\,dx$

**d** Use graphical evidence and known area facts to find $\displaystyle\int_0^2 \left(-\sqrt{4-x^2}\right)dx$.

# D INTEGRATION

Earlier, we showed that the **antiderivative** of $x^2$ is $\frac{1}{3}x^3$, and that any function of the form $\frac{1}{3}x^3+c$ where $c$ is a constant, has derivative $x^2$.

We say that the **indefinite integral** or **integral** of $x^2$ is $\frac{1}{3}x^3+c$, and write $\int x^2\,dx = \frac{1}{3}x^3+c$.

We read this as "the integral of $x^2$ with respect to $x$ is $\frac{1}{3}x^3+c$, where $c$ is a constant".

If $F'(x)=f(x)$ then $\int f(x)\,dx = F(x)+c$.

This process is known as **indefinite integration**. It is indefinite because it is not being applied to a particular interval.

## DISCOVERING INTEGRALS

Since integration is the reverse process of differentiation we can sometimes discover integrals by differentiation. For example:

- if $F(x)=x^4$ then $F'(x)=4x^3$
  $$\therefore \int 4x^3\,dx = x^4+c$$
- if $F(x)=\sqrt{x}=x^{\frac{1}{2}}$ then $F'(x)=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$
  $$\therefore \int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x}+c$$

The following rules may prove useful:

- Any constant may be written in front of the integral sign.

$$\int k\,f(x)\,dx = k\int f(x)\,dx, \quad k \text{ is a constant}$$

**Proof:** Consider differentiating $kF(x)$ where $F'(x)=f(x)$.

$$\frac{d}{dx}(k\,F(x)) = k\,F'(x) = k\,f(x)$$
$$\therefore \int k\,f(x)\,dx = k\,F(x)$$
$$= k\int f(x)\,dx$$

- The integral of a sum is the sum of the separate integrals. This rule enables us to integrate term by term.

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

### Example 5 — Self Tutor

If $y = x^4 + 2x^3$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int (2x^3 + 3x^2)\, dx$.

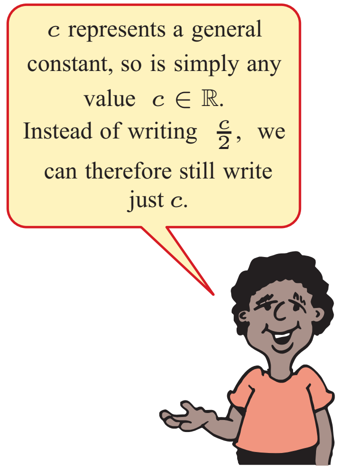
If $y = x^4 + 2x^3$ then $\dfrac{dy}{dx} = 4x^3 + 6x^2$

$$\therefore \int (4x^3 + 6x^2)\, dx = x^4 + 2x^3 + c$$

$$\therefore \int 2(2x^3 + 3x^2)\, dx = x^4 + 2x^3 + c$$

$$\therefore 2\int (2x^3 + 3x^2)\, dx = x^4 + 2x^3 + c$$

$$\therefore \int (2x^3 + 3x^2)\, dx = \tfrac{1}{2}x^4 + x^3 + c$$




## EXERCISE 15D

**1** If $y = x^7$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int x^6\, dx$.

**2** If $y = x^3 + x^2$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int (3x^2 + 2x)\, dx$.

**3** If $y = e^{2x+1}$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int e^{2x+1}\, dx$.

**4** If $y = (2x+1)^4$ find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int (2x+1)^3\, dx$.

### Example 6 — Self Tutor

Suppose $y = \sqrt{5x - 1}$.

**a** Find $\dfrac{dy}{dx}$.

**b** Hence find $\displaystyle\int \frac{1}{\sqrt{5x-1}}\, dx$.

**a**
$$\begin{aligned} y &= \sqrt{5x-1} \\ &= (5x-1)^{\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= \tfrac{1}{2}(5x-1)^{-\frac{1}{2}}(5) \quad \{\text{chain rule}\} \\ &= \frac{5}{2\sqrt{5x-1}} \end{aligned}$$

**b** Using **a**, $\displaystyle\int \frac{5}{2\sqrt{5x-1}}\, dx = \sqrt{5x-1} + c$

$$\therefore \frac{5}{2}\int \frac{1}{\sqrt{5x-1}}\, dx = \sqrt{5x-1} + c$$

$$\therefore \int \frac{1}{\sqrt{5x-1}}\, dx = \tfrac{2}{5}\sqrt{5x-1} + c$$

**5** If $y = x\sqrt{x}$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int \sqrt{x}\, dx$.

**6** If $y = \dfrac{1}{\sqrt{x}}$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int \frac{1}{x\sqrt{x}}\, dx$.

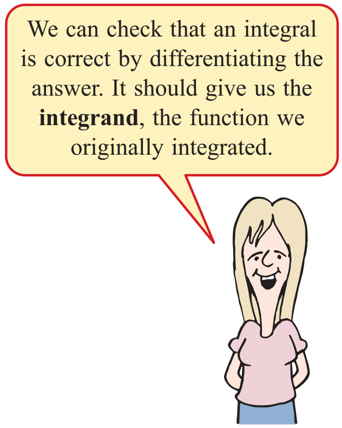
**7** If $y = \cos 2x$, find $\frac{dy}{dx}$. Hence find $\int \sin 2x\, dx$.

**8** If $y = \sin(1 - 5x)$, find $\frac{dy}{dx}$. Hence find $\int \cos(1 - 5x)\, dx$.

**9** By considering $\frac{d}{dx}(x^2 - x)^3$, find $\int (2x - 1)(x^2 - x)^2\, dx$.

**10** Prove the rule $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$.

**11** Find $\frac{dy}{dx}$ if $y = \sqrt{1 - 4x}$. Hence find $\int \frac{1}{\sqrt{1 - 4x}}\, dx$.




## E RULES FOR INTEGRATION

In **Chapter 13** we developed a set of rules to help us differentiate functions more efficiently:

| *Function* | *Derivative* | *Name* |
|---|---|---|
| $c$, a constant | $0$ | |
| $mx + c$, $m$ and $c$ are constants | $m$ | |
| $x^n$ | $nx^{n-1}$ | **power rule** |
| $cu(x)$ | $cu'(x)$ | |
| $u(x) + v(x)$ | $u'(x) + v'(x)$ | **addition rule** |
| $u(x)v(x)$ | $u'(x)v(x) + u(x)v'(x)$ | **product rule** |
| $\frac{u(x)}{v(x)}$ | $\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$ | **quotient rule** |
| $y = f(u)$ where $u = u(x)$ | $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ | **chain rule** |
| $e^x$ | $e^x$ | |
| $e^{f(x)}$ | $e^{f(x)} f'(x)$ | |
| $\ln x$ | $\frac{1}{x}$ | |
| $\ln f(x)$ | $\frac{f'(x)}{f(x)}$ | |
| $[f(x)]^n$ | $n[f(x)]^{n-1} f'(x)$ | |
| $\sin x$ | $\cos x$ | |
| $\cos x$ | $-\sin x$ | |
| $\tan x$ | $\sec^2 x$ | |

These rules or combinations of them can be used to differentiate all of the functions we consider in this course. Given an algebraic formula, we can repeatedly apply these rules until we get to basic functions such as $x^n$ or $\sin x$, which we know how to differentiate.

However, the task of finding **antiderivatives** is not so easy. Given an algebraic formula there is no simple list of rules to find the antiderivative.

The problem was finally solved in 1968 by Robert Henry Risch. He devised a method for deciding if a function has an elementary antiderivative, and if it does, finding it. The original summary of his method took over 100 pages. Later developments from this are now used in all computer algebra systems.

Fortunately, our course is restricted to a few special cases.

## RULES FOR INTEGRATION

For $k$ a constant, $\frac{d}{dx}(kx+c) = k$ $\quad\therefore\quad$ $\boldsymbol{\int k\,dx = kx + c}$

If $n \neq -1$, $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + c\right) = \frac{(n+1)x^n}{n+1} = x^n$ $\quad\therefore\quad$ $\boldsymbol{\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \ n \neq -1}$

$\frac{d}{dx}(e^x + c) = e^x$ $\quad\therefore\quad$ $\boldsymbol{\int e^x\,dx = e^x + c}$

$\frac{d}{dx}(\sin x + c) = \cos x$ $\quad\therefore\quad$ $\boldsymbol{\int \cos x\,dx = \sin x + c}$

$\frac{d}{dx}(-\cos x + c) = \sin x$ $\quad\therefore\quad$ $\boldsymbol{\int \sin x\,dx = -\cos x + c}$

| *Function* | *Integral* |
|---|---|
| $k$, a constant | $kx + c$ |
| $x^n$, $n \neq -1$ | $\frac{x^{n+1}}{n+1} + c$ |
| $e^x$ | $e^x + c$ |
| $\cos x$ | $\sin x + c$ |
| $\sin x$ | $-\cos x + c$ |

$c$ is an arbitrary constant called the **constant of integration** or **integrating constant**.

Remember that you can always check your integration by differentiating the resulting function.

### Example 7

Self Tutor

Find:

**a** $\displaystyle\int (x^3 - 2x^2 + 5)\,dx$ **b** $\displaystyle\int \left(\frac{1}{x^3} - \sqrt{x}\right) dx$ **c** $\displaystyle\int (2\sin x - \cos x)\,dx$

**a**
$$\int (x^3 - 2x^2 + 5)\,dx$$
$$= \frac{x^4}{4} - \frac{2x^3}{3} + 5x + c$$

**b**
$$\int \left(\frac{1}{x^3} - \sqrt{x}\right) dx$$
$$= \int (x^{-3} - x^{\frac{1}{2}})\,dx$$
$$= \frac{x^{-2}}{-2} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= -\frac{1}{2x^2} - \tfrac{2}{3}x^{\frac{3}{2}} + c$$

**c**
$$\int (2\sin x - \cos x)\,dx$$
$$= 2(-\cos x) - \sin x + c$$
$$= -2\cos x - \sin x + c$$

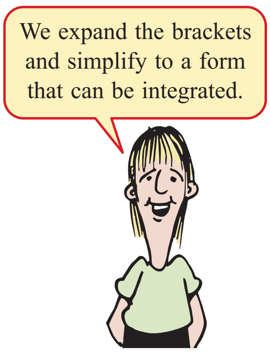
There is no product or quotient rule for integration. Consequently we often have to carry out multiplication or division before we integrate.

### Example 8

Self Tutor

Find: **a** $\displaystyle\int \left(3x + \frac{2}{x}\right)^2 dx$ **b** $\displaystyle\int \left(\frac{x^2 - 2}{\sqrt{x}}\right) dx$

**a**
$$\int \left(3x + \frac{2}{x}\right)^2 dx$$
$$= \int \left(9x^2 + 12 + \frac{4}{x^2}\right) dx$$
$$= \int (9x^2 + 12 + 4x^{-2})\,dx$$
$$= \frac{9x^3}{3} + 12x + \frac{4x^{-1}}{-1} + c$$
$$= 3x^3 + 12x - \frac{4}{x} + c$$




**b**
$$\int \left(\frac{x^2 - 2}{\sqrt{x}}\right) dx$$
$$= \int \left(\frac{x^2}{\sqrt{x}} - \frac{2}{\sqrt{x}}\right) dx$$
$$= \int (x^{\frac{3}{2}} - 2x^{-\frac{1}{2}})\,dx$$
$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \tfrac{2}{5}x^2\sqrt{x} - 4\sqrt{x} + c$$

## EXERCISE 15E.1

**1** Find:

**a** $\displaystyle\int (x^4 - x^2 - x + 2)\,dx$ **b** $\displaystyle\int (5x^4 - 4x^3 - 6x^2 - 7)\,dx$ **c** $\displaystyle\int (\sqrt{x} + e^x)\,dx$

**d** $\displaystyle\int (3e^x + x^2)\,dx$ **e** $\displaystyle\int (x\sqrt{x} - 2)\,dx$ **f** $\displaystyle\int \left(\frac{1}{x\sqrt{x}} + 4x\right) dx$

**g** $\displaystyle\int \left(\tfrac{1}{2}x^3 - x^4 + x^{\frac{1}{3}}\right) dx$ **h** $\displaystyle\int \left(\frac{x}{2} + x^2 - e^x\right) dx$ **i** $\displaystyle\int \left(5e^x + \tfrac{1}{3}x^3 - \sqrt{x}\right) dx$

**2** Integrate with respect to $x$:

**a** $3\sin x - 2$ **b** $4x - 2\cos x$ **c** $\sin x - 2\cos x + e^x$

**d** $x^2\sqrt{x} - 10\sin x$ **e** $\dfrac{x(x-1)}{3} + \cos x$ **f** $-\sin x + 2\sqrt{x}$

**3** Find:

**a** $\displaystyle\int (x^2 + 3x - 2)\, dx$ **b** $\displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$ **c** $\displaystyle\int \left(2e^x - \frac{1}{x^2}\right) dx$

**d** $\displaystyle\int \frac{1-4x}{x\sqrt{x}}\, dx$ **e** $\displaystyle\int (2x+1)^2\, dx$ **f** $\displaystyle\int \left(x + \frac{1}{x}\right)^2 dx$

**g** $\displaystyle\int \frac{2x-1}{\sqrt{x}}\, dx$ **h** $\displaystyle\int \frac{x^2 - 4x + 10}{x^2\sqrt{x}}\, dx$ **i** $\displaystyle\int (x+1)^3\, dx$

**4** Find:

**a** $\displaystyle\int \left(\sqrt{x} + \tfrac{1}{2}\cos x\right) dx$ **b** $\displaystyle\int (2e^t - 4\sin t)\, dt$ **c** $\displaystyle\int (3\cos t - \sin t)\, dt$

**5** Find $y$ if:

**a** $\dfrac{dy}{dx} = 6$ **b** $\dfrac{dy}{dx} = 4x^2$ **c** $\dfrac{dy}{dx} = 5\sqrt{x} - x^2$

**d** $\dfrac{dy}{dx} = \dfrac{1}{x^2}$ **e** $\dfrac{dy}{dx} = 2e^x - 5$ **f** $\dfrac{dy}{dx} = 4x^3 + 3x^2$

**6** Find $f(x)$ if:

**a** $f'(x) = (1-2x)^2$ **b** $f'(x) = \sqrt{x} - \dfrac{2}{\sqrt{x}}$ **c** $f'(x) = \dfrac{x^2 - 5}{x^2}$

## PARTICULAR VALUES

We can find the constant of integration $c$ if we are given a particular value of the function.

### Example 9 — Self Tutor

Find $f(x)$ given that:

**a** $f'(x) = x^3 - 2x^2 + 3$ and $f(0) = 2$ **b** $f'(x) = 2\sin x - \sqrt{x}$ and $f(0) = 4$.

**a** Since $f'(x) = x^3 - 2x^2 + 3$,

$$f(x) = \int (x^3 - 2x^2 + 3)\, dx$$

$$\therefore\ f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + c$$

But $f(0) = 2$, so $c = 2$

Thus $f(x) = \dfrac{x^4}{4} - \dfrac{2x^3}{3} + 3x + 2$

**b** $$f(x) = \int \left(2\sin x - x^{\frac{1}{2}}\right) dx$$

$$\therefore\ f(x) = 2 \times (-\cos x) - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\therefore\ f(x) = -2\cos x - \tfrac{2}{3}x^{\frac{3}{2}} + c$$

But $f(0) = 4$,

so $-2\cos 0 - 0 + c = 4$

$\therefore\ c = 6$

Thus $f(x) = -2\cos x - \tfrac{2}{3}x^{\frac{3}{2}} + 6$.

If we are given the second derivative we need to integrate twice to find the function. This creates two integrating constants, so we need two other facts about the curve in order to determine these constants.

**Example 10** Self Tutor

Find $f(x)$ given that $f''(x) = 12x^2 - 4$, $f'(0) = -1$, and $f(1) = 4$.

If $f''(x) = 12x^2 - 4$

then $f'(x) = \frac{12x^3}{3} - 4x + c$ {integrating with respect to $x$}

$\therefore$ $f'(x) = 4x^3 - 4x + c$

But $f'(0) = -1$, so $c = -1$

Thus $f'(x) = 4x^3 - 4x - 1$

$\therefore$ $f(x) = \frac{4x^4}{4} - \frac{4x^2}{2} - x + d$ {integrating again}

$\therefore$ $f(x) = x^4 - 2x^2 - x + d$

But $f(1) = 4$, so $1 - 2 - 1 + d = 4$ and hence $d = 6$

Thus $f(x) = x^4 - 2x^2 - x + 6$

## EXERCISE 15E.2

**1** Find $f(x)$ given that:

**a** $f'(x) = 2x - 1$ and $f(0) = 3$

**b** $f'(x) = 3x^2 + 2x$ and $f(2) = 5$

**c** $f'(x) = e^x + \frac{1}{\sqrt{x}}$ and $f(1) = 1$

**d** $f'(x) = x - \frac{2}{\sqrt{x}}$ and $f(1) = 2$

**2** Find $f(x)$ given that:

**a** $f'(x) = x^2 - 4\cos x$ and $f(0) = 3$

**b** $f'(x) = 2\cos x - 3\sin x$ and $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

**3** Find $f(x)$ given that:

**a** $f''(x) = 2x + 1$, $f'(1) = 3$, and $f(2) = 7$

**b** $f''(x) = 15\sqrt{x} + \frac{3}{\sqrt{x}}$, $f'(1) = 12$, and $f(0) = 5$

**c** $f''(x) = \cos x$, $f'(\frac{\pi}{2}) = 0$, and $f(0) = 3$

**d** $f''(x) = 2x$ and the points $(1, 0)$ and $(0, 5)$ lie on the curve.

# F INTEGRATING $f(ax + b)$

In this section we deal with integrals of functions which are composite with the linear function $ax + b$.

Notice that $\frac{d}{dx}\left(\frac{1}{a}\,e^{ax+b}\right) = \frac{1}{a}\,e^{ax+b} \times a = e^{ax+b}$

$$\therefore \quad \boxed{\int e^{ax+b}\,dx = \frac{1}{a}\,e^{ax+b} + c \quad \text{for } a \neq 0}$$

Likewise if $n \neq -1$,

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1}\right) &= \frac{1}{a(n+1)}(n+1)(ax+b)^n \times a,\\ &= (ax+b)^n\end{aligned}$$

$$\therefore \quad \boxed{\int (ax+b)^n\,dx = \frac{1}{a}\,\frac{(ax+b)^{n+1}}{(n+1)} + c \quad \text{for } n \neq -1}$$

We can perform the same process for the circular functions:

$$\frac{d}{dx}(\sin(ax+b)) = a\cos(ax+b)$$

$$\therefore \quad \int a\cos(ax+b)\,dx = \sin(ax+b) + c$$

$$\therefore \quad a\int \cos(ax+b)\,dx = \sin(ax+b) + c$$

So, $$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c \quad \text{for } a \neq 0.$$

Likewise we can show $$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c \quad \text{for } a \neq 0.$$

For $a$, $b$ constants with $a \neq 0$, we have:

| *Function* | *Integral* |
|---|---|
| $e^{ax+b}$ | $\frac{1}{a}\,e^{ax+b} + c$ |
| $(ax+b)^n, \quad n \neq -1$ | $\frac{1}{a}\,\frac{(ax+b)^{n+1}}{n+1} + c$ |
| $\cos(ax+b)$ | $\frac{1}{a}\sin(ax+b) + c$ |
| $\sin(ax+b)$ | $-\frac{1}{a}\cos(ax+b) + c$ |

## Example 11 — Self Tutor

Find: **a** $\int (2x+3)^4\,dx$ **b** $\int \frac{1}{\sqrt{1-2x}}\,dx$

**a**
$$\begin{aligned}&\int (2x+3)^4\,dx\\ &= \tfrac{1}{2} \times \frac{(2x+3)^5}{5} + c\\ &= \tfrac{1}{10}(2x+3)^5 + c\end{aligned}$$

**b**
$$\begin{aligned}&\int \frac{1}{\sqrt{1-2x}}\,dx\\ &= \int (1-2x)^{-\frac{1}{2}}\,dx\\ &= \tfrac{1}{-2} \times \frac{(1-2x)^{\frac{1}{2}}}{\frac{1}{2}} + c\\ &= -\sqrt{1-2x} + c\end{aligned}$$

## Example 12

Self Tutor

Integrate with respect to $x$:

**a** $2e^{2x} - e^{-3x}$

**b** $2\sin(3x) + \cos(4x + \pi)$

**a** $\int (2e^{2x} - e^{-3x})\, dx$

$= 2(\frac{1}{2})e^{2x} - (\frac{1}{-3})e^{-3x} + c$

$= e^{2x} + \frac{1}{3}e^{-3x} + c$

**b** $\int (2\sin(3x) + \cos(4x + \pi))\, dx$

$= 2 \times -\frac{1}{3}\cos(3x) + \frac{1}{4}\sin(4x + \pi) + c$

$= -\frac{2}{3}\cos(3x) + \frac{1}{4}\sin(4x + \pi) + c$

## EXERCISE 15F

**1** Find:

**a** $\int (2x+5)^3\, dx$ **b** $\int \frac{1}{(3-2x)^2}\, dx$ **c** $\int \frac{4}{(2x-1)^4}\, dx$

**d** $\int (4x-3)^7\, dx$ **e** $\int \sqrt{3x-4}\, dx$ **f** $\int \frac{10}{\sqrt{1-5x}}\, dx$

**g** $\int 3(1-x)^4\, dx$ **h** $\int \frac{4}{\sqrt{3-4x}}\, dx$

**2** Integrate with respect to $x$:

**a** $\sin(3x)$ **b** $2\cos(-4x) + 1$ **c** $3\cos\left(\frac{x}{2}\right)$

**d** $3\sin(2x) - e^{-x}$ **e** $2\sin\left(2x + \frac{\pi}{6}\right)$ **f** $-3\cos\left(\frac{\pi}{4} - x\right)$

**g** $\cos(2x) + \sin(2x)$ **h** $2\sin(3x) + 5\cos(4x)$ **i** $\frac{1}{2}\cos(8x) - 3\sin x$

**3** Find $y = f(x)$ given $\frac{dy}{dx} = \sqrt{2x-7}$ and that $y = 11$ when $x = 8$.

**4** The function $f(x)$ has gradient function $f'(x) = \frac{4}{\sqrt{1-x}}$, and the curve $y = f(x)$ passes through the point $(-3, -11)$.
Find the point on the graph of $y = f(x)$ with $x$-coordinate $-8$.

**5** Find:

**a** $\int 3(2x-1)^2\, dx$ **b** $\int (x^2 - x)^2\, dx$ **c** $\int (1-3x)^3\, dx$

**d** $\int (1-x^2)^2\, dx$ **e** $\int 4\sqrt{5-x}\, dx$ **f** $\int (x^2+1)^3\, dx$

**6** Find:

**a** $\int \left(2e^x + 5e^{2x}\right) dx$ **b** $\int \left(3e^{5x-2}\right) dx$ **c** $\int \left(e^{7-3x}\right) dx$

**d** $\int (e^x + e^{-x})^2\, dx$ **e** $\int (e^{-x} + 2)^2\, dx$ **f** $\int \left(x - \frac{5}{(1-x)^2}\right) dx$

**7** Find an expression for $y$ given that $\frac{dy}{dx} = (1 - e^x)^2$, and that the graph has $y$-intercept 4.

**8** Suppose $f'(x) = p\sin\left(\frac{1}{2}x\right)$ where $p$ is a constant. $f(0) = 1$ and $f(2\pi) = 0$. Find $p$ and hence $f(x)$.

**9** Consider a function $g$ such that $g''(x) = -\sin 2x$.
Show that the gradients of the tangents to $y = g(x)$ when $x = \pi$ and $x = -\pi$ are equal.

**10** Find $f(x)$ given $f'(x) = 2e^{-2x}$ and $f(0) = 3$.

**11** A curve has gradient function $\sqrt{x} + \frac{1}{2}e^{-4x}$ and passes through $(1, 0)$. Find the equation of the function.

## G DEFINITE INTEGRALS

Earlier we saw the **fundamental theorem of calculus**:

If $F(x)$ is the antiderivative of $f(x)$ where $f(x)$ is continuous on the interval $a \leqslant x \leqslant b$, then the **definite integral** of $f(x)$ on this interval is $\boldsymbol{\int_a^b f(x)\,dx = F(b) - F(a)}$.

$\int_a^b f(x)\,dx$ reads "the integral from $x = a$ to $x = b$ of $f(x)$ with respect to $x$"
or "the integral from $a$ to $b$ of $f(x)$ with respect to $x$".

It is called a **definite** integral because there are lower and upper limits for the integration, and it therefore results in a numerical answer.

When calculating definite integrals we can omit the constant of integration $c$ as this will always cancel out in the subtraction process.

It is common to write $F(b) - F(a)$ as $[F(x)]_a^b$, and so

$$\boldsymbol{\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)}$$

Earlier in the chapter we proved the following properties of definite integrals using the fundamental theorem of calculus:

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, $c$ is any constant
- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

### Example 13

Self Tutor

Find:

**a** $\displaystyle\int_1^3 (x^2+2)\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{3}} \sin x\,dx$

**a** $\displaystyle\int_1^3 (x^2+2)\,dx$

$= \left[\frac{x^3}{3}+2x\right]_1^3$

$= \left(\frac{3^3}{3}+2(3)\right)-\left(\frac{1^3}{3}+2(1)\right)$

$= (9+6)-(\frac{1}{3}+2)$

$= 12\frac{2}{3}$

**b** $\displaystyle\int_0^{\frac{\pi}{3}} \sin x\,dx$

$= [-\cos x]_0^{\frac{\pi}{3}}$

$= (-\cos\frac{\pi}{3})-(-\cos 0)$

$= -\frac{1}{2}+1$

$= \frac{1}{2}$

## EXERCISE 15G

Use questions **1** to **4** to check the properties of definite integrals.

**1** Find: **a** $\displaystyle\int_1^4 \sqrt{x}\,dx$ and $\displaystyle\int_1^4 (-\sqrt{x})\,dx$ **b** $\displaystyle\int_0^1 x^7\,dx$ and $\displaystyle\int_0^1 (-x^7)\,dx$

**2** Find: **a** $\displaystyle\int_0^1 x^2\,dx$ **b** $\displaystyle\int_1^2 x^2\,dx$ **c** $\displaystyle\int_0^2 x^2\,dx$ **d** $\displaystyle\int_0^1 3x^2\,dx$

**3** Find: **a** $\displaystyle\int_0^2 (x^3-4x)\,dx$ **b** $\displaystyle\int_2^3 (x^3-4x)\,dx$ **c** $\displaystyle\int_0^3 (x^3-4x)\,dx$

**4** Find: **a** $\displaystyle\int_0^1 x^2\,dx$ **b** $\displaystyle\int_0^1 \sqrt{x}\,dx$ **c** $\displaystyle\int_0^1 (x^2+\sqrt{x})\,dx$

**5** Evaluate:

**a** $\displaystyle\int_0^1 x^3\,dx$ **b** $\displaystyle\int_0^2 (x^2-x)\,dx$ **c** $\displaystyle\int_0^1 e^x\,dx$

**d** $\displaystyle\int_0^{\frac{\pi}{6}} \cos x\,dx$ **e** $\displaystyle\int_1^4 \left(x-\frac{3}{\sqrt{x}}\right)dx$ **f** $\displaystyle\int_4^9 \frac{x-3}{\sqrt{x}}\,dx$

**g** $\displaystyle\int_1^3 \frac{1}{x}\,dx$ **h** $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\,dx$ **i** $\displaystyle\int_1^2 (e^{-x}+1)^2\,dx$

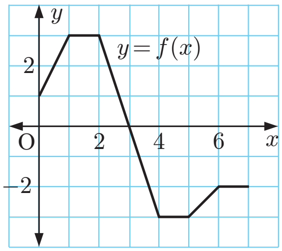
**j** $\displaystyle\int_2^6 \frac{1}{\sqrt{2x-3}}\,dx$ **k** $\displaystyle\int_0^1 e^{1-x}\,dx$ **l** $\displaystyle\int_0^{\frac{\pi}{6}} \sin(3x)\,dx$

**6** Find $m$ such that $\displaystyle\int_m^{2m} (2x-1)\,dx = 4$.

**7** **a** Use the identity $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$ to help evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 x\, dx$.

**b** Use the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$ to help evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$.

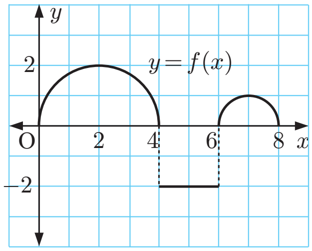
**8** Evaluate the following integrals using area interpretation:




**a** $\displaystyle\int_0^3 f(x)\, dx$ **b** $\displaystyle\int_3^7 f(x)\, dx$

**c** $\displaystyle\int_2^4 f(x)\, dx$ **d** $\displaystyle\int_0^7 f(x)\, dx$

**9** Evaluate the following integrals using area interpretation:




**a** $\displaystyle\int_0^4 f(x)\, dx$ **b** $\displaystyle\int_4^6 f(x)\, dx$

**c** $\displaystyle\int_6^8 f(x)\, dx$ **d** $\displaystyle\int_0^8 f(x)\, dx$

**10** Write as a single integral:

**a** $\displaystyle\int_2^4 f(x)\, dx + \int_4^7 f(x)\, dx$ **b** $\displaystyle\int_1^3 g(x)\, dx + \int_3^8 g(x)\, dx + \int_8^9 g(x)\, dx$

**11** **a** If $\displaystyle\int_1^3 f(x)\, dx = 2$ and $\displaystyle\int_1^6 f(x)\, dx = -3$, find $\displaystyle\int_3^6 f(x)\, dx$.

**b** If $\displaystyle\int_0^2 f(x)\, dx = 5$, $\displaystyle\int_4^6 f(x)\, dx = -2$, and $\displaystyle\int_0^6 f(x)\, dx = 7$, find $\displaystyle\int_2^4 f(x)\, dx$.

**12** Given that $\displaystyle\int_{-1}^1 f(x)\, dx = -4$, determine the value of:

**a** $\displaystyle\int_1^{-1} f(x)\, dx$ **b** $\displaystyle\int_{-1}^1 (2 + f(x))\, dx$ **c** $\displaystyle\int_{-1}^1 2f(x)\, dx$

**d** $k$ such that $\displaystyle\int_{-1}^1 kf(x)\, dx = 7$

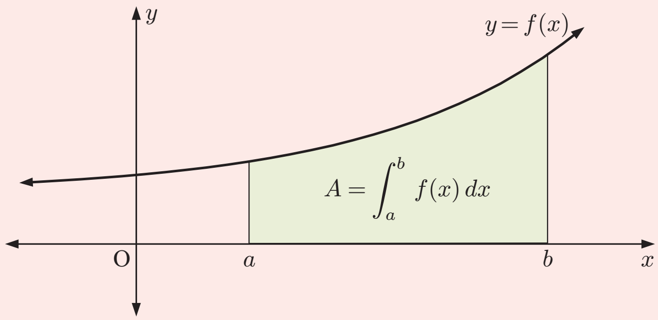
**13** If $g(2) = 4$ and $g(3) = 5$, calculate $\displaystyle\int_2^3 (g'(x) - 1)\, dx$.

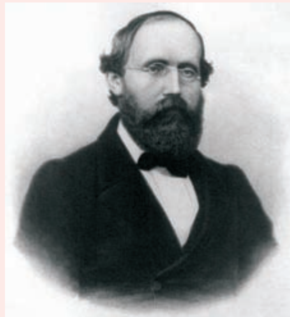
## Historical note

Following the work of Newton and Leibniz, integration was rigorously formalised using limits by the German mathematician **Bernhard Riemann** (1826 - 1866).

If $f(x) \geqslant 0$ on the interval $a \leqslant x \leqslant b$, we have seen that the area under the curve is $A = \int_a^b f(x)\,dx$.

This is known as the **Riemann integral**.



***Bernhard Riemann***

## Review set 15A

**1** **a** Sketch the region between the curve $y = \dfrac{4}{1+x^2}$ and the $x$-axis for $0 \leqslant x \leqslant 1$.

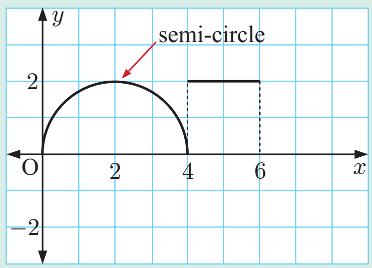
Divide the interval into 5 equal parts and display the 5 upper and lower rectangles.

**b** Use the **area finder** software to find the lower and upper rectangle sums for $n = 5$, 50, 100, and 500.


**c** Give your best estimate for $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx$ and compare this answer with $\pi$.

**2** The graph of $y = f(x)$ is illustrated:

Evaluate the following using area interpretation:




**a** $\displaystyle\int_0^4 f(x)\,dx$ **b** $\displaystyle\int_4^6 f(x)\,dx$

**3** Integrate with respect to $x$:

**a** $\dfrac{4}{\sqrt{x}}$ **b** $\sin(4x-5)$ **c** $e^{4-3x}$

**4** Find the exact value of:

**a** $\displaystyle\int_{-5}^{-1} \sqrt{1-3x}\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{2}} \cos\left(\frac{x}{2}\right)\,dx$

**5** By differentiating $y = \sqrt{x^2-4}$, find $\displaystyle\int \frac{x}{\sqrt{x^2-4}}\,dx$.

**6** Find the values of $b$ such that $\int_0^b \cos x \, dx = \frac{1}{\sqrt{2}}, \quad 0 < b < \pi$.

**7** Find $y$ if:

**a** $\frac{dy}{dx} = (x^2 - 1)^2$ **b** $\frac{dy}{dx} = 400 - 20e^{-\frac{x}{2}}$

**8** A curve $y = f(x)$ has $f''(x) = 18x + 10$. Find $f(x)$ if $f(0) = -1$ and $f(1) = 13$.

**9** If $\int_0^a e^{1-2x} \, dx = \frac{e}{4}$, find $a$ in the form $\ln k$.

**10** Suppose $f''(x) = 3x^2 + 2x$ and $f(0) = f(2) = 3$. Find:

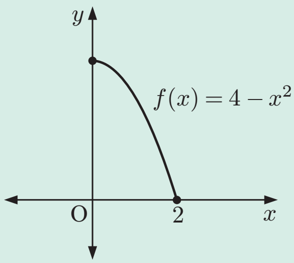
**a** $f(x)$ **b** the equation of the normal to $y = f(x)$ at $x = 2$.

**11** **a** Find $(e^x + 2)^3$ using the binomial expansion.

**b** Hence find the exact value of $\int_0^1 (e^x + 2)^3 \, dx$.

## Review set 15B

**1**




**a** Use *four* upper and lower rectangles to find rational numbers $A$ and $B$ such that:
$$A < \int_0^2 (4 - x^2) \, dx < B.$$

**b** Hence, find a good estimate for
$$\int_0^2 (4 - x^2) \, dx.$$

**2** Find:

**a** $\int (2e^{-x} + 3) \, dx$ **b** $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$ **c** $\int \left(3 + e^{2x-1}\right)^2 dx$

**3** Given that $f'(x) = x^2 - 3x + 2$ and $f(1) = 3$, find $f(x)$.

**4** Find the exact value of $\int_2^3 \frac{1}{\sqrt{3x - 4}} \, dx$.

**5** By differentiating $(3x^2 + x)^3$, find $\int (3x^2 + x)^2 (6x + 1) \, dx$.

**6** If $\int_1^4 f(x) \, dx = 3$, determine:

**a** $\int_1^4 (f(x) + 1) \, dx$ **b** $\int_1^2 f(x) \, dx - \int_4^2 f(x) \, dx$

**7** Given that $f''(x) = 2\sin(2x)$, $f'(\frac{\pi}{2}) = 0$, and $f(0) = 3$, find the exact value of $f(\frac{\pi}{2})$.

**8** Find $\frac{d}{dx}(e^{-2x} \sin x)$ and hence find $\int_0^{\frac{\pi}{2}} \left[e^{-2x}(\cos x - 2\sin x)\right] dx$

**9** Find $\int (2x+3)^n \, dx$ for all integers $n \neq -1$.

**10** A function has gradient function $2\sqrt{x} + \dfrac{a}{\sqrt{x}}$ and passes through the points $(0, 2)$ and $(1, 4)$.
Find $a$ and hence explain why the function $y = f(x)$ has no stationary points.

**11** $\displaystyle\int_a^{2a} (x^2 + ax + 2) \, dx = \frac{73a}{2}$. Find $a$.

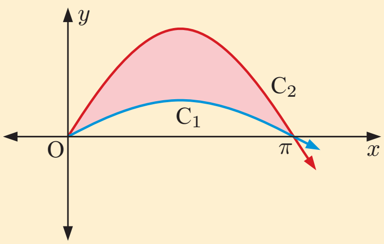
# 16

# Applications of integration

**Contents:**

## Opening problem

The illustrated curves are those of $y = \sin x$ and $y = 3\sin x$.




**Things to think about:**

**a** Can you identify each curve?

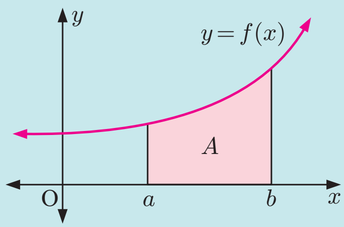
**b** Can you find the shaded area enclosed by $C_1$ and $C_2$ for $0 \leqslant x \leqslant \pi$?

We have already seen how definite integrals can be related to the areas between functions and the $x$-axis. In this chapter we explore this relationship further, and consider other applications of integral calculus including kinematics.

# A THE AREA UNDER A CURVE

We have already established in **Chapter 15** that:

If $f(x)$ is positive and continuous on the interval $a \leqslant x \leqslant b$, then the area bounded by $y = f(x)$, the $x$-axis, and the vertical lines $x = a$ and $x = b$ is given by $A = \int_a^b f(x)\,dx$ or $\int_a^b y\,dx$.

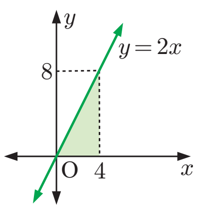



### Example 1 — Self Tutor

Find the area of the region enclosed by $y = 2x$, the $x$-axis, $x = 0$, and $x = 4$ by using:

**a** a geometric argument **b** integration.

**a**




$$\text{Area} = \tfrac{1}{2} \times 4 \times 8 = 16 \text{ units}^2$$

**b**

$$\begin{aligned}\text{Area} &= \int_0^4 2x\,dx \\ &= \left[x^2\right]_0^4 \\ &= 4^2 - 0^2 \\ &= 16 \text{ units}^2\end{aligned}$$

## EXERCISE 16A

**1** Find the area of each of the regions described below by using:

**i** a geometric argument **ii** integration

**a** $y = 5$, the $x$-axis, $x = -6$, and $x = 0$

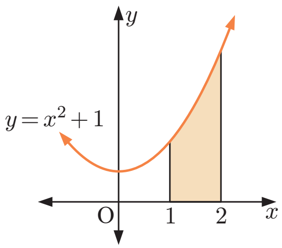
**b** $y = x$, the $x$-axis, $x = 4$, and $x = 5$

**c** $y = -3x$, the $x$-axis, $x = -3$, and $x = 0$

**d** $y = -x$, the $x$-axis, $x = 0$, and $x = 2$

### Example 2 — Self Tutor

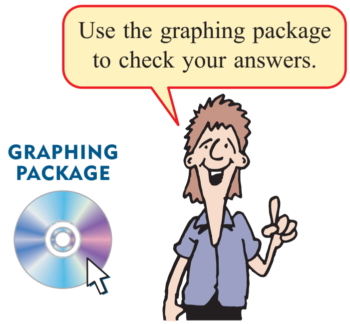
Find the area of the region enclosed by $y = x^2 + 1$, the $x$-axis, $x = 1$, and $x = 2$.




$$\text{Area} = \int_1^2 (x^2 + 1)\, dx$$
$$= \left[\frac{x^3}{3} + x\right]_1^2$$
$$= \left(\tfrac{8}{3} + 2\right) - \left(\tfrac{1}{3} + 1\right)$$
$$= 3\tfrac{1}{3} \text{ units}^2$$


**2** Find the area of the region bounded by:

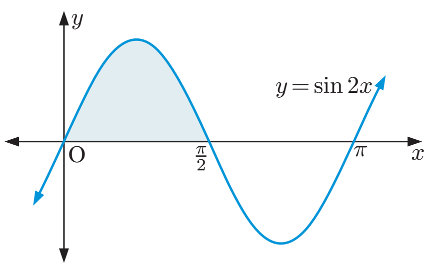
- **a** $y = x^2$, the $x$-axis, and $x = 1$
- **b** $y = \sin x$, the $x$-axis, $x = 0$, and $x = \pi$
- **c** $y = x^3$, the $x$-axis, $x = 1$, and $x = 4$
- **d** $y = e^x$, the $x$-axis, the $y$-axis, and $x = 1$
- **e** the $x$-axis and the part of $y = 6 + x - x^2$ above the $x$-axis
- **f** the axes and $y = \sqrt{9 - x}$
- **g** $y = \dfrac{1}{x^2}$, the $x$-axis, $x = 1$, and $x = 2$
- **h** $y = 2 - \dfrac{1}{\sqrt{x}}$, the $x$-axis, and $x = 4$
- **i** $y = e^x + e^{-x}$, the $x$-axis, $x = -1$, and $x = 1$




### Example 3 — Self Tutor

Find the area enclosed by one arch of the curve $y = \sin 2x$ and the $x$-axis.

The period of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$, so the first positive $x$-intercept is $\frac{\pi}{2}$.




$$\text{The required area} = \int_0^{\frac{\pi}{2}} \sin 2x\, dx$$
$$= \left[\tfrac{1}{2}(-\cos 2x)\right]_0^{\frac{\pi}{2}}$$
$$= -\tfrac{1}{2}\left[\cos 2x\right]_0^{\frac{\pi}{2}}$$
$$= -\tfrac{1}{2}(\cos \pi - \cos 0)$$
$$= 1 \text{ unit}^2$$

**3** Find the area enclosed by one arch of the curve $y = \cos 3x$ and the $x$-axis.

### Discovery $\int_a^b f(x)\,dx$ and areas

Does $\int_a^b f(x)\,dx$ always give us an area?

**What to do:**

1. Find $\int_0^1 x^3\,dx$ and $\int_{-1}^1 x^3\,dx$.
2. Using a graph, explain why the first integral in **1** gives an area, whereas the second integral does not.
3. Find $\int_{-1}^0 x^3\,dx$ and explain why the answer is negative.
4. Show that $\int_{-1}^0 x^3\,dx + \int_0^1 x^3\,dx = \int_{-1}^1 x^3\,dx$.
5. Find $\int_0^{-1} x^3\,dx$ and interpret its meaning.
6. Suppose $f(x)$ is a function such that $f(x) \leqslant 0$ for all $a \leqslant x \leqslant b$. Suggest an expression for the area between the curve and the function for $a \leqslant x \leqslant b$.

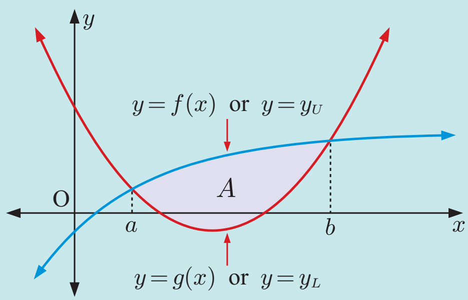
## B THE AREA BETWEEN TWO FUNCTIONS

If two functions $f(x)$ and $g(x)$ intersect at $x = a$ and $x = b$, and $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$, then the area of the shaded region between their points of intersection is given by

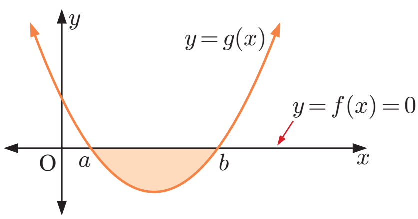
$$A = \int_a^b [f(x) - g(x)]\,dx.$$

Alternatively, if the upper and lower functions are $y = y_U$ and $y = y_L$ respectively, then the area is

$$A = \int_a^b [y_U - y_L]\,dx.$$




We can see immediately that if $f(x)$ is the $x$-axis $f(x) = 0$, then the enclosed area is $\int_a^b [-g(x)]\,dx$ or $-\int_a^b g(x)\,dx$.



### Example 4

Self Tutor

Use $\int_a^b [y_U - y_L]\,dx$ to find the area bounded by the $x$-axis and $y = x^2 - 2x$.

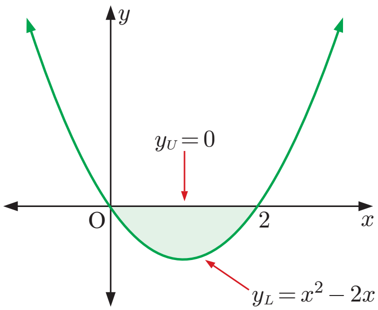
The curve cuts the $x$-axis when $y = 0$

$\therefore\ x^2 - 2x = 0$

$\therefore\ x(x-2) = 0$

$\therefore\ x = 0$ or $2$

$\therefore$ the $x$-intercepts are 0 and 2.




$$\begin{aligned}\text{Area} &= \int_0^2 [y_U - y_L]\,dx \\ &= \int_0^2 [0 - (x^2 - 2x)]\,dx \\ &= \int_0^2 (2x - x^2)\,dx \\ &= \left[x^2 - \frac{x^3}{3}\right]_0^2 \\ &= \left(4 - \tfrac{8}{3}\right) - (0)\end{aligned}$$

$\therefore$ the area is $\frac{4}{3}$ units$^2$.

### Example 5

Self Tutor

Find the area of the region enclosed by $y = x + 2$ and $y = x^2 + x - 2$.

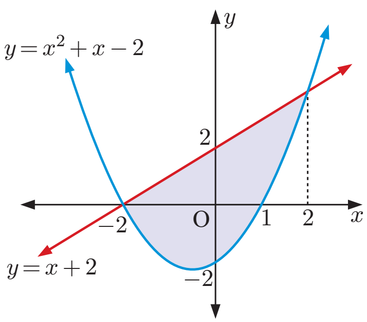
$y = x + 2$ meets $y = x^2 + x - 2$

where $x^2 + x - 2 = x + 2$

$\therefore\ x^2 - 4 = 0$

$\therefore\ (x+2)(x-2) = 0$

$\therefore\ x = \pm 2$




$$\begin{aligned}\text{Area} &= \int_{-2}^{2} [y_U - y_L]\,dx \\ &= \int_{-2}^{2} [(x+2) - (x^2 + x - 2)]\,dx \\ &= \int_{-2}^{2} (4 - x^2)\,dx \\ &= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} \\ &= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right) \\ &= 10\tfrac{2}{3} \text{ units}^2\end{aligned}$$

$\therefore$ the area is $10\frac{2}{3}$ units$^2$.

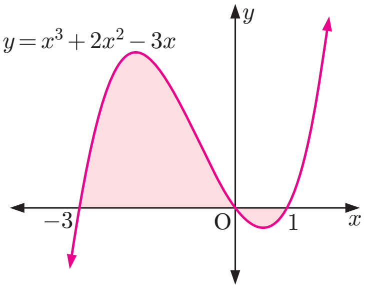
### Example 6

Self Tutor

Find the total area of the regions contained by $y = f(x)$ and the $x$-axis for $f(x) = x^3 + 2x^2 - 3x$.

$$\begin{aligned} f(x) &= x^3 + 2x^2 - 3x \\ &= x(x^2 + 2x - 3) \\ &= x(x-1)(x+3) \end{aligned}$$

$\therefore$ $y = f(x)$ cuts the $x$-axis at 0, 1, and $-3$.




Total area

$$\begin{aligned} &= \int_{-3}^{0} (x^3 + 2x^2 - 3x)\, dx - \int_{0}^{1} (x^3 + 2x^2 - 3x)\, dx \\ &= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{-3}^{0} - \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{0}^{1} \\ &= \left(0 - -11\tfrac{1}{4}\right) - \left(-\tfrac{7}{12} - 0\right) \\ &= 11\tfrac{5}{6} \text{ units}^2 \end{aligned}$$

## EXERCISE 16B

**1** Find the exact value of the area bounded by:

**a** the $x$-axis and $y = x^2 + x - 2$

**b** the $x$-axis, $y = e^{-x} - 1$, and $x = 2$

**c** the $x$-axis and the part of $y = 3x^2 - 8x + 4$ below the $x$-axis

**d** $y = \cos x$, the $x$-axis, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$

**e** $y = x^3 - 4x$, the $x$-axis, $x = 1$, and $x = 2$

**f** $y = \sin x - 1$, the $x$-axis, $x = 0$, and $x = \frac{\pi}{2}$

**2** Find the area of the region enclosed by $y = x^2 - 2x$ and $y = 3$.

**3** Consider the graphs of $y = x - 3$ and $y = x^2 - 3x$.

**a** Sketch the graphs on the same set of axes.

**b** Find the coordinates of the points where the graphs meet.

**c** Find the area of the region enclosed by the two graphs.

**4** Determine the area of the region enclosed by $y = \sqrt{x}$ and $y = x^2$.

**5** **a** On the same set of axes, graph $y = e^x - 1$ and $y = 2 - 2e^{-x}$, showing axes intercepts and asymptotes.

**b** Find algebraically the points of intersection of $y = e^x - 1$ and $y = 2 - 2e^{-x}$.

**c** Find the area of the region enclosed by the two curves.

**6** Find the area of the region bounded by $y = 2e^x$, $y = e^{2x}$, and $x = 0$.

**7** On the same set of axes, sketch $y = 2x$ and $y = 4x^2$.
Find the area of the region enclosed by these functions.

**8** Sketch the circle with equation $x^2 + y^2 = 9$.

**a** Explain why the upper half of the circle has equation $y = \sqrt{9 - x^2}$.

**b** Hence, determine $\displaystyle\int_0^3 \sqrt{9 - x^2}\,dx$ without actually integrating the function.

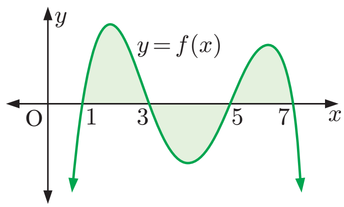
**9** Find the area enclosed by the function $y = f(x)$ and the $x$-axis for:

**a** $f(x) = x^3 - 9x$ **b** $f(x) = -x(x-2)(x-4)$ **c** $f(x) = x^4 - 5x^2 + 4$.

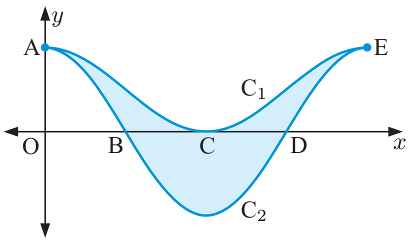
**10** Answer the **Opening Problem** on page **438**.

**11** **a** Explain why the total area shaded is *not* equal to $\displaystyle\int_1^7 f(x)\,dx$.

**b** Write an expression for the total shaded area in terms of integrals.




**12**




The illustrated curves are $y = \cos(2x)$ and $y = \frac{1}{2} + \frac{1}{2}\cos(2x)$.

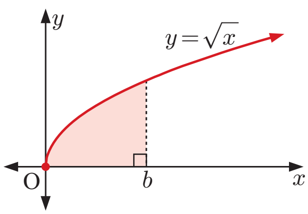
**a** Identify each curve as $C_1$ or $C_2$.

**b** Determine the coordinates of A, B, C, D, and E.

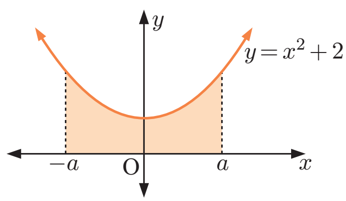
**c** Show that the area of the shaded region is $\frac{\pi}{2}$ units$^2$.

**13** Explain why the area between two functions $f(x)$ and $g(x)$ on the interval $a \leqslant x \leqslant b$ is given by $A = \displaystyle\int_a^b |f(x) - g(x)|\,dx$.

**14** The shaded area is 1 unit$^2$.
Find $b$, correct to 4 decimal places.




**15** The shaded area is $6a$ units$^2$.
Find the exact value of $a$.



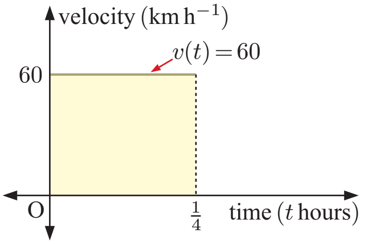
# C KINEMATICS

## DISTANCES FROM VELOCITY GRAPHS

Suppose a car travels at a constant positive velocity of 60 km h$^{-1}$ for 15 minutes.

We know the distance travelled = speed × time

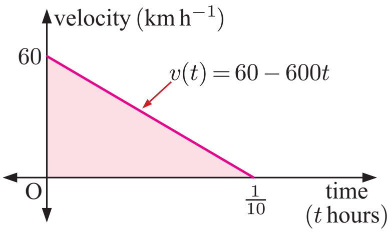
$$= 60 \text{ km h}^{-1} \times \tfrac{1}{4} \text{ h}$$
$$= 15 \text{ km}.$$




When we graph *velocity* against *time*, the graph is a horizontal line, and we can see that the distance travelled is the area shaded.

So, the distance travelled can also be found by the definite integral $\displaystyle\int_0^{\frac{1}{4}} 60\, dt = 15$ km.

Now suppose the velocity decreases at a constant rate, so that the car, initially travelling at 60 km h$^{-1}$, stops in 6 minutes or $\frac{1}{10}$ hour.




In this case the *average* speed is 30 km h$^{-1}$, so the distance travelled $= 30 \text{ km h}^{-1} \times \frac{1}{10}$ h

$= 3$ km

But the triangle has area $= \frac{1}{2} \times \text{base} \times \text{altitude}$

$$= \tfrac{1}{2} \times \tfrac{1}{10} \times 60 = 3$$

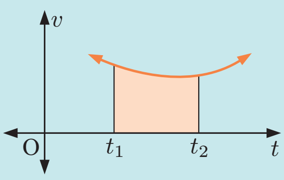
So, once again the shaded area gives us the distance travelled, and we can find it using the definite integral

$$\int_0^{\frac{1}{10}} (60 - 600t)\, dt = 3.$$

These results suggest that:

For a velocity-time function $v(t)$ where $v(t) \geqslant 0$ on the interval $t_1 \leqslant t \leqslant t_2$,

$$\textbf{distance travelled} = \int_{t_1}^{t_2} v(t)\, dt.$$

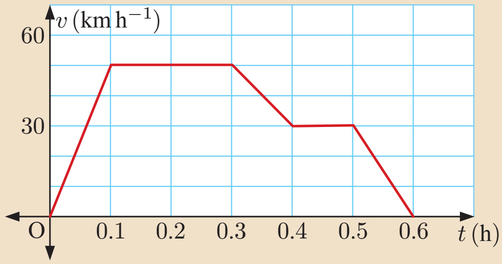



If we have a change of direction within the time interval then the velocity will change sign. We therefore need to add the components of area above and below the $t$-axis to find the total distance travelled.

## Example 7

Self Tutor

The velocity-time graph for a train journey is illustrated in the graph alongside. Find the total distance travelled by the train.




Total distance travelled

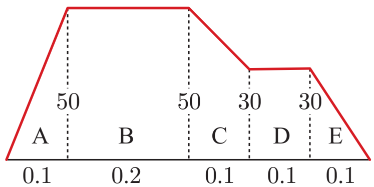
$= $ total area under the graph

$= $ area A $+$ area B $+$ area C $+$ area D $+$ area E

$= \frac{1}{2}(0.1)50 + (0.2)50 + \left(\frac{50+30}{2}\right)(0.1) + (0.1)30 + \frac{1}{2}(0.1)30$

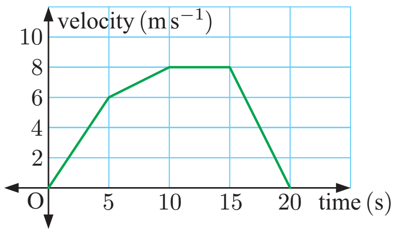
$= 2.5 + 10 + 4 + 3 + 1.5$

$= 21$ km

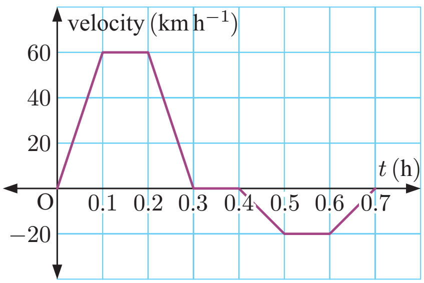



## EXERCISE 16C.1

**1** A runner has the velocity-time graph shown. Find the total distance travelled by the runner.




**2**




A car travels along a straight road with the velocity-time function illustrated.

**a** What is the significance of the graph:

  **i** above the $t$-axis

  **ii** below the $t$-axis?

**b** Find the total *distance* travelled by the car.

**c** Find the final *displacement* of the car from its starting point.

**3** A cyclist rides off from rest, accelerating at a constant rate for 3 minutes until she reaches 40 km h$^{-1}$. She then maintains a constant speed for 4 minutes until reaching a hill. She slows down at a constant rate over one minute to 30 km h$^{-1}$, then continues at this rate for 10 minutes. At the top of the hill she reduces her speed uniformly and is stationary 2 minutes later.

**a** Draw a graph to show the cyclist's motion.

**b** How far has the cyclist travelled?

## DISPLACEMENT AND VELOCITY FUNCTIONS

In this section we are concerned with **motion in a straight line**.

For some displacement function $s(t)$, the velocity function is $v(t) = s'(t)$.

So, given a velocity function we can determine the displacement function by the integral

$$s(t) = \int v(t)\, dt$$

The constant of integration determines the **initial position** on the line where the object begins.

Using the displacement function we can determine the change in displacement in a time interval $t_1 \leqslant t \leqslant t_2$ using the integral:

$$\textbf{Displacement} = s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\, dt$$

## TOTAL DISTANCE TRAVELLED

To determine the total distance travelled in a time interval $t_1 \leqslant t \leqslant t_2$, we need to account for any changes of direction in the motion.

To find the total distance travelled given a velocity function $v(t) = s'(t)$ on $t_1 \leqslant t \leqslant t_2$:

- Draw a sign diagram for $v(t)$ so we can determine any changes of direction.
- Determine $s(t)$ by integration, including a constant of integration.
- Find $s(t_1)$ and $s(t_2)$. Also find $s(t)$ at each time the direction changes.
- Draw a motion diagram.
- Determine the total distance travelled from the motion diagram.

## VELOCITY AND ACCELERATION FUNCTIONS

We know that the acceleration function is the derivative of the velocity function, so $a(t) = v'(t)$.

So, given an acceleration function, we can determine the velocity function by integration:

$$v(t) = \int a(t)\, dt$$

### Summary

$$\begin{tikzcd}[column sep=huge]
\begin{array}{c} s(t) \\ \textbf{displacement} \end{array} \arrow[r, bend left, "\text{differentiate}"] & \begin{array}{c} v(t) = \dfrac{ds}{dt} \\ \textbf{velocity} \end{array} \arrow[l, bend left, "\text{integrate}"] \arrow[r, bend left, "\text{differentiate}"] & \begin{array}{c} a(t) = \dfrac{dv}{dt} = \dfrac{d^2 s}{dt^2} \\ \textbf{acceleration} \end{array} \arrow[l, bend left, "\text{integrate}"]
\end{tikzcd}$$

**Example 8** — Self Tutor

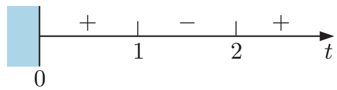
A particle P moves in a straight line with velocity function $v(t) = t^2 - 3t + 2$ $\text{m s}^{-1}$.

**a** How far does P travel in the first 4 seconds of motion?

**b** Find the displacement of P after 4 seconds.

**a** $v(t) = s'(t) = t^2 - 3t + 2$
$\qquad\qquad\quad = (t-1)(t-2)$

$\therefore$ the sign diagram of $v$ is:



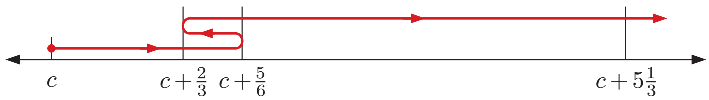

Since the signs change, P reverses direction at $t = 1$ and $t = 2$ seconds.

Now $s(t) = \int (t^2 - 3t + 2)\,dt = \frac{t^3}{3} - \frac{3t^2}{2} + 2t + c$

Hence $s(0) = c$ $\qquad s(1) = \frac{1}{3} - \frac{3}{2} + 2 + c = c + \frac{5}{6}$

$s(2) = \frac{8}{3} - 6 + 4 + c = c + \frac{2}{3}$ $\qquad s(4) = \frac{64}{3} - 24 + 8 + c = c + 5\frac{1}{3}$

Motion diagram:




$\therefore$ total distance travelled $= (c + \frac{5}{6} - c) + (c + \frac{5}{6} - [c + \frac{2}{3}]) + (c + 5\frac{1}{3} - [c + \frac{2}{3}])$
$= \frac{5}{6} + \frac{5}{6} - \frac{2}{3} + 5\frac{1}{3} - \frac{2}{3}$
$= 5\frac{2}{3}$ m

**b** Displacement $=$ final position $-$ original position
$= s(4) - s(0)$
$= c + 5\frac{1}{3} - c$
$= 5\frac{1}{3}$ m

So, the displacement is $5\frac{1}{3}$ m to the right.

## EXERCISE 16C.2

**1** A particle has velocity function $v(t) = 1 - 2t$ $\text{cm s}^{-1}$ as it moves in a straight line. The particle is initially 2 cm to the right of O.

**a** Write a formula for the displacement function $s(t)$.

**b** Find the total distance travelled in the first second of motion.

**c** Find the displacement of the particle at the end of one second.

**2** Particle P is initially at the origin O. It moves with the velocity function $v(t) = t^2 - t - 2$ $\text{cm s}^{-1}$.

**a** Write a formula for the displacement function $s(t)$.

**b** Find the total distance travelled in the first 3 seconds of motion.

**c** Find the displacement of the particle at the end of three seconds.

**3** An object has velocity function $v(t) = \cos(2t)$ $\text{m s}^{-1}$. If $s(\frac{\pi}{4}) = 1$ m, determine $s(\frac{\pi}{3})$ exactly.

**4** The velocity of a moving object is given by $v(t) = 32 + 4t$ $\text{m s}^{-1}$.

**a** If $s = 16$ m when $t = 0$ seconds, find the displacement function.

**b** Explain why the displacement of the object and its total distance travelled in the interval $0 \leqslant t \leqslant t_1$, can both be represented by the definite integral $\int_0^{t_1} (32 + 4t)\, dt$.

**c** Show that the object is travelling with constant acceleration.

**5** A particle moves along the $x$-axis with velocity function $s'(t) = 16t - 4t^3$ units per second. Find the total distance travelled in the time interval:

**a** $0 \leqslant t \leqslant 3$ seconds **b** $1 \leqslant t \leqslant 3$ seconds.

**6** A particle moves in a straight line with velocity function $v(t) = \cos t$ $\text{m s}^{-1}$.

**a** Show that the particle oscillates between two points.

**b** Find the distance between the two points in **a**.

**7** The velocity of a particle travelling in a straight line is given by $v(t) = 50 - 10e^{-0.5t}$ $\text{m s}^{-1}$, where $t \geqslant 0$, $t$ in seconds.

**a** State the initial velocity of the particle.

**b** Find the velocity of the particle after 3 seconds.

**c** How long will it take for the particle's velocity to increase to 45 $\text{m s}^{-1}$?

**d** Discuss $v(t)$ as $t \to \infty$.

**e** Show that the particle's acceleration is always positive.

**f** Draw the graph of $v(t)$ against $t$.

**g** Find the total distance travelled by the particle in the first 3 seconds of motion.

## Example 9

Self Tutor

A particle is initially at the origin and moving to the right at 5 $\text{cm s}^{-1}$. It accelerates with time according to $a(t) = 4 - 2t$ $\text{cm s}^{-2}$.

**a** Find the velocity function of the particle, and sketch its graph for $0 \leqslant t \leqslant 6$ s.

**b** For the first 6 seconds of motion, determine the:

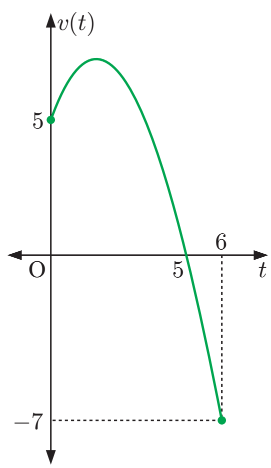
**i** displacement of the particle **ii** total distance travelled.

**a** $v(t) = \int a(t)\, dt = \int (4 - 2t)\, dt$

$= 4t - t^2 + c$

But $v(0) = 5$, so $c = 5$

$\therefore$ $v(t) = -t^2 + 4t + 5$ $\text{cm s}^{-1}$




**b** $s(t) = \int v(t)\, dt = \int (-t^2 + 4t + 5)\, dt$

$= -\frac{1}{3}t^3 + 2t^2 + 5t + c$ cm

But $s(0) = 0$, so $c = 0$

$\therefore$ $s(t) = -\frac{1}{3}t^3 + 2t^2 + 5t$ cm

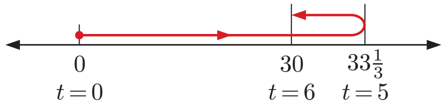
**i** Displacement $= s(6) - s(0)$

$= -\frac{1}{3}(6)^3 + 2(6)^2 + 5(6)$

$= 30$ cm

**ii** The particle changes direction when $t = 5$ s.
Now $s(5) = -\frac{1}{3}(5)^3 + 2(5)^2 + 5(5) = 33\frac{1}{3}$ cm

Motion diagram:




$\therefore$ the total distance travelled $= 33\frac{1}{3} + 3\frac{1}{3}$
$= 36\frac{2}{3}$ cm

**8** A particle is initially stationary at the origin. It accelerates according to the function $a(t) = \dfrac{2}{(t+1)^3}$ m s$^{-2}$.

**a** Find the velocity function $v(t)$ for the particle.
**b** Find the displacement function $s(t)$ for the particle.
**c** Describe the motion of the particle at the time $t = 2$ seconds.

**9** A train moves along a straight track with acceleration $\dfrac{t}{10} - 3$ m s$^{-2}$. The initial velocity of the train is 45 m s$^{-1}$.

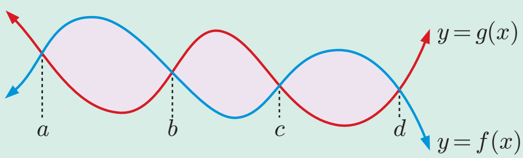
**a** Determine the velocity function $v(t)$.
**b** Evaluate $\displaystyle\int_0^{60} v(t)\,dt$ and explain what this value represents.

**10** An object has initial velocity 20 m s$^{-1}$ as it moves in a straight line with acceleration function $4e^{-\frac{t}{20}}$ m s$^{-2}$.

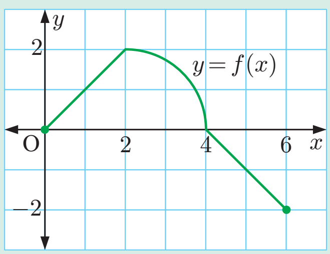
**a** Show that as $t$ increases the object approaches a limiting velocity.
**b** Find the total distance travelled in the first 10 seconds of motion.

## Review set 16A

**1** Write an expression for the total shaded area.




**2**



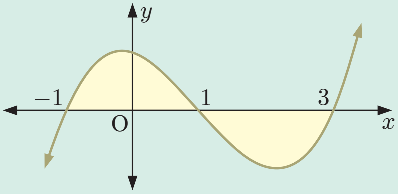

Find:

**a** $\displaystyle\int_0^4 f(x)\,dx$

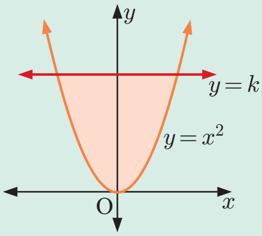
**b** $\displaystyle\int_4^6 f(x)\,dx$

**c** $\displaystyle\int_0^6 f(x)\,dx$

**3** Does $\int_{-1}^{3} f(x)\,dx$ represent the area of the shaded region?
Explain your answer briefly.




**4** Determine $k$ if the enclosed region has area $5\frac{1}{3}$ units$^2$.




**5** Find the area of the region enclosed by $y = x^2 + 4x + 1$ and $y = 3x + 3$.

**6** A particle moves in a straight line with velocity $v(t) = t^2 - 6t + 8$ m s$^{-1}$, for $t \geqslant 0$ seconds.

- **a** Draw a sign diagram for $v(t)$.
- **b** Describe what happens to the particle in the first 5 seconds of motion.
- **c** After 5 seconds, how far is the particle from its original position?
- **d** Find the total distance travelled in the first 5 seconds of motion.

**7** Determine the area enclosed by the axes and $y = 4e^x - 1$.

**8** A particle moves in a straight line with velocity given by $v(t) = \sin t$ m s$^{-1}$, where $t \geqslant 0$ seconds. Find the total distance travelled by the particle in the first 4 seconds of motion.

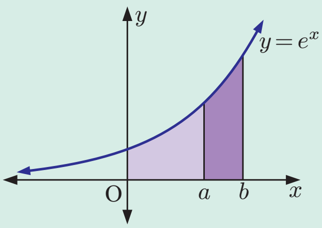
## Review set 16B

**1** At time $t = 0$ a particle passes through the origin with velocity 27 cm s$^{-1}$. Its acceleration $t$ seconds later is $6t - 30$ cm s$^{-2}$.

- **a** Write an expression for the particle's velocity.
- **b** Calculate the displacement from the origin after 6 seconds.

**2**
- **a** Sketch the graphs of $y = \frac{1}{2} - \frac{1}{2}\cos 2x$ and $y = \sin x$ on the same set of axes for $0 \leqslant x \leqslant \pi$.
- **b** Verify that both graphs pass through the points $(0, 0)$ and $(\frac{\pi}{2}, 1)$.
- **c** Find the area enclosed by these curves for $0 \leqslant x \leqslant \frac{\pi}{2}$.

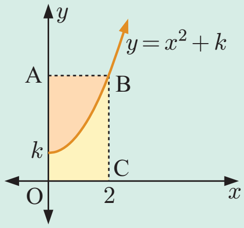
**3** Find $a$ given that the area of the region between $y = e^x$ and the $x$-axis from $x = 0$ to $x = a$ is 2 units$^2$.
Hence determine $b$ such that the area of the region from $x = a$ to $x = b$ is also 2 units$^2$.



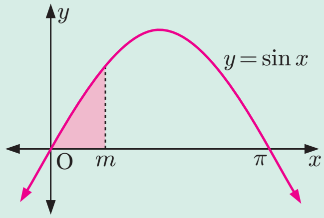

**4** A particle moves in a straight line with velocity $v(t) = 2t - 3t^2$ m s$^{-1}$, for $t \geqslant 0$ seconds.

- **a** Find a formula for the acceleration function $a(t)$.
- **b** Find a formula for the displacement function $s(t)$.
- **c** Find the change in displacement after two seconds.

**5** OABC is a rectangle and the two shaded regions are equal in area. Find $k$.




**6** The shaded region has area $\frac{1}{2}$ unit$^2$. Find the value of $m$.




**7** Find the area of the region enclosed by $y = x$ and $y = \sin\left(\frac{\pi x}{2}\right)$.

**8** A boat travelling in a straight line has its engine turned off at time $t = 0$. Its velocity at time $t$ seconds thereafter is given by $v(t) = \dfrac{100}{(t+2)^2}$ m s$^{-1}$.

**a** Find the initial velocity of the boat, and its velocity after 3 seconds.

**b** Discuss $v(t)$ as $t \to \infty$.

**c** Sketch the graph of $v(t)$ against $t$.

**d** Find how long it takes for the boat to travel 30 metres from when the engine is turned off.

**e** Find the acceleration of the boat at any time $t$.

**f** Show that $\dfrac{dv}{dt} = -kv^{\frac{3}{2}}$, and find the value of the constant $k$.

# ANSWERS

## EXERCISE 1A

**1** **a** $5 \in D$ **b** $6 \notin G$ **c** $d \notin \{a, e, i, o, u\}$
**d** $\{2, 5\} \subseteq \{1, 2, 3, 4, 5, 6\}$
**e** $\{3, 8, 6\} \not\subseteq \{1, 2, 3, 4, 5, 6\}$

**2** **a** **i** $\{9\}$ **ii** $\{5, 6, 7, 8, 9, 10, 11, 12, 13\}$
**b** **i** $\varnothing$ **ii** $\{1, 2, 3, 4, 5, 6, 7, 8\}$
**c** **i** $\{1, 3, 5, 7\} = A$ **ii** $\{1, 2, 3, 4, 5, 6, 7, 8, 9\} = B$

**3** **a** 5 **b** 6 **c** 2 **d** 9

**4** **a** true **b** true **c** true **d** true
**e** false **f** true **g** true **h** false

**5** **a** finite **b** infinite **c** infinite **d** infinite

**6** **a** true **b** true **c** false **d** true

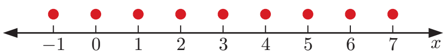
**7** **a** disjoint **b** not disjoint **8** true

**9** **a** 15 subsets **b** $2^n - 1, \ n \in \mathbb{Z}^+$

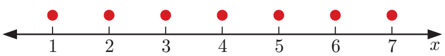
## EXERCISE 1B

**1** **a** finite **b** infinite **c** infinite **d** infinite
**e** infinite **f** infinite **g** infinite

**2** **a** **i** The set of all integers $x$ such that $x$ is between $-1$ and 7, including $-1$ and 7.
**ii** $\{-1, 0, 1, 2, 3, 4, 5, 6, 7\}$ **iii** 9
**iv**




**b** **i** The set of all natural numbers $x$ such that $x$ is between $-2$ and 8.
**ii** $\{1, 2, 3, 4, 5, 6, 7\}$ **iii** 7
**iv**

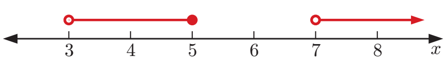



**c** **i** The set of all real numbers $x$ such that $x$ is between 0 and 1, including 0 and 1.
**ii** not possible **iii** infinite
**iv**


**d** **i** The set of all rational numbers $x$ such that $x$ is between 5 and 6, including 5 and 6.
**ii** not possible **iii** infinite
**iv** cannot be illustrated

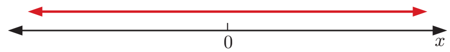
**e** **i** The set of all real numbers $x$ such that $x$ is between $-1$ and 5, including $-1$.
**ii** not possible **iii** infinite
**iv**


**f** **i** The set of all real numbers $x$ such that $x$ is between 3 and 5 (including 5), or greater than 7.
**ii** not possible **iii** infinite
**iv**




**g** **i** The set of all real numbers $x$ such that $x$ is less than or equal to 1, or greater than 2.
**ii** not possible **iii** infinite
**iv**


**h** **i** The set of all real numbers $x$ such that $x$ is less than 2, or greater than or equal to 1. (So, $A$ is the set of all real numbers.)
**ii** not possible **iii** infinite
**iv**



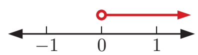

**3** **a** $A = \{x \in \mathbb{Z} : -100 < x < 100\}$
**b** $A = \{x \in \mathbb{R} : x > 1000\}$
**c** $A = \{x \in \mathbb{Q} : 2 \leqslant x \leqslant 3\}$

**4** **a** $A = \{x \in \mathbb{Z} : -2 \leqslant x \leqslant 3\}$
**b** $A = \{x \in \mathbb{Z} : x \leqslant -3\}$
**c** $A = \{x \in \mathbb{R} : -3 \leqslant x < 2\}$
**d** $A = \{x \in \mathbb{R} : 1 \leqslant x \leqslant 3 \cup x > 5\}$

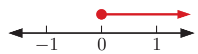
**5** **a** $A \subseteq B$ **b** $A \not\subseteq B$ **c** $A \subseteq B$ **d** $A \subseteq B$
**e** $A \not\subseteq B$ **f** $A \not\subseteq B$

**6** **a** neither **b** open **c** neither **d** open
**e** closed **f** neither

**7** **a** There are infinitely many rational numbers within any given interval, so we cannot represent $\mathbb{Q}$ as a series of dots like we can with $\mathbb{Z}$. We cannot represent $\mathbb{Q}$ with a continuous line either (like we do with $\mathbb{R}$), as this would imply that irrational numbers are part of $\mathbb{Q}$.
**b** **i** the set of positive real numbers, $\{x \in \mathbb{R} : x > 0\}$

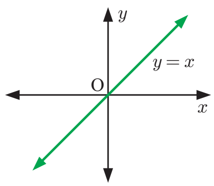



**ii** the set of positive real numbers and zero, $\{x \in \mathbb{R} : x \geqslant 0\}$

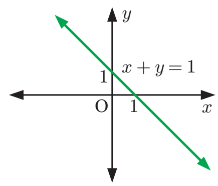



## EXERCISE 1C

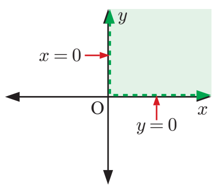
**1** **a** infinite



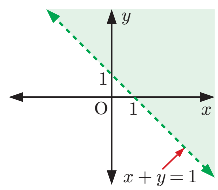

**b** infinite




**c** infinite



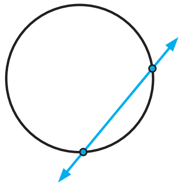

**d** infinite



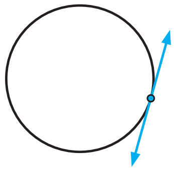

**2** **a** infinite **b** finite **c** infinite

**3** **a** **i** The set of all points of intersection between the line and the circle.
**ii** The set of all points that lie on either the straight line or the circle.

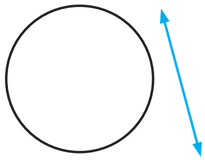
**b** **i** There are two points of intersection between the straight line and the circle.



**ii** There is one point of intersection between the straight line and the circle (that is, the straight line is a tangent to the circle).



**iii** The straight line and the circle do not intersect.



## EXERCISE 1D

**1** **a** $C' = \{\text{consonants}\}$ **b** $C' = \{x \in \mathbb{Z} : x \geqslant 0\}$
**c** $C' = \{x \in \mathbb{Z} : x \geqslant -4\}$ **d** $C' = \{x \in \mathbb{Q} : 2 < x < 8\}$

**2** **a** $\{2, 3, 4, 5, 6, 7\}$ **b** $\{0, 1, 8\}$ **c** $\{5, 6, 7, 8\}$
**d** $\{0, 1, 2, 3, 4\}$ **e** $\{5, 6, 7\}$ **f** $\{2, 3, 4, 5, 6, 7, 8\}$
**g** $\{2, 3, 4\}$ **h** $\{0, 1, 2, 3, 4, 8\}$

**3** **a** 9 **b** 11 **4** **a** false **b** true

**5** **a** $\{1, 2, 10, 11, 12\}$ **b** $\{1, 2, 3, 4, 12\}$
**c** $\{1, 8, 9, 10, 11, 12\}$ **d** $\{3, 4, 5, 6, 7\}$
**e** $\{1, 2, 8, 9, 10, 11, 12\}$ **f** $\{8, 9, 10, 11\}$
**g** $\{1, 2, 5, 6, 7, 8, 9, 10, 11, 12\}$ **h** $\{2, 10, 11\}$

**6** **a** $[0, \infty)$ **b** $(-\infty, 1)$ **c** $(-\infty, 3) \cup [2, \infty)$
**d** $(-\infty, -5] \cup (7, \infty)$ **e** $[1, 3)$
**f** $(-\infty, -5) \cup [0, 1]$

## EXERCISE 1E

**1** **a** $P = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$ **b** $\{2, 5, 11\}$
**c** $\{2, 3, 4, 5, 7, 11, 12, 13, 15, 17, 19, 23\}$
**d** $12 = 9 + 6 - 3$ ✓

**2** **a** $P = \{1, 2, 4, 7, 14, 28\}$, $Q = \{1, 2, 4, 5, 8, 10, 20, 40\}$
**b** $\{1, 2, 4\}$ **c** $\{1, 2, 4, 5, 7, 8, 10, 14, 20, 28, 40\}$
**d** $11 = 6 + 8 - 3$ ✓

**3** **a** $M = \{32, 36, 40, 44, 48, 52, 56\}$, $N = \{36, 42, 48, 54\}$
**b** $\{36, 48\}$ **c** $\{32, 36, 40, 42, 44, 48, 52, 54, 56\}$
**d** $9 = 7 + 4 - 2$ ✓

**4** **a** $R = \{-2, -1, 0, 1, 2, 3, 4\}$, $S = \{0, 1, 2, 3, 4, 5, 6\}$
**b** $\{0, 1, 2, 3, 4\}$ **c** $\{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$
**d** $9 = 7 + 7 - 5$ ✓

**5** **a** $C = \{-4, -3, -2, -1\}$
$D = \{-7, -6, -5, -4, -3, -2, -1\}$
**b** $\{-4, -3, -2, -1\}$ **c** $\{-7, -6, -5, -4, -3, -2, -1\}$
**d** $7 = 4 + 7 - 4$ ✓

**6** **a** $P = \{1, 2, 3, 4, 6, 12\}$, $Q = \{1, 2, 3, 6, 9, 18\}$,
$R = \{1, 3, 9, 27\}$
**b** **i** $\{1, 2, 3, 6\}$ **ii** $\{1, 3\}$ **iii** $\{1, 3, 9\}$
**iv** $\{1, 2, 3, 4, 6, 9, 12, 18\}$ **v** $\{1, 2, 3, 4, 6, 9, 12, 27\}$
**vi** $\{1, 2, 3, 6, 9, 18, 27\}$

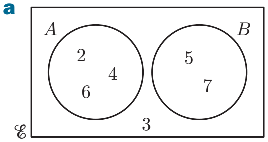
**c** **i** $\{1, 3\}$ **ii** $\{1, 2, 3, 4, 6, 9, 12, 18, 27\}$

**7** **a** $A = \{4, 8, 12, 16, 20, 24, 28, 32, 36\}$
$B = \{6, 12, 18, 24, 30, 36\}$, $C = \{12, 24, 36\}$
**b** **i** $\{12, 24, 36\}$ **ii** $\{12, 24, 36\}$
**iii** $\{12, 24, 36\}$ **iv** $\{12, 24, 36\}$
**v** $\{4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36\}$
**c** $12 = 9 + 6 + 3 - 3 - 3 - 3 + 3$ ✓

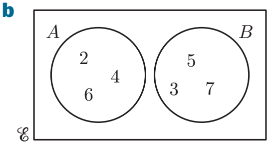
**8** **a** $A = \{6, 12, 18, 24, 30\}$, $B = \{1, 2, 3, 5, 6, 10, 15, 30\}$
$C = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$
**b** **i** $\{6, 30\}$ **ii** $\{2, 3, 5\}$ **iii** $\varnothing$ **iv** $\varnothing$
**v** $\{1, 2, 3, 5, 6, 7, 10, 11, 12, 13, 15, 17, 18, 19, 23, 24, 29, 30\}$
**c** $18 = 5 + 8 + 10 - 2 - 3 - 0 + 0$ ✓

## EXERCISE 1F.1

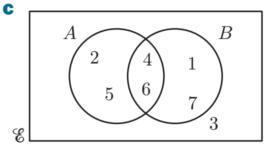
**1** **a**



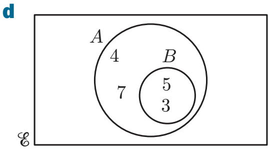

**b**




**c**

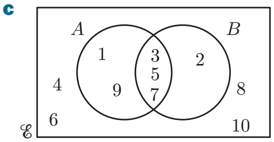



**d**



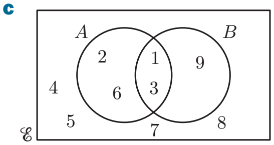

**2** **a** $A = \{1, 3, 5, 7, 9\}$
$B = \{2, 3, 5, 7\}$
**b** $A \cap B = \{3, 5, 7\}$
$A \cup B = \{1, 2, 3, 5, 7, 9\}$

**c**



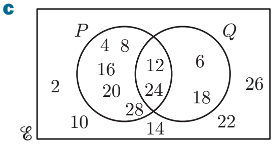

**3** **a** $A = \{1, 2, 3, 6\}$
$B = \{1, 3, 9\}$
**b** $A \cap B = \{1, 3\}$
$A \cup B = \{1, 2, 3, 6, 9\}$

**c**



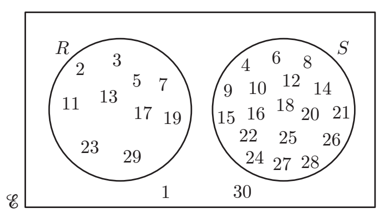

**4** **a** $P = \{4, 8, 12, 16, 20, 24, 28\}$
$Q = \{6, 12, 18, 24\}$
**b** $P \cap Q = \{12, 24\}$
$P \cup Q = \{4, 6, 8, 12, 16, 18, 20, 24, 28\}$

**c**




**5** **a** $R = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$
$S = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28\}$
**b** $R \cap S = \varnothing$
$R \cup S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\}$

**c**



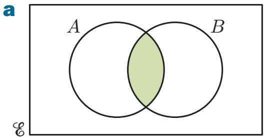
**6** **a** $\{b, d, e, h\}$ **b** $\{e, f, h, i, j\}$ **c** $\{a, c, f, g, i, j, k\}$
**d** $\{a, b, c, d, g, k\}$ **e** $\{e, h\}$ **f** $\{b, d, e, f, h, i, j\}$
**g** $\{a, c, g, k\}$ **h** $\{a, b, c, d, f, g, i, j, k\}$

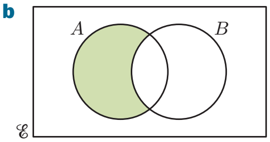
**7** **a** **i** $\{a, b, c, d, h, j\}$ **ii** $\{a, c, d, e, f, g, k\}$
**iii** $\{a, b, e, f, i, l\}$ **iv** $\{a, c, d\}$
**v** $\{a, b, c, d, e, f, g, h, j, k\}$ **vi** $\{a, e, f\}$
**vii** $\{a\}$ **viii** $\{a, b, c, d, e, f, g, h, i, j, k, l\}$

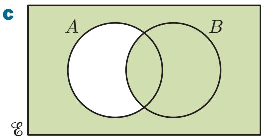
**b** **i** 12 **ii** 12

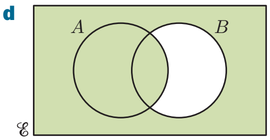
**c** $n(A \cup B \cup C)$
$= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C)$
$- n(B \cap C) + n(A \cap B \cap C)$

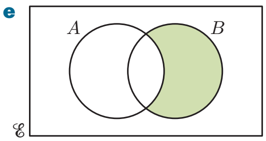
## EXERCISE 1F.2

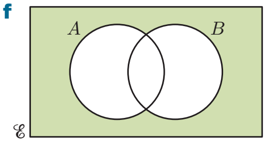
**1** **a**


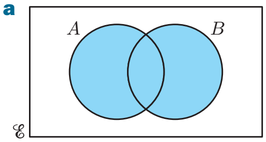

**b**


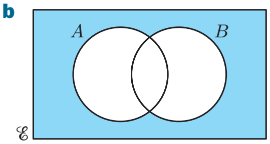

**c**


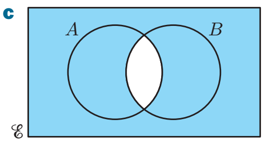

**d**


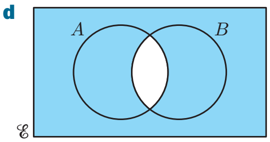

**e**


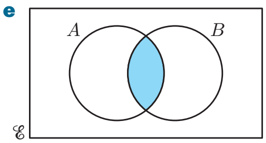

**f**


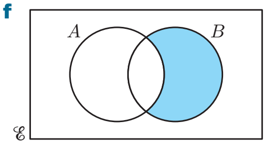

**2** **a**


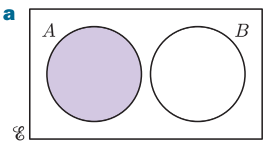

**b**


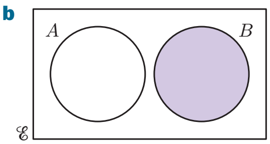

**c**


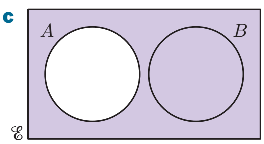

**d**


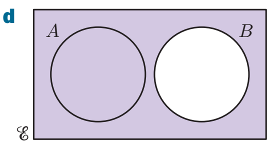

**e**


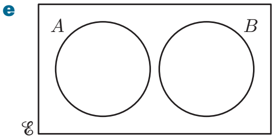

**f**


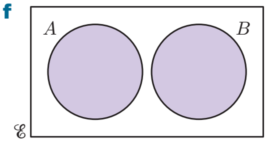

**3** **a**


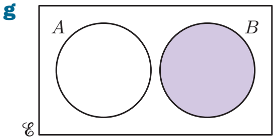

**b**


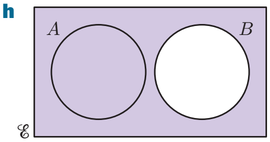

**c**


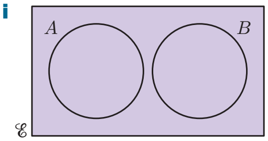

**d**


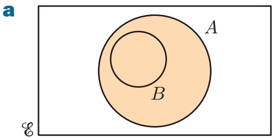

**e**


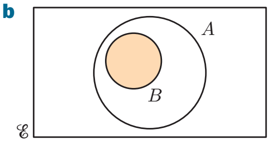

**f**


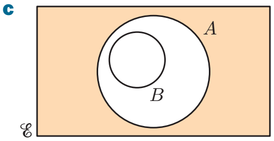

**g**


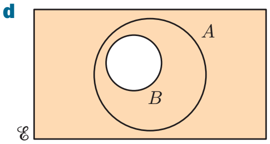

**h**


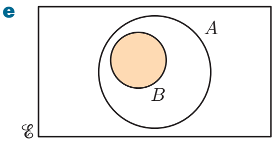

**i**


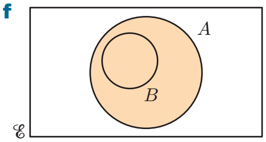

**4** **a**


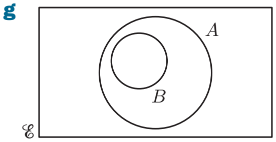

**b**


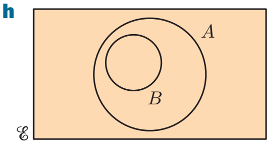

**c**


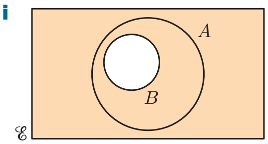

**d**


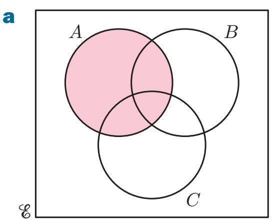

**e**


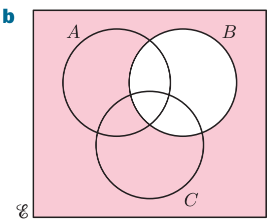

**f**


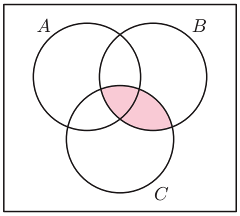

**g**


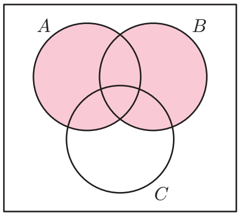

**h**


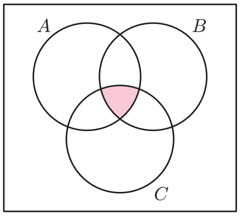

**i**


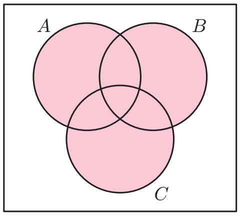

**5** **a**


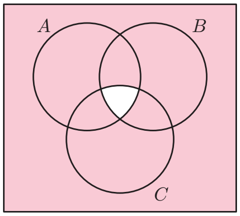

**b**


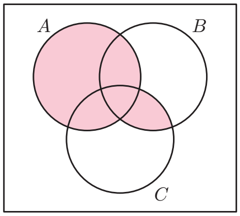
**c** 

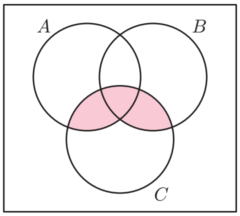
**d** 

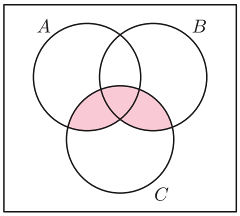
**e** 

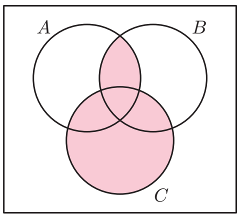
**f** 

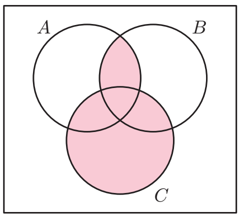
**g** 

**h** 

**i** 

**j** 

**k** 

**l** 

## EXERCISE 1G

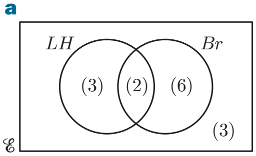
**1** **a** 7 **b** 14 **c** 14 **d** 7 **e** 5 **f** 9

**2** **a** $b+c$ **b** $c+d$ **c** $b$
**d** $a+b+c$ **e** $a+c+d$ **f** $d$

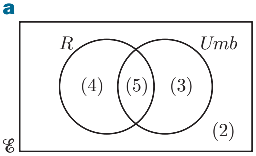
**3** **a** **i** $2a+4$ **ii** $4a+4$ **iii** $3a-5$ **iv** $5a-1$
**b** **i** $a=6$ **ii** $a=\frac{32}{5}$
Since $a \in \mathbb{N}$, there cannot be 31 elements in $\mathscr{E}$, but it is possible to have 29 elements.

**5** **a** 15 **b** 4 **6** **a** 18 **b** 6 **7** **a** 7 **b** 23

## EXERCISE 1H

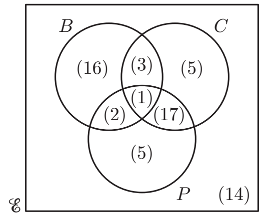
**1** **a** 


**b** **i** 9 **ii** 3 **iii** 3

**2** **a** 


**b** **i** 4 **ii** 2

**3** 13 players **4** 20 people

**5** **a** 


**b** **i** 16 **ii** 33 **iii** 14 **iv** 7

**6** **a** 29 **b** 6 **c** 1 **d** 11

**7** **a** 3 **b** 5 **c** 5 **d** 21

**8** **a** 3 **b** 4 **c** 9

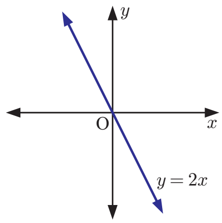
## REVIEW SET 1A

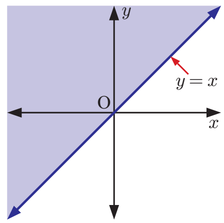
**1** **a** $S=\{3, 4, 5, 6, 7\}$ **b** 5 **c** 31

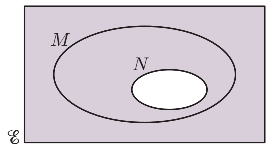
**2** **a** yes **b** yes **c** no **d** yes

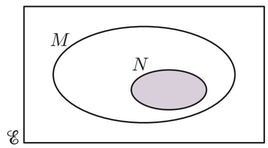
**3** **a** $X' = \{$orange, yellow, green, blue$\}$
**b** $X' = \{-5, -3, -2, 0, 1, 2, 5\}$
**c** $X' = \{x \in \mathbb{Q} : x \geqslant -8\}$
**d** $X' = \{x \in (-\infty, -3) \cup [1, 4]\}$

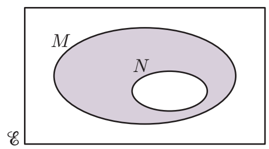
**4** **a** $\{x \in \mathbb{R} : -2 \leqslant x < 3\}$, neither
**b** $\{x \in \mathbb{R} : x < 3\}$, open

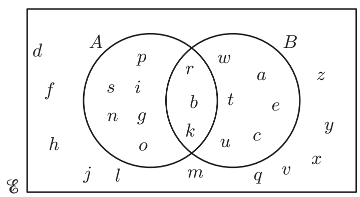
**5** **a** 


**b** 


**6** **a** 

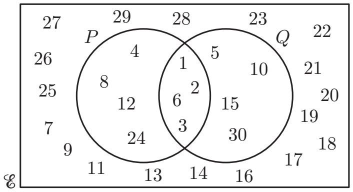
**b** 

**c** 

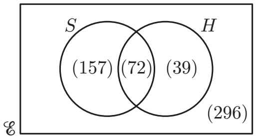
**7** **a** **i** $\{s, p, r, i, n, g, b, o, k, w, a, t, e, u, c\}$
**ii** $\{r, b, k\}$ **iii** $\{g, i, n, o, p, s\}$
**b** **i** {the letters in 'springbok' or 'waterbuck'}
**ii** {the letters common to both 'springbok' and 'waterbuck'}
**iii** {the letters in 'springbok' but not 'waterbuck'}
**c** 

**8** **a** **i** $\{1, 2, 3, 4, 6, 8, 12, 24\}$
**ii** $\{1, 2, 3, 5, 6, 10, 15, 30\}$ **iii** $\{1, 2, 3, 6\}$
**iv** $\{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 24, 30\}$

**b**




**9** **a**

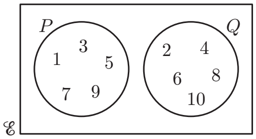



**b** **i** 72 **ii** 39 **iii** 268

**10** 8 **11** **a** 9 **b** 7 **c** 17

**12** **a** $P = \{1, 3, 5, 7, 9\}$
$Q = \{2, 4, 6, 8, 10\}$
**b** They are disjoint.

**c**




## REVIEW SET 1B

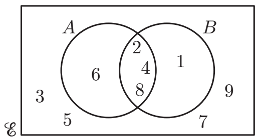
**1** **a** true **b** false **c** true **d** false **e** false

**2** **a** **i** $\{x \in \mathbb{R} : 5 < x < 12\}$ **ii** $\{x \in \mathbb{Z} : -4 \leqslant x < 7\}$
**iii** $\{x \in \mathbb{N} : x > 45\}$
**b** **i** infinite **ii** finite **iii** infinite

**3** $\varnothing$, $\{1\}$, $\{3\}$, $\{5\}$, $\{1, 3\}$, $\{3, 5\}$, $\{1, 5\}$, $\{1, 3, 5\}$

**4** **a** **i** $\{2, 4, 6, 8\}$
**ii** $\{2, 4, 8\}$
**iii** $\{3, 5, 7, 9\}$

**b**



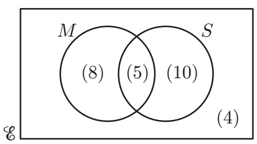

**5** **a** $\varnothing$ **b** $s + t$

**6** **a** neither **b** closed **c** neither

**7** **a** **i** The set of points which lie on both $A$ and $B$ (that is, the point(s) of intersection of line $A$ and line $B$).
**ii** The set of points which lie on line $A$ or line $B$.
**b** No. If the lines are coincident (so, $A$ and $B$ describe the same line), then $A \cap B$ will be infinite.
**c** $n(A \cap B) = 0$ or $1$

**8** **a** $C'$ **b** $(A \cap B) \cup (A \cap C)$ or $A \cap (B \cup C)$

**9** **a**

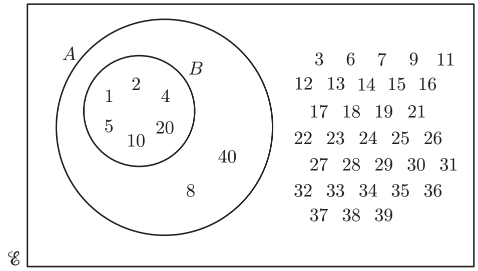



**b** **i** 27 **ii** 8 **iii** 14

**10** 4

**11** **a** $A = \{1, 2, 4, 5, 8, 10, 20, 40\}$, $B = \{1, 2, 4, 5, 10, 20\}$
**b** $B \subset A$

**c**




**12** **a** 1 **b** 7 **c** 15

## EXERCISE 2A.1

**1** **a**, **d**, **e** **2** **a**, **b**, **c**, **e**, **g**, **i**

**3** No, for example $(0, 4)$ and $(0, -4)$ satisfy the relation.

## EXERCISE 2A.2

**1** **a**, **c**, **f**

**2** **a** not a function **b** function, one-one
**c** function, not one-one

**3** **a** **i** \$13 **ii** yes **iii** yes
**b** **i** no **ii** no

## EXERCISE 2B

**1** **a** 2 **b** 2 **c** $-16$ **d** $-68$ **e** $\frac{17}{4}$

**2** **a** $-3$ **b** 3 **c** 3 **d** $-3$ **e** $\frac{15}{2}$

**3** **a** **i** 1 **ii** $-1$ **b** $x = -4$

**4** **a** $7 - 3a$ **b** $7 + 3a$ **c** $-3a - 2$ **d** $10 - 3b$
**e** $1 - 3x$ **f** $7 - 3x - 3h$

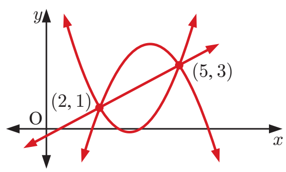
**5** **a** $2x^2 + 19x + 43$ **b** $2x^2 - 11x + 13$
**c** $2x^2 - 3x - 1$ **d** $2x^4 + 3x^2 - 1$
**e** $2x^4 - x^2 - 2$ **f** $2x^2 + (4h + 3)x + 2h^2 + 3h - 1$

**6** **a** **i** $-\frac{7}{2}$ **ii** $-\frac{3}{4}$ **iii** $-\frac{4}{9}$
**b** $x = 4$ **c** $\dfrac{2x + 7}{x - 2}$ **d** $x = \frac{9}{5}$

**7** $f$ is the function which converts $x$ into $f(x)$ whereas $f(x)$ is the value of the function at any value of $x$.

**8** **a** $V(4) = 6210$, the value in dollars after 4 years
**b** $t = 4.5$, the time in years for the photocopier to reach a value of 5780 dollars.
**c** 9650 dollars

**9**




**10** $f(x) = -2x + 5$

**11** $a = 3$, $b = -2$

**12** $a = 3$, $b = -1$, $c = -4$

## EXERCISE 2C

**1** **a** Domain $= \{x : x \geqslant -1\}$, Range $= \{y : y \leqslant 3\}$
**b** Domain $= \{x : -1 < x \leqslant 5\}$, Range $= \{y : 1 < y \leqslant 3\}$
**c** Domain $= \{x : x \neq 2\}$, Range $= \{y : y \neq -1\}$
**d** Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{y : 0 < y \leqslant 2\}$
**e** Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{y : y \geqslant -1\}$
**f** Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{y : y \leqslant \frac{25}{4}\}$
**g** Domain $= \{x : x \geqslant -4\}$, Range $= \{y : y \geqslant -3\}$

**h** Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{y : y > -2\}$

**i** Domain $= \{x : x \neq \pm 2\}$, Range $= \{y : y \leqslant -1 \text{ or } y > 0\}$

**2** **a** $f(x)$ defined for $x \geqslant -6$, Domain $= \{x : x \geqslant -6\}$

**b** $f(x)$ defined for $x \neq 0$, Domain $= \{x : x \neq 0\}$

**c** $f(x)$ defined for $x < \frac{3}{2}$, Domain $= \{x : x < \frac{3}{2}\}$

**3** **a** Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{y : y \in \mathbb{R}\}$

**b** Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{3\}$

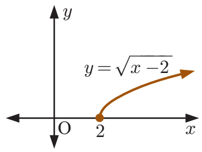
**c** Domain $= \{x : x \geqslant 0\}$, Range $= \{y : y \geqslant 0\}$

**d** Domain $= \{x : x \neq -1\}$, Range $= \{y : y \neq 0\}$

**e** Domain $= \{x : x > 0\}$, Range $= \{y : y < 0\}$

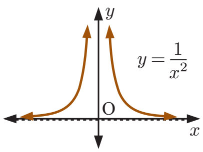
**f** Domain $= \{x : x \neq 3\}$, Range $= \{y : y \neq 0\}$

**4** **a**



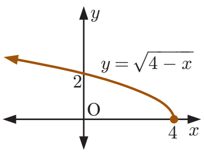

Domain $= \{x : x \geqslant 2\}$
Range $= \{y : y \geqslant 0\}$

**b**



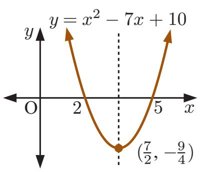

Domain $= \{x : x \neq 0\}$
Range $= \{y : y > 0\}$

**c**



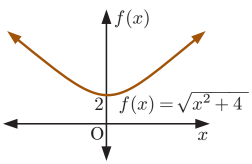

Domain $= \{x : x \leqslant 4\}$
Range $= \{y : y \geqslant 0\}$

**d**



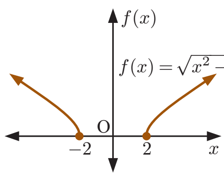

Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \geqslant -2\frac{1}{4}\}$

**e**



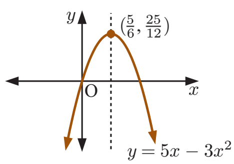

Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \geqslant 2\}$

**f**



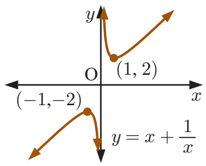

Domain $= \{x : x \leqslant -2 \text{ or } x \geqslant 2\}$
Range $= \{y : y \geqslant 0\}$

**g**



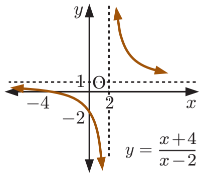

Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \leqslant \frac{25}{12}\}$

**h**



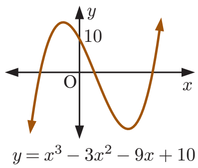

Domain $= \{x : x \neq 0\}$
Range $= \{y : y \leqslant -2 \text{ or } y \geqslant 2\}$

**i**



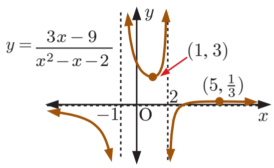

Domain $= \{x : x \neq 2\}$
Range $= \{y : y \neq 1\}$

**j**



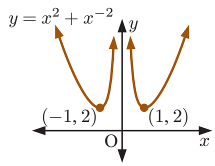

Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \in \mathbb{R}\}$

**k**



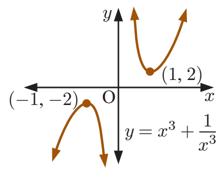

Domain $= \{x : x \neq -1 \text{ and } x \neq 2\}$
Range $= \{y : y \leqslant \frac{1}{3} \text{ or } y \geqslant 3\}$

**l**



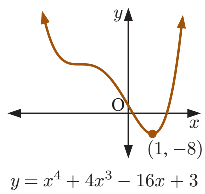

Domain $= \{x : x \neq 0\}$
Range $= \{y : y \geqslant 2\}$

**m**




Domain $= \{x : x \neq 0\}$
Range $= \{y : y \leqslant -2 \text{ or } y \geqslant 2\}$

**n**




Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \geqslant -8\}$

## EXERCISE 2D.1

**1** **a** 5 **b** 5 **c** 4 **d** 4 **e** 6 **f** 0 **g** $\frac{2}{7}$ **h** $\frac{8}{27}$

**2** **a** 1 **b** 6 **c** 4 **d** 3

**3** **a** 2 **b** $-4$ **c** $-6$ **d** $-5$

## EXERCISE 2D.2

**1** **a** $x = \pm 3$ **b** no solution **c** $x = 0$ **d** $x = 4$ or $-2$ **e** $x = -1$ or $7$ **f** no solution

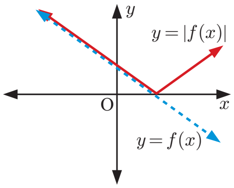
**g** $x = 1$ or $\frac{1}{3}$ **h** $x = 0$ or $3$ **i** $x = -2$ or $\frac{14}{5}$

**2** **a** $x = -\frac{1}{4}$ or $\frac{3}{2}$ **b** $x = -6$ or $-\frac{4}{3}$ **c** $x = \frac{1}{2}$

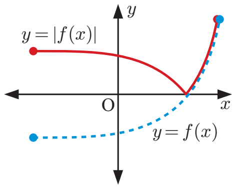
**d** $x = \frac{5}{2}$ **e** $x = 0$ or $\frac{2}{5}$ **f** $x = -2$ or $0$

## EXERCISE 2D.3

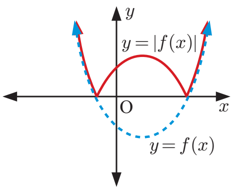
**1** **a**



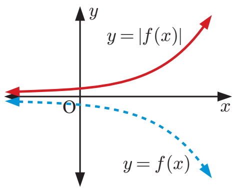

**b**



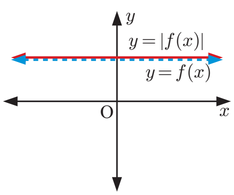

**c**



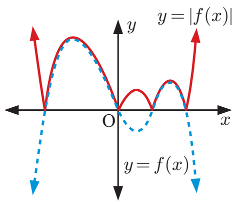

**d**




**e**



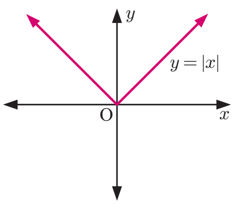

**f**



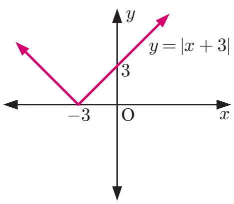

**2** function **d** **3** $\{y : 0 \leqslant y \leqslant 6\}$

**4** **a** false **b** true **c** true **d** false

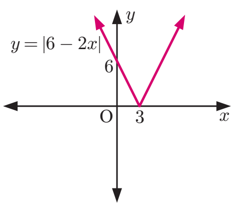
**5** **a**



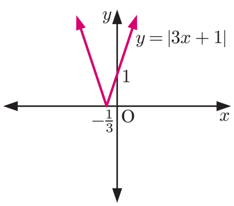

**b**



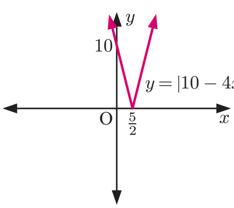

**c**



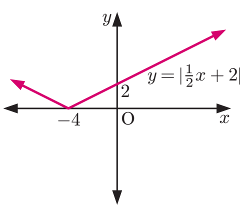

**d**




**e**




**f**




## EXERCISE 2E

**1** **a** $5 - 2x$ **b** $-2x - 2$ **c** $11$

**2** **a** $5 - x$ **b** $1 - x$ **c** $4 + x$

**3** **a** $25x - 42$ **b** $\sqrt{8}$ **c** $-7$

**4** $f(g(x)) = (2 - x)^2$,
Domain $= \{x : x \in \mathbb{R}\}$,
Range $= \{y : y \geqslant 0\}$

$g(f(x)) = 2 - x^2$,
Domain $= \{x : x \in \mathbb{R}\}$,
Range $= \{y : y \leqslant 2\}$

**5** **a** $(f \circ g)(x) = 6x - 4$ **b** $x = -\frac{3}{4}$

**6** **a** **i** $x^2 - 6x + 10$ **ii** $2 - x^2$ **b** $x = \pm\frac{1}{\sqrt{2}}$

**7** **a** Let $x = 0$, $\therefore\ b = d$ and so
$ax + b = cx + b$
$\therefore\ ax = cx$ for all $x$
Let $x = 1$, $\therefore\ a = c$

**b** $(f \circ g)(x) = [2a]x + [2b + 3] = 1x + 0$ for all $x$
$\therefore\ 2a = 1$ and $2b + 3 = 0$

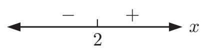
**c** Yes, $\{(g \circ f)(x) = [2a]x + [3a + b]\}$

**8** **a** $(f \circ g)(x) = \sqrt{1 - x^2}$

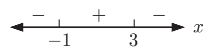
**b** Domain $= \{x : -1 \leqslant x \leqslant 1\}$, Range $= \{y : 0 \leqslant y \leqslant 1\}$

## EXERCISE 2F

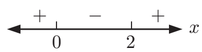
**1** **a**



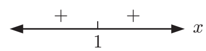

**b**



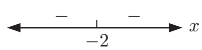

**c**



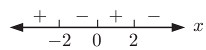

**d**



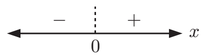

**e**



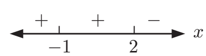

**f**



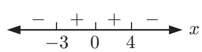

**g**




**h**

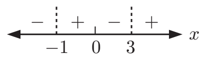



**i**

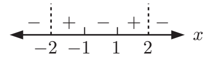



**j**

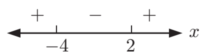

**k**



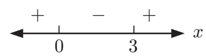

**l**



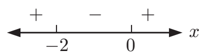

**2** **a**



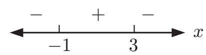

**b**



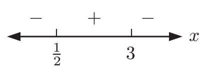

**c**



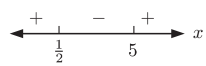

**d**



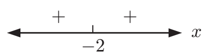

**e**



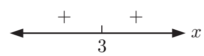

**f**



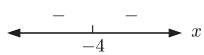

**g**



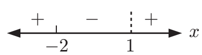

**h**



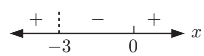

**i**



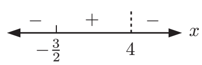

**3** **a**



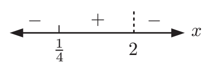

**b**

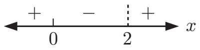


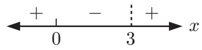

**c**

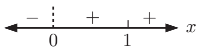


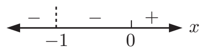

**d**

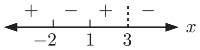


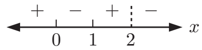
**e** 

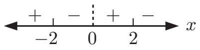

**f** 

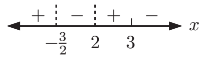

**g** 

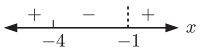

**h** 

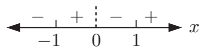

**i** 

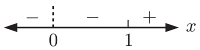

**j** 


**k** 

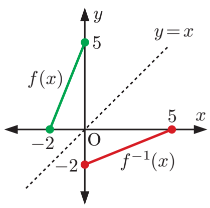

**l** 


**4** **a** 


**b** 


**c** 


## EXERCISE 2G

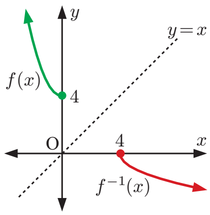
**1** **a** 


Domain of $f(x)$ is $\{x : -2 \leqslant x \leqslant 0\}$

Range of $f(x)$ is $\{y : 0 \leqslant y \leqslant 5\}$

Domain of $f^{-1}(x)$ is $\{x : 0 \leqslant x \leqslant 5\}$

Range of $f^{-1}(x)$ is $\{y : -2 \leqslant y \leqslant 0\}$

**b** 

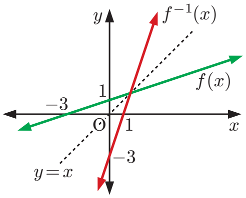

Domain of $f(x)$ is $\{x : x \leqslant 0\}$

Range of $f(x)$ is $\{y : y \geqslant 4\}$

Domain of $f^{-1}(x)$ is $\{x : x \geqslant 4\}$

Range of $f^{-1}(x)$ is $\{y : y \leqslant 0\}$

**c** The function does not have an inverse, as it is not one-one.

**d** 

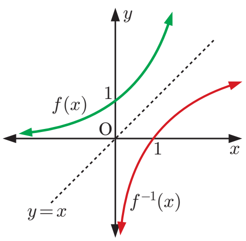

Domain of $f(x)$ is $\{x : x \in \mathbb{R}\}$

Range of $f(x)$ is $\{y : y \in \mathbb{R}\}$

Domain of $f^{-1}(x)$ is $\{x : x \in \mathbb{R}\}$

Range of $f^{-1}(x)$ is $\{y : y \in \mathbb{R}\}$

**e** The function does not have an inverse, as it is not one-one.

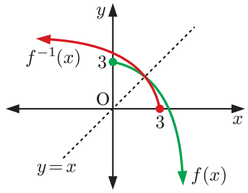
**f** 


Domain of $f(x)$ is $\{x : x \in \mathbb{R}\}$

Range of $f(x)$ is $\{y : y > 0\}$

Domain of $f^{-1}(x)$ is $\{x : x > 0\}$

Range of $f^{-1}(x)$ is $\{y : y \in \mathbb{R}\}$

**g** 

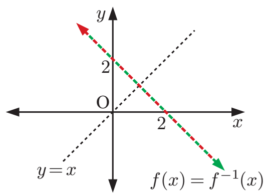

Domain of $f(x)$ is $\{x : x \geqslant 0\}$

Range of $f(x)$ is $\{y : y \leqslant 3\}$

Domain of $f^{-1}(x)$ is $\{x : x \leqslant 3\}$

Range of $f^{-1}(x)$ is $\{y : y \geqslant 0\}$

**h** The function does not have an inverse, as it is not one-one.

**i** 

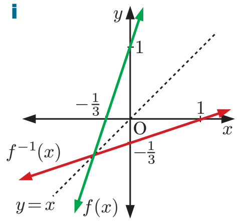

Domain of $f(x)$ is $\{x : x \in \mathbb{R}\}$

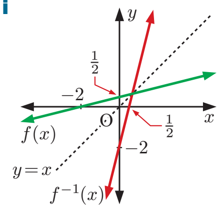
Range of $f(x)$ is $\{y : y \in \mathbb{R}\}$

Domain of $f^{-1}(x)$ is $\{x : x \in \mathbb{R}\}$

Range of $f^{-1}(x)$ is $\{y : y \in \mathbb{R}\}$

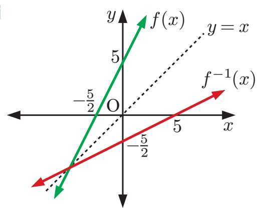
**2** function **i** **3** Range $= \{y : -2 \leqslant y < 3\}$

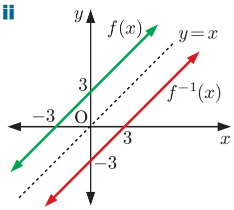
**4** **a** **i** 


**b** **i** 

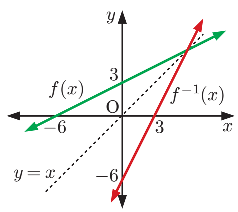

**ii** $f^{-1}(x) = \dfrac{x-1}{3}$ **ii** $f^{-1}(x) = 4x - 2$

**5** **a** **i** $f^{-1}(x) = \dfrac{x-5}{2}$ **b** **i** $f^{-1}(x) = x - 3$

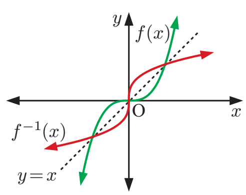
**ii** 


**ii** 


**c** **i** $f^{-1}(x) = 2x - 6$

**ii** 


**6** $f(x)$ is the same as $(f^{-1})^{-1}(x)$

**7** 

**8** $f^{-1}(x)=\dfrac{1}{x}$ and $f(x)=\dfrac{1}{x}$ $\therefore$ $f=f^{-1}$

$\therefore$ $f$ is a self-inverse function

**9** **a** $f^{-1}(x)=2x+2$

**b** **i** $(f\circ f^{-1})(x)=x$ **ii** $(f^{-1}\circ f)(x)=x$

**10** **a** 10

**b** $f^{-1}(x)=\dfrac{x-5}{2}$ and $f^{-1}(-3)=-4$

$g^{-1}(x)=8-2x$ and $g^{-1}(6)=-4$

$\therefore$ $f^{-1}(-3)-g^{-1}(6)=0$

**c** $x=3$

**11** **a** **i** 25 **ii** 16 **b** $x=1$

**12** **a** Is not **b** Is **c** Is **d** Is **e** Is **f** Is not

**13** $(f^{-1}\circ g^{-1})(x)=\dfrac{x+3}{8}$ and $(g\circ f)^{-1}(x)=\dfrac{x+3}{8}$

## REVIEW SET 2A

**1** **a** function **b** function **c** not a function **d** function

**2** $a=-6,\ b=13$

**3** **a** $x=-2$ or $12$ **b** $x=-5$ or $1$

**4** **a** 10 **b** $x^2-x-2$

**5** **a** **i** Domain is $\{x: x\geqslant -3\}$, Range is $\{y: y\geqslant 2\}$

**ii** function is one-one

**b** **i** Domain is $\{x: x\in\mathbb{R}\}$, Range is $\{y: y\geqslant -5\}$

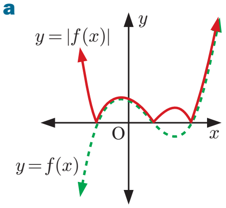
**ii** function is not one-one

**c** **i** Domain is $\{x: x\in\mathbb{R}\}$,

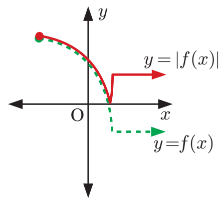
Range is $\{y: y=-3 \text{ or } y=1\}$

**ii** function is not one-one

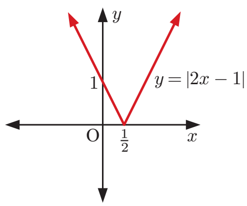
**6** **a**



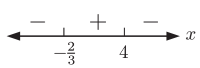

**b**



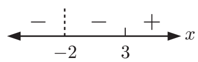

**7**




**8** **a**

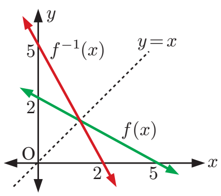



**b**

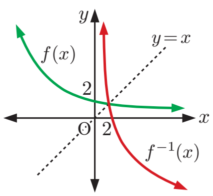



**9** **a** $2x^2+1$ **b** $4x^2-12x+11$

**10** **a**




**b**




**c** The function does not have an inverse.

**11** **a** $f^{-1}(x)=\dfrac{x-2}{4}$ **b** $f^{-1}(x)=\dfrac{3-4x}{5}$

**12** **a** $f(-3)=(-3)^2=9$ **b** 169 **c** $x=-4$

$g(-\frac{4}{3})=1-6(-\frac{4}{3})=9$

**13** $(f^{-1}\circ h^{-1})(x)=(h\circ f)^{-1}(x)=x-2$

## REVIEW SET 2B

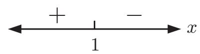
**1** **a** not a function **b** function, one-one

**c** function, not one-one

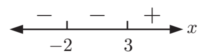
**2** **a** 12 **b** $x=\pm 1$

**3** **a** $x=-5$ or $6$ **b** $x=1$ or $3$

**4** **a**



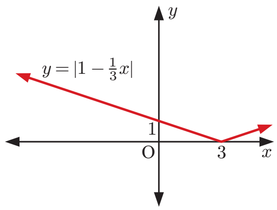

**b**




**5** **a** $10-6x$ **b** $9x-14$ **c** $-23$

**6** $\{y: 3\leqslant y\leqslant 7\}$

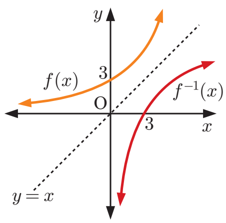
**7**



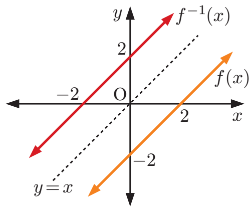

**8** **a** **i** $1-10x$ **ii** $5-10x$ **b** $x=-\frac{3}{5}$

**9** $a=1,\ b=-6,\ c=5$

**10** **a**




**c**




**b** The function does not have an inverse.

**11** **a** $f^{-1}(x)=\dfrac{7-x}{4}$ **b** $f^{-1}(x)=\dfrac{5x-3}{2}$

**12** $(f^{-1}\circ h^{-1})(x)=(h\circ f)^{-1}(x)=\dfrac{4x+6}{15}$ **13** 16

## EXERCISE 3A.1

**1** **a** $x=0,\ -\frac{7}{4}$ **b** $x=0,\ -\frac{1}{3}$ **c** $x=0,\ \frac{7}{3}$

**d** $x=0,\ \frac{11}{2}$ **e** $x=0,\ \frac{8}{3}$ **f** $x=0,\ \frac{3}{2}$

**g** $x=3,\ 2$ **h** $x=4,\ -2$ **i** $x=3,\ 7$

**j** $x=3$ **k** $x=-4,\ 3$ **l** $x=-11,\ 3$

**2** **a** $x=\frac{2}{3}$ **b** $x=-\frac{1}{2},\ 7$ **c** $x=-\frac{2}{3},\ 6$

**d** $x=\frac{1}{3},\ -2$ **e** $x=\frac{3}{2},\ 1$ **f** $x=-\frac{2}{3},\ -2$

**g** $x=-\frac{2}{3},\ 4$ **h** $x=\frac{1}{2},\ -\frac{3}{2}$ **i** $x=-\frac{1}{4},\ 3$

**j** $x=-\frac{3}{4},\ \frac{5}{3}$ **k** $x=\frac{1}{7},\ -1$ **l** $x=-2,\ \frac{28}{15}$

**3** **a** $x=2,\ 5$ **b** $x=-3,\ 2$ **c** $x=0,\ -\frac{3}{2}$

**d** $x=1,\ 2$ **e** $x=\frac{1}{2},\ -1$ **f** $x=3$

## EXERCISE 3A.2

**1** **a** $x = -5 \pm \sqrt{2}$ **b** no real solutions **c** $x = 4 \pm 2\sqrt{2}$
**d** $x = 8 \pm \sqrt{7}$ **e** $x = -3 \pm \sqrt{5}$ **f** $x = 2 \pm \sqrt{6}$
**g** $x = -1 \pm \sqrt{10}$ **h** $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$ **i** $x = \frac{1}{3} \pm \frac{\sqrt{7}}{3}$

**2** **a** $x = 2 \pm \sqrt{3}$ **b** $x = -3 \pm \sqrt{7}$ **c** $x = 7 \pm \sqrt{3}$
**d** $x = 2 \pm \sqrt{7}$ **e** $x = -3 \pm \sqrt{2}$ **f** $x = 1 \pm \sqrt{7}$
**g** $x = -3 \pm \sqrt{11}$ **h** $x = 4 \pm \sqrt{6}$ **i** no real solns.

**3** **a** $x = -1 \pm \frac{1}{\sqrt{2}}$ **b** $x = \frac{5}{2} \pm \frac{\sqrt{19}}{2}$ **c** $x = -2 \pm \sqrt{\frac{7}{3}}$
**d** $x = 1 \pm \sqrt{\frac{7}{3}}$ **e** $x = \frac{3}{2} \pm \sqrt{\frac{37}{20}}$ **f** $x = -\frac{1}{2} \pm \frac{\sqrt{6}}{2}$

**4** **a** $x = \frac{2}{3} \pm \frac{\sqrt{10}}{3}$ **b** $x = -\frac{1}{10} \pm \frac{\sqrt{21}}{10}$ **c** $x = -\frac{5}{6} \pm \frac{\sqrt{13}}{6}$

**5** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

## EXERCISE 3A.3

**1** **a** $x = 2 \pm \sqrt{7}$ **b** $x = -3 \pm \sqrt{2}$ **c** $x = 2 \pm \sqrt{3}$
**d** $x = -2 \pm \sqrt{5}$ **e** $x = 2 \pm \sqrt{2}$ **f** $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}$
**g** $x = \frac{5}{6} \pm \frac{\sqrt{37}}{6}$ **h** $x = 2 \pm \sqrt{10}$ **i** $x = \frac{7}{4} \pm \frac{\sqrt{33}}{4}$

**2** **a** $x = -2 \pm 2\sqrt{2}$ **b** $x = -\frac{5}{8} \pm \frac{\sqrt{57}}{8}$ **c** $x = \frac{5}{2} \pm \frac{\sqrt{13}}{2}$
**d** $x = -\frac{4}{9} \pm \frac{\sqrt{7}}{9}$ **e** $x = -\frac{7}{4} \pm \frac{\sqrt{97}}{4}$ **f** $x = \frac{1}{8} \pm \frac{\sqrt{145}}{8}$
**g** $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}$ **h** $x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$ **i** $x = \frac{3}{4} \pm \frac{\sqrt{17}}{4}$

## EXERCISE 3B

**1** **a** $x \leqslant -3$ or $x \geqslant 2$ **b** $-1 < x < 4$
**c** $x < -\frac{1}{2}$ or $x > 3$ **d** $x \leqslant 0$ or $x \geqslant 1$
**e** $x \leqslant 0$ or $x \geqslant 3$ **f** $-\frac{2}{3} < x < 0$
**g** $-2 < x < 2$ **h** $x \leqslant -3$ or $x \geqslant 3$
**i** $x \neq -2$ **j** $x < -5$ or $x > 3$
**k** $4 \leqslant x \leqslant 7$ **l** $-6 < x < -4$
**m** $x \leqslant -2$ or $x \geqslant 15$ **n** $x \leqslant -1$ or $x \geqslant \frac{3}{2}$
**o** no solutions **p** $-\frac{3}{2} < x < \frac{1}{3}$
**q** $x < -\frac{4}{3}$ or $x > 4$ **r** no solutions **s** $\frac{1}{3} \leqslant x \leqslant \frac{1}{2}$
**t** $x \in \mathbb{R}$ **u** $-\frac{11}{8} - \frac{\sqrt{73}}{8} < x < -\frac{11}{8} + \frac{\sqrt{73}}{8}$

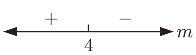
**2** **a** $\square = <$ **b** $\square = \leqslant$ **c** $\square = \geqslant$ or $>$

## EXERCISE 3C

**1** **a** 2 distinct irrational roots **b** 2 distinct rational roots
**c** 2 distinct rational roots **d** 2 distinct irrational roots
**e** no real roots **f** a repeated root

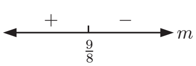
**2** **a**, **c**, **d**, **f**

**3** **a** $\Delta = 16 - 4m$



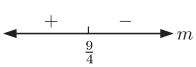

**i** $m = 4$ **ii** $m < 4$ **iii** $m > 4$

**b** $\Delta = 9 - 8m$



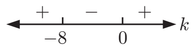

**i** $m = \frac{9}{8}$ **ii** $m < \frac{9}{8}$, $m \neq 0$ **iii** $m > \frac{9}{8}$

**c** $\Delta = 9 - 4m$



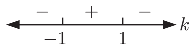

**i** $m = \frac{9}{4}$ **ii** $m < \frac{9}{4}$, $m \neq 0$ **iii** $m > \frac{9}{4}$

**4** **a** $\Delta = k^2 + 8k$



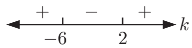

**i** $k < -8$ or $k > 0$ **ii** $k \leqslant -8$ or $k \geqslant 0$
**iii** $k = -8$ or $0$ **iv** $-8 < k < 0$

**b** $\Delta = 4 - 4k^2$



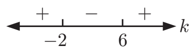

**i** $-1 < k < 1$, $k \neq 0$ **ii** $-1 \leqslant k \leqslant 1$, $k \neq 0$
**iii** $k = \pm 1$ **iv** $k < -1$ or $k > 1$

**c** $\Delta = k^2 + 4k - 12$



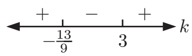

**i** $k < -6$ or $k > 2$ **ii** $k \leqslant -6$ or $k \geqslant 2$
**iii** $k = -6$ or $2$ **iv** $-6 < k < 2$

**d** $\Delta = k^2 - 4k - 12$



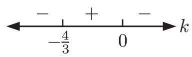

**i** $k < -2$ or $k > 6$ **ii** $k \leqslant -2$ or $k \geqslant 6$
**iii** $k = 6$ or $-2$ **iv** $-2 < k < 6$

**e** $\Delta = 9k^2 - 14k - 39$




**i** $k < -\frac{13}{9}$ or $k > 3$ **ii** $k \leqslant -\frac{13}{9}$ or $k \geqslant 3$
**iii** $k = -\frac{13}{9}$ or $3$ **iv** $-\frac{13}{9} < k < 3$

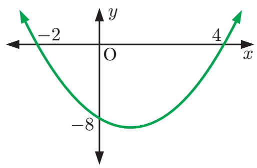
**f** $\Delta = -3k^2 - 4k$



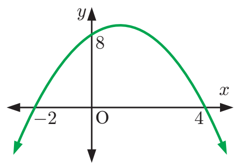

**i** $-\frac{4}{3} < k < 0$, $k \neq -1$ **ii** $-\frac{4}{3} \leqslant k \leqslant 0$, $k \neq -1$
**iii** $k = -\frac{4}{3}$ or $0$ **iv** $k < -\frac{4}{3}$ or $k > 0$

## EXERCISE 3D.1

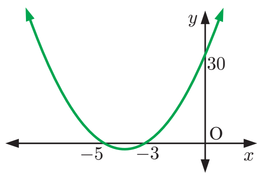
**1** **a** $y = (x-4)(x+2)$



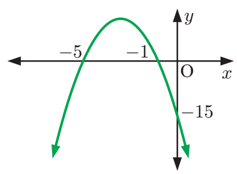

**b** $f(x) = -(x-4)(x+2)$



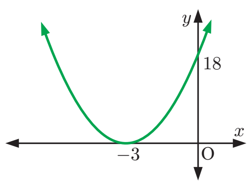

**c** $y = 2(x+3)(x+5)$



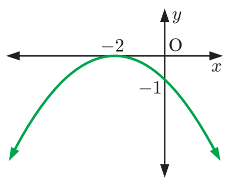

**d** $f(x) = -3(x+1)(x+5)$




**e** $f(x) = 2(x+3)^2$



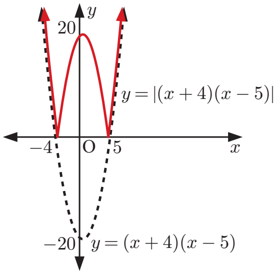

**f** $y = -\frac{1}{4}(x+2)^2$



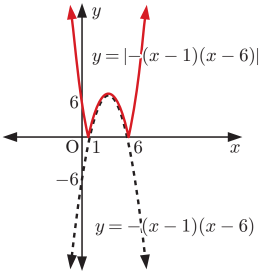

**2** **a** $x = 1$ **b** $x = 1$ **c** $x = -4$
**d** $x = -3$ **e** $x = -3$ **f** $x = -2$

**3** **a** C **b** E **c** B **d** F **e** G **f** H **g** A **h** D

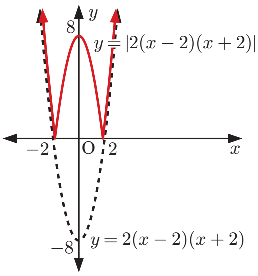
**4** **a**



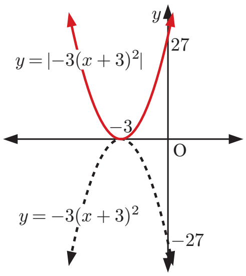

**b**



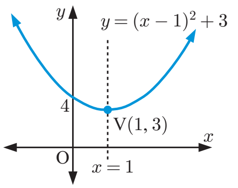

**c**



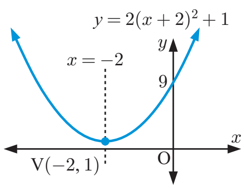

**d**



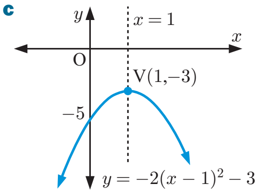

**5** **a**



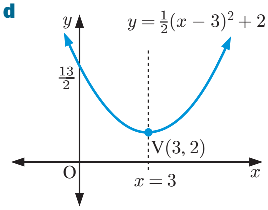

**b**



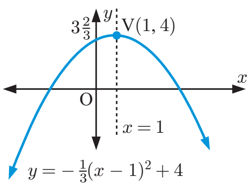

**c**



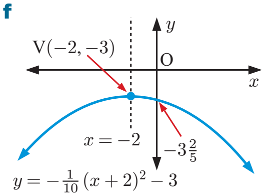

**d**




**e**



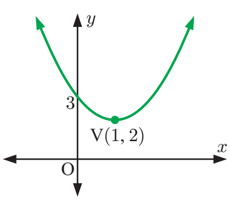

**f**



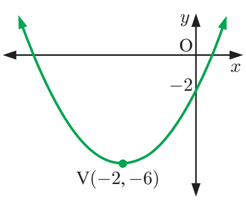

**6** **a** G **b** A **c** E **d** B **e** I
**f** C **g** D **h** F **i** H

## EXERCISE 3D.2

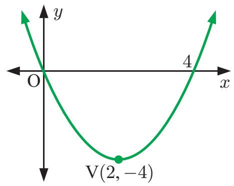
**1** **a** $y = (x-1)^2 + 2$



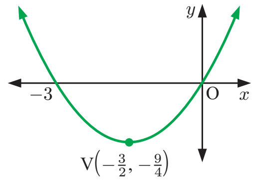

**b** $y = (x+2)^2 - 6$



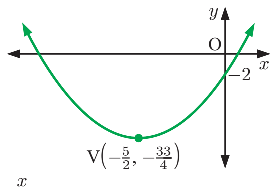

**c** $y = (x-2)^2 - 4$



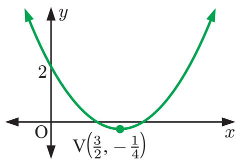

**d** $y = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$



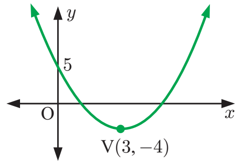

**e** $y = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$



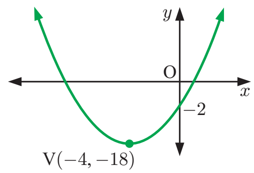

**f** $y = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$



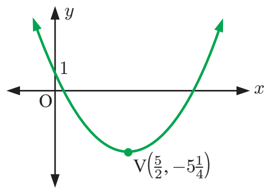

**g** $y = (x-3)^2 - 4$




**h** $y = (x+4)^2 - 18$

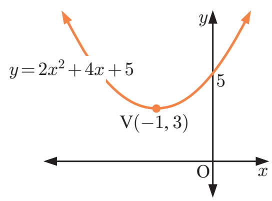



**i** $y = \left(x - \frac{5}{2}\right)^2 - 5\frac{1}{4}$



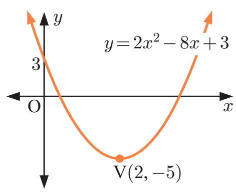

**2** **a** **i** $y = 2(x+1)^2 + 3$ **ii** $(-1, 3)$ **iii** 5

**iv**



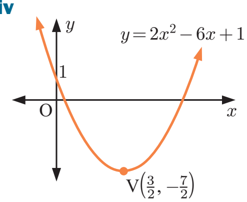

**b** **i** $y = 2(x-2)^2 - 5$ **ii** $(2, -5)$ **iii** 3

**iv**



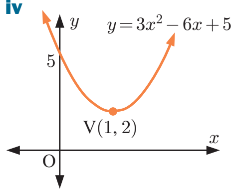

**c** **i** $y = 2(x - \frac{3}{2})^2 - \frac{7}{2}$ **ii** $(\frac{3}{2}, -\frac{7}{2})$ **iii** 1

**iv**

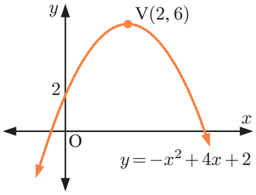



**d** **i** $y = 3(x-1)^2 + 2$ **ii** $(1, 2)$ **iii** 5

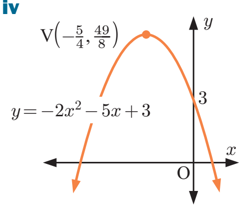
**iv**



**e** **i** $y = -(x-2)^2 + 6$
**ii** $(2, 6)$ **iii** $2$
**iv**



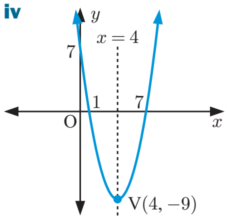

**f** **i** $y = -2(x + \frac{5}{4})^2 + \frac{49}{8}$
**ii** $(-\frac{5}{4}, \frac{49}{8})$ **iii** $3$
**iv**




## EXERCISE 3D.3

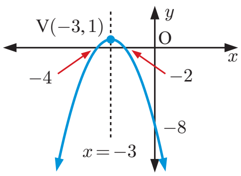
**1** **a** $(2, -2)$ **b** $(-1, -4)$ **c** $(0, 4)$
**d** $(0, 1)$ **e** $(-2, -15)$ **f** $(-2, -5)$
**g** $(-\frac{3}{2}, -\frac{11}{2})$ **h** $(\frac{5}{2}, -\frac{19}{2})$ **i** $(1, -\frac{9}{2})$

**2** **a** **i** $x = 4$
**ii** $(4, -9)$
**iii** $x$-intercepts 1, 7, $y$-intercept 7
**iv**



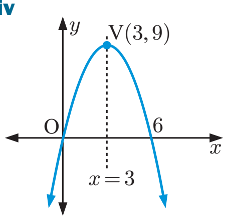

**v** $\{y : y \geqslant -9\}$

**b** **i** $x = -3$
**ii** $(-3, 1)$
**iii** $x$-int. $-2, -4$, $y$-intercept $-8$
**iv**



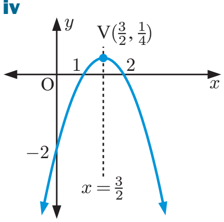

**v** $\{y : y \leqslant 1\}$

**c** **i** $x = 3$
**ii** $(3, 9)$
**iii** $x$-intercepts 0, 6, $y$-intercept 0
**iv**



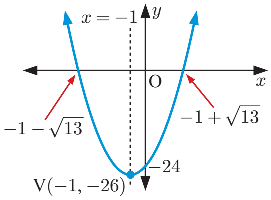

**v** $\{y : y \leqslant 9\}$

**d** **i** $x = \frac{3}{2}$
**ii** $(\frac{3}{2}, \frac{1}{4})$
**iii** $x$-intercepts 1, 2, $y$-intercept $-2$
**iv**



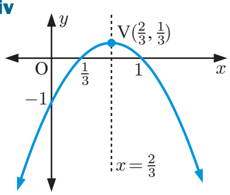

**v** $\{y : y \leqslant \frac{1}{4}\}$

**e** **i** $x = -1$
**ii** $(-1, -26)$
**iii** $x$-int. $-1 \pm \sqrt{13}$, $y$-intercept $-24$
**iv**



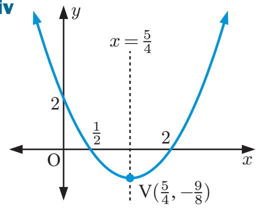

**v** $\{y : y \geqslant -26\}$

**f** **i** $x = \frac{2}{3}$
**ii** $(\frac{2}{3}, \frac{1}{3})$
**iii** $x$-intercepts $\frac{1}{3}$, 1, $y$-intercept $-1$
**iv**



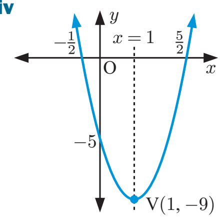

**v** $\{y : y \leqslant \frac{1}{3}\}$

**g** **i** $x = \frac{5}{4}$
**ii** $(\frac{5}{4}, -\frac{9}{8})$
**iii** $x$-intercepts $\frac{1}{2}$, 2, $y$-intercept 2
**iv**



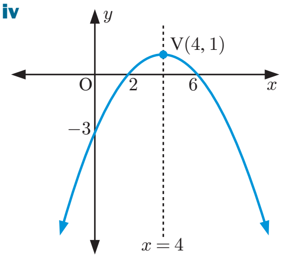

**v** $\{y : y \geqslant -\frac{9}{8}\}$

**h** **i** $x = 1$
**ii** $(1, -9)$
**iii** $x$-intercepts $-\frac{1}{2}$, $\frac{5}{2}$, $y$-intercept $-5$
**iv**



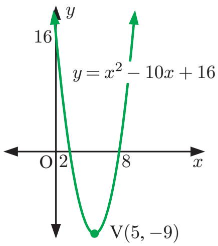

**v** $\{y : y \geqslant -9\}$

**i** **i** $x = 4$
**ii** $(4, 1)$
**iii** $x$-intercepts 2, 6, $y$-intercept $-3$
**iv**

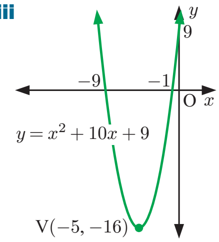



**v** $\{y : y \leqslant 1\}$

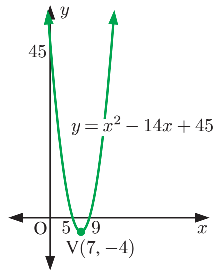
**3** **a** **i** $y = (x-2)(x-8)$, roots are 2 and 8
**ii** $y = (x-5)^2 - 9$, vertex is $(5, -9)$
**iii**



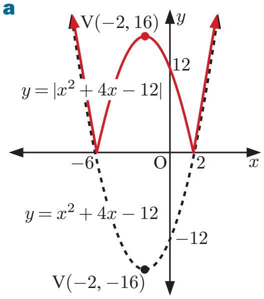

**b** **i** $y = (x+1)(x+9)$, roots are $-1$ and $-9$
**ii** $y = (x+5)^2 - 16$, vertex is $(-5, -16)$
**iii**



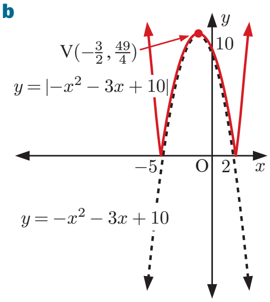

**c** **i** $y = (x-5)(x-9)$, roots are 5 and 9
**ii** $y = (x-7)^2 - 4$, vertex is $(7, -4)$
**iii**



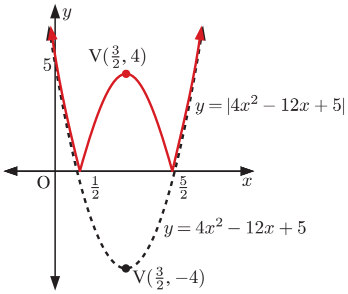

**4** **a**




**b**



**c**



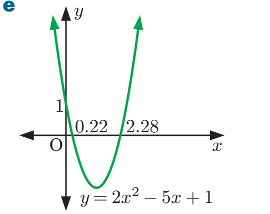

**5** **a** $\{y : -10 \leqslant y \leqslant 15\}$ **b** $\{y : 3 \leqslant y \leqslant 19\}$
**c** $\{y : -13 \leqslant y \leqslant 37\}$ **d** $\{y : -8 \leqslant y \leqslant \frac{49}{4}\}$

## EXERCISE 3D.4

**1** **a** cuts $x$-axis twice, concave up
**b** cuts $x$-axis twice, concave up
**c** lies entirely below the $x$-axis, concave down, negative definite
**d** cuts $x$-axis twice, concave up
**e** touches $x$-axis, concave up
**f** cuts $x$-axis twice, concave down
**g** cuts $x$-axis twice, concave up
**h** cuts $x$-axis twice, concave down
**i** touches $x$-axis, concave up

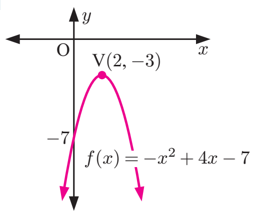
**2** **a** concave up
**b** $\Delta = 17$ which is $> 0$
**c** $x$-intercepts $\approx 0.22$ and $2.28$
**d** $y$-intercept $= 1$

**e**




**3** **a** $\Delta = -12$ which is $< 0$
**b** negative definite
**c** vertex is $(2, -3)$, $y$-intercept $= -7$

**d**




**4** **a** $a = 1$ which is $> 0$ and $\Delta = -15$ which is $< 0$ so is entirely above the $x$-axis.
**b** $a = -1$ which is $< 0$ and $\Delta = -8$ which is $< 0$ so is entirely below the $x$-axis.
**c** $a = 2$ which is $> 0$ and $\Delta = -40$ which is $< 0$ so is entirely above the $x$-axis.
**d** $a = -2$ which is $< 0$ and $\Delta = -23$ which is $< 0$ so is entirely below the $x$-axis.

**5** $a = 3$ which is $> 0$ and $\Delta = k^2 + 12$ which is always $> 0$ {as $k^2 \geqslant 0$ for all $k$} $\therefore$ always cuts $x$-axis twice.

**6** $-4 < k < 4$

## EXERCISE 3E

**1** **a** $y = 2(x-1)(x-2)$ **b** $y = 2(x-2)^2$
**c** $y = (x-1)(x-3)$ **d** $y = -(x-3)(x+1)$
**e** $y = -3(x-1)^2$ **f** $y = -2(x+2)(x-3)$

**2** **a** $y = \frac{3}{2}(x-2)(x-4)$ **b** $y = -\frac{1}{2}(x+4)(x-2)$
**c** $y = -\frac{4}{3}(x+3)^2$

**3** **a** $y = 3x^2 - 18x + 15$ **b** $y = -4x^2 + 6x + 4$
**c** $y = -x^2 + 6x - 9$ **d** $y = 4x^2 + 16x + 16$

**4** **a** $y = \frac{3}{2}x^2 - 6x + \frac{9}{2}$ **b** $y = -\frac{1}{3}x^2 + \frac{2}{3}x + 5$

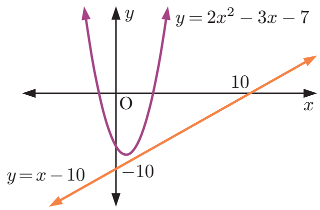
**5** **a** $y = -(x-2)^2 + 4$ **b** $y = 2(x-2)^2 - 1$
**c** $y = -2(x-3)^2 + 8$ **d** $y = \frac{2}{3}(x-4)^2 - 6$
**e** $y = -2(x-2)^2 + 3$ **f** $y = 2(x-\frac{1}{2})^2 - \frac{3}{2}$

## EXERCISE 3F

**1** **a** (1, 7) and (2, 8) **b** (4, 5) and (−3, −9)
**c** (3, 0) (touching) **d** graphs do not meet

**2** $c = -9$ **3** $m = 0$ or $-8$ **4** −1 or 11

**5** **a** $c < -9$
**b** example: $c = -10$




**6** **a** $c > -2$ **b** $c = -2$ **c** $c < -2$

**7** **a** $m < -1$ or $m > 7$ **b** $m = -1$ or $m = 7$
**c** $-1 < m < 7$

**8** **Hint:** A straight line through (0, 3) will have an equation of the form $y = mx + 3$.

## EXERCISE 3G

**1** 7 and −5 or −7 and 5 **2** 5 or $\frac{1}{5}$ **3** 14
**4** 18 and 20 or −18 and −20 **5** 15 sides
**6** 3.48 cm **7** **b** 6 cm by 6 cm by 7 cm
**8** 11.2 cm square **9** no

**11** **a** $y = -\frac{8}{9}x^2 + 8$
**b** No, as the tunnel is only 4.44 m high when it is the same width as the truck.

**12** **b** The graph is a parabola. **c** 21.25 m
**d** $f(x) = -0.05x^2 + 2x + 1.25$ **e** yes

## EXERCISE 3H

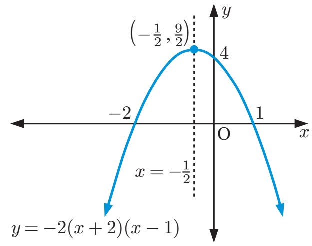
**1** **a** min. −1, when $x = 1$ **b** max. 8, when $x = -1$
**c** max. $8\frac{1}{3}$, when $x = \frac{1}{3}$ **d** min. $-1\frac{1}{8}$, when $x = -\frac{1}{4}$
**e** min. $4\frac{15}{16}$, when $x = \frac{1}{8}$ **f** max. $6\frac{1}{8}$, when $x = \frac{7}{4}$

**2** **a** 40 refrigerators **b** $4000
**4** 500 m by 250 m **5** **c** 100 m by 112.5 m
**6** **a** $41\frac{2}{3}$ m by $41\frac{2}{3}$ m **b** 50 m by $31\frac{1}{4}$ m
**7** **b** $3\frac{1}{8}$ units **8** **a** $y = 6 - \frac{3}{4}x$ **b** 3 cm by 4 cm

## REVIEW SET 3A

**1** **a** −2, 1
**b** $x = -\frac{1}{2}$
**c** 4
**d** $(-\frac{1}{2}, \frac{9}{2})$
**f** $\{y : y \leqslant \frac{9}{2}\}$

**e**

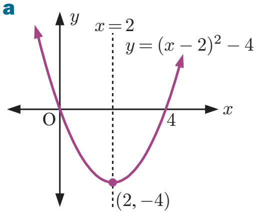


**2** **a** $x = 0$ or $4$ **b** $x = -\frac{5}{3}$ or $2$ **c** $x = 15$ or $-4$

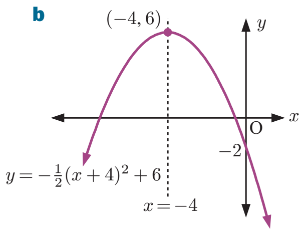
**3** **a** $x = -\frac{5}{2} \pm \frac{\sqrt{13}}{2}$ **b** $x = -\frac{11}{6} \pm \frac{\sqrt{145}}{6}$

**4** **a** $-3 < x < 7$ **b** $x \leqslant -\frac{1}{3}$ or $x \geqslant 2$

**5** **a**




**b**




**6** **a** $y = 3x^2 - 24x + 48$ **b** $y = \frac{2}{5}x^2 + \frac{16}{5}x + \frac{37}{5}$

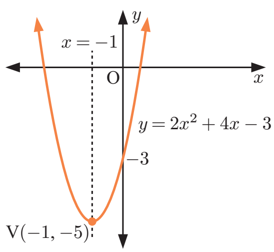
**7** $a = -2$ which is $< 0$ $\therefore$  a max.
max. $= 5$ when $x = 1$

**8** $(4, 4)$ and $(-3, 18)$ **9** $k < -3\frac{1}{8}$

**10** **a** $m = \frac{9}{8}$ **b** $m < \frac{9}{8}$ **c** $m > \frac{9}{8}$ **11** $\frac{6}{5}$ or $\frac{5}{6}$

**12** **Hint:** Let the line have equation $y = mx + 10$.

**13** **a** $y = 2(x+1)^2 - 5$ **b**



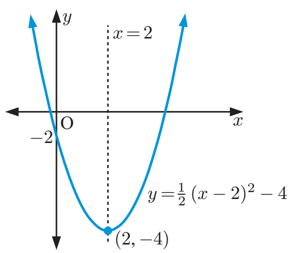

**14** **a** $y = \frac{20}{9}(x-2)^2 - 20$ **b** $y = -\frac{2}{7}(x-1)(x-7)$
**c** $y = \frac{2}{9}(x+3)^2$

**15** $\{y : -13 \leqslant y \leqslant 12\}$ **16** 21 m

## REVIEW SET 3B

**1** **a** $x = 2$
**b** $(2, -4)$
**c** $-2$
**e** $\{y : y \geqslant -4\}$

**d**

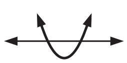


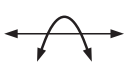

**2** **a** $x = \frac{5}{2} \pm \frac{\sqrt{37}}{2}$ **b** $x = \frac{7}{4} \pm \frac{\sqrt{73}}{4}$

**3** **a** $-7 \leqslant x \leqslant 2$ **b** $x < -4$ or $x > \frac{3}{2}$

**4** $x = \frac{4}{3}$, $V(\frac{4}{3}, 12\frac{1}{3})$

**5** **a** graph cuts $x$-axis twice **b** graph cuts $x$-axis twice



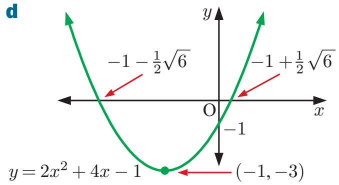


**6** **a** $a < 0$, $\Delta > 0$, neither
**b** $a > 0$, $\Delta < 0$, positive definite

**7** $y = -6(x-2)^2 + 25$ **8** $m < -5$ or $m > 19$

**9** **a** $x = -1$
**b** $(-1, -3)$
**c** $y$-intercept $-1$,
$x$-ints. $-1 \pm \frac{1}{2}\sqrt{6}$

**d**




**10** $\{y : -55 \leqslant y \leqslant \frac{11}{2}\}$

**11** **a** $k = -8$ **b** $k < -8$ or $k > 0$ **c** $-8 < k < 0$

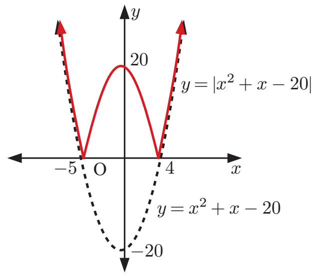
**12** **a** $c > -6$
**b** example: $c = -2$, $(-1, -5)$ and $(3, 7)$

**13** **a** $y = -\frac{2}{5}(x+5)(x-1)$ **b** $(-2, 3\frac{3}{5})$

**14** **a** min. $= 5\frac{2}{3}$ when $x = -\frac{2}{3}$
**b** max. $= 5\frac{1}{8}$ when $x = -\frac{5}{4}$

**15** **b** $37\frac{1}{2}$ m by $33\frac{1}{3}$ m **c** $1250\ m^2$

**16**




## EXERCISE 4A.1

**1** **a** 11 **b** $\sqrt{15}$ **c** 3 **d** $\sqrt{30}$
**e** 4 **f** 12 **g** 42 **h** 45
**i** $\sqrt{6}$ **j** $\sqrt{6}$ **k** 2 **l** $\sqrt{5}$

**2** **a** $2\sqrt{2}$ **b** $2\sqrt{3}$ **c** $2\sqrt{5}$ **d** $4\sqrt{2}$
**e** $3\sqrt{3}$ **f** $3\sqrt{5}$ **g** $4\sqrt{3}$ **h** $3\sqrt{6}$
**i** $5\sqrt{2}$ **j** $4\sqrt{5}$ **k** $4\sqrt{6}$ **l** $6\sqrt{3}$

## EXERCISE 4A.2

**1** **a** $5\sqrt{2}$ **b** $-\sqrt{2}$ **c** $2\sqrt{5}$ **d** $8\sqrt{5}$
**e** $-2\sqrt{5}$ **f** $9\sqrt{3}$ **g** $-3\sqrt{6}$ **h** $3\sqrt{2}$

**2** **a** $3\sqrt{2} - 2$ **b** $5 + \sqrt{5}$ **c** $3\sqrt{10} + 20$
**d** $21 - 4\sqrt{7}$ **e** $-5\sqrt{3} - 3$ **f** $12 - 14\sqrt{6}$
**g** $-8 + 5\sqrt{8}$ **h** $-12\sqrt{2} + 36$

**3** **a** $22 + 9\sqrt{2}$ **b** $34 + 15\sqrt{3}$ **c** $22 + 14\sqrt{7}$
**d** $-7 - \sqrt{3}$ **e** $34 - 15\sqrt{8}$ **f** $-47 + 30\sqrt{5}$

**4** **a** $11 + 6\sqrt{2}$ **b** $39 - 12\sqrt{3}$ **c** $6 + 2\sqrt{5}$
**d** $17 - 6\sqrt{8}$ **e** $28 + 16\sqrt{3}$ **f** $46 + 6\sqrt{5}$
**g** $89 - 28\sqrt{10}$ **h** $166 - 40\sqrt{6}$

**5** **a** 2 **b** $-23$ **c** 13 **d** 7
**e** $-56$ **f** 218

## EXERCISE 4A.3

**1** **a** $\frac{\sqrt{3}}{3}$ **b** $\sqrt{3}$ **c** $3\sqrt{3}$ **d** $\frac{11\sqrt{3}}{3}$ **e** $\frac{\sqrt{6}}{9}$
**f** $\sqrt{2}$ **g** $3\sqrt{2}$ **h** $6\sqrt{2}$ **i** $\frac{\sqrt{6}}{2}$ **j** $\frac{\sqrt{2}}{8}$

**2** **a** $\sqrt{5}$ **b** $3\sqrt{5}$ **c** $-\frac{3\sqrt{5}}{5}$ **d** $40\sqrt{5}$ **e** $\frac{\sqrt{5}}{15}$
**f** $\sqrt{7}$ **g** $3\sqrt{7}$ **h** $\frac{2\sqrt{11}}{11}$ **i** $2\sqrt{13}$ **j** $\frac{\sqrt{3}}{9}$

**3** **a** $\frac{3-\sqrt{2}}{7}$ **b** $\frac{6+2\sqrt{2}}{7}$ **c** $-2+\sqrt{5}$
**d** $1+\sqrt{2}$ **e** $2+2\sqrt{6}$ **f** $\frac{\sqrt{21}-2\sqrt{3}}{3}$
**g** $-3-2\sqrt{2}$ **h** $\frac{3+4\sqrt{3}}{13}$ **i** $4+2\sqrt{2}$
**j** $-7-3\sqrt{5}$ **k** $\frac{5+3\sqrt{3}}{2}$ **l** $\frac{-38+11\sqrt{10}}{6}$

**4** **a** $-\frac{9}{7}-\frac{3}{7}\sqrt{2}$ **b** $4-2\sqrt{2}$ **c** $-\frac{2}{23}-\frac{5}{23}\sqrt{2}$
**d** $-4+2\sqrt{2}$

**5** **a** $-2-2\sqrt{3}$ **b** $12-6\sqrt{3}$ **c** $3+2\sqrt{3}$ **d** $-\frac{1}{2}+\frac{5}{6}\sqrt{3}$

**6** **a** $(a+b\sqrt{c})(a-b\sqrt{c})=a^2-b^2c$
which is an integer as $a$, $b$, and $c$ are integers.
**b** **i** $\frac{-1+2\sqrt{3}}{11}$ **ii** $\frac{-6-5\sqrt{2}}{7}$ **iii** $1+\sqrt{2}$

**7** **a** $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$
which is an integer as $a$ and $b$ are integers.
**b** **i** $\sqrt{3}-\sqrt{2}$ **ii** $\frac{-3-\sqrt{15}}{2}$ **iii** $\frac{2\sqrt{154}-25}{3}$

**8** $x=-7+5\sqrt{3}$ **9** $x=\frac{10}{19}+\frac{1}{19}\sqrt{5}$

## EXERCISE 4B

**1** **a** $2^1=2,\ 2^2=4,\ 2^3=8,\ 2^4=16,\ 2^5=32,\ 2^6=64$
**b** $3^1=3,\ 3^2=9,\ 3^3=27,\ 3^4=81,\ 3^5=243,$ $3^6=729$
**c** $4^1=4,\ 4^2=16,\ 4^3=64,\ 4^4=256,\ 4^5=1024,$ $4^6=4096$

**2** **a** $5^1=5,\ 5^2=25,\ 5^3=125,\ 5^4=625$
**b** $6^1=6,\ 6^2=36,\ 6^3=216,\ 6^4=1296$
**c** $7^1=7,\ 7^2=49,\ 7^3=343,\ 7^4=2401$

**3** **a** $-1$ **b** $1$ **c** $1$ **d** $-1$ **e** $1$
**f** $-1$ **g** $-1$ **h** $-32$ **i** $-32$ **j** $-64$
**k** $625$ **l** $-625$

**4** **a** $16\,384$ **b** $2401$ **c** $-3125$
**d** $-3125$ **e** $262\,144$ **f** $262\,144$
**g** $-262\,144$ **h** $902.436\,039\,6$ **i** $-902.436\,039\,6$
**j** $-902.436\,039\,6$

**5** **a** $0.\overline{1}$ **b** $0.\overline{1}$ **c** $0.0\overline{27}$ **d** $0.0\overline{27}$
**e** $0.012\,345\,679$ **f** $0.012\,345\,679$ **g** $1$ **h** $1$
Notice that $a^{-n}=\frac{1}{a^n}$

**6** $3$ **7** $7$

## EXERCISE 4C

**1** **a** $5^{11}$ **b** $d^8$ **c** $k^5$ **d** $\frac{1}{7}$ **e** $x^{10}$ **f** $3^{16}$
**g** $p^{-4}$ **h** $n^{12}$ **i** $5^{3t}$ **j** $7^{x+2}$ **k** $10^{3-q}$ **l** $c^{4m}$

**2** **a** $2^2$ **b** $2^{-2}$ **c** $2^3$ **d** $2^{-3}$ **e** $2^5$ **f** $2^{-5}$
**g** $2^1$ **h** $2^{-1}$ **i** $2^6$ **j** $2^{-6}$ **k** $2^7$ **l** $2^{-7}$

**3** **a** $3^2$ **b** $3^{-2}$ **c** $3^3$ **d** $3^{-3}$ **e** $3^1$ **f** $3^{-1}$
**g** $3^4$ **h** $3^{-4}$ **i** $3^0$ **j** $3^5$ **k** $3^{-5}$

**4** **a** $2^{a+1}$ **b** $2^{b+2}$ **c** $2^{t+3}$ **d** $2^{2x+2}$ **e** $2^{n-1}$
**f** $2^{c-2}$ **g** $2^{2m}$ **h** $2^{n+1}$ **i** $2^1$ **j** $2^{3x-1}$

**5** **a** $3^{p+2}$ **b** $3^{3a}$ **c** $3^{2n+1}$ **d** $3^{d+3}$ **e** $3^{3t+2}$
**f** $3^{y-1}$ **g** $3^{1-y}$ **h** $3^{2-3t}$ **i** $3^{3a-1}$ **j** $3^3$

**6** **a** $4a^2$ **b** $27b^3$ **c** $a^4b^4$ **d** $p^3q^3$ **e** $\frac{m^2}{n^2}$
**f** $\frac{a^3}{27}$ **g** $\frac{b^4}{c^4}$ **h** $1,\ a,\, b\neq 0$ **i** $\frac{m^4}{81n^4}$ **j** $\frac{x^3y^3}{8}$

**7** **a** $4a^2$ **b** $36b^4$ **c** $-8a^3$ **d** $-27m^6n^6$
**e** $16a^4b^{16}$ **f** $\frac{-8a^6}{b^6}$ **g** $\frac{16a^6}{b^2}$ **h** $\frac{9p^4}{q^6}$
**i** $4x^3y^2$ **j** $32a^5b$ **k** $\frac{5a^{12}}{b^2}$ **l** $\frac{-2x^{18}}{y^3}$

**8** **a** $\frac{a}{b^2}$ **b** $\frac{1}{a^2b^2}$ **c** $\frac{4a^2}{b^2}$ **d** $\frac{9b^2}{a^4}$ **e** $\frac{a^2}{bc^2}$
**f** $\frac{a^2c^2}{b}$ **g** $a^3$ **h** $\frac{b^3}{a^2}$ **i** $\frac{2}{ad^2}$ **j** $12am^3$

**9** **a** $a^{-n}$ **b** $b^n$ **c** $3^{n-2}$ **d** $a^nb^m$ **e** $a^{-2n-2}$

**10** **a** $1$ **b** $\frac{4}{7}$ **c** $6$ **d** $27$ **e** $\frac{9}{16}$ **f** $\frac{5}{2}$
**g** $\frac{27}{125}$ **h** $\frac{151}{5}$

**11** **a** $3^{-2}$ **b** $2^{-4}$ **c** $5^{-3}$ **d** $3^1\times 5^{-1}$ **e** $2^2\times 3^{-3}$
**f** $2^{c-3}\times 3^{-2}$ **g** $3^{2k}\times 2^{-1}\times 5^{-1}$ **h** $2^p\times 3^{p-1}\times 5^{-2}$

**12** **a** $5^3=21+23+25+27+29$
**b** $7^3=43+45+47+49+51+53+55$
**c** $12^3=133+135+137+139+141+143+145+147$ $+149+151+153+155$

## EXERCISE 4D

**1** **a** $2^{\frac{1}{5}}$ **b** $2^{-\frac{1}{5}}$ **c** $2^{\frac{3}{2}}$ **d** $2^{\frac{5}{2}}$ **e** $2^{-\frac{1}{3}}$
**f** $2^{\frac{4}{3}}$ **g** $2^{\frac{3}{2}}$ **h** $2^{\frac{3}{2}}$ **i** $2^{-\frac{4}{3}}$ **j** $2^{-\frac{3}{2}}$

**2** **a** $3^{\frac{1}{3}}$ **b** $3^{-\frac{1}{3}}$ **c** $3^{\frac{1}{4}}$ **d** $3^{\frac{3}{2}}$ **e** $3^{-\frac{5}{2}}$

**3** **a** $7^{\frac{1}{3}}$ **b** $3^{\frac{3}{4}}$ **c** $2^{\frac{4}{5}}$ **d** $2^{\frac{5}{3}}$ **e** $7^{\frac{2}{7}}$
**f** $7^{-\frac{1}{3}}$ **g** $3^{-\frac{3}{4}}$ **h** $2^{-\frac{4}{5}}$ **i** $2^{-\frac{5}{3}}$ **j** $7^{-\frac{2}{7}}$

**4** **a** $2.28$ **b** $1.83$ **c** $0.794$ **d** $0.435$ **e** $1.68$
**f** $1.93$ **g** $0.523$

**5** **a** $8$ **b** $32$ **c** $8$ **d** $125$ **e** $4$
**f** $\frac{1}{2}$ **g** $\frac{1}{27}$ **h** $\frac{1}{16}$ **i** $\frac{1}{81}$ **j** $\frac{1}{25}$

## EXERCISE 4E.1

**1** **a** $x^5+2x^4+x^2$ **b** $4^x+2^x$ **c** $x+1$
**d** $49^x+2(7^x)$ **e** $2(3^x)-1$ **f** $x^2+2x+3$
**g** $1+5(2^{-x})$ **h** $5^x+1$ **i** $x^{\frac{3}{2}}+x^{\frac{1}{2}}+1$

**2** **a** $4^x+2^{x+1}-3$ **b** $9^x+7(3^x)+10$
**c** $25^x-6(5^x)+8$ **d** $4^x+6(2^x)+9$
**e** $9^x-2(3^x)+1$ **f** $16^x+14(4^x)+49$

**3** **a** $x-4$ **b** $4^x-9$ **c** $x-x^{-1}$ **d** $x^2+4+\frac{4}{x^2}$
**e** $7^{2x}-2+7^{-2x}$ **f** $25-10(2^{-x})+4^{-x}$
**g** $x^{\frac{4}{3}}+2x+x^{\frac{2}{3}}$ **h** $x^3-2x^2+x$ **i** $4x-4+x^{-1}$

## EXERCISE 4E.2

**1** **a** $5^x(5^x+1)$ **b** $10(3^n)$ **c** $7^n(1+7^{2n})$
**d** $5(5^n-1)$ **e** $6(6^{n+1}-1)$ **f** $16(4^n-1)$

**2** **a** $(3^x+2)(3^x-2)$ **b** $(2^x+5)(2^x-5)$
**c** $(4+3^x)(4-3^x)$ **d** $(5+2^x)(5-2^x)$
**e** $(3^x+2^x)(3^x-2^x)$ **f** $(2^x+3)^2$
**g** $(3^x+5)^2$ **h** $(2^x-7)^2$ **i** $(5^x-2)^2$

**3** **a** $(2^x+3)(2^x+6)$ **b** $(2^x+4)(2^x-5)$
**c** $(3^x+2)(3^x+7)$ **d** $(3^x+5)(3^x-1)$
**e** $(5^x+2)(5^x-1)$ **f** $(7^x-4)(7^x-3)$

**4** **a** $2^n$ **b** $10^a$ **c** $3^b$ **d** $\dfrac{1}{5^n}$ **e** $5^x$
**f** $(\frac{3}{4})^a$ **g** $5$ **h** $5^n$

**5** **a** $3^m+1$ **b** $1+6^n$ **c** $4^n+2^n$ **d** $4^x-1$
**e** $6^n$ **f** $5^n$ **g** $4$ **h** $2^n-1$ **i** $\frac{1}{2}$

**6** **a** $n\,2^{n+1}$ **b** $-3^{n-1}$

## EXERCISE 4F

**1** **a** $x=3$ **b** $x=2$ **c** $x=4$ **d** $x=0$
**e** $x=-1$ **f** $x=\frac{1}{2}$ **g** $x=-3$ **h** $x=2$
**i** $x=-3$ **j** $x=-4$ **k** $x=2$ **l** $x=1$

**2** **a** $x=\frac{5}{3}$ **b** $x=-\frac{3}{2}$ **c** $x=-\frac{3}{2}$ **d** $x=-\frac{1}{2}$
**e** $x=-\frac{2}{3}$ **f** $x=-\frac{5}{4}$ **g** $x=\frac{3}{2}$ **h** $x=\frac{5}{2}$
**i** $x=\frac{1}{8}$ **j** $x=\frac{9}{2}$ **k** $x=-4$ **l** $x=-4$
**m** $x=0$ **n** $x=\frac{7}{2}$ **o** $x=-2$ **p** $x=-6$

**3** **a** $x=\frac{1}{7}$ **b** has no solutions **c** $x=2\frac{1}{2}$

**4** **a** $x=1$ **b** $x=2$ **c** $x=1$
**d** $x=\frac{5}{4}$ **e** $x=2$ **f** $x=-\frac{9}{7}$

**5** **a** $x=3$ **b** $x=2$ **c** $x=2$
**d** $x=2$ **e** $x=-2$ **f** $x=-2$

**6** **a** $x=1$ or $2$ **b** $x=1$ **c** $x=1$ or $2$
**d** $x=1$ **e** $x=2$ **f** $x=0$

## EXERCISE 4G

**1** **a** $1.4$ **b** $1.7$ **c** $2.8$ **d** $0.4$

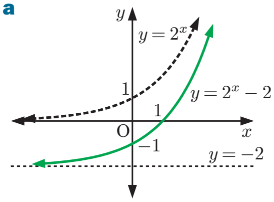
**2** **a** $x\approx 1.6$ **b** $x\approx -0.7$ **c** $x\approx 2.1$ **d** $x\approx -1.7$

**3** $y=2^x$ has a horizontal asymptote of $y=0$

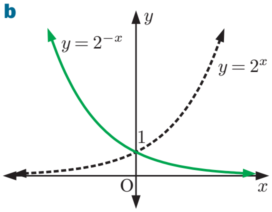
**4** **a** $2$ **b** $54$ **c** $\frac{2}{9}$

**5** **a** $g(0)=3,\ \ g(-1)=\frac{11}{5}$ **b** $a=2$

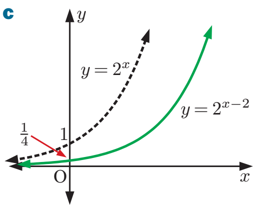
**6** **a**



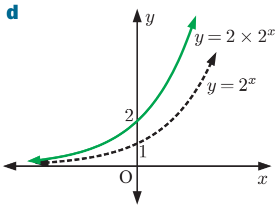

**b**



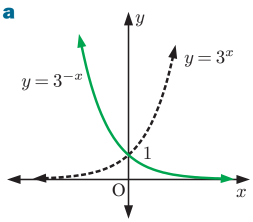

**c**



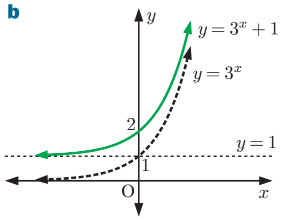

**d**



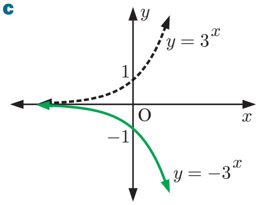

**7** **a**



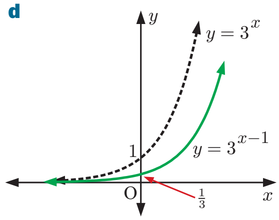

**b**



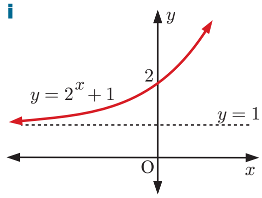

**c**




**d**




**8** **a** **i**



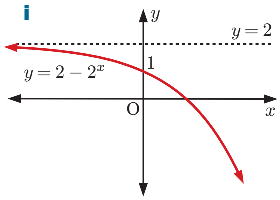

**ii** Domain: $\{x : x\in\mathbb{R}\}$
Range: $\{y : y>1\}$

**iii** $y\approx 3.67$

**iv** As $x\to\infty,\ \ y\to\infty$
As $x\to-\infty,\ \ y\to 1$ from above

**v** $y=1$

**b** **i**



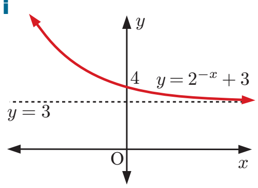

**ii** Domain: $\{x : x\in\mathbb{R}\}$
Range: $\{y : y<2\}$

**iii** $y\approx -0.665$

**iv** As $x\to\infty,\ \ y\to-\infty$
As $x\to-\infty,\ \ y\to 2$ from below

**v** $y=2$

**c** **i**



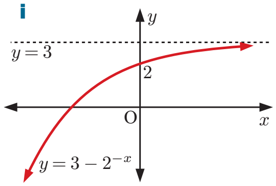

**ii** Domain: $\{x : x\in\mathbb{R}\}$
Range: $\{y : y>3\}$

**iii** $y\approx 3.38$

**iv** As $x\to\infty,\ \ y\to 3$ from above
As $x\to-\infty,\ \ y\to\infty$

**v** $y=3$

**d** **i**




**ii** Domain: $\{x : x\in\mathbb{R}\}$
Range: $\{y : y<3\}$

**iii** $y\approx 2.62$

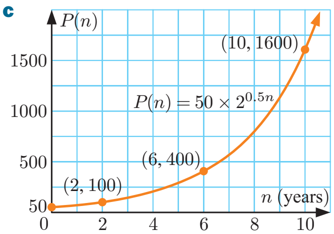
**iv** As $x\to\infty,\ \ y\to 3$ from below
As $x\to-\infty,\ \ y\to-\infty$

**v** $y=3$

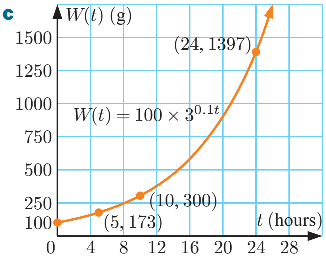
**9** **a** $P_0=50$

**b** **i** 100 possums
**ii** 400 possums
**iii** 1600 possums

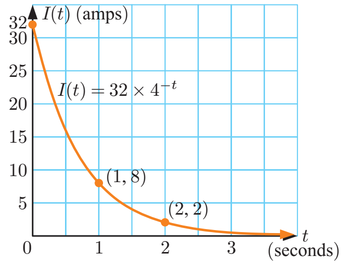
**c**



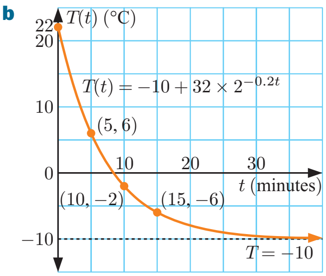

**d** 8 years

**10** **a** 100 g
**b** **i** $\approx 173$ g
**ii** 300 g
**iii** $\approx 1397$ g
**c** 

**d** 20 hours

**11** **a** 32 amps
**b** **i** 8 amps
**ii** 2 amps
**c** 

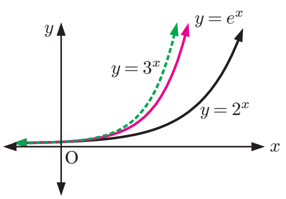
**d** 3 seconds

**12** **a** **i** $22°C$
**ii** $6°C$
**iii** $-2°C$
**iv** $-6°C$
**b** 

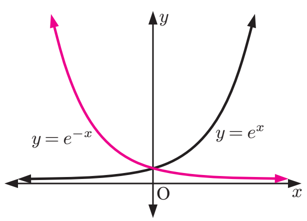
**c** The temperature will not reach $-10°C$ according to this model, as the model has a horizontal asymptote at $T = -10$.

## EXERCISE 4H

**1** The graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$.




**2** One is the other reflected in the $y$-axis.

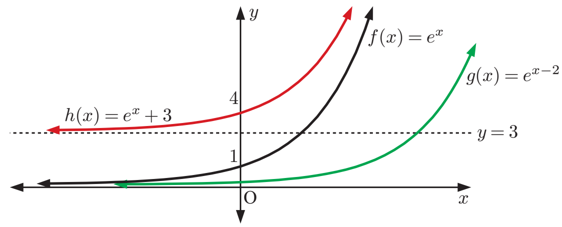



**3** $a$

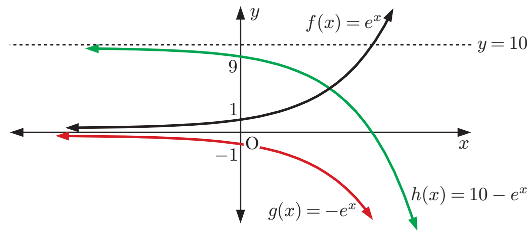
**4** **a** $e^x > 0$ for all $x$
**b** **i** 0.000 000 004 12 **ii** 970 000 000

**5** **a** $\approx 7.39$ **b** $\approx 20.1$ **c** $\approx 2.01$ **d** $\approx 1.65$
**e** $\approx 0.368$

**6** **a** $e^{\frac{1}{2}}$ **b** $e^{-\frac{1}{2}}$ **c** $e^{-2}$ **d** $e^{\frac{3}{2}}$

**7** 


Domain of $f$, $g$, and $h$ is $\{x : x \in \mathbb{R}\}$
Range of $f$ is $\{y : y > 0\}$, Range of $g$ is $\{y : y > 0\}$
Range of $h$ is $\{y : y > 3\}$

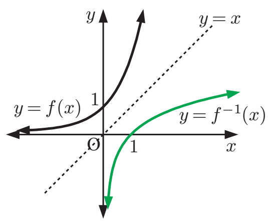
**8** 


Domain of $f$, $g$, and $h$ is $\{x : x \in \mathbb{R}\}$
Range of $f$ is $\{y : y > 0\}$, Range of $g$ is $\{y : y < 0\}$
Range of $h$ is $\{y : y < 10\}$

**9** **a** $e^{2x} + 2e^x + 1$ **b** $1 - e^{2x}$ **c** $1 - 3e^x$

**10** **a** $x = \frac{1}{2}$ **b** $x = -4$

**11** **a** $fg(x) = e^{3x+2}$, $gf(x) = 3e^x + 2$ **b** $x = -1$

**12** **a** 

**b** Domain of $f^{-1}$ is $\{x : x > 0\}$,
Range of $f^{-1}$ is $\{y : y \in \mathbb{R}\}$

## REVIEW SET 4A

**1** **a** $-15 + 20\sqrt{3}$ **b** $86 - 60\sqrt{2}$

**2** **a** $\frac{2\sqrt{3}}{3}$ **b** $\frac{\sqrt{35}}{5}$ **c** $\frac{\sqrt{7}}{28}$

**3** **a** $a^6b^7$ **b** $\dfrac{2}{3x}$ **c** $\dfrac{y^2}{5}$

**4** **a** **i** 81 **ii** $\frac{1}{3}$ **b** $k = 9$

**5** **a** $\dfrac{1}{x^5}$ **b** $\dfrac{2}{a^2b^2}$ **c** $\dfrac{2a}{b^2}$

**6** **a** $3^{3-2a}$ **b** $3^{\frac{5}{2}-\frac{9}{2}x}$ **7** **a** 4 **b** $\frac{1}{9}$

**8** **a** $\dfrac{m}{n^2}$ **b** $\dfrac{1}{m^3n^3}$ **c** $\dfrac{m^2p^2}{n}$ **d** $\dfrac{16n^2}{m^2}$

**9** **a** $9 - 6e^x + e^{2x}$ **b** $x - 4$ **c** $2^x + 1$

**10** $x = \frac{9}{34} + \frac{1}{34}\sqrt{13}$

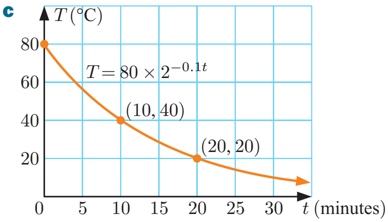
**11** **a** $x = -2$ **b** $x = \frac{3}{4}$ **c** $x = -\frac{1}{4}$

**12** **a** **C** **b** **E** **c** **A** **d** **B** **e** **D**

**13** **a** 3 **b** 24 **c** $\frac{3}{4}$

**14** **a** Range of $f$ is $\{y : y > -3\}$ **b** $-2$ **c** $x = \frac{1}{2}$

**15** **a** $80°\text{C}$
**b** **i** $40°\text{C}$
**ii** $20°\text{C}$
**d** 30 minutes
**c**




## REVIEW SET 4B

**1** **a** $17 - 11\sqrt{3}$ **b** $28$

**2** **a** $\dfrac{5+\sqrt{3}}{22}$ **b** $\dfrac{\sqrt{77}+2\sqrt{11}}{3}$ **c** $\dfrac{26+11\sqrt{2}}{7}$
**d** $\dfrac{-33-14\sqrt{5}}{3}$

**3** **a** $x \approx 1.45$ **b** $x \approx -0.6$ **c** $x \approx 1.1$

**4** **a** $3-2\sqrt{2}$ **b** $3-2\sqrt{2}$ **c** $3-2\sqrt{2}$ **d** $3-2\sqrt{2}$

**5** **a** $a^{21}$ **b** $p^4q^6$ **c** $\dfrac{4b}{a^3}$

**6** **a** $2^{-3}$ **b** $2^7$ **c** $2^{12}$

**7** **a** $4m^6$ **b** $\dfrac{-a^9}{b^3}$ **c** $3x^3y^2$ **d** $16ab^{\frac{4}{5}}$

**8** $2^{2x}$ **9** **a** $5^0$ **b** $5^{\frac{3}{2}}$ **c** $5^{-\frac{1}{4}}$ **d** $5^{2a+6}$

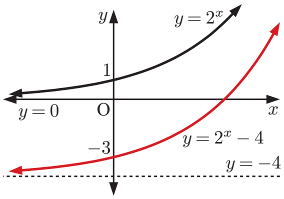
**10** **a** $1+e^{2x}$ **b** $2^{2x}+10(2^x)+25$ **c** $x-49$

**11** **a** $x=5$ **b** $x=-4$

**12** **a** $x=-\frac{2}{5}$ **b** $x=1$ **c** $x=\frac{7}{11}$

**13** **a** $\frac{1}{\sqrt{2}}+1 \approx 1.71$
**b** $a=-1$

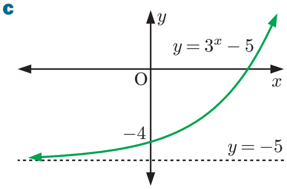
**14**




**15** **a**

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | $-4\frac{8}{9}$ | $-4\frac{2}{3}$ | $-4$ | $-2$ | $4$ |

**b** as $x \to \infty$, $y \to \infty$;
as $x \to -\infty$, $y \to -5$ (above)
**c**




**d** $\{y : y > -5\}$

**16** **a** Range of $f$ is $\{y : y > 0\}$ **b** $g(\sqrt{2}) = e^2$
**c** $1+\frac{1}{2}\sqrt{2}$

## EXERCISE 5A

**1** **a** $4$ **b** $-3$ **c** $1$ **d** $0$ **e** $\frac{1}{2}$ **f** $\frac{1}{3}$
**g** $-\frac{1}{4}$ **h** $1\frac{1}{2}$ **i** $\frac{2}{3}$ **j** $1\frac{1}{2}$ **k** $1\frac{1}{3}$ **l** $3\frac{1}{2}$

**2** **a** $n$ **b** $a+2$ **c** $1-m$ **d** $a-b$

**3** **a** $\lg 41 \approx 1.6128$ **b** $41 \approx 10^{1.6128}$

**4** **a** $10^{0.7782}$ **b** $10^{1.7782}$ **c** $10^{3.7782}$ **d** $10^{-0.2218}$
**e** $10^{-2.2218}$ **f** $10^{1.1761}$ **g** $10^{3.1761}$ **h** $10^{0.1761}$
**i** $10^{-0.8239}$ **j** $10^{-3.8239}$

**5** A negative number cannot be written in the form $10^b$ where $b \in \mathbb{R}$, so its logarithm cannot be found.

**6** **a** **i** $0.477$ **ii** $2.477$ **b** $\lg 300 = \lg(3 \times 10^2)$

**7** **a** **i** $0.699$ **ii** $-1.301$ **b** $\lg 0.05 = \lg(5 \times 10^{-2})$

**8** **a** $x=100$ **b** $x=10$ **c** $x=1$
**d** $x=\frac{1}{10}$ **e** $x=10^{\frac{1}{2}}$ **f** $x=10^{-\frac{1}{2}}$
**g** $x=10\,000$ **h** $x=0.000\,01$ **i** $x \approx 6.84$
**j** $x \approx 140$ **k** $x \approx 0.0419$ **l** $x \approx 0.000\,631$

## EXERCISE 5B

**1** **a** $10^2 = 100$ **b** $10^4 = 10\,000$ **c** $10^{-1} = 0.1$
**d** $10^{\frac{1}{2}} = \sqrt{10}$ **e** $2^3 = 8$ **f** $3^2 = 9$
**g** $2^{-2} = \frac{1}{4}$ **h** $3^{1.5} = \sqrt{27}$ **i** $5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$

**2** **a** $\log_2 4 = 2$ **b** $\log_4 64 = 3$ **c** $\log_5 25 = 2$
**d** $\log_7 49 = 2$ **e** $\log_2 64 = 6$ **f** $\log_2(\frac{1}{8}) = -3$
**g** $\log_{10} 0.01 = -2$ **h** $\log_2(\frac{1}{2}) = -1$ **i** $\log_3(\frac{1}{27}) = -3$

**3** **a** $5$ **b** $-2$ **c** $\frac{1}{2}$ **d** $3$ **e** $6$ **f** $7$ **g** $2$
**h** $3$ **i** $-3$ **j** $\frac{1}{2}$ **k** $2$ **l** $\frac{1}{2}$ **m** $5$ **n** $\frac{1}{3}$
**o** $n$, $a > 0$ **p** $\frac{1}{3}$ **q** $-1$, $t > 0$ **r** $\frac{3}{2}$ **s** $0$
**t** $1$

**4** **a** $\approx 2.18$ **b** $\approx 1.40$ **c** $\approx 1.87$ **d** $\approx -0.0969$

**5** **a** $x=8$ **b** $x=2$ **c** $x=3$ **d** $x=14$

**6** **a** $2$ **b** $2$ **c** $-1$ **d** $\frac{3}{4}$ **e** $-\frac{1}{2}$ **f** $\frac{5}{2}$
**g** $-\frac{3}{2}$ **h** $-\frac{3}{4}$ **i** $2$, $x > 0$ **j** $\frac{1}{2}$, $x > 0$
**k** $3$, $m > 0$ **l** $\frac{3}{2}$, $x > 0$ **m** $-1$, $n > 0$
**n** $-2$, $a > 0$ **o** $-\frac{1}{2}$, $a > 0$ **p** $\frac{5}{2}$, $m > 0$

## EXERCISE 5C

**1** **a** $\lg 16$ **b** $\lg 20$ **c** $\lg 8$ **d** $\lg \dfrac{p}{m}$
**e** $1$ **f** $\lg 2$ **g** $\lg 24$ **h** $\log_2 6$
**i** $\lg 0.4$ **j** $1$ **k** $\lg 200$
**l** $\lg(10^t \times w)$ **m** $\log_m \left(\dfrac{40}{m^2}\right)$ **n** $0$
**o** $\lg(0.005)$ **p** $\log_5(\frac{5}{2})$ **q** $2$ **r** $\lg 28$

**2** **a** $\lg 96$ **b** $\lg 72$ **c** $\lg 8$ **d** $\log_3(\frac{25}{8})$
**e** $1$ **f** $\lg \frac{1}{2}$ **g** $\lg 20$ **h** $\lg 25$
**i** $\log_n \left(\dfrac{n^2}{10}\right)$

**3** **a** $2$ **b** $\frac{3}{2}$ **c** $3$ **d** $\frac{1}{2}$ **e** $-2$ **f** $-\frac{3}{2}$

**4** For example, for **a**, $\lg 9 = \lg 3^2 = 2\lg 3$

**5** **a** $2$ **b** $-1$ **c** $1$

**6** **a** $x+z$ **b** $z+2y$ **c** $x+z-y$ **d** $2x+\frac{1}{2}y$
**e** $3y-\frac{1}{2}z$ **f** $2z+\frac{1}{2}y-3x$

**7** **a** $p+q$ **b** $2q+r$ **c** $2p+3q$ **d** $r+\frac{1}{2}q-p$
**e** $r-5p$ **f** $p-2q$

**8** **a** $0.86$ **b** $2.15$ **c** $1.075$ **9** $\log_b Q = 3$

**10** **a** $\log_t A + 3\log_t B = 15$, $2\log_t A - \log_t B = 9$
**b** $\log_t A = 6$, $\log_t B = 3$ **c** $\log_t(B^5\sqrt{A}) = 18$
**d** $B = t^3$

## EXERCISE 5D.1

**1** **a** $\lg y = x \lg 2$ **b** $\lg y = 3\lg x$ **c** $\lg M = 4\lg d$ **d** $\lg T = x\lg 5$ **e** $\lg y = \frac{1}{2}\lg x$ **f** $\lg y = \lg 7 + x\lg 3$ **g** $\lg S = \lg 9 - \lg t$ **h** $\lg M = 2 + x\lg 7$ **i** $\lg T = \lg 5 + \frac{1}{2}\lg d$ **j** $\lg F = 3 - \frac{1}{2}\lg n$ **k** $\lg S = \lg 200 + t\lg 2$ **l** $\lg y = \frac{1}{2}\lg 15 - \frac{1}{2}\lg x$

**2** **a** $y = 7^x$ **b** $D = 2x$ **c** $F = \dfrac{5}{t}$ **d** $y = 6 \times 2^x$ **e** $P = \sqrt{x}$ **f** $N = \dfrac{1}{\sqrt[3]{p}}$ **g** $P = 10x^3$ **h** $y = \dfrac{10^x}{2}$ **i** $y = \dfrac{x^2}{10}$ **j** $T = 2k^5$ **k** $P = \dfrac{n^4}{9}$ **l** $y = 8 \times 16^x$

**3** **a** $y = \dfrac{x^3}{2}$ **b** **i** $y = 4$ **ii** $y = 32$

**4** **a** $y = 100(10^{\frac{1}{3}x})$ **b** **i** $y = 100$ **ii** $y = 1000$

**5** **a** If there is a *power* relationship between $y$ and $x$, for example $y = 5x^3$, then there is a *linear* relationship between $\lg y$ and $\lg x$.
**b** If there is an *exponential* relationship between $y$ and $x$, for example $y = 4 \times 2^x$, then there is a *linear* relationship between $\lg y$ and $x$.

## EXERCISE 5D.2

**1** **a** $x = 25$ **b** $x = 67$ **c** $x = 20$ **d** $x = \frac{125}{64}$ **e** $x = 5$ **f** no solution **g** $x = \frac{9}{8}$ **h** no solution

**2** **a** $x = 5$ **b** $x = 3$ or $6$ **c** $x = 2$ or $4$ **d** $x = 2$ **e** $x = 1$ **f** no solution **g** $x = 2$ **h** $x = 4$

**3** **a** $x = 8$ **b** $x = 3$ **c** $x = 6$ **d** $x = 4$

## EXERCISE 5E.1

**1** **a** 2 **b** 3 **c** $\frac{1}{2}$ **d** 0 **e** $-1$ **f** $\frac{1}{3}$ **g** $-2$ **h** $-\frac{1}{2}$

**2** **a** 3 **b** 9 **c** $\frac{1}{5}$ **d** $\frac{1}{4}$

**3** $x$ does not exist such that $e^x = -2$ or $0$

**4** **a** $a$ **b** $a+1$ **c** $a+b$ **d** $ab$ **e** $a-b$

**5** **a** $e^{1.7918}$ **b** $e^{4.0943}$ **c** $e^{8.6995}$ **d** $e^{-0.5108}$ **e** $e^{-5.1160}$ **f** $e^{2.7081}$ **g** $e^{7.3132}$ **h** $e^{0.4055}$ **i** $e^{-1.8971}$ **j** $e^{-8.8049}$

**6** **a** $x \approx 20.1$ **b** $x = e \approx 2.72$ **c** $x = 1$ **d** $x = \dfrac{1}{e} \approx 0.368$ **e** $x \approx 0.006\,74$ **f** $x \approx 2.30$ **g** $x \approx 8.54$ **h** $x \approx 0.0370$

## EXERCISE 5E.2

**1** **a** $\ln 45$ **b** $\ln 5$ **c** $\ln 4$ **d** $\ln 24$ **e** $\ln 1 = 0$ **f** $\ln 30$ **g** $\ln(4e)$ **h** $\ln\left(\dfrac{6}{e}\right)$ **i** $\ln 20$ **j** $\ln(4e^2)$ **k** $\ln\left(\dfrac{20}{e^2}\right)$ **l** $\ln 1 = 0$

**2** **a** $\ln 972$ **b** $\ln 200$ **c** $\ln 1 = 0$ **d** $\ln 16$ **e** $\ln 6$ **f** $\ln\left(\frac{1}{3}\right)$ **g** $\ln\left(\frac{1}{2}\right)$ **h** $\ln 2$ **i** $\ln 16$

**3** For example, for **a**, $\ln 27 = \ln 3^3 = 3\ln 3$

**4** **Hint:** In **d**, $\ln\left(\dfrac{e^2}{8}\right) = \ln e^2 - \ln 2^3$

**5** **a** $D = ex$ **b** $F = \dfrac{e^2}{p}$ **c** $P = 5e^{2x}$ **d** $M = e^3y^2$ **e** $B = \frac{1}{4}e^{3t}$ **f** $N = \dfrac{1}{\sqrt[3]{g}}$ **g** $Q \approx 8.66x^3$ **h** $D \approx 0.518n^{0.4}$ **i** $T \approx \dfrac{4.85}{e^x}$

## EXERCISE 5F

**1** **a** $x \approx 3.32$ **b** $x \approx 2.73$ **c** $x \approx 3.32$ **d** $x = 4$ **e** $x \approx 8.00$ **f** $x = -5$

**2** **a** $x \approx 1.43$ **b** $x \approx 1.56$ **c** $x \approx 3.44$ **d** $x \approx 5.82$ **e** $x \approx -1.34$ **f** $x \approx 2.37$ **g** $x \approx 0.275$ **h** $x \approx 1.81$ **i** $x \approx 9.64$

**3** **a** $x = \ln 10$ **b** $x = \ln 1000$ **c** $x = \ln 0.15$ **d** $x = 2\ln 5$ **e** $x = \frac{1}{2}\ln 18$ **f** $x = 0$

**4** **a** $x = \frac{1}{2}\ln 300$ **b** $x \approx 2.85$

**5** **a** $x = -\dfrac{\lg(0.03)}{\lg 2}$ **b** $x = \dfrac{10\lg\left(\frac{10}{3}\right)}{\lg 5}$ **c** $x = \dfrac{-4\lg\left(\frac{1}{8}\right)}{\lg 3}$

**6** **a** 3.90 hours **b** 15.5 hours **7** **b** $t \approx 6.93$ hours

**8** **a** 50 g **b** $\approx 13\,200$ years

**9** **a** $x = \ln 2$ **b** $x = 0$ **c** $x = \ln 2$ or $\ln 3$ **d** $x = 0$ **e** $x = \ln 4$ **f** $x = \ln\left(\frac{3+\sqrt{5}}{2}\right)$ or $\ln\left(\frac{3-\sqrt{5}}{2}\right)$

**10** **a** $(\ln 3, 3)$ **b** $(\ln 2, 5)$ **c** $(0, 2)$ and $(\ln 5, -2)$

## EXERCISE 5G

**1** **a** $\approx 2.26$ **b** $\approx -10.3$ **c** $\approx -2.46$ **d** $\approx 5.42$

**2** **a** $x \approx -4.29$ **b** $x \approx 3.87$ **c** $x \approx 0.139$

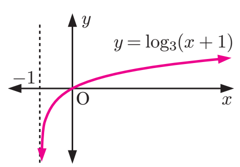
**3** **a** $\log_9 26 = \frac{1}{2}\log_3 26$ **b** $\log_2 11 = 2\log_4 11$ **c** $\dfrac{6}{\log_7 25} = 3\log_5 7$

**4** **a** $x = \sqrt[3]{50}$ **b** $x = \sqrt{13}$ **c** $x = 49$ **d** $x = 5$ **e** $x = 8$ **f** $x = 16$

**5** **b** **i** $x = \frac{1}{9}$ or $9$ **ii** $x = \frac{1}{2}$ or $32$ **iii** $x = 2$ or $64$

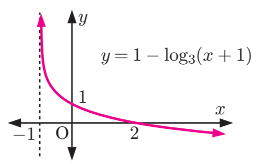
## EXERCISE 5H

**1** **a** **i** Domain is $\{x : x > -1\}$, Range is $\{y : y \in \mathbb{R}\}$
**ii** VA is $x = -1$, $x$ and $y$-intercepts 0
**iii**



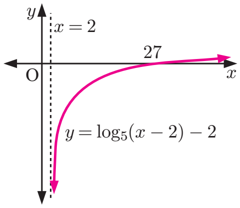

**iv** $x = -\frac{2}{3}$
**v** $f^{-1}(x) = 3^x - 1$

**b** **i** Domain is $\{x : x > -1\}$, Range is $\{y : y \in \mathbb{R}\}$
**ii** VA is $x = -1$, $x$-intercept 2, $y$-intercept 1
**iii**



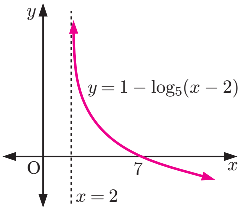

**iv** $x = 8$
**v** $f^{-1}(x) = 3^{1-x} - 1$

**c** **i** Domain is $\{x : x > 2\}$, Range is $\{y : y \in \mathbb{R}\}$
**ii** VA is $x = 2$, $x$-intercept 27, no $y$-intercept
**iii**



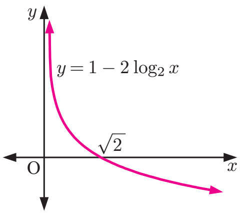

**iv** $x = 7$
**v** $f^{-1}(x) = 5^{2+x} + 2$

**d** **i** Domain is $\{x : x > 2\}$, Range is $\{y : y \in \mathbb{R}\}$
**ii** VA is $x = 2$, $x$-intercept 7, no $y$-intercept
**iii**



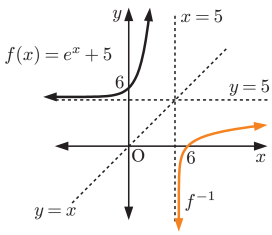

**iv** $x = 27$
**v** $f^{-1}(x) = 5^{1-x} + 2$

**e** **i** Domain is $\{x : x > 0\}$, Range is $\{y : y \in \mathbb{R}\}$
**ii** VA is $x = 0$, $x$-intercept $\sqrt{2}$, no $y$-intercept
**iii**



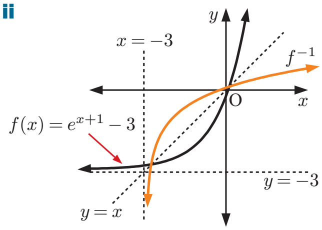

**iv** $x = 2$
**v** $f^{-1}(x) = 2^{\frac{1-x}{2}}$

**2** **a** **i** $f^{-1}(x) = \ln(x-5)$
**ii**



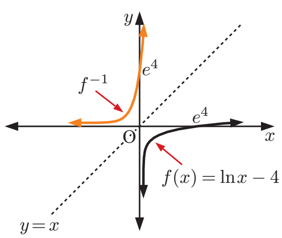

**iii** Domain of $f$ is $\{x : x \in \mathbb{R}\}$, Range is $\{y : y > 5\}$
Domain of $f^{-1}$ is $\{x : x > 5\}$, Range is $\{y : y \in \mathbb{R}\}$
**iv** $f$ has a HA $y = 5$, $f$ has $y$-int 6
$f^{-1}$ has a VA $x = 5$, $f^{-1}$ has $x$-int 6

**b** **i** $f^{-1}(x) = \ln(x+3) - 1$
**ii**



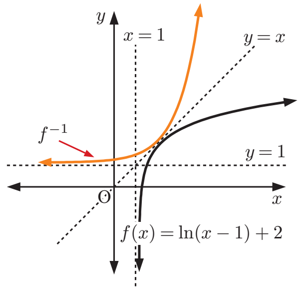

**iii** Domain of $f$ is $\{x : x \in \mathbb{R}\}$, Range is $\{y : y > -3\}$
Domain of $f^{-1}$ is $\{x : x > -3\}$, Range is $\{y : y \in \mathbb{R}\}$
**iv** $f$ has a HA $y = -3$, $x$-int $\ln 3 - 1$, $y$-int $e - 3$
$f^{-1}$ has a VA $x = -3$, $x$-int $e - 3$, $y$-int $\ln 3 - 1$

**c** **i** $f^{-1}(x) = e^{x+4}$
**ii**



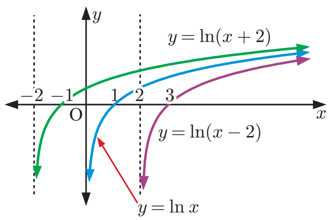

**iii** Domain of $f$ is $\{x : x > 0\}$, Range of $f$ is $\{y : y \in \mathbb{R}\}$
Domain of $f^{-1}$ is $\{x : x \in \mathbb{R}\}$, Range is $\{y : y > 0\}$
**iv** $f$ has a VA $x = 0$, $x$-int $e^4$
$f^{-1}$ has a HA $y = 0$, $y$-int $e^4$

**d** **i** $f^{-1}(x) = 1 + e^{x-2}$
**ii**




**iii** Domain of $f$ is $\{x : x > 1\}$, Range is $\{y : y \in \mathbb{R}\}$
Domain of $f^{-1}$ is $\{x : x \in \mathbb{R}\}$, Range is $\{y : y > 1\}$
**iv** $f$ has a VA $x = 1$, $x$-int $1 + e^{-2}$
$f^{-1}$ has a HA $y = 1$, $y$-int $1 + e^{-2}$

**3** **a** A is $y = \ln x$ as its $x$-intercept is 1
**b**




**c** $y = \ln x$ has VA $x = 0$
$y = \ln(x-2)$ has VA $x = 2$
$y = \ln(x+2)$ has VA $x = -2$

**4** $y = \ln(x^2) = 2\ln x$, so she is correct.
This is because the $y$-values are twice as large for $y = \ln(x^2)$ as they are for $y = \ln x$.

**5** **a** $f^{-1} : x \mapsto \ln(x-2) - 3$
**b** **i** $x < -5.30$ **ii** $x < -7.61$ **iii** $x < -9.91$
**iv** $x < -12.2$ Conjecture HA is $y = 2$
**c** as $x \to \infty$, $f(x) \to \infty$,
as $x \to -\infty$, $e^{x+3} \to 0$ and $f(x) \to 2$
$\therefore$ HA is $y = 2$
**d** VA of $f^{-1}$ is $x = 2$, Domain of $f^{-1}$ is $\{x : x > 2\}$

**6** **a** **i** $f(5) = 3$ **ii** $f(x^2) = \log_2(x^2 + 3)$
**iii** $f(2x-1) = 1 + \log_2(x+1)$
**b** Domain of $f(x)$ is $\{x : x > -3\}$ **c** $x = \pm 5$

**7** **a** Range is $\{y : y > 1\}$ **b** $f^{-1}(x) = \frac{1}{3}\ln(x-1)$
**c** $f^{-1}(10) = \frac{1}{3}\ln 9$
**d** Domain of $f^{-1}(x)$ is $\{x : x > 1\}$
**e** $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$

**8** **a** $f^{-1}(x) = \frac{1}{2}\ln x$
**i** $(f^{-1} \circ g)(x) = \frac{1}{2}\ln(2x-1)$
**ii** $(g \circ f)^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{2}\right)$
**b** $x = 13$

**9** **a** $f(1) = \dfrac{10}{e}$, $g(6) = \ln 3$ **b** $x$-intercept of $g(x)$ is 4
**c** $fg(x) = \dfrac{10}{x-3}$ **d** $x = \ln 2$

**10** **a** Domain of $f(x)$ is $\{x : x > -6\}$
**b** $f^{-1}(x) = e^x - 6$
**c** $x$-intercept is $-5$, $y$-intercept is $\ln 6$ **d** $x = -\frac{8}{3}$ or 3

## REVIEW SET 5A

**1** **a** 3 **b** 8 **c** $-2$ **d** $\frac{1}{2}$ **e** 0
**f** $\frac{1}{4}$ **g** $-1$ **h** $\frac{1}{2}$, $k > 0$

**2** **a** $\frac{1}{2}$ **b** $-\frac{1}{3}$ **c** $a+b+1$

**3** **a** $\ln 144$ **b** $\ln\left(\frac{3}{2}\right)$ **c** $\ln\left(\dfrac{25}{e}\right)$ **d** $\ln 3$

**4** **a** $\frac{3}{2}$ **b** $-3$ **c** $2x$ **d** $1-x$

**5** **a** $\lg 144$ **b** $\log_2\left(\frac{16}{9}\right)$ **c** $\log_4 80$

**6** **a** $\lg P = \lg 3 + x\lg 7$ **b** $\lg m = 3\lg n - \lg 5$

**7** **a** $x=3$ **b** $x=5$

**8** **Hint:** Use change of base rule.

**9** **a** $T = \dfrac{x^2}{5}$ **b** $K = 3\times 2^x$

**10** **a** $5\ln 2$ **b** $3\ln 5$ **c** $6\ln 3$

**11**

| *Function* | $y=\log_2 x$ | $y=\ln(x+5)$ |
|---|---|---|
| *Domain* | $x>0$ | $x>-5$ |
| *Range* | $y\in\mathbb{R}$ | $y\in\mathbb{R}$ |

**12** **a** $2A+2B$ **b** $A+3B$ **c** $3A+\frac{1}{2}B$
**d** $4B-2A$ **e** $3A-2B$

**13** **a** $x=0$ or $\ln\left(\frac{2}{3}\right)$ **b** $x=e^2$

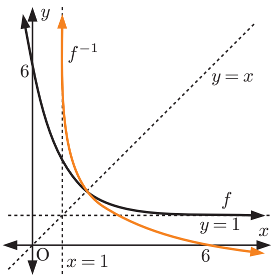
**14** **a** $x\approx 2.46$ **b** $x\approx 1.88$ **15** $\approx 6.97$ years

**16** **a** Range of $f$ is $\{y: y>1\}$

**b** **i** $f^{-1}(x) = \ln\left(\dfrac{5}{x-1}\right)$ **ii** $f^{-1}(2)=\ln 5$

**c** Domain of $f^{-1}$ is $\{x: x>1\}$ **d** $x=6$

**e**




## REVIEW SET 5B

**1** **a** $\frac{3}{2}$ **b** $\frac{2}{3}$ **c** $a+b$

**2** **a** $\approx 10^{1.5051}$ **b** $\approx 10^{-2.8861}$ **c** $\approx 10^{-4.0475}$

**3** **a** $x=\frac{1}{8}$ **b** $x\approx 82.7$ **c** $x\approx 0.0316$

**4** **a** $k\approx 3.25\times 2^x$ **b** $Q=5P^3$ **c** $A=6\times 2^x$

**5** **a** $x=\dfrac{\lg 7}{\lg 5}$ **b** $x=2$

**6** $-1$ **7** $\log_8 30 = \frac{1}{3}\log_2 30$

**8** **a** $x=8$ **b** $x=3$ **9** **a** $9$ **b** $\ln 5$

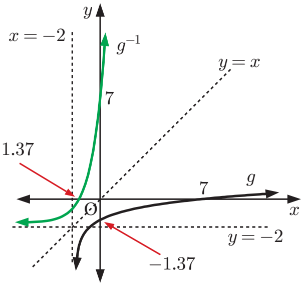
**10** **a** $\ln 3$ **b** $\ln 4$ **c** $\ln 125$

**11** **a** $\lg M = \lg 5 + x\lg 6$ **b** $\lg T = \lg 5 - \frac{1}{2}\lg l$
**c** $\lg G = \lg 4 - \lg c$

**12** **a** $x=\ln 3$ **b** $x=\ln 3$ or $\ln 4$

**13** **a** Domain is $\{x: x>-2\}$, Range is $\{y: y\in\mathbb{R}\}$
**b** VA is $x=-2$, $x$-intercept is 7, $y$-intercept is $\approx -1.37$
**c** $g^{-1}(x) = 3^{x+2}-2$

**d**




**14** 13.9 weeks

**15** **a** $x=5$ **b** $x=32$ or $\frac{1}{32}$ **c** $x=9$ or $81$

**16** **a** Domain is $\{x: x>4\}$, Range is $\{y: y\in\mathbb{R}\}$
**b** $x$-intercept is 5, no $y$-intercept **c** $x=4+\sqrt{6}$
**d** $x=0$

## EXERCISE 6A.1

**1** **a** $3x^2+6x+9$ **b** $5x^2+7x+9$ **c** $-7x^2-8x-9$
**d** $4x^4+13x^3+28x^2+27x+18$

**2** **a** $x^3+x^2-4x+7$ **b** $x^3-x^2-2x+3$
**c** $3x^3+2x^2-11x+19$ **d** $2x^3-x^2-x+5$
**e** $x^5-x^4-x^3+8x^2-11x+10$
**f** $x^4-2x^3+5x^2-4x+4$

**3** **a** $2x^3-3x^2+4x+3$ **b** $x^4+x^3-7x^2+7x-2$
**c** $x^3+6x^2+12x+8$ **d** $4x^4-4x^3+13x^2-6x+9$
**e** $16x^4-32x^3+24x^2-8x+1$
**f** $18x^4-87x^3+56x^2+20x-16$

**4** **a** $6x^3-11x^2+18x-5$ **b** $8x^3+18x^2-x+10$
**c** $-2x^3+7x^2+13x+10$ **d** $2x^3-7x^2+4x+4$
**e** $2x^4-2x^3-9x^2+11x-2$
**f** $15x^4+x^3-x^2+7x-6$
**g** $x^4-2x^3+7x^2-6x+9$
**h** $4x^4+4x^3-15x^2-8x+16$
**i** $8x^3+60x^2+150x+125$
**j** $x^6+2x^5+x^4-4x^3-4x^2+4$

## EXERCISE 6A.2

**1** **a** $Q(x)=x$, $R=-3$, $x^2+2x-3 = x(x+2)-3$
**b** $Q(x)=x-4$, $R=-3$,
$x^2-5x+1=(x-4)(x-1)-3$
**c** $Q(x)=2x^2+10x+16$, $R=35$,
$2x^3+6x^2-4x+3=(2x^2+10x+16)(x-2)+35$

**2** **a** $x^2-3x+6=(x+1)(x-4)+10$
**b** $x^2+4x-11=(x+1)(x+3)-14$
**c** $2x^2-7x+2=(2x-3)(x-2)-4$
**d** $2x^3+3x^2-3x-2=(x^2+x-2)(2x+1)$
**e** $3x^3+11x^2+8x+7=(x^2+4x+4)(3x-1)+11$
**f** $2x^4-x^3-x^2+7x+4$
$=(x^3-2x^2+\frac{5}{2}x-\frac{1}{4})(2x+3)+\frac{19}{4}$

**3** **a** $x+2+\dfrac{9}{x-2}$ **b** $2x+1-\dfrac{1}{x+1}$
**c** $3x-4+\dfrac{3}{x+2}$ **d** $x^2+3x-2$
**e** $2x^2-8x+31-\dfrac{124}{x+4}$ **f** $x^2+3x+6+\dfrac{7}{x-2}$