5 Find exact values for $\sin x$ and $\cos x$ given that:

a $\tan x = \frac{2}{3}$ and $0 < x < \frac{\pi}{2}$

b $\tan x = -\frac{4}{3}$ and $\frac{\pi}{2} < x < \pi$

c $\tan x = \frac{\sqrt{5}}{3}$ and $\pi < x < \frac{3\pi}{2}$

d $\tan x = -\frac{12}{5}$ and $\frac{3\pi}{2} < x < 2\pi$

6 Suppose $\tan\theta = k$ where k is a constant and $\pi < \theta < \frac{3\pi}{2}$. Write expressions for $\sin\theta$ and $\cos\theta$ in terms of k.

FINDING ANGLES WITH PARTICULAR TRIGONOMETRIC RATIOS

From **Exercise 8C** you should have discovered that:

For θ in radians:

- $\boldsymbol{\sin(\pi - \theta) = \sin\theta}$
- $\boldsymbol{\cos(\pi - \theta) = -\cos\theta}$
- $\boldsymbol{\cos(2\pi - \theta) = \cos\theta}$

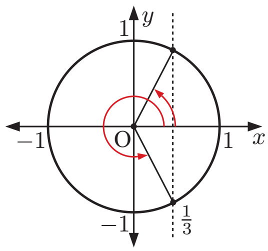
We need results such as these, and also the periodicity of the trigonometric ratios, to find angles which have a particular sine, cosine, or tangent.

Example 10 — Self Tutor

Find the two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\cos\theta = \frac{1}{3}$ **b** $\sin\theta = \frac{3}{4}$ **c** $\tan\theta = 2$

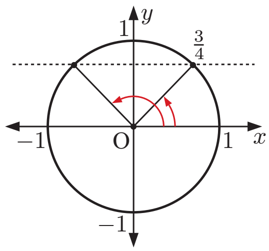
a $\cos^{-1}(\frac{1}{3}) \approx 1.23$



$\therefore \theta \approx 1.23$ or $2\pi - 1.23$

$\therefore \theta \approx 1.23$ or 5.05

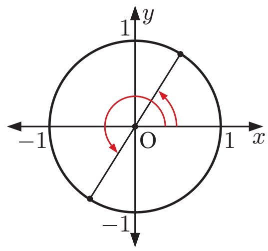
b $\sin^{-1}(\frac{3}{4}) \approx 0.848$



$\therefore \theta \approx 0.848$ or $\pi - 0.848$

$\therefore \theta \approx 0.848$ or 2.29

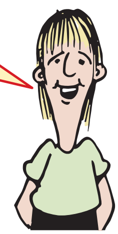
c $\tan^{-1}(2) \approx 1.11$



$\therefore \theta \approx 1.11$ or $\pi + 1.11$

$\therefore \theta \approx 1.11$ or 4.25

If $\cos\theta$, $\sin\theta$, or $\tan\theta$ is positive, your calculator will give θ in the domain $0 < \theta < \frac{\pi}{2}$.



EXERCISE 8D.2

1 Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\tan\theta = 4$ **b** $\cos\theta = 0.83$ **c** $\sin\theta = \frac{3}{5}$

d $\cos\theta = 0$ **e** $\tan\theta = 1.2$ **f** $\cos\theta = 0.7816$

g $\sin\theta = \frac{1}{11}$ **h** $\tan\theta = 20.2$ **i** $\sin\theta = \frac{39}{40}$

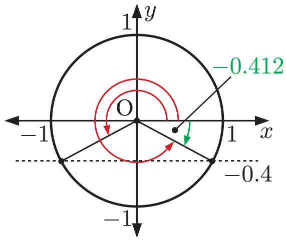
Example 11

Self Tutor

Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\sin\theta = -0.4$ **b** $\cos\theta = -\frac{2}{3}$ **c** $\tan\theta = -\frac{1}{3}$

a $\sin^{-1}(-0.4) \approx -0.412$

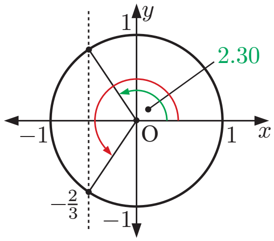


But $0 \leqslant \theta \leqslant 2\pi$

$\therefore\ \theta \approx \pi + 0.412$ or $2\pi - 0.412$

$\therefore\ \theta \approx 3.55$ or 5.87

b $\cos^{-1}(-\frac{2}{3}) \approx 2.30$

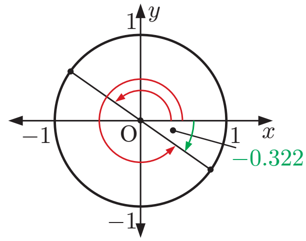


But $0 \leqslant \theta \leqslant 2\pi$

$\therefore\ \theta \approx 2.30$ or $2\pi - 2.30$

$\therefore\ \theta \approx 2.30$ or 3.98

c $\tan^{-1}(-\frac{1}{3}) \approx -0.322$



But $0 \leqslant \theta \leqslant 2\pi$

$\therefore\ \theta \approx \pi - 0.322$ or $2\pi - 0.322$

$\therefore\ \theta \approx 2.82$ or 5.96

If $\sin\theta$ or $\tan\theta$ is negative, your calculator will give θ in the domain $-\frac{\pi}{2} < \theta < 0$.

The green arrow shows the angle that your calculator gives.

2 Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\cos\theta = -\frac{1}{4}$ **b** $\sin\theta = 0$ **c** $\tan\theta = -3.1$

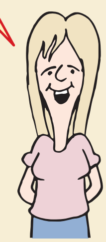
d $\sin\theta = -0.421$ **e** $\tan\theta = -6.67$ **f** $\cos\theta = -\frac{2}{17}$

g $\tan\theta = -\sqrt{5}$ **h** $\cos\theta = -\frac{1}{\sqrt{3}}$ **i** $\sin\theta = -\frac{\sqrt{2}}{\sqrt{5}}$

Discovery 2

Parametric equations

The use of parametric equations is not required for the syllabus.



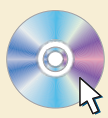
Usually we write functions in the form $y = f(x)$.

For example: $y = 3x + 7$, $y = x^2 - 6x + 8$, $y = \sin x$

However, sometimes it is useful to express **both** x and y in terms of another variable t, called the **parameter**. In this case we say we have **parametric equations**.

What to do:

PARAMETRIC PLOTTER



1 **a** Use the graphing package to plot $\{(x, y) : x = \cos t,\ y = \sin t,\ 0^\circ \leqslant t \leqslant 360^\circ\}$. Use the same scale on both axes.

b Describe the resulting graph. Is it the graph of a function?

c Evaluate $x^2 + y^2$. Hence determine the equation of this graph in terms of x and y only.

2 Use the graphing package to plot:

a $\{(x, y) : x = 2\cos t, \quad y = \sin(2t), \quad 0° \leqslant t \leqslant 360°\}$

b $\{(x, y) : x = 2\cos t, \quad y = 2\sin(3t), \quad 0° \leqslant t \leqslant 360°\}$

c $\{(x, y) : x = 2\cos t, \quad y = \cos t - \sin t, \quad 0° \leqslant t \leqslant 360°\}$

d $\{(x, y) : x = \cos^2 t + \sin 2t, \quad y = \cos t, \quad 0° \leqslant t \leqslant 360°\}$

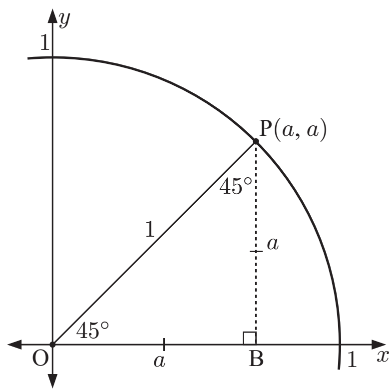
e $\{(x, y) : x = \cos^3 t, \quad y = \sin t, \quad 0° \leqslant t \leqslant 360°\}$

E MULTIPLES OF $\frac{\pi}{6}$ AND $\frac{\pi}{4}$

Angles which are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ occur frequently, so it is important for us to write their trigonometric ratios exactly.

MULTIPLES OF $\frac{\pi}{4}$ OR 45°

Triangle OBP is isosceles as angle OPB also measures 45°.



Letting OB = BP = a,

$a^2 + a^2 = 1^2$ {Pythagoras}

$\therefore \quad 2a^2 = 1$

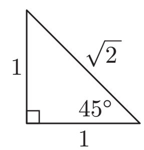
$\therefore \quad a^2 = \frac{1}{2}$

$\therefore \quad a = \frac{1}{\sqrt{2}}$ {as $a > 0$}

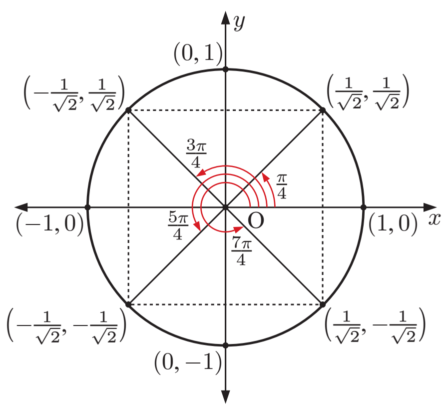
So, P is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ where $\frac{1}{\sqrt{2}} \approx 0.707$.

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

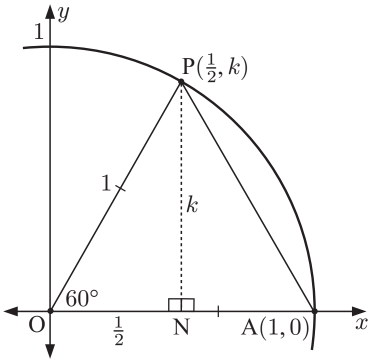
You should remember these values. If you forget, draw a right angled isosceles triangle with equal sides of length 1.



For **multiples of** $\frac{\pi}{4}$, we have:



MULTIPLES OF $\frac{\pi}{6}$ OR $30°$



Since OA = OP, triangle OAP is isosceles.

The remaining angles are therefore also $60°$, and so triangle AOP is equilateral.

The altitude [PN] bisects base [OA], so $\text{ON} = \frac{1}{2}$.

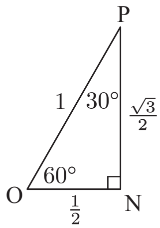
If P is $(\frac{1}{2}, k)$, then $(\frac{1}{2})^2 + k^2 = 1$ {Pythagoras}

$\therefore \; k^2 = \frac{3}{4}$

$\therefore \; k = \frac{\sqrt{3}}{2}$ {as $k > 0$}

So, P is $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ where $\frac{\sqrt{3}}{2} \approx 0.866$.

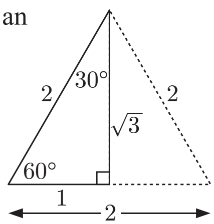
$$\cos\frac{\pi}{3} = \frac{1}{2} \quad \text{and} \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$



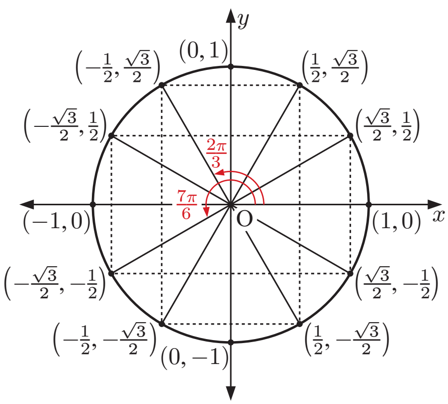
Now $\widehat{\text{NPO}} = \frac{\pi}{6} = 30°$.

Hence $$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\frac{\pi}{6} = \frac{1}{2}$$

You should remember these values. If you forget, divide in two an equilateral triangle with side length 2.



For **multiples of** $\frac{\pi}{6}$, we have:



Summary

- For **multiples of** $\frac{\pi}{2}$, the coordinates of the points on the unit circle involve 0 and ± 1.
- For *other* **multiples of** $\frac{\pi}{4}$, the coordinates involve $\pm\frac{1}{\sqrt{2}}$.
- For *other* **multiples of** $\frac{\pi}{6}$, the coordinates involve $\pm\frac{1}{2}$ and $\pm\frac{\sqrt{3}}{2}$.
- The signs of the coordinates are determined by which quadrant the angle is in.

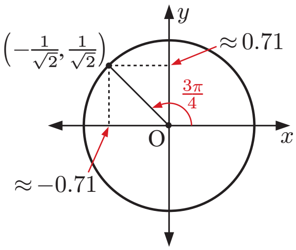
You should be able to use this summary to find the trigonometric ratios for angles which are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$.

Example 12 — Self Tutor

Find the exact values of $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ for:

a $\alpha = \frac{3\pi}{4}$ **b** $\alpha = \frac{4\pi}{3}$

a $\frac{3\pi}{4}$ is a multiple of $\frac{\pi}{4}$.
The angle lies in quadrant 2, so only $\sin\frac{3\pi}{4}$ is positive.

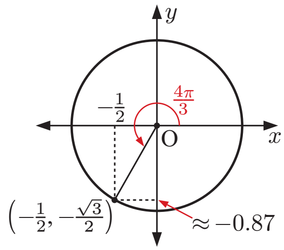


$\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$\tan\left(\frac{3\pi}{4}\right) = -1$

b $\frac{4\pi}{3}$ is a multiple of $\frac{\pi}{6}$.
The angle lies in quadrant 3, so only $\tan\frac{4\pi}{3}$ is positive.



$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$

$\tan\left(\frac{4\pi}{3}\right) = \dfrac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$

EXERCISE 8E

1 Use a unit circle diagram to find exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$, for θ equal to:

a $\frac{\pi}{4}$ **b** $\frac{\pi}{2}$ **c** $\frac{7\pi}{4}$ **d** π **e** $\frac{-3\pi}{4}$

2 Use a unit circle diagram to find exact values for $\sin\beta$, $\cos\beta$, and $\tan\beta$, for β equal to:

a $\frac{\pi}{6}$ **b** $\frac{2\pi}{3}$ **c** $\frac{7\pi}{6}$ **d** $\frac{5\pi}{3}$ **e** $\frac{11\pi}{6}$

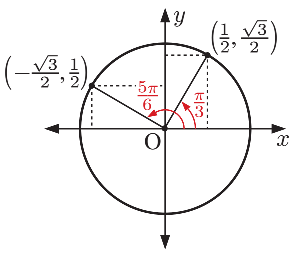
3 Find the exact values of:

a $\cos 120°$, $\sin 120°$, and $\tan 120°$ **b** $\cos(-45°)$, $\sin(-45°)$, and $\tan(-45°)$

4 **a** Find the exact values of $\cos 270°$ and $\sin 270°$.
b What can you say about $\tan 270°$?

Example 13 — Self Tutor

Without using a calculator, show that $8\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{5\pi}{6}\right) = -6$.



$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$$\begin{aligned}\therefore\ 8\sin\left(\tfrac{\pi}{3}\right)\cos\left(\tfrac{5\pi}{6}\right) &= 8\left(\tfrac{\sqrt{3}}{2}\right)\left(-\tfrac{\sqrt{3}}{2}\right)\\ &= 2(-3)\\ &= -6\end{aligned}$$

5 Without using a calculator, evaluate:

a $\sin^2 60°$ **b** $\sin 30° \cos 60°$ **c** $4 \sin 60° \cos 30°$

d $1 - \cos^2(\frac{\pi}{6})$ **e** $\sin^2(\frac{2\pi}{3}) - 1$ **f** $\cos^2(\frac{\pi}{4}) - \sin(\frac{7\pi}{6})$

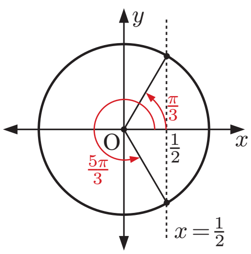
g $\sin(\frac{3\pi}{4}) - \cos(\frac{5\pi}{4})$ **h** $1 - 2\sin^2(\frac{7\pi}{6})$ **i** $\cos^2(\frac{5\pi}{6}) - \sin^2(\frac{5\pi}{6})$

j $\tan^2(\frac{\pi}{3}) - 2\sin^2(\frac{\pi}{4})$ **k** $2\tan(-\frac{5\pi}{4}) - \sin(\frac{3\pi}{2})$ **l** $\dfrac{2\tan 150°}{1 - \tan^2 150°}$

Check your answers using your calculator.

Example 14 Self Tutor

Find all angles $0 \leqslant \theta \leqslant 2\pi$ with a cosine of $\frac{1}{2}$.



Since the cosine is $\frac{1}{2}$, we draw the vertical line $x = \frac{1}{2}$.

Because $\frac{1}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

6 Find all angles between $0°$ and $360°$ with:

a a sine of $\frac{1}{2}$ **b** a sine of $\frac{\sqrt{3}}{2}$ **c** a cosine of $\frac{1}{\sqrt{2}}$

d a cosine of $-\frac{1}{2}$ **e** a cosine of $-\frac{1}{\sqrt{2}}$ **f** a sine of $-\frac{\sqrt{3}}{2}$

7 Find all angles between 0 and 2π (inclusive) which have:

a a tangent of 1 **b** a tangent of -1 **c** a tangent of $\sqrt{3}$

d a tangent of 0 **e** a tangent of $\frac{1}{\sqrt{3}}$ **f** a tangent of $-\sqrt{3}$

8 Find all angles between 0 and 4π with:

a a cosine of $\frac{\sqrt{3}}{2}$ **b** a sine of $-\frac{1}{2}$ **c** a sine of -1

9 Find θ if $0 \leqslant \theta \leqslant 2\pi$ and:

a $\cos\theta = \frac{1}{2}$ **b** $\sin\theta = \frac{\sqrt{3}}{2}$ **c** $\cos\theta = -1$ **d** $\sin\theta = 1$

e $\cos\theta = -\frac{1}{\sqrt{2}}$ **f** $\sin^2\theta = 1$ **g** $\cos^2\theta = 1$ **h** $\cos^2\theta = \frac{1}{2}$

i $\tan\theta = -\frac{1}{\sqrt{3}}$ **j** $\tan^2\theta = 3$

10 Find *all* values of θ for which $\tan\theta$ is: **a** zero **b** undefined.

F RECIPROCAL TRIGONOMETRIC RATIOS

We define the reciprocal trigonometric functions cosec θ, secant θ, and cotangent θ as:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \text{and} \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Using these definitions we can derive the identities:

$$\tan^2\theta + 1 = \sec^2\theta \quad \text{and} \quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

Proof: Using $\sin^2\theta + \cos^2\theta = 1$,

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \quad \{\text{dividing each term by } \cos^2\theta\}$$

$$\therefore \tan^2\theta + 1 = \sec^2\theta$$

Also using $\sin^2\theta + \cos^2\theta = 1$,

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \quad \{\text{dividing each term by } \sin^2\theta\}$$

$$\therefore 1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

EXERCISE 8F

1 Without using a calculator, find:

a $\operatorname{cosec}\left(\frac{\pi}{3}\right)$ **b** $\cot\left(\frac{2\pi}{3}\right)$ **c** $\sec\left(\frac{5\pi}{6}\right)$ **d** $\cot(\pi)$

e $\operatorname{cosec}\left(\frac{4\pi}{3}\right)$ **f** $\sec\left(\frac{7\pi}{4}\right)$

2 Without using a calculator, find $\operatorname{cosec} x$, $\sec x$, and $\cot x$ for:

a $\sin x = \frac{3}{5}$, $0 \leqslant x \leqslant \frac{\pi}{2}$ **b** $\cos x = \frac{2}{3}$, $\frac{3\pi}{2} < x < 2\pi$

3 Find the other *five* trigonometric ratios if:

a $\cos\theta = \frac{3}{4}$ and $\frac{3\pi}{2} < \theta < 2\pi$ **b** $\sin x = -\frac{2}{3}$ and $\pi < x < \frac{3\pi}{2}$

c $\sec x = 2\frac{1}{2}$ and $0 < x < \frac{\pi}{2}$ **d** $\operatorname{cosec}\theta = 2$ and $\frac{\pi}{2} < \theta < \pi$

e $\tan\beta = \frac{1}{2}$ and $\pi < \beta < \frac{3\pi}{2}$ **f** $\cot\theta = \frac{4}{3}$ and $\pi < \theta < \frac{3\pi}{2}$

4 Find *all* values of θ for which:

a $\operatorname{cosec}\theta$ is undefined **b** $\sec\theta$ is undefined

c $\cot\theta$ is zero **d** $\cot\theta$ is undefined.

Review set 8A

1 Convert these to radians in terms of π:

a $120°$ **b** $225°$ **c** $150°$ **d** $540°$

2 Find the acute angles that would have the same:

a sine as $\frac{2\pi}{3}$ **b** sine as $165°$ **c** cosine as $276°$.

3 Find:

a $\sin 159°$ if $\sin 21° \approx 0.358$
b $\cos 92°$ if $\cos 88° \approx 0.035$
c $\cos 75°$ if $\cos 105° \approx -0.259$
d $\sin(-133°)$ if $\sin 47° \approx 0.731$

4 Determine the area of a sector of angle $\frac{5\pi}{12}$ and radius 13 cm.

5 Use the unit circle to find θ such that $\cos\theta = -\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$.

6 Find exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$ for θ equal to:

a $360°$ **b** $\frac{2\pi}{3}$ **c** $-\pi$ **d** $\frac{8\pi}{3}$

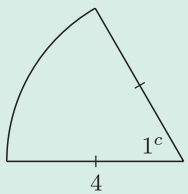
7 If $\cos\theta = \frac{3}{4}$ find the possible values of $\sin\theta$.

8 Evaluate:

a $2\sin(\frac{\pi}{3})\cos(\frac{\pi}{3})$ **b** $\tan^2(\frac{\pi}{4}) - 1$ **c** $\cos^2(\frac{\pi}{6}) - \sin^2(\frac{\pi}{6})$

9 Given $\tan x = -\frac{3}{2}$ and $\frac{3\pi}{2} < x < 2\pi$, find: **a** $\cos x$ **b** $\sin x$.

10



Find the perimeter and area of the sector.

11 Suppose $\cos\theta = \frac{\sqrt{11}}{\sqrt{17}}$ and θ is acute. Find the exact value of $\tan\theta$.

12 Find all angles between $0°$ and $360°$ which have:

a a cosine of $-\frac{\sqrt{3}}{2}$ **b** a secant of $\sqrt{2}$ **c** a cotangent of $-\frac{1}{\sqrt{3}}$

13 Find θ for $0 \leqslant \theta \leqslant 2\pi$ if:

a $\cos\theta = -1$ **b** $\sin^2\theta = \frac{3}{4}$

14 If $\sin x = -\frac{1}{4}$ and $\pi < x < \frac{3\pi}{2}$, find the other *five* trigonometric ratios exactly.

Review set 8B

1 Convert these radian measurements to degrees:

a $\frac{2\pi}{5}$ **b** $\frac{5\pi}{4}$ **c** $\frac{7\pi}{9}$ **d** $\frac{11\pi}{6}$

2 Illustrate the regions where $\sin\theta$ and $\cos\theta$ have the same sign.

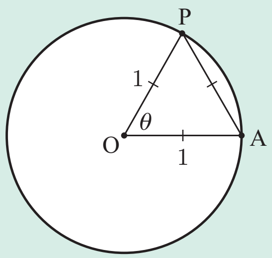
3 Use a unit circle diagram to find:

a $\cos(\frac{3\pi}{2})$ and $\sin(\frac{3\pi}{2})$
b $\cos(-\frac{\pi}{2})$ and $\sin(-\frac{\pi}{2})$

4 Suppose $m = \sin p$, where p is acute. Write an expression in terms of m for:

a $\sin(\pi - p)$ **b** $\sin(p + 2\pi)$ **c** $\cos p$ **d** $\tan p$

5



a State the value of θ in:

i degrees **ii** radians.

b State the arc length AP.

c State the area of the minor *sector* OAP.

6 Show that $\cos(\frac{3\pi}{4}) - \sin(\frac{3\pi}{4}) = -\sqrt{2}$.

7 If $\cos\theta = -\frac{2}{5}$, $\frac{\pi}{2} < \theta < \pi$ find the other *five* trigonometric ratios exactly.

8 Without using a calculator, evaluate:

a $\tan^2 60° - \sin^2 45°$ **b** $\cos^2(\frac{\pi}{4}) + \sin(\frac{\pi}{2})$ **c** $\cos(\frac{5\pi}{3}) - \tan(\frac{5\pi}{4})$

9 Find two angles on the unit circle with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\cos\theta = \frac{2}{3}$ **b** $\sin\theta = -\frac{1}{4}$ **c** $\tan\theta = 3$

10 Find the perimeter and area of a sector of radius 11 cm and angle 63°.

11 Find the radius and area of a sector of perimeter 36 cm with an angle of $\frac{2\pi}{3}$.

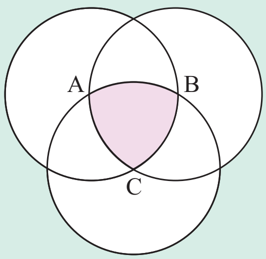
12 Simplify:

a $\sin(\pi - \theta) - \sin\theta$ **b** $\cos\theta\tan\theta$

13 If $\sec\alpha = -3\frac{1}{3}$ and $0 < \alpha < \pi$, find the other *five* trigonometric ratios exactly.

14 Three circles with radius r are drawn as shown, each with its centre on the circumference of the other two circles. A, B, and C are the centres of the three circles.

Prove that an expression for the area of the shaded region is $A = \dfrac{r^2}{2}(\pi - \sqrt{3})$.



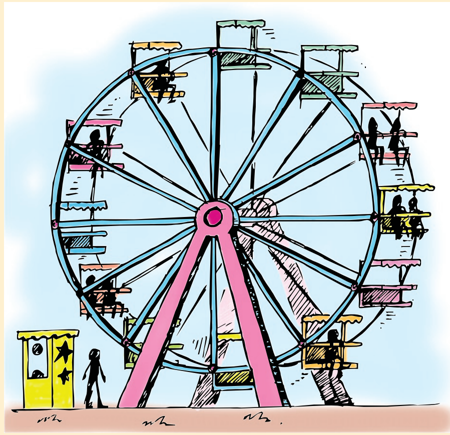
9

Trigonometric functions

Contents:

Opening problem

A Ferris wheel rotates at a constant speed. The wheel's radius is 10 m and the bottom of the wheel is 2 m above ground level. From a point in front of the wheel, Andrew is watching a green light on the perimeter of the wheel. Andrew notices that the green light moves in a circle. He estimates how high the light is above ground level at two second intervals, and draws a scatter diagram of his results.



Things to think about:

- **a** What will Andrew's scatter diagram look like?
- **b** What function can be used to model the data?
- **c** How could this function be used to find:
  - **i** the light's position at any point in time
  - **ii** the times when the light is at its maximum and minimum heights?
- **d** What part of the function indicates the time for one full revolution of the wheel?

Click on the icon to visit a simulation of the Ferris wheel. You will be able to view the light from:

DEMO

- in front of the wheel
- a side-on position
- above the wheel.

You can then observe the graph of the green light's position as the wheel rotates at a constant rate.

A PERIODIC BEHAVIOUR

Periodic phenomena occur all the time in the physical world. Their behaviour repeats again and again over time.

We see periodic behaviour in:

- seasonal variations in our climate
- variations in average maximum and minimum monthly temperatures
- the number of daylight hours at a particular location
- tidal variations in the depth of water in a harbour
- the phases of the moon
- animal populations.

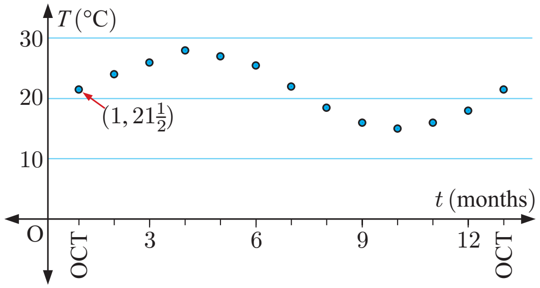
In this chapter we will see how trigonometric functions can be used to model periodic phenomena.

OBSERVING PERIODIC BEHAVIOUR

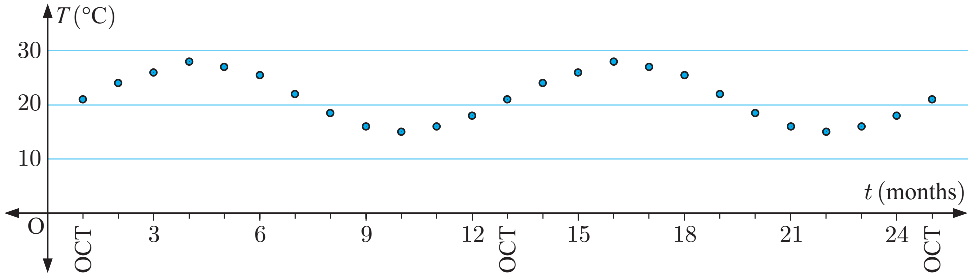
The table below shows the mean monthly maximum temperature for Cape Town, South Africa.

Month	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep
Temperature T (°C)	$21\frac{1}{2}$	24	26	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18

On the scatter diagram alongside we plot the temperature T on the vertical axis. We assign October as $t = 1$ month, November as $t = 2$ months, and so on for the rest of the year.

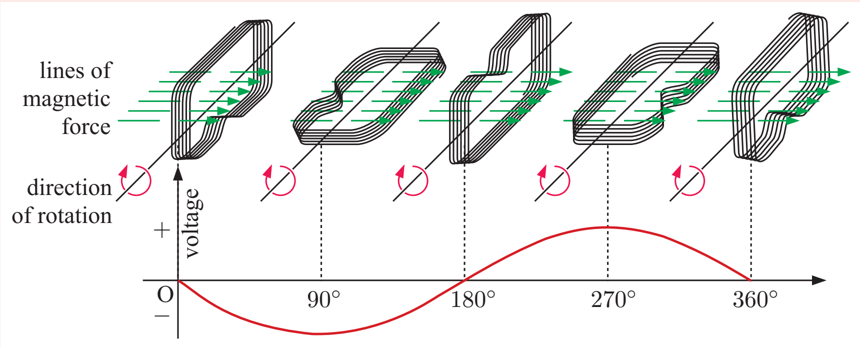


The cycle will approximately repeat itself for each subsequent 12 month period. By the end of the chapter we will be able to establish a **periodic function** which approximately fits this set of points.



Graphs with this basic shape, where the cycle is repeated over and over, are called **sine waves**.

Historical note



In 1831 **Michael Faraday** discovered that an electric current was generated by rotating a coil of wire in a magnetic field. The electric current produced showed a voltage which varied between positive and negative values as the coil rotated through 360°.

GATHERING PERIODIC DATA

Data on a number of periodic phenomena can be found online or in other publications. For example:

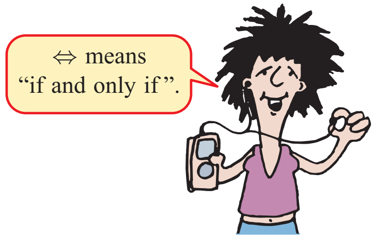
- Maximum and minimum monthly temperatures can be found at www.weatherbase.com
- Tidal details can be obtained from daily newspapers or internet sites such as http://tidesandcurrents.noaa.gov or http://www.bom.gov.au/oceanography

TERMINOLOGY USED TO DESCRIBE PERIODICITY

A **periodic function** is one which repeats itself over and over in a horizontal direction, in intervals of the same length.

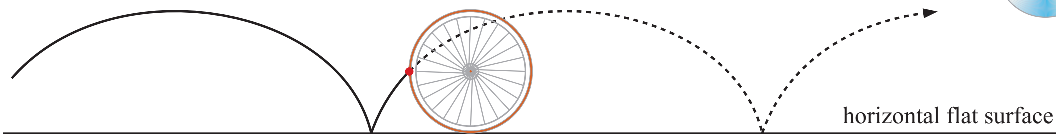
The **period** of a periodic function is the length of one repetition or cycle.

$f(x)$ is a periodic function with period $p \Leftrightarrow \boldsymbol{f(x+p) = f(x)}$ for all x, and p is the smallest positive value for this to be true.



A **cycloid** is an example of a periodic function. It is the curve traced out by a point on a circle as the circle rolls across a flat surface in a straight line.

DEMO



Use a **graphing package** to examine the function $f(x) = x - [x]$ where $[x]$ is "the largest integer less than or equal to x".
Is $f(x)$ periodic? What is its period?

GRAPHING PACKAGE

WAVES

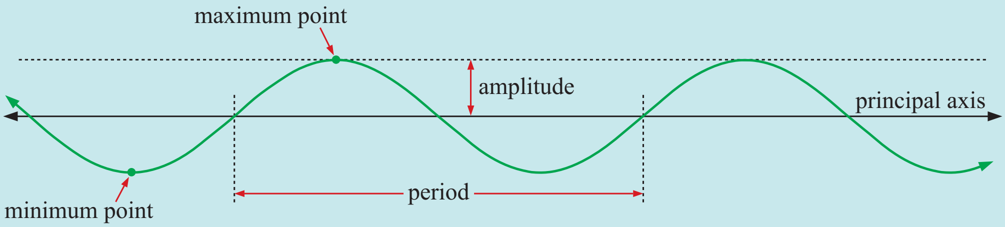
In this course we are mainly concerned with periodic phenomena which show a wave pattern.

The wave oscillates about a horizontal line called the **principal axis** or **mean line** which has equation $\boldsymbol{y = \dfrac{\textbf{max} + \textbf{min}}{2}}$.

A **maximum point** occurs at the top of a crest, and a **minimum point** at the bottom of a trough.

The **amplitude** is the distance between a maximum (or minimum) point and the principal axis.

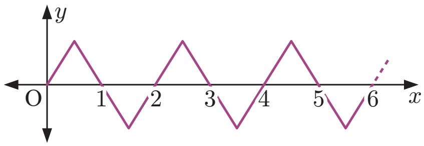
$$\textbf{amplitude} = \frac{\textbf{max} - \textbf{min}}{\textbf{2}}$$



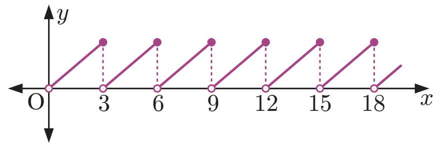
EXERCISE 9A

1 Which of these graphs show periodic behaviour?

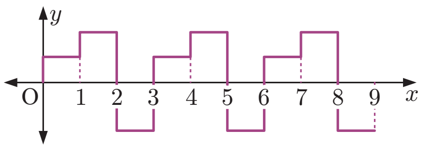
a



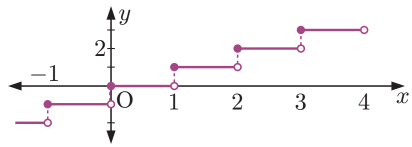
b



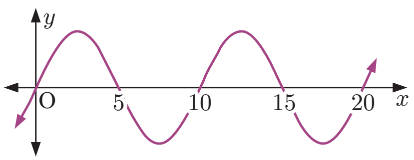
c



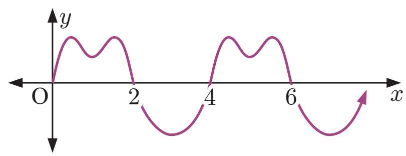
d



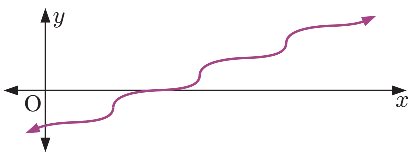
e



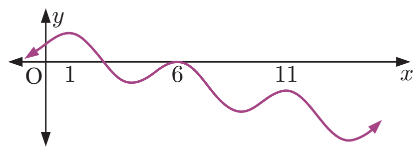
f



g



h



2 The table below shows the height above the ground of a point on a bicycle wheel as it is rolled along a flat surface.

Distance travelled (cm)	0	20	40	60	80	100	120	140	160	180	200
Height above ground (cm)	0	6	23	42	57	64	59	43	23	7	1

Distance travelled (cm)	220	240	260	280	300	320	340	360	380	400
Height above ground (cm)	5	27	40	55	63	60	44	24	9	3

a Plot the graph of height against distance.

b Is it reasonable to fit a curve to this data, or should we leave it as discrete points?

c Is the data periodic? If so, estimate:

i the equation of the principal axis
ii the maximum value
iii the period
iv the amplitude.

3 Draw a scatter diagram for each set of data below. Is there evidence to suggest the data is periodic?

a

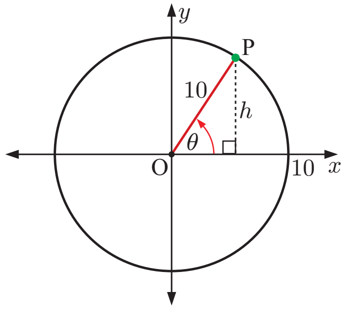
x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	1	1.4	1	0	−1	−1.4	−1	0	1	1.4	1	0

b

x	0	2	3	4	5	6	7	8	9	10	12
y	0	4.7	3.4	1.7	2.1	5.2	8.9	10.9	10.2	8.4	10.4

B THE SINE FUNCTION

In previous studies of trigonometry we have only considered static situations where an angle is fixed. However, when an object moves around a circle, the situation is dynamic. The angle θ between the radius [OP] and the positive x-axis continually changes with time.



Consider again the **Opening Problem** in which a Ferris wheel of radius 10 m revolves at constant speed. We let P represent the green light on the wheel.

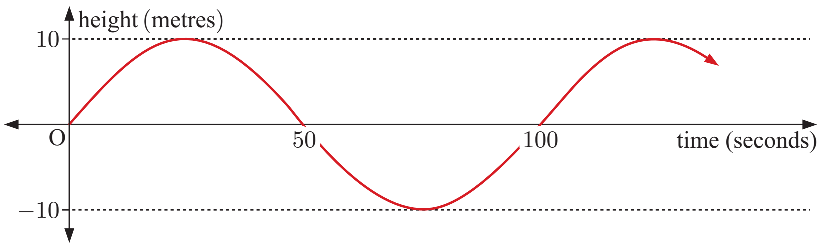
The height of P relative to the x-axis can be determined using right angled triangle trigonometry:

$$\sin\theta = \frac{h}{10}, \quad \text{so} \quad h = 10\sin\theta.$$

As time goes by, θ changes and so does h.

So, we can write h as a function of θ, or alternatively we can write h as a function of time t.

For example, suppose the Ferris wheel observed by Andrew takes 100 seconds for a full revolution. The graph below shows the height of the light above or below the principal axis against the time in seconds.

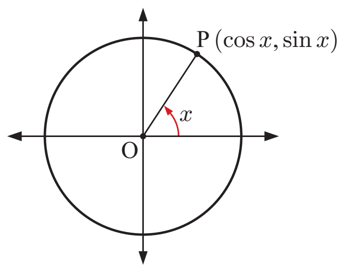


DEMO

We observe that the amplitude is 10 metres and the period is 100 seconds.

THE BASIC SINE CURVE $y = \sin x$

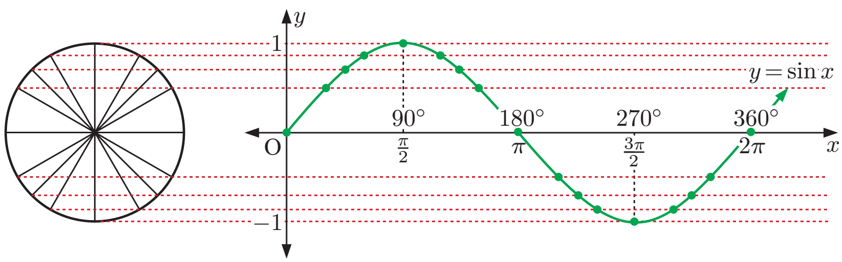
Suppose point P moves around the unit circle so the angle [OP] makes with the positive horizontal axis is x. In this case P has coordinates $(\cos x, \sin x)$.



If we project the values of $\sin x$ from the unit circle to a set of axes alongside, we can obtain the graph of $y = \sin x$.

Note carefully that x on the unit circle diagram is an *angle*, and becomes the horizontal coordinate of the sine function.

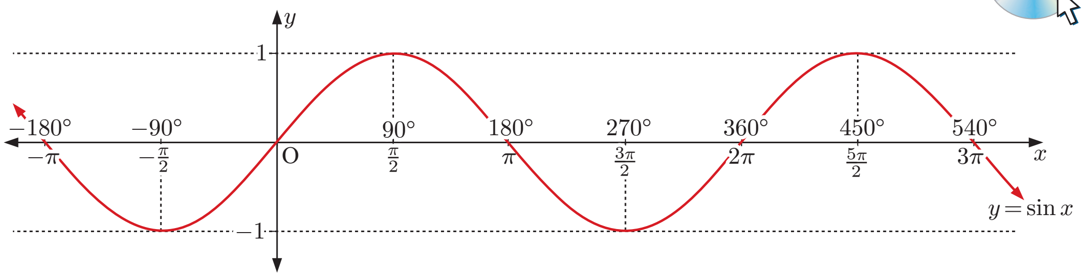
Unless indicated otherwise, you should assume that x is measured in radians. Degrees are only included on this graph for the sake of completeness.



Click on the icon to generate the sine function for yourself.

SINE FUNCTION

You should observe that the sine function can be continued beyond $0 \leqslant x \leqslant 2\pi$ in either direction.



The unit circle repeats itself after one full revolution, so the **period** of $y = \sin x$ is 2π.

The *maximum* value is 1 and the *minimum* is -1, as $-1 \leqslant y \leqslant 1$ on the unit circle.

The **amplitude** of $y = \sin x$ is 1.

TRANSFORMATIONS OF THE SINE CURVE

In the **Discoveries** that follow, we will consider different transformations of the sine curve $y = \sin x$. We will hence be able to generate the curve for the general sine function $y = a \sin bx + c$, $a > 0$, $b > 0$.

Discovery 1 — The family $y = a \sin x$, $a > 0$

Click on the icon to explore the family $y = a \sin x$, $a > 0$.

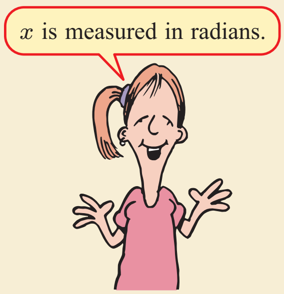
DYNAMIC SINE FUNCTION

What to do:

1 Use the slider to vary the value of a. Observe the changes to the graph of the function.

2 Use the software to help complete the table:

a	*Function*	*Maximum*	*Minimum*	*Period*	*Amplitude*
1	$y = \sin x$	1	-1	2π	1
2	$y = 2 \sin x$				
3	$y = 3 \sin x$				
0.5	$y = 0.5 \sin x$				
a	$y = a \sin x$				



3 How does a affect the function $y = a \sin x$?

Discovery 2 — The family $y = \sin bx$, $b > 0$

Click on the icon to explore the family $y = \sin bx$, $b > 0$.

DYNAMIC SINE FUNCTION

What to do:

1 Use the slider to vary the value of b. Observe the changes to the graph of the function.

2 Use the software to help complete the table:

b	*Function*	*Maximum*	*Minimum*	*Period*	*Amplitude*
1	$y = \sin x$	1	-1	2π	1
2	$y = \sin 2x$				
3	$y = \sin 3x$				
$\frac{1}{2}$	$y = \sin(\frac{1}{2}x)$				
b	$y = \sin bx$				

3 How does b affect the function $y = \sin bx$?

Discovery 3 — *The family* $y = \sin x + c$

Click on the icon to explore the family $y = \sin x + c$.

DYNAMIC SINE FUNCTION

What to do:

1 Use the slider to vary the value of c. Observe the changes to the graph of the function.

2 Use the software to help complete the table:

d	*Function*	*Maximum*	*Minimum*	*Period*	*Amplitude*
0	$y = \sin x$	1	-1	2π	1
3	$y = \sin x + 3$				
-2	$y = \sin x - 2$				
d	$y = \sin x + c$				

3 How does c affect the function $y = \sin x + c$?

THE GENERAL SINE FUNCTION

The **general sine function** is

$$\boldsymbol{y = a \sin bx + c} \quad \text{where } a > 0, \; b > 0.$$

affects **amplitude** | affects **period** | affects **vertical translation**

The **principal axis** of the general sine function is $\boldsymbol{y = c}$.

The **period** of the general sine function is $\boldsymbol{\dfrac{2\pi}{b}}$.

The **amplitude** of the general sine function is $\boldsymbol{a}$.

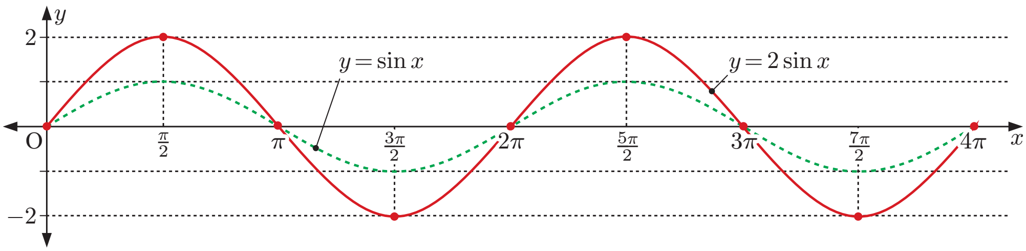
Example 1

Self Tutor

Without using technology, sketch the following graphs for $0 \leqslant x \leqslant 4\pi$:

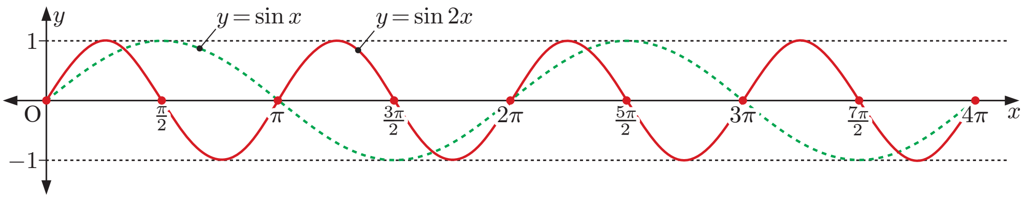
a $y = 2\sin x$ **b** $y = \sin 2x$ **c** $y = \sin x - 1$

a The amplitude is 2 and the period is 2π.



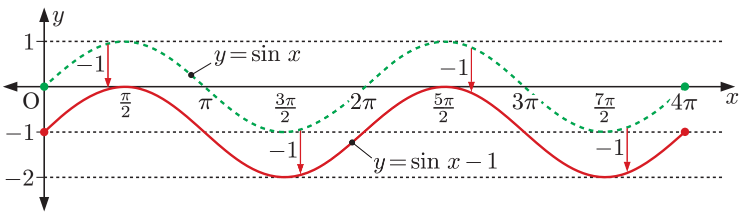
b The period is $\frac{2\pi}{2} = \pi$.

$\therefore$ the maximum values are π units apart.



Since $\sin 2x$ has half the period of $\sin x$, the first maximum is at $\frac{\pi}{4}$ not $\frac{\pi}{2}$.

c This is a vertical translation of $y = \sin x$ downwards by 1 unit. The principal axis is now $y = -1$.



EXERCISE 9B

GRAPHING PACKAGE

1 Without using technology, sketch the following graphs for $0 \leqslant x \leqslant 4\pi$:

a $y = 3\sin x$ **b** $y = 4\sin x$ **c** $y = \sin 3x$

d $y = \sin 4x$ **e** $y = \sin x + 2$ **f** $y = \sin x - 3$

Check your answers using technology.

2 Find the value of a given that the function $y = a\sin x$, $a > 0$, has amplitude:

a 2 **b** 5 **c** 11

3 Find the value of b given that the function $y = \sin bx$, $b > 0$, has period:

a $\frac{2\pi}{3}$ **b** $\frac{2\pi}{5}$ **c** $\frac{\pi}{3}$ **d** $\frac{\pi}{2}$

4 Find the value of c given that the function $y = \sin x + c$ has principal axis:

a $y = 3$ **b** $y = -1$ **c** $y = 5$

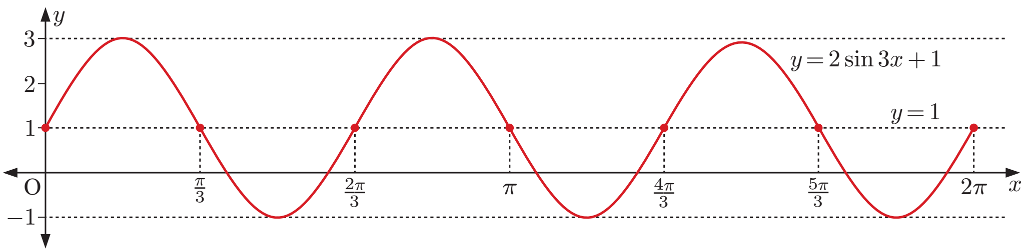
Example 2

Self Tutor

Without using technology, sketch $y = 2\sin 3x + 1$ for $0 \leqslant x \leqslant 2\pi$.

We start with $y = \sin x$. We then:

- double the amplitude to produce $y = 2\sin x$, then
- divide the period by 3 to produce $y = 2\sin 3x$, then
- translate the graph 1 unit upwards to produce $y = 2\sin 3x + 1$, so the principal axis is now $y = 1$.



5 Without using technology, sketch the following graphs for $0 \leqslant x \leqslant 2\pi$:

a $y = 3\sin x - 1$ **b** $y = 2\sin 3x$ **c** $y = \sin 2x + 3$

d $y = 3\sin 2x - 1$ **e** $y = 5\sin 2x + 3$ **f** $y = 4\sin 3x - 2$

Check your answers using technology.

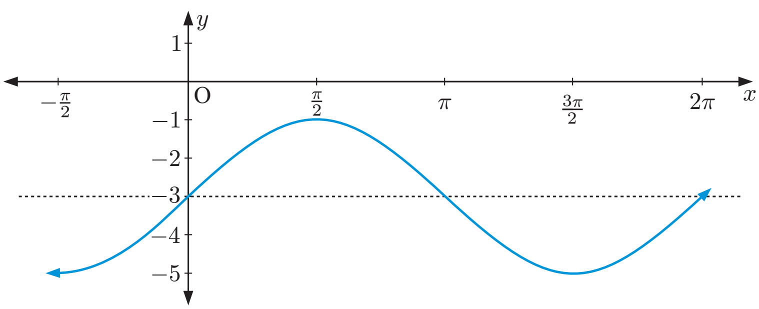
6 Find a, b, and c given that the function $y = a\sin bx + c$, $a > 0$, $b > 0$, has:

a amplitude 3, period 2π, and principal axis $y = 0$

b amplitude 2, period $\frac{2\pi}{5}$, and principal axis $y = 6$

c amplitude 5, period $\frac{2\pi}{3}$, and principal axis $y = -2$.

7 Find m and n given the following graph of the function $y = m\sin x + n$.



8 On the same set of axes, sketch for $0 \leqslant x \leqslant 2\pi$:

a $y = \sin x$ and $y = |\sin x|$ **b** $y = 3\sin 2x$ and $y = |3\sin 2x|$

Discovery 4 — Modelling using sine functions

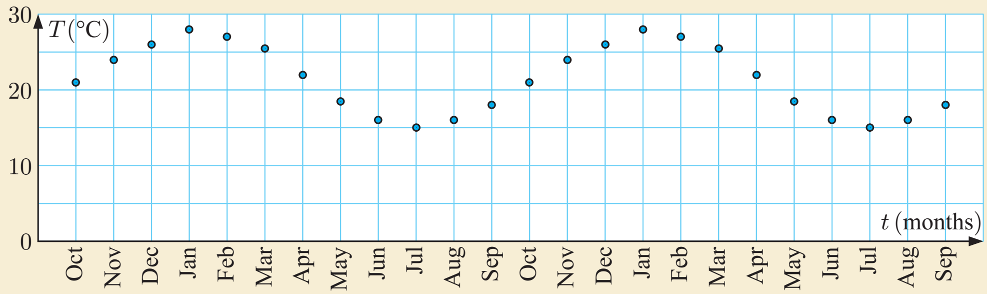
When patterns of variation can be identified and quantified using a formula or equation, predictions may be made about behaviour in the future. Examples of this include tidal movement which can be predicted many months ahead, and the date of a future full moon.

What to do:

1 Consider again the mean monthly maximum temperature for Cape Town:

Month	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep
Temperature T (°C)	$21\frac{1}{2}$	24	26	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18

The graph over a two year period is shown below:



We attempt to model this data using the general sine function $y = a\sin bx + c$,
or in this case $T = a\sin bt + c$.

a State the period of the function. Hence show that $b = \frac{\pi}{6}$.

b Use the amplitude to show that $a \approx 6.5$.

c Use the principal axis to show that $c \approx 21.5$.

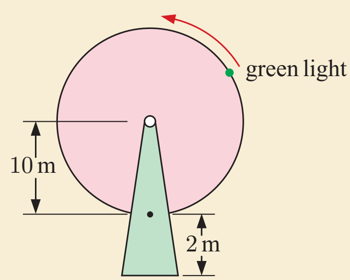
d Superimpose the model $T \approx 6.5\sin(\frac{\pi}{6}t) + 21.5$ on the original data to confirm its accuracy.

2 Some of the largest tides in the world are observed in Canada's Bay of Fundy. The difference between high and low tides is 14 metres, and the average time difference between high tides is about 12.4 hours.
Suppose the mean tide occurs at midnight.

a Find a sine model for the height of the tide H in terms of the time t.

b Sketch the graph of the model over one period.

3 Revisit the **Opening Problem** on page **226**.
The wheel takes 100 seconds to complete one revolution. Find the sine model which gives the height of the light above the ground at any point in time. Assume that at time $t = 0$, the light is at its mean position.



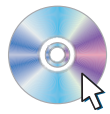
C THE COSINE FUNCTION

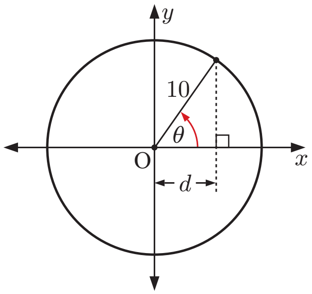
We return to the Ferris wheel and now view the movement of the green light from above.

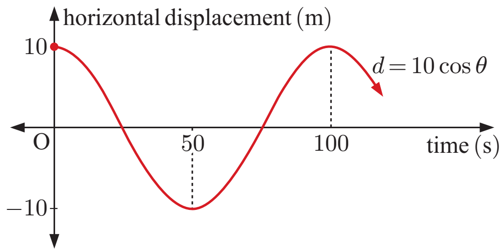
Now $\cos\theta = \frac{d}{10}$ so $d = 10\cos\theta$.

The graph being generated over time is therefore a **cosine function**.

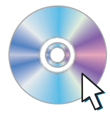
DEMO







GRAPHING PACKAGE



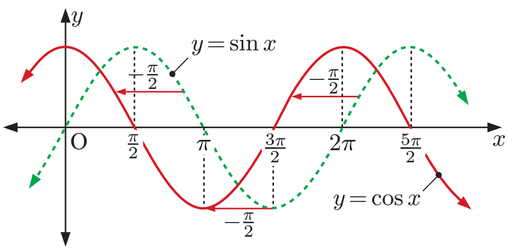
Use the graphing package to graph $y = \cos x$ and $y = \sin x$ on the same set of axes.

Like the sine curve $y = \sin x$, the cosine curve $y = \cos x$ has a **period** of 2π, an **amplitude** of 1, and its **range** is $-1 \leqslant y \leqslant 1$.

You should observe that $y = \cos x$ and $y = \sin x$ are identical in shape, but the cosine function is $\frac{\pi}{2}$ units left of the sine function.

Use the graphing package to graph $y = \cos x$ and $y = \sin\left(x + \frac{\pi}{2}\right)$ on the same set of axes.

You should observe that $\boldsymbol{\cos x = \sin\left(x + \frac{\pi}{2}\right)}$.

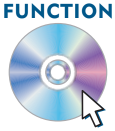


THE GENERAL COSINE FUNCTION

The **general cosine function** is $y = a\cos bx + c$ where $a > 0$, $b > 0$.

Since the cosine function is a horizontal translation of the sine function, the constants a, b, and c have the same effects as for the general sine function. Click on the icon to check this.

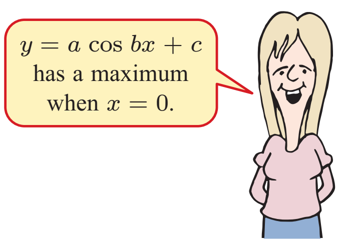
DYNAMIC COSINE FUNCTION



The **principal axis** of the general cosine function is $\boldsymbol{y = c}$.

The **period** of the general cosine function is $\boldsymbol{\frac{2\pi}{b}}$.

The **amplitude** of the general cosine function is $\boldsymbol{a}$.



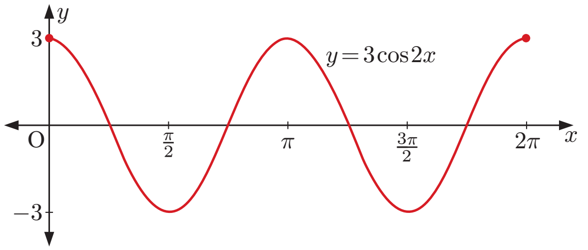
Example 3

Self Tutor

Without using technology, sketch the graph of $y = 3\cos 2x$ for $0 \leqslant x \leqslant 2\pi$.

$a = 3$, so the amplitude is 3.

$b = 2$, so the period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$.



EXERCISE 9C

1 Without using technology, sketch the following graphs for $0 \leqslant x \leqslant 2\pi$:

a $y = 3\cos x$ **b** $y = 5\cos x$ **c** $y = \cos 2x$

d $y = \cos 3x$ **e** $y = \cos x + 2$ **f** $y = \cos x - 1$

g $y = 2\cos 2x$ **h** $y = \cos 3x + 1$ **i** $y = 4\cos x + 10$

j $y = 2\cos 3x + 4$ **k** $y = 4\cos 2x - 2$ **l** $y = 3\cos 2x + 5$

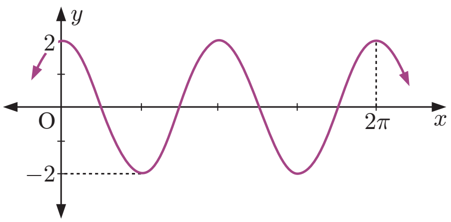
2 Find a, b, and c given that the function $y = a\cos bx + c$, $a > 0$, $b > 0$, has:

a amplitude 4, period $\frac{2\pi}{3}$, and principal axis $y = -1$

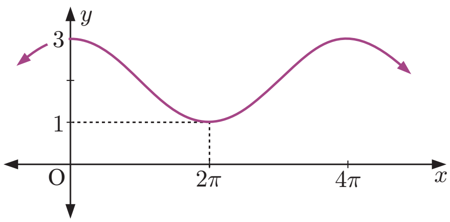
b amplitude 3, period $\frac{2\pi}{5}$, and principal axis $y = 3$.

3 Find the cosine function shown in the graph:

a



b

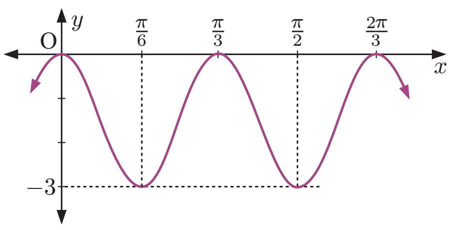


4 The function $y = a\cos bx + c$, $a > 0$, $b > 0$, has amplitude 5, period 2π, and principal axis $y = 1$.

a Find the values of a, b, and c.

b Sketch the function for $0 \leqslant x \leqslant 2\pi$.

5 The graph shown has the form $y = a\cos bx + c$ where $a > 0$, $b > 0$.



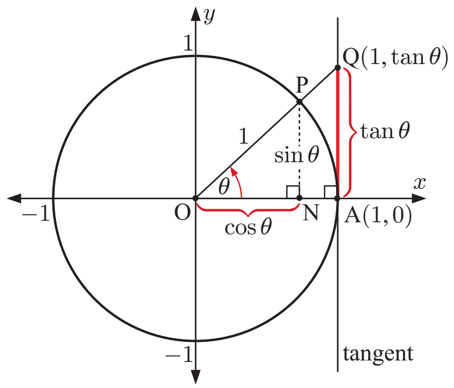
a Find the values of a, b, and c.

b Sketch the reflection of the function in the x-axis.

c Write down the equation of the reflection in **b**.

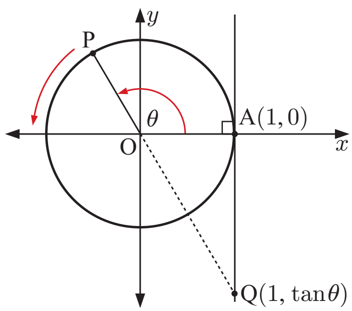
D THE TANGENT FUNCTION

We have seen that if $P(\cos\theta, \sin\theta)$ is a point which is free to move around the unit circle, and if [OP] is extended to meet the tangent at $A(1, 0)$, the intersection between these lines occurs at $Q(1, \tan\theta)$.



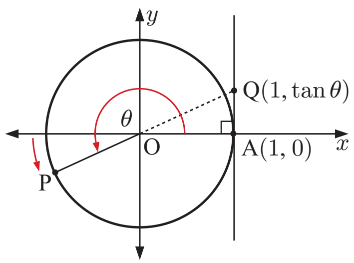
This enables us to define the **tangent function**

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

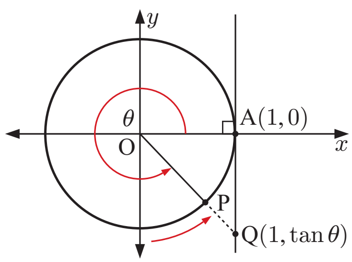


For θ in quadrant 2, $\sin\theta$ is positive and $\cos\theta$ is negative and so $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is negative.

As before, [OP] is extended to meet the tangent at A at $Q(1, \tan\theta)$. We see that Q is below the x-axis.



For θ in quadrant 3, $\sin\theta$ and $\cos\theta$ are both negative and so $\tan\theta$ is positive. This is clearly demonstrated as Q is back above the x-axis.



For θ in quadrant 4, $\sin\theta$ is negative and $\cos\theta$ is positive. $\tan\theta$ is again negative. We see that Q is below the x-axis.

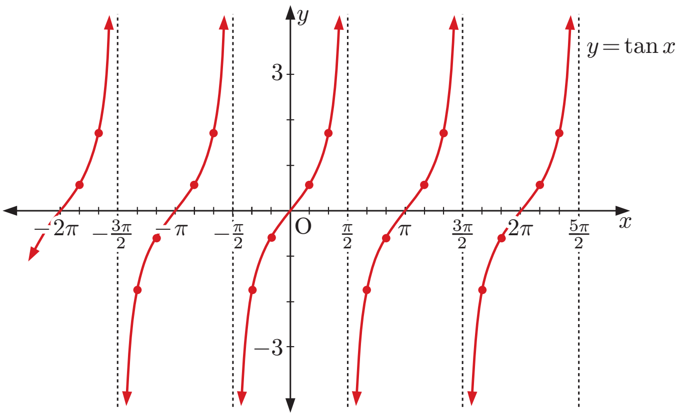
Discussion

What happens to $\tan\theta$ when P is at $(0, 1)$ and $(0, -1)$?

THE GRAPH OF $y = \tan x$

Since $\tan x = \dfrac{\sin x}{\cos x}$, $\tan x$ will be undefined whenever $\cos x = 0$.

The zeros of the function $y = \cos x$ correspond to vertical asymptotes of the function $y = \tan x$.



DEMO

We observe that $y = \tan x$ has:

- **period** π
- **range** $y \in \mathbb{R}$
- **vertical asymptotes** $x = \frac{\pi}{2} + k\pi$ for all $k \in \mathbb{Z}$.

TANGENT FUNCTION

Click on the icon to explore how the tangent function is produced from the unit circle.

THE GENERAL TANGENT FUNCTION

The **general tangent function** is $y = a\tan bx + c$, $a > 0$, $b > 0$.

- The **principal axis** is $y = c$.
- The **period** of this function is $\dfrac{\pi}{b}$.
- The **amplitude** of this function is undefined.

DYNAMIC TANGENT FUNCTION

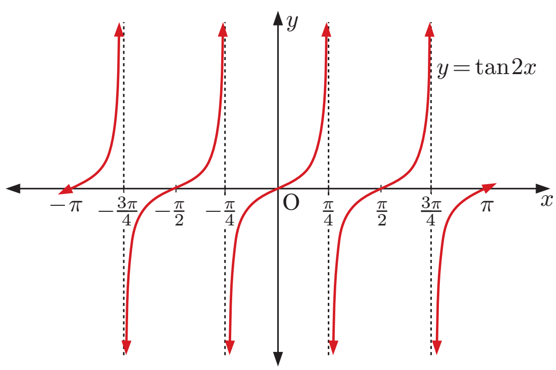
Click on the icon to explore the properties of this function.

Example 4 — Self Tutor

Without using technology, sketch the graph of $y = \tan 2x$ for $-\pi \leqslant x \leqslant \pi$.

Since $b = 2$, the period is $\frac{\pi}{2}$.

The vertical asymptotes are
$x = \pm\frac{\pi}{4}$, $x = \pm\frac{3\pi}{4}$,
and the x-axis intercepts are at
$0, \pm\frac{\pi}{2}, \pm\pi$.



Discussion

- Discuss how to find the x-intercepts of $y = \tan x$.
- How can we simplify $\tan(x - \pi)$?
- How many solutions does the equation $\tan x = 2$ have?

EXERCISE 9D

1 *Sketch* the following functions for $-\pi \leqslant x \leqslant \pi$:

GRAPHING PACKAGE

a $y = 2\tan x$ **b** $y = \tan 3x$ **c** $y = \tan x + 2$

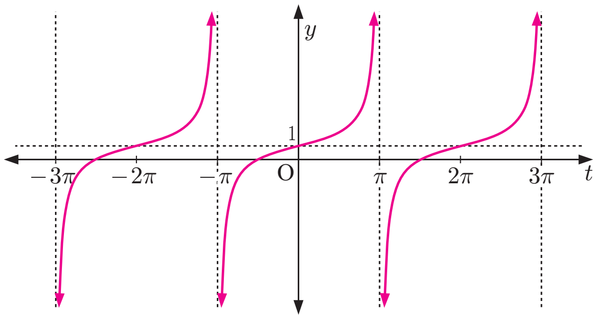
d $y = 3\tan 2x$ **e** $y = 2\tan x - 1$ **f** $y = 2\tan 3x + 2$

Use technology to check your answers.

2 Find b and c given that the function $y = \tan bx + c$, $b > 0$, has:

a period $\frac{2\pi}{3}$ and principal axis $y = 2$ **b** period $\frac{\pi}{2}$ and principal axis $y = -3$.

3 Find p and q given the following graph of the function $y = \tan pt + q$.



Activity

Click on the icon to run a card game for trigonometric functions.

CARD GAME

E TRIGONOMETRIC EQUATIONS

Linear equations such as $2x + 3 = 11$ have exactly one solution. Quadratic equations of the form $ax^2 + bx + c = 0$, $a \neq 0$ have at most two real solutions.

Trigonometric equations generally have infinitely many solutions unless a restricted domain such as $0 \leqslant x \leqslant 3\pi$ is given.

For example, suppose that Andrew in the **Opening Problem** wants to know when the green light will be 16 metres above the ground. To find out, he will need to solve a trigonometric equation. If the wheel keeps rotating, the equation would have infinitely many solutions. Andrew may therefore specify that he is interested in the *first* time the green light is 16 metres above the ground.

If a periodic function $f(x)$ has period p then the domain $0 \leqslant x < p$ is called the **principal domain**. By solving an equation on the principal domain, all the other solutions can be found using the periodic behaviour.

If $x = a$ is a solution, then $x = a + kp$ will also be a solution for all $k \in \mathbb{Z}$.

For example, $\sin x$ has period 2π, so it is normal to consider the domain $0 \leqslant x < 2\pi$.

Discussion

What would you choose as the principal domain for:

- $y = \cos x$
- $y = \sin(2x)$
- $y = \tan x$?

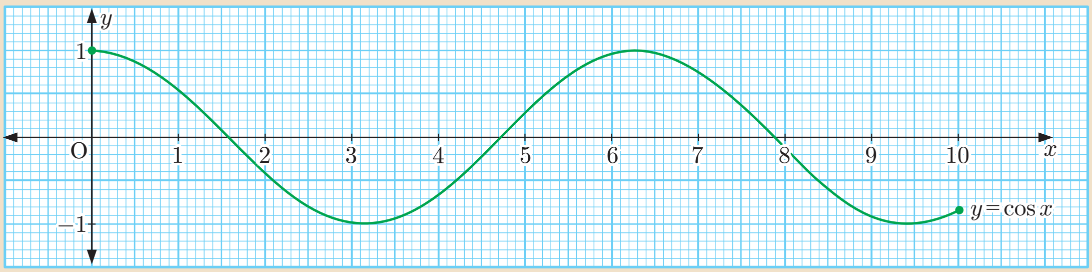
GRAPHICAL SOLUTION OF TRIGONOMETRIC EQUATIONS

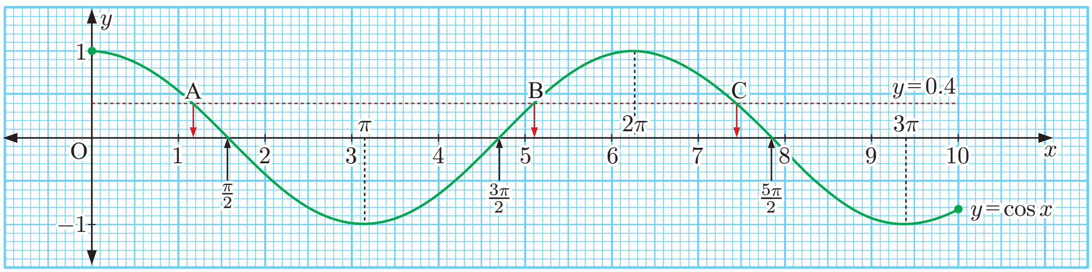
Sometimes simple trigonometric graphs are available on grid paper. In such cases we can estimate solutions straight from the graph.

Example 5

Self Tutor

Solve $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$ radians using the graph of $y = \cos x$.





$y = 0.4$ meets $y = \cos x$ at A, B, and C. Hence $x \approx 1.2$, 5.1, or 7.4.

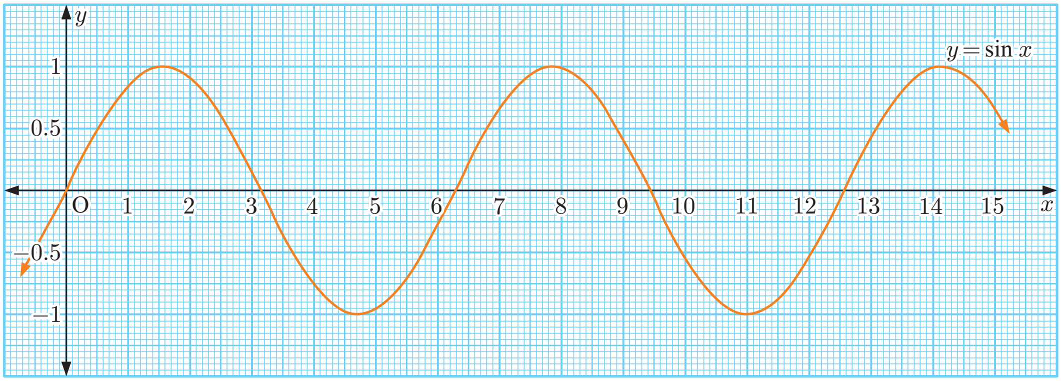
The solutions of $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$ radians are 1.2, 5.1, and 7.4.

Trigonometric equations may also be solved using the graphing package.

GRAPHING PACKAGE

EXERCISE 9E.1

1

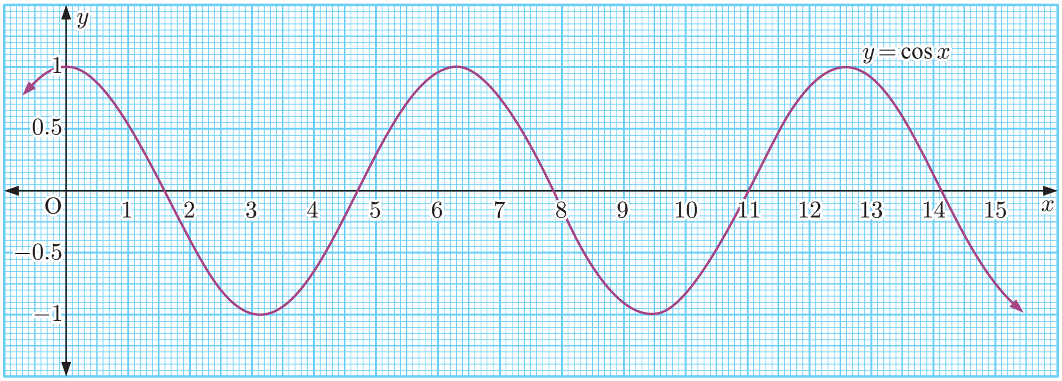


Use the graph of $y = \sin x$ to find, correct to 1 decimal place, the solutions of:

a $\sin x = 0.3$ for $0 \leqslant x \leqslant 15$

b $\sin x = -0.4$ for $5 \leqslant x \leqslant 15$.

2

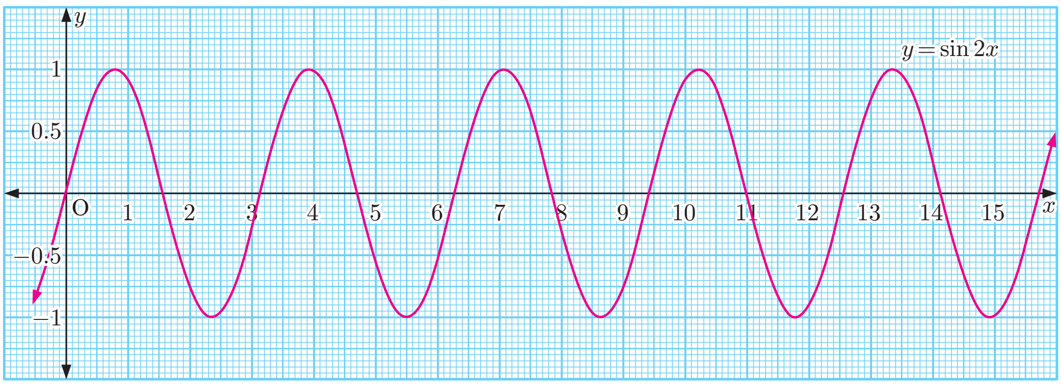


Use the graph of $y = \cos x$ to find, correct to 1 decimal place, the solutions of:

a $\cos x = 0.6$ for $0 \leqslant x \leqslant 10$

b $\cos x = -0.3$ for $4 \leqslant x \leqslant 12$.

3

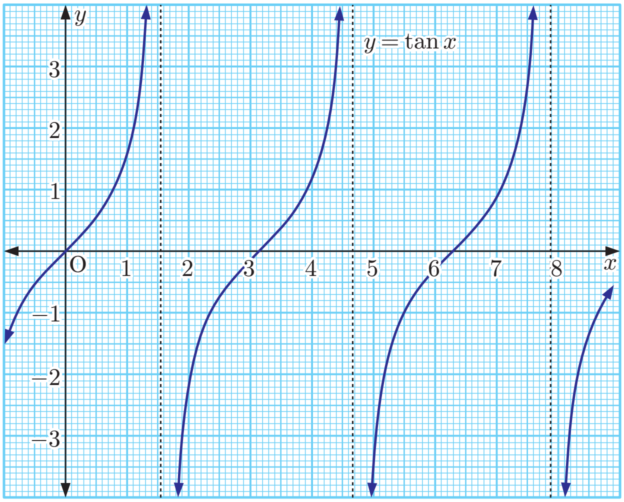


Use the graph of $y = \sin 2x$ to find, correct to 1 decimal place, the solutions of:

a $\sin 2x = 0.7$ for $0 \leqslant x \leqslant 16$

b $\sin 2x = -0.3$ for $0 \leqslant x \leqslant 16$.

4



The graph of $y = \tan x$ is illustrated.

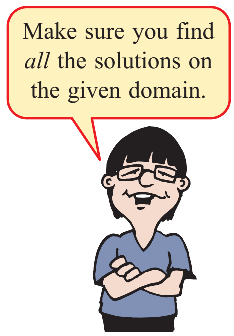
a Use the graph to estimate: **i** $\tan 1$ **ii** $\tan 2.3$
Check your answers using a calculator.

b Find, correct to 1 decimal place, the solutions of:
i $\tan x = 2$ for $0 \leqslant x \leqslant 8$ **ii** $\tan x = -1.4$ for $2 \leqslant x \leqslant 7$.

5 Use the graphing package to solve for x on the domain $0 < x < 4\pi$:
a $\sin x = 0.431$ **b** $\cos x = -0.814$ **c** $3\tan x - 2 = 0$

6 Use the graphing package to solve for x on the domain $-5 \leqslant x \leqslant 5$:
a $5\cos x - 4 = 0$ **b** $2\tan x + 13 = 0$ **c** $8\sin x + 3 = 0$

7 **a** Use the graphing package to solve $\sin^2 x + \sin x - 2 = 0$ for $0 \leqslant x \leqslant 2\pi$.
b Solve for m: $m^2 + m - 2 = 0$.
c Hence explain your answer in **a**.



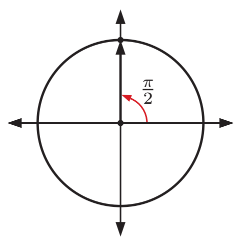
SOLVING TRIGONOMETRIC EQUATIONS USING ALGEBRA

Using a graph we get approximate decimal or **numerical** solutions to trigonometric equations.

Sometimes exact solutions are needed in terms of π, and these arise when the solutions are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$. Exact solutions obtained using algebra are called **analytical** solutions.

We use the periodicity of the trigonometric functions to give us all solutions in the required domain.

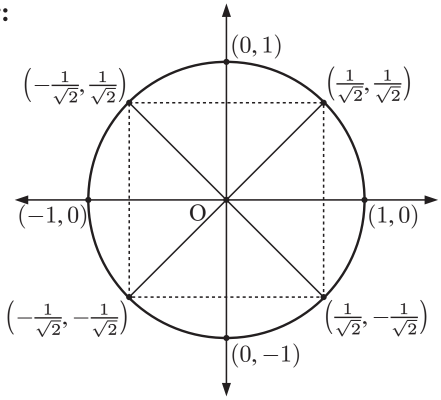
For example, consider $\sin x = 1$. We know from the unit circle that a solution is $x = \frac{\pi}{2}$. However, since the period of $\sin x$ is 2π, there are infinitely many solutions spaced 2π apart.

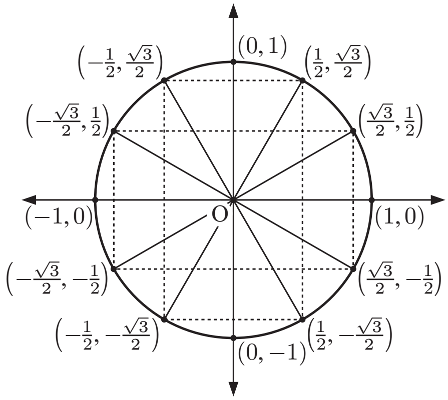


Hence $x = \frac{\pi}{2} + k2\pi$ is a solution for any $k \in \mathbb{Z}$.

In this course we will be solving equations on a fixed domain. This means there will be a finite number of solutions.

Reminder:





Example 6 — Self Tutor

Solve for x: $\quad 2\sin x - 1 = 0, \quad 0 \leqslant x \leqslant \pi$

$2\sin x - 1 = 0$

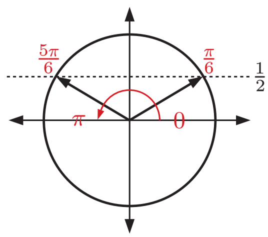
$\therefore \quad \sin x = \frac{1}{2}$

There are two points on the unit circle with sine $\frac{1}{2}$.

They correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

These are the only solutions on the domain $0 \leqslant x \leqslant \pi$, so

$x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.



Since the tangent function is periodic with period π we see that $\tan(x+\pi) = \tan x$ for all values of x. This means that equal $\tan$ values are π units apart.

Example 7 — Self Tutor

Solve $\tan x + \sqrt{3} = 0$ for $0 < x < 4\pi$.

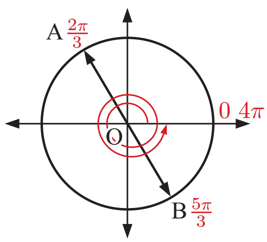
$\tan x + \sqrt{3} = 0$

$\therefore \quad \tan x = -\sqrt{3}$

There are two points on the unit circle with tangent $-\sqrt{3}$.

They correspond to angles $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

For the domain $0 < x < 4\pi$ we have 4 solutions: $x = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}$, or $\frac{11\pi}{3}$.



Start at angle 0 and work around to 4π, noting down the angle every time you reach points A and B.

EXERCISE 9E.2

1 Solve for x on the domain $0 \leqslant x \leqslant 4\pi$:

a $2\cos x - 1 = 0$ **b** $\sqrt{2}\sin x = 1$ **c** $\tan x = 1$

2 Solve for x on the domain $-2\pi \leqslant x \leqslant 2\pi$:

a $2\sin x - \sqrt{3} = 0$ **b** $\sqrt{2}\cos x + 1 = 0$ **c** $\tan x = -1$

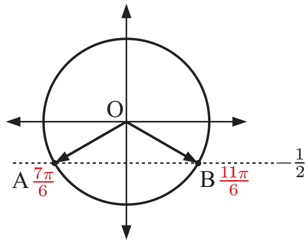
Example 8 Self Tutor

Solve exactly for $0 \leqslant x \leqslant 3\pi$: **a** $\sin x = -\frac{1}{2}$ **b** $\sin 2x = -\frac{1}{2}$

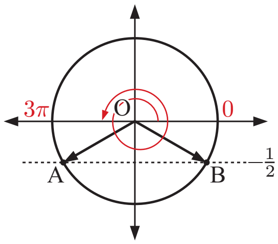
The equations both have the form $\sin\theta = -\frac{1}{2}$.

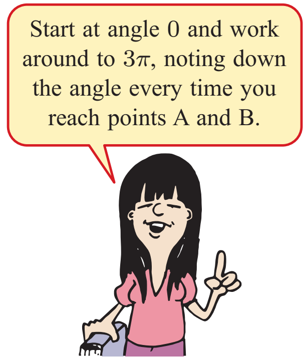
There are two points on the unit circle with sine $-\frac{1}{2}$.

They correspond to angles $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.



a In this case θ is simply x, so we have the domain $0 \leqslant x \leqslant 3\pi$.
The only solutions for this domain are $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.





b In this case θ is $2x$.
If $0 \leqslant x \leqslant 3\pi$ then $0 \leqslant 2x \leqslant 6\pi$.

$\therefore \; 2x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{31\pi}{6}$, or $\frac{35\pi}{6}$

$\therefore \; x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \frac{31\pi}{12}$, or $\frac{35\pi}{12}$

3 Solve exactly for $0 \leqslant x \leqslant 3\pi$: **a** $\cos x = \frac{1}{2}$ **b** $\cos 2x = \frac{1}{2}$

4 Solve exactly for $0 \leqslant x \leqslant 2\pi$: **a** $\sin x = -\frac{1}{\sqrt{2}}$ **b** $\sin 3x = -\frac{1}{\sqrt{2}}$

5 Find the exact solutions of:

a $\cos x = -\frac{1}{2}, \; 0 \leqslant x \leqslant 5\pi$

b $2\sin x - 1 = 0, \; -360° \leqslant x \leqslant 360°$

c $2\cos x + \sqrt{3} = 0, \; 0 \leqslant x \leqslant 3\pi$

d $3\cos 2x + 3 = 0, \; 0 \leqslant x \leqslant 3\pi$

e $4\cos 3x + 2 = 0, \; -\pi \leqslant x \leqslant \pi$

Example 9 Self Tutor

Solve $\tan 2x + 1 = 2$ for $-\pi \leqslant x \leqslant \pi$.

$\tan 2x = 1$

There are two points on the unit circle which have tangent 1.

Since $-\pi \leqslant x \leqslant \pi$,

$-2\pi \leqslant 2x \leqslant 2\pi$

So, $2x = -\frac{7\pi}{4}, \; -\frac{3\pi}{4}, \; \frac{\pi}{4}$, or $\frac{5\pi}{4}$

$\therefore \; x = -\frac{7\pi}{8}, \; -\frac{3\pi}{8}, \; \frac{\pi}{8}$, or $\frac{5\pi}{8}$