STAGE 5.3

MATHSCAPE

Working Mathematically

10

EXTENSION

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Preface

Mathscape 10 extension is a comprehensive teaching and learning resource that has been written to address the new Stage 5.1/5.2/5.3 Mathematics syllabus in NSW. Our aim was to write a book that would allow more able students to grow in confidence, to improve their understanding of Mathematics and to develop a genuine appreciation of its inherent beauty. Teachers who wish to inspire their students will find this an exciting, yet very practical resource. The text encourages a deeper exploration of mathematical ideas through substantial, well-graded exercises that consolidate students' knowledge, understanding and skills. It also provides opportunities for students to explore the history of Mathematics and to address many practical applications in contexts that are both familiar and relevant.

From a teaching perspective, we sought to produce a book that would adhere as strictly as possible to both the content and spirit of the new syllabus. Together with *Mathscape 9 extension*, this book allows teachers to confidently teach the Stage 5.1/5.2/5.3 courses knowing that they are covering all of the mandatory outcomes.

Mathscape 10 extension has embedded cross-curriculum content, which will support students in achieving the broad learning outcomes defined by the Board of Studies. The content also addresses the important key competencies of the Curriculum Framework, which requires students to collect, analyse and organise information; to communicate mathematical ideas; to plan and organise activities; to work with others in groups; to use mathematical ideas and techniques; to solve problems; and to use technology.

A feature of each chapter which teachers will find both challenging and interesting for their students is the 'Focus on working mathematically' section. Although the processes of working mathematically are embedded throughout the book, these activities are specifically designed to provoke curiosity and deepen mathematical insight. Most begin with a motivating real-life context, such as television advertising, or the gradient of a ski run, but on occasion they begin with a purely mathematical question. (These activities can also be used for assessment purposes.)

In our view, there are many legitimate, time-proven ways to teach Mathematics successfully. However, if students are to develop a deep appreciation of the subject, they will need more than traditional methods. We believe that all students should be given the opportunity to appreciate Mathematics as an essential and relevant part of life. They need to be given the opportunity to begin a Mathematical exploration from a real-life context that is meaningful to them. To show interest and enjoyment in enquiry and the pursuit of mathematical knowledge, students need activities where they can work with others and listen to their arguments, as well as work individually. To demonstrate confidence in applying their mathematical knowledge and skills to the solution of everyday problems, they will need experience of this in the classroom. If they are to learn to persevere with difficult and challenging problems, they will need to experience these sorts of problems as well. Finally, to recognise that mathematics has been developed in many cultures in response to human needs, students will need experiences of what other cultures have achieved mathematically.

We have tried to address these values and attitudes in this series of books. Our best wishes to all teachers and students who are part of this great endeavour.

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How to use this book

Mathscape 10 extension is a practical resource that can be used by teachers to supplement their teaching program. The exercises in this book and the companion text (*Mathscape 9 extension*) provide a complete and thorough coverage of all content and skills in the Stage 5.1/5.2/5.3 course. The great number and variety of questions allow for the effective teaching of more able students. Each chapter contains:

- a set of chapter outcomes directed to the student
- all relevant theory and explanations, with important definitions and formulae boxed and coloured
- step-by-step instructions for standard questions
- a large number of fully worked examples preceding each exercise
- extensive, thorough and well-graded exercises that cover each concept in detail
- chapter-related, problem-solving activities called 'Try this' integrated throughout
- a language skills section linked to the *Macquarie Learners Dictionary*
- novel learning activities focusing on the processes of working mathematically
- a thorough chapter review.

Explanations and examples

The content and skills required to complete each exercise have been introduced in a manner and at a level that is appropriate to the students in this course. Important definitions and formulae have been boxed and coloured for easy reference. For those techniques that require a number of steps, the steps have been listed in point form, boxed and coloured. Each exercise is preceded by several fully worked examples. This should enable the average student to independently complete the majority of relevant exercises if necessary.

The exercises

The exercises have been carefully graded into three distinct sections:

- Introduction. The questions in this section are designed to introduce students to the most basic concepts and skills associated with the outcome(s) being covered in the exercise. Students need to have mastered these ideas before attempting the questions in the next section.
- Consolidation. This is a major part of the exercise. It allows students to consolidate their understanding of the basic ideas and apply them in a variety of situations. Students may need to use content learned or skills acquired in previous exercises or topics to answer some of these questions. The average student should be able to complete most of the questions in this section, although the last few questions may be a little more difficult.
- Further applications. Some questions presented in this section will be accessible to the average student; however, the majority of questions are difficult. They might require a reverse procedure, the use of algebra, more sophisticated techniques, a proof, or simply time-consuming research. The questions can be open-ended, requiring an answer with a justification. They may also involve extension or off-syllabus material. In some questions, alternative techniques and methods of solution other than the standard method(s) may be introduced, which may confuse some students.

Teachers need to be selective in the questions they choose for their students. Some students may not need to complete all of the questions in the Introduction or Consolidations sections of each exercise, while only the most able students should usually be expected to attempt the questions in the Further applications section. Those questions not completed in class might be set as homework at the teacher's discretion. It is not intended that any student would attempt to answer every possible question in each exercise.

Focus on working mathematically

The Working Mathematically strand of the syllabus requires a deeper understanding of Mathematics than do the other strands. As such, it will be the most challenging strand for students to engage with and for teachers to assess. The Working Mathematically outcomes listed in the syllabus have been carefully integrated into each chapter of the book; however, we also decided to include learning activities in each chapter that will enable teachers to focus sharply on the processes of working mathematically. Each activity begins with a real-life context and the Mathematics emerges naturally. Teachers are advised to work through them before using them in class. Answers have not been provided, but notes for teachers have been included on the *Mathscape 10/10 Extension School CD-ROM*, with suggested weblinks. Teachers may wish to select and use the Learning activities in 'Focus on working mathematically' for purposes of assessment. This too is encouraged. The Extension activities will test the brightest students. Suggestions are also provided to assess the outcomes regarding Communication and Reflection.

Problem solving

Each chapter contains a number of small, chapter-related, problem-solving activities called 'Try this'. They may be of some historical significance, or require an area outside the classroom, or require students to conduct research, or involve the use of algebra, while others relate the chapter content to real-life context. Teachers are advised to work through these exercises before using them in class.

Technology

The use of technology is a clear emphasis in the new syllabus. Innovative technology for supporting the growth of understanding of mathematical ideas is provided on the *Mathscape 10/10 Extension School CD-ROM*, which is fully networkable and comes free-of-charge to schools adopting *Mathscape 10 extension* for student use. Key features of the CD-ROM include:
- spreadsheet activities
- dynamic geometry
- animations
- executables
- student worksheets
- weblinks for 'Focus on working mathematically'.

Language

The consistent use of correct mathematical terms, symbols and conventions is emphasised strongly in this book, while being mindful of the students' average reading age. Students will only learn to use and spell correct mathematical terms if they are required to use them frequently in appropriate contexts. A language section has also been included at the end of each chapter titled 'Language link with Macquarie', where students can demonstrate their understanding of important mathematical terms. This might, for example, include explaining the difference between the mathematical meaning and the everyday meaning of a word. Most chapters include a large number of worded problems. Students are challenged to read and interpret the problem, translate it into mathematical language and symbols, solve the problem, then give the answer in an appropriate context.

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1 Consumer arithmetic

This chapter at a glance

Stage 5.1/5.2/5.3

After completing this chapter, you should be able to:

- calculate the simple interest earned on an investment
- calculate the principal and interest rate in problems involving simple interest
- explain the difference between simple interest and compound interest
- calculate compound interest by repeated use of the simple interest formula
- calculate compound interest by use of the compound interest formula
- calculate compound interest by repeated multiplication on a calculator
- calculate compound interest where the interest does not compound annually
- calculate the principal given the interest rate, number of interest payments and interest earned
- apply the compound interest formula to solve practical problems involving appreciation
- calculate the depreciated value of an item by repeated use of percentage reduction
- calculate the depreciated value of an item by use of the depreciation formula
- calculate the depreciated value of an item by repeated multiplication on a calculator
- calculate the original value of a depreciating item
- solve problems involving consecutive discounts
- calculate and compare the cost of purchasing goods by paying cash, lay-by, deferred payment and buying on terms
- convert between annual and monthly or daily rates of simple interest
- solve problems that involve the purchase of items by use of a credit card
- calculate the monthly repayments and interest on a loan using a given table
- compare the cost of loans using flat and reducible interest.

1.1 Simple interest

Interest is money that a bank or other financial institution pays to its customers for lending the bank their money.

Banks also lend money to other customers as home loans or personal loans so that they can buy a house, car, computer or take a holiday. The bank charges the borrower interest on the loan. The borrower must repay the amount of the loan plus the interest. Banks always charge higher interest rates for loans than they pay on savings? Why?

The terms **principal** and **capital** refer to an amount of money that has been invested (or borrowed) for a period of time and on which interest is paid or charged. Interest that is calculated on the principal only is called **simple interest**.

The amount of simple interest $\$I$ to be paid on an investment or loan at $r\%$ p.a. is given by the formula:

$I = PRT$ **where**
- **$\$P$ is the amount invested or borrowed**
- **R is the percentage rate of interest, i.e. $R = \dfrac{r}{100}$**
- **T is the time period of the investment or loan, in years.**

Example 1

a Calculate the simple interest earned on an investment of \$350 at 6% p.a. over 7 years.
b Find the total balance of the account after 7 years.

Solutions

a $P = \$350, \quad R = \frac{6}{100}, \quad T = 7$
$\qquad\qquad\quad = 0.06$

$I = PRT$
$\;\; = \$350 \times 0.06 \times 7$
$\;\; = \$147$

$\therefore$ The simple interest earned is \$147.

b Balance = principal + interest
$\qquad\qquad = \$350 + \$147$
$\qquad\qquad = \$497$

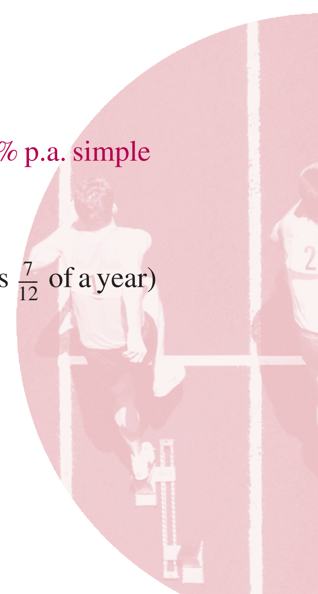
Example 2

Calculate the amount of simple interest that must be paid on a loan of \$400 at 5.4% p.a. simple interest over 7 months.

Solution

$P = \$400, \quad R = \frac{5.4}{100}, \quad T = \frac{7}{12}$ (since the time must be given in years and 7 months is $\frac{7}{12}$ of a year)
$\qquad\qquad\quad = 0.054$

$I = PRT$
$\;\; = \$400 \times 0.054 \times \frac{7}{12}$
$\;\; = \$12.60$



Exercise 1.1

1 Calculate the simple interest earned on savings of:
- a $200 at 7% p.a. for 1 year
- b $800 at 5% p.a. for 6 years
- c $650 at 4% p.a. for 3 years
- d $450 at 8% p.a. for 5 years
- e $1040 at 5% p.a. for $3\frac{1}{2}$ years
- f $1200 at 4% p.a. for $5\frac{1}{2}$ years
- g $1540 at 7% p.a. for $6\frac{1}{4}$ years
- h $2780 at 9% p.a. for $2\frac{3}{4}$ years

2 Find the total balance of an account after:
- a 3 years if $1500 is invested at 6% p.a.
- b 5 years if $2720 is invested at 8% p.a.

3 Calculate the simple interest that would be earned on the following investments.
- a $4000 at 1.5% per month for 7 months
- b $2700 at 0.95% per month for 4 months
- c $5500 at 0.023% per day for 65 days
- d $9475 at 0.049% per day for 213 days

4 Calculate the amount of interest that would be earned on the following investments.
- a $800 at 4% p.a. for 6 months
- b $900 at 5% p.a. for 4 months
- c $1300 at 9% p.a. for 7 months
- d $1725 at 7% p.a. for 18 months

Consolidation

5 Find the total balance of an account if the following amounts were invested.
- a $720 for 3 years at 8.5% p.a.
- b $940 for 5 years at $6\frac{1}{2}$% p.a.
- c $1680 for 8 months at $7\frac{1}{4}$% p.a.
- d $2695 for 4 months at 4.6% p.a.

6 Calculate the simple interest earned on the following investments.
- a $3800 at 7.1% p.a. for 98 days
- b $15 270 at 5.9% p.a. for 125 days
- c $920 at $3\frac{1}{2}$% p.a. for 240 days
- d $13 850 at $4\frac{3}{4}$% p.a. for 202 days

7 Calculate the simple interest earned on these investments.
- a $6210 at 1.6% per quarter for 15 months
- b $8040 at 1.35% per quarter for $2\frac{1}{2}$ years

8 Laurie borrowed $9000 from a finance company at 12% p.a. simple interest over 3 years to buy some new furniture for his house.
- a How much interest will Laurie pay on this loan?
- b How much will he repay altogether?

9 Leanne invested $500 with the Eastpac bank for 3 years and earned simple interest at the rate of 4% p.a. At the end of the 3 years, she transferred the entire amount (i.e. capital plus interest) into a new account with the St Patrick bank for 3 more years at 5% p.a.
- a How much will Leanne have in the account after the first 3 years?
- b How much will Leanne have in the account at the end of the 6 years?
- c How much interest did she earn altogether?

10 Carlos invested $5400 for 6 years at 5.2% p.a. simple interest. At the end of the 6 years, he withdrew the capital and reinvested the interest only for another 4 years. How much interest did he receive in total over this 10 year period?

11 Liam deposited a sum of money into a new credit union account. The account paid simple interest at the rate of 6% p.a. and after 3 years he had earned $129.60 interest. How much money did Liam invest in the account?

12 Mateya invested a sum of money in a savings account on which simple interest was paid at the rate of 7.5% p.a. After 4 years, Mateya had earned $180 in interest. How much money did she invest?

13 Meryn earned $140 simple interest on an investment of $500 at 4% p.a. For how many years was the money invested?

14 Brett invested $360 at 5% p.a. and earned simple interest of $13.50. For how many months was the money invested?

15 Wade was paid $300 simple interest on an investment of $2000 over 5 years. Calculate the annual rate of interest.

16 Greta invested $850 for 5 years and earned simple interest of $276.25. What was the interest rate per annum?

■ Further applications

17 Terry invested an amount of money for 6 years at 4% p.a. simple interest. What would the interest rate need to be in order for Terry to earn the same amount of interest in:
a 1 year? **b** 2 years? **c** 3 years? **d** 12 years?

18 For how many months would I need to invest $2500 at 7.2% p.a. simple interest to earn interest of $75?

19 Rhonda invested $1750 at 6.4% p.a. simple interest. For how many days was the money invested if she earned $56.15 interest?

1.2 Compound interest

Compound interest tables

When simple interest is calculated on an investment, the interest is only calculated on the principal. The amount of interest earned in each time period, therefore, is always the same.

When compound interest is calculated on an investment, the interest is calculated on the principal as well as on any interest that has been earned previously. The amount of interest earned in each time period increases, because the value of the investment on which it is being calculated is increasing.

To calculate the compound interest that will accrue on an investment:
- **calculate the interest on the principal**
- **add the interest to the principal to give the new balance**
- **calculate the interest on the new balance then add this to the previous balance**
- **repeat this procedure for each time period that the interest is to be paid.**

For example, \$5000 is invested at 8% p.a. compound interest for 3 years with the interest compounding annually. Use a table to find the final value of the investment and the amount of interest that will accrue.

Year	Opening balance	Interest	Closing balance
1st	\$5000	$\$5000 \times 0.08 = \$400$	\$5400
2nd	\$5400	$\$5400 \times 0.08 = \$432$	\$5832
3rd	\$5832	$\$5832 \times 0.08 = \$466.56$	\$6298.56

After 3 years, the original investment of \$5000 has grown to \$6298.56.

$$\begin{aligned}\text{Interest} &= \$6298.56 - \$5000\\ &= \$1298.56\end{aligned} \qquad \text{or} \qquad \begin{aligned}\text{Interest} &= \$400 + \$432 + \$466.56\\ &= \$1298.56\end{aligned}$$

The compound interest formula

In the previous example, the value of the investment after each year could have been calculated as follows:

$$\begin{aligned}\text{Value after 1 year} &= \$5000(1 + 0.08)^1\\ &= \$5400\\ \text{Value after 2 years} &= \$5000(1 + 0.08)^2\\ &= \$5832\\ \text{Value after 3 years} &= \$5000(1 + 0.08)^3\\ &\doteqdot \$6298.56\end{aligned}$$

From this example, we can see that if $\$P$ was invested at $r\%$ p.a. compounding annually, then its value $\$A$, after n years, would be given by the formula $A = P\left(1 + \frac{r}{100}\right)^n$. This formula can more simply be written as $A = P(1 + R)^n$, where the interest rate is expressed as a decimal, i.e. $R = \frac{r}{100}$.

If the interest is paid half-yearly, quarterly or monthly instead of annually, then we must adjust the values of R and n accordingly.

If an amount of money is invested and earns compound interest for a period of time, then its value at the end of this time is given by:

$$\boldsymbol{A = P(1 + R)^n}$$

where
- ***P*** **is the principal, or amount invested**
- ***R*** **is the interest rate per time period, expressed as a decimal**
- ***n*** **is the number of time periods**
- ***A*** **is the value of the investment after** ***n*** **time periods.**

Repeated multiplication

The value of an investment that earns compound interest over a period of time can be found by performing repeated multiplications on a calculator.

Again, using the above example, we could find the value of the investment by multiplying the principal, \$5000, by 1.08 for each year that it earns interest.

That is, the value of the investment after 3 years $= \$5000 \times 1.08 \times 1.08 \times 1.08$
$= \$5000 \times 1.08^3$
$= \$6298.56$

Example 1

Craig invested \$20 000 at 7% p.a. compound interest, with interest compounding annually. Find:

a the value of this investment after 4 years **b** the amount of interest earned.

Solutions

a $P = \$20\,000$, $R = 0.07$, $n = 4$

$$\begin{aligned} A &= P(1 + R)^n \\ &= \$20\,000(1 + 0.07)^4 \\ &= \$20\,000(1.07)^4 \\ &\doteqdot \$26\,215.92 \end{aligned}$$

b $$\begin{aligned} \text{Interest} &= \$26\,215.92 - \$20\,000 \\ &= \$6215.92 \end{aligned}$$

Example 2

A sum of \$10 000 is invested at 8% p.a. compound interest. Find the value of this investment after 3 years if the interest is compounded:

a half-yearly **b** quarterly

Solutions

The amount does not compound annually; therefore, we need to adjust the interest rate, R, and the number of time periods, n.

a The interest compounds every 6 months. The annual interest rate of 8% p.a. is equivalent to a rate of 4% every 6 months. Interest is paid twice a year for 3 years, therefore there are 6 time periods.

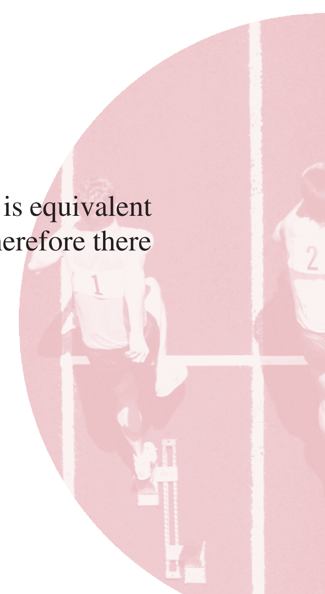
$P = \$10\,000$, $R = 0.04$, $n = 6$

$$\begin{aligned} A &= P(1 + R)^n \\ &= \$10\,000(1 + 0.04)^6 \\ &= \$10\,000(1.04)^6 \\ &\doteqdot \$12\,653.19 \end{aligned}$$

b The interest compounds every 3 months. The annual interest rate of 8% p.a. is equivalent to a rate of 2% every 3 months. Interest is paid 4 times a year for 3 years, therefore there are 12 time periods.

$P = \$10\,000$, $R = 0.02$, $n = 12$

$$\begin{aligned} A &= P(1 + R)^n \\ &= \$10\,000(1 + 0.02)^{12} \\ &= \$10\,000(1.02)^{12} \\ &\doteqdot \$12\,682.42 \end{aligned}$$



Exercise 1.2

Give all answers in this exercise correct to the nearest cent, unless otherwise directed.

1 A sum of \$10 000 was invested for 3 years at 6% p.a., compounded annually.

a Copy and complete this table.

Year	Opening balance	Interest	Closing balance
1st	\$10 000	\$10 000 × 0.06 = \$600	\$10 600
2nd			
3rd			

b Find the interest that is earned on this investment.

2 Draw up a table similar to that in question 1 to find the account balance and compound interest earned on these investments, if the interest is compounded annually.

a \$800 at 5% p.a. for 3 years
b \$2000 at 3.5% p.a. for 2 years

3 Use a compound interest table to find the amount of compound interest that will accrue on each investment, if the interest is compounded monthly. [HINT: First change the annual interest rate to a monthly rate].

a \$500 at 12% p.a. for 3 months
b \$3000 at 6% p.a. for 2 months

4 Use the compound interest formula to find the final value of each investment if the interest is compounded annually.

a \$1000 at 5% p.a. for 2 years
b \$750 at 4% p.a. for 3 years
c \$1300 at 6% p.a. for 5 years
d \$1950 at 8% p.a. for 4 years
e \$3000 at 3% p.a. for 8 years
f \$12 000 at 9% p.a. for 5 years
g \$9500 at 7% p.a. for 4 years
h \$15 000 at 6% p.a. for 10 years

Consolidation

5 Use the compound interest formula to find the amount of interest earned on each investment if the interest is compounded annually.

a \$7000 at 8.5% p.a. for 3 years
b \$12 000 at 6.3% p.a. for 4 years
c \$10 500 at 7.1% p.a. for 6 years
d \$9200 at 5.8% p.a. for 10 years
e \$25 000 at $9\frac{1}{2}$% p.a. for 5 years
f \$18 000 at $7\frac{1}{2}$% p.a. for 6 years
g \$31 500 at $6\frac{1}{4}$% p.a. for 11 years
h \$52 750 at $8\frac{3}{4}$% p.a. for 15 years

6 Bridie invests \$3500 at 12% p.a. compound interest. Find the value of her investment after 4 years if the interest is compounded:

a annually
b half-yearly
c quarterly
d monthly

7 Use the compound interest formula to calculate the amount of interest that will accrue on the following investments.

a \$2000 at 8% p.a. for 1 year, compounded every 6 months
b \$1500 at 10% p.a. for 7 years, compounded every 6 months

c \$2400 at 12% p.a. $5\frac{1}{2}$ for years, compounded every 6 months
d \$4000 at 8% p.a. for 1 year, compounded quarterly
e \$7200 at 12% p.a. for 2 years, compounded quarterly
f \$3800 at 4% p.a. $3\frac{1}{2}$ for years, compounded quarterly
g \$9000 at 12% p.a. for 2 years, compounded monthly
h \$15 000 at 3% p.a. for 5 years, compounded monthly
i \$50 000 at 9% p.a. for 3 years, compounded monthly

8 Heath invested \$8000 at 6% p.a. compounded monthly. Find:
a the value of the investment after 5 years **b** the amount of interest earned

9 Paul invested \$150 000 at 9% p.a. for 5 years, with the interest compounding every 4 months. How much interest will he earn?

10 A finance company invested \$250 000 at a daily interest rate of 0.02%. Find:
a the value of the investment after 20 days **b** the amount of interest earned
c the equivalent annual rate of interest.

11 Merv invested \$1000 at 7% p.a. simple interest for 4 years, while Christine invested \$1000 at 7% p.a. compound interest for 4 years, compounding annually. Who received the greater amount of interest, and by how much?

12 Which of the following pays more interest on an investment of \$4000?
A 8% p.a. simple interest for 9 years, or
B 6% p.a. compound interest for 10 years, compounded annually

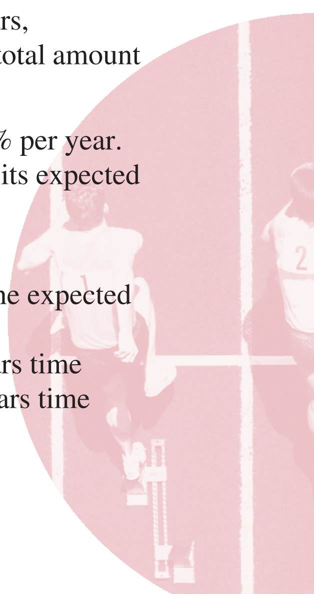
13 A building society advertises the following investment plans:
A 7.2% p.a. compounded quarterly, or **B** 6.6% p.a compounded monthly
Which plan will provide the greater amount of interest on an investment of \$30 000 for 5 years and by how much?

14 On her first birthday, Emma's parents invested \$5000 in a 20 year savings bond. The bond pays interest at 9.2% p.a., compounding every 6 months. What will be the value of the bond when it matures on Emma's 21st birthday?

15 A woman invested \$40 000 at 7.5% p.a. simple interest for 3 years. At the end of this term, she re-invested the capital and interest at 6% p.a. compound interest for 4 years, compounding half-yearly. Calculate the final value of the investment and the total amount of interest earned over the 7 years.

16 House prices in a certain suburb are appreciating in value by an average of 9% per year. If a house is purchased today for \$820 000, find, correct to the nearest dollar, its expected value in:
a 2 years **b** 5 years **c** 10 years

17 If the national inflation rate is 3.5% p.a., find, correct to the nearest 5 cents, the expected cost of:
a a 90c carton of milk in 3 years time **b** a \$2 bag of potatoes in 5 years time
c a \$3 loaf of bread in 2 years time **d** a \$40 tank of petrol in 10 years time



18 The population of a country town was 8000 at the end of 2003. If, on average, the town's population is increasing at the rate of 3% per year, find the expected population at the end of 2010.

19 The present population of a country is 25.7 million people. If this is expected to increase by 1.8% p.a. over the next decade, find the population of the country in 10 years time.

20 A painting was purchased in 2001 for $34 000. If it appreciates in value by 4.5% p.a., what would be its value in 2006? Answer correct to the nearest dollar.

21 Samantha earns $400 per week as a personal assistant. Her employer has agreed to increase her salary by 3% each year. Calculate her annual salary in 5 years time, correct to the nearest dollar.

Further applications

22 Barry invested a sum of money at 5% p.a., compounded annually. After 3 years his investment had grown to $694.58. Find the amount of money that he invested, correct to the nearest dollar.

23 Calculate the amount of money, correct to the nearest dollar, that will grow to:

a $676.63 if invested for 7 years at 6% p.a., compounded annually
b $2478.61 if invested for 5 years at 14% p.a., compounded half-yearly
c $1902.36 if invested for 3 years at 8% p.a., compounded quarterly
d $3868.62 if invested for $2\frac{1}{2}$ years at 18% p.a., compounded monthly

TRY THIS **Inflation**

Michelle, a year 10 student, is 15 years-old and has just begun her first part-time job. When she tells her grandmother that she is to be paid $15 per hour her grandmother recalls that for her first job she only received 2 pence an hour (about $1\frac{1}{2}$ cents). Michelle wonders what the wages will be when she is 75 years-old and has 15 year-old grandchildren. Assuming an inflation rate of 5% per annum, what amount will $15 have grown to in 60 years? How much difference would it make if the inflation rate was 3% or 10%?

1.3 Depreciation

Depreciation tables

If an item loses value over a period of time, then it is said to **depreciate** in value. The rate at which it depreciates is often expressed as a percentage. In most cases the depreciating value of an item is calculated annually. Therefore, the depreciating value of an item each year is related to its value in the previous year.

To calculate the depreciating value of an item:
- **calculate the loss during the first year**
- **subtract the loss from the original value of the item to give its new value**
- **calculate the loss on this value during the second year, then subtract this from the value in the previous year**
- **repeat this procedure for each year that the item depreciates.**

For example, a car is bought for \$20 000 and depreciates at the rate of 15% p.a. Find the expected value of the car after 3 years. Answer correct to the nearest dollar.

Year	Opening balance	Loss	Closing balance
1st	\$20 000	\$20 000 × 0.15 = \$3000	\$17 000
2nd	\$17 000	\$17 000 × 0.15 = \$2550	\$14 450
3rd	\$14 450	\$14 450 × 0.15 = \$2167.50	\$12 282.50

After 3 years, the car would have a value of \$12 283, correct to the nearest dollar.

The depreciation formula

In the example above, the value of the car after each year could have been calculated as follows:

$$\begin{aligned}
\text{Value after 1 year} &= \$20\,000(1-0.15)^1 \\
&= \$17\,000 \\
\text{Value after 2 years} &= \$20\,000(1-0.15)^2 \\
&= \$14\,450 \\
\text{Value after 3 years} &= \$20\,000(1-0.15)^3 \\
&= \$12\,283 \text{ (to the nearest dollar).}
\end{aligned}$$

From this example, we can see that if an item with an initial value of $\$P$ depreciates by $r\%$ p.a., then its value, $\$V$, after n years would be given by the formula $V = P\left(1-\frac{r}{100}\right)^n$. This formula can more simply be written as $V = P(1-R)^n$, where the interest rate is expressed as a decimal, i.e. $R = \frac{r}{100}$.

If an item depreciates over a period of time, then its value at the end of this time is given by:

$$V = P(1-R)^n$$

where
- P **is the original value of the item**
- R **is the annual rate of depreciation, expressed as a decimal**
- n **is the number of years the item depreciates**
- V **is the value of the item after** n **years.**

Example 1

EG +S

A machine was purchased in 2004 for \$80 000 and depreciates by 12% p.a. Find the expected value of the machine in 2010. Give your answer correct to the nearest dollar.

Solution

$P = \$80\,000$
$R = 0.12$
$n = 6$

$$\begin{aligned} V &= P(1 - R)^n \\ &= \$80\,000(1 - 0.12)^6 \\ &= \$80\,000(0.88)^6 \\ &\doteqdot \$37\,152 \end{aligned}$$

Example 2

EG +S

A school purchased a new photocopier. The photocopier depreciated by 25% p.a. and had a value of \$3006 after 4 years. Find the purchase price of the photocopier, correct to the nearest dollar.

Solution

$P = ?$
$R = 0.25$
$n = 4$
$V = \$3006$

$$\begin{aligned} V &= P(1 - R)^n \\ 3006 &= P(1 - 0.25)^4 \\ \frac{3006}{0.75^4} &= \frac{P(0.75)^4}{0.75^4} \\ \therefore P &= \frac{3006}{0.75^4} \\ &\doteqdot \$9500 \end{aligned}$$

Exercise 1.3

In this exercise, give all answers correct to the nearest dollar, unless otherwise indicated.

1 A new car was purchased for \$30 000. If it depreciates at a rate of 20% p.a., use a depreciation table to find the value of the car after 3 years.

2 Use the depreciation formula to find the value of a computer that was purchased for:

- **a** \$4000 and depreciated at a rate of 10% p.a. for 3 years
- **b** \$2700 and depreciated at a rate of 12% p.a. for 4 years
- **c** \$5900 and depreciated at a rate of 15% p.a. for 6 years
- **d** \$7450 and depreciated at a rate of 25% p.a. for 5 years

3 Paul paid \$16 500 to buy equipment for his new glass repair business. If the rate of depreciation is 14% p.a., find the value of the equipment after 6 years.

4 Find the value of a teacher's professional library in 4 years time, if the present value is \$3200, and the books are depreciating at the rate of 30% p.a.

5 The population of Warraga is falling at the rate of 5% p.a. due to the closure of the local zinc mine. In January 2005, the town had a population of 9170. What is the population of the town expected to be in January 2012?

Consolidation

6 The number of serious crimes reported in a suburb fell by 7.5% each year for the previous 5 years. If 800 crimes were reported in 2000, find the number of crimes that are expected to be reported in 2005.

7 Reanne bought exercise equipment valued at \$1100 and estimated that it would depreciate in value at the rate of 12.5% p.a. What price should Reanne ask for the equipment if she wanted to sell it after 4 years?

8 A DVD player purchased for \$750 depreciates at the rate of 16% p.a.

a What would be the value of the DVD player after 5 years?

b By how much would the DVD player depreciate during this time?

9 A pond in a large suburban park held 50 000 kilolitres of water after a period of sustained rainfall. The volume of water in the pond then decreased by 2% per week.

a Find the volume of water in the pond after 10 weeks.

b How much water was lost during that time?

10 A farmer had 6000 head of cattle on his property. During a severe drought, 8% of the herd died each month. How many head of cattle died during the first year of the drought?

11 A stockholder purchased \$400 000 worth of shares in Consolidated Iron. The shares fell in value by 5% during the first month after the purchase, and continued to fall by 2% per month after that.

a Calculate the value of the shares after 6 months.

b How much money did the stockholder lose on this investment?

12 Keith purchased a share portfolio for \$3000 at the start of the year. The shares rose in value by 5% per month for the first 6 months, then fell in value by 5% per month for the next 6 months. Find the value of the shares at the end of the year.

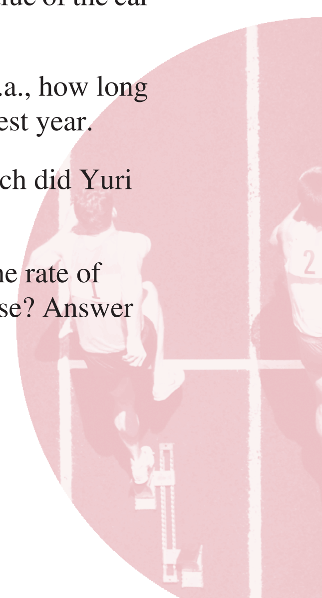
13 A machine purchased for \$400 000 depreciates at a rate of 20% p.a. After how many years will the machine have a value of \$131 072?

14 Enrique bought a secondhand car for \$9000. After how many years will the value of the car first fall below \$3000, if the annual rate of depreciation is 18%?

15 Marianne bought a diamond ring for \$1600. If the ring depreciates by 10% p.a., how long is it before the ring loses half its value? Give your answer correct to the nearest year.

16 Yuri's car depreciated at the rate of 16% p.a. If he sold it for \$4500, how much did Yuri pay for the car when he bought it 7 years ago?

17 A textile machine with a present value of \$15 000 has been depreciating at the rate of 5% p.a. for 6 years. What was the value of the machine at the time of purchase? Answer correct to the nearest \$100.



Further applications

18 A lawnmower was purchased for \$480 and 4 years later its value had depreciated to \$275. Find the annual rate of depreciation.

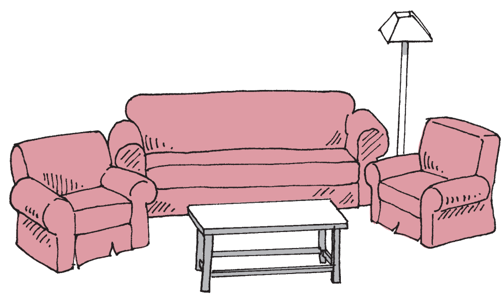
19 A fleet of dump trucks depreciated in value by r% p.a. from \$1.2 million to \$425 000 in 11 years.
- **a** Find the annual rate of depreciation.
- **b** Hence, find the value of the fleet at the end of the 5th year, correct to the nearest \$10 000.

TRY THIS **Depreciating value**

The furniture's value is currently \$7200. If it has been depreciating at 10% p.a. for the last 4 years, find its original purchase price, if we assume:

- **a** linear depreciation
- **b** reducing balance depreciation.

NOTE: With reducing balance depreciation, the value of the item depreciates by a percentage of the previous year's value, whereas with linear depreciation the item depreciates by a percentage of the original value.



1.4 Buying major items

Ways of paying

There are many ways by which major items can be purchased.

- **Cash:** Goods are paid for and received immediately. Discounts are sometimes given for cash purchases.
- **Lay-by:** A deposit is paid up-front, and the balance of the purchase price is paid in instalments over a period of time. The goods are not received until the entire purchase price is paid off. Interest is not charged on lay-by purchases.
- **On terms:** A deposit is paid up-front and the balance is paid by regular instalments over a period of time. The goods are received immediately. Interest is usually charged on the balance owing and is included in each of the instalments.
- **Deferred payment:** A deposit is usually required. The goods can then be taken, with the balance to be paid before some agreed time in the future. Interest is paid if the balance is not paid on time.

- **Credit card:** The card has a set credit limit, or amount that can be spent. It is a form of loan from a financial institution, which can be used at any time to purchase goods. High rates of interest are usually charged on the outstanding debt. An interest-free period may be available. Goods can be taken immediately when purchased by a credit card.
- **Loans:** Money is borrowed from a financial institution such as a bank, building society or credit union. Interest is charged on the outstanding debt, which must be paid off by regular instalments. There may be a waiting time before the loan is approved, as credit checks need to be made.

Consecutive discounts

Retail items in stores are discounted for many reasons, such as to sell excess or out-of-date stock to make room for new stock. A further discount is sometimes offered to customers who pay for their purchases by cash. Two discounts offered one after the other on an item are called **consecutive discounts**. When consecutive discounts are given on an item, they must be worked out one at a time; the percentages cannot be added to give a single equivalent discount.

For example, consecutive discounts of 10% and 5% on an item are not equivalent to a single 15% discount. This is because the second discount of 5% is calculated on a different value than the first discount of 10%.

Finding the annual interest rate

When paying for an item on terms, it is important to note that interest is only charged on the outstanding balance of the purchase price of the item. For example, if a deposit of $50 is paid on a table priced at $200, the outstanding balance is $150. In a sense, the purchaser is borrowing the $150 and repaying it and the interest by making regular instalments. Interest is only charged on the $150, not on the entire purchase price of $200.

To calculate the annual interest rate on a loan or hire purchase:
- **calculate the amount borrowed or outstanding, after any deposit is paid**
- **calculate the amount of interest that is paid each year on the amount borrowed**
- **express the interest paid per year as a percentage of the amount borrowed.**

EG +S

Example 1

A sewing machine is priced for sale at $480. The sewing machine is discounted by 15% during the mid-year sale, and a customer is given a further 5% discount for paying by cash. Find the purchase price of the sewing machine.

Solution

i $85\% \times \$480$
$= 0.85 \times \$480$
$= \$408$

ii $95\% \times \$408$
$= 0.85 \times \$408$
$= \$387.60$

Example 2

Samantha bought a surround-sound television and stereo system valued at \$12 000. She paid a deposit of 20%, followed by equal monthly payments of \$320 over 4 years.

a Find the deposit and the balance owing.
b Calculate the total amount that Samantha paid for the sound system.
c How much interest did she pay?
d What is the annual interest rate that Samantha was charged?

Solutions

a
$$\begin{aligned}\text{Deposit} &= 20\% \times \$12\,000\\ &= 0.2 \times \$12\,000\\ &= \$2400\end{aligned}$$

$$\begin{aligned}\text{Balance owing} &= \text{cash price} - \text{deposit}\\ &= \$12\,000 - \$2400\\ &= \$9600\end{aligned}$$

b
$$\begin{aligned}\text{Total paid} &= \text{deposit} + \text{monthly instalments}\\ &= \$2400 + (48 \times \$320)\\ &= \$17\,760\end{aligned}$$

c
$$\begin{aligned}\text{Interest} &= \text{total paid} - \text{cash price}\\ &= \$17\,760 - \$12\,000\\ &= \$5760\end{aligned}$$

d
$$\begin{aligned}\text{One year's interest} &= \$5760 \div 4\\ &= \$1440\end{aligned}$$

$$\begin{aligned}\text{Interest rate} &= \frac{\text{one year's interest}}{\text{balance owing}} \times 100\%\\ &= \frac{1440}{9600} \times 100\%\\ &= 15\%\ \text{p.a.}\end{aligned}$$

Exercise 1.4

1 Denise wants to reserve on lay-by a hat priced at \$52. The store requires a deposit of \$17.50. How much more does Denise have to pay before she can take the hat home?

2 Matthew bought a \$350 surfboard on lay-by. He paid a deposit of \$50 and paid off the balance at \$25 per week. After how many weeks will Matthew be able to take the surfboard home?

3 Leanne saw a table for sale at \$145. The store required a lay-by deposit of 15%. The balance must then be paid off in five equal weekly payments before Leanne may take the table home.
a Calculate the size of the lay-by deposit. **b** Find the amount of the weekly payments.

4 A store offers refrigerators for sale for a cash price of \$720 on the following interest-free terms: one-third down, with the balance owing to be paid in three equal instalments, the first payment due 12 months after the date of purchase. Each subsequent payment is due 4 months after the previous payment. A penalty fee equivalent to 5% of the balance owing applies for late payments.
a How much is required as a deposit?
b Calculate the penalty fee payable if the second instalment after the deposit is late.

5 The retail price of a washing machine was \$700. A discount of 15% was offered on the sale because the machine had some minor scratches. A further discount of 5% was offered to customers who paid cash.

- **a** Can the cash price of the washing machine be calculated by discounting the marked price by 20%? Explain.
- **b** Calculate the cash price of the washing machine.

6 A recliner chair was advertised for sale with a marked price of \$1150. All items of furniture in the store were discounted by 20% during January, and a further 3% discount was offered to customers who paid cash. How much would a customer pay for this chair in January, if they paid cash?

7 Simon bought an air-conditioning unit priced at \$1800. He paid a deposit of \$600 followed by 12 equal monthly payments of \$120.

- **a** How much did he pay altogether for the air-conditioner?
- **b** How much interest did he pay?

Consolidation

8 Gina purchased a digital camera priced at \$450, to take photos on her overseas holiday. She paid a deposit of 20% of the cash price and \$70 per month for 6 months.

- **a** How much interest did she pay?
- **b** Express the interest as a percentage of the cash price. Answer correct to 1 decimal place.

9 How much interest would Michaela pay on the purchase of a sewing machine priced at \$520, if she paid a deposit of \$150 followed by equal monthly payments of \$25 over 2 years?

10 A car is advertised for sale at \$17 990. A customer wants to purchase the car on terms and is given a choice of two payment plans.
Plan A: a deposit of \$3500 and equal monthly payments of \$450 for 4 years, or
Plan B: a deposit of 20% and equal monthly payments of \$350 over 5 years
Which is the cheaper payment plan, and by how much?

11 Luke borrowed \$15 000 over 3 years to renovate his kitchen. He was charged simple interest of 9% p.a. on this amount. The loan and interest was to be repaid in 36 equal monthly instalments.

- **a** Calculate the interest that was charged on this loan.
- **b** How much will Luke repay altogether?
- **c** Find the size of his monthly instalments.

12 A woman borrowed \$9500 from a finance company to build a swimming pool. She was charged 10.5% p.a. interest and agreed to repay the loan plus interest over 2 years. Calculate:

- **a** the total amount to be repaid
- **b** the amount of each monthly instalment.

13 A plasma screen TV was advertised for sale at $9200. A customer decided to purchase the TV on terms by paying a deposit of 15%, with interest charged on the balance at 12% p.a. over 3 years.

a Calculate the size of the deposit.
b Find the balance owing after the deposit has been paid.
c Calculate the interest charged on the balance.
d How much will the customer pay altogether in monthly instalments?
e Find the amount of each monthly instalment.

14 Eamon bought on terms a surround-sound system with a marked price of $11 000. He paid a 20% deposit, with interest charged on the balance at 9.6% p.a. over 4 years.

a How much is the deposit?
b Find the balance owing after the deposit has been paid.
c How much interest does Eamon pay on this purchase?
d Calculate the total amount to be repaid in monthly instalments.
e Find the amount of each monthly instalment.

Further applications

15 A man was quoted $5600 for the construction of a garage at the front of his house. He decided to pay for the construction on terms by paying equal monthly instalments of $294 for 2 years.

a How much did he pay for the construction of the garage?
b How much interest did he pay?
c Find the annual interest rate that was charged.

16 Anjee bought a boat priced at $25 000, and paid it off by making equal monthly payments of $615 over 5 years. Find the annual interest rate that was charged.

17 A car with a sale price of $23 990 was purchased on the following terms: a deposit of 20% with equal monthly instalments of $632 over 5 years. Find the annual interest rate charged, correct to 1 decimal place. *Note*: Interest is only charged on the balance owing.

1.5 Credit cards

Credit cards are issued by financial institutions such as banks, building societies and credit unions, as well as by private credit providers such as American Express. They provide people with a source of credit that can be used to purchase goods or pay bills.

High rates of interest are usually charged on outstanding debts; however, many cards come with an **interest-free period**. This is usually up to a maximum period of 55 days. The amount on which the interest is charged varies between financial institutions. For example, some credit providers may only charge interest on the amount that is outstanding, while others may charge interest on the entire debt, even though some of it may have been repaid within the interest-free period.

Credit cards have a set **credit limit**. This is the maximum amount that can be outstanding at any time. Credit limits are put in place in order to prevent people from running up very large debts that they cannot afford to repay.

Customers receive a **monthly statement**, which provides a complete list of all transactions that occurred during the previous month, the balance owing and the minimum payment due. The minimum payment must be made by the due date, otherwise further interest or charges may be applied.

Example 1

Find the equivalent annual interest rate for each of the following.

a 0.034% per day

b 0.85% per month

Solutions

a $0.034\% \times 365 = 12.41\%$ p.a.

b $0.85\% \times 12 = 10.2\%$ p.a.

Example 2

Anita's credit card comes with a 55 day interest-free period. Interest is charged on the balance outstanding after this period at an interest rate of 14% p.a. Anita used her credit card to pay for her holiday to Hamilton Island, which cost \$1260. She repaid \$450 of this amount within the interest-free period and the balance after 75 days. How much interest will she pay?

Solution

i The credit card debt after the interest-free period

$= \$1260 - \$450$

$= \$810$

ii The credit card has an interest-free period.

Therefore interest is only calculated for the final 20 days.

$$\text{Interest} = \$810 \times 0.14 \times \frac{20}{365}$$
$$\doteqdot \$6.21$$

Exercise 1.5

1 Here is a copy of Mrs Walsh's monthly credit card statement. Some of the details have been left out.

Monthly credit statement				
Account number	1344 2956 0012 7849	**Statement date**		1 May 2004
Date	**Transaction details**	**Debits**	**Credits**	**Balance**
1 Apr	Opening balance			\$930
4 Apr	Payment — thank you		\$25	\$905
5 Apr	Motor registry — Bondi Junction	\$344		\$1249
9 Apr	Salary — ABC Radio		\$1056	\$193
12 Apr	Woolworths — Double Bay	\$105.70		\$298.70
17 Apr	El Nino Restaurant	\$83.55		\$382.25
20 Apr	Grace Bros — Bondi Junction	\$839		\$1221.25
23 Apr	Salary — ABC Radio		\$1056	
28 Apr	Shell service station — Edgecliffe	\$47.35		\$212.60
Credit limit \$3000	**Credit charges** 21.65% p.a. 0.0593% daily		**Closing balance**	
			Available credit	
Overdue amount \$0.00	**Minimum payment due**		**Due date** 15 May 2004	

a During which month did the transactions in this statement occur?
b How many days after the statement date is the minimum payment due?
c What is the daily interest rate?
d What is Mrs Walsh's credit limit?
e What was her account balance at the end of the previous month?
f Find the account balance on 23 April.
g What was the closing balance at the end of this month.
h What is the available credit at the end of this month?
i The minimum payment due is 5% of the closing balance or \$15, whichever is the greater amount. What is the minimum payment due?
j Calculate the interest charged if Mrs Walsh paid the account 5 days after the due date?

2 Find the equivalent annual interest rate for each of the following.

a 0.036% per day
b 0.0456% per day
c 0.055 48% per day
d 0.65% per month
e 0.705% per month
f 0.8495% per month

Consolidation

3 Calculate the amount of simple interest that is payable on each of these credit card debts.

- **a** \$80 for 10 days at 0.06% per day
- **b** \$125 for 21 days at 0.054% per day
- **c** \$392 for 38 days at 0.052 25% per day
- **d** \$1040 for 53 days at 0.058 72% per day

4 Calculate the amount of simple interest that is payable on each of these credit card debts.

- **a** \$150 for 20 days at 21% p.a.
- **b** \$575 for 46 days at 18% p.a.
- **c** \$1610 for 14 days at 20.25% p.a.
- **d** \$3725 for 45 days at 23.75% p.a.

5 Tim bought a jumper priced at \$120 on 5 July, using his credit card which has no interest free period. He paid this off on 27 July and was charged interest at the rate of 0.056 44% per day from the date of purchase. How much interest did he pay on this purchase?

6 Freda accumulated a credit card debt of \$870 on 12 October. She was charged interest on this debt at the rate of 0.062% per day from the date of purchase. On 3 December, Freda paid off her debt and the interest charge.

- **a** Calculate the amount of interest that she was required to pay.
- **b** How much did she pay altogether on 3 December?

7 Jacinta has a credit card with no annual fee. Interest is charged at 15.33% p.a. on all purchases at a daily rate, from the date of purchase.

- **a** Calculate the equivalent daily rate of interest.
- **b** Jacinta used her credit card on 17 June to pay for her car registration. The cost of registering the car was \$318. She paid the credit card account on 8 July. How many days was she charged interest?
- **c** How much did she pay altogether, including the interest charge?

8 Josh's credit card has no interest-free period and interest is charged at 0.048% per day, from the date of purchase. He used his credit card on 13 November to buy a coffee table for \$115. He used the card again on 21 November to pay for petrol. The petrol cost \$53. Josh paid the account in full on 3 December.

- **a** How much interest was he charged for the coffee table purchased on 13 November?
- **b** How much interest was he charged for the petrol purchased on 21 November?
- **c** How much did Josh pay altogether on 3 December?

9 The ABC credit card offers customers a 55 day interest-free credit period. That is, if the balance is paid in full before the due date, no interest will be charged. If the account is not paid by the due date, the following conditions will apply:

- Interest is charged at the rate of 16.5% p.a.
- One month's interest is charged immediately on the balance owing.
- Daily interest is charged on the balance until the balance is paid.

The due date on Lee's credit card statement was 15 September and the balance owing was \$285.

- **a** If Lee paid the account on 12 September, how much interest would he have been charged?
- **b** Calculate the interest charged if Lee paid the account on 20 September.

10 Tracy has a credit limit of $2500 on her credit card. If she exceeds this limit at any time, the card company charges an extra fee of $25 plus 0.0584% per day on the amount owing above her credit limit. On 16 May, Tracy had a credit card debt of $2370. The following day, she used the card to pay her car insurance of $560.

a Has she exceeded her credit limit? If so, by how much?

b If she pays this amount 7 days later, calculate the extra fee that she will be charged.

Further applications

11 Sean has a credit card with no interest-free period. Interest is charged at 18% p.a. from the date of purchase. Sean made the following purchases during January. Calculate the total amount of interest charged if Sean paid the account in full on 15 February.

Date	Item Purchased	Cost
1 Jan	Petrol	$54.00
5 Jan	CD Player	$285.00
16 Jan	Shoes	$104.00
23 Jan	Dinner	$83.50
31 Jan	Groceries	$75.20

1.6 Loans

A **loan** is an amount of money that is borrowed, usually from a bank, credit union or finance company. Interest on loans is usually charged as **reducible interest**. That is, interest is only charged on the amount of the loan that is still outstanding. The amount of interest that will be paid can be minimised by paying off the loan as soon as possible. The borrower repays the loan and interest by making regular payments called instalments. Each instalment includes a portion of the original loan amount as well as an interest payment.

Home loans can be taken out at variable rates or at fixed rates. For **variable rate** loans, the financial institution can vary or change the interest rate at any time, as changes in the economy occur. **Fixed rates** are slightly higher than variable rates, but cannot change for the agreed period. This gives the borrower certainty that their repayments will not rise.

Applicants are charged a number of fees and charges by the financial institution and by the government when a loan is taken out. These charges may amount to several hundred dollars.

The table shows the monthly repayments required to repay each $1000 of a home loan at various monthly reducible interest rates.

Interest rate p.a.	10 years	15 years	20 years	25 years	30 years
5%	10.61	7.91	6.60	5.85	5.37
5.25%	10.73	8.04	**6.74**	5.99	5.52
5.5%	10.85	8.17	6.88	6.14	5.68
5.75%	10.98	8.30	7.02	6.29	5.84
6%	11.10	8.44	7.16	6.44	6.00

Example

A couple borrowed \$350 000 from a bank to buy a house. The bank charged reducible interest on the loan at the rate of 5.25% p.a. monthly. The borrowers agreed to repay the loan and interest over a period of 20 years. Use the table above to answer the following questions.

a What are the monthly repayments?
b How much will they repay altogether on this loan?
c How much interest will they pay?
d Calculate the equivalent flat yearly interest rate, correct to 1 decimal place.

Solutions

a To calculate the monthly repayments on a loan of \$350 000, first look up the monthly repayments on \$1000 at 5.25% p.a. over 20 years, then multiply this by 350.

$$\begin{aligned}\text{Monthly repayments} &= \$6.74 \times 350\\ &= \$2359\end{aligned}$$

b
$$\begin{aligned}\text{Total repaid over 20 years} &= \$2359 \times 12 \times 20\\ &= \$566\ 160\end{aligned}$$

c
$$\begin{aligned}\text{Interest} &= \text{amount repaid} - \text{amount borrowed}\\ &= \$566\ 160 - \$350\ 000\\ &= \$216\ 160\end{aligned}$$

d
$$\begin{aligned}\text{One year's interest} &= \$216\ 160 \div 20\\ &= \$10\ 808\end{aligned}$$

$$\begin{aligned}\text{Flat interest rate} &= \frac{\text{one year's interest}}{\text{amount borrowed}} \times 100\%\\ &= \frac{10\ 808}{350\ 000} \times 100\%\\ &= 3.1\%\ \text{p.a. (to 1 decimal place)}\end{aligned}$$

Exercise 1.6

The table below shows the monthly payments required to repay a loan of \$1000 over periods of 1 to 5 years at monthly reducible interest rates. Use this table to answer questions 1 to 3.

Interest rate p.a.	1 year	2 years	3 years	4 years	5 years
9%	87.45	45.68	31.80	24.89	20.76
9.5%	87.68	45.91	32.03	25.12	21.00
10%	87.92	46.14	32.27	25.36	21.25
10.5%	88.15	46.38	32.50	25.60	21.49
11%	88.38	46.61	32.74	25.85	21.74
11.5%	88.62	46.84	32.98	26.09	21.99
12%	88.85	47.07	33.21	26.33	22.24

1 Find the monthly repayments for a loan of \$1000 at:

a 11% p.a. for 1 year **b** 9.5% p.a. for 1 year **c** 10% p.a. for 2 years
d 11.5% p.a. for 4 years **e** 10.5% p.a. for 3 years **f** 12% p.a. for 5 years

2 Find the monthly repayments on each of the following loans.
   a $3000 at 10% p.a. for 1 year
   b $7000 at 9% p.a. for 2 years
   c $12 000 at 11.5% p.a. for 5 years
   d $15 000 at 9.5% p.a. for 1 year
   e $2500 at 12% p.a. for 2 years
   f $6400 at 10.5% p.a. for 4 years

3 For each of the loans below, find:
   i the total amount to be repaid
   ii the amount of interest charged
   a $1000 at 10.5% p.a. for 3 years
   b $5000 at 9% p.a. for 4 years
   c $20 000 at 11% p.a. for 5 years
   d $35 000 at 12% p.a. for 2 years

■ Consolidation

The table below shows the monthly repayments required to repay each $1000 of a housing loan at various monthly reducible rates of interest. Use this table to answer questions 4 to 10.

Interest rate p.a.	10 years	15 years	20 years	25 years	30 years
4.5%	10.36	7.65	6.33	5.56	5.07
4.75%	10.48	7.78	6.46	5.70	5.22
5%	10.61	7.91	6.60	5.85	5.37
5.25%	10.73	8.04	6.74	5.99	5.52
5.5%	10.85	8.17	6.88	6.14	5.68
5.75%	10.98	8.30	7.02	6.29	5.84
6%	11.10	8.44	7.16	6.44	6.00
6.25%	11.23	8.57	7.31	6.60	6.16
6.5%	11.35	8.71	7.46	6.75	6.32

4 Peter borrows $200 000 to buy a home unit. The bank charges 5% p.a. monthly reducible interest and the term of the loan is 15 years.
   a What are Peter's monthly repayments?
   b How much will he repay altogether on this loan?
   c How much interest will he pay?
   d Express the interest charged as a percentage of the amount borrowed.

5 Keira borrows $280 000 from a building society to buy a townhouse. She is charged 6.25% p.a. monthly reducible interest over 25 years.
   a How much will Keira repay altogether on this loan?
   b How much interest will she pay?
   c Calculate the flat yearly interest rate, correct to 2 decimal places

6 Sinead and Patrick borrowed $350 000 from a bank to build a house on their block of land. The loan was taken over a period of 30 years and the interest rate quoted was 5.75% p.a. monthly reducible.
   a How much interest will they pay on this loan?
   b Express the interest as a percentage of the amount borrowed.
   c Calculate the flat yearly interest rate, correct to 2 decimal places.

7 Xander wanted to borrow \$400 000 to buy a house and land package in a new housing estate. He compared the home loans on offer from two financial institutions. Sunshine Bank offered a loan at 6% p.a. monthly reducible interest over 25 years, while the Cape York Building Society loan was at 4.25% p.a. simple interest over 25 years. Which institution offered the cheaper loan, and by how much?

8 A couple borrowed \$270 000 at 4.75% p.a. monthly reducible interest over 15 years. A week later, the interest rate was increased to 5% p.a. How much more will this couple have to pay over the term of the loan as a result of the rate rise?

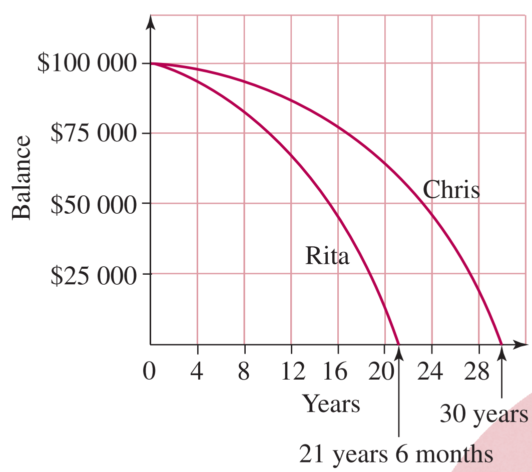
9 Geraldine borrowed \$185 000 at 5.25% p.a. monthly reducible interest to buy a half share in a home unit with her sister. How much money would Geraldine save if she paid off the loan over 15 years rather than over 20 years?

10 William took out a housing loan at 6.5% p.a. monthly reducible interest over 20 years. Over the term of the loan, he repaid the bank \$572 928.

- **a** How much did William borrow?
- **b** How much interest did he pay?
- **c** Find the equivalent flat yearly interest rate.

11 The graphs on the next page show the benefit of making more frequent repayments on a loan. In each case, the loan amount is \$100 000 and the interest rate is 9.25% p.a. monthly reducible. Chris makes one payment of \$822 per month, while his sister Rita makes separate payments of \$411 each fortnight.

- **a** Does each person repay the same amount each month?
- **b** How much sooner than Chris does Rita pay off her loan?
- **c** How much interest does she save over the term of the loan?
- **d** What is the effect of making fortnightly rather than monthly repayments?



Further applications

12 Maree borrows \$250 000 at 0.5% per month, monthly reducible interest. At the end of each month, the interest on the outstanding amount is added, then Maree makes a payment of \$1500.

- **a** How much does Maree still owe on the loan after the following periods?
  i 1 month **ii** 2 months **iii** 3 months **iv** 4 months
- **b** Will she eventually pay off this loan? Why?
- **c** If her monthly repayments were only \$1200, would she eventually pay off this loan? Why?

13 Wendy borrows $300 000 at 9% p.a. monthly reducible interest over 25 years. At the end of each month, the interest on the outstanding amount is added, then Wendy makes a payment of $2500.

- **a** How much of the loan will she have paid off after 6 months?
- **b** What could she do to improve this situation?

TRY THIS Housing loans

Each month the Reserve Bank of Australia announces any change to its rate of interest. The rate adopted is very important because it influences all other banks and lending agencies, most of which adjust their rates by the same margin as the Reserve Bank. The day of the announcement is eagerly awaited by financiers and shareholders, as even small changes can result in major changes to their financial position. Rises or falls are also of considerable importance to many families who have home loans to repay.

Investigate the effect of a small interest rate increase, say 0.25%, on a couple who plan to borrow $500 000 to allow them to purchase a house. Use the tables on pages 23 and 24 to check how much extra the house would cost them. This will vary according to the length of the loan and the interest rate, so you should do your calculations using several different scenarios to enable you to form a general view of the results of interest rate changes.

Focus on working mathematically

The cost of living in Australia

Introduction

In this chapter you have learned about earning and saving money. You have learned the different ways to earn interest from your savings and how to make wise choices in the purchase of goods to get the 'best buy'.

The focus in this activity is on the ways in which your savings and the goods you own depreciate, that is, lose their value. The value of money generally depreciates through *inflation*. What you can actually buy for your dollar decreases over time, so you need more money to maintain your standard of living. Governments try to limit inflation by various methods including control of interest rates.

In Australia, interest rates were increased twice in 2003 and again in March 2005 to try to discourage people from borrowing too much. Nevertheless, inflation in Australia is low and we have a very strong economy.

Learning activities

How do we measure inflation?

The first activities are designed for discussion in class or in small groups.

1 The **Consumer Price Index (CPI)** is the official measure of inflation in Australia. The CPI is measured every 3 months and published by the Australian Bureau of Statistics. It is a measure of *the cost of living* based on the prices of goods and services in 10 main groups. Three of these groups are food, clothing and footwear, and transport. Discuss other items you would include in the CPI and make a list. What might the other main groups be?

2 Every item in a group has a weighting based on its relative importance. For example in the food category, takeaway and fast foods might have a weighting of 1.2%, while vegetables might be 0.4%. What might this imply about the relative importance of these two items in the CPI at that time?

3 The price index of an item in any group is given by the formula

$$\text{price index} = \frac{\text{new price}}{\text{old price}} \times 100$$

Find the price index of a cheeseburger which cost \$2.50 in 2004 and \$3.50 in 2005.

4 In question 3, the year in which the old price was taken is the *base year*. What was the base year for the price index of the cheeseburger? Explain why the price index of an item in the base year is always 100.

5 The Australian Bureau of Statistics computes the CPI every quarter for each capital city and then computes a national weighted average. It is the *change* in the CPI that measures inflation, and it may be up or down on the base year. Here is the published percentage change in the CPI for the March quarter 2004 to the March quarter 2005.

Weighted average of eight capital cites	% change in CPI Mar 2004–2005
Food	0.8
Alcohol & tobacco	3.7
Clothing & footwear	–1.9
Housing	4.0
Household furnishings, supplies & services	–0.8
Health	5.7
Transportation	2.9
Communication	1.5
Recreation	1.6
Education	6.2
Miscellaneous	2.6
All groups	**2.4**

What was the rate of inflation for this period? What particular groups contributed most to the change? How might you account for the rise of the CPI for some items and a fall for others?

Challenge activities

1 The effect of inflation on savings

In 2002 Finn invested \$5000 for 3 years at 4.5% compound interest. Assuming an inflation rate of 2.4% p.a. over the 3 years, what was the real value of her investment at maturity?

Compare the solutions to the problem below and argue a case for your choice.

Solution 1

Using the compound interest formula $A = P(1 + R)^n$, $P = 5000$, $R = 0.045$, $n = 3$

By calculator: $A = \$5705.83$

However, because of inflation the value of her money is worth less by 2.4%.

So the real value is only 97.6% of \$5705.83.

By calculator, $\boldsymbol{A} = \mathbf{\$5568.89}$

Solution 2

Finn earns 4.5% interest each year but loses 2.4% each year because of inflation.

Her overall gain is only 2.1%. Using the formula, $P = 5000$, $R = 0.021$, $n = 3$

$\boldsymbol{A} = \mathbf{\$5321.66}$

Solution 3

Finn's balance on paper at the end of 3 years is \$5705.83 (see solution 1).

However over 3 years \$5000 depreciates 2.4% each year. The depreciation is given by the formula $A = P(1 - R)^n$ where $P = 5000$, $R = 0.024$ and $n = 3$. By calculator this comes to \$4648.57. Her loss due to inflation is therefore \$351.43

Answer: $\boldsymbol{A} = \mathbf{\$5705.83 - \$351.43 = \$5354.40}$

2 **Visit the Australian Bureau of Statistics** website <www.abs.gov.au> and click on Consumer Price Index Australia. Compare the capital cities for inflation rates and the items which had the most effect on the CPI for the latest data available. Why does it cost more to live in some cities than others?

Let's communicate

Make a class poster to illustrate what you have learned about inflation in this exercise. Summarise the important points in your workbook.

Reflecting

It is really important to understand how you can use mathematics to manage your finances. How you manage your income and your spending will determine your lifestyle, whether you earn a lot or have a modest income. Your future happiness will depend on it.

Language link with Macquarie

MACQUARIE

1 *Without using formulas explain simply the difference between* ***simple*** *and* ***compound*** *interest.*
2 *What is a* ***credit card****? Is the interest charged simple or compound?*
3 *What is the CPI? What does it measure?*
4 *What is the difference between a* ***wage*** *and a* ***salary****? Name two occupations where people are paid wages, and two that are paid salaries.*
5 *Read the* Macquarie Learners Dictionary *entry for* ***budget****:*

budget *noun* **1.** a plan showing how much money a person, organisation or country will earn and how it will be spent
–*verb* **2.** to make such a plan: *We budgeted on the basis that we would both have full-time jobs next year.*
–*adjective* **3.** not costing much: *budget clothes.*
❒ ***Word Family:*** **budgetary** *adjective*

Does the mathematical meaning differ from that in everyday speech?

CHAPTER REVIEW

1 Calculate the simple interest earned on the following investments.
 a \$300 at 5% p.a. for 2 years
 b \$450 at 7.5% for $3\frac{1}{2}$ years
 c \$785 at 6.2% p.a. for 8 years
 d \$1200 at $4\frac{3}{4}$% for $5\frac{1}{2}$ years

2 Find the total balance of an account after 4 years if \$2000 is invested at 6% p.a. simple interest.

3 Calculate the simple interest that would be earned on the following investments.
 a \$700 at 8% p.a. for 6 months
 b \$840 at 6.5% p.a. for 9 months
 c \$1500 at 7.25% p.a. for 5 months

4 Brian invested \$600 with a building society for 3 years, with simple interest paid at 5% p.a. He then transferred the principal and interest to a bank account for 2 years and was paid simple interest at the rate of 5.8% p.a.
 a How much money did Brian have at the end of the 5th year?
 b How much interest did he earn altogether?

5 Louise invested an amount of money for 6 years at 4% p.a. simple interest. She earned \$192 in interest on this investment. How much did she invest?

6 Rachel deposited \$12 000 in a savings account. After 8 years she had earned \$6720 in simple interest. Find the annual interest rate.

7 A sum of \$20 000 was invested for 3 years at 5% p.a. compounded annually. Draw up a compound interest table and hence find the account balance after:
 a 1 year **b** 2 years **c** 3 years

8 Theo invested \$50 000 at 4.5% p.a. interest, compounding annually. Draw up a compound interest table and hence determine the account balance and interest earned after 4 years.

9 Use the compound interest formula to find the value of each investment and the amount of interest earned, if the interest compounds annually.
- **a** \$2000 at 4% p.a. for 2 years
- **b** \$1500 at 6% p.a. for 4 years
- **c** \$3750 at 8.5% p.a. for 5 years
- **d** \$16 000 at 7.25% p.a. for 3 years
- **e** \$25 000 at $5\frac{1}{2}$% p.a. for 6 years
- **f** \$40 000 at $6\frac{3}{4}$% p.a. for 8 years

10 Bill invests \$7500 at 12% p.a. compound interest. Find the value of this investment after 5 years if the interest compounds:
- **a** annually
- **b** half-yearly
- **c** quarterly
- **d** monthly

11 Find the value of the following investments.
- **a** \$4000 after 2 years at 5% p.a., compounded half-yearly
- **b** \$6500 after 3 years at 6% p.a., compounded half-yearly
- **c** \$10 000 after 2 years at 8% p.a., compounded quarterly
- **d** \$25 000 after 7 years at 6% p.a., compounded quarterly
- **e** \$15 000 after 2 years at 12% p.a., compounded monthly
- **f** \$32 000 after $1\frac{1}{2}$ years at 6% p.a., compounded monthly.

12 An investor deposited \$150 000 at 7% p.a., with interest compounding annually.
- **a** How much interest will have accrued after 4 years?
- **b** Find the equivalent simple interest rate, correct to 2 decimal places.

13 Which pays the greater amount of interest on an investment of \$5000 and by how much?
- **A** 9% p.a. simple interest for 7 years
- **B** 8.5% p.a. compound interest for 6 years, compounded annually

14 A bank advertises the following savings plans:
- **A** 8.8% p.a. compounded half-yearly
- **B** 8.4% p.a. compounded quarterly

Which plan will provide the greater interest on an investment of \$10 000 for 3 years, and by how much?

15 House prices in a certain suburb are expected to rise by an average of 12% per year for the next 5 years. Penny bought a house in 2005 for \$360 000. What will be the expected value of the house in 2010?

16 If the national inflation rate is 3.8% p.a., find the expected cost of a \$3 loaf of bread in 10 years time. Answer correct to the nearest 5 cents.

17 Find the amount of money, correct to the nearest dollar, that will grow to:
- **a** \$1072 if invested for 6 years at 5% p.a., compounded annually
- **b** \$7110.77 if invested for 4 years at 7% p.a., compounded half-yearly
- **c** \$22 289.21 if invested for 5 years at 8% p.a., compounded quarterly

18 A new car is purchased for \$20 000 and depreciates by 10% p.a. Find, to the nearest dollar, the value of the car after:
- **a** 1 year
- **b** 2 years
- **c** 5 years
- **d** 10 years

CHAPTER REVIEW

19 A printing machine purchased for \$55 000 depreciates at the rate of 7% p.a. Find, to the nearest dollar, the value of the printing machine after 6 years.

20 A stockholder purchased 300 000 shares at \$1.80 each. The shares rose in value by 5% during the first month, then fell in value by 3% each month after that. Find, to the nearest dollar, the value of the shares after:

a 1 month **b** 6 months

21 Karen bought a computer scanner for \$1500. It depreciated by 5% p.a. After how many years will the scanner have lost more than half its value?

22 A television was advertised for sale with a marked price of \$800. The store offered a discount of 10%, plus an additional discount of 5% on the discounted price to customers who paid cash.

a Find the cash purchase price.

b Are the successive discounts equivalent to a single discount of 15%?

23 Shaun bought a new bed priced at \$980. He paid a deposit of \$140, followed by 12 equal monthly instalments of \$78.75.

a How much did he pay altogether for the bed?

b How much interest did he pay?

c What was the rate of simple interest charged?

24 Anthony borrowed \$18 000 at 11% p.a. simple interest to build a new pool and garage. The loan plus interest was to be repaid in equal monthly instalments over 4 years.

a How much interest was charged on the loan?

b How much will Anthony repay altogether?

c Calculate the monthly repayments.

25 A couple purchased new carpet to the value of \$12 000 for their house. They paid a deposit of 30%, with interest charged on the balance at 10% p.a. The balance plus interest is to be repaid in equal monthly instalments over 3 years.

a Calculate the deposit.

b Find the balance owing.

c Calculate the interest charged.

d Find the amount of each monthly instalment.

26 Melissa borrowed \$6000 to renovate her kitchen. She repaid the loan by making monthly payments of \$290 over 2 years.

a How much will she repay altogether on this loan?

b Calculate the interest charged.

c Find the annual interest rate.

27 Find the amount of interest payable on a credit card debt of:

a \$520 for 13 days at 0.052% per day

b \$952 for 38 days at 15.4% p.a.

28 Ray bought a jumper and three pairs of pants on 8 August for \$390 using his credit card. He paid this off on 4 September and was charged interest at the rate of 0.051 72% per day. How much interest did he pay? Ray's credit card has no interest-free period.

The table on the next page shows the monthly repayments on a loan of \$1000 at monthly reducible interest rates. Use the table to answer questions 29 and 30.

Interest rate p.a.	1 year	2 years	3 years
9%	87.45	45.68	31.80
9.5%	87.68	45.91	32.03
10%	87.92	46.14	32.27
10.5%	88.15	46.38	32.50
11%	88.38	46.61	32.74
11.5%	88.62	46.84	32.98
12%	88.85	47.07	33.21

29 Find the monthly repayments on a loan of:

a $1000 at 10.5% p.a. for 1 year

b $7000 at 9% p.a. for 3 years

c $4600 at 11.5% p.a. for 2 years

d $21 300 at 10% p.a. for 3 years

30 For each loan below, find the amount of interest charged

a $1000 at 9% p.a. for 1 year

b $5000 at 12% p.a. for 2 years

c $8300 at 10.5% p.a. for 3 years

The table below shows the monthly payments required to repay each $1000 of a housing loan at monthly reducible interest rates. Use this table to answer questions 31–33.

Interest rate p.a.	20 years	25 years	30 years
4.5%	6.33	5.56	5.07
4.75%	6.46	5.70	5.22
5%	6.60	5.85	5.37
5.25%	6.74	5.99	5.52
5.5%	6.88	6.14	5.68
5.75%	7.02	6.29	5.84
6%	7.16	6.44	6.00
6.25%	7.31	6.60	6.16
6.5%	7.46	6.75	6.32

31 A couple borrowed $400 000 over 25 years at 5.75% p.a. monthly reducible interest, to buy a house.

a What are the monthly repayments on this loan?

b How much will the couple repay altogether?

c How much interest will they pay?

d Calculate the equivalent flat yearly interest rate.

32 Amanda borrowed $270 000 over 20 years at 6% p.a., monthly reducible interest. Later that day, the interest rate rose to 6.5% p.a. How much extra will she have to pay over the course of the loan as a result of the rate rise?

33 Gabriel took out a housing loan over 30 years at 5.25% p.a., monthly reducible interest. Over the term of the loan he will repay $476 928.

a How much did Gabriel borrow?

b How much interest will he pay?

c Calculate the equivalent flat yearly interest rate, correct to 1 decimal place.

CHAPTER REVIEW

2 Trigonometry

This chapter at a glance

Stage 5.1/5.2/5.3

After completing this chapter, you should be able to:

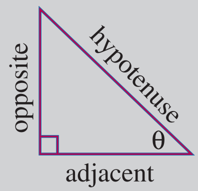
- use trigonometry to find lengths of sides in right-angled triangles where the given angle is measured in degrees and minutes
- use trigonometry to find sizes of angles measured in degrees and minutes in right-angled triangles
- solve practical problems that involve finding the lengths of sides and sizes of angles in right-angled triangles, where the angles are measured in degrees and minutes
- solve practical problems that involve angles of elevation and depression, where the angles are measured in degrees and minutes
- express the tangent ratio in terms of the sine and cosine ratios and use this result to solve problems
- state the complementary results for the sine and cosine ratios and use these results to solve problems
- state the exact values and use them to solve problems
- draw diagrams showing the compass bearing of one point from another point
- draw diagrams showing the true bearing of one point from another point
- find the bearing of A from B given the bearing of B from A
- solve practical trigonometry problems involving bearings.

2.1 The trigonometric ratios

The trigonometric ratios are the ratios of pairs of sides in right-angled triangles. These ratios in relation to an angle θ are the sine ratio (sin θ), the cosine ratio (cos θ) and the tangent ratio (tan θ).

The definitions of the trigonometric ratios are:

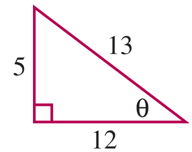
- $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
- $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
- $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$



NOTE: The abbreviations SOH CAH TOA can be used to help you remember these definitions.

Example 1

Find values for sin θ, cos θ and tan θ in the given triangle.

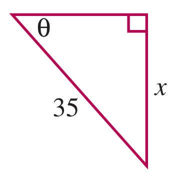


Solutions

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5}{13} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{12}{13} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12}$$

Example 2

Find value of x, given that $\sin\theta = \dfrac{3}{7}$.



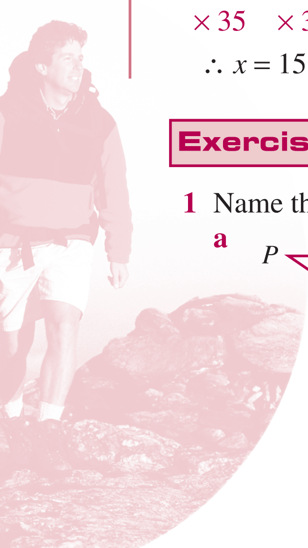
Solution

$$\sin\theta = \frac{3}{7}$$

$$\frac{x}{35} = \frac{3}{7}$$

$$\times 35 \quad \times 35$$

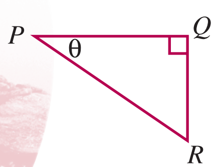
$$\therefore x = 15$$



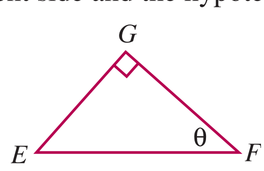
Exercise 2.1

1 Name the opposite side, the adjacent side and the hypotenuse in each triangle.

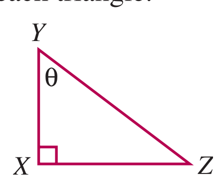
a



b



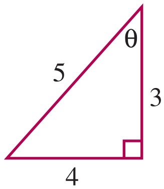
c



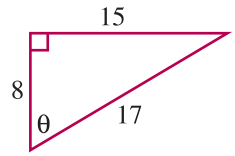
2 For each of the following triangles, state as a fraction the value of:

i sin θ **ii** cos θ **iii** tan θ

a



b



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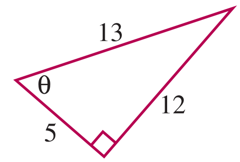

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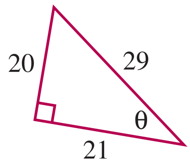

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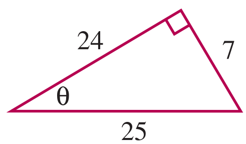

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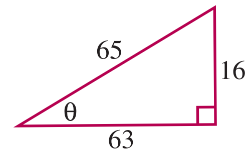
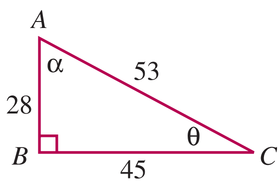


Consolidation

3



Name the angle that has a:

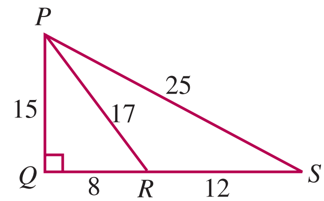
a tangent of $\frac{45}{28}$ **b** sine of $\frac{28}{53}$

c cosine of $\frac{45}{53}$ **d** tangent of $\frac{28}{45}$

e cosine of $\frac{28}{53}$ **f** sine of $\frac{45}{53}$

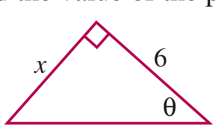
4 Find, without simplifying, the value of:

a sin ∠*PRQ* **b** tan ∠*PSQ*
c cos ∠*SPQ* **d** sin ∠*QPR*
e tan ∠*QPR* **f** cos ∠*PRQ*
g sin ∠*PSQ* **h** tan ∠*SPQ*
i cos ∠*QPR* **j** sin ∠*SPQ*
k tan ∠*PRQ* **l** cos ∠*PSQ*



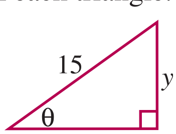
5 Find the value of the pronumeral in each triangle.

a



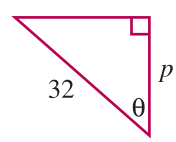
$\tan\theta = \frac{1}{2}$

b



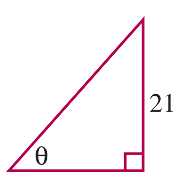
$\sin\theta = \frac{3}{5}$

c



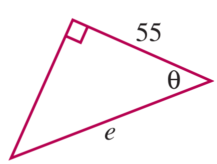
$\cos\theta = \frac{5}{8}$

d



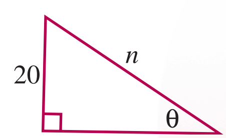
$\tan\theta = \frac{7}{9}$

e



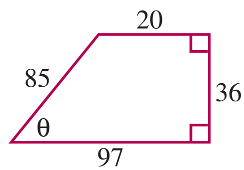
$\cos\theta = \frac{11}{12}$

f

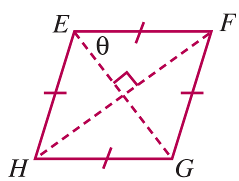


$\sin\theta = \frac{4}{7}$

6 Find the values for sin θ, cos θ, tan θ in this trapezium.



7



EFGH is a rhombus with diagonals *EG* and *FH* of length 32 cm and 126 cm respectively.

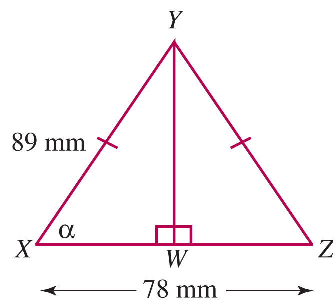
a Find the side length of the rhombus.

b Find values for sin θ, cos θ, tan θ.

8 In the isosceles triangle *XYZ*, *W* is a point on *XZ* such that $YW \perp XZ$. If $XY = YZ = 89$ mm and $XZ = 78$ mm:

a find the length of *YW*

b find values for sin α, cos α and tan α.



Further applications

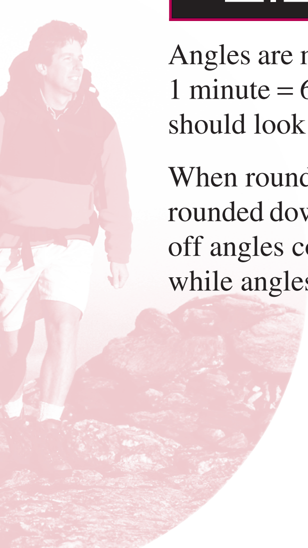
9 **a** If $\tan\theta = \frac{20}{21}$, find values for sin θ, cos θ.

b If $\sin\theta = \frac{24}{25}$, find values for cos θ, tan θ.

c If $\cos\theta = \frac{55}{73}$, find values for tan θ, sin θ.

10 If $\tan\theta = \frac{x}{y}$, find expressions for sin θ, cos θ.

2.2 Degrees and minutes



Angles are measured in degrees, minutes and seconds (1 degree = 60 minutes, i.e. $1° = 60'$ and 1 minute = 60 seconds, i.e. $1' = 60''$). The degrees, minutes and seconds key on your calculator should look like either DMS or ° ′ ″.

When rounding off angles correct to the nearest degree, angles with less than 30 minutes are rounded down, while angles with 30 minutes or more are rounded up. Similarly, when rounding off angles correct to the nearest minute, angles with less than 30 seconds are rounded down, while angles with 30 seconds or more are rounded up.

Evaluating trigonometric expressions

A calculator can be used to find the value of a trigonometric expression. The order in which you press the keys will vary between calculator models.

> **To evaluate a trigonometric expression:**
> - **press the appropriate trigonometric ratio key, then enter the angle.**

Finding an angle

The inverse key $\boxed{\text{INV}}$, shift key $\boxed{\text{shift}}$, or second function key $\boxed{\text{2nd F}}$, can be used to undo the process of finding the sine, cosine or tangent of an angle, and hence to find the size of an angle.

> **To find the size of an angle given either a fraction or a decimal:**
> - **press the $\boxed{\textbf{INV}}$, $\boxed{\textbf{shift}}$ or $\boxed{\textbf{2nd F}}$ key followed by the appropriate trigonometric function key**
> - **enter the fraction or decimal into the calculator, then press $\boxed{=}$**
> - **round off the angle as required.**

Example 1

Round off each angle correct to the nearest degree.

a $15°26'$ **b** $37°43'$ **c** $75°30'$

Solutions

a $26'$ is less than $30'$, so we round down: $15°26' \doteqdot 15°$.
b $43'$ is greater than $30'$, so we round up: $37°43' \doteqdot 38°$.
c $30'$ is halfway, so we round up: $75°30' \doteqdot 76°$.

Example 2

Round off each angle correct to the nearest minute.

a $21°38'19''$ **b** $52°13'49''$ **c** $5°56'30''$

Solutions

a $19''$ is less than $30''$, so we round down: $21°38'19'' \doteqdot 21°38'$.
b $49''$ is greater than $30''$, so we round up: $52°13'49'' \doteqdot 52°14'$.
c $30''$ is halfway, so we round up: $5°56'30'' \doteqdot 5°57'$.

Example 3

Evaluate each expression, correct to 2 decimal places.

a $\sin 23°16'$ **b** $19 \cos 8°39'$ **c** $\dfrac{31}{\tan 12°42'}$

Solutions

	Calculator steps	Calculator readout	Answer
a	$\boxed{\sin}$ 23 $\boxed{\circ\,'\,''}$ 16 $\boxed{\circ\,'\,''}$ $\boxed{=}$	0.395011105	0.40
b	19 $\boxed{\cos}$ 8 $\boxed{\circ\,'\,''}$ 39 $\boxed{\circ\,'\,''}$ $\boxed{=}$	18.7838847	18.78
c	31 $\boxed{\div}$ $\boxed{\tan}$ 12 $\boxed{\circ\,'\,''}$ 42 $\boxed{\circ\,'\,''}$ $\boxed{=}$	137.5578485	137.56

Example 4

Evaluate $\dfrac{\cos 42°9'}{\tan 8° + \sin 54°}$ correct to 3 significant figures.

Solution

Calculator steps: $\boxed{\cos}$ 42 $\boxed{\circ\,'\,''}$ 9 $\boxed{\circ\,'\,''}$ $\boxed{\div}$ $\boxed{(}$ $\boxed{\tan}$ 8 $\boxed{+}$ $\boxed{\sin}$ 54 $\boxed{)}$ $\boxed{=}$

Calculator readout: 0.780774459

Answer: 0.781

Example 5

Find θ, correct to the nearest minute.

a $\tan\theta = 1.3759$ **b** $\sin\theta = 0.1382$

Solutions

	Calculator steps	Calculator readout	Answer
a	$\boxed{\text{shift}}$ $\boxed{\tan}$ 1.3759 $\boxed{=}$ $\boxed{\circ\,'\,''}$	53°59′25.65″	53°59′
b	$\boxed{\text{shift}}$ $\boxed{\sin}$ 0.1382 $\boxed{=}$ $\boxed{\circ\,'\,''}$	7°56′37.32″	7°57′

Exercise 2.2

1 Round off each angle, correct to the nearest degree.

a 19°56′ **b** 36°15′ **c** 20°30′ **d** 71°24′
e 62°50′ **f** 84°9′ **g** 109°35′ **h** 137°52′

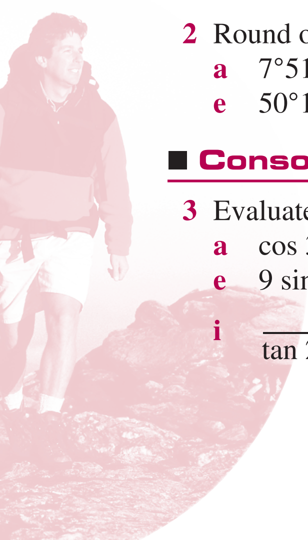
2 Round off each angle, correct to the nearest minute.

a 7°51′35″ **b** 22°6′13″ **c** 14°25′30″ **d** 76°32′49″
e 50°16′8″ **f** 68°4′30″ **g** 119°12′37″ **h** 164°2′10″

■ Consolidation

3 Evaluate these trigonometric expressions, correct to 2 decimal places.

a cos 37°8′ **b** tan 6°29′ **c** sin 21°54′ **d** cos 43°35′
e 9 sin 57°18′ **f** 25 cos 8°26′ **g** 16.3 tan 49°50′ **h** 45.7 sin 20°32′
i $\dfrac{13}{\tan 25°11'}$ **j** $\dfrac{29.8}{\sin 5°27'}$ **k** $\dfrac{1}{\cos 44°20'}$ **l** $\dfrac{1}{\tan 72°55'}$



4 Evaluate each expression, correct to 4 significant figures.

a $\dfrac{9 \sin 53°28'}{8 \cos 8°14'}$ **b** $\dfrac{17 \tan 73°45'}{21 \sin 12°29'}$ **c** $\dfrac{23 \cos 15°34'}{5 \tan 9°56'}$

5 Evaluate each expression, correct to 4 significant figures.

a $\dfrac{\sin 40°18' + \tan 26°11'}{\cos 47°19'}$ **b** $\dfrac{\cos 4°37'}{\sin 52°25' - \tan 9°14'}$

6 Find the acute angle θ, correct to the nearest minute.

a $\tan \theta = 0.3675$ **b** $\cos \theta = 0.7173$ **c** $\sin \theta = 0.1247$
d $\cos \theta = 0.2662$ **e** $\sin \theta = 0.9641$ **f** $\tan \theta = 0.0899$
g $\sin \theta = 0.5863$ **h** $\tan \theta = 1.4085$ **i** $\cos \theta = 0.9999$
j $\tan \theta = 2.7891$ **k** $\cos \theta = 0.6124$ **l** $\sin \theta = 0.4207$

7 Find the acute angle θ, correct to the nearest minute.

a $\tan \theta = \frac{2}{5}$ **b** $\sin \theta = \frac{9}{10}$ **c** $\cos \theta = \frac{3}{4}$ **d** $\sin \theta = \frac{6}{11}$

Further applications

8 **a** If $\cos \theta = 0.7923$, find $\tan \theta$ and $\sin \theta$, correct to 4 decimal places.
b If $\tan \theta = 4.0672$, find $\sin \theta$ and $\cos \theta$, correct to 4 decimal places.

9 Find the acute angle α, correct to the nearest minute.

a $3 \sin \alpha + 5 = 6$ **b** $\dfrac{7 \tan \alpha}{3} = 4$ **c** $\dfrac{2}{\cos \alpha} = 9$

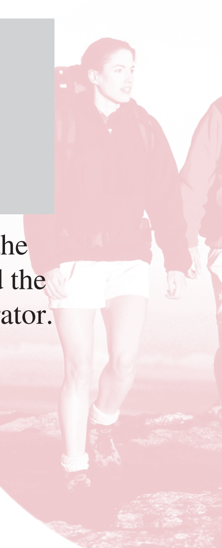
2.3 Finding the length of a side

Pythagoras' theorem is used to find the length of a side in a right-angled triangle when the lengths of the other two sides are known. Trigonometry is used to find the length of a side when the length of one other side and the size of one angle are known.

Finding the length of a short side

To find the length of the opposite or adjacent sides:
- **determine which ratio is to be used**
- **write down a trigonometric equation**
- **multiply both sides by the denominator**
- **evaluate using a calculator.**

NOTE: In those questions involving the tangent ratio where the pronumeral would be in the denominator, it is often easier to find the other acute angle in the triangle and use it to find the required side length. By using the other angle, the pronumeral should then be in the numerator.



Finding the length of the hypotenuse

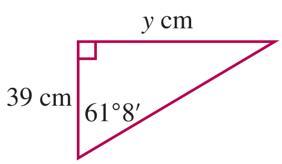
To find the length of the hypotenuse:
- **determine whether the sine or cosine ratio is to be used**
- **write down a trigonometric equation**
- **take the reciprocal of both sides**
- **multiply both sides by the denominator under the pronumeral**
- **evaluate using a calculator.**

Example 1

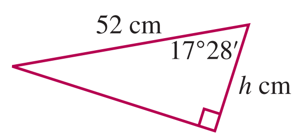
EG +S

Find the value of the pronumeral in each of the following, correct to 1 decimal place.

a



b



Solutions

a

$$\tan 61°8' = \frac{y}{39}$$

$$\times 39 \quad \times 39$$

$$\begin{aligned} y &= 39 \times \tan 61°8' \\ &= 70.74562525 \\ &= 70.7 \text{ (1 decimal place)} \end{aligned}$$

b

$$\cos 17°28' = \frac{h}{52}$$

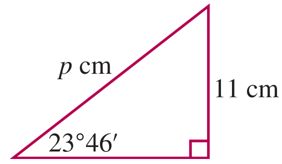
$$\times 52 \quad \times 52$$

$$\begin{aligned} h &= 52 \times \cos 17°28' \\ &= 49.60237011 \\ &= 49.6 \text{ (1 decimal place)} \end{aligned}$$

Example 2

EG +S

Find the length of the hypotenuse, correct to 1 decimal place.



Solution

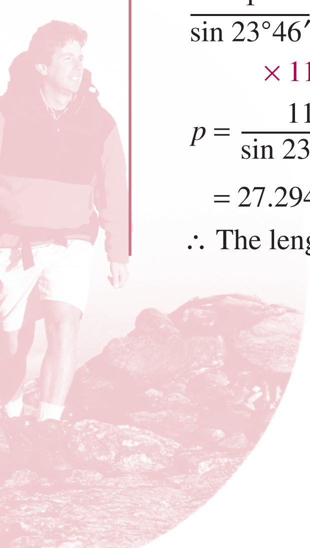
$$\sin 23°46' = \frac{11}{p}$$

$$\frac{1}{\sin 23°46'} = \frac{p}{11}$$

$$\times 11 \quad \times 11$$

$$\begin{aligned} p &= \frac{11}{\sin 23°46'} \\ &= 27.2944099 \end{aligned}$$

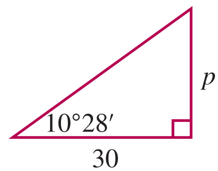
∴ The length of the hypotenuse is 27.3 cm (1 decimal place).



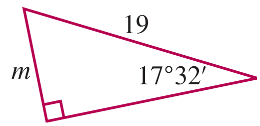
Exercise 2.3

1 Find the value of the pronumeral in each of these, correct to 1 decimal place. All lengths are in centimetres.

a



b



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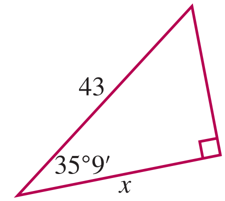

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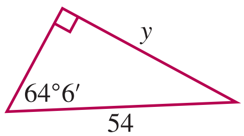

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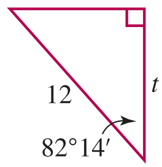

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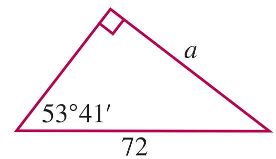

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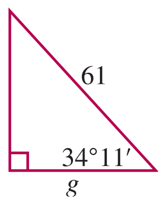

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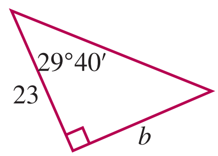

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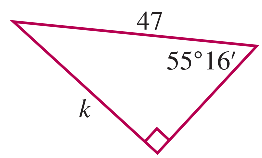

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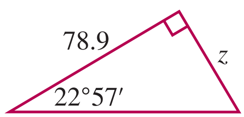

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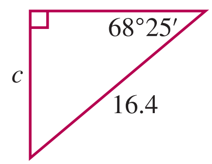

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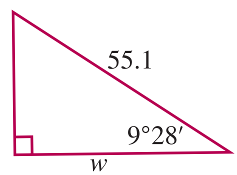

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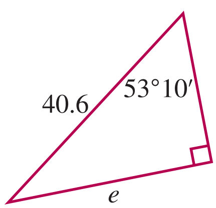

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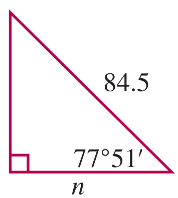

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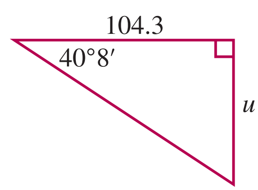
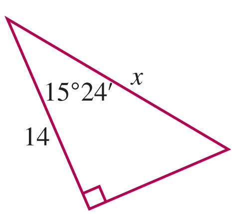


2 Find the value of the pronumeral in each triangle, correct to 2 decimal places. All lengths are in metres.

a



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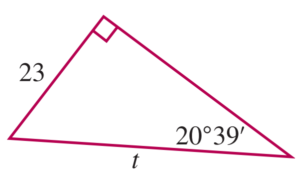

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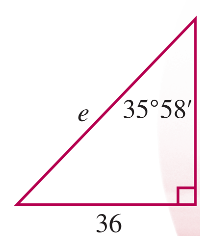

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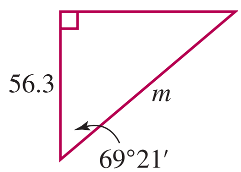

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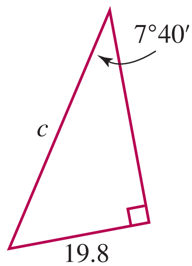

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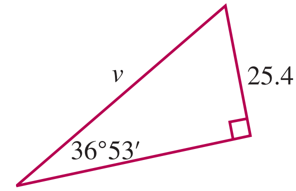


Consolidation

3 Draw a diagram and mark on it all of the given information, to answer each of the following.

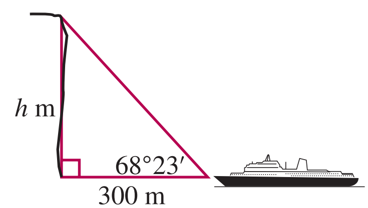
a In ΔXYZ, $\angle Y = 90°$, $\angle X = 54°17'$ and $XZ = 25$ cm. Find the length of XY, correct to 1 decimal place.

b In ΔPQR, $\angle R = 90°$, $\angle Q = 21°4'$ and $QR = 43$ m. Find the length of PR, correct to 4 significant figures.

c In ΔLMN, $\angle L = 90°$, $\angle N = 16°45'$ and $MN = 37$ cm. Find the length of LM, correct to the nearest millimetre.

d In ΔABC, $\angle A = 90°$, $\angle B = 47°50'$ and $AC = 85$ cm. Find the length of BC, correct to 3 significant figures.

4 A cruise ship is anchored 300 m from the base of a vertical cliff. A passenger stood on the bow of the ship and looked up to the top of the cliff at an angle of 68°23′. Calculate the height of the cliff, correct to the nearest metre.



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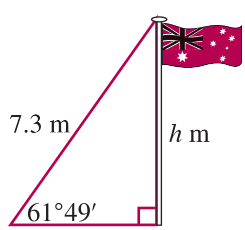
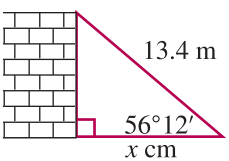
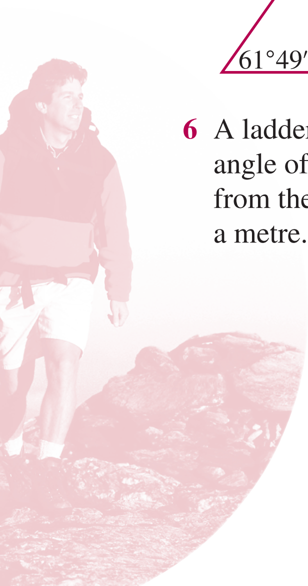
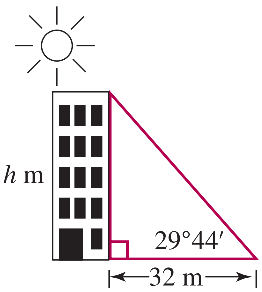


A wire of length 7.3 m attached to the top of a flagpole is inclined to the ground at an angle of 61°49′. Find the height of the flagpole, correct to the nearest tenth of a metre.

6 A ladder of length 13.4 m leans against a wall and makes an angle of 56°12′ with the ground. How far is the foot of the ladder from the base of the wall? Answer correct to the nearest tenth of a metre.

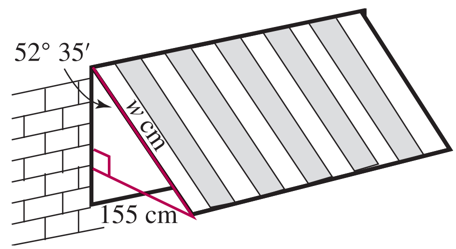


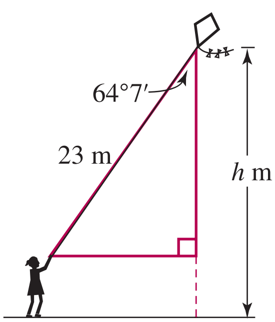


7 

A school building casts a shadow 32 m long on the playground when the sun has an altitude of 29°44′. Calculate the height of the building, correct to the nearest metre.

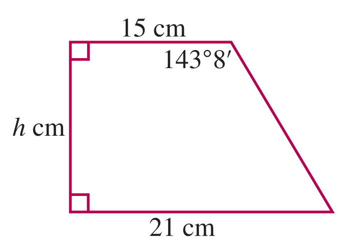
8 The window awning outside a butcher shop is attached to the front wall at an angle of 52°35′. Find the width of the awning given that it reaches 155 cm out from the wall. Answer correct to the nearest centimetre.

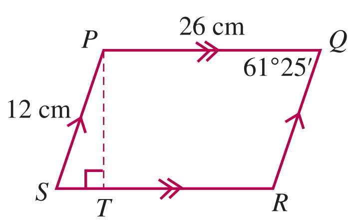


9 

Elicia is flying a kite at the local park. She is 1.4 m tall and the hand holding the string is just above eye level. The string is 23 m long and makes an angle of 64°7′ with the vertical. How high is the kite above the ground? Answer correct to the nearest tenth of a metre.

10 Find the height, *h* cm, of this trapezium, correct to 1 decimal place.

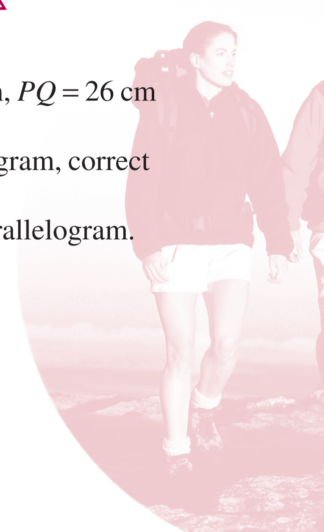


11 

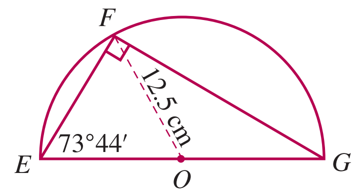
In the parallelogram *PQRS*, *PS* = 12 cm, *PQ* = 26 cm and ∠*PQR* = 61°25′.

a Find *PT*, the height of the parallelogram, correct to 1 decimal place.

b Hence, calculate the area of the parallelogram.



12 In the semicircle, centre O, $OF = 12.5$ cm and $\angle FEG = 73°44'$.



a Write down the length of the diameter EG.

b Find the length of the chord FG, correct to the nearest centimetre.

13 The diagonal in a rectangle makes an angle of 26°39′ with the length. If the rectangle has a length of 15 cm, find the following correct to 1 decimal place:

a the width of the rectangle

b the length of a diagonal

14 A straight road rises at an angle of 19°16′ to the horizontal and has a vertical rise of 115 m. Find the length of the road, correct to the nearest metre.

15 When the sun has an altitude of 67°42′, a pine tree casts a shadow 8.3 m long. Find the height of the tree. Answer correct to the nearest tenth of a metre.

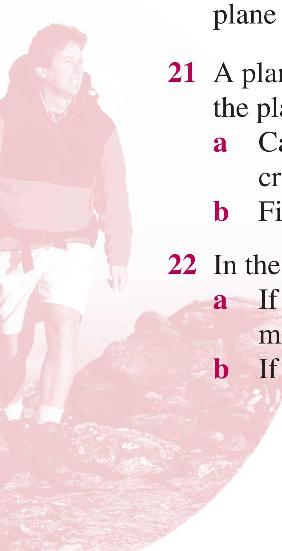
16 A wedge-shaped ramp is set up for a car stunt on a movie set. The ramp has an angle of inclination of 21°37′ and a vertical rise of 2.45 m. Find the length of the ramp, correct to the nearest centimetre.

17 A surveyor sights the top of a mountain from a distance of 1500 m. The angle of elevation of the top of the mountain is 51°17′. Find the height of the mountain, correct to the nearest metre.

18 A wall casts a shadow of length 24.6 m along the ground. If the angle of elevation of the sun is 18°23′ from the end of the shadow, find the height of the wall, correct to the nearest centimetre.

19 From a window on the fifth floor of an office tower, a worker observes a ferry that has just moored across the street at Circular Quay. The window is 20 m above the street and the angle of depression of the ferry from the window is 21°40′. How far is the ferry from the base of the building? Answer correct to 3 significant figures.

20 From the top of the control tower at the airport, the angle of depression of a plane on the tarmac is 39°25′. If the height of the control tower is 47 m, find the distance between the plane and the tower, correct to the nearest metre.



21 A plane takes off at a speed of 300 km/h and rises at an angle of 36°49′. After 3 minutes the plane levels off and continues flying at a constant altitude.

a Calculate the distance in metres that the plane flies through the air, before reaching its cruising altitude.

b Find the cruising altitude of the plane in metres, correct to 1 significant figure.

22 In the isosceles ΔPQR, $PQ = PR$ and S is a point on QR such that $PS \perp QR$.

a If $\angle PQR = 74°53'$ and $PQ = 85$ mm, find the length of QR, correct to the nearest millimetre.

b If $\angle PQR = 62°35'$ and $QR = 126$ mm, find the length of PQ, correct to 1 decimal place.

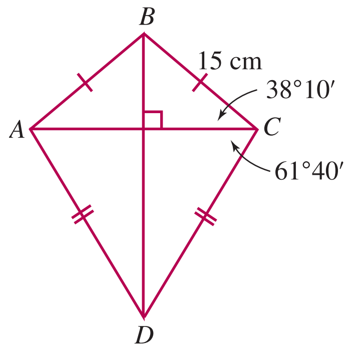
c If $\angle QPR = 104°28'$ and $PS = 93$ mm, find the length of QR, correct to 3 significant figures.

d If $\angle QPR = 81°36'$ and $QR = 138$ mm, find the length of PS, correct to 2 significant figures.

23 A cone has a diameter of 31.6 cm and the angle between the vertical axis and the slant height is $15°52'$. Find the perpendicular height of the cone, correct to 4 significant figures.

Further applications

24



In the kite $ABCD$, $AB = BC = 15$ cm, $AD = DC$, $\angle ACB = 38°10'$ and $\angle ACD = 61°40'$.

a Find the length of the diagonal AC, correct to 1 decimal place.

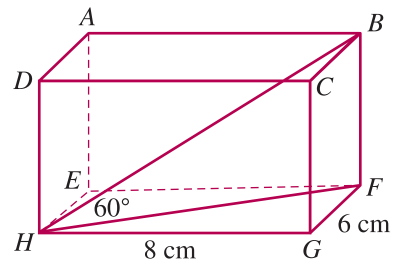
b Find the length of the diagonal BD, correct to 1 decimal place.

c Hence, calculate the area of the kite.

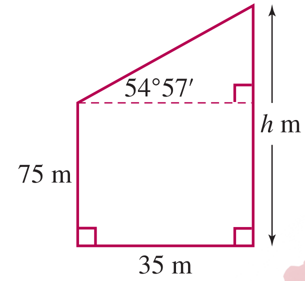
25 In the rectangular prism shown, $FG = 6$ cm, $HG = 8$ cm and $\angle BHF = 60°$.

a Find the length of FH.

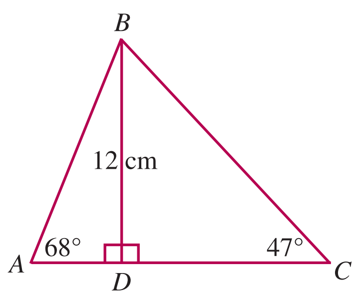
b Hence, find the length of the diagonal BH.



26 Two buildings of different heights stand on opposite sides of the street, 35 m apart. A man standing on the roof of the shorter building measures the angle of elevation of the top of the taller building as $54°57'$. The shorter building is 75 m tall. Calculate the height of the taller building, correct to the nearest metre.

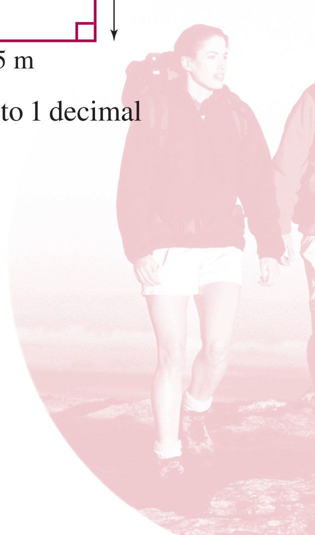


27



a Find the length of AD and DC, correct to 1 decimal place.

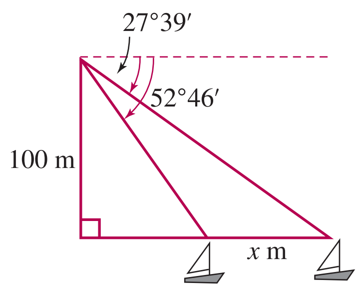
b Hence, find the length of AC.



28 **a** Find the length of QS, correct to 1 decimal place.

b Hence, find the length of QR, correct to 1 decimal place.

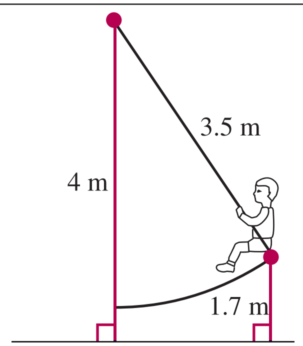
29



From the top of a lighthouse situated 100 m above sea level, the angles of depression of two boats at sea are 27°39′ and 52°46′ respectively. Calculate the distance between the boats, correct to the nearest metre.

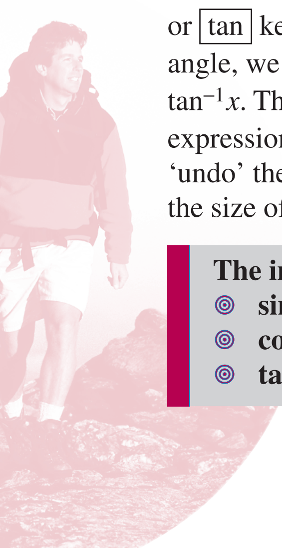
TRY THIS **Swing**

The supporting bar of a playground swing is 4 m above the ground and the swing itself is 3.5 m long. A man places his child on the swing and pulls it back until the seat is 1.7 m above the ground. Through what angle will the child swing? Give your answer to the nearest degree.



2.4 Finding the size of an angle

Trigonometry can also be used to find sizes of angles in triangles. In exercise 2.2 we used the inverse key INV, shift key shift, or second function key 2nd F, followed by the sin, cos or tan key to find the size of an angle given a fraction or decimal value. When we find an angle, we are actually using the **inverse trigonometric functions**—$\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$. The notation $\sin^{-1}x$, read as the *inverse sine of x*, means 'the angle whose sine is x'. The expressions $\cos^{-1}x$ and $\tan^{-1}x$ have similar meanings. The inverse trigonometric functions 'undo' the basic trigonometric functions of the sine, cosine and tangent ratios, and hence give the size of the angle.



The inverse trigonometric functions are used to find the size of an angle, where:
- **$\sin^{-1}x$ means 'the angle whose sine is x'**
- **$\cos^{-1}x$ means 'the angle whose cosine is x'**
- **$\tan^{-1}x$ means 'the angle whose tangent is x'.**

To find the size of an angle:

- **press the $\boxed{\text{INV}}$, $\boxed{\text{shift}}$ or $\boxed{\text{2nd F}}$ key followed by the appropriate trigonometric function key**
- **enter the given fraction or decimal, then press $\boxed{=}$**
- **round off the angle as required.**

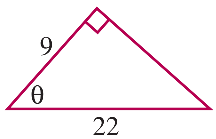
NOTE: If the side lengths are decimals you may need to use the division and grouping symbols keys on the calculator rather than the fraction key.

EG +S

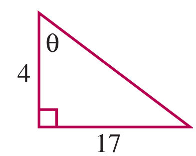
Example 1

Find the size of the angle θ, correct to the nearest minute.

a



b



Solutions

a $\cos\theta = \dfrac{9}{22}$

$\theta = \cos^{-1}\left(\dfrac{9}{22}\right)$

Press $\boxed{\text{shift}}\ \boxed{\cos}\ 9\ \boxed{a\tfrac{b}{c}}\ 22\ \boxed{=}\ \boxed{\circ\,'\,''}$

$\theta = 65°51'8.14''$

$= 65°51'$ (to the nearest minute)

b $\tan\theta = \dfrac{17}{4}$

$\theta = \tan^{-1}\left(\dfrac{17}{4}\right)$

Press $\boxed{\text{shift}}\ \boxed{\tan}\ 17\ \boxed{a\tfrac{b}{c}}\ 4\ \boxed{=}\ \boxed{\circ\,'\,''}$

$\theta = 76°45'34.13''$

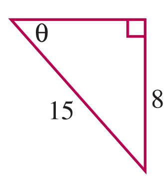
$= 76°46'$ (to the nearest minute)

Exercise 2.4

In this exercise, find all angles correct to the nearest minute, unless otherwise stated.

1 Find the size of the angle θ. All lengths are in metres.

a



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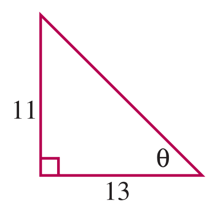

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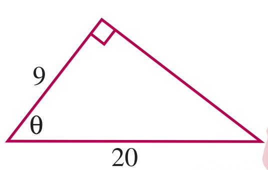

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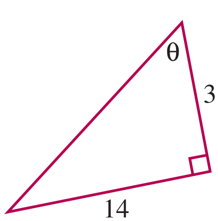

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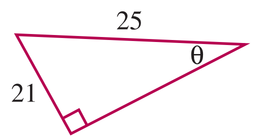

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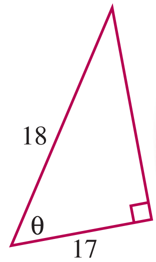

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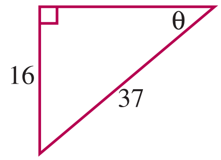

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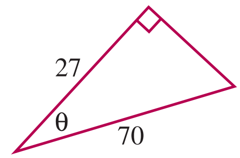
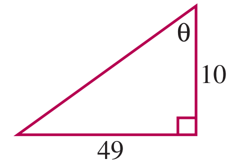


i



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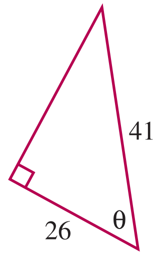

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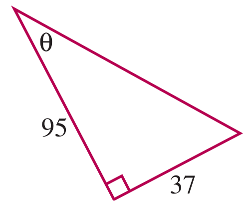

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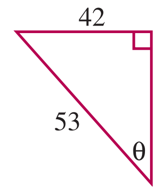
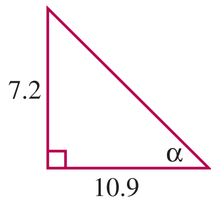
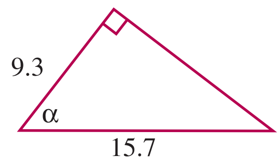


2 Find the size of the angle α. All lengths are in millimetres.

a



b



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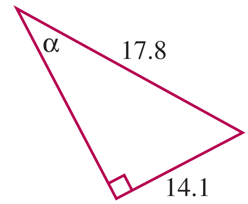

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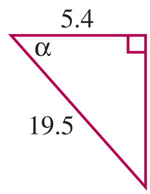

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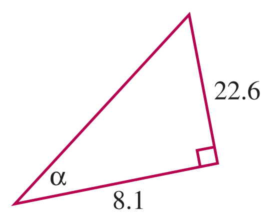

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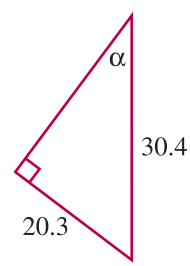

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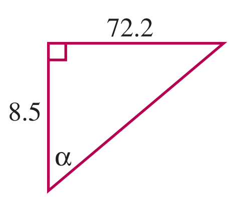

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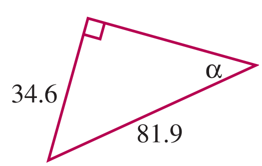

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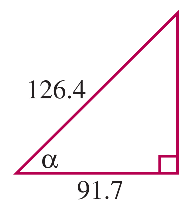
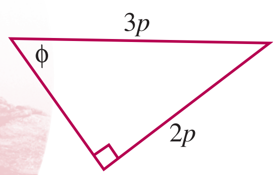


Consolidation

3 Find the size of the angle ϕ. All lengths are in centimetres.

a



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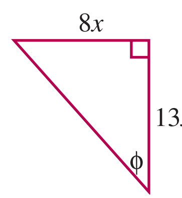

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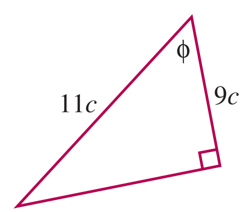


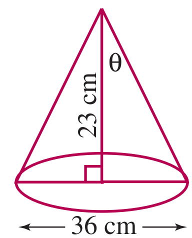
4 Draw a diagram and mark on it all of the given information, to answer each of the following.

a In ΔCDE, $\angle D = 90°$, $CD = 9$ cm and $DE = 17$ cm. Find $\angle C$.

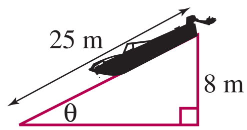
b In ΔOPQ, $\angle Q = 90°$, $OQ = 13$ mm and $OP = 14.8$ mm. Find $\angle P$.

c In ΔIJK, $\angle I = 90°$, $IK = 12.7$ m and $JK = 15.9$ m. Find $\angle K$.

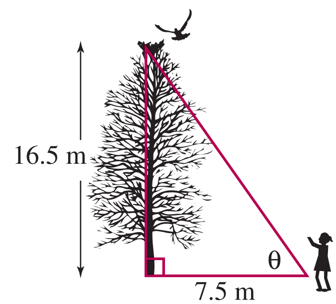
5 A cone has a diameter of 36 cm and a perpendicular height of 23 cm. Find the size of the semi-vertical angle θ.



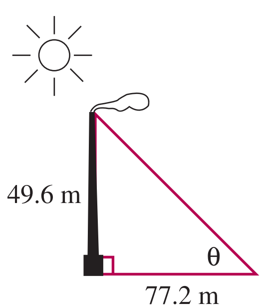
6 A boat is to be launched down a 25 m ramp with a vertical drop of 8 m. At what angle is the ramp inclined to the horizontal?



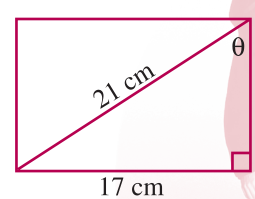
7 A girl observes a bird's nest at the top of a tree. She stands 7.5 m from the base of the tree, which is 16.5 m tall. At what angle must the girl look up in order to see the nest? (Ignore the girl's height.)

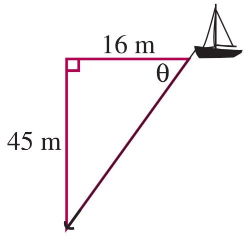


8 A chimney stack 49.6 m tall casts a shadow 77.2 m long on the ground. Calculate the angle at which the sun's rays strike the ground.



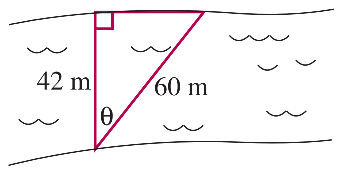
9 A rectangle has a length of 17 cm and diagonals of length 21 cm. Find the angle between the width and a diagonal.



10 

The anchor of a boat was dropped 45 m to the sea floor. During the next hour, the boat drifted 16 m from this spot. Find the angle between the anchor line and the surface of the water.

11 While trying to swim across a river, Brad was swept downstream by the current. The river is 42 m wide, but Brad had to swim 60 m to get across. At what angle was he dragged downstream?



12 **a** A ladder of length 6 m leans against a wall. The foot of the ladder is 1.3 m from the base of the wall. Find the angle formed between the ladder and the ground.

b A ladder of length 7.3 m reaches 5.95 m up the side of a wall. What angle does the ladder make with the wall?

13 A straight stretch of railway track has a gradient of $\frac{4}{15}$. Calculate the angle at which the track rises.

14 A grass ski run falls 125 m vertically over a sloping run of 160 m. Find the angle at which the ski run is inclined to the horizontal.

15 In the isosceles ΔRST, $RS = ST$ and U is a point on RT such that $SU \perp RT$.

a If $SU = 17$ cm and $RT = 26$ cm, find the size of $\angle SRT$.

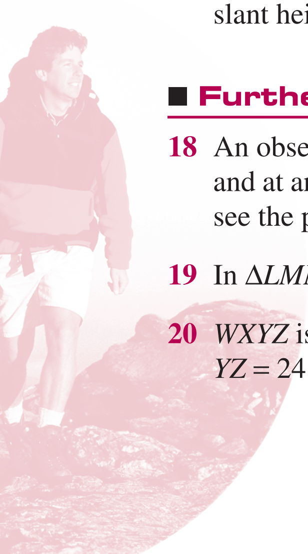
b If $ST = 29$ cm and $RT = 43$ cm, find the size of $\angle STR$.

c If $RS = 33$ cm and $SU = 21$ cm, find the size of $\angle RST$.

d If $RT = 50$ cm and $SU = 37$ cm, find the size of $\angle RST$.

16 A pendulum of length 85 cm swings through a horizontal distance of 60 cm before it stops and swings back again. Find the vertical angle through which the pendulum swings.

17 A cone has a diameter of 46 cm and a slant height of 63 cm. Find the angle between the slant height and the base of the cone.



■ Further applications

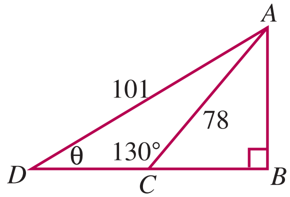
18 An observer on the ground sees a plane flying directly overhead at a speed of 360 km/h, and at an altitude of 5000 m. At what angle would the observer need to look up in order to see the plane 5 minutes later?

19 In ΔLMN, $\angle M = 90°$, $MN = 2LM$ and X is the midpoint of LM. Find the size of $\angle LNX$.

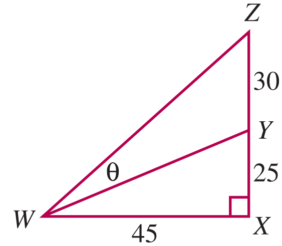
20 $WXYZ$ is a quadrilateral in which $\angle X = 90°$, $\angle Z = 90°$, $WX = 15$ mm, $XY = 20$ mm, $YZ = 24$ mm and $WZ = 7$ mm. Find the size of $\angle W$.

21 Find the angle θ in each of the following.

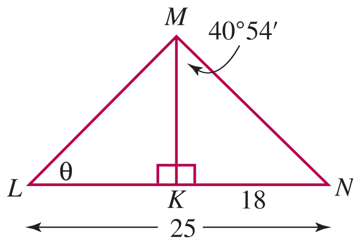
a



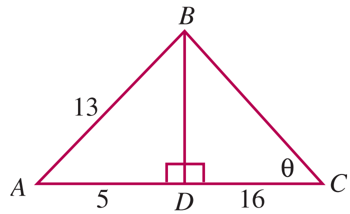
b



c



d



TRY THIS High flying

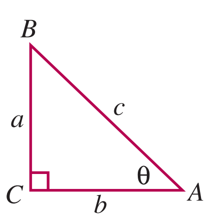
An airliner flying at a height of 8000 m needs to descend to a height of 200 m when it is 5 km from the airport. If its angle of descent is 3°, how far from the airport should the pilot begin his descent?

2.5 The tangent ratio

The tangent ratio can be expressed as the quotient of the sine and cosine ratios. For any angle θ:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Proof:



In ΔABC, let $\angle CAB = \theta$.

$$\therefore \sin\theta = \frac{a}{c},\ \cos\theta = \frac{b}{c} \text{ and } \tan\theta = \frac{a}{b}$$

$$\text{Now, } \frac{\sin\theta}{\cos\theta} = \frac{\frac{a}{c} \times c}{\frac{b}{c} \times c}$$

$$= \frac{a}{b}$$

$$= \tan\theta$$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Example 1

a Find the value of $\tan\theta$, where θ is an acute angle, if $\sin\theta = 0.36$ and $\cos\theta = 0.4$.

b Hence, find θ correct to the nearest minute.

Solutions

a
$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta}\\ &= \frac{0.36}{0.4}\\ &= 0.9\end{aligned}$$

b
$$\begin{aligned}\tan\theta &= 0.9\\ \theta &= \tan^{-1} 0.9\\ &= 41°59' \text{ (to the nearest minute)}\end{aligned}$$

Example 2

Express each equation in terms of $\tan\theta$, where θ is an acute angle, then solve for θ. Answer correct to the nearest minute.

a $\sin\theta = 4\cos\theta$ **b** $5.2\sin\theta = 2.3\cos\theta$

Solutions

a
$$\begin{aligned}\sin\theta &= 4\cos\theta\\ \div\cos\theta \quad & \quad \div\cos\theta\\ \frac{\sin\theta}{\cos\theta} &= 4\\ \tan\theta &= 4\\ \therefore\ \theta &= \tan^{-1} 4\\ &= 75°58'\end{aligned}$$
(to the nearest minute)

b
$$\begin{aligned}5.2\sin\theta &= 2.3\cos\theta\\ \div 5.2\cos\theta \quad & \quad \div 5.2\cos\theta\\ \frac{\sin\theta}{\cos\theta} &= \frac{2.3}{5.2}\\ \tan\theta &= \frac{2.3}{5.2}\\ \therefore\ \theta &= \tan^{-1}\left(\frac{2.3}{5.2}\right)\\ &= 23°52'\end{aligned}$$
(to the nearest minute)

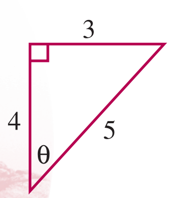
Exercise 2.5

1 Verify each of the following by using a calculator.

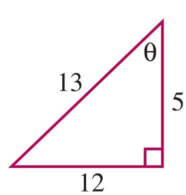
a $\dfrac{\sin 40°}{\cos 40°} = \tan 40°$ **b** $\dfrac{\sin 17°38'}{\cos 17°38'} = \tan 17°38'$ **c** $\dfrac{\sin 115°}{\cos 115°} = \tan 115°$

2 Find values for $\sin\theta$, $\cos\theta$ and $\tan\theta$, then verify that $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$ in each triangle.

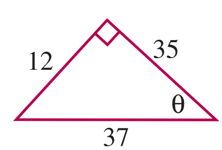
a



b



c



Consolidation

3 Find the value of tan θ in each of the following, where θ is an acute angle. Hence, find the size of the angle θ, correct to the nearest minute.

a $\sin\theta = \frac{8}{17}$ and $\cos\theta = \frac{15}{17}$

b $\sin\theta = 0.5466$ and $\cos\theta = 0.8374$

c $\cos\theta = 0.6118$ and $\sin\theta = 0.7910$

d $\cos\theta = \frac{\sqrt{2}}{3}$ and $\sin\theta = \frac{\sqrt{7}}{3}$

4 If $\tan\theta = \frac{\sin\theta}{\cos\theta}$, show that:

a $\sin\theta = \cos\theta\tan\theta$

b $\cos\theta = \frac{\sin\theta}{\tan\theta}$

5 **a** If $\cos\theta = \frac{7}{25}$ and $\tan\theta = \frac{24}{7}$, find the value of sin θ.

b If $\tan\theta = \frac{9}{40}$ and $\cos\theta = \frac{40}{41}$, find the value of sin θ.

c If $\sin\theta = \frac{20}{29}$ and $\tan\theta = \frac{20}{21}$, find the value of cos θ.

d If $\tan\theta = \frac{63}{16}$ and $\sin\theta = \frac{63}{65}$, find the value of cos θ.

6 Solve the equation $\sin\theta = \cos\theta$, where θ is an acute angle.

7 In the equations below, θ is an acute angle. Express each equation in terms of tan θ, then solve for θ, correct to the nearest minute.

a $\sin\theta = 2\cos\theta$

b $\cos\theta = 3\sin\theta$

c $5\sin\theta = 8\cos\theta$

d $13\cos\theta = 7\sin\theta$

e $6.4\cos\theta = 3.9\sin\theta$

f $21.5\sin\theta = 16.1\cos\theta$

Further applications

8 In the equations below, θ is an acute angle. Express each equation in terms of tan θ, then solve for θ, correct to the nearest minute.

a $\frac{8}{\sin\theta} = \frac{1}{\cos\theta}$

b $\frac{\sin\theta}{3} = \frac{\cos\theta}{4}$

c $\frac{\sqrt{5}}{\cos\theta} = \frac{2}{\sin\theta}$

9 Prove each of the following identities.

a $\frac{\sin^2 A}{\cos^2 A} = \tan^2 A$

b $\sin A\cos A\tan A = \sin^2 A$

c $\frac{\sin A\cos A}{\tan A} = \cos^2 A$

d $\frac{\sin A}{\cos A\tan A} = 1$

e $\frac{\tan A\cos A}{\sin A} = 1$

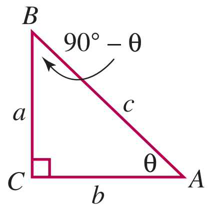
f $\frac{\sin^2 A + \sin A\cos A}{\cos^2 A + \sin A\cos A} = \tan A$

2.6 The complementary results

In any right-angled triangle, the sine of an acute angle is equal to the cosine of its complement, and the cosine of an acute angle is equal to the sine of its complement. That is,

$$\mathbf{\sin\theta = \cos(90^\circ - \theta) \quad and \quad \cos\theta = \sin(90^\circ - \theta)}$$

Proof:



In ΔABC, $\angle C = 90^\circ$, $\angle A = \theta$ and $\angle B = 90^\circ - \theta$.

Now, $\sin\theta = \frac{a}{c}$ and $\cos(90^\circ - \theta) = \frac{a}{c}$, $\therefore \sin\theta = \cos(90^\circ - \theta)$

Also, $\cos\theta = \frac{b}{c}$ and $\sin(90^\circ - \theta) = \frac{b}{c}$, $\therefore \cos\theta = \sin(90^\circ - \theta)$

Example 1

EG +S

Find the value of x in each of the following.

a $\cos x^\circ = \sin 70^\circ$ **b** $\sin x^\circ = \cos 55^\circ$

Solutions

a $\cos x^\circ = \sin 70^\circ$
$\cos x^\circ = \cos 20^\circ$
$\therefore x = 20$

b $\sin x^\circ = \cos 55^\circ$
$\sin x^\circ = \sin 35^\circ$
$\therefore x = 35$

Example 2

EG +S

Find the value of x in each of these.

a $\cos(x - 20)^\circ = \sin 40^\circ$ **b** $\sin(2x)^\circ = \cos 10^\circ$

Solutions

a $\cos(x - 20)^\circ = \sin 40^\circ$
$\cos(x - 20)^\circ = \cos 50^\circ$
$\therefore x - 20 = 50$
$\therefore x = 70$

b $\sin(2x)^\circ = \cos 10^\circ$
$\sin(2x)^\circ = \sin 80^\circ$
$\therefore 2x = 80$
$\therefore x = 40$

Example 3

EG +S

Simplify $\frac{3\sin 75^\circ}{\cos 15^\circ}$ without using a calculator.

Solution

$$\frac{3\sin 75^\circ}{\cos 15^\circ} = \frac{3\sin 75^\circ}{\sin 75^\circ}$$
$$= 3$$

Exercise 2.6

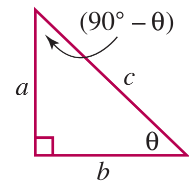
1 a Find values for each of the following, using the given triangle.

i $\sin\theta$ **ii** $\cos\theta$

iii $\sin(90° - \theta)$ **iv** $\cos(90° - \theta)$

b Use your results from part **a** to complete the following statements.

i $\sin\theta =$ .... **ii** $\cos\theta =$ ....



2 Verify the following results by using a calculator.

a $\sin 60° = \cos 30°$ **b** $\cos 60° = \sin 30°$ **c** $\cos 20° = \sin 70°$

d $\sin 20° = \cos 70°$ **e** $\sin 35° = \cos 55°$ **f** $\cos 55° = \sin 35°$

3 a Simplify $27°39' + 62°21'$.

b Verify that $\sin 27°39' = \cos 62°21'$.

c Would $\sin 62°21' = \cos 27°39'$? Why?

4 Complete the following statements.

a If $\cos 60° = 0.5$, then sin _____ = 0.5

b If $\sin 40° = 0.643$, then cos _____ = 0.643

c If $\cos 15° = 0.966$, then _____ 75° = 0.966

d If $\sin 72° = 0.951$, then _____ 18° = 0.951

Consolidation

5 Find the value of x in each of these.

a $\sin 50° = \cos x°$ **b** $\cos 35° = \sin x°$ **c** $\sin x° = \cos 65°$

d $\cos x° = \sin 12°$ **e** $\cos x° = \sin 48°$ **f** $\sin 53° = \cos x°$

g $\sin 17° = \cos x°$ **h** $\sin x° = \cos 84°$ **i** $\cos x° = \sin 5°$

j $\cos 64° = \sin x°$ **k** $\sin 71° = \cos x°$ **l** $\sin x° = \cos 29°$

6 Simplify without using a calculator:

a $\dfrac{\sin 10°}{\cos 80°}$ **b** $\dfrac{\cos 74°}{\sin 16°}$ **c** $\dfrac{2\sin 57°}{\cos 33°}$ **d** $\dfrac{3\cos 29°}{4\sin 61°}$

7 Solve each of these equations.

a $\sin(x + 30)° = \cos 20°$ **b** $\sin 35° = \cos(x + 10)°$

c $\cos 30° = \sin(x - 23)°$ **d** $\cos(x - 16)° = \sin 26°$

e $\sin(2x)° = \cos 36°$ **f** $\cos 42° = \sin(3x)°$

g $\sin\left(\dfrac{x}{4}\right)° = \cos 30°$ **h** $\cos\left(\dfrac{x}{2}\right)° = \sin 40°$

i $\cos(2x + 56)° = \sin 14°$ **j** $\sin x° = \cos x°$

k $\cos(2x)° = \sin x°$ **l** $\sin(x + 18)° = \cos(x - 8)°$

8 a If $\cos 75° = 0.26$, find the approximate value of $\cos 75° + \sin 15°$.

b If $\sin 4° = 0.07$, find the approximate value of $3\cos 86°$.

■ Further applications

9 Simplify the following expressions.

a $\dfrac{\cos(90° - \theta)}{\sin\theta}$ **b** $\dfrac{\cos(90° - \theta)}{\cos\theta}$ **c** $\dfrac{\cos\theta}{\sin(90° - \theta)}$ **d** $\dfrac{\sin\theta}{\sin(90° - \theta)}$

e $\sin\theta\cos(90° - \theta)$ **f** $\cos\theta\sin(90° - \theta)$ **g** $\tan\theta\sin(90° - \theta)$

h $\dfrac{\cos(90° - \theta)}{\tan\theta}$ **i** $\sin(90° - \theta)\cos(90° - \theta)\tan(90° - \theta)$

10 Prove the following identities.

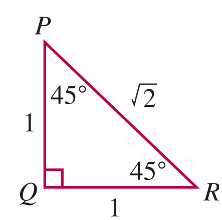
a $\dfrac{\sin\theta\cos(90° - \theta)}{\cos\theta\sin(90° - \theta)} = \tan^2\theta$ **b** $\dfrac{\sin\theta\sin(90° - \theta)}{\cos\theta\cos(90° - \theta)} = 1$

c $\sqrt{\cos\theta\tan\theta\cos(90° - \theta)} = \sin\theta$ **d** $\sqrt{\dfrac{\sin\theta\sin(90° - \theta)}{\tan\theta}} = \cos\theta$

2.7 The exact values

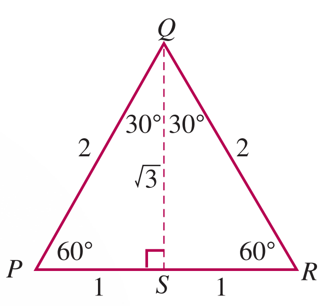
The triangles below can be used to derive exact values for the trigonometric ratios involving angles of 30°, 45° and 60°. This means that the answers to many questions can now be given in exact form as fractions or as surds, rather than as the decimal approximations that are usually obtained by using a calculator.

A right-angled isosceles triangle is used to find the exact values of the trigonometric ratios involving angles of 45°.



In ΔPQR, $\angle Q = 90°$, $PQ = QR = 1$ unit and $\angle P = \angle R = 45°$.
$PR = \sqrt{2}$ units (by Pythagoras' theorem).

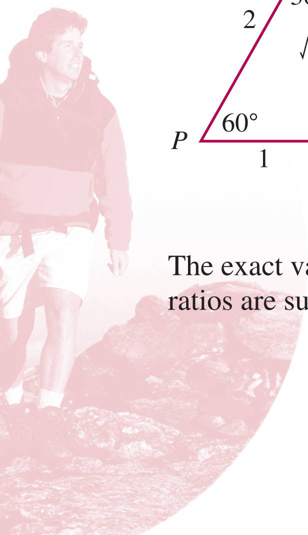
An equilateral triangle is used to find the exact values of the trigonometric ratios involving angles of 30° and 60°.



In ΔPQR, $PQ = QR = RS = 2$ units and $\angle P = \angle Q = \angle R = 60°$.
Construct $QS \perp PR$. Now QS bisects both PR and $\angle PQR$.
Therefore, $PS = SQ = 1$ unit and $\angle PQS = \angle RQS = 30°$.
$QS = \sqrt{3}$ (by Pythagoras' theorem).

The exact values for the trigonometric ratios are summarised in the table.

θ	**30°**	**45°**	**60°**
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$



Example 1

Evaluate each expression without using a calculator.

a tan 30° sin 60°

b $\tan^2 60° - \sin^2 60°$

Solutions

a tan 30° sin 60°

$$= \frac{1}{\cancel{\sqrt{3}}} \times \frac{\cancel{\sqrt{3}}}{2}$$

$$= \tfrac{1}{2}$$

b $\tan^2 60° - \sin^2 60°$

$$= (\tan 60°)^2 - (\sin 60°)^2$$

$$= (\sqrt{3})^2 - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= 3 - \frac{3}{4}$$

$$= 2\tfrac{1}{4}$$

Example 2

Find the exact value of each expression. Give the answers in simplest surd form with a rational denominator.

a cos 30° + sin 45°

b $\dfrac{\sin 45°}{\tan 60°}$

Solutions

a cos 30° + sin 45°

$$= \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}}{2} + \left(\frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}}\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{3} + \sqrt{2}}{2}$$

b $\dfrac{\sin 45°}{\tan 60°}$

$$= \frac{\frac{1}{\sqrt{2}} \times \sqrt{2}}{\sqrt{3} \times \sqrt{2}}$$

$$= \frac{1 \times \sqrt{6}}{\sqrt{6} \times \sqrt{6}}$$

$$= \frac{\sqrt{6}}{6}$$

Example 3

Solve the equation $\tan\theta = \dfrac{1}{\sqrt{3}}$ for θ, where $0° < \theta < 90°$.

Solution

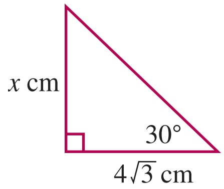
$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= 30°$$

Example 4

Find the value of x.



Solution

$$\tan 30° = \frac{x}{4\sqrt{3}}$$

$$\frac{1}{\sqrt{3}} = \frac{x}{4\sqrt{3}}$$

$$\times 4\sqrt{3} \qquad \times 4\sqrt{3}$$

$$\therefore x = 4$$

Exercise 2.7

1 Write down the exact value of each expression.

a sin 30°	**b** cos 30°	**c** tan 30°
d sin 45°	**e** cos 45°	**f** tan 45°
g sin 60°	**h** cos 60°	**i** tan 60°

2 Find the exact value of:

a $\tan^2 45°$	**b** $\sin^2 30°$	**c** $\tan^2 60°$
d $\cos^2 45°$	**e** $\tan^2 30°$	**f** $\cos^2 30°$

3 Find the exact value of each expression. Give your answers in simplest surd form where necessary, with a rational denominator.

a sin 30° cos 60°	**b** sin 45° cos 45°	**c** tan 60° tan 30°
d sin 60° cos 30°	**e** cos 30° sin 45°	**f** cos 45° tan 60°
g sin 45° tan 30°	**h** cos 45° cos 60°	**i** tan 30° cos 30°
j sin 30° cos 45° sin 45°	**k** tan 60° cos 30° tan 45°	**l** sin 60° tan 30° cos 30°

Consolidation

4 Find the exact value of each expression. Give your answers in simplest surd form where necessary, with a rational denominator.

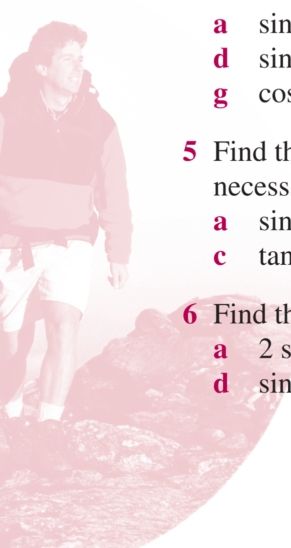
a sin 30° + cos 60°	**b** tan 60° + tan 30°	**c** sin 45° + cos 45°
d sin 60° + cos 30°	**e** cos 45° + sin 30°	**f** sin 60° + sin 45°
g cos 30° + tan 60°	**h** tan 45° + sin 60°	**i** tan 45° + sin 30°

5 Find the exact value of each expression. Give your answers in simplest surd form where necessary, with a rational denominator.

a sin 30° + cos 30° + tan 30°	**b** tan 30° + cos 45° + sin 60°
c tan 45° + tan 30° + tan 60°	**d** tan 60° + sin 45° + cos 60°

6 Find the exact value of each expression.

a $2\sin^2 45° - 1$	**b** $1 - 2\sin^2 60°$	**c** $\tan^2 60° - \tan^2 45°$
d $\sin^2 60° - \sin^2 30°$	**e** $4(\sin^2 45° + \cos^2 45°)$	**f** $\tan^2 60° - 2\cos^2 30°$



7 Find the exact value of each expression. Give your answers in simplest surd form where necessary, with a rational denominator.

a $\dfrac{\sin 30°}{\cos 60°}$ **b** $\dfrac{\tan 45°}{\tan 30°}$ **c** $\dfrac{\sin 60°}{\tan 30°}$ **d** $\dfrac{\sin 60°}{\cos 60°}$

e $\dfrac{\tan 60°}{\tan 30°}$ **f** $\dfrac{\tan 45°}{\cos 30°}$ **g** $\dfrac{\cos 30°}{\cos 45°}$ **h** $\dfrac{\sin 45°}{\sin 60°}$

8 Show that:

a $\sin 60° \cos 30° + \cos 60° \sin 30° = 1$

b $\cos 45° \cos 30° + \sin 45° \sin 30° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

9 Solve each equation for θ, where $0° < \theta < 90°$.

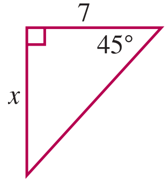
a $\sin \theta = \dfrac{1}{2}$ **b** $\tan \theta = \sqrt{3}$ **c** $\cos \theta = \dfrac{1}{\sqrt{2}}$

d $\sin \theta = \dfrac{\sqrt{3}}{2}$ **e** $\cos \theta = \dfrac{1}{2}$ **f** $\tan \theta = \dfrac{1}{\sqrt{3}}$

g $\cos \theta = \dfrac{\sqrt{3}}{2}$ **h** $\tan \theta = 1$ **i** $\sin \theta = \dfrac{1}{\sqrt{2}}$

10 Find the exact value of the pronumeral in each of these. All lengths are in metres.

a



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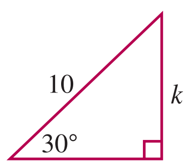

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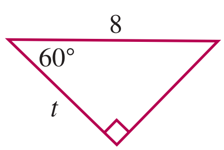

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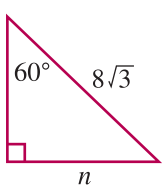

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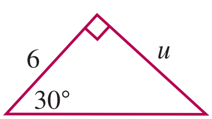

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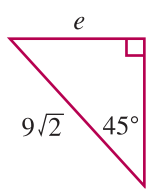
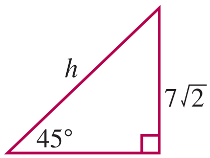


11 Find the exact value of the pronumeral in each of these. All lengths are in cm.

a



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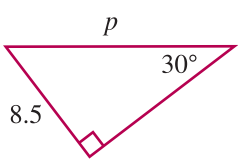

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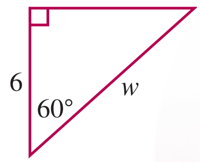
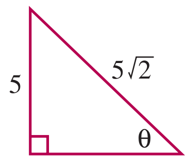


12 Find the size of the angle θ in each of these. All lengths are in mm.

a



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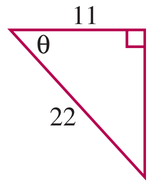

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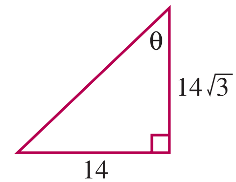

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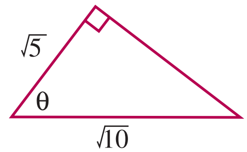

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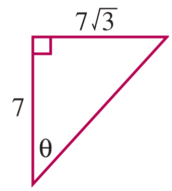

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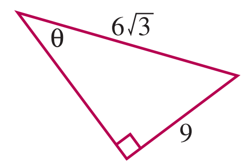
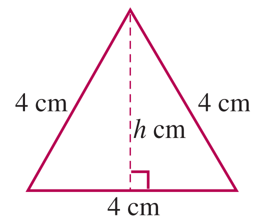


13 **a** Use the exact values to find the height, h cm, of this equilateral triangle.

b Hence, find the exact area of the triangle.



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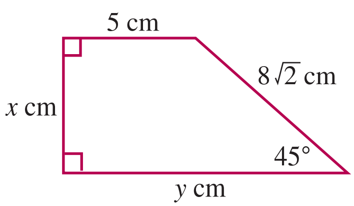


a Use the exact values to find the value of x.

b Find the value of y.

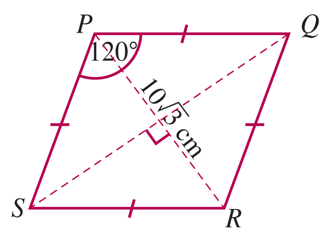
c Hence, find the area of the trapezium.

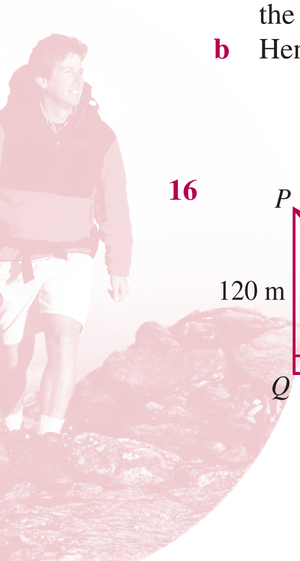
■ Further applications

15 In the rhombus $PQRS$, the diagonal PR has a length of $10\sqrt{3}$ cm and $\angle SPQ = 120°$.

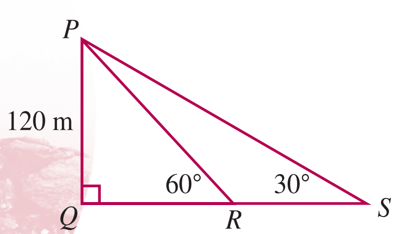
a Use the exact values to find the length of the diagonal SQ.

b Hence, find the area of the rhombus.





16



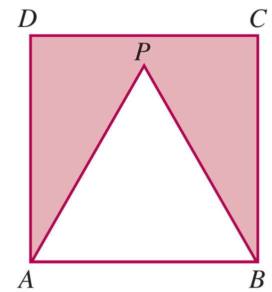
A man walked from S along level ground towards Q, the foot of a vertical cliff PQ. From S, the angle of elevation to the top of the cliff is 30°. After walking to R, he noted that the angle of elevation was then 60°. If the height of the cliff is 120 metres, find the distance RS, without using a calculator.

TRY THIS

Square area

ABCD is a square and *APB* is an equilateral triangle.

If *AB* = 2 units, what is the exact shaded area?

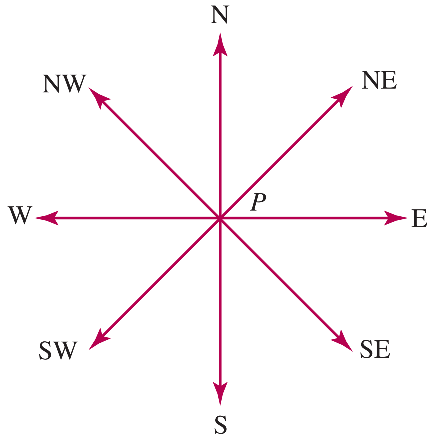


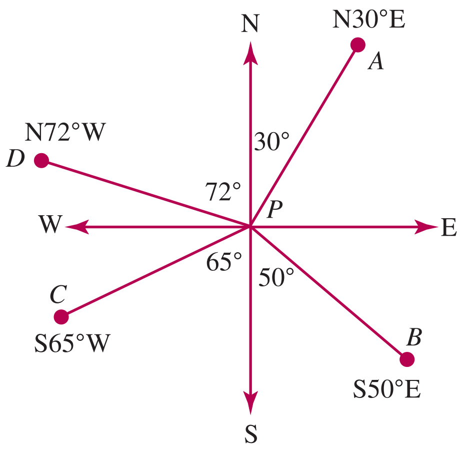
2.8 Bearings Review

A **bearing** is a measure of the direction of one point from another point. There are two types of bearings: compass bearings and true bearings.

Compass bearings

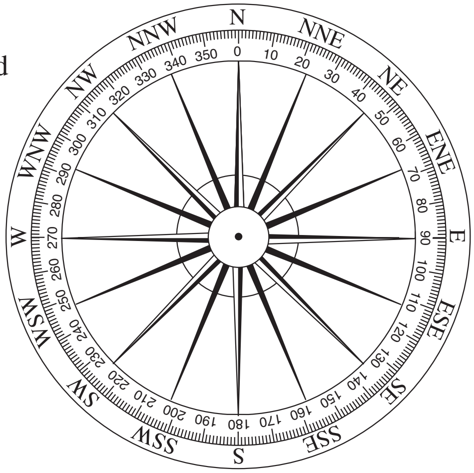
A compass bearing is a deviation from north or south and towards east or west. For example, a bearing of S30°W means a deviation of 30° from the south towards the west. The diagram to the right shows the compass bearings of four pints *A*, *B*, *C*, *D* from a point *P*.





A bearing such as NE means N45°E. Similarly, NW means N45°W, SE means S45°E and SW means S45°W.

The 16 point compass



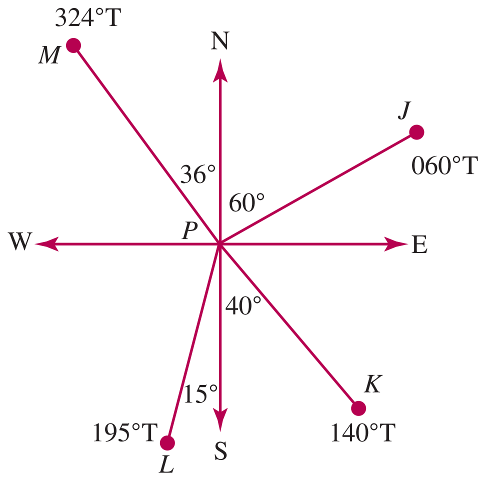
The 16 point compass, or mariner's compass, was used by mariners to determine directions while at sea. It is still used today in many aspects of everyday life, such as for the wind directions on daily weather reports.

The directions on the 16 point compass are given as deviations from the four cardinal directions (i.e. N, S, E, W) and towards the four intermediate directions (i.e. NE, NW, SE, SW).

For example, the bearing between north and north-east is north north east (NNE). The bearing between north east and east is east north east (ENE), not north east east (NEE).

True bearings

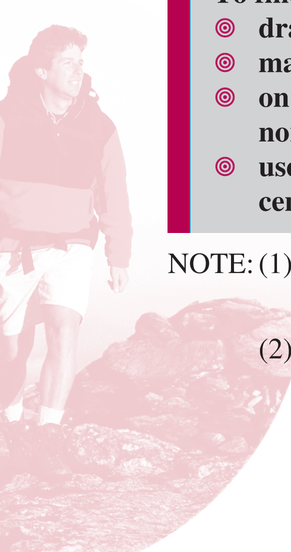
A true bearing is a deviation from north, measured in a clockwise direction. True bearings are written using three digits. For example, a clockwise rotation from north of 52° is written as 052°. The diagram to the right shows the true bearings of four points *J*, *K*, *L*, *M* from a point *P*.



Opposite bearings

The **opposite bearing** of *B* from *A* is the bearing of *A* from *B*. To find the opposite bearing (or any change in direction) it will be necessary to draw a new compass at the end of the ray.

To find the bearing of *A* from *B* given the bearing of *B* from *A*:
- **draw a compass at *B***
- **mark on this compass the angle from north around to the ray *BA***
- **on the compass with centre *A*, find the acute angle between *BA* and the north–south axis**
- **use parallel line properties to find the required bearing on the compass with centre *B*.**

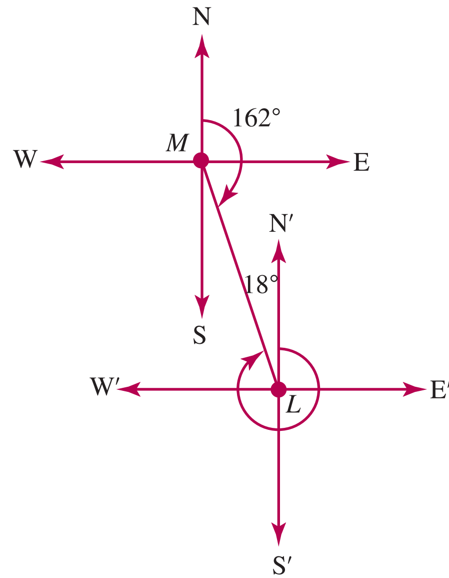


NOTE: (1) Opposite bearings always differ by 180°. That is, the new bearing will be either 180° more than the original bearing, or 180° less.

(2) The reference given after the word 'FROM' should be at the centre of the *active compass* (i.e. the compass in which you are working).

Example 1

The bearing of L from M is 162°.
Find the bearing of M from L.



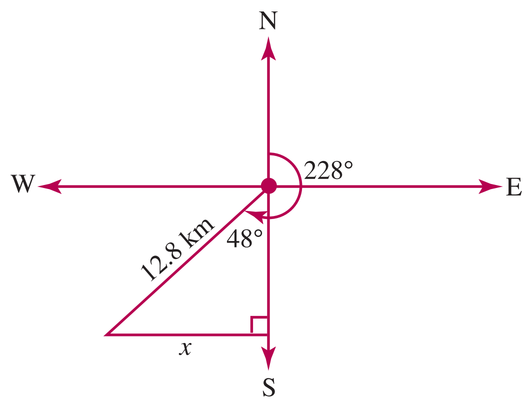
Solution

i $\angle MLN' = 18°$ (Co-interior $\angle$s, $NM \parallel N'L$)

ii Bearing of M from $L = 360° - 18°$
$= 342°$

Example 2

Susie drove 12.8 km on a bearing of 228°.
How far west did she drive, correct to 1 decimal place?



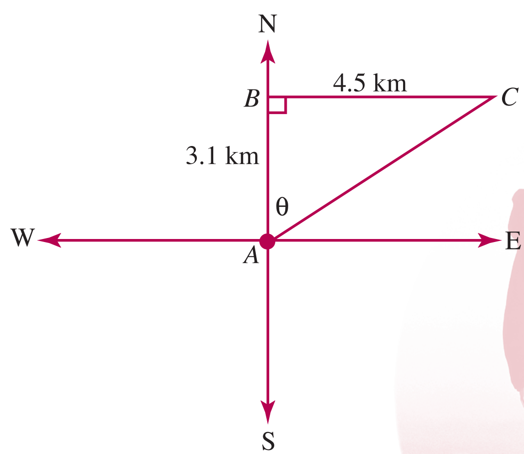
Solution

$$\sin 48° = \frac{x}{12.8}$$

$$\therefore\ x = 12.8 \times \sin 48°$$
$$= 9.512253766$$
$$= 9.5 \text{ km (correct to 1 decimal place)}$$

Example 3

Daniel walked due north for 3.1 km from A to B, then turned and walked 4.5 km due east to C. Find the bearing of C from A.



Solution

$$\tan \theta = \frac{4.5}{3.1}$$

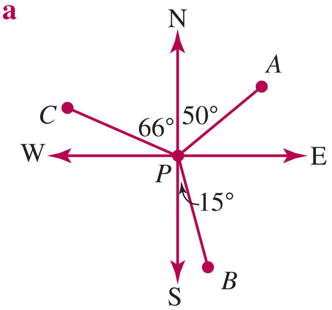
$$\therefore\ \theta = \tan^{-1}\left(\frac{4.5}{3.1}\right)$$
$$\doteqdot 55°$$

$\therefore$ The bearing of C from A is 055°.

Exercise 2.8

1 **i** Find the compass bearings from P of the points A, B and C.
ii Find the true bearings from P of the points A, B and C.

a



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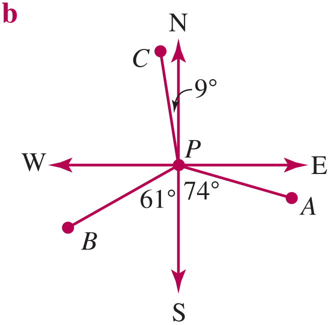
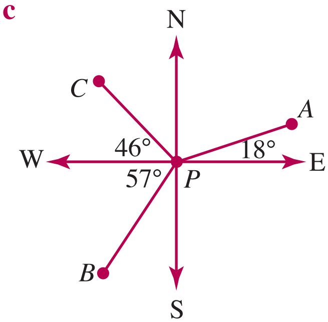
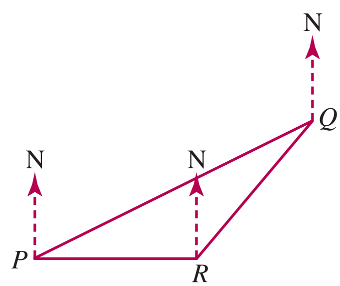


c

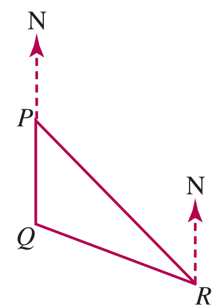


2 Use a pair of alternate angles to find the bearings of X from Y, given that the bearing of Y from X is:
a 030° **b** 235° **c** 148° **d** 303°

3 Find the size of $\angle PQR$ given that:
a **i** the bearing of Q from P is 034°
ii the bearing of Q from R is 025°



b **i** the bearing of R from P is 163°
ii the bearing of Q from R is 317°



■ Consolidation

4 Answer the following questions, correct to 1 decimal place.
a Albert drove 54 km on a bearing of 065°. How far east was he then from his starting point?
b Nerida flew a light plane 116 km on a bearing of 304°. How far north was she then from her starting point?
c A car travels 115 km due south from L to M, then turns and travels due east to N. If the bearing of N from L is 133°, find the distance MN.
d A cruise ship leaves port and sails due west from P to Q then turns and sails 95 km due south to R. If the bearing of R from P is 224°, find the distance PQ.

5 Find the following distances, correct to 3 significant figures.
a A helicopter flew 26.4 km due north from A to B, then turned and flew due west to C. If the bearing of C from A is 287°, find the distance AC.
b A hiker walked 6.8 km due south from E to F. After a short rest, he then walked due west to G, which is on a bearing of 216° from E. Find the distance between E and G.

c During a cross-country rally, Marcus drove his race car due north from X to Y. He then drove 21.4 km due east to Z, which is on a bearing of 067° from X. Calculate the distance XZ.

d A yacht sailed 11.2 km due east from T, then turned around a buoy at U and sailed due south to V. The bearing of V from T is 138°. How far was the yacht then from its starting point?

6 Give the following bearings as three-figure bearings, correct to the nearest degree.

a A woman drove 25 km due east from F to G, then drove 29 km due south to H. Find the bearing of:
i H from F **ii** F from H

b A man rode due north from I to J, then rode 10 km due west to K, which is 14 km from I. Find the bearing of:
i K from I **ii** I from K

c Marta flew her glider 28 km due south from A to B. She then turned and flew due west to C, which is 36 km from A. Find the bearing of:
i C from A **ii** A from C

d Keiran sailed 19 km due east from R to S, then sailed 22 km due north to T. Find the bearing of:
i T from R **ii** R from T

7 **a** Alice walked from home (H) to the cinema (C) on a bearing of 043°. After the movie, she walked on a bearing of 133° to a friend's house (F) 320 m due east of her home.
i Show that $\angle HCF = 90°$.
ii Find the distance between Alice's home and the cinema, correct to the nearest m.

b Cruz drove from home (H) to the beach (B) on a bearing of 254° to pick up his children. He then drove to the supermarket (S) on a bearing of 344°, which is 9.6 km due west of his home.
i Show that $\angle HBS = 90°$.
ii Find the distance between the beach and the supermarket, correct to the nearest tenth of a km.

Further applications

8 Two cars A and B left home at the same time. Car A travelled due west at 70 km/h whilst car B travelled due north at 90 km/h. Find, after 2 hours:
a the distance between the cars **b** the bearing of B from A.

9 During a yacht race, a yacht sailed from a harbour (H) due west for 16 nautical miles to buoy X. After sailing around the buoy, the yacht then proceeded to another buoy Y on a course of 288° for 13 nautical miles.

a How far west has the yacht sailed from the harbour, correct to 2 decimal places?

b How far north has the ship sailed from the harbour, correct to 2 decimal places?

c Find the distance between the second buoy (Y) and the harbour, correct to the nearest nautical mile.

d On what bearing must the yacht then sail in order to return directly to the harbour?

10 From a point R, two planes P and Q flying due west and in the same horizontal line are on bearings of 115° and 157° respectively. If P is 200 km from R and Q is 150 km from R, find the distance between the planes, correct to the nearest km.

FOCUS ON WORKING MATHEMATICALLY

ORIENTEERING

Introduction

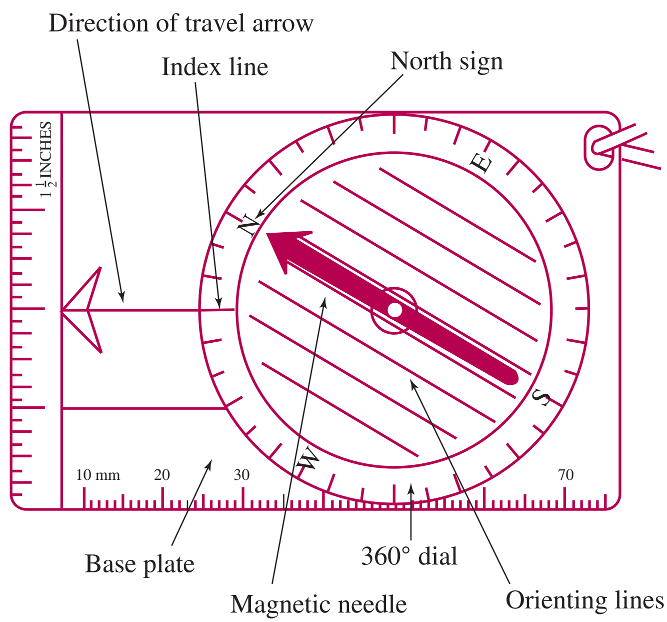
Orienteering is a sport in which competitors pass through a number of check points called controls, to complete a course aided only by a map and a compass. On the route, orange-and-white control markers are set in the places that correspond to the marked control points on the map. Participants punch their control cards (or use an electronic equivalent) at each control point. Orienteering on foot can be thought of as running navigation. You need to be able to accurately read maps and find direction quickly to be good at it. You also need to be able to make quick decisions and concentrate under stress. There are various types of orienteering including ski, mountain bike and street orienteering but foot orienteering is easily the most popular.

In competitive orienteering the winner is the person who has used the shortest time to visit the control points in the correct order. Fast running alone does not make you a winner. You must also choose the best route between control points and find the markers without wasting unnecessary time. At elite level Australia enters teams in the world championships but it is popular with people of all ages and fitness levels.

Learning activities

The baseplate compass

The base plate compass was invented in 1930 in Sweden for the sport of orienteering. It enabled a great saving in time, as competitors could transfer compass readings to maps directly. The diagram below shows the essential features of the compass. It is not possible to explain how it works here. Check with your teacher or the internet. It is not necessary for the learning activity to follow.



An orienteering problem

The map on the next page shows the leg between two control points 17(P) and 18(Q) which competitors must pass through on an orienteering course in the Australian bush. The leg means crossing a wide stream, shallow at the banks but deep enough in the centre to mean getting wet to the waist wading across. North lines on an orienteering map point to magnetic north and are usually spaced 500 m apart. The map is not drawn to scale.

The shortest route from P to Q, a straight line, means traversing mud flats and the sand hills beyond. The terrain marked on the map shows the river bank on the other side is also covered with areas of thick bush and makes a less direct route seem more attractive. The south side of the stream looks easier to navigate, but there is also an area of thick bush to get around. Downstream at R the map shows a flat rock which gives a better view of the area surrounding control point Q.

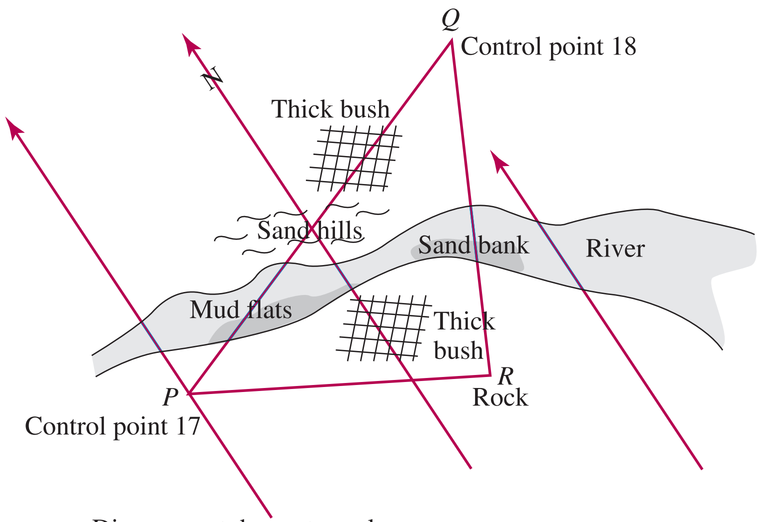


Diagram not drawn to scale

The team members decide to run downstream to R and cross the river at the sand bank. Estimate the additional distance this route entails.

1 Using the side of the baseplate compass, the distance on the map from P to R is 5.4 cm and the scale on the map is 1:15 000. Make a rough copy of the map in your book and calculate the actual distance from P to R. Show it on your map.

2 The bearing of Q from P is 071° and the bearing of R from P is 120°. Show these angles on your map.

3 The bearing of Q from R is 011° and the bearing of P from R is 282°. Show these angles on your map.

4 Calculate the size of angle PRQ and angle QPR. Show these angles on your map.

5 Given the size of angle PRQ, is it reasonable to estimate the distance RQ by using a trigonometric ratio? What ratio would you choose? Estimate RQ (nearest 10 m).

6 Use a suitable trigonometric ratio to estimate the distance PQ (nearest 10 m).

7 Calculate the extra distance for the route via R. Is the answer reasonable?

Challenge activities

1 Make a scale drawing of the map and measure the lengths of PR, RQ and PQ. Calculate the extra distance travelled by the team. How does the answer compare with the estimate calculated from trigonometric ratios?

2 If possible learn to use a base plate compass. The class could repeat the exercise above by choosing three distinctive landmarks P, Q and R in the playground, take bearings and measure distance PR with a tape. Why will the angle PRQ need to be approximately 90° to get a good estimate of RQ and PQ using trigonometric ratios?

3 Make an orienteering map using directions from the internet. For example the Orienteering Association of Western Australia has produced an excellent *Parkland Mapping Manual* which schools can download from the Orienteering Australia website <www.orienteering.asn.au>. Click on the links for mapping and then park mapping. Park maps are typically designed to fit A4 or A3 sized paper so that orienteers, schools, community groups and individuals can photocopy them and enjoy an introduction to orienteering in a familiar environment.

Let's communicate

In about half a page of writing, explain the sport of orienteering and what mathematical skills are needed to participate.

Reflecting

Reflect on the importance of a map and a compass to find direction in the physical world. Why were maps originally made?

Language link with Macquarie

MACQUARIE

1 Give a meaning for:
- ***a compass bearing***
- ***b angle of elevation***
- ***c sin A**, **tan A** and **cos A***

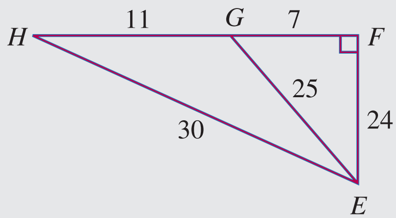
*2 Explain the sixteen **points of the compass** with a drawing.*

3 Read the Macquarie Learners Dictionary *for **bearing**:*

bearing *noun* **1.** the way you stand or behave: *a man of proud bearing* **2.** a supporting part of machine **3. bearings**, direction or position: *We lost our bearings in the dark.*
–phrase **4. have a bearing on**, to have a connection or relevance to: *This information has no bearing on the problem.*

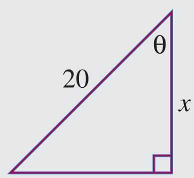
Do you think this explains the mathematical meaning *of the word adequately? What is missing?*

1 Find without simplifying, the value of each ratio below.



a $\sin \angle FEG$ **b** $\cos \angle FHE$
c $\tan \angle FGE$ **d** $\sin \angle FHE$
e $\tan \angle FEH$ **f** $\cos \angle FEG$

2 Find the value of x if $\cos \theta = \frac{3}{4}$.



3 If $\tan \theta = \frac{65}{72}$, draw a right-angled triangle and hence find the value of $\sin \theta$ and $\cos \theta$.

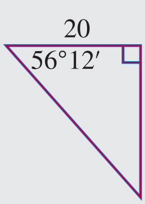
4 Evaluate $\dfrac{\tan 76°19'}{\cos 12°36' - \sin 64°10'}$, correct to 2 decimal places.

5 Find the acute angle θ, correct to the nearest minute.
a $\tan \theta = 3.6816$ **b** $\sin \theta = \frac{8}{21}$

6 If $\tan \theta = 1.0729$, find the value of $\sin \theta$, correct to 4 decimal places.

7 Find the value of each pronumeral, correct to 1 decimal place.

a



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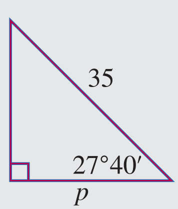

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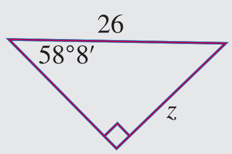
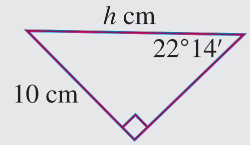


8 Find the length of the hypotenuse in each of these, correct to 1 decimal place.

a



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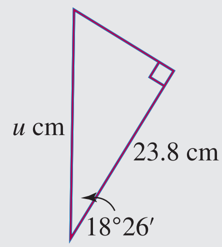
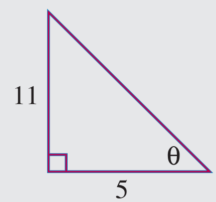


9 Find the angle θ, correct to the nearest minute.

a



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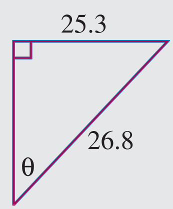

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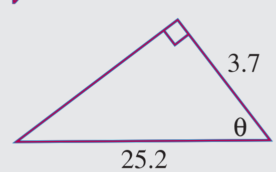


10 **a** In ΔABC, $\angle A = 90°$, $\angle C = 62°45'$ and $BC = 70$ cm. Find the length of AB, correct to 3 significant figures.
b In ΔLMN, $\angle M = 90°$, $\angle L = 73°21'$ and $LM = 36.7$ cm. Find the length of LN, correct to the nearest mm.
c In ΔFGH, $\angle H = 90°$, $GH = 19$ cm and $FH = 10$ cm. Find $\angle F$, correct to the nearest minute.

CHAPTER REVIEW

Chapter Review

11 A building casts a shadow 35 m long on level ground when the altitude of the sun is 51°17′. Find the height of the building, correct to the nearest metre.

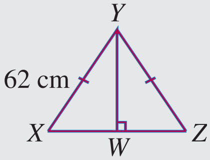
12 A wheelchair ramp is inclined at an angle of 15°43′ and has a vertical rise of 1.64 m. Find the length of the ramp, correct to the nearest cm.

13 A surveyor sights the top of a mountain from a distance of 2400 m. The angle of elevation of the peak is 47°29′. Find the height of the mountain, correct to the nearest metre.

14 A man standing on top of a cliff of height 155 m looks down to a boat that is anchored 115 m from the base of the cliff. Find the angle of depression of the boat from the top of the cliff, correct to the nearest minute.

15 A water pipe runs along the slope of a 295 m high hill. The pipe is 372 m long. At what angle is the pipe inclined to the horizontal? Answer correct to the nearest minute.

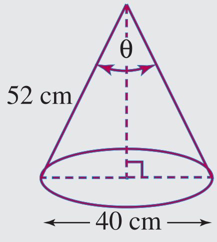
16 In the isosceles triangle XYZ, $YW \perp XZ$, $XY = YZ = 62$ cm and $\angle XYZ = 104°18'$.



a Find the altitude YW, correct to the nearest centimetre.

b Find the length of XZ, correct to the nearest centimetre.

17



An inverted cone has a slant height of 52 cm and a diameter of 40 cm. Find the vertical angle θ, correct to the nearest minute.

18 Which one of the following statements is true?

A $\sin\theta = \dfrac{\cos\theta}{\tan\theta}$ **B** $\cos\theta = \dfrac{\tan\theta}{\sin\theta}$

C $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ **D** $\tan\theta = \dfrac{\cos\theta}{\sin\theta}$

19 Simplify:

a $\cos\theta\tan\theta$ **b** $\dfrac{\sin\theta}{\tan\theta}$

20 Solve each equation for θ, where $0° < \theta < 90°$. Answer correct to the nearest minute.

a $\sin\theta = 4\cos\theta$

b $\cos\theta = 2\sin\theta$

c $5\cos\theta = 3\sin\theta$

21 Find x, if $0° < x° < 90°$.

a $\sin x° = \cos 70°$

b $\cos x° = \sin 25°$

c $\cos (2x)° = \sin 58°$

d $\sin (x - 23)° = \cos 40°$

e $\sin\left(\dfrac{x}{2}\right)° = \cos 78°$

f $\sin x° = \cos x°$

22 Simplify:

a $\dfrac{\cos(90° - \theta)}{\sin\theta}$ **b** $\dfrac{\cos(90° - \theta)}{\cos\theta}$

c $\cos\theta\sin(90° - \theta)$

d $\tan\theta\sin(90° - \theta)$

23 Simplify each of these without the use of a calculator. Give your answers in simplest surd form with a rational denominator.

a $\cos 45°\cos 60°$

b $\sin 60°\cos 30°$

c $\sin 45° + \cos 45°$

d $\tan 30° + \sin 60°$

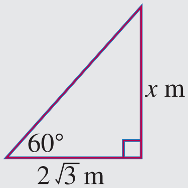
e $1 - 2\sin^2 30°$

f $2\cos^2 30° + \tan^2 60°$

g $\dfrac{\tan 45°}{\sin 30°}$ **h** $\dfrac{\cos 45°}{\sin 60°}$

24 Find the exact value of x.

a



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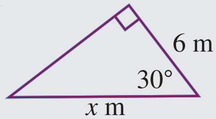
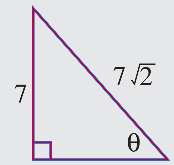
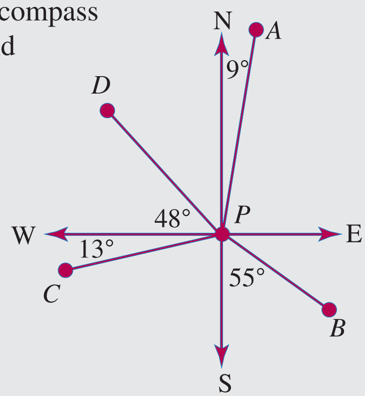


25 Find θ, without the use of a calculator.



26 Find **i** the compass bearing and **ii** the true bearing of each point from P.

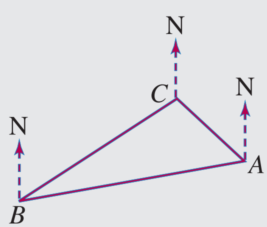


a A **b** B **c** C **d** D

27 Find the bearing of X from Y, given that the bearing of Y from X is:

a 112° **b** 237°

28 Find the size of $\angle ABC$, given that the bearing of B from A is 235° and the bearing of B from C is 217°.



29 a Amber drove 126 km on a bearing of 137°. How far south did she drive? Answer correct to 1 decimal place.

b Ken rode his horse 8.7 km on a bearing of 295°. How far west did he ride? Answer correct to 1 decimal place.

30 Answer each of the following, correct to 4 significant figures.

a A ship leaves port and sails 60 nautical miles due east from L to M, then turns and sails due north to N. If the bearing of N from L is N24°E, find the distance MN.

b Beaumont Hills is due west of Coleville. Coleville is 11 km due north of Allentown. The bearing of Allentown from Beaumont Hills is 136°. Find the distance between Beaumont Hills and Allentown.

31 Answer each of the following as a true bearing, correct to the nearest degree.

a A ferry sailed 12 km due south from T to U, then turned and sailed 19 km due east to V. Find the bearing of:

i V from T **ii** T from V

b Peter drove 94 km due north from C to D, then turned and drove 52 km due west to E. Find the bearing of:

i E from C **ii** C from E

32 Xian walked 250 m from I to J on a bearing of 043°. He then turned and walked on a bearing of 133° to K, which is due east of I.

a Show that $\angle IJK = 90°$.

b Find the distance JK, correct to the nearest metre.

33 Angus drove from P to Q on a bearing of 211°. He then turned and drove 48 km on a bearing of 301° to R, which is due west of P.

a Show that $\angle PQR = 90°$.

b Find the distance PR, correct to the nearest kilometre.

CHAPTER REVIEW

3 Volume and surface area

This chapter at a glance

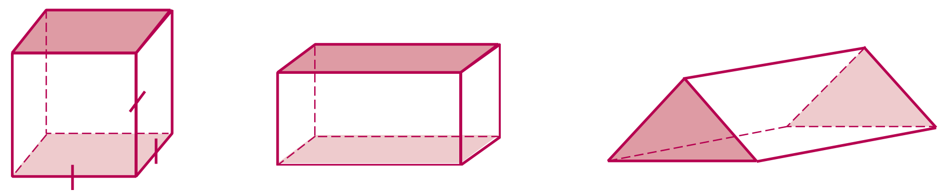
Stage 5.1/5.2/5.3

After completing this chapter, you should be able to:

- calculate the surface area of a variety of prisms and pyramids
- calculate the surface area and curved surface area of a cylinder
- calculate the surface area of a cone
- calculate the surface area of a sphere
- calculate the surface area of composite solids
- calculate the volume of a variety of prisms
- calculate the volume of a cylinder
- solve problems involving the volume and surface area of prisms and cylinders
- calculate the volume of a variety of pyramids
- calculate the volume of a cone
- calculate the volume of a sphere
- calculate the volume of composite solids
- find unknown dimensions of solids given the other dimensions and either the volume or surface area
- solve problems involving the volume of pyramids, cones and spheres.

3.1 Surface area of a prism

The **surface area** of a solid with plane faces is the sum of the areas of its faces. When calculating surface area, we need to ensure that the areas of all faces are found and that no areas are counted more than once. This is best done by finding the areas in a systematic way. For example, in a rectangular prism, find the areas of the top and bottom faces, the front and back faces and then the left and right side faces.



It is also important to consider whether the figure is open or closed, such as in the case of a swimming pool, or a room that is to have its walls and ceiling painted.

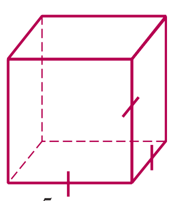
The surface area of a solid is the sum of the areas of its faces.

NOTE: If a solid is a prism, then all faces other than the bases must be rectangles.

Example 1

Find the surface area of each prism.

a



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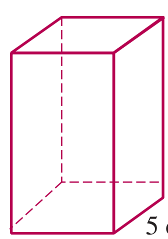

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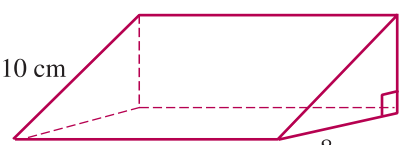


Solutions

a A cube has six sides that are equal in area.

i Area of a side $= 5^2$
$= 25 \text{ cm}^2$

ii Surface area $= 6 \times$ area of one face
$= 6 \times 25$
$= 150 \text{ cm}^2$

b The opposite sides of a rectangular prism have the same area.

Surface area = (area of bottom face × 2) + (area of right face × 2) + (area of front face × 2)
$= (4 \times 5 \times 2) + (5 \times 7 \times 2) + (4 \times 7 \times 2)$
$= 40 + 70 + 56$
$= 166 \text{ cm}^2$

c The prism consists of two triangular faces with equal areas and three rectangles.

Surface area = (area of triangular face × 2) + (area of bottom face) + (area of back face) + (area of front face)

$$= \left(\frac{6 \times 8}{2} \times 2\right) + (15 \times 8) + (15 \times 6) + (15 \times 10)$$

$$= 48 + 120 + 90 + 150$$

$$= 408 \text{ cm}^2$$

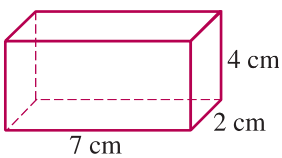
Exercise 3.1

1 Find the surface area of a cube with sides of length:

a 2 cm **b** 3 cm **c** 5.4 cm **d** 9.7 cm

2 Find the surface area of each rectangular prism.

a



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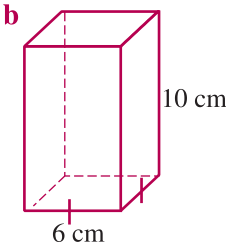

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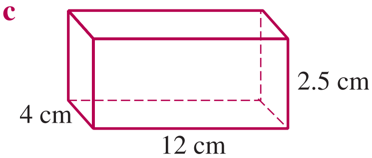
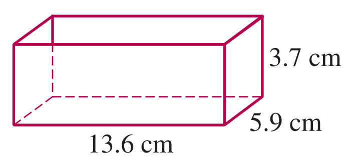
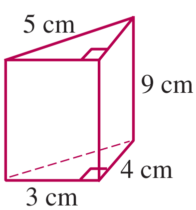


3 Find the surface area of this rectangular prism, which is open at the top.



4 Find the surface area of each triangular prism.

a



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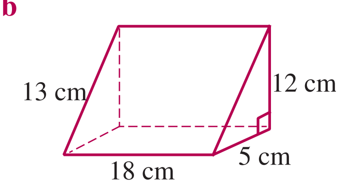

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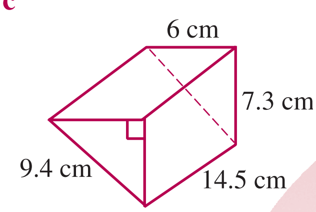
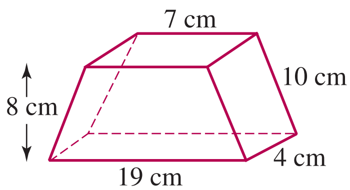


5 Find the surface area of each trapezoidal prism.

a



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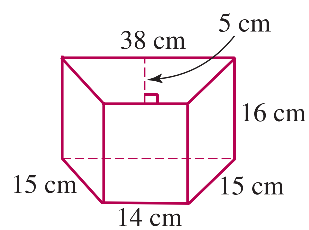
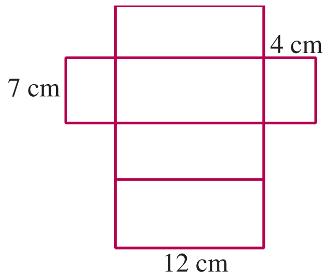


6 Find the total surface area of the solids with the following nets.

a



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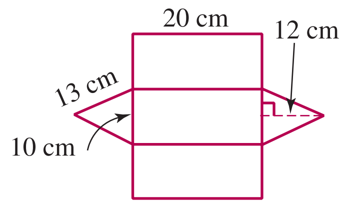
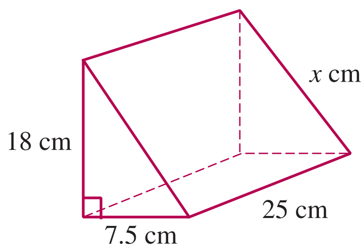


7 **i** Find the value of the pronumeral in each triangular prism.

ii Hence, calculate the surface area.

a



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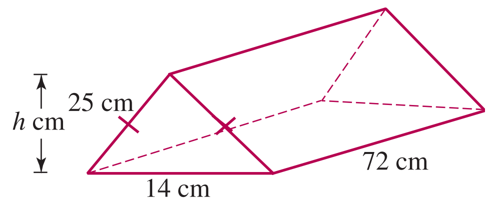
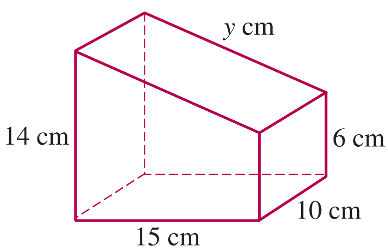


8 Find the side length of a cube whose surface area is:

a 96 cm^2 **b** 294 cm^2 **c** 1014 cm^2

9



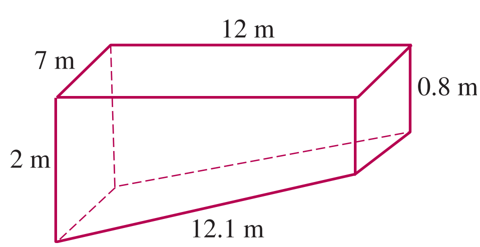
a Find the value of y.

b Hence, find the surface area of this trapezoidal prism.

10 The sides and floor of this swimming pool are to be tiled. The tiles cost $9 per square metre and there is a further charge of $750 for labour.

a Calculate the area to be tiled.

b Find the cost of tiling the pool.



11 Marco painted the walls and ceiling of his new games room. The windows and door have a combined area of 8.4 m^2. The ceiling measures 6 m × 4.6 m and the walls are 2.5 m high. The cost of painting the room was $613.70.

a Calculate the area that has been painted.

b Find the cost, per square metre, of painting the room.

■ Further applications

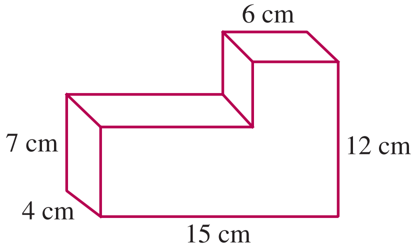
12 Write down a possible set of dimensions for a rectangular prism whose surface area is 236 cm^2.

13 Find the width of a rectangular prism whose length is 13 cm, height is 7 cm and surface area is 542 cm^2.

14 A rectangular prism, open at the top, has a length of 15 cm, width of 11 cm and a surface area of 789 cm^2. Find the height of the prism.

15 Calculate the total surface area of each solid.

a



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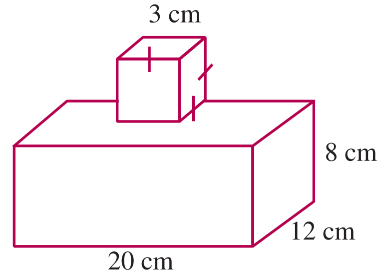

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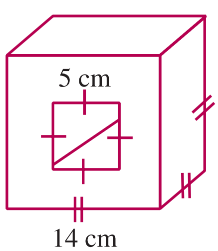
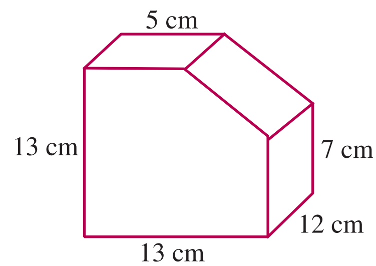


d



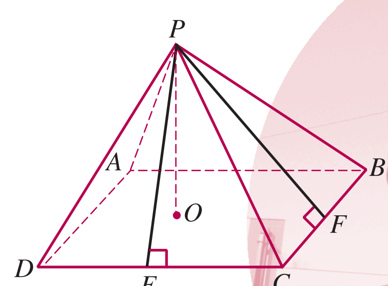
3.2 Surface area of a pyramid

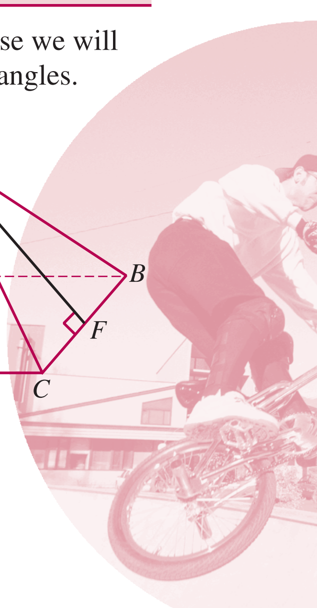
In a right pyramid, the apex is directly above the centre of the base. In this exercise we will consider only right pyramids. In any pyramid, all faces other than the base are triangles.

The perpendicular height of a pyramid is the distance from the apex to the centre of the base. The slant heights are the distances from the apex to the midpoints of the base edges.

PO is the perpendicular height, *PE* and *PF* are slant heights.

The lengths of the slant edges *PA*, *PB*, *PC*, *PD* are all equal.



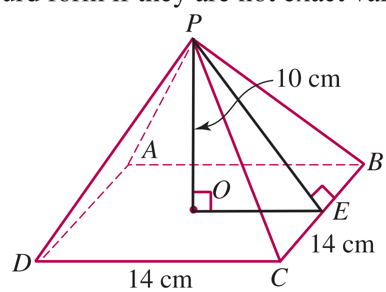


To find the surface area of a pyramid:
- **find the slant heights, if necessary, by using Pythagoras' theorem**
- **find the sum of the areas of the base and the triangular faces.**

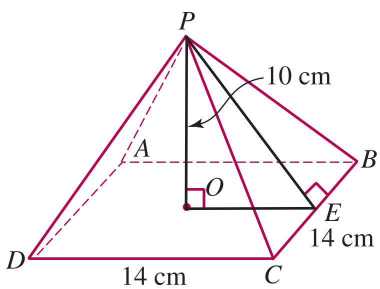
NOTE: The slant heights should be left in exact surd form if they are not exact values.

Example 1

Find the surface area of this square pyramid.



Solution



i Find PE, the perpendicular height of the triangular faces.

$PE^2 = 10^2 + 7^2$ (by Pythagoras' theorem)

$PE^2 = 149$

$\therefore PE = \sqrt{149}$

ii S.A. = (area of base) + (area of 4 triangular faces)

$= (14 \times 14) + (4 \times \frac{1}{2} \times 14 \times \sqrt{149})$

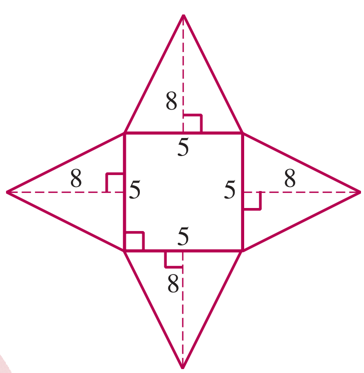
$= 196 + 28\sqrt{149}$

$\doteqdot 537.8 \text{ cm}^2$

Exercise 3.2

1 Find the surface area of the pyramids that have the following nets. All lengths are in cm.

a



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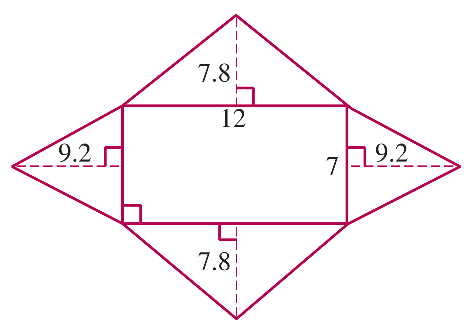
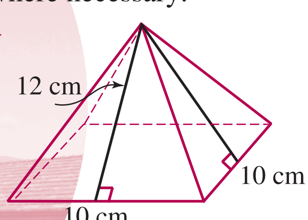


2 Find the surface area of these square and rectangular pyramids, correct to the nearest cm^2 where necessary.

a



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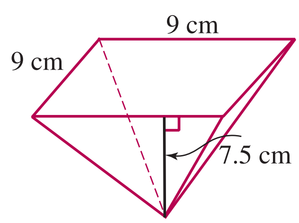
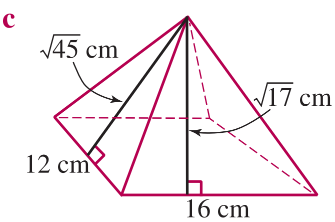
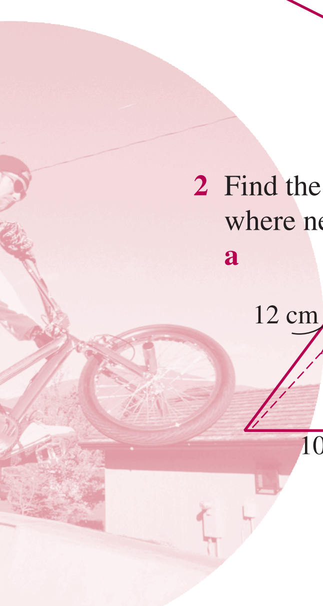


c

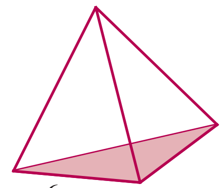




■ Consolidation

3 **a** Show that an equilateral triangle of side 6 cm has a height of $3\sqrt{3}$ cm.

b Hence, find the exact surface area of this tetrahedron.

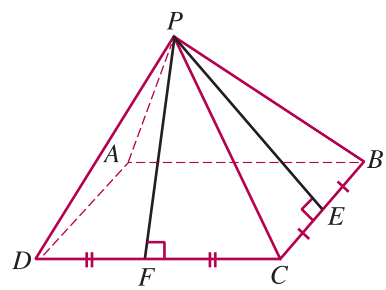


4 $PABCD$ is a rectangular pyramid.
$PE = 24$ cm, $PF = 20$ cm, $BC = 14$ cm.

a Find PC

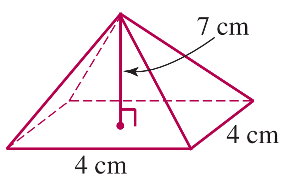
b Find CD

c Hence, find the surface area of the pyramid.

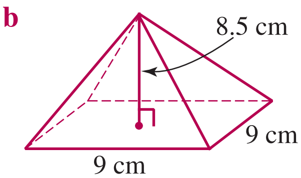


5 Find the exact height of each triangular face in the following square or rectangular pyramids. Hence, calculate the surface area of each pyramid, correct to 1 decimal place.

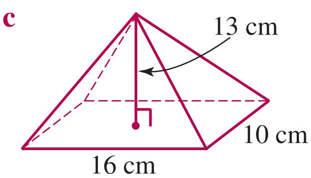
a



b



c



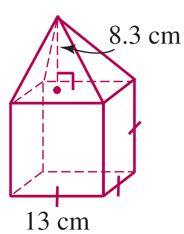
6 Find correct to 1 decimal place, the surface area of:

a a square pyramid that has base edges of 8 cm and a perpendicular height of 12 cm.

b a rectangular pyramid that has base edges of 14 cm and 18 cm and a perpendicular height of 21 cm.

7 Find the surface area of each solid. Answer correct to 1 decimal place.

a



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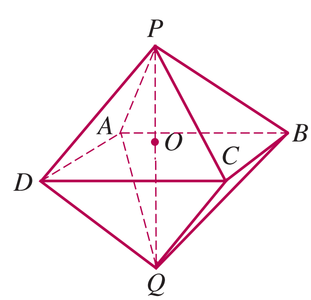


$PO = 8$ cm, $OQ = 11$ cm,
$AB = BC = CD = AD = 14$ cm

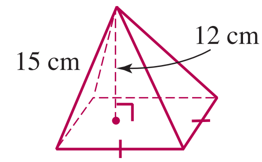
Further applications

8 A square pyramid has slant edges of 15 cm and a perpendicular height of 12 cm.

a Show that the diagonals in the base have length 18 cm.

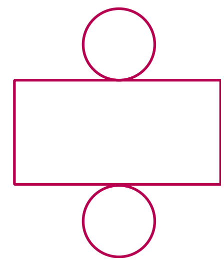
b Find the exact length of the base edges.

c Hence, find the surface area of the pyramid, correct to 1 decimal place.



3.3 Surface area of a cylinder

The surface area of a cylinder is made up of two circles and a curved surface. In order to find the area of the curved surface, we need to examine the net of the cylinder. The net shown here shows that the surface of a cylinder is actually made up of one rectangle and two circles.



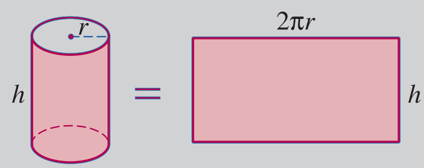
Curved surface area

The **curved surface area** of a cylinder is equal to the area of the rectangle in its net. The area of this rectangle is found by multiplying the length by the breadth. However, the length of the rectangle is equal to the circumference of the base circle. Therefore, the curved surface area can be calculated by multiplying the circumference of the base circle by the height of the cylinder.

The curved surface area of a cylinder is given by:

$$A = 2\pi rh$$

where • ***r*** **is the radius of the base**
• ***h*** **is the height of the cylinder.**



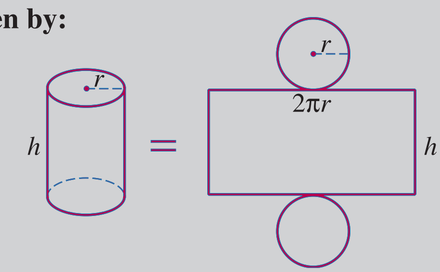
Total surface area

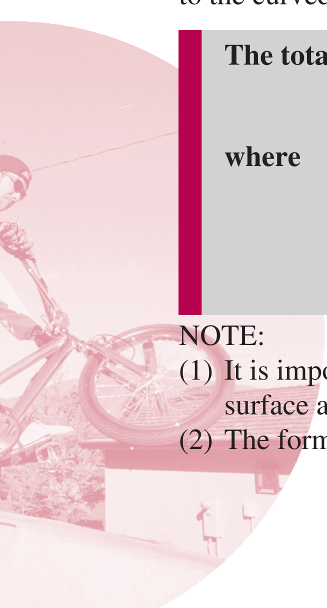
The total surface area of a closed cylinder is found by adding the areas of the two base circles to the curved surface area.

The total surface area of a closed cylinder is given by:

$$A = 2\pi r^2 + 2\pi rh$$

where • ***r*** **is the radius of the base**
• ***h*** **is the height of the cylinder.**





NOTE:

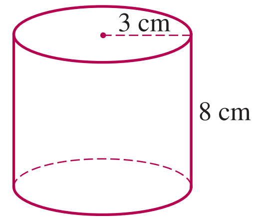
(1) It is important to consider whether the cylinder is open or closed before calculating its surface area.

(2) The formula for total surface area is sometimes factorised and given as $A = 2\pi r(r + h)$.

Example 1

For the cylinder shown, find correct to 1 decimal place:

a the curved surface area **b** the total surface area



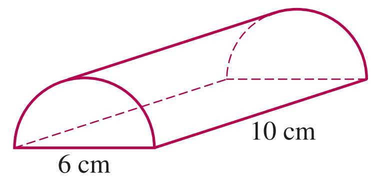
Solutions

a $A = 2\pi rh$
$= 2 \times \pi \times 3 \times 8$
$\doteqdot 150.8 \text{ cm}^2$

b $A = 2\pi r^2 + 2\pi rh$
$= (2 \times \pi \times 3^2) + (2 \times \pi \times 3 \times 8)$
$\doteqdot 207.3 \text{ cm}^2$

Example 2

Find the total surface area of this half-cylinder, correct to 1 decimal place.



Solution

The surface area consists of the curved surface (A_1), two semicircles (A_2) and a rectangle (A_3).

i $A_1 = 2\pi rh \times \frac{1}{2}$
$= 2 \times \pi \times 3 \times 10 \times \frac{1}{2}$
$= 30\pi \text{ cm}^2$

ii $A_2 = \pi r^2$
$= \pi \times 3^2$
$= 9\pi \text{ cm}^2$

iii $A_3 = l \times b$
$= 6 \times 10$
$= 60 \text{ cm}^2$

iv $A = A_1 + A_2 + A_3$
$= 30\pi + 9\pi + 60$
$\doteqdot 182.5 \text{ cm}^2$

Exercise 3.3

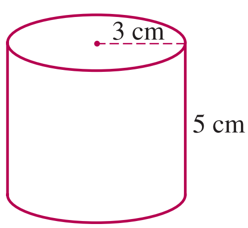
1 Find correct to 1 decimal place, the curved surface area of a cylinder with:

a radius 7 cm and height 11 cm **b** radius 5.3 cm and height 13.4 cm
c diameter 16 cm and height 6 cm **d** diameter 23.6 cm and height 17.5 cm

2 Find the exact curved surface area of a cylinder with:

a radius 5 cm and height 8 cm **b** diameter 9 cm and height 22 cm

3 Find the total surface area of each cylinder, correct to 1 decimal place.

a


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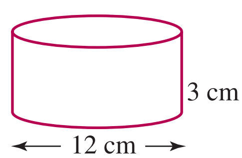

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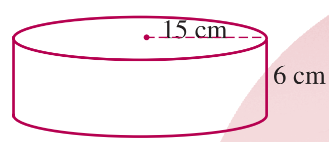

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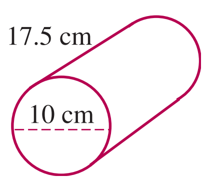

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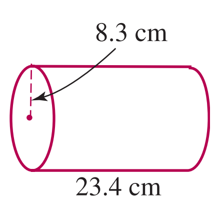

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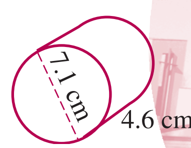


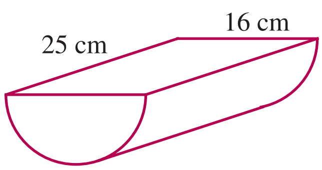
■ Consolidation

4 a Find the length of the radius of a cylinder with a height 11 cm and a curved surface area of 1320 cm^2. Answer correct to 3 significant figures.

b Find the length of the radius in a cylinder with a height 6.7 cm and a curved surface area of 3584 cm^2. Answer correct to 3 significant figures.

c Find the height of a cylinder that has a radius of 7.5 cm and a curved surface area of 120π cm^2.

5



The solid is a half cylinder. Find:

a the area of the top face

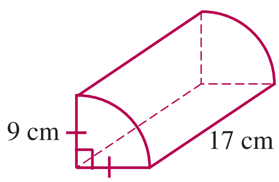
b the exact area of the two semicircular ends

c the exact area of the curved surface

d the total surface area of the cylinder, correct to 1 decimal place

6 Find the total surface area of each solid, correct to 1 decimal place.

a



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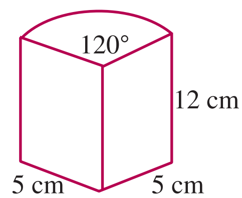
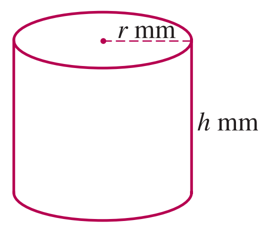


7 A metal tube in the shape of a cylinder is open at the top. The outside of the tube, including the base, is to be painted blue.

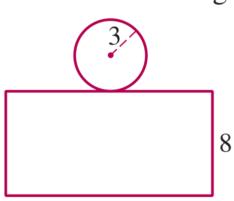
a Write down a formula for the area that is to be painted.

b Find this area if the diameter is 28 mm and the height is 40 mm. Answer correct to the nearest mm^2.

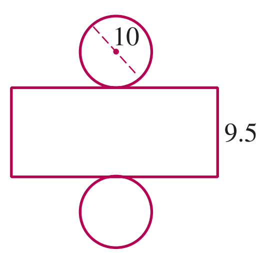


8 Find the surface area of the cylinders that have the following nets. Answer correct to 3 significant figures. All lengths are in cm.

a



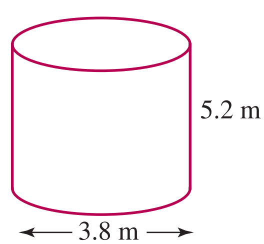
b

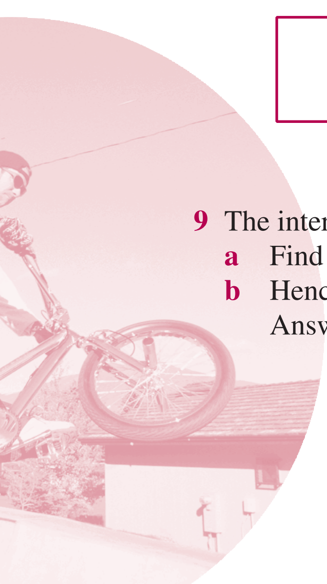


9 The interior of a cylindrical water tank is to be rustproofed.

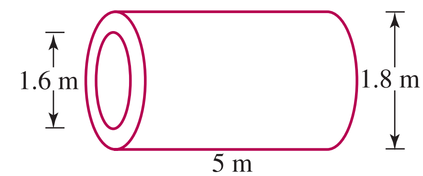
a Find the required area correct to 2 decimal places.

b Hence, find the cost of rustproofing the tank at \$2.95/m^2. Answer correct to the nearest 5 cents.





10 A 5 m length of pipe has an inner diameter of 1.6 m and an outer diameter of 1.8 m.



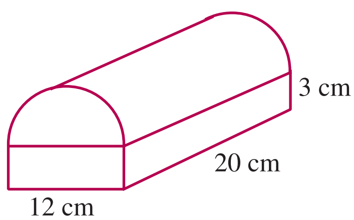
a Find the outer curved surface area in terms of π.
b Find the inner curved surface area in terms of π.
c Find the area of the two ends of the pipe in terms of π.
d Hence, find the total surface area of the pipe, correct to 1 decimal place.

11 A company is to manufacture 500 aluminium tubes. The tubes are cylindrical in shape and are closed at both ends. Each tube has a radius of 6 cm and a length of 24 cm. Find the cost of manufacturing the tubes if aluminium costs 0.253 cents per cm^2.

■ Further applications

12 Find the total surface area of each solid, correct to the nearest cm^2.

a



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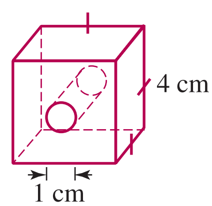
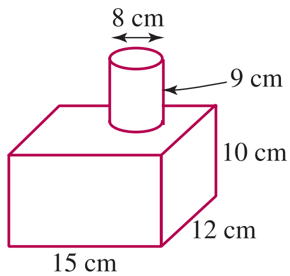


c



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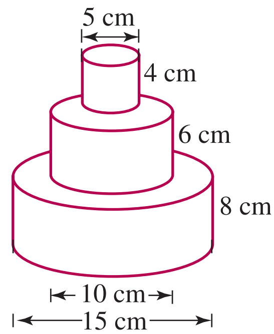
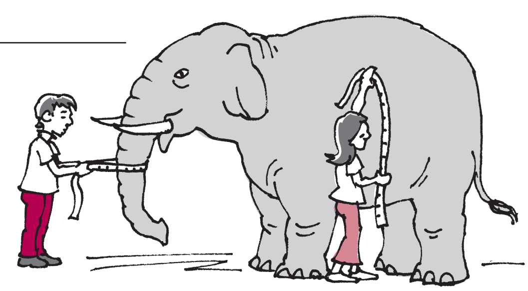


13 The surface area of a cylinder is 88π cm^2 and the radius is 4 cm. Find the height of the cylinder.

14 A closed cylinder has a curved surface area of 72π cm^2 and a height of 4 cm. Find the total surface area of the cylinder in terms of π.

TRY THIS Elephants

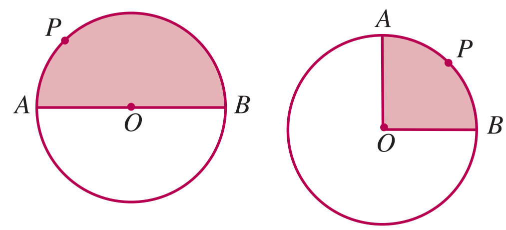
With the help of a picture or otherwise, what is the surface area of an elephant?



3.4 Surface area of a cone and sphere

Surface area of a cone

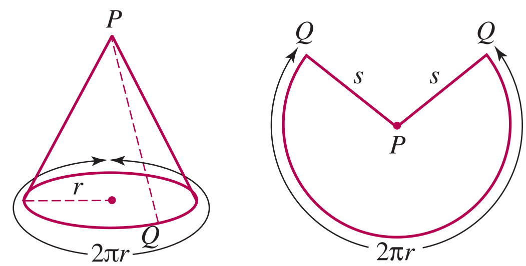
Consider the semi-circle and quadrant shown.



In each case, the ratio area of sector *OAPB* : area of circle is equivalent to the ratio length of arc *APB* : circumference of the circle.

That is, $\dfrac{\text{area of sector}}{\text{area of circle}} = \dfrac{\text{length of arc}}{\text{circumference of circle}}$

Now, consider a cone with base radius *r* units and a slant height *s* units. The circumference of the base is $2\pi r$ units. If the cone is cut along the line *PQ* and flattened out, then the curved surface would be a sector of a circle with radius *s* units and arc length $2\pi r$ units.



Now, $\dfrac{\text{area of sector}}{\text{area of circle}} = \dfrac{\text{length of arc}}{\text{circumference of circle}}$

$$\therefore \frac{\text{area of sector}}{\pi s^2} = \frac{2\pi r}{2\pi s}$$

$$\therefore \text{area of sector} = \frac{\cancel{2\pi} r}{\cancel{2\pi}\cancel{s}} \times \pi \cancel{s}^{\cancel{2}}$$

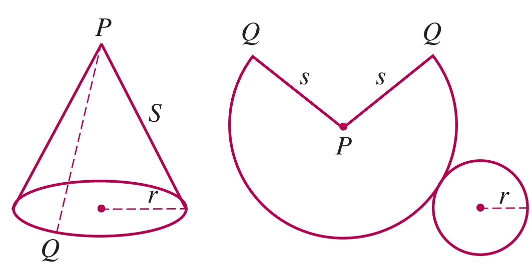
$$= \pi r s$$

The curved surface area of a cone is given by:

$$\mathbf{A = \pi r s}$$

where ◎ ***r* is the length of the radius**
◎ ***s* is the slant height of the cone.**

The total surface area of a cone is then found by adding the area of the circular base to the curved surface area.



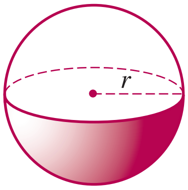
The total surface area of a cone is given by:

$$\mathbf{A = \pi r^2 + \pi r s}$$

where ◎ ***r* is the length of the radius**
◎ ***s* is the slant height of the cone.**

Surface area of a sphere

The proof of the following formula for the surface area of a sphere is beyond the scope of this course.



The surface area of a sphere is given by:

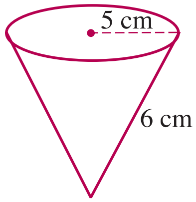
$$A = 4\pi r^2$$

where r is the length of the radius.

Example 1

a Find the exact curved surface area of the cone.

b Hence, find the total surface area, correct to 1 decimal place.



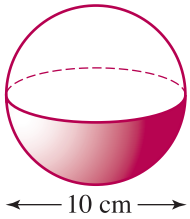
Solutions

a $A = \pi rs$
$= \pi \times 5 \times 6$
$= 30\pi \text{ cm}^2$

b $A = \pi r^2 + \pi rs$
$= (\pi \times 5^2) + 30\pi$
$= 25\pi + 30\pi$
$= 55\pi$
$\doteqdot 172.8 \text{ cm}^2$

Example 2

Find the surface area of this sphere, correct to 1 decimal place.

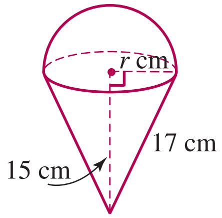


Solution

$A = 4\pi r^2$
$= 4 \times \pi \times 5^2$
$= 100\pi$
$\doteqdot 314.2 \text{ cm}^2$

Example 3

Find the total surface area of this solid, correct to 1 decimal place.



Solution

i $r^2 + 15^2 = 17^2$ (by Pythagoras' theorem)

$r^2 + 225 = 289$

$r^2 = 64$

$\therefore r = 8$

ii Area of hemisphere

$= 4\pi r^2 \times \frac{1}{2}$

$= 4 \times \pi \times 8^2 \times \frac{1}{2}$

$= 128\pi \text{ cm}^2$

iii Area of cone

$= \pi rs$

$= \pi \times 8 \times 17$

$= 136\pi \text{ cm}^2$

iv Area of solid

$= 128\pi + 136\pi$

$= 264\pi$

$\doteqdot 829.4 \text{ cm}^2$

Example 4

EG +S

Find the radius of a sphere whose surface area is 196π cm^2.

Solution

$A = 4\pi r^2$

$196\pi = 4\pi r^2$

$\div 4\pi \quad \div 4\pi$

$\therefore r^2 = 49$

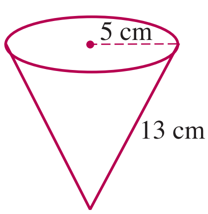
$\therefore r = 7$

$\therefore$ the radius is 7 cm.

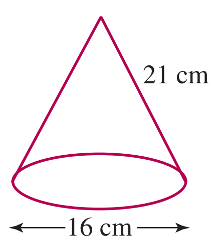
Exercise 3.4

1 Find the exact curved surface area of each cone.

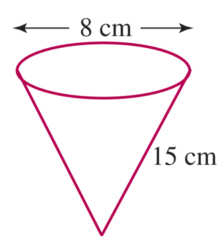
a



b

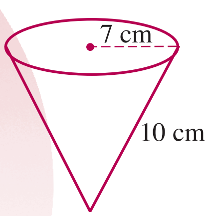


c



2 Find the surface area of each cone, correct to 1 decimal place.

a



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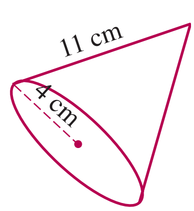

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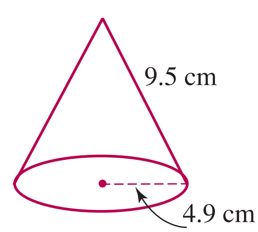
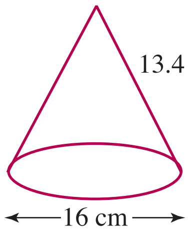


d



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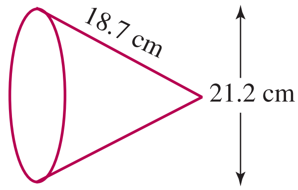

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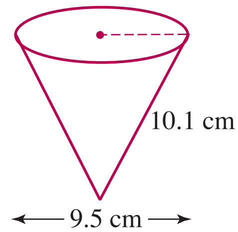
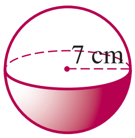
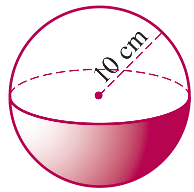


3 Find the surface area of each sphere, correct to the nearest cm^2.

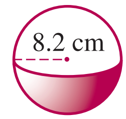
a



b



c



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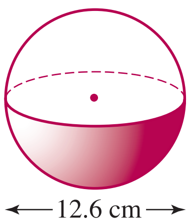

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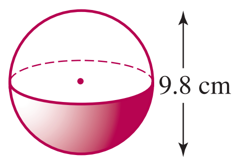

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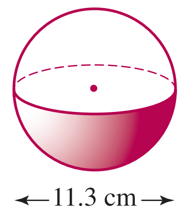


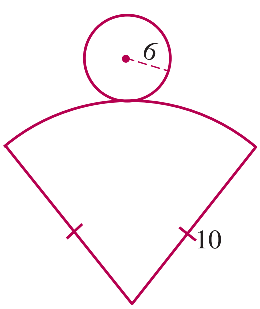
4 Find the exact surface area of a sphere with:

a radius 3 cm **b** radius 2 cm **c** diameter 12 cm **d** diameter 20 cm

■ Consolidation

5 Find the surface area of the cone with each net below. All lengths are in cm. Answer correct to 1 decimal place.

a



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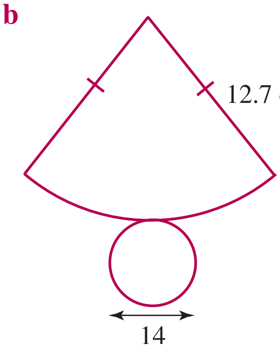

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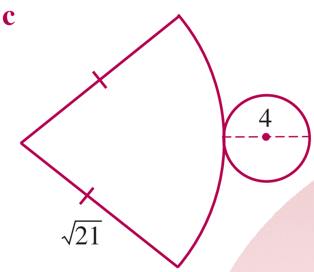
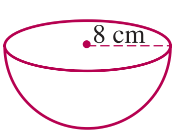
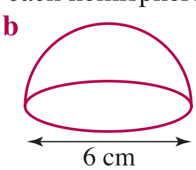


6 Find the exact surface area of each hemisphere.

a



b



7 Find the radius of a sphere whose surface area is:

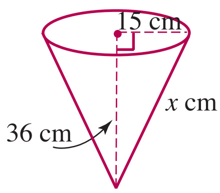
a 324π cm^2 **b** 484π cm^2 **c** 2704π cm^2

8 Find the radius of a sphere whose surface area is given below. Answer correct to 2 decimal places.

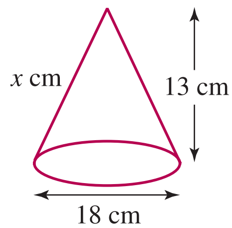
a 100 cm^2 **b** 325 cm^2 **c** 2150 cm^2

9 Find the value of x (as a surd if necessary). Hence find the surface area of each cone correct to the nearest cm^2.

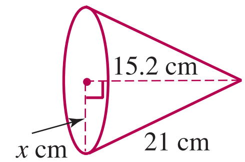
a



b



c



10 The radius of the Earth is approximately 6400 km. Find the surface area, correct to the nearest km^2.

11 A cone with radius 9 cm has a curved surface area of 135π cm^2.

a Find the slant height.

b Find the total surface area of the cone, correct to 3 significant figures.

12 The circular end of a cone has an area of 49π cm^2 and the perpendicular height of the cone is 24 cm.

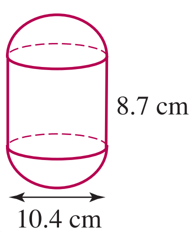
a Find the slant height. **b** Hence, find the exact curved surface area.

13 A cone has a slant height of 6 cm and a surface area of 40π cm^2. Find the radius.

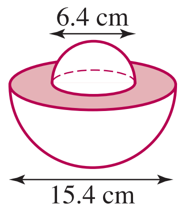
14 A solid sphere is cut in half. Find in simplest form, the ratio of the surface area of the hemisphere to the surface area of the sphere.

15 Find the total surface area of each solid, correct to the nearest cm^2.

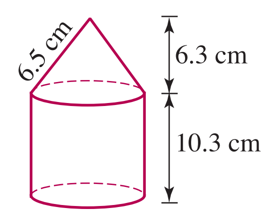
a



b



c



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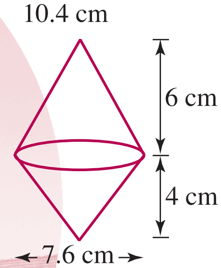
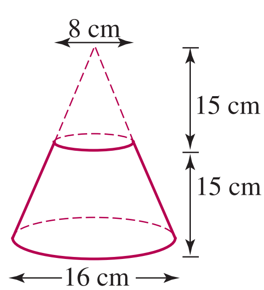


e



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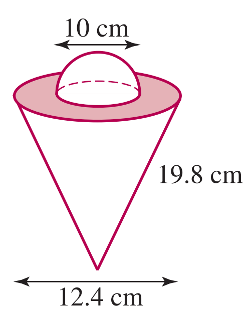
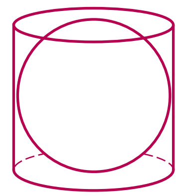


Further applications

16 A hemisphere of radius 7 cm is cut from a larger hemisphere of radius 10 cm. Find the exact surface area of the remaining solid.

17 A cone has the same radius and surface area as a sphere. Show that the slant height of the cone must be 3 times the radius.

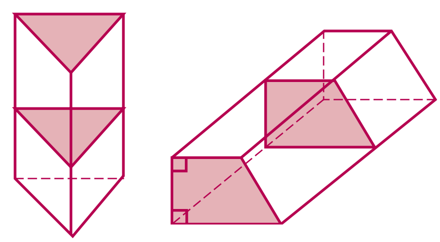
18 A sphere fits tightly into a cylinder as shown. Show that the ratio of the surface area of the sphere to the surface area of the cylinder is 2 : 3.



3.5 Volume of a prism

Definition of a prism

A **prism** is a solid that has a **uniform cross-section**. The two faces parallel to this cross-section are called bases. If the solid is sliced several times parallel to a base, the resulting cross-sections would all have the same size and shape.

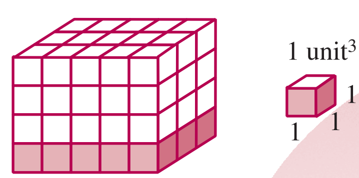


The base of a prism can be any polygon, but all of the other faces in the solid must be rectangles. The prism is named according to the shape of the cross-section. For example, if the prism has triangular cross-sections, the solid is referred to as a triangular prism.

Volume of a prism

Volume is the space inside a three-dimensional figure. It is measured in cubic units such as mm^3, cm^3 or m^3. The volume of a rectangular prism could be found by counting the number of unit cubes in one layer, then multiplying this by the number of layers.

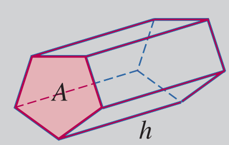
For example, in this rectangular prism, there are $5 \times 3 = 15$ cubes in each layer and there are 4 layers. Therefore, the total number of unit cubes is $15 \times 4 = 60$. That is, the prism has a volume of 60 units^3. This is equivalent to calculating the cross-sectional area and multiplying it by the height of the prism.



The volume of a prism is given by:

$$V = Ah$$

where
- A **is the cross-sectional area**
- h **is the height of the prism.**

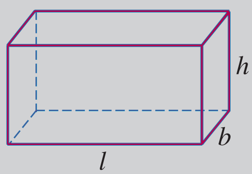


NOTE: The height of a prism, h units, is the distance between the bases. The height of a prism is not necessarily vertical.

Volume of a rectangular prism

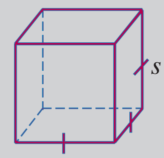
The rectangular prism and the cube have special volume formulae, because their volumes are calculated so often.

Rectangular prism



$V = l \times b \times h$

Cube

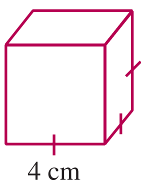


$V = s^3$

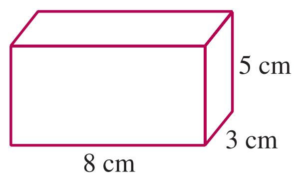
Example 1

Find the volume of each prism.

a



b



Solutions

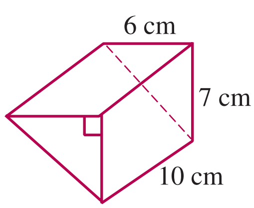
a $V = s^3$
$= 4^3$
$= 64 \text{ cm}^3$

b $V = lbh$
$= 8 \times 3 \times 5$
$= 120 \text{ cm}^3$

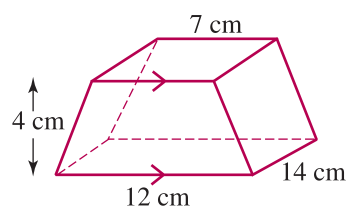
Example 2

Find the volume of each prism.

a



b



Solutions

a **i** $A = \frac{1}{2}bh$
$= \frac{1}{2} \times 6 \times 7$
$= 21 \text{ cm}^2$

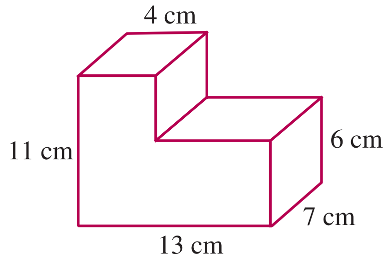
b **ii** $V = Ah$
$= 21 \times 10$
$= 210 \text{ cm}^3$

b **i** $A = \frac{h}{2}(a + b)$
$= \frac{4}{2}(7 + 12)$
$= 38 \text{ cm}^2$

ii $V = Ah$
$= 38 \times 14$
$= 532 \text{ cm}^3$

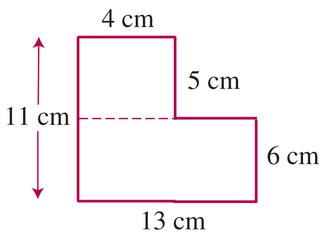
Example 3

Find the volume of this solid.



Solution

i



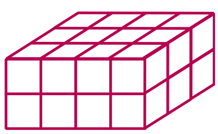
$A = (4 \times 5) + (13 \times 6)$
$\quad = 98 \text{ cm}^2$

ii $V = Ah$
$\quad = 98 \times 7$
$\quad = 686 \text{ cm}^3$

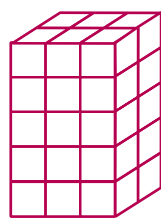
Exercise 3.5

1 Find the volume of each prism in cubic units.

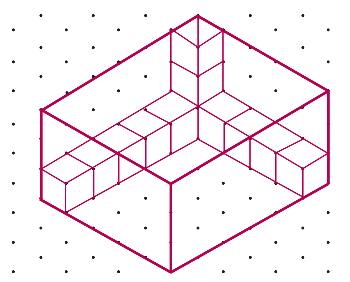
a



b

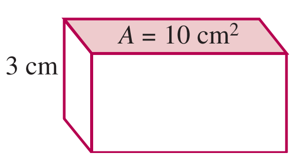


c

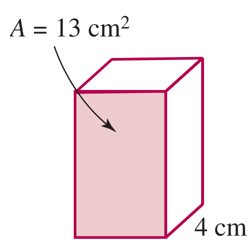


2 Find the volume of each prism.

a



b



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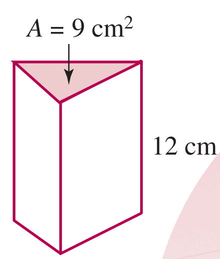

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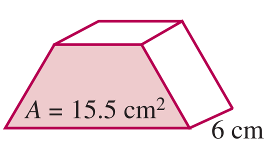
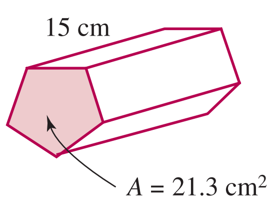


e



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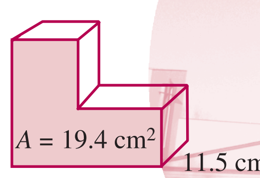
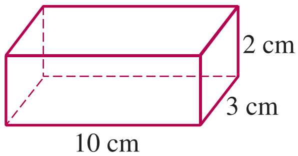


3 Find the volume of each rectangular prism.

a



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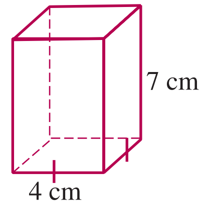

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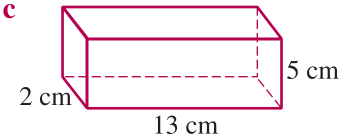


4 Find the volume of a cube with sides of length:

a 2 cm **b** 5 cm **c** 9 cm **d** 12 cm

■ Consolidation

5 Find, correct to 2 decimal places, the volume of a rectangular prism with dimensions:

a length = 4.7 m, breadth = 2.8 m, height = 5.9 m

b length = 15.2 m, breadth = 8.4 m, height = 9.6 m

c length = 1.54 m, breadth = 0.35 m, height = 0.96 m

6 Find, correct to 1 decimal place where necessary, the side length of a cube whose volume is:

a 512 cm^3 **b** 234 cm^3 **c** 776 cm^3 **d** 1350 cm^3

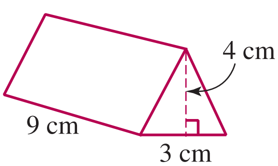
7 The tables below show the volume and two dimensions of a rectangular prism. Find the missing dimensions.

	l	b	h	V
a	5 m		2 m	30 m^3
b		4 m	5 m	180 m^3
c	7 m	6 m		126 m^3

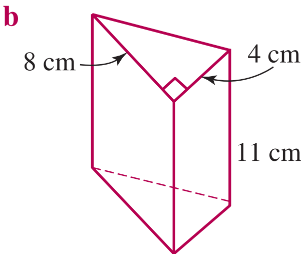
	l	b	h	V
d	15.8 m		4.5 m	853.2 m^3
e	17.6 m	5.2 m		1052.48 m^3
f		13.75 m	12.6 m	4227.3 m^3

8 Find the volume of these triangular prisms.

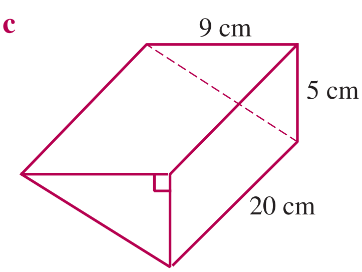
a



b

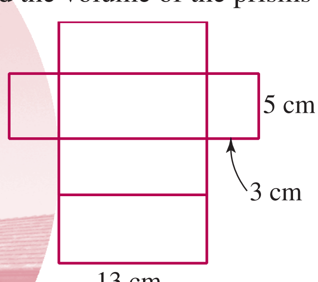


c



9 Find the volume of the prisms that have the following nets.

a



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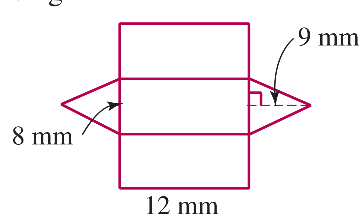
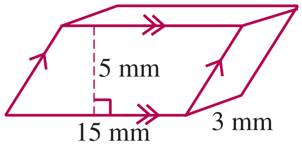


10 The cross-section in each prism is a parallelogram. Find the volume.

a



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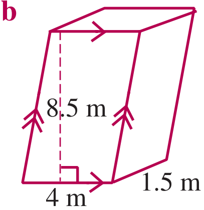

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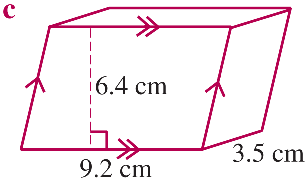
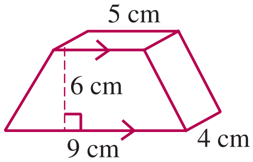
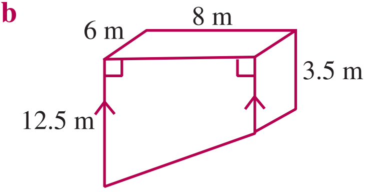


11 Find the volume of each trapezoidal prism.

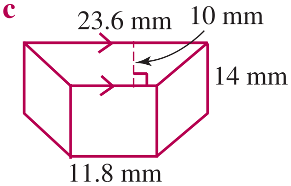
a



b



c



12 **a** Find the height of a trapezoidal prism whose volume is 117 cm^3 and cross-sectional area is 13 cm^2.

b Find the cross-sectional area of a pentagonal prism whose volume is 322 mm^3 and height is 14 mm.

c Find the height of an octagonal prism whose volume is 148.5 cm^3 and cross-sectional area is 19.8 cm^2.

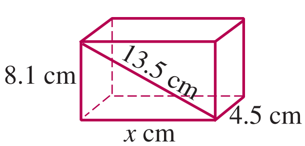
d Find the cross-sectional area of a triangular prism whose volume is 310.34 m^3 and height is 11.8 m.

13 For each of the following prisms:

i find the value of the pronumeral, using Pythagoras' theorem

ii calculate the volume

a



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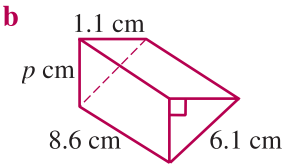

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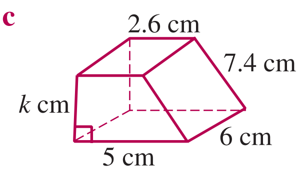
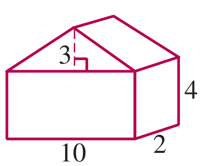


14 Find the volume of each solid. All measurements are in metres.

a



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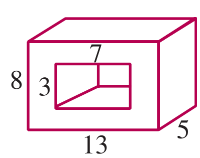

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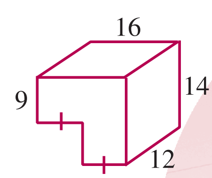

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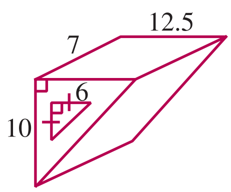

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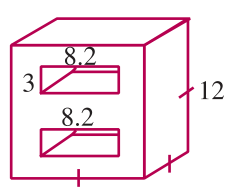

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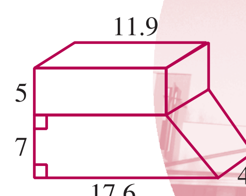

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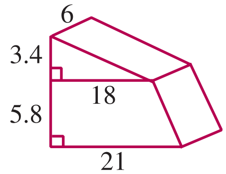

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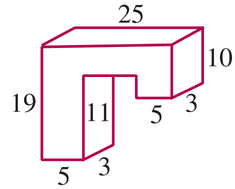

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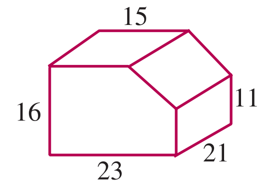


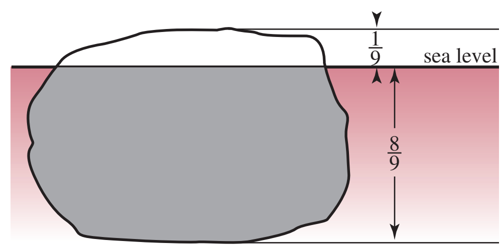
Further applications

15 a Find the surface area of a cube whose volume is 729 cm^3.

b Find the volume of a cube whose surface area is 726 cm^2.

16 What would be the side length of a cube whose volume is numerically equal to its surface area?

TRY THIS Can we use an iceberg?



An iceberg is made from snow, so it is fresh water. Very large icebergs with flat tops—table bergs—are formed in parts of Antarctica. The largest recorded table berg was 335 km long and 95 km across at its widest part. Its height above the water is not recorded. These flat bergs have $\frac{8}{9}$ths of their volume below the surface. In other words, they are much bigger than they look. Because Australia is a dry continent, the question has sometimes been asked whether a large iceberg could supply parts of our continent with fresh water.

Let us investigate a proposal to tow a medium berg to, say, the dry north-west of Australia. (Without considering how it is to be towed, or the cost of the tow.) We will investigate what quantity of water is available.

Imagine towing a berg 8 km long × 3 km wide with a table top 10 metres above the sea level. Assume that 45% of the berg is lost in the towing process.

Answer the following questions.

1 What is the volume measured in m^3 of the iceberg which arrives at the destination?

2 How many megalitres of fresh water are there in the berg?

3 In your opinion, is the volume of water obtained from the berg large enough to be of value? (For comparison the Woronora Dam holds 71.79 million cubic metres; the Warragamba Dam holds 2031 million m^3.)

4 What area of cropland, to the nearest hectare, could be supplied with water if 4.4 ha requires 1 ML each year?

NOTES: **1** A litre of ice contains 0.9 litres of fresh water.

2 A megalitre = 1 million litres, which occupies 1000 cubic metres.

3 1 m^3 = 1000 litres.

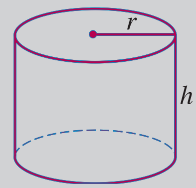
3.6 Volume of a cylinder

Although a cylinder is not a prism, it does have a uniform cross-section. Therefore, like a prism, its volume can also be determined by finding the product of the cross-sectional area and the height. That is, the volume could be found by using the formula $V = Ah$. However, the cross-section is a circle, whose area is given by $A = \pi r^2$. Therefore, in the case of a cylinder, we could re-write the formula $V = Ah$ as $V = \pi r^2 h$.

The volume of a cylinder is given by:

$$V = \pi r^2 h$$

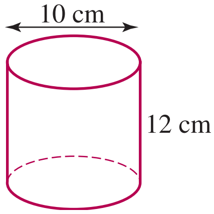
where
- **r is the radius of the base**
- **h is the height of the cylinder.**



Example 1

Find the volume of each solid, correct to the nearest cm^3.

a



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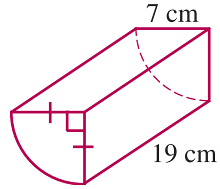


Solutions

a $V = \pi r^2 h$

$= \pi \times 5^2 \times 12$

$\doteqdot 942 \text{ cm}^3$

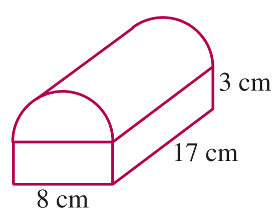
b $V = \pi r^2 h \times \frac{1}{4}$

$= \pi \times 7^2 \times 19 \times \frac{1}{4}$

$\doteqdot 731 \text{ cm}^3$

Example 2

Find the total volume of this solid, correct to 1 decimal place.



Solution

The solid consists of a half-cylinder and a rectangular prism.

i $V_1 = \pi r^2 h \times \frac{1}{2}$

$= \pi \times 4^2 \times 17 \times \frac{1}{2}$

$\doteqdot 427.3 \text{ cm}^3$

ii $V_2 = lbh$

$= 8 \times 17 \times 3$

$= 408 \text{ cm}^3$

iii $V = V_1 + V_2$

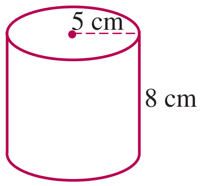
$= 427.3 + 408$

$= 835.3 \text{ cm}^3$

Exercise 3.6

1 Find the volume of each cylinder, correct to the nearest cm^3.

a



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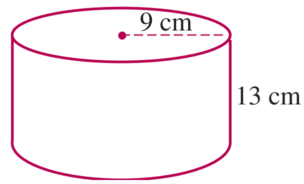

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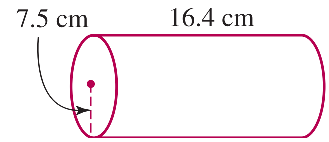

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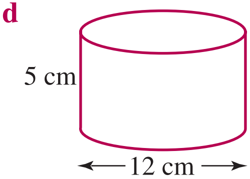

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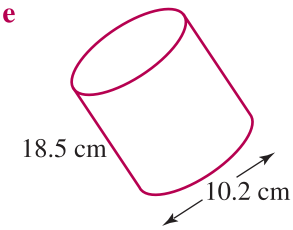

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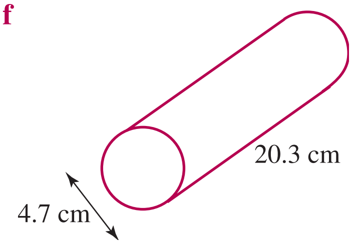


2 Find the volume of a cylinder with the following dimensions. Answer correct to 1 decimal place.

a radius 4 cm and height 10 cm
b radius 12.8 cm and height 15.4 cm
c diameter 32 cm and height 12.5 cm
d diameter 50.8 cm and height 11.9 cm

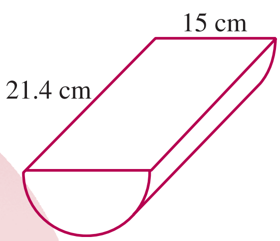
■ Consolidation

3 Find the exact volume of a cylinder with:

a radius 2 mm and height 5 mm
b radius 9 mm and height 4 mm
c diameter 12 mm and height 1.5 mm
d diameter 8 mm and height 2.75 mm

4 Find the volume of each solid, correct to 4 significant figures.

a



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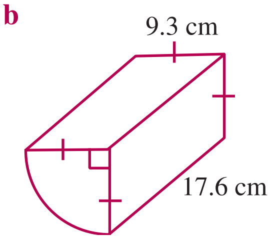

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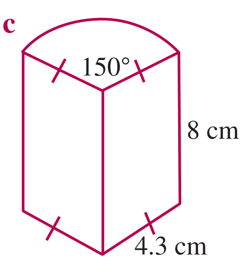

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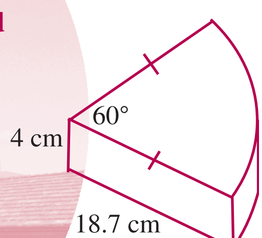

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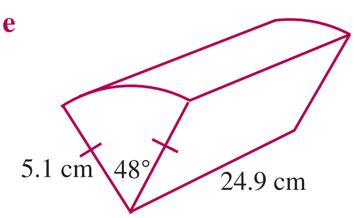

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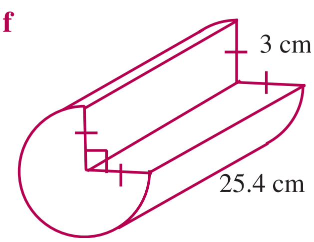
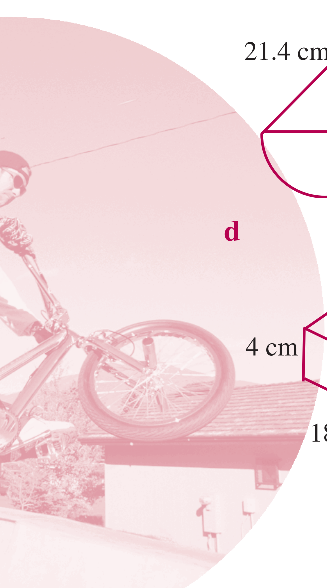




5 Find, correct to the nearest metre, the height of a cylinder with:

a volume of 1846 m^3 and radius of 7 m

b volume of 3817 m^3 and diameter of 18 m

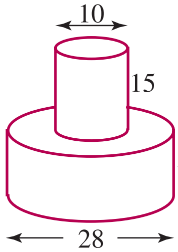
6 Find, correct to the nearest metre, the radius of a cylinder with:

a volume of 8743 m^3 and height of 23 m

b volume of 4825 m^3 and height of 6 m

7 Find the volume of each solid, correct to the nearest cm^3. All lengths are in cm.

a



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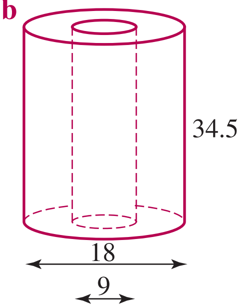

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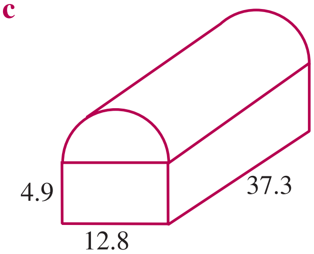

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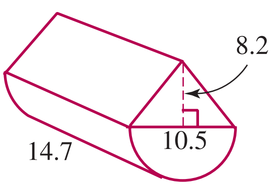

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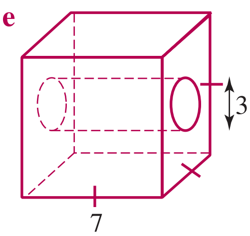

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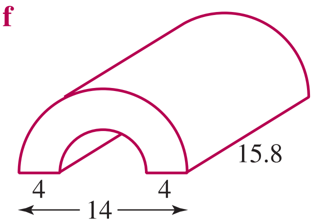
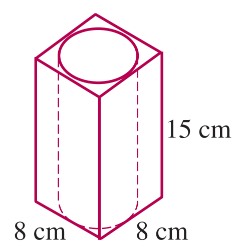


8 A cylindrical can of fruit just fits into a box as shown.

a Find the exact volume of the can.

b Find the volume of the box.

c What percentage of the volume of the box is occupied by the can? Answer correct to 1 decimal place.



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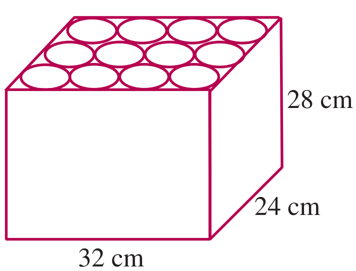


A box contains 24 cans of Cola. There are 2 layers of cans and each layer holds 3 rows of 4 cans. The box is 32 cm long, 24 cm wide and 28 cm high.

a Find the height and radius of the cans.

b Find the total volume of Cola in the box. Answer correct to the nearest cm^3.

c How many litres of Cola are in the box? Answer correct to 1 decimal place.

10 Find the capacity of a saucepan with a diameter of 19 cm and a height of 8 cm. Answers in litres, correct to 1 decimal place. (1000 cm^3 = 1 L)

11 Richard purchased a large cylindrical rainwater tank so that he could water his garden during the water restrictions. The tank has a diameter of 2.8 m and a height of 2.5 m.
- **a** Find the volume of the tank, correct to 3 decimal places.
- **b** Hence, find the capacity of the tank in litres, correct to the nearest litre. (1 m^3 = 1000 L)

Further applications

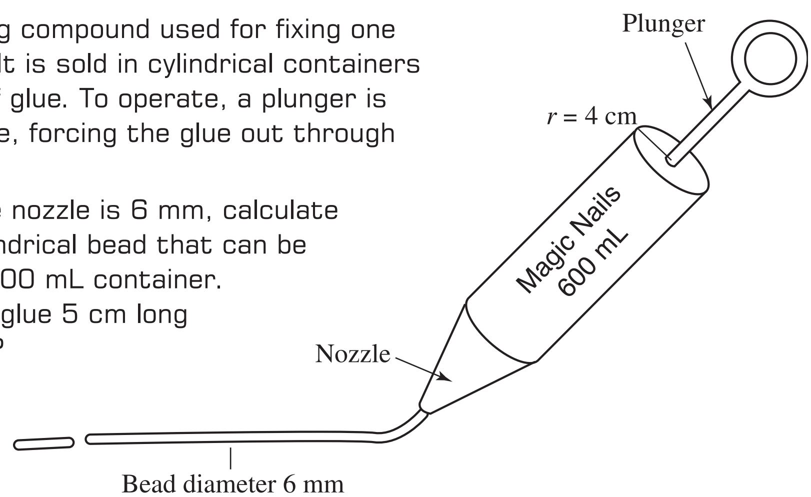
12 Samantha's fish tank is cylindrical in shape. The tank holds 24 L of water when it is filled to a height of 30 cm. Find the radius of the tank, correct to the nearest cm.

13 **a** A cylinder has a volume of 252π cm^3 and a radius of 6 cm. Find the exact surface area of the cylinder.
b A cylinder has a curved surface area of 130π cm^3 and a diameter of 26 cm. Find the exact volume of the cylinder.

TRY THIS Magic Nails

Magic Nails is a gluing compound used for fixing one material to another. It is sold in cylindrical containers containing 600 mL of glue. To operate, a plunger is inserted into the tube, forcing the glue out through a nozzle.

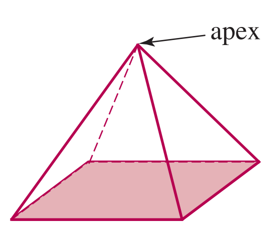
If the diameter of the nozzle is 6 mm, calculate the length of the cylindrical bead that can be 'squeezed-out' of a 600 mL container. How many squirts of glue 5 cm long can be squeezed out? (1 mL = 1 cm^3)



3.7 Volume of a pyramid

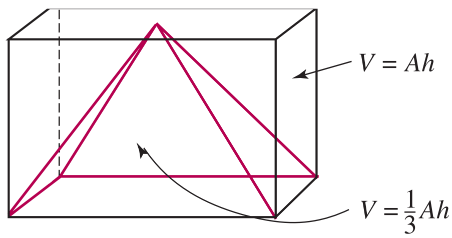
Definition of a pyramid

A **pyramid** is a solid that has cross-sections that are similar. That is, the cross-sections are the same shape, but not the same size. The largest cross-section is called the base. A pyramid has straight edges and all edges not in the base meet at a common point called the **apex**.



The base of a pyramid can be any polygon, however, all the other faces must be triangles. The pyramid is named according to the shape of the base. For example, if the pyramid has a rectangular base, then the solid is referred to as a rectangular pyramid.

Volume of a pyramid

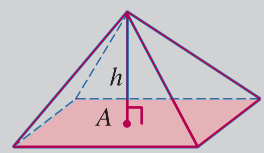


If a pyramid and a prism have equal base areas and equal heights, then the volume of the pyramid will be one-third of the volume of the prism.

The volume of a pyramid is given by:

$$V = \frac{1}{3}Ah$$

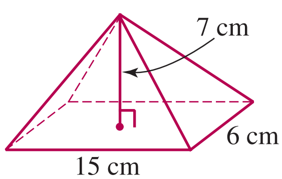
where • ***A* is the area of the base**
• ***h* is the height of the pyramid.**



Example 1

Find the volume of each pyramid.

a



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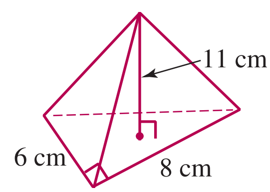


Solutions

a **i** $A = l \times b$
$= 15 \times 6$
$= 90 \text{ cm}^2$

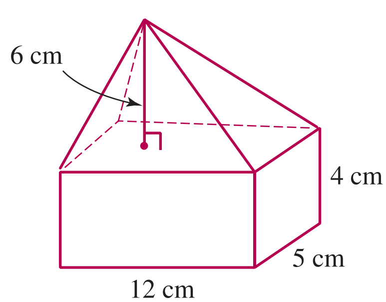
ii $V = \frac{1}{3}Ah$
$= \frac{1}{3} \times 90 \times 7$
$= 210 \text{ cm}^3$

b **i** $A = \frac{1}{2}bh$
$= \frac{1}{2} \times 6 \times 8$
$= 24 \text{ cm}^2$

ii $V = \frac{1}{3}Ah$
$= \frac{1}{3} \times 24 \times 11$
$= 88 \text{ cm}^3$

Example 2

Find the volume of this solid.



Solution

The solid consists of a rectangular pyramid (V_1) and a rectangular prism (V_2).

i $A = 12 \times 5$
$= 60\text{ cm}^2$

ii $V_1 = \frac{1}{3}Ah$
$= \frac{1}{3} \times 60 \times 6$
$= 120\text{ cm}^3$

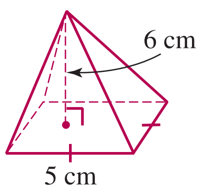
iii $V_2 = l \times b \times h$
$= 12 \times 5 \times 4$
$= 240\text{ cm}^3$

iv $V = V_1 + V_2$
$= 120 + 240$
$= 360\text{ cm}^3$

Exercise 3.7

1 Find the volumes of these pyramids.

a



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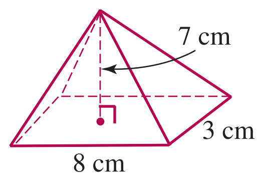

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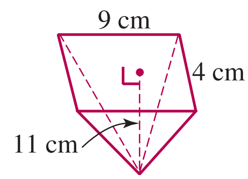

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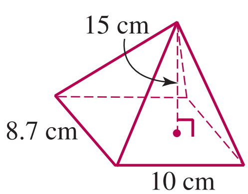

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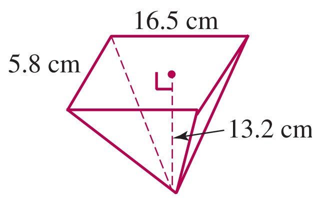

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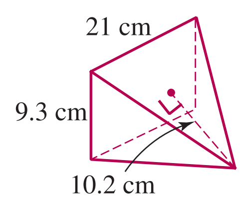
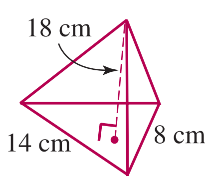


2 Calculate the volume of these triangular pyramids.

a



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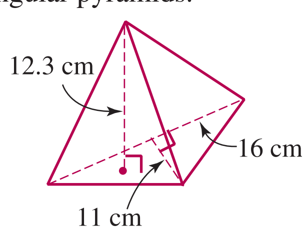

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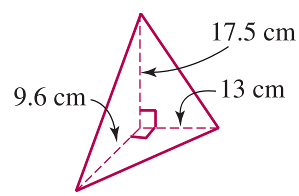
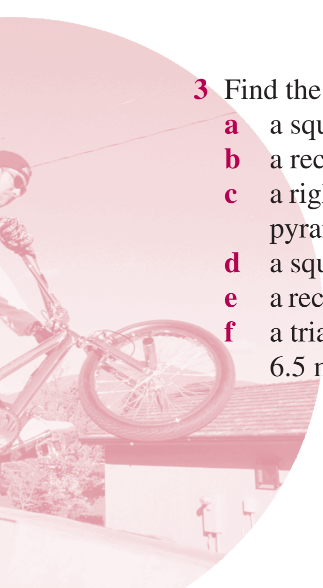


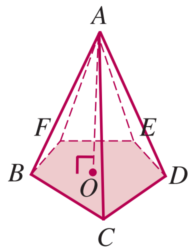
3 Find the volume of a pyramid in which the base is:

- **a** a square of side 3.5 cm and the height of the pyramid is 6 cm
- **b** a rectangle with length 9 m and width 5 m and the height of the pyramid is 7 m
- **c** a right-angled triangle with perpendicular sides 12 mm and 8 mm and the height of the pyramid is 11 mm
- **d** a square of side 7.2 cm and the height of the pyramid is 12.25 cm
- **e** a rectangle with length 19.6 m and width 11.4 m and the height of the pyramid is 10.5 m
- **f** a triangle with one edge 13 mm and altitude 4.8 mm and the height of the pyramid is 6.5 mm



Consolidation

4

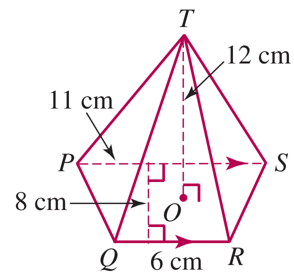


The solid shown is a pentagonal pyramid. The pentagonal base has an area of 112.8 cm^2 and $AO = 9$ cm. Find the volume of the pyramid.

5 The solid shown is a trapezoidal pyramid.

a Find the area of the base $PQRS$.

b Hence, find the volume of the pyramid.



6 **a** A square pyramid has a volume of 32 cm^3 and a base of side 4 cm. Find the height of the pyramid.

b A square pyramid has a volume of 30 cm^3 and a height of 3.6 cm. Find the side length of the base.

c A rectangular pyramid has a volume of 140 cm^3. The base has dimensions 10.5 cm and 8 cm. Find the height of the pyramid.

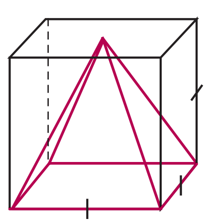
d A rectangular pyramid has a volume of 308 cm^3. The height of the pyramid is 11 cm and the width of the base is 7 cm. Find the length of the base.

7 A square pyramid made of iron with base edges of 8.4 cm and height 11.5 cm is melted down and recast in the shape of a cube.

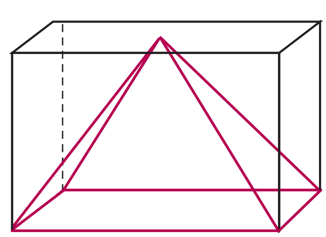
a Find the volume of the pyramid.

b Find the side length of the cube, correct to 1 decimal place.

8 **a** The cube has a volume of 60 cm^3. What is the volume of the pyramid?



b The rectangular pyramid has a volume of 12 cm^3. What is the volume of the prism?

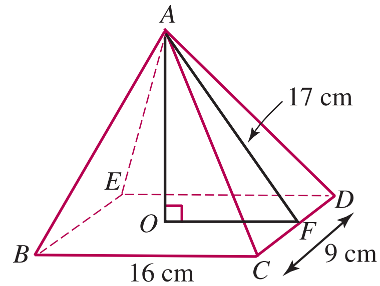


9 In the rectangular pyramid shown, $BC = 16$ cm, $CD = 9$ cm and $AF = 17$ cm.

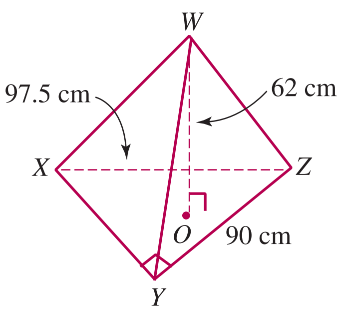
a Write down the length of OF.

b Find AO, the height of the pyramid.

c Hence, find the volume of the pyramid.



10



In the triangular pyramid shown, $XY \perp YZ$, $XZ = 97.5$ cm, $YZ = 90$ cm and $WO = 62$ cm.

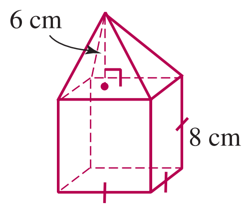
a Find the length of XY.

b Find the area of the base, ΔXYZ.

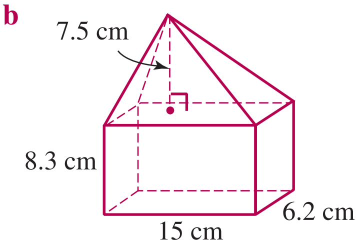
c Hence, find the volume of the pyramid.

11 Find the volume of these solids.

a



b



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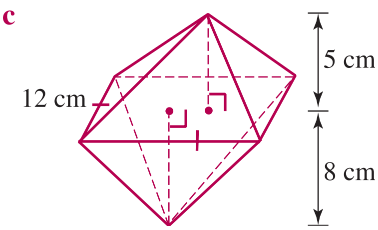

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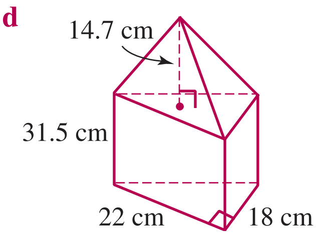
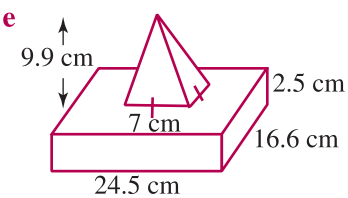


e



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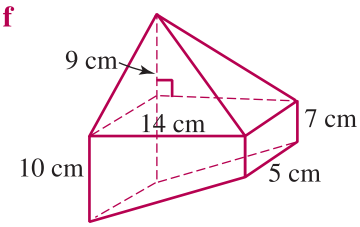
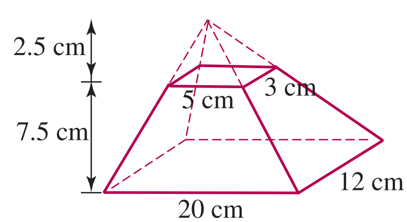


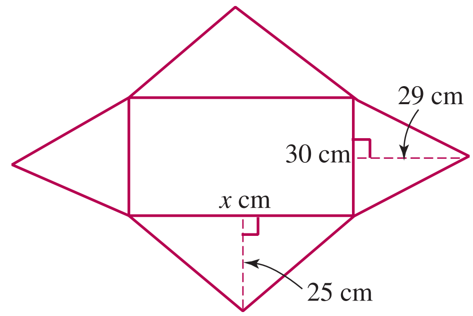
12 A rectangular pyramid has its top removed as shown. Find the volume of the remaining solid.



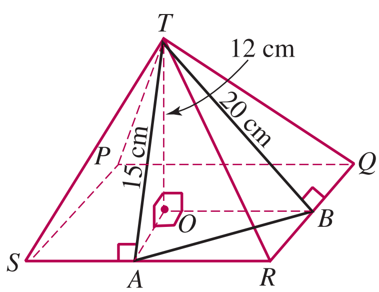
Further applications

13 The net shown is that of a rectangular pyramid.

- **a** Find the height of the pyramid.
- **b** Find the value of x.
- **c** Hence, calculate the volume of the pyramid.



14



In the diagram shown, $PQRST$ is a rectangular pyramid, $TO = 12$ cm, $TA = 15$ cm and $TB = 20$ cm.

- **a** Find the lengths OA and OB.
- **b** Find the volume of the rectangular pyramid $PQRST$.
- **c** Find the volume of the triangular prism $AOBT$.
- **d** What fraction of the rectangular pyramid $PQRST$ is occupied by the triangular pyramid $AOBT$?

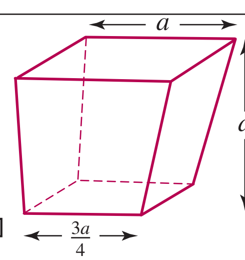
TRY THIS Popcorn

Popcorn is sold in a tapering cardboard container with a square cross section and dimensions as shown in the diagram.

1 If all dimensions were halved, by what fraction would the volume be reduced?

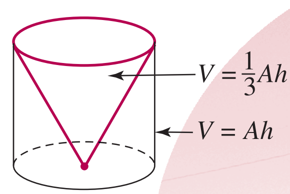
2 Find a formula for the volume, V, of the container.

[HINT: The volume of a pyramid is given by the formula $V = \frac{Ah}{3}$]



3.8 Volume of a cone

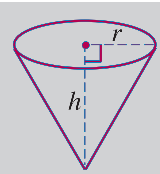
If a cylinder and a cone have equal base areas and equal heights, then the volume of the cone will be one-third of the volume of the cylinder.

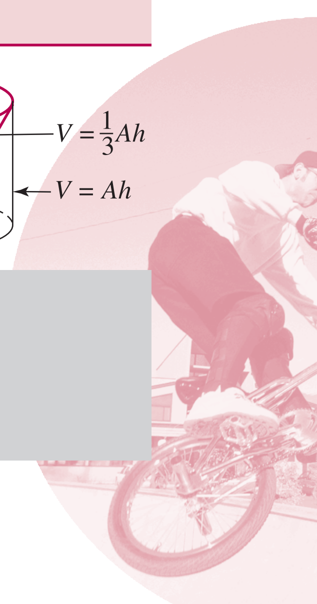


The volume of a cone is given by:

$$V = \frac{1}{3}\pi r^2 h$$

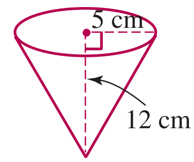
where
- ***r* is the radius of the base**
- ***h* is the height of the cone.**





Example 1

Find the volume of this cone, correct to 1 decimal place.



Solution

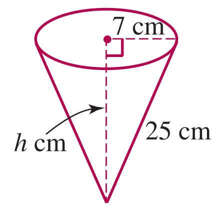
$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \pi \times 5^2 \times 12$$
$$= 314.2 \text{ cm}^3 \text{ (to 1 decimal place)}$$

Example 2

A cone has a radius of 7 cm and a slant height of 25 cm.

a Find the perpendicular height, h cm.

b Hence, find the volume of the cone, correct to 1 decimal place.



Solutions

a By Pythagoras' theorem,

$$h^2 + 7^2 = 25^2$$
$$h^2 + 49 = 625$$
$$h^2 = 576$$
$$h = \sqrt{576}$$
$$= 24$$

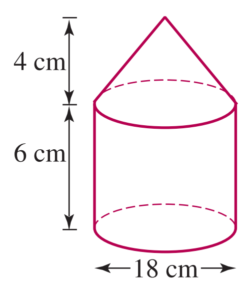
$\therefore$ The perpendicular height is 24 cm.

b $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times \pi \times 7^2 \times 24$$
$$= 1231.5 \text{ cm}^3 \text{ (to 1 decimal place)}$$

Example 3

Find the exact volume of this solid.



Solution

The solid consists of a cone (V_1) and a cylinder (V_2).

i $V_1 = \frac{1}{3}\pi r^2 h$

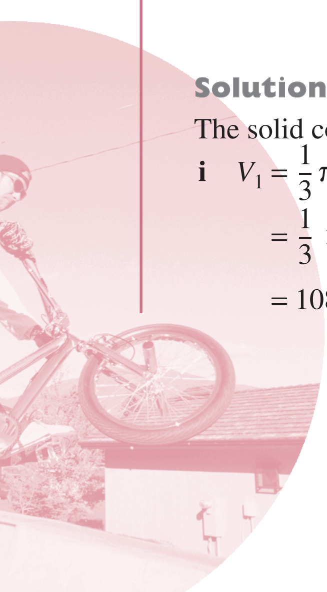
$$= \frac{1}{3} \times \pi \times 9^2 \times 4$$
$$= 108\pi \text{ cm}^3$$

ii $V_2 = \pi r^2 h$

$$= \pi \times 9^2 \times 6$$
$$= 486\pi \text{ cm}^3$$

iii $V = V_1 + V_2$

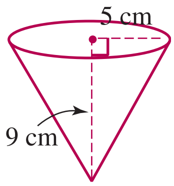
$$= 108\pi + 486\pi$$
$$= 594\pi \text{ cm}^3$$



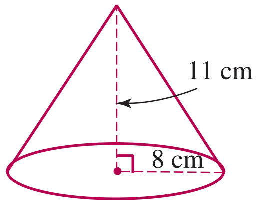
Exercise 3.8

1 Find the volume of each cone, correct to 1 decimal place.

a



b



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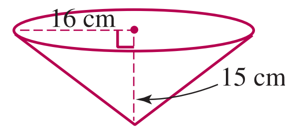

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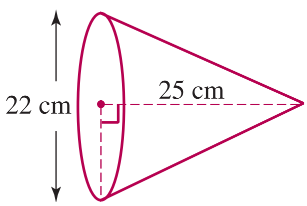
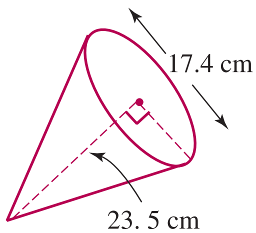


e



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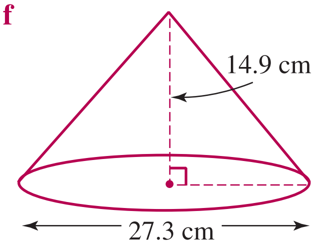


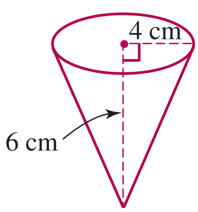
2 Find, correct to the nearest cm^3, the volume of a cone with the following dimensions.

a radius 5 cm and height 7.1 cm
b radius 6.5 cm and height 11.4 cm
c diameter 12 cm and height 9.3 cm
d diameter 11.9 cm and height 18.4 cm

Consolidation

3 Find the exact volume of each cone.

a



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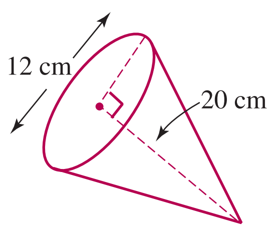

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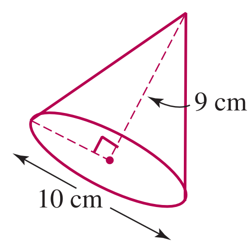
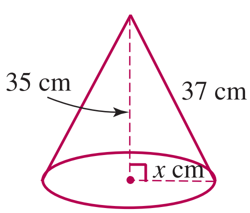


4 In each of the following:

i find the value of x
ii calculate the volume of the cone, correct to 1 decimal place

a



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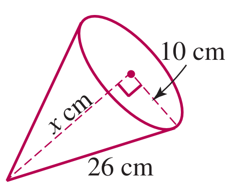

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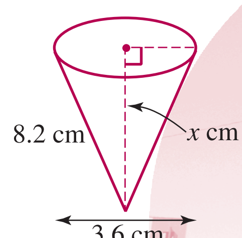
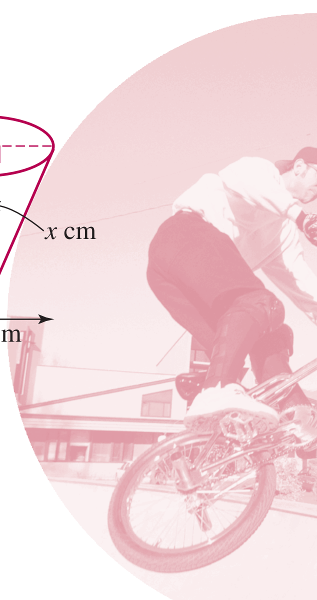
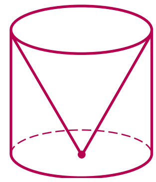




5 A cone has been inscribed in a cylinder as shown.

a If the volume of the cylinder is 30 mm^3, what is the volume of the cone?

b If the volume of the cone is 25 mm^3, what is the volume of the cylinder?



6 **a** A cone has a volume of 564.4 mm^3. Find, correct to the nearest mm, the height of the cone if the area of the base is 154 mm^2.

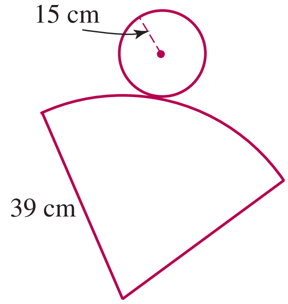
b A cone has a volume of 615.8 mm^3. Find, correct to the nearest mm, the area of the base if the cone has a height of 12 mm.

7 For each of the following nets:

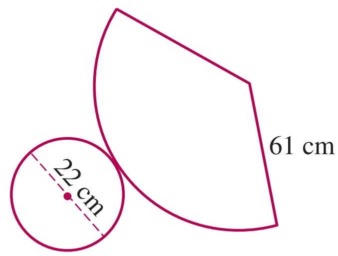
i sketch the cone and calculate its perpendicular height

ii find the volume of the cone, correct to the nearest cm^3

a



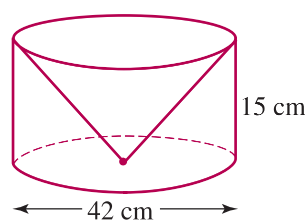
b



8 A wooden cylinder has a cone cut out as shown.

a What fraction of the volume of the cylinder remains?

b Calculate the remaining volume, correct to the nearest cm^3.



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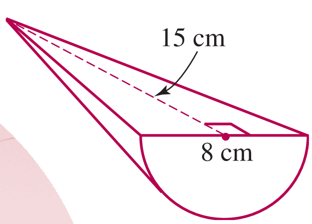
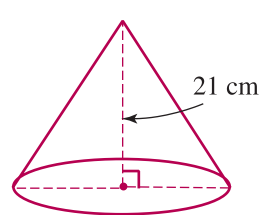


The solid shown is a half-cone.
Find its volume, correct to 1 decimal place.

10 The base of this cone has a circumference of 26π cm.
The perpendicular height of the cone is 21 cm.

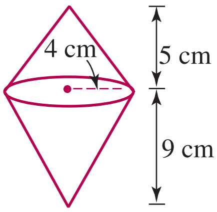
a Find the diameter of the base.

b Calculate the volume, correct to 1 decimal place.

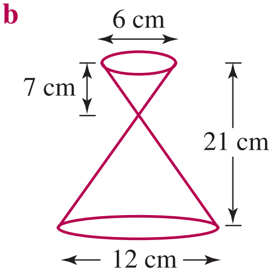


11 Calculate the volume of each solid, correct to the nearest cm^3.

a



b



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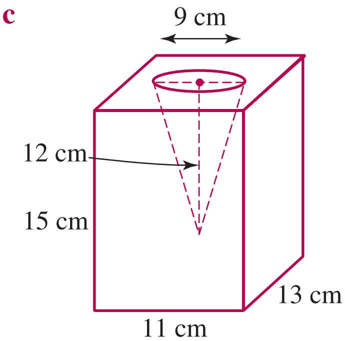

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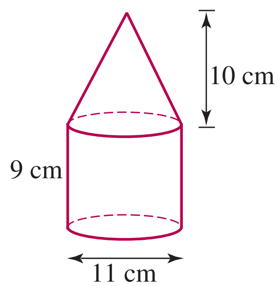

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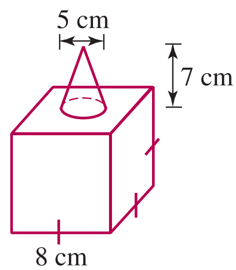

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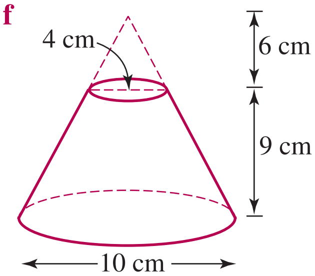


12 **a** Find the height of a cone with the following measurements.
- **i** volume 21π cm^3, radius 3 cm
- **ii** volume 192π cm^3, diameter 16 cm

b Find the radius of a cone that has a volume of 750π cm^3 and height of 10 cm.

13 Find, in litres correct to 1 decimal place, the capacity of a cone with the following dimensions. (1000 cm^3 = 1 L)

a radius 15 cm and height 22 cm **b** diameter 21 cm and height 14 cm

Further applications

14 A cone with radius 9 cm has a volume of 2.4 L. Find the height of the cone, correct to 3 significant figures.

15 Cones A and B have equal volumes. Cone A has a radius of 18 cm and a height of 13 cm, while cone B has a radius of 15 cm. Find the height of cone B.

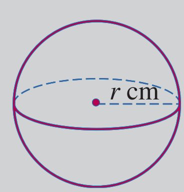
3.9 Volume of a sphere

The proof for the volume of a sphere is beyond the scope of this course.

The volume of a sphere is given by:

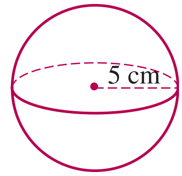
$$V = \frac{4}{3}\pi r^3$$

where • r **is the radius of the sphere.**



Example 1

Find the volume of this sphere, correct to 1 decimal place.

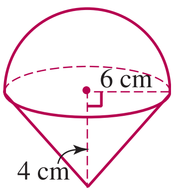


Solution

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times \pi \times 5^3$$
$$= 523.6 \text{ cm}^3 \text{ (to 1 decimal place)}$$

Example 2

Find the exact volume of this solid.



Solution

The solid consists of a hemisphere (V_1) and a cone (V_2).

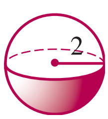
i $V_1 = \frac{4}{3}\pi r^3 \times \frac{1}{2}$
$= \frac{4}{3} \times \pi \times 6^3 \times \frac{1}{2}$
$= 144\pi \text{ cm}^3$

ii $V_2 = \frac{1}{3}\pi r^2 h$
$= \frac{1}{3} \times \pi \times 6^2 \times 4$
$= 48\pi \text{ cm}^3$

iii $V = V_1 + V_2$
$= 144\pi + 48\pi$
$= 192\pi \text{ cm}^3$

Exercise 3.9

1 Find the volume of each sphere, correct to the nearest cm^3. All lengths are in cm.

a


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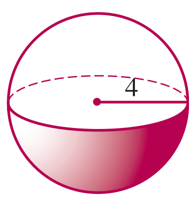
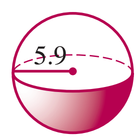


c


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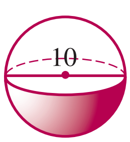
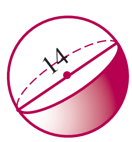


e


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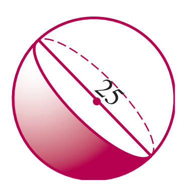


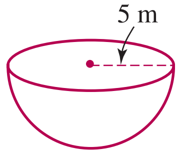
2 Find, correct to 1 decimal place, the volume of a sphere with:
a radius 6.3 mm **b** radius 1.7 mm **c** diameter 4 mm **d** diameter 9.5 mm

Consolidation

3 Find the exact volume of a sphere with:
a radius 3 cm **b** radius 9 cm **c** radius 21 cm
d diameter 24 cm **e** diameter 30 cm **f** diameter 36 cm

4 Find the volume of each of these hemispheres, correct to the nearest m^3.

a



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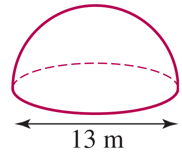

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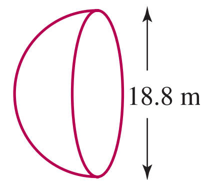
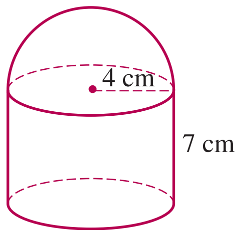


5 Calculate the volume of each solid, correct to the nearest cm^3.

a



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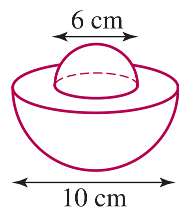

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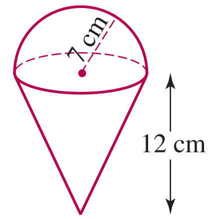

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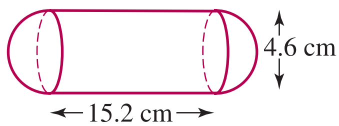

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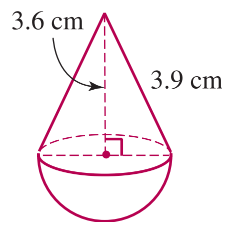

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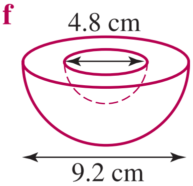


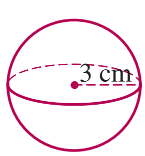
6 Find the weight of a cast-iron ball with radius 2.5 cm, if 1 cm^3 of cast iron weighs 8 grams. Answer correct to the nearest gram.

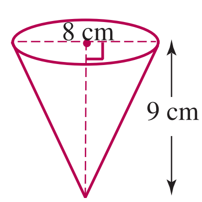
7 A sphere of radius 9 cm fits tightly inside a cylinder as shown.

a Find the exact volume of the sphere.
b Find the exact volume of the cylinder.
c Hence, calculate the empty space that remains in the cylinder, correct to 1 decimal place.

8 A cubic box has a volume of 32 800 cm^3.
a Find the side length of the cube, correct to the nearest cm.
b Hence, find the volume of the largest ball that can fit inside the box. Answer correct to the nearest cm^3.

9 Find in simplest form, the ratio of the volume of the sphere to the volume of the cone.





10 Find the radius of a sphere whose volume is:

a 36π cm^3 **b** 288π cm^3 **c** 4500π cm^3

11 Find, correct to 1 decimal place, the radius of a sphere whose volume is 1000 cm^3.

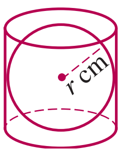
Further applications

12 A glass test tube has a cylindrical section of diameter 1.6 cm and a hemispherical base. The test tube is filled with a chemical solution to a depth of 5.8 cm.

a Find the volume of the test tube in cm^3, correct to the nearest cm^3.

b Find the volume of the solution, in mL.

13



A sphere of radius r cm fits tightly into a cylinder.

a Find an expression for the height of the cylinder in terms of r.

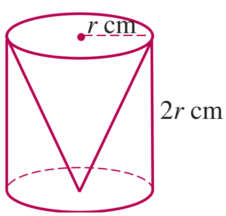
b Show that the volume of the cylinder is $2\pi r^3$ cm^3.

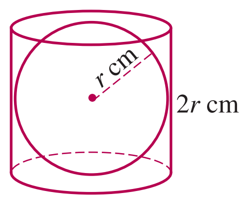
c Write down the volume of the sphere in terms of r.

d Find, in simplest form, the ratio of the volume of the sphere to the volume of the cylinder.

e If the cylinder had a volume of 24π cm^3, what would be the volume of the sphere?

TRY THIS Cones, spheres and cylinders





A sphere and cone fit tightly into two identical cylinders. Show that when the volume of the cone is added to the volume of the sphere, the result is equal to the volume of the cylinder.

THE SURFACE AREA OF A SOCCER BALL

Introduction

Measuring the area of a curved surface of a sphere is an intriguing mathematical problem. Is it related to the area of a circle? Is there an exact formula? Archimedes (287–212 BC), one of the greatest mathematicians of antiquity, answered yes to both questions with brilliant mathematical reasoning. It was just a small part of his geometrical discoveries about the volumes and surface areas of spheres, cones and cylinders. Interestingly they were discovered from the study of mechanics, and he used original ideas which were developed into the concepts of calculus centuries later by Newton and Leibnitz.

In this activity we will measure the area of a soccer ball, assumed to be a sphere, to test a hypothesis about the formula. The Challenge activity will enable you to explore how Archimedes solved it, and his most famous theorem.

LEARNING ACTIVITY 1

Practical measurement

Plan for a double period. The first activity may take a single 50 min period and is best carried out in groups. Each group will need a soccer ball, suitable grid paper, scissors, calculator, tape and callipers. If soccer balls are not available and time is limited go straight to question 4.

1 Examine the shapes which cover the surface. Does it surprise you that the surface can be covered in this way? Is this a *tessellation*?

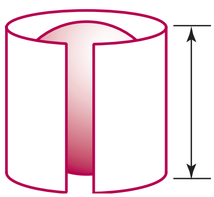
2 Decide how you will count and measure the area of each basic shape to get an estimate of the area of the ball. Decide on a unit of measurement, say square centimetres.

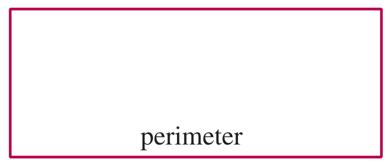
3 Draw up a table to show your results. Change the table if you have different shapes. Record the number of each shape on the surface of the ball and multiply to get the total area.

Shape	**Area**	**Number**	**Total area (cm^2)**
Pentagon			
Hexagon			

Surface area of ball = $\qquad$ cm^2

4 A quicker alternative is to wrap a rectangular piece of fairly stiff paper around a suitable ball so that it just fits. The width of the paper is cut so that it is equal to the diameter of the ball. Unwrap the paper and calculate the rectangular area. This will approximate the surface area of the ball.





Learning activity 2

Reasoning and hypothesising

The second activity is to make a hypothesis about the formula and to test it against the calculated value of the surface area of the ball.

1 Firstly, consider the cross sectional area of the ball through the centre. What shape do you get? Does it seem reasonable that the surface area is some how connected to the area of this circle? Why?

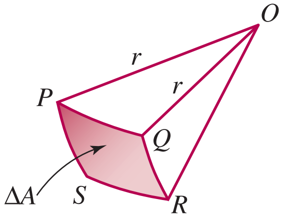
2 Calculate the area of the great circle whose centre is the centre of the sphere. Note that it is easier to measure the diameter of the ball than the radius.

3 Compare the area of the great circle with the calculated surface area of the ball. How many times greater is it, roughly?

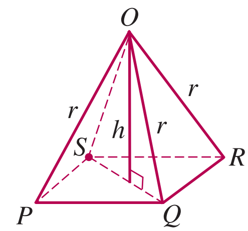
4 If your measurements were accurate you should have found that the surface area is close to 4 times the area of the circular cross section whose centre is the centre of the sphere. Make a hypothesis about the formula for the surface area of a sphere in terms of its radius r. Record this in your book and draw a diagram to illustrate it.

CHALLENGE ACTIVITIES

1 The big question now is how did Archimedes prove that the surface area of a sphere of radius r is exactly $4\pi r^2$? It came as a consequence of his discovery of the *volume* of a sphere. Copy the figure shown into your book and write down the formula for the volume V of the whole sphere.



2 The idea is to imagine the volume of the sphere as made up of small polyhedra with base area ΔA. A typical one is shown in the diagram. The volume of this 'element of volume' is approximately $\frac{1}{3}$ (area of the base *x* perpendicular height). As the subdivision is made smaller and smaller, the perpendicular height h gets closer to the radius r, and the sum of all the elements approaches the volume of the sphere. Furthermore the sum of the base areas is the surface area A of the sphere.

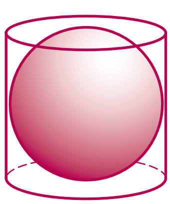


Using the symbol $\sum$ to mean the sum of the elements of volume, in the limit, as the subdivision is made finer and finer, h approaches r, and $\sum(\frac{1}{3} \times \Delta A \times h)$ gets closer and closer to the volume of the sphere $\frac{4}{3}\pi r^3$.

That is $\quad \frac{1}{3} \times A \times r = \frac{4}{3}\pi r^3,$

and so $\quad A = 4\pi r^2.$

3 Now copy this figure into your book. The cylinder is solid and the base and top are tangential to the sphere.



4 Using known formulae, prove that the volume of a sphere is $\frac{2}{3}$ the volume the cylinder which encloses it. Prove that surface area of a sphere is also $\frac{2}{3}$ of the cylinder which encloses it. That the ratio of volumes and the ratio of surface areas were identical was considered by Archimedes as his most profound theorem.

LET'S COMMUNICATE

Either

a Search the internet to find out more about Archimedes, the circumstances of his death, where he was buried, and his wish to have the theorem of question 4 recorded on his tomb. Who was the Roman poet who found Archimedes' grave after it had fallen into disrepair? Tell the story in your own words.

or

b Record at least three different scientific discoveries or inventions that Archimedes made which came from applying mathematical ideas to human needs in the Greco-Roman world of his day.

REFLECTING

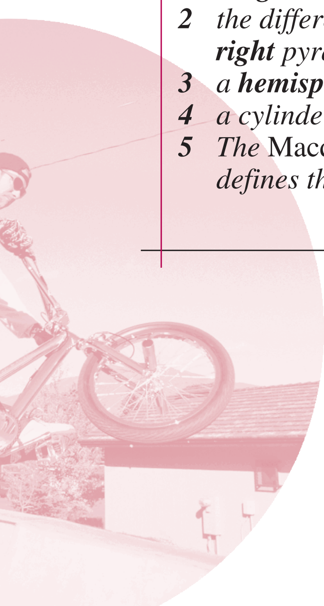
The volume and surface area of a sphere, cylinder and cone is hardly new knowledge. They have been known for about 2000 years. However the results were not formally proved until centuries later with the invention of *calculus* by Isaac Newton. Even then, Newton developed ideas originally used by Archimedes. What does this suggest about the importance of history in learning mathematics?

In a short sentence explain:

1. *a **right** cone*
2. *the difference between an **oblique** and **right** pyramid*
3. *a **hemisphere***
4. *a cylinder **circumscribed** about a sphere*
5. *The* Macquarie Learners Dictionary *defines the word **composite** as:*

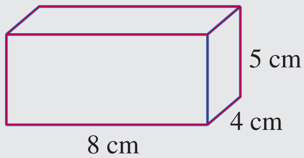
composite *noun* **1.** something made up of different parts
–*adjective* **2.** made up of various parts: *a composite picture*

*Give a mathematical example of a **composite** shape and a **composite** solid.*



1 Find the surface area of each prism.

a



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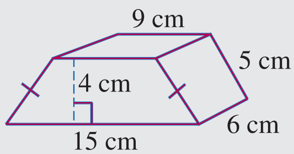
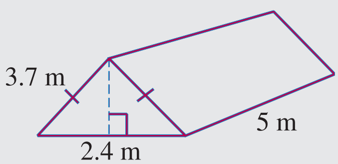


2 Find the side length of a cube whose surface area is 1176 cm^2.

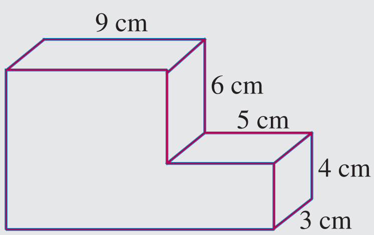
3



a Find the height of the triangular face.

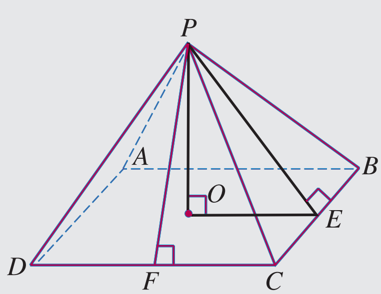
b Find the surface area of the prism.

4



Find the surface area of this prism.

5



PABCD is a rectangular pyramid. $PO = 11$ cm, $DC = 20$ cm and $BC = 14$ cm.

a Find the slant heights *PE* and *PF* as surds.

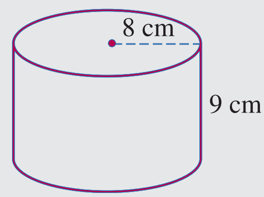
b Hence, find the surface area of the pyramid.

6 Find the surface area of a square pyramid with a perpendicular height of 14 cm and base edges of 19 cm. Answer correct to 1 decimal place.

7 Find, correct to the nearest cm, the curved surface area of a cylinder with diameter 22 cm and height 17 cm.

8 The curved surface area of a cylinder is 42π cm^2. Find the height if the radius is 3 cm.

9 Find the surface area of this cylinder, correct to 1 decimal place.



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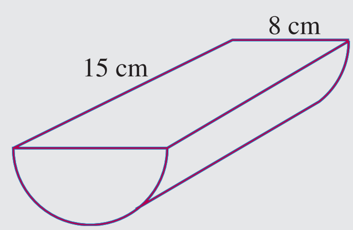
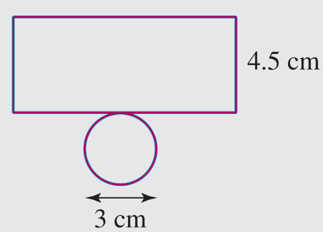


Find the surface area of this half-cylinder, correct to 1 decimal place.

11



The net shown is that of a cylinder, open at the top. Find the surface area of the cylinder, correct to 1 decimal place.

12 A cone has a diameter of 10 cm and a slant height of 12 cm.

a Find the exact curved surface area.

b Find the surface area of the cone, correct to 1 decimal place.

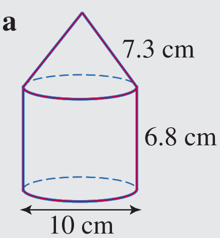
13 Find the surface area of a sphere with radius 4.3 cm. Answer correct to 1 decimal place.

14 The surface area of a sphere is 36π cm^2. Find the radius.

15 A cone has a surface area of 78π cm^2 and a diameter of 12 cm. Find the slant height.

16 Find the surface area of each solid, correct to 1 decimal place.

a



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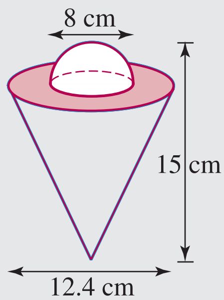
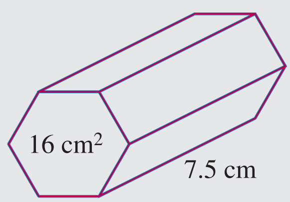


17 Find the volume of this hexagonal prism.



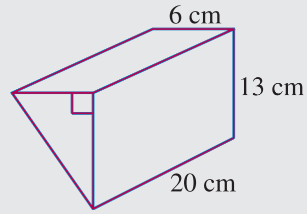
18 Find the volume of:

a a cube of side 7 cm

b a rectangular prism with length 7 cm, breadth 6 cm and height 2.5 cm

19 Find the volume of each prism.

a



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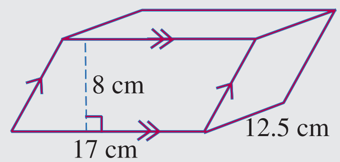

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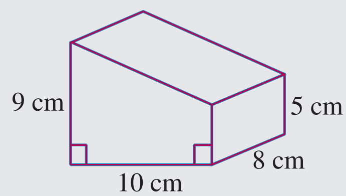
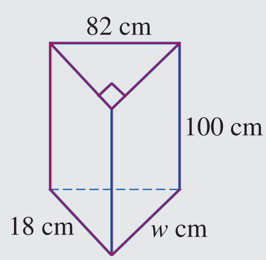


20 The volume of an octagonal prism is 204.6 mm^3. If the prism has a perpendicular height of 12.4 mm, find the cross-sectional area.

21



a Use Pythagoras' theorem to find the value of w.

b Find the volume of the prism.

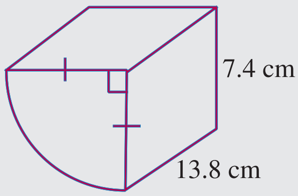
c Find the capacity of the prism, in litres (1 L = 1000 cm^3).

22 Find, correct to 1 decimal place, the volume of a cylinder with diameter 23 cm and height 14.6 cm

CHAPTER REVIEW

23 Find the volume of each solid, correct to the nearest cm^3.

a



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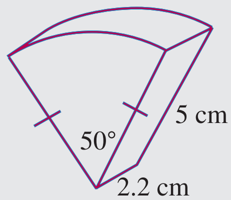
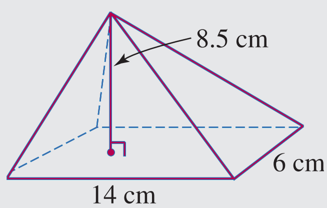


24 Find the volume of each pyramid.

a



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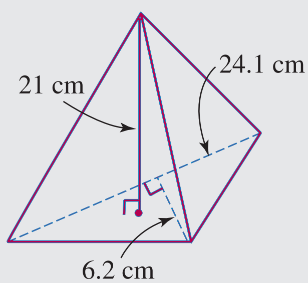
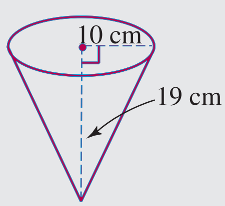


25 Find the volume of each solid, correct to the nearest cm^3.

a



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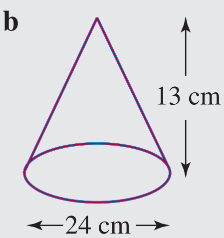

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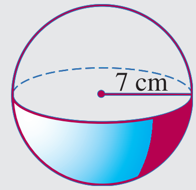

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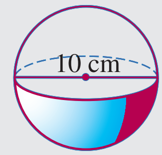


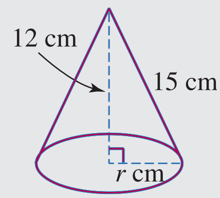
26 Find the exact volume of:

a a cylinder with radius 6 cm and height 4.5 cm

b a cone with diameter 10 cm and height 9 cm

c a sphere with radius 3 cm

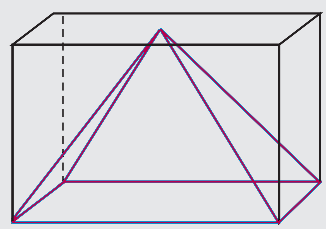
27



a Find the length of the radius.

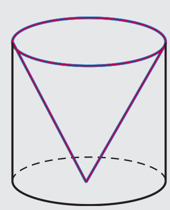
b Find the volume of the cone, correct to 1 decimal place.

28 a



The prism has a volume of 18 cm^3. What is the volume of the pyramid?

b



The cone has a volume of 7π cm^3. What is the volume of the cylinder?

29 a A square pyramid has a height of 6 cm and a volume of 50 cm^3. Find the side length of the base.

b A rectangular pyramid has a height of 11 cm and a volume of 352 cm^3. If the length of the base is 12 cm, find the breadth.

30 **a** A cylinder has a volume of 120π cm^3 and a diameter of 10 cm. Find the height.

b A cone has a volume of 242π cm^3 and a height of 6 cm. Find the radius.

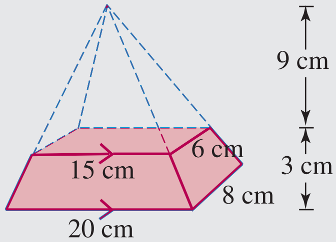
c A sphere has a volume of 2304π cm^3. Find the radius.

31 **a** The volume of a cube is 64 cm^3. Find the surface area.

b The surface area of a cube is 1350 cm^2. Find the volume.

32 Find the volume of each solid. Answer correct to the nearest cm^3 where necessary.

a



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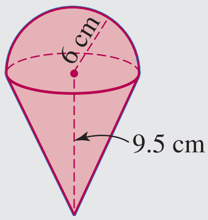

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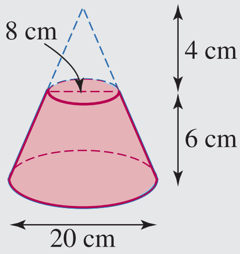

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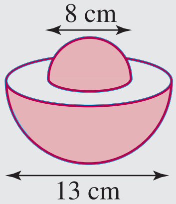


CHAPTER REVIEW

4 Deductive geometry

This chapter at a glance

Stage 5.1/5.2/5.3

After completing this chapter, you should be able to:

- apply the properties of complementary, supplementary and vertically opposite angles and angles at a point to find unknown angles, giving reasons
- find unknown angles on parallel lines, giving reasons
- use the angle sum of a triangle to find unknown angles, giving reasons
- use the exterior angle property of triangles to find unknown angles, giving reasons
- use the angle sum of a quadrilateral to find unknown sides and angles, giving reasons
- use the properties of the special quadrilaterals to find unknown sides and angles, giving reasons
- name a polygon according to the number of sides
- classify a polygon as either convex or non-convex, regular or irregular
- find the interior angle sum of a polygon
- find the size of the interior and exterior angles of a regular polygon
- use geometric properties to find unknown angles in diagrams involving more than one step
- prove unfamiliar geometric results by using formal reasoning
- apply the congruence tests to justify that two triangles are congruent
- apply the congruence tests to establish properties of triangles and quadrilaterals
- write formal deductive proofs involving the tests for congruent triangles
- use Pythagoras' theorem to prove unfamiliar results
- use the converse of Pythagoras' theorem to prove that a triangle is right-angled

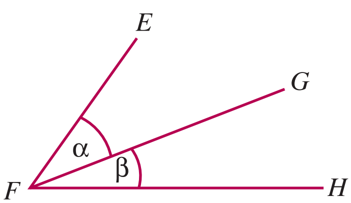
4.1 Simple numerical exercises

The questions in this exercise will review the basic geometrical properties of angles, triangles, quadrilaterals and general polygons that have been covered in previous years. The emphasis in this exercise is on the giving of correct reasons to justify each answer. Formal proofs of results and definitions of the special quadrilaterals are left to later exercises.

Adjacent angles

Two angles are adjacent if they:

- **have a common vertex, and**
- **have a common ray, and**
- **lie on opposite sides of this common ray.**



For example, $\angle EFG$ is adjacent to $\angle GFH$ because:

1 F is a common vertex, and

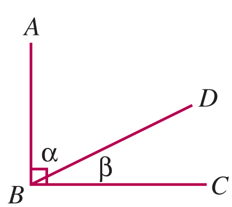
2 FG is a common ray, and

3 the angles lie on opposite sides of FG.

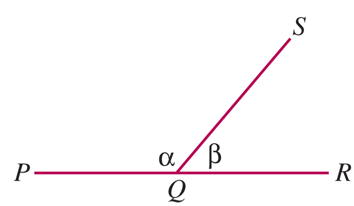
Complementary and supplementary angles

Complementary angles have a sum of 90°.
Supplementary angles have a sum of 180°.

For example:



$\angle ABD$ and $\angle DBC$ are complementary angles: $\alpha + \beta = 90°$



$\angle PQS$ and $\angle SQR$ are supplementary angles: $\alpha + \beta = 180°$

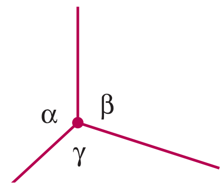
NOTE: When giving reasons, the terms complementary angles and supplementary angles must not be used. These terms simply mean that the angles have a sum of 90° or 180°. In geometric reasoning, you need to explain *why* the angles have that sum (for example angles in a right angle, or angles on a straight line).

Angles at a point

Angles at a point are two or more angles that have a common vertex and whose sum is 360°, or one complete revolution.

Angles at a point have a sum of 360°.

For example:



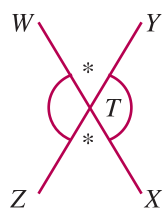
$\alpha + \beta + \gamma = 360°$.

Vertically opposite angles

Vertically opposite angles are formed by the intersection of two straight lines. Vertically opposite angles must be equal because they are adjacent and supplementary to a common angle.

Vertically opposite angles are equal.

For example:

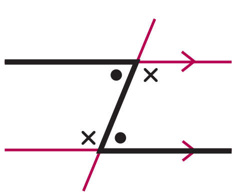


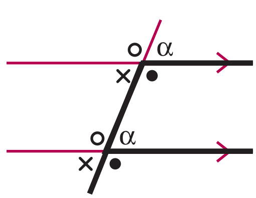
$\angle WTZ$ and $\angle YTX$ are vertically opposite angles.
$\angle WTY$ and $\angle ZTX$ are vertically opposite angles.

Parallel lines

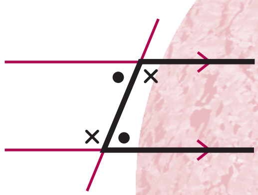
Parallel lines are two or more lines that have been drawn in the same plane and never meet. The notation || means 'is parallel to'. A line that cuts two or more parallel lines is called a transversal.

When a pair of parallel lines is cut by a transversal, 8 angles are formed. These angles can be classified into 3 special pairs of angles: alternate angles, corresponding angles and co-interior angles.


alternate angles

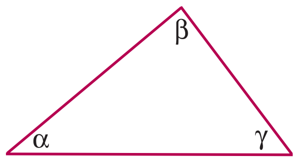


corresponding angles


co-interior angles

The angle sum of a triangle

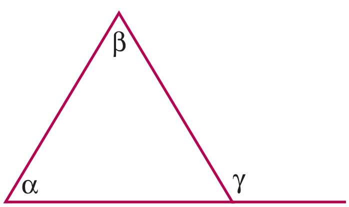
The angle sum of a triangle is 180°.



That is, $\alpha + \beta + \gamma = 180°$.

The exterior angle of a triangle

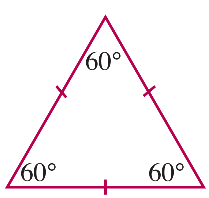
The exterior angle of a triangle is equal to the sum of the two interior opposite angles.

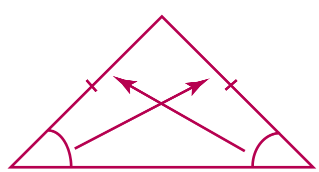


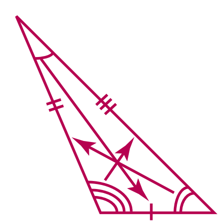
That is, $\gamma = \alpha + \beta$.

Some other properties of triangles

- In an equilateral triangle, all angles are 60°.
- In an isosceles triangle, the equal angles are opposite the equal sides.
- In any triangle, the longest side is opposite the largest angle and the shortest side is opposite the smallest angle.

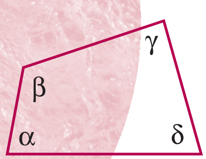






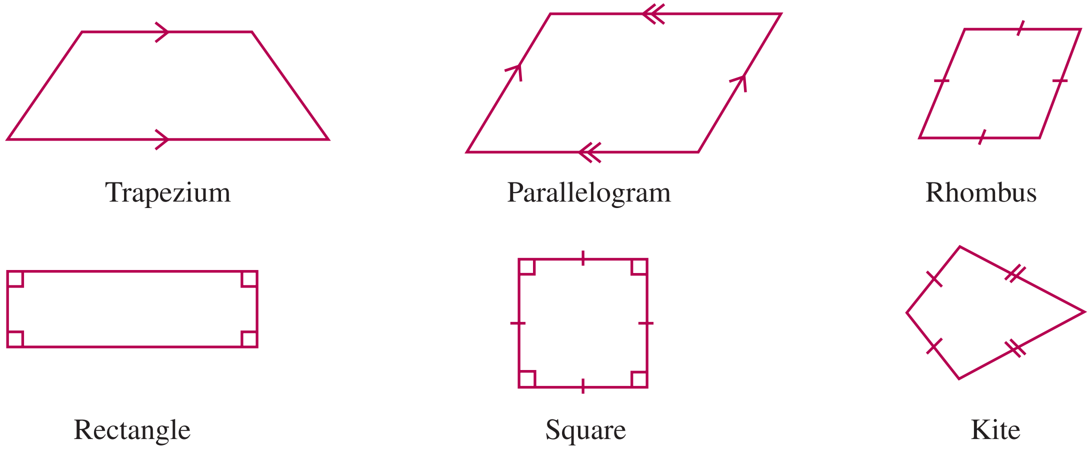
The angle sum of a quadrilateral

The angle sum of a quadrilateral is 360°.



That is, $\alpha + \beta + \gamma + \delta = 360°$.

Properties of the special quadrilaterals



Properties of a parallelogram:
- opposite sides are parallel
- opposite sides are equal
- opposite angles are equal
- diagonals bisect each other

Properties of a rhombus
- all properties of a parallelogram
- all sides are equal
- diagonals are perpendicular
- diagonals bisect the angles at the vertices

Properties of a rectangle:
- all properties of a parallelogram
- all angles are right angles
- diagonals are equal

Properties of a square:
- all properties of a rectangle
- all sides are equal
- diagonals bisect the angles at the vertices
- diagonals are perpendicular

Properties of a trapezium:
- one pair of opposite sides are parallel

Properties of a kite:
- two pairs of adjacent sides are equal

Exercise 4.1

1 Find the value of each pronumeral, giving reasons.

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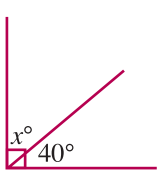

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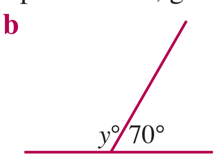

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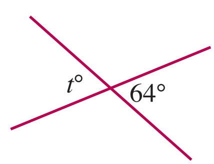

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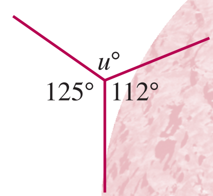

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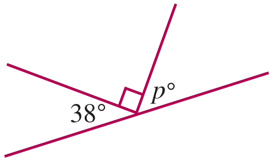

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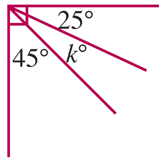

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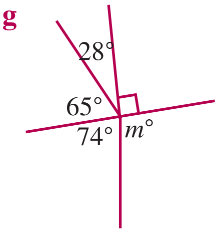

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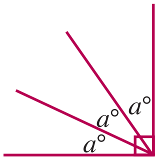

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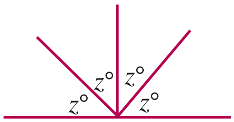

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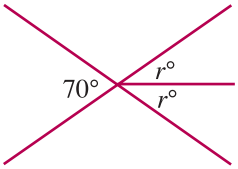

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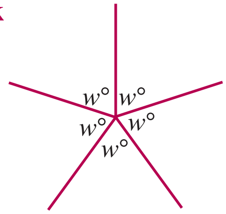

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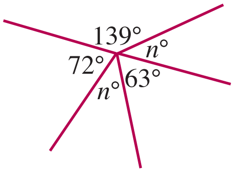

2 Find the value of each pronumeral, giving reasons.

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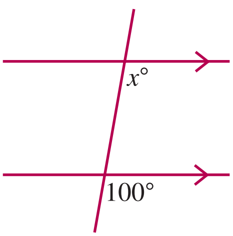

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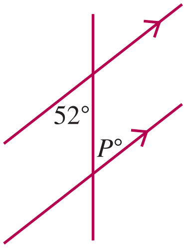

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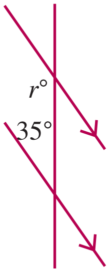

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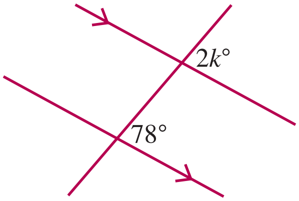

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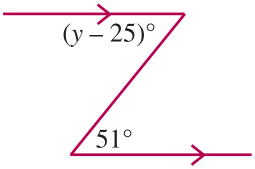

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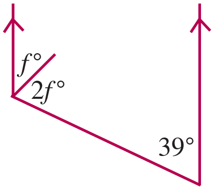
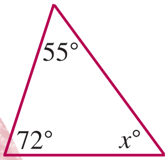


3 Find the value of each pronumeral, giving reasons.

a



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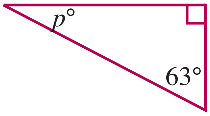

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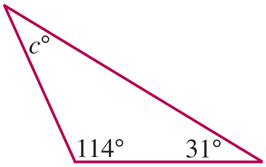

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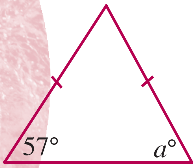

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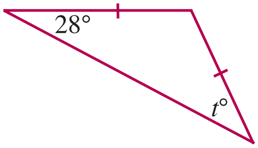

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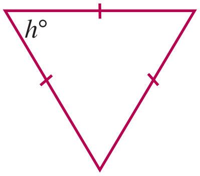

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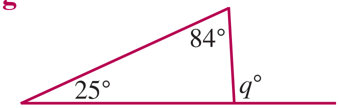

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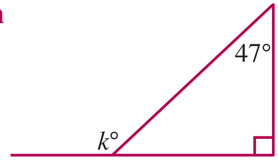

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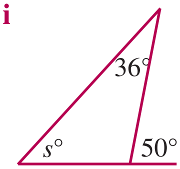
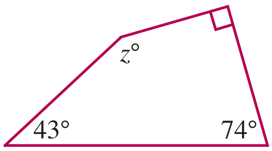


4 Find the value of all pronumerals, giving reasons. All lengths are in cm.

a



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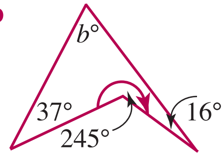

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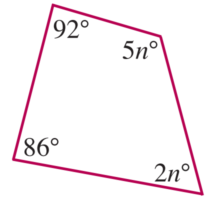

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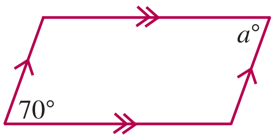

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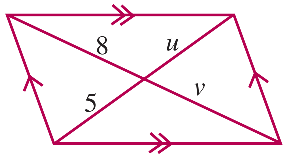

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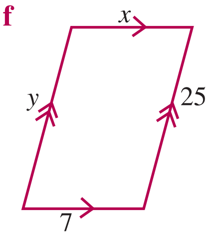

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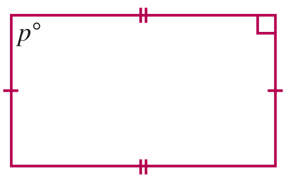

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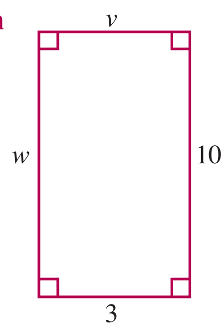

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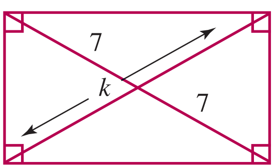

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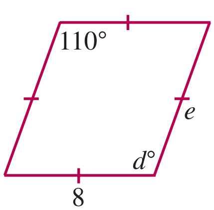

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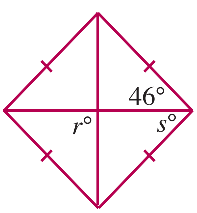

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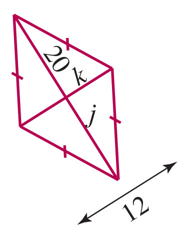

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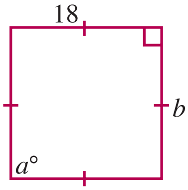

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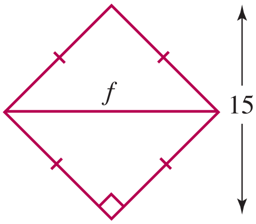

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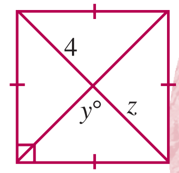
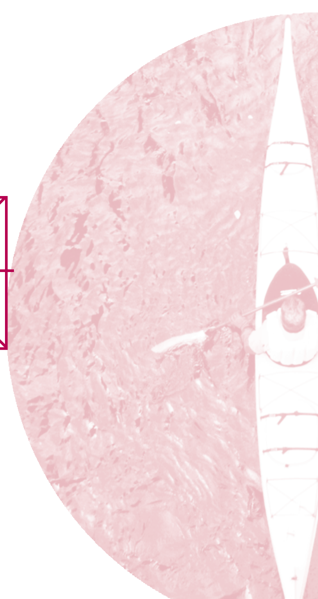
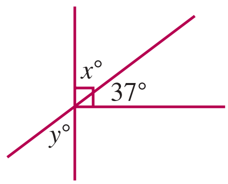




5 Find the value of all pronumerals, giving reasons.

a



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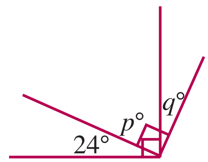

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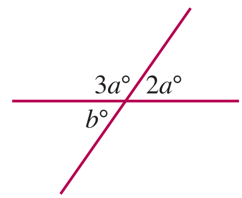

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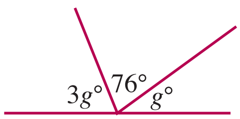

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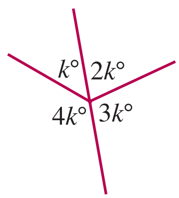

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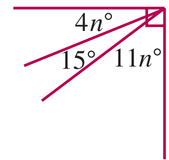

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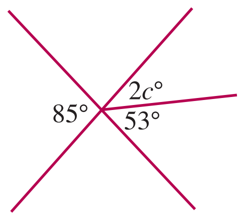

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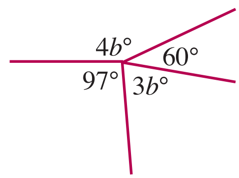

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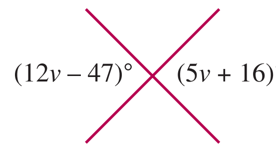
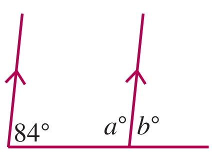


6 Find the value of all pronumerals, giving reasons.

a



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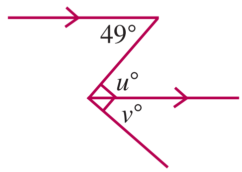

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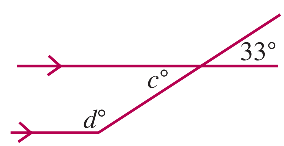

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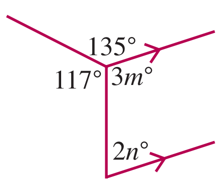

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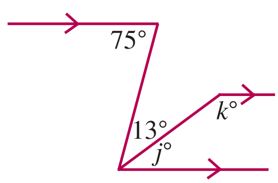

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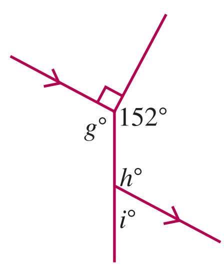
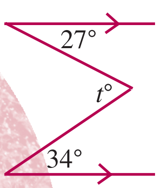


7 Find the value of each pronumeral. (You do not need to give reasons.)

a



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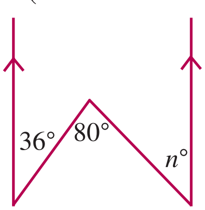

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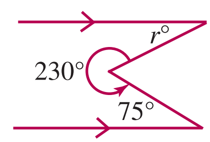

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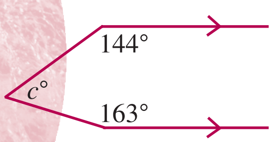

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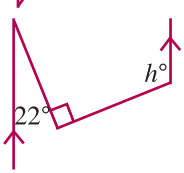

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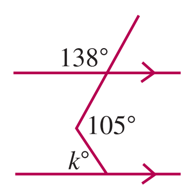
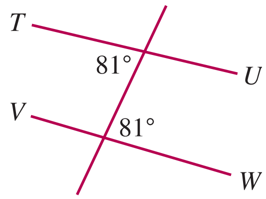


8 State whether $TU \parallel VW$ in each of these. If the lines are parallel, give a reason.

a


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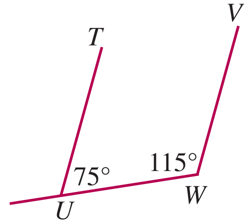

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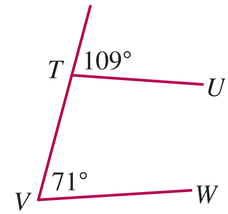
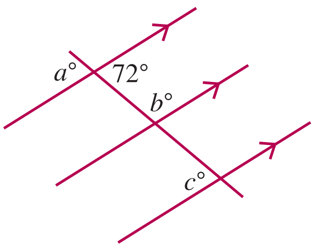


9 Find the value of all pronumerals. Do not give reasons.

a


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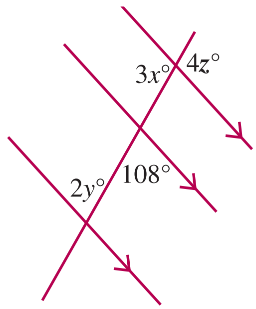

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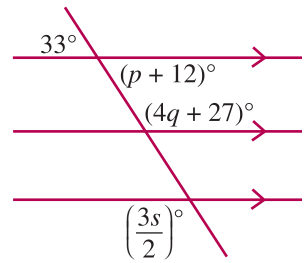
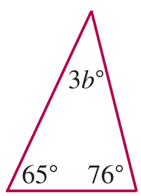


10 Find the value of the pronumeral in each of these, giving reasons.

a


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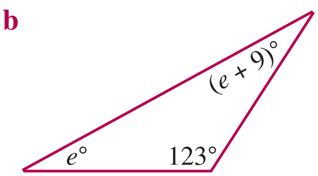

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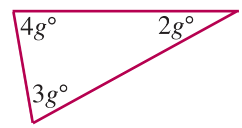

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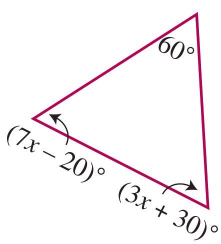

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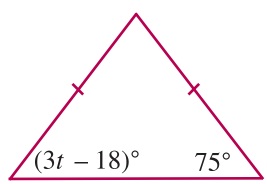

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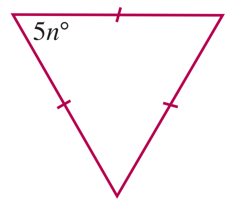

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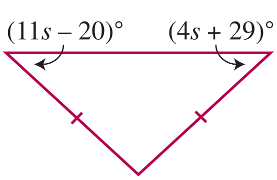

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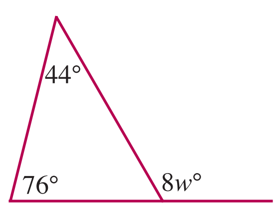

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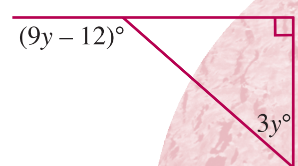
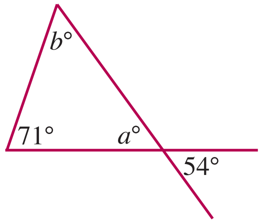


11 Find the value of all pronumerals, giving reasons.

a



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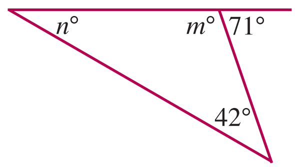

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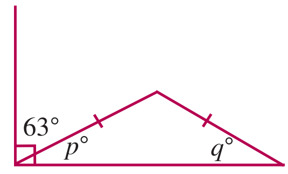

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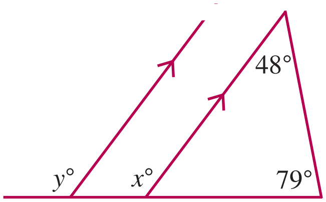

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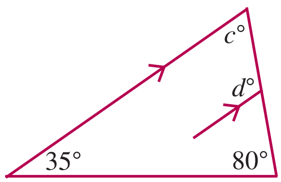

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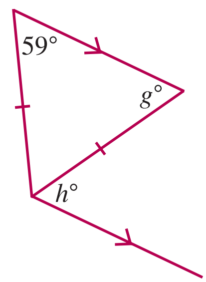

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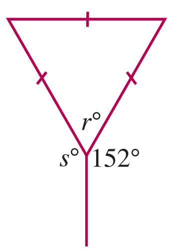

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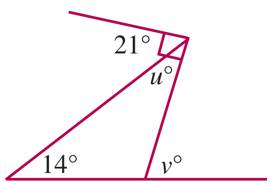

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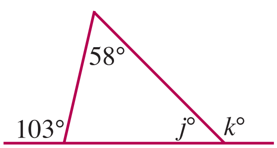

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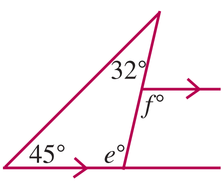

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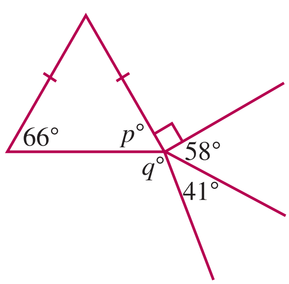

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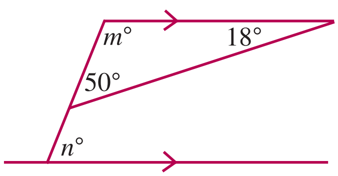
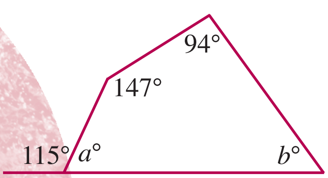


12 Find the value of all pronumerals, giving reasons.

a



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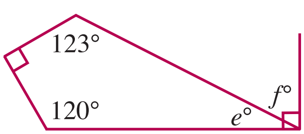

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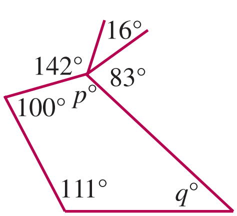

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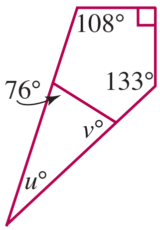

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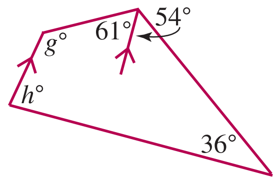

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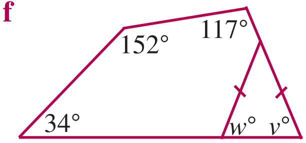

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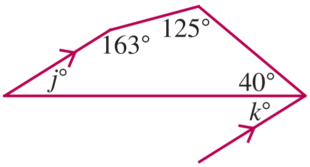

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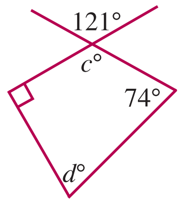

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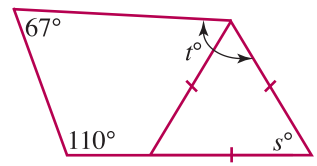
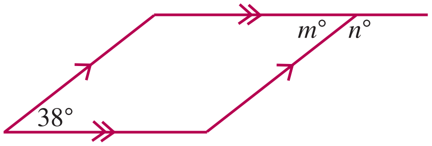


13 Find the value of all pronumerals, giving reasons.

a



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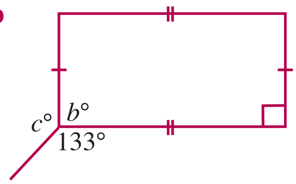

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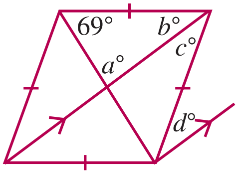

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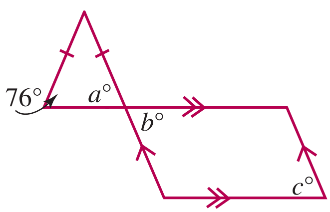

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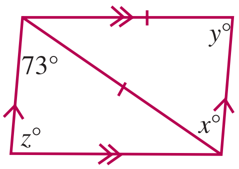

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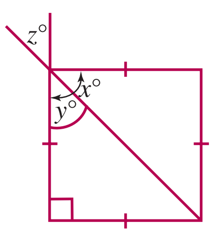
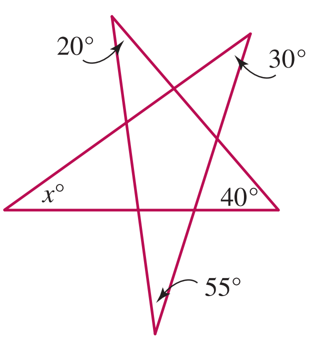
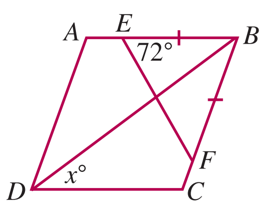


■ Further applications

14 Find the value of x. (You do not need to give reasons.)



15 $ABCD$ is a rhombus and $BE = BF$. Find the value of x. (You do not have to give reasons.)



4.2 Polygons

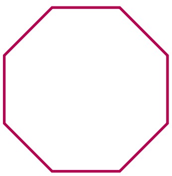
Common polygons

A **polygon** is a closed figure bounded by only straight sides. The name of a polygon is based on the number of sides that make up the boundary of the figure. The names of the first 10 polygons are shown below.

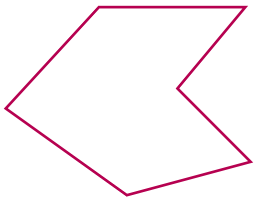
Sides	Polygon	Sides	Polygon
3	Triangle	8	Octagon
4	Quadrilateral	9	Nonagon
5	Pentagon	10	Decagon
6	Hexagon	11	Undecagon
7	Heptagon	12	Dodecagon

Convex and non-convex polygons

A polygon can be convex or non-convex. A **convex polygon** is a polygon in which all of the diagonals lie within the figure. All interior angles are less than 180°. A **non-convex polygon** is a polygon in which at least one diagonal does not lie completely within the figure. One or more interior angles is greater than 180°.



Convex polygon



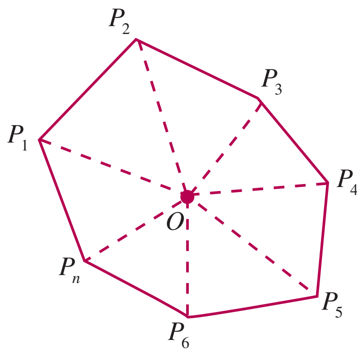
Non-convex polygon

The interior angle sum of a polygon

The sum S of the interior angles of any n-sided polygon is given by $S = (n - 2) \times 180°$.

Proof: Let $P_1P_2P_3P_4 \ldots P_n$ be a convex polygon with n sides. Choose any point O inside the polygon and join it to each of the vertices, forming n triangles.

The angle sum of each triangle is 180°, therefore, the sum of the angles in n triangles is $180n°$. However, this includes the angles around O whose sum is 360°. These angles must be subtracted from the angles around the boundary of the polygon to give the interior angle sum S.



Hence, $S = 180n° - 360°$

$= 180°(n - 2)$, on factorising

The exterior angle sum of a polygon

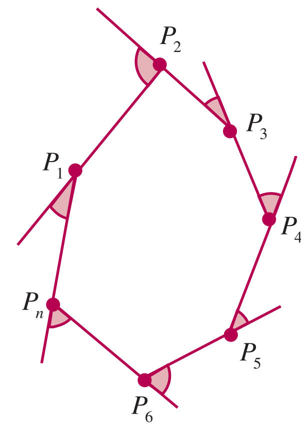
When one side of a polygon is produced, the angle between this produced side and an adjacent side of the polygon is called an exterior angle of the polygon.

The sum S of the exterior angles of any convex polygon is 360°.

Proof: Let $P_1P_2P_3P_4 \ldots P_n$ be a convex polygon with n sides. If each side of the polygon is produced as shown, then the sum of the interior and exterior angles at each vertex is 180°. As there are n interior angles and n exterior angles, the total sum of these angles is $180n°$.

We know that the sum of the interior angles is $180°(n - 2)$. The interior angles must be subtracted from the total angle sum to find S, the sum of the exterior angles of the polygon.

Hence, $S = 180n° - 180°(n - 2)$
$= 180n° - 180n° + 360°$
$= 360°$



Regular polygons

A **regular polygon** is a polygon in which all of the sides are equal and all of the angles are equal. The size of the interior and exterior angles in a regular polygon can be found by dividing the sum of these angles by the number of angles.

In any regular n-sided convex polygon:

- **each interior angle measures $\dfrac{180°(n-2)}{n}$**
- **each exterior angle measures $\dfrac{360°}{n}$**

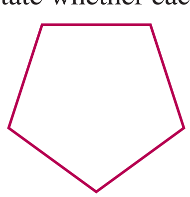
Exercise 4.2

1 How many sides has each polygon?

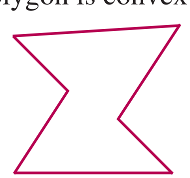
a quadrilateral **b** hexagon **c** octagon
d decagon **e** pentagon **f** heptagon
g nonagon **h** dodecagon **i** undecagon

2 State whether each polygon is convex or non-convex.

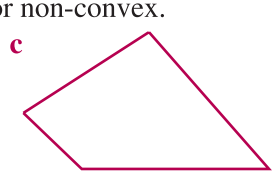
a



b



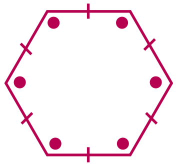
c



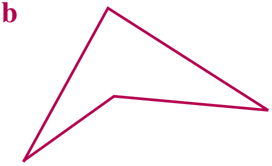
d

3 State whether each polygon is regular or irregular.

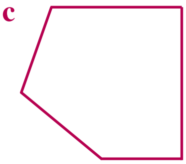
a



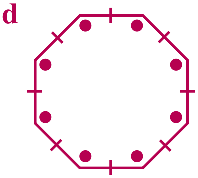
b



c



d



4 **a** Name a quadrilateral in which:

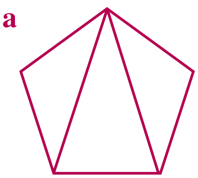
i the angles are equal but the sides are not.

ii the sides are equal but the angles are not.

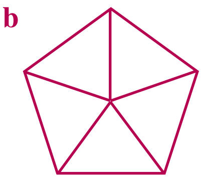
b Are the quadrilaterals in **a** regular? Explain.

5 Find the angle sum of each pentagon by dividing the figure into triangles, as shown.

a



b



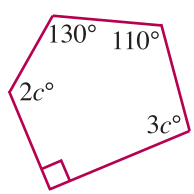
■ Consolidation

6 Find the size of the interior angles in a regular:

a pentagon **b** hexagon **c** octagon **d** decagon

7 Find the size of the interior angles in a regular polygon with 20 sides.

8 Form an equation and solve it to find the value of c.



9 Use the formula $\theta = \dfrac{360°}{n}$ to find the size of each exterior angle in a regular:

a hexagon **b** pentagon **c** dodecagon

10 Find the size of the exterior angles of an equilateral triangle.

11 Find the size of the exterior angles in a regular polygon with 24 sides.

12 How many sides has a regular polygon whose exterior angles measure:

a 45° **b** 36° **c** 20° **d** 40°

13 Show that a regular polygon cannot have interior angles measuring 100°.

14 Construct a regular pentagon in a circle using a ruler and compasses. Measure the angles in the pentagon and verify that each angle is 108°.

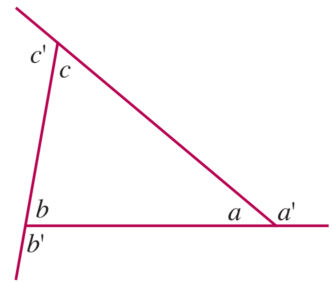
■ Further applications

15 Find the interior angle sum of a regular polygon that has:

a exterior angles measuring 72° **b** interior angles measuring 156°

TRY THIS Ratio of exterior angles

If the exterior angles (a', b', c') are in the ratio 4 : 5 : 6, find the ratio of the interior angles $a : b : c$.

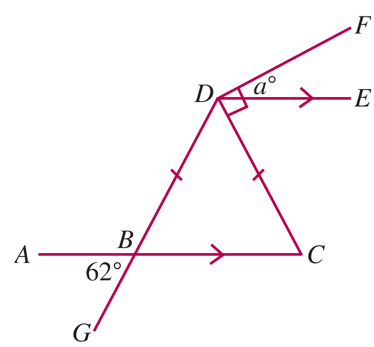


4.3 Harder numerical problems

The logical order of the steps in these questions needs to be thought through before a written solution is attempted. Full reasons must be given for each step in the argument.

Example 1

In the diagram, $DB = DC$, $DE \parallel AC$ and $CD \perp DF$. Find the value of a, giving reasons.



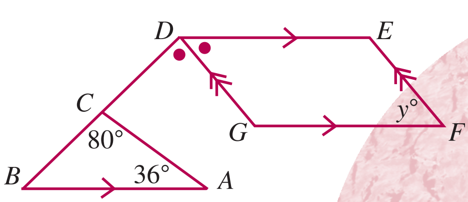
Solution

- $\angle DBC = 62°$ (vertically opposite $\angle$s)
- $\angle DCB = 62°$ (base $\angle$s of an isosceles Δ, $DB = DC$)
- $\angle EDC = 62°$ (alternate $\angle$s, $DE \parallel AC$)
- $\angle FDE = 28°$ (adjacent $\angle$s in a right angle)

$\therefore a = 28$

Example 2

In the diagram, $DE \parallel BA$, $DEFG$ is a parallelogram and GD bisects $\angle BDE$. Find the value of y, giving reasons.



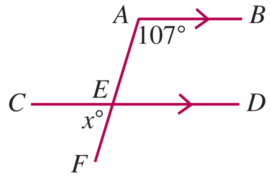
Solution

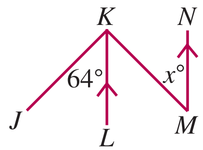
- $\angle ABC = 64°$ ($\angle$ sum of ΔABC is 180°)
- $\angle BDE = 116°$ (co-interior $\angle$s, $DE \parallel BA$)
- $\angle EDG = 58°$ (GD bisects $\angle BDE$)
- $\angle EFG = 58°$ (opposite $\angle$s of a parallelogram)

$\therefore y = 58$

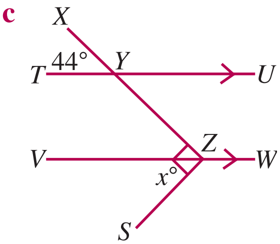
Exercise 4.3

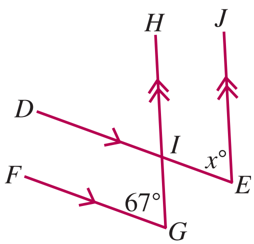
1 Find the value of x in each of the following, giving reasons.

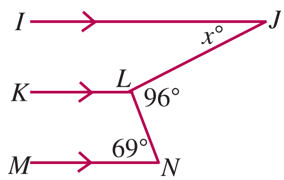
a


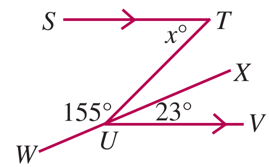
b


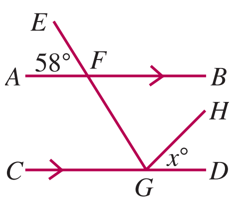
KL bisects ∠*JKM*

c


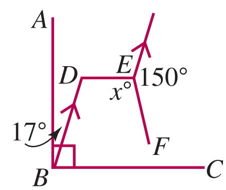
d


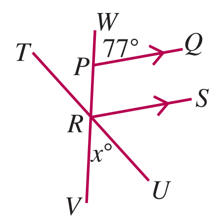
e


f


g


HG bisects ∠*EGD*

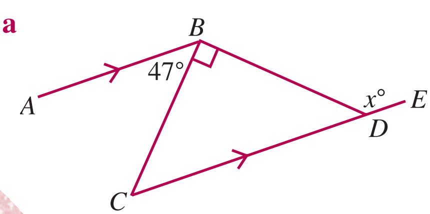
h


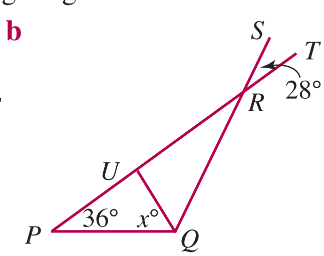
i


WV bisects ∠*TRS*

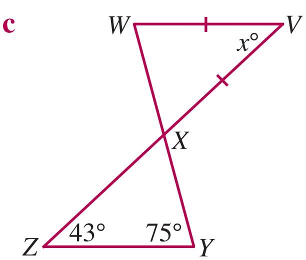
Consolidation

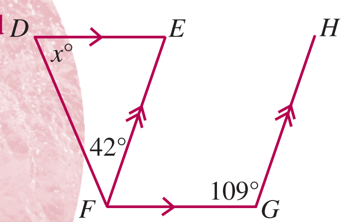
2 Find the value of x in each of these, giving reasons.

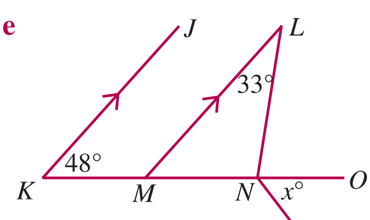
a


b


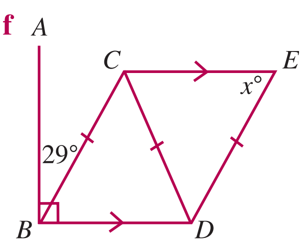
UQ bisects ∠*PQR*

c


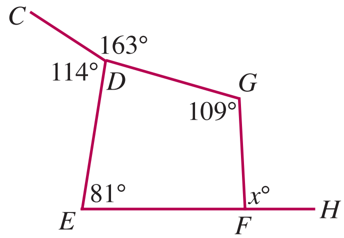
d


e


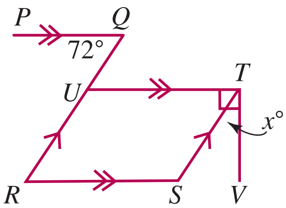
OK bisects ∠*LNP*

f


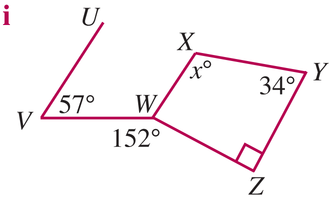
g



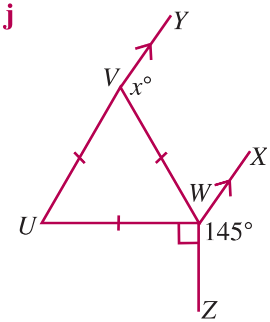
h



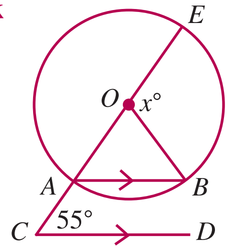
i



j

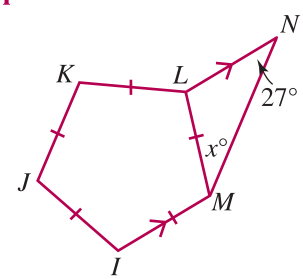


k



O is the centre of the circle

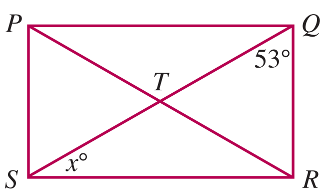
l



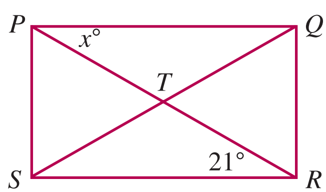
$IJKLM$ is a regular pentagon

3 In each of the following, $PQRS$ is a rectangle. Find the value of x, giving reasons.

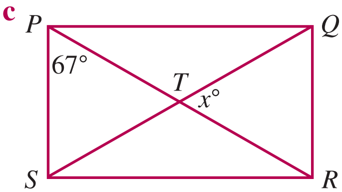
a



b



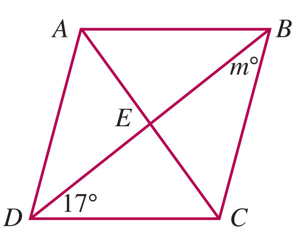
c



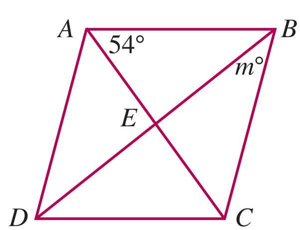
■ Further applications

4 In each of the following, $ABCD$ is a rhombus. Find the value of m, giving reasons.

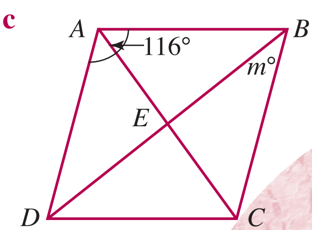
a



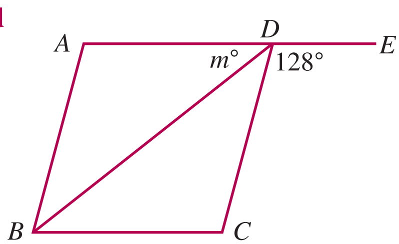
b



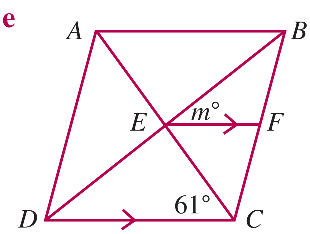
c



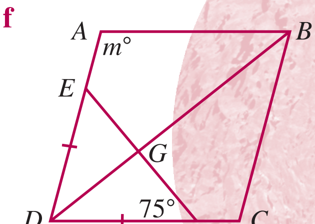
d



e



f



$DE = DF$

4.4 Deductive proofs involving angles

The questions in this exercise involve the proof of general results in figures by the use of angle relationships.

In some questions it is necessary to begin by choosing one particular angle and writing a statement such as, 'let $\angle ABC = \alpha$'.

In other questions it may be necessary to choose two particular angles and write a statement such as 'let $\angle ABC = \alpha$ and $\angle DEF = \beta$'.

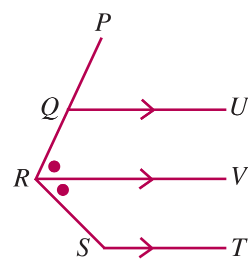
When the size of an angle is referred to by a Greek letter, the degrees symbol is not written, by convention.

Full and correct reasons need to be given for each step in the argument.

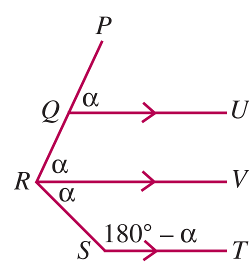
EG +S

Example 1

In the diagram, $QU \parallel RV \parallel ST$ and RV bisects $\angle PRS$.
Prove that $\angle PQU$ and $\angle RST$ are supplementary angles.



Solution



Let $\angle PQU = \alpha$

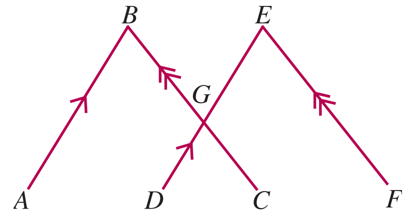
- $\angle PRV = \angle PQU$ (corresponding $\angle$s, $QU \parallel RV$)
  $\therefore \angle PRV = \alpha$
- $\angle VRS = \angle PRV$ (RV bisects $\angle PRS$)
  $\therefore \angle VRS = \alpha$
- $\angle VRS + \angle RST = 180°$ (co-interior $\angle$s, $RV \parallel ST$)
  $\therefore \alpha + \angle RST = 180°$
  $\therefore \angle RST = 180° - \alpha$

Now, $\angle PQU + \angle RST$
$= \alpha + 180° - \alpha$
$= 180°$

$\therefore \angle PQU$ and $\angle RST$ are supplementary angles

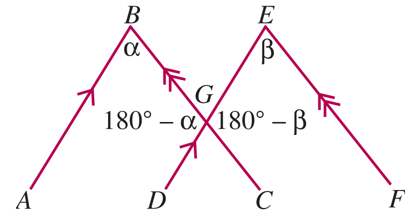
Example 2

In the diagram, $AB \parallel DE$ and $BC \parallel EF$.
Prove that $\angle ABC = \angle DEF$.



Solution

Let $\angle ABC = \alpha$ and $\angle DEF = \beta$

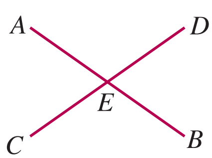


- $\angle ABG + \angle BGD = 180°$ (co-interior $\angle$s, $AB \parallel DE$)
  $\therefore \alpha + \angle BGD = 180°$
  $\therefore \angle BGD = 180° - \alpha$
- $\angle GEF + \angle EGC = 180°$ (co-interior $\angle$s, $EF \parallel GC$)
  $\therefore \beta + \angle EGC = 180°$
  $\therefore \angle EGC = 180° - \beta$
- $\angle BGD = \angle EGC$ (vertically opposite $\angle$s)

$\therefore 180° - \alpha = 180° - \beta$
$\therefore \alpha = \beta$
$\therefore \angle ABC = \angle DEF$

Exercise 4.4

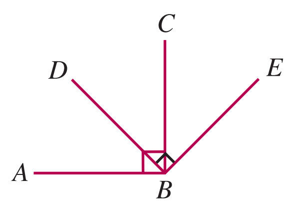
1



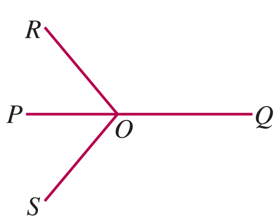
- *AB* and *CD* are straight lines which intersect at *E*.
  Prove that $\angle AEC = \angle DEB$.

2
- $CB \perp AB$
- $DB \perp BE$

a Prove that $\angle ABD = \angle EBC$.
b Prove that $\angle ABE + \angle DBC = 180°$.



3

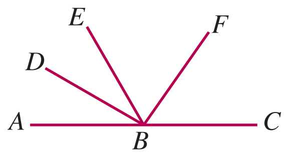


- *PQ* is a straight line through *O*.
- *PO* bisects $\angle ROS$.

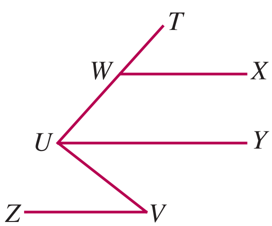
Prove that $\angle ROQ = \angle SOQ$.

Consolidation

4 • AC is a straight line through B.
• DB bisects $\angle ABE$.
• FB bisects $\angle EBC$.
Prove that $DB \perp BF$.

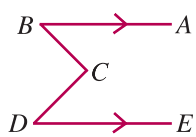


5



• $WX \parallel UY \parallel ZV$
• UY bisects $\angle TUV$
Prove that $\angle TWX = \angle UVZ$.

6 • $BA \parallel DE$
• Prove that $\angle BCD = \angle ABC + \angle CDE$



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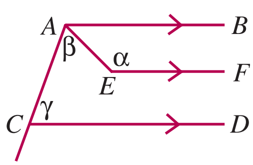
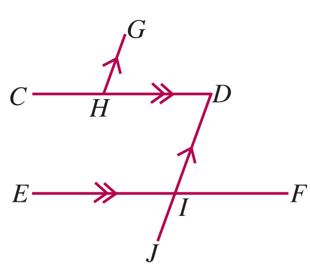


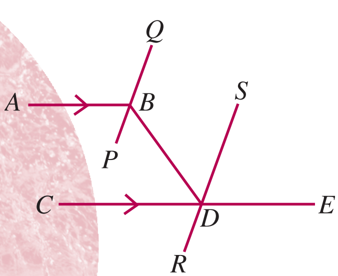
• $AB \parallel CD \parallel EF$.
Prove that $\alpha = \beta + \gamma$.

8 • $CD \parallel EF$
• $GH \parallel DJ$
Prove that $\angle CHG = \angle JIF$



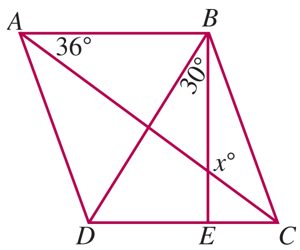
Further applications

9



• $AB \parallel CE$
• PQ bisects $\angle ABD$
• RS bisects $\angle BDE$
Prove that $PQ \parallel RS$.

TRY THIS Angles in a rhombus



ABCD is a rhombus.
$\angle DBE = 30°$.
$\angle BAC = 36°$.
Find the value of x.

4.5 Deductive proofs involving triangles

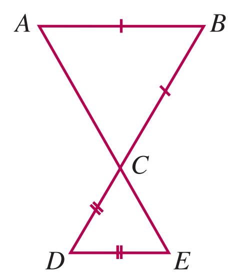
The questions in this exercise involve the proof of general results in figures by the use of geometrical properties of triangles and angle relationships.

As with the questions in the previous exercise, you should begin each question by choosing one or two angles and labelling them as α or β. Congruent triangles are not required in this exercise.

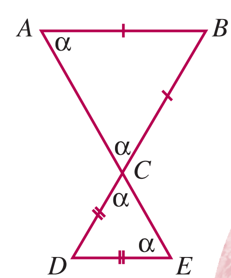
EG +S

Example 1

In the diagram, $AB = BC$ and $CD = DE$. Prove that $AB \parallel DE$.



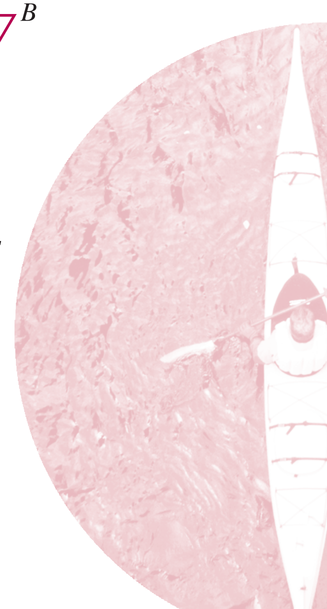
Solution



Let $\angle BAC = \alpha$

- $\angle BCA = \angle BAC$ (base $\angle$s of an isosceles Δ, $AB = BC$)
  $\therefore \angle BCA = \alpha$
- $\angle DCE = \angle BCA$ (vert. opp. $\angle$s)
  $\therefore \angle DCE = \alpha$
- $\angle CED = \angle DCE$ (base $\angle$s of an isosceles Δ, $CD = DE$)
  $\therefore \angle CED = \alpha$

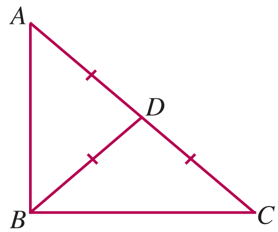
Now, $\angle BAC = \angle CED$ (both equal to α)
$\therefore AB \parallel DE$ (alternate $\angle$s are equal)



EG +S

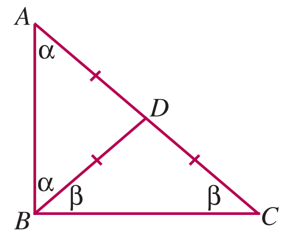
Example 2

In the diagram, $DA = DB = DC$.
Prove that $AB \perp BC$.



Solution

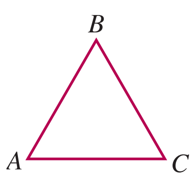
Let $\angle DAB = \alpha$ and $\angle DCB = \beta$.



- $\angle DBA = \angle DAB$ (base $\angle$s of isosceles ΔABD, $DA = DB$)
  $\therefore \angle DBA = \alpha$
- $\angle DBC = \angle DCB$ (base $\angle$s of isosceles ΔBCD, $DB = DC$)
  $\therefore \angle DBC = \beta$
- $\angle DAB + \angle ABC + \angle DCB = 180°$ ($\angle$ sum of ΔABC is 180°)
  $\therefore \alpha + (\alpha + \beta) + \beta = 180°$
  $2\alpha + 2\beta = 180°$
  $\therefore \alpha + \beta = 90°$
  $\therefore \angle ABC = 90°$
  $\therefore AB \perp BC$.

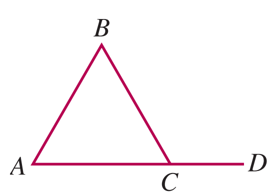
Exercise 4.5

1



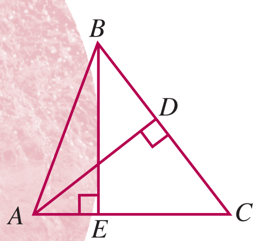
ABC is an any triangle. Prove that the angle sum of the triangle is 180°. [Hint: Construct DE through B, parallel to AC.]

2 In ΔABC, AC is produced to D.
Prove that $\angle BCD = \angle BAC + \angle ABC$.
[Hint: Construct CE parallel to AB.]



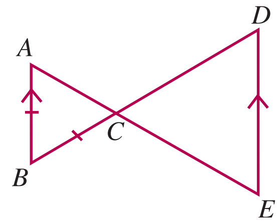
3 In ΔABC, $\angle C = \angle A + \angle B$. Prove that the triangle is right-angled.

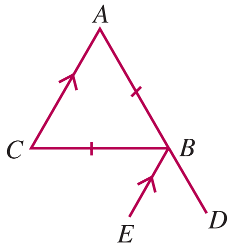
4



In ΔABC, D and E are points on BC and AC respectively such that $AD \perp BC$ and $BE \perp AC$. Prove that $\angle DAC = \angle EBC$.

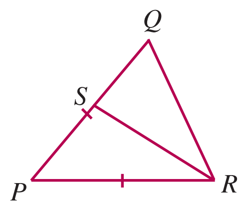
5 In the diagram, $AB \parallel DE$ and $AB = BC$. Prove that ΔCDE is isosceles.

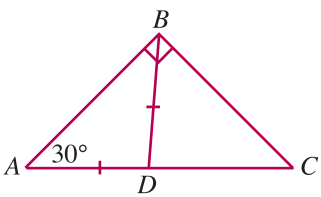


6 

In the diagram, $AB = BC$ and $AC \parallel BE$. Prove that EB bisects $\angle CBD$.

7 In ΔPQR, $PQ = PR$. S is a point on PQ such that SR bisects $\angle PRQ$. Prove that $\angle PSR = 3\angle PRS$.



8 

In ΔABC, $AB \perp BC$ and $\angle BAC = 30°$. BD is drawn to AC such that $AD = BD$. Prove that ΔBCD is equilateral.

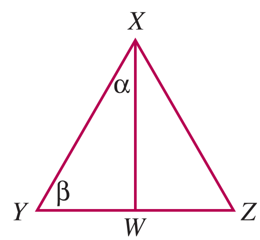
Consolidation

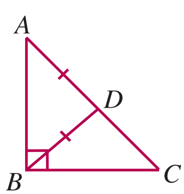
9 In ΔXYZ, $XY = XZ$. The bisector of $\angle X$ meets the base YZ at W. Let $\angle YXW = \alpha$ and $\angle XYW = \beta$.

a Explain why $\angle XWZ = \alpha + \beta$.

b Show that $\angle XWY = \alpha + \beta$.

c Hence, prove that $XW \perp YZ$.



10 

In ΔABC, $AB \perp BC$. D is a point on AC such that $AD = DB$.

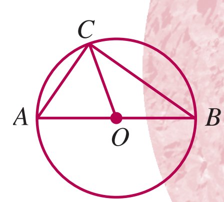
a Prove that ΔBCD is isosceles.

b Hence, prove that D is the midpoint of AC.

11 C is a point on the circumference of a circle with centre O and diameter AB.

a Explain why $OA = OB = OC$.

b Hence, prove that $\angle ACB = 90°$.



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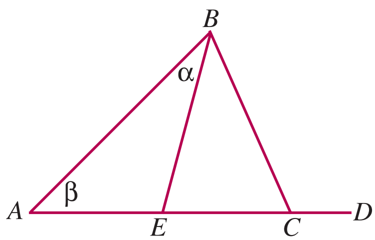
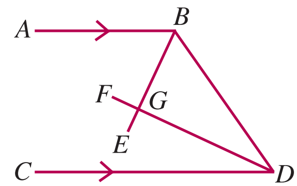


In ΔABC, AC is produced to D. E is a point on AC such that BE bisects $\angle ABC$. Let $\angle ABE = \alpha$ and $\angle BAC = \beta$.

a Find expressions for $\angle BEC$ and $\angle BCD$, giving reasons.

b Hence, prove that $\angle BAC + \angle BCD = 2\angle BEC$.

13 In the diagram, $AB \parallel CD$, EB bisects $\angle ABD$ and FD bisects $\angle BDC$. EB and FD meet at G. Prove that $EB \perp FD$.



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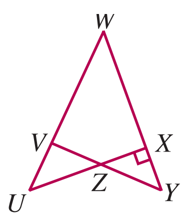
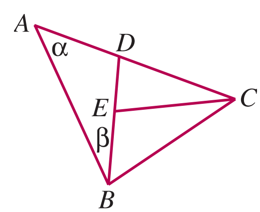


In the diagram, $VW = VY$ and $UX \perp WY$. Prove that ΔUVZ is isosceles.

15 In ΔABC, D is a point on AC such that BD bisects $\angle ABC$. E is a point on BD such that $\angle BCE = \angle BAD$. Let $\angle BAC = \alpha$ and $\angle ABD = \beta$.

a Explain why $\angle BDC = \alpha + \beta$.

b Hence, prove that $CD = CE$.



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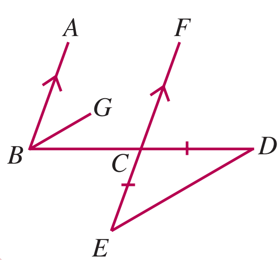
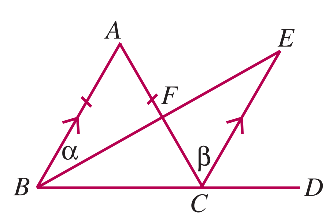


In the diagram, $CD = CE$, $AB \parallel FE$ and GB bisects $\angle ABC$. Prove that $GB \parallel DE$.

■ Further applications

17 In the diagram, $AB = AC$ and $AB \parallel EC$. EB bisects $\angle ABC$ and EC bisects $\angle ACD$. Prove that $AC \perp BE$. [Hint: Let $\angle ABF = \alpha$ and $\angle ACE = \beta$.]



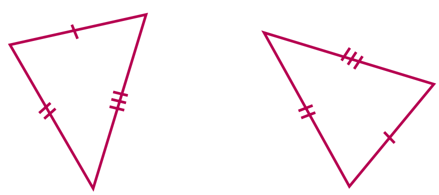
4.6 Congruent triangles review

Two triangles are said to be **congruent** if they have exactly the same size and shape. Each triangle can be obtained from the other by performing one or more of the following transformations—translation, rotation or reflection. The sides and angles that are in the same positions relative to other sides and angles are called **matching sides** and **matching angles**. The symbols ≡ and ≅ are used to mean ‘is congruent to’.

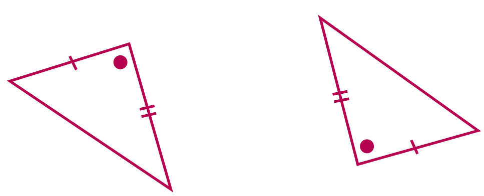
If two triangles are congruent, then:
- **the matching sides are equal in length**
- **the matching angles are equal in size**
- **the figures are equal in area.**

There are four standard tests that can be used to determine whether two triangles are congruent.

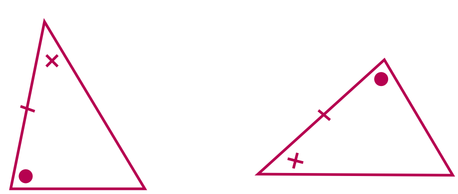
1 If the three sides of one triangle are equal to the three sides of another triangle, then the two triangles are congruent (SSS).



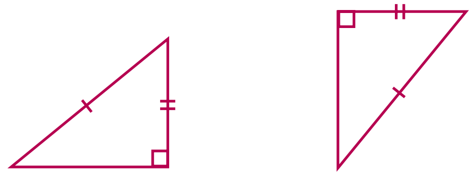
2 If two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, then the two triangles are congruent (SAS).



3 If two angles and one side of one triangle are equal to two angles and the matching side of another triangle, then the two triangles are congruent (AAS).

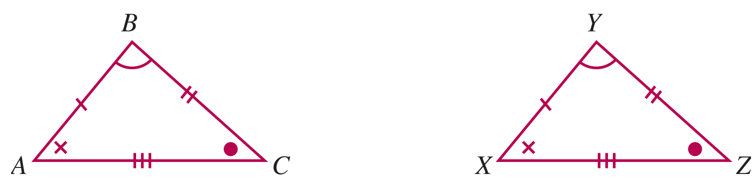


4 If the hypotenuse and a second side of one right-angled triangle are equal to the hypotenuse and a second side of another right-angled triangle, then the two triangles are congruent (RHS).



NOTE: **1** If the three angles of one triangle are equal to the three angles of another triangle, then the triangles are not necessarily congruent. That is, **AAA** is *not* a test for congruent triangles.

2 When naming congruent figures, the vertices must be given in matching order. Thus, for the triangles below, we would write $\Delta ABC \equiv \Delta XYZ$.



The standard congruence proof for triangles has five steps.

To prove that two triangles are congruent:
- **identify the triangles that are being used in the proof**
- **name the three pairs of equal sides or angles**
- **name the congruent triangles, giving the vertices of the triangles in matching order, and state the congruence test used.**

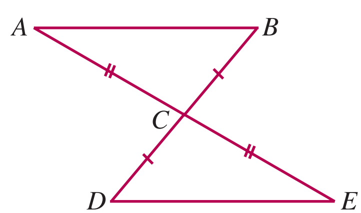
NOTE: By convention, the sides or angles on the LHS of the proof should belong to one triangle and the sides or angles on the RHS should belong to the other triangle.

Example 1

- AE and BD bisect each other at C.

a Prove that $\Delta ABC \equiv \Delta EDC$.

b Hence, show that $AB \parallel DE$.



Solutions

a In ΔABC and ΔEDC
- $AC = CE$ (BD bisects AE)
- $\angle ACB = \angle DCE$ (vert. opp. $\angle$s)
- $BC = CD$ (AE bisects BD)

$\therefore \Delta ABC \equiv \Delta EDC$ (SAS)

b $\angle ABC = \angle CDE$ (matching $\angle$s of congruent Δs)

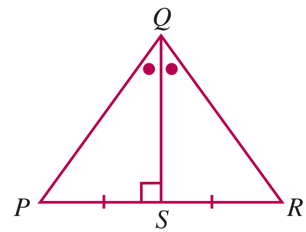
$\therefore AB \parallel DE$ (alternate $\angle$s are equal).

Example 2

- $QS \perp PR$
- QS bisects $\angle PQR$.

a Prove that $\Delta PQS \equiv \Delta RQS$.

b Hence, show that ΔPQR is isosceles.



Solutions

a In ΔPQS and ΔRQS

- $\angle PQS = \angle RQS$ (QS bisects $\angle PQR$)
- $\angle QSP = \angle QSR = 90°$ ($QS \perp PR$)
- QS is a common side

$\therefore$ $\Delta PQS \equiv \Delta RQS$ (AAS)

b $PQ = QR$ (matching sides of congruent Δs)

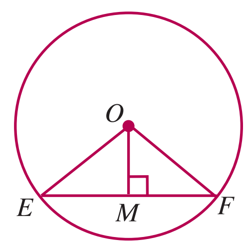
$\therefore$ ΔPQR is isosceles.

Example 3

- O is the centre of the circle
- $OM \perp EF$.

a Prove that $\Delta OEM \equiv \Delta OFM$.

b Hence, show that OM bisects $\angle EOF$.



Solutions

a In ΔOEM and ΔOFM

- $\angle OME = \angle OMF = 90°$ ($OM \perp EF$)
- $OE = OF$ (equal radii)
- OM is a common side

$\therefore$ $\Delta OEM \equiv \Delta OFM$ (RHS)

b $\angle EOM = \angle FOM$ (matching $\angle$s of congruent Δs)

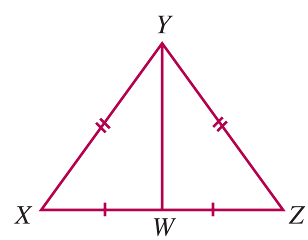
$\therefore$ OM bisects $\angle EOF$.

Example 4

- $XY = YZ$
- YW bisects XZ.

a Prove that $\Delta YXW \equiv \Delta YZW$.

b Hence, show that $YW \perp XZ$.



Solutions

a In ΔYXW and ΔYZW

- $XY = YZ$ (given)
- $XW = WZ$ (YW bisects XZ)
- YW is a common side

$\therefore$ $\Delta YXW \equiv \Delta YZW$ (SSS)

b $\angle YWX = \angle YWZ$ (matching $\angle$s of congruent Δs)

But, $\angle YWX + \angle YWZ = 180°$

(adjacent $\angle$s on a straight line)

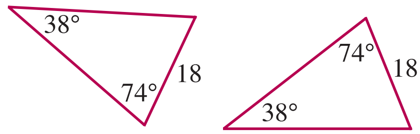
$\therefore$ $\angle YWX = \angle YWZ = 90°$

$\therefore$ $YW \perp XZ$.

Exercise 4.6

1 State the test that could be used to prove that each pair of triangles are congruent.

a



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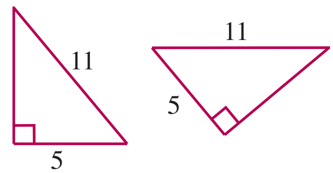

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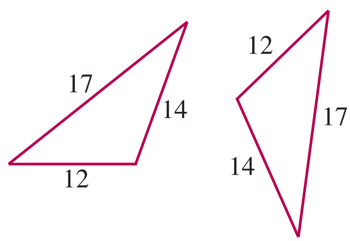

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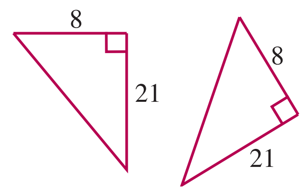
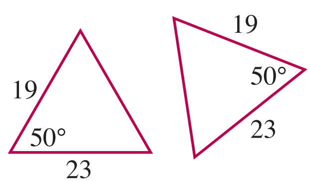
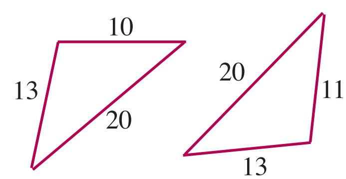


2 State whether each pair of triangles are congruent. If they are congruent, state the test used.

a



b



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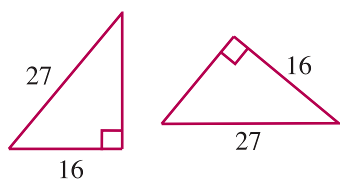

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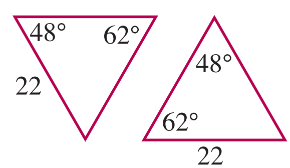

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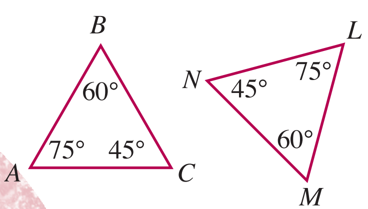
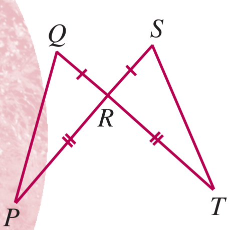


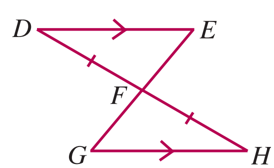
Is $\Delta ABC \equiv \Delta LMN$? Explain.

4 a



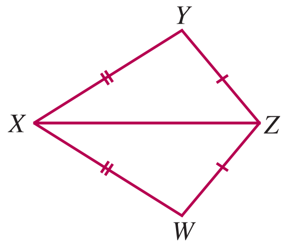
Prove that $\Delta PQR \equiv \Delta TSR$.

b



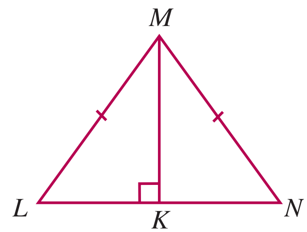
Prove that $\Delta DEF \equiv \Delta HGF$.

c



Prove that $\Delta XYZ \equiv \Delta XWZ$.

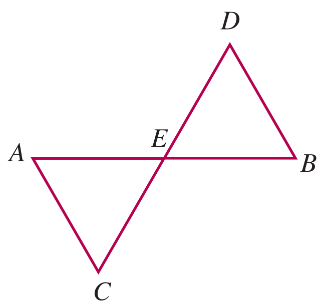
d



Prove that $\Delta MLK \equiv \Delta MNK$.

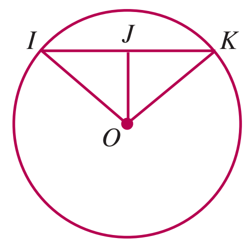
Consolidation

5 **a**



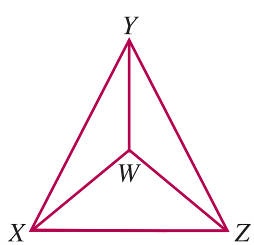
AB and CD bisect each other at E.
Prove that $AC = BD$.

b



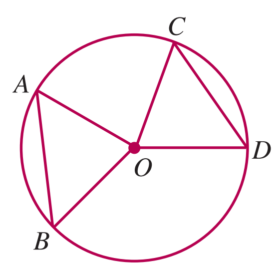
O is the centre of the circle and $OJ \perp IK$.
Prove that OJ bisects $\angle IOK$.

c



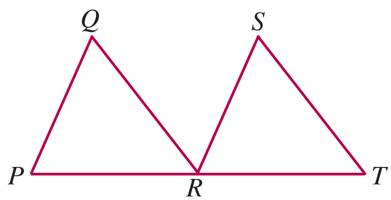
$\angle WXY = \angle WZY$ and WY bisects $\angle XYZ$.
Prove that ΔXWZ is isosceles.

d



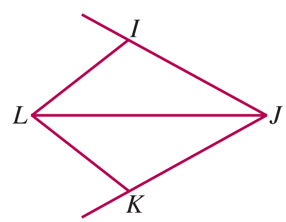
O is the centre of the circle and $AB = CD$.
Prove that $\angle AOB = \angle COD$.

e



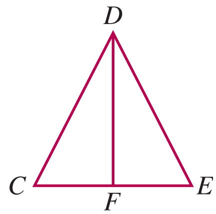
$PQ \parallel RS$, $QR \parallel ST$ and QR bisects PT.
Prove that $PQ = RS$.

f



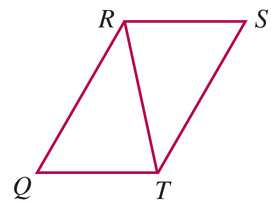
$IJ \parallel LK$ and $LI \parallel KJ$.
Prove that LJ bisects $\angle IJK$.

g



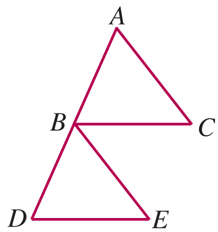
$CD = DE$ and $DF \perp CE$.
Prove that DF bisects CE.

h



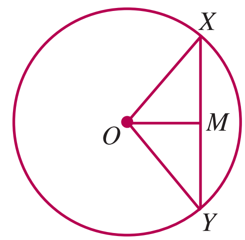
$RS = QT$ and $RS \parallel QT$.
Prove that $QR \parallel TS$.

i



BC bisects AD, $BC \parallel DE$ and $BC = DE$.
Prove that $AC \parallel BE$.

j

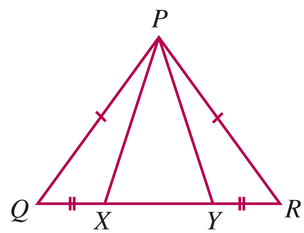


O is the centre of the circle and OM bisects XY. Prove that $OM \perp XY$.

6 In the isosceles triangle PQR, $PQ = PR$. $QX = RY$.

a Prove that $\Delta PQX \equiv \Delta PRY$.

b Hence, show that ΔPXY is isosceles.



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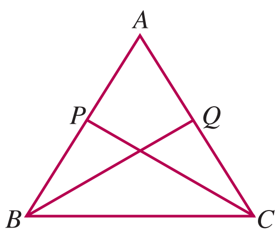


In the isosceles triangle ABC, $AB = AC$.
P and Q are the midpoints of AB and AC respectively.

a Prove that $PB = QC$.

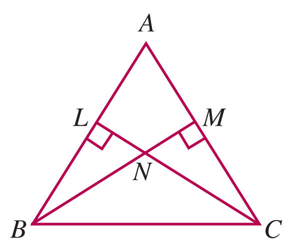
b Hence, prove that $PC = BQ$.

8 In the isosceles triangle ABC, $AB = AC$.
$CL \perp AB$ and $BM \perp AC$.

a Prove that $\Delta BLC \equiv \Delta CMB$.

b Prove that $\Delta BLN \equiv \Delta CMN$.

c Hence show that $LN = MN$.



9 **a** In ΔABC, $AB = AC$. Prove that $\angle ABC = \angle ACB$.

b In ΔABC, $\angle ABC = \angle ACB$. Prove that $AB = AC$. (This is the converse of **a**.)

10 ΔABC is equilateral.

a Construct CD, the bisector of $\angle C$. Hence, prove that $\angle A = \angle B$.

b Construct AE, the bisector of $\angle A$. Hence, prove that $\angle B = \angle C$.

c Hence show that each angle in an equilateral triangle is 60°.

Further applications

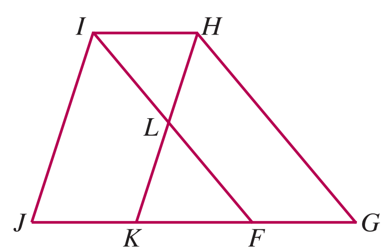
11 ABC is a triangle with $AB = AC$. P is a point inside the triangle so that $\angle PBC = \angle PCB$. Prove that PA bisects $\angle BAC$.

12 In a triangle LMN the sides LM and LN are equal and greater than MN. P is any point on MN. On ML cut off $MQ = NP$, and on NL cut off $NS = MP$. Prove that $\angle PQS = \angle PSQ$.

13 ABC is an equilateral triangle. The perpendicular to BC at C meets BA produced at D, and the perpendicular to AB at B meets AC produced at E. Prove that $CE = AD$.

TRY THIS Intersecting parallelograms

HIJK and *FGHI* are both parallelograms.
L is the midpoint of *HK*.
Prove that $GJ = 3 \times HI$.



4.7 Deductive proofs involving quadrilaterals

Definitions of the special quadrilaterals

- **A trapezium is a quadrilateral with at least one pair of opposite sides parallel.**
- **A parallelogram is a quadrilateral with both pairs of opposite sides parallel.**
- **A rhombus is a parallelogram with two adjacent sides equal in length.**
- **A rectangle is a parallelogram in which one angle is a right angle.**
- **A square is a rectangle with two adjacent sides equal in length.**
- **A kite is a quadrilateral with two pairs of adjacent sides equal in length.**

Tests for the special quadrilaterals

- **Parallelogram**
  A quadrilateral is a parallelogram if:
  - **the opposite sides are equal, or**
  - **the opposite angles are equal, or**
  - **one pair of opposite sides are equal and parallel, or**
  - **the diagonals bisect each other.**
- **Rhombus**
  A quadrilateral is a rhombus if:
  - **all sides are equal, or**
  - **the diagonals bisect each other at right angles.**
- **Rectangle**
  A quadrilateral is a rectangle if:
  - **all angles are equal**
  - **the diagonals are equal and bisect each other.**
- **Square**
  A quadrilateral is a square if:
  - **all sides are equal and one angle is a right angle, or**
  - **all angles are right angles and two adjacent sides are equal, or**
  - **the diagonals are equal and bisect each other at right angles.**

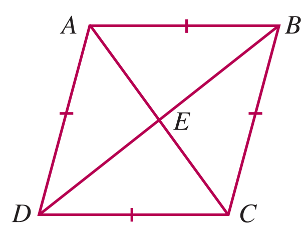
EG +S

Example 1

ABCD is a rhombus.

a Prove that $\Delta ABE \equiv \Delta CBE$.

b Hence, show that the diagonals of a rhombus are perpendicular.



Solutions

a In ΔABE and ΔCBE

- $AB = BC$ (sides of a rhombus are equal)
- $\angle ABE = \angle CBE$ (diagonals of a rhombus bisect the angles at the vertices)
- BE is a common side

$\therefore \Delta ABE \equiv \Delta CBE$ (SAS)

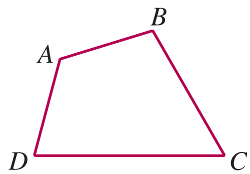
b $\angle AEB = \angle CEB$ (matching $\angle$s of congruent Δs)

But, $\angle AEB + \angle CEB = 180°$ (adj. $\angle$s on a st. line)

$\therefore \angle AEB = \angle CEB = 90°$

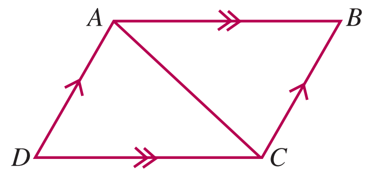
$\therefore$ the diagonals of a rhombus are perpendicular.

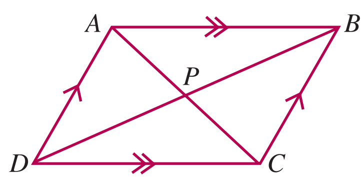
Exercise 4.7

1 

$ABCD$ is any quadrilateral.
Prove that the angle sum of a quadrilateral is 360°.
[Hint: Construct the diagonal AC.]

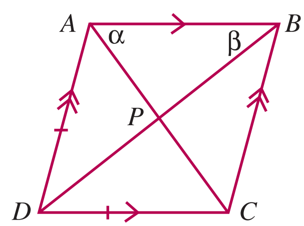
2 $ABCD$ is a parallelogram.
- **a** Prove that $\Delta ABC \equiv \Delta CDA$.
- **b** Hence show that $AB = DC$ and $AD = BC$.
- **c** Show that $\angle ABC = \angle ADC$.
- **d** What property of a parallelogram have you proven?

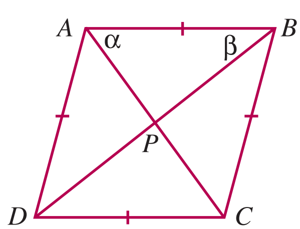


3 

$ABCD$ is a parallelogram. The diagonals AC and BD meet at P.
- **a** Prove that $\Delta APB \equiv \Delta CPD$.
- **b** Hence show that $AP = PC$ and $DP = PB$.
- **c** What property of a parallelogram have you proven?

4 $ABCD$ is a rhombus. The diagonals AC and BD meet at P. Let $\angle CAB = \alpha$ and $\angle ABD = \beta$.
- **a** Explain why $\angle BCA = \alpha$ and $\angle DAC = \alpha$.
- **b** Similarly, explain why $\angle BDC = \beta$ and $\angle CBD = \beta$.
- **c** What property of a rhombus have you proven?

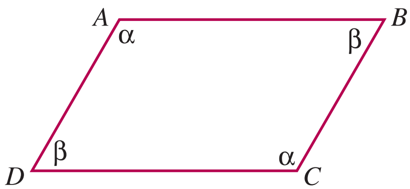


5 

$ABCD$ is a rhombus. The diagonals AC and BD meet at P. Let $\angle CAB = \alpha$ and $\angle ABD = \beta$.
- **a** Explain why $\angle BCA = \alpha$ and $\angle CBD = \beta$.
- **b** Find $\alpha + \beta$.
- **c** Hence, explain why $AC \perp BD$.
- **d** What property of a rhombus have you proven?

6 $ABCD$ is a rectangle.
- **a** Prove that $\Delta ABC \equiv \Delta DCB$.
- **b** Hence show that $AC = BD$.
- **c** What property of a rectangle have you proven?

7 The proofs in this question verify the four standard tests for a parallelogram.



a $ABCD$ is a quadrilateral in which the opposite angles are equal. Let $\angle A = \angle C = \alpha$ and $\angle B = \angle D = \beta$.

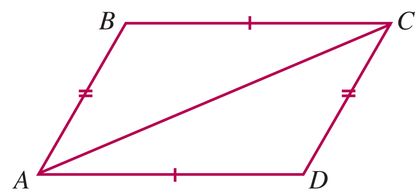
  i Find the value of $\alpha + \beta$.

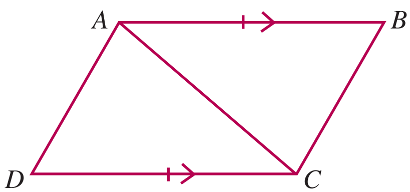
  ii Hence show that $AB \parallel DC$ and $AD \parallel BC$.

b $ABCD$ is a quadrilateral in which the opposite sides are equal.

  i Prove that $\Delta ABC \equiv \Delta CDA$.

  ii Hence show that $AB \parallel DC$ and $BC \parallel AD$.





c $ABCD$ is a quadrilateral in which $AB = DC$ and $AB \parallel DC$.

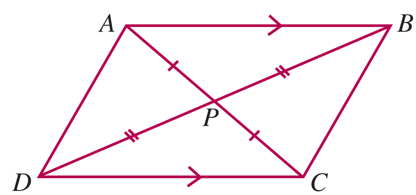
  i Prove that $\Delta ABC \equiv \Delta CDA$.

  ii Hence show that $AD \parallel BC$.

d $ABCD$ is a quadrilateral in which the diagonals AC and BD bisect each other at P.

  i Prove that $\Delta APB \equiv \Delta CPD$.

  ii Hence show that $AB = DC$ and $AB \parallel DC$.



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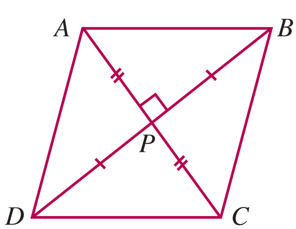


$ABCD$ is a quadrilateral in which the diagonals bisect each other at right angles at P.

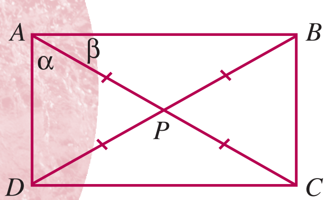
a Prove that $\Delta ABP \equiv \Delta CBP$.

b Hence prove that $AB = BC$.

c Explain why $ABCD$ is a rhombus.

9 $ABCD$ is a quadrilateral in which all angles are equal. Prove that $ABCD$ is a rectangle.

10



$ABCD$ is a quadrilateral in which the diagonals are equal and bisect each other. Let $\angle DAP = \alpha$ and $\angle PAB = \beta$.

a Show that $\angle ADP = \alpha$ and $\angle ABP = \beta$.

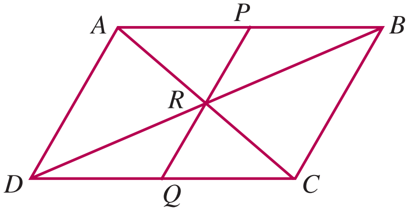
b Find the value of $\alpha + \beta$. Hence prove that $\angle DAB = 90°$.

c Explain why $ABCD$ is a rectangle.

11 *ABCD* is a parallelogram. The diagonals *AC* and *BD* meet at *R*. A line *PQ* is drawn through *R*, where *P* lies on *AB* and *Q* lies on *DC*.

a Prove that $\Delta BPR \equiv \Delta DQR$.

b Hence show that $PB = DQ$ and $AP = QC$.



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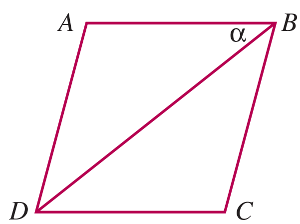


ABCD is a parallelogram. The diagonal *BD* bisects $\angle ADC$. Let $\angle ABD = \alpha$.

a Prove that $AB = AD$.

b Explain why *ABCD* is a rhombus.

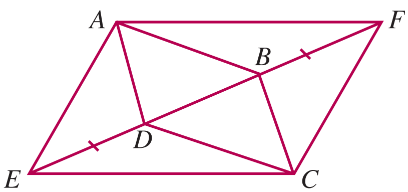
Further applications

13 *ABCD* is a parallelogram. *BD* is produced to *E* and *DB* is produced to *F* such that $DE = BF$.

a Show that $\angle FBC = \angle ADE$.

b Prove that $\Delta FBC \equiv \Delta EDA$.

c Hence prove that *AFCE* is a parallelogram.

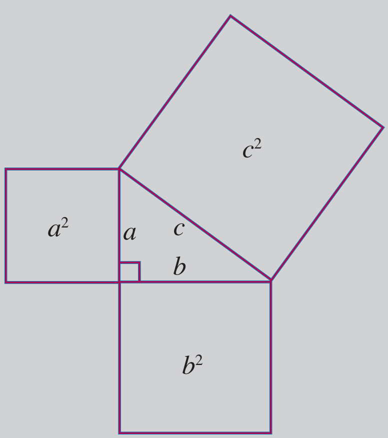


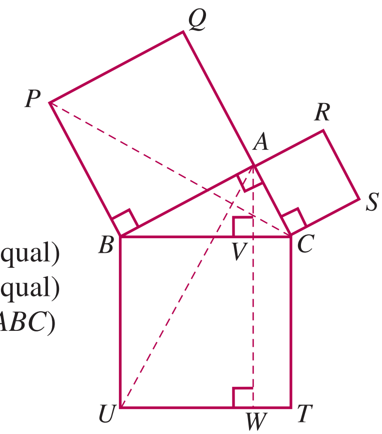
4.8 Pythagoras' theorem

Pythagoras' theorem has been proven in more ways than any other theorem in geometry. In this exercise we will consider both numerical and geometric applications of the theorem.

In any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

That is, $c^2 = a^2 + b^2$





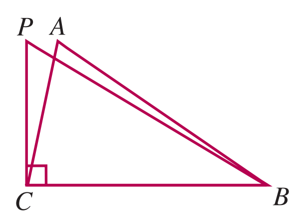
PROOF:
Data: ΔABC is right-angled at A, and squares are drawn on all sides.
Aim: To prove that the square on side BC = sum of the squares on the other 2 sides.
Construction: Join PC, AU and AVW as shown.
Proof: In ΔABU and ΔPBC,

$AB = PB$ (sides of a square are equal)
$BU = BC$ (sides of a square are equal)
$\angle ABU = \angle PBC$ (right $\angle$ + common $\angle ABC$)
$\therefore\ \Delta ABU \equiv \Delta PBC$ (SAS)
Now $\angle BAC = \angle BAQ$ (= 90°)
$\therefore$ QAC is a straight line

$\Delta ABU = \frac{1}{2}$ rectangle $BVWU$ (same base and height)
$\Delta PBC = \frac{1}{2}$ square $PQAB$ (same base and height)
$\therefore$ rectangle $BVWU$ = square $PQAB$ (double equal triangles)
Similarly, rectangle $VCTW$ = square $ARSC$
$\therefore$ square on BC = sum of squares on AB and AC

The converse of Pythagoras' theorem

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, then the angle formed between these two sides is a right angle.

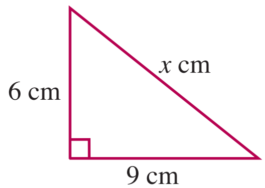


PROOF:
Data: ΔABC such that $AB^2 = AC^2 + BC^2$.
Aim: To prove that $\angle ACB$ is a right angle.
Construction: Assume $\angle ACB$ is 90° and draw $PC = AC$ so that $\angle PCB = 90°$.
Proof: $PC^2 + CB^2 = PB^2$ (Pythagorean theorem)
$\therefore\ AC^2 + CB^2 = PB^2$ (since $PC = AC$)
But $AC^2 + CB^2 = AB^2$ (given)
$\therefore\ AB^2 = PB^2$
$\therefore\ AB = PB$
Also, BC is common
and $AC = PC$ (by construction)
$\therefore\ \Delta ACB \equiv \Delta PCB$ (SSS)
$\therefore\ \angle ACB = \angle PCB$
$\therefore\ \angle ACB = 90°$

Example 1

Find the value of x in each triangle, correct to 1 decimal place.

a



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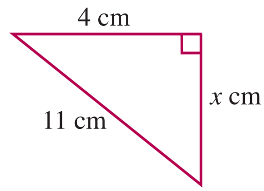


Solutions

a By Pythagoras' theorem,

$x^2 = 6^2 + 9^2$

$x^2 = 117$

$\therefore x = \sqrt{117}$

$= 10.8$ (correct to 1 decimal place)

b By Pythagoras' theorem,

$x^2 + 4^2 = 11^2$

$x^2 + 16 = 121$

$x^2 = 105$

$\therefore x = \sqrt{105}$

$= 10.2$ (correct to 1 decimal place)

Example 2

Show that a triangle with sides 35 cm, 84 cm, 91 cm is right-angled.

Solution

$35^2 + 84^2 = 1225 + 7056$

$= 8281$

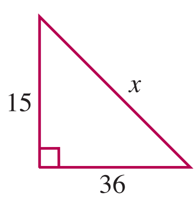
$= 91^2$

$\therefore$ the triangle is right-angled (converse of Pythagoras' theorem).

Exercise 4.8

1 Find the value of the pronumeral in each triangle. Answer in simplest surd form, where necessary.

a



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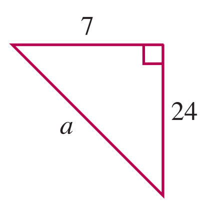

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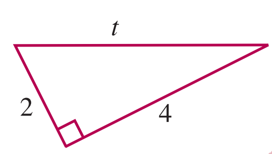

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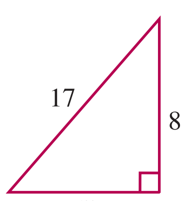

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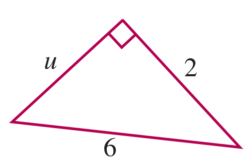

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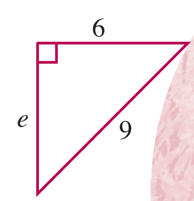
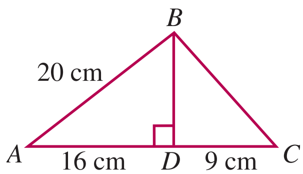


2 Which of the following could be the sides of a right-angled triangle? All lengths are in mm.

a 4, 5, 6 **b** 9, 12, 15 **c** 7, 10, 13 **d** 20, 21, 29

3 

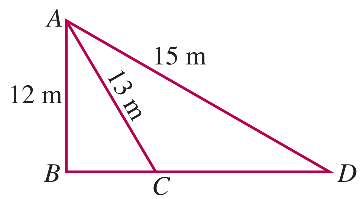
In ΔABC, D is a point on BC such that $BD \perp AC$.

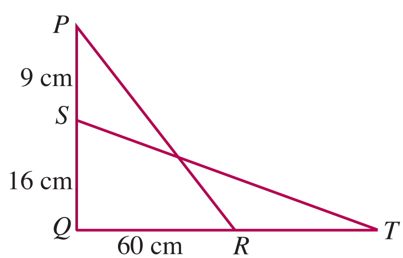
a Find BD and BC.

b Hence show that ΔABC is right-angled.

4 An isosceles triangle has a base of 70 cm and congruent sides of 37 cm. Find the altitude of the triangle.

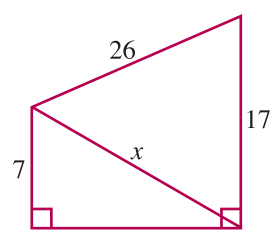
5 Find the length of CD.



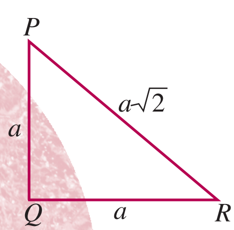
6 

In the diagram, $PQ \perp QT$, $PR = ST$, $PS = 9$ cm, $SQ = 16$ cm and $QR = 60$ cm. Find RT.

7 Find the value of x.

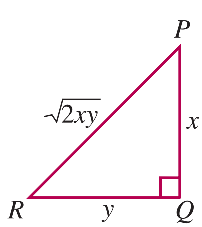


8 In a right-angled triangle, the hypotenuse is 8 cm longer than the shortest side and 1 cm longer than the third side. Find the length of the sides.

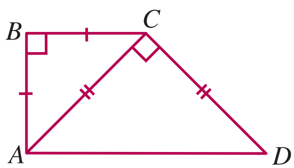
9 

Show that ΔPQR is right-angled.

10 Show that ΔPQR is isosceles.



11 The diagonals of a rhombus are 32 cm and 60 cm. Find the perimeter.

12 

In the diagram $AB = BC$ and $AC = CD$. Show that $AD = 2AB$.

13 Prove that each set of expressions is a Pythagorean triad.

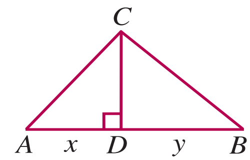
a $2x, x^2 - 1, x^2 + 1$ **b** $2xy, x^2 - y^2, x^2 + y^2$

14 In ΔABC, CD is an altitude and $\dfrac{AD}{CD} = \dfrac{CD}{DB}$.
Let $AD = x$ and $DB = y$.

a Show that $CD = \sqrt{xy}$.

b Find expressions for AC^2 and BC^2.

c Hence, show that ΔABC is right-angled.

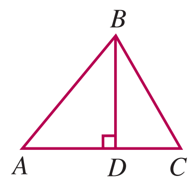


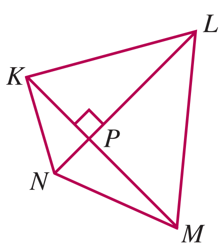
15 In ΔABC, D is a point on AC such that $BD \perp AC$.

a Find an expression for AB^2 in ΔABD.

b Find an expression for CD^2 in ΔBCD.

c Hence, show that $AB^2 + CD^2 = BC^2 + AD^2$.



16 

In the quadrilateral $KLMN$, the diagonals KM and LN meet at P.

a Show that $KL^2 - KN^2 = PL^2 - PN^2$.

b Show that $LM^2 - MN^2 = PL^2 - PN^2$.

c Hence show that $KL^2 + MN^2 = KN^2 + LM^2$.

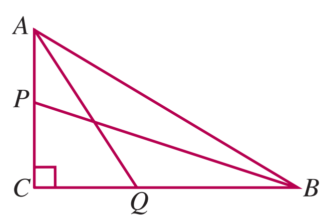
d What geometric property of quadrilaterals have you proven?

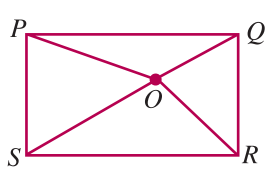
Further applications

17 In ΔABC, $AC \perp BC$. P is any point on AC and Q is any point on BC.

a Prove that $AQ^2 + BP^2 = AB^2 + PQ^2$.

b If P and Q are the midpoints of AC and BC respectively, prove that $AQ^2 + BP^2 = 5PQ^2$. [Hint: Let $AC = 2x$ and $BC = 2y$.]



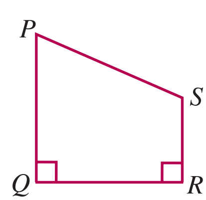
18 

$PQRS$ is a rectangle and O is any point in the interior. Prove that $OP^2 + OR^2 = OQ^2 + OS^2$. [Hint: Construct the altitudes OX, OY to PQ, RS respectively.]

19 In the quadrilateral $PQRS$, $PQ \perp QR$ and $SR \perp QR$.

a Prove that $PR^2 - QS^2 = PQ^2 - RS^2$.

b Hence, prove that $PS^2 - QR^2 = (PQ - RS)^2$.

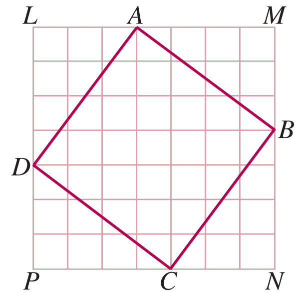


TRY THIS — An unusual proof of Pythagoras' theorem

Within a large square $LMNP$ of side 7 units, a smaller square $ABCD$ is drawn.

a Can you use this figure to show that $3^2 + 4^2 = 5^2$?

b Can you generalise from this argument to prove that $AB^2 = AM^2 + MB^2$?



Focus on working mathematically

An exploration of an equilateral triangle

Introduction

In this chapter you have been developing your skills in deductive reasoning. In working mathematically our aim is to fine tune these skills, by giving you an opportunity to make, refine and test your conjectures. You will also have the opportunity to make a generalisation from some specific cases and to prove a result.

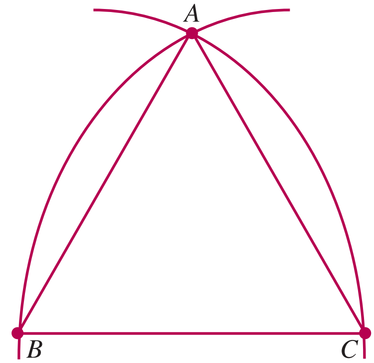
In these activities dynamic geometry software such as Cabri or Geometer's Sketchpad are highly recommended. However if they are not available, geometrical instruments can be used.

FOCUS ON WORKING MATHEMATICALLY

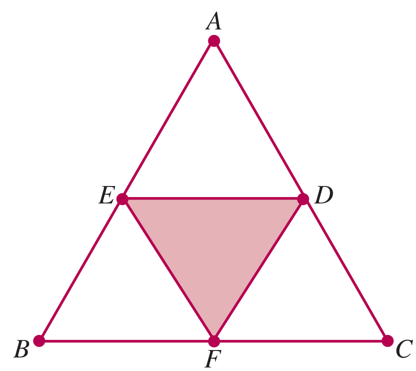
The equilateral triangle

Our starting point is the equilateral triangle *ABC*. Draw it in your book and write down as many geometrical facts about it as you can.

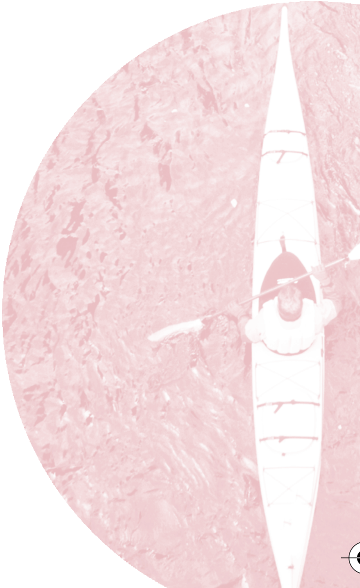
If you are using a software program like Geometer's Sketchpad or Cabri think carefully about how you will construct it. Part of a Sketchpad diagram using the circle by centre and radius tool is shown opposite. You can select and hide the circles.



Learning activities

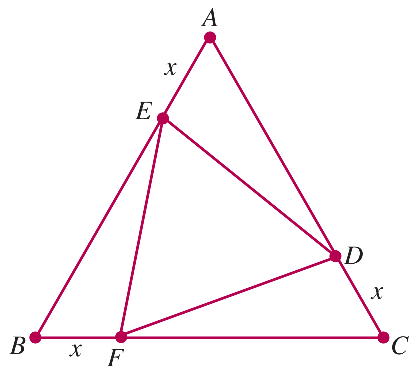


1. On your diagram construct and join the midpoints *D*, *E* and *F* of the sides.
2. Prove that the triangles *AED*, *BFE* and *CDF* are congruent. Explain why triangle *DEF* is equilateral.
3. Explain why triangle *DEF* is $\frac{1}{4}$ of the area of the original equilateral triangle.
4. Now draw another equilateral triangle *ABC*. Construct the points *D*, *E* and *F* such that $AE = BF = CD = \frac{1}{3}AB$. Prove that the triangles *AED*, *BFE* and *CDF* are again congruent. What do you notice about the sizes of the angles of these triangles?
5. Explain why triangle *DEF* is again equilateral.
6. What fraction of triangle *ABC* is triangle *DEF*?
7. Make a conjecture about triangle *DEF* as a fraction of triangle *ABC* when $AE = BF = CD = \frac{1}{4}AB$. Will it be greater or less than $\frac{1}{3}$?



Challenge activities

1 We will now consider the general case where AE, BF and CD are all equal to some length x. Draw the diagram opposite and prove that the triangles AED, BFE and CDF are again congruent and that triangle DEF is equilateral. In the usual notation, the side length $a = b = c$ and the length of $AE = CD = BF = a - x$.



2 Now consider the size of triangle DEF as a fraction of triangle ABC.

- **a** Write down an expression for the area of triangle ABC
- **b** Use the cosine rule to find an expression for the length of FE^2.
- **c** Write down an expression for the area of triangle DEF.
- **d** Show that the ratio of the areas is

$$\frac{\Delta DEF}{\Delta ABC} = \frac{a^2 - 3x(a-x)}{a^2}$$

- **e** Confirm that the value of the ratio when $x = \frac{a}{2}$ is $\frac{1}{4}$.
- **f** Confirm that the value of the ratio when $x = \frac{a}{3}$ is $\frac{1}{3}$.
- **g** Calculate the value when $x = \frac{a}{4}$ and compare with your conjecture in question 7 above.

3 Were you surprised about the result? Disappointed it was not so simple? Given that x is a fraction of a, will the ratio always be an exact rational number? Why?

Let's communicate

Discuss in class what you have learned from this activity about using special cases to help clarify a general rule.

Reflecting

Think over the power of algebra to generalise when we are tackling problems in geometry.

Language link with Macquarie

MACQUARIE

1 *In a short sentence explain*
   - *a* ***Equilateral*** *triangles*
   - *b* ***Congruent*** *triangles*
   - *c* ***A conjecture***
   - *d* ***Deductive*** *reasoning*
   - *e* *The* ***specific*** *as opposed to the* ***general*** *case.*

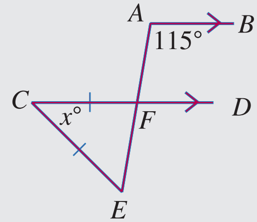
2 *The* Macquarie Learners' Dictionary *defines the word* ***justify*** *as follows:*

justify *verb* (**justified**, **justifying**) to show (an action, argument, etc.) to be right or reasonable: *She can justify her decision to leave.*

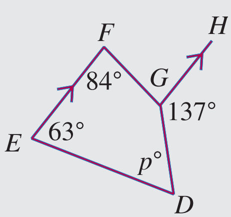
How does the meaning of justify as given here "to show to be right or reasonable" differ from the mathematical use of the word?

1 Find the value of each pronumeral, giving reasons.

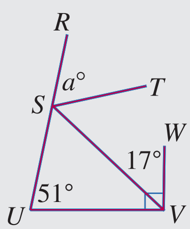
a



b

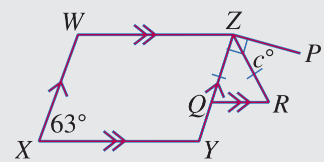


c

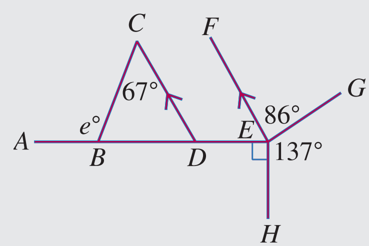


ST bisects $\angle RSV$

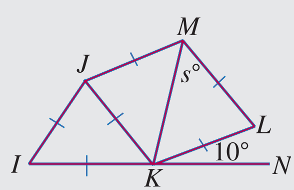
d



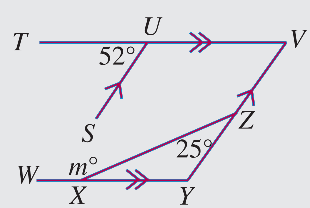
e



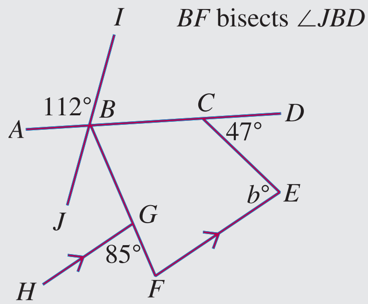
f



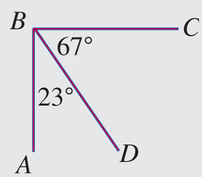
g



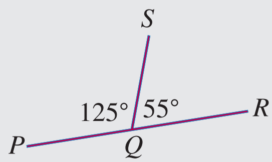
h BF bisects $\angle JBD$



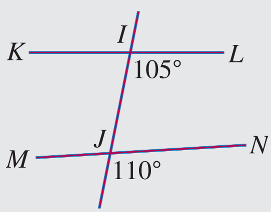
2 a Show that $AB \perp BC$.



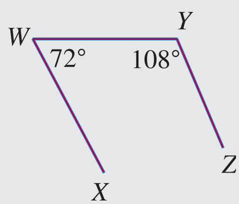
b Show that P, Q, R are collinear points.



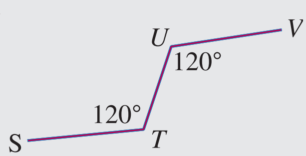
3 a Is $KL \parallel MN$? Explain.



b Is $WX \parallel YZ$? Explain.



c Is $ST \parallel UV$? Explain.



4 How many sides has:

a a heptagon? **b** an undecagon?

5 Find the angle sum and interior angles of a regular:

a hexagon **b** decagon

c pentagon **d** nonagon

e octagon **f** dodecagon

6 Find the exterior angles of a regular:

a pentagon **b** octagon

c hexagon

7 How many sides are there in a regular polygon whose:

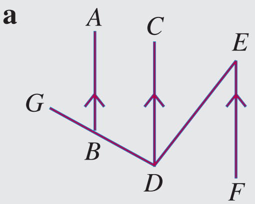
a exterior angles measure 36°?

b interior angles measure 165°?

c angle sum is 2880°?

8 Can a regular polygon have interior angles of 124°? Explain.

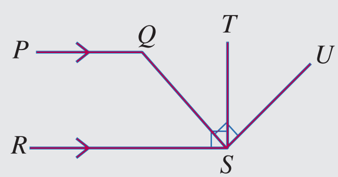
9 a



- $AB \parallel CD \parallel EF$
- $\angle ABG = \angle DEF$

Prove that CD bisects $\angle GDE$.

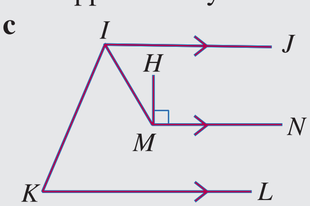
b



- $PQ \parallel RS$
- $TS \perp RS$
- $QS \perp SU$

Prove that $\angle PQS$ and $\angle TSU$ are supplementary.

c



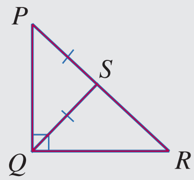
- $IJ \parallel KL \parallel MN$
- $HM \perp MN$
- IM bisects $\angle KIJ$

Prove that $\angle IKL = 2\angle IMH$.

CHAPTER REVIEW

CHAPTER REVIEW

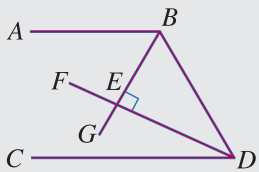
d



- $PQ \perp QR$
- $PS = SQ$

Prove that ΔQRS is isosceles.

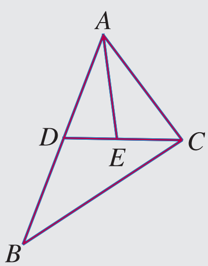
e



- $BG \perp FD$
- BG bisects $\angle ABD$
- FD bisects $\angle BDC$

Prove that $AB \parallel CD$.

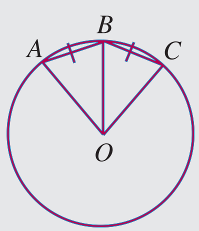
f



- DC bisects $\angle ACB$
- $\angle EAC = \angle ABC$

Prove that ΔADE is isosceles.

10 a

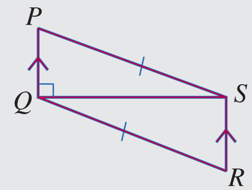


- O is the centre
- $AB = BC$

i Prove that $\Delta OAB \equiv \Delta OCB$.

ii Hence show that OB bisects $\angle AOC$.

b

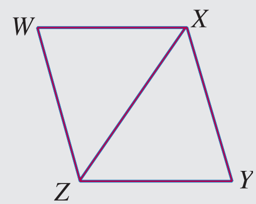


- $PQ \parallel SR$
- $PS = QR$
- $PQ \perp QS$

i Prove that $\Delta PQS \equiv \Delta RSQ$.

ii Hence show that $PQRS$ is a parallelogram.

c

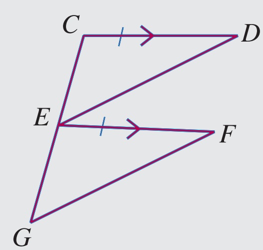


- $\angle XWZ = \angle ZXY$
- XZ bisects $\angle WZY$

i Prove that $\Delta WXZ \equiv \Delta YXZ$.

ii Hence show that ΔYXZ is isosceles.

d

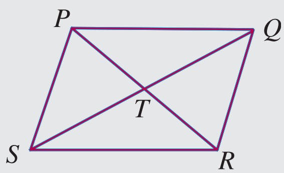


- $CD \parallel EF$
- $CD = EF$
- EF bisects CG

i Prove that $\Delta CDE \equiv \Delta EFG$.

ii Hence show that $DE \parallel FG$.

11

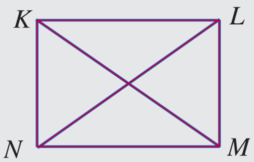


PQRS is a parallelogram.

a Prove that $\Delta PQT \equiv \Delta RST$.

b Hence show that the diagonals bisect each other.

12

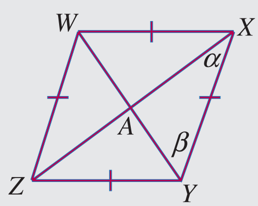


KLMN is a rectangle.

a Prove that $\Delta KNM \equiv \Delta LMN$.

b Hence show that the diagonals of a rectangle are equal.

13

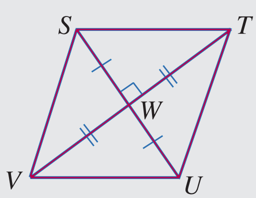


WXYZ is a rhombus. Let $\angle AXY = \alpha$ and $\angle AYX = \beta$.

a Prove that *ZX* bisects $\angle WXY$, without the use of congruent triangles.

b Prove that *XZ* bisects *WY* at right angles, without the use of congruent triangles.

14



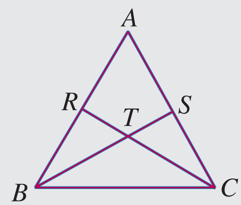
STUV is a quadrilateral in which the diagonals bisect each other at right angles.

a Explain why *STUV* is a parallelogram.

b Prove that $\Delta SVW \equiv \Delta STW$.

c Hence show that *STUV* is a rhombus.

15



- $AB = AC$
- *R* and *S* are midpoints of *AB* and *AC* respectively.

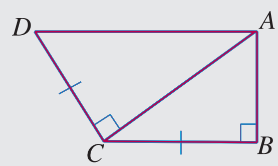
a Show that $BR = CS$.

b Prove that $\Delta RBC \equiv \Delta SCB$.

c Hence show that $CR = BS$.

d If $TR = TS$, show that ΔBTC is isosceles.

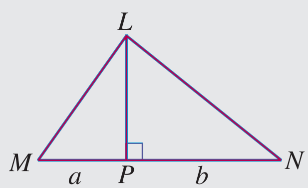
16



- $AB \perp BC$, $DC \perp CA$
- $BC = CD = 2AB$

Prove that $AD = 3AB$.

17



- $LP \perp MN$
- $\dfrac{MP}{LP} = \dfrac{LP}{PN}$

a Show that $LP = \sqrt{ab}$.

b Find expressions for LM^2 and LN^2.

c Hence show that ΔLMN is right-angled.

Chapter Review

5 Factorisation and algebraic fractions

This chapter at a glance

Stage 5.1/5.2/5.3

After completing this chapter, you should be able to:

- expand binomial products
- expand perfect squares using the special identities
- determine whether a given expression is a perfect square
- complete a perfect square
- expand expressions using the difference of two squares identity
- expand expressions that involve a combination of algebraic techniques
- factorise expressions by taking out the highest common factor
- factorise expressions using the difference of two squares identity
- factorise expressions by grouping in pairs
- factorise monic quadratic trinomials
- factorise general quadratic trinomials
- factorise expressions that require a combination of factorisation techniques
- simplify individual algebraic fractions by factorising then cancelling common factors
- multiply and divide algebraic fractions by factorising then cancelling common factors
- add and subtract algebraic fractions by factorising the denominators and finding the lowest common denominator.

5.1 Binomial products review

A binomial product is the product of two binomial expressions. For example, $(x + 3)(x - 1)$, $(2a + 5)(a + 2)$, $(4m - 3n)(2m + 5n)$.

The distributive law

The **distributive law** can be used to expand expressions such as $a(b + c)$. When using the distributive law, the term outside the grouping symbols is multiplied by each term inside the grouping symbols. That is, $a(b + c) = ab + ac$.

To expand a binomial product using the distributive law:
- **multiply the first term in the first factor by each term in the second factor**
- **multiply the second term in the first factor by each term in the second factor.**

$$\begin{aligned}(a + b)(c + d) &= a(c + d) + b(c + d)\\ &= ac + ad + bc + bd\end{aligned}$$

NOTE: The distributive law can be used to expand expressions in which the factors contain more than two terms.

The FOIL method

The **FOIL method** is simply the use of the distributive law without writing the first line of working. The acronym FOIL stands for **F**irst, **O**utside, **I**nside and **L**ast, which is the order in which the terms in the grouping symbols should be multiplied together. This method is also referred to as expansion by **inspection**.

To expand a binomial product using the FOIL method:
- **multiply the first terms**
- **multiply the outside terms**
- **multiply the inside terms**
- **multiply the last terms**
- **collect any like terms.**

$$(a + b)(c + d) = ac + ad + bc + bd$$

Perfect squares

A **perfect square** is the product of two identical expressions. Some examples of perfect squares are:

$$a^2,\ 9k^2,\ (a+b)^2,\ (3m-4n)^2$$

To expand a perfect square of the form $(a+b)^2$ or $(a-b)^2$:
- **square the first term**
- **add or subtract twice the product of the two terms, depending on the sign in the expression**
- **add the square of the last term.**

$$(a+b)^2 = a^2 + 2ab + b^2 \quad \text{and} \quad (a-b)^2 = a^2 - 2ab + b^2$$

NOTE: In the expanded form of the perfect square, the first sign is the same as the sign in the grouping symbols and the last sign is always +.

Proofs:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b) + b(a+b)\\ &= a^2 + ab + ba + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a(a-b) - b(a-b)\\ &= a^2 - ab - ba + b^2\\ &= a^2 - 2ab + b^2\end{aligned}$$

Difference of two squares

When the sum of two terms is multiplied by their difference, the resulting expression is called a difference of two squares.

To expand an expression of the form $(a+b)(a-b)$:
- **square the first term**
- **subtract the square of the second term.**

$$(a+b)(a-b) = a^2 - b^2$$

Proof:

$$\begin{aligned}(a+b)(a-b) &= a(a-b) + b(a-b)\\ &= a^2 - ab + ba - b^2\\ &= a^2 - b^2\end{aligned}$$

Example 1

EG +S

Expand and simplify:

a $(2x+3)(x+5)$ **b** $(x+7)^2$ **c** $(3p-2q)^2$ **d** $(m+4)(m-4)$

Solutions

a $(2x+3)(x+5)$
$= 2x(x+5) + 3(x+5)$
$= 2x^2 + 10x + 3x + 15$
$= 2x^2 + 13x + 15$

b $(x+7)^2$
$= x^2 + (2 \times x \times 7) + 7^2$
$= x^2 + 14x + 49$

c $(3p - 2q)^2$
$= (3p)^2 - (2 \times 3p \times 2q) + (2q)^2$
$= 9p^2 - 12pq + 4q^2$

d $(m + 4)(m - 4)$
$= m^2 - 4^2$
$= m^2 - 16$

Example 2

Complete each perfect square.

a $a^2 + \_\_\_\_\_ + 36 = (\_\_\_\_\_)^2$

b $4e^2 + 12e + \_\_\_\_\_ = (\_\_\_\_\_)^2$

Solutions

a The first and last terms in the grouping symbols are $\sqrt{a^2} = a$ and $\sqrt{36} = 6$, respectively.
The middle term in the trinomial is twice the product of the terms in the grouping symbols.
So, middle term $= 2 \times a \times 6$
$= 12a$
$\therefore\ a^2 + 12a + 36 = (a + 6)^2$

b The first term in the grouping symbols is $\sqrt{4e^2} = 2e$.
The middle term in the trinomial is twice the product of the terms in the grouping symbols.
$\therefore\ 12e = 2 \times 2e \times$ last term
$12e = 4e \times$ last term
$\therefore$ the last term is 3.
The last term in the grouping symbols is $3^2 = 9$.
$\therefore\ 4e^2 + 12e + 9 = (2e + 3)^2$.

Example 3

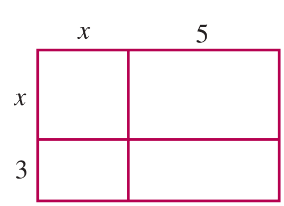
Expand and simplify $(x + 5)(x - 5) - (x - 3)^2$.

Solution

$(x + 5)(x - 5) - (x - 3)^2$
$= x^2 - 25 - (x^2 - 6x + 9)$
$= x^2 - 25 - x^2 + 6x - 9$
$= 6x - 34$

Exercise 5.1

1 Expand $(x + 5)(x + 3)$ by using the area diagram.



2 Expand each binomial product by using the distributive law.

a $(x + 2)(y + 3)$ **b** $(p + 5)(q - 2)$ **c** $(u - 4)(v - 6)$

3 Expand these binomial products by using the distributive law, then simplify.

a $(x+2)(x+4)$ **b** $(m+1)(m+6)$ **c** $(u+8)(u-5)$
d $(b+9)(b-2)$ **e** $(a-8)(a+3)$ **f** $(t-1)(t-2)$
g $(c-4)(c-6)$ **h** $(z-6)(z-7)$ **i** $(d+12)(d-5)$
j $(x+3)(2x+1)$ **k** $(3m+4)(m+2)$ **l** $(5a+2)(a-4)$
m $(2g-5)(g-3)$ **n** $(3t+1)(t-6)$ **o** $(4n-7)(n+5)$
p $(5r+3)(2r+7)$ **q** $(2k+9)(3k-5)$ **r** $(7v-2)(3v-8)$

■ Consolidation

4 Prove the identity $(x+a)(x+b) = x^2 + (a+b)x + ab$.

5 Expand the following expressions by using the identity in question 4.

a $(x+2)(x+1)$ **b** $(p+3)(p+2)$ **c** $(a+7)(a+3)$
d $(t-5)(t-4)$ **e** $(b-6)(b-2)$ **f** $(c-3)(c-4)$
g $(z+4)(z-1)$ **h** $(d+3)(d-6)$ **i** $(s-2)(s+10)$
j $(e+7)(e-2)$ **k** $(u-3)(u+9)$ **l** $(k-11)(k+4)$
m $(f+5)(f+7)$ **n** $(w-6)(w+5)$ **o** $(r-8)(r-4)$
p $(g+8)(g-6)$ **q** $(h+11)(h-5)$ **r** $(v+3)(v+8)$
s $(q-9)(q-4)$ **t** $(m-6)(m+9)$ **u** $(i+3)(i-12)$
v $(l+10)(l-6)$ **w** $(y-13)(y+3)$ **x** $(j-8)(j-7)$

6 Expand and simplify $3(2a+5)(a-4)$.

7 Expand:

a $(a+b)^2$ **b** $(m+n)^2$ **c** $(g-h)^2$ **d** $(p-q)^2$

8 Expand:

a $(a+3)^2$ **b** $(p-4)^2$ **c** $(c+6)^2$ **d** $(t-1)^2$
e $(u-5)^2$ **f** $(k+10)^2$ **g** $(s+7)^2$ **h** $(p-9)^2$

9 Expand:

a $(2x+3)^2$ **b** $(3t-5)^2$ **c** $(4m+1)^2$
d $(5c-2)^2$ **e** $(7-2g)^2$ **f** $(5+6r)^2$
g $(3p-q)^2$ **h** $(3a+4b)^2$ **i** $(2e-11f)^2$
j $(4c+5d)^2$ **k** $(7g-3h)^2$ **l** $(12j+5k)^2$
m $(pq+r)^2$ **n** $(6ab-7cd)^2$ **o** $\left(x+\frac{1}{x}\right)^2$

10 State whether each expression is a perfect square.

a k^2+4 **b** $(a+5)^2$ **c** $u^2+4u+16$
d $t^2+26t+169$ **e** a^2b^2 **f** $c^2+12c-36$
g n^2-1 **h** $4e^2-6e+9$ **i** x^2+y^2+2xy

11 Complete these perfect squares.

a $(m + 5)^2 = m^2 + 10m +$ _____
b $(p - 3)^2 = p^2 -$ _____ $+ 9$
c $(\_\_\_\_\_)^2 = y^2 +$ _____ $+ 36$
d $(\_\_\_\_\_)^2 = a^2 -$ _____ $+ 49$
e $(\_\_\_\_\_)^2 = k^2 + 8k +$ _____
f $(\_\_\_\_\_)^2 = u^2 - 2u +$ _____
g $(2b + 3)^2 = 4b^2 +$ _____ $+ 9$
h $(\_\_\_\_\_)^2 = 9e^2 +$ _____ $+ 25$
i $(\_\_\_\_\_)^2 = 25n^2 + 60n +$ _____
j $(\_\_\_\_\_)^2 = 49z^2 - 42z +$ _____

12 Use the difference of two squares identity to expand each of the following.

a $(m + n)(m - n)$
b $(p - 2)(p + 2)$
c $(r + 5)(r - 5)$
d $(3 - g)(3 + g)$
e $(y - 7)(y + 7)$
f $(4 - w)(4 + w)$
g $(t + 1)(t - 1)$
h $(k - 6)(k + 6)$
i $(10 - b)(10 + b)$
j $(2e + 3)(2e - 3)$
k $(5r - 2)(5r + 2)$
l $(3c - 1)(3c + 1)$
m $(4 - 7h)(4 + 7h)$
n $(6 + 5n)(6 - 5n)$
o $(2p - 9q)(2p + 9q)$
p $(10y + 3z)(10y - 3z)$
q $(7s + 6t)(7s - 6t)$
r $(ab - 8c)(ab + 8c)$

Further applications

13 Expand and simplify the following expressions.

a $(a + 7)(a + 2) + 4$
b $(t - 6)(t + 6) - 10$
c $4p + (2p - 3)(p - 2)$
d $3e^2 + (e - 4)^2 - 6$
e $(a + 5)^2 - 2(a - 5)$
f $(x + 4)(x + 6) + (x + 1)^2$
g $(m + 4)(m - 4) - (m + 3)^2$
h $(y + 7)^2 - (y + 7)(y - 7)$
i $(3k - 4)(k - 2) - (2k - 1)(k + 6)$
j $(5c + 6)^2 - (5c - 6)^2$
k $(2h + 7)^2 - (2h + 3)(2h - 5)$
l $(4m - 3)(4m + 3) - (4m - 3)^2$

14 Expand and simplify:

a $(x + 2)(x^2 + 3x + 4)$
b $(a - 4)(a^2 - 2a + 5)$
c $(k + 3)(k + 2)(k + 1)$
d $(n - 7)(n + 3)(n - 2)$
e $(p + 1)(p + 7)^2$
f $(y - 5)(y - 3)^2$
g $(t + 2)^3$
h $(e - 3)^3$

5.2 The highest common factor

To **factorise** an expression means to write the expression as the product of its factors. This is the same as reversing or undoing the expansion process.

Expanding

$$\boldsymbol{a(b + c) = ab + ac}$$

Factorising

Many expressions can be factorised in several ways. For example, we can factorise $6a + 12$ as $1(6a + 12)$ or $2(3a + 6)$ or $3(2a + 4)$ or $6(a + 2)$. However, by convention, we take out the **highest common factor (HCF)** when factorising an expression. That is, we take out the *greatest possible factor* that is *common to every term* in the expression. In this example, the HCF of $6a$ and 12 is 6. Therefore, the correct factorisation of $6a + 12$ is $6(a + 2)$.

To factorise an algebraic expression:
- **write the HCF of the terms outside the grouping symbols**
- **divide each term in the expression by the HCF to find the terms inside the grouping symbols.**

$$ab + ac = a(b + c) \quad \text{and} \quad ab - ac = a(b - c)$$

NOTE:
- If the first term of an expression is negative, then by convention, the HCF is also negative.
- Factorisations should be checked by expanding the answers.

EG +S

Example 1

Factorise:

a $5p + 20$ **b** $8n - 20$ **c** $m^2 + 3m$
d $12h^2 + 27h$ **e** $c^2d - cd^2$ **f** $-uv + vw$

Solutions

a $5p + 20 = 5(p + 4)$

b $8n - 20 = 4(2n - 5)$

c $m^2 + 3m = m(m + 3)$

d $12h^2 + 27h = 3h(4h + 9)$

e $c^2d - cd^2 = cd(c - d)$

f $-uv + vw = -v(u - w)$

Exercise 5.2

1 Factorise each of these expressions by taking out the highest common factor.

a $2c + 6$ **b** $5m + 20$ **c** $21 + 3e$ **d** $49 + 7x$
e $3g - 15$ **f** $6k - 6$ **g** $27 - 3r$ **h** $55 - 5t$
i $2x + 2y$ **j** $3m - 12n$ **k** $24p + 8q$ **l** $12f - 48g$
m $xy + xz$ **n** $ab - bc$ **o** $m^2 + 7m$ **p** $c^2 - c$

2 Factorise by removing the highest common factor.

a $4n + 6$ **b** $9b + 15$ **c** $12y - 8$ **d** $10u - 25$
e $21p + 28$ **f** $15g - 18$ **g** $14w + 21$ **h** $24z - 30$
i $44h - 33$ **j** $35d + 49$ **k** $20q - 50$ **l** $44f - 77$
m $25 - 45k$ **n** $56 + 63v$ **o** $40 - 64a$ **p** $24 - 84s$

Consolidation

3 Factorise each expression completely.

a $3xy + 12yz$ **b** $2pq + 10pr$ **c** $5fg - 15gh$ **d** $7cd - 42ce$

e $6ij + 9jk$ **f** $12mn - 20np$ **g** $22uv - 33vw$ **h** $32st + 24tu$
i $6b^2 + 24b$ **j** $21a^2 - 7a$ **k** $14q^2 - 20q$ **l** $25u + 30u^2$
m $pqr + pqs$ **n** $cde - def$ **o** $x^2y + xy^2$ **p** $abc - ab^2$
q $fg^2 - f^2gh$ **r** $6jk + 15k^2m$ **s** $16tu^2 + 20t^2uv$ **t** $28g^2h^2 - 49ghi$

4 Factorise:
a $4p + 4q + 4r$ **b** $ab + ac - ad$ **c** $x^2 - xy + xz$
d $3e + 6f + 15g$ **e** $6m^2 - 10m + 14n$ **f** $5 - 15v - 30v^2$
g $30c^2 - 12c + 18$ **h** $11ab - a + ab^2$ **i** $3r^2 - 3rs - 9r$
j $20y - 25yz + 10y^2$ **k** $7ij + 14i^2 - 42ik$ **l** $24z - 60yz - 96z^2$
m $p^2q + pq + pq^2$ **n** $9rs - rs^2 - r^2s$ **o** $a^2bc + ab^2c - abc^2$

5 Factorise by taking out the greatest negative common factor.
a $-2k - 8$ **b** $-3n - 15$ **c** $-10c - 25$ **d** $-14w - 21$
e $-4x + 4$ **f** $-9d + 18$ **g** $-16m + 20$ **h** $-9g + 21$
i $-18 - 27y$ **j** $-36 + 15e$ **k** $-32 - 40z$ **l** $-44 + 99t$
m $-cd - de$ **n** $-ij + jk$ **o** $-a^2 - 7a$ **p** $-11v + v^2$
q $-8n^2 + 20n$ **r** $-21b - 24b^2$ **s** $-20f + 36ef$ **t** $-26c^2d - 39cd^2$

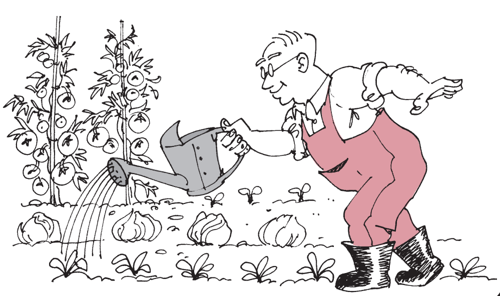
Further applications

6 Factorise by taking out the highest common factor only.
a $m^2 + m^3$ **b** $x^6 + x^2$ **c** $t^3 - t^8$
d $y^{10} - y^3$ **e** $a^4 + 3a^7$ **f** $2g^3 - g^8$
g $7u^9 + 4u^5$ **h** $5h^3 - 4h^4$ **i** $3c^{10} + 9c^5$
j $22g^8 + 11g^3$ **k** $16q^7 - 24q^{13}$ **l** $45z^9 - 20z^2$

TRY THIS **Market garden**

A market gardener has two vegetable plots, one square and the other a rectangle with one side 3 m shorter than the side of the square, and the other side 4 m longer than the side of the square. Both plots have the same area.

1 Draw a diagram
2 Find the dimensions of each plot.



5.3 Difference of two squares

In exercise 5.1 it was shown that sum by difference products such as $(a - b)(a + b)$ could be expanded, giving a difference of two squares, $a^2 - b^2$. This procedure can be used in reverse to factorise a difference of two squares.

$$\mathbf{a^2 - b^2 = (a - b)(a + b)}$$

Example 1

Factorise each of the following.

a $x^2 - 16$ **b** $9q^2 - 25$ **c** $121u^2 - 49v^2$

Solutions

a $x^2 - 16$
$= (x - 4)(x + 4)$

b $9q^2 - 25$
$= (3q)^2 - 5^2$
$= (3q - 5)(3q + 5)$

c $121u^2 - 49v^2$
$= (11u)^2 - (7v)^2$
$= (11u - 7v)(11u + 7v)$

Example 2

Factorise completely:

a $5y^2 - 45$ **b** $7c^3 - 28cd^2$

Solutions

In general, when factorising expressions, always remove the highest common factor first.

a $5y^2 - 45 = 5(y^2 - 9)$
$= 5(y - 3)(y + 3)$

b $7c^3 - 28cd^2 = 7c(c^2 - 4d^2)$
$= 7c(c - 2d)(c + 2d)$

Exercise 5.3

1 **a** Expand $(x + 3)(x - 3)$. Hence, factorise $x^2 - 9$.
b Expand $(2x + 5)(2x - 5)$. Hence, factorise $4x^2 - 25$.

2 Factorise each of these as a difference of two squares.
a $p^2 - q^2$ **b** $c^2 - d^2$ **c** $m^2 - n^2$ **d** $u^2 - v^2$

3 Factorise each expression as a difference of two squares.
a $x^2 - 4$ **b** $a^2 - 25$ **c** $p^2 - 1$ **d** $y^2 - 36$
e $z^2 - 64$ **f** $c^2 - 121$ **g** $t^2 - 49$ **h** $b^2 - 81$
i $16 - k^2$ **j** $64 - g^2$ **k** $100 - m^2$ **l** $144 - u^2$

4 Factorise:
a $e^2 - 169$ **b** $h^2 - 256$ **c** $529 - s^2$ **d** $361 - j^2$

■ Consolidation

5 Factorise:
a $4a^2 - 9$ **b** $9p^2 - 25$ **c** $49q^2 - 4$ **d** $16c^2 - 1$
e $36 - 49y^2$ **f** $25 - 16r^2$ **g** $1 - 121u^2$ **h** $100 - 81t^2$
i $9a^2 - b^2$ **j** $x^2 - 4y^2$ **k** $36e^2 - f^2$ **l** $j^2 - 64k^2$
m $4g^2 - 49h^2$ **n** $25m^2 - 144n^2$ **o** $100p^2 - 9q^2$ **p** $36s^2 - 121t^2$
q $a^2b^2 - c^2$ **r** $p^2 - q^2r^2$ **s** $4x^2 - 9y^2z^2$ **t** $16e^2f^2 - 81g^2h^2$

6 Factorise these expressions completely by first taking out the highest common factor.
a $2m^2 - 18$ **b** $3a^2 - 12$ **c** $11t^2 - 11$ **d** $5y^2 - 500$
e $75 - 3x^2$ **f** $63 - 7p^2$ **g** $128 - 2e^2$ **h** $54 - 6z^2$

i	$8n^2 - 18$	**j**	$12c^2 - 27$	**k**	$45 - 20f^2$	**l**	$18k^2 - 50$
m	$a^3 - a$	**n**	$n^2 - n^4$	**o**	$49d^3 - d$	**p**	$16u - 25u^3$
q	$8h^3 - 32h$	**r**	$54w^3 - 24w$	**s**	$72s - 32s^3$	**t**	$80j^3 - 125j$

7 **a** Factorise $4a^2 - 36$ as a difference of two squares. Has it been completely factorised? Why?
b Discuss the correct method for factorising this expression, then factorise it completely.

8 Factorise completely each of these expressions.
a $9k^2 - 36$ **b** $4c^2 - 100$ **c** $25x^2 - 100y^2$ **d** $4e^2 - 144f^2$

9 Evaluate $16^2 - 15^2$ by first expressing it as $(16 - 15)(16 + 15)$.

10 Use the method in question 9 to evaluate:
a $17^2 - 49$ **b** $99^2 - 1$ **c** $45^2 - 25$

Further applications

11 Factorise each expression as a difference of two squares.
a $(a + b)^2 - c^2$ **b** $(m - n)^2 - p^2$ **c** $(x + 3)^2 - y^2$
d $(j - k)^2 - 36$ **e** $(b + 5)^2 - 49$ **f** $(p + 2)^2 - 4$
g $4(m + n)^2 - 9$ **h** $9(c - d)^2 - 25$ **i** $81(3p + 2q)^2 - 64r^2$

5.4 Grouping in pairs

Expressions that contain four terms can often be factorised by grouping the terms in pairs.

To factorise a four-term expression:
- **group the expression in pairs such that each pair has a common factor**
- **factorise both pairs of expressions**
- **factorise this overall expression by taking out a binomial common factor.**

$$\begin{aligned} ac + bc + ad + bd &= c(a + b) + d(a + b) \\ &= (a + b)(c + d) \end{aligned}$$

NOTE: In some questions it may be necessary to first re-arrange the terms into a different order.

Example 1

EG +S

Factorise:
a $pq + 7p + 3q + 21$ **b** $mn - mp - 4n + 4p$ **c** $xy - yz + x - z$

Solutions

a $pq + 7p + 3q + 21$
$= p(q + 7) + 3(q + 7)$
$= (q + 7)(p + 3)$

b $mn - mp - 4n + 4p$
$= m(n - p) - 4(n - p)$
$= (n - p)(m - 4)$

c $xy - yz + x - z$
$= y(x - z) + 1(x - z)$
$= (x - z)(y + 1)$

Example 2

Factorise $e^2 + 5g + eg + 5e$ by grouping in pairs.

Solution

The expression must be re-arranged since the first pair of terms does not have a common factor.

$$\begin{aligned} e^2 + 5g + eg + 5e &= e^2 + eg + 5e + 5g \\ &= e(e + g) + 5(e + g) \\ &= (e + g)(e + 5) \end{aligned}$$

Exercise 5.4

1 Factorise each of the following by taking out the binomial common factor.

a $a(c + d) + b(c + d)$	**b** $x(x + 3) + 2(x + 3)$	**c** $3n(p - 4) + 7(p - 4)$
d $w(x + y) - z(x + y)$	**e** $t(t - 1) - 9(t - 1)$	**f** $4g(a - 8) - 5h(a - 8)$
g $p(q + r) + s(r + q)$	**h** $5(a^2 + 2) - c(a^2 + 2)$	**i** $mn\,(y + 4) - 8(4 + y)$
j $u(d + e) + (d + e)$	**k** $(p - 2q) + w(p - 2q)$	**l** $6h(7i - 2j) - (7i - 2j)$

Consolidation

2 Factorise each expression by taking out the highest common factor from each pair of terms.

a $km + kn + 5m + 5n$	**b** $ac + ad + bc + bd$	**c** $p^2 + pq + 2p + 2q$
d $wx + 3x + 2wy + 6y$	**e** $ce - 2c + 4de - 8d$	**f** $2gh - 2g + 3h - 3$
g $4uv + 28u + 5v + 35$	**h** $3x + 6y + 8ax + 16ay$	**i** $5mn - 15m + 6pn - 18p$
j $3gk + 18gh + 4k + 24h$	**k** $pqr + 6pq + 7r + 42$	**l** $mn + np + m + p$
m $a - 5b + ac - 5bc$	**n** $3e + f + 3e^2 + ef$	**o** $a^3 + a^2 + a + 1$

3 Factorise each of these by taking out a negative common factor from the second pair of terms.

a $2c + 2d - ce - de$	**b** $pq + ps - qr - rs$	**c** $p^2 - pq - 8p + 8q$
d $xy - 3x - y^2 + 3y$	**e** $gh + gi - h - i$	**f** $3u - 2v - 3uw + 2vw$
g $x^3 - 4x^2 - xy + 4y$	**h** $k^2 + 7k - 2km - 14m$	**i** $jkm - 4jk - 2m + 8$
j $35np - 14nq - 5p + 2q$	**k** $3z^2 - 3wz - uz + uw$	**l** $10x + 15y - 4xz - 6yz$

4 Factorise each expression by first re-arranging the terms into a more suitable order.

a $xy + wz + xz + wy$	**b** $pr + 2q + qr + 2p$	**c** $3mn + kp + kn + 3mp$
d $4yz + 15 + 3z + 20y$	**e** $cd + 7 + 7c + d$	**f** $6 + 5ef + 6e + 5f$

Further applications

5 Explain why $a(b - c) = -a(c - b)$. Hence, factorise each of the following.

a $2x - xy + y^2 - 2y$	**b** $yz - 7y + 7z - z^2$	**c** $ab - ac - bc + b^2$
d $a^3 - a^2b + b^2 - ab$	**e** $11m - mn - n^2 + 11n$	**f** $m - 4n - 20pn + 5mp$
g $6uv - 6uw - 5w^2 + 5vw$	**h** $14de - 7ce + c - 2d$	**i** $2wx - 2wy + y^2 - xy$
j $c^3 - c^2d - d^2 + cd$	**k** $10pq - 25pr + 30qr - 12q^2$	**l** $18rs - 63rt - 28st + 8s^2$

6 Factorise:

a $(x+3)(x+4)+y(x+4)$
b $(m-2)(m-5)+n(m-2)$
c $(k+6)(k-4)-m(k-4)$
d $(a+b)^2+c(a+b)$
e $(x-7)^2-w(x-7)$
f $f(g+h)+(g+h)^2$

7 Factorise:

a $(2c-d)(c+3d)-5(d-2c)$
b $(p-q)^2-3r(q-p)$

TRY THIS **Squaring fives**

Consider the following:

$15^2 = 10 \times 20 + 25 = 225$, $25^2 = 20 \times 30 + 25 = 625$, $35^2 = 30 \times 40 + 25 = 1225$.
Why does this work?

Use algebra to show why this works. (NOTE: A number *mn* equals $10 \times m + n$).
Does the rule work for numbers greater than 125, e.g. $(125)^2$?

5.5 Factorising monic quadratic trinomials

Expressions such as $2a + b - 5c$, which have three terms, are called **trinomials**. Trinomials such as $x^2 + 4x + 7$ and $3x^2 - 2x + 8$ are called **quadratic trinomials** because the highest power of the variable is 2. If the co-efficient of x^2 is 1, the trinomial is said to be **monic**. Hence, $x^2 + 4x + 7$ is a monic quadratic trinomial.

$$\begin{aligned} \text{Let } x^2 + bx + c &= (x+p)(x+q) \\ \therefore\ x^2 + bx + c &= x^2 + px + qx + pq \\ &= x^2 + (p+q)x + pq \end{aligned}$$

Equating the co-efficients of x we have $p + q = b$ and equating the constants we have $pq = c$.

$$(x+p)(x+q) = x^2 + (p+q)x + pq$$

To factorise a monic quadratic trinomial $x^2 + bx + c$:
- **find by inspection two numbers p, q such that $p + q = b$ and $pq = c$**
- **factorise the trinomial as $(x+p)(x+q)$.**

NOTE: In the trinomial $x^2 + bx + c$:
1. if $c > 0$, then the numbers p, q both have the same sign as b
2. if $c < 0$, then the numbers p, q have different signs, with the larger number having the same sign as b.

Example 1

Factorise:

a $x^2 + 9x + 20$ **b** $x^2 - 7x + 12$ **c** $x^2 + 2x - 3$ **d** $x^2 - 4x - 12$

Solutions

a $4 + 5 = 9$ and $4 \times 5 = 20$
$\therefore x^2 + 9x + 20 = (x + 4)(x + 5)$

b $(-3) + (-4) = -7$ and $(-3) \times (-4) = 12$
$\therefore x^2 - 7x + 12 = (x - 3)(x - 4)$

c $(-1) + 3 = 2$ and $(-1) \times 3 = -3$
$\therefore x^2 + 2x - 3 = (x - 1)(x + 3)$

d $(-6) + 2 = -4$ and $(-6) \times 2 = -12$
$\therefore x^2 - 4x - 12 = (x - 6)(x + 2)$

Example 2

Factorise $3x^2 + 21x + 30$

Solution

$$\begin{aligned} 3x^2 + 21x + 30 &= 3(x^2 + 7x + 10) \\ &= 3(x + 5)(x + 2) \end{aligned}$$

Exercise 5.5

1 Find two integers a and b such that:

a $a + b = 8$, $ab = 15$
b $a + b = 9$, $ab = 14$
c $a + b = 10$, $ab = 24$
d $a + b = 13$, $ab = 30$
e $a + b = -7$, $ab = 10$
f $a + b = -3$, $ab = 2$
g $a + b = -11$, $ab = 28$
h $a + b = -17$, $ab = 72$
i $a + b = 4$, $ab = -12$
j $a + b = 1$, $ab = -20$
k $a + b = -2$, $ab = -35$
l $a + b = -3$, $ab = -54$

2 **a** Show that $(x + a)(x + b) = x^2 + (a + b)x + ab$.
b What can you say about the signs of a and b, where $a > b$ if:
i $ab > 0$ and $a + b > 0$? **ii** $ab > 0$ and $a + b < 0$?
iii $ab < 0$ and $a + b > 0$? **iv** $ab < 0$ and $a + b < 0$?

Consolidation

3 Factorise each of these monic trinomials.

a $x^2 + 5x + 6$ **b** $y^2 + 7y + 12$ **c** $u^2 + 3u + 2$ **d** $m^2 + 10m + 24$
e $a^2 + 10a + 21$ **f** $t^2 + 9t + 18$ **g** $k^2 + 9k + 14$ **h** $p^2 + 8p + 7$
i $n^2 + 14n + 45$ **j** $d^2 + 12d + 32$ **k** $s^2 + 11s + 30$ **l** $b^2 + 13b + 36$
m $e^2 + 15e + 50$ **n** $c^2 + 17c + 60$ **o** $r^2 + 19r + 88$ **p** $z^2 + 20z + 96$

4 Factorise each of the following:

a $m^2 - 8m + 15$ **b** $q^2 - 7q + 10$ **c** $d^2 - 12d + 20$ **d** $a^2 - 12a + 35$
e $u^2 - 11u + 18$ **f** $e^2 - 9e + 8$ **g** $n^2 - 13n + 40$ **h** $w^2 - 9w + 20$
i $h^2 - 16h + 63$ **j** $v^2 - 11v + 24$ **k** $t^2 - 7t + 6$ **l** $s^2 - 15s + 44$
m $k^2 - 13k + 30$ **n** $j^2 - 18j + 77$ **o** $x^2 - 15x + 54$ **p** $f^2 - 14f + 48$

5 Factorise each of these.

a $y^2 + 4y - 12$	**b** $d^2 + 2d - 15$	**c** $a^2 + 3a - 4$	**d** $p^2 + 6p - 16$
e $v^2 - 3v - 10$	**f** $u^2 - 7u - 18$	**g** $m^2 - m - 30$	**h** $x^2 - 4x - 5$
i $f^2 + 4f - 21$	**j** $w^2 - 2w - 63$	**k** $k^2 - 6k - 27$	**l** $c^2 + 4c - 32$
m $z^2 - 3z - 40$	**n** $i^2 + i - 12$	**o** $r^2 - 2r - 99$	**p** $e^2 - 5e - 14$
q $s^2 + 3s - 70$	**r** $h^2 + h - 56$	**s** $b^2 - 6b - 40$	**t** $t^2 + 3t - 108$

6 Factorise:

a $n^2 - 6n + 8$	**b** $c^2 + 13c + 42$	**c** $x^2 - x - 2$	**d** $d^2 + 7d - 44$
e $q^2 + 13q + 22$	**f** $t^2 - 11t + 10$	**g** $v^2 + 14v + 48$	**h** $j^2 - 6j - 72$
i $g^2 - 8g - 20$	**j** $b^2 + 15b + 36$	**k** $r^2 - 3r - 54$	**l** $u^2 - 13u + 12$
m $e^2 + 22e + 120$	**n** $l^2 - l - 42$	**o** $y^2 - 14y + 24$	**p** $p^2 + 6p - 55$
q $z^2 - 2z - 80$	**r** $a^2 + 4a - 60$	**s** $f^2 - 16f + 39$	**t** $m^2 + 21m + 110$
u $w^2 - 5w - 36$	**v** $k^2 + 15k + 26$	**w** $h^2 - 19h + 84$	**x** $i^2 + i - 132$

7 Factorise each of these perfect squares.

a $p^2 + 6p + 9$	**b** $c^2 + 10c + 25$	**c** $g^2 - 8g + 16$	**d** $y^2 - 14y + 49$
e $t^2 + 2t + 1$	**f** $r^2 - 12r + 36$	**g** $x^2 - 18x + 81$	**h** $j^2 + 22j + 121$

8 Factorise each expression completely by first taking out the highest common factor.

a $2m^2 - 20m + 32$	**b** $3k^2 + 24k + 36$	**c** $4a^2 + 24a - 64$
d $5c^2 - 30c - 135$	**e** $3t^2 - 27t + 24$	**f** $7x^2 - 7x - 84$
g $4d^2 - 32d + 60$	**h** $5n^2 + 40n - 100$	**i** $10y^2 - 80y - 330$

9 Find what:

a $x + 3$ must be multiplied by to give $x^2 + 9x + 18$

b $g - 3$ must be multiplied by to give $g^2 + 2g - 15$

Further applications

10 Factorise each expression completely.

a $x^4 - 5x^2 + 4$	**b** $x^4 - 29x^2 + 100$	**c** $x^4 - 11x^2 + 18$	**d** $x^4 + x^2 - 20$
e $x^4 + 7x^2 - 8$	**f** $x^4 - 10x^2 + 9$	**g** $x^4 - 19x^2 + 48$	**h** $x^4 - 13x^2 + 36$
i $x^4 - 17x^2 + 16$	**j** $x^4 - 7x^2 + 6$	**k** $x^4 + 3x^2 - 28$	**l** $x^4 - 25x^2 + 144$

5.6 Factorising general quadratic trinomials

In quadratic trinomials such as $3x^2 + 5x + 2$ and $4 + 7x - 5x^2$, the co-efficient of x^2 is not 1. Such trinomials are often referred to as **general quadratic trinomials**. The identity $x^2 + (a + b)x + ab = (x + a)(x + b)$ cannot be used to factorise quadratic trinomials where the leading co-efficient is a number other than 1. In these situations we must resort to a *guess and check* approach. This is made easier by the use of the cross-method.

To factorise a general quadratic trinomial $ax^2 + bx + c$ using the cross method:

- **draw a cross**
- **on the left-hand side of the cross, write two factors that multiply to give ax^2**
- **on the right-hand side of the cross, write two factors that multiply to give c**
- **check that the sum of the cross-products gives the middle term bx**
- **if the cross-products do not give the correct sum, try different combinations until the correct sum is obtained**
- **factorise the trinomial.**

$$ax^2 + bx + c = (px + r)(qx + s)$$

where • $a = pq$ **and** $c = rs$

• $b = ps + qr$

Example 1

Factorise $3x^2 + 14x + 8$ using the cross method.

Solution

$3x$ $+1$ x $+8$	$(3x \times 8) + (x \times 1)$ $= 24x + x$ $= 25x$	This combination does not give the middle term in the trinomial, so it is not correct.
$3x$ $+8$ x $+1$	$(3x \times 1) + (x \times 8)$ $= 3x + 8x$ $= 11x$	This combination does not give the middle term in the trinomial, so it is not correct.
$3x$ $+4$ x $+2$	$(3x \times 2) + (x \times 4)$ $= 6x + 4x$ $= 10x$	This combination does not give the middle term in the trinomial, so it is not correct.
$3x$ $+2$ x $+4$	$(3x \times 4) + (x \times 2)$ $= 12x + 2x$ $= 14x$	This combination gives the correct middle term in the trinomial.

$\therefore 3x^2 + 14x + 8 = (3x + 2)(x + 4)$

Example 2

Factorise:

a $2x^2 - 13x + 21$ **b** $5x^2 - 4x - 12$

Solutions

a $\begin{array}{cc} 2x & -7 \\ x & -3 \end{array}$

$(2x \times -3) + (x \times -7)$
$= -6x + (-7x)$
$= -13x$

This is equal to the middle term in the trinomial.
$\therefore\ 2x^2 - 13x + 21 = (2x - 7)(x - 3)$

b $\begin{array}{cc} 5x & +6 \\ x & -2 \end{array}$

$(5x \times -2) + (x \times 6)$
$= -10x + 6x$
$= -4x$

This is equal to the middle term in the trinomial.
$\therefore\ 5x^2 - 4x - 12 = (5x + 6)(x - 2)$

Exercise 5.6

1 Which diagram correctly shows the factors of $2x^2 + 13x + 21$?

A $\begin{array}{cc} 2x & +3 \\ x & +7 \end{array}$ **B** $\begin{array}{cc} 2x & +7 \\ x & +3 \end{array}$ **C** $\begin{array}{cc} 2x & +1 \\ x & +21 \end{array}$ **D** $\begin{array}{cc} 2x & +21 \\ x & +1 \end{array}$

2 Which diagram correctly shows the factors of $3x^2 - 17x + 10$?

A $\begin{array}{cc} 3x & +5 \\ x & -2 \end{array}$ **B** $\begin{array}{cc} 3x & -5 \\ x & -2 \end{array}$ **C** $\begin{array}{cc} 3x & -2 \\ x & -5 \end{array}$ **D** $\begin{array}{cc} 3x & +2 \\ x & -5 \end{array}$

3 Which diagram correctly shows the factors of $5x^2 + 7x - 6$?

A $\begin{array}{cc} 5x & +6 \\ x & -1 \end{array}$ **B** $\begin{array}{cc} 5x & -6 \\ x & +1 \end{array}$ **C** $\begin{array}{cc} 5x & +3 \\ x & -2 \end{array}$ **D** $\begin{array}{cc} 5x & -3 \\ x & +2 \end{array}$

4 Which diagram correctly shows the factors of $2x^2 - 3x - 27$?

A $\begin{array}{cc} 2x & -9 \\ x & +3 \end{array}$ **B** $\begin{array}{cc} 2x & +9 \\ x & -3 \end{array}$ **C** $\begin{array}{cc} 2x & -3 \\ x & +9 \end{array}$ **D** $\begin{array}{cc} 2x & +3 \\ x & -9 \end{array}$

Consolidation

5 Factorise each of these trinomials.

a $2x^2 + 7x + 3$ **b** $3x^2 + 14x + 8$ **c** $4x^2 + 11x + 7$
d $2x^2 + 11x + 5$ **e** $5x^2 + 22x + 8$ **f** $7x^2 + 36x + 5$
g $2x^2 + 21x + 54$ **h** $7x^2 + 27x + 18$ **i** $3x^2 + 20x + 32$
j $2x^2 - 5x + 2$ **k** $3x^2 - 11x + 6$ **l** $5x^2 - 17x + 6$

m $2x^2 - 13x + 15$ **n** $3x^2 - 16x + 16$ **o** $5x^2 - 16x + 12$
p $3x^2 - 26x + 35$ **q** $2x^2 - 19x + 44$ **r** $7x^2 - 41x + 30$

6 Factorise each of these trinomials.
a $3x^2 + 5x - 2$ **b** $2x^2 + x - 10$ **c** $2x^2 + 7x - 15$
d $3x^2 + 2x - 21$ **e** $2x^2 - 5x - 25$ **f** $3x^2 - 22x - 16$
g $5x^2 - 11x - 36$ **h** $11x^2 - 52x - 15$ **i** $4x^2 + 27x - 7$
j $6x^2 - 17x - 3$ **k** $5x^2 + 4x - 12$ **l** $4x^2 - 17x - 42$
m $7x^2 + 51x - 40$ **n** $7x^2 + 36x - 36$ **o** $4x^2 + 9x - 28$

7 Factorise:
a $2k^2 + 21k + 40$ **b** $2c^2 + 3c - 27$ **c** $4n^2 - 17n + 18$
d $3y^2 - 17y - 28$ **e** $7p^2 - 31p + 12$ **f** $6a^2 - 23a - 4$
g $4b^2 + 31b + 42$ **h** $5u^2 + 3u - 36$ **i** $2w^2 - 9w - 110$
j $5h^2 + 19h + 12$ **k** $3j^2 + 2j - 40$ **l** $6l^2 - 59l + 45$

8 Factorise the following trinomials.
a $6a^2 + 13a + 6$ **b** $4n^2 - 20n + 21$ **c** $8k^2 + 14k - 9$
d $9p^2 - 9p - 10$ **e** $6c^2 + 19c + 15$ **f** $4e^2 + 8e - 45$
g $12t^2 - 28t + 15$ **h** $10b^2 + 29b - 72$ **i** $12m^2 - 47m + 40$
j $20y^2 + 33y + 10$ **k** $27w^2 - 12w - 55$ **l** $20q^2 + 7q - 6$

9 Factorise:
a $28 + 3x - x^2$ **b** $15 - a - 2a^2$ **c** $24 - 14p - 3p^2$
d $15 - 28m + 12m^2$ **e** $21 + 43g + 20g^2$ **f** $56 + 19w - 10w^2$

10 Factorise each expression completely by first taking out the highest common factor.
a $4k^2 + 18k + 20$ **b** $18p^2 - 33p + 9$ **c** $20a^2 + 2a - 6$
d $12v^2 - 24v - 63$ **e** $24f^2 - 44f + 12$ **f** $10e^2 + 15e - 100$

11 Find what:
a $a + 9$ must be multiplied by to give $3a^2 + 31a + 36$
b $3n - 8$ must be multiplied by to give $6n^2 + 11n - 72$

■ Further applications

Quadratic trinomials can be factorised using the following identity:

$$ax^2 + bx + c = \frac{(ax + m)(ax + n)}{a}, \text{ where } m + n = b \text{ and } mn = ac.$$

For example, to factorise $2x^2 + 7x + 6$, we first find m, n such that $m + n = 7$ and $mn = 2 \times 6$, i.e. $mn = 12$. Therefore, $m = 4$, $n = 3$ (or $m = 3$, $n = 4$).

$$\begin{aligned}\text{So, } 2x^2 + 7x + 6 &= \frac{(2x + 4)(2x + 3)}{2} \\ &= \frac{2(x + 2)(2x + 3)}{2} \\ &= (x + 2)(2x + 3)\end{aligned}$$

12 Use this method to factorise the following trinomials.
a $3x^2 + 10x + 3$ **b** $2x^2 - 11x + 12$ **c** $3x^2 + 4x - 4$
d $4x^2 - 17x - 15$ **e** $7x^2 - 15x + 2$ **f** $5x^2 + 22x - 15$

5.7 Miscellaneous factorisations

To factorise an expression:

- **take out the HCF first if there is one**
- **look for a difference of two squares if there are 2 terms**
- **use the sum/product identity or cross method if there are 3 terms**
- **group the terms in pairs if there are 4 terms.**

Example

EG +S

Factorise:

a $p^2q - pq^2$ **b** $2w^2 - 18$ **c** $4a^2 + 20a - 24$

d $40 - s - 6s^2$ **e** $4t^2 - 36t + 81$ **f** $7cd + 28ce + 2de + 8e^2$

Solutions

a $p^2q - pq^2$

$= pq(p - q)$

b $2w^2 - 18$

$= 2(w^2 - 9)$

$= 2(w - 3)(w + 3)$

c $4a^2 + 20a - 24$

$= 4(a^2 + 5a - 6)$

$= 4(a + 6)(a - 1)$

d $40 - s - 6s^2$

$5 \quad -2s$

$8 \quad +3s$

$= (5 - 2s)(8 + 3s)$

e $4t^2 - 36t + 81$

$2t \quad -9$

$2t \quad -9$

$= (2t - 9)(2t - 9)$

$= (2t - 9)^2$

f $7cd + 28ce + 2de + 8e^2$

$= 7c(d + 4e) + 2e(d + 4e)$

$= (d + 4e)(7c + 2e)$

Exercise 5.7

1 Factorise each of these expressions.

a $10e + 30$ **b** $a^2 - 4$ **c** $m^2 + 7m + 12$

d $n^2 + 6n + np + 6p$ **e** $2x^2 + 13x + 20$ **f** $25 - k^2$

g $x^2 - x$ **h** $t^2 - 6t + 8$ **i** $w^2 + 4w + 4$

j $ap - 3a + 5p - 15$ **k** $15y + 35$ **l** $2q^2 - 7q + 3$

m $g^2 - h^2$ **n** $-7j + 14k$ **o** $u^2 - 2u - 15$

p $3b^2 + 26b + 16$ **q** $28t^2 - 63tu$ **r** $4c^2 - 9$

s $4xy - 8x + 9y - 18$ **t** $efg + fgh$ **u** $h^2 + 8h - 20$

v $25v^2 - 36w^2$ **w** $5gm - 30m - 2g + 12$ **x** $2f^2 - 3f - 5$

Consolidation

2 Factorise each expression completely.

a $27pq + 45p^2$ **b** $x^2 - 4x - 77$ **c** $9e^2 - 64f^2$

d $3b^2 - b - 44$ **e** $5g + 15h - 2gi - 6hi$ **f** $60xy - 84y$

g $z^2 + 13z - 30$ **h** $p^2q^2 - r^2$ **i** $40 + 11k - 2k^2$
j $64 - k^2l^2$ **k** $m^3 + 7m^2 - mn - 7n$ **l** $10s^2 - 53s + 36$
m $9h^2 - 36$ **n** $3x^2 + 3x - 60$ **o** $5 - 3j - 20k + 12jk$
p $n^3 - n$ **q** $(a + b)^2 + c(a + b)$ **r** $1 + t + t^2 + t^3$
s $2u^2 + 20u + 50$ **t** $-2f^2 + 32$ **u** $10a - 25b + 12ac - 30bc$
v $10a^2 + 14ab - a$ **w** $2k^3 + 12k^2 - 54k$ **x** $4u^2v - 17uv - 42v$

Further applications

3 Factorise each expression completely.
a $(x + y)^2 + z(x + y)$ **b** $t^4 - 1$ **c** $w^4 - 26w^2 + 25$
d $25a^2b^2 - 100b^2c^2$ **e** $16 - n^4$ **f** $y^2 + 14y - 72$
g $(a + b)^2 - (c + d)^2$ **h** $a^3 - 9a - a^2b + 9b$ **i** $4gx^2 - 100g - 75 + 3x^2$

TRY THIS Difference of two squares

Factorise $x^4 + x^2 + 1$.
[HINT: Express $x^4 + x^2 + 1$ as the difference of two squares.]

5.8 Simplifying algebraic fractions

When simplifying numerical fractions, we divide the numerator and denominator by their highest common factor. Algebraic fractions can be simplified in a similar manner.

To simplify algebraic fractions:
- **factorise both the numerator and the denominator if possible**
- **cancel any common factors.**

EG +S

Example

Factorise and simplify:

a $\dfrac{8x + 20}{6x^2 + 15x}$ **b** $\dfrac{d^2 - 49}{3d - 21}$ **c** $\dfrac{b^2 - 11b + 30}{b^2 - 3b - 18}$

Solutions

a
$$\frac{8x + 20}{6x^2 + 15x} = \frac{4(2x + 5)}{3x(2x + 5)} = \frac{4}{3x}$$

b
$$\frac{d^2 - 49}{3d - 21} = \frac{(d + 7)(d - 7)}{3(d - 7)} = \frac{d + 7}{3}$$

c
$$\frac{b^2 - 11b + 30}{b^2 - 3b - 18} = \frac{(b - 5)(b - 6)}{(b + 3)(b - 6)} = \frac{b - 5}{b + 3}$$

Exercise 5.8

1 Express these algebraic fractions in simplest form.

a $\frac{3a}{6}$ **b** $\frac{4t}{20}$ **c** $\frac{6k}{8}$ **d** $\frac{12m}{15}$ **e** $\frac{15y}{10}$

f $\frac{28c}{21}$ **g** $\frac{5}{5d}$ **h** $\frac{4}{16g}$ **i** $\frac{ab}{bc}$ **j** $\frac{pq}{5p}$

k $\frac{pqr}{qrs}$ **l** $\frac{6ab}{18a}$ **m** $\frac{25gh}{30h}$ **n** $\frac{42x}{49xy}$ **o** $\frac{36ef}{27fg}$

p $\frac{u^2}{u}$ **q** $\frac{s}{s^2}$ **r** $\frac{v^2w}{vw^2}$ **s** $\frac{36r^2}{45r}$ **t** $\frac{60cd^2}{35c^2d}$

2 Factorise and simplify:

a $\frac{3x + 3}{3}$ **b** $\frac{4k - 8}{4}$ **c** $\frac{10m + 15}{5}$ **d** $\frac{24a - 32}{16}$

e $\frac{40t - 50}{20}$ **f** $\frac{ab + ac}{a}$ **g** $\frac{u}{uv - uw}$ **h** $\frac{x^2 + 6x}{x}$

i $\frac{5t}{10t - 25t^2}$ **j** $\frac{14c^2 - 7cd}{21c}$ **k** $\frac{8mn + 16m^2}{24mp}$ **l** $\frac{15g^2 - 20gh}{55g^2}$

m $\frac{x + y}{4x + 4y}$ **n** $\frac{x^2 + xy}{xz + yz}$ **o** $\frac{12ac - 14bc}{18ac - 21bc}$ **p** $\frac{15c^2 + 6cd}{18cd + 45c^2}$

Consolidation

3 Factorise and simplify:

a $\frac{m^2 - 49}{m - 7}$ **b** $\frac{a^2 - 25}{4a + 20}$ **c** $\frac{x^2 - x}{x^2 - 1}$ **d** $\frac{n + 12}{n^2 - 144}$

e $\frac{9t^2 - 25}{3t - 5}$ **f** $\frac{5h^2 - 5}{5h - 5}$ **g** $\frac{6k + 18}{4k^2 - 36}$ **h** $\frac{(p - q)^2}{p^2 - q^2}$

4 Factorise and simplify:

a $\frac{x^2 + 4x + 3}{x + 3}$ **b** $\frac{a^2 - 10a + 24}{a - 6}$ **c** $\frac{w^2 - 6w - 40}{2w - 20}$

d $\frac{n + 8}{n^2 + 3n - 40}$ **e** $\frac{c^2 - 6c + 9}{5c^2 - 15c}$ **f** $\frac{e^2 + 14e + 33}{e^2 - 121}$

g $\frac{z^2 - 81}{z^2 - z - 90}$ **h** $\frac{s^2 - 11s + 28}{s^2 + 4s - 77}$ **i** $\frac{2d^2 + 22d + 48}{4d^2 - 36}$

5 Factorise and simplify:

a $\frac{2x^2 + 17x + 30}{x + 6}$ **b** $\frac{2r - 5}{10r^2 - 29r + 10}$ **c** $\frac{6p^2 + 17p + 12}{21p + 28}$

d $\frac{25k^2 - 64}{5k^2 + 12k - 32}$ **e** $\frac{4h^2 - 19h - 63}{h^2 - 13h + 42}$ **f** $\frac{3v^2 - 7v - 6}{4v^2 - 13v + 3}$

6 Factorise and simplify:

a $\dfrac{px + qx + py + qy}{x + y}$ **b** $\dfrac{km - 5k + 3m - 15}{m^2 - 25}$ **c** $\dfrac{u^2 - 49}{uv - uw - 7v + 7w}$

d $\dfrac{x^2 + xy - 5x - 5y}{x^2 - 8x + 15}$ **e** $\dfrac{ab - ac - 4b + 4c}{2a^2 - 32}$ **f** $\dfrac{n^2 + 2n - pn - 2p}{n^2 + 2n + pn + 2p}$

Further applications

7 Show that $\dfrac{a - b}{b - a} = -1$. Hence, simplify each of these fractions.

a $\dfrac{h - 3}{3 - h}$ **b** $\dfrac{8y - 16}{2 - y}$ **c** $\dfrac{e - f}{5f - 5e}$

d $\dfrac{4m - 4n}{6n - 6m}$ **e** $\dfrac{a^2 - 169}{13 - a}$ **f** $\dfrac{c^2 - 12c + 36}{12 - 2c}$

g $\dfrac{s^2 - 17s + 72}{24 - 3s}$ **h** $\dfrac{g^2 - 12g + 27}{9g^2 - g^3}$ **i** $\dfrac{48 - 3k^2}{k^2 - 2k - 8}$

5.9 Multiplying and dividing algebraic fractions

To multiply or divide algebraic fractions:

- **change divisions to multiplications and take reciprocals of any fractions that follow division signs**
- **factorise each expression if possible**
- **cancel common factors between the numerator and denominator**
- **multiply the numerators and multiply the denominators.**

Example 1

Factorise and simplify $\dfrac{3z + 15}{9z^2 + 63z} \times \dfrac{z^2 + 2z - 35}{z^2 - 25}$.

Solution

$$\frac{3z + 15}{9z^2 + 63z} \times \frac{z^2 + 2z - 35}{z^2 - 25} = \frac{{}^{1}\cancel{3}\cancel{(z+5)}}{{}_{3}\cancel{9}z\cancel{(z+7)}} \times \frac{\cancel{(z+7)}\cancel{(z-5)}}{\cancel{(z+5)}\cancel{(z-5)}}$$

$$= \frac{1}{3z}$$

EG +S

Example 2

Factorise and simplify $\dfrac{a^2+5a-24}{a^2-a-6} \div \dfrac{a^2+8a}{a^2-4}$.

Solution

$$\begin{aligned}\frac{a^2+5a-24}{a^2-a-6} \div \frac{a^2+8a}{a^2-4} &= \frac{a^2+5a-24}{a^2-a-6} \times \frac{a^2-4}{a^2+8a} \\ &= \frac{\cancel{(a+8)}\cancel{(a-3)}}{\cancel{(a-3)}\cancel{(a+2)}} \times \frac{(a-2)\cancel{(a+2)}}{a\cancel{(a+8)}} \\ &= \frac{a-2}{a}\end{aligned}$$

Exercise 5.9

1 Simplify each of the following.

a $\dfrac{x}{2} \times \dfrac{y}{7}$ **b** $\dfrac{2}{m} \times \dfrac{2m}{9}$ **c** $\dfrac{15a}{16b} \times \dfrac{24bc}{25a}$ **d** $\dfrac{15de}{4p^2q} \times \dfrac{pq^2}{9cd}$

e $\dfrac{m}{6} \div \dfrac{2}{n}$ **f** $\dfrac{5}{x} \div \dfrac{10y}{3x}$ **g** $\dfrac{7r}{12t} \div \dfrac{21r}{16s}$ **h** $\dfrac{36ad}{35bc} \div \dfrac{42a^2}{55ab}$

2 Simplify:

a $\dfrac{a+3}{2} \times \dfrac{8}{a+3}$ **b** $\dfrac{p-1}{5q} \times \dfrac{10q}{3(p-1)}$

c $\dfrac{4b(2b+3)}{(2b+3)^2} \times \dfrac{(2b+3)(b-4)}{6b}$ **d** $\dfrac{12}{m+8} \div \dfrac{27}{2(m+8)}$

e $\dfrac{(x-y)^2}{15} \div \dfrac{7(x-y)}{10}$ **f** $\dfrac{(k+2)(k-3)}{12(k+1)} \div \dfrac{(k-3)(k+4)}{9(k+4)}$

3 Factorise and simplify each of the following.

a $\dfrac{2y+10}{15} \times \dfrac{9}{3y+15}$ **b** $\dfrac{4n+12}{12k-24} \times \dfrac{9k-18}{7n+21}$

c $\dfrac{2a^2+3a}{c^2-3c} \times \dfrac{bc^2}{2ab+3b}$ **d** $\dfrac{4a+4b}{12} \div \dfrac{3}{ac+bc}$

e $\dfrac{10t-30}{21t} \div \dfrac{8t-24}{35u}$ **f** $\dfrac{18m+12n}{35mn^2} \div \dfrac{27m+18n}{20m^2n}$

Consolidation

4 Factorise and simplify:

a $\dfrac{a^2-9}{5b+10} \times \dfrac{b+2}{a-3}$ **b** $\dfrac{s^2-2s}{7} \times \dfrac{s+2}{s^2-4}$

c $\dfrac{t+4}{2t-8} \times \dfrac{3t^2-12t}{t^2-16}$ **d** $\dfrac{x^2+8x+15}{x+3} \times \dfrac{9x}{x^2-25}$

e $\dfrac{k^2-4}{k^2+7k+10}\times\dfrac{5k+25}{k^2-2k}$

f $\dfrac{c^2+5c+6}{c^2+7c}\times\dfrac{c^2+5c}{c^2+8c+15}$

g $\dfrac{n^2-9n+20}{n^2-6n+5}\times\dfrac{n^2-n-2}{n^2-6n+8}$

h $\dfrac{r^2-49}{5r^2-15r}\times\dfrac{r^2-9}{r^2-4r-21}$

i $\dfrac{a^2+2ab+b^2}{a^2-b^2}\times\dfrac{a^2b-ab^2}{a^2b^2}$

j $\dfrac{b^2-14b+45}{2b^2-10b}\times\dfrac{b^3-4b^2}{b-9}$

k $\dfrac{2w^2+19w-33}{16w^2-24w}\times\dfrac{2w^2-242}{6w-66}$

l $\dfrac{a^2-4}{3a^2-2a-8}\times\dfrac{3a^2+22a+24}{a^2+6a}$

m $\dfrac{2e^2+12e+16}{e^2-e}\times\dfrac{e^2-1}{6e^2+18e+12}$

n $\dfrac{cd+ce+d^2+de}{d^2-df}\times\dfrac{d^2-f^2}{d^2+df+de+ef}$

5 Factorise and simplify:

a $\dfrac{k^2-36}{k^2-6k}\div\dfrac{5k+30}{35}$

b $\dfrac{n^2-8n}{4n}\div\dfrac{n^2-64}{4n+8}$

c $\dfrac{h+3}{18h}\div\dfrac{h^2+4h+3}{9h^2+9h}$

d $\dfrac{a^2-1}{10a+25}\div\dfrac{a^2+a}{24a^2+60a}$

e $\dfrac{z^2-13z+30}{z^2-3z}\div\dfrac{z^2-100}{(z+10)^2}$

f $\dfrac{2t^2-6t}{7t^2-28t}\div\dfrac{t^2+3t-18}{t^2+2t-24}$

g $\dfrac{xy-y^2}{(x-y)^2}\div\dfrac{4xy+4y^2}{x^2-y^2}$

h $\dfrac{r^2+12r+35}{r^2-2r-35}\div\dfrac{r^2+r-42}{r^2-14r+48}$

i $\dfrac{m^2-144}{m^2+15m+36}\div\dfrac{2m^2-24m}{(m+3)^2}$

j $\dfrac{2x^2+7x+5}{x^2-1}\div\dfrac{4x^2-25}{6x^2-15x}$

k $\dfrac{3q^2-13q+14}{q^2-64}\div\dfrac{3q^2+2q-21}{q^2-5q-24}$

l $\dfrac{2u^2-18}{7u^2v-21uv}\div\dfrac{4u^2+48u+108}{7u^2+63u}$

m $\dfrac{x^2+8x+xy+8y}{x^2-4x+xy-4y}\div\dfrac{x^2+12x+32}{2x^2-32}$

n $\dfrac{49c^2-121}{2c^2+17c-9}\div\dfrac{7c^2+3c-22}{c^2+11c+18}$

Further applications

6 Factorise and simplify:

a $\dfrac{2-2a}{5}\times\dfrac{4}{3a-3}$

b $\dfrac{10n-20}{15mn+30m}\times\dfrac{mn^2+2mn}{12-6n}$

c $\dfrac{2xy+2y^2}{21y^2z-42xyz}\div\dfrac{5x^2+5xy}{28x-14y}$

d $\dfrac{3p^2-6pq}{27p^2q+9pq^2}\div\dfrac{8q^2-4pq}{-36p-12q}$

5.10 Adding and subtracting algebraic fractions

Algebraic fractions, like numerical fractions, can only be added or subtracted if they have a common denominator. Consider the addition $\frac{1}{6} + \frac{7}{10}$. The lowest common denominator (LCD) is not necessarily found by multiplying the denominators. Rather, it is found by factorising the individual denominators into primes, with each different factor then taken once and multiplied together.

For example, $\frac{1}{6} + \frac{7}{10} = \frac{1}{3 \times 2} + \frac{7}{2 \times 5}$

$= \frac{(5 \times 1) + (3 \times 7)}{3 \times 2 \times 5}$

- (3×2) divides into $(3 \times 2 \times 5)$ 5 times, $5 \times 1 = 5$
- (2×5) divides into $(3 \times 2 \times 5)$ 3 times, $3 \times 7 = 21$

$= \frac{26}{30}$

$= \frac{13}{15}$

The same method is used to add or subtract algebraic fractions.

To add or subtract algebraic fractions with binomial or trinomial denominators:

- **factorise the denominator in each fraction if possible**
- **form the LCD by taking each different factor in the individual denominators once and finding their product**
- **divide the denominators into the LCD then multiply by the numerators**
- **add or subtract the numerators**
- **check whether the resulting fraction can be simplified by factorising and cancelling.**

NOTE: If one of the denominators is a perfect square, both factors must be included in the LCD.

Example 1

EG +S

Simplify $\frac{3}{x+4} + \frac{5}{x-1}$

Solution

$$\frac{3}{x+4} + \frac{5}{x-1} = \frac{3(x-1) + 5(x+4)}{(x+4)(x-1)}$$

$$= \frac{3x - 3 + 5x + 20}{(x+4)(x-1)}$$

$$= \frac{8x + 17}{(x+4)(x-1)}$$

Example 2

Simplify:

a $\dfrac{1}{x^2-4}+\dfrac{1}{x^2+13x+22}$

b $\dfrac{3}{2x^2-5x}-\dfrac{2}{4x^2-25}$

Solutions

a $\dfrac{1}{x^2-4}+\dfrac{1}{x^2+13x+22}$

$$= \frac{1}{(x-2)(x+2)}+\frac{1}{(x+2)(x+11)}$$
$$= \frac{x+11+x-2}{(x-2)(x+2)(x+11)}$$
$$= \frac{2x+9}{(x-2)(x+2)(x+11)}$$

b $\dfrac{3}{2x^2-5x}-\dfrac{2}{4x^2-25}$

$$= \frac{3}{x(2x-5)}-\frac{2}{(2x-5)(2x+5)}$$
$$= \frac{3(2x+5)-2(x)}{x(2x-5)(2x+5)}$$
$$= \frac{6x+15-2x}{x(2x-5)(2x+5)}$$
$$= \frac{4x+15}{x(2x-5)(2x+5)}$$

Example 3

Simplify $\dfrac{1}{(x+3)^2}+\dfrac{x}{x+4}$

Solution

$$\frac{1}{(x+3)^2}+\frac{x}{x+4} = \frac{x+4+x(x+3)^2}{(x+3)^2(x+4)}$$
$$= \frac{x+4+x(x^2+6x+9)}{(x+3)^2(x+4)}$$
$$= \frac{x+4+x^3+6x^2+9x}{(x+3)^2(x+4)}$$
$$= \frac{x^3+6x^2+10x+4}{(x+3)^2(x+4)}$$

Exercise 5.10

1 Simplify each of the following.

a $\dfrac{11z}{15}-\dfrac{z}{15}$ **b** $\dfrac{5a}{8}+\dfrac{7a}{8}$ **c** $\dfrac{2m}{5}+\dfrac{m}{10}$ **d** $\dfrac{11x}{14}-\dfrac{2x}{7}$

e $\dfrac{2k}{3}-\dfrac{k}{4}$ **f** $\dfrac{3f}{5}+\dfrac{f}{8}$ **g** $\dfrac{5}{6y}+\dfrac{11}{12y}$ **h** $\dfrac{5}{9u}+\dfrac{4}{6u}$

2 Simplify:

a $\dfrac{x+1}{8}+\dfrac{x+3}{4}$ **b** $\dfrac{n+5}{7}+\dfrac{n-4}{2}$ **c** $\dfrac{3t-2}{6}+\dfrac{t+6}{4}$

d $\dfrac{b+3}{2}-\dfrac{b+2}{10}$ **e** $\dfrac{k+7}{3}-\dfrac{k-1}{4}$ **f** $\dfrac{3z-5}{8}-\dfrac{2z-3}{12}$

Consolidation

3 Simplify:

a $\dfrac{1}{x}+\dfrac{1}{x+1}$ **b** $\dfrac{1}{x+1}+\dfrac{1}{x-1}$ **c** $\dfrac{1}{x+2}+\dfrac{1}{x+3}$

d $\dfrac{3}{x}+\dfrac{2}{x+5}$ **e** $\dfrac{2}{x+3}+\dfrac{3}{x+2}$ **f** $\dfrac{5}{x-1}+\dfrac{2}{x+6}$

g $\dfrac{1}{x}-\dfrac{1}{x+1}$ **h** $\dfrac{1}{x+2}-\dfrac{1}{x}$ **i** $\dfrac{1}{x-1}-\dfrac{1}{x+1}$

j $\dfrac{4}{x}-\dfrac{2}{x+2}$ **k** $\dfrac{7}{x+5}-\dfrac{4}{x+3}$ **l** $\dfrac{5}{2x-1}-\dfrac{3}{3x-4}$

4 Simplify:

a $\dfrac{x}{x+1}+\dfrac{1}{x+2}$ **b** $\dfrac{2x}{x+3}-\dfrac{x}{x-1}$ **c** $\dfrac{x+1}{x+3}+\dfrac{x+2}{x+1}$

5 Simplify each of the following.

a $\dfrac{1}{x(x+1)}+\dfrac{1}{x+1}$ **b** $\dfrac{1}{(x+1)(x+5)}+\dfrac{1}{x+5}$

c $\dfrac{1}{(x+1)(x+2)}+\dfrac{1}{(x+2)(x+5)}$ **d** $\dfrac{1}{(x+4)(x-2)}+\dfrac{1}{(x+5)(x-2)}$

e $\dfrac{1}{x(x-4)}-\dfrac{1}{(x+6)(x-4)}$ **f** $\dfrac{1}{(x-1)(x+1)}-\dfrac{1}{(x+1)(x+2)}$

g $\dfrac{3}{(x+2)(x+4)}+\dfrac{2}{(x+1)(x+4)}$ **h** $\dfrac{5}{(x-3)(x+2)}+\dfrac{4}{(x+2)(x+5)}$

i $\dfrac{7}{2x(2x-1)}-\dfrac{1}{(2x-1)(x-4)}$ **j** $\dfrac{4}{(3x+2)(x-2)}-\dfrac{2}{(x+6)(x-2)}$

k $\dfrac{x}{(x-3)(x-4)}+\dfrac{2x}{(x-4)(x+1)}$ **l** $\dfrac{x+1}{x(x-1)}+\dfrac{x+2}{x(x+1)}$

Further applications

6 Factorise the denominator in each fraction, then express the fractions with a common denominator and simplify.

a $\dfrac{1}{2x+8}+\dfrac{1}{x^2+4x}$ **b** $\dfrac{1}{4x+12}+\dfrac{1}{3x+9}$

c $\dfrac{3}{x^2-4}+\dfrac{1}{x-2}$

d $\dfrac{4}{x^2-9}+\dfrac{2}{5x-15}$

e $\dfrac{1}{x^2-1}+\dfrac{1}{x^2+3x+2}$

f $\dfrac{1}{x^2-7x+6}+\dfrac{1}{x^2-36}$

g $\dfrac{1}{x^2+8x+15}+\dfrac{1}{x^2+11x+30}$

h $\dfrac{1}{x^2-2x-63}+\dfrac{1}{x^2-81}$

i $\dfrac{3}{x^2-25}+\dfrac{4}{x^2-7x+10}$

j $\dfrac{2}{x^2-7x-30}+\dfrac{5}{x^2-14x+40}$

k $\dfrac{6}{x^2-9}+\dfrac{1}{3x^2-5x-12}$

l $\dfrac{2}{2x^2+19x+35}+\dfrac{3}{x^2-49}$

7 Factorise the denominator in each fraction, then express the fractions with a common denominator and simplify.

a $\dfrac{1}{x^2-6x}-\dfrac{1}{3x-18}$

b $\dfrac{1}{2x+8}-\dfrac{1}{5x+20}$

c $\dfrac{1}{x^2-16}-\dfrac{2}{x+4}$

d $\dfrac{3}{x^2-64}-\dfrac{4}{5x-40}$

e $\dfrac{1}{9x^2-16}-\dfrac{1}{21x-28}$

f $\dfrac{1}{x^2-2x-35}-\dfrac{1}{x^2-25}$

g $\dfrac{1}{x^2-13x+22}-\dfrac{1}{x^2-14x+33}$

h $\dfrac{1}{7x^2-14x}-\dfrac{1}{x^2+4x-12}$

i $\dfrac{2}{x^2-144}-\dfrac{3}{x^2+9x-36}$

j $\dfrac{5}{x^2-5x-50}-\dfrac{2}{x^2-4x-45}$

k $\dfrac{4x}{4x^2-49}-\dfrac{2x}{2x^2+11x+14}$

l $\dfrac{x-3}{4x^2+37x-30}-\dfrac{x+2}{x^2-100}$

FOCUS ON WORKING MATHEMATICALLY

TAXICAB NUMBERS AND THE SUM OF TWO CUBES

Introduction

What are ‘taxicab numbers’? How did they come be studied? Who were the mathematicians involved? We can only relate part of the story here. It began during the first world war when the British mathematician GH Hardy (1877–1947) went to visit his protégé and colleague Srinivasa Ramanujan (1877–1920) who lay dying in hospital in London. Hardy had gone out to the hospital in Putney by taxi.

He was a shy and self conscious man in situations like this, not knowing how to open a conversation easily, despite the fact that he had worked with Ramanujan for years. CP Snow in the foreword to Hardy's book (*A Mathematicians Apology*, Cambridge University Press, 2000, page 37) records the conversation. Without a greeting, and certainly as his first remark, Hardy blurted out:

> 'I thought the number of my taxicab was 1729. It seemed to me a rather dull number.'
> To which Ramanujan replied, 'No Hardy! No Hardy! It is a very interesting number.
> It is the smallest number expressible as the sum of two cubes in two different ways.'

The number 1729 can be written as $1^3 + 12^3$ and also as $9^3 + 10^3$. Today such numbers have become known as 'taxicab numbers'. Mathematicians define the smallest number expressible as the sum of two cubes in n different ways as the nth taxicab number, denoted by taxicab(n). Thus taxicab(2) = 1729. Taxicab(3) was discovered in 1957 to be 87 539 319 and taxicab(4) in 1991 to be 6 963 472 309 248. Taxicab(5) was discovered in 1997. As you can imagine, computers played a major role in the discovery of taxicab numbers but little else is known about them.

Learning activities

1. What number is taxicab(1)?
2. Let's take a closer look at taxicab(2) = 1729. Using a calculator verify that 1729 can be written as the sum of the two cubes $1^3 + 12^3$ and $9^3 + 10^3$.
3. In this chapter you have learnt to factorise algebraic expressions. By multiplying out, verify that the factors of the expression $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
4. Let $a = 9$ and $b = 10$. Use your calculator to verify that the right hand side is 1729. Repeat with $a = 1$ and $b = 12$.
5. Equations of the form $c = a^3 + b^3$ are called Diophantine equations (named after Diophantus, around 250 AD) where a, b, and c are integers. Solve the Diophantine equation $28 = a^3 + b^3$ for a and b. Why is 28 *not* a taxicab number?

Challenge activities

1. Two cubes have side lengths a and b (whole numbers with $a < b$). The sum of their volumes is equal to the sum of the lengths of their edges. Find a and b.
2. Given that $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ deduce the factors of $a^3 - b^3$ by setting $-b$ for b.
3. Two cubes have side lengths a and b (whole numbers with $a > b$). The difference in their volumes is equal to the difference of the total lengths of their edges. Investigate whether integer solutions for a and b can be found.

Let's communicate

Reasoning means making logical statements in a sequence. To combine logical statements, we can use 'linking words' such as *and, if, when, however, because* or *but* to name a few. Write a short paragraph using linking words to communicate the meaning of *taxicab numbers*. If you have tried this you will immediately see the power of algebra to present meaning in symbolic form.

Reflecting

There are two ways to describe mathematical thinking on which you should reflect. One is through a search for specific patterns which may suggest a general rule. This type of thinking is *inductive*. The second concerns the need for proof. In this case the result we suspect to be true is put to the test of *deductive* reasoning, that is a rigorous chain of argument that leads to an inevitable conclusion.

In their book *An Introduction to the Theory of Numbers* (Oxford University Press, 1954), GH Hardy and his colleague EM Wright proved a theorem to show that taxicab numbers, denoted by taxicab(n), exist for any value of $n \geq 1$. What type of mathematical reasoning do you think they used?

Language link with Macquarie

MACQUARIE

Use each of the following in a simple sentence:

1. *Binomial factor*
2. *Difference of two squares*
3. *The sum of two cubes*
4. *The difference between inductive and deductive reasoning*
5. *The* Macquarie Learners Dictionary *entry for* ***factor****:*

factor *noun* **1.** one of the things that brings about a result: *Hard work was a factor in her success.*
2. *Specialised* one of two or more numbers which, when multiplied together, give the product: *Factors of 18 are 3 and 6.*

Note the special mathematical meaning of ***factor*** *and that it applies in algebra as well as in arithmetic.*

1 Expand and simplify:
a $(y + 4)(y + 5)$ **b** $(m - 7(m - 3)$
c $(t + 8)(t - 2)$ **d** $(a - 11)(a + 4)$

2 Expand and simplify:
a $(2p - 3)(3p - 8)$
b $(4 - 3r)(2 + r)$

3 Expand these perfect squares.
a $(m - 7)^2$ **b** $(2c + 5)^2$

4 Expand and simplify $2x(9x - 2y)^2$.

5 Complete these perfect squares.
a $(\_\_\_\_)^2 = t^2 + 22t + \_\_\_\_$
b $(\_\_\_\_)^2 = n^2 - \_\_\_\_ + 81$
c $(\_\_\_\_)^2 = 9x^2 + \_\_\_\_ + 25$
d $(\_\_\_\_)^2 = 16u^2 - 88u + \_\_\_\_$

6 State whether each expression is a perfect square.
a $n^2 + 4$ **b** $k^2 + 6k - 9$
c $a^2 + 2a + 1$ **d** $e^2 - 36$
e $q^2 - 10q + 100$ **f** $9c^2 + 24c + 16$

7 Expand and simplify:
a $(a + 3)(a^2 + 5a + 4)$
b $(n + 6)(n + 2) + (3n - 4)^2$
c $(a + 10)(a - 10) - (a - 7)^2$
d $(2u + 9)(u - 4) - (u - 3)(u + 6)$

8 Factorise:
a $7x + 28$ **b** $a^2 + a$
c $10m + 15$ **d** $9p - 24q$
e $12rs + 20st$ **f** $pqr - pqs$
g $27y^2 - 36y$ **h** $g^2h - gh^2$
i $-ab + 3a$ **j** $-18c^2 - 14cd$

9 Factorise:
a $m^2 - n^2$ **b** $z^2 - 9$
c $1 - p^2$ **d** $49 - r^2$
e $w^2 - 529$ **f** $4a^2 - 25$
g $x^2 - 16y^2$ **h** $81u^2 - 100v^2$
i $a^2b^2 - c^2$ **j** $36p^2q^2 - 121r^2s^2$

10 Factorise:
a $x(z + 3) + y(z + 3)$
b $m^3 + m^2 + m + 1$
c $e^2 + ef - 2e - 2f$
d $6jk - 14j - 15k + 35$
e $4p - pq + q^2 - 4q$
f $10c - cd - d^2 + 10d$

11 Factorise:
a $x^2 + 4x + 3$ **b** $b^2 - 10b + 21$
c $e^2 - e - 12$ **d** $p^2 + 3p - 40$
e $a^2 + 10a + 25$ **f** $q^2 - 18q + 81$

12 Factorise:
a $3t^2 + 14t + 8$ **b** $2m^2 - 13m + 21$
c $3c^2 + 7c - 6$ **d** $7b^2 - 23b - 20$
e $6s^2 + 31s + 35$ **f** $12d^2 - 19d - 18$

13 Factorise:
a $n^2 - 36$
b $v^2 + 8v + 15$
c $20k + 35$
d $3e^2 + 14e + 15$
e $a^2 + 4a + ab + 4b$
f $9h^2 - 25$
g $p^2 - 5p - 24$
h $-21xy - 35yz$
i $1 + u + u^2 + u^3$
j $16a^2 - 121b^2$
k $5m^2 - 14m - 24$
l $y^2 - 9y - 10$
m $2cd + 6ce - 5d - 15e$
n $rs^2 - r^2s + rs$
o $30g^2 + 7g - 15$

14 Factorise completely:
a $3x^2 - 12$
b $2a^2 + 18a + 36$
c $n^3 - n$
d $4h^2 - 36$
e $ab^2 - 7ab + 12a$
f $6u^2 + 26u + 8$
g $12z^2 - 75$
h $x^3 - 2x^2y - 3x^2 + 6xy$

CHAPTER REVIEW

15 Simplify:

a $\frac{3}{3d}$ **b** $\frac{21tu}{28uv}$

c $\frac{25a^2}{10a}$ **d** $\frac{p^2q}{pq^2}$

16 Factorise and simplify:

a $\frac{15e + 25}{5}$ **b** $\frac{a}{a^2 + 9a}$

c $\frac{9x^2 - 15xy}{6x}$ **d** $\frac{24m - 16n}{36m - 24n}$

e $\frac{u^2 - 4}{u - 2}$ **f** $\frac{16h^2 - 25}{8h + 10}$

g $\frac{x^2 + 7x + 10}{x^2 + 5x + 6}$ **h** $\frac{k^2 + 3k - 28}{k^2 - 16}$

i $\frac{3p^2 - 19p + 20}{(p - 5)^2}$

j $\frac{ab - ac - 3b + 3c}{2a^2 - 13a + 21}$

k $\frac{5w - 10}{2 - w}$ **l** $\frac{c^2 - 64}{24 - 3c}$

17 Factorise and simplify:

a $\frac{a + 3}{2a - 6} \times \frac{4a^2 - 12a}{a^2 - 9}$

b $\frac{c^2 + 9c + 20}{c^2 + 5c} \times \frac{(c - 4)^2}{c^2 - 16}$

c $\frac{y^2 - 100}{2y^2 - 20y} \div \frac{y^2 + 14y + 40}{y^3 + 4y^2}$

d $\frac{x^2 - 13x + 42}{x^2 - 49} \div \frac{2x^2 - 11x - 6}{x^2 + 7x}$

18 Simplify:

a $\frac{a + 8}{3} + \frac{a - 2}{4}$ **b** $\frac{2h - 3}{5} - \frac{3h + 7}{10}$

c $\frac{1}{x + 3} + \frac{1}{x}$ **d** $\frac{x}{x - 1} - \frac{2}{x + 1}$

e $\frac{x + 1}{x + 2} + \frac{x + 5}{x + 1}$

19 Simplify:

a $\frac{1}{3x + 6} + \frac{1}{x^2 - 4}$

b $\frac{1}{x^2 - 25} + \frac{1}{x^2 + 2x - 35}$

c $\frac{3}{x^2 - 11x + 18} - \frac{2}{x^2 - 12x + 27}$

d $\frac{x}{4x^2 - 9} - \frac{4}{2x^2 + 13x + 15}$

6

Quadratic equations

This chapter at a glance

Stage 5.1/5.2/5.3

After completing this chapter, you should be able to:

- determine whether a given expression is a quadratic expression
- determine a quadratic relationship given a table of values
- solve a quadratic equation by inspection
- solve a quadratic equation by factorisation
- solve a quadratic equation by completing the square
- solve a quadratic equation by using the quadratic formula
- form a quadratic equation given its solutions
- determine the most appropriate method that should be used to solve a quadratic equation
- solve a pair of simultaneous equations where one equation is linear and the other is quadratic
- solve a quadratic equation by the use of a suitable substitution
- solve practical problems that involve quadratic equations.

6.1 Quadratic relationships

A **quadratic expression** is an expression of the form $ax^2 + bx + c$, where a, b, c are constants and $a \neq 0$. That is, the highest power of the variable x is 2. Hence, $ax^2 + bx + c = 0$ is a quadratic equation.

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where a, b, c are constants and $a \neq 0$.

In Year 9 we formed equations to describe linear relationships between two variables, given either a table of values or a straight line graph. We will now extend this idea to form equations to describe quadratic relationships between two variables.

To find a quadratic relationship of the form $y = x^2 + bx + c$ given a table of values:
- **substitute the ordered pair (x, y) in which the x-value is 0 to find the value of c**
- **substitute any other ordered pair from the table to find the value of b**
- **write the equation in the form $y = x^2 + bx + c$.**

Example 1

State whether or not each of the following is a quadratic expression.

a $x^2 + 5x + 3$ **b** $2x^2 - 7$ **c** $(x - 3)^2$ **d** $\dfrac{1}{x^2 + 4}$

Solutions

a $x^2 + 5x + 3$ is a quadratic expression because it is of the form $ax^2 + bx + c$, where $a = 1$, $b = 5$, $c = 3$.

b $2x^2 - 7$ is a quadratic expression because it is of the form $ax^2 + bx + c$, where $a = 2$, $b = 0$, $c = -7$.

c $(x - 3)^2$ is a quadratic expression because $(x - 3)^2 = x^2 - 6x + 9$, which is of the form $ax^2 + bx + c$, where $a = 1$, $b = -6$, $c = 9$.

d $\dfrac{1}{x^2 + 4}$ is not a quadratic expression because it is not of the form $ax^2 + bx + c$.

Example 2

Find an equation of the form $y = x^2 + bx + c$ to describe the relationship between the x and y values in this table.

x	0	1	2	3	4	5	6
y	−2	2	8	16	26	38	52

Solution

When $x = 0$, $y = -2$:

$$\begin{aligned} y &= x^2 + bx + c \\ -2 &= 0^2 + b(0) + c \\ \therefore c &= -2 \end{aligned}$$

When $x = 1$, $y = 2$:

$$\begin{aligned} y &= x^2 + bx - 2 \\ 2 &= 1^2 + b(1) - 2 \\ 2 &= b - 1 \\ \therefore b &= 3 \end{aligned}$$

$\therefore$ The equation is $y = x^2 + 3x - 2$.

EG +S

Example 3

Find two solutions for the equation $x(x - 2) = 24$, using a guess, check and refine approach.

Solution

When $x = 6$:

$$\begin{aligned} \text{LHS} &= x(x - 2) \\ &= 6(6 - 2) \\ &= 6 \times 4 \\ &= 24 \\ &= \text{RHS} \end{aligned}$$

When $x = -4$:

$$\begin{aligned} \text{LHS} &= x(x - 2) \\ &= -4(-4 - 2) \\ &= -4 \times (-6) \\ &= 24 \\ &= \text{RHS} \end{aligned}$$

$\therefore$ The solutions are $x = 6, -4$.

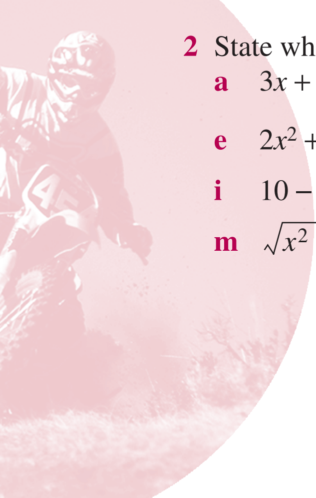
Exercise 6.1

1 Find the value of each quadratic expression, using the given substitutions.

a	$x^2 + 1$	**i** $x = 4$	**ii** $x = -3$	**iii** $x = 0$
b	$2x^2$	**i** $x = 5$	**ii** $x = -2$	**iii** $x = 1.5$
c	$2x - x^2$	**i** $x = 3$	**ii** $x = -1$	**iii** $x = \frac{1}{2}$
d	$20 - 3x^2$	**i** $x = 0$	**ii** $x = 2$	**iii** $x = -4$
e	$x^2 + 5x + 4$	**i** $x = 3$	**ii** $x = -6$	**iii** $x = \frac{1}{4}$
f	$2x^2 - 5x + 12$	**i** $x = -1$	**ii** $x = \frac{2}{3}$	**iii** $x = -\frac{3}{5}$

2 State whether or not each of the following is a quadratic expression.

a $3x + 1$	**b** x^2	**c** $3x^2$	**d** $\dfrac{1}{x^2}$
e $2x^2 + 9$	**f** $x^2 - x$	**g** $x^2 + 7x + 4$	**h** $x^2 + \dfrac{1}{x}$
i $10 - 3x^2$	**j** $x^3 + x^2$	**k** $(x^2 + 1)^2$	**l** $4x^2 - 3x - 8$
m $\sqrt{x^2 + 9}$	**n** $1 - 5x - 4x^2$	**o** $(x + 5)^2$	**p** $(x^2 - 1)^3$



Consolidation

3 Write down a quadratic relationship in the form $y = ax^2 + c$ for each table of values.

a

x	0	1	2	3	4	5
y	1	2	5	10	17	26

b

x	0	1	2	3	4	5
y	–3	–2	1	6	13	22

c

x	0	1	2	3	4	5
y	0	2	8	18	32	50

d

x	0	1	2	3	4	5
y	0	–3	–12	–27	–48	–75

e

x	0	1	2	3	4	5
y	5	7	13	23	37	55

f

x	0	1	2	3	4	5
y	–2	1	10	25	46	73

4 Find a quadratic relationship of the form $y = x^2 + bx + c$ for each table of values.

a

x	0	1	2	3	4	5
y	3	6	11	18	27	38

b

x	0	1	2	3	4	5
y	2	9	18	29	42	57

c

x	0	1	2	3	4	5
y	–1	4	11	20	31	44

d

x	0	1	2	3	4	5
y	8	4	2	2	4	8

e

x	0	1	2	3	4	5
y	–10	–1	10	23	38	55

f

x	0	1	2	3	4	5
y	–7	–7	–5	–1	5	13

g

x	0	1	2	3	4	5
y	11	15	21	29	39	51

h

x	0	1	2	3	4	5
y	20	14	10	8	8	10

5 Find the positive integer solution for each quadratic equation, using the guess, check and refine approach.

a $x(x - 1) = 6$ **b** $x(x + 2) = 24$ **c** $x(x - 7) = 0$

d $x(x + 3) = 40$ **e** $x(x + 4) = 60$ **f** $x(x - 5) = 84$

6 Find a positive solution and a negative solution for each quadratic equation, using the guess, check and refine approach.

a $x(x - 2) = 3$ **b** $x(x + 1) = 20$ **c** $x(x - 3) = 28$

7 Find two solutions for each of these quadratic equations, using the guess, check and refine approach.

a $x^2 - 7x + 12 = 0$ **b** $x^2 - 3x - 4 = 0$ **c** $x^2 + 2x - 15 = 0$

Further applications

8 The number of diagonals, D, in a polygon with n sides is given by the formula $D = \frac{n}{2}(n-3)$. Find the number of sides in a polygon that has 77 diagonals.

9 The area of a rectangle is 120 cm^2 and the length of the rectangle is 7 cm more than the width.
  - **a** If the width of the rectangle is x cm, find an expression for the length and hence show that $x(x + 7) = 120$.
  - **b** Find the positive integer solution for this quadratic equation, using a guess, check and refine approach.
  - **c** Hence, find the dimensions of the rectangle.

6.2 Solving quadratic equations using factors

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where a, b, c are constants and $a \neq 0$. Many quadratic equations can be solved by making use of the following principle.

If $pq = 0$, then either $p = 0$ or $q = 0$, or both p and q are equal to zero.

This is called the **null factor law**.

To solve a quadratic equation by factorising:
- **take all terms to one side of the equation such that $a > 0$**
- **factorise the quadratic expression**
- **solve each factor equal to zero.**

NOTE: A quadratic trinomial of the form $ax^2 + bx + c$ may have two different factors, two like factors (if it is a perfect square), or no factors. Therefore a quadratic equation of the form $ax^2 + bx + c = 0$ may have two solutions, one solution or no solutions. These solutions, if there are any, can be checked by substitution, as for linear equations.

Example 1

EG +S

Solve the following quadratic equations.

a $2x(x - 8) = 0$ **b** $(x - 3)(x + 7) = 0$ **c** $(3x - 2)(x - 4) = 0$

Solutions

a If $2x(x - 8) = 0$,
then either
$2x = 0$ or $x - 8 = 0$
$\therefore x = 0$ or $x = 8$

b If $(x - 3)(x + 7) = 0$,
then either
$x - 3 = 0$ or $x + 7 = 0$
$\therefore x = 3$ or $x = -7$

c If $(3x - 2)(x - 4) = 0$,
then either
$3x - 2 = 0$ or $x - 4 = 0$
$\therefore x = \frac{2}{3}$ or $x = 4$

Example 2

Solve each of these quadratic equations.

a $x^2 - 16 = 0$ **b** $x^2 - 3x - 10 = 0$ **c** $2x^2 + 3x - 20 = 0$

Solutions

a
$x^2 - 16 = 0$
$(x - 4)(x + 4) = 0$
$\therefore x - 4 = 0$ or $x + 4 = 0$
$\therefore x = 4, -4$

b
$x^2 - 3x - 10 = 0$
$(x - 5)(x + 2) = 0$
$\therefore x - 5 = 0$ or $x + 2 = 0$
$x = 5, -2$

c
$2x^2 + 3x - 20 = 0$
$(2x - 5)(x + 4) = 0$
$\therefore 2x - 5 = 0$ or $x + 4 = 0$
$\therefore x = 2\frac{1}{2}, -4$

Example 3

Solve for x.

a $3x^2 = 6x$ **b** $x^2 = 5x - 6$

Solutions

a
$3x^2 = 6x$
$3x^2 - 6x = 0$
$3x(x - 2) = 0$
$\therefore 3x = 0$ or $x - 2 = 0$
$\therefore x = 0, 2$

b
$x^2 = 5x - 6$
$x^2 - 5x + 6 = 0$
$(x - 2)(x - 3) = 0$
$\therefore x - 2 = 0$ or $x - 3 = 0$
$\therefore x = 2, 3$

Exercise 6.2

1 Solve these quadratic equations.

a $x(x - 2) = 0$ **b** $a(a + 5) = 0$ **c** $t(t - 4) = 0$
d $5m(m - 1) = 0$ **e** $2k(k + 3) = 0$ **f** $4c(c - 6) = 0$
g $(x - 1)(x - 2) = 0$ **h** $(e + 5)(e + 3) = 0$ **i** $(p + 2)(p - 4) = 0$
j $(n - 7)(n + 1) = 0$ **k** $(z + 8)(z + 6) = 0$ **l** $(q - 11)(q + 11) = 0$
m $w(2w - 1) = 0$ **n** $2f(3f - 2) = 0$ **o** $4u(5u + 3) = 0$
p $(2a - 1)(a - 4) = 0$ **q** $(4q + 3)(q - 1) = 0$ **r** $(3c - 5)(c + 6) = 0$
s $(5t - 2)(2t - 5) = 0$ **t** $(3v + 7)(2v + 3) = 0$ **u** $(9b - 4)(11b + 2) = 0$
v $(x - 3)^2 = 0$ **w** $(y + 9)^2 = 0$ **x** $(4m - 7)^2 = 0$

Consolidation

2 In each of the following, factorise the expression on the left-hand side, then solve the equation.

a $a^2 - 3a = 0$ **b** $y^2 + 7y = 0$ **c** $p^2 - p = 0$
d $6q^2 + 6q = 0$ **e** $2e^2 - 8e = 0$ **f** $5g^2 + 10g = 0$
g $n^2 - 9 = 0$ **h** $d^2 - 36 = 0$ **i** $r^2 - 1 = 0$
j $h^2 - 81 = 0$ **k** $f^2 - 64 = 0$ **l** $t^2 - 121 = 0$
m $k^2 + 4k + 3 = 0$ **n** $m^2 - 7m + 12 = 0$ **o** $x^2 + 11x + 30 = 0$

p $y^2 + 2y - 15 = 0$ **q** $a^2 - 4a - 21 = 0$ **r** $z^2 + z - 72 = 0$
s $n^2 + 2n - 24 = 0$ **t** $j^2 - 3j - 40 = 0$ **u** $r^2 - 13r + 36 = 0$
v $p^2 + 6p + 9 = 0$ **w** $h^2 - 10h + 25 = 0$ **x** $v^2 + 24v + 144 = 0$

3 Solve:
a $5y - y^2 = 0$ **b** $8c - c^2 = 0$ **c** $9d - d^2 = 0$
d $4k - 5k^2 = 0$ **e** $7e - 3e^2 = 0$ **f** $2m - 11m^2 = 0$
g $(2 - x)^2 = 0$ **h** $(10 + 3u)^2 = 0$ **i** $(5 - 6t)^2 = 0$

4 Solve these quadratic equations by at first factorising the expression on the left-hand side.
a $2x^2 + 5x + 2 = 0$ **b** $3a^2 + 13a + 4 = 0$ **c** $2p^2 + 7p + 6 = 0$
d $3u^2 - 11u + 10 = 0$ **e** $5y^2 - 22y + 8 = 0$ **f** $7c^2 - 36c + 5 = 0$
g $3e^2 + 2e - 8 = 0$ **h** $4k^2 + 11k - 3 = 0$ **i** $2m^2 - 15m - 27 = 0$
j $5v^2 - 4v - 12 = 0$ **k** $7g^2 + 31g - 20 = 0$ **l** $4d^2 - 9d - 28 = 0$
m $8t^2 + 14t + 3 = 0$ **n** $9h^2 - 9h + 2 = 0$ **o** $2n^2 - 13n - 24 = 0$
p $6w^2 + 7w + 2 = 0$ **q** $12u^2 - u - 20 = 0$ **r** $10s^2 - 21s + 9 = 0$
s $4a^2 + 4a - 35 = 0$ **t** $9j^2 - 30j + 16 = 0$ **u** $15z^2 - 16z - 15 = 0$

■ Further applications

5 Take all terms to the left-hand side, then solve each of these equations.
a $a^2 = 2a$ **b** $p^2 = -4p$ **c** $s^2 = s$
d $q^2 = 16$ **e** $b^2 = 49$ **f** $u^2 = 100$
g $x^2 + 5x = 6$ **h** $u^2 - 2u = 8$ **i** $y^2 - 11y = -28$
j $n^2 - 12 = 4n$ **k** $p^2 + 45 = 14p$ **l** $r^2 - 42 = -r$
m $c^2 = 8c + 33$ **n** $u^2 = 70 - 3u$ **o** $m^2 = 17m - 60$
p $3z^2 + 14 = 23z$ **q** $5w^2 = 17w + 12$ **r** $2k^2 = 29k - 90$

6 Form the quadratic equation in the form $ax^2 + bx + c = 0$ whose solutions are:
a $x = 2, 3$ **b** $x = 4, 1$ **c** $x = -3, -5$ **d** $x = -6, -4$
e $x = 6, -2$ **f** $x = 1, -7$ **g** $x = 11, -4$ **h** $x = -12, 3$

TRY THIS Forming a quadratic equation

Find a quadratic equation in the form $ax^2 + bx + c = 0$, where a, b and c are integers whose roots are $\frac{2}{3}$ and $-\frac{3}{4}$.

6.3 Completing the square

In the previous exercise we learned to solve quadratic equations by factorising the expression on the left-hand (or right-hand) side. However, not all quadratic expressions can be factorised. This means that some quadratic equations may have solutions, but they will not be *rational* solutions. In these cases we give the solutions either as surds in simplest form, or as decimals rounded to an appropriate degree of accuracy. One method of solving such equations relies on completing a perfect square.

To solve a quadratic equation of the form $x^2 + bx + c = 0$ by completing the square:
- **take the constant term c to the other side of the equation**
- **add $\left(\frac{b}{2}\right)^2$ to both sides of the equation to make the expression on the LHS a perfect square**
- **factorise the perfect square**
- **take the square root of both sides of the equation (take both the positive and negative square roots on the RHS)**
- **solve the two resulting equations**
- **give the solutions either as decimals or as surds in simplest form.**

NOTE: To solve an equation of the form $ax^2 + bx + c = 0$ by completing the square, we must first divide both sides of the equation by a, the co-efficient of x^2.

Example 1

Solve the following quadratic equations by completing the perfect square. Give the solutions in simplest surd form.

a $x^2 - 2x - 7 = 0$

b $x^2 + 5x + 2 = 0$

Solutions

a

$$\begin{aligned} x^2 - 2x - 7 &= 0 \\ +7 \quad &\quad +7 \\ x^2 - 2x + 1 &= 7 + 1 \\ (x - 1)^2 &= 8 \\ \sqrt{} \quad &= \sqrt{} \\ x - 1 &= \pm\sqrt{8} \\ +1 \quad &\quad +1 \\ \therefore x &= 1 \pm \sqrt{8} \\ &= 1 \pm 2\sqrt{2} \end{aligned}$$

b

$$\begin{aligned} x^2 + 5x + 2 &= 0 \\ -2 \quad &\quad -2 \\ x^2 + 5x + \left(\frac{5}{2}\right)^2 &= -2 + \left(\frac{5}{2}\right)^2 \\ \left(x + \frac{5}{2}\right)^2 &= \frac{17}{4} \\ \sqrt{} \quad &\quad \sqrt{} \\ x + \frac{5}{2} &= \pm\frac{\sqrt{17}}{2} \\ -\frac{5}{2} \quad &\quad -\frac{5}{2} \\ \therefore x &= \frac{-5 \pm \sqrt{17}}{2} \end{aligned}$$

Exercise 6.3

1 Complete each of the following perfect squares.

a $x^2 + 6x + \_\_\_ = (\_\_\_\_)^2$ **b** $x^2 + 8x + \_\_\_ = (\_\_\_\_)^2$
c $x^2 - 12x + \_\_\_ = (\_\_\_\_)^2$ **d** $x^2 - 2x + \_\_\_ = (\_\_\_\_)^2$
e $x^2 + 14x + \_\_\_ = (\_\_\_\_)^2$ **f** $x^2 - 10x + \_\_\_ = (\_\_\_\_)^2$
g $x^2 - 22x + \_\_\_ = (\_\_\_\_)^2$ **h** $x^2 + 16x + \_\_\_ = (\_\_\_\_)^2$

2 Complete each of the following perfect squares.

a $x^2 + 3x + \_\_\_ = (\_\_\_\_)^2$ **b** $x^2 + 5x + \_\_\_ = (\_\_\_\_)^2$
c $x^2 - 7x + \_\_\_ = (\_\_\_\_)^2$ **d** $x^2 - x + \_\_\_ = (\_\_\_\_)^2$
e $x^2 + 11x + \_\_\_ = (\_\_\_\_)^2$ **f** $x^2 - 9x + \_\_\_ = (\_\_\_\_)^2$

Consolidation

3 Solve each of these quadratic equations. All solutions are integers.

a $(x - 2)^2 = 9$ **b** $(x + 3)^2 = 25$ **c** $(x - 1)^2 = 4$
d $(x + 4)^2 = 16$ **e** $(x + 7)^2 = 1$ **f** $(x - 5)^2 = 49$
g $(x + 6)^2 = 121$ **h** $(x - 3)^2 = 36$ **i** $(x + 8)^2 = 64$

4 Solve the following equations, giving the solutions in simplest surd form.

a $(x - 1)^2 = 2$ **b** $(x - 2)^2 = 5$ **c** $(x + 4)^2 = 3$
d $(x + 3)^2 = 6$ **e** $(x - 7)^2 = 11$ **f** $(x + 5)^2 = 17$
g $(x - 1)^2 = 8$ **h** $(x + 3)^2 = 12$ **i** $(x - 2)^2 = 18$
j $(x + 5)^2 = 20$ **k** $(x - 4)^2 = 27$ **l** $(x + 7)^2 = 50$

5 Solve each of these equations by first taking the constant term to the right-hand side, then completing the square. Answer correct to 2 decimal places.

a $x^2 + 2x - 5 = 0$ **b** $x^2 - 2x - 1 = 0$ **c** $x^2 - 4x - 3 = 0$
d $x^2 + 6x - 5 = 0$ **e** $x^2 + 8x + 3 = 0$ **f** $x^2 - 6x + 4 = 0$
g $x^2 - 10x - 5 = 0$ **h** $x^2 + 12x - 11 = 0$ **i** $x^2 - 16x + 50 = 0$
j $x^2 + 3x - 2 = 0$ **k** $x^2 - 5x + 3 = 0$ **l** $x^2 + x - 1 = 0$
m $x^2 + 7x + 5 = 0$ **n** $x^2 + 3x - 1 = 0$ **o** $x^2 - 9x - 6 = 0$
p $x^2 - x - 4 = 0$ **q** $x^2 + 15x + 12 = 0$ **r** $x^2 - 11x + 7 = 0$

Further applications

6 Complete each of these perfect squares.

a $x^2 + \frac{x}{2} + \_\_\_ = (\_\_\_\_)^2$ **b** $x^2 - \frac{x}{3} + \_\_\_ = (\_\_\_\_)^2$

c $x^2 + \frac{x}{5} + \_\_\_ + (\_\_\_\_)^2$ **d** $x^2 + \frac{2x}{3} + \_\_\_ = (\_\_\_\_)^2$

e $x^2 - \frac{6x}{5} + \_\_\_ = (\_\_\_\_)^2$ **f** $x^2 - \frac{4x}{7} + \_\_\_ = (\_\_\_\_)^2$

g $x^2 + \frac{3x}{2} + \_\_\_ = (\_\_\_\_)^2$ **h** $x^2 + \frac{5x}{4} + \_\_\_ = (\_\_\_\_)^2$

i $x^2 - \frac{3x}{5} + \_\_\_ = (\_\_\_\_)^2$ **j** $x^2 - \frac{7x}{3} + \_\_\_ = (\_\_\_\_)^2$

7 In each of the following equations, divide through by the co-efficient of x^2, then solve for x by completing the square. Give the solutions correct to 2 decimal places.

a $2x^2 + 8x + 1 = 0$ **b** $2x^2 - 10x - 7 = 0$ **c** $3x^2 - 9x + 4 = 0$

d $4x^2 + 7x - 5 = 0$ **e** $3x^2 - 2x - 6 = 0$ **f** $5x^2 + 6x - 9 = 0$

6.4 The quadratic formula

The method of completing the square can be used to solve quadratic equations when the quadratic trinomial cannot be easily factorised, if at all. However, the method, as we have seen, can be quite difficult at times, particularly if the co-efficient of x^2 is not 1.

By re-arranging the equation $ax^2 + bx + c = 0$ so that the subject is x, we can derive a new formula called the **quadratic formula**. It can be used to solve quadratic equations quite easily, even where the co-efficient of x^2 is not 1. The proof relies on the method of completing the square.

The solutions to the equation $ax^2 + bx + c = 0$ are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof:

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{(dividing through by } a\text{)}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{(subtracting } \tfrac{c}{a} \text{ from both sides)}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{(completing the square)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\therefore\ x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{(subtracting } \tfrac{b}{2a} \text{ from both sides)}$$

Example 1

EG +S

Solve the following equations by using the quadratic formula. Answer correct to 2 decimal places where necessary.

a $2x^2 - 11x + 5 = 0$ **b** $3x^2 + 7x - 2 = 0$

Solutions

a The equation $2x^2 - 11x + 5 = 0$ is of the form $ax^2 + bx + c = 0$,
$\therefore\ a = 2, b = -11, c = 5.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{11 \pm \sqrt{(-11)^2 - 4 \times 2 \times 5}}{2 \times 2}$$
$$= \frac{11 \pm \sqrt{81}}{4}$$
$$= \frac{11 \pm 9}{4}$$
$$\therefore\ x = 5, \tfrac{1}{2}$$

b The equation $3x^2 + 7x - 2 = 0$ is of the form $ax^2 + bx + c = 0$,
$\therefore\ a = 3, b = 7, c = -2.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-7 \pm \sqrt{7^2 - 4 \times 3 \times -2}}{2 \times 3}$$
$$= \frac{-7 \pm \sqrt{73}}{6}$$
$$= \frac{-7 + \sqrt{73}}{6}, \frac{-7 - \sqrt{73}}{6}$$
$\therefore\ x = 0.26, -2.59$ (to 2 decimal places)

Example 2

Solve $x^2 + 8x - 4 = 0$ by using the quadratic formula, giving the solutions in simplest surd form.

Solution

The equation $x^2 + 8x - 4 = 0$ is of the form $ax^2 + bx + c = 0$, $\therefore\ a = 1, b = 8, c = -4.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-8 \pm \sqrt{8^2 - 4 \times 1 \times -4}}{2 \times 1}$$
$$= \frac{-8 \pm \sqrt{80}}{2}$$
$$= \frac{-8 \pm 4\sqrt{5}}{2}$$
$$\therefore\ x = -4 \pm 2\sqrt{5}$$

Example 3

Show that the equation $3x^2 - 4x + 8 = 0$ has no solutions.

Solution

The equation $3x^2 - 4x + 8 = 0$ is of the form $ax^2 + bx + c = 0$, $\therefore\ a = 3, b = -4, c = 8.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{4 \pm \sqrt{(-4)^2 - 4 \times 3 \times 8}}{2 \times 3}$$
$$= \frac{4 \pm \sqrt{16 - 96}}{6}$$
$$= \frac{4 \pm \sqrt{-80}}{6}$$

As we cannot find the square root of a negative number, the equation has no solutions.

Exercise 6.4

1 Solve these equations by using the quadratic formula. All solutions are integers.

a $x^2 + 3x + 2 = 0$	**b** $x^2 + 8x + 12 = 0$	**c** $x^2 + 7x + 10 = 0$
d $x^2 - 8x + 15 = 0$	**e** $x^2 + 4x - 12 = 0$	**f** $x^2 - 10x + 24 = 0$
g $x^2 - 3x - 10 = 0$	**h** $x^2 + 7x - 8 = 0$	**i** $x^2 + 8x - 33 = 0$
j $x^2 + x - 42 = 0$	**k** $x^2 - 3x - 28 = 0$	**l** $x^2 - 14x + 40 = 0$

■ Consolidation

2 Solve these equations by using the quadratic formula. All solutions are rational.

a $2x^2 + 7x + 3 = 0$	**b** $2x^2 + 13x + 15 = 0$	**c** $3x^2 + 10x + 8 = 0$
d $2x^2 - 9x + 10 = 0$	**e** $3x^2 - 14x + 15 = 0$	**f** $4x^2 + x - 3 = 0$
g $5x^2 + 4x - 12 = 0$	**h** $7x^2 - 17x - 12 = 0$	**i** $4x^2 - 23x - 35 = 0$

3 Use the quadratic formula to solve each of these equations. Give your solutions in surd form.

a $x^2 + 3x + 1 = 0$	**b** $x^2 + 5x + 2 = 0$	**c** $x^2 - 7x + 4 = 0$
d $x^2 + 7x - 2 = 0$	**e** $x^2 - 9x - 5 = 0$	**f** $x^2 - 5x - 3 = 0$
g $2x^2 + 9x + 5 = 0$	**h** $2x^2 - 5x - 2 = 0$	**i** $3x^2 + 3x - 4 = 0$
j $4x^2 - 7x + 1 = 0$	**k** $3x^2 - 5x + 1 = 0$	**l** $5x^2 + x - 3 = 0$

4 Solve these equations by using the quadratic formula. Give your solutions in simplest surd form.

a $x^2 + 4x + 1 = 0$	**b** $x^2 - 6x + 2 = 0$	**c** $x^2 + 2x - 1 = 0$
d $x^2 - 4x - 2 = 0$	**e** $x^2 - 8x + 6 = 0$	**f** $x^2 + 4x - 3 = 0$
g $x^2 - 10x + 3 = 0$	**h** $x^2 + 6x - 3 = 0$	**i** $x^2 - 10x + 5 = 0$
j $2x^2 + 6x + 1 = 0$	**k** $3x^2 - 6x + 2 = 0$	**l** $3x^2 - 4x - 2 = 0$
m $6x^2 + 4x - 1 = 0$	**n** $2x^2 + 8x + 5 = 0$	**o** $5x^2 - 12x + 3 = 0$
p $4x^2 - 2x - 1 = 0$	**q** $3x^2 - 10x + 5 = 0$	**r** $6x^2 + 4x - 3 = 0$

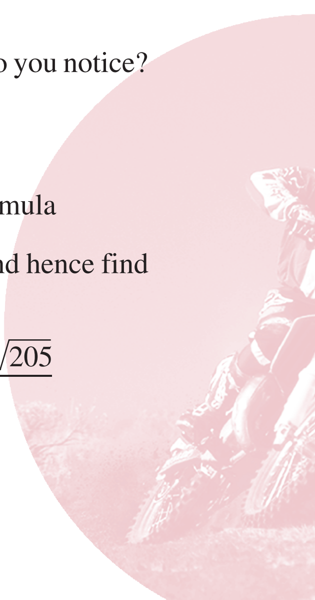
5 **a** Solve the equation $4x^2 + 12x + 9 = 0$ by using the quadratic formula.
b How many solutions are there?
c What does this tell you about the expression $4x^2 - 12x + 9$?

6 **a** Solve the equation $2x^2 - 3x + 6 = 0$ by using the quadratic formula. What do you notice?
b How could you determine whether a quadratic equation has solutions?

■ Further applications

7 Equations of the form $ax^2 + bx + c = 0$ were solved by using the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. By equating the relevant parts, find values for a, b, c and hence find the equation that has been solved in each case.

a $x = \dfrac{7 \pm \sqrt{29}}{10}$	**b** $x = \dfrac{-5 \pm \sqrt{97}}{4}$	**c** $x = \dfrac{11 \pm \sqrt{205}}{6}$



TRY THIS **Chocolate time!**

182 chocolates are equally divided among a number of people at a function. If the number of chocolates that each person receives is 1 more than the number of people, find how many people were at the function.

6.5 Miscellaneous equations

We have now seen that there are three methods that can be used to solve quadratic equations. They are:

- factorisation
- completing the square
- the quadratic formula.

Before solving a quadratic equation, it must first be written in the form $ax^2 + bx + c = 0$. The equation can then be solved by using the most appropriate method.

To solve a quadratic equation:
- **write the equation in the form $ax^2 + bx + c = 0$**
- **factorise the expression on the left-hand side if it factorises easily, or**
- **use the method of completing the square if the co-efficient of x^2 is 1 and the co-efficient of x is even, or**
- **use the quadratic formula if the co-efficient of x^2 is not equal to 1 and/or the co-efficient of x is odd.**

Example 1

Solve the following equations using the most appropriate method. Answer correct to 2 decimal places where necessary.

a $x^2 = 7x - 12$ **b** $x(x - 2) = 10$ **c** $3x^2 + 4x + 3 = x^2 - 3x - 1$

Solutions

a
$$\begin{aligned} x^2 &= 7x - 12 \\ x^2 - 7x + 12 &= 0 \\ (x - 3)(x - 4) &= 0 \\ \therefore x &= 3, 4 \end{aligned}$$

b
$$\begin{aligned} x(x - 2) &= 10 \\ x^2 - 2x &= 10 \\ x^2 - 2x + 1 &= 10 + 1 \\ (x - 1)^2 &= 11 \\ \sqrt{} \quad & \quad \sqrt{} \\ x - 1 &= \pm\sqrt{11} \\ \therefore x &= 1 \pm \sqrt{11} \\ &\doteqdot 4.32, -2.32 \end{aligned}$$

c $3x^2 + 4x + 3 = x^2 - 3x - 1$

Take all terms to LHS.

$$2x^2 + 7x + 4 = 0$$
$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times 4}}{2 \times 2} \\ &= \frac{-7 \pm \sqrt{17}}{4} \\ &\doteqdot -0.72, -2.78 \end{aligned}$$

EG +S

Example 2

Solve the following equations. Give the solutions in simplest surd form where necessary.

a $x + \frac{1}{x} = 5$ **b** $\frac{4}{x} - x = \frac{16}{3}$

Solutions

We need to multiply both sides of the equation by the lowest common denominator.

a
$$x + \frac{1}{x} = 5$$
$$\times x \qquad \times x$$
$$x^2 + 1 = 5x$$
$$x^2 - 5x + 1 = 0$$
$$x = \frac{5 \pm \sqrt{(-5)^2 - 4 \times 1 \times 1}}{2 \times 1}$$
$$\therefore x = \frac{5 \pm \sqrt{21}}{2}$$

b
$$\frac{4}{x} - x = \frac{16}{3}$$
$$\times 3x \qquad \times 3x$$
$$12 - 3x^2 = 16x$$
$3x^2 + 16x - 12 = 0$ (taking all terms to the RHS)
$$(3x - 2)(x + 6) = 0$$
$$\therefore x = \tfrac{2}{3}, -6$$

Exercise 6.5

1 Solve each of the following equations. Give your answers correct to 2 decimal places where necessary.

a $x^2 - 2x = 0$ **b** $x^2 - 25 = 0$ **c** $4x^2 - 36 = 0$
d $(x - 3)^2 = 4$ **e** $(x + 1)^2 = 7$ **f** $(2x - 1)^2 = 10$
g $x^2 + 5x + 4 = 0$ **h** $x^2 - 9x + 20 = 0$ **i** $x^2 - 4x - 12 = 0$
j $x^2 + 7x + 2 = 0$ **k** $x^2 - 2x - 5 = 0$ **l** $x^2 - 9x + 4 = 0$
m $2x^2 + 11x + 12 = 0$ **n** $3x^2 + 13x - 10 = 0$ **o** $6x^2 - 7x + 2 = 0$
p $3x^2 - 8x + 3 = 0$ **q** $4x^2 - 17x - 5 = 0$ **r** $10x^2 + x - 1 = 0$

2 Solve each of these quadratic equations, correct to 2 decimal places where necessary.

a $x^2 = 5x$ **b** $2x = x^2$ **c** $3x^2 = 12x$
d $x^2 + x = 12$ **e** $x^2 + 30 = 11x$ **f** $x^2 = 5x - 6$
g $3x^2 + 8x = 3$ **h** $2x^2 + 20 = 13x$ **i** $4x^2 = x + 14$
j $x^2 - 6x = 4$ **k** $x^2 = 12 - 5x$ **l** $10x = 3 + 6x^2$

■ Consolidation

3 Solve each of these quadratic equations.

a $x^2 + 11x = 3x - 15$ **b** $x^2 + 6 = 9x - 8$
c $x^2 + x = 10 - 2x$ **d** $x^2 - 5x - 20 = 8 - 2x$
e $x^2 + 2x = 9 - 5x - x^2$ **f** $4x^2 - 13x + 11 = x^2 + 6x - 9$
g $x^2 - 5x = x - 1$ **h** $x^2 + 4 = 12x - 5$
i $x^2 + 7x + 3 = 6 + 4x$ **j** $5 + x^2 = 13 + 3x - x^2$
k $3x^2 + 4x + 3 = x^2 - 5x + 7$ **l** $5x - 4x^2 = 8 - 11x + 3x^2$

4 Solve the following equations correct to 2 decimal places.

a $x(x + 5) = 24$
b $x(x - 2) = 8$
c $x^2 = 3(4x - 9)$
d $2(x^2 + 6) = 11x$
e $(x - 4)(x + 4) = 6x$
f $9x = (6 + x)(6 - x)$
g $x(x + 2) = 4$
h $x^2 = 2(3x - 1)$
i $3(4 + x^2) = -14x$
j $(x + 1)(x - 1) = 9x$
k $(2x + 1)(x + 5) = 2x$
l $(3x - 1)(3x + 1) = x$

5 Solve for x, giving the solutions in simplest surd form, where necessary.

a $x = \dfrac{7x - 12}{x}$
b $x = \dfrac{3x + 40}{x}$
c $\dfrac{35 - x^2}{x} = 2$
d $x = 9 - \dfrac{18}{x}$
e $x - \dfrac{20}{x} = 8$
f $\dfrac{5}{x} + 9 = 2x$
g $x = \dfrac{7x - 3}{x}$
h $x = \dfrac{3}{x - 1}$
i $\dfrac{1 + 4x}{2x} = x$
j $x + \dfrac{1}{x} = 4$
k $x - 6 = \dfrac{2}{x}$
l $\dfrac{3}{x} = 8 - 2x$

6 Solve for x, giving the solutions correct to 2 decimal places.

a $x - \dfrac{7}{x} = \dfrac{2}{3}$
b $\dfrac{15}{x} - x = \dfrac{17}{2}$
c $2x + \dfrac{1}{x} = 3\tfrac{2}{3}$
d $\dfrac{1}{x} - x = \dfrac{1}{4}$
e $x - \dfrac{2}{5} = \dfrac{2}{x}$
f $\dfrac{3}{x} - \dfrac{7x}{2} = 4$

■ Further applications

7 Solve the following equations simultaneously.

a $y = x^2$ and $y = 4$
b $y = x^2$ and $y = -3x$
c $y = x^2$ and $y = 2x + 3$
d $y = x^2$ and $y = 9x - 20$
e $y = x^2 + x$ and $y = x + 1$
f $y= x^2 + 3$ and $y = 6x - 2$
g $y = x^2 - x$ and $y = 5x - 9$
h $y = x^2 + 2x - 11$ and $y = 4$
i $y = x^2 - x - 2$ and $y = x + 6$
j $y = x^2 - 5x + 8$ and $y = 2x - 4$
k $y = 3x^2$ and $y = 10 - x$
l $y = 2x^2$ and $y = 9x - 10$

8 Solve the following equations using the substitution $u = x^2$.

a $x^4 - 5x^2 + 4 = 0$
b $x^4 - 10x^2 + 9 = 0$
c $x^4 - 13x^2 + 36 = 0$
d $x^4 + 3x^2 - 4 = 0$
e $x^4 - 2x^2 - 8 = 0$
f $x^4 - 6x^2 - 27 = 0$
g $x^4 - 17x^2 + 16 = 0$
h $x^4 - 8x^2 - 9 = 0$
i $x^4 - 29x^2 + 100 = 0$
j $x^4 - 23x^2 - 50 = 0$
k $x^4 - 50x^2 + 49 = 0$
l $x^4 - 20x^2 + 64 = 0$

9 Solve the following equations using the substitution $u = x^2$.

a $3x^4 - x^2 - 2 = 0$
b $2x^4 - 17x^2 - 9 = 0$
c $4x^4 -19x^2 + 12 = 0$

10 Solve each of these equations by using a suitable substitution.

a $(2^x)^2 - 5 \times 2^x + 4 = 0$
b $(2^x)^2 - 12 \times 2^x + 32 = 0$
c $(3^x)^2 - 28 \times 3^x + 27 = 0$

6.6 Practical problems involving quadratic equations

Many practical problems can be solved by forming and solving quadratic equations. An important point to note, however, is that although the equation usually results in two solutions, they may not both be solutions to the original problem. Each solution must be examined in the context of the problem. In many questions, a negative solution will not be a valid solution to a practical problem.

Example 1

The product of two consecutive positive numbers is 20. Find the numbers.

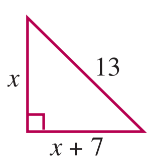
Solution

Let the numbers be x and $x + 1$. [$x > 0$]

$$\begin{aligned} x(x+1) &= 20 \\ x^2 + x &= 20 \\ x^2 + x - 20 &= 0 \\ (x+5)(x-4) &= 0 \\ \therefore x &= -5, 4 \end{aligned}$$

But $x > 0$, $\therefore$ we reject the solution $x = -5$.
That is, $x = 4$ only.
$\therefore$ The numbers are 4, 5. [If $x = 4$, then $x + 1 = 5$.]

Example 2



Find the value of x.

Solution

$$\begin{aligned} x^2 + (x+7)^2 &= 13^2 \text{ (by Pythagoras' theorem)} \\ x^2 + x^2 + 14x + 49 &= 169 \\ 2x^2 + 14x - 120 &= 0 \\ x^2 + 7x - 60 &= 0 \text{ (dividing both sides by 2)} \\ (x+12)(x-5) &= 0 \\ \therefore x &= -12, 5 \end{aligned}$$

But, $x \neq -12$, otherwise two sides of the triangle would have negative lengths.
That is, $x = 5$ only.

Exercise 6.6

1 Form a quadratic equation and solve it to answer the following problems.

- **a** Find two consecutive positive numbers whose product is 30.
- **b** The product of two consecutive positive numbers is 56. Find the numbers.
- **c** The product of two consecutive even numbers is 24. Find the numbers.
- **d** The product of two consecutive odd numbers is 63. Find the numbers.
- **e** Find two positive numbers whose product is 88, if the larger number is 3 more than the smaller number.

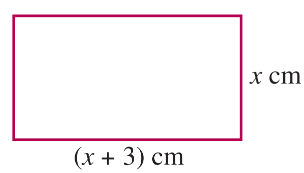
2 **a** When a positive integer is added to its square, the result is 72. Find the number.
- **b** The difference between a positive integer and its square is 42. Find the number.

3 When a number is doubled and then subtracted from its square, the result is 15. Find the two possible numbers.

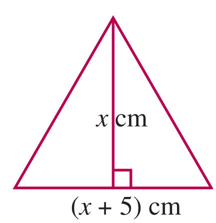
4 **a** Two positive numbers differ by 3 and the square of their sum is 169. What are the numbers?
- **b** Two positive numbers differ by 5 and the sum of their squares is 53. What are the numbers?
- **c** Find two consecutive positive integers such that the sum of their squares is 61.
- **d** Find four consecutive positive integers such that the sum of the squares of the second and fourth numbers is 202.

Consolidation

5 The length of a rectangle is 3 cm greater than the breadth. If the area of the rectangle is 180 cm^2, find its dimensions.



6 The base of a certain triangle is 5 cm longer than the altitude, and the area is 42 cm^2. Find the length of the base.



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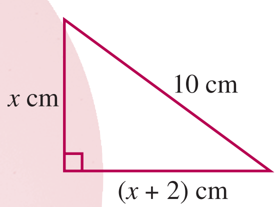
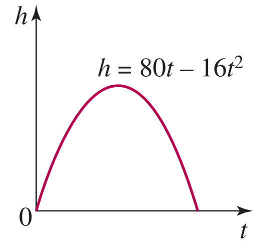


In a right-angled triangle, the length of the hypotenuse is 10 cm. Of the two shorter sides, one side is 2 cm longer than the other side.

- **a** Find the value of x.
- **b** Hence, find the area of the triangle.

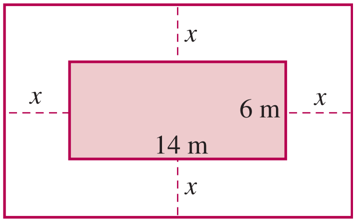
8 The sum S of the first n counting numbers is given by the formula $S = \frac{n}{2}(n + 1)$. How many numbers must be added for the sum to equal 66?

9 Daniel throws a stone vertically upwards and after t seconds its height h metres above the ground is given by the formula $h = 80t - 16t^2$.



a At what time(s) will the stone be 96 metres above the ground?
b After how many seconds will the stone hit the ground?
c Can the stone ever reach a height of 128 metres?

10 A rectangular swimming pool is 14 m long and 6 m wide. It is surrounded by a pebble path of uniform width x m. If the area of the path is 96 m^2, find the width.



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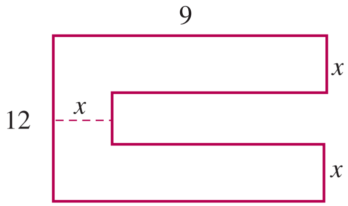


The figure shown has an area of 88 cm^2. Find the value of x.

12 The sum of the areas of two circles is 106π cm^2 and the radius of the larger circle is 4 cm longer than the radius of the smaller circle. Find the lengths of the radii.

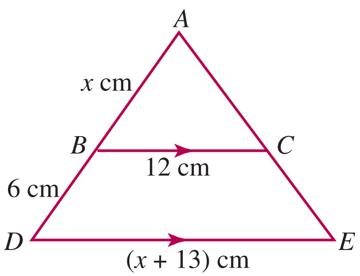
13 The sum of the terms in the series 3 + 7 + 11 + 15 + 19 + . . . can be found by using the formula $S = \frac{n}{2}[2a + (n - 1)d]$, where:

- S is the sum of the terms,
- n is the number of terms and
- a is the first term,
- d is the common difference between the terms.

a What is the value of a?
b By how much are the terms increasing each time? This is called the common difference.
c How many terms of this series would need to be added for the sum to be 820?

Further applications

14



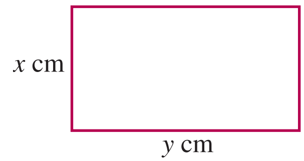
In this diagram, ΔABC is similar to ΔADE. Find the value of x.

15 The sum of a number and its reciprocal is $2\frac{9}{10}$. What are the numbers?

16 The perimeter of a rectangle is 22 cm and the area is 28 cm^2.

a Show that $y = 11 - x$ and $x^2 - 11x + 28 = 0$.

b Hence, find the dimensions of the rectangle.

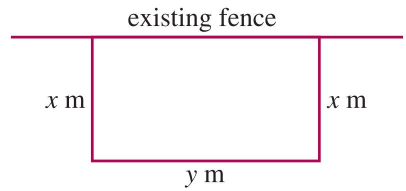


17 Find the dimensions of a rectangle that has a perimeter of 46 cm and an area of 102 cm^2.

18 A farmer used 100 metres of fencing to build a new enclosure as shown, by using an existing fence as one side.

a Find an expression for the area of the enclosure in terms of x.

b Find all possible dimensions of the enclosure if it has an area of 1200 m^2.

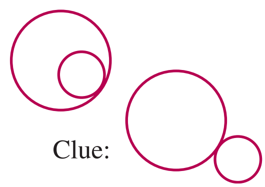


TRY THIS Touching circles

Can you draw three circles each of which touches the other two? (It can be done in two ways!)

Now can you draw a fourth circle that touches each of the three circles? (It can be done in two ways for each of the ways of drawing the three circles!)

Clue:



A **very** peculiar 2nd degree (quadratic) equation relates the radii for the four circles:

$$\left(\frac{1}{r_1}\right)^2 + \left(\frac{1}{r_2}\right)^2 + \left(\frac{1}{r_3}\right)^2 + \left(\frac{1}{r_4}\right)^2 = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right)^2$$

If the first three circles have radii 1 cm, $\frac{1}{2}$ cm and $\frac{1}{3}$ cm, find the radius of the fourth circle.

Naturally, being a quadratic equation it will have two solutions fitting two different diagrams. One of the answers is negative. Why?

THE LONG DISTANCE FLIGHT OF ARROWS UNDER GRAVITY

Introduction

Legolas, one of the main characters in *The Lord of the Rings*, was an expert marksman with his bow and arrows. He was able to kill orcs and other evil creatures of Mordor at a great distance by firing upwards at an angle, to allow the arrow to fly through the air in a curve. The speed of the arrow leaving the bow enabled him to shoot horizontally as well, so that the path of the arrow was very close to a straight line. In the battle scenes there were hundreds of archers in the armies of Sauron who fired their arrows together with great effect. In medieval times bows and arrows were used extensively throughout Europe as weapons of war. Robin Hood was an expert with the bow too.

In this activity we are going to look at the long distance flight of an arrow fired at speed from a bow.

LEARNING ACTIVITIES

1 An arrow is fired at high speed from a bow at an acute angle to the horizontal. Draw a rough sketch of the path of the arrow through the air. What is the name of the curve traced out by arrows and other missiles fired at high speed in such a way?

2 The height y (metres) after time t (seconds) of an arrow fired at an acute angle to the horizontal is modelled by the equation

$$y = 40t - 5t^2.$$

Draw up a table of values and plot the height of the arrow from $t = 0$ to $t = 8$. Use a scale of 1 unit = 1 on the x-axis and 1 unit = 20 on the y-axis. Draw a smooth curve through the points.

3 What is the time taken to reach the greatest height? What is this height?

4 Write down the time of the flight—the time that has elapsed before the arrow hits the ground.

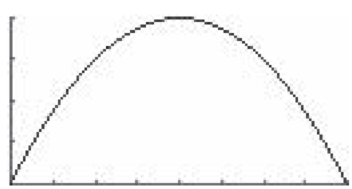
5 Why do you think the graph is symmetrical? In free flight under gravity, does the time to reach the greatest height always equal the time to come down?

6 How high is the arrow after 1 second? Look at the graph. After how many seconds will the arrow again be at this height?

7 If you have access to a graphics calculator draw the graph of $y = 40t - 5t^2$. Here is the key sequence for the **Casio CFX-9850G** or **CFX-9850GB Plus** (ask your teacher if you are using another calculator). Note that the graphics calculator uses x instead of t for the independent variable, so it will draw $y = 40x - 5x^2$, which has exactly the same shape.

- **a** On the Main Menu select the **GRAPH** icon and press **EXE**.
- **b** Specify **the range of the *x*- and *y*-axes**. It will be useful if you use the same ranges as on the hand drawn graph. Hold down the yellow **SHIFT** key and press **F3** to display the **View Window**. Enter the value 0 for *X*min and press the **EXE** key. The calculator automatically selects the next parameter. Now enter 8 for *X*max press **EXE** and continue: *X*scale 1, **EXE**, *Y*min 0, **EXE**, *Y*max 80, **EXE**, *Y* scale 20 **EXE**. The screen will look like this:

- **c** When you have finished press the **EXIT** key.
- **d** You will now be returned to the **GRAPH** function screen. Type in $40x - 5x^2$ as follows: $\boxed{4}$, $\boxed{0}$, $\boxed{\text{X}, \theta, \text{T}}$, $\boxed{-}$, $\boxed{5}$, $\boxed{\text{X}, \theta, \text{T}}$, $\boxed{x^2}$ and press **EXE**. Now press **F6** and the graph is drawn.



- **e** Once the graph has been drawn, press F1 (TRACE) and use the cursor arrow keys to move along the graph. The co-ordinates of the point are displayed. Check out the highest point and the point where the arrow hits the ground.

Challenge activities

1 Draw the tangent to your graph at the highest point to show the direction of the speed of the arrow. What do you notice?

2 Draw a tangent to the curve to show the direction of the speed of the arrow at $t = 0$. Now draw the tangent to the curve at $t = 8$. What do you notice? If the arrow is fired at a speed of 56 m s^{-1} what would you expect its speed to be when it hits the ground?

3 The distance x (metres) travelled horizontally after time t is modelled by the equation $x = 40t$. Find the range of the flight, the distance travelled before it hits the ground.

4 Show by substituting t in terms of x into the equation $y = 40t - 5t^2$ show that

$$y = x - \frac{x^2}{320}$$

Draw the graph on your graphics calculator from $x = 0$ to 320, using a scale of 1 unit = 80 on the x-axis. Show that it is also a parabola, and read off the range. Does it agree with your value in question 3?

Let's communicate

Create a poster to show what you have learned about the flight of arrows under gravity. Include the symmetry and the name of the curve, and show the angle of projection, the greatest height, the time of flight, and the range.

Reflecting

Quadratic equations often crop up in the study of the natural world. The motion of projectiles under gravity is just one example. Reflect on how mathematics enables us to explain and predict the physical world.

Language link with Macquarie

MACQUARIE

1 Give a brief explanation of:

*a a **quadratic equation***

*b the **roots of a quadratic equation***

c how, given values for a *and* c, *the quadratic equation* $ax^2 = c$ *has no root, one root or two roots. Give an example in each case.*

2 The Macquarie Learners Dictionary *does not define the word* quadratic. *However the prefix* quad *means four—think of* quad*rilateral for example. Write down some other mathematical words with the prefix* quad.

3 So what is the origin of the word? One explanation is that quadratic *is derived from the Latin word* quadrare, *which means 'to make square'. So part of the word is connected to the Latin word for 'four', though not in a way in which one might expect. It refers to squaring, and a square is a regular four-sided figure.*

1 State whether or not each expression is a quadratic expression.

a $4x^2$ b $2x^2 - 3x + 1$

c $\frac{1}{x^2}$ d $(x + 3)^2$

e $2 - 5x - x^2$ f $\sqrt{x^2 - 4}$

2 Determine a quadratic relationship in the form $y = ax^2 + c$ for each table of values.

a

x	0	1	2	3	4
y	−4	−3	0	5	12

b

x	0	1	2	3	4
y	3	5	11	21	35

3 Determine a quadratic relationship in the form $x^2 + bx + c$ for each table of values.

a

x	0	1	2	3	4
y	−2	2	8	16	26

b

x	0	1	2	3	4
y	7	4	3	4	7

4 Solve:

a $m(m - 6) = 0$

b $x(x + 8) = 0$

c $3t(t + 1) = 0$

d $4n(3n - 2) = 0$

e $(p - 3)(p + 5) = 0$

f $(3k - 2)(2k + 5) = 0$

5 Solve each equation by first factorising the expression on the LHS.

a $p^2 - 4p = 0$

b $5u^2 + 15u = 0$

c $n^2 - 49 = 0$

d $9 - e^2 = 0$

e $x^2 + 10x + 24 = 0$

f $c^2 - c - 12 = 0$

g $y^2 - 12y + 36 = 0$

h $a^2 + 4a - 32 = 0$

i $2q^2 - 17q + 21 = 0$

j $5t^2 + 21t - 20 = 0$

6 Solve:

a $a^2 = 16$ b $t^2 = 5t$

c $b^2 - 2b = 15$ d $y^2 - 8 = 2y$

e $k^2 = 13k - 22$ f $7w = 4 - 2w^2$

7 Solve the following equations. Give the solutions in simplest surd form where necessary.

a $(x - 10)^2 = 36$ b $(x + 6)^2 = 50$

8 Solve each equation by completing the square. Give the solutions correct to 2 decimal places.

a $x^2 + 2x - 7 = 0$

b $x^2 - 6x - 15 = 0$

c $x^2 + 3x - 2 = 0$

d $x^2 - 5x + 3 = 0$

9 Solve each equation by using the quadratic formula. Give the solutions in simplest surd form where necessary.

a $x^2 + 5x + 6 = 0$

b $3x^2 + 13x - 10 = 0$

c $x^2 - x - 5 = 0$

d $3x^2 - 9x + 4 = 0$

10 Solve for x by using the most appropriate method. Give the solutions correct to 2 decimal places where necessary.

a $x^2 - 25 = 2x + 10$

b $2x^2 + 7x = 5 - 3x - x^2$

c $(x + 4)(x - 4) = 12(x - 3)$

d $(3x - 2)(x + 9) = 5x$

e $x = \frac{x + 30}{x}$

f $x + \frac{28}{x} = 11$

g $x + \frac{9}{2x} = \frac{27}{4}$

CHAPTER REVIEW

CHAPTER REVIEW

11 Form a quadratic equation and solve it to answer the following problems.

a The product of two consecutive positive integers is 20. Find the numbers.

b The product of two positive integers is 60 and the larger number is 4 more than the smaller number. Find the numbers.

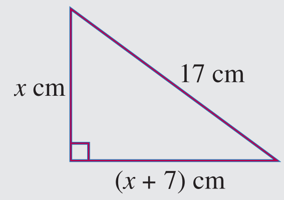
c Two positive integers differ by 7 and the sum of their squares is 85. Find the numbers.

12 The length of a rectangle is 5 cm greater than the breadth. If the area of the rectangle is 84 cm^2, find:

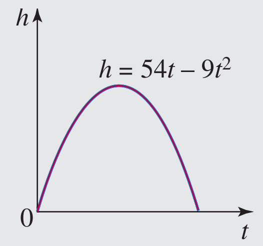
a the dimensions of the rectangle

b the perimeter of the rectangle

13 Find the value of x.



14 A rock is thrown vertically upwards and after t seconds its height h metres above the ground is given by $h = 54t - 9t^2$.



a How long will it take the rock to hit the ground?

b After how many seconds will the rock be 72 m above the ground?

c After how many seconds will the rock reach its greatest height?

7 Graphs in the number plane

This chapter at a glance

Stage 5.1/5.2/5.3

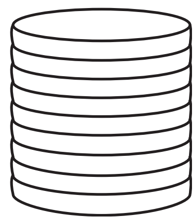
After completing this chapter, you should be able to:

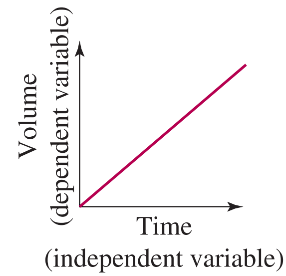
- use the relative positions of two points on a graph to interpret information
- interpret and draw distance–time graphs involving both constant and variable speed
- sketch informal graphs to model unfamiliar events
- draw and interpret travel graphs
- find the gradient and intercepts with the co-ordinate axes of a straight line
- graph a linear equation by plotting points, finding the x- and y-intercepts or by plotting the y-intercept and using the gradient to plot other points
- rearrange the equation of a line from the general form to the gradient–intercept form and vice-versa
- determine by substitution whether a given point lies on a line or curve
- explain the effect of the constants a, b and c on the shape and position of the parabola $y = ax^2 + bx + c$
- sketch parabolas with equations of the form $y = ax^2$, $y = ax^2 + c$, $y = ax^2 + bx + c$ and $y = a(x - h)^2 + k$
- find the equation of the axis of symmetry of a parabola and the co-ordinates of the vertex
- sketch cubic curves with equations of the form $y = ax^3 + c$
- sketch hyperbolas with equations of the form $y = \frac{k}{x}$ and $xy = k$
- sketch exponential curves with equations of the form $y = a^x$ and $y = a^{-x}$
- sketch circles with equations of the form $x^2 + y^2 = r^2$
- compare and contrast a mixed set of graphs and determine possible equations using their key features.

7.1 Graphs of physical phenomena

The relationship between two variables can be represented in the form of a graph. Graphs show very clearly how one quantity varies or changes with respect to another quantity.

For example, a large cylindrical tank is being filled with water at a constant rate. The volume of water that is in the tank depends on the length of time that has passed. Time passes whether or not water is being poured into the tank. That is, the passing of time is independent of the volume of water in the tank. Hence, in this case, time is said to be an **independent variable**. However, the volume of water in the tank depends directly on the length of time that has elapsed. Therefore, in this case, volume is said to be a **dependent variable**.





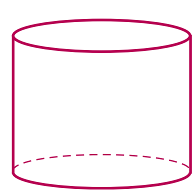
When drawing graphs of physical phenomena, it is customary to place the independent variable on the horizontal axis and the dependent variable on the vertical axis.

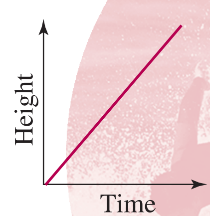
The slope or gradient of a graph indicates the **rate of change**. That is, the rate at which the quantity on the vertical axis is changing with respect to the quantity that is shown on the horizontal axis.

- **If the graph is a straight line, then the rate of change is constant.**
- **If the graph is curved, then the rate of change is not constant.**
- **If the line or curve is increasing from left to right, then the dependent variable is increasing.**
- **If the line or curve is decreasing from left to right, then the dependent variable is decreasing.**

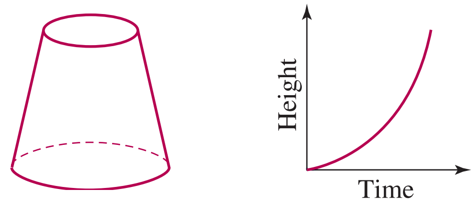
NOTE: The graph represents the relationship between two variables. It is not a physical representation of the actual situation. For example, a distance–time graph that represents a journey downhill does not decrease from left to right. The graph shows the relationship between the distance travelled and the time taken. It does not show the shape of the terrain. The graph actually increases from left to right, since the distance travelled increases with time.

Consider a cylindrical container that is filled with water. The graph of height against time would be a straight line, increasing from left to right, because the water would be rising at a constant rate.





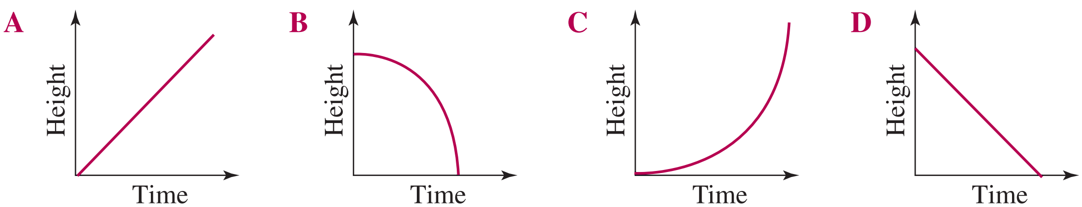
If the container is not cylindrical, but instead gradually becomes narrower, the water level would rise at an increasing rate, not a constant rate.



Example 1

The graphs below show the change in height against time as a container is either filled with water, or emptied. Which graph shows that the height is:

a increasing at a constant rate? **b** decreasing at a constant rate?
c increasing at a variable rate? **d** decreasing at a variable rate?

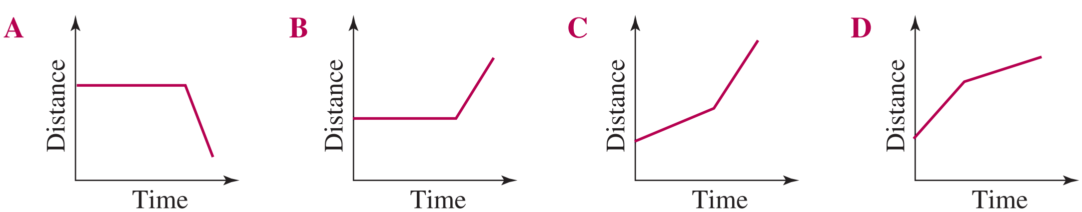


Solutions

a A **b** D **c** C **d** B

Example 2

A man is driving his car along the highway at a constant speed. He then decreases his speed because road work is being carried out. Which graph best represents the relationship between distance travelled and time taken?

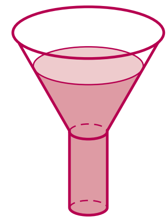


Solution

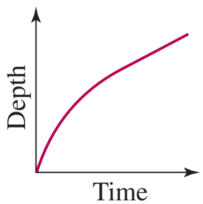
As the speed is constant initially, the distance travelled must be increasing at a constant rate. Hence, the first section of the graph must be a straight line that is increasing from left to right. When the driver slows down, the distance travelled would continue to increase, but at a decreased rate. Hence, the line would be flatter in the second section of the graph. Graph D best represents the situation.

Example 3

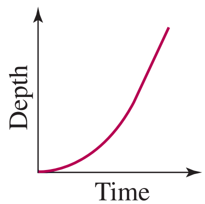
A funnel was closed at the base with a stopper, then filled with water at a constant rate. Which graph best shows the change in depth against time?



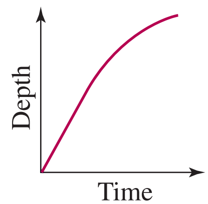
A



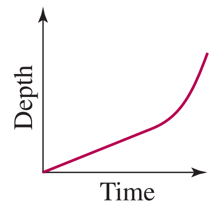
B



C



D



Solution

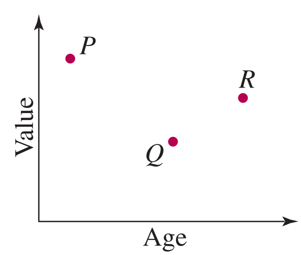
The water fills the straight narrow section of the funnel first. Since the sides are vertical the water rises at a constant rate. Hence the first section of the graph is a steep, increasing straight line.

The water then fills out gradually in the top section of the funnel. It rises at a decreasing rate because the funnel is wider towards the top. Therefore, the second section of the graph is a concave down increasing curve.

The graph which best illustrates the change in depth of the water against time is *C*.

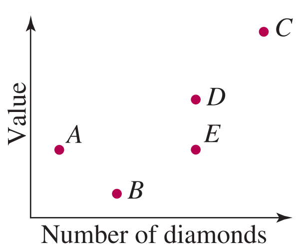
Exercise 7.1

1 The graph shows the relative ages and values of three blocks of units.



a Which building is the oldest?
b Which building is the youngest?
c Which building has the greatest value?
d Which building has the least value?

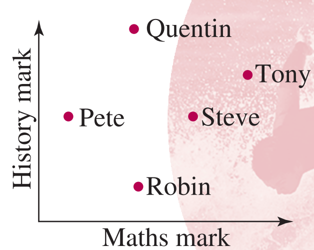
2



a Which ring is the cheapest?
b Which ring has the most diamonds?
c Which two rings have the same value?
d Which two rings have the same number of diamonds?

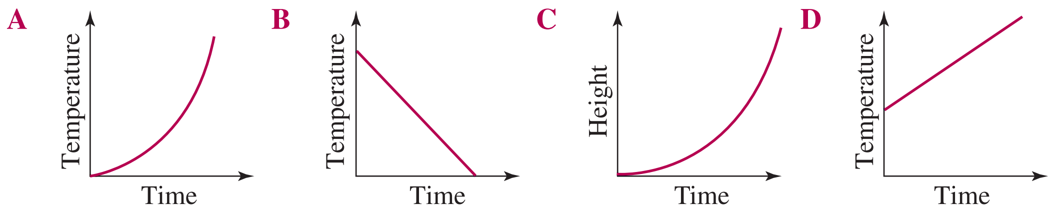
3 a Which student had:
i the highest Maths mark? **ii** the lowest Maths mark?
iii the highest History mark? **iv** the lowest History mark?

b Which two students had equal marks in:
i Maths? **ii** History?



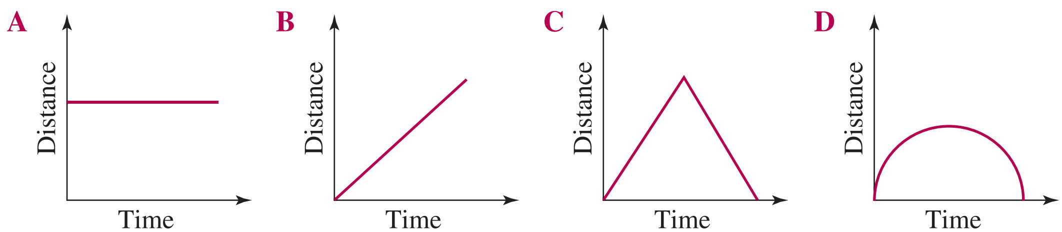
4 These graphs show the changing temperatures of four objects. Which graph(s) show that the temperature is:

a increasing?
b decreasing?
c increasing at a constant rate?
d decreasing at a constant rate?

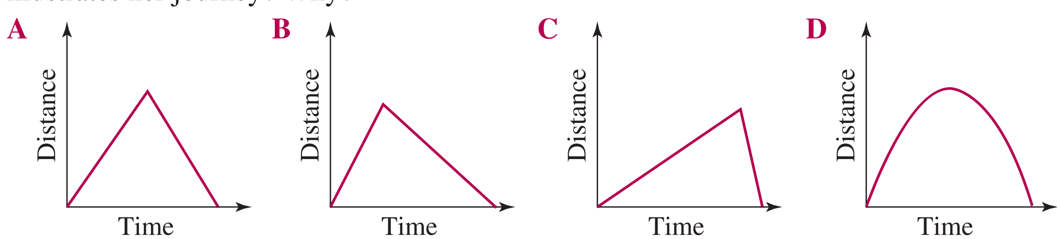


Consolidation

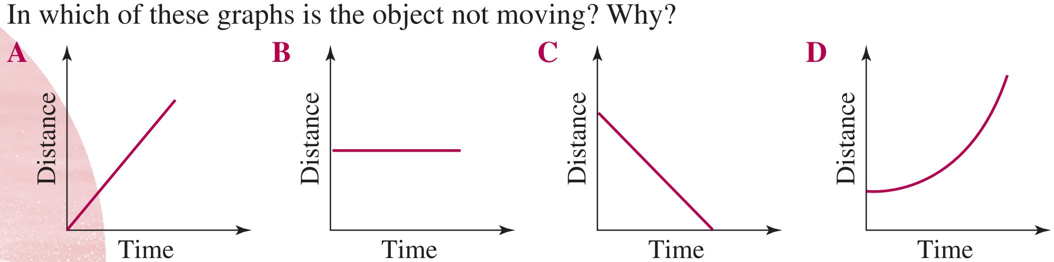
5 Yuri is an Olympic athlete. He trains every day by running laps around a circular track at a constant speed. Which graph best represents the distance that he has run at various times? Why?



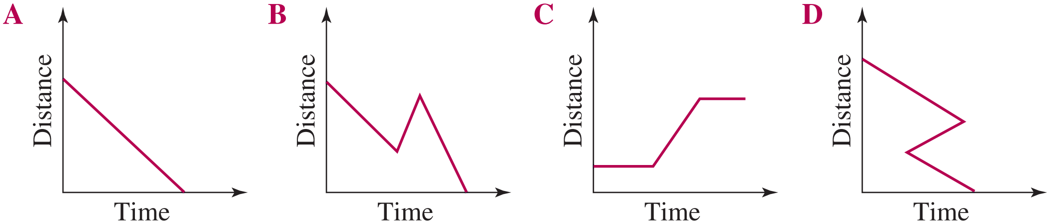
6 Karen walked up to the top of a snow-covered hill then skied back down. Which graph best illustrates her journey? Why?



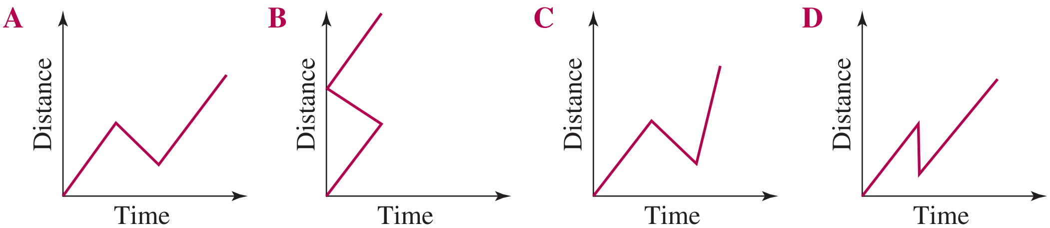
7 In which of these graphs is the object not moving? Why?



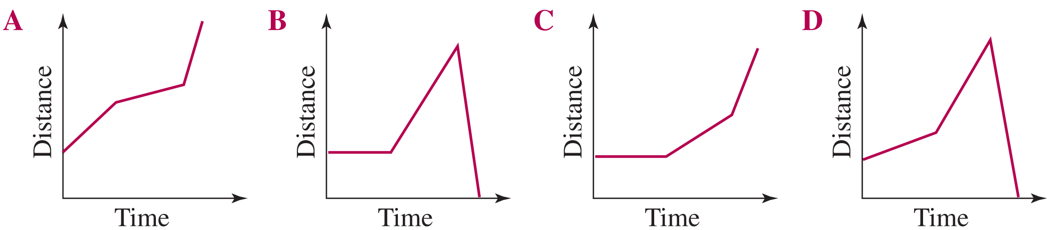
8 Which one of these distance–time graphs is impossible? Why?



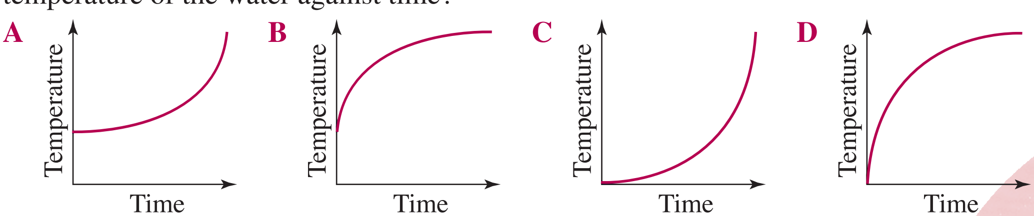
9 Helen left home and drove at constant speed along a straight stretch of highway. She was forced to take a detour back towards home due to local flooding, then continued on the highway again at her original speed. Which graph best illustrates her journey?



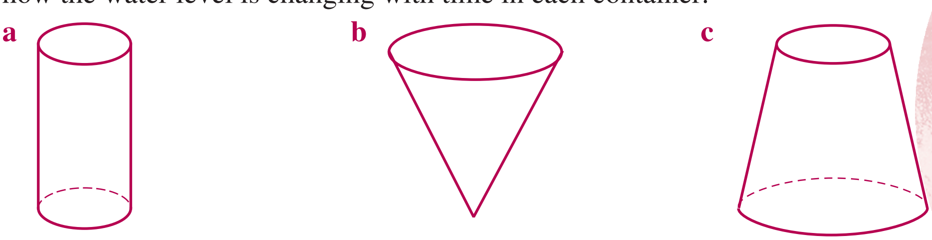
10 Grant was riding home on his bicycle from the shopping centre at a constant speed until he reached a steep hill. He rode up the hill at a steady pace, then rode very fast down the other side. Which graph best illustrates his journey?

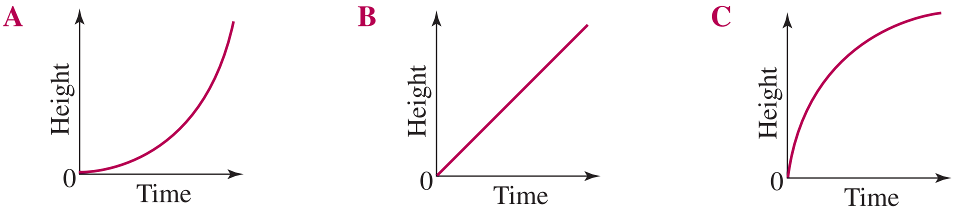


11 A kettle is filled with cold tap water and brought to the boil. Which graph best shows the temperature of the water against time?

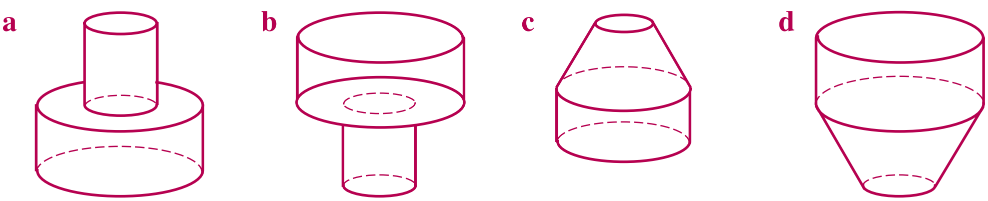


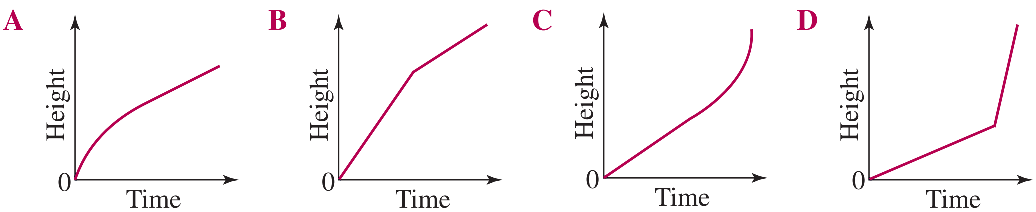
12 The containers below are filled with water at a constant rate. Choose the graph that shows how the water level is changing with time in each container.



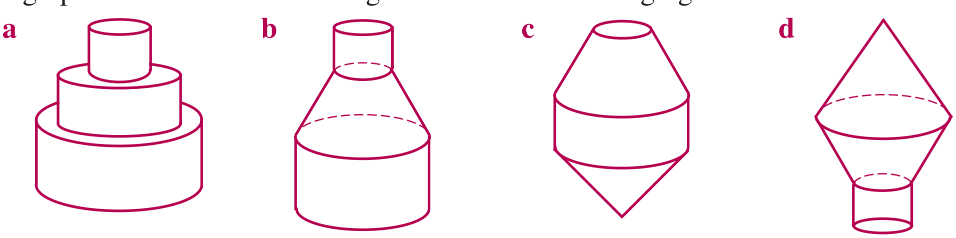


13 The containers below are to be filled with water at a uniform rate. For each container, select the graph that best shows the change in the height of the water against time, and justify your choice.

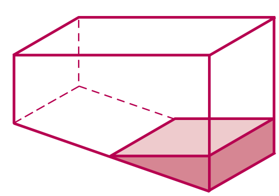




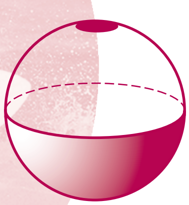
14 The containers below are to be filled with water at a constant rate. For each container, draw a graph that shows how the height of the water is changing with time.



15 A serious bacterial infection was detected in the water at a local council swimming pool. As shown in the diagram, the water was drained and the pool disinfected. Draw a graph that shows the depth of the water against time, as the pool is emptied.



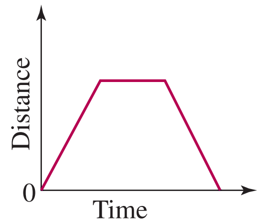
16



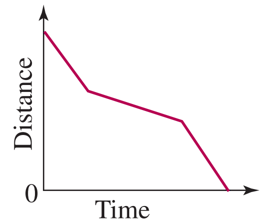
A remote controlled diving bell in the shape of a sphere was damaged after hitting rocks. It filled with water through a small hole that had opened at the top. Draw a graph showing the height of the water against time.

17 Write a short story to describe what is happening in each of these distance–time graphs.

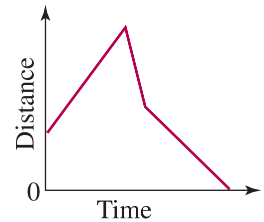
a



b

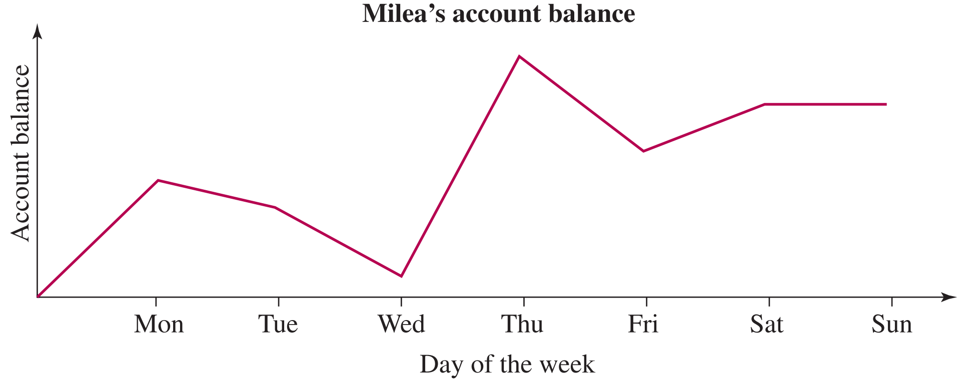


c



18 This graph shows Milea's account balance at the end of each day, for a period of one week.

Milea's account balance



- **a** Do you think that Milea had an existing bank account, or did she just open a new account? Explain?
- **b** On which days did she withdraw money?
- **c** On which day did she make her largest withdrawal?
- **d** Which day of the week do you think is pay day for Milea? Why?
- **e** On which day did she not make any deposits or withdrawals?

19 Draw a graph on 5 mm grid paper to represent each of the following stories.

- **a** A woman ran a bath for 5 minutes then got in. After 15 minutes she got out to answer the phone. She spoke on the phone for 10 minutes then got back into the bath and added more hot water for 5 minutes, at the same rate at which she filled the bath initially. After another 10 minutes she got out of the bath and emptied it, which took 5 minutes.
- **b** A woman ran a bath for 5 minutes. She put one child in and then, after 5 minutes, a second child got in. After 15 minutes, both children got out of the bath, and 5 minutes later another child got into the bath. The third child got out after 10 minutes. The bath was then emptied, which took 5 minutes.

■ Further applications

20 Draw an informal graph to model the rise and fall of the water level in a dam between January and July. Show the water level on the vertical axis and the months on the horizontal axis. Values are not required on the vertical axis.

During January it is very hot and the water level in the dam fell. There was a heat wave in February and no rain fell. The water level did not change appreciably in March due to light rains. There were moderate falls of rain during April; however, May was unseasonally hot and dry. During June, there was moderate rainfall. It rained for 15 days during July, with heavy falls leading to flooding in some areas. At the end of July, the dam levels were slightly higher than at the start of the year.

21 Draw an informal graph to model the following classroom situation. Show the noise level on the vertical axis and time on the horizontal axis. Values are not required to be shown on the axes.

The teacher, Mrs Brookes, enters a very noisy classroom for the start of her Maths lesson. She asks the class to be quiet, takes the roll, then discusses the previous night's homework with the students. Mrs Brookes then explains the new work and sets the class an exercise to complete. After a while, she notices that the level of noise in the room is excessive and asks the students to work quietly. The students comply, however, later on, the noise level creeps up again. Just then, the bell goes and the students pack up. Mrs Brookes insists on silence, says good morning to the class, then leaves the room.

22 Move along a measuring tape for 30 seconds at a constant rate in the manner suggested below. Record the distance at fixed time intervals, then draw a graph.

a Walk slowly from the beginning of the tape towards the other end.

b Walk slowly for 10 seconds, stop for 10 seconds, then continue walking at the same rate for 10 seconds.

c Walk for 10 seconds, stop for 10 seconds, then turn around and walk back to the beginning of the tape for 10 seconds.

d Start at the other end of the tape and walk back towards the beginning at a constant rate.

7.2 Travel graphs

A **travel graph** is a type of line graph. It is used to represent a journey and compares the distance travelled with the time taken. Time is always shown on the horizontal axis and distance is always shown on the vertical axis. The **slope** or steepness of the line is related to the speed of the object.

- **The slope of the line indicates the speed at which the object is moving.**
- **The steeper the line, the faster the speed. The flatter the line, the slower the speed.**
- **A horizontal line indicates that the object is stationary, i.e. not moving.**

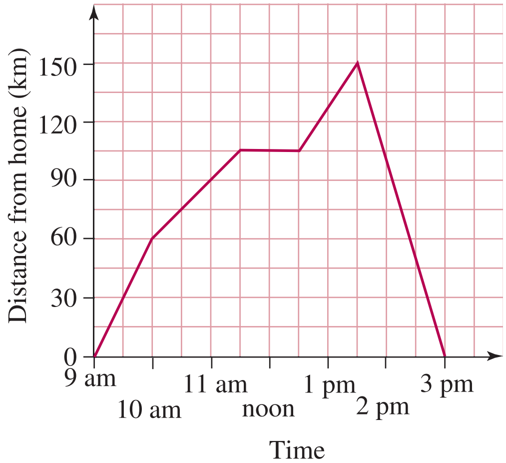
The formulae for speed, distance and time will be used frequently in this exercise.

- **Speed** $= \dfrac{\textbf{distance}}{\textbf{time}}$
- **Distance = speed × time**
- **Time** $= \dfrac{\textbf{distance}}{\textbf{speed}}$

NOTE: Travel graphs show the relationship between distance travelled and time taken. They do not represent the physical nature of the journey uphill, downhill or around corners.

Example 1

The graph shows Scott's distance from home at various times throughout the day.



a How far from home is Scott at 11 am?
b At what times is he 75 km from home?
c Between what times did he stop for lunch?
d What is the furthest distance he reached from home?
e At what time did Scott begin to drive home?
f At what time did he arrive home?
g How far did he travel between 12:30 pm and 1:30 pm?
h How fast did Scott travel on the return journey?
i Did his speed increase or decrease at 10 am? How do you know?
j How far did he travel altogether?

Solutions

a 90 km

b 10:30 am and 2:15 pm

c 11:30 am and 12:30 pm

d 150 km

e 1:30 pm

f 3 pm

g 150 km − 105 km = 45 km

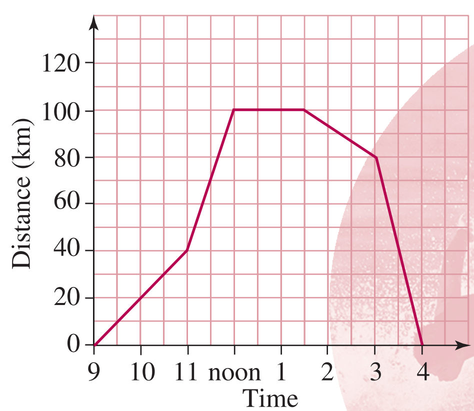
h $S = \frac{D}{T}$
$= \frac{150}{1.5}$
$= 100$ km/h

i The speed must have decreased at 10 am because the line became less steep at that time.

j 150 km + 150 km = 300 km

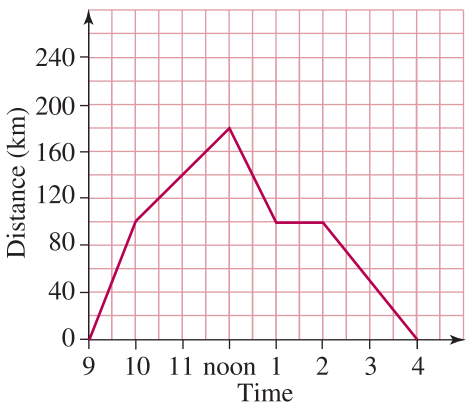
Exercise 7.2

1 The graph shows the journey taken by a woman from her home.



a How far did the woman travel in the first 2 hours?
b What happened to her speed at 11 am? How do you know?
c At what time did she stop for lunch?
d How long did she spend at lunch?
e What is the furthest distance she travelled from her home?
f For how long did she travel after lunch?
g How far did she travel altogether?

2



The graph shows the distance driven by a man from his hotel while on a business trip.

a At what time did he leave the hotel?

b How far did he drive between 10 am and noon?

c Calculate his average speed between:

i 9 am and 10 am

ii 9 am and 12 noon

d When did he change direction and begin travelling back towards the hotel?

e How far was he from the hotel when he stopped for lunch?

f How far had he driven by 9.15 am?

■ Consolidation

3 The travel graph shows how far Ryan is from town A at various times of the day.

a How far was Ryan from A when he left home at 9 am?

b How far had he travelled by 10 am?

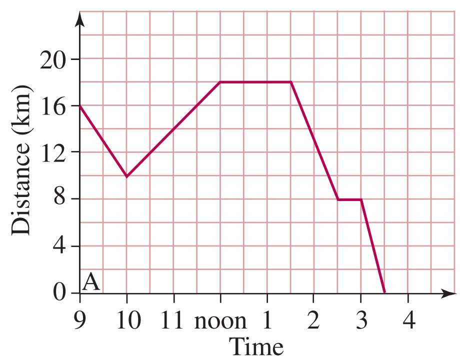
c Between what times was he travelling away from A?

d What is the furthest distance from A that Ryan travelled?

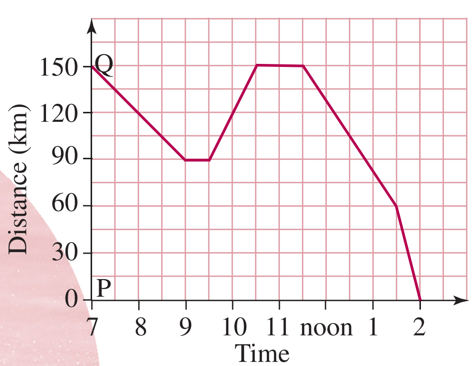
e How far did he travel between 11 am and 2.30 pm?

f At what time did he reach town A?

g How far did he travel altogether during the day?



4



The travel graph shows the trip taken by a woman from town Q to town P.

a How far apart are the two towns?

b Without calculating the actual speeds, write down the times between which her speed was greatest.

c The woman returned to Q during the day. At what time was this?

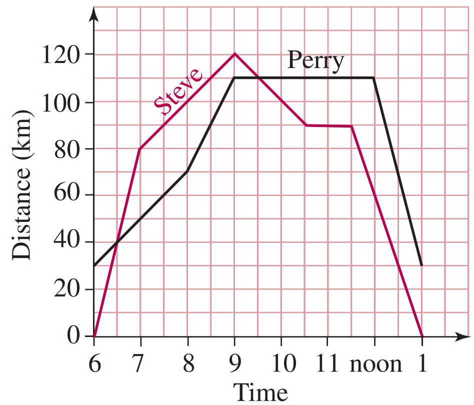
d At what times was she 45 km from Q?

e Between what times was she stopped?

f For how long was she driving during the day?

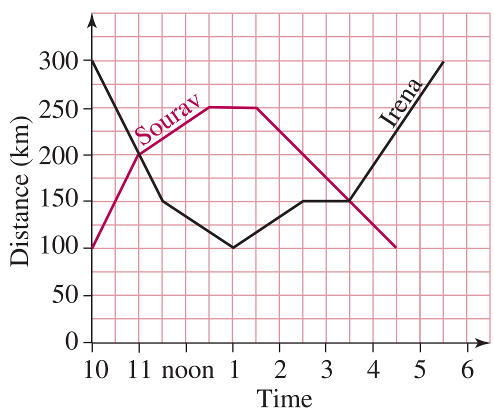
g How far did she travel between 7 am and 12.30 pm?

5 Steve and Perry undertook separate journeys during the day. Both men began their journeys from home.



- **a** When did each man leave home?
- **b** At what times did their paths cross?
- **c** How far apart do the men live?
- **d** Who was travelling faster during the first hour?
- **e** How far apart were the men when Perry's speed first increased?
- **f** How far had Steve travelled when Perry began his return trip?
- **g** Calculate each man's average speed for his trip, excluding stops.

6



Sourav and Irena deliver letters and parcels for two different express couriers. They begin each day at their respective dispatch centres. The graph shows their distance from Sydney at various times during the day.

- **a** How far apart are the dispatch centres?
- **b** At what times did they pass each other?
- **c** How far apart were they when Irena had travelled 200 km?
- **d** Who returned to their dispatch centre first?
- **e** Who drove further and by what distance?
- **f** Between what times was Irena driving at a speed of 75 km/h?
- **g** Calculate Sourav's average speed during the day, excluding any rest stops. Answer correct to 1 decimal place.

■ Further applications

7 Warren left home at 10 am and drove to his friend's house, which is 80 km away. He reached the house at noon and stayed for two hours. He then drove for 30 minutes to the shopping centre, which was 20 km away, and spent half an hour buying groceries. Warren had lunch there for 1 hour then drove home at a speed of 40 km/h for $1\frac{1}{2}$ hours, because of peak hour traffic.

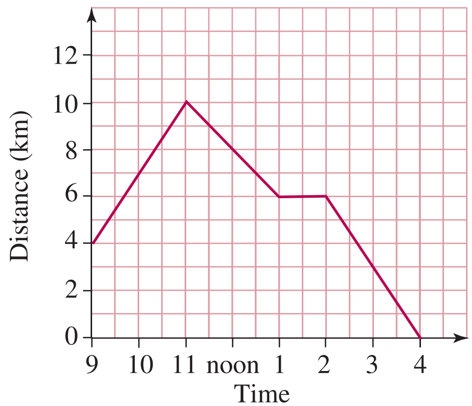
- **a** Draw a travel graph to show Warren's trip.
- **b** At what speed did he drive on the way to his friend's house?
- **c** When did he leave the shopping centre?
- **d** At what time did Warren reach home after lunch?
- **e** At what times during the day was Warren 50 km from home?

8 Michelle lives on a farm 60 km from Gunnedah. She left home at 9 am and drove into town at a speed of 30 km/h. She shopped for 1 hour, then drove back towards home. Having driven for 30 minutes at 60 km/h, Michelle arrived at her parents' house, where she stayed for 2 hours. At 2.30 pm she left and drove home, arriving 1 hour later.

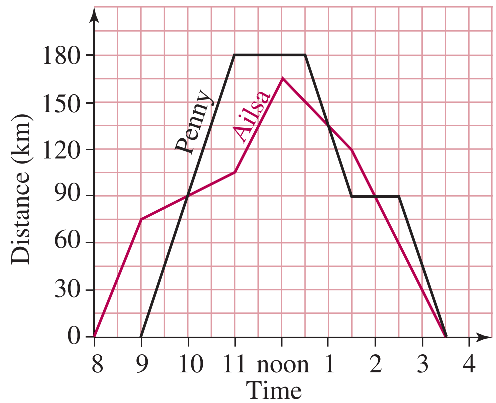
a Draw a travel graph to show Michelle's journey.
b How long did it take her to drive into town in the morning?
c At what time did she reach her parents' house?
d How fast did Michelle drive between her parents' house and home?
e How far did Michelle drive altogether during the day?

9 Write a short story that matches the information shown in each travel graph.

a



b



7.3 Straight line graphs

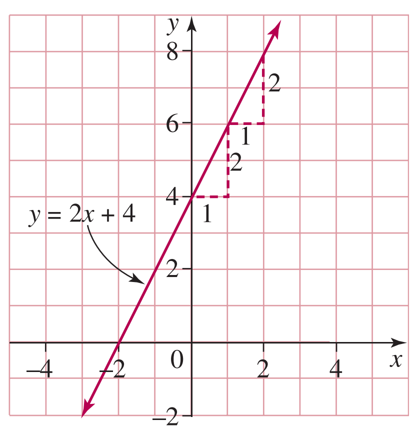
The equation of a line

The equation of a line can be written in several different forms. The two most commonly used forms are the gradient–intercept form and the general form.

When the equation of a line is written in the form $y = mx + b$, it is said to be in the **gradient–intercept form**, where m is the gradient and b is the y-intercept.

$\left(\textit{NOTE: } m = \dfrac{\text{vertical rise}}{\text{horizontal run}}\right)$

For example, if a line has a gradient of 2 and a y-intercept of 4, then its equation is $y = 2x + 4$.



The gradient–intercept form of the linear equation is $y = mx + b$, where:
- **m is the gradient, and**
- **b is the y-intercept.**

When the equation of a line is written in the form $ax + by + c = 0$, where a, b, c are integers and $a > 0$, then it is said to be in the general form. For example, the equation $y = 2x + 4$ could be written as $2x - y + 4 = 0$ in the general form.

The general form of the linear equation is:
$ax + by + c = 0$, where a, b, c are integers and $a > 0$.

Graphing straight lines

In Year 9 we studied three different methods that could be used to graph straight lines on a number plane. They are:

1 completing a table of values
2 finding the x- and y-intercepts
3 using the gradient and the y-intercept.

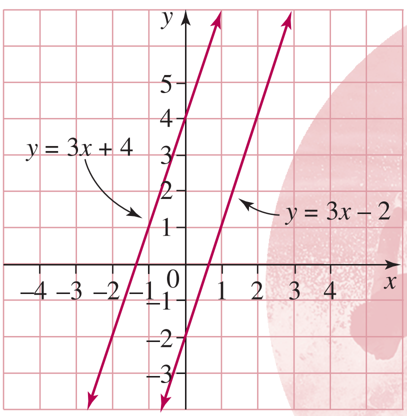
To graph a straight line by using a table of values:
- **substitute each x-value into the equation to find the corresponding y-value**
- **plot the points on a number plane**
- **draw a straight line through the points.**

To graph a straight line by finding the x- and y-intercepts:
- **substitute $x = 0$ into the equation to find the y-intercept**
- **substitute $y = 0$ into the equation to find the x-intercept**
- **draw a straight line through the intercepts.**

To graph a straight line by using the gradient and the y-intercept:
- **mark the y-intercept on the y-axis**
- **use the gradient to find at least two other points that lie on the line**
- **draw a straight line through the points.**

Parallel lines

Two lines are parallel if they have the same gradient. For example, consider the lines with equations $y = 3x - 2$ and $y = 3x + 4$. Both equations are written in the form $y = mx + b$, so the gradient of each line is clearly 3. The lines have the same gradient, so they are parallel.

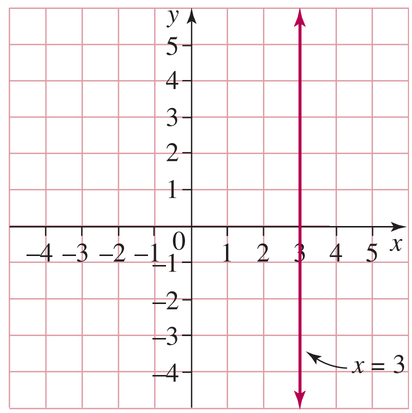


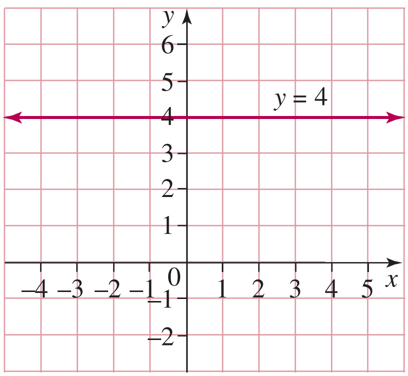
Vertical and horizontal lines

Consider the line that passes through the given points.

x	3	3	3	3	3
y	0	1	2	3	4

The x- and y-values have no direct relationship, but the x-values for all these points are 3s. Hence, the equation of the line is $x = 3$. By plotting the points, it is clear that the line must be vertical and cut the x-axis at 3.





Consider the line that passes through the given points.

x	0	1	2	3	4
y	4	4	4	4	4

The x- and y-values have no direct relationship, but the y-values for all these points are 4s. Hence, the equation of the line is $y = 4$. By plotting the points, it is clear that the line must be horizontal and cut the y-axis at 4.

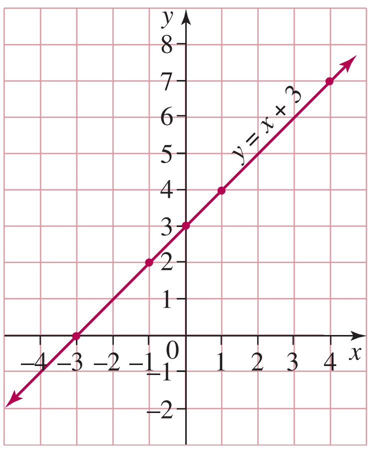
- **$x = a$ is the equation of a vertical line, cutting the x-axis at a.**
- **$y = b$ is the equation of a horizontal line, cutting the y-axis at b.**

The condition for a point to lie on a line

To determine whether a point lies on a line, we substitute the co-ordinates of the point into the equation of the line. If the co-ordinates of the point satisfy the equation, then the point lies on the line. If the co-ordinates do not satisfy the equation, then the point does not lie on the line. It is not necessary to draw the graph of the line.

A point lies on a line if its co-ordinates satisfy the equation of the line.

For example, the points (0, 3), (1, 4), (4, 7), (−1, 2) and (−3, 0) would all lie on the line $y = x + 3$ because in each point, the y-value is 3 more than the x-value. Would the point (4, 6) lie on this line? Why?



Example 1

State the gradient and y-intercept of each line.

a $y = 2x - 3$ **b** $y = x$ **c** $y = 6 - x$

Solutions

a $m = 2, b = -3$ **b** $m = 1, b = 0$ **c** $m = -1, b = 6$

Example 2

Write the equation $2x - 3y + 12 = 0$ in the gradient–intercept form, then write down the gradient and y-intercept of the line.

Solution

$$2x - 3y + 12 = 0$$
$$+3y \qquad +3y$$
$$2x + 12 = 3y$$
$$\div 3 \qquad \div 3$$
$$\frac{2}{3}x + 4 = y$$
$$\therefore y = \frac{2}{3}x + 4$$
$$\therefore m = \frac{2}{3}, b = 4$$

Example 3

Graph the line $y = \frac{1}{2}x + 1$ by:

a using a table of values

b finding the x- and y-intercepts

c using the gradient and y-intercept

Solutions

a

x	0	2	4
y	1	2	3

$x = 0$: $\quad y = (\frac{1}{2} \times 0) + 1$

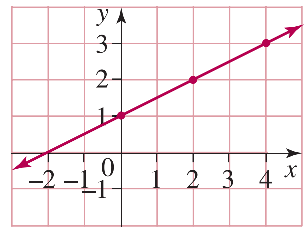
$\therefore y = 1$

$x = 2$: $\quad y = (\frac{1}{2} \times 2) + 1$

$\therefore y = 2$

$x = 3$: $\quad y = (\frac{1}{2} \times 4) + 1$

$\therefore y = 3$



b $x = 0$: $\quad y = (\frac{1}{2} \times 0) + 1$

$\therefore y = 1$

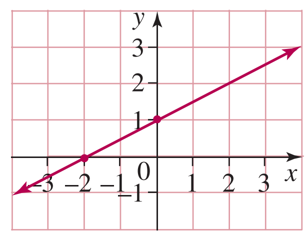
$y = 0$: $\quad 0 = (\frac{1}{2}\ x + 1$

$-1 \qquad -1$

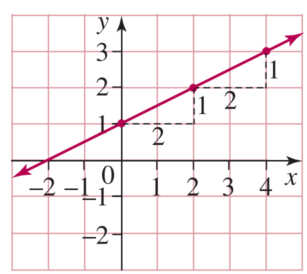
$-1 = \frac{1}{2}x$

$\times 2 \quad \times 2$

$\therefore x = -2$



c The y-intercept is 1 and the gradient is $\frac{1}{2}$.
Starting at 1 on the y-axis, count
2 units to the right and 1 unit up.



Example 4

Determine by substitution whether the point $P(4, -3)$ lies on the line $3x - 4y - 24 = 0$.

Solution

Substitute $x = 4$ and $y = -3$ into the equation $3x - 4y - 24 = 0$.

$$3x - 4y - 24 = 0$$
$$(3 \times 4) - (4 \times -3) - 24 = 0$$
$$12 + 12 - 24 = 0$$
$$0 = 0$$

The co-ordinates satisfy the equation,
$\therefore$ the point A lies on the line.

Exercise 7.3

1 Complete these tables of values, then graph each line on a number plane.

a $y = 3x$

x	0	1	2
y			

b $y = 2x + 3$

x	0	1	2
y			

c $y = 5 - 3x$

x	0	1	2
y			

d $2x + y = 6$

x	0	1	2
y			

2 Find the x- and y-intercepts of each line, then draw its graph on a number plane.

a $y = x - 3$ **b** $y = x + 4$ **c** $y = 2 - x$ **d** $y = 3x - 6$
e $y = 10 - 2x$ **f** $y = 2x - 3$ **g** $x - 2y = 4$ **h** $2x + 3y = 6$
i $3x - 4y = 24$ **j** $y = \frac{1}{3}x + 1$ **k** $y = 2 - \frac{1}{2}x$ **l** $y = \frac{2}{3}x + 4$

3 Write down the gradient and y-intercept of each line.

a $y = x + 7$ **b** $y = 2x - 5$ **c** $y = 4 + 3x$ **d** $y = -x - 6$
e $y = 4x$ **f** $y = 8 - 5x$ **g** $y = \frac{1}{2}x + 4$ **h** $y = \frac{5x}{6}$

4 Write down the equation of a line that has:

a a gradient of 2 and a y-intercept of 3
b a gradient of -1 and y-intercept of 4
c a slope of 3 and cuts the y-axis at -2
d a slope of $-\frac{1}{2}$ and cuts the y-axis at -6
e a gradient of $\frac{2}{3}$ and passes through the origin

Consolidation

5 Write each equation in the gradient–intercept form, then write down the gradient and y-intercept.

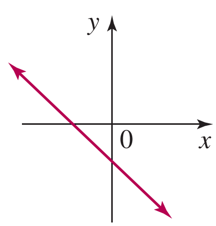
a $4x - 2y + 3 = 0$ **b** $2x - 6y - 1 = 0$ **c** $3x + 4y + 12 = 0$ **d** $5x + 3y - 11 = 0$

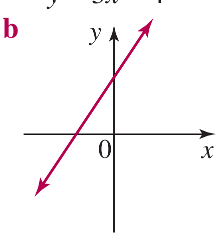
6 Graph the following lines by first plotting the y-intercept, then using the gradient to find two other points that lie on the line.

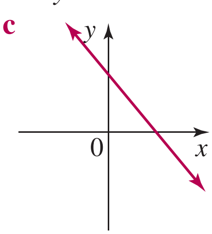
a $y = x + 3$ **b** $y = 3x$ **c** $y = 2x + 1$ **d** $y = 4 - x$
e $y = 3 - 2x$ **f** $y = -3x - 2$ **g** $y = \frac{1}{2}x$ **h** $y = \frac{1}{4}x + 1$
i $y = -\frac{1}{3}x + 3$ **j** $y = \frac{2}{3}x - 4$ **k** $y = -\frac{3}{2}x - 1$ **l** $y = 2 - \frac{3}{4}x$

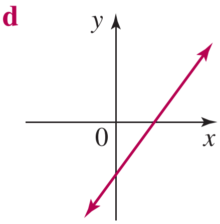
7 Choose the equation that best describes each of the lines below.

- $y = 3x + 4$
- $y = 3x - 4$
- $y = 4 - 3x$
- $y = -3x - 4$

a


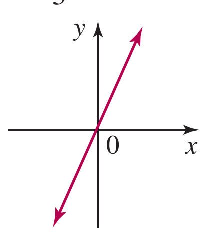
b


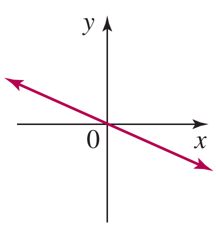
c


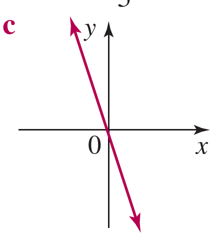
d


8 Choose the equation that best describes each of the lines below.

- $y = \frac{1}{3}x$
- $y = 3x$
- $y = -\frac{1}{3}x$
- $y = -3x$

a


b


c


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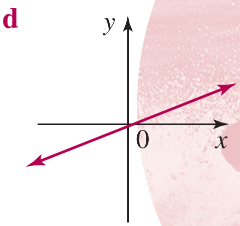
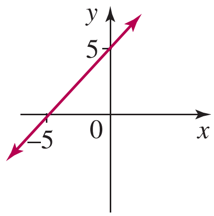
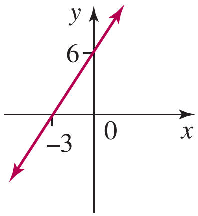


9 Find the gradient and y-intercept of each line, and hence write down its equation.

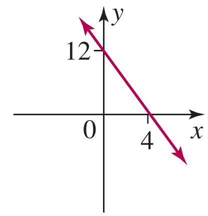
a



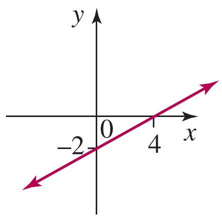
b



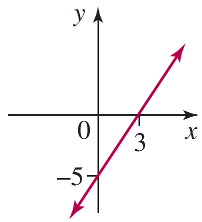
c



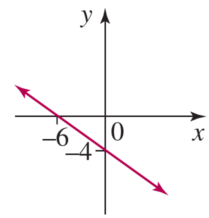
d



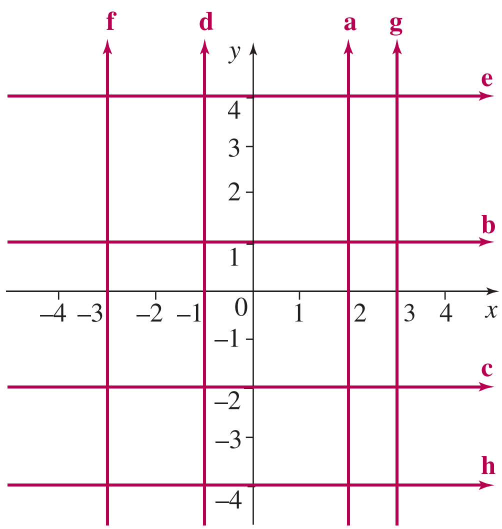
e



f



10 Write down the equation of each line.



11 Write down the equation of the straight line that is:

a parallel to the x-axis and passes through (1, 5)
b parallel to the y-axis and passes through (2, –4)
c perpendicular to the x-axis and passes through (–3, –8)
d perpendicular to the y-axis and passes through (–6, –5)

12 State whether each of these lines is parallel to the line $y = 2x - 3$.

a $y = 2x$ **b** $y = x + 2$ **c** $y = x - 3$
d $y = 2x + 5$ **e** $y = 1 - 2x$ **f** $2x - y + 13 = 0$

13 Determine whether each of the following points lies on the line $y = 4x - 1$.

a (0, 1) **b** (0, –1) **c** (2, 7) **d** (3, 12)
e (–1, –5) **f** (–2, –7) **g** $(\frac{1}{2}, 1)$ **h** $(-\frac{3}{4}, -4)$

14 Determine whether each of the following lines passes through the point (3, −5).

a $y = x - 2$ **b** $y = 2x - 11$ **c** $y = -x - 2$ **d** $y = 8 - x$

e $y = \frac{1}{3}x - 4$ **f** $y = \frac{2}{3}x - 7$ **g** $2x + y = 1$ **h** $3x - 2y + 1 = 0$

15 Find the value of the pronumeral if the point:

a $(2, c)$ lies on the line $y = x - 8$ **b** $(g, 5)$ lies on the line $y = 2x - 9$

c $(-3, m)$ lies on the line $y = 2 - 3x$ **d** $(z, -1)$ lies on the line $2x - 3y + 13 = 0$

16 Find the equation of the line that passes through each set of points.

a

x	0	1	2	3	4
y	7	8	9	10	11

b

x	0	1	2	3	4
y	0	5	10	15	20

c

x	0	1	2	3	4
y	3	5	7	9	11

d

x	0	1	2	3	4
y	−1	3	7	11	15

e

x	0	1	2	3	4
y	8	7	6	5	4

f

x	0	1	2	3	4
y	−5	4	13	22	31

g

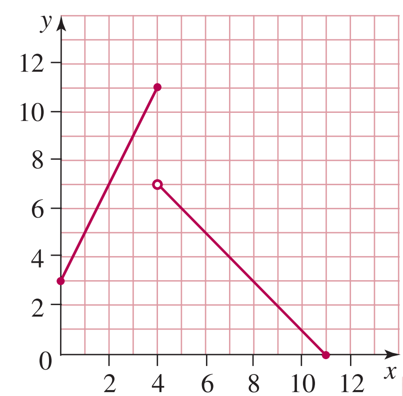
x	0	1	2	3	4
y	7	5	3	1	−1

h

x	0	2	4	6	8
y	−2	−1	0	1	2

■ Further applications

17 This piecemeal graph is made up of two straight line graphs, as shown.



Find the value of y if:

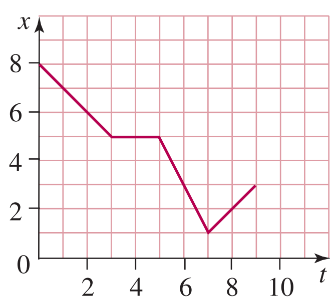
a **i** $x = 3$ **ii** $x = 4$ **iii** $x = 11$

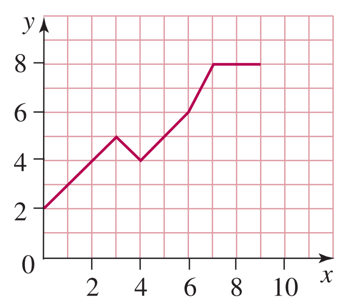
Find the value(s) of x if:

b **i** $y = 2$ **ii** $y = 5$ **iii** $y = 7$

c Find the equation of each section of the graph and state the restrictions on the value of x in each case.

18 The graphs below show the relationship between t and x, and between x and y.





a Find the value of y if:

i $t = 1$
ii $t = 6$
iii $t = 8$

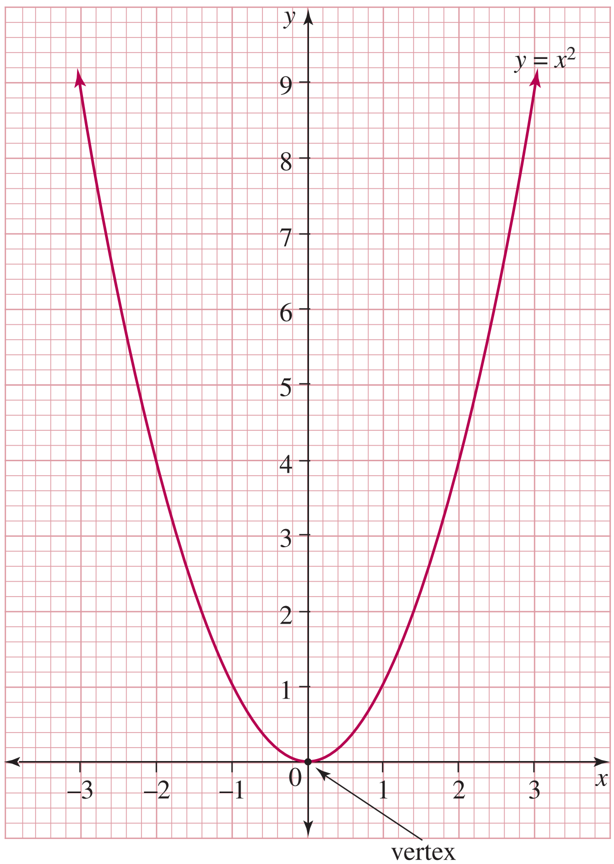
b Find the value(s) of t if:

i $y = 3$
ii $y = 4$
iii $y = 5$

7.4 The parabola $y = ax^2$

Until now, all of the equations that we have graphed have resulted in straight line graphs. However, some equations produce curved graphs. The simplest example of this is the graph of $y = x^2$. As with straight line graphs, we need to draw up a table, choose a number of x-values and substitute these into the equation. The corresponding y-values are found by squaring each of the x-values.

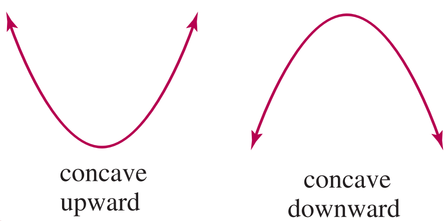
x	–3	–2	–1	0	1	2	3
y	9	4	1	0	1	4	9



The points in the table are then plotted on a number plane and joined to form a smooth curve. The resulting graph is called a **parabola**.

The turning point of the graph is called the **vertex**. The vertex in this case is a minimum point. Its co-ordinates are (0, 0).

The curve is pointing upwards at the ends. We say that this curve is **concave upward**. If the curve was pointing downwards at the ends, then we would say that the curve was **concave downward**.



Features of the graph of $y = x^2$:

- **The graph is a smooth curve.**
- **The curve is concave upward.**
- **The vertex has co-ordinates (0, 0).**
- **The curve is symmetric about the y-axis.**

Example

Use the graph of $y = x^2$ above to evaluate each of the following, correct to 1 decimal place.

a 1.6^2 **b** $\sqrt{5.8}$

Solutions

a Start at 1.6 on the x-axis. Move up to the curve and across to the y-axis.
$\therefore 1.6^2 \doteqdot 2.6$

b Start at 5.8 on the y-axis. Move to the right to the curve and down to the x-axis.
$\therefore \sqrt{5.8} \doteqdot 2.4$

Exercise 7.4

1 **a** Copy and complete this table of values for the equation $y = x^2$.

x	−3	−2	−1	0	1	2	3

b Plot these points on a number plane. Draw a smooth curve through the points.
c Are the y-values ever negative?
d What is the equation of the axis of symmetry?
e Write down the co-ordinates of the vertex.
f Are there any limitations on the possible x-values for this curve?
g What is the minimum y-value?
h Is the curve concave upward or downward?

2 **a** How many y-values are there for each x-value, on the curve $y = x^2$?
b How many x-values are there for each y-value other than 0, on the curve $y = x^2$?

3 Use the graph on page 239 to evaluate each of the following, correct to 1 decimal place.
a 0.4^2 **b** 1.6^2 **c** 2.2^2 **d** 2.7^2
e $(-0.6)^2$ **f** $(-1.3)^2$ **g** $(-1.8)^2$ **h** $(-2.1)^2$

4 Use the graph on page 239 to evaluate each of these square roots, correct to 1 decimal place.
a $\sqrt{2}$ **b** $\sqrt{5}$ **c** $\sqrt{7}$ **d** $\sqrt{8}$
e $\sqrt{0.8}$ **f** $\sqrt{3.3}$ **g** $\sqrt{6.2}$ **h** $\sqrt{7.4}$

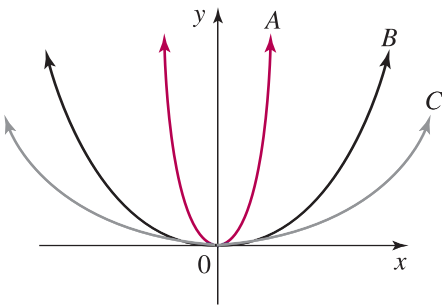
5 Use the graph on page 239 to find the two possible x-values on the graph of $y = x^2$ for which:
a $y = 3$ **b** $y = 8$ **c** $y = 1.8$ **d** $y = 5.8$

Consolidation

6 **a** Copy and complete the table below for each of these equations.
i $y = x^2$ **ii** $y = 2x^2$ **iii** $y = 3x^2$

x	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
y									

b Graph the three curves on the same set of axes.
c What is the effect on the graph of $y = ax^2$ when the value of a increases, if $a > 0$?

7 

Three parabolas $y = x^2$, $y = \frac{1}{2}x^2$ and $y = 4x^2$ have been drawn on the same number plane. State the graph whose equation is:

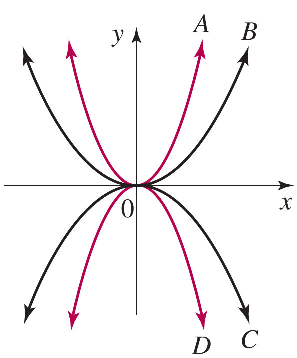
a $y = x^2$ **b** $y = \frac{1}{2}x^2$ **c** $y = 4x^2$

8 **a** Copy and complete this table of values for the equation $y = -x^2$.

x	−3	−2	−1	0	1	2	3
y							

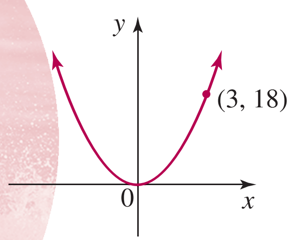
b Plot these points on a number plane. Draw a smooth curve through the points.
c Are the y-values ever positive?
d What is the equation of the axis of symmetry?
e Write down the co-ordinates of the vertex.
f Are there any limitations on the possible x-values for this curve?
g What is the maximum y-value?
h Is the curve concave upward or downward?

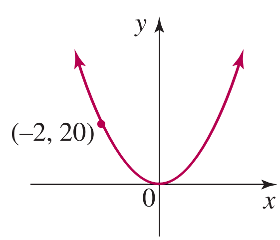
9 What can you say about the concavity of a parabola $y = ax^2$ if:
a a is positive? **b** a is negative?

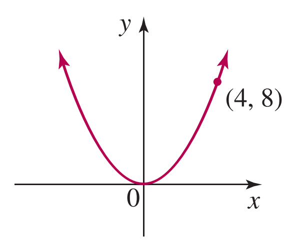
10 

Four parabolas $y = x^2$, $y = -x^2$, $y = 2x^2$ and $y = -2x^2$ have been drawn on the same number plane. State the graph whose equation is:
a $y = x^2$ **b** $y = -x^2$
c $y = 2x^2$ **d** $y = -2x^2$

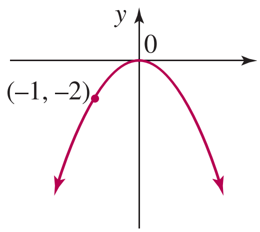
11 The curves below are parabolas with equations of the form $y = ax^2$, where a is a constant. For each curve, find the value of a and hence determine its equation.

a 

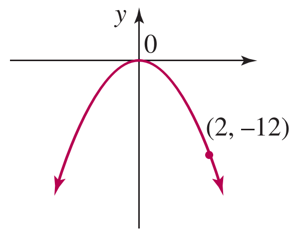
b 

c 

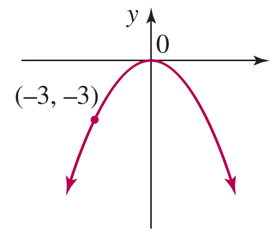
d



e



f



Further applications

12 Galileo (1564–1642) showed that objects fall from rest under gravity according to the equation $s = 5t^2$, where s is the distance travelled, in metres, and t is the time, in seconds.

- **a** What limitation must be placed on the possible values of t? Why?
- **b** Sketch a distance–time graph for this equation. Values are not required on the axes.
- **c** Find how far an object would fall from rest in:
  i 1 second **ii** 2 seconds **iii** 7 seconds
- **d** Find how long it would take for an object to fall from rest a distance of:
  i 45 metres **ii** 980 metres **iii** 2 kilometres
- **e** How far would an object fall during the third second?

13 In a certain rectangle, the length is four times the width.

- **a** If the width is x centimetres, find an expression for the length.
- **b** Write down an equation for the area, A, of the rectangle.
- **c** What limitation must be placed on the possible values of x? Why?
- **d** Draw an accurate graph of this area equation on a number plane, using 2 mm grid paper ($0 \leqslant x \leqslant 5$).
- **e** Use your graph to find the area of the rectangle if the width is:
  i 3 cm **ii** 4 cm **iii** 5 cm
- **f** Verify your answers to part e by substituting the widths into the area equation.
- **g** Use your graph to find the dimensions of the rectangle if the area is 49 cm^2.
- **h** Verify your answer to part g by substituting the area into the area equation.

TRY THIS The Parkes telescope

Radio telescopes that collect radio waves from outer space are parabolic in shape.

The famous radio telescope near Parkes in NSW is 64 m in diameter.

The equation of the parabola is $y = \dfrac{x^2}{108}$.

What is the depth of the dish at its centre? Answer correct to the nearest cm.

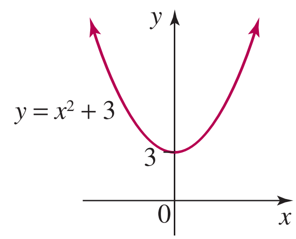
7.5 The parabola $y = ax^2 + c$

When graphing straight lines with equations of the form $y = mx + b$, we have seen that the constant term b is the y-intercept of the line. This is because at every point on the y-axis, $x = 0$. When we substitute $x = 0$ into the equation $y = mx + b$, the result is $y = b$. Hence, the constant term in the equation corresponds to the y-intercept of the line.

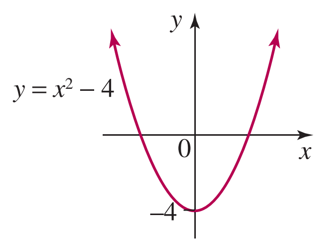
This is also the case with the parabola when the equation is written in the form $y = ax^2 + c$. If we substitute $x = 0$ into the equation, the result is $y = c$. Hence, the constant term in the equation corresponds to the y-intercept of the parabola.

We can also think of the constant term as the number of units by which the graph is moved up or down the y-axis. For example:

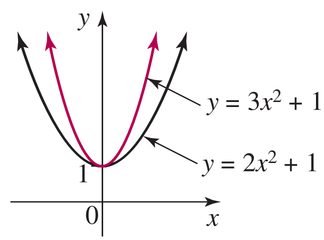
- the equation $y = x^2 + 3$ represents the parabola $y = x^2$ after it has been shifted up by 3 units. The curve has a y-intercept of 3. The co-ordinates of the vertex are (0, 3).



- the equation $y = x^2 - 4$ represents the parabola $y = x^2$ after it has been shifted down by 4 units. The curve has a y-intercept of –4. The co-ordinates of the vertex are (0, –4).



NOTE: If more than one graph is to be drawn on the same number plane, the larger the value of a (where $a > 0$), the narrower is the parabola. For example, the graph of $y = 3x^2 + 1$ would be narrower than the graph of $y = 2x^2 + 1$ because $3 > 2$.



Features of the graph of $y = ax^2 + c$:
- **The graph is symmetric about the y-axis.**
- **The curve is concave upward if $a > 0$ and concave downward if $a < 0$.**
- **The vertex has co-ordinates $(0, c)$.**
- **The greater the value of a, (where $a > 0$), the narrower is the parabola.**

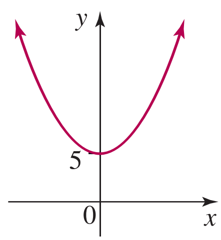
Example 1

Sketch the following graphs on separate number planes. State the co-ordinates of the vertex in each case.

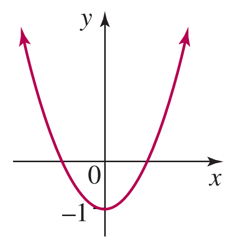
a $y = x^2 + 5$ **b** $y = x^2 - 1$ **c** $y = 3 - x^2$ **d** $y = -x^2 - 2$

Solutions

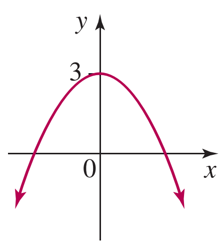
a The graph of $y = x^2 + 5$ is obtained by shifting the graph of $y = x^2$ up by 5 units. The vertex is (0, 5).



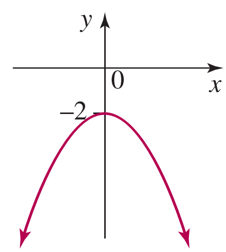
b The graph of $y = x^2 - 1$ is obtained by shifting the graph of $y = x^2$ down by 1 unit. The vertex is (0, −1).



c The graph of $y = 3 - x^2$ is obtained by shifting the graph of $y = -x^2$ up by 3 units. The vertex is (0, 3).



d The graph of $y = -x^2 - 2$ is obtained by shifting the graph of $y = x^2$ down by 2 units. The vertex is (0, −2).

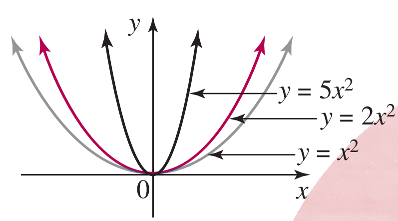


Example 2

Sketch the graphs of $y = 5x^2$, $y = 2x^2$ and $y = x^2$ on the same number plane. You are not required to show any values on the axes.

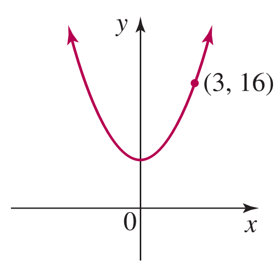
Solution

The larger the co-efficient of x^2, the narrower is the graph. Therefore, the graph of $y = 5x^2$ is closest to the y-axis and the graph of $y = x^2$ is furthest from the y-axis.



Example 3

The curve shown is a parabola. It has an equation of the form $y = x^2 + c$. Find the value of c and hence write down the equation of the curve.



Solution

To find the value of c, we substitute the co-ordinates of the given point into the equation of the curve.

$$y = x^2 + c$$
$$16 = 3^2 + c$$
$$16 = 9 + c$$
$$\therefore c = 7$$

$\therefore$ The equation of the parabola is $y = x^2 + 7$.

Exercise 7.5

1 Copy and complete the following tables of values, then graph the curves on the same number plane.

a $y = x^2$

x	-2	-1	0	1	2
y					

b $y = x^2 + 1$

x	-2	-1	0	1	2
y					

c $y = x^2 + 3$

x	-2	-1	0	1	2
y					

d $y = x^2 - 2$

x	-2	-1	0	1	2
y					

2 How can graphs of $y = x^2 + c$ and $y = x^2 - c$ $(c > 0)$ be obtained from the graph of $y = x^2$?

3 How could the following graphs be obtained from the graph of $y = x^2$?
a $y = x^2 + 4$ **b** $y = x^2 - 5$ **c** $y = x^2 + 7$ **d** $y = x^2 - 12$

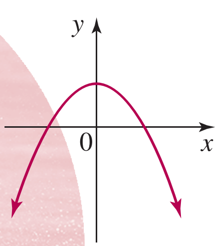
4 Sketch each of the following parabolas, showing the intercept with the y-axis. Other values are not required on the axes.
a $y = x^2 + 1$ **b** $y = x^2 - 4$ **c** $y = 2x^2 + 5$ **d** $y = 3x^2 - 7$
e $y = -x^2 - 3$ **f** $y = 10 - x^2$ **g** $y = 4 - 3x^2$ **h** $y = -2x^2 - 9$

5 Write down the co-ordinates of the vertex of the parabola $y = 5x^2 - 8$.

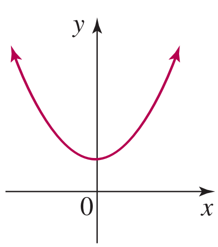
Consolidation

6 Match each of these equations with one of the graphs below.
- $y = x^2 + 3$
- $y = x^2 - 3$
- $y = 3 - x^2$
- $y = -x^2 - 3$

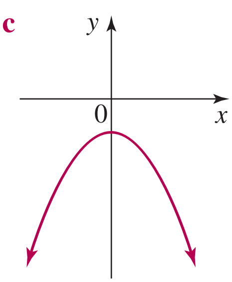
a



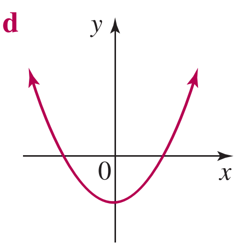
b



c



d

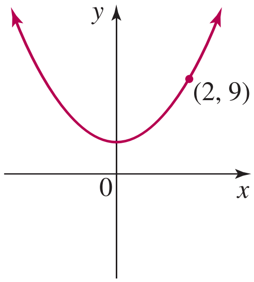


7 Find the equation of the new parabola if the curve $y = x^2 + 2$ is translated:
a 6 units up **b** 2 units down **c** 5 units down

8 Find the two x-intercepts and the y-intercept of each parabola, then sketch the graph.
a $y = x^2 - 4$ **b** $y = 1 - x^2$ **c** $y = x^2 - 16$ **d** $y = 9 - x^2$

9 The parabola $y = x^2$ is reflected in the x-axis, then translated 9 units up. What is the equation of this new parabola?

10 The curves below are parabolas with equations of the form $y = x^2 + c$ or $y = -x^2 + c$. For each curve, find the value of c and hence determine its equation.

a


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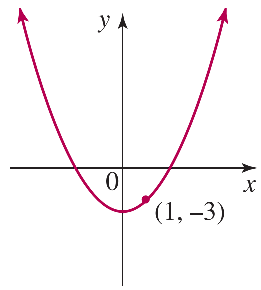

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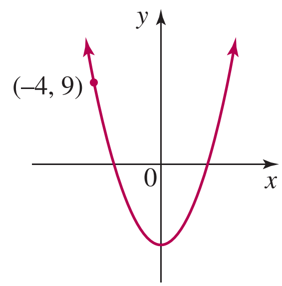

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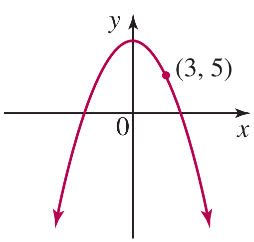
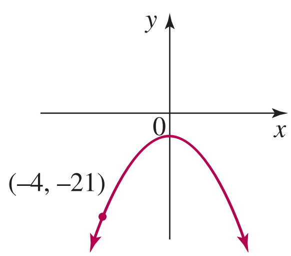
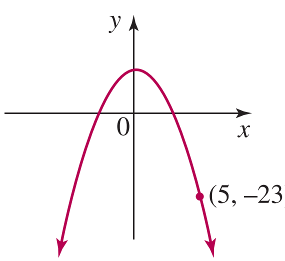
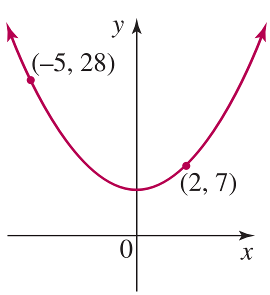


e


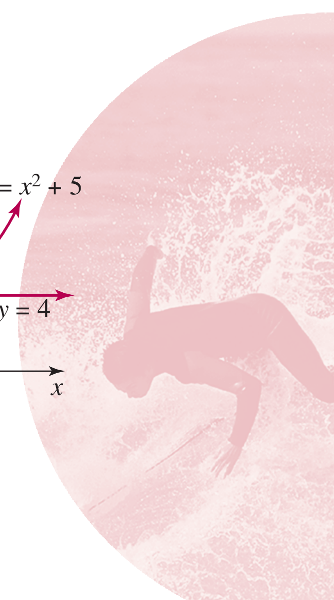
f


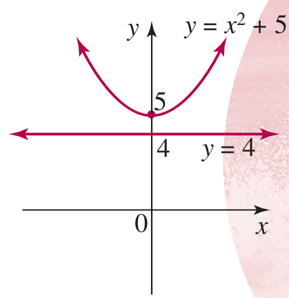
■ Further applications

11 Find the equation of a parabola in the form $y = ax^2 + c$ given that:
a the vertex has co-ordinates $(0, -3)$ and the curve passes through the point $(2, 5)$
b the vertex has co-ordinates $(0, 7)$ and the curve passes through the point $(-1, 3)$

12


The parabola shown has an equation of the form $y = ax^2 + c$. Form a pair of simultaneous equations and hence find the equation of the parabola.

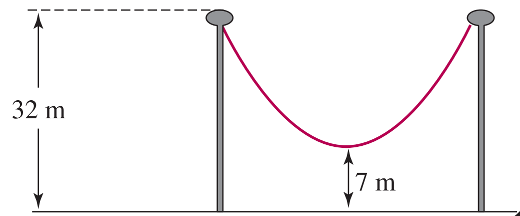




13 What would be the equation of the new parabola if the curve $y = x^2 + 5$ is reflected in the line $y = 4$?

TRY THIS Flagpoles

Two flagpoles are 32 m high. A rope is strung between the tops of the flagpoles. The lowest point of the rope is 7 m from the ground. How far apart are the poles? (Assume that the rope has the parabolic shape $y = x^2$.)



7.6 The parabola $y = ax^2 + bx + c$

Many, but not all, parabolas with equations of the form $y = ax^2 + bx + c$ cut the x-axis at two distinct points. These parabolas can be sketched by finding the x and y intercepts and the vertex, then drawing a smooth curve through these points.

The x- and y-intercepts

Every point on the x-axis is of the form $(x, 0)$. Therefore, the x-intercepts of a curve can be found by substituting $y = 0$ into the equation of the curve. Similarly, every point on the y-axis is of the form $(0, y)$. Therefore, the y-intercept of a curve can be found by substituting $x = 0$ into the equation of the curve.

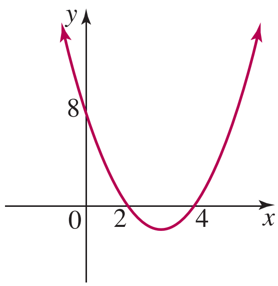
For example, consider the parabola with equation $y = x^2 - 6x + 8$.

When $x = 0$: $\quad y = 0^2 - 6(0) + 8$
$\qquad = 8$

$\therefore$ The y-intercept is 8.

When $y = 0$: $\quad x^2 - 6x + 8 = 0$
$(x - 2)(x - 4) = 0$
$\therefore x = 2, 4$

$\therefore$ The x-intercepts are 2, 4.

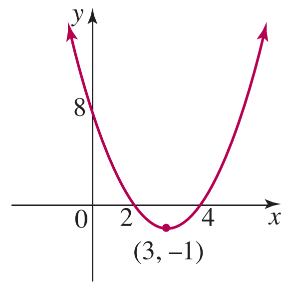


In the equation, the co-efficient of x^2 is positive. Therefore, the curve is concave up.

The axis of symmetry and the vertex

The parabola has a vertical axis of symmetry. It meets the x-axis halfway between the two x-intercepts of the curve (if there are two x-intercepts). In the example above, the equation of the axis of symmetry is $x = 3$.

The axis of symmetry meets the parabola at the vertex. Therefore, the x co-ordinate of the vertex is 3. To find the y co-ordinate of the vertex, we substitute $x = 3$ into the equation of the curve.



$$\begin{aligned} y &= 3^2 - 6(3) + 8 \\ &= 9 - 18 + 8 \\ &= -1 \end{aligned}$$

$\therefore$ The co-ordinates of the vertex are $(3, -1)$.

In this exercise we will only consider parabolas which cut the x-axis at two distinct points.

EG +S

Example 1

For the parabola with equation $y = -x^2 + 6x + 7$:

a find the x-intercepts

b find the y-intercept

c find the equation of the axis of symmetry

d find the co-ordinates of the vertex

e sketch the curve.

Solutions

a $y = 0$:
$$\begin{aligned} -x^2 + 6x + 7 &= 0 \\ -(x^2 - 6x - 7) &= 0 \\ -(x - 7)(x + 1) &= 0 \\ \therefore x &= 7, -1 \end{aligned}$$
$\therefore$ The x-intercepts are $7, -1$.

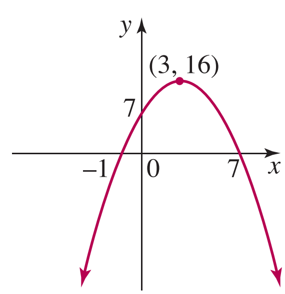
b $x = 0$:
$$\begin{aligned} y &= -(0)^2 + 6(0) + 7 \\ &= 0 + 0 + 7 \\ &= 7 \end{aligned}$$
$\therefore$ The y-intercept is 7.

c The equation of the axis of symmetry is $x = \dfrac{-1 + 7}{2}$

$\therefore x = 3$

d Substitute $x = 3$ into the equation.
$$\begin{aligned} y &= -(3)^2 + 6(3) + 7 \\ &= -9 + 18 + 7 \\ &= 16 \end{aligned}$$
$\therefore$ The vertex is at $(3, 16)$.

e The co-efficient of x^2 is negative. Therefore the curve is concave down.



Exercise 7.6

1 Find the x- and y-intercepts of each parabola, then sketch the curve.

a $y = x(x - 6)$

b $y = x(3 - x)$

c $y = (x - 2)(x + 2)$

d $y = (5 - x)(5 + x)$

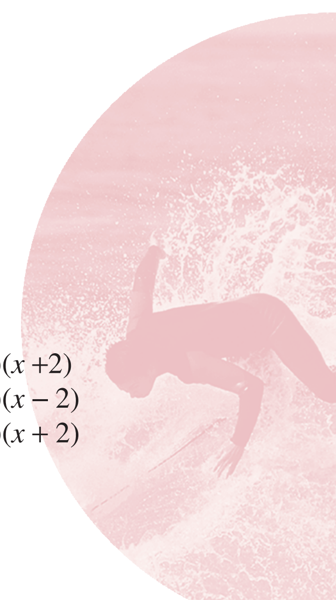
e $y = (x - 2)(x - 5)$

f $y = (x + 1)(x - 2)$

g $y = -(x + 3)(x + 4)$

h $y = (1 - x)(x + 4)$

i $y = (x - 6)(x + 2)$



2 For each of the following parabolas,

i find the x-intercepts **ii** write down the y-intercept

iii find the equation of the axis of symmetry and the co-ordinates of the vertex

iv sketch the curve.

a $y = x^2 - 2x$ **b** $y = 10x - 2x^2$

Consolidation

3 Sketch the following curves showing the x- and y-intercepts and the vertex.

a $y = x^2 - 4x + 3$ **b** $y = x^2 + 6x + 5$ **c** $y = x^2 + 2x - 3$

d $y = x^2 - 2x - 8$ **e** $y = x^2 - 7x + 12$ **f** $y = x^2 - x - 2$

4 Sketch the following curves showing the x- and y-intercepts and the vertex.

a $y = -x^2 + 4x - 3$ **b** $y = -x^2 - 10x - 16$ **c** $y = 5 + 4x - x^2$

d $y = 15 - 2x - x^2$ **e** $y = -x^2 + x + 6$ **f** $y = 18 + 3x - x^2$

5 Sketch the following curves showing the x- and y-intercepts and the vertex.

a $y = 2x^2 - 5x + 3$ **b** $y = 2x^2 + x - 1$ **c** $y = -3x^2 + 11x - 10$

6 a Sketch the graph of $y = x^2 - 4x$, showing the x- and y-intercepts and the vertex.

b Hence, sketch the graph of $y = x^2 - 4x + 7$ on the same number plane.

7 a Sketch the graph of $y = 4x - 2x^2$, showing the x- and y-intercepts and the vertex.

b Hence, sketch the graph of $y = 5 + 4x - 2x^2$ on the same number plane.

8 a Sketch the graph of $y = (x - 1)(x - 5)$ showing the x- and y-intercepts and the vertex.

b Hence, sketch the graph of $y = 2(x - 1)(x - 5)$ on the same number plane.

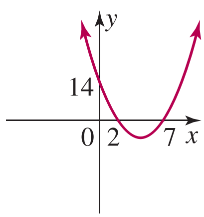
9 Find the x-intercepts of each parabola. Answer in simplest surd form.

a $y = x^2 - 10x + 22$ **b** $y = x^2 - 9x - 3$ **c** $y = x^2 + 6x + 4$

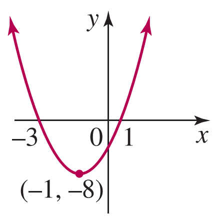
Further applications

10 Find the equation of each parabola in the form $y = k(x - a)(x - b)$.

a



b



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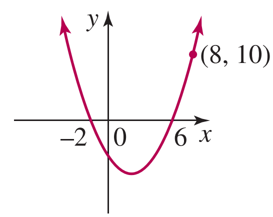
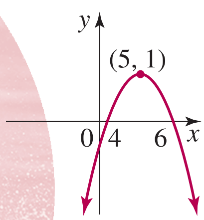
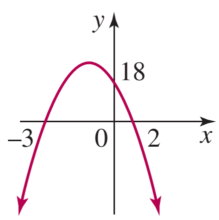


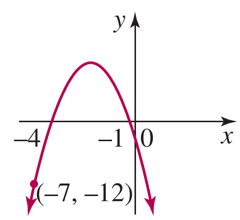
d



e



f



11 Find the equation of the parabola that passes through the points (0, 4), (1, 5) and (–3, 25).
[Hint: Start $y = ax^2 + bx + c$ and find c first.]

7.7 The axis of symmetry of the parabola

The equation of the axis of symmetry

Consider a parabola with equation $y = ax^2 + bx + c$, which cuts the x-axis at two points. The x-intercepts can be found by substituting $y = 0$ into the equation, then solving it by the use of the quadratic formula.

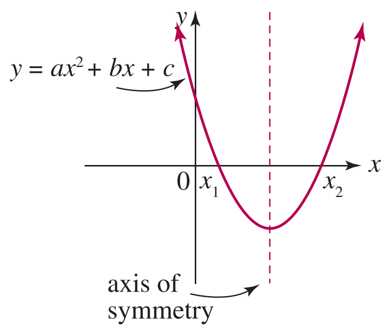
$y = ax^2 + bx + c = 0$

$y = 0$: $\quad ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{i.e. } x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\text{and } x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$



The x co-ordinate of the point on the x-axis halfway between the two x-intercepts is found by averaging the x-intercepts.

That is,
$$\begin{aligned} x &= \frac{x_1 + x_2}{2} \\ &= \frac{\dfrac{-b - \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}}{2} \\ &= \frac{-\cancel{2}b}{2a} \times \frac{1}{\cancel{2}} \\ &= \frac{-b}{2a} \end{aligned}$$

It follows then that the equation of the axis of symmetry of the parabola is given by $x = \dfrac{-b}{2a}$. Although this result is proven here for parabolas which cut the x-axis at two distinct points, the result holds for all parabolas, including those which do not intersect the x-axis at all, or which intersect it at one point only.

The equation of the axis of symmetry of the parabola with equation $y = ax^2 + bx + c$ is:

$$x = \frac{-b}{2a}$$

To find the y co-ordinate of the vertex:

- **substitute $x = \dfrac{-b}{2a}$ into the equation of the parabola.**

The minimum and maximum value of a quadratic function

The quadratic function $y = ax^2 + bx + c$ is represented graphically by a parabola. If $a > 0$, the curve is concave up and the vertex is a minimum turning point. The minimum value of the quadratic function is given by the y-value of the vertex of the parabola.

If $a < 0$, the curve is concave down and the vertex is a maximum turning point. The maximum value of the quadratic function is given by the y-value of the vertex of the parabola.

Example 1

For the parabola with equation $y = x^2 - 6x + 14$:

a find the equation of the axis of symmetry
b find the co-ordinates of the vertex
c sketch the curve showing the vertex and y-intercept
d state the minimum value of the quadratic function.

Solutions

a In $y = x^2 - 6x + 14$,
$a = 1, b = -6, c = 14$.

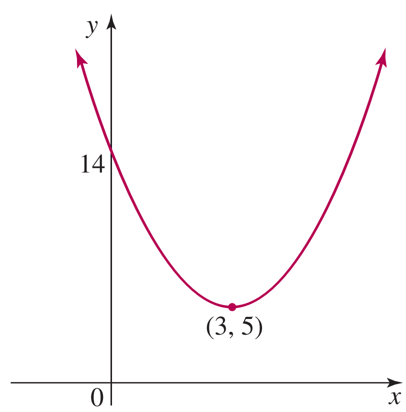
$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(-6)}{2 \times 1} \\ &= 3 \end{aligned}$$

$\therefore$ The equation of the axis of symmetry is $x = 3$.

b $x = 3$:
$$\begin{aligned} y &= 3^2 - 6(3) + 14 \\ &= 9 - 18 + 14 \\ &= 5 \end{aligned}$$

$\therefore$ The co-ordinates of the vertex are (3, 5).

c The co-efficient of x^2 is positive, so the parabola is concave up.



d The minimum value of the function is 5, when $x = 3$.

Exercise 7.7

1 For the parabola $y = x^2 - 6x$:
a find the equation of the axis of symmetry
b determine the co-ordinates of the vertex
c find the x-intercepts
d sketch the curve showing the vertex and the x-intercepts.

2 For each of the following equations:

i find the equation of the axis of symmetry
ii find the co-ordinates of the vertex
iii find the y-intercept
iv sketch the curve showing the vertex and the y-intercept.

a $y = x^2 - 6x + 11$ **b** $y = x^2 - 2x + 7$ **c** $y = x^2 + 4x - 3$
d $y = x^2 - 10x - 6$ **e** $y = x^2 + 3x + 20$ **f** $y = x^2 + 7x - 5$
g $y = -x^2 + 2x + 9$ **h** $y = -x^2 - 6x + 4$ **i** $y = -x^2 + 10x - 17$
j $y = -x^2 + 5x + 2$ **k** $y = -x^2 - 3x - 10$ **l** $y = -x^2 - 9x + 6$

■ Consolidation

3 Find the co-ordinates of the vertex for each of the following.

a $y = 2x^2 + 8x + 7$ **b** $y = 3x^2 - 6x - 2$ **c** $y = -2x^2 + 16x + 13$
d $y = -3x^2 - 12x - 8$ **e** $y = 2x^2 + 10x - 5$ **f** $y = 12 + 20x - 5x^2$

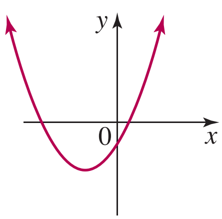
4 Find the maximum or minimum value of each of the following.

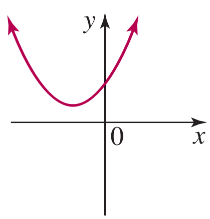
a $y = 2x^2 - 3x + 7$ **b** $y = 3x^2 + 5x - 2$ **c** $y = -2x^2 + 9x + 6$
d $y = 5x^2 - 15x - 12$ **e** $y = -4x^2 - 11x - 12$ **f** $y = 5 - 2x - 3x^2$

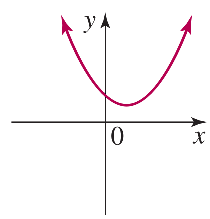
5 Find the minimum value of the quadratic function $y = \frac{1}{2}x^2 + 4x + 3$.

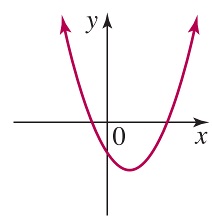
6 Match each equation to one of the graphs below by considering the axis of symmetry and the y-intercept.

- $y = x^2 - 4x + 10$
- $y = x^2 + 4x - 10$
- $y = x^2 - 4x - 10$
- $y = x^2 + 4x + 10$

a


b


c


d


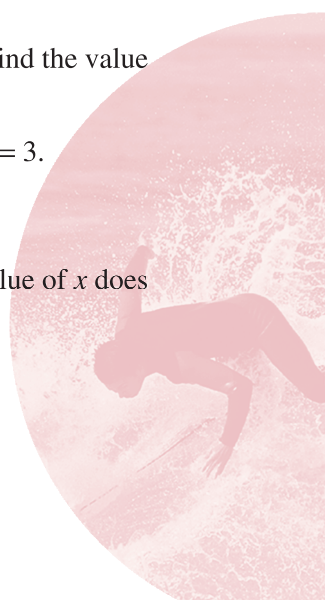
■ Further applications

7 The parabola $y = 2x^2 + kx - 7$ has an axis of symmetry with equation $x = 3$. Find the value of k.

8 The quadratic function $y = 3x^2 + bx + c$ has a minimum value of -16 when $x = 3$.

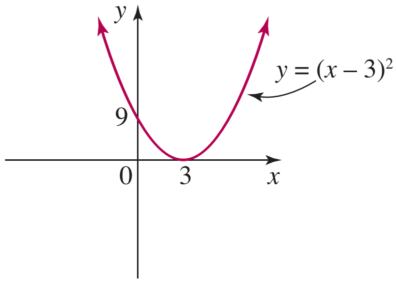
a Find values for b and c.
b Hence, find the value of y when $x = -2$.

9 What is the maximum value of the function $y = 15 - (x - 4)^2$ and for what value of x does it occur?



7.8 The parabola $y = (x - h)^2 + k$

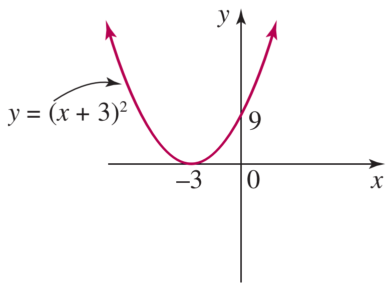
The graph of $y = (x - h)^2$



Consider the parabola with equation $y = (x - 3)^2$.
To find any x-intercepts, we let $y = 0$.

$$\therefore (x - 3)^2 = 0$$
$$x - 3 = 0$$
$$\therefore x = 3$$

That is, the parabola cuts the x-axis at one point only.
Therefore, the vertex has co-ordinates (3, 0).
This curve could have been sketched by shifting the graph of $y = x^2$ 3 units to the right.



Consider the parabola with equation $y = (x + 3)^2$.
To find any x-intercepts, we again let $y = 0$.

$$\therefore (x + 3)^2 = 0$$
$$x + 3 = 0$$
$$\therefore x = -3$$

This parabola also cuts the x-axis at one point only.
Therefore, the vertex has co-ordinates (–3, 0).
This curve could have been sketched by shifting the graph of $y = x^2$ 3 units to the left.

To sketch the graph of $y = (x - h)^2$:
- **shift the graph of $y = x^2$ to the right if $h > 0$. The vertex is then $(h, 0)$**
- **shift the graph of $y = x^2$ to the left if $h < 0$. The vertex is then $(-h, 0)$.**

The graph of $y = a(x - h)^2 + k$

We have seen previously that a parabola is concave up if $a > 0$ and concave down if $a < 0$. Also, the larger the value of a (in absolute terms), the narrower is the parabola. We have also seen that adding a constant term k to the equation has the effect of shifting the graph upward or downward.

To sketch the graph of $y = a(x - h)^2 + k$:
- **determine whether the parabola is concave up (if $a > 0$) or concave down (if $a < 0$)**
- **shift the parabola $y = ax^2$ so that its vertex is (h, k).**

Completing the square

Equations of the form $y = x^2 + bx + c$ can be expressed in the form $y = (x - h)^2 + k$ by completing the square.

To complete the perfect square for $y = x^2 + bx + c$:

- **add $\left(\frac{b}{2}\right)^2$ to $(x^2 + bx)$ to give $\left(x + \frac{b}{2}\right)^2$**
- **subtract $\left(\frac{b}{2}\right)^2$ from c.**

Example 1

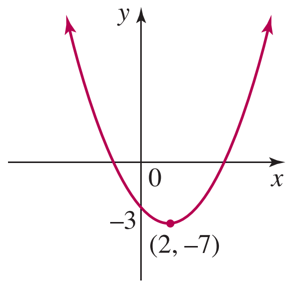
For each of the following, state whether the parabola is concave up or down, find the vertex and y-intercept, then sketch the curve.

a $y = (x - 2)^2 - 7$ **b** $y = 4 - (x + 1)^2$ **c** $y = 2(x - 3)^2 - 6$

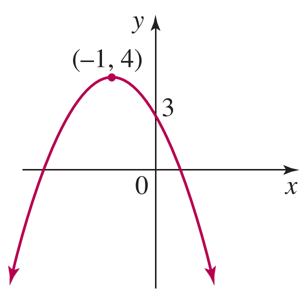
Solutions

Each equation is of the form $y = a(x - h)^2 + k$.

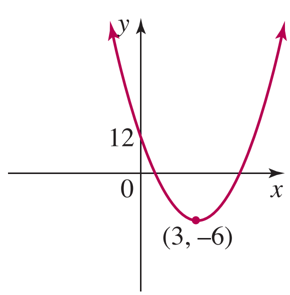
a $a = 1$, so the parabola is concave up.
The vertex has co-ordinates $(2, -7)$.
When $x = 0$: $y = (0 - 2)^2 - 7$
$= 4 - 7$
$= -3$
$\therefore$ the y-intercept is -3.



b $a = -1$, so the parabola is concave down.
The vertex has co-ordinates $(-1, 4)$.
When $x = 0$: $y = 4 - (0 + 1)^2$
$= 4 - 1$
$= 3$
$\therefore$ the y-intercept is 3.



c $a = 2$, so the parabola is concave up.
The vertex has co-ordinates $(3, -6)$.
When $x = 0$: $y = 2(0 - 3)^2 - 6$
$= 2 \times 9 - 6$
$= 12$
$\therefore$ the y-intercept is 12.



EG +S

Example 2

Express the equation $y = x^2 + 6x + 4$ in the form $y = (x - h)^2 + k$. Hence, sketch the graph of $y = x^2 + 6x + 4$.

Solution

$y = x^2 + 6x + 4$

$y = x^2 + 6x + \left(\frac{6}{2}\right)^2 + 4 - \left(\frac{6}{2}\right)^2$

$y = (x^2 + 6x + 9) + 4 - 9$

$\therefore\ y = (x + 3)^2 - 5$

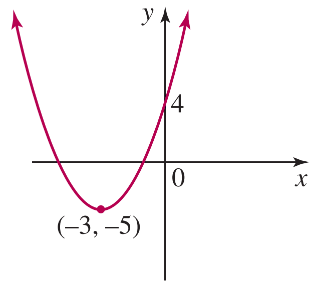
$\therefore$ the co-ordinates of the vertex are $(-3, -5)$.

The co-efficient of x^2 is positive,
therefore the parabola is concave up.

When $x = 0$: $\quad y = 0^2 + 6(0) + 4$

$\qquad\qquad\qquad = 4$

$\therefore$ the y-intercept is 4.



Exercise 7.8

1 Complete each table of values using the given equation, then graph the curves on the same number plane.

a **i** $y = (x - 1)^2$

x	–2	–1	0	1	2	3	4
y							

ii $y = (x + 1)^2$

x	–4	–3	–2	–1	0	1	2
y							

b **i** $y = -(x - 2)^2$

x	–1	0	1	2	3	4	5
y							

ii $y = -(x + 2)^2$

x	–5	–4	–3	–2	–1	0	1
y							

2 Look at the graphs that you have drawn in question 1. Describe how each graph could have been drawn from the graph of $y = x^2$ in a or $y = -x^2$ in b.

3 Explain how the following graphs could be drawn from the graph of $y = x^2$. Sketch each curve showing the y-intercept and the co-ordinates of the vertex.

a $y = (x - 3)^2 + 4$ **b** $y = (x + 2)^2 + 1$ **c** $y = (x - 1)^2 - 3$
d $y = (x + 4)^2 - 2$ **e** $y = -(x - 2)^2 + 7$ **f** $y = -(x + 3)^2 + 5$
g $y = -(x - 4)^2 - 1$ **h** $y = -(x + 1)^2 - 4$

4 Sketch the following curves showing the y-intercept and the vertex.

a $y = (x + 2)^2 + 5$ **b** $y = (x + 3)^2 - 6$ **c** $y = -(x - 1)^2 + 4$
d $y = -(x - 4)^2 - 8$ **e** $y = (x - 5)^2 - 2$ **f** $y = -(x + 1)^2 + 3$
g $y = -(x + 4)^2 - 7$ **h** $y = (x - 9)^2 + 4$ **i** $y = -(x + 2)^2 + 10$
j $y = (x - 7)^2 - 5$ **k** $y = 7 - (x - 1)^2$ **l** $y = 4 - (x + 2)^2$

5 Express each equation in the form $y = (x - h)^2 + k$, then state the co-ordinates of the vertex.

a $y = x^2 + 2x + 4$ **b** $y = x^2 + 6x + 13$ **c** $y = x^2 - 4x + 9$
d $y = x^2 - 2x + 5$ **e** $y = x^2 + 6x + 8$ **f** $y = x^2 - 8x + 11$
g $y = x^2 + 4x - 3$ **h** $y = x^2 - 8x + 21$ **i** $y = x^2 - 4x - 5$
j $y = x^2 + 6x + 10$ **k** $y = x^2 - 10x + 13$ **l** $y = x^2 - 8x - 4$
m $y = x^2 + 2x - 6$ **n** $y = x^2 + 10x + 32$ **o** $y = x^2 - 12x + 39$

6 Express each equation in the form $y = (x - h)^2 + k$, then state the co-ordinates of the vertex.

a $y = x^2 + 3x + 4$ **b** $y = x^2 - 5x + 9$ **c** $y = x^2 + x - 5$

7 For each parabola below:

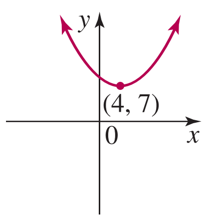
i state the co-ordinates of the vertex **ii** find the y-intercept
iii find the x-intercepts **iv** sketch the curve

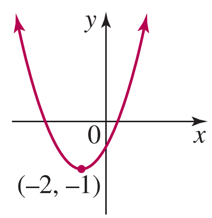
a $y = (x - 1)^2 - 9$ **b** $y = 4 - (x + 3)^2$

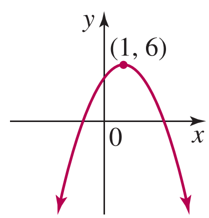
8 Sketch each pair of graphs on the same number plane showing the vertex and y-intercept.

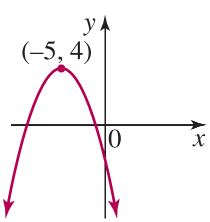
a $y = (x - 1)^2 + 4$ and $y = 2(x - 1)^2 + 4$
b $y = (x + 2)^2 - 8$ and $y = 3(x + 2)^2 - 8$

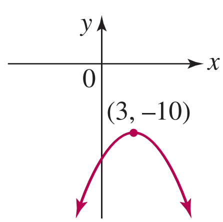
9 Find the equation of each curve in the form $y = a(x - h)^2 + k$, where $a = 1$ or -1.

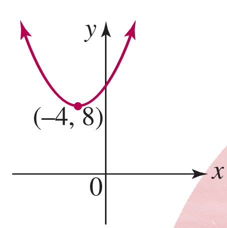
a


b


c


d


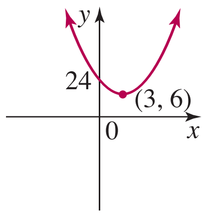
e


f


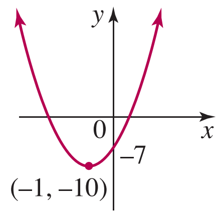
Further applications

10 Find the equation of each parabola in the form $y = a(x - h)^2 + k$.

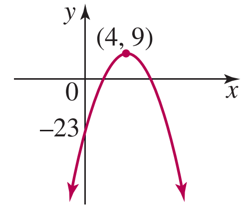
a



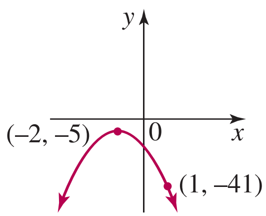
b



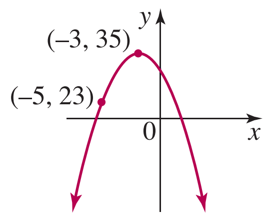
c



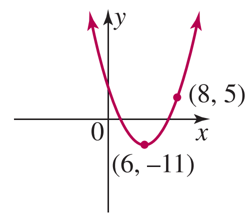
d



e



f



7.9 The cubic curve $y = ax^3 + d$

The graph of $y = x^3$

To determine the shape of the cubic curve, we can complete a table of values.

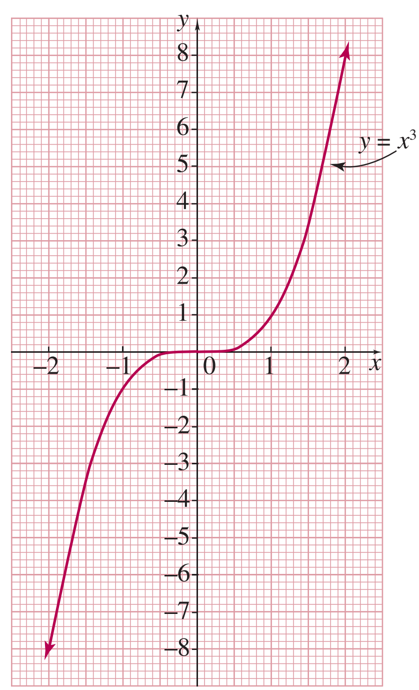
$y = x^3$

x	–2	–1.5	–1	–0.5	0	0.5	1	1.5	2
y	–8	–3.4	–1	–0.1	0	0.1	1	3.4	8

The y-values are given correct to 1 decimal place, where necessary.

The point where the graph changes concavity is called a point of inflexion. Its co-ordinates are (0, 0).

As the x-values become larger, the y-values also become larger.

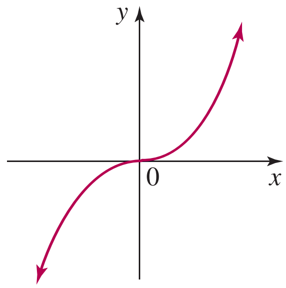


Features of the graph $y = x^3$:
- **The graph is a smooth, continuous curve.**
- **The curve increases from left to right.**
- **The point of inflexion has co-ordinates (0, 0).**
- **The curve has point symmetry about the origin.**

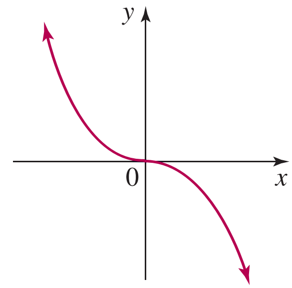
The graph of $y = -x^3$

The shape of the graph of the equation $y = -x^3$ can be determined as follows:
- if x is positive, then x^3 is positive, $\therefore -x^3$ is negative, so y is negative.
- if x is negative, then x^3 is negative, $\therefore -x^3$ is positive, so y is positive.
- if $x = 0$, $y = -0^3$, $\therefore y = 0$.

We can see that the graph of $y = -x^3$ is a reflection in the x-axis of the graph of $y = x^3$.



$y = x^3$



$y = -x^3$

The graph of $y = ax^3 + d$

We know that for parabolas with equations $y = ax^2 + bx + c$ or $y = a(x - h)^2 + k$, the curves become steeper as the value of a increases. This is also the case with cubic curves.

As with parabolas, the effect of adding a constant to the cubic equation $y = ax^3$ is to move the cubic curve up or down.

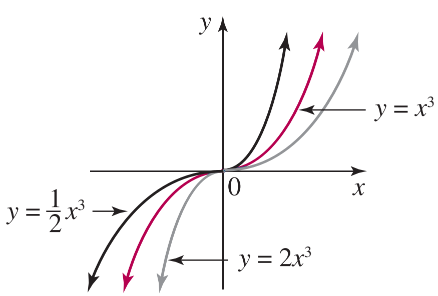
Example 1

Sketch these cubic curves on the same number plane.

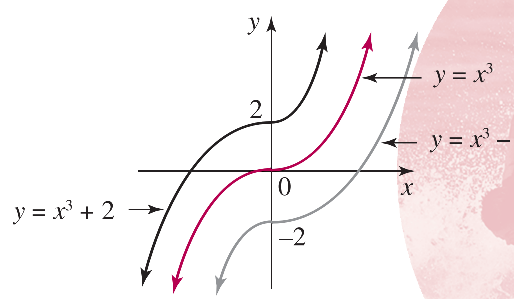
a $y = x^3$, $y = 2x^3$, $y = \frac{1}{2}x^3$ **b** $y = x^3$, $y = x^3 + 2$, $y = x^3 - 2$

Solutions

a



b



Exercise 7.9

1 Copy and complete the following tables of values. Graph the curves on separate number planes.

a $y = x^3$

x	-2	-1	0	1	2
y					

b $y = -x^3$

x	-2	-1	0	1	2
y					

2 How could the following graphs be obtained from the graph of $y = x^3$?

a $y = x^3 + 1$ b $y = x^3 - 2$ c $y = x^3 + 4$ d $y = x^3 - 7$

3 Sketch the following cubic curves. Show the intercept on the y-axis.

a $y = x^3 + 2$ b $y = x^3 - 5$ c $y = -x^3 + 1$ d $y = -x^3 - 3$

4 Sketch the following curves on the same number plane.

a $y = x^3$ and $y = 2x^3$ b $y = x^3$ and $y = \frac{1}{2}x^3$

c $y = -x^3$ and $y = -2x^3$ d $y = -x^3$ and $y = -\frac{1}{2}x^3$

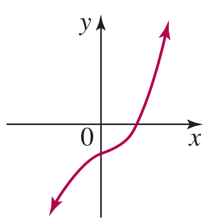
Consolidation

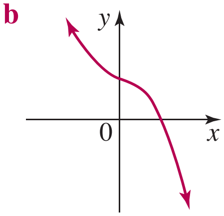
5 Write down the co-ordinates of the point of inflexion for each curve.

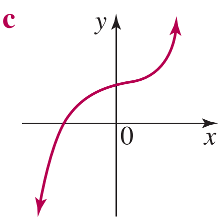
a $y = 2x^3$ b $y = x^3 + 6$ c $y = 3 - x^3$ d $y = -4x^3 - 5$

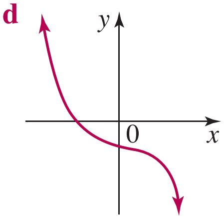
6 Match each of these equations with one of the graphs below.

- $y = x^3 + 2$
- $y = x^3 - 2$
- $y = -x^3 + 2$
- $y = -x^3 - 2$

a


b


c


d


7 Find the equation of the new cubic if the curve $y = x^3 + 4$ is translated:

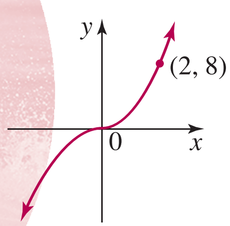
a 3 units up b 4 units down c 6 units down

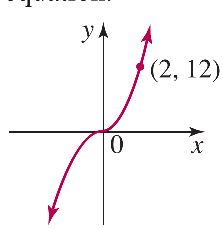
8 Sketch these cubic curves showing the x and y-intercepts.

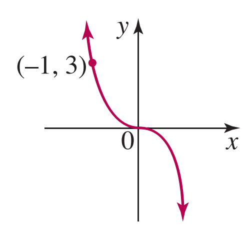
a $y = x^3 - 1$ b $y = x^3 + 27$ c $y = 8 - x^3$ d $y = -x^3 - 64$

9 The cubic curve $y = x^3$ is reflected in the x-axis then translated 5 units up. What is the equation of the new curve?

10 The curves below are cubics with equations of the form $y = ax^3$. For each curve, find the value of a and hence determine its equation.

a


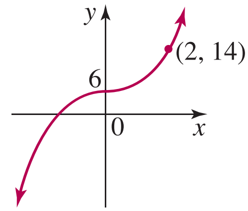
b


c


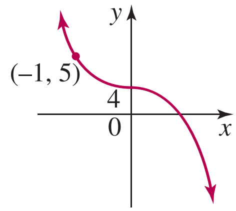
Further applications

11 The curves below are cubics with equations of the form $y = ax^3 + d$. For each curve, find values for a, d and hence determine its equation.

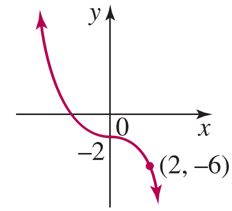
a



b



c



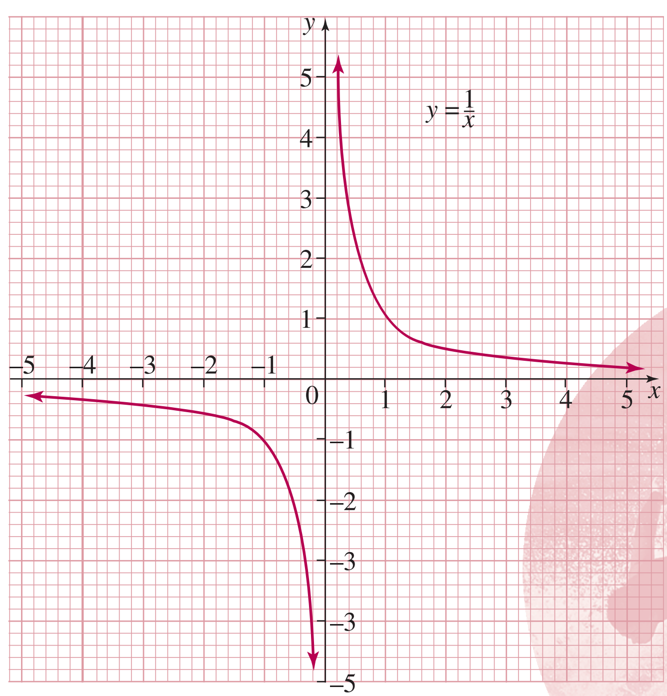
7.10 The hyperbola

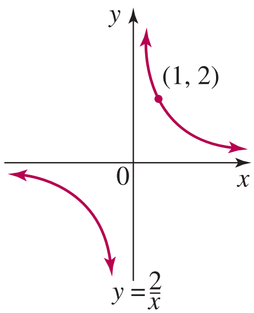
In some practical situations we see that one quantity varies as the inverse of another quantity. That is, as one quantity becomes larger, the other becomes smaller. This situation can be modelled by graphing the equation $y = \frac{1}{x}$. The resulting graph is called a **hyperbola**.

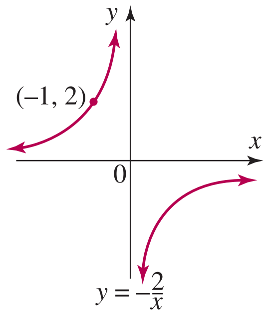
The curve consists of two separate branches, so a large number of points must be plotted in order to get a complete view of the graph.

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-3	∞	3	2	1	$\frac{1}{2}$	$\frac{1}{3}$

When any equation of the form $y = \frac{k}{x}$ is graphed on a number plane, where k is a constant ($k \neq 0$), the resulting graph will be a hyperbola. This equation is also commonly written as $xy = k$. The graph does not exist when $x = 0$ (or $y = 0$), because division by zero is not possible.







If the value of k is positive, then the curve lies entirely within the first and third quadrants. If the value of k is negative, then the curve lies entirely within the second and fourth quadrants. The further the value of k is from zero, the further is the curve from the origin.

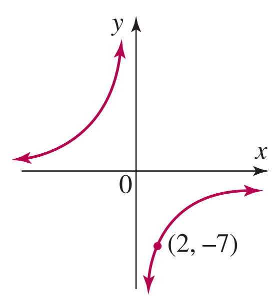
An **asymptote** is a line that is approached by a curve, but never touched by it. The hyperbola approaches both the x- and y-axes, but never touches them. Therefore, in this case, we say that the axes are asymptotes and that the hyperbola is an *asymptotic* curve.

Features of the hyperbola $y = \frac{k}{x}$:

- **The graph is a smooth, discontinuous curve consisting of two separate branches.**
- **The curve has point symmetry about the origin.**
- **The lines $y = x$ and $y = -x$ are axes of symmetry.**
- **The curve lies in the first and third quadrants if $k > 0$, and in the second and fourth quadrants if $k < 0$.**
- **The x- and y-axes are asymptotes.**

Example 1

The curve shown is a hyperbola. It has an equation of the form $xy = k$. Find the value of k and hence write down the equation of the hyperbola.



Solution

To find the value of k, we substitute the co-ordinates of the given point into the equation of the curve.

$$xy = k$$
$$2 \times -7 = k$$
$$\therefore k = -14$$

$\therefore$ The equation of the hyperbola is $xy = -14$.

Exercise 7.10

1 a Copy and complete this table of values for the equation $y = \frac{1}{x}$.

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3
y											

b Plot these points on a number plane. Draw a smooth curve through the points.
c In which two quadrants does the curve lie?
d Look at the branch of the hyperbola that lies in the first quadrant. Describe what happens to the y-values as the x-values:
i increase **ii** decrease
e What can you say about the value of y if:
i x is positive? **ii** x is negative?
f Does the graph ever cut the axes?
g What name is given to the axes in this situation?
h The graph has two axes of symmetry. What are their equations?

2 Copy and complete each table of values, then graph the equations on separate number planes.

a $y = \frac{2}{x}$

x	-4	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	4
y									

b $y = \frac{3}{x}$

x	-6	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	6
y											

3 **a** Copy and complete this table of values for the equation $y = -\frac{1}{x}$.

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3
y											

b Plot these points on a number plane. Draw a smooth curve through the points.
c In which two quadrants does the curve lie?
d What you can say about the value of y if:
i x is positive? **ii** x is negative?
e Does the graph have any asymptotes? If so, what are they?
f The graph has two axes of symmetry. What are their equations?

Consolidation

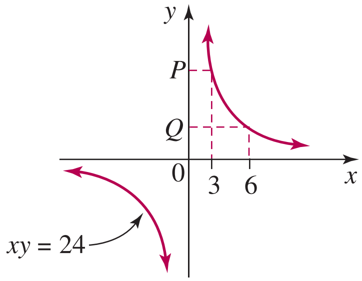
4 Sketch the following hyperbolas on separate number planes. Values are not required on the axes; however, it is customary to show one point that lies on the curve.

a $y = \frac{4}{x}$ **b** $y = -\frac{3}{x}$ **c** $y = \frac{7}{x}$ **d** $y = -\frac{5}{x}$
e $xy = 2$ **f** $xy = 6$ **g** $xy = -10$ **h** $xy = -12$

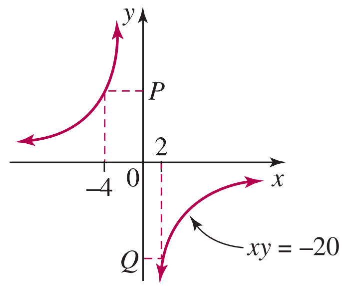
5 Find the value of the pronumeral in each point, if the points all lie on the hyperbola $xy = 12$.
a $(2, p)$ **b** $(-3, a)$ **c** $(k, -12)$ **d** $(u, 8)$

6 Find the length of the interval PQ in each diagram.

a

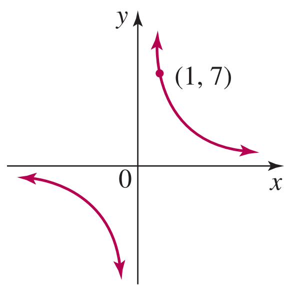


b



7 Find the equation of each hyperbola. Give the equations in the form $xy = k$.

a



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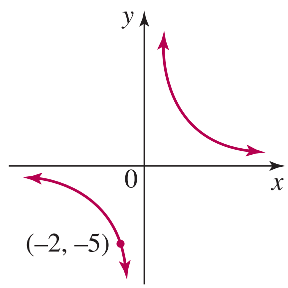
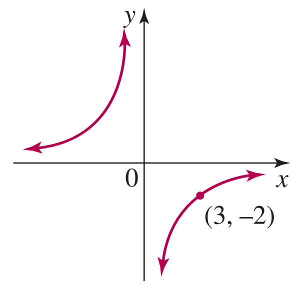
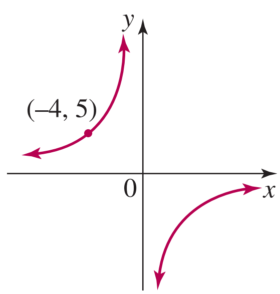


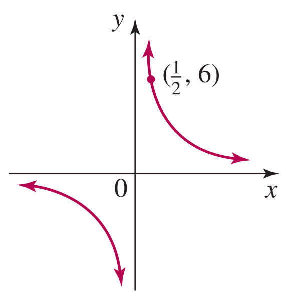
c



d



e



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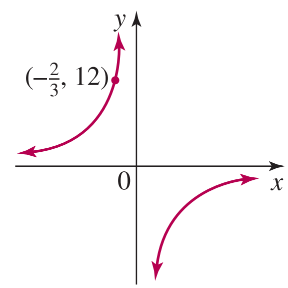


Further applications

8 A rectangle has dimensions such that the area remains constant at 20 cm^2.

a If the length is x cm, find an expression for the width.

b Copy and complete this table of values for the length (x) and width (y) of the rectangle.

x	1	2	4	5	10	20
y						

c Plot the points from the table on a number plane and join them to form a curve.

d What kind of curve have you drawn?

9 The speed s km/h of a moving object is given by the formula $s = \frac{d}{t}$, where d is the distance, in kilometres, and t is the time, in hours.
Irfan is planning to drive 120 km to see a friend.

a Copy and complete this table of values to calculate the required speeds that he could drive at in order to arrive at his friend's house in the given times.

t	1	2	4	5	6	8	10	12
s								

b Draw a speed–time graph using the values in the table. Show the times on the horizontal axis and the speeds on the vertical axis.

c From your graph, find the speed at which Irfan should drive if he wants to arrive at his friend's house in exactly 3 hours.

TRY THIS Gas under pressure

An experiment was carried out on a given mass of gas enclosed at constant temperature. The results are given in the table.

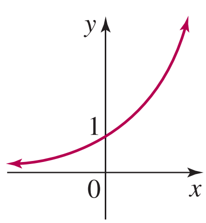
1. Graph the results. (Show P on the horizontal axis and V on the vertical axis.)
2. Does an inverse relationship exist?
3. What pressure would be required to halve the volume to 12 cm^3?
4. What is the formula relating P and V?

Pressure (P) in cm of Hg	Volume (V) in cm^3
200	24
160	30
150	32
100	48
80	60
50	96

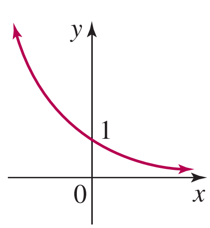
7.11 The exponential curve

In the real world, many quantities increase or decrease at an exponential rate. For example, the number of bacteria in a Petri dish may increase 10-fold every minute, or the level of radioactivity present in the soil after a nuclear explosion may be halved every 100 years.

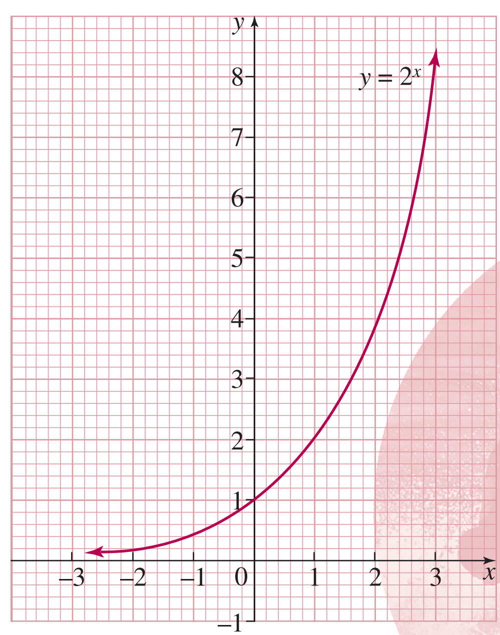
Situations involving exponential change can be modelled by equations of the form $y = a^x$ (for growth) and $y = a^{-x}$ (for decay), where a is a positive integer.



$y = a^x$



$y = a^{-x}$



If the equation is of the form $y = a^x$, then the curve will increase from left to right. If the equation is of the form $y = a^{-x}$, then the curve will decrease from left to right.

The exponential curves $y = a^x$ and $y = a^{-x}$ approach the x-axis but never touch it. That is, the x-axis is an asymptote and, like the hyperbola, the exponential curve is an asymptotic curve.

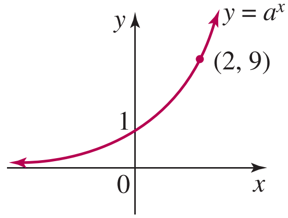
Features of the exponential curve $y = a^x$:
- **The graph is a smooth, continuous curve.**
- **The curve increases from left to right.**
- **The curve lies entirely above the x-axis.**
- **The x-axis is a horizontal asymptote.**

Features of the exponential curve $y = a^{-x}$:
- **The graph is a smooth, continuous curve.**
- **The curve decreases from left to right.**
- **The curve lies entirely above the x-axis.**
- **The x-axis is a horizontal asymptote.**

EG +S

Example 1

The curve shown is an exponential curve. It has an equation of the form $y = a^x$ ($a > 0$). Find the value of a and hence write down the equation of the curve.



Solution

To find the value of a, we substitute the co-ordinates of the given point into the equation of the curve.

$$y = a^x$$
$$9 = a^2$$
$$\therefore a = 3 \qquad \text{(note: } a > 0\text{)}$$

$\therefore$ The equation of the curve is $y = 3^x$.

Exercise 7.11

1 a Copy and complete this table of values for the equation $y = 2^x$.

x	–3	–2	–1	0	1	2	3
y							

b Plot these points on a number plane. Draw a smooth curve through the points.
c Describe what happens to the y values as the x values:
i increase **ii** decrease
d Does the curve ever cut the x-axis?
e What name is given to the x-axis in this situation?
f What can you say about the y values of every point on the curve?
g Where does the curve cut the y-axis?
h Does the curve have any axes of symmetry?

2 Copy and complete this table of values, then graph the equation $y = 3^x$ on a number plane.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y									

3 **a** Copy and complete this table of values for the equation $y = 2^{-x}$.

x	-3	-2	-1	0	1	2	3
y							

b Plot these points on a number plane. Draw a smooth curve through the points.

c Describe what happens to the y values as the x values:
i increase **ii** decrease

d Does the curve ever cut the x-axis?

e What can you say about the y values of every point on the curve?

f Where does the curve cut the y-axis?

g Does the curve have any axes of symmetry?

4 Do all exponential curves with equations of the form $y = a^x$ ($a > 0$) have the same y-intercept? Explain.

Consolidation

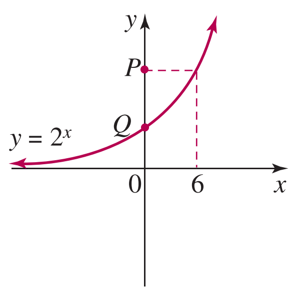
5 Sketch the following exponential curves on separate number planes. Values are not required on the axes; however, it is customary to show one point that lies on the curve.

a $y = 4^x$ **b** $y = 3^{-x}$ **c** $y = 6^x$ **d** $y = 5^{-x}$

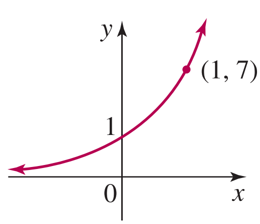
6 Find the value of the pronumeral in each point, if the points all lie on the curve $y = 2^x$.

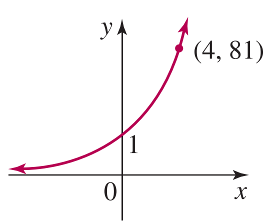
a $(4, m)$ **b** $(c, 8)$ **c** $(-1, w)$ **d** $(h, \frac{1}{4})$

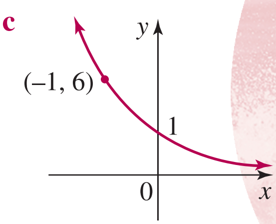
7 Find the length of the interval PQ.

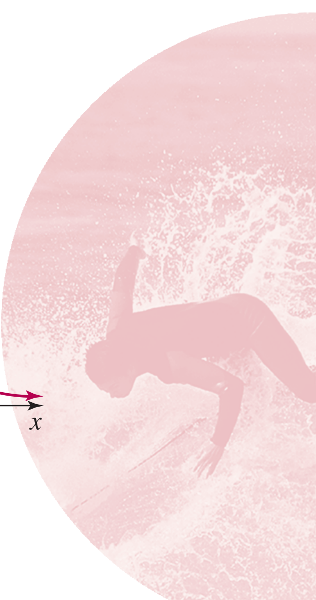


8 Find the equation of each curve in the form $y = a^x$, where a is an integer.

a


b


c




9 The expression -2^x means -1×2^x, not $(-2)^x$.
  a How could the graph of $y = -2^x$ be obtained from the graph of $y = 2^x$?
  b Sketch the curves $y = 2^x$ and $y = -2^x$ on the same number plane.

10 Sketch the curves $y = 5^{-x}$ and $y = -5^{-x}$ on the same number plane.

11 Would the graph of $y = 2 \times 3^x$ be the same as the graph of $y = 6^x$? Explain.

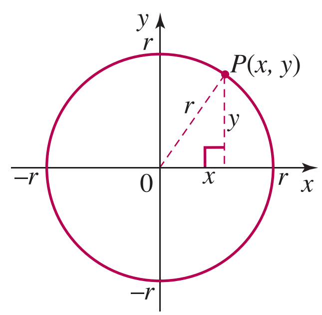
12 How could the graph of $y = 7^{-x}$ be obtained from the graph of $y = 7^x$?

Further applications

13 Sketch the following curves clearly showing any intercepts with the co-ordinate axes and the equation of the asymptote.
  a $y = 4^x + 1$ b $y = 2^x + 3$
  c $y = 3^x + 2$ d $y = 5^x - 1$
  e $y = 2^x - 8$ f $y = 3^x - 9$

14 Sketch the graph of $y = 1^x$.

7.12 The circle



A circle is the set of all points that are equidistant from a fixed point (the centre).

Let $P(x, y)$ be a point on the circle with centre (0, 0) and radius r units.

By Pythagoras' theorem, $x^2 + y^2 = r^2$. This condition holds for every point P on the circle. Hence it is the equation of the circle.

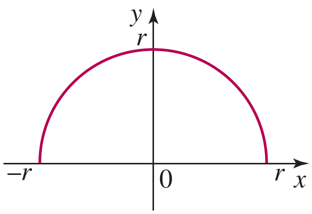
The equation of a circle with centre (0, 0) and radius r units is:

$$x^2 + y^2 = r^2$$

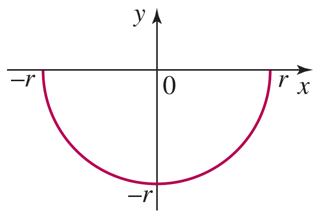
If a point satisfies the inequality $x^2 + y^2 < r^2$, then the point lies inside the circle. If the point satisfies the inequality $x^2 + y^2 > r^2$, then the point lies outside the circle.

The semi-circle (Extension)

Re-arranging the equation $x^2 + y^2 = r^2$, we obtain the equations $y = \pm\sqrt{r^2 - x^2}$. That is, $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$. These equations represent the upper and lower halves of the circle.



$y = \sqrt{r^2 - x^2}$



$y = -\sqrt{r^2 - x^2}$

- **The equation of an upper semi-circle is:**

$$y = \sqrt{r^2 - x^2}$$

- **The equation of a lower semi-circle is:**

$$y = -\sqrt{r^2 - x^2}$$

Example 1

State the equation of the circle with centre (0, 0) and radius:

a 7 units **b** $2\sqrt{6}$ units **c** $1\frac{2}{3}$ units

Solutions

a $x^2 + y^2 = 7^2$

$\therefore x^2 + y^2 = 49$

b $x^2 + y^2 = (2\sqrt{6})^2$

$\therefore x^2 + y^2 = 24$

c $x^2 + y^2 = \left(\frac{5}{3}\right)^2$

$x^2 + y^2 = \frac{25}{9}$

$\therefore 9x^2 + 9y^2 = 25$

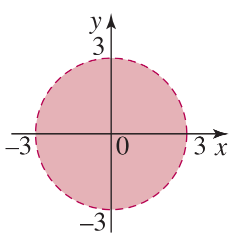
Example 2

Shade the following regions.

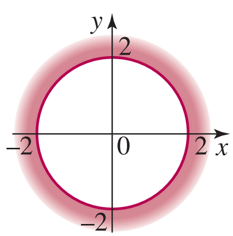
a $x^2 + y^2 < 9$ **b** $x^2 + y^2 \geqslant 4$ **c** $4 \leqslant x^2 + y^2 \leqslant 25$

Solutions

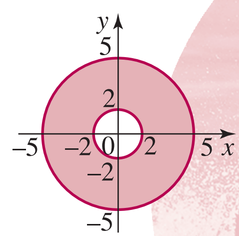
a



b

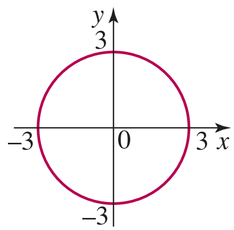


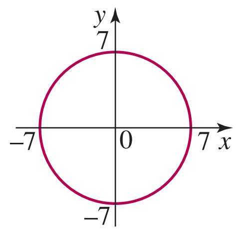
c

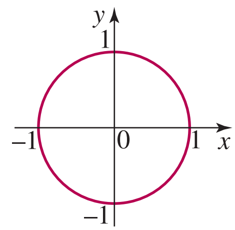


Exercise 7.12

1 State the equation of each circle.

a


b


c


2 Find the equation of the circle with centre (0, 0) and radius:

a 2 units **b** 5 units **c** 4 units **d** $\sqrt{3}$ units
e $\sqrt{7}$ units **f** $\sqrt{10}$ units **g** $2\sqrt{2}$ units **h** $4\sqrt{3}$ units
i $3\sqrt{7}$ units **j** 1.2 units **k** 1.9 units **l** 2.5 units

3 Find the length of the radius in each circle. Answer in simplest surd form where appropriate.

a $x^2 + y^2 = 9$ **b** $x^2 + y^2 = 49$ **c** $x^2 + y^2 = 121$ **d** $x^2 + y^2 = 5$
e $x^2 + y^2 = 14$ **f** $x^2 + y^2 = 22$ **g** $x^2 + y^2 = 12$ **h** $x^2 + y^2 = 27$
i $x^2 + y^2 = 45$ **j** $x^2 + y^2 = 1.69$ **k** $x^2 + y^2 = 2.89$ **l** $x^2 + y^2 = 12.25$

4 Sketch these circles.

a $x^2 + y^2 = 36$ **b** $x^2 + y^2 = 3$ **c** $x^2 + y^2 = 18$

Consolidation

5 Find, in the form $ax^2 + ay^2 = b^2$ where a, b are integers, the equation of a circle with centre (0, 0) and radius:

a $1\frac{1}{2}$ units **b** $2\frac{3}{4}$ units **c** $3\frac{2}{5}$ units

6 Find, as a fraction or mixed numeral, the radius of each circle.

a $4x^2 + 4y^2 = 9$ **b** $9x^2 + 9y^2 = 4$ **c** $4x^2 + 4y^2 = 25$
d $49x^2 + 49y^2 = 16$ **e** $16x^2 + 16y^2 = 9$ **f** $25x^2 + 25y^2 = 121$

7 Find the co-ordinates of the two points on the circle $x^2 + y^2 = 100$ that have:

a an x co-ordinate of 6 **b** a y co-ordinate of -8
c an x co-ordinate of 7 **d** a y co-ordinate of -5

8 Find the equation of the circle centre (0, 0) which passes through each of these points.

a (3, 4) **b** (5, –3) **c** $(-\sqrt{5}, 2\sqrt{3})$

9 State whether each equation is the equation of a circle.

a $y = 2x + 7$ **b** $x^2 + y^2 = 64$ **c** $y = x^2 + 4$
d $x^2 + y^2 = 10$ **e** $2x^2 + 2y^2 = 7$ **f** $x^2 - y^2 = 9$
g $y^2 = 8 - x^2$ **h** $x^2 = 15 + y^2$ **i** $y^2 = x^2 - 4x$

10 Determine whether each point lies inside, outside or on the circle $x^2 + y^2 = 40$.

a $(4, 3)$ **b** $(-5, 4)$ **c** $(2, 6)$ **d** $(-3, -5)$
e $(6.5, 0)$ **f** $(-6, -1.5)$ **g** $(4, 2\sqrt{6})$ **h** $(4\sqrt{2}, -2\sqrt{3})$

11 Shade the following regions.

a $x^2 + y^2 \leqslant 4$ **b** $x^2 + y^2 > 9$ **c** $x^2 + y^2 < 81$
d $x^2 + y^2 \geqslant 12.25$ **e** $x^2 + y^2 \leqslant 20$ **f** $x^2 + y^2 > 32$

Further applications

12 The circle $x^2 + y^2 = r^2$ can be divided into an upper semi-circle $y = \sqrt{r^2 - x^2}$ and a lower semi-circle $y = -\sqrt{r^2 - x^2}$. Sketch the following semi-circles.

a $y = \sqrt{4 - x^2}$ **b** $y = -\sqrt{25 - x^2}$ **c** $y = \sqrt{121 - x^2}$
d $y = -\sqrt{2 - x^2}$ **e** $y = -\sqrt{28 - x^2}$ **f** $y = \sqrt{50 - x^2}$

13 Shade the region where both inequalities hold simultaneously.

a $x^2 + y^2 \leqslant 16$ and $y \geqslant 2$ **b** $x^2 + y^2 < 49$ and $x \geqslant -3$
c $x^2 + y^2 \geqslant 4$ and $x > 1$ **d** $x^2 + y^2 > 25$ and $y \leqslant 0$
e $x^2 + y^2 < 8$ and $y > x$ **f** $4 < x^2 + y^2 \leqslant 9$

7.13 Miscellaneous graphs

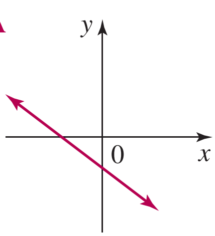
In order to sketch a graph, we must first be able to state what kind of graph is described by an equation.

Example 1

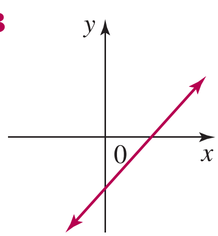
Match each equation with one of the graphs below.

a $y = 2x + 7$ **b** $y = 7 - 2x$ **c** $y = -2x - 7$ **d** $y = 2x - 7$

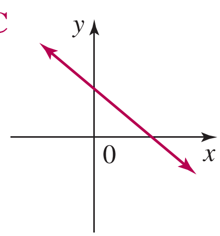
A



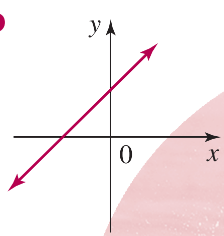
B



C



D



Solutions

a The gradient is positive ($m = 2$), so the line is increasing from left to right. The y-intercept is positive ($b = 7$). Therefore the equation describes graph D.

b The gradient is negative ($m = -2$), so the line is decreasing from left to right. The y-intercept is positive ($b = 7$). Therefore the equation describes graph C.

c The gradient is negative ($m = -2$), so the line is decreasing from left to right. The y-intercept is negative ($b = -7$). Therefore the equation describes graph A.

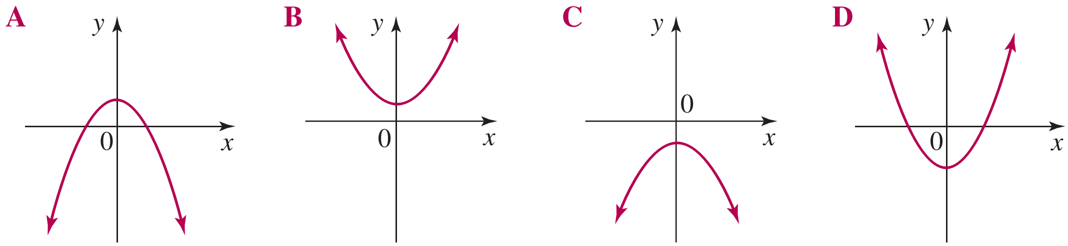
d The gradient is positive ($m = 2$), so the line is increasing from left to right. The y-intercept is negative ($b = -7$). Therefore the equation describes graph B.

Example 2

EG +S

Match each equation with one of the graphs below.

a $y = x^2 + 1$ **b** $y = 1 - x^2$ **c** $y = x^2 - 1$ **d** $y = -x^2 - 1$



Solutions

a The co-efficient of x^2 is positive, so the curve is concave upward. The y-intercept is positive. Therefore, the equation describes graph B.

b The co-efficient of x^2 is negative, so the curve is concave downward. The y-intercept is positive. Therefore, the equation describes graph A.

c The co-efficient of x^2 is positive, so the curve is concave upward. The y-intercept is negative. Therefore, the equation describes graph D.

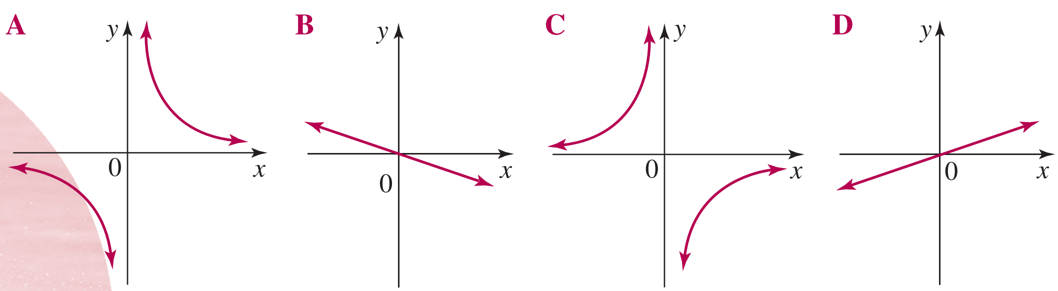
d The co-efficient of x^2 is negative, so the curve is concave downward. The y-intercept is negative. Therefore, the equation describes graph C.

Example 3

EG +S

Match each equation with one of the graphs below.

a $y = \frac{x}{3}$ **b** $y = \frac{3}{x}$ **c** $y = -\frac{x}{3}$ **d** $y = -\frac{3}{x}$



Solutions

a The equation $y = \frac{x}{3}$ can be re-written as $y = \frac{1}{3}x$. This is the equation of a straight line that passes through the origin and has a gradient of $\frac{1}{3}$. Therefore, the equation describes graph D.

b The equation $y = \frac{3}{x}$ is of the form $y = \frac{k}{x}$, where k is a positive constant. This is the equation of a hyperbola that lies in the first and third quadrants. Therefore, the equation describes graph A.

c The equation $y = -\frac{x}{3}$ can be re-written as $y = -\frac{1}{3}x$. This is the equation of a straight line that passes through the origin and has a gradient of $-\frac{1}{3}$. Therefore, the equation describes graph B.

d The equation $y = -\frac{3}{x}$ is of the form $y = \frac{k}{x}$, where k is a negative constant. This is the equation of a hyperbola that lies in the second and fourth quadrants. Therefore, the equation describes graph C.

Exercise 7.13

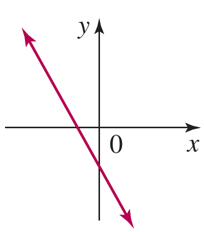
1 State whether each equation is the equation of a straight line (L), a parabola (P), a cubic (CU), a hyperbola (H), an exponential curve (E) or a circle (C).

a $y = x$	**b** $y = \frac{1}{x}$	**c** $y = x^2$	**d** $y = 2^x$
e $x^2 + y^2 = 1$	**f** $y = x^3$	**g** $y = 3x^2$	**h** $y = 1$
i $y = 3^{-x}$	**j** $xy = 2$	**k** $y = 5 - x$	**l** $x^2 + y^2 = 9$
m $y = x^3 - 1$	**n** $x = 7$	**o** $y = \frac{6}{x}$	**p** $y = -5^x$
q $x^2 + y^2 = 2$	**r** $y = 2x^3 + 9$	**s** $y = \frac{x}{5}$	**t** $y = x^2 - 5x + 4$
u $y = 0$	**v** $xy = -3$	**w** $y = -4^{-x}$	**x** $3x^2 + 3y^2 = 5$

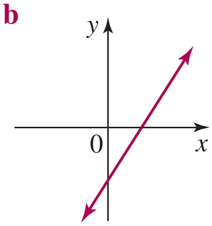
2 Match each equation with one of the graphs below.

- $y = 3x + 1$
- $y = 1 - 3x$
- $y = 3x - 1$
- $y = -3x - 1$

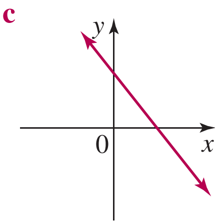
a



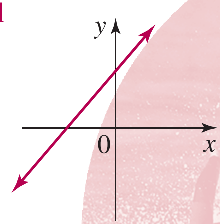
b



c

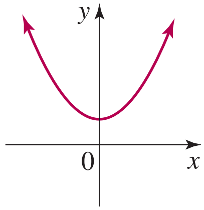


d



3 Match each equation with one of the graphs below.

- $y = x^2 - 5$
- $y = 5 - x^2$
- $y = -x^2 - 5$
- $y = x^2 + 5$

a


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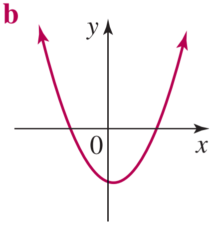

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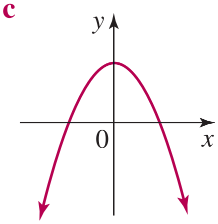

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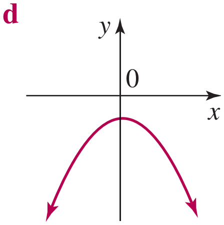
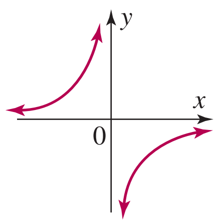


4 Match each equation with one of the graphs below.

- $y = \frac{2}{x}$
- $y = \frac{x}{2}$
- $y = -\frac{2}{x}$
- $y = -\frac{x}{2}$

a


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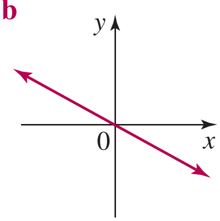

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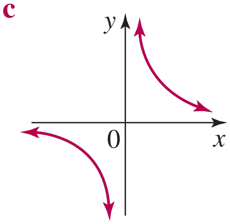

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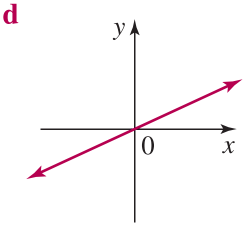
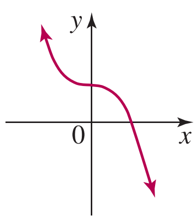


5 Match each equation with one of the graphs below.

- $y = x^3 - 1$
- $y = x^3 + 1$
- $y = -x^3 - 1$
- $y = 1 - x^3$

a


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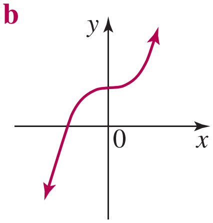

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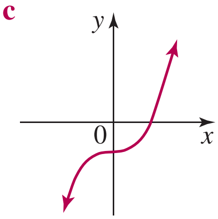

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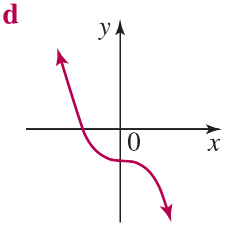
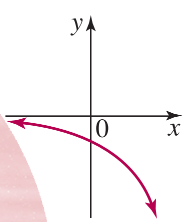


6 Match each equation with one of the graphs below.

- $y = 2^x$
- $y = 2^{-x}$
- $y = -2^x$
- $y = -2^{-x}$

a


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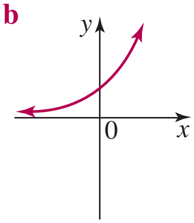

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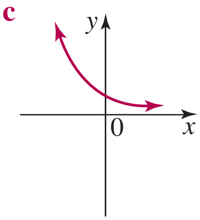

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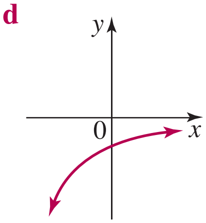
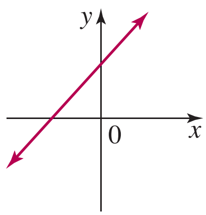


Consolidation

7 Choose the graph that best represents each equation.

a $y = x^2$	**b** $y = \dfrac{1}{x}$	**c** $y = x$
d $y = x^3$	**e** $x^2 + y^2 = 4$	**f** $y = 2^x$
g $x = 4$	**h** $y = x^2 + 1$	**i** $xy = -4$
j $y = -x^2 - 6$	**k** $y = x^3 + 5$	**l** $y = 3$
m $y = 2^{-x}$	**n** $y = x^2 - 2$	**o** $y = 1 - x^3$
p $y = -x^2$	**q** $y = x + 7$	**r** $y = -x^3$
s $y = -x - 4$	**t** $y = x^3 - 2$	**u** $y = x - 2$
v $y = -x^3 - 6$	**w** $y = 3 - x$	**x** $y = -x$

A


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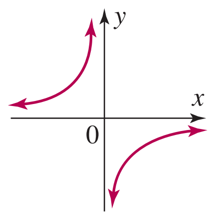

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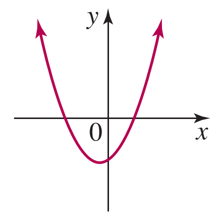

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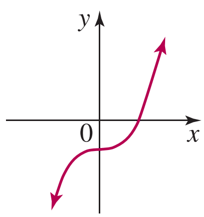

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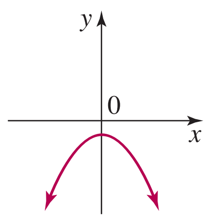

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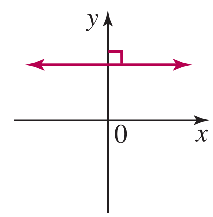

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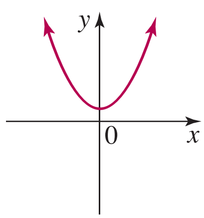

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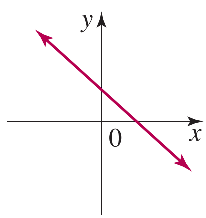

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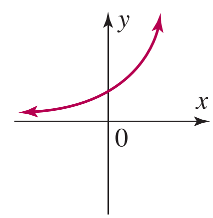

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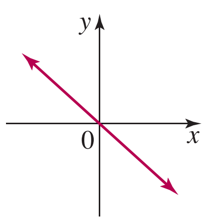

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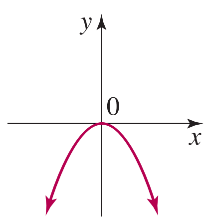

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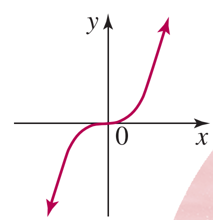

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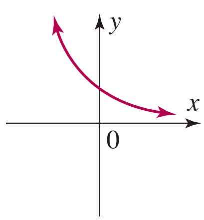

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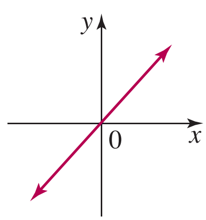

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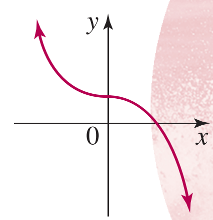

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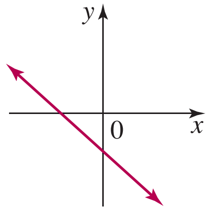

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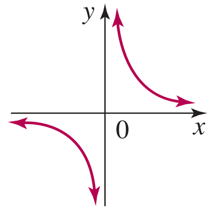

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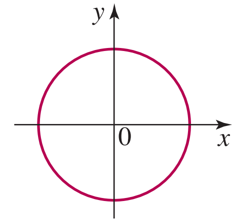

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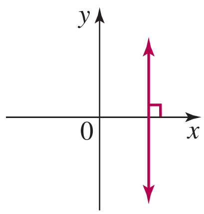

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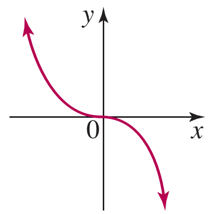

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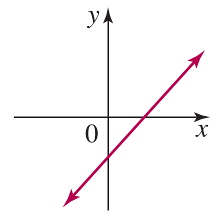

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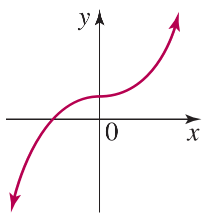

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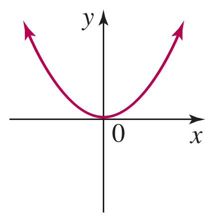

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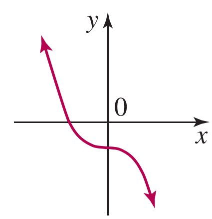
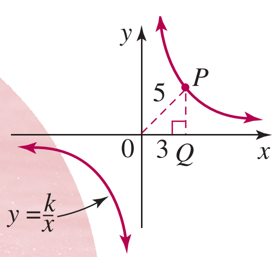


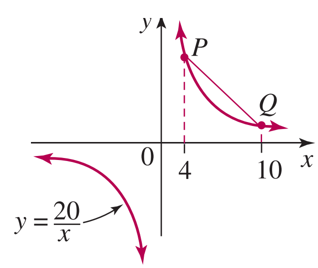
8 Draw a neat sketch of the graph of each equation.

a $y = 2x + 5$	**b** $x^2 + y^2 = 16$	**c** $xy = 6$
d $y = 3x^2$	**e** $y = -2$	**f** $y = 8 - x^2$
g $y = 4^x$	**h** $y = \dfrac{4}{x}$	**i** $y = x^3 + 3$
j $x^2 + y^2 = 20$	**k** $y = 2x^2 - 4$	**l** $y = (x - 2)^2$
m $y = -\dfrac{8}{x}$	**n** $y = 6^{-x}$	**o** $y = 7 - x^3$
p $y = 9 - 2x$	**q** $y = \dfrac{1}{4}x^2 + 1$	**r** $y = (x + 3)^2 - 8$
s $y = -3^x$	**t** $xy = -12$	**u** $y = \dfrac{1}{2}x - 3$
v $x = -1$	**w** $x^2 + y^2 = 63$	**x** $y = 6 - (x - 4)^2$

9



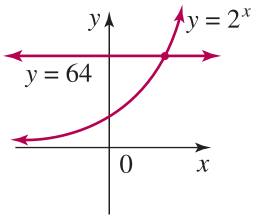
a Find the length of the interval PQ.
b Write down the co-ordinates of P.
c Find the value of k and hence find the equation of the hyperbola.



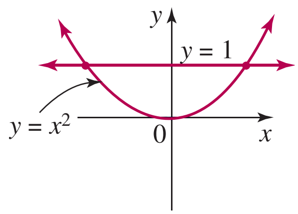
10 **a** Find the co-ordinates of P and Q.
b Hence, find the gradient of the interval PQ.

11 Find the co-ordinates of all points of intersection in each of the following.

a



b



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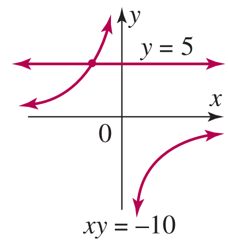

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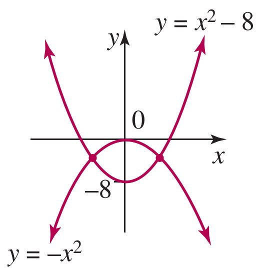

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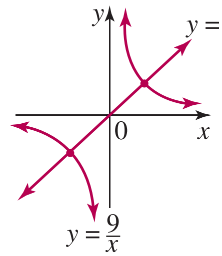

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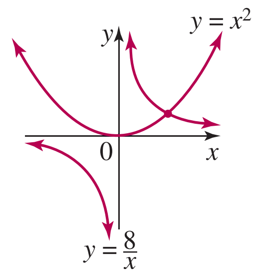

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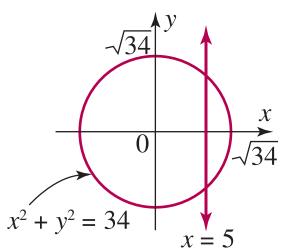

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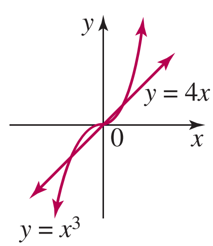

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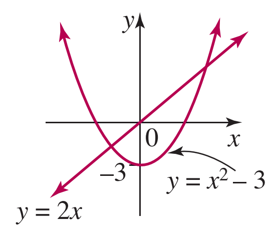
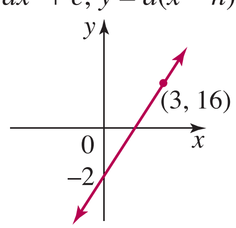


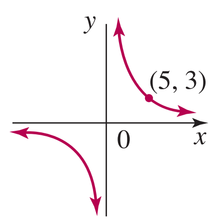
■ Further applications

12 Find the equation of each graph. Give your answers in the form $y = mx + b$, $y = \dfrac{k}{x}$, $y = ax^2 + c$, $y = a(x-h)^2 + k$, $y = a(x-p)(x-q)$, $y = ax^3 + d$, $x^2 + y^2 = r^2$, $y = a^x$ or $y = a^{-x}$.

a



b



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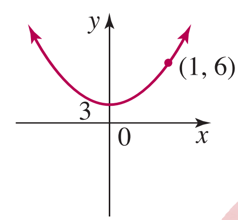

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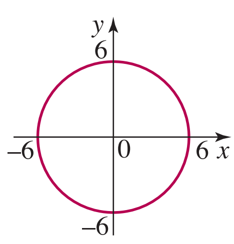

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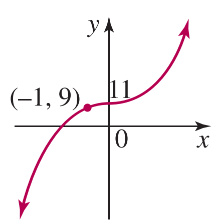
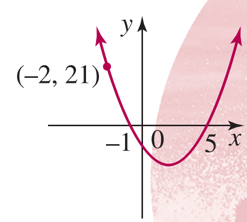
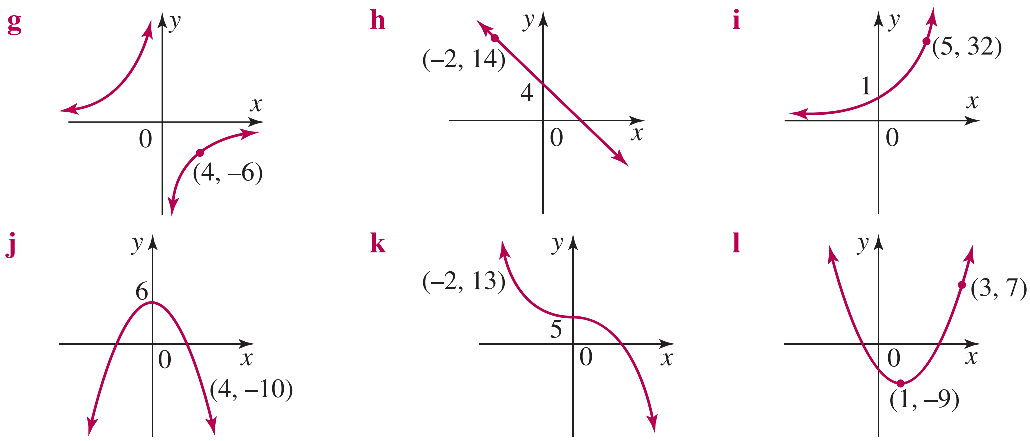


f





TRY THIS **Maximum area**

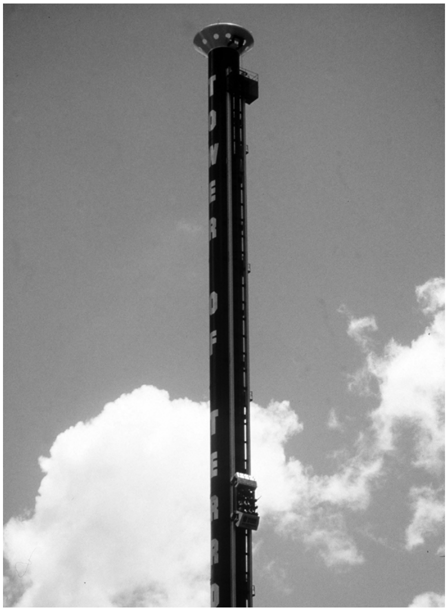
If the perimeter of a rectangle remains fixed at 16 cm, draw up a table for length (l) against area (A) and graph the results. Find the dimensions that give the maximum area.

Focus on working mathematically

The Tower of Terror

Introduction

One of the fastest and tallest rides in the world is the tower of terror at Dreamworld on the Gold Coast in Queensland. Riders climb 11 m up to the ride which starts inside a darkened skull. The pod accelerates from the darkness into daylight, reaching 160 km/h in just 7 seconds along a horizontal track 200 m long. It continues at constant speed on the horizontal track which then begins a steep upward rise attached to a tower 115 m high (the equivalent of a 38 storey building). The vertical part of the track is 1 m from the tower all the way up. After coming to rest at the top of the ride, the pod then falls freely backwards down the track. The tower of terror lives up to its name as truly terrifying.



LEARNING ACTIVITIES

1 Copy the diagram of the ride into your book.

The Tower of Terror

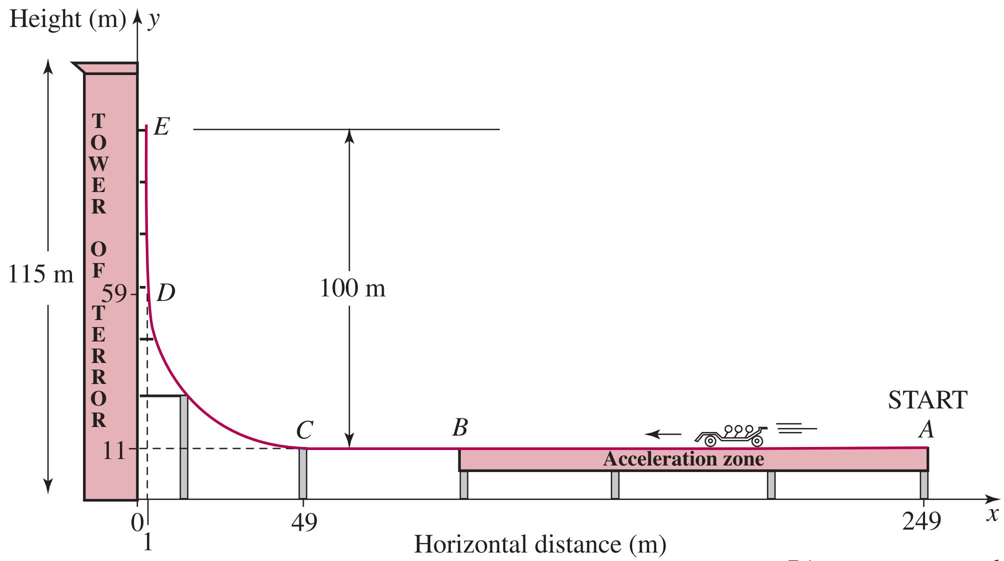


Diagram not to scale

2 What curve does the overall shape remind you of?

3 The ride can be thought of in two parts, going up and coming down. Starting from *A*, riders are accelerated along part of the horizontal track from *A* to *B*. The pod continues at constant speed to *C*. Using the information in the introduction write down the speed of the riders at *A*, *B* and *C*.

4 Change the speed (v) at *B* from km/h to m/s (1 decimal place).

5 How long does it take to get from *A* to *B*? (see the introduction).

6 The equation $v = u + at$ models the straight line motion from *A* to *B*. In this equation, v is the velocity of the riders at *B*; u is the speed of the riders at the start *A*; a is the constant acceleration; and t the time taken in seconds to move from *A* to *B*. Using your values for v, u and t above calculate the acceleration a (2 decimal places).

7 Using the equation $s = ut + \frac{1}{2}at^2$ and your values for u, a and t show that the distance s from *A* to *B* is 156 m (nearest m).

8 Find the distance from *B* to *C*. Using the equation $s = vt$ show that the time taken to reach *C* from *B* is approximately 1 s.

9 If the time taken from *C* to *E* is 5.5 s. What is the total time for the upward ride?

10 If the point *E* is 100 m above the horizontal part of the track, calculate the highest point of the ride.

CHALLENGE ACTIVITIES

We are now going to model the ride with equations using coordinate geometry.

1 Using a coordinate system as shown in the diagram, write down the equation of the horizontal line *CA* and the vertical line *DE*.

2 What might be the shape of the curve *CD*? Justify your answer using the graph.

3 Why is the hyperbola $y = \frac{k}{x}$ where *k* is some constant, **not** a good model for the whole track? Explain.

4 What do you think is the time taken for the whole ride? Justify your answer.

5 During the ride, people experience a sensation of "weightlessness". Find out what this means and where it occurs in the ride.

6 In modelling the ride for this focus on working mathematically, several physical factors were not taken into account (see questions 1 to 10 of the learning activities above). What might they be? How might they effect the answers?

LET'S COMMUNICATE

In about half a page of writing write an account of the ride describing the shape of the track, the speeds reached, the acceleration and the time taken for the ride up and down.

REFLECTING

When you climb aboard the ride of the Tower of Terror you put your faith in the engineers who built it. Reflect on the mathematics required to build and test the track and the pod for safety. What does this say about the importance of mathematics for safe amusement parks?

Language link with Macquarie

MACQUARIE

1 Use a single word to describe the shape of the graph:

a The shape of $y = ax^2 + c$ where a and c are constants is a …

b The shape of $y = \frac{k}{x}$ where k is a constant is a …

c The shape of $y = mx + b$ where m and b are constants is a …

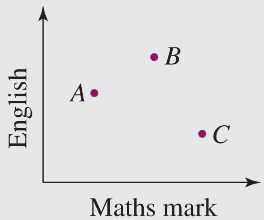
2 Explain the difference between the graphs of $y = x^2$ and $y = 2^x$ with a diagram.

3 The definition of the word ***graph*** *in the* Macquarie Learners Dictionary *is as follows:*

graph *noun* a diagram which shows the relationship between two or more things by dots, lines or bars

Do you feel the definition is adequate for mathematical purposes? If not, how would you change it?

1

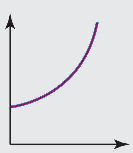


The graph shown compares the Maths and English results of three students. State which student had:

a the highest Maths mark
b the highest English mark
c the lowest Maths mark
d the lowest English mark

2 These graphs show the changing temperatures of four objects.

A



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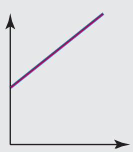

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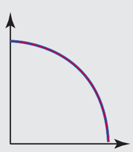
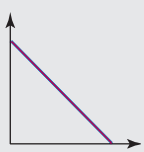


D

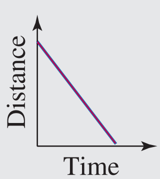


State which graph shows the temperature:

a increasing at a constant rate
b decreasing at a changing rate

3 Which distance–time graph is impossible?

A



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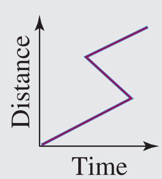

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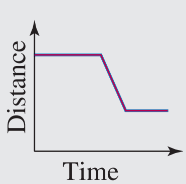

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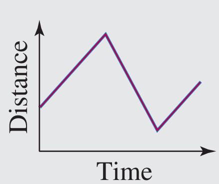
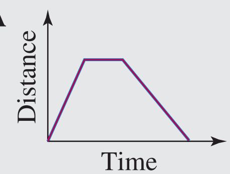


4 Michael walked up a steep hill, rested for 10 min, then ran down the other side. Which graph best illustrates the relationship between time taken and distance travelled?

A



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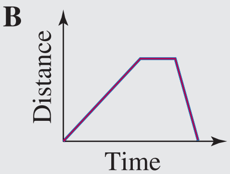

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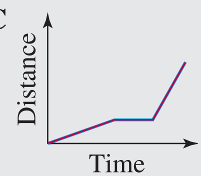

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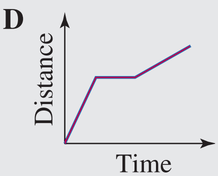
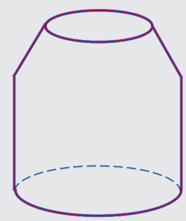
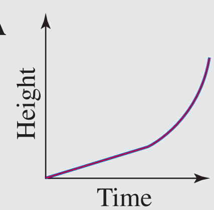


5 Water is poured at a constant rate into this jar. Which graph best represents the situation?



A



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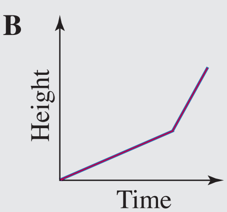

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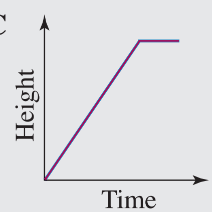

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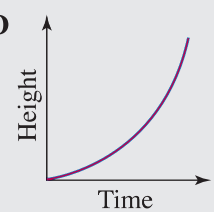
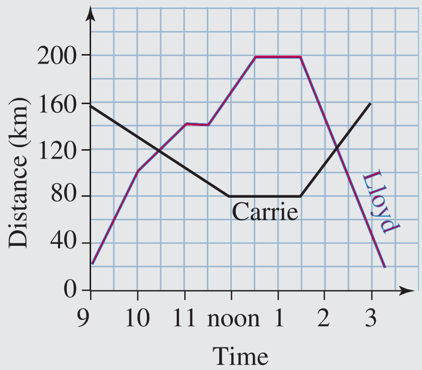


6 The graph shows the journeys taken by Carrie and Lloyd. They left home at the same time.



- **a** At what time did Lloyd leave home?
- **b** How far apart do Carrie and Lloyd live?
- **c** At what time did they first pass each other?
- **d** When did Lloyd stop for lunch?
- **e** How far had Lloyd travelled when Carrie stopped for lunch?
- **f** How far did Carrie travel altogether during the day?
- **g** Did Lloyd's speed increase or decrease at 10 am? How do you know?
- **h** How far apart were Carrie and Lloyd at noon?
- **i** What was Lloyd's speed between 11:30 am and 12:30 pm?
- **j** At what time had Lloyd travelled 40 km?

7 Complete this table of values for each equation, then graph the lines on separate number planes.

x	0	1	2	3	4
y					

a $y = \frac{1}{2}x - 2$ **b** $2x + y = 4$

8 Find the x- and y-intercepts for each line, then graph the lines on separate number planes.

a $3x - 2y = 12$ **b** $y = \frac{1}{2}x + 2$

9 Write down the equation of a line that has:

- **a** a gradient of 4 and a y-intercept of -3
- **b** a slope of -2 and passes through the origin

10 Find the gradient of the line $3x + 4y - 8 = 0$.

11 Graph the following lines by first plotting the y-intercept, then using the gradient to find two other points that lie on the line.

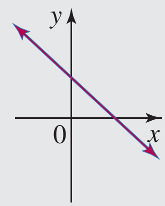
a $y = 2x + 3$ **b** $y = 4 - x$

c $y = \frac{1}{2}x - 4$

12 Choose the equation that best describes each graph.

- $y = 2x + 7$
- $y = -2x - 7$
- $y = 2x - 7$
- $y = 7 - 2x$

a



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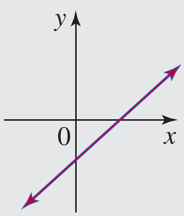

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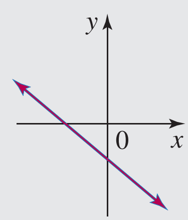
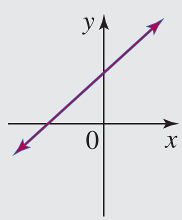


d

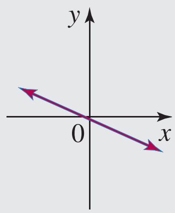


13 Choose the equation that best describes each graph.

- $y = 5x$
- $y = -5x$
- $y = \frac{1}{5}x$
- $y = -\frac{1}{5}x$

CHAPTER REVIEW

a



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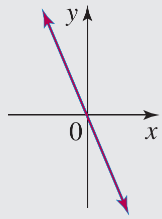

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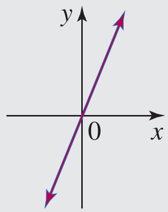

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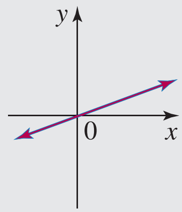
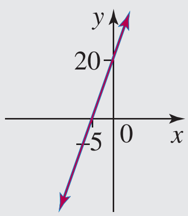


14 Find the equation of each line.

a



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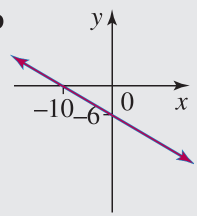


15 Sketch these lines on the same number plane.

a $x = 3$ **b** $y = -1$

16 Find the equation of the line that is parallel to:

a the x-axis and passes through $(-1, 5)$

b the y-axis and passes through $(4, 3)$

17 Determine whether the following points lie on the line $y = 5 - 2x$.

a $(-7, 19)$ **b** $(0, 3)$

18 The point $(k, -3)$ lies on the line $3x - 5y - 21 = 0$. Find the value of k.

19 Find the equation of the line that passes through these points.

a

x	0	1	2	3	4
y	–6	–5	–4	–3	–2

b

x	0	1	2	3	4
y	4	6	8	10	12

c

x	0	1	2	3	4
y	5	4	3	2	1

d

x	0	3	6	9	12
y	1	2	3	4	5

20 Sketch the graphs of $y = x^2$, $y = 2x^2$, $y = \frac{1}{2}x^2$ on the same number plane.

21 Sketch the graphs of $y = x^2$ and $y = -x^2$ on the same number plane.

22 Sketch the following graphs on separate number planes. Show the y-intercept. Other values on the axes are not required.

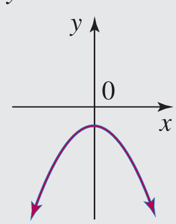
a $y = x^2 - 1$ **b** $y = x^2 + 2$

c $y = 6 - x^2$ **d** $y = -x^2 - 7$

23 Match each of these equations with one of the graphs below.

- $y = x^2 + 1$
- $y = x^2 - 1$
- $y = 1 - x^2$
- $y = -x^2 - 1$

a



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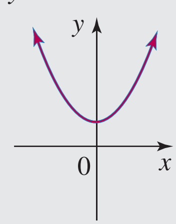

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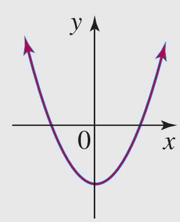

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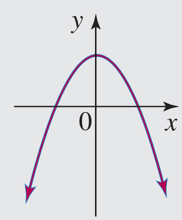


24 Sketch each parabola showing the x- and y-intercepts and the vertex.

a $y = 6x - x^2$ **b** $y = x^2 + 2x - 8$

c $y = 3x^2 - 7x + 2$

25 Find the equation of the axis of symmetry and the vertex for each parabola.

a $y = x^2 - 8x + 10$

b $y = -3x^2 + 12x - 7$

26 Find the minimum value of the function $y = \frac{1}{2}x^2 + 6x - 3$.

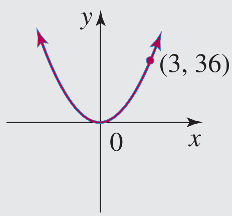
27 Sketch these parabolas showing the vertex and the y-intercept.

a $y = (x - 2)^2$ **b** $y = -(x + 3)^2$
c $y = (x + 4)^2 + 3$ **d** $y = 9 - (x - 1)^2$

28 Express the equation $y = x^2 - 6x + 11$ in the form $y = (x - h)^2 + k$, then sketch the curve.

29 Find the equation of each parabola in the form $y = ax^2$ or $y = x^2 + c$.

a



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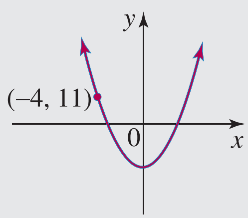


30 A parabola with equation of the form $y = ax^2 + c$ has its vertex at $(0, -7)$ and passes through $(4, 1)$. Find values for a, c and hence find the equation of the parabola.

31 Sketch:

a $y = x^3$ **b** $y = -2x^3$
c $y = x^3 + 4$ **d** $y = 3 - x^3$

32 Draw an accurate graph of the curve $y = \frac{4}{x}$ on a number plane by first completing the following table of values.

x	-4	-2	-1	0	1	2	4
y							

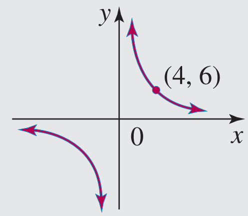
33 Sketch the following curves on separate number planes. Show one point on each curve. Values are not required on the axes.

a $y = \frac{1}{x}$ **b** $xy = -6$

34 The points $(2, t)$ and $(u, 12)$ lie on the curve $xy = -8$. Find values for t and u.

35 Find the equation of each hyperbola.

a



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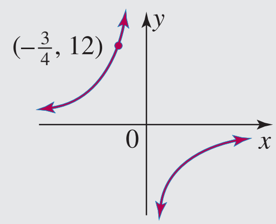
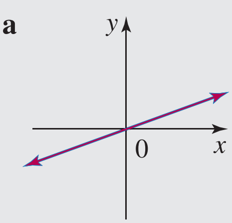


36 Match each equation with one of the graphs below.

- $y = \frac{3}{x}$
- $y = \frac{x}{3}$
- $y = -\frac{x}{3}$
- $y = -\frac{3}{x}$

a



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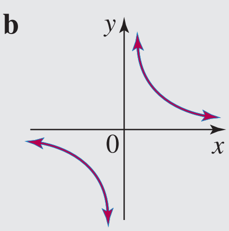
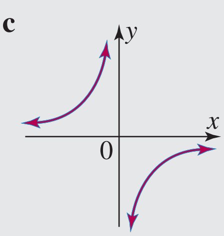
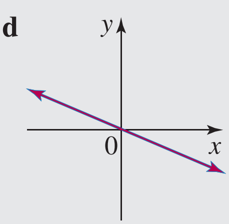


c



d



37 Sketch the following curves on separate number planes. Values are not required on the axes.

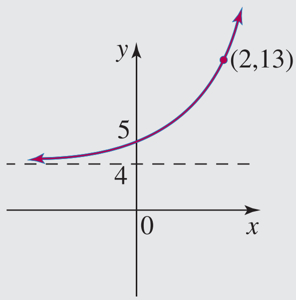
a $y = 3^x$ **b** $y = 3^{-x}$

38 Sketch the following curves showing the asymptote and intercepts with the axes.

a $y = 5^{-x} + 2$ **b** $y = 2^x - 4$

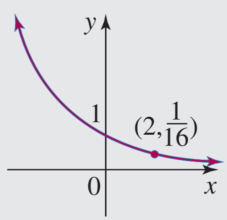
39 Find the equation of each exponential curve, in the given form.

a



$y = a^x + c$

b



$y = a^{-x}$

40 Find the equation of the circle with centre (0, 0) and radius:

a 5 units **b** $3\sqrt{2}$ units

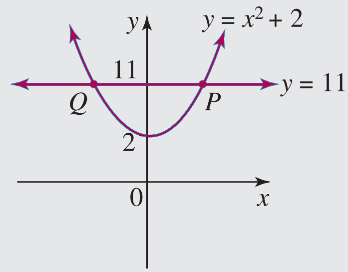
41 Sketch the graph of $x^2 + y^2 = 16$.

42 Shade these regions.

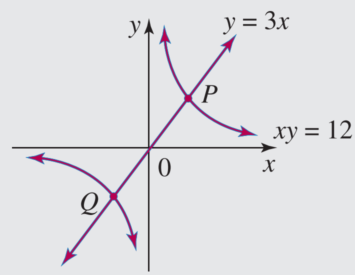
a $x^2 + y^2 \leqslant 9$ **b** $x^2 + y^2 > 49$

43 Find the co-ordinates of P and Q in each of these.

a



b



8

Data analysis and evaluation

This chapter at a glance

Stage 5.1/5.2/5.3

After completing this chapter, you should be able to:

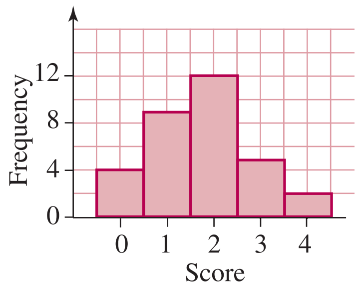
- state that the mean, median and mode are measures of location
- calculate the mean, median and mode given a set of individual scores, frequency distribution table, frequency histogram, frequency polygon, dot plot or stem and leaf plot
- solve problems involving the mean, median and mode
- construct and interpret a cumulative frequency histogram and cumulative frequency polygon (ogive)
- find the mean and modal class given a set of grouped data
- state that the range, inter-quartile range and standard deviation are measures of spread
- find the lower and upper quartiles for a set of scores
- calculate the range, inter-quartile range and standard deviation given a set of individual scores, frequency distribution table, frequency histogram, frequency polygon, dot plot or stem and leaf plot
- construct a box and whisker plot using a five-point summary
- solve problems involving the range, inter-quartile range and standard deviation
- calculate equivalent marks in different subjects using the mean and standard deviation
- determine the percentage of scores that would lie within a given number of standard deviations either side of the mean
- use the mean and standard deviation to compare two sets of data
- choose an appropriate way to display two sets of data in order to make comparisons
- compare the relative merits of the measures of spread
- determine whether a given distribution is positively skewed, negatively skewed or symmetrical.

8.1 Data analysis review

Graphs and tables

Graphs and tables are used to represent both numerical and categorical data in a way that makes the data easier to understand and analyse. A graph must have a title, the axes must be clearly labelled and the scale chosen must be appropriate.

The frequency histogram



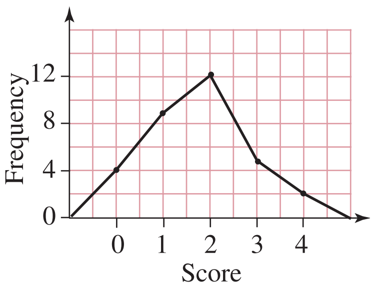
The frequency histogram is a type of column graph.
In a histogram:

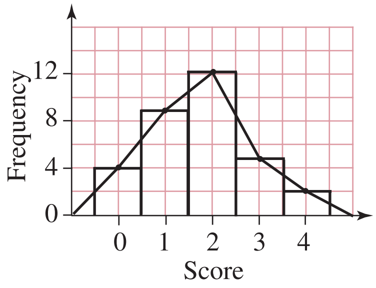
- the scores are placed on the horizontal axis
- the frequencies are shown on the vertical axis
- the columns straddle the scores and are drawn next to each other without a gap
- a space of half the width of one column is left on the horizontal axis before the first column is drawn.

The frequency polygon

The frequency polygon is a type of line graph.
In a polygon:

- the scores are placed on the horizontal axis
- the frequencies are shown on the vertical axis
- the polygon begins and ends on the horizontal axis
- the first score is marked one full unit away from the vertical axis.





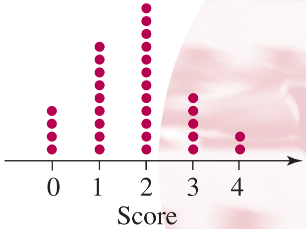
When a polygon and histogram are drawn on the same set of axes, the polygon joins the midpoints of the tops of the columns. The area under the histogram is equal to the area under the polygon.

The dot plot

The dot plot is a simplified version of the histogram.
In the dot plot:

- the scores are placed along a horizontal line
- one dot is placed above the score in a vertical line each time that score occurs.

Clusters or bunches are easily seen, as well as any **outliers**, that is, scores that are a long way from the other scores.



The stem-and-leaf plot

The stem-and-leaf plot is similar to a histogram that has been drawn on its side, except that the rows are made up of digits. In the stem-and-leaf plot:

- the first part of each score is called the **stem**, and is written on the left-hand side of the plot
- the remaining part of the number is called the **leaf**, and is written on the right-hand side of the plot.

Stem	Leaf
2	4 7
3	2 4 5 8
4	0 1 2 4
5	3 6 7

For example, the ordered stem-and-leaf plot above shows the scores 24, 27, 32, 34, 35, 38, 40, 41, 42, 44, 53, 56, 57.

When a large number of scores begins with the same digit(s), the scores can be written in class intervals of 5. For example, this stem-and-leaf plot shows the scores 70, 72, 73, 75, 76, 76, 77, 81, 82, 83, 86, 89.

Stem	Leaf
$7^{(0)}$	0 2 3
$7^{(5)}$	5 6 6 7
$8^{(0)}$	1 2 3
$8^{(5)}$	6 9

Measures of location

Much of our work in data analysis is concerned with finding a score that is in some way typical or representative of all of the scores in a distribution. The **measures of location** or **central tendency** are often used for this purpose. They are the mean, the median and the mode.

The mean

The **mean** is the sum of the scores divided by the number of scores. The symbol for the mean is $\bar{x}$. The term 'average' is most commonly used to refer to the mean.

$$\text{Mean} = \frac{\text{sum of the scores}}{\text{number of scores}}$$

i.e. $\bar{x} = \dfrac{\sum x}{n}$ **where**

- $\bar{x}$ **is the mean**
- $\sum x$ **is the sum of the scores**
- n **is the number of scores.**

The mean of the data in a frequency distribution table is given by:

$\bar{x} = \dfrac{\sum fx}{\sum f}$ **where**

- $\bar{x}$ **is the mean**
- $\sum fx$ **is the sum of the scores**
- $\sum f$ **is the number of scores.**

The median

The **median** is the central value in a distribution, when the scores have been arranged in ascending order. If the number of scores is odd, then the median is the middle score. If the number of scores is even, then the median is the average of the two middle scores.

When a set of n scores has been arranged in ascending order, the median is:

- **the $\left(\frac{n+1}{2}\right)$th score if n is odd**
- **the average of the $\frac{n}{2}$th and $\left(\frac{n}{2}+1\right)$th scores if n is even.**

The mode

The **mode** is the score with the highest frequency. That is, it is the score that occurs more times than any other. If a set of scores has two or more scores each with the highest frequency, then there would be two or more modes. If all the scores have the same frequency, there is no mode.

The mode is the score with the highest frequency.

The range

The **range** is a measure of spread. It is the difference between the highest and lowest scores in a distribution. This and other measures of spread will be discussed in more detail in Exercises 7.2–7.4.

The range is the difference between the highest and lowest scores.

Cumulative frequency

The cumulative frequency column in a frequency distribution table gives a progressive total of the frequencies. A cumulative frequency histogram and a cumulative frequency polygon, or **ogive**, can be drawn by using the cumulative frequencies as the heights of the columns.

To draw an ogive:

- **draw a cumulative frequency histogram with the columns having heights equal to the cumulative frequencies of the scores**
- **join the top right-hand corners of the columns with a line graph, starting with the bottom left-hand corner of the first column.**

NOTE: The ogive finishes at the top of the last column and is not drawn back down to the horizontal axis as for the frequency polygon.

The median can also be determined graphically from the ogive.

To find the median of a distribution by using an ogive:
- **draw a horizontal line from the halfway mark (50th percentile) on the vertical axis to the ogive**
- **draw a vertical line down to the horizontal axis**
- **read off the median.**

NOTE: For discrete data, the median can be read directly from the horizontal axis. However, for continuous data, the median must be estimated from the reading on the horizontal axis.

Grouped data

If the number of individual scores is large, it is not practical to show all of the scores in a frequency distribution table or in a histogram. In these cases we group the data into class intervals such as 1–5, 6–10 etc. When data is grouped in this way, we no longer have access to individual scores.

In a grouped data frequency distribution table, class intervals are used in place of individual scores and the class centres are given. The class centres are the values that lie halfway between the upper and lower limits of each class interval. With single data we use an fx column, where fx is the product of the scores and their frequencies. With grouped data we use an $(f \times cc)$ column, where $f \times cc$ is the product of the class centres and the frequencies.

As the individual scores are not known, it is not possible to calculate exactly the mean, median, mode or range. The mode is replaced with the modal class. If an ogive is used to find the median of a grouped data distribution, then the median that is read off the horizontal axis is also an estimate. The mean can be found by dividing the totals; however, it is only an approximation.

The approximate mean of the data in a grouped data frequency distribution table is given by:

$$\bar{x} = \frac{\sum(f \times cc)}{\sum f}$$

Example 1

The data below shows the number of overtime hours worked by a group of employees during a particular week.

5 8 9 3 8 6 4 9 7

a Find the mean, correct to 1 decimal place.
b Find the median.
c Find the mode.
d Find the range.

Solutions

a $\bar{x} = \frac{\sum x}{n}$

$= \frac{5 + 8 + 9 + 3 + \ldots + 7}{9}$

$= \frac{59}{9}$

$= 6.6$ (to 1 decimal place)

b We first arrange the scores in ascending order.

3 4 5 6 7 8 8 9 9

Median $= \left(\frac{9+1}{2}\right)$th score

$=$ 5th score

$= 7$

c There are two scores, 8 and 9, which occur more than any other score. Therefore 8 and 9 are both modes.

d Range = highest score − lowest score

$= 9 - 3$

$= 6$

Example 2

The number of students in each class in a primary school is shown below.

23	25	23	26	29	24	27	24	26	27
26	27	23	28	27	24	26	25	28	25
27	24	26	25	29	27	23	27	28	24

a Draw a frequency distribution table for the data. Use the following column headings: Number of students (x), tally, frequency (f), cumulative frequency and fx.

b Draw a frequency histogram and polygon for the data.

c Find the mean, correct to 1 decimal place.

d Find the median.

e Find the mode.

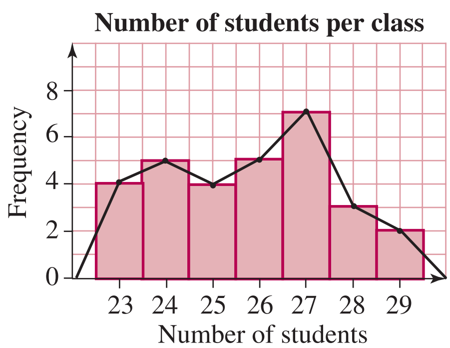
f Find the range.

Solutions

a

Number of students (x)	Tally	Frequency (f)	Cumulative frequency	fx
23	||||	4	4	92
24	~~||||~~	5	9	120
25	||||	4	13	100
26	~~||||~~	5	18	130
27	~~||||~~ ||	7	25	189
28	|||	3	28	84
29	||	2	30	58
		30		773

b



c $\bar{x} = \dfrac{\sum fx}{\sum f}$

$= \dfrac{773}{30}$

$= 25.8$ (to 1 decimal place)

d Median $= \dfrac{\text{15th score + 16th score}}{2}$

$= \dfrac{26 + 26}{2}$

$= 26$

e The mode is the score with the highest frequency. Therefore, the mode is 27.

f Range = highest score − lowest score

= 29 − 23

= 6

Example 3

Draw a cumulative frequency histogram and ogive for the scores in each table. Use the ogives to find the median in each case.

a

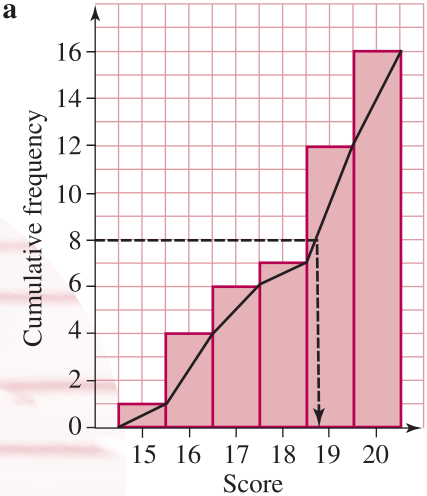
x	15	16	17	18	19	20
f	1	3	2	1	5	4

b

x	20	21	22	23	24	25
f	6	6	2	4	4	6

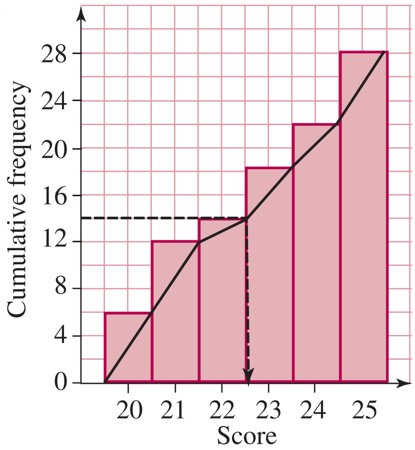
Solutions

a



The median is 19.

b



The median is 22.5.

Example 4

The data below shows the golf scores of a group of 36 golfers in the first round of a golf tournament.

72	76	70	73	70	77	73	69	66	75	78	74
69	73	75	74	72	72	71	68	70	76	80	74
67	70	77	76	72	69	71	72	76	79	67	73

a Draw a grouped data frequency distribution table for these scores. Use the following column headings: Class, class centre (cc), tally, frequency (f), $f \times cc$. Use classes of 66–68, 69–71, etc.

b Find the modal class.

c Use the table to find the approximate mean golf score, correct to the nearest whole number.

Solutions

a

Class	Class centre (cc)	Tally	Frequency (f)	$f \times cc$
66–68	67	||||	4	268
69–71	70	~~||||~~ ||||	9	630
72–74	73	~~||||~~ ~~||||~~ ||	12	876
75–77	76	~~||||~~ |||	8	608
78–80	79	|||	3	237
			36	2619

b The modal class is 72–74, since it is the class with the highest frequency.

c $\bar{x} = \dfrac{\sum(f \times cc)}{\sum f}$

$= \dfrac{2619}{36}$

$= 73$ (to the nearest whole number)

Example 5

A set of 47 scores has a mean of 22. Find the new mean after a score of 37 is added. Answer correct to 1 decimal place.

Solution

$\bar{x} = \dfrac{\sum x}{n}$

$= \dfrac{(47 \times 22) + 37}{47 + 1}$

$= \dfrac{1071}{48}$

$= 22.3$ (to 1 decimal place)

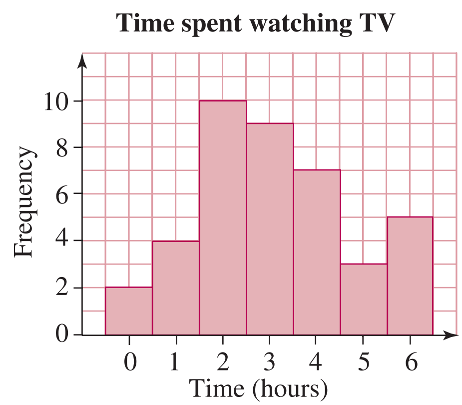
Exercise 8.1

1 The data below shows the ages of the members of a book club.

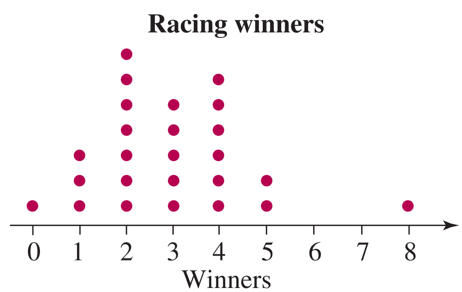
45	47	43	44	49	52	50	48	44	46
45	43	46	44	52	46	48	51	46	50
45	49	48	51	50	50	52	46	45	47

a Organise the data into a frequency distribution table with score (x), tally, frequency (f), cumulative frequency and fx columns.
b Draw a frequency histogram and polygon for the data.
c How many members are 48 years or younger?
d What percentage of the members are 45 years old? Answer correct to 1 decimal place.

2 The histogram shows the number of hours that a group of children spend watching television each day.
a How many children watch television for 4 hours each day?
b How many children watch for 3 hours or less per day?
c How many more children watch television for 2 hours per day than watch for 5 hours?
d How many children were surveyed?
e What percentage of the children watch television for 6 hours per day?



3 Clarence, the racing commentator, gives his predictions for the winners of the Sydney horse races each week. The number of winners that he has predicted each week is shown in the dot plot.
a In how many weeks did Clarence predict exactly 3 winners?
b What was the greatest number of winners that he predicted?
c How many times was Clarence not able to pick any winners?
d Are there any outliers? If so, what are they?
e Are there any clusters? If so, what are they?
f What percentage of the time did Clarence predict at least 4 winners?



4 The stem-and-leaf plot shows the heights (in cm) of people who had been arrested by the police at a protest march.
a How many people were arrested?
b What are the heights of the tallest and shortest protesters?
c What was the most common height?
d How many protesters were taller than 1.8 metres?

Stem	Leaf
16	4 5 7 8
17	0 1 1 2 3 5 6 8
18	1 2 2 3 4 4 4 5 7 9
19	3 4 6 6 7 8
20	0 2

5 Find the mean of these scores, correct to 1 decimal place.

a 8, 7, 21, 13, 28, 4, 25, 19
b 24, 16, 33, 15, 24, 56, 18, 51, 29
c 1.8, 1.2, 5.2, 9.4, 3.9, 7.8, 4.6
d 49, −12, 7, 65, −29, 0, −3, 40, −11

6 Find the median of these scores.

a 5, 6, 8, 11, 13, 14, 19
b 32, 23, 31, 25, 18, 29, 24, 28, 32
c 44, 41, 32, 35, 49, 33, 37, 42
d 77.6, 80.2, 73.2, 85.8, 71.3, 89.4

7 Find the mode of these scores.

a 1, 4, 3, 2, 6, 5, 4
b 9, 10, 12, 7, 9, 8, 11, 10, 10
c 37, 36, 29, 31, 38, 30, 37, 31
d 60.4, 61.8, 64.9, 66.7, 69.8, 70.3, 72.5

8 Find the range of these scores.

a 25, 34, 12, 7, 29, 18, 21
b 24, 28, 42, 9, 37, 16, 22, 28
c 41.2, 52.9, 35.1, 56.4, 69.3, 37.4
d 8, −6, 36, −11, −1, 21, 0, −15

■ Consolidation

9 Find the mean, median, mode and range for each set of scores. Answer correct to 1 decimal place where necessary.

a 2, 4, 5, 7, 10, 12, 16, 16, 27
b 15, 17, 20, 28, 29, 29, 34, 40
c 52, 54, 55, 59, 62, 62, 71
d 31, 31, 32, 32, 32, 35, 38, 39, 41, 46
e 114, 117, 117, 125, 132, 134, 134
f 91, 92, 94, 94, 94, 96, 97, 97, 97, 99
g 7.3, 7.5, 7.6, 7.9, 8.3, 8.3, 8.3, 8.8
h 13.1, 13.5, 13.8, 14.0, 14.1, 14.3, 14.7

10 Find the mean, median, mode and range for each set of scores, correct to 1 decimal place where necessary.

a

x	1	2	3	4	5	6
f	5	2	10	14	17	19

b

x	12	13	14	15	16	17
f	15	13	6	4	7	4

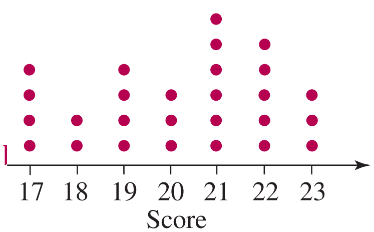
c

x	7	10	15	23	27	31
f	8	9	15	12	13	7

d

x	28	33	38	42	45	50
f	10	14	19	17	15	11

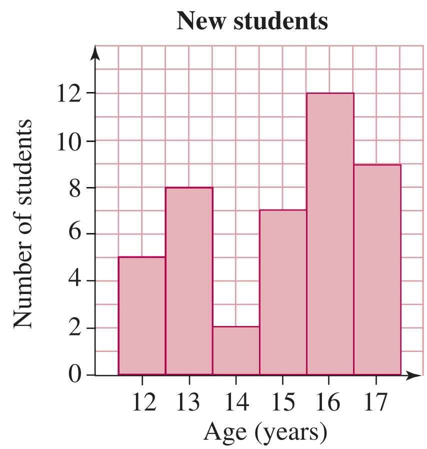
11



a Draw a frequency distribution table for the data in the dot plot. Include score (x), frequency (f), cumulative frequency and fx columns.

b Find the mean, median, mode and range for these scores.

12 This histogram shows the ages of the new students who joined a Maths tutoring centre in 2004.



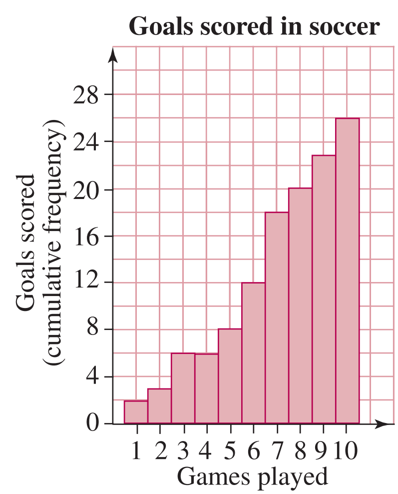
a How many new students joined the centre in 2004?
b How many of these students were 15 years old?
c Which age is the mode?
d Find the range of the ages of these students.
e What is the median age?
f Calculate the average age of the new students, correct to 1 decimal place.

13 The Nevapay Insurance Company paid out the following amounts ($\times$ \$1000) for stolen cars during one month.

25	31	16	9	12	23	34	30	19	22
18	25	20	27	19	14	32	29	27	17
26	24	25	29	36	24	8	10	15	16

a Draw a stem and leaf plot for the data with stems $0^{(5)}$, $1^{(0)}$, $1^{(5)}$, etc.
b How many insurance claims were made during the month?
c What was the value of the most expensive car that was stolen?
d Find the range of the insurance claims.
e Which value is the mode?
f What was the median value of the cars?
g Find the average value of the cars. Answer correct to the nearest \$100.
h What percentage of the cars were above the average value? Answer correct to the nearest whole percentage.

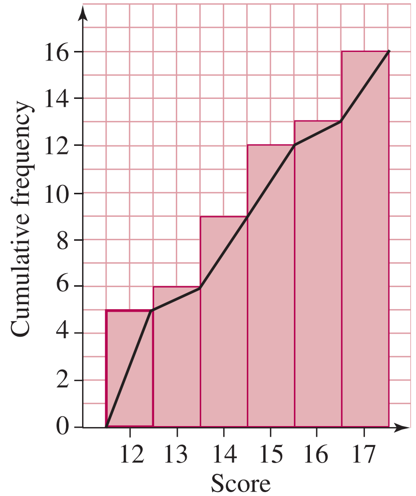
14 The number of goals scored each game through the season by a junior soccer team was recorded in a cumulative frequency histogram.



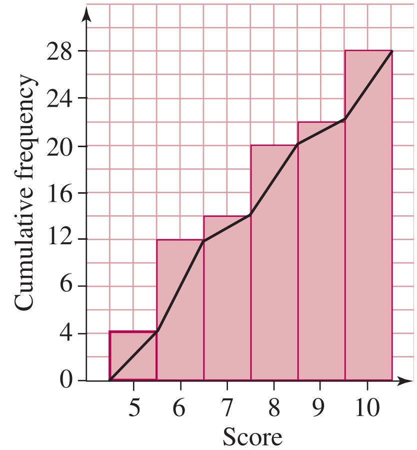
a How many games did the team play?
b How many goals were scored during the season?
c Find the average number of goals scored per game.
d How many goals were scored in the 6th game?
e In which game did the team score the most goals for the season?
f In which game did the team fail to score?

15 Use the ogives below to find the median of each set of scores.

a



b



16 The students at a certain high school wanted to convince their principal that they needed airconditioners in the demountable classrooms. They measured the temperature in a classroom at 1 pm each day during February and recorded the results. The temperatures (in °C) are shown below.

32	31	28	35	34	26	29	33	37	38
30	32	36	31	27	24	26	29	33	35
29	29	25	31	32	35	36	37	33	30

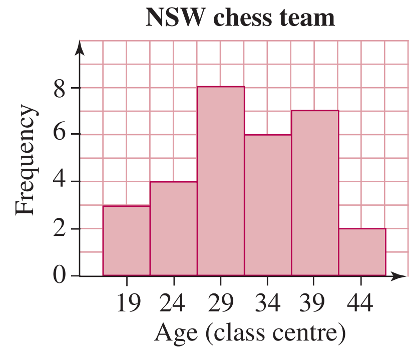
a Copy and complete this grouped data frequency distribution table.

Class	Class centre (cc)	Tally	Frequency (f)	$f \times cc$
24–26				
27–29				
30–32				
33–35				
36–38				

b What is the modal class?
c Use the frequency distribution table to find the average temperature.
d Draw a cumulative frequency histogram and ogive.
e Use the ogive to estimate the median temperature.
f Does the data support the students' claim that the classrooms should be airconditioned?

17 The histogram shows the ages of the team (to the nearest year) chosen to represent NSW at the national chess championships.

a Write down the class intervals.
b How many people were selected in the team?
c What is the modal age class?
d How many members of the team are aged 36 years or under?
e Find the average age of the team. Answer correct to the nearest whole year.



18 What number should replace the box so that the median score is 66?

Stem	Leaf
4	2 5 7
5	1 3 4 5 8
6	0 2 3 □ 8 9
7	4 6 7 8
8	0 1 2 4 5 6

19 **a** A set of 11 scores has a sum of 187. Find the mean of the scores.
b The mean of a set of 24 scores is 9. Find the sum of the scores.
c The mean of a set of scores is 13.5 and their sum is 378. Find the number of scores.

20 **a** The mean of three scores is 16. If two of the scores are 15 and 8, find the third score.
b The mean of four scores is 12.5. If three of the scores are 17, 19 and 6, find the fourth score.

21 A set of scores has a mean of 50. What would be the effect on the mean if the following scores were added to the set?
a 37 **b** 68 **c** 50

22 A set of scores has a mean of 75. What would be the effect on the mean if the following scores were taken out of the set?
a 75 **b** 40 **c** 96

23 A set of 43 scores has a mean of 62. Find, correct to 1 decimal place, the new mean after:
a a score of 25 is added to the set **b** a score of 39 is taken out of the set.

24 After 14 games this season, Juliana's goal average in netball is 4.5 goals per game. How many goals has she scored altogether this season?

25 Joshua has an average of 76% after three tests this semester. What mark does he need in the next test in order to raise his average to 80%?

26 Sourav has an average of 65% on the first three tests of the semester. Find the highest possible mark that he can score this semester if there are two more tests to go and each test contributes equally to his report mark.

Further applications

27 The Intermediate 1 class averaged 75 on the half-yearly exam, while the Intermediate 2 class averaged 62. There are 28 students in Intermediate 1 and 23 students in Intermediate 2. What is the overall average for the two classes on this exam? Answer correct to 1 decimal place.

28 **a** The mean of a set of 12 scores is 27. After a new score is added, the mean rises to 29. Find the new score.

b The mean of a set of 25 scores is 68. After a new score is added, the mean falls to 66.5. Find the new score.

29 **a** The mean of a set of 38 scores is 53. When one of the scores is taken out, the new mean is 54. Find the score that was taken out.

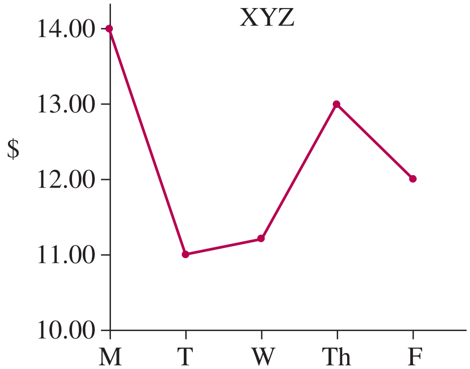
b The mean of a set of 47 scores is 26. When one of the scores is taken out, the new mean is then 24.5. Find the score that was taken out.

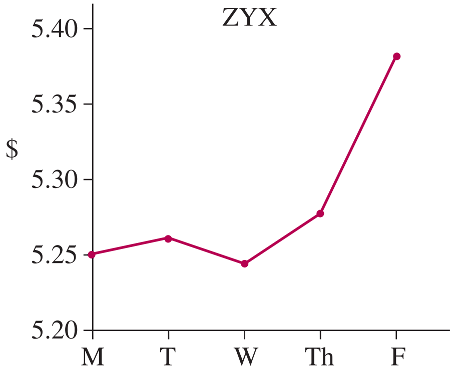
30 The mean of the scores is 7 and the mode is 8. Find the values for x and y.

Score	Frequency
x	6
y	12

TRY THIS Misleading graphs

Newspapers often use graphs, like those above, which show the share prices of two companies XYZ and ZYX over a period of one week. Explain how these graphs could be misleading for some readers. Examine a number of daily newspapers to find any other graphs which could be misleading.





8.2 The interquartile range

The mean, median and mode are measures of location. In particular, the mean and median provide information about the centre of a distribution. **Measures of spread** or **dispersion** tell us to what degree the scores in a distribution are spread out. In this chapter, we will consider the following measures of spread:

- the range
- the interquartile range
- the standard deviation

The range

The **range** is the difference between the highest and lowest scores in a distribution. It is the only measure of spread that we have studied so far. It tells us how far apart the highest and lowest scores are, but nothing about the scores in between. We do not know, for example, whether the scores are widely spread, or whether they are bunched up around the mean. The range can also be greatly affected by any outliers, thus giving a false impression of the spread of the data. For these reasons, the range has only limited use as a measure of spread.

The range is the difference between the highest and lowest scores.

The interquartile range

The **lower** or **first quartile (Q_1)**, is the value up to which one-quarter of the data lies. That is, 25% of the scores in the distribution lie at or below the lower quartile. If the scores are arranged in ascending order, the lower quartile is the central value of those scores that are less than the median. The median is also called the **second quartile (Q_2)**.

The **upper** or **third quartile (Q_3)**, is the score up to which three-quarters of the data lies. That is, 75% of the scores in the distribution lie at or below the upper quartile. If the scores are arranged in ascending order, the upper quartile is the central value of those scores that are greater than the median.

The **interquartile range** is the difference between the upper and lower quartiles. It is the range of the middle 50% of the scores in the distribution. Unlike the range, the interquartile range is not affected by outliers. Hence, it is a better measure of spread than the range.

For example, if the scores 8, 14, 16, 7, 15, 10, 18, 12 are arranged in ascending order, we can find the interquartile range as follows.

$\boxed{7, 8}$ $\boxed{10, 12}$ $\boxed{14, 15}$ $\boxed{16, 18}$

lower quartile

median

upper quartile

lower quartile (Q_1)
$= \dfrac{8 + 10}{2}$
$= 9$

median (Q_2)
$= \dfrac{12 + 14}{2}$
$= 13$

upper quartile (Q_3)
$= \dfrac{15 + 16}{2}$
$= 15.5$

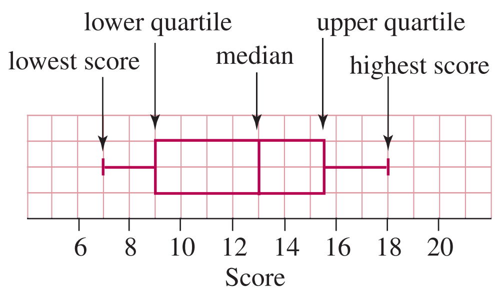
interquartile range $= Q_3 - Q_1$
$= 15.5 - 9$
$= 6.5$

The interquartile range is the difference between the upper and lower quartiles.

Box-and-whisker plots

The extreme values (i.e. the highest and lowest scores), the median and the upper and lower quartiles are often referred to as a **five-point summary**. This data can be shown graphically in the form of a **box-and-whisker plot**, or more simply a **boxplot**. The quartiles and the median are shown by the box, while the extreme values are indicated by the 'whiskers'.

For example, the five-point summary above could be represented as follows.



Example 1

Consider the scores 21, 24, 26, 29, 30, 31, 35.

a Find the range.

b Find the interquartile range.

c Draw a box-and-whisker plot for the data.

Solutions

a Range = highest score − lowest score
= 35 − 21
= 14

b 21 24 26 29 30 31 35

24 → lower quartile; 29 → median; 31 → upper quartile

Interquartile range = upper quartile − lower quartile
= 31 − 24
= 7

c

Example 2

A dart was thrown 20 times at a dartboard. The results were recorded in the form of a stem-and-leaf plot.

a Find the range.

b Find the median.

c Find the interquartile range.

d Draw a box-and-whisker plot for the data.

Stem	Leaf
3	2 5
4	0 1 5 7
5	3 4 6 7 9
6	1 2 3 5 9
7	3 4 4 6

Solutions

a Range = highest score − lowest score
= 76 − 32
= 44

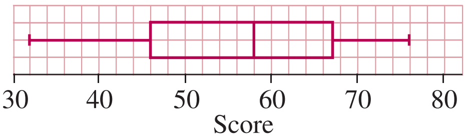
b Median = $\frac{\text{10th score + 11th score}}{2}$
$= \frac{57 + 59}{2}$
= 58

c Lower quartile = $\frac{\text{5th score + 6th score}}{2}$
$= \frac{45 + 47}{2}$
= 46

Upper quartile = $\frac{\text{15th score + 16th score}}{2}$
$= \frac{65 + 69}{2}$
= 67

Interquartile range = upper quartile − lower quartile
= 67 − 46
= 21

d



Exercise 8.2

1 Find the range of each set of scores.

a 17, 9, 54, 21, 75, 38, 41

b 15, −4, 23, 7, 19, −10, 0, −14

2 The range of a set of scores is 25.

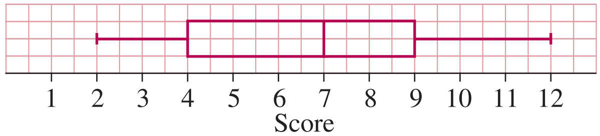
a Find the highest score if the lowest score is 17.

b Find the lowest score if the highest score is 90.

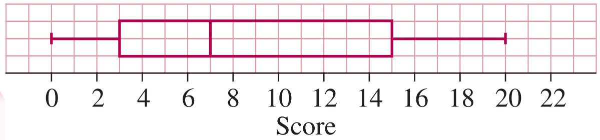
3 For each of these box-and-whisker plots, write down:

i the lowest and highest scores
ii the range
iii the median
iv the lower and upper quartiles
v the interquartile range

a



b



4 For each set of scores below, find:

i the range
ii the median
iii the lower and upper quartiles
iv the inter-quartile range

a 9, 10, 12, 13, 14, 14, 17

b 17, 18, 20, 25, 27, 28, 30, 33, 37

c 32, 33, 38, 40, 45,
49, 50, 52, 57, 59

d 27, 28, 28, 33, 34, 38,
41, 43, 46, 52, 55, 56

e 51, 52, 54, 55, 58, 60, 63, 66, 67, 67, 68, 70, 79, 80, 83

f 34, 35, 37, 37, 38, 38, 39, 40, 42, 45, 46, 51, 52, 54, 57, 63, 64, 67, 67, 70

5 Find the median and the lower and upper quartiles, then draw a box-and-whisker plot for each set of scores.

a 13, 14, 15, 17, 18, 20, 22

b 5, 6, 7, 7, 8, 10, 13, 15, 20

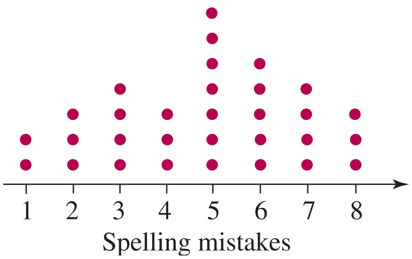
c 25, 28, 28, 29, 30, 31, 32, 34, 35, 36, 38, 39

d 5.3, 5.4, 5.7, 5.7, 5.8, 6.0, 6.2, 6.5, 6.9, 7.1, 7.1, 7.2, 7.4, 7.5, 7.7

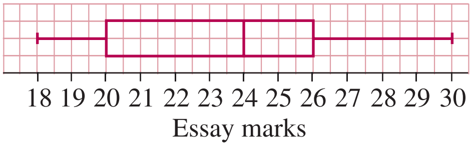
6 This dot plot shows the number of mistakes made by a class of Year 5 students on their weekly spelling test.

a Find the median and the interquartile range.

b Draw a box-and-whisker plot for the data.



7 This box-and-whisker plot shows the essay marks obtained by a large group of Year 10 English students.



a Find the percentage of marks that lie between:

i 18 and 20 ii 24 and 26 iii 20 and 26 iv 20 and 30

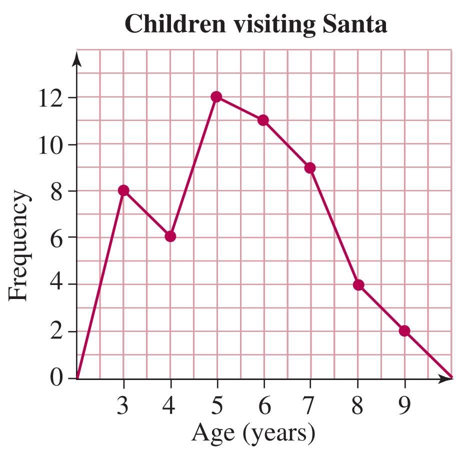
b If there are 28 students in the class, find the number of students who scored a mark between 26 and 30.

8 In a set of scores, the lower quartile is 24 and the interquartile range is 60. What is the upper quartile?

9 The number of pages written each fortnight by an author is shown in this stem-and-leaf plot. Find the interquartile range for the data.

Stem	Leaf
1	7 9
2	2 3 4 5 7
3	0 1 1 2 4 5 6 8
4	2 2 3 5 6 9
5	3 8 9

10



The polygon shows the ages of children who visited Santa in a department store leading up to Christmas.

a Find the extremes and the range of the children's ages.

b What is the median age?

c Find the lower and upper quartiles.

d What is the interquartile range?

11 Find the interquartile range of the scores in each frequency table.

a

x	8	9	10	11	12	13	14	15	16	17
f	3	5	1	6	0	9	4	11	12	5

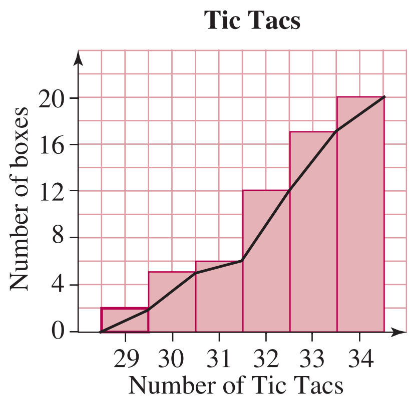
b

x	21	22	23	24	25	26	27	28	29	30
f	4	11	3	6	13	1	2	4	8	9

12 In a set of scores, the lower quartile is 10 and the interquartile range is 6. Adam stated that the median must be 13. Is he correct? Explain.

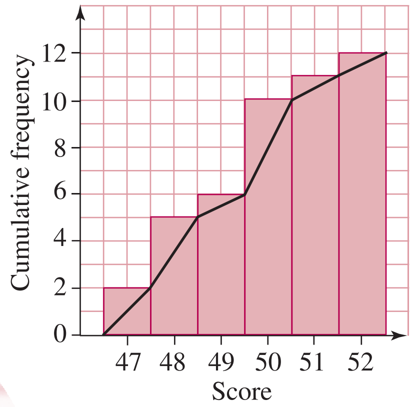
13 This cumulative frequency histogram and ogive show the number of Tic Tacs per box in a sample.

a What is the median?
b Find the lower and upper quartiles.
c Find the interquartile range.
d Draw a box-and-whisker plot for the data.

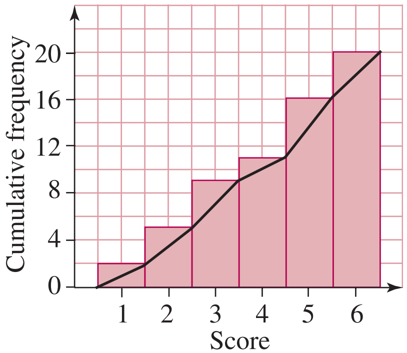


14 Find the median and interquartile range for these sets of discrete scores.

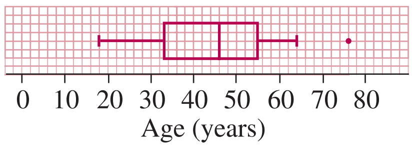
a



b



15 This box-and-whisker plot shows the ages of the employees of a small business.



a What is the median age?
b One person's age has not been included in the box-and-whisker plot. What is the outlier age?
c Between which two ages are:
i the youngest 25% of employees? **ii** the middle 50% of employees?

16 Write true (T) or false (F) for each of the following statements.
- **a** The median always lies halfway between the quartiles.
- **b** Half of the scores lie between the quartiles.
- **c** The interquartile range must be half the size of the range because it contains the middle 50% of scores.
- **d** The lower quartile could be equal to the lowest score.
- **e** The interquartile range is a measure of location.

17 The table below shows the weekly pay of the employees at Wally's Warehouse.

Weekly pay ($)	200–299	300–399	400–499	500–599	600–699
Number of employees	7	11	9	7	2

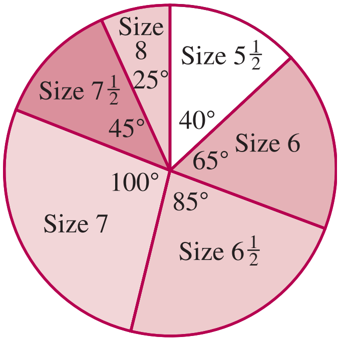
- **a** Construct a cumulative frequency histogram and ogive from the data in the table. Use values of 200, 300, 400 etc. on the horizontal axis.
- **b** Use the ogive to estimate the median weekly pay.
- **c** Use the ogive to estimate the interquartile range, correct to the nearest $10.

■ Further applications

18 Write down a set of nine scores in which the median is 15 and the interquartile range is 20.

19 This sector graph shows the shoe sizes of the 144 students in Year 8 at a certain high school. Find the interquartile range of the students' shoe sizes.

Shoe size of Year 8 students



TRY THIS **Cliometrics**

What is cliometrics?

After whom is it named?

8.3 The standard deviation

The **standard deviation** (σ_n) is a measure of the spread of the scores about the mean. The larger the standard deviation, the more widely spread are the scores from the mean. The smaller the standard deviation, the more bunched up are the scores about the mean.

The standard deviation is defined by the following formula.

$$\sigma_n = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

where
- **σ_n = the standard deviation**
- **x = each individual score**
- **$\bar{x}$ = the mean**
- **n = the number of scores.**

To find the standard deviation for a set of scores:
- **calculate the mean**
- **subtract the mean from each individual score**
- **square each of these differences**
- **find the average of these squares**
- **take the square root.**

The formula above is only used initially to provide an understanding of what the standard deviation is and how it is calculated. After the introduction, however, the standard deviation will be found by using a calculator. Students are not expected to memorise the above formula.

To find the standard deviation of a set of scores by using a calculator:
- **set the calculator to the statistics mode**
- **enter each score followed by the $\boxed{\text{M+}}$ key**
- **press the $\boxed{\sigma_n}$ key.**

Example 1

EG +S

Find the standard deviation of the scores below, without the use of a calculator.

17 18 20 21 24 25 29

Solution

i $\bar{x} = \frac{\sum x}{n}$

$= 22$

ii

Score (x)	$x - \bar{x}$	$(x - \bar{x})^2$
17	–5	25
18	–4	16
20	–2	4
21	–1	1
24	2	4
25	3	9
29	7	49

iii $\sigma_n = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$

$= \sqrt{\frac{108}{7}}$

$= 3.9$ (to 1 decimal place)

Example 2

The table below shows the scores obtained by 10 boys and 10 girls on a spelling test.

Boys	57	62	84	85	88	66	59	77	78	93
Girls	68	74	51	82	85	75	70	73	79	69

a Which group of students achieved the higher mean?

b Which group of students achieved the more consistent results?

Solutions

When the scores are entered into a calculator, the following results are obtained:

	$\bar{x}$	σ_n
Boys	74.9	12.3
Girls	72.6	8.9

a The boys had a slightly higher mean than the girls.

b The standard deviation for the girls' results is lower than that of the boys. Therefore, the girls achieved the more consistent results.

Exercise 8.3

1 For each of the following:

i find the mean

ii copy and complete the table

iii find the standard deviation, correct to 1 decimal place, using $\sigma_n = \sqrt{\dfrac{\sum(x-\bar{x})^2}{n}}$

a

Score (x)	$x - \bar{x}$	$(x - \bar{x})^2$
13		
17		
19		
24		
27		
	Total =	

b

Score (x)	$x - \bar{x}$	$(x - \bar{x})^2$
52		
55		
57		
61		
64		
65		
	Total =	

2 Use a calculator to find the mean and standard deviation of each set of scores. Answer correct to 1 decimal place.

a 4 9 3 8 6 5 7 5

b 21 19 12 8 23 6 30

c 23 19 25 11 21
13 16 34 18 22

d 31 37 32 39 46 28
37 32 30 36 28 35

e 14.6 11.2 19.4 20.5 9.8
17.6 14.1 37.2 18.6 6.9

f 5.3 6.4 9.1 8.3 4.2 0.9
0.6 5.2 7.4 1.6 2.8 4.5

Consolidation

3 For each set of scores, find correct to 1 decimal place where necessary:
i the mean **ii** the median **iii** the mode **iv** the standard deviation

a

Score	Freq.
9	6
10	2
11	5
12	8
13	13
14	1

b

Score	Freq.
95	1
96	15
97	10
98	9
99	4
100	12

c

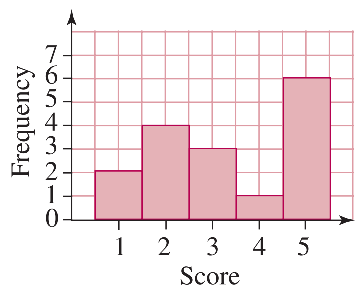
Score	Freq.
22	4
25	7
29	3
30	5
35	13
43	6

4 Find the range and standard deviation of each set of scores, correct to 2 decimal places where necessary.

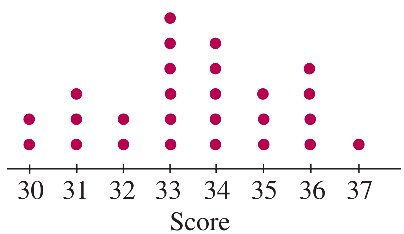
a

Stem	Leaf
$5^{(0)}$	1 2 4
$5^{(5)}$	5 6 6 7
$6^{(0)}$	0 1 3 3 4
$6^{(5)}$	6 7 7 8
$7^{(0)}$	2 2 3

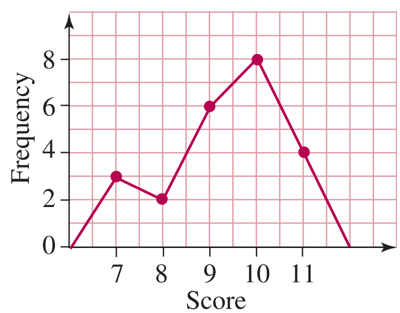
b



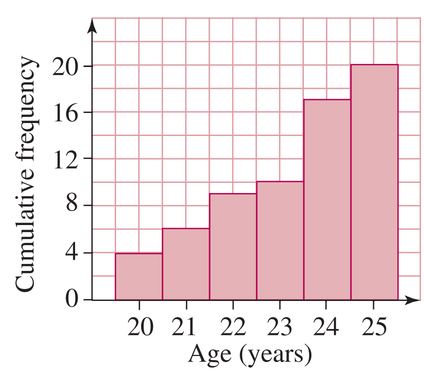
c



d



5 This cumulative frequency histogram shows the ages of people at a party.
a Find the range of the ages.
b Which age is the mode?
c Find the median age.
d Find the mean age.
e Find the standard deviation of the ages, correct to 1 decimal place.



6 Find the mean and standard deviation for each set of grouped data. Answer correct to 1 decimal place.

a

Class	Class centre	Frequency
1–10	5.5	6
11–20	15.5	9
21–30	25.5	10
31–40	35.5	17
41–50	45.5	3
51–60	55.5	4

b

Class	Class centre	Frequency
1–7	4	5
8–14	11	4
15–21	18	8
22–28	25	14
29–35	32	11
36–42	39	10

7 The maximum daily temperatures (°C) recorded in a city over a period of 20 days are given below.

25 24 27 28 26 30 28 25 29 30
31 34 32 30 26 23 24 26 30 27

a Find the range of the temperatures.
b Find the interquartile range of the temperatures.
c Find the standard deviation, correct to 1 decimal place.
d What would be the two most appropriate measures of spread for these temperatures? Why?

8 The number of 'hits' per day on a certain website over a period of 14 days is shown below.

75 62 81 59 83 9 58 61 57 54 66 72 145 60

a Find the range.
b Find the interquartile range.
c Find the standard deviation.
d Which would be the most appropriate measure of spread for this data? Why?

9 Van's class had a mean of 60 and standard deviation of 12 on the half-yearly Maths exam. Lucy's class had a mean of 60 and standard deviation of 8 on the same exam. In which class were the marks more tightly bunched around the mean? Explain.

10 Write true (T) or false (F) for the following statements.
a Standard deviation is a measure of location.
b The standard deviation can be greater than the range.
c The greater the spread of the scores, the greater is the standard deviation.

11 If all of the scores in a distribution are equal, what would be the standard deviation? Why?

12 **a** Find the standard deviation of the following scores, correct to 1 decimal place.

5, 7, 9, 10, 12, 15, 16, 16, 19

b If an outlier of 40 was included, what effect do you think this would have on the standard deviation? Use a calculator to verify your answer.

13 Two basketball teams scored the following number of points in their first 10 games of the season.

Wombats:	94	96	103	85	99	101	105	97	90	82
Ferrets:	96	97	99	101	93	90	87	94	95	103

a Find the mean and standard deviation for each team, correct to 1 decimal place.
b Which team has been the more consistent point scorer? Explain.

14 The two History classes in Year 11 were set a research assignment on the causes of World War II. The students' results are given below, as percentages.

Mrs Bremner's class:	84	79	89	92	53	90	74	68	85	65	77	83
Mrs Spencer's class:	69	65	78	93	84	73	79	82	66	92	81	72

a Find the mean and standard deviation for the students' results in each class, correct to 1 decimal place.
b Which class performed better on the assignment? Why?
c Which class had the more consistent results? Why?

Further applications

15 Consider the scores 1, 2, 3, 4, 5, 6, 7.
a Find the mean and standard deviation of the scores.
b If each score was increased by 5, what effect would this have on the mean and standard deviation? Why? Verify your answers using a calculator.
c If each score was multiplied by 5, what effect would this have on the mean and standard deviation? Why? Verify your answers using a calculator.

16 Consider the scores 2, 4, 6, 8, 10, 12, 14. If each score was multiplied by −1, what would be the effect on the standard deviation? Why? Verify your answer using a calculator.

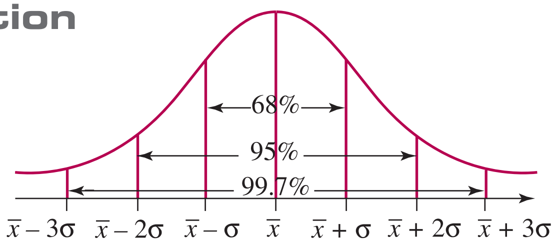
8.4 Applications of the standard deviation

The standard deviation can be used to compare the relative performances of two or more people on different tasks such as tests or exams. This is done by finding the number of standard deviations that each result is above or below the mean.

	Mark	$\bar{x}$	σ_n
Economics	72	60	6
Geography	75	65	10

For example, this table shows Shyla's exam results in Economics and Geography, compared to those of the other students in each course. At first glance it would appear that she performed better in Geography because her mark in that subject (75) was higher than her mark in Economics (72). However, this is not the case. In Geography, Shyla's mark was only 1 standard deviation above the mean (i.e. 75 = 65 + 10), while her mark in Economics was 2 standard deviations above the mean (i.e. 72 = 60 + 6 + 6). Therefore, relative to the other students in each course, Shyla actually performed better in Economics than she did in Geography, even though her Economics mark was slightly lower.

The normal distribution



If there is a large number of scores, then the shape of the distribution will approximate a bell-shaped curve called the normal curve. The normal curve is smooth and is symmetrical about the mean. Approximately two-thirds of the scores are clustered about the mean, with the curve tapering away at each end. The greater the number of scores in the distribution, the closer the curve will be to the normal curve. Mathematicians have shown that the following results are true for large populations.

For a normal distribution it has been shown that:
- **approximately 68% of the scores lie within 1 standard deviation either side of the mean**
- **approximately 95% of the scores lie within 2 standard deviations either side of the mean**
- **approximately 99.7% of the scores lie within 3 standard deviations either side of the mean.**

NOTE: In a perfect normal curve, the mean, the median and the mode are all equal.

If we knew the height of every adult in Australia and calculated the mean and standard deviation, we would find that approximately two-thirds of the adult population would have a height that lies within the range $\bar{x} - \sigma < \bar{x} < \bar{x} + \sigma$. This could also apply to other variables such as weight, quality of sight and hearing, shoe size and IQ.

We say that if a score is chosen at random from a distribution, then that score will:
- ***probably* lie within one standard deviation of the mean**
- ***very probably* lie within two standard deviations of the mean**
- ***almost certainly* lie within three standard deviations of the mean.**

The standard deviation and the normal curve are used widely by:
- manufacturers to reject sub-quality products
- universities to determine pass marks in courses
- insurance companies to set insurance premiums.

Example 1

EG +S

The mean of a set of scores is 70 and the standard deviation is 12. Find:

a $\bar{x} + \sigma$ **b** $\bar{x} - 2\sigma$ **c** $\bar{x} + \frac{1}{2}\sigma$ **d** $\bar{x} - 1\frac{3}{4}\sigma$

Solutions

a $\bar{x} + \sigma$
$= 70 + 12$
$= 82$

b $\bar{x} - 2\sigma$
$= 70 - (2 \times 12)$
$= 70 - 24$
$= 46$

c $\bar{x} + \frac{1}{2}\sigma$
$= 70 + (\frac{1}{2} \times 12)$
$= 70 + 6$
$= 76$

d $\bar{x} - 1\frac{3}{4}\sigma$
$= 70 - (1\frac{3}{4} \times 12)$
$= 70 - 21$
$= 49$

Example 2

The table shows Martin's results on consecutive Computing tests.

	Mark	$\bar{x}$	σ_n
Test 1	69	60	12
Test 2	82	72	10

a On which test did Martin perform better?

b What mark in test 2 corresponds to each of the following marks in test 1?

i 48 **ii** 78

Solutions

a Test 1

$$\frac{x-\bar{x}}{\sigma} = \frac{69-60}{12} = \frac{9}{12} = \frac{3}{4}$$

Test 2

$$\frac{x-\bar{x}}{\sigma} = \frac{82-72}{10} = \frac{10}{10} = 1$$

On test 1, Martin's mark was $\frac{3}{4}$ of a standard deviation above the mean. On test 2, Martin's mark was 1 standard deviation above the mean. Therefore, he performed better on test 2.

b i Test 1

$$\frac{x-\bar{x}}{\sigma} = \frac{48-60}{12} = \frac{-12}{12} = -1$$

$\therefore$ A mark of 48 is 1 standard deviation below the mean.

Test 2: The equivalent mark on test 2 must also be 1 standard deviation below the mean.

$\therefore$ Test 2 mark $= 72 - 10 = 62$

b ii Test 1

$$\frac{x-\bar{x}}{\sigma} = \frac{78-60}{12} = \frac{18}{12} = 1\frac{1}{2}$$

$\therefore$ A mark of 78 is $1\frac{1}{2}$ standard deviations above the mean.

Test 2: The equivalent mark on test 2 must also be $1\frac{1}{2}$ standard deviations above the mean.

$\therefore$ Test 2 mark $= 72 + (1\frac{1}{2} \times 10) = 72 + 15 = 87$

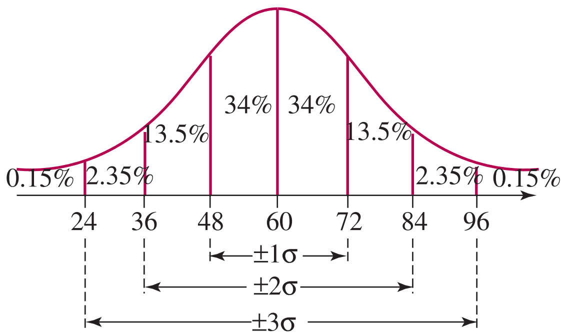
Example 3

For a set of marks (out of 100), the mean is 60 and the standard deviation is 12. If the distribution of the marks is normal, find the percentage of students that should have scored a mark between:

a 48 and 72 **b** 36 and 84 **c** 24 and 96
d 60 and 72 **e** 36 and 48 **f** 84 and 100

Solutions

The graph shows the percentage of marks that should lie in each section for a normal distribution.



a 68% **b** 95%
c 99.7% **d** 34%
e 13.5% **f** 2.5%

Exercise 8.4

1 The end of year Science exam in Year 10 had a mean of 60 and a standard deviation of 8. Write down the mark that is:

a 1 standard deviation above the mean **b** 1 standard deviation below the mean
c 2 standard deviations above the mean **d** 2 standard deviations below the mean
e 3 standard deviations above the mean **f** 3 standard deviations below the mean

2 The table shows the mean and standard deviation for the mid-year results in English and Drama. Find the mark in each subject that is:

	$\bar{x}$	σ_n
English	60	14
Drama	53	8

a 1 standard deviation below the mean
b 2 standard deviations above the mean
c 0.5 standard deviations above the mean
d 1.5 standard deviations below the mean

3 The mean ($\bar{x}$) of a set of scores is 57 and the standard deviation (σ_n) is 12. Find:

a $\bar{x} - \sigma$ **b** $\bar{x} + \sigma$ **c** $\bar{x} - 2\sigma$ **d** $\bar{x} + 2\sigma$

e $\bar{x} + \frac{1}{2}\sigma$ **f** $\bar{x} - \frac{1}{4}\sigma$ **g** $\bar{x} + \frac{2}{3}\sigma$ **h** $\bar{x} - \frac{5}{6}\sigma$

i $\bar{x} - 1\frac{1}{2}\sigma$ **j** $\bar{x} + 1\frac{1}{6}\sigma$ **k** $\bar{x} - 2\frac{3}{4}\sigma$ **l** $\bar{x} + 2\frac{2}{3}\sigma$

Consolidation

4 Aimee scored 61 on her music practical. Her mark was two standard deviations below the class mean of 75. What was the standard deviation?

5 Tim scored 59 on a Year 7 numeracy test, which was 1 standard deviation below the mean. On the same test, Francis scored 77, which was 2 standard deviations above the mean.
- **a** What was the standard deviation?
- **b** Find the mean.
- **c** Keith's mark was 1.5 standard deviations above the mean. What was his mark?
- **d** Helen's mark was 2.5 standard deviations below the mean. What was her mark?

6 How many standard deviations above or below the mean is a mark of 60, given that:
a $\bar{x} = 55$ and $\sigma_n = 5$ **b** $\bar{x} = 76$ and $\sigma_n = 8$ **c** $\bar{x} = 75$ and $\sigma_n = 10$

7

	Mark	$\bar{x}$	σ_n
Maths	77	70	7
Science	75	60	10

Jason compared his end of year results in Maths and Science. In which subject did he perform better? Explain.

8 In each of the following, use the mean and standard deviation to determine which is the better result.

a

	Mark	$\bar{x}$	σ_n
Test 1	86	80	6
Test 2	83	75	4

b

	Mark	$\bar{x}$	σ_n
Test 1	70	63	7
Test 2	76	81	5

c

	Mark	$\bar{x}$	σ_n
Test 1	55	60	10
Test 2	55	63	8

d

	Mark	$\bar{x}$	σ_n
Test 1	63	67	8
Test 2	62	59	6

e

	Mark	$\bar{x}$	σ_n
Test 1	70	88	12
Test 2	69	83	7

f

	Mark	$\bar{x}$	σ_n
Test 1	81	72	12
Test 2	81	72	9

g

	Mark	$\bar{x}$	σ_n
Test 1	69	59	15
Test 2	64	57	14

h

	Mark	$\bar{x}$	σ_n
Test 1	50	56	8
Test 2	54	65	11

9 The mean and standard deviation are shown for the exam results in Art and Music.

	$\bar{x}$	σ_n
Art	70	8
Music	68	12

- **a** Find the mark in Music that corresponds to a mark in Art of:
  i 78 **ii** 62 **iii** 86 **iv** 54
- **b** Find the mark in Art that corresponds to a mark in Music of:
  i 74 **ii** 62 **iii** 86 **iv** 53

10 A university lecturer decides that the pass mark in his course will be two standard deviations below the mean. Antonella's final mark after two assignments and an exam is 41. If the mean is 65 and the standard deviation is 11.3, determine whether Antonella will pass the course.

11 Each potato in a new batch was washed and weighed as part of the quality control process. The mean weight of the potatoes was found to be 65 grams and the standard deviation was 7. Assuming that the weights of the potatoes are normally distributed, between what weights should we expect to find:

a 68% of the potatoes? **b** 95% of the potatoes? **c** 99.7% of the potatoes?

12 The raw marks obtained by the Year 12 students at a certain high school had a mean of 70 and a standard deviation of 8. If the distribution of the marks is approximately normal, find the percentage of students that should have scored a mark between:

a 62 and 78 **b** 54 and 86 **c** 70 and 78
d 54 and 62 **e** 62 and 86 **f** 78 and 100

13 Four hundred people were weighed as part of a free community health check. The mean weight was 76 kg and the standard deviation was 11 kg. How many people would be expected to have a weight:

a between 65 kg and 87 kg? **b** between 54 kg and 98 kg?
c between 76 kg and 87 kg? **d** between 54 kg and 87 kg?
e that is more than 87 kg? **f** that is less than 54 kg?

14 The average weekly wage at Custom Computers is \$570, with a standard deviation of \$115. One employee is chosen at random. Within what limits would his weekly wage **very probably** lie if the weekly wages are normally distributed?

15 A car manufacturer states that its new model, the Galaxy, has an average petrol consumption rate of 12 km per litre, with a standard deviation of 600 metres per litre. Within what limits will the car's rate of petrol consumption **almost certainly** lie if the consumption rates are normally distributed?

16 Ivan scored 48 on a test in which the year average was 63, the standard deviation was 7.5 and the marks were normally distributed.

a What percentage of students in the year had a lower mark than Ivan?
b Is it likely that any student scored a mark of 93 or more on this test? Why?

■ Further applications

17 The mean of a set of scores is 50 and the standard deviation is 10. Would the mean and standard deviation increase, decrease or stay the same if the following scores were added to the set?

a 80 **b** 20 **c** 50

18 The mean of a set of scores is 60 and the standard deviation is 12. Would the mean and standard deviation increase, decrease or stay the same if the following scores were taken out of the set?

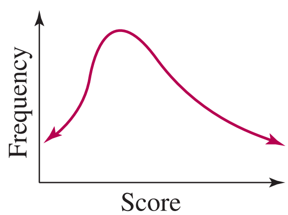
a 90 **b** 30 **c** 60

TRY THIS Two standard deviations

There are two standard deviation keys on your calculator — σ_n and σ_{n-1}.
What is σ_{n-1} and when it is used?

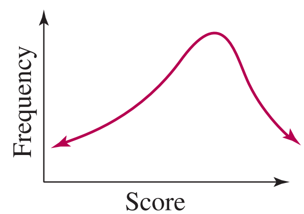
8.5 The shape of a distribution

A distribution is **symmetrical** if the scores are spread evenly about the mean. If a distribution is not symmetrical, then it is said to be **skewed**.



If most of the scores in a distribution are relatively low, then the distribution is **positively skewed**. The longer tail of the frequency curve will lie to the right.

If most of the scores in a distribution are relatively high, then the distribution is **negatively skewed**. The longer tail of the frequency curve will lie to the left.

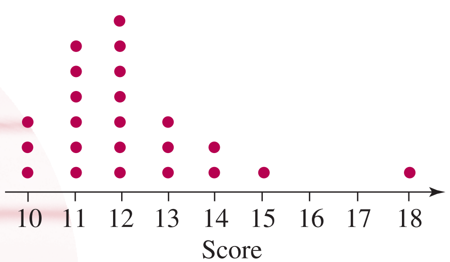


A distribution is:
- **symmetrical if the scores are spread evenly about the mean**
- **positively skewed if most of the scores are low**
- **negatively skewed if most of the scores are high.**

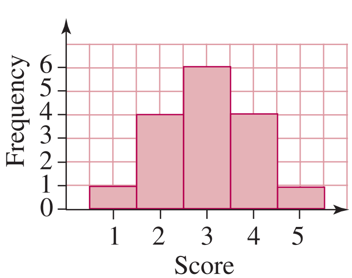
Example 1

Describe the shape of each distribution. Are there any outliers?

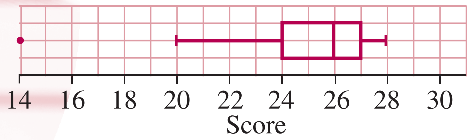
a



b



c



Solutions

a The scores are mostly low, therefore the distribution is positively skewed; 18 is an outlier.

b The scores are evenly spread about the mean, therefore the distribution is symmetrical. There are no outliers.

c The scores are mostly high, therefore the distribution is negatively skewed; 14 is an outlier.

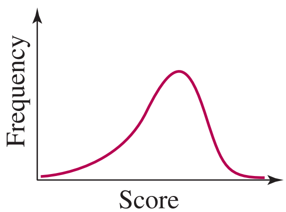
Exercise 8.5

1 Draw a curve which represents a statistical display that is:

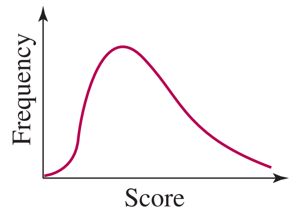
a symmetrical **b** positively skewed **c** negatively skewed

2 State whether each display is either symmetrical, positively skewed or negatively skewed.

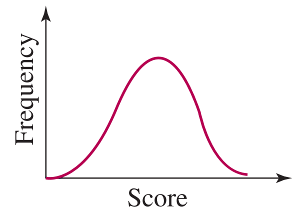
a



b



c

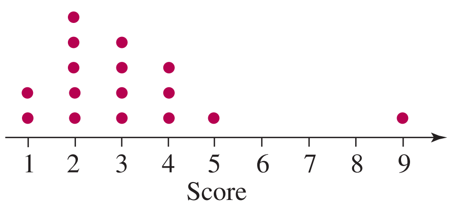


3 For each of the graphs below:

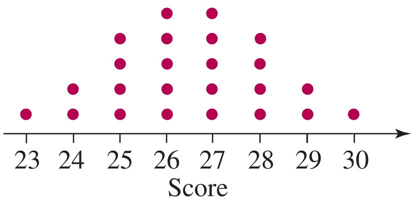
i describe the shape of the distribution

ii state whether there are any outliers

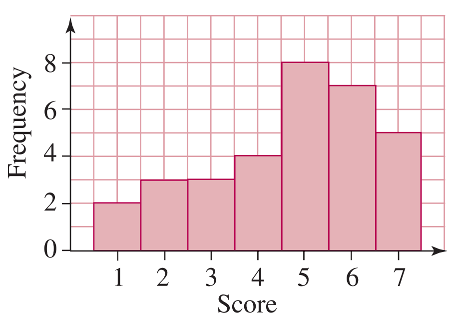
a



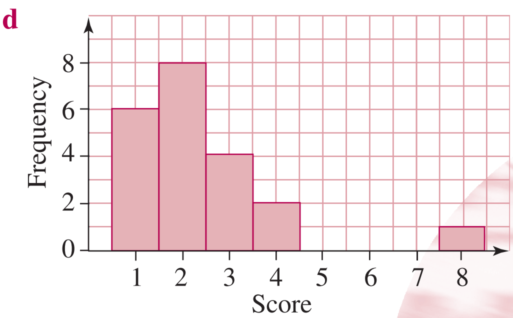
b



c



d



Consolidation

4 Mrs Spiteri gave her class a test that consisted of 20 questions worth 1 mark each. After marking the test, she announced that the mode was 7 and the median was 9. If a column graph was drawn to represent these results, would it most likely be symmetrical, positively skewed or negatively skewed?

5 **a** Draw a dot plot for these scores.
b How many peaks are there?
c Which score is the mode?

51	47	53	54	50	51
53	48	47	53	51	49
47	54	53	48	52	50
49	47	54	51	51	48
53	51	50	47	53	51

6 State whether each distribution is symmetrical, positively skewed or negatively skewed.

a

Stem	Leaf
1	2 5
2	0 1 4
3	3 4 7
4	1 3 6 7 8 9
5	0 2 4 5 6 6 7 8
6	4 5 5 6 9 9

b

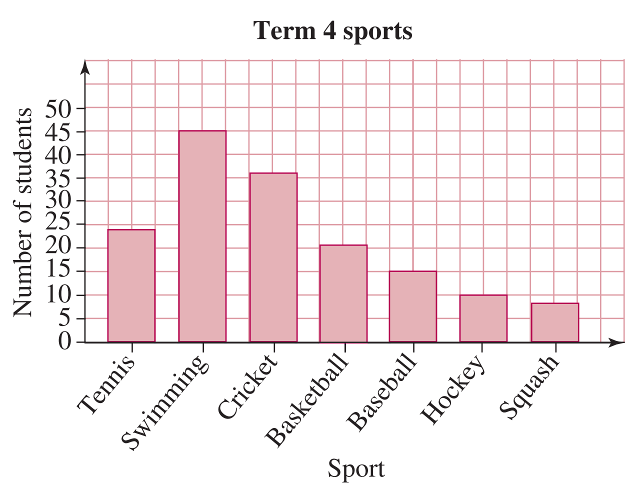
Stem	Leaf
10	4 5
11	1 2 4 6
12	0 5 6 6 7 9
13	3 3 8 9
14	2 6 7

7 A police radar unit recorded drivers travelling at the following speeds in a 60 km/h zone.

52	57	68	56	62	54	62	58	50	57
65	54	61	55	58	60	74	59	51	56
52	71	60	88	67	57	53	55	61	53

a Draw a dot plot for the data by grouping the speeds into classes 50–54, 55–59, 60–64, etc.
b Could any of the recorded speeds be considered outliers?
c Describe the shape of the distribution.
d A 4 km/h tolerance is allowed, so the camera is set to take a photograph of any vehicle which is travelling at 65 km/h or above. How many of these drivers will be booked for speeding?

8 This column graph shows the number of Year 10 students who want to play each of the sports that are available in term 4. Morgan stated that this distribution is positively skewed. Is she correct?

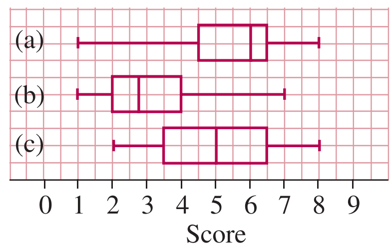


9 Peter is a 16-year-old cricketer who made the following scores last season.

24	17	34	6	45	27	20
39	39	4	52	24	40	15
28	41	13	8	54	22	35
3	21	13	27	12	61	16

a Draw a stem-and-leaf plot to represent this data.
b Are there any outliers?
c Describe the shape of the distribution.
d Would you say that Peter has had a good season? Why?

10 Describe the shape of the distribution represented by each of these box-and-whisker plots.



11 The data below shows the number of Year 10 students who were absent each day during term 3.

2	5	8	10	11	9	7	10	8	11	6	7
12	9	10	11	3	11	11	9	4	5	10	12
1	10	9	11	8	3	6	10	10	12	11	7
11	10	4	9	5	12	6	0	2	10	12	12

a Copy and complete this frequency table for the given data.

Absentees	0	1	2	3	4	5	6	7	8	9	10	11	12
Frequency													

b Construct a five-point summary for the data.
c Draw a box-and-whisker plot.
d Describe the shape of the distribution.

12 The maximum price of petrol at a certain petrol station was recorded each day in cents per litre, correct to the nearest cent.

84	86	92	90	91	96	93	84	78	77
81	82	90	92	97	86	85	83	82	87
91	92	84	84	83	85	94	95	99	96

a Show these petrol prices in this stem-and-leaf plot.
b Describe the shape of the distribution.
c Are there any outliers?

Stem	**Leaf**
$7^{(5)}$	
$8^{(0)}$	
$8^{(5)}$	
$9^{(0)}$	
$9^{(5)}$	

Further applications

13 Karl Pearson, a famous statistician, derived the following formula to measure the degree of skew, or skewness, in a distribution.

$$\text{Skewness} = \frac{\text{mean} - \text{mode}}{\text{standard deviation}}$$

A distribution is positively skewed if the skewness is positive and is negatively skewed if the skewness is negative.

a Would a distribution be positively skewed or negatively skewed if:

i the mean is greater than the mode? **ii** the mean is less than the mode?

b What would be the skewness of a distribution if the mean was equal to the mode?

14 Calculate the skewness of the following sets of scores.

a

x	13	14	15	16	17	18
f	10	17	12	6	4	1

b

x	70	71	72	73	74	75
f	2	5	8	15	19	16

TRY THIS Correlation

1 Survey everyone in your class to find the average shoe size of your fellow students.

2 Survey everyone in your class to find the length of their arms and their height. Is there a relationship between the length of a person's arms and their height?

8.6 Comparing two data sets

A number of statistical graphs and tables can be used to compare two sets of data. These include back-to-back column graphs, double bar graphs, area charts and line graphs. The back-to-back stem-and-leaf plot is also useful for comparing two data sets. It has an advantage over graphs in that it shows all of the scores for both data sets. Box-and-whisker plots give an instant picture of the spread of the scores in each distribution.

The mean, median and mode can be found to give a typical or representative score for each distribution. The range, interquartile range and standard deviation can be used to compare the spread of the scores in each distribution.

Together, graphs, tables, measures of location and measures of spread enable us to compare two or more sets of data. We can then make judgements and predictions based on this information.

Example 1

The systolic blood pressure of some students was measured after a 100 m sprint.

Heat 1:	105	105	110	110	115	115	120	120	120	125	130
Heat 2:	100	105	105	110	110	110	120	120	125	125	130

a For the students in each heat, calculate correct to 1 decimal place:
i the mean **ii** the median **iii** the mode
iv the range **v** the interquartile range

b Draw a box-and-whisker plot for each set of data.

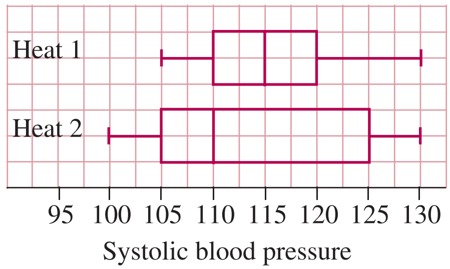
c Comment on the blood pressure of the two groups of students.

Solutions

a

		Heat 1	Heat 2
i	mean	115.9	114.5
ii	median	115	110
iii	mode	120	110
iv	range	25	30
v	interquartile range	10	20

b



c The students in heat 1 generally have higher pulse rates. The pulse rates of the students in heat 2 have a greater spread.

Example 2

The labels on soft drink cans state that they contain 375 mL. Samples of 30 cans produced by two different companies were chosen at random. The volume of drink actually contained in each can was measured. The results, in millilitres, are shown below.

Company A					*Company B*				
375	360	380	377	375	375	368	376	375	368
374	373	371	370	372	374	379	378	370	377
381	385	380	378	382	375	380	368	372	381
362	380	369	377	378	375	376	369	377	370
384	375	377	374	375	371	370	369	380	375
387	379	369	364	370	374	380	378	373	381

a Draw a back-to-back stem-and-leaf plot for the data set.

b Find the mean, median, range and standard deviation for each distribution.

c What percentage of cans from each company contained less than 375 mL of drink?

d Which company provides the more consistent volume of drink in its cans?

Solutions

a

Company A	Stem	Company B
4 2 0	$36^{(0)}$	
9 9	$36^{(5)}$	8 8 8 9 9
4 4 3 2 1 0 0	$37^{(0)}$	0 0 0 1 2 3 4 4
9 8 8 7 7 7 5 5 5 5	$37^{(5)}$	5 5 5 5 5 6 6 7 7 8 8 9
4 2 1 0 0 0	$38^{(0)}$	0 0 0 1 1
7 5	$38^{(5)}$	

b

	Company A	Company B
mean	375.1	374.5
median	375	375
range	27	13
standard deviation	6.3	4.1

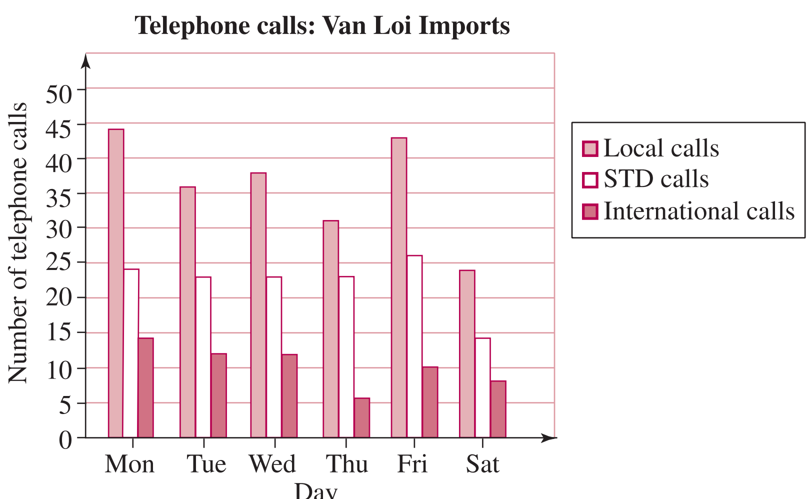
c Company A: $\frac{12}{30} \times \frac{100}{1}\%$
$= 40\%$

Company B: $\frac{13}{30} \times \frac{100}{1}\%$
$= 43.3\%$ (to 1 decimal place)

d The range and standard deviation are lower for company B.
Therefore, company B provides the more consistent volume of drink in its cans.

Exercise 8.6

1 A business kept a record of the number of telephone calls made by its employees each day for a week.



a Find the number of local calls made on:
  i Tuesday
  ii Friday

b How many more STD calls were made on Friday than on Saturday?

c On which day were the most international calls made?

d What was the total number of calls made on Wednesday?

e On which day were the greatest number of calls made?

f What percentage of the calls made on Thursday were STD calls?

2 A climatic graph is actually a combination of two graphs. The column graph shows rainfall (in mm) and is measured on the left-hand scale. The line graph shows temperature (in °C) and is measured on the right-hand scale.

a Name the month that has:

i the highest average maximum temperature

ii the lowest average maximum temperature

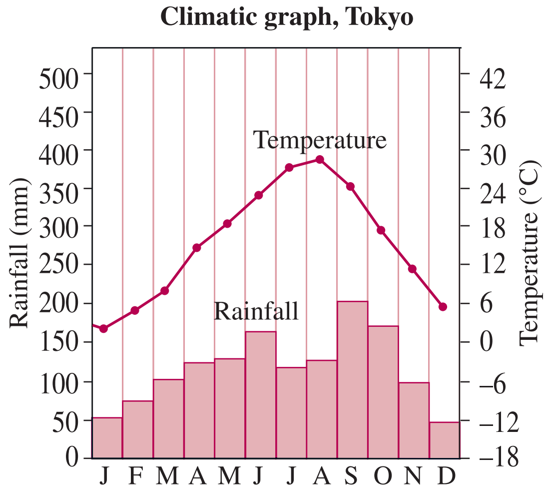
iii the highest average rainfall

iv the lowest average rainfall

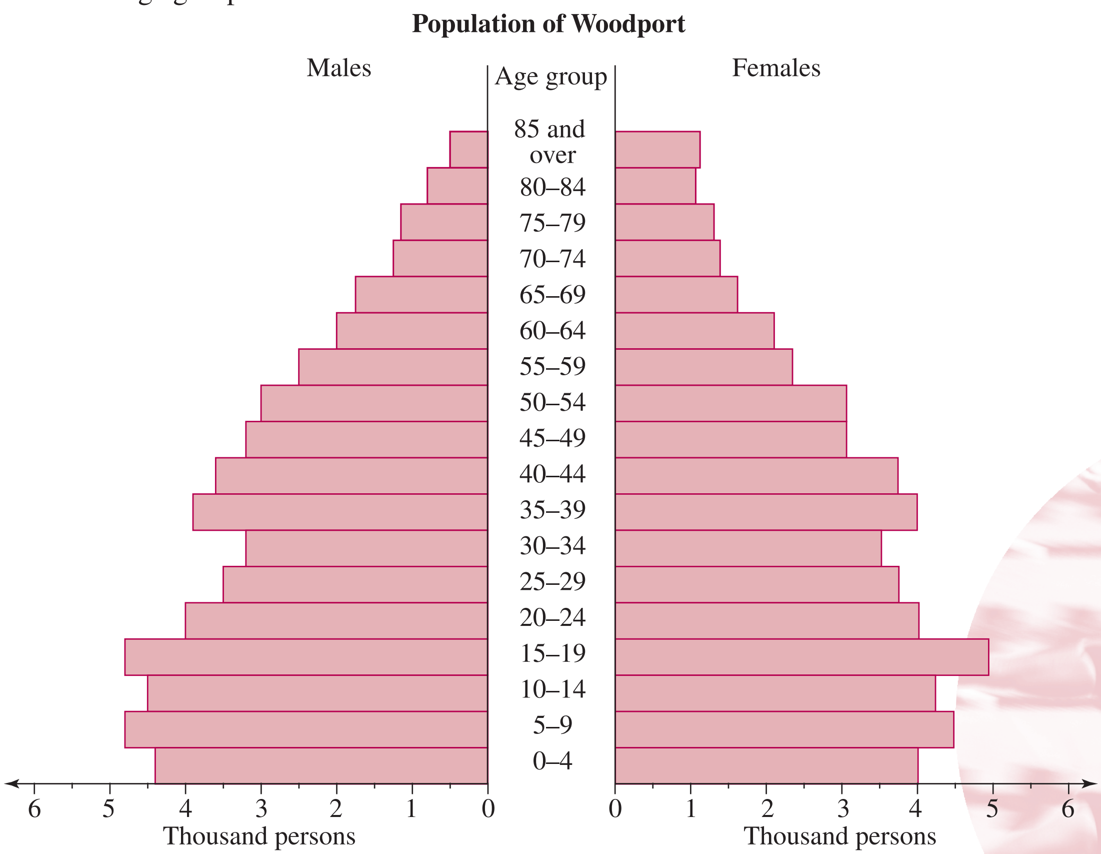
b Which three months have the most rain?

c Which months have an average maximum temperature less than 12 °C?

d Which months have an average rainfall of less than 100 mm?



3 This graph shows the population profile of Woodport. The population of the town has been divided into age groups of 5 years. All persons aged 85 years and over have been counted in one age group.



a Find approximately how many male persons in Woodport are aged:
i 20–24 **ii** 10–14 **iii** 65–69
b Find approximately how many female persons in Woodport are aged:
i 45–49 **ii** 5–9 **iii** 25–29
c In Woodport, are there more males or females in the age group:
i 15–19? **ii** 45–49?
d Find approximately how many children in Woodport are:
i less than 5 years old **ii** less than 10 years old
e Find approximately how many people in Woodport are aged 85 and over.

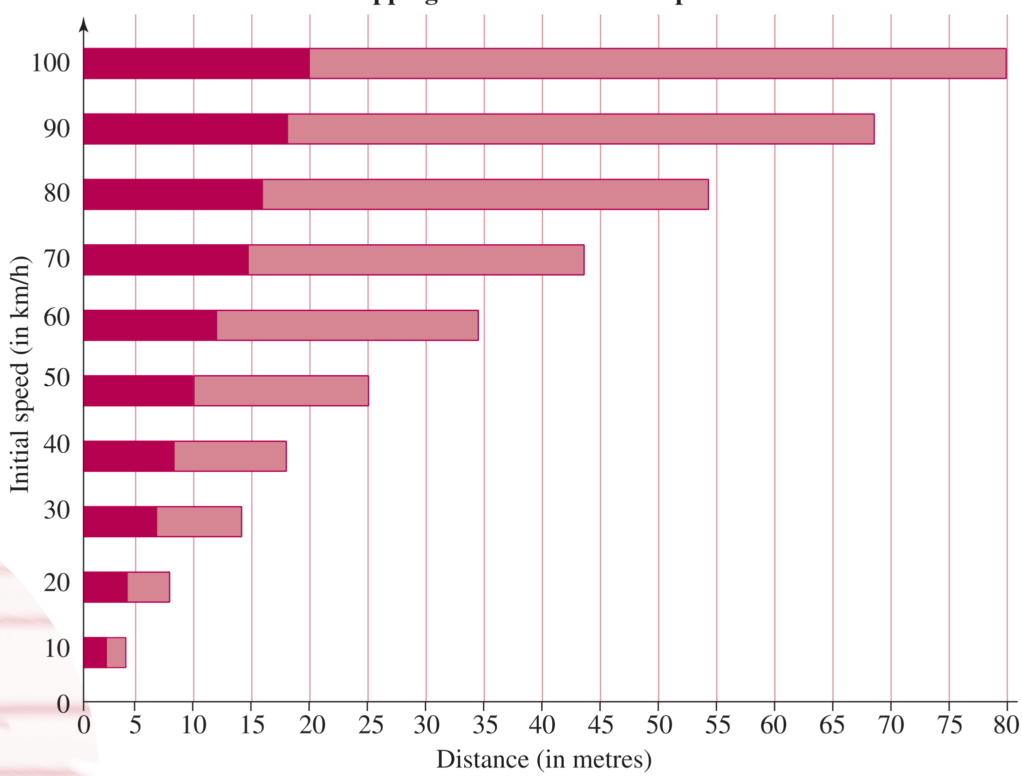
Consolidation

4 The graph shows the stopping distance for a car being driven at various speeds. Each bar is divided into two parts: the reaction distance and the braking distance.

The reaction distance is the distance travelled by the car while the driver's brain responds to the signal to apply the brakes.

The braking distance is the distance travelled from the time the brakes are applied to the time the car comes to a stop.

Stopping distance at various speeds



a How far does the car travel until the brakes are applied at 50 km/h?
b How far does the car travel after the brakes are applied at 50 km/h?
c Find the total stopping distance when the car is travelling at:
i 50 km/h **ii** 80 km/h **iii** 100 km/h

d How much greater is the reaction distance when the car is travelling at 80 km/h than when it is travelling at 60 km/h?

e How much greater is the total stopping distance when the car is travelling at 100 km/h than when it is travelling at 60 km/h?

5 The mean monthly temperatures for Alice Springs and Darwin are shown in the table below.

	J	F	M	A	M	J	J	A	S	O	N	D
Alice Springs	36.0	34.8	32.4	27.9	22.8	19.8	19.3	22.3	26.5	30.6	33.4	35.3
Darwin	31.9	30.9	31.4	32.3	31.3	29.9	29.8	30.8	32.2	32.9	33.3	32.6

a For each city, calculate the mean, median, range and interquartile range.

b The residents of Alice Springs claim that Darwin is a much hotter city than their own. Is this statement correct? Explain.

6 These box-and-whisker plots summarise Simone's progress in Mathematics from years 7 to 10. In which year was:

a the highest mark scored?

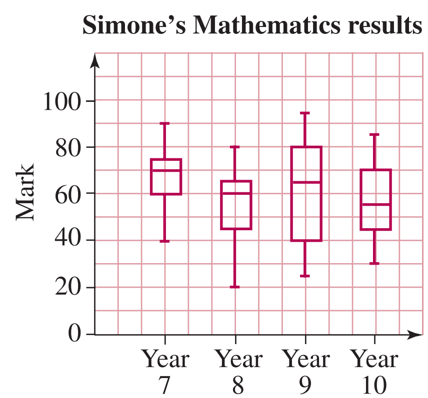
b the lowest mark scored?

c the median mark greatest?

d the lower quartile 55?

e the range of marks greatest?

f the interquartile range 20?



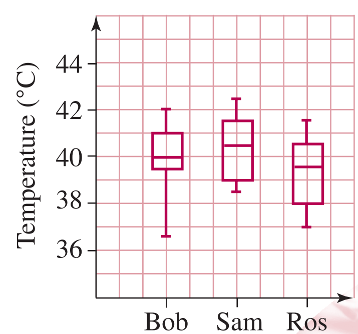
7 Three patients in hospital had their temperatures taken every 4 hours over a period of 3 days. Which patient had:

a the highest temperature?

b the lowest temperature?

c the lowest median temperature?

d the greatest temperature range?

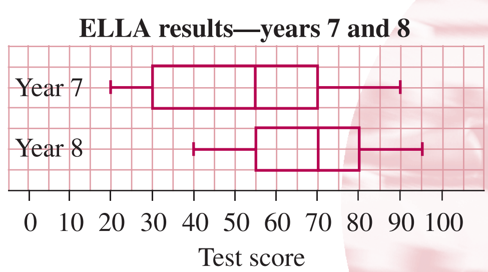


8 These box-and-whisker plots show the results obtained by the same group of students on the ELLA tests, when they were in years 7 and 8.

a Construct a five-point summary for each data set.

b Calculate the interquartile range for each data set.

c Comment on the change that can be seen in the students' results.



9 The scores obtained by 24 players in the first two rounds of a golf tournament are given below.

First round

73	75	79	69	70	72
78	84	68	75	78	80
72	72	74	77	70	76
83	67	72	75	81	74

Second round

70	76	71	72	65	73
75	79	73	74	67	70
71	67	77	85	82	79
76	71	67	74	78	75

a Organise the data into a back-to-back stem-and-leaf plot.

First round		Second round
	$6^{(5)}$	
	$7^{(0)}$	
	$7^{(5)}$	
	$8^{(0)}$	
	$8^{(5)}$	

b Find the mean, median and mode for each round of scores.

c Which measure of location is the most appropriate to compare the scores in each round? Why?

d In which round did the players play better, on average? (Lower scores are better in golf.)

10 The table below shows the number of days on which it rained in two towns, over a 12 month period.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Castledon	11	3	7	6	7	7	5	6	8	6	3	5
Renville	3	1	4	8	10	11	10	5	9	7	7	5

a Draw a box-and-whisker plot for each data set, on the same scale.

b Find the standard deviation for each data set, correct to 1 decimal place.

c The residents of Renville claim that their town has many more rainy days than Castledon. Does the data support this argument?

11 A football coach analysed the team's defensive performance in the previous season by considering the number of points that were scored against them in each game.

First half		Second half
8 8 8 6 6 4	0	4 8
8 5 4 2 2 1 0	1	0 6 6 8 9
6 6 5 4 2	2	2 4 6 7 8 8
2 0	3	4 4 5 6 8
	4	2 6

a How many games were played during the season?

b Find, correct to 2 decimal places, the mean and standard deviation for:

i the first half **ii** the second half.

c Find the standard deviation of each data set, correct to 1 decimal place.

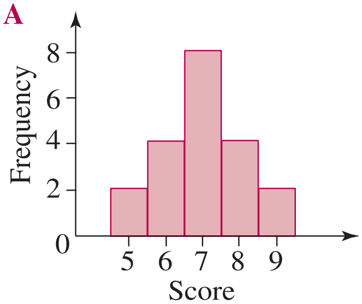
d Draw box-and-whisker plots for both data sets, on the same scale.

e What conclusions can you draw about the team's defensive performance throughout the season?

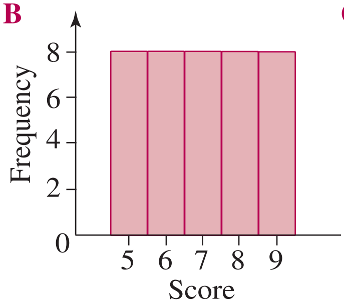
Further applications

12 Consider the following histograms.

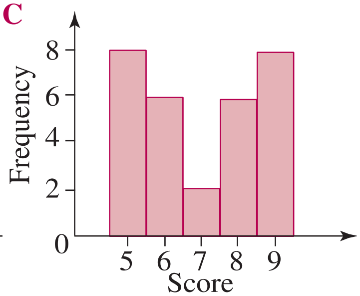
A



B



C



a Comment on the means of these distributions, without the use of a calculator.

b Without using a calculator, determine which distribution has:

  i the highest standard deviation **ii** the lowest standard deviation

c Find the mean and standard deviation of each distribution by using a calculator and verify your answers to parts a and b.

Focus on working mathematically

FOCUS ON WORKING MATHEMATICALLY
FOCUS ON WORKING MATHEMATICALLY

The imprisonment of Indigenous people

Introduction

The 1992 National Prison Census showed that 14% of the total prison population of 15 559 were Indigenous people. The figures showed that Aboriginal people were in prison at 13 times the rate of non-Aboriginal people. Since that time the rate of imprisonment of Aboriginal people has increased. Despite a Royal Commission into deaths in custody, public apologies to the stolen generation, and millions of dollars spent on health and education, the situation of Australia's Indigenous people is still considered a national disgrace. The riots in February 2004 in Redfern, following the death of an Aboriginal teenager, were a symptom of the underlying anger and despair felt by Aboriginal people.

In this exercise we will see how a mathematical analysis can help clarify the extent of the issue.

Learning activities

1 How many people in Australia identify as Aboriginal? What is this as a proportion of the entire population? Use the census data below to estimate the answer as a %.

Census data for the number of Aboriginal people in Australia 1996 and 2001

Census year	Number of Aboriginal people	Total population
1996	352 763	17 892 423
2001	410 003	18 972 350

2 The table below shows the number of Indigenous prisoners and the imprisonment rates per 100 000 Indigenous people aged 18 years and over from 1992 to 1998. Which is the most useful for comparison, the number of prisoners or the rate of imprisonment? Choose a graph to represent the data and draw it. What statistics might be useful? What conclusions can you make?

Number of Indigenous prisoners and imprisonment rates per 100 000 Indigenous people aged 18 years and over 1992–1998

Year	Number of prisoners	Rate per 100 000
1992	2223	1112
1994	2800	1340
1996	3273	1517
1998	3750	1557

3 In June 2003 the total number of people in prison was 23 555. Of these, 4818 were Indigenous people. What percentage is this? What is the increase since 1992 (see introduction)? Given that the percentage of Indigenous people in the Australian community was roughly 2% at the time, what conclusion can you draw?

4 The table on the next page shows the number of times an Indigenous young person was more likely to be held in juvenile custody than a non-Indigenous young person in 1993. What statistics could you use and what graph would you draw to describe the data? What do you conclude? How do you account for such large interstate differences?

Over-representation of Indigenous 10–17 year-olds in juvenile custody 1993

State	Times more likely
NSW	20.8
Vic	37.2
Qld	26.5
WA	48.4
SA	23.6
NT	4.0
All states	24.2

Challenge activities

Activity 1

The table below shows the age distribution of the Indigenous and non-Indigenous prison population on prison census night 30th June 1998. The data for each age group is shown as a percentage.

Age composition as a % of Indigenous and non-Indigenous prisoners in Australia 1998

Age group	Midpoint	Indigenous %	Non-indigenous %
17–19*	18	9	5
20–24	22	28	20
25–29		26	22
30–34		18	16
35–39		11	13
40–44		5	9
45–49		2	6
50–54		1	5
55–59		0	2
60–64		0	1
≥ 65#	65	0	1
Total		100%	100%

* The lowest age groups were collapsed to 17–19 with a class width of 3.
# This is an open interval with no midpoint. For calculations assume it is 65.

1 Copy the table and complete the column showing the midpoint of each class.

2 Look down the columns to compare the two groups of prisoners. What do you immediately notice?

3 Write down the modal class for each group of prisoners.

4 Find the age group in which the median for each group of prisoners would lie.

5 Find the range of the ages for each group of prisoners. Assume the oldest non-Indigenous person is 65.

6 Use the class centres to estimate the mean age of each group of prisoners.

7 What would be the best way to display the data on the same graph? What graph would you draw? If you do *not* have a graphics calculator or access to a spreadsheet you can draw it now in your workbook. Otherwise go to the next question.

8 Type the data into a graphics calculator or spreadsheet. Compare the data by drawing a double histogram. What do you notice?

9 Now draw a box-and-whisker plot for each group and compare them. What do you notice?

10 Compare the age distributions of Indigenous and non-Indigenous prisoners and make a conclusion. How do you account for the differences?

Calculator instructions

If you have a **CASIO CFX-9850G** or **GB Plus**, enter **STAT** mode. Our objective is to compare the data for each group by drawing a box-and-whisker plot for each one and then view them simultaneously.

1 Enter the midpoints into list 1, the percentages for the Indigenous group into list 2 and the percentages for the non-Indigenous group into list 3.

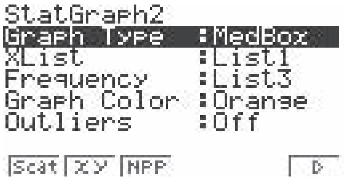
2 Select **GRPH(F1)** followed by **SET(F6)** to determine the type of graph. To set the first graph **Statgraph1**, scroll down to Graph Type to highlight it. Now press F6 followed by F2 to select **MedBox** (a Median Box Plot).

3 Scroll down to **XList** and check that it is **List 1**.
If not press F1 to select it.

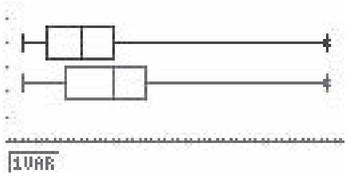
4 Scroll down to **Frequency** and select **List 2 (F3)**.

5 Scroll down to **Graph Colour**. Select blue, orange or green.

6 Return to the top and press F2 to select **Statgraph2**. Scroll down and set the Graph Type to **MedBox**, Xlist to **List 1** and frequency to **List 3 (F4)**. Select a different Graph Colour for the display.



7 Press **EXIT** and then the F4 **(SEL)** key to draw both graphs at the same time. Turn StatGraph1 and StatGraph2 on, using F1. Press F6 DRAW.



8 Trace the Graphs **(SHIFT F1)** to record and compare the 5 statistics MinX, Q1, Med (Median), Q3 and MaxX.

Activity 2

The table below compares the number of indigenous and non-indigenous prisoners who had completed secondary school before entering prison. The ratios in the far right column are calculated from the base rate for non-indigenous prisoners who completed secondary school. Since this is 17, the ratio for non-indigenous prisoners who did not complete secondary school is 176/17 = 10 (nearest whole number). This means that those non-indigenous people who did not complete secondary school are 10 times more likely to be in prison than those who did complete secondary school. Other ratios are calculated in the same way.

Educational Attainment of Prisoners, by Aboriginality, Australia, 30 June 1992

	Prisoners at 30.6.92	Prison rate per 100 000 persons aged 15+	Relative over-representation ratios
Indigenous people			
Completed school	88	164	10
Not completed	1394	2217	130
Non-indigenous people			
Completed school	918	17	1
Not completed	5229	176	10

Sources: Walker & Salloom 1993; ABS 1992.
Note: Table excludes NSW as no data were available on educational status of prisoners.

Complete the following statements:

1 Indigenous prisoners who did not complete secondary school are … times more likely to be in prison than those non-indigenous prisoners who completed school.

2 Indigenous prisoners who completed secondary school are … times more likely to be in prison than those non-indigenous prisoners who completed secondary school.

3 Indigenous prisoners who did not complete secondary school are … times more likely to be in prison than those indigenous prisoners who did complete secondary school.

4 Make a conclusion about the effect of education on the likelihood that a person will someday serve a prison sentence.

Activity 3

In this activity the employment of prisoners prior to arrest is compared for indigenous and non-indigenous people. The table below is taken from the same 1992 data source. The base for comparison is the prison rate of employed non-indigenous people prior to arrest.

For example, unemployed non-indigenous people are 720/25 = 29 times more likely to be in prison than employed non-indigenous people.

Prior Employment Status of Prisoners, by Aboriginality, Australia, 30 June 1992

	Prisoners at 30.6.92	Prison rate per 100 000 persons aged 15+	Relative over-representation ratios
Indigenous people			
Unemployed	1143	6495	259
Other	328	332	13
Non-indigenous people			
Unemployed	4163	720	29
Other	1923	25	1

Sources: Walker & Salloom 1993; ABS 1992.
Notes: "Other" includes employed, running a business, in full or part-time education, occupied with "home duties", or retired. Table excludes NSW as no data were available on employment status of prisoners.

Copy and complete:

1 Unemployed indigenous people are … times more likely to be in prison than non-indigenous people who were employed before arrest.

2 Employed indigenous people are … times more likely to be in prison than non-indigenous people who were employed before arrest.

3 Unemployed indigenous prisoners are … times more likely to be in prison than indigenous prisoners who were employed.

4 Make a conclusion about the effect of unemployment on the likelihood that a person will serve a prison sentence.

Let's communicate

The over-representation of Indigenous people in prison is a striking social issue. List the factors that the data analysis in this activity suggests are vitally important for governments to address. List other factors that you think are important. Use the internet to find out what is being done about it.

Reflecting

Mathematics can make an important contribution to the development of social policy. Reflect on the data gathering techniques and analysis that can be used to inform debate on social issues, such as the plight of Indigenous people or the effects of addictive substances like alcohol and tobacco on the health of the nation.

Language link with Macquarie

MACQUARIE

1 *Explain as simply as you can the meaning of:*
   a *a* **box and whisker** *plot*
   b *the* **mean** *and* **standard deviation** *of a set of scores*
   c *a* **skewed** *distribution*
   d *a* **symmetrical** *distribution.*

2 *Read the* Macquarie Learners Dictionary *entries for* **data** *and* **database***:*

data *plural noun* facts or information: *I'm gathering data for a report on schools in Australia.*
Although **data** is plural in form (its singular is **datum**), you can use a singular or plural verb with it: *When the computer crashed all the data was lost.* (singular) *The data have been collected from all over Australia.* (plural)

database *noun* a collection of information stored in a computer and organised in a meaningful structure of categories.

What is the difference between them?

CHAPTER REVIEW

1 The data below shows the times, to the nearest second, recorded by 30 swimmers over 100 m.

58	56	62	59	61	63
60	57	58	58	62	61
59	58	58	59	62	63
57	59	61	62	60	58
56	60	59	59	57	58

   a Organise the data into a frequency distribution table with score (x), tally, frequency (f), cumulative frequency and fx columns.
   b Draw a frequency histogram and polygon for the data.
   c How many swimmers had a time under 1 minute?
   d What percentage of swimmers had a time of 59 seconds?
   e What was the slowest time?
   f What was the most common time?

2 The stem-and-leaf plot below shows the heights, in cm, of the nurses at a small private hospital.

Stem	Leaf
14	8 9
15	0 2 6 8
16	1 1 3 5 6 9
17	2 3 3 3 4 7 8 9
18	0 2 2 3 4 6 9

   a How many nurses work at the hospital?
   b What are the heights of the shortest and tallest nurses?
   c How many nurses are 1.8 m or taller?
   d What is the average height of these nurses? Answer correct to the nearest mm.

3 The number of students who were absent each day during February and March at a certain high school is shown below.

38	43	41	35	37	36	36
50	42	42	36	37	38	38
39	40	43	37	36	39	38
42	38	35	36	36	37	41
43	37	37	39	36	37	36
38	39	38	36	37	38	36

a Represent the data on a dot plot.
b Are there any clusters? If so, what are they?
c Are there any outliers? If so, what are they?

4 Find the mean, median, mode, range and standard deviation for each set of scores. Answer correct to 1 decimal place, where necessary.
a 9, 18, 4, 21, 15, 18, 13
b 42, 59, 37, 65, 15, 80, 59, 12, 45
c 33, 30, 37, 30, 36, 37, 39, 34
d 5.2, 6.7, 3.3, 7.1, 4.8, 5.0

5 Find the mean, median, mode, range and standard deviation for each set of scores. Answer correct to 1 decimal place where necessary.

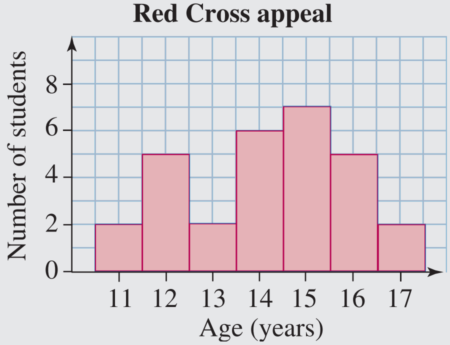
a

x	1	2	3	4	5
f	3	1	2	8	9

b

x	21	23	27	32	36
f	15	25	17	13	11

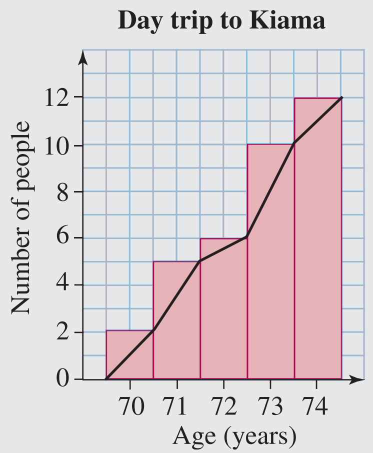
6 The histogram shows the ages of the students from a certain school that took part in the annual Red Cross appeal.



a How many 14-year-olds took part in the appeal?
b What was the age of the oldest student who took part?
c How many students took part?
d What percentage of the students were 13 years old?
e What is the modal age?
f Find the median age?
g Calculate the mean age, correct to 1 decimal place.

7 The range of a set of scores is 35.
a If the lowest score is 47, find the highest score.
b If the highest score is 89, find the lowest score.

8



The cumulative frequency histogram and ogive below show the ages of a group of senior citizens who travelled to Kiama for a day out.

a How many people went on the trip?
b How many 72-year-olds were there?
c What is the range of the ages?
d What is the median age?
e What is the modal age?
f Calculate the mean age, correct to 1 decimal place.
g Find the standard deviation of their ages, correct to 1 decimal place.

9 The scores obtained by a group of Year 10 students on their Statistics test are shown below. The test was marked out of 30.

18	24	13	21	25	28
16	19	19	22	27	25
22	24	17	14	20	21
23	11	6	15	17	23
3	28	27	25	24	20

a Organise the data into a grouped data frequency distribution table with class (x), class centre (cc), tally, frequency (f), and ($f \times cc$) columns. Use classes of 1–5, 6–10, …
b What is the modal class?
c How many students scored less than 70%?
d Use the frequency table to find the mean and standard deviation, correct to 1 decimal place.

10 **a** A set of 14 scores has a mean of 8. What is the sum of the scores?
b The mean of a set of scores is 27 and their sum is 864. How many scores are there?

11 The mean of four scores is 18.5. If three of the scores are 27, 32 and 15, what is the fourth score?

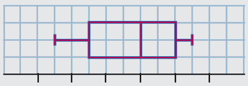
12 A set of scores has a mean of 60. What is the effect on the mean if the following scores are:
i added to the set?
ii taken out of the set?
a 82 **b** 35 **c** 60

13 A set of 47 scores has a mean of 71. Find the new mean, correct to 1 decimal place, after a score of:
a 30 is added to the set
b 52 is taken out of the set

14 Peddar has an average of 76% after three tests. What mark does she need to score in the next test to raise her average to 80%?

15 The mean of a set of 14 scores is 35. After a new score is added, the mean rises to 36. Find the score that was added.

16 This box-and-whisker plot shows the scores obtained by a group of boys on a spelling quiz.



a Find:
- **i** the extreme scores
- **ii** the median
- **iii** the range
- **iv** the lower quartile
- **v** the upper quartile
- **vi** the interquartile range

b Find the percentage of scores that lie between:
- **i** 1 and 3
- **ii** 3 and 6
- **iii** 3 and 8
- **iv** 8 and 9

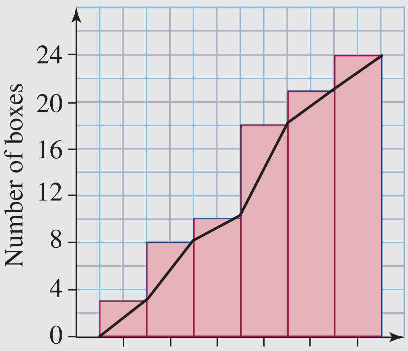
17 For each set of scores below:
- **i** find the median
- **ii** find the lower and upper quartiles
- **iii** find the interquartile range
- **iv** draw a box-and-whisker plot

a 4, 6, 7, 10, 14, 15, 18
b 21, 23, 26, 30, 35, 36, 39, 43, 45, 47
c 52, 54, 55, 55, 58, 60, 61, 62, 64, 67, 67, 69, 73, 74, 75, 76, 78, 81, 84, 85, 87, 90, 91, 96

18 Find the upper and lower quartiles and the interquartile range for this set of scores.

x	1	2	3	4	5	6	7
f	5	9	13	20	3	14	4

19 This cumulative frequency histogram and ogive shows the number of Smarties per box.



a What is the median?
b Find the upper and lower quartiles.
c What is the interquartile range?

20 If every score in a distribution is 5, what would be the standard deviation?
A 0 **B** 1
C 2.5 **D** 5

21 Which of the following is NOT a measure of spread?
A range
B mean
C standard deviation
D interquartile range

22 Two cricket teams scored the following innings totals in the first four games of the season.

Petersham

195 198 265 223 248 301 154 211

Manly

252 375 136 300 258 294 103 224

a Find the mean and standard deviation for each team.

b Which team was the more consistent during the first four games? Why?

23 The students in Year 10 had a mean of 50 and a standard deviation of 12 on their mid-year History exam. Write down the mark that is:

a 1 standard deviation above the mean

b 2 standard deviations below the mean

c 0.5 standard deviations above the mean

d 1.5 standard deviations below the mean

24 Gerard scored 66 on a Maths test in which the class mean was 78. His mark was 2 standard deviations below the mean. What was the standard deviation?

25 For each of the following, use the mean and standard deviation to determine which is the better mark.

a

	Mark	$\bar{x}$	σ_n
Test 1	90	80	5
Test 2	92	84	8

b

	Mark	$\bar{x}$	σ_n
Test 1	61	75	7
Test 2	59	68	6

26 The table shows the mean and standard deviation for the Year 11 report marks in Physics and Chemistry.

	$\bar{x}$	σ_n
Physics	70	8
Chemistry	63	10

a Find the mark in Physics that corresponds to a mark in Chemistry of:

i 73 **ii** 43

b Find the mark in Chemistry that corresponds to a mark in Physics of:

i 66 **ii** 90

27 A group of people took part in an IQ test. The mean of this group was found to be 120 and the standard deviation was 9. If the distribution of the IQ scores is approximately normal, find the percentage of people in this group that we expect to have an IQ that is:

a between 111 and 129

b between 102 and 138

c between 93 and 147

d between 120 and 129

e between 102 and 111

f above 138

28 Two hundred people were asked their ages as part of a survey. The mean age is 42 and the standard deviation of the ages is 13. If the ages of the survey group are normally distributed, find the number of people who should be aged:

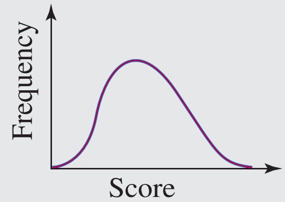
a between 29 and 55

b between 16 and 68

c under 16 and over 68

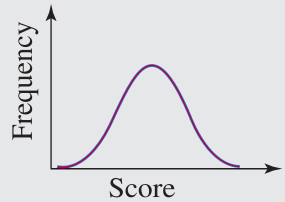
29 State whether each display is symmetrical, positively skewed or negatively skewed?

a



b

c



30 The scores obtained by the boys and girls in class 10C on a fitness test are shown in this back-to-back stem and leaf plot. The higher the score, the greater is the student's level of fitness.

Boys		Girls
4	$6^{(0)}$	1 2 3
7 6 6	$6^{(5)}$	5 5 7
4 3 2 1	$7^{(0)}$	0 3 4 4
8 7 6 5	$7^{(5)}$	6 6 7
4 3	$8^{(0)}$	1 2
9 7	$8^{(5)}$	7

a Find the mean, median and standard deviation for each group.

b Which group is fitter, the boys or the girls?

9 Probability

This chapter at a glance

Stage 5.1/5.2/5.3

After completing this chapter, you should be able to:

- repeat an experiment a number of times to determine the relative frequency of an event
- estimate the probability of an event from experimental data
- use the formula to calculate probabilities for simple events
- distinguish between dependent and independent events
- apply relative frequency to predict future experimental outcomes
- design a device to produce a specified relative frequency
- recognise and explain differences between relative frequency and theoretical probability
- sample with and without replacement in two-stage experiments
- analyse two-stage events through constructing organised lists, tables and/or tree diagrams
- solve two-stage probability problems including instances of sampling with and without replacement
- find probabilities of compound events using organised lists, tables or diagrams
- solve probability problems involving compound events
- critically evaluate statements on chance and probability
- evaluate the likelihood of winning a prize in lotteries and other competitions
- evaluate games for fairness
- identify common misconceptions related to chance events
- recognise the use of probability by governments and companies
- use Venn diagrams as a tool in solving some probability problems.

9.1 Probability review

There are many situations in our everyday lives where we are uncertain what will happen. For example my train may be delayed and I will be late for work or it may rain on Saturday and netball will be cancelled. While we may be uncertain what will occur, it is often very important that we consider all the possibilities, assess their likelihoods and prepare for all eventualities. As we have seen in previous years the branch of mathematics that we use for this purpose is called probability.

The following terms have been used in our previous study of probability.

An **experiment** is an activity, such as tossing a coin or rolling a die, a **trial** is a single occurrence of an experiment.

Each possible result of an experiment is called an **outcome**. In one roll of a die the outcomes are 1, 2, 3, 4, 5 or 6. If each outcome has the same likelihood of occuring they are **equally likely** outcomes.

An **event** is a particular collection of possible outcomes. For the roll of a die one event, E, would be that the result is an even number. This event would occur with any of the outcomes 2, 4 or 6. An event is **certain** if it is sure to occur and **impossible** if it cannot occur.

Two events are **mutually exclusive** if they cannot both occur in the same trial. For example rolling a die the events A, a number less than 3, and B, a number greater than 4, are mutually exclusive since there is no outcome which can satisfy both A and B.

The probability of an event, A, is given by the formula:

$$P(A) = \frac{\text{number of outcomes favourable to } A}{\text{total number of possible outcomes}}$$

Note: with this formula all outcomes must be equally likely.

Experimental probability

In experimental probability an experiment is conducted a number of times (usually a large number) and the results are used to estimate the probabilities of the different outcomes.

For example, a die is rolled 100 times for the following results 6—14 times, 5—18 times, 4—17 times, 3—17 times, 2—18 times, 1—16 times.

The experimental probabilities in this case would be $P(6) = 0.14$, $P(5) = 0.18$, $P(4) = 0.17$, $P(3) = 0.17$, $P(2) = 0.18$, $P(1) = 0.16$.

Theoretical probability

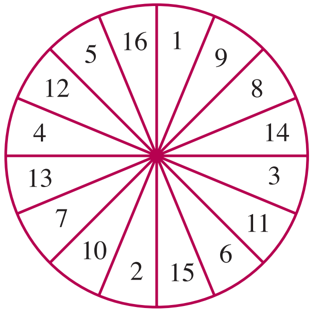
In some cases we are able to construct a theoretical description which allows us to calculate the probabilities of events. For example, a card is drawn from a pack of ten numbered 1 to 10, but otherwise identical cards. We would expect all cards to have the same chance of being drawn so we theorise that the probabilities are all $\frac{1}{10}$.

Both experimental and theoretical probability are very important in our lives, as there are situations in which each is the only possible practical method. Practical probability situations often involve so many variables and are so complicated that a theoretical approach is not possible. Equally, it is not desirable to repeat an experiment many times if the experiment is very costly or dangerous.

EG +S

Example 1

What is the probability of spinning a number greater than 3 on the spinner pictured?



Solution

All 16 numbers are equally likely and there are 13 numbers greater than 3.

The probability is $\frac{13}{16}$.

EG +S

Example 2

The tables shows the numbers of males and females in the year 10 classes in a school.

	Male	Female	Class Total
10M1	11	21	32
10M2	20	12	32
10M3	13	17	30
10M4	15	12	27
10M5	14	15	29
Gender Total	73	77	150

a What is the probability that a student selected at random will be:

i in 10M3? **ii** a female? **iii** a male in 10M1?

b What is the probability that a student who is in 10M4 is male?

Solutions

There are 150 students altogether in year 10.

a **i** Since there are 30 students in 10M3 the probability $= \frac{30}{150} = \frac{1}{5}$.

ii There are 77 girls so the probability $= \frac{77}{150}$.

iii There are 11 males in 10M1 so the probability $= \frac{11}{150}$.

b There are 27 students in 10M4 and 15 of them are male. The probability $= \frac{15}{27} = \frac{5}{9}$.

Complementary events

An event, E, has a number of possible outcomes.

The complement of E is an event which has *all* the outcomes that are *not* included in E. Simply put the complement of E is *not* E.

We use the symbol $\tilde{E}$ for the complement of E.

$$P(\tilde{E}) = 1 - P(E)$$

Example 3

A die is rolled. U is the event that either a 2 or a 5 result.

a What is the complement of U ?

b What is the probability of $\tilde{U}$?

Solutions

a The complement is all possible outcomes apart from 2 and 5.

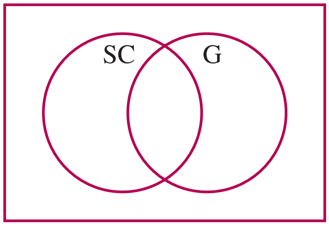
b Thus $\tilde{U}$ = 1, 3, 4 or 6.

$$P(\tilde{U}) = \frac{2}{3}.$$

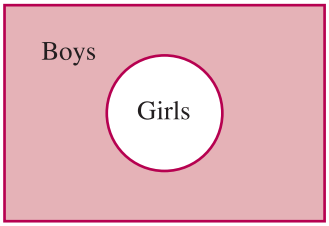
Venn diagrams

Venn diagrams provide a pictorial representation, and are especially helpful where the numbers are large.

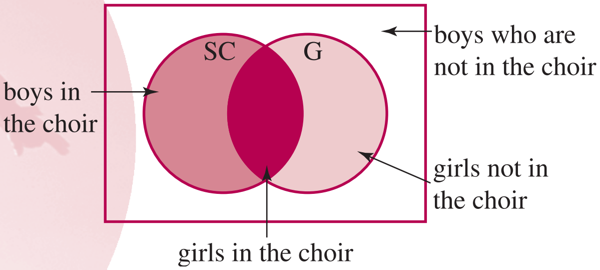
This Venn diagram represents the children at a school.



The circle labelled G represents all the girls at the school, with each girl being represented by a point within the circle. We do not usually mark the points for the girls, since the numbers are too great, but write the number of girls inside the circle. Of course the region outside the G circle will contain the boys.

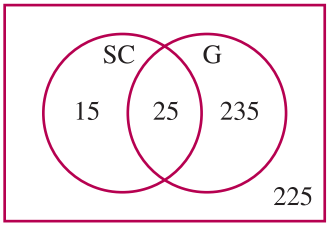


The circle labelled SC contains all members of the school choir. The girls in the choir will be in the region which is common to both circles. Similarly the remainder of the SC circle contains the boys in the choir.



Example 4

This Venn diagram represents the school situation described in example 2, with the number of students in each section shown.



a How many girls attend the school?
b How many students attend the school?
c How many boys sing in the choir?
d What is the probability that a student chosen at random sings in the choir?
e What is the probability that a girl chosen at random sings in the choir?

Solutions

a The girls are all inside the G circle. 25 of them are also in the SC circle (sing in the choir) and 235 are not.
Altogether there are $25 + 235 = 260$ girls.

b There are four regions in the Venn diagram. We add all four numbers to find the total number of students.
Number of students $= 15 + 25 + 235 + 225 = 500$

c We need the section of the SC circle which doesn't overlap the G circle.
There are 15 boys.

d There are $15 + 25 = 40$ students in the choir, and 500 students at the school.

$$\text{The probability} = \frac{40}{500} = \frac{2}{25}.$$

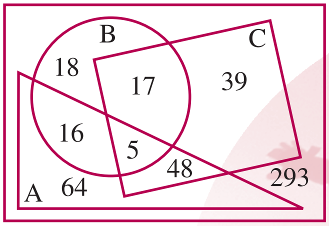
e There are 260 girls and 25 sing in the choir.

$$\text{The probability} = \frac{25}{260} = \frac{5}{52}.$$

Where a situation involves three sets we use three circles (or other shapes) making sure that each circle intersects both the others.

Example 5

The regions A, B and C in this Venn diagram show school representatives in athletics, basketball and cross country running, respectively.



a Find the number of students who:
- **i** do not represent in any of the three sports.
- **ii** are athletics representatives.
- **iii** represent in basketball and athletics but not cross country.

b Find the probability that a particular student:
- **i** represents in basketball.
- **ii** represents in all three sports.
- **iii** who represents in athletics also represents in cross country.

Solutions

a **i** There are 293 students not inside any of the shapes.

ii $16 + 5 + 48 + 64 = 133$ students are athletics representatives.

iii We need to check the region which is *in* the circle B and *in* the triangle A but *not* in the rectangle C.

There are 16 students in this category.

b **i** The total for basketball is $17 + 16 + 5 + 18 = 56$.

The probability $= \frac{56}{500} = \frac{14}{125}$.

ii There are only 5 students in the central region where the triangle, circle and rectangle overlap.

The probability $= \frac{5}{500} = \frac{1}{100}$.

iii There are 133 athletics representatives. Those who also represent at basketball are in the region where triangle and rectangle overlap. There are $5 + 48 = 53$ of them.

The probability $= \frac{53}{133}$.

Exercise 9.1

1 A card is drawn from a normal deck of 52 cards (Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King of each of the four suits spades (black), hearts (red), diamonds (red) and clubs (black)). What is the probability that the card is:

a a Jack?

b red?

c the 5 of clubs or the 6 of clubs?

d a 7 of any suit or a diamond (or both)?

e a 'court' card (Jack, Queen or King)?

2 A 20 sided die with faces labeled 1 to 20 is rolled. What is the probability of a number:

a divisible by 3?

b between 11 and 18 ?

c divisible by 4 or 5 (or both)?

d that is prime?

3 A company produces boxes of individually wrapped chocolates. They intend each box to have 30 chocolates but suspect that the packing process is not entirely accurate, so they conduct a random sample of 100 boxes and check the contents. The table shows the results obtained.

Number of chocolates	28	29	30	31
Number of boxes	3	15	72	10

a What is the probability of buying a box with:

i the correct number of chocolates?

ii more than 30 chocolates?

iii fewer chocolates than intended?

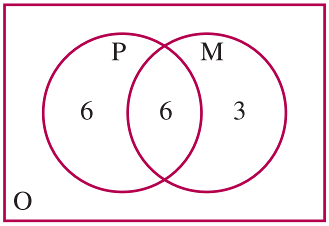
b Do you consider that this situation is fair to the purchaser?

4 Kevin has a set of cards labelled 1, 2, 3, …, 10, and conducts an experiment in which he draws one card at random. Consider the following events.

$A = \{ 2, 4, 6, 8, 10 \}$ = an even card is drawn
$B = \{ 1, 3, 5, 7, 9 \}$ = an odd card is drawn
$C = \{ 2, 3, 5, 7 \}$ = a prime number is drawn
$D = \{ 5, 6, 7, 8, 9, 10 \}$ = a number greater than 4
$E = \{ 8, 9, 10 \}$ = a number greater than 7
$F = \{ 1, 2, 3, 4 \}$ = a number less than 5

a Name any pairs of:
  i complementary events **ii** mutually exclusive events
b Find the probability of each event.
c Write the description of an event which would be:
  i impossible **ii** certain

5 This Venn diagram shows the choices made by a group of 15 friends who were all keen to go to a particular movie, *M*, on Friday night and were also invited to a party, *P*, on Saturday night. What is the probability that a person chosen at random:



a went to both party and movie?
b went to the movie but not the party?
c went to the party?
d did not attend either movie or party?

■ Consolidation

6 Nadia has appeared on a TV quiz and won the contest. She now has a chance to spin a wheel to see which prize she receives. The wheel has 30 identical sections labelled with the prizes, as shown in the table.

Prize	Car	TV	Holiday	Shopping voucher	DVD player	Mobile phone	Beauty pack	Movie passes
Value	\$42 000	\$9000 each	\$4500 each	\$400 each	\$300 each	\$180 each	\$150 each	\$50 each
Number of sections on the wheel	1	2	2	3	3	4	5	10

What is the probability that Nadia wins:

a the car?
b a DVD player?
c a prize worth more than \$1000?
d a prize worth less than \$200?
e a holiday or a beauty pack?

7 This diagram shows the choices of two electives made by a class of 28 students. They were required to choose either Music or Art *and* either French or Japanese.

	Art	**Music**
French	7	4
Japanese	6	11

a A student is chosen at random. What is the probability that he/she:

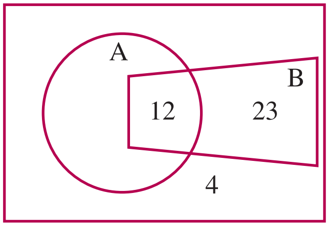
i chose Art and Japanese? **ii** did not choose Music?

iii chose French? **iv** did not choose Japanese or Art?

b What is the probability that a student:

i who chose French also does Art? **ii** who doesn't do Art does study Japanese?

8 The Venn diagram contains a total of 40 members.
Find the probabilities:



a $P(B)$

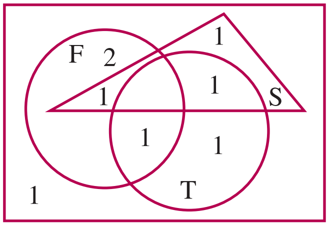
b $P(A)$

c $P(\tilde{A})$

d $P(B$ but not $A)$

e $P($neither B nor $A)$.

9 The Australian Netball Team played Jamaica in three tests. Due to injuries and form some players in the 12 member team were unavailable or were dropped for various tests. This Venn diagram illustrates the situation, where F = first test, S = second test, T = third test.
If a squad member was chosen at random, what is the probability that she:



a only played in the third test?

b played in all three tests?

c did not play a test?

d played in the first and second tests but not in the third?

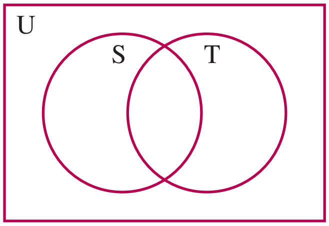
e played in the first test and in either the second or the third tests (but not both)?

f played only two tests?

g played only one test?

10 Using the information in this Venn diagram find the probabilities $P(T)$, $P(S)$, $P(S$ and $T)$.
Total number in S and/or $T = 50$

$$n(T) = 36$$
$$n(S) = 20.$$



11 In a group of 20 families, 5 families took overseas holidays between 2006 and 2010. 16 of the families took interstate holidays within Australia during this time, and two families did not travel overseas nor interstate. Draw a Venn diagram to illustrate this information.

12 In the USA the game 'craps' is popular. In craps a player rolls two dice with the results as shown in the table. On the second roll the player must roll the same sum as they rolled in the first roll to continue.

sum of 7 or 11	win
sum of 2, 3 or 12	lose
any other sum	roll again

a What is the probability of:

i winning on the first roll? **ii** losing on the first roll?

b What is the probability of neither winning nor losing on the first throw?

c If a player scores 5 on the first roll, what is the probability that they will again score that sum on the second roll to allow them to continue?

d Apart from 7 or 11 which is the next best sum to score on the first roll?

13 Anastasia received a shiny new fifty cent coin in change at a shop and wondered if it really was unbiased. She decided to test it and tossed the coin 20 times for 13 tails and 7 heads.

a What is the experimental probability of each outcome?

b What conclusion could Anastasia draw from this result?

Thinking that the coin really should be unbiased she decided to toss some more times. In another 20 tosses she had 10 tails and 10 heads.

c Is she now justified in claiming that the coin is unbiased?

d What advice would you give Anastasia?

Further applications

14 In a lottery 100 000 tickets (numbered 1 to 100 000) are sold for \$20 each. First prize is \$80 000, second \$20 000, third \$12 000 and there are many smaller prizes. A man buys a single ticket but is dismayed to find that it is number 17. "When did you ever see a number that small win the lottery?" he complained, tearing the ticket into small pieces. Discuss.

15 A soccer squad of twenty players has been chosen to represent Australia. In analysing the squad the coach finds that:

- only 3 can play as forwards, in the midfield or as a defender
- 6 can play forward or in the midfield
- 1 can play forward or as a defender (but not in the midfield)
- there are 11 players who can play forward
- 7 of the 11 midfielders can play as defenders
- 1 player is injured and will not be able to play.

Draw a Venn diagram to illustrate this situation.

9.2 Independent compound events

Our study of probability up to this point has provided a theoretical framework which allows us to calculate the probabilities of certain events occurring. These have been events comprising just a single stage, for example rolling a 3 with a normal die, but in many practical situations events involve more than one stage. We will now consider the calculation of probabilities of compound events, those involving two or more stages, in which the two stages are independent (do not affect each other).

Two events A and B are independent if the result in either event does not affect the outcome in the other event.

Example 1

William plays a game in which he draws a card from a normal pack of playing cards and rolls a die. He wins if the card is a diamond and the die shows an even number.

What is the probability that William will win?

Solution

This is a two-stage event since there are two requirements for William to win. The main consideration in calculating William's chance of winning is to ensure that we consider all possible equally likely outcomes. This sounds simple enough but it is often easy to overlook some possibilities so we need to be systemmatic in making our list.

One method of ensuring that we don't omit any possibilities in our list of outcomes is to use a **tree diagram**. A tree diagram for William's game is shown.

Card draw	Coin toss	Outcome	Result
heart	even	h-e	lose
	odd	h-o	lose
diamond	even	d-e	win
	odd	d-o	lose
club	even	c-e	lose
	odd	c-o	lose
spade	even	s-e	lose
	odd	s-o	lose

In this diagram the first column shows the four possible suits (heart, diamond, club or spade) when the card is drawn. The second column shows the possible results when the die is thrown, and the third column gives a list of all the combinations of suit and number.

There are 8 possible combinations (all equally likely) and only d-e will allow William to win.

His probability of winning is $\frac{1}{8}$.

Example 2

Samantha takes four identical cards and labels them 1, 2, 3 and 4. She shuffles the cards, draws one at random and records the number drawn. She returns the card, shuffles again, selects a second card and records its number.

What is the probability that Samantha has written two 3s?

Solution

The tree diagram for this activity is shown.

There is just one 3, 3 in the list of 16 possible outcomes so the probability = $\frac{1}{16}$.

First draw	Second draw	Result
1	1	1, 1
	2	1, 2
	3	1, 3
	4	1, 4
2	1	2, 1
	2	2, 2
	3	2, 3
	4	2, 4
3	1	3, 1
	2	3, 2
	3	3, 3
	4	3, 4
4	1	4, 1
	2	4, 2
	3	4, 3
	4	4, 4

While tree diagrams are useful for small numbers of outcomes they become unwieldy when there is a larger number of stages, or more outcomes at each stage. An alternative means of representing all possible outcomes of a two-stage situation is to use a table.

This table illustrates William's game which we have already represented as a tree diagram. The card selections have been listed down the left hand side and the die outcomes, even and odd, across the top. Each cell in the table represents one outcome, for example heart-even.

	Even	Odd
Heart	h-e	h-o
Diamond	d-e	d-o
Club	c-e	c-o
Spade	s-e	s-o

Example 3

EG +S

In a game of Monopoly a player rolls a pair of dice and adds the two numbers to determine how many spaces they move around the board. Sam is six spaces from an opponent's hotel and he is very keen not to land on it.

a What is the probability that Sam rolls a total of 6?

b What is his chance of avoiding the hotel on this throw?

c Which sum is the most likely with a pair of dice?

d Which sums have a probability of $\frac{1}{9}$?

Solutions

We use the table where each cell shows the sum of the two numbers on the dice.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

a Five of the 36 cells have a total of 6. Therefore the probability is $\frac{5}{36}$.

b 31 cells have totals other than six. The probability is $\frac{31}{36}$.

c More cells contain 7 than any other number. 7 is the most likely total with a probability of $\frac{1}{6}$.

d $\frac{1}{9} = \frac{4}{36}$. Both 5 and 9 appear in four cells.

You may have realized that it is not necessary to know the details of each of the possible outcomes of a compound experiment, or of those outcomes that will yield the event being considered. What we need to know is the total number of possible outcomes and *how many* outcomes yield the event.

For a two-stage event we label the events A and B, the number of favourable trials $n(A)$ and $n(B)$, and the total number of trials $n(N_A)$ and $n(N_B)$.

The total number of compound events $= n(N_A) \times n(N_B)$

The number of favourable events $= n(A) \times n(B)$

Therefore, the probability of the compound event $= \dfrac{n(A) \times n(B)}{n(N_A) \times n(N_B)}$

$$= \frac{n(A)}{n(N_A)} \times \frac{n(B)}{n(N_B)}$$

$$= P(A) \times P(B)$$

Thus $P(AB) = P(A) \times P(B)$

This formula, the **Product Rule**, is easily applied where the events A and B are independent, that is the outcome of either does not affect the possible outcomes of the other.

If A and B are two independent events, then the probability that the event A will occur, followed by event B, is given by:

$$P(AB) = P(A) \times P(B)$$

where
- **$P(A)$ is the probability that the event A will occur**
- **$P(B)$ is the probability that the event B will occur**
- **$P(AB)$ is the probability that events A and B will occur in that order.**

Using our example of William's game again we have:

probability of drawing a diamond $= \dfrac{1}{4}$

probability of rolling an even number $= \dfrac{1}{2}$

Therefore, the probability of both, $P(AB) = P(A) \times P(B) = \dfrac{1}{4} \times \dfrac{1}{2} = \dfrac{1}{8}$.

With understanding, application of the product rule is much quicker than compiling an exhaustive list or drawing a tree diagram.

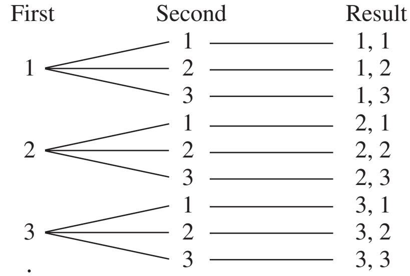
Using the product rule for example 2 we have

$P(3, 3) = P(3_F) \times P(3_S)$ where 3_F = drawing a 3 at the first draw

$= \dfrac{1}{4} \times \dfrac{1}{4}$ and 3_S = drawing a 3 at the second draw.

$= \dfrac{1}{16}$.

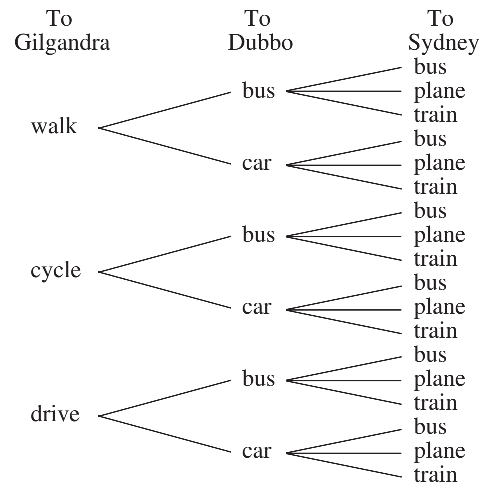
Exercise 9.2



1 Use the tree diagrams provided to calculate the required probabilities.

a Three cards are labeled 1, 2 and 3. A card is selected at random and the number recorded. The card is replaced and a second selection is made. The two umbers are added. Find the probability that the sum is:

i 5
ii 4
iii greater than 5
iv less than 5



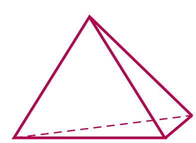
b Luke, who lives a few kilometres from Gilgandra, wishes to travel to Sydney. He can either walk, cycle or drive to Gilgandra. From Gilgandra he can take a bus to Dubbo, or get a lift in a friend's car. From Dubbo he can travel to Sydney by train, plane or bus. If all Luke's choices are random, find the probability that:

i he travels in a bus, train and car
ii he travels in two buses
iii he travels in two cars
iv he doesn't travel by train
v he doesn't travel by car

2 Draw tree diagrams to create a list of all the possible outcomes for each experiment.

a A student chooses her summer sport from cricket, swimming and tennis, and her winter sport from soccer and basketball.

b The sides of a four sided die are marked A, B, C and D. The die is rolled twice and the outcomes recorded.



c A man travels from Dubbo to Orange twice, each time choosing to go by car, bus or train.

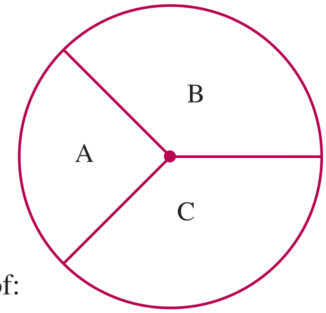
d A normal coin is tossed three times and the results recorded.

e At the Year 10 Formal students have a three course meal.

Entrée	Smoked salmon **or** Turkey
Main	Roast beef **or** Apricot chicken **or** Sauteed pork
Dessert	Caramel profiterole **or** Chocolate mudcake

Each person chooses one entrée, one main and one dessert.

3 Draw tree diagrams to create a list of all the possible outcomes for each experiment. Use your tree diagram to help you calculate the probabilities required.

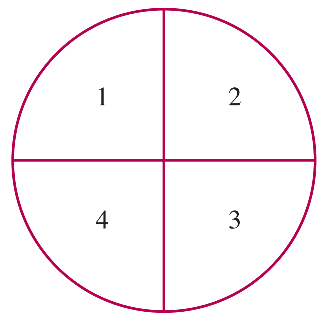


a A circular spinner has three sectors labelled A, B and C. If the spinner is spun twice what is the probability of:

i two As? **ii** one B and one C?
iii no Cs? **iv** at least one B?

b An unbiased coin is tossed twice. What is the probability of:

i two heads? **ii** two tails?
iii one head and one tail?



c A coin is tossed and then the spinner at the right is spun. If the result is a head and an even number Brandon wins. If it is a tail and an odd number Jordan wins. Otherwise Daniel wins.

i What probability does each boy have of winning?
ii What is the probability of a head and a prime number?
iii What is the probability of a head or an odd number, or both?

d A man leaves Ashfield to drive to Newcastle. He can travel to Hornsby via the Gladesville Bridge (no toll), the Sydney Harbour Bridge (toll) or the Sydney Harbour Tunnell (toll). From Hornsby he can travel by Freeway (no toll) or by the Pacific Highway (no toll). If both his choices are random find the probability that:

i he crosses the Gladesville Bridge.
ii he uses the Sydney Harbour Tunnel and the Freeway.
iii he doesn't pay any tolls.

4 Use the tables provided to calculate the required probabilities.

a A pair of regular dice are rolled and the two numbers recorded.

	1	**2**	**3**	**4**	**5**	**6**
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

What is the probability that:

i the numbers rolled are 6 and 3? **ii** the numbers are both 2s?
iii the product of the numbers is 12? **iv** the sum of the numbers is 13?
v the numbers differ by more than two? **vi** the sum of the numbers is less than 9?

b Seven cards are labeled 1, 2, 3 …, 7. Another eight cards are labeled A, B, C, …, H. One card is selected at random from each set.

	1	2	3	4	5	6	7
A	(A, 1)	(A, 2)	(A, 3)	(A, 4)	(A, 5)	(A, 6)	(A, 7)
B	(B, 1)	(B, 2)	(B, 3)	(B, 4)	(B, 5)	(B, 6)	(B, 7)
C	(C, 1)	(C, 2)	(C, 3)	(C, 4)	(C, 5)	(C, 6)	(C, 7)
D	(D, 1)	(D, 2)	(D, 3)	(D, 4)	(D, 5)	(D, 6)	(D, 7)
E	(E, 1)	(E, 2)	(E, 3)	(E, 4)	(E, 5)	(E, 6)	(E, 7)
F	(F, 1)	(F, 2)	(F, 3)	(F, 4)	(F, 5)	(F, 6)	(F, 7)
G	(G, 1)	(G, 2)	(G, 3)	(G, 4)	(G, 5)	(G, 6)	(G, 7)
H	(H, 1)	(H, 2)	(H, 3)	(H, 4)	(H, 5)	(H, 6)	(H, 7)

Find the probability of:

i one card being a D.
ii the cards being G and 4.
iii the cards being an A and an even number.
iv the cards being a prime number and a letter other than G or H.
v the letter being composed entirely of straight lines and the number being greater than 3.

5 For each part, draw a table to show all possible outcomes and, hence, calculate the required probabilities.

a Bronwyn has four different skirts (black, white, cream and pink) and six tops (white, red, gold, blue, green and orange). If she chooses at random how many different outfits does she have? What is the probability of her wearing:

i a cream skirt and a blue top?
ii either her white skirt or her pink skirt?
iii her black skirt with either her orange or white top?
iv her cream skirt with her red top or her pink skirt with any of her tops?
v neither her pink skirt nor her gold top?

b A pair of regular dice, one white and the other cream, are rolled. What is the probability of:

i two 6s?
ii a white 6 and a cream 5?
iii a 1 and a 2 of either colour?
iv a white number greater than 2?
v the product of the numbers being less than 24?

c A regulation pack of cards has the four suits hearts (red), diamonds (red), clubs (black) and spades (black). Sophie draws one card at random, replaces it, then draws a second card at random. Calculate the probability that:

i the cards are both hearts.
ii the cards are both red.
iii one card is red and the other black.
iv one card is red and the other a spade.
v neither card is black nor a diamond.
vi at least one of the cards is a club.

d Olivia travels from Sydney to Melbourne by bus, train, car or plane (all being equally likely) and then from Melbourne to Adelaide by plane, car, bus, train, motorbike or bicycle (all equally likely). Calculate the probability that:
  - **i** she completes the whole trip by train.
  - **ii** she travels by bus for at least part of the trip.
  - **iii** she doesn't use her motorbike or bicycle.
  - **iv** she travels by bus for part of the trip and by train for the other part.
  - **v** she doesn't travel by car or bus.
  - **vi** she doesn't travel by road for at least part of the trip.

■ Consolidation

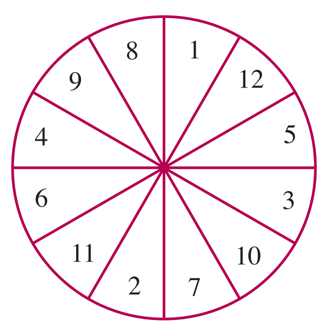
6 Angelique and Ramon were married in 2001. They were both keen to have children and hoped to have three, with at least one of each gender. By 2007 they had three lovely children but all three were boys. Angelique and Ramon considered three children the ideal size of family but they were very keen to have a daughter. Taking probability into account, what advice would you give Ramon and Angelique?

7 Use the product rule to calculate the probability in each of the following.

a The Australian Netball Team (7 players) is to play two tests against New Zealand. For each test one player is chosen at random to lead the team onto the court. If the same seven players are in the team for both tests, find the probability that:
  - **i** the Centre leads the team on for both tests.
  - **ii** the Goal Shooter leads for the first test and the Wing Defense for the second.
  - **iii** the Goal Attack leads for the first but not for the second test.
  - **iv** the Goalkeeper is not chosen to lead for either test.
  - **v** neither Goal Defense nor Centre lead on for the second test.

b Antoine creates a two-digit number by randomly selecting each digit from the numbers 1, 3, 5, 7 and 9.
What is the chance that the number is:
  - **i** odd?
  - **ii** 57 or 91?
  - **iii** greater than 50?
  - **iv** less than 42?
  - **v** divisible by 11?

c The spinner illustrated is spun twice.
What is the probability of scoring:
  - **i** 1 on each spin?
  - **ii** the same number on each spin?
  - **iii** a number less than 6 on the first spin and one greater than 6 on the second spin?
  - **iv** a number divisible by 3 on both spins?
  - **v** two numbers only one of which is prime?
  - **vi** a total greater than 3?



d Jabad and Chris each have a regular pack of cards. Each of them selects a card at random from their pack. Find the probability that:

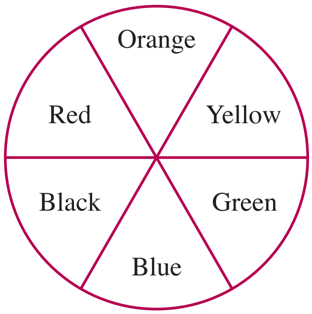
i their cards are of different colours.
ii Chris has an 8 and Jabad a club.
iii Chris has an 8 or 9 and Jabad has a spade.
iv Jabad doesn't have a club and Chris doesn't have a court card (Jack, Queen or King).
v Jabad has a 9 of diamonds and Chris has a diamond or a 9.

8 Annabel has a regulation pack of cards (Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King in each of the four suits). She chooses a card at random, records its details, and replaces it. She repeats the process twice, selecting three cards in total. Find the probability that:

a all three cards are spades.
b all three cards are court cards (Jack, Queen or King).
c the first two cards are the 10 of diamonds and the 3 of clubs in that order.
d no Ace is chosen.
e the first card is a black Jack, the second is a number between 2 and 10, and the third is a court card.

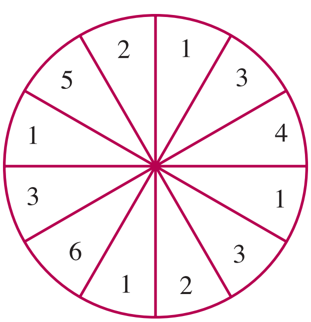
9 The spinner shown is spun four times.
What is the probability that:

a all four results are either red or black?
b the first and last spins are orange?
c only the first and last spins are orange?
d the first spin is black, the second is orange or yellow, and the third and fourth spins are red, black or blue?
e none of the spins is red or green?



10 The spinner illustrated is spun three times.
What is the probability of:

a three 2s?
b 6, 5, 4 in that order?
c 3, 2, 1 in that order?
d no 1s?
e at least two different scores?



■ Further applications

11 Rachael purchases a townhouse 'off the plan' and the builder offers her a choice of colours for interior painting. The walls can be cream, light blue or light green, and the ceilings vivid white or off-white. Different colours may be chosen for different rooms but only two colours can be used in any room (one for the ceiling and one for the walls). The townhouse consists of four rooms; living room, bedroom, kitchen and bathroom. It is all so confusing that Rachael makes random selections for each room.

a What is the number of possible colour schemes for Rachael's townhouse?

b What is the probability that:

i the bedroom has cream walls and a vivid white ceiling?

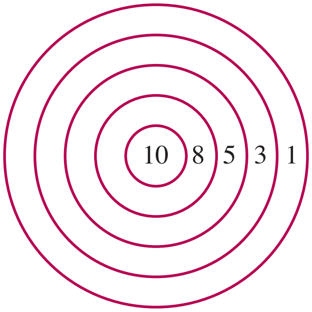
ii the living room and the bedroom have the same colour schemes?

iii no ceiling is off-white?

iv all the walls are of the same colour?

v at least two rooms have identical colour schemes?

12 Salam is an archer. In a competition she fires just two arrows at the target and her scores are added. Over many competitions she has found that her probabilities are 10—1/20, 8—1/10, 5—1/5, 3—3/10, 1—7/20 (she never misses!). What is the probability that:



a she scores better than 2?

b her total is 6?

c her total is more than 10?

d at least one arrow scores 10?

13 Basheer studies four subjects each semester for six semesters in his university degree. There are five possible results for each subject and Basheer's chances of earning each result are:

High distinction	7%	Distinction	34%
Credit	40%	Pass	16%
Fail	3%		

a How many different sets of results are possible for the complete degree if a student has no failures?

b What is the chance (to nearest 0.1%) that in a particular semester Basheer earns four grades of distinction or better?

c What is the probability (to nearest 0.1%) of earning no credits in one year?

d Basheer's friend Simon earns four passes in one semester. What is Basheer's likelihood (to nearest 0.1%) of not failing any subject and doing better than Simon?

TRY THIS **Party time**

A party of 16 girls is divided at random into 4 groups of 4. What is the probability that 4 particular girls are in 4 different groups?

9.3 Dependent events

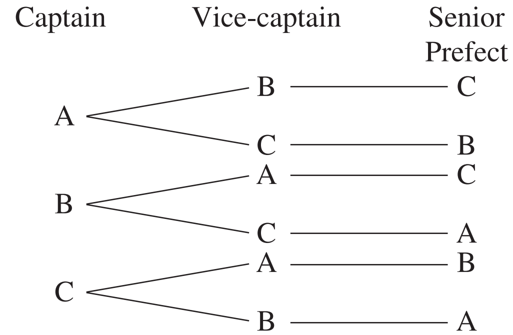
In the previous section we limited our consideration of multiple stage events to cases where the stages were independent, so the outcome of one stage did not affect the possible outcomes at the next or subsequent stages. We will now consider cases where the stages are not independent. Our use of tree diagrams, tables and lists will remain the same but we will need to be very careful in listing the possible outcomes at each stage.

Example 1

Three students A, B, and C are randomly assigned to the positions School Captain, School Vice-captain and Senior Prefect. Construct a tree diagram to show all possible outcomes.

Solution

Clearly a person cannot fill more than one position, therefore the choice of Captain affects who is left to fill the other positions. We can see that there are three choices for the first stage, two for the second and only one for the third. The number of possible outcomes is $3 \times 2 \times 1 = 6$.



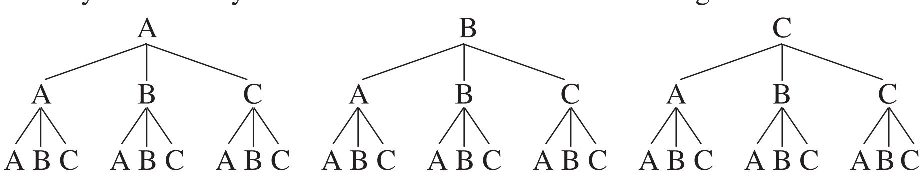
Situations such as that in example 1 are selections **without replacement** since each person can only be selected once. Situations where a person can be chosen more than once are described as selections **with replacement**.

Example 2

A teacher randomly selects one of A, B, C to ring the school bell at recess, lunch and end of school. Construct a tree diagram to show all possible outcomes.

Solution

Since any student may be chosen more than once the tree diagram would be:



There are 27 possible outcomes when replacement is possible. The complete list is:

AAA	AAB	AAC	ABA	ABB	ABC	ACA	ACB	ACC
BAA	BAB	BAC	BBA	BBB	BBC	BCA	BCB	BCC
CAA	CAB	CAC	CBA	CBB	CBC	CCA	CCB	CCC

Example 3

A woman owns four motor vehicles: a sedan, station wagon, sports car and 4WD. She will be driving on both Saturday and Sunday next weekend.

a Draw tree diagrams to show all possible driving arrangements if she:

  i wants to drive a different vehicle each day

  ii has no restrictions on the vehicles she drives.

b Construct a table for each case in part a.

c For each of **i** and **ii** find the probability that the sports car is used at some time over the weekend.

Solutions

a **i** In this case the selection is without replacement.

Saturday	Sunday	
Sports car	Sedan	*
	4WD	*
	Station Wagon	*
Sedan	Sports car	*
	4WD	
	Station Wagon	
4WD	Sports car	*
	Sedan	
	Station wagon	
Station wagon	Sports car	*
	Sedan	
	4WD	

ii

Saturday	Sunday	
Sports car	Sports car	*
	Sedan	*
	4WD	*
	Station Wagon	*
Sedan	Sports car	*
	Sedan	
	4WD	
	Station Wagon	
4WD	Sports car	*
	Sedan	
	4WD	
	Station wagon	
Station wagon	Sports car	*
	Sedan	
	4WD	
	Station wagon	

b **i** The cells which are not possible are shaded with the dark colour.

	Sports car	Sedan	4WD	Station wagon
Sports car		*	*	*
Sedan	*			
4WD	*			
Station wagon	*			

ii

	Sports car	Sedan	4WD	Station wagon
Sports car	*	*	*	*
Sedan	*			
4WD	*			
Station wagon	*			

c **i** All the outcomes marked * include use of the sports car.

The probability $= \frac{6}{12} = \frac{1}{2}$.

ii The outcomes which include use of the sports car are again marked *.

The probability $= \frac{7}{16}$.

As we have already noted, as the number of stages increases or the number of outcomes at each stage increases, the number of possible multistage outcomes becomes very large. In such cases tree diagrams and tables may be impractical and instead we focus on the numbers of outcomes without describing each one.

Example 4

EG +S

a A card is selected at random from a normal pack of 52 cards, its details recorded and the card replaced. This is repeated four times so that five cards are selected. What is the total number of possible outcomes?

b A hand of five cards is dealt from a normal pack of 52 cards. What is the total number of possible outcomes?

c A normal coin is tossed 30 times. How many possible outcomes are there?

Solutions

a This is a case of selection with replacement. There are 52 possibilities for each drawing.
Number of outcomes $= 52 \times 52 \times 52 \times 52 \times 52 = 380\ 204\ 032$

b This is a case of selection without replacement. There will only be 51 cards left after the first card is dealt, 50 after the second, etc.
Number of outcomes $= 52 \times 51 \times 50 \times 49 \times 48 = 311\ 875\ 200$

c This is selection with replacement. There are 30 tosses with two possibilities for each one.
Number of outcomes $= 2 \times 2 \times 2 \times \ldots \times 2 = 2^{30} = 1\ 073\ 741\ 824$

Example 5

EG +S

a Two cards are dealt from a regular pack of cards. What is the probability that they are both spades?

b A hand of five cards is dealt. What is the probability that they are all red cards?

Solutions

a The number of possibilities for two cards $= 52 \times 51$

The number of possibilities for two spades $= 13 \times 12$

The probability of two spades $= \dfrac{13 \times 12}{52 \times 51} = \dfrac{1}{17}$

OR

The probability of the first card being a spade $= \dfrac{13}{52}$

With one spade gone there are 12 remaining in 51 cards.

The probability of the second card being a spade $= \dfrac{12}{51}$

The probability of two spades $= \dfrac{13}{52} \times \dfrac{12}{51} = \dfrac{1}{17}$

b The number of possibilities for five cards $= 52 \times 51 \times 50 \times 49 \times 48$

The number of possibilities for five red cards $= 26 \times 25 \times 24 \times 23 \times 22$

The probability of five red cards $= \dfrac{26 \times 25 \times 24 \times 23 \times 22}{52 \times 51 \times 50 \times 49 \times 48}$

$= \dfrac{253}{9996}$

Exercise 9.3

1 A Davis Cup tennis team has four playing members *A*, *B*, *C* and *D*, and the non-playing captain has to select two players to play the opening two singles matches. His choice is random.

- **a** Draw a tree diagram to illustrate this situation and list all possible choices.
- **b** What is the probability that:
  - **i** the two players are C and D?
  - **ii** B is one of the singles players?
  - **iii** A is not selected?

2 Lachie has five cards labelled 0, 1, 2, 3, and 4. He chooses two cards at random and places them on his desk forming a number.

- **a** Draw a tree diagram for this situation.
- **b** Calculate the probability that the number formed is:
  - **i** 24.
  - **ii** a two digit number (one that doesn't begin with 0).
  - **iii** greater than 24.
  - **iv** divisible by 5.
  - **v** divisible by 3.

3 Lachie is forming two digit numbers again (see question 1), this time with ten cards labelled 0, 1, 2, … , 9. The possible numbers are represented in this table.

Units digit

	0	1	2	3	4	5	6	7	8	9
0		01	02	03	04	05	06	07	08	09
1	10		12	13	14	15	16	17	18	19
2	20	21		23	24	25	26	27	28	29
3	30	31	32		34	35	36	37	38	39
4	40	41	42	43		45	46	47	48	49
5	50	51	52	53	54		56	57	58	59
6	60	61	62	63	64	65		67	68	69
7	70	71	72	73	74	75	76		78	79
8	80	81	82	83	84	85	86	87		89
9	90	91	92	93	94	95	96	97	98	

a How many numbers can Lachie form?
b How many two digit numbers can Lachie form?
c He forms one two digit number. What is the probability that
 i the number is divisible by 5?
 ii the number is greater than 50?
 iii the digits of the number have a sum of 10?
 iv the number is between (not including) 23 and 72?

4 Two girls are chosen at random from a group of five—Annabelle, Becky, Catriona, Jess and Mieka. The possible selections are as shown. Use the table to help you calculate the probability that:
a Becky is selected
b Jess and Catriona are chosen
c Mieka is not chosen but Annabelle is
d neither Becky nor Catriona is chosen
e either Mieka or Annabelle, or both, is not selected.

AB	AC	AJ	AM
BA	BC	BJ	BM
CA	CB	CJ	CM
JA	JB	JC	JM
MA	MB	MC	MJ

■ Consolidation

5 Emma has 48 books in her bookcase.
a In how many ways can she choose two books to read?
b If she has 15 novels, what is the probability that:
 i both books chosen are novels? **ii** neither book is a novel?
 iii at least one of the books is a novel?
c If the books are arranged with 24 on each of two shelves, what is the probability that the two books are taken from different shelves?

6 A bag contains 4 red, 7 blue, 2 yellow and 2 white discs of identical shape and size.

a Two discs are drawn without replacement. What is the probability that:

i both discs are blue? **ii** neither disc is yellow? **iii** one is red and the other white?

b Three discs are drawn without replacement. What is the probability that:

i all three are yellow? **ii** all three are red?

iii no disc is blue? **iv** at least one disc is white?

7 One hundred tickets are sold in a raffle.

a Sasha buys one ticket. What is her chance of winning:

i first prize? **ii** second prize?

b Alexis buys five tickets. What is her probability of winning:

i first prize? **ii** second prize? **iii** third prize?

8 Two letters are chosen randomly *without replacement* from the word MATHEMATICAL.

a What is the probability that:

i both letters are Ms? **ii** neither letter is an A? **iii** neither E nor T is chosen

b If three letters were chosen, what is the probability that:

i they are three As? **ii** there are no As chosen?

iii the letters are A, L, I in that order? **iv** the letters are A, L, I in any order?

c If three letters were chosen *with replacement*, find the probability of:

i no As chosen **ii** the letters A, L, I in that order

iii only two of the letters being M

9 A die has three green faces, two red faces and one blue. The die is rolled twice. What is the probability of the two results being red and green (order is not important)?

10 A sandwich bar has only six sandwiches left, three egg and lettuce, two ham and pineapple and one chicken and avocado. Three friends *A*, *B*, and *C* buy two sandwiches each with *A* choosing first, then *B* then *C*.

a *C* is hoping to buy the chicken sandwich. What is her chance of getting it?

b If *A* chooses two egg sandwiches, what is *C*'s chance of buying the chicken?

c What is the probability that each person buys an egg sandwich?

d What is the probability that *C* buys an egg sandwich and either another egg or a ham?

11 A choir of 51 members elects a president each year and all members have an equal chance of election.

a If a president can not be re-elected for a period of five years, what is the probability that:

i Eddie is elected in the first year of the choir's existence?

ii Eddie is elected for the first time in the third year?

iii Erin and Josh each serve as president in the first two years?

iv David and Samantha are each elected in the first three years?

b Calculate the probabilities for i to iv in part a if there are no restrictions on who is elected.

Further applications

12 A 'point' in tennis is begun by one player serving. If the player serves a 'fault' (the ball does not land in the correct part of the court), a second serve is allowed. If both serves are faults the player loses the point.

Jeff uses either a fast serve or a slow serve.

When he serves his fast serve the chance of a fault is $\frac{1}{2}$ but if the serve is 'good' his probability of winning the point is $\frac{4}{5}$.

When he serves his slow serve the chance of a fault is $\frac{1}{5}$ but if the serve is 'good' his probability of winning the point is only $\frac{1}{2}$.

What are the best tactics for Jeff to use: hard-hard, hard-soft, soft-hard or soft-soft?

13 A five card hand is dealt from a regulation pack of cards. Find the probability of:

a a flush (all five cards are of the same suit).
b an Ace high straight (Ace, King, Queen, Jack and ten of any suits).
c four Aces and any other card.
d three Kings and two Queens.

14 Three numbers are chosen at random from the numbers 1 to 100. Calculate the probability of:

a the first two both being divisible by 5
b the first number being divisible by 8 and the second being even
c the first number being even and the second divisible by eight
d the three numbers are divisible by 3, 6 and 12 in any order
e two of the numbers being greater than 50

9.4 Games and other applications

Many games involve an element of chance. The theory of probability allows us to analyse the games we play and assess our chances of winning. Most games involve more than one stage, so the techniques we have used in sections 2 and 3 of this chapter will often be of use. Since the numbers involved are usually extremely large tree diagrams, tables and lists are impractical, and instead it is necessary to use the product rule.

EG +S

Example 1

Shane plays Snakes and Ladders rolling a single die. If he has ten turns how many sets of scores are possible?

Solution

There are 6 possible outcomes at each roll of the die so in 10 rolls there are 6^{10} outcomes, or $6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6$. ($6^{10} = 60\ 466\ 176$!)

If Shane plays with two friends then after each player has had 10 rolls the number of possible outcomes is 6^{30}.

$6^{30} \doteqdot 221\ 000\ 000\ 000\ 000\ 000\ 000\ 000$!

In discussing music it is quite common for people who have not studied probability to think that we must soon exhaust the set of possible melodies. "After all," they argue, "millions of composers have written many millions of pieces of music over thousands of years. We must be running out of possibilities by now!"

Example 2

A composer writes a short refrain of 20 crotchet beats choosing from just 8 notes (do, re, me, … , do i.e. just the one octave). How many possibilities exist?

Solution

With 8 choices for each of the 20 crotchets the number is 8^{20}.
($8^{20} \doteqdot 1\,153\,000\,000\,000\,000\,000$)

Of course there are far more than 8 notes from which to choose, songs are much longer than 20 notes, and notes need not all be crotchets. Clearly the number of possible songs is immense. Our Sun would die and the Earth cease to exist in its present state before we had done more than barely begin using the possibilities.
Note: The sun is expected to live for about 10 000 000 000 years!

Example 3

The card game Bridge is played with four players and the whole pack of cards is dealt with each player receiving 13 cards. How many different deals of all 52 cards are possible?

Solution

This case is without replacement so the number is
$52 \times 51 \times 49 \times \ldots \times 3 \times 2 \times 1 = 52!$ (this is called factorial 52).
$52! \doteqdot 80\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000$
If each game took 10 minutes, 10 million games were in progress at any moment around the world and each game resulted in a different deal, it would take more than
1 500 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 years to exhaust all the possibilities.

Example 4

Ngaire and Mairaid play a game in which the winner is the first to toss a head with an unbiased coin. Ngaire tosses first, Mairaid has the second and third tosses. If the first three tosses are all tails the game is drawn. What is the probability that:

a Ngaire wins?
b Mairaid wins?
c the game is drawn?
d Do you feel that this game is fair to both players?

Solutions

A tree diagram for this game is shown.

First Toss	Second Toss	Third Toss	Result
Head			Ngaire wins
Tail	Head		Mairaid wins
	Tail	Head	Mairaid wins
		Tail	Draw

a Ngaire can only win at the first toss. $P(\text{Ngaire}) = \frac{1}{2}$

b Mairaid can win at the second toss $\text{Probability} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

or at the third toss. $\text{Probability} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

$P(\text{Mairaid}) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

c Three successive tails yields a draw $P(\text{draw}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

d The two players do not have equal probabilities of winning so the game does not seem fair.

Games are considered fair if the outcomes are equal for all players. Where there are prizes the amount of the prize must be taken into account and we use a measure known as the **expected value** or **expectation**.

The expected value is the *average* return one would expect if the game was played many times.

Expected value = value of prize × probability of prize

If the game is fair then the expected value will be the same as the amount invested.

Example 5

EG +S

Gladys pays $1 to play a game. She rolls a normal die and wins $6 if she rolls a 1. She receives no prize for any other score. What is her expected return? Is the game fair?

Solution

$$\begin{aligned}\text{Expected value} &= \text{value of prize} \times \text{probability of prize}\\ &= \$6 \times \frac{1}{6}.\\ &= \$1\end{aligned}$$

Gladys has paid $1 to play and, *on average*, she expects a return of $1. Her expected return is equal to her outlay. The game is fair.

Where there are more than just one prize, the expected return is the sum of all the separate expected returns for the various prizes.

Example 6

EG +S

Letitia pays \$5 to play a game. She rolls a normal die and wins \$15 if she rolls a 6, and \$9 for a 1. She receives no prize for any other score. What is her expected return? Is this a fair game?

Solution

$$\text{Expected return} = \$15 \times \frac{1}{6} + \$9 \times \frac{1}{6}.$$
$$= \$2.50 + \$1.50$$
$$= \$4$$

Letitia's expected return, \$4, is not equal to her outlay, \$5, so the game is not fair.

Exercise 9.4

1 A hand of three cards is dealt from a pack of 52 cards.
- **a** How many different hands are possible?
- **b** What is the probability that:
  - **i** the first card dealt is the Ace of spades?
  - **ii** all cards are hearts?

2 Ben pays \$10 to play a game in which he rolls a regular die twice. If he rolls the same number on both rolls he receives \$50, otherwise he receives nothing.
- **a** Calculate Ben's chance of winning the \$50.
- **b** What is his expected return for this game?
- **c** Do you consider this a fair game?

3 You are offered the opportunity to play any of a number of games. In each case you pay \$10 to play and roll a pair of dice.
Game 1 If the sum of the numbers is 8, 9, 10, 11 or 12 you receive \$20.
Game 2 If the sum of the numbers is 2, 3, 4, 10, 11 or 12 you receive \$20.
Game 3 If the sum of the numbers is 2, 4, 6, 8, 10 or 12 you receive \$20.
- **a** Would you choose to play? Explain why or why not.
- **b** Which game gives you the best probability of winning?
- **c** Are any of the games fair?

4 In small groups, or as directed by your teacher, consider how probability may have been influential in determining the following restrictions which apply in our community.
- **a** Motor vehicle insurance premiums are higher and/or the amount of coverage is less where the vehicle is driven by younger drivers (under 26).
- **b** L and P plate drivers have more restrictions placed on their driving, for example lower speed limits.
- **c** It is illegal to sell alcoholic beverages to people less than 18 years of age.
- **d** Drivers over 85 years of age have a driving test every year.
- **e** Some elderly drivers have restricted licenses, for example they may only be able to drive within 50 kilometres of their home.

f Organisers of games of chance, for example lotteries and marketing schemes with prizes are required to provide full details of the number of tickets, number and value of prizes and other information to anyone entering the game or competition.
g Some professional sports have "salary caps" for the teams in major competitions.
h Some members of our community have been declared bankrupt.
i Some people who have broken the law are in gaol.
j Most insurance companies will not issue contents policies for residents of nursing homes or hostels.

Consolidation

5 A hand of five cards is dealt from a pack of 52 cards.
a How many different hands are possible?
b What is the probability that:
i the first card dealt is an Ace? **ii** all cards are hearts?

6 You are offered the opportunity to play any of a number of games. In each case you pay $10 to play and roll a pair of dice.
Game 1 If the sum of the numbers is 7 you receive $50.
Game 2 If the sum of the numbers is 6 or 8 you receive $30.
Game 3 If the sum of the numbers is 2, 3, 4 or 5 you receive $30.
a Would you choose to play? Explain why or why not.
b Which game gives you the best probability of winning?
c Are any of the games fair?

7 Jae and Jemima play a game in which Jae has the first roll of a regular die. If the number rolled is even she wins. If Jae does not win Jemima rolls the die. If Jemima rolls a number greater than 2 she wins, otherwise the game is drawn. Calculate each player's probability of winning and decide whether the game is fair.

8 Blake and Cynthia play a game in which Blake has the first roll of a regular die. If the number rolled is divisible by 3 he wins. If Blake does not win Cynthia rolls the die. If Cynthia rolls an even number she wins, otherwise the game is drawn. Calculate each player's probability of winning and decide whether the game is fair.

9 A musician composes a piece of music consisting of just six crotchet beats. Each note in the composition is a crotchet and only eight different pitches (do, re, … , do) can be used for any particular note.
a How many possible pieces can she create?
b What is the probability that:
i the composition begins with so?
ii the first three notes are do, so, so?
iii neither re nor fa are used at any point in the piece?
iv the last two notes are te and do (upper)?

10 A group of cricketers attend a restaurant for their team's Christmas party.

a A, B, C, D, E and F sit at a circular table for six. What is the probability that:

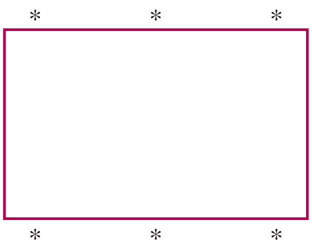
i C and E sit in adjacent places?

ii B and F are separated by just one other person?

b G, H, I, J, K and L sit at a rectangular table for six (see diagram). What is the probability that:

i I and J sit on opposite sides of the table?

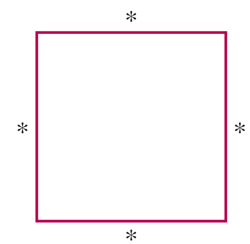
ii G and L sit opposite each other?



c M, N, O and P sit at a square table for four (see diagram). What is the probability that:

i M and N sit opposite each other?

ii P sits to M's left?



11 Parents of children at a school are offered an insurance policy to cover their child against certain specified accidents in the school playground over a one-year period. The accidents to be covered, their historical probabilities of occurrence, and the benefits provided are shown in the table.

Nature of injury	Benefit ($)	Probability	Expected return
Broken limb	1 500	0.01	
Loss of eyesight	100 000	0.0005	
Loss of hearing	50 000	0.0002	
Accidental death	10 000	0.0001	

Copy and complete the table. If the annual premium is $150, do you consider the policy offers fair value for money? Discuss.

Further applications

12 With its catalogue a mail-order company includes a ticket which the recipient is told has already won a prize. The recipient is advised that to receive their prize they need to order an item from the catalogue. The prize will be one of the following:

- $10 000 cash
- a TV valued at $900
- a DVD player value at $120
- your choice of a silver bangle or a gold plated tiepin.

In a small print section labeled “Conditions” the following information is included.

i 1 500 000 catalogues have been mailed

ii the number of prizes available and their values are:

Prize	Money	TV	DVD	Jewellery
Number	1	1	1	60 000
Value	$10 000	$900	$120	$10

a How much is the company spending in prizes?

b What proportion of recipients would you expect to place an order? Give reasons for your answer.

c What is your probability of receiving the money, the TV, the DVD or the piece of jewellery?

d If you place an order what is your expected prize value?

e How is the company able to justify this expenditure?

f Do you feel that this type of promotion is fair to consumers?

13 Ann is playing Monopoly and rolling two dice at each turn. She is very keen to land on a square that is five spaces ahead of her present position. What is the chance that she will land on the square on her present trip around the board?

14 Research the use of probability in our community. You could consider major categories such as government, major public companies, small business, household finance, the insurance industry, sport and leisure and health.
You should consider questions such as:

- In what aspects of our lives is probability used?
- How often is it used?
- How important is its use?
- How would we be affected if there was no mathematical approach to probability?

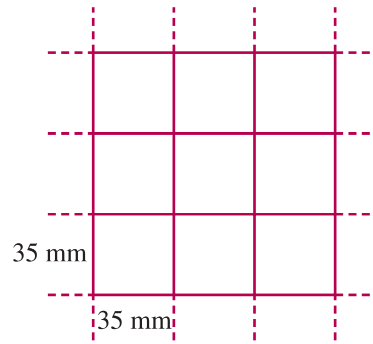
One starting point for your research could be to review a daily newspaper and assess how many items would have some relationship to probability.

15 Ngaire and Mairaid play a game in which the first to toss a head with an unbiased coin wins. Ngaire has first toss and Mairaid has all the subsequent tosses. Is this a fair game?

16 Jae and Jemima play a game in which Jae has the first roll of a regular die. If the number rolled is even she wins. If Jae does not win Jemima rolls the die. If Jemima rolls a number greater than 2 she wins, otherwise the game is repeated. This sequence of steps is repeated until there is a winner. What is each player’s chance of winning? Is the game fair?

TRY THIS 'Fair' games

In one of the games at a fair players throw 20-cent coins (one at a time) onto a large grid like the one shown here. If the coin lands touching a line, the player loses the coin. However, if the coin lands completely inside a square, the player wins the amount of money stated inside that square.



A 20-cent coin has a diameter of 29 mm.

(a) What is the probability (on each throw) of the player winning?

(b) How large would the grid squares have to be for the probability of winning to equal $\frac{1}{2}$?

Focus on working mathematically

Focus on Working Mathematically

Playing cricket with dice

Introduction

In this activity we are going to strengthen your understanding of probability by simulating a game of cricket. There are 11 players on a team. The whole team is out when 10 players are out. The other team then bats.

You will need a partner and two dice, preferably of different colours. The idea is to roll the dice and add the numbers on the uppermost faces. This will be the number of runs scored for that throw. However if you roll a double, the batsman is out. Each player rolls the dice 6 times for an over.

Copy and complete the table on page 370 in your book. Roll the dice until the ten batsmen on the first team are out. Repeat the experiment for the second team. The winner is the team who scores the most runs. Each player can name and own a team to make it fun. We have filled in possible runs for the first few batsmen and used the symbol W when a double was thrown to show when they got out.

In your table record the total number of throws to dismiss the team and the total runs scored.

Table 1: Number of throws and runs scored in an innings

Batsmen	Scorebook	No. of throws	Score
1	3, 6, 9, 4, W	5	22
2	7, 7, 11, 5, 10, W	6	40
3	W	1	0
4			
5			
6			
7			
8			
9			
10			
11			
Totals			

Focus question

How many overs would you expect to have to make to complete a game?

Learning activities

1 Copy and complete the possible outcomes of throwing the dice and adding the numbers of the uppermost faces.

Table 2: Sum of the numbers on the dice

		Die 1					
	+	**1**	**2**	**3**	**4**	**5**	**6**
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
Die 2	**4**						
	5						
	6						

2 Use your table to show that the probability of the outcome "throwing a double", that is of getting a batsman out, is $\frac{1}{6}$.

3 Write down the probability of the event "*not* throwing a double", that is of the batsman scoring runs, on any throw of the dice. What is this event called?

4 For each innings, work out

$$\frac{\text{the number of doubles}}{\text{the total number of throws}}$$

and use a calculator to express this fraction as a decimal (the number of doubles will be 10, as the innings closes when 10 batsmen are out). What do you notice?

5 Make an estimate of the number of throws you would have to make to get a whole team out. If the whole class carries out the experiment in groups, you could average the results. Answer the focus question.

6 Is each throw a dependent or independent event? Why?

7 A person throws the dice 10 times in a row without throwing a double. Her friend expects that on the next throw she will be more likely to get a double. Is this true?

Challenge activities

1 Repeat the game but increase the odds of getting out. This time the batsman is out if the sum of the numbers on the die is 7 or 9. Draw up a new innings table and record your results. What did you find?

2 What is the theoretical probability of getting out? How does this compare with what you found from the experiment? Answer the focus question again.

3 You can make up your own rules and play the game as much as you wish. You can use three dice instead of two to make it more complex. Check with your teacher when you report the outcomes.

4 Explore how the game could be played on a computer using a spreadsheet. There are notes for teachers on the *Mathscape10/10 extension School CD-ROM* to help you.

5 As a real brainteaser, see if you can work out the expected number of runs in an innings. Make an estimate from the class results first and check with your teacher.

Let's communicate

In small groups first, discuss the difference between experimental probability and theoretical probability. That is, what you find in carrying out an experiment such as throwing dice, and what could be expected from theory. How are they linked? Then discuss as a whole class with your teacher.

Reflecting

Reflect on the connections between what you have learned about probability and the topics of fractions and decimals. Could a probability be expressed as a percentage also? Are there any links with algebra? Geometry? Think it over and discuss with your teacher.

Language link with Macquarie

MACQUARIE

1. *In your own words describe the difference between a* ***dependent*** *and an* ***independent*** *event.*
2. *Give an example of a* ***compound*** *event.*
3. *Give an example of* ***sampling with replacement****.*
4. *Give an example of* ***sampling without replacement****.*
5. *Read the* Macquarie Learners Dictionary *entry for the word* ***chance****:*

chance *noun* **1.** (*uncount*) the absence of any known reason for something happening: *They met by chance.* **2.** risk: *to take a chance.* **3.** opportunity: *Now is your chance to tell him.*

Are any of these meanings the ***mathematical*** *meaning of the word? Explain.*

CHAPTER **REVIEW**

1 A card is drawn from a standard deck of 52 cards.

J = a black Jack is drawn
K = a court card is drawn
L = the card drawn is red
M = the card is either red or a court card but not both
N = the card is neither black nor an Ace

a Name a pair of mutually exclusive events.
b Describe $\tilde{J}$, the complement of J.
c Calculate the probabilities:
  i $P(L)$
  ii $P(M)$
  iii $P(N)$

2 A pair of dice are rolled and the numbers added. What is the probability of:
a 6
b an even number
c a number less than 7

3 A roulette wheel lies horizontally and spins around a vertical axis. The wheel has 38 slots around its circumference and while the wheel is spinning a ball is rolled in the opposite direction around the rim finally falling into one of the 38 slots. All slots are equally likely.

The slots are numbered 1 to 36, 0 and 00. Half of the 1 to 36 slots are red and the other half black. The 0 and 00 slots are green.

Find the probability of:
a red
b green
c 32
d an odd number
e a square number or a green number

4 The table on the next page shows the number of vehicles insured and the number of claims made on an insurance company during one year.

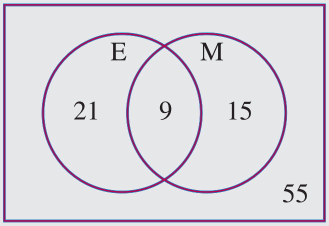
Type of vehicle	Sedan	Station wagon	Sports car	4WD	Utility
Number insured	120 000	28 000	8 000	49 000	33 000
Number of claims	2 439	603	302	2 175	2 317

a What is the probability that a vehicle chosen at random has been the subject of a claim if it is:
  i a utility?
  ii a sedan?
  iii a 4WD or a sports car?

b Which category of vehicle has the worst claim rate?

c What is the probability of a claim for all the types of vehicles taken together?

5 The Venn diagram shows the number of students electing to study extension English and/or extension Maths in a year 11 group.



E = students taking extension English
M = students taking extension Mathematics

a What is the total number of students?

b What is the probability that a student chosen at random
  i takes extension Mathematics?
  ii does not study extension English?

c What is the chance that a student who is attempting extension English also does extension Maths?

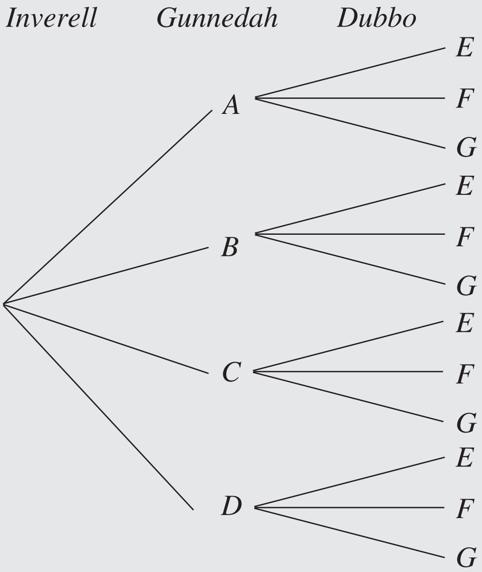
d What is the probability that an extension Maths student is also enrolled in extension English?

6 In a class of 30 Year 12 students 25 are licensed drivers. 15 of the 30 students travel to school by public transport and 11 of them have licences. Draw a Venn diagram to illustrate this information.

a What is the probability that a student chosen at random from this class:
  i has yet to get a drivers's licence?
  ii has a licence but still travels by public transport?
  iii has no licence but does not use public transport?

b What is the probability that a student who travels by public transport has a licence?

7 There are four reasonable routes from Inverell to Gunnedah by car, and three from Gunnedah to Dubbo. This information is illustrated on the tree diagram.



If Virginia makes random choices in deciding how to drive from Inverell to Dubbo

a by how many different routes can she travel?

b What is the probability of:
 i using road *C*?
 ii using roads *A* and *G*?
 iii using road *E*?
 iv using either road *B* or road *F* (or both)?
 v not using either road *D* or road *G*?

8 A woman chooses to live in one of four suburbs: Ashfield, Belmore, Cronulla or Denistone. She chooses to travel to work by bus or train or car.
a Draw a tree diagram to illustrate all the possible outcomes for her two choices.
b Assuming that both her choices are random what is the probability that:
 i she lives in Cronulla?
 ii she lives in Belmore and travels by bus?
 iii she travels to work by train?
 iv she does not live in Ashfield or Belmore and doesn't travel by car?

9 Estelle tosses an unbiassed coin and notes the result. She then rolls a die and records whether the number is even or odd. From a bag containing 3 red, 3 black and 3 white discs she draws a disc at random.
a Draw a tree diagram to illustrate all the possible outcomes for Estelle's three-stage experiment.
b What is the probability that Estelle will:
 i roll an even number?
 ii toss a head and draw a red disc?
 iii toss a tail, roll an odd number and draw white?
 iv roll an even number but not draw a black disc?
 v either roll a head or draw red but *not both*?

10 Marette rolls two dice, one of which is white and the other green, and *multiplies* the two numbers that result.
a Copy and complete this table.

Green Die (columns); White die (rows)

	1	2	3	4	5	6
1	1	2	3			
2	2	4				
3	3	6				
4						
5						
6						

b What is the probability that Marette's product is:
 i 5?
 ii 6?
 iii greater than 12?
 iv less than 20?
 v either 12 or greater than 16?

11 A wheel with sections coloured red, blue and green is spun. A ball is then randomly drawn from a bag containing 20 balls which may be coloured white, yellow or black. This two-stage experiment is repeated for 100 trials and the results are shown in the table below.

Wheel (columns); Bag (rows)

	Red	Blue	Green
White	30	16	16
Yellow	14	6	9
Black	4	3	2

a Using the experimental results in the table estimate the probabilities of spinning:
 i red **ii** blue **iii** green
b Estimate the probabilities of drawing:
 i white **ii** yellow **iii** black

c Given that there are exactly 20 balls in the bag, estimate the number of each colour.

d Using your answer to part **a**, what size sectors do you think red, blue and green would have on the wheel (estimate the angles at the centre)?

12 Your teacher comes to class for a lesson on probability and produces a die which she says is unbiassed. One of the students rolls the die and scores 5. A second student also rolls 5 and so does a third.

a Would you still believe that the die was unbiassed?

b How many consecutive 5s would be required to convince you that the die was not unbiassed?

13 A roulette wheel has 38 slots, all equally likely, and 18 of these are black.

a What is the probability of:

i black on one spin of the wheel?

ii black on two consecutive spins?

iii black on three consecutive spins?

iv black on ten consecutive spins?

b At a particular Monte Carlo casino the black came up in 26 consecutive spins on 18 August 1913.

i What was the probability of this occurrence?

ii During this amazing run of black some people bet heavily against black (on red) assuming that red had to 'catch up'. Were they wise? Explain.

14 Four friends, W, X, Y, Z, sing in a barbershop quartet. They choose one of the four to be leader and another person to be publicity agent.

a Draw a tree diagram to illustrate this situation and list all possible outcomes.

b If the two positions are filled by a random selection, what is the probability that:

i the two chosen are W and Y (without taking the particular positon into account)?

ii Z fills one of the positions?

iii W is the president and Y the publicity agent?

iv X is not chosen for either position?

15 The coach of a school netball team with 7 members randomly selects one member to be captain and another to be vice-captain.

a Construct a table to illustrate the possible outcomes of the coach's choices. Use labels A, B, C, D, E, F and G for the team members.

b Using your table calculate the probability that:

i A is chosen for one of the positions

ii B and E are both chosen

iii F is not chosen for either position

iv C becomes captain and D vice-captain

v G is not captain but F is vice-captain

vi either C or D is captain and either A or E is vice-captain

16 A student has to choose two elective subjects from five that are available: Computing, Art, Music, Japanese and Personal Health.

a List all possible choices the student could make (order of choice is not important).

b If all possible choices are equally likely, find the probability that the student:

i chooses Art

CHAPTER REVIEW

ii chooses Japanese but not Music
iii does not choose Computing or Japanese, but does choose Personal Health or Music

17 Katie has 20 books arranged on one shelf above her desk. She chooses two books, at random, to take away on a holiday. What is the chance that the two books were alongside each other?

18 Two letters are chosen at random from the word PROBABILITY. What is the chance that:

a they are *P* and *T* in that order?
b they are *P* and *T* in either order?
c they are both *B*s?
d they are not the same?
e one is a *B* or an *I* and the other is not?

19 After a federal election in Australia the new Prime Minister has to choose a Deputy Prime Minister and a Treasurer (two very important portfolios) from his ten most senior colleagues (*A, B, C, D, E, F, G, H, I* and *J*). If his choice is random what is the chance that:

a he chooses *A* for Deputy Prime Minister and *B* for Treasurer?
b he chooses *D* and *E* for the two posts, but defers deciding which position each will have?
c neither *F* nor *C* is selected?
d *G* receives a post but *I* does not?
e *J* becomes Treasurer and the Deputy Prime Minister is *F*, *C* or *B*?

20 **a** A poker hand consists of five cards dealt from a regular pack of 52 cards. How many such hands are possible?
b Our alphabet contains 26 letters. Assuming that any collection of letters forms a word (for example zcptl), how many 6 letter words can be formed?
c A boy watches TV for 5 hours. Assuming that all programs last $\frac{1}{2}$ hour and there are five channels available to him, how many different sequences of programs are possible for him?

21 Jemima and Sam play a game in which Sam has first turn. Sam draws a card from a pack of 52. If the card is a spade, Sam wins. If not, it becomes Jemima's turn. Jemima rolls a die and wins if the number rolled is divisible by 3. Calculate each player's chance of winning. Is this a fair game?

22 **a** Gomez pays \$10 to play a game in which he rolls a die and wins if he rolls a 6. The prize is \$50. What is Gomez's expected return? Is this a fair game?
b Gomez again pays \$10 to play the game but there are two prizes: \$30 for a 6, and \$20 for a 5. What is Gomez's expected return? Is this a fair game?
c Gomez plays again with a 'prize' for every number: 1—\$2, 2—\$4, 3—\$7, 4—\$12, 5—\$15, 6—\$20.
i What is the probability that Gomez rolls a winning score?
ii What is his expected return in this game?
iii Is this a fair game?

23 David and Samantha play a game in which David has first turn and rolls a regular die. If the die shows 1 or 2, David wins. If the die shows 3 to 6 the game continues with Samantha's turn. Samantha tosses an unbiassed coin and wins if the result is a head. If she tosses a tail it becomes David's turn again and the first two steps are repeated. The game continues until one of the players achieves a win. Is this a fair game?

CHAPTER REVIEW

10 Further trigonometry

This chapter at a glance

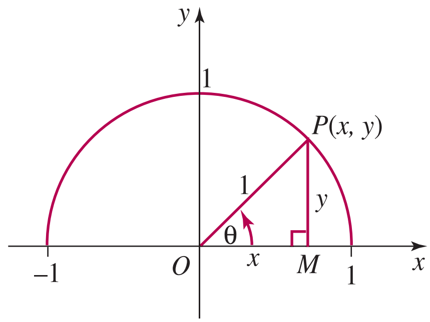
Stage 5.1/5.2/5.3

After completing this chapter, you should be able to:

- find values for trigonometric expressions that contain obtuse angles
- use the second quadrant identities to find equivalent trigonometric expressions containing acute angles for those that contain obtuse angles
- solve trigonometric equations for θ, where θ lies between $0°$ and $180°$
- use the Sine Rule to find the lengths of sides in non-right-angled triangles
- use the Sine Rule to find the sizes of angles in non-right-angled triangles
- solve practical problems that involve the use of the Sine Rule
- use the Cosine Rule to find the lengths of sides in non-right-angled triangles
- use the Cosine Rule to find the sizes of angles in non-right-angled triangles
- solve practical problems that involve the use of the Cosine Rule
- find the area of triangles by trigonometry
- solve practical problems that involve the area of a triangle by trigonometry.

10.1 Redefining the trigonometric ratios

Let $P(x, y)$ be any point in the first quadrant on the unit circle with centre O. Construct PM such that PM is perpendicular to the positive x-axis. The radius OP makes an angle θ with the x-axis.



Now, $\cos\theta = \dfrac{OM}{OP}$ and $\sin\theta = \dfrac{PM}{OP}$

$$\begin{aligned}\cos\theta &= \frac{OM}{OP} & \sin\theta &= \frac{PM}{OP}\\ &= \frac{x}{1} & &= \frac{y}{1}\\ &= x & &= y\end{aligned}$$

That is, the x co-ordinate of P can also be written as $\cos\theta$ and the y co-ordinate of P can be written as $\sin\theta$. Therefore the point $P(x, y)$ can also be written as $P(\cos\theta, \sin\theta)$.

Further, we know that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\therefore \tan\theta = \frac{y}{x}$$

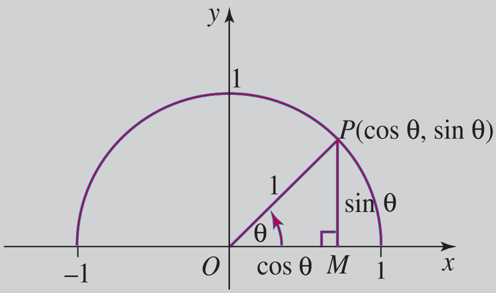
That is, $\tan\theta = \dfrac{\text{the } y \text{ co-ordinate of } P}{\text{the } x \text{ co-ordinate of } P}$.

If P is a point on the unit circle, centre O, and the radius OP makes an angle θ with the positive x-axis, then:

$\cos\theta$ = the x co-ordinate of P,

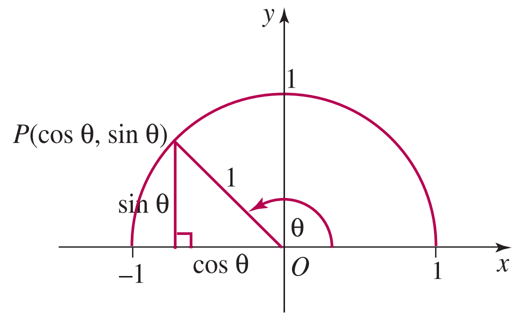
$\sin\theta$ = the y co-ordinate of P,

$\tan\theta = \dfrac{\text{the } y \text{ co-ordinate of } P}{\text{the } x \text{ co-ordinate of } P}$.



The trigonometric ratios can now be defined in terms of the anticlockwise rotation of point P on a unit circle with centre O, through an angle θ, measured from the positive x-axis. If the point P moves around the circle so as to be in the second quadrant, then the angle of rotation, θ, will be obtuse. The above definitions obtained in the first quadrant still holds for obtuse angles.

We have shown that cos $\theta = x$. However, in the second quadrant, the x-value of the point P will be negative. Therefore, the value of cos θ will be negative if θ is an obtuse angle.



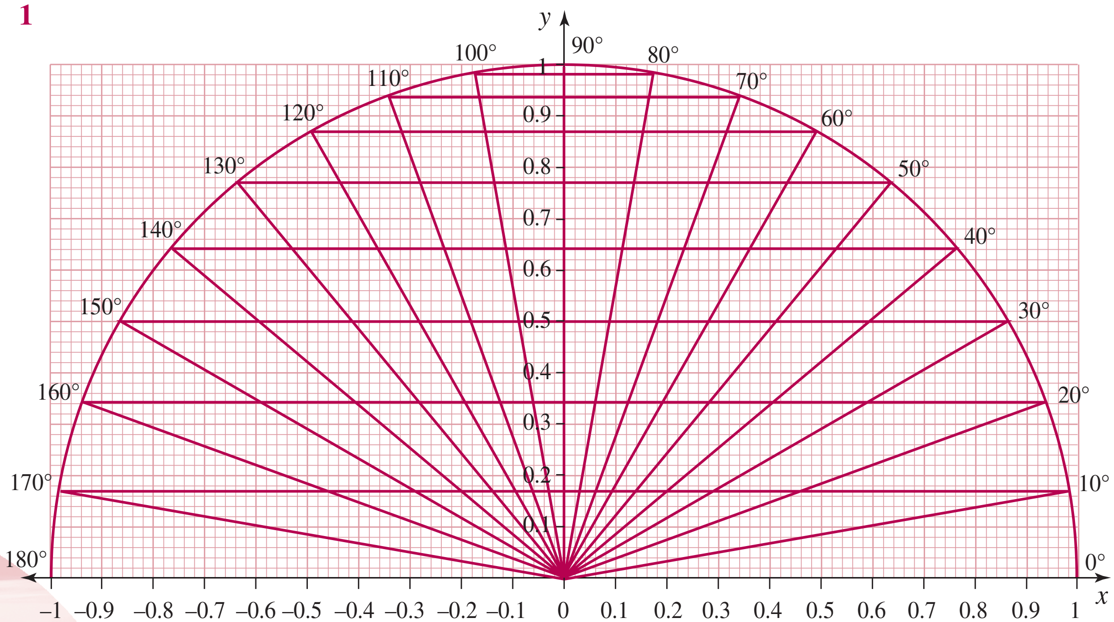
We have also shown that sin $\theta = y$. Now, in the second quadrant, the y-value of the point P will be positive. Therefore, the value of sin θ will be positive if θ is an obtuse angle.

We also know that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. Now, in the second quadrant, the value of sin θ is positive and the value of cos θ is negative. Therefore, the value of tan θ will be negative if θ is an obtuse angle.

The following exercise is an investigation activity to further explore and verify these results.

Exercise 10.1

1



Use the graph above to find values for the following trigonometric expressions. Answer correct to 2 decimal places.

a sin 40° and sin 140°

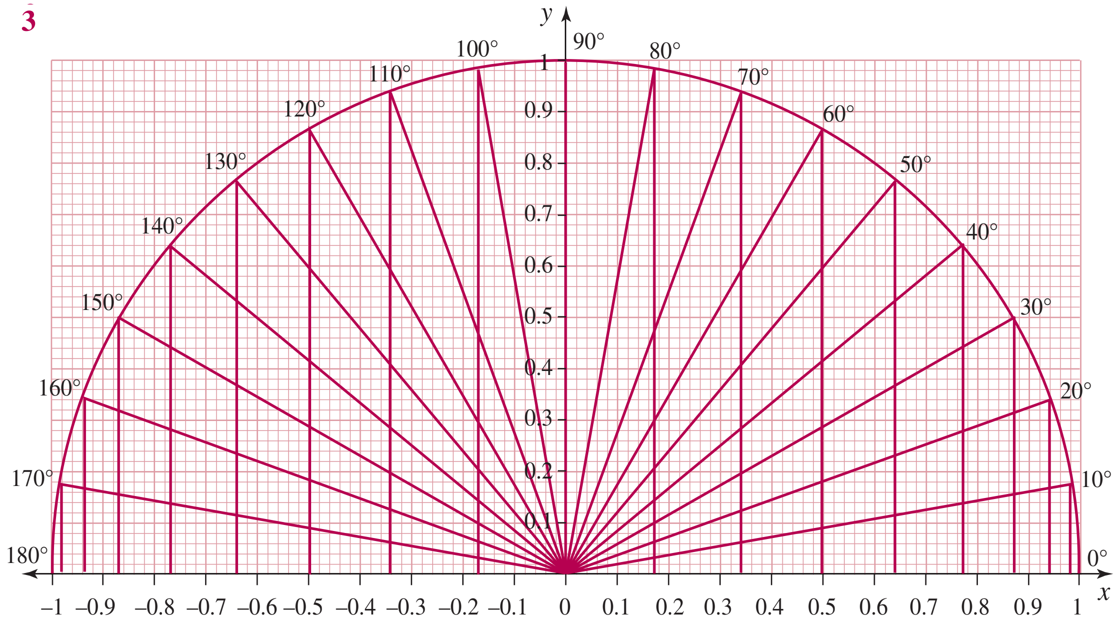
b sin 20° and sin 160°

c sin 60° and sin 120°

d sin 80° and sin 100°

2 Look at your answers to question 1. Is there a relationship between sin θ and sin $(180° - \theta)$, where θ is an acute angle? If so, what is it?

3



Use the graph above to find values for the following trigonometric expressions. Answer correct to 2 decimal places.

a cos 50° and cos 130° **b** cos 10° and cos 170°
c cos 40° and cos 140° **d** cos 80° and cos 100°

4 Look at your answers to question 3. Is there a relationship between cos θ and cos (180° – θ), where θ is an acute angle? If so, what is it?

■ Consolidation

5 Use your result from question 2 to write down a trigonometric expression that would have the same value as each of the following.

a sin 30° **b** sin 50° **c** sin 135° **d** sin 105°
e sin 73° **f** sin 112° **g** sin 99° **h** sin 167°

6 Use your result from question 4 to write down the trigonometric expression that would have the opposite value to each of the following.

a cos 20° **b** cos 70° **c** cos 120° **d** cos 155°
e cos 85° **f** cos 102° **g** cos 41° **h** cos 137°

7 Use the graphs in questions 1 and 3, the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and your calculator to find the value of tan 50°.

8 Use the graph in question 1 to find two possible values of θ, correct to the nearest degree, if:

a $\sin\theta = 0.5$ **b** $\sin\theta = 0.8$ **c** $\sin\theta = 0.2$
d $\sin\theta = 0.56$ **e** $\sin\theta = 0.14$ **f** $\sin\theta = 0.74$
g $\sin\theta = 0.36$ **h** $\sin\theta = 0.82$ **i** $\sin\theta = 0.48$

9 Use the graph in question 3 to find the value of θ between 0° and 180° if:

a $\cos\theta = 0.5$	**b** $\cos\theta = 0.34$	**c** $\cos\theta = -0.94$
d $\cos\theta = -0.77$	**e** $\cos\theta = 0.42$	**f** $\cos\theta = -0.28$
g $\cos\theta = 0.85$	**h** $\cos\theta = 0.25$	**i** $\cos\theta = -0.54$

Further applications

10 Find as a decimal, correct to 2 decimal places where necessary, the value of $\sin\theta$, where $0° < \theta < 180°$ if:

a $\cos\theta = 0.6$	**b** $\cos\theta = 0.34$	**c** $\cos\theta = 0.28$	**d** $\cos\theta = 0.84$
e $\cos\theta = -0.9$	**f** $\cos\theta = -0.64$	**g** $\cos\theta = -0.38$	**h** $\cos\theta = -0.22$

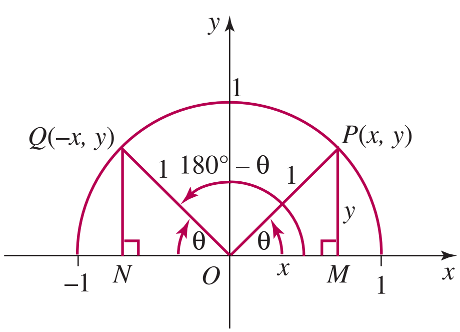
10.2 Trigonometric ratios of obtuse angles

In the previous exercise, we found by inspection that there are two angles between 0° and 180° that have the same sine. These angles are supplements of one another. For example, $\sin 120° = \sin 60°$ and $\sin 100° = \sin 80°$.

We also found that there are two angles between 0° and 180° which have cosines that are equal in value but have opposite signs. These angles are also supplements of one another. For example, $\cos 120° = -\cos 60°$ and $\cos 100° = -\cos 80°$.

We will now prove these results in a more formal manner.

Construct the points $P(x, y)$ and $Q(-x, y)$ on the unit circle centre O, where P lies in the first quadrant and Q lies in the second quadrant. The radii OP and OQ are equally inclined to the positive and negative x-axes respectively. Now, the angle of rotation from the positive x-axis to OP is θ. In the notes on the previous exercise, we proved that the co-ordinates of the point P could be redefined as $(\cos\theta, \sin\theta)$.



The angle of rotation from the positive x-axis to OQ is $180° - \theta$. Therefore, we can redefine the co-ordinates of the point Q as $(\cos(180° - \theta), \sin(180° - \theta))$.

The co-ordinates of the point P can be written as (x, y) or $(\cos\theta, \sin\theta)$. The co-ordinates of the point Q can be written as $(-x, y)$ or $(\cos(180° - \theta), \sin(180° - \theta))$. By equating the respective x- and y-values of Q, we can see that:

$$\sin(180° - \theta) = y = \sin\theta \qquad \cos(180° - \theta) = -x = -\cos\theta \qquad \tan(180° - \theta) = \frac{y}{-x} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta$$

If θ is an acute angle, then:

- **$\sin(180° - \theta) = \sin\theta$**
- **$\cos(180° - \theta) = -\cos\theta$**
- **$\tan(180° - \theta) = -\tan\theta$**

That is, the sine of an obtuse angle is positive, while the cosine and the tangent of an obtuse angle are negative.

Example 1

Express each of the following in terms of θ, where $0° < \theta < 90°$, then evaluate correct to 2 decimal places.

a $\sin 130°$ **b** $\cos 165°$ **c** $\tan 107°20'$

Solutions

a
$$\begin{aligned}\sin 130° &= \sin(180° - 50°)\\ &= \sin 50° \text{ (since } \sin(180° - \theta) = \sin\theta)\\ &= 0.77 \text{ (2 decimal places)}\end{aligned}$$

b
$$\begin{aligned}\cos 165° &= \cos(180° - 15°)\\ &= -\cos 15° \text{ (since } \cos(180° - \theta) = -\cos\theta)\\ &= -0.97 \text{ (2 decimal places)}\end{aligned}$$

c
$$\begin{aligned}\tan 107°20' &= \tan(180° - 72°40')\\ &= -\tan 72°40' \text{ (since } \tan(180° - \theta) = -\tan\theta)\\ &= -3.20\end{aligned}$$

Example 2

If $\sin\theta = 0.2924$, find two possible angles θ, where $0° < \theta < 180°$. Answer correct to the nearest degree.

Solution

$$\begin{aligned}\sin\theta &= 0.2924\\ \theta\text{ (acute)} &= \sin^{-1} 0.2924\\ &= 17° \text{ (to the nearest degree)}\\ \therefore\ \theta &= 17°, 180° - 17°\\ &= 17°, 163°\end{aligned}$$

Example 3

For each of the following, find θ, where $0° < \theta < 180°$. Answer correct to the nearest degree.

a $\cos\theta = -0.9511$ **b** $\tan\theta = -2.3559$

Solutions

a $\cos\theta$ is negative,

$\therefore$ θ is an obtuse angle.

$$\cos\theta = -0.9511$$
$$\theta\ (\text{acute}) = \cos^{-1} 0.9511$$
$$= 18°\ (\text{to the nearest degree})$$
$$\therefore \theta = 180° - 18°$$
$$= 162°$$

b $\tan\theta$ is negative,

$\therefore$ θ is an obtuse angle.

$$\tan\theta = -2.3559$$
$$\theta\ (\text{acute}) = \tan^{-1} 2.3559$$
$$= 67°\ (\text{to the nearest degree})$$
$$\therefore \theta = 180° - 67°$$
$$= 113°$$

Exercise 10.2

1 State whether each of these trigonometric ratios is positive or negative. Use a calculator to verify your answers.

a sin 70° **b** cos 25° **c** tan 110° **d** cos 104°
e tan 18° **f** sin 156° **g** tan 163° **h** sin 108°
i sin 15°20′ **j** tan 81°17′ **k** cos 142°35′ **l** tan 119°26′
m cos 54°6′ **n** sin 91°43′ **o** cos 174°30′ **p** sin 122°55′

2 State whether the angle θ is acute or obtuse, where $0° < \theta < 180°$, if:

a $\sin\theta > 0$ and $\tan\theta > 0$ **b** $\sin\theta > 0$ and $\cos\theta < 0$
c $\tan\theta < 0$ and $\cos\theta < 0$ **d** $\cos\theta > 0$ and $\tan\theta > 0$
e $\sin\theta > 0$ and $\tan\theta < 0$ **f** $\sin\theta > 0$ and $\cos\theta > 0$

3 Express each of the following trigonometric ratios in terms of an acute angle, then evaluate correct to 2 decimal places.

a sin 100° **b** cos 140° **c** tan 160° **d** cos 125°
e tan 134° **f** sin 152° **g** tan 101° **h** sin 115°
i cos 108° **j** sin 164° **k** tan 122° **l** cos 171°
m tan 159° **n** cos 153° **o** sin 137° **p** cos 94°
q sin 96° **r** tan 114° **s** cos 132° **t** sin 169°

4 Express each of the following trigonometric ratios in terms of an acute angle, then evaluate correct to 4 decimal places.

a tan 128°15′ **b** sin 105°27′ **c** cos 164°8′
d sin 136°20′ **e** cos 113°51′ **f** tan 157°44′
g cos 149°36′ **h** tan 99°7′ **i** sin 165°32′

5 Find two possible angles θ, where $0° < \theta < 180°$, correct to the nearest degree.

a $\sin\theta = 0.1564$ **b** $\sin\theta = 0.2657$ **c** $\sin\theta = 0.8371$
d $\sin\theta = 0.6049$ **e** $\sin\theta = 0.0695$ **f** $\sin\theta = 0.4224$
g $\sin\theta = 0.1827$ **h** $\sin\theta = 0.7599$ **i** $\sin\theta = 0.3914$

6 Find the obtuse angle θ, correct to the nearest degree.

a $\cos\theta = -0.5218$ **b** $\tan\theta = -0.1243$ **c** $\cos\theta = -0.1921$
d $\tan\theta = -0.9443$ **e** $\tan\theta = -1.8724$ **f** $\cos\theta = -0.8407$
g $\tan\theta = -3.4060$ **h** $\cos\theta = -0.6906$ **i** $\cos\theta = -0.3793$

7 Find all possible values for θ, where $0° < \theta < 180°$, correct to the nearest degree.

a $\sin\theta = 0.5246$	**b** $\tan\theta = 0.7154$	**c** $\cos\theta = -0.6238$
d $\tan\theta = -1.9287$	**e** $\cos\theta = 0.4712$	**f** $\sin\theta = 0.1863$
g $\cos\theta = -0.8247$	**h** $\sin\theta = 0.8921$	**i** $\tan\theta = 2.1661$
j $\sin\theta = 0.7466$	**k** $\tan\theta = -4.0975$	**l** $\cos\theta = -0.6044$

8 Find the exact value of each trigonometric ratio, without using a calculator.

a sin 150°	**b** cos 135°	**c** tan 120°
d cos 150°	**e** sin 120°	**f** tan 135°
g sin 135°	**h** tan 150°	**i** cos 120°

9 **a** Complete the following tables of values by using a calculator. Give your answers correct to 1 decimal place.

x	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$\sin x$										
$\cos x$										
$\tan x$										

x	100°	110°	120°	130°	140°	150°	160°	170°	180°
$\sin x$									
$\cos x$									
$\tan x$									

b Use your tables of values to draw the following graphs on 2 mm grid paper.
i $y = \sin x$ **ii** $y = \cos x$ **iii** $y = \tan x$

10 Use your graph of $y = \sin x$ to answer the following questions for $0° \le x \le 180°$.
a What is the maximum value of $\sin x$?
b What is the minimum value of $\sin x$?
c Find the values of x for which $\sin x$ is: **i** positive **ii** negative
d Find the values of x for which $\sin x = 0$.

11 Use your graph of $y = \cos x$ to answer the following questions for $0° \le x \le 180°$.
a What is the maximum value of $\cos x$?
b What is the minimum value of $\cos x$?
c Find the values of x for which $\cos x$ is: **i** positive **ii** negative
d Find the values of x for which $\cos x = 0$.

12 Use your graph of $y = \tan x$ to answer the following questions for $0° \le x \le 180°$.
a Does $\tan x$ have a maximum value or a minimum value?
b Find the value of x for which $\tan x$ is not defined.
c Find the values of x for which $\tan x$ is: **i** positive **ii** negative
d Find the values of x for which $\tan x = 0$.

■ Further applications

13 Prove the following identities.

a $\cos(180° - \theta)\tan(180° - \theta) = \sin\theta$

b $\dfrac{\cos(90° - \theta)}{\cos(180° - \theta)} = -\tan\theta$

c $\tan(180° - \theta)\sin(90° - \theta) = -\sin\theta$

d $\dfrac{\cos(180° - \theta)\sin\theta}{\sin(180° - \theta)} = -\cos\theta$

e $\sin(180° - \theta)\cos(180° - \theta)\tan(180° - \theta) = \sin^2\theta$

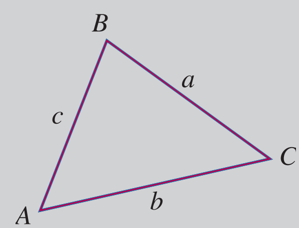
f $\tan(90° - \theta)\tan(180° - \theta) = -1$

10.3 The Sine Rule

Until now we have only used trigonometry to find the length of a side or the size of an angle in right-angled triangles. However, definitions such as $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ can only apply to right-angled triangles, as only right-angled triangles have a hypotenuse. We will now derive a rule that can be used to find sides and angles in non-right-angled triangles. It is called the **Sine Rule**.

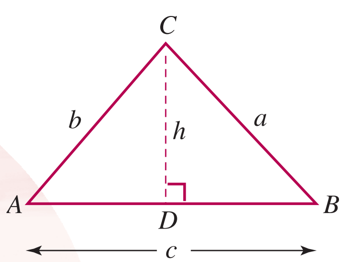
The Sine Rule states that in any triangle *ABC*:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Proof:
We need to consider the use of the Sine Rule in acute-angled and obtuse-angled triangles.

1

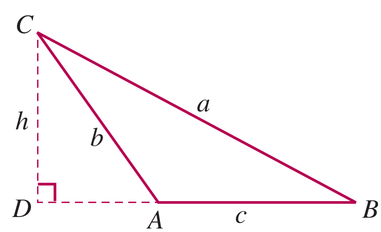


ABC is any acute-angled triangle and *D* is the point on *AB* such that $CD \perp AB$. Let $CD = h$.

In ΔBCD, $\sin B = \dfrac{h}{a}$

$\therefore\ h = a\sin B$.

2



ABC is any obtuse-angled triangle ($\angle A$ is obtuse) and *D* is the point on *BA* (produced) such that $CD \perp AB$. Let $CD = h$.

In ΔBCD, $\sin B = \dfrac{h}{a}$

$\therefore\ h = a\sin B$

In ΔACD, $\sin A = \dfrac{h}{b}$
$\therefore\ h = b \sin A$.

In ΔACD, $\sin(180° - A) = \dfrac{h}{b}$
$\therefore\ h = b \sin(180° - A)$
$\therefore\ h = b \sin A$
[*Note*: $(\sin 180° - A) = \sin A$)]

Equating these results for each triangle, we have:

$$a \sin B = b \sin A$$

$\div \sin A \sin B \quad \div \sin A \sin B$

$$\frac{a \sin B}{\sin A \sin B} = \frac{b \sin A}{\sin A \sin B}$$

$$\therefore\ \frac{a}{\sin A} = \frac{b}{\sin B}$$

In each case, we could draw a perpendicular from A to BC and hence show that $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

$$\therefore\ \frac{a}{\sin A} = \frac{b}{\sin b} = \frac{c}{\sin C}$$

To find an angle by using the Sine Rule, it is best to write the sines of the angles in the numerator and the side lengths in the denominator.

- **To find the length of a side by using the Sine Rule, use** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.
- **To find the size of an angle by using the Sine Rule, use** $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

The Sine Rule can be used to find:
- **the length of a side if given one side and two angles**
- **the size of an angle if given one angle and two sides.**

NOTE: The Sine Rule can only be used when at least one side and the angle opposite it are known.

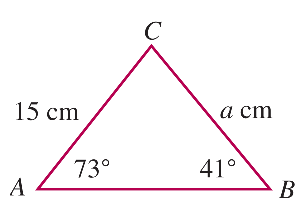
The ambiguous case

When the Sine Rule is used to find an angle, it is possible that there will be two angles. [See example 3.]

EG +S

Example 1

Find the value of a, correct to 1 decimal place.



Solution

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 73°} = \frac{15}{\sin 41°}$$

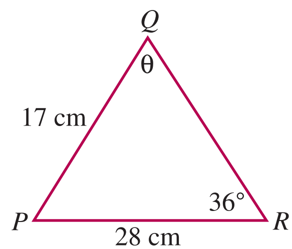
$\times \sin 73° \quad \times \sin 73°$

$$\therefore a = \frac{15 \sin 73°}{\sin 41°}$$

$$= 21.9 \text{ (to 1 decimal place)}$$

Example 2

Find the size of the acute angle θ, correct to the nearest minute.



Solution

$$\frac{\sin Q}{q} = \frac{\sin R}{r}$$

$$\frac{\sin \theta}{28} = \frac{\sin 36°}{17}$$

$\times 28 \quad \times 28$

$$\sin \theta = \frac{28 \sin 36°}{17}$$

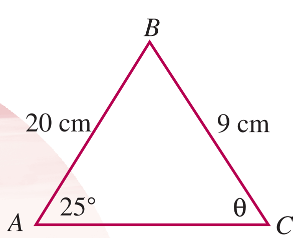
$$\therefore \theta = \sin^{-1}\left(\frac{28 \sin 36°}{17}\right)$$

$$= 75°30' \text{ (to the nearest minute)}$$

Example 3

In $\triangle ABC$, $\angle A = 25°$, $BC = 9$ cm and $AB = 20$ cm. Find the two possible values for $\angle C$, correct to the nearest degree. Hence show that there are two possible triangles.

Solution



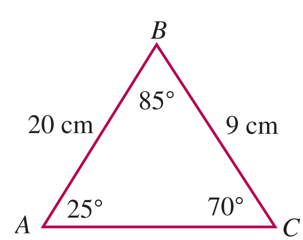
$$\frac{\sin \theta}{20} = \frac{\sin 25°}{9}$$

$$\sin \theta = \frac{20 \sin 25°}{9}$$

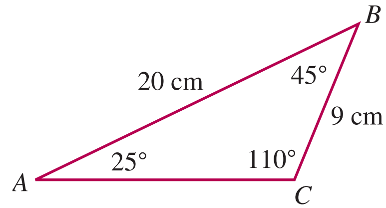
$$\therefore \theta \doteqdot 70° \text{ or } 180° - 70°$$

$$\therefore \angle C = 70° \text{ or } 110° \text{ (to the nearest degree).}$$

- If $\angle C = 70°$, $\angle B = 180° - (25° + 70°)$
  $= 85°$



- If $\angle C = 110°$, $\angle B = 180° - (25° + 110°)$
  $= 45°$



Thus there are two possible values for $\angle C$, and hence, two possible triangles.

Exercise 10.3

1 Answer the following questions, without using a calculator.

a In ΔABC, $c = 21$, $\sin A = 0.4$ and $\sin C = 0.8$. Find a.

b In ΔUVW, $w = 15$, $\sin V = 0.9$ and $\sin W = 0.6$. Find v.

c In ΔLMN, $l = 24$, $\sin L = \frac{1}{2}$ and $\sin M = \frac{2}{3}$. Find m.

2 Answer the following questions, without using a calculator.

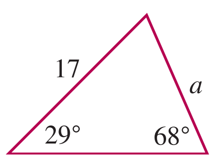
a In ΔDEF, $d = 36$ mm, $e = 30$ mm and $\sin D = \frac{4}{5}$. Find $\sin E$.

b In ΔXYZ, $x = 16$ mm, $y = 20$ mm and $\sin Y = 0.75$. Find $\sin X$.

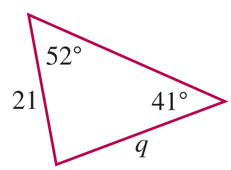
c In ΔQRS, $r = \sqrt{7}$ mm, $s = 3\sqrt{7}$ mm and $\sin S = \frac{1}{4}$. Find $\sin R$.

3 Find the value of the pronumeral in each triangle, correct to 1 decimal place. All lengths are in cm.

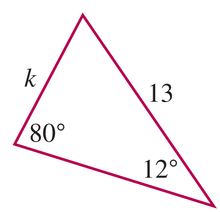
a



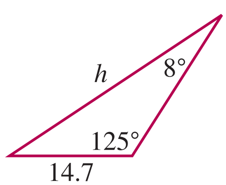
b



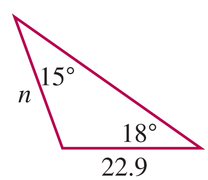
c



d



e



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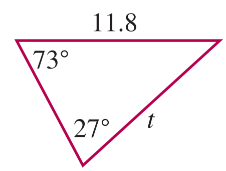
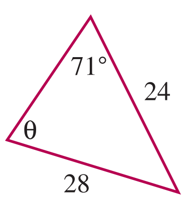


4 Find the size of the acute angle θ, correct to the nearest degree.

a



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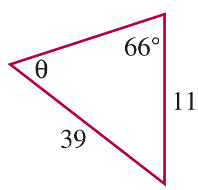

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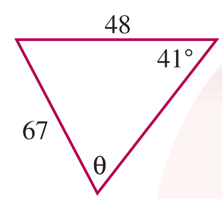

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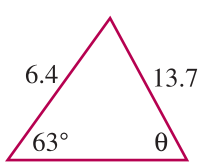

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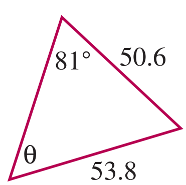
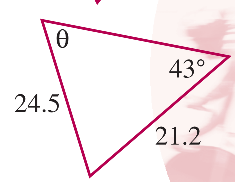
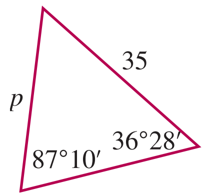


f



Consolidation

5 Find the value of each pronumeral, correct to 1 decimal place. All lengths are in mm.

a


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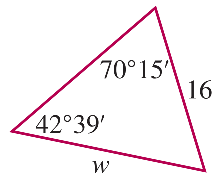

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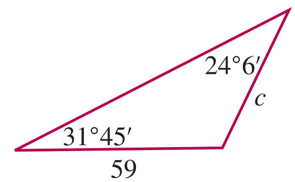

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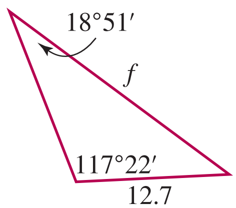
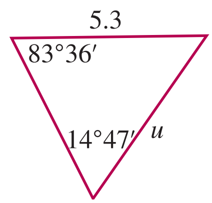


e


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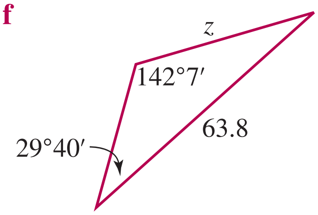
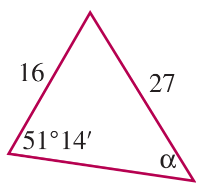


6 Find the size of the acute angle α, correct to the nearest minute.

a


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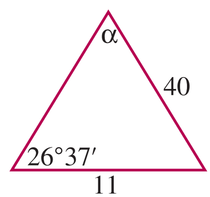

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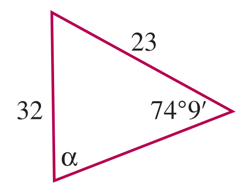

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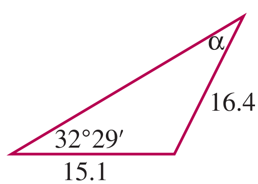

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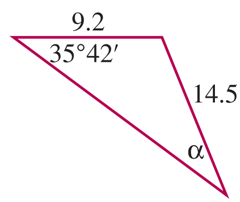

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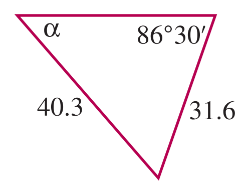
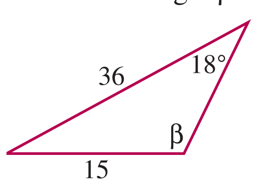


7 Find the obtuse angle β in each triangle, correct to the nearest degree.

a


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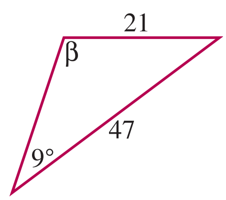

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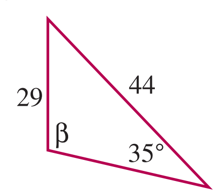

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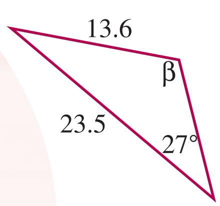

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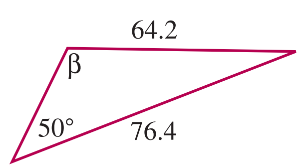

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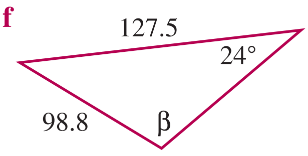


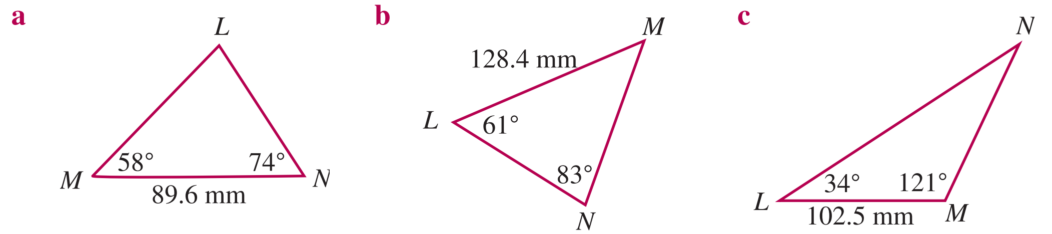
8 In ΔPQR, $\angle P = 47°$, $\angle Q = 62°$ and $PR = 132$ cm.

a Draw a diagram and mark on it all of the given information.

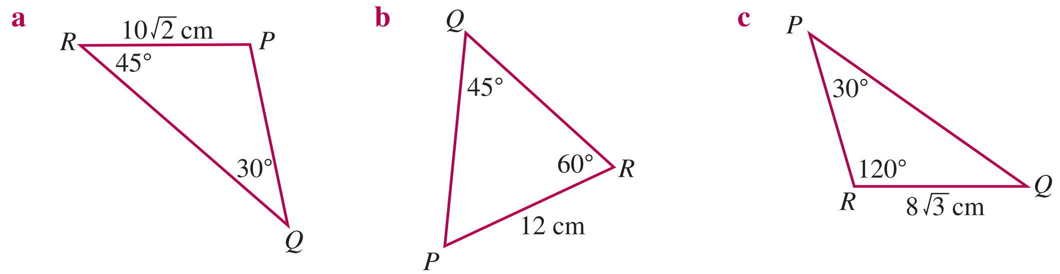
b Find the size of $\angle R$.

c Hence, find the length of PQ, correct to 1 decimal place.

9 Find the length of LN in each triangle, correct to 3 significant figures.

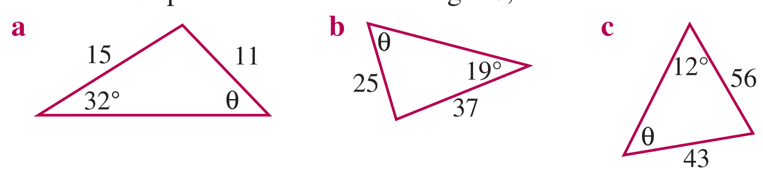


10 Find the exact length of PQ in each triangle. Answer in simplest surd form where necessary.



Further applications

11 Find the two possible sizes for the angle θ, correct to the nearest degree.

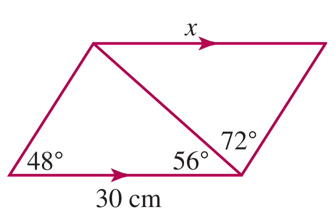


12 In ΔABC, $\angle A = 16°28'$, $BC = 29$ cm and $AB = 42$ cm.

- **a** Draw a diagram showing the given information.
- **b** Find the two possible sizes of $\angle C$, correct to the nearest minute.
- **c** Hence, find the two possible sizes of $\angle B$, correct to the nearest minute.

13 In ΔTUV, $\angle V = 71°$, $TV = 10$ cm and $TU = 12$ cm. Find the size of $\angle U$, correct to the nearest degree.

TRY THIS Double trouble



Find the value of x, to the nearest millimetre.

10.4 Applications of the Sine Rule

Many practical problems can be solved by using the Sine Rule.

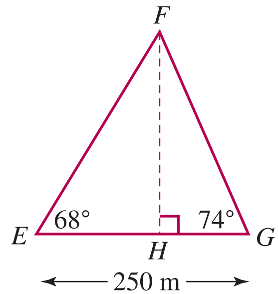
Example 1

EG +S

A triangular field EFG is to be sub-divided into two paddocks by constructing a new fence FH, perpendicular to EG.

a Find the length of the fence EF, correct to the nearest metre.

b Hence, find FH, the length of the new fence, correct to the nearest metre.



Solutions

a In ΔEFG, $\angle EFG = 180° - (68° + 74°)$

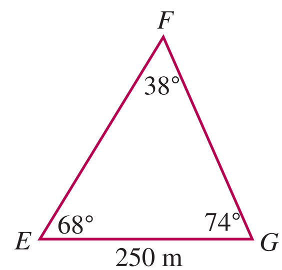
$= 38°$

By the Sine Rule, $\dfrac{EF}{\sin 74°} = \dfrac{250}{\sin 38°}$

$\times \sin 74° \quad \times \sin 74°$

$\therefore EF = \dfrac{250 \sin 74°}{\sin 38°}$

$= 390$ m (to the nearest metre).

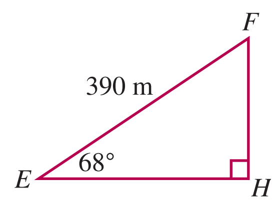


b In ΔEFH, $\sin 68° = \dfrac{FH}{390}$

$\times 390 \quad \times 390$

$\therefore FH = 390 \sin 68°$

$= 362$ m (to the nearest metre).

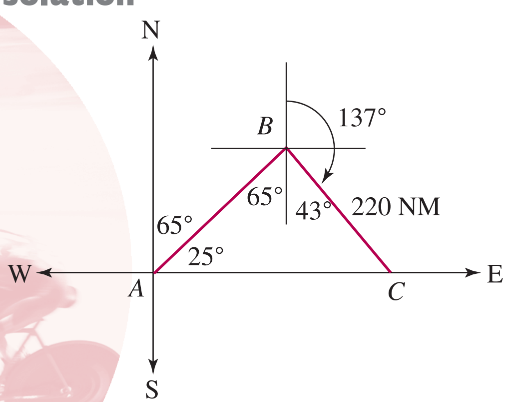


Example 2

EG +S

A ship sailed from A to B on a bearing of 065°. It then turned and sailed 220 nautical miles on a bearing of 137° to C, which is due east of A. Find the distance AC, correct to the nearest nautical mile.

Solution



Now: • $\angle ABC = 65° + 43°$

$= 108°$

• $\angle BAC = 90° - 65°$

$= 25°$

By the Sine Rule, $\dfrac{AC}{\sin 108°} = \dfrac{220}{\sin 25°}$

$\times \sin 108° \quad \times \sin 108°$

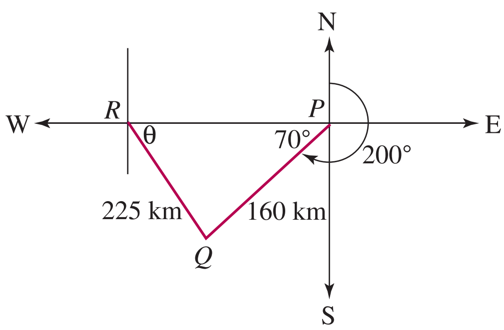
$\therefore AC = \dfrac{220 \sin 108°}{\sin 25°}$

$= 495$ nautical miles (to the nearest nautical mile).

Example 3

A plane flew 160 km from P to Q on a bearing of 200°. It then turned and flew 225 km to R, which is due west of P. Find, correct to the nearest degree, the bearing of Q from R.

Solution



Now, $\angle RPQ = 270° - 200°$
$= 70°$

Let $\angle PRQ = \theta$.

By the Sine Rule, $\dfrac{\sin\theta}{160} = \dfrac{\sin 70°}{225}$
$\times 160 \quad \times 160$

$\sin\theta = \dfrac{160 \sin 70°}{225}$

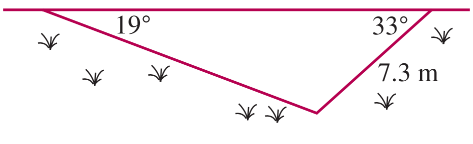
$\therefore \theta = 42°$ (to the nearest degree).

The bearing of Q from $R = 90° + 42°$
$= 132°$.

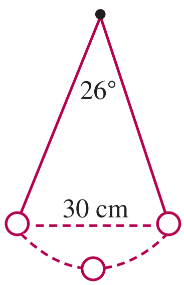
Exercise 10.4

1 **a** In ΔABC, $\angle A = 60°$, $\angle B = 85°$ and $b = 12$. Find a, correct to 1 decimal place.
b In ΔTUV, $\angle T = 38°$, $\angle U = 129°$ and $t = 56$. Find u, correct to 3 significant figures.
c In ΔLMN, $\angle M = 106°$, $m = 84$ and $n = 71$. Find $\angle N$, in degrees, correct to 1 decimal place.
d In ΔXYZ, $\angle Z = 67°$, $x = 45$ and $z = 110$. Find $\angle X$, correct to the nearest minute.

2 A bridge passes over a gorge which is inclined at 19° on one side and 33° on the other side. The length of the steeper incline is 7.3 m. Find the length of the bridge, correct to 1 decimal place.

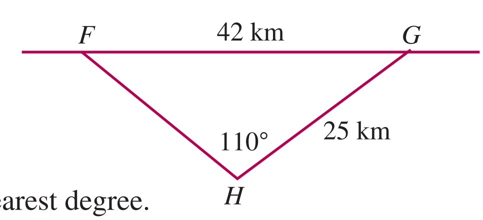


3



A pendulum swings in an arc through an angle of 26°. The horizontal distance between rest positions is 30 cm. Find the length of the pendulum, correct to 2 significant figures.

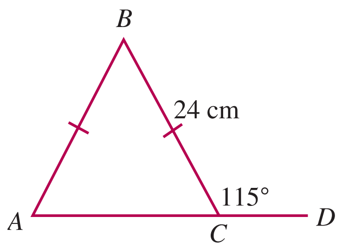
4 Two towns F and G are 42 km apart. The road that joins the towns is closed while it is being resurfaced. A detour between the towns requires that drivers travel through a third town H, where $GH = 25$ km and $\angle FHG = 110°$.



a Calculate the size of $\angle GFH$, correct to the nearest degree.
b Find the distance FH, correct to the nearest tenth of a kilometre.
c How much further must drivers travel as a result of the road closure?

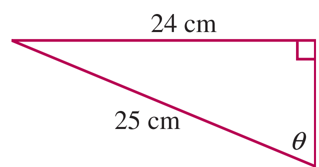
Consolidation

5



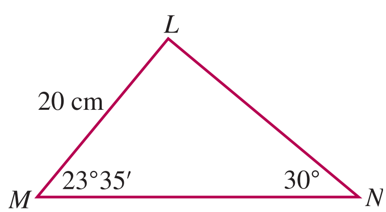
In the diagram, $AB = BC = 24$ cm, $\angle BCD = 115°$. Find the length of AC, correct to 1 decimal place.

6 Find the angle θ, correct to the nearest minute by using:



a right-angled trigonometry
b the Sine Rule.

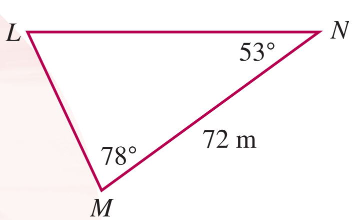
7



a Evaluate sin 23°35′, correct to 1 decimal place.
b Hence, find the length of LN, without further use of a calculator.

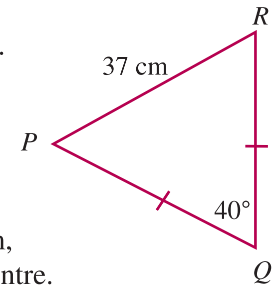
8 **a** In ΔXYZ, $\angle X = 51°$, $\angle Y = 45°$ and the longest side is 17 cm. Find the length of the shortest side, correct to 3 significant figures.
b In ΔTUV, $TU = 9$ mm, $UV = 7$ mm, $TV = 12$ mm and $\angle V = 42°$. Find the size of the smallest angle in the triangle, correct to the nearest minute.

9

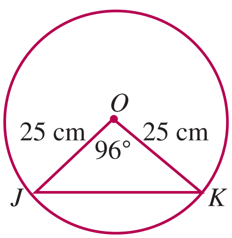


A triangular field LMN is to be fenced. The side MN is 72 m long, $\angle LMN = 78°$ and $\angle LNM = 53°$.
a Find the lengths of the two unknown sides, correct to the nearest tenth of a metre.
b Calculate the cost of fencing the field, if the fence wire costs \$52 per metre.

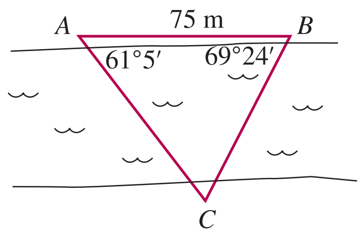
10 In this isosceles triangle, $PQ = QR$, $\angle PQR = 40°$ and $PR = 37$ cm. Find the length of the equal sides, correct to the nearest centimetre.



11 In a circle with centre O and radius 25 cm, a chord subtends an angle of 96° at the centre. Find the length of the chord, correct to 3 significant figures.



12 Two girls, Anna and Betty are standing at A and B, respectively, 75 metres apart, on one bank of a river. A boy is standing at C on the other side of the river, such that $\angle CAB = 61°5'$ and $\angle CBA = 69°24'$. Which girl is closer to the boy and by what distance? Answer correct to the nearest metre.



13 In ΔPQR, $PQ = QR = 4$ cm and $\angle PQR = 120°$. S is a point on QR such that SP bisects $\angle QPR$.

- **a** Draw a diagram showing the given information.
- **b** Find the size of $\angle PSQ$ without the use of trigonometry.
- **c** Show that $PS = 2\sqrt{6}$ cm.
- **d** Show that $PR = 4\sqrt{3}$ cm.

■ Further applications

14 **a** Sonia drove from T to U on a bearing of 042°. She then turned and drove 35 km to V on a bearing of 105° until she was due east of T. Find the distance TU, correct to the nearest tenth of a kilometre.

b Kim, who is standing at X, sees her brother standing at Y, on a bearing of 231°. Kim's sister is standing at Z, which is due west of X and on a bearing of 342° from Y. If Kim's brother and sister are presently 48 m apart, find how far Kim is from her brother, correct to the nearest metre.

c Three towers A, B and C stand on level ground. The bearing of B from A is 250°, C is due south of A and the bearing of C from B is 118°. If the towers B and C are 41 m apart, find the distance AB, correct to 3 significant figures.

d Megan and Jane were taking part in an orienteering activity. Megan walked on a bearing of 057° from C to D, while Jane walked 295 m due north from C to E. If the bearing of E from D is 284°, find the distance between the girls, correct to the nearest metre.

15 a Karl rode his horse 3.2 km from A to B on a bearing of 037°. He then turned and rode 4.1 km to C, which is due east of A. Find the size of $\angle ACB$, correct to the nearest degree. Hence, find the bearing of C from B.

b Three fire towers at P, Q, R stand on level ground in a national park. The towers at P and Q are 4.3 km apart and the bearing of Q from P is 145°. The tower at R is due south of P and 6.4 km from Q. Find the size of $\angle PRQ$, correct to the nearest degree. Hence, find the bearing of R from Q.

c A cruise ship sailed 160 nautical miles from a port F on a bearing of 214° to port G. More passengers boarded and the ship sailed 220 nautical miles to a holiday resort at H, which is due west of F. Find the size of $\angle FHG$, correct to the nearest degree. Hence, find the bearing of H from G.

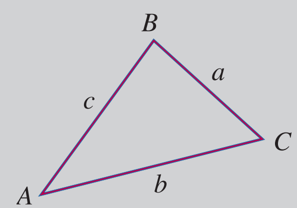
d A surveyor is standing at a point X and observes a hill, Y, 19 km away on a bearing of 307°. A second surveyor at Y observes a rock formation 21 km away at Z, which is due north of X. Find the size of $\angle YZX$, correct to the nearest degree. Hence, find the bearing of Z from Y.

10.5 The Cosine Rule

As with the Sine Rule, the Cosine Rule can be used to find the length of a side or the size of an angle in a non-right-angled triangle. The Cosine Rule is used in situations where the Sine Rule cannot be applied.

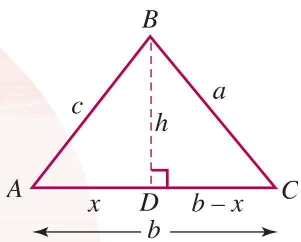
The Cosine Rule states that in any triangle ABC:

$$a^2 = b^2 + c^2 - 2bc \cos A$$



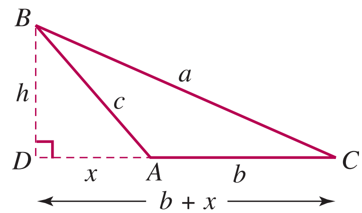
Proof:
We need to consider the use of the Cosine Rule in acute-angled and obtuse-angled triangles.

1



ABC is any acute-angled triangle and D is the point on AC such that $BD \perp AC$. Let $BD = h$.

2



ABC is any obtuse-angled triangle ($\angle A$ is obtuse) and D is the point on CA (produced) such that $BD \perp CA$. Let $BD = h$.

In ΔBCD, $a^2 = (b - x)^2 + h^2$ … ①
In ΔABD, $c^2 = h^2 + x^2$
$\therefore h^2 = c^2 - x^2$ … ②
Substitute ② into ①:
$a^2 = (b - x)^2 + c^2 - x^2$
$a^2 = b^2 - 2bx + x^2 + c^2 - x^2$
$\therefore a^2 = b^2 + c^2 - 2bx$ … ③
But, in ΔABD, $\cos A = \frac{x}{c}$
$\therefore x = c \cos A$ … ④
Substitute ④ into ③:
$\therefore a^2 = b^2 + c^2 - 2bc \cos A$

In ΔBCD, $a^2 = (b + x)^2 + h^2$ … ①
In ΔABD, $c^2 = h^2 + x^2$
$\therefore h^2 = c^2 - x^2$ … ②
Substitute ② into ①:
$a^2 = (b + x)^2 + c^2 - x^2$
$a^2 = b^2 + 2bx + x^2 + c^2 - x^2$
$\therefore a^2 = b^2 + c^2 + 2bx$ … ③
But, in ΔABD, $\cos (180° - A) = \frac{x}{c}$
$x = c \cos (180° - A)$
$= c \times -\cos A$
$\therefore x = -c \cos A$ … ④
Substitute ④ into ③:
$\therefore a^2 = b^2 + c^2 - 2bc \cos A$

To find an angle we rearrange the Cosine Rule so that the cosine of the angle is the subject.

- **To find the length of a side by using the Cosine Rule, use $a^2 = b^2 + c^2 - 2bc \cos A$.**
- **To find the size of an angle by using the Cosine Rule, use $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.**

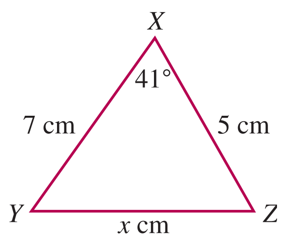
The Cosine Rule can be used to find:
- **the length of a side if given two sides and the included angle**
- **the size of an angle if given three sides.**

Example 1

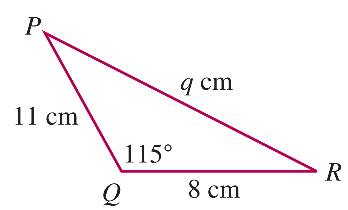
EG +S

Find the value of each pronumeral, correct to 1 decimal place.

a



b



Solutions

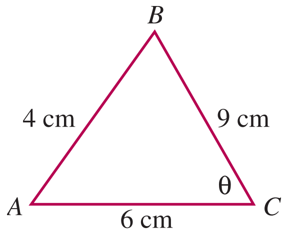
a
$$\begin{aligned} x^2 &= y^2 + z^2 - 2yz \cos X \\ &= 5^2 + 7^2 - (2 \times 5 \times 7 \times \cos 41°) \\ &= 25 + 49 - 70 \cos 41° \\ &= 74 - 70 \cos 41° \\ &= 21.170\,329\,38 \\ \therefore x &= \sqrt{21.170\,329\,38} \\ &= 4.6 \text{ (to 1 decimal place)} \end{aligned}$$

b
$$\begin{aligned} q^2 &= p^2 + r^2 - 2pr \cos Q \\ &= 8^2 + 11^2 - (2 \times 8 \times 11 \times \cos 115°) \\ &= 64 + 121 - 176 \cos 115° \\ &= 185 - 176 \cos 115° \\ &= 259.380\,814\,1 \\ \therefore q &= \sqrt{259.380\,814\,1} \\ &= 16.1 \text{ (to 1 decimal place)} \end{aligned}$$

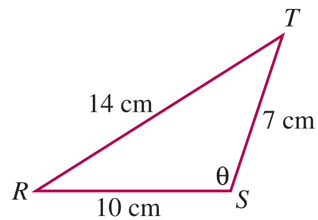
Example 2

Find the angle θ in each triangle, correct to the nearest minute.

a



b



Solutions

a
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos \theta = \frac{9^2 + 6^2 - 4^2}{2 \times 9 \times 6}$$
$$= \frac{101}{108}$$
$$\therefore \theta = \cos^{-1}\left(\frac{101}{108}\right)$$
$$= 20°45' \text{ (to the nearest minute)}$$

b
$$\cos S = \frac{r^2 + t^2 - s^2}{2rt}$$
$$\cos \theta = \frac{7^2 + 10^2 - 14^2}{2 \times 7 \times 10}$$
$$= \frac{-47}{140}$$
$$\therefore \theta = \cos^{-1}\left(\frac{-47}{140}\right)$$
$$= 109°37' \text{ (to the nearest minute)}$$

Exercise 10.5

1 Answer the following questions, without using a calculator.

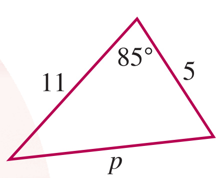
- **a** In ΔFGH, $f = 4$, $h = 3$ and $\cos G = \frac{7}{12}$. Find g.
- **b** In ΔXYZ, $x = 2$, $y = 5$ and $\cos Z = \frac{4}{5}$. Find z.
- **c** In ΔPQR, $q = 3$, $r = 6$ and $\cos P = \frac{5}{9}$. Find p.

2 Answer the following questions, without using a calculator.

- **a** In ΔSTU, $s = 6$ mm, $t = 10$ mm and $u = 11$ mm. Find $\cos U$.
- **b** In ΔBCD, $b = 5$ mm, $c = 6$ mm and $d = 7$ mm. Find $\cos B$.
- **c** In ΔIJK, $i = 4$ mm, $j = 5$ mm and $k = 7$ mm. Find $\cos K$.

3 Find the value of the pronumeral in each triangle, correct to 1 decimal place.

a



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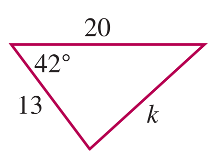
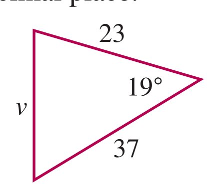
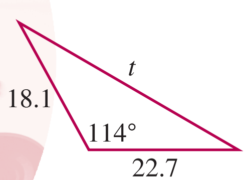


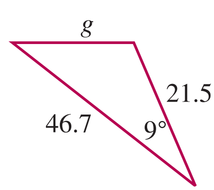
c



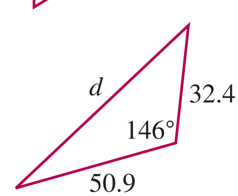
d



e

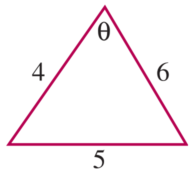


f



4 Find the size of the angle θ, correct to the nearest degree.

a



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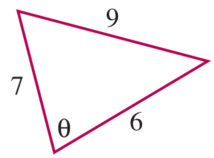
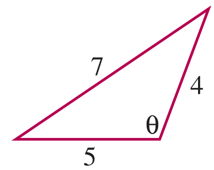


c



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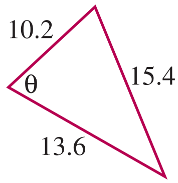

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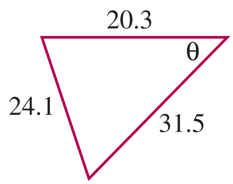
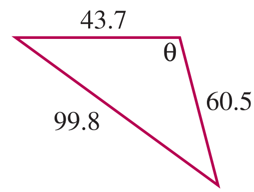


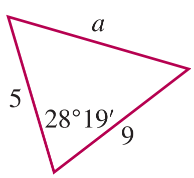
f



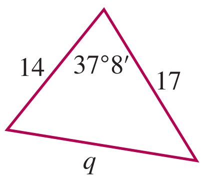
■ Consolidation

5 Find the value of the pronumeral, correct to 1 decimal place.

a



b



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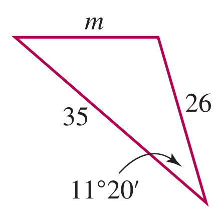

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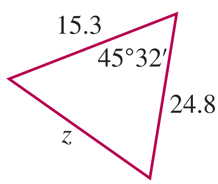

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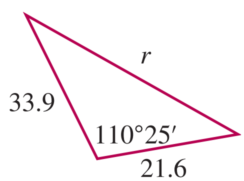

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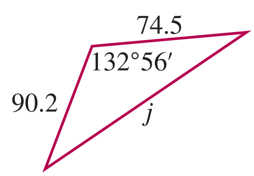
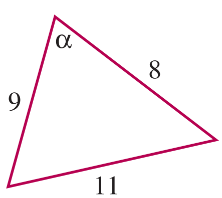
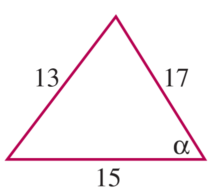


6 Find the size of the angle α, correct to the nearest minute.

a



b



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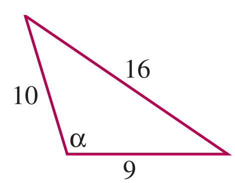

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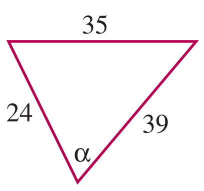

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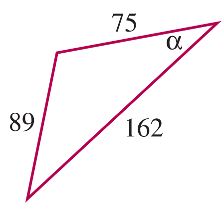

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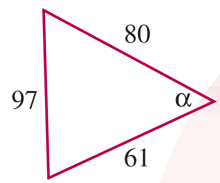
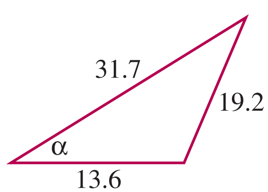
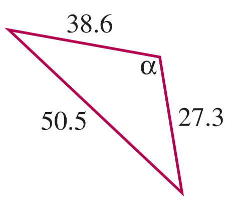


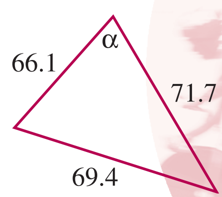
g

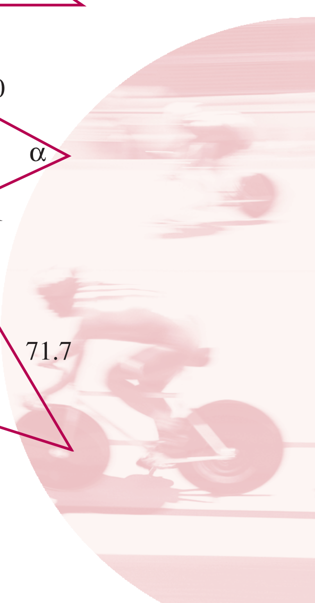


h



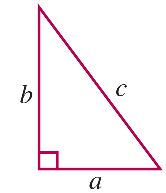
i



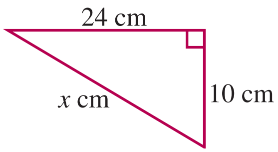


7 a Use the Cosine Rule to find an expression for c^2 in terms of a and b.

b What happens to the Cosine Rule for sides when the included angle is a right angle? Why?



8 Use the Cosine Rule to find the value of x.



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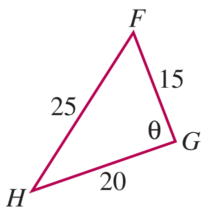


Show that ΔFGH is right-angled at G by using:

a Pythagoras' theorem

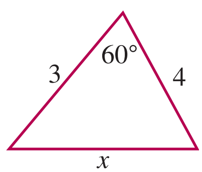
b the Cosine Rule

■ Further applications

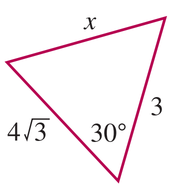
10 The sides of a certain triangle are in the ratio 4 : 5 : 8. Find the sizes of the angles, correct to the nearest degree.

11 Find the value of x in each of these, without the use of a calculator. All lengths are in metres.

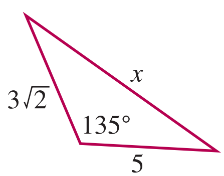
a



b

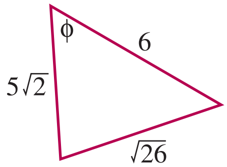


c

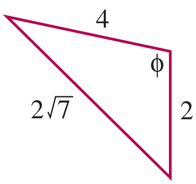


12 Find the size of the angle ϕ in each of these, without the use of a calculator. All lengths are in metres.

a



b



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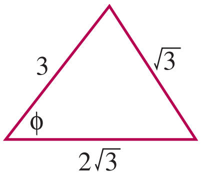
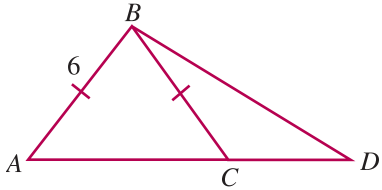


TRY THIS Exact length



In ΔABC, $AB = BC = 6$ cm. AC is produced to D so that $CD = 5$ cm and $BD = 9$ cm. Find the exact length of AC.

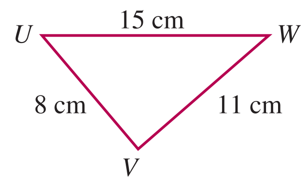
10.6 Applications of the Cosine Rule

As with the Sine Rule, many practical problems can be solved by the use of the Cosine Rule.

Example 1

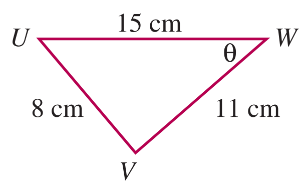
EG +S

Find the size of the smallest angle in this triangle, correct to the nearest minute.



Solution

The smallest angle is opposite the shortest side, $\therefore$ $\angle W$ is the smallest angle.



$$\cos\theta = \frac{11^2 + 15^2 - 8^2}{2 \times 11 \times 15}$$

$$= \frac{282}{330}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{282}{330}\right)$$

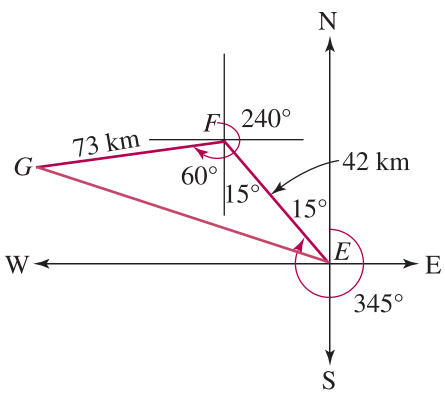
$$= 31°17' \text{ (to the nearest minute).}$$

Example 2

EG +S

Alana drove 42 km from E to F on a bearing of 345°. She then turned and drove 73 km on a bearing of 240° to G. Find the distance EG, correct to 1 decimal place.

Solution



Now, $\angle EFG = 60° + 15°$

$= 75°$

By the Cosine Rule, $EG^2 = 42^2 + 73^2 - (2 \times 42 \times 73 \times \cos 75°)$

$= 1764 + 5329 - 6132 \cos 75°$

$= 7093 - 6132 \cos 75°$

$= 5505.921\ 615$

$\therefore\ EG = \sqrt{5505.921\ 615}$

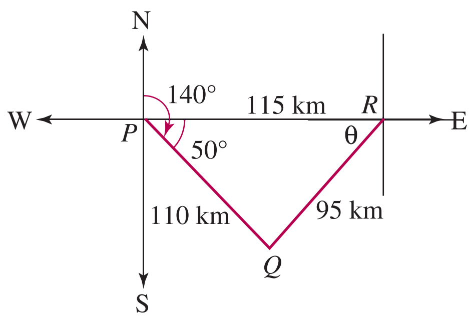
$= 74.2$ km (to 1 decimal place).

Example 3

EG +S

Three towns P, Q, R are situated such that the bearing of Q from P is 140° and R is due east of P. If $PQ = 110$ km, $QR = 95$ km and $PR = 115$ km, find correct to the nearest degree, the bearing of Q from R.

Solution



Now, $\angle RPQ = 140° - 90°$
$= 50°$

Let $\angle PRQ = \theta$.

By the Cosine Rule,

$$\cos\theta = \frac{115^2 + 95^2 - 110^2}{2 \times 115 \times 95}$$
$$= \frac{10\,150}{21\,850}$$
$$\therefore \theta = \cos^{-1}\left(\frac{10\,150}{21\,850}\right)$$
$$= 62° \text{ (to the nearest degree)}$$

$\therefore$ The bearing of Q from $R = 270° - 62°$
$= 208°$.

Exercise 10.6

1 **a** In ΔXYZ, $XZ = 28$ cm, $YZ = 19$ cm and $\angle XZY = 13°$. Find XY, correct to the nearest millimetre.

b In ΔCDE, $CE = 11$ m, $DE = 6$ m and $\angle CED = 84°$. Find CD, correct to 4 significant figures.

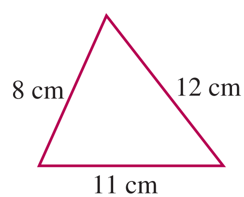
c In ΔKLM, $KL = 7$ cm, $LM = 9$ cm and $KM = 6$ cm. Find $\angle L$ in degrees, correct to 2 decimal places.

d In ΔHIJ, $HI = 14$ mm, $IJ = 16$ mm and $HJ = 27$ mm. Find $\angle I$, correct to the nearest minute.

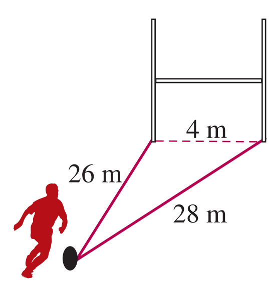
2 Find, correct to the nearest minute, the size of:

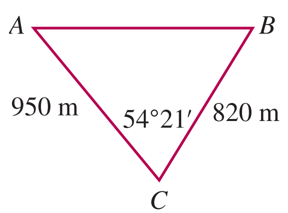
a the largest angle

b the smallest angle.



3 In a game of rugby, a goal kicker is to attempt a penalty goal. The ball is placed 28 m away from one upright and 26 m away from the other. The goal posts are 4 m apart. Within what angle must he kick the ball in order to score a goal? Answer correct to the nearest degree.

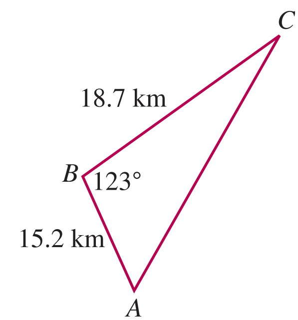


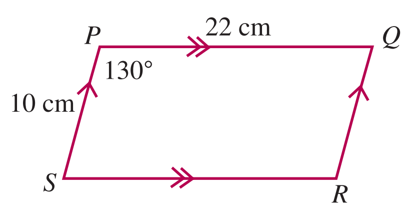
4 

A, B and C are three forts. The soldiers in A want to fire cannonballs at B, but don't know the distance between the forts. However, they know that a third fort C is 950 m from A and 820 m from B, and that $\angle ACB = 54°21'$. How far would a cannonball need to be fired from A in order to reach the fort at B? Answer correct to the nearest metre.

■ Consolidation

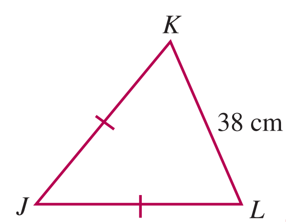
5 A yacht race is sailed over a triangular course. The yachts commence at A and sail around two buoys, B and C, then sail back to A. The first leg of the race is 15.2 km, the second leg is 18.7 km and the angle between these legs is 123°. Over what distance is the race sailed? Answer correct to 1 decimal place.

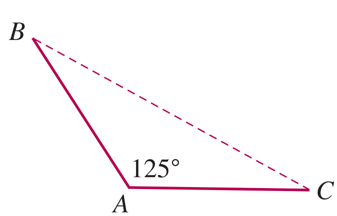


6 

In the parallelogram $PQRS$, $PQ = 22$ cm, $PS = 10$ cm and $\angle SPQ = 130°$. Find the lengths of the diagonals, correct to the nearest millimetre.

7 In ΔJKL, $JK = JL = 38$ cm and the perimeter is 104 cm. Find the size of $\angle J$, correct to the nearest minute.



8 

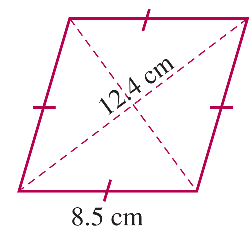
Amelia and Elizabeth leave A at noon and travel along separate roads AB and AC, which are inclined at an angle of 125°. Amelia travels at a speed of 80 km/h, while Elizabeth travels at 70 km/h.

a What distance would each woman have travelled by 3 am?

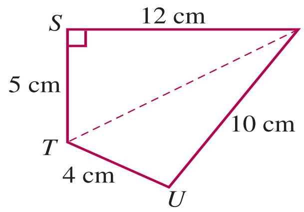
b How far apart would they be then, correct to the nearest kilometre?

9 A rhombus of side length 8.5 cm has its longer diagonal of length 12.4 cm.

- **a** Find the obtuse angle in the rhombus, correct to the nearest minute.
- **b** Find the acute angle in the rhombus, without using trigonometry.
- **c** Hence, calculate the length of the shorter diagonal, correct to 1 decimal place.



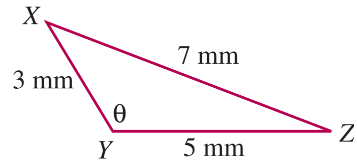
10



- **a** Find the length of the diagonal TV.
- **b** Hence, find the size of $\angle TUV$, correct to the nearest minute.

11 In ΔXYZ, $XY = 3$ mm, $YZ = 5$ mm and $XZ = 7$ mm.

- **a** Show that $\cos\theta = -\frac{1}{2}$.
- **b** Hence, find the size of $\angle XYZ$, without using a calculator.



Further applications

12 In each of the following problems, give the required distance correct to 1 decimal place.

- **a** Mai drove 37 km from A to B on a bearing of 062°. She then turned and drove for 54 km on a bearing of 112° to C. Find the distance AC.
- **b** A man sailed 16 km from L to M on a bearing of 237°. He then turned and sailed for 21 km on a bearing of 314° to N. Find the distance LN.
- **c** A group of scouts set up three campsites at I, J and K. The bearing of J from I is 102°, the bearing of K from J is 246°. If J is 15 km from I and 21 km from K, find the distance between the campsites at I and K.
- **d** Two towns U and V are situated on opposite sides of a deep canyon and V is due north of U. To travel from U to V, a person must travel 6.4 km to a crossing at W, on a bearing of 317°, then 9.1 km to V on a bearing of 029°. How far apart are the towns?

13 Use the Cosine Rule to answer each of the following.

- **a** Mitchell flew his light plane 28 km from A to B on a bearing of 138°, then turned and flew 25 km to C, which is 33 km due east of A. Find the size of $\angle ABC$, correct to the nearest degree. Hence, find the bearing of C from B.
- **b** Inzamam sailed his boat 11 km from P to Q on a bearing of 306° until the engine cut out. The boat then drifted 13 km to R, which is 20 km due west of P. Find the size of $\angle PQR$, correct to the nearest degree. Hence, find the bearing of R from Q.
- **c** A pod of whales travelled 14 km from X to Y on a bearing of 242°, then 35 km to Z which is 39 km due south of X. Find the size of $\angle XYZ$, correct to the nearest degree. Hence, find the bearing of Z from Y.

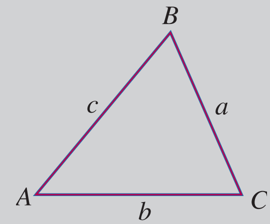
d The weather bureau tracked a severe storm front on radar. It moved 35 km from E to F on a bearing of 054°, then turned and moved 41 km to G, which is 50 km due north of E. Find the size of $\angle EFG$, correct to the nearest degree. Hence, find the bearing of G from F.

10.7 Area of a triangle

The area of a triangle can be found by trigonometry if we know two sides and the included angle.

The area of any triangle ABC is given by:

$$A = \frac{1}{2}ab\sin C$$



Proof:
ABC is any triangle and D is the point on BC such that $AD \perp BC$. Let $AD = h$.

In ΔABC, area $= \frac{1}{2} \times \text{base} \times \text{height}$

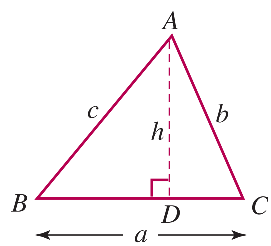
$= \frac{1}{2}ah \ldots (1)$

In ΔACD, $\sin C = \frac{h}{b}$

$\therefore h = b\sin C \ldots (2)$

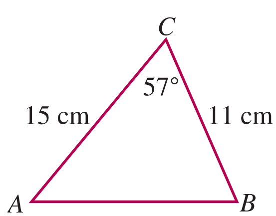
Substitute (2) into (1):

$\therefore$ Area $= \frac{1}{2}ab\sin C$



Example 1

Find the area of this triangle, correct to 1 decimal place.



Solution

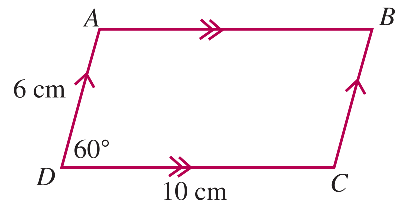
$A = \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 11 \times 15 \times \sin 57°$

$= 69.2\text{ cm}^2$ (to 1 decimal place).

Example 2

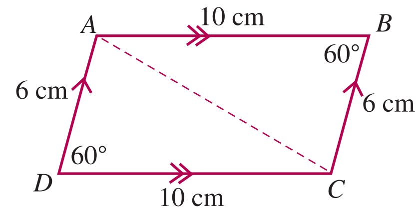
Find the exact area of this parallelogram.



Solution

Construct the diagonal AC.

Now, $\Delta ADC \equiv \Delta CBA$ (SSS)



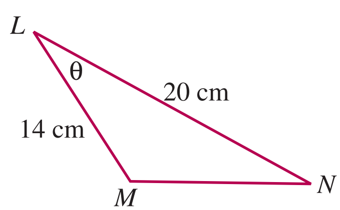
$$\therefore A = \frac{1}{2} \times 10 \times 6 \times \sin 60° \times 2$$

(there are 2 equal triangles)

$$= \frac{1}{2} \times 10 \times 6 \times \frac{\sqrt{3}}{2} \times 2$$

$$= 30\sqrt{3}\ \text{cm}^2$$

Example 3



In ΔLMN, $\angle M$ is obtuse.
The area of the triangle is 133 cm^2.
Find the angle θ, correct to the nearest minute.

Solution

$$A = \frac{1}{2}mn \sin L$$

$$133 = \frac{1}{2} \times 20 \times 14 \times \sin \theta$$

$$133 = 140 \sin \theta$$

$$\div 140 \quad \div 140$$

$$\sin \theta = \frac{133}{140}$$

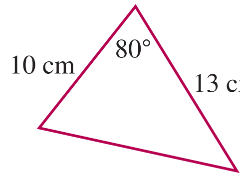
$$\therefore \theta = \sin^{-1}\left(\frac{133}{140}\right)$$

$$= 71°48' \text{ (to the nearest minute).}$$

Exercise 10.7

1 Find the area of each triangle, correct to 1 decimal place.

a



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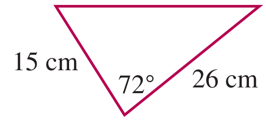

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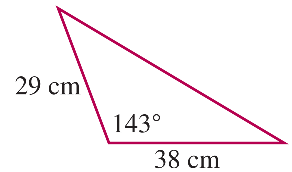

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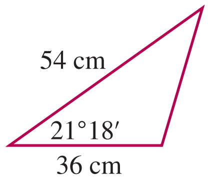
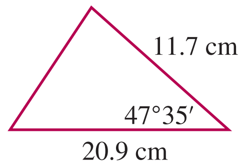


e



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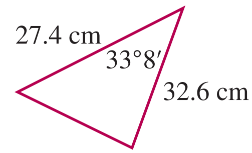


2 **a** In ΔPQR, $PQ = 15$ mm, $PR = 12$ mm and $\angle P = 38°$. Find the area of ΔPQR, correct to 2 decimal places.

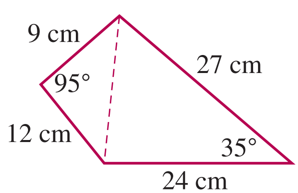
b In ΔEFG, $EF = 21$ cm, $FG = 27$ cm and $\angle F = 129°$. Find the area of ΔEFG, correct to 4 significant figures.

c In ΔXYZ, $XZ = 50.1$ m, $YZ = 34.5$ m and $\angle Z = 46°17'$. Find the area of ΔXYZ, correct to the nearest square metre.

■ Consolidation

3 Find the area of each quadrilateral, correct to 1 decimal place.

a



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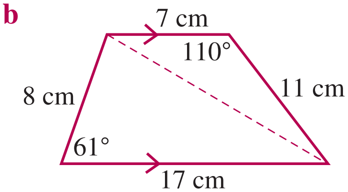

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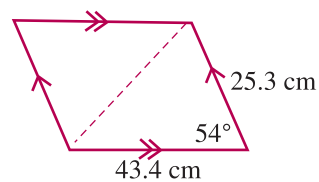
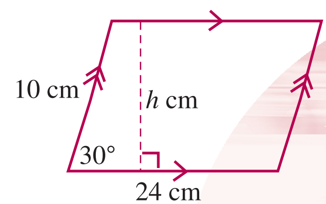


4 **a** Find the area of the parallelogram, using trigonometry.

b Hence, find the perpendicular height, h cm.



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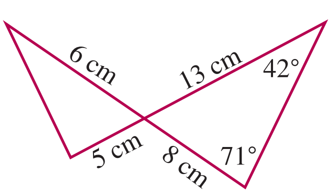
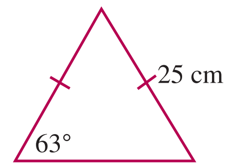


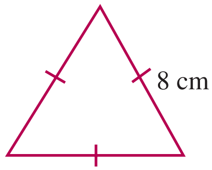
Find the total area of this figure, correct to the nearest cm^2.

6 Find the area of this isosceles triangle, correct to 3 significant figures.

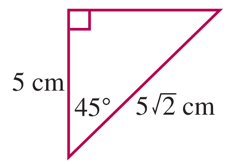


7 Find the area of each triangle, without the use of a calculator.

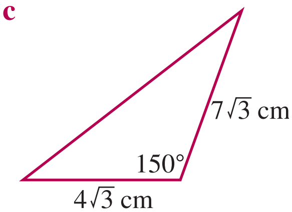
a



b

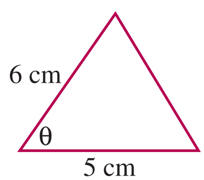


c



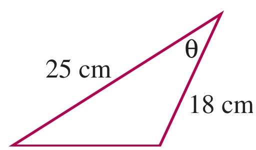
8 Find the angle θ in each triangle, correct to the nearest degree. Angles that appear obtuse are obtuse.

a



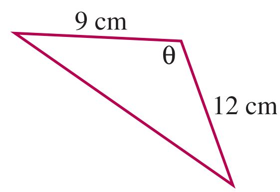
Area = 10 cm^2

b



Area = 91.5 cm^2

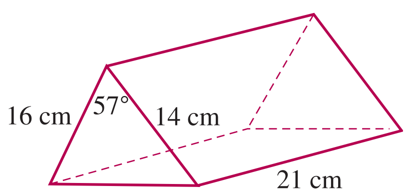
c



Area = 25.4 cm^2

9 Find the volume of each solid, correct to 4 significant figures.

a



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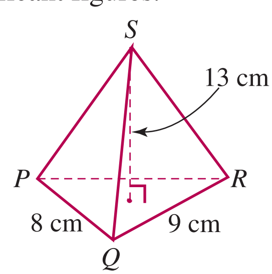
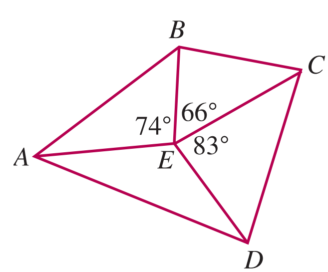
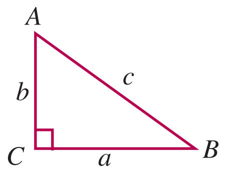


$\angle PQR = 100°$

10 An irregularly shaped field has been divided into four paddocks as shown. If $AE = 205$ m, $BE = 153$ m, $CE = 178$ m and $DE = 164$ m, find the total area of the field, in hectares. Answer correct to 2 decimal places.



11



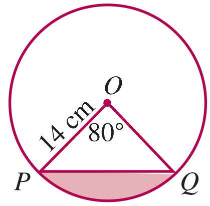
a Find the area of this triangle by using the formula $A = \frac{1}{2}ab \sin C$.

b What happens to this formula when it is used to find the area of a right-angled triangle, and the included angle is 90°?

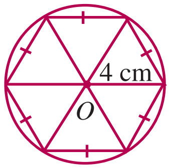
Further applications

12 This circle with centre O has a radius of 14 cm. The chord PQ subtends an angle of 80° at the centre.

- **a** Find the area of the sector POQ, correct to 1 decimal place.
- **b** Find the area of ΔPOQ, correct to 1 decimal place.
- **c** Hence, find the area of the shaded segment.

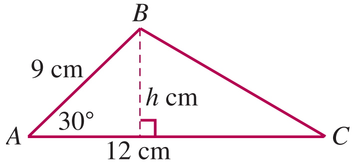


13



A regular hexagon has been inscribed in a circle with centre O and radius 4 cm. Find the area of the hexagon, without the use of a calculator.

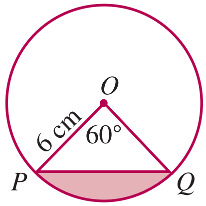
14 Find the value of h in this triangle, without the use of a calculator or the definition $\sin \theta = \frac{OPP}{HYP}$.



TRY THIS Exact area of a segment

In the circle with centre O, $OP = OQ = 6$ cm and $\angle POQ = 60°$.

Show that the area of the shaded segment is $3(2\pi - 3\sqrt{3})$ cm^2.



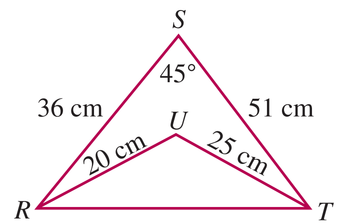
10.8 Miscellaneous problems

The questions in this exercise require the use of right-angled trigonometry, the Sine Rule, the Cosine Rule and the area of a triangle. You will need to determine which approach to take in each question depending on the given information.

- **Use the Sine Rule if given two sides and the non-included angle or two angles and one side.**
- **Use the Cosine Rule if given two sides and the included angle or three sides.**

EG +S

Example 1



a Find the length of RT, correct to 1 decimal place.

b Hence, find the size of $\angle RUT$, correct to the nearest minute.

Solutions

a In ΔRST, we are given two sides (RS and ST) and the included angle ($\angle RST$), $\therefore$ we use the Cosine Rule to find RT.

$$\begin{aligned} RT^2 &= 36^2 + 51^2 - (2 \times 36 \times 51 \times \cos 45°) \\ &= 1296 + 2601 - 3672 \cos 45° \\ &= 3897 - 3672 \cos 45° \\ &= 1300.503\,899 \end{aligned}$$

$$\begin{aligned} \therefore RT &= \sqrt{1300.503\,899} \\ &= 36.1 \text{ cm (to 1 decimal place)} \end{aligned}$$

b In ΔRUT, we are given three sides, $\therefore$ we use the Cosine Rule to find $\angle RUT$.

Let $\angle RUT = \theta$

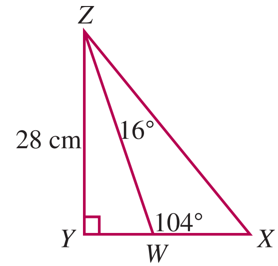
$$\begin{aligned} \cos\theta &= \frac{20^2 + 25^2 - 36.1^2}{2 \times 20 \times 25} \\ &= \frac{-278.21}{1000} \end{aligned}$$

$$\begin{aligned} \therefore \theta &= \cos^{-1}\left(\frac{-278.21}{1000}\right) \\ &= 106°9' \end{aligned}$$

Example 2

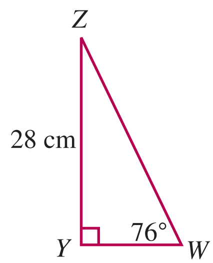
a Find the size of $\angle YWZ$ and hence calculate the length of WZ, correct to 1 decimal place.

b Find the length of XW, correct to 1 decimal place.



Solutions

a In ΔWYZ, $\angle ZWY = 180° - 104°$
$= 76°$



$$\sin 76° = \frac{28}{ZW}$$

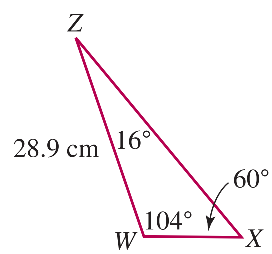
$$\frac{1}{\sin 76°} = \frac{ZW}{28}$$

$$\times 28 \quad \times 28$$

$$\therefore ZW = \frac{28}{\sin 76°}$$

$= 28.9$ cm (to 1 decimal place).

b In ΔXWZ, we now have two angles and one side,
$\therefore$ we use the Sine Rule to find WX.
Now, $\angle WXZ = 180° - (16° + 104°)$
$= 60°$



$$\text{By the Sine Rule, } \frac{WX}{\sin 16°} = \frac{28.9}{\sin 60°}$$

$$\times \sin 16° \quad \times \sin 16°$$

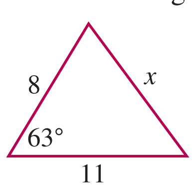
$$\therefore WX = \frac{28.9 \sin 16°}{\sin 60°}$$

$= 9.2$ cm (to 1 decimal place).

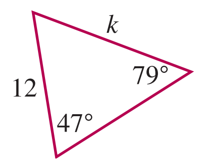
Exercise 10.8

1 Find the value of each pronumeral, correct to 1 decimal place. Angles that appear obtuse are obtuse. All lengths are in centimetres.

a



b



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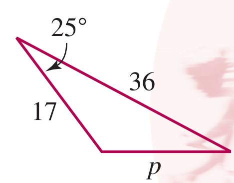

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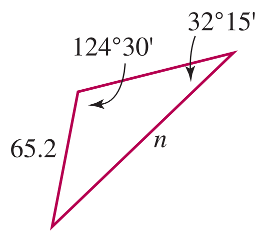

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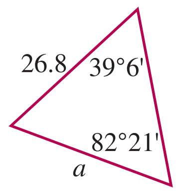

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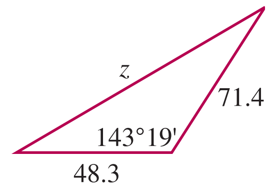
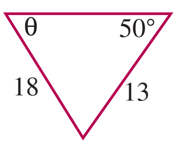


2 Find the size of the angle marked θ, correct to the nearest minute. Angles that appear obtuse are obtuse. All lengths are in metres.

a



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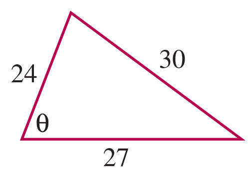

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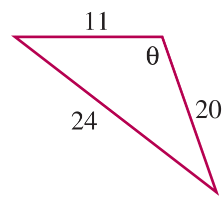

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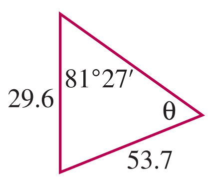

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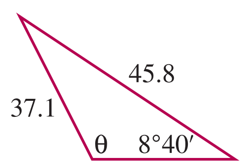

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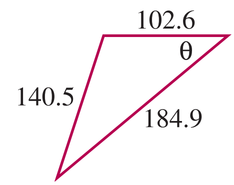
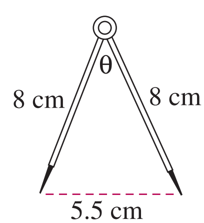


■ Consolidation

3 A pair of compasses of length 8 cm are opened to a width of 5.5 cm. Find the angle between the arms, correct to the nearest degree.

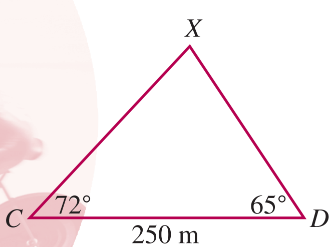


4 In ΔCDE, the lengths of the sides DE and CE are in the ratio 3 : 1. If $\sin D = \frac{1}{6}$, find the size of $\angle C$, where $\angle C$ is obtuse.

5 **a** Two angles in a triangle are 28° and 134°. If the shortest side is 11 cm, find the length of the longest side, correct to 2 decimal places.

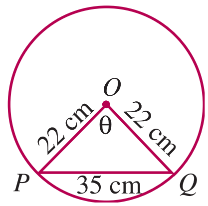
b Find, correct to the nearest minute, the smallest angle in a triangle with sides of length 4.5 cm, 6 cm and 7.2 cm.

6



Two tanks are situated at C and D, 250 m apart on level ground. They site a target X such that $\angle XCD = 72°$ and $\angle XDC = 65°$. Find the distance between the tank at C and the target, correct to the nearest metre.

7 A chord of length 35 cm subtends an angle θ at the centre of a circle with radius 22 cm. Find the angle θ, correct to the nearest minute.



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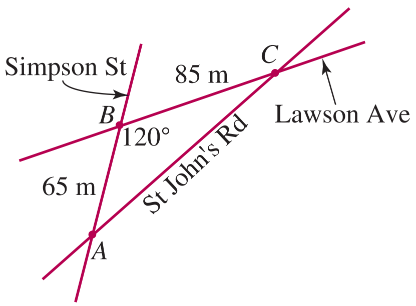
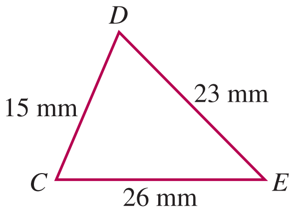


Zachary walked along St John's Rd from A to C instead of walking down Simpson St and then Lawson Avenue. Calculate the distance that he saved, correct to the nearest metre, by taking this route.

9 In ΔCDE, $CD = 15$ mm, $DE = 23$ mm and $CE = 26$ mm.

a Find the size of $\angle CDE$, correct to the nearest minute.

b Find the area of ΔCDE, correct to 3 significant figures.



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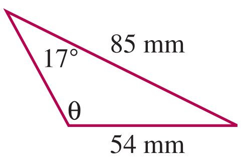


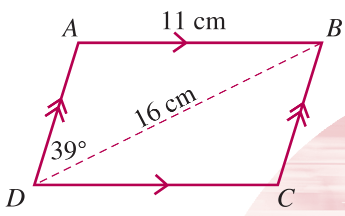
a Find the size of the obtuse angle θ, correct to the nearest degree.

b Hence, find the area of the triangle, correct to the nearest square millimetre.

11 In the parallelogram $ABCD$, $AB = 11$ cm, $BD = 16$ cm, $\angle ADB = 39°$ and $\angle DAB$ is obtuse.

a Find the size of $\angle DAB$, correct to the nearest degree.

b Find the length of AD, correct to the nearest centimetre.



12 In ΔABC, $\angle A = 69°$, $\angle B = 47°$ and $AC = 15$ cm.

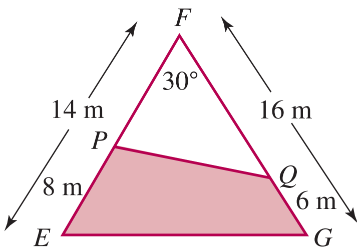
a Draw a diagram and mark on it all of the given information.

b Calculate the perimeter of the triangle, correct to 1 decimal place.

13 In ΔPQR, $\angle P = 60°$, $\angle R = 75°$ and $PQ = 8$ cm.

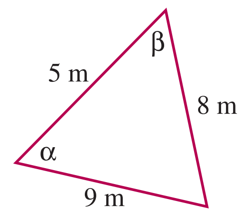
a Draw a diagram and mark on it all of the given information.

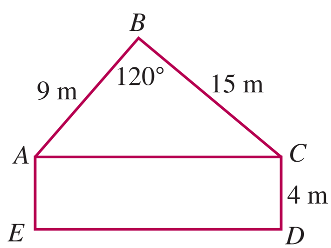
b Find the exact length of QR.

14 

In the diagram shown, $EF = 14$ m, $FG = 16$ m, $EP = 8$ m, $QG = 6$ m and $\angle EFG = 30°$. Find the shaded area.

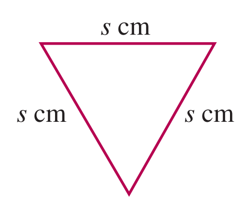
15 **a** **i** Find the angle α, correct to the nearest minute.
ii Hence, find the area of the triangle, correct to 1 decimal place.
b **i** Find the angle β, correct to the nearest minute.
ii Hence, find the area of the triangle, correct to 1 decimal place.

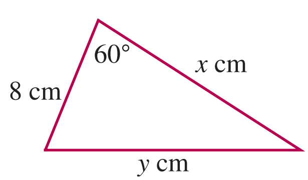


16 

a Find the length of AC.
b Calculate the total area of this figure.

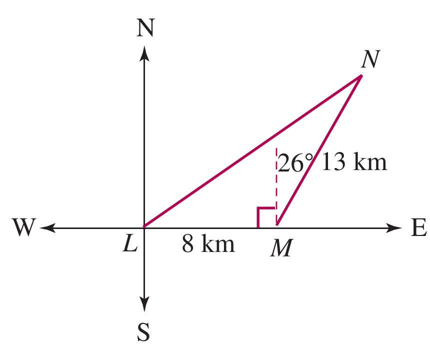
17 Show, by trigonometry, that the area of an equilateral triangle of side s cm is given by the formula $A = \frac{\sqrt{3}}{4}s^2$.



18 

The area of this triangle is $22\sqrt{3}$ cm^2.
a Find the value of x.
b Hence, find the exact value of y.

19 A bushwalker walks 8 km due east from L to M, then proceeds to walk a further 13 km to N on a bearing of 026°.
a Calculate his distance from L, correct to the nearest kilometre.
b Find his bearing from L, correct to the nearest degree.



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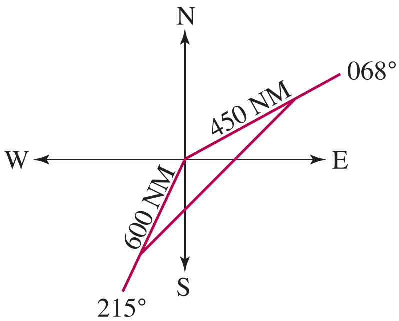
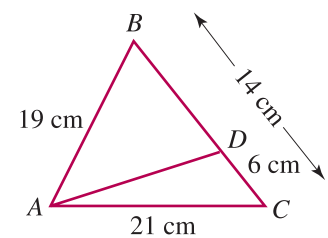


Two ships leave the same port simultaneously. One ship sails 450 nautical miles on a bearing of 068°, while the other ship sails 600 nautical miles on a bearing of 215°. How far apart are the ships then, correct to the nearest nautical mile.

■ Further applications

21 In ΔABC, $AB = 19$ cm, $BC = 14$ cm and $AC = 21$ cm. D is a point on BC such that $DC = 6$ cm.

- **a** Use the Cosine Rule in ΔABC to find the size of $\angle ABC$, correct to the nearest minute.
- **b** Use the Cosine Rule in ΔABD to find the length of AD, correct to 1 decimal place.



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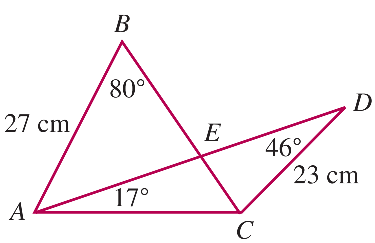
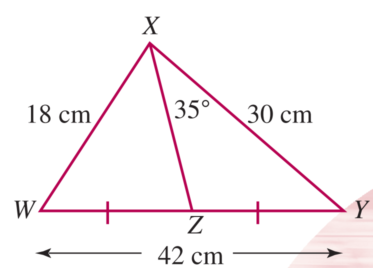


In the diagram, $AB = 27$ cm, $CD = 23$ cm, $\angle ABC = 80°$, $\angle ADC = 46°$ and $\angle DAC = 17°$.

- **a** Find the length of AC, correct to 3 significant figures.
- **b** Hence, find the size of $\angle ACB$, correct to the nearest degree.

23 In ΔWXY, Z is a point on WY such that XZ bisects WY.

- **a** Use the Sine Rule in ΔXYZ to find the size of $\angle XZY$, correct to the nearest degree.
- **b** Hence, find the size of $\angle WXZ$, correct to the nearest degree.



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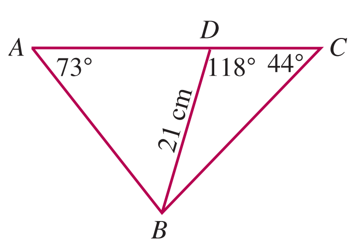
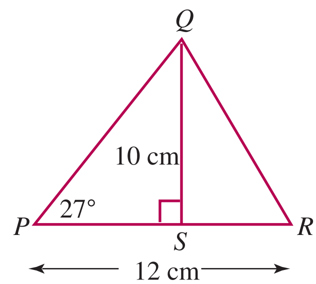


In ΔABC, $\angle BAC = 73°$ and $\angle ACB = 44°$. D is a point on AC such that $\angle BDC = 118°$ and $BD = 21$ cm.

- **a** Use the Sine Rule in ΔBCD to find the length of BC, correct to 1 decimal place.
- **b** Use the Sine Rule in ΔABC to find the length of AB, correct to 1 decimal place.

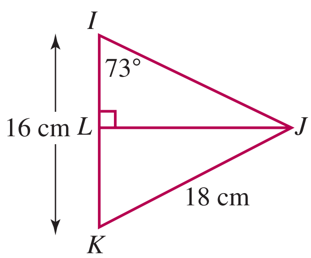
25 In ΔPQR, $PR = 12$ cm and $\angle QPR = 27°$. S is a point on PR such that $QS \perp PR$ and $QS = 10$ cm.



a Find the length of PQ in ΔPQS, correct to the nearest centimetre.

b Use the Cosine Rule in ΔPQR to find the length of QR, correct to the nearest centimetre.

26



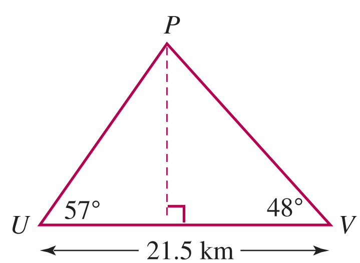
In ΔIJK, $IK = 16$ cm, $JK = 18$ cm and $\angle JIK = 73°$. L is a point on IK such that $JL \perp IK$.

a Use the Sine Rule in ΔIJK to find the size of $\angle IJK$, correct to the nearest degree.

b Write down the size of $\angle IKJ$.

c Hence, find the altitude JL, correct to the nearest millimetre.

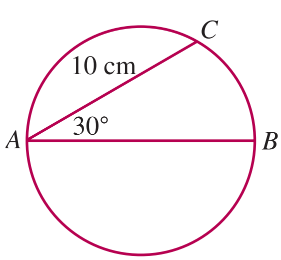
27 Two radar stations U and V are situated on level ground, 21.5 km apart. A plane P is sighted simultaneously by the two stations. The angles of elevation of the plane from U and V at this time are 57° and 48° respectively.



a Find the distance PU, correct to 2 decimal places.

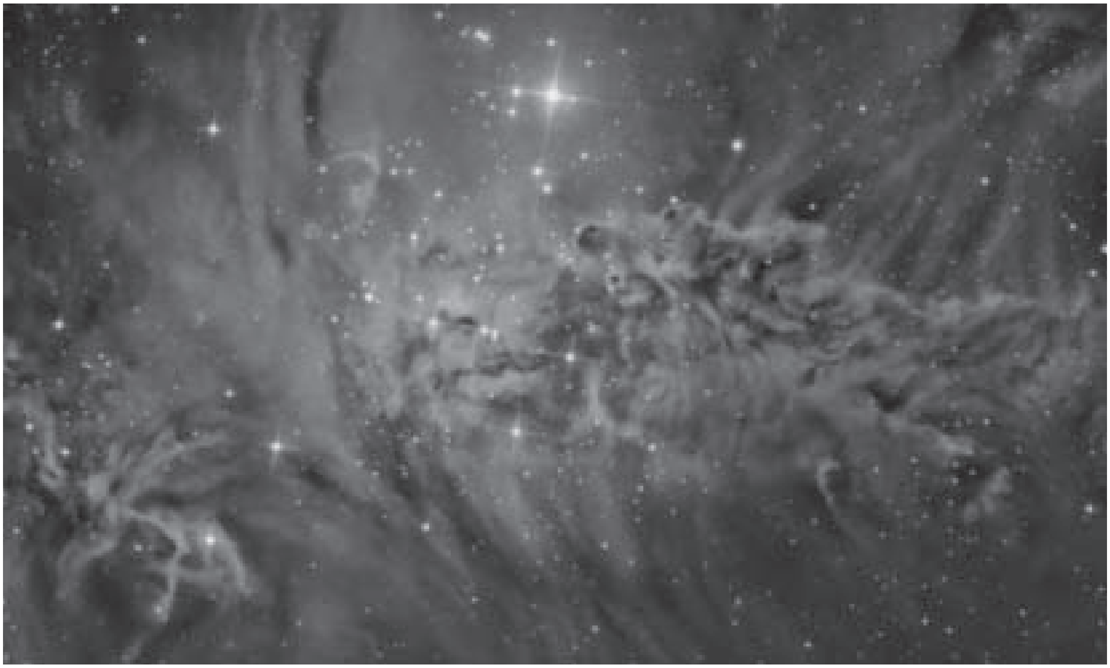
b Calculate the altitude of the plane, in metres, correct to the nearest 100 metres.

TRY THIS Chord and radius



If AB is the diameter of a circle, and AC is a chord 10 cm long, what is the exact length of the radius of the circle if $\angle BAC = 30°$?

MEASURING THE DISTANCE TO THE STARS



Introduction

The origin of the universe, how it came to be formed, and our place in it as human beings, has occupied the thoughts of many people through the ages. We know that the stars we can see are part of a huge galaxy of which our solar system and, inside it planet Earth, are a very small part. Photography plays a major role in astronomical research. High resolution images taken through a powerful telescope not only reveal the colour and structure of a universe largely hidden from us, they are also of exceptional beauty. Further, they can reveal the vast emptiness of space by taking us to galaxies of stars far beyond our own.

In this focus on working mathematically we will follow in the steps of the mathematician and astronomer who first measured the distance to the stars of our galaxy with accuracy, taking into account the motion of the Earth and many other variables which effected his observations. His name was Friedrich Wilhelm Bessel. Bessel used the phenomenon of *parallax* to determine the distance to the star called 61Cygni in the constellation of the Swan as 10.1 light years, announcing his result in 1838. Up until then parallax had been used successfully to determine the distances from Earth to the planets, but the deep space beyond was a mystery.

Parallax

Hold up a pen or pencil in front of your face so that it obscures an object in your vision. Now close each eye in turn. Each eye looks at the pen from a slightly different direction. As you open and close each eye the pen appears to move to the left or right. The closer the object the more apparent the movement. This phenomenon is called *parallax*.

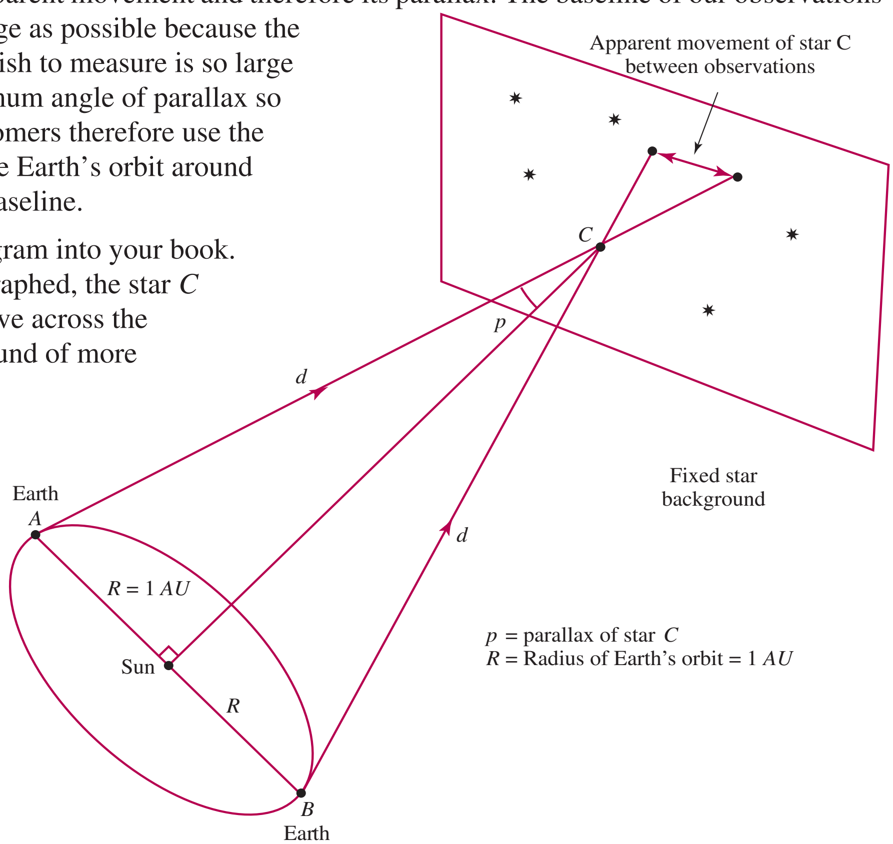
Now imagine the pen is a distant star. When looked at through a telescope from two different positions, the movement of the star against a fixed background of more distant stars is very small and very difficult to detect. This is why we tend to see the stars as fixed in the sky. Bessel measured the parallax of 61Cygni as 0.314 *arcseconds*. An arcsecond (symbol ″ or *as*) is 1/3600 of a degree, so you can see how small the movement is. This was a brilliant effort. With modern techniques such as high resolution photography and very powerful telescopes the correct value is 0.294″.

Learning activities

Observing parallax

1 We need to observe the star relative to a fixed background from two different positions to observe its apparent movement and therefore its parallax. The baseline of our observations must be as large as possible because the distance we wish to measure is so large and the maximum angle of parallax so small. Astronomers therefore use the diameter of the Earth's orbit around the Sun as a baseline.

Copy this diagram into your book. When photographed, the star C appears to move across the fixed background of more distant stars.



Measuring parallax

2 This is the difficult part! You have to make sure you take account of every variable that could interfere with the observations. Bessel had to accurately calculate the positions of the stars on fixed dates of observation and eliminate the effects of the Earth's motion and other sources of error.

Look at your diagram. If you made your first observation from A on a certain date, how long would you have to wait before being able to observe the maximum parallax? Where would the Earth then be?

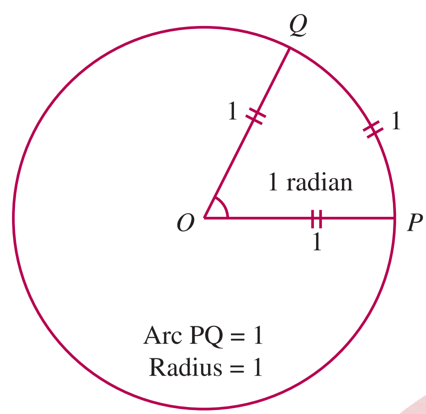
3 If you have access to the internet you can observe stellar parallax at <www.astro.washington.edu/labs/parallax>. Click on the Parallax Java demo to observe an animation. You can also measure parallax from photographs.

The calculation of the distance of the star from Earth

4 The angle of parallax p is defined as *half* of the observed movement as shown in the diagram. This makes it easy to directly apply trigonometry to ΔABC. Write down sin p and derive an expression for d. Note that in astronomy, R is a standard unit called the Astronomical Unit or AU. 1 AU is 150 million km.

5 Bessel observed the parallax p as 0.314″. However before we do the calculation it is necessary to express the angle p in *radians*. This takes us ahead of year 10 work. If you wish skip this question and go to question 7. Check with your teacher.

One radian is the angle at the centre of a circle subtended by an arc equal in length to the radius. Copy the diagram into your book and record the conversion below.



The circumference of the circle is 2π so the arc PQ is $\frac{1}{2\pi}$ of the circumference. Expressed in degrees, the angle POQ is therefore $\frac{1}{2\pi}$ of 360° (about 57. 296° or 57° 18′).

$$1 \text{ radian} = \frac{1}{2\pi} \text{ of } 360°$$

$$\text{so} \quad \pi \text{ radians} = 180°$$

6 Using the information in question 5 show that the parallax

$$p = 1.5223149 \times 10^{-6} \text{ radians}$$

7 Now use the equation for d in question 4 to calculate d in AU. Note that $R = 1$AU and using a calculator show that sin $p = p = 1.5223149 \times 10^{-6}$. Hence show that d is approximately 656 894 AU. (Check with your teacher as to why the value of sin p is close to p for small angles measured in radians.)

8 The distance to 61Cygni from Earth is over 656 000 times the radius of Earth's orbit around the Sun. This is a huge distance. A *light year,* the distance light travels in one year, is another convenient unit in astronomy. Show that the value of d above is equivalent to 10.4 light years (take the speed of light as 300 000 km/s).

Challenge activities

1 Bessel's work turned astronomers' interest away from the solar system to the stars beyond. For the first time the dimensions of the universe beyond our solar system became known. It was not long before astronomers found that 61Cygni was 19th in order of distance from Earth. The closest star is Alpha Centauri in the constellation of the Southern Cross, about $4\frac{1}{3}$ light years away.

   a Given the parallax of 61Cygni as 0.29″ at a distance of 11 light years, estimate the parallax of Alpha Centauri. Give a reason why it is larger than the parallax of 61Cygni.

   b Using the method of questions 4 to 8 show that Alpha's parallax is approximately 0.75″.

2 Search the internet to find out how parallax could be used to calculate the distance from Earth to a closer object like the moon. In this case two observations from Earth are made simultaneously. Find out why this is necessary. It turns out that the moon moves in an ellipse around the Earth, not a circle, and that its parallax varies between 53.9 and 61.5 minutes. Find out the mean distance to the moon and different techniques used to measure it today.

3 For those of you planning to study mathematics at university you will come across *Bessel functions*, another legacy left by this great mathematician from his work in astronomy. Check out how they came to be important.

Let's communicate

In no more than half a page, give some examples of how trigonometry plays a vital role in astronomy.

Reflecting

Clever measurement (photography still plays a strong role) and simple mathematics has made it possible to study the far reaches of the universe. Reflect on the power of mathematics as a tool in uncovering the nature of the physical world.

Language link with Macquarie

MACQUARIE

1 *In a short sentence explain*
- **a** *the abbreviation* **sin**, **cos** *and* **tan**
- **b** *what the* **sine rule** *is used for*
- **c** *what the* **cosine rule** *isused for*
- **d** *a* **compass bearing**
- **e** *an* **angle of elevation**

2 *The* Macquarie Learners Dictionary *defines the word* **tangent** *as follows:*

tangent *noun* **1.** *Specialised* in geometry, a straight line which touches a curve. **2.** a sudden new direction: *I find it hard to follow his conversation because he keeps flying off at a tangent.*

Knowing the precise mathematical meaning of a **tangent**, *how do you think the phrase* **"flying off at a tangent"** *might have come into common usage?*

1 State whether the angle θ is acute or obtuse, where $0° < \theta < 180°$, if:
- **a** $\cos\theta > 0$ and $\tan\theta > 0$
- **b** $\sin\theta > 0$ and $\cos\theta < 0$.

2 Express each of these trigonometric ratios in terms of an acute angle, then evaluate correct to 2 decimal places.
- **a** sin 110° **b** cos 130°
- **c** tan 165° **d** sin 125°48′
- **e** tan 142°11′ **f** cos 153°22′

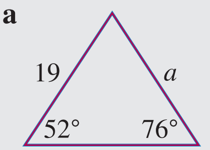
3 Find two possible angles θ, where $0° < \theta < 180°$, correct to the nearest degree.
- **a** $\sin\theta = 0.7124$ **b** $\sin\theta = 0.3867$

4 Find θ, where $0° < \theta < 180°$, correct to the nearest degree.
- **a** $\cos\theta = -0.4052$
- **b** $\tan\theta = -2.8665$

5 Find the exact value of:
- **a** sin 135° **b** cos 120°
- **c** tan 150°

6 Find the value of each pronumeral, correct to 1 decimal place. All lengths are in cm.

a



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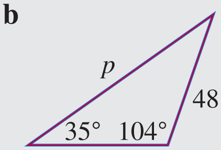

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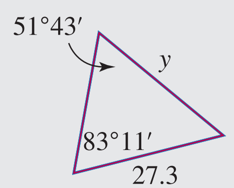

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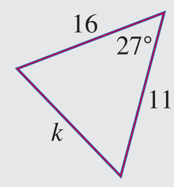

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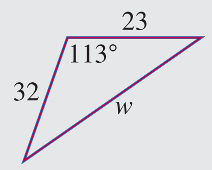

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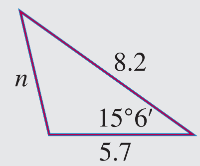
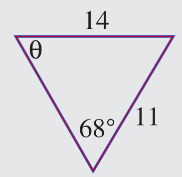


7 Find the angle θ in each triangle, correct to the nearest minute. All lengths are in mm. Angles that appear obtuse are obtuse.

a



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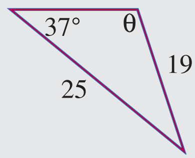

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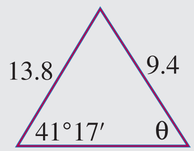

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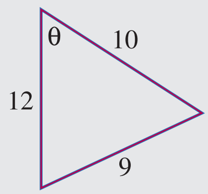

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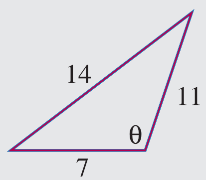

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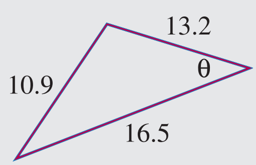


8 Two straight roads PQ and QR meet at an angle of 58°. The roads PR and QR meet at an angle of 73°.
If $PQ = 712$ m, find the length of the road QR, correct to the nearest metre.

9 In ΔLMN, $LM = 12$ cm, $MN = 13$ cm and $LN = 10$ cm. Find, correct to the nearest minute, the size of the largest angle.

10 Libby drove 105 km from T to U on a bearing of 305°. She then turned and drove on a bearing of 209° to V, which is due west of T.

a Draw a diagram and mark on it all of the given information.

b Find $\angle UTV$ and $\angle TUV$.

c Hence, find the distance VT, correct to the nearest km.

11 Ross sailed 24 nautical miles on a bearing of 057° from C to D. He then turned and sailed 30 nautical miles to E, which is due east of C.

a Draw a diagram and mark on it all of the given information.

b Find the size of $\angle CED$, correct to the nearest degree.

c Hence, find the bearing of E from D.

12 Harrison walked 450 m from X to Y on a bearing of 214° while orienteering. He then turned and walked 520 m to Z, which is 610 m due west of X.

a Draw a diagram and mark on it all of the given information.

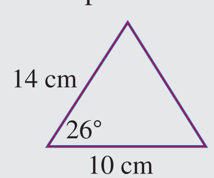
b Use the Cosine Rule to find $\angle ZXY$, correct to the nearest degree.

c Find the bearing of Y from Z.

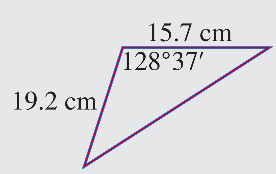
d Find the bearing of Z from Y.

13 Find the area of each triangle, correct to 1 decimal place.

a



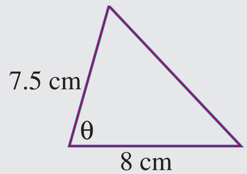
b



CHAPTER REVIEW

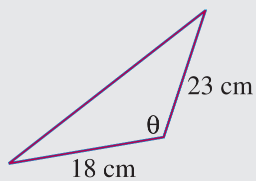
14 Find the angle θ in each of these, correct to the nearest minute. Angles that appear obtuse are obtuse.

a



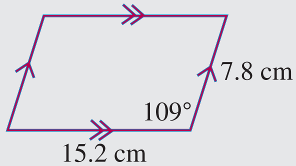
Area = 28.4 cm^2

b



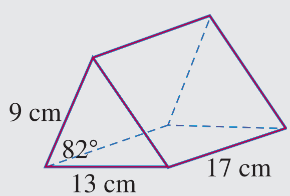
Area = 122 cm^2

15 Find the area of this parallelogram, correct to 1 decimal place.



16 Find the volume of each solid, correct to 1 decimal place.

a



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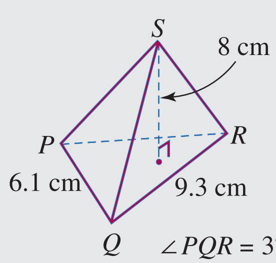
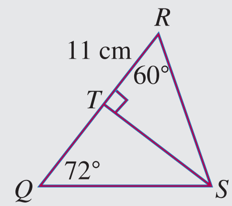


$\angle PQR = 37°15'$

17 A triangle has sides of length 19 mm, 24 mm and 27 mm. Find the area of the triangle, correct to the nearest mm^2.

18 In ΔQRS, T is a point on QR such that $ST \perp QR$, $\angle Q = 72°$, $\angle R = 60°$ and $TR = 11$ cm.

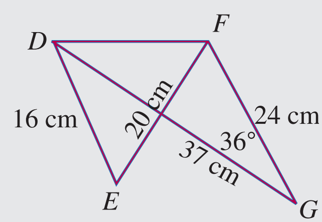


a Find the length of RS.

b Write down the size of $\angle QSR$.

c Use the Sine Rule in ΔQRS to find the length of QT, correct to 1 decimal place.

19 In the diagram, $DE = 16$ cm, $EF = 20$ cm, $DG = 37$ cm, $FG = 24$ cm and $\angle DGF = 36°$.



a Use the Cosine rule in ΔDFG to find the length of DF, correct to 3 significant figures.

b Hence, find the size of $\angle DEF$, correct to the nearest minute.

CHAPTER REVIEW

11 Similarity

This chapter at a glance

Stage 5.1/5.2/5.3

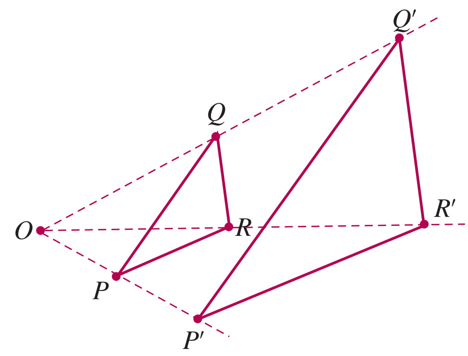
After completing this chapter, you should be able to:

- enlarge or reduce a figure using a given scale factor
- explain the difference between congruent figures and similar figures
- determine the scale factor for a pair of similar figures
- determine whether two polygons are similar
- identify the matching sides and angles in similar figures
- calculate dimensions of similar figures using the enlargement or reduction factor
- name a pair of similar triangles, giving their vertices in matching order
- use the tests for similar triangles to prove that two triangles are similar
- apply geometrical facts, properties and relationships to find sides and angles in diagrams containing similar triangles
- solve practical problems that involve similar triangles
- write formal deductive proofs involving the tests for similar triangles
- prove and apply similarity theorems involving the midpoints of sides of triangles.

11.1 Similar figures

Enlargement

Figures can be enlarged or reduced by choosing a fixed point O, called the **centre of enlargement**. If all of the sides in a figure are doubled, then we say that the **scale factor** is 2. This is often written as $k = 2$. The new figure resulting from the enlargement is called the **image**.



When a figure PQR is enlarged or reduced by a factor of k, the resulting image is $P'Q'R'$.

In the previous diagram, the figure PQR has been enlarged by a factor of 2 by doubling the distances OP, OQ and OR. That is, $OP' = 2 \times OP$, $OQ' = 2 \times OQ$ and $OR' = 2 \times OR$. It then follows that $P'Q' = 2 \times PQ$, $Q'R' = 2 \times QR$ and $P'R' = 2 \times PR$.

- **If $k > 1$, then the figure will be enlarged. The scale factor in this case is an *enlargement factor*.**
- **If $0 < k < 1$, then the figure will be reduced. The scale factor in this case is a *reduction factor*.**

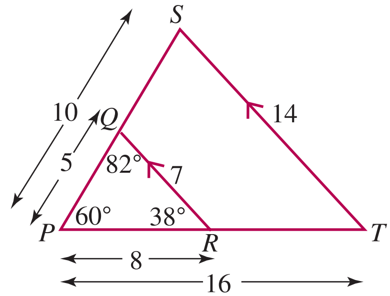
For example, if $k = 3$, the figure will be enlarged and all lengths on the image will be 3 times the corresponding lengths on the original figure. If $k = \frac{1}{3}$, then the figure will be reduced and all lengths on the image will be $\frac{1}{3}$ of the corresponding lengths on the original figure.

Similar figures

Similar figures are figures that have the same shape but not necessarily the same size. For example, when a picture is enlarged or reduced on a photocopier, the picture that results is similar to the original. If one figure is an enlargement or reduction of another, then the two figures are similar.

Similar figures have the same shape but not necessarily the same size.

Angles are not affected by the enlargement transformation. Consider, for example, a triangle with sides 5 cm, 7 cm and 8 cm. The angles in this triangle are approximately 60°, 82° and 38°.



From the diagram it follows that:

- $\angle PST = \angle PQR$ (Corresponding angles, $QR \parallel ST$)
- $\angle PTS = \angle PRQ$ (Corresponding angles, $QR \parallel ST$)
- $\angle P$ is a common angle.

While the lengths of the sides have doubled, the sizes of the angles have not changed. The angle sum of any triangle is 180°, no matter its shape or size. Therefore, the angle sum cannot increase or decrease if the sides are extended to, say, twice their original length. That is, doubling the lengths of the sides does not result in a doubling of the individual angles in the triangle.

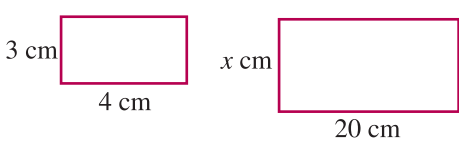
In similar figures, all pairs of matching sides are in the same ratio and all pairs of matching angles are equal.

Example 1

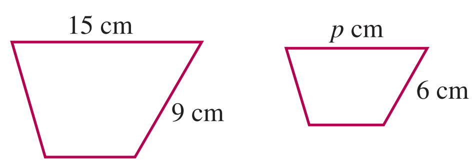
i State the enlargement or reduction factor for these similar figures.

ii Use the scale factor to find the value of the pronumeral.

a



b



Solutions

a Enlargement factor $= \dfrac{20}{4}$

$= 5$

$\therefore x = 5 \times 3$

$= 15$

b Reduction factor $= \dfrac{6}{9}$

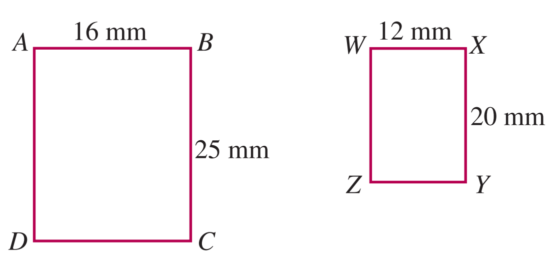
$= \dfrac{2}{3}$

$\therefore p = \dfrac{2}{3} \times 15$

$= 10$

Example 2

Find the ratios of the matching sides of these rectangles.
Are the rectangles similar?



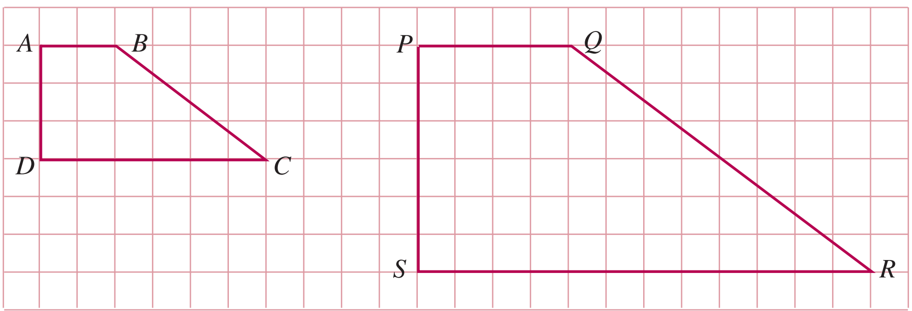
Solution

$\frac{WX}{AB} = \frac{12}{16}$ $\qquad$ $\frac{XY}{BC} = \frac{20}{25}$ $\qquad$ $\frac{WX}{AB} \neq \frac{XY}{BC}$

$= \frac{3}{4}$ $\qquad$ $= \frac{4}{5}$ $\qquad$ $\therefore$ The rectangles are not similar.

Exercise 11.1

1 The quadrilaterals shown are similiar.



a State the lengths of the sides in each figure in mm.

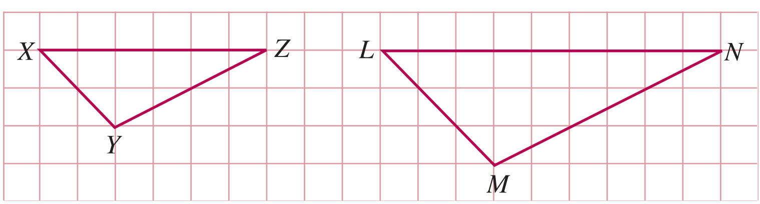
b Find the following ratios in simplest form.

i $AB:PQ$ **ii** $BC:QR$ **iii** $CD:RS$ **iv** $DA:SP$

c What can you say about the ratios of the matching sides?

d What is the scale factor?

2 The triangles shown are similar.

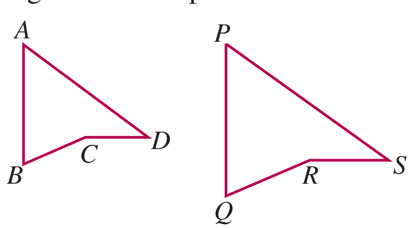


a Measure the angles in each triangle.

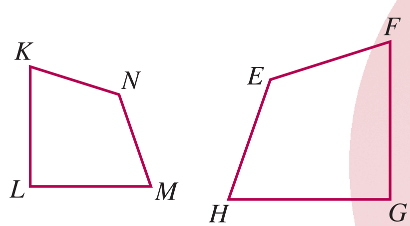
b Are the angles in ΔXYZ equal to the matching angles in ΔLMN?

3 The figures in each pair are similar. Write down all pairs of matching sides.

a

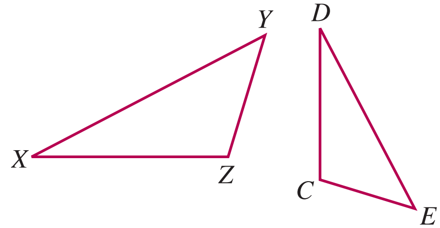


b



4 The figures in each pair are similar. Write down all pairs of matching angles.

a



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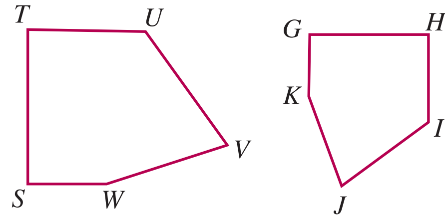


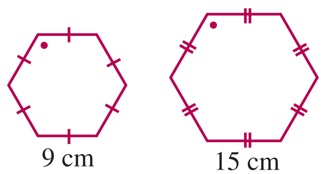
■ Consolidation

5 State whether the following figures are always similar.

a equilateral triangles **b** isosceles triangles **c** squares
d rectangles **e** parallelograms **f** rhombuses

6 State the enlargement factor for these similar figures.

a



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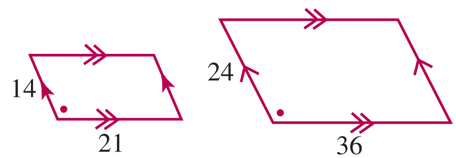
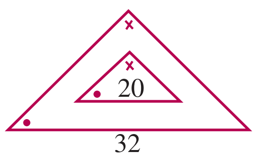


c



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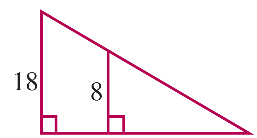
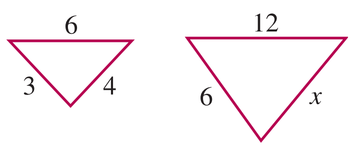


7 Find the scale factor for each pair of similar figures. Hence find the value of the pronumeral. All lengths are in cm.

a



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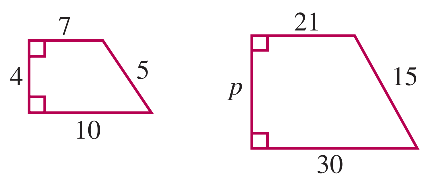
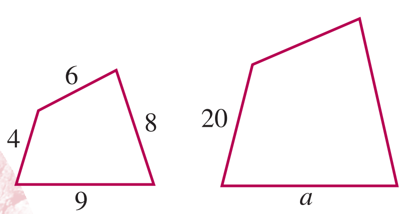


c



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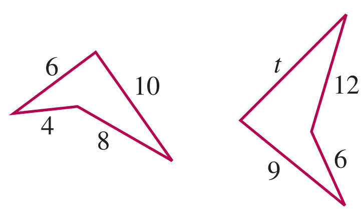
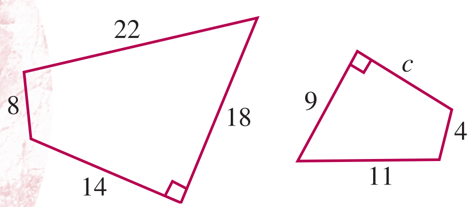
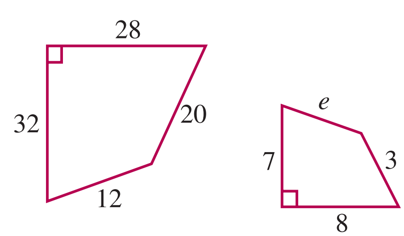


e



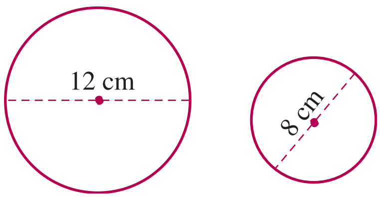
f



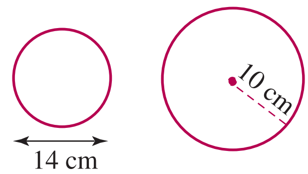
8 **a** Are all circles similar?
b Are all semicircles similar?

9 Find the scale factor for each pair of circles.

a

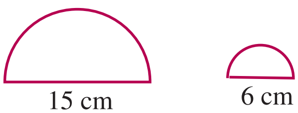


b



10 Find the scale factor for each pair of semicircles.

a



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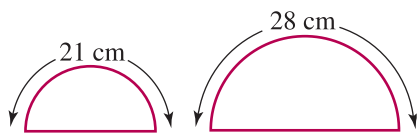
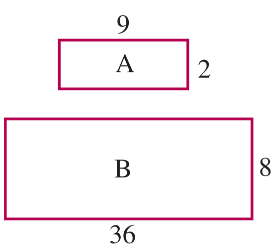


11 **a** Are all congruent figures similar?
b Are all similar figures congruent?

12 For each pair of rectangles below, find the ratios of the matching sides (figure A : figure B). Hence determine whether the rectangles are similar. All lengths are in metres.

a



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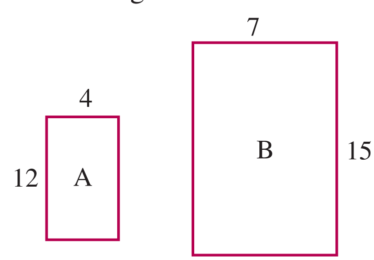
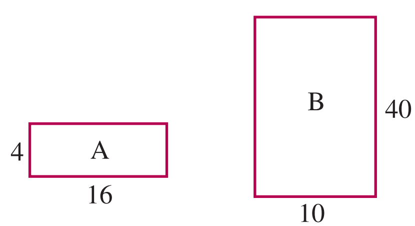


c



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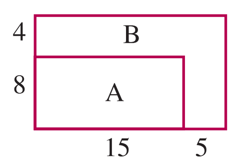
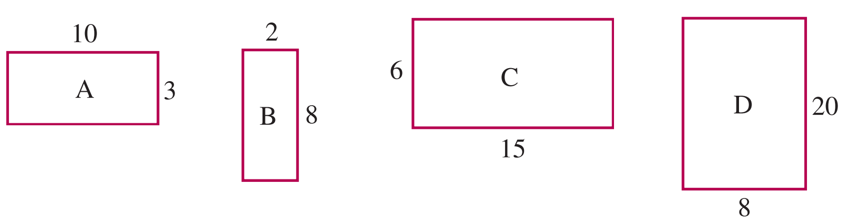


13 Find the similar figures in each set.

a



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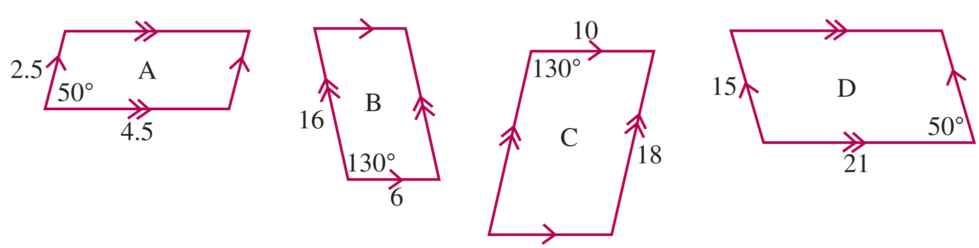

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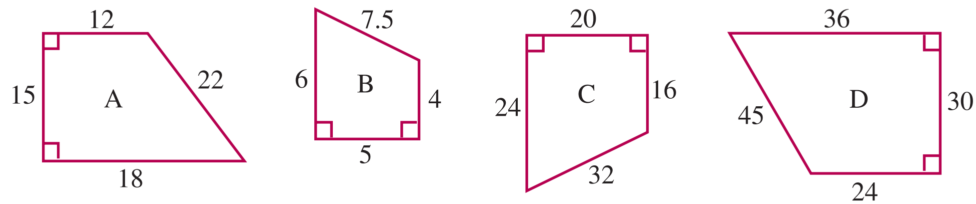
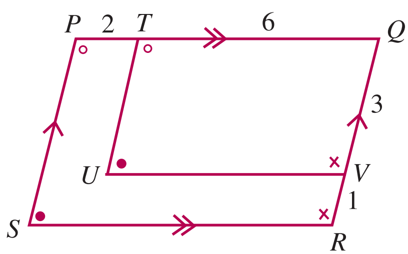
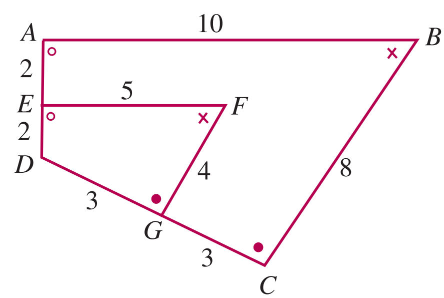


■ Further applications

14 a Prove that *PQRS* ||| *TQVU*.



b Prove that *ABCD* ||| *EFGD*.



TRY THIS **Investigation**

1 For each of the following sets of measurements:

i construct ΔABC and ΔPQR

ii state whether the triangles are equiangular

iii state whether the triangles are similar.

a In ΔABC, $AB = 3$ cm, $BC = 4$ cm, $AC = 6$ cm.
In ΔPQR, $PQ = 6$ cm, $QR = 8$ cm, $PR = 12$ cm.

b In ΔABC, $AB = 4$ cm, $\angle ABC = 70°$, $BC = 6$ cm.
In ΔPQR, $PQ = 8$ cm, $\angle PQR = 70°$, $QR = 12$ cm.

c In ΔABC, $\angle ABC = 90°$, $AB = 3$ cm, $AC = 5$ cm.
In ΔPQR, $\angle PQR = 90°$, $PQ = 6$ cm, $PR = 10$ cm.

2 Make a summary of your findings about similar triangles from this investigation.

11.2 Tests for similar triangles

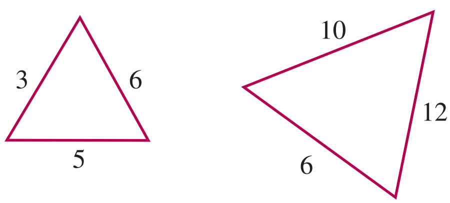
Two triangles are said to be similar if they have the same shape but not necessarily the same size. One triangle can be obtained from the other by either an enlargement or a reduction transformation. The matching sides in the triangles are in proportion (in the same ratio) and the matching angles are equal. The symbol ||| is used to mean 'is similar to'.

If two triangles are similar, then:
- **the matching sides are in proportion**
- **the matching angles are equal.**

As is the case with congruent triangles, it is not necessary to know the measurements of every side and every angle in a pair of triangles in order for us to be certain that the triangles are similar. Having completed the previous investigation, you should have deduced the following four tests for similar triangles. These tests represent the *minimum amount of information* that is needed to prove that two triangles are similar.

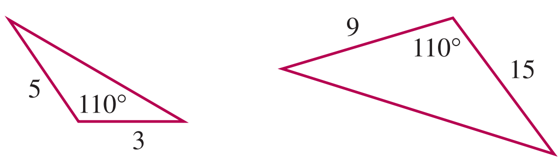
1 If the three sides of one triangle are respectively proportional to the three sides of another triangle, then the triangles are similar.

For example,



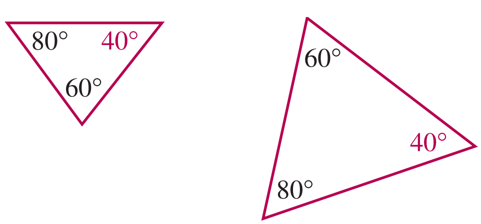
2 If two sides of one triangle are respectively proportional to two sides of another triangle, and the included angles are equal, then the triangles are similar.

For example,



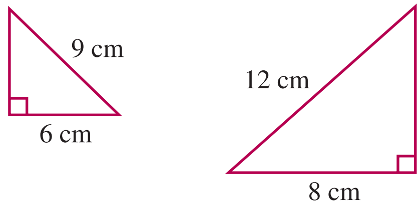
3 If two angles of one triangle are respectively equal to two angles of another triangle, then the triangles are similar.

For example,



4 If the hypotenuse and a second side of a right-angled triangle are respectively proportional to the hypotenuse and a second side of another right-angled triangle, then the triangles are similar.

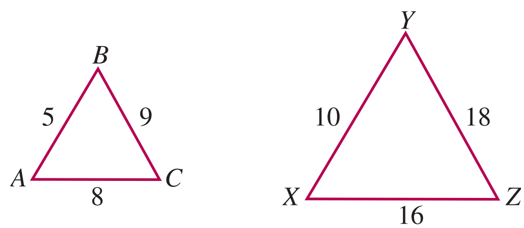
For example,



NOTE: (1) If in two triangles, two pairs of matching angles are equal, then the third pair of angles must also be equal, because the angle sum of each triangle is 180°. Therefore, in order to prove that two triangles are similar, it is sufficient to show that two pairs of angles are equal.

(2) When naming similar figures, the vertices must be given in matching order. That is, for the triangles below, we would write $\Delta ABC ||| \Delta XYZ$.

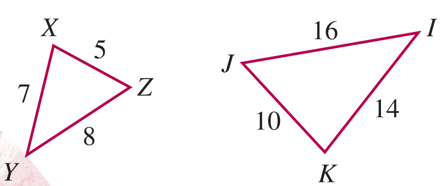
(3) By convention, abbreviations such as SSS, SAS, AA and RHS are not used in proofs involving similar triangles.



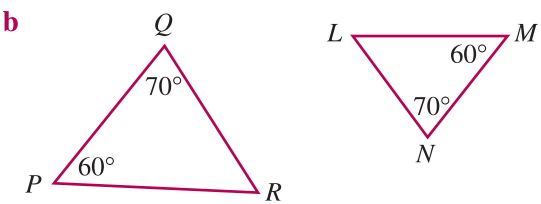
Example 1

Show that each pair of triangles are similar.

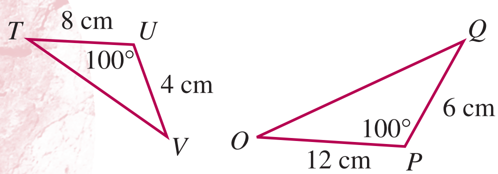
a



b



c



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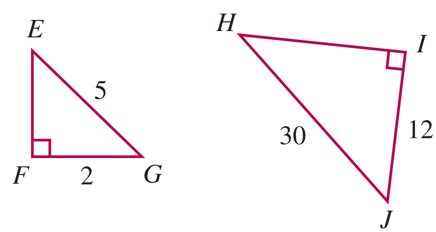


Solutions

a In ΔXYZ and ΔIJK

- $\frac{XZ}{JK} = \frac{5}{10} = \frac{1}{2}$
- $\frac{XY}{IK} = \frac{7}{14} = \frac{1}{2}$
- $\frac{YZ}{IJ} = \frac{8}{16} = \frac{1}{2}$

$\therefore \Delta XYZ \ ||| \ \Delta KIJ$

(three pairs of matching sides are in proportion)

b In ΔPQR and ΔLMN

- $\angle P = \angle M$
- $\angle Q = \angle N$

$\therefore \Delta PQR \ ||| \ \Delta MNL$

(two pairs of matching angles are equal)

c In ΔTUV and ΔOPQ

- $\frac{TU}{OP} = \frac{8}{12} = \frac{2}{3}$
- $\frac{UV}{OP} = \frac{4}{6} = \frac{2}{3}$
- $\angle TUV = \angle OPQ$

$\therefore \Delta TUV \ ||| \ \Delta OPQ$

(two pairs of matching sides are in proportion and the included angles are equal)

d In ΔEFG and ΔHIJ

- $\frac{FG}{IJ} = \frac{2}{12} = \frac{1}{6}$
- $\frac{EG}{HJ} = \frac{5}{30} = \frac{1}{6}$
- $\angle EFG = \angle HIJ = 90°$

$\therefore \Delta EFG \ ||| \ \Delta HIJ$

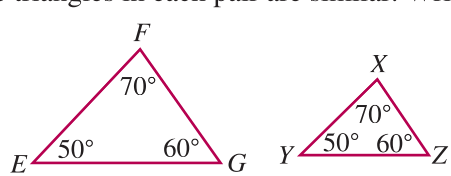
(the hypotenuse and a second side of each right-angled triangle are in proportion)

Exercise 11.2

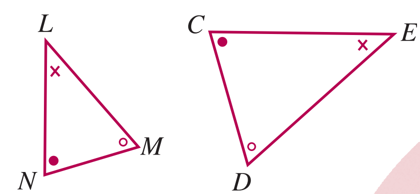
The questions in this exercise are designed to be done without formal proofs. These will be covered in exercise 11.4.

1 The triangles in each pair are similar. Write down all pairs of matching sides.

a

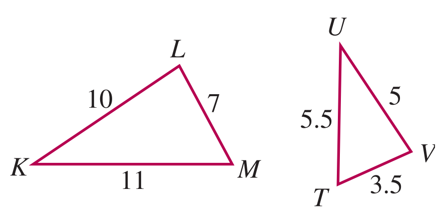


b

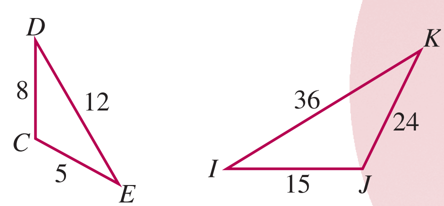


2 The triangles in each pair are similar. Write down all pairs of matching angles. All lengths are in metres.

a

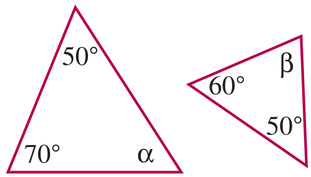


b



3 Determine whether the triangles below are similar by first finding the missing angles.

a



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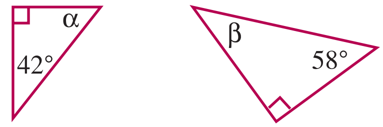
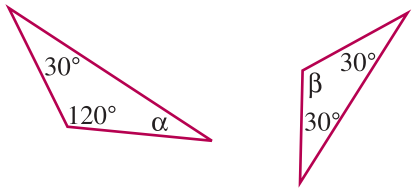
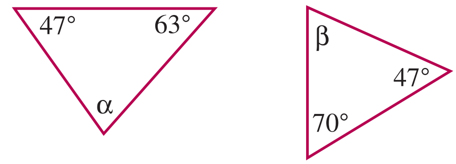


c

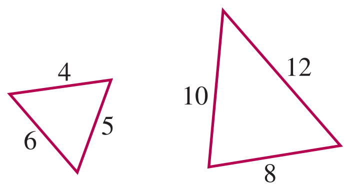


d

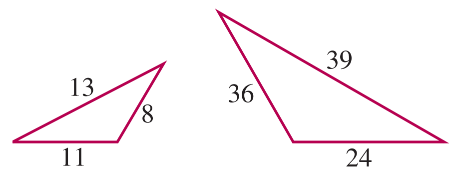


4 Find the ratio of the matching sides in each pair of triangles. Hence determine whether the triangles are similar.

a



b



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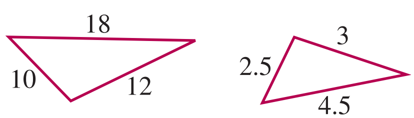
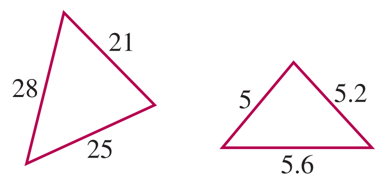
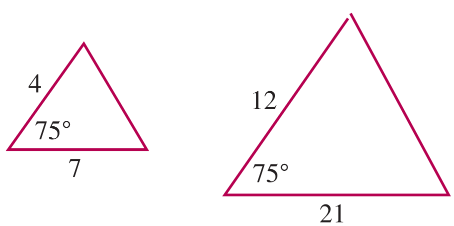


d

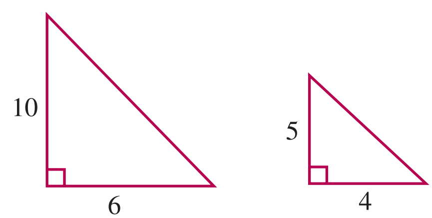


5 Find the ratio of the matching sides in each pair of triangles. Hence determine whether the triangles are similar.

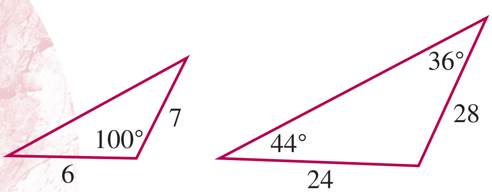
a



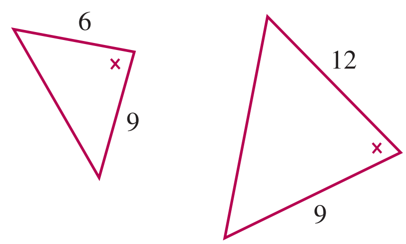
b



c

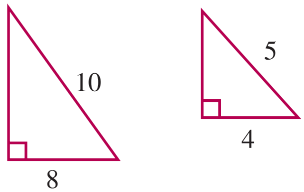


d

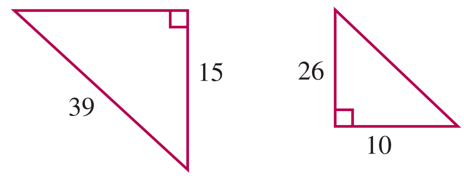


6 Find the ratio of the matching sides in these right-angled triangles. Hence determine whether the triangles are similar.

a



b



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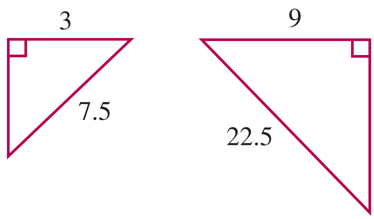

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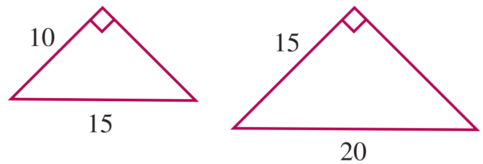
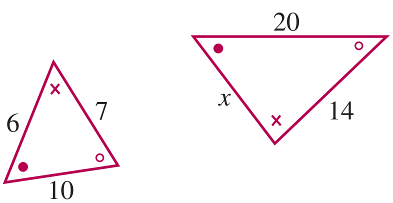


Consolidation

7 The triangles in each pair are similar. Find the value of the pronumeral. Give your answers as fractions or mixed numerals where necessary. All lengths are in mm.

a



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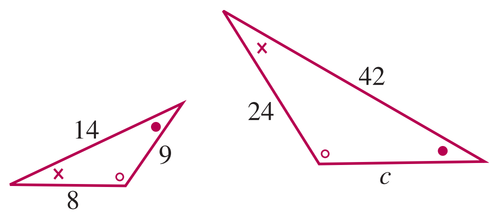

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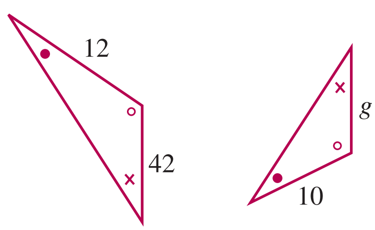

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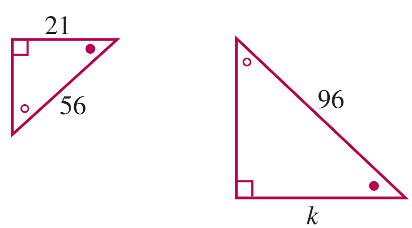
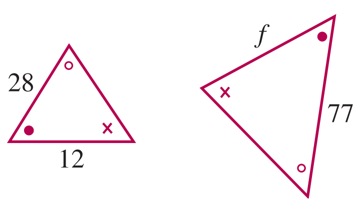


e



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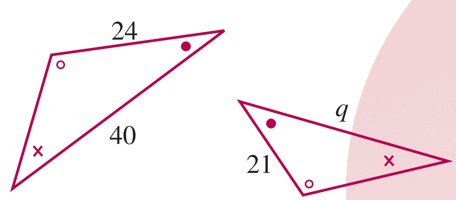
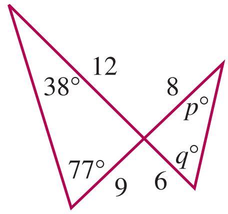
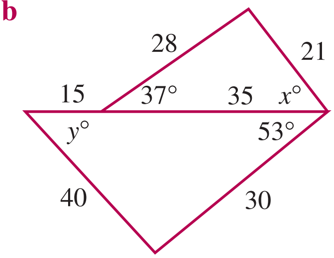


8 Explain why the triangles in each figure are similar. Hence find the value of all pronumerals.

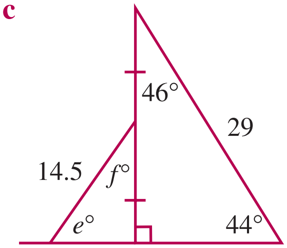
a



b

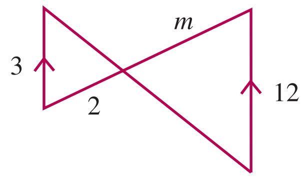


c



9 Explain why the triangles in each figure are similar. Hence find the value of the pronumeral.

a



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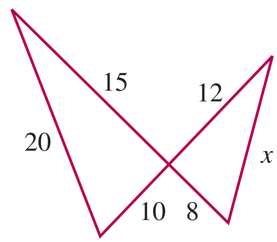

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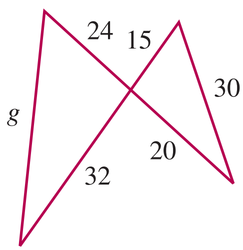

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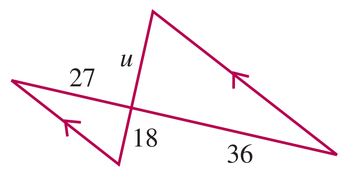
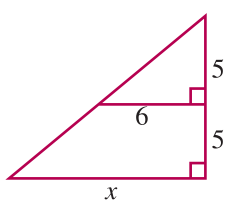


10 Show by marking equal angles that the triangles in each figure are similar. Hence find the value of the pronumeral.

a



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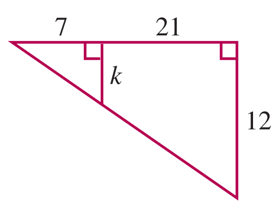

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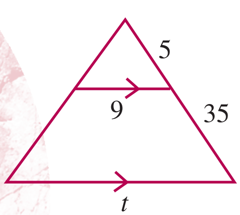

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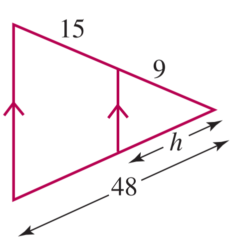

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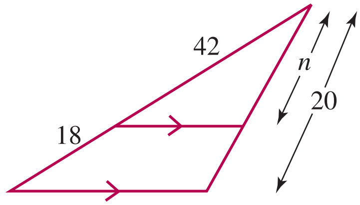

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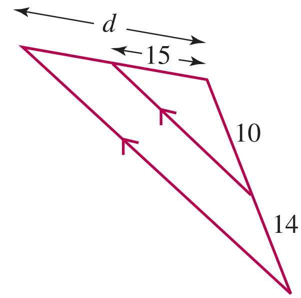
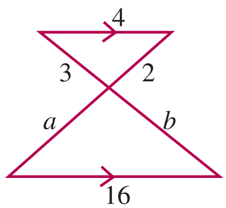
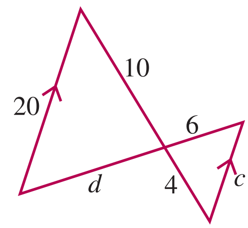


11 Find the value of all pronumerals in the similar triangles below. All measurements are in cm.

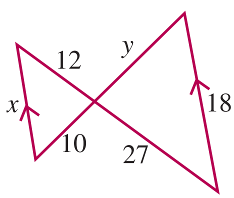
a



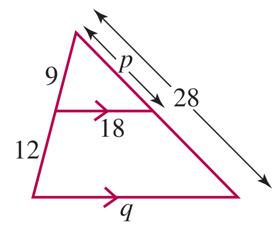
b



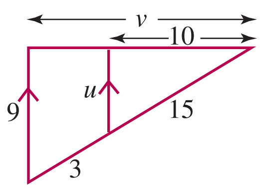
c



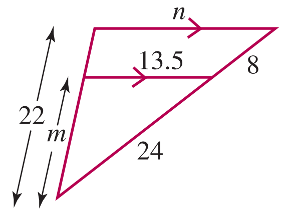
d



e



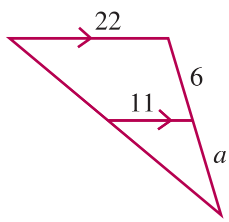
f



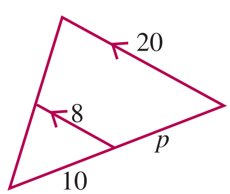
■ Further applications

12 Find the value of all pronumerals in the similar triangles below. All measurements are in metres.

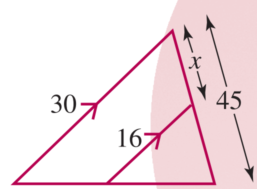
a



b



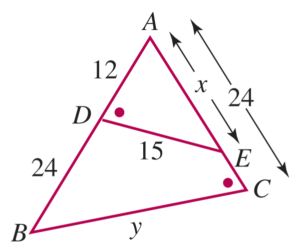
c



13 In each of the following:

- **i** redraw the triangles separately in the same orientation, showing the equal angles and any sides marked
- **ii** verify that the triangles are similar by marking the equal angles
- **iii** use the scale factor to find the value of all pronumerals.

a



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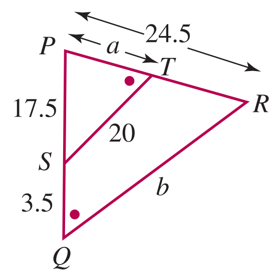

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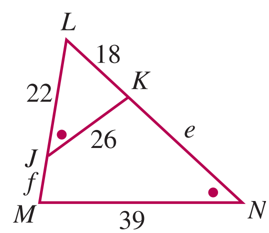
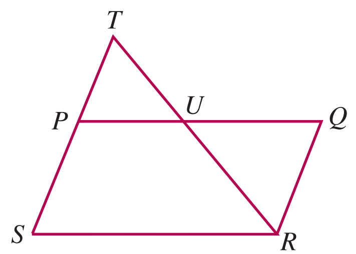


TRY THIS **Similar triangles 1**

PQRS is a parallelogram, with *SP* produced to *T*. If $TP = 2$ cm, $QR = 3$ cm and $SR = 8$ cm, find the length of *UQ*.

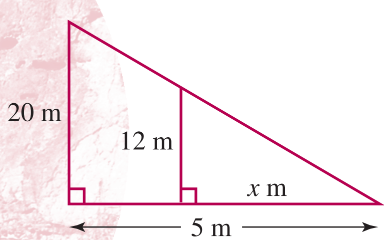


11.3 Solving problems that involve similar triangles

The practical problems in this exercise can be solved by using the properties of similar triangles. In each question you need to determine the scale factor between the triangles, then use it to find the required length or distance. You are not required to prove that the triangles are similar.

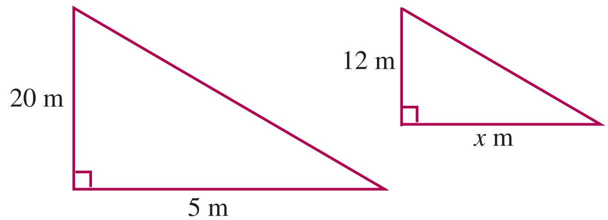
Example

EG +S



A building 20 m tall casts a shadow of length 5 m when the sun is almost directly overhead. At the same time, a tree of height 12 m casts a shadow in the same line as the shadow cast by the building. Find the length of the shadow cast by the tree.

Solution



$$\frac{x}{5} = \frac{12}{20}$$

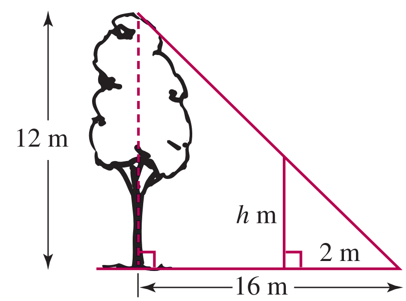
$$\times 5 \quad \times 5$$

$$\therefore x = 3$$

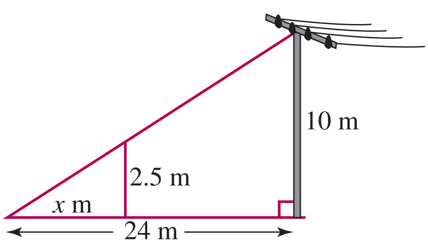
$\therefore$ The tree casts a shadow of length 3 m.

Exercise 11.3

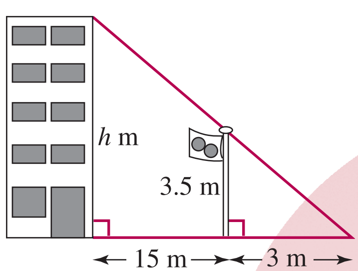
1 Annaleis used a tree and its shadow to measure her height. The tree of height 12 m casts a shadow of length 16 m. Annaleis stands 2 m from the end of the shadow so that her shadow finishes at the same point as that of the tree. How tall is Annaleis?



2 A telegraph pole is 10 m high. At 4 pm the pole casts a shadow of length 24 m. A small cherry tree of height 2.5 m casts a shadow in the same line as that made by the pole. What is the length of the shadow cast by the cherry tree?

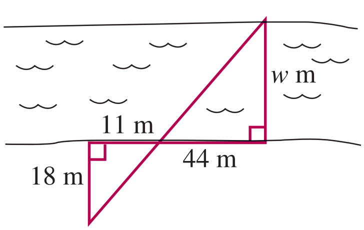


3 A flagpole of height 3.5 m casts a shadow of length 3 m. At the same time, the clubhouse which is 15 m away, casts a shadow in the same line as that of the flagpole. Find the height of the clubhouse.

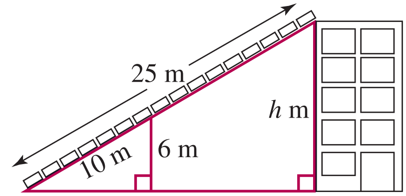


■ Consolidation

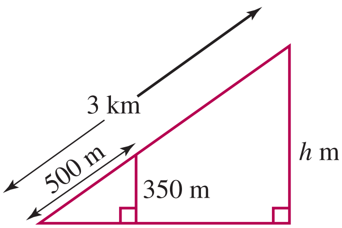
4 A surveyor needed to calculate the width of a river. He was unable to cross it and so made the measurements shown in the diagram. Find the width of the river.



5 A conveyor belt of length 25 m carries bricks from the ground to the top of a building that is under construction. When the bricks have travelled 10 m along the conveyor belt, they are 6 m above the ground. Find the height of the building.

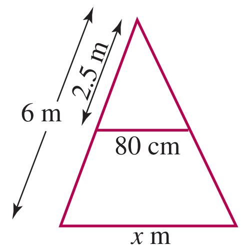


6

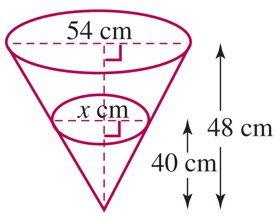


A cable car travels from a village up to a ski resort at the top of a mountain. When the cable car has travelled 500 m, it is 350 m above the ground. To reach the resort, the car must travel 3 km along the cable. How high is the mountain?

7 The legs of a 6 m ladder are supported by a strut of length 80 cm at a point 2.5 m from the top. Find the maximum distance between the feet of the ladder. (Watch the units!)



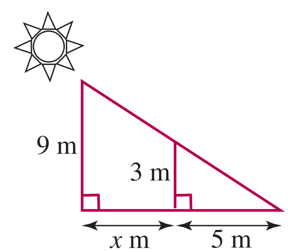
8



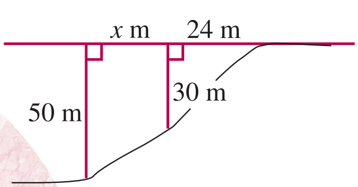
A cone of height 48 cm is filled to a height of 40 cm. If the cone has a diameter of 54 cm, find the diameter of the cone at the surface of the water.

■ Further applications

9 Two trees on a street have heights of 3 m and 9 m respectively. At a certain time of day the shorter tree casts a shadow of length 5 m. How far apart are the trees?

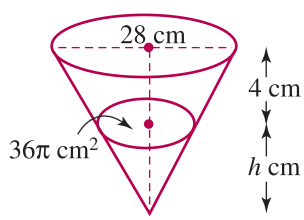


10



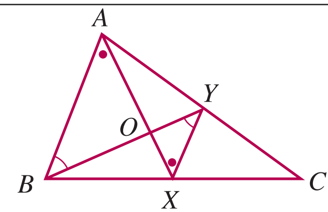
A railway bridge across a gorge is supported by a number of concrete pylons. The first pylon has a height of 30 m and is 24 m from the edge of the gorge. The second pylon has a height of 50 m. How far apart are the pylons?

11 A cone with diameter 28 cm is filled with water to a point 4 cm from the top of the cone. The surface of the water has an area of 36π cm^2. Find the height of the water.



TRY THIS Similar triangles 2

In the triangle ABC, AX and BY are medians. Use the fact that X and Y are the midpoints of BC and AC respectively to prove $\Delta AOB \,|||\, \Delta XOY$. Hence prove that O divides AX and BY in the ratio 2 : 1.

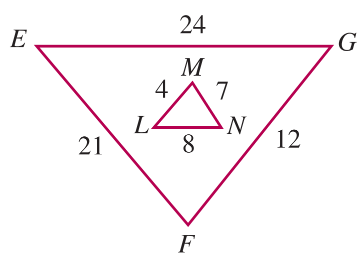


11.4 Proofs involving similar triangles

Proofs involving similar triangles should be set out in the same format as proofs involving congruent triangles.

Example 1

Prove that $\Delta LMN \,|||\, \Delta GFE$.



Solution

In ΔLMN and ΔGFE

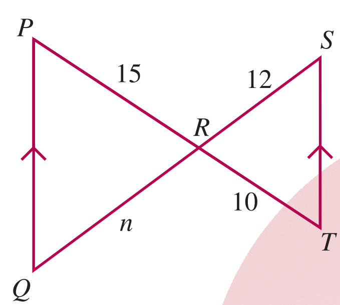
- $\dfrac{LM}{GF} = \dfrac{4}{12} = \dfrac{1}{3}$ (given)
- $\dfrac{MN}{EF} = \dfrac{7}{21} = \dfrac{1}{3}$ (given)
- $\dfrac{LN}{EG} = \dfrac{8}{24} = \dfrac{1}{3}$ (given)

$\therefore \Delta LMN \,|||\, \Delta GFE$ (three pairs of matching sides are in proportion)

Example 2

a Prove that $\Delta PQR \,|||\, \Delta TSR$.

b Hence, find the value of n.



Solutions

a In ΔPQR and ΔTSR

- $\angle PQR = \angle RST$ (alternate angles, $PQ \parallel ST$)
- $\angle PRQ = \angle SRT$ (vertically opposite angles)

$\therefore \Delta PQR \,|||\, \Delta TSR$ (two pairs of matching angles are equal)

b $\dfrac{QR}{SR} = \dfrac{PR}{TR}$ (matching sides of similar Δs are in proportion)

$\dfrac{n}{12} = \dfrac{15}{10}$

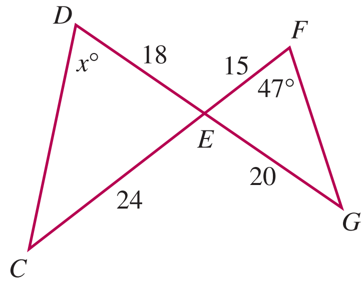
$\times 12 \quad \times 12$

$\therefore n = 18$

Example 3

a Prove that $\Delta CDE \; ||| \; \Delta GFE$.

b Hence, find the value of x.



Solutions

a In ΔCDE and ΔGFE

- $\dfrac{DE}{EF} = \dfrac{18}{15} = \dfrac{6}{5}$ (given)
- $\dfrac{CE}{EG} = \dfrac{24}{20} = \dfrac{6}{5}$ (given)
- $\angle DEC = \angle FEG$ (vertically opposite angles)

$\therefore \Delta CDE \; ||| \; \Delta GFE$

(two pairs of matching sides are in proportion and the included angles are equal)

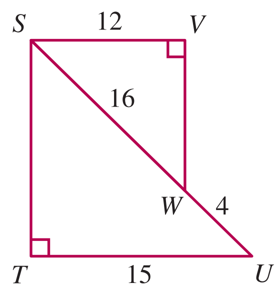
b $\angle CDE = \angle GFE$ (matching angles in similar Δs are equal)

$\therefore x = 47$

Example 4

a Prove that $\Delta SVW \; ||| \; \Delta UTS$.

b Hence, prove that $SV \parallel TU$.



Solutions

a In ΔSVW and ΔUTS

- $\angle SVW = \angle UTS = 90°$ (given)
- $\dfrac{SV}{TU} = \dfrac{12}{15} = \dfrac{4}{5}$ (given)
- $\dfrac{SW}{SU} = \dfrac{16}{20} = \dfrac{4}{5}$ (given)

$\therefore \Delta SVW \; ||| \; \Delta UTS$ (the hypotenuse and one side of a right-angled triangle are respectively proportional to the hypotenuse and one side of another right-angled triangle).

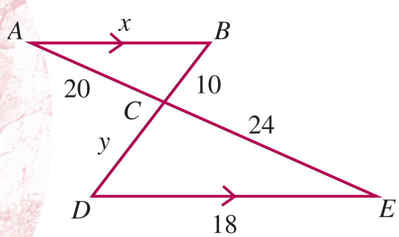
b $\angle VSW = \angle TUS$ (matching angles in similar Δs are equal)

$\therefore SV \parallel TU$ (alternate angles are equal).

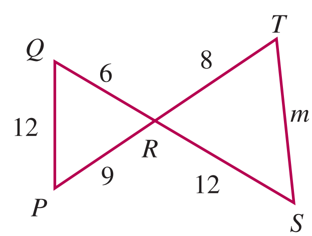
Exercise 11.4

1 Prove that the triangles are similar in each of the following. Hence, find the value of each pronumeral.

a



b



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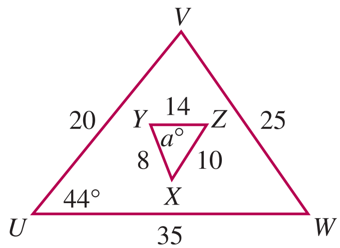

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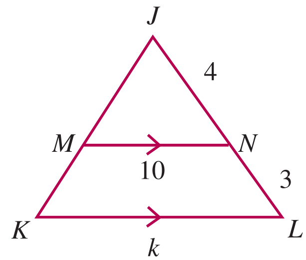
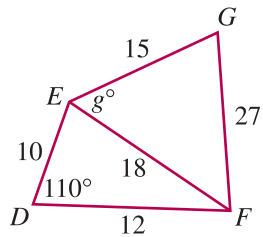


e



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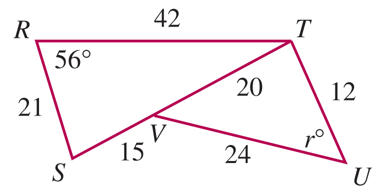
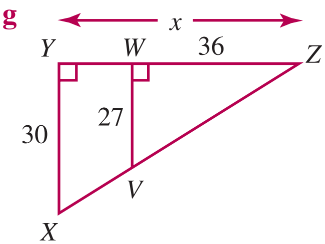


g



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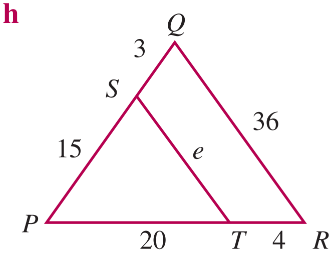
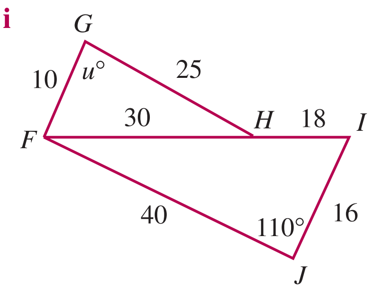


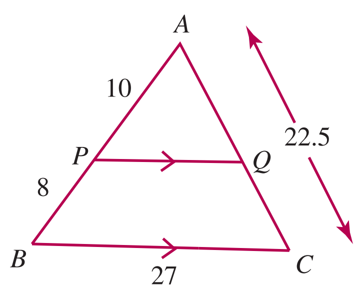
i



■ Consolidation

2 **a** Prove that $\Delta APQ \,|||\, \Delta ABC$.

b Hence find PQ, AQ, QC.



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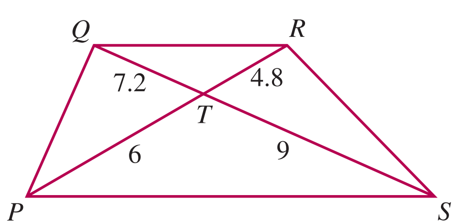
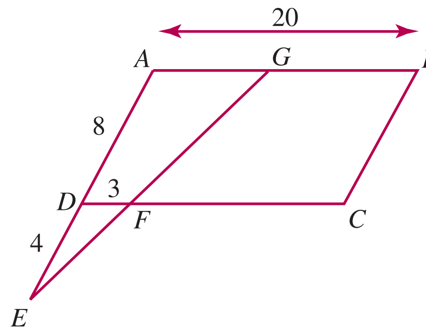


a Prove that $\Delta QTR \,|||\, \Delta STP$.

b Hence prove that $PQRS$ is a trapezium.

4 $ABCD$ is a parallelogram.

- **a** Prove that $\Delta DEF \,|||\, \Delta AEG$.
- **b** Hence find AG.



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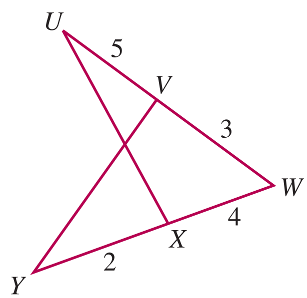
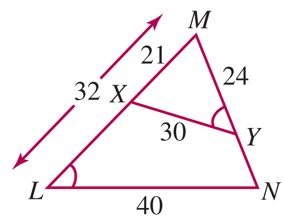


- **a** Prove that $\Delta WUX \,|||\, \Delta WYV$.
- **b** If $UX = 7$, find VY.

6

- **a** Prove that $\Delta LMN \,|||\, \Delta YMX$.
- **b** Hence find MN and YN.



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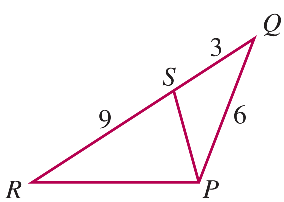
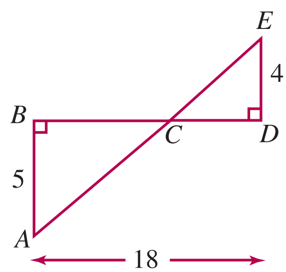


- **a** Show that $\Delta PQS \,|||\, \Delta RQP$.
- **b** Hence prove that $\angle PSQ = \angle RPQ$.

8

- **a** Prove that $\Delta ABC \,|||\, \Delta EDC$.
- **b** Find BC and CD.



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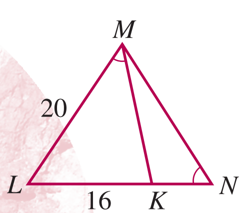
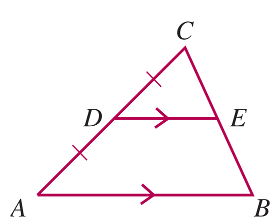


- **a** Prove that $\Delta KLM \,|||\, \Delta MLN$.
- **b** Hence find LN and KN.

10 ABC is any triangle with D the midpoint of AC and $DE \parallel AB$. Prove $CE = EB$.



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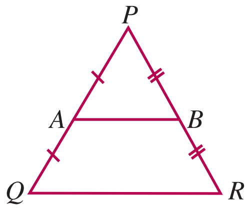


A and B are the midpoints of PQ and PR respectively.

a Prove that $\Delta APB \;|||\; \Delta QPR$.

b Hence prove that QR is parallel to AB and twice its length.

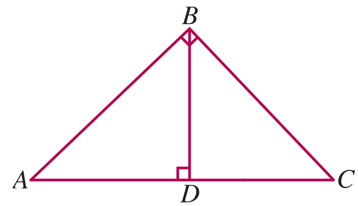
12 In the diagram shown, $AB \perp BC$ and $BD \perp AC$.

a Prove that $\angle BAD = \angle CBD$.

b Prove that $\Delta ABD \;|||\; \Delta CDB$.

c Show that $BD^2 = AD \,.\, DC$.

d If $AD = 16$ cm and $DC = 9$ cm, find BD.



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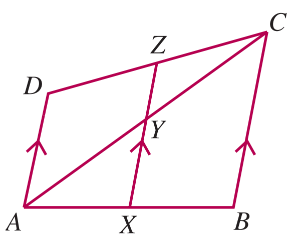


$AD \parallel XZ \parallel BC$ and X is the midpoint of AB.
$AD = 6$ cm and $BC = 10$ cm.

a Prove that $\Delta AXY \;|||\; \Delta ABC$.

b Show that Y is the midpoint of AC.

c Find XY.

d Prove that $\Delta CZY \;|||\; \Delta CDA$.

e Find YZ.

f Hence find XZ.

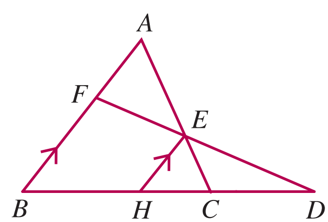
Further applications

14 In the diagram shown, $AB = AC$.

a Show that $EH = EC$.

b Prove that $\Delta BFD \;|||\; \Delta HED$.

c Hence find the ratio $FB : EC$.



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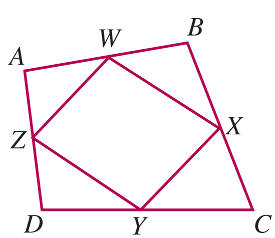
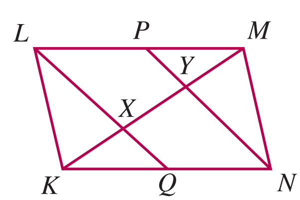
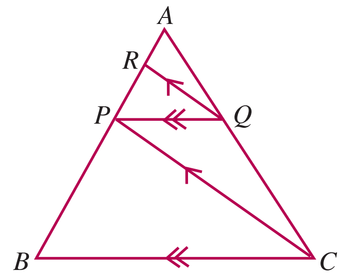


$ABCD$ is any quadrilateral with midpoints W, X, Y, Z as shown. Prove that $WXYZ$ is a parallelogram.

16 $KLMN$ is a parallelogram. P and Q are the midpoints of LM and KN respectively. Prove that LQ and PN trisect KM at X and Y.



TRY THIS Ratio in triangles



In the diagram $PQ \parallel BC$ and $RQ \parallel PC$.
If $AR = 4$ and $RP = 6$:

1 show that $\Delta RPQ \ ||| \ \Delta PBC$

2 show that $\Delta APQ \ ||| \ \Delta ABC$

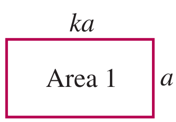
3 show that $\frac{RP}{PB} = \frac{PQ}{BC}$

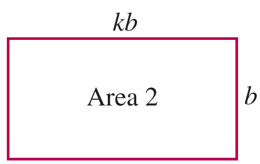
4 show that $\frac{AP}{AB} = \frac{PQ}{BC}$

5 hence, find the length of PB

11.5 Area of similar figures

Consider two similar rectangles with matching sides in the ratio $a : b$.





$$\text{Area 1} = ka \times a = ka^2$$

$$\text{Area 2} = kb \times b = kb^2$$

$$\text{Area 1} : \text{Area 2} = ka^2 : kb^2 = a^2 : b^2$$

That is, if the matching sides of the figures are in the ratio $a : b$, then their corresponding areas are in the ratio $a^2 : b^2$. This is the case for all similar figures.

In any similar figures, if:
the ratio of the matching sides is $a : b$,
then the ratio of the corresponding areas is $a^2 : b^2$.

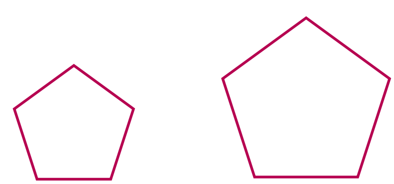
Example 1

EG +S

Two similar pentagons have sides of length 10 cm and 14 cm.

a Find the ratio of the corresponding areas.

b If the area of the smaller pentagon is 100 cm^2, find the area of the larger pentagon.



10 cm 14 cm

Solutions

a **i** The ratio of the sides

$= 10 : 14$

$= 5 : 7$

The ratio of the corresponding areas

$= 5^2 : 7^2$

$= 25 : 49$

ii $\dfrac{\text{Area 1}}{\text{Area 2}} = \dfrac{25}{49}$

$\dfrac{100}{\text{Area 2}} = \dfrac{25}{49}$

$\dfrac{\text{Area 2}}{100} = \dfrac{49}{25}$

$\therefore \text{Area 2} = \dfrac{49}{25} \times 100$

$= 196 \text{ cm}^2$

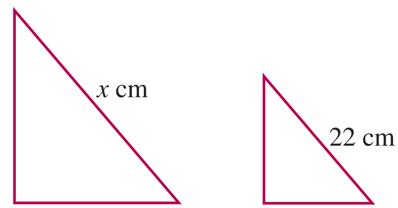
Example 2

EG +S

Two similar triangles have areas of 108 cm^2 and 48 cm^2. The hypotenuse of the smaller triangle is 22 cm.

a Find the ratio of the matching sides.

b Hence, find the value of x.



Solutions

i $\dfrac{\text{Area 1}}{\text{Area 2}} = \dfrac{108}{48}$

$= \dfrac{9}{4}$

$= \dfrac{3^2}{2^2}$

$\therefore$ the sides are in the ratio $3 : 2$.

ii $\dfrac{x}{22} = \dfrac{3}{2}$

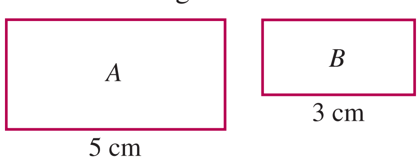
$\therefore x = \dfrac{3}{2} \times 22$

$= 33$

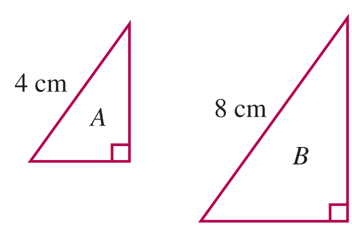
Exercise 11.5

1 The figures in each pair below are similar. Find, in simplest form, the ratio area of figure A : area of figure B.

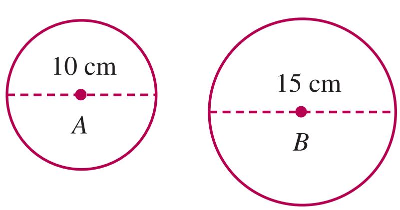
a



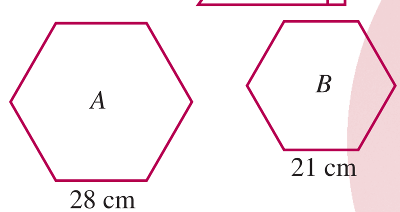
b



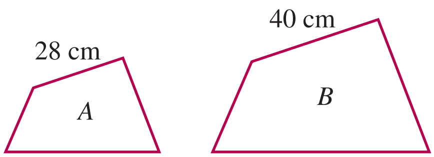
c



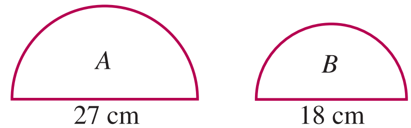
d



e



f



2 **a** The perimeters of two regular decagons are in the ratio 6 : 5. In what ratio are the corresponding areas?

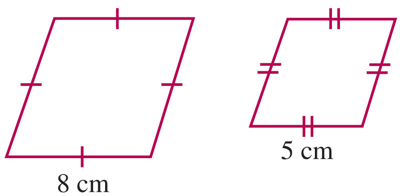
b The areas of two kites are in the ratio 9 : 16. In what ratio are the corresponding perimeters?

c The circumferences of two circles are in the ratio 9 : 5. In what ratio are the corresponding areas?

d The areas of two similar sectors are in the ratio 100 : 169. In what ratio are the corresponding perimeters?

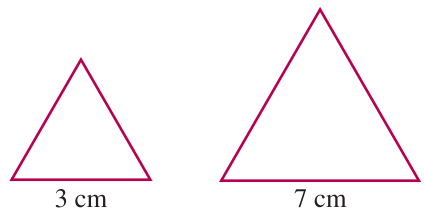
3 The figures in each pair below are similar.

a



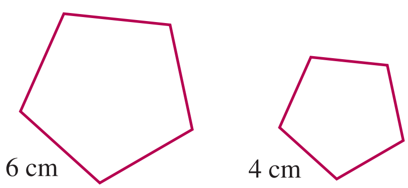
The area of the larger rhombus is 128 cm^2. Find the area of the smaller rhombus.

b



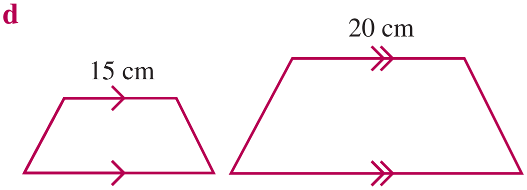
The area of the smaller triangle is 36 cm^2. Find the area of the larger triangle.

c



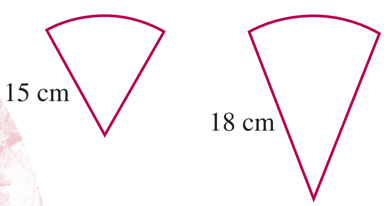
The area of the larger pentagon is 117 cm^2. Find the area of the smaller pentagon.

d



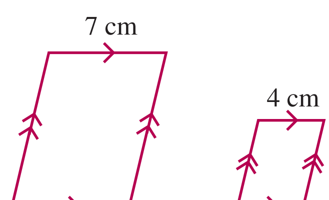
The area of the smaller trapezium is 67.5 cm^2. Find the area of the larger trapezium.

e



The area of the smaller sector is 115 cm^2. Find the area of the larger sector.

f

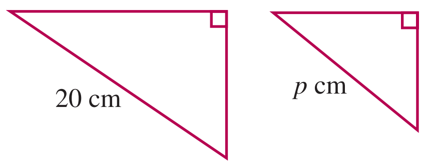


The area of the larger parallelogram is 306.25 cm^2. Find the area of the smaller parallelogram.

■ Consolidation

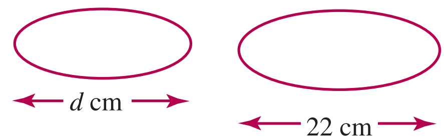
4 The figures in each pair below are similar. Find the ratio of the sides and hence find the value of the pronumeral.

a



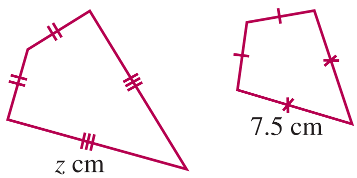
Area = 175 cm^2 Area = 112 cm^2

b



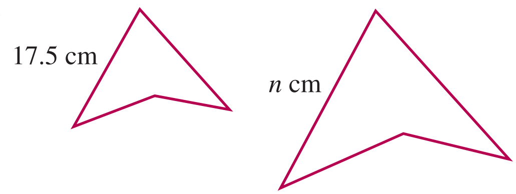
Area = 54 cm^2 Area = 726 cm^2

c



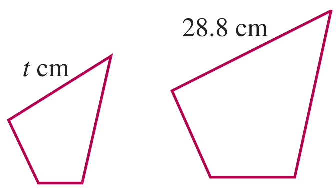
Area = 210.6 cm^2 Area = 65 cm^2

d



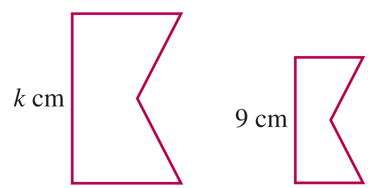
Area = 73.5 cm^2 Area = 216 cm^2

e



Area = 245 cm^2 Area = 320 cm^2

f



Area = 126.75 cm^2 Area = 27 cm^2

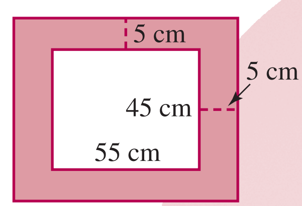
5 The sectors shown are similar.

a If the radii of the sectors are 5 cm and 12.5 cm, and the smaller sector has an area of 20π cm^2, find the area of the larger sector.

b If the sectors have areas of 196π cm^2 and 400π cm^2, and the radius of the larger sector is 15 cm, find the radius of the smaller sector.

6 A framed print has a length of 24 cm and a perimeter of 80 cm. A similar print has a length of 42 cm. Find its area.

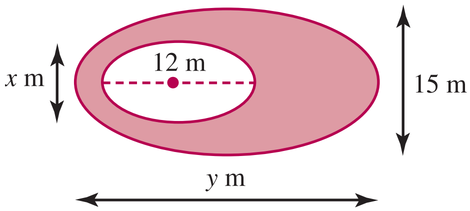
7 A painting with dimensions 55 cm × 45 cm is surrounded by a frame of width 5 cm.



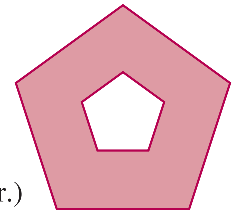
a Find the ratio of the area of the unframed painting to the area of the framed painting.

b Is the unframed painting similar to the framed painting?

Further applications

8 

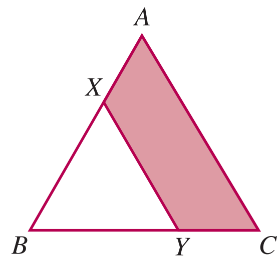
The area of the inner ellipse is 27π m^2 and the shaded area is 48 π m^2. Find values for x and y if the inner and outer ellipses are similar.



9 The ratio of the shaded area to the area of the larger pentagon is 21 : 25. If the larger pentagon has sides of length 22.5 cm, find the length of the sides of the smaller pentagon. (The pentagons are regular.)

TRY THIS **Area ratio in triangles**

ABC is an equilateral triangle of side length 3 cm.
BX = *BY* = 2 cm.
What fraction is *AXYC* of ΔABC?



11.6 Volume and surface area of similar solids

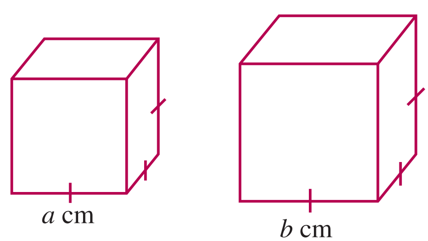
Similar solids have the same shape, their matching angles are equal and the lengths of their matching edges are in proportion.

Surface area of similar solids

Consider two cubes with sides in the ratio $a : b$.

Surface area$_1$ $= a^2 \times 6$ $= 6a^2$

Surface area$_2$ $= b^2 \times 6$ $= 6b^2$



Surface area$_1$: Surface area$_2$
$= 6a^2 : 6b^2$
$= a^2 : b^2$

That is, if the matching sides of the solid are in the ratio $a : b$, then their corresponding surface areas are in the ratio $a^2 : b^2$. This is the case for all similar solids.

> **In any similar solids, if:**
> **the ratio of the matching sides is $a:b$,**
> **then the ratio of the corresponding surface areas is $a^2:b^2$.**

Volume of similar solids

Consider the cubes in the previous example.

$$\text{Volume}_1 = a \times a \times a = a^3$$

$$\text{Volume}_2 = b \times b \times b = b^3$$

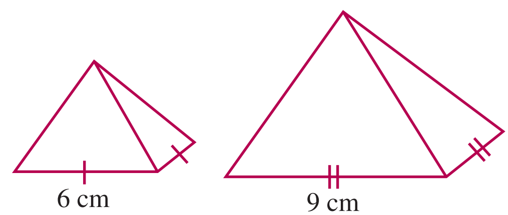
$$\text{Volume}_1 : \text{Volume}_2 = a^3 : b^3$$

That is, if the matching sides of the solid are in the ratio $a:b$, then the corresponding volumes are in the ratio $a^3:b^3$. This is the case for all similar solids.

> **In any similar solids, if:**
> **the ratio of the matching sides is $a:b$,**
> **then the ratio of the corresponding volumes is $a^3:b^3$.**

Example 1

Two similar square pyramids have base edges of length 6 cm and 9 cm.



a Find the ratio of the corresponding surface areas.

b Find the ratio of the corresponding volumes.

c If the larger pyramid has a surface area of 45 cm^2, find the surface area of the smaller pyramid.

d If the smaller pyramid has a volume of 56 cm^3, find the volume of the larger pyramid.

Solutions

a The ratio of the sides $= 6:9 = 2:3$

The ratio of the surface areas $= 2^2:3^2 = 4:9$

b The ratio of the volumes $= 2^3:3^3 = 8:27$

c
$$\frac{\text{Area}_1}{\text{Area}_2} = \frac{4}{9}$$
$$\frac{\text{Area}_1}{45} = \frac{4}{9}$$
$$\therefore \text{Area}_1 = \frac{4}{9} \times 45 = 20 \text{ cm}^2$$

d
$$\frac{\text{Volume}_1}{\text{Volume}_2} = \frac{8}{27}$$
$$\frac{56}{\text{Volume}_2} = \frac{8}{27}$$
$$\frac{\text{Volume}_2}{56} = \frac{27}{8}$$
$$\therefore \text{Volume}_2 = \frac{27}{8} \times 56 = 189 \text{ cm}^3$$

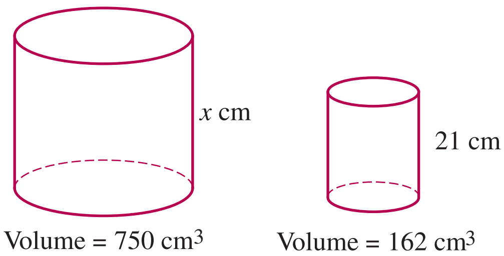
EG+S

Example 2

Two similar cylinders have volumes of 750 cm^3 and 162 cm^3.

a Find the ratio of the corresponding heights.

b Hence, find the value of x.



Solutions

a $$\frac{\text{Volume}_1}{\text{Volume}_2} = \frac{750}{162} = \frac{125}{27} = \frac{5^3}{3^3}$$

$\therefore$ The ratio of the corresponding heights is $5:3$.

b $$\frac{x}{21} = \frac{5}{3}$$

$$\therefore x = \frac{5}{3} \times 21 = 35 \text{ cm}^2$$

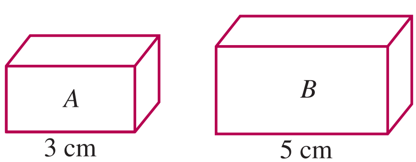
Exercise 11.6

1 The solids in each pair below are similar. Find the following ratios, in simplest form.

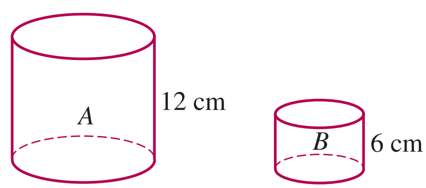
i Surface area of solid A : surface area of solid B.

ii Volume of solid A : volume of solid B.

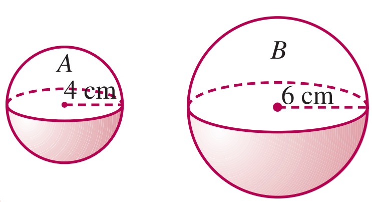
a



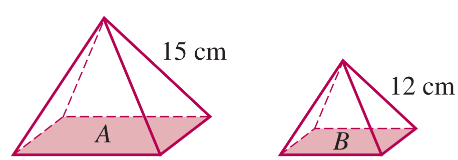
b



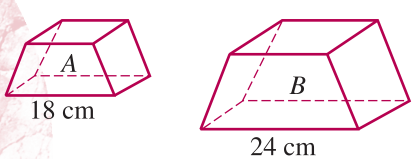
c



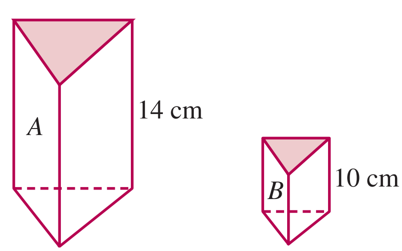
d



e

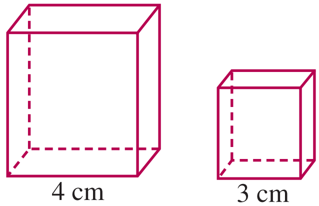


f



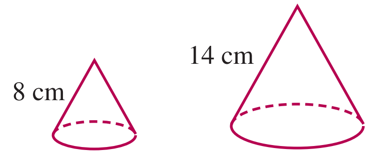
2 The solids in each pair below are similar.

a



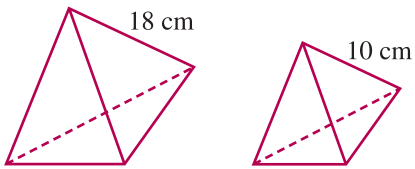
The area of the top face of the larger prism is 80 cm^2. Find the area of the top face of the smaller prism.

b



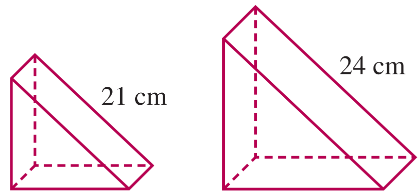
The area of the base of the smaller cone is 48 cm^2. Find the area of the base of the larger cone.

c



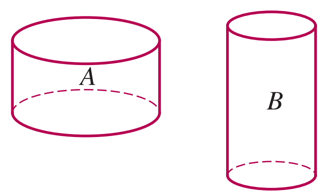
The surface area of the larger pyramid is 364.5 cm^2. Find the surface area of the smaller pyramid.

d



The surface area of the smaller prism is 122.5 cm^2. Find the surface area of the larger prism.

3 Two cylinders A and B are such that cylinder A has twice the diameter of cylinder B and cylinder B has twice the height of cylinder A. Are the two cylinders similar?



4 **a** Two similar rectangular prisms have matching heights 2 cm and 5 cm. The smaller prism has a volume of 104 cm^3. Find the volume of the larger prism.

b Two similar rectangular pyramids have perpendicular heights of 12 cm and 9 cm. The larger pyramid has a volume of 236.8 cm^3. Find the volume of the smaller pyramid.

c Two similar trapezoidal prisms have matching edges of 19 cm and 11.4 cm. The smaller prism has a volume of 178.2 cm^3. Find the volume of the larger prism.

d Two similar cylinders have diameters of 17.6 cm and 19.2 cm. The larger cylinder has a volume of 2592 cm^3. Find the volume of the smaller cylinder.

■ Consolidation

5 Two similar cones have surface areas in the ratio 49 : 4.

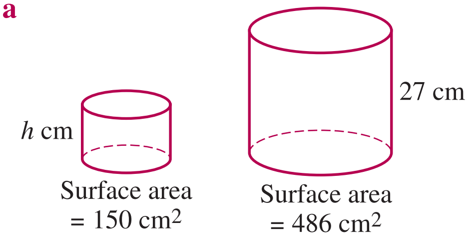
a What is the ratio of the corresponding slant heights?

b What is the ratio of the corresponding volumes?

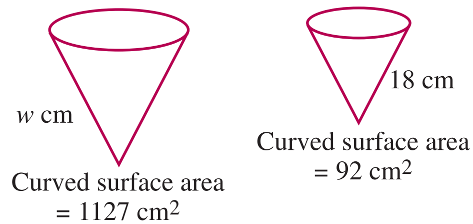
6 Two similar chocolate teddy bears have volumes in the ratio 512 : 729. What is the ratio of the corresponding surface areas?

7 The solids in each pair below are similar. Find the value of each pronumeral.

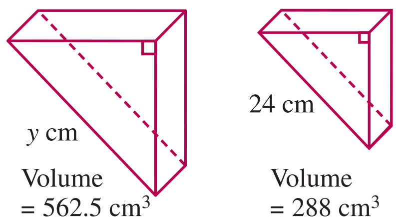
a



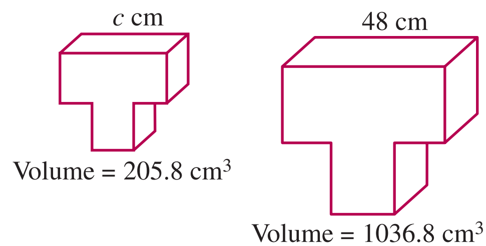
b



c

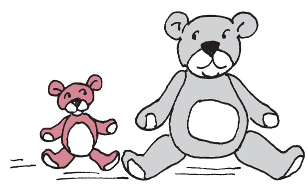


d



8 Two similar cones have base radii of 7.5 cm and 9 cm. The larger cone has a volume of 198π cm^3. Find the volume of the smaller cone correct to the nearest cm^3.

9 The two soft toys shown are similar. One toy is three times the height of the other.

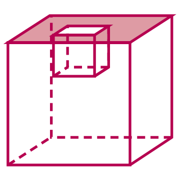


a How much more material is required to make it?
b How much more filling is required?

10 A lead ball of radius 6 cm is melted down and re-cast into ball bearings of radius 1.5 cm. How many ball bearings can be produced?

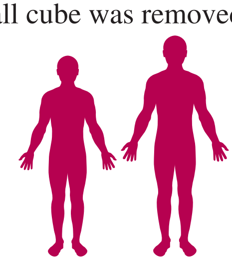
Further applications

11



A small cube is cut from a large cube, as shown. The ratio of the remaining volume to the original volume is 19 : 27. If the small cube has sides of length 14 cm, find the length of the sides of the large cube (before the small cube was removed).

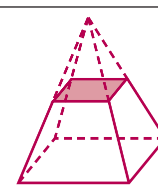
12 The corresponding dimensions of figure B are approximately 10% larger than those of figure A.



a What is the percentage increase in body area?
b What is the percentage increase in body volume?

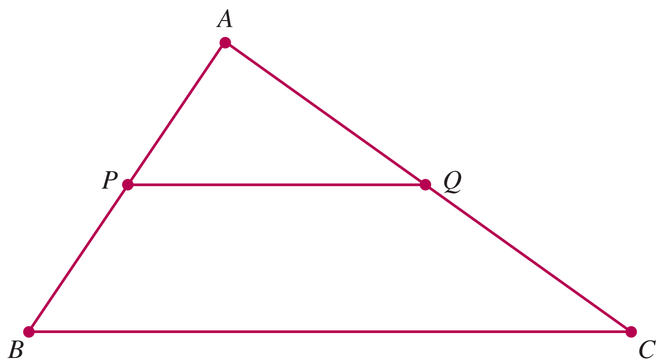
TRY THIS Volume ratio

A square-based pyramid has the top $\frac{1}{3}$ (height) removed to form a pedestal. What fraction is the pedestal of the original pyramid?



USING SIMILAR TRIANGLES TO SOLVE A CONSTRUCTION PROBLEM

The midpoint theorem



In triangle ABC, P is the midpoint of AB, and Q is the midpoint of AC. It can be proved that PQ is parallel to BC and is half its length.

Introduction

In this chapter you have been learning about similar triangles and developing your skills in deductive reasoning. The theorem above is a good example of how useful similar triangles can be in establishing proof.

In working mathematically our aim is to fine tune these skills, by giving you an opportunity to make, refine and test conjectures. Another aim is to enable you to think about possible converses of a known result and examine whether or not they are true. Finally, we get you to apply your knowledge of a theorem and its converse to solve a problem.

In the challenge activities dynamic geometry software such as Cabri or Geometer's Sketchpad are highly recommended. However if they are not available, geometrical instruments can be used.

The mid-points of a triangle theorem

An important theorem from the syllabus states that:

> *The interval joining the midpoints of two sides of a triangle is parallel to the third side and equal to half its length.*

We will use this as our starting point for the activities to follow. If you are not familiar with this theorem please check with your teacher.

Learning activities

1 Draw a diagram and write the above theorem using an "if—then" format. Copy and complete the following "*If* an interval is drawn to join the mid-points of two sides of a triangle, *then* …."

2 State a possible converse of this theorem using an *if* … *then* format. Draw a diagram to show the data.

3 See if you can prove your converse (rough outline of the proof only).

4 Now consider this possible converse "*If* a line is drawn through the midpoint of one side of a triangle parallel to another side *then* it bisects the third side".

5 See if you can prove this statement. Draw a diagram and use similar triangles.

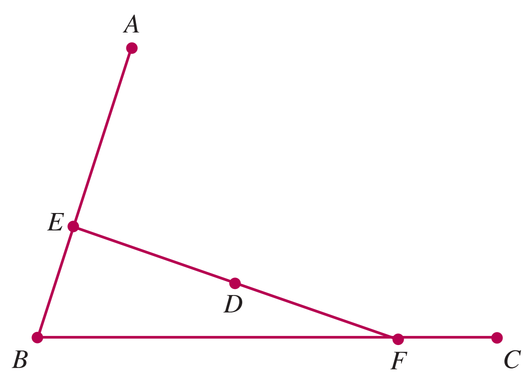
6 Deduce that the line drawn through the midpoint of one side of a triangle parallel to another side, is indeed half the length of the third side.

7 In mathematics we use the phrase "if and only if" to indicate when a theorem and its converse are both true. Discuss in class the truth of the following statement:

A line drawn through the midpoint of one side of a triangle bisects another side *if and only if* it is parallel to the third side.

Challenge activities

Here is a construction problem. It begins with an acute angle *ABC* say and a point *D* which lies anywhere between the arms of the angle. We wish to construct two points *E* and *F* on the arms *BA* and *BC* such that *D* is the midpoint of *EF*. Make a rough copy of the figure in your book. The points *E* and *F* are to determined. *EF* has been joined to make the task clear.



Constructing the points *E* and *F*

Construct the figure above using dynamic geometry software, or geometrical instruments, showing only the angle *ABC* and any point *D* between the arms.

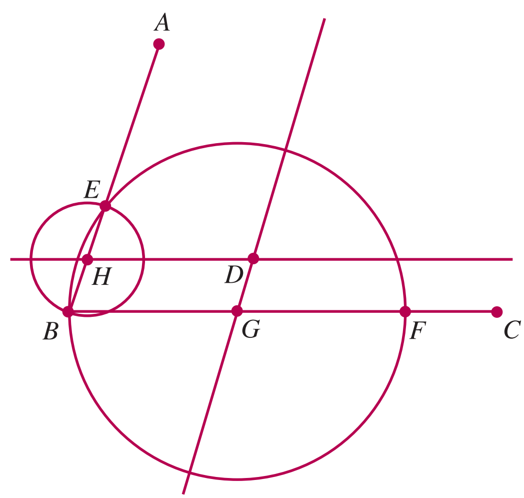
1 Using your knowledge of the midpoint theorem and/or its converse, see if you can construct the points E and F. Feel free to use any other geometrical knowledge you have to solve the problem. There are many ways to do it.

2 Write out a proof to justify your answer.

3 Now consider the following proof. Draw the construction lines on a rough diagram as you go. Compare it with your proof in questions 1 and 2.

- **a** Construct a line parallel to *BA* passing through *D*. Let it cut *BC* at *G*.
- **b** Construct a line parallel to *BC* passing through *D*. Let it cut *BA* at *H*.
- **c** According to the converse of the midpoint theorem, a line drawn parallel to *BA* through *D* will bisect *BF* at *G*. Similarly, a line drawn parallel to *BC* through *D* will bisect *BE* at *H*. Therefore to construct the points *E* and *F*, mark off $HE = BH$ on *BA*, and $GF = BG$ on *BC*.
- **d** As an alternative to the last step, could you simply join *HG* and construct the parallel through *D* to find *E* and *F*? What difference would this make to your proof?

4 Complete an accurate construction using Geometer's Sketchpad or Cabri. One solution is shown below. Select *D* and the segment *BA* and construct the parallel through *D*. Repeat to draw the line through *D* parallel to *BC*. Select *D* and the segment *BA* and construct the intersection point. Label it *H*. Repeat to find and label *G*. To construct *E* and *F*, select *H* and *B* and construct the circle centre *H* radius *HB*. Repeat with *G* and *B* to find and label *F*. Drag the point *D* anywhere between the arms and note the result. You could also drag *A* and *C* to vary the size of the angle.



Let's communicate

Discuss in class what you have learned about deductive reasoning from this activity. How did you start the construction? What theorem(s) did you use? How did you justify your solution?

Reflecting

It has been claimed that "geometry is a central part of mathematics and that geometrical thinking is a fundamental way to engage with mathematics" (from Teaching and learning geometry 11–19, Report to the Royal Society, London 2001). Would you agree? Why?

Language link with Macquarie

MACQUARIE

1 In a short sentence explain
- *a* ***Similar** triangles*
- *b* ***Similar** polygons*
- *c* ***Deductive** reasoning*
- *d How does a **conjecture** differ from a **proof**?*

2 The Macquarie Learners Dictionary *defines the word **proof** as follows:*

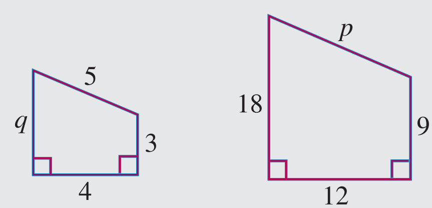
proof *noun* **1.** something that shows a thing is true: *Can you give me any proof that you were at home at the time of the robbery?*
–*adjective* **2.** strong enough to resist: *It is proof against fire.*
–*verb* **3.** to treat or coat something to make it resistant to damage, etc.: *This carpet has been proofed against stains.*

How does the concept of proof in a court of law differ from mathematical proof?

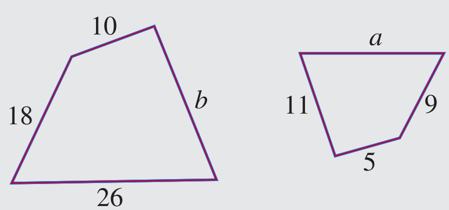
Chapter Review

1 Find the scale factor for each pair of similar figures. Hence, find the values of the pronumerals. All lengths are in cm.

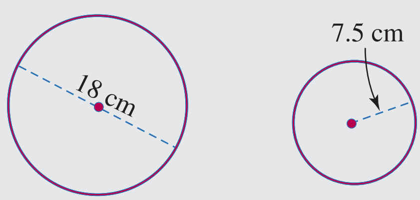
a



b

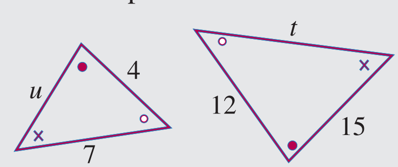


2 Find the scale factor for these circles.

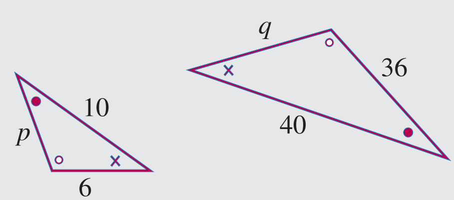


3 The triangles below are similar. Find the value of each pronumeral.

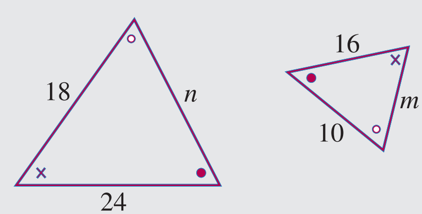
a



b



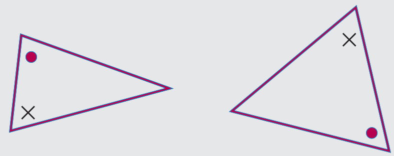
c



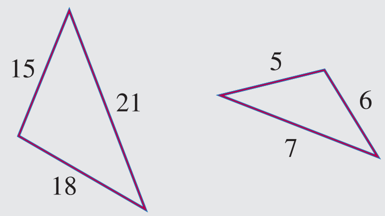
CHAPTER REVIEW

4 Are the triangles in each pair similar? If they are, give a reason.

a



b



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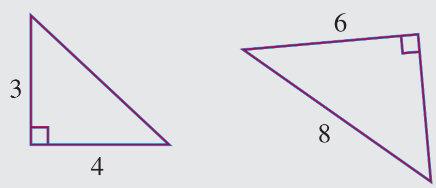
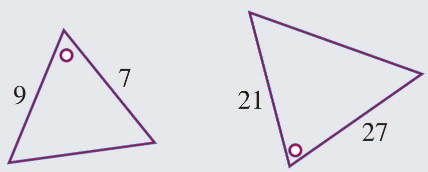


d



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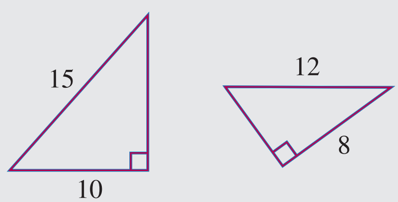

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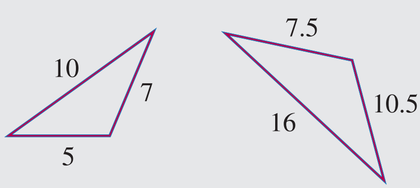
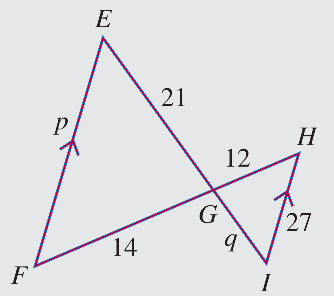


5 Prove that the triangles in each pair are similar. Hence find the value of all pronumerals.

a



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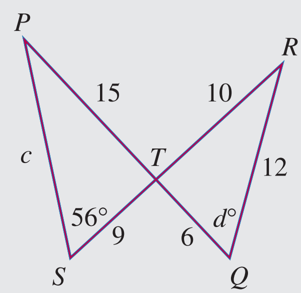

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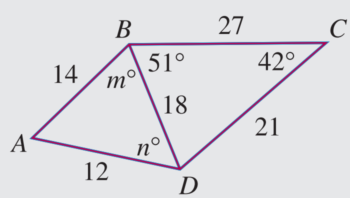

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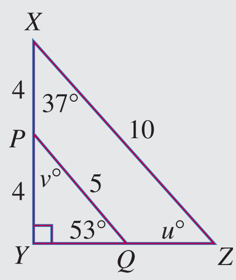

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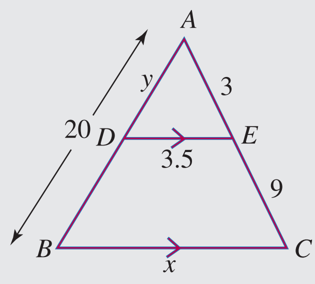

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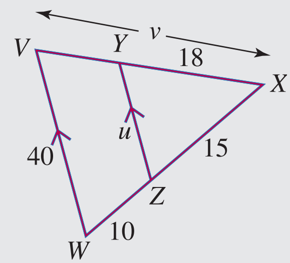

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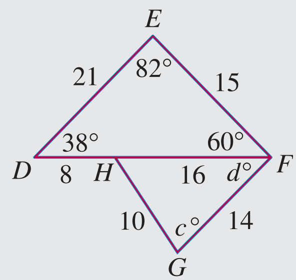

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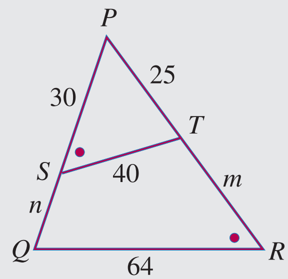

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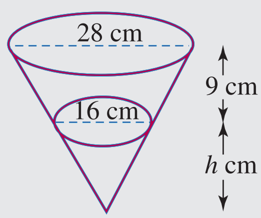
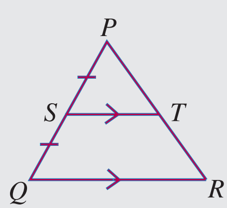


A cone with diameter 28 cm is partially filled with water to a point 9 cm from the top. The surface of the water has a diameter of 16 cm. Find the height of the water, h cm.

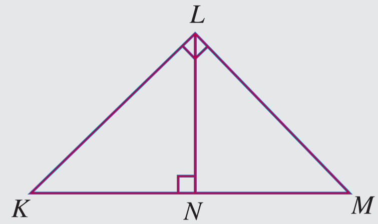
7 In ΔPQR, S is the midpoint of PQ and $ST \parallel QR$.



a Prove that $\Delta PST \,|||\, \Delta PQR$.

b Hence, prove that T is the midpoint of PR.

8 In ΔKLM, $KL \perp LM$ and $LN \perp KM$.



a Prove that $\Delta KLN \,|||\, \Delta LMN$.

b Hence, show that $LN^2 = KN \,.\, NM$.

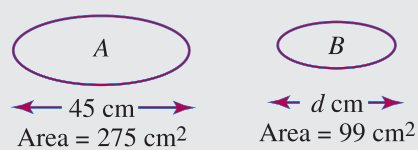
c If $KN = 25$ cm and $NM = 9$ cm, find LN.

9 Two similar sectors A, B have radii 12 cm and 21 cm respectively.

a In what ratio is area of sector A to area of sector B?

b If sector A has an area of 80 cm^2, find the area of sector B.

10



The ellipses above are similar.

a Find the ratio of the major diameters ($A : B$).

b Hence, find the diameter of B.

11 Two similar square pyramids A and B have perpendicular heights of 25 cm and 30 cm respectively.

a If the surface area of A is 200 cm^2, find the surface area of B.

b If the volume of B is 864 cm^3, find the volume of A.

12 Two similar solids have surface areas of 128 cm^2 and 72 cm^2. Find the corresponding ratio of their volumes.

12 Functions and logarithms

This chapter at a glance

Stage 5.1/5.2/5.3

After completing this chapter, you should be able to:

- define the terms relation, function, dependent and independent variables
- use the vertical line test to determine whether a given graph represents a function
- use the notation $f(x)$ and evaluate $f(x)$ at $x = a$
- find the permissible x- and y-values for a variety of functions and relations
- determine the inverse functions for a variety of functions
- determine whether the inverse of a function is also a function
- place restrictions on a function such that its inverse will be a function
- use symmetry about the line $y = x$ to sketch the graph of $y = f^{-1}(x)$ given the graph of $y = f(x)$
- sketch the graph of $y = f(x) + k$ and $y = f(x - a)$ given the graph of $y = f(x)$
- solve simple exponential equations
- convert between index form and logarithmic form using the definition of a logarithm
- evaluate simple logarithmic expressions using the definition of a logarithm
- solve simple logarithmic equations using the definition of a logarithm
- evaluate numerical expressions by using the logarithm laws
- simplify algebraic expressions by using the logarithm laws
- solve further exponential equations by taking the logarithm to base 10 of both sides
- sketch the graph of $y = \log_a x$
- explain the symmetry between the graphs of $y = a^x$ and $y = \log_a x$

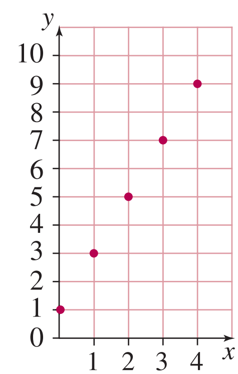
12.1 Functions and relations

Relations

A relation is any set of ordered pairs.

The variables x and y are usually used to represent the first and second numbers respectively in an ordered pair. That is, (x, y) is a common ordered pair of a relation. Tables or diagrams cannot always include every ordered pair that belongs to a relation. In many cases a rule or algebraic formula is used to show the relationship between the variables in a relation.

For example, consider an infinite set of collinear points which includes the points (0, 1), (1, 3), (2, 5), (3, 7), (4, 9).



For each point (x, y), the y co-ordinate is 1 more than twice the x co-ordinate. Therefore this relation could be described by the rule $y = 2x + 1$.

Since the y-value depends directly upon the value of x that was input, y is called the **dependent variable**. Each x-value can be chosen independent of the relationship between x and y. Hence, x is called the **independent variable**.

Functions

A function is a set of ordered pairs (x, y) in which for each x co-ordinate there is a unique y co-ordinate.

In a function, no two ordered pairs can have the same x co-ordinate.

For example,

1 (−3, 2), (1, 6), (2, 7), (6, 11) is a function since no two ordered pairs have the same x co-ordinate.

2 (−3, 2), (1, 6), (1, 7), (6, 11) is not a function since the ordered pairs (1, 6) and (1, 7) have the same x co-ordinate. In this case, the four ordered pairs are a relation.

NOTE: Every function is a relation, but not every relation is a function.

The vertical line test

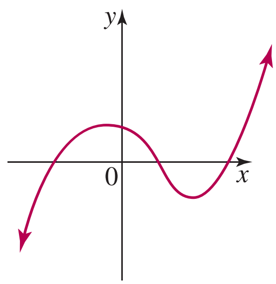
If a vertical line cuts a graph more than once, then there must be at least two points on the graph that have the same x co-ordinate. Therefore, the graph cannot represent a function.

If a graph can be cut by a vertical line:
- **in one point only, then the graph represents a function**
- **in two or more points, then the graph does not represent a function.**

Example 1

State whether each graph represents a function.

a



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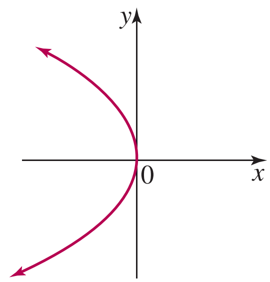

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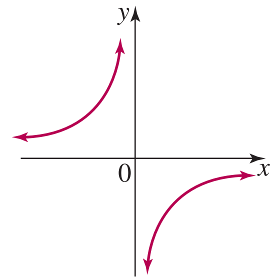


Solutions

a There is no vertical line that can be drawn to cut the graph more than once. Therefore, the graph represents a function.

b It is possible to draw a vertical line that cuts the graph more than once. Therefore, the graph does not represent a function.

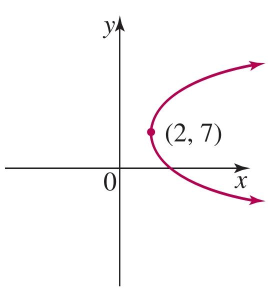
c The graph is discontinuous. However, it is not possible to draw a vertical line that cuts the graph more than once. Therefore, the graph represents a function.

Example 2

For each of the following graphs, state:

i the permissible x-values **ii** the permissible y-values.

a



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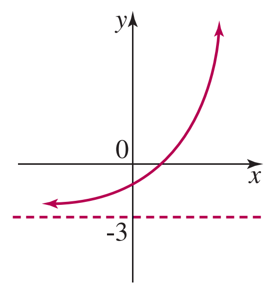

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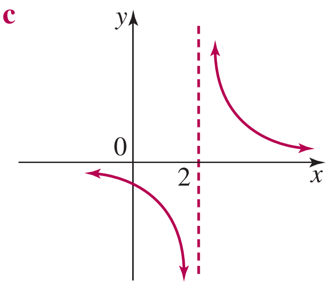


Solutions

a **i** $x \geqslant 2$ **ii** all real y

b **i** all real x **ii** $y > -3$

c **i** all real x, $x \neq 2$ **ii** all real y, $y \neq 0$

Exercise 12.1

1 State whether each set of ordered pairs represents a function.

a (0, 2), (1, 3), (2, 4), (3, 5), (4, 6)

b (1, 2), (2, 4), (3, 6), (4, 8), (5, 10)

c (2, 5), (2, –5), (3, 8), (4, –2), (5, 1)

d (3, 10), (4, 9), (5, 8), (6, 7), (7, 6)

e (0, 4), (0, –4), (1, 2), (1, –2), (2, 0)

f (–6, 0), (6, 0), (–4, 3), (4, 3), (0, 7)

2 Which table of values represents a function?

A:

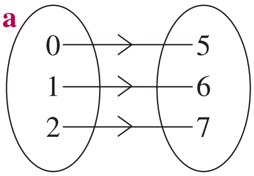
x	2	2	2	2	2
y	3	4	5	6	7

B:

x	0	1	2	3	4
y	5	5	5	5	5

3 State whether these mapping diagrams represent functions.

a



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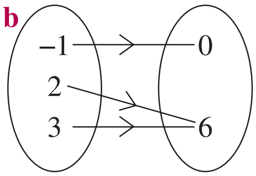

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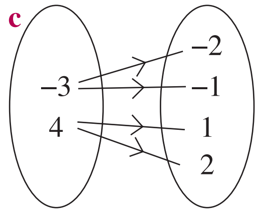

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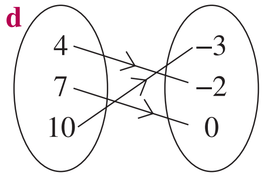
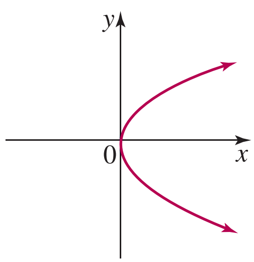


Consolidation

4 Use the vertical line test to determine whether each graph represents a function.

a



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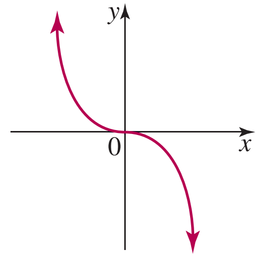

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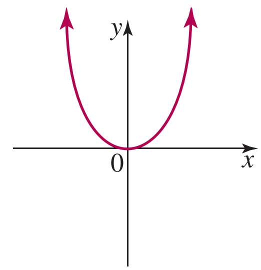

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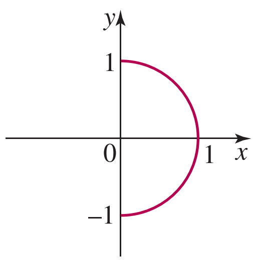

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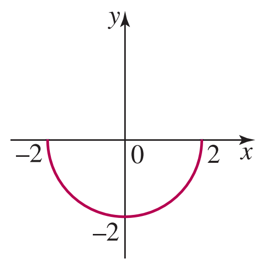

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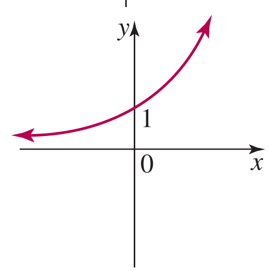

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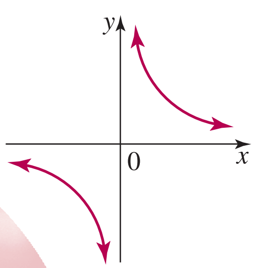

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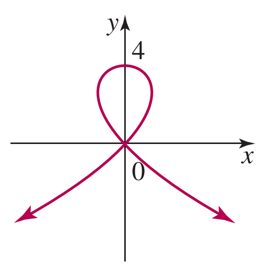

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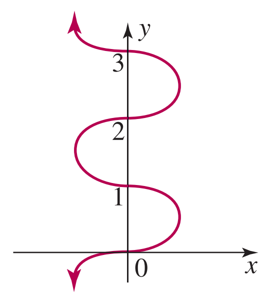

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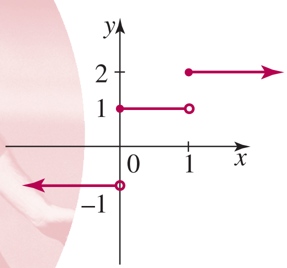

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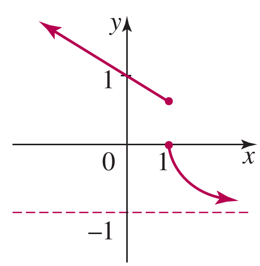

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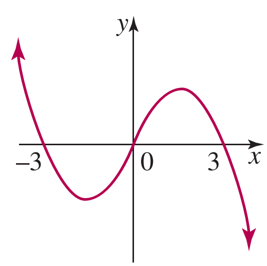


5 Explain the circumstances under which a straight line:

a must be a function **b** cannot be a function.

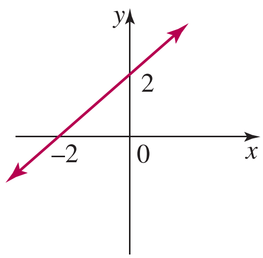
6 Determine whether each of the following equations represents a function.

a $y = 2x + 3$ **b** $x = 4$ **c** $y = -2$ **d** $y = x^2$

e $y = 1 - x^2$ **f** $x^2 + y^2 = 4$ **g** $y = \dfrac{1}{x}$ **h** $y = \sqrt{9 - x^2}$

7 Find the permissible x- and y-values for each of the following.

a



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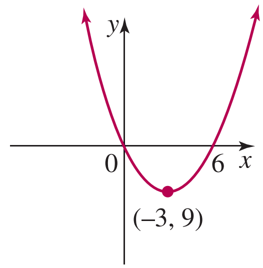

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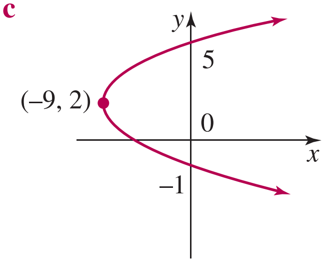

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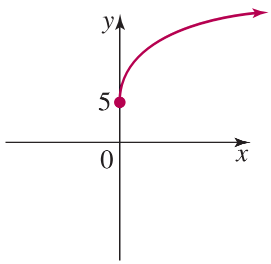

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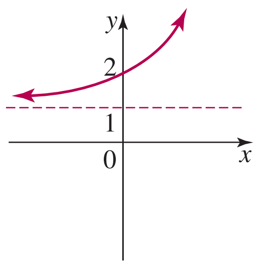

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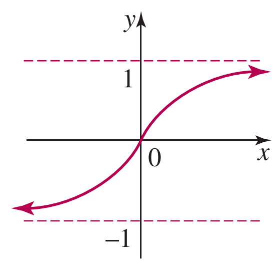

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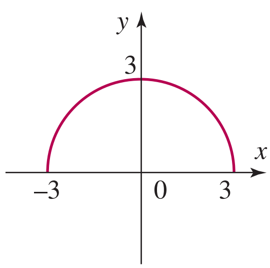

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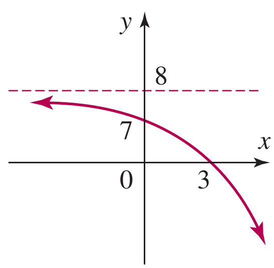

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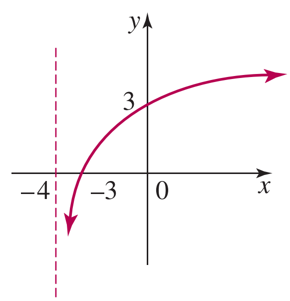

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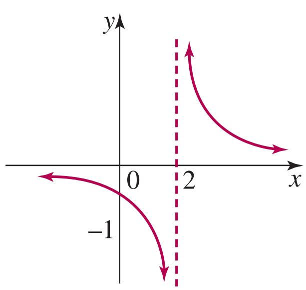

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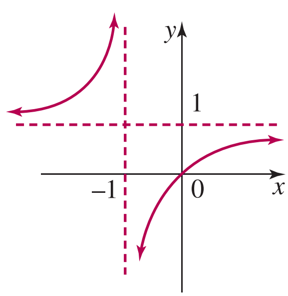

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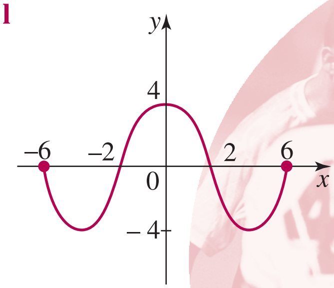

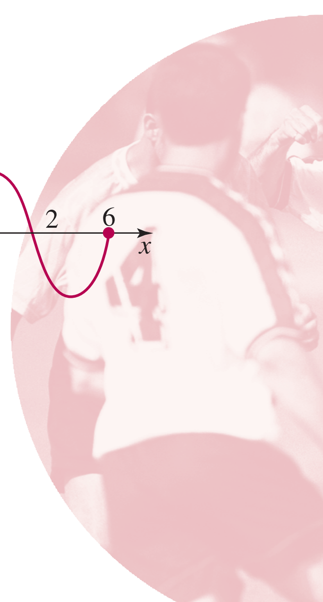

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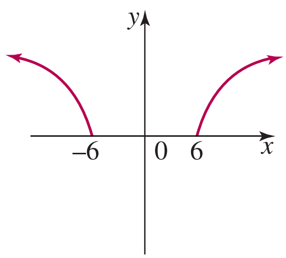

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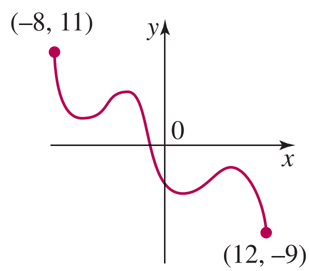

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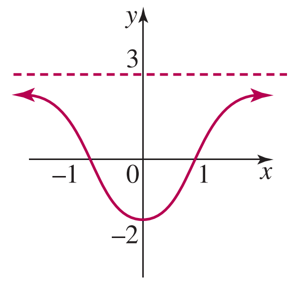
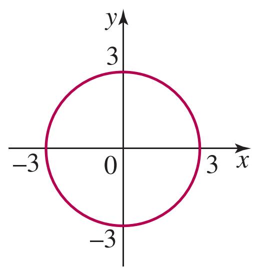


8 State the permissible x- and y-values for these circles.

a



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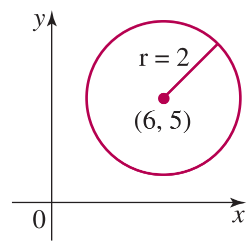


Further applications

9 Sketch a possible graph to model each of the following practical situations. Discuss your graphs with other students.

- **a** The temperature of a roast after it is removed from a hot oven (temperature vs time).
- **b** The share price of a particular company over 7 days (share price vs time).
- **c** The cost of making a mobile phone call (cost vs time).
- **d** The weekly earnings of a telemarketer who is paid by commission (income vs time).
- **e** The braking distance of a car (braking distance vs speed).
- **f** The temperature of a bucket of water after a hot iron bar is placed in it (temperature vs time).

12.2 Function notation

A function can be thought of as a machine. An input value is fed into the "function machine", which produces an output value.

The output value depends directly on the input value and the given rule, or function. If we call the function f, then the relationship between each input value and the corresponding output value can be represented by a statement such as $f(x) = 3x + 2$. That is, for each x-value that is input into the function f, the output value will be 2 more than 3 times this number.

For example, if $f(x) = 3x + 2$:

$$f(0) = (3 \times 0) + 2 \\ = 2$$

$$f(1) = (3 \times 1) + 2 \\ = 5$$

$$f(-4) = (3 \times -4) + 2 \\ = -10$$

A statement such as $f(7) = 23$, therefore, means that when $x = 7$ is input into a function f, the output is 23.

Different functions can be represented by different letters, such as g, h, G, H, F.

For example, if $g(x) = (x + 2)^2$,

$$\begin{aligned} \text{then } g(5) &= (5 + 2)^2 \\ &= 7^2 \\ &= 49 \end{aligned}$$

Example 1

If $f(x) = x^2 - 3x + 7$, find:

a $f(-4)$ **b** $f(\sqrt{2})$ **c** $f\left(\frac{2}{3}\right)$

Solutions

a
$$\begin{aligned} f(x) &= x^2 - 3x + 7 \\ f(-4) &= (-4)^2 - 3(-4) + 7 \\ &= 16 + 12 + 7 \\ &= 35 \end{aligned}$$

b
$$\begin{aligned} f(x) &= x^2 - 3x + 7 \\ f(\sqrt{2}) &= (\sqrt{2})^2 - 3(\sqrt{2}) + 7 \\ &= 2 - 3\sqrt{2} + 7 \\ &= 9 - 3\sqrt{2} \end{aligned}$$

c
$$\begin{aligned} f(x) &= x^2 - 3x + 7 \\ f\left(\frac{2}{3}\right) &= \left(\frac{2}{3}\right)^2 - 3\left(\frac{2}{3}\right) + 7 \\ &= \frac{4}{9} - 2 + 7 \\ &= 5\frac{4}{9} \end{aligned}$$

Example 2

If $g(x) = x^2 - 2x$, solve $g(x) = 8$.

Solution

$$\begin{aligned} g(x) &= x^2 - 2x \\ 8 &= x^2 - 2x \\ x^2 - 2x - 8 &= 0 \\ (x - 4)(x + 2) &= 0 \\ \therefore x &= 4, -2 \end{aligned}$$

Exercise 12.2

1 a If $f(x) = 3x + 5$, find: **i** $f(2)$ **ii** $f(-4)$ **iii** $f(0)$

b If $g(x) = 11 - 2x$, find: **i** $g(-3)$ **ii** $g\left(\frac{3}{2}\right)$ **iii** $g(6.5)$

c If $h(x) = x^2 - 4$, find: **i** $h(5)$ **ii** $h(2)$ **iii** $h(\sqrt{2})$

d If $F(x) = 2x^3$, find: **i** $F(0)$ **ii** $F(4)$ **iii** $F(-2)$

e If $u(x) = x^2 - 4x + 1$, find: **i** $u(3)$ **ii** $u(-5)$ **iii** $u\left(\frac{3}{4}\right)$

f If $G(x) = x(x + 5)$, find: **i** $G(4)$ **ii** $G(-5)$ **iii** $G(2\sqrt{3})$

g If $\phi(x) = 2^x$, find: **i** $\phi(0)$ **ii** $\phi(4)$ **iii** $\phi(-3)$

h If $v(x) = (x + 1)^2$, find: **i** $v(4)$ **ii** $v(-7)$ **iii** $v(\sqrt{2})$

i If $H(x) = \sqrt{25 - x^2}$, find: **i** $H(0)$ **ii** $H(3)$ **iii** $H(\sqrt{7})$

j If $p(x) = x + \dfrac{1}{x}$, find: **i** $p(2)$ **ii** $p\left(\dfrac{1}{2}\right)$ **iii** $p(-5)$

k If $k(x) = (x - 1)(x + 4)$, find: **i** $k(1)$ **ii** $k(-6)$ **iii** $k\left(\dfrac{1}{2}\right)$

l If $L(x) = \dfrac{x - 12}{x + 4}$, find: **i** $L(0)$ **ii** $L(4)$ **iii** $L\left(-2\dfrac{1}{2}\right)$

2 If $T(x) = \dfrac{1}{x - 1}$, find $T\left(\dfrac{1}{2}\right)$.

■ Consolidation

3 a If $f(x) = 3x - 8$, find the value of x for which $f(x) = 25$.

b If $F(x) = x^2 + 6$, find all values of x for which $F(x) = 15$.

c If $h(x) = \dfrac{1}{x - 2}$, solve $h(x) = \dfrac{2}{3}$.

d If $\phi(x) = (x - 3)^2$, solve $\phi(x) = 49$.

e If $g(x) = 2^{x-1}$, find x such that $g(x) = 64$.

f If $E(x) = x^2 + 4x - 10$, find all values of x such that $E(x) = 11$.

g If $t(x) = \sqrt{3x - 7}$, solve $t(x) = 2\sqrt{5}$.

h If $u(x) = \dfrac{5 + 4x}{3 - 2x}$, solve $u(x) = -1$.

4 If $f(x) = 2^x + 2^{-x}$, show that $f(1) = f(-1)$.

5 If $F(x) = 6x - 2$, find an expression for:

a $F(3a)$ **b** $F(a + 1)$ **c** $F\left(\dfrac{a}{2}\right)$

6 If $g(x) = (x - a)(x - b)$, show that $g(a) = g(b)$.

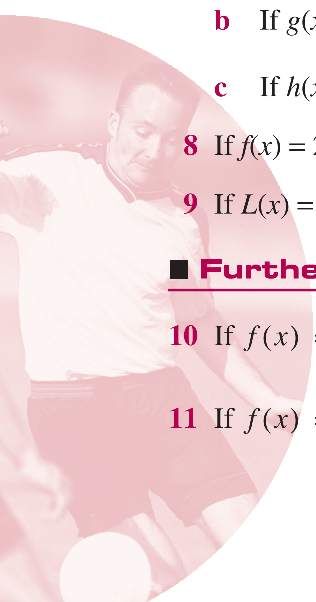
7 a If $f(x) = x^2$, find the simplest form of $\dfrac{f(x) - f(3)}{x - 3}$.

b If $g(x) = 3x^2$, find the simplest form of $\dfrac{g(x) - g(2)}{x - 2}$.

c If $h(x) = x^2 + 5x + 3$, find the simplest form of $\dfrac{h(x) - h(4)}{x - 4}$.

8 If $f(x) = 2^x$, explain why $f(x) = 0$ has no solutions.

9 If $L(x) = 3 - 2x$, solve $L(x) < 15$.



■ Further applications

10 If $f(x) = \dfrac{1}{x}$, show that $\dfrac{f(x) - f(1)}{x - 1} = -\dfrac{1}{x}$.

11 If $f(x) = \sqrt{x}$, show that $\dfrac{f(x) - f(1)}{x - 1} = \dfrac{1}{1 + \sqrt{x}}$.

12 a **i** Sketch $f(x) = \begin{cases} x^2 \text{ if } x < 0 \\ 3x \text{ if } 0 \leqslant x < 4 \\ 12 \text{ if } x \geqslant 4 \end{cases}$

ii Evaluate $f(2) + f(5) + f(-3)$.

b **i** Sketch $f(x) = \begin{cases} \dfrac{8}{x} \text{ if } x < 0 \\ 7 \text{ if } 0 \leqslant x \leqslant 2 \\ 2x + 3 \text{ if } x > 2 \end{cases}$

ii Evaluate $f(4) + f(-2) + f(0)$.

c **i** Sketch $f(x) = \begin{cases} x^2 \text{ if } x < 1 \\ x \text{ if } 1 \leqslant x < 4 \\ \dfrac{16}{x} \text{ if } x \geqslant 4 \end{cases}$

ii Evaluate $f(-2) - f(1) + f(8)$.

d **i** Sketch $f(x) = \begin{cases} -\dfrac{12}{x} \text{ if } x < 0 \\ 5 \text{ if } 0 \leqslant x \leqslant 3 \\ 3x - 4 \text{ if } x > 3 \end{cases}$

ii Evaluate $f(-3) + f(0) - f(5)$.

13 a Sketch $f(x) = \begin{cases} -4x \text{ if } x < 0 \\ x^2 \text{ if } 0 \leqslant x < 3 \\ 9 \text{ if } x \geqslant 3 \end{cases}$ **b** Solve $f(x) = 4$ **c** Solve $f(x) = 9$

TRY THIS **Minimum value**

Find the minimum value of the function $f(x) = 2^{x^2 - 2x}$.

12.3 Inverse functions

Inverse operations

For many years now, you have been familiar with inverse operations. An **inverse operation** reverses, or undoes another operation.

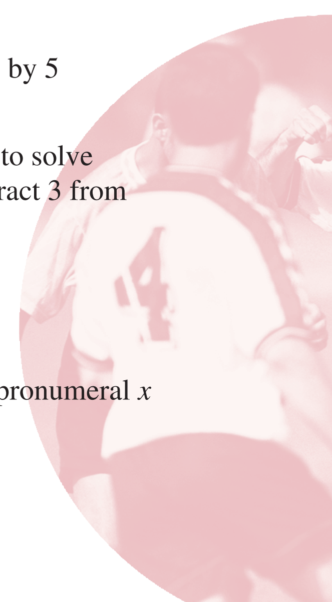
Some examples of inverse operations are:

- adding 3 and subtracting 3
- doubling and halving
- multiplying by 5 and dividing by 5
- cubing and taking a cube root

Inverse operations have been used extensively to solve equations. For example, to solve $x + 3 = 7$ we need to remove the operation (+3) on the LHS. To do this, we subtract 3 from both sides of the equation.

$$\begin{aligned} \text{i.e. } x + 3 &= 7 \\ -3 \quad &\;\; -3 \\ \therefore x &= 4 \end{aligned}$$

The inverse operation (−3) undoes the original operation (+3), leaving only the pronumeral x and its value.



Inverse functions

An inverse function is a function which reverses, or undoes another function.

That is, the inverse of a function returns the output values back to their original input values.

For example, the inverse of the function $y = 2x$ is $y = \frac{1}{2}x$, or $y = \frac{x}{2}$.

$y = 2x$

x	y
0	0
1	2
2	4
3	6
4	8

$y = \frac{1}{2}x$

x	y
0	0
2	1
4	2
6	3
8	4

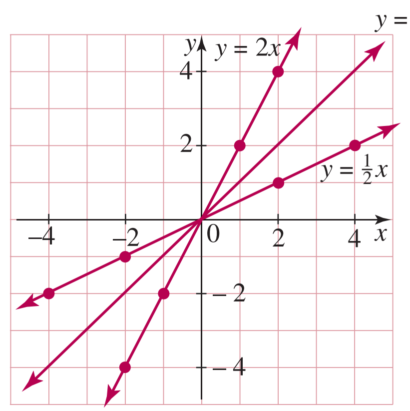
The input and output values are inter-changed for this function and its inverse.

The inverse of a function (or any relation) is found by reversing the values in each ordered pair.

Graphing inverse relations and functions

The geometrical effect of swapping the x- and y-values in each ordered pair of a relation is to produce a reflection of the graph of the relation in the line $y = x$.

For example, $y = 2x$ and $y = \frac{1}{2}x$ are inverse functions. Each graph is a reflection of the other in the line $y = x$.



To graph the inverse of a function (or any relation):
- **reflect the graph of the function (or relation) in the line $y = x$.**

NOTE: If the graphs of a function (or relation) and its inverse intersect, they do so only on the line $y = x$.

The equation of the inverse

Since the inverse of a function (or any relation) is obtained by interchanging the values in each ordered pair, the equation which describes this inverse can be found by interchanging the x- and y-variables in the original equation.

To find the equation of the inverse of a function (or any relation):
- **interchange the variables x and y in the original equation**
- **make y the subject of the new equation.**

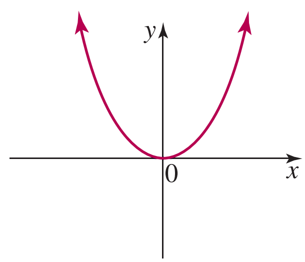
For example, the inverse of the function $y = 2x + 1$ is $x = 2y + 1$, which becomes $y = \frac{x-1}{2}$, on re-arranging.

The horizontal line test

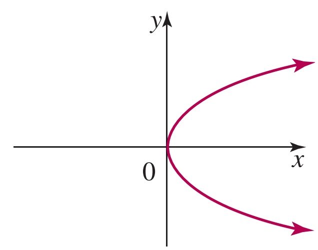
It does not necessarily follow that the inverse of a function is itself a function.

For example, consider the function $y = x^2$ and its inverse $x = y^2$.

Whilst the graph of the original function $y = x^2$ passes the vertical line test, its inverse $x = y^2$ fails the test. Hence, although $y = x^2$ is a function, its inverse is not a function.



$y = x^2$



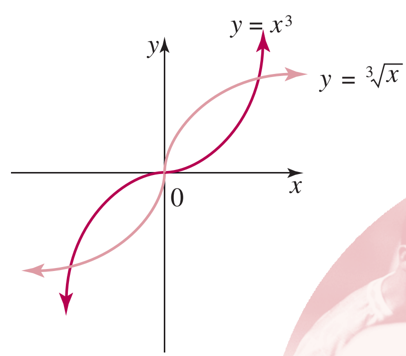
$x = y^2$

In some cases, however, the inverse of a function is also a function.

For example, consider the function $y = x^3$ and its inverse $x = y^3$, or $y = \sqrt[3]{x}$.

Both graphs pass the vertical line test. Therefore the inverse of the function $y = x^3$ is also a function.

Since the inverse is obtained by swapping the x and y co-ordinates in each ordered pair, it follows that if any horizontal line can only cut the graph of the original function in one point, then a vertical line can only cut the graph of the inverse in one point. Hence, the inverse is also a function.



The inverse of a function (or any relation) is a function if no horizontal line can be drawn to cut the graph of the original function (or relation) more than once.

The notation $y = f^{-1}(x)$

If $f(x)$ is a function whose inverse is also a function, then the inverse function is written as $f^{-1}(x)$. The notation $f^{-1}(x)$ is read as “the inverse function of x”.

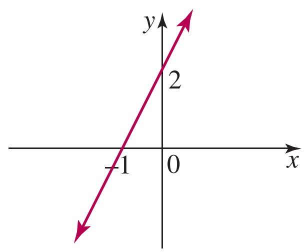
For example, if $f(x) = 2x$, then $f^{-1}(x) = \frac{1}{2}x$.

NOTE: The use of the raised −1 in this situation must not be confused with the use of −1 as an index, which refers to the reciprocal.

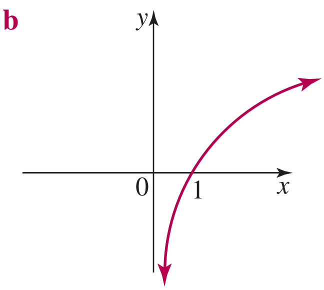
Example 1

Sketch the inverse of each of the following.

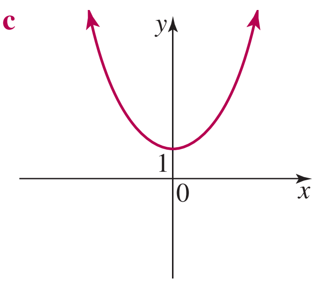
a



b



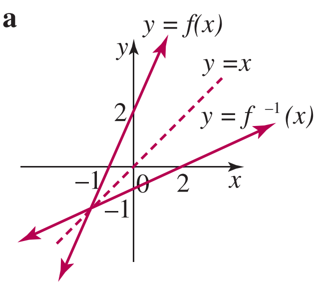
c



Solutions

To sketch each inverse, reflect the original graph in the line $y = x$.

a



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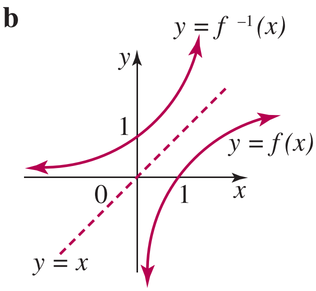

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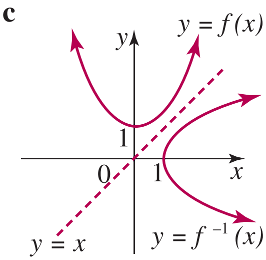


Example 2

Find the inverse function of each function.

a $y = \dfrac{x}{2} + 3$ **b** $y = \dfrac{1}{x-2}$

Solutions

To find the inverse, interchange x and y in each equation.

a f: $y = \dfrac{x}{2} + 3$

f^{-1}: $x = \dfrac{y}{2} + 3$

$x - 3 = \dfrac{y}{2}$

$\therefore y = 2x - 6$

b f: $y = \dfrac{1}{x-2}$

f^{-1}: $x = \dfrac{1}{y-2}$

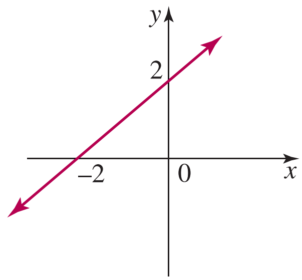
$\dfrac{1}{x} = y - 2$

$\therefore y = \dfrac{1}{x} + 2$

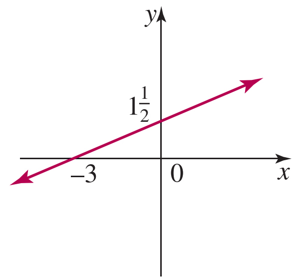
Exercise 12.3

1 Sketch the inverse relation for each of the following.

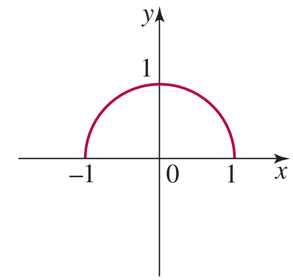
a



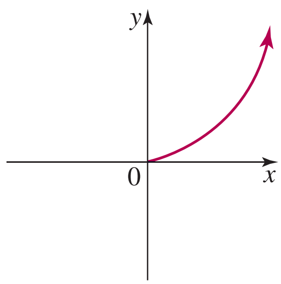
b



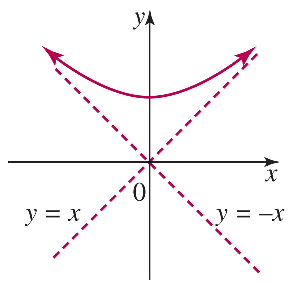
c



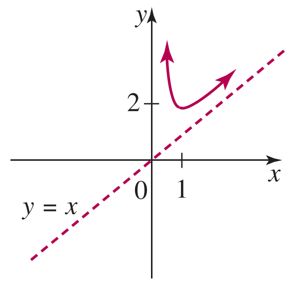
d



e



f



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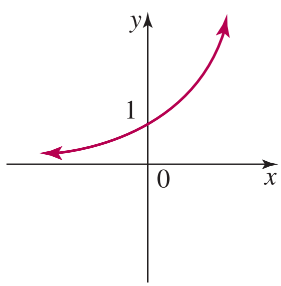

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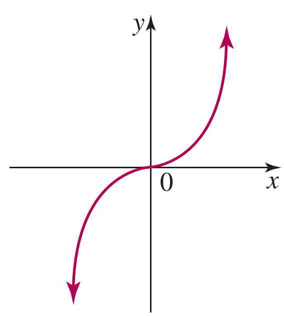

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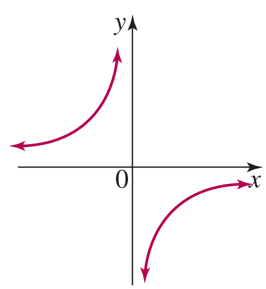

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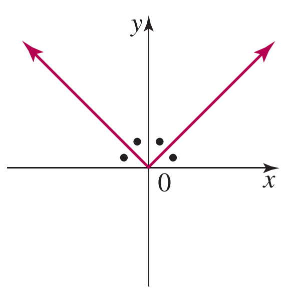

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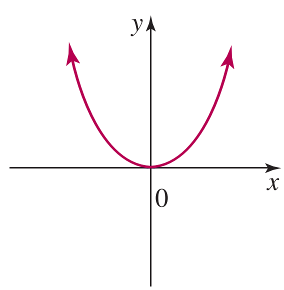

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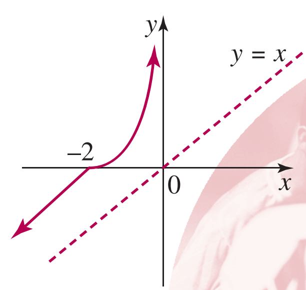


2 Which of the relations in question 1 have an inverse that is a function?

Consolidation

3 Find the inverse function for each of these linear functions. Sketch the function, its inverse and the line $y = x$ on the same number plane.

a $y = x + 3$ **b** $y = \frac{x}{2}$ **c** $y = 2x + 4$

d $y = 6 - 3x$ **e** $y = 5 - \frac{x}{4}$ **f** $x - 2y + 8 = 0$

4 Find the inverse function for each of the following functions. For each inverse, make y the subject.

a $y = \frac{1}{x} - 2$ **b** $y = \frac{1}{x - 1}$ **c** $y = \frac{x - 3}{x + 3}$ **d** $y = \frac{2x}{5 - x}$

5 **a** Find the inverse of $y = \frac{7 - 2x}{x + 2}$. What do you notice?

b What is the geometric significance of this result?

6 Write down a restricted set of permissible x-values, so that each function will have an inverse that is also a function.

a $y = x^2$ **b** $y = 2x^2 - 5$ **c** $y = (x - 3)^2$

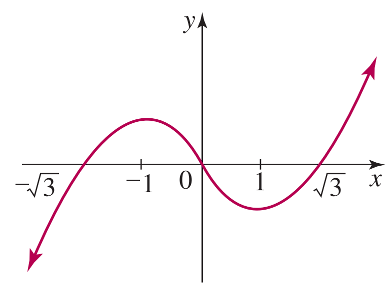
d $y = (x + 4)^2$ **e** $y = (x - 1)^2 + 2$ **f** $y = x(x - 2)$

7 The graph of $y = x^3 - 3x$ is shown.

a Is $y = x^3 - 3x$ a function?

b Is the inverse of $y = x^3 - 3x$ a function?

c Write down the largest possible set of permissible x-values containing the origin such that the inverse is also a function.



Further applications

8 The domain of a relation is the set of permissible x-values for which it is defined. Similarly the range of a relation is the set of permissible y-values for which it is defined. The domain of a relation is the same as the range of its inverse and the range of a relation is the same as the domain of its inverse. Why?

9 For each function below:

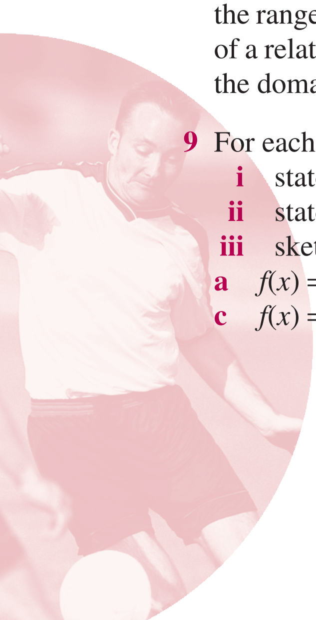
i state the domain and range of the given function

ii state the domain and range of the inverse function

iii sketch the function and its inverse on the same number plane.

a $f(x) = 2x + 4$, $(-2 \leqslant x \leqslant 0)$ **b** $f(x) = x^2$, $(x \geqslant 0)$

c $f(x) = (x - 2)^2$, $(x \geqslant 2)$ **d** $f(x) = 2^x$



10 Show that the functions $f(x)$ and $g(x)$ are inverses of each other by showing that $f(g(x)) = x$ and $g(f(x)) = x$.

a $f(x) = x + 7$ and $g(x) = x - 7$
b $f(x) = 5x$ and $g(x) = \frac{x}{5}$
c $f(x) = 2x + 2$ and $g(x) = \frac{1}{2}x - 1$
d $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$
e $f(x) = \frac{1}{x+3}$ and $g(x) = \frac{1}{x} - 3$
f $f(x) = \frac{x-1}{x+2}$ and $g(x) = \frac{2x+1}{1-x}$

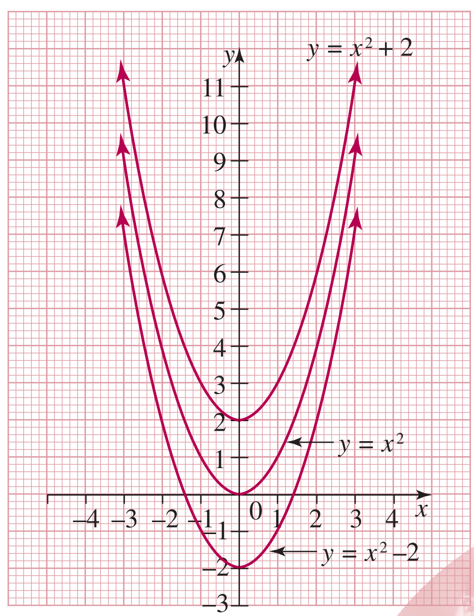
12.4 Translating graphs of functions

Given the graph of $y = f(x)$, we can sketch the graphs of $y = f(x) + k$ and $y = f(x - a)$ by performing vertical and horizontal translations respectively.

The graph of $y = f(x) + k$

Consider the function $y = x^2$.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 + 2$	11	6	3	2	3	6	11
$x^2 - 2$	7	2	-1	-2	-1	2	7



From this table, we can see that the y-values for the curve $y = x^2 + 2$ are each 2 more than the y-values for the curve $y = x^2$.

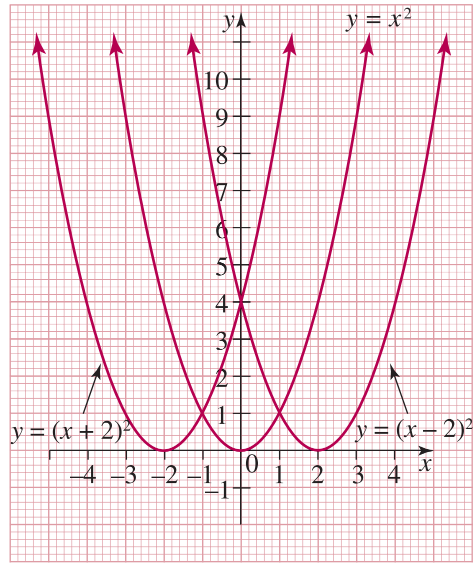
The graph of $y = x^2 + 2$, therefore, can be obtained by shifting the graph of $y = x^2$ up by 2 units. Similarly, the y-values for the curve $y = x^2 - 2$ are each 2 less than the y-values for the curve $y = x^2$. The graph of $y = x^2 - 2$, therefore, can be obtained by shifting the graph of $y = x^2$ down by 2 units.

To sketch the graph of:
- **$y = f(x) + k$, $(k > 0)$, shift the graph of $y = f(x)$ up k units**
- **$y = f(x) - k$, $(k > 0)$, shift the graph of $y = f(x)$ down k units.**

The graph of $y = f(x - a)$

Consider, again, the function $y = x^2$.

x	–4	–3	–2	–1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
$(x - 2)^2$	36	25	16	9	4	1	0	1	4
$(x + 2)^2$	4	1	0	1	4	9	16	25	36



From this table and the corresponding graphs, we can see that the graph of $y = (x - 2)^2$ could have been obtained by shifting the graph of $y = x^2$ 2 units to the right. Similarly, the graph of $y = (x + 2)^2$ could have been obtained by shifting the graph of $y = x^2$ 2 units to the left.

To sketch the graph of:
- **$y = f(x - a)$, $(a > 0)$, shift the graph of $y = f(x)$ a units to the right**
- **$y = f(x + a)$, $(a > 0)$, shift the graph of $y = f(x)$ a units to the left.**

Example 1

Write down the equation of the new function if

a $y = x^3$ has been translated:
  i 3 units to the right **ii** 1 unit down

b $y = \frac{1}{x}$ has been translated:
  i 4 units to the left **ii** 5 units up

c $y = 3^x$ has been translated:
  i 1 unit to the right **ii** 2 units down

Solutions

a **i** $y = (x - 3)^3$ **ii** $y = x^3 - 1$

b **i** $y = \frac{1}{x + 4}$ **ii** $y = \frac{1}{x} + 5$

c **i** $y = 3^{x - 1}$ **ii** $y = 3^x - 2$

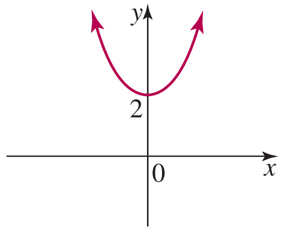
Example 2

If $f(x) = x^2$, sketch each of the following graphs.

a $y = f(x) + 2$ **b** $y = f(x) - 2$ **c** $y = f(x - 2)$ **d** $y = f(x + 2)$

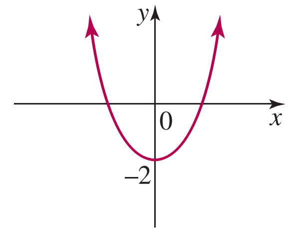
Solutions

a



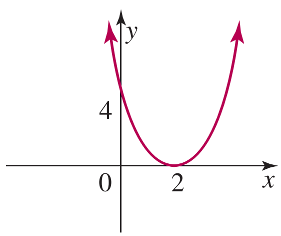
$y = f(x) + 2$, i.e. $y = x^2 + 2$

b



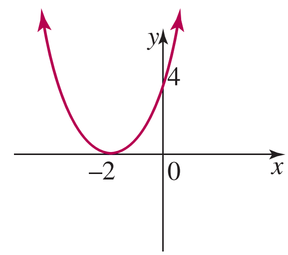
$y = f(x) - 2$, i.e. $y = x^2 - 2$

c



$y = f(x - 2)$, i.e. $y = (x - 2)^2$

d



$y = f(x + 2)$, i.e. $y = (x + 2)^2$

Exercise 12.4

1 Consider the function $y = x^2$. Write down the new equation of this function after it has been translated:
a up 2 units **b** down 3 units **c** right 1 unit **d** left 4 units

2 Consider the function $y = x^3$. Write down the new equation of this function after it has been translated:
a left 2 units **b** up 1 unit **c** down 6 units **d** right 4 units

3 Consider the function $y = 2^x$. Write down the new equation of this function after it has been translated:
a down 1 unit **b** right 3 units **c** up 5 units **d** left 2 units

4 Consider the function $y = \frac{1}{x}$. Write down the new equation of this function after it has been translated:
a right 5 units **b** down 2 units **c** left 3 units **d** up 7 units

5 Find the new equation of each function after the given translations have occurred.
a $y = x^2 + 5$; down 2 units **b** $y = 3^x - 4$; up 6 units **c** $y = \frac{1}{x} + 3$; down 2 units
d $y = 2^x + 1$; right 3 units **e** $y = x^2 - 7$; left 1 unit **f** $y = \frac{1}{x} - 2$; right 5 units

Consolidation

6 Find the new equation of each function after the given translations have occurred.
a $y = 2x^2$ **i** down 5 units **ii** right 3 units

b $y = \frac{1}{2}x^3$ **i** up 2 units **ii** left 4 units

c $y = \frac{3}{x}$ **i** up 7 units **ii** right 1 unit

7 Find the new equation of each function after the given translations have occurred.

a $y = -x^2$ **i** up 4 units **ii** right 3 units

b $y = 5^{-x}$ **i** down 1 unit **ii** left 2 units

c $y = -\frac{1}{x}$ **i** right 5 units **ii** down 7 units

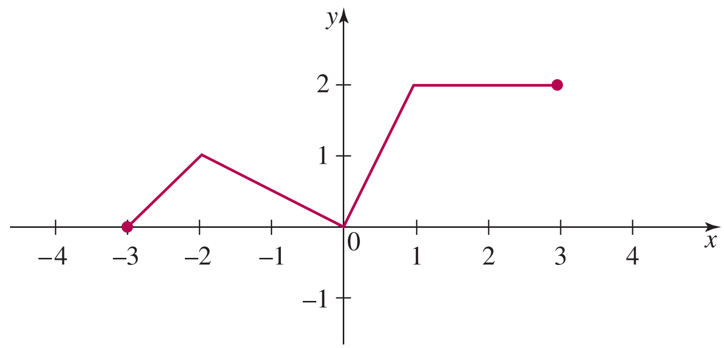
8 The graph of $y = f(x)$ is shown. Sketch the following graphs.

a $y = f(x) + 1$

b $y = f(x) - 2$

c $y = f(x - 1)$

d $y = f(x + 2)$



9 If $f(x) = x^2$, sketch:

a $y = f(x)$ **b** $y = f(x - 3)$ **c** $y = f(x) + 2$

10 If $f(x) = 2^x$, sketch:

a $y = f(x)$ **b** $y = f(x + 2)$ **c** $y = f(x) - 4$

11 If $f(x) = \frac{1}{x}$, sketch:

a $y = f(x)$ **b** $y = f(x - 2)$ **c** $y = f(x) + 1$

12 If $f(x) = -x^2$, sketch:

a $y = f(x)$ **b** $y = f(x) + 9$ **c** $y = f(x - 2)$

13 If $f(x) = 3^{-x}$, sketch:

a $y = f(x)$ **b** $y = f(x) + 2$ **c** $y = f(x + 1)$

14 If $f(x) = -\frac{1}{x}$, sketch:

a $y = f(x)$ **b** $y = f(x) - 3$ **c** $y = f(x + 4)$

15 If $f(x) = x^2$, sketch: **a** $y = f(x - 2) + 3$ **b** $y = f(x + 1) - 4$

16 If $f(x) = \frac{1}{x}$, sketch: **a** $y = f(x + 2) + 1$ **b** $y = f(x - 3) - 2$

■ Further applications

17 If $f(x) = x(x - 2)$, sketch:

a $y = f(x) + 1$ **b** $y = f(x) - 3$ **c** $y = f(x + 2)$ **d** $y = f(x - 1)$

18 a If $f(x) = \frac{1}{2}x - 1$, find $f^{-1}(x)$. **b** Sketch the graph of $y = f^{-1}(x) + 1$.

19 a If $g(x) = \frac{2}{x+1} + 1$, find $g^{-1}(x)$. **b** Sketch the graph of $y = g^{-1}(x - 1)$.

12.5 Solving simple exponential equations

Index laws

The following index laws will be used extensively in the rest of this topic.

$a^m \times a^n = a^{m+n}$	$a^m \div a^n = a^{m-n}$	$(a^m)^n = a^{mn}$
$a^{\circ} = 1$	$a^{-1} = \dfrac{1}{a}$	$a^{-n} = \dfrac{1}{a^n}$
$\left(\dfrac{a}{b}\right)^{-1} = \dfrac{b}{a}$	$\left(\dfrac{a}{b}\right)^{-n} = \dfrac{b^n}{a^n}$	$a^{\frac{1}{2}} = \sqrt{a}$
$a^{\frac{1}{3}} = \sqrt[3]{a}$	$a^{\frac{1}{n}} = \sqrt[n]{a}$	$a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$

Exponential equations

An exponential equation is an equation in which the power or index is the unknown. For example, $3^x = 9$, $8^x = 4$, $2^{x-3} = 16$.

To solve an exponential equation:
- **express both sides of the equation with the same base**
- **equate the indices**
- **solve the resulting equation.**

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Example

Solve the following equations.

a $2^x = 8$ **b** $3^x = \frac{1}{27}$ **c** $36^x = 6$

d $4^x = 32$ **e** $7^{x-4} = 1$ **f** $5^{3x-1} = 125$

Solutions

a
$$\begin{aligned} 2^x &= 8 \\ 2^x &= 2^3 \\ \therefore x &= 3 \end{aligned}$$

b
$$\begin{aligned} 3^x &= \frac{1}{27} \\ 3^x &= \frac{1}{3^3} \\ 3^x &= 3^{-3} \\ \therefore x &= -3 \end{aligned}$$

c
$$\begin{aligned} 36^x &= 6 \\ (6^2)^x &= 6^1 \\ 6^{2x} &= 6^1 \\ 2x &= 1 \\ \therefore x &= \frac{1}{2} \end{aligned}$$

d
$$\begin{aligned} 4^x &= 32 \\ (2^2)^x &= 2^5 \\ 2^{2x} &= 2^5 \\ 2x &= 5 \\ \therefore x &= \frac{5}{2} \end{aligned}$$

e
$$\begin{aligned} 7^{x-4} &= 1 \\ 7^{x-4} &= 7^0 \\ \therefore x - 4 &= 0 \\ \therefore x &= 4 \end{aligned}$$

f
$$\begin{aligned} 5^{3x-1} &= 125 \\ 5^{3x-1} &= 5^3 \\ \therefore 3x - 1 &= 3 \\ 3x &= 4 \\ \therefore x &= \frac{4}{3} \end{aligned}$$

Exercise 12.5

1 Solve the following equations.

a $3^x = 9$ b $2^x = 16$ c $7^x = 7$ d $10^x = 1000$
e $6^x = 1$ f $3^x = 81$ g $2^x = 64$ h $5^x = 625$
i $2^x = 256$ j $3^x = 243$ k $7^x = 2401$ l $6^x = 7776$

2 Solve for x:

a $25^x = 5$ b $49^x = 7$ c $27^x = 3$ d $16^x = 2$
e $1000^x = 10$ f $121^x = 11$ g $32^x = 2$ h $81^x = 3$

3 Solve for x:

a $5^x = \frac{1}{5}$ b $3^x = \frac{1}{9}$ c $2^x = \frac{1}{8}$ d $10^x = \frac{1}{10\,000}$
e $4^x = \frac{1}{16}$ f $2^x = \frac{1}{32}$ g $5^x = \frac{1}{625}$ h $3^x = \frac{1}{729}$

Consolidation

4 Solve:

a $4^x = 8$ b $27^x = 9$ c $8^x = 16$ d $125^x = 25$
e $16^x = 32$ f $1000^x = 100$ g $64^x = 32$ h $32^x = 128$
i $8^x = \frac{1}{4}$ j $9^x = \frac{1}{27}$ k $32^x = \frac{1}{8}$ l $125^x = \frac{1}{625}$

5 Solve the following equations.

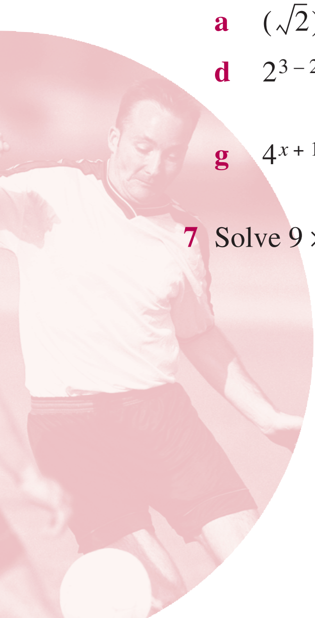
a $2^{x+1} = 16$ b $7^{x-3} = 49$ c $5^{2-x} = 125$
d $2^{5x} = 8$ e $3^{2x-5} = 27$ f $5^{2x-6} = 1$
g $4^{1-x} = 32^x$ h $8^{5x} = 16^{2x-1}$ i $27^{x-4} = 9^{4-x}$

Further applications

6 Solve for x:

a $(\sqrt{2})^x = 8$ b $5^{x-2} = \sqrt{5}$ c $9^{x+1} = \sqrt{3}$
d $2^{3-2x} = 4\sqrt{2}$ e $\frac{1}{5^x} = \sqrt{125}$ f $7^{1-x} = \frac{1}{7\sqrt{7}}$
g $4^{x+1} = \frac{1}{16\sqrt{2}}$ h $\left(\frac{1}{4}\right)^{2-x} = \sqrt{8}$ i $\left(\frac{1}{3}\right)^{x-1} = \frac{\sqrt{12}}{18}$

7 Solve $9 \times 2^x = 6^x$. [Hint: $(ab)^n = a^n b^n$.]



12.6 Definition of a logarithm

When indices are used to perform calculations, the indices are called logarithms.

The logarithm of a number to any base is the index to which the base must be raised to give that number.

$$y = a^x \Leftrightarrow \log_a y = x$$

That is, by the definition of a logarithm, $y = a^x$ and $\log_a y = x$ are equivalent statements.

NOTE: x can be any number, but $a > 1$ and $y > 0$.

Example 1

EG +S

a Write $3^4 = 81$ in logarithmic form. **b** Write $\log_4 16 = 2$ in index form.

Solutions

a If $a^x = y$, then $\log_a y = x$,
$\therefore$ if $3^4 = 81$, then $\log_3 81 = 4$.

b If $\log_a y = x$, then $a^x = y$,
$\therefore$ if $\log_4 16 = 2$, then $4^2 = 16$.

Example 2

EG +S

a Evaluate $\log_3 9$ **b** Solve $\log_7 x = 2$ **c** Solve $\log_x 36 = 2$

Solutions

a Let $\log_3 9 = x$
$\therefore 3^x = 9$
$\therefore x = 2$
$\therefore \log_3 9 = 2$

b $\log_7 x = 2$
$\therefore 7^2 = x$
$\therefore x = 49$

c $\log_x 36 = 2$
$\therefore x^2 = 36$
$\therefore x = 6\ (x > 1)$

Exercise 12.6

1 Write each expression in logarithmic form.

a $9 = 3^2$ **b** $8 = 2^3$ **c** $6^2 = 36$ **d** $2^5 = 32$
e $8^1 = 8$ **f** $5^0 = 1$ **g** $64 = 4^3$ **h** $3^5 = 243$
i $5^{-1} = \frac{1}{5}$ **j** $2^{-2} = \frac{1}{4}$ **k** $\sqrt{3} = 3^{\frac{1}{2}}$ **l** $\sqrt[3]{7} = 7^{\frac{1}{3}}$
m $9^{\frac{3}{2}} = 27$ **n** $\sqrt[5]{8} = 2^{\frac{3}{5}}$ **o** $144^{-\frac{1}{2}} = \frac{1}{12}$ **p** $16^{-\frac{3}{4}} = \frac{1}{8}$

2 Write each expression in index form.

a $\log_5 25 = 2$ **b** $\log_2 16 = 4$ **c** $\log_7 7 = 1$ **d** $\log_2 8 = 3$
e $\log_3 81 = 4$ **f** $\log_{10} 100 = 2$ **g** $\log_2 32 = 5$ **h** $\log_6 1 = 0$
i $\log_3 243 = 5$ **j** $\log_2\left(\frac{1}{2}\right) = -1$ **k** $\log_3\left(\frac{1}{9}\right) = -2$ **l** $\log_4 2 = \frac{1}{2}$
m $\log_{27} 3 = \frac{1}{3}$ **n** $\log_5 \sqrt{5} = \frac{1}{2}$ **o** $\log_8 4 = \frac{2}{3}$ **p** $\log_{1000}\left(\frac{1}{100}\right) = -\frac{2}{3}$

Consolidation

3 Evaluate:

a $\log_{10} 10$ **b** $\log_4 16$ **c** $\log_3 27$ **d** $\log_2 32$
e $\log_9 1$ **f** $\log_{10} 1000$ **g** $\log_6 36$ **h** $\log_3 81$
i $\log_2 \sqrt{2}$ **j** $\log_8 2$ **k** $\log_4\left(\frac{1}{4}\right)$ **l** $\log_3\left(\frac{1}{9}\right)$
m $\log_5\left(\frac{1}{\sqrt{5}}\right)$ **n** $\log_2 0.25$ **o** $\log_8 4$ **p** $\log_4 \sqrt{2}$

4 Solve the following equations.

a $\log_2 x = 4$ **b** $\log_6 x = 2$ **c** $\log_5 x = 3$ **d** $\log_2 x = 5$
e $\log_{12} x = 0$ **f** $\log_4 x = 1$ **g** $\log_7 x = 4$ **h** $\log_4 x = 5$
i $\log_6 x = -1$ **j** $\log_2 x = -3$ **k** $\log_5 x = -2$ **l** $\log_{10} x = \frac{1}{2}$
m $\log_{64} x = \frac{1}{3}$ **n** $\log_{16} x = \frac{3}{4}$ **o** $\log_{49} x = -\frac{1}{2}$ **p** $\log_{32} x = -\frac{3}{5}$

5 Solve for x.

a $\log_x 8 = 3$ **b** $\log_x 9 = 2$ **c** $\log_x 13 = 1$ **d** $\log_x 64 = 6$
e $\log_x 121 = 2$ **f** $\log_x 64 = 3$ **g** $\log_x 243 = 5$ **h** $\log_x 10\,000 = 4$
i $\log_x\left(\frac{1}{5}\right) = -1$ **j** $\log_x\left(\frac{1}{36}\right) = -2$ **k** $\log_x\left(\frac{1}{16}\right) = -4$ **l** $\log_x \sqrt{6} = \frac{1}{2}$
m $\log_x 2 = \frac{1}{3}$ **n** $\log_x 4 = \frac{2}{3}$ **o** $\log_x 27 = \frac{3}{2}$ **p** $\log_x 64 = \frac{6}{5}$

Further applications

6 Solve $\log_5[\log_5(\log_5 x)] = 0$.

7 Prove that $\log_b a \times \log_a b = 1$. [Hint: let $\log_b a = x$ and $\log_a b = y$.]

12.7 The logarithm laws

The index laws and the definition of a logarithm can be used to deduce the logarithm laws.

$$\log_a xy = \log_a x + \log_a y$$

Proof: Let $x = a^m$ and $y = a^n$

$\therefore\ m = \log_a x$ and $n = \log_a y$ (definition of a logarithm)

Now, $xy = a^m \times a^n$

$\therefore\ xy = a^{m+n}$

$\therefore\ \log_a xy = m + n$

$= \log_a x + \log_a y$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

Proof: Let $x = a^m$ and $y = a^n$

$\therefore\ m = \log_a x$ and $n = \log_a y$ (definition of a logarithm)

$$\text{Now, } \frac{x}{y} = \frac{a^m}{a^n}$$

$$\therefore\ \frac{x}{y} = a^{m-n}$$

$$\begin{aligned}\therefore\ \log_a \left(\frac{x}{y}\right) &= m - n \\ &= \log_a x - \log_a y\end{aligned}$$

$$\log_a x^n = n \log_a x$$

Proof: Let $x = a^m$

$\therefore\ m = \log_a x$ (definition of a logarithm)

Now, $x^n = (a^m)^n$

$\therefore\ x^n = a^{mn}$

$$\begin{aligned}\therefore\ \log_a x^n &= mn \\ &= n \log_a x\end{aligned}$$

The results below follow directly from the definition of a logarithm.

$$\log_a 1 = 0 \qquad \log_a a = 1 \qquad \log_a a^x = x$$

Example 1

Evaluate each of the following expressions.

a $\log_{10} 25 + \log_{10} 4$

b $\log_2 40 - \log_2 5$

c $\log_{10} 12 + \log_{10} 5 - \log_{10} 6$

d $2 \log_6 3 + \frac{1}{2} \log_6 16$

Solutions

a $$\begin{aligned}&\log_4 25 + \log_{10} 4 \\ &= \log_{10} (25 \times 4) \\ &= \log_{10} 100 \\ &= 2\end{aligned}$$

b $$\begin{aligned}&\log_2 40 - \log_2 5 \\ &= \log_2 \left(\frac{40}{5}\right) \\ &= \log_2 8 \\ &= 3\end{aligned}$$

c $$\begin{aligned}&\log_{10} 12 + \log_{10} 5 - \log_{10} 6 \\ &= \log_{10} \left(\frac{12 \times 5}{6}\right) \\ &= \log_{10} 10 \\ &= 1\end{aligned}$$

d $$\begin{aligned}&2 \log_6 3 + \frac{1}{2} \log_6 16 \\ &= \log_6 3^2 + \log_6 16^{\frac{1}{2}} \\ &= \log_6 9 + \log_6 4 \\ &= \log_6 36 \\ &= 2\end{aligned}$$

Example 2

If $\log_a 2 = 0.301$ and $\log_a 3 = 0.477$, evaluate each of the following.

a $\log_a 6$ **b** $\log_a 1.5$ **c** $\log_a 8$

d $\log_a \left(\frac{1}{2}\right)$ **e** $\log_a \sqrt{3}$ **f** $\log_a 24$

Solutions

a
$$\begin{aligned}&\log_a 6\\&= \log_a (3 \times 2)\\&= \log_a 3 + \log_a 2\\&= 0.477 + 0.301\\&= 0.778\end{aligned}$$

b
$$\begin{aligned}&\log_a 1.5\\&= \log_a \left(\frac{3}{2}\right)\\&= \log_a 3 - \log_a 2\\&= 0.477 - 0.301\\&= 0.176\end{aligned}$$

c
$$\begin{aligned}&\log_a 8\\&= \log_a 2^3\\&= 3 \log_a 2\\&= 3 \times 0.301\\&= 0.903\end{aligned}$$

d
$$\begin{aligned}&\log_a \left(\frac{1}{2}\right)\\&= \log_a 2^{-1}\\&= -\log_a 2\\&= -0.301\end{aligned}$$

e
$$\begin{aligned}&\log_a \sqrt{3}\\&= \log_a 3^{\frac{1}{2}}\\&= \frac{1}{2} \log_a 3\\&= \frac{1}{2} \times 0.477\\&= 0.2385\end{aligned}$$

f
$$\begin{aligned}&\log_a 24\\&= \log_a (8 \times 3)\\&= \log_a 8 + \log_a 3\\&= \log_a 2^3 + \log_a 3\\&= 3 \log_a 2 + \log_a 3\\&= (3 \times 0.301) + 0.477\\&= 1.38\end{aligned}$$

Example 3

a Simplify $3 \log x^4 - 2 \log x^5$

b Simplify $\dfrac{3 \log a^8}{4 \log \sqrt{a}}$

Solutions

a
$$\begin{aligned}&3 \log x^4 - 2 \log x^5\\&= 12 \log x - 10 \log x\\&= 2 \log x\end{aligned}$$

b
$$\begin{aligned}&\frac{3 \log a^8}{4 \log \sqrt{a}}\\&= \frac{3 \log a^8}{4 \log a^{\frac{1}{2}}}\\&= \frac{24 \cancel{\log a}}{2 \cancel{\log a}}\\&= 12\end{aligned}$$

Example 4

Solve $\log 4x - \log 3 = \log (x + 7)$.

Solution

$$\begin{aligned}\log 4x - \log 3 &= \log (x + 7)\\ \log \left(\frac{4x}{3}\right) &= \log (x + 7)\\ \therefore \frac{4x}{3} &= x + 7\\ 4x &= 3x + 21\\ \therefore x &= 21\end{aligned}$$

Exercise 12.7

1 Evaluate each of the following using the logarithm laws.

a $\log_{12} 3 + \log_{12} 4$ **b** $\log_3 15 - \log_3 5$ **c** $\log_6 12 + \log_6 3$

d $\log_2 48 - \log_2 6$ **e** $\log_2 80 - \log_2 5$ **f** $\log_7 2 + \log_7 \left(\frac{1}{2}\right)$

g $\log_4 36 - 2\log_4 3$ **h** $3\log_{10} 5 + \log_{10} 8$ **i** $\log_{100} 40 - \log_{100} 4$

j $\log_2 5 - \log_2 40$ **k** $4\log_8 2 + \frac{1}{2}\log_8 16$ **l** $2\log_{10} 5 - \log_{10}\left(\frac{1}{4}\right)$

2 Evaluate:

a $\log_{12} 3 + \log_{12} 6 + \log_{12} 8$ **b** $\log_4 10 + \log_4 8 - \log_4 5$

c $\log_4 14 - \log_4 7 + \log_4 32$ **d** $\log_3 54 - (\log_3 22 - \log_3 11)$

e $\log_{10} 18 + \log_{10} 15 - 3\log_{10} 3$ **f** $2\log_5 10 + \frac{1}{2}\log_5 100 - 3\log_5 2$

Consolidation

3 If $\log_a 2 = 0.631$ and $\log_a 5 = 1.465$, find the value of:

a $\log_a 10$ **b** $\log_a 2.5$ **c** $\log_a 4$ **d** $\log_a \left(\frac{1}{2}\right)$

e $\log_a \sqrt{2}$ **f** $\log_a 125$ **g** $\log_a 50$ **h** $\log_a 40$

i $\log_a 100$ **j** $\log_a 0.2$ **k** $\log_a 0.25$ **l** $\log_a \sqrt{20}$

4 If $\log_a 3 = 0.477$, $\log_a 4 = 0.602$ and $\log_a 6 = 0.778$, find the value of:

a $\log_a 12$ **b** $\log_a 18$ **c** $\log_a 2$ **d** $\log_a \left(\frac{2}{3}\right)$

e $\log_a 0.75$ **f** $\log_a 36$ **g** $\log_a 27$ **h** $\log_a \left(\frac{1}{4}\right)$

i $\log_a \sqrt{6}$ **j** $\log_a \left(\frac{1}{\sqrt{3}}\right)$ **k** $\log_a 8$ **l** $\log_a 54$

m $\log_a 4.5$ **n** $\log_a 32$ **o** $\log_a \left(\frac{1}{24}\right)$ **p** $\log_a \sqrt{48}$

5 Simplify each of the following.

a $\log 2a + \log 5b$ **b** $\log 4p - \log 3q$ **c** $3\log x + 2\log y$

d $5\log m - 4\log n$ **e** $\log a + \frac{1}{2}\log b$ **f** $2\log p - \log\left(\frac{1}{q}\right)$

g $\log p + \log q + \log r$ **h** $\log a + \log b - \log c$ **i** $\log e - \log f - \log g$

j $2\log x - 3\log y + \log z$ **k** $\frac{1}{2}\log m + \frac{1}{2}\log n + 2\log p$ **l** $\log x + \log\frac{1}{x} - 3\log y$

6 Expand:

a $\log(abc)$ **b** $\log\left(\frac{pq}{r}\right)$ **c** $\log\left(\frac{x}{yz}\right)$

d $2\log\sqrt{gh}$ **e** $\log\sqrt[3]{pq}$ **f** $\log\left(\frac{1}{mn}\right)$

g $\log\left(\frac{a^2}{\sqrt{b}}\right)$ **h** $\log\sqrt{\frac{c}{d}}$ **i** $\log\left(\frac{\sqrt[3]{x^2}}{y^4}\right)$

7 Simplify the following expressions. Give your answers in the form $a \log x$.

a $4 \log x + 3 \log x$ **b** $6 \log x - \log x$ **c** $\log x^5 + 2 \log x$

d $5 \log x^3 - 4 \log x^2$ **e** $\frac{1}{2} \log x^{12} + 2 \log x$ **f** $\frac{1}{2} \log x^5 + \frac{1}{2} \log x^3$

8 Simplify:

a $\frac{\log a^2}{\log a}$ **b** $\frac{\log a^6}{\log a^2}$ **c** $\frac{\log a^2}{\log a^3}$

d $\frac{\log \sqrt{a}}{\log a}$ **e** $\frac{\log a}{\log \sqrt[3]{a}}$ **f** $\frac{\log a^3}{5 \log a}$

g $\frac{\log a^4}{\log \sqrt{a}}$ **h** $\frac{5 \log a^3}{6 \log \sqrt{a}}$ **i** $\frac{\log \left(\frac{1}{a^2}\right)}{\log a}$

9 If $\log_a 5 = 1.16$ and $\log_a 10 = 1.66$, evaluate:

a $\log_a 5a$ **b** $\log_a \left(\frac{10}{a}\right)$ **c** $\log_a \left(\frac{a}{5}\right)$

d $\log_a 25a$ **e** $\log_a 10a^2$ **f** $\log_a 125a^3$

g $\log_a a\sqrt{5}$ **h** $\log_a \sqrt{5a}$ **i** $\log_a \left(\frac{a^2}{10}\right)$

j $\log_a 2a$ **k** $\log_a \left(\frac{a}{2}\right)$ **l** $\log_a \left(\frac{1}{2a}\right)$

10 Solve the following equations.

a $\log x + \log 2 = \log 14$ **b** $\log x - \log 3 = \log 5$

c $\log 4 + \log x = \log (x + 21)$ **d** $\log 2x - \log 5 = \log 4$

e $\log 3x - \log 4 = \log (x - 6)$ **f** $\log (x - 6) - \log 2 = \log (x + 2) - \log 3$

■ Further applications

11 Solve for x:

a $\log x + \log (x - 3) = \log (7x - 25)$ **b** $\log 16 - \log x = \log (8 - x)$

c $\log x + \log (x - 2) = \log (5x - 12)$ **d** $\log x - \log (x - 3) = \log 12 - \log (x - 1)$

e $2 \log x = \log 2 + \log (x + 4)$ **f** $\log x = \log 10 - \log (x + 3)$

12 Solve:

a $\log_5 (3x + 13) - \log_5 (x - 1) = 1$ **b** $\log_2 (2x + 6) - \log_2 (x - 3) = 3$

c $\log_3 x + \log_3 (x + 8) = 2$

TRY THIS Logarithm relation

Given that $a^2 + b^2 = 7ab$, prove that $\log_x \frac{a + b}{3} = \frac{1}{2} (\log_x a + \log_x b)$

12.8 Solving further exponential equations

Equations of the form $a^x = b$ can be solved relatively easily when b is a power of a. If b is not a power of a we can either solve the equation by using a guess, check and refine approach, or by taking the logarithm to base 10 of both sides.

To solve an equation of the form $a^x = b$ where b is not a power of a:
- **take the logarithm to base 10 of both sides**
- **re-write the LHS in the form $x \log_{10} a$**
- **divide both sides by $\log_{10} a$.**

Example 1

Solve each of these equations, correct to 3 decimal places.

a $2^x = 11$ **b** $3^{2x+1} = 45$

Solutions

a

$$\begin{aligned} 2^x &= 11 \\ \log_{10} 2^x &= \log_{10} 11 \\ x \log_{10} 2 &= \log_{10} 11 \\ \therefore x &= \frac{\log_{10} 11}{\log_{10} 2} \\ &\doteqdot 3.459 \end{aligned}$$

b

$$\begin{aligned} 3^{2x+1} &= 45 \\ \log_{10} 3^{2x+1} &= \log_{10} 45 \\ (2x+1) \log_{10} 3 &= \log_{10} 45 \\ 2x + 1 &= \frac{\log_{10} 45}{\log_{10} 3} \\ x &= \left(\frac{\log_{10} 45}{\log_{10} 3} - 1\right) \div 2 \\ &\doteqdot 1.232 \end{aligned}$$

Example 2

Solve $7^{x-1} = 3^{x+1}$, correct to 3 decimal places.

Solution

$$\begin{aligned} 7^{x-1} &= 3^{x+1} \\ \log_{10} 7^{x-1} &= \log_{10} 3^{x+1} \\ (x-1) \log_{10} 7 &= (x+1) \log_{10} 3 \\ x \log_{10} 7 - \log_{10} 7 &= x \log_{10} 3 + \log_{10} 3 \\ x \log_{10} 7 - x \log_{10} 3 &= \log_{10} 7 + \log_{10} 3 \\ x(\log_{10} 7 - \log_{10} 3) &= \log_{10} 7 + \log_{10} 3 \\ \therefore x &= \frac{\log_{10} 7 + \log_{10} 3}{\log_{10} 7 - \log_{10} 3} \\ &\doteqdot 3.593 \end{aligned}$$

Exercise 12.8

1 Solve the following equations by using a guess, check and refine approach. Give your solutions correct to 2 decimal places.

a $2^x = 7$ **b** $5^x = 12$ **c** $3^x = 115$

Consolidation

2 Solve the following equations by first taking the log to base 10 of both sides. Answer correct to 3 decimal places.

a $2^x = 5$ **b** $3^x = 4$ **c** $5^x = 9$ **d** $7^x = 3$
e $10^x = 2$ **f** $6^x = 41$ **g** $3^x = 37$ **h** $2^x = 50$
i $4^x = 78.3$ **j** $5^x = 214.7$ **k** $3^x = 106.2$ **l** $2^x = 531.4$

3 Solve for x, correct to 3 decimal places.

a $3^{x+1} = 13$ **b** $2^{x-2} = 6$ **c** $5^{x+4} = 43$
d $2^{3x} = 14$ **e** $3^{2x} = 25$ **f** $5^{4x} = 3$
g $2^{2x-1} = 30$ **h** $7^{3x+2} = 151$ **i** $4^{1-2x} = 112$

Further applications

4 Solve for x, correct to 3 decimal places.

a $5^{x-1} = 2^x$ **b** $3^{x+2} = 7^x$ **c** $2^{3x} = 6^{x+2}$
d $2^{x+1} = 3^{x-2}$ **e** $5^{x+4} = 7^{2x}$ **f** $13^{2x-1} = 6^{x+3}$

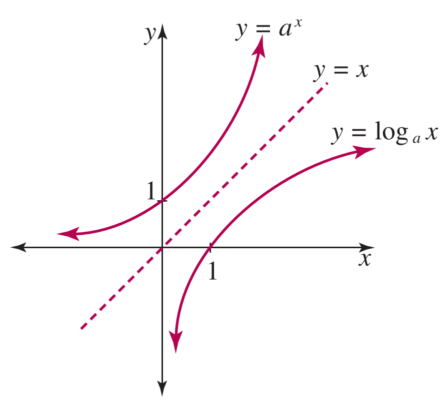
5 Solve $0.5^x > 3$, correct to 3 decimal places.

12.9 The logarithmic graph

Earlier in this chapter we found that the inverse of a function can be derived by swapping the x and y variables in the equation of the curve.

Consider the exponential function $y = a^x$. Interchanging x and y, we have $x = a^y$, that is $y = \log_a x$.

Therefore, $y = a^x$ and $y = \log_a x$ are inverses of each other. The inverse of a function can be drawn by reflecting the graph of the original function in the line $y = x$.



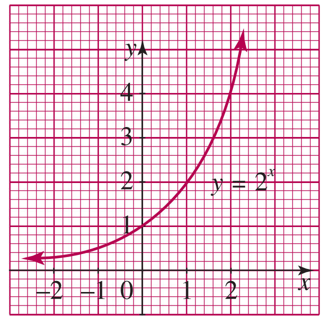
The inverse of $y = a^x$ is $y = \log_a x$.

Example 1

Draw up a table of values to draw the exponential graph $y = 2^x$. Hence, draw the graph of $y = \log_2 x$.

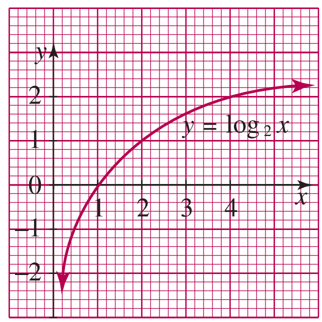
Solution

x	-2	-1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
y	$\frac{1}{2}$	$\frac{1}{2}$	1	1.4	2	2.8	4



Now by using the same table but interchanging x and y we obtain values for $y = \log_2 x$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	1.4	2	2.8	4
y	-2	-1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2



Exercise 12.9

1 **a** Copy and complete this table of values for $y = \log_{10} x$ using the $\boxed{\text{log}}$ key on your calculator. Give the y-values correct to 2 decimal places.

x	0.2	0.4	0.6	0.8	1	2	3	4	5	6	7	8	9	10
y														

b Plot these points on 2 mm grid paper and join them to draw the graph of $y = \log_{10} x$.

c Where does the graph cut the x-axis?

d For what values of x is:

i y positive? **ii** y negative?

e Does the curve cut the y-axis?

f What happens to the value of y as:

i the value of x increases? **ii** the value of x decreases?

g Use your graph to find approximate values for each of the following. Answer correct to 1 decimal place.

i $\log_{10} 2.6$ **ii** $\log_{10} 4.4$ **iii** $\log_{10} 7.8$

h For what value of x would:

i $y = 2$? **ii** $y = 3$? **iii** $y = 4$?

Consolidation

2 a Copy and complete the following table of values for $y = 2^x$.

x	−3	−2	−1	0	1	2	3
y							

b Explain why $y = \log_2 x$ and $x = 2^y$ are equivalent statements.

c Copy and complete the following table of values for $y = \log_2 x$ (i.e. $x = 2^y$).

x							
y	−3	−2	−1	0	1	2	3

d Draw the graphs of $y = 2^x$ and $y = \log_2 x$ on the same number plane.

e Draw the line $y = x$ on this number plane. What do you notice about the graphs of $y = 2^x$ and $y = \log_2 x$?

3 a Sketch the graphs of $y = 2^x$ and $y = 3^x$ on the same number plane.

b Hence, sketch the graphs of $y = \log_2 x$ and $y = \log_3 x$.

c Which is larger, $\log_2 x$ or $\log_3 x$, when:

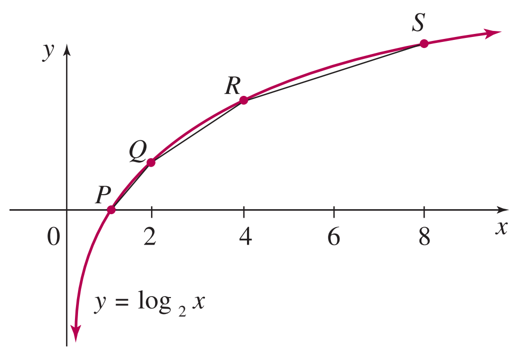
i $x > 1$? **ii** $0 < x < 1$?

4 The graph of $y = \log_2 x$ is shown.

a Find the co-ordinates of P, Q, R, S.

b Find the gradient of the intervals PQ, QR, RS.

c What can you say about the gradient of the curve as the value of x increases?



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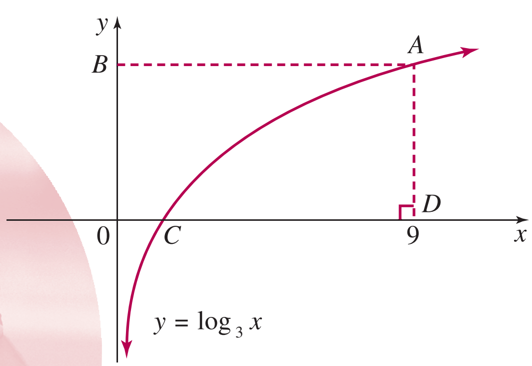


The graph of $y = \log_3 x$ is shown.

a Find the co-ordinates of A, B, C.

b Find the area of the region ACD given that the area of $ABOC$ is 7.3 units^2.

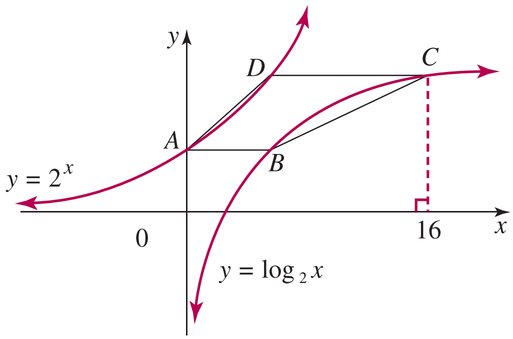
Further applications

6 The graphs of $y = 2^x$ and $y = \log_2 x$ are shown.

a Find the co-ordinates of A, B and hence find the length of the interval AB.

b Find the co-ordinates of C, D and hence find the length of the interval CD.

c Find the area of the region $ABCD$, (i.e. the region bounded by the intervals AB, BC, CD, DA).



TRY THIS Log challenges!

1 If $\log_b a = c$ and $\log_x b = c$, show that $\log_a x = \dfrac{1}{c^2}$.

2 Determine all the values of x for which $\log_5 (x - 6) + \log_5 (x - 2) - 1 = 0$.

3 The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are 2 points on the graph $y = \log_a x$. Through the midpoint of the line segment PQ, a horizontal line is drawn to cut the curve at $M(x_3, y_3)$. Prove that $x_3{}^2 = x_1x_2$.

Focus on working mathematically

How do you measure the magnitude of an earthquake?

The Indian Ocean earthquake of 2004

On 26 December 2004 there was a massive undersea earthquake below the Indian ocean, off the west coast of northern Sumatra in Indonesia. The earthquake, which measured 9.0 on the Richter scale, caused the sudden vertical rise of the seabed by about 15 m along 1200 km of the faultline. In turn this displaced massive volumes of water, generating fast moving waves which sped across the ocean. In deep waters such waves are small in height (amplitude) and are barely noticeable and harmless. But they move at high speed, usually from 500 to 1000 km/h. In shallow water near coastlines they slow down to only tens of km/h but form devastating heights. In Aceh for example they reached a height of 24 m as they came ashore. These waves, known as *tsunami* (pronounced "soo-nam-ee") waves, devastated the shores of many countries including Indonesia, Sri Lanka, South India and Thailand. They caused serious damage and deaths as far away as the east coast of Africa, and as far south as Port Elizabeth in South Africa, some 8000 km away. Anywhere between 228 000 and 310 000 people are thought to have died.

The problem

How do you measure the magnitude of an earthquake? The energy released is massive, but it also varies enormously between earthquakes. You would need a scale which handles large variation but something simple which people can readily understand.

In this chapter you have learned that a logarithm is simply an index. For example the logarithm of 1 000 000 to the base 10 is 6 because $1\,000\,000 = 10^6$. Similarly, 10 000 000 to the base 10 is 7 because $10\,000\,000 = 10^7$. On a base 10 logarithmic scale, an increase of 1 unit (from say 6 to 7) corresponds to a 10 fold increase in the size of the measurement itself. Here is a base 10 logarithmic scale from −1 to 8:

Scale value	−1	0	1	2	3	4	5	6	7	8
Actual value	10^{-1}	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8

The Richter scale

Dr Charles F. Richter was a seismologist, a scientist who studies and measures earthquakes, working at the California Institute of Technology. He recognised that the seismic waves generated by all earthquakes could provide good estimates of their magnitudes. The magnitude of an earthquake is a measure of the amount of energy released. The larger the total or intrinsic energy of an earthquake, the larger the amplitude of the ground vibrations at a given distance.

Richter recorded the seismic waves of many earthquakes and in 1935, in collaboration with his colleague Beno Gutenberg, developed a scale known as the *Richter scale* to describe the magnitude of an earthquake. This scale is a base 10 logarithmic scale, ranging from 0 to 10, although negative values are possible. There are also limitations to the scale which we will not discuss here. More modern techniques in use today have overcome these deficiencies. The important thing to note is that a unit increase in the scale value means a 10 fold increase in the amplitude of the seismic waves on the seismograph.

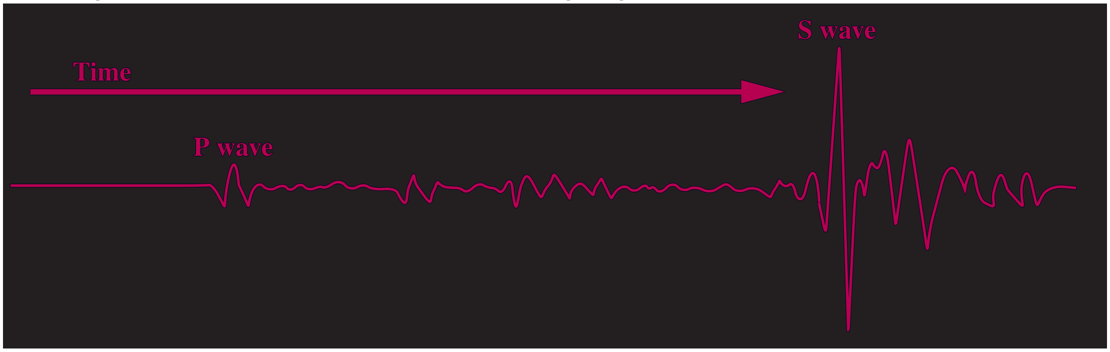
The Richter equation

Richter realised that the amplitude of the seismic waves he recorded would be affected by the distance of the seismograph from the epicentre of the earthquake. From his many observations, he developed a table to allow for this. If M is the magnitude of an earthquake on his scale, and A the amplitude in mm of the largest seismic shear waves recorded on his seismograph, then

$$M = \log_{10} A + (\text{distance correction factor})$$

To calculate the correction factor for a given earthquake, Richter measured the time difference for the arrival of two types of seismic waves to be recorded on his seismograph. These waves are known as primary (p) and secondary (s) waves.

The diagram below shows what such a recording might look like:

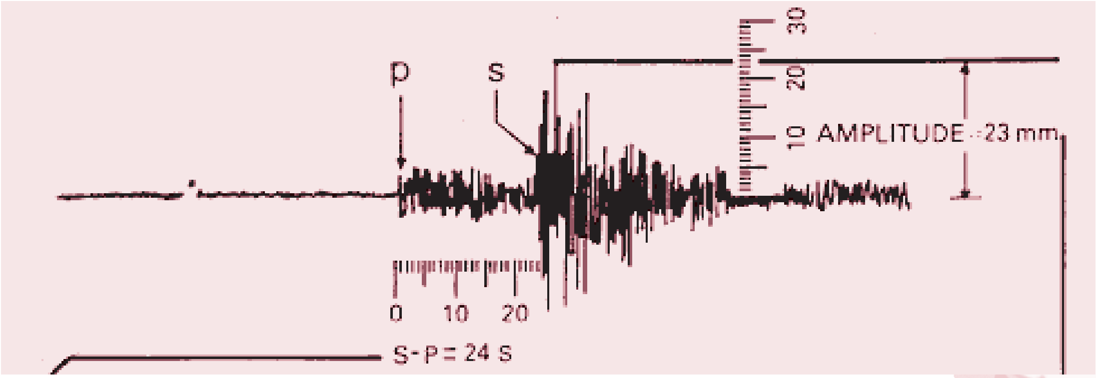


The difference in the time of arrival in seconds (Δt) of the faster p waves from the slower s waves enabled Richter to calculate the correction factor. For his particular seismograph and his data from earthquakes in California he obtained the equation

$$M = \log_{10} A + 3 \log_{10} (8\,\Delta t) - 2.92$$

Example

On the seismograph record below, the amplitude A of the largest s wave is 23 mm, and the time difference of the p and s waves Δt is 24 s.



For this earthquake, $M = \log_{10} 23 + 3 \log_{10} (8 \times 24) - 2.92$, giving a Richter scale magnitude of 5.3 (to the nearest tenth).

Learning activities

1 Using the example above, calculate the Richter scale magnitude of an earthquake whose amplitude A measures 30 mm, where the time difference of the p and s waves $\Delta t = 18$ s.

2 The time difference Δt of the p and s waves of an earthquake measuring 5.5 on the Richter scale is 29 seconds. Show that the amplitude A of the earthquake is 21mm.

3 Here is a table from the internet showing the range of the Richter scale, the general effects for earthquakes of that size and the number of times they are recorded around the world. Copy it into your workbook.

Magnitude	Earthquake Effects	Estimated No. /yr
2.5 or less	Usually not felt, but can be recorded by seismograph.	900 000
2.5 to 5.4	Often felt, but only causes minor damage.	30 000
5.5 to 6.0	Slight damage to buildings and other structures.	500
6.1 to 6.9	May cause a lot of damage in very populated areas.	100
7.0 to 7.9	Major earthquake. Serious damage.	20
8.0 to 8.9	Great earthquake. Can totally destroy communities nearby.	1 every 5 to 10 years
$\geqslant 9.0$	Rare/great earthquake—as above.	1 per 20 years

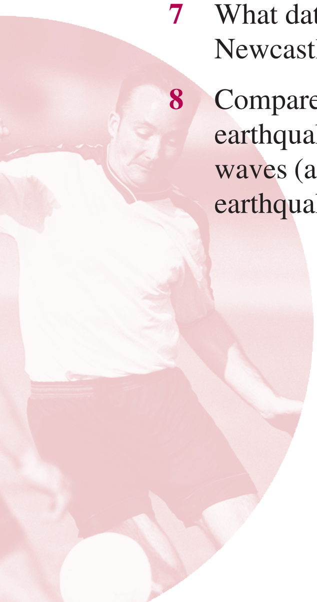
4 Into which category does the Indian Ocean earthquake of 2004 fall? When might you expect another earthquake of this magnitude to occur?

5 Approximately how many earthquakes occur each day but are not felt around the world? What does tell you about seismic activity within our planet?

6 On 28 December 1989 at 10.27 am the City of Newcastle in Australia was devastated by an earthquake measured as 5.6 on the Richter scale. Into which category would this earthquake fall in the above table? Find out the extent of the damage using the internet.

7 What data would you need to estimate when another earthquake of the magnitude of the Newcastle earthquake could occur in Australia?

8 Compare the Newcastle earthquake (5.6 on the Richter scale) with the Indian Ocean earthquake (9.0 on the Richter scale). How many times greater in amplitude are the seismic waves (as measured on a seismograph) of the Indian Ocean earthquake than the Newcastle earthquake?



Focus on working mathematically

Seismic energy

The amount of energy radiated from an earthquake as seismic waves (the waves recorded on the seismograph) is only a small fraction of the total or intrinsic energy transferred during an earthquake. Nevertheless, the estimation of this energy is an important part of understanding energy changes during an earthquake. Richter and Gutenberg developed a mathematical model to show the relationship between the *seismic energy* E_s of an earthquake (in ergs) and the *magnitude* M of the earthquake as measured on the Richter scale.

$$\log_{10} E_s = 11.8 + 1.5M \qquad (1)$$

1 Draw a rough sketch of the graph of $\log_{10} E_s$ against M. Take M as the independent variable with values from 0 to 10. What sort of function is $\log_{10} E_s$ with respect to M?

2 Now draw a rough sketch of E_s against M. What sort of function is E_s with respect to M? (Note that E_s is in ergs, a vary small unit of energy.)

3 Using equation (1), show that when M is increased by 1 unit, the seismic energy E_s is increased by a factor of 32 (to the nearest whole number).

4 The table below shows the *seismic energy* released by various earthquakes and other phenomena, as classified on the Richter scale. The units have been changed to approximate tonne equivalents of the explosive TNT so that you can more easily comprehend the massive energies involved. Recall that these are only *seismic* energies. Copy the table into your book (the prefix kilo = 1000, mega = 1 million or 10^6 and giga = 1 billion or 10^9).

Richter magnitude	Approximate seismic energy in TNT	Example
4.0	1 kilotonne	small atomic bomb
5.0	32 kilotonnes	Nagasaki atomic bomb
6.0	1 megatonne	Nevada earthquake 1994
7.0	32 megatonnes	largest thermonuclear weapon
8.0	1 gigatonne	San Francisco earthquake 1906
9.0	32 gigatonnes	Indian Ocean earthquake 2004

5 Confirm that a 1 unit increase in the Richter value represents a 32 fold increase in the seismic energy released. Note the enormous energy of the Indian ocean earthquake, 32 billion tonnes of TNT is a frightening figure.

Let's communicate

Make a chart to highlight the major features of the Richter scale.

Reflecting

Under what circumstances might it be a good idea to use a logarithmic scale for a measurement? Use the internet to explore other examples where logarithmic scales are used in scientific measurement, for example the pH scale for measuring acidity or the decibel scale for measuring sound intensity. You may need to consult your science teacher.

Language link with Macquarie

MACQUARIE

1 Give a brief explanation and an example of

a a ***function***

b a ***logarithm***

c the ***notation f(x)***

d ***inverse*** *functions*

2 The Macquarie Learners Dictionary *defines the word* ***inverse*** *as follows:*

inverse *adjective* **1.** turned in the opposite position or direction
–noun (uncount) **2.** the opposite: *The result was the inverse of what they had expected.*
❐ **Word Family: inversely** *adverb: inversely proportional*

How does the meaning differ from the specific mathematical meaning?

Chapter Review

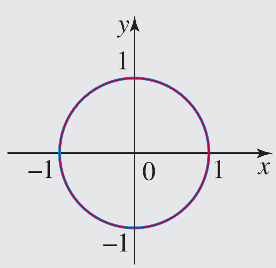
1 State whether each set of ordered pairs represents a function.

a (2, 3), (5, 4), (7, 11), (8, 1)

b (1, 5), (2, 7), (2, –3), (3, 4)

2 State whether each graph represents a function.

a



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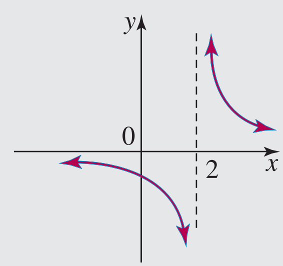

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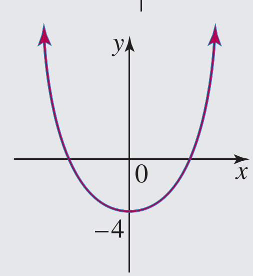

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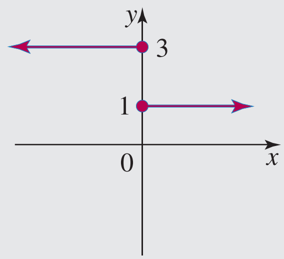

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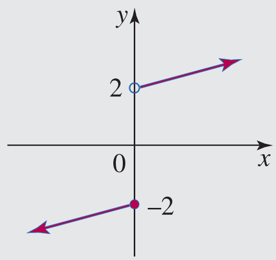

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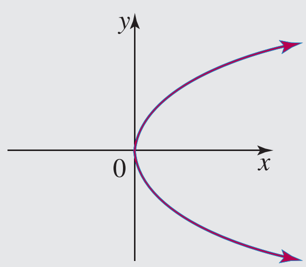


3 State the permissible x-values and y-values for each graph in question 2.

4 **a** If $f(x) = x^2 - 3x$, find:

i $f(4)$ **ii** $f(-5)$

iii $f(\sqrt{2})$ **iv** $f(1\frac{1}{2})$

b If $g(x) = 2^{x+3}$, find:

i $g(2)$ **ii** $g(-3)$

iii $g(0)$ **iv** $g(-2\frac{1}{2})$

5 **a** If $f(x) = x(x - 1)$, find x when $f(x) = 12$.

b If $h(x) = 1 - 3x$, solve $h(x) > 7$.

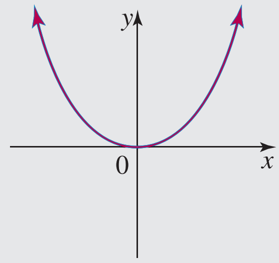
c If $g(x) = \log_2 x$, find x when $g(x) = 6$.

6 If $f(x) = x^2$, find in simplest form

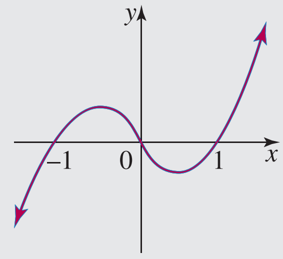
$$\frac{f(x) - f(5)}{x - 5} \quad (x \neq 5).$$

7 State whether each of the following has an inverse that is a function.

a



b



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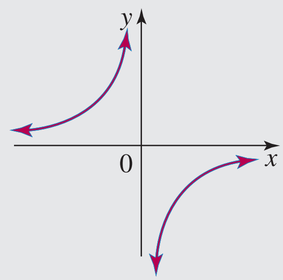

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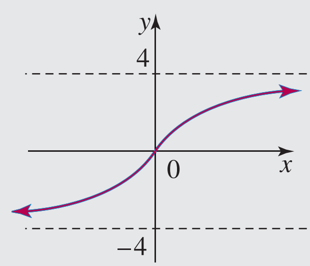


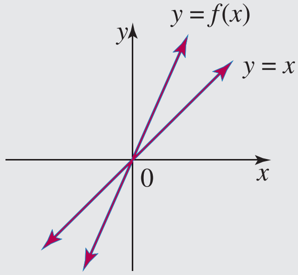
8 Find the inverse function for each of the following.

a $y = 3x$ **b** $y = x - 7$

c $y = \frac{x}{5} - 2$ **d** $y = \frac{3}{x - 1}$

9 Sketch the graph of $y = f^{-1}(x)$ for each of the following.

a



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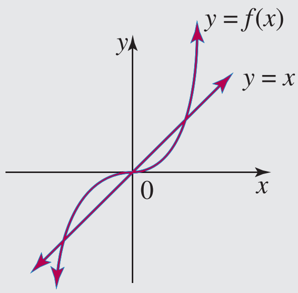


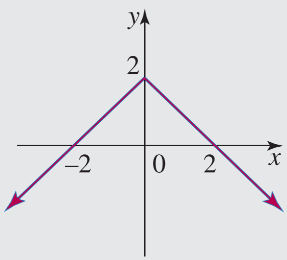
10 Given the graph of $y = f(x)$, sketch:

a $y = f(x - 1)$

b $y = f(x + 1)$

c $y = f(x) + 1$

d $y = f(x) - 1$



11 Solve:

a $2^x = 32$ **b** $25^x = 5$

c $4^x = \frac{1}{4}$ **d** $2^x = \frac{1}{8}$

e $4^x = 8$ **f** $9^{x-2} = 27$

g $2^{3-x} = \sqrt{8}$ **h** $25^{3x} = 125^{x-1}$

12 a Write $3^4 = 81$ in logarithmic form.

b Write $\log_{10} 100 = 2$ in index form.

13 Solve:

a $\log_3 27 = x$ **b** $\log_{16} 8 = x$

c $\log_5 (\frac{1}{25}) = x$ **d** $\log_4 \sqrt{2} = x$

14 Solve:

a $\log_x 49 = 2$ **b** $\log_x \frac{1}{3} = -1$

c $\log_x 6 = 1$ **d** $\log_x 2 = \frac{1}{3}$

15 Evaluate:

a $\log_6 36$ **b** $\log_5 1$

c $\log_7 \frac{1}{49}$ **d** $\log_4 32$

16 Evaluate each of the following by using the logarithm laws.

a $\log_{10} 20 + \log_{10} 5$

b $\log_3 18 - \log_3 6$

c $\log_5 250 - \log_5 2$

d $\log_6 48 - 3 \log_6 2$

e $\log_2 5 + \log_2 16 - \log_2 10$

17 If $\log_a 3 = 0.477$ and $\log_a 4 = 0.602$, find:

a $\log_a 12$ **b** $\log_a 0.75$

c $\log_a 16$ **d** $\log_a 36$

e $\log_a (\frac{1}{3})$ **f** $\log_a 2$

g $\log_a 4a$ **h** $\log_a 3a^2$

18 Simplify:

a $5 \log_a a + 2 \log_a a^3$

b $\dfrac{\log_a x^6}{\log_a x^2}$

c $\log_a \left(\dfrac{m}{n}\right) + \log_a \left(\dfrac{n}{m}\right)$

19 Solve for x:

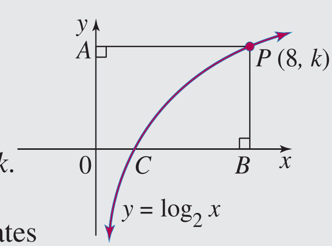
a $\log_a x = 2 \log_a 5 + \log_a 3$

b $\log_a 3 + \log_a (x - 1) = \log_a 21$

c $\log_a (2x - 3) - \log_a 3 = \log_a 11$

d $\log_a x + \log_a (x - 2) = \log_a 8$

20 The graph of $y = \log_2 x$ is shown.



a Find the value of k.

b Find the co-ordinates of A, B, C.

c Find the length of AC, in surd form.

EXTENSION

21 Solve for x, correct to 2 decimal places.

a $5^x = 22$ **b** $3^{x-1} = 45$ **c** $8^x = 3^{x+1}$

13 Curve sketching and polynomials

This chapter at a glance

Stage 5.1/5.2/5.3

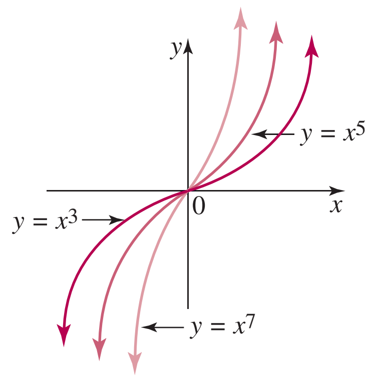
After completing this chapter, you should be able to:

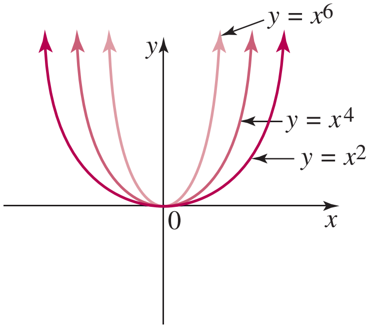
- sketch graphs with equations of the form $y = ax^n$ $(n > 0)$ and explain the effect of n being odd or even on the shape of the curve
- sketch graphs with equations of the form $y = ax^n + k$ and $y = a(x - r)^n$ by using vertical and horizontal translations
- sketch circles with equations of the form $(x - h)^2 + (y - k)^2 = r^2$
- find the centre and radius of a circle whose equation is of the form $x^2 + gx + y^2 + hy + c = 0$ by completing the squares for x and y
- determine both graphically and algebraically the points of intersection of a variety of graphs
- determine whether an expression is a polynomial
- identify the leading term, leading co-efficient, degree and constant term in a given polynomial
- determine whether a polynomial is monic
- use the polynomial notation $P(x)$ and evaluate $P(x)$ at $x = a$
- add, subtract and multiply polynomials
- divide polynomials by linear expressions and hence express the polynomial in the form: dividend = divisor × quotient + remainder
- use the remainder theorem to find the remainder when a polynomial is divided by a linear divisor
- solve algebraic problems using the remainder theorem
- use the factor theorem to determine whether a linear expression is a factor of a polynomial
- use the factor theorem to factorise a polynomial
- use the factor theorem to solve a polynomial equation
- solve algebraic problems using the factor theorem
- sketch the graphs of polynomial functions up to degree four, including those with double and triple roots
- use the graph of $y = P(x)$ to sketch the graph of $y = -P(x)$, $y = P(-x)$, $y = P(x) + c$, $y = P(x - a)$ and $y = aP(x)$

13.1 Sketching $y = ax^n + k$ and $y = a(x - r)^n$

The graph of $y = ax^n + k$

Curves with equations of the form $y = ax^n$, where n is odd and $n \geqslant 5$, look like the cubic curve $y = x^3$. As the values of a and n increase, the curve becomes steeper.





Curves with equations of the form $y = ax^n$, where n is even and $n \geqslant 4$, look like the parabola $y = x^2$. As the values of a and n increase, the curve becomes steeper.

The graph of $y = ax^n + k$ $(k > 0)$ is the graph of $y = ax^n$ shifted up by k units. Similarly, the graph of $y = ax^n - k$ $(k > 0)$ is the graph of $y = ax^n$ shifted down by k units.

The graph of $y = a(x - r)^n$

The graph of $y = a(x - r)^n$ $(r > 0)$ is the graph of $y = ax^n$ shifted to the right by r units. Similarly, the graph of $y = a(x + r)^n$ $(r > 0)$ is the graph of $y = ax^n$ shifted to the left by r units.

Example 1

EG +S

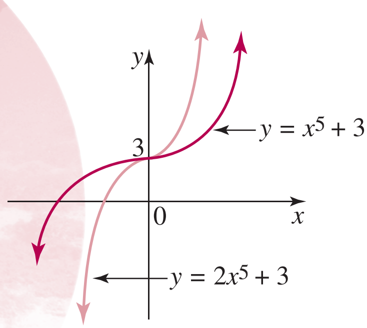
Sketch the graphs below on the same number plane.

a $y = x^5 + 3$ and $y = 2x^5 + 3$

b $y = x^4 - 2$ and $y = \frac{1}{2}x^4 - 2$

Solutions

a



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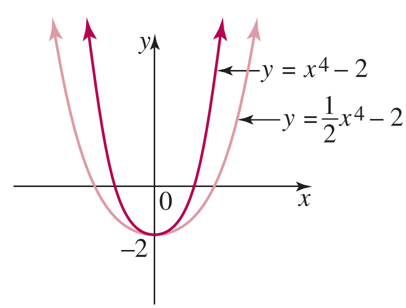


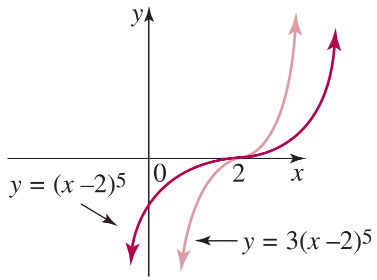
Example 2

Sketch the graphs below on the same number plane.

a $y = (x - 2)^5$ and $y = 3(x - 2)^5$ **b** $y = (x + 3)^6$ and $y = \frac{1}{2}(x + 3)^6$

Solutions

a



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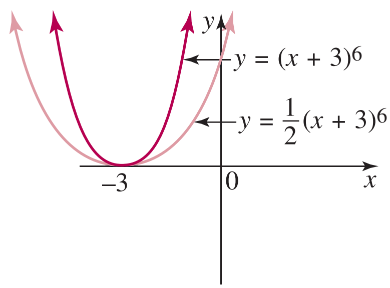


Exercise 13.1

1 Sketch the following graphs on the same number plane.
- **a** $y = x^2, y = x^4, y = x^6$
- **b** $y = x^3, y = x^5, y = x^7$
- **c** $y = -x^2, y = -x^4, y = -x^6$
- **d** $y = -x^3, y = -x^5, y = -x^7$

2 Sketch these graphs on the same number plane.
- **a** $y = x^2, y = 2x^2, y = \frac{1}{2}x^2$
- **b** $y = x^3, y = \frac{1}{2}x^3, y = 3x^3$
- **c** $y = x^4, y = \frac{3}{4}x^4, y = 2x^4$
- **d** $y = x^5, y = 3x^5, y = \frac{2}{3}x^5$

3 Sketch the following graphs on separate number planes.
- **a** $y = x^2 + 1$
- **b** $y = x^2 - 1$
- **c** $y = (x - 1)^2$
- **d** $y = (x + 1)^2$
- **e** $y = x^3 + 2$
- **f** $y = x^3 - 2$
- **g** $y = (x - 2)^3$
- **h** $y = (x + 2)^3$

■ Consolidation

4 Sketch these curves on the same number plane.
- **a** $y = x^4$ and $y = x^4 + 2$
- **b** $y = x^5$ and $y = x^5 - 1$
- **c** $y = 2x^6$ and $y = 2x^6 - 3$
- **d** $y = \frac{1}{2}x^7$ and $y = \frac{1}{2}x^7 + 4$
- **e** $y = -x^6$ and $y = -x^6 + 2$
- **f** $y = -x^7$ and $y = -x^7 - 5$
- **g** $y = -\frac{1}{3}x^4$ and $y = -\frac{1}{3}x^4 - 6$
- **h** $y = -3x^5$ and $y = -3x^5 + 1$

5 Sketch on separate number planes:
- **a** $y = (x - 1)^4$
- **b** $y = (x + 2)^5$
- **c** $y = (x + 1)^6$
- **d** $y = (x - 2)^7$
- **e** $y = -(x - 1)^5$
- **f** $y = -(x - 2)^4$
- **g** $y = -(x + 1)^7$
- **h** $y = -(x + 2)^6$

6 Find the x- and y-intercepts for each curve.
- **a** $y = 2(x + 2)^4$
- **b** $y = 3(x - 2)^5$
- **c** $y = -3(x - 1)^6$
- **d** $y = -2(x + 1)^7$

Further applications

7 Sketch the following curves.

a $y = 3(x+1)^2 + 3$ **b** $y = 2(x-1)^3 + 5$ **c** $y = 5(x-1)^4 - 2$

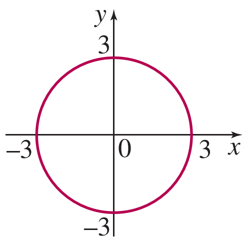
d $y = 4(x+1)^5 - 3$ **e** $y = -3(x+1)^6 + 4$ **f** $y = -2(x-1)^7 - 6$

13.2 The circle

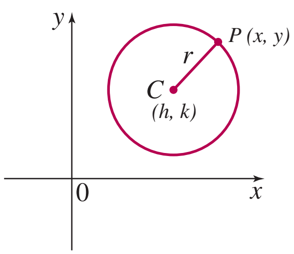
A circle is defined as the set of all points that are equidistant from a fixed point (the centre).

The equation of the circle with centre (0, 0) and radius r units is $x^2 + y^2 = r^2$. This equation was derived in chapter 7 by using Pythagoras' theorem.

For example, a circle with centre (0, 0) and radius 3 units has equation $x^2 + y^2 = 9$.



$x^2 + y^2 = 9$



The equation of a general circle with centre (h, k) and radius r units can be derived by the use of the distance formula.

Let $(x_1, y_1) = (h, k)$ and $(x_2, y_2) = (x, y)$

$$CP = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

$\therefore (x-h)^2 + (y-k)^2 = r^2$ (squaring both sides)

The equation of a circle with centre (h, k) and radius r units is:

$$(x-h)^2 + (y-k)^2 = r^2$$

Example 1

Find the centre and radius of each circle.

a $(x-2)^2 + (y+3)^2 = 49$ **b** $x^2 + (y-1)^2 = 5$ **c** $(x+8)^2 + y^2 = 12$

Solutions

a $(x-2)^2 + (y+3)^2 = 49$

$(x-2)^2 + (y - -3)^2 = 7^2$

$\therefore$ the centre is $(2, -3)$ and the radius is 7 units.

b $x^2 + (y-1)^2 = 5$

$x^2 + (y-1)^2 = (\sqrt{5})^2$

$\therefore$ the centre is (0, 1) and the radius is $\sqrt{5}$ units.

c $(x+8)^2 + y^2 = 12$

$(x - -8)^2 + y^2 = (2\sqrt{3})^2$

$\therefore$ the centre is $(-8, 0)$ and the radius is $2\sqrt{3}$ units.

Example 2

Find the centre and radius of the circle $x^2 + y^2 + 6x - 8y - 11 = 0$.

Solution

$x^2 + y^2 + 6x - 8y - 11 = 0$
$x^2 + 6x + y^2 - 8y = 11$
Complete the squares for x and y.
$x^2 + 6x + 9 + y^2 - 8y + 16 = 11 + 9 + 16$
$(x + 3)^2 + (y - 4)^2 = 36$
$\therefore$ the centre is $(-3, 4)$ and the radius is 6 units.

Exercise 13.2

1 Write down the equation of the circle with centre the origin and radius:
a 3 units **b** 7 units **c** $\sqrt{2}$ units **d** $2\sqrt{5}$ units

2 State the length of the radius in each circle.
a $x^2 + y^2 = 25$ **b** $x^2 + y^2 = 64$ **c** $x^2 + y^2 = 144$
d $x^2 + y^2 = 3$ **e** $x^2 + y^2 = 18$ **f** $x^2 + y^2 = 32$

3 The equation of a circle is $4x^2 + 4y^2 = 49$. Express the equation in the form $x^2 + y^2 = r^2$. Hence, find the centre and radius.

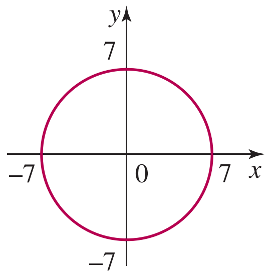
4 Find the centre and radius in each circle.
a $(x - 1)^2 + (y - 5)^2 = 9$ **b** $(x + 6)^2 + (y + 2)^2 = 4$
c $(x + 2)^2 + (y - 3)^2 = 25$ **d** $(x - 4)^2 + (y + 5)^2 = 1$
e $x^2 + (y - 7)^2 = 36$ **f** $(x + 3)^2 + y^2 = 100$
g $(x + 4)^2 + (y + 6)^2 = 5$ **h** $(x - 2)^2 + (y + 1)^2 = 8$
i $x^2 + (y - 4)^2 = 40$ **j** $(x - 10)^2 + y^2 = 45$

5 Find the equation of the circle with centre C and radius r units.
a $C(1, 3)$, $r = 5$ units **b** $C(-2, 7)$, $r = 6$ units
c $C(-4, -1)$, $r = 3$ units **d** $C(6, -3)$, $r = 9$ units
e $C(2, 1)$, $r = \sqrt{7}$ units **f** $C(-3, -2)$, $r = \sqrt{15}$ units
g $C(7, -5)$, $r = 3\sqrt{2}$ units **h** $C(-8, 11)$, $r = 2\sqrt{7}$ units

Consolidation

6 Find the equation of each circle.

a



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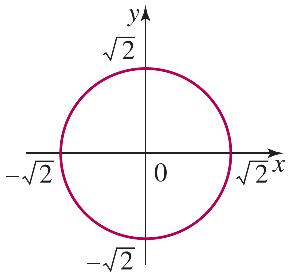

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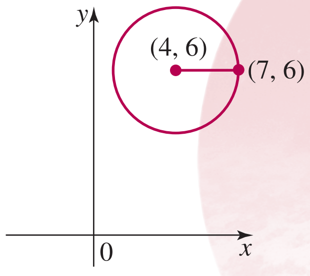

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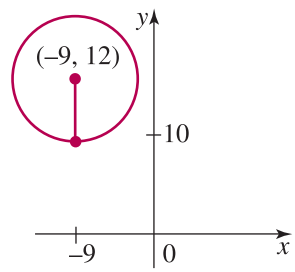

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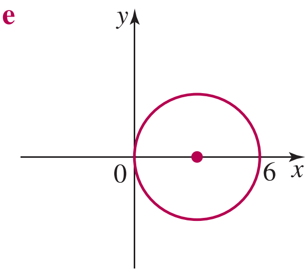

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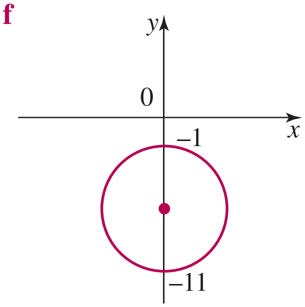


7 Sketch the following circles showing the centre and radius.

a $(x-2)^2+(y-2)^2=4$ **b** $(x+2)^2+y^2=1$

c $x^2+(y-4)^2=25$ **d** $(x+1)^2+(y-2)^2=9$

8 Sketch these circles showing the centre, radius and x, y intercepts.

a $(x+3)^2+(y-4)^2=25$ **b** $(x+1)^2+(y-2)^2=9$

9 State whether each of the following equations describes a circle.

a $x^2+y^2=10$ **b** $x^2-y^2=4$ **c** $x^2+y^2=-25$

d $x^2+y^2=0$ **e** $\dfrac{x^2}{2}+\dfrac{y^2}{2}=3$ **f** $4x^2+4y^2=9$

g $\dfrac{x^2}{9}+\dfrac{y^2}{3}=1$ **h** $x^2+9y^2=16$ **i** $(x+y)^2=36$

j $(x-3)^2+(y+1)^2=4$ **k** $x^2-(y+2)^2=9$ **l** $(x+3)^2=121-(y-2)^2$

10 Express each equation in the form $(x-h)^2+(y-k)^2=r^2$ by completing the square for x and y. Hence find the centre and radius for each circle.

a $x^2+y^2-2x-4y-20=0$ **b** $x^2+y^2+12x+10y+52=0$

c $x^2+y^2-18x+14y+66=0$ **d** $x^2+y^2+4x-18y-36=0$

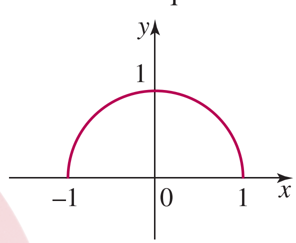
e $x^2+y^2+6y-40=0$ **f** $x^2+y^2-14x-15=0$

g $x^2+y^2+6x-10y+7=0$ **h** $x^2+y^2-20x+4y+60=0$

■ Further applications

11 Write down the equation of each semi-circle.

a



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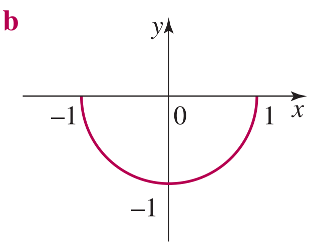

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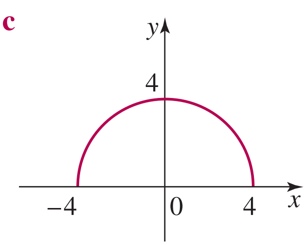

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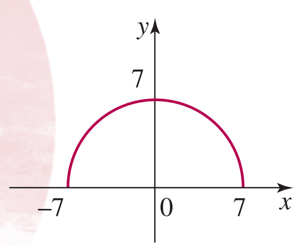

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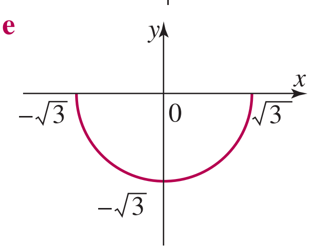

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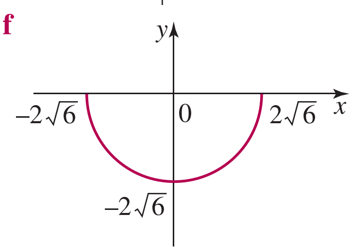


12 Find the length of the radius for each semi-circle.

a $y = \sqrt{9 - x^2}$ **b** $y = -\sqrt{4 - x^2}$ **c** $y = \sqrt{25 - x^2}$ **d** $y = -\sqrt{36 - x^2}$

e $y = -\sqrt{100 - x^2}$ **f** $y = \sqrt{6 - x^2}$ **g** $y = -\sqrt{40 - x^2}$ **h** $y = \sqrt{63 - x^2}$

TRY THIS **Two circle problems**

1 Find the equation of the circle that passes through the points (1, 4), (5, 10) and (7, 6).

2 Find a relationship between a, b and c if the equation $x^2 + y^2 + ax + by + c = 0$ always represents a circle.

13.3 The intersection of graphs

If two straight lines are not parallel, then they must intersect in exactly one point. A line and a curve, however, can intersect in several points, or not at all. Points of intersection of graphs can be found graphically by drawing accurate graphs and reading off the points of intersection, or algebraically by solving a pair of simultaneous equations.

EG +S

Example 1

Graph the parabola $y = x^2$ and the line $y = x + 2$ on the same number plane. Hence find their points of intersection.

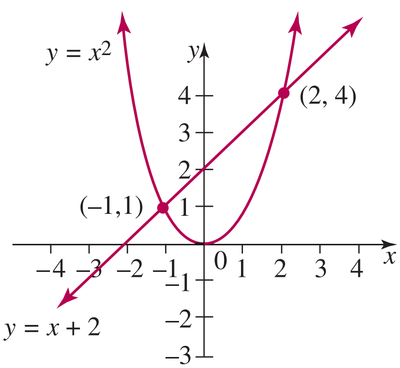
Solution

$y = x^2$

x	−2	−1	0	1	2
y	4	1	0	1	4

$y = x + 2$

x	−2	−1	0	1	2
y	0	1	2	3	4



The points of intersection are (−1, 1) and (2, 4).

EG +S

Example 2

Find, algebraically, the points of intersection of the hyperbola $xy = 6$ and $y = 5 - x$ and then sketch their graphs.

Solution

i $xy = 6 \ldots. ①$

$y = 5 - x \ldots. ②$

Substitute ② into ①

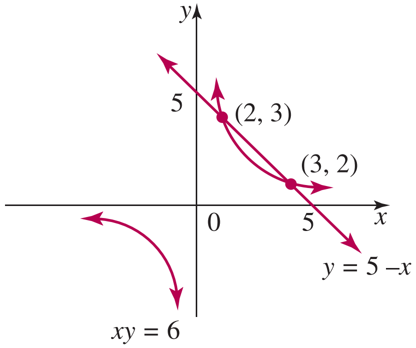
$$x(5 - x) = 6$$
$$5x - x^2 = 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$\therefore x = 2, 3$$

ii when $x = 2$, $y = 5 - 2$
$= 3$

when $x = 3$, $y = 5 - 3$
$= 2$

$\therefore$ the points of intersection are (2, 3) and (3, 2).

iii



Exercise 13.3

1 Graph each pair of equations accurately on the same number plane. Hence find all points of intersection of these graphs.

a $y = x^2$ and $y = 4$

b $y = x^2$ and $y = x + 2$

c $y = x^2 - 2x$ and $y = 2x - 3$

d $y = \dfrac{4}{x}$ and $y = x$

e $xy = 6$ and $y = x + 1$

f $y = x^3$ and $y = x$

g $x^2 + y^2 = 25$ and $y = 4$

h $x^2 + y^2 = 16$ and $y = x - 4$

Consolidation

2 Find, algebraically, the points of intersection of the given line and parabola.

a $y = x^2$ and $y = 25$

b $y = x^2$ and $y = 3x$

c $y = x^2 + 2x$ and $y = 3$

d $y = x(x - 1)$ and $y = 7x - 15$

3 Find, algebraically, the points of intersection of the given line and hyperbola.

a $y = \dfrac{6}{x}$ and $y = 2$

b $xy = 8$ and $y = 2x$

c $xy = 12$ and $y = \dfrac{x}{3}$

d $y = \dfrac{3}{x}$ and $y = 4 - x$

4 Find, algebraically, the points of intersection of the given line and circle.

a $x^2 + y^2 = 34$ and $y = 3$

b $x^2 + y^2 = 50$ and $y = x$

c $x^2 + y^2 = 20$ and $y = -2x$

d $x^2 + y^2 = 40$ and $y = x - 4$

Further applications

5 Find, algebraically, all points of intersection for each pair of curves. Sketch the curves showing any intercepts with the co-ordinate axes and the points of intersection.

a $y = x^2$ and $y = \dfrac{8}{x}$ **b** $y = x^3$ and $xy = 16$ **c** $y = (x - 2)^2$ and $y = 10 - x^2$

6 Find the points of intersection of the curves $y = x^3$ and $y = x^2 + x - 1$.

7 Find, algebraically, the points of intersection of:

a the circle $x^2 + y^2 = 20$ and the parabola $y = x^2$.

b the circle $x^2 + y^2 = 5$ and the hyperbola $y = \dfrac{2}{x}$.

13.4 Definition of a polynomial

A polynomial is an expression of the form
$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$
where $a_n, a_{n-1}, a_{n-2}, \ldots a_1, a_0$ are the co-efficients of the powers of x.

NOTE: In a polynomial, the powers of x must be positive integers or zero.

For example, the expression $3x^4 + 5x^3 - 7x^2 - 2x + 1$ is a polynomial. However, $2x^2 - 5x^{\frac{1}{2}} + x^{-1}$ is not a polynomial, because two of the indices are not integers ($\frac{1}{2}$ and -1).

In any polynomial $P(x)$:
- **the leading term is the term with the highest index**
- **the leading co-efficient is the co-efficient of the leading term**
- **the degree is the highest power of the variable**
- **the constant term is the term which is independent of x.**

A **monic polynomial** is a polynomial in which the leading co-efficient is 1. For example, $x^3 - 3x + 1$ is monic because the co-efficient of the leading term x^3 is 1. However, $2x^3 - 3x + 1$ is not monic because the co-efficient of x^3 is 2, not 1.

The **zero polynomial** is a special polynomial where all of the co-efficients are zero. That is, $a_n = a_{n-1} = a_{n-2} = \ldots = a_1 = a_0 = 0$. Therefore, $P(x) = 0$ is the zero polynomial.

Polynomial notation

Polynomials are functions, since for each value of x, there is a unique value of y in the equation $y = P(x)$. The notation $y = P(x)$ refers to a polynomial function and thus it may be treated in the same way as $y = f(x)$.

For example, if $P(x) = x^3 + x^2 - 4x - 3$,

$$\begin{aligned} P(2) &= 2^3 + 2^2 - 4(2) - 3 \\ &= 8 + 4 - 8 - 3 \\ &= 1 \end{aligned}$$

Example 1

State the degree, leading term, leading co-efficient and constant term for each polynomial.

a $P(x) = 2x^6 + 3x^5 - 6x + 9$ **b** $P(x) = 3 - 4x^2 - x^4$ **c** $P(x) = x^3 + 11x^2 - 8x$

Solutions

a The degree is 6, the leading term is $2x^6$, the leading co-efficient is 2 and the constant term is 9.

b The degree is 4, the leading term is $-x^4$, the leading co-efficient is -1 and the constant term is 3.

c The degree is 3, the leading term is x^3, the leading co-efficient is 1 and the constant term is 0.

Example 2

Which of the following expressions are polynomials?

a $3x + \frac{1}{x} - 7$ **b** $6x^5 + 5\sqrt{x}$ **c** 3 **d** $\frac{3}{4}x^4 - \pi x^3 + 2\sqrt{3}$

Solutions

a $3x + \frac{1}{x} - 7$ is not a polynomial since $\frac{1}{x} = x^{-1}$, and hence all of the powers are not positive integers or zero.

b $6x^5 + 5\sqrt{x}$ is not a polynomial since $5\sqrt{x} = 5x^{\frac{1}{2}}$, and hence all of the powers are not positive integers or zero.

c 3 is a polynomial since $3 = 3x^0$, and hence has degree zero.

d $\frac{3}{4}x^4 - \pi x^3 + 2\sqrt{3}$ is a polynomial since all powers are positive integers or zero and all co-efficients are real numbers. (Note: the co-efficients need only be real, they do not have to be rational.)

Exercise 13.4

1 State whether each of the following expressions is a polynomial.

a $x^2 + 5x$ **b** $3x^4 - 12x^2 + 7$ **c** $x^3 + \frac{1}{x^3}$

d $\frac{1}{x^2 - 5x - 3}$ **e** 3 **f** $1 + \sqrt{x}$

g $\sqrt{2}x^4 - \sqrt{3}x^3$ **h** $(x + 1)^2$ **i** $(\sqrt{x} + 1)^2$

j $x^3 - x^2 + \sqrt{x^3}$ **k** $(2x + 3)(x - 4)^3$ **l** $x^2 + 2^x$

m $\frac{3}{4}x^3 + 2x + 11$ **n** $4x^{\frac{3}{2}} - 9$ **o** $5x^{-1} - x^4$

p $\frac{x^2 - 3x}{4}$ **q** $\frac{4}{x^2 - 3x}$ **r** $\frac{x + 4}{x - 2}$

s 0 **t** $\frac{2}{3}x^5 - 2\sqrt{2}x$ **u** $x\sqrt{x} + 7$

2 Which of the following polynomials are monic?

A $3x^2 + 4x + 1$ **B** $x^2 - 7$ **C** $x^4 - 2x^3 + 6x^2$
D $1 + 2x - 5x^3$ **E** $4 + x + x^5$ **F** $10 - 3x - x^2$
G $x^3(3x^4 - 2)$ **H** $(x + 2)^5$ **I** $(5 - x)(2 - x^2)$

3 For each of the polynomials below, state:

i the degree **ii** the leading term **iii** the leading co-efficient **iv** the constant term.

a $x^3 + 7x + 4$ **b** $2x^4 + x^3 - 5x^2 - 1$ **c** $5x^6 + 12$
d $3x^5$ **e** $7 - 4x$ **f** $11 - 2x + x^4$
g $10x - 3x^2$ **h** 4 **i** $\sqrt{2}x^3 - 9x^2 - 10$
j $\frac{1}{2}x^4 + 6x - 8$ **k** $\frac{x^3}{2} + \frac{x^2}{4} - 5x$ **l** $\frac{5x^6}{3} - \frac{2x^2}{7} + 1$

■ Consolidation

4 Find the following values for the given polynomials.

a $P(x) = x^2 + 3x - 7$ **i** $P(1)$ **ii** $P(3)$ **iii** $P(-2)$
b $P(x) = x^3 + 5x^2 + 2$ **i** $P(2)$ **ii** $P(-4)$ **iii** $P(0)$
c $P(x) = 2x^3 + 3x$ **i** $P(5)$ **ii** $P(-3)$ **iii** $P\left(\frac{1}{2}\right)$
d $P(x) = 10 - 6x - 3x^2$ **i** $P(0)$ **ii** $P(-1)$ **iii** $P\left(\frac{2}{3}\right)$

5 **a** If $P(x) = x^2 - 9$, solve $P(x) = 0$. **b** If $P(x) = x^2 - 2x - 3$, solve $P(x) = 0$.
c If $P(x) = x^2 + 2x$, solve $P(x) = 15$. **d** If $P(x) = x^3 - x$, solve $P(x) = 0$.
e If $P(x) = 3x^3 - 12x + 1$, solve $P(x) = 1$.

6 Without expanding the polynomials below, state:

i the degree **ii** the leading term **iii** the leading co-efficient **iv** the constant term.

a $(3x - 4)^2$ **b** $5x(x + 3)^2$
c $x^2(2x - 7)(2x + 7)$ **d** $(x - 2)^3$
e $(2x^3 - 4x - 1)^4$ **f** $(x + 2)(x - 3)(x - 4)$
g $(x^3 - 5)(2x^2 + 3x - 2)$ **h** $(x + 7)(x + 3) + (x + 4)^2$
i $(x + 6)^2 - (x - 4)^2$ **j** $(2x^2 + 5)^2 + (3x^2 - 2)^2$

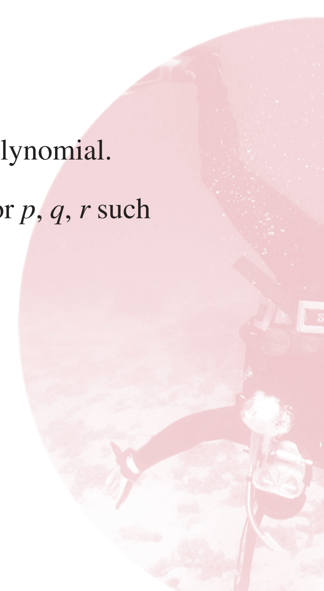
■ Further applications

7 State the values of k for which the expression $x^{k-3} + 5x^2 + 4$ would be:

a a polynomial of degree 4 **b** a polynomial of degree 2 **c** a monic polynomial.

8 Consider the expression $(p - 1)x^7 + (q + 4)x^3 + (2r - 10)$. Determine values for p, q, r such that the expression is:

a a polynomial of degree 7 **b** a polynomial of degree 3
c a non-zero constant polynomial **d** the zero polynomial.



13.5 Addition, subtraction and multiplication of polynomials

Addition and subtraction of polynomials

To add or subtract polynomials:
- **add or subtract the like terms (the terms with the same powers).**

When adding or subtracting polynomials vertically, keep the like terms underneath each other. Leave a space for any missing powers of the pronumeral.

The sum or difference of two polynomials is also a polynomial.

Multiplication of polynomials

To multiply two polynomials:
- **multiply each term in the first polynomial by each term in the second polynomial**
- **collect any like terms.**

The product of two polynomials is also a polynomial.

If $P(x)$ is a polynomial of degree m and $Q(x)$ is a polynomial of degree n, then the degree of $P(x) \, . \, Q(x)$ is $m + n$.

Example 1

If $P(x) = 2x^3 + 2x^2 - 7x - 3$ and $Q(x) = x^3 - 3x + 11$, find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$

Solutions

a
$$\begin{array}{rrrrl} 2x^3 & + 2x^2 & - 7x & - 3 & \\ x^3 & & - 3x & + 11 & + \\ \hline 3x^3 & + 2x^2 & - 10x & + 8 & \end{array}$$

b
$$\begin{array}{rrrrl} 2x^3 & + 2x^2 & - 7x & - 3 & \\ x^3 & & - 3x & + 11 & - \\ \hline x^3 & + 2x^2 & - 4x & - 14 & \end{array}$$

Example 2

If $P(x) = 3x - 2$ and $Q(x) = x^3 + 4x^2 - 5$, find $P(x) \, . \, Q(x)$.

Solution

$$\begin{aligned} P(x) \, . \, Q(x) &= (3x - 2)(x^3 + 4x^2 - 5) \\ &= 3x(x^3 + 4x^2 - 5) - 2(x^3 + 4x^2 - 5) \\ &= 3x^4 + 12x^3 - 15x - 2x^3 - 8x^2 + 10 \\ &= 3x^4 + 10x^3 - 8x^2 - 15x + 10 \end{aligned}$$

Exercise 13.5

1 If $P(x) = x^3 + 5x^2 + 2x + 9$, $Q(x) = 2x^2 - 3x + 7$ and $R(x) = x^3 + 3x - 6$, find:

a $P(x) + Q(x)$ **b** $Q(x) + R(x)$ **c** $R(x) + P(x)$

d $P(x) - Q(x)$ **e** $R(x) - Q(x)$ **f** $P(x) - R(x)$

g $P(x) + Q(x) + R(x)$ **h** $R(x) - P(x) + Q(x)$ **i** $Q(x) - P(x) - R(x)$

2 Simplify:

a $(x^3 + 4x^2 - 2x + 10) + (2x^2 + 5x - 3)$ **b** $(x^3 + 6x - 11) + (3x^3 - 5x^2 - 2x + 1)$

c $(5x^3 - x^2 + x - 6) + (2x^3 + x^2 - x - 4)$ **d** $(x^3 + 7x^2 + 12x - 8) - (x^2 - 3x + 5)$

e $(2x^3 - 9x^2 - 15) - (x^3 + 4x^2 - 7x - 6)$ **f** $(x^4 - 2x^3 - 7x^2 + 11x + 10) - (3x^3 - 5x - 4)$

g $(7x^4 + 6x^2 - 13) - (6x^4 + 4x^3 - 2x^2 + 5)$ **h** $(x^3 + 8x + 3) - (5 - 2x + 6x^2 - x^3)$

i $(3x^5 + 4x^3 - 9x) + (1 + 7x + 4x^2 - 2x^3 - 3x^5)$

3 Expand and simplify the following products.

a $(x + 1)(x^2 + 4x + 2)$ **b** $(x - 2)(x^2 - 3x - 4)$

c $(3 - x)(x^2 - 7x + 3)$ **d** $(3x + 4)(x^2 + x - 2)$

e $(2x + 5)(3x^2 - 4x - 2)$ **f** $(4x - 3)(2x^3 + 8x^2 - 1)$

■ Consolidation

4 $P(x)$ is a polynomial of degree 5 and $Q(x)$ is a polynomial of degree 3. State the degree of:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $P(x) \,.\, Q(x)$

d $[P(x)]^2$ **e** $2 \,.\, P(x)$ **f** $(2x^4 + 7) \,.\, Q(x)$

5 $P(x) = 2x^5 + 6x^4 - x^3 + 10$ and $Q(x) = 2x^5 - 3x^4$.

a State the degree of $P(x) - Q(x)$. **b** Find the constant term in $P(x) \,.\, Q(x)$.

6 $P(x)$ is a polynomial of degree m and $Q(x)$ is a polynomial of degree n, where $m > n$. Find the degree of each of the following.

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $P(x) \,.\, Q(x)$

7 $P(x) = 2x^2 + 5$, $Q(x) = x^3 - 7x + 4$ and $R(x) = P(x) \,.\, Q(x)$.

a Find $R(x)$. **b** Show that $P(1) \,.\, Q(1) = R(1)$.

8 **a** If $P(x) = x^4 + x^3 - 2x$, find $P(-x)$.

b Hence, find:

i $P(x) + P(-x)$ **ii** $P(x) - P(-x)$ **iii** $P(x) \,.\, P(-x)$

9 If $\dfrac{P(x)}{x-4} = x^2 + 2x + 7$, find $P(x)$.

10 $P(x) = 3x^4 - 5x^2 - 4$, $Q(x) = x^m + 2x^2 + n$ and $R(x) = P(x) \,.\, Q(x)$. Find values for m, n $(m > 2)$ if $R(x)$ is of degree 9 and has a constant term of 24.

■ Further applications

11 If $P(x) = x - 4$, $Q(x) = x^2 + 2x - 5$ and $R(x) = x^2 + 10$, show that:

a $[P(x) + Q(x)] + R(x) = P(x) + [Q(x) + R(x)]$

b $[P(x) \,.\, Q(x)] \,.\, R(x) = P(x) \,.\, [Q(x) \,.\, R(x)]$

c $P(x)[Q(x) + R(x)] = P(x) \,.\, Q(x) + P(x) \,.\, R(x)$.

13.6 The division transformation

Long division with polynomials is performed in a similar manner as that for integers.

For example, a numerical division such as $445 \div 13$ would be set out as follows.

$$\begin{array}{r} 34 \\ 13\overline{)445} \\ \underline{39} \\ 55 \\ \underline{52} \\ 3 \end{array}$$

- 13 into 44 goes 3 times
- $3 \times 13 = 39$
- subtract and bring down the 5
- 13 into 55 goes 4 times
- $4 \times 13 = 52$
- subtract

$\therefore 445 \div 13 = 34$ remainder 3, or $34\frac{3}{13}$.

In this division, we say that:

- 445 is the dividend
- 13 is the divisor
- 34 is the quotient
- 3 is the remainder

We can thus express the dividend in the form
DIVIDEND = DIVISOR × QUOTIENT + REMAINDER.

So, in this example, $445 = 13 \times 34 + 3$. This is called the division transformation for integers. We note that the remainder must be less than the divisor, otherwise we would divide further.

The division transformation for polynomials states that if the polynomial $P(x)$ is divided by another polynomial $A(x)$, then

$$P(x) = A(x) \,.\, Q(x) + R(x)$$

where $A(x)$ is the divisor, $Q(x)$ is the quotient and $R(x)$ is the remainder.

NOTE: The degree of the remainder $R(x)$ must be less than the degree of the divisor $A(x)$.

The division transformation is outlined in the following examples.

Example 1

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Find $P(x) \div A(x)$ and hence express $P(x)$ in the form $A(x) \,.\, Q(x) + R(x)$ given that $P(x) = x^3 + 2x^2 - 11x + 27$ and $A(x) = x + 5$.

Solution

$$\begin{array}{rl}
 & x^2 - 3x + 4 \\
x + 5\,\big) & x^3 + 2x^2 - 11x + 27 \\
 & \underline{x^3 + 5x^2} \\
 & \quad -3x^2 - 11x + 27 \\
 & \quad \underline{-3x^2 - 15x} \\
 & \qquad\qquad 4x + 27 \\
 & \qquad\qquad \underline{4x + 20} \\
 & \qquad\qquad\qquad 7
\end{array}$$

- $x^3 \div x = x^2$
- $x^2(x + 5) = x^3 + 5x^2$
- subtract and bring down $-11x + 27$
- $-3x^2 \div x = -3x$
- $-3x(x + 5) = -3x^2 - 15x$
- subtract and bring down 27
- $4x \div x = 4$
- $4(x + 5) = 4x + 20$
- subtract

$\therefore\ x^3 + 2x^2 - 11x + 27 = (x + 5)(x^2 - 3x + 4) + 7.$

Exercise 13.6

1 Simplify the following fractions using the division algorithm.

a $\dfrac{x^2 + 7x + 12}{x + 3}$ **b** $\dfrac{x^2 + 5x - 24}{x + 8}$ **c** $\dfrac{2x^2 - 13x + 21}{x - 3}$ **d** $\dfrac{5x^2 - 28x - 12}{x - 6}$

2 Perform the following divisions. Express the result in the form dividend = divisor × quotient + remainder.

a $(x^2 + 7x + 14) \div (x + 2)$ **b** $(x^2 + 4x - 29) \div (x - 4)$
c $(6x^2 + 23x + 27) \div (3x + 4)$ **d** $(10x^2 - 29x + 6) \div (2x - 5)$

Consolidation

3 Perform the following divisions. Express each result in the form dividend = divisor × quotient + remainder.

a $(x^3 + 7x^2 + 10x + 15) \div (x + 1)$ **b** $(x^3 + 12x^2 + 37x + 32) \div (x + 3)$
c $(x^3 - 9x^2 + 16x - 3) \div (x - 2)$ **d** $(x^3 - 4x^2 - 9x + 34) \div (x - 5)$
e $(2x^3 + 15x^2 + 32x + 15) \div (x + 4)$ **f** $(3x^3 - 7x^2 - 11x + 8) \div (3x + 5)$
g $(3x^3 - 4x^2 + 12x + 28) \div (3x - 4)$ **h** $(6x^3 + 19x^2 - 13x - 10) \div (2x + 7)$
i $(x^4 - 5x^3 + 3x^2 + 12x - 30) \div (x + 2)$ **j** $(2x^4 + x^3 - 16x^2 + 11x - 2) \div (2x - 3)$

4 Find the quotient $Q(x)$ and remainder $R(x)$ for each of the following divisions.

a $(x^3 + 5x^2 + 8) \div (x + 1)$ **b** $(x^3 - 3x + 4) \div (x - 2)$
c $(x^4 - 3x^2 + 7) \div (x - 1)$ **d** $(x^4 + 4x - 6) \div (x + 3)$

Further applications

5 Find the quotient $Q(x)$ and remainder $R(x)$ for each of the following divisions.

a $(x^4 + 2x^3 - 5x^2 + 10x - 13) \div (x^2 + 3)$ **b** $(x^4 + 5x^3 - 2x^2 - 7x + 4) \div (x^2 - 3x + 1)$
c $(2x^4 + 7x^3 - 6x^2 + x - 4) \div (x^2 - x + 1)$ **d** $(x^4 + 5x^3 + x - 2) \div (x^2 + 3)$
e $(8x^4 + 6x^3 - 7x + 12) \div (2x^2 - 5)$ **f** $(x^4 + 6x^2 + 5) \div (x^2 + 2x + 3)$
g $(2x^4 - 17x^2 - 3) \div (x^2 + 4x + 3)$ **h** $(12x^4 + 11x^2 + 3x) \div (2x^2 - x + 1)$

6 a If $P(x) = x^3 + 5x^2 + kx + 2k$ and $A(x) = x - 1$, determine the quotient $Q(x)$ and remainder $R(x)$ when $P(x)$ is divided by $A(x)$.

b Find the value of k for which $A(x)$ is a factor of $P(x)$.

c Find the value of k for which $P(x) \div A(x)$ leaves a remainder of 21.

13.7 The remainder theorem

The remainder theorem for polynomials states that:

If a polynomial $P(x)$ is divided by $(x - a)$, then the remainder is $P(a)$.

Proof:
If a polynomial $P(x)$ is divided by the linear polynomial $(x - a)$, then the remainder must be a constant, since the degree of the remainder must be less than the degree of the divisor. Let the remainder be r.

If $P(x) = A(x) \,.\, Q(x) + R(x)$,
then $P(x) = (x - a) \,.\, Q(x) + r$.
Now, $P(a) = (a - a) \,.\, Q(a) + r$
$\therefore P(a) = r$.

So, when $P(x)$ is divided by $(x - a)$, the remainder is given by $P(a)$.

EG +S

Example 1

Find the remainder when $P(x) = x^3 - 2x^2 + 5$ is divided by:

a $x - 1$ **b** $x + 3$

Solutions

a
$$P(x) = x^3 - 2x^2 + 5$$
$$P(1) = 1^3 - 2(1)^2 + 5$$
$$\therefore P(1) = 4$$
So, when $P(x)$ is divided by $(x - 1)$, the remainder is 4.

b
$$P(x) = x^3 - 2x^2 + 5$$
$$P(-3) = (-3)^3 - 2(-3)^2 + 5$$
$$\therefore P(-3) = -40$$
So, when $P(x)$ is divided by $(x + 3)$, the remainder is -40.

EG +S

Example 2

Find the value of k such that when $P(x) = x^3 + 3x^2 + kx - 1$ is divided by $(x + 2)$, the remainder is -5.

Solution

$$P(x) = x^3 + 3x^2 + kx - 1$$
$$P(-2) = (-2)^3 + 3(-2)^2 + k(-2) - 1$$
$$-5 = -8 + 12 - 2k - 1$$
$$-5 = 3 - 2k$$
$$2k = 8$$
$$\therefore k = 4$$

Example 3

When the polynomial $P(x)$ is divided by $(x-4)$ the remainder is 1 and when it is divided by $(x-6)$ the remainder is 5. Find the remainder when $P(x)$ is divided by $(x-4)(x-6)$.

Solution

$P(x) = (x-4)(x-6) \,.\, Q(x) + R(x)$. Since the divisor is of degree 2, the remainder must be of the form $(ax + b)$, that is either linear or constant.

Now, $R(x) = ax + b$

$\therefore\ 1 = a(4) + b$ (since $P(4) = 1$).

So, $4a + b = 1 \ldots.$ ①

and $5 = a(6) + b$ (since $P(6) = 5$).

So, $6a + b = 5 \ldots.$ ②

Solving ① and ② simultaneously,

$$\left.\begin{aligned} 4a + b &= 1 \\ 6a + b &= 5 \end{aligned}\right\}\ \text{subtract}$$

$$-2a = -4$$

$$\therefore\ a = 2$$

Substitute $a = 2$ into ①

$$4a + b = 1$$

$$4(2) + b = 1$$

$$\therefore\ b = -7$$

$\therefore$ the remainder when $P(x)$ is divided by $(x-4)(x-6)$ is $(2x-7)$.

Exercise 13.7

1 $P(x) = x^3 + 5x^2 - 2x - 1$ and $A(x) = x - 1$.

- **a** Use the remainder theorem to find the remainder when $P(x)$ is divided by $A(x)$.
- **b** Verify your answer by dividing $A(x)$ into $P(x)$.

2 Use the remainder theorem to find the remainder when $P(x)$ is divided by $A(x)$.

- **a** $P(x) = x^3 - 5x^2 + 6x + 9$ and $A(x) = x - 1$
- **b** $P(x) = x^3 + 3x^2 - 6x - 14$ and $A(x) = x + 1$
- **c** $P(x) = x^3 - 3x + 2$ and $A(x) = x - 2$
- **d** $P(x) = x^4 + 5x^3 + x^2 - 17x + 4$ and $A(x) = x + 3$
- **e** $P(x) = 2x^3 + 4x^2 - x + 2$ and $A(x) = x - 1$
- **f** $P(x) = 3x^4 + 10x^3 + 3x^2 - 12x - 3$ and $A(x) = x + 1$
- **g** $P(x) = 2x^3 - x^2 - 32x + 30$ and $A(x) = x - 4$
- **h** $P(x) = 5x^4 + 25x^3 - 4x^2 - 22x - 25$ and $A(x) = x + 5$

Consolidation

3 When $x^3 + x^2 + 9x + k$ is divided by $(x - 1)$, the remainder is 4. Find the value of k.

4 Find the value of k in each of these.
- **a** When $x^3 - 2x^2 + kx + 8$ is divided by $(x + 1)$, the remainder is 12.
- **b** When $x^3 - x^2 - 11x + (k + 2)$ is divided by $(x - 3)$, the remainder is -6.
- **c** When $3x^3 + 3kx^2 + x + 28$ is divided by $(x - 4)$, the remainder is -16.

5 When $P(x) = x^3 + ax^2 + ax - 6$ is divided by $(x - 2)$, the remainder is 8. Find the value of a and hence find the polynomial $P(x)$.

6 Find the remainder when $P(x) = x^3 + 12x^2 + 2$ is divided by:
- **a** $2x - 1$
- **b** $3x + 2$

7 When the quadratic polynomial $x^2 + 2x - 13$ is divided by $(x - k)$, the remainder is 2. Find two possible values for k.

8 A polynomial $P(x)$ of degree 3 is divisible by $(x + 1)$, $(x - 2)$ and $(x - 4)$, but leaves a remainder of -8 when divided by $(x - 3)$. Find $P(x)$. [Hint: Let $P(x) = a(x + 1)(x - 2)(x - 4)$.]

9
- **a** A polynomial $P(x)$ of degree 4 is divisible by $(x + 5)$, $(x + 2)$, $(x - 1)$ and $(x - 3)$, but leaves a remainder of 28 when divided by $(x - 2)$. Find $P(x)$.
- **b** A polynomial $P(x)$ of degree 3 is divisible by $(2x + 1)$, $(3x - 2)$ and $(x - 2)$, but leaves a remainder of 6 when divided by $(x - 1)$. Find $P(x)$.

Further applications

10 When the polynomial $P(x)$ is divided by $(x - 1)$, the remainder is 8. When $P(x)$ is divided by $(x - 2)$, the remainder is 11.
- **a** If $P(x)$ is divided by $(x - 1)(x - 2)$, explain why the remainder $R(x)$ is of the form $(ax + b)$.
- **b** Find values for a and b. Hence find the remainder $R(x)$.

11 When the polynomial $P(x)$ is divided by $(x - 3)$, the remainder is 2. When $P(x)$ is divided by $(x + 2)$, the remainder is -8. Find the remainder when $P(x)$ is divided by $(x - 3)(x + 2)$.

TRY THIS Quadratic remainder

When the polynomial $P(x)$ is divided by $(x - 1)$, $(x - 2)$ and $(x - 3)$ the respective remainders are 9, 10 and 23.

1. If $P(x)$ is divided by $(x - 1)$, $(x - 2)$ and $(x - 3)$ explain why the remainder $R(x)$ is of the form $ax^2 + bx + c$.
2. Find values for a, b and c and hence find the remainder $R(x)$.

13.8 The factor theorem

The factor theorem for polynomials states that:

If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

Proof:

$P(x) = (x - a) \,.\, Q(x) + r$

$P(a) = (a - a) \,.\, Q(a) + r$

$\therefore P(a) = r$

If $P(a) = 0$, then $r = 0$.

$\therefore P(x) = (x - a) \,.\, Q(x)$

$\therefore (x - a)$ is a factor of $P(x)$.

The converse of this theorem is also true.

$P(x) = (x - a) \,.\, Q(x)$

$P(a) = (a - a) \,.\, Q(a)$

$\therefore P(a) = 0.$

That is, if $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$.

Example 1

Show that $(x + 2)$ is a factor of the polynomial $P(x) = x^3 - 4x^2 - 3x + 18$.

Solution

If $(x + 2)$ is a factor of $P(x)$, then $P(-2) = 0$.

$$\begin{aligned} P(-2) &= (-2)^3 - 4(-2)^2 - 3(-2) + 18 \\ &= -8 - 16 + 6 + 18 \\ &= 0 \end{aligned}$$

$\therefore (x + 2)$ is a factor of $P(x)$.

Example 2

Solve the equation $x^3 - 2x^2 - 11x + 12 = 0$.

Solution

We use trial and error to find one of the linear factors of $P(x)$ and then divide by this factor to find the other factors.

$$\begin{aligned} \text{Now, } P(1) &= 1^3 - 2(1)^2 - 11(1) + 12 \\ &= 0 \end{aligned}$$

$\therefore (x - 1)$ is a factor of $P(x)$.

$$\begin{array}{r|l} & x^2 - x - 12 \\ \hline x - 1 & x^3 - 2x^2 - 11x + 12 \\ & x^3 - x^2 \\ \hline & \quad - x^2 - 11x + 12 \\ & \quad - x^2 + x \\ \hline & \qquad -12x + 12 \\ & \qquad -12x + 12 \\ \hline & \qquad\qquad 0 \end{array}$$

$\therefore (x - 1)(x^2 - x - 12) = 0$

$\therefore (x - 1)(x - 4)(x + 3) = 0$

$\therefore x = 1, 4, -3.$

Example 3

The polynomial $P(x) = x^3 + ax^2 + bx + 20$ has a factor of $(x - 5)$ and leaves a remainder of -10 when divided by $(x - 3)$. Find the values of a and b.

Solution

Now, $(x - 5)$ is a factor of $P(x)$, therefore $P(5) = 0$. Also, when $P(x)$ is divided by $(x - 3)$ the remainder is -10, therefore $P(3) = -10$.

$$P(5) = 5^3 + a(5)^2 + b(5) + 20$$
$$0 = 125 + 25a + 5b + 20$$
$$25a + 5b = -145$$
$$\therefore\ 5a + b = -29 \ldots.\ (1)$$
$$P(3) = 3^3 + a(3)^2 + b(3) + 20$$
$$-10 = 27 + 9a + 3b + 20$$
$$9a + 3b = -57$$
$$\therefore\ 3a + b = -19 \ldots.\ (2)$$

Solving (1) and (2) simultaneously,

$$\left.\begin{aligned} 5a + \mathrm{b} &= -29 \\ 3a + b &= -19 \end{aligned}\right\} \text{ subtract}$$
$$2a = -10$$
$$\therefore\ a = -5$$

Substitute $a = -5$ into (1).

$$5a + b = -29$$
$$5(-5) + b = -29$$
$$\therefore\ b = -4$$

Exercise 13.8

1 $P(x) = x^3 + x^2 - x - 10$ and $A(x) = x - 2$.

- **a** Use the factor theorem to show that $A(x)$ is a factor of $P(x)$.
- **b** Verify your result by dividing $A(x)$ into $P(x)$.

2 Determine by use of the factor theorem whether each of the following expressions is a factor of $x^3 + 5x^2 + 2x - 8$.

- **a** $x - 1$
- **b** $x + 1$
- **c** $x - 2$
- **d** $x + 2$
- **e** $x - 4$
- **f** $x + 4$

3 Show that $A(x)$ is a factor of $P(x)$ in each of the following. Hence, express $P(x)$ as the product of 3 linear factors.

- **a** $P(x) = x^3 + x^2 - 10x + 8;\ A(x) = x - 1$
- **b** $P(x) = x^3 - 3x^2 - 25x - 21;\ A(x) = x + 1$
- **c** $P(x) = x^3 + 6x^2 - x - 6;\ A(x) = x + 6$
- **d** $P(x) = x^3 - 4x^2 - 11x + 30;\ A(x) = x - 2$
- **e** $P(x) = x^3 - x^2 - 10x - 8;\ A(x) = x + 1$
- **f** $P(x) = x^3 - 7x + 6;\ A(x) = x + 3$

Consolidation

4 For each of the following, show that $A(x)$ is a factor of $P(x)$. Hence, express $P(x)$ as the product of 3 linear factors.

a $P(x) = 3x^3 - x^2 - 22x + 24;\ A(x) = x - 2$ **b** $P(x) = 2x^3 - 9x^2 - 20x + 12;\ A(x) = x + 2$

5 Factorise these polynomials completely.

a $x^3 + 8x^2 + 11x - 20$ **b** $x^3 + 6x^2 - 9x - 14$ **c** $x^3 - 3x^2 - 4x + 12$

d $x^3 + 10x^2 + 31x + 30$ **e** $2x^3 + 7x^2 + x - 10$ **f** $3x^3 - 5x^2 - 34x + 24$

6 Factorise $x^3 - 3x^2 + 4$ completely.

7 Solve these equations.

a $x^3 - 6x^2 - 19x + 24 = 0$ **b** $x^3 + 4x^2 - 4x - 16 = 0$ **c** $x^3 - 5x^2 - 9x + 45 = 0$

d $x^3 - 12x^2 + 41x - 42 = 0$ **e** $2x^3 - 7x^2 - 5x + 4 = 0$ **f** $3x^3 - 19x^2 + 16x + 20 = 0$

8 Solve these equations, giving the irrational roots in simplest surd form where necessary.

a $x^3 - x^2 - 4x - 2 = 0$ **b** $x^3 - 4x^2 + 8 = 0$

9 Show that the polynomial $P(x) = x^3 - 3x^2 + x + 5$ has only one real zero. What is it?

10 Find the value of k in each of the following given that:

a $(x - 3)$ is a factor of $x^3 - 19x + k$ **b** $(x - 2)$ is a factor of $x^3 + x^2 + kx + 10$

c $(x + 1)$ is a factor of $x^3 + kx^2 + 3x + 7$ **d** $(x + 4)$ is a factor of $2x^3 + 13x^2 + kx - 32$

Further applications

11 Solve the equation $(x - 3)^3 + 3(x - 3)^2 - 6(x - 3) - 8 = 0$. [Hint: Let $u = x - 3$.]

12 Find all values of a for which the polynomial $P(x) = x^3 - 2x^2 - 31x - 28$ is divisible by $(x - a)$.

13 The polynomial $P(x) = x^3 + ax^2 - x + b$ is divisible by both $(x + 2)$ and $(x - 5)$. Form a pair of simultaneous equations and solve them to find values for a and b.

14 The polynomial $P(x) = x^3 + px^2 + qx - 18$ is divisible by $(x + 6)$, but leaves a remainder of -28 when divided by $(x - 1)$. Find values for p and q.

TRY THIS Algebraic remainder

1 Find the remainder when the quadratic polynomial $P(x) = ax^2 + bx + c$ is divided by $(x - 1)$.

2 What relationship exists between a, b and c if $(x - 1)$ is a factor of $P(x)$?

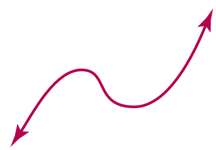
13.9 Sketching graphs of polynomial functions

Polynomial graphs are continuous curves. That is, they can be drawn completely without taking the pen off the page. For very large and very small values of x, the sign of the polynomial is determined by the leading term $a_n x^n$.

For example, consider the cubic polynomial $y = x^3 - 7x^2 + 6$, where the leading term is x^3.

As $x \to +\infty$, $x^3 \to +\infty$, $\therefore\ y \to +\infty$.
And, as $x \to -\infty$, $x^3 \to -\infty$, $\therefore\ y \to -\infty$.

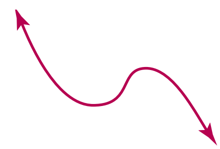
Hence, the graph would have this shape.



Consider also the cubic polynomial $y = -x^3 - 4x + 3$, where the leading term is $-x^3$.

As $x \to +\infty$, $x^3 \to +\infty$, $-x^3 \to -\infty$, $\therefore\ y \to -\infty$.
And, as $x \to -\infty$, $x^3 \to -\infty$, $-x^3 \to +\infty$, $\therefore\ y \to +\infty$.

Hence, the graph would have this shape.



To sketch the graph of a polynomial function:
- **find the x-intercepts by letting $y = 0$**
- **find the y-intercept by letting $x = 0$**
- **determine the shape of the curve by examining the degree of the polynomial and the sign of the leading term as $x \to +\infty$ and $x \to -\infty$**
- **sketch the graph of the polynomial.**

The following points should also be noted when sketching the graph of a polynomial function.

- **A polynomial $P(x)$ of odd degree has at least one real zero. That is, the graph of $y = P(x)$ must cut the x-axis in at least one point.**
- **If $P(x)$ has a double zero (k), then the graph $y = P(x)$ touches the x-axis at $x = k$.**
- **If $P(x)$ has a triple zero (k), then the graph of $y = P(x)$ has a point of inflexion on the x-axis at $x = k$.**
- **If $P(x)$ is of odd degree, then the ends of the curve will point in opposite directions.**
- **If $P(x)$ is of even degree, then the ends of the curve will point in the same direction.**

Example 1

EG +S

Sketch the following graphs.

a $y = (x + 2)(x - 1)(x - 3)$ **b** $y = -x(x - 4)^2$

Solutions

a **i** when $x = 0$:

$y = (0 + 2)(0 - 1)(0 - 3)$
$= 2 \times -1 \times -3$
$= 6$

$\therefore$ the y-intercept is 6.

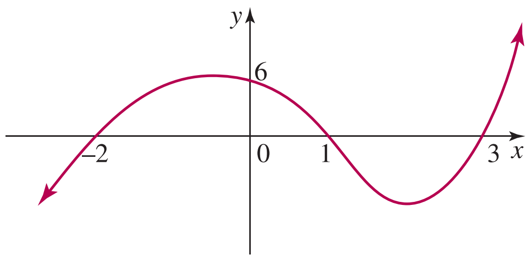
ii when $y = 0$:

$(x + 2)(x - 1)(x - 3) = 0$
$\therefore x = -2, 1, 3$

$\therefore$ the x-intercepts are -2, 1, 3.

iii The leading term of the polynomial is x^3.
As $x \to +\infty$, $y \to +\infty$
and as $x \to -\infty$, $y \to -\infty$.

iv



$y = (x + 2)(x - 1)(x - 3)$

b **i** when $x = 0$:

$y = -0(0 - 4)^2$
$= 0 \times 16$
$= 0$

$\therefore$ the y-intercept is 0.

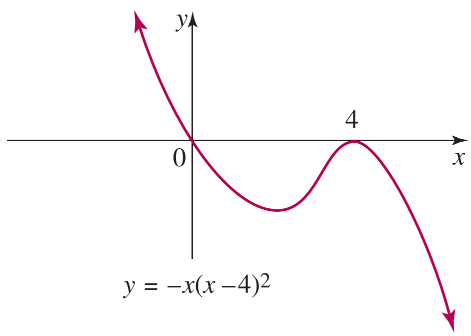
ii when $y = 0$:

$-x(x - 4)^2 = 0$
$\therefore -x = 0$ or $(x - 4)^2 = 0$
$\therefore x = 0, 4$

However, 4 is a double zero.
Therefore, the curve touches the x-axis at 4.

iii The leading term of the polynomial is $-x^3$.
As $x \to +\infty$, $y \to -\infty$
and as $x \to -\infty$, $y \to +\infty$.

iv



$y = -x(x - 4)^2$

Example 2

EG +S

Sketch the following graphs.

a $y = (x + 3)(x + 1)(x - 1)(x - 2)$ **b** $y = (x + 2)^2(x - 1)^2$ **c** $y = -x(x - 2)^3$

Solutions

a **i** when $x = 0$:

$y = (0 + 3)(0 + 1)(0 - 1)(0 - 2)$
$= 3 \times 1 \times (-1) \times (-2)$
$= 6$

$\therefore$ the y-intercept is 6.

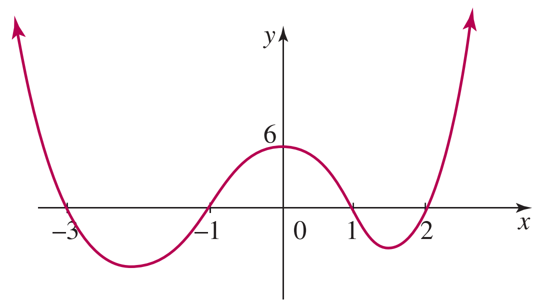
ii when $y = 0$:

$(x + 3)(x + 1)(x - 1)(x - 2) = 0$
$\therefore x = -3, -1, 1, 2$

$\therefore$ the x-intercepts are -3, -1, 1, 2.

iii The leading term of the polynomial is x^4.
As $x \to +\infty$, $y \to +\infty$
and as $x \to -\infty$, $y \to +\infty$.

iv



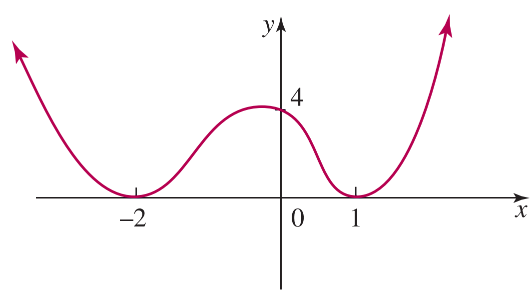
$y = (x + 3)(x + 1)(x - 1)(x - 2)$

b **i** when $x = 0$:
$y = (0 + 2)^2(0 - 1)^2$
$= 4 \times 1$
$= 4$
$\therefore$ the y-intercept is 4.

ii when $y = 0$:
$(x + 2)^2(x - 1)^2 = 0$
$\therefore x = -2, 1.$
However, –2 and 1 are both double zeros.
Therefore, the curve touches the x-axis at –2 and 1.

iii The leading term of the polynomial is x^4.
As $x \to +\infty$, $y \to +\infty$
and as $x \to -\infty$, $y \to +\infty$.

iv



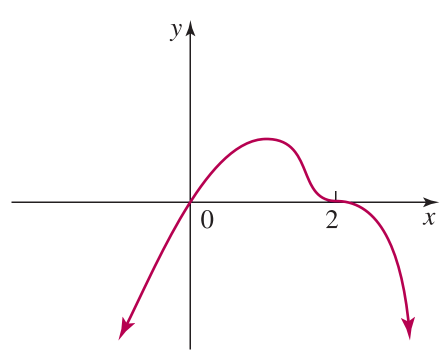
$y = (x + 2)^2(x - 1)^2$

c **i** when $x = 0$:
$y = -0(0 - 2)^3$
$= 0 \times -8$
$= 0$
$\therefore$ the y-intercept is 0.

ii when $y = 0$:
$-x(x - 2)^3 = 0$
$\therefore x = 0, 2.$
$\therefore$ the x-intercepts are 0, 2.
However, 2 is a triple zero.
Therefore there is a point of inflexion on the x-axis at 2.

iii The leading term of the polynomial is $-x^4$.
As $x \to +\infty$, $y \to -\infty$
and as $x \to -\infty$, $y \to -\infty$.

iv



$y = -x(x - 2)^3$

Exercise 13.9

1 Sketch the graphs of the following quadratic polynomials. Show all intercepts with the co-ordinate axes.

a $P(x) = x(x-2)$ **b** $P(x) = -x(x+3)$ **c** $P(x) = (x-3)(x+1)$
d $P(x) = (x+4)(5-x)$ **e** $P(x) = -(x+2)(x+1)$ **f** $P(x) = (2x-1)(x-2)$

2 Find all zeros for these cubic polynomials then sketch their graphs. Show all intercepts with the co-ordinate axes.

a $P(x) = (x-1)(x-2)(x-4)$ **b** $P(x) = (x+3)(x+1)(x-1)$
c $P(x) = x(x+2)(x-2)$ **d** $P(x) = -x(x+1)(x+4)$
e $P(x) = (1-x)(x-3)(x-4)$ **f** $P(x) = (x+3)(x+1)(2-x)$

3 Find all zeros for these quartic polynomials then sketch their graphs. Show all intercepts with the co-ordinate axes.

a $P(x) = (x+3)(x+2)(x-1)(x-3)$ **b** $P(x) = x(x+2)(x-2)(x-4)$
c $P(x) = (x+1)(x-1)(x+2)(2-x)$ **d** $P(x) = -x(x+4)(x+3)(x+1)$

■ Consolidation

4 Sketch the graph of each polynomial.

a $y = (2x+1)(x-1)(x-2)$ **b** $y = (3x-2)(x+1)(2-x)$

5 Sketch the graphs of these quadratic polynomials. Show the x, y intercepts on each graph.

a $y = (x-3)^2$ **b** $y = (x+1)^2$ **c** $y = -(x-2)^2$ **d** $y = -(x+4)^2$

6 Find all zeros for these cubic polynomials then sketch their graphs.

a $P(x) = (x-1)^2(x-2)$ **b** $P(x) = x^2(x+3)$
c $P(x) = x(x+2)^2$ **d** $P(x) = (x-2)^2(x+1)$
e $P(x) = -x^2(x+4)$ **f** $P(x) = -(x+1)^2(x+3)$
g $P(x) = (x+1)^2(2-x)$ **h** $P(x) = (1-x)(x-4)^2$

7 Find all zeros for these quartic polynomials then sketch their graphs.

a $P(x) = x^2(x-2)^2$ **b** $P(x) = -(x+3)^2(x-1)^2$
c $P(x) = x(x-1)(x-2)^2$ **d** $P(x) = (x+2)^2(x+4)(1-x)$
e $P(x) = x^2(x+3)(4-x)$ **f** $P(x) = (x-5)^2(x-2)^2$

8 Find all zeros for these quartic polynomials then sketch their graphs.

a $P(x) = x^3(x-2)$ **b** $P(x) = (x-1)^3(x+3)$
c $P(x) = -x(x+2)^3$ **d** $P(x) = (4-x)^3(x+1)$

9 Factorise each of these polynomials then sketch their graphs.

a $P(x) = x^3 - x$ **b** $P(x) = x^3 - x^2$ **c** $P(x) = 4x - x^3$
d $P(x) = x^4 - x^3$ **e** $P(x) = x^2 - x^4$ **f** $P(x) = x^4 - 9x^2$
g $P(x) = x^3 - 7x^2 + 10x$ **h** $P(x) = -x^3 + 6x^2 - 9x$ **i** $P(x) = x^4 + x^3 - 2x^2$

10 Factorise these polynomials using the factor theorem, then draw a neat sketch showing the x, y intercepts.

a $y = x^3 + x^2 - 14x - 24$ **b** $y = -x^3 - x^2 + 10x - 8$

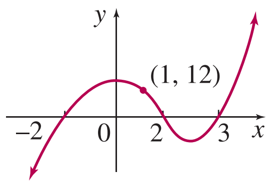
11 a Show that $P(x) = (x + 1)(x^2 - 4x + 6)$ has only 1 zero.

b How many times would the graph of this polynomial cut the x-axis?

■ Further applications

12 Find the equation of each of these cubic curves. Leave the equation in factorised form.

a



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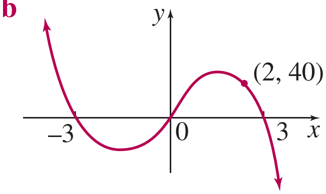

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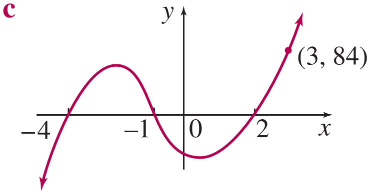

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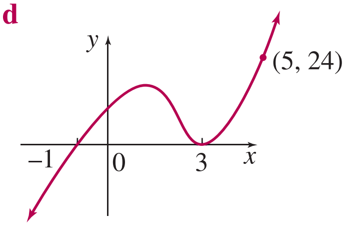

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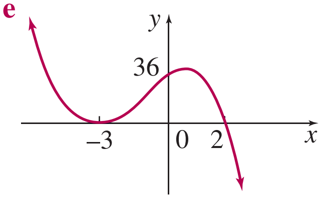

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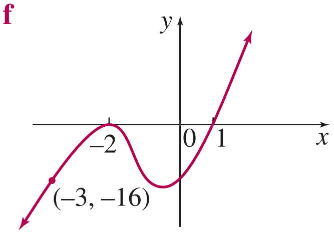


TRY THIS **Quadratic factor**

The polynomial $P(x) = x^4 + 2x^3 + x^2 + ax + b$ is divisible by $(x^2 - 4)$. Find values for a and b.

13.10 Transformations of polynomial graphs

Given the graph of $y = P(x)$, the graph of:

- **$y = P(x) + c$ $(c > 0)$ is sketched by shifting the curve $y = P(x)$ up by c units**
- **$y = P(x) - c$ $(c > 0)$ is sketched by shifting the curve $y = P(x)$ down by c units**
- **$y = P(x - a)$ $(a > 0)$ is sketched by shifting the curve $y = P(x)$ to the right by a units**
- **$y = P(x + a)$ $(a > 0)$ is sketched by shifting the curve $y = P(x)$ to the left by a units**
- **$y = aP(x)$ $(a > 1)$ is sketched by stretching the curve $y = P(x)$ vertically by a factor of a**
- **$y = aP(x)$ $(0 < a < 1)$ is sketched by shrinking the curve $y = P(x)$ vertically by a factor of a**
- **$y = -P(x)$ is sketched by reflecting the curve $y = P(x)$ in the x-axis**
- **$y = P(-x)$ is sketched by reflecting the curve $y = P(x)$ in the y-axis.**

Example 1

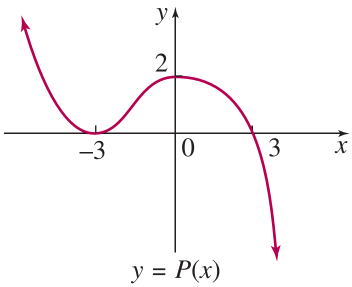
Given the graph of $y = P(x)$ to the right, sketch the following graphs.

a $y = P(x) + 2$ **b** $y = P(x) - 2$

c $y = P(x - 3)$ **d** $y = P(x + 3)$

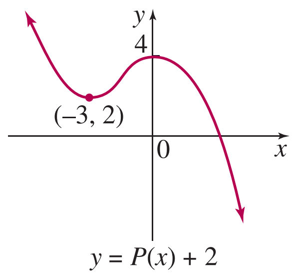
e $y = 2P(x)$ **f** $y = \frac{1}{2}P(x)$

g $y = -P(x)$ **h** $y = P(-x)$

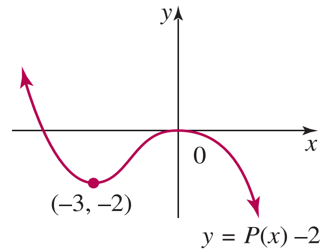


Solutions

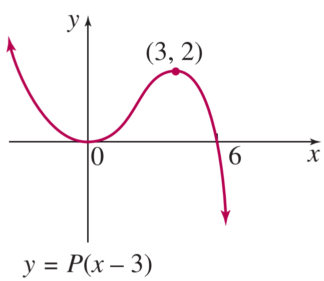
a The graph of $y = P(x) + 2$ is sketched by shifting the curve $y = P(x)$ up by 2 units.



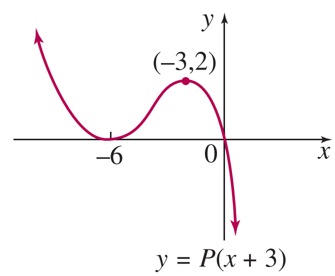
b The graph of $y = P(x) - 2$ is sketched by shifting the curve $y = P(x)$ down by 2 units.



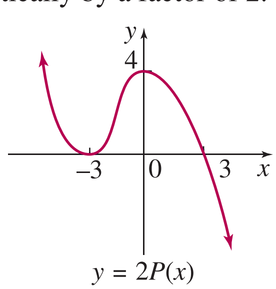
c The graph of $y = P(x - 3)$ is sketched by shifting the curve $y = P(x)$ to the right by 3 units.



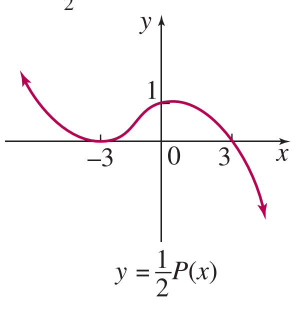
d The graph of $y = P(x + 3)$ is sketched by shifting the curve $y = P(x)$ to the left by 3 units.



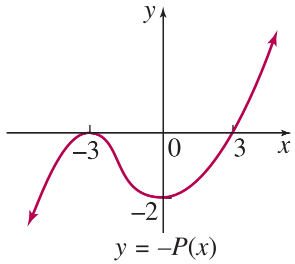
e The graph of $y = 2P(x)$ is sketched by stretching the curve $y = P(x)$ vertically by a factor of 2.



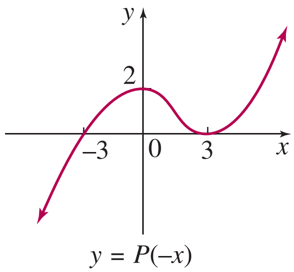
f The graph of $y = \frac{1}{2}P(x)$ is sketched by shrinking the curve $y = P(x)$ by a factor of $\frac{1}{2}$.



g The graph of $y = -P(x)$ is sketched by reflecting the curve $y = P(x)$ in the x-axis.



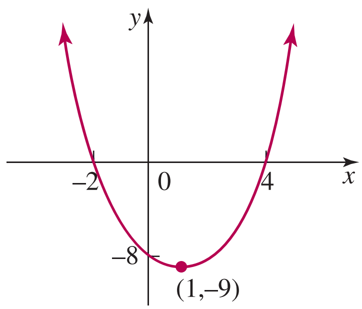
h The graph of $y = P(-x)$ is sketched by reflecting the curve $y = P(x)$ in the y-axis.



Exercise 13.10

NOTE: In exercise 13.10 it may not be possible to find x-intercepts for each graph.

1



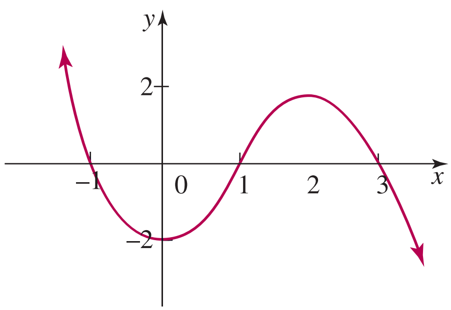
Given the graph of $y = P(x)$, sketch:

a $y = P(x) + 9$ **b** $y = P(x - 2)$
c $y = 2P(x)$ **d** $y = \frac{1}{2}P(x)$
e $y = P(-x)$ **f** $y = -P(x)$

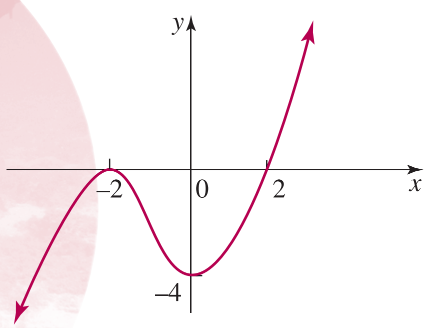
Consolidation

2 Given the graph of $y = P(x)$, sketch:

a $y = P(x) + 1$ **b** $y = P(x) - 1$
c $y = 2P(x)$ **d** $y = \frac{1}{2}P(x)$
e $y = P(x - 1)$ **f** $y = P(x + 2)$
g $y = -P(x)$ **h** $y = P(-x)$



3



Given the graph of $y = P(x)$, sketch:

a $y = P(x + 2)$ **b** $y = P(x - 2)$
c $y = -P(x)$ **d** $y = P(-x)$
e $y = P(x) + 4$ **f** $y = P(x) - 1$
g $y = 3P(x)$ **h** $y = \frac{1}{4}P(x)$

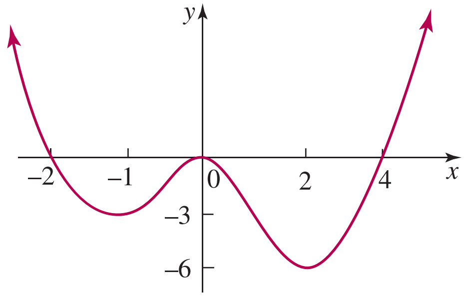
4 Given the graph of $y = P(x)$, sketch:

a $y = P(x) + 3$ **b** $y = P(x) - 2$

c $y = P(-x)$ **d** $y = -P(x)$

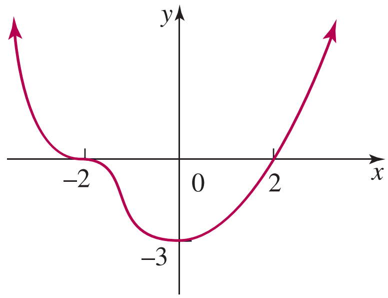
e $y = 2P(x)$ **f** $y = \frac{1}{3}P(x)$

g $y = P(x - 2)$ **h** $y = P(x + 1)$



Further applications

5



Given the graph of $y = P(x)$, sketch:

a $y = -P(x - 2)$ **b** $y = P(-x) + 3$

c $y = -2P(x)$ **d** $y = 1 - P(-x)$

FOCUS ON WORKING MATHEMATICALLY

Focus on working mathematically

Lawyers, a curious number problem and a family of hyperbolas

Introduction

In this chapter you have drawn many different types of curves, including polynomials like $y = x^3 - 3x^2 + 3x - 27$, quadratics like $y = x^2 - x - 2$ and circles like $x^2 + y^2 = 1$. The circle is an example of an *indeterminate* quadratic equation because the solutions are not fixed or *determined*.

For a given value of x there may be two values of y. A similar situation arises with the curve $x^2 - y^2 = 1$ which you may not have yet seen.

The lawyers

In 1657 Pierre de Fermat, the outstanding French lawyer, also recognised throughout Europe as a brilliant mathematician, proposed a challenge to his colleagues which included friends such as John Wallis in England and Bernard Frenicle de Bessy a fellow lawyer in Paris. The plague of 1651 had killed thousands of people in Toulouse where he lived and he narrowly escaped death. The following year Fermat was promoted to the highest criminal court as Councillor of the King and it was at this time that he worked on *number theory*. Fermat made an outstanding contribution to mathematics and is best remembered for his famous *last theorem* which was not finally proved until 1994 by the British mathematician Andrew Wiles. He died in 1665 in Castres France.

Here was the challenge:

> *Given a positive integer n, find a positive integer y such that $ny^2 + 1$ is a perfect square.*

In this activity we will explore some solutions and their graphical representation.

The problem

Let $x^2 = ny^2 + 1$ so that, given n, the problem is to find positive integers x and y which satisfy the equation. This equation later became known as **Pell's equation** because Leonhard Euler in the mid 18th century accidentally called it Pell's equation and the name stuck even though the Englishman John Pell (1611–1685) had nothing to do with it.

Learning Activities

1. Show that when $n = 1$ the equation reduces to $x^2 = y^2 + 1$ and that $y = 0$ is the only value which gives a solution. However zero is not a positive integer so this is a special case. Note however that (1, 0) will lie on the curve $x^2 = y^2 + 1$. In this activity we will consider $n > 1$.
2. Suppose $n = 8$. By inspection write down a value for y so that $8y^2 + 1$ is a perfect square and find x. Find another solution by testing values of y up to 10.
3. Verify that $y = 35$, $y = 204$ and $y = 1189$ also work and find x in each case. Notice the solutions get increasingly large.
4. European mathematicians were not the only ones to be interested in the problem. The Indian mathematician Brahmagupta investigated such equations as long ago as 628 AD. For example he showed that the smallest solution to the Pell equation when $n = 61$ is $y = 226\ 153\ 980$. Verify this solution with a calculator and find x.
5. Bragmagupta found that if (a, b) is a solution of $x^2 = ny^2 + 1$ then so is $(a^2 + nb^2, 2ab)$. Using your results from question 1, show that if (3, 1) is a solution of $x^2 = 8y^2 + 1$, so also is (17, 6). Apply the theorem again using the solution (17, 6) to find another solution.

6 Following question 1, are there any other restrictions on n for integer solutions to be possible? What happens when n is a perfect square? See if you can find an integer solution to $x^2 = 4y^2 + 1$. Make a conclusion about your deductions.

7 Hypothesise the number of solutions to Pell's equation for a given $n > 1$.

Challenge activities

It is now time to graph our results. First we will graph the equation for all values of x and y and then think about the positive integer solutions.

1 Consider the case $n = 8$ in questions 2, 3 and 5. Show that the graph must be symmetrical about the y-axis and that it crosses the x-axis at $x = 1$ and $x = -1$.

2 By making y the subject of the equation, show that as x gets large (and is positive) the value of y approaches $\frac{x}{\sqrt{8}}$.

3 Show that as x gets large (and is negative) the value of y approaches $\frac{-x}{\sqrt{8}}$.

4 What are the lines in questions 2 and 3 above called? Make a rough sketch of the curve to show its shape.

5 On what part of the curve do positive integer solutions of the equation lie? Highlight this on your sketch.

6 What type of curve does this particular equation represent? What can you conclude about the shape of Pell's equation in general?

7 Pell's equation $x^2 = ny^2 + 1$ is a special case of the general family of hyperbola's $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The numbers a and b are the x and y intercepts on the axes. Check this. Also the straight lines $y = \frac{bx}{a}$ and $y = \frac{-bx}{a}$ are the asymptotes. Make a rough sketch of $x^2 - y^2 = 1$ (refer back to question 1) to get the idea.

Let's communicate

Make a summary in your workbook of what you have learned about positive integer solutions to Pell's equation and what the solutions look like on a graph.

Reflecting

Fermat was considered an amateur mathematician because his primary occupation was the law and he had little formal mathematical training. Reflect on the way people of all occupations and backgrounds can be very good at mathematics and enjoy the subject as a life long pleasure.

Language link with Macquarie

MACQUARIE

1 *Give a brief explanation and an example of*
   - **a** *a* ***polynomial*** *of degree 3*
   - **b** *the* ***zeros*** *of a polynomial*
   - **c** *the* ***factor theorem***

2 *The* Macquarie Learners Dictionary *does not define specialised words of the mathematics register like* ***binomial***, ***trinomial*** *and* ***polynomial*** *expressions. How are these words linked? What is the significance of the prefixes* ***bi***, ***tri*** *and* ***poly***?

Chapter Review

1 Sketch these graphs on the same number plane.
- **a** $y = x^3$, $y = \frac{1}{2}x^3$, $y = -x^3$
- **b** $y = x^4$, $y = 2x^4$, $y = -x^4$
- **c** $y = x^5$, $y = x^5 + 2$, $y = x^5 - 2$
- **d** $y = x^6$, $y = (x - 1)^6$, $y = -(x + 1)^6$

2 Sketch the following circles.
- **a** $x^2 + y^2 = 16$ **b** $9x^2 + 9y^2 = 4$

3 Find the centre and radius of each circle.
- **a** $(x - 2)^2 + (y + 3)^2 = 49$
- **b** $(x + 6)^2 + (y + 1)^2 = 4$
- **c** $x^2 + (y - 4)^2 = 8$

4 Write down the equation of the circle with:
- **a** centre $(3, -5)$ and radius 8 units.
- **b** centre $(-2, 0)$ and radius $2\sqrt{3}$ units.

5 State whether each equation describes a circle.
- **a** $x^2 + y^2 = 1$ **b** $x^2 - y^2 = 4$
- **c** $9x^2 + 9y^2 = 25$ **d** $\frac{x^2}{4} + \frac{y^2}{16} = 1$
- **e** $(x - 3)^2 + (y - 6)^2 = 100$

6 Find the centre and radius for each circle.
- **a** $x^2 + y^2 - 2x + 6y - 6 = 0$
- **b** $x^2 + y^2 - 10x - 20 = 0$

7 Find all points of intersection for the following graphs.
- **a** $y = 2x^2 - 8x$ and $y = 5x - 15$
- **b** $y = \frac{12}{x}$ and $y = x + 1$
- **c** $x^2 + y^2 = 45$ and $y = 2x$

8 State whether each expression is a polynomial.
- **a** $x^3 - 3x^2 + 5$ **b** $3^x + 2$
- **c** $\sqrt{x} - 4x$ **d** $2x$
- **e** $x^3 + \frac{1}{x}$ **f** 5
- **g** 0 **h** $\sqrt{2}x^4 + 7x^3$

9 For each polynomial below, state:
- **i** the leading term
- **ii** the leading co-efficient
- **iii** the degree
- **iv** the constant term
- **a** $5x^4 - x^3 + 7$ **b** $6 + 2x^2 - x^5$
- **c** $x^2 + 11x$ **d** $\frac{x^3}{5} - \frac{x^2}{4} - 1$

10 Explain why $x^3 - 4x$ is a monic polynomial, but $4x - x^3$ is not.

11 If $P(x) = x^3 - 5x^2 + 6x + 4$, find:
- **a** $P(2)$ **b** $P(-1)$ **c** $P(0)$
- **d** $P\left(\frac{2}{3}\right)$ **e** $P(\sqrt{2})$ **f** $P(3a)$

CHAPTER REVIEW

12 If $P(x) = x^3 + 5x^2 - 4x + 7$ and $Q(x) = x^3 - 2x^2 - 6x + 3$, find:
a $P(x) + Q(x)$ **b** $P(x) - Q(x)$

13 If $A(x) = 2x^3 - 3x + 5$ and $B(x) = 3x + 4$, find $A(x) . B(x)$.

14 If $P(x) = 7x^2 + 2x - 11$, $Q(x) = 4x^3 + 5x$ and $R(x) = 12 - 7x^2$, find the degree of:
a $P(x) + Q(x)$ **b** $P(x) + R(x)$
c $P(x) - R(x)$ **d** $P(x) - Q(x) + R(x)$
e $P(x) . Q(x)$ **f** $Q(x)[P(x) + R(x)]$

15 Perform the following divisions and express each result in the form dividend = divisor × quotient + remainder.
a $(x^2 - 4x + 5) \div (x - 1)$
b $(x^3 + 5x^2 - x + 6) \div (x + 2)$
c $(2x^3 + 3x^2 - 7x - 10) \div (2x - 3)$
d $(x^4 + 6x^2 + 10) \div (x - 2)$

16 Determine the remainder when $P(x) = x^3 + 2x^2 - 8x + 3$ is divided by:
a $x - 1$ **b** $x + 1$
c $x - 2$ **d** $x + 3$

17 a When $x^3 - 3x^2 + 6x + k$ is divided by $(x - 3)$, the remainder is 22. Find k.
b When $x^3 + x^2 + kx - 7$ is divided by $(x + 2)$, the remainder is -3. Find k.

18 Show that $A(x)$ is a factor of $P(x)$ in each of these, without long division.
a $P(x) = x^3 - 3x^2 - 2x - 8$; $A(x) = x - 4$
b $P(x) = x^3 - x^2 - 12x - 12$; $A(x) = x + 2$

19 Find the value of k if $(x - 5)$ is a factor of $x^3 + kx^2 - 7x - 15$.

20 a Show that $(x + 2)$ is a factor of $P(x) = x^3 + 3x^2 - 10x - 24$.
b Hence express $P(x)$ as the product of 3 linear factors.

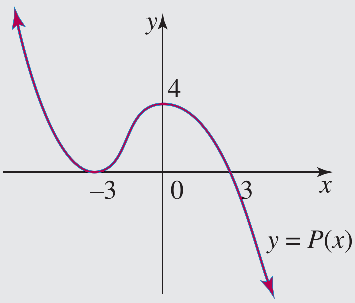
21 Factorise into 3 linear factors:
a $x^3 + 2x^2 - 29x + 42$
b $2x^3 - 7x^2 + 2x + 3$

22 Solve the equation $3x^3 + 7x^2 - 4 = 0$.

23 $P(x) = x^3 + ax^2 + bx - 30$ is divisible by $(x - 3)$, but leaves a remainder of -8 when divided by $(x - 1)$. Find values for a and b.

24 Sketch the following graphs.
a $y = x(x - 2)(x + 2)$
b $y = (x + 3)(x + 1)(2 - x)$
c $y = (x - 1)^2(x - 4)$
d $y = -(x + 3)(x + 2)^2$
e $y = (x + 2)^2(x - 2)^2$
f $y = (x - 1)^3(x + 3)$
g $y = -x^3(x - 4)$

25



Given the graph of $y = P(x)$, sketch:
a $y = P(x + 3)$ **b** $y = P(x - 3)$
c $y = -P(x)$ **d** $y = P(-x)$
e $y = P(x) + 1$ **f** $y = P(x) - 4$
g $y = 2P(x)$ **h** $y = \frac{1}{2}P(x)$

Circle geometry

14

This chapter at a glance

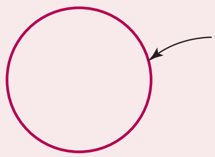
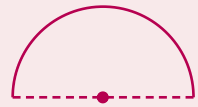
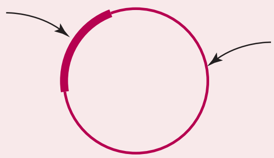
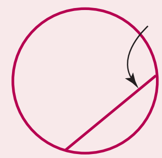
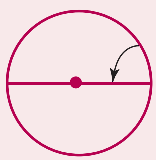
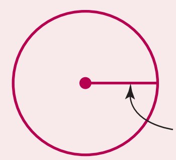
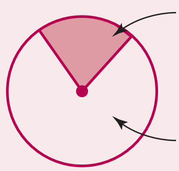
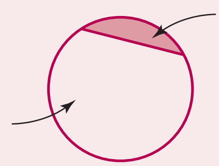
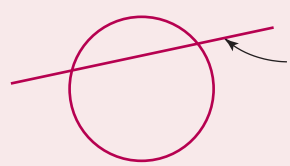
Stage 5.1/5.2/5.3

After completing this chapter, you should be able to:

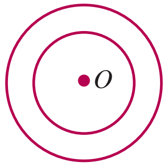
- identify and name the parts of a circle
- use correct terminology associated with circle geometry
- use the chord properties to solve numerical problems
- use the chord properties to solve simple deductive problems, giving reasons
- use the angle properties to solve numerical problems
- use the angle properties to solve simple deductive problems, giving reasons
- use the properties of cyclic quadrilaterals to solve numerical problems
- use the properties of cyclic quadrilaterals to solve simple deductive problems, giving reasons
- use the tangent properties to solve numerical problems
- use the tangent properties to solve simple deductive properties, giving reasons
- use the further circle properties involving chords, tangents and secants to solve numerical problems
- use the further circle properties involving chords, tangents and secants to solve simple deductive problems, giving reasons
- write formal deductive proofs involving circle properties

14.1 Circle terminology

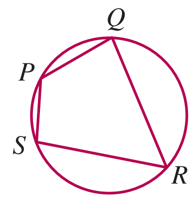
The following table shows the various parts of a circle and lists their definitions.

 **Circumference:** the boundary of a circle.	 **Semicircle:** half the boundary of a circle.
 **Arc:** part of the circumference. The smaller arc is called the minor arc and the larger arc is called the major arc.	 **Chord:** an interval that joins two points on the circumference.
 **Diameter:** a chord that passes through the centre.	 **Radius:** an interval that joins the centre to a point on the circumference. It is half the length of the diameter.
 **Sector:** the area bounded by an arc and two radii. The smaller sector is called the minor sector and the larger sector is called the major sector.	 **Segment:** the area bounded by an arc and a chord. The smaller segment is called the minor segment and the larger segment is called the major segment.
 **Tangent:** a line that touches a circle at one point.	 **Secant:** a line that cuts a circle in two points.

Some other important terms are:



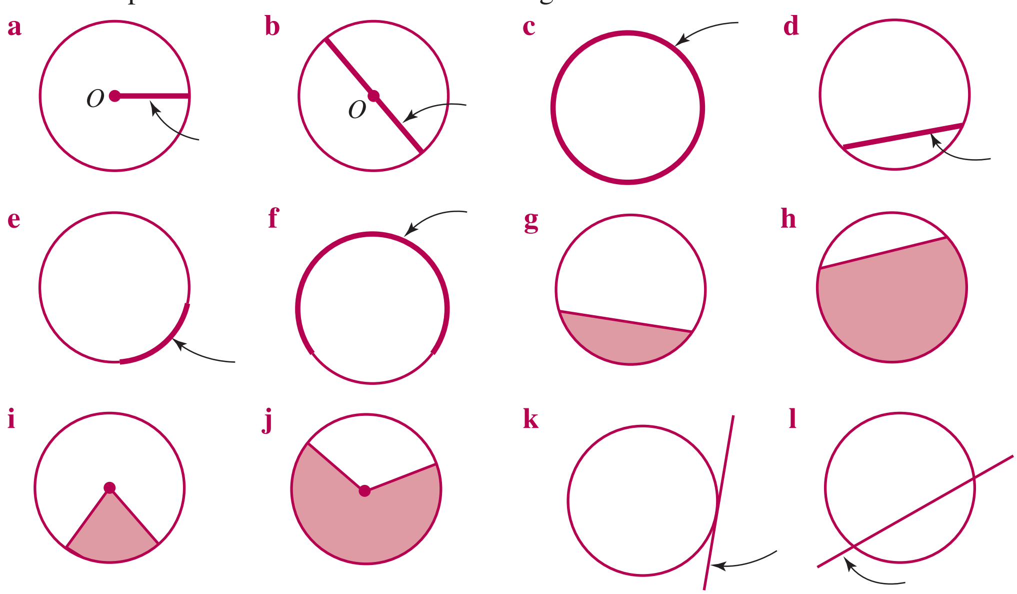
Concentric circles are two or more circles that have the same centre.



A **cyclic quadrilateral** is a quadrilateral in which all of the vertices lie on a circle. The points P, Q, R, S are **concyclic points**.

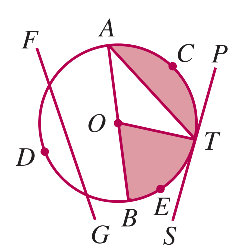
Exercise 14.1

1 Name the parts of the circle shown on the diagrams below.



2 Name the following features that are shown on the diagram.

a	the interval OT	**b**	the interval AB
c	the line PS	**d**	the interval AT
e	the arc ADB	**f**	the line FG
g	the region $OBET$	**h**	the region ACT



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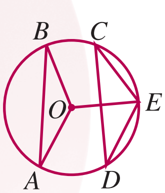


On the given diagram, name all the:

a radii

b chords

4 State the circle feature whose definition is given below.

a Half of the circumference.
b A section of the circumference.
c An interval that joins two points on the circumference.
d An interval that joins the centre of a circle to the circumference.
e A line that touches a circle in exactly one point.
f A chord that passes through the centre of a circle.
g The area bounded by a chord and the circumference.
h A line that cuts a circle in two points.
i The area bounded by two radii and the circumference.

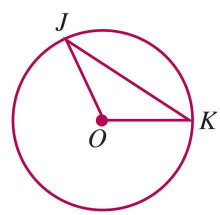
■ Consolidation

5 A circle has a radius of length 12 cm.

a What is the length of the longest chord in the circle?
b What name is given to this chord?

6 ΔJOK is drawn in a circle with centre O.

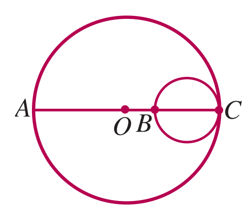
a Explain why $OJ = OK$.
b What kind of triangle is OJK?
c Which angles are equal?



7 C and D are two points on a circle with centre O such that $\angle COD = 60°$. What kind of triangle is COD? Why?

8 Two concentric circles have diameters 22 cm and 14 cm. How far apart are the circles?

9 The circles shown touch internally at C. The larger circle has centre O. If the diameters of these circles are 20 cm and 7 cm, find OB.



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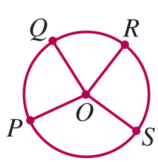
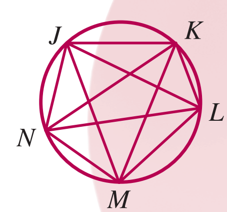
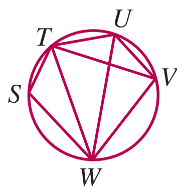


Name the angle at the centre of this circle that is:

a standing on the minor arc PQ
b subtended by the minor arc RS.

11 **a** Name the 3 angles at the circumference standing on the arc:
i JK **ii** LM
b Name the 3 angles at the circumference standing on the arc:
i JM **ii** LN



12 

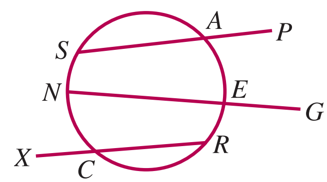
a Name the 2 angles standing on the minor arc TW.

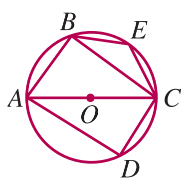
b Name the angle standing on the major arc:

i TW **ii** TV **iii** SU

13 PA and PS are the intercepts of the secant drawn to the circle from P. Name the intercepts of the secant drawn from

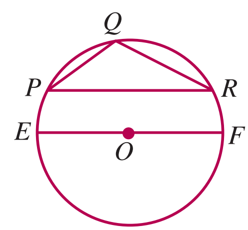
a G **b** X.

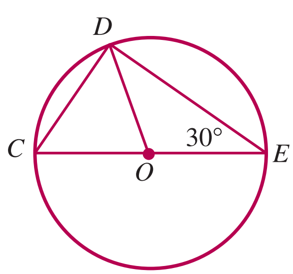


14 

$\angle ABC$ is an angle in a semi-circle, since the angle is subtended at the circumference by the diameter. Name two other angles in a semi-circle.

15 Is $\angle PQR$ an angle in a semi-circle? Explain.



16 

Find the size of each angle in the given diagram.

a $\angle ODE$

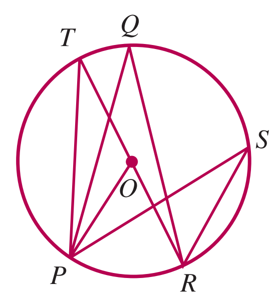
b $\angle COD$

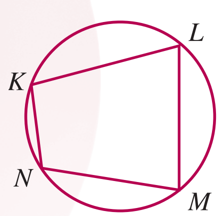
c $\angle OCD$

d $\angle CDE$

17 **a** Name the angle at the centre, standing on the minor arc PR.

b Name all angles at the circumference, standing on the minor arc PR.



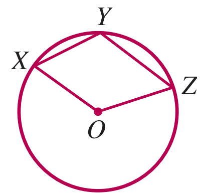
18 

a Are the points K, L, M, N concyclic points?

b What name is given to the quadrilateral $KLMN$?

c Which angle is opposite $\angle K$?

19 Is $OXYZ$ a cyclic quadrilateral? Explain.



20 $PQRS$ is a cyclic quadrilateral. If PQ is produced to T, name the exterior angle at Q.

21 Two intersecting circles with centres O and C have a common chord AB. What kind of quadrilateral is $OACB$ if:

a the circles are the same size?

b the circles are different sizes?

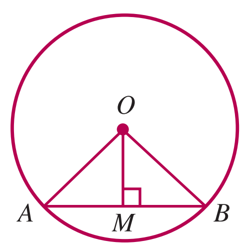
■ Further applications

22 Draw a circle of radius 3 cm and draw any chord in the circle. Using a ruler and compasses, construct the perpendicular bisector of the chord. Does it pass through the centre of the circle?

23 Draw a circle of radius 3 cm and any two non-intersecting chords. Outline the constructions that could be used to locate the centre of the circle.

14.2 Chord properties of circles

The line drawn through the centre of a circle perpendicular to a chord bisects the chord.



Data: O is the centre, $OM \perp AB$.

Aim: To prove that OM bisects AB.

Construction: Join OA and OB.

Proof: In Δs OAM, OBM

- $\angle OMA = \angle OMB = 90°$ ($OM \perp AB$)
- $OA = OB$ (equal radii)
- OM is a common side

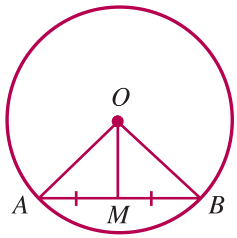
$\therefore \Delta OAM \equiv \Delta OBM$ (RHS)

$\therefore AM = MB$ (matching sides of congruent Δs)

$\therefore OM$ bisects AB.

That is, the line drawn through the centre of a circle perpendicular to a chord bisects the chord.

The line drawn through the centre of a circle to the midpoint of a chord is perpendicular to the chord.



Data: O is the centre, M is the midpoint of AB.
Aim: To prove that OM is perpendicular to AB.
Construction: Join OA and OB.
Proof: In Δs OAM, OBM

- $OA = OB$ (equal radii)
- $AM = MB$ (M is the midpoint of AB)
- OM is a common side

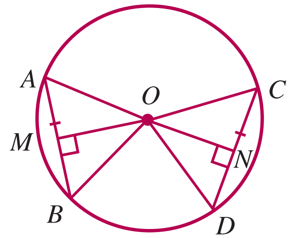
$\therefore \Delta OAM \equiv \Delta OBM$ (SSS)
$\therefore \angle OMA = \angle OMB$ (matching $\angle$s of congruent Δs)
But, $\angle OMA + \angle OMB = 180°$ (adjacent $\angle$s on a straight line)
$\therefore \angle OMA = \angle OMB = 90°$
$\therefore OM \perp AB$.

That is, the line drawn through the centre of a circle to the midpoint of a chord is perpendicular to the chord.

The perpendicular bisector of a chord of a circle passes through the centre.

In the first proof above, it was proven that the perpendicular drawn through the centre O to a chord bisects the chord. Hence, this line is the perpendicular bisector of the chord. Therefore, the perpendicular bisector of the chord passes through the centre of the circle.

Equal chords of a circle subtend equal angles at the centre and are equidistant from the centre.



Data: O is the centre, $AB = CD$, $OM \perp AB$, $ON \perp CD$.
Aim: **i** To prove that $\angle AOB = \angle COD$.
ii To prove that $OM = ON$.
Construction: Join OA, OB, OC and OD.
Proof: **i**

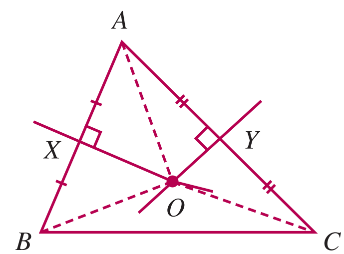
- $OA = OD$ (equal radii)
- $OB = OC$ (equal radii)
- $AB = CD$ (given)

$\therefore \Delta AOB \equiv \Delta DOC$ (SSS)
$\therefore \angle AOB = \angle DOC$ (matching $\angle$s of congruent Δs).

That is, equal chords of a circle subtend equal angles at the centre.

ii $OM = ON$ (altitudes of congruent Δs are equal).

Given any three non-collinear points, the point of intersection of the perpendicular bisectors of any two sides formed by the three points is the centre of the circle through all three points.



Data: ABC is any triangle, O is the point of intersection of OX and OY, the perpendicular bisectors of AB and AC respectively.

Aim: To prove that the circle with centre O and radius OA also passes through B and C.

Construction: Join OA, OB, OC.

Proof: In Δs AOX and BOX

- $AX = XB$ (OX bisects AB)
- $\angle AXO = \angle BXO = 90°$ ($OX \perp AB$)
- OX is a common side

$\therefore \Delta AOX \equiv \Delta BOX$ (SAS)

$\therefore OA = OB$ (matching sides of congruent Δs)

In Δs AOY and COY

- $AY = YC$ (OY bisects AC)
- $\angle AYO = \angle CYO = 90°$ ($OY \perp AC$)
- OY is a common side

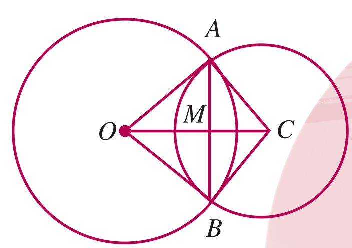
$\therefore \Delta AOY \equiv \Delta COY$ (SAS)

$\therefore OA = OC$ (matching sides of congruent Δs)

Now, $OA = OB$ and $OA = OC$, $\therefore OA = OB = OC$.

Hence, the circle with centre O and radius OA also passes through B and C. That is, given any three non-collinear points, the point of intersection of the perpendicular bisectors of any two sides formed by the three points is the centre of the circle through all three points.

When two circles intersect, the line joining their centres bisects their common chord at right angles.



Data: O, C are the centres, AB is a common chord.

Aim: To prove that AB bisects OC at right angles.

Construction: Join OA, OB, CA, CB.

Proof: In Δs OAC and OBC

- $OA = OB$ (equal radii)
- $CA = CB$ (equal radii)
- OC is a common side

$\therefore \Delta OAC \equiv \Delta OBC$ (SSS)

$\therefore \angle AOC = \angle BOC$ (matching $\angle$s of congruent Δs)

In Δs AOM and BOM

- $OA = OB$ (equal radii)
- $\angle AOC = \angle BOC$ (proven above)
- OM is a common side

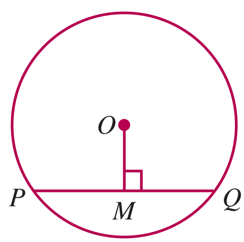
$\therefore \Delta AOM \equiv \Delta BOM$ (SAS)
$\therefore AM = MB$ (matching sides of congruent Δs)
$\therefore OM$ bisects AB.
Also, $\angle AMO = \angle BMO$ (matching $\angle$s of congruent Δs).
But, $\angle AMO + \angle BMO = 180°$ (adjacent $\angle$s on a straight line).
$\therefore \angle AMO = \angle BMO = 90°$
$\therefore OM \perp AB$.

That is, when two circles intersect, the line joining their centres bisects their common chord at right angles.

Example 1

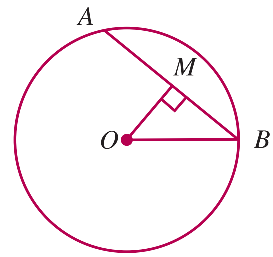
EG +S

a



$PQ = 12$ cm. Find PM, giving reasons.

b



$OB = 15$ cm and $OM = 12$ cm. Find AB, giving reasons.

Solutions

a $PM = \frac{1}{2} PQ$ (line through centre perpendicular to a chord bisects the chord)

$\therefore PM = \frac{1}{2} \times 12$ cm

$= 6$ cm

b **i** $OB^2 = MB^2 + OM^2$ (Pythagoras' theorem)

$15^2 = MB^2 + 12^2$

$MB^2 = 81$

$\therefore MB = 9$ cm

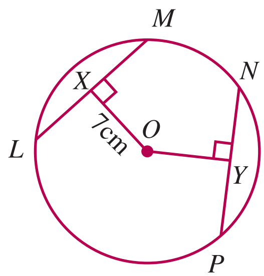
ii $AB = 2MB$ (line through centre perpendicular to a chord bisects the chord)

$\therefore AB = 2 \times 9$ cm

$= 18$ cm

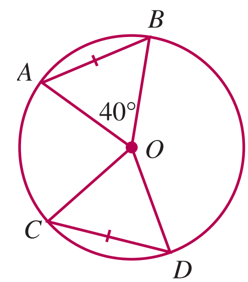
Example 2

a



$LM = NP$ and $OX = 7$ cm.
Find OY, giving reasons.

b



$AB = CD$ and $\angle AOB = 40°$.
Find $\angle COD$, giving reasons.

Solutions

a $OY = OX$ (equal chords are equidistant from the centre)

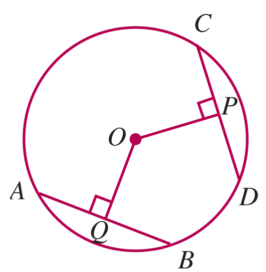
$\therefore OY = 7$ cm

b $\angle COD = \angle AOB$ (equal chords subtend equal angles at the centre)

$\therefore \angle COD = 40°$

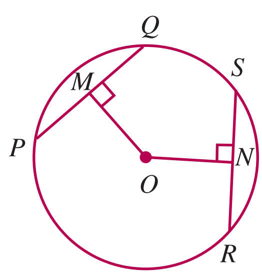
Exercise 14.2

1 a



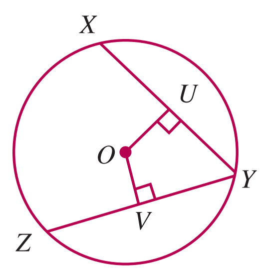
$AB = CD$ and $OP = 7$ cm. Find OQ.

b



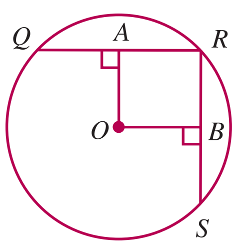
$OM = ON$ and $PQ = 10$ cm. Find RS.

c



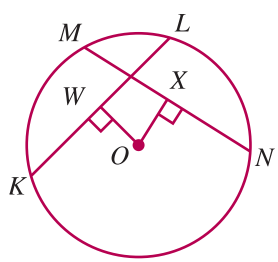
$OU = OV = 6$ cm and $XY = 9$ cm. Find YZ.

d



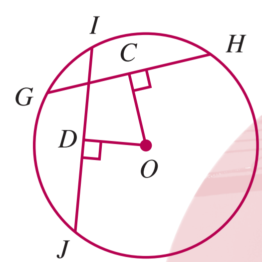
$QR = RS$ and $OA = 6.5$ cm. Find OB.

e



$OW = OX = 9$ cm and $KL = 25$ cm. Find MN.

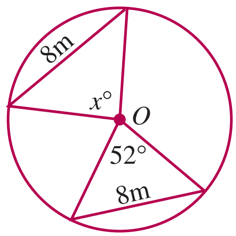
f



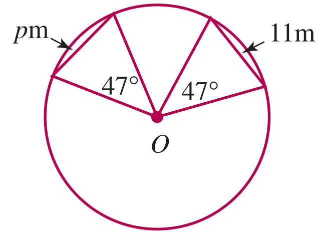
$GH = IJ = 20$ cm and $OD = 8$ cm. Find OC.

2 Find the value of the pronumeral in each of these.

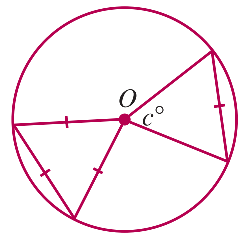
a



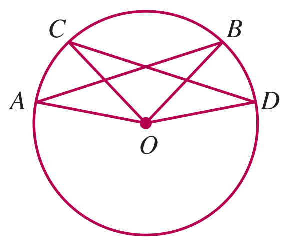
b



c

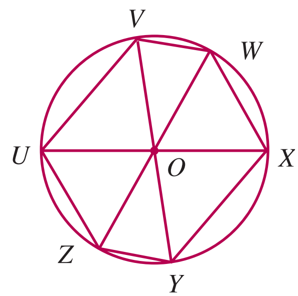


3 a



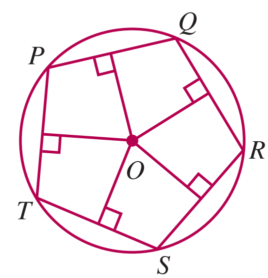
$AB = CD$, $\angle AOB = 110°$, $\angle COB = 65°$. Find $\angle BOD$.

b



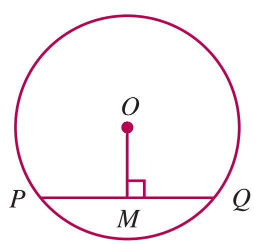
$UV = 7$ cm, $YZ = 4$ cm, $WX = 6$ cm. Find VW, XY, UZ.

c



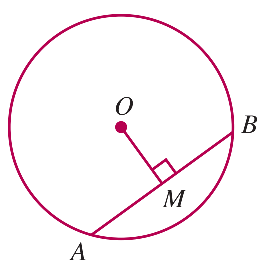
$PQ = QR = RS = ST = TP$. Find $\angle PQR$.

4 a



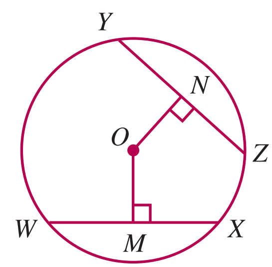
$PQ = 10$ cm. Find PM.

b



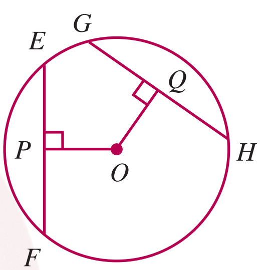
$AM = 7$ cm. Find AB.

c



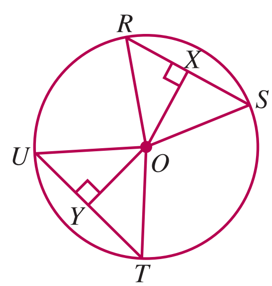
$OM = ON$, $YZ = 22$ cm. Find WX, WM.

d



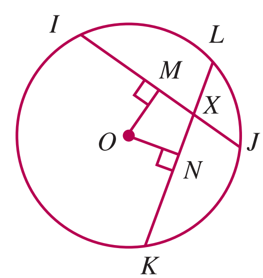
$EF = 15$ cm, $GQ = 7.5$ cm, $OP = 9$ cm. Find GH, OQ.

e



$\angle ROS = \angle UOT$, $UT = 13$ cm. Find XS.

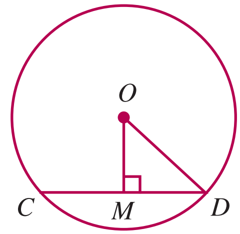
f



$IJ = 20$ cm, $OM = ON$, $LX = 4$ cm. Find XN.

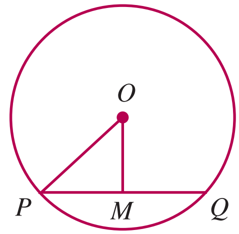
Consolidation

5 **a**



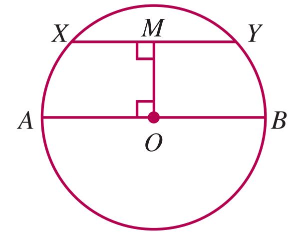
$OD = 13$ cm, $OM = 5$ cm. Find MD, CD.

b



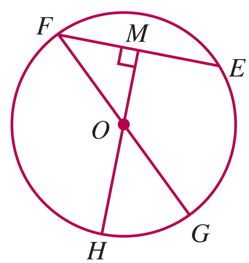
$PM = MQ = 20$ cm, $OM = 21$ cm. Find OP.

c



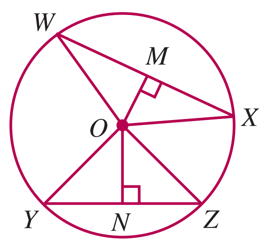
$AB = 52$ cm, $OM = 10$ cm. Find XY.

d



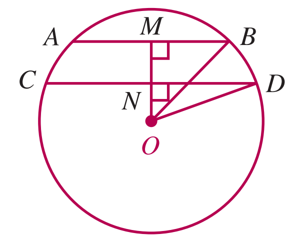
$EF = 24$ cm, $FG = 30$ cm. Find OM, HM.

e



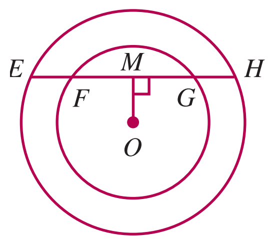
$OM = 15$ cm, $YZ = 14$ cm, $ON = 24$ cm. Find OZ, WX.

f



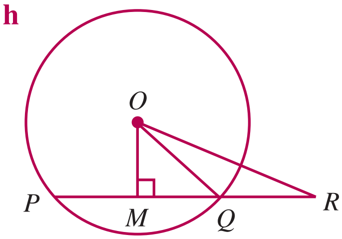
$AB = 80$ cm, $OD = 50$ cm, $CD = 96$ cm. Find OM, MN.

g



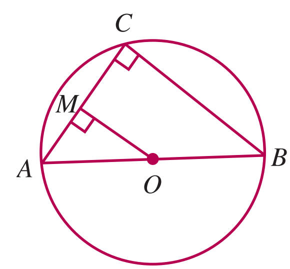
$FG = 12$ cm, $OG = 10$ cm, $OH = 17$ cm. Find GH.

h



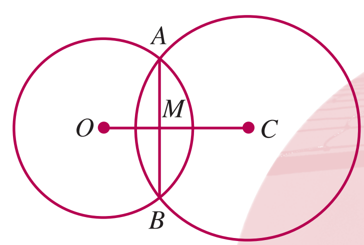
$OR = 39$ cm, $OQ = 25$ cm, $OM = 15$ cm. Find QR, PR.

i



$AB = 74$ cm, $BC = 70$ cm. Find OM.

6 Two circles with centres O and C intersect at A and B as shown. Let M be the point of intersection of the common chord and the line joining the centres of the circles.



a If $OC = 42$ cm, $OA = 20$ cm and $AB = 32$ cm, find AC, the radius of the larger circle.

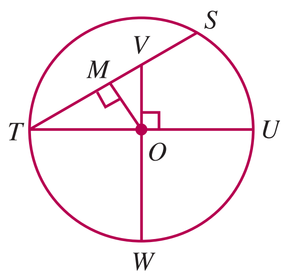
b If $OA = 29$ cm, $AC = 35$ cm and $AB = 42$ cm, find OC, the distance between the centres.

Further applications

7 A chord of length 18 cm is 12 cm from the centre of a circle. How far is a chord of length 10 cm from the centre? Answer in simplest surd form.

8 Two chords are drawn parallel to each other in a circle of radius 17 cm. The chords have lengths 16 cm and 30 cm. Find the distance between the chords if they lie:

a on the same side of the centre **b** on opposite sides of the centre.

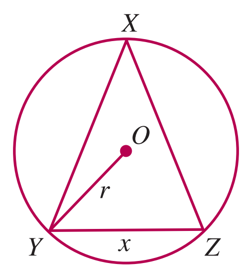
9 

In the diagram shown, $TU = 120$ cm, $VW = 105$ cm and $OM = 36$ cm. Find SV.

10 Two circles with centres O and C intersect at P and Q. OC and PQ meet at M. If the circles have radii of 25 cm, 26 cm and the centres are 17 cm apart, find the length of the common chord PQ. [Hint: Let $OM = x$, $PM = y$.]

TRY THIS **Circumcircle symmetry**

XYZ is an equilateral triangle of side length x cm, drawn in a circle centre O with radius r cm. Prove $x^2 = 3r^2$.



14.3 Angle properties of circles

The angle at the centre of a circle is twice the angle at the circumference, standing on the same arc.

Data: O is the centre.

Aim: To prove that $\angle AOC = 2\angle ABC$.

Construction: Join BO and produce BO to P.

Proof: (In both figure 1 and figure 2)

In ΔAOB, let $\angle OBA = \alpha$

- $OA = OB$ (equal radii)

$\therefore$ ΔAOB is isosceles

- $\angle OAB = \alpha$ (base $\angle$s of isosceles Δ, $OA = OB$)
- $\angle AOP = 2\alpha$ (external $\angle$ of ΔAOB)

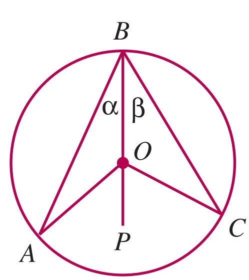


figure 1

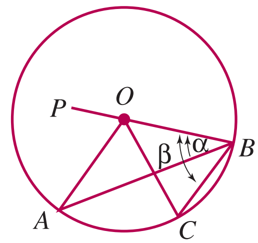


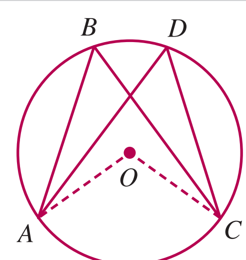
figure 2

In ΔBOC, let $\angle OBC = \beta$
- $OB = OC$ (equal radii)

$\therefore \Delta BOC$ is isosceles
- $\angle OCB = \beta$ (base $\angle$s of isosceles Δ, $OB = OC$)
- $\angle POC = 2\beta$ (external $\angle$ of ΔBOC)

Now, in figure 1, $\angle ABC = \angle OBA + \angle OBC$
$$= \alpha + \beta$$
And, $\angle AOC = \angle AOP + \angle POC$
$$= 2\alpha + 2\beta$$
$$= 2(\alpha + \beta)$$
$$= 2\angle ABC$$
Also, in figure 2, $\angle ABC = \angle OBC - \angle OBA$
$$= \beta - \alpha$$
And, $\angle AOC = \angle POC - \angle AOP$
$$= 2\beta - 2\alpha$$
$$= 2(\beta - \alpha)$$
$$= 2\angle ABC$$
That is, the angle at the centre of a circle is twice the angle at the circumference, standing on the same arc.

Angles at the circumference, standing on the same arc, are equal.

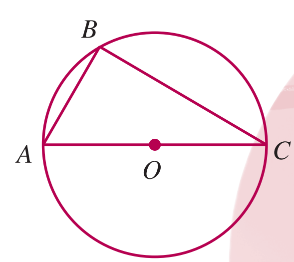


Data: O is the centre.
Aim: To prove that $\angle ABC = \angle ADC$.
Construction: Join OA and OC.
Proof: Let $\angle ABC = \alpha$
- $\angle AOC = 2\alpha$ (angle at centre is twice angle at circumference)
- $\angle ADC = \alpha$ (angle at centre is twice angle at circumference)

$\therefore \angle ABC = \angle ADC$

That is, angles at the circumference, standing on the same arc, are equal.

The angle in a semi-circle is a right angle.



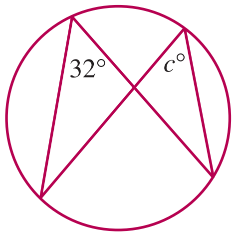
Data: AC is a diameter.
Aim: To prove that $\angle ABC = 90°$.
Construction: Join AB and BC.
Proof:
- $\angle AOC = 180°$ (AC is a straight line)
- $\angle ABC = 90°$ (angle at centre is twice angle at the circumference)

That is, the angle in a semi-circle is a right angle.

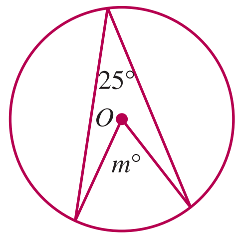
Example 1

Find the value of the pronumeral in each of these, giving reasons.

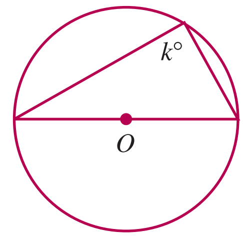
a



b



c



Solutions

a $c = 32$ (angles at circumference standing on the same arc are equal)

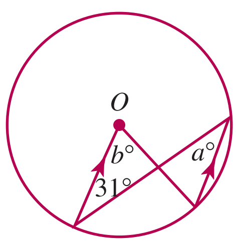
b $m = 50$ (angle at the centre is twice the angle at the circumference)

c $k = 90$ (angle in a semi-circle)

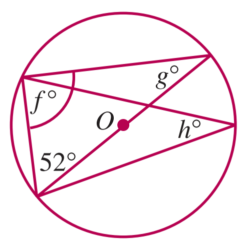
Example 2

Find the value of all pronumerals in each of these, giving reasons.

a



b



Solutions

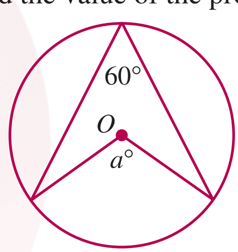
a
- $a = 31$ (alternate angles in parallel lines)
- $b = 62$ (angle at centre is twice the angle at the circumference)

b
- $f = 90$ (angle in a semi-circle is a right angle)
- $g = 38$ (angle sum of a triangle)
- $h = 38$ (angles at circumference standing on the same arc are equal)

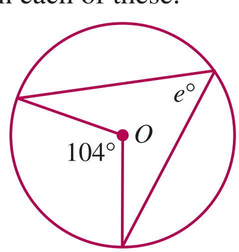
Exercise 14.3

1 Find the value of the pronumeral in each of these.

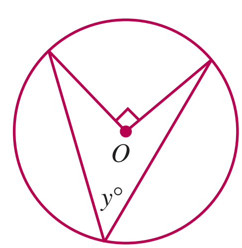
a



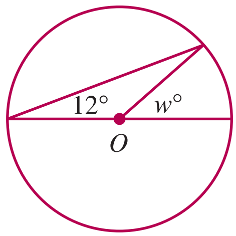
b



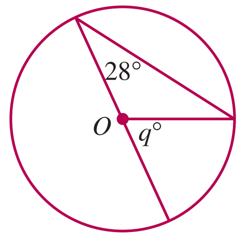
c



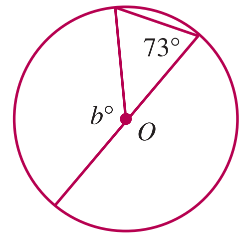
d



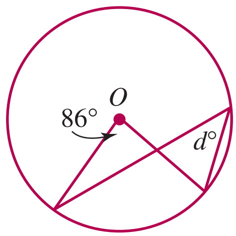
e



f



g



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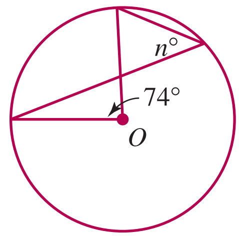
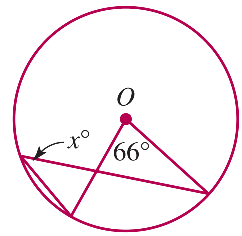
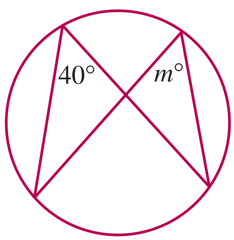


i

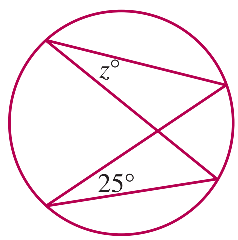


2 Find the value of each pronumeral.

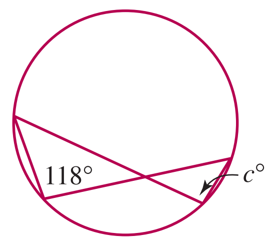
a



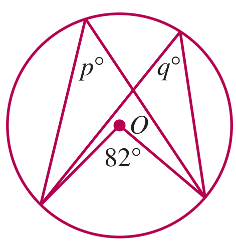
b



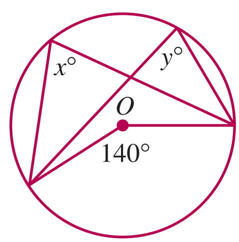
c



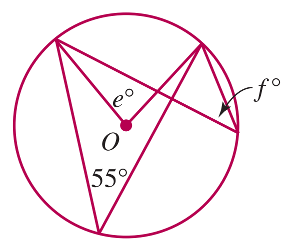
d



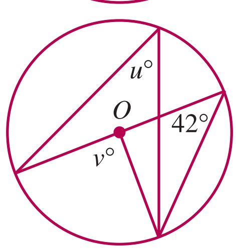
e



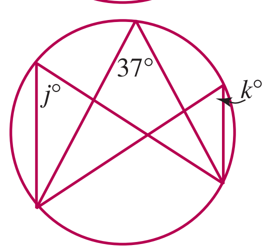
f



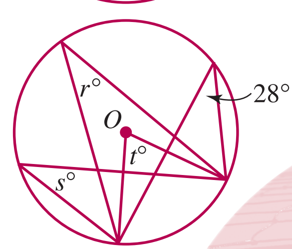
g



h

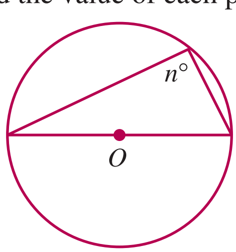


i



3 Find the value of each pronumeral.

a



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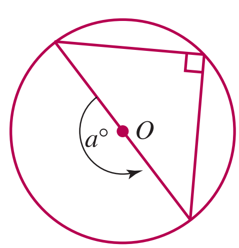
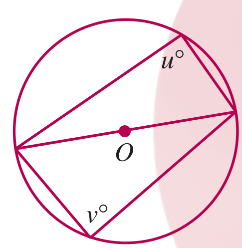


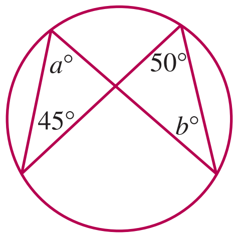
c



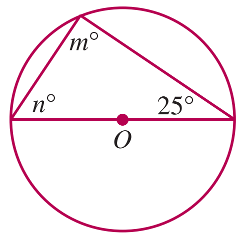
Consolidation

4 Find the value of all pronumerals in each of the following.

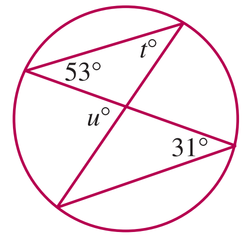
a



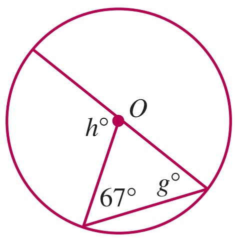
b



c



d



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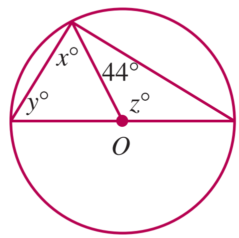

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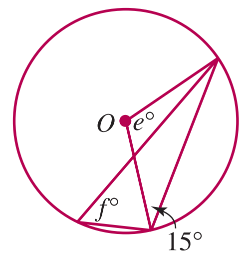
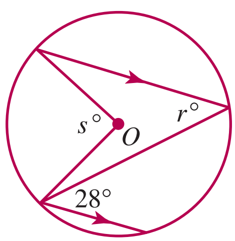


g



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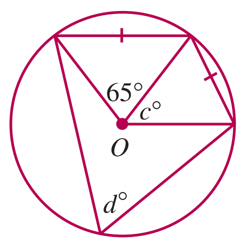

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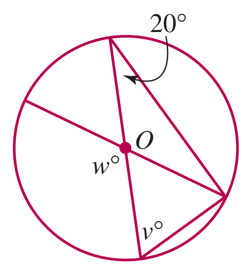

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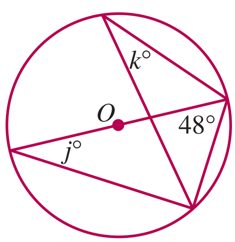

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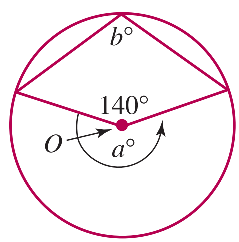

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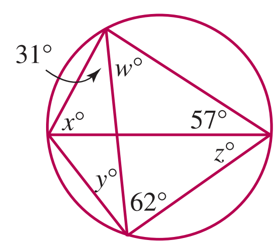
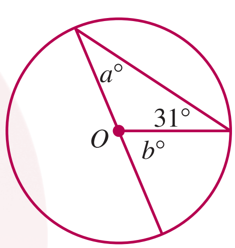


m



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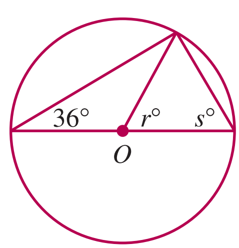

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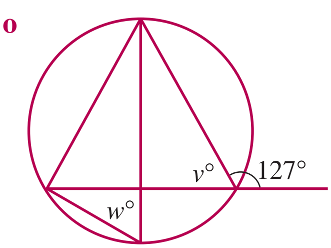
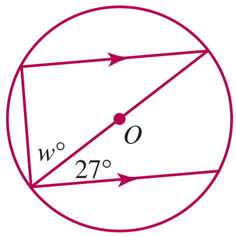


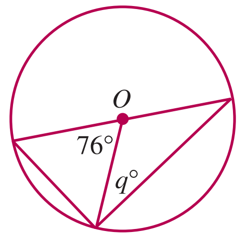
■ Further applications

5 Find the value of all pronumerals.

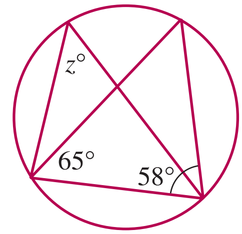
a



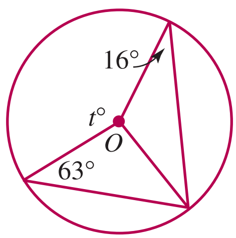
b



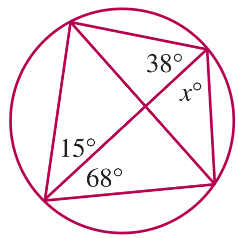
c



d



e



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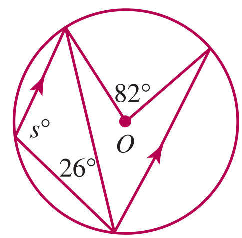
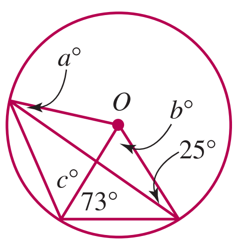
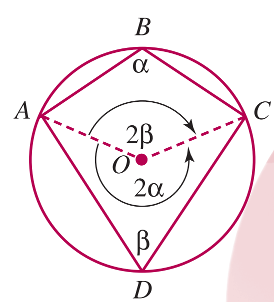


6 Find values for *a*, *b*, *c*.



14.4 Cyclic quadrilaterals

The opposite angles of a cyclic quadrilateral are supplementary.



Data: *O* is the centre, *ABCD* is a cyclic quadrilateral.

Aim: To prove that $\angle ABC + \angle ADC = 180°$.

Construction: Join *OA* and *OC*.

Proof: Let $\angle ABC = \alpha$ and $\angle ADC = \beta$.

- $\angle AOC$ (reflex) $= 2\alpha$ (angle at centre is twice angle at circumference)
- $\angle AOC$ (obtuse) $= 2\beta$ (angle at centre is twice angle at circumference)

Now, $2\alpha + 2\beta = 360°$ (angles at a point)

$\therefore\ \alpha + \beta = 180°$

$\therefore\ \angle ABC + \angle ADC = 180°$

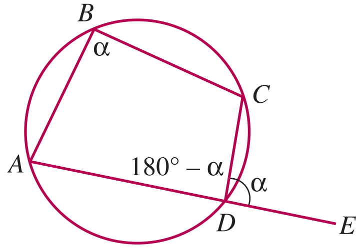
That is, the opposite angles of a cyclic quadrilateral are supplementary.

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Data: *ABCD* is a cyclic quadrilateral. *AD* is produced to *E*.
Aim: To prove that $\angle CDE = \angle ABC$.
Proof: Let $\angle ABC = \alpha$.

- $\angle ADC = 180° - \alpha$ (opposite angles of a cyclic quadrilateral are supplementary)
- $\angle CDE = 180° - (180° - \alpha)$ (adjacent $\angle$s on a straight line)
  $= \alpha$

$\therefore \angle CDE = \angle ABC$

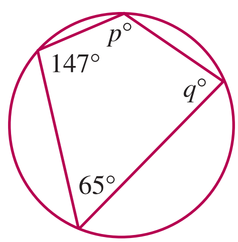


That is, the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

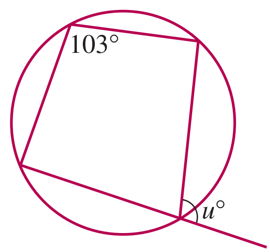
Example 1

Find the value of each pronumeral, giving reasons.

a



b



Solutions

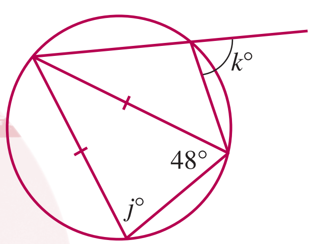
a • $p + 65 = 180$ (opposite angles of a cyclic quadrilateral are supplementary)
$\therefore p = 115$
• $q + 147 = 180$ (opposite angles of a cyclic quadrilateral are supplementary)
$\therefore q = 33$

b $u = 103$ (exterior angle of a cyclic quadrilateral is equal to interior opposite angle).

Example 2

Find the value of all pronumerals, giving reasons.

a



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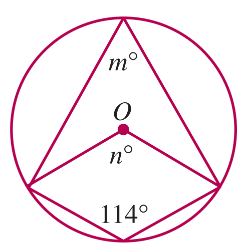


Solutions

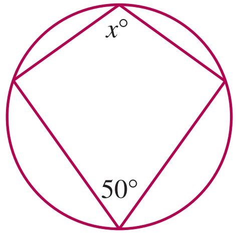
a • $j = 48$ (base angles of an isosceles Δ)
• $k = 48$ (exterior angle of a cyclic quadrilateral is equal to the interior opposite angle)

b • $m = 66$ (opposite angles of a cyclic quadrilateral are supplementary)
• $n = 132$ (angle at centre is twice the angle at the circumference)

Exercise 14.4

1 Find the value of each pronumeral.

a



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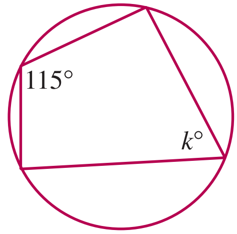

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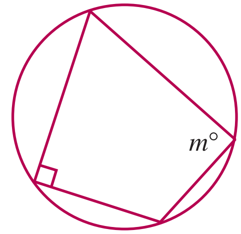
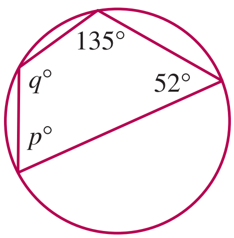
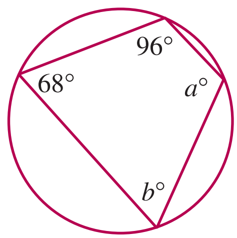


d



e



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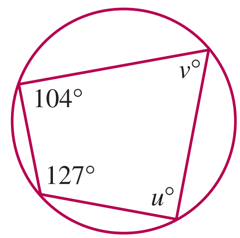
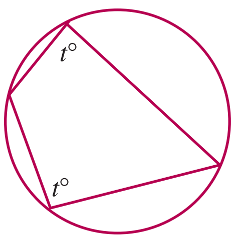


g



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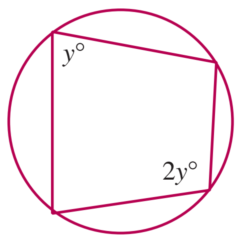
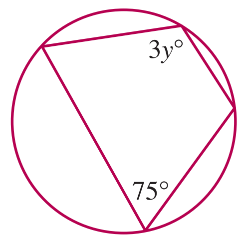
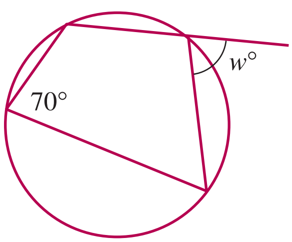


i

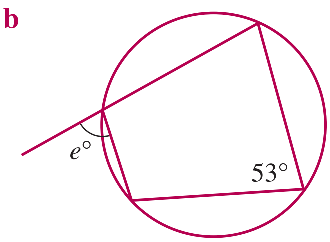


2 Find the value of the pronumeral in each of these.

a



b



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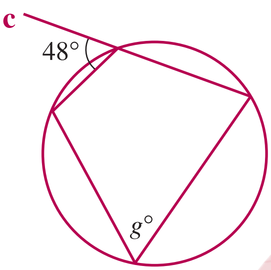

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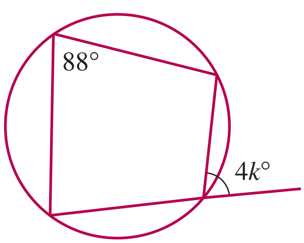

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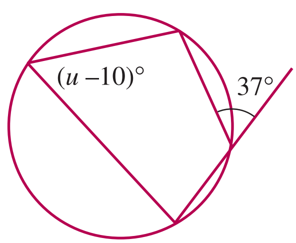

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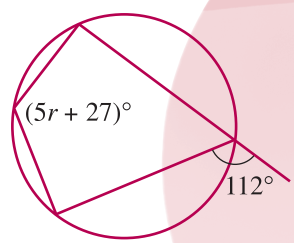
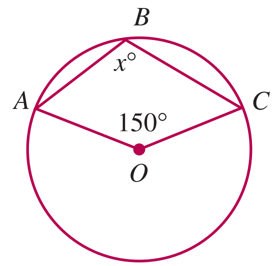


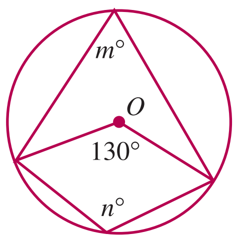
Consolidation

3 **a** Is $OABC$ a cyclic quadrilateral? Explain.
b Are the opposite angles in $OABC$ supplementary?
c Find the reflex $\angle AOC$.
d Find the value of x.



4 Find the value of the pronumerals in each of the following.

a



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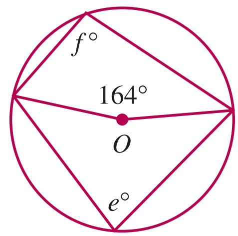

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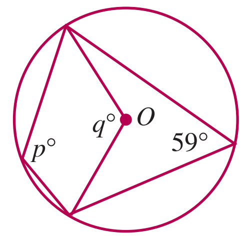

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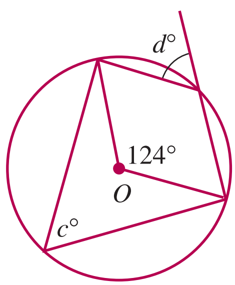

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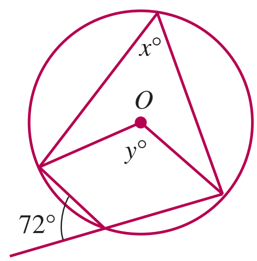

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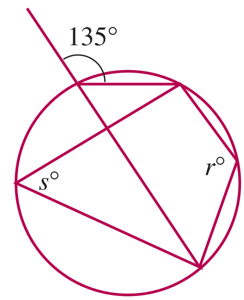
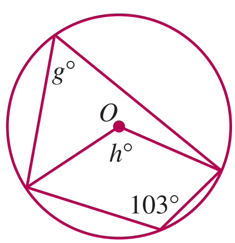


5 Find the value of all pronumerals.

a



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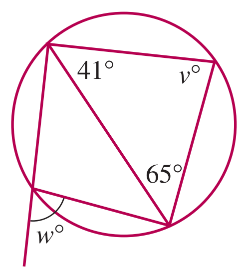

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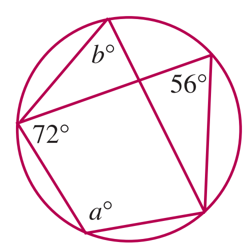

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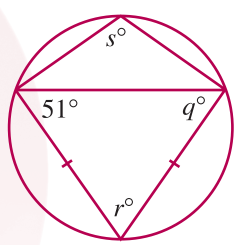

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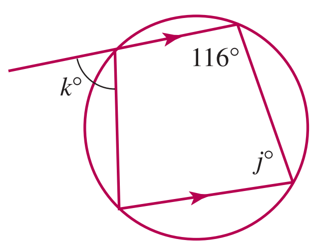

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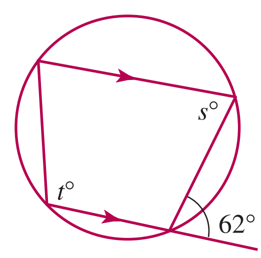

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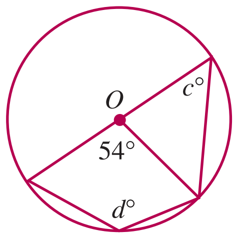

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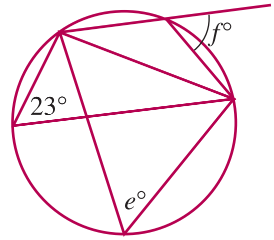

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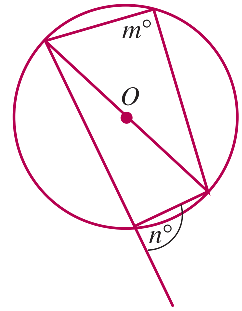
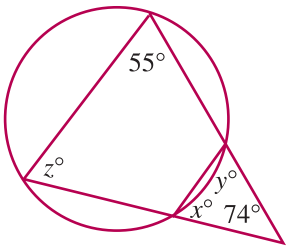
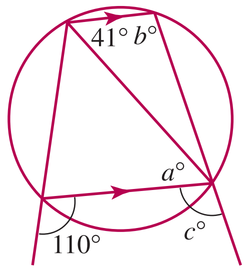


6 Find the value of all pronumerals in each of the following.

a



b



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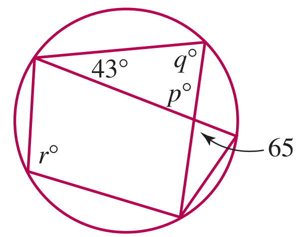
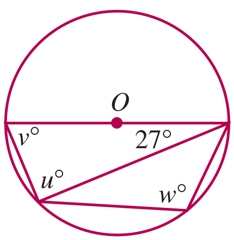
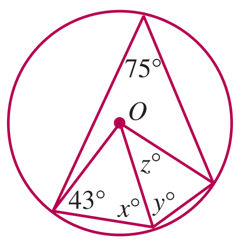


d



e



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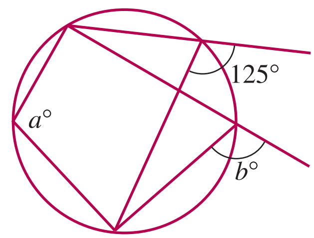
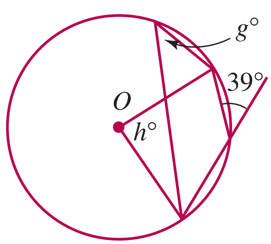


g



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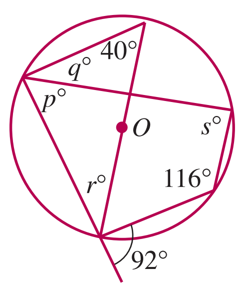
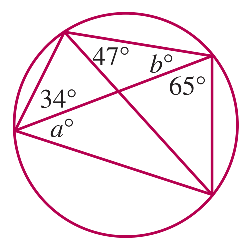


i



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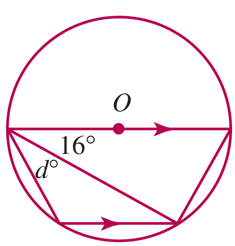

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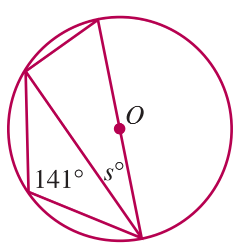

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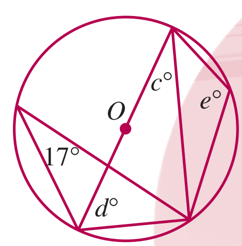

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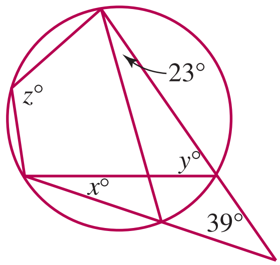

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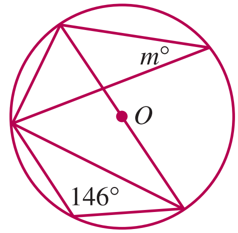
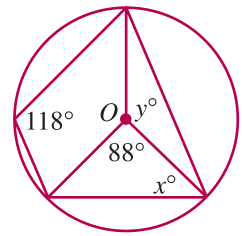
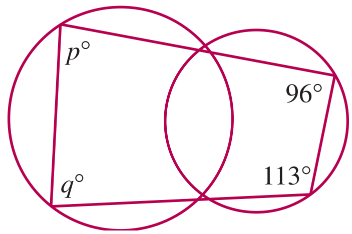


o

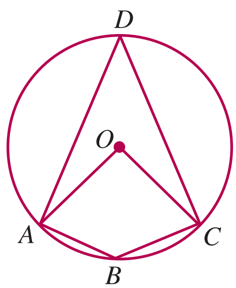


Further applications

7 Find values for p and q.

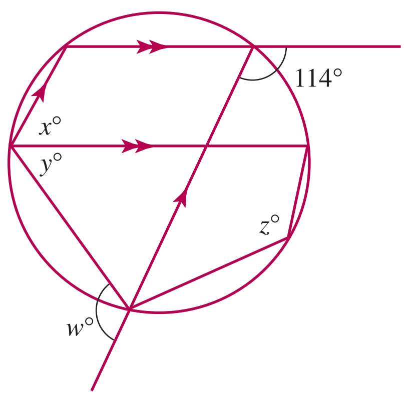


8



$\angle ABC = 2\angle AOC$. Find $\angle ADC$.

9 Find values for w, x, y, z.

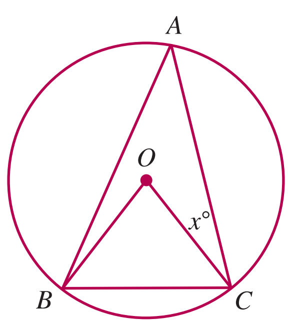


TRY THIS Angle tangle

O is the centre of this circle.

$\angle ABO = 5°$ and $\angle OCB = 40°$.

What is the value of x?



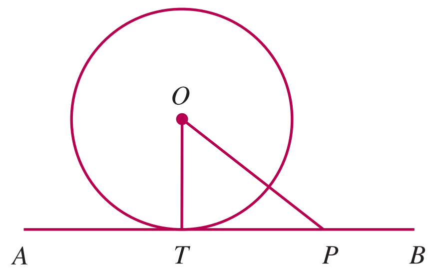
14.5 Tangent properties of circles

The tangent to a circle is perpendicular to the radius at the point of contact.

Data: O is the centre, AB is a tangent to the circle at T.

Aim: To prove that $OT \perp AB$.

Proof: Let P be any point on the tangent AB, other than T. Now, since P lies outside the circle, OP must be greater than the length of the radius OT, for all positions of P on AB.
That is, the shortest distance between O and the tangent AB is OT. However, the shortest distance between a point and a line is the perpendicular distance. Therefore, $OT \perp AB$.



That is, the tangent to a circle is perpendicular to the radius at the point of contact.

The two tangents drawn to a circle from an external point are equal in length.

Data: O is the centre. PA and PB are tangents.

Aim: To prove that $PA = PB$.

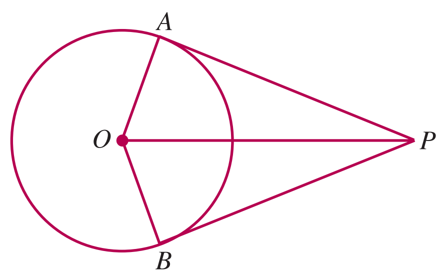
Construction: Join OA, OB, OP.

Proof: In Δs OAP and OBP

- $\angle OAP = \angle OBP = 90°$ (tangents PA, PB are perpendicular to radii OA, OB)
- OP is a common side
- $OA = OB$ (equal radii)

$\therefore \Delta OAP \equiv \Delta OBP$ (RHS)

$\therefore PA = PB$ (matching sides of congruent Δs)



That is, the two tangents drawn to a circle from an external point are equal in length.

The angle between a tangent and a chord drawn to the point of contact is equal to the angle in the alternate segment.

Data: TP is a tangent, O is the centre.

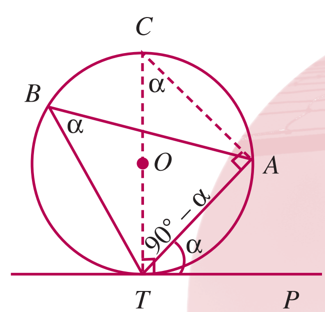
Aim: To prove that $\angle ATP = \angle ABT$.

Construction: Draw in the diameter CT and the chord AC.

Proof: Let $\angle ATP = \alpha$

- $\angle CTP = 90°$ (tangent is perpendicular to a radius)
- $\angle CTA = 90° - \alpha$ (adjacent angles in a right angle)
- $\angle CAT = 90°$ (angle in a semi-circle)
- $\angle ACT = \alpha$ (angle sum of ΔACT is 180°)
- $\angle ABT = \alpha$ (angles at circumference standing on same arc are equal)

$\therefore \angle ATP = \angle ABT$



That is, the angle between a tangent and a chord drawn to the point of contact is equal to the angle in the alternate segment.

When two circles touch, their centres and the point of contact are collinear.

Data: Two circles with centres O, C touch at T.
Aim: To prove that O, T, C are collinear points.
Construction: Construct the common tangent AB at T.
Proof: (for figure 1):

- $\angle ATO = 90°$ (tangent is perpendicular to a radius)
- $\angle ATC = 90°$ (tangent is perpendicular to a radius)
- $\angle ATO + \angle ATC = 90° + 90°$
  $= 180°$

$\therefore$ O, T, C are collinear.

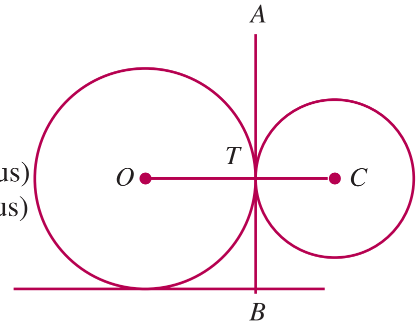


figure 1

Proof: (for figure 2):

- $\angle ATO = 90°$ (tangent is perpendicular to a radius)
- $\angle ATC = 90°$ (tangent is perpendicular to a radius)
- $\angle ATO - \angle ATC = 90° - 90°$
  $= 0°$

$\therefore$ O, T, C are collinear.

That is, when two circles touch (internally or externally), their centres and the point of contact are collinear.

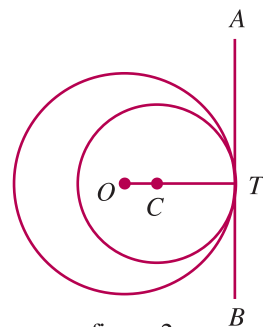
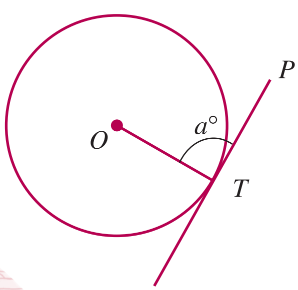


figure 2

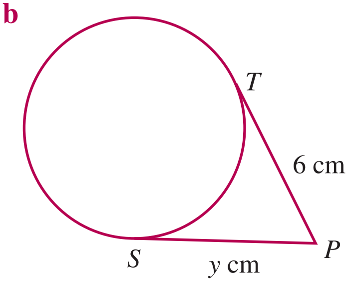
Example 1

Find the value of each pronumeral, giving reasons. PT and PS are tangents.

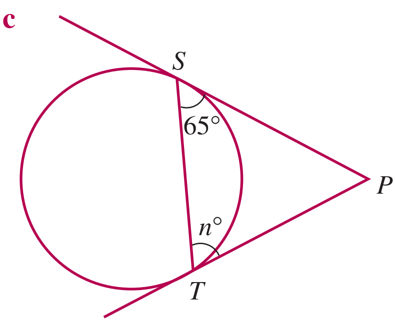
a



b



c



Solutions

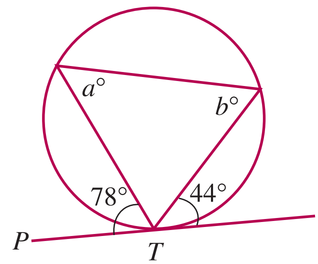
a $a = 90$ (tangent is perpendicular to a radius)

b $y = 6$ (tangents drawn from an external point are equal)

c
- $PS = PT$ (tangents drawn from an external point are equal)
- $n = 65$ (base angles of an isosceles triangle)

Example 2

PT is a tangent. Find values for a and b, giving reasons.

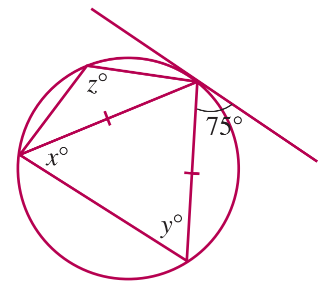


Solutions

- $a = 44$ (angle in the alternate segment)
- $b = 78$ (angle in the alternate segment)

Example 3

Find values for x, y, z, giving reasons.



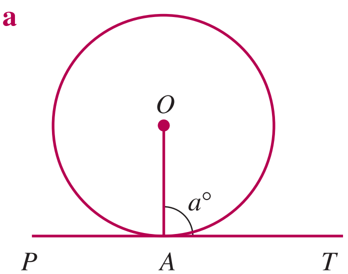
Solutions

- $x = 75$ (angle in the alternate segment)
- $y = 75$ (base angles of an isosceles triangle)
- $z = 105$ (opposite angles of a cyclic quadrilateral are supplementary)

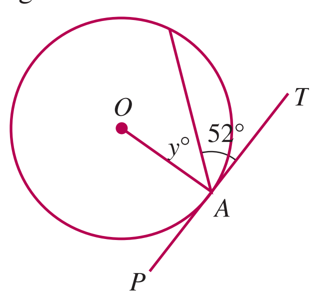
Exercise 14.5

1 In each of the following, *PT* is a tangent at *A*. Find the value of each pronumeral.

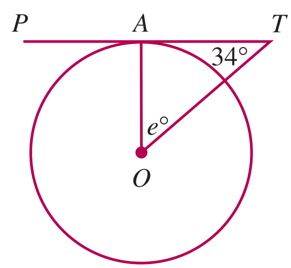
a



b

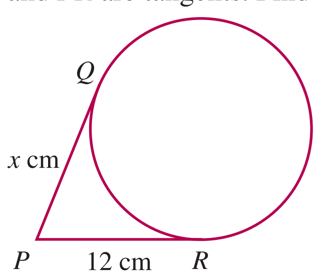


c

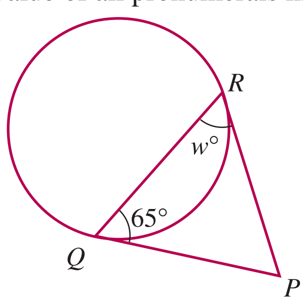


2 *PQ* and *PR* are tangents. Find the value of all pronumerals in each of these.

a



b



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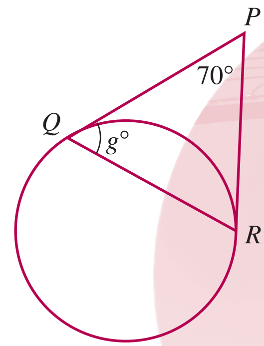
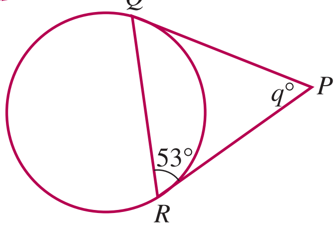
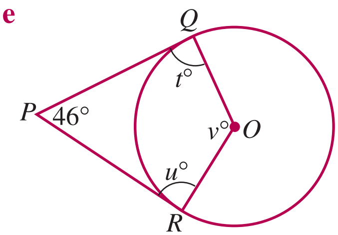


d



e



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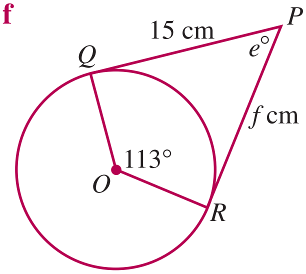
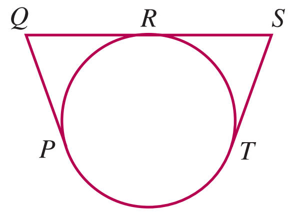


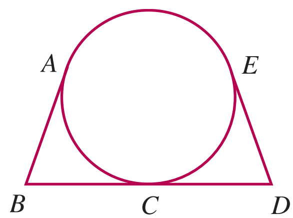
Consolidation

3 a



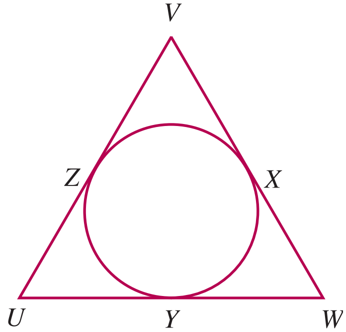
QP, QR, SR, ST are tangents. $QP = 7$ cm and $QS = 18$ cm. Find ST.

b



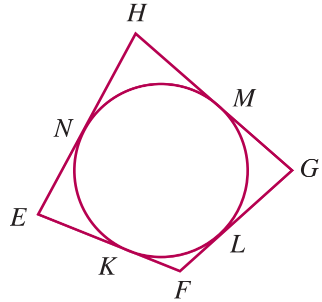
BA, BC, DC, DE are tangents. $BA = 6$ cm and $DE = 8$ cm. Find BD.

c



UV, VW, UW are tangents. $VZ = 5$ cm, $UY = 8$ cm and $VW = 14$ cm. Find the perimeter of ΔUVW.

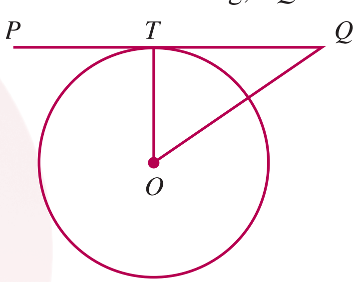
d



EF, FG, GH, EH are tangents. $EK = 7$ cm, $FK = 5$ cm, $FG = 8$ cm and the perimeter of $EFGH$ is 48 cm. Find HM.

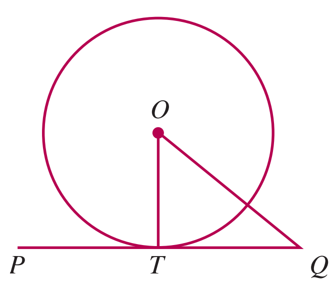
4 In each of the following, PQ is a tangent at T.

a



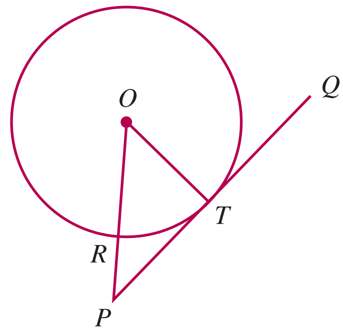
$OT = 12$ cm and $TQ = 35$ cm. Find OQ.

b



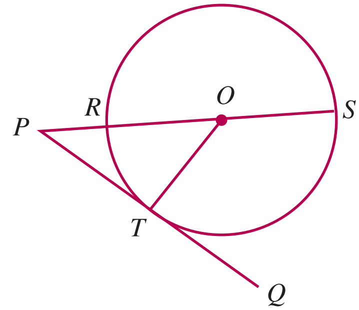
$OQ = 26$ cm and $OT = 10$ cm. Find TQ.

c



$PT = 36$ cm and $OP = 39$ cm. Find PR.

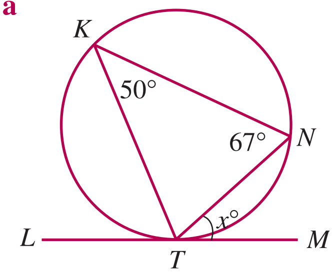
d



$RS = 28$ cm and $PT = 48$ cm. Find PS.

5 LM is a tangent at T in each of the following. Find the value of all pronumerals.

a



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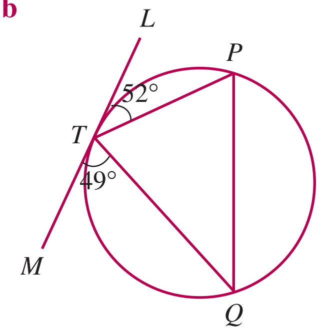

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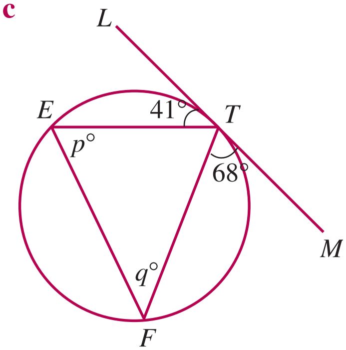

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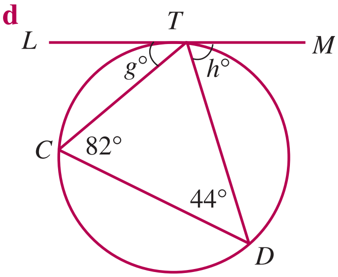

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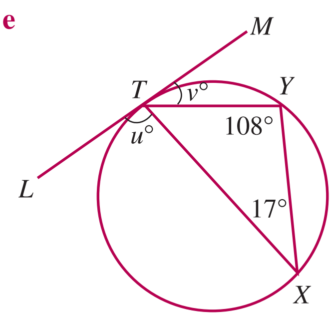

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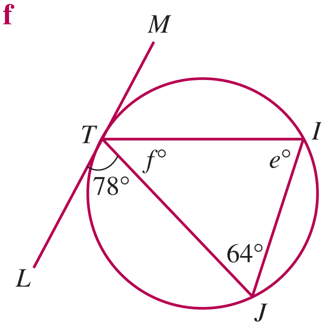

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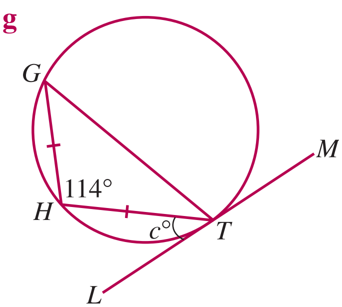

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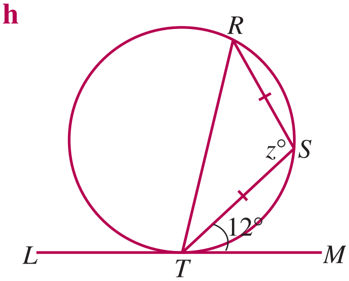

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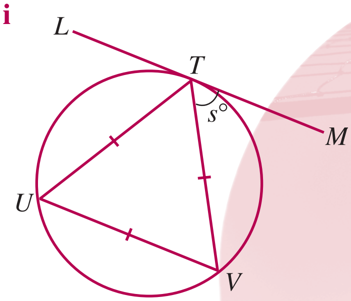
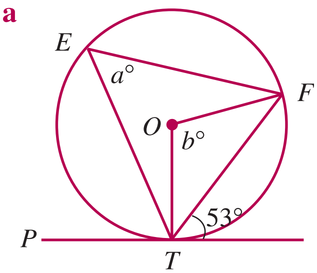
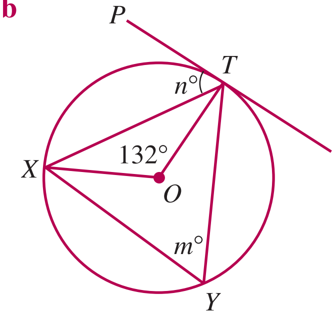


6 *PT* is a tangent in each of the following. Find the value of all pronumerals.

a



b



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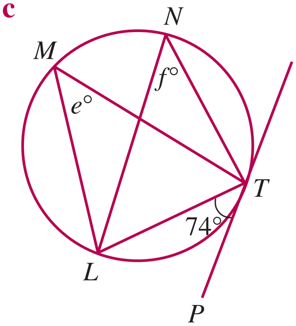

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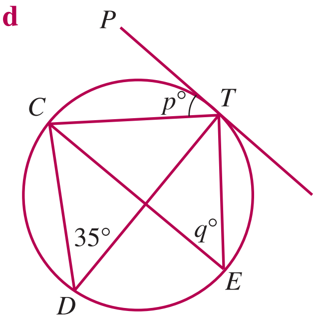

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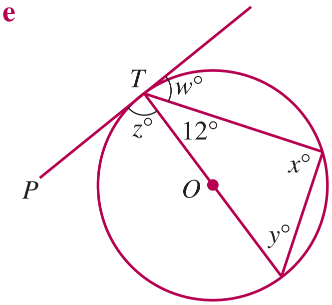

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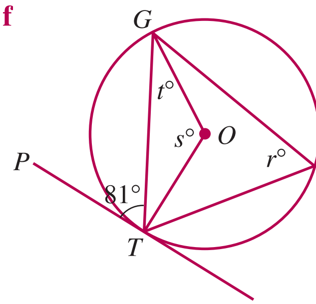

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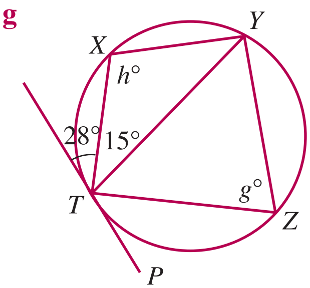

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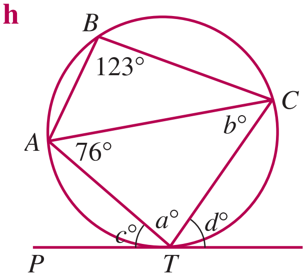

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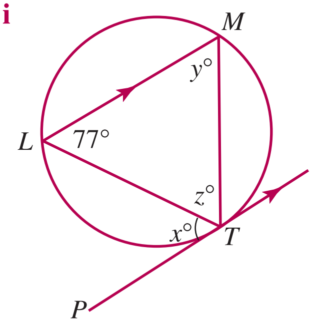
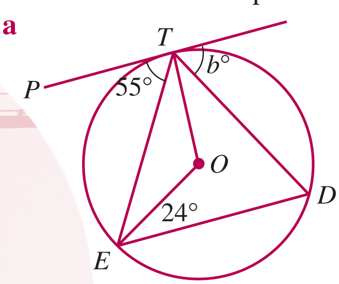
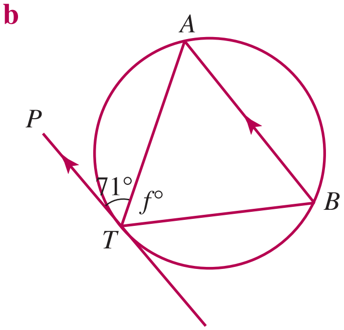


7 Find the value of the pronumeral in each of these. *PT*, *PS*, *QR* are tangents.

a



b



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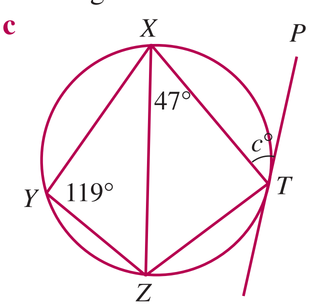

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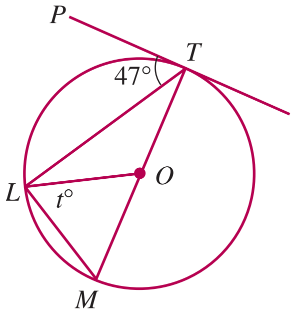

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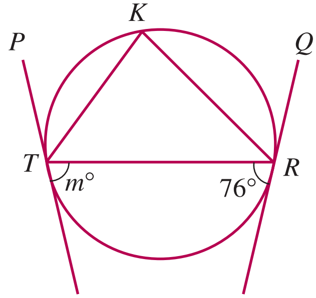

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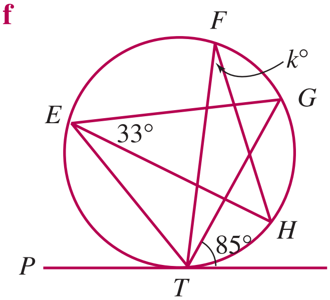

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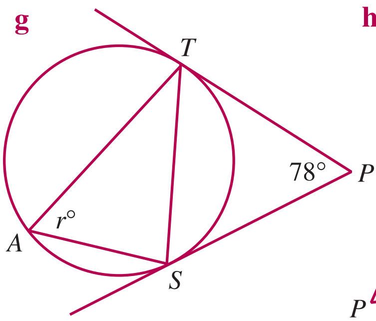

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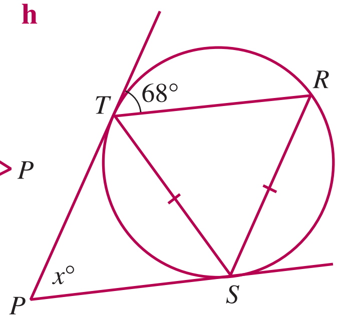

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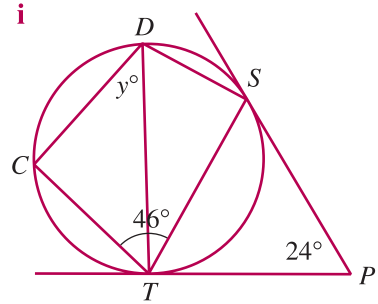

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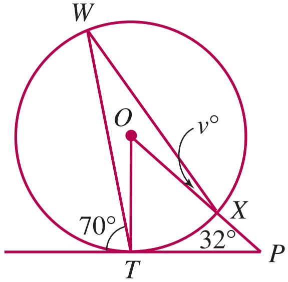

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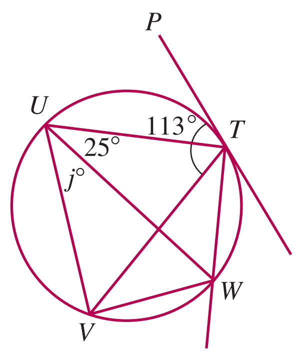

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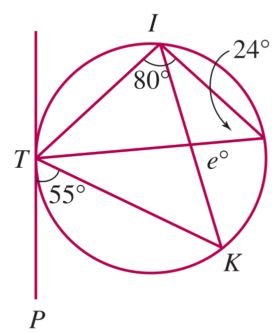

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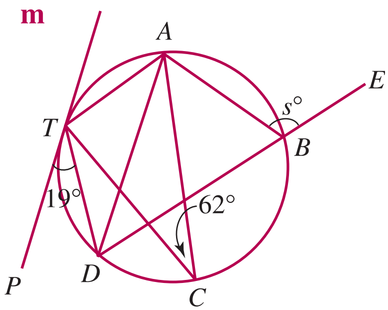

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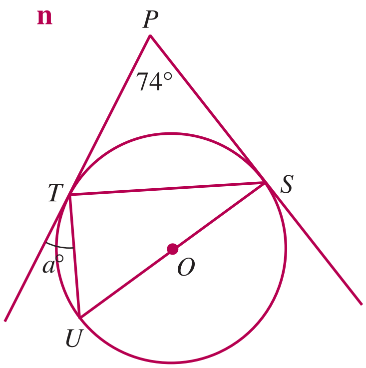

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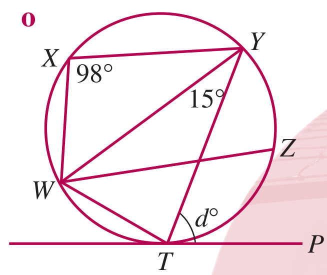
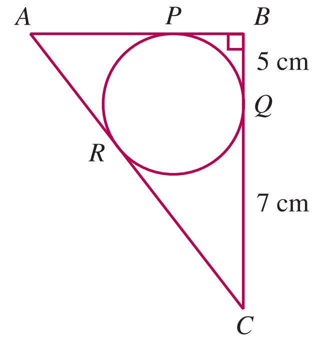
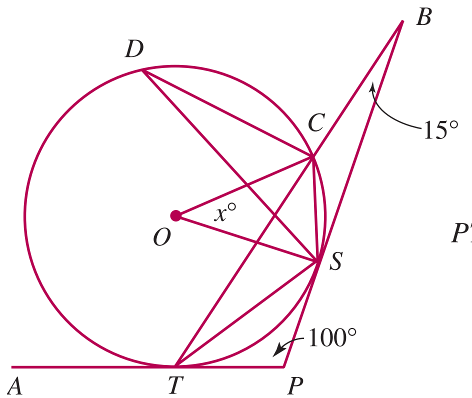


■ Further applications

8 A circle is inscribed in a right-angled triangle as shown. $BQ = 5$ cm and $QC = 7$ cm. Find the length of AC.

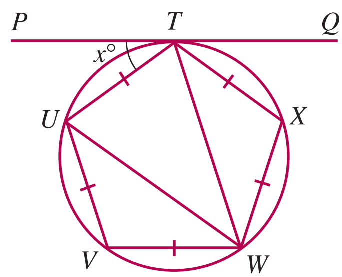


9



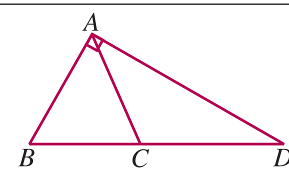
PT and PS are tangents. Find the value of x.

10 PQ is a tangent at T. Find the value of x.



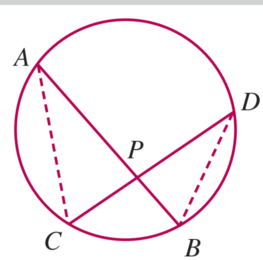
TRY THIS Length of a median

ABD is a right-angled triangle and C is the midpoint of BD. If $AB = 6$ cm and $AD = 8$ cm, find the exact length of AC.



14.6 Further circle properties

The product of the intercepts of two intersecting chords of a circle are equal.



Data: Two chords AB and CD intersect at P.
Aim: To prove that $AP \cdot PB = CP \cdot PD$
Construction: Join AC and BD.
Proof: In Δs APC and DPB

- $\angle CAP = \angle BDP$ (angles at circumference standing on same arc are equal)
- $\angle APC = \angle DPB$ (vertically opposite angles)

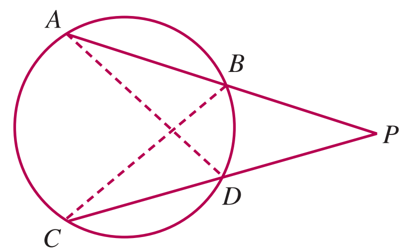
$\therefore \Delta APC \mathbin{|||} \Delta DPB$ (equiangular)

$\therefore \dfrac{AP}{PD} = \dfrac{CP}{PB}$ (matching sides of similar Δs are in proportion)

$\therefore AP \cdot PB = CP \cdot PD$

That is, the products of the intercepts of two intersecting chords of a circle are equal.

The products of the intercepts of two intersecting secants to a circle from an external point are equal.



Data: The secants PA and PC cut the circle at B and D respectively.
Aim: To prove that $PA \,.\, PB = PC \,.\, PD$
Construction: Join AD and BC.
Proof: In Δs ADP and CBP

- $\angle BAD = \angle BCD$ (angles at circumference standing on same arc are equal)
- $\angle APD = \angle CPB$ (common angle)

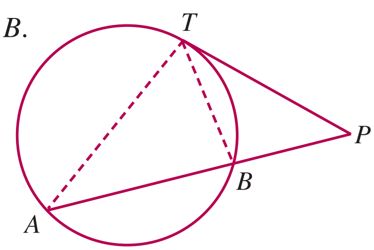
$\therefore \Delta ADP \mathbin{|||} \Delta CBP$ (equiangular)

$\therefore \dfrac{PA}{PC} = \dfrac{PD}{PB}$ (matching sides of similar Δs are in proportion)

$\therefore PA \,.\, PB = PC \,.\, PD$

That is, the products of the intercepts of two intersecting secants to a circle from an external point are equal.

The square of a tangent to a circle from an external point is equal to the product of the intercepts of a secant from the point.



Data: PT is a tangent, PA is a secant, which cuts the circle at B.
Aim: To prove that $PT^2 = PA \,.\, PB$
Construction: Join AT and BT.
Proof: In Δs BPT and TPA

- $\angle PTB = \angle TAB$ (angle in the alternate segment)
- $\angle TPB = \angle TPA$ (common angle)

$\therefore \Delta BPT \mathbin{|||} \Delta TPA$ (equiangular)

$\therefore \dfrac{PT}{PA} = \dfrac{PB}{PT}$ (matching sides of similar Δs are in proportion)

$\therefore PT^2 = PA \,.\, PB$

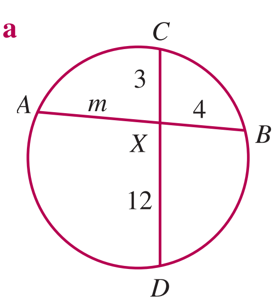
That is, the square of a tangent to a circle from an external point is equal to the product of the intercepts of any secants from the point.

Example 1

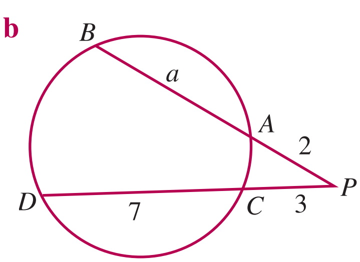
EG +S

Find the value of the pronumeral in each of the following, giving reasons. PT is a tangent in (c).

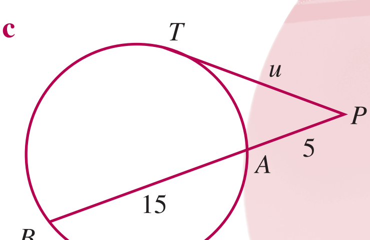
a



b



c



Solutions

a $AX \,.\, XB = CX \,.\, XD$ (products of intercepts of intersecting chords)

$\therefore\ 4 \times m = 3 \times 12$

$4m = 36$

$\therefore\ m = 9$

b $PA \,.\, PB = PC \,.\, PD$ (products of intercepts of intersecting secants)

$2(a + 2) = 3(3 + 7)$

$2a + 4 = 30$

$2a = 26$

$\therefore\ a = 13$

c $PT^2 = PA \,.\, PB$ (square of tangent is equal to the product of the intercepts of the secant)

$u^2 = 5(5 + 15)$

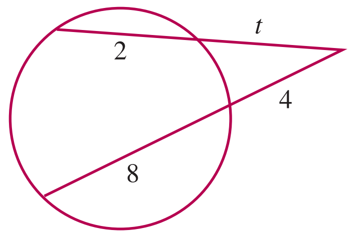
$u^2 = 100$

$\therefore\ u = 10\ (u > 0)$

EG +S

Example 2

Find the value of t.



Solution

$t(t + 2) = 4(4 + 8)$

$t^2 + 2t = 48$

$t^2 + 2t - 48 = 0$

$(t + 8)(t - 6) = 0$

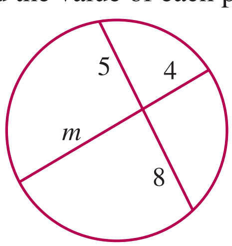
$\therefore\ t = -8, 6$

But $t > 0$, $\therefore\ t = 6$.

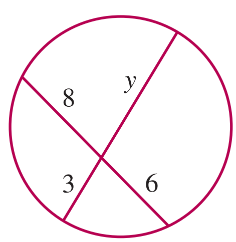
Exercise 14.6

1 Find the value of each pronumeral.

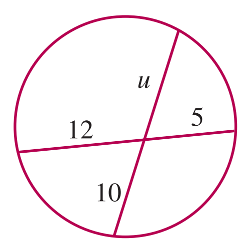
a



b

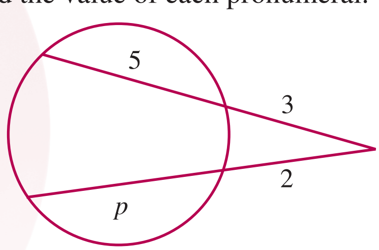


c



2 Find the value of each pronumeral.

a



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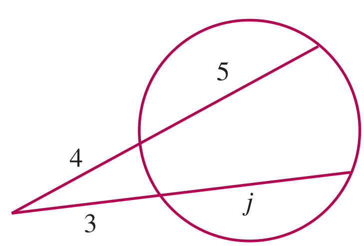

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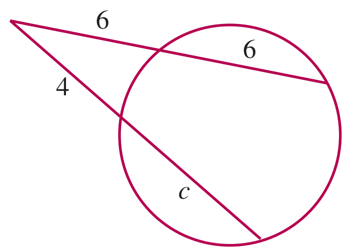
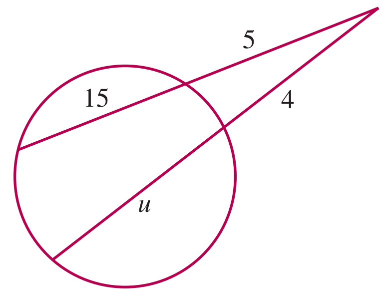
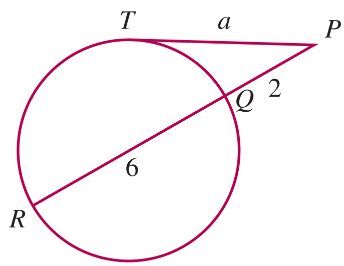


d

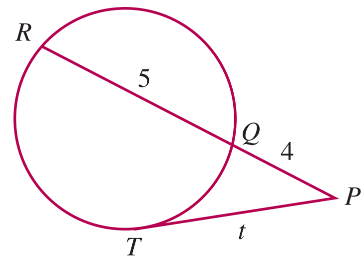


3 PT is a tangent in each diagram. Find the value of each pronumeral.

a



b



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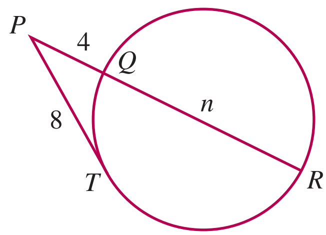

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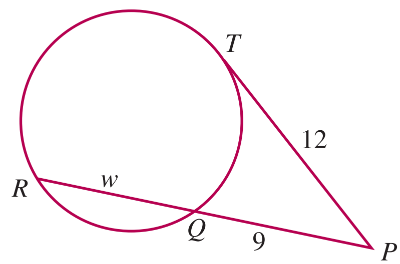
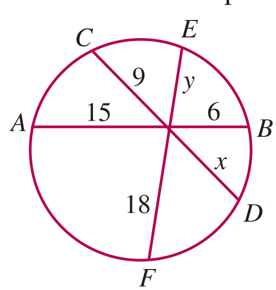


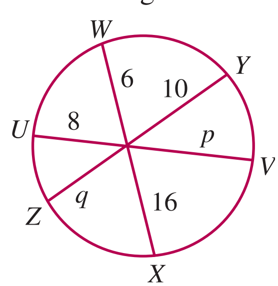
■ Consolidation

4 Find the value of each pronumeral. PT is a tangent in **d** and **e**.

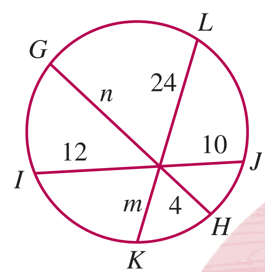
a



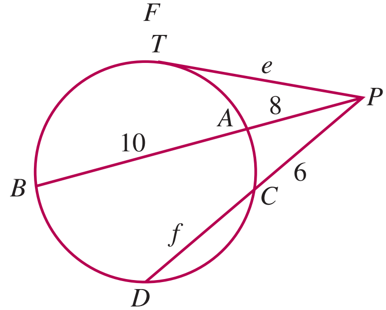
b



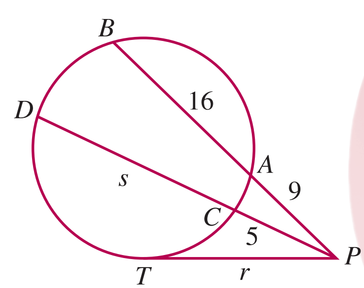
c



d



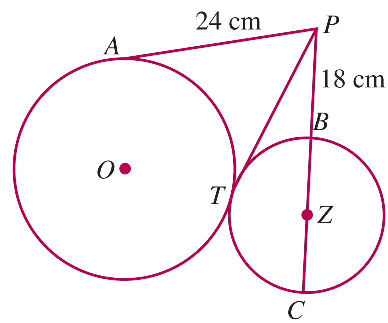
e



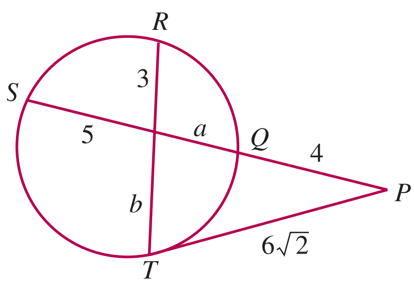
5 O and Z are centres, PA is a tangent to the circle with centre O. PT is a common tangent to both circles. $AP = 24$ cm and $PB = 18$ cm.

a How long is PT? Why?

b Find the length of BC.



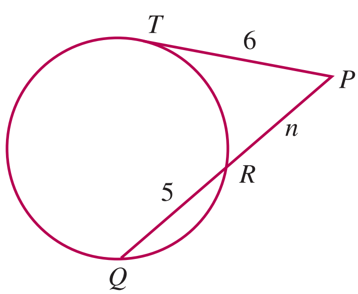
6



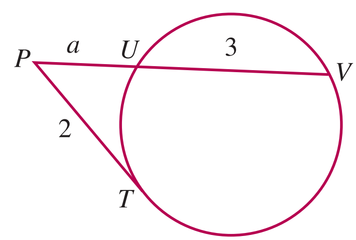
PT is a tangent. Find values for a and b.

7 Find the value of each pronumeral. PT is a tangent in **a** and **b**.

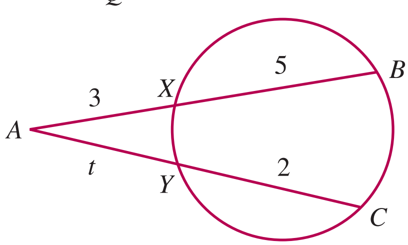
a



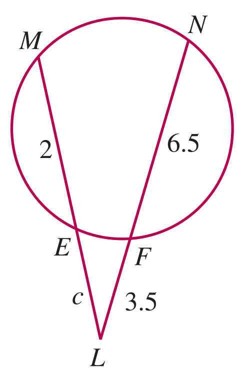
b



c

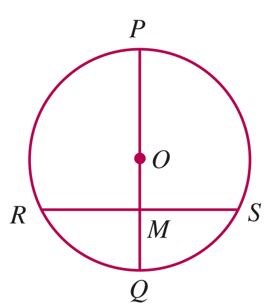


d



■ Further applications

8 In the diagram, $PQ \perp RS$, $RS = 16$ cm and $MQ = 4$ cm. Find the length of the radius.



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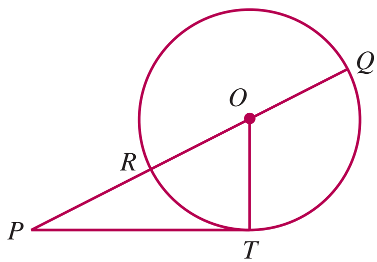


PT is a tangent, $PR = 4$ cm and $PT = 8$ cm.
Find the length of the radius:

a by using Pythagoras' theorem
b by using a circle property

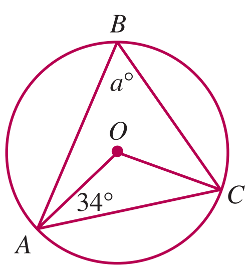
14.7 Deductive proofs involving circle properties

The emphasis in this exercise is on the justification of each step in an argument. Full reasons must therefore be given in all questions. O is the centre of the circle, unless otherwise stated.

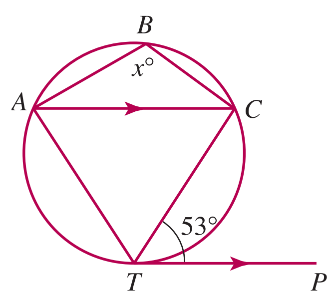
Example 1

Find the value of the pronumeral in each of the following, giving reasons.

a



b



PT is a tangent.

Solutions

a $OA = OC$ (equal radii)

$\therefore \Delta OAC$ is isosceles

$\angle OCA = 34°$ (base $\angle$s of an isosceles Δ, $OA = OC$)

- $\angle AOC = 112°$ ($\angle$sum of ΔAOC)
- $\angle ABC = 56°$ ($\angle$ at centre is twice $\angle$ at circumference)

$\therefore a = 56$

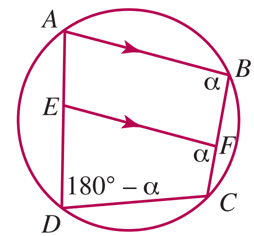
b

- $\angle TAC = 53°$ ($\angle$ in the alternate segment)
- $\angle ACT = 53°$ (alternate $\angle$s, $AC \parallel TP$)
- $\angle ATC = 74°$ ($\angle$ sum of ΔACT)
- $\angle ABT = 106°$ (opposite $\angle$s of a cyclic quad. are supplementary)

$\therefore x = 106$

Example 2

ABCD is a cyclic quadrilateral. *E* and *F* are points on *AD* and *BC* respectively such that $AB \parallel EF$. Prove that *C*, *D*, *E*, *F* are concyclic points.



Solution

Let $\angle ABC = \alpha$

- $\angle ADC = 180° - \alpha$ (opposite $\angle$s of a cyclic quadrilateral are supplementary)
- $\angle EFC = \alpha$ (corresponding $\angle$s, $AB \parallel EF$)

Now, $\angle ADC + \angle EFC = 180° - \alpha + \alpha$

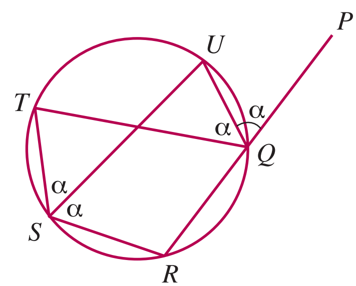
$= 180°$

$\therefore$ *CDEF* is a cyclic quadrilateral

$\therefore$ *C*, *D*, *E*, *F* are concyclic points.

Example 3

US bisects $\angle RST$. Prove that *UQ* bisects $\angle PQT$.



Solution

Let $\angle TSU = \alpha$

- $\angle USR = \alpha$ (*US* bisects $\angle RST$)
- $\angle TQU = \alpha$ ($\angle$s at circumference standing on same arc are equal)
- $\angle PQU = \alpha$ (external $\angle$ of cyclic quadrilateral *QRSU* is equal to interior opposite $\angle$)

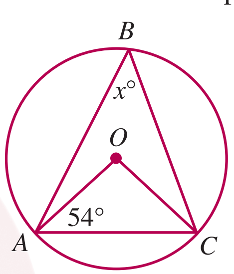
$\therefore \angle TQU = \angle PQU$

$\therefore$ *UQ* bisects $\angle PQT$

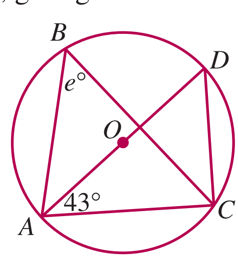
Exercise 14.7

1 Find the value of each pronumeral, giving reasons. *PT* and *PS* are tangents.

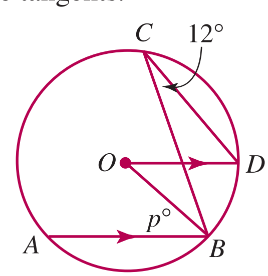
a



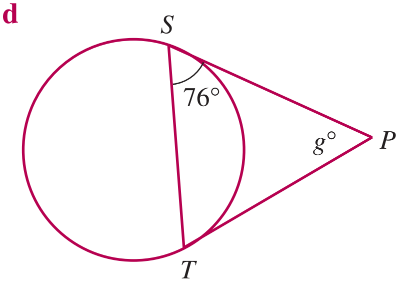
b



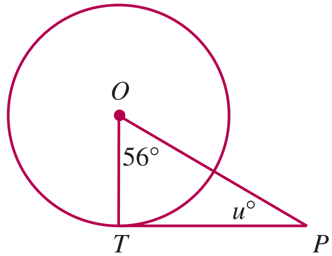
c



d



e



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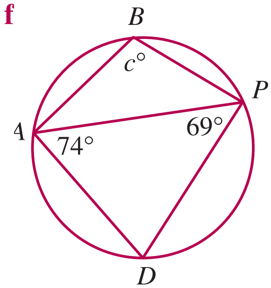
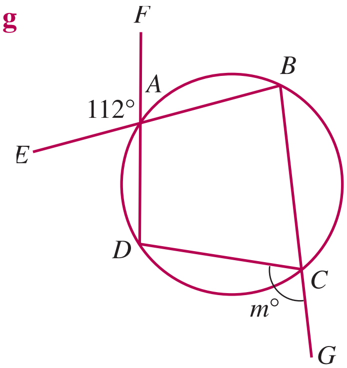
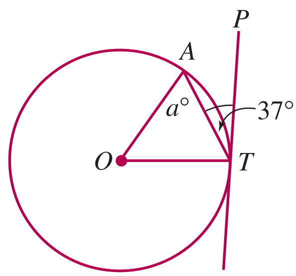


g



h



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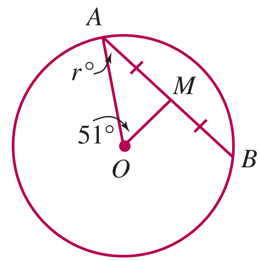

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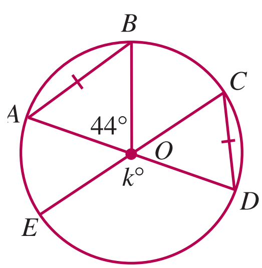

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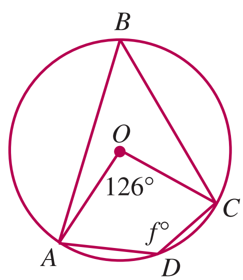

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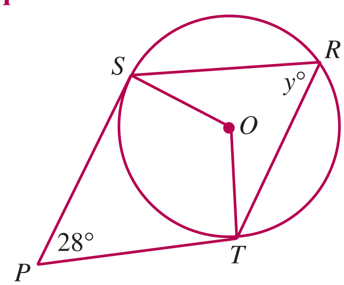

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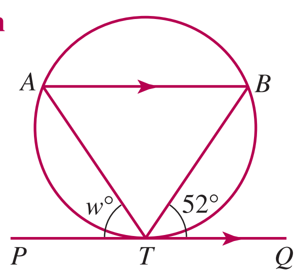

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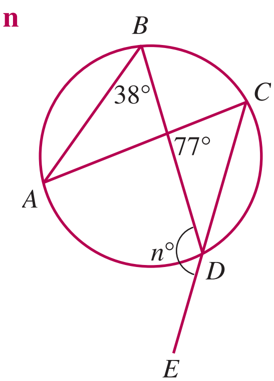

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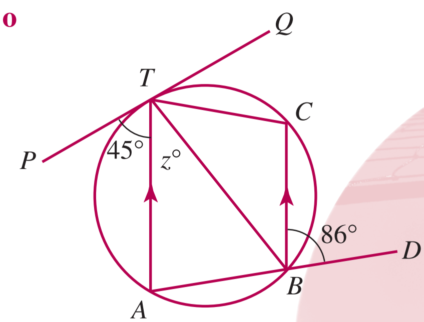

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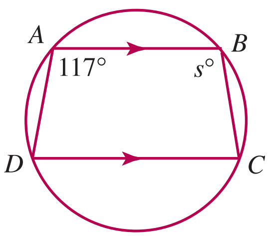

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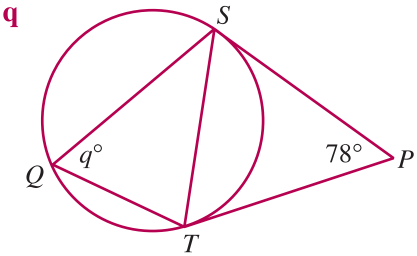

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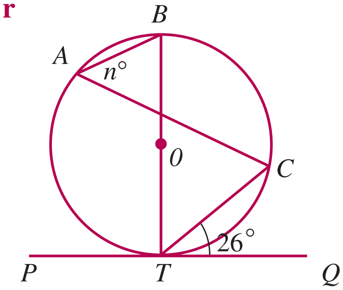

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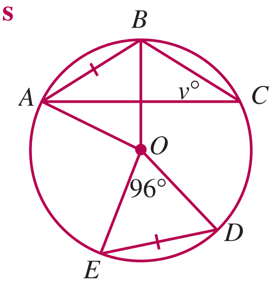

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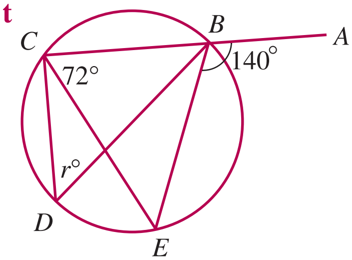

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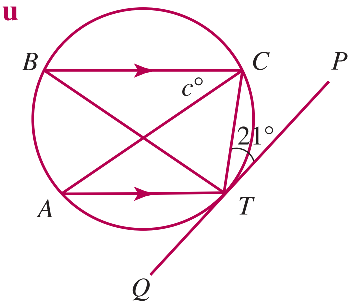

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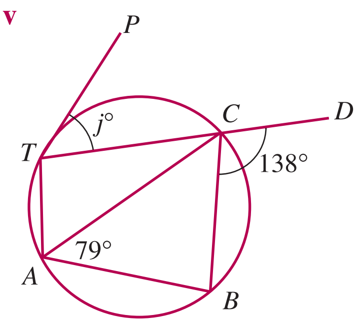

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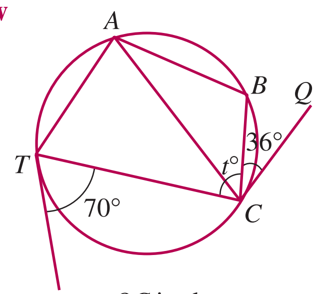
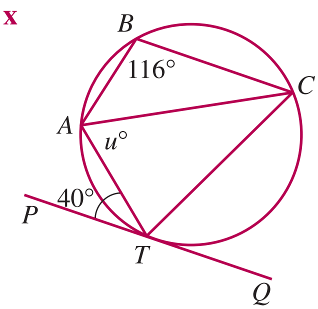


QC is also a tangent

x

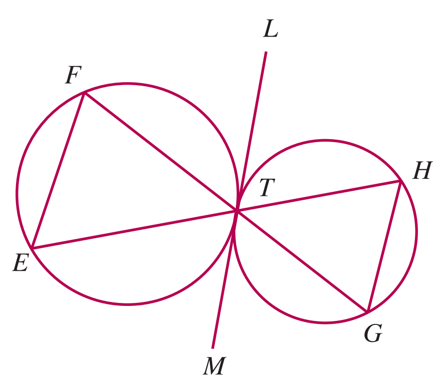


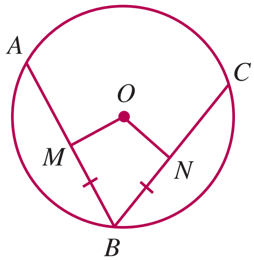
■ Consolidation

2 *LM* is a common tangent. *LM*, *FG* and *EH* meet at *T*, the point of contact of the circles.

a Explain why $\angle FTL = \angle MTG$.

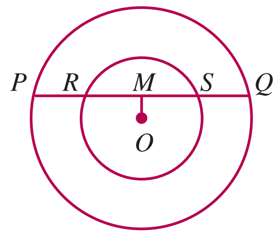
b Hence, prove that $EF \parallel GH$.

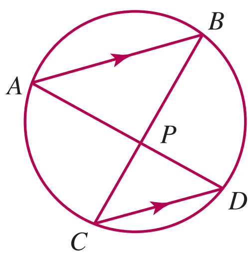


3 

$OM \perp AB$, $ON \perp BC$ and $BM = BN$.
Prove that $OM = ON$.

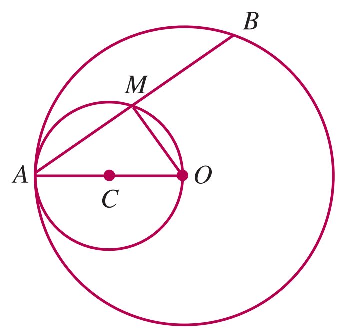
4 The circles shown are concentric and $OM \perp PQ$.
Prove that $PR = SQ$.

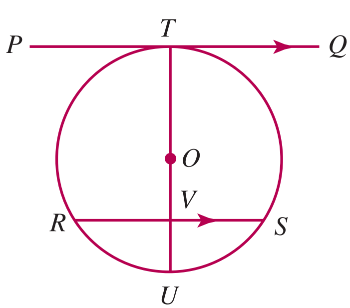


5 

In the diagram shown, $AB \parallel CD$.
Prove that ΔAPB is isosceles.

6 Two circles with centres O and C touch internally at A.
Prove that M is the midpoint of AB.



7 

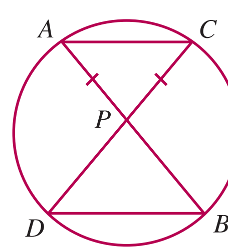
In the diagram shown, PQ is a tangent and $PQ \parallel RS$.
Prove that TU bisects RS.

8 Two chords AB and CD meet at P. $PA = PC$.

a Prove that ΔDPB is isosceles.

b Hence, prove that $AB = CD$.

c Prove that $AC \parallel DB$.



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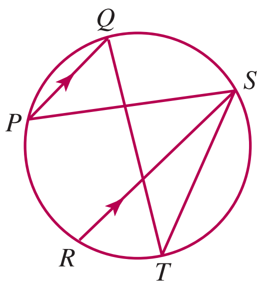
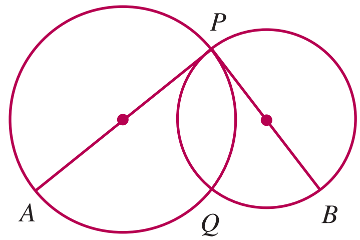
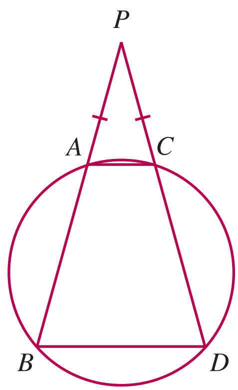


In the diagram, $PQ \parallel RS$.
Prove that $\angle QTS = \angle PSR$.

10 Two circles intersect at P and Q.
PA and PB are diameters.
Prove that A, Q, B are collinear.

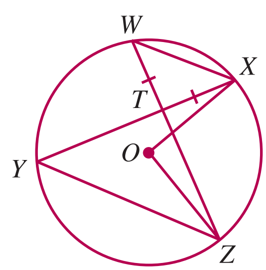


11



From an external point P, two secants are drawn to a circle as shown. $PA = PC$. Prove that $AB = CD$.

12 In the diagram, $TW = TX$ and $\angle XOZ = \angle XTZ$.
Prove that $\angle XOZ = \angle XTZ$.



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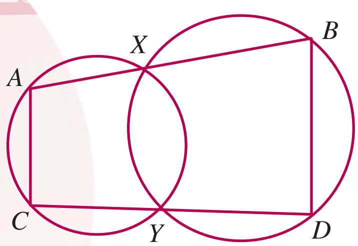
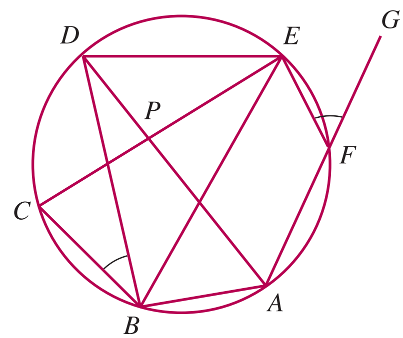


Two circles intersect at X and Y.
AB and CD are straight lines.
Prove that $AC \parallel BD$.

14 In the diagram, $\angle CBD = \angle EFG$.

a Prove that $\angle CBD = \angle EBA$.

b Hence, prove that $PD = PE$.



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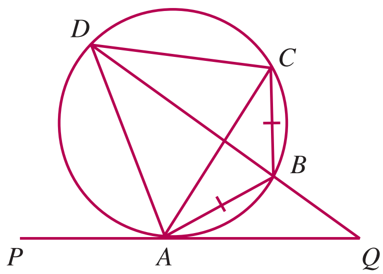
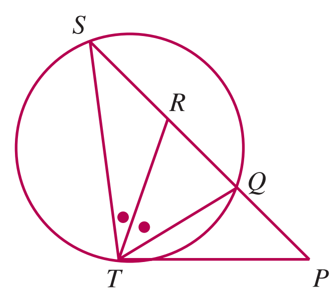


PQ is a tangent and $AB = BC$.

a Prove that *AB* bisects $\angle CAQ$.

b Prove that *DB* bisects $\angle ADC$.

16 *PT* is a tangent and *RT* bisects $\angle STQ$. Prove that $PR = PT$.



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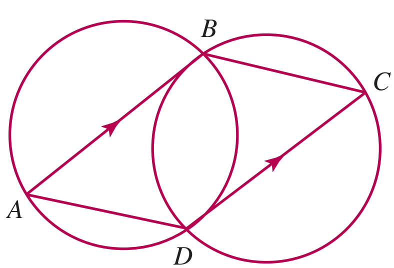
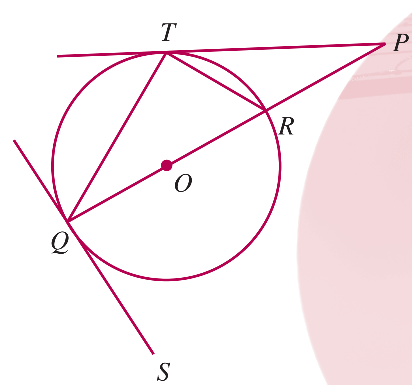


Two circles intersect at *B* and *D*.
AB and *DC* are tangents and $AB \parallel DC$.
Prove that $BC \parallel AD$.

18 *PT* and *SQ* are tangents. Prove that $\angle TQS = \angle PRT$.



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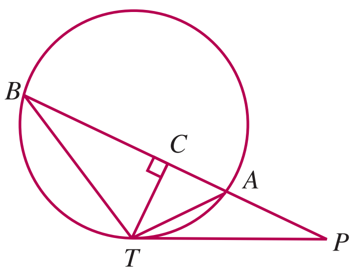


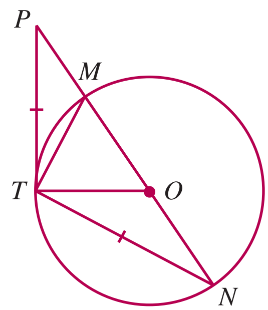
PT is a tangent.
AT bisects $\angle CTP$ and $TC \perp AB$.
Prove that AB is a diameter.

Further applications

20 PT is a tangent and $PT = TN$.

a Prove that $PM = MT$.

b Hence, prove that ΔMOT is equilateral.



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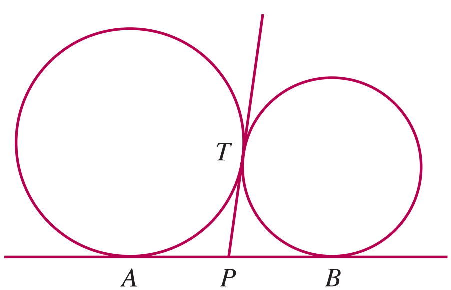
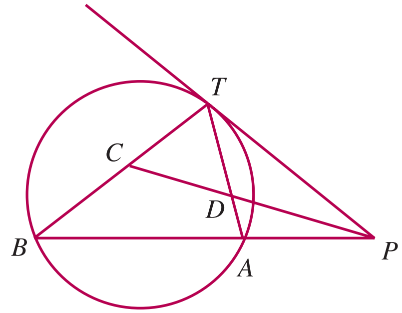


AB is a tangent to both circles and PT is a common tangent. Prove that A, B, T are concyclic points.

22 PT is a tangent and CP bisects $\angle BPT$.
Prove that ΔTCD is isosceles.



AN INTERESTING PROOF OF THE SINE RULE USING CIRCLE GEOMETRY

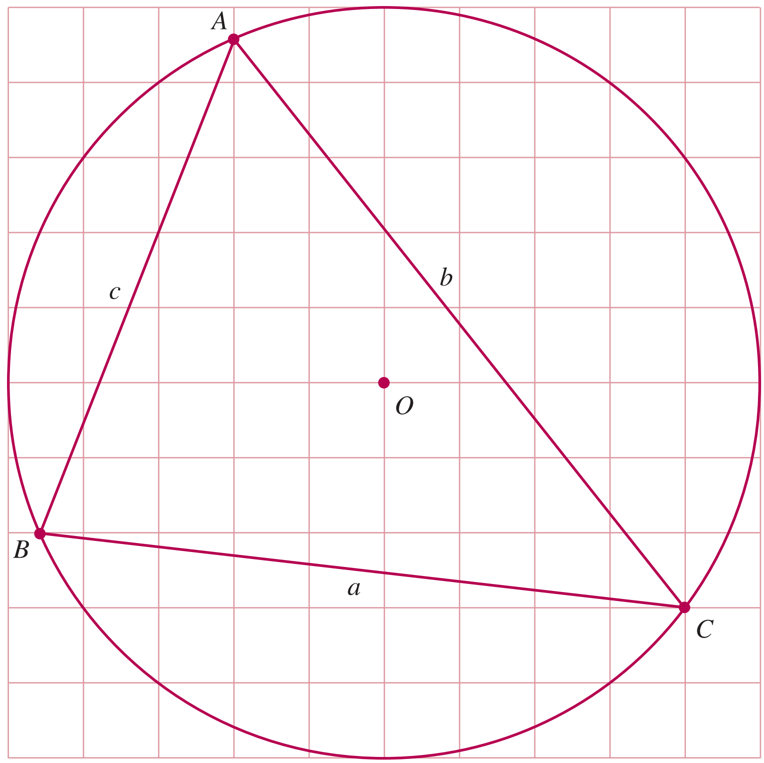


figure 1 $\qquad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Introduction

Circle geometry provides a very useful way to deepen your understanding of important trigonometric results. In the diagram above a triangle *ABC* has been drawn inside a circle centre O. In this focus on working mathematically we will use two circle theorems to prove the sine rule.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

In the process we will discover an interesting link between the sine rule and circle geometry.

1 Copy figure 1 into your book. If you can, construct the diagram using Geometer's Sketchpad. This will enable you to drag the triangle around and get a better idea of why the proof works no matter where A, B and C are on the circle. Instructions for this are set out below.

2 We now need to draw some construction lines. Join B to O and produce the line to meet the circle at D. Now join DC. Label the radius of the circle R. Your figure will now look like this:

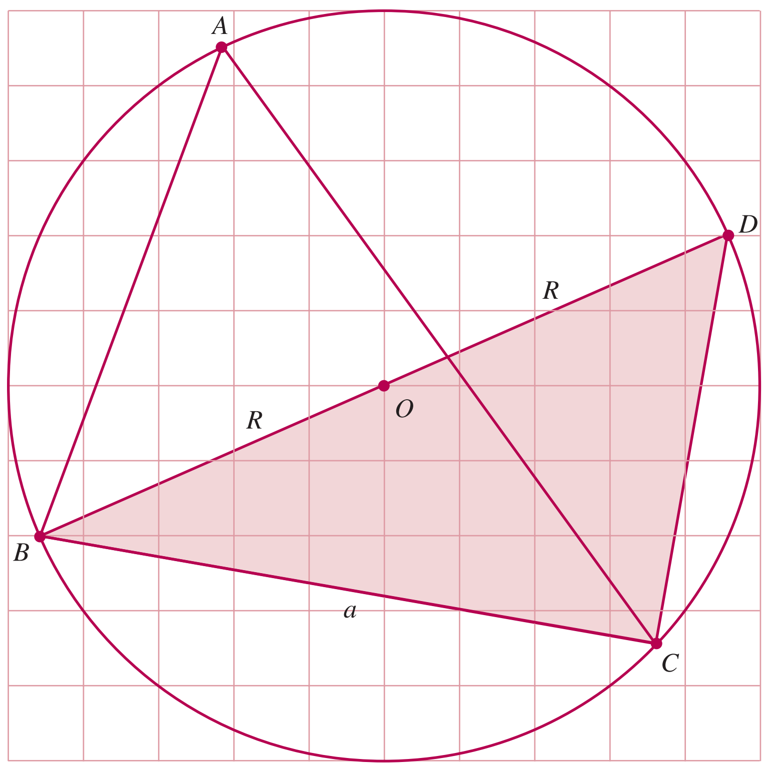


figure 2

3 What special name is given to the straight line BD passing through O?

4 Look at the shaded triangle BDC. What angle in this triangle is equal to angle A in triangle BAC? Why? What circle theorem did you use? Mark the equal angles on your diagram.

5 What do you notice about the size of angle BCD? Why? What circle theorem did you use? Mark the size of the angle on your diagram.

6 Write down an expression for $\sin D$ in terms of a and R.

7 Hence show that $\sin A = \dfrac{a}{2R}$ and that $\dfrac{a}{\sin A} = 2R$.

8 Repeat the argument, this time showing that, similarly, $\dfrac{b}{\sin B} = 2R$. It is a good idea to draw a new diagram. Decide what diameter you will need to draw. Discuss this in class.

9 Can you see that the method will work for $\dfrac{c}{\sin C}$? Discuss this in class. Complete the proof.

10 Complete this statement: in any given triangle *ABC*, each ratio in the sine rule $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ represents …

CHALLENGE ACTIVITIES

1 In the diagram on page 575, $a = 10$ cm and $A = 60°$. Calculate the diameter of the circumcircle of the triangle *ABC* (leave the answer in surd form).

2 From the result of question 1, calculate the length of *DC* (leave the answer in surd form).

3 If angle *ACD* is 45° show that angle *ABC* is 75°. Give reasons for each step of your argument.

4 Calculate *b*, the length of *AC*, and hence find the area of triangle *ABC* (answer to 2 decimal places).

Instructions for using Geometer's Sketchpad

1 Drawing figure 1.

- **a** Select the **Square Grid** from the **Graph** menu. Then hide the axes and reference points by selecting them and choosing **Hide Objects** from the **Display** menu.
- **b** Use the **Compass (circle) tool** to draw the circle of radius 5 units. Select the point at the end of the radius and hide it. Choose **Hide Point** from the **Display** menu.
- **c** Select the circle and then choose **Point on Circle** from the **Construct** menu. Repeat to create the points *A*, *B* and *C* on the circle. (relabel the centre and other points as necessary by double clicking on the label). Drag *A*, *B* and *C* around the circle to suitable positions as shown. Then select *A*, *B* and *C* and choose **Segments** from the **Construct** menu to construct the triangle.
- **d** Select each side and use the **Text tool** to label the length of the sides *a*, *b* and *c*.

2 Drawing figure 2.

- **a** Select *B* and then *O* (in that order) and choose **Line** from the **Construct** menu.
- **b** Select the circle and the line and choose **Intersections** from the **Construct** menu. Hide the intersection point which appears next to *B* and label the other intersection point *D*.
- **c** Now select and hide the line. As you drag *B* around the circle you will see that *D* moves in the same way. It is always at the end of the diameter.

d Now select B, O and D and choose **Segments** from the **Construct** menu. B, O and D are all in a straight line, so BD is a diameter. Now join D to C. Select and label the radii R.

e Finally select the points D, B and C and choose **Triangle Interior** from the **Construct** menu. Choose a colour from the range in the **Display** menu.

f Drag the points A, B and C around the circle to see the effect on the diagram. What geometrical properties remain constant? Can you see that the result is true no matter where A, B and C are on the circle?

Let's communicate

In no more than half a page, summarise what new mathematics you have learned from this activity. Use a diagram to illustrate.

Reflecting

Did it surprise you that the sine rule is so closely connected to the diameter of the circumcircle of the triangle? Why is trigonometry so closely linked to geometry?

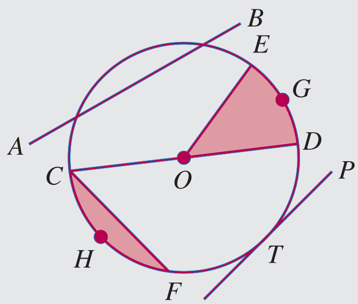
1 Draw a diagram to clearly illustrate the meaning of each of the following:
- ***a*** *An angle at the centre of a circle*
- ***b*** *An angle at the circumference of a circle*
- ***c*** *An angle in a semicircle*
- ***d*** *An angle between a tangent and a chord*
- ***e*** *An angle in the alternate segment*

2 *Why is it so much easier to draw the angles above rather than describe them in words? Are there times when the English language actually appears to be a hindrance in communicating mathematical ideas?*

3 *Do you think it is possible for mathematicians who speak different languages, such as English and Chinese, to communicate mathematically? Why?*

O is the centre of the circle in all questions, unless otherwise specified.

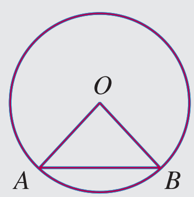
1



Name the following parts of the circle.

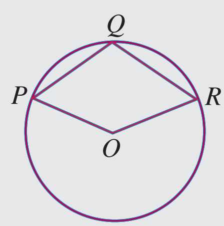
a the interval OE
b the interval CD
c the interval CF
d the line AB
e the line PT
f the region $EODG$
g the region CFH
h the curved interval EGD

2 **a** Explain why $OA = OB$.
b What kind of triangle is OAB?
c Which angles are equal?

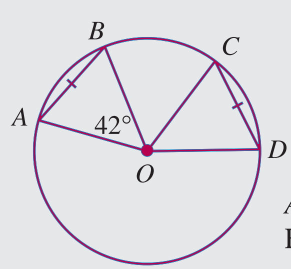


3 What are concentric circles?

4 Is $OPQR$ a cyclic quadrilateral? Explain.

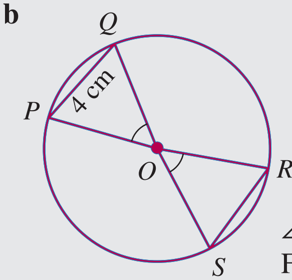


5 **a**



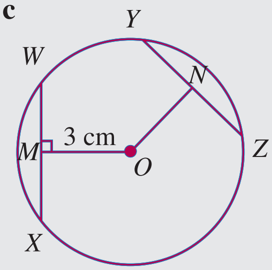
$AB = CD$.
Find $\angle COD$.

b



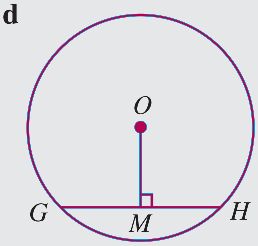
$\angle POQ = \angle ROS$.
Find RS.

c



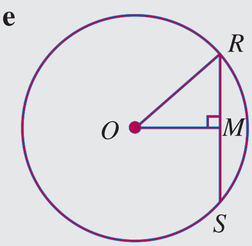
$WX = YZ$.
Find ON.

d



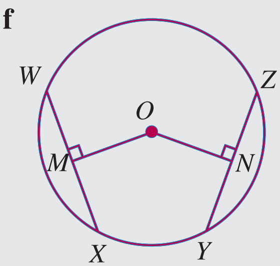
$GM = 5$ cm.
Find GH.

e



$OR = 13$ cm,
$OM = 5$ cm.
Find RS.

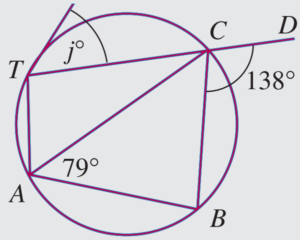
f



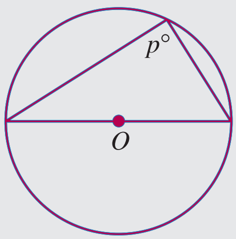
$OM = ON$,
$WX = 14$ cm.
Find NZ.

6 Find the value of each pronumeral.

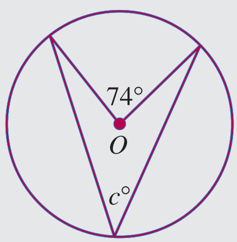
a



b

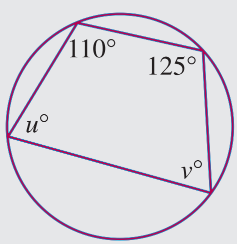


c

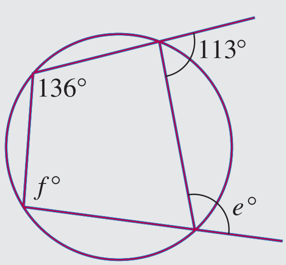


7 Find the value of all pronumerals.

a



b



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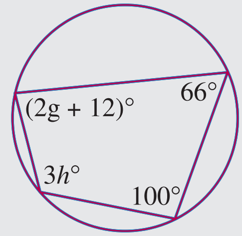
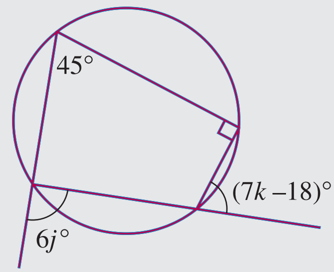
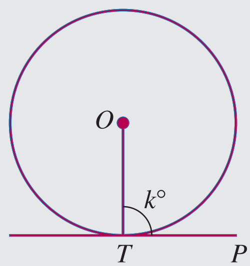


d



8 Find the value of all pronumerals. *PT* and *PS* are tangents.

a



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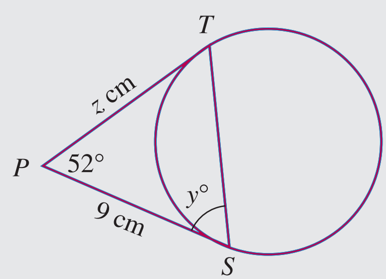

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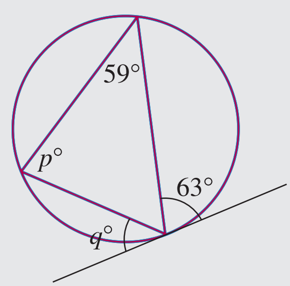
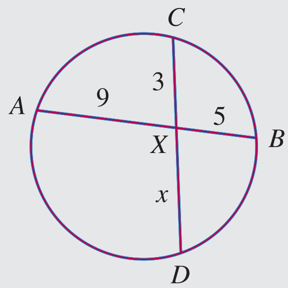


9 Find the value of each pronumeral. PT is a tangent.

a



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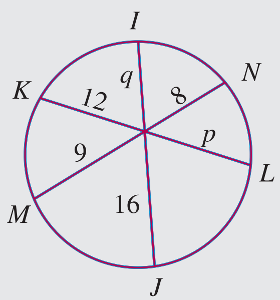

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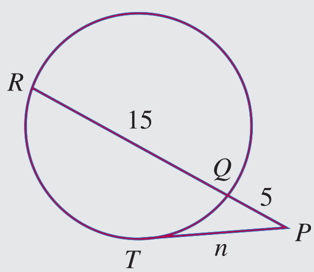

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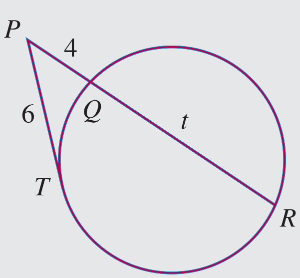

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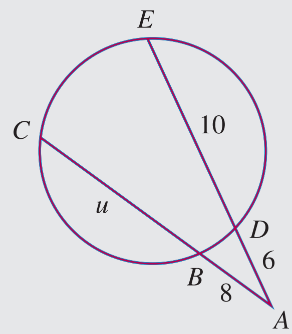

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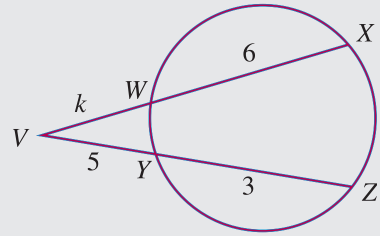
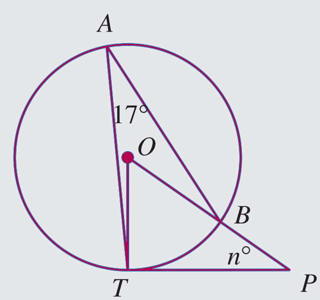
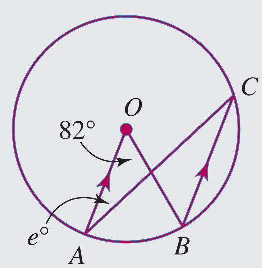


10 Find the value of all pronumerals, giving reasons. PT and PS are tangents.

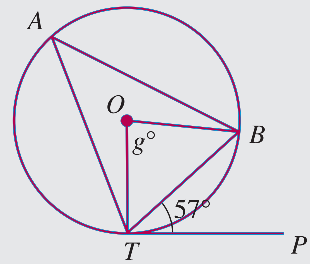
a



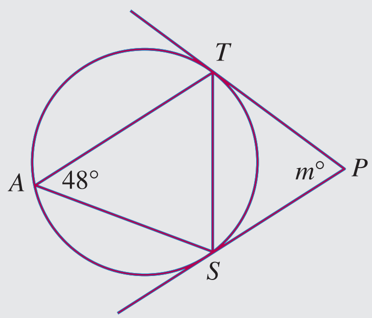
b



c



d



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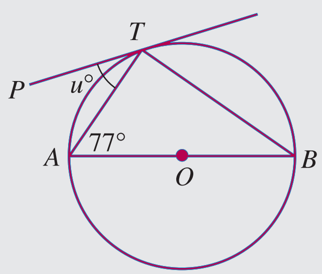

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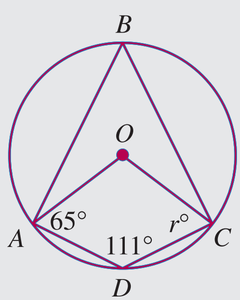
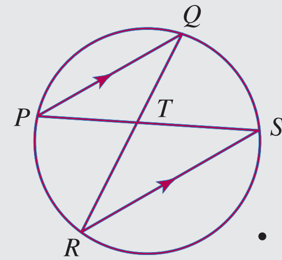


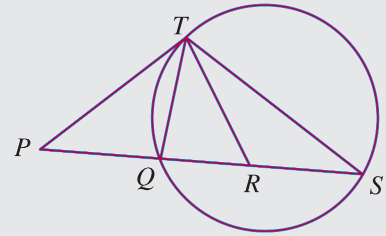
11 a



- $PQ \parallel RS$

Prove that ΔRST is isosceles.

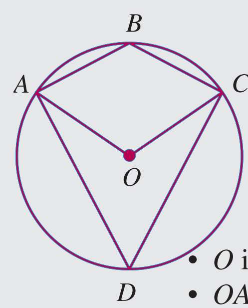
b



- PT is a tangent
- $PT = PR$

Prove that TR bisects $\angle QTS$.

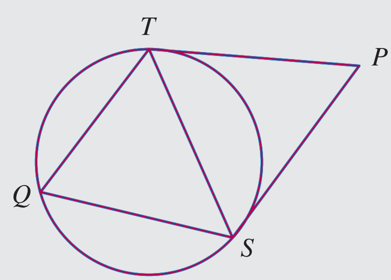
c



- O is the centre
- $OABC$ is a rhombus

i Prove that $\angle AOC = 2\angle OAB$.
ii Find $\angle AOC$.

d



- PT, PS are tangents
- TS bisects $\angle QTP$

i Prove that $TQ \parallel PS$.
ii Prove that ΔQST is isosceles.

CHAPTER REVIEW

Answers

1 Consumer arithmetic

Exercise 1.1

1 a $14, **b** $240, **c** $78, **d** $180, **e** $182, **f** $264, **g** $673.75, **h** $688.05 **2 a** $1770, **b** $3808 **3 a** $420, **b** $102.60, **c** $82.23, **d** $988.91 **4 a** $16, **b** $15, **c** $68.25, **d** $181.13 **5 a** $903.60, **b** $1245.50, **c** $1761.20, **d** $2736.32 **6 a** $72.44, **b** $308.54, **c** $21.17, **d** $364.08 **7 a** $496.80, **b** $1085.40 **8 a** $3240, **b** $12 240 **9 a** $560, **b** $644, **c** $144 **10** $2035.24 **11** $720 **12** $600 **13** 7 years **14** 9 months **15** 3% p.a. **16** 6.5% p.a. **17 a** 24% p.a., **b** 12% p.a., **c** 8% p.a., **d** 2% p.a. **18** 5 months **19** 183 days

Exercise 1.2

1 a

Year	Opening balance	Interest	Closing balance
1st	$10 000	$10 000 × 0.06	$10 600
2nd	$10 600	$10 600 × 0.06	$11 236
3rd	$11 236	$11 236 × 0.06	$11 910.16

b $1910.16 **2 a** Balance = $926.10; Interest = $126.10, **b** Balance = $2142.45; Interest = $142.45 **3 a** $515.15, **b** $30.08 **4 a** $1102.50, **b** $843.65, **c** $1739.69, **d** $2652.95, **e** $3800.31, **f** $18 463.49, **g** $12 452.56, **h** $26 862.72 **5 a** $1941.02, **b** $3321.96, **c** 5346.24, **d** $6967.56, **e** $14 355.97, **f** $9779.43, **g** $29 866.15, **h** $132 885.68 **6 a** $5507.32, **b** $5578.47, **c** $5616.47, **d** $5642.79 **7 a** $163.20, **b** $1469.90, **c** $2155.92, **d** $329.73, **e** $1920.74, **f** $568.00, **g** $2427.61, **h** $2424.25, **i** $15 432.27 **8 a** $10 790.80, **b** $2790.80 **9** $83 695.11 **10 a** $251 001.90, **b** $1001.90 **11** Christine, $30.80 **12** B **13** A, $1171.10 **14** $30 216.03 **15** Value = $62 071.73; Interest = $22 071.73 **16 a** $974 242, **b** $1 261 672, **c** $1 941 238 **17 a** $1.00, **b** $2.40, **c** $3.20, **d** $56.40 **18** 9839 **19** 30 719 271 **20** $42 370 **21** $24 113 **22** $600 **23 a** $450, **b** $1260, **c** $1500, **d** $2475

Exercise 1.3

1 $15 360 **2 a** $2916, **b** $1619, **c** $2225, **d** $1768 **3** $6675 **4** $768 **5** 6404 **6** 542 **7** $645 **8 a** $314, **b** $436 **9 a** 40 854 kL, **b** 9146 kL **10** 3794 **11 a** $343 490, **b** $56 510 **12** $2955 **13** 5 years **14** 6 years **15** 7 years **16** $15 250 **17** $20 400 **18** 13% **19 a** 9%, **b** $750 000

Exercise 1.4

1 $34.50 **2** 12 **3 a** $21.75, **b** $24.65 **4 a** $240, **b** $16 **5 a** no, **b** $565.25 **6** $892.40 **7 a** $2040, **b** $240 **8 a** $60, **b** 13.3% **9** $230 **10** B, $502 **11 a** $4050, **b** $19 050, **c** $529.17 **12 a** $11 495, **b** $478.96 **13 a** $1380, **b** $7820, **c** $2815.20, **d** $10 635.20, **e** $295.42 **14 a** $2200, **b** $8800, **c** $3379.20, **d** $12 179.20, **e** $253.73 **15 a** $7056, **b** $1456, **c** 13% **16** 9.52% **17** 19.5%

Exercise 1.5

1 a April, **b** 14, **c** 0.0593%, **d** $3000, **e** $930, **f** $165.25, **g** $212.60, **h** $2787.40, **i** $15, **j** $0.63 **2 a** 13.14% p.a., **b** 16.64% p.a., **c** 20.25% p.a., **d** 7.80% p.a., **e** 8.46% p.a., **f** 10.19% p.a. **3 a** $0.48, **b** $1.42, **c** $7.78, **d** $32.37 **4 a** $1.73, **b** $13.04, **c** $12.51, **d** $109.07 **5** $1.56 **6 a** $28.59, **b** $898.59 **7 a** 0.042%, **b** 22, **c** $320.94 **8 a** $1.16, **b** $0.33, **c** $169.49 **9 a** $0, **b** $4.56 **10 a** Yes, by $430, **b** $26.76 **11** $10.30

Exercise 1.6

1 a \$88.38, **b** \$87.68, **c** \$46.14, **d** \$26.09, **e** \$32.50, **f** \$22.24 **2 a** \$263.76, **b** \$319.76, **c** \$263.88, **d** \$1315.20, **e** \$117.68, **f** \$163.84 **3 a i** \$1170, **ii** \$170, **b i** \$5973.60, **ii** \$973.60, **c i** \$26 088, **ii** \$6088, **d i** \$39 538.80, **ii** \$4538.80 **4 a** \$1582, **b** \$284 760, **c** \$84 760, **d** 42.38% **5 a** \$554 400, **b** \$274 400, **c** 3.92% p.a. **6 a** \$385 840, **b** 110.24%, **c** 3.67% p.a. **7 a** Sunshine Bank, \$52 200 **8** \$6318 **9** \$31 524 **10 a** \$320 000, **b** \$252 928, **c** 3.95% p.a. **11 a** No, a month is longer than 2 fortnights. **b** $8\frac{1}{2}$ years, **c** \$83 844, **d** The loan is paid off more quickly and the amount of interest paid is reduced.

12 a i \$249 750, **ii** \$249 498.75, **iii** \$249 246.24, **iv** \$248 992.47, **b** Yes, the repayment is greater than the monthly interest. **c** No, the repayment is less than the monthly interest. **13 a** \$1528.41, **b** Increase the size of the repayments.

Chapter 1 Review

1 a \$30, **b** \$118.13, **c** \$389.36, **d** \$313.50 **2** \$2480 **3 a** \$28, **b** \$40.95, **c** \$45.31 **4 a** \$770.04, **b** \$170.04 **5** \$800 **6** 7% p.a. **7 a** \$21 000, **b** \$22 050, **c** \$23 152.50 **8** Balance = \$59 625.93, Interest = \$9625.93 **9 a** \$2163.20, \$163.20, **b** \$1893.72, \$393.72, **c** \$5638.71, \$1888.71, **d** \$19 738.40, \$3738.40, **e** \$34 471.07, \$9471.07, **f** \$67 453.28, \$27 453.28 **10 a** \$13 217.56, **b** \$13 431.36, **c** \$13 545.83, **d** \$13 625.23 **11 a** \$4415.25, **b** \$7761.34, **c** \$11 716.59, **d** \$37 930.55, **e** \$19 046.02, **f** \$35 005.73 **12 a** \$46 619.40, **b** 7.77% p.a. **13** B, by \$7.34 **14** A, by \$115.58 **15** \$634 443 **16** \$4.36 **17 a** \$800, **b** \$5400, **c** \$15 000 **18 a** \$18 000, **b** \$16 200, **c** \$11 810, **d** \$6974 **19** \$35 584 **20 a** \$567 000, **b** \$486 902 **21** 14 years **22 a** \$684, **b** no **23 a** \$1085, **b** \$105, **c** 12.5% p.a. **24 a** \$7920, **b** \$25 920, **c** \$540 **25 a** \$3600, **b** \$8400, **c** \$2520, **d** \$303.33 **26 a** \$6960, **b** \$960, **c** 8% p.a. **27 a** \$3.52, **b** \$15.26 **28** \$5.45 **29 a** \$88.15, **b** \$222.60, **c** \$215.46, **d** \$687.35 **30 a** \$49.40, **b** \$648.40, **c** \$1411 **31 a** \$2516, **b** \$754 800, **c** \$354 800, **d** 3.548% p.a. **32** \$19 440 **33 a** \$240 000, **b** \$236 928, **c** \$3.3% p.a.

2 Trigonometry

Exercise 2.1

1 a opp = QR, adj = PQ, hyp = PR, **b** opp = EG, adj = FG, hyp = EF, **c** opp = XZ, adj = XY, hyp = YZ

2 a i $\frac{4}{5}$, **ii** $\frac{3}{5}$, **iii** $\frac{4}{3}$, **b i** $\frac{15}{17}$, **ii** $\frac{8}{17}$, **iii** $\frac{15}{8}$, **c i** $\frac{12}{13}$, **ii** $\frac{5}{13}$, **iii** $\frac{12}{5}$, **d i** $\frac{20}{29}$, **ii** $\frac{21}{29}$, **iii** $\frac{20}{21}$, **e i** $\frac{7}{25}$, **ii** $\frac{24}{25}$, **iii** $\frac{7}{24}$, **f i** $\frac{16}{65}$, **ii** $\frac{63}{65}$, **iii** $\frac{16}{63}$ **3 a** α, **b** θ, **c** θ, **d** θ, **e** α, **f** α **4 a** $\frac{15}{17}$, **b** $\frac{15}{20}$, **c** $\frac{15}{25}$, **d** $\frac{8}{17}$, **e** $\frac{8}{15}$, **f** $\frac{8}{17}$, **g** $\frac{15}{25}$, **h** $\frac{20}{15}$, **i** $\frac{15}{17}$, **j** $\frac{20}{25}$, **k** $\frac{15}{8}$, **l** $\frac{20}{25}$ **5 a** $x = 3$, **b** $y = 9$, **c** $p = 20$, **d** $c = 27$, **e** $e = 60$, **f** $n = 35$ **6 a** $\sin\theta = \frac{36}{85}$, $\cos\theta = \frac{77}{85}$, $\tan\theta = \frac{36}{77}$ **7 a** 65 cm, **b** $\sin\theta = \frac{63}{65}$, $\cos\theta = \frac{16}{65}$, $\tan\theta = \frac{63}{16}$ **8 a** 80 mm, **b** $\sin\alpha = \frac{80}{89}$, $\cos\alpha = \frac{39}{89}$, $\tan\alpha = \frac{80}{39}$ **9 a** $\sin\theta = \frac{20}{29}$, $\cos\theta = \frac{21}{29}$, **b** $\cos\theta = \frac{7}{25}$, $\tan\theta = \frac{24}{7}$, **c** $\tan\theta = \frac{48}{55}$, $\sin\theta = \frac{48}{73}$ **10** $\sin\theta = \frac{x}{\sqrt{x^2+y^2}}$, $\cos\theta = \frac{y}{\sqrt{x^2+y^2}}$

Exercise 2.2

1 a 20°, **b** 36°, **c** 21°, **d** 71°, **e** 63°, **f** 84°, **g** 110°, **h** 138° **2 a** 7°52′, **b** 22°6′, **c** 14°26′, **d** 76°33′, **e** 50°16′, **f** 68°5′, **g** 119°13′ **h** 164°2′ **3 a** 0.80, **b** 0.11, **c** 0.37, **d** 0.72, **e** 7.57, **f** 24.73, **g** 19.31, **h** 16.03, **i** 40.38, **j** 313.76, **k** 1.40, **l** 0.31 **4 a** 0.9134, **b** 12.85, **c** 25.30 **5 a** 1.679, **b** 1.582 **6 a** 20°11′, **b** 44°10′, **c** 7°10′, **d** 74°34′, **e** 74°36′, **f** 5°8′, **g** 35°54′, **h** 54°38′, **i** 0°49′, **j** 70°17′, **k** 52°14′, **l** 24°53′ **7 a** 21°48′, **b** 64°9′, **c** 41°25′, **d** 33°3′ **8 a** 0.7701, 0.6101, **b** 0.9711, 0.2388 **9 a** 19°28′, **b** 59°45′, **c** 77°10′

Exercise 2.3

1 a $p = 5.5$, **b** $m = 5.7$, **c** $x = 35.2$, **d** $y = 48.6$, **e** $t = 1.6$, **f** $a = 58.0$, **g** $g = 50.5$, **h** $b = 13.1$, **i** $k = 38.6$, **j** $z = 33.4$, **k** $c = 15.3$, **l** $w = 54.3$, **m** $e = 32.5$, **n** $n = 17.8$, **o** $u = 87.9$ **2 a** $x = 14.52$, **b** $t = 65.22$, **c** $e = 61.30$, **d** $m = 159.64$, **e** $c = 148.42$, **f** $v = 42.32$ **3 a** 14.6 cm, **b** 16.56 m, **c** 10.7 cm, **d** 115 cm **4** 757 m

5 6.4 m **6** 7.5 m **7** 18 m **8** 195 cm **9** 11.4 m **10** 4.5 cm **11 a** 10.5 cm, **b** 273 cm^2
12 a 25 cm, **b** 24 cm **13 a** 7.5 cm, **b** 16.8 cm **14** 349 m **15** 20.2 m **16** 6.65 m **17** 1871 m
18 8.18 m **19** 50.3 m **20** 57 m **21 a** 15 000 m, **b** 9000 m **22 a** 44 mm, **b** 136.8 mm, **c** 240 mm,
d 80 mm **23** 55.59 cm **24 a** 23.6 cm, **b** 31.2 cm,, **c** 368.16 cm^2 **25 a** 10 cm, **b** 20 cm **26** 125 m
27 a AD = 4.8 cm, DC = 11.2 cm, **b** 16 cm **28 a** 16.9 cm, **b** 69.9 cm **29** 115 m

Exercise 2.4

1 a 32°14′, **b** 40°14′, **c** 63°15′, **d** 77°54′, **e** 57°8′, **f** 19°11′, **g** 25°37′, **h** 67°19′, **i** 78°28′, **j** 50°39′, **k** 21°17′, **l** 52°25′ **2 a** 33°27′, **b** 53°41′, **c** 52°23′, **d** 73°55′, **e** 70°17′, **f** 41°54′, **g** 83°17′, **h** 24°59′, **i** 43°29′
3 a 41°49′, **b** 31°36′, **c** 35°6′ **4 a** 62°6′, **b** 61°27′, **c** 36°59′ **5** 38°3′ **6** 18°40′ **7** 65°33′
8 32°43′ **9** 54°3′ **10** 70°26′ **11** 45°34′ **12 a** 77°29′, **b** 35°24′ **13** 14°56′ **14** 51°23′
15 a 52°36′, **b** 42°9′, **c** 100°57′, **d** 68°6′ **16** 41°20′ **17** 68°35′ **18** 9°28′ **19** 12°32′ **20** 126°52′
21 a 36°16′, **b** 21°39′, **c** 71°23′, **d** 36°52′

Exercise 2.5

2 a $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, **b** $\sin\theta = \frac{12}{13}$, $\cos\theta = \frac{5}{13}$, $\tan\theta = \frac{12}{5}$, **c** $\sin\theta = \frac{12}{37}$, $\cos\theta = \frac{35}{37}$, $\tan\theta = \frac{12}{35}$

3 a 28°4′, **b** 33°8′, **c** 52°17′, **d** 61°52′ **5 a** $\frac{24}{25}$, **b** $\frac{9}{41}$, **c** $\frac{21}{29}$, **d** $\frac{16}{65}$ **6** 45° **7 a** 63°26′, **b** 18°26′, **c** 58°0′, **d** 61°42′, **e** 58°39′, **f** 36°50′ **8 a** 82°52′, **b** 36°52′, **c** 41°49′

Exercise 2.6

1 a i $\frac{a}{c}$, **ii** $\frac{b}{c}$, **iii** $\frac{b}{c}$, **iv** $\frac{a}{c}$, **b i** $\sin\theta = \cos(90° - \theta)$, **ii** $\cos\theta = \sin(90° - \theta)$ **3 a** 90°, **c** Yes, $\sin\theta = \cos(90° - \theta)$.
4 a sin 30° = 0.5, **b** cos 50° = 0.643, **c** sin 75° = 0.966, **d** cos 18° = 0.951 **5 a** $x = 40$, **b** $x = 55$, **c** $x = 25$, **d** $x = 78$, **e** $x = 42$, **f** $x = 37$, **g** $x = 73$, **h** $x = 6$, **i** $x = 85$, **j** $x = 26$, **k** $x = 19$, **l** $x = 61$ **6 a** 1, **b** 1, **c** 2, **d** $\frac{3}{4}$
7 a $x = 40$, **b** $x = 45$, **c** $x = 83$, **d** $x = 80$, **e** $x = 27$, **f** $x = 16$, **g** $x = 240$, **h** $x = 100$, **i** $x = 10$, **j** $x = 45$, **k** $x = 30$, **l** $x = 40$ **8 a** 0.52, **b** 0.21 **9 a** 1, **b** $\tan\theta$, **c** 1, **d** $\tan\theta$, **e** $\sin^2\theta$, **f** $\cos^2\theta$, **g** $\sin\theta$, **h** $\cos\theta$, **i** $\cos^2\theta$

Exercise 2.7

1 a $\frac{1}{2}$, **b** $\frac{\sqrt{3}}{2}$, **c** $\frac{1}{\sqrt{3}}$, **d** $\frac{1}{\sqrt{2}}$, **e** $\frac{1}{\sqrt{2}}$, **f** 1, **g** $\frac{\sqrt{3}}{2}$, **h** $\frac{1}{2}$, **i** $\sqrt{3}$ **2 a** 1, **b** $\frac{1}{4}$, **c** 3, **d** $\frac{1}{2}$, **e** $\frac{1}{3}$, **f** $\frac{3}{4}$ **3 a** $\frac{1}{4}$, **b** $\frac{1}{2}$, **c** 1, **d** $\frac{3}{4}$, **e** $\frac{\sqrt{6}}{4}$, **f** $\frac{\sqrt{6}}{2}$, **g** $\frac{\sqrt{6}}{6}$, **h** $\frac{\sqrt{2}}{4}$, **i** $\frac{1}{2}$, **j** $\frac{1}{4}$, **k** $1\frac{1}{2}$, **l** $\frac{\sqrt{3}}{4}$ **4 a** 1, **b** $\frac{4\sqrt{3}}{3}$, **c** $\sqrt{2}$, **d** $\sqrt{3}$, **e** $\frac{1+\sqrt{2}}{2}$, **f** $\frac{\sqrt{3}+\sqrt{2}}{2}$, **g** $\frac{3\sqrt{3}}{2}$, **h** $\frac{2+\sqrt{3}}{2}$, **i** $1\frac{1}{2}$ **5 a** $\frac{3+5\sqrt{3}}{6}$, **b** $\frac{3\sqrt{2}+5\sqrt{3}}{6}$, **c** $\frac{3+4\sqrt{3}}{3}$, **d** $\frac{1+\sqrt{2}+2\sqrt{3}}{2}$ **6 a** 0, **b** $-\frac{1}{2}$, **c** 2, **d** $\frac{1}{2}$, **e** 4, **f** $1\frac{1}{2}$ **7 a** 1, **b** $\sqrt{3}$, **c** $1\frac{1}{2}$, **d** $\sqrt{3}$, **e** 3, **f** $\frac{2\sqrt{3}}{3}$, **g** $\frac{\sqrt{6}}{2}$, **h** $\frac{\sqrt{6}}{3}$ **9 a** 30°, **b** 60°, **c** 45°, **d** 60°, **e** 60°, **f** 30°, **g** 30°, **h** 45°, **i** 45° **10 a** $x = 7$, **b** $k = 5$, **c** $t = 4$, **d** $n = 12$, **e** $u = 2\sqrt{3}$, **f** $e = 9$ **11 a** $h = 14$, **b** $p = 17$, **c** $w = 12$ **12 a** 45°, **b** 60°, **c** 30°, **d** 45°, **e** 60°, **f** 60° **13 a** $2\sqrt{3}$ cm, **b** $4\sqrt{3}$ cm^2 **14 a** $x = 8$, **b** $y = 13$, **c** 72 cm^2 **15 a** 30 cm, **b** $150\sqrt{3}$ cm^2
16 $80\sqrt{3}$ m

Exercise 2.8

1 a i A = N50°E, B = S15°E, C = N66°W, **ii** A = 050°, B = 165°, C = 294°, **b i** A = S74°E, B = S61°W, C = N9°W, **ii** A = 106°, B = 241°, C = 351°, **c i** A = N72°E, B = S33°W, C = N44°W, **ii** A = 072°, B = 213°, C = 316°
2 a 210°, **b** 055°, **c** 328°, **d** 123° **3 a** 9°, **b** 137° **4 a** 48.9 km, **b** 64.9 km, **c** 123.3 km, **d** 91.7 km
5 a 90.3 km, **b** 8.41 km, **c** 23.2 km, **d** 16.7 km **6 a i** 139°, **ii** 319°, **b i** 314°, **ii** 134°, **c i** 219°, **ii** 039°, **d i** 041°, **ii** 221° **7 a i** $\angle HSP$ = 43° + 47° = 90°, **ii** 218 m, **b i** $\angle HBS$ = 16° + 74° = 90°, **ii** 2.6 km
8 a 228 km, **b** 038° **9 a** 28.36 NM, **b** 4.02 NM, **c** 29 NM, **d** 098° **10** 123 km

Chapter 2 Review

1 a $\frac{7}{25}$, **b** $\frac{18}{30}$, **c** $\frac{24}{7}$, **d** $\frac{24}{30}$, **e** $\frac{18}{24}$, **f** $\frac{24}{25}$ **2** $x = 15$ **3** $\sin\theta = \frac{65}{97}$, $\cos\theta = \frac{72}{97}$ **4** 54.15
5 a 74°48′, **b** 22°24′ **6** 0.7315 **7 a** $a = 29.9$, **b** $p = 31.0$, **c** $z = 22.1$ **8 a** 26.4 cm, **b** 25.1 cm
9 a 65°33,′ **b** 70°44′, **c** 81°33′ **10 a** 62.2 cm, **b** 128.1 cm, **c** 62°14′ **11** 44 m **12** 6.05 m
13 2618 m **14** 53°26′ **15** 52°28′ **16 a** 38 cm, **b** 98 cm **17** 45°14′ **18** C **19 a** $\sin\theta$, **b** $\cos\theta$
20 a 75°58′, **b** 26°34′, **c** 59°2′ **21 a** $x = 20$, **b** $x = 65$, **c** $x = 16$, **d** $x = 73$, **e** $x = 24$, **f** $x = 45$ **22 a** 1, **b** $\tan\theta$,
c $\cos^2\theta$, **d** $\sin\theta$ **23 a** $\frac{\sqrt{2}}{4}$, **b** $\frac{3}{4}$, **c** $\sqrt{2}$, **d** $\frac{5\sqrt{3}}{6}$, **e** $\frac{1}{2}$, **f** $4\frac{1}{2}$, **g** 2, **h** $\frac{\sqrt{6}}{3}$ **24 a** $x = 6$, **b** $x = 4\sqrt{3}$
25 45° **26 a i** N9°E, **ii** 009°, **b i** S55°E, **ii** 125°, **c i** S77°W, **ii** 257°, **d i** N42°W, **ii** 318°
27 a 292°, **b** 057° **28** 18° **29 a** 92.2 km, **b** 7.9 km **30 a** 134.8 NM, **b** 15.29 km **31 a i** 122°,
ii 302°, **b i** 331°, **ii** 151° **32 a** $\angle IJK = 43° + 47° = 90°$, **b** 268 m **33 a** $\angle PQR = 59° + 31° = 90°$, **b** 56 km

3 Volume and surface area

Exercise 3.1

1 a 24 cm^2, **b** 54 cm^2, **c** 174.96 cm^2, **d** 564.54 cm^2 **2 a** 100 cm^2, **b** 312 cm^2, **c** 176 cm^2 **3** 224.54 cm^2
4 a 120 cm^2, **b** 600 cm^2, **c** 372.95 cm^2 **5 a** 392 cm^2, **b** 1540 cm^2 **6 a** 320 cm^2, **b** 840 cm^2
7 a i $x = 19.5$, **ii** 1260 cm^2, **b i** $h = 24$, **ii** 4944 cm^2 **8 a** 4 cm, **b** 7 cm, **c** 13 cm **9 a** $y = 17$, **b** 820 cm^2
10 a 137.9 m^2, **b** \$1991.10 **11 a** 72.2 m^2, **b** \$8.50 **12** 8 cm, 6 cm, 5 cm **13** 9 cm **14** 12 cm
15 a 486 cm^2, **b** 1028 cm^2, **c** 1406 cm^2 **d** 866 cm^2

Exercise 3.2

1 a 105 cm^2, **b** 242 cm^2 **2 a** 340 cm^2, **b** 216 cm^2, **c** 338 cm^2 **3 b** $36\sqrt{3}$ cm^2 **4 a** 25 cm, **b** 30 cm,
c 1356 cm^2 **5 a** 74.2 cm^2, **b** 254.1 cm^2, **c** 535.5 cm^2 **6 a** 266.4 cm^2, **b** 970.3 cm^2 **7 a** 1119.1 cm^2,
b 662.7 cm^2 **8 b** $9\sqrt{2}$ cm, **c** 507.8 cm^2

Exercise 3.3

1 a 483.8 cm^2, **b** 446.2 cm^2, **c** 301.6 cm^2, **d** 1297.5 cm^2 **2 a** 80π cm^2, **b** 198π cm^2 **3 a** 150.8 cm^2,
b 339.3 cm^2, **c** 1979.2 cm^2, **d** 706.9 cm^2, **e** 1653.2 cm^2, **f** 181.8 cm^2 **4 a** 19.1 cm, **b** 85.1 cm, **c** 8 cm
5 a 400 cm^2, **b** 64π cm^2, **c** 200π cm^2, **d** 1229.4 cm^2 **6 a** 673.6 cm^2, **b** 298.0 cm^2 **7 a** $A = \pi r^2 + 2\pi rh$,
b 4134 mm^2 **8 a** 179 cm^2, **b** 456 cm^2 **9 a** 73.42 m^2, **b** \$216.60 **10 a** 9π m^2, **b** 8π m^2, **c** 0.34π m^2,
d 54.5 m^2 **11** \$1430.68 **12 a** 922 cm^2, **b** 107 cm^2, **c** 1126 cm^2, **d** 982 cm^2 **13** 7 cm **14** 234π cm^2

Exercise 3.4

1 a 65π cm^2, **b** 168π cm^2, **c** 60π cm^2 **2 a** 373.8 cm^2, **b** 188.5 cm^2, **c** 221.7 cm^2, **d** 537.8 cm^2, **e** 975.7 cm^2,
f 221.6 cm^2 **3 a** 616 cm^2, **b** 1257 cm^2, **c** 845 cm^2, **d** 499 cm^2, **e** 302 cm^2, **f** 401 cm^2 **4 a** 36π cm^2,
b 16π cm^2, **c** 144π cm^2, **d** 400π cm^2 **5 a** 301.6 cm^2, **b** 433.2 cm^2, **c** 41.4 cm^2 **6 a** 192π cm^2, **b** 27π cm^2
7 a 9 cm, **b** 11 cm, **c** 26 cm **8 a** 2.82 cm, **b** 5.09 cm, **c** 13.08 cm **9 a** $x = 39$, S.A. = 2545 cm^2,
b $x = \sqrt{250}$, S.A. = 702 cm^2, **c** $x = \sqrt{209.96}$, S.A. = 1616 cm^2 **10** 514 718 540 km^2 **11 a** 15 cm,
b 679 cm^2 **12 a** 25 cm, **b** 175π cm^2 **13** 4 cm **14** 3 : 4 **15 a** 624 cm^2, **b** 591 cm^2, **c** 144 cm^2,
d 151 cm^2, **e** 837 cm^2, **f** 585 cm^2 **16** 349π cm^2

Exercise 3.5

1 a 24 units^3, **b** 30 units^3, **c** 90 units^3 **2 a** 30 cm^3, **b** 52 cm^3, **c** 108 cm^3, **d** 93 cm^3, **e** 319.5 cm^3, **f** 223.1 cm^3
3 a 60 cm^3, **b** 112 cm^3, **c** 130 cm^3 **4 a** 8 cm^3, **b** 125 cm^3, **c** 729 cm^3, **d** 1728 cm^3 **5 a** 77.64 m^3,
b 1225.73 m^3, **c** 0.52 m^3 **6 a** 8 cm, **b** 6.2 cm, **c** 9.2 cm, **d** 11.1 cm **7 a** 3 m, **b** 9 m, **c** 3 m, **d** 12 m,
e 11.5 m, **f** 24.4 m **8 a** 54 cm^3, **b** 176 cm^3, **c** 450 cm^3 **9 a** 195 cm^3, **b** 432 mm^3 **10 a** 225 mm^3,
b 51 m^3, **c** 206.08 cm^3 **11 a** 168 cm^3, **b** 384 m^3, **c** 2478 mm^3 **12 a** 9 cm, **b** 23 mm^2, **c** 7.5 cm, **d** 26.3 m^2
13 a i $x = 10.8$, **ii** 393.66 cm^3, **b i** $p = 6$, **ii** 28.38 cm^3, **c i** $k = 7$, **ii** 159.6 cm^3 **14 a** 110 m^3, **b** 415 m^3,
c 2208 m^3, **d** 311.5 m^3, **e** 1137.6 m^3, **f** 651 m^3, **g** 862.2 m^3, **h** 795 m^3, **i** 7308 m^3 **15 a** 486 cm^2, **b** 1331 cm^3
16 6 units

Exercise 3.6

1 a 628 cm^3, **b** 3308 cm^3, **c** 2898 cm^3, **d** 565 cm^3, **e** 1512 cm^3, **f** 352 cm^3 **2 a** 502.7 cm^3, **b** 7926.7 cm^3, **c** 10 053.1 cm^3, **d** 24 119.3 cm^3 **3 a** 20π mm^3, **b** 324π mm^3, **c** 54π mm^3, **d** 44π mm^3 **4 a** 1891 cm^3, **b** 1196 cm^3, **c** 193.6 cm^3, **d** 732.4 cm^3, **e** 271.3 cm^3, **f** 538.6 cm^3 **5 a** 12 m, **b** 15 m **6 a** 11 m, **b** 16 m **7 a** 8567 cm^3, **b** 6584 cm^3, **c** 4739 cm^3, **d** 1269 cm^3, **e** 294 cm^3, **f** 993 cm^3 **8 a** 240π cm^3, **b** 960 cm^3, **c** 78.5% **9 a** $h = 14$ cm, $r = 4$ cm, **b** 16 889 cm^3, **c** 16.9 L **10** 2.3 L **11 a** 15.394 m^3, **b** 15 394 L **12** 16 cm **13 a** 156π cm^2, **b** 845π cm^3

Exercise 3.7

1 a 50 cm^3, **b** 56 cm^3, **c** 132 cm^3, **d** 435 cm^3, **e** 421.08 cm^3, **f** 664.02 cm^3 **2 a** 336 cm^3, **b** 360.8 cm^3, **c** 364 cm^3 **3 a** 24.5 cm^3, **b** 105 m^3, **c** 176 mm^3, **d** 211.68 cm^3, **e** 782.04 m^3, **f** 67.6 mm^3 **4** 338.4 cm^3 **5 a** 68 cm^2, **b** 272 cm^3 **6 a** 6 cm, **b** 5 cm, **c** 5 cm, **d** 12 cm **7 a** 270.48 cm^3, **b** 6.5 cm **8 a** 20 cm^3, **b** 36 cm^3 **9 a** 8 cm, **b** 15 cm, **c** 720 cm^3 **10 a** 37.5 cm, **b** 1687.5 cm^2, **c** 34 875 cm^3 **11 a** 640 cm^3, **b** 1004.4 cm^3, **c** 624 cm^3, **d** 7207.2 cm^3, **e** 1178.45 cm^3, **f** 805 cm^3 **12** 787.5 cm^3 **13 a** 20 cm, **b** $x = 42$, **c** 8400 cm^3 **14 a** $OA = 9$ cm, $OB = 16$ cm, **b** 2304 cm^3, **c** 288 cm^3, **d** $\frac{1}{8}$

Exercise 3.8

1 a 235.6 cm^3, **b** 737.2 cm^3, **c** 4021.2 cm^3, **d** 3167.8 cm^3, **e** 1862.7 cm^3, **f** 2907.2 cm^3 **2 a** 186 cm^3, **b** 504 cm^3, **c** 351 cm^3, **d** 682 cm^3 **3 a** 32π cm^3, **b** 240π cm^3, **c** 75π cm^3 **4 a i** $x = 12$, **ii** 5277.9 cm^3, **b i** $x = 24$, **ii** 2513.3 cm^3, **c i** $x = 8$, **ii** 27.1 cm^3 **5 a** 10 mm^3, **b** 75 mm^3 **6 a** 11 mm, **b** 154 mm^2 **7 a i** 36 cm, **ii** 8482 cm^3, **b i** 60 cm, **ii** 7603 cm^3 **8 a** $\frac{2}{3}$, **b** 13 854 cm^3 **9** 125.7 cm^3 **10 a** 26 cm, **b** 3716.5 cm^3 **11 a** 235 cm^3, **b** 594 cm^3, **c** 1891 cm^3, **d** 1172 cm^3, **e** 558 cm^3, **f** 368 cm^3 **12 a i** 7 cm, **ii** 9 cm, **b** 15 cm **13 a** 5.2 L, **b** 1.6 L **14** 28.3 cm **15** 18.72 cm

Exercise 3.9

1 a 34 cm^3, **b** 268 cm^3, **c** 860 cm^3, **d** 524 cm^3, **e** 1437 cm^3, **f** 8181 cm^3 **2 a** 1047.4 mm^3, **b** 20.6 mm^3 **c** 33.5 mm^3, **d** 448.9 mm^3 **3 a** 36π cm^3, **b** 972π cm^3, **c** $12\,348\pi$ cm^3, **d** 2304π cm^3, **e** 4500π cm^3, **f** 7776π cm^3 **4 a** 262 m^3, **b** 575 m^3, **c** 1740 m^3 **5 a** 486 cm^3, **b** 318 cm^3, **c** 1334 cm^3, **d** 304 cm^3, **e** 16 cm^3, **f** 175 cm^3 **6** 524 g **7 a** 972π cm^3, **b** 1458π cm^3, **c** 1526.8 cm^3 **8 a** 32 cm, **b** 17 157 cm **9** 3 : 4 **10 a** 3 cm, **b** 6 cm, **c** 15 cm **11** 6.2 cm **12 a** 11 cm^3, **b** 11 mL **13 a** $2r$ cm, **b** 5 cm, **c** $\frac{4}{3}\pi r^3$ cm^3, **d** 2 : 3, **e** 16π cm^3

Chapter 3 Review

1 a 184 cm^2, **b** 300 cm^2 **2** 14 cm **3 a** $x = 15$, **b** 5430 cm^2 **4** 364 cm^2 **5 a** $PE = \sqrt{221}$ cm, $PF = \sqrt{170}$ cm, **b** 749 cm^2 **6** 1004 cm^2 **7** 1175 cm^2 **8** 7 cm **9** 854.5 cm^2 **10** 358.8 cm^2 **11** 49.5 cm^2 **12 a** 60π cm^2, **b** 267 cm^2 **13** 232.4 cm^2 **14** 3 cm **15** 7 cm **16 a** 406.8 cm^2, **b** 417.0 cm^2 **17** 120 cm^3 **18 a** 343 cm^3, **b** 105 cm^3 **19 a** 780 cm^3, **b** 1700 cm^3, **c** 560 cm^3 **20** 16.5 mm^2 **21 a** $w = 80$, **b** 72 000 cm^3, **c** 72 L **22** 6065.9 cm^3 **23 a** 594 cm^3, **b** 24 cm^3 **24 a** 320 cm^3, **b** 522.97 cm^3 **25 a** 1990 cm^3, **b** 1960 cm^3, **c** 1437 cm^3, **d** 524 cm^3 **26 a** 162π cm^3, **b** 75π cm^3, **c** 36π cm^3 **27 a** 9 cm, **b** 1017.9 cm^3 **28 a** 6 cm^3, **b** 21π cm^3 **29 a** 5 cm, **b** 8 cm **30** a 4.8 cm, **b** 11 cm, **c** 12 cm **31 a** 96 cm^2, **b** 3375 cm^3 **32 a** 370 cm^3, **b** 811 cm^3, **c** 980 cm^3, **d** 709 cm^3

4 Deductive geometry

Exercise 4.1

1 a $x = 50$, **b** $y = 110$, **c** $t = 64$, **d** $u = 123$, **e** $p = 52$, **f** $k = 20$, **g** $m = 103$, **h** $a = 30$, **i** $z = 45$, **j** $r = 35$, **k** $w = 72$, **l** $n = 43$ **2 a** $x = 100$, **b** $p = 52$, **c** $r = 145$, **d** $k = 39$, **e** $y = 76$, **f** $f = 47$ **3 a** $x = 53$, **b** $p = 27$, **c** $c = 35$, **d** $a = 57$, **e** $t = 28$, **f** $h = 60$, **g** $q = 109$, **h** $k = 137$, **i** $s = 14$ **4 a** $z = 153$, **b** $b = 62$, **c** $n = 26$, **d** $a = 70$, **e** $u = 5, v = 8$, **f** $x = 7, y = 25$, **g** $p = 90$, **h** $v = 3, w = 10$, **i** $k = 14$, **j** $d = 110, e = 8$, **k** $r = 90, s = 46$, **l** $j = 20, k = 6$, **m** $a = 90, b = 18$, **n** $f = 15$, **o** $y = 90, z = 4$ **5 a** $x = 53, y = 53$, **b** $p = 66, q = 24$, **c** $a = 36, b = 72$, **d** $g = 26$, **e** $k = 36$, **f** $n = 5$, **g** $c = 16$, **h** $b = 29$, **i** $v = 9$ **6 a** $a = 96, b = 84$, **b** $u = 49, v = 41$, **c** $c = 33, d = 147$,

ANSWERS

d $m = 36, n = 36$, **e** $j = 62, k = 118$, **f** $g = 118, h = 118, i = 62$ **7 a** $t = 61$, **b** $n = 44$, **c** $r = 55$, **d** $c = 53$, **e** $h = 112$, **f** $k = 63$ **8 a** yes, **b** no, **c** no **9 a** $a = 72, b = 108, c = 72$, **b** $x = 24, y = 54, z = 18$, **c** $p = 21, q = 30, r = 98$ **10 a** $b = 13$, **b** $e = 24$, **c** $g = 20$, **d** $x = 11$, **e** $t = 31$, **f** $n = 12$, **g** $s = 7$, **h** $w = 15$, **i** $y = 17$ **11 a** $a = 54, b = 55$, **b** $m = 109, n = 29$, **c** $p = 27, q = 27$, **d** $x = 127, y = 127$, **e** $c = 65, d = 115$, **f** $g = 59, h = 59$, **g** $r = 60, s = 148$, **h** $u = 69, v = 83$, **i** $j = 45, k = 135$, **j** $e = 103, f = 103$, **k** $p = 66, q = 105$, **l** $m = 112, n = 68$ **12 a** $a = 65, b = 54$, **b** $e = 27, f = 63$, **c** $p = 119, q = 30$, **d** $u = 29, v = 75$, **e** $g = 119, h = 90$, **f** $v = 57, w = 57$, **g** $j = 32, k = 32$, **h** $c = 121, d = 75$, **i** $s = 60, t = 123$ **13 a** $m = 38, n = 142$, **b** $b = 90, c = 137$, **c** $a = 90, b = 21, c = 21, d = 21$, **d** $a = 76, b = 76, c = 76$, **e** $x = 73, y = 73, z = 73$, **f** $x = 90, y = 45, z = 45$ **14** $x = 35$ **15** $x = 18$

Exercise 4.2

1 a 4, **b** 6, **c** 8, **d** 10, **e** 5, **f** 7, **g** 9, **h** 12, **i** 11 **2 a** convex, **b** non-convex, **c** convex, **d** non-convex **3 a** regular, **b** irregular, **c** irregular, **d** regular **4 a i** rectangle, **ii** rhombus, **b** no — all sides and all angles are equal in regular polygons. **5** 540° **6 a** 108°, **b** 120°, **c** 135°, **d** 144° **7** 162° **8** $c = 42$ **9 a** 60°, **b** 72°, **c** 30° **10** 120° **11** 15° **12 a** 8, **b** 10, **c** 18, **d** 9 **15 a** 540°, **b** 2340°

Exercise 4.3

1 a $\angle AED = 73°$ (co-interior $\angle$s, $AB \parallel CD$)
$\angle CEF = 73°$ (vertically opposite $\angle$s)
$\therefore x = 73$

b $\angle LKM = 64°$ ($\angle K$ bisects $\angle JKM$)
$\angle KMN = 64°$ (alternate $\angle$s, $KL \parallel NM$)
$\therefore x = 64$

c $\angle XZV = 44°$ (corresponding $\angle$s, $TU \parallel VW$)
$\angle VZS = 46°$ (adjacent $\angle$s in a right angle)
$\therefore x = 46$

d $\angle DIH = 67°$ (corresponding $\angle$s, $DE \parallel FG$)
$\angle DEJ = 67°$ (corresponding $\angle$s, $GH \parallel EJ$)
$\therefore x = 67$

e $\angle KLN = 111°$ (co-interior $\angle$s, $KL \parallel MN$)
$\angle KLJ = 153°$ ($\angle$s at a point)
$\angle IJL = 27°$ (co-interior $\angle$s, $IJ \parallel KL$)
$\therefore x = 27$

f $\angle TUX = 25°$ (adjacent $\angle$s on a straight line)
$\angle TUV = 48°$ (sum of adjacent $\angle$s)
$\angle STU = 48°$ (alternate $\angle$s, $ST \parallel UV$)
$\therefore x = 48$

g $\angle BFG = 58°$ (vertically opposite $\angle$s)
$\angle EGD = 122°$ (co-interior $\angle$s, $AB \parallel CD$)
$\angle HGD = 61°$ (HG bisects $\angle EGD$)
$\therefore x = 61$

h $\angle DBC = 73°$ (adjacent $\angle$s in a right angle)
$\angle BDE = 107°$ (co-interior $\angle$s, $DE \parallel BC$)
$\angle DEG = 107°$ (alternate $\angle$s, $BD \parallel EG$)
$\angle DEF = 103°$ ($\angle$s at a point)
$\therefore x = 103$

i $\angle WRS = 77°$ (corresponding $\angle$s, $PQ \parallel RS$)
$\angle TRP = 77°$ (WV bisects $\angle TRS$)
$\angle VRU = 77°$ (vertically opposite $\angle$s)
$\therefore x = 77$

2 a $\angle BCD = 47°$ (alternate $\angle$s, $AB \parallel CE$)
$\angle BDE = 137°$ (exterior $\angle$ of a Δ)
$\therefore x = 137$

b $\angle PRQ = 28°$ (vertically opposite $\angle$s)
$\angle PQR = 116°$ ($\angle$ sum of a Δ)
$\angle PQU = 58°$ (UQ bisects $\angle PQR$)
$\therefore x = 58$

c $\angle ZXY = 62°$ ($\angle$ sum of a Δ)
$\angle WXV = 62°$ (vertically opposite $\angle$s)
$\angle VWX = 62°$ (base $\angle$s of isosceles Δ, $VW = VX$)
$\angle WVX = 56°$ ($\angle$ sum of a Δ)
$\therefore x = 56$

d $\angle EFG = 71°$ (co-interior $\angle$s, $EF \parallel HG$)
$\angle DEF = 71°$ (alternate $\angle$s, $DE \parallel FG$)
$\angle EDF = 67°$ ($\angle$ sum of a Δ)
$\therefore x = 67$

e $\angle LMN = 48°$ (corresponding $\angle$s, $JK \parallel LM$)
$\angle LNO = 81°$ (exterior $\angle$ of a Δ)
$\angle ONP = 81°$ (OK bisects $\angle LNP$)
$\therefore x = 81$

f $\angle CBD = 61°$ (adjacent $\angle$s in a right angle)
$\angle CDB = 61°$ (base $\angle$s of an isosceles Δ, $CB = CD$)
$\angle DCE = 61°$ (alternate $\angle$s, $CE \parallel BD$)
$\angle CED = 61°$ (base $\angle$s of an isosceles Δ, $DC = DE$)
$\therefore x = 61$

g $\angle EDG = 83°$ ($\angle$s at a point)
$\angle EFG = 87°$ ($\angle$ sum of a quadrilateral)
$\angle GFH = 93°$ (adjacent $\angle$s on a straight line)
$\therefore x = 93$

h $\angle QRS = 72°$ (alternate $\angle$s, $PQ \parallel RS$)
$\angle UTS = 72°$ (opposite $\angle$s of a parallelogram)
$\angle STV = 18°$ (adjacent $\angle$s in a right angle)
$\therefore x = 18$

j $\angle UWV = 60°$ ($\angle$ in an equilateral Δ)
$\angle VWX = 65°$ ($\angle$s at a point)
$\angle WVY = 115°$ (co-interior $\angle$s, $VY \parallel WX$)
$\therefore x = 115$

l $\angle IML = 108°$ ($\angle$ in a regular pentagon)
$\angle MLN = 108°$ (alternate $\angle$s, $LN \parallel IM$)
$\angle LMN = 45°$ ($\angle$ sum of a Δ)
$\therefore x = 45$

i $\angle VWX = 123°$ (co-interior $\angle$s, $UV \parallel XW$)
$\angle XWZ = 85°$ ($\angle$s at a point)
$\angle WXY = 151°$ ($\angle$ sum of a quadrilateral)
$\therefore x = 151$

k $OA = OB$ (equal radii)
$\angle OAB = 55°$ (corresponding $\angle$s, $AB \parallel CD$)
$\angle OBA = 55°$ (base $\angle$s of an isosceles Δ, $OA = OB$)
$\angle EOB = 110°$ (exterior $\angle$ of a Δ)
$\therefore x = 110$

3 a $\angle QRS = 90°$ ($\angle$ in a rectangle)
$\angle QSR = 37°$ ($\angle$ sum of a Δ)
$\therefore x = 37$

b $PQ \parallel SR$ (opposite sides of a rectangle are parallel)
$\angle QPR = 21°$ (alternate $\angle$s, $PQ \parallel SR$)
$\therefore x = 21$

c $PT = TS$ (diagonals of a rectangle are equal and bisect each other)
$\angle PST = 67°$ (base $\angle$s of an isosceles Δ, $PT = TS$)
$\angle PTS = 46°$ ($\angle$ sum of a Δ)
$\angle QTR = 46°$ (vertically opposite $\angle$s)
$\therefore x = 46$

4 a $BC = CD$ (sides of a rhombus)
$\angle CBD = 17°$ (base $\angle$s of an isosceles Δ, $BC = CD$)
$\therefore m = 17$

b $\angle AEB = 90°$ (diagonals of a rhombus are perp.)
$\angle ABE = 36°$ ($\angle$ sum of a Δ)
$\angle CBE = 36°$ (diagonals of a rhombus bisect $\angle$s at the vertices)
$\therefore m = 36$

c $AD \parallel BC$ (opposite sides of a rhombus are parallel)
$\angle ABC = 64°$ (co-interior $\angle$s, $AD \parallel BC$)
$\angle EBC = 32°$ (diagonals of a rhombus bisect $\angle$s at the vertices)
$\therefore m = 32$

d $\angle ADC = 52°$ (adjacent $\angle$s on a st. line)
$\angle ADB = 26°$ (diagonals of a rhombus bisect $\angle$s at the vertices)
$\therefore m = 26$

e $\angle BEC = 90°$ (diagonals of a rhombus are perp.)
$\angle FEC = 61°$ (alternate $\angle$s, $EF \parallel DC$)
$\angle BEF = 29°$ (adjacent $\angle$s in a right angle)
$\therefore m = 29$

f $\angle DEF = 75°$ (base $\angle$s of an isosceles Δ, $DE = DF$)
$\angle EDF = 30°$ ($\angle$ sum of a Δ)
$AB \parallel DC$ (opposite sides of a rhombus are parallel)
$\angle DAB = 150°$ (co-interior $\angle$s, $AB \parallel DC$)
$\therefore m = 150$

Exercise 4.4

1 Let $\angle AEC = \alpha$
$\angle AED = 180° - \alpha$ (adjacent $\angle$s on a straight line)
$\angle DEB = 180° - (180° - \alpha) = \alpha$ (adjacent $\angle$s on a straight line)
$\therefore \angle AEC = \angle DEB$ (both equal to α)

3 Let $\angle POR = \alpha$
$\angle POS = \alpha$ (PO bisects $\angle ROS$)
$\angle ROQ = 180° - \alpha$ (adjacent $\angle$s on a straight line)
$\angle SOQ = 180° - \alpha$ (adjacent $\angle$s on a straight line)
$\therefore \angle ROQ = \angle SOQ$ (both equal to $180° - \alpha$)

2 a Let $\angle ABD = \alpha$
$\angle DBC = 90° - \alpha$ (adjacent $\angle$s in a right angle)
$\angle EBC = \alpha$ (adjacent $\angle$s in a right angle)
$\therefore \angle ABD = \angle EBC$ (both equal to α)

b $\angle ABE + \angle DBC = 90° + \alpha + 90° - \alpha$
$= 180°$

4 Let $\angle ABD = \alpha$ and $\angle FBC = \beta$
$\angle DBE = \alpha$ (DB bisects $\angle ABE$)
$\angle EBF = \beta$ (FB bisects $\angle EBC$)
$2x + 2\beta = 180°$ ($\angle$s on a straight line)
$\therefore \alpha + \beta = 90°$
$\therefore \angle DBF = \alpha + \beta$ (sum of adjacent $\angle$s)
$= 90°$
$\therefore DB \perp BF$

ANSWERS

5 Let $\angle TWX = \alpha$
$\angle TUY = \alpha$ (corresponding $\angle$s, $WX \parallel UY$)
$\angle YUV = \alpha$ (UY bisects $\angle TUV$)
$\angle UVZ = \alpha$ (alternate $\angle$s, $UY \parallel ZV$)
$\therefore \angle TWX = \angle UVZ$ (both equal to α)

6 Let $\angle ABC = \alpha$ and $\angle CDE = \beta$
Construct FC, parallel to BA and DE
$\angle BCF = \alpha$ (alternate $\angle$s, $BA \parallel FC$)
$\angle DCF = \beta$ (alternate $\angle$s, $FC \parallel DE$)
$\angle BCD = \alpha + \beta$ (sum of adjacent $\angle$s)
$\therefore \angle BCD = \angle ABC + \angle CDE$

7 Let $\angle AEF = \alpha$, $\angle CAE = \beta$ and $\angle ACD = \gamma$
$\angle BAE = 180° - \alpha$ (co-interior $\angle$s, $AB \parallel EF$)
$(180° - \alpha) + \beta + \gamma = 180°$ (co-interior $\angle$s, $AB \parallel CD$)
$\therefore \alpha = \beta + \gamma$

8 Let $\angle CHG = \alpha$
$\angle GHD = 180° - \alpha$ (adjacent $\angle$s on a straight line)
$\angle CDI = 180° - \alpha$ (alternate $\angle$s, $GH \parallel DJ$)
$\angle DIE = 180° - (180° - \alpha) = \alpha$ (co-interior $\angle$s, $CD \parallel EF$)
$\angle JIF = \alpha$ (vertically opposite $\angle$s)
$\therefore \angle CHG = \angle JIF$ (both equal to α)

9 Let $\angle ABP = \alpha$
$\angle PBD = \alpha$ (PQ bisects $\angle ABD$)
$\angle ABD = 2\alpha$ (sum of adjacent $\angle$s)
$\angle BDE = 2\alpha$ (alternate $\angle$s, $AB \parallel CE$)
$\angle BDS = \alpha$ (RS bisects $\angle BDE$)
$\therefore \angle PBD = \angle BDS$ (both equal to α)
$\therefore PQ \parallel RS$ (alternate $\angle$s are equal)

Exercise 4.5

1 Construct DE, through B, parallel to AC
$\angle DBA = \angle BAC$ (alternate $\angle$s, $DE \parallel AC$)
$\angle EBC = \angle BCA$ (alternate $\angle$s, $DE \parallel AC$)
$\angle DBA + \angle ABC + \angle EBC = 180°$ ($\angle$s on a straight line)
$\therefore \angle BAC + \angle ABC + \angle BCA = 180°$

2 Construct CE, parallel to AB
$\angle ECD = \angle BAC$ (corresponding $\angle$s, $AB \parallel CE$)
$\angle BCE = \angle ABC$ (alternate $\angle$s, $AB \parallel CE$)
$\angle BCD = \angle ECD + \angle BCE$ (sum of adjacent $\angle$s)
$\therefore \angle BCD = \angle BAC + \angle ABC$

3 $\angle C = \angle A + \angle B$ (given)
$\angle A + \angle B + \angle C = 180°$ ($\angle$ sum of a Δ)
$\therefore \angle A + \angle B + (\angle A + \angle B) = 180°$
$2(\angle A + \angle B) = 180°$
$\therefore \angle A + \angle B = 90°$
$\therefore \angle C = 90°$
$\therefore \Delta ABC$ is right-angled

4 Let $\angle BCA = \alpha$
$\angle DAC = 90° - \alpha$ ($\angle$ sum of ΔACD)
$\angle EBC = 90° - \alpha$ ($\angle$ sum of ΔBCE)
$\therefore \angle DAC = \angle EBC$ (both equal to $90° - \alpha$)

5 Let $\angle BAC = \alpha$
$\angle BCA = \alpha$ (base $\angle$s of isosceles Δ, $AB = BC$)
$\angle DCE = \alpha$ (vertically opposite $\angle$s)
$\angle DEC = \alpha$ (alternate $\angle$s, $AB \parallel DE$)
$\therefore \Delta CDE$ is isosceles ($\angle DCE = \angle DEC$)

6 Let $\angle CAB = \alpha$
$\angle ACB = \alpha$ (base $\angle$s of isosceles Δ, $AB = BC$)
$\angle EBD = \alpha$ (corresponding $\angle$s, $AC \parallel BE$)
$\angle CBE = \alpha$ (alternate $\angle$s, $AC \parallel BE$)
$\therefore \angle CBE = \angle EBD$ (both equal to α)
$\therefore EB$ bisects $\angle CBD$

7 Let $\angle PRS = \alpha$
$\angle QRS = \alpha$ (SR bisects $\angle PRQ$)
$\angle PRQ = 2\alpha$ (sum of adjacent $\angle$s)
$\angle PQR = 2\alpha$ (base $\angle$s of isosceles Δ, $PQ = PR$)
$\angle PSR = 3\alpha$ (exterior $\angle$ of ΔQRS)
$\therefore \angle PSR = 3\angle PRS$

8 $\angle BAD = 30°$ (given)
$\angle ABD = 30°$ (base $\angle$s of isosceles Δ, $AD = BD$)
$\angle DBC = 60°$ (adjacent $\angle$s in a right angle)
$\angle BCD = 60°$ ($\angle$ sum of ΔABC)
$\angle BDC = 60°$ (exterior $\angle$ of ΔABD)
$\therefore \Delta BCD$ is equilateral (all $\angle$s are $60°$)

9 Let $\angle YXW = \alpha$ and $\angle XYW = \beta$

a $\angle XWZ = \alpha + \beta$ (exterior $\angle$ of ΔXYW)

b $\angle WXZ = \alpha$ (XW bisects $\angle YXZ$)

$\angle XZY = \beta$ (base $\angle$s of isosceles Δ, $XY = XZ$)

$\angle XWY = \alpha + \beta$ (exterior $\angle$ of ΔXWZ)

c $\angle XWY + \angle XWZ = 180°$ (adjacent $\angle$s on a straight line)

$(\alpha + \beta) + (\alpha + \beta) = 180°$

$2\alpha + 2\beta = 180°$

$\therefore \alpha + \beta = 90°$

$\therefore \angle XWZ = \angle XWY = 90°$

$\therefore XW \perp YZ$

10 a Let $\angle BAD = \alpha$

$\angle ABD = \alpha$ (base $\angle$s of isosceles Δ, $AD = DB$)

$\angle DBC = 90° - \alpha$ (adjacent $\angle$s in a right angle)

$\angle ACB = 90° - \alpha$ ($\angle$ sum of ΔABC)

$\therefore \angle DBC = \angle ACB$ (both equal to $90° - \alpha$)

$\therefore \Delta BCD$ is isosceles

b $AD = DB$ (given)

$DB = DC$ (equal sides lie opposite equal $\angle$s, $\angle DBC = \angle ACB$)

$\therefore AD = DC$ (both equal to DB)

$\therefore D$ is the midpoint of AC

11 a $OA = OB = OC$ (equal radii)

b Let $\angle OAC = \alpha$ and $\angle OBC = \beta$

$\angle OCA = \alpha$ (base $\angle$s of isosceles Δ, $OA = OC$)

$\angle OCB = \beta$ (base $\angle$s of isosceles Δ, $OB = OC$)

$\therefore 2\alpha + 2\beta = 180°$ ($\angle$ sum of ΔABC)

$\therefore \alpha + \beta = 90°$

$\angle ACB = \alpha + \beta$ (sum of adjacent $\angle$s)

$\therefore \angle ACB = 90°$

12 a Let $\angle ABE = \alpha$ and $\angle BAC = \beta$

$\angle BEC = \alpha + \beta$ (exterior $\angle$ of ΔABE)

$\angle EBC = \alpha$ (BE bisects $\angle ABC$)

$\angle BCD = (\alpha + \beta) + \alpha = 2\alpha + \beta$ (exterior $\angle$ of ΔBCD)

b $\angle BAC + \angle BCD = \beta + 2\alpha + \beta$

$= 2(\alpha + \beta)$

$= 2\angle BEC$

13 Let $\angle ABG = \alpha$ and $\angle CDG = \beta$

$\angle GBD = \alpha$ (EB bisects $\angle ABD$)

$\angle GDB = \beta$ (FD bisects $\angle BDC$)

$\therefore 2\alpha + 2\beta = 180°$ (co-interior $\angle$s, $AB \parallel CD$)

$\therefore \alpha + \beta = 90°$

$\angle BGD = 180° - (\alpha + \beta)$ ($\angle$ sum of a Δ)

$= 180° - 90°$

$= 90°$

$\therefore EB \perp FD$

14 Let $\angle UWY = \alpha$

$\angle WUX = 90° - \alpha$ ($\angle$ sum of ΔUWX)

$\angle VYW = \alpha$ (base $\angle$s of isosceles Δ, $VW = VY$)

$\angle XZY = 90° - \alpha$ ($\angle$ sum of ΔXYZ)

$\angle UZV = 90° - \alpha$ (vertically opposite $\angle$s)

$\therefore \angle WUX = \angle UZV$ (both equal to $90° - \alpha$)

$\therefore \Delta UVZ$ is isosceles

15 a $\angle BDC = \alpha + \beta$ (exterior $\angle$ of ΔABD)

b $\angle BCE = \angle BAD$ (given)

$= \alpha$

$\angle DBC = \beta$ (BD bisects $\angle ABC$)

$\angle DEC = \alpha + \beta$ (exterior $\angle$ of ΔBCE)

$\therefore \angle BDC = \angle DEC$ (both equal to $\alpha + \beta$)

$\therefore CD = CE$ (equal sides lie opposite equal $\angle$s)

16 Let $\angle ABG = \alpha$ and $\angle CDE = \beta$

$\angle GBC = \alpha$ (GB bisects $\angle ABC$)

$\angle ABC = 2\alpha$ (sum of adjacent $\angle$s)

$\angle BCF = 180° - 2\alpha$ (co-interior $\angle$s, $AB \parallel FE$)

$\angle CED = \beta$ (base $\angle$s of isosceles Δ, $CD = CE$)

$\angle ECD = 180° - 2\beta$ ($\angle$ sum of ΔCDE)

$\angle BCF = \angle ECD$ (vertically opposite $\angle$s)

$\therefore 180° - 2\alpha = 180° - 2\beta$

$\therefore \alpha = \beta$

$\therefore \angle GBC = \angle CDE$

$\therefore GB \parallel DE$ (alternate $\angle$s are equal)

17 Let $\angle ABF = \alpha$ and $\angle ACE = \beta$

$\angle EBC = \alpha$ (EB bisects $\angle ABC$)

$\angle ACB = 2\alpha$ (base $\angle$s of isosceles Δ, $AB = AC$)

$\angle BEC = \alpha$ (alternate $\angle$s, $AB \parallel EC$)

$\angle ECD = \beta$ (EC bisects $\angle ACD$)

$2\alpha + 2\beta = 180°$ (adjacent $\angle$s on a straight line)

$\therefore \alpha + \beta = 90°$

$\angle BFC = \alpha + \beta$ (exterior $\angle$ of ΔCEF)

$\therefore \angle BFC = 90°$

$\therefore AC \perp BE$

ANSWERS

Exercise 4.6

1 **a** AAS, **b** RHS, **c** SSS, **d** SAS **2** **a** yes, SAS, **b** no, **c** yes, RHS, **d** no **3** no, AAA is not a congruence test

4 **a** $QR = SR$ (given)
$PR = TR$ (given)
$\angle QRP = \angle SRT$ (vertically opposite $\angle$s)
$\therefore \Delta PQR \equiv \Delta TSR$ (SAS)

b $DF = FH$ (given)
$\angle DEF = \angle FGH$ (alternate $\angle$s, $DE \parallel GH$)
$\angle DFE = \angle GFH$ (vertically opposite $\angle$s)
$\therefore \Delta DEF \equiv \Delta HGF$ (AAS)

c $XY = XW$ (given)
$YZ = WZ$ (given)
XZ is a common side
$\therefore \Delta XYZ \equiv \Delta XWZ$ (SSS)

d $LM = MN$ (given)
MK is a common side
$\angle MKL = \angle MKN = 90°$ ($MK \perp LN$)
$\therefore \Delta MLK \equiv \Delta MNK$ (RHS)

5 **a** $AE = BE$ (CD bisects AB)
$CE = DE$ (AB bisects CD)
$\angle AEC = \angle BED$ (vertically opposite $\angle$s)
$\therefore \Delta ACE \equiv \Delta BDE$ (SAS)
$\therefore AC = BD$ (matching sides of congruent Δs)

b $OI = OK$ (equal radii)
OJ is a common side
$\angle OJI = \angle OJK = 90°$ ($OJ \perp IK$)
$\therefore \Delta OIJ \equiv \Delta OKJ$ (RHS)
$\therefore \angle IOJ = \angle KOJ$ (matching $\angle$s of congruent Δs)
$\therefore OJ$ bisects $\angle IOK$

c $\angle WXY = \angle WZY$ (given)
$\angle XYW = \angle ZYW$ (WY bisects $\angle XYZ$)
WY is a common side
$\therefore \Delta XYW \equiv \Delta ZYW$ (AAS)
$\therefore WX = WZ$ (matching sides of congruent Δs)
$\therefore \Delta XWZ$ is isosceles

d $OA = OD$ (equal radii)
$OB = OC$ (equal radii)
$AB = CD$ (given)
$\therefore \Delta AOB \equiv \Delta DOC$ (SSS)
$\therefore \angle AOB = \angle COD$ (matching $\angle$s of congruent Δs)

e $\angle QPR = \angle SRT$ (corresponding $\angle$s, $PQ \parallel RS$)
$\angle QRP = \angle STR$ (corresponding $\angle$s, $QR \parallel ST$)
$PR = RT$ (QR bisects PT)
$\therefore \Delta PQR \equiv \Delta RST$ (AAS)
$\therefore PQ = RS$ (matching sides of congruent Δs)

f $\angle IJL = \angle JLK$ (alternate $\angle$s, $IJ \parallel LK$)
$\angle ILJ = \angle LJK$ (alternate $\angle$s, $LI \parallel KJ$)
LJ is a common side
$\therefore \Delta IJL \equiv \Delta KLJ$ (AAS)
$\therefore \angle IJL = \angle KJL$ (matching $\angle$s of congruent Δs)
$\therefore LJ$ bisects $\angle IJK$

g $CD = DE$ (given)
DF is a common side
$\angle DFC = \angle DFE = 90°$ ($DF \perp CE$)
$\therefore \Delta CDF \equiv \Delta EDF$ (RHS)
$\therefore CF = FE$ (matching sides of congruent Δs)
$\therefore DF$ bisects CE

h $RS = QT$ (given)
RT is a common side
$\angle SRT = \angle RTQ$ (alternate $\angle$s, $RS \parallel QT$)
$\therefore \Delta RST \equiv \Delta TQR$ (SAS)
$\therefore \angle RTS = \angle QRT$ (matching $\angle$s of congruent Δs)
$\therefore QR \parallel TS$ (alternate $\angle$s are equal)

i $AB = BD$ (BC bisects AD)
$BC = DE$ (given)
$\angle ABC = \angle BDE$ (corresponding $\angle$s, $BC \parallel DE$)
$\therefore \Delta ABC \equiv \Delta BDE$ (SAS)
$\therefore \angle BAC = \angle DBE$ (matching $\angle$s of congruent Δs)
$\therefore AC \parallel BE$ (corresponding $\angle$s are equal)

j $OX = OY$ (equal radii)
$XM = MY$ (OM bisects XY)
OM is a common side
$\therefore \Delta OMX \equiv \Delta OMY$ (SSS)
$\therefore \angle OMX = \angle OMY$ (matching $\angle$s of congruent Δs)
But, $\angle OMX + \angle OMY = 180°$ (adjacent $\angle$s on a straight line)
$\therefore \angle OMX = \angle OMY = 90°$
$\therefore OM \perp XY$

6 **a** $PQ = PR$ (given)
$QX = RY$ (given)
$\angle PQX = \angle PRY$ (base $\angle$s of isosceles Δ, $PQ = PR$)
$\therefore \Delta PQX \equiv \Delta PRY$ (SAS)

b $PX = PY$ (matching sides of congruent Δs)
$\therefore \Delta$PXY is isosceles

7 **a** $AB = AC$ (given)
$PB = QC$ (P, Q are midpoints of equal sides)

b $PB = QC$ (proven above)
BC is a common side
$\angle ABC = \angle ACB$ (base $\angle$s of isosceles Δ, $AB = AC$)
$\therefore \Delta PBC \equiv \Delta QCB$ (SAS)
$\therefore PC = QB$ (matching sides of congruent Δs)

8 **a** $\angle ABC = \angle ACB$ (base $\angle$s of isosceles Δ, $AB = AC$)
$\angle BLC = \angle CMB = 90°$ ($CL \perp AB$, $BM \perp AC$)
BC is a common side
$\therefore \Delta BLC \equiv \Delta CMB$ (AAS)

b $BL = CM$ (matching sides of congruent Δs)
$\angle BLN = \angle CMN = 90°$ ($CL \perp AB$, $BM \perp AC$)
$\angle BNL = \angle CNM$ (vertically opposite $\angle$s)
$\therefore \Delta BLN \equiv \Delta CMN$ (AAS)

c $LN = MN$ (matching sides of congruent Δs)

9 a Construct AD, the angle bisector of $\angle A$

$AB = AC$ (given)

$\angle BAD = \angle CAD$ (AD bisects $\angle BAC$)

AD is a common side

$\therefore \Delta BAD \equiv \Delta CAD$ (SAS)

$\therefore \angle ABC = \angle ACB$ (matching $\angle$s of congruent Δs)

b Construct AD, the angle bisector of $\angle A$

$\angle ABC = \angle ACB$ (given)

$\angle BAD = \angle CAD$ (AD bisects $\angle BAC$)

AD is a common side

$\therefore \Delta ABD \equiv \Delta ACD$ (AAS)

$\therefore AB = AC$ (matching sides of congruent Δs)

10 a $AC = CB$ (sides of an equilateral Δ)

$\angle ACD = \angle BCD$ (CD bisects $\angle ACB$)

CD is a common side

$\therefore \Delta ACD \equiv \Delta BCD$ (SAS)

$\therefore \angle A = \angle B$ (matching $\angle$s of congruent Δs)

b $AB = AC$ (sides of an equilateral Δ)

$\angle BAE = \angle CAE$ (AE bisects $\angle BAC$)

AE is a common side

$\therefore \Delta ABE \equiv \Delta ACE$ (SAS)

$\therefore \angle B = \angle C$ (matching $\angle$s of congruent Δs)

c $\angle A = B$ (proven above)

$\angle B = \angle C$ (proven above)

$\therefore \angle A = \angle B = \angle C$

But, $\angle A + \angle B + \angle C = 180°$ ($\angle$ sum of a Δ)

$\therefore \angle A = \angle B = \angle C = 60°$

11 i Let $\angle ABC = \alpha$ and $\angle PBC = \beta$

$\angle ABP = \alpha - \beta$ (by subtraction)

$\angle ACB = \alpha$ (base $\angle$s of isosceles Δ, $AB = AC$)

$\angle PCB = \beta$ (base $\angle$s of isosceles Δ, $PB = PC$)

$\angle ACP = \alpha - \beta$ (by subtraction)

$\therefore \angle ABP = \angle ACP$ (both equal to $\alpha - \beta$)

ii $AB = AC$ (given)

$\angle ABP = \angle ACP$ (proven above)

$PB = PC$ (given)

$\therefore \Delta ABP \equiv \Delta ACP$ (SAS)

$\angle BAP = \angle CAP$ (matching $\angle$s of congruent Δs)

$\therefore PA$ bisects $\angle ABC$

12 $QM = PN$ (given)

$PM = SN$ (given)

$\angle LMN = \angle LNM$ (base $\angle$s of isosceles Δ, $LM = LN$)

$\therefore \Delta QMP \equiv \Delta PNS$ (SAS)

$\therefore PQ = PS$ (matching sides of congruent Δs)

$\therefore \angle PQS = \angle PSQ$ (base $\angle$s of isosceles Δ, $PQ = PS$)

13 i $\angle BAC = 60°$ ($\angle$ in an equilateral Δ)

$\angle DAC = 120°$ (adjacent $\angle$s on a straight line)

$\angle ACB = 60°$ ($\angle$ in an equilateral Δ)

$\angle BCE = 120°$ (adjacent $\angle$s on a straight line)

$\angle ACD = 30°$ (adjacent $\angle$s in a right angle)

$\angle ABC = 60°$ ($\angle$ in an equilateral Δ)

$\angle CBE = 30°$ (adjacent $\angle$s in a right angle)

ii $AC = BC$ (sides of an equilateral Δ)

$\angle ACD = \angle CBE = 30°$ (proven above)

$\angle DAC = \angle BCE = 120°$ (proven above)

$\therefore \Delta ACD \equiv \Delta CBE$ (AAS)

$\therefore CE = AD$ (matching sides of congruent Δs)

Exercise 4.7

1 $\angle BAC + \angle ABC + \angle BCA = 180°$ ($\angle$ sum of ΔABC)

$\angle DAC + \angle ADC + \angle DCA = 180°$ ($\angle$ sum of ΔADC)

$\therefore (\angle BAC + \angle DAC) + \angle ABC + (\angle BCA + \angle DCA)$

$+ \angle ADC = 360°$ (by addition)

$\therefore \angle A + \angle B + \angle C + \angle D = 360°$

2 a $\angle BAC = \angle ACD$ (alternate $\angle$s, $AB \parallel DC$)

$\angle BCA = \angle CAD$ (alternate $\angle$s, $BC \parallel AD$)

AC is a common side

$\therefore \Delta ABC ||| \Delta CDA$ (AAS)

b $AB = DC$ (matching sides of congruent Δs)

$AD = BC$ (matching sides of congruent Δs)

c $\angle ABC = \angle ADC$ (matching $\angle$s of congruent Δs)

d the opposite sides of a parallelogram are equal, and the opposite angles of a parallelogram are equal.

3 a $\angle ABP = \angle PDC$ (alternate $\angle$s, $AB \parallel DC$)

$\angle BAP = \angle PCD$ (alternate $\angle$s, $AB \parallel DC$)

$AB = DC$ (opp. sides of a parallelogram)

$\therefore \Delta APB \equiv \Delta CPD$ (AAS)

b $AP = PC$ (matching sides of congruent Δs)

$BP = PD$ (matching sides of congruent Δs)

c the diagonals of a parallelogram bisect each other.

4 a $\angle BCA = \alpha$ (base $\angle$s of isosceles Δ, $AB = BC$)

$\angle DAC = \alpha$ (alternate $\angle$s, $AD \parallel BC$)

b $\angle BDC = \beta$ (alternate $\angle$s, $AB \parallel DC$)

$BC = CD$ (sides of a rhombus are equal)

$\angle CBD = \beta$ (base $\angle$s of isosceles Δ, $BC = CD$)

c the diagonals of a rhombus bisect the angles at the vertices.

5 a $\angle BCA = \alpha$ (base $\angle$s of isosceles Δ, $AB = BC$)
$\angle CBD = \beta$ (diagonal of a rhombus bisects the $\angle$ at the vertex)
b $2\alpha + 2\beta = 180°$ ($\angle$ sum of ΔABC)
$\therefore \alpha + \beta = 90°$
c $\angle BPC = \alpha + \beta$ (exterior $\angle$ of ΔABP)
$\therefore \angle BPC = 90°$
$\therefore AC \perp BD$
d the diagonals of a rhombus are perpendicular.

7 a i $2\alpha + 2\beta = 360°$ ($\angle$ sum of a quadrilateral)
$\therefore \alpha + \beta = 180°$
ii $\angle BAD + \angle ADC = \alpha + \beta = 180°$
$\therefore AB \parallel DC$ (co-interior $\angle$s are supplementary)
$\angle BAD + \angle ABC = \alpha + \beta = 180°$
$\therefore AD \parallel BC$ (co-interior $\angle$s are supplementary)
b i $AB = CD$ (given)
$BC = AD$ (given)
AC is a common side
$\therefore \Delta ABC \equiv \Delta CDA$ (SSS)
ii $\angle BAC = \angle ACD$ (matching $\angle$s of congruent Δs)
$\therefore AB \parallel CD$ (alternate $\angle$s are equal)
$\angle BCA = \angle CAD$ (matching $\angle$s of congruent Δs)
$\therefore BC \parallel AD$ (alternate $\angle$s are equal)
c i $AB = DC$ (given)
$\angle BAC = \angle ACD$ (alternate $\angle$s, $AB \parallel DC$)
AC is a common side
$\therefore \Delta ABC \equiv \Delta CDA$ (SAS)
ii $\angle BCA = \angle CAD$ (matching $\angle$s of congruent Δs)
$\therefore AD \parallel BC$ (alternate $\angle$s are equal)
d i $AP = PC$ (BD bisects AC)
$BP = PD$ (AC bisects BD)
$\angle APB = \angle CPD$ (vertically opposite $\angle$s)
$\therefore \Delta APB \equiv \Delta CPD$ (SAS)
ii $AB = DC$ (matching sides of congruent Δs)
$\angle ABP = \angle PDC$ (matching $\angle$s of congruent Δs)
$\therefore AB \parallel DC$ (alternate $\angle$s are equal)

10 a $\angle ADP = \alpha$ (base $\angle$s of isosceles Δ, $PA = PD$)
$\angle ABP = \beta$ (base $\angle$s of isosceles Δ, $PA = PB$)
b $2\alpha + 2\beta = 180°$ ($\angle$ sum of ΔABD)
$\therefore \alpha + \beta = 90°$
$\angle DAB = \alpha + \beta$ (sum of adjacent $\angle$s)
$\therefore \angle DAB = 90°$
c $ABCD$ is a parallelogram (diagonals bisect each other)
But, $\angle DAB = 90°$ (proven above)
$\therefore ABCD$ is a rectangle (parallelogram with one angle a right angle).

6 a $AB = CD$ (opposite sides of a rectangle are equal)
$\angle ABC = \angle BCD = 90°$ (angles in a rectangle)
BC is a common side
$\therefore \Delta ABC \equiv \Delta DCB$ (SAS)
b $AC = BD$ (matching sides of congruent Δs)
c the diagonals of a rectangle are equal.

8 a $AP = PC$ (BD bisects AC)
$\angle BPA = \angle BPC = 90°$ ($BD \perp AC$)
BP is a common side
$\therefore \Delta ABP \equiv \Delta CBP$ (SAS)
b $AB = BC$ (matching sides of congruent Δs)
c $ABCD$ is a parallelogram (diagonals bisect each other)
$AB = BC$ (proven above)
$\therefore ABCD$ is a rhombus (parallelogram with a pair of adjacent sides equal)

9 i $4\alpha = 360°$ ($\angle$ sum of a quadrilateral)
$\therefore \alpha = 90°$
ii $\angle BAD + \angle ADC = \alpha + \alpha = 180°$
$\therefore AB \parallel DC$ (co-interior $\angle$s are supplementary)
$\angle BAD + \angle ABC = \alpha + \alpha = 180°$
$\therefore AD \parallel BC$ (co-interior $\angle$s are supplementary)
$\therefore ABCD$ is a parallelogram (two pairs of opposite sides parallel)
But, $\angle ABC = 90°$
$\therefore ABCD$ is a rectangle (a parallelogram with one angle a right angle).

11 a $\angle PBR = \angle RDC$ (alternate $\angle$s, $AB \parallel DC$)
$\angle BPR = \angle RQD$ (alternate $\angle$s, $AB \parallel DC$)
$BR = RD$ (AC bisects BD)
$\therefore \Delta BPR \equiv \Delta DQR$ (AAS)
b $PB = DQ$ (matching sides of congruent Δs)
$AB = DC$ (opposite sides of a parallelogram are equal)
$\therefore AB - PB = DC - DQ$
$\therefore AP = QC$

12 a $\angle BDC = \alpha$ (alternate $\angle$s, $AB \parallel DC$)
$\angle ADB = \alpha$ (BD bisects $\angle ADC$)
$\therefore AB = AD$ (equal sides lie opposite equal $\angle$s)
b $ABCD$ is a rhombus (parallelogram with a pair of adjacent sides equal).

13 a Let $\angle DBC = \alpha$
$\angle ADB = \alpha$ (alternate $\angle$s, $AB \parallel DC$)
$\angle FBC = 180° - \alpha$ (adjacent $\angle$s on a st. line)
$\angle ADE = 180° - \alpha$ (adjacent $\angle$s on a st. line)
$\therefore \angle FBC = \angle ADE$ (both equal to $180° - \alpha$)
b $\angle FBC = \angle ADE$ (proven above)
$BC = AD$ (opposite sides of a parallelogram are equal)
$BF = ED$ (given)
$\therefore \Delta FBC \equiv \Delta EDA$ (SAS)
c $\angle BFC = \angle DEA$ (matching $\angle$s of congruent Δs)
$\therefore FC \parallel AE$ (alternate $\angle$s are equal)
$FC = AE$ (matching sides of congruent Δs)
$\therefore AFCE$ is a parallelogram (one pair of opposite sides equal and parallel).

Exercise 4.8

1 a $x = 39$, **b** $a = 25$, **c** $t = 2\sqrt{5}$, **d** $m = 15$, **e** $u = 4\sqrt{2}$, **f** $e = 3\sqrt{5}$
2 a no, **b** yes, **c** no, **d** yes **3 a** $BD = 12$ cm, $BC = 15$ cm, **b** $AC^2 = AB^2 + BC^2$; $\therefore \Delta ABC$ is right-angled.
4 12 cm **5** 4 m **6** 3 cm **7** $x = 25$ **8** 5 cm, 12 cm, 13 cm **11** 136 cm
14 b $AC^2 = x^2 + xy$, $BC^2 = y^2 + xy$, **c** $AB^2 = AC^2 + BC^2$ **15 a** $AB^2 = AD^2 + BD^2$, **b** $CD^2 = BC^2 - BD^2$
16 d In a quadrilateral in which the diagonals are perpendicular, the sum of the squares on the opposite sides are equal.

Chapter 4 Review

1 a $x = 50$, **b** $p = 74$, **c** $a = 62$, **d** $c = 36$, **e** $e = 114$, **f** $s = 55$, **g** $m = 153$, **h** $b = 86$
2 a $\angle ABD + \angle DBC = 90°$, $\therefore AB \perp BC$, **b** $\angle PQS + \angle SQR = 180°$, $\therefore P, Q, R$ are collinear.
3 a no, corresponding $\angle$s are not equal, **b** yes, co-interior $\angle$s are supplementary, **c** yes, alternate $\angle$s are equal
4 a 7, **b** 11 **5 a** 720°, 60°, **b** 1440°, 144°, **c** 540°, 108°, **d** 1260°, 140°, **e** 1080°, 135°, **f** 1800°, 150°
6 a 72°, **b** 45°, **c** 60° **7 a** 10, **b** 24, **c** 18 **8** no

9 a Let $\angle ABG = \alpha$
$\angle CDG = \angle ABG$ (corresponding $\angle$s, $AB \parallel CD$)
$= \alpha$
$\angle DEF = \angle ABG$ (given)
$= \alpha$
$\angle CDE = \angle DEF$ (alternate $\angle$s, $CD \parallel EF$)
$= \alpha$
Now, $\angle CDE = \angle CDG$ (both equal to α)
$\therefore CD$ bisects $\angle GDE$

b Let $\angle QSR = \alpha$
$\angle PQS = 180° - \alpha$ (co-interior $\angle$s, $PQ \parallel RS$)
$\angle QST = 90° - \alpha$ ($TS \perp RS$)
$\angle TSU = 90° - (90° - \alpha)$ ($QS \perp SU$)
$= \alpha$
Now, $\angle PQS + \angle TSU = 180°$, $\therefore \angle PQS$ and $\angle TSU$ are supplementary.

c Let $\angle IMH = \alpha$
$\angle IMN = 90° + \alpha$ (by addition)
$\angle JIM = 180° - (90° + \alpha)$ (co-interior $\angle$s, $IJ \parallel MN$)
$= 90° - \alpha$
$\angle JIK = 2\angle JIM$ (IM bisects $\angle JIK$)
$= 180° - 2\alpha$
$\angle IKL = 2\alpha$ (co-interior $\angle$s, $IJ \parallel KL$)
$\therefore \angle IKL = 2\angle IMH$

d Let $\angle PQS = \alpha$
$\angle SPQ = \angle PQS$ (base $\angle$s of isosceles Δ, $PS = SQ$)
$= \alpha$
$\angle SQR = 90° - \alpha$ ($PQ \perp QR$)
$\angle PRQ = 90° - \alpha$ ($\angle$ sum of ΔPQR)
Now, $\angle SQR = \angle PRQ$ (both equal to $90° - \alpha$)
$\therefore \Delta QRS$ is isosceles

e Let $\angle GBD = \alpha$
$\angle ABG = \angle GBD$ (BG bisects $\angle ABD$)
$= \alpha$
$\angle BDF = 90° - \alpha$ ($\angle$ sum of ΔBED)
$\angle CDF = \angle BDF$ (FD bisects $\angle BDC$)
$= 90° - \alpha$
Now, $\angle ABD + \angle BDC = 2\alpha + 2(90° - \alpha)$
$= 180°$
$\therefore AB \parallel CD$ (co-interior $\angle$s are supplementary)

f Let $\angle EAC = \alpha$ and $\angle ECA = \beta$
$\angle ABC = \angle EAC$ (given)
$= \alpha$
$\angle BCD = \angle ECA$ (DC bisects $\angle ACB$)
$= \beta$
$\angle AED = \alpha + \beta$ (ext. $\angle$ of ΔACE)
$\angle ADE = \alpha + \beta$ (ext. $\angle$ of ΔBCD)
Now, $\angle AED = \angle ADE$ (both equal to $\alpha + \beta$)
$\therefore \Delta ADE$ is isosceles

10 a i $AB = BC$ (given)
$OA = OC$ (equal radii)
OB is a common side
$\therefore \Delta OAB \equiv \Delta OCB$ (SSS)

ii $\angle AOB = \angle COB$ (matching $\angle$s of congruent Δs)
$\therefore OB$ bisects $\angle AOC$

b i $\angle PQS = \angle QSR = 90°$ (alternate $\angle$s, $PQ \parallel SR$)
$PS = QR$ (given)
QS is a common side
$\therefore \Delta PQS \equiv \Delta RSQ$ (RHS)

ii $\angle PSQ = \angle RQS$ (matching $\angle$s of congruent Δs)
$\therefore PS \parallel QR$ (alternate $\angle$s are equal)
$PQ \parallel SR$ (given)
$\therefore PQRS$ is a parallelogram (opposite sides are parallel)

c i $\angle XWZ = \angle XYZ$ (given)
$\angle XZW = \angle XZY$ (XZ bisects $\angle WZY$)
XZ is a common side
$\therefore \Delta XWZ \equiv \Delta XYZ$ (AAS)

ii $XW = XY$ (matching sides of congruent Δs)
$\therefore \Delta WXY$ is isosceles

d i $CD = EF$ (given)
$\angle DCE = \angle FEG$ (corresponding $\angle$s, $CD \parallel EF$)
$CE = EG$ (EF bisects CG)
$\therefore \Delta CDE \equiv \Delta EFG$ (SAS)

ii $DE = FG$ (matching sides of congruent Δs)

11 a $PQ = SR$ (opposite sides of a parallelogram)
$\angle PQT = \angle RST$ (alternate $\angle$s, $PQ \parallel SR$)
$\angle PTQ = \angle STR$ (vert. opp. $\angle$s)
$\therefore \Delta PQT \equiv \Delta RST$ (AAS)

b $PT = TR$ (matching sides of congruent Δs)
$QT = TS$ (matching sides of congruent Δs)
$\therefore$ the diagonals bisect each other

12 a $KN = LM$ (opposite sides of a rectangle)
$\angle KNM = \angle LMN = 90°$ ($\angle$s in a rectangle)
NM is a common side
$\therefore \Delta KNM \equiv \Delta LMN$ (SAS)

b $KM = LN$ (matching sides of congruent Δs)
$\therefore$ diagonals of a rectangle are equal

13 a $XY = YZ$ (sides of a rhombus)
$\angle XZY = \angle ZXY$ (base $\angle$s of isosceles Δ, $XY = YZ$)
$= \alpha$
$\angle WXZ = \angle XZY$ (alternate $\angle$s, $WX \parallel ZY$)
$= \alpha$
Now, $\angle WXZ = \angle ZXY$ (both equal to α)
$\therefore ZX$ bisects $\angle WXY$

b $WX = XY$ (sides of a rhombus)
$\angle XWY = \angle XYW$ (base $\angle$s of isosceles Δ, $WX = XY$)
$= \beta$
$2\alpha + 2\beta = 180°$ ($\angle$ sum of ΔWXY)
$\therefore \alpha + \beta = 90°$
$\angle WAX = \alpha + \beta$ (exterior $\angle$ of ΔXAY)
$= 90°$
$\therefore XZ \perp WY$

14 a diagonals bisect each other

b SW is a common side
$\angle SWV = \angle SWT = 90°$ ($SU \perp VT$)
$VW = WT$ (SU bisects VT)
$\therefore \Delta SVW \equiv \Delta STW$ (SAS)

c $SV = ST$ (matching sides of congruent Δs)
$\therefore STUV$ is a rhombus (parallelogram with a pair of adjacent sides equal)

15 a $AB = AC$ (given)
$BR = CS$ (R, S are midpoints of AB, AC)

b $BR = CS$ (proven above)
$\angle ABC = \angle ACB$ (base $\angle$s of isosceles Δ, $AB = AC$)
BC is a common side
$\therefore \Delta RBC \equiv \Delta SCB$ (SAS)

c $CR = BS$ (matching sides of congruent Δs)

d $TR = TS$ (given)
$\therefore CR - TR = BS - TS$
$\therefore TB = TC$
$\therefore \Delta BTC$ is isosceles

16 $AC^2 = AB^2 + BC^2$
$= AB^2 + (2AB)^2$
$= 5AB^2$

$AD^2 = AC^2 + CD^2$
$= 5AB^2 + (2AB)^2$
$= 9AB^2$
$\therefore AD = 3AB$

17 a $\dfrac{MP}{LP} = \dfrac{LP}{PN}$ (given)

$\therefore \dfrac{a}{LP} = \dfrac{LP}{b}$

$LP^2 = ab$

$\therefore LP = \sqrt{ab}$

b $LM^2 = a^2 + ab$, $LN^2 = b^2 + ab$

c $LM^2 + LN^2 = a^2 + ab + b^2 + ab$
$= a^2 + 2ab + b^2$
$= (a + b)^2$
$= MN^2$
$\therefore \Delta LMN$ is right-angled (converse of Pythagoras' theorem)

ANSWERS

5 Factorisation and algebraic fractions

Exercise 5.1

1 $x^2 + 8x + 15$ **2 a** $xy + 3x + 2y + 6$, **b** $pq - 2p + 5q - 10$, **c** $uv - 6u - 4v + 24$ **3 a** $x^2 + 6x + 8$, **b** $m^2 + 7m + 6$, **c** $u^2 + 3u - 40$, **d** $b^2 + 7b - 18$, **e** $a^2 - 5a - 24$, **f** $t^2 - 3t + 2$, **g** $c^2 - 10c + 24$, **h** $z^2 - 13z + 42$, **i** $d^2 + 7d - 60$, **j** $2x^2 + 7x + 3$, **k** $3m^2 + 10m + 8$, **l** $5a^2 - 18a - 8$, **m** $2g^2 - 11g + 15$, **n** $3t^2 - 17t - 6$, **o** $4n^2 + 13n - 35$, **p** $10r^2 + 41r + 21$, **q** $6k^2 + 17k - 45$, **r** $21v^2 - 62v + 16$ **5 a** $x^2 + 3x + 2$, **b** $p^2 + 5p + 6$, **c** $a^2 + 10a + 21$, **d** $t^2 - 9t + 20$, **e** $b^2 - 8b + 12$, **f** $c^2 - 7c + 12$, **g** $z^2 + 3z - 4$, **h** $d^2 - 3d - 18$, **i** $s^2 + 8s - 20$, **j** $e^2 + 5e - 14$, **k** $u^2 + 6u - 27$, **l** $k^2 - 7k - 44$, **m** $f^2 + 12f + 35$, **n** $w^2 - w - 30$, **o** $r^2 - 12r + 32$, **p** $g^2 + 2g - 48$, **q** $h^2 + 6h - 55$, **r** $v^2 + 11v + 24$, **s** $q^2 - 13q + 36$, **t** $m^2 + 3m - 54$, **u** $i^2 - 9i - 36$, **v** $l^2 + 4l - 60$, **w** $y^2 - 10y - 39$, **x** $j^2 - 15j + 56$ **6** $6a^2 - 9a - 60$ **7 a** $a^2 + 2ab + b^2$, **b** $m^2 + 2mn + n^2$, **c** $g^2 - 2gh + h^2$, **d** $p^2 - 2pq + q^2$ **8 a** $a^2 + 6a + 9$, **b** $p^2 - 8p + 16$, **c** $c^2 + 12c + 36$, **d** $t^2 - 2t + 1$, **e** $u^2 - 10u + 25$, **f** $k^2 + 20k + 100$, **g** $s^2 + 14s + 49$, **h** $p^2 - 18p + 81$ **9 a** $4x^2 + 12x + 9$, **b** $9t^2 - 30t + 25$, **c** $16m^2 + 8m + 1$, **d** $25c^2 - 20c + 4$, **e** $49 - 28g + 4g^2$, **f** $25 + 60r + 36r^2$, **g** $9p^2 - 6pq + q^2$, **h** $9a^2 + 24ab + 16b^2$, **i** $4e^2 - 44ef + 121f^2$, **j** $16c^2 + 40cd + 25d^2$, **k** $49g^2 - 42gh + 9h^2$, **l** $144j^2 + 120jk + 25k^2$, **m** $p^2q^2 + 2pqr + r^2$, **n** $36a^2b^2 - 84abcd + 49c^2d^2$, **o** $x^2 + 2 + \frac{1}{x^2}$ **10 a** no, **b** yes, **c** no, **d** yes, **e** yes, **f** no, **g** no, **h** no, **i** yes **11 a** 25, **b** $6p$, **c** $y + 6$, $12y$, **d** $a - 7$, $14a$, **e** $k + 4$, 16, **f** $u - 1$, 1, **g** $12b$, **h** $3e + 5$, $30e$, **i** $5n + 6$, 36, **j** $7z - 3$, 9 **12 a** $m^2 - n^2$, **b** $p^2 - 4$, **c** $r^2 - 25$, **d** $9 - g^2$, **e** $y^2 - 49$, **f** $16 - w^2$, **g** $t^2 - 1$, **h** $k^2 - 36$, **i** $100 - b^2$, **j** $4e^2 - 9$, **k** $25r^2 - 4$, **l** $9c^2 - 1$, **m** $16 - 49h^2$, **n** $36 - 25n^2$, **o** $4p^2 - 81q^2$, **p** $100y^2 - 9z^2$, **q** $49s^2 - 36t^2$, **r** $a^2b^2 - 64c^2$ **13 a** $a^2 + 9a + 18$, **b** $t^2 - 46$, **c** $2p^2 - 3p + 6$, **d** $4e^2 - 8e + 10$, **e** $a^2 + 8a + 35$, **f** $2x^2 + 12x + 25$, **g** $-6m - 25$, **h** $14y + 98$, **i** $k^2 - 21k + 14$, **j** $120c$, **k** $32h + 64$, **l** $24m - 18$ **14 a** $x^3 + 5x^2 + 10x + 8$, **b** $a^3 - 6a^2 + 13a - 20$, **c** $k^3 + 6k^2 + 11k + 6$, **d** $n^3 - 6n^2 - 13n + 42$, **e** $p^3 + 15p^2 + 63p + 49$, **f** $y^3 - 11y^2 + 39y - 45$, **g** $t^3 + 6t^2 + 12t + 8$, **h** $e^3 - 9e^2 + 27e - 27$

Exercise 5.2

1 a $2(c + 3)$, **b** $5(m + 4)$, **c** $3(7 + e)$, **d** $7(7 + x)$, **e** $3(g - 5)$, **f** $6(k - 1)$, **g** $3(9 - r)$, **h** $5(11 - t)$, **i** $2(x + y)$, **j** $3(m - 4n)$, **k** $8(3p + q)$, **l** $12(f - 4g)$, **m** $x(y + z)$, **n** $b(a - c)$, **o** $m(m + 7)$, **p** $c(c - 1)$ **2 a** $2(2n + 3)$, **b** $3(3b + 5)$, **c** $4(3y - 2)$, **d** $5(2u - 5)$, **e** $7(3p + 4)$, **f** $3(5g - 6)$, **g** $7(2w + 3)$, **h** $6(4z - 5)$, **i** $11(4h - 3)$, **j** $7(5d + 7)$, **k** $10(2q - 5)$, **l** $11(4f - 7)$, **m** $5(5 - 9k)$, **n** $7(8 + 9v)$, **o** $8(5 - 8a)$, **p** $12(2 - 7s)$ **3 a** $3y(x + 4z)$, **b** $2p(q + 5r)$, **c** $5g(f - 3h)$, **d** $7c(d - 6e)$, **e** $3j(2i + 3k)$, **f** $4n(3m - 5p)$, **g** $11v(2u - 3w)$, **h** $8t(4s + 3u)$, **i** $6b(b + 4)$, **j** $7a(3a - 1)$, **k** $2q(7q - 10)$, **l** $5u(5 + 6u)$, **m** $pq(r + s)$, **n** $de(c - f)$, **o** $xy(x + y)$, **p** $ab(c - b)$, **q** $fg(g - fh)$, **r** $3k(2j + 5km)$, **s** $4tu(4u + 5tv)$, **t** $7gh(4gh - 7i)$ **4 a** $4(p + q + r)$, **b** $a(b + c - d)$, **c** $x(x - y + z)$, **d** $3(e + 2f + 5g)$, **e** $2(3m^2 - 5m + 7n)$, **f** $5(1 - 3v - 6v^2)$, **g** $6(5c^2 - 2c + 3)$, **h** $a(11b - 1 + b^2)$, **i** $3r(r - s - 3)$, **j** $5y(4 - 5z + 2y)$, **k** $7i(j + 2i - 6k)$, **l** $12z(2 - 5y - 8z)$, **m** $pq(p + 1 + q)$, **n** $rs(9 - s - r)$, **o** $abc(a + b - c)$ **5 a** $-2(k + 4)$, **b** $-3(n + 5)$, **c** $-5(2c + 5)$, **d** $-7(2w + 3)$, **e** $-4(x - 1)$, **f** $-9(d - 2)$, **g** $-4(4m - 5)$, **h** $-3(3g - 7)$, **i** $-9(2 + 3y)$, **j** $-3(12 - 5e)$, **k** $-8(4 + 5z)$, **l** $-11(4 - 9t)$, **m** $-d(c + e)$, **n** $-j(i - k)$, **o** $-a(a + 7)$, **p** $-v(11 - v)$, **q** $-4n(2n - 5)$, **r** $-3b(7 + 8b)$, **s** $-4f(5 - 9e)$, **t** $-13cd(2c + 3d)$ **6 a** $m^2(1 + m)$, **b** $x^2(x^4 + 1)$, **c** $t^3(1 - t^5)$, **d** $y^3(y^7 - 1)$, **e** $a^4(1 + 3a^3)$, **f** $g^3(2 - g^5)$, **g** $u^5(7u^4 + 4)$, **h** $h^3(5 - 4h)$, **i** $3c^5(c^5 + 3)$, **j** $11g^3(2g^5 + 1)$, **k** $8q^7(2 - 3q^6)$, **l** $5z^2(9z^7 - 4)$

Exercise 5.3

1 a $x^2 - 9$, $(x + 3)(x - 3)$, **b** $4x^2 - 25$, $(2x + 5)(2x - 5)$ **2 a** $(p - q)(p + q)$, **b** $(c - d)(c + d)$, **c** $(m - n)(m + n)$, **d** $(u - v)(u + v)$ **3 a** $(x - 2)(x + 2)$, **b** $(a - 5)(a + 5)$, **c** $(p - 1)(p + 1)$, **d** $(y - 6)(y + 6)$, **e** $(z - 8)(z + 8)$, **f** $(c - 11)(c + 11)$, **g** $(t - 7)(t + 7)$, **h** $(b - 9)(b + 9)$, **i** $(4 - k)(4 + k)$, **j** $(8 - g)(8 + g)$, **k** $(10 - m)(10 + m)$, **l** $(12 - u)(12 + u)$ **4 a** $(e - 13)(e + 13)$, **b** $(h - 16)(h + 16)$, **c** $(23 - s)(23 + s)$, **d** $(19 - j)(19 + j)$ **5 a** $(2a - 3)(2a + 3)$, **b** $(3p - 5)(3p + 5)$, **c** $(7q - 2)(7q + 2)$, **d** $(4c - 1)(4c + 1)$, **e** $(6 - 7y)(6 + 7y)$, **f** $(5 - 4r)(5 + 4r)$, **g** $(1 - 11u)(1 + 11u)$, **h** $(10 - 9t)(10 + 9t)$, **i** $(3a - b)(3a + b)$, **j** $(x - 2y)(x + 2y)$, **k** $(6e - f)(6e + f)$, **l** $(j - 8k)(j + 8k)$, **m** $(2g - 7h)(2g + 7h)$, **n** $(5m - 12n)(5m + 12n)$, **o** $(10p - 3q)(10p + 3q)$, **p** $(6s - 11t)(6s + 11t)$, **q** $(ab - c)(ab + c)$, **r** $(p - qr)(p + qr)$, **s** $(2x - 3yz)(2x + 3yz)$, **t** $(4ef - 9gh)(4ef + 9gh)$ **6 a** $2(m - 3)(m + 3)$, **b** $3(a - 2)(a + 2)$, **c** $11(t - 1)(t + 1)$, **d** $5(y - 10)(y + 10)$, **e** $3(5 - x)(5 + x)$, **f** $7(3 - p)(3 + p)$, **g** $2(8 - e)(8 + e)$, **h** $6(3 - z)(3 + z)$, **i** $2(2n - 3)(2n + 3)$, **j** $3(2c - 3)(2c + 3)$, **k** $5(3 - 2f)(3 + 2f)$, **l** $2(3k - 5)(3k + 5)$, **m** $a(a - 1)(a + 1)$, **n** $n^2(1 - n)(1 + n)$, **o** $d(7d - 1)(7d + 1)$, **p** $u(4 - 5u)(4 + 5u)$, **q** $8h(h - 2)(h + 2)$, **r** $6w(3w - 2)(3w + 2)$, **s** $8s(3 - 2s)(3 + 2s)$, **t** $5j(4j - 5)(4j + 5)$ **7 a** $(2a - 6)(2a + 6)$, no — each factor still has a common factor of 2. **b** $4(a - 3)(a + 3)$ **8 a** $9(k - 2)(k + 2)$, **b** $4(c - 5)(c + 5)$, **c** $25(x - 2y)(x + 2y)$, **d** $4(e - 6f)(e + 6f)$ **9** 31 **10 a** 240, **b** 9800, **c** 2000 **11 a** $(a + b - c)(a + b + c)$, **b** $(m - n - p)(m - n + p)$, **c** $(x + 3 - y)(x + 3 + y)$, **d** $(j - k - 6)(j - k + 6)$, **e** $(b - 2)(b + 12)$, **f** $p(p + 4)$, **g** $(2m + 2n - 3)(2m + 2n + 3)$, **h** $(3c - 3d - 5)(3c - 3d + 5)$, **i** $(27p + 18q - 8r)(27p + 18q + 8r)$

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Exercise 5.4

1 **a** $(c+d)(a+b)$, **b** $(x+3)(x+2)$, **c** $(p-4)(3n+7)$, **d** $(x+y)(w-z)$, **e** $(t-1)(t-9)$, **f** $(a-8)(4g-5h)$, **g** $(q+r)(p+s)$, **h** $(a^2+2)(5-c)$, **i** $(y+4)(mn-8)$, **j** $(d+e)(u+1)$, **k** $(p-2q)(1+w)$, **l** $(7i-2j)(6h-1)$, **2** **a** $(m+n)(k+5)$, **b** $(c+d)(a+b)$, **c** $(p+q)(p+2)$, **d** $(w+3)(x+2y)$, **e** $(e-2)(c+4d)$, **f** $(h-1)(2g+3)$, **g** $(v+7)(4u+5)$, **h** $(x+2y)(3+8a)$, **i** $(n-3)(5m+6p)$, **j** $(k+6h)(3g+4)$, **k** $(r+6)(pq+7)$, **l** $(m+p)(n+1)$, **m** $(a-5b)(1+c)$, **n** $(3e+f)(1+e)$, **o** $(a+1)(a^2+1)$ **3** **a** $(c+d)(2-e)$, **b** $(q+s)(p-r)$, **c** $(p-q)(p-8)$, **d** $(y-3)(x-y)$, **e** $(h+i)(g-1)$, **f** $(3u-2v)(1-w)$, **g** $(x-4)(x^2-y)$, **h** $(k+7)(k-2m)$, **i** $(m-4)(jk-2)$, **j** $(5p-2q)(7n-1)$, **k** $(z-w)(3z-u)$, **l** $(2x+3y)(5-2z)$ **4** **a** $(y+z)(x+w)$, **b** $(p+q)(r+2)$, **c** $(n+p)(3m+k)$, **d** $(4y+3)(z+5)$, **e** $(d+7)(c+1)$, **f** $(e+1)(5f+6)$ **5** **a** $(2-y)(x-y)$, **b** $(z-7)(y-z)$, **c** $(b-c)(a+b)$, **d** $(a-b)(a^2-b)$, **e** $(11-n)(m+n)$, **f** $(m-4n)(1+5p)$, **g** $(v-w)(6u+5w)$, **h** $(2d-c)(7e-1)$, **i** $(x-y)(2w-y)$, **j** $(c-d)(c^2+d)$, **k** $(2q-5r)(5p-6q)$, **l** $(2s-7t)(9r+4s)$ **6** **a** $(x+4)(x+3+y)$, **b** $(m-2)(m-5+n)$, **c** $(k-4)(k+6-m)$, **d** $(a+b)(a+b+c)$, **e** $(x-7)(x-7-w)$, **f** $(g+h)(f+g+h)$ **7** **a** $(2c-d)(c+3d+5)$, **b** $(p-q)(p-q+3r)$

Exercise 5.5

1 **a** 3, 5, **b** 2, 7, **c** 4, 6, **d** 3, 10, **e** –2, –5, **f** –1, –2, **g** –4, –7, **h** –8, –9, **i** 6, –2, **j** 5, –4, **k** –7, 5, **l** –9, 6 **2** **b** **i** $a>0, b>0$, **ii** $a<0, b<0$, **iii** $a>0, b<0$, **iv** $a<0, b>0$ **3** **a** $(x+2)(x+3)$, **b** $(y+3)(y+4)$, **c** $(u+1)(u+2)$, **d** $(m+4)(m+6)$, **e** $(a+3)(a+7)$, **f** $(t+3)(t+6)$, **g** $(k+2)(k+7)$, **h** $(p+1)(p+7)$, **i** $(n+5)(n+9)$, **j** $(d+4)(d+8)$, **k** $(s+5)(s+6)$, **l** $(b+4)(b+9)$, **m** $(e+5)(e+10)$, **n** $(c+5)(c+12)$, **o** $(r+8)(r+11)$, **p** $(z+8)(z+12)$ **4** **a** $(m-3)(m-5)$, **b** $(q-2)(q-5)$, **c** $(d-2)(d-10)$, **d** $(a-5)(a-7)$, **e** $(u-2)(u-9)$, **f** $(e-1)(e-8)$, **g** $(n-5)(n-8)$, **h** $(w-4)(w-5)$, **i** $(h-7)(h-9)$, **j** $(v-3)(v-8)$, **k** $(t-1)(t-6)$, **l** $(s-4)(s-11)$, **m** $(k-3)(k-10)$, **n** $(j-7)(j-11)$, **o** $(x-6)(x-9)$, **p** $(f-6)(f-8)$ **5** **a** $(y+6)(y-2)$, **b** $(d+5)(d-3)$, **c** $(a+4)(a-1)$, **d** $(p+8)(p-2)$, **e** $(v-5)(v+2)$, **f** $(u-9)(u+2)$, **g** $(m-6)(m+5)$, **h** $(x-5)(x+1)$, **i** $(f+7)(f-3)$, **j** $(w-9)(w+7)$, **k** $(k-9)(k+3)$, **l** $(c+8)(c-4)$, **m** $(z-8)(z+5)$, **n** $(i+4)(i-3)$, **o** $(r-11)(r+9)$, **p** $(e-7)(e+2)$, **q** $(s+10)(s-7)$, **r** $(h+8)(h-7)$, **s** $(b-10)(b+4)$, **t** $(t+12)(t-9)$ **6** **a** $(n-2)(n-4)$, **b** $(c+6)(c+7)$, **c** $(x-2)(x+1)$, **d** $(d+11)(d-4)$, **e** $(q+2)(q+11)$, **f** $(t-1)(t-10)$, **g** $(v+6)(v+8)$, **h** $(j-12)(j+6)$, **i** $(g-10)(g+2)$, **j** $(b+3)(b+12)$, **k** $(r-9)(r+6)$, **l** $(u-1)(u-12)$, **m** $(e+10)(e+12)$, **n** $(l-7)(l+6)$, **o** $(y-2)(y-12)$, **p** $(p+11)(p-5)$, **q** $(z-10)(z+8)$, **r** $(a+10)(a-6)$, **s** $(f-3)(f-13)$, **t** $(m+10)(m+11)$, **u** $(w-9)(w+4)$, **v** $(k+2)(k+13)$, **w** $(h-7)(h-12)$, **x** $(i+12)(i-11)$ **7** **a** $(p+3)^2$, **b** $(c+5)^2$, **c** $(g-4)^2$, **d** $(y-7)^2$, **e** $(t+1)^2$, **f** $(r-6)^2$, **g** $(x-9)^2$, **h** $(j+11)^2$ **8** **a** $2(m-2)(m-8)$, **b** $3(k+2)(k+6)$, **c** $4(a+8)(a-2)$, **d** $5(c-9)(c+3)$, **e** $3(t-1)(t-8)$, **f** $7(x-4)(x+3)$, **g** $4(d-3)(d-5)$, **h** $5(n+10)(n-2)$, **i** $10(y-11)(y+3)$ **9** **a** $x+6$, **b** $g+5$ **10** **a** $(x-1)(x+1)(x-2)(x+2)$, **b** $(x-2)(x+2)(x-5)(x+5)$, **c** $(x^2-2)(x-3)(x+3)$, **d** $(x^2+5)(x-2)(x+2)$, **e** $(x^2+8)(x-1)(x+1)$, **f** $(x-1)(x+1)(x-3)(x+3)$, **g** $(x^2-3)(x-4)(x+4)$, **h** $(x-2)(x+2)(x-3)(x+3)$, **i** $(x-1)(x+1)(x-4)(x+4)$, **j** $(x^2-6)(x-1)(x+1)$, **k** $(x^2+7)(x-2)(x+2)$, **l** $(x-3)(x+3)(x-4)(x+4)$

Exercise 5.6

1 B **2** C **3** D **4** A **5** **a** $(2x+1)(x+3)$, **b** $(3x+2)(x+4)$, **c** $(4x+7)(x+1)$, **d** $(2x+1)(x+5)$, **e** $(5x+2)(x+4)$, **f** $(7x+1)(x+5)$, **g** $(2x+9)(x+6)$, **h** $(7x+6)(x+3)$, **i** $(3x+8)(x+4)$, **j** $(2x-1)(x-2)$, **k** $(3x-2)(x-3)$, **l** $(5x-2)(x-3)$, **m** $(2x-3)(x-5)$, **n** $(3x-4)(x-4)$, **o** $(5x-6)(x-2)$, **p** $(3x-5)(x-7)$, **q** $(2x-11)(x-4)$, **r** $(7x-6)(x-5)$ **6** **a** $(3x-1)(x+2)$, **b** $(2x+5)(x-2)$, **c** $(2x-3)(x+5)$, **d** $(3x-7)(x+3)$, **e** $(2x+5)(x-5)$, **f** $(3x+2)(x-8)$, **g** $(5x+9)(x-4)$, **h** $(11x+3)(x-5)$, **i** $(4x-1)(x+7)$, **j** $(6x+1)(x-3)$, **k** $(5x-6)(x+2)$, **l** $(4x+7)(x-6)$, **m** $(7x-5)(x+8)$, **n** $(7x-6)(x+6)$, **o** $(4x-7)(x+4)$, **7** **a** $(2k+5)(k+8)$, **b** $(2c+9)(c-3)$, **c** $(4n-9)(n-2)$, **d** $(3y+4)(y-7)$, **e** $(7p-3)(p-4)$, **f** $(6a+1)(a-4)$, **g** $(4b+7)(b+6)$, **h** $(5u-12)(u+3)$, **i** $(2w+11)(w-10)$, **j** $(5h+4)(h+3)$, **k** $(3j-10)(j+4)$, **l** $(6l-5)(l-9)$ **8** **a** $(3a+2)(2a+3)$, **b** $(2n-7)(2n-3)$, **c** $(4k+9)(2k-1)$, **d** $(3p-5)(3p+2)$, **e** $(3c+5)(2c+3)$, **f** $(2e-5)(2e+9)$, **g** $(2t-3)(6t-5)$, **h** $(5b-8)(2b+9)$, **i** $(3m-8)(4m-5)$, **j** $(5y+2)(4y+5)$, **k** $(9w+11)(3w-5)$, **l** $(5q-2)(4q+3)$ **9** **a** $(7-x)(4+x)$, **b** $(5-2a)(3+a)$, **c** $(4-3p)(6+p)$, **d** $(5-6m)(3-2m)$, **e** $(7+5g)(3+4g)$ **f** $(7-2w)(8+5w)$ **10** **a** $2(2k+5)(k+2)$, **b** $3(3p-1)(2p-3)$, **c** $2(5a+3)(2a-1)$, **d** $3(2v+3)(2v-7)$, **e** $4(3f-1)(2f-3)$, **f** $5(2e-5)(e+4)$ **11** **a** $3a+4$, **b** $2n+9$ **12** **a** $(3x+1)(x+3)$, **b** $(2x-3)(x-4)$, **c** $(3x-2)(x+2)$, **d** $(4x+3)(x-5)$, **e** $(7x-1)(x-2)$, **f** $(5x-3)(x+5)$

Exercise 5.7

1 **a** $10(e+3)$, **b** $(a-2)(a+2)$, **c** $(m+3)(m+4)$, **d** $(n+6)(n+p)$, **e** $(2x+5)(x+4)$, **f** $(5-k)(5+k)$, **g** $x(x-1)$, **h** $(t-2)(t-4)$, **i** $(w+2)^2$, **j** $(p-3)(a+5)$, **k** $5(3y+7)$, **l** $(2q-1)(q-3)$, **m** $(g-h)(g+h)$, **n** $-7(j-2k)$, **o** $(u-5)(u+3)$, **p** $(3b+2)(b+8)$, **q** $7t(4t-9u)$, **r** $(2c-3)(2c+3)$, **s** $(y-2)(4x+9)$, **t** $fg(e+h)$, **u** $(h+10)(h-2)$, **v** $(5v-6w)(5v+6w)$, **w** $(g-6)(5m-2)$, **x** $(2f-5)(f+1)$ **2** **a** $9p(3q+5p)$, **b** $(x-11)(x+7)$, **c** $(3e-8f)(3e+8f)$, **d** $(3b+11)(b-4)$, **e** $(g+3h)(5-2i)$, **f** $12y(5x-7)$, **g** $(z+15)(z-2)$, **h** $(pq-r)(pq+r)$, **i** $(8-k)(5+2k)$, **j** $(8-kl)(8+kl)$, **k** $(m+7)(m^2-n)$, **l** $(5s-4)(2s-9)$, **m** $9(h-2)(h+2)$, **n** $3(x+5)(x-4)$, **o** $(5-3j)(1-4k)$,

p $n(n-1)(n+1)$, **q** $(a+b)(a+b+c)$, **r** $(1+t)(1+t^2)$, **s** $2(u+5)^2$, **t** $-2(f-4)(f+4)$, **u** $(2a-5b)(5+6c)$, **v** $a(10a+14b-1)$, **w** $2k(k+9)(k-3)$, **x** $v(4u+7)(u-6)$ **3 a** $(x+y)(x+y+z)$, **b** $(t^2+1)(t-1)(t+1)$, **c** $(w-1)(w+1)(w-5)(w+5)$, **d** $25b^2(a-2c)(a+2c)$, **e** $(4+n^2)(2-n)(2+n)$, **f** $(y+18)(y-4)$, **g** $(a+b+c+d)(a+b-c-d)$, **h** $(a-3)(a+3)(a-b)$, **i** $(x-5)(x+5)(4g+3)$

Exercise 5.8

1 a $\frac{a}{2}$, **b** $\frac{t}{5}$, **c** $\frac{3k}{4}$, **d** $\frac{4m}{5}$, **e** $\frac{3y}{2}$, **f** $\frac{4c}{3}$, **g** $\frac{1}{d}$, **h** $\frac{1}{4g}$, **i** $\frac{a}{c}$, **j** $\frac{q}{5}$, **k** $\frac{p}{s}$, **l** $\frac{b}{3}$, **m** $\frac{5g}{6}$, **n** $\frac{6}{7y}$, **o** $\frac{4e}{3g}$, **p** u, **q** $\frac{1}{s}$, **r** $\frac{v}{w}$, **s** $\frac{4r}{5}$, **t** $\frac{12d}{7c}$ **2 a** $x+1$, **b** $k-2$, **c** $2m+3$, **d** $\frac{3a-4}{2}$, **e** $\frac{4t-5}{2}$, **f** $b+c$, **g** $\frac{1}{v-w}$, **h** $x+6$, **i** $\frac{1}{2-5t}$, **j** $\frac{2c-d}{3}$, **k** $\frac{n+2m}{3p}$, **l** $\frac{3g-4h}{11g}$, **m** $\frac{1}{4}$, **n** $\frac{x}{z}$, **o** $\frac{2}{3}$, **p** $\frac{1}{3}$ **3 a** $m+7$, **b** $\frac{a-5}{4}$, **c** $\frac{x}{x+1}$, **d** $\frac{1}{n-12}$, **e** $3t+5$, **f** $h+1$, **g** $\frac{3}{2(k-3)}$, **h** $\frac{p-q}{p+q}$ **4 a** $x+1$, **b** $a-4$, **c** $\frac{w+4}{2}$, **d** $\frac{1}{n-5}$, **e** $\frac{c-3}{5c}$, **f** $\frac{e+3}{e-11}$, **g** $\frac{z-9}{z-10}$, **h** $\frac{s-4}{s+11}$, **i** $\frac{d+8}{2(d-3)}$ **5 a** $2x+5$, **b** $\frac{1}{5r-2}$, **c** $\frac{2p+3}{7}$, **d** $\frac{5k+8}{k+4}$, **e** $\frac{4h+9}{h-6}$, **f** $\frac{3v+2}{4v-1}$ **6 a** $p+q$, **b** $\frac{k+3}{m+5}$, **c** $\frac{u+7}{v-w}$, **d** $\frac{x+y}{x-3}$, **e** $\frac{b-c}{2(a+4)}$, **f** $\frac{n-p}{n+p}$ **7 a** -1, **b** -8, **c** $-\frac{1}{5}$, **d** $-\frac{2}{3}$, **e** $-a-13$, **f** $\frac{6-c}{2}$, **g** $\frac{9-s}{3}$, **h** $\frac{3-g}{g^2}$, **i** $\frac{-3(4+k)}{k+2}$

Exercise 5.9

1 a $\frac{xy}{14}$, **b** $\frac{4}{9}$, **c** $\frac{9c}{10}$, **d** $\frac{5eq}{12cp}$, **e** $\frac{mn}{12}$, **f** $\frac{3}{2y}$, **g** $\frac{4s}{9t}$, **h** $\frac{66d}{49c}$ **2 a** 4, **b** $\frac{2}{3}$, **c** $\frac{2(b-4)}{3}$, **d** $\frac{8}{9}$, **e** $\frac{2(x-y)}{21}$, **f** $\frac{3(k+2)}{4(k+1)}$ **3 a** $\frac{2}{5}$, **b** $\frac{3}{7}$, **c** $\frac{ac}{c-3}$, **d** $\frac{c(a+b)^2}{9}$, **e** $\frac{25u}{12t}$, **f** $\frac{8m}{21n}$ **4 a** $\frac{a+3}{5}$, **b** $\frac{s}{7}$, **c** $\frac{3t}{2(t-4)}$, **d** $\frac{9x}{x-5}$, **e** $\frac{5}{k}$, **f** $\frac{c+2}{c+7}$, **g** $\frac{n+1}{n-1}$, **h** $\frac{r+7}{5r}$, **i** $\frac{a+b}{ab}$, **j** $\frac{b(b-4)}{2}$, **k** $\frac{(w+11)^2}{24w}$, **l** $\frac{a+2}{a}$, **m** $\frac{e+4}{3e}$, **n** $\frac{c+d}{d}$
5 a $\frac{7}{k}$, **b** $\frac{n+2}{n+8}$, **c** $\frac{1}{2}$, **d** $\frac{12(a-1)}{5}$, **e** $\frac{z+10}{z}$, **f** $\frac{2}{7}$, **g** $\frac{1}{4}$, **h** $\frac{r-8}{r-7}$, **i** $\frac{m+3}{2m}$, **j** $\frac{3x}{x-1}$, **k** $\frac{q-2}{q+8}$, **l** $\frac{1}{2v}$, **m** 2, **n** $\frac{7c+11}{2c-1}$ **6 a** $-\frac{8}{15}$, **b** $-\frac{n}{9}$, **c** $-\frac{4}{15xz}$, **d** $\frac{1}{q^2}$

Exercise 5.10

1 a $\frac{2z}{3}$, **b** $\frac{3a}{2}$, **c** $\frac{m}{2}$, **d** $\frac{x}{2}$, **e** $\frac{5k}{12}$, **f** $\frac{29f}{40}$, **g** $\frac{7}{4y}$, **h** $\frac{11}{9u}$ **2 a** $\frac{3x+7}{8}$, **b** $\frac{9n-18}{14}$, **c** $\frac{9t+14}{12}$, **d** $\frac{4b+13}{10}$, **e** $\frac{k+31}{12}$, **f** $\frac{5z-9}{24}$ **3 a** $\frac{2x+1}{x(x+1)}$, **b** $\frac{2x}{(x+1)(x-1)}$, **c** $\frac{2x+5}{(x+2)(x+3)}$, **d** $\frac{5x+15}{x(x+5)}$, **e** $\frac{5x+13}{(x+3)(x+2)}$, **f** $\frac{7x+28}{(x-1)(x+6)}$, **g** $\frac{1}{x(x+1)}$, **h** $\frac{-2}{x(x+2)}$, **i** $\frac{2}{(x-1)(x+1)}$, **j** $\frac{2x+8}{x(x+2)}$, **k** $\frac{3x+1}{(x+5)(x+3)}$, **l** $\frac{9x-17}{(2x-1)(3x-4)}$
4 a $\frac{x^2+3x+1}{(x+1)(x+2)}$, **b** $\frac{x^2-5x}{(x+3)(x-1)}$, **c** $\frac{2x^2+7x+7}{(x+3)(x+1)}$ **5 a** $\frac{1}{x}$, **b** $\frac{x+2}{(x+1)(x+5)}$, **c** $\frac{2x+6}{(x+1)(x+2)(x+5)}$, **d** $\frac{2x+9}{(x+4)(x-2)(x+5)}$, **e** $\frac{6}{x(x-4)(x+6)}$, **f** $\frac{3}{(x-1)(x+1)(x+2)}$, **g** $\frac{5x+7}{(x+2)(x+4)(x+1)}$, **h** $\frac{9x+13}{(x-3)(x+2)(x+5)}$, **i** $\frac{5x-28}{2x(2x-1)(x-4)}$, **j** $\frac{20-2x}{(3x+2)(x-2)(x+6)}$, **k** $\frac{3x^2-5x}{(x-3)(x-4)(x+1)}$, **l** $\frac{2x^2+3x-1}{x(x-1)(x+1)}$ **6 a** $\frac{x+2}{2x(x+4)}$, **b** $\frac{7}{12(x+3)}$, **c** $\frac{x+5}{(x-2)(x+2)}$, **d** $\frac{2x+26}{5(x-3)(x+3)}$, **e** $\frac{2x+1}{(x-1)(x+1)(x+2)}$, **f** $\frac{2x+5}{(x-1)(x-6)(x+6)}$, **g** $\frac{2x+9}{(x+3)(x+5)(x+6)}$, **h** $\frac{2x+16}{(x-9)(x+7)(x+9)}$,

i $\dfrac{7x+14}{(x+5)(x-5)(x-2)}$, **j** $\dfrac{7x+7}{(x-10)(x+3)(x-4)}$, **k** $\dfrac{19x+27}{(x-3)(x+3)(3x+4)}$, **l** $\dfrac{8x+1}{(2x+5)(x+7)(x-7)}$
7 a $\dfrac{3-x}{3x(x-6)}$, **b** $\dfrac{3}{10(x+4)}$, **c** $\dfrac{9-2x}{(x-4)(x+4)}$, **d** $\dfrac{-4x-17}{5(x-8)(x+8)}$, **e** $\dfrac{3-3x}{7(3x-4)(3x+4)}$,
f $\dfrac{2}{(x-7)(x+5)(x-5)}$, **g** $\dfrac{-1}{(x-2)(x-11)(x-3)}$, **h** $\dfrac{6-6x}{7x(x-2)(x+6)}$, **i** $\dfrac{30-x}{(x-12)(x+12)(x-3)}$,
j $\dfrac{3x-25}{(x-10)(x+5)(x-9)}$, **k** $\dfrac{22x}{(2x-7)(2x+7)(x+2)}$, **l** $\dfrac{36-18x-3x^2}{(4x-3)(x+10)(x-10)}$

Chapter 5 Review

1 a $y^2+9y+20$, **b** $m^2-10m+21$, **c** $t^2+6t-16$, **d** $a^2-7a-44$ **2 a** $6p^2-25p+24$, **b** $8-2r-3r^2$ **3 a** $m^2-14m+49$, **b** $4c^2+20c+25$, **4** $162x^3-72x^2y+8xy^2$ **5 a** $(t+11)^2=t^2+22t+121$, **b** $(n-9)^2=n^2-18n+81$, **c** $(3x+5)^2=9x^2+30x+25$, **d** $(4u-11)^2=16u^2-88u+121$ **6 a** no, **b** no, **c** yes, **d** no, **e** no, **f** yes **7 a** $a^3+8a^2+19a+12$, **b** $10n^2-16n+28$, **c** $14a-59$, **d** $u^2-2u-18$ **8 a** $7(x+4)$, **b** $a(a+1)$, **c** $5(2m+3)$, **d** $3(3p-8q)$, **e** $4s(3r+5t)$, **f** $pq(r-s)$, **g** $9y(3y-4)$, **h** $gh(g-h)$, **i** $-a(b-3)$, **j** $-2c(9c+7d)$ **9 a** $(m-n)(m+n)$, **b** $(z-3)(z+3)$, **c** $(1-p)(1+p)$, **d** $(7-r)(7+r)$, **e** $(w-23)(w+23)$, **f** $(2a-5)(2a+5)$, **g** $(x-4y)(x+4y)$, **h** $(9u-10v)(9u+10v)$, **i** $(ab-c)(ab+c)$, **j** $(6pq-11rs)(6pq+11rs)$ **10 a** $(z+3)(x+y)$, **b** $(q-r)(p-1)$, **c** $(e+f)(e-2)$, **d** $(3k-7)(2j-5)$, **e** $(4-q)(p-q)$, **f** $(10-d)(c+d)$ **11 a** $(x+1)(x+3)$, **b** $(b-3)(b-7)$, **c** $(e-4)(e+3)$, **d** $(p+8)(p-5)$, **e** $(a+5)^2$, **f** $(q-9)^2$ **12 a** $(3t+2)(t+4)$, **b** $(2m-7)(m-3)$, **c** $(3c-2)(c+3)$, **d** $(7b+5)(b-4)$, **e** $(3s+5)(2s+7)$, **f** $(4d-9)(3d+2)$ **13 a** $(n-6)(n+6)$, **b** $(v+3)(v+5)$, **c** $5(4k+7)$, **d** $(3e+5)(e+3)$, **e** $(a+4)(a+b)$, **f** $(3h-5)(3h+5)$, **g** $(p-8)(p+3)$, **h** $-7x(3y+5z)$, **i** $(1+u)(1+u^2)$, **j** $(4a-11b)(4a+11b)$, **k** $(5m+6)(m-4)$, **l** $(y-10)(y+1)$, **m** $(d+3e)(2c-5)$, **n** $rs(s-r+1)$, **o** $(6g+5)(5g-3)$ **14 a** $3(x-2)(x+2)$, **b** $2(a+3)(a+6)$, **c** $n(n-1)(n+1)$, **d** $4(h-3)(h+3)$, **e** $a(b-3)(b-4)$, **f** $2(3u+1)(u+4)$, **g** $3(2z-5)(2z+5)$, **h** $x(x-2y)(x-3)$ **15 a** $\dfrac{1}{d}$, **b** $\dfrac{3t}{4v}$, **c** $\dfrac{5a}{2}$, **d** $\dfrac{p}{q}$
16 a $3e+5$, **b** $\dfrac{1}{a+9}$, **c** $\dfrac{3x-5y}{2}$, **d** $\dfrac{2}{3}$, **e** $u+2$, **f** $\dfrac{4h-5}{2}$, **g** $\dfrac{x+5}{x+3}$, **h** $\dfrac{k+7}{k+4}$, **i** $\dfrac{3p-4}{p-5}$, **j** $\dfrac{b-c}{2a-7}$, **k** -5,
l $\dfrac{-c-8}{3}$ **17 a** $\dfrac{2a}{a-3}$, **b** $\dfrac{c-4}{c}$, **c** $\dfrac{y}{2}$, **d** $\dfrac{x}{2x+1}$ **18 a** $\dfrac{7a+26}{12}$, **b** $\dfrac{h-13}{10}$, **c** $\dfrac{2x+3}{x(x+3)}$, **d** $\dfrac{x^2-x+2}{(x-1)(x+1)}$,
e $\dfrac{2x^2+9x+11}{(x+2)(x+1)}$ **19 a** $\dfrac{x+1}{3(x+2)(x-2)}$, **b** $\dfrac{2x+12}{(x-5)(x+5)(x+7)}$, **c** $\dfrac{x-5}{(x-2)(x-9)(x-3)}$,
d $\dfrac{x^2-3x+12}{(2x-3)(2x+3)(x+5)}$

6 Quadratic equations

Exercise 6.1

1 a i 17, **ii** 10, **iii** 1, **b i** 50, **ii** 8, **iii** 4.5, **c i** –3, **ii** –3, **iii** $\frac{3}{4}$, **d i** 20, **ii** 8, **iii** –28, **e i** 28, **ii** 10, **iii** $5\frac{5}{16}$, **f i** 19, **ii** $9\frac{5}{9}$, **iii** $15\frac{18}{25}$ **2 a** no, **b** yes, **c** yes, **d** no, **e** yes, **f** yes, **g** yes, **h** no, **i** yes, **j** no, **k** no, **l** yes, **m** no, **n** yes, **o** yes, **p** no **3 a** $y=x^2+1$, **b** $y=x^2-3$, **c** $y=2x^2$, **d** $y=-3x^2$, **e** $y=2x^2+5$, **f** $y=3x^2-2$ **4 a** $y=x^2+2x+3$, **b** $y=x^2+6x+2$, **c** $y=x^2+4x-1$, **d** $y=x^2-5x+8$, **e** $y=x^2+8x-10$, **f** $y=x^2-x-7$, **g** $y=x^2+3x+11$, **h** $y=x^2-7x+20$ **5 a** $x=3$, **b** $x=4$, **c** $x=7$, **d** $x=5$, **e** $x=6$, **f** $x=12$ **6 a** $x=-1, 3$, **b** $x=-5, 4$, **c** $x=-4, 7$ **7 a** $x=3, 4$, **b** $x=-1, 4$, **c** $x=-5, 3$ **8** 14 **9 a** $(x+7)$ cm, **b** $x=8$, **c** 15 cm × 8 cm

Exercise 6.2

1 a $x=0, 2$, **b** $a=0, -5$, **c** $t=0, 4$, **d** $m=0, 1$, **e** $k=0, -3$, **f** $c=0, 6$, **g** $x=1, 2$, **h** $e=-5, -3$, **i** $p=-2, 4$, **j** $n=7, -1$, **k** $z=-8, -6$, **l** $q=11, -11$, **m** $w=0, \frac{1}{2}$, **n** $f=0, \frac{2}{3}$, **o** $u=0, -\frac{3}{5}$, **p** $a=\frac{1}{2}, 4$, **q** $q=-\frac{3}{4}, 1$, **r** $c=\frac{5}{3}, -6$, **s** $t=\frac{2}{5}, \frac{5}{2}$, **t** $v=-\frac{7}{3}, -\frac{3}{2}$, **u** $b=\frac{4}{9}, -\frac{2}{11}$, **v** $x=3$, **w** $y=-9$, **x** $m=\frac{7}{4}$ **2 a** $a=0, 3$, **b** $y=0, -7$, **c** $p=0, 1$, **d** $q=0, -1$, **e** $e=0, 4$, **f** $g=0, -2$, **g** $n=-3, 3$, **h** $d=-6, 6$, **i** $r=-1, 1$, **j** $h=-9, 9$, **k** $f=-8, 8$, **l** $t=-11, 11$, **m** $k=-1, -3$, **n** $m=3, 4$, **o** $x=-6, -5$, **p** $y=-5, 3$, **q** $a=7, -3$, **r** $z=-9, 8$, **s** $n=-6, 4$, **t** $j=8, -5$, **u** $r=4, 9$,

v $p = -3$, **w** $h = 5$, **x** $v = -12$ **3** **a** $y = 0, 5$, **b** $c = 0, 8$, **c** $d = 0, 9$, **d** $k = 0, \frac{4}{5}$, **e** $e = 0, \frac{7}{3}$, **f** $m = 0, \frac{2}{11}$, **g** $x = 2$, **h** $u = -\frac{10}{3}$, **i** $t = \frac{5}{6}$ **4** **a** $x = -\frac{1}{2}, -2$, **b** $a = -\frac{1}{3}, -4$, **c** $p = -\frac{3}{2}, -2$, **d** $u = \frac{5}{3}, 2$, **e** $y = \frac{2}{5}, 4$, **f** $c = \frac{1}{7}, 5$, **g** $e = \frac{4}{3}, -2$, **h** $k = \frac{1}{4}, -3$, **i** $m = -\frac{3}{2}, 9$, **j** $v = -\frac{6}{5}, 2$, **k** $g = \frac{4}{7}, -5$, **l** $d = -\frac{7}{4}, 4$, **m** $t = -\frac{1}{4}, -\frac{3}{2}$, **n** $h = \frac{1}{3}, \frac{2}{3}$, **o** $n = -\frac{3}{2}, 8$, **p** $w = -\frac{2}{3}, -\frac{1}{2}$, **q** $u = \frac{4}{3}, -\frac{5}{4}$, **r** $s = \frac{3}{5}, \frac{3}{2}$, **s** $a = \frac{5}{2}, -\frac{7}{2}$, **t** $j = \frac{2}{3}, \frac{8}{3}$, **u** $z = -\frac{3}{5}, \frac{5}{3}$ **5** **a** $a = 0, 2$, **b** $p = 0, -4$, **c** $s = 0, 1$, **d** $q = -4, 4$, **e** $b = -7, 7$, **f** $u = -10, 10$, **g** $x = -6, 1$, **h** $u = 4, -2$, **i** $y = 4, 7$, **j** $n = 6, -2$, **k** $p = 5, 9$, **l** $r = -7, 6$, **m** $c = 11, -3$, **n** $u = -10, 7$, **o** $m = 5, 12$, **p** $z = \frac{2}{3}, 7$, **q** $w = -\frac{3}{5}, 4$, **r** $k = \frac{9}{2}, 10$ **6** **a** $x^2 - 5x + 6 = 0$, **b** $x^2 - 5x + 4 = 0$, **c** $x^2 + 8x + 15 = 0$, **d** $x^2 + 10x + 24 = 0$, **e** $x^2 - 4x - 12 = 0$, **f** $x^2 + 6x - 7 = 0$, **g** $x^2 - 7x - 44 = 0$, **h** $x^2 + 9x - 36 = 0$

Exercise 6.3

1 **a** $9, x + 3$, **b** $16, x + 4$, **c** $36, x - 6$, **d** $1, x - 1$, **e** $49, x + 7$, **f** $25, x - 5$, **g** $121, x - 11$, **h** $64, x + 8$ **2** **a** $\frac{9}{4}, x + \frac{3}{2}$, **b** $\frac{25}{4}, x + \frac{5}{2}$, **c** $\frac{49}{4}, x - \frac{7}{2}$, **d** $\frac{1}{4}, x - \frac{1}{2}$, **e** $\frac{121}{4}, x + \frac{11}{2}$, **f** $\frac{81}{4}, x - \frac{9}{2}$ **3** **a** $x = -1, 5$, **b** $x = -8, 2$, **c** $x = -1, 3$, **d** $x = 0, -8$, **e** $x = -6, -8$, **f** $x = -2, 12$, **g** $x = -17, 5$, **h** $x = -3, 9$, **i** $x = 0, -16$ **4** **a** $x = 1 \pm \sqrt{2}$, **b** $x = 2 \pm \sqrt{5}$, **c** $x = -4 \pm \sqrt{3}$, **d** $x = -3 \pm \sqrt{6}$, **e** $x = 7 \pm \sqrt{11}$, **f** $x = -5 \pm \sqrt{17}$, **g** $x = 1 \pm 2\sqrt{2}$, **h** $x = -3 \pm 2\sqrt{3}$, **i** $x = 2 \pm 3\sqrt{2}$, **j** $x = -5 \pm 2\sqrt{5}$, **k** $x = 4 \pm 3\sqrt{3}$, **l** $x = -7 \pm 5\sqrt{2}$ **5** **a** $x = 1.45, -3.45$, **b** $x = 2.41, -0.41$, **c** $x = 4.65, -0.65$, **d** $x = 0.74, -6.74$, **e** $x = -0.39, -7.61$, **f** $x = 5.24, 0.76$, **g** $x = 10.48, -0.48$, **h** $x = 0.86, -12.86$, **i** $x = 11.74, 4.26$, **j** $x = 0.56, -3.56$, **k** $x = 4.30, 0.70$, **l** $x = 0.62, -1.62$, **m** $x = -0.81, -6.19$, **n** $x = 0.30, -3.30$, **o** $x = 9.62, -0.62$, **p** $x = 2.56, -1.56$, **q** $x = -0.85, -14.15$, **r** $x = 10.32, 0.68$ **6** **a** $\frac{1}{16}, x + \frac{1}{4}$, **b** $\frac{1}{36}, x - \frac{1}{6}$, **c** $\frac{1}{100}, x + \frac{1}{10}$, **d** $\frac{1}{9}, x + \frac{1}{3}$, **e** $\frac{9}{25}, x - \frac{3}{5}$, **f** $\frac{4}{49}, x - \frac{2}{7}$, **g** $\frac{9}{16}, x + \frac{3}{4}$, **h** $\frac{25}{64}, x + \frac{5}{8}$, **i** $\frac{9}{100}, x - \frac{3}{10}$, **j** $\frac{49}{36}, x - \frac{7}{6}$ **7** **a** $x = -0.13, -3.87$, **b** $x = 5.62, -0.62$, **c** $x = 2.46, 0.54$, **d** $x = 0.54, -2.29$, **e** $x = 1.79, -1.12$, **f** $x = 0.87, -2.07$

Exercise 6.4

1 **a** $x = -1, -2$, **b** $x = -2, -6$, **c** $x = -2, -5$, **d** $x = 3, 5$, **e** $x = -6, 2$, **f** $x = 4, 6$, **g** $x = 5, -2$, **h** $x = -8, 1$, **i** $x = -11, 3$, **j** $x = -7, 6$, **k** $x = 7, -4$, **l** $x = 4, 10$ **2** **a** $x = -\frac{1}{2}, -3$, **b** $x = -\frac{3}{2}, -5$, **c** $x = -\frac{4}{3}, -2$, **d** $x = \frac{5}{2}, 2$, **e** $x = \frac{5}{3}, 3$, **f** $x = \frac{3}{4}, -1$, **g** $x = \frac{6}{5}, -2$, **h** $x = -\frac{4}{7}, 3$, **i** $x = -\frac{5}{4}, 7$ **3** **a** $x = \frac{-3 \pm \sqrt{5}}{2}$, **b** $x = \frac{-5 \pm \sqrt{17}}{2}$, **c** $x = \frac{7 \pm \sqrt{33}}{2}$, **d** $x = \frac{-7 \pm \sqrt{57}}{2}$, **e** $x = \frac{9 \pm \sqrt{101}}{2}$, **f** $x = \frac{5 \pm \sqrt{37}}{2}$, **g** $x = \frac{-9 \pm \sqrt{41}}{4}$, **h** $x = \frac{5 \pm \sqrt{41}}{4}$, **i** $x = \frac{-3 \pm \sqrt{57}}{6}$, **j** $x = \frac{7 \pm \sqrt{33}}{8}$, **k** $x = \frac{5 \pm \sqrt{13}}{6}$, **l** $x = \frac{-1 \pm \sqrt{61}}{10}$ **4** **a** $x = -2 \pm \sqrt{3}$, **b** $x = 3 \pm \sqrt{7}$, **c** $x = -1 \pm \sqrt{2}$, **d** $x = 2 \pm \sqrt{6}$, **e** $x = 4 \pm \sqrt{10}$, **f** $x = -2 \pm \sqrt{7}$, **g** $x = 5 \pm \sqrt{22}$, **h** $x = -3 \pm 2\sqrt{3}$, **i** $x = 5 \pm 2\sqrt{5}$, **j** $x = \frac{-3 \pm \sqrt{7}}{2}$, **k** $x = \frac{3 \pm \sqrt{3}}{3}$, **l** $x = \frac{2 \pm \sqrt{10}}{3}$, **m** $x = \frac{-2 \pm \sqrt{10}}{6}$, **n** $x = \frac{-4 \pm \sqrt{6}}{2}$, **o** $x = \frac{6 \pm \sqrt{21}}{5}$, **p** $x = \frac{1 \pm \sqrt{5}}{4}$, **q** $x = \frac{5 \pm \sqrt{10}}{3}$, **r** $x = \frac{-2 \pm \sqrt{22}}{6}$ **5** **a** $x = -\frac{3}{2}$, **b** 1, **c** It is a perfect square. **6** **a** There are no solutions. **b** $b^2 - 4ac \geqslant 0$ if the equation has solutions. **7** **a** $5x^2 - 7x + 1 = 0$, **b** $2x^2 + 5x - 9 = 0$, **c** $3x^2 - 11x - 7 = 0$

Exercise 6.5

1 **a** $x = 0, 2$, **b** $x = -5, 5$, **c** $x = -3, 3$, **d** $x = 1, 5$, **e** $x = 1.65, -3.65$, **f** $x = 2.08, -1.08$, **g** $x = -1, -4$, **h** $x = 4, 5$, **i** $x = -2, 6$, **j** $x = -0.30, -6.70$, **k** $x = 3.45, -1.45$, **l** $x = 8.53, 0.47$, **m** $x = -\frac{3}{2}, -4$, **n** $x = \frac{2}{3}, -5$, **o** $x = \frac{1}{2}, \frac{2}{3}$,

p $x = 2.22, 0.45$, **q** $x = 4.53, -0.28$, **r** $x = 0.27, -0.37$ **2 a** $x = 0, 5$, **b** $x = 0, 2$, **c** $x = 0, 4$, **d** $x = -4, 3$, **e** $x = 5, 6$, **f** $x = 2, 3$, **g** $x = \frac{1}{3}, -3$, **h** $x = \frac{5}{2}, 4$, **i** $x = -\frac{7}{4}, 2$, **j** $x = 6.61, -0.61$, **k** $x = 1.77, -6.77$, **l** $x = 1.27, 0.39$ **3 a** $x = -3, -5$, **b** $x = 2, 7$, **c** $x = -5, 2$, **d** $x = -4, 7$, **e** $x = -\frac{9}{2}, 1$, **f** $x = \frac{4}{3}, 5$, **g** $x = 5.83, 0.17$, **h** $x = 11.20, 0.80$, **i** $x = 0.79, -3.79$, **j** $x = 2.89, -1.39$, **k** $x = 0.41, -4.91$, **l** $x = 1.55, 0.74$ **4 a** $x = -8, 3$, **b** $x = -2, 4$, **c** $x = 3, 9$, **d** $x = \frac{3}{2}, 4$, **e** $x = -2, 8$, **f** $x = -12, 3$, **g** $x = 1.24, -3.24$, **h** $x = 5.65, 0.35$, **i** $x = -1.13, -3.54$, **j** $x = 9.11, -0.11$, **k** $x = -0.65, -3.85$, **l** $x = 0.39, -0.28$ **5 a** $x = 3, 4$, **b** $x = -5, 8$, **c** $x = -7, 5$, **d** $x = 3, 6$, **e** $x = -2, 10$, **f** $x = -\frac{1}{2}, 5$, **g** $x = \frac{7 \pm \sqrt{37}}{2}$, **h** $x = \frac{1 \pm \sqrt{13}}{2}$, **i** $x = \frac{2 \pm \sqrt{6}}{2}$, **j** $x = 2 \pm \sqrt{3}$, **k** $x = 3 \pm \sqrt{11}$, **l** $x = \frac{4 \pm \sqrt{10}}{2}$

6 a $x = -\frac{7}{3}, 3$, **b** $x = \frac{3}{2}, -10$, **c** $x = \frac{3}{2}, \frac{1}{3}$, **d** $x = 0.88, -1.13$, **e** $x = 1.63, -1.23$, **f** $x = 0.52, -1.66$
7 a $x = 2, y = 4$ and $x = -2, y = 4$, **b** $x = 0, y = 0$ and $x = -3, y = 9$, **c** $x = 3, y = 9$ and $x = -1, y = 1$, **d** $x = 4, y = 16$ and $x = 5, y = 25$, **e** $x = 1, y = 2$ and $x = -1, y = 0$, **f** $x = 1, y = 4$ and $x = 5, y = 28$, **g** $x = 3, y = 6$, **h** $x = 3, y = 4$ and $x = -5, y = 4$, **i** $x = 4, y = 10$ and $x = -2, y = 4$, **j** $x = 3, y = 2$ and $x = 4, y = 4$, **k** $x = 1\frac{2}{3}, y = 8\frac{1}{3}$ and $x = -2, y = 12$, **l** $x = 2\frac{1}{2}, y = 12\frac{1}{2}$ and $x = 2, y = 8$ **8 a** $x = -1, 1, -2, 2$, **b** $x = -1, 1, -3, 3$, **c** $x = -2, 2, -3, 3$, **d** $x = -1, 1$, **e** $x = -2, 2$, **f** $x = -3, 3$, **g** $x = -1, 1, -4, 4$, **h** $x = -3, 3$, **i** $x = -2, 2, -5, 5$, **j** $x = -5, 5$, **k** $x = -1, 1, -7, 7$, **l** $x = -2, 2, -4, 4$ **9 a** $x = -1, 1$, **b** $x = -3, 3$, **c** $x = -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, -2, 2$
10 a $x = 0, 2$, **b** $x = 2, 3$, **c** $x = 0, 3$

Exercise 6.6

1 a 5, 6, **b** 7, 8, **c** 4, 6, **d** 7, 9, **e** 8, 11 **2 a** 8, **b** 7 **3** –3, 5 **4 a** 5, 8, **b** 2, 7, **c** 5, 6, **d** 8, 9, 10, 11
5 15 cm × 12 cm **6** 12 cm **7 a** $x = 6$, **b** 24 cm^2 **8** 11 **9 a** 2 s, 3 s, **b** 5 s, **c** No, the greatest height is 100 m when $t = 2.5$ s. **10** 2 m **11** $x = 4$ **12** 5 cm, 9 cm **13 a** 3, **b** 4, **c** 20 **14** $x = 8$
15 $\frac{2}{5}, \frac{5}{2}$ **16 b** 7 cm × 4 cm **17** 17 cm × 6 cm **18 a** $A = 100x - 2x^2$, **b** 40 m × 30 m, 60 m × 20 m

Chapter 6 Review

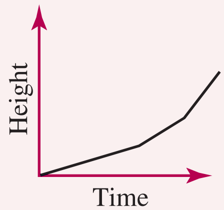
1 a yes, **b** yes, **c** no, **d** yes, **e** yes, **f** no **2 a** $y = x^2 - 4$, **b** $y = 2x^2 + 3$ **3 a** $y = x^2 + 3x - 2$, **b** $y = x^2 - 4x + 7$
4 a $m = 0, 6$, **b** $x = 0, -8$, **c** $t = 0, -1$, **d** $n = 0, \frac{2}{3}$, **e** $p = 3, -5$, **f** $k = \frac{2}{3}, -\frac{5}{2}$ **5 a** $p = 0, 4$, **b** $u = 0, -3$, **c** $n = -7, 7$, **d** $e = -3, 3$, **e** $x = -4, -6$, **f** $c = 4, -3$, **g** $y = 6$, **h** $a = -8, 4$, **i** $q = \frac{3}{2}, 7$, **j** $t = \frac{4}{5}, -5$
6 a $a = -4, 4$, **b** $t = 0, 5$, **c** $b = 5, -3$, **d** $y = 4, -2$, **e** $k = 2, 11$, **f** $w = \frac{1}{2}, -4$ **7 a** $x = 4, 16$, **b** $x = -6 \pm 5\sqrt{2}$
8 a $x = -3.83, 1.83$, **b** $x = 7.90, -1.90$, **c** $x = 0.56, -3.56$, **d** $x = 4.30, 0.70$ **9 a** $x = -2, -3$, **b** $x = \frac{2}{3}, -5$, **c** $x = \frac{1 \pm \sqrt{21}}{2}$, **d** $x = \frac{9 \pm \sqrt{33}}{6}$ **10 a** $x = 7, -5$, **b** $x = 0.44, -3.77$, **c** $x = 2, 10$, **d** $x = 0.80, -7.47$, **e** $x = 6, -5$, **f** $x = 4, 7$, **g** $x = \frac{3}{4}, -6$ **11 a** 4, 5, **b** 6, 10, **c** 2, 9 **12 a** 12 cm × 7 cm, **b** 38 cm **13** $x = 8$ **14 a** 6 s, **b** 2 s, 4 s, **c** 3 s

7 Graphs in the number plane

Exercise 7.1

1 a R, **b** P, **c** P, **d** Q **2 a** B, **b** C, **c** A, E, **d** D, E **3 a i** Tony, **ii** Pete, **iii** Quentin, **iv** Robin, **b i** Quentin, Robin, **ii** Pete, Steve **4 a** A, D, **b** B, C **c** D, **d** B **5** B **6** C **7** B **8** D **9** A
10 A **11** A **12 a** B, **b** C, **c** A **13 a** D, **b** B, **c** C, **d** A

14 a



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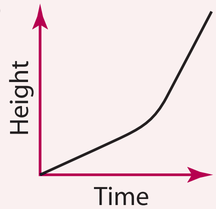

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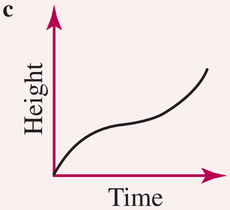

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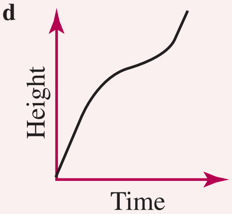

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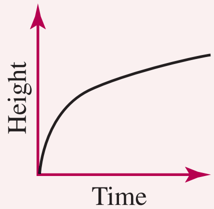
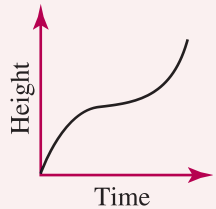
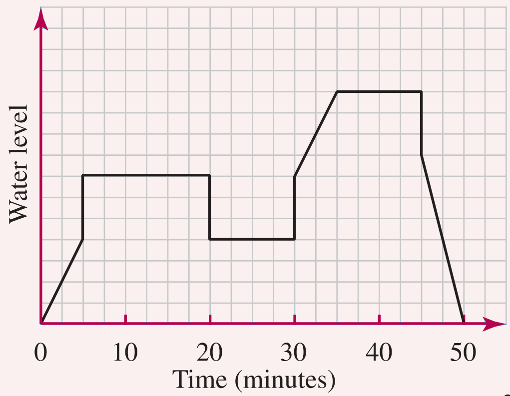


16

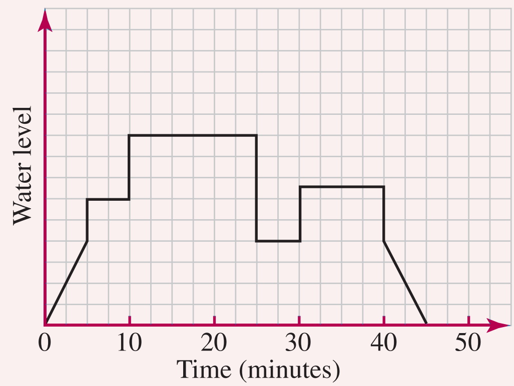


18 a She probably opened a new account, since the opening balance is $0.
b Tuesday, Wednesday, Friday,
c Friday,
d Thursday,
e Sunday

19 a



b



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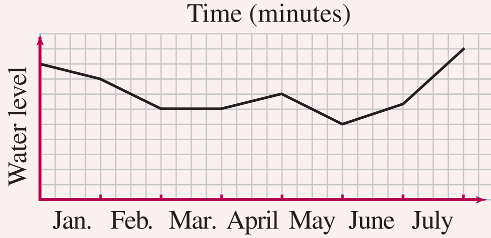
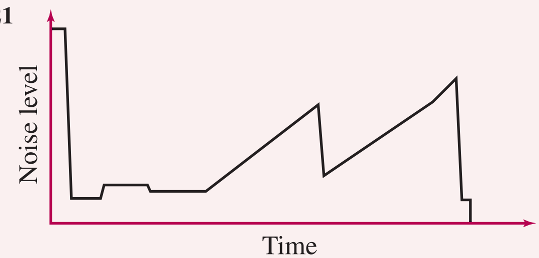
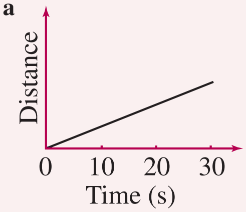


21



22 a



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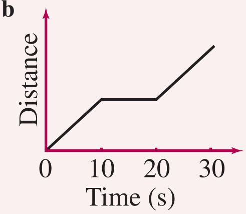
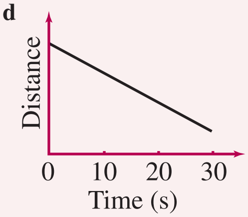


c

d

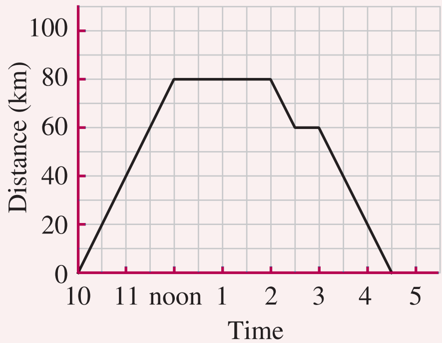


Exercise 7.2

1 a 40 km, **b** Her speed increased, the line became steeper. **c** noon, **d** $1\frac{1}{2}$ h, **e** 100 km, **f** $2\frac{1}{2}$ h, **g** 200 km
2 a 9 am, **b** 80 km, **c i** 100 km/h, **ii** 60 km/h, **d** noon, **e** 100 km, **f** 25 km **3 a** 16 km, **b** 6 km, **c** 10 am–noon, **d** 18 km, **e** 14 km, **f** 3.30 pm, **g** 32 km **4 a** 150 km, **b** 1.30 pm–2 pm, **c** 10.30 am, **d** 8.30 am, 9.45 am, 12.30 pm, **e** 9am–9.30 am, 10.30 am–11.30 am, **f** $5\frac{1}{2}$ h, **g** 165 km
5 a 6 am, **b** 6.30 am, 9.30 am, **c** 30 km, **d** Steve, **e** 30 km, **f** 180 km, **g** Steve — 40 km/h, Perry — 80 km/h
6 a 200 km, **b** 11 am, 3.30 pm, **c** 150 km, **d** Sourav, **e** Irena, by 100 km, **f** 3.30 pm–5.30 pm, **g** 54.5 km/h

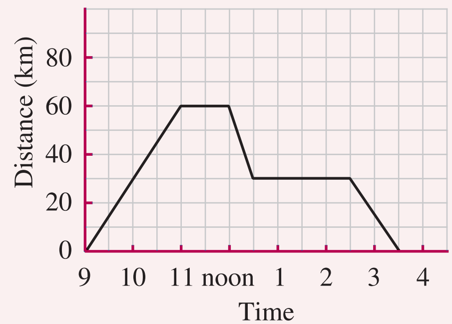
ANSWERS

7 a



b 40 km/h, **c** 3 pm, **d** 4.30 pm, **e** 11.15 am, 3.15 pm

8 a

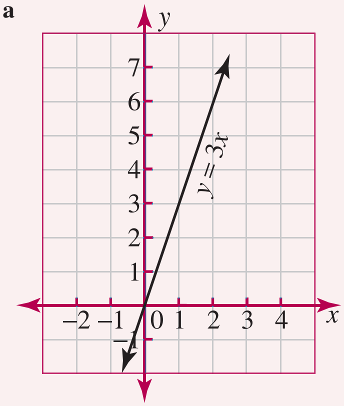


b 2 h, **c** 12.30 pm, **d** 30 km/h, **e** 120 km

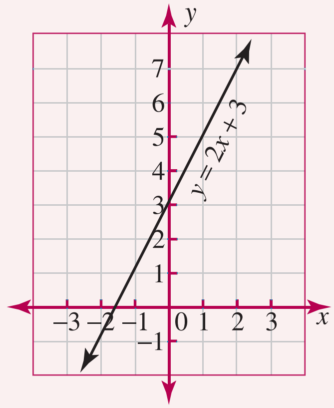
22 a

Exercise 7.3

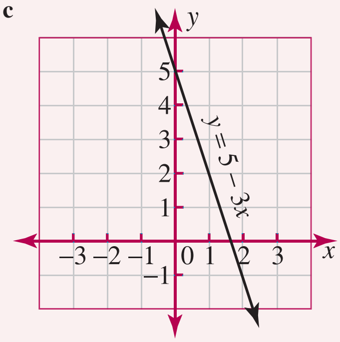
1 a



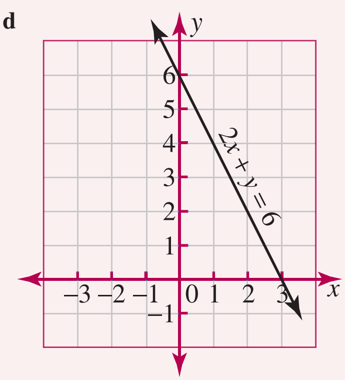
b



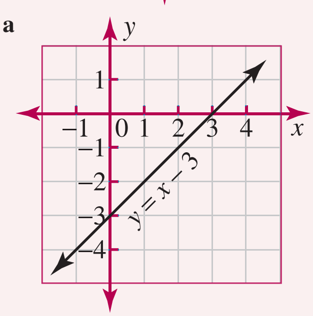
c



d



2 a



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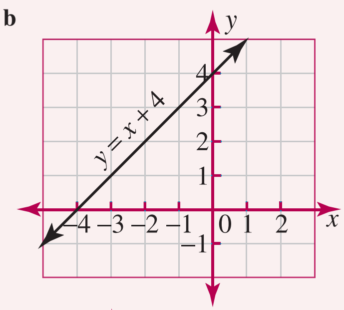

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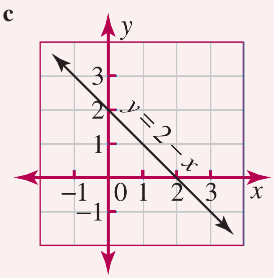

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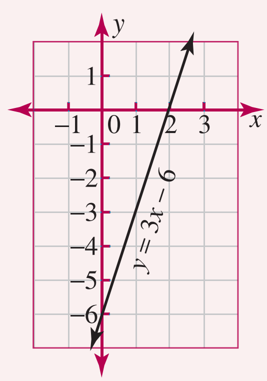
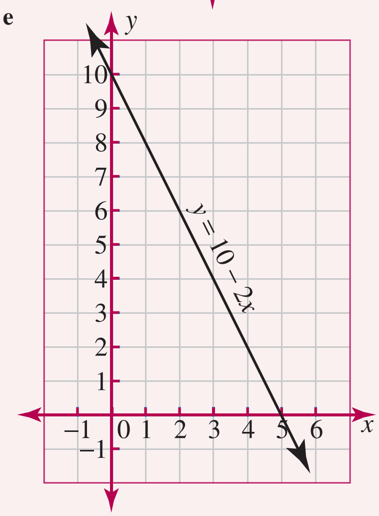
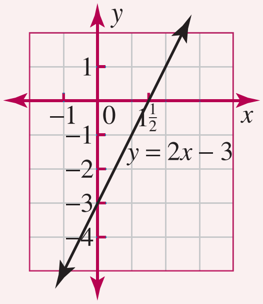


e



ANSWERS

f



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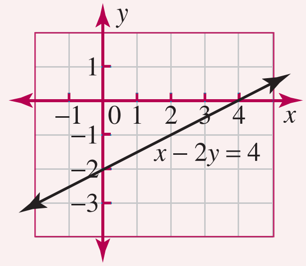

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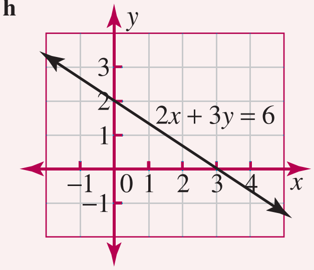

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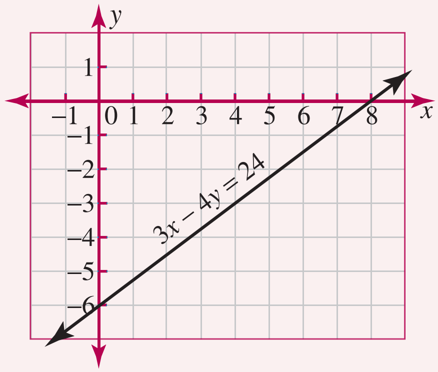

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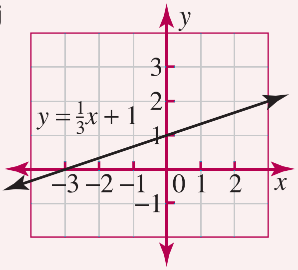

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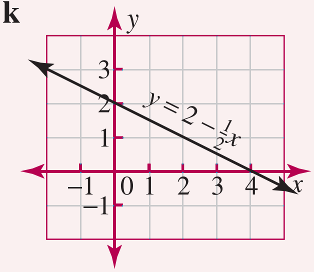

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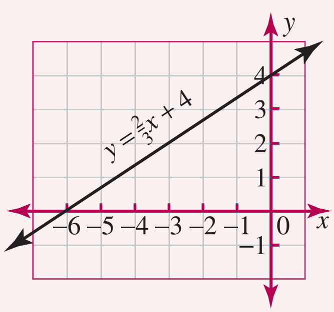
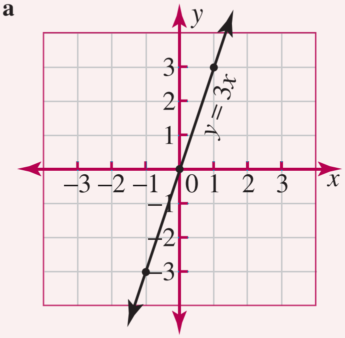


3 **a** $m = 1, b = 7$, **b** $m = 2, b = -5$, **c** $m = 3, b = 4$, **d** $m = -1, b = -6$, **e** $m = 4, b = 0$, **f** $m = -5, b = 8$, **g** $m = \frac{1}{2}, b = 4$, **h** $m = \frac{5}{6}, b = 0$

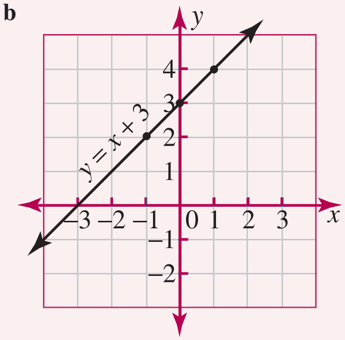
4 **a** $y = 2x + 3$, **b** $y = -x + 4$, **c** $y = 3x - 2$, **d** $y = -\frac{1}{2}x - 6$, **e** $y = \frac{2}{3}x$

5 **a** $y = 2x + \frac{3}{2}, m = 2, b = \frac{3}{2}$, **b** $y = \frac{1}{3}x - \frac{1}{6}, m = \frac{1}{3}, b = -\frac{1}{6}$, **c** $y = -\frac{3}{4}x - 3, m = -\frac{3}{4}, b = -3$, **d** $y = -\frac{5}{3}x + \frac{11}{3}, m = -\frac{5}{3}, b = \frac{11}{3}$

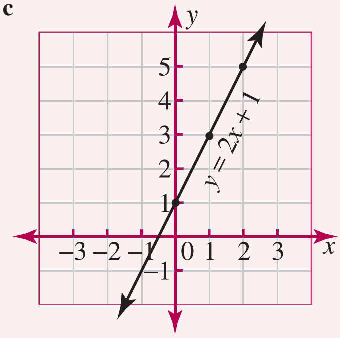
6 **a**



b

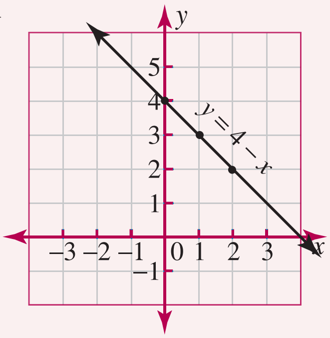


c



ANSWERS

d



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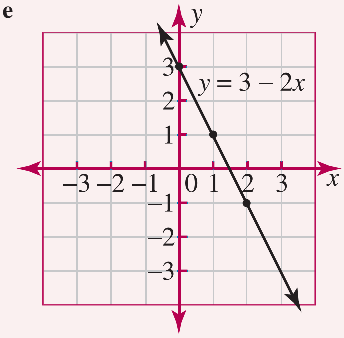

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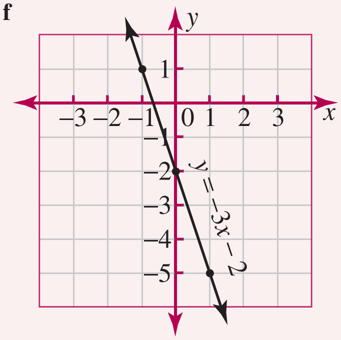
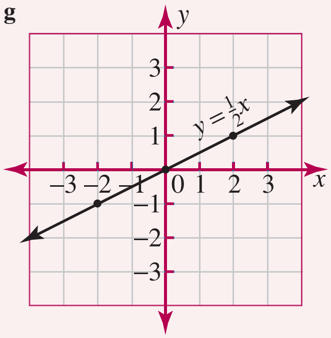


g



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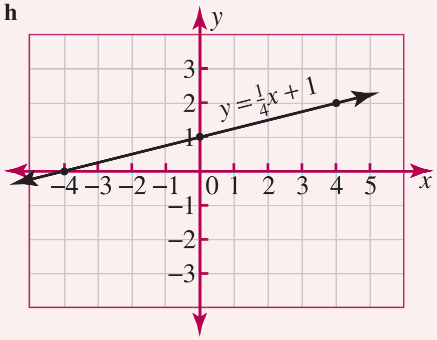
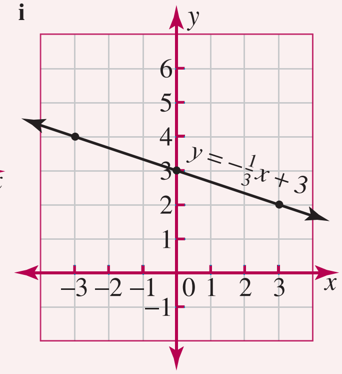


i



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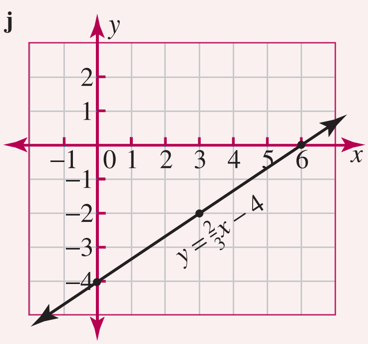

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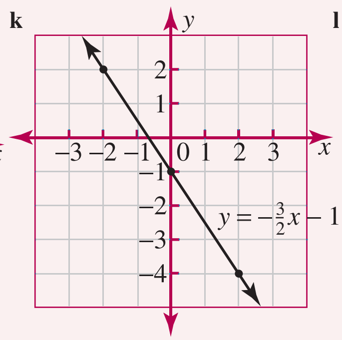

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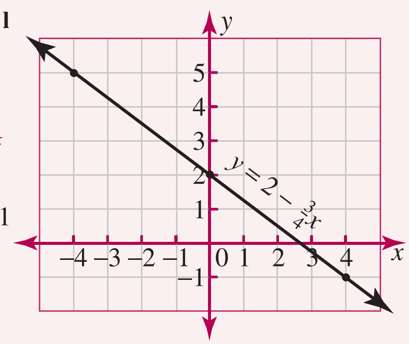


7 a $y = -3x - 4$, **b** $y = 3x + 4$, **c** $y = 4 - 3x$, **d** $y = 3x - 4$ **8 a** $y = 3x$, **b** $y = -\frac{1}{3}x$, **c** $y = -3x$, **d** $y = \frac{1}{3}x$
9 a $y = x + 5$, **b** $y = 2x + 6$, **c** $y = -3x + 12$, **d** $y = \frac{1}{2}x - 2$, **e** $y = \frac{5}{3}x - 5$, **f** $y = -\frac{2}{3}x - 4$ **10 a** $x = 2$, **b** $y = 1$, **c** $y = -2$, **d** $x = -1$, **e** $y = 4$, **f** $x = -3$, **g** $x = 3$, **h** $y = -4$ **11 a** $y = 5$, **b** $x = 2$, **c** $x = -3$, **d** $y = -5$ **12 a** yes, **b** no, **c** no, **d** yes, **e** no, **f** yes **13 a** no, **b** yes, **c** yes, **d** no, **e** yes, **f** no, **g** yes, **h** yes **14 a** no, **b** yes, **c** yes, **d** no, **e** no, **f** yes, **g** yes, **h** no **15 a** $c = -6$, **b** $g = 7$, **c** $m = 11$, **d** $z = -8$ **16 a** $y = x + 7$, **b** $y = 5x$, **c** $y = 2x + 3$, **d** $y = 4x - 1$, **e** $y = -x + 8$, **f** $y = 9x - 5$, **g** $y = -2x + 7$, **h** $y = \frac{1}{2}x - 2$ **17 a i** $y = 9$, **ii** $y = 11$, **iii** $y = 0$, **b i** $x = 9$, **ii** $x = 1, 6$, **iii** $x = 2$, **c** $y = 2x + 3$ for $0 \leqslant x \leqslant 4$, $y = -x + 11$ for $4 < x \leqslant 11$ **18 a i** $y = 8$, **ii** $y = 5$, **iii** $y = 4$, **b i** $t = 7$, **ii** $y = 5\frac{1}{2}, 6\frac{1}{2}, 8$, **iii** $3 \leqslant t \leqslant 5$, $t = 6, 9$

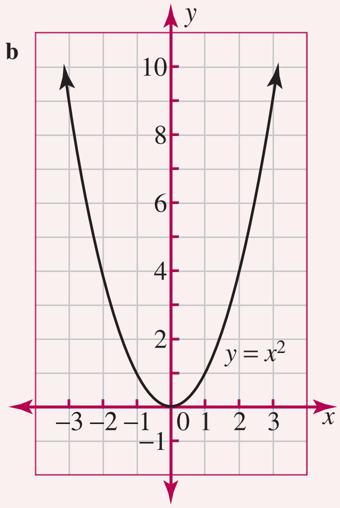
ANSWERS

Exercise 7.4

1 a

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

b



c no, **d** $x = 0$, **e** (0, 0), **f** no, **g** $y = 0$, **h** concave upward
2 a 1, **b** 2 **3 a** 0.2, **b** 2.6, **c** 4.8, **d** 7.3, **e** 0.4, **f** 1.7, **g** 3.2, **h** 4.4
4 a 1.4, **b** 2.2, **c** 2.6, **d** 2.8, **e** 0.9, **f** 1.8, **g** 2.5, **h** 2.7
5 a $x = \pm 1.7$, **b** $x = \pm 2.8$, **c** $x = \pm 1.3$, **d** $x = \pm 2.4$

6 a i

x	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
y	4	2.25	1	0.25	0	0.25	1	2.25	4

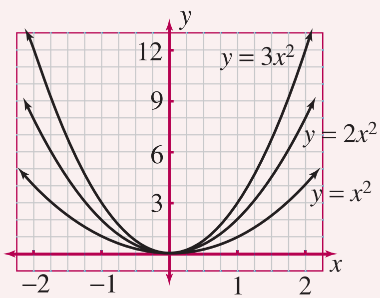
ii

x	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
y	8	4.5	2	0.5	0	0.5	2	4.5	8

iii

x	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
y	12	6.75	3	0.75	0	0.75	3	6.75	12

b

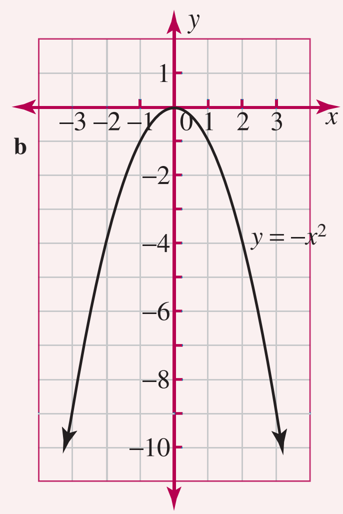


c The curve becomes steeper as the value of a decreases.
7 a B, **b** C, **c** A

8 a

x	−3	−2	−1	0	1	2	3
y	−9	−4	−1	0	1	4	9

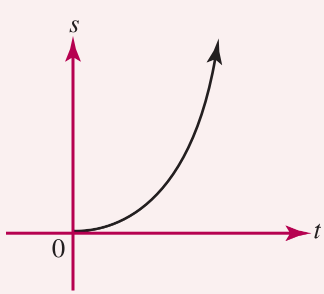
b



c no, **d** $x = 0$, **e** (0, 0), **f** no, **g** $y = 0$, **h** concave downward

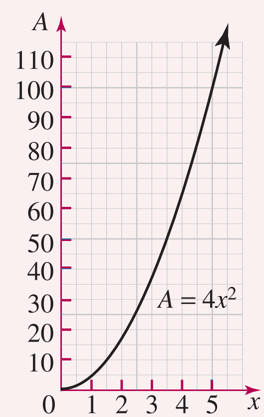
9 a If $a > 0$, the parabola is concave upward. **b** If $a < 0$, the parabola is concave downward.
10 a B, **b** C, **c** A, **d** D
11 a $y = 2x^2$, **b** $y = 5x^2$, **c** $y = \frac{1}{2}x^2$, **d** $y = -2x^2$, **e** $y = -3x^2$, **f** $y = -\frac{1}{3}x^2$
12 a $t \geqslant 0$, time cannot be negative.

b



c i 5 m, **ii** 20 m, **iii** 245 m, **d i** 3 s, **ii** 14 s, **iii** 20 s, **e** 25 m
13 a $4x$, **b** $A = 4x^2$, **c** $x > 0$, the dimensionscannot be negativeor zero. **e i** 36 cm², **ii** 64 cm², **iii** 100 cm², **g** 14 cm × 3.5 cm

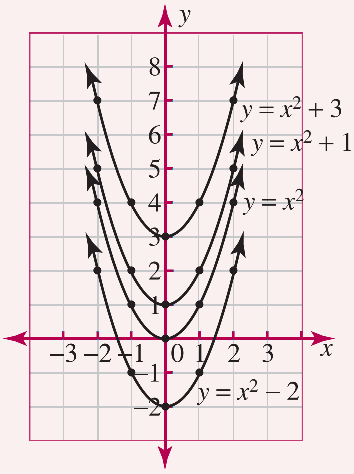
d



ANSWERS

Exercise 7.5

1



a

x	−2	−1	0	1	2
y	4	1	0	1	4

b

x	−2	−1	0	1	2
y	5	2	1	2	5

c

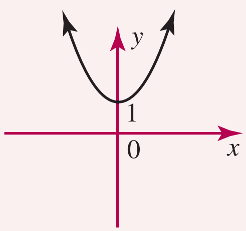
x	−2	−1	0	1	2
y	7	4	3	4	7

d

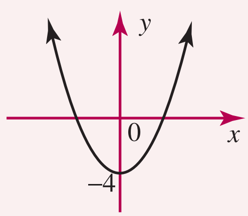
x	−2	−1	0	1	2
y	2	−1	−2	−1	2

2 The graphs of $y = x^2 + c$ and $y = x^2 - c$ $(c > 0)$ are obtained by translating the parabola $y = x^2$ upward or downward by c units respectively.
3 **a** translate $y = x^2$ upward by 4 units, **b** translate $y = x^2$ downward by 5 units, **c** translate $y = x^2$ upward by 7 units, **d** translate $y = x^2$ downward by 12 units

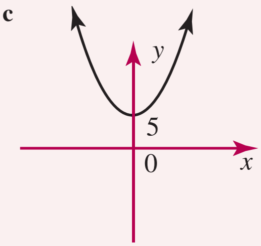
4 **a**



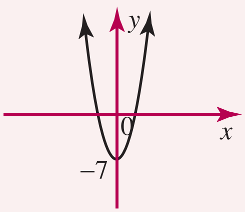
b



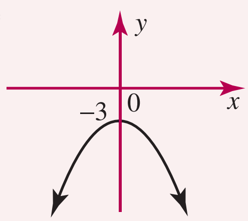
c



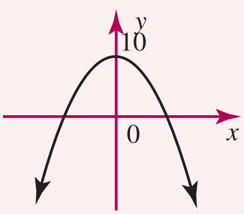
d



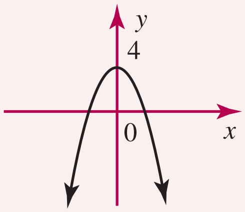
e



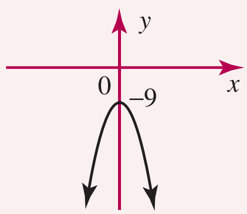
f



g

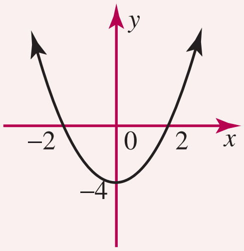


h

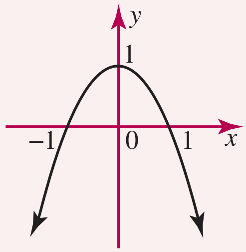


5 (0, −8) **6** **a** $y = 3 - x^2$, **b** $y = x^2 + 3$, **c** $y = -x^2 - 3$, **d** $y = x^2 - 3$ **7** **a** $y = x^2 + 8$, **b** $y = x^2$, **c** $y = x^2 - 3$

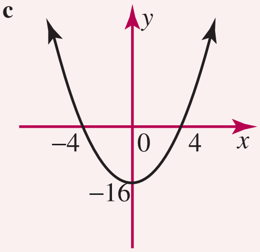
8 **a**



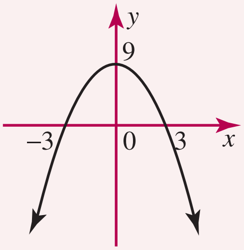
b



c



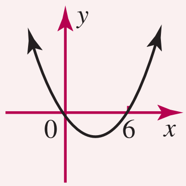
d



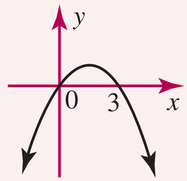
9 $y = 9 - x^2$ **10** **a** $y = x^2 + 5$, **b** $y = x^2 - 4$, **c** $y = x^2 - 7$, **d** $y = 14 - x^2$, **e** $y = -x^2 - 5$, **f** $y = 2 - x^2$
11 **a** $y = 2x^2 - 3$, **b** $y = 7 - 4x^2$ **12** $y = x^2 + 3$ **13** $y = 3 - x^2$

Exercise 7.6

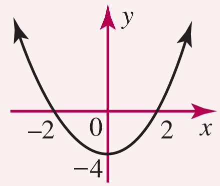
1 a



b



c



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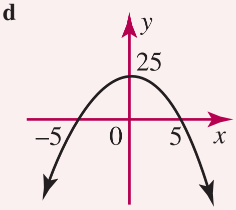

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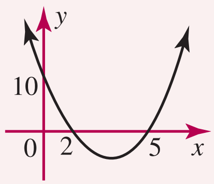

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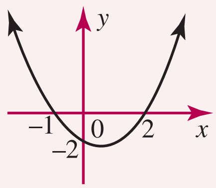

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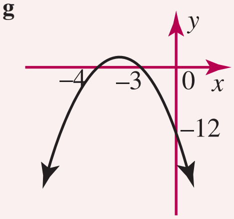

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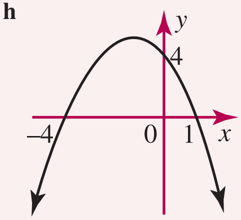

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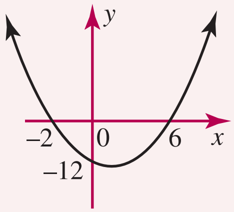
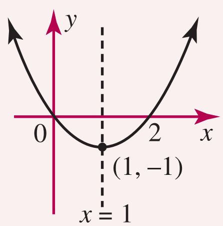
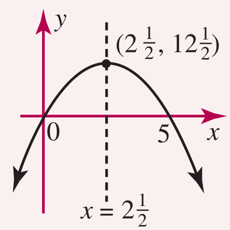


2 a i $x = 0, 2,$
ii $y = 0,$
iii $x = 1,\ V(1, -1),$
iv



b i $x = 0, 5$
ii $y = 0$
iii $x = 2\frac{1}{2},\ V(2\frac{1}{2}, 12\frac{1}{2})$
iv



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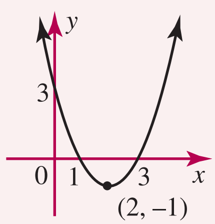

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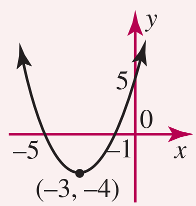

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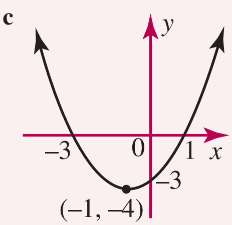

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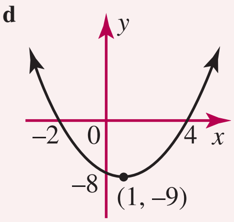

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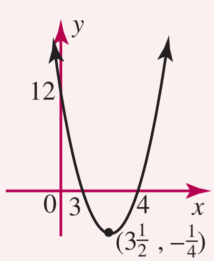

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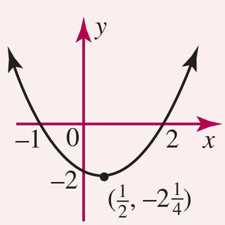

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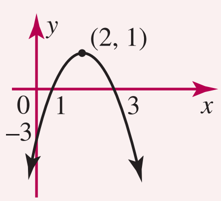

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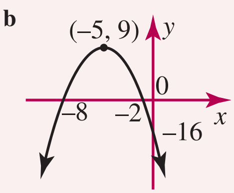

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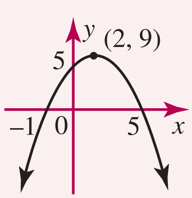

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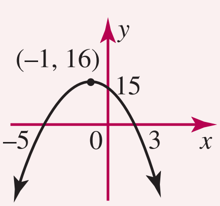

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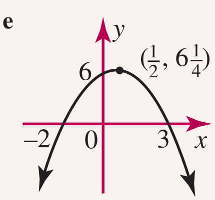

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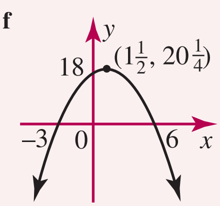
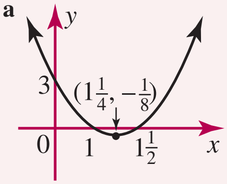


ANSWERS

5 a


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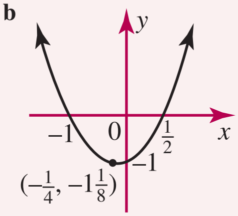

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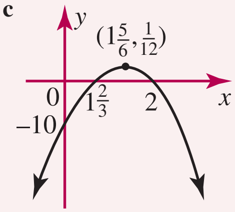

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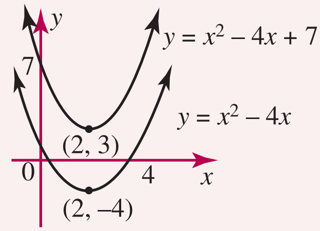

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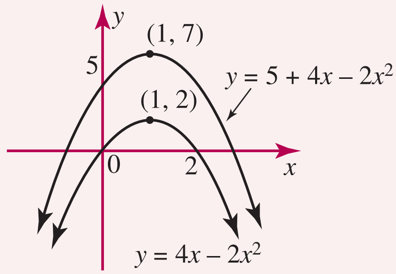

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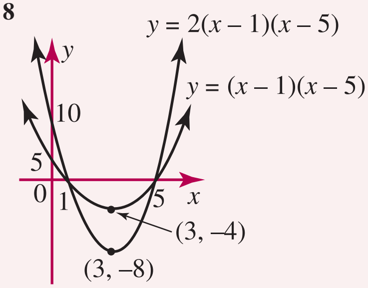
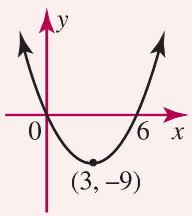


9 a $x = 5 \pm \sqrt{3}$, **b** $x = \dfrac{9 \pm \sqrt{93}}{2}$, **c** $x = -3 \pm \sqrt{5}$ **10 a** $y = (x - 2)(x - 7)$, **b** $y = 2(x + 3)(x - 1)$, **c** $y = \frac{1}{2}(x + 2)(x - 6)$, **d** $y = -(x - 4)(x - 6)$, **e** $y = -3(x + 3)(x - 2)$, **f** $y = -\frac{2}{3}(x + 4)(x + 1)$ **11** $y = 2x^2 - x + 4$

Exercise 7.7

1 a $x = 3$, **b** $(3, -9)$, **c** $x = 0, 6$

d


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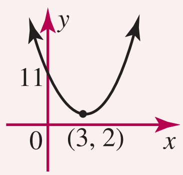

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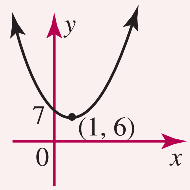

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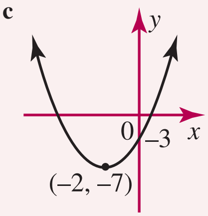

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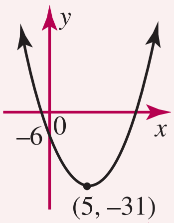

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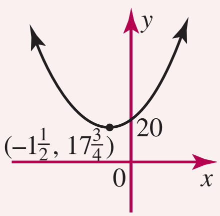

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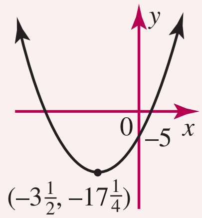

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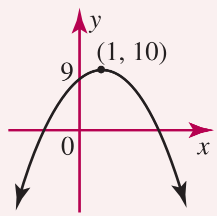

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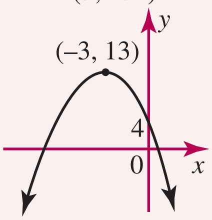

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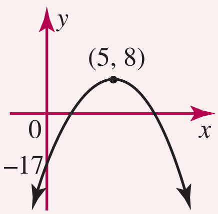

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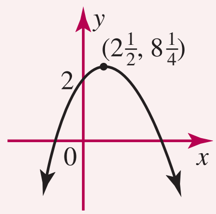

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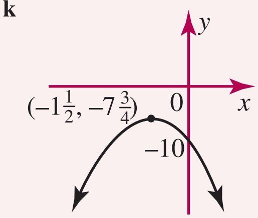

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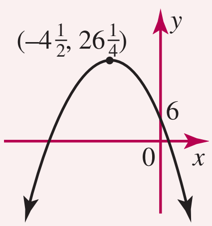


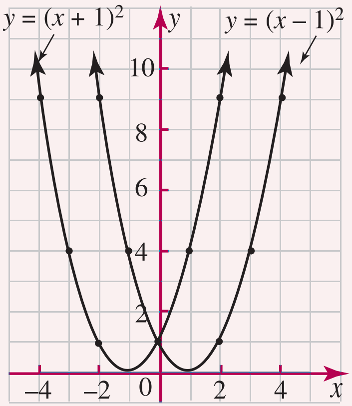
3 a $(-2, -1)$, **b** $(1, -5)$, **c** $(4, 45)$, **d** $(-2, 4)$, **e** $(-2\frac{1}{2}, -17\frac{1}{2})$, **f** $(2, 32)$

4 a min. $= 5\frac{7}{8}$, **b** min. $= -4\frac{1}{12}$, **c** max. $= 16\frac{1}{8}$, **d** min. $= -23\frac{1}{4}$, **e** max. $= -4\frac{7}{16}$, **f** max. $= 5\frac{1}{3}$ **5** -5 **6 a** $y = x^2 + 4x - 10$, **b** $y = x^2 + 4x + 10$, **c** $y = x^2 - 4x + 10$, **d** $y = x^2 - 4x - 10$ **7** $k = -12$ **8 a** $b = -18$, $c = 11$, **b** 59

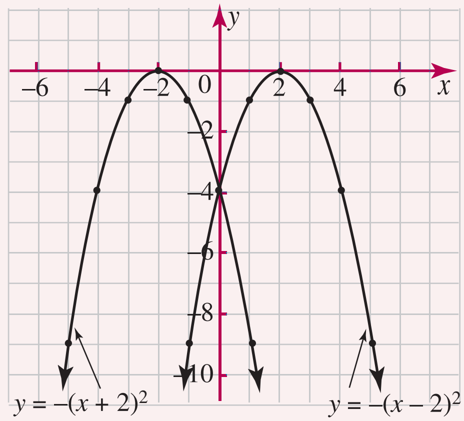
9 max. value = 15, when $x = 4$

Exercise 7.8

1 a

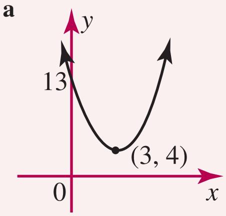


b



2 a $y = (x - 1)^2$ – shift $y = x^2$ one unit to the right; $y = (x + 1)^2$ – shift $y = x^2$ one unit to the left.
b $y = -(x - 2)^2$ – shift $y = -x^2$ two units to the right; $y = -(x + 2)^2$ – shift $y = -x^2$ two units to the left.

3 a



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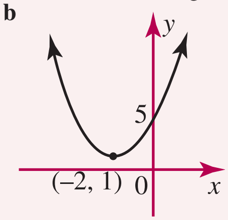

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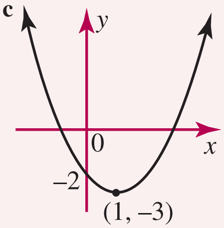
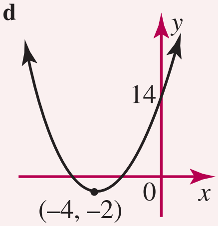


d



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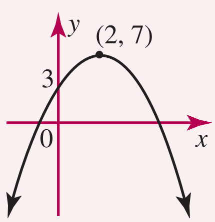

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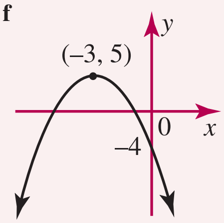

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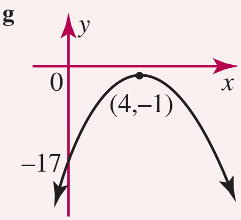

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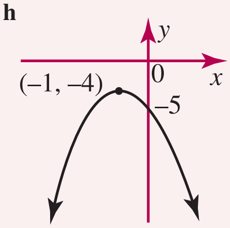

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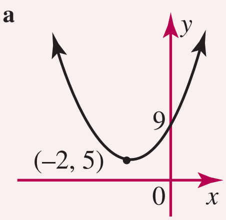

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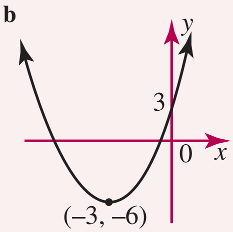

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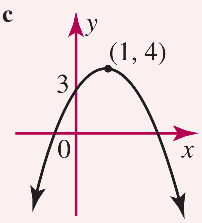

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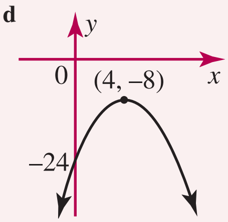

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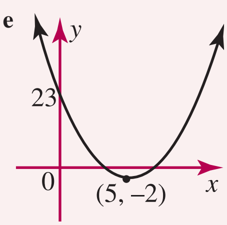

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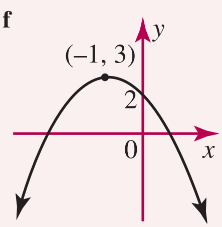

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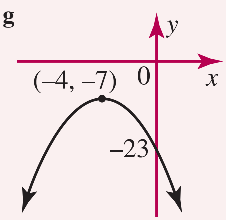

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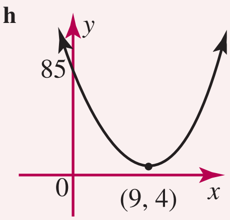

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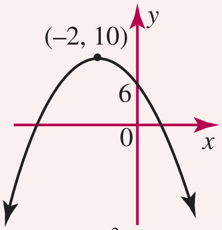

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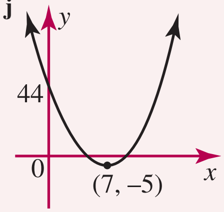

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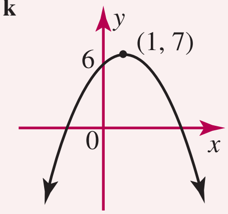

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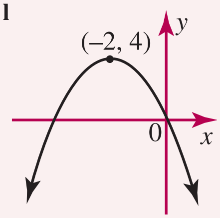
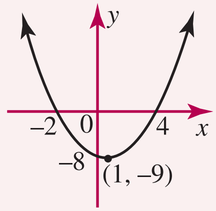


5 **a** $y = (x+1)^2 + 3$, $(-1, 3)$, **b** $y = (x+3)^2 + 4$, $(-3, 4)$, **c** $y = (x-2)^2 + 5$, $(2, 5)$, **d** $y = (x-1)^2 + 4$, $(1, 4)$, **e** $y = (x+3)^2 - 1$, $(-3, -1)$, **f** $y = (x-4)^2 - 5$, $(4, -5)$, **g** $y = (x+2)^2 - 7$, $(-2, -7)$, **h** $y = (x-4)^2 + 5$, $(4, 5)$, **i** $y = (x-2)^2 - 9$, $(2, -9)$, **j** $y = (x+3)^2 + 1$, $(-3, 1)$, **k** $y = (x-5)^2 - 12$, $(5, -12)$, **l** $y = (x-4)^2 - 20$, $(4, -20)$, **m** $y = (x+1)^2 - 7$, $(-1, -7)$, **n** $y = (x+5)^2 + 7$, $(-5, 7)$, **o** $y = (x-6)^2 + 3$, $(6, 3)$

6 **a** $y = (x + 1\frac{1}{2})^2 + 1\frac{3}{4}$, $V(-1\frac{1}{2}, 1\frac{3}{4})$, **b** $y = (x - 2\frac{1}{2})^2 + 2\frac{3}{4}$, $V(2\frac{1}{2}, 2\frac{3}{4})$, **c** $y = (x + \frac{1}{2})^2 - 5\frac{1}{4}$, $V(-\frac{1}{2}, -5\frac{1}{4})$

7 **a**


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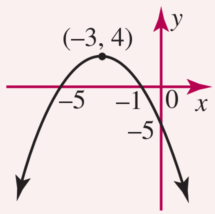
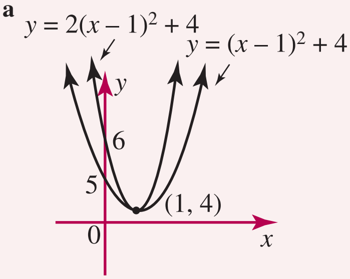
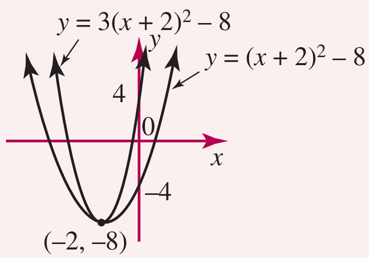


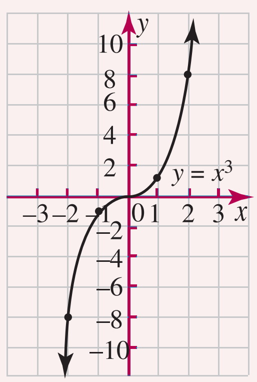
8 **a**


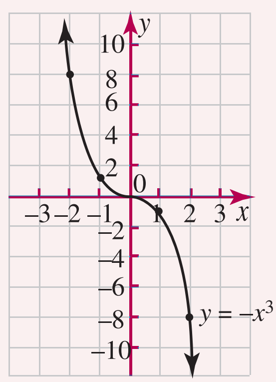
b


9 **a** $y = (x-4)^2 + 7$, **b** $y = (x+2)^2 - 1$, **c** $y = -(x-1)^2 + 6$, **d** $y = -(x+5)^2 + 4$, **e** $y = -(x-3)^2 - 10$, **f** $y = (x+4)^2 + 8$

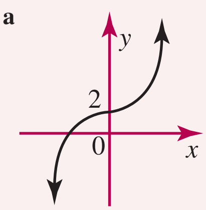
10 **a** $y = 2(x-3)^2 + 6$, **b** $y = 3(x+1)^2 - 10$, **c** $y = -2(x-4)^2 + 9$, **d** $y = -4(x+2)^2 - 5$, **e** $y = -3(x+3)^2 + 35$, **f** $y = 4(x-6)^2 - 11$

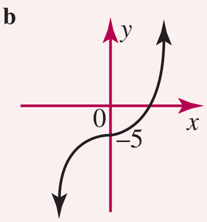
Exercise 7.9

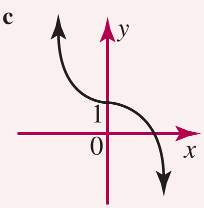
1 **a**


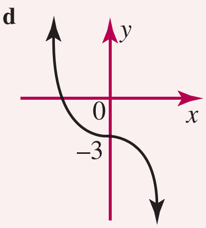
b


2 **a** translate up 1 unit
b translate down 2 units
c translate up 4 units
d translate down 7 units

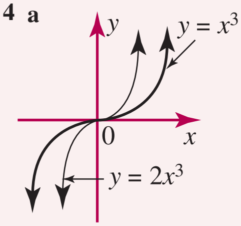
3 **a**


b


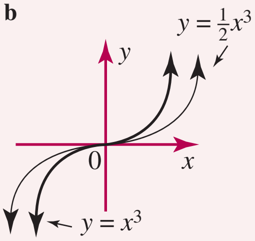
c


d


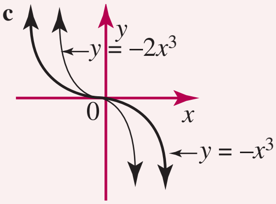
4 a



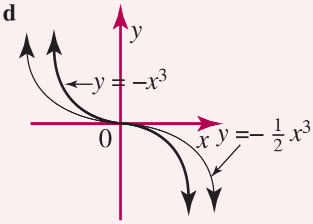
b



c



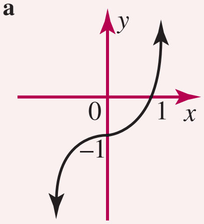
d



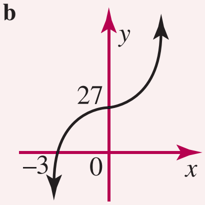
5 a (0, 0), **b** (0, 6), **c** (0, 3), **d** (0, –5) **6 a** $y = x^3 - 2$, **b** $y = -x^3 + 2$, **c** $y = x^3 + 2$, **d** $y = -x^3 - 2$

7 a $y = x^3 + 7$, **b** $y = x^3$, **c** $y = x^3 - 2$

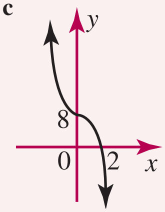
8 a



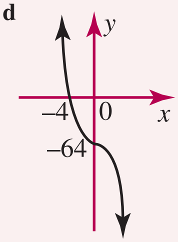
b



c



d



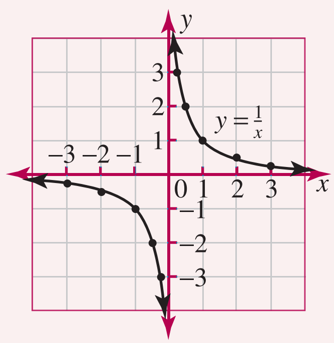
9 $y = -x^3 + 5$ **10 a** $y = x^3$, **b** $y = \frac{3}{2}x^3$, **c** $y = -3x^3$ **11 a** $y = x^3 + 6$, **b** $y = -x^3 + 4$, **c** $y = -\frac{1}{2}x^3 - 2$

Exercise 7.10

1 a

x	–3	–2	–1	$-\frac{1}{2}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	–1	–2	–3	–	3	2	1	$\frac{1}{2}$	$\frac{1}{3}$

b



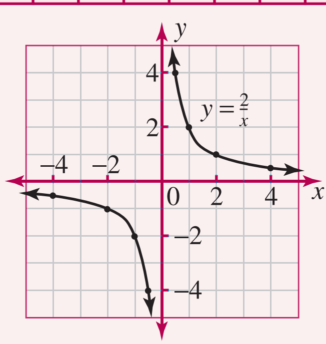
c 1st, 3rd, **d i** the y-values decrease, **ii** the y-values increase, **e i** y is positive, **ii** y is negative, **f** no, **g** asymptotes, **h** $y = x$, $y = -x$

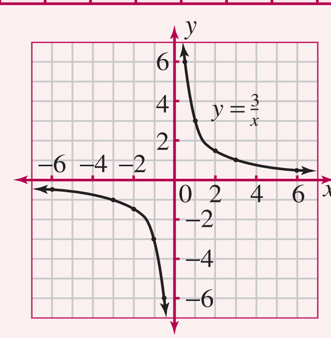
2 a

x	–4	–2	–1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	4
y	$-\frac{1}{2}$	–1	–2	–4	–	4	2	1	$\frac{1}{2}$

b

x	–6	–3	–2	–1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	6
y	$-\frac{1}{2}$	–1	$-1\frac{1}{2}$	–3	–6	–	6	3	$1\frac{1}{2}$	1	$\frac{1}{2}$



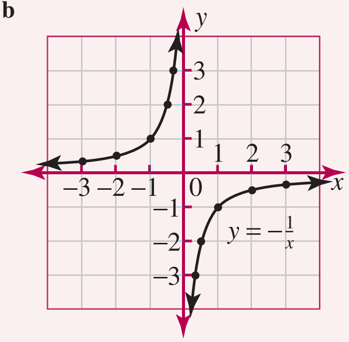


ANSWERS

3 a

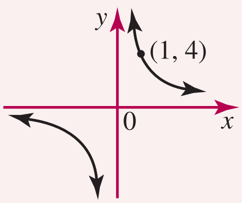
x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3
y	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	–	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$

b

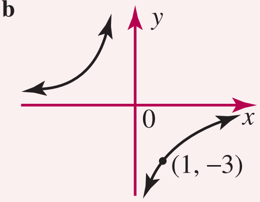


c 2nd, 4th, **d i** y is negative, **ii** y is positive, **e** yes, the x, y axes, **f** $y = x$, $y = -x$

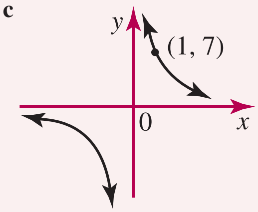
4 a



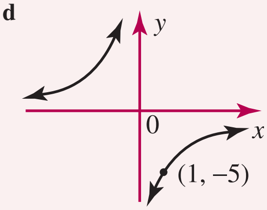
b



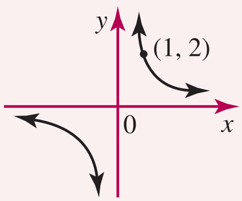
c



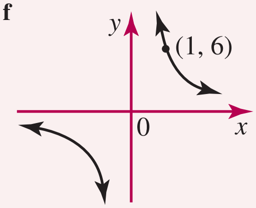
d



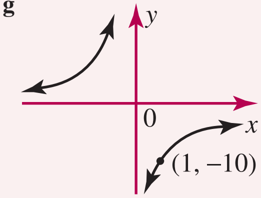
e



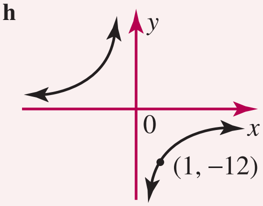
f



g



h

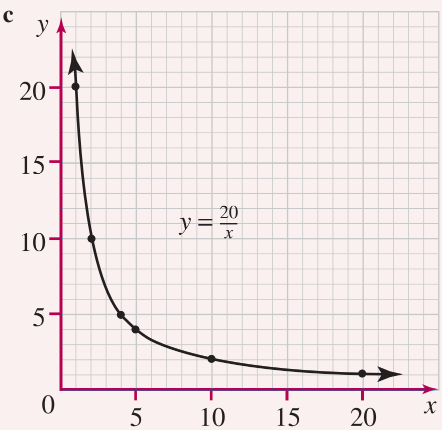


5 a $p = 6$, **b** $a = -4$, **c** $k = -1$, **d** $u = \frac{3}{2}$ **6 a** 4 units, **b** 15 units **7 a** $xy = 7$, **b** $xy = 10$, **c** $xy = -6$, **d** $xy = -20$, **e** $xy = 3$, **f** $xy = -8$ **8 a** $\frac{20}{x}$ cm,

b

x	1	2	4	5	10	20
y	20	10	5	4	2	1

c

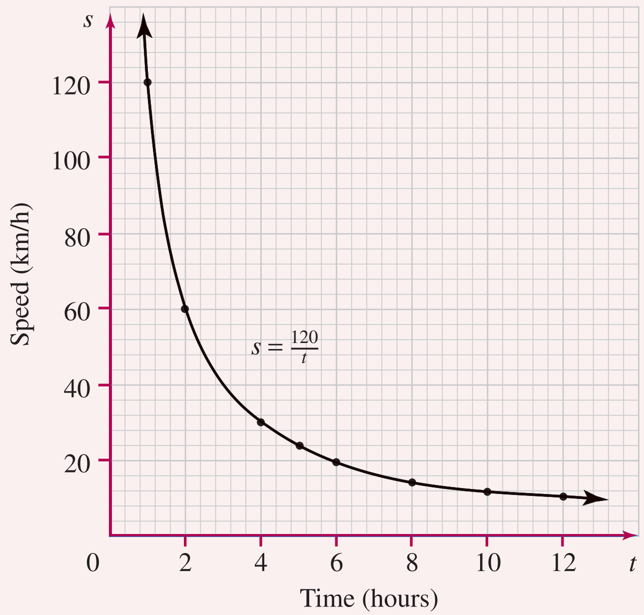


d Hyperbola ($x > 0$)

9 a

t	1	2	4	5	6	8	10	12
s	120	60	30	24	20	15	12	10

b



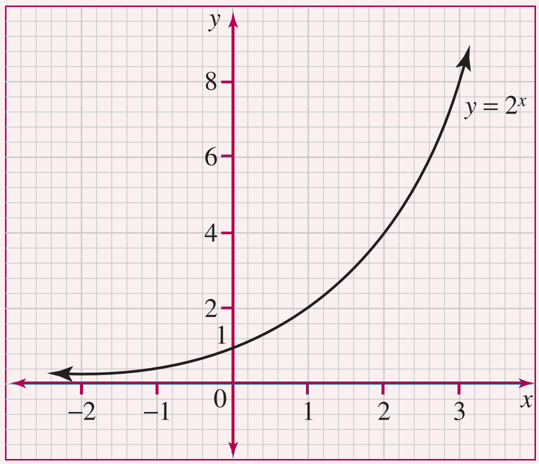
c 40 km/h

Exercise 7.11

1 a

x	−2	−1	0	1	2	3
y	0.25	0.5	1	2	4	8

b

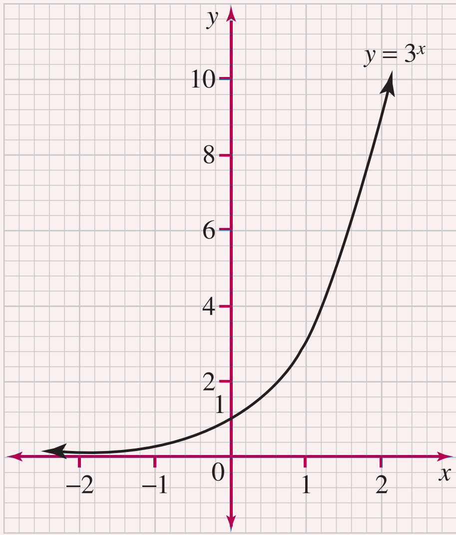


c **i** the y-values increase, **ii** the y-values decrease, **d** no, **e** asymptote, **f** the y-values are positive, **g** (0, 1), **h** no

ANSWERS

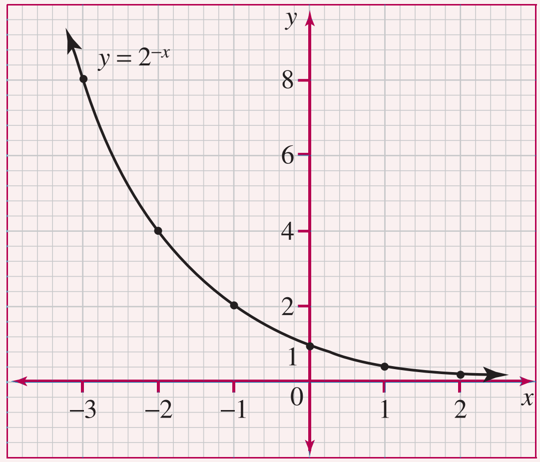
2

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y	0.1	0.2	0.3	0.6	1	1.7	3	5.2	9



3

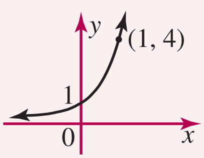
x	-3	-2	-1	0	1	2
y	8	4	2	1	0.5	0.25



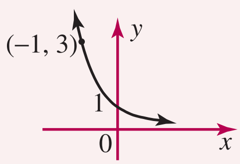
c **i** the y-values decrease, **ii** the y-values increase, **d** no, **e** the y-values are positive, **f** (0, 1), **g** no

4 Yes, $a^0 = 1$ for all values of a, where $a > 0$.

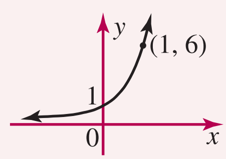
5 **a**



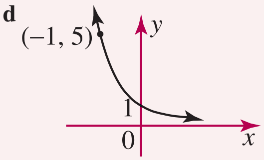
b



c

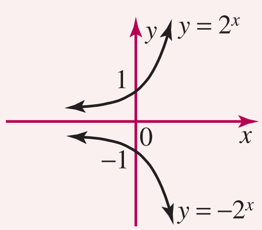


d

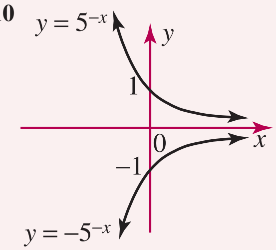


6 **a** $m = 16$, **b** $c = 3$, **c** $w = \frac{1}{2}$, **d** $h = -2$ **7** 63 units **8** **a** $y = 7^x$, **b** $y = 3^x$, **c** $y = 6^{-x}$

9 **a** Reflect the graph of $y = 2^x$ in the x-axis. **b**

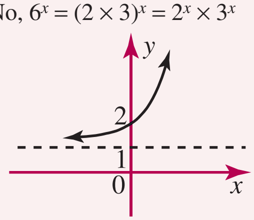


10

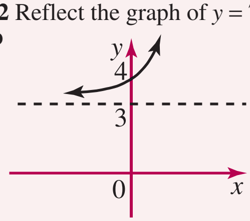


11 No, $6^x = (2 \times 3)^x = 2^x \times 3^x$ **12** Reflect the graph of $y = 7^x$ in the y-axis.

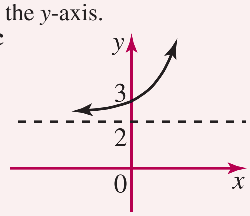
13 **a**



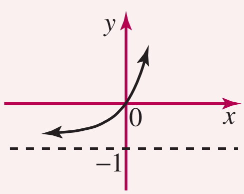
b



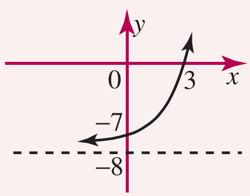
c



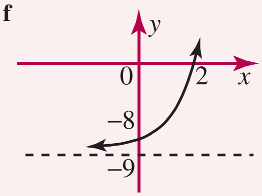
d

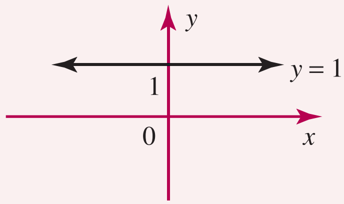


e



f

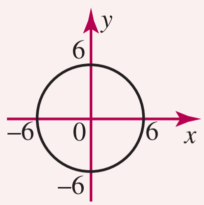


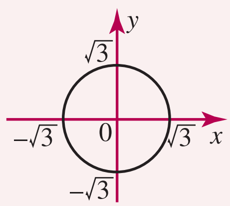
14 

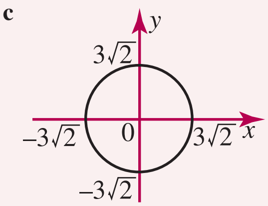
The graph of $y = 1^x$ is the graph of $y = 1$.

Exercise 7.12

1 **a** $x^2 + y^2 = 9$, **b** $x^2 + y^2 = 49$, **c** $x^2 + y^2 = 1$ **2** **a** $x^2 + y^2 = 4$, **b** $x^2 + y^2 = 25$, **c** $x^2 + y^2 = 16$, **d** $x^2 + y^2 = 3$, **e** $x^2 + y^2 = 7$, **f** $x^2 + y^2 = 10$, **g** $x^2 + y^2 = 8$, **h** $x^2 + y^2 = 48$, **i** $x^2 + y^2 = 63$, **j** $x^2 + y^2 = 1.44$, **k** $x^2 + y^2 = 3.61$, **l** $x^2 + y^2 = 6.25$ **3** **a** 3, **b** 7, **c** 11, **d** $\sqrt{5}$, **e** $\sqrt{14}$, **f** $\sqrt{22}$, **g** $2\sqrt{3}$, **h** $3\sqrt{3}$, **i** $3\sqrt{5}$, **j** 1.3, **k** 1.7, **l** 3.5

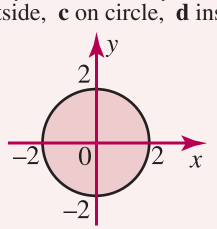
4 **a** 

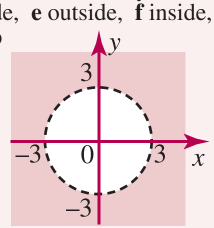
b 

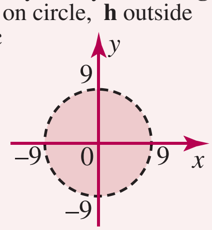
c 

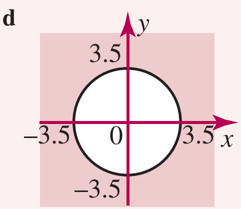
5 **a** $4x^2 + 4y^2 = 9$, **b** $16x^2 + 16y^2 = 121$, **c** $25x^2 + 25y^2 = 289$ **6** **a** $1\frac{1}{2}$, **b** $\frac{2}{3}$, **c** $2\frac{1}{2}$, **d** $\frac{4}{7}$, **e** $\frac{3}{4}$, **f** $2\frac{1}{5}$

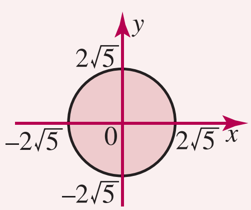
7 **a** (6, 8), (6, –8), **b** (6, –8), (–6, –8), **c** $(7, \sqrt{51})$, $(7, -\sqrt{51})$, **d** $(5\sqrt{3}, -5)$, $(-5\sqrt{3}, -5)$ **8** **a** $x^2 + y^2 = 25$, **b** $x^2 + y^2 = 34$, **c** $x^2 + y^2 = 17$ **9** **a** no, **b** yes, **c** no, **d** yes, **e** yes, **f** no, **g** yes, **h** no, **i** no **10** **a** inside, **b** outside, **c** on circle, **d** inside, **e** outside, **f** inside, **g** on circle, **h** outside

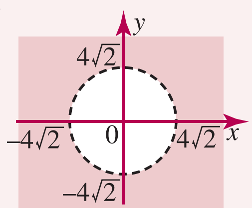
11 **a** 

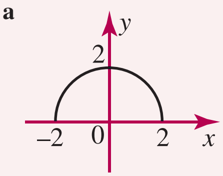
b 

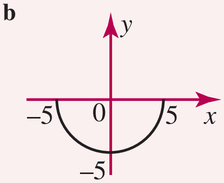
c 

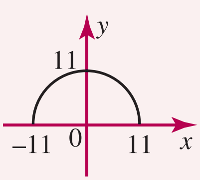
d 

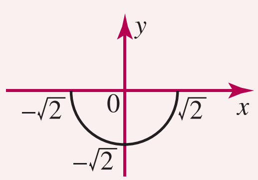
e 

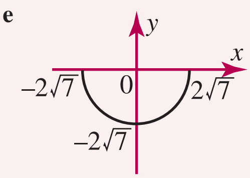
f 

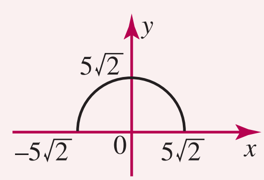
12 **a** 

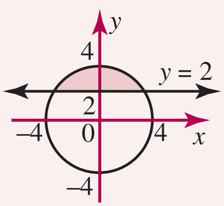
b 

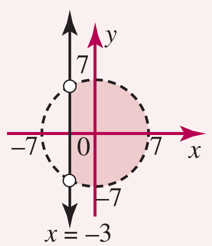
c 

d 

e 

f 

13 **a** 

b 

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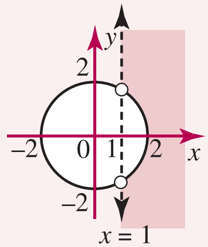

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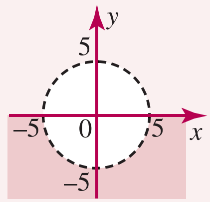

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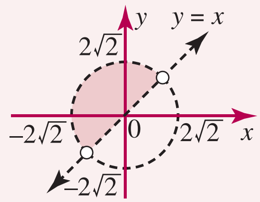

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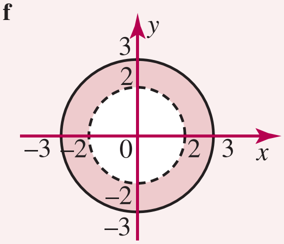
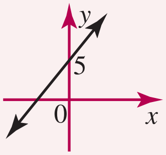


Exercise 7.13

1 **a** *L*, **b** *H*, **c** *P*, **d** *E*, **e** *C*, **f** *CU*, **g** *P*, **h** *L*, **i** *E*, **j** *H*, **k** *L*, **l** *C*, **m** *CU*, **n** *L*, **o** *H*, **p** *E*, **q** *C*, **r** *CU*, **s** *L*, **t** *P*, **u** *L*, **v** *H*, **w** *E*, **x** *C* **2** **a** $y = -3x - 1$, **b** $y = 3x - 1$, **c** $y = 1 - 3x$, **d** $y = 3x + 1$ **3** **a** $y = x^2 + 5$, **b** $y = x^2 - 5$, **c** $y = 5 - x^2$, **d** $y = -x^2 - 5$ **4** **a** $y = -\frac{2}{x}$, **b** $y = -\frac{x}{2}$, **c** $y = \frac{2}{x}$, **d** $y = \frac{x}{2}$ **5** **a** $y = 1 - x^3$, **b** $y = x^3 + 1$, **c** $y = x^3 - 1$, **d** $y = -x^3 - 1$ **6** **a** $y = -2^x$, **b** $y = 2^x$, **c** $y = 2^{-x}$, **d** $y = -2^{-x}$ **7** **a** *W*, **b** *Q*, **c** *N*, **d** *L*, **e** *R*, **f** *I*, **g** *S*, **h** *G*, **i** *B*, **j** *E*, **k** *V*, **l** *F*, **m** *M*, **n** *C*, **o** *O*, **p** *K*, **q** *A*, **r** *T*, **s** *P*, **t** *D*, **u** *U*, **v** *X*, **w** *H*, **x** *J*

8 **a**


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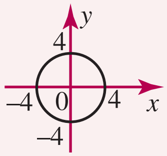

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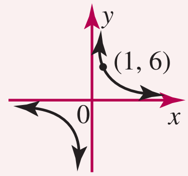

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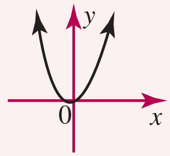

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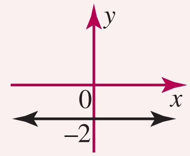

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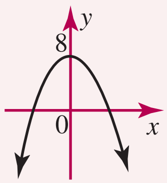

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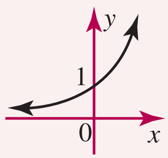

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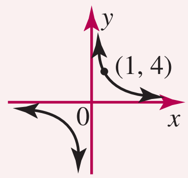

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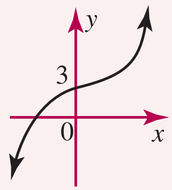

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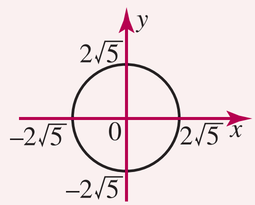

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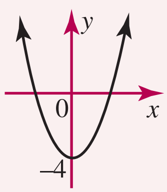

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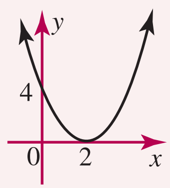

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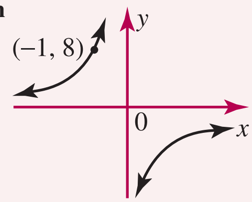

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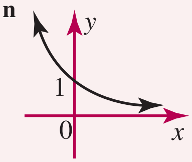

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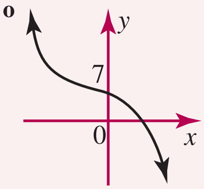

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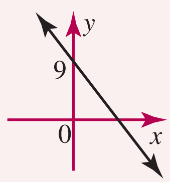

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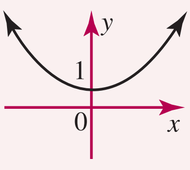

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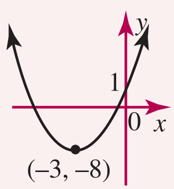

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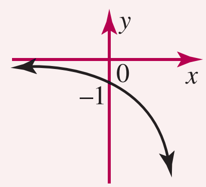

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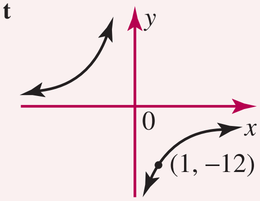

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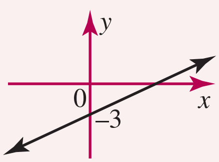

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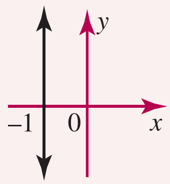

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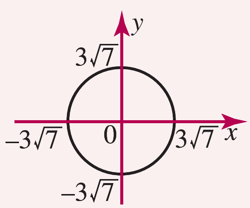
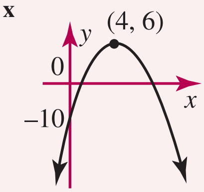


x


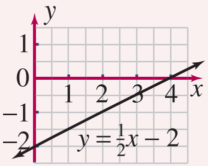
9 **a** 4 units, **b** (3, 4), **c** $k = 12, y = \frac{12}{x}$ **10** **a** $P(4, 5), Q(10, 2)$, **b** $m = -\frac{1}{2}$ **11** **a** (6, 64), **b** (1, 1), (−1, 1), **c** (−2, 5), **d** (2, −4), (−2, −4), **e** (3, 3), (−3, −3), **f** (2, 4), **g** (5, 3), (5, −3), **h** (0, 0), (2, 2), (−2, −2), **i** (−1, −2), (3, 6)
12 **a** $y = 6x - 2$, **b** $y = \frac{15}{x}$, **c** $y = 3x^2 + 3$, **d** $x^2 + y^2 = 36$, **e** $y = 2x^3 + 11$, **f** $y = 3(x + 1)(x - 5)$, **g** $y = -\frac{24}{x}$, **h** $y = -5x + 4$, **i** $y = 2^x$, **j** $y = 6 - x^2$, **k** $y = 5 - x^3$, **l** $y = 4(x - 1)^2 - 9$

Chapter 7 Review

1 **a** C, **b** B, **c** A, **d** C **2** **a** B, **b** C **3** B **4** C **5** A **6** **a** 9 am, **b** 140 km, **c** 10.30 am, **d** 12.30 pm, **e** 150 km, **f** 160 km, **g** Lloyd's speed decreased, the line is less steep, **h** 90 km, **i** 60 km/h, **j** 9.30 am

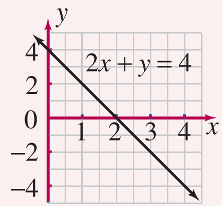
7 **a**

x	0	1	2	3	4
y	−2	$-1\frac{1}{2}$	−1	$-\frac{1}{2}$	0

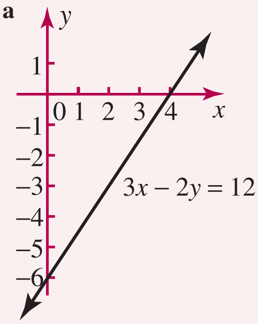


b

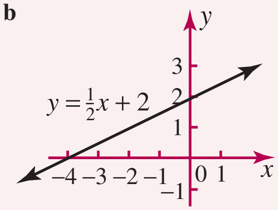
x	0	1	2	3	4
y	4	2	0	−2	−4



8 **a**

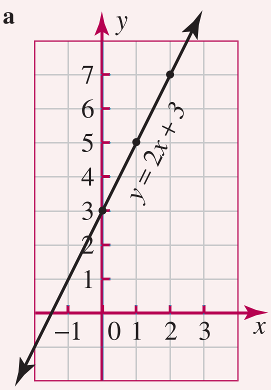


b

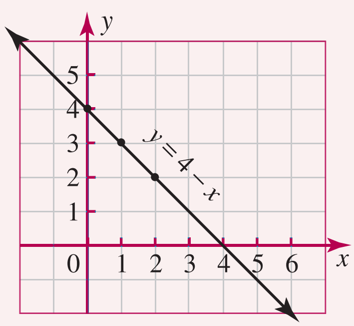


9 **a** $y = 4x - 3$, **b** $y = -2x$ **10** $-\frac{3}{4}$

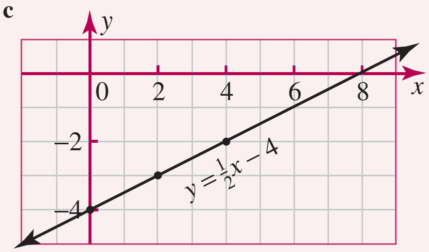
11 **a**



b

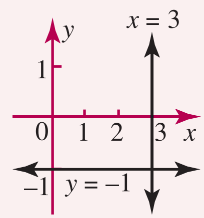


c

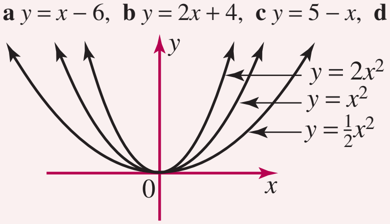


12 **a** $y = 7 - 2x$, **b** $y = 2x - 7$, **c** $y = -2x - 7$, **d** $y = 2x + 7$
13 **a** $y = -\frac{1}{5}x$, **b** $y = -5x$, **c** $y = 5x$, **d** $y = \frac{1}{5}x$
14 **a** $y = 4x + 20$, **b** $y = -\frac{3}{5}x - 6$

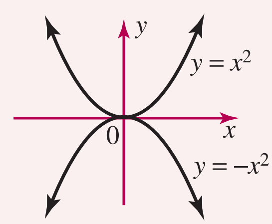
15



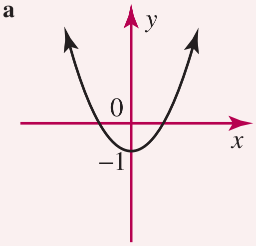
16 **a** $y = 5$, **b** $x = 4$
17 **a** yes, **b** no
18 $k = 2$
19 **a** $y = x - 6$, **b** $y = 2x + 4$, **c** $y = 5 - x$, **d** $y = \frac{1}{3}x + 1$
20



21



ANSWERS

22 **a**


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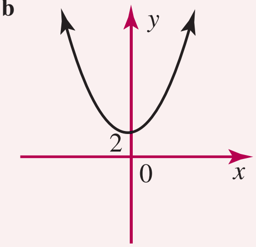

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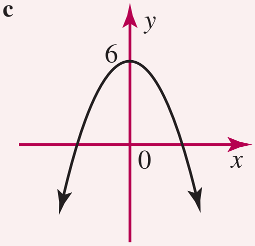

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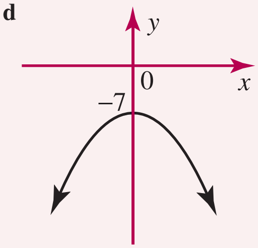
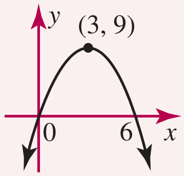


23 **a** $y = -x^2 - 1$, **b** $y = x^2 + 1$, **c** $y = x^2 - 1$, **d** $y = 1 - x^2$

24 **a**


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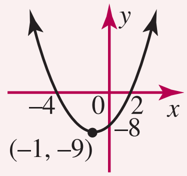

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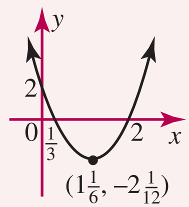
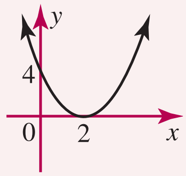


25 **a** $x = 4$, $(4, -6)$, **b** $x = 2$, $(2, 5)$ **26** -21

27 **a**


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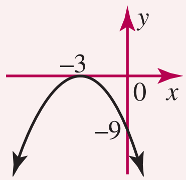

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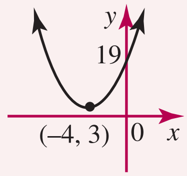

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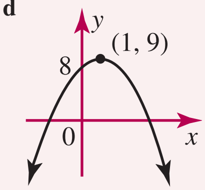
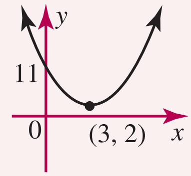
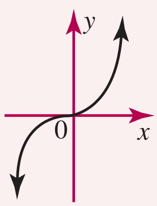


28


29 **a** $y = 4x^2$, **b** $y = x^2 - 5$ **30** $y = \frac{1}{2}x^2 - 7$

31 **a**


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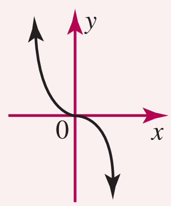
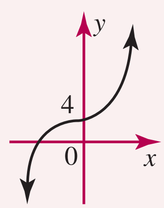
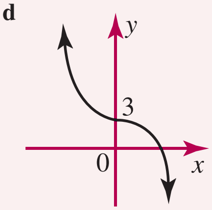
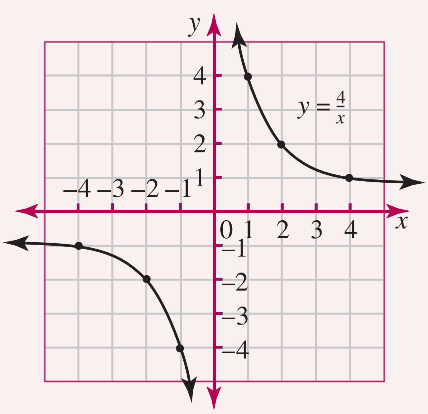


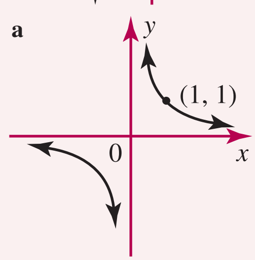
c


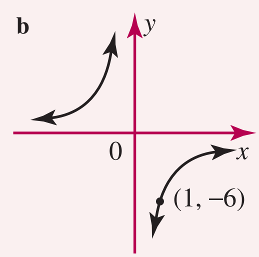
d


32

x	−4	−2	−1	0	1	2	4
y	−1	−2	−4	–	4	2	1

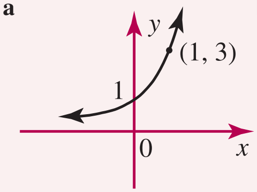


33 **a**


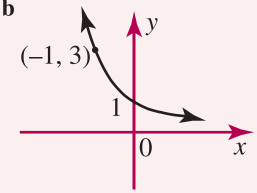
b


34 $t = -4$, $u = -\frac{2}{3}$ **35** **a** $y = \frac{24}{x}$, **b** $y = -\frac{9}{x}$ **36** **a** $y = \frac{x}{3}$, **b** $y = \frac{3}{x}$, **c** $y = -\frac{3}{x}$, **d** $y = -\frac{x}{3}$

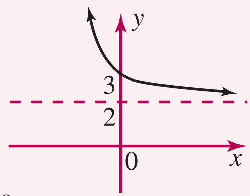
37 a



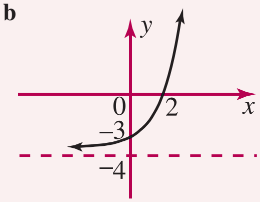
b



38 a

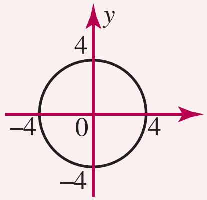


b

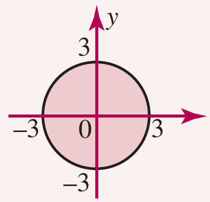


39 a $y = 3^x + 4$, **b** $y = 4^{-x}$ **40 a** $x^2 + y^2 = 25$, **b** $x^2 + y^2 = 18$

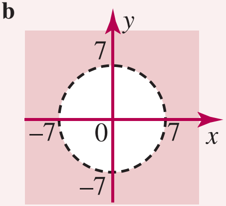
41



42 a



b



43 a $P(3, 11)$, $Q(-3, 11)$, **b** $P(2, 6)$, $Q(-2, -6)$

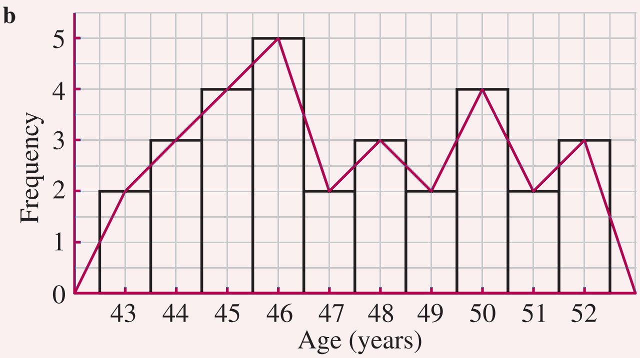
8 Data analysis and evaluation

Exercise 8.1

1 a

x	Tally	*f*	*c.f.*	*fx*
43	II	2	2	86
44	III	3	5	132
45	IIII	4	9	180
46	~~IIII~~	5	14	230
47	II	2	16	94
48	III	3	19	144
49	II	2	21	98
50	IIII	4	25	200
51	II	2	27	102
52	III	3	30	156
		30		1422

b



c 19, **d** 13.3%

2 a 7, **b** 25, **c** 7, **d** 40, **e** 12.5% **3 a** 5, **b** 8, **c** 1, **d** Yes, 8, **e** Yes, 2–4, **f** 36% **4 a** 30, **b** 202 cm, 164 cm, **c** 184 cm, **d** 18
5 a 15.6, **b** 29.6, **c** 4.8, **d** 11.8 **6 a** 11, **b** 28, **c** 39, **d** 78.9
7 a 4, **b** 10, **c** 31, 37, **d** no mode **8 a** 27, **b** 33, **c** 34.2, **d** 51
9 a mean = 11, median = 10, mode = 16, range = 25, **b** mean = 26.5, median = 28.5, mode = 29, range = 25, **c** mean = 59.3, median = 59, mode = 62, range = 19, **d** mean = 35.7, median = 33.5, mode = 32, range = 15, **e** mean = 124.7, median = 125, mode = 117, 134, range = 20, **f** mean = 95.1, median = 95, mode = 94, 97, range = 8, **g** mean = 8, median = 8.1, mode = 8.3, range = 1.5, **h** mean = 13.9, median = 14.0, no mode, range = 1.6
10 a mean = 4.4, median = 5, mode = 6, range = 5, **b** mean = 13.7, median = 13, mode = 12, range = 5, **c** mean = 19.0, median = 19, mode = 15, range = 24, **d** mean = 39.6, median = 40, mode = 38, range = 22

11 a

x	*f*	*c.f.*	*fx*
17	4	4	68
18	2	6	36
19	4	10	76
20	3	13	60
21	6	19	126
22	5	24	110
23	3	27	69
	27		545

b mean = 20.2, median = 21, mode = 21, range = 6

12 a 43, **b** 7, **c** 16 yrs, **d** 5 yrs, **e** 15 yrs, **f** 14.9 yrs

13 a

Stem	Leaf
$0^{(5)}$	8 9
$1^{(0)}$	0 2 4
$1^{(5)}$	5 6 6 7 8 9 9
$2^{(0)}$	0 2 3 4 4
$2^{(5)}$	5 5 5 6 7 7 9 9
$3^{(0)}$	0 1 2 4
$3^{(5)}$	6

b 30, **c** \$36 000, **d** \$28 000, **e** \$25 000, **f** \$23 500, **g** \$22 100, **h** 53%

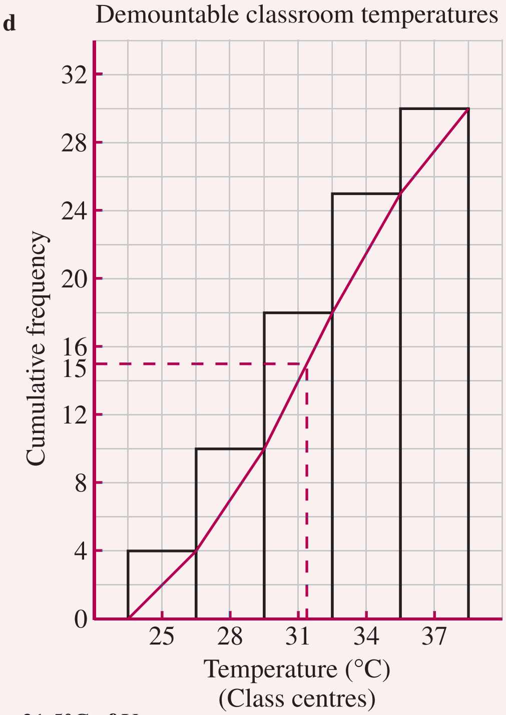
14 a 10, **b** 26, **c** 2.6, **d** 4, **e** 7th, **f** 4th
15 a 14, **b** 7.5

16 **a**

Class	Class centre (*c.c.*)	Tally	Frequency (*f*)	$f \times c.c.$
24–26	25	||||	4	100
27–29	28	卌 |	6	168
30–32	31	卌 |||	8	248
33–35	34	卌 ||	7	238
36–38	37	卌	5	185
			30	939

b 30–32, **c** 31.3°C

d



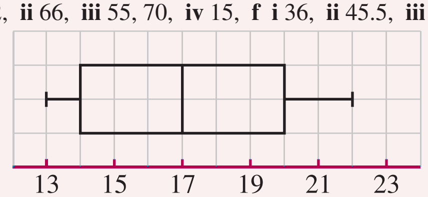
e 31.5°C, **f** Yes

17 **a** 17–21, 22–26, 27–31, 32–36, 37–41, 42–46, **b** 30, **c** 27–31, **d** 21, **e** 32 yrs **18** 4
19 **a** 17, **b** 216, **c** 28 **20** **a** 25, **b** 8
21 **a** decrease, **b** increase, **c** stay the same
22 **a** stay the same, **b** increase, **c** decrease
23 **a** 61.2, **b** 62.5 **24** 63 **25** 92%
26 79% **27** 69.1 **28** **a** 53, **b** 29
29 **a** 16, **b** 95 **30** $x = 5$, $y = 8$

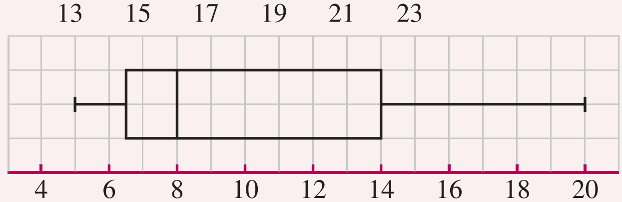
Exercise 8.2

1 **a** 66, **b** 37 **2** **a** 42, **b** 65 **3** **a** **i** 2, 12, **ii** 10, **iii** 7, **iv** 4, 9, **v** 5, **b** **i** 0, 20, **ii** 20, **iii** 7, **iv** 3, 15, **v** 12,
4 **a** **i** 8, **ii** 13, **iii** 10, 14, **iv** 4, **b** **i** 20, **ii** 27, **iii** 19, 31.5, **iv** 12.5, **c** **i** 27, **ii** 47, **iii** 38, 52, **iv** 14, **d** **i** 29, **ii** 39.5, **iii** 30.5, 49, **iv** 18.5, **e** **i** 32, **ii** 66, **iii** 55, 70, **iv** 15, **f** **i** 36, **ii** 45.5, **iii** 38, 60, **iv** 22

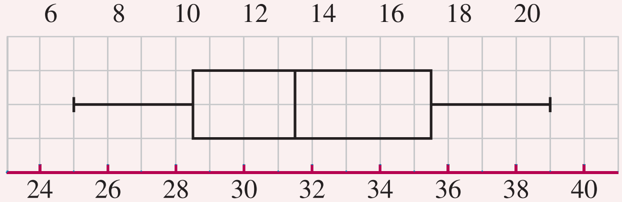
5 **a** median = 17, quartiles = 14, 20



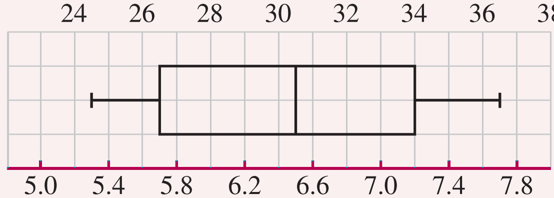
b median = 8, quartiles = 6.5, 14



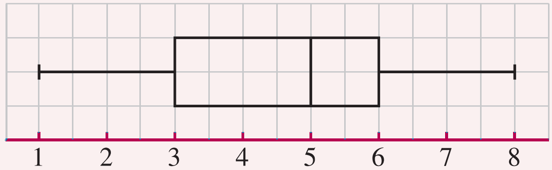
c median = 31.5, quartiles = 28.5, 35.5



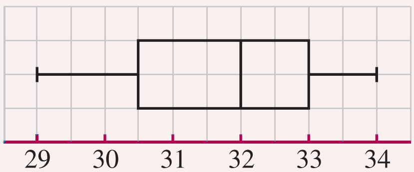
d median = 6.5, quartiles = 5.7, 7.2



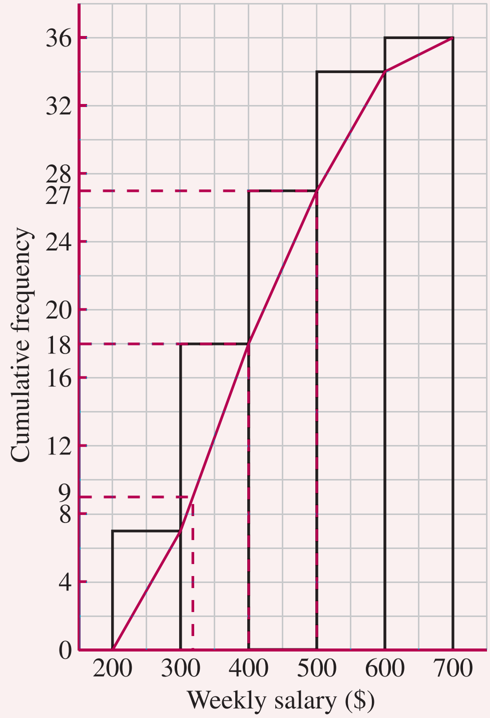
6 a median = 5, IQR = 3, **b**



7 a i 25%, **ii** 25%, **iii** 50%, **iv** 75%, **b** 7 **8** 84 **9** 18 **10 a** extremes = 3 yrs, 9 yrs, range = 6 yrs, **b** 5.5 yrs, **c** 4 yrs, 7 yrs, **d** 3 yrs **11 a** 5, **b** 6.5 **12** No, the median is the middle score of all the scores not the middle score between the quartiles. **13 a** 32, **b** 30.5, 33, **c** 2.5,
d



14 a median = 49.5, IQR = 2, **b** median = 4, IQR = 2.5 **15 a** 46 yrs, **b** 76 yrs, **c i** 18–33 yrs, **ii** 33–55 yrs
16 a F, **b** T, **c** F, **d** T, **e** F **17 a**



b \$400, **c** \$180 **19** 1

Exercise 8.3

1 a i 20, **ii** $\sum(x-\bar{x})^2 = 124$, **iii** 5.0, **b i** 59, **ii** $\sum(x-\bar{x})^2 = 134$, **iii** 4.7 **2 a** $\bar{x} = 5.9, \sigma_n = 1.9$, **b** $\bar{x} = 17, \sigma_n = 8.0$, **c** $\bar{x} = 20.2, \sigma_n = 6.2$, **d** $\bar{x} = 34.3, \sigma_n = 5.0$, **e** $\bar{x} = 17.0, \sigma_n = 7.9$, **f** $\bar{x} = 4.7, \sigma_n = 2.7$ **3 a i** 11.7, **ii** 12, **iii** 13, **iv** 1.5, **b i** 97.7, **ii** 97, **iii** 96, **iv** 1.6, **c i** 31.9, **ii** 32.5, **iii** 35, **iv** 6.6 **4 a** range = 22, $\sigma_n = 6.7$, **b** range = 4, $\sigma_n = 1.5$, **c** range = 7, $\sigma_n = 1.9$, **d** range = 4, $\sigma_n = 1.2$ **5 a** 5 yrs, **b** 24 yrs, **c** 23.5 yrs, **d** 22.7 yrs, **e** 1.8 yrs **6 a** $\bar{x} = 28.4, \sigma_n = 13.9$, **b** $\bar{x} = 25, \sigma_n = 10.6$ **7 a** 11°C, **b** 4.5°C, **c** 2.9°C, **d** The range and standard deviation, since there are no outliers. **8 a** 136, **b** 17, **c** 27.4, **d** The interquartile range, since the range and standard deviation are affected by outliers (9 and 145). **9** In Lucy's class — the standard deviation is smaller in her class. **10 a** F, **b** F, **c** T **11** $\sigma_n = 0$, the scores are all equal to the mean (i.e. $\sum(x-\bar{x})^2 = 0$, $\therefore \sigma_n = 0$). **12 a** 4.4, **b** The standard deviation would increase because the extra score is a long way from the mean. **13 a** Wombats: $\bar{x} = 95.2, \sigma_n = 7.2$, Ferrets: $\bar{x} = 95.5, \sigma_n = 4.6$, **b** The Ferrets, the standard deviation of the Ferrets scores is less than the standard deviation of the Wombats scores. **14 a** Mrs Bremner's class: $\bar{x} = 78.3, \sigma_n = 11.1$, Mrs Spencer's class: $\bar{x} = 77.8, \sigma_n = 8.8$, **b** Mrs Bremner's class — the mean is higher for her class.

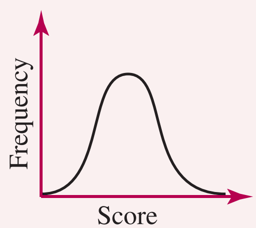
c Mrs Spencer's class — the standard deviation is lower for her class. **15** **a** $\bar{x} = 4, \sigma_n = 2$, **b** The mean would increase by 5, the standard deviation would not change. **c** The mean and standard deviation would both be 5 times greater. **16** The standard deviation would not change since the spread of both sets of scores is the same. That is, the average distance from the mean is the same in each set.

Exercise 8.4

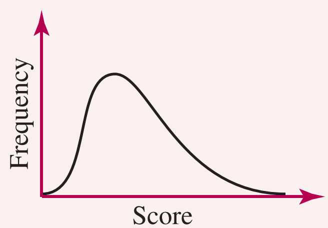
1 **a** 68, **b** 52, **c** 76, **d** 44, **e** 84, **f** 36 **2** **a** English: 46, Drama: 45, **b** English: 88, Drama: 69, **c** English: 67, Drama: 57, **d** English: 39, Drama: 41 **3** **a** 45, **b** 69, **c** 33, **d** 81, **e** 63, **f** 54, **g** 65, **h** 47, **i** 39, **j** 71, **k** 24, **l** 89 **4** 7 **5** **a** 6, **b** 65, **c** 74, **d** 50 **6** **a** 1 s.d. above $\bar{x}$, **b** 2 s.d. below $\bar{x}$, **c** $1\frac{1}{2}$ s.d. below $\bar{x}$ **7** Science, the Science mark is $1\frac{1}{2}$ s.d. above the mean but the Maths mark is only 1 s.d. above the mean. **8** **a** Test 2, **b** Test 1, **c** Test 1, **d** Test 2, **e** Test 1, **f** Test 2, **g** Test 1, **h** Test 1 **9** **a** **i** 80, **ii** 56, **iii** 92, **iv** 44, **b** **i** 74, **ii** 66, **iii** 82, **iv** 60 **10** No, she will fail. **11** **a** 58 g, 72 g, **b** 51 g, 79 g, **c** 44 g, 86 g **12** **a** 68%, **b** 95%, **c** 34%, **d** 13.5%, **e** 81.5%, **f** 16% **13** **a** 272, **b** 380, **c** 136, **d** 326, **e** 64, **f** 10 **14** \$340–\$800 **15** 10.2–13.8 km/L **16** **a** 2.5%, **b** No, the mark would be 4 s.d. above the mean. **17** **a** $\bar{x}$ will increase, σ_n will increase, **b** $\bar{x}$ will decrease, σ_n will increase, **c** $\bar{x}$ will stay the same, σ_n will decrease **18** **a** $\bar{x}$ will decrease, σ_n will decrease, **b** $\bar{x}$ will increase, σ_n will decrease, **c** $\bar{x}$ will stay the same, σ_n will increase

Exercise 8.5

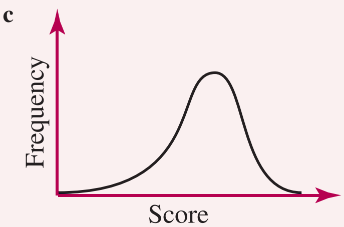
1 **a**



b

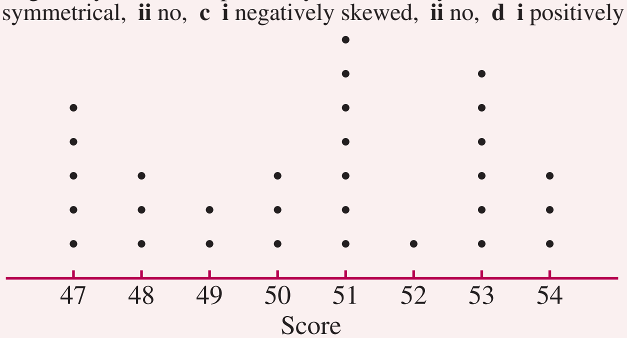


c



2 **a** negatively skewed, **b** positively skewed, **c** symmetrical **3** **a** **i** positively skewed, **ii** yes, 9 **b** **i** symmetrical, **ii** no, **c** **i** negatively skewed, **ii** no, **d** **i** positively skewed, **ii** yes, 8 **4** Positively skewed

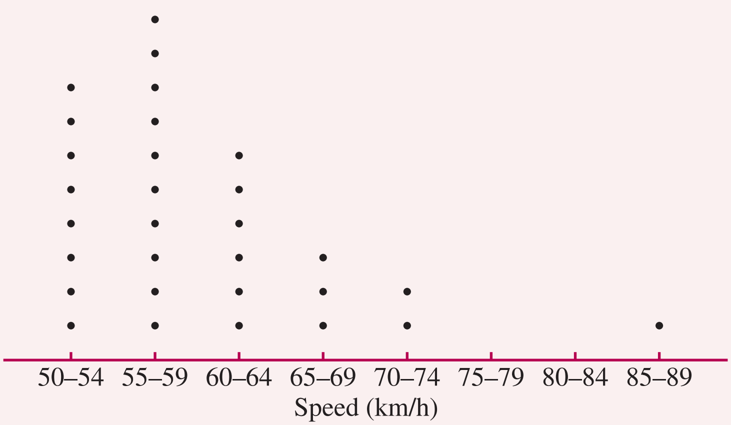
5 **a**



b 3, **c** 51

6 **a** negatively skewed, **b** symmetrical

7 **a**



b Yes, 88,
c positively skewed,
d 6

8 No, skewness only applies to numerical or quantitative data, not categorical data.

9 a

Stem	Leaf
0	3 4 6 8
1	2 3 3 5 6 7
2	0 1 2 4 4 7 7 8
3	4 5 6 9
4	0 1 5
5	2 4
6	1

b No, **c** Positively skewed, **d** No, most of his scores are low.

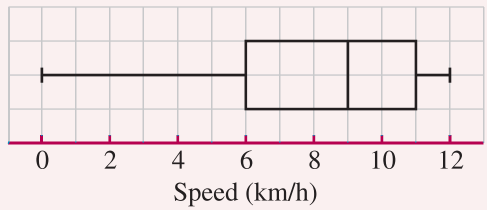
10 a Negatively skewed, **b** Positively skewed, **c** Symmetrical

11 a

Absentees	0	1	2	3	4	5	6	7	8	9	10	11	12
Frequency	1	1	2	2	2	3	3	3	3	5	9	8	6

b lowest score = 0, highest score = 12, median = 9, lower quartile = 6, upper quartile = 11.

c



d negatively skewed

12 a

Stem	Leaf
$7^{(5)}$	7 8
$8^{(0)}$	1 2 2 3 3 4 4 4 4
$8^{(5)}$	5 5 6 6 7
$9^{(0)}$	0 0 1 1 2 2 2 3 4
$9^{(5)}$	5 6 6 7 9

b Bimodal, **c** No

13 a i positively skewed, **ii** negatively skewed, **b** no skew

14 a 0.47, **b** –0.43

Exercise 8.6

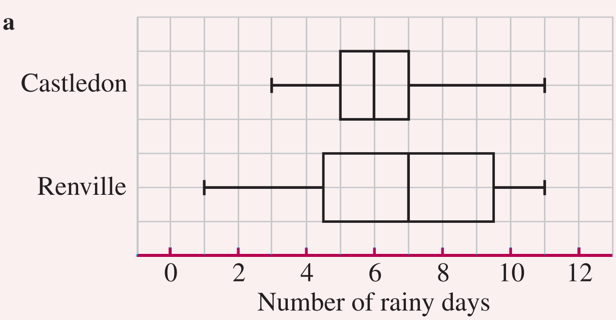
1 a i 36, **ii** 43, **b** 12, **c** Monday, **d** 73, **e** Monday, **f** 10% **2 a i** August, **ii** January, **iii** September, **iv** December, **b** September, October, July, **c** January, February, March, December, **d** January, February, December **3 a i** 4000, **ii** 4500, **iii** 1750, **b i** 3000, **ii** 4500, **iii** 3750, **c i** Females, **ii** Males, **d i** 8250, **ii** 17 500, **e** 1500 **4 a** 10 m, **b** 15 m, **c i** 25 m, **ii** 54 m, **iii** 80 m, **d** 4 m, **e** 46 m **5 a** Alice Springs: mean = 28.4°C, median = 29.3°C, range = 16.7°C, IQR = 11.5°C; Darwin: mean = 31.6°C, median = 31.7°C, range = 3.5°C, IQR = 1.6°C. **b** Yes, although Jan, Feb, Dec are hotter months in Alice Springs, the temperatures are more consistently high in Darwin throughout the year. **6 a** Yr 9, **b** Yr 8, **c** Yr 7, **d** Yr 10, **e** Yr 9, **f** Yr 8 **7 a** Sam, **b** Bob, **c** Ros, **d** Bob **8 a** Yr 7: lowest score = 20, highest score is 90, median = 55, lower quartile = 30, upper quartile = 70; Yr 8: lowest score = 40, highest score is 95, median = 70, lower quartile = 55, upper quartile = 80. **b** Yr 7 — 40, Yr 8 — 25, **c** The students results have improved greatly. In Yr 7 50% of the marks were greater than 55, in Yr 8 75% of the marks were greater than 55.

9 a

First round	Stem	Second round
9 8 7	$6^{(5)}$	5 7 7 7
4 4 3 2 2 2 2 0 0	$7^{(0)}$	0 0 1 1 1 2 3 3 4 4
9 8 8 7 6 5 5 5	$7^{(5)}$	5 5 6 6 7 8 9 9
4 3 1 0	$8^{(0)}$	2
	$8^{(5)}$	5

b Round 1: mean = 74.75, median = 74.5, mode = 72; Round 2: mean = 73.63, median = 74, mode = 67, 71 **c** Mean, since there are no outliers. **d** Second round

10 a

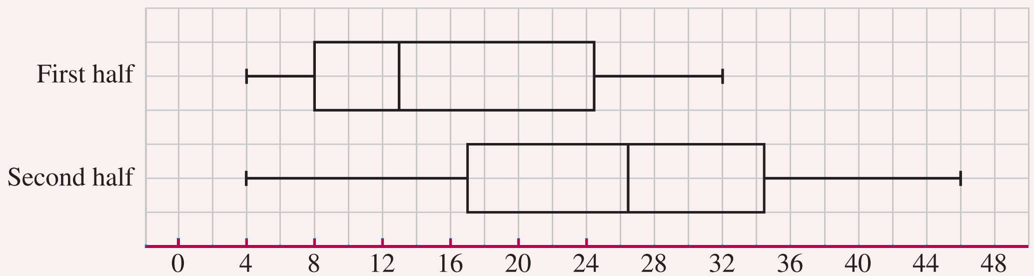


b Castledon — 2.1, Renville — 3.0, **c** Renville has 6 more rainy days (80) compared to Castledon (74), and the rainfall pattern is more wide spread.

11 a 20, **b i** $\bar{x} = 15.85$, $\sigma_n = 8.57$, **ii** $\bar{x} = 25.55$, $\sigma_n = 11.20$

ANSWERS

c



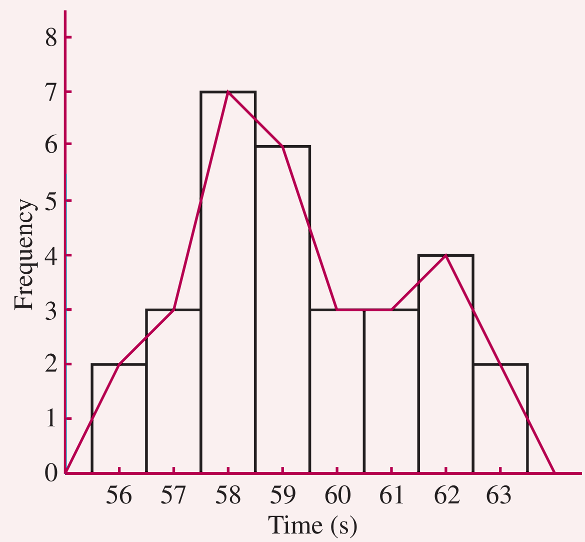
d Overall, the team concedes more points in the second half. **12 a** The means are all 7, **b i** C, **ii** A
c A: $\bar{x} = 7, \sigma_n = 1.1$; B: $\bar{x} = 7, \sigma_n = 1.4$; C: $\bar{x} = 7, \sigma_n = 1.6$

Chapter 8 Review

1 a

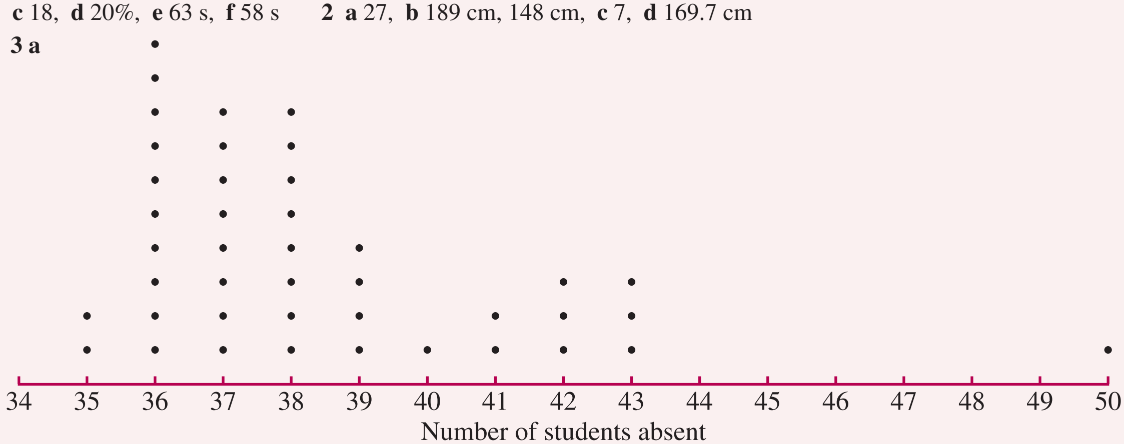
x	Tally	*f*	*c.f.*	*fx*
56	II	2	2	112
57	III	3	5	171
58	~~IIII~~ II	7	12	406
59	~~IIII~~ I	6	18	354
60	III	3	21	180
61	III	3	24	183
62	IIII	4	28	248
63	II	2	30	126
		30		1780

b



c 18, **d** 20%, **e** 63 s, **f** 58 s **2 a** 27, **b** 189 cm, 148 cm, **c** 7, **d** 169.7 cm

3 a



b yes, 36–38, **c** yes, 50
4 a mean = 14, median = 15, mode = 18, range = 17, $\sigma_n = 5.5$, **b** mean = 46, median = 45, mode = 59, range = 68, $\sigma_n = 21.3$, **c** mean = 34.5, median = 35, mode = 30, 37, range = 9, $\sigma_n = 3.1$, **d** mean = 5.4, median = 5.1, no mode, range = 3.8, $\sigma_n = 1.3$ **5 a** mean = 3.8, median = 4, mode = 5, range = 4, $\sigma_n = 1.3$, **b** mean = 26.7, median = 27, mode = 23, range = 15, $\sigma_n = 5.1$ **6 a** 6, **b** 17 yrs, **c** 29, **d** 6.9%, **e** 15 yrs, **f** 14 yrs, **g** 14.2 yrs **7 a** 82, **b** 54

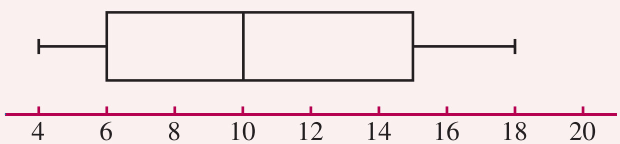
8 **a** 12, **b** 1, **c** 4 yrs, **d** 72.5 yrs, **e** 73 yrs, **f** 72.1 yrs, **g** 1.4 yrs

9 **a**

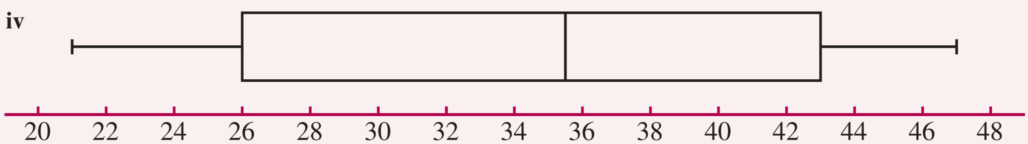
Class	Class centre (*c.c.*)	Tally	Frequency (*f*)	$f \times c.c.$
1–5	3	|	1	3
6–10	8	|	1	8
11–15	13	||||	4	52
16–20	18	𝍸 |||	8	144
21–25	23	𝍸 𝍸 ||	12	276
26–30	28	||||	4	112
			30	595

b 21–25, **c** 14, **d** $\bar{x} = 19.8$, $\sigma_n = 5.8$
10 **a** 112, **b** 32 **11** 10
12 **a** **i** increase, **ii** decrease,
b **i** decrease, **ii** increase,
c **i** stay the same, **ii** stay the same
13 **a** 70.1, **b** 71.4 **14** 92% **15** 50
16 **a** **i** L.S. = 1, H.S. = 9, **ii** 6, **iii** 8, **iv** 3, **v** 8, **vi** 5, **b** **i** 25%, **ii** 25%, **iii** 50%, **iv** 25%

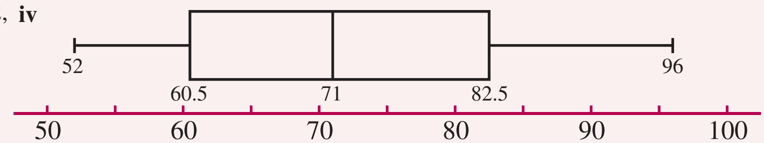
17 **a** **i** 10, **ii** $Q_1 = 6$, $Q_3 = 15$, **iii** 9, **iv**



b **i** 35.5, **ii** $Q_1 = 26$, $Q_3 = 43$, **iii** 17,
iv



c **i** 71, **ii** $Q_1 = 60.5$, $Q_3 = 82.5$, **iii** 22, **iv**

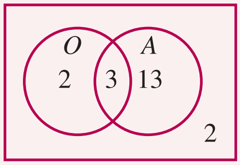


18 $Q_1 = 3$, $Q_3 = 6$, I.Q.R. = 3 **19** **a** 46, **b** $Q_1 = 44$, $Q_3 = 46.5$, **c** 2.5 **20** A **21** B
22 **a** Petersham: $\bar{x} = 224.4$, $\sigma_n = 43.0$; Manly: $\bar{x} = 242.8$, $\sigma_n = 82.9$, **b** The Petersham team had a lower standard deviation, so they were the more consistent team. **23** **a** 62, **b** 26, **c** 56, **d** 32 **24** 6
25 **a** test 1, **b** test 2 **26** **a** **i** 78, **ii** 54, **b** **i** 58, **ii** 88 **27** **a** 68%, **b** 95%, **c** 99.7%, **d** 34%, **e** 13.5%, **f** 2.5%
28 **a** 136, **b** 190, **c** 5% **29** **a** positively skewed, **b** negatively skewed, **c** symmetrical
30 **a** Boys: $\bar{x} = 75.1$, median $= 74.5$, $\sigma_n = 7.4$; Girls: $\bar{x} = 72.1$, median $= 73.5$, $\sigma_n = 7.5$
b The boys have a higher mean and median so they are slightly fitter than the girls.

9 Probability

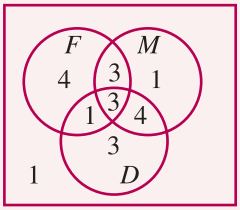
Exercise 9.1

1 **a** $\frac{1}{13}$, **b** $\frac{1}{2}$, **c** $\frac{1}{26}$, **d** $\frac{4}{13}$, **e** $\frac{3}{13}$ **2** **a** $\frac{3}{10}$, **b** $\frac{3}{10}$, **c** $\frac{2}{5}$, **d** $\frac{2}{5}$ **3** **a** **i** $\frac{18}{25}$, **ii** $\frac{1}{10}$, **iii** $\frac{9}{50}$, **b** yes
4 **a** **i** *A* and *B*, *D* and *F*, **ii** *A* and *B*, *C* and *E*, *D* and *F*, *E* and *F*, **b** $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{2}{5}$, $P(D) = \frac{3}{5}$, $P(E) = \frac{3}{10}$, $P(F) = \frac{2}{5}$ **5** **a** $\frac{2}{5}$, **b** $\frac{1}{5}$, **c** $\frac{4}{5}$, **d** 0 **6** **a** $\frac{1}{30}$, **b** $\frac{1}{10}$, **c** $\frac{1}{6}$, **d** $\frac{19}{30}$, **e** $\frac{7}{30}$ **7** **a** **i** $\frac{3}{14}$, **ii** $\frac{13}{28}$, **iii** $\frac{11}{28}$, **iv** $\frac{1}{7}$, **b** **i** $\frac{7}{11}$, **ii** $\frac{11}{15}$ **8** **a** $\frac{7}{8}$, **b** $\frac{13}{40}$, **c** $\frac{27}{40}$, **d** $\frac{23}{40}$, **e** $\frac{1}{10}$ **9** **a** $\frac{1}{12}$, **b** $\frac{1}{3}$, **c** $\frac{1}{12}$, **d** $\frac{1}{12}$, **e** $\frac{1}{6}$, **f** $\frac{1}{4}$, **g** $\frac{1}{3}$ **10** $P(T) = \frac{18}{25}$, $P(S) = \frac{2}{5}$, $P(S \text{ and } T) = \frac{3}{25}$ **11**



12 **a** **i** $\frac{2}{9}$, **ii** $\frac{1}{9}$, **b** $\frac{2}{3}$, **c** $\frac{1}{9}$,
d 6 or 8
13 **a** $P(T) = \frac{13}{20}$, $P(H) = \frac{7}{20}$, the coin is biased towards tails.
c toss the coin a large number of times.

15

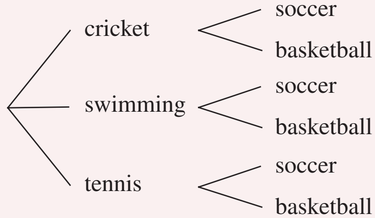


ANSWERS

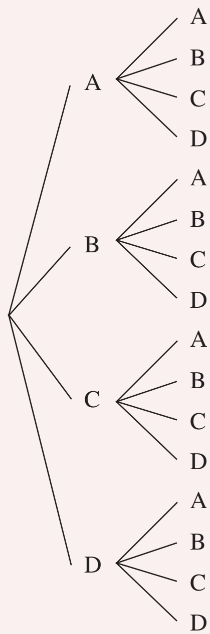
Exercise 9.2

1 a i $\frac{2}{9}$, **ii** $\frac{1}{3}$, **iii** $\frac{1}{9}$, **iv** $\frac{2}{3}$, **b i** $\frac{1}{18}$, **ii** $\frac{1}{6}$, **iii** $\frac{1}{6}$, **iv** $\frac{2}{3}$, **v** $\frac{1}{3}$

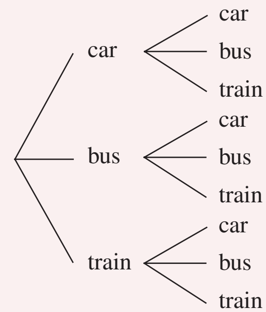
2 a



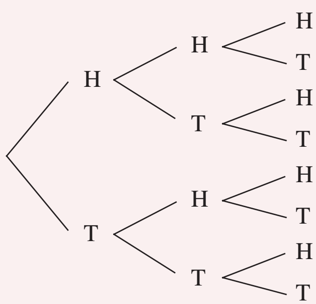
b



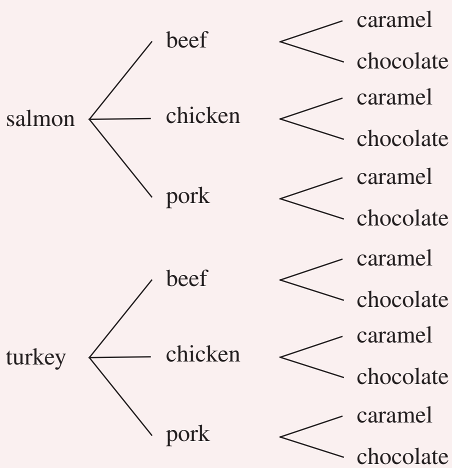
c



d



e

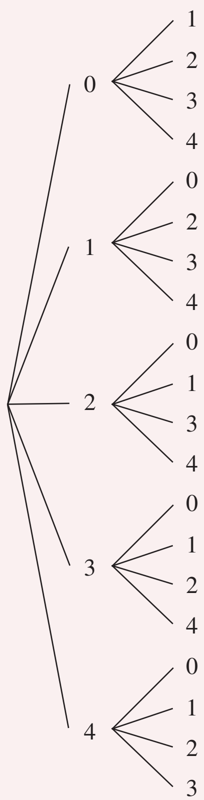


3 a i $\frac{1}{9}$, **ii** $\frac{2}{9}$, **iii** $\frac{4}{9}$, **iv** $\frac{5}{9}$, **b i** $\frac{1}{4}$, **ii** $\frac{1}{4}$, **iii** $\frac{1}{2}$, **c i** $P(B) = \frac{1}{4}, P(J) = \frac{1}{4}, P(D) = \frac{1}{2}$, **ii** $\frac{1}{8}$, **iii** $\frac{3}{4}$, **d i** $\frac{1}{3}$, **ii** $\frac{1}{6}$, **iii** $\frac{1}{3}$
4 a i $\frac{1}{18}$, **ii** $\frac{1}{36}$, **iii** $\frac{1}{9}$, **iv** 0, **v** $\frac{1}{3}$, **vi** $\frac{13}{18}$, **b i** $\frac{1}{8}$, **ii** $\frac{1}{56}$, **iii** $\frac{3}{56}$, **iv** $\frac{3}{7}$, **v** $\frac{2}{7}$ **5 a** 24, **i** $\frac{1}{24}$, **ii** $\frac{1}{2}$, **iii** $\frac{1}{12}$, **iv** $\frac{7}{24}$, **v** $\frac{5}{8}$
b i $\frac{1}{36}$, **ii** $\frac{1}{36}$, **iii** $\frac{1}{18}$, **iv** $\frac{2}{3}$, **v** $\frac{5}{6}$, **c i** $\frac{1}{16}$, **ii** $\frac{1}{4}$, **iii** $\frac{1}{2}$, **iv** $\frac{1}{4}$, **v** $\frac{1}{16}$, **vi** $\frac{7}{16}$, **d i** $\frac{1}{24}$, **ii** $\frac{3}{8}$, **iii** $\frac{2}{3}$, **iv** $\frac{1}{12}$, **v** $\frac{1}{3}$, **vi** $\frac{2}{3}$
7 a i $\frac{1}{49}$, **ii** $\frac{1}{49}$, **iii** $\frac{6}{49}$, **iv** $\frac{36}{49}$, **v** $\frac{5}{7}$, **b i** 1, **ii** $\frac{2}{25}$, **iii** $\frac{3}{5}$, **iv** $\frac{2}{5}$, **v** $\frac{1}{5}$, **c i** $\frac{1}{144}$, **ii** $\frac{1}{12}$, **iii** $\frac{5}{24}$, **iv** $\frac{1}{9}$, **v** $\frac{35}{72}$, **vi** $\frac{47}{48}$,
d i $\frac{1}{2}$, **ii** $\frac{1}{52}$, **iii** $\frac{1}{26}$, **iv** $\frac{15}{26}$, **v** $\frac{1}{169}$ **8 a** $\frac{1}{64}$, **b** $\frac{27}{2197}$, **c** $\frac{1}{2704}$, **d** $\frac{1728}{2197}$, **e** $\frac{27}{4394}$ **9 a** $\frac{1}{81}$, **b** $\frac{1}{36}$, **c** $\frac{25}{1296}$, **d** $\frac{1}{72}$,
e $\frac{16}{81}$ **10 a** $\frac{1}{216}$, **b** $\frac{1}{1728}$, **c** $\frac{1}{72}$, **d** $\frac{8}{27}$, **e** $\frac{271}{288}$ **11 a** 1296, **b i** $\frac{1}{6}$, **ii** $\frac{1}{6}$, **iii** $\frac{1}{16}$, **iv** $\frac{1}{27}$, **v** $\frac{13}{18}$ **12 a** $\frac{351}{400}$,
b $\frac{23}{100}$, **c** $\frac{83}{400}$, **d** $\frac{39}{400}$ **13 a** 281 474 976 710 656, **b** 2.8%, **c** 1.7%, **d** 88.5%

Exercise 9.3

1 a AB, AC, AD, BC, BD, CD, **b i** $\frac{1}{6}$, **ii** $\frac{1}{2}$, **iii** $\frac{1}{2}$

2 a



, **b i** $\frac{1}{20}$, **ii** $\frac{4}{5}$, **iii** $\frac{2}{5}$, **iv** $\frac{1}{5}$, **v** $\frac{3}{10}$

3 a 90, **b** 81, **c i** $\frac{1}{5}$, **ii** $\frac{22}{45}$, **iii** $\frac{4}{45}$, **iv** $\frac{22}{45}$

4 a $\frac{2}{5}$, **b** $\frac{1}{10}$, **c** $\frac{3}{10}$, **d** $\frac{3}{10}$, **e** $\frac{9}{10}$

5 a 1128, **b i** $\frac{35}{376}$, **ii** $\frac{22}{47}$, **iii** $\frac{25}{47}$, **c** $\frac{24}{47}$

6 a i $\frac{1}{5}$, **ii** $\frac{26}{35}$, **iii** $\frac{8}{105}$, **b i** 0, **ii** $\frac{4}{455}$, **iii** $\frac{8}{65}$, **iv** $\frac{13}{35}$

7 a i $\frac{1}{100}$, **ii** $\frac{1}{100}$, **b i** $\frac{1}{20}$, **ii** $\frac{1}{20}$, **iii** $\frac{1}{20}$

8 a i $\frac{1}{66}$, **ii** $\frac{6}{11}$, **iii** $\frac{6}{11}$, **b i** $\frac{1}{220}$, **ii** $\frac{21}{55}$, **iii** $\frac{1}{440}$, **iv** $\frac{3}{220}$, **c i** $\frac{27}{64}$, **ii** $\frac{1}{576}$, **iii** $\frac{11}{144}$

9 $\frac{1}{3}$

10 a $\frac{1}{3}$, **b** $\frac{1}{2}$, **c** $\frac{2}{5}$, **d** $\frac{3}{5}$

11 a i $\frac{1}{51}$, **ii** $\frac{1}{51}$, **iii** $\frac{1}{1275}$, **iv** $\frac{1}{425}$, **b i** $\frac{1}{51}$, **ii** $\frac{2500}{132\,651}$, **iii** $\frac{2}{2601}$, **iv** $\frac{100}{44\,217}$

12 hard-hard or hard-soft

13 a $\frac{33}{16\,660}$, **b** $\frac{64}{162\,435}$, **c** $\frac{1}{54\,145}$, **d** $\frac{1}{108\,290}$

14 a $\frac{14}{495}$, **b** $\frac{49}{825}$, **c** $\frac{49}{825}$, **d** $\frac{62}{2695}$,
e Assuming the question means only two of the numbers are greater than 50 the probability is $\frac{75}{198}$. If the question means at least two, then the probability is $\frac{1}{2}$.

Exercise 9.4

1 a 22 100, **b i** $\frac{1}{52}$, **ii** $\frac{11}{850}$ **2 a** $\frac{1}{6}$, **b** \$8.33, **c** no **3 b** game 3, **c** yes, game 3

5 a 2 598 960, **b i** $\frac{1}{13}$, **ii** $\frac{33}{66\,640}$ **6 a** no, **b** games 2 and 3, **c** no, the expected return is \$8.33 for each game

7 $P(\text{Jae wins}) = \frac{1}{2}$, $P(\text{Jemima wins}) = \frac{1}{3}$, $P(\text{Draw}) = \frac{1}{6}$. The game is not fair.

8 $P(\text{Blake wins}) = \frac{1}{3}$, $P(\text{Cynthia wins}) = \frac{1}{3}$, $P(\text{Draw}) = \frac{1}{3}$. The game is fair.

9 a $8^6 = 262\,144$, **b i** $\frac{1}{8}$, **ii** $\frac{1}{512}$, **iii** $\frac{729}{4096}$, **iv** $\frac{1}{64}$ **10 a i** $\frac{2}{5}$, **ii** $\frac{2}{5}$, **b i** $\frac{3}{5}$, **ii** $\frac{1}{5}$, **c i** $\frac{1}{3}$, **ii** $\frac{1}{3}$

11 Limb = \$15, Eyesight = \$50, Hearing = \$10, Death = \$1.

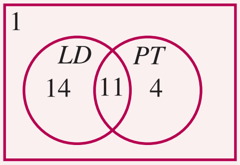
15 $P(\text{Ngaire wins}) = \frac{1}{2}$, $P(\text{Mairaid wins}) = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots < \frac{1}{2}$. The game is not fair.

16 $P(\text{Jae wins}) = \frac{3}{5}$, $P(\text{Jemima wins}) = \frac{2}{5}$. The game is not fair.

Chapter 9 Review

1 **a** J and L, **b** the card is not a black Jack, **c** **i** $\frac{1}{2}$, **ii** $\frac{1}{2}$, **iii** $\frac{6}{13}$ **2** **a** $\frac{5}{36}$, **b** $\frac{1}{2}$, **c** $\frac{5}{12}$ **3** **a** $\frac{9}{19}$, **b** $\frac{1}{19}$, **c** $\frac{1}{38}$, **d** $\frac{9}{19}$, **e** $\frac{4}{19}$ **4** **a** **i** $\frac{2317}{33\,000}$, **ii** $\frac{813}{40\,000}$, **iii** $\frac{2477}{57\,000}$, **b** Utility, **c** $\frac{1959}{59\,500}$ **5** **a** 100, **b** **i** $\frac{6}{25}$, **ii** $\frac{7}{10}$, **c** $\frac{3}{10}$, **d** $\frac{3}{8}$

6



a **i** $\frac{1}{6}$, **ii** $\frac{11}{30}$, **iii** $\frac{1}{30}$, **b** $\frac{11}{15}$ **7** **a** 12, **b** **i** $\frac{1}{4}$, **ii** $\frac{1}{12}$, **iii** $\frac{1}{3}$, **iv** $\frac{1}{2}$, **v** $\frac{1}{2}$

8 **a** AB, AT, AC, BB, BT, BC, CB, CT, CC, DB, DT, DC, **b** **i** $\frac{1}{4}$, **ii** $\frac{1}{12}$, **iii** $\frac{1}{3}$, **iv** $\frac{1}{3}$

9 **a** HOR, HOB, HOW, HER, HEB, HEW, TOR, TOB, TOW, TER, TEB, TEW **b** **i** $\frac{1}{2}$, **ii** $\frac{1}{6}$, **iii** $\frac{1}{12}$, **iv** $\frac{1}{3}$, **v** $\frac{1}{2}$ **10** **b** **i** $\frac{1}{18}$, **ii** $\frac{1}{9}$, **iii** $\frac{13}{36}$, **iv** $\frac{7}{9}$, **v** $\frac{7}{18}$

11 **a** **i** $\frac{12}{25}$, **ii** $\frac{1}{4}$, **iii** $\frac{27}{100}$, **b** **i** $\frac{31}{50}$, **ii** $\frac{29}{100}$, **iii** $\frac{9}{100}$, **c** best estimate is 12 white, 6 yellow, 2 black, **d** best estimate is red — 173°, blue — 90°, green — 97° **13** **a** **i** $\frac{9}{19}$, **ii** $\frac{81}{361}$, **iii** $\frac{729}{6859}$, **iv** $\frac{3\,486\,784\,401}{6\,131\,066\,257\,801}$, **b** **i** $\left(\frac{9}{19}\right)^{26}$ **14** **a** WX, WY, WZ, XW, XY, XZ, YW, YX, YZ, ZW, ZX, ZY, **b** **i** $\frac{1}{6}$, **ii** $\frac{1}{2}$, **iii** $\frac{1}{12}$, **iv** $\frac{1}{2}$

15 **a**

VICE-CAPTAIN (columns); CAPTAIN (rows)

	A	B	C	D	E	F	G
A	—	AB	AC	AD	AE	AF	AG
B	BA	—	BC	BD	BE	BF	BG
C	CA	CB	—	CD	CE	CF	CG
D	DA	DB	DC	—	DE	DF	DG
E	EA	EB	EC	ED	—	EF	EG
F	FA	FB	FC	FD	FE	—	FG
G	GA	GB	GC	GD	GE	GF	—

b **i** $\frac{2}{7}$, **ii** $\frac{1}{21}$, **iii** $\frac{5}{7}$, **iv** $\frac{1}{42}$, **v** $\frac{5}{42}$, **vi** $\frac{2}{21}$ **16** **a** CA, CM, CJ, CH, AM, AJ, AH, MJ, MH, JH **b** **i** $\frac{2}{5}$, **ii** $\frac{3}{10}$, **iii** $\frac{3}{10}$ **17** $\frac{1}{10}$ **18** **a** $\frac{1}{110}$, **b** $\frac{1}{55}$, **c** $\frac{1}{55}$, **d** $\frac{53}{55}$, **e** $\frac{28}{55}$ **19** **a** $\frac{1}{90}$, **b** $\frac{1}{45}$, **c** $\frac{28}{45}$, **d** $\frac{8}{45}$, **e** $\frac{1}{30}$ **20** **a** 2 598 960, **b** 308 915 776, **c** 9 765 625 **21** P(Jemima wins) = $\frac{1}{4}$, P(Sam wins) = $\frac{1}{4}$, the game is fair **22** **a** \$8.33, not fair, **b** \$8.33, not fair, **c** **i** $\frac{1}{2}$, **ii** \$10, **iii** yes **23** P(David wins) = $\frac{1}{2}$, P(Samantha wins) = $\frac{1}{2}$, the game is fair

10 Further trigonometry

Exercise 10.1

1 **a** 0.64, **b** 0.34, **c** 0.87, **d** 0.98 **2** sin (180° – θ) = sin θ **3** **a** cos 50° = 0.64, cos 130° = –0.64, **b** cos 10° = 0.98, cos 170° = –0.98, **c** cos 40° = 0.77, cos 140° = –0.77, **d** cos 80° = 0.17, cos 100° = –0.17 **4** cos (180° – θ) = –cos θ **5** **a** sin 150°, **b** sin 130°, **c** sin 45°, **d** sin 75°, **e** sin 107°, **f** sin 68°, **g** sin 81°, **h** sin 13° **6** **a** cos 160°, **b** cos 110°, **c** cos 60°, **d** cos 25°, **e** cos 95°, **f** cos 78°, **g** cos 139°, **h** cos 43° **7** 1.20 **8** **a** 30°, 150°, **b** 53°, 127°, **c** 12°, 168°, **d** 34°, 146°, **e** 8°, 172°, **f** 48°, 132°, **g** 21°, 159°, **h** 55°, 125°, **i** 29°, 151° **9** **a** 60°, **b** 70°, **c** 160°, **d** 140°, **e** 65°, **f** 106°, **g** 32°, **h** 76°, **i** 123° **10** **a** 0.8, **b** 0.94, **c** 0.96, **d** 0.54, **e** 0.44, **f** 0.77, **g** 0.92, **h** 0.98

Exercise 10.2

1 **a** positive, **b** positive, **c** negative, **d** negative, **e** positive, **f** positive, **g** negative, **h** positive, **i** positive, **j** positive, **k** negative, **l** negative, **m** positive, **n** positive, **o** negative, **p** positive **2** **a** acute, **b** obtuse, **c** obtuse, **d** acute, **e** obtuse, **f** acute **3** **a** sin 80° = 0.98, **b** –cos 40° = –0.77, **c** –tan 20° = –0.36, **d** –cos 55° = –0.57, **e** –tan 46° = –1.04, **f** sin 28° = 0.47, **g** –tan 79° = –5.14, **h** sin 65° = 0.91, **i** –cos 72° = –0.31, **j** sin 16° = 0.28, **k** –tan 58° = –1.60, **l** –cos 9° = –0.99, **m** –tan 21° = –0.38, **n** –cos 27° = –0.89, **o** sin 43° = 0.68, **p** –cos 86° = –0.07, **q** sin 84° = 0.99, **r** –tan 66° = –2.25, **s** –cos 48° = –0.67, **t** sin 11° = 0.19 **4** **a** –tan 51°45′ = –1.2685, **b** sin 74°33′ = 0.9639, **c** –cos 15°52′ = –0.9619, **d** sin 43°40′ = 0.6905, **e** –cos 66°9′ = –0.4043, **f** –tan 22°16′ = –0.4095, **g** –cos 30°24′ = –0.8625, **h** –tan 80°53′ = –6.2316, **i** sin 14°28′ = 0.2498 **5** **a** 9°, 171°, **b** 15°, 165°, **c** 57°, 123°, **d** 37°, 143°, **e** 4°, 176°, **f** 25°, 155°, **g** 11°, 169°, **h** 49°, 131°, **i** 23°, 157° **6** **a** 121°, **b** 173°, **c** 101°, **d** 137°, **e** 118°, **f** 147°, **g** 106°, **h** 134°, **i** 112° **7** **a** 32°, 148°, **b** 36°, **c** 129°, **d** 117°, **e** 62°, **f** 11°, 169°, **g** 146°, **h** 63°, 117, **i** 65°, **j** 48°, 132°, **k** 104°, **l** 127°

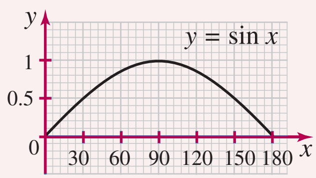
8 **a** $\frac{1}{2}$, **b** $-\frac{1}{\sqrt{2}}$, **c** $-\sqrt{3}$, **d** $-\frac{\sqrt{3}}{2}$, **e** $\frac{\sqrt{3}}{2}$, **f** -1, **g** $\frac{1}{\sqrt{2}}$, **h** $-\frac{1}{\sqrt{3}}$, **i** $-\frac{1}{2}$

9 **a**

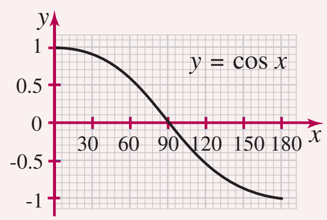
x	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
sin x	0.0	0.2	0.3	0.5	0.6	0.8	0.9	0.9	1.0	1.0
cos x	1.0	1.0	0.9	0.9	0.8	0.6	0.5	0.3	0.2	0.0
tan x	0.0	0.2	0.4	0.6	0.8	1.2	1.7	2.7	5.7	—

x	100°	110°	120°	130°	140°	150°	160°	170°	180°
sin x	1.0	0.9	0.9	0.8	0.6	0.5	0.3	0.2	0.0
cos x	–0.2	–0.3	–0.5	–0.6	–0.8	–0.9	–0.9	–1.0	–1.0
tan x	–5.7	–2.7	–1.7	–1.2	–0.8	–0.6	–0.4	–0.2	0.0

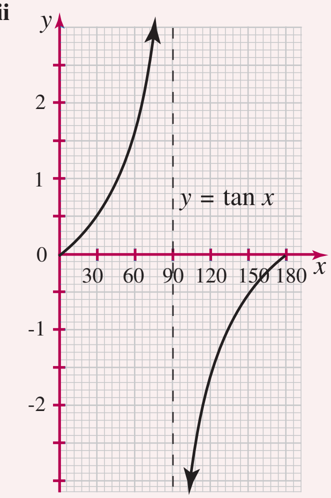
9 **b** **i**



ii



iii



10 **a** 1, **b** 0, **c** **i** $0° < x < 180°$, **ii** none, **d** $x = 0°, 180°$
11 **a** 1, **b** -1, **c** **i** $0° \leqslant x < 90°$, **ii** $90° < x \leqslant 180°$, **d** $x = 90°$
12 **a** no, **b** $x = 90°$, **c** **i** $0° < x < 90°$, **ii** $90° < x < 180°$, **d** $x = 0°, 180°$

Exercise 10.3

1 **a** $a = 10.5$, **b** $v = 22.5$, **c** $m = 32$ **2** **a** $\frac{2}{3}$, **b** 0.6, **c** $\frac{1}{12}$ **3** **a** $a = 8.9$, **b** $q = 25.2$, **c** $k = 2.7$, **d** $h = 86.5$, **e** $n = 27.3$, **f** $t = 24.9$ **4** **a** 54°, **b** 15°, **c** 28°, **d** 25°, **e** 68°, **f** 36° **5** **a** $p = 20.8$, **b** $w = 22.2$, **c** $c = 76.0$, **d** $f = 34.9$, **e** $u = 20.6$, **f** $z = 51.4$ **6** **a** 27°31′, **b** 7°5′, **c** 43°45′, **d** 29°38′, **e** 21°44′, **f** 51°30′ **7** **a** 132°, **b** 160°, **c** 120°, **d** 128°, **e** 114°, **f** 148° **8** **b** 71°, **c** 141.4 cm **9** **a** 102 mm, **b** 76 mm, **c** 208 mm **10** **a** 20 cm, **b** $6\sqrt{6}$ cm, **c** 24 cm **11** **a** 46°, 134°, **b** 29°, 151°, **c** 16°, 164° **12** **b** 24°14′, 155°46′, **c** 139°18′, 7°46′ **13** 52°

Exercise 10.4

1 **a** $a = 10.4$, **b** $u = 70.7$, **c** 54.3°, **d** 22°7′ **2** 17.7 m **3** 67 cm **4** **a** 34°, **b** 26.3 km, **c** 9.3 km **5** 20.3 cm **6** 73°44′ **7** **b** 16 cm **8** **a** 12.1 cm, **b** 31°22′ **9** **a** $LM = 76.2$ m, $LN = 93.3$ m, **b** $12 558 **10** 54 cm **11** 37.2 cm **12** Betty is closer by 6 m **13** **b** 45° **14** **a** 12.2 km, **b** 73 m, **c** 38.5 m, **d** 338 m **15** **a** $\angle ACB = 39°$, bearing = 129°, **b** $\angle PRQ = 23°$, bearing = 203°, **c** $\angle FHG = 37°$, bearing = 307°, **d** $\angle YZX = 46°$, bearing = 046°

Exercise 10.5

1 **a** $g = \sqrt{11}$, **b** $z = \sqrt{13}$, **c** $p = 5$ **2** **a** $\frac{1}{8}$, **b** $\frac{5}{7}$, **c** $-\frac{1}{5}$ **3** **a** $p = 11.7$, **b** $k = 13.5$, **c** $v = 17.0$, **d** $t = 34.3$, **e** $g = 25.7$, **f** $d = 79.8$ **4** **a** 56°, **b** 87°, **c** 102°, **d** 79°, **e** 50°, **f** 146° **5** **a** $a = 5.2$, **b** $q = 10.3$, **c** $m = 10.8$, **d** $z = 17.8$, **e** $r = 46.1$, **f** $j = 151.1$ **6** **a** 80°24′, **b** 47°26′, **c** 114°37′, **d** 62°14′, **e** 9°46′, **f** 85°49′, **g** 17°45′, **h** 98°36′, **i** 60°19′ **7** **a** $c^2 = a^2 + b^2$, **b** Pythagoras' Theorem is derived. **8** $x = 26$ **10** 24°, 31°, 125° **11** **a** $x = \sqrt{13}$, **b** $x = \sqrt{21}$, **c** $x = \sqrt{73}$ **12** **a** 45°, **b** 120°, **c** 30°

ANSWERS

Exercise 10.6

1 a 10.4 cm, **b** 11.97 m, **c** 41.75°, **d** 128°12′ **2 a** 76°32′, **b** 40°25 **3** 7° **4** 817 m **5** 63.7 km **6** QS = 29.4 cm, PR = 17.4 cm **7** 70°18′ **8 a** Amelia — 240 km, Elizabeth — 210 km, **b** 399 km **9 a** 93°40′, **b** 86°20′, **c** 11.6 cm **10 a** 13 cm, **b** 131°29′ **11 b** 120° **12 a** 82.8 km, **b** 29.1 km, **c** 12.5 km, **d** 12.6 km **13 a** $\angle ABC = 77°$, bearing = 035°, **b** $\angle PQR = 113°$, bearing = 239°, **c** $\angle XYZ = 96°$, bearing = 158°, **d** $\angle EFG = 82°$, bearing = 316°

Exercise 10.7

1 a 64.0 cm^2, **b** 185.5 cm^2, **c** 331.6 cm^2, **d** 353.1 cm^2, **e** 90.3 cm^2, **f** 244.1 cm^2 **2 a** 55.41 mm^2, **b** 220.3 cm^2, **c** 625 m^2 **3 a** 239.6 cm^2, **b** 95.7 cm^2, **c** 888.3 cm^2 **4 a** 120 cm^2, **b** 5 cm **5** 62 cm^2 **6** 253 cm^2 **7 a** $16\sqrt{3}$ cm^2, **b** 12.5 cm^2, **c** 21 cm^2 **8 a** 42°, **b** 24°, **c** 152° **9 a** 1973 cm^3, **b** 153.6 cm^3 **10** 5.35 ha **11 a** $A = \frac{1}{2}ab$, **b** The standard formula for the area of a triangle is derived. **12 a** 136.8 cm^2, **b** 96.5 cm^2, **c** 40.3 cm^2 **13** $24\sqrt{3}$ cm^2 **14** $h = 4.5$

Exercise 10.8

1 a $x = 10.3$, **b** $k = 8.9$, **c** $p = 21.8$, **d** $n = 100.7$, **e** $a = 17.1$, **f** $z = 113.9$ **2 a** 33°35′, **b** 71°47′, **c** 97°11′, **d** 33°2′, **e** 169°17′, **f** 48°50′ **3** 40° **4** 150° **5 a** 25.61 cm, **b** 38°32′ **6** 332 m **7** 105°24′ **8** 20 m **9 a** 83°31′, **b** 171 mm^2 **10 a** 153°, **b** 399 m^2 **11 a** 114°, **b** 8 cm **12 b** 52.6 cm **13 b** $4\sqrt{6}$ cm **14** 41 m^2 **15 a i** 62°11′, **ii** 19.9 m^2, **b i** 84°16′, **ii** 19.9 m^2 **16 a** 21 m, **b** 142.5 m^2 **18 a** $x = 11$, **b** $y = \sqrt{97}$ **19 a** 18 km, **b** 050° **20** 1008 NM **21 a** 77°24′, **b** 18.9 cm **22 a** 56.6 cm, **b** 28° **23 a** 125°, **b** 73° **24 a** 26.7 cm, **b** 19.4 cm **25 a** 22 cm, **b** 13 cm **26 a** 58°, **b** 49°, **c** 13.6 cm **27 a** 16.54 km, **b** 13 900 m

Chapter 10 Review

1 a acute, **b** obtuse **2 a** 0.94, **b** –0.64, **c** –0.27, **d** 0.81, **e** –0.78, **f** –0.89 **3 a** 45°, 135°, **b** 23°, 157°, **4 a** 114°, **b** 109° **5 a** $\frac{1}{\sqrt{2}}$, **b** $-\frac{1}{2}$, **c** $-\frac{1}{\sqrt{3}}$ **6 a** $a = 15.4$, **b** $p = 81.2$, **c** $y = 34.5$, **d** $k = 8.0$, **e** $w = 46.1$, **f** $n = 3.1$ **7 a** 46°46′, **b** 127°38′, **c** 75°36′, **d** 47°13′, **e** 99°43′, **f** 41°13′ **8** 562 m **9** 71°47′ **10 b** $\angle UTV = 35°$, $\angle TUV = 84°$, **c** 119 km **11 b** 26°, **c** 116° **12 b** 46°, **c** 136°, **d** 316° **13 a** 30.7 cm^2, **b** 117.8 cm^2 **14 a** 71°12′, **b** 143°53′ **15** 112.1 cm^2 **16 a** 984.8 cm^3, **b** 45.8 cm^3 **17** 222 mm^2 **18 a** 22 cm, **b** 48°, **c** 6.2 cm **19 a** 22.5 cm, **b** 76°28′

11 Similarity

Exercise 11.1

1 a AB = 10 mm, BC = 25 mm, CD = 30 mm, DA = 15 mm; PQ = 20 mm, QR = 50 mm, RS = 60 mm, SP = 30 mm **b i** 1 : 2, **ii** 1 : 2, **iii** 1 : 2, **iv** 1 : 2, **c** They are equal. **d** 2 **2 a** $\angle X = 45°$, $\angle Y = 108°$, $\angle Z = 27°$; $\angle L = 45°$, $\angle M = 108°$, $\angle N = 27°$, **b** Yes **3 a** AB and PQ, BC and QR, CD and RS, DA and SP, **b** KL and FG, LM and GH, MN and HE, NK and EF. **4 a** $\angle X$ and $\angle D$, $\angle Y$ and $\angle E$, $\angle Z$ and $\angle C$, **b** $\angle S$ and $\angle G$, $\angle T$ and $\angle H$, $\angle U$ and $\angle I$, $\angle V$ and $\angle J$, $\angle W$ and $\angle K$ **5 a** yes, **b** no, **c** yes, **d** no, **e** no, **f** no **6 a** $\frac{5}{3}$, **b** $\frac{12}{7}$, **c** $\frac{8}{5}$, **d** $\frac{9}{4}$ **7 a** 2, $x = 8$, **b** 3, $p = 12$, **c** 5, $a = 45$, **d** $\frac{3}{2}$, $t = 15$, **e** $\frac{1}{2}$, $c = 7$, **f** $\frac{1}{4}$, $e = 5$ **8 a** yes, **b** yes **9 a** $\frac{2}{3}$, **b** $\frac{10}{7}$ **10 a** $\frac{2}{5}$, **b** $\frac{4}{3}$ **11 a** yes, **b** no **12 a** 1 : 4, 1 : 4, yes, **b** 4 : 7, 4 : 5, no, **c** 2 : 5, 2 : 5, yes, **d** 2 : 3, 3 : 4, no **13 a** C and D, **b** A and C, **c** B and D **14 a** $\frac{PQ}{TQ} = \frac{QR}{QV} = \frac{RS}{VU} = \frac{SP}{UT} = \frac{4}{3}$, $\angle P = \angle T$, $\angle Q$ is common, $\angle R = \angle V$, $\angle S = \angle U$, $\therefore$ $PQRS \mathbin{|||} TQVU$ (matching sides are in proportion and matching angles are equal). **b** $\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GD} = \frac{DA}{DE} = 2$, $\angle A = \angle E$, $\angle B = \angle F$, $\angle C = \angle G$, $\angle D$ is common, $\therefore$ $ABCD \mathbin{|||} EFGD$ (matching sides are in proportion and matching angles are equal).

Exercise 11.2

1 a EF and YX, FG and XZ, EG and YZ, **b** LM and DE, MN and CD, LN and CE **2 a** $\angle K$ and $\angle U$, $\angle L$ and $\angle V$, $\angle M$ and $\angle T$, **b** $\angle C$ and $\angle J$, $\angle D$ and $\angle K$, $\angle E$ and $\angle I$ **3 a** $\alpha = 60°$, $\beta = 70°$, yes **b** $\alpha = 48°$, $\beta = 32°$, no **c** $\alpha = 30°$, $\beta = 120°$, yes **d** $\alpha = 70°$, $\beta = 63°$, yes **4 a** $\frac{8}{4} = \frac{10}{5} = \frac{12}{6} = 2$, yes, **b** $\frac{39}{13} = 3$, $\frac{24}{8} = 3$, $\frac{36}{11} = 3\frac{3}{11}$, no, **c** $\frac{2.5}{10} = \frac{3}{12} = \frac{4.5}{18} = \frac{1}{4}$, yes, **d** $\frac{5}{21} = 0.208$, $\frac{5.2}{25} = 0.208$, $\frac{5.6}{28} = 0.2$, no **5 a** yes, **b** no, **c** yes, **d** no **6 a** yes, **b** no, **c** yes, **d** no **7 a** $x = 12$, **b** $c = 27$, **c** $g = 35$, **d** $k = 36$, **e** $f = 33$, **f** $q = 35$ **8 a** $p = 38$, $q = 77$, **b** $x = 53$, $y = 37$, **c** $e = 44$, $f = 46$ **9 a** $m = 8$, **b** $x = 16$, **c** $g = 48$, **d** $u = 24$ **10 a** $x = 12$, **b** $k = 3$, **c** $t = 72$, **d** $h = 18$, **e** $n = 14$, **f** $d = 36$ **11 a** $a = 8$, $b = 12$, **b** $c = 8$, $d = 15$, **c** $x = 8$, $y = 22.5$, **d** $p = 12$, $q = 42$, **e** $u = 7.5$, $v = 12$, **f** $m = 16.5$, $n = 18$ **12 a** $a = 6$, **b** $p = 15$, **c** $x = 21$ **13 a iii** $x = 18$, $y = 30$, **b iii** $a = 15$, $b = 28$, **c iii** $e = 15$, $f = 5$

Exercise 11.3

1 1.5 m **2** 6 m **3** 21 m **4** 72 m **5** 15 m **6** 2100 m **7** 1.92 m **8** 45 cm **9** 10 m **10** 16 m **11** 3 cm

Exercise 11.4

1 a $\angle BAC = \angle CED$ (alternate $\angle$s, $AB \parallel DE$)
$\angle ACB = \angle DCE$ (vertically opposite $\angle$s)
$\therefore \Delta ABC \,|||\, \Delta EDC$ (equiangular); $x = 15$, $y = 12$

c $\frac{XY}{UV} = \frac{8}{20} = \frac{2}{5}$
$\frac{XZ}{VW} = \frac{10}{25} = \frac{2}{5}$
$\frac{YZ}{UW} = \frac{14}{35} = \frac{2}{5}$
$\therefore \Delta XYZ \,|||\, \Delta VUW$ (matching sides in proportion); $a = 44$

e $\frac{DE}{EG} = \frac{10}{15} = \frac{2}{3}$
$\frac{DE}{EF} = \frac{12}{18} = \frac{2}{3}$
$\frac{EF}{FG} = \frac{18}{27} = \frac{2}{3}$
$\therefore \Delta DEF \,|||\, \Delta EGF$ (matching sides in proportion); $g = 110$

g $\angle VWZ = \angle XYZ = 90°$ (given)
$\angle Z$ is a common angle
$\therefore \Delta VWZ \,|||\, \Delta XYZ$ (equiangular); $x = 40$

i $\frac{FG}{JI} = \frac{10}{16} = \frac{5}{8}$
$\frac{GH}{FJ} = \frac{25}{40} = \frac{5}{8}$
$\frac{FH}{FI} = \frac{30}{48} = \frac{5}{8}$
$\therefore \Delta FGH \,|||\, \Delta IJF$ (matching sides are in proportion); $u = 110$

b $\frac{QR}{RT} = \frac{6}{8} = \frac{3}{4}$
$\frac{PR}{RS} = \frac{9}{12} = \frac{3}{4}$
$\angle QRP = \angle TRS$ (vertically opposite $\angle$s)
$\therefore \Delta PQR \,|||\, \Delta STR$ (sides about equal $\angle$s are in proportion); $m = 16$

d $\angle JMN = \angle JKL$ (corresponding $\angle$s, $MN \parallel KL$)
$\angle J$ is a common angle
$\therefore \Delta JMN \,||\, \Delta JKL$ (equiangular); $k = 17.5$

f $\frac{TU}{RS} = \frac{12}{21} = \frac{4}{7}$
$\frac{TV}{ST} = \frac{20}{35} = \frac{4}{7}$
$\frac{UV}{RT} = \frac{24}{42} = \frac{4}{7}$
$\therefore \Delta TUV \,|||\, \Delta SRT$ (matching sides are in proportion); $r = 56$

h $\frac{PS}{PQ} = \frac{15}{18} = \frac{5}{6}$
$\frac{PT}{PR} = \frac{20}{24} = \frac{5}{6}$
$\angle P$ is a common angle
$\therefore \Delta PST \,|||\, \Delta PQR$ (sides about equal $\angle$s are in proportion); $e = 30$

2 a $\angle APQ = \angle ABC$ (corresponding $\angle$s, $PQ \parallel BC$)
$\angle A$ is a common angle
$\therefore \Delta APQ \,|||\, \Delta ABC$ (equiangular)

b $PQ = 15$, $AQ = 12.5$, $QC = 10$

3 a $\frac{QT}{TS} = \frac{7.2}{9} = \frac{4}{5}$

$\frac{RT}{TP} = \frac{4.8}{6} = \frac{4}{5}$

$\angle QTR = \angle PTS$ (vertically opposite $\angle$s)

$\therefore \Delta QTR \mathbin{|||} \Delta STP$ (sides about equal $\angle$s are in proportion)

b $\angle RQT = \angle PST$ (matching $\angle$s in similar Δs)

$\therefore QR \parallel PS$ (alternate $\angle$s are equal)

$\therefore PQRS$ is a trapezium (one pair of opposite sides are parallel)

4 a $\angle EDF = \angle EAG$ (corresponding $\angle$s, $DC \parallel AB$)

$\angle E$ is a common angle

$\therefore \Delta DEF \mathbin{|||} \Delta AEG$ (equiangular)

b $AG = 9$

5 a $\frac{VW}{XW} = \frac{3}{4}$

$\frac{WY}{UW} = \frac{6}{8} = \frac{3}{4}$

$\angle W$ is a common angle

$\therefore \Delta WUX \mathbin{|||} \Delta WYV$ (sides about equal $\angle$s are in proportion)

b $VY = 5\frac{1}{4}$

6 a $\angle MLN = \angle MYX$ (given)

$\angle M$ is a common angle

$\therefore \Delta LMN \mathbin{|||} \Delta YMX$ (equiangular)

b $MN = 28$, $YN = 4$

7 a $\frac{PQ}{QR} = \frac{6}{12} = \frac{1}{2}$

$\frac{QS}{PQ} = \frac{3}{6} = \frac{1}{2}$

$\angle Q$ is a common angle

$\therefore \Delta PQS \mathbin{|||} \Delta RQP$ (sides about equal $\angle$s are in proportion)

b $\angle PSQ = \angle RPQ$ (matching $\angle$s in similar Δs)

8 a $\angle ABC = \angle CDE = 90°$ (given)

$\angle ACB = \angle DCE$ (vertically opposite $\angle$s)

$\therefore \Delta ABC \mathbin{|||} \Delta EDC$ (equiangular)

b $BC = 10$, $CD = 8$

9 a $\angle KML = \angle LNM$ (given)

$\angle L$ is a common angle

$\therefore \Delta KLM \mathbin{|||} \Delta MLN$ (equiangular)

b $LN = 25$, $KN = 9$

10 $\angle CDE = \angle CAB$ (corresponding $\angle$s, $DE \parallel AB$

$\angle C$ is a common angle

$\therefore \Delta CDE \mathbin{|||} \Delta CAB$ (equiangular)

$\frac{CE}{CB} = \frac{CD}{CA}$ (matching sides are in proportion)

$\therefore \frac{CE}{CB} = \frac{1}{2}$

$\therefore CB = 2CE$

$\therefore E$ is the midpoint of CB

$\therefore CE = EB$

11 a $\dfrac{PA}{PQ} = \dfrac{1}{2}$ (A is the midpoint of PQ)

$\dfrac{PB}{PR} = \dfrac{1}{2}$ (B is the midpoint of PR)

$\angle P$ is a common angle

$\therefore$ ΔAPB ||| ΔQPR (sides about equal $\angle$s are in proportion)

b $\dfrac{AB}{QR} = \dfrac{PA}{PQ}$ (matching sides are in proportion)

$\therefore \dfrac{AB}{QR} = \dfrac{1}{2}$

$\therefore QR = 2AB$

$\angle PAB = \angle PQR$ (matching $\angle$s in similar Δs)

$\therefore AB \parallel$ QR(corresponding $\angle$s are equal)

$\therefore QR$ is parallel to AB and twice its length

12 a Let $\angle BAD = \alpha$

$\angle ABD = 90° - \alpha$ ($\angle$ sum of ΔABD)

$\angle CBD = \alpha$ (adjacent $\angle$s in a right angle)

$\therefore \angle BAD = \angle CBD$ (both equal to α)

b $\angle BAD = \angle CBD$ (proven above)

$\angle BDA = \angle BDC = 90°$ ($BD \perp AC$)

$\therefore \Delta ABD$ ||| ΔBCD (equiangular)

c $\dfrac{BD}{DC} = \dfrac{AD}{BD}$ (matching sides are in proportion)

$\therefore BD^2 = AD.\ DC$

d $BD = 12$ cm

13 a $\angle AXY = \angle ABC$ (corresponding $\angle$s, $XZ \parallel BC$)

$\angle A$ is a common angle

$\therefore \Delta AXY$ ||| ΔABC (equiangular)

b $\dfrac{AY}{AC} = \dfrac{AX}{AB}$ (matching sides are in proportion)

$\therefore \dfrac{AY}{AC} = \dfrac{1}{2}$ (X is midpoint of AB)

$\therefore Y$ is the midpoint of AC

c $\dfrac{XY}{BC} = \dfrac{AX}{AB}$ (matching sides are in proportion)

$\dfrac{XY}{10} = \dfrac{1}{2}$

$\therefore XY = 5$ cm

d $\angle CZY = \angle CDA$ (corresponding $\angle$s, $XY \parallel AD$)

$\angle C$ is a common angle

$\therefore \Delta CZY$ ||| ΔCDA (equiangular)

e $\dfrac{YZ}{AD} = \dfrac{CY}{CA}$ (matching sides are in proportion)

$\therefore \dfrac{YZ}{6} = \dfrac{1}{2}$ (Y is the midpoint of AC)

$\therefore YZ = 3$ cm

f $XZ = 5$ cm + 3 cm = 8 cm

14 a $\angle EHC = \angle ABC$ (corresponding $\angle$s, $AB \parallel EH$)

$\angle ACB = \angle ABC$ (base $\angle$s of isosceles Δ, $AB = AC$)

$\therefore \angle EHC = \angle ACB$ (both equal to $\angle ABC$)

$\therefore EH = EC$ (equal sides lie opposite equal $\angle$s)

b $\angle FBD = \angle EHD$ (corresponding $\angle$s, $AB \parallel EH$)

$\angle A$ is a common angle

$\therefore \Delta BFD$ ||| ΔHED (equiangular)

c $\dfrac{FB}{EH} = \dfrac{FD}{ED}$ (matching sides are in proportion)

$= \dfrac{2}{1}$ (E is midpoint of FD)

But, $EH = EC$ (proven above)

$\therefore \dfrac{FB}{EC} = \dfrac{2}{1}$, i.e. $FB : EC = 2 : 1$

ANSWERS

15 **i** Construct BD

$\frac{AW}{AB} = \frac{1}{2}$ (W is midpoint of AB)

$\frac{AZ}{AD} = \frac{1}{2}$ (Z is midpoint of AD)

$\angle A$ is a common angle

$\therefore \Delta ZAW \mathbin{|||} \Delta DAB$ (sides about equal $\angle$s are in proportion)

ii $\frac{WZ}{BD} = \frac{AW}{AB}$ (matching sides are in proportion)

$= \frac{1}{2}$

$\therefore BD = 2WZ$

$\angle AZW = \angle ADB$ (matching $\angle$s in similar Δs)

$\therefore ZW \parallel BD$ (corresponding $\angle$s are equal)

iii $\frac{CX}{CB} = \frac{1}{2}$ (X is midpoint of BC)

$\frac{CY}{CD} = \frac{1}{2}$ (Y is midpoint of CD)

$\angle C$ is a common angle

$\therefore \Delta CXY \mathbin{|||} \Delta CBD$ (sides about equal $\angle$s are in proportion)

iv $\frac{XY}{BD} = \frac{CX}{CB}$ (matching sides are in proportion)

$= \frac{1}{2}$

$\therefore BD = 2XY$

$\angle CXY = \angle CBD$ (matching $\angle$s in similar Δs)

$\therefore XY \parallel BD$ (corresponding $\angle$s are equal)

v $2WZ = BD$ and $2XY = BD$, $\therefore WZ = XY$

$WZ \parallel BD$ and $XY \parallel BD$, $\therefore WZ \parallel XY$

$\therefore WXYZ$ is a parallelogram (one pair of opposite sides are equal and parallel)

16 **i** $\angle PMY = \angle YKN$ (alternate $\angle$s, $LM \parallel KN$)

$\angle PYM = \angle KYN$ (vertically opposite $\angle$s)

$\therefore \Delta PMY \mathbin{|||} \Delta NKY$ (equiangular)

ii $\frac{PM}{LM} = \frac{1}{2}$ (P is midpoint of LM)

$LM = KN$ (opposite sides of a parallelogram)

$\therefore \frac{PM}{KN} = \frac{1}{2}$

$\frac{MY}{YK} = \frac{PM}{KN}$ (matching sides are in proportion)

$\therefore \frac{MY}{YK} = \frac{1}{2}$

iii $\angle XKQ = \angle LMK$ (alternate $\angle$s, $LM \parallel KN$)

$\angle KXQ = \angle LXM$ (vertically opposite $\angle$s)

$\therefore \Delta XKQ \mathbin{|||} \Delta XML$ (equiangular)

iv $\frac{KQ}{KN} = \frac{1}{2}$ (Q is midpoint of KN)

$LM = KN$ (opposite sides of a parallelogram)

$\therefore \frac{KQ}{LM} = \frac{1}{2}$

$\frac{KX}{XM} = \frac{KQ}{LM}$ (matching sides are in proportion)

$\therefore \frac{KX}{XM} = \frac{1}{2}$

v $\frac{MY}{YK} = \frac{KX}{XM} = \frac{1}{2}$, $\therefore KX = XY = YM$

$\therefore LQ$ and PN trisect KM at X and Y

Exercise 11.5

1 **a** 25 : 9, **b** 1 : 4, **c** 4 : 9, **d** 16 : 9, **e** 49 : 100, **f** 9 : 4 **2** **a** 36 : 25, **b** 3 : 4, **c** 81 : 25, **d** 10 : 13 **3** **a** 50 cm^2, **b** 196 cm^2, **c** 52 cm^2, **d** 120 cm^2, **e** 165.6 cm^2, **f** 100 cm^2 **4** **a** $p = 16$, **b** $d = 6$, **c** $z = 13.5$, **d** $n = 30$, **e** $t = 25.2$, **f** $k = 19.5$ **5** **a** 125π cm^2, **b** 10.5 cm **6** 1176 cm^2 **7** **a** 9 : 13, **b** no **8** $x = 9$, $y = 20$ **9** 9 cm

Exercise 11.6

1 **a** **i** 9 : 25, **ii** 27 : 125, **b** **i** 4 : 1, **ii** 8 : 1, **c** **i** 4 : 9, **ii** 8 : 27, **d** **i** 25 : 16, **ii** 125 : 64, **e** **i** 9 : 16, **ii** 27 : 64, **f** **i** 49 : 25, **ii** 343 : 125 **2** **a** 45 cm^2, **b** 147 cm^2, **c** 112.5 cm^2, **d** 160 cm^2 **3** no **4** **a** 1625 cm^3, **b** 99.9 cm^3, **c** 825 cm^3, **d** 1996.5 cm^3 **5** **a** 7 : 2, **b** 343 : 8 **6** 64 : 81 **7** **a** $h = 15$, **b** $w = 63$, **c** $y = 30$, **d** $c = 28$ **8** 360 cm^3 **9** **a** 9 times, **b** 27 times **10** 64 **11** 21 cm **12** **a** 21%, **b** 33.1%

Chapter 11 Review

1 **a** scale factor = 3, $p = 15$, $q = 6$, **b** scale factor = $\frac{1}{2}$, $a = 13$, $b = 22$ **2** $\frac{5}{6}$ **3** **a** $t = 21$, $u = 5$, **b** $p = 9$, $q = 24$, **c** $m = 12$, $n = 15$ **4** **a** yes, equiangular Δs, **b** yes, matching sides are in proportion, **c** no, **d** yes, sides about equal $\angle$s are in proportion, **e** yes, hypotenuses and another pair of matching sides in 2 right-angled Δs are in proportion, **f** no

5 **a** $\angle EGF = \angle HGI$ (vertically opposite $\angle$s)
$\angle EFG = \angle GHI$ (alternate $\angle$s, $EF \parallel HI$)
$\therefore \Delta EFG \parallel\!| \Delta IHG$ (equiangular)
$p = 31.5$, $q = 18$

b $\frac{ST}{TQ} = \frac{9}{6} = \frac{3}{2}$
$\angle PTS = \angle RTQ$ (vertically opposite $\angle$s)
$\frac{PT}{TR} = \frac{15}{10} = \frac{3}{2}$
$\therefore \Delta PTS \parallel\!| \Delta RTQ$ (sides about equal $\angle$s are in proportion)
$c = 18$, $d = 56$

c $\frac{AD}{BD} = \frac{12}{18} = \frac{2}{3}$
$\frac{AB}{CD} = \frac{14}{21} = \frac{2}{3}$
$\frac{BD}{BC} = \frac{18}{27} = \frac{2}{3}$
$\therefore \Delta ABD \parallel\!| \Delta DCB$ (matching sides are in proportion)
$m = 42$, $n = 51$

d $\frac{PY}{XY} = \frac{4}{8} = \frac{1}{2}$
$\frac{PQ}{XZ} = \frac{5}{10} = \frac{1}{2}$
$\angle PYQ = \angle XYZ = 90°$ (common angle)
$\therefore \Delta PYQ \parallel\!| \Delta XYZ$ (hypotenuses and another pair of matching sides in 2 right-angled Δs are in proportion)
$u = 53$, $v = 37$

e $\angle ADE = \angle ABC$ (corresponding $\angle$s, $DE \parallel BC$)
$\angle A$ is a common angle
$\therefore \Delta ADE \parallel\!| \Delta ABC$ (equiangular)
$x = 14$, $y = 5$

f $\angle XYZ = \angle XVW$ (corresponding $\angle$s, $YZ \parallel VW$)
$\angle X$ is a common angle
$\therefore \Delta XYZ \parallel\!| \Delta XVW$ (equiangular)
$u = 24$, $v = 30$

g $\frac{HG}{EF} = \frac{10}{15} = \frac{2}{3}$
$\frac{FG}{ED} = \frac{14}{21} = \frac{2}{3}$
$\frac{FH}{DF} = \frac{16}{24} = \frac{2}{3}$
$\therefore \Delta FGH \parallel\!| \Delta DEF$ (matching sides are in proportion)
$c = 82$, $d = 38$

h $\angle PST = \angle PRQ$ (given)
$\angle P$ is a common angle
$\therefore \Delta PST \parallel\!| \Delta PRQ$ (equiangular)
$m = 23$, $n = 10$

6 12 cm

7 **a** $\angle PST = \angle PQR$ (corresponding $\angle$s, $ST \parallel QR$)
$\angle P$ is a common angle
$\therefore \Delta PST \parallel\!| \Delta PQR$ (equiangular)

b $\frac{PT}{PR} = \frac{PS}{PQ}$ (matching sides are in proportion)
$\therefore \frac{PT}{PR} = \frac{1}{2}$
$\therefore PR = 2PT$
$\therefore PT = TR$
$\therefore T$ is the midpoint of PR

8 **a** Let $\angle KLN = \alpha$
$\angle LKN = 90° - \alpha$ ($\angle$ sum of ΔKLN)
$\angle MLN = 90° - \alpha$ (adjacent $\angle$s in a right angle)
$\angle LMN = \alpha$ ($\angle$ sum of ΔLMN)
Now, $\angle KLN = \angle LMN$ (both equal to α)
and $\angle LKN = \angle MLN$ (both equal to $90° - \alpha$)
$\therefore \Delta KLN \parallel\!| \Delta LMN$ (equiangular)

b $\frac{LN}{NM} = \frac{KN}{LN}$ (matching sides are in proportion)
$\therefore LN^2 = KN.\ NM$

c 15 cm

9 **a** 16 : 49, **b** 245 cm^2 **10** **a** 5 : 3, **b** 27 cm **11** **a** 288 cm^2, **b** 500 cm^3 **12** 64 : 27

12 Functions and logarithms

Exercise 12.1

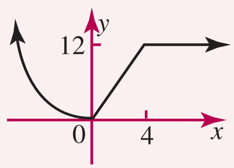
1 **a** yes, **b** yes, **c** no, **d** yes, **e** no, **f** yes **2** B **3** **a** yes, **b** yes, **c** no, **d** yes **4** **a** no, **b** yes, **c** yes, **d** no, **e** yes, **f** yes, **g** yes, **h** no, **i** no, **j** yes, **k** no, **l** yes **5** **a** the line is not vertical, **b** the line is vertical
6 **a** yes, **b** no, **c** yes, **d** yes, **e** yes, **f** no, **g** yes, **h** yes **7** **a** all real x; all real y, **b** all real x; $y \geqslant -9$,

ANSWERS

c $x \geqslant -9$; all real y, **d** $x \geqslant 0$; $y \geqslant 5$, **e** all real x; $y > 1$, **f** all real x; $-1 < y < 1$, **g** $-3 \leqslant x \leqslant 3$; $0 \leqslant y \leqslant 3$, **h** all real x; $y < 8$, **i** $x > -4$; all real y, **j** all real x, $x \neq 2$; all real y, $y \neq 0$, **k** all real x, $x \neq -1$; all real y, $y \neq 1$, **l** $-6 \leqslant x \leqslant 6$; $-4 \leqslant y \leqslant 4$, **m** $x \leqslant -6$ or $x \geqslant 6$; $y \geqslant 0$, **n** $-8 \leqslant x \leqslant 12$; $-9 \leqslant y \leqslant 11$, **o** all real x; $-2 \leqslant y < 3$
8 a $-3 \leqslant x \leqslant 3$; $-3 \leqslant y \leqslant 3$, **b** $4 \leqslant x \leqslant 8$; $3 \leqslant y \leqslant 7$

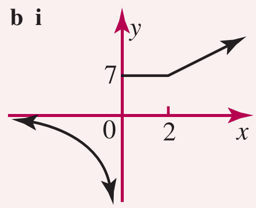
Exercise 12.2

1 a i 11, **ii** -7, **iii** 5, **b i** 17, **ii** 8, **iii** -2, **c i** 21, **ii** 0, **iii** -2, **d i** 0, **ii** 128, **iii** -16, **e i** -2, **ii** 46, **iii** $-1\frac{7}{16}$, **f i** 36, **ii** 0, **iii** $12 + 10\sqrt{3}$, **g i** 1, **ii** 16, **iii** $\frac{1}{8}$, **h i** 25, **ii** 36, **iii** $3 + 2\sqrt{2}$, **i i** 5, **ii** 4, **iii** $3\sqrt{2}$, **j i** $2\frac{1}{2}$, **ii** $2\frac{1}{2}$, **iii** $-5\frac{1}{5}$, **k i** 0, **ii** 14, **iii** $-2\frac{1}{4}$, **l i** -3, **ii** -1, **iii** $-9\frac{2}{3}$ **2** -2 **3 a** $x = 11$, **b** $x = \pm 3$, **c** $x = 3\frac{1}{2}$, **d** $x = 10, -4$, **e** $x = 7$, **f** $x = -7, 3$, **g** $x = 9$, **h** $x = -4$ **4** $f(1) = f(-1) = 2\frac{1}{2}$ **5 a** $18a - 2$, **b** $6a + 4$, **c** $3a - 2$ **6** $g(a) = g(b) = 0$
7 a $x + 3$, **b** $3x + 6$, **c** $x + 9$ **8** Powers of 2 are all positive (also, the graph of $y = 2^x$ never crosses the x-axis).
9 $x > -6$
12 a i



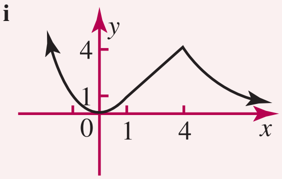
ii 27

b i



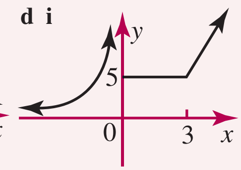
ii 14

c i



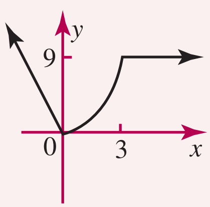
ii 5

d i



ii -2

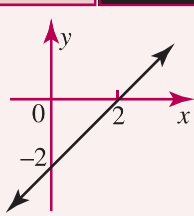
13 a



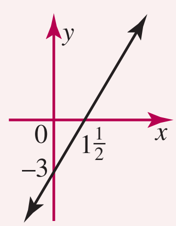
b $x = -1, 2$, **c** $x = -2\frac{1}{4}, x \geqslant 3$

Exercise 12.3

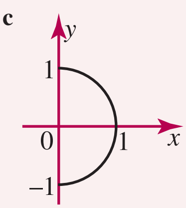
1 a



b



c



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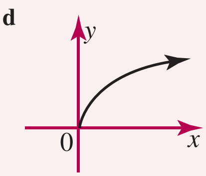

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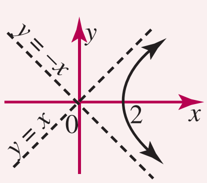

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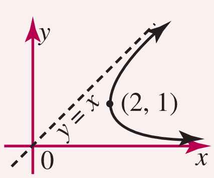
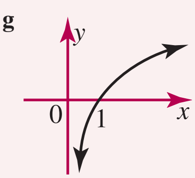


g



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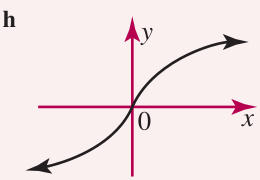
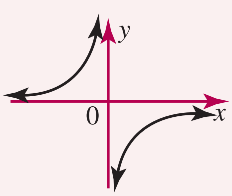
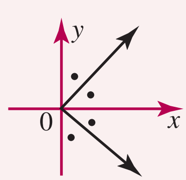


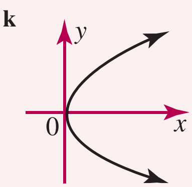
i



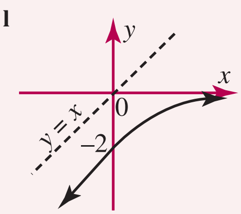
j



k

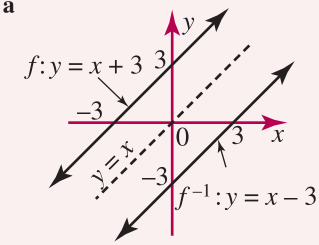


l

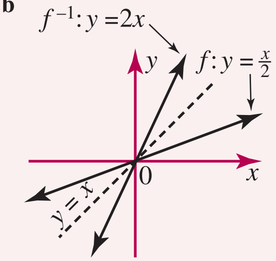


2 a, b, d, g, h, i, l

3 a



b



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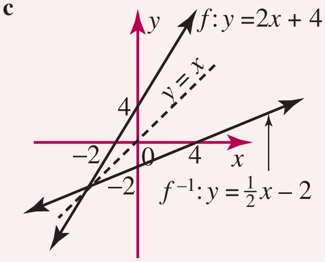

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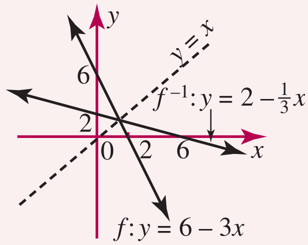

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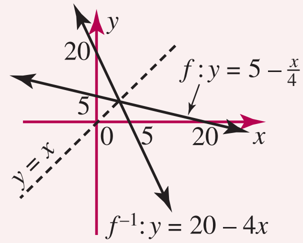
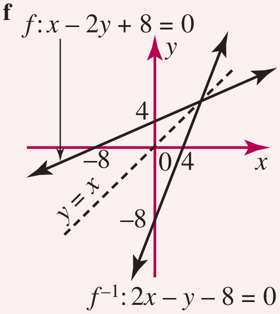
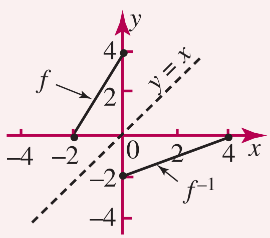


f

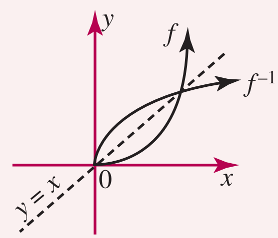


4 a $y = \frac{1}{x+2}$, **b** $y = \frac{1}{x} + 1$, **c** $y = \frac{3x+3}{1-x}$, **d** $y = \frac{5x}{x+2}$ **5 a** $y = \frac{7-2x}{x+2}$, $f(x) = f^{-1}(x)$, **b** The graph is symmetric about the line $y = x$. **6** Other answers are possible. **a** $x \geqslant 0$, **b** $x \geqslant 0$, **c** $x \geqslant 3$, **d** $x \geqslant -4$, **e** $x \geqslant 1$, **f** $x \geqslant 1$ **7 a** yes, **b** no, **c** $-1 \leqslant x \leqslant 1$ **8** Because the x and y values are interchanged when finding the inverse.

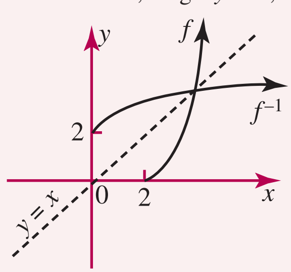
9 a i domain: $-2 \leqslant x \leqslant 0$, range: $0 \leqslant y \leqslant 4$,
ii domain: $0 \leqslant x \leqslant 4$, range: $-2 \leqslant y \leqslant 0$,
iii



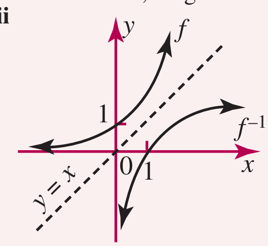
b i domain: $x \geqslant 0$, range: $y \geqslant 0$,
ii domain: $x \geqslant 0$, range: $y \geqslant 0$
iii



c i domain: $x \geqslant 2$, range: $y \geqslant 0$,
ii domain: $x \geqslant 0$, range: $y \geqslant 2$,
iii



d i domain: all real x, range: $y > 0$,
ii domain: $x > 0$, range: all real y
iii



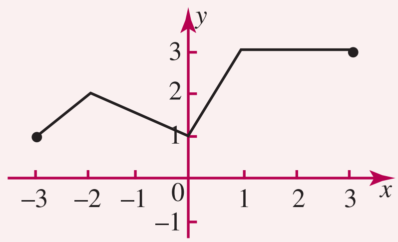
Exercise 12.4

1 a $y = x^2 + 2$, **b** $y = x^2 - 3$, **c** $y = (x-1)^2$, **d** $y = (x+4)^2$ **2 a** $y = (x+2)^3$, **b** $y = x^3 + 1$, **c** $y = x^3 - 6$, **d** $y = (x-4)^3$

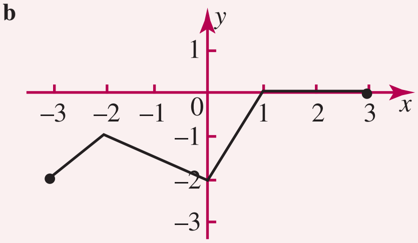
3 a $y = 2^x - 1$, **b** $y = 2^{x-3}$, **c** $y = 2^x + 5$, **d** $y = 2^{x+2}$ **4 a** $y = \frac{1}{x-5}$, **b** $y = \frac{1}{x} - 2$, **c** $y = \frac{1}{x+3}$, **d** $y = \frac{1}{x} + 7$

5 a $y=x^2+3$, **b** $y=3^x+2$, **c** $y=\frac{1}{x}+1$, **d** $y=2^{x-3}+1$, **e** $y=(x+1)^2-7$, **f** $y=\frac{1}{x-5}-2$ **6 a i** $y=2x^2-5$, **ii** $y=2(x-3)^2$, **b i** $y=\frac{1}{2}x^3+2$, **ii** $y=\frac{1}{2}(x+4)^3$, **c i** $y=\frac{3}{x}+7$, **ii** $y=\frac{3}{x-1}$ **7 a i** $y=-x^2+4$, **ii** $y=-(x-3)^2$, **b i** $y=5^{-x}-1$, **ii** $y=5^{-(x+2)}$, **c i** $y=-\frac{1}{x-5}$, **ii** $y=-\frac{1}{x}-7$

8 a



b



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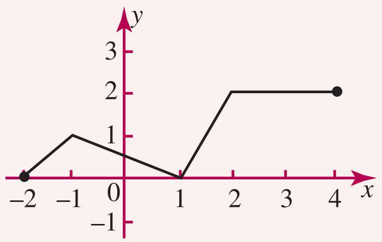

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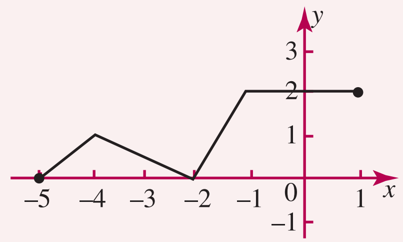

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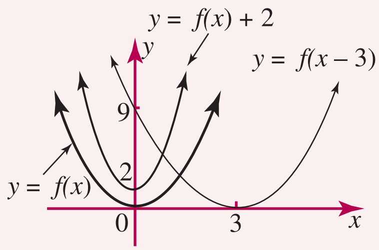

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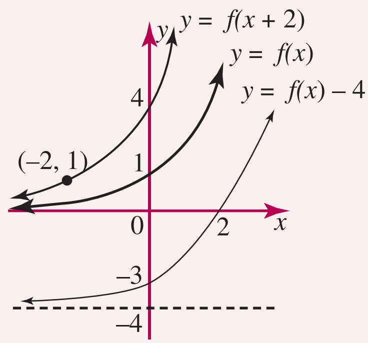

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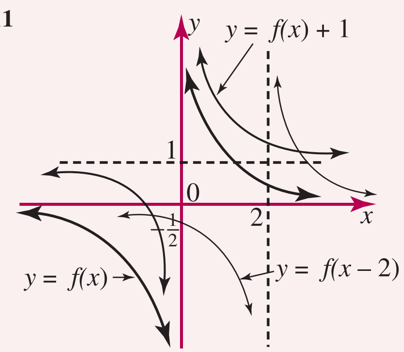

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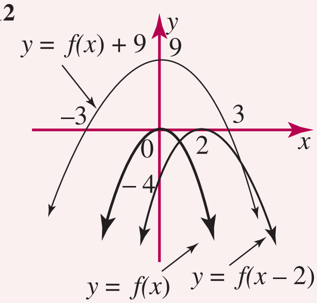

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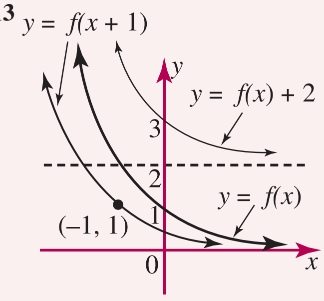
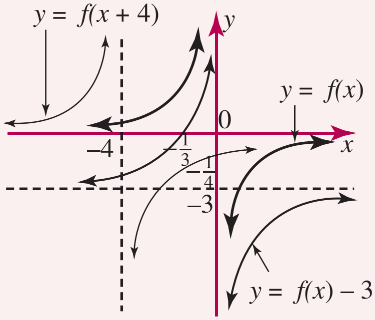
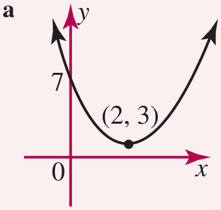
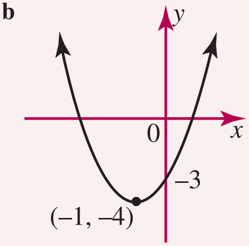
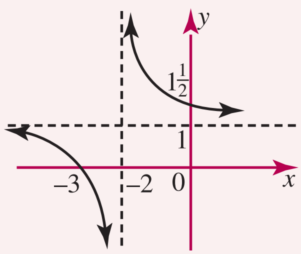
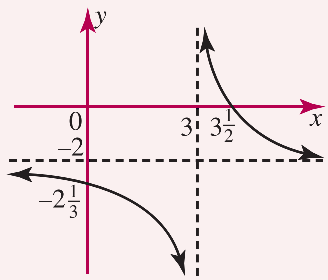


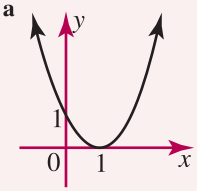
14 

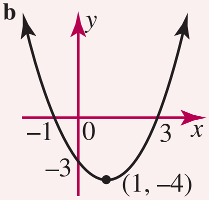
15 a 

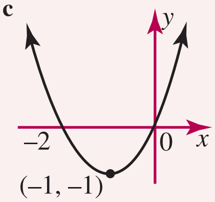
b 

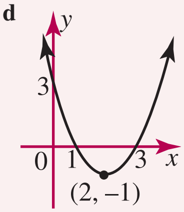
16 a 

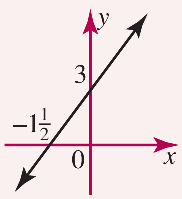
b 

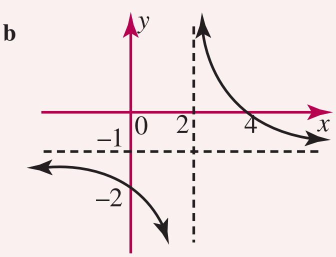
17 a 

b 

c 

d 

18 a $y = 2x + 2$, **b** 

19 a $y = \dfrac{2}{x-1} - 1$, **b** 

Exercise 12.5

1 **a** $x = 2$, **b** $x = 4$, **c** $x = 1$, **d** $x = 3$, **e** $x = 0$, **f** $x = 4$, **g** $x = 6$, **h** $x = 4$, **i** $x = 8$, **j** $x = 5$, **k** $x = 4$, **l** $x = 5$
2 **a** $x = \frac{1}{2}$, **b** $x = \frac{1}{2}$, **c** $x = \frac{1}{3}$, **d** $x = \frac{1}{4}$, **e** $x = \frac{1}{3}$, **f** $x = \frac{1}{2}$, **g** $x = \frac{1}{5}$, **h** $x = \frac{1}{4}$ **3** **a** $x = -1$, **b** $x = -2$, **c** $x = -3$, **d** $x = -4$, **e** $x = -2$, **f** $x = -5$, **g** $x = -4$, **h** $x = -6$ **4** **a** $x = \frac{3}{2}$, **b** $x = \frac{2}{3}$, **c** $x = \frac{4}{3}$, **d** $x = \frac{2}{3}$, **e** $x = \frac{5}{4}$, **f** $x = \frac{2}{3}$, **g** $x = \frac{5}{6}$, **h** $x = \frac{7}{5}$, **i** $x = -\frac{2}{3}$, **j** $x = -\frac{3}{2}$, **k** $x = -\frac{3}{5}$, **l** $x = -\frac{4}{3}$ **5** **a** $x = 3$, **b** $x = 5$, **c** $x = -1$, **d** $x = \frac{3}{5}$, **e** $x = 4$, **f** $x = 3$, **g** $x = \frac{2}{7}$, **h** $x = -\frac{4}{7}$, **i** $x = 4$ **6** **a** $x = 6$, **b** $x = \frac{5}{2}$, **c** $x = -\frac{3}{4}$, **d** $x = \frac{1}{4}$, **e** $x = -\frac{3}{2}$, **f** $x = \frac{5}{2}$, **g** $x = -\frac{13}{4}$, **h** $x = \frac{11}{4}$, **i** $x = \frac{5}{2}$ **7** $x = 2$

Exercise 12.6

1 **a** $\log_3 9 = 2$, **b** $\log_2 8 = 3$, **c** $\log_6 36 = 2$, **d** $\log_2 32 = 5$, **e** $\log_8 8 = 1$, **f** $\log_5 1 = 0$, **g** $\log_4 64 = 3$, **h** $\log_3 243 = 5$, **i** $\log_5(\frac{1}{5}) = -1$, **j** $\log_2(\frac{1}{4}) = -2$, **k** $\log_3 \sqrt{3} = \frac{1}{2}$, **l** $\log_7(\sqrt[3]{7}) = \frac{1}{3}$, **m** $\log_9 27 = \frac{3}{2}$, **n** $\log_2(\sqrt[5]{8}) = \frac{3}{5}$, **o** $\log_{144}(\frac{1}{12}) = -\frac{1}{2}$, **p** $\log_{16}(\frac{1}{8}) = -\frac{3}{4}$ **2** **a** $5^2 = 25$, **b** $2^4 = 16$, **c** $7^1 = 7$, **d** $2^3 = 8$, **e** $3^4 = 81$, **f** $10^2 = 100$, **g** $2^5 = 32$, **h** $6^0 = 1$,

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i $3^5 = 243$, **j** $2^{-1} = \frac{1}{2}$, **k** $3^{-2} = \frac{1}{9}$, **l** $4^{\frac{1}{2}} = 2$, **m** $27^{\frac{1}{3}} = 3$, **n** $5^{\frac{1}{2}} = \sqrt{5}$, **o** $8^{\frac{2}{3}} = 4$, **p** $1000^{-\frac{2}{3}} = \frac{1}{100}$ **3 a** 1, **b** 2, **c** 3, **d** 5, **e** 0, **f** 3, **g** 2, **h** 4, **i** $\frac{1}{2}$, **j** $\frac{1}{3}$, **k** −1, **l** −2, **m** $-\frac{1}{2}$, **n** −2, **o** $\frac{2}{3}$, **p** $\frac{1}{4}$ **4 a** $x = 16$, **b** $x = 36$, **c** $x = 125$, **d** $x = 32$, **e** $x = 1$, **f** $x = 4$, **g** $x = 2401$, **h** $x = 1024$, **i** $x = \frac{1}{6}$, **j** $x = \frac{1}{8}$, **k** $x = \frac{1}{25}$, **l** $x = \sqrt{10}$, **m** $x = 4$, **n** $x = 8$, **o** $x = \frac{1}{7}$, **p** $x = \frac{1}{8}$ **5 a** $x = 2$, **b** $x = 3$, **c** $x = 13$, **d** $x = 2$, **e** $x = 11$, **f** $x = 4$, **g** $x = 3$, **h** $x = 10$, **i** $x = 5$, **j** $x = 6$, **k** $x = 2$, **l** $x = 6$, **m** $x = 8$, **n** $x = 8$, **o** $x = 9$, **p** $x = 32$ **6** $x = 3125$

Exercise 12.7

1 a 1, **b** 1, **c** 2, **d** 3, **e** 4, **f** 0, **g** 1, **h** 3, **i** $\frac{1}{2}$, **j** −3, **k** 2, **l** 2 **2 a** 2, **b** 2, **c** 3, **d** 3, **e** 1, **f** 3 **3 a** 2.096, **b** 0.834, **c** 1.262, **d** −0.631, **e** 0.3155, **f** 4.395, **g** 3.561, **h** 3.358, **i** 4.192, **j** −1.465, **k** −1.262, **l** 1.3635 **4 a** 1.079, **b** 1.255, **c** 0.301, **d** −0.176, **e** −0.125, **f** 1.556, **g** 1.431, **h** −0.602, **i** 0.389, **j** −0.2385, **k** 0.903, **l** 1.732, **m** 0.653, **n** 1.505, **o** −1.38, **p** 0.8405 **5 a** $\log(10ab)$, **b** $\log\left(\frac{4p}{3q}\right)$, **c** $\log(x^3y^2)$, **d** $\log\left(\frac{m^5}{n^4}\right)$, **e** $\log(a\sqrt{b})$, **f** $\log(p^2q)$, **g** $\log(pqr)$, **h** $\log\left(\frac{ab}{c}\right)$, **i** $\log\left(\frac{e}{fg}\right)$, **j** $\log\left(\frac{x^2z}{y^3}\right)$, **k** $\log(p^2\sqrt{mn})$, **l** $\log\left(\frac{1}{y^3}\right)$

6 a $\log a + \log b + \log c$, **b** $\log p + \log q - \log r$, **c** $\log x - \log y - \log z$, **d** $\log g + \log h$, **e** $\frac{1}{3}\log p + \frac{1}{3}\log q$, **f** $-\log m - \log n$, **g** $2\log a - \frac{1}{2}\log b$, **h** $\frac{1}{2}\log c - \frac{1}{2}\log d$, **i** $\frac{2}{3}\log x - 4\log y$ **7 a** $7\log x$, **b** $5\log x$, **c** $7\log x$, **d** $7\log x$, **e** $8\log x$, **f** $4\log x$ **8 a** 2, **b** 3, **c** $\frac{2}{3}$, **d** $\frac{1}{2}$, **e** 3, **f** $\frac{3}{5}$, **g** 8, **h** 5, **i** −2 **9 a** 2.16, **b** 0.66, **c** −0.16, **d** 3.32, **e** 3.66, **f** 6.48, **g** 1.58, **h** 1.08, **i** 0.34, **j** 1.5, **k** 0.5, **l** −1.5 **10 a** $x = 7$, **b** $x = 15$, **c** $x = 7$, **d** $x = 10$, **e** $x = 24$, **f** $x = 22$ **11 a** $x = 5$, **b** $x = 4$, **c** $x = 3, 4$, **d** $x = 4, 9$, **e** $x = 4$, **f** $x = 2$ **12 a** $x = 9$, **b** $x = 5$, **c** $x = 1$

Exercise 12.8

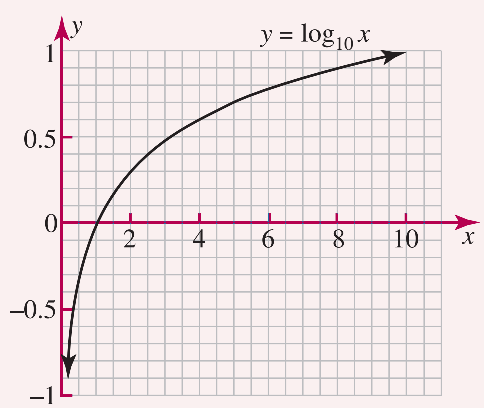
1 a $x = 2.81$, **b** $x = 1.54$, **c** $x = 4.32$ **2 a** $x = 2.322$, **b** $x = 1.262$, **c** $x = 1.365$, **d** $x = 0.565$, **e** $x = 0.301$, **f** $x = 2.073$, **g** $x = 3.287$, **h** $x = 5.644$, **i** $x = 3.145$, **j** $x = 3.336$, **k** $x = 4.247$, **l** $x = 9.054$ **3 a** $x = 1.335$, **b** $x = 4.585$, **c** $x = -1.663$, **d** $x = 1.269$, **e** $x = 1.465$, **f** $x = 0.171$, **g** $x = 2.953$, **h** $x = 0.193$, **i** $x = -1.202$ **4 a** $x = 1.756$, **b** $x = 2.593$, **c** $x = 12.457$, **d** $x = 7.129$, **e** $x = 2.821$, **f** $x = 2.379$ **5** $x < -1.585$

Exercise 12.9

1 a

x	0.2	0.4	0.6	0.8	1	2	3	4	5	6	7	8	9	10
y	−0.70	−0.40	−0.22	−0.10	0	0.30	0.48	0.60	0.70	0.78	0.85	0.90	0.95	1

b



c $x = 1$,
d i $x > 1$, **ii** $0 < x < 1$,
e no,
f i the y-values increase,
ii the y-values decrease,
g i 0.4, **ii** 0.6, **iii** 0.9,
h i 100, **ii** 1000, **iii** 10 000

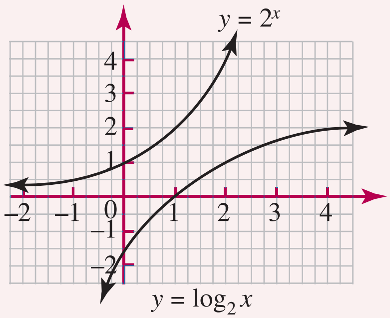
2 a

x	−3	−2	−1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

b definition of a logarithm, **c**

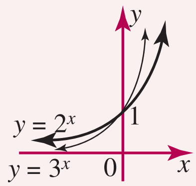
x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	−3	−2	−1	0	1	2	3

d

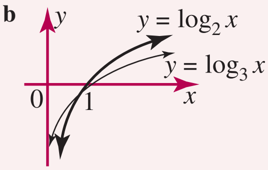


e The graphs are reflections of each other in the line $y = x$. (That is the functions are inverses of each other.)

3 **a**



b

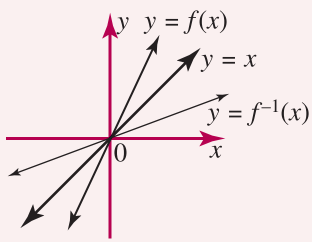


c **i** $\log_2 x$, **ii** $\log_3 x$ **4** **a** $P(1, 0)$, $Q(2, 1)$, $R(4, 2)$, $S(8, 3)$
b $m_{PQ} = 1$, $m_{QR} = \frac{1}{2}$, $m_{RS} = \frac{1}{4}$, **c** the gradient decreases
5 **a** $A(9, 2)$, $B(0, 2)$, $C(1, 0)$, **b** 10.7 units^2
6 **a** $A(0, 1)$, $B(2, 1)$, $AB = 2$ units, **b** $C(16, 4)$, $D(2, 4)$, $CD = 14$ units
c 24 units^2

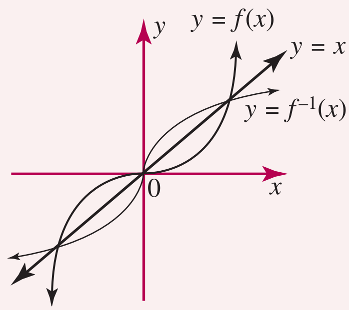
Chapter 12 Review

1 **a** yes, **b** no **2** **a** no, **b** yes, **c** yes, **d** no, **e** yes, **f** no **3** **a** $-1 \leqslant x \leqslant 1$; $-1 \leqslant y \leqslant 1$, **b** all real x, $x \neq 2$; all real y, $y \neq 0$, **c** all real x; $y \geqslant -4$, **d** all real x; $y = 1, 3$, **e** all real x; $y \leqslant -2$ or $y > 2$, **f** $x \geqslant 0$; all real y **4** **a** **i** 4, **ii** 40, **iii** $2 - 3\sqrt{2}$, **iv** $-2\frac{1}{4}$, **b** **i** 32, **ii** 1, **iii** 8, **iv** $\sqrt{2}$ **5** **a** $x = -3, 4$, **b** $x < -2$, **c** $x = 64$ **6** $x + 5$ **7** **a** no, **b** no, **c** yes, **d** yes **8** **a** $y = \frac{1}{3}x$, **b** $y = x + 7$, **c** $y = 5x + 10$, **d** $y = 1 + \frac{3}{x}$

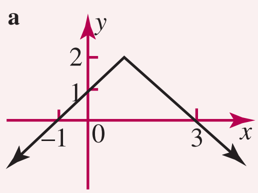
9 **a**



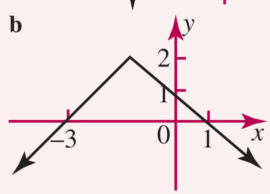
b



10 **a**



b



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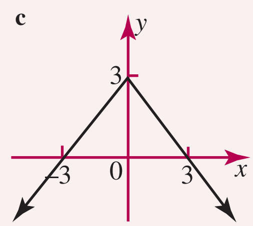

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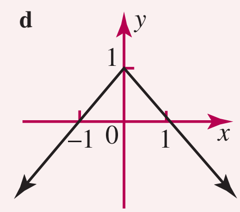


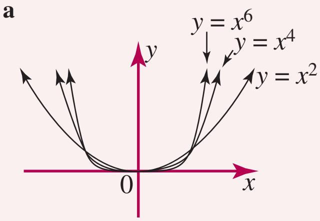
11 **a** $x = 5$, **b** $x = \frac{1}{2}$, **c** $x = -1$, **d** $x = -3$, **e** $x = \frac{3}{2}$, **f** $x = \frac{7}{2}$, **g** $x = \frac{3}{2}$, **h** $x = -1$ **12** **a** $\log_3 81 = 4$, **b** $10^2 = 100$,
13 **a** $x = 3$, **b** $x = \frac{3}{4}$, **c** $x = -2$, **d** $x = \frac{1}{4}$ **14** **a** $x = 7$, **b** $x = 3$, **c** $x = 6$, **d** $x = 8$ **15** **a** 2, **b** 0, **c** –2, **d** $\frac{5}{2}$
16 **a** 2, **b** 1, **c** 3, **d** 1, **e** 3 **17** **a** 1.079, **b** –0.125, **c** 1.204, **d** 1.556, **e** –0.477, **f** 0.301, **g** 1.602, **h** 2.477
18 **a** 11, **b** 3, **c** 0 **19** **a** $x = 75$, **b** $x = 8$, **c** $x = 18$, **d** $x = 4$ **20** **a** $k = 3$, **b** $A(0, 3)$, $B(8, 0)$, $C(1, 0)$, **c** $\sqrt{10}$ units
21 **a** $x = 1.92$, **b** $x = 4.46$, **c** $x = 1.12$

ANSWERS

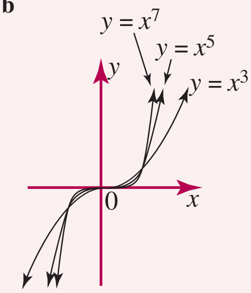
13 Curve sketching and polynomials

Exercise 13.1

1 a



b



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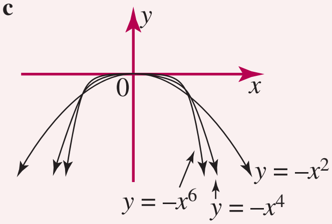

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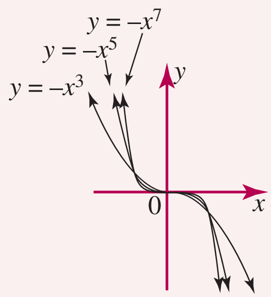
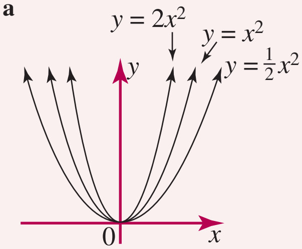
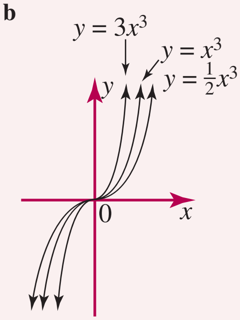


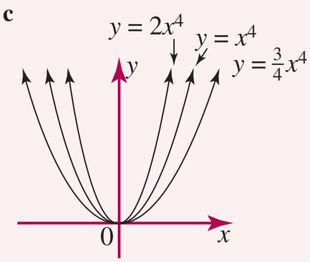
2 a



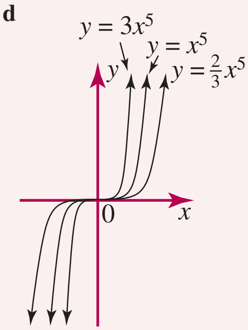
b



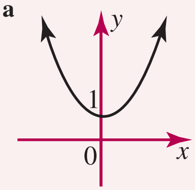
c



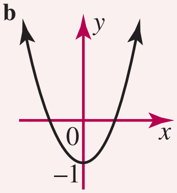
d



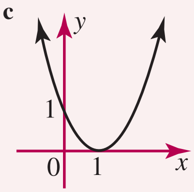
3 a



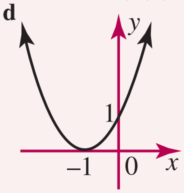
b



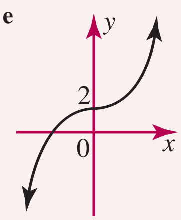
c



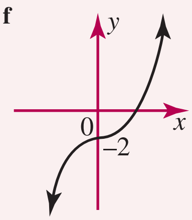
d



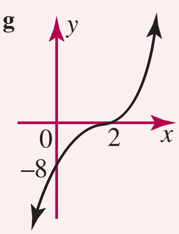
e



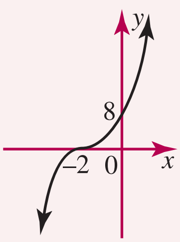
f



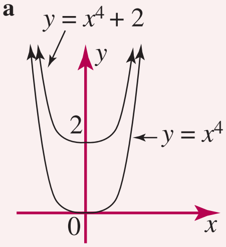
g



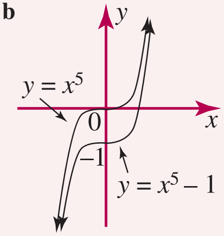
h



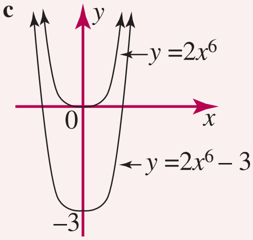
4 a



b



c



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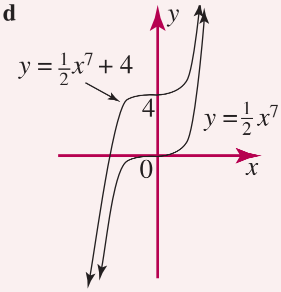

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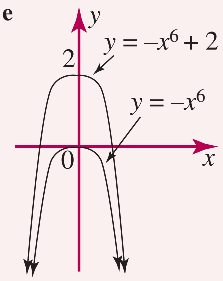

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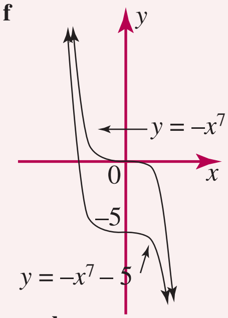
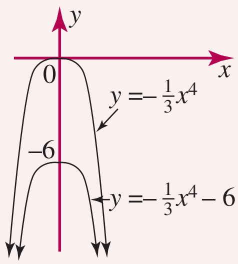
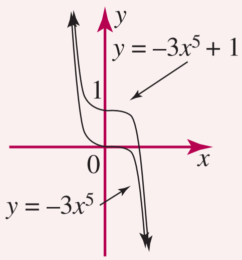


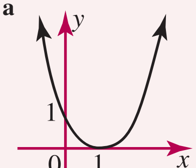
g



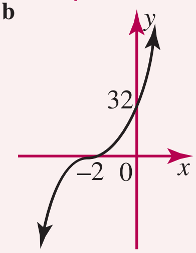
h



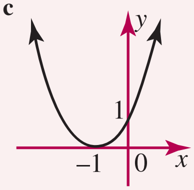
5 a



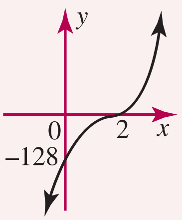
b



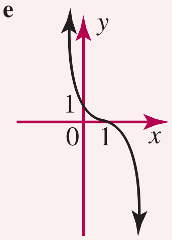
c



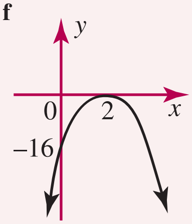
d



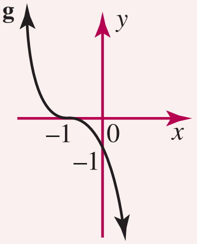
e



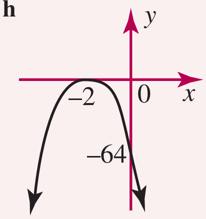
f



g

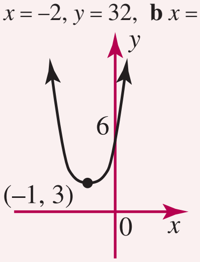


h

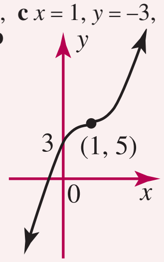


6 **a** $x = -2, y = 32$, **b** $x = 2, y = -96$, **c** $x = 1, y = -3$, **d** $x = -1, y = -2$

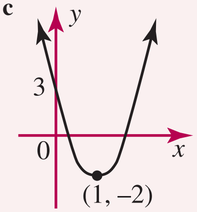
7 a



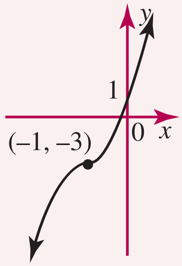
b



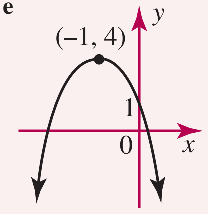
c



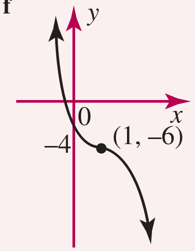
d



e



f



Exercise 13.2

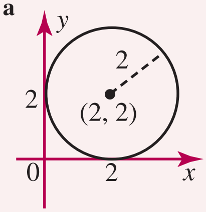
1 **a** $x^2 + y^2 = 9$, **b** $x^2 + y^2 = 49$, **c** $x^2 + y^2 = 2$, **d** $x^2 + y^2 = 20$ **2** **a** 5, **b** 8, **c** 12, **d** $\sqrt{3}$, **e** $3\sqrt{2}$, **f** $4\sqrt{2}$

3 $x^2 + y^2 = \dfrac{49}{4}$, centre is (0, 0), radius is $3\frac{1}{2}$ units. **4** **a** (1, 5), $r = 3$, **b** (–6, –2), $r = 2$, **c** (–2, 3), $r = 5$,

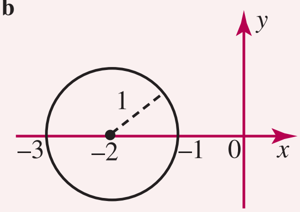
d (4, –5), $r = 1$, **e** (0, 7), $r = 6$, **f** (–3, 0), $r = 10$, **g** (–4, –6), $r = \sqrt{5}$, **h** (2, –1), $r = 2\sqrt{2}$, **i** (0, 4), $r = 2\sqrt{10}$,

j (10, 0), $r = 3\sqrt{5}$ **5** **a** $(x-1)^2+(y-3)^2=25$, **b** $(x+2)^2+(y-7)^2=36$, **c** $(x+4)^2+(y+1)^2=9$, **d** $(x-6)^2+(y+3)^2=81$, **e** $(x-2)^2+(y-1)^2=7$, **f** $(x+3)^2+(y+2)^2=15$, **g** $(x-7)^2+(y+5)^2=18$, **h** $(x+8)^2+(y-11)^2=28$ **6** **a** $x^2+y^2=49$, **b** $x^2+y^2=2$, **c** $(x-4)^2+(y-6)^2=9$, **d** $(x+9)^2+(y-12)^2=4$, **e** $(x-3)^2+y^2=9$, **f** $x^2+(y+6)^2=25$

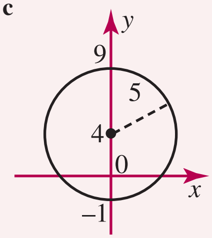
7 **a**



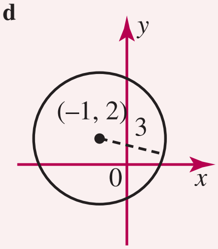
b



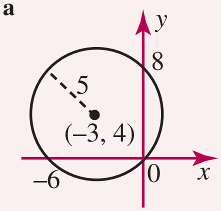
c



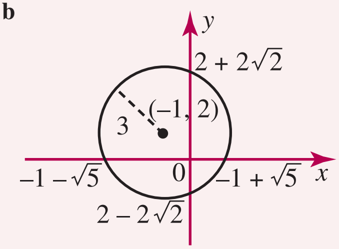
d



8 **a**



b

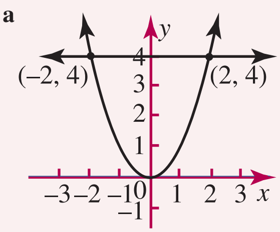


9 **a** yes, **b** no, **c** no, **d** no, **e** yes, **f** yes, **g** no, **h** no, **i** no, **j** yes, **k** no, **l** yes
10 **a** (1, 2), $r = 5$, **b** (–6, –5), $r = 3$, **c** (9, –7), $r = 8$, **d** (–2, 9), $r = 11$, **e** (0, –3), $r = 7$, **f** (7, 0), $r = 8$, **g** (–3, 5), $r = 3\sqrt{3}$, **h** (10, –2), $r = 2\sqrt{11}$

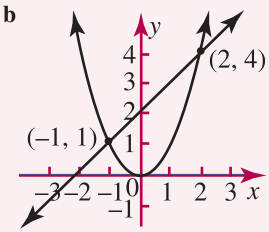
11 **a** $y = \sqrt{1-x^2}$, **b** $y = -\sqrt{1-x^2}$, **c** $y = \sqrt{16-x^2}$, **d** $y = \sqrt{49-x^2}$, **e** $y = -\sqrt{3-x^2}$, **f** $y = -\sqrt{24-x^2}$
12 **a** 3, **b** 2, **c** 5, **d** 6, **e** 10, **f** $\sqrt{6}$, **g** $2\sqrt{10}$, **h** $3\sqrt{7}$

Exercise 13.3

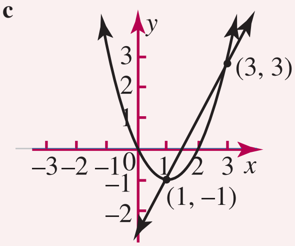
1 **a**



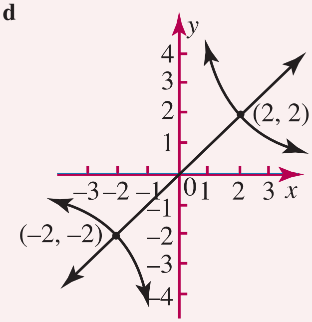
b



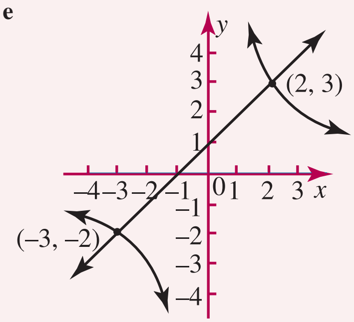
c



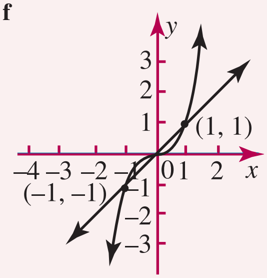
d



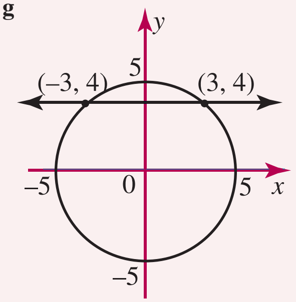
e



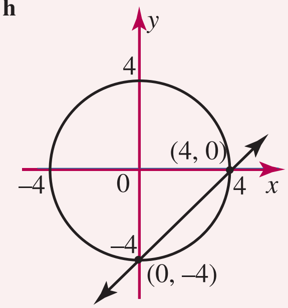
f



g

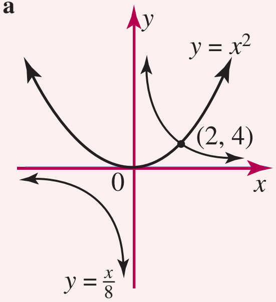


h

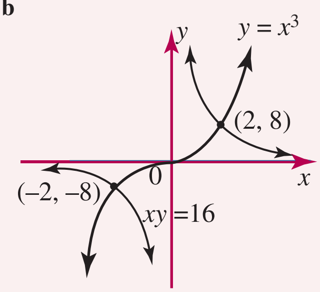


2 **a** (5, 25), (−5, 25), **b** (0, 0), (3, 9), **c** (−3, 3), (1, 3), **d** (3, 6), (5, 20) **3** **a** (3, 2), **b** (−2, −4), (2, 4), **c** (−6, −2), (6, 2), **d** (1, 3), (3, 1) **4** **a** (−5, 3), (5, 3), **b** (−5, −5), (5, 5), **c** (−2, 4), (2, −4), **d** (−2, −6), (6, 2)

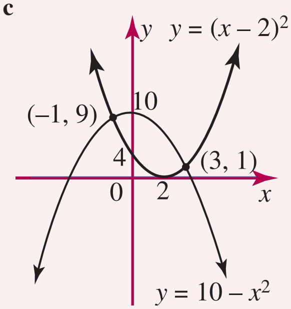
5 **a**



b



c



6 (−1, −1), (1, 1) **7** **a** (−2, 4), (2, 4), **b** (1, 2), (−1, −2), (2, 1), (−2, −1)

Exercise 13.4

1 **a** yes, **b** yes, **c** no, **d** no, **e** yes, **f** no, **g** yes, **h** yes, **i** no, **j** no, **k** yes, **l** no, **m** yes, **n** no, **o** no, **p** yes, **q** no, **r** no, **s** yes, **t** yes, **u** no **2** B, C, E, H, I **3** **a** **i** 3, **ii** x^3, **iii** 1, **iv** 4, **b** **i** 4, **ii** $2x^4$, **iii** 2, **iv** −1, **c** **i** 6, **ii** $5x^6$, **iii** 5, **iv** 12, **d** **i** 5, **ii** $3x^5$, **iii** 3, **iv** 0, **e** **i** 1, **ii** $-4x$, **iii** −4, **iv** 7, **f** **i** 4, **ii** x^4, **iii** 1, **iv** 11, **g** **i** 2, **ii** $-3x^2$, **iii** −3, **iv** 0, **h** **i** 0, **ii** 4, **iii** 4, **iv** 4, **i** **i** 3, **ii** $\sqrt{2}x^3$, **iii** $\sqrt{2}$, **iv** −10, **j** **i** 4, **ii** $\frac{1}{2}x^4$, **iii** $\frac{1}{2}$, **iv** −8, **k** **i** 3, **ii** $\frac{x^3}{2}$, **iii** $\frac{1}{2}$, **iv** 0, **l** **i** 6, **ii** $\frac{5x^6}{3}$, **iii** $\frac{5}{3}$, **iv** 1 **4** **a** **i** −3, **ii** 11, **iii** −9, **b** **i** 30, **ii** 18, **iii** 2, **c** **i** 265, **ii** −63, **iii** $1\frac{3}{4}$, **d** **i** 10, **ii** 13, **iii** $4\frac{2}{3}$ **5** **a** $x = -3, 3$, **b** $x = -1, 3$, **c** $x = -5, 3$, **d** $x = 0, 1, -1$, **e** $x = 0, 2, -2$, **6** **a** **i** 2, **ii** $9x^2$, **iii** 9, **iv** 16, **b** **i** 3, **ii** $5x^3$, **iii** 5, **iv** 0, **c** **i** 4, **ii** $4x^4$, **iii** 4, **iv** 0, **d** **i** 3, **ii** x^3, **iii** 1, **iv** −8, **e** **i** 12, **ii** $16x^{12}$, **iii** 16, **iv** 1, **f** **i** 3, **ii** x^3, **iii** 1, **iv** 24, **g** **i** 5, **ii** $2x^5$, **iii** 2, **iv** 10, **h** **i** 2, **ii** $2x^2$, **iii** 2, **iv** 37, **i** **i** 1, **ii** $20x$, **iii** 20, **iv** 20, **j** **i** 4, **ii** $13x^4$, **iii** 13, **iv** 29 **7** **a** $k = 7$, **b** $k = 3$, 4 or 5, **c** $k \geqslant 6$ **8** **a** $p \neq 1$, q and r = any value, **b** $p = 1$, $q \neq -4$, r = any value, **c** $p = 1$, $q = -4$, $r \neq 5$, **d** $p = 1$, $q = -4$, $r = 5$

Exercise 13.5

1 **a** $x^3 + 7x^2 - x + 16$, **b** $x^3 + 2x^2 + 1$, **c** $2x^3 + 5x^2 + 5x + 3$, **d** $x^3 + 3x^2 + 5x + 2$, **e** $x^3 - 2x^2 + 6x - 13$, **f** $5x^2 - x + 15$, **g** $2x^3 + 7x^2 + 2x + 10$, **h** $-3x^2 - 2x - 8$, **i** $4 - 8x - 3x^2 - 2x^3$ **2** **a** $x^3 + 6x^2 + 3x + 7$, **b** $4x^3 - 5x^2 + 4x - 10$, **c** $7x^3 - 10$, **d** $x^3 + 6x^2 + 15x - 13$, **e** $x^3 - 13x^2 + 7x - 9$, **f** $x^4 - 5x^3 - 7x^2 + 16x + 14$, **g** $x^4 - 4x^3 + 8x^2 - 18$, **h** $2x^3 - 6x^2 + 10x - 2$, **i** $2x^3 + 4x^2 - 2x + 1$ **3** **a** $x^3 + 5x^2 + 6x + 2$, **b** $x^3 - 5x^2 + 2x + 8$, **c** $9 - 24x + 10x^2 - x^3$, **d** $3x^3 + 7x^2 - 2x - 8$, **e** $6x^3 + 7x^2 - 24x - 10$, **f** $8x^4 + 26x^3 - 24x^2 - 4x + 3$ **4** **a** 5, **b** 5, **c** 8, **d** 10, **e** 5, **f** 7 **5** **a** 4, **b** 0 **6** **a** m, **b** m, **c** $m + n$ **7** **a** $2x^5 - 9x^3 + 8x^2 - 35x + 20$ **8** **a** $x^4 - x^3 + 2x$, **b** **i** $2x^4$, **ii** $2x^3 - 4x$, **iii** $x^8 - x^6 + 4x^4 - 4x^2$ **9** $x^3 - 2x^2 - x - 28$, **10** $m = 5$, $n = -6$

Exercise 13.6

1 **a** $x + 4$, **b** $x - 3$, **c** $2x - 7$, **d** $5x + 2$ **2** **a** $(x + 2)(x + 5) + 4$, **b** $(x - 4)(x + 8) + 3$, **c** $(3x + 2)(2x + 5) + 7$, **d** $(2x - 5)(5x - 2) - 4$ **3** **a** $(x + 1)(x^2 + 6x + 4) + 11$, **b** $(x + 3)(x^2 + 9x + 10) + 2$, **c** $(x - 2)(x^2 - 7x + 2) + 1$, **d** $(x - 5)(x^2 + x - 4) + 14$, **e** $(x + 4)(2x^2 + 7x + 4) - 1$, **f** $(3x + 5)(x^2 - 4x + 3) - 7$, **g** $(3x - 4)(x^2 + 4) + 12$, **h** $(2x + 7)(3x^2 - x - 3) + 11$, **i** $(x + 2)(x^3 - 7x^2 + 17x - 22) + 14$, **j** $(2x - 3)(x^3 + 2x^2 - 5x - 2) - 8$ **4** **a** $Q(x) = x^2 + 4x - 4$, $R(x) = 12$, **b** $Q(x) = x^2 + 2x + 1$, $R(x) = 6$, **c** $Q(x) = x^3 + x^2 - 2x - 2$, $R(x) = 5$, **d** $Q(x) = x^3 - 3x^2 + 9x - 23$, $R(x) = 63$ **5** **a** $Q(x) = x^2 + 2x - 8$, $R(x) = 4x + 11$,

b $Q(x) = x^2 + 8x + 21$, $R(x) = 48x - 17$, **c** $Q(x) = 2x^2 + 9x + 1$, $R(x) = -7x - 5$. **d** $Q(x) = x^2 + 5x - 3$, $R(x) = -14x + 7$, **e** $Q(x) = 4x^2 + 3x + 10$, $R(x) = 8x + 62$, **f** $Q(x) = x^2 - 2x + 7$, $R(x) = -8x - 16$, **g** $Q(x) = 2x^2 - 8x + 9$, $R(x) = -12x - 30$, **h** $Q(x) = 6x^2 + 3x + 4$, $R(x) = 4x - 4$ **6 a** $Q(x) = x^2 + 6x + (k + 6)$, $R(x) = 3k + 6$, **b** $k = -2$, **c** $k = 5$

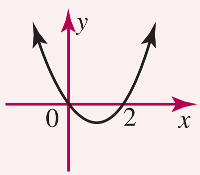
Exercise 13.7

1 a 3 **2 a** 11, **b** –6, **c** 4, **d** 10, **e** 7, **f** 5, **g** 14, **h** –15 **3** $k = -7$ **4 a** $k = -7$, **b** $k = 7$, **c** $k = -5$
5 $a = 1$, $P(x) = x^3 + x^2 + x - 6$ **6 a** $5\frac{1}{8}$, **b** $7\frac{1}{27}$ **7** $k = -5, 3$ **8** $P(x) = 2(x + 1)(x - 2)(x - 4)$
9 a $P(x) = -(x + 5)(x + 2)(x - 1)(x - 3)$, **b** $P(x) = -2(2x + 1)(3x - 2)(x - 2)$ **10 a** the divisor is of degree 2, so any remainder must be of degree 1 or 0. **b** $a = 3$, $b = 5$, $R(x) = 3x + 5$ **11** $R(x) = 2x - 4$

Exercise 13.8

1 a $P(2) = 0$ **2 a** yes, **b** no, **c** no, **d** yes, **e** no, **f** yes **3 a** $(x - 1)(x - 2)(x + 4)$, **b** $(x + 1)(x - 7)(x + 3)$, **c** $(x + 6)(x - 1)(x + 1)$, **d** $(x - 2)(x - 5)(x + 3)$, **e** $(x + 1)(x - 4)(x + 2)$, **f** $(x + 3)(x - 2)(x - 1)$
4 a $(x - 2)(3x - 4)(x + 3)$, **b** $(x + 2)(2x - 1)(x - 6)$ **5 a** $(x - 1)(x + 4)(x + 5)$, **b** $(x + 1)(x - 2)(x + 7)$, **c** $(x - 2)(x + 2)(x - 3)$, **d** $(x + 2)(x + 3)(x + 5)$, **e** $(2x + 5)(x - 1)(x + 2)$, **f** $(3x - 2)(x + 3)(x - 4)$
6 $(x - 2)^2(x + 1)$ **7 a** $x = 1, 8, -3$, **b** $x = 2, -2, -4$, **c** $x = 3, 5, -3$, **d** $x = 2, 3, 7$, **e** $x = -1, 4, \frac{1}{2}$, **f** $x = 2, 5, -\frac{2}{3}$
8 a $x = -1, 1 \pm \sqrt{3}$, **b** $x = 2, 1 \pm \sqrt{5}$ **9** $x = -1$ **10 a** $k = 30$, **b** $k = -11$, **c** $k = -3$, **d** $k = 12$ **11** $x = -1, 2, 5$
12 $a = -1, -4, 7$ **13** $a = -6$, $b = 30$ **14** $p = 4$, $q = -15$

Exercise 13.9

1 a


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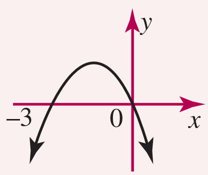

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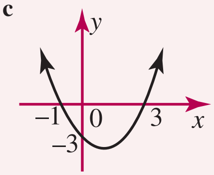

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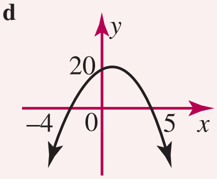

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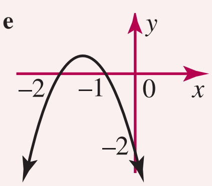

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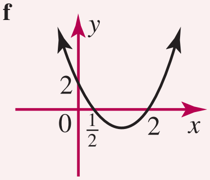

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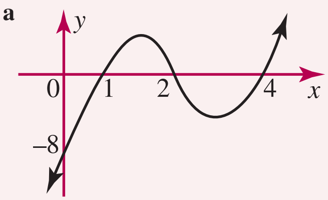

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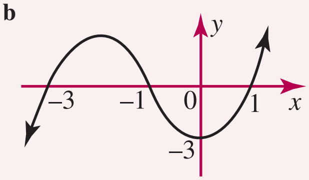

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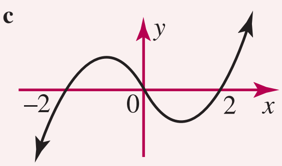

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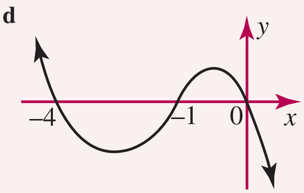

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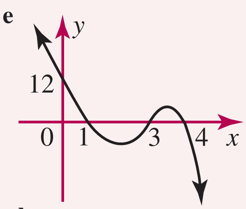

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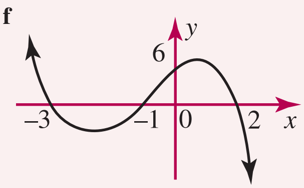

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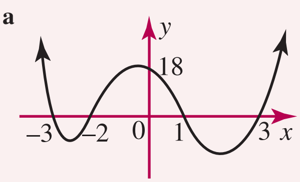

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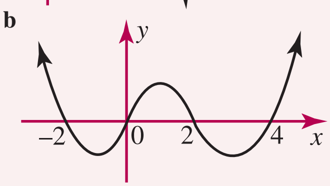

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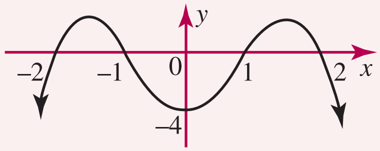

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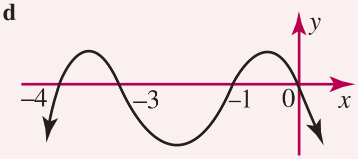

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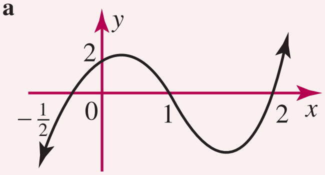

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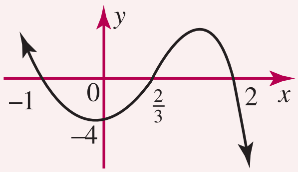

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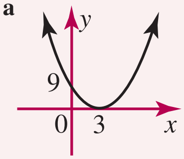

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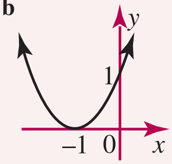

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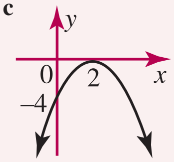

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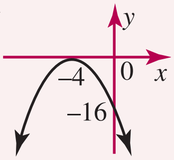

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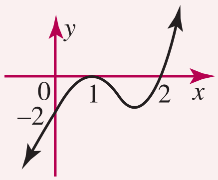

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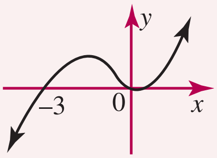

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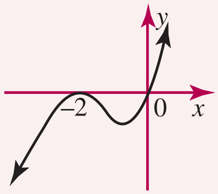

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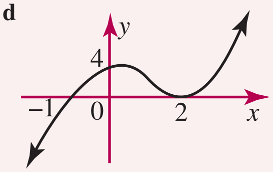

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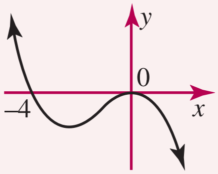

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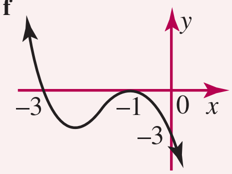

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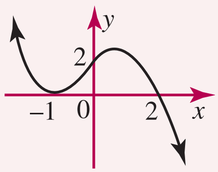

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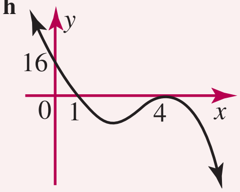

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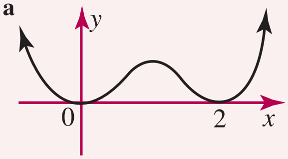

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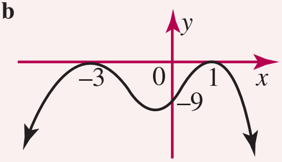

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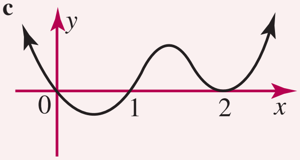

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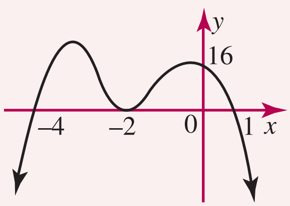

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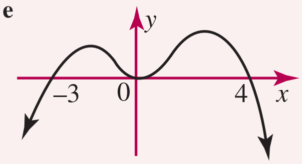

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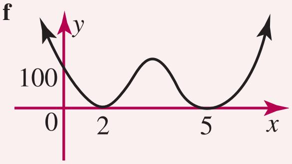

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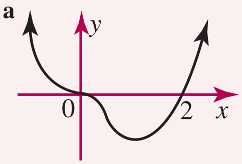

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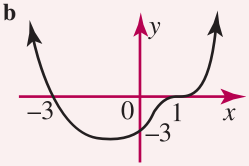

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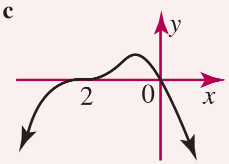

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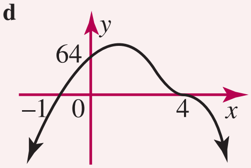

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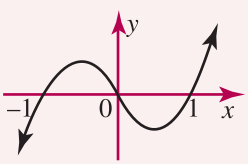

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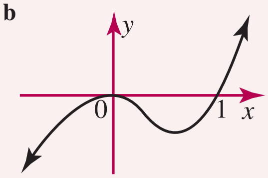

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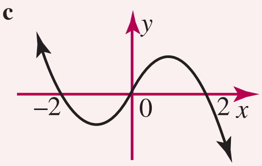
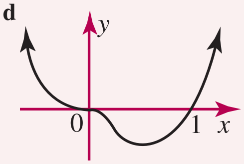


ANSWERS

d


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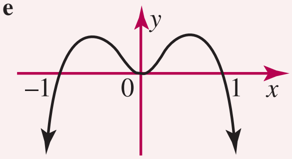

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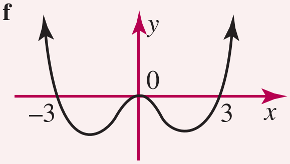

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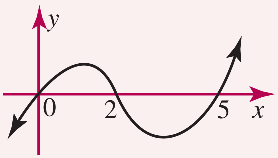

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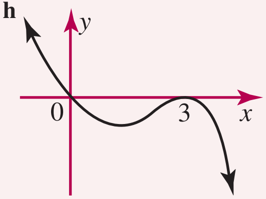

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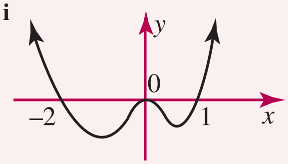

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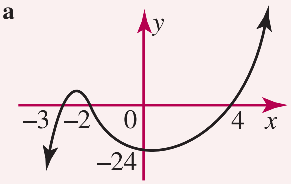

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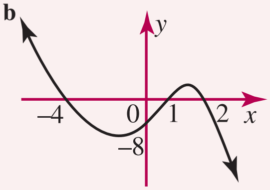
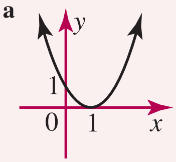


11 b once
12 a $y = 2(x + 2)(x - 2)(x - 3)$,
b $y = -4x(x + 3)(x - 3)$,
c $y = 3(x + 4)(x + 1)(x - 2)$,
d $y = (x + 1)(x - 3)^2$,
e $y = -2(x + 3)^2(x - 2)$,
f $y = 4(x + 2)^2(x - 1)$

Exercise 13.10

1 a


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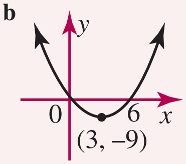

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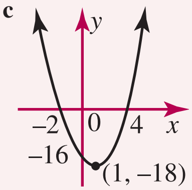

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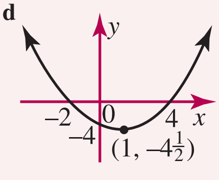

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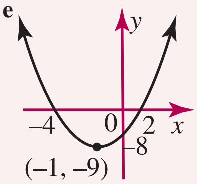

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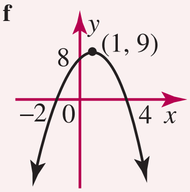

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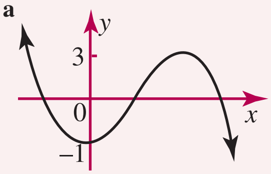

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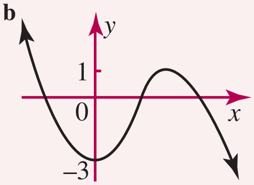

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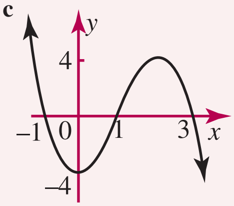

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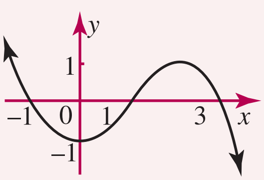

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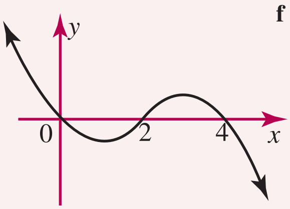

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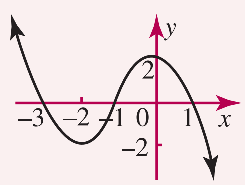

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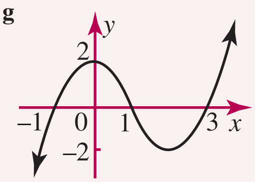

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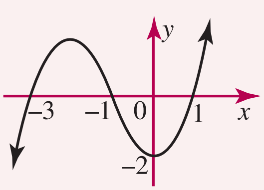

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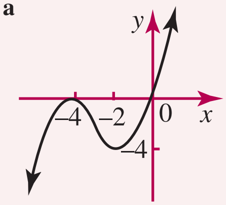

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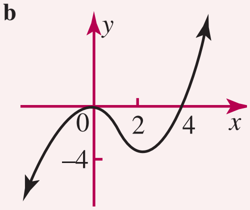

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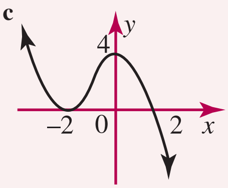

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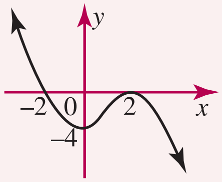

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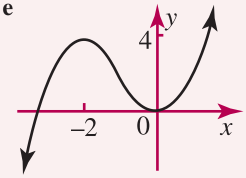

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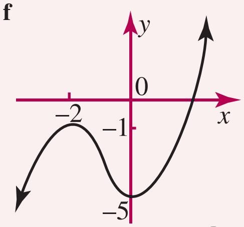

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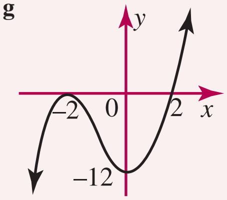

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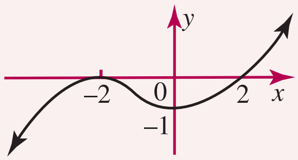

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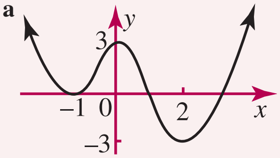

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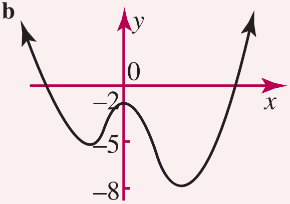

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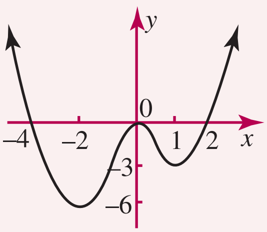

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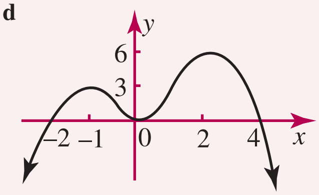

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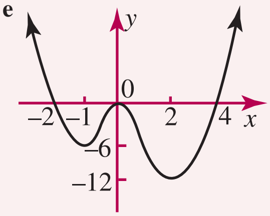

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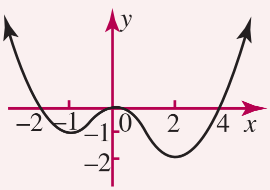

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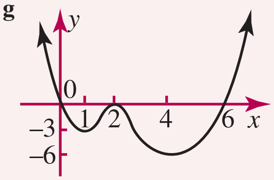

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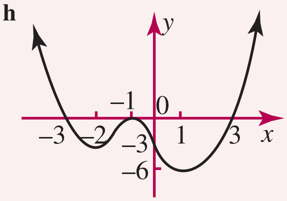

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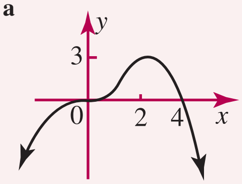

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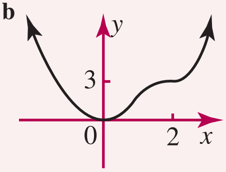

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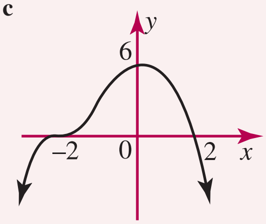

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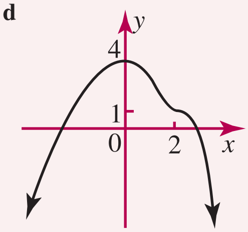

Chapter 13 Review

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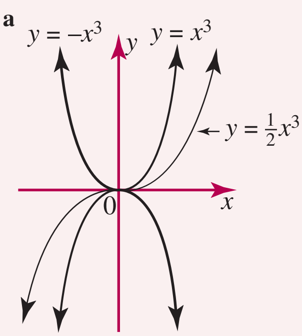

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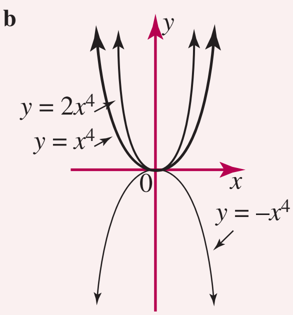

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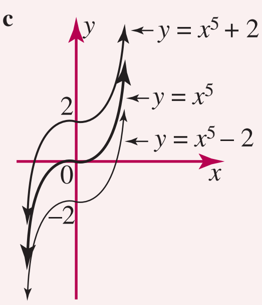

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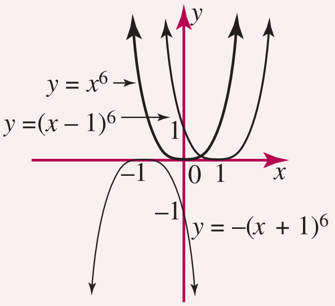

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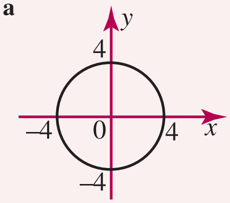

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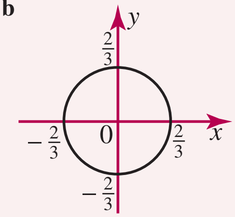
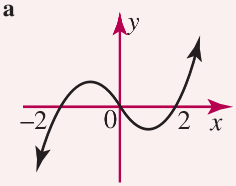


3 a $(2, -3)$, $r = 7$, **b** $(-6, -1)$, $r = 2$, **c** $(0, 4)$, $r = 2\sqrt{2}$ **4 a** $(x - 3)^2 + (y + 5)^2 = 64$, **b** $(x + 2)^2 + y^2 = 12$
5 a yes, **b** no, **c** yes, **d** no, **e** yes **6 a** $(1, -3)$, $r = 4$, **b** $(5, 0)$, $r = 3\sqrt{5}$ **7 a** $(1\frac{1}{2}, -7\frac{1}{2})$, $(5, 10)$, **b** $(-4, -3)$, $(3, 4)$, **c** $(3, 6)$, $(-3, -6)$ **8 a** yes, **b** no, **c** no, **d** yes, **e** no, **f** yes, **g** yes, **h** yes **9 a i** $5x^4$, **ii** 5, **iii** 4, **iv** 7, **b i** $-x^5$, **ii** −1, **iii** 5, **iv** 6, **c i** x^2, **ii** 1, **iii** 2, **iv** 0, **c i** $\frac{x^3}{5}$, **ii** $\frac{1}{5}$, **iii** 3, **iv** −1
10 the leading co-efficient is 1 in $x^3 - 4x$, but not in $4x - x^3$. **11 a** 4, **b** −8, **c** 4, **d** $6\frac{2}{27}$, **e** $8\sqrt{2} - 6$, **f** $27a^3 - 45a^2 + 18a + 4$ **12 a** $2x^3 + 3x^2 - 10x + 10$, **b** $7x^2 + 2x + 4$ **13** $6x^4 + 8x^3 - 9x^2 + 3x + 20$
14 a $4x^3 + 7x^2 + 7x - 11$, **b** $2x + 1$, **c** $14x^2 + 2x - 23$, **d** $1 - 3x - 4x^2$, **e** $28x^5 + 8x^4 - 9x^3 + 10x^2 - 55x$, **f** $8x^4 + 48x^3 + 10x^2 + 60x$ **15 a** $(x - 1)(x - 3) + 2$, **b** $(x + 2)(x^2 + 3x - 7) + 20$, **c** $(2x - 3)(x^2 + 3x + 1) - 7$, **d** $(x - 2)(x^3 + 2x^2 + 10x + 20) + 50$ **16 a** −2, **b** 12, **c** 3, **d** 18 **17 a** $k = 4$, **b** $k = -4$ **18 a** $P(4) = 0$, **b** $P(-2) = 0$ **19** $k = -3$ **20 a** $P(-2) = 0$, **b** $(x + 2)(x + 4)(x - 3)$ **21 a** $(x - 2)(x - 3)(x + 7)$, **b** $(2x + 1)(x - 1)(x - 3)$ **22** $x = \frac{2}{3}, -1, -2$ **23** $a = -10$, $b = 31$

24 a



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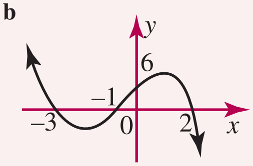

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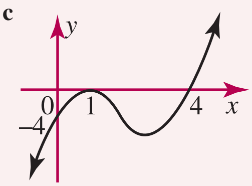

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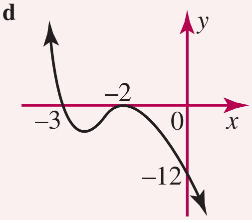

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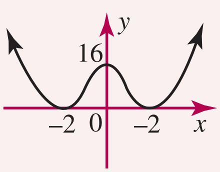

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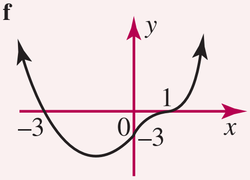

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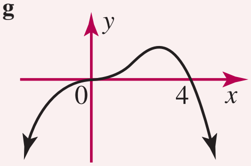

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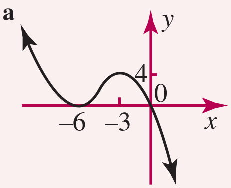

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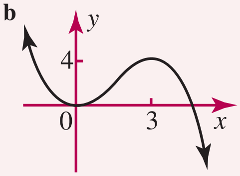

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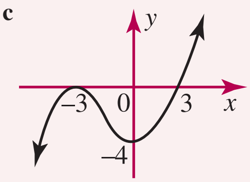

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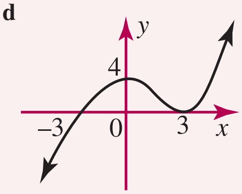

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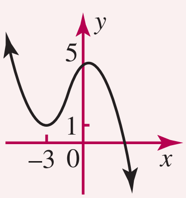

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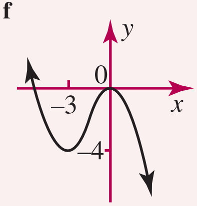

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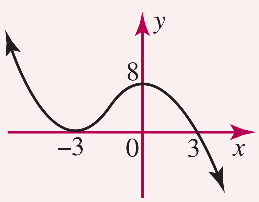

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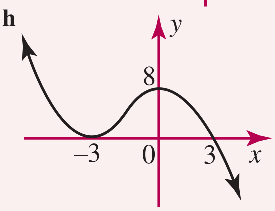

14 Circle geometry

Exercise 14.1

1 **a** radius, **b** diameter, **c** circumference, **d** chord, **e** minor arc, **f** major arc, **g** minor segment, **h** major segment, **i** minor sector, **j** major sector, **k** tangent, **l** secant **2** **a** radius, **b** diameter, **c** tangent, **d** chord, **e** semi-circle, **f** secant, **g** sector, **h** segment **3** **a** *OA*, *OB*, *OE*, **b** *AB*, *CD*, *CE*, *DE* **4** **a** semi-circle, **b** arc, **c** chord, **d** radius, **e** tangent, **f** diameter, **g** segment, **h** secant, **i** sector **5** **a** 24 cm, **b** diameter **6** **a** equal radii, **b** isosceles, **c** $\angle OJK = \angle OKJ$ **7** equilateral, all angles are 60° **8** 4 cm **9** 3 cm **10** **a** $\angle POQ$, **b** $\angle ROS$ **11** **a** **i** $\angle JNK$, $\angle JMK$, $\angle JLK$, **ii** $\angle LNM$, $\angle LJM$, $\angle LKM$, **b** **i** $\angle JKM$, $\angle JLM$, $\angle JNM$ **ii** $\angle LJN$, $\angle LKN$, $\angle LMN$ **12** **a** $\angle TUW$, $\angle TVW$, **b** **i** $\angle TSW$, **ii** $\angle TUV$, **iii** $\angle STU$ **13** **a** *GE*, *GN*, **b** *XC*, *XR* **14** $\angle AEC$, $\angle ADC$ **15** no, *PR* is not a diameter **16** **a** 30°, **b** 60°, **c** 60°, **d** 90° **17** **a** $\angle POR$, **b** $\angle PTR$, $\angle PQR$, $\angle PSR$ **18** **a** yes, **b** cyclic quadrilateral, **c** $\angle M$ **19** no, the points *O*, *X*, *Y*, *Z* are not concyclic **20** $\angle RQT$ **21** **a** rhombus, **b** kite **22** yes **23** construct the perpendicular bisectors of the chords and find their point of intersection

Exercise 14.2

1 **a** 7 cm, **b** 10 cm, **c** 9 cm, **d** 6.5 cm, **e** 25 cm, **f** 8 cm **2** **a** $x = 52$, **b** $p = 11$, **c** $c = 60$ **3** **a** 45°, **b** $VW = 4$ cm, $XY = 7$ cm, $UZ = 6$ cm, **c** 108° **4** **a** 5 cm, **b** 14 cm, **c** $WX = 22$ cm, $WM = 11$ cm, **d** $GH = 15$ cm, $OQ = 9$ cm, **e** 6.5 cm, **f** 6 cm **5** **a** $MD = 12$ cm, $CD = 24$ cm, **b** 29 cm, **c** 48 cm, **d** $OM = 9$ cm, $HM = 24$ cm, **e** $OZ = 25$ cm, $WX = 40$ cm, **f** $OM = 30$ cm, $MN = 16$ cm, **g** 9 cm, **h** $QR = 16$ cm, $PR = 56$ cm, **i** 35 cm

6 **a** 34 cm, **b** 48 cm **7** $10\sqrt{2}$ cm **8** **a** 7 cm, **b** 23 cm **9** 21 cm **10** 48 cm

Exercise 14.3

1 **a** $a = 120$, **b** $e = 52$, **c** $y = 45$, **d** $w = 24$, **e** $q = 56$, **f** $b = 146$, **g** $d = 43$, **h** $n = 37$, **i** $x = 33$ **2** **a** $m = 40$, **b** $z = 25$, **c** $c = 118$, **d** $p = 41, q = 41$, **e** $x = 70, y = 70$, **f** $e = 110, f = 55$, **g** $u = 42, v = 84$, **h** $j = 37, k = 37$, **i** $r = 28, s = 28, t = 56$ **3** **a** $n = 90$, **b** $a = 180$, **c** $u = 90, v = 90$ **4** **a** a = 50, $b = 45$, **b** $m = 90, n = 65$, **c** $t = 31, u = 84$, **d** $g = 67, h = 134$, **e** $x = 46, y = 46, z = 92$, **f** $e = 150, f = 75$, **g** $r = 28$, s = 56, **h** $c = 65, d = 65$, **i** $v = 70, w = 140$, **j** $j = 42, k = 42$, **k** a = 220, $b = 110$, **l** $x = 62, y = 57, z = 31, w = 30$, **m** $a = 31, b = 62$, **n** $r = 72, s = 54$, **o** $v = 53, w = 53$ **5** **a** $w = 63$, **b** $q = 38$, **c** $z = 57$, **d** $t = 158$, **e** $x = 59$, **f** $s = 113$ **6** $a = 25, b = 34, c = 42$

Exercise 14.4

1 **a** $x = 130$, **b** $k = 65$, **c** $m = 90$, **d** $p = 45, q = 128$, **e** $a = 112, b = 84$, **f** $u = 76, v = 53$, **g** $t = 90$, **h** $y = 60$, **i** $y = 35$ **2** **a** $w = 70$, **b** $e = 53$, **c** $g = 48$, **d** $k = 22$, **e** $u = 47$, **f** $r = 17$ **3** **a** no, the points *O*, *A*, *B*, *C* are not concyclic, **b** no, **c** 210°, **d** $x = 105$ **4** **a** $m = 65, n = 115$, **b** $e = 82, f = 98$, **c** $p = 121, q = 118$, **d** $c = 62, d = 62$, **e** $x = 72, y = 144$, **f** $r = 135, s = 45$ **5** **a** $g = 77, h = 154$, **b** $v = 74, w = 74$, **c** $a = 124, b = 56$, **d** $q = 51, r = 78, s = 102$, **e** $j = 64, k = 64$, **f** $s = 62, t = 118$, **g** $c = 27, d = 153$, **h** $e = 23, f = 23$, **i** $m = 90, n = 90$ **6** **a** $x = 55, y = 51, z = 51$, **b** $a = 41, b = 110, c = 110$, **c** $p = 65, q = 72, r = 108$, **d** $u = 90, v = 63, w = 117$, **e** $x = 43, y = 62, z = 56$, **f** $a = 125, b = 125$, **g** $g = 39, h = 78$, **h** $p = 64, q = 26, r = 50, s = 92$, **i** $a = 47, b = 34$, **j** $d = 58$, **k** $s = 51$, **l** $c = 17, d = 73, e = 107$, **m** $x = 23, y = 62, z = 118$, **n** $m = 56$, **o** $x = 46, y = 148$ **7** $p = 84, q = 67$ **8** 36° **9** $w = 114, x = 66, y = 48, z = 132$

Exercise 14.5

1 **a** $a = 90$, **b** $y = 38$, **c** $e = 56$ **2** **a** $x = 12$, **b** $w = 65$, **c** $g = 55$, **d** $q = 74$, **e** $t = 90, u = 90, v = 134$, **f** $e = 67, f = 15$ **3** **a** 11 cm, **b** 14 cm, **c** 44 cm, **d** 9 cm **4** **a** 37 cm, **b** 24 cm, **c** 24 cm, **d** 64 cm **5** **a** $x = 50$, **b** $c = 49, d = 52$, **c** $p = 68, q = 41$, **d** $g = 44, h = 82$, **e** $u = 108, v = 17$, **f** $e = 78, f = 38$, **g** $c = 33$, **h** $z = 156$, **i** $s = 60$ **6** **a** $a = 53, b = 106$, **b** $m = 66, n = 66$, **c** $e = 74, f = 74$, **d** $p = 35, q = 35$, **e** $x = 90, y = 78, z = 90, w = 78$, **f** $r = 81, s = 162, t = 9$, **g** $g = 43, h = 137$, **h** $a = 57, b = 47, c = 47, d = 76$, **i** $x = 77, y = 77, z = 26$ **7** **a** $b = 59$, **b** $f = 38$, **c** $c = 72$, **d** $t = 47$, **e** $m = 76$, **f** $k = 52$, **g** $r = 51$, **h** $x = 68$, **i** $y = 56$, **j** $v = 9$, **k** $j = 42$, **l** $e = 49$, **m** $s = 99$, **n** $a = 37$, **o** $d = 83$ **8** 37 cm **9** $x = 50$ **10** $x = 36$

Exercise 14.6

1 **a** $m = 10$, **b** $y = 16$, **c** $u = 6$ **2** **a** $p = 10$, **b** $j = 9$, **c** $c = 14$, **d** $u = 21$ **3** **a** $a = 4$, **b** $t = 6$, **c** $n = 12$, **d** $w = 7$ **4** **a** $x = 10, y = 5$, **b** $p = 12, q = 9.6$, **c** $m = 5, n = 30$, **d** $e = 12, f = 18$, **e** $r = 15, s = 40$ **5** **a** 24 cm, **b** 14 cm **6** $a = 9, b = 15$ **7** **a** $n = 4$, **b** $a = 1$, **c** $t = 4$, **d** $c = 5$ **8** 10 cm **9** **a** 6 cm, **b** 6 cm

Exercise 14.7

1 **a** $x = 36$, **b** $e = 47$, **c** $p = 24$, **d** $g = 28$, **e** $u = 34$, **f** $c = 143$, **g** $m = 112$, **h** $a = 53$, **i** $r = 39$, **j** $k = 136$, **k** $f = 117$, **l** $y = 76$, **m** $w = 52$, **n** $n = 115$, **o** $z = 49$, **p** $s = 117$, **q** $q = 51$, **r** $n = 64$, **s** $v = 48$, **t** $r = 68$, **u** $c = 21$, **v** $j = 59$, **w** $t = 74$, **x** $u = 76$

2 **a** $\angle FTL = \angle MTG$ (vertically opposite $\angle$s)
b Let $\angle FTL = \alpha$
$\angle MTG = \angle FTL$ (proven above)
$= \alpha$
$\angle FET = \angle FTL$ ($\angle$ in alternate segment)
$= \alpha$
$\angle GHT = \angle MTG$ ($\angle$ in alternate segment)
$= \alpha$
$\angle FET = \angle GHT$ (both equal to α)
$\therefore EF \parallel GH$ (alternate $\angle$s are equal)

3 $AB = 2BM$ (line from centre perpendicular to a chord bisects the chord)
$BC = 2BN$ (line from centre perpendicular to a chord bisects the chord)
$BM = BN$ (given)
$\therefore AB = BC$ (equal chords are equidistant from centre)

4 $MP = MQ$ (line from centre perpendicular to a chord bisects the chord)
$MR = MS$ (line from centre perpendicular to a chord bisects the chord)
$\therefore MP - MR = MQ - MS$ (by subtraction)
$\therefore PR = SQ$

5 Let $\angle BAD = \alpha$
$\angle BCD = \angle BAD$ ($\angle$s at circumference are equal)
$= \alpha$
$\angle ABC = \angle BCD$ (alternate $\angle$s, $AB \parallel CD$)
$= \alpha$
$\angle BAD = \angle BCD$ (both equal to α)
$\therefore \Delta APB$ is isosceles

6 $\angle AMO = 90°$ ($\angle$ in a semi-circle)
$AM = MB$ (line from centre perpendicular to a chord bisects the chord)
$\therefore M$ is the midpoint of AB

7 $\angle UTQ = 90°$ (tangent is perpendicular to a radius)
$\angle TVS = 90°$ (co-interior $\angle$s, $PQ \parallel RS$)
$RV = VS$ (line from centre perpendicular to a chord bisects the chord)
$\therefore TU$ bisects RS

8 **a** Let $\angle CAB = \alpha$
$\angle CDB = \angle CAB$ ($\angle$s at circumference are equal)
$= \alpha$
$\angle ACD = \angle CAB$ (base $\angle$s of isosceles Δ, $PA = PC$)
$= \alpha$
$\angle ABD = \angle ACD$ ($\angle$s at circumference are equal)
$= \alpha$
$\angle CDB = \angle ABD$ (both equal to α)
$\therefore \Delta DPB$ is isosceles
b $PA = PC$ (given)
$PB = PD$ (equal sides lie opposite equal $\angle$s)
$\therefore PA + PB = PC + PD$ (by addition)
$\therefore AB = CD$
c $\angle CAB = \angle ABD$ (both equal to α)
$\therefore AC \parallel DB$ (alternate $\angle$s are equal)

9 Let $\angle QTS = \alpha$
$\angle QPS = \angle QTS$ ($\angle$s at circumference are equal)
$= \alpha$
$\angle PSR = \angle QPS$ (alternate $\angle$s, $PQ \parallel RS$)
$= \alpha$
$\therefore \angle QTS = \angle PSR$ (both equal to α)

10 Construct AQ, PQ, QB
$\angle PQA = 90°$ ($\angle$ in a semi-circle)
$\angle PQB = 90°$ ($\angle$ in a semi-circle)
$\angle AQB = 90° + 90°$
$= 180°$
$\therefore A, Q, B$ are collinear

11 Let $\angle PAC = \alpha$
$\angle PDB = \angle PAC$ (ext. $\angle$ of a cyclic quadrilateral)
$= \alpha$
$\angle PCA = \angle PAC$ (base $\angle$s of isosceles Δ, $PA = PC$)
$= \alpha$
$\angle PBD = \angle PCA$ (exterior $\angle$ of a cyclic quad.)
$= \alpha$
$\angle PBD = \angle PDB$ (both equal to α)
$\therefore PB = PD$ (equal sides lie opposite equal $\angle$s)
$PA = PC$ (given)
$\therefore PB - PA = PD - PC$ (by subtraction)
$\therefore AB = CD$

12 Let $\angle XWZ = \alpha$
$\angle WXT = \angle XWZ$ (base $\angle$s of isosceles Δ, $TW = TX$)
$= \alpha$
$\angle XTZ = 2\alpha$ (ext. $\angle$ of ΔWTX)
$\angle XYZ = \angle XWZ$ ($\angle$s at circumference are equal)
$= \alpha$
$\angle XOZ = 2\alpha$ ($\angle$ at centre is twice the $\angle$ at circumference)
$\therefore \angle XOZ = \angle XTZ$ (both equal to 2α)

13 Construct XY
Let $\angle CAX = \alpha$
$\angle XYD = \alpha$ (exterior angle of a cyclic quadrilateral)
$\angle XBD = 180° - \alpha$ (opposite $\angle$s of a cyclic quadrilateral are supplementary)
$\angle CAX + \angle XBD = \alpha + (180° - \alpha)$
$= 180°$
$\therefore AC \parallel BD$ (co-interior $\angle$s are supplementary)

14 a Let $\angle CBD = \alpha$
$\angle EFG = \angle CBD$ (given)
$= \alpha$
$\angle EBA = \angle EFG$ (exterior $\angle$ of a cyclic quadrilateral)
$= \alpha$
$\therefore \angle CBD = \angle EBA$ (both equal to α)

b $\angle CED = \angle CBD$ ($\angle$'s at circumference are equal)
$= \alpha$
$\angle ADE = \angle CED$ (both equal to α)
$\therefore PD = PE$ (equal sides lie opposite equal $\angle$s)

15 a Let $\angle BAQ = \alpha$
$\angle ACB = \angle BAQ$ ($\angle$ in alternate segment)
$= \alpha$
$\angle CAB = \angle ACB$ (base $\angle$s of isosceles Δ, $AB = BC$)
$= \alpha$
$\therefore \angle CAB = \angle BAQ$ (both equal to α)
$\therefore AB$ bisects $\angle CAQ$

b $\angle ADB = \angle BAQ$ ($\angle$ in alternate segment)
$= \alpha$
$\angle BDC = \angle CAB$ ($\angle$s at circumference are equal)
$= \alpha$
$\therefore \angle ADB = \angle BDC$ (both equal to α)
$\therefore DB$ bisects $\angle ADC$

16 Let $\angle RTQ = \alpha$ and $\angle QTP = \beta$
$\angle STR = \angle RTQ$ (RT bisects $\angle STQ$)
$= \alpha$
$\angle TSR = \angle QTP$ ($\angle$ in alternate segment)
$= \beta$
$\angle TRP = \alpha + \beta$ (exterior $\angle$ of ΔRST)
$\angle RTP = \alpha + \beta$ (by addition)
$\therefore \angle TRP = \angle RTP$ (both equal to $\alpha + \beta$)
$\therefore PR = PT$ (equal sides lie opposite equal $\angle$s)

17 Let $\angle ABD = \alpha$
$\angle BDC = \angle ABD$ (alternate $\angle$s, $AB \parallel DC$)
$= \alpha$
$\angle BCD = \angle ABD$ ($\angle$ in alternate segment)
$= \alpha$
$\angle BAD = \angle BDC$ ($\angle$ in alternate segment)
$= \alpha$
$\angle ADB = 180° - 2\alpha$ ($\angle$ sum of ΔABD)
$\angle DBC = 180° - 2\alpha$ ($\angle$ sum of ΔBCD)
$\therefore \angle ADB = \angle DBC$ (both equal to $180° - 2\alpha$)
$\therefore BC \parallel AD$ (alternate $\angle$s are equal)

18 Let $\angle PTR = \alpha$
$\angle TQR = \angle PTR$ ($\angle$ in alternate segment)
$= \alpha$
$\angle QTR = 90°$ ($\angle$ in a semi-circle)
$\angle OQS = 90°$ (tangent is perpendicular to a radius)
$\angle PRT = 90° + \alpha$ (ext. $\angle$ of ΔQRT)
$\angle TQS = \angle TQR + \angle OQS$ (by addition)
$= 90° + \alpha$
$\therefore \angle TQS = \angle PRT$ (both equal to $90° + \alpha$)

19 Let $\angle ATP = \alpha$
$\angle TBA = \angle ATP$ ($\angle$ in alternate segment)
$= \alpha$
$\angle BTC = 90° - \alpha$ ($\angle$ sum of ΔBCT)
$\angle ATC = \angle ATP$ (AT bisects $\angle CTP$)
$= \alpha$
$\angle BTA = (90° - \alpha) + \alpha$ (by addition)
$= 90°$
$\therefore AB$ is a diameter ($\angle$ in a semi-circle is $90°$)

20 a Let $\angle PTM = \alpha$
$\angle TNP = \angle PTM$ ($\angle$ in alternate segment)
$= \alpha$
$\angle TPN = \angle TNP$ (base $\angle$s of isosceles Δ, $TP = TN$)
$= \alpha$
$\angle TPN = \angle PTM$ (both equal to α)
$\therefore PM = MT$ (equal sides lie opposite equal $\angle$s)

b $\angle TMO = 2\alpha$ (ext. $\angle$ of ΔPTM)
$\angle PTO = 90°$ (tangent is perpendicular to a radius)
$\angle MTO = 90° - \alpha$ (by subtraction)
$OM = OT$ (equal radii)
$\therefore \angle TMO = \angle MTO$ (base $\angle$'s of isosceles Δ)
$\therefore 2\alpha = 90° - \alpha$
$\therefore \alpha = 30°$
$\therefore \angle TMO = \angle MTO = 60°$
$\angle MOT = 60°$ ($\angle$ sum of ΔMOT)
$\therefore \Delta MOT$ is equilateral (all $\angle$s are $60°$)

21 Construct TA and TB
Let $\angle TAP = \alpha$ and $\angle TBP = \beta$
$PA = PT = PB$ (tangents from an exterior point are equal)
$\angle ATP = \angle TAP$ (base $\angle$s of isosceles Δ, $PA = PT$)
$= \alpha$
$\angle BTP = \angle TBP$ (base $\angle$s of isosceles Δ, $PB = PT$)
$= \beta$
$2\alpha + 2\beta = 180°$ ($\angle$ sum of ΔABT)
$\therefore \alpha + \beta = 90°$
$\therefore \angle ATB = 90°$
$\therefore A, B, T$ lie on a semi-circle with diameter AB
$\therefore A, B, T$ are concyclic points

22 Let $\angle APC = \alpha$ and $\angle ATP = \beta$
$\angle TPC = \angle APC$ (CP bisects $\angle BPT$)
$= \alpha$
$\angle TBP = \angle ATP$ ($\angle$ in alternate segment)
$= \beta$
$\angle TDC = \alpha + \beta$ (exterior $\angle$ of ΔTDP)
$\angle TCD = \alpha + \beta$ (exterior $\angle$ of ΔBCP)
$\angle TDC = \angle TCD$ (both equal to $\alpha + \beta$)
$\therefore TC = TD$ (equal sides lie opposite equal $\angle$s)
$\therefore \Delta TCD$ is isosceles

Chapter 14 Review

1 **a** radius, **b** diameter, **c** chord, **d** secant, **e** tangent, **f** sector, **g** segment, **h** arc **2** **a** equal radii, **b** isosceles, **c** $\angle OAB$, $\angle OBA$ **3** circles with the same centre **4** no; O, P, Q, R are not concyclic points **5** **a** 42°, **b** 4 cm, **c** 3 cm, **d** 10 cm, **e** 24 cm, **f** 7 cm **6** **a** $j = 59$, **b** $p = 90$, **c** $c = 37$ **7** **a** $u = 55$, $v = 70$, **b** $e = 136$, $f = 113$, **c** $g = 34$, $h = 38$, **d** $j = 15$, $k = 9$ **8** **a** $k = 90$, **b** $y = 64$, $z = 9$, **c** $p = 63$, $q = 59$ **9** **a** $x = 15$, **b** $p = 6$, $q = 4.5$, **c** $n = 10$, **d** $t = 5$, **e** $u = 4$, **f** $k = 4$ **10** **a** $n = 56$, **b** $e = 41$, **c** $g = 114$, **d** $m = 84$, **e** $u = 13$, **f** $r = 46$

11 **a** Let $\angle QRS = \alpha$
$\angle PQR = \angle QRS$ (alternate $\angle$s, $PQ \parallel RS$)
$= \alpha$
$\angle PSR = \angle PQR$ ($\angle$s at circumference are equal)
$= \alpha$
Now, $\angle QRS = \angle PSR$ (both equal to α)
$\therefore \Delta RST$ is isosceles

b Let $\angle RTS = \alpha$ and $\angle RST = \beta$
$\angle TRP = \alpha + \beta$ (exterior Δ of ΔRST)
$\angle PTR = \angle TRP$ (base $\angle$s of isosceles Δ, $PT = PR$)
$= \alpha + \beta$
$\angle PTQ = \angle RST$ ($\angle$ in alternate segment)
$= \beta$
$\angle QTR = \angle PTR - \angle PTQ$ (by subtraction)
$= \alpha$
Now, $\angle QTR = \angle RTS$ (both equal to α)
$\therefore TR$ bisects $\angle QTS$

c **i** Let $\angle ADC = \alpha$
$\angle AOC = 2\alpha$ ($\angle$ at centre is twice $\angle$ at circumference)
$\angle ABC = 180° - \alpha$ (opposite $\angle$s in a cyclic quadrilateral)
$\angle A = \alpha$ (co-interior $\angle$s, $BC \parallel AO$)
Now, $\angle AOC = 2\alpha$ and $\angle OAB = \alpha$
$\therefore \angle AOC = 2\angle AOB$
ii $\angle AOC + \angle AOB = 180°$ (co-interior $\angle$s, $AB \parallel OC$)
$\therefore 2\alpha + \alpha = 180°$
$\therefore \alpha = 60°$
$\therefore \angle AOC = 120°$

d **i** Let $\angle PST = \alpha$
$\angle PTS = \angle PST$ (tangents from an exterior point)
$= \alpha$
$\angle QTS = \angle PTS$ (TS bisects $\angle QTP$)
$= \alpha$
Now, $\angle QTS = \angle PST$ (both equal to α)
$\therefore TQ \parallel PS$ (alternate $\angle$s are equal)
ii $\angle TQS = \angle PST$ ($\angle$ in alternate segment)
Now, $\angle TQS = \angle QTS$ (both equal to α)
$\therefore \Delta QST$ is isosceles