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Generalized eigenvalue for even order tensors via Einstein product and its applications in multilinear control systems

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Abstract
This paper devotes to the generalized eigenvalues for even order tensors. We extend classical spectral theory for matrix pairs to the multilinear case, including the generalized Schur decomposition, the Geršgorin circle theorem, and the Bauer–Fike theorem for regular tensor pairs of even order. We introduce the backward errors and ϵ-pseudospectrums for generalized tensor eigenvalues in normwise and componentwise, respectively, and particularize the application in stability analysis for the generalized multilinear systems. By the normwise pseudospectral theory, we obtain a lower bound for the distance from a regular tensor pair to singularity, and a formulation of the distance from a reachable multilinear time invariant control system to unreachability is given.

Keywords Generalized eigenvalue · Pseudospectrum · Schur decomposition · Einstein product · Multilinear time invariant control systems

Mathematics Subject Classification 15A18 · 15A69 · 65F15 · 65F10

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1 Introduction

Tensors are multidimensional generalizations of matrices, and arise in many high-dimensional problems, for example, in high-order Markov chain (Li and Ng 2014), high-dimensional partial differential equations (Dolgov et al. 2021; Khoromskij 2015), dimensionality reduction (Vasilescu and Terzopoulos 2003), and signal processing (Cardoso 1999; De Lathauwer et al. 2007). An N-th-order tensor $\mathcal{X}$ of size $I_1 \times I_2 \times \cdots \times I_N$ is an N-dimensional array

$$\mathcal{X} = \left(\mathcal{X}_{i_1 i_2 \ldots i_N}\right), \quad \mathcal{X}_{i_1 i_2 \ldots i_N} \in \mathbb{C},\ 1 \le i_k \le I_k,\ k = 1, 2, \ldots, N.$$

Einstein product as a tensor multiplicative operation has been naturally emerged and widely used in many fields. For two even order tensors $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times J_1 \times J_2 \times \cdots \times J_N}$ and $\mathcal{B} \in \mathbb{C}^{J_1 \times J_2 \times \cdots \times J_N \times K_1 \times K_2 \times \cdots \times K_N}$, their Einstein product $\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times K_1 \times K_2 \times \cdots \times K_N}$ is defined by

$$(\mathcal{A} *_N \mathcal{B})_{i_1 i_2 \ldots i_N k_1 k_2 \ldots k_N} = \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \cdots \sum_{j_N=1}^{J_N} \mathcal{A}_{i_1 i_2 \ldots i_N j_1 j_2 \ldots j_N} \mathcal{B}_{j_1 j_2 \ldots j_N k_1 k_2 \ldots k_N},$$

where $\mathbb{C}^{L_1 \times L_2 \times \cdots \times L_N}$ denotes the set of all N-th-order tensors of size $L_1 \times L_2 \times \cdots \times L_N$ over complex field.

In recent years, Brazell et al. (2013) found that the tensor group endowed with Einstein product is isomorphic to the general linear group, and they defined the inverse of even order tensors to address multilinear systems, etc. Subsequently, a great deal of efforts has been devoted to the tensor algebraic structure and numerical multilinear algebra. For instance, the theory of generalized inverses for high-order tensors has been proposed (Ji and Wei 2018; Liang and Zheng 2019; Ma et al. 2019; Sun et al. 2016). The notions extended from linear algebra such as elementary tensor transformations, tensor unfolding rank, and tensor unfolding determinant were introduced by Liang et al. (2019). Recently, Chen et al. (2019; 2021) proposed a class of multilinear time invariant (MLTI) control systems based on Einstein product and discussed their stability, reachability, and observability.

Eigenvalue problems are all the time of great significance in computational mathematics and engineering technology. The tensor eigenvalue based on tensor–vector mode product was proposed independently by Qi (2005) and Lim (2005) in 2005.

Definition 1.1 (Qi 2005; Lim 2005) Let $\mathcal{A} \in \mathbb{R}^{n \times n \times \cdots \times n}$ be an m-th-order n-dimensional tensor. If a nonzero vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $\lambda \in \mathbb{C}$ satisfy

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},$$

then we call λ an eigenvalue of $\mathcal{A}$, and $\mathbf{x}$ its corresponding eigenvector. Here

$$\mathcal{A}\mathbf{x}^{m-1} = \left(\sum_{i_2,\ldots,i_m=1}^{n} \mathcal{A}_{i_1 i_2 \ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m}\right)_{1 \le i_1 \le n} \quad \text{and } \mathbf{x}^{[m-1]} = \left(x_i^{m-1}\right)_{1 \le i \le n}.$$

From then on, the tensor eigenvalue becomes a hot topic due to its extensive applications. In 2009, Chang et al. (2009) introduced the generalized tensor eigenvalue and gave a positive answer to the conjecture posed by Qi (2005). Later, Ding and Wei (2015) contributed to the properties and perturbations of the spectra of regular tensor pairs and particularized some related applications. Che et al. (2017) investigated the pseudospectral theory of complex tensors and tensor polynomials. He et al. (2020) and Li et al. (2019) researched the pseudospectra localization sets of tensor eigenvalues and generalized tensor eigenvalues, respectively. Generally, the eigen-problems in these literature are NP-hard (Hillar and Lim 2013). In addition, the tensor T-eigenvalue for third-order tensors based on T-product (Miao et al. 2020, 2021) is also studied in some literature; one can see, e.g., Cao and Xie (2022), Chang and Wei (2022a, b, c), Liu and Jin (2021) for more details.

However, it is worth noting that another class of tensor eigenvalue problems via Einstein product numerously arises in elastic mechanics (Itskov 2000; Mehrabadi and Cowin 1990), which can be formulated as follows.

Definition 1.2 (Cui et al. 2016; Liang et al. 2019) Let $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times I_1 \times I_2 \times \cdots \times I_N}$ be a $2N$-th-order tensor. If an N-th-order nonzero tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$ and $\lambda \in \mathbb{C}$ satisfy

$$\mathcal{A} *_N \mathcal{X} = \lambda \mathcal{X}, \tag{1}$$

then λ and $\mathcal{X}$ are eigenvalue and eigen-tensor of $\mathcal{A}$, respectively.

The tensor eigenvalue (1) via Einstein product differs from the one in Definition 1.1 and tensor T-eigenvalue in Liu and Jin (2021). For the eigen-problem (1), Cui et al. (2016) revealed the relationship with higher order singular value decomposition using the matricization of tensors, and analyzed the lower and upper bounds of eigenvalues of Toeplitz tensors. The unfolding determinant, characteristic polynomial, and the Schur decomposition for even order tensors were introduced by Liang et al. (2019). The stability of MLTI control systems is studied by the spectral radius of the coefficient tensors (Chen et al. 2021). Recently, Miao et al. (2022) defined the $\mathcal{M}$-tensor under Einstein product using the spectral radius to study the existence of the solutions for tensor Riccati equations. Chandra Rout et al. (2022) introduced the numerical range and numerical radius of even order tensors via Einstein product and related fundamental theories, and showed that the spectrum of a tensor always lies in its numerical range.

In this paper, we pay attention to the generalized eigenvalue problem via Einstein product for even order tensors.

Definition 1.3 Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times I_1 \times I_2 \times \cdots \times I_N}$. If N-th-order nonzero tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$ and $\lambda \in \mathbb{C}$ satisfy

$$\mathcal{A} *_N \mathcal{X} = \lambda \mathcal{B} *_N \mathcal{X}, \tag{2}$$

we call λ an eigenvalue of the tensor pair $\{\mathcal{A}, \mathcal{B}\}$ and $\mathcal{X}$ its corresponding eigen-tensor.

Clearly, the tensor eigenvalue (1) is exactly a special case of the generalized eigenvalue problem (2) when $\mathcal{B}$ is the identity tensor, and the generalized matrix eigenvalue is also a special case of (2) when $N = 1$. Based on the developed multilinear algebraic theory, our contributions are mainly to the spectral and pseudospectral theories for even order tensor pairs, including some elementary theorems extended from the ones for matrix pairs and some novel notions such as the componentwise ϵ-pseudospectrum of tensor pairs, and an application in a class of MLTI control systems.

The remainder of this paper is organized as follows. In Sect. 2, we present the basic theory and involved block tensors. In Sect. 3, we introduce the tensor generalized Schur decomposition and regular tensor pairs, and show that the spectral norm and Frobenius norm of tensors are compatible, and then, we generalize the Geršgorin circle theorem and the Bauer–Fike theorem for regular tensor pairs. In Sect. 4, some results of normwise and componentwise backward errors and ϵ-pseudospectrum for regular tensor pairs are presented, respectively, and a lower bound for the distance from a regular tensor pair to the nearest singular one is obtained. Section 5 devotes to an application in a class of MLTI control systems; concretely, a formulation for the distance from a reachable MLTI system to unreachability is obtained by the pesudospectral theory.

2 Preliminary

Tensors are written by calligraphy of capital letters; for instance, $\mathcal{I}$ represents the identity tensors and $\mathcal{O}$ denotes zero tensors. The sizes of tensors are indicated by boldface of capital letters, such as $\mathbf{I} = \{I_1, I_2, \ldots, I_N\}$, $\mathbf{J} = \{J_1, J_2, \ldots, J_N\}$. $|\mathbf{I}|$ and $|\mathbf{J}|$ represent the product of all elements in the sets $\mathbf{I}$ and $\mathbf{J}$, respectively. The sets of indexed indices are denoted by boldfaced small letters, such as $\mathbf{i} = \{i_1, i_2, \ldots, i_N\}$, $\mathbf{j} = \{j_1, j_2, \ldots, j_N\}$ and $\mathbf{k} = \{k_1, k_2, \ldots, k_N\}$. For the sake of simplicity, $\mathbf{1} \leq \mathbf{i} \leq \mathbf{I}$ represents $1 \leq i_k \leq I_k$ for all $k = 1, 2, \ldots, N$, $\mathbf{i} \neq \mathbf{j}$ has the meaning that $i_k \neq j_k$ for some $1 \leq k \leq N$, $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{|\mathbf{I}|})$ denotes the diagonal tensor with $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{|\mathbf{I}|})_{i_1 i_2 \ldots i_N i_1 i_2 \ldots i_N} = \lambda_{\mathrm{ivec}(\mathbf{i},\mathbf{I})}$ for all $\mathbf{1} \leq \mathbf{i} \leq \mathbf{I}$.

2.1 Basic notions

Tensor matricization is a frequently used operation in both tensor algebraic analysis and numerical computations (Brazell et al. 2013; Kolda and Bader 2009). In this paper, we denote by $\mathrm{ivec}(\cdot, \cdot) : \mathbb{Z}_+^N \times \mathbb{Z}_+^N \longrightarrow \mathbb{Z}_+$ the index mapping function in the tensor unfolding process; concretely

$$\mathrm{ivec}(\mathbf{i}, \mathbf{I}) = i_1 + \sum_{k=2}^{N} (i_k - 1) \prod_{l=1}^{k-1} I_l,$$

the unfolding process of $2N$-th-order tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times J_1 \times J_2 \times \cdots \times J_N}$ represents the transformation

$$\begin{cases} \phi : & \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times J_1 \times J_2 \times \cdots \times J_N} \longrightarrow \mathbb{C}^{|\mathbf{I}| \times |\mathbf{J}|}, \\ & \mathcal{A} \longrightarrow A \text{ with } A_{ij} = \mathcal{A}_{i_1 i_2 \ldots i_N j_1 j_2 \ldots j_N}, \end{cases}$$

where the subscripts $i = \mathrm{ivec}(\mathbf{i}, \mathbf{I})$, $j = \mathrm{ivec}(\mathbf{j}, \mathbf{J})$.

In Brazell et al. (2013), Brazell et al. showed that the above bijective mapping ϕ is an isomorphism between the general linear group $GL(\mathbb{R})$ and the fourth-order tensor group under Einstein product, wherefore some basic concepts from the matrix theory such as the orthogonality and the invertibility are generalized to tensors. These notions can also be introduced for even order tensors over complex field (Liang et al. 2019; Sun et al. 2016).

For a tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times J_1 \times J_2 \times \cdots \times J_N}$, $\mathcal{B} \in \mathbb{C}^{J_1 \times J_2 \times \cdots \times J_N \times I_1 \times I_2 \times \cdots \times I_N}$ is the *conjugate transpose* of $\mathcal{A}$ if $\mathcal{B}_{j_1 j_2 \ldots j_N i_1 i_2 \ldots i_N} = \overline{\mathcal{A}_{i_1 i_2 \ldots i_N j_1 j_2 \ldots j_N}}$ and is represented by $\mathcal{A}^H$. $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times J_1 \times J_2 \times \cdots \times J_N}$ is called a square tensor if $\mathbf{I} = \mathbf{J}$. A square tensor $\mathcal{A}$ satisfying $\mathcal{A}^H = \mathcal{A}$ is called a *Hermitian tensor*, $\mathcal{D}$ is said to be *diagonal* if all entries are zero except for its diagonal entries, the *identity tensor* $\mathcal{I}$ is a special diagonal tensor whose all diagonal entries are 1, $\mathcal{U}$ is *unitary* if $\mathcal{U} *_N \mathcal{U}^H = \mathcal{U}^H *_N \mathcal{U} = \mathcal{I}$, $\mathcal{T}$ is *upper triangular* if $\mathcal{T}_{i_1 i_2 \cdots i_N j_1 j_2 \ldots j_N} = 0$ for $\mathrm{ivec}(\mathbf{i}, \mathbf{I}) > \mathrm{ivec}(\mathbf{j}, \mathbf{I})$, $\mathcal{O}$ is a *zero tensor* whose all entries are 0. For a square tensor $\mathcal{A}$, if there exists a tensor $\mathcal{X}$ of the same size, such that $\mathcal{A} *_N \mathcal{X} = \mathcal{X} *_N \mathcal{A} = \mathcal{I}$, then $\mathcal{A}$ is said to be *invertible or nonsingular*, and $\mathcal{X}$ is the *inverse* of $\mathcal{A}$ denoted by $\mathcal{A}^{-1}$.

Definition 2.1 (*Moore–Penrose inverse* Sun et al. 2016; Wei et al. 2018) Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$. If $\mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_N\times I_1\times\cdots\times I_N}$ satisfies the four equations

$$\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A}, \qquad \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}, \qquad (\mathcal{A} *_N \mathcal{X})^H = \mathcal{A} *_N \mathcal{X},$$
$$(\mathcal{X} *_N \mathcal{A})^H = \mathcal{X} *_N \mathcal{A},$$

$\mathcal{X}$ is called the Moore–Penrose inverse of $\mathcal{A}$, denoted by $\mathcal{A}^\dagger$.

Definition 2.2 (*Null space and range space* Ji and Wei 2018) The null space and range space of $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$ are $\mathcal{N}(\mathcal{A}) := \{\mathcal{X} \in \mathbb{C}^{J_1\times J_2\times\cdots\times J_N} \big| \mathcal{A} *_N \mathcal{X} = \mathcal{O}\}$ and $\mathcal{R}(\mathcal{A}) := \{\mathcal{A} *_N \mathcal{X} \big| \mathcal{X} \in \mathbb{C}^{J_1\times J_2\times\cdots\times J_N}\}$, respectively.

Lemma 2.1 (Ji and Wei 2018) *Let $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times J_1\times J_2\times\cdots\times J_N}$ be an even order tensor. Then*

$$\dim\mathcal{N}(\mathcal{A}^H) + \dim\mathcal{R}(\mathcal{A}) = |\mathbf{I}|\,, \quad \dim\mathcal{N}(\mathcal{A}) + \dim\mathcal{R}(\mathcal{A}) = |\mathbf{J}|\,.$$

Definition 2.3 (*Tensor rank* Chen et al. 2019) For a tensor $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times J_1\times J_2\times\cdots\times J_N}$, the rank of $\mathcal{A}$ is

$$\operatorname{rank}_U(\mathcal{A}) := \dim\mathcal{R}(\mathcal{A}).$$

Remark 2.1 The above tensor rank is equivalent to the tensor unfolding rank defined as the rank of its unfolding matrix (Chen et al. 2019; Liang et al. 2019), i.e., $\operatorname{rank}_U(\mathcal{A}) = \operatorname{rank}(\phi(\mathcal{A}))$.

Definition 2.4 (*Tensor unfolding determinant* Liang et al. 2019) Let $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$. The unfolding determinant of $\mathcal{A}$ is $\det_U(\mathcal{A}) := \det(\phi(\mathcal{A}))$, or equivalently

$$\det_U(\mathcal{A}) := \sum_{\pi\in\mathbb{S}_{|\mathbf{I}|}} (-1)^c \prod_{k=1}^{|\mathbf{I}|} \mathcal{A}_{i_1^{(k)} i_2^{(k)} \ldots i_N^{(k)} j_1^{\pi(k)} j_2^{\pi(k)} \ldots j_N^{\pi(k)}},$$

where $\mathbb{S}_{|\mathbf{I}|}$ is the symmetric group, the row index $\mathbf{i}^{(k)} = \{i_1^{(k)}, i_2^{(k)}, \ldots, i_N^{(k)}\}$ and column index $\mathbf{j}^{(\pi(k))} = \{j_1^{\pi(k)}, j_2^{\pi(k)}, \ldots, j_N^{\pi(k)}\}$ satisfy $\operatorname{ivec}(\mathbf{i}^{(k)}, \mathbf{I}) = k$ and $\operatorname{ivec}(\mathbf{j}^{(\pi(k))}, \mathbf{I}) = \pi(k)$, respectively. The superscript c denotes the total number of the inversions in the sequence $\mathbf{j}^{(\pi(1))}, \mathbf{j}^{(\pi(2))}, \ldots, \mathbf{j}^{(\pi(|\mathbf{I}|))}$ contrasting to the natural sequence $\mathbf{i}^{(1)}, \mathbf{i}^{(2)}, \ldots, \mathbf{i}^{(|\mathbf{I}|)}$.

Lemma 2.2 (Liang et al. 2019) *Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$. Then*

(1) $\det_U\left(\mathcal{A}^H\right) = \det_U(\mathcal{A})\,,\ \det_U(\mathcal{A} *_N \mathcal{B}) = \det_U(\mathcal{A})\det_U(\mathcal{B})$,
(2) $\mathcal{A}$ *is invertible* $\Longleftrightarrow \det_U(\mathcal{A}) \neq 0 \Longleftrightarrow \operatorname{rank}_U(\mathcal{A}) = |\mathbf{I}|$.

The singular value decomposition (SVD) of a fourth-order square tensor has been proposed in Brazell et al. (2013). For any even order tensors, we can perform their SVD according to the following lemma.

Lemma 2.3 (Tensor SVD Sun et al. 2016) *For a tensor $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times J_1\times J_2\times\cdots\times J_N}$, the* SVD *of $\mathcal{A}$ has the form*

$$\mathcal{A} = \mathcal{U} *_N \mathcal{D} *_N \mathcal{V}^H,$$

where $\mathcal{U} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ and $\mathcal{V} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ are unitary tensors, $\mathcal{D} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$ is a diagonal tensor with entries $\mathcal{D}_{i_1\ldots i_N i_1\ldots i_N}$ called the singular values.

The outer product of two tensors $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$, $\mathcal{Y} \in \mathbb{C}^{J_1 \times J_2 \times \cdots \times J_N}$ is the $2N$-th-order tensor $\mathcal{X} \circ \mathcal{Y} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times J_1 \times J_2 \times \cdots \times J_N}$, whose elements

$$(\mathcal{X} \circ \mathcal{Y})_{i_1 i_2 \dots i_N j_1 j_2 \dots j_N} = \mathcal{X}_{i_1 i_2 \dots i_N} \mathcal{Y}_{j_1 j_2 \dots j_N}, \quad \mathbf{1} \leq \mathbf{i} \leq \mathbf{I}, \mathbf{1} \leq \mathbf{j} \leq \mathbf{J}.$$

The inner product of two tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{J_1 \times J_2 \times \dots \times J_N}$ is

$$\langle \mathcal{X}, \mathcal{Y} \rangle = \sum_{\mathbf{1} \leq \mathbf{j} \leq \mathbf{J}} \overline{\mathcal{X}_{j_1 j_2 \dots j_N}} \cdot \mathcal{Y}_{j_1 j_2 \dots j_N},$$

the Frobenius norm of $\mathcal{X} \in \mathbb{C}^{J_1 \times J_2 \times \cdots \times J_N}$ is $\sqrt{\langle \mathcal{X}, \mathcal{X} \rangle}$, that is

$$\|\mathcal{X}\|_F = \sqrt{\sum_{\mathbf{1} \leq \mathbf{j} \leq \mathbf{J}} \left|\mathcal{X}_{j_1 j_2 \dots j_N}\right|^2}.$$

Remark 2.2 The tensor Frobenius norm is invariant under Einstein product by an unitary tensor, i.e., $\|\mathcal{U} *_N \mathcal{X}\|_F = \|\mathcal{X}\|_F$ if $\mathcal{U}$ is a $2N$-th-order unitary tensor.

Definition 2.5 (*Spectral norm* Ma et al. 2019) Let $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times J_1 \times J_2 \times \cdots \times J_N}$. Then, the spectral norm of $\mathcal{A}$ is defined as

$$\|\mathcal{A}\|_2 := \sigma_{\max}(\mathcal{A}) = \sqrt{\lambda_{\max}\left(\mathcal{A}^H *_N \mathcal{A}\right)},$$

where $\lambda_{\max}(\cdot)$ and $\sigma_{\max}(\cdot)$ denote the largest eigenvalue and largest singular value, respectively.

Lemma 2.4 (Ma et al. 2019) *Let $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times J_1 \times J_2 \times \cdots \times J_N}$. Then, the spectral norm is invariant under the unfolding isomorphism ϕ, i.e., $\|\mathcal{A}\|_2 = \|\phi(\mathcal{A})\|_2$, the spectral norm of unfolding matrix.*

Definition 2.6 (*Semi-positive definite tensor* Chen et al. 2021) A Hermitian tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times I_1 \times I_2 \times \cdots \times I_N}$ is semi-positive definite if

$$\mathcal{X}^H *_N \mathcal{A} *_N \mathcal{X} = \langle \mathcal{X}, \mathcal{A} *_N \mathcal{X} \rangle \geq 0$$

for any nonzero $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$, positive definite if the above inequality is strict.

2.2 Block tensors

Ragnarsson and Van Loan (2012) developed basic notations and operations for block tensors that are suitable for the development of block tensor algorithms, and analyzed the unfolding patterns of block tensors. Sun et al. (2016) and Miao et al. (2022) proposed an approach of concatenation of block tensors which might introduce lots of zero entries. Here, we use a more compact concatenated mode of subtensors referring to Chen et al. (2019) but mildly weaken the requirement that two subtensors must be of the same size.

Definition 2.7 Let $2N$-th-order tensors $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times J_1 \times J_2 \times \cdots \times J_N}$, $\mathcal{B} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times \overline{J}_1 \times J_2 \times \cdots \times J_N}$, whose $(N+1)$-th dimension can be different. Then, the row block tensor

$$\left[\mathcal{A}\ \mathcal{B}\right] \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times (J_1 + \overline{J}_1) \times J_2 \times \cdots \times J_N}$$

concatenated by $\mathcal{A}$ and $\mathcal{B}$ is

$$[\mathcal{A}\ \mathcal{B}]_{i_1\dots i_N j_1\dots j_N} := \begin{cases} \mathcal{A}_{i_1\dots i_N j_1\dots j_N}, & i_k = 1,\dots,I_k,\ j_k = 1,\dots,J_k\ \forall k, \\ \mathcal{B}_{i_1\dots i_N (j_1-J_1) j_2\dots j_N}, & i_k = 1,\dots,I_k,\ \forall k,\ j_k = 1,\dots,J_k \text{ for } k \neq 1, \\ & j_1 = J_1+1,\dots,J_1+\overline{J}_1. \end{cases}$$

Let $\mathcal{C} \in \mathbb{C}^{J_1\times J_2\times\cdots\times J_N\times I_1\times I_2\times\cdots\times I_N}$, $\mathcal{D} \in \mathbb{C}^{\overline{J}_1\times J_2\times\cdots\times J_N\times I_1\times I_2\times\cdots\times I_N}$. The column block tensor of $\mathcal{C}$ and $\mathcal{D}$ is defined as

$$\begin{bmatrix}\mathcal{C}\\ \mathcal{D}\end{bmatrix} = \left[\mathcal{C}^H\ \mathcal{D}^H\right]^H \in \mathbb{C}^{(J_1+\overline{J}_1)\times J_2\times\cdots\times J_N\times I_1\times I_2\times\cdots\times I_N}.$$

Remark 2.3 Let N-th-order tensors $\mathcal{X} \in \mathbb{C}^{J_1\times J_2\times\cdots\times J_N}$, $\mathcal{Y} \in \mathbb{C}^{\overline{J}_1\times J_2\cdots\times J_N}$. Their column block tensor $\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix} \in \mathbb{C}^{(J_1+\overline{J}_1)\times J_2\times\cdots\times J_N}$ is analogous to the above $2N$-th-order column block tensor by setting $\mathbf{I} = \{1, 1, \dots, 1\}$.

Let $s = qr$, and $\Pi_{q,r} \in \mathbb{C}^{s\times s}$ be the perfect shuffle permutation (Golub and Van Loan 2013; Ragnarsson and Van Loan 2012) defined by

$$\Pi_{q,r} z = \begin{bmatrix} z(1:r:s) \\ z(2:r:s) \\ \vdots \\ z(r:r:s) \end{bmatrix}$$

for any $z \in \mathbb{C}^s$. We present the following proposition as a corollary of Ragnarsson and Van Loan (2012, Theorem 3.3).

Proposition 2.1 *Let* $2N$-th-*order tensors* $\mathcal{A}, \mathcal{B}$ *be stated as in Definition 2.7. For* $k = 1, 2, \dots, N$, *set* $L_k = (J_1 + \overline{J}_1)J_2 \dots J_k$, *and we define*

$$Q_1 = I_{L_N}, \quad Q_k = I_{L_N/L_k} \otimes \left(\begin{bmatrix} \Pi_{J_1\dots J_{k-1}, J_k} & O \\ O & \Pi_{\overline{J}_1\dots J_{k-1}, J_k} \end{bmatrix} \Pi_{J_k,(J_1+\overline{J}_1)\dots J_{k-1}}\right),\ k = 2, 3, \dots, N,$$

where '⊗' is the Kronecker product (Golub and Van Loan 2013). Then, there is a permutation matrix $P = Q_N \dots Q_2 Q_1$, *such that*

$$\phi\left([\mathcal{A}\ \mathcal{B}]\right) = \left[\phi(\mathcal{A})\ \phi(\mathcal{B})\right] P.$$

Accordingly

$$\operatorname{rank}_U([\mathcal{A}\ \mathcal{B}]) = \operatorname{rank}(\left[\phi(\mathcal{A})\ \phi(\mathcal{B})\right]).$$

For example, $\mathcal{A} \in \mathbb{C}^{2\times 2\times 2\times 2}$, $\mathcal{B} \in \mathbb{C}^{2\times 2\times 1\times 2}$

$$\phi(\mathcal{A}) = \begin{bmatrix} \mathcal{A}_{1111} & \mathcal{A}_{1121} & \mathcal{A}_{1112} & \mathcal{A}_{1122} \\ \mathcal{A}_{2111} & \mathcal{A}_{2121} & \mathcal{A}_{2112} & \mathcal{A}_{2122} \\ \mathcal{A}_{1211} & \mathcal{A}_{1221} & \mathcal{A}_{1212} & \mathcal{A}_{1222} \\ \mathcal{A}_{2211} & \mathcal{A}_{2221} & \mathcal{A}_{2212} & \mathcal{A}_{2222} \end{bmatrix}, \quad \phi(\mathcal{B}) = \begin{bmatrix} \mathcal{B}_{1111} & \mathcal{B}_{1112} \\ \mathcal{B}_{2111} & \mathcal{B}_{2112} \\ \mathcal{B}_{1211} & \mathcal{B}_{1212} \\ \mathcal{B}_{2211} & \mathcal{B}_{2212} \end{bmatrix};$$

from Definition 2.7 of the row block tensor, we obtain

$$\phi\left([\mathcal{A}\ \mathcal{B}]\right) = \begin{bmatrix} \mathcal{A}_{1111} & \mathcal{A}_{1121} & \mathcal{B}_{1111} & \mathcal{A}_{1112} & \mathcal{A}_{1122} & \mathcal{B}_{1112} \\ \mathcal{A}_{2111} & \mathcal{A}_{2121} & \mathcal{B}_{2111} & \mathcal{A}_{2112} & \mathcal{A}_{2122} & \mathcal{B}_{2112} \\ \mathcal{A}_{1211} & \mathcal{A}_{1221} & \mathcal{B}_{1211} & \mathcal{A}_{1212} & \mathcal{A}_{1222} & \mathcal{B}_{1212} \\ \mathcal{A}_{2211} & \mathcal{A}_{2221} & \mathcal{B}_{2211} & \mathcal{A}_{2212} & \mathcal{A}_{2222} & \mathcal{B}_{2212} \end{bmatrix} = \left[\phi(\mathcal{A})\ \phi(\mathcal{B})\right] \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \end{bmatrix}.$$

Associating with Einstein product, there are some elegant product properties for block tensors.

Proposition 2.2 *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{X}, \mathcal{Y}$ be stated as in Definition 2.7. Then*

1. $\left[\mathcal{P} *_N \mathcal{A}\ \mathcal{P} *_N \mathcal{B}\right] = \mathcal{P} *_N \left[\mathcal{A}\ \mathcal{B}\right]$, for $\mathcal{P} \in \mathbb{C}^{J_1 \times J_2 \times \cdots \times J_N \times I_1 \times I_2 \times \cdots \times I_N}$.
2. $\begin{bmatrix} \mathcal{C} *_N \mathcal{Q} \\ \mathcal{D} *_N \mathcal{Q} \end{bmatrix} = \begin{bmatrix} \mathcal{C} \\ \mathcal{D} \end{bmatrix} *_N \mathcal{Q}$, for $\mathcal{Q} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times J_1 \times J_2 \times \cdots \times J_N}$.
3. $\left[\mathcal{A}\ \mathcal{B}\right] *_N \begin{bmatrix} \mathcal{C} \\ \mathcal{D} \end{bmatrix} = \mathcal{A} *_N \mathcal{C} + \mathcal{B} *_N \mathcal{D}$, $\left[\mathcal{A}\ \mathcal{B}\right] *_N \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} = \mathcal{A} *_N \mathcal{X} + \mathcal{B} *_N \mathcal{Y}$.

Remark 2.4 When $N = 1$, the pattern of tensor concatenation and these operative properties are consistent with the classical block matrices.

3 Spectral theory for tensor pairs

In this section, we extend some classical fundamental theorems for matrix pairs to the generalized tensor eigenvalue (2), such as the generalized Schur decomposition (Moler and Stewart 1973; Zhang et al. 2022), regular matrix pairs (Stewart and Sun 1990), the Geršgorin circle theorem (Golub and Van Loan 2013; Stewart and Sun 1990; Varga 2004), and the Bauer–Fike theorem (Bauer and Fike 1960; Elsner and Sun 1982; Shi and Wei 2012).

Proposition 3.1 *The spectral norm of an even order tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times J_1 \times J_2 \times \cdots \times J_N}$ can be equivalently defined as the operator norm induced from the Frobenius norm $\|\cdot\|_F$, that is*

$$\|\mathcal{A}\|_2 = \max_{\|\mathcal{X}\|_F = 1} \|\mathcal{A} *_N \mathcal{X}\|_F\,.$$

Proof According to Lemma 2.3, $\mathcal{A}$ has the tensor SVD form $\mathcal{A} = \mathcal{U} *_N \mathcal{D} *_N \mathcal{V}^H$, where $\mathcal{U}, \mathcal{V}$ are unitary tensors. Thus, for all $\mathcal{X} \in \mathbb{C}^{J_1 \times \cdots \times J_N}$ with $\|\mathcal{X}\|_F = 1$

$$\|\mathcal{A} *_N \mathcal{X}\|_F = \left\|\mathcal{U} *_N \mathcal{D} *_N \mathcal{V}^H *_N \mathcal{X}\right\|_F = \left\|\mathcal{D} *_N \mathcal{V}^H *_N \mathcal{X}\right\|_F.$$

Let $\mathcal{Y} = \mathcal{V}^H *_N \mathcal{X}$, then

$$\begin{aligned} \|\mathcal{A} *_N \mathcal{X}\|_F = \|\mathcal{D} *_N \mathcal{Y}\|_F &= \left(\sum_{\mathbf{1} \le \mathbf{j} \le \mathbf{J}} \left|\mathcal{D}_{j_1 \ldots j_N j_1 \cdots j_N} \cdot \mathcal{Y}_{j_1 \ldots j_N}\right|^2\right)^{\frac{1}{2}} \\ &\le \max_{\mathbf{1} \le \mathbf{j} \le \mathbf{J}} \mathcal{D}_{j_1 \ldots j_N j_1 \ldots j_N} \cdot \|\mathcal{Y}\|_F = \sigma_{\max}(\mathcal{A}). \end{aligned}$$

On the other hand, suppose $\mathcal{D}_{k_1 \ldots k_N k_1 \ldots k_N} = \sigma_{\max}(\mathcal{A})$. Set $\mathcal{Y} \in \mathbb{C}^{J_1 \times \cdots \times J_N}$ with $\mathcal{Y}_{k_1 \ldots k_N} = 1$ and $\mathcal{Y}_{j_1 \ldots j_N} = 0$ for all $\mathbf{j} \neq \mathbf{k}$, $\mathcal{X} = \mathcal{V} *_N \mathcal{Y}$, we have

$$\|\mathcal{A} *_N \mathcal{X}\|_F = \|\mathcal{D} *_N \mathcal{Y}\|_F = \sigma_{\max}(\mathcal{A}).$$

Therefore, $\max_{\|\mathcal{X}\|_F = 1} \|\mathcal{A} *_N \mathcal{X}\|_F = \sigma_{\max}(\mathcal{A}) = \|\mathcal{A}\|_2\,.$ □

Remark 3.1 As a result, the tensor spectral norm and the Frobenius norm are compatible, i.e., $\|\mathcal{A} *_N \mathcal{X}\|_F \le \|\mathcal{A}\|_2 \|\mathcal{X}\|_F$ for any $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times J_1 \times J_2 \times \cdots \times J_N}$, $\mathcal{X} \in \mathbb{C}^{J_1 \times J_2 \times \cdots \times J_N}$, and this inequality is frequently used in the sequel.

Remark 3.2 Similar to the above proof, it is easily verified that $\min_{\|\mathcal{X}\|_F = 1} \|\mathcal{A} *_N \mathcal{X}\|_F = \frac{1}{\|\mathcal{A}^\dagger\|_2} = \sigma_{\min}(\mathcal{A})$, the smallest singular value of $\mathcal{A}$.

Tensor generalized Schur decomposition

For generalized eigenvalue (2), it follows from Lemmas 2.1 and 2.2 that:

$$\begin{aligned}\exists \mathcal{X} \neq \mathcal{O},\ (\mathcal{A}-\lambda\mathcal{B}) *_N \mathcal{X} = \mathcal{O}. \ &\Leftrightarrow \mathcal{N}(\mathcal{A}-\lambda\mathcal{B}) \neq \{\mathcal{O}\},\ \mathrm{rank}_U(\mathcal{A}-\lambda\mathcal{B}) < |\mathbf{I}|\,.\\ &\Leftrightarrow \det{}_U\,(\mathcal{A}-\lambda\mathcal{B}) = 0.\end{aligned}$$

Therefore, the finite spectrum of a tensor pair $\{\mathcal{A}, \mathcal{B}\}$ constituted of its finite eigenvalues

$$\Lambda(\mathcal{A}, \mathcal{B}) := \{\lambda \in \mathbb{C} \text{ satisfies (2)}\} = \{\lambda \in \mathbb{C} |\ \det{}_U\,(\mathcal{A}-\lambda\mathcal{B}) = 0\}\,.$$

The Schur decomposition for square tensors of even order has been established (Liang et al. 2019). To get a better understanding of the generalized tensor eigenvalue (2), we extend the generalized Schur decomposition to the tensor pairs.

Theorem 3.1 (Tensor generalized Schur decomposition) *Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$ be two square tensors. Then, there exist unitary tensors $\mathcal{P}$ and $\mathcal{Q}$, such that $\mathcal{P}^H *_N \mathcal{A} *_N \mathcal{Q} = \mathcal{T}$ and $\mathcal{P}^H *_N \mathcal{B} *_N \mathcal{Q} = \mathcal{S}$ are upper triangular. If for some $\mathbf{1} \le \mathbf{i} \le \mathbf{I}$, $\mathcal{T}_{i_1 i_2 \ldots i_N i_1 i_2 \ldots i_N}$ and $\mathcal{S}_{i_1 i_2 \ldots i_N i_1 i_2 \ldots i_N}$ are both zero, then $\Lambda(\mathcal{A}, \mathcal{B}) = \mathbb{C}$. Otherwise*

$$\Lambda(\mathcal{A}, \mathcal{B}) = \left\{ \frac{\mathcal{T}_{i_1 i_2 \ldots i_N i_1 i_2 \ldots i_N}}{\mathcal{S}_{i_1 i_2 \ldots i_N i_1 i_2 \ldots i_N}} \middle| \mathcal{S}_{i_1 i_2 \ldots i_N i_1 i_2 \ldots i_N} \neq 0 \right\}.$$

Proof Let $A = \phi(\mathcal{A}),\ B = \phi(\mathcal{B}) \in \mathbb{C}^{|\mathbf{I}|\times|\mathbf{I}|}$. From the generalized Schur decomposition for the matrix pair $\{A, B\}$, there exist unitary matrices P and Q, such that $P^H A Q = T$ and $P^H B Q = S$ are upper triangular. We can fold P and Q into tensors by the inverse of unfolding process, i.e., $\mathcal{P} = \phi^{-1}(P)$, $\mathcal{Q} = \phi^{-1}(Q)$, and let $\mathcal{T} = \phi^{-1}(T)$, $\mathcal{S} = \phi^{-1}(S)$. Since the mapping ϕ^{-1} is an isomorphism, we obtain

$$\mathcal{P}^H *_N \mathcal{A} *_N \mathcal{Q} = \mathcal{T}, \quad \mathcal{P}^H *_N \mathcal{B} *_N \mathcal{Q} = \mathcal{S},$$

where $\mathcal{P}, \mathcal{Q} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ are unitary, $\mathcal{T}, \mathcal{S} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ are upper triangular tensors. From Lemma 2.2 (1), we further obtain

$$\det{}_U\,(\mathcal{A}-\lambda\mathcal{B}) = \det{}_U\left(\mathcal{P} *_N (\mathcal{T}-\lambda\mathcal{S}) *_N \mathcal{Q}^H\right) = \det{}_U(\mathcal{P})\det{}_U(\mathcal{T}-\lambda\mathcal{S})\det{}_U(\mathcal{Q});$$

therefore, $\Lambda(\mathcal{A}, \mathcal{B}) = \Lambda(\mathcal{T}, \mathcal{S})$. Moreover, from the definition of tensor unfolding determinant

$$\det{}_U(\mathcal{T}-\lambda\mathcal{S}) = \prod_{\mathbf{1}\le\mathbf{i}\le\mathbf{I}} \left(\mathcal{T}_{i_1\ldots i_N i_1\ldots i_N} - \lambda\mathcal{S}_{i_1\ldots i_N i_1\ldots i_N}\right).$$

If for some $\mathbf{1} \le \mathbf{i} \le \mathbf{I}$, $\mathcal{T}_{i_1\ldots i_N i_1\ldots i_N}$ and $\mathcal{S}_{i_1\ldots i_N i_1\ldots i_N}$ are both zero, then $\det_U\,(\mathcal{A}-\lambda\mathcal{B}) \equiv 0$, we have $\Lambda(\mathcal{A}, \mathcal{B}) = \mathbb{C}$; otherwise, $\Lambda(\mathcal{A}, \mathcal{B}) = \left\{\mathcal{T}_{i_1\cdots i_N i_1\cdots i_N}/\mathcal{S}_{i_1\ldots i_N i_1\ldots i_N} | \mathcal{S}_{i_1\ldots i_N i_1\ldots i_N} \neq 0\right\}$. □

The generalized Schur decomposition sheds light on that the spectrum might be the whole complex plane $\mathbb{C}$, then we define the following two classes of tensor pairs to distinguish such cases.

Definition 3.1 (*Regular tensor pair*) Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$. If

$$\det{}_U(\mathcal{A}-\lambda\mathcal{B}) \not\equiv 0, \quad \lambda \in \mathbb{C},$$

then we call $\{\mathcal{A}, \mathcal{B}\}$ a regular tensor pair. Otherwise

$$\det_U(\mathcal{A} - \lambda\mathcal{B}) \equiv 0, \quad \lambda \in \mathbb{C},$$

we call $\{\mathcal{A}, \mathcal{B}\}$ a singular tensor pair.

In the following, we focus on the regular tensor pairs, and extend the Geršgorin-type theorem and the Bauer–Fike theorem for the regular tensor pairs.

Geršgorin circle theorem

From the tensor generalized Schur decomposition, the number of finite eigenvalues counting multiplicities of a regular tensor pair maybe smaller than $|\mathbf{I}|$. However, it is reasonable to take into account these infinite eigenvalues, i.e., consider the spectrum over the extended complex plane $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. To this end, we rewrite the generalized eigenvalue problem (2) to the homogeneous form

$$\beta\mathcal{A} *_N \mathcal{X} = \alpha\mathcal{B} *_N \mathcal{X}, \tag{3}$$

and introduce some notions in Elsner and Sun (1982), Stewart and Sun (1990) to cope with the possible infinite eigenvalues.

The projective complex plane (Stewart and Sun 1990)

$$\mathbb{G}_{1,2} := \{[\alpha, \beta] \neq [0, 0] |\ \alpha, \beta \in \mathbb{C}\},$$

where $[\alpha, \beta]$ is a representative element in $\mathbb{G}_{1,2}$, that is, $[\tilde{\alpha}, \tilde{\beta}_1]$ and $[\alpha, \beta]$ denote the same point in $\mathbb{G}_{1,2}$, if there is a nonzero complex number γ, such that $[\tilde{\alpha}, \tilde{\beta}] = \gamma[\alpha, \beta]$.

Remark 3.3 Every point $z \in \bar{\mathbb{C}}$ can be represented by homogeneous coordinate $[\alpha, \beta] \neq [0, 0]$, $[\alpha, \beta]$ denotes a finite point $z = \alpha/\beta$ when $\beta \neq 0$, the infinite point ∞ when $\beta = 0$ (Stewart and Sun 1990).

Definition 3.2 (*Chordal metric* Elsner and Sun 1982) The chordal metric measuring the distance of two points on the projective complex plane $\mathbb{G}_{1,2}$ is

$$\rho\left([\alpha_1, \beta_1], [\alpha_2, \beta_2]\right) := \frac{|\alpha_1\beta_2 - \beta_1\alpha_2|}{\sqrt{|\alpha_1|^2 + |\beta_1|^2}\sqrt{|\alpha_2|^2 + |\beta_2|^2}}.$$

Taking the infinite eigenvalues into account, we denote the spectrum of a regular tensor pair $\{\mathcal{A}, \mathcal{B}\}$ by

$$\bar{\Lambda}(\mathcal{A}, \mathcal{B}) := \{[\alpha, \beta] \in \mathbb{G}_{1,2} \text{ satisfies (3)}\} = \{[\alpha, \beta] \in \mathbb{G}_{1,2} |\ \det_U\left(\beta\mathcal{A} - \alpha\mathcal{B}\right) = 0\}.$$

Next, we derive the Geršgorin-type theorems for generalized eigenvalue (3).

Theorem 3.2 *Let $\{\mathcal{A}, \mathcal{B}\}$ and $\{\mathcal{C}, \mathcal{D}\}$ be regular tensor pairs, and*

$$\begin{aligned}\mathfrak{D}(\mathcal{A}, \mathcal{B}) := \Big\{[\alpha, \beta] \in \mathbb{G}_{1,2} |\ &\left\|(\beta\mathcal{C} - \alpha\mathcal{D})^{-1} *_N [\beta(\mathcal{C} - \mathcal{A}) - \alpha(\mathcal{D} - \mathcal{B})]\right\|_2 \\ &\geq 1, \text{ or } \det_U(\beta\mathcal{C} - \alpha\mathcal{D}) = 0\Big\}.\end{aligned}$$

Then

$$\bar{\Lambda}(\mathcal{A}, \mathcal{B}) \subset \mathfrak{D}(\mathcal{A}, \mathcal{B}).$$

Proof Take any $[\alpha, \beta] \in \bar{\Lambda}(\mathcal{A}, \mathcal{B})$. If $[\alpha, \beta] \in \bar{\Lambda}(\mathcal{C}, \mathcal{D})$, $\det_U(\beta\mathcal{C} - \alpha\mathcal{D}) = 0$, the conclusion obviously holds. Suppose $[\alpha, \beta] \notin \bar{\Lambda}(\mathcal{C}, \mathcal{D})$, $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ is an eigen-tensor of $\{\mathcal{A}, \mathcal{B}\}$ belonging to $[\alpha, \beta]$. Then, $\beta\mathcal{C} - \alpha\mathcal{D}$ is invertible and

$$\beta\mathcal{C} *_N \mathcal{X} - \alpha\mathcal{D} *_N \mathcal{X} = [\beta(\mathcal{C} - \mathcal{A}) - \alpha(\mathcal{D} - \mathcal{B})] *_N \mathcal{X};$$

thus

$$\mathcal{X} = (\beta\mathcal{C} - \alpha\mathcal{D})^{-1} *_N [\beta(\mathcal{C} - \mathcal{A}) - \alpha(\mathcal{D} - \mathcal{B})] *_N \mathcal{X}.$$

Taking the Frobenius norm of both sides of above equation, we obtain from Proposition 3.1 that

$$\left\|(\beta\mathcal{C} - \alpha\mathcal{D})^{-1} *_N [\beta(\mathcal{C} - \mathcal{A}) - \alpha(\mathcal{D} - \mathcal{B})]\right\|_2 \geq 1.$$

□

Theorem 3.3 *Let $\{\mathcal{A}, \mathcal{B}\}$ be a regular tensor pair, and $\left(\mathcal{A}_{i_1 \ldots i_N i_1 \ldots i_N}, \mathcal{B}_{i_1 \ldots i_N i_1 \ldots i_N}\right) \neq (0, 0)$ for all $\mathbf{1} \leq \mathbf{i} \leq \mathbf{I}$. Denote $\mathfrak{D}_{\mathbf{i}}(\mathcal{A}, \mathcal{B}) := \{[\alpha, \beta] \in \mathbb{G}_{1,2} |\ \left|\beta\mathcal{A}_{i_1 \ldots i_N i_1 \ldots i_N} - \alpha\mathcal{B}_{i_1 \ldots i_N i_1 \ldots i_N}\right| \leq \sum_{\mathbf{j} \neq \mathbf{i}} \left|\beta\mathcal{A}_{i_1 \cdots i_N j_1 \ldots j_N} - \alpha\mathcal{B}_{i_1 \cdots i_N j_1 \ldots j_N}\right|\}$, then*

$$\bar{\Lambda}(\mathcal{A}, \mathcal{B}) \subseteq \bigcup_{\mathbf{1} \leq \mathbf{i} \leq \mathbf{I}} \mathfrak{D}_{\mathbf{i}}(\mathcal{A}, \mathcal{B}).$$

Proof For any $[\alpha, \beta] \in \bar{\Lambda}(\mathcal{A}, \mathcal{B})$, there exists a nonzero eigen-tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$, such that

$$\beta\mathcal{A} *_N \mathcal{X} = \alpha\mathcal{B} *_N \mathcal{X};$$

in the componentwise form

$$\beta \sum_{\mathbf{1} \leq \mathbf{j} \leq \mathbf{I}} \mathcal{A}_{k_1 \ldots k_N j_1 \ldots j_N} \mathcal{X}_{j_1 \ldots j_N} = \alpha \sum_{\mathbf{1} \leq \mathbf{j} \leq \mathbf{I}} \mathcal{B}_{k_1 \ldots k_N j_1 \ldots j_N} \mathcal{X}_{j_1 \ldots j_N}, \quad \mathbf{1} \leq \mathbf{k} \leq \mathbf{I}.$$

Since $\mathcal{X}$ is nonzero, there is a row index $\mathbf{i} = \{i_1 \ldots i_N\}$, such that $\left|\mathcal{X}_{i_1 \ldots i_N}\right| = \max_{\mathbf{1} \leq \mathbf{j} \leq \mathbf{I}} \left|\mathcal{X}_{j_1 \ldots j_N}\right| > 0$. For this index $\mathbf{i}$, we have

$$\beta \sum_{\mathbf{1} \leq \mathbf{j} \leq \mathbf{I}} \mathcal{A}_{i_1 \ldots i_N j_1 \ldots j_N} \mathcal{X}_{j_1 \ldots j_N} = \alpha \sum_{\mathbf{1} \leq \mathbf{j} \leq \mathbf{I}} \mathcal{B}_{i_1 \ldots i_N j_1 \ldots j_N} \mathcal{X}_{j_1 \ldots j_N};$$

equivalently,

$$\left(\beta\mathcal{A}_{i_1 \cdots i_N i_1 \cdots i_N} - \alpha\mathcal{B}_{i_1 \cdots i_N i_1 \cdots i_N}\right) \mathcal{X}_{i_1 \cdots i_N} = \sum_{\mathbf{j} \neq \mathbf{i}} \left(\alpha\mathcal{B}_{i_1 \cdots i_N j_1 \cdots j_N} - \beta\mathcal{A}_{i_1 \cdots i_N j_1 \cdots j_N}\right) \mathcal{X}_{j_1 \cdots j_N};$$

taking the absolute value in both sides of the above equation and using the triangle inequality gives

$$\left|\beta\mathcal{A}_{i_1 \ldots i_N i_1 \ldots i_N} - \alpha\mathcal{B}_{i_1 \ldots i_N i_1 \ldots i_N}\right| \left|\mathcal{X}_{i_1 \ldots i_N}\right| \leq \sum_{\mathbf{j} \neq \mathbf{i}} \left|\alpha\mathcal{B}_{i_1 \ldots i_N j_1 \ldots j_N} - \beta\mathcal{A}_{i_1 \ldots i_N j_1 \ldots j_N}\right| \left|\mathcal{X}_{j_1 \ldots j_N}\right|,$$

divided by $\left|\mathcal{X}_{i_1 \ldots i_N}\right|$, it clearly follows that $[\alpha, \beta] \in \mathfrak{D}_{\mathbf{i}}(\mathcal{A}, \mathcal{B})$. By the arbitrariness of $[\alpha, \beta]$ taken in $\bar{\Lambda}(\mathcal{A}, \mathcal{B})$, the conclusion follows. □

Moreover, by the triangle inequality and Cauchy's inequality

$$\begin{aligned}
&\sum_{\mathbf{j}\neq\mathbf{i}} |\beta\mathcal{A}_{i_1\ldots i_N j_1\ldots j_N} - \alpha\mathcal{B}_{i_1\ldots i_N j_1\ldots j_N}| \\
&\quad \leq |\beta| \sum_{\mathbf{j}\neq\mathbf{i}} \left|\mathcal{A}_{i_1\ldots i_N j_1\ldots j_N}\right| + |\alpha| \sum_{\mathbf{j}\neq\mathbf{i}} \left|\mathcal{B}_{i_1\ldots i_N j_1\ldots j_N}\right| \\
&\quad \leq \sqrt{|\alpha|^2 + |\beta|^2} \sqrt{\left(\sum_{\mathbf{j}\neq\mathbf{i}} \left|\mathcal{A}_{i_1\ldots i_N j_1\ldots j_N}\right|\right)^2 + \left(\sum_{\mathbf{j}\neq\mathbf{i}} \left|\mathcal{B}_{i_1\ldots i_N j_1\ldots j_N}\right|\right)^2}.
\end{aligned}$$

Denote the disks

$$G_{\mathbf{i}}(\mathcal{A}, \mathcal{B}) := \left\{[\alpha, \beta] \in \mathbb{G}_{1,2} |\ \rho\left([\alpha, \beta], [\mathcal{A}_{i_1\ldots i_N i_1\ldots i_N}, \mathcal{B}_{i_1\ldots i_N i_1\ldots i_N}]\right) \leq \theta_{\mathbf{i}}(\mathcal{A}, \mathcal{B})\right\}, \ \mathbf{1} \leq \mathbf{i} \leq \mathbf{I},$$

where

$$\theta_{\mathbf{i}}(\mathcal{A}, \mathcal{B}) = \sqrt{\frac{\left(\sum_{\mathbf{j}\neq\mathbf{i}} \left|\mathcal{A}_{i_1\ldots i_N j_1\ldots j_N}\right|\right)^2 + \left(\sum_{\mathbf{j}\neq\mathbf{i}} \left|\mathcal{B}_{i_1\ldots i_N j_1\ldots j_N}\right|\right)^2}{\left|\mathcal{A}_{i_1\ldots i_N i_1\ldots i_N}\right|^2 + \left|\mathcal{B}_{i_1\ldots i_N i_1\ldots i_N}\right|^2}};$$

then from Theorem 3.3, we have the following tensor Geršgorin circle theorem.

Theorem 3.4 (Geršgorin circle theorem for generalized tensor eigenvalues) *Let* $\{\mathcal{A}, \mathcal{B}\}$ *be a regular tensor pair, and suppose that* $(\mathcal{A}_{i_1 i_2\ldots i_N i_1 i_2\cdots i_N}, \mathcal{B}_{i_1 i_2\ldots i_N i_1 i_2\ldots i_N}) \neq (0, 0)$ *for all* $\mathbf{1} \leq \mathbf{i} \leq \mathbf{I}$. *Then*

$$\bar{\Lambda}(\mathcal{A}, \mathcal{B}) \subseteq \bigcup_{\mathbf{1}\leq\mathbf{i}\leq\mathbf{I}} G_{\mathbf{i}}(\mathcal{A}, \mathcal{B}).$$

Bauer–Fike theorem

Definition 3.3 (*Square root of positive definite tensor*) Let $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$ be a positive definite tensor. According to the tensor Schur decomposition (Theorem 4.9 in Liang et al. 2019), there is a unitary tensor $\mathcal{Q} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$, such that $\mathcal{A} = \mathcal{Q}^H *_N \mathcal{D} *_N \mathcal{Q}$, where $\mathcal{D}$ is a diagonal tensor whose diagonal entries are greater than 0. Define $\mathcal{D}^{\frac{1}{2}} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$ by the square root

$$\mathcal{D}^{\frac{1}{2}}_{i_1 i_2\ldots i_N j_1 j_2\ldots j_N} := \sqrt{\mathcal{D}_{i_1 i_2\ldots i_N j_1 j_2\ldots j_N}}, \quad \text{for } \mathbf{1} \leq \mathbf{i}, \mathbf{j} \leq \mathbf{I}.$$

We define the square root tensor of $\mathcal{A}$ as $\mathcal{A}^{\frac{1}{2}} := \mathcal{Q}^H *_N \mathcal{D}^{\frac{1}{2}} *_N \mathcal{Q}$.

Definition 3.4 (*Diagonalizable tensor pair*) We call a regular tensor pair $\{\mathcal{A}, \mathcal{B}\}$ diagonalizable, if there are nonsingular tensors $\mathcal{P}$ and $\mathcal{Q}$, such that

$$\mathcal{A} = \mathcal{P} *_N \Lambda *_N \mathcal{Q}, \quad \mathcal{B} = \mathcal{P} *_N \Omega *_N \mathcal{Q},$$

where $\Lambda = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_{|\mathbf{I}|})$, $\Omega = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_{|\mathbf{I}|})$ are diagonal tensors.

Remark 3.4 From $\det_U(\beta\Lambda - \alpha\Omega) = \det_U(\mathcal{P})\det_U(\beta\mathcal{A} - \alpha\mathcal{B})\det_U(\mathcal{Q})$, an immediate consequence is that $\bar{\Lambda}(\mathcal{A}, \mathcal{B}) = \bar{\Lambda}(\Lambda, \Omega)$, that is, $\bar{\Lambda}(\mathcal{A}, \mathcal{B}) = \{[\alpha_i, \beta_i],\ 1 \leq i \leq |\mathbf{I}|\}$.

Lemma 3.1 *Let $\mathcal{M}, \mathcal{N} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times I_1 \times I_2 \times \cdots \times I_N}$ be positive definite. If $\mathcal{M} - \mathcal{N}$ is semi-positive definite, then $\mathcal{N}^{-1} - \mathcal{M}^{-1}$ is also semi-positive definite.*

Proof Based on the unfolding property that ϕ is an isomorphism, $\phi(\mathcal{M}), \phi(\mathcal{N}) \in \mathbb{C}^{|\mathbf{I}| \times |\mathbf{I}|}$ is positive definite, then there exists an invertible matrix $G \in \mathbb{C}^{|\mathbf{I}| \times |\mathbf{I}|}$, such that $G^H \phi(\mathcal{M}) G = I$, $G^H \phi(\mathcal{N}) G = D$, where D is an invertible diagonal matrix. Let $\mathcal{G} = \phi^{-1}(G), \mathcal{D} = \phi^{-1}(D)$, we have

$$\mathcal{G}^H *_N \mathcal{M} *_N \mathcal{G} = \mathcal{I}, \qquad \mathcal{G}^H *_N \mathcal{N} *_N \mathcal{G} = \mathcal{D}.$$

By the positive definiteness of $\mathcal{N}$ and semi-positive definiteness of $\mathcal{M} - \mathcal{N}$, we draw a conclusion that $0 < \mathcal{D}_{i_1 \ldots i_N i_1 \ldots i_N} \leq 1$ for all $\mathbf{1} \leq \mathbf{i} \leq \mathbf{I}$. Hence, $\mathcal{D}^{-1} \geq \mathcal{I}$. Thereby, $\mathcal{N}^{-1} - \mathcal{M}^{-1} = \mathcal{G} *_N \left(\mathcal{D}^{-1} - \mathcal{I}\right) *_N \mathcal{G}^H$ is semi-positive definite. □

Proposition 3.2 *If $\{\mathcal{A}, \mathcal{B}\}$ is a regular tensor pair, then $\left[\mathcal{A}\ \mathcal{B}\right]$ is of full row unfolding rank, i.e.,* $\mathrm{rank}_U \left(\left[\mathcal{A}\ \mathcal{B}\right]\right) = |\mathbf{I}|$.

Proof Since $\{\mathcal{A}, \mathcal{B}\}$ is a regular tensor pair, then there exists $\lambda \in \mathbb{C}$

$$\det{}_U(\mathcal{A} - \lambda \mathcal{B}) \neq 0.$$

From Lemma 2.2 (2), we obtain

$$\mathrm{rank}_U(\mathcal{A} - \lambda \mathcal{B}) = |\mathbf{I}|.$$

Therefore

$$\mathrm{rank}\,(\phi(\mathcal{A}) - \lambda \phi(\mathcal{B})) = \mathrm{rank}\,(\phi(\mathcal{A} - \lambda \mathcal{B})) = |\mathbf{I}|;$$

according to Proposition 2.1, we have

$$\mathrm{rank}_U \left(\left[\mathcal{A}\ \mathcal{B}\right]\right) = \mathrm{rank}\left(\left[\phi(\mathcal{A})\ \phi(\mathcal{B})\right]\right) \geq \mathrm{rank}\,(\phi(\mathcal{A}) - \lambda \phi(\mathcal{B})) = |\mathbf{I}|.$$

Hence, $\mathrm{rank}_U \left(\left[\mathcal{A}\ \mathcal{B}\right]\right) = |\mathbf{I}|$. □

Let $\{\mathcal{A}, \mathcal{B}\}$ and $\{\mathcal{C}, \mathcal{D}\}$ be two regular tensor pairs, and

$$\bar{\Lambda}(\mathcal{A}, \mathcal{B}) = \{[\alpha_j, \beta_j], 1 \leq j \leq |\mathbf{I}|\}, \quad \bar{\Lambda}(\mathcal{C}, \mathcal{D}) = \{[\gamma_i, \delta_i],\ 1 \leq i \leq |\mathbf{I}|\}.$$

We establish the Bauer–Fike theorem for regular tensor pairs to bound the spectral variation (Stewart and Sun 1990)

$$s_{\{\mathcal{A},\mathcal{B}\}}\{\mathcal{C}, \mathcal{D}\} := \max_i \min_j \rho\left([\alpha_j, \beta_j], [\gamma_i, \delta_i]\right)$$

between $\bar{\Lambda}(\mathcal{A}, \mathcal{B})$ and $\bar{\Lambda}(\mathcal{C}, \mathcal{D})$. Denote by $\mathcal{Z} = \left[\mathcal{A}\ \mathcal{B}\right]$ the row block tensor, it is easily verified from Proposition 3.2 that the Moore–Penrose inverse

$$\mathcal{Z}^\dagger = \mathcal{Z}^H *_N \left(\mathcal{Z} *_N \mathcal{Z}^H\right)^{-1},$$

then the orthogonal projector associated to $\mathcal{Z}^H$ is

$$P_{\mathcal{Z}^H} := \mathcal{Z}^\dagger *_N \mathcal{Z} = \mathcal{Z}^H *_N \left(\mathcal{Z} *_N \mathcal{Z}^H\right)^{-1} *_N \mathcal{Z}. \tag{4}$$

Next, the result for perturbation of generalized eigenvalues extended from Elsner and Sun (1982, Theorem 2.1) can be derived.

Theorem 3.5 (Bauer–Fike theorem for generalized tensor eigenvalues) *Let $\{\mathcal{A}, \mathcal{B}\}$ be a diagonalizable regular tensor pair having the eigenvalue decomposition as shown in Definition 3.4, and $\{\mathcal{C}, \mathcal{D}\}$ be a regular tensor pair, $\mathcal{W} = \left[\mathcal{C}\ \mathcal{D}\right]$. Then*

$$s_{\{\mathcal{A},\mathcal{B}\}}\{\mathcal{C}, \mathcal{D}\} \leq \|\mathcal{Q}^{-1}\|_2\, \|\mathcal{Q}\|_2\, d_2(\mathcal{Z}, \mathcal{W}), \tag{5}$$

where $d_2(\mathcal{Z}\Leftrightarrow\mathcal{W}) = \left\| P_{\mathcal{Z}^H} - P_{\mathcal{W}^H} \right\|_2$.

Proof Let

$$\mathcal{Z}_1 = (\mathcal{Z} *_N \mathcal{Z}^H)^{-\frac{1}{2}} *_N \mathcal{Z} =: \left[\mathcal{A}_1\ \mathcal{B}_1\right], \quad \mathcal{W}_1 = (\mathcal{W} *_N \mathcal{W}^H)^{-\frac{1}{2}} *_N \mathcal{W} =: \left[\mathcal{C}_1\ \mathcal{D}_1\right],$$

where $\left(\mathcal{Z} *_N \mathcal{Z}^H\right)^{-\frac{1}{2}} := \left(\left(\mathcal{Z} *_N \mathcal{Z}^H\right)^{\frac{1}{2}}\right)^{-1}$, and

$$\begin{cases} \mathcal{A}_1 = \left(\mathcal{Z} *_N \mathcal{Z}^H\right)^{-\frac{1}{2}} *_N \mathcal{A}, & \mathcal{B}_1 = \left(\mathcal{Z} *_N \mathcal{Z}^H\right)^{-\frac{1}{2}} *_N \mathcal{B}, \\ \mathcal{C}_1 = (\mathcal{W} *_N \mathcal{W}^H)^{-\frac{1}{2}} *_N \mathcal{C}, & \mathcal{D}_1 = (\mathcal{W} *_N \mathcal{W}^H)^{-\frac{1}{2}} *_N \mathcal{D}. \end{cases}$$

From Proposition 2.2 (1), $\{\mathcal{A}_1, \mathcal{B}_1\}$ has the decomposition

$$\mathcal{A}_1 = \mathcal{P}_1 *_N \varLambda *_N \mathcal{Q}, \qquad \mathcal{B}_1 = \mathcal{P}_1 *_N \varOmega *_N \mathcal{Q}$$

with $\mathcal{P}_1 = (\mathcal{Z} *_N \mathcal{Z}^H)^{-\frac{1}{2}} *_N \mathcal{P}$. Choose any $(\gamma, \delta) \in \bar{\Lambda}(\mathcal{C}, \mathcal{D}) = \bar{\Lambda}(\mathcal{C}_1, \mathcal{D}_1)$, without lose of generality, we assume that $|\gamma|^2 + |\delta|^2 = 1$. Suppose $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ is a unit eigen-tensor of (γ, δ), that is

$$\delta\mathcal{C}_1 *_N \mathcal{X} = \gamma\mathcal{D}_1 *_N \mathcal{X}, \quad \|\mathcal{X}\|_F = 1.$$

Then, from Proposition 2.2, we obtain

$$\begin{aligned} &\delta\mathcal{A}_1 *_N \mathcal{X} - \gamma\mathcal{B}_1 *_N \mathcal{X} \\ &\quad= \delta\left(\mathcal{A}_1 - \mathcal{Z}_1 *_N \mathcal{W}_1^H *_N \mathcal{C}_1\right) *_N \mathcal{X} - \gamma\left(\mathcal{B}_1 - \mathcal{Z}_1 *_N \mathcal{W}_1^H *_N \mathcal{D}_1\right) *_N \mathcal{X} \\ &\quad= \left[\mathcal{A}_1 - \mathcal{Z}_1 *_N \mathcal{W}_1^H *_N \mathcal{C}_1\ \mathcal{B}_1 - \mathcal{Z}_1 *_N \mathcal{W}_1^H *_N \mathcal{D}_1\right] *_N \begin{bmatrix} \delta\mathcal{X} \\ -\gamma\mathcal{X} \end{bmatrix} \\ &\quad= \left(\left[\mathcal{A}_1\ \mathcal{B}_1\right] - \left[(\mathcal{Z}_1 *_N \mathcal{W}_1^H) *_N \mathcal{C}_1\ (\mathcal{Z}_1 *_N \mathcal{W}_1^H) *_N \mathcal{D}_1\right]\right) *_N \begin{bmatrix} \delta\mathcal{X} \\ -\gamma\mathcal{X} \end{bmatrix} \\ &\quad= \left(\left[\mathcal{A}_1\ \mathcal{B}_1\right] - \left(\mathcal{Z}_1 *_N \mathcal{W}_1^H\right) *_N \left[\mathcal{C}_1\ \mathcal{D}_1\right]\right) *_N \begin{bmatrix} \delta\mathcal{X} \\ -\gamma\mathcal{X} \end{bmatrix} \\ &\quad= \mathcal{Z}_1 *_N \left(\mathcal{Z}_1^H *_N \mathcal{Z}_1 - \mathcal{W}_1^H *_N \mathcal{W}_1\right) *_N \begin{bmatrix} \delta\mathcal{X} \\ -\gamma\mathcal{X} \end{bmatrix} \\ &\quad= \mathcal{Z}_1 *_N \left(P_{\mathcal{Z}^H} - P_{\mathcal{W}^H}\right) *_N \begin{bmatrix} \delta\mathcal{X} \\ -\gamma\mathcal{X} \end{bmatrix}. \end{aligned} \tag{6}$$

Since $\mathcal{Z}_1 *_N \mathcal{Z}_1^H = \mathcal{I}$, we have $\|\mathcal{Z}_1\|_2 = 1$; in addition $\left\|\begin{bmatrix} \delta\mathcal{X} \\ -\gamma\mathcal{X} \end{bmatrix}\right\|_F = \sqrt{\|\delta\mathcal{X}\|_F^2 + \|\gamma\mathcal{X}\|_F^2} = 1$. Hence, taking the Frobenius norm on both sides of equation (6) gives that

$$\|\delta\mathcal{A}_1 *_N \mathcal{X} - \gamma\mathcal{B}_1 *_N \mathcal{X}\|_F \leq \left\| P_{\mathcal{Z}^H} - P_{\mathcal{W}^H} \right\|_2 = d_2(\mathcal{Z}, \mathcal{W}). \tag{7}$$

On the other hand, from Proposition 2.2

$$\mathcal{Z}_1 = \left[\mathcal{A}_1\ \mathcal{B}_1\right] = \left[\mathcal{P}_1 *_N \varLambda *_N \mathcal{Q}\ \mathcal{P}_1 *_N \varOmega *_N \mathcal{Q}\right] = \mathcal{P}_1 *_N \left[\varLambda *_N \mathcal{Q}\ \varOmega *_N \mathcal{Q}\right],$$

it follows:

$$\mathcal{I} = \mathcal{Z}_1 *_N \mathcal{Z}_1^H = \mathcal{P}_1 *_N \left(\Lambda *_N \mathcal{Q} *_N \mathcal{Q}^H *_N \bar{\Lambda} + \Omega *_N \mathcal{Q} *_N \mathcal{Q}^H *_N \bar{\Omega} \right) *_N \mathcal{P}_1^H.$$

Thus, we have

$$\left(\mathcal{P}_1^H *_N \mathcal{P}_1 \right)^{-1} = \left(\Lambda *_N \mathcal{Q} *_N \mathcal{Q}^H *_N \bar{\Lambda} + \Omega *_N \mathcal{Q} *_N \mathcal{Q}^H *_N \bar{\Omega} \right).$$

It follows from Proposition 3.1 that for all $\mathcal{Y} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$:

$$\begin{aligned} \mathcal{Y}^H *_N \left(\mathcal{P}_1^H *_N \mathcal{P}_1 \right)^{-1} *_N \mathcal{Y} &= \left\| \mathcal{Q}^H *_N \bar{\Lambda} *_N \mathcal{Y} \right\|_F^2 + \left\| \mathcal{Q}^H *_N \bar{\Omega} *_N \mathcal{Y} \right\|_F^2 \\ &\leq \|\mathcal{Q}\|_2^2 \; \mathcal{Y}^H *_N \left(\Lambda *_N \bar{\Lambda} + \Omega *_N \bar{\Omega} \right) *_N \mathcal{Y}. \end{aligned}$$

Obviously, $\left(\mathcal{P}_1^H *_N \mathcal{P}_1 \right)^{-1}$ is positive definite, since $\{\mathcal{A}, \mathcal{B}\}$ is a regular tensor pair, then

$$\Lambda *_N \bar{\Lambda} + \Omega *_N \bar{\Omega} = \mathrm{diag}(|\alpha_1|^2 + |\beta_1|^2, |\alpha_2|^2 + |\beta_2|^2, \ldots, |\alpha_{|\mathbf{I}|}|^2 + |\beta_{|\mathbf{I}|}|^2) > 0$$

is also positive definite. Thereby, from Lemma 3.1, we infer

$$\mathcal{Y}^H *_N \left(\mathcal{P}_1^H *_N \mathcal{P}_1 \right) *_N \mathcal{Y} \geq \frac{1}{\|\mathcal{Q}\|_2^2} \mathcal{Y}^H *_N \left(\Lambda *_N \bar{\Lambda} + \Omega *_N \bar{\Omega} \right)^{-1} *_N \mathcal{Y}, \;\; \forall \mathcal{Y} \in \mathbb{C}^{I_1 \times \cdots \times I_N}.$$

Therefore

$$\begin{aligned} &\|\delta \mathcal{A}_1 *_N \mathcal{X} - \gamma \mathcal{B}_1 *_N \mathcal{X}\|_F = \|\mathcal{P}_1 *_N (\delta \Lambda - \gamma \Omega) *_N \mathcal{Q} *_N \mathcal{X}\|_F \\ &\quad = \left(\mathcal{X}^H *_N \mathcal{Q}^H *_N (\delta \Lambda - \gamma \Omega)^H *_N \mathcal{P}_1^H *_N \mathcal{P}_1 *_N (\delta \Lambda - \gamma \Omega) *_N \mathcal{Q} *_N \mathcal{X} \right)^{\frac{1}{2}} \\ &\quad \geq \frac{1}{\|\mathcal{Q}\|_2} \left(\mathcal{X}^H *_N \mathcal{Q}^H *_N (\delta \Lambda - \gamma \Omega)^H *_N \left(\Lambda *_N \bar{\Lambda} + \Omega *_N \bar{\Omega} \right)^{-1} *_N (\delta \Lambda - \gamma \Omega) *_N \mathcal{Q} *_N \mathcal{X} \right)^{\frac{1}{2}} \\ &\quad \geq \frac{1}{\|\mathcal{Q}\|_2} \min_{1 \leq i \leq |\mathbf{I}|} \frac{|\delta \alpha_i - \gamma \beta_i|}{\sqrt{|\alpha_i|^2 + |\beta_i|^2}} \cdot \left(\mathcal{X}^H *_N \mathcal{Q}^H *_N \mathcal{Q} *_N \mathcal{X} \right)^{\frac{1}{2}} \\ &\quad \geq \frac{\sigma_{\min}(\mathcal{Q})}{\|\mathcal{Q}\|_2} \min_{1 \leq i \leq |\mathbf{I}|} \rho\left([\alpha_i, \beta_i], [\gamma, \delta] \right). \\ &\quad = \frac{1}{\|\mathcal{Q}\|_2 \left\| \mathcal{Q}^{-1} \right\|_2} \min_{1 \leq i \leq |\mathbf{I}|} \rho\left([\alpha_i, \beta_i], [\gamma, \delta] \right). \end{aligned} \tag{8}$$

Combining inequalities (7) and (8), we obtain

$$\min_{1 \leq i \leq |\mathbf{I}|} \rho\left([\alpha_i, \beta_i], [\gamma, \delta] \right) \leq \|\mathcal{Q}^{-1}\|_2 \, \|\mathcal{Q}\|_2 \, d_2(\mathcal{Z}, \mathcal{W}).$$

By the arbitrariness of $[\gamma, \delta] \in \Lambda(\mathcal{C}, \mathcal{D})$, (5) holds. □

4 Pseudospectral theory of regular tensor pairs

In this section, we propose the definitions of backward error and ϵ-pseudospectrum for regular tensor pairs in normwise sense and componentwise sense, respectively.

4.1 Normwise backward error and pseudospectrum

The normwise backward error of an approximate eigenvalue for a matrix pair has been analyzed in Bai et al. (2000), Frayssé and Toumazou (1998). Under Einstein product, these results can be extended to the generalized tensor eigenvalue (2).

Definition 4.1 Let $\tilde{\lambda} \in \mathbb{C}$ be an approximate eigenvalue of the generalized tensor eigenproblem (2). The normwise backward error of $\tilde{\lambda}$ is

$$\begin{aligned}\eta(\tilde{\lambda}) = \min\big\{\gamma > 0 |\ \exists\, \mathcal{U} \neq \mathcal{O},\ (\mathcal{A} + \Delta\mathcal{A}) *_N \mathcal{U} \\ = \tilde{\lambda}(\mathcal{B} + \Delta\mathcal{B}) *_N \mathcal{U} \text{ with } \|\Delta\mathcal{A}\|_2 \leq \alpha\gamma,\ \|\Delta\mathcal{B}\|_2 \leq \beta\gamma\big\},\end{aligned} \tag{9}$$

where α and β are nonnegative numbers allowing different measurements of the perturbations on $\mathcal{A}$ and $\mathcal{B}$ independently. For example, the perturbations are in an absolute sense when $\alpha = \beta = 1$, or a relative sense by setting $\alpha = \|\mathcal{A}\|_2$ and $\beta = \|\mathcal{B}\|_2$.

Lemma 4.1 *The normwise backward error has the following characterization:*

$$\eta\left(\tilde{\lambda}\right) = \frac{\min\limits_{\|\mathcal{U}\|_F=1} \left\|\left(\mathcal{A} - \tilde{\lambda}\mathcal{B}\right) *_N \mathcal{U}\right\|_F}{\alpha + |\tilde{\lambda}|\beta}. \tag{10}$$

Proof By Definition 4.1, $\exists\, \mathcal{U} \neq \mathcal{O}$, $\|\mathcal{U}\|_F = 1$, such that

$$\left(\mathcal{A} - \tilde{\lambda}\mathcal{B}\right) *_N \mathcal{U} = -\left(\Delta\mathcal{A} - \tilde{\lambda}\Delta\mathcal{B}\right) *_N \mathcal{U},$$

with $\|\Delta\mathcal{A}\|_2 \leq \alpha\eta(\tilde{\lambda})$ and $\|\Delta\mathcal{B}\|_2 \leq \beta\eta(\tilde{\lambda})$. Takeing the Frobenius norm in above equality, we obtain

$$\begin{aligned}\left\|\left(\mathcal{A} - \tilde{\lambda}\mathcal{B}\right) *_N \mathcal{U}\right\|_F &\leq \left\|-\left(\Delta\mathcal{A} - \tilde{\lambda}\Delta\mathcal{B}\right)\right\|_2 \cdot \|\mathcal{U}\|_F \leq \|\Delta\mathcal{A}\|_2 + |\tilde{\lambda}|\, \|\Delta\mathcal{B}\|_2 \\ &\leq \eta\left(\tilde{\lambda}\right)\left(\alpha + |\tilde{\lambda}|\beta\right).\end{aligned}$$

Hence

$$\eta(\tilde{\lambda}) \geq \frac{\min\limits_{\|\mathcal{U}\|_F=1} \|(\mathcal{A} - \tilde{\lambda}\mathcal{B}) *_N \mathcal{U}\|_F}{\alpha + |\tilde{\lambda}|\beta}.$$

On the other hand, suppose that $\tilde{\mathcal{U}} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ satisfies $\|\tilde{\mathcal{U}}\|_F = 1$ and

$$\left\|(\mathcal{A} - \tilde{\lambda}\mathcal{B}) *_N \tilde{\mathcal{U}}\right\|_F = \min_{\|\mathcal{U}\|_F=1} \left\|(\mathcal{A} - \tilde{\lambda}\mathcal{B}) *_N \mathcal{U}\right\|_F.$$

Let $\mathcal{Y} = (\mathcal{A} - \tilde{\lambda}\mathcal{B}) *_N \tilde{\mathcal{U}}$, $\mathcal{H} = \frac{1}{\|\mathcal{Y}\|_F}\mathcal{Y} \circ \tilde{\mathcal{U}}^H \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$, then $\mathcal{H} *_N \tilde{\mathcal{U}} = \mathcal{Y} / \|\mathcal{Y}\|_F$. From $\mathcal{H}^H *_N \mathcal{H} = \tilde{\mathcal{U}} \circ \tilde{\mathcal{U}}^H$, we infer $\|\mathcal{H}\|_2 = 1$. Let

$$\Delta\mathcal{A} := -\frac{\alpha\mathcal{H}\,\|\mathcal{Y}\|_F}{\alpha + |\tilde{\lambda}|\beta}, \quad \Delta\mathcal{B} := \frac{\bar{\tilde{\lambda}}\beta\mathcal{H}\,\|\mathcal{Y}\|_F}{|\tilde{\lambda}|(\alpha + |\tilde{\lambda}|\beta)}.$$

Then

$$\|\Delta\mathcal{A}\|_2 \leq \frac{\|(\mathcal{A} - \tilde{\lambda}\mathcal{B}) *_N \tilde{\mathcal{U}}\|_F}{\alpha + |\tilde{\lambda}|\beta}\alpha, \quad \|\Delta\mathcal{B}\|_2 \leq \frac{\|(\mathcal{A} - \tilde{\lambda}\mathcal{B}) *_N \tilde{\mathcal{U}}\|_F}{\alpha + |\tilde{\lambda}|\beta}\beta,$$

and

$$(\mathcal{A} + \Delta\mathcal{A}) *_N \tilde{\mathcal{U}} = \tilde{\lambda}(\mathcal{B} + \Delta\mathcal{B}) *_N \tilde{\mathcal{U}}.$$

Therefore

$$\eta(\tilde{\lambda}) \leq \frac{\min\limits_{\|\mathcal{U}\|_F=1} \|(\mathcal{A} - \tilde{\lambda}\mathcal{B}) *_N \mathcal{U}\|_F}{\alpha + |\tilde{\lambda}|\beta}.$$

Thus, the equality (10) holds. □

From Lemma 4.1 and Remark 3.2, a corollary can be given immediately.

Corollary 4.1 *The normwise backward error can be computed as*

$$\eta(\tilde{\lambda}) = \frac{1}{\left(\alpha + |\tilde{\lambda}|\beta\right) \|(\mathcal{A} - \tilde{\lambda}\mathcal{B})^{-1}\|_2} = \frac{\sigma_{\min}\left(\mathcal{A} - \tilde{\lambda}\mathcal{B}\right)}{\alpha + |\tilde{\lambda}|\beta},$$

where we employ the convention that $\|(\mathcal{A} - \tilde{\lambda}\mathcal{B})^{-1}\|_2 = +\infty$ *for* $\tilde{\lambda} \in \Lambda(\mathcal{A}, \mathcal{B})$.

Normwise pseudospectral theory has been extensively applied in numerical analysis, optimal control systems, and stability analysis of dynamical systems (Du and Wei 2006; Lancaster and Psarrakos 2005; Trefethen and Embree 2005). In recent years, there are also several papers on pseudospectra for tensors (Che et al. 2017; He et al. 2020; Li et al. 2019). We give the definition of normwise ϵ-pseudospectrum for regular tensor pairs, which is a generalized version of normwise ϵ-pseudospectrum of matrix pairs in Higham and Tisseur (2002), Lancaster and Psarrakos (2005).

Definition 4.2 Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times I_1 \times I_2 \times \cdots \times I_N}$, and $\{\mathcal{A}, \mathcal{B}\}$ be a regular tensor pair. We define the normwise ϵ-pseudospectrum

$$\begin{aligned}\Lambda_\epsilon^{\alpha,\beta}(\mathcal{A}, \mathcal{B}) = \{z \in \mathbb{C} |\ \exists\, \mathcal{U} \neq \mathcal{O},\ (\mathcal{A} + \Delta\mathcal{A}) *_N \mathcal{U} \\ = z(\mathcal{B} + \Delta\mathcal{B}) *_N \mathcal{U} \text{ with } \|\Delta\mathcal{A}\|_2 \leq \alpha\epsilon,\ \|\Delta\mathcal{B}\|_2 \leq \beta\epsilon\}.\end{aligned} \tag{11}$$

As a special case, the normwise ϵ-pseudospectrum for tensor eigenvalue (1) is

$$\Lambda_\epsilon(\mathcal{A}) := \Lambda_\epsilon^{1,0}(\mathcal{A}, \mathcal{I}) = \{z \in \mathbb{C} |\ \exists\, \mathcal{U} \neq \mathcal{O},\ (\mathcal{A} + \Delta\mathcal{A}) *_N \mathcal{U} = z\mathcal{U} \text{ with } \|\Delta\mathcal{A}\|_2 \leq \epsilon\},$$

which is a generalization of the classical well-known ϵ-pseudospectrum of matrices (Trefethen and Embree 2005).

Remark 4.1 The normwise ϵ-pseudospectrum in (11) can be represented by the normwise backward error

$$\Lambda_\epsilon^{\alpha,\beta}(\mathcal{A}, \mathcal{B}) = \{z \in \mathbb{C} \big|\ \eta(z) \leq \epsilon\}.$$

Theorem 4.1 *The pseudospectrum of tensor pair* $\{\mathcal{A}, \mathcal{B}\}$ *defined in (11) has the following equivalent definitions:*

$$\Lambda_\epsilon^{\alpha,\beta}(\mathcal{A}, \mathcal{B}) = \left\{z \in \mathbb{C} \Big|\ \min_{\|\mathcal{U}\|_F=1} \|(\mathcal{A} - z\mathcal{B}) *_N \mathcal{U}\|_F \leq \epsilon(\alpha + |z|\,\beta)\right\} \tag{12}$$

$$= \left\{z \in \mathbb{C} \Big|\ \left\|(\mathcal{A} - z\mathcal{B})^{-1}\right\|_2^{-1} \leq \epsilon(\alpha + |z|\,\beta)\right\} \tag{13}$$

$$= \left\{z \in \mathbb{C} \big|\ \sigma_{\min}(\mathcal{A} - z\mathcal{B}) \leq \epsilon(\alpha + |z|\,\beta)\right\}. \tag{14}$$

Corollary 4.2 $\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{A},\mathcal{B})=\mathbb{C}$ *if and only if* $\max\limits_{z\in\mathbb{C}}\min\limits_{\|\mathcal{U}\|_F=1}\frac{\|(\mathcal{A}-z\mathcal{B})*_N\mathcal{U}\|_F}{\alpha+|z|\beta}\le\epsilon$.

The pseudospectrum of a regular tensor pair has some elegant properties.

Proposition 4.1 *For any regular tensor pair* $\{\mathcal{A},\mathcal{B}\}$*, its pseudospectrum has the following properties:*

1. $\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{A},\mathcal{B})$ *are monotonically increasing with respect to* ϵ*, i.e.,* $\Lambda_{\epsilon_1}^{\alpha,\beta}(\mathcal{A},\mathcal{B})\subset\Lambda_{\epsilon_2}^{\alpha,\beta}(\mathcal{A},\mathcal{B})$ *if* $\epsilon_1<\epsilon_2$.
2. $\Lambda(\mathcal{A},\mathcal{B})=\Lambda_{0}^{\alpha,\beta}(\mathcal{A},\mathcal{B})\subset\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{A},\mathcal{B})$*, specially,* $\Lambda(\mathcal{A})\subset\Lambda_{\epsilon}(\mathcal{A})$ *for* $\epsilon>0$.
3. $\Lambda_{\epsilon}^{|\theta|\alpha,|\gamma|\beta}(\theta\mathcal{A},\gamma\mathcal{B})=\frac{\theta}{\gamma}\cdot\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{A},\mathcal{B})$*, for nonzero* $\theta,\gamma\in\mathbb{C}$.
4. *If* $\mathcal{Q},\mathcal{Z}$ *are invertible tensors,* $\mathcal{Q}*_N\mathcal{A}*_N\mathcal{Z}=\mathcal{T}$ *and* $\mathcal{Q}*_N\mathcal{B}*_N\mathcal{Z}=\mathcal{S}$*, then* $\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{A},\mathcal{B})\subset\Lambda_{\epsilon\|\mathcal{Q}\|_2\|\mathcal{Z}\|_2}^{\alpha,\beta}(\mathcal{T},\mathcal{S})$*. Furthermore,* $\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{A},\mathcal{B})=\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{T},\mathcal{S})$ *if the tensors* $\mathcal{Q}$ *and* $\mathcal{Z}$ *are unitary.*

Proof Properties 1 and 2 are immediate consequences from Definition 4.2.

Property 3. From Eq. (13), we get

$$\begin{aligned}\Lambda_{\epsilon}^{|\theta|\alpha,|\gamma|\beta}(\theta\mathcal{A},\gamma\mathcal{B})&=\left\{z\in\mathbb{C}\Big|\ \left\|(\theta\mathcal{A}-z\gamma\mathcal{B})^{-1}\right\|_2\ge\frac{1}{\epsilon(|\theta|\,\alpha+|z|\,|\gamma|\,\beta)}\right\}\\&=\left\{z\in\mathbb{C}\Big|\ \left\|\left(\mathcal{A}-z\frac{\gamma}{\theta}\mathcal{B}\right)^{-1}\right\|_2\ge\frac{1}{\epsilon\left(\alpha+\left|z\frac{\gamma}{\theta}\right|\beta\right)}\right\}\\&=\frac{\theta}{\gamma}\cdot\left\{z\in\mathbb{C}\Big|\ \left\|(\mathcal{A}-z\mathcal{B})^{-1}\right\|_2\ge\frac{1}{\epsilon\,(\alpha+|z|\,\beta)}\right\}\\&=\frac{\theta}{\gamma}\cdot\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{A},\mathcal{B}).\end{aligned}$$

Property 4. Suppose $z\in\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{A},\mathcal{B})$, then

$$\frac{1}{\epsilon(\alpha+|z|\,\beta)}\le\|\,(\mathcal{A}-z\mathcal{B})^{-1}\,\|_2\le\|\mathcal{Z}*_N(\mathcal{T}-z\mathcal{S})^{-1}*_N\mathcal{Q}\|_2\le\|\mathcal{Z}\|_2\|\,(\mathcal{T}-z\mathcal{S})^{-1}\,\|_2\|\mathcal{Q}\|_2,$$

it follows that $\|\,(\mathcal{T}-z\mathcal{S})^{-1}\,\|_2\ge\frac{1}{\epsilon\|\mathcal{Q}\|_2\|\mathcal{Z}\|_2(\alpha+|z|\beta)}$. Therefore, $z\in\Lambda_{\epsilon\|\mathcal{Q}\|_2\|\mathcal{Z}\|_2}^{\alpha,\beta}(\mathcal{T},\mathcal{S})$.

If $\mathcal{Q}$ and $\mathcal{Z}$ are unitary, then

$$\|\,(\mathcal{A}-z\mathcal{B})^{-1}\,\|_2=\|\,(\mathcal{T}-z\mathcal{S})^{-1}\,\|_2;$$

from (13), we deduce $\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{A},\mathcal{B})=\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{T},\mathcal{S})$. □

Remark 4.2 By Theorem 3.1, we can gain an upper triangular tensor pair $\{\mathcal{T},\mathcal{S}\}$ by performing the generalized Schur decomposition for $\{\mathcal{A},\mathcal{B}\}$, then $\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{A},\mathcal{B})=\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{T},\mathcal{S})$.

Theorem 4.2 *Let* $\mathcal{A},\mathcal{B}$ *be both real or Hermitian. Then, for any* $\epsilon\ge0$ *and* $\alpha,\beta\ge0$*, the normwise* ϵ*-pseudospectrum* $\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{A},\mathcal{B})$ *is symmetric with respect to the real axis in complex plane.*

Proof Suppose $z\in\Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{A},\mathcal{B})$, then from definition (11), there are $\Delta\mathcal{A}$ with $\|\Delta\mathcal{A}\|_2\le\epsilon\alpha$ and $\Delta\mathcal{B}$ with $\|\Delta\mathcal{B}\|_2\le\epsilon\beta$, such that

$$\det_U[\mathcal{A}+\Delta\mathcal{A}-z(\mathcal{B}+\Delta\mathcal{B})]=0.$$

If $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ are real, then

$$\begin{aligned}\det_U\left[\mathcal{A}+\overline{\Delta\mathcal{A}}-\bar{z}(\mathcal{B}+\overline{\Delta\mathcal{B}})\right] &= \det_U\left[\overline{\mathcal{A}+\Delta\mathcal{A}-z(\mathcal{B}+\Delta\mathcal{B})}\right]\\ &= \overline{\det_U\left[\mathcal{A}+\Delta\mathcal{A}-z(\mathcal{B}+\Delta\mathcal{B})\right]}=0,\end{aligned}$$

where $\|\overline{\Delta\mathcal{A}}\|_2 = \|\Delta\mathcal{A}\|_2 \le \epsilon\alpha$, $\|\overline{\Delta\mathcal{B}}\|_2 = \|\Delta\mathcal{B}\|_2 \le \epsilon\beta$.

If $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ are Hermitian, then it follows from Lemma 2.2 (1) that:

$$\det_U\left[\mathcal{A}+\Delta\mathcal{A}^H-\bar{z}\left(\mathcal{B}+\Delta\mathcal{B}^H\right)\right] = \det_U\left[\left(\mathcal{A}+\Delta\mathcal{A}-z\left(\mathcal{B}+\Delta\mathcal{B}\right)\right)^H\right]=0,$$

where $\|\Delta\mathcal{A}^H\|_2 = \|\Delta\mathcal{A}\|_2 \le \epsilon\alpha$, $\|\Delta\mathcal{B}^H\|_2 = \|\Delta\mathcal{B}\|_2 \le \epsilon\beta$, hence $\bar{z} \in \Lambda_\epsilon^{\alpha,\beta}(\mathcal{A},\mathcal{B})$. □

Corollary 4.2 shows that the normwise ϵ-pseudospectrum $\Lambda_\epsilon^{\alpha,\beta}(\mathcal{A},\mathcal{B})$ might be the whole complex plane $\mathbb{C}$. Next, we give a necessary condition for the boundedness of normwise ϵ-pseudospectrum.

Theorem 4.3 *Let $\{\mathcal{A},\mathcal{B}\}$ be a regular tensor pair. If $\Lambda_\epsilon^{\alpha,\beta}(\mathcal{A},\mathcal{B})$ is bounded, then $\mathcal{B}+\Delta\mathcal{B}$ is nonsingular for all $\Delta\mathcal{B}$ with $\|\Delta\mathcal{B}\|_2 < \epsilon\beta$.*

Proof Assume that $\Lambda_\epsilon^{\alpha,\beta}(\mathcal{A},\mathcal{B})$ is bounded, but there exists a perturbed tensor pair $\{\mathcal{A}+\Delta\mathcal{A}, \mathcal{B}+\Delta\mathcal{B}\}$ with $\|\Delta\mathcal{A}\|_2 \le \epsilon\alpha$ and $\|\Delta\mathcal{B}\|_2 < \epsilon\beta$, but $\det_U(\mathcal{B}+\Delta\mathcal{B}) = 0$. From Theorem 3.1, there exist unitary tensors $\mathcal{P}$, $\mathcal{Q}$, such that

$$\mathcal{P}^H *_N (\mathcal{A}+\Delta\mathcal{A}) *_N \mathcal{Q} = \mathcal{T}, \qquad \mathcal{P}^H *_N (\mathcal{B}+\Delta\mathcal{B}) *_N \mathcal{Q} = \mathcal{S}$$

are upper triangular tensors. From $\det_U(\mathcal{S}) = \det_U(\mathcal{P})\det_U(\mathcal{Q})\det_U(\mathcal{B}+\Delta\mathcal{B}) = 0$, it follows that $\mathcal{S}_{i_1\cdots i_N i_1\ldots i_N} = 0$ for some $\mathbf{1} \le \mathbf{i} \le \mathbf{I}$. We infer that $\mathcal{T}_{i_1\ldots i_N i_1\ldots i_N} \neq 0$ from the boundedness of $\Lambda_\epsilon^{\alpha,\beta}(\mathcal{A},\mathcal{B})$ or the regularity of tensor pair $\{\mathcal{A},\mathcal{B}\}$. Construct a nonnegative sequence $\{\xi_k \in \mathbb{R}^+ : \xi_k \le \epsilon\beta - \|\Delta\mathcal{B}\|_2, k \in \mathbb{N}^+\}$, such that $\lim_{k\to\infty}\xi_k = 0$. Meanwhile, we construct a tensor sequence

$$\Delta\mathcal{B}_k = \Delta\mathcal{B} + \mathcal{P} *_N \mathcal{S}_k *_N \mathcal{Q}^H,$$

where the entries of $\mathcal{S}_k \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ are 0 except $(\mathcal{S}_k)_{i_1\ldots i_N i_1\ldots i_N} = \xi_k$ for all $k \in \mathbb{N}^+$, it is clear that

$$\|\Delta\mathcal{B}_k\|_2 \le \|\Delta\mathcal{B}\|_2 + \xi_k \le \epsilon\beta.$$

Furthermore

$$\mathcal{P}^H *_N (\mathcal{B}+\Delta\mathcal{B}_k) *_N \mathcal{Q} = (\mathcal{S}+\mathcal{S}_k).$$

Therefore, $\frac{\mathcal{T}_{i_1\ldots i_N i_1\ldots i_N}}{(\mathcal{S}+\mathcal{S}_k)_{i_1\ldots i_N i_1\ldots i_N}} = \frac{\mathcal{T}_{i_1\ldots i_N i_1\ldots i_N}}{\xi_k}$ is an eigenvalue of the perturbed pair $\{\mathcal{A}+\Delta\mathcal{A}, \mathcal{B}+\Delta\mathcal{B}_k\}$; hence, $\{\mathcal{T}_{i_1\ldots i_N i_1\ldots i_N}/\xi_k : k \in \mathbb{N}^+\} \subset \Lambda_\epsilon^{\alpha,\beta}(\mathcal{A},\mathcal{B})$, but $\lim_{k\to+\infty} \frac{\mathcal{T}_{i_1\ldots i_N i_1\ldots i_N}}{\xi_k} \longrightarrow \infty$ is a contradiction. □

Distance to the nearest singular tensor pair

Regularity is an often required condition for applications involving matrix pencils; for instance, the regularity of the coefficient matrix pair of a generalized linear system guarantees the existence and uniqueness of its solution. The distance from a regular matrix pair to the nearest singular one has therefore aroused the interest of researchers (Byers et al. 1998; Higham and Tisseur 2002). Here, we define the distance from a given regular matrix pair to the nearest singular tensor pair with respect to the perturbation measurements α, β as

$$\begin{aligned}\delta_s(\mathcal{A},\mathcal{B}) := \min\{\epsilon>0|\ \det_U[\mathcal{A}+\Delta\mathcal{A}-z(\mathcal{B}+\Delta\mathcal{B})]\equiv 0,\ \text{for all } z\in\mathbb{C} \text{ for some}\\ \|\Delta\mathcal{A}\|_2\le\epsilon\alpha,\ \|\Delta\mathcal{B}\|_2\le\epsilon\beta\}.\end{aligned}$$

From Definition 4.2 of normwise ϵ-pseudospectrum, we get

$$\delta_s(\mathcal{A},\mathcal{B}) \ge \min\{\epsilon>0|\ \Lambda_\epsilon^{\alpha,\beta}(\mathcal{A},\mathcal{B})=\mathbb{C}\};$$

according to Corollary 4.2, this lower bound can be rewritten as

$$\delta_s(\mathcal{A},\mathcal{B}) \ge \max_{z\in\mathbb{C}}\min_{\|\mathcal{U}\|_F=1}\frac{\|(\mathcal{A}-z\mathcal{B})*_N\mathcal{U}\|_F}{\alpha+|z|\,\beta} = \max_{z\in\mathbb{C}}\frac{\sigma_{\min}(\mathcal{A}-z\mathcal{B})}{\alpha+|z|\,\beta}.$$

4.2 Componentwise backward error and pseudospectrum

Componentwise perturbation for matrices is an interesting hot topic in past few decades. Rump (1999, 2003) developed Perron–Frobenius theories for real and complex matrices, respectively, and employed it to analyze almost sharp bounds for the componentwise distance to the nearest singular matrix (Demmel 1992). Higham and Higham (1998) evaluated the componentwise backward errors and condition numbers of generalized matrix eigenvalues.

In the sequel, $\rho(\mathcal{A}) := \max\{|\lambda|\mid\exists\,\mathcal{U}\neq\mathcal{O},\ \mathcal{A}*_N\mathcal{U}=\lambda\mathcal{U}\}$ denotes the spectral radius of $\mathcal{A}$, $|\mathcal{A}| = (|\mathcal{A}_{i_1i_2\cdots i_Nj_1j_2\cdots j_N}|)$ is the componentwise modulus of $\mathcal{A}$, and $\mathcal{A}^m = \underbrace{\mathcal{A}*_N\mathcal{A}*_N*\cdots*_N\mathcal{A}}_{m}$.

Definition 4.3 The componentwise backward error of an approximate $\tilde{\lambda}\in\mathbb{C}$ for generalized tensor eigenvalue (2) is

$$\omega(\tilde{\lambda}) = \min\left\{\gamma>0|\ \exists\,\mathcal{U}\neq\mathcal{O},\ (\mathcal{A}+\Delta\mathcal{A})*_N\mathcal{U}=\tilde{\lambda}(\mathcal{B}+\Delta\mathcal{B})*_N\mathcal{U}\ \text{with}\ |\Delta\mathcal{A}|\le\gamma\mathcal{E},\ |\Delta\mathcal{B}|\le\gamma\mathcal{F}\right\},$$

where nonnegative weighted tensors $\mathcal{E},\mathcal{F}\in\mathbb{R}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$ describing the perturbed weight on $\mathcal{A}$ and $\mathcal{B}$, respectively, the modulus and inequalities between tensors hold entrywise.

Lemma 4.2 *The componentwise backward error has the characterization*

$$\omega(\tilde{\lambda}) = \frac{1}{\max\limits_{|\Delta\mathcal{A}|\le\mathcal{E},|\Delta\mathcal{B}|\le\mathcal{F}}\rho\left((\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1}*_N\left(\Delta\mathcal{A}-\tilde{\lambda}\Delta\mathcal{B}\right)\right)}; \tag{15}$$

here, we employ the convention that $\rho\left((\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1}*_N\left(\Delta\mathcal{A}-\tilde{\lambda}\Delta\mathcal{B}\right)\right)=\infty$ *for* $\tilde{\lambda}\in\Lambda(\mathcal{A},\mathcal{B})$.

Proof If $\tilde{\lambda}\in\Lambda(\mathcal{A},\mathcal{B})$, i.e., $\mathcal{A}-\tilde{\lambda}\mathcal{B}$ is singular, then the equality obviously holds by $0=1/\infty$. Suppose $\mathcal{A}-\tilde{\lambda}\mathcal{B}$ is nonsingular. For any tensors $|\Delta\mathcal{A}|\le\mathcal{E}$, $|\Delta\mathcal{B}|\le\mathcal{F}$, let $\mu\in\mathbb{C}$ be an eigenvalue of $(\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1}*_N(\Delta\mathcal{A}-\tilde{\lambda}\Delta\mathcal{B})$, and $|\mu|=\rho\left((\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1}*_N(\Delta\mathcal{A}-\tilde{\lambda}\Delta\mathcal{B})\right)>0$. The tensor $\mu\mathcal{I}-(\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1}*_N(\Delta\mathcal{A}-\tilde{\lambda}\Delta\mathcal{B})$ is singular, which yields that

$$(\mathcal{A}-\mu^{-1}\Delta\mathcal{A})-\tilde{\lambda}(\mathcal{B}-\mu^{-1}\Delta\mathcal{B}) = \mu^{-1}(\mathcal{A}-\tilde{\lambda}\mathcal{B})*_N\left(\mu\mathcal{I}-(\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1}*_N(\Delta\mathcal{A}-\lambda\Delta\mathcal{B})\right)$$

is singular with $|\mu^{-1}\Delta\mathcal{A}| \le |\mu|^{-1}\mathcal{E}$, $|\mu^{-1}\Delta\mathcal{B}| \le |\mu|^{-1}\mathcal{F}$. Therefore, $\omega(\tilde{\lambda}) \le |\mu|^{-1}$ and

$$\omega(\tilde{\lambda}) \le \frac{1}{\max\limits_{|\Delta\mathcal{A}|\le\mathcal{E},|\Delta\mathcal{B}|\le\mathcal{F}} \rho\left((\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1} *_N (\Delta\mathcal{A}-\tilde{\lambda}\Delta\mathcal{B})\right)}.$$

On the other hand, suppose $\mathcal{A}-\tilde{\lambda}\mathcal{B}+\tilde{\mathcal{E}}-\tilde{\lambda}\tilde{\mathcal{F}}$ is singular for $|\tilde{\mathcal{E}}| \le \alpha\mathcal{E}$, $|\tilde{\mathcal{F}}| \le \alpha\mathcal{F}$ with $\alpha := \omega(\tilde{\lambda})$. Nonsingularity of $\mathcal{A}-\tilde{\lambda}\mathcal{B}$ assures that $\alpha > 0$, then $\alpha^{-1}\mathcal{I} + (\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1} *_N (\alpha^{-1}\tilde{\mathcal{E}} - \tilde{\lambda}\alpha^{-1}\tilde{\mathcal{F}}) = \alpha^{-1}(\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1} *_N (\mathcal{A}-\tilde{\lambda}\mathcal{B}+\tilde{\mathcal{E}}-\tilde{\lambda}\tilde{\mathcal{F}})$ is singular; therefore

$$\rho\left(\left(\mathcal{A}-\tilde{\lambda}\mathcal{B}\right)^{-1} *_N \left(\alpha^{-1}\tilde{\mathcal{E}} - \tilde{\lambda}\alpha^{-1}\tilde{\mathcal{F}}\right)\right) \ge \alpha^{-1},$$

that is, $\omega(\tilde{\lambda}) = \alpha \ge \left[\rho\left((\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1} *_N (\alpha^{-1}\tilde{\mathcal{E}} - \tilde{\lambda}\alpha^{-1}\tilde{\mathcal{F}})\right)\right]^{-1}$ with $|\alpha^{-1}\tilde{\mathcal{E}}| \le \mathcal{E}$, $|\alpha^{-1}\tilde{\mathcal{F}}| \le \mathcal{F}$. Based on the above discussions, Eq. (15) holds. □

Lemma 4.3 (Gelfand formula Conway 2019; Song and Qi 2013) *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$ *and* $\|\cdot\|$ *be a tensor norm. Then,* $\rho(\mathcal{A}) = \lim\limits_{k\to\infty} \|\mathcal{A}^k\|^{\frac{1}{k}}$.

The conclusion about the monotonicity of spectral radius of nonnegative matrices (Horn and Johnson 2012) can be extend to tensors.

Proposition 4.2 *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$, $\mathcal{B} \in \mathbb{R}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$, *and suppose that* $\mathcal{B}$ *is nonnegative. If* $|\mathcal{A}| \le \mathcal{B}$, *then* $\rho(\mathcal{A}) \le \rho(|\mathcal{A}|) \le \rho(\mathcal{B})$.

Proof Using the triangular inequality, we obtain by $|\mathcal{A}| \le \mathcal{B}$ that

$$|\mathcal{A}^m| \le |\mathcal{A}|^m \le \mathcal{B}^m,\ m = 1, 2, \ldots;$$

taking Frobenius norm gives

$$\|\mathcal{A}^m\|_F \le \||\mathcal{A}|^m\|_F \le \|\mathcal{B}^m\|_F, \text{ and } \|\mathcal{A}^m\|_F^{\frac{1}{m}} \le \||\mathcal{A}|^m\|_F^{\frac{1}{m}} \le \|\mathcal{B}^m\|_F^{\frac{1}{m}}$$

for all $m = 1, 2, \ldots$, let $m \to \infty$ and we apply the Gelfand formula; the conclusion holds. □

Theorem 4.4 *The componentwise backward error has the property that*

$$\omega(\tilde{\lambda}) \ge \frac{1}{\rho\left(|(\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1}| *_N (\mathcal{E}+|\tilde{\lambda}|\mathcal{F})\right)}. \tag{16}$$

Proof For any tensors $|\Delta\mathcal{A}| \le \mathcal{E}$, $|\Delta\mathcal{B}| \le \mathcal{F}$, according to Proposition 4.2

$$\begin{aligned}\rho\left((\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1} *_N \left(\Delta\mathcal{A}-\tilde{\lambda}\Delta\mathcal{B}\right)\right) &\le \rho\left(|(\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1}| *_N \left(|\Delta\mathcal{A}|+|\tilde{\lambda}\Delta\mathcal{B}|\right)\right)\\ &\le \rho\left(|(\mathcal{A}-\tilde{\lambda}\mathcal{B})^{-1}| *_N \left(\mathcal{E}+|\tilde{\lambda}|\mathcal{F}\right)\right).\end{aligned}$$

By Lemma 4.2, Eq. (16) holds. □

Analogous to the normwise pseudospectrum, it is reasonable to define the componentwise ϵ-pseudospectrum of a regular tensor pair $\{\mathcal{A}, \mathcal{B}\}$ as follows:

$$\begin{aligned}\Lambda_\epsilon^{\mathcal{E},\mathcal{F}}(\mathcal{A},\mathcal{B}) = \Big\{ & z \in \mathbb{C} \mid \exists \Delta\mathcal{A}, \Delta\mathcal{B} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N} \text{ such that } |\Delta\mathcal{A}| \le \epsilon\mathcal{E}, |\Delta\mathcal{B}| \le \epsilon\mathcal{F},\\ & \text{and } \mathcal{A}+\Delta\mathcal{A} - z(\mathcal{B}+\Delta\mathcal{B}) \text{ is singular}\Big\}.\end{aligned} \tag{17}$$

In other words, $\Lambda_{\epsilon}^{\mathcal{E},\mathcal{F}}(\mathcal{A},\mathcal{B}) = \{z \in \mathbb{C} \big| \, \omega(z) \le \epsilon\}$, but it turned out that the computation is NP-hard, when $N = 1$, $\mathcal{B} = I$ and $\mathcal{F} = \mathcal{O}$ (Poljak and Rohn 1993; Rump 2003). Aspired by the definition of componentwise ϵ-pseudospectrum of matrices (Malyshev and Sadkane 2004), we introduce the componentwise ϵ-pseudospectrum of a regular tensor pair $\{\mathcal{A},\mathcal{B}\}$ as

$$\Gamma_{\epsilon}^{\mathcal{E},\mathcal{F}}(\mathcal{A},\mathcal{B}) := \left\{ z \in \mathbb{C} | \; \left[\rho \left(|(\mathcal{A} - z\mathcal{B})^{-1}| *_N (\mathcal{E} + |z|\mathcal{F}) \right) \right]^{-1} \le \epsilon \right\}.$$

Set $\mathcal{B} = \mathcal{I}$ and the weight tensor $\mathcal{F} = \mathcal{O}$, the componentwise ϵ-pseudospectrum of a single tensor is

$$\Gamma_{\epsilon}(\mathcal{A}) := \Gamma_{\epsilon}^{\mathcal{E},\mathcal{O}}(\mathcal{A},\mathcal{I}) = \left\{ z \in \mathbb{C} | \; \left[\rho \left(|(\mathcal{A} - z\mathcal{I})^{-1}| *_N \mathcal{E} \right) \right]^{-1} \le \epsilon \right\};$$

when $N = 1$, the above definition is consistent with the one for matrices (Malyshev and Sadkane 2004).

Remark 4.3 Based on the inequality in Theorem 4.4, we get $\Lambda_{\epsilon}^{\mathcal{E},\mathcal{F}}(\mathcal{A},\mathcal{B}) \subset \Gamma_{\epsilon}^{\mathcal{E},\mathcal{F}}(\mathcal{A},\mathcal{B})$.

4.3 Numerical examples

In this subsection, we perform two numerical examples to show the backward errors and ϵ-pseudospectrum. Let $A = \phi(\mathcal{A})$, $B = \phi(\mathcal{B})$, $E = \phi(\mathcal{E})$, $F = \phi(\mathcal{F})$. From the definition in (13) and Lemma 2.4

$$\begin{aligned} \Lambda_{\epsilon}^{\alpha,\beta}(\mathcal{A},\mathcal{B}) &= \left\{ z \in \mathbb{C} \big| \; \left\| \phi\left((\mathcal{A} - z\mathcal{B})^{-1} \right) \right\|_2^{-1} \le \epsilon(\alpha + |z|\,\beta) \right\} \\ &= \left\{ z \in \mathbb{C} \big| \; \| (A - zB)^{-1} \|_2^{-1} \le \epsilon(\alpha + |z|\,\beta) \right\} \\ &= \left\{ z \in \mathbb{C} \Big| \; \frac{\sigma_{\min}(A - zB)}{\alpha + |z|\,\beta} \le \epsilon \right\}. \end{aligned} \tag{18}$$

Accordingly, we can portray the normwise ϵ-pseudospectrum of $\{\mathcal{A},\mathcal{B}\}$ by computing the smallest singular value of the unfolding matrix $(A - zB)$ on a grid region, then plot their contour figure, for which there are many efficient algorithms (Trefethen and Embree 2005), specially the toolbox “Eigtool” in Matlab for the visual pseudospectra computation of a single matrix (Thomas 2002). The computational process is in Algorithm 1, where step 2 is based on Remark 4.2 to reduce the amount of computations. Additionally, it holds from the isomorphic map ϕ that an even order tensor has the same spectrum as its unfolding matrix (Cui et al. 2016), then

$$\rho\left(\left|(\mathcal{A} - z\mathcal{B})^{-1}\right| *_N (\mathcal{E} + |z|\mathcal{F}) \right) = \rho\left(\left|(A - zB)^{-1}\right| (E + |z|F) \right).$$

Thereby, we compute the componentwise ϵ-pseudospectrum by Algorithm 2.

Example 4.1 (Brazell et al. 2013) The well-known Poisson equation

$$\begin{cases} -\nabla^2 v = f & \text{in} \quad \Omega, \\ v = 0 & \text{on} \; \partial\Omega, \end{cases} \tag{19}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : 1 < x, y < 1\}$ with boundary $\partial\Omega$, f is a given function, and

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2},$$

Algorithm 1 Normwise backward error and pseudospectra contour

Input: Regular tensor pair $\{\mathcal{A}, \mathcal{B}\}$ and grid region D
Output: Normwise pseudospectra contour on D
1: Unfolding process: $A = reshape(\mathcal{A}, [|\mathbf{I}|\ \ |\mathbf{I}|])$, $B = reshape(\mathcal{B}, [|\mathbf{I}|\ \ |\mathbf{I}|])$
2: Generalized Schur decomposition: $(T, S) = qz(A, B)$
3: Inverse power iteration to compute $\lambda_{\min}\left((T - zS)^H (T - zS)\right)$ on D
4: Compute $h(z) := \frac{\sigma_{\min}(A - zB)}{\alpha + |z|\beta} = \frac{\sqrt{\lambda_{\min}((T - zS)^H (T - zS))}}{\alpha + |z|\beta}$, plot the contour for $h(z)$ on D.

Algorithm 2 Componentwise pseudospectra contour

Input: Regular tensor pair $\{\mathcal{A}, \mathcal{B}\}$, weighted nonnegative tensors $\mathcal{E}$ and $\mathcal{F}$, grid region D
Output: Componentwise pseudospectra contour on D
1: Unfolding process: $A = reshape(\mathcal{A}, [|\mathbf{I}|\ \ |\mathbf{I}|])$, $B = reshape(\mathcal{B}, [|\mathbf{I}|\ \ |\mathbf{I}|])$, $E = reshape(\mathcal{E}, [|\mathbf{I}|\ \ |\mathbf{I}|])$, $F = reshape(\mathcal{F}, [|\mathbf{I}|\ \ |\mathbf{I}|])$
2: Computation of spectral radius $g(z) := \rho\left(|(A - zB)^{-1}|(E + |z|F)\right)$ on D
3: Plot the contour for $1/g(z)$ on D.

models several problems in physics and mechanics. The unknown function $v(x, y)$ can be solved approximately by finite difference method, where the mesh points are obtained by discretizing the unit square domain with step sizes, Δx in the x-direction and Δy in the y-direction. If assume that $\Delta x = \Delta y = h = \frac{1}{M+1}$, after the central difference approximations, we obtain the five-point difference scheme

$$4v_{ij} - v_{i-1j} - v_{i+1j} - v_{ij-1} - v_{ij+1} = h^2 f_{ij}, \quad i, j = 1, 2, \ldots, M. \tag{20}$$

Here, the Dirichlet boundary conditions are imposed, so $v_{i0} = v_{iM+1} = v_{0j} = v_{M+1j} = 0$ for $i, j = 1, 2, \ldots, M$.

The higher order tensor representation of the 2D discretized Poisson problem (20) is

$$\mathcal{A} *_2 V = F, \tag{21}$$

where $V = (v_{ij})$, $F = (f_{ij}) \in \mathbb{C}^{M \times M}$, $\mathcal{A} \in \mathbb{C}^{M \times M \times M \times M}$ is the fourth-order tensor with nonzero entries

$$\begin{cases} \mathcal{A}(m, n, m, n) = \frac{4}{h^2}, \quad m, n = 1, 2, \ldots, M, \\ \mathcal{A}(m, n, m, n+1) = \mathcal{A}(m, n+1, m, n) = \frac{-1}{h^2}, \quad m = 1, 2, \ldots, M, n = 1, 2, \ldots, M-1, \\ \mathcal{A}(m, n, m+1, n) = \mathcal{A}(m+1, n, m, n) = \frac{-1}{h^2}, \quad m = 1, 2, \ldots, M-1, n = 1, 2, \ldots, M. \end{cases}$$

The high-order Jacobi iterative method for solving the multilinear system (21) has the following tensor-based format:

$$V^{(k+1)} = \mathcal{J} *_2 V^{(k)} + \mathcal{D}^{-1} *_2 \mathcal{F},$$

where $\mathcal{D}$ is the diagonal tensor obtained from setting the off-diagonal elements of $\mathcal{A}$ to zero, $\mathcal{J} = \mathcal{D}^{-1} *_2 (\mathcal{D} - \mathcal{A})$ is the high-order Jacobi iterative tensor.

In the numerical test, we take $M = 4$. Figure 1 shows the normwise and componentwise ϵ-pseudospectrums in (a) and (c), respectively. As we can see, the normwise ϵ-pseudospectrum is exactly monotonically increasing with respect to ϵ, bounded and symmetric about the real axis, all of which are visualizations of the foregoing theoretical results. Besides, these pseudospectra have only small variance to the exact eigenvalues whenever the perturbations $\|\Delta\mathcal{J}\|_2 \leq 10^{-2.5}$, or $|\Delta\mathcal{J}| \leq 10^{-2.5}|\mathcal{J}|$.

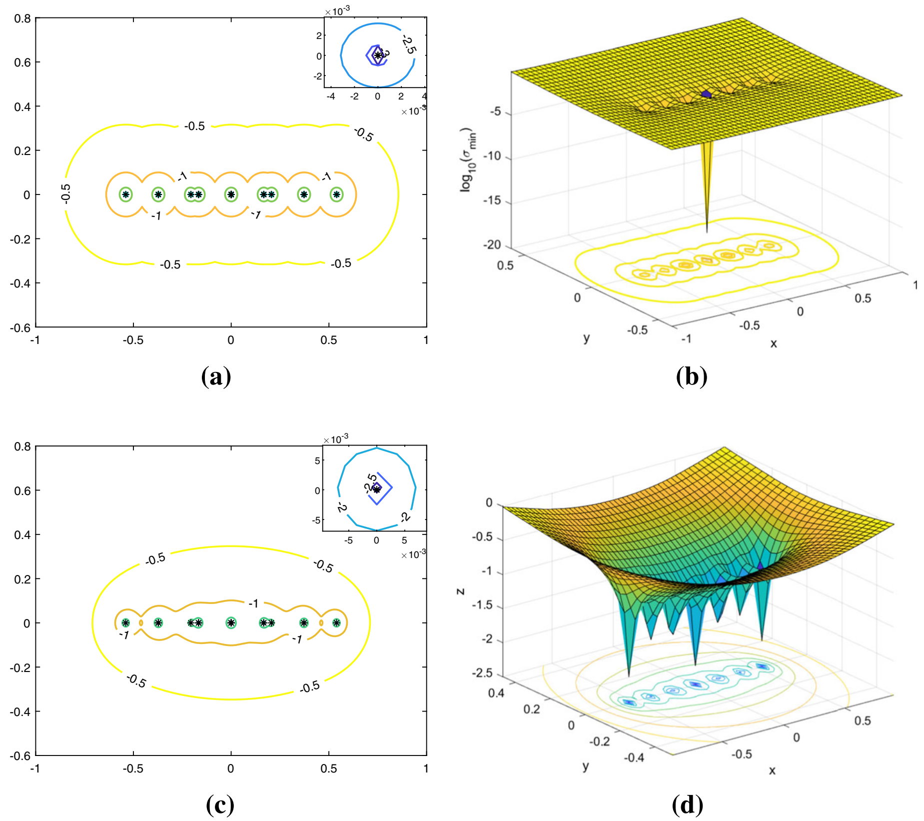


Fig. 1 **a** Boundaries of normwise pseudospectrum $\Lambda_\epsilon(\mathcal{J})$ for $\epsilon = 10^{-3.5}, 10^{-3}, \ldots, 10^{-0.5}$, the black stars $*$ are the positions of eigenvalues of $\mathcal{J}$. **b** Surface of the logarithmic value of normwise backward errors $\eta(x+\iota y) = \sigma_{\min}(\mathcal{J} - (x+\iota y)\mathcal{I})$ to the base 10. **c** Boundaries of componentwise pseudospectrum $\Gamma_\epsilon(\mathcal{J})$ for $\epsilon = 10^{-3.5}, 10^{-3}, \ldots, 10^{-0.5}$ by setting $\mathcal{E} = |\mathcal{J}|$, the black stars $*$ are the positions of eigenvalues of $\mathcal{J}$. **d** Surface of the logarithmic value of $[\rho\left(\left|(\mathcal{J} - \lambda\mathcal{I})^{-1}\right| *_N |\mathcal{J}|\right)]^{-1}$ to the base 10

Example 4.2 The generalized linear systems have been extensively applied in many engineering and technological fields. Under the tensor algebraic structure via Einstein product, we introduce the generalized multilinear (GML) systems

$$\mathcal{B} *_N \frac{\mathrm{d}\mathcal{X}(t)}{\mathrm{d}t} = \mathcal{A} *_N \mathcal{X}(t), \tag{22}$$

where $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times I_1 \times I_2 \times \cdots \times I_N}$ are square tensors, and $\mathcal{X}(t)$ is a tensor function to be solved. Ansatz $\mathcal{X}(t) = \mathcal{X}_0 e^{\lambda t}$ as a solution of this system, and then, it leads to a generalized eigenvalue problem $\mathcal{A} *_N \mathcal{X}_0 = \lambda \mathcal{B} *_N \mathcal{X}_0$. As a result, the asymptotic stability of the solution is closely related to the spectra of $\{\mathcal{A}, \mathcal{B}\}$, and we can analyze the stability of perturbed systems by ϵ-pseudospectrum. In the numerical computation, $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{3\times 4\times 3\times 4}$ have nonzero entries

$$\begin{cases} \mathcal{A}(m,n,m,n) = \frac{(16-m-3n)+\iota(m+3n-4)}{12}, & m = 1,2,3, n = 1,2,3,4, \\ \mathcal{A}(m,n,m+1,n) = \frac{(m+3n-3)-\iota}{6}, & m = 1,2, n = 1,2,3,4, \\ \mathcal{A}(3,n,1,n+1) = 1-\iota, & n = 1,2,3, \end{cases} \tag{23}$$

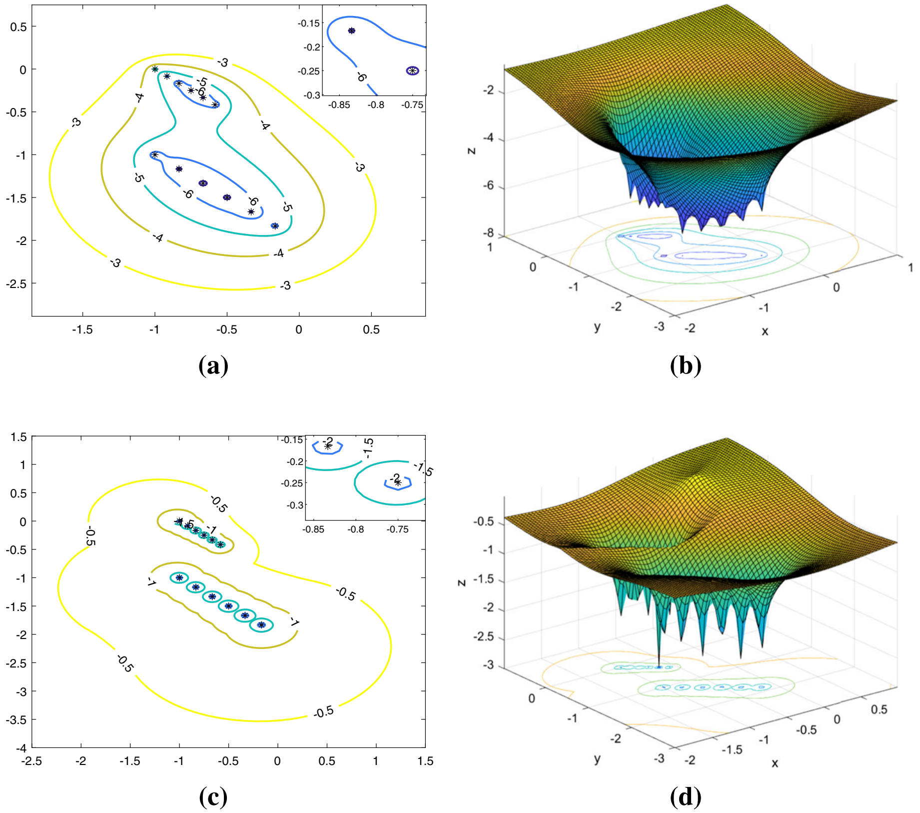


Fig. 2 **a** Boundaries of normwise pseudospectrum $\Lambda_{\epsilon}^{1,1}(\mathcal{A}, \mathcal{B})$ for $\epsilon = 10^{-7}, 10^{-6}, \ldots, 10^{-3}$, the black stars $*$ are the positions of generalized eigenvalues of $\{\mathcal{A}, \mathcal{B}\}$. **b** Surface of the logarithmic value of normwise backward error $\eta(x + \iota y)$ to the base 10. **c** Boundaries of componentwise pseudospectrum $\Gamma_{\epsilon}^{|\mathcal{A}|,|\mathcal{B}|}(\mathcal{A}, \mathcal{B})$ for $\epsilon = 10^{-3}, 10^{-2.5}, \ldots, 10^{-0.5}$, the black stars $*$ are the positions of generalized eigenvalues of $\{\mathcal{A}, \mathcal{B}\}$. (d) Surface of the logarithmic value of $\left[\rho\left(|(\mathcal{A} - \lambda\mathcal{B})^{-1}| *_N (|\mathcal{A}| + |\lambda||\mathcal{B}|)\right)\right]^{-1}$ to the base 10

$$\begin{cases} \mathcal{B}(m, n, m, n) = -1, \; m = 1, 2, 3, n = 1, 2, \\ \mathcal{B}(m, n, m, n) = -\frac{1}{2}, \; m = 1, 2, 3, n = 3, 4, \end{cases} \tag{24}$$

where ι denotes the imaginary unit $\sqrt{-1}$. We investigate the pseudospectrum of regular tensor pair $\{\mathcal{A}, \mathcal{B}\}$ presented in Fig. 2, and the experimental results are consistent with our pseudospectral theories. We can get more informations for the GML system (22)–(24) from the pseudospectra contour figure, $\Lambda(\mathcal{A}, \mathcal{B}) \subset \Lambda_{10^{-5}}^{1,1}(\mathcal{A}, \mathcal{B}) \subset \{z \in \mathbb{C} | \, \mathrm{Real}(z) < 0\}$, and $\Gamma_{10^{-1.5}}^{|\mathcal{A}|,|\mathcal{B}|}(\mathcal{A}, \mathcal{B}) \subset \{z \in \mathbb{C} | \, \mathrm{Real}(z) < 0\}$; in other words, all perturbed generalized eigenvalues in these pseudospectrum of tensor pair $\{\mathcal{A} + \Delta\mathcal{A}, \mathcal{B} + \Delta\mathcal{B}\}$ are located in the left-half complex plane, and the solutions of the GML systems (22)–(24) are still asymptotically stable when the perturbations $\|\Delta\mathcal{A}\|_2 \leq 10^{-5}$, $\|\Delta\mathcal{B}\|_2 \leq 10^{-5}$, or $|\Delta\mathcal{A}| \leq 10^{-1.5}|\mathcal{A}|$, $|\Delta\mathcal{B}| \leq 10^{-1.5}|\mathcal{B}|$.

5 Applications in multilinear control systems

The control theory of linear time invariant (LTI) systems has been applied in many engineering technology fields. In recent years, a rather bit of research on multilinear time invariant (MLTI) systems emerged. For instance, Rogers et al. (2013) introduced a class of MLTI systems, in which the states and outputs are set as tensors, and the evolution of systems is generated by the action of tensors on matrices using Tucker product. Chen et al. (2019, 2021) generalized these MLTI systems via Einstein product of even order tensors, which can be formulated as

$$\begin{cases} \mathcal{X}_{t+1} = \mathcal{A} *_N \mathcal{X}_t + \mathcal{B} *_N \mathcal{U}_t, \\ \mathcal{Y}_t = \mathcal{C} *_N \mathcal{X}_t, \end{cases} \tag{25}$$

where $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$, $\mathcal{B} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times J_1\times J_2\times\cdots\times J_N}$, $\mathcal{C} \in \mathbb{C}^{K_1\times K_2\times\cdots\times K_N\times I_1\times I_2\times\cdots\times I_N}$. Meanwhile, they studied the stability, reachability, and observability for the MLTI systems, etc.

Proposition 5.1 (Chen et al. 2019, 2021) *The pair* $\{\mathcal{A}, \mathcal{B}\}$ *is reachable if and only if the* $I_1 \times \cdots \times I_N \times I_1 J_1 \times \cdots \times I_N J_N$ *even order reachability tensor*

$$\mathcal{R} = \left(\mathcal{B}\ \mathcal{A} *_N \mathcal{B} \cdots \mathcal{A}^{|\mathbf{I}|-1} *_N \mathcal{B}\right)$$

spans $\mathbb{C}^{I_1\times I_2\times\cdots\times I_N}$*, where the row block tensor* $\mathcal{R}$ *is recursively concatenated by Definition 2.7 in Chen et al. (2019). In other words,* $\mathrm{rank}_U(\mathcal{R}) = |\mathbf{I}|$.

Let $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$, $\mathcal{B} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times \bar{I}_1\times I_2\times\cdots\times I_N}$. Then, we have the following necessary and sufficient condition for reachability.

Proposition 5.2 *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times I_1\times I_2\times\cdots\times I_N}$, $\mathcal{B} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times \bar{I}_1\times I_2\times\cdots\times I_N}$. *The pair* $\{\mathcal{A}, \mathcal{B}\}$ *is reachable if and only if*

$$\mathrm{rank}_U([\mathcal{A} - \lambda\mathcal{I}\ \mathcal{B}]) = |\mathbf{I}| \quad \text{for } \forall\lambda \in \mathbb{C}.$$

Proof Using the unfolding transformation ϕ, one can express

$$\phi(\mathcal{R}) = \left[B\ AB\ \cdots\ A^{|\mathbf{I}|-1}B\right] P,$$

where $A = \phi(\mathcal{A})$, $B = \phi(\mathcal{B})$ and P is some permutation matrix (Chen et al. 2019). Hence

$$\mathrm{rank}_U(\mathcal{R}) = \mathrm{rank}\left(\left[B\ AB\ \cdots\ A^{|\mathbf{I}|-1}B\right]\right).$$

On the other hand, from Proposition 2.1, we have

$$\mathrm{rank}_U([\mathcal{A} - \lambda\mathcal{I}\ \mathcal{B}]) = \mathrm{rank}\left(\left[\phi(\mathcal{A} - \lambda\mathcal{I})\ \phi(\mathcal{B})\right]\right) = \mathrm{rank}\left(\left[A - \lambda I\ \ B\right]\right).$$

From classical control theory for LTI systems, it holds that

$$\mathrm{rank}\left(\left[B\ AB\ \cdots\ A^{|\mathbf{I}|-1}B\right]\right) = |\mathbf{I}| \Leftrightarrow \mathrm{rank}_U\left(\left[A - \lambda I\ \ B\right]\right) = |\mathbf{I}| \text{ for } \forall\lambda \in \mathbb{C}.$$

Therefore

$$\mathrm{rank}_U(\mathcal{R}) = |\mathbf{I}| \Leftrightarrow \mathrm{rank}_U\left(\left[\mathcal{A} - \lambda\mathcal{I}\ \mathcal{B}\right]\right) = |\mathbf{I}| \text{ for } \forall\lambda \in \mathbb{C}.$$

□

In classical linear control theory for LTI systems, the distance to uncontrollability (Eising 1984) is of great interest. For the discrete-time MLTI systems (25), we analyze the distance from a reachable pair to unreachability using the normwise pseudospectral theory.

The normwise ϵ-pseudospectrum adapts to the nonsquare tensor pairs, and the proof of Lemma 4.1 still holds; accordingly, we have the following result.

Corollary 5.1 *Let* $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N\times J_1\times J_2\times\cdots\times J_N}$, $|\mathbf{I}| \geq |\mathbf{J}|$. *Then*

$$\Lambda_\epsilon^{\alpha,\beta}(\mathcal{A}, \mathcal{B}) \neq \emptyset \Leftrightarrow \min_{z\in\mathbb{C}} \min_{\|\mathcal{U}\|_F=1} \frac{\|(\mathcal{A} - z\mathcal{B}) *_N \mathcal{U}\|_F}{\alpha + |z|\,\beta} \leq \epsilon.$$

Referring to the definition of the distance to uncontrollability for LTI systems (Eising 1984), we define the distance from a reachable MLTI system to unreachability

$$\delta_u(\mathcal{A}, \mathcal{B}) := \min\left\{\epsilon > 0 \middle| \{\mathcal{A} + \Delta\mathcal{A}, \mathcal{B} + \Delta\mathcal{B}\} \text{ is unreachable, for all } \left\|\left[\Delta\mathcal{A}\ \Delta\mathcal{B}\right]\right\|_2 \leq \epsilon\right\};$$

using Proposition 5.2 gives

$$\delta_u(\mathcal{A}, \mathcal{B}) = \min\{\epsilon > 0 \big| \operatorname{rank}_U\left(\left[\mathcal{A} + \Delta\mathcal{A} - z\mathcal{I}\ \mathcal{B} + \Delta\mathcal{B}\right]\right) < |\mathbf{I}|$$
$$\text{for some } z \in \mathbb{C} \text{ for all } \left\|\left[\Delta\mathcal{A}\ \Delta\mathcal{B}\right]\right\|_2 \leq \epsilon\}.$$

It follows from Definition 2.3 and Lemma 2.1 that:

$$\delta_u(\mathcal{A}, \mathcal{B}) = \min\{\epsilon > 0 \big| \exists\, \mathcal{U} \neq \mathcal{O},\ (\mathcal{C} + \Delta\mathcal{C} - z\mathcal{D}) *_N \mathcal{U} = \mathcal{O}$$
$$\text{for some } z \in \mathbb{C} \text{ for all } \left\|\left[\Delta\mathcal{A}\ \Delta\mathcal{B}\right]\right\|_2 \leq \epsilon\},$$

where

$$\mathcal{C} = \begin{bmatrix} \mathcal{A}^H \\ \mathcal{B}^H \end{bmatrix},\ \mathcal{D} = \begin{bmatrix} \mathcal{I} \\ \mathcal{O} \end{bmatrix},\ \Delta\mathcal{C} = \begin{bmatrix} \Delta\mathcal{A}^H \\ \Delta\mathcal{B}^H \end{bmatrix}.$$

By the language of normwise ϵ-pseudospectrum, we have

$$\delta_u(\mathcal{A}, \mathcal{B}) = \min\{\epsilon > 0 \big| \Lambda_\epsilon^{1,0}(\mathcal{C}, \mathcal{D}) \neq \emptyset\}.$$

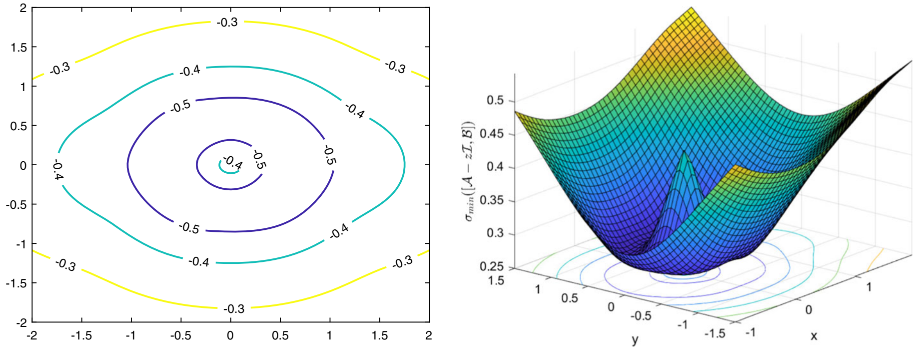
It follows from Corollary 5.1 that:

$$\begin{aligned}\delta_u(\mathcal{A}, \mathcal{B}) &= \min_{z\in\mathbb{C}} \min_{\|\mathcal{U}\|_F=1} \|(\mathcal{C} - z\mathcal{D}) *_N \mathcal{U}\|_F \\ &= \min_{z\in\mathbb{C}} \min_{\|\mathcal{U}\|_F=1} \left\| \begin{bmatrix} \mathcal{A}^H - z\mathcal{I} \\ \mathcal{B}^H \end{bmatrix} *_N \mathcal{U} \right\|_F \\ &= \min_{z\in\mathbb{C}} \sigma_{\min}\left(\left[\mathcal{A} - z\mathcal{I}\ \mathcal{B}\right]\right).\end{aligned}$$

According to Lemma 2.4 in Ma et al. (2019), the tensor $\left[\mathcal{A} - z\mathcal{I}\ \mathcal{B}\right]$ possesses the same singular values as its unfolding matrix $\phi\left(\left[\mathcal{A} - z\mathcal{I}\ \mathcal{B}\right]\right)$, then has the same singular values as $\left[\phi(\mathcal{A}) - zI\ \phi(\mathcal{B})\right]$ from Proposition 2.1. Therefore, we can estimate the distance by the formulation $\delta_u(\mathcal{A}, \mathcal{B}) = \min_{z\in\mathbb{C}} \sigma_{\min}\left(\left[\phi(\mathcal{A}) - zI\ \phi(\mathcal{B})\right]\right)$, for which there have been efficient numerical methods (Burke et al. 2004; Gu 2000).

For example, we consider the following reachable MLTI system where $\mathcal{A} \in \mathbb{C}^{3\times2\times3\times2}$, $\mathcal{B} \in \mathbb{C}^{3\times2\times1\times2}$. The pseudospectral contour and the surface of smallest singular values of $\left[\mathcal{A} - z\mathcal{I}\ \mathcal{B}\right]$ are presented in Fig. 3, which show a upper bound $10^{-0.5}$ for the distance $\delta_u(\mathcal{A}, \mathcal{B})$.

$$\mathcal{A}(:, :, 1, 1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad \mathcal{A}(:, :, 2, 1) = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \\ 0 & 0.25 \end{bmatrix},$$



(a) (b)

Fig. 3 **a** Boundaries of normwise pseudospectrum $\Lambda_{\epsilon}^{1,0}(\mathcal{C}, \mathcal{D})$ for $\epsilon = 10^{-0.5}, 10^{-0.4}, 10^{-0.3}$. **b** Surface of the minimum singular value of $\left[\mathcal{A} - (x + \iota y)\mathcal{I}\ \mathcal{B}\right]$

$$\mathcal{A}(:,:,1,1) = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \\ 0 & 0.4 \end{bmatrix}, \quad \mathcal{A}(:,:,1,2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.2 & 0 \end{bmatrix},$$
$$\mathcal{A}(:,:,2,2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0.5 & 0 \end{bmatrix}, \quad \mathcal{A}(:,:,3,2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0.8 & 0 \end{bmatrix},$$
$$\mathcal{B}(:,:,1,1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{B}(:,:,1,2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

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Declarations

Conflict of interest The authors have not disclosed any competing interests.

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