The University of Austin at Texas

those who know

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Contest (1)

template.cpp 14 lines

```
#include <bits/stdc++.h>
using namespace std;

#define rep(i, a, b) for(int i = a; i < (b); ++i)
#define all(x) begin(x), end(x)
#define sz(x) (int)(x).size()
#define pb push_back
typedef long long ll;
typedef pair<int, int> pii;
typedef vector<int> vi;

int main() {
  cin.tie(0)->sync_with_stdio(0);
}
```

.bashrc 10 lines

```
run () {
  ok=1
  if [[ ! -f $1 || $1 -ot $1.cpp ]]
  then
    g++ $1.cpp -O2 -o $1 -std=c++17 -Wall -Wextra -Wshadow
         -Wconversion -fsanitize=undefined,address || ok=0
  fi
  [[ $ok -eq 1 ]] && ./$1
}

xmodmap -e 'clear Lock' -e 'keycode 0x42 = Escape'
```

.vimrc 5 lines

```
set cin aw ai is ts=4 sw=4 tm=50 rnu noeb bg=dark ru cul
     mouse=a
sy on | no ; :
" Select region and then type :Hash to hash your selection.
ca Hash w !cpp -dD -P -fpreprocessed \| tr -d '[:space:]' \
 \| md5sum \| cut -c-6
```

brute.sh 12 lines

```
#!/bin/zsh

sz=100
for ((i=1;;i++)); do
    echo "$i"
    ./gen "$i" "$sz" > input
    ./sol < input > output1
    ./brute < input > output2
    if (! diff output1 output2); then
        break
    fi
done
```

Mathematics (2)

2.1 Equations

$$\begin{matrix} ax+by=e \\ cx+dy=f \end{matrix} \Rightarrow \begin{matrix} x=\dfrac{ed-bf}{ad-bc} \\ y=\dfrac{af-ec}{ad-bc} \end{matrix}$$

$$x_i=\frac{\det A_i'}{\det A}$$

2.2 Trigonometry

$$\sin(v+w)=\sin v\cos w+\cos v\sin w$$
$$\cos(v+w)=\cos v\cos w-\sin v\sin w$$
$$\tan(v+w)=\frac{\tan v+\tan w}{1-\tan v\tan w}$$
$$\sin v+\sin w=2\sin\frac{v+w}{2}\cos\frac{v-w}{2}$$
$$\cos v+\cos w=2\cos\frac{v+w}{2}\cos\frac{v-w}{2}$$

$$(V+W)\tan(v-w)/2=(V-W)\tan(v+w)/2$$

where V, W are lengths of sides opposite angles v, w.

$$a\cos x+b\sin x=r\cos(x-\phi)$$
$$a\sin x+b\cos x=r\sin(x+\phi)$$

where $r=\sqrt{a^2+b^2}, \phi=\operatorname{atan2}(b,a)$.

2.3 Geometry

2.3.1 Triangles

Circumradius: $R=abc/4A$
Inradius: $r=A/p$
Length of median (divides triangle into two equal-area triangles): $m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$
Length of bisector (divides angles in two):

$$s_a=\sqrt{bc\left[1-\left(\frac{a}{b+c}\right)^2\right]}$$

Law of sines: $\dfrac{\sin\alpha}{a}=\dfrac{\sin\beta}{b}=\dfrac{\sin\gamma}{c}=\dfrac{1}{2R}$
Law of cosines: $a^2=b^2+c^2-2bc\cos\alpha$

Law of tangents: $\dfrac{a+b}{a-b}=\dfrac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$

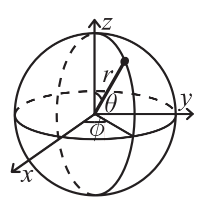
2.3.2 Quadrilaterals

With side lengths a, b, c, d, diagonals e, f, diagonals angle θ, area A and magic flux $F=b^2+d^2-a^2-c^2$:

$$4A=2ef\cdot\sin\theta=F\tan\theta=\sqrt{4e^2f^2-F^2}$$

For cyclic quadrilaterals the sum of opposite angles is 180°, $ef=ac+bd$, and $A=\sqrt{(p-a)(p-b)(p-c)(p-d)}$.

2.3.3 Spherical coordinates



$$\begin{matrix} x=r\sin\theta\cos\phi & r=\sqrt{x^2+y^2+z^2} \\ y=r\sin\theta\sin\phi & \theta=\operatorname{acos}(z/\sqrt{x^2+y^2+z^2}) \\ z=r\cos\theta & \phi=\operatorname{atan2}(y,x) \end{matrix}$$

2.4 Derivatives/Integrals

$$\frac{d}{dx}\arcsin x=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x=-\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan x=1+\tan^2 x \qquad \frac{d}{dx}\arctan x=\frac{1}{1+x^2}$$
$$\int\tan ax=-\frac{\ln|\cos ax|}{a} \qquad \int x\sin ax=\frac{\sin ax-ax\cos ax}{a^2}$$
$$\int e^{-x^2}=\frac{\sqrt{\pi}}{2}\operatorname{erf}(x) \qquad \int xe^{ax}dx=\frac{e^{ax}}{a^2}(ax-1)$$

Integration by parts:

$$\int_a^b f(x)g(x)dx=[F(x)g(x)]_a^b-\int_a^b F(x)g'(x)dx$$

2.5 Sums

$$c^a+c^{a+1}+\cdots+c^b=\frac{c^{b+1}-c^a}{c-1}, c\neq 1$$

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$
$$1^2+2^2+3^2+\cdots+n^2=\frac{n(2n+1)(n+1)}{6}$$
$$1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$$
$$1^4+2^4+3^4+\cdots+n^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

2.6 Series

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots,(-\infty<x<\infty)$$
$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots,(-1<x\leq 1)$$
$$\sqrt{1+x}=1+\frac{x}{2}-\frac{x^2}{8}+\frac{2x^3}{32}-\frac{5x^4}{128}+\ldots,(-1\leq x\leq 1)$$
$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots,(-\infty<x<\infty)$$
$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots,(-\infty<x<\infty)$$

2.7 Probability theory

$$\sigma^2=V(X)=\mathbb{E}(X^2)-(\mathbb{E}(X))^2$$
$$\mathbb{E}(aX+bY)=a\mathbb{E}(X)+b\mathbb{E}(Y)$$
$$\text{ind. } X, Y, V(aX+bY)=a^2V(X)+b^2V(Y).$$

2.7.1 Discrete distributions

Binomial distribution

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = np,\ \sigma^2 = np(1-p)$$

$\mathrm{Bin}(n,p)$ is approximately $\mathrm{Po}(np)$ for small p.

Geometric distribution

$$p(k) = p(1-p)^{k-1},\ k = 1,2,\ldots$$

$$\mu = \frac{1}{p},\ \sigma^2 = \frac{1-p}{p^2}$$

Poisson distribution

$$p(k) = e^{-\lambda}\frac{\lambda^k}{k!}, k = 0,1,2,\ldots$$

$$\mu = \lambda,\ \sigma^2 = \lambda$$

2.7.2 Continuous distributions

Uniform distribution

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \frac{a+b}{2},\ \sigma^2 = \frac{(b-a)^2}{12}$$

Exponential distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mu = \frac{1}{\lambda},\ \sigma^2 = \frac{1}{\lambda^2}$$

Normal distribution

Most real random values with mean μ and variance σ^2 are well described by $\mathcal{N}(\mu,\sigma^2)$, $\sigma > 0$.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If $X_1 \sim \mathcal{N}(\mu_1,\sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2,\sigma_2^2)$ then

$$aX_1 + bX_2 + c \sim \mathcal{N}(\mu_1+\mu_2+c, a^2\sigma_1^2 + b^2\sigma_2^2)$$

2.8 Markov chains

A *Markov chain* is a discrete random process with the property that the next state depends only on the current state. Let $X_1, X_2, \ldots$ be a sequence of random variables generated by the Markov process. Then there is a transition matrix $\mathbf{P} = (p_{ij})$, with $p_{ij} = \Pr(X_n = i | X_{n-1} = j)$, and $\mathbf{p}^{(n)} = \mathbf{P}^n\mathbf{p}^{(0)}$ is the probability distribution for X_n (i.e., $p_i^{(n)} = \Pr(X_n = i)$), where $\mathbf{p}^{(0)}$ is the initial distribution.

π is a stationary distribution if $\pi = \pi\mathbf{P}$. If the Markov chain is *irreducible* (it is possible to get to any state from any state), then $\pi_i = \frac{1}{\mathbb{E}(T_i)}$ where $\mathbb{E}(T_i)$ is the expected time between two visits in state i. π_j/π_i is the expected number of visits in state j between two visits in state i.

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors, π_i is proportional to node i's degree.

A Markov chain is *ergodic* if the asymptotic distribution is independent of the initial distribution. A finite Markov chain is ergodic iff it is irreducible and *aperiodic* (i.e., the gcd of cycle lengths is 1). $\lim_{k\to\infty}\mathbf{P}^k = \mathbf{1}\pi$.

A Markov chain is an A-chain if the states can be partitioned into two sets $\mathbf{A}$ and $\mathbf{G}$, such that all states in $\mathbf{A}$ are absorbing ($p_{ii} = 1$), and all states in $\mathbf{G}$ leads to an absorbing state in $\mathbf{A}$. The probability for absorption in state $i \in \mathbf{A}$, when the initial state is j, is $a_{ij} = p_{ij} + \sum_{k\in\mathbf{G}} a_{ik}p_{kj}$. The expected time until absorption, when the initial state is i, is $t_i = 1 + \sum_{k\in\mathbf{G}} p_{ki}t_k$.

Data structures (3)

OrderStatisticTree.h

Description: A set (not multiset!) with support for finding the n'th element, and finding the index of an element. To get a map, change `null_type`.
Time: $\mathcal{O}(\log N)$

782797, 16 lines

```
#include <bits/extc++.h>//893
using namespace __gnu_pbds;

template<class T>
using Tree = tree<T, null_type, less<T>, rb_tree_tag,
  tree_order_statistics_node_update>;//988

void example() {
  Tree<int> t, t2;
  auto it = t.insert(10).first;
  assert(it == t.lower_bound(9));//6bd
  assert(t.order_of_key(10) == 1);
  assert(t.order_of_key(11) == 2);
  assert(*t.find_by_order(0) == 8);
  t.join(t2); // assuming T < T2 or T > T2, merge t2 into t
}//cbb
```

HashMap.h

Description: Hash map with mostly the same API as unordered_map, but ~3x faster. Uses 1.5x memory. Initial capacity must be a power of 2 (if provided).

d77092, 7 lines

```
#include <bits/extc++.h>//1e4
// To use most bits rather than just the lowest ones:
struct chash { // large odd number for C
  const uint64_t C = ll(4e18 * acos(0)) | 71;
  ll operator()(ll x) const { return __builtin_bswap64(x*C)
      ; }
};//198
__gnu_pbds::gp_hash_table<ll,int,chash> h({},{},{},{},{
    1<<16});
```

Matrix.h

Description: Basic operations on square matrices.
Usage: `Matrix<int, 3> A;`
`A.d = {{{{1,2,3}}, {{4,5,6}}, {{7,8,9}}}};`
`vector<int> vec = {1,2,3};`
`vec = (A^N) * vec;`

c43c7d, 26 lines

```
template<class T, int N> struct Matrix {//1aa
  typedef Matrix M;
  array<array<T, N>, N> d{};
  M operator*(const M& m) const {
    M a;
    rep(i,0,N) rep(j,0,N)//683
      rep(k,0,N) a.d[i][j] += d[i][k]*m.d[k][j];
    return a;
  }
  vector<T> operator*(const vector<T>& vec) const {
    vector<T> ret(N);//9bd
    rep(i,0,N) rep(j,0,N) ret[i] += d[i][j] * vec[j];
    return ret;
  }
  M operator^(ll p) const {
    assert(p >= 0);//358
    M a, b(*this);
    rep(i,0,N) a.d[i][i] = 1;
    while (p) {
      if (p&1) a = a*b;
      b = b*b;//1d8
      p >>= 1;
    }
    return a;
  }
};//214
```

LineContainer.h

Description: Container where you can add lines of the form kx+m, and query maximum values at points x. Useful for dynamic programming ("convex hull trick").
Time: $\mathcal{O}(\log N)$

8ec1c7, 30 lines

```
struct Line {//7e3
  mutable ll k, m, p;
  bool operator<(const Line& o) const { return k < o.k; }
  bool operator<(ll x) const { return p < x; }
};
//d77
struct LineContainer : multiset<Line, less<>> {
  // (for doubles, use inf = 1/.0, div(a,b) = a/b)
  static const ll inf = LLONG_MAX;
  ll div(ll a, ll b) { // floored division
    return a / b - ((a ^ b) < 0 && a % b); }//66e
  bool isect(iterator x, iterator y) {
    if (y == end()) return x->p = inf, 0;
    if (x->k == y->k) x->p = x->m > y->m ? inf : -inf;
    else x->p = div(y->m - x->m, x->k - y->k);
    return x->p >= y->p;//bec
  }
  void add(ll k, ll m) {
```

```
    auto z = insert({k, m, 0}), y = z++, x = y;
    while (isect(y, z)) z = erase(z);
    if (x != begin() && isect(--x, y)) isect(x, y = erase(y
        ));//890
    while ((y = x) != begin() && (--x)->p >= y->p)
      isect(x, erase(y));
  }
  ll query(ll x) {
    assert(!empty());//b07
    auto l = *lower_bound(x);
    return l.k * x + l.m;
  }
};
```

Treap.h

Description: A short self-balancing tree. It acts as a sequential container with log-time splits/joins, and is easy to augment with additional data.
Time: $\mathcal{O}(\log N)$ 30f532, 55 lines

```
struct Node {//e9f
  Node *l = 0, *r = 0;
  int val, y, c = 1;
  Node(int v) : val(v), y(rand()) {}
  void recalc();
};//3ef

int cnt(Node* n) { return n ? n->c : 0; }
void Node::recalc() { c = cnt(l) + cnt(r) + 1; }

template<class F> void each(Node* n, F f) {//5d5
  if (n) { each(n->l, f); f(n->val); each(n->r, f); }
}

pair<Node*, Node*> split(Node* n, int k) {
  if (!n) return {};//ca5
  if (cnt(n->l) >= k) { // "n->val >= k" for lower_bound(k)
    auto pa = split(n->l, k);
    n->l = pa.second;
    n->recalc();
    return {pa.first, n};//b54
  } else {
    auto pa = split(n->r, k - cnt(n->l) - 1); // and just "
        k"
    n->r = pa.first;
    n->recalc();
    return {n, pa.second};//86d
  }
}

Node* merge(Node* l, Node* r) {
  if (!l) return r;//fbf
  if (!r) return l;
  if (l->y > r->y) {
    l->r = merge(l->r, r);
    l->recalc();
    return l;//780
  } else {
    r->l = merge(l, r->l);
    r->recalc();
    return r;
  }//96d
}

Node* ins(Node* t, Node* n, int pos) {
  auto pa = split(t, pos);
  return merge(merge(pa.first, n), pa.second);//99b
}

// Example application: move the range [l, r) to index k
void move(Node*& t, int l, int r, int k) {
  Node *a, *b, *c;//99c
  tie(a,b) = split(t, l); tie(b,c) = split(b, r - l);
  if (k <= l) t = merge(ins(a, b, k), c);
  else t = merge(a, ins(c, b, k - r));
}
```

FenwickTree.h

Description: Computes partial sums a[0] + a[1] + ... + a[pos - 1], and updates single elements a[i], taking the difference between the old and new value.
Time: Both operations are $\mathcal{O}(\log N)$. e62fac, 22 lines

```
struct FT {//711
  vector<ll> s;
  FT(int n) : s(n) {}
  void update(int pos, ll dif) { // a[pos] += dif
    for (; pos < sz(s); pos |= pos + 1) s[pos] += dif;
  }//cc4
  ll query(int pos) { // sum of values in [0, pos)
    ll res = 0;
    for (; pos > 0; pos &= pos - 1) res += s[pos-1];
    return res;
  }//477
  int lower_bound(ll sum) {// min pos st sum of [0, pos] >=
      sum
    // Returns n if no sum is >= sum, or -1 if empty sum is
      .
    if (sum <= 0) return -1;
    int pos = 0;
    for (int pw = 1 << 25; pw; pw >>= 1) {//fc5
      if (pos + pw <= sz(s) && s[pos + pw-1] < sum)
        pos += pw, sum -= s[pos-1];
    }
    return pos;
  }//e03
};
```

FenwickTree2d.h

Description: Computes sums a[i,j] for all i<I, j<J, and increases single elements a[i,j]. Requires that the elements to be updated are known in advance (call fakeUpdate() before init()).
Time: $\mathcal{O}(\log^2 N)$. (Use persistent segment trees for $\mathcal{O}(\log N)$.)
"FenwickTree.h" e2f703, 22 lines

```
struct FT2 {//4ce
  vector<vi> ys; vector<FT> ft;
  FT2(int limx) : ys(limx) {}
  void fakeUpdate(int x, int y) {
    for (; x < sz(ys); x |= x + 1) ys[x].pb(y);
  }//57f
  void init() {
    for (vi& v : ys) sort(all(v)), ft.emplace_back(sz(v));
  }
  int ind(int x, int y) {
    return (int)(lower_bound(all(ys[x]), y) - ys[x].begin()
        ); }//358
  void update(int x, int y, ll dif) {
    for (; x < sz(ys); x |= x + 1)
      ft[x].update(ind(x, y), dif);
  }
  ll query(int x, int y) {//688
    ll sum = 0;
    for (; x; x &= x - 1)
      sum += ft[x-1].query(ind(x-1, y));
    return sum;
  }//e03
};
```

RMQ.h

Description: Range Minimum Queries on an array. Returns min(V[a], V[a + 1], ... V[b - 1]) in constant time.
Usage: `RMQ rmq(values);`
`rmq.query(inclusive, exclusive);`
Time: $\mathcal{O}(|V| \log |V| + Q)$ 510c32, 16 lines

```
template<class T>//722
struct RMQ {
  vector<vector<T>> jmp;
  RMQ(const vector<T>& V) : jmp(1, V) {
    for (int pw = 1, k = 1; pw * 2 <= sz(V); pw *= 2, ++k)
        {
      jmp.emplace_back(sz(V) - pw * 2 + 1);//f6c
      rep(j,0,sz(jmp[k]))
        jmp[k][j] = min(jmp[k - 1][j], jmp[k - 1][j + pw]);
    }
  }
  T query(int a, int b) {//a3d
    assert(a < b); // or return inf if a == b
    int dep = 31 - __builtin_clz(b - a);
    return min(jmp[dep][a], jmp[dep][b - (1 << dep)]);
  }
};//214
```

MoQueries.h

Description: Answer interval or tree path queries by finding an approximate TSP through the queries, and moving from one query to the next by adding/removing points at the ends. If values are on tree edges, change `step` to add/remove the edge (a, c) and remove the initial `add` call (but keep `in`).
Time: $\mathcal{O}(N\sqrt{Q})$ a12ef4, 49 lines

```
void add(int ind, int end) { ... } // add a[ind] (end = 0
    or 1)//342
void del(int ind, int end) { ... } // remove a[ind]
int calc() { ... } // compute current answer

vi mo(vector<pii> Q) {
  int L = 0, R = 0, blk = 350; // ~N/sqrt(Q)//cb0
  vi s(sz(Q)), res = s;
#define K(x) pii(x.first/blk, x.second ^ -(x.first/blk & 1)
    )
  iota(all(s), 0);
  sort(all(s), [&](int s, int t){ return K(Q[s]) < K(Q[t]);
      });
  for (int qi : s) {//623
    pii q = Q[qi];
    while (L > q.first) add(--L, 0);
    while (R < q.second) add(R++, 1);
    while (L < q.first) del(L++, 0);
    while (R > q.second) del(--R, 1);//d22
    res[qi] = calc();
  }
  return res;
}
//842
vi moTree(vector<array<int, 2>> Q, vector<vi>& ed, int root
    =0){
  int N = sz(ed), pos[2] = {}, blk = 350; // ~N/sqrt(Q)
  vi s(sz(Q)), res = s, I(N), L(N), R(N), in(N), par(N);
  add(0, 0), in[0] = 1;
  auto dfs = [&](int x, int p, int dep, auto& f) -> void {
      //263
    par[x] = p;
    L[x] = N;
    if (dep) I[x] = N++;
    for (int y : ed[x]) if (y != p) f(y, x, !dep, f);
    if (!dep) I[x] = N++;//23e
    R[x] = N;
  };
  dfs(root, -1, 0, dfs);
#define K(x) pii(I[x[0]] / blk, I[x[1]] ^ -(I[x[0]] / blk &
    1))
  iota(all(s), 0);//064
```

```
  sort(all(s), [&](int s, int t){ return K(Q[s]) < K(Q[t]);
      });
  for (int qi : s) rep(end,0,2) {
    int &a = pos[end], b = Q[qi][end], i = 0;
#define step(c) { if (in[c]) { del(a, end); in[a] = 0; } \
                  else { add(c, end); in[c] = 1; } a = c; }
                       //440
    while (!(L[b] <= L[a] && R[a] <= R[b]))
      I[i++] = b, b = par[b];
    while (a != b) step(par[a]);
    while (i--) step(I[i]);
    if (end) res[qi] = calc();//695
  }
  return res;
}
```

LazySegmentTree.h

Description: Range set and add lazy segtree
Time: $\mathcal{O}(\log N)$.

6f1390, 50 lines

```
struct segtree {//b1e
#define lc 2*v
#define rc 2*v+1
#define m (l + r) / 2
  int n;
  vector<ll> st;//68e
  vector<bool> toset;
  vector<ll> lazy;
  segtree(int _n) : n(_n), st(4 * n), toset(4 * n), lazy(4
      * n) {}
  void push(int v, int l, int r) {
    if (toset[v]) {//dcb
      if (l != r) {
        toset[lc] = toset[rc] = true;
        lazy[lc] = lazy[v];
        lazy[rc] = lazy[v];
      }//c8c
      st[v] = lazy[v];
      toset[v] = false;
    }
  }
  void upd(int l, int r, ll val) { upd(1, 0, n-1, l, r, val
      ); }//a32
  void upd(int v, int l, int r, int x, int y, ll val) {
    push(v, l, r);
    if (r < x || y < l) return;
    if (x <= l && r <= y) {
      toset[v] = true;//50b
      lazy[v] = val;
      push(v, l, r);
    } else {
      upd(lc, l, m, x, y, val);
      upd(rc, m+1, r, x, y, val);//aef
      st[v] = max(st[lc], st[rc]);
    }
  }
  ll qry(int x, int y) { return qry(1, 0, n-1, x, y); }
  ll qry(int v, int l, int r, int x, int y) {//901
    push(v, l, r);
    if (r < x || y < l) return 0;
    if (x <= l && r <= y) {
      return st[v];
    } else {//f9e
      auto v1 = qry(lc, l, m, x, y);
      auto v2 = qry(rc, m+1, r, x, y);
      return max(v1, v2);
    }
  }//f2e
#undef lc
#undef rc
#undef m
};
```

UnionFindRollback.h

Description: Disjoint-set data structure with undo. If undo is not needed, skip st, time() and rollback().
Usage: `int t = uf.time(); ...; uf.rollback(t);`
Time: $\mathcal{O}(\log(N))$

84e98b, 21 lines

```
struct RollbackUF {//f73
  vi e; vector<pii> st;
  RollbackUF(int n) : e(n, -1) {}
  int size(int x) { return -e[find(x)]; }
  int find(int x) { return e[x] < 0 ? x : find(e[x]); }
  int time() { return sz(st); }//cbd
  void rollback(int t) {
    for (int i = time(); i --> t;)
      e[st[i].first] = st[i].second;
    st.resize(t);
  }//e73
  bool join(int a, int b) {
    a = find(a), b = find(b);
    if (a == b) return false;
    if (e[a] > e[b]) swap(a, b);
    st.pb({a, e[a]});//0d8
    st.pb({b, e[b]});
    e[a] += e[b]; e[b] = a;
    return true;
  }
};//214
```

Numerical (4)

4.1 Polynomials and recurrences

Polynomial.h

c9b7b0, 17 lines

```
struct Poly {//1b7
  vector<double> a;
  double operator()(double x) const {
    double val = 0;
    for (int i = sz(a); i--;) (val *= x) += a[i];
    return val;//06d
  }
  void diff() {
    rep(i,1,sz(a)) a[i-1] = i*a[i];
    a.pop_back();
  }//b82
  void divroot(double x0) {
    double b = a.back(), c; a.back() = 0;
    for(int i=sz(a)-1; i--;) c = a[i], a[i] = a[i+1]*x0+b,
        b=c;
    a.pop_back();
  }//e03
};
```

PolyRoots.h

Description: Finds the real roots to a polynomial.
Usage: `polyRoots({{2,-3,1}},-1e9,1e9) // solve x^2-3x+2 = 0`
Time: $\mathcal{O}\left(n^2 \log(1/\epsilon)\right)$

"Polynomial.h" fbf066, 23 lines

```
vector<double> polyRoots(Poly p, double xmin, double xmax)
    {//840
  if (sz(p.a) == 2) { return {-p.a[0]/p.a[1]}; }
  vector<double> ret;
  Poly der = p;
  der.diff();
  auto dr = polyRoots(der, xmin, xmax);//ec1
  dr.pb(xmin-1);
  dr.pb(xmax+1);
  sort(all(dr));
  rep(i,0,sz(dr)-1) {
    double l = dr[i], h = dr[i+1];//189
    bool sign = p(l) > 0;
    if (sign ^ (p(h) > 0)) {
      rep(it,0,60) { // while (h - l > 1e-8)
        double m = (l + h) / 2, f = p(m);
        if ((f <= 0) ^ sign) l = m;//810
        else h = m;
      }
      ret.pb((l + h) / 2);
    }
  }//808
  return ret;
}
```

PolyInterpolate.h

Description: Given n points (x[i], y[i]), computes an n-1-degree polynomial p that passes through them: $p(x) = a[0]*x^0 + ... + a[n-1]*x^{n-1}$. For numerical precision, pick $x[k] = c*\cos(k/(n-1)*\pi), k = 0 \dots n-1$.
Time: $\mathcal{O}\left(n^2\right)$

08bf48, 13 lines

```
typedef vector<double> vd;//159
vd interpolate(vd x, vd y, int n) {
  vd res(n), temp(n);
  rep(k,0,n-1) rep(i,k+1,n)
    y[i] = (y[i] - y[k]) / (x[i] - x[k]);
  double last = 0; temp[0] = 1;//746
  rep(k,0,n) rep(i,0,n) {
    res[i] += y[k] * temp[i];
    swap(last, temp[i]);
    temp[i] -= last * x[k];
  }//0e1
  return res;
}
```

BerlekampMassey.h

Description: Recovers any n-order linear recurrence relation from the first $2n$ terms of the recurrence. Useful for guessing linear recurrences after brute-forcing the first terms. Should work on any field, but numerical stability for floats is not guaranteed. Output will have size $\le n$.
Usage: `berlekampMassey({0, 1, 1, 3, 5, 11}) // {1, 2}`
Time: $\mathcal{O}\left(N^2\right)$

"../number-theory/ModPow.h" 96548b, 20 lines

```
vector<ll> berlekampMassey(vector<ll> s) {//b21
  int n = sz(s), L = 0, m = 0;
  vector<ll> C(n), B(n), T;
  C[0] = B[0] = 1;

  ll b = 1;//4c7
  rep(i,0,n) { ++m;
    ll d = s[i] % mod;
    rep(j,1,L+1) d = (d + C[j] * s[i - j]) % mod;
    if (!d) continue;
    T = C; ll coef = d * modpow(b, mod-2) % mod;//1b2
    rep(j,m,n) C[j] = (C[j] - coef * B[j - m]) % mod;
    if (2 * L > i) continue;
    L = i + 1 - L; B = T; b = d; m = 0;
  }
//255
  C.resize(L + 1); C.erase(C.begin());
  for (ll& x : C) x = (mod - x) % mod;
  return C;
}
```

LinearRecurrence.h

Description: Generates the k'th term of an n-order linear recurrence $S[i] = \sum_j S[i-j-1]tr[j]$, given $S[0 \dots \ge n-1]$ and $tr[0 \dots n-1]$. Faster than matrix multiplication. Useful together with Berlekamp–Massey.
Usage: `linearRec({0, 1}, {1, 1}, k) // k'th Fibonacci number`

Time: $\mathcal{O}\left(n^2 \log k\right)$

f4e444, 26 lines

```
typedef vector<ll> Poly;//bb1
ll linearRec(Poly S, Poly tr, ll k) {
  int n = sz(tr);

  auto combine = [&](Poly a, Poly b) {
    Poly res(n * 2 + 1);//251
    rep(i,0,n+1) rep(j,0,n+1)
      res[i + j] = (res[i + j] + a[i] * b[j]) % mod;
    for (int i = 2 * n; i > n; --i) rep(j,0,n)
      res[i - 1 - j] = (res[i - 1 - j] + res[i] * tr[j]) %
          mod;
    res.resize(n + 1);//12f
    return res;
  };

  Poly pol(n + 1), e(pol);
  pol[0] = e[1] = 1;//df7

  for (++k; k; k /= 2) {
    if (k % 2) pol = combine(pol, e);
    e = combine(e, e);
  }//c0e

  ll res = 0;
  rep(i,0,n) res = (res + pol[i + 1] * S[i]) % mod;
  return res;
}//cbb
```

4.2 Optimization

GoldenSectionSearch.h

Description: Finds the argument minimizing the function f in the interval $[a, b]$ assuming f is unimodal on the interval, i.e. has only one local minimum and no local maximum. The maximum error in the result is *eps*. Works equally well for maximization with a small change in the code. See TernarySearch.h in the Various chapter for a discrete version.

Usage: `double func(double x) { return 4+x+.3*x*x; }`
`double xmin = gss(-1000,1000,func);`

Time: $\mathcal{O}\left(\log((b-a)/\epsilon)\right)$

d7b114, 15 lines

```
template<class F>//5c6
double gss(double a, double b, F f) {
  double r = (sqrt(5)-1)/2, eps = 1e-7;
  double x1 = b - r*(b-a), x2 = a + r*(b-a);
  double f1 = f(x1), f2 = f(x2);
  while (b-a > eps)//905
    if (f1 < f2) { //change to > to find maximum
      b = x2; x2 = x1; f2 = f1;
      x1 = b - r*(b-a); f1 = f(x1);
    } else {
      a = x1; x1 = x2; f1 = f2;//00c
      x2 = a + r*(b-a); f2 = f(x2);
    }
  return a;
}
```

Integrate.h

Description: Simple integration of a function over an interval using Simpson's rule. The error should be proportional to h^4, although in practice you will want to verify that the result is stable to desired precision when epsilon changes.

4756fc, 7 lines

```
template<class F>//e93
double quad(double a, double b, F f, const int n = 1000) {
  double h = (b - a) / 2 / n, v = f(a) + f(b);
  rep(i,1,n*2)
    v += f(a + i*h) * (i&1 ? 4 : 2);
  return v * h / 3;//2d2
}
```

IntegrateAdaptive.h

Description: Fast integration using an adaptive Simpson's rule.

Usage: `double sphereVolume = quad(-1, 1, [](double x) {`
`return quad(-1, 1, [&](double y) {`
`return quad(-1, 1, [&](double z) {`
`return x*x + y*y + z*z < 1; });});});`

92dd79, 15 lines

```
typedef double d;//e70
#define S(a,b) (f(a) + 4*f((a+b) / 2) + f(b)) * (b-a) / 6

template <class F>
d rec(F& f, d a, d b, d eps, d S) {
  d c = (a + b) / 2;//b17
  d S1 = S(a, c), S2 = S(c, b), T = S1 + S2;
  if (abs(T - S) <= 15 * eps || b - a < 1e-10)
    return T + (T - S) / 15;
  return rec(f, a, c, eps / 2, S1) + rec(f, c, b, eps / 2,
      S2);
}//836
template<class F>
d quad(d a, d b, F f, d eps = 1e-8) {
  return rec(f, a, b, eps, S(a, b));
}
```

Simplex.h

Description: Solves a general linear maximization problem: maximize $c^T x$ subject to $Ax \le b$, $x \ge 0$. Returns -inf if there is no solution, inf if there are arbitrarily good solutions, or the maximum value of $c^T x$ otherwise. The input vector is set to an optimal x (or in the unbounded case, an arbitrary solution fulfilling the constraints). Numerical stability is not guaranteed. For better performance, define variables such that $x = 0$ is viable.

Usage: `vvd A = {{1,-1}, {-1,1}, {-1,-2}};`
`vd b = {1,1,-4}, c = {-1,-1}, x;`
`T val = LPSolver(A, b, c).solve(x);`

Time: $\mathcal{O}\left(NM * \#pivots\right)$, where a pivot may be e.g. an edge relaxation. $\mathcal{O}\left(2^n\right)$ in the general case.

aa8530, 68 lines

```
typedef double T; // long double, Rational, double + mod<P
    >...//629
typedef vector<T> vd;
typedef vector<vd> vvd;

const T eps = 1e-8, inf = 1/.0;
#define MP make_pair//94e
#define ltj(X) if(s == -1 || MP(X[j],N[j]) < MP(X[s],N[s]))
    s=j

struct LPSolver {
  int m, n;
  vi N, B;//282
  vvd D;

  LPSolver(const vvd& A, const vd& b, const vd& c) :
    m(sz(b)), n(sz(c)), N(n+1), B(m), D(m+2, vd(n+2)) {
      rep(i,0,m) rep(j,0,n) D[i][j] = A[i][j];//108
      rep(i,0,m) { B[i] = n+i; D[i][n] = -1; D[i][n+1] = b[
          i];}
      rep(j,0,n) { N[j] = j; D[m][j] = -c[j]; }
      N[n] = -1; D[m+1][n] = 1;
    }
//9c3
  void pivot(int r, int s) {
    T *a = D[r].data(), inv = 1 / a[s];
    rep(i,0,m+2) if (i != r && abs(D[i][s]) > eps) {
      T *b = D[i].data(), inv2 = b[s] * inv;
      rep(j,0,n+2) b[j] -= a[j] * inv2;//d0d
      b[s] = a[s] * inv2;
    }
    rep(j,0,n+2) if (j != s) D[r][j] *= inv;
    rep(i,0,m+2) if (i != r) D[i][s] *= -inv;
    D[r][s] = inv;//aa5
    swap(B[r], N[s]);
  }

  bool simplex(int phase) {
    int x = m + phase - 1;//c51
    for (;;) {
      int s = -1;
      rep(j,0,n+1) if (N[j] != -phase) ltj(D[x]);
      if (D[x][s] >= -eps) return true;
      int r = -1;//bc0
      rep(i,0,m) {
        if (D[i][s] <= eps) continue;
        if (r == -1 || MP(D[i][n+1] / D[i][s], B[i])
                     < MP(D[r][n+1] / D[r][s], B[r])) r = i
                         ;
      }//00c
      if (r == -1) return false;
      pivot(r, s);
    }
  }
//d2f
  T solve(vd &x) {
    int r = 0;
    rep(i,1,m) if (D[i][n+1] < D[r][n+1]) r = i;
    if (D[r][n+1] < -eps) {
      pivot(r, n);//f81
      if (!simplex(2) || D[m+1][n+1] < -eps) return -inf;
      rep(i,0,m) if (B[i] == -1) {
        int s = 0;
        rep(j,1,n+1) ltj(D[i]);
        pivot(i, s);//866
      }
    }
    bool ok = simplex(1); x = vd(n);
    rep(i,0,m) if (B[i] < n) x[B[i]] = D[i][n+1];
    return ok ? D[m][n+1] : inf;//401
  }
};
```

4.3 Matrices

Determinant.h

Description: Calculates determinant of a matrix. Destroys the matrix.

Time: $\mathcal{O}\left(N^3\right)$

bd5cec, 15 lines

```
double det(vector<vector<double>>& a) {//309
  int n = sz(a); double res = 1;
  rep(i,0,n) {
    int b = i;
    rep(j,i+1,n) if (fabs(a[j][i]) > fabs(a[b][i])) b = j;
    if (i != b) swap(a[i], a[b]), res *= -1;//454
    res *= a[i][i];
    if (res == 0) return 0;
    rep(j,i+1,n) {
      double v = a[j][i] / a[i][i];
      if (v != 0) rep(k,i+1,n) a[j][k] -= v * a[i][k];//07b
    }
  }
  return res;
}
```

IntDeterminant.h

Description: Calculates determinant using modular arithmetics. Modulos can also be removed to get a pure-integer version.

Time: $\mathcal{O}\left(N^3\right)$

3313dc, 18 lines

```
const ll mod = 12345;//cab
ll det(vector<vector<ll>>& a) {
  int n = sz(a); ll ans = 1;
  rep(i,0,n) {
    rep(j,i+1,n) {
      while (a[j][i] != 0) { // gcd step//c65
```

```
        ll t = a[i][i] / a[j][i];
        if (t) rep(k,i,n)
          a[i][k] = (a[i][k] - a[j][k] * t) % mod;
        swap(a[i], a[j]);
        ans *= -1;//bc6
      }
    }
    ans = ans * a[i][i] % mod;
    if (!ans) return 0;
  }//b19
  return (ans + mod) % mod;
}
```

SolveLinear.h

Description: Solves $A * x = b$. If there are multiple solutions, an arbitrary one is returned. Returns rank, or -1 if no solutions. Data in A and b is lost.
Time: $\mathcal{O}\left(n^2 m\right)$

44c9ab, 38 lines

```
typedef vector<double> vd;//2cf
const double eps = 1e-12;

int solveLinear(vector<vd>& A, vd& b, vd& x) {
  int n = sz(A), m = sz(x), rank = 0, br, bc;
  if (n) assert(sz(A[0]) == m);//940
  vi col(m); iota(all(col), 0);

  rep(i,0,n) {
    double v, bv = 0;
    rep(r,i,n) rep(c,i,m)//ddb
      if ((v = fabs(A[r][c])) > bv)
        br = r, bc = c, bv = v;
    if (bv <= eps) {
      rep(j,i,n) if (fabs(b[j]) > eps) return -1;
      break;//de0
    }
    swap(A[i], A[br]);
    swap(b[i], b[br]);
    swap(col[i], col[bc]);
    rep(j,0,n) swap(A[j][i], A[j][bc]);//328
    bv = 1/A[i][i];
    rep(j,i+1,n) {
      double fac = A[j][i] * bv;
      b[j] -= fac * b[i];
      rep(k,i+1,m) A[j][k] -= fac*A[i][k];//af1
    }
    rank++;
  }

  x.assign(m, 0);//3c5
  for (int i = rank; i--;) {
    b[i] /= A[i][i];
    x[col[i]] = b[i];
    rep(j,0,i) b[j] -= A[j][i] * b[i];
  }//807
  return rank; // (multiple solutions if rank < m)
}
```

SolveLinear2.h

Description: To get all uniquely determined values of x back from SolveLinear, make the following changes:

"SolveLinear.h" 08e495, 7 lines

```
rep(j,0,n) if (j != i) // instead of rep(j,i+1,n)//22b
// ... then at the end:
x.assign(m, undefined);
rep(i,0,rank) {
  rep(j,rank,m) if (fabs(A[i][j]) > eps) goto fail;
  x[col[i]] = b[i] / A[i][i];//4e3
fail:; }
```

SolveLinearBinary.h

Description: Solves $Ax = b$ over $\mathbb{F}_2$. If there are multiple solutions, one is returned arbitrarily. Returns rank, or -1 if no solutions. Destroys A and b.
Time: $\mathcal{O}\left(n^2 m\right)$

fa2d7a, 34 lines

```
typedef bitset<1000> bs;//d90

int solveLinear(vector<bs>& A, vi& b, bs& x, int m) {
  int n = sz(A), rank = 0, br;
  assert(m <= sz(x));
  vi col(m); iota(all(col), 0);//2c9
  rep(i,0,n) {
    for (br=i; br<n; ++br) if (A[br].any()) break;
    if (br == n) {
      rep(j,i,n) if(b[j]) return -1;
      break;//13e
    }
    int bc = (int)A[br]._Find_next(i-1);
    swap(A[i], A[br]);
    swap(b[i], b[br]);
    swap(col[i], col[bc]);//b88
    rep(j,0,n) if (A[j][i] != A[j][bc]) {
      A[j].flip(i); A[j].flip(bc);
    }
    rep(j,i+1,n) if (A[j][i]) {
      b[j] ^= b[i];//76c
      A[j] ^= A[i];
    }
    rank++;
  }
//7a7
  x = bs();
  for (int i = rank; i--;) {
    if (!b[i]) continue;
    x[col[i]] = 1;
    rep(j,0,i) b[j] ^= A[j][i];//df7
  }
  return rank; // (multiple solutions if rank < m)
}
```

MatrixInverse.h

Description: Invert matrix A. Returns rank; result is stored in A unless singular (rank < n). Can easily be extended to prime moduli; for prime powers, repeatedly set $A^{-1} = A^{-1}(2I - AA^{-1})$ (mod p^k) where A^{-1} starts as the inverse of A mod p, and k is doubled in each step.
Time: $\mathcal{O}\left(n^3\right)$

ebfff6, 35 lines

```
int matInv(vector<vector<double>>& A) {//9a9
  int n = sz(A); vi col(n);
  vector<vector<double>> tmp(n, vector<double>(n));
  rep(i,0,n) tmp[i][i] = 1, col[i] = i;

  rep(i,0,n) {//214
    int r = i, c = i;
    rep(j,i,n) rep(k,i,n)
      if (fabs(A[j][k]) > fabs(A[r][c]))
        r = j, c = k;
    if (fabs(A[r][c]) < 1e-12) return i;//e5b
    A[i].swap(A[r]); tmp[i].swap(tmp[r]);
    rep(j,0,n)
      swap(A[j][i], A[j][c]), swap(tmp[j][i], tmp[j][c]);
    swap(col[i], col[c]);
    double v = A[i][i];//afc
    rep(j,i+1,n) {
      double f = A[j][i] / v;
      A[j][i] = 0;
      rep(k,i+1,n) A[j][k] -= f*A[i][k];
      rep(k,0,n) tmp[j][k] -= f*tmp[i][k];//c80
    }
    rep(j,i+1,n) A[i][j] /= v;
    rep(j,0,n) tmp[i][j] /= v;
    A[i][i] = 1;
  }//bfb

  for (int i = n-1; i > 0; --i) rep(j,0,i) {
    double v = A[j][i];
    rep(k,0,n) tmp[j][k] -= v*tmp[i][k];
  }//e74

  rep(i,0,n) rep(j,0,n) A[col[i]][col[j]] = tmp[i][j];
  return n;
}
```

MatrixInverse-mod.h

Description: Invert matrix A modulo a prime. Returns rank; result is stored in A unless singular (rank < n). For prime powers, repeatedly set $A^{-1} = A^{-1}(2I - AA^{-1})$ (mod p^k) where A^{-1} starts as the inverse of A mod p, and k is doubled in each step.
Time: $\mathcal{O}\left(n^3\right)$

"../number-theory/ModPow.h" 0b7b13, 37 lines

```
int matInv(vector<vector<ll>>& A) {//ebd
  int n = sz(A); vi col(n);
  vector<vector<ll>> tmp(n, vector<ll>(n));
  rep(i,0,n) tmp[i][i] = 1, col[i] = i;

  rep(i,0,n) {//79d
    int r = i, c = i;
    rep(j,i,n) rep(k,i,n) if (A[j][k]) {
      r = j; c = k; goto found;
    }
    return i;//4e3
found:
    A[i].swap(A[r]); tmp[i].swap(tmp[r]);
    rep(j,0,n)
      swap(A[j][i], A[j][c]), swap(tmp[j][i], tmp[j][c]);
    swap(col[i], col[c]);//416
    ll v = modpow(A[i][i], mod - 2);
    rep(j,i+1,n) {
      ll f = A[j][i] * v % mod;
      A[j][i] = 0;
      rep(k,i+1,n) A[j][k] = (A[j][k] - f*A[i][k]) % mod;//
          9f7
      rep(k,0,n) tmp[j][k] = (tmp[j][k] - f*tmp[i][k]) %
          mod;
    }
    rep(j,i+1,n) A[i][j] = A[i][j] * v % mod;
    rep(j,0,n) tmp[i][j] = tmp[i][j] * v % mod;
    A[i][i] = 1;//e3d
  }

  for (int i = n-1; i > 0; --i) rep(j,0,i) {
    ll v = A[j][i];
    rep(k,0,n) tmp[j][k] = (tmp[j][k] - v*tmp[i][k]) % mod;
        //4b2
  }

  rep(i,0,n) rep(j,0,n)
    A[col[i]][col[j]] = tmp[i][j] % mod + (tmp[i][j] < 0)*
        mod;
  return n;//400
}
```

Tridiagonal.h

Description: $x = \text{tridiagonal}(d, p, q, b)$ solves the equation system

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} d_0 & p_0 & 0 & 0 & \cdots & 0 \\ q_0 & d_1 & p_1 & 0 & \cdots & 0 \\ 0 & q_1 & d_2 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n-3} & d_{n-2} & p_{n-2} \\ 0 & 0 & \cdots & 0 & q_{n-2} & d_{n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

This is useful for solving problems on the type

$$a_i = b_i a_{i-1} + c_i a_{i+1} + d_i,\ 1 \le i \le n,$$

where a_0, a_{n+1}, b_i, c_i and d_i are known. a can then be obtained from

$$\{a_i\} = \text{tridiagonal}(\{1, -1, -1, ..., -1, 1\}, \{0, c_1, c_2, \ldots, c_n\}, \{b_1, b_2, \ldots, b_n, 0\}, \{a_0, d_1, d_2, \ldots, d_n, a_{n+1}\}).$$

Fails if the solution is not unique.

If $|d_i| > |p_i| + |q_{i-1}|$ for all i, or $|d_i| > |p_{i-1}| + |q_i|$, or the matrix is positive definite, the algorithm is numerically stable and neither `tr` nor the check for `diag[i] == 0` is needed.

Time: $\mathcal{O}(N)$

8f9fa8, 26 lines

```
typedef double T;//399
vector<T> tridiagonal(vector<T> diag, const vector<T>&
    super,
    const vector<T>& sub, vector<T> b) {
  int n = sz(b); vi tr(n);
  rep(i,0,n-1) {
    if (abs(diag[i]) < 1e-9 * abs(super[i])) { // diag[i]
        == 0//464
      b[i+1] -= b[i] * diag[i+1] / super[i];
      if (i+2 < n) b[i+2] -= b[i] * sub[i+1] / super[i];
      diag[i+1] = sub[i]; tr[++i] = 1;
    } else {
      diag[i+1] -= super[i]*sub[i]/diag[i];//d50
      b[i+1] -= b[i]*sub[i]/diag[i];
    }
  }
  for (int i = n; i--;) {
    if (tr[i]) {//054
      swap(b[i], b[i-1]);
      diag[i-1] = diag[i];
      b[i] /= super[i-1];
    } else {
      b[i] /= diag[i];//20b
      if (i) b[i-1] -= b[i]*super[i-1];
    }
  }
  return b;
}//cbb
```

4.4 Fourier transforms

FastFourierTransform.h

Description: fft(a) computes $\hat{f}(k) = \sum_x a[x] \exp(2\pi i \cdot kx/N)$ for all k. N must be a power of 2. Useful for convolution: `conv(a, b) = c`, where $c[x] = \sum a[i]b[x-i]$. For convolution of complex numbers or more than two vectors: FFT, multiply pointwise, divide by n, reverse(start+1, end), FFT back. Rounding is safe if $(\sum a_i^2 + \sum b_i^2)\log_2 N < 9 \cdot 10^{14}$ (in practice 10^{16}; higher for random inputs). Otherwise, use NTT/FFTMod.

Time: $\mathcal{O}(N \log N)$ with $N = |A| + |B|$ ($\sim$1s for $N = 2^{22}$)

00ced6, 35 lines

```
typedef complex<double> C;//1ec
typedef vector<double> vd;
void fft(vector<C>& a) {
  int n = sz(a), L = 31 - __builtin_clz(n);
  static vector<complex<long double>> R(2, 1);
  static vector<C> rt(2, 1);  // (^ 10% faster if double)//
      c50
  for (static int k = 2; k < n; k *= 2) {
    R.resize(n); rt.resize(n);
    auto x = polar(1.0L, acos(-1.0L) / k);
    rep(i,k,2*k) rt[i] = R[i] = i&1 ? R[i/2] * x : R[i/2];
  }//292
  vi rev(n);
  rep(i,0,n) rev[i] = (rev[i / 2] | (i & 1) << L) / 2;
  rep(i,0,n) if (i < rev[i]) swap(a[i], a[rev[i]]);
  for (int k = 1; k < n; k *= 2)
    for (int i = 0; i < n; i += 2 * k) rep(j,0,k) {//577
      C z = rt[j+k] * a[i+j+k]; // (25% faster if hand-
          rolled)
      a[i + j + k] = a[i + j] - z;
      a[i + j] += z;
    }
}//15f
vd conv(const vd& a, const vd& b) {
  if (a.empty() || b.empty()) return {};
  vd res(sz(a) + sz(b) - 1);
  int L = 32 - __builtin_clz(sz(res)), n = 1 << L;
  vector<C> in(n), out(n);//d93
  copy(all(a), begin(in));
  rep(i,0,sz(b)) in[i].imag(b[i]);
  fft(in);
  for (C& x : in) x *= x;
  rep(i,0,n) out[i] = in[-i & (n - 1)] - conj(in[i]);//36e
  fft(out);
  rep(i,0,sz(res)) res[i] = imag(out[i]) / (4 * n);
  return res;
}
```

FastFourierTransformMod.h

Description: Higher precision FFT, can be used for convolutions modulo arbitrary integers as long as $N \log_2 N \cdot \text{mod} < 8.6 \cdot 10^{14}$ (in practice 10^{16} or higher). Inputs must be in $[0, \text{mod})$.

Time: $\mathcal{O}(N \log N)$, where $N = |A| + |B|$ (twice as slow as NTT or FFT)

"FastFourierTransform.h" b82773, 22 lines

```
typedef vector<ll> vl;//2c4
template<int M> vl convMod(const vl &a, const vl &b) {
  if (a.empty() || b.empty()) return {};
  vl res(sz(a) + sz(b) - 1);
  int B=32-__builtin_clz(sz(res)), n=1<<B, cut=int(sqrt(M))
      ;
  vector<C> L(n), R(n), outs(n), outl(n);//c4f
  rep(i,0,sz(a)) L[i] = C((int)a[i] / cut, (int)a[i] % cut)
      ;
  rep(i,0,sz(b)) R[i] = C((int)b[i] / cut, (int)b[i] % cut)
      ;
  fft(L), fft(R);
  rep(i,0,n) {
    int j = -i & (n - 1);//3eb
    outl[j] = (L[i] + conj(L[j])) * R[i] / (2.0 * n);
    outs[j] = (L[i] - conj(L[j])) * R[i] / (2.0 * n) / 1i;
  }
  fft(outl), fft(outs);
  rep(i,0,sz(res)) {//58f
    ll av = ll(real(outl[i])+.5), cv = ll(imag(outs[i])+.5)
        ;
    ll bv = ll(imag(outl[i])+.5) + ll(real(outs[i])+.5);
    res[i] = ((av % M * cut + bv) % M * cut + cv) % M;
  }
  return res;//510
}
```

NumberTheoreticTransform.h

Description: ntt(a) computes $\hat{f}(k) = \sum_x a[x]g^{xk}$ for all k, where $g = \text{root}^{(mod-1)/N}$. N must be a power of 2. Useful for convolution modulo specific nice primes of the form $2^a b + 1$, where the convolution result has size at most 2^a. For arbitrary modulo, see FFTMod. `conv(a, b) = c`, where $c[x] = \sum a[i]b[x-i]$. For manual convolution: NTT the inputs, multiply pointwise, divide by n, reverse(start+1, end), NTT back. Inputs must be in [0, mod).

Time: $\mathcal{O}(N \log N)$

"../number-theory/ModPow.h" ced03d, 35 lines

```
const ll mod = (119 << 23) + 1, root = 62; // =
    998244353//0ca
// For p < 2^30 there is also e.g. 5 << 25, 7 << 26, 479 <<
    21
// and 483 << 21 (same root). The last two are > 10^9.
typedef vector<ll> vl;
void ntt(vl &a) {
  int n = sz(a), L = 31 - __builtin_clz(n);//cc5
  static vl rt(2, 1);
  for (static int k = 2, s = 2; k < n; k *= 2, s++) {
    rt.resize(n);
    ll z[] = {1, modpow(root, mod >> s)};
    rep(i,k,2*k) rt[i] = rt[i / 2] * z[i & 1] % mod;//4a0
  }
  vi rev(n);
  rep(i,0,n) rev[i] = (rev[i / 2] | (i & 1) << L) / 2;
  rep(i,0,n) if (i < rev[i]) swap(a[i], a[rev[i]]);
  for (int k = 1; k < n; k *= 2)//ed7
    for (int i = 0; i < n; i += 2 * k) rep(j,0,k) {
      ll z = rt[j + k] * a[i + j + k] % mod, &ai = a[i + j
          ];
      a[i + j + k] = ai - z + (z > ai ? mod : 0);
      ai += (ai + z >= mod ? z - mod : z);
    }//dfc
}
vl conv(const vl &a, const vl &b) {
  if (a.empty() || b.empty()) return {};
  int s = sz(a) + sz(b) - 1, B = 32 - __builtin_clz(s),
    n = 1 << B;//d58
  int inv = modpow(n, mod - 2);
  vl L(a), R(b), out(n);
  L.resize(n), R.resize(n);
  ntt(L), ntt(R);
  rep(i,0,n)//f18
    out[-i & (n - 1)] = (ll)L[i] * R[i] % mod * inv % mod;
  ntt(out);
  return {out.begin(), out.begin() + s};
}
```

FastSubsetTransform.h

Description: Transform to a basis with fast convolutions of the form $c[z] = \sum_{z=x\oplus y} a[x] \cdot b[y]$, where $\oplus$ is one of AND, OR, XOR. The size of a must be a power of two.

Time: $\mathcal{O}(N \log N)$

464cf3, 16 lines

```
void FST(vi& a, bool inv) {//ae8
  for (int n = sz(a), step = 1; step < n; step *= 2) {
    for (int i = 0; i < n; i += 2 * step) rep(j,i,i+step) {
      int &u = a[j], &v = a[j + step]; tie(u, v) =
        inv ? pii(v - u, u) : pii(v, u + v); // AND
        inv ? pii(v, u - v) : pii(u + v, u); // OR//0af
        pii(u + v, u - v);                   // XOR
    }
  }
  if (inv) for (int& x : a) x /= sz(a); // XOR only
}//dc4
vi conv(vi a, vi b) {
  FST(a, 0); FST(b, 0);
  rep(i,0,sz(a)) a[i] *= b[i];
  FST(a, 1); return a;
}//cbb
```

4.5 Polynomial

PolyBase.h

Description: A FFT based Polynomial class.

"../number-theory/ModularArithmetic.h", "FastFourierTransform.h", "FastFourierTransformMod.h", "NumberTheoreticTransform.h" dd1be7, 35 lines

```
typedef Mod num;//810
typedef vector<num> poly;
poly &operator+=(poly &a, const poly &b) {
  a.resize(max(sz(a), sz(b)));
  rep(i, 0, sz(b)) a[i] = a[i] + b[i];
  return a;//8a9
}
poly &operator-=(poly &a, const poly &b) {
  a.resize(max(sz(a), sz(b)));
  rep(i, 0, sz(b)) a[i] = a[i] - b[i];
  return a;//e10
}

poly &operator*=(poly &a, const poly &b) {
  if (sz(a) + sz(b) < 100){
    poly res(sz(a) + sz(b) - 1);//025
    rep(i,0,sz(a)) rep(j,0,sz(b))
      res[i + j] = (res[i + j] + a[i] * b[j]);
    return (a = res);
  }
  // auto res = convMod<mod>(vl(all(a)), vl(all(b)));//0cb
  auto res = conv(vl(all(a)), vl(all(b)));
  return (a = poly(all(res)));
}
poly operator*(poly a, const num b) {
  poly c = a;//41c
  for(auto& i : c) i = i * b;
  return c;
}
#define OP(o, oe) \
  poly operator o(poly a, poly b) { \//f19
    poly c = a; \
    return c o##= b; \
  }
OP(*, *=) OP(+, +=) OP(-, -=);
```

PolyEvaluate.h

Description: Multi-point evaluation. Evaluates a given polynomial A at $A(x_0), \ldots A(x_n)$.
Time: $\mathcal{O}\left(n \log^2 n\right)$

"PolyBase.h", "PolyMod.h" dc2cdf, 14 lines

```
vector<num> eval(const poly &a, const vector<num> &x) {//9
    fa
  int n = sz(x);
  if (!n) return {};
  vector<poly> up(2 * n);
  rep(i, 0, n) up[i + n] = poly({num(0) - x[i], 1});
  for (int i = n - 1; i > 0; i--)//923
    up[i] = up[2 * i] * up[2 * i + 1];
  vector<poly> down(2 * n);
  down[1] = a % up[1];
  rep(i, 2, 2 * n) down[i] = down[i / 2] % up[i];
  vector<num> y(n);//835
  rep(i, 0, n) y[i] = down[i + n][0];
  return y;
}
```

PolyIntegDeriv.h

Description: Calculate $\frac{da}{dx}$ and $\int a\,dx$.
Time: $\mathcal{O}(|a|)$

"PolyBase.h" 803fd5, 14 lines

```
poly deriv(poly a) {//aba
  if (a.empty()) return {};
  poly b(sz(a) - 1);
  rep(i, 1, sz(a)) b[i - 1] = a[i] * num(i);
  return b;
}//6f9
poly integr(poly a) {
  if (a.empty()) return {0};
  poly b(sz(a) + 1);
  b[1] = num(1);
  rep(i, 2, sz(b)) b[i] = b[mod%i]*Mod(-mod/i+mod);//176
  rep(i, 1 ,sz(b)) b[i] = a[i-1] * b[i];
  return b;
}
```

PolyInterpolate2.h

Description: Given n points (x[i], y[i]), computes an n-1-degree polynomial p that passes through them: $p(x) = a[0] \cdot x^0 + \ldots + a[n-1] \cdot x^{n-1}$.
Time: $\mathcal{O}\left(n \log^2 n\right)$

"PolyBase.h", "PolyIntegDeriv.h", "PolyEvaluate.h" b911f5, 11 lines

```
poly interp(vector<num> x, vector<num> y) {//74d
  int n=sz(x);
  vector<poly> up(n*2);
  rep(i,0,n) up[i+n] = poly({num(0)-x[i], num(1)});
  for(int i=n-1; i>0;i--) up[i] = up[2*i]*up[2*i+1];
  vector<num> a = eval(deriv(up[1]), x);//6dd
  vector<poly> down(2*n);
  rep(i,0,n) down[i+n] = poly({y[i]*(num(1)/a[i])});
  for(int i=n-1;i>0;i--) down[i] = down[i*2] * up[i*2+1] +
      down[i*2+1] * up[i*2];
  return down[1];
}//cbb
```

PolyInverse.h

Description: Calculate the first $|a|$ coefficients of a^{-1}.
Time: $\mathcal{O}(n \log n)$.

"PolyBase.h" 703c16, 7 lines

```
poly modK(poly a, int k) { return {a.begin(), a.begin() +
    min(k, sz(a))}; }//140
poly inverse(poly A) {
  poly B = poly({num(1) / A[0]});
  while (sz(B) < sz(A))
    B = modK(B * (poly({num(2)}) - modK(A, 2*sz(B)) * B), 2
        * sz(B));
  return modK(B, sz(A));//556
}
```

PolyLogExp.h

Description: Calculate the first $|a|$ coefficients of a^{-1} of $\log a$ and $\exp a$.
Time: $\mathcal{O}(n \log n)$.

"PolyBase.h", "PolyInverse.h", "PolyIntegDeriv.h" 83ea75, 14 lines

```
poly log(poly a) {//9c1
  return modK(integr(deriv(a) * inverse(a)), sz(a));
}
poly exp(poly a) {
  poly b(1, num(1));
  if (a.empty())//8ff
    return b;
  while (sz(b) < sz(a)) {
    b.resize(sz(b) * 2);
    b *= (poly({num(1)}) + modK(a, sz(b)) - log(b));
    b.resize(sz(b) / 2 + 1);//1f2
  }
  return modK(b, sz(a));
}
```

PolyMod.h

Description: Calculate the remainder and quotient of the Euclidean division $\frac{a}{b}$.
Time: $\mathcal{O}(n \log n)$.

"PolyBase.h", "PolyInverse.h" 264551, 20 lines

```
poly &operator/=(poly &a, poly b) {//b9b
  if (sz(a) < sz(b))
    return a = {};
  int s = sz(a) - sz(b) + 1;
  reverse(all(a)), reverse(all(b));
  a.resize(s), b.resize(s);//e15
  a = a * inverse(b);
  a.resize(s), reverse(all(a));
  return a;
}
OP(/, /=)//9fe
poly &operator%=(poly &a, poly &b) {
  if (sz(a) < sz(b))
    return a;
  poly c = (a / b) * b;
  a.resize(sz(b) - 1);//f62
  rep(i, 0, sz(a)) a[i] = a[i] - c[i];
  return a;
}
OP(%, %=)
```

PolyPow.h

Description: Calculate the first $|a|$ coeffients of a^m.
Time: $\mathcal{O}(n \log n)$.

"PolyBase.h", "PolyLogExp.h" f0005c, 13 lines

```
poly pow(poly a, ll m) {//760
  int p = 0, n = sz(a);
  while (p < sz(a) && a[p].v == 0)
    ++p;
  if (ll(m)*p >= sz(a)) return poly(sz(a));
  num j = a[p];//a78
  a = {a.begin() + p, a.end()};
  a = a * (num(1) / j);
  a.resize(n);
  auto res = exp(log(a) * num(m)) * (j ^ m);
  res.insert(res.begin(), p*m, 0);//6a6
  return {res.begin(), res.begin()+n};
}
```

PolyRoots.h

Description: Finds the real roots to a polynomial.
Usage: polyRoots({{2,-3,1}},-1e9,1e9) // solve x^2-3x+2 = 0
Time: $\mathcal{O}\left(n^2 \log(1/\epsilon)\right)$

"Polynomial.h" fbf066, 23 lines

```
vector<double> polyRoots(Poly p, double xmin, double xmax)
    {//840
  if (sz(p.a) == 2) { return {-p.a[0]/p.a[1]}; }
  vector<double> ret;
  Poly der = p;
  der.diff();
  auto dr = polyRoots(der, xmin, xmax);//ec1
  dr.pb(xmin-1);
  dr.pb(xmax+1);
  sort(all(dr));
  rep(i,0,sz(dr)-1) {
    double l = dr[i], h = dr[i+1];//189
    bool sign = p(l) > 0;
    if (sign ^ (p(h) > 0)) {
      rep(it,0,60) { // while (h - l > 1e-8)
        double m = (l + h) / 2, f = p(m);
        if ((f <= 0) ^ sign) l = m;//810
        else h = m;
      }
      ret.pb((l + h) / 2);
    }
  }//808
  return ret;
}
```

Number theory (5)

5.1 Modular arithmetic

ModInverse.h

Description: Pre-computation of modular inverses. Assumes LIM $\leq$ mod and that mod is a prime.

b4a981, 2 lines

```
ll* inv = new ll[LIM] - 1; inv[1] = 1;//b4a
rep(i,2,LIM) inv[i] = mod - (mod / i) * inv[mod % i] % mod;
```

ModPow.h

7b7908, 8 lines

```
const int mod = 1000000007; // faster if const//dce

ll modpow(ll b, ll e) {
  ll ans = 1;
  for (; e; b = b * b % mod, e /= 2)
    if (e & 1) ans = ans * b % mod;//7e5
  return ans;
}
```

ModLog.h

Description: Returns the smallest $x > 0$ s.t. $a^x = b \pmod m$, or -1 if no such x exists. modLog(a,1,m) can be used to calculate the order of a.
Time: $\mathcal{O}\left(\sqrt{m}\right)$

c040b8, 11 lines

```
ll modLog(ll a, ll b, ll m) {//260
  ll n = (ll) sqrt(m) + 1, e = 1, f = 1, j = 1;
  unordered_map<ll, ll> A;
  while (j <= n && (e = f = e * a % m) != b % m)
    A[e * b % m] = j++;
  if (e == b % m) return j;//d16
  if (__gcd(m, e) == __gcd(m, b))
    rep(i,2,n+2) if (A.count(e = e * f % m))
      return n * i - A[e];
  return -1;
}//cbb
```

ModSum.h

Description: Sums of mod'ed arithmetic progressions.
modsum(to, c, k, m) $= \sum_{i=0}^{\mathrm{to}-1} (ki+c)\%m$. divsum is similar but for floored division.
Time: $\log(m)$, with a large constant.

5c5bc5, 16 lines

```
typedef unsigned long long ull;//df3
ull sumsq(ull to) { return to / 2 * ((to-1) | 1); }

ull divsum(ull to, ull c, ull k, ull m) {
  ull res = k / m * sumsq(to) + c / m * to;
  k %= m; c %= m;//e1a
  if (!k) return res;
  ull to2 = (to * k + c) / m;
  return res + (to - 1) * to2 - divsum(to2, m-1 - c, m, k);
}
//1ae
ll modsum(ull to, ll c, ll k, ll m) {
  c = ((c % m) + m) % m;
  k = ((k % m) + m) % m;
  return to * c + k * sumsq(to) - m * divsum(to, c, k, m);
}//cbb
```

ModMulLL.h

Description: Calculate $a \cdot b \bmod c$ (or $a^b \bmod c$) for $0 \leq a, b \leq c \leq 7.2 \cdot 10^{18}$.
Time: $\mathcal{O}(1)$ for modmul, $\mathcal{O}(\log b)$ for modpow

bbbd8f, 11 lines

```
typedef unsigned long long ull;//a9c
ull modmul(ull a, ull b, ull M) {
  ll ret = a * b - M * ull(1.L / M * a * b);
  return ret + M * (ret < 0) - M * (ret >= (ll)M);
}
ull modpow(ull b, ull e, ull mod) {//51d
  ull ans = 1;
  for (; e; b = modmul(b, b, mod), e /= 2)
    if (e & 1) ans = modmul(ans, b, mod);
  return ans;
}//cbb
```

ModSqrt.h

Description: Tonelli-Shanks algorithm for modular square roots. Finds x s.t. $x^2 = a \pmod p$ ($-x$ gives the other solution).
Time: $\mathcal{O}\left(\log^2 p\right)$ worst case, $\mathcal{O}(\log p)$ for most p

"ModPow.h" 19a793, 24 lines

```
ll sqrt(ll a, ll p) {//473
  a %= p; if (a < 0) a += p;
  if (a == 0) return 0;
  assert(modpow(a, (p-1)/2, p) == 1); // else no solution
  if (p % 4 == 3) return modpow(a, (p+1)/4, p);
  // a^(n+3)/8 or 2^(n+3)/8 * 2^(n-1)/4 works if p % 8 ==
      5//a48
  ll s = p - 1, n = 2;
  int r = 0, m;
  while (s % 2 == 0)
    ++r, s /= 2;
  while (modpow(n, (p - 1) / 2, p) != p - 1) ++n;//c4b
  ll x = modpow(a, (s + 1) / 2, p);
  ll b = modpow(a, s, p), g = modpow(n, s, p);
  for (;; r = m) {
    ll t = b;
    for (m = 0; m < r && t != 1; ++m)//faf
      t = t * t % p;
    if (m == 0) return x;
    ll gs = modpow(g, 1LL << (r - m - 1), p);
    g = gs * gs % p;
    x = x * gs % p;//a28
    b = b * g % p;
  }
}
```

5.2 Primality

FastEratosthenes.h

Description: Prime sieve for generating all primes smaller than LIM.
Time: LIM=1e9 $\approx$ 1.5s

9ac0a0, 20 lines

```
const int LIM = 1e6;//058
bitset<LIM> isPrime;
vi eratosthenes() {
  const int S = (int)round(sqrt(LIM)), R = LIM / 2;
  vi pr = {2}, sieve(S+1); pr.reserve(int(LIM/log(LIM)*1.1)
      );
  vector<pii> cp;//86b
  for (int i = 3; i <= S; i += 2) if (!sieve[i]) {
    cp.pb({i, i * i / 2});
    for (int j = i * i; j <= S; j += 2 * i) sieve[j] = 1;
  }
  for (int L = 1; L <= R; L += S) {//62d
    array<bool, S> block{};
    for (auto &[p, idx] : cp)
      for (int i=idx; i < S+L; idx = (i+=p)) block[i-L] =
          1;
    rep(i,0,min(S, R - L))
      if (!block[i]) pr.pb((L + i) * 2 + 1);//0b9
  }
  for (int i : pr) isPrime[i] = 1;
  return pr;
}
```

MillerRabin.h

Description: Deterministic Miller-Rabin primality test. Guaranteed to work for numbers up to $7 \cdot 10^{18}$; for larger numbers, use Python and extend A randomly.
Time: 7 times the complexity of $a^b \mod c$.

"ModMulLL.h" 60dcd1, 12 lines

```
bool isPrime(ull n) {//60a
  if (n < 2 || n % 6 % 4 != 1) return (n | 1) == 3;
  ull A[] = {2, 325, 9375, 28178, 450775, 9780504,
      1795265022},
    s = __builtin_ctzll(n-1), d = n >> s;
  for (ull a : A) {   // ^ count trailing zeroes
    ull p = modpow(a%n, d, n), i = s;//81c
    while (p != 1 && p != n - 1 && a % n && i--)
      p = modmul(p, p, n);
    if (p != n-1 && i != s) return 0;
  }
  return 1;//84a
}
```

Factor.h

Description: Pollard-rho randomized factorization algorithm. Returns prime factors of a number, in arbitrary order (e.g. 2299 -> {11, 19, 11}).
Time: $\mathcal{O}\left(n^{1/4}\right)$, less for numbers with small factors.

"ModMulLL.h", "MillerRabin.h" 21a173, 18 lines

```
ull pollard(ull n) {//c81
  ull x = 0, y = 0, t = 30, prd = 2, i = 1, q;
  auto f = [&](ull a) { return modmul(a, a, n) + i; };
  while (t++ % 40 || __gcd(prd, n) == 1) {
    if (x == y) x = ++i, y = f(x);
    if ((q = modmul(prd, max(x,y) - min(x,y), n))) prd = q;
        //049
    x = f(x), y = f(f(y));
  }
  return __gcd(prd, n);
}
vector<ull> factor(ull n) {//c19
  if (n == 1) return {};
  if (isPrime(n)) return {n};
  ull x = pollard(n);
  auto l = factor(x), r = factor(n / x);
  l.insert(l.end(), all(r));//363
  return l;
}
```

5.3 Divisibility

euclid.h

Description: Finds two integers x and y, such that $ax + by = \gcd(a, b)$. If you just need gcd, use the built in __gcd instead. If a and b are coprime, then x is the inverse of $a \pmod b$.

33ba8f, 5 lines

```
ll euclid(ll a, ll b, ll &x, ll &y) {//33b
  if (!b) return x = 1, y = 0, a;
  ll d = euclid(b, a % b, y, x);
  return y -= a/b * x, d;
}
```

CRT.h

Description: Chinese Remainder Theorem.
crt(a, m, b, n) computes x such that $x \equiv a \pmod m$, $x \equiv b \pmod n$. If $|a| < m$ and $|b| < n$, x will obey $0 \leq x < \mathrm{lcm}(m, n)$. Assumes $mn < 2^{62}$.
Time: $\log(n)$

"euclid.h" 04d93a, 7 lines

```
ll crt(ll a, ll m, ll b, ll n) {//eae
  if (n > m) swap(a, b), swap(m, n);
  ll x, y, g = euclid(m, n, x, y);
  assert((a - b) % g == 0); // else no solution
  x = (b - a) % n * x % n / g * m + a;
  return x < 0 ? x + m*n/g : x;//6ac
```

```
}
```

5.3.1 Bézout's identity

For $a \neq, b \neq 0$, then $d = gcd(a, b)$ is the smallest positive integer for which there are integer solutions to

$$ax + by = d$$

If (x, y) is one solution, then all solutions are given by

$$\left(x + \frac{kb}{\gcd(a,b)}, y - \frac{ka}{\gcd(a,b)}\right), \quad k \in \mathbb{Z}$$

phiFunction.h

Description: *Euler's* ϕ function is defined as $\phi(n) := \#$ of positive integers $\leq n$ that are coprime with n. $\phi(1) = 1$, p prime $\Rightarrow \phi(p^k) = (p-1)p^{k-1}$, m, n coprime $\Rightarrow \phi(mn) = \phi(m)\phi(n)$. If $n = p_1^{k_1} p_2^{k_2} ... p_r^{k_r}$ then $\phi(n) = (p_1 - 1)p_1^{k_1 - 1} ... (p_r - 1)p_r^{k_r - 1}$. $\phi(n) = n \cdot \prod_{p|n}(1 - 1/p)$. $\sum_{d|n} \phi(d) = n$, $\sum_{1 \leq k \leq n, \gcd(k,n)=1} k = n\phi(n)/2, n > 1$

Euler's thm: a, n coprime $\Rightarrow a^{\phi(n)} \equiv 1 \pmod{n}$.

Fermat's little thm: p prime $\Rightarrow a^{p-1} \equiv 1 \pmod{p}\ \forall a$.

cf7d6d, 8 lines

```
const int LIM = 5000000;//70b
int phi[LIM];

void calculatePhi() {
  rep(i,0,LIM) phi[i] = i&1 ? i : i/2;
  for (int i = 3; i < LIM; i += 2) if(phi[i] == i)//103
    for (int j = i; j < LIM; j += i) phi[j] -= phi[j] / i;
}
```

5.4 Fractions

ContinuedFractions.h

Description: Given N and a real number $x \geq 0$, finds the closest rational approximation p/q with $p, q \leq N$. It will obey $|p/q - x| \leq 1/qN$. For consecutive convergents, $p_{k+1}q_k - q_{k+1}p_k = (-1)^k$. ($p_k/q_k$ alternates between $> x$ and $< x$.) If x is rational, y eventually becomes ∞; if x is the root of a degree 2 polynomial the a's eventually become cyclic.

Time: $\mathcal{O}(\log N)$

ea42e2, 21 lines

```
typedef double d; // for N ~ 1e7; long double for N ~ 1e9
    //32b
pair<ll, ll> approximate(d x, ll N) {
  ll LP = 0, LQ = 1, P = 1, Q = 0, inf = LLONG_MAX; d y = x
    ;
  for (;;) {
    ll lim = min(P ? (N-LP) / P : inf, Q ? (N-LQ) / Q : inf
      ),
      a = (ll)floor(y), b = min(a, lim),//5ad
      NP = b*P + LP, NQ = b*Q + LQ;
    if (a > b) {
      // If b > a/2, we have a semi-convergent that gives
        us a
      // better approximation; if b = a/2, we *may* have
        one.
      // Return {P, Q} here for a more canonical
        approximation.//fcb
      return (abs(x - (d)NP / (d)NQ) < abs(x - (d)P / (d)Q)
        ) ?
        make_pair(NP, NQ) : make_pair(P, Q);
    }
    if (abs(y = 1/(y - (d)a)) > (d)N*3) {
      return {NP, NQ};//5c7
    }
    LP = P; P = NP;
    LQ = Q; Q = NQ;
  }
}//cbb
```

FracBinarySearch.h

Description: Given f and N, finds the smallest fraction $p/q \in [0, 1]$ such that $f(p/q)$ is true, and $p, q \leq N$. You may want to throw an exception from f if it finds an exact solution, in which case N can be removed.

Usage: `fracBS([](Frac f) { return f.p>=3*f.q; }, 10); // {1,3}`

Time: $\mathcal{O}(\log(N))$

27ab3e, 25 lines

```
struct Frac { ll p, q; };//386

template<class F>
Frac fracBS(F f, ll N) {
  bool dir = 1, A = 1, B = 1;
  Frac lo{0, 1}, hi{1, 1}; // Set hi to 1/0 to search (0, N
    ]//262
  if (f(lo)) return lo;
  assert(f(hi));
  while (A || B) {
    ll adv = 0, step = 1; // move hi if dir, else lo
    for (int si = 0; step; (step *= 2) >>= si) {//7e2
      adv += step;
      Frac mid{lo.p * adv + hi.p, lo.q * adv + hi.q};
      if (abs(mid.p) > N || mid.q > N || dir == !f(mid)) {
        adv -= step; si = 2;
      }//bf0
    }
    hi.p += lo.p * adv;
    hi.q += lo.q * adv;
    dir = !dir;
    swap(lo, hi);//f58
    A = B; B = !!adv;
  }
  return dir ? hi : lo;
}
```

5.5 Pythagorean Triples

The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), \quad b = k \cdot (2mn), \quad c = k \cdot (m^2 + n^2),$$

with $m > n > 0$, $k > 0$, $m \perp n$, and either m or n even.

5.6 Primes

$p = 962592769$ is such that $2^{21} \mid p - 1$, which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than 1 000 000.

Primitive roots exist modulo any prime power p^a, except for $p = 2, a > 2$, and there are $\phi(\phi(p^a))$ many. For $p = 2, a > 2$, the group $\mathbb{Z}_{2^a}^\times$ is instead isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$.

5.7 Mobius Function

$\sum_{d|n} \mu(d) = [n = 1]$ (very useful)

$g(n) = \sum_{n|d} f(d) \Leftrightarrow f(n) = \sum_{n|d} \mu(d/n) g(d)$

$g(n) = \sum_{1 \leq m \leq n} f(\lfloor \frac{n}{m} \rfloor) \Leftrightarrow f(n) = \sum_{1 \leq m \leq n} \mu(m) g(\lfloor \frac{n}{m} \rfloor)$

Combinatorial (6)

6.1 Permutations

6.1.1 Factorial

IntPerm.h

Description: Permutation -> integer conversion. (Not order preserving.) Integer -> permutation can use a lookup table.

Time: $\mathcal{O}(n)$

044568, 6 lines

```
int permToInt(vi& v) {//cf9
  int use = 0, i = 0, r = 0;
  for(int x:v) r = r * ++i + __builtin_popcount(use & -(1<<
      x)),
    use |= 1 << x;                     // (note: minus, not
          ~!)
  return r;
}//cbb
```

6.1.2 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1) + D(n-2)) = nD(n-1) + (-1)^n = \left\lfloor \frac{n!}{e} \right\rceil$$

6.1.3 Burnside's lemma

Given a group G of symmetries and a set X, the number of elements of X *up to symmetry* equals

$$\frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where X^g are the elements fixed by g $(g.x = x)$.

If $f(n)$ counts "configurations" (of some sort) of length n, we can ignore rotational symmetry using $G = \mathbb{Z}_n$ to get

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n, k)) = \frac{1}{n} \sum_{k|n} f(k)\phi(n/k).$$

6.2 Partitions and subsets

6.2.1 Partition function

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, \ p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k-1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

n	5	10	15	20	25	30	35	40	50	100
$p(n)$	7	42	176	627	1958	5604	14883	37338	~2e5	~2e8

6.2.2 Lucas' Theorem

Let n, m be non-negative integers and p a prime. Write $n = n_k p^k + ... + n_1 p + n_0$ and $m = m_k p^k + ... + m_1 p + m_0$. Then $\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod{p}$.

6.3 General purpose numbers

6.3.1 Bernoulli numbers

EGF of Bernoulli numbers is $B(t) = \frac{t}{e^t - 1}$ (FFT-able). $B[0, \ldots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots]$

Sums of powers:

$$\sum_{i=1}^{n} n^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k \cdot (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\sum_{i=m}^{\infty} f(i) = \int_m^{\infty} f(x)dx - \sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(m)$$

$$\approx \int_m^{\infty} f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m))$$

6.3.2 Stirling numbers of the first kind

Number of permutations on n items with k cycles.

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k), \; c(0,0) = 1$$
$$\textstyle\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\ldots(x+n-1)$$

$c(8,k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$
$c(n,2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \ldots$

6.3.3 Eulerian numbers

Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j:s s.t. $\pi(j) \geq j$, k j:s s.t. $\pi(j) > j$.

$$E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k)$$

$$E(n,0) = E(n,n-1) = 1$$

$$E(n,k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n$$

6.3.4 Stirling numbers of the second kind

Partitions of n distinct elements into exactly k groups.

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

$$S(n,1) = S(n,n) = 1$$

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

6.3.5 Bell numbers

Total number of partitions of n distinct elements. $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \ldots$. For p prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

6.3.6 Labeled unrooted trees

\# on n vertices: n^{n-2}
\# on k existing trees of size n_i: $n_1 n_2 \cdots n_k n^{k-2}$
\# with degrees d_i: $(n-2)!/((d_1-1)! \cdots (d_n-1)!)$

6.3.7 Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1, \; C_{n+1} = \frac{2(2n+1)}{n+2} C_n, \; C_{n+1} = \sum C_i C_{n-i}$$

$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$

- sub-diagonal monotone paths in an $n \times n$ grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with with $n+1$ leaves (0 or 2 children).
- ordered trees with $n+1$ vertices.
- ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.
- permutations of $[n]$ with no 3-term increasing subseq.

6.3.8 Gambler's Ruin

$$P_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq q \\ i/N & \text{if } p = q = 0.5 \end{cases}$$

Graph (7)

7.1 Network flow

PushRelabel.h

Description: Push-relabel using the highest label selection rule and the gap heuristic. Quite fast in practice. To obtain the actual flow, look at positive values only.
Time: $\mathcal{O}\left(V^2\sqrt{E}\right)$

91e61c, 48 lines

```
struct PushRelabel {//d82
  struct Edge {
    int dest, back;
    ll f, c;
  };
  vector<vector<Edge>> g;//bef
  vector<ll> ec;
  vector<Edge*> cur;
  vector<vi> hs; vi H;
  PushRelabel(int n) : g(n), ec(n), cur(n), hs(2*n), H(n) {
      }
//a7b
  void addEdge(int s, int t, ll cap, ll rcap=0) {
    if (s == t) return;
    g[s].pb({t, sz(g[t]), 0, cap});
    g[t].pb({s, sz(g[s])-1, 0, rcap});
  }//7a1

  void addFlow(Edge& e, ll f) {
    Edge &back = g[e.dest][e.back];
    if (!ec[e.dest] && f) hs[H[e.dest]].pb(e.dest);
    e.f += f; e.c -= f; ec[e.dest] += f;//124
    back.f -= f; back.c += f; ec[back.dest] -= f;
  }
  ll calc(int s, int t) {
    int v = sz(g); H[s] = v; ec[t] = 1;
    vi co(2*v); co[0] = v-1;//a96
    rep(i,0,v) cur[i] = g[i].data();
    for (Edge& e : g[s]) addFlow(e, e.c);

    for (int hi = 0;;) {
      while (hs[hi].empty()) if (!hi--) return -ec[s];//e2e
      int u = hs[hi].back(); hs[hi].pop_back();
      while (ec[u] > 0)  // discharge u
        if (cur[u] == g[u].data() + sz(g[u])) {
          H[u] = 1e9;
          for (Edge& e : g[u]) if (e.c && H[u] > H[e.dest
              ]+1)//9ff
            H[u] = H[e.dest]+1, cur[u] = &e;
          if (++co[H[u]], !--co[hi] && hi < v)
            rep(i,0,v) if (hi < H[i] && H[i] < v)
              --co[H[i]], H[i] = v + 1;
          hi = H[u];//7ed
        } else if (cur[u]->c && H[u] == H[cur[u]->dest]+1)
          addFlow(*cur[u], min(ec[u], cur[u]->c));
        else ++cur[u];
    }
  }//a5b
  bool leftOfMinCut(int a) { return H[a] >= sz(g); }
};
```

MinCostMaxFlow.h

Description: Min-cost max-flow. If costs can be negative, call setpi before maxflow, but note that negative cost cycles are not supported. To obtain the actual flow, look at positive values only.
Time: $\mathcal{O}(FE\log(V))$ where F is max flow. $\mathcal{O}(VE)$ for setpi.

<bits/extc++.h> bb147c, 77 lines

```
const ll INF = numeric_limits<ll>::max() / 4;//d4e

struct MCMF {
  struct edge {
    int from, to, rev;
    ll cap, cost, flow;//309
  };
  int N;
  vector<vector<edge>> ed;
  vi seen;
  vector<ll> dist, pi;//16a
  vector<edge*> par;

  MCMF(int _N) : N(_N), ed(N), seen(N), dist(N), pi(N), par
      (N) {}

  void addEdge(int from, int to, ll cap, ll cost) {//a9c
    if (from == to) return;
    ed[from].pb(edge{ from,to,sz(ed[to]),cap,cost,0 });
    ed[to].pb(edge{ to,from,sz(ed[from])-1,0,-cost,0 });
  }
//635
  void path(int s) {
    fill(all(seen), 0);
    fill(all(dist), INF);
    dist[s] = 0; ll di;
//17c
    __gnu_pbds::priority_queue<pair<ll, int>> q;
    vector<decltype(q)::point_iterator> its(N);
    q.push({ 0, s });

    while (!q.empty()) {//95a
      s = q.top().second; q.pop();
      seen[s] = 1; di = dist[s] + pi[s];
      for (edge& e : ed[s]) if (!seen[e.to]) {
        ll val = di - pi[e.to] + e.cost;
        if (e.cap - e.flow > 0 && val < dist[e.to]) {//c63
          dist[e.to] = val;
          par[e.to] = &e;
```

```
        if (its[e.to] == q.end())
          its[e.to] = q.push({ -dist[e.to], e.to });
        else//dbc
          q.modify(its[e.to], { -dist[e.to], e.to });
      }
    }
  }
  rep(i,0,N) pi[i] = min(pi[i] + dist[i], INF);//02d
}

pair<ll, ll> maxflow(int s, int t) {
  ll totflow = 0, totcost = 0;
  while (path(s), seen[t]) {//aa6
    ll fl = INF;
    for (edge* x = par[t]; x; x = par[x->from])
      fl = min(fl, x->cap - x->flow);

    totflow += fl;//21b
    for (edge* x = par[t]; x; x = par[x->from]) {
      x->flow += fl;
      ed[x->to][x->rev].flow -= fl;
    }
  }//cd4
  rep(i,0,N) for(edge& e : ed[i]) totcost += e.cost * e.
       flow;
  return {totflow, totcost/2};
}

// If some costs can be negative, call this before
     maxflow://7c7
void setpi(int s) { // (otherwise, leave this out)
  fill(all(pi), INF); pi[s] = 0;
  int it = N, ch = 1; ll v;
  while (ch-- && it--)
    rep(i,0,N) if (pi[i] != INF)//42d
      for (edge& e : ed[i]) if (e.cap)
        if ((v = pi[i] + e.cost) < pi[e.to])
          pi[e.to] = v, ch = 1;
  assert(it >= 0); // negative cost cycle
}//e03
};
```

GlobalMinCut.h

Description: Find a global minimum cut in an undirected graph, as represented by an adjacency matrix.
Time: $\mathcal{O}(V^3)$

8b0e19, 21 lines

```
pair<int, vi> globalMinCut(vector<vi> mat) {//f64
  pair<int, vi> best = {INT_MAX, {}};
  int n = sz(mat);
  vector<vi> co(n);
  rep(i,0,n) co[i] = {i};
  rep(ph,1,n) {//c8f
    vi w = mat[0];
    size_t s = 0, t = 0;
    rep(it,0,n-ph) { // O(V^2) -> O(E log V) with prio.
        queue
      w[t] = INT_MIN;
      s = t, t = max_element(all(w)) - w.begin();//0bb
      rep(i,0,n) w[i] += mat[t][i];
    }
    best = min(best, {w[t] - mat[t][t], co[t]});
    co[s].insert(co[s].end(), all(co[t]));
    rep(i,0,n) mat[s][i] += mat[t][i];//a2c
    rep(i,0,n) mat[i][s] = mat[s][i];
    mat[0][t] = INT_MIN;
  }
  return best;
}//cbb
```

GomoryHu.h

Description: Given a list of edges representing an undirected flow graph, returns edges of the Gomory-Hu tree. The max flow between any pair of vertices is given by minimum edge weight along the Gomory-Hu tree path.
Time: $\mathcal{O}(V)$ Flow Computations

"PushRelabel.h" 1ec6c8, 13 lines

```
struct Edge { int from, to; ll cap; };//81a
vector<Edge> gomoryHu(int N, vector<Edge> ed) {
  vector<Edge> tree;
  vi par(N);
  rep(i,1,N) {
    PushRelabel D(N); // Dinic also works//489
    for (Edge t : ed) D.addEdge(t.from, t.to, t.cap, t.cap)
        ;
    tree.pb({i, par[i], D.calc(i, par[i])});
    rep(j,i+1,N)
      if (par[j] == par[i] && D.leftOfMinCut(j)) par[j] = i
          ;
  }//eec
  return tree;
}
```

FlowWithDemands.h

Description: Add a new source s' and sink t', new edges from s' to everything, and new edges from everything to t'. Define:

- $c'((s',v)) = \sum_{u\in V} d((u,v))$ for each edge (s',v)
- $c'((v,t')) = \sum_{w\in V} d((v,w))$ for each edge (v,t')
- $c'((u,v)) = c((u,v)) - d((u,v))$ for each old edge (u,v)
- $c'((t,s)) = \infty$

d41d8c, 1 lines

```
//d41
```

7.1.1 Flow Notes

Kőnig's Theorem: For bipartite graph,
|minimum node cover| = |maximum matching|

Nodes not in a minimum node cover form a maximum independent set

Node-disjoint Path Cover: Each node belongs to exactly one path

- $\{a_1, \ldots, a_n, b_1, \ldots b_n\}$ where $(a_u \to b_v)$ exists for each edges $(u \to v)$ in DAG G

(General) Path Cover: Each node belongs to at least one path

- Same construction as node-disjoint but with transitive closure edges

Dilworth's Theorem: In a DAG,
|minimum general path cover| = |maximum antichain|

7.2 Matching

hopcroftKarp.h

Description: Fast bipartite matching algorithm. Graph g should be a list of neighbors of the left partition, and *btoa* should be a vector full of -1's of the same size as the right partition. Returns the size of the matching. *btoa*[*i*] will be the match for vertex i on the right side, or -1 if it's not matched.
Usage: vi btoa(m, -1); hopcroftKarp(g, btoa);
Time: $\mathcal{O}\left(\sqrt{V}E\right)$

d93347, 42 lines

```
bool dfs(int a, int L, vector<vi>& g, vi& btoa, vi& A, vi&
     B) {//d9e
  if (A[a] != L) return 0;
  A[a] = -1;
  for (int b : g[a]) if (B[b] == L + 1) {
    B[b] = 0;
    if (btoa[b] == -1 || dfs(btoa[b], L + 1, g, btoa, A, B)
        )//613
      return btoa[b] = a, 1;
  }
  return 0;
}
//ad4
int hopcroftKarp(vector<vi>& g, vi& btoa) {
  int res = 0;
  vi A(g.size()), B(btoa.size()), cur, next;
  for (;;) {
    fill(all(A), 0);//d58
    fill(all(B), 0);
    cur.clear();
    for (int a : btoa) if(a != -1) A[a] = -1;
    rep(a,0,sz(g)) if(A[a] == 0) cur.pb(a);
    for (int lay = 1;; lay++) {//559
      bool islast = 0;
      next.clear();
      for (int a : cur) for (int b : g[a]) {
        if (btoa[b] == -1) {
          B[b] = lay;//1ca
          islast = 1;
        }
        else if (btoa[b] != a && !B[b]) {
          B[b] = lay;
          next.pb(btoa[b]);//c66
        }
      }
      if (islast) break;
      if (next.empty()) return res;
      for (int a : next) A[a] = lay;//4f3
      cur.swap(next);
    }
    rep(a,0,sz(g))
      res += dfs(a, 0, g, btoa, A, B);
  }//67c
}
```

DFSMatching.h

Description: Simple bipartite matching algorithm. Graph g should be a list of neighbors of the left partition, and *btoa* should be a vector full of -1's of the same size as the right partition. Returns the size of the matching. *btoa*[*i*] will be the match for vertex i on the right side, or -1 if it's not matched.
Usage: vi btoa(m, -1); dfsMatching(g, btoa);
Time: $\mathcal{O}(VE)$

522b98, 22 lines

```
bool find(int j, vector<vi>& g, vi& btoa, vi& vis) {//400
  if (btoa[j] == -1) return 1;
  vis[j] = 1; int di = btoa[j];
  for (int e : g[di])
    if (!vis[e] && find(e, g, btoa, vis)) {
      btoa[e] = di;//a0e
      return 1;
    }
  return 0;
}
int dfsMatching(vector<vi>& g, vi& btoa) {//52f
  vi vis;
  rep(i,0,sz(g)) {
    vis.assign(sz(btoa), 0);
    for (int j : g[i])
      if (find(j, g, btoa, vis)) {//e5b
        btoa[j] = i;
```

```
        break;
      }
  }
  return sz(btoa) - (int)count(all(btoa), -1);//ff5
}
```

MinimumVertexCover.h

Description: Finds a minimum vertex cover in a bipartite graph. The size is the same as the size of a maximum matching, and the complement is a maximum independent set.

"DFSMatching.h" 62c4ec, 20 lines

```
vi cover(vector<vi>& g, int n, int m) {//60f
  vi match(m, -1);
  int res = dfsMatching(g, match);
  vector<bool> lfound(n, true), seen(m);
  for (int it : match) if (it != -1) lfound[it] = false;
  vi q, cover;//2da
  rep(i,0,n) if (lfound[i]) q.pb(i);
  while (!q.empty()) {
    int i = q.back(); q.pop_back();
    lfound[i] = 1;
    for (int e : g[i]) if (!seen[e] && match[e] != -1) {//4
        ed
      seen[e] = true;
      q.pb(match[e]);
    }
  }
  rep(i,0,n) if (!lfound[i]) cover.pb(i);//a72
  rep(i,0,m) if (seen[i]) cover.pb(n+i);
  assert(sz(cover) == res);
  return cover;
}
```

WeightedMatching.h

Description: Given a weighted bipartite graph, matches every node on the left with a node on the right such that no nodes are in two matchings and the sum of the edge weights is minimal. Takes cost[N][M], where cost[i][j] = cost for L[i] to be matched with R[j] and returns (min cost, match), where L[i] is matched with R[match[i]]. Negate costs for max cost. Requires $N \leq M$.

Time: $\mathcal{O}\left(N^2M\right)$ df0677, 31 lines

```
pair<int, vi> hungarian(const vector<vi> &a) {//64f
  if (a.empty()) return {0, {}};
  int n = sz(a) + 1, m = sz(a[0]) + 1;
  vi u(n), v(m), p(m), ans(n - 1);
  rep(i,1,n) {
    p[0] = i;//0b5
    int j0 = 0; // add "dummy" worker 0
    vi dist(m, INT_MAX), pre(m, -1);
    vector<bool> done(m + 1);
    do { // dijkstra
      done[j0] = true;//bd1
      int i0 = p[j0], j1 = -1, delta = INT_MAX;
      rep(j,1,m) if (!done[j]) {
        auto cur = a[i0 - 1][j - 1] - u[i0] - v[j];
        if (cur < dist[j]) dist[j] = cur, pre[j] = j0;
        if (dist[j] < delta) delta = dist[j], j1 = j;//865
      }
      rep(j,0,m) {
        if (done[j]) u[p[j]] += delta, v[j] -= delta;
        else dist[j] -= delta;
      }//aa1
      j0 = j1;
    } while (p[j0]);
    while (j0) { // update alternating path
      int j1 = pre[j0];
      p[j0] = p[j1], j0 = j1;//88f
    }
  }
  rep(j,1,m) if (p[j]) ans[p[j] - 1] = j - 1;
  return {-v[0], ans}; // min cost
}//cbb
```

GeneralMatching.h

Description: Matching for general graphs. Fails with probability N/mod.

Time: $\mathcal{O}\left(N^3\right)$

"../numerical/MatrixInverse-mod.h" 1e40dd, 40 lines

```
vector<pii> generalMatching(int N, vector<pii>& ed) {//19e
  vector<vector<ll>> mat(N, vector<ll>(N)), A;
  for (pii pa : ed) {
    int a = pa.first, b = pa.second, r = rand() % mod;
    mat[a][b] = r, mat[b][a] = (mod - r) % mod;
  }//063

  int r = matInv(A = mat), M = 2*N - r, fi, fj;
  assert(r % 2 == 0);

  if (M != N) do {//f88
    mat.resize(M, vector<ll>(M));
    rep(i,0,N) {
      mat[i].resize(M);
      rep(j,N,M) {
        int rr = rand() % mod;//b47
        mat[i][j] = rr, mat[j][i] = (mod - rr) % mod;
      }
    }
  } while (matInv(A = mat) != M);
//92b
  vi has(M, 1); vector<pii> ret;
  rep(it,0,M/2) {
    rep(i,0,M) if (has[i])
      rep(j,i+1,M) if (A[i][j] && mat[i][j]) {
        fi = i; fj = j; goto done;//e0a
    } assert(0); done:
    if (fj < N) ret.emplace_back(fi, fj);
    has[fi] = has[fj] = 0;
    rep(sw,0,2) {
      ll a = modpow(A[fi][fj], mod-2);//b7f
      rep(i,0,M) if (has[i] && A[i][fj]) {
        ll b = A[i][fj] * a % mod;
        rep(j,0,M) A[i][j] = (A[i][j] - A[fi][j] * b) % mod
            ;
      }
      swap(fi,fj);//3c7
    }
  }
  return ret;
}
```

MatroidIntersection.h

Description: Given two matroids, finds the largest common independent set. Pass the matroid with more expensive add/clear operations to M1.

Time: $R^2N(M2.add + M1.check + M2.check) + R^3M1.add + R^2M1.clear + RNM2.clear$ 02fb70, 52 lines

```
struct ColorMat {//b9c
  vi cnt, clr;
  ColorMat(int n, vector<int> clr) : cnt(n), clr(clr) {}
  bool check(int x) { return !cnt[clr[x]]; }
  void add(int x) { cnt[clr[x]]++; }
  void clear() { fill(all(cnt), 0); }//540
};
template <class M1, class M2> struct MatroidIsect {
  int n;
  vector<char> iset;
  M1 m1; M2 m2;//286
  MatroidIsect(M1 m1, M2 m2, int n) : n(n), iset(n + 1), m1
      (m1), m2(m2) {}
  vi solve() {
    rep(i,0,n) if (m1.check(i) && m2.check(i))
      iset[i] = true, m1.add(i), m2.add(i);
    while (augment());//b25
    vi ans;
    rep(i,0,n) if (iset[i]) ans.push_back(i);
    return ans;
  }
  bool augment() {//110
    vector<int> frm(n, -1);
    queue<int> q({n}); // starts at dummy node
    auto fwdE = [&](int a) {
      vi ans;
      m1.clear();//2d7
      rep(v, 0, n) if (iset[v] && v != a) m1.add(v);
      rep(b, 0, n) if (!iset[b] && frm[b] == -1 && m1.check
          (b))
        ans.push_back(b), frm[b] = a;
      return ans;
    };//20c
    auto backE = [&](int b) {
      m2.clear();
      rep(cas, 0, 2) rep(v, 0, n)
        if ((v == b || iset[v]) && (frm[v] == -1) == cas) {
          if (!m2.check(v))//45e
            return cas ? q.push(v), frm[v] = b, v : -1;
          m2.add(v);
        }
      return n;
    };//c4c
    while (!q.empty()) {
      int a = q.front(), c; q.pop();
      for (int b : fwdE(a))
        while((c = backE(b)) >= 0) if (c == n) {
          while (b != n) iset[b] ^= 1, b = frm[b];//d9b
          return true;
        }
    }
    return false;
  }//e03
};
```

7.2.1 Matroid Notes

$\text{rank}(A) + \text{rank}(B) \geq \text{rank}(A \cup B) + \text{rank}(A \cap B)$

Matroid Intersection steps:

- Build exchange graph $D_{M_1,M_2}(S)$ - Find "free to include" sets Y_1 and Y_2 - Find augmenting path P **without shortcuts (BFS)** from an element in Y_1 to an element in Y_2 - Alternate inclusion into S of all element in P

Exchange Graph construction:

- Left side contains nodes in S, right side contains nodes not in S
- Edge $(x \to y)$ where $x \in S$ and $y \notin S$ exists iff removing $S \setminus \{x\} \cup \{y\}$ is independent in M_1
- Edge $(x \leftarrow y)$ where $x \in S$ and $y \notin S$ exists iff removing $S \setminus \{x\} \cup \{y\}$ is independent in M_2
- $y \in Y_1$ iff $S \cup \{y\}$ is independent in M_1
- $y \in Y_2$ iff $S \cup \{y\}$ is independent in M_2

Weighted Matroid Intersection:

- Assign weights to **nodes** ($w(x)$ for $x \in S$ and $-w(x)$ o.w.)

- Find shortest augmenting path first by sum of weights, then by # of edges

7.3 DFS algorithms

SCC.h

Description: Finds strongly connected components in a directed graph. If vertices u, v belong to the same component, we can reach u from v and vice versa.

Usage: scc(graph, [&](vi& v) { ... }) visits all components in reverse topological order. comp[i] holds the component index of a node (a component only has edges to components with lower index). ncomps will contain the number of components.

Time: $\mathcal{O}(E+V)$

c16b3c, 24 lines

```
template<class F> vi scc(const vector<vi> &adj, F f) {//496
  int n = sz(adj);
  vi val(n), comp(n, -1), z, cont;
  int time = 0, ncomps = 0;
  auto dfs = [&](auto &&self, int u) -> int {
    int low = val[u] = ++time, x; z.push_back(u);//ad5
    for (auto e : adj[u]) if (comp[e] < 0)
      low = min(low, val[e] ?: self(self, e));
    if (low == val[u]) {
      do {
        x = z.back(); z.pop_back();//4f1
        comp[x] = ncomps;
        cont.push_back(x);
      } while (x != u);
      f(cont); cont.clear();
      ncomps++;//cda
    }
    return val[u] = low;
  };
  rep(i, 0, n) {
    if (comp[i] < 0) dfs(dfs, i);//418
  }
  return comp;
}
```

BiconnectedComponents.h

Description: Finds all biconnected components in an undirected graph, and runs a callback for the edges in each. In a biconnected component there are at least two distinct paths between any two nodes. Note that a node can be in several components. An edge which is not in a component is a bridge, i.e., not part of any cycle.

Usage: int eid = 0; ed.resize(N);
for each edge (a,b) {
ed[a].emplace_back(b, eid);
ed[b].emplace_back(a, eid++); }
bicomps([&](const vi& edgelist) {...});

Time: $\mathcal{O}(E+V)$

389e66, 28 lines

```
template<class F>//c2c
void bicomps(vector<vector<pii>> &ed, F f) {
  vi num(sz(ed)), st;
  int t=0;
  auto dfs = [&](auto &&self, int at, int par) -> int {
    int me = num[at] = ++t, top = me;//b12
    for (auto [y, e] : ed[at]) if (e != par) {
      if (num[y]) {
        top = min(top, num[y]);
        if (num[y] < me)
          st.pb(e);//6e9
      } else {
        int si = sz(st);
        int up = self(self, y, e);
        top = min(top, up);
        if (up == me) {//2cf
          st.pb(e);
          f(vi(st.begin() + si, st.end()));
          st.resize(si);
        }
        else if (up < me) st.pb(e);//51c
        else { /* e is a bridge */ }
      }
    }
    return top;
  };//835
  rep(i,0,sz(ed)) if (!num[i]) dfs(dfs, i, -1);
}
```

Articulation.h

Description: Finds articulation points (removal separates graph)

Time: $\mathcal{O}(n+m)$

a7b0ba, 25 lines

```
vector<bool> cutpoints(const vector<vi> &adj) {//259
  int timer=0, n=sz(adj);
  vi tin(n,-1), low(n,-1);
  vector<bool> vis(n);
  vector<bool> iscut(n);
  auto dfs = [&](auto &&self, int v, int p) -> void {//7a1
    vis[v] = true;
    tin[v] = low[v] = timer++;
    int ch = 0;
    for (int to : adj[v]) {
      if (to == p) continue;//b9d
      if (vis[to])
        low[v] = min(low[v], tin[to]);
      else {
        self(self, to, v);
        low[v] = min(low[v], low[to]);//914
        if (low[to] >= tin[v] && p!=-1) iscut[v]=1;
        ++ch;
      }
    }
    if(p == -1 && ch > 1) iscut[v]=1;//4e2
  };
  rep(i,0,n) if (!vis[i]) dfs(dfs, i, -1);
  return iscut;
}
```

2sat.h

Description: Calculates a valid assignment to boolean variables a, b, c,... to a 2-SAT problem, so that an expression of the type $(a||b)\&\&(!a||c)\&\&(d||!b)\&\&\ldots$ becomes true, or reports that it is unsatisfiable. Negated variables are represented by bit-inversions (~x).

Usage: TwoSat ts(number of boolean variables);
ts.either(0, ~3); // Var 0 is true or var 3 is false
ts.setValue(2); // Var 2 is true
ts.atMostOne({0,~1,2}); // <= 1 of vars 0, ~1 and 2 are true
ts.solve(); // Returns true iff it is solvable
ts.values[0..N-1] holds the assigned values to the vars

Time: $\mathcal{O}(N+E)$, where N is the number of boolean variables, and E is the number of clauses.

35fbf7, 56 lines

```
struct TwoSat {//7c0
  int N;
  vector<vi> gr;
  vi values; // 0 = false, 1 = true

  TwoSat(int n = 0) : N(n), gr(2*n) {}//54e

  int addVar() { // (optional)
    gr.emplace_back();
    gr.emplace_back();
    return N++;//662
  }
  void either(int f, int j) {
    f = max(2*f, -1-2*f);
    j = max(2*j, -1-2*j);//2d3
    gr[f].pb(j^1);
    gr[j].pb(f^1);
  }
  void setValue(int x) { either(x, x); }
//41c
  void atMostOne(const vi& li) { // (optional)
    if (sz(li) <= 1) return;
    int cur = ~li[0];
    rep(i,2,sz(li)) {
      int next = addVar();//f5e
      either(cur, ~li[i]);
      either(cur, next);
      either(~li[i], next);
      cur = ~next;
    }//276
    either(cur, ~li[1]);
  }

  vi val, comp, z; int time = 0;
  int dfs(int i) {//1e9
    int low = val[i] = ++time, x; z.pb(i);
    for(int e : gr[i]) if (!comp[e])
      low = min(low, val[e] ?: dfs(e));
    if (low == val[i]) do {
      x = z.back(); z.pop_back();//0c0
      comp[x] = low;
      if (values[x>>1] == -1)
        values[x>>1] = x&1;
    } while (x != i);
    return val[i] = low;//749
  }

  bool solve() {
    values.assign(N, -1);
    val.assign(2*N, 0); comp = val;//4fa
    rep(i,0,2*N) if (!comp[i]) dfs(i);
    rep(i,0,N) if (comp[2*i] == comp[2*i+1]) return 0;
    return 1;
  }
};//214
```

EulerWalk.h

Description: Eulerian undirected/directed path/cycle algorithm. Input should be a vector of (dest, global edge index), where for undirected graphs, forward/backward edges have the same index. Returns a list of nodes in the Eulerian path/cycle with src at both start and end, or empty list if no cycle/path exists. To get edge indices back, add .second to s and ret.

Time: $\mathcal{O}(V+E)$

3e0eb1, 15 lines

```
vi eulerWalk(vector<vector<pii>>& gr, int nedges, int src
    =0) {//fda
  int n = sz(gr);
  vi D(n), its(n), eu(nedges), ret, s = {src};
  D[src]++; // to allow Euler paths, not just cycles
  while (!s.empty()) {
    int x = s.back(), y, e, &it = its[x], end = sz(gr[x]);
        //39a
    if (it == end){ ret.pb(x); s.pop_back(); continue; }
    tie(y, e) = gr[x][it++];
    if (!eu[e]) {
      D[x]--, D[y]++;
      eu[e] = 1; s.pb(y);//f91
    }}
  for (int x : D) if (x < 0 || sz(ret) != nedges+1) return
      {};
  return {ret.rbegin(), ret.rend()};
}
```

7.4 Coloring

EdgeColoring.h

Description: Given a simple, undirected graph with max degree D, computes a $(D+1)$-coloring of the edges such that no neighboring edges share a color. (D-coloring is NP-hard, but can be done for bipartite graphs by repeated matchings of max-degree nodes.)
Time: $\mathcal{O}(NM)$

ca07a0, 31 lines

```
vi edgeColoring(int N, vector<pii> eds) {//d26
  vi cc(N + 1), ret(sz(eds)), fan(N), free(N), loc;
  for (pii e : eds) ++cc[e.first], ++cc[e.second];
  int u, v, ncols = *max_element(all(cc)) + 1;
  vector<vi> adj(N, vi(ncols, -1));
  for (pii e : eds) {//945
    tie(u, v) = e;
    fan[0] = v;
    loc.assign(ncols, 0);
    int at = u, end = u, d, c = free[u], ind = 0, i = 0;
    while (d = free[v], !loc[d] && (v = adj[u][d]) != -1)//
        665
      loc[d] = ++ind, cc[ind] = d, fan[ind] = v;
    cc[loc[d]] = c;
    for (int cd = d; at != -1; cd ^= c ^ d, at = adj[at][cd
        ])
      swap(adj[at][cd], adj[end = at][cd ^ c ^ d]);
    while (adj[fan[i]][d] != -1) {//f7e
      int left = fan[i], right = fan[++i], x = cc[i];
      adj[u][x] = left;
      adj[left][x] = u;
      adj[right][x] = -1;
      free[right] = x;//e59
    }
    adj[u][d] = fan[i];
    adj[fan[i]][d] = u;
    for (int y : {fan[0], u, end})
      for (int& z = free[y] = 0; adj[y][z] != -1; z++);//
          b06
  }
  rep(i,0,sz(eds))
    for (tie(u, v) = eds[i]; adj[u][ret[i]] != v;) ++ret[i
        ];
  return ret;
}//cbb
```

7.5 Heuristics

MaximalCliques.h

Description: Runs a callback for all maximal cliques in a graph (given as a symmetric bitset matrix; self-edges not allowed). Callback is given a bitset representing the maximal clique.
Time: $\mathcal{O}\left(3^{n/3}\right)$, much faster for sparse graphs

b0d5b1, 12 lines

```
typedef bitset<128> B;//abb
template<class F>
void cliques(vector<B>& eds, F f, B P = ~B(), B X={}, B R={
    }) {
  if (!P.any()) { if (!X.any()) f(R); return; }
  auto q = (P | X)._Find_first();
  auto cands = P & ~eds[q];//7d8
  rep(i,0,sz(eds)) if (cands[i]) {
    R[i] = 1;
    cliques(eds, f, P & eds[i], X & eds[i], R);
    R[i] = P[i] = 0; X[i] = 1;
  }//67c
}
```

MaximumClique.h

Description: Quickly finds a maximum clique of a graph (given as symmetric bitset matrix; self-edges not allowed). Can be used to find a maximum independent set by finding a clique of the complement graph.
Time: Runs in about 1s for n=155 and worst case random graphs (p=.90). Runs faster for sparse graphs.

108bd4, 49 lines

```
typedef vector<bitset<200>> vb;//b92
struct Maxclique {
  double limit=0.025, pk=0;
  struct Vertex { int i, d=0; };
  typedef vector<Vertex> vv;
  vb e;//5b2
  vv V;
  vector<vi> C;
  vi qmax, q, S, old;
  void init(vv& r) {
    for (auto& v : r) v.d = 0;//dab
    for (auto& v : r) for (auto j : r) v.d += e[v.i][j.i];
    sort(all(r), [](auto a, auto b) { return a.d > b.d; });
    int mxD = r[0].d;
    rep(i,0,sz(r)) r[i].d = min(i, mxD) + 1;
  }//a6a
  void expand(vv& R, int lev = 1) {
    S[lev] += S[lev - 1] - old[lev];
    old[lev] = S[lev - 1];
    while (sz(R)) {
      if (sz(q) + R.back().d <= sz(qmax)) return;//62e
      q.pb(R.back().i);
      vv T;
      for(auto v:R) if (e[R.back().i][v.i]) T.pb({v.i});
      if (sz(T)) {
        if (S[lev]++ / ++pk < limit) init(T);//feb
        int j = 0, mxk = 1, mnk = max(sz(qmax) - sz(q) + 1,
             1);
        C[1].clear(), C[2].clear();
        for (auto v : T) {
          int k = 1;
          auto f = [&](int i) { return e[v.i][i]; };//94f
          while (any_of(all(C[k]), f)) k++;
          if (k > mxk) mxk = k, C[mxk + 1].clear();
          if (k < mnk) T[j++].i = v.i;
          C[k].pb(v.i);
        }//08b
        if (j > 0) T[j - 1].d = 0;
        rep(k,mnk,mxk + 1) for (int i : C[k])
          T[j].i = i, T[j++].d = k;
        expand(T, lev + 1);
      } else if (sz(q) > sz(qmax)) qmax = q;//15f
      q.pop_back(), R.pop_back();
    }
  }
  vi maxClique() { init(V), expand(V); return qmax; }
  Maxclique(vb conn) : e(conn), C(sz(e)+1), S(sz(C)), old(S
      ) {//83c
    rep(i,0,sz(e)) V.pb({i});
  }
};
```

7.6 Trees

CompressTree.h

Description: Given a rooted tree and a subset S of nodes, compute the minimal subtree that contains all the nodes by adding all (at most $|S|-1$) pairwise LCA's and compressing edges. Returns a list of (par, orig_index) representing a tree rooted at 0. The root points to itself.
Time: $\mathcal{O}(|S| \log |S|)$

"LCA.h" cea406, 21 lines

```
typedef vector<pair<int, int>> vpi;//386
vpi compressTree(LCA& lca, const vi& subset) {
  static vi rev; rev.resize(sz(lca.time));
  vi li = subset, &T = lca.time;
  auto cmp = [&](int a, int b) { return T[a] < T[b]; };
  sort(all(li), cmp);//3b2
  int m = sz(li)-1;
  rep(i,0,m) {
    int a = li[i], b = li[i+1];
    li.pb(lca.lca(a, b));
  }//c76
  sort(all(li), cmp);
  li.erase(unique(all(li)), li.end());
  rep(i,0,sz(li)) rev[li[i]] = i;
  vpi ret = {pii(0, li[0])};
  rep(i,0,sz(li)-1) {//ff8
    int a = li[i], b = li[i+1];
    ret.emplace_back(rev[lca.lca(a, b)], b);
  }
  return ret;
}//cbb
```

HLD.h

Description: Decomposes a tree into vertex disjoint heavy paths and light edges such that the path from any leaf to the root contains at most $\log(n)$ light edges. Takes as input the full adjacency list. op edges being true means that values are stored in the edges, as opposed to the nodes.
Time: $\mathcal{O}(\log N)$

f6f4ce, 55 lines

```
template<bool op_edges = false>//1ab
struct hld {
  vector<vi> adj;
  vi par, size, in, head, d;
  hld(int n) : adj(n), par(n), size(n), in(n), head(n), d(n
      ) {}
//c22
  void add_edge(int u, int v) {
    adj[u].pb(v);
    adj[v].pb(u);
  }
//9ee
  void dfs_size(int v = 0, int p = 0) {
    size[v] = 1;
    for (int &e : adj[v]) {
      if (e != p) {
        d[e] = d[v] + 1;//a74
        par[e] = v;
        dfs_size(e, v);
        size[v] += size[e];
        if (size[e] > size[adj[v][0]] || adj[v][0] == p)
          swap(e, adj[v][0]);
      }//96d
    }
  }

  void dfs_hld(int v = 0, int p = 0) {
    static int t = 0;//978
    in[v] = t++;
    for (int e : adj[v]) {
      if (e != p) {
        if (e == adj[v][0]) {
          head[e] = head[v];//c3e
        } else {
          head[e] = e;
        }
        dfs_hld(e, v);
      }//47a
    }
  }

  template<typename F>
  void op_path(int x, int y, F op) {//b81
    while (head[x] != head[y]) {
      if (d[head[x]] > d[head[y]]) swap(x, y);
      op(in[head[x]], in[y] + 1);
      y = par[head[y]];
    }//387
    if (d[x] > d[y]) swap(x, y);
    op(in[x] + (op_edges ? 1 : 0), in[y] + 1);
```

```
  }

  template<typename F>//d25
  void op_subtree(int x, F op) {
    op(in[x] + (op_edges ? 1 : 0), in[x] + size[x]);
  }
};
```

LinkCutTree.h

Description: Represents a forest of unrooted trees. You can add and remove edges (as long as the result is still a forest), and check whether two nodes are in the same tree.

Time: All operations take amortized $\mathcal{O}(\log N)$.

0fb462, 90 lines

```
struct Node { // Splay tree. Root's pp contains tree's
    parent.//a4e
  Node *p = 0, *pp = 0, *c[2];
  bool flip = 0;
  Node() { c[0] = c[1] = 0; fix(); }
  void fix() {
    if (c[0]) c[0]->p = this;//b8f
    if (c[1]) c[1]->p = this;
    // (+ update sum of subtree elements etc. if wanted)
  }
  void pushFlip() {
    if (!flip) return;//dfd
    flip = 0; swap(c[0], c[1]);
    if (c[0]) c[0]->flip ^= 1;
    if (c[1]) c[1]->flip ^= 1;
  }
  int up() { return p ? p->c[1] == this : -1; }//3a9
  void rot(int i, int b) {
    int h = i ^ b;
    Node *x = c[i], *y = b == 2 ? x : x->c[h], *z = b ? y :
        x;
    if ((y->p = p)) p->c[up()] = y;
    c[i] = z->c[i ^ 1];//eb7
    if (b < 2) {
      x->c[h] = y->c[h ^ 1];
      y->c[h ^ 1] = x;
    }
    z->c[i ^ 1] = this;//430
    fix(); x->fix(); y->fix();
    if (p) p->fix();
    swap(pp, y->pp);
  }
  void splay() {//4c8
    for (pushFlip(); p; ) {
      if (p->p) p->p->pushFlip();
      p->pushFlip(); pushFlip();
      int c1 = up(), c2 = p->up();
      if (c2 == -1) p->rot(c1, 2);//9e8
      else p->p->rot(c2, c1 != c2);
    }
  }
  Node* first() {
    pushFlip();//828
    return c[0] ? c[0]->first() : (splay(), this);
  }
};

struct LinkCut {//d99
  vector<Node> node;
  LinkCut(int N) : node(N) {}

  void link(int u, int v) { // add an edge (u, v)
    assert(!connected(u, v));//166
    makeRoot(&node[u]);
    node[u].pp = &node[v];
  }
  void cut(int u, int v) { // remove an edge (u, v)
    Node *x = &node[u], *top = &node[v];//0b9
    makeRoot(top); x->splay();
    assert(top == (x->pp ?: x->c[0]));
    if (x->pp) x->pp = 0;
    else {
      x->c[0] = top->p = 0;//158
      x->fix();
    }
  }
  bool connected(int u, int v) { // are u, v in the same
      tree?
    Node* nu = access(&node[u])->first();//781
    return nu == access(&node[v])->first();
  }
  void makeRoot(Node* u) {
    access(u);
    u->splay();//09d
    if(u->c[0]) {
      u->c[0]->p = 0;
      u->c[0]->flip ^= 1;
      u->c[0]->pp = u;
      u->c[0] = 0;//41e
      u->fix();
    }
  }
  Node* access(Node* u) {
    u->splay();//4e7
    while (Node* pp = u->pp) {
      pp->splay(); u->pp = 0;
      if (pp->c[1]) {
        pp->c[1]->p = 0; pp->c[1]->pp = pp; }
      pp->c[1] = u; pp->fix(); u = pp;//f4d
    }
    return u;
  }
};
```

DirectedMST.h

Description: Finds a minimum spanning tree/arborescence of a directed graph, given a root node. If no MST exists, returns -1.

Time: $\mathcal{O}(E \log V)$

"../data-structures/UnionFindRollback.h" 057d96, 60 lines

```
struct Edge { int a, b; ll w{}; };//4d9
struct Node {
  Edge key;
  Node *l=0, *r=0;
  ll delta{};
  void prop() {//936
    key.w += delta;
    if (l) l->delta += delta;
    if (r) r->delta += delta;
    delta = 0;
  }//5dc
  Edge top() { prop(); return key; }
};
Node *merge(Node *a, Node *b) {
  if (!a || !b) return a ?: b;
  a->prop(), b->prop();//72a
  if (a->key.w > b->key.w) swap(a, b);
  swap(a->l, (a->r = merge(b, a->r)));
  return a;
}
void pop(Node*& a) { a->prop(); a = merge(a->l, a->r); }//8
    e9

pair<ll, vi> dmst(int n, int r, vector<Edge>& g) {
  RollbackUF uf(n);
  vector<Node*> heap(n);
  for (Edge e : g) heap[e.b] = merge(heap[e.b], new Node{e}
      );//0f3
  ll res = 0;
  vi seen(n, -1), path(n), par(n);
  seen[r] = r;
  vector<Edge> Q(n), in(n, {-1,-1}), comp;
  deque<tuple<int, int, vector<Edge>>> cycs;//4c6
  rep(s,0,n) {
    int u = s, qi = 0, w;
    while (seen[u] < 0) {
      if (!heap[u]) return {-1,{}};
      Edge e = heap[u]->top();//2b0
      heap[u]->delta -= e.w, pop(heap[u]);
      Q[qi] = e, path[qi++] = u, seen[u] = s;
      res += e.w, u = uf.find(e.a);
      if (seen[u] == s) {
        Node* cyc = 0;//fff
        int end = qi, time = uf.time();
        do cyc = merge(cyc, heap[w = path[--qi]]);
        while (uf.join(u, w));
        u = uf.find(u), heap[u] = cyc, seen[u] = -1;
        cycs.push_front({u, time, {&Q[qi], &Q[end]}});//984
      }
    }
    rep(i,0,qi) in[uf.find(Q[i].b)] = Q[i];
  }
//eba
  for (auto& [u,t,cc] : cycs) { // restore sol (optional)
    uf.rollback(t);
    Edge inEdge = in[u];
    for (auto& e : cc) in[uf.find(e.b)] = e;
    in[uf.find(inEdge.b)] = inEdge;//ffd
  }
  rep(i,0,n) par[i] = in[i].a;
  return {res, par};
}
```

LCA.h

Description: Data structure for computing lowest common ancestors in a tree (with 0 as root). C should be an adjacency list of the tree, either directed or undirected.

Time: $\mathcal{O}(N \log N + Q)$

"../data-structures/RMQ.h" aa4761, 21 lines

```
struct LCA {//169
  int T = 0;
  vi time, path, ret;
  RMQ<int> rmq;

  LCA(vector<vi>& C) : time(sz(C)), rmq((dfs(C,0,-1), ret))
      {}//1e9
  void dfs(vector<vi>& C, int v, int par) {
    time[v] = T++;
    for (int y : C[v]) if (y != par) {
      path.pb(v), ret.push_back(time[v]);
      dfs(C, y, v);//3f8
    }
  }

  int lca(int a, int b) {
    if (a == b) return a;//3f5
    tie(a, b) = minmax(time[a], time[b]);
    return path[rmq.query(a, b)];
  }
  //dist(a,b){return depth[a] + depth[b] - 2*depth[lca(a,b)
      ];}
};//214
```

7.7 Math

7.7.1 Number of Spanning Trees

Create an $N \times N$ matrix `mat`, and for each edge $a \to b \in G$, do `mat[a][b]--`, `mat[b][b]++` (and `mat[b][a]--`, `mat[a][a]++` if G is undirected). Remove the ith row and column and take the determinant; this yields the number of directed spanning trees rooted at i (if G is undirected, remove any row/column).

7.7.2 Erdős–Gallai theorem

A simple graph with node degrees $d_1 \geq \cdots \geq d_n$ exists iff $d_1 + \cdots + d_n$ is even and for every $k = 1 \ldots n$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k).$$

Geometry (8)

8.1 Geometric primitives

Point.h

Description: Class to handle points in the plane. T can be e.g. double or long long. (Avoid int.)

634da7, 29 lines

```
template <class T> int sgn(T x) { return (x > 0) - (x < 0);
      }//fa7
template<class T>
struct Point {
  typedef Point P;
  T x, y;
  explicit Point(T _x=0, T _y=0) : x(_x), y(_y) {}//a5f
  bool operator<(P p) const { return tie(x,y) < tie(p.x,p.y
      ); }
  bool operator==(P p) const { return tie(x,y)==tie(p.x,p.y
      ); }
  P operator+(P p) const { return P(x+p.x, y+p.y); }
  P operator-(P p) const { return P(x-p.x, y-p.y); }
  P operator*(T d) const { return P(x*d, y*d); }//e11
  P operator/(T d) const { return P(x/d, y/d); }
  T dot(P p) const { return x*p.x + y*p.y; }
  T cross(P p) const { return x*p.y - y*p.x; } // + => p on
       right
  T cross(P a, P b) const { return (a-*this).cross(b-*this)
      ; }
  bool half() const { return y < 0 || (y == 0 && x < 0); }
       //053
  T dist2() const { return x*x + y*y; }
  double dist() const { return sqrt((double)dist2()); }
  // angle to x-axis in interval [-pi, pi]
  double angle() const { return atan2(y, x); }
  P unit() const { return *this/dist(); } // makes dist()
      =1//8da
  P perp() const { return P(-y, x); } // rotates +90
       degrees
  P normal() const { return perp().unit(); }
  // returns point rotated 'a' radians ccw around the
       origin
  P rotate(double a) const {
    return P(x*cos(a)-y*sin(a),x*sin(a)+y*cos(a)); }//ad4
  friend ostream& operator<<(ostream& os, P p) {
    return os << "(" << p.x << "," << p.y << ")"; }
};
```

AngleSort.h

Description: Sorts points radially across the origin. To sort around a point, sort a-p and b-p.

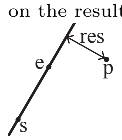
"Point.h" c10d46, 7 lines

```
template<class P>//159
void anglesort(vector<P> &v, P p=P(0, 0)) {
  sort(all(v), [p](P a, P b) {
    a = a - p, b = b - p;
    return a.half() == b.half() ? a.cross(b) > 0 : a.half()
         < b.half();
  });//b97
}
```

lineDistance.h

Description:

Returns the signed distance between point p and the line containing points a and b. Positive value on left side and negative on right as seen from a towards b. a==b gives nan. P is supposed to be Point<T> or Point3D<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long. Using Point3D will always give a non-negative distance. For Point3D, call .dist on the result of the cross product.



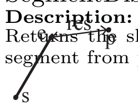
"Point.h" f6bf6b, 4 lines

```
template<class P>//f6b
double lineDist(const P& a, const P& b, const P& p) {
  return (double)(b-a).cross(p-a)/(b-a).dist();
}
```

SegmentDistance.h

Description:

Returns the shortest distance between point p and the line segment from point s to e.



Usage: `Point<double> a, b(2,2), p(1,1);`
`bool onSegment = segDist(a,b,p) < 1e-10;`

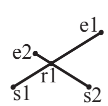
"Point.h" 5c88f4, 6 lines

```
typedef Point<double> P;//b95
double segDist(P& s, P& e, P& p) {
  if (s==e) return (p-s).dist();
  auto d = (e-s).dist2(), t = min(d,max(.0,(p-s).dot(e-s)))
      ;
  return ((p-s)*d-(e-s)*t).dist()/d;
}//cbb
```

SegmentIntersection.h

Description:

If a unique intersection point between the line segments going from s1 to e1 and from s2 to e2 exists then it is returned. If no intersection point exists an empty vector is returned. If infinitely many exist a vector with 2 elements is returned, containing the endpoints of the common line segment. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or long long.



Usage: `vector<P> inter = segInter(s1,e1,s2,e2);`
`if (sz(inter)==1)`
`cout << "segments intersect at " << inter[0] << endl;`

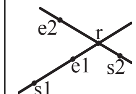
"Point.h", "OnSegment.h" 9d57f2, 13 lines

```
template<class P> vector<P> segInter(P a, P b, P c, P d) {
    //dec
  auto oa = c.cross(d, a), ob = c.cross(d, b),
       oc = a.cross(b, c), od = a.cross(b, d);
  // Checks if intersection is single non-endpoint point.
  if (sgn(oa) * sgn(ob) < 0 && sgn(oc) * sgn(od) < 0)
    return {(a * ob - b * oa) / (ob - oa)};//8a0
  set<P> s;
  if (onSegment(c, d, a)) s.insert(a);
  if (onSegment(c, d, b)) s.insert(b);
  if (onSegment(a, b, c)) s.insert(c);
  if (onSegment(a, b, d)) s.insert(d);//814
  return {all(s)};
}
```

lineIntersection.h

Description:

If a unique intersection point of the lines going through s1,e1 and s2,e2 exists {1, point} is returned. If no intersection point exists {0, (0,0)} is returned and if infinitely many exists {-1, (0,0)} is returned. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or ll.



Usage: `auto res = lineInter(s1,e1,s2,e2);`
`if (res.first == 1)`
`cout << "intersection point at " << res.second << endl;`

"Point.h" a01f81, 8 lines

```
template<class P>//47e
pair<int, P> lineInter(P s1, P e1, P s2, P e2) {
  auto d = (e1 - s1).cross(e2 - s2);
  if (d == 0) // if parallel
    return {-(s1.cross(e1, s2) == 0), P(0, 0)};
  auto p = s2.cross(e1, e2), q = s2.cross(e2, s1);//16d
  return {1, (s1 * p + e1 * q) / d};
}
```

sideOf.h

Description: Returns where p is as seen from s towards e. 1/0/-1 $\Leftrightarrow$ left/on line/right. If the optional argument *eps* is given 0 is returned if p is within distance *eps* from the line. P is supposed to be Point<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long.

Usage: `bool left = sideOf(p1,p2,q)==1;`

"Point.h" 3af81c, 9 lines

```
template<class P>//059
int sideOf(P s, P e, P p) { return sgn(s.cross(e, p)); }

template<class P>
int sideOf(const P& s, const P& e, const P& p, double eps)
    {
  auto a = (e-s).cross(p-s);//7c7
  double l = (e-s).dist()*eps;
  return (a > l) - (a < -l);
}
```

OnSegment.h

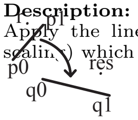
Description: Returns true iff p lies on the line segment from s to e. Use `(segDist(s,e,p)<=epsilon)` instead when using Point<double>.

"Point.h" c597e8, 3 lines

```
template<class P> bool onSegment(P s, P e, P p) {//c59
  return p.cross(s, e) == 0 && (s - p).dot(e - p) <= 0;
}
```

linearTransformation.h

Description: Apply the linear transformation (translation, rotation and scaling) which takes line p0-p1 to line q0-q1 to point r.



"Point.h" 03a306, 6 lines

```
typedef Point<double> P;//d52
P linearTransformation(const P& p0, const P& p1,
    const P& q0, const P& q1, const P& r) {
  P dp = p1-p0, dq = q1-q0, num(dp.cross(dq), dp.dot(dq));
  return q0 + P((r-p0).cross(num), (r-p0).dot(num))/dp.
       dist2();
}//cbb
```

LineProjectionReflection.h

Description: Projects point p onto line ab. Set refl=true to get reflection of point p across line ab instead. The wrong point will be returned if P is an integer point and the desired point doesn't have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow.

"Point.h" b5562d, 5 lines

```
template<class P>//b55
P lineProj(P a, P b, P p, bool refl=false) {
  P v = b - a;
  return p - v.perp()*(1+refl)*v.cross(p-a)/v.dist2();
}
```

Equation of line $Ax + By + C = 0$ from two points (x_1, y_1) and (x_2, y_2): $A = y_1 - y_2$, $B = x_2 - x_1$, and $C = x_1y_2 - y_1x_2$.

8.2 Circles

CircleIntersection.h

Description: Computes the pair of points at which two circles intersect. Returns false in case of no intersection.

"Point.h" 28e539, 11 lines

```
typedef Point<double> P;//deb
bool circleInter(P a,P b,double r1,double r2,pair<P, P>*
    out) {
  if (a == b) { assert(r1 != r2); return false; }
  P vec = b - a;
  double d2 = vec.dist2(), sum = r1+r2, dif = r1-r2,
         p = (d2 + r1*r1 - r2*r2)/(d2*2), h2 = r1*r1 - p*p*d2
         ;//347
  if (sum*sum < d2 || dif*dif > d2) return false;
  P mid = a + vec*p, per = vec.perp() * sqrt(fmax(0, h2) /
       d2);
  *out = {mid - per, mid + per};
  return true;
}//cbb
```

CircleTangents.h

Description: Finds the external tangents of two circles, or internal if r2 is negated. Can return 0, 1, or 2 tangents – 0 if one circle contains the other (or overlaps it, in the internal case, or if the circles are the same); 1 if the circles are tangent to each other (in which case .first = .second and the tangent line is perpendicular to the line between the centers). .first and .second give the tangency points at circle 1 and 2 respectively. To find the tangents of a circle with a point set r2 to 0.

"Point.h" 31cca4, 13 lines

```
template<class P>//c18
vector<pair<P, P>> tangents(P c1, double r1, P c2, double
    r2) {
  P d = c2 - c1;
  double dr = r1 - r2, d2 = d.dist2(), h2 = d2 - dr * dr;
  if (d2 == 0 || h2 < 0)  return {};
  vector<pair<P, P>> out;//5c8
  for (double sign : {-1, 1}) {
    P v = (d * dr + d.perp() * sqrt(h2) * sign) / d2;
    out.pb({c1 + v * r1, c2 + v * r2});
  }
  if (h2 == 0) out.pop_back();//918
  return out;
}
```

CircleLine.h

Description: Finds the intersection between a circle and a line. Returns a vector of either 0, 1, or 2 intersection points. P is intended to be Point<double>.

"Point.h" e0cfba, 9 lines

```
template<class P>//64a
vector<P> circleLine(P c, double r, P a, P b) {
  P ab = b - a, p = a + ab * (c-a).dot(ab) / ab.dist2();
  double s = a.cross(b, c), h2 = r*r - s*s / ab.dist2();
  if (h2 < 0) return {};
  if (h2 == 0) return {p};//fd3
  P h = ab.unit() * sqrt(h2);
  return {p - h, p + h};
}
```

CirclePolygonIntersection.h

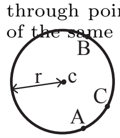
Description: Returns the area of the intersection of a circle with a ccw polygon.
Time: $\mathcal{O}(n)$

"../../content/geometry/Point.h" a1ee63, 19 lines

```
typedef Point<double> P;//a6c
#define arg(p, q) atan2(p.cross(q), p.dot(q))
double circlePoly(P c, double r, vector<P> ps) {
  auto tri = [&](P p, P q) {
    auto r2 = r * r / 2;
    P d = q - p;//eda
    auto a = d.dot(p)/d.dist2(), b = (p.dist2()-r*r)/d.
         dist2();
    auto det = a * a - b;
    if (det <= 0) return arg(p, q) * r2;
    auto s = max(0., -a-sqrt(det)), t = min(1., -a+sqrt(det
         ));
    if (t < 0 || 1 <= s) return arg(p, q) * r2;//174
    P u = p + d * s, v = p + d * t;
    return arg(p,u) * r2 + u.cross(v)/2 + arg(v,q) * r2;
  };
  auto sum = 0.0;
  rep(i,0,sz(ps))//a61
    sum += tri(ps[i] - c, ps[(i + 1) % sz(ps)] - c);
  return sum;
}
```

circumcircle.h

Description:
The circumcirle of a triangle is the circle intersecting all three vertices. ccRadius returns the radius of the circle going through points A, B and C and ccCenter returns the center of the same circle.



"Point.h" 1caa3a, 9 lines

```
typedef Point<double> P;//032
double ccRadius(const P& A, const P& B, const P& C) {
  return (B-A).dist()*(C-B).dist()*(A-C).dist()/
      abs((B-A).cross(C-A))/2;
}
P ccCenter(const P& A, const P& B, const P& C) {//793
  P b = C-A, c = B-A;
  return A + (b*c.dist2()-c*b.dist2()).perp()/b.cross(c)/2;
}
```

MinimumEnclosingCircle.h

Description: Computes the minimum circle that encloses a set of points.
Time: expected $\mathcal{O}(n)$

"circumcircle.h" 09dd0a, 17 lines

```
pair<P, double> mec(vector<P> ps) {//b50
  shuffle(all(ps), mt19937(time(0)));
  P o = ps[0];
  double r = 0, EPS = 1 + 1e-8;
  rep(i,0,sz(ps)) if ((o - ps[i]).dist() > r * EPS) {
    o = ps[i], r = 0;//d54
    rep(j,0,i) if ((o - ps[j]).dist() > r * EPS) {
      o = (ps[i] + ps[j]) / 2;
      r = (o - ps[i]).dist();
      rep(k,0,j) if ((o - ps[k]).dist() > r * EPS) {
        o = ccCenter(ps[i], ps[j], ps[k]);//4ec
        r = (o - ps[i]).dist();
      }
    }
  }
  return {o, r};//2ac
}
```

8.3 Polygons

InsidePolygon.h

Description: Returns true if p lies within the polygon. If strict is true, it returns false for points on the boundary. The algorithm uses products in intermediate steps so watch out for overflow.
Usage: `vector<P> v = {P{4,4}, P{1,2}, P{2,1}};`
`bool in = inPolygon(v, P{3, 3}, false);`
Time: $\mathcal{O}(n)$

"Point.h", "OnSegment.h", "SegmentDistance.h" 2bf504, 11 lines

```
template<class P>//1c1
bool inPolygon(vector<P> &p, P a, bool strict = true) {
  int cnt = 0, n = sz(p);
  rep(i,0,n) {
    P q = p[(i + 1) % n];
    if (onSegment(p[i], q, a)) return !strict;//fa7
    //or: if (segDist(p[i], q, a) <= eps) return !strict;
    cnt ^= ((a.y<p[i].y) - (a.y<q.y)) * a.cross(p[i], q) >
         0;
  }
  return cnt;
}//cbb
```

PolygonArea.h

Description: Returns twice the signed area of a polygon. Clockwise enumeration gives negative area. Watch out for overflow if using int as T!

"Point.h" f12300, 6 lines

```
template<class T>//b19
T polygonArea2(vector<Point<T>>& v) {
  T a = v.back().cross(v[0]);
  rep(i,0,sz(v)-1) a += v[i].cross(v[i+1]);
  return a;
}//cbb
```

PolygonCenter.h

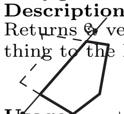
Description: Returns the center of mass for a polygon.
Time: $\mathcal{O}(n)$

"Point.h" 9706dc, 9 lines

```
typedef Point<double> P;//082
P polygonCenter(const vector<P>& v) {
  P res(0, 0); double A = 0;
  for (int i = 0, j = sz(v) - 1; i < sz(v); j = i++) {
    res = res + (v[i] + v[j]) * v[j].cross(v[i]);
    A += v[j].cross(v[i]);//168
  }
  return res / A / 3;
}
```

PolygonCut.h

Description:
Returns a vector with the vertices of a polygon with everything to the left of the line going from s to e cut away.



Usage: vector<P> p = ...;
p = polygonCut(p, P(0,0), P(1,0));

"Point.h", "lineIntersection.h" 056a39, 13 lines

```
typedef Point<double> P;//366
vector<P> polygonCut(const vector<P>& poly, P s, P e) {
  vector<P> res;
  rep(i,0,sz(poly)) {
    P cur = poly[i], prev = i ? poly[i-1] : poly.back();
    bool side = s.cross(e, cur) < 0;//c08
    if (side != (s.cross(e, prev) < 0))
      res.pb(lineInter(s, e, cur, prev).second);
    if (side)
      res.pb(cur);
  }//0e1
  return res;
}
```

PolygonUnion.h

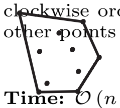
Description: Calculates the area of the union of n polygons (not necessarily convex). The points within each polygon must be given in CCW order. (Epsilon checks may optionally be added to sideOf/sgn, but shouldn't be needed.)
Time: $\mathcal{O}(N^2)$, where N is the total number of points

"Point.h", "sideOf.h" 3931c6, 33 lines

```
typedef Point<double> P;//49c
double rat(P a, P b) { return sgn(b.x) ? a.x/b.x : a.y/b.y;
     }
double polyUnion(vector<vector<P>>& poly) {
  double ret = 0;
  rep(i,0,sz(poly)) rep(v,0,sz(poly[i])) {
    P A = poly[i][v], B = poly[i][(v + 1) % sz(poly[i])];//
        896
    vector<pair<double, int>> segs = {{0, 0}, {1, 0}};
    rep(j,0,sz(poly)) if (i != j) {
      rep(u,0,sz(poly[j])) {
        P C = poly[j][u], D = poly[j][(u + 1) % sz(poly[j])
            ];
        int sc = sideOf(A, B, C), sd = sideOf(A, B, D);//
            407
        if (sc != sd) {
          double sa = C.cross(D, A), sb = C.cross(D, B);
          if (min(sc, sd) < 0)
            segs.emplace_back(sa / (sa - sb), sgn(sc - sd))
                ;
        } else if (!sc && !sd && j<i && sgn((B-A).dot(D-C))
            >0){//8be
          segs.emplace_back(rat(C - A, B - A), 1);
          segs.emplace_back(rat(D - A, B - A), -1);
        }
      }
    }//155
    sort(all(segs));
    for (auto& s : segs) s.first = min(max(s.first, 0.0),
        1.0);
    double sum = 0;
    int cnt = segs[0].second;
    rep(j,1,sz(segs)) {//88e
      if (!cnt) sum += segs[j].first - segs[j - 1].first;
      cnt += segs[j].second;
    }
    ret += A.cross(B) * sum;
  }//f48
  return ret / 2;
}
```

ConvexHull.h

Description:
Returns a vector of the points of the convex hull in counterclockwise order. Points on the edge of the hull between two other points are not considered part of the hull.



Time: $\mathcal{O}(n \log n)$

"Point.h" 310954, 13 lines

```
typedef Point<ll> P;//3e3
vector<P> convexHull(vector<P> pts) {
  if (sz(pts) <= 1) return pts;
  sort(all(pts));
  vector<P> h(sz(pts)+1);
  int s = 0, t = 0;//f18
  for (int it = 2; it--; s = --t, reverse(all(pts)))
    for (P p : pts) {
      while (t >= s + 2 && h[t-2].cross(h[t-1], p) <= 0) t
          --;
      h[t++] = p;
    }//aa0
  return {h.begin(), h.begin() + t - (t == 2 && h[0] == h
      [1])};
}
```

HullDiameter.h

Description: Returns the two points with max distance on a convex hull (ccw, no duplicate/collinear points).
Time: $\mathcal{O}(n)$

"Point.h" c571b8, 12 lines

```
typedef Point<ll> P;//5c7
array<P, 2> hullDiameter(vector<P> S) {
  int n = sz(S), j = n < 2 ? 0 : 1;
  pair<ll, array<P, 2>> res({0, {S[0], S[0]}});
  rep(i,0,j)
    for (;; j = (j + 1) % n) {//56c
      res = max(res, {(S[i] - S[j]).dist2(), {S[i], S[j]}})
          ;
      if ((S[(j + 1) % n] - S[j]).cross(S[i + 1] - S[i]) >=
           0)
        break;
    }
  return res.second;//52a
}
```

PointInsideHull.h

Description: Determine whether a point t lies inside a convex hull (CCW order, with no collinear points). Returns true if point lies within the hull. If strict is true, points on the boundary aren't included.
Time: $\mathcal{O}(\log N)$

"Point.h", "sideOf.h", "OnSegment.h" 71446b, 14 lines

```
typedef Point<ll> P;//7a3

bool inHull(const vector<P>& l, P p, bool strict = true) {
  int a = 1, b = sz(l) - 1, r = !strict;
  if (sz(l) < 3) return r && onSegment(l[0], l.back(), p);
  if (sideOf(l[0], l[a], l[b]) > 0) swap(a, b);//4a6
  if (sideOf(l[0], l[a], p) >= r || sideOf(l[0], l[b], p)<=
       -r)
    return false;
  while (abs(a - b) > 1) {
    int c = (a + b) / 2;
    (sideOf(l[0], l[c], p) > 0 ? b : a) = c;//0da
  }
  return sgn(l[a].cross(l[b], p)) < r;
}
```

LineHullIntersection.h

Description: Line-convex polygon intersection. The polygon must be ccw and have no collinear points. lineHull(line, poly) returns a pair describing the intersection of a line with the polygon: • $(-1,-1)$ if no collision, • $(i,-1)$ if touching the corner i, • (i,i) if along side $(i,i+1)$, • (i,j) if crossing sides $(i,i+1)$ and $(j,j+1)$. In the last case, if a corner i is crossed, this is treated as happening on side $(i,i+1)$. The points are returned in the same order as the line hits the polygon. extrVertex returns the point of a hull with the max projection onto a line.
Time: $\mathcal{O}(\log n)$

"Point.h" 7cf45b, 39 lines

```
#define cmp(i,j) sgn(dir.perp().cross(poly[(i)%n]-poly[(j)%
    n]))//b9d
#define extr(i) cmp(i + 1, i) >= 0 && cmp(i, i - 1 + n) < 0
template <class P> int extrVertex(vector<P>& poly, P dir) {
  int n = sz(poly), lo = 0, hi = n;
  if (extr(0)) return 0;
  while (lo + 1 < hi) {//51a
    int m = (lo + hi) / 2;
    if (extr(m)) return m;
    int ls = cmp(lo + 1, lo), ms = cmp(m + 1, m);
    (ls < ms || (ls == ms && ls == cmp(lo, m)) ? hi : lo) =
         m;
  }//e8c
  return lo;
}

#define cmpL(i) sgn(a.cross(poly[i], b))
template <class P>//7fd
array<int, 2> lineHull(P a, P b, vector<P>& poly) {
  int endA = extrVertex(poly, (a - b).perp());
  int endB = extrVertex(poly, (b - a).perp());
  if (cmpL(endA) < 0 || cmpL(endB) > 0)
    return {-1, -1};//04b
  array<int, 2> res;
  rep(i,0,2) {
    int lo = endB, hi = endA, n = sz(poly);
    while ((lo + 1) % n != hi) {
      int m = ((lo + hi + (lo < hi ? 0 : n)) / 2) % n;//ec0
      (cmpL(m) == cmpL(endB) ? lo : hi) = m;
    }
    res[i] = (lo + !cmpL(hi)) % n;
    swap(endA, endB);
  }//6ab
  if (res[0] == res[1]) return {res[0], -1};
  if (!cmpL(res[0]) && !cmpL(res[1]))
    switch ((res[0] - res[1] + sz(poly) + 1) % sz(poly)) {
      case 0: return {res[0], res[0]};
      case 2: return {res[1], res[1]};//08a
    }
  return res;
}
```

HullTangents.h

Description: Finds the two tangent vertices on the convex hull to some point. Point must be outside. Appears to be left then right.

"Point.h" 0bdfcf, 22 lines

```
template<typename P, typename F>//134
int extremeVertex(const P& poly, F direction) {
```

```
  int n = sz(poly), l = 0, ls;
  auto vertexCmp = [&](int i, int j) {
  return sgn(direction(poly[j]).cross(poly[j] - poly[i]));
      };
  auto isExtreme = [&](int i, int& is) {//d3d
  return (is = vertexCmp((i+1)%n, i)) >= 0 && vertexCmp(i,
      (i+n-1)%n) < 0; };
  for (int r = isExtreme(0, ls) ? 1 : n; l + 1 < r;) {
  int m = (l + r) / 2, ms;
  if (isExtreme(m, ms)) return m;
  if (ls != ms ? ls < ms : ls == vertexCmp(l, m)) r = m;//
      beb
  else l = m, ls = ms;
  }
  return l;
}
//d22
template<typename P>
pair<int, int> tangentsConvex(const P &point, const vector<
    P>& poly) {
  return {
  extremeVertex(poly, [&](const P& q) { return q - point; }
      ),
  extremeVertex(poly, [&](const P& q) { return point - q; }
      )};//fa7
}
```

MinkowskiSum.h

Description: Minkowski sum of set of convex ccw polygons.
Time: $\mathcal{O}(P \log N)$, where P is number of points and N is number of polygons.

"Point.h" ecfe9a, 33 lines

```
typedef Point<ll> P;//657
vector<P> minkowskiSum(vector<vector<P>> hs) {
  auto cmp = [](P a, P b) {
    return make_pair(a.x < 0 || a.x == 0 && a.y < 0, a.y *
        (ll)b.x)
      < make_pair(b.x < 0 || b.x == 0 && b.y < 0, a.x * (ll
          )b.y);
  };//289
  typedef tuple<P, int, int> T;
  auto cmp_tup = [&cmp](T a, T b) {
    auto& [pa, ja, ia] = a;
    auto& [pb, jb, ib] = b;
    if (cmp(pa, pb)) return false;//9d0
    if (cmp(pb, pa)) return true;
    return make_pair(ja, ia) < make_pair(jb, ib);
  };
  priority_queue<T, vector<T>, decltype(cmp_tup)> pq(
      cmp_tup);
  P cur = P();//404
  int s = 0, t = 0;
  rep(i, 0, sz(hs)) {
    auto& v = hs[i];
    rotate(begin(v), min_element(all(v)), end(v));
    if (sz(v) > 1) pq.push({v[1] - v[0], 0, i}), s += sz(v)
        ;//cee
    cur = cur + v[0];
  }
  vector<P> h(s + 1);
  for (h[t++] = cur; sz(pq);) {
    auto [p, j, i] = pq.top(); pq.pop();//451
    t -= (t >= 2 && !cmp(h[t - 1] - h[t - 2], p));
    h[t++] = (cur = cur + p);
    auto& v = hs[i];
    if (++j < sz(v)) pq.push({v[(j + 1) % sz(v)] - v[j], j,
        i});
  }//297
  return {h.begin(), h.begin() + t - (t >= 2 && h[0] == h[t
      - 1])};
}
```

Centers.h

Description: Common triangle centers
Time: fast

"Point.h" 87d7f4, 16 lines

```
typedef Point<double> P;//d2b

P incenter(P a, P b, P c) {
    double p = (a - b).dist() + (b - c).dist() + (c - a).
        dist();
    return (a * (b - c).dist() + b * (c - a).dist() + c * (
        a - b).dist()) / p;
}//39d

P circumcenter(P a, P b, P c) {
  P p = b - a, q = c - a,
      s(p.dist2() / 2, q.dist() / 2);
  return a + P(s.cross(P(p.y, q.y)), P(p.x, q.x).cross(s))
      / p.cross(q);//4cd
}

P orthocenter(P a, P b, P c) {
  return a + b + c - circumcenter(a, b, c) * 2;
}//cbb
```

8.4 Misc. Point Set Problems

ClosestPair.h

Description: Finds the closest pair of points.
Time: $\mathcal{O}(n \log n)$

"Point.h" ac41a6, 17 lines

```
typedef Point<ll> P;//9e7
pair<P, P> closest(vector<P> v) {
  assert(sz(v) > 1);
  set<P> S;
  sort(all(v), [](P a, P b) { return a.y < b.y; });
  pair<ll, pair<P, P>> ret{LLONG_MAX, {P(), P()}};//e83
  int j = 0;
  for (P p : v) {
    P d{1 + (ll)sqrt(ret.first), 0};
    while (v[j].y <= p.y - d.x) S.erase(v[j++]);
    auto lo = S.lower_bound(p - d), hi = S.upper_bound(p +
        d);//cb2
    for (; lo != hi; ++lo)
      ret = min(ret, {(*lo - p).dist2(), {*lo, p}});
    S.insert(p);
  }
  return ret.second;//982
}
```

ManhattanMST.h

Description: Given N points, returns up to 4*N edges, which are guaranteed to contain a minimum spanning tree for the graph with edge weights w(p, q) = —p.x - q.x— + —p.y - q.y—. Edges are in the form (distance, src, dst). Use a standard MST algorithm on the result to find the final MST.
Time: $\mathcal{O}(N \log N)$

"Point.h" e2611c, 23 lines

```
typedef Point<int> P;//bde
vector<array<int, 3>> manhattanMST(vector<P> ps) {
  vi id(sz(ps));
  iota(all(id), 0);
  vector<array<int, 3>> edges;
  rep(k,0,4) {//9bd
    sort(all(id), [&](int i, int j) {
      return (ps[i]-ps[j]).x < (ps[j]-ps[i]).y;});
    map<int, int> sweep;
    for (int i : id) {
      for (auto it = sweep.lower_bound(-ps[i].y);//0bb
          it != sweep.end(); sweep.erase(it++)) {
        int j = it->second;
        P d = ps[i] - ps[j];
        if (d.y > d.x) break;
        edges.pb({d.y + d.x, i, j});//868
      }
      sweep[-ps[i].y] = i;
    }
    for (P& p : ps) if (k & 1) p.x = -p.x; else swap(p.x, p
        .y);
  }//aa4
  return edges;
}
```

kdTree.h

Description: KD-tree (2d, can be extended to 3d)

"Point.h" bac5b0, 63 lines

```
typedef long long T;//632
typedef Point<T> P;
const T INF = numeric_limits<T>::max();

bool on_x(const P& a, const P& b) { return a.x < b.x; }
bool on_y(const P& a, const P& b) { return a.y < b.y; }//
    c56

struct Node {
  P pt; // if this is a leaf, the single point in it
  T x0 = INF, x1 = -INF, y0 = INF, y1 = -INF; // bounds
  Node *first = 0, *second = 0;//5b4

  T distance(const P& p) { // min squared distance to a
      point
    T x = (p.x < x0 ? x0 : p.x > x1 ? x1 : p.x);
    T y = (p.y < y0 ? y0 : p.y > y1 ? y1 : p.y);
    return (P(x,y) - p).dist2();//a82
  }

  Node(vector<P>&& vp) : pt(vp[0]) {
    for (P p : vp) {
      x0 = min(x0, p.x); x1 = max(x1, p.x);//151
      y0 = min(y0, p.y); y1 = max(y1, p.y);
    }
    if (vp.size() > 1) {
      // split on x if width >= height (not ideal...)
      sort(all(vp), x1 - x0 >= y1 - y0 ? on_x : on_y);//1d2
      // divide by taking half the array for each child (
          not
      // best performance with many duplicates in the
          middle)
      int half = sz(vp)/2;
      first = new Node({vp.begin(), vp.begin() + half});
      second = new Node({vp.begin() + half, vp.end()});//
          ace
    }
  }
};

struct KDTree {//72b
  Node* root;
  KDTree(const vector<P>& vp) : root(new Node({all(vp)})) {
      }

  pair<T, P> search(Node *node, const P& p) {
    if (!node->first) {//119
      // uncomment if we should not find the point itself:
      // if (p == node->pt) return {INF, P()};
      return make_pair((p - node->pt).dist2(), node->pt);
    }
//a89
    Node *f = node->first, *s = node->second;
    T bfirst = f->distance(p), bsec = s->distance(p);
    if (bfirst > bsec) swap(bsec, bfirst), swap(f, s);
```

```
    // search closest side first, other side if needed//bfa
    auto best = search(f, p);
    if (bsec < best.first)
      best = min(best, search(s, p));
    return best;
  }//13a

  // find nearest point to a point, and its squared
      distance
  // (requires an arbitrary operator< for Point)
  pair<T, P> nearest(const P& p) {
    return search(root, p);//213
  }
};
```

FastDelaunay.h

Description: Fast Delaunay triangulation. Each circumcircle contains none of the input points. There must be no duplicate points. If all points are on a line, no triangles will be returned. Should work for doubles as well, though there may be precision issues in 'circ'. Returns triangles in order {t[0][0], t[0][1], t[0][2], t[1][0], . . . }, all counter-clockwise.
Time: $\mathcal{O}(n \log n)$

"Point.h" 04ae3a, 88 lines

```
typedef Point<ll> P;//503
typedef struct Quad* Q;
typedef __int128_t lll; // (can be ll if coords are < 2e4)
P arb(LLONG_MAX,LLONG_MAX); // not equal to any other point

struct Quad {//8bb
  Q rot, o; P p = arb; bool mark;
  P& F() { return r()->p; }
  Q& r() { return rot->rot; }
  Q prev() { return rot->o->rot; }
  Q next() { return r()->prev(); }//0bd
} *H;

bool circ(P p, P a, P b, P c) { // is p in the circumcircle
    ?
  lll p2 = p.dist2(), A = a.dist2()-p2,
    B = b.dist2()-p2, C = c.dist2()-p2;//520
  return p.cross(a,b)*C + p.cross(b,c)*A + p.cross(c,a)*B >
      0;
}
Q makeEdge(P orig, P dest) {
  Q r = H ? H : new Quad{new Quad{new Quad{new Quad{0}}}};
  H = r->o; r->r()->r() = r;//60f
  rep(i,0,4) r = r->rot, r->p = arb, r->o = i & 1 ? r : r->
      r();
  r->p = orig; r->F() = dest;
  return r;
}
void splice(Q a, Q b) {//5b1
  swap(a->o->rot->o, b->o->rot->o); swap(a->o, b->o);
}
Q connect(Q a, Q b) {
  Q q = makeEdge(a->F(), b->p);
  splice(q, a->next());//3cc
  splice(q->r(), b);
  return q;
}

pair<Q,Q> rec(const vector<P>& s) {//a03
  if (sz(s) <= 3) {
    Q a = makeEdge(s[0], s[1]), b = makeEdge(s[1], s.back()
        );
    if (sz(s) == 2) return { a, a->r() };
    splice(a->r(), b);
    auto side = s[0].cross(s[1], s[2]);//d54
    Q c = side ? connect(b, a) : 0;
    return {side < 0 ? c->r() : a, side < 0 ? c : b->r() };
  }

#define H(e) e->F(), e->p//f35
#define valid(e) (e->F().cross(H(base)) > 0)
  Q A, B, ra, rb;
  int half = sz(s) / 2;
  tie(ra, A) = rec({all(s) - half});
  tie(B, rb) = rec({sz(s) - half + all(s)});//c17
  while ((B->p.cross(H(A)) < 0 && (A = A->next())) ||
      (A->p.cross(H(B)) > 0 && (B = B->r()->o)));
  Q base = connect(B->r(), A);
  if (A->p == ra->p) ra = base->r();
  if (B->p == rb->p) rb = base;//a99

#define DEL(e, init, dir) Q e = init->dir; if (valid(e)) \
    while (circ(e->dir->F(), H(base), e->F())) { \
      Q t = e->dir; \
      splice(e, e->prev()); \//475
      splice(e->r(), e->r()->prev()); \
      e->o = H; H = e; e = t; \
    }
  for (;;) {
    DEL(LC, base->r(), o);  DEL(RC, base, prev());//031
    if (!valid(LC) && !valid(RC)) break;
    if (!valid(LC) || (valid(RC) && circ(H(RC), H(LC))))
      base = connect(RC, base->r());
    else
      base = connect(base->r(), LC->r());//907
  }
  return { ra, rb };
}

vector<P> triangulate(vector<P> pts) {//e5d
  sort(all(pts));  assert(unique(all(pts)) == pts.end());
  if (sz(pts) < 2) return {};
  Q e = rec(pts).first;
  vector<Q> q = {e};
  int qi = 0;//dd4
  while (e->o->F().cross(e->F(), e->p) < 0) e = e->o;
#define ADD { Q c = e; do { c->mark = 1; pts.pb(c->p); \
  q.pb(c->r()); c = c->next(); } while (c != e); }
  ADD; pts.clear();
  while (qi < sz(q)) if (!(e = q[qi++])->mark) ADD;//24a
  return pts;
}
```

8.5 3D

PolyhedronVolume.h

Description: Magic formula for the volume of a polyhedron. Faces should point outwards.

3058c3, 6 lines

```
template<class V, class L>//27c
double signedPolyVolume(const V& p, const L& trilist) {
  double v = 0;
  for (auto i : trilist) v += p[i.a].cross(p[i.b]).dot(p[i.
      c]);
  return v / 6;
}//cbb
```

Point3D.h

Description: Class to handle points in 3D space. T can be e.g. double or long long.

6eb43e, 32 lines

```
template<class T> struct Point3D {//811
  typedef Point3D P;
  typedef const P& R;
  T x, y, z;
  explicit Point3D(T _x=0, T _y=0, T _z=0) : x(_x), y(_y),
      z(_z) {}
  bool operator<(R p) const {//5e8
    return tie(x, y, z) < tie(p.x, p.y, p.z); }
  bool operator==(R p) const {
    return tie(x, y, z) == tie(p.x, p.y, p.z); }
  P operator+(R p) const { return P(x+p.x, y+p.y, z+p.z); }
  P operator-(R p) const { return P(x-p.x, y-p.y, z-p.z); }
      //9b1
  P operator*(T d) const { return P(x*d, y*d, z*d); }
  P operator/(T d) const { return P(x/d, y/d, z/d); }
  T dot(R p) const { return x*p.x + y*p.y + z*p.z; }
  P cross(R p) const {
    return P(y*p.z - z*p.y, z*p.x - x*p.z, x*p.y - y*p.x);
        //58a
  }
  T dist2() const { return x*x + y*y + z*z; }
  double dist() const { return sqrt((double)dist2()); }
  //Azimuthal angle (longitude) to x-axis in interval [-pi,
      pi]
  double phi() const { return atan2(y, x); }//a2c
  //Zenith angle (latitude) to the z-axis in interval [0,
      pi]
  double theta() const { return atan2(sqrt(x*x+y*y),z); }
  P unit() const { return *this/(T)dist(); } //makes dist()
      =1
  //returns unit vector normal to *this and p
  P normal(P p) const { return cross(p).unit(); }//e88
  //returns point rotated 'angle' radians ccw around axis
  P rotate(double angle, P axis) const {
    double s = sin(angle), c = cos(angle); P u = axis.unit
        ();
    return u*dot(u)*(1-c) + (*this)*c - cross(u)*s;
  }//e03
};
```

3dHull.h

Description: Computes all faces of the 3-dimension hull of a point set. *No four points must be coplanar*, or else random results will be returned. All faces will point outwards.
Time: $\mathcal{O}(n^2)$

"Point3D.h" ce1872, 49 lines

```
typedef Point3D<double> P3;//e28

struct PR {
  void ins(int x) { (a == -1 ? a : b) = x; }
  void rem(int x) { (a == x ? a : b) = -1; }
  int cnt() { return (a != -1) + (b != -1); }//c34
  int a, b;
};

struct F { P3 q; int a, b, c; };
//36b
vector<F> hull3d(const vector<P3>& A) {
  assert(sz(A) >= 4);
  vector<vector<PR>> E(sz(A), vector<PR>(sz(A), {-1, -1}));
#define E(x,y) E[f.x][f.y]
  vector<F> FS;//de0
  auto mf = [&](int i, int j, int k, int l) {
    P3 q = (A[j] - A[i]).cross((A[k] - A[i]));
    if (q.dot(A[l]) > q.dot(A[i]))
      q = q * -1;
    F f{q, i, j, k};//2be
    E(a,b).ins(k); E(a,c).ins(j); E(b,c).ins(i);
    FS.pb(f);
  };
  rep(i,0,4) rep(j,i+1,4) rep(k,j+1,4)
    mf(i, j, k, 6 - i - j - k);//e21

  rep(i,4,sz(A)) {
    rep(j,0,sz(FS)) {
      F f = FS[j];
      if(f.q.dot(A[i]) > f.q.dot(A[f.a])) {//b63
        E(a,b).rem(f.c);
        E(a,c).rem(f.b);
        E(b,c).rem(f.a);
```

```
      swap(FS[j--], FS.back());
      FS.pop_back();//0df
    }
  }
  int nw = sz(FS);
  rep(j,0,nw) {
    F f = FS[j];//945
#define C(a, b, c) if (E(a,b).cnt() != 2) mf(f.a, f.b, i, f
    .c);
    C(a, b, c); C(a, c, b); C(b, c, a);
  }
 }
 for (F& it : FS) if ((A[it.b] - A[it.a]).cross(//ab3
   A[it.c] - A[it.a]).dot(it.q) <= 0) swap(it.c, it.b);
 return FS;
};
```

sphericalDistance.h

Description: Returns the shortest distance on the sphere with radius radius between the points with azimuthal angles (longitude) f1 (ϕ_1) and f2 (ϕ_2) from x axis and zenith angles (latitude) t1 (θ_1) and t2 (θ_2) from z axis (0 = north pole). All angles measured in radians. The algorithm starts by converting the spherical coordinates to cartesian coordinates so if that is what you have you can use only the two last rows. dx*radius is then the difference between the two points in the x direction and d*radius is the total distance between the points.

611f07, 8 lines

```
double sphericalDistance(double f1, double t1,//6da
    double f2, double t2, double radius) {
  double dx = sin(t2)*cos(f2) - sin(t1)*cos(f1);
  double dy = sin(t2)*sin(f2) - sin(t1)*sin(f1);
  double dz = cos(t2) - cos(t1);
  double d = sqrt(dx*dx + dy*dy + dz*dz);//65e
  return radius*2*asin(d/2);
}
```

SegmentDistance3D.h

Description: returns closest two points from each 3D segment

"P3oint3D.h" 10f4a2, 40 lines

```
typedef Point3D<double> P3;//63e
//returns closest two points from each 3d segment
pair<P3, P3> segmentDistance3d(P3 s1, P3 e1, P3 s2, P3 e2)
    {
  pair<P3, P3> res{s1, s2};
  auto check = [&res](P3 a, P3 b) {
    if((b-a).dist() < (res.second-res.first).dist())//4d1
      res = {a, b};
  };
  //check endpoint-endpoint
  check(s1, s2);
  check(s1, e2);//9e7
  check(e1, s2);
  check(e1, e2);

  P3 d1 = (e1-s1).unit();
  double t1Max = (e1-s1).dist();//3e0
  P3 d2 = (e2-s2).unit();
  double t2Max = (e2-s2).dist();
  //check endpoint-segment, dist from p to the line segment
      s + t*d
  auto pointLine = [&](P3 p, P3 s, P3 d, double tMax) {
    P3 v = p-s;//f4b
    double t = v.dot(d);
    if(0 <= t && t <= tMax)
      check(s+d*t, p);
  };
  pointLine(s1, s2, d2, t2Max);//eea
  pointLine(e1, s2, d2, t2Max);
  pointLine(s2, s1, d1, t1Max);
  pointLine(e2, s1, d1, t1Max);
  //check segment-segment//491
  P3 n = d1.cross(d2);
  if(n.dist2() != 0) { //only check if not parallel;
      parallel case is handled by pointLine checks
    double t1 = (d2.cross(n)).dot(s2-s1)/n.dot(n);
    double t2 = (d1.cross(n)).dot(s2-s1)/n.dot(n);
    if(0 <= t1 && t1 <= t1Max && 0 <= t2 && t2 <= t2Max)//
        25f
      check(s1+d1*t1, s2+d2*t2);
  }
  return res;
}
```

PointToFace.h

Description: Finds the distance between a point and a 3d hull face

"Point3D.h", "3dHull.h" a9d101, 6 lines

```
typedef Point3D<double> P3;//f1f
double pointToFace(P3 p, F f) {
  auto dir = f.q.unit() * -1;
  auto vec = p - pts[f.a];
  return vec.dot(dir);
}//cbb
```

To project points onto a plane and map plane to xy-plane, project point onto the plane, use normalized $(B-A)$ and $(C-A)$ as unit vectors, and find projected point as linear combination of basis vectors on xy-plane. $(B-A)$ maps to $(1,0)$, $(C-A)$ maps to $(\cos\theta, \sin\theta)$, where θ is the angle between $B-A$ and $C-A$.

Strings (9)

KMP.h

Description: pi[x] computes the length of the longest prefix of s that ends at x, other than s[0...x] itself (abacaba -> 0010123). Can be used to find all occurrences of a string.
Time: $\mathcal{O}(n)$

d4f9aa, 16 lines

```
vi pi(const string& s) {//f6d
  vi p(sz(s));
  rep(i,1,sz(s)) {
    int g = p[i-1];
    while (g && s[i] != s[g]) g = p[g-1];
    p[i] = g + (s[i] == s[g]);//0ff
  }
  return p;
}

vi match(const string& s, const string& pat) {//9e6
  vi p = pi(pat + '\0' + s), res;
  rep(i,sz(p)-sz(s),sz(p))
    if (p[i] == sz(pat)) res.pb(i - 2 * sz(pat));
  return res;
}//cbb
```

Zfunc.h

Description: z[i] computes the length of the longest common prefix of s[i:] and s, except z[0] = 0. (abacaba -> 0010301)
Time: $\mathcal{O}(n)$

ee09e2, 12 lines

```
vi Z(const string& S) {//fc3
  vi z(sz(S));
  int l = -1, r = -1;
  rep(i,1,sz(S)) {
    z[i] = i >= r ? 0 : min(r - i, z[i - l]);
    while (i + z[i] < sz(S) && S[i + z[i]] == S[z[i]])//8ec
      z[i]++;
    if (i + z[i] > r)
      l = i, r = i + z[i];
  }
  return z;//939
}
```

Manacher.h

Description: For each position in a string, computes p[0][i] = half length of longest even palindrome around pos i, p[1][i] = longest odd (half rounded down).
Time: $\mathcal{O}(N)$

e7ad79, 13 lines

```
array<vi, 2> manacher(const string& s) {//510
  int n = sz(s);
  array<vi,2> p = {vi(n+1), vi(n)};
  rep(z,0,2) for (int i=0,l=0,r=0; i < n; i++) {
    int t = r-i+!z;
    if (i<r) p[z][i] = min(t, p[z][l+t]);//f50
    int L = i-p[z][i], R = i+p[z][i]-!z;
    while (L>=1 && R+1<n && s[L-1] == s[R+1])
      p[z][i]++, L--, R++;
    if (R>r) l=L, r=R;
  }//291
  return p;
}
```

MinRotation.h

Description: Finds the lexicographically smallest rotation of a string.
Usage: rotate(v.begin(), v.begin()+minRotation(v), v.end());
Time: $\mathcal{O}(N)$

d07a42, 8 lines

```
int minRotation(string s) {//20f
  int a=0, N=sz(s); s += s;
  rep(b,0,N) rep(k,0,N) {
    if (a+k == b || s[a+k] < s[b+k]) {b += max(0, k-1);
        break;}
    if (s[a+k] > s[b+k]) { a = b; break; }
  }//3a8
  return a;
}
```

SuffixArray.h

Description: Builds suffix array for a string. `sa[i]` is the starting index of the suffix which is i'th in the sorted suffix array. The returned vector is of size $n+1$, and `sa[0] = n`. The `lcp` array contains longest common prefixes for neighbouring strings in the suffix array: `lcp[i] = lcp(sa[i], sa[i-1])`, `lcp[0] = 0`. The input string must not contain any zero bytes.
Time: $\mathcal{O}(n \log n)$

bc716b, 22 lines

```
struct SuffixArray {//7a7
  vi sa, lcp;
  SuffixArray(string& s, int lim=256) { // or basic_string<
      int>
    int n = sz(s) + 1, k = 0, a, b;
    vi x(all(s)), y(n), ws(max(n, lim));
    x.push_back(0), sa = lcp = y, iota(all(sa), 0);//7c9
    for (int j = 0, p = 0; p < n; j = max(1, j * 2), lim =
        p) {
      p = j, iota(all(y), n - j);
      rep(i,0,n) if (sa[i] >= j) y[p++] = sa[i] - j;
      fill(all(ws), 0);
      rep(i,0,n) ws[x[i]]++;//f08
      rep(i,1,lim) ws[i] += ws[i - 1];
      for (int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
      swap(x, y), p = 1, x[sa[0]] = 0;
      rep(i,1,n) a = sa[i - 1], b = sa[i], x[b] =
        (y[a] == y[b] && y[a + j] == y[b + j]) ? p - 1 : p
            ++;//726
    }
    for (int i = 0, j; i < n - 1; lcp[x[i++]] = k)
```

```
    for (k && k--, j = sa[x[i] - 1];
        s[i + k] == s[j + k]; k++);
  }//e03
};
```

Hashing.h

Description: Self-explanatory methods for string hashing. 3966ff, 44 lines

```
// Arithmetic mod 2^64-1. 2x slower than mod 2^64 and more
    //d41
// code, but works on evil test data (e.g. Thue-Morse,
    where
// ABBA... and BAAB... of length 2^10 hash the same mod 2^
    64).
// "typedef ull H;" instead if you think test data is
    random,
// or work mod 10^9+7 if the Birthday paradox is not a
    problem.
typedef uint64_t ull;//98c
struct H {
  ull x; H(ull x=0) : x(x) {}
  H operator+(H o) { return x + o.x + (x + o.x < x); }
  H operator-(H o) { return *this + ~o.x; }
  H operator*(H o) { auto m = (__uint128_t)x * o.x;//884
    return H((ull)m) + (ull)(m >> 64); }
  ull get() const { return x + !~x; }
  bool operator==(H o) const { return get() == o.get(); }
  bool operator<(H o) const { return get() < o.get(); }
};//7dd
static const H C = (ll)1e11+3; // (order ~ 3e9; random also
     ok)

struct HashInterval {
  vector<H> ha, pw;
  HashInterval(string& str) : ha(sz(str)+1), pw(ha) {//c1e
    pw[0] = 1;
    rep(i,0,sz(str))
      ha[i+1] = ha[i] * C + str[i],
      pw[i+1] = pw[i] * C;
  }//b8f
  H hashInterval(int a, int b) { // hash [a, b)
    return ha[b] - ha[a] * pw[b - a];
  }
};
//4b7
vector<H> getHashes(string& str, int length) {
  if (sz(str) < length) return {};
  H h = 0, pw = 1;
  rep(i,0,length)
    h = h * C + str[i], pw = pw * C;//6b3
  vector<H> ret = {h};
  rep(i,length,sz(str)) {
    ret.pb(h = h * C + str[i] - pw * str[i-length]);
  }
  return ret;//413
}

H hashString(string& s){H h{}; for(char c:s) h=h*C+c;return
     h;}
```

AhoCorasick.h

Description: Constructs Aho-Corasick automaton for given list of words

Time: Construction is $\mathcal{O}(N)$, move() is amortized $\mathcal{O}(N)$ 2d48ca, 34 lines

```
struct AhoCorasick {//be4
  struct Node {
    map<char, int> nxt;
    int lnk = 0, cnt = 0;
  };
  vector<Node> t;//b46
  AhoCorasick(vector<string> words) {
    t.pb({});
    for (string s : words) {
      int cur = 0;
      for (char c : s) {//638
        if (!t[cur].nxt[c]) {
          t[cur].nxt[c] = sz(t);
          t.pb({});
        }
        cur = t[cur].nxt[c];//05c
      }
      t[cur].cnt++;
    }
    queue<int> q; q.push(0);
    while (!q.empty()) {//b39
      int u = q.front(); q.pop();
      assert(0 <= u && u < sz(t));
      for (auto &p : t[u].nxt) {
        if (u) t[p.second].lnk = move(t[u].lnk, p.first);
        q.push(p.second);//e67
      }
    }
  }
  int move(int u, int c) {
    if (t[u].nxt[c]) return t[u].nxt[c];//1f8
    return u == 0 ? 0 : t[u].nxt[c] = move(t[u].lnk, c);
  }
};
```

SuffixAutomaton.h

Description: Constructs a suffix automaton on string s cnt() constructs an array of equivalence class sizes first() constructs an array of first occurences for each node

Time: All functions are $\mathcal{O}(N)$. 24974f, 55 lines

```
struct SuffixAutomaton {//3d3
  struct Node {
    int len = 0, lnk = 0;
    int nxt[26];
  };
  string s;//f20
  vector<Node> t; int last = 0;
  SuffixAutomaton(string s = "") {
    t.pb({0, -1, {}});
    for (char c : s) add(c);
  }//a4a
  void add(char c) { s += c; c -= 'a';
    int u = last; int v = last = sz(t);
    t.pb({t[u].len + 1, 0, {}});
    while (u >= 0 && !t[u].nxt[c])
      t[u].nxt[c] = v, u = t[u].lnk;//b21
    if (u == -1) return;
    int q = t[u].nxt[c];
    if (t[u].len + 1 == t[q].len)
      { t[v].lnk = q; return; }
    int cpy = sz(t); t.pb(t[q]);//aa4
    t[cpy].len = t[u].len + 1;
    while (u >= 0 && t[u].nxt[c] == q)
      t[u].nxt[c] = cpy, u = t[u].lnk;
    t[v].lnk = t[q].lnk = cpy;
  }//2f2
  vector<int> cnt() {
    vector<int> res(sz(t), 0);
    int cur = 0;
    for (char c : s)
      res[cur = t[cur].nxt[c - 'a']]++;//82c
    vector<pair<int, int>> srt;
    for (int i = 1; i < sz(t); i++)
      srt.pb({-t[i].len, i});
    sort(all(srt));
    for (auto &p : srt)//cc8
      res[t[p.second].lnk] += res[p.second];
    return res;
  }
  vector<int> first() {
    vector<int> res(sz(t), sz(s));//c3b
    int cur = 0;
    for (int i = 0; i < sz(s); i++) {
      cur = t[cur].nxt[s[i] - 'a'];
      res[cur] = min(res[cur], i);
    }//268
    vector<pair<int, int>> srt;
    for (int i = 1; i < sz(t); i++)
      srt.pb({-t[i].len, i});
    sort(all(srt));
    for (auto &p : srt)//23d
      res[t[p.second].lnk] = min(res[t[p.second].lnk], res[
          p.second]);
    return res;
  }
};
```

9.0.1 Lyndon Decomposition

Duval.h

Description: Returns a list of starting indices of the strings in the Lyndon factorization of s

Time: $\mathcal{O}(n)$ 125a69, 14 lines

```
vi duval(string s) {//04b
  int i = 0;
  vi res;
  while (i < sz(s)) {
    int j = i, k = i + 1;
    while (k < sz(s) && s[k] >= s[j]) {//7e1
      if (s[k] > s[j]) j = i;
      else j++;
      k++;
    }
    while (i <= j) res.pb(i), i += k - j;//552
  }
  return res;
}
```

s is a Lyndon word $\iff$ s is strictly smaller than all of its proper suffixes $\iff$ s is strictly smaller than all of its nontrivial rotations

Lyndon factorization is the unique set of $s = w_1 + ... + w_k$ such that w is non-increasing and each w_i is a Lyndon word

w_k is the smallest suffix of s (different from smallest rotation)

9.0.2 Run Enumeration

A run is a tuple (l, r, p) where p is the (partial) period of $s[l, r]$ and $[l, r]$ is maximal

A string of length n has at most n runs

Sum of $(r - l)/p$ is at most $3n$

Number of two-repeats is at most $n \log n$

RunEnumeration.h

Description: Returns tuples (p, l, r), representing the run [l, r) with period p

Time: $\mathcal{O}(n \log n)$ 2f002c, 42 lines

```
struct LCP {//897
    SuffixArray sa;
    vector<int> pos;
    RMQ<int> rmq;
    LCP(string s) : sa(s), rmq(sa.lcp) {
        pos = vector<int>(sz(s) + 1);//cd8
        rep(i, 0, sz(s) + 1) pos[sa.sa[i]] = i;
    }
    int lcp(int i, int j) {
        if (i == j) return sz(pos) - i;
        int l = pos[i], r = pos[j];//0b3
        if (l > r) swap(l, r);
        return rmq.query(l+1, r+1);
    }
};
//658
vector<vector<int>> run_enumerate(string s) {
    string sI = s;
    for (char &c : sI) c = 127 - c;
    set<vector<int>> runs;
    auto part = [&](string s) {//bb0
        string sR = s; reverse(all(sR));
        LCP lcp(s), lcpR(sR);
        vector<int> suf = lcp.sa.sa;
        vector<int> prefix(sz(s));
        set<int> st; st.insert(sz(s));//9b6
        rep(i, 1, sz(s)+1) {
          prefix[suf[i]] = *st.lower_bound(suf[i]) - suf[i
              ];
          st.insert(suf[i]);
        }
        rep(i, 0, sz(s)) {//852
          int j = i + prefix[i];
            int k = j + (j == sz(s) ? 0 : lcp.lcp(i, j));
            int l = i - (i == 0 ? 0 : lcpR.lcp(sz(s) - j,
                sz(s) - i));
            if (k - l >= 2 * (j - i)) runs.insert({j - i, l
              , k});
        }//bf7
    };
    part(s); part(sI);
    vector<vector<int>> res;
    for (auto &v : runs) res.pb(v);
    return res;//510
}
```

Various (10)

10.1 Intervals

IntervalContainer.h

Description: Add and remove intervals from a set of disjoint intervals. Will merge the added interval with any overlapping intervals in the set when adding. Intervals are [inclusive, exclusive).

Time: $\mathcal{O}(\log N)$

edce47, 23 lines

```
set<pii>::iterator addInterval(set<pii>& is, int L, int R)
    {//ba1
  if (L == R) return is.end();
  auto it = is.lower_bound({L, R}), before = it;
  while (it != is.end() && it->first <= R) {
    R = max(R, it->second);
    before = it = is.erase(it);//ea6
  }
  if (it != is.begin() && (--it)->second >= L) {
    L = min(L, it->first);
    R = max(R, it->second);
    is.erase(it);//05d
  }
  return is.insert(before, {L,R});
}

void removeInterval(set<pii>& is, int L, int R) {//858
  if (L == R) return;
  auto it = addInterval(is, L, R);
  auto r2 = it->second;
  if (it->first == L) is.erase(it);
  else (int&)it->second = L;//61f
  if (R != r2) is.emplace(R, r2);
}
```

ConstantIntervals.h

Description: Split a monotone function on [from, to) into a minimal set of half-open intervals on which it has the same value. Runs a callback g for each such interval.

Usage: `constantIntervals(0, sz(v), [&](int x){return v[x];}, [&](int lo, int hi, T val){...});`

Time: $\mathcal{O}\left(k \log \frac{n}{k}\right)$

753a4c, 19 lines

```
template<class F, class G, class T>//570
void rec(int from, int to, F& f, G& g, int& i, T& p, T q) {
  if (p == q) return;
  if (from == to) {
    g(i, to, p);
    i = to; p = q;//05f
  } else {
    int mid = (from + to) >> 1;
    rec(from, mid, f, g, i, p, f(mid));
    rec(mid+1, to, f, g, i, p, q);
  }//729
}
template<class F, class G>
void constantIntervals(int from, int to, F f, G g) {
  if (to <= from) return;
  int i = from; auto p = f(i), q = f(to-1);//a6c
  rec(from, to-1, f, g, i, p, q);
  g(i, to, q);
}
```

10.2 Misc. algorithms

TernarySearch.h

Description: Find the smallest i in $[a, b]$ that maximizes $f(i)$, assuming that $f(a) < \ldots < f(i) \geq \cdots \geq f(b)$. To reverse which of the sides allows non-strict inequalities, change the < marked with (A) to <=, and reverse the loop at (B). To minimize f, change it to >, also at (B).

Usage: `int ind = ternSearch(0,n-1,[&](int i){return a[i];});`

Time: $\mathcal{O}(\log(b-a))$

9155b4, 11 lines

```
template<class F>//7d4
int ternSearch(int a, int b, F f) {
  assert(a <= b);
  while (b - a >= 5) {
    int mid = (a + b) / 2;
    if (f(mid) < f(mid+1)) a = mid; // (A)//ec4
    else b = mid+1;
  }
  rep(i,a+1,b+1) if (f(a) < f(i)) a = i; // (B)
  return a;
}//cbb
```

10.3 Dynamic programming

KnuthDP.h

Description: When doing DP on intervals: $a[i][j] = \min_{i<k<j}(a[i][k] + a[k][j]) + f(i, j)$, where the (minimal) optimal k increases with both i and j, one can solve intervals in increasing order of length, and search $k = p[i][j]$ for $a[i][j]$ only between $p[i][j-1]$ and $p[i+1][j]$. This is known as Knuth DP. Sufficient criteria for this are if $f(b, c) \leq f(a, d)$ and $f(a, c) + f(b, d) \leq f(a, d) + f(b, c)$ for all $a \leq b \leq c \leq d$. Consider also: LineContainer (ch. Data structures), monotone queues, ternary search.

Time: $\mathcal{O}\left(N^2\right)$

d41d8c, 1 lines

```
//d41
```

DivideAndConquerDP.h

Description: Given $a[i] = \min_{lo(i) \leq k < hi(i)}(f(i, k))$ where the (minimal) optimal k increases with i, computes $a[i]$ for $i = L..R-1$.

Time: $\mathcal{O}((N + (hi - lo)) \log N)$

d38d2b, 18 lines

```
struct DP { // Modify at will://ff9
  int lo(int ind) { return 0; }
  int hi(int ind) { return ind; }
  ll f(int ind, int k) { return dp[ind][k]; }
  void store(int ind, int k, ll v) { res[ind] = pii(k, v);
      }
//ec8
  void rec(int L, int R, int LO, int HI) {
    if (L >= R) return;
    int mid = (L + R) >> 1;
    pair<ll, int> best(LLONG_MAX, LO);
    rep(k, max(LO,lo(mid)), min(HI,hi(mid)))//680
      best = min(best, make_pair(f(mid, k), k));
    store(mid, best.second, best.first);
    rec(L, mid, LO, best.second+1);
    rec(mid+1, R, best.second, HI);
  }//a30
  void solve(int L, int R) { rec(L, R, INT_MIN, INT_MAX); }
};
```

10.4 Optimization tricks

`__builtin_ia32_ldmxcsr(40896);` disables denormals (which make floats 20x slower near their minimum value).

10.4.1 Bit hacks

- `x & -x` is the least bit in `x`.
- `for (int x = m; x; ) { --x &= m; ... }` loops over all subset masks of `m` (except `m` itself).
- `c = x&-x, r = x+c; (((r^x) >> 2)/c) | r` is the next number after `x` with the same number of bits set.
- `rep(b,0,K) rep(i,0,(1 << K))`
  `if (i & 1 << b) D[i] += D[i^(1 << b)];`
  computes all sums of subsets.

10.4.2 Pragmas

- `#pragma GCC optimize ("Ofast")` will make GCC auto-vectorize loops and optimizes floating points better.
- `#pragma GCC target ("avx2")` can double performance of vectorized code, but causes crashes on old machines.

- **#pragma** GCC optimize ("trapv") kills the program on integer overflows (but is really slow).

Extra Stuff (11)

Pruefer.h

Description: Helps construct random tree Choose random n-2 length array, values [0, n-1]
Time: $\mathcal{O}(n)$

295d68, 23 lines

```
vector<pii> pruefer_decode(const vi &code) {//865
  int n = sz(code) + 2;
  vi degree(n, 1);
  for (int i : code)
    degree[i]++;
//c77
  set<int> leaves;
  rep(i, 0, n)
    if (degree[i] == 1)
      leaves.insert(i);
//f6a
  vector<pii> edges;
  for (int v : code) {
    int leaf = *leaves.begin();
    leaves.erase(leaves.begin());
//1b3
    edges.emplace_back(leaf, v);
    if (--degree[v] == 1)
      leaves.insert(v);
  }
  edges.emplace_back(*leaves.begin(), n-1);//062
  return edges;
}
```

Dylan's Templates (12)

WaveletTree.h

Description: Range K-th Smallest
Time: $\mathcal{O}(n\log^2(n))$ I think

ccdc27, 39 lines

```
struct WaveletTree {//b57
  struct Node {
    vector<int> a, b;
    int l = -1, r = -1;
    bool leaf = 0;
  };//188
  int bits = 30;
  vector<int> arr;
  vector<Node> tree = {{}};
  void construct(int u, int lo, int hi) {
    if (lo == hi) { tree[u].leaf = 1; return; }//5bb
    int l = sz(tree); tree.pb({});
    int r = sz(tree); tree.pb({});
    tree[u].l = l; tree[u].r = r;
    int mid = (lo + hi) / 2;
    int pre = 0;//16b
    for (int i = 0; i < sz(tree[u].a); i++) {
      if (arr[tree[u].a[i]] <= mid) {
        tree[l].a.pb(tree[u].a[i]);
        pre++;
      } else tree[r].a.pb(tree[u].a[i]);//b04
      tree[u].b.pb(pre);
    }
    if (!tree[l].a.empty()) construct(l, lo, mid);
    if (!tree[r].a.empty()) construct(r, mid + 1, hi);
  }//102
  WaveletTree(vector<int> a) : arr(a) {
    for (int i = 0; i < sz(arr); i++) tree[0].pb(i);
    construct(0, 0, (1 << bits) - 1);
  }
  int query(int l, int r, int k, int u = 0) {//07f
    if (u == -1) return 0;
    int lp = lb(tree[u].a, l), rp = lb(tree[u].a, r + 1) -
        1;
    if (tree[u].leaf) return tree[u].a[lp + k - 1];
    int n = tree[u].b[rp] - (lp == 0 ? 0 : tree[u].b[lp -
        1]);
    if (n >= k) return query(l, r, k, tree[u].l);//3a4
    return query(l, r, k - n, tree[u].r);
  }
};
```

SMAWK.h

Description: Outputs row minima of nxm totally monotone matrix f(r, c1, c2) should output 1 if a[r][c1] <= a[r][c2]
Time: $\mathcal{O}(n)$

3ff481, 77 lines

```
template<typename F>//95b
vector<int> smawk(F f, vector<int> rows, vector<int> cols)
    {
  vector<int> row_mins;
  if (sz(rows) <= 2 && sz(cols) <= 2) {
    for (int r : rows) {
      int mn = -1;//7ea
      for (int c : cols) {
        if (mn == -1 || f(r, c, mn)) mn = c;
      }
      row_mins.pb(mn);
    }//1e0
  } else if (sz(rows) < sz(cols)) {
    // Reduce
    vector<int> st;
    for (int j : cols) {
      while (!st.empty()) {//f89
        if (f(rows[sz(st) - 1], j, st.back())) st.pop_back
            ();
        else if (sz(st) == sz(rows)) break;
        else { st.push_back(j); break; }
      }
      if (st.empty()) st.pb(j);//97e
    }
    row_mins = smawk(f, rows, st);
  } else {
    // Interpolate
    vector<int> half_rows;//420
    for (int i = 1; i < sz(rows); i += 2) half_rows.pb(rows
        [i]);
    vector<int> half_row_mins = smawk(f, half_rows, cols);
    vector<int> mn_index;
    for (int i = 0, j = 0; j < sz(cols); j++) {
      while (i < sz(half_row_mins) && cols[j] ==
          half_row_mins[i]) {//943
        mn_index.pb(j);
        i++;
      }
    }
    for (int i = 0; i < sz(rows); i++) {//605
      if (i % 2) row_mins.pb(half_row_mins[i / 2]);
      else {
        int l = i == 0 ? 0 : mn_index[i / 2 - 1];
        int r = i == sz(rows) - 1 ? sz(cols) - 1 : mn_index
            [i / 2];
        int mn = cols[l];//57d
        for (int j = l + 1; j <= r; j++) {
          if (f(rows[i], cols[j], mn)) mn = cols[j];
        }
        row_mins.pb(mn);
      }//049
    }
  }
  return row_mins;
}
//fc4
// Min-plus Convolution (b must be convex)
vector<ll> min_plus(vector<ll> a, vector<ll> b) {
  int n = sz(a), m = sz(b);
  auto get = [&](int r, int c) { return a[c] + b[r - c]; };
  auto comp = [&](int r, int c1, int c2) {//792
    if (r - max(c1, c2) < 0) return c1 < c2;
    if (r - min(c1, c2) >= m) return c2 < c1;
    return get(r, c1) <= get(r, c2);
  };
  vector<int> rows, cols;//e94
  for (int i = 0; i < n + m - 1; i++) rows.pb(i);
  for (int j = 0; j < n; j++) cols.pb(j);
  vector<int> row_mins = smawk(comp, rows, cols);
  vector<ll> res;
  for (int i = 0; i < n + m - 1; i++) res.pb(get(i,
      row_mins[i]));//332
  return res;
}

// Min-plus Convolution (b must be concave)
vector<ll> max_plus(vector<ll> a, vector<ll> b) {//325
  for (ll &i : a) i = -i;
  for (ll &i : b) i = -i;
  vector<ll> res = min_plus(a, b);
  for (ll &i : res) i = -i;
  return res;//510
}
```

XORConvolution.h

Description: Bitwise XOR Convolution
Time: $\mathcal{O}(n\log(n))$

c3f3d6, 19 lines

```
void xorfft(vector<ll> &v, bool inv = 0) {//475
  for (int k = 0; k < bits; k++) {
    for (int i = 0; i < 1 << bits; i++, i += i & 1 << k) {
      ll a = v[i], b = v[i ^ 1 << k];
      v[i] = a + b, v[i ^ 1 << k] = a - b;
    }//aa7
  }
  for (int i = 0; i < 1 << bits; i++) v[i] = (v[i] % mod +
      mod) % mod;
  if (inv) {
    ll n = 1; for (int i = 0; i < bits; i++) n = n * 2 %
        mod;
    n = mInv(n);//e2a
    for (int i = 0; i < 1 << bits; i++) v[i] = v[i] * n %
        mod;
  }
}
vector<ll> mult(vector<ll> a, vector<ll> b) {
  xorfft(a); xorfft(b); vector<ll> c(1 << bits);//703
  for (int i = 0; i < 1 << bits; i++) c[i] = a[i] * b[i] %
      mod;
  xorfft(c, 1); return c;
}
```