Paul's Online Notes

Section 10.12 : Strategy For Series

Now that we've got all of our tests out of the way it's time to think about organizing all of them into a general set of guidelines to help us determine the convergence of a series.

Note that these are a general set of guidelines and because some series can have more than one test applied to them we will get a different result depending on the path that we take through this set of guidelines. In fact, because more than one test may apply, you should always go completely through the guidelines and identify all possible tests that can be used on a given series. Once this has been done you can identify the test that you feel will be the easiest for you to use.

With that said here is the set of guidelines for determining the convergence of a series.

1. With a quick glance does it look like the series terms don't converge to zero in the limit, *i.e.* does $\lim\limits_{n \to \infty} a_n \neq 0$? If so, use the Divergence Test. Note that you should only do the Divergence Test if a quick glance suggests that the series terms may not converge to zero in the limit.

2. Is the series a p-series ($\sum \frac{1}{n^p}$) or a geometric series ($\sum\limits_{n=0}^{\infty} ar^n$ or $\sum\limits_{n=1}^{\infty} ar^{n-1}$)? If so use the fact that p-series will only converge if $p > 1$ and a geometric series will only converge if $|r| < 1$. Remember as well that often

some algebraic manipulation is required to get a geometric series into the correct form.

3. Is the series similar to a p-series or a geometric series? If so, try the Comparison Test.

4. Is the series a rational expression involving only polynomials or polynomials under radicals (*i.e.* a fraction involving only polynomials or polynomials under radicals)? If so, try the Comparison Test and/or the Limit Comparison Test. Remember however, that in order to use the Comparison Test and the Limit Comparison Test the series terms all need to be positive.

5. Does the series contain factorials or constants raised to powers involving n? If so, then the Ratio Test may work. Note that if the series term contains a factorial then the only test that we've got that will work is the Ratio Test.

6. Can the series terms be written in the form $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$? If so, then the Alternating Series Test may work.

7. Can the series terms be written in the form $a_n = (b_n)^n$? If so, then the Root Test may work.

8. If $a_n = f(n)$ for some positive, decreasing function and $\int_a^\infty f(x)\, dx$ is easy to evaluate then the Integral Test may work.

Again, remember that these are only a set of guidelines and not a set of hard and fast rules to use when trying to determine the best test to use on a series. If more than one test can be used try to use the test that will be the easiest for you to use and remember that what is easy for someone else may not be easy for you!

Also, just so we can put all the tests into one place here is a quick listing of all the tests that we've got.

Divergence Test

If $\lim\limits_{n \to \infty} a_n \neq 0$ then $\sum a_n$ will diverge

Integral Test

Suppose that $f(x)$ is a positive, decreasing function on the interval $[k, \infty)$ and that $f(n) = a_n$ then,

1. If $\int_{k}^{\infty} f(x)\ dx$ is convergent then so is $\sum\limits_{n=k}^{\infty} a_n$.

2. If $\int_{k}^{\infty} f(x)\ dx$ is divergent then so is $\sum\limits_{n=k}^{\infty} a_n$.

Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n. Then,

1. If $\sum b_n$ is convergent then so is $\sum a_n$.
2. If $\sum a_n$ is divergent then so is $\sum b_n$.

Limit Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n. Define,

$$c = \lim_{n \to \infty} \frac{a_n}{b_n}$$

If c is positive (*i.e.* $c > 0$) and is finite (*i.e.* $c < \infty$) then either both series converge or both series diverge.

Alternating Series Test

Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all n. Then if,

1. $\lim\limits_{n \to \infty} b_n = 0$ and,

2. $\{b_n\}$ is eventually a decreasing sequence

the series $\sum a_n$ is convergent

Ratio Test

Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).

2. if $L > 1$ the series is divergent.

3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

Root Test

Suppose that we have the series $\sum a_n$. Define,

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} |a_n|^{\frac{1}{n}}$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).

2. if $L > 1$ the series is divergent.

3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.