Tenta 2018-08-23

(a)

$$f'(x) = e^{\sqrt{x+1}} \ln(x+1) \cdot \frac{1}{2\sqrt{x+1}} + e^{\sqrt{x+1}} \cdot \frac{1}{x+1}$$

$$f'(0) = e \ln(1) \cdot \frac{1}{2} + e \cdot \frac{1}{1}$$

$$\boxed{f'(0) = e}$$

(b) $f(0) = 0$

$$y - 0 = e(x-0)$$

$$\boxed{y = ex}$$

2.

$$\lim_{x \to 0} \frac{x - \sin x}{\tan x - x} = [\text{l'Hospital}]$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{1 + \tan^2 x - 1} = \lim_{x \to 0} \frac{1 - \cos x}{\tan^2 x}$$

$$= [\text{l'Hospital}] = \lim_{x \to 0} \frac{\sin x}{2\tan x (1 + \tan^2 x)}$$

$$= [\text{l'Hospital}] = \cancel{\lim_{x \to 0}}$$

$$= \lim_{x \to 0} \frac{\cos x}{2(1+\tan^2 x)^2 + 2\tan x \left(2\tan x (1 + \tan^2 x)\right)}$$

$$= \frac{1}{2}$$

3.

$$2y^3 - 3x^2 = 2$$

$$2 \cdot 3y^2 \cdot y' - 6x = 0$$

$$y' = \frac{6x}{6y^2} = \frac{x}{y^2}$$

$$y'(0) = 0$$

$$y'' = \left(\frac{x}{y^2}\right)' = \frac{1 \cdot y^2 - x \cdot 2y \cdot y'}{y^4}$$

$$y''(0) = \frac{1 - 0}{1^2} = 1$$

$$P_2(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2$$

$$= 1 + \frac{1}{2}x^2$$

Svar $\boxed{P_2(x) = 1 + \frac{x^2}{2}}$

4. (a)

$$\int_0^2 x^2\sqrt{4-x^2}\,dx = \left[\begin{array}{l} x = 2\sin t \\ dx = 2\cos t\,dt \\ x=2 \Rightarrow t=\frac{\pi}{2} \\ x=0 \Rightarrow t=0 \\ \sqrt{4-x^2} = 2\cos t \end{array}\right]$$

$$= \int_0^{\pi/2} 4\sin^2 t\, 2\cos t \cdot 2\cos t\,dt$$

$$= 16\int_0^{\pi/2} \sin^2 t\cos^2 t\,dt$$

$$= 16\int_0^{\pi/2} \left(\frac{\sin 2t}{2}\right)^2 dt$$

$$= 4\int_0^{\pi/2} \sin^2 2t\,dt$$

$$\sin^2 2t = \frac{1-\cos 4t}{2}$$

$$4\int_0^{\pi/2} \sin^2 2t\,dt =$$

$$= 4\int_0^{\pi/2} \frac{1-\cos 4t}{2}\,dt$$

$$= 2\int_0^{\pi/2} (1-\cos 4t)\,dt$$

$$= 2\left[t - \frac{\sin 4t}{4}\right]_0^{\pi/2}$$

$$= 2\cdot\frac{\pi}{2} = \pi.$$

(b)

$$x^3 - x^2 + 4x + 6$$

$$= (x+1)(x^2 - 2x + 6)$$

$$\frac{3x+6}{(x+1)(x^2-2x+6)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+6}$$

Bestäm A, B, C ger

$$A = -1, \quad B = 1, \quad C = 0$$

$$\int_1^{1+\sqrt{5}} \frac{3x-6}{x^3 - x^2 + 4x + 6}\,dx$$

$$= \int_1^{1+\sqrt{5}} \frac{x}{x^2-2x+6}\,dx - \int_1^{1+\sqrt{5}} \frac{1}{x+1}\,dx$$

$$= I_1 - I_2$$

$$I_1 = \int_1^{1+\sqrt{5}} \frac{x}{(x-1)^2+5}\,dx$$

$$= \int_0^{\sqrt{5}} \frac{t+1}{t^2+5}\,dx$$

$$= \left[\frac{1}{2} \ln(t^2+5) \right]_0^{\sqrt{5}}$$

$$+ \int_0^{\sqrt{5}} \frac{1}{t^2+5}\,dt$$

$$= \frac{1}{2}\ln(10) - \frac{1}{2}\ln 5$$

$$+ \int_0^{\sqrt{5}} \frac{1}{t^2+5}\,dt$$

$$\int_0^{\sqrt{5}} \frac{1}{t^2+5}\,dt = \left[\frac{1}{5}\arctan\frac{t}{\sqrt{5}} \cdot \sqrt{5}\right]_0^{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{\pi}{4}$$

$$\therefore\ I_1 = \frac{1}{2}\ln 10 - \frac{1}{2}\ln 5 + \frac{1}{\sqrt{5}} \cdot \frac{\pi}{4}$$

$$= \frac{1}{2}\ln 2 + \frac{\pi}{4\sqrt{5}}$$

$$I_2 = \left[\ln(x+1)\right]_1^{1+\sqrt{5}}$$

$$= \ln(2+\sqrt{5}) - \ln 2$$

$$I_1 - I_2 = \frac{3}{2}\ln 2 + \frac{\pi}{4\sqrt{5}}$$

$$- \ln(2+\sqrt{5})$$

5(a)

$$y' + \frac{x}{x^2-1} y = \frac{2x}{x^2-1}$$

I.F. $$e^{\int \frac{x}{x^2-1} dx} = e^{\frac{1}{2}\ln(x^2-1)}$$
$$= \sqrt{x^2-1}$$

Mult. med. IF. ger

$$\sqrt{x^2-1}\, y' + \frac{x}{\sqrt{x^2-1}} y = \frac{2x}{\sqrt{x^2-1}}$$

$$\Leftrightarrow \left(\sqrt{x^2-1}\, y\right)' = \frac{2x}{\sqrt{x^2-1}}$$

$$\Rightarrow \sqrt{x^2-1}\, y = \int \frac{2x}{\sqrt{x^2-1}} dx$$

$$\int \frac{2x}{\sqrt{x^2-1}} dx = \left[\begin{array}{l} t = x^2-1 \\ dt = 2x\, dx \end{array}\right] \Rightarrow$$

$$= \int \frac{1}{\sqrt{t}} dt$$

$$= 2\sqrt{t} + C$$

$$= 2\sqrt{x^2-1} + C$$

$$\Rightarrow$$

$$y = 2 + \frac{C}{\sqrt{x^2-1}}$$

$\therefore$ Allmän lösning

$$y = 2 + \frac{C}{\sqrt{x^2-1}}$$

$y(?) = 2 \Rightarrow \quad C = 0$

Svar $y = 2$ är lösningen

5(b)

$$x^2 \frac{dy}{dx} = y^2 + 2y$$

$$\Leftrightarrow \quad \frac{dy}{y^2+2y} = \frac{1}{x^2}\,dx$$

$$\Rightarrow \quad \int \frac{1}{y^2+2y}\,dy = \int \frac{1}{x^2}\,dx$$

$$\frac{1}{y^2+2y} = \frac{A}{y} + \frac{B}{y+2}$$

Sai $\quad \dfrac{A(y+2)+By}{y(y+2)}$

$$= \frac{(A+B)y+2A}{y(y+2)} = \frac{1}{y^2+2y}$$

$$\Rightarrow A = 1/2, \quad B = -1/2$$

$$\int \frac{1}{y^2+2y}\,dy = \int \left(\frac{1/2}{y} - \frac{1/2}{y+2}\right) dy$$

$$= \frac{1}{2}\ln|y| - \frac{1}{2}\ln|y+2|$$

$$= \frac{1}{2}\ln\left|\frac{y}{y+2}\right|$$

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

$$\Rightarrow \ln\left|\frac{y}{y+2}\right| = -\frac{2}{x} + C$$

$$\frac{y}{y+2} = Ce^{-2/x} \quad (*)$$

$$\Rightarrow \frac{y+2}{y} = Ce^{2/x}$$

$$y+2 = Ce^{2/x}y$$

$$\Leftrightarrow \quad y(1 - Ce^{2/x}) = -2$$

$$\Rightarrow \quad y = \frac{2}{Ce^{2/x} - 1}$$

Skrivfel i uppgift, ska vara $y(1) = 1$

Full poäng för (*)

(c)

$r^2 - 3r + 2 = 0$

$\Leftrightarrow (r-2)(r-1) = 0$

$\Rightarrow$

$y_h = C_1 e^{2x} + C_2 e^{x}$.

Ansats för y_p:

$y_p = Ae^{2x}$

$y_p' = 2Ae^{2x}, \; y_p'' = 4Ae^{2x}$

$y_p'' - 3y_p' + 2y_p = (4A - 6A + 2A)e^{2x}$
$= 0\, e^{2x} = 0$

går ej med $y_p = Ae^{2x}$
(varför?)

Ny ansats

$y_p = Axe^{2x}$

$y_p' = Ae^{2x} + 2Axe^{2x}$

$y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$
$= 4Ae^{2x} + 4Axe^{2x}$

$y_p'' - 3y_p' + 2y_p$
$= 4Ae^{2x} + 4Axe^{2x} - 3Ae^{2x} - 6Axe^{2x}$
$+ 2Axe^{2x}$
$= Ae^{2x} \Rightarrow A = 1$

$\boxed{y_p = xe^{2x}}$

Svar: $\boxed{C_1 e^{2x} + C_2 e^{x} + xe^{2x}}$

6. $\pi r^2 h = 10 \Leftrightarrow h = \frac{10}{\pi r^2}$

$$A(r) = 2\pi r^2 + 2\pi r h$$

$$= 2\pi r^2 + 2\pi r \cdot \frac{10}{\pi r^2}$$

$$= 2\pi r^2 + \frac{20}{r}$$

$$A'(r) = 4\pi r - \frac{20}{r^2}$$

$$A'(r) = 0 \Leftrightarrow 4\pi r = \frac{20}{r^2}$$

$$\Leftrightarrow r^3 = \frac{5}{\pi} \Rightarrow r = \sqrt[3]{\frac{5}{\pi}}$$

eller $r = \frac{5^{1/3}}{\pi^{1/3}}$

(alt. skrivsätt)

Teckenschema:

$$A'(r) = 4\pi r - \frac{20}{r^2}$$

$$= \frac{4\pi r^3 - 20}{r^2}$$

		$\sqrt[3]{\frac{5}{\pi}}$	
$A'(r)$	$- - -$	0	$+ + + +$
A	$\searrow$	$\_$	$\nearrow$

$\therefore$ $r = \sqrt[3]{\frac{5}{\pi}}$

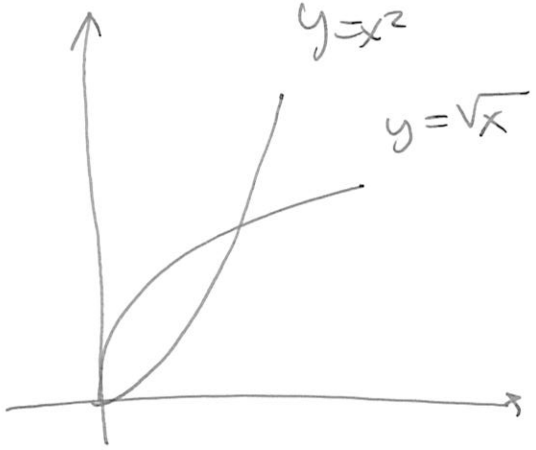
$$h = \frac{10}{\pi \left(\frac{5}{\pi}\right)^{2/3}} = \frac{10}{5^{2/3}\pi^{1/3}}$$

$$= \frac{2 \cdot 5^{1/3}}{\pi^{1/3}}$$

<u>Svar</u>: $r = \frac{5^{1/3}}{\pi^{1/3}}$, $h = \frac{2 \cdot 5^{1/3}}{\pi^{1/3}}$

7.

(a)



$$V = \pi \int_0^1 \left[(\sqrt{x})^2 - (x^2)^2\right] dx$$

$$= \pi \int_0^1 (x - x^4)\, dx$$

$$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= \pi \left(\frac{1}{2} - \frac{1}{5}\right) = \pi \cdot \left(\frac{5}{10} - \frac{2}{10}\right)$$

$$= \boxed{\frac{3\pi}{10}}$$

(b)

$$V = 2\pi \int_0^1 x\left(\sqrt{x} - x^2\right) dx$$

$$= 2\pi \int_0^1 \left(x\sqrt{x} - x^3\right) dx$$

$$= 2\pi \int_0^1 \left(x^{3/2} - x^3\right) dx$$

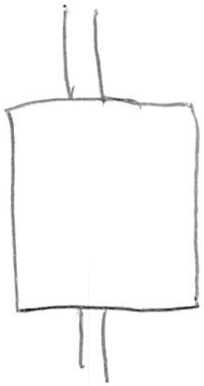
$$= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^1$$

$$= 2\pi \left[\frac{2}{5} - \frac{1}{4}\right]$$

$$= 2\pi \left(\frac{8}{20} - \frac{5}{20}\right)$$

$$= \frac{6\pi}{20} = \boxed{\frac{3\pi}{10}}$$

8.



$m(t)$ = mängd salt (mätt i g) efter t minuter

mängd in = 50 g/min

mängd ut = $\frac{m(t)}{1000} \cdot 10$ g/min

$$\begin{cases} \frac{dm}{dt} = 50 - \frac{m}{100} \\ m(0) = 20\,000 \end{cases}$$

$$m' + \frac{1}{100} m = 50$$

$$\left(m e^{\frac{1}{100}t}\right)' = 50 e^{\frac{1}{100}t}$$

$$m e^{\frac{1}{100}t} = 5000 e^{\frac{1}{100}t} + C$$

$$m(t) = 5000 + Ce^{-t/100}$$

$$\boxed{m(t) = 5000 + 15\,000 e^{-t/100}}$$