1. Transformation.

Given Point $P = (8, 4, 4)$.

1) Rotate around z by $30°$.

Rotation matrix.

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \\ z' &= z. \end{aligned}$$

$$\Rightarrow \begin{aligned} x' &= 8\cos 30° - 4\sin 30° = \frac{8\sqrt{3}}{2} - 2 = 4\sqrt{3} - 2 \\ y' &= 8\sin 30° + 4\cos 30° = \frac{8}{2} + \frac{4\sqrt{3}}{2} = 4 + 2\sqrt{3} \\ z' &= 4 \end{aligned}$$

2) $(4\sqrt{3}-2, 4+2\sqrt{3}, 4)$ Around y axis by $45°$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow \begin{aligned} x' &= x\cos\theta + z\sin\theta \\ y' &= y \\ z' &= -x\sin\theta + z\cos\theta \end{aligned}$$

$$\Rightarrow \begin{aligned} x' &= (4\sqrt{3}-2)\cos 45 + 4\sin 45° = 2\sqrt{6} - \sqrt{2} + 2\sqrt{2} = 2\sqrt{6} + \sqrt{2}. \\ y' &= 4 + 2\sqrt{3}. \\ z' &= -(4\sqrt{3}-2)\sin 45° + 4\cos 45° = \frac{2 - 4\sqrt{3}}{2}\sqrt{2} + 2\sqrt{2} = \sqrt{2} - 2\sqrt{6} + 2\sqrt{2} = \\ &= 3\sqrt{2} - 2\sqrt{6} \end{aligned}$$

$T = TP. \qquad T = (5, -8, 0)$

$$P' = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{6}+\sqrt{2} \\ 4+2\sqrt{3} \\ 3\sqrt{2}-2\sqrt{6} \\ 1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{6}+\sqrt{2}+5 \\ 4+2\sqrt{3}-8 \\ 3\sqrt{2}-2\sqrt{6} \\ 1 \end{bmatrix}.$$

For quaternions:

Rotation Matrix:

$$M = 2 \cdot \begin{bmatrix} q_0^2+q_1^2+0.5 & q_1q_2-q_0q_3 & q_0q_2+q_1q_3 \\ q_0q_3+q_1q_2 & q_0^2+q_2^2-0.5 & q_2q_3-q_0q_1 \\ q_1q_3-q_0q_2 & q_0q_1+q_2q_3 & q_0^2+q_3^2-0.5 \end{bmatrix} \Rightarrow$$

$|q_0| = \cos\frac{\theta}{2}.$

$|q_1| = |q_2| = 0.$

$|q_3| = \sin\frac{\theta}{2}.$

1) First rotation:

$$M = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow |q_0| = \frac{\sqrt{6}+\sqrt{2}}{4}, \quad |q_1| = |q_2| = 0, \quad |q_3| = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

$$q_R = \frac{\sqrt{6}+\sqrt{2}}{4} + \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)k.$$

The point is $(8, 4, 4) \Rightarrow$

$q_A = 0 + 8i + 4j + 4k. \Rightarrow$

$q_R \cdot q_A \cdot q_R^* =$

$$\left(\frac{\sqrt{6}+\sqrt{2}}{4} + \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)k\right)(8i+4j+4k)\left(\frac{\sqrt{6}+\sqrt{2}}{4} - \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)k\right).$$

$$= \left(2(\sqrt{6}+\sqrt{2})i + 2(\sqrt{6}-\sqrt{2})j + (\sqrt{6}+\sqrt{2})j - (\sqrt{6}-\sqrt{2})i + (\sqrt{6}+\sqrt{2})k + \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2\right)\left(\frac{\sqrt{6}+\sqrt{2}}{4} - \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)k\right)$$

$$= \left((\sqrt{6}+\sqrt{2})i + \left(2(\sqrt{6}-\sqrt{2}) + (\sqrt{6}+\sqrt{2})\right)j + (\sqrt{6}+\sqrt{2})k + \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2\right)\left(\frac{\sqrt{6}+\sqrt{2}}{4} - \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)k\right)$$