Womack

Morgan Fissel S352 Final

1. $E(X_i) = \theta$

$Var(x_i) = v\theta$

$\hat{\theta} = \frac{n}{n+1}\bar{x} + \frac{2}{n+1}$

$\hat{\theta}$ is biased estimator of θ

$$E[\hat{\theta}] = E\left[\frac{n}{n+1}X + \frac{2}{n+1}\right]$$

$$= \frac{\overbrace{E[X_1] + \dots E[X_n]}^{n \text{ times}}}{n+1} + \frac{2}{n+1}$$

$$= \frac{\overbrace{\theta + \dots + \theta}^{n \text{ times}}}{n+1} + \frac{2}{n+1}$$

$$= \frac{n \times \theta}{n+1} + \frac{2}{n+1}$$

$$= \frac{(n \times \theta) + 2}{n+1}$$

Expectation $\rightarrow E[\hat{\theta}] = \frac{n\theta + 2}{n+1}$

bias $= \frac{n\theta + 2}{n+1} - \theta = \frac{n\theta + 2}{n+1} - \frac{n\theta + \theta}{n+1} = \frac{2-\theta}{n+1}$ = Bias

except when $\theta = 2$

$Var(X_i) = v\theta$

$Var[\hat{\theta}] = \frac{n}{n+1}(v\theta) + \frac{2}{n+1}$

Variance $\frac{nv\theta}{n+1} + \frac{2}{n+1} = \frac{nv\theta + 2}{n+1}$

Risk $bias^2$ + variance

$$Risk[\hat{\theta}] = \left(\frac{2-\theta}{n+1}\right)^2 + \frac{nv\theta + 2}{n+1}$$

$$R[\hat{\theta}] = \left(\frac{2-\theta}{n+1}\right)\left(\frac{2-\theta}{n+1}\right) + \frac{nv\theta + 2}{n+1}$$

$$Risk[\hat{\theta}] = \frac{4 - 4\theta - \theta^2}{(n+1)^2} + \frac{nv\theta + 2}{n+1}$$

2. $f(x_i;\theta) = \frac{1}{4} \exp(-\sqrt{|x_i-\theta|})$

$\log(f(x_i;\theta)) = \log(\frac{1}{4} \exp(-\sqrt{|x_i-\theta|}))$

$= -\sqrt{|x_i-\theta|} \log(\frac{1}{4})$

$\ell(\theta, \underbrace{x_1, \ldots, x_n}_{\bar{x}}) = \sum_{i=1}^{n} (-\sqrt{|\bar{x}-\theta|} \log(\frac{1}{4})$

$= \boxed{n[-\sqrt{|\bar{x}-\theta|} \log(\frac{1}{4})]}$

3. $\ell(\theta_a, \theta_b; x_1, \ldots x_n) = n_a \log(\theta_a) + n_b \log(\theta_b) + (n-n_a-n_b) \log(1-\theta_a - \theta_b)$

$n_a = \sum_1^n I(x_i = a)$; $n_b = \sum_1^n I(x_i = b)$

MLE: $\hat{\theta}_a = \frac{n_a}{n}$; $\hat{\theta}_b = \frac{n_b}{n}$ $\quad \hat{\theta}_a = \hat{\theta}_b = \hat{\theta}_0 = \frac{n_a + n_b}{2n}$

$H_0: \theta_a = \theta_b$

$H_A: \theta_a \neq \theta_b$

$\lambda = 2\ell(\hat{\theta}_a, \hat{\theta}_b; x_1, \ldots, x_n) - 2\ell(\hat{\theta}_0; x_1, \ldots, x_n)$

H_0: $n_a \log(\hat{\theta}_a) + n_b \log(\hat{\theta}_b) + (n-n_a-n_b) \log(1-\hat{\theta}_a - \hat{\theta}_b)$

$n_a \log(\frac{n_a+n_b}{2n}) + n_b \log(\frac{n_a+n_b}{2n}) + (n-n_a-n_b) \log(1-\frac{n_a+n_b}{2n} - \frac{n_a+n_b}{2n})$

$2n \times [n_a \log(\frac{n_a+n_b}{2n}) + n_b \log(\frac{n_a+n_b}{2n})]$ $\quad \log(1) = 0$

H_A: $n_a \log(\frac{n_a}{n}) + n_b \log(\frac{n_b}{n}) + (n-n_a-n_b) \log(1-\frac{n_a-n_b}{n})$

$2n \times [n_a \log(\frac{n_a}{n}) + n_b \log(\frac{n_b}{n}) + (n-n_a-n_b) \log(1-\frac{n_a-n_b}{n})]$

$\rightarrow$ H_0 v.s. H_A H_A has an additional factor but it is quite similar to H_0.

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4. $X|Z$ X conditioned on Z

$Y|Z$ Y conditioned on Z

$P(XY|Z) = P(X|Z)P(Y|Z)$

5. $f(x|y) = (y+1)x^y$ $\quad 0<X<1$ $\quad$ Y-non-negative int

$X|Y \sim Beta(y+1, 1)$

$f(y) = \frac{1}{(y+1)!} \frac{1}{e-1}$ $\quad e = \exp(1)$

$f(x) = \frac{f(x|y)f(y)}{f(y|x)}$

$f(y|x) \propto f(x|y)f(y)$

$\propto (y+1)x^y \cdot \frac{1}{(y+1)!} \frac{1}{e-1}$

$f(y|x) \propto \frac{(y+1)x^y}{(y+1)!} \frac{(y+1)x^y}{e-1}$

$$f(x) = \frac{(y+1)x^y \cdot \frac{1}{(y+1)!} \frac{1}{e-1}}{\frac{(y+1)x^y}{(y+1)!} \frac{y+1\ x^y}{e-1}} = \frac{\frac{(y+1)x^y}{(y+1)!} \frac{(y+1)x^y}{e-1}}{\text{same}}$$

$\longrightarrow$ same

$\frac{\text{same}}{\text{same}} = 1$

6. $f(x_i|\theta) = \theta^{x_i}(1-\theta)$

$f(\theta|x_i) \propto f(x_i|\theta)\,f(\theta)$

$\propto \theta^{x_i}(1-\theta)\,\theta^{p-1}(1-\theta)$

$\propto \left(\theta^{x_i} - \theta^{x_i+1}\right)\left(\theta^{p-1} - \theta^{p}\right)$

not sure $\propto \theta^{x_i+p-1} - \theta^{x_i+p} - \theta^{x_i+p} + \theta^{x_i+1+p}$

about this $\propto 2\theta^{x_i+p+1} - 2\theta^{x_i+p}$

stuff

$\theta|x_i \sim \text{Geometric}(x_i+p)$

our p we are using as a placeholder holds the information about θ from the data in to the post.