Problem Set 8: Derivatives

The junction $y = \frac{1}{2}x^2$

1. a, We will apply Fomula (8) with $j(x) = \frac{1}{2}x^2$, $x_0 = 3$, and $x_1 = 4$. This yields.

$$r_{ave} = \frac{j(x_1) - j(x_0)}{x_1 - x_0} = \frac{j(4) - j(3)}{4-3} = \frac{8 - 4,5}{1} = 3,5$$

Thus, y increases an average of 3,5 units per unit increase in x over the interval [3, 4]

b, We will apply Fomula (9) with $j(x) = \frac{1}{2}x^2$, $x_0 = 3$. This yields.

$$r_{inst} = \lim_{x_1 \to x_0} \frac{j(x_1) - j(x_0)}{x_1 - x_0} = \lim_{x_1 \to 3} \frac{j(x_1) - j(3)}{x_1 - 3}$$

$$= \lim_{x_1 \to 3} \frac{\frac{1}{2}x_1^2 - \frac{9}{2}}{x_1 - 3} = \lim_{x_1 - 3} \frac{x_1^2 - 9}{2(x_1 - 3)}$$

$$= \lim_{x_1 \to 3} \frac{(x_1-3)(x_1+3)}{2(x_1 - 3)} = \lim_{x_1 \to 3} \frac{x_1 + 3}{2} = 3.$$

Thus, a small increase in x from $x = 3$ will produce ~~a~~ approximatelly at 3- jold increase in y.

c, We will apply Fomula (9) with $j(x) = \frac{1}{2}x^2$, $x_0 = a$. This yields.

$$r_{inst} = \lim_{x_1 \to a} \frac{j(x_1) - j(x_0)}{x_1 - x_0} = \lim_{x_1 \to a} \frac{\frac{1}{2}x_1^2 - \frac{1}{2}a^2}{x_1 - a}$$

$$= \lim_{x_1 \to a} \frac{x^2 - a^2}{2(x_1 - a)} = \lim_{x_1 \to a} \frac{(x_1 - a)(x_1 + a)}{2(x_1 - a)}$$

$$= \lim_{x_1 \to a} \frac{x_1 + a}{2} = \frac{2a}{2} = a.$$

2 We have :

$$r_{ave} = \frac{3.(h+1)^{2,5} + 580h - 3}{10h}$$

$\Rightarrow$ The instantaneous velocity of the car at $t = 1$ is:

$$\lim_{h \to 0} \frac{3(h+1)^{2,5} + 580h - 3}{10h}$$

3 a, $y = \sqrt{9-4x}$

$$y' = (\sqrt{9-4x})' = \frac{(9-4x)'}{2\sqrt{9-4x}} = \frac{-4}{2\sqrt{9-4x}} = \frac{-2}{\sqrt{9-4x}}$$

b, $y = \frac{x}{x+1}$

$$y' = \left(\frac{x}{x+1}\right)' = \frac{x'(x+1) - (x+1)'.x}{(x+1)^2}$$

$$= \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

4. $f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ k(x-1), & x > 1 \end{cases}$

a, continuous

When $x = 1$ then $x^2 - 1 = 0$ and $k(x-1) = 0$

So the function is continuous for all value of k.

b, Differentiable :

$$f'(x) = \begin{cases} (x^2-1)' \\ [k(x-1)]' \end{cases} \Leftrightarrow f'(x) = \begin{cases} 2x \\ x \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 2 \text{ when } x = 1 \\ 2 \text{ when } k = 2 \end{cases}$$

So the function is differentiable at $x = 1$ only when $k = 2$.

a, $f(x) = x^2 \sin x$

5 $\Rightarrow f'(x) = (x^2 \sin x)' = (x^2)' . \sin x + x^2 (\sin x)'$

$= 2x . \sin x + x^2 . \cos x$.

b, $f(x) = \sqrt{x} + \cos^2 x$

$\rightarrow f'(x) = (\sqrt{x} + \cos^2 x)' = (\sqrt{x})' + (\cos^2 x)'$

$= \frac{1}{2\sqrt{x}} + \left(\frac{1 + \cos 2x}{2}\right)'$

$= \frac{1}{2\sqrt{x}} + \frac{(1 + \cos 2x)' . 2 - (1 + \cos 2x) . 2'}{4}$

$= \frac{1}{2\sqrt{x}} + \frac{-2 \sin 2x . 2}{4}$

$= \frac{1}{2\sqrt{x}} - \sin 2x$.

c, $f(x) = \frac{2x^2 - x + 5}{3x + 2}$

$\rightarrow f'(x) = \frac{(2x^2 - x + 5)' . (3x + 2) - (2x^2 - x + 5) . (3x + 2)'}{(3x + 2)^2}$

$= \frac{(4x - 1)(3x + 2) - (2x^2 - x + 5) . 3}{(3x + 2)^2}$

$= \frac{12x^2 + 8x - 3x - 2 - 6x^2 + 3x - 15}{(3x + 2)^2}$

$= \frac{6x^2 + 8x - 17}{(3x + 2)^2}$

d, $f(x) = \frac{\tan x}{1 + x^2}$

$\rightarrow f'(x) = \left(\frac{\tan x}{1 + x^2}\right)' = \frac{(\tan x)' . (1 + x^2) - \tan x (1 + x^2)'}{(1 + x^2)^2}$

$= \frac{\frac{1}{\cos^2 x} . (1 + x^2) - \tan x . 2x}{(1 + x^2)^2}$

$= \frac{\frac{1}{\cos^2 x} + \frac{x^2}{\cos^2 x} - 2x \tan x}{(1 + x^2)^2} = \frac{\frac{1 + x^2}{\cos^2 x} - 2x \tan x}{(1 + x^2)^2}$

6 We have:

$$\lim_{h\to 0} \frac{f(1+h)}{h} = 5 \Rightarrow \lim_{h\to 0} f(1+h) = 0 \Rightarrow f(1) = 0$$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{f(1+h)}{h} = 5$$

7 We have:

$$\lim_{x\to 2} \frac{x^3 . f(x) - 24}{x-2} = 28 \Rightarrow \lim_{x\to 2} \left(x^3 . f(x) - 24\right) = 0$$

$$\Rightarrow 8f(2) - 24 = 0 \Rightarrow f(2) = 3$$

Set $g(x) = x^3 . f(x)$, so

$\rightarrow g(2) = 8 . f(2)$

$g'(2) = 3x^2 . f'(x) + x^3 . f'(x)$

$\rightarrow g'(2) = 12 f(2) + 8f'(2)$

We consider:

$$\lim_{x\to 2} \frac{x^3 . f(x) - 24}{x-2} = 28.$$

$$\Leftrightarrow \lim_{x\to 2} \frac{g(x) - 8f(2)}{x-2} = 28.$$

$$\Leftrightarrow \lim_{x\to 2} \frac{g(x) - g(2)}{x-2} = 28$$

$\Leftrightarrow g'(2) = 28$

So $12f(2) + 8f'(2) = 28$

$\Leftrightarrow 8f'(2) = -8 \quad \Leftrightarrow f'(2) = -1.$

8 $y = x^3 - 9x^2 - 16x \quad (1)$

+) $y' = 3x^2 - 18x - 16 = M$

+) Equation passes through the origin : $M = \frac{y-0}{x-0} = \frac{y}{x}$

$$\Rightarrow 3x^2 - 18x - 16 = \frac{y}{x}$$

$\Leftrightarrow y = 3x^3 - 18x^2 - 16x \quad (2)$

(1) & (2) $\Rightarrow x^3 - 9x^2 - 16x = 3x^3 - 18x^2 - 16x$

$\Leftrightarrow -2x^3 + 9x^2 = 0 \Leftrightarrow \begin{cases} x = 0 \\ x = \frac{9}{2} \end{cases}$

+) When $x = 0 \Rightarrow y' = -16$ then $y - 0 = m(x - 0) \Leftrightarrow y = -16x$.

+) When $x = \frac{9}{2} \Rightarrow y' = \frac{-145}{4}$ then $y - 0 = m(x - 0) \Leftrightarrow y = -\frac{145}{4}x$

So the equations are $\begin{cases} y = -16x \\ y = -\frac{145}{4}x \end{cases}$

9 +) $x + 4y = 10 \Rightarrow 4y = 10 - x$

$\Rightarrow y = -\frac{1}{4}x + \frac{10}{4}$

$\Rightarrow m = -\frac{1}{4}$

$\Rightarrow k = 4 \ (m.k = -1)$

+) $y = 2x^3 . x^2 \Rightarrow y' = 6x^2 - 2x = 4$

$\Leftrightarrow 6x^2 - 2x - 4 = 0$

$\Leftrightarrow \begin{cases} x = 1 \\ x = \frac{-2}{3} \end{cases} \Rightarrow \begin{cases} y = 1 \\ y = \frac{-19}{27} \end{cases}$.

10 $f(1) = 1$, $g(1) = -2$, $f'(1) = 3$, $g'(1) = -1$.

a, $\frac{d}{dx}\left[f(x) . g(x)\right]\Big|_{x=1}$

$\Rightarrow [f(x) . g(x)]' = f'(x) . g(x) + f(x) . g'(x)$

$\Rightarrow m'(1) = f'(1) . g(1) + f(1) . g'(1) = 3.(-2) + 1.(-1) = -7$.

b, $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]\Bigg|_{x=1}$

$\Rightarrow \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)} = m'(x)$

$m'(1) = \frac{f'(1) \cdot g(1) - f(1) \cdot g'(1)}{g^2(1)} = \frac{3 \cdot (-2) - 1(-1)}{(-2)^2} = \frac{-5}{4}$

c, $\frac{d}{dx}\left[\sqrt{f(x)}\right]\Bigg|_{x=1}$

$\Rightarrow \frac{f'(x)}{2\sqrt{x}} = m'(x)$

$m'(1) = \frac{f'(1)}{2\sqrt{1}} = \frac{3}{2}$

d, $\frac{d}{dx}\left[f(1) \cdot g'(1)\right]$

$\Rightarrow m'(1) = f'(1) \cdot g'(1) + f(1) \cdot g''(1)$

$= 3 \cdot (-1) + 1 \cdot 0 = -3.$

11 a, $f(x) = x^8 - 3\sqrt{x} + 5x^{-3}$

$\rightarrow f'(x) = 8x^7 - \frac{3}{2\sqrt{x}} + (-15)x^{-4}$

b, $f(x) = (2x+1)^{101}(5x^2-7)$

$\rightarrow f'(x) = \left[(2x+1)^{101}\right]' \cdot (5x^2-7) + (2x+1)^{101} \cdot (5x^2-7)'$

$= 101 \cdot (2x+1)^{100} \cdot (5x^2-7) + (2x+1)^{101} \cdot 10x$

c, $f(x) = \sin x + 2\cos^3 x$

$\Leftrightarrow f(x) = \sin x + 2 \cdot \frac{3\cos x + \cos 3x}{4}$

$\rightarrow f'(x) = \cos x + 2 \cdot \frac{(3\cos x + \cos 3x)' \cdot 4 - (3\cos x + \cos 3x) \cdot 4'}{16}$

$= \cos + 2 \cdot \frac{(-3\sin x - 3\sin 3x) \cdot 4}{16}$

$= \cos + \frac{-3\sin x - 3\sin 3x}{2}$

d, $f(x) = (1 + \sec x)(x^2 - \tan x)$

$\rightarrow f'(x) = (1 + \sec x)' . (x^2 - \tan x) + (1 + \sec x) . (x^2 - \tan x)'$

$= (\sec x . \tan x) . (x^2 - \tan x) + (1 + \sec x) . (2x - \sec^2 x)$

e, $f(x) = \sqrt{3x+1} . (x-1)^2$

$\rightarrow f'(x) = (\sqrt{3x+1})' . (x-1)^2 + \sqrt{3x+1} \left[(x-1)^2\right]'$

$= \frac{(3x+1)'}{2\sqrt{3x+1}} . (x-1)^2 + \sqrt{3x+1} . 2(x-1)$

$= \frac{3}{2\sqrt{3x+1}} . (x-1)^2 + 2\sqrt{3x+1} . (x-1)$

f) $f(x) = \left(\frac{3x+1}{x^2}\right)^3$

$\rightarrow f'(x) = 3 . \left(\frac{3x+1}{x^2}\right)^2 . \left(\frac{3x+1}{x^2}\right)'$

$= 3 . \left(\frac{3x+1}{x^2}\right)^2 . \frac{(3x+1)' . x^2 - (3x+1) . (x^2)'}{x^4}$

$= 3 . \left(\frac{3x+1}{x^2}\right)^2 . \frac{3x^2 - (3x+1) . 2x}{x^4}$

$= 3 . \left(\frac{3x+1}{x^2}\right)^2 . \frac{3x^2 - 6x^2 - 2x}{x^4}$

$= 3 . \left(\frac{3x+1}{2}\right)^2 . \frac{3x - 6x - 2}{x^3}$

g, $f(x) = \frac{1}{2x + \sin^3 x}$

$\rightarrow f'(x) = \frac{1' . (2x + \sin^3 x) - 1 . (2x + \sin^3 x)'}{(2x + \sin^3 x)^2}$

$= \frac{-(2 + 3\sin^2 x . \cos x)}{(2x + \sin^3 x)^2}$

12 a, $f(x) = (2x+7)^6 \cdot (x-2)^5$

$\rightarrow f'(x) = [(2x+7)^6]' \cdot (x-2)^5 + (2x+7)^6 \cdot [(x-2)^5]'$

$= 6(2x+7)^5 \cdot 2 \cdot (x-2)^5 + (2x+7)^6 \cdot 5(x-2)^4$

$= (2x+7)^5 \cdot (x-2)^4 \cdot [12(x-2) + 5(2x+7)]$

$= (2x+7)^5 \cdot (x-2)^4 \cdot (22x+11)$

+) $f'(x) = 0 \Leftrightarrow (2x+7)^5 \cdot (x-2)^4 \cdot (22x+11) = 0$

$\Leftrightarrow \left[\begin{array}{l} 2x+7=0 \\ x-2=0 \\ 22x+11=0 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = -7/2 \\ x = 2 \\ x = -1/2 \end{array}\right.$.

b, $f(x) = \dfrac{(x-3)^4}{x^2+2x}$ $(x \setminus \{0; -2\})$.

$\rightarrow f'(x) = \dfrac{[(x-3)^4]' \cdot (x^2+2x) - (x-3)^4 \cdot (x^2+2x)'}{(x^2+2x)^2}$

$= \dfrac{4(x-3)^3 \cdot (x^2+2x) - (x-3)^4 \cdot (2x+2)}{(x^2+2x)^2}$

$= \dfrac{(x-3)^3 \cdot [4(x^2+2x) - (x-3) \cdot (2x+2)]}{(x^2+2x)^2}$

+) $f'(x) = 0 \Leftrightarrow (x-3)^3 \cdot [4(x^2+2x) - (x-3)(2x+2)] = 0$.

$\Leftrightarrow (x-3)^3 \cdot [4x^2 + 8x - 2x^2 - 2x + 6x + 6] = 0$.

$\Leftrightarrow (x-3)^3 \cdot 8(2x^2 + 12x + 6) = 0$.

$\Leftrightarrow \left[\begin{array}{l} x-3=0 \\ 2x^2+12x+6=0 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = 3. \\ x = -3 \pm \sqrt{6}. \end{array}\right.$

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+) $y = 3x - \tan x$

$\rightarrow y' = 3 - \sec^2 x \quad (1)$

+) $y - 2 = x \rightarrow y = x + 2$

$\rightarrow y' = 1 \quad (2)$

+) (1) & (2) $\Rightarrow 3 - \sec^2 x = 1$

$\Leftrightarrow \sec^2 x = 2$

$\Leftrightarrow \dfrac{1}{\cos^2 x} = 2.$

$\Leftrightarrow \cos^2 x = \dfrac{1}{2}$

$\Leftrightarrow \cos x = \pm \dfrac{\sqrt{2}}{2}$

$\Leftrightarrow x = \pm \dfrac{\pi}{4} + k\pi \quad (k \in \mathbb{Z})$

14 & 15

$f(x) = M \sin x + N \cos x$

+) $f\left(\dfrac{\pi}{4}\right) = 3 \Leftrightarrow M \sin \dfrac{\pi}{4} + N \cos \dfrac{\pi}{4} = 3$

$\Leftrightarrow \dfrac{\sqrt{2}}{2} M + \dfrac{\sqrt{2}}{2} N = 3 \quad (1)$

+) $f'(x) = M \cos x - N \sin x$

$\rightarrow f'\left(\dfrac{\pi}{4}\right) = 1 \Leftrightarrow M \cos \dfrac{\pi}{4} - N \sin \dfrac{\pi}{4} = 1$

$\Leftrightarrow \dfrac{\sqrt{2}}{2} M - \dfrac{\sqrt{2}}{2} N = 1 \quad (2)$

(1) & (2) $\Leftrightarrow \begin{cases} \dfrac{\sqrt{2}}{2} M + \dfrac{\sqrt{2}}{2} N = 3 \\ \dfrac{\sqrt{2}}{2} M - \dfrac{\sqrt{2}}{2} N = 1 \end{cases} \Leftrightarrow \begin{cases} M = 2\sqrt{2} \\ N = \sqrt{2} \end{cases}.$

$\Rightarrow f(x) = 2\sqrt{2} \sin x + \sqrt{2} \cos x.$

$f'(x) = 2\sqrt{2} \cos x - \sqrt{2} \sin x$

+) $f\left(\dfrac{\pi}{4}\right) = 2\sqrt{2} \sin \dfrac{\pi}{4} + \sqrt{2} \cos \dfrac{\pi}{4} = -3$

+) $f'\left(\dfrac{\pi}{4}\right) = 2\sqrt{2} \cos \dfrac{\pi}{4} - \sqrt{2} \sin \dfrac{\pi}{4} = 1.$

Equation for the tangent line to $y = f(x)$ at $x = \frac{3\pi}{4}$ is $y = -3\left(x - \frac{3\pi}{4}\right) + 1$.

16. $f(x) = M\tan x + N\sec x$

$\rightarrow f'(x) = M\sec^2 x + N\sec x \cdot \tan x$.

- $f\left(\frac{\pi}{4}\right) = 2 \Leftrightarrow M\tan\frac{\pi}{4} + N\sec\frac{\pi}{4} = 2$.

  $\Leftrightarrow M + \sqrt{2}N = 2 \qquad (1)$

- $f'\left(\frac{\pi}{4}\right) = 0 \Leftrightarrow M\sec^2 x + N\sec x \cdot \tan x = 0$

  $\Leftrightarrow 2M + \sqrt{2}N = 0 \qquad (2)$

- $(1) \& (2) \Rightarrow \begin{cases} M + \sqrt{2}N = 2 \\ 2M + \sqrt{2}N = 0 \end{cases} \Leftrightarrow \begin{cases} M = -2 \\ N = 2\sqrt{2}. \end{cases}$

$\Rightarrow f(x) = -2\tan x + N\sec x \Rightarrow f(0) = -2$.

$f'(x) = -2\sec^2 x + N\sec x \cdot \tan x \Rightarrow f'(0) = 2\sqrt{2}$.

Equation for the tangent line $y = f(x)$ at $x = 0$ is

$y = -2(x - 0) + 2\sqrt{2} = -2x + 2\sqrt{2}$.

17. $f'(x) = 2x \cdot f(x)$

$\Rightarrow \dfrac{f'(x)}{f(x)} = 2x$

$\Rightarrow \displaystyle\int \frac{f'(x)}{f(x)} = \int 2x$

$\Rightarrow \ln f(x) = x^2 + c$

$\Rightarrow f(x) = e^{x^2 + c} = C \cdot e^{x^2}$

+) $f(2) = 5 \Rightarrow f(2) = C \cdot e^4$

$\Rightarrow C = \dfrac{5}{e^4}$

$\Rightarrow f(x) = \dfrac{5}{e^4} \cdot e^{x^2} = 5 \cdot e^{x^2 - 4}$

a, $g(x) = f(\sec x) = 5 . e^{\sec^2 x - 4}$

$\Rightarrow g'(x) = 5 . e^{\sec^2 x - 4} . (\sec^2 x - 4)'$

$= 5 . e^{\sec^2 - 4} . 2\sec^2 x . \tan x$

$= 10 e^{\sec^2 x - 4} . \sec^2 x . \tan x .$

$g'\left(\frac{\pi}{3}\right) = 10 . e^{\sec^2 \frac{\pi}{2} - 4} . \sec^2 \frac{\pi}{3} . \tan \frac{\pi}{3} = 40\sqrt{3} .$

b, $h(x) = \left[\frac{f(x)}{x-1}\right]^4$

$\rightarrow h' = 4 . \left[\frac{f(x)}{x-1}\right]^3 . \left[\frac{f(x)}{x-1}\right]'$

$= 4 . \left[\frac{5e^{x^2-4}}{x-1}\right]^3 . \left[\frac{5e^{x^4-4}}{x-1}\right]'$

$= 4 . \left[\frac{5e^{x^2-4}}{x-1}\right]^3 . \frac{(5e^{x^2})' . (x-1) - 5e^{x^2-4} . (x-1)'}{(x-1)^2}$

$= 4 . \left[\frac{5e^{x^2-4}}{x-1}\right]^3 . \frac{5e^{x^2} . 2x . (x-1) - 5e^{x^2-4}}{(x-1)^2}$

$= 4 . \left[\frac{5e^{x^2-4}}{x-1}\right]^3 . \frac{5e^{x^2-4} . [2x(x-1) - 1]}{(x-1)^2}$

$= 4 . \left[\frac{5e^{x^2-4}}{x-1}\right]^3 . \frac{5e^{x^2-4} . (2x^2 - 2x - 1)}{(x-1)^2}$

$\Rightarrow h'(2) = 4 . \left(\frac{5e^0}{1}\right)^3 . \frac{5e^0 . (4^2 - 4 - 1)}{1^2}$

$= 4 . 5^3 . 5 . 11 = 27500 .$