Section 3.1

1. +) Expanding the first row:

$$\begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix} = A$$

$$\det A = a(-1)^{1+1} . 3 . \det A_{11} + (-1)^{1+2} . 0 . \det A_{12} + (-1)^{1+3} . 4 . \det A_{13}$$

$$= 3 . \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix} + 0 + 4 . \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix}$$

$$= 3.(-13) + 4.10 = 1.$$

2. $B = \begin{bmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

+) Expanding the first row:

$$\det B = 0 + (-1)^{1+2} \cdot 4 \cdot \begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} + (-1)^{1+3} \cdot 1 \cdot \begin{vmatrix} 5 & -3 \\ 2 & 3 \end{vmatrix}$$

$$= -4.5 + 21$$

$$= 1.$$

+) Expanding the second column:

$$\det B = -4 \cdot \begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 3 \cdot \begin{vmatrix} 0 & 1 \\ 5 & 0 \end{vmatrix}$$

$$= -4.5 - 3.(-2) + 15$$

$$= 1.$$

9 $$\begin{bmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{bmatrix} = C$$

+) Epanding the third row, then expanding the first row of $\det C_{13}$.

$$\det C = (-1)^{3+1} \cdot 3 \cdot \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{vmatrix} + 0 + 0 + 0$$

$$= -3 \cdot (-1)^{1+3} \cdot 5 \cdot \begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 3.5.1 = 15$$

10 $$\begin{bmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{bmatrix} = D$$

Firstly, expanding the second row of D, then expanding the first column of $\det D_{23}$.

$$\det D = -3 \cdot \begin{vmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ 2 & 0 & 5 \end{vmatrix}$$

$$= -3 \cdot 1 \cdot \begin{vmatrix} -4 & 5 \\ 0 & 5 \end{vmatrix} - 2 \cdot \begin{vmatrix} -2 & 2 \\ 0 & 5 \end{vmatrix} + 2 \cdot \begin{vmatrix} -2 & 2 \\ -4 & 5 \end{vmatrix}$$

$$= -3 \cdot 1 \cdot (-20) - 2 \cdot (-10) + 2 \cdot (-2)$$
$$= 76.$$

12. $$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{bmatrix} = E$$

Firstly, expanding the first row of E, then expanding the first row of E_{11}.

$$\det E = 3 \cdot E_{11}$$
$$= 3 \cdot \begin{vmatrix} -2 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -3 \end{vmatrix} = 3 \cdot (-2) \cdot \begin{vmatrix} 3 & 0 \\ 4 & -3 \end{vmatrix}$$
$$= 3 \cdot (-2) \cdot (-9) = 54.$$

13. $$\begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix} = F.$$

Firstly, expanding the second row of F, then expanding the second column of F_{23}.

$$\det F = -2 \cdot F_{23}$$

$$= -2 \cdot \begin{bmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$= -2 \cdot 3 \cdot \begin{bmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= -2 \cdot 3 \cdot (-1) \cdot \begin{vmatrix} 3 & -5 \\ 2 & -3 \end{vmatrix} + 2 \cdot \begin{vmatrix} 4 & 3 \\ 5 & 2 \end{vmatrix}$$

$$= -2 \cdot 3 \cdot (-1) \cdot 1 \cdot 2 + (-7) = 5.$$

24 $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 4 & -4 \\ 2 & -3 & 1 \end{bmatrix} = G$; $\begin{bmatrix} k & 0 & k \\ -3 & 4 & -3 \\ 2 & -3 & 1 \end{bmatrix} = H$.

$$\det G = 1 \cdot \begin{vmatrix} 4 & -4 \\ -3 & 1 \end{vmatrix} + 0 + 1 \cdot \begin{vmatrix} -3 & 4 \\ 2 & -3 \end{vmatrix}$$

$$= 20 + 1 = 21.$$

$$\det H = k \cdot \begin{vmatrix} 4 & -4 \\ -3 & 1 \end{vmatrix} + 0 + k \begin{vmatrix} -3 & 4 \\ 2 & -3 \end{vmatrix}$$

$$= 20k + k = 21k.$$

The row operation scale row 1 by k, and the determinant is multiplied by k.

33. $\det A = ad - bc$; $EA = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$

$\det E = 1$. $\Rightarrow \det EA = (a+kc)d - c(b+kd)$

$\det A \cdot \det E$ $= ad + kcd - cb - kcd = ab - bc$

$= ad - bc$. Thus $\det EA = \det E \cdot \det A$.

35. $\det A = ad - bc$; $EA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$\det E = -1$

$\Rightarrow \det E . \det A = -ad + bc$ $\qquad \Rightarrow \det EA = cb - ad$.

Thus $\det EA = \det E . \det A$.

Section 3.2.

5. Using row ~~de~~ reduce to echolon jorm, we have:

$$\begin{vmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix} = -3$$

7. Using row reduce to echolon jorm, we have:

$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 30 & 27 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 1.1.30.0 = 0.$$

8. Using row reduce to echolon jorm, we have:

$$\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & -11 \\ 0 & -1 & -1 & -10 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 1 & -15 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -15 \\ 0 & 0 & 0 & -6 \end{vmatrix} = -(-6) = 6 .$$

10. Using row reduce to echolon jorm, we have:

$$\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & 4 & -2 & 6 \\ -2 & -6 & 2 & 3 & 10 \\ 1 & 5 & -6 & 2 & -3 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & 4 & -2 & -6 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 2 & -7 & 2 & -5 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & -1 & 0 & 2 \\ 0 & 2 & 4 & -2 & -6 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & -11 & 4 & 1 \\ 0 & 0 & -8 & 7 & 15 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 0 & 2 \\ 0 & 2 & 4 & -2 & -6 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & -3/2 & -31/2 \\ 0 & 0 & 0 & -1 & -9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & -1 & 0 & 2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & -3/2 & -31/2 \\ 0 & 0 & 0 & 0 & 4/3 \end{vmatrix} = 1 \cdot 2 \cdot 0 \cdot \left(\frac{-3}{2}\right) \cdot \frac{4}{3} = 0 .$$

11 Using row reduction and cojactor expansion, we have:

$$\begin{vmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 0 & 0 & 2 & 1 \end{vmatrix} = (-1)^{3} \cdot 4 \begin{vmatrix} 3 & 1 & -3 \\ -6 & -4 & 3 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 4 \cdot \begin{vmatrix} 3 & 1 & -3 \\ 0 & -2 & -3 \\ 0 & 2 & 1 \end{vmatrix} = -4 \cdot 3 \cdot \begin{vmatrix} -2 & -3 \\ 2 & 1 \end{vmatrix}$$

$$= -4 \cdot 3 \cdot (-2+6) = -4 \cdot 3 \cdot 4 = -48.$$

12. $$\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 11 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 3 & 0 & -2 & 0 \\ 4 & 2 & 4 & 3 \end{vmatrix}$$

$$= 3 \cdot \begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ 3 & 0 & -2 \end{vmatrix} = 3 \cdot \begin{vmatrix} -1 & 2 & 3 \\ 5 & 0 & -3 \\ 3 & 0 & -2 \end{vmatrix}$$

$$= 3 \cdot (-2) \cdot \begin{vmatrix} 5 & -3 \\ 3 & -2 \end{vmatrix} = 3 \cdot (-2) \cdot (-10+9) = 6.$$

13 $$\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 0 \\ 6 & -2 & -4 & 1 \\ -6 & 7 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 0 \\ 4 & -7 & -8 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 4 & 7 & 6 \\ 4 & -7 & -8 \\ -6 & 7 & 7 \end{vmatrix} = \begin{vmatrix} 4 & 7 & 6 \\ 8 & 0 & -2 \\ -10 & 0 & 1 \end{vmatrix} = (-7) \cdot \begin{vmatrix} 8 & -2 \\ 10 & 1 \end{vmatrix}$$

$$= (-7) \cdot (8-20) = 84.$$

15. det $= 3.7 = 21$

17. det $= 7$

18 det $= -7$

19 det $= 2.7 = 14$

20 det $= 7$

22 $\begin{vmatrix} 5 & 1 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 16 & 9 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = (-1) \cdot \begin{vmatrix} 16 & 9 \\ 5 & 3 \end{vmatrix}$

$= (-1) \cdot 3 = -3$

Since det $= -3 \neq 0 \Rightarrow$ the matrix is invertible.