Section 2.5.

We have:

5

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -4 & 3 & -5 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $b = \begin{bmatrix} 1 \\ 7 \\ 0 \\ 3 \end{bmatrix}$

+) $Ly = b \Leftrightarrow \begin{bmatrix} L & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 7 \\ -1 & 0 & 1 & 0 & 0 \\ 4 & 3 & -5 & 1 & 3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 3 & -5 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -5 & 1 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 1 \\ 5 \\ 1 \\ -3 \end{bmatrix}$$

+) $Ux = y \Leftrightarrow \begin{bmatrix} U & y \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & -4 & -3 & 1 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -2 & -4 & 0 & -8 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 & 0 & -8 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -2 \\ -1 \\ 2 \\ -3 \end{bmatrix}$$

10. We have:

$$A = \begin{bmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & 14 \end{bmatrix} \sim \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix} = U$$

+) $\begin{bmatrix} -5 \\ 10 \\ 15 \end{bmatrix} \begin{bmatrix} -2 \\ 10 \end{bmatrix} \begin{bmatrix} 9 \end{bmatrix}$ $\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{bmatrix}$

Section 2.8.

6. The vector u is in the subspace of $\mathbb{R}^4$ generated by $\{v_1, v_2, v_3\}$ if and only if the vector equation $x_1 v_1 + x_2 v_2 + x_3 v_3 = u$ is consistent.

We have the row operations:

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -4 \\ -2 & -7 & -8 & 10 \\ 4 & 9 & 6 & -7 \\ -3 & 7 & 5 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -4 \\ 0 & 1 & 2 & 2 \\ 0 & -7 & -14 & 9 \\ 0 & -5 & -10 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 23 \\ 0 & 0 & 0 & 17 \end{bmatrix}$$

$\Rightarrow$ The vector equation is inconsistant.

$\Rightarrow$ u is not in the subspace generated by $\{v_1, v_2, v_3\}$

8. We have:

$$\begin{bmatrix} A & p \end{bmatrix} \cong \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & 3 & 3 & -9 \end{bmatrix} \sim \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & -1 & 3 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The augmented matrix above consponds to a consistend

$\Rightarrow$ p is in col A.

9. To determine whether p is in the Nul A, we have to compute Ap.

$$Ap = \begin{bmatrix} 2 & -3 & -4 \\ -8 & 8 & 6 \\ 6 & -7 & -7 \end{bmatrix} \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix} = \begin{bmatrix} -2 \\ -62 \\ 29 \end{bmatrix}$$

Since $Ap \neq 0 \Rightarrow p$ is not in Nul A.

10. To determine whether u is in Nul A, we have to compute Au.

$$Au = \begin{bmatrix} -3 & -2 & 0 \\ 0 & 2 & -6 \\ 6 & 3 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $Au = 0 \Rightarrow u$ is in Nul A.

18. Place the three vectors into a 3×3 matrix A and determine whether A is invertible.

$$A = \begin{bmatrix} 1 & -5 & 7 \\ 1 & -1 & 0 \\ -2 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 7 \\ 0 & 4 & -7 \\ 0 & -8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 7 \\ 0 & 4 & -7 \\ 0 & 0 & -5 \end{bmatrix}$$

The matrix A has three pivots, so A is invertible by the IMT and its columns form a basis for $\mathbb{R}^3$.

25. $A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Basis for col A : $\begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}$

For Nul A, obtain the reduced echelon form for $Ax=0$

$$[A\ 0] \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 7 & 0 \\ 0 & 1 & 2{,}5 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 - 2x_3 + 7x_5 = 0 \\ x_2 + 2{,}5x_3 - 5x_5 = 0 \\ x_4 + 4x_5 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2x_3 - 7x_5 \\ x_2 = -2{,}5x_3 + 5x_5 \\ x_4 = -4x_5 \\ 0 = 0. \end{cases}$$

The solution of $Ax=0$ in parametric vector form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - 7x_5 \\ -2{,}5x_3 + 5x_5 \\ x_3 \\ -4x_5 \\ x_5 \end{bmatrix} = x_3 \cdot \underbrace{\begin{bmatrix} 2 \\ -2{,}5 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{u} + x_5 \underbrace{\begin{bmatrix} -7 \\ 5 \\ 0 \\ -4 \\ 1 \end{bmatrix}}_{v}$$

$\Rightarrow$ Basis for Nul $A = \{u, v\}$

Section 2.9.

10. The information $A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 9 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ show that columns 1, 2.

and 4 of A form a basis for col A: $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix}$

For Nul A, $\begin{bmatrix} A & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} x_1 + 3x_3 = 0 \\ x_2 - 3x_3 - 7x_5 = 0 \\ x_4 - 2x_5 = 0 \\ 0 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -3x_3 \\ x_2 = 3x_3 + 7x_5 \\ x_4 = 2x_5 \\ x_3, x_5 \text{ are free variables.} \end{cases}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 3x_3 + 7x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Basis for Nul A: $\begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}$

From this, dim Col A = 3 and dim Nul A = 2.

13. The 4 vectors span the column space H of a matrix that can be reduced to echelon form:

$$\begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 10 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1, 3 and 4 of the original matrix form a basis for H, so dim H = 3.

15. Col A = $\mathbb{R}^3$, because A has a pivot in each row and so the column of A span $\mathbb{R}^3$. Nul A cannot equal $\mathbb{R}^2$, because Nul A is a subspace of $\mathbb{R}^5$. It is true, however, that Nul A is two-dimensional. Reason: the equation $Ax = 0$ has two free variables, because A has 5 columns and only three of them are pivot columns.

16. Col A cannot be $\mathbb{R}^3$, because the column of A has 4 entries.
Since A has 7 columns and 3 pivot columns, the equation $Ax = 0$ has 4 free variables.
So $\dim \text{Nul } A = 4$.

20. A 4×5 matrix A has 5 columns. By the Rank Theorem, rank $A = 5 - \dim \text{Nul } A$. Since the null space is three-dimensional, rank $A = 2$.