# Problem Set 10 : Integration.

1 Evaluate the integral :

a, $\int \left( \frac{1}{2x^3} + 4\sqrt{x} \right) dx$

$= \int \frac{1}{2x^3} dx + \int 4\sqrt{x}\, dx = \frac{1}{2} \int x^{-3} dx + 4 \int x^{1/2} dx$

$= \frac{1}{2} \cdot \frac{1}{-3+1} x^{-3+1} + 4 \cdot \frac{1}{\frac{1}{2}+1} x^{1/2+1} + C$

$= -\frac{1}{4} x^{-2} + \frac{8}{3} x^{3/2} + C$

$= -\frac{1}{4x^2} + \frac{8}{3x\sqrt{x}} + C.$

b, $\int (u^3 - 2u + 7)\, du$.

$= \int u^3 du - \int 2u\, du + \int 7 du$.

$= \frac{1}{4} u^4 - 2 \cdot \frac{1}{2} u^2 + 7u + C$

$= \frac{u^4}{4} - u^2 + 7u + C.$

c, $\int (4 \sin x + 2 \cos x)\, dx$.

~~$= \int (4 \sin x + 2 \cos x)$~~

$= \int 4 \sin dx + \int 2 \cos x\, dx$

$= 4 \int \sin x\, dx + 2 \int \cos x\, dx$

$= -4 \cos x + 2 \sin x + C$

$(\sec x)' = \sec x \,.\, \tan x$

$\sec x = \dfrac{1}{\cos x}$

d, $\int \sec x \,(\tan x + \cos x)\, dx$

$= \int \sec x . \tan x \, dx + \int \sec x . \cos x \, dx$

$= \sec x + c + \int \dfrac{1}{\cos x} . \cos x \, dx$

$= \sec x + c + \int 1 \, dx$

$= \sec x + x + c.$

e, $\int (x^{-2/3} - 5e^x)\, dx$

$= \int x^{-2/3} dx - \int 5e^x dx$

$= \int x^{-2/3} dx - 5 \int e^x dx$

$= \dfrac{1}{-2/3 + 1} x^{-2/3+1} - 5e^x + c.$

$= 3x^{1/3} - 5e^x + c$

$= 3\sqrt[3]{x} - 5e^x + c.$

f, $\int \left(\dfrac{3}{4x} - \sec^2 x\right) dx$

$= \int \dfrac{3}{4x} dx - \int \sec^2 x \, dx$

$= \dfrac{3}{4} \int \dfrac{1}{x} dx - \int \sec^2 x \, dx$

$= \dfrac{3}{4} \ln |x| - \tan x + c.$

g, $\int \left(\dfrac{1}{1+x^2} + \dfrac{2}{\sqrt{1-x^2}}\right) dx$

$= \int \dfrac{1}{1+x^2} dx + \int \dfrac{2}{\sqrt{1-x^2}} dx$

$= \int \dfrac{1}{1+x^2} dx + 2 \int \dfrac{1}{\sqrt{1-x^2}} dx$

$= \arctan x + 2 \arcsin x + c$

$$\boxed{\begin{aligned} (\arcsin x)' &= \frac{1}{\sqrt{1-x^2}} \\ (\arccos x)' &= \frac{-1}{\sqrt{1-x^2}} \\ (\arctan x)' &= \frac{1}{x^2+1} \end{aligned}}$$

h, $\int \left( \frac{12}{x\sqrt{x^2-1}} + \frac{1-x^4}{1+x^2} \right) dx$

$= \int \frac{12}{x\sqrt{x^2-1}} dx + \int \frac{1-x^4}{1+x^2} dx$

$= 12 \int \frac{1}{x\sqrt{x^2-1}} dx + \int \frac{(1-x^2)(1+x^2)}{(1+x^2)} dx$

$= 12 \int \frac{1}{x\sqrt{x^2-1}} dx + \int (1-x^2) dx$

$= 12 \text{ arc sec} x + x - \frac{x^3}{3} + C$

2 Solve the initial problems.

a, $\frac{dy}{dx} = \frac{1-x}{\sqrt{x}}$ , $y(1) = 0$.

+) $\frac{dy}{dx} = \frac{1-x}{\sqrt{x}} \Rightarrow dy = \frac{1-x}{\sqrt{x}} dx$

+) $\int dy = \int \frac{1-x}{\sqrt{x}} dx = \int \frac{1}{\sqrt{x}} dx - \int \frac{x}{\sqrt{x}} dx$

$= \int x^{-1/2} dx - \int x^{1/2} dx$

$= \frac{1}{-\frac{1}{2}+1} x^{-1/2+1} - \frac{1}{\frac{1}{2}+1} x^{1/2+1} + C$

$= 2x^{1/2} - \frac{2}{3} x^{3/2} + C$

+) $y(1) = 0 \Leftrightarrow 2 . 1^{1/2} - \frac{2}{3} . 1^{3/2} + C = 0$

$\Leftrightarrow \frac{4}{3} + C = 0$

$\Leftrightarrow C = -\frac{4}{3}$

$\Rightarrow y = 2\sqrt{x} - \frac{2}{3} x^{2/3} - \frac{4}{3}$.

b, $\frac{dy}{dx} = \cos x - 5e^x \quad , y(0) = 0.$

+, $\frac{dy}{dx} = \cos x - 5e^x \Rightarrow dy = (\cos x - 5e^x)\, dx$

+, $\int dy = \int (\cos x - 5e^x)\, dx = \int \cos x\, dx - 5\int e^x dx$

$= \sin x - 5e^x + c.$

+, $y(0) = 0 \Leftrightarrow \sin 0 - 5e^0 + c = 0.$

$\Leftrightarrow c = 5.$

$\Rightarrow y = \sin x - 5e^x + 5.$

c, $\frac{dy}{dx} = \sqrt[3]{x} \quad , y(1) = 2.$

+, $\frac{dy}{dx} = \sqrt[3]{x} \Rightarrow dy = \sqrt[3]{x}\, dx$

+, $\int dy = \int \sqrt[3]{x}\, dx = \int x^{1/3}\, dx$

$= \frac{1}{\frac{1}{3}+1} x^{1/3+1} + c = \frac{3}{4} x^{4/3} + c$

+, $y(1) = 2 \Leftrightarrow \frac{3}{4} \cdot 1^{4/3} + c = 2.$

$\Leftrightarrow \frac{3}{4} + c = 2 \Rightarrow c = \frac{5}{4}.$

$\Rightarrow y = \frac{3}{4} x^{4/3} + \frac{5}{4}$

d, $\frac{dy}{dx} = x.e^{x^2} \quad , y(0) = 0.$

+, $\frac{dy}{dx} = x.e^{x^2} \Rightarrow dy = x.e^{x^2} dx.$

+, $\int dy = \int x.e^{x^2} dx$

Let $u = x^2$

$\Rightarrow du = 2x\,dx$

$\Rightarrow x\,dx = \frac{1}{2}\,du$

$\Rightarrow \int dy = \int x \cdot e^{x^2}\,dx = \int e^u \cdot \frac{1}{2}\,du = \frac{1}{2}e^u + c = \frac{1}{2}e^{x^2} + c.$

+, $y(0) = 0 \Leftrightarrow \frac{1}{2}e^0 + c = 0$

$\Leftrightarrow c = -\frac{1}{2}.$

$\Rightarrow y = \frac{1}{2}e^{x^2} - \frac{1}{2}.$

3 $I = \int \sec^2 x \tan x\,dx$

a, +, Let $u = \sec x$

$\Rightarrow du = \sec x \cdot \tan x\,dx$

$\Rightarrow I = \int \sec^2 x \cdot \tan x\,dx = \int u\,du = \frac{u^2}{2} + c$

$= \frac{\sec^2 x}{2} + c$

+, Let $u = \tan x$

$\Rightarrow du = \sec^2 x\,dx$

$\Rightarrow I = \int \sec^2 x \tan x\,dx = \int u\,du = \frac{1}{2}u^2 + c = \frac{\tan^2 x}{2} + c.$

b, Explain

$I = \frac{\sec^2 x}{2} + c = \frac{1}{2} \cdot \frac{1}{\cos^2 x} + c$

$= \frac{\sin^2 x + \cos^2 x}{2\cos^2 x} + c$

$= \frac{\sin^2 x}{2\cos^2 x} + \frac{\cos^2 x}{2\cos^2 x} + c$

$= \frac{\tan^2 x}{2} + \frac{1}{2} + c = \frac{\tan^2 x}{2} + c$

**4** $\displaystyle\int_0^{\pi/4} \sec^2 x \cdot \tan x \, dx$

+) Let $u = \sec x \Rightarrow$

| $x$ | $0 \to \pi/4$ |
|---|---|
| $u$ | $1 \to \sqrt{2}$ |

use the substitution in Ex 3,

$$\int_0^{\pi/4} \sec^2 x \cdot \tan x \, dx = \int_1^{\sqrt{2}} u \, du = \left(\frac{\sec^2 x}{2}\right)\Bigg|_1^{\sqrt{2}} = \frac{1}{2} \cdot \quad (1)$$

+) Let $u = \tan x \Rightarrow$

| $x$ | $0 \to \pi/4$ |
|---|---|
| $u$ | $0 \to 1$ |

use the substitution is Ex 3,

$$\int_0^{\pi/4} \sec^2 x \cdot \tan x \, dx = \int_0^1 u \, du = \left(\frac{\tan^2 x}{2}\right)\Bigg|_0^1 = \frac{1}{2} \cdot \quad (2)$$

+) (1) & (2) $\Rightarrow$ the same result.

**5.** $\displaystyle\int \frac{x}{(x^2-1)\sqrt{x^4-2x^2}} \, dx$

Let +) $u = x^2 - 1$

$\rightarrow du = 2x \, dx \Rightarrow x dx = \dfrac{du}{2}$

+) $u = x^2 - 1$

$\Rightarrow x^2 = u + 1$

$\Rightarrow x^4 - 2x^2 = (u+1)^2 - 2(u+1)$

$= u^2 + 2u + 1 - 2u - 2$

$= u^2 - 1$

So $\displaystyle I = \int \frac{x}{(x^2-1)\sqrt{x^4-2x^2}} \, dx$

$\displaystyle = \int \frac{1}{u\sqrt{u^2-1}} \cdot \frac{du}{2}$

$\displaystyle = \frac{1}{2} \int \frac{1}{u\sqrt{u^2-1}} \, du = \frac{1}{2} \sec^{-1} u + c$

$\displaystyle = \frac{1}{2} \sec^{-1}(x^2-1) + c$

6 $\int \sqrt{1+x^{-2/3}}\,dx$

+) Let $u = 1 + x^{-2/3}$

$\Rightarrow du = \frac{2}{3} x^{-1/3}\,dx$

$\Rightarrow x^{-1/3}\,dx = \frac{2}{3}\,du.$ (1)

+) $\int \sqrt{1+x^{-2/3}}\,dx$

$= \int \sqrt{1+\frac{1}{x^{2/3}}}\,dx = \int \sqrt{\frac{1+x^{2/3}}{x^{2/3}}}\,dx$

$= \int \sqrt{1+x^{2/3}}\,\frac{dx}{x^{1/3}}$ (2)

From (1) and (2), we have:

$I = \frac{3}{2}\int \sqrt{u}\,du = \frac{3}{2}\cdot\frac{2}{3}\,u^{3/2} + c.$

$= u^{3/2} + c$

$= (1 + x^{2/3})^{3/2} + c$

9 Use the geometric argument to evaluate.

$\int_0^1 |2x-1|\,dx.$

Solution:

We have: $|2x-1| = \begin{cases} 2x-1, & 0 \le x \le \frac{1}{2} \\ 1-2x, & \frac{1}{2} \le x \le 1. \end{cases}$

$\Rightarrow \int_0^1 |2x+1|\,dx = \int_0^{1/2} (2x-1)\,dx + \int_{1/2}^1 (1-2x)\,dx$

$= (x^2 - x)\Big|_0^{1/2} + (x - x^2)\Big|_{1/2}^1$

$= \frac{1}{4} - \frac{1}{2} + 1 - \frac{1}{4} = \frac{1}{2}.$

Suppose that:

10

$$\int_0^1 f(x)\,dx = \frac{1}{2}$$

$$= F(1) - F(0) = \frac{1}{2}$$

$$\int_1^2 f(x)\,dx = \frac{1}{4}$$

$$= F(2) - F(1) = \frac{1}{4}$$

$$\int_0^3 f(x)\,dx = -1$$

$$= F(3) - F(0) = -1$$

$$\int_0^1 g(x)\,dx = 2$$

$$= G(1) - G(0) = 2.$$

a, $$\int_0^2 f(x)\,dx = F(2) - F(0)$$

$$= \frac{1}{4} + F(1) - \left[F(1) - \frac{1}{2}\right]$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

b, $$\int_1^3 f(x)\,dx = F(3) - F(1)$$

$$= -1 + F(0) - \left(\frac{1}{2} + F(0)\right)$$

$$= -1 - \frac{1}{2} = -\frac{3}{2}.$$

c, $$\int_2^3 5f(x)\,dx = 5F(3) - 5F(2)$$

$$= 5.(-1 + F(0)) - 5.\left(\frac{1}{4} + F(1)\right)$$

$$= -5 + 5F(0) - 5F(1) - \frac{5}{4}$$

$$= -5 + 5F(0) - 5\left(\frac{1}{2} + F(0)\right) - \frac{5}{4}$$

$$= -5 + 5F(0) - 5F(0) - \frac{5}{2} - \frac{5}{4}$$

$$= -\frac{35}{4}.$$

d, $\int_1^0 g(x)\,dx = -\int_0^1 g(x)\,dx = -2.$

e, $\int_0^1 g(2x)\,dx$

Let $u = 2x$ $\Rightarrow$ $du = 2dx$ $\Rightarrow$

| $x$ | $0 \to 1$ |
|---|---|
| $u$ | $0 \to 2$ |

$\Rightarrow \int_0^1 g(2x)\,dx = \int_0^2 g(\ )\cdot\frac{du}{2} = \frac{1}{2}\int_0^2 g(\ )\,du.$

f, $\int_0^1 [g(x)]\,dx =$

g, $\int_0^1 [f(x) + g(x)]\,dx$

$= \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx = \frac{1}{2} + 2 = \frac{5}{2}.$

h, $\int_0^1 f(x)\cdot g(x)\,dx.$

i, $\int_0^1 \frac{f(x)}{g(x)}\,dx.$

k, $\int_0^1 [4g(x) - 3f(x)]\,dx$

$= \int_0^1 4g(x)\,dx - \int_0^1 3f(x)\,dx = 4.2 - 3.\frac{1}{2} = \frac{13}{2}$

11 Exaluate the integral, using Fundamental Theorem of Calculus and properties of the definite integral.

a, $\int_{-3}^0 (x^2 - 4x + 7)\,dx = \int_{-3}^0 x^2\,dx - \int_{-3}^0 4x\,dx + \int_{-3}^0 7\,dx$

$= \frac{x^3}{3}\Big|_{-3}^0 - 2x^2\Big|_{-3}^0 + 7x\Big|_{-3}^0 = 9 + 18 + 21 = 48.$

b) $\int_{-1}^{2} x(1+x^3)\,dx$

$= \int_{-1}^{2} (x+x^4)\,dx$

$= \int_{-1}^{2} x\,dx + \int_{-1}^{2} x^4\,dx = \frac{x^2}{2}\Big|_{-1}^{2} + \frac{x^5}{5}\Big|_{-1}^{2}$

$= 2 - \frac{1}{2} + \frac{32}{5} + \frac{1}{5} = \frac{81}{10}$.

c) $\int_{1}^{3} \frac{1}{x^2}\,dx = \int_{1}^{3} x^{-2}\,dx = -x^{-1}\Big|_{1}^{3} = -\frac{1}{3} + 1 = \frac{2}{3}$.

d) $\int_{1}^{8} (5x^{2/3} - 4x^{-2})\,dx$

$= \int_{1}^{8} 5x^{2/3}\,dx - \int_{1}^{8} 4x^{-2}\,dx = 5 \cdot \frac{3}{5} x^{5/3}\Big|_{1}^{8} + 4 \cdot x^{-1}\Big|_{1}^{8}$

$= 96 - 3 + \frac{1}{2} - 4 = \frac{179}{2}$.

e) $\int_{0}^{1} (x - \sec x \tan x)\,dx$

$= \int_{0}^{1} x\,dx - \int_{0}^{1} \sec x \tan x\,dx \qquad (\sec x)' = \sec x \tan x$

$= \frac{x^2}{2}\Big|_{0}^{1} - \sec x\Big|_{0}^{1}$

$= \frac{x^2}{2}\Big|_{0}^{1} - \frac{1}{\cos x}\Big|_{0}^{1}$

$= \frac{1}{2} - 0 - \frac{1}{\cos(1)} + 1 = \frac{3}{2} - \frac{1}{\cos 1} = \frac{3\cos(1) - 2}{2\cos(1)}$

f) $\int_{1}^{4} \left(\frac{3}{\sqrt{t}} - 5\sqrt{t} - t^{-3/2}\right) dt$

$= 3\int_{1}^{4} t^{-1/2}\,dt - 5\int_{1}^{4} t^{1/2}\,dt - \int_{1}^{4} t^{-3/2}\,dt$

$= 3 \cdot 2t^{1/2}\Big|_{1}^{4} - 5 \cdot \frac{2}{3} t^{3/2}\Big|_{1}^{4} + 2t^{-1/2}\Big|_{1}^{4}$

$= 12 - 6 - \frac{80}{3} + \frac{10}{3} + 1 - 2 = \frac{-55}{3}$.

$g, \int_0^2 |2x-3|\,dx$

$$|2x-3| = \begin{cases} 2x-3 & , 0 \le x \le \frac{3}{2} \\ 3-2x & , \frac{3}{2} \le x \le 2 \end{cases}$$

$$\Rightarrow \int_0^2 |2x-3|\,dx = \int_0^{3/2} (2x-3)\,dx + \int_{3/2}^{2} (3-2x)\,dx$$

$$= (x^2 - 3x)\Big|_0^{3/2} + (3x - x^2)\Big|_{3/2}^{2}$$

$$= -\frac{9}{4} + 2 - \frac{9}{4} = -\frac{5}{2}.$$

$h, \int_0^{\pi/2} \left| \frac{1}{2} - \sin x \right| dx$

$$\left| \frac{1}{2} - \sin x \right| = \begin{cases} \frac{1}{2} - \sin x & , \\ \sin x - \frac{1}{2} & , \end{cases}$$

**12** a, $f(x) = \sqrt{x}$ $[1,9]$

$$S = \int_1^9 \sqrt{x}\,dx = \int_1^9 x^{1/2} = \frac{52}{3}$$

b, $f(x) = e^x$ $[1;3]$

$$S = \int_1^3 e^x\,dx = e^3 - e.$$

**13** $y = (1-x)(x-2)$

$$y = 0 \Leftrightarrow \left[ \begin{array}{l} x = 1 \\ x = 2 \end{array} \right.$$

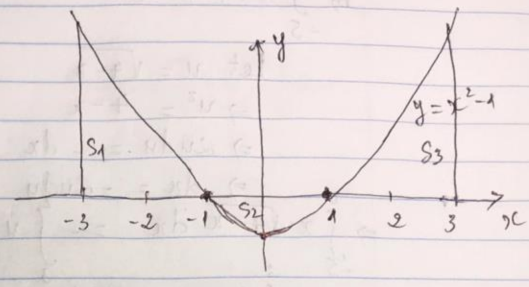
$$\Rightarrow \int_1^2 (1-x)(x-2)\,dx = \int_1^2 (-x^2 + 3x - 2)\,dx = \frac{1}{6}.$$

14 a, $y = x^2 - 1 \quad [0;3]$

$S_1 = \int_{-3}^{-1} (x^2 - 1)\,dx = \frac{20}{3}$

$S_2 = \int_{-1}^{1} (1 - x^2)\,dx = \frac{4}{3}$

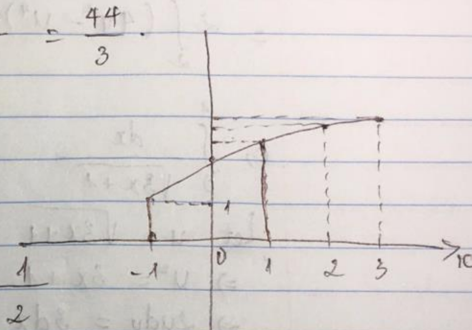
$S_3 = \int_{1}^{3} (x^2 - 1)\,dx = \frac{20}{3}$




$\Rightarrow S = S_1 + S_2 + S_3 = \frac{20}{3} + \frac{4}{3} + \frac{20}{3} = \frac{44}{3}$.

b, $g = \sqrt{x+1} - 1 \quad [-1;1]$

$S = \int_{-1}^{1} (\sqrt{x+1} - 1)\,dx$




Let $u = x + 1 \quad \Rightarrow$

$\Rightarrow du = dx$

| $x$ | $-1 \to 1$ |
|---|---|
| $u$ | $0 \to 2$ |

$\Rightarrow \int_{-1}^{1} (\sqrt{x+1} - 1)\,dx = \int_{0}^{2} (\sqrt{u} - 1)\,du$

$= \int_{0}^{2} \sqrt{u}\,du - \int_{0}^{2} 1\,dx$

$= \int_{0}^{2} u^{1/2}\,du - \int_{0}^{2} du = \frac{2}{3} u^{3/2} \Big|_0^2 - u\Big|_0^2 = \frac{2}{3} \cdot 2^{3/2} - 2$.

15 a, $\int_{0}^{1} (2x+1)^4\,dx$

Let $u = 2x + 1 \quad \Rightarrow$

$\Rightarrow du = 2\,dx$

$\Rightarrow dx = \frac{1}{2} du$

| $x$ | $0 \to 1$ |
|---|---|
| $u$ | $1 \to 3$ |

$\Rightarrow \int_{0}^{1} (2x+1)^4\,dx = \int_{1}^{3} u^4 \cdot \frac{1}{2}\,du = \frac{1}{2} \int_{1}^{3} u^4\,du$

$= \frac{1}{2} \cdot \frac{242}{5} = \frac{121}{5}$

b, $\int_{-5}^{0} x\sqrt{4-x}\, dx$

Let $u = \sqrt{4-x}$

$\Rightarrow u^2 = 4 - x$

$\Rightarrow 2udu = -dx$

$\Rightarrow dx = -2udu$

$\Rightarrow$

| $x$ | $-5 \to 0$ |
|---|---|
| $u$ | $3 \to 2$ |

$\Rightarrow \int_{-5}^{0} x\sqrt{4-x}\, dx = -\int_{3}^{2} u.(4-u^2)\, 2udu.$

$= 2\int_{2}^{3} (4u^2 - u^4)\, du = \frac{152}{3} - \frac{422}{5} = -\frac{506}{15}$

c, $\int_{0}^{1} \frac{dx}{\sqrt{3x+1}}$

Let $u = \sqrt{3x+1}$

$\Rightarrow u^2 = 3x + 1$

$\Rightarrow 2udu = 3dx$

$\Rightarrow dx = \frac{2}{3} u\, du$

$\Rightarrow$

| $x$ | $0 \to 1$ |
|---|---|
| $u$ | $1 \to 2$ |

$\Rightarrow \int_{0}^{1} \frac{dx}{\sqrt{3x+1}} = \int_{0}^{1} \frac{1}{u} . \frac{2}{3} u du = \frac{2}{3}\int_{1}^{2} du = \frac{2}{3}.$

d, $\int_{0}^{\sqrt{\pi}} x \sin x^2\, dx$

Let $u = x^2$

$\Rightarrow du = 2xdx$

$\Rightarrow$

| $x$ | $0 \to \sqrt{\pi}$ |
|---|---|
| $u$ | $0 \to \pi$ |

$\Rightarrow \int_{0}^{\sqrt{\pi}} x \sin x^2 dx = \int_{0}^{\pi} \sin u . \frac{du}{2}.$

$= \frac{1}{2}\int_{0}^{\pi} \sin u\, du = 1.$

e, $\int_0^1 \sin^2(\pi x)\cos(\pi x)\, dx$

Let $\sin(\pi x) = u$ $\quad\Rightarrow \begin{array}{c|c} x & 0 \to 1 \\ \hline u & 0 \to 0 \end{array}$

$\Rightarrow \pi \cos(\pi x)\, dx = du$

$\Rightarrow \cos(\pi x)\, dx = \frac{1}{\pi} du$

$$\Rightarrow \int_0^1 \sin^2(\pi x)\cos(\pi x)\, dx = \int_0^0 \frac{1}{\pi} u^2\, du = 0.$$

f, $\displaystyle\int_e^{e^2} \frac{dx}{x \ln x} = \int_e^{e^2} \frac{1}{\ln x}\, d(\ln x) = \ln 2.$

g, $\displaystyle\int_0^1 \frac{dx}{\sqrt{e^x}} = \int_0^1 -\frac{1}{2} e^x dx = 2 - \frac{2}{\sqrt{e}}$

h, $\displaystyle\int_0^{2/\sqrt{3}} \frac{1}{4+9x^2}\, dx$

Let $u = \frac{3x}{2}$ $\quad\Rightarrow \begin{array}{c|c} x & 0 \to 2/\sqrt{3} \\ \hline u & 0 \to \sqrt{3} \end{array}$

$\Rightarrow d = \frac{3}{2} dx$

$\Rightarrow dx = \frac{2}{3} du$

$$\Rightarrow \int_0^{2/\sqrt{3}} \frac{1}{4+9x^2}\, dx = \frac{1}{6}\int_0^{\sqrt{3}} \frac{1}{u^2+1}\, du = \frac{\pi}{18}$$