Calculus.

Section 4.1.

4.1 a, If u and v in V, then their entries are nonnegative
Since sum of nonnegative number is nonnegative
$\Rightarrow$ Vector $u + v$ has nonnegative entries.
$\Rightarrow$ $u + v$ is in V.

b, If $u = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $c = -1$, then u is in V but cu is not in V.

4.2 a, If $u = \begin{bmatrix} x \\ y \end{bmatrix}$ in W, then the vector $cu = c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$
is in W because $(cx)(cy) = c^2(xy) \geq 0$.

b, If $u = \begin{bmatrix} -1 \\ -7 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then u and v are in W
but $u + v$ is not in W.

4.13. a, The vector w is not in the set $\{v_1, v_2, v_3\}$. There are 3 vectors in the set $\{v_1, v_2, v_3\}$

b, The set Span $\{v_1, v_2, v_3\}$ contains infinities many vectors.

c, The vector w is in the subspace spanned by $\{v_1, v_2, v_3\}$ if and only if the equation $x_1v_1 + x_2v_2 + x_3v_3 = w$ has a solution. Row reducing the augmented matrix for this system of linear equation gives:

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the equation has a solution and w is in the subspace spanned by $\{v_1, v_2, v_3\}$.

Section 4.2

2. One calculates that :

$$Aw = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So w in Nul A.

5. First find the general solution of $Ax=0$ in terms of the free variables. Since

$$[A \ 0] = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 2x_2 - 4x_4 \\ x_2 \text{ is free} \\ x_3 = 9x_4 \\ x_4 \text{ is free} \\ x_5 = 0 . \end{cases}$$

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - 4x_4 \\ x_2 \\ 9x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}$$

A spanning set for Nul A is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix} \right\}$

10. The set W ~~R~~ is the set of all solutions to the homogenous system of equation $a + 3b - c = 0$, $a + b + c - d = 0$. Thus $W = \text{Nul } A$, where

$$A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$. Thus W is a subspace of $\mathbb{R}^4$ by Theorem 2, and is a vector space.

16. An element in this set ma be written as

$$b \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b \\ c \\ d \end{bmatrix}$$

where b, c and d are any real numbers.

So the set is Col A where $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$

18. The matrix A is a 4x matrix. Thus
(a) Nul A is a subspace of $\mathbb{R}$.
(b) Col A is a subspace of $\mathbb{R}^4$.

24. Consider the system with augmented matrix $\begin{bmatrix} A & w \end{bmatrix}$

Since

$$\begin{bmatrix} A & w \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1/2 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consider and w is in Col A. Also, since:

$$Aw = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ $\rightarrow$ w is in Nul A.

Section 4.3

3. Consider the matrix whose columns are the given set of vectors. The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9/2 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

So the matrix has only two pivot positions. Thus its columns do not form a basis for $\mathbb{R}^3$; the set of vectors is neither linearly independent nor does it span $\mathbb{R}^3$.

13. Since B is a row echelon form of A, we see that the first and second columns of A are its pivot columns.
Thus a basis for Col A is:

$$\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$$

To find a basis for Nul A, we find the general solution of $Ax = 0$ in terms of the free variables:
$x_1 = -6x_3 - 5x_4$; $x_2 = -5/2\, x_3 - 3/2\, x_4$, with x_3, x_4 free.
So:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$$

and a basis for Nul A is:

$$\left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

15. This problem is equivalent to finding a basis for Col A, where $A = [v_1 . v_2 \quad v_3 \quad v_4 \quad v_5]$. Since the reduced echelon form of A is:

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -4 & 1 \\ -3 & 2 & 1 & -6 \\ 2 & -3 & 6 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the first; second, and fourth columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}$$

19. Since $4v_1 + 5v_2 - 3v_1 = 0$, we see that each of vectors is a linear combination of the others. Thus the sets $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_2, v_3\}$ all span H. Since we may confirm that non of the three vectors is a multiple of any of the others, the set $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$ are linearly - independent and thus each forms a basis for H.