# Calculus.
# Problem Set 9.

**1** a, $x^3 + xy - 2x = 1$.

(i) $\frac{d}{dx}(x^3 + xy - 2x) = \frac{d}{dx}$ (1)

$\Leftrightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(xy) - \frac{d}{dx}(2x) = 0$

$\Leftrightarrow 3x^2 + x\frac{dy}{dx} - 2 + y = 0$.

$\Leftrightarrow \frac{dy}{dx} = \frac{2 - 3x^2 - y}{x}$

(ii) $x^3 + xy - 2x = 1$.

$\Leftrightarrow xy = 1 + 2x - x^3$

$\Leftrightarrow y = x^{-1} + 2 - x^3$ (1)

$\Leftrightarrow \frac{dy}{dx} = -x^{-2} - 2x$

(iii) $3x^2 + x\frac{dy}{dx} + y - 2 = 0$

$\Leftrightarrow x\frac{dy}{dx} = -3x^2 - y + 2$.

$\Leftrightarrow \frac{dy}{dx} = -3x - y.x^{-1} + 2.x^{-1}$

Sub (1) in (b) into :

$\frac{dy}{dx} = -3x - x^{-1}.(x^{-1} + 2 - x^3) + 2.x^{-1}$

$= 3x - x^{-2} - 2x^{-1} + x + 2.x^{-1}$

$= -2x - x^{-2}$

We can see (ii) and (iii) each give $\frac{dy}{dx} = -x^{-2} - 2x$.

b, $xy = x - y$

(i) $\frac{d}{dx}(xy) = \frac{d}{dx}(x-y)$

$\Leftrightarrow x.\frac{dy}{dx} + y = 1 - \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1-y}{x+1}$

(ii) $xy = x - y$

$\Leftrightarrow y(x+1) = x$

$\Leftrightarrow y = \frac{x}{x+1}$

$\Leftrightarrow \frac{dy}{dx} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$

(iii) $x.\frac{dy}{dx} + y = 1 - \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1-y}{1+x} = \frac{1}{(x+1)^2}$

We can see (ii) and (iii) each give $\frac{dy}{dx} = \frac{1}{(x+1)^2}$.

2 a, $\frac{1}{y} + \frac{1}{x} = 1.$

$\Leftrightarrow y^{-1} + x^{-1} = 1.$

$\Leftrightarrow -y^{-2}\frac{dy}{dx} - x^{-2} = 0.$

$\Leftrightarrow \frac{dy}{dx} = \frac{-x^{-2}}{y^{-2}} = -\frac{y^2}{x^2}$

b) $x^3 - y^3 = 6xy$

$\Leftrightarrow \frac{d}{dx}(x^3 - y^3) = \frac{d}{dx}(6xy)$

$\Leftrightarrow 3x^2 - 3y^2 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx}$

$\Leftrightarrow (-3y^2 - 6x) \frac{dy}{dx} = 6y - 3x^2$

$\Leftrightarrow \frac{dy}{dx} = \frac{3x^2 - 6y}{3y^2 - 6x}$

c) $\sec(xy) = y$

$\Leftrightarrow \frac{d}{dx}(\sec(xy)) = \frac{d}{dx}(y)$

$\Leftrightarrow \tan(xy(x)) \sec(xy(x)) \left(\frac{d}{dx}(xy(x))\right) = \tan(xy(x)) \sec(xy(x)) \left(\frac{d}{dx}(x)\, y(x) + x \frac{d}{dx}(y(x))\right)$

$\Leftrightarrow x \frac{d}{dx}(y(x)) + y(x) \left(\frac{d}{dx}(x)\right) \tan(xy(x)) \sec(xy(x))$

$= x \frac{d}{dx}(y(x)) + y(x) \tan(xy(x)) \sec(xy(x))$

$\Rightarrow \frac{d}{dx}(\sec(xy(x))) = \left(x \frac{d}{dx}(y(x)) + y(x) \tan(xy(x)) \sec(xy(x))\right)$

$\Leftrightarrow \frac{dy}{dx} = \left(x \frac{dy}{dx} + y\right) \tan(xy(x)) \sec(xy)$

$\Leftrightarrow \frac{dy}{dx} = -\frac{y \sec(xy) \tan xy}{x \sec(xy) \tan xy - 1}$

d, $x^2 = \dfrac{\cot y}{1 + \csc y}$

$$\frac{d}{dx}(x^2) = \frac{d}{dx}\left(\frac{\cot y}{1+\csc y}\right)$$

$$RHS = \frac{\frac{d}{dx}(\cot y(x))(\csc(x)+1) - \cot(y(x))\frac{d}{dx}(\csc(y(x))+1)}{(\csc(y(x))+1)^2}$$

$$= \frac{(\csc(y(x))+1)(-\csc^2(y(x)))\frac{d}{dx}(y(x)) - \cot(y(x))\frac{d}{dx}(\csc(y(x))+1)}{(\csc(y(x))+1)^2}$$

$$= \frac{(-\cot(y(x)))\csc((y(x))\frac{d}{dx}(y(x))) - (\csc(y(x)+1)\csc^2(y(x))\frac{d}{dx}(y(x)))}{(\csc(y(x))+1)^2}$$

$$\Leftrightarrow \frac{(-\csc(y(x))+1)\csc^2(y(x))\frac{d}{dx}(y(x)) + \cot^2(y(x))\csc(y(x))\frac{d}{dx}(y(x))}{(\csc(y(x))+1)^2}$$

$$= -\frac{\frac{d}{dx}(y(x))}{\sin(y(x))+1}$$

$$\Leftrightarrow \frac{d}{dx}\left(\frac{\cot(y(x))}{\csc(y(x)+1)}\right) = -\frac{\frac{d}{dx}(y(x))}{\sin(y(x))+1}$$

$$\Leftrightarrow 2x = \frac{-\frac{dy}{dx}}{\sin(y)+1}$$

$$\Leftrightarrow \frac{dy}{dx} = -2x(\sin(y)+1).$$

a, $3x^2 - 4y^2 = 7.$

3 $\frac{d}{dx}(3x^2 - 4y^2) = \frac{d}{dx}(7)$

$\Leftrightarrow 6x - 8y\frac{dy}{dx} = 0.$

$\Rightarrow \frac{dy}{dx} = \frac{3x}{4y}$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{3x}{4y}\right)$

$\frac{d^2y}{dx^2} = \frac{3.4y - 3x \cdot 4\frac{dy}{dx}}{16y^2}$

$= \frac{12y - 9x^2/y}{16y^2}$

$= \frac{3(4y^2 - 3x^2)}{16y^3}$

b, $2xy - y^2 = 3.$

$\frac{d}{dx}(2xy - y^2) = \frac{d}{dx}(3)$

$2y + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0.$

$\Leftrightarrow \frac{dy}{dx} = \frac{-y}{x-y} = \frac{y}{y-x}$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y}{y-x}\right)$

$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy}{dx}\right)(y-x) - y\cdot\left(\frac{dy}{dx} - 1\right)}{(y-x)^2}$

$= \frac{y - y\left(\frac{y+x-y}{y-x}\right)}{(y-x)^2}$

$= \frac{y - \frac{yx}{y-x}}{(y-x)^2} = \frac{y^2 - 2xy}{(y-x)^3}.$

10

a, $y = \ln 2x$

$$\frac{dy}{dx} = \frac{1}{x}$$

b, $y = \sqrt[3]{\ln x + 1} = (\ln x + 1)^{1/3}$

$$\rightarrow \frac{dy}{dx} = \frac{1}{3}(\ln x + 1)^{-2/3} \cdot \frac{1}{x}$$

c, $y = \log(\ln x)$

$$\frac{dy}{dx} = \frac{1}{\ln x} \cdot \frac{d}{dx}(\ln(x))$$

$$= \frac{\frac{1}{x}}{\ln x} = \frac{1}{x \ln x}$$

d, $y = \ln\left(x^{3/2}\sqrt{1+x^4}\right)$

$$= \frac{3}{2} \cdot \ln x + \frac{1}{2}\ln(1+x^4)$$

$$\frac{dy}{dx} = \frac{3}{2x} + \frac{2x^3}{1+x^4}$$

$$= \frac{7x^4+3}{2x+2x^5}$$

e, $y = e^{\ln(x^2+1)}$

$$\frac{dy}{dx} = \frac{2x}{x^2+1} \cdot e^{\ln(x^2+1)}$$

f, $y = 2xe^{\sqrt{x}}$

$$\frac{dy}{dx} = 2e^{\sqrt{x}} + 2x \cdot \frac{1}{2\sqrt{x}} \cdot e^{\sqrt{x}}$$

g, $y = x^{(e^x)}$

$$\frac{dy}{dx} = e^x \cdot x^{e^x}$$

h, $y = (\ln x)^2$

$$\rightarrow \frac{dy}{dx} = 2\ln x \cdot \frac{1}{x}$$

i) $y = \ln\left(\sqrt[3]{x+1}\right)$

$\rightarrow \frac{dy}{dx} = \frac{1/3\,(x+1)^{-2/3}}{\sqrt[3]{x+1}}$

j) $y = \frac{1+\log x}{1-\log x}$

$\rightarrow \frac{dy}{dx} = \frac{\frac{1}{x}(1-\log x) - (1+\log x)\cdot\left(\frac{-1}{x}\right)}{(1-\log x)^2}$

$= \frac{\frac{\ln x + 1}{x} + \frac{1-\ln x}{x}}{(1-\ln x)^2}$

$= \frac{\ln x + 1}{x(1-\ln x)^2} + \frac{1}{x(1-\ln x)}$

$= \frac{2}{x(1-\ln x)^2}$

k) $y = \ln\left(\frac{\sqrt{x}\cos x}{1+x^2}\right)$

$= \ln\sqrt{x} + \ln\cos x - \ln(1+x^2)$

$\rightarrow \frac{dy}{dx} = \frac{1}{2x} + \frac{-\sin x}{\cos x} - \frac{2x}{1+x^2} = \frac{1-3x^2}{2x+2x^3} - \tan x.$

l) $y = \ln\left(\frac{1+e^x+e^{2x}}{1-e^{3x}}\right)$

$= \ln(1+e^x+e^{2x}) - \ln(1-e^{3x}).$

$\rightarrow \frac{dy}{dx} = \frac{e^x+2e^{2x}}{1+e^x+e^{2x}} - \frac{-3e^{3x}}{1-e^{3x}}$

$= \frac{-(e^x+2e^{2x})(1-e^x)+3e^{3x}}{1-e^{3x}}$

$= \frac{e^x+e^{2x}+e^{3x}}{1-e^{3x}} = \frac{e^x}{1-e^x}$

n, $y = (1+x)^{1/x}$

$$\to \frac{dy}{dx} = (1+x)^{1/x} \cdot \frac{d}{dx}\left(\ln(x+1) \cdot \frac{1}{x}\right)$$

$$= (1+x)^{1/x} \cdot \frac{d}{dx}\left(\frac{\ln(x+1)}{x}\right)$$

$$= \frac{(x+1)^{1/x} \cdot \left(\frac{x}{x+1} - \ln(x+1)\right)}{x^2}$$

$$= (x+1)^{1/x} \cdot \left(\frac{1}{x(x+1)} - \ln\frac{(x+1)}{x^2}\right).$$

9 a, $\ln\left(\frac{(x+1)(x+2)^2}{(x+3)^3(x+4)^4}\right)$

$= \ln(x+1) + 2\ln(x+2) - 3\ln(x+3) - 4\ln(x+4).$

$$\to \frac{dy}{dx} = \frac{1}{x+1} + \frac{2}{x+2} - \frac{3}{x+3} - \frac{4}{x+4}$$

b, $\ln\left(\frac{\sqrt{x}\sqrt[3]{x+1}}{\sin x \cdot \sec x}\right)$

$$= \frac{1}{2}\ln x + \frac{1}{3}\ln(x+1) - \ln(\sin x) - \ln(\sec x)$$

$$\to \frac{dy}{dx} = \frac{1}{2x} + \frac{1}{3(x+1)} - \cot x - \tan x$$

11 a, $y = \frac{x^3}{\sqrt{x^2+1}}$

$$\log y = \log\frac{x^3}{\sqrt{x^2+1}}$$

$$= 3\log x - \frac{1}{2}\log(x^2+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - \frac{x}{x^2+1} = \frac{2x^2+3}{x^3+x}$$

$$\frac{dy}{dx} = \frac{(2x^2+3)y}{x^3+x}$$

b, $y = \sqrt[3]{\dfrac{x^2-1}{x^2+1}}$

$\log y = \dfrac{1}{3} \log \dfrac{x^2-1}{x^2+1}$

$= \dfrac{1}{3} \log(x^2-1) - \dfrac{1}{3} \log(x^2+1)$

$\dfrac{1}{y} \cdot \dfrac{dy}{dx} = \dfrac{2x}{3(x^2-1)} - \dfrac{2x}{3(x^2+1)} = \dfrac{x^3+x-2x^3+2x}{3(x^4-1)}$

$\rightarrow \dfrac{dy}{dx} = \dfrac{4xy}{3(x^4-1)}$

12 +, Let $y(x) = e^{3x} \rightarrow y'(x) = 3e^{3x}$

+, The tangent line to $e^{3x}$ at $x = a \rightarrow k = 3e^{3a}$

$\Rightarrow y - e^{3a} = 3e^{3a}(x-a) = 3e^{3a}x - 3ae^{3a}$

$\Rightarrow y = 3e^{3a}x - 3ae^{3a} + e^{3a}$

$= e^{3a}(3x - 3a + 1)$.

+, Pass through the origin $\rightarrow e^{3a}(1-3a) = 0$.

$\Rightarrow 1 - 3a = 0 \quad (e^{3a} \neq 0)$.

$\Leftrightarrow a = \dfrac{1}{3}$.

$y\left(\dfrac{1}{3}\right) = e$.

+, The point whose tangent line passes through $(0;0)$ is $A\left(\dfrac{1}{3}; e\right)$.

13. $y = e^{ax} \sin bx$

+, $y' = ae^{ax} \sin bx + e^{ax} . b\cos bx$.

+, $y'' = a^2 . e^{ax} \sin bx + ae^{ax} . b\cos bx + ae^{ax} b\cos bx - e^{ax} b^2 \sin bx$.

$= (a^2 - b^2) e^{ax} \sin bx + 2ab\, e^{ax} . \cos bx$.

+, $y'' - 2ay' + (a^2+b^2)y$

$= (a^2-b^2)e^{ax} \sin bx + 2abe^{ax} . \cos bx - 2a(ae^{ax} \sin bx + e^{ax} b\cos bx) + (a^2+b^2)e^{ax} \sin bx = 0$ (đpcm).

14 a, $\lim_{h\to 0} \frac{(1+h)^{\pi}-1}{h}$

$= \lim_{h\to 0} \frac{\pi(h+1)^{\pi-1}}{1}$

$= \frac{\pi(0+1)^{\pi-1}}{1} = \pi.$

b, $\lim_{x\to e} \frac{1-\ln x}{(x-e)\ln x}$

$= \lim_{x\to e} \left(\frac{1-\ln x}{x-e} \cdot \frac{1}{\ln x}\right)$

$= \lim_{x\to e} \left(\frac{1-\ln x}{x-e}\right) \cdot \lim_{x\to e}\left(\frac{1}{\ln x}\right)$

$= \frac{-1}{e}.$

15 a, $\lim_{x\to +\infty} (e^x - x^2)$

$= \lim_{x\to +\infty} \left(e^x\left(1-\frac{x^2}{e^x}\right)\right) = \infty.$

b, $\lim_{x\to 1} \sqrt{\frac{\ln x}{x^4-1}}$

$= \sqrt{\lim_{x\to 1} \frac{\ln x}{x^4-1}} = \sqrt{\lim_{x\to 1} \frac{\frac{1}{x}}{4x^3}} = \frac{1}{2}$

c, $\lim_{x\to 0} \frac{x^2 \cdot e^x}{\sin^2 3x}$

$= \lim_{x\to 0} \frac{2xe^x + e^x \cdot x^2}{3\sin 6x} = \lim_{x\to 0} \frac{e^x \cdot x^2 + 4e^x \cdot x + 2e^x}{18\cos 6x} = \frac{1}{9}.$

d, $\lim_{x\to e} \frac{a^x-1}{x}$

$= \frac{a^e-1}{e}$

4 +) $y = x\tan\left(\frac{\pi x}{2}\right)$

$y' = \tan\frac{\pi x}{2} + x \cdot \frac{\pi \sec^2\left(\frac{\pi x}{2}\right)}{2}$

+) $\left(-\frac{1}{2}; \frac{1}{2}\right) \rightarrow y' = \tan\frac{\pi}{4} + \frac{\pi}{4} \cdot \sec^2\left(\frac{\pi}{4}\right)$.

$= \frac{\pi}{2} + 1$.

5 +) $y^2 = 2x^3$

$\rightarrow y = \sqrt{2x^3}$

$\rightarrow \frac{dy}{dx} = \frac{6x^2}{2\sqrt{2x^3}} = \frac{3x^2}{\sqrt{2x^3}}$

+) $4x - 3y + 1 = 0$.

$\rightarrow y = \frac{4}{3}x + \frac{1}{3}$.

+) $\frac{3x^2}{\sqrt{2x^3}} \cdot \frac{4}{3} = -1$.

$\rightarrow 4x^2 = -2\sqrt{x^3}$

$\rightarrow x = 0$.

+) $A\left(0; \frac{1}{3}\right)$.

6 +) $x^3 - xy + y^3 = 0$.

$\rightarrow \frac{dy}{dx}(x^3 - xy + y^3) = 0$.

$\Leftrightarrow 3x^2 - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$.

$\Leftrightarrow (3y^2 - x)\frac{dy}{dx} = y - 3x^2$.

$\Leftrightarrow \frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}$

+) The tangent line is parallel to the x exist y the slope of the tangent is zero.

$\Rightarrow y - 3x^2 = 0$.
$\Leftrightarrow 3x^2 = y$.

+) $A(t; 3t^2)$
$\rightarrow t^3 - 3t^3 + 27t^6 = 0$.
$\Leftrightarrow 27t^6 - 2t^3 = 0 \quad \Leftrightarrow \begin{cases} t = 0 \\ t = \dfrac{\sqrt[3]{2}}{3} \end{cases} \rightarrow x = \dfrac{\sqrt[3]{2}}{3}, \; y = \sqrt[3]{4}$.

+) So $A\left(\dfrac{\sqrt[3]{2}}{3}; \sqrt[3]{4}\right)$

7 $\dfrac{dy}{dx} = \dfrac{y - 3x^2}{3y^2 - x}$

+) $3y^2 - x = 0 \quad \rightarrow x = 3y$.
+) $A(3t^2; t)$.

$\rightarrow 27t^6 - 2t^3 = 0 \quad \Rightarrow t = \dfrac{\sqrt[3]{2}}{3}$

+) So $A\left(\sqrt[3]{4}; \dfrac{\sqrt[3]{2}}{3}\right)$